On the relation between geometric and deformation quantization

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Abstract

In this paper we investigate the possibility of constructing a complete quantization procedure consisting of geometric and deformation quantization. The latter assigns a noncommutative algebra to a symplectic manifold, by deforming the ordinary pointwise product of functions, whereas geometric quantization is a prescription for the construction of a Hilbert space and a few quantum operators, starting from a symplectic manifold. We determine under which conditions it is possible to define a representation of the deformed algebra on this Hilbert space, thereby to extend the small class of quantizable observables in geometric quantization to all smooth functions, as well as to give a natural representation of the algebra. In particular we look at the special cases of a cotangent bundle and a Kähler manifold.
1 Introduction

There are two important concepts for quantization of arbitrary symplectic manifolds \((M,\omega)\), or phase spaces. Geometric quantization was developed in an attempt to unify several special quantization schemes that arose in applications (see [24] and references therein), and like in ordinary quantum mechanics on a flat phase space, physical states are elements of a Hilbert space \(\mathcal{H}\), explicitly represented as wave functions on a subspace of phase space. This subspace is not uniquely determined, but depends on the choice of a polarization, although for finite-dimensional spaces different polarizations are supposed to lead to equivalent theories, like position and momentum space representations. There is a problem with the observables though, because for a fixed polarization only a very limited class of them can be quantized.

On the other hand there is deformation quantization [2, 3, 10], whose philosophy is to introduce a new product \(\ast\), called a star product, on some function algebra over \(M\), say \(C^\infty(M)\). The condition for the star product to be a deformation of the ordinary product is expressed by demanding

\[
    f \ast g = fg + \mathcal{O}(\hbar)
\]

for all \(f, g \in C^\infty(M)\), so that Planck’s constant \(\hbar\) plays the role of a deformation parameter. Further the condition

\[
    [f, g]_\ast := f \ast g - g \ast f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)
\]

is imposed, where \(\{\cdot, \cdot\}\) denotes the classical Poisson bracket, generalizing Dirac’s quantum condition [9]. Kontsevich [18] has proved that any Poisson manifold admits a star product, but for the case of a symplectic manifold there exists a much simpler construction, due to Fedosov [11, 12]. We will consider only the symplectic case here. It is notable that the star product is not uniquely determined by \((M,\omega)\) either, see the brief review of Fedosov’s construction in section 3.

Giving the full construction of the algebra of quantum observables \(\mathcal{A}\), deformation quantization has the drawback not to offer a physical interpretation of the states. As for \(C^\ast\) algebras states can be defined as positive linear functionals on the algebra [5], but they do not have a natural identification as square roots of volume elements on ’configuration space’, as in geometric quantization.

Therefore it would be desirable to have a representation of the deformed algebra \(\mathcal{A}\) on the Hilbert space \(\mathcal{H}\), thus offering a physical interpretation of states in deformation quantization, and extending the algebra of observables in geometric quantization. As the latter already defines a quantization map \(\rho’\) for a subset of \(C^\infty(M)\), one might hope to find a unique extension \(\rho\) of \(\rho’\) to all of \(\mathcal{A}\) by requiring it to be an algebra homomorphism. There is then an obvious compatibility condition to be fulfilled, viz. that the original \(\rho’\) respects the star product:

\[
    \rho'(f \ast g) = \rho'(f) \ast \rho'(g)
\]

for any \(f, g \in C^\infty(M)\) with the property that \(f, g\) and also \(f \ast g\) are geometric quantizable. As neither geometric nor deformation quantization are uniquely determined by \(\omega\), but depend on further structure, one should not expect equation (1.1) to hold for any arbitrary choice of these structures, but rather to give a compatibility condition for them. The best possible result would then be to get a unique (class of) solution(s) to the compatibility condition, thus fixing the ambiguities in the separate quantization procedures.

Usually deformation and geometric quantization are considered as competing theories, and deformation quantization is often referred to as the more promising candidate for a consistent theory of quantization, e.g. in [10]. A point of view similar to ours is taken by Landsman in [20].
We will restrict our attention to two important special cases, a cotangent bundle $T^*Q$ with the canonical two-form, and a Kähler manifold. For cotangent bundles, equation (1.1) is a direct consequence of a result in [4], whereas for Kähler manifolds with their most natural polarization and connection, a full proof remains out of reach. Therefore, we restrict ourselves to providing some evidence in favor of compatibility.

2 Geometric quantization

This section contains a concise summary of the theory, as it is presented in the monograph [24]. Starting with a symplectic manifold $(M, \omega)$, our aim is the construction of a Hilbert space $\mathcal{H}$ and a quantization map $f \mapsto \hat{f}$, assigning an operator on $\mathcal{H}$ to a smooth function $f$ on $M$. The following conditions are to be satisfied:

(i) linearity: $\lambda \hat{f} + \hat{g} = \lambda \hat{f} + \hat{g}$, $\lambda \in \mathbb{C}$.

(ii) $f$ constant implies $\hat{f} = f 1$,

(iii) the correspondence principle: $[\hat{f}, \hat{g}] = i\hbar \{f, g\}$.

There is a simple construction obeying these conditions, consisting of a Hermitian line bundle $B \to M$ with metric connection of curvature $-\frac{i}{\hbar} \omega$, the Hilbert space $\mathcal{H} = L^2(M, B)$ and

$$\hat{f} = -i\hbar \nabla_{X_f} f + f 1, \quad f \in C^\infty(M). \quad (2.1)$$

The Hamiltonian vector field $X_f = df^\sharp$ is defined by means of the musical isomorphism

$$\sharp : T^*M \to TM, \quad \eta(\phi^\sharp) = \omega^{-1}(\eta, \phi) = \omega^{ab} \eta_a \phi_b,$$

and $\nabla$ is the connection on $B$. In a trivialization it is expressed as $\nabla = d - \frac{i}{\hbar} \theta$, where $\theta$ is a symplectic potential, obeying $d\theta = \omega$. $B$ is called a pre-quantum bundle.

It turns out that the Hilbert space $L^2(M, B)$ is too large, as it contains functions of both $q$ and $p$ in the flat case, and gives rise to a reducible representation of the Heisenberg algebra. A method to reduce degrees of freedoms is needed, and this is provided by a polarization:

**Definition 2.1** (polarization). A polarization of a symplectic manifold $(M, \omega)$ is an integrable subbundle of $T^*_CM$ (also called an involutive distribution), with the properties that

1. Every subspace $P_m \subset T_m^*M$ is a Lagrange subspace, i.e. the symplectic complement

   $$P_m^\perp = \{ v \in T_m^*M \mid \omega(v, p) = 0 \forall p \in P_m \}$$

   satisfies $P^\perp = P$. In particular this implies $\dim \mathbb{C}P = \dim \mathbb{R}M/2$.

2. The involutive distribution $D = (P \cup \overline{P}) \cap TM$ has constant real dimension $d$, which is called the real index of $P$.

3. The distribution $E = (P + \overline{P}) \cap TM$ is involutive.

A function $f \in C^\infty(M)$ is called polarized if $X(f) = 0$ holds for any $X \in \Gamma(P)$. A symplectic potential $\theta$ is called adapted to $P$, if $\theta(X) = 0$ for $X \in \Gamma(P)$.

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It is non-standard to include the requirement that \( E \) be involutive into the definition, usually one speaks of a strongly integrable polarization [24], or a Nirenberg integrable subbundle [14] in case it is satisfied. However, this condition is essential for our purposes, and implies the existence of an adapted symplectic potential. If \( d = n = \dim M/2 \) the polarization is called real, and is the complexification of a real subbundle of \( TM \). At the other extreme there are the totally complex polarizations with \( d = 0 \). They have the property \( E = TM \), and any vector \( X \in T_m M \) can be uniquely decomposed as \( X = Z + \overline{Z} \), where \( Z \in P_m \). Then
\[
J : TM \to TM, \quad Z + \overline{Z} \mapsto iZ - i\overline{Z}
\]
defines a complex structure on \( TM \), which is compatible with \( \omega \), i.e. is a symplectic transformation. \( J \) in turn induces a semi-Riemannian metric
\[
g(X, Y) = \omega(X, JY)
\]
on \( TM \), which, if positive definite, defines a Kähler structure on \( M \). On the other hand, every Kähler manifold has a canonical polarization: \( P_m = \{ X - iJX \mid X \in T_m M \} \). The main example of a real polarization is given by the vertical foliation of a cotangent bundle, where \( P_m \) is the vertical, or fibre-parallel subspace of \( T^*_m (T^*Q) \).

**Definition 2.2** (adapted coordinates). Let \( P \subset T\mathbb{C}M \) be a polarization, then in a neighborhood of any point of \( M \) there are coordinates \( q^i, p_j, z^\alpha \), where \( q, p \) are real and \( z \) complex, such that \( P \) is spanned by the \( \partial p_j \) and \( \partial z^\alpha \). Further, a standard form for \( \omega \) can be obtained ([24], p. 97), which for a real polarization takes the form \( \omega = dp_i \wedge dq^i \) (then \( q, p \) are called Darboux coordinates), and for a Kähler polarization
\[
\omega = i \frac{\partial^2 K}{\partial z^\alpha \partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta.
\]
The real function \( K \) is called a Kähler potential. These coordinates are said to be adapted to \( P \).

In the real case a polarized function depends only on the \( q \)s, whereas a polarized function on a Kähler manifold is holomorphic.

**Definition 2.3** (canonical bundle). Let \( P \) be a polarization of \( (M, \omega) \), \( \dim M = 2n \), then the canonical bundle \( K_P \to M \) is a complex line bundle over \( M \), with fibre
\[
(K_P)_m = \{ \alpha \in \Lambda^n T^*_m \mathbb{C}M \mid \iota_{\overline{X}} \alpha = 0 \ \forall X \in \Gamma(P) \}.
\]
(2.2)
\( \iota_{\overline{X}} \) denotes contraction with the vector \( \overline{X} \).

We also consider \( K \) as a bundle over the space \( LM \) of all nonnegative polarizations of \( M \) (see [24]), with fibre \( K_P \) at \( P \in LM \). A square root of \( K \to LM \) is a line bundle \( \delta \to LM \) satisfying
\[
\delta^2 := \delta \otimes \delta = K.
\]
According to Kostant such a bundle exists iff \( M \) admits a metaplectic structure [19]. In the following we will assume that \( M \) carries a metaplectic structure, and hence a square root \( \delta \) of \( K \). To a given (nonnegative) polarization \( P \) we assign the so called half-form bundle \( \delta P \to M \), with fibre \((\delta P)_m \) at \( m \in M \). Here \( LM \) is considered as a bundle over \( M \), whose fibre at \( m \) consists of all polarizations of \( T_m \mathbb{C}M \).

The Lie derivative of a vector field \( X \) preserves sections of \( K_P \) iff the flow of \( X \) preserves the
polarization $P$, which is equivalent to $[X,Y] \in \Gamma(P)$ for any $Y \in \Gamma(P)$. In this case the Lie derivative $\mathcal{L}_X$ is transferred from $K_P$ to $\delta_P$ by
\[
2(\mathcal{L}_X \nu) = \mathcal{L}_X \nu^2, \quad \forall \nu \in \Gamma(\delta_P).
\]
On $K_P$ the partial connection $\nabla$ is defined by
\[
\nabla_X \beta = \iota_X d\beta, \quad \forall X \in \Gamma(P), \quad \beta \in \Gamma(K_P),
\]
which is also transferred to $\delta_P$ by the requirement $2(\nabla_X \nu) = \nabla_X \nu^2$. There is a natural sesquilinear product $(\cdot, \cdot)$ on $\delta_P$, taking values in the densities on $M/D$, for the construction see [21], p.230 (assuming that $M/D$ is a Hausdorff manifold). We consider the bundle $B_P = B \otimes \delta_P$, equipped with the partial connection induced by the connection on $B$ and the partial connection on $\delta_P$, and take as state space the set of 'polarized wave functions'
\[
\Gamma_P(B_P) = \{ s \in \Gamma(B_P) \mid \nabla_X s = 0 \forall X \in \Gamma(P) \}.
\]
The prescription
\[
\langle \psi \otimes \mu, \phi \otimes \nu \rangle = \int_{M/D} (\psi, \phi)(\mu, \nu)
\]
defines an inner product on $\Gamma_P(B_P)$. Our Hilbert space $\mathcal{H}_P$ is the set of square integrable elements of $\Gamma_P(B_P)$. The quantization of an observable $f$ becomes
\[
\hat{f}(\psi \otimes \nu) = (\hat{f} \psi) \otimes \nu - i\hbar \psi \otimes \mathcal{L}_X \nu,
\]
in case $X_f$ preserves the polarization. For $P$ real, these are functions of the form
\[
f(q,p) = a^i(q)p_i + b(q),
\]
in adapted coordinates, whereas on Kähler manifolds with holomorphic polarization the condition implies
\[
f(z, \bar{z}) = a^a(z) \frac{\partial K}{\partial z^a} + v(z).
\]
**Example 2.4.** On a cotangent bundle $M = T^*Q$ over a (semi-)Riemannian manifold $(Q, g)$, with coordinates $q$ on $Q$ and $(q,p = dq^i)$ on $M$, an adapted potential is given by $\theta = p_i dq^i$. In this gauge, wave functions can be expressed as $\psi \otimes \sqrt{\mu}$, where $\psi = \psi(q)$ is polarized, and $\mu = \sqrt{|\det g(q)|} dq^i$. Then we have for $f$ as in (2.8):
\[
\hat{f}(\psi \otimes \sqrt{\mu}) = \frac{\hbar}{i} \left( a^i \partial_j \psi + \left( b + \frac{1}{2} \text{div}(a) \right) \psi \right) \otimes \sqrt{\mu},
\]
where div($a$) is defined by $\mathcal{L}_a \mu = d(u_a \mu) =: \text{div}(a) \mu$. In particular this gives (neglecting $\sqrt{\mu}$)
\[
\tilde{q}^a \psi(q) = q^a \psi(q), \quad \text{and} \quad \tilde{p}_a \psi(q) = -i\hbar \left( \partial_a + \frac{1}{4} g^{bc} \partial_a g_{bc} \right) \psi(q).
\]
A monomial $q^{i_1} \ldots q^{i_k} p_j$ is mapped to the symmetrized product of the operators $\tilde{q}^{i_1}, \ldots, \tilde{q}^{i_k}, \tilde{p}_j$, reproducing the Weyl ordering convention.

**Example 2.5.** On a Kähler manifold with adapted potential $\theta = -i \frac{\partial K}{\partial z^a} dz^a$ and half-form $\nu = \sqrt{\mu} dz^a$, wave functions are holomorphic, and the metric on the pre-quantum bundle is $\langle \psi, \phi \rangle = e^{-K/\hbar} \psi \phi$. One gets $\tilde{z}^a \psi(z) = z^a \psi(z)$, and the function $f(z, \bar{z}) = \frac{\partial K}{\partial z^a}$ is quantized as:
\[
\hat{f}(z) = \hbar \partial_a \psi(z).
\]
Again, an observable linear in $\partial_a K$ gives rise to a symmetrized operator.
3 Deformation quantization

The purpose of deformation quantization is to construct a star product on $C^\infty(M)$, i.e. a deformation of the algebra of classical observables on a symplectic manifold. Our main references are [11, 12]. The following conditions are supposed to hold, where $h$ is considered as a formal deformation parameter [2, 10]:

1. the coefficients $c_k$ of the product of $f = \sum_k h^k f_k$ and $g = \sum_k h^k g_k$:
   \[ c = f \ast g = \sum_{k=0}^\infty h^k c_k(f, g) \]
   are bi-differential operators (of finite order).
2. $c_0(x) = f_0(x)g_0(x)$, i.e. $\ast$ is a deformation of the ordinary pointwise product on $C^\infty(M)$.
3. the correspondence principle
   \[ [f, g]_\ast = f \ast g - g \ast f = i\hbar\{f_0, g_0\} + O(h^2), \]
   holds, where $\{f, g\} = \omega^{ab} \partial_a f \partial_b g$ denotes the poisson bracket defined by $\omega$.

Fedosovs construction of $\ast$ makes use of a bijection from $C^\infty(M)$ to the set of flat, smooth sections of a vector bundle, we give a brief review of the method.

First we need a symplectic (torsion-free) connection on the tangent bundle $TM$, i.e. a covariant derivative $\nabla$ respecting $\omega$:

\[ d(\omega(X, Y)) = \omega(\nabla X, Y) + \omega(X, \nabla Y), \]

or $\nabla \omega = 0$ for the induced connection on $T^*M \otimes T^*M$. Such a connection always exists, but contrary to the Riemannian (symmetric) case it is not unique. We will later comment on the correct choice of $\nabla$. In local Darboux coordinates it takes the form $\nabla = d + \Gamma$, where the connection form $\Gamma$ is a 1-form with values in the symplectic Lie algebra, i.e. $\Gamma \in \Omega^1(U; \mathfrak{sp}(2n))$. As usual, its components are defined by $\Gamma^k_{ij} e_k = \Gamma(e_i) \cdot e_j$, where $e_i (i = 1, \ldots, 2n)$ is the local Darboux basis of $TM$, $e_i = \partial_{q^i}$, and $e_{n+i} = \partial_{p^i}$ for $i = 1, \ldots, n$.

Further we set $\Gamma_{ijk} := \omega_{il} \Gamma^l_{jk}$, which is totally symmetric in its three indices due to $\Gamma$ being torsion-free and symplectic. The components of the curvature tensor are

\[ R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km} \Gamma^m_{lj} - \Gamma^i_{lm} \Gamma^m_{kj}, \]

and $R_{ijkl} = \omega_{lm} R^{lm}_{ijkl}$ is symmetric in the first two (Lie algebra) indices, and antisymmetric in the last two (differential form) indices. $\nabla$ also induces a connection on $T^*M$, with connection form $-\Gamma^T$, and on the whole tensor algebra through the Leibniz formula. Particularly we are interested in the symmetric algebra $\mathcal{W} := \text{Sym}_C(T^*M)$; instead of the common $dx^i$ we use the symbols $y^i$ to denote the local basis of $T^*M$, and suppress the symmetric tensor product sign: $y^i y^j = y^j y^i := \frac{1}{2}(y^i \otimes y^j + y^j \otimes y^i)$. In some cases we need the product bundle $\mathcal{W} \otimes \Lambda := \Lambda(\mathcal{W})$ and $\Omega(\mathcal{W}) = \Gamma(\mathcal{W}) \otimes \Lambda$, where the basis of $T^*M$ considered as a subset of $\Lambda(T^*M)$ is denoted by $dx^i$. Now we introduce a new product $\circ$ on $W_m = \text{Sym}_C(T^*_m M)$, through the relations

\[ [v, w]_\circ = v \circ w - w \circ v = i\hbar \omega^{-1}(v, w), \quad v \circ w + w \circ v = 2vw, \quad \forall v, w \in T^*_m M. \tag{3.1} \]

Then we have $[y^i, y^j] = i\hbar \omega^{ij}$, so that the $y^i$ can be identified as local position and momentum operators. $(W_m, \circ)$ is called the Weyl algebra associated to $(T^*_m M, \omega^{-1}_m)$, it is the symplectic analog of a Clifford algebra. On $\Omega(\mathcal{W})$ we write $\circ$ for $\circ \otimes \Lambda$ and use the graded commutator

\[ [\xi, \eta] = \xi \circ \eta - (-1)^{pq} \eta \circ \xi \]
for $\xi \in \Omega^q(W)$ and $\eta \in \Omega^p(W)$. The connection is extended to $\Omega(W)$ by

$$\nabla(\xi \circ \eta) = \nabla \xi \circ \eta + (-1)^q \xi \circ \nabla \eta, \quad \xi \in \Gamma(W \otimes \Lambda^q)$$

$$\nabla(\phi \wedge \eta) = d\phi \wedge \eta + (-1)^q \phi \wedge \nabla \eta, \quad \phi \in \Omega^q(M).$$

Explicitly we have

$$\nabla y^{i_1} \ldots y^{i_k} = - \sum_j \Gamma_{ab}^{ij} y^{i_1} \ldots \tilde{y}^{ij} \ldots y^{i_k} y^a dx^b,$$

where $\tilde{y}^{ij}$ means omitting the element, and which can be written in the form

$$\nabla = d + \{dU(\Gamma), \cdot\} := d - \frac{i}{2\hbar} \Gamma_{ijk} y^i y^j \cdot dx^k.$$  (3.2)

Here $dU$ is the isomorphism between the symplectic and metaplectic Lie algebras $[13]$:

$$dU : \mathfrak{sp}(2n) \to \mathfrak{mp}(2n), \quad A \mapsto -\frac{i}{2\hbar} \omega_{ij} A^j k y^i y^k.$$  (3.3)

Now we introduce two further operators on $\Omega(W)$:

$$\delta = dx^k \wedge \frac{\partial}{\partial y^k}, \quad \delta^* = y^k \imath \left( \frac{\partial}{\partial y^k} \right),$$

where $y^k$ denotes (commutative) multiplication with $y^k$, and the contraction $\imath \left( \frac{\partial}{\partial y^k} \right)$ acts only on the form part. In brief, $\delta$ replaces one of the $y^i$ by $dx^i$, whereas $\delta^*$ replaces $dx^i$ by $y^i$. One easily checks

**Lemma 3.1.**

1. $\delta^2 = (\delta^*)^2 = 0$

2. Applied to $y^{i_1} \ldots y^{i_l} dx^{j_1} \wedge \cdots \wedge dx^{j_p}$ the following identity holds:

$$\delta \delta^* + \delta^* \delta = (l + p)\text{id.}$$  (3.4)

We also define $\delta^{-1}$ by

$$\delta^{-1} = \frac{1}{l + p} \delta^*$$  (3.5)

for $l + p > 0$, and $\delta^{-1} = 0$ otherwise. If the projection of an element $\xi \in \Omega(W)$ to its part in $C^\infty(M) \subset \Omega(W)$ is denoted by $\xi_{00}$, then the following decomposition holds, analogously to the Hodge-de Rahm decomposition of forms:

$$\xi = \delta \delta^{-1} \xi + \delta^{-1} \delta \xi + \xi_{00}.$$  (3.6)

From now on we consider elements of $\Omega(W)$ as formal series in $\hbar$ (replace $W = \text{Sym}_C(T^*M)$ by $W = \text{Sym}_C(T^*M)[[\hbar]]$, and also allow for an infinite number of homogeneous elements in the $y^i$. Then elements of $\Omega(W)$ have the expansion

$$\xi = \sum_{k,l,m \geq 0} \hbar^k \xi_{k;i_1 \ldots i_l;k_1 \ldots k_m} y^{i_1} \ldots y^{i_l} dx^{j_1} \wedge \cdots \wedge dx^{j_m}.$$  (3.7)
Definition 3.2. The $h$-degree of a homogeneous element

$$\sum_m h^k \xi_{k; i_1, \ldots, i_n} y^{i_1} \cdots y^{i_n} dx^{j_1} \wedge \cdots \wedge dx^{j_m}$$  \hspace{1cm} (3.7)

(no sum over $k, l$) is defined as $k + l/2$.

Due to the relations (3.1) $\circ$ respects the gradation defined by the $h$-degree. The subspaces of homogeneous elements of $h$-degree $j$ are denoted by $\mathcal{W}_j$. Now $\mathcal{W}$ and $\Omega(\mathcal{W})$ carry two gradations, the other one being defined by the degree in $\text{Sym}^k(T^*M)$, which is respected by the symmetric tensor product, but not by $\circ$. The corresponding projections are

$$\pi^j_h : \mathcal{W} \rightarrow \mathcal{W}_j, \quad \text{and} \quad \pi^k_\circ : \mathcal{W} \rightarrow \text{Sym}^k(T^*M)[[h]],$$  \hspace{1cm} (3.8)

we also adopt the convention to denote $\pi^0_\circ(\xi)$ as $\xi_0$. It is important to note that $\delta$ decreases the $h$-degree by $1/2$, whereas $\delta^*$ increases it.

Now we come to the construction of a flat connection on $\mathcal{W}$, with curvature $\Omega = \frac{i}{\hbar}[\omega, \cdot] = 0$. Making the ansatz

$$D = \nabla + \frac{i}{\hbar}[\gamma, \cdot] = d + [dU(\Gamma) + \frac{i}{\hbar}\gamma, \cdot]$$  \hspace{1cm} (3.9)

with an as yet undetermined 1-form $\gamma \in \Omega^1(\mathcal{W})$, we obtain for the curvature $\Omega = D^2 = \frac{i}{\hbar}[\tilde{\Omega}, \cdot]$, with

$$\tilde{\Omega} = \frac{\hbar}{i} dU(R) + \nabla \gamma + \frac{i}{\hbar} \gamma^2.$$  \hspace{1cm} (3.10)

Here $R$ denotes the curvature 2-form of $\nabla$, $dU(R) = -\frac{1}{2m} R_{ijkl} y^i y^j dx^k \wedge dx^l$, and $\gamma^2 = \gamma \circ \gamma$. Of course, $\gamma$ is only determined up to addition of a scalar form. We require the normalization $\gamma_0 = 0$.

In the flat case $M = \mathbb{R}^{2n}$ with standard symplectic form $\omega = dp_i \wedge dq^i$ we can choose $\Gamma = 0$, and $\gamma = \omega_{ab} y^b dx^a$ leads to $\tilde{\Omega} = \frac{i}{\hbar} \omega$. Therefore, in the general case we split $\gamma$ as

$$\gamma = \omega_{ab} y^b dx^a + r.$$  \hspace{1cm} (3.11)

Observing that $\delta$ can be written in the form $\delta \xi = -\frac{i}{\hbar} \omega_{ab} dx^a [y^b, \xi]$, we obtain for the curvature

$$\tilde{\Omega} = \omega + \frac{\hbar}{i} dU(R) - \delta r + \nabla r + \frac{i}{\hbar} r^2,$$

so that $\tilde{\Omega} = \frac{i}{\hbar} \omega$ becomes equivalent to

$$\delta r = \hat{R} + \nabla r + \frac{i}{\hbar} r^2,$$  \hspace{1cm} (3.12)

where $\hat{R} = \frac{\hbar}{i} dU(R) = -\frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l$ has been introduced.

Theorem 3.3 (Fedosov). Under the condition $\delta^{-1} r = 0$, eq. (3.12) has exactly one solution $r$.

Sketch of proof. For a 1-form we have $r_{00} = 0$, together with the condition $\delta^{-1} r = 0$ this implies that the decomposition (3.6) takes the form $r = \delta^{-1} \delta r$. Applying $\delta^{-1}$ to (3.12) leads to

$$r = \delta^{-1} \hat{R} + \delta^{-1} \left( \nabla r + \frac{i}{\hbar} r^2 \right).$$  \hspace{1cm} (3.13)

As $\nabla$ preserves the $h$-filtration, whereas $\delta^{-1}$ increases the degree by $1/2$, it follows by iteration that (3.13) has exactly one solution (as a formal power series in $h$ and $y^i$). The iteration steps are

$$r^{(3)} = \delta^{-1} \hat{R}, \quad r^{(n+1)} = \delta^{-1} \hat{R} + \delta^{-1} \left( \nabla r^{(n)} + \frac{i}{\hbar} (r^{(n)})^2 \right),$$  \hspace{1cm} (3.14)

and $r = \lim_{n \rightarrow \infty} r^{(n)}$. The condition $\delta^{-1} r = 0$ is fulfilled due to $(\delta^{-1})^2 = 0$. For the proof that $r$ indeed solves (3.12) we refer to Fedosovs texts [11, 12].
The first iteration steps are

\begin{align}
\gamma^{(3)} &= \delta^{-1} \hat{R} = \frac{1}{16} R_{ijkl} y^i y^j (y^k dx^l - y^l dx^k) \\
&= -\frac{1}{8} R_{ijkl} y^i y^j y^k dx^l \\
\nabla \delta^{-1} \hat{R} &= -\frac{1}{8} \nabla_m R_{ijkl} y^i y^j y^k dx^m \wedge dx^l \\
\delta^{-1} \nabla \gamma^{(3)} &= -\frac{1}{40} \left( \nabla_m R_{ijkl} - \nabla_l R_{ijkm} + R_{ijkm} \Gamma^m_{ml} \right) y^i y^j y^k y^l dx^m \\
\gamma^{(4)} &= -\frac{1}{40} \tilde{\nabla}_m R_{ijkl} y^i y^j y^m dx^l + O(\hbar^{5/2})
\end{align}

where \( \tilde{\nabla} \) is the product connection on \( W \otimes \Lambda \), thus acts the same way on \( y^i \) and \( dx^i \). In the next to last line, \( \nabla \) also acts on the \( y^i \) outside the brackets, and the term \( \nabla_m R_{ijkl} y^i y^j y^k y^l dx^m \) vanishes due to the antisymmetry of \( R_{ijkl} \) under exchange of \( k \) and \( l \).

We have thus constructed a (formal) flat connection on \( W \), and want to identify the quantum operators with the set of flat sections of \( W \) with respect to \( D \), i.e. those satisfying \( D \hat{f} = 0 \). As \( D = \nabla - \delta + \frac{i}{\hbar} [r, \cdot] \), this equation can be written in the form

\[ \delta \hat{f} = \nabla \hat{f} + \frac{i}{\hbar} [r, \hat{f}] . \]  

We denote the set of flat sections by \( \Gamma_D(W) \).

**Theorem 3.4 (Fedosov).** To every \( f \in C^\infty(M)[[\hbar]] \) there is exactly one \( \hat{f} \in \Gamma_D(W) \) such that \( \hat{f}_0 = f \).

**Sketch of proof.** For a 0-form \( \hat{f} \) we have \( \delta^{-1} \hat{f} = 0 \) and \( \hat{f}_{00} = \hat{f}_0 \), so that the decomposition (3.6) becomes \( \hat{f} = \hat{f}_0 + \delta^{-1} \delta \hat{f} \). Then equation (3.16) implies

\[ \hat{f} = \hat{f}_0 + \delta^{-1} \left( \nabla \hat{f} + \frac{i}{\hbar} [r, \hat{f}] \right) . \]  

Again this equation has a unique solution, which can be determined by iteration:

\[ \hat{f}^{(0)} = \hat{f}_0 = f, \quad \hat{f}^{(n+1)} = \hat{f}_0 + \delta^{-1} \left( \nabla \hat{f}^{(n)} + \frac{i}{\hbar} [r, \hat{f}^{(n)}] \right) . \]  

For the proof that \( \hat{f} \) indeed solves (3.16) we again refer to [11, 12].
The first iterations give (we always determine \( \hat{f}^{(n)} \) up to order \( h^n/2 \) only, because higher order terms are not stable under iteration):

\[
\hat{f}^{(1)} + \mathcal{O}(h^1) = f + \delta^{-1} \nabla f = f + y^k \nabla_k f = f + \partial_k f y^k
\]

\[
\delta^{-1} \nabla \partial_k f y^k = \delta^{-1} (dx^j \nabla_j y^k \nabla_k f) = \frac{1}{2} y^j \nabla_j y^k \nabla_k f
\]

\[
\hat{f}^{(2)} = f + \partial_k f y^k + \frac{1}{2} \nabla_j \partial_k f y^j y^k + \mathcal{O}(h^{3/2})
\]

\[
[f^{(0)}, \hat{f}^{(1)}] = -\frac{1}{8} R_{abcd} \partial_k f [y^a y^b \omega_c, y^c] dx^d
\]

\[
-\frac{i}{h} R_{abcd} \partial_k f y^a y^b \omega_y dx^d
\]

\[
\frac{i}{h} \delta^{-1}[f^{(0)}, \hat{f}^{(1)}] = \frac{1}{24} R_{abcd} \omega_y \partial_k f y^a y^b dx^d
\]

\[
\hat{f}^{(3)} = f + \partial_k f y^k + \frac{1}{2} \omega_x (\nabla_k X_f)^l y^i y^k + \frac{1}{6} [\omega_x (\nabla_i \nabla_k X_f)^l + \frac{1}{4} R_{ijkl} X_f^l] y^i y^j y^k,
\]

which only contains covariant derivatives of the vector field \( X_f \), thus the \( \nabla \)-operators do not act on the \( y^i \). The quantization map

\[
(\pi_0^{-1}) : C^\infty(M)[[h]] \to \Gamma_D(W), \ f \mapsto \hat{f}
\]

allows for the definition of a star product on \( C^\infty(M)[[h]] \), namely

\[
f \ast g := \pi_0(\hat{f} \circ \hat{g}).
\]

Using the explicit expression \([3,20]\), as well as the relation

\[
\pi_0(y^{a_1} \ldots y^{a_k} \circ y^{i_1} \ldots y^{i_k}) = \left( \frac{ih}{2} \right)^k \sum_{\pi \in S_k} \omega^{a_1 i_{\pi(1)}} \ldots \omega^{a_k i_{\pi(k)}},
\]

we arrive at

\[
f \ast g = fg - \frac{ih}{2} \omega(X_f, X_g) + \frac{h^2}{4} (\nabla_j X_f)^b (\nabla_b X_g)^j + \mathcal{O}(h^3).
\]

The conditions for a star product mentioned at the beginning of the section are easily checked, using the fact that the Poisson bracket can be expressed as \( \{f, g\} = \omega(X_g, X_f) \). For \( M = \mathbb{R}^{2n} \) with \( \Gamma = 0 \) one obtains

\[
\hat{f} = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \ldots \partial_{i_k} f) y^{i_1} \ldots y^{i_k},
\]

and thus the Groenewold-Moyal product \([15, 21]\)

\[
f \ast g(x) = \exp \left( \frac{ih}{2} \omega_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x) g(y) \bigg|_{y=x}.
\]
In general the result seems to depend on the symplectic connection chosen, but different connections lead to equivalent star products, where two algebras $\mathcal{A}_1, \mathcal{A}_2$ are considered equivalent if there is a formal sum
\[
S = 1 + \sum_{k \geq 1} \hbar^k S_k
\]
of differential operators $S_k : \mathcal{A}_1 \to \mathcal{A}_2$, with $Sf \ast_2 Sg = S(f \ast_1 g)$. Still it is possible to obtain nonequivalent star products by allowing for higher order terms in the curvature $\Omega = \frac{i}{\hbar} [\tilde{\Omega}, \cdot]$:
\[
\tilde{\Omega} = \omega + \sum_{k \geq 1} \hbar^k \omega_k,
\]
with closed 2-forms $\omega_k$. The equivalence class of the resulting star product depends on the cohomology class of the $\omega_k$, and as every star product can be constructed this way, $H^2(M, \mathbb{C})[[\hbar]]$ parametrizes inequivalent star products on $(M, \omega)$. The class $[\omega] + \sum_k \hbar^k [\omega_k]$ is called Fedosov class.

4 Representations of the deformed algebra

**Example 4.1** (The symplectic Clifford algebra). We consider a vector space case $M = V$ with constant symplectic form $\omega$, and $\Gamma = 0$. The following construction is in complete analogy to the construction of representations of the Clifford algebra over a metric vector space [23]. The Weyl algebra is $W(V^*) = \text{Sym}(V^*)[[\hbar]]$ with commutation relation $[v, w] = i\hbar \omega^{-1}(v, w)$. A given polarization $P \subset V^C$ is transferred to the dual space $V^*^C$ by means of the musical isomorphism $\flat : V \to V^*^C$, $v \flat(\omega) = \omega(v, \cdot) = \omega_{ab} w^a v^b$.

For a real polarization we consider $P$ as a subset of $V$, and choose a Darboux basis $q^i, p_j$ of $V^*$ such that $P^\flat = \text{span}\{q^1, \ldots, q^n\}$. A representation of $W(V^*)$ is defined on $\text{Sym}(P^\flat)[[\hbar]]$, i.e. on complex polynomials in the $q^i$, by
\[
\sigma(q)\psi = q\psi, \quad \sigma(p)\psi = [p, \psi] = i\hbar u(p)\psi,
\]
for $q \in P^\flat, p \in \text{span}\{p_1, \ldots, p_n\}$, and
\[
\iota(p)q^{i_1} \cdots q^{i_k} = \sum_{j=1}^k \omega^{-1}(p, q^{i_j})q^{i_1} \cdots \hat{q}^{i_j} \cdots q^{i_k}.
\]
For a basis element $p_i$ we obtain for $\sigma(p_i)$ the well-known Schrödinger operator $\frac{i\hbar}{\imath} \partial_{q_i}$. Allowing in a slight generalization for arbitrary functions $\psi$ in $q$, we get the Schrödinger (or position space) representation of quantum mechanics.

Consider now a totally complex polarization $P \subset V^C$, such that $D = (P \cap \overline{P}) \cap V = 0$. We choose a complex basis $\{z^1, \bar{z}^1\}$ of $V^*_C$ such that $P^\flat = \text{span} \{z^1, \ldots, z^n\}$. Now the representation space is $\text{Sym}(\overline{P})[[\hbar]]$, i.e. polynomials in $z$, or holomorphic functions, and
\[
\rho(z)\psi = z\psi, \quad \rho(\bar{z})\psi = [\bar{z}, \psi] = i\hbar u(\bar{z})\psi,
\]
for \( z \in \mathcal{P}^\flat, \bar{z} \in \mathbb{P}^h \), and
\[
\iota(z)z^{i_1} \ldots z^{i_k} = \sum_{j=1}^k \omega^{-1}(z, z^{i_j})z^{i_1} \ldots \hat{z}^{i_j} \ldots z^{i_k}.
\]
This way we get the Fock representation of \( W(V^*) \), with \( \rho(z) = 2\hbar \partial_{z^j} \), in case \( \omega^{-1} \) assumes the normal form \( \omega^{-1}(z^i, z^j) = 2i\delta^{ij} \). The case of a general polarization \( P \subset V_C \) is treated analogously, with representation space \( \text{Sym}(\mathcal{P})[[\hbar]] \), which can also be thought of as a space of complex functions on \( V/D \), where \( D = (P \cap \mathcal{P}) \cap V \).

The example suggests that in order to obtain classical quantum mechanics on a symplectic vector space one needs not only the star product, but also a polarization (though all polarizations are equivalent), which usually occurs in the framework of geometric quantization. It therefore strongly hints at the necessity to consider ‘polarized deformation quantization’ (a term apparently introduced by Bressler and Donin, [6]) also in the general case of a symplectic manifold, and thus to unify geometric and deformation quantization.

It is our aim to define a representation of the deformed algebra \( \mathcal{A} = (C^\infty(M)[[\hbar]], *) \) on the Hilbert space \( \mathcal{H}_P \) of geometric quantization. We already remarked that it is not obvious whether this is possible, and if so, whether the undetermined structures (symplectic connection and polarization) have to satisfy some compatibility condition. Therefore, we restrict our attention to the special cases of a cotangent bundle and a Kähler manifold, where we have a natural choice for these structures. For a few observables the representation on \( \mathcal{H}_P \) has already been defined by geometric quantization in (2.7) as
\[
\rho(f) = -i\hbar(\nabla_{X_f} \otimes 1 + 1 \otimes L_{X_f}) + f 1 \otimes 1,
\]
(which was denoted by \( \tilde{f} \) before) in case the flow of \( X_f \) preserves the polarization. In order to extend \( \rho \) to an algebra homomorphism we must make sure that
\[
\rho(f)\rho(g) = \rho(f \ast g)
\]
is satisfied whenever the flows of the Hamiltonian vector fields of \( f, g \) and \( f \ast g \) preserve the polarization. Then we can define \( \rho(f \ast g) \) by \( \rho(f)\rho(g) \) if \( f \) and \( g \) satisfy the quantization condition but \( f \ast g \) does not, and iterate this procedure to obtain the operators for a large class of functions.

**Cotangent bundle** Let \((Q,g)\) be an \( n \)-dimensional, orientable semi-Riemannian manifold, and \( M = T^*Q \). \( M \) carries a natural symplectic form, given by \( \omega = dp_i \wedge dq^i \), where \( \{q^i\} \) are coordinates on \( Q \), and \( \{q^i, p_j = dq^i\} \) the induced coordinates on \( T^*Q \). The vertical polarization of \((M, \omega)\) is spanned by the vector fields \( \partial_{p_i} \), so that polarized functions on \( M \) can be identified as functions on \( Q \). For the symplectic potential given by \( \theta = p_i dq^i \), wave functions are just polarized functions. If \( f \) and \( g \) are polarized, then so is \( fg \), and we have \( \rho(f)\rho(g) = \rho(fg) \), where the operators are given by multiplication with the corresponding functions. In this case the compatibility condition (4.4) becomes
\[
f \ast g = fg.
\]
Further we consider the case that \( f \) is linear in \( p \), \( f(q,p) = a^i(q)p_i \), and \( g \) polarized. Then we have \( \rho(f) = \text{Sym}(a^i(q)p_i) \), which we can be reordered as
\[
\rho(f) = \bar{p}^i a_i(q) + \frac{i\hbar}{2}(\partial_i a^i)(\bar{q}).
\]
This gives

\[ \rho(f)\rho(g) = \left[ \hat{p}_i a^i(\tilde{q}) + \frac{i\hbar}{2}(\partial_i a^i)(\tilde{q}) \right] g(\tilde{q}) \]
\[ = \hat{p}_i a^i(\tilde{q})g(\tilde{q}) + \frac{i\hbar}{2}(\partial_i a^i g)(\tilde{q}) - \frac{i\hbar}{2} a^i(\tilde{q})\partial_i g(\tilde{q}) \]  \hspace{1cm} (4.6)
\[ = \rho(fg - \frac{i\hbar}{2}\partial_p f\partial_q g), \]

and condition (4.4) becomes \( f * g = fg + \frac{i\hbar}{2}\{f, g\} \). Analogously one obtains \( g * f = gf + \frac{i\hbar}{2}\{g, f\} \).

In case \( fg \) is a polynomial (or power series) of degree at least \( 2 \) in \( p \), then also \( f * g = fg + O(h^1) \) is a polynomial (or power series) of degree at least \( 2 \) in \( p \), and it follows that \( f * g \) is not directly quantizable. Thus the compatibility condition only applies to the cases considered so far, and we have

**Theorem 4.2.** A star product (defined by a symplectic connection) on \((T^*Q, \omega)\) is compatible with geometric quantization of the vertical polarization in the sense that (4.4) holds, iff for any polarized functions \( f, g \) and any function \( h \) such that \( X_h \) preserves the polarization, the following are satisfied

1. \( f * g = fg \),
2. \( f * h = fh + \frac{i\hbar}{2}\{f, h\} \), \hspace{0.5cm} \( h * f = hf + \frac{i\hbar}{2}\{h, f\} \).

We recall that the first terms of the star product (3.23) are \( f * g = fg + \frac{i\hbar}{2}\{f, g\} + O(h^2) \), so that the conditions are equivalent to the vanishing of all higher order terms. Bordemann et. al. [1] have shown that there is a natural lift of any torsion-free connection on \( Q \) to \( T^*Q \), such that the resulting connection is torsion-free, symplectic, and leads to a homogeneous star product, i.e. for two homogeneous polynomials of degree \( k \) and \( l \) in \( p \), their star product is a homogeneous polynomial of degree \( k + l \) in \( p \) and \( h \). More precisely, \( \mathcal{H} = p_i \frac{\partial}{\partial p_i} + h \frac{\partial}{\partial \mathcal{H}} \) satisfies

\[ \mathcal{H}(f * g) = (\mathcal{H}f) * g + f * (\mathcal{H}g). \]

In particular, this implies that the conditions of theorem 4.2 are satisfied. Further, any function polynomial in the momenta can be expressed as a finite sum of star products of functions that are affine-linear in \( p \). Explicitly the lift is given by

\[
\Gamma^k_{ij} = -\Gamma^k_{ji} = -\Gamma^k_{ij}, \\
\Gamma^k_{ij} = \frac{p^a}{3} \left( 2\Gamma^a_{ji} \tilde{\Gamma}^i_k + \partial_k \tilde{\Gamma}^a_i + \text{cycl.}(ijk) \right), \\
R^k_{ij} = -R^k_{ji} = \tilde{R}^k_{ij}, \\
R^k_{ij} = \frac{1}{3} (\tilde{R}^l_{ikl} + \tilde{R}^l_{jkl}), \\
\tilde{R}^k_{jkl} = \frac{p^a}{3} \left( \nabla_i \tilde{\Gamma}^a_{jkl} - 3\tilde{\Gamma}^a_{iml} \tilde{\Gamma}^m_{jik} - \tilde{\Gamma}^a_{im} \tilde{\Gamma}^m_{ijk} + \tilde{\Gamma}^a_{km} \tilde{\Gamma}^m_{ijl} + (i \leftrightarrow j) \right),
\]

where \( \Gamma^k_{ij} \) and \( \tilde{R}^k_{jkl} \) are the Christoffel symbols and curvature tensor on \( Q \), and \( \Gamma^k_{ij}, R^k_{ij} \) the lifted objects on \( T^*Q \). The indices run from \( 1 \) to \( n \), an unbarred index \( i \) stands for \( q_i \), a barred index \( \bar{i} \) for \( p_i \). Components not listed vanish. This lift has the property that every geodesic on \( T^*Q \) is projected to a geodesic on \( Q \) [22]. As \( Q \) is a semi-Riemannian manifold there is a natural connection on \( Q \) to start with, the Levi-Civita connection of \( g \).

Thus condition (4.4) is satisfied for the most natural choice of star product on \( T^*Q \), and every function polynomial in the momenta can be quantized, although the explicit calculation of the corresponding operator becomes tedious for high order polynomials. It should also be emphasized that the quantization really depends on the metric \( g \) on \( Q \) (or the choice of a torsion-free connection), and not just on \( \omega \).
Example 4.3 (Quadratic observables). We determine the kinetic energy operator, i.e. the quantization of $g^{ab}(q)p_ap_b$, which we split into two parts as $g^{ab}p_ap_b = f_a h^a$, where

$$f_a(q,p) = p_a, \quad h^a(q,p) = g^{ab}p_b.$$

(4.8)

Due to the homogeneity of the star product we have

$$h^a \ast f_a = h^a f_a + \frac{i\hbar}{2} \{h^a, f_a\} + \frac{\hbar^2}{4} (\nabla_\mu X_{h^a})^\mu (\nabla_\mu X_{f_a})^\nu,$$

which, in the representation $\rho$, allows us to solve for $\rho(h^a f_a)$:

$$\rho(g^{ab}p_ap_b) = \rho(h^a)\rho(f_a) - \frac{i\hbar}{2} \rho(\{h^a, f_a\}) - \frac{\hbar^2}{4} \rho((\nabla_\mu X_{h^a})^\mu (\nabla_\mu X_{f_a})^\nu).$$

(4.9)

A simple calculation yields

$$\{h^a, f_a\} = p_b \partial_q g^{ab},$$

$$(\nabla_\mu X_{h^a})^\mu (\nabla_\mu X_{f_a})^\nu = 2g^{ab} \tilde{\Gamma}_{ja} \tilde{\Gamma}_{ib},$$

where (4.7) has been used for the second equation. Now the operators on the right hand side of (4.9) are determined by geometric quantization:

$$\rho(f_a) = \tilde{p}_a, \quad \rho(h^a) = g^{ab}(\tilde{q}) \tilde{p}_b - \frac{i\hbar}{2} (\partial_\mu g^{ab})(\tilde{q}),$$

$$\rho(\{h^a, f_a\}) = (\partial_\mu g^{ab})(\tilde{q}) \tilde{p}_b - \frac{i\hbar}{2} (\partial_\mu \partial_\nu g^{ab})(\tilde{q}),$$

$$\rho((\nabla_\mu X_{h^a})^\mu (\nabla_\mu X_{f_a})^\nu) = 2g^{ab} \tilde{\Gamma}_{ja} \tilde{\Gamma}_{ib}(\tilde{q}).$$

Inserting these expressions into (4.9) gives

$$\rho(g^{ab}p_ap_b) = g^{ab}(\tilde{q}) \tilde{p}_b \tilde{p}_a - \frac{i\hbar}{2} ((\partial_\mu g^{ab})(\tilde{q}) \tilde{p}_a + (\partial_\mu g^{ab})(\tilde{q}) \tilde{p}_b)$$

$$- \frac{\hbar^2}{4} [(\partial_\mu \partial_\nu g^{ab})(\tilde{q}) + 2g^{ab} \tilde{\Gamma}_{ja} \tilde{\Gamma}_{ib}(\tilde{q})].$$

(4.10)

Here we make use of the explicit form of $\tilde{q}^a$ and $\tilde{p}_b$ (2.10):

$$\tilde{q}^a = \tilde{q}^a, \quad \tilde{p}_a = \hat{p}_a - \frac{i\hbar}{4} g^{bc} \partial_\mu g_{bc}(\tilde{q}),$$

where $\tilde{q}^a$ is multiplication by $q^a$ and $\hat{p}_b = -i\hbar \partial_{q^b}$. $\tilde{p}_a$ can also be written as $\tilde{p}_a = \hat{p}_a - \frac{i\hbar}{4} g^{-1/2} \partial_\alpha g^{1/2}$, with $g := |\det g|$. Then the first term becomes

$$g^{ab} \tilde{p}_a \tilde{p}_b = g^{ab} \hat{p}_a \hat{p}_b - i\hbar g^{1/2} \partial_\alpha g^{1/2} \hat{p}_a \hat{p}_b - \hbar^2 g^{ab} g^{-1/2} \partial_\alpha \partial_\beta g^{1/2},$$

and we obtain

$$\rho(g^{ab}p_ap_b) = g^{ab} \hat{p}_a \hat{p}_b - i\hbar g^{-1/2} \partial_\alpha (g^{1/2} g^{ab}) \hat{p}_b -$$

$$- \frac{\hbar^2}{4} [\partial_\alpha \partial_\beta g^{ab} + 2g^{-1/2} \partial_\alpha (g^{ab} \partial_\beta g^{1/2}) + 2g^{ab} \tilde{\Gamma}_{ja} \tilde{\Gamma}_{ib}].$$

(4.11)
The first two terms yield \(-\hbar^2\) times the Laplace-Beltrami operator \(\Delta = g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b\). In order to simplify the last four terms we use normal coordinates, which satisfy \(\partial_a g_{ij} = \partial_a g^{ij} = \Gamma^k_{ij} = 0\) in a fixed point, and are left with \(-\hbar^2/4\) times
\[
\partial_a \partial_b g^{ab} + 2g^{-1/2}g^{ab}\partial_a \partial_b g^{1/2}.
\]
Using \(\partial_a g^{1/2} = \frac{1}{2} g^{1/2} g^{kl} \partial_a g_{kl}\) and the relation \(\partial_a g^{kl} = -g^{km} g^{ln} \partial_a g_{mn}\), which directly follows from \(g^{kl} g_{lm} = \delta^k_m\), we arrive at
\[
\rho(g^{ab}p_a p_b) = -\hbar^2 \Delta - \frac{\hbar^2}{4} g^{ac} g^{bd} [\partial_a \partial_b g_{cd} - \partial_a \partial_c g_{bd}].
\]
Here we recognize the scalar curvature of \(Q\) (in normal coordinates):
\[
\tilde{R} = g^{ij} \tilde{R}_{ijkl} = \frac{1}{2} g^{ik} g^{jl} (\partial_i \partial_j g_{jk} + \partial_j \partial_k g_{jl} - \partial_i \partial_k g_{jl} - \partial_j \partial_l g_{ik}) = g^{ik} g^{jl} (\partial_i \partial_j g_{jk} - \partial_i \partial_j g_{jl}).
\]
(4.12)
Thus the final result reads
\[
\rho(g^{ab}p_a p_b) = -\hbar^2 \left( \Delta - \frac{\tilde{R}}{4} \right).
\]
(4.13)
It is of the well-known form \(\Delta - \alpha \tilde{R}\); there has been a long debate over the correct value of \(\alpha\) though. E.g. our result \(\alpha = 1/3\) was obtained by a variational principle in [17], whereas Cheng got \(\alpha = 1/6\) from a path integral formalism [7], and Woodhouse gives an argument for \(\alpha = 1/6\) in the framework of geometric quantization [21]. In the formalism advocated here, we see that some inevitable arbitrariness lies in the choice of the half-form (here \(\sqrt{g^{1/2}d^nq}\)), and that different symplectic connections would lead to different results. Still, our choices are the most natural ones.

**Kähler manifolds** For the choice \(\theta = -i \frac{\partial K}{\partial z^k} dz^k\) and \(\mu = d^n z\) wave functions are polarized, the situation is thus very similar to the cotangent bundle case. Functions of the form \(u^k(z) \frac{\partial K}{\partial z^k} + v(z)\) can be quantized directly in geometric quantization, and
\[
\tilde{z}^a \psi(z) = z^a \psi(z), \quad (\tilde{\partial}_{\tilde{z}^a} K) \psi(z) = \hbar \partial_{\tilde{z}^a} \psi(z).
\]
Theorem (4.2) continues to hold if we replace \(T^*Q\) by a Kähler manifold \(M\) and the vertical polarization by the holomorphic Kähler polarization. Thus compatibility is fulfilled if for \(f, g\) holomorphic also \(f \ast g\) is holomorphic, and the star product of a function affine-linear in \(\frac{\partial K}{\partial z^k}\) with a holomorphic function is affine-linear again. There is a natural symplectic connection on Kähler manifolds, given by the Levi-Civita connection of the Kähler metric \(g = \frac{1}{2} \frac{\partial^2 K}{\partial z^j \partial \bar{z}^k} (dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j)\). Defining
\[
A_{\tilde{z}^k} = \frac{\partial^2 K}{\partial \bar{z}^j \partial z^k}, \quad A_j A_k = A_{\tilde{z}^k} A_{\tilde{z}^l} = \delta_j^l, \quad (4.14)
\]
the symplectic form becomes \(\omega_{\tilde{z}^k} = -\omega_{\bar{z}^j} = iA_{\tilde{z}^k}\), and its inverse \(\omega^{\tilde{z}^k} = -\omega_j = iA^{\bar{z}^j}\). The only non-vanishing connection coefficients are given by [10]
\[
\Gamma^{\tilde{z}^k}_{ij} = A^{\tilde{z}^k} \partial_j A_{\tilde{z}^l}, \quad \Gamma^i_{j \bar{z}^m} = A^{\bar{z}^l} \partial_j A_{\bar{z}^m}, \quad (4.15)
\]
where \( \partial_j = \partial_{z_j} \) and \( \partial^j = \partial_{\bar{z}_j} \), and the curvature coefficients are determined by

\[
R_{k\ell ij} = i \partial_i \partial_T A_{k\ell j} - i A^{\bar{m}m} \partial_i A_{k\bar{m}} \partial_T A_{m\bar{j}}
\]  

(4.16)

plus the symmetries

\[
R_{k\ell ij} = -R_{k\ell ji} = -R_{j\ell ik} = R_{j\ell kj}.
\]  

(4.17)

Other components vanish. Here \( R_{\mu\nu\kappa\lambda} = i \omega_{\mu\nu} R^\tau_{\kappa\lambda} \) whereas usually in the context of Kähler manifolds the index is lowered with the metric. Due to the obvious relation \( \partial_i A_{j\bar{k}} = \partial_j A_{i\bar{k}} \) and (4.16), or the first Bianchi identity, also the following symmetries hold:

\[
\begin{align*}
R_{Tj\ell j} &= R_{Tj\ell j}, \\
R_{k\ell ij} &= R_{k\ell ji}.
\end{align*}
\]  

(4.18)

This allows us to determine \( r^{(3)} \) (3.15):

\[
\begin{align*}
r^{(3)} &= -\frac{1}{8} R_{\kappa\lambda\mu\nu} y^\kappa y^\lambda y^\mu dx^\nu \\
&= -\frac{1}{8} \left[ R_{k\ell ij} \hat{z}^k \hat{z}^\ell \hat{z}^i dx^j + R_{k\ell ij} \hat{z}^j \hat{z}^i \hat{z}^\ell dx^k + R_{k\ell ij} \hat{z}^j \hat{z}^i \hat{z}^k dx^\ell \right] \\
&= -\frac{1}{4} R_{k\ell ij} \hat{z}^k \hat{z}^j \hat{z}^i (\hat{z}^j dz^i - \hat{z}^i dz^j),
\end{align*}
\]  

(4.19)

where \( \hat{z}^k, \hat{z}^j \) are the local generators of \( T^* M \), collectively denoted \( y^\mu \) before, and satisfying the canonical commutation relation \([\hat{z}^k, \hat{z}^j] = i \hbar \delta^k_j \) in \( \hbar A^k \). As before for \( y^\mu \), here \( \hat{z}^j \hat{z}^k \) denotes the symmetrized (!) product of \( \hat{z}^j \) and \( \hat{z}^k \), whereas simple composition of operators is \( \hat{z}^j \circ \hat{z}^k \).

We calculate the first terms of the local operators \( \hat{f}, \hat{h} \in \Gamma_D(W) \) corresponding to the functions \( f(z, \bar{z}) = w_a z^a, h(z, \bar{z}) = -i \partial K \); according to (3.18) and (3.21), in order to prove that there are no contributions to \( f \circ h \) in order \( \hbar^2 \) and \( \hbar^3 \). We could as well use the solution (3.19) to (3.18) directly, but performing the iteration explicitly allows us to drop a few terms that do not contribute to the star product in third order. Introducing the \( h \)-homogeneous elements \( \hat{f}_n = f(n) - f(n-1) \) (mod \( \hbar^{(n+1)/2} \)), we have \( \hat{f}_1 = w_a \hat{z}^a \), and

\[
\begin{align*}
\hat{f}_{(2)} &= -\frac{1}{2} w_a R_{bc} \hat{z}^b \hat{z}^c \\
\hat{f}_{(3)} &\sim -\frac{1}{6} w_a \partial_d \Gamma^a_{bc} \hat{z}^b \hat{z}^c \hat{z}^d + i \frac{\hbar}{\hbar^3} [r^{(3)}, \hat{f}_{(1)}] \\
&= -\frac{1}{6} w_a \partial_d \Gamma^a_{bc} \hat{z}^b \hat{z}^c \hat{z}^d + \frac{1}{12} w_a R_{bc} \hat{z}^b \hat{z}^c \hat{z}^d.
\end{align*}
\]  

(4.20)

\( \sim \) means equality up to terms not containing any operator \( \hat{z}^i \), which will not contribute. Observing that \( \partial_d \Gamma^a_{bc} = R^a_{cab} = -R^a_{bca} \), we obtain

\[
\hat{f}_{(3)} \sim -\frac{1}{4} w_a R^a_{bc} \hat{z}^b \hat{z}^c \hat{z}^d.
\]  

(4.21)

Further \( \hat{h}_{(1)} = -i \partial_n \partial_a K \hat{z}^a - i A_{mb} \hat{z}^b \), and

\[
\hat{h}_{(2)} \sim \frac{1}{2} \partial_a A_{mb} \hat{z}^a \hat{z}^b + \frac{1}{2} \hat{z}^a \nabla_a (A_{mb} \hat{z}^b) + \frac{1}{2} \partial_a A_{mb} \hat{z}^a \hat{z}^b.
\]  

(4.22)
The vanishing of $\nabla_\pi(\omega_{m\bar{b}}z^b)$ is a consequence of the connection being symplectic: $\nabla \omega = 0$, and the vanishing of some connection coefficients (4.15), and is easily checked directly. We are left with $\hat{h}(2) \sim -i \partial_a A_{m\bar{b}}z^a z^b$. Now we define $\approx$ to be equality mod terms containing less than two operators $\partial^\tau$, and get

$$\hat{h}(3) \approx -\frac{i}{3} z^c \nabla_\pi(\partial_a A_{mb} z^a z^b) + \frac{i}{\hbar} \delta^{-1}[r(3), \hat{h}(1)].$$

$$\approx \frac{1}{3} R_{\mu c\bar{a}b} z^b z^c a + \frac{1}{12} R_{\mu b\bar{a}c} z^b z^c = -\frac{1}{4} R_{\mu b\bar{a}c} z^b z^c (4.22)$$

From the definition of the star product (3.21), relation (3.22) and the results for $\hat{f}, \hat{h}$ we can read off that there is no contribution to $f \ast h$ in order $h^2$, in agreement with the compatibility condition, and in order $h^3$ we get the following result:

$$\pi_\otimes(\hat{f}(3) \circ \hat{h}(3)) = -\frac{\hbar^3}{64} \tilde{w}_a R_{\alpha \beta \gamma} R_{\mu \nu \lambda} F^{\gamma \nu \lambda} A^{z^b} A^{m^b} A_{z^c}.$$ (4.23)

But there are further contributions in order $h^3$, from $\hat{f}(5) \circ \hat{h}(1)$ and $\hat{f}(1) \circ \hat{h}(5)$, due to the term $r^2$ in (3.12). $(r^{(n)})^2$ contains terms of the form $y^i \ldots y^{i_n} \circ y^{j_1} \ldots y^{j_n}$, which can be reordered, using $y^i \circ y^j = y^j y^i + \frac{i}{2} \omega_{ij}$, such that only fully symmetric expressions $y^{i_1} \ldots y^{i_k}$, $k = 0, \ldots, 2n$, occur. Obviously, only such terms in $\hat{f}(5)$ contribute to $\pi_\otimes(\hat{f}(5) \circ \hat{h}(1))$ that contain exactly one operator $y^i$. The only term of this kind in $\hat{f}(5)$ comes from $\delta^{-1}[r(5), \hat{f}(1)]$ in the iteration formula for $\hat{f}$ (3.18), and there only the $\delta^{-1} \frac{1}{\hbar} \pi_\otimes((r^{(3)})^2)$ part of $r^{(5)}$ contributes. Neglecting those terms not contributing, we have

$$(r^{(3)})^2 = \frac{ih^3}{32} R_{k\ell j} R_{ab\gamma \omega} \tilde{f}_a \omega_{j \gamma \ell \omega} \tilde{f}_d d z^d \wedge d z^i,$$

$$r(5) = \delta^{-1} \frac{i}{\hbar} (r^{(3)})^2 = -\frac{h^2}{32} R_{k\ell j} R_{ab\gamma \omega} \tilde{f}_a \omega_{j \gamma \ell \omega} \tilde{f}_d d z^d - \frac{i}{\hbar} d z^i,$$

which leads to

$$\delta^{-1} \frac{i}{\hbar} [r(5), \hat{f}(1)] = -\frac{h^2}{32} R_{k\ell j} R_{ab\gamma \omega} \tilde{f}_a \omega_{j \gamma \ell \omega} \tilde{f}_d \omega_{i d} \partial_m f \tilde{z}^i,$$

$$\delta^{-1} \frac{i}{\hbar} [r(5), \hat{h}(1)] = -\frac{h^2}{32} R_{k\ell j} R_{ab\gamma \omega} \tilde{f}_a \omega_{j \gamma \ell \omega} \tilde{f}_d \omega_{i d} \partial_m h \tilde{z}^i,$$

where we have also neglected a contribution containing $\partial_m h \omega_{\bar{m} \bar{a} b}$. According to the discussion above, the last two lines give the only components of $\hat{f}(5)$ and $\hat{h}(5)$ that contribute to $f \ast h$ in order $h^3$.

$$\pi_\otimes(\hat{f}(5) \circ \hat{h}(1)) = -\frac{ih^3}{128} R_{k\ell j} \omega_{ab} R_{\gamma \omega} \tilde{f}_a \omega_{j \gamma \ell \omega} \tilde{f}_d \omega_{i d} \partial_m f \omega_{\bar{i} \bar{m}}$$

$$= \pi_\otimes(\hat{f}(1) \circ \hat{h}(5)).$$

Adding up, we get

$$\pi_\otimes(\hat{f}(5) \circ \hat{h}(1) + \hat{f}(1) \circ \hat{h}(5)) = \frac{\hbar^3}{64} \tilde{w}_a R_{\alpha \beta \gamma} R_{\mu \nu \lambda} F^{\gamma \nu \lambda} A^{z^b} A^{m^b} A_{z^c},$$ (4.24)

which exactly cancels the contribution of $\pi_\otimes(\hat{f}(3) \circ \hat{h}(3))$ (4.23), and we conclude

$$f \ast h = f h + \frac{ih}{2} [f, h] + O(h^4).$$ (4.25)
For two holomorphic functions $f, g$ the operators $\hat{f}(1)$ and $\hat{g}(1)$ both contain only unbarred operators, the same is true for $\hat{f}(2), \hat{g}(2)$, and the summands of $\hat{f}(3)$ contain at least two unbarred and at most one barred operator. Therefore the contraction $\pi_0^0(f^{(3)} \circ \hat{g}^{(3)})$ vanishes. Further, $\pi_0^1(f^{(5)} \circ \hat{g}(1))$ vanishes, as $\pi_1^0(\hat{f}(5))$ contains only unbarred operators as well. Thus

$$f * g = fg + O(h^4). \quad (4.26)$$

The vanishing of the $h^2$ component in $f * h$ can be clearly attributed to the choice of connection; it is a result of $\nabla_{\bar{c}}(A_{mb} \bar{z}^b) = \nabla_a \bar{z}^b = \nabla_{\bar{z}} z^b = 0$. Compatibility in order $h^3$ appears somewhat mysterious however, resulting from a cancelation of quite different contributions.

We have thus established compatibility up to order $h^3$ for a large class of functions, which, together with compatibility for cotangent bundles, provides strong evidence for exact compatibility. For a complete proof one could try to imitate the proof of Bordemann et. al. for cotangent bundles $[4]$, in order to establish homogeneity also for Kähler manifolds. It appears, though, that the Kähler connection is not homogeneous in the sense of $[4]$, so that at least some modifications are unavoidable. If the star product is compatible with the holomorphic polarization for general Kähler manifolds, then it must be compatible with the antiholomorphic polarization as well, for symmetry reasons. Further, if it is homogeneous in $\partial_a K$, then it must be homogeneous in $\partial_{\bar{z}} K$ too.

It can easily be seen from the iteration formula and the vanishing of connection coefficients with both barred and unbarred indices, that compatibility holds exactly if the curvature vanishes.

5 Conclusion

We have proposed an interpretation of geometric and deformation quantization as complementary theories, in an attempt to overcome the difficulties arising in geometric quantization in quantizing the observables, and to define a natural representation of the deformed algebra.

It turned out to work well if phase space is a cotangent bundle, as a consequence of homogeneity of the star product, which was established in $[4]$. Still, even after restriction to the vertical polarization the quantization map is not uniquely determined. One further needs a metric or, alternatively, a torsion-free connection on configuration space. This explains the mentioned difficulties in geometric quantization quite naturally; although the framework for classical Hamiltonian mechanics is a symplectic manifold $(M, \omega)$ $[1, 24]$, quantization is not determined by $\omega$ alone. One should keep in mind that in the typical situation of classical mechanics one has a configuration space $(Q, g)$ and as phase-space $T^*Q$, where the metric is needed to specify the kinetic term in the Lagrangian: $L_{kin} = g^{ij} p_i p_j$. Therefore, from a physical point of view the occurrence of the metric in the quantization process is not too surprising.

Then it is tempting to assign a similar role to the metric and its associated Levi-Civita connection in the important case of general Kähler manifolds. But for these, we could only establish compatibility of the two theories up to a finite order in $h$. The vanishing of the obstructions has partly been shown to be related to the special choice of connection.

The case of general polarizations was not treated in this paper.

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