ON THE HILBERT PROPERTY AND THE FUNDAMENTAL GROUP
OF ALGEBRAIC VARIETIES

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Abstract. In this paper we review, under a perspective which appears different from
previous ones, the so-called Hilbert Property (HP) for an algebraic variety (over a number
field); this is linked to Hilbert’s Irreducibility Theorem and has important implications,
for instance towards the Inverse Galois Problem.

We shall observe that the HP is in a sense ‘opposite’ to the Chevalley-Weil Theorem,
which concerns unramified covers; this link shall immediately entail the result that the
HP can possibly hold only for simply connected varieties (in the appropriate sense). In
turn, this leads to new counterexamples to the HP, involving Enriques surfaces.

In this view, we shall also formulate an alternative related property, possibly holding
in full generality, and some conjectures which unify part of what is known in the topic.
These predict that for a variety with a Zariski-dense set of rational points, the validity
of the HP, which is defined arithmetically, is indeed of purely topological nature. Also,
a consequence of these conjectures would be a positive solution of the Inverse Galois
Problem.

In the paper we shall also prove the HP for a K3 surface related to the above Enriques
surface, providing what appears to be the first example of a non-rational variety for which
the HP can be proved. In an Appendix we shall further discuss, among other things, the
HP also for Kummer surfaces. All of this basically exhausts the study of the HP for
surfaces with a Zariski-dense set of rational points.

1. Introduction

This paper is mainly concerned with a kind of ‘re-reading,’ so to say, of a circle of issues
around the so-called ‘Hilbert Property’, concerning the set $X(k)$ of $k$-rational points for
an algebraic variety $X/k$, where $k$ is a field (which for us shall be a number field). The
Hilbert Property (abbreviated to HP in the sequel) is well known to admit important
implications, for instance towards the Inverse Galois Problem (see [29], [37]).

In particular, we shall relate this property with the Chevalley-Weil theorem, which
usually, in books or expositions, is dealt with separately from the above. Further, we
shall provide (in the shape of theorems) a number of examples, concerning surfaces,
which we have not found in the literature: we shall consider new cases in which the HP
holds and also shall provide new counterexamples. Moreover we shall state some general
conjectures relating this kind of property with purely topological aspects of the varieties
in question.

As a consequence of this viewpoint, we shall also re-obtain some known results, and
of course this article does not claim novelty in these cases. However, it seems to us that
seldom the links alluded to above are presented together within the theory of the HP;
and it is also in this sense that we hope to contribute to any small extent.

For convenience and completeness, we start by recalling in short some definitions and
known facts about these topics, which shall also serve to illustrate simultaneously our
purposes.

Throughout, algebraic varieties shall be understood to be integral over $k$ and quasi-
projective. By cover of algebraic varieties $\pi : Y \to X$ we shall mean a dominant rational
map of finite degree. In particular, we must have $\dim Y = \dim X$, but $\pi$ need not be a
morphism.
Any possible additional property of \( \pi \) shall be specified explicitly. (In particular, in §1.2 we shall recall some facts concerning ramification of a cover, a concept especially relevant in this paper.)

1.1. The Hilbert Property. We mainly refer to [29] (see especially Ch. 3) for the theory, limiting ourselves to recall only a few definitions and properties.

**Definition.** We say that \( X/k \) has the Hilbert Property, abbreviated HP in the sequel, if, given any finite number of covers \( \pi_i : Y_i \to X, i = 1, \ldots, r \), each of degree \( > 1 \), the set \( X(k) \setminus \bigcup_{i=1}^r \pi_i(Y_i(k)) \) is Zariski-dense in \( X(k) \).

This property is invariant by birational isomorphism over \( k \).

Any finite union of a set \( \bigcup_{i=1}^r \pi_i(Y_i(k)) \) as above, together with another finite union \( \bigcup_{j=1}^s Z_j(k) \) for proper subvarieties \( Z_j \) of \( X \), is usually called thin in \( X(k) \). The HP may then be restated (as e.g. in [29]) by saying that \( X(k) \) is not thin.

Of course, for given \( X \), this strongly depends on the ground field \( k \) (for instance, if \( k \) is algebraically closed the HP trivially fails); however, it is preserved by finite extensions, as in [29], Prop. 3.2.1 (but the converse is not necessarily true).

In the sequel, we shall often omit explicit reference to \( k \), which is supposed to be fixed.

The relevant fields for the HP are those with ‘arithmetical’ restrictions, so to say, and a field \( k \) is called Hilbertian if the HP holds for some variety \( X/k \) (in which case it is known that it holds for any \( \mathbb{P}_n \)).

In the present paper we shall consider only number fields.

In this case (or more generally when \( k \) is finitely generated over the prime field) the HP is known to hold for projective spaces (Hilbert [20]), for which several proofs are available (see [5], [15], [22], [27], [28], [29], [37]). Colliot-Thélène and Sansuc [8] proved the HP for connected reductive algebraic groups over a Hilbertian field. Other examples may sometimes be constructed starting from these ones: for instance recently it has been shown in [1] that if the HP holds for \( X,Y \) it holds for \( X \times Y \). In this paper an example of a non (unirational) surface with the HP, which appears to be new and not a direct consequence of previous results, shall be given in Theorem 1.4.

In the converse direction, a ‘trivial’ reason for the failure of HP occurs when \( X(k) \) is not Zariski-dense. Of course it can be of the utmost difficulty to establish this: Faltings’ celebrated theorems cover e.g. the case of curves of genus \( \geq 2 \) (and more generally subvarieties of abelian varieties which are not translates of abelian subvarieties); otherwise, little is known unconditionally, and the Vojta conjectures give in general the expected geometrical conditions (see e.g. [5]).

But, concerning the HP itself, the most interesting failures occur when \( X(k) \) may be Zariski-dense and still the HP does not hold. In this sense, the typical examples that are usually presented are curves of genus 1 over number fields: the (weak) Mordell-Weil Theorem provides, for every integer \( m > 1 \), finitely many points \( p_i \in E(k) \) such that every \( p \in E(k) \) is of the shape \( p = p_i + mq \) for a \( q \in E(k) \), so \( E(k) \) is covered by the sets \( \{ p_i + mq \} \), which are images of covers of degree \( (= m^2) > 1 \).

The same argument applies to any abelian variety (and in turn to any curve of genus \( > 1 \) embedded in its Jacobian, even forgetting about Faltings’ theorem).

Further examples, in higher dimensions, occur in the shape of varieties \( X \) mapping nontrivially to an abelian variety; for smooth \( X \), this happens when the Albanese variety of \( X \) is not trivial, or, equivalently, when the irregularity \( q = \dim \mathcal{H}^0(X, \Omega_X^1) \) is positive.

In these cases it may be shown that the components of the pullback of multiplication by \( m \) produce covers of degree \( > 1 \) for suitable \( m \), and the failure of the HP for the abelian variety transfers to the variety. (We may agree to say that the examples obtained in this way are not primitive.)

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1. In fact, it suffices to assume that this set is always nonempty; indeed, if the set happens to be contained in a proper Zariski closed subset, say defined by \( f = 0 \), it suffices to consider a further cover with function field \( k(X)(f^{1/n}) \) for suitable \( n \) to lift all remaining points in \( X(k) \), thus making empty the complement with respect to this further cover.

2. Such algebraic groups are rational varieties, but possibly not over \( k \).
In dimension 2, explicit instances in this direction are ruled surfaces with elliptic base (for which there is a map $f : X \to E$, $E$ elliptic curve, with rational generic fiber), as for instance the symmetric square $E^{(2)}$ of an elliptic curve $E$, obtained on identifying $(x,y), (y,x) \in E^2$. One may also take surfaces $(E \times F)/G$ where $E, F$ are elliptic curves and $G$ is a finite group acting without fixed points. Again, these surfaces admit non-constant maps to elliptic curves. More generally, quotients of abelian varieties by groups without fixed points also fall in this pattern (it may be shown that the abelian variety is isogenous to a product and that there is a nontrivial map from the quotient to a factor).

In this paper we give an example which escapes from all of these constructions: a smooth surface with a Zariski-dense set of rational points, trivial Albanese variety and failing the HP (see Theorem 1.3).

Further, our aim here is to point out certain simple links of this type of property with topological features of algebraic varieties, especially their fundamental group; in particular, it is this connection which suggests and leads to the said new examples of violation of the HP.

Simultaneously, it shall appear that it is equally natural to study a certain modified (weakened) form of the HP, stated below in §2.2. On the one hand, such property has been implicitly already proved for some varieties for which the HP fails; on the other hand, perhaps this property always holds. It is also to be remarked that this different property would admit in principle much the same applications as the HP.

All of this depends ultimately on ramification, and then we pause for recalling some simple concepts in this realm.

1.2. Fundamental groups and unramified covers. In this subsection we review some definitions and facts concerning ramification, and we define an algebraic notion, useful for us, of being simply connected. (Here we could have relied on the well-known notion of the profinite completion of the usual fundamental group; see e.g. Mézard’s paper [24] for this and more. However for simplicity we have preferred to repeat here very briefly some relevant constructions and definitions; this shall also allow us to work with singular varieties, and shall make the article more self-contained.)

Let $X/k$ be a (quasi-projective) algebraic variety over a number field $k$. We say that a finite map $Y \to X$, from a variety $Y$, is unramified (at a point $y \in Y$) if its differential at $y$ is an isomorphism between the corresponding tangent spaces. Otherwise we say that the map is ramified at $y$, and we call $y$ a ramification point and $x = \pi(y)$ a branch point.

The set of ramification points is a proper closed subset of $Y$. Its image in $X$ is generally called the branch locus. \( ^3 \)

This notion does not immediately apply for covers in our general sense, that are merely maps of finite degree. To cope with this, we argue as follows.

We note that given any cover $\pi : Y \to X$ in the previous sense, there exists a normal variety $Y'$ with a birational map $f_Y : Y' \to Y$, a normalization $f_X : X' \to X$ and a finite map $\pi' : Y' \to X'$ such that $f_X \circ \pi' = \pi \circ f_Y$. \( ^4 \)

Then we may define the ramification of the original cover as the ramification of the finite morphism of normal varieties $\pi' : Y' \to X'$. This is well defined because the normalization (of $X$) is unique up to isomorphism.

Note that if e.g. $X$ is smooth, so in particular $X' = X$, $y \in Y'$ is a ramification point if and only if the number of inverse images $|\pi'^{-1}(\pi(y))|$ is strictly smaller than the degree of $\pi$, whereas otherwise it equals this degree.

\( ^3 \)It is easy to see that $E^{(2)}$ is birationally isomorphic to $E \times \mathbb{P}_1$.

\( ^4 \)As remarked in [16], there are seven possibilities for a nontrivial such $G$.

\( ^5 \)However some authors adopt the converse terminology.

\( ^6 \)The variety $Y'$ may be obtained e.g. by taking on affine pieces the integral closure of the affine ring of $X$ in the function field $k(Y)$, and then by gluing these affine parts.
Moreover, after Zariski’s Purity Theorem, in case $X$ is smooth the branch locus is of pure codimension 1. So in this case we may speak of the ramification divisor in $Y'$ and of branch divisor in $X$.

**Definition.** We say that a variety $X/k$ is algebraically simply connected if any cover of degree $> 1$ of its normalization $X'$ is ramified (somewhere).

We note that, according to our definition, the algebraic simply connectedness must be checked by passing to the normal model.

After fixing an embedding $k \hookrightarrow \mathbb{C}$, we can associate to $X$ the topological space $X(\mathbb{C})$ of its complex points and so define its topological fundamental group $\pi_1(X)$.\footnote{We do not consider here multiplicities; see \cite{6} for this.}

In case $X$ is smooth, if the ramification divisor of a cover is empty, then there is no ramification at all and hence the morphism $Y' \rightarrow X$ defines a topological covering of the space $X(\mathbb{C})$.

In general, we have the following

**Proposition 1.1.** If $X$ is normal then $X$ is algebraically simply connected if and only if $\pi_1(X)$ has no subgroup of finite index $> 1$.

**Remark 1.2.** We note at once that the assumption that $X$ is normal cannot be omitted, as shown by the example of the nodal curve $yz^2 = x^3 + z^2$ in $\mathbb{P}_2$, obtained by identifying two points of $\mathbb{P}_1$. The normalisation of this curve is $\mathbb{P}_1$, so it is algebraically simply connected in our sense, however the topological fundamental group of the nodal curve over $\mathbb{C}$ is $\mathbb{Z}$.

We owe to F. Catanese examples showing that the normality assumption cannot be omitted even for the converse implication: indeed, there are simply connected surfaces whose normalisation is the product of a line by a hyperelliptic curve.\footnote{It was proved by Serre \cite{22} that different embeddings can produce different fundamental groups; however the properties relevant for us are independent of this embedding, and the same holds for the profinite completion of the fundamental group.}

**Proof of Proposition.** Let us start by assuming $X$ normal and algebraically simply connected. Suppose by contradiction that $H$ is a subgroup of $\pi_1(X)$ of finite index $d > 1$. This corresponds to a connected topological cover $Y$ of $X(\mathbb{C})$ of degree $d$. By a theorem of Chow (in the compact case) and of Grauert and Remmert (in the general case) such a topological cover can be endowed with a (unique) structure of algebraic variety (see \cite{29}, Ch. 6). We contend that $Y$ is an irreducible variety. This follows from a theorem of Zariski (see \cite{25}, page 52) asserting that under the normality assumption each point of $X$ is topologically unibranch (in the sense of Definition (3.9) in \cite{29}, page 43). (Indeed, if $Y$ were reducible, then removing singular points would disconnect it in a neighbourhood of any singular point.)

But now the cover provided by $Y$ would contradict the property of being algebraically simply connected.

Let us now consider the converse implication. If $X$ is normal and not algebraically simply connected, let $\pi : Y \rightarrow X$ be a finite cover with no ramification. This would correspond to a topological cover, and thus to a subgroup of finite index $> 1$ in the fundamental group.

This concludes the proof. \hfill $\Box$

It is important to notice that the notion of algebraic simply connectedness for projective varieties is not a birational invariant, the same as for the topological notion. (Actually, each projective algebraic variety is birationally isomorphic to a hypersurface of a projective space, and by the Lefschetz hyperplane section theorem, each hypersurface of $\mathbb{P}_n$, for $n \geq 3$, is simply connected.)

\footnote{One of Catanese’s examples consists of a product $\mathbb{P}_1 \times E$ where $E$ is an elliptic curve. Taking the quotient by the equivalence relation $(t_1, p_1) \sim (t_2, p_2)$ if and only if $t_1 = t_2 = 0$ and $p_2 = -p_1$ or $(t_1, p_1) = (t_2, p_2)$, we obtain a variety $X$ which may be shown to be simply connected. Its normalization is just the original $\mathbb{P}_1 \times E$, whose fundamental group is $\mathbb{Z}^2$.}
However it is a birational invariant for smooth projective varieties: so if one smooth projective model of an algebraic variety is algebraically simply connected, then every other smooth model is.

Further, it may happen that a normal model is simply connected, and a smooth model is not: an instance is provided by the normal but non smooth model of the Enriques surface of Theorem 1.3. This is an irreducible hypersurface in \( \mathbb{P}_3 \) and therefore the space of its complex points is simply connected by the mentioned theorem of Lefschetz. Also, it has only isolated points as singularities, and hence is normal (e.g. by a criterion of Serre). However a smooth model is an Enriques surface, admitting an unramified cover of degree 2, which shall be implicitly described in our construction below.\(^1\)

Let us now state some results in detail.

1.3. Some results. We start with counterexamples.

**New failures of the HP.** We do not know of any prior example of a variety over a number field with a Zariski-dense set of rational points but without the HP, which cannot be reduced as above to abelian varieties. We give such an example in dimension 2, originated from some considerations below, and we shall prove in §3.2 the following

**Theorem 1.3.** The (irreducible) surface defined in \( \mathbb{P}_3 \) by the equation \( x_0x_4^4 + x_1x_3^4 = x_2^2x_1^4 + x_3^2x_1^4 \) has a Zariski-dense set of rational points, has not the Hilbert Property over \( \mathbb{Q} \), and does not admit non-constant rational maps to abelian varieties.

This surface is simply connected but not smooth. A smooth model of it (which is an Enriques surface\(^1\)) is not simply connected, which fits with the viewpoint of this paper. In §3.4 we shall add more comments on the nature of this surface (and in Remark 3.5(ii) we shall point out that the same arguments work over any number field.)

**New examples of the HP.** Together with this example, we shall prove in §3.1 the HP for a K3 surface (which admits the former surface as a quotient); namely, the following

**Theorem 1.4.** The surface defined in \( \mathbb{P}_3 \) by \( x^4 + y^4 = z^4 + w^4 \) is not (uni)rational (over \( \mathbb{C} \)) and has the Hilbert Property over \( \mathbb{Q} \).

We do not know of any previous example in the literature when the HP is proved unconditionally for a non-(uni)rational (even over \( \mathbb{C} \)) variety. Recall that non-rational curves never have the HP, so this attains the minimal possible dimension in such kind of example. (A criterion for producing varieties with the Hilbert Property has been recently proved in \[1\]; however, in the case of surfaces this would work only for rational ones over \( \mathbb{C} \), so our example is new also in this respect.)

Together with the HP one can formulate (as in §2 below) a similar property for \( S \)-integral points, rather than rational points. One can surely find examples also for this case, and we shall recall some results for tori \( \mathbb{G}_m^n \).

1.4. The Chevalley-Weil Theorem. A fundamental result linked to the HP, but usually presented in separate discussions, is the Chevalley-Weil theorem. Let us briefly recall (the basic case of) this theorem, abbreviated to CWT in the sequel, which can be seen as an arithmetical analogue of the lifting of maps in homotopy theory. (See \[2\] for extended statements, e.g. Thm. 10.3.11.)

**CWT:** Let \( \pi : Y \to X \) be a(n irreducible) cover which is also an unramified finite morphism of projective varieties over the number field \( k \). Then there is a finite extension \( k'/k \) such that \( X(k) \subset \pi(Y(k')) \).

\(^{10}\)On the contrary, if a normal model is not simply connected, the same holds for a desingularization, as can be seen on taking the pullback of a possible cover to the smooth model.

\(^{11}\)These surfaces were first exhibited by Enriques: in 1894 he communicated in a letter to Castelnuovo a relevant example of a sextic, which showed that the vanishing of the irregularity and geometric genus were not sufficient for rationality; see \[13\]. Afterwards, in 1896, Castelnuovo found his celebrated rationality criterion, that the vanishing of the irregularity and the second plurigenus are indeed sufficient. We thank P. Oliverio for these references.
Note that this goes in a direction opposite to the HP, because it asserts that rational points can all be lifted to a single cover and number field, albeit possibly larger than $k$.

We do not pause to recall a proof of this, which depends on Hermite’s finiteness theorem for number fields of given discriminant. Instead, we now give a (simple) proof of the following proposition, that immediately leads to a modified version of CWT, which allows to maintain the ground field.\footnote{This version is probably known, though usually not explicitly mentioned.}

**Proposition 1.5.** Let $\pi: Y \to X$ be a cover of degree $> 1$ and defined over $k$. Let $k'/k$ be a finite extension and let $T \subseteq Y(k')$ be a set of points over $k'$ such that $\pi(T) \subseteq X(k)$. Then there exist finitely many covers $\pi_i: Y_i \to X$, each of degree $> 1$, defined and irreducible over $k$, such that $\pi(T) \subseteq \bigcup \pi_i(Y_i(k))$.

In particular, we see that if $X(k)$ can be lifted to finitely many covers over $k'$, then it may be lifted already over $k$ at the cost of adding finitely many further covers.

Then, putting together the CWT with this proposition, we obtain the following variant (a refinement for what concerns the field of definition):

**Alternative CWT:** Let $\pi: Y \to X$ be a(n irreducible) cover of degree $> 1$ which is also an unramified finite morphism of projective varieties over the number field $k$. Then there exist finitely many covers $\pi_i: Y_i \to X$ of degree $> 1$ such that $X(k) \subseteq \bigcup \pi_i(Y_i(k))$.

Now, note that the number field $k$ does not increase to $k'$, so this expresses something that is literally opposite to the HP; therefore the HP is violated for the varieties in question, i.e. those which admit an unramified cover as previously defined. See Theorem 1.6 below for an explicit statement.

This fact also gives rise to the new example of violation of the HP alluded to above.

**Proof of Proposition.** The proof is simple. We let $\tilde{Y}$ be the restriction of scalars from $k'/k$ (see [29]). If $d = [k': k]$, this is a variety of dimension $d \cdot \dim Y$, isomorphic over $k'$ to $Y^d$.

The morphism $\pi$ yields a morphism, denoted in the same way, $\pi: \tilde{Y} \to \tilde{X}$. Now, $X$ embeds diagonally as a variety $\Delta \subseteq X$ into $\tilde{X}$ and similarly for $X(k) \subseteq \tilde{X}(k)$. We know that the points in $\pi(T) \subseteq X(k)$ lift to $T \subseteq Y(k')$ and hence to $\tilde{Y}(k)$. Now it suffices to define the $Y_i$ as the irreducible (over $k$) components of $\pi^{-1}(\Delta)$.

Note that no such component can have degree 1, for otherwise it would have a rational section (over some finite extension of $k$), and the same would be true for the original cover.

This concludes the proof. \qed

We remark that, if the original cover is Galois, the covers so obtained are all isomorphic to $Y$ over $k'$.

The variant of the CWT just stated shows that to violate the HP for $X/k$ it is sufficient to have a finite unramified cover of $X$, i.e. that $X$ is not algebraically simply connected.

In practice, we assert that:

**Theorem 1.6.** Let $X/k$ be a projective algebraic variety with the Hilbert Property. Then $X$ is algebraically simply connected. Also, the fundamental group of every normal projective model of $X$ admits no subgroup of finite index $> 1$.

**Proof.** We have already remarked that the HP is a birational invariant. So we may suppose that $X$ is projective, normal, non algebraically simply-connected and prove that the HP fails. Take a finite unramified cover $Y \to X$ of degree $> 1$, possibly defined over a finite extension $k'$ of $k$. Since finite extension of the ground field preserves the HP, as in [29], Prop. 3.2.1., the failure of HP over $k'$ implies the failure of HP over $k$. Now, by our refined version of CWT, we obtain finitely many covers $\pi_i: Z_i \to X$ over $k'$ such that $X(k') = \cup \pi_i(Z_i(k'))$, so the HP does not hold for $X/k'$, as wanted. We have then proved the first part of the theorem.

Taking this into account, the second part is already in part (i) of Proposition 1.5 above. \qed
This shows a basic feature which forbids the HP, and links it with a purely topological property; also, this immediately explains the above mentioned examples related to abelian varieties. For instance, one also obtains Corollary 4.7 of [1] where it is proved that the only algebraic groups which are of Hilbert type are the linear ones. But Theorem 1.6 leads also to new examples of failure of HP (still with a Zariski-dense set of rational points), like the mentioned one of Theorem 1.3: the projective surface appearing in the Theorem is simply-connected, but its smooth models are not. It is also tempting to formulate a kind of converse of the above assertion, as we shall do in §2 below.

Actually, we shall formulate more general conjectural statements (also for $S$-integral points). These statements lead to the study of what happens on considering only ramified covers, omitting unramified ones: such omission is not a matter of taste, but has the good reason that, as we have just observed, the alternative CWT predicts lifting of all rational points in all the unramified cases.

This motivates the definition of a new property, including in a sense the HP, and seemingly with much the same applications as the HP. See §2 for further comments, especially §2.2 for an explicit statement, and see the papers [9], [14], [41] for some results regarding implicitly this different property.

1.5. Weak approximation and the Hilbert Property. Below we let $k_v$ denote the completion of the number field $k$ with respect to a place $v$.

We start by recalling the definition of the weak-weak approximation property, denoted WWAP in the sequel:

WWAP: A variety $X/k$ has the WWAP if there exists a finite set $S_0$ of places of $k$ such that, for every finite set $S$ of places of $k$, $S$ disjoint from $S_0$, the diagonal embedding $X(k) \hookrightarrow \prod_{v \in S} X(k_v)$ is dense.

In other words, there are so many rational points on $X$ to approximate simultaneously within $S$ to any finite set of $k_v$-rational points. There is an analogous notion for integral points. Also, we speak of a weak approximation property WAP if $S_0$ may be taken to be empty.

It is not difficult to see that for smooth varieties (not necessarily complete) these properties are invariant by birational isomorphism over $k$, as happens for the HP.

Actually, a precise link with the HP was established by Colliot-Thélène and Ekedahl, who observed that the WWAP implies the HP: see [29]. (In fact, those arguments show a bit more, i.e. that it is sufficient that ‘most’ local points can be approximated.)

It shall appear as a consequence of conjectures below that $k$-unirational varieties have the HP. Through the result just mentioned, this would follow also from a conjecture of Colliot-Thélène asserting that a unirational variety has the WWAP,

The implication established by Colliot-Thélène and Ekedahl, combined with Theorem 1.6 results in particular in the following further implication:

If the projective variety $X/k$ has the WWAP then $X$ is algebraically simply connected.

This conclusion appears (in slightly different form) already in the 1989 paper [24] by Minchev, where it is proved independently of the above circle of observations (but at bottom with similar arguments). See also Harari’s paper [17] for a survey discussion presenting also several examples related to (weak) approximation properties. The paper [16] by Harari also discusses interesting relations between obstructions to weak approximation and the fundamental group; these go in a direction similar to the present paper, even if the HP is not directly considered.

These results represent instances, different from e.g. the well-known ones involving zeta functions, of how an arithmetical aspect may happen to reveal and actually prove a topological one.

Concerning mutual implications among these properties, apart from the mentioned ones, we shall discuss in §3.3 below a cubic surface, unirational but not rational over $\mathbb{Q}$,

13There are also concepts of strong a. p. - different from WAP only for integral points - and of hyper-weak a. p., this last one being introduced by Harari, see [18].
PIETRO CORVAJA AND UMBERTO ZANNIER

He proved that it has not the WAP; we shall reproduce his argument and slightly expand it for a similar surface. We shall also prove the HP for such a surface, so we can conclude that

**Theorem 1.7.** The Hilbert Property for a surface over $\mathbb{Q}$ does not generally imply the Weak Approximation Property for its smooth part.

(Note that the analogous statement for curves would not be true, since the only curves with the HP are rational and thus their smooth part has the WAP.) However we do not know whether the HP generally implies at least the WWAP; we tend to believe this should not be the case, but for this we have neither a proof nor real evidence.

The Appendix 2 below shall discuss the WWAP for the above cubic surface, essentially proving (a slightly different form of) it: see Theorem 12.15 (This is also sufficient to yield a different proof of the HP for that surface.)

Other examples in this direction are possible for integral points. We just pause for a few words on the celebrated Markoff-surface $x^2 + y^2 + z^2 = 3xyz$, in which the nonzero integral points are Zariski-dense and all in a single orbit of $(1, 1, 1)$ by the (discrete) group of automorphisms, generated by $(x, y, z) \mapsto (x, y, 3xy - z)$ and the permutations of coordinates. The problem of WAP for integral points on this surface is open, but Bourgain, Gamburd and Sarnak [6] proved much in this direction, showing in particular that for ‘almost all’ primes (in a strong sense) we may approximate by integral points any nonzero point over $\mathbb{F}_p$. (Their methods should be amply sufficient for instance to derive the HP for integral points of this surface.)

In the next section we shall raise some questions and conjectures originated from the links that we have alluded to. In [3] we shall give detailed proofs of the mentioned results, which concern surfaces of K3 and Enriques type. In the Appendices we shall sketch proofs of other assertions, for Kummer surfaces and relations with approximation properties. In view of the Bombieri-Lang conjecture that surfaces of general type should never have a Zariski-dense set of rational points, all of this basically exhausts the study of the HP for surfaces with a Zariski-dense set of rational points.

2. SOME QUESTIONS, CONJECTURES AND VARIATIONS

In this section, in the direction of what we raised before, we put forward some questions and tentative conjectures, and also state some further results. We shall consider also the case of integral points on affine varieties.

We start with questions.

2.1. Questions and conjectures. We have pointed out that a projective variety non-algebraically simply connected cannot have the Hilbert Property and we start by asking a converse of this:

**Question-Conjecture 1:** Any smooth algebraically simply connected projective variety (over $k$) with a Zariski-dense set of rational points has the Hilbert Property (over $k$).

Note that the smoothness assumption cannot be omitted, a relevant example coming (once more) from the surface in Theorem 13.

A proof of this statement, if at all true, would be most probably very difficult.

Since the HP is a birational invariant, any projective variety with a smooth model which is simply connected should have the HP as soon as its rational points are dense. This would include for instance unirational varieties (over $k$) since after a result of Serre [31] any smooth model of a unirational variety is simply connected and of course their $k$-rational points are Zariski-dense. By a more general result - see [12], Corollary 4.18 - this would also imply that a rationally connected variety over $k$ has the HP [3].

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14Actually, an independent and more general proof of the WWAP for this surface appears already in the paper [7]; we thank Colliot-Thélène for this (and related) reference(s).

15This case is also linked with other questions concerning the Brauer Manin obstruction: it is not known whether for rationally connected varieties this is the only obstruction to weak approximation: if
We also remark that in turn this would imply a positive answer to the Inverse Galois problem. Indeed, as is well known (see [29] and [37]), there is a strong link between the HP and the Inverse Galois Problem, provided by the method introduced by E. Noether of taking a quotient e.g. of $\mathbb{P}_n$ by the action of a linear representation of a given finite group $G$ (for instance by permutation of coordinates), obtaining a unirational variety.

However, this statement looks very optimistic (only little evidence is provided by Theorem [1.4], and as in some conjectures of Vojta it is perhaps necessary to weaken the conclusion, and to allow (at least) an enlargement of the number field in order to ensure the Hilbert Property.

To appreciate the strength of the statement, we note that the above version would have striking (unknown) consequences as the following one:

Let $A$ be an abelian variety over $\mathbb{Q}$ with a Zariski dense set of rational points. Then there exists a non-rational point $p \in A$, defined over some quadratic field such that $-p$ equals the quadratic conjugate of $p$.

This is obtained on applying the statement of Question-Conjecture 1 to the quotient $X := A/\pm 1$, which may be checked to be algebraically simply connected (it is $\mathbb{P}_1$ when $A$ is an elliptic curve). Since $A$ is a double cover of $X$, the statement predicts rational points on $X$ which do not come from $A(\mathbb{Q})$, and this translates in the existence of $p$. Actually, one would get a Zariski-dense set of such points.

Already the cases when $A$ is a product of several elliptic curves lead to arithmetical questions such as:

Let $f_1, \ldots, f_r \in \mathbb{Q}[x]$ be $r \geq 2$ cubic polynomials such that all curves $y^2 = f_i(x)$ are elliptic with infinitely many rational points. Is it true that necessarily there are rational numbers $u_1, \ldots, u_r$ such that no $f_1(u_i)$ is a square but all products $f_1(u_i)f_j(u_j)$ are squares in $\mathbb{Q}^+$?

A positive answer would follow from the above.

We do not know how to treat this problem in general, but in Appendix 1 below we shall give a positive answer for the cases $\dim A = 2$. In these cases the surfaces $X$ which appear are called Kummer surfaces, a classical object of study.

Anyway, apart for these instances, we have no real evidence even for the validity of these hypothetical statements (not to mention the full conjecture). On the other hand, at least the said implication for the quotients $A/\pm 1$ may be obtained on enlarging the ground field, for which we sketch a possible argument: on taking a dominant map, say of degree $d$, from $X$ to $\mathbb{P}_n$ ($n = \dim X$), just by Hilbert’s theorem we may find rational points on $\mathbb{P}_n$ which lift to points $p$ of degree degree $2d$ on $A$, and degree $d$ on $X$. If $k$ is the field generated by the image of $p$ in $X$, then $p$ is defined over a quadratic extension $k'$ of $k$, but not over $k$. By taking multiples of $p$, in general we obtain a Zariski-dense set of such points on $A$ (with the same $k, k'$), which give what is needed. (These multiples shall be Zariski-dense for a suitable choice of the starting rational point on $\mathbb{P}_n$.)

In any case, forgetting the hypothesis on simply connectedness, we can also extend the statement to all varieties, asking for instance the following variant.

**Question-Conjecture 2:** Let $X/k$ be a variety with a Zariski-dense set of rational points. If $\pi_i : Y_i \rightarrow X$, $i = 1, \ldots, m$, are finitely many covers of degree $> 1$, each with non-empty ramification, then $X(k) = \bigcup_{i=1}^{m} \pi_i(Y_i(k))$ is still Zariski-dense in $X$.

Again, a weaker and more plausible version is obtained by allowing a number field larger than $k$ in the conclusion, and/or also by adding more stringent requirements on the branch locus, or else the conclusion could maybe hold generally only after bounding the dimension.

\[ \text{this was the case, then HP for smooth rationally connected varieties would hold; we thank Borovoi for explaining us this link} \]
In the case of curves, Question-Conjecture 2 is settled: the crucial cases arise from curves of genus zero (solved by Hilbert Irreducibility Theorem) and curves of genus one (which follows from Faltings’ Theorem\(^1\)).

Apart from the above remarks, some evidence for these statements comes from the celebrated conjectures of Vojta, which, very roughly speaking, predict that (possibly at the cost of a finite extension of the ground field) the distribution of rational points is governed by the ‘bigness’ of the canonical class: *the bigger \(K_X\) is, the smaller chance we have to find many rational points on \(X\).* (See [5], Ch. 14, for a discussion of several of these conjectures.)

Now, if \(X\) is (smooth and) simply connected, any cover \(\pi: Y \to X\) of degree \(> 1\) has ramification, and the ramification divisor \(R\) (considered now with appropriate multiplicities, see [5], p. 473) contributes to \(K_Y\), for we have \(K_Y = \pi^*(K_X) + R\). Hence, if \(R \neq 0\), and especially of \(R\) is big, we expect to find less rational points on \(Y\) than those which would be accounted if most rational points on \(X\) would lift.

Actually, this principle is made much more explicit in Manin’s conjectures on the distribution of rational points on Fano varieties, i.e. those with \(-K_X\) ample. After some variants of Manin’s conjecture, one could expect to find ‘more’ points on \(X(\mathbb{Q})\) than those coming from \(Y(\mathbb{Q})\) (or from intermediate covers).

Perhaps a version of Manin’s conjecture for singular Fano varieties, as formulated by Batyrev, Manin and Tschinkel (see e.g. [2], [4]) might provide the HP for suitable quotients of projective spaces, so implying a positive solution to the Inverse Galois Problem by the approach of Noether mentioned above. Some steps in this direction have been done by T. Yasuda in [39], [40].

### 2.2. A modified HP

The property implicit in the Question-Conjecture 2 is a modified natural form of the HP, which we shall call *Weak Hilbert Property* abbreviated WHP, i.e.:

**Definition.** We say that a normal variety \(X/k\) has the *Weak Hilbert Property* WHP if, given finitely many covers \(\pi_i: Y_i \to X\), \(i = 1, \ldots, m\), each ramified above a non-empty divisor, \(X(k) - \bigcup_{i=1}^m \pi_i(Y_i(k))\) is Zariski-dense in \(X\).

Namely, it is the same as the HP, except that we consider only covers ramified (above a divisor). And hence, in particular, for algebraically simply connected varieties it coincides with the HP.

In fact, we have seen that there is a very good reason to disregard unramified covers, because the (varied) CWT shows that as soon as we have such a cover then all rational points can be lifted to finitely many covers.

As expressed by the above Question-Conjecture 2, this new property WHP, differently from the HP, could possibly hold for *all varieties with a Zariski-dense set of rational points*.

For instance, it holds for all curves with infinitely many rational points (*a posteriori* of genus \(\leq 1\)), since it holds for rational curves and since a ramified cover of a curve of genus 1 has genus \(> 1\) and one can apply Faltings’ theorem (or even a previous estimate on heights by Mumford). For surfaces, the results of this paper combined with the results of [41] for covers of some abelian surfaces and with the Bombieri-Lang conjecture for surfaces of general type, also provide substantial evidence of a general validity.

Regarding abelian varieties of any dimension \(> 1\), unfortunately there are no general results for rational points on covers of them as strong as the results for subvarieties (although the mentioned Vojta’s Conjecture predicts degeneracy of rational points on each ramified cover of an abelian variety). The issue of covers is treated in [41] with different methods compared to the use of Faltings’ or similar theorems. This is sufficient to prove the WHP for products of elliptic curves with the (unessential) restriction that we deal with a Zariski-dense cyclic group of rational points. Those methods in principle

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\(^{16}\)Historically, the case of curves of genus one was settled previously by Néron, [26], via a method based on heights estimates on abelian varieties, subsequently much improved by Mumford; still another proof for this case can be deduced from the results in [41].
should extend to any abelian variety, provided certain results on the Galois action on torsion points are available.

### 2.3. Integral points

One can formulate in a rather obvious way analogous statements for $S$-integral points, in which the variety $X$ can be non complete, and the ($S$-) integral points are meant e.g. with respect to a compactification. In this case the analogue of the HP is that not all $S$-integral points on $X$ can be lifted to rational points of finitely many covers of degree $> 1$.

Now, there is a version of the Chevalley-Weil Theorem true for $S$-integral points. We state it, denoting by $\mathcal{O}_S$ the ring of $S$-integers of $k$ and by $X(\mathcal{O}_S)$ set of $S$-integral points of the affine variety $X$:

**CWT for integral points:** Let $\pi : Y \to X$ be an irreducible cover which is also an unramified finite morphism of affine varieties over the number field $k$. Then there is a finite extension $k'/k$ such that $X(\mathcal{O}_S) \subset \pi(Y(k'))$.

Much the same considerations as above apply for this case, and one is led to a WHP on considering only covers with a divisor of ramification not contained in the divisor at infinity.

A general version of Question-Conjecture 1 could be:

**Question-Conjecture 1bis.** Let $X/k$ be a smooth affine variety topologically simply connected and such that the set of $S$-integral points are Zariski-dense. Let $\pi_i : Y_i \to X$ be morphisms of finite degree from algebraic varieties $Y_1, \ldots, Y_n$. Then $X(\mathcal{O}_S) - \bigcup_{i=1}^n \pi_i(Y(k))$ is Zariski-dense.

We can naturally formulate also an analogue of Question-Conjecture 2. For brevity we do not repeat this, and only recall that some results in this direction are proved in [9], [14], [41] for multiplicative tori $X = \mathbb{G}_m^n$. In this toric case the $S$-integral points are those with $S$-unit coordinates, and they form a finitely generated group. In the quoted papers it is shown in particular that a Zariski-dense subgroup of integral points cannot be lifted to rational points of a finite number of ramified covers, confirming the statement of Question-Conjecture 2 (i.e. the WHP) for these cases.

### 2.4. The Hilbert Property and Nevanlinna Theory

We conclude this introductory part by mentioning that these topics admit natural analogues in the context of Nevanlinna Theory (where deep links with Diophantine Geometry have been recognised since long ago, especially by Vojta, see [38] and [5]). For instance, the analogue of Chevalley-Weil is well known: since $\mathbb{C}$ is simply connected, given an unramified cover of algebraic varieties $Y \to X$ over $\mathbb{C}$, every holomorphic map $\mathbb{C} \to X$ lifts to $Y$.

Concerning the HP, an analogue of Question-Conjecture 1 could be as follows:

**Let $X$ be a simply connected smooth projective algebraic variety over $\mathbb{C}$ with an analytic map $f : \mathbb{C} \to X$ with Zariski-dense image. Then for every cover $Y \to X$ (in the present sense) of degree $> 1$ there is an analytic map $g : \mathbb{C} \to X$ that does not lift to $Y$.**

Similarly, there are analogies also with other issues raised in this paper.

### 3. Proof of main assertions

#### 3.1. A non-rational surface with the Hilbert Property and the proof of Theorem [1,4]

In this subsection we provide an example of a non-rational surface with the Hilbert Property, proving in particular Theorem [1,4].

It seems that such examples, where one can actually prove the HP and not merely fit it into conjectures, are not so common: on the one hand we know from the above that we

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17 This happens in considerable generality, as shown e.g. by Serre, after Faltings’ proof of certain Tate conjectures.

18 They may be defined as the points which do not reduce to the divisor at infinity for any prime outside $S$.

19 It is immaterial that we require that the points are integral also in the covers, since one can change anyway slightly a cover to get a finite map and ensure integrality of rational points sent to integral ones.
must start with a simply connected variety, on the other hand we have to provide ‘many’ rational points.

Our example is furnished by the Fermat quartic smooth surface (K3) defined in \( \mathbb{P}_3 \) by

\[
F: \quad x^4 + y^4 = z^4 + w^4.
\]

It is well known that this is not (uni)rational (because e.g. the canonical class vanishes, hence it has global sections) and that it is simply connected, because e.g. it is a smooth complete intersection of dimension > 1.

Also, it has a Zariski-dense set of rational points; this is due to Swinnerton-Dyer \[1] by the authors: see Thm. 3 and Cor. 2 therein.

(Being simply connected and with a Zariski-dense set of rational points, the HP is actually predicted by the conjectures above.)

We reproduce a proof of this density here, borrowing from Swinnerton-Dyer’s construction, for completeness and usefulness for the sequel.

This surface \( F \) contains forty-eight lines, of which eight are defined over \( \mathbb{Q} \), and given by \( x = \pm z, y = \pm w, \) or \( x = \pm w, y = \pm z \). A first idea is, for one of these lines, to consider the pencil of planes through this line and intersect it with \( F \); this shall produce a curve which is a union of the line with a cubic in the plane, which turns out to be generically smooth. Now, any other of the lines which does not intersect the former one shall intersect the plane in one point necessarily on the cubic. This yields a section of the pencil of cubics (i.e. a rational map from any point of \( \mathbb{P}_1 \) to the corresponding cubic); in turn, choosing this point as an origin on the cubic, shall yield an elliptic curve. By changing this last line we shall obtain another section and thus a point on the elliptic curve, distinct from the origin. \[2\] It turns out that this point is not (identically) torsion.

Now, restricting the pencil to rational points on \( \mathbb{P}_1 \), we obtain a family of elliptic curves defined over \( \mathbb{Q} \), with a rational point which can be torsion at most for finitely many values (by a theorem of Silverman or a direct argument). This provides a Zariski-dense set of rational points.

To be explicit, say that we start with the line \( L : x = z, y = w, \) with corresponding pencil of planes defined by \( \Pi_\lambda : w - y = \lambda (x - z), \lambda \in \mathbb{P}_1 \). We find for the cubic the (further) equation

\[
x^3 + x^2 z + x z^2 + z^3 = 4 \lambda y^3 + 6 \lambda^2 y^3 (x - z) + 4 \lambda^3 y (x - z)^2 + \lambda^4 (x - z)^3.
\]

Call \( E_\lambda \) this cubic (understood on the said plane). It is singular only if \( \lambda (\lambda^3 - 1) = 0 \) (or \( \lambda = \infty \)), and we shall tacitly disregard these points.

Note that the \( E_\lambda \cup L \) are the fibers for the rational map \( \lambda : F \rightarrow \mathbb{P}_1 \), given by \( \lambda := \frac{w - y}{x - z} \). Actually, this map is well defined on \( F - L \) \[23\] with fibers \( E_\lambda - L = E_\lambda - \{ (\eta : 1 : \eta : 1) : \eta^3 = \lambda \} \), and of course restricts to a regular map on the whole \( E_l \), provided \( l \) is such that this last curve is smooth.

If we intersect the plane \( \Pi_\lambda \) with the different line \( L' : x = -z, y = -w, \) we get the point \( (1 : -\lambda : -1 : \lambda) \in E_\lambda \). As above, we equip \( E_\lambda \) with a structure of elliptic curve by prescribing this point as the origin.

Intersecting now the same plane with the line \( x = w, y = -z, \) we get the new point \( (\lambda + 1 : 1 - \lambda : -1 : \lambda + 1) \in E_\lambda \).

20It is worth noticing that, somewhat in the converse direction, no example seems to be known of a smooth simply connected projective surface where the rational points can be proved to be not Zariski-dense; this should be expected e.g. for smooth hypersurfaces in \( \mathbb{P}_3 \) of large enough degree (which are of general type). Instead, for integral points on affine surfaces such an example may be found in the paper [33] by the authors: see Thm. 3 and Cor. 2 therein.

21He actually proved the density of rational points in the euclidean topology, inside the set of real points.

22Here we could also use the ‘tangent process’, i.e. intersecting the cubic with the tangent at the previous point; or else we could intersect the cubic with the line. However this last method, though sufficient for our purposes, would produce rational points only for a ‘thin’ subset of the pencil.

23In fact, if we define \( \lambda \) by the same formula on the whole \( \mathbb{P}_3 \), then it is not defined at any point of \( L \). However, its restriction to \( F \) can be continued to a regular map on the whole surface.
We omit here the verification that the difference of the two points is not torsion identically in $\lambda$, and refer to [33]; we only add that this may be done by using either specialization at some $\lambda \in \mathbb{Q}$ (and then any of the usual arithmetic methods) or also on considering functional heights, on viewing $\lambda$ as a variable.

In particular, by a well-known result of Silverman, the values of $\lambda$ such that this point is torsion on $E_\lambda$ are algebraic and have bounded height, hence the rational values of this type are finite in number. This implies that $E_l(\mathbb{Q})$ has positive rank for all but finitely many $l \in \mathbb{Q}$, proving that $F(\mathbb{Q})$ is Zariski-dense in $F$.

**Remark 3.1.** The paper [33] of Swinnerton-Dyer gives a Weierstrass equation and uses in practice the Lutz-Nagell theorem for any rational value of $\lambda$, avoiding the appeal to Silverman’s result. Also, it does not only verify that the rational points are Zariski-dense in $F$, but proves that they are dense in the real topology. This is a ‘piece’ of the WAP for $F$. On the other hand, we do not know whether $F$ has the WAP or even the WWAP. This is studied by Swinnerton-Dyer also in the more recent paper [36], where a double elliptic fibration is used, as we shall also do for our (different) purposes.

For future reference, we note that starting with e.g. the different line $L': x = w, y = -z$ and intersecting $F$ with the pencil of planes $\Pi'_\mu: x - w = \mu(y + z)$, we obtain another pencil of elliptic curves $E'_{\mu}$; essentially fibers of the rational map $\mu = \frac{w - x}{y + z}: F \to \mathbb{P}_4$. The rational points on $F$ produce rational values of both maps $\lambda, \mu$ (when they are defined).

We shall now prove the HP for $F$, using both of the above fibrations by elliptic curves. We note that even with the help of the Vojta’s conjecture an immediate deduction of the HP seems not obvious. Indeed, since $K_F = 0$, the canonical class of a cover $Y$ of $F$ is given by the ramification divisor $R$ (considered with multiplicities). In case this is a ‘big’ divisor, the Vojta’s conjecture would imply that $Y(k)$ is not Zariski-dense in $Y$ and then this cover would be negligible. But the bigness of $R$ is not guaranteed, and to take care of this possibility appears implicitly also in our argument below.

If the Hilbert Property fails for $F/\mathbb{Q}$, then let $\pi_i: Y_i \to F$ be finitely many covers each of degree $\geq 1$ such that $F(\mathbb{Q}) = \bigcup \pi_i(Y_i(\mathbb{Q}))$ is not Zariski-dense. We may suppose that the $Y_i, \pi_i$ are defined over $\mathbb{Q}$, for when this does not happen the corresponding map produces a non-dense set of rational points on $F$, as is easy to see by conjugating over $\mathbb{Q}$.

We shall inspect these covers by restricting them above the elliptic curves coming from the above fibrations.

Let $Y, \pi$ be one of these covers, where we can assume $Y$ to be projective and $\pi$ finite. Note that $F$ is smooth and simply connected, hence algebraically simply connected.

Therefore the branch locus of $\pi$ contains some irreducible curve $B$ on $F$.

Now, a first case occurs when $\lambda$ is not constant on $B$ (including the case $B = L$), and let us call ‘of the first type’ the covers containing such a $B$.

This fact implies that $B$ shall generically meet the elliptic curves $E_l$.

A subcase occurs when the cover is generically reducible above $E_l$ for $l \in \mathbb{C}$. Then it becomes reducible after a base change to $C \times \mathbb{P}_1 F$, where $C$ is a suitable curve (the fiber product being understood for the map $\lambda$ on $F$). But then, since the cover is irreducible over $F$, the components above $E_l$ may be defined over $\mathbb{Q}$ only for a thin set of rational values of $l$. That is, outside a thin set of $l \in \mathbb{Q}$, the cover restricted above $E_l$ is irreducible over $\mathbb{Q}$ but reducible over $\overline{\mathbb{Q}}$. It is a well known easy fact that this implies that the cover

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24 This double fibration by elliptic curves yields two sets $\{m_i : m \in \mathbb{Z}\}$, $i = 1, 2$ of rational endomorphisms of $F$ obtained by integer multiplication on the two families; on using the compositions $[m_i] \circ [m_2], m, n \in \mathbb{Z}$, applied to a given rational point, we again obtain a Zariski-dense set of rational points.

25 We note that considering a single fibration (and merely the rational points coming from the above sections) would not suffice in absence of additional information. Indeed, under e.g. the first fibration, $F$ may be seen as an elliptic curve over $\mathbb{Q}(\lambda)$, with a point of infinite order defined over $\mathbb{Q}(\lambda)$; then the group generated by this point would lift to the union of the two covers of this elliptic curve obtained by division by 2 followed by suitable translations (as in the weak Mordell-Weil).
restricted to such a rational \( l \) has only finitely many rational points (see e.g. [29], p. 20, first Remark).

In conclusion, in this subcase we have only finitely many rational points above \( E_l \), except possibly for a thin set of \( l \in \mathbb{Q} \).

The other subcase occurs when the cover is generically irreducible over \( E_l \), hence irreducible for all but finitely many \( l \in \mathbb{C} \); now the (irreducible) curve above \( E_l \) will have genus \( > 1 \) and shall contain only finitely many rational points by Faltings’ theorem. So we still have the previous conclusions, actually for all but finitely many \( l \).

We also conclude that if all the covers in question would be of this type then for ‘most’ \( E_l \) only finitely many rational points of \( E_l \) would lift to the cover, and we would have a contradiction.

Let us then study separately the covers such that the whole branch locus (apart from finitely many points) consists entirely of curves on which \( \lambda \) is constant (so curves which are components of some \( E_l \)); let us refer to these covers as being ‘of the second type’.

Again we have two subcases, the first one being when the cover is generically reducible above \( E_l \). As before, apart from a thin set of rational \( l \), only finitely many points may lift to the cover, and we may disregard these covers as well, and put them together with the ones of the first type.

We denote by \( T \) the union of the (exceptional) thin sets of rational numbers \( l \) that we have just described; it is still a thin set (in \( \mathbb{Q} \)).

Suppose then that \( Y \) is of the second type and that it remains generically irreducible above \( E_l \) for \( l \in \mathbb{C} \); we denote by \( S \) the set of these covers.

Then for a cover in \( S \), \( \pi^{-1}(E_l) \) will be irreducible for all but finitely many \( l \) (and we may assume these exceptional ones to be in \( T \)), unramified, and hence will become an elliptic curve \( E_l \) (after a choice of origin), and the map \( \pi|_{E_l} \) will be a translate of an isogeny. For \( l \in \mathbb{Q} - T \), our assumptions and the above arguments imply that all but finitely many rational points of \( E_l \) lift to rational points of some \( E_l \) (i.e. for a suitable \( (Y, \pi) \) in \( S \)).

Now, for all of these \( l \in \mathbb{Q} \), the group \( E_l(\mathbb{Q}) \) is a finitely generated abelian group of positive rank (by the above construction); and of course also the involved groups \( E_l(\mathbb{Q}) \) are finitely generated, and the \( \pi(E_l(\mathbb{Q})) \) shall be translates of subgroups of \( E_l(\mathbb{Q}) \). We now appeal to the following simple

**Lemma 3.2.** Let \( G \) be a finitely generated abelian group of positive rank, let, for \( u \in \mathbb{Q} \) in a finite set \( U \), \( H_u \) be subgroups of \( G \) and \( h_u \in G \). Suppose that \( G - \bigcup_{u \in U}(h_u + H_u) \) is finite. Then this complement is actually empty.

*Proof of Lemma.* Let \( U' \) be the subset of \( u \in U \) such that \( H_u \) has the same rank \( r > 0 \) of \( G \). The intersection \( H := \bigcap_{u \in U'} H_u \) is clearly of rank \( r \). If \( h_u \) denotes the image of \( h_u \) in \( G/H \) then \( \bigcup_{u \in U'}(h_u + (H_u/H)) \) either covers \( G/H \) or not.

In the first case we have \( \bigcup_{u \in U'}(h_u + H_u) = G \), proving the conclusion of the lemma (actually with a possibly smaller set of \( u \) than needed).

In the second case, the complement of \( \bigcup_{u \in U'}(h_u + H_u) \) in \( G \) contains some coset of \( H \), hence cannot be covered up to a finite set by finitely many cosets of subgroups of smaller rank, a contradiction which proves the lemma. \( \square \)

Using the above remarks and the lemma we conclude that for \( l \in \mathbb{Q} - T \) actually all the points in \( E_l(\mathbb{Q}) \) are lifted to rational points on some cover in \( S \).

Now, consider a curve \( E_m' \), for some general enough \( m \in \mathbb{Q} \), supposing also that \( m \) has large enough height so that \( E_m'(\mathbb{Q}) \) is infinite.

Since \( E_m' \) is not among the \( E_l \), it shall intersect each of the \( E_l \) in a finite set, which shall be nonempty for general \( m \). So, each cover in \( S \) shall be somewhere ramified above \( E_m' \).

By the same argument as given above for the covers of the first type, and for \( m \) outside a suitable thin set \( T' \) of \( \mathbb{Q} \), only finitely many points of \( E_m'(\mathbb{Q}) \) can lift to rational points of covers in \( S \).
However any rational point \( p \) in \( E'_m(Q) \) lies also on some \( E_l \), where \( l \) is just the value \( \lambda(p) \) of the map \( \lambda \) at the rational point.

Therefore, if we can find \( m \in Q - T' \) such that \( \lambda(p) \) is not in \( T \) for infinitely many \( p \in E'_m(Q) \), we have a contradiction with Lemma 3.2 which would say that all such points lift to some cover in \( S \).

We can then suppose that, for any given \( m \in Q - T' \) all but finitely many points \( p \) in \( E'_m(Q) \) (which is an infinite set) are such that \( \lambda(p) \in T \). Since \( T \) is a thin set, it is a finite union of images \( \varphi(Z(Q)) \), for morphisms \( \varphi : Z \to \mathbb{P}_1 \), of degree \( > 1 \), from certain curves \( Z \).

Each such morphism will be branched over a nonempty finite subset of \( \mathbb{P}_1 \) (actually containing at least two points) and let us denote the union of these finite sets of branch points by \( \mathcal{R} \subset \mathbb{P}_1(\overline{Q}) \).

The set of branch points of \( \lambda \) restricted to \( E'_m \) depends \textit{a priori} on \( m \). Indeed, it turns out that each branch point depends on \( m \), as we prove in the following

**Lemma 3.3.** For all \( m \) such that \( E'_m \) is smooth, the restriction to \( E'_m \) of the rational function \( \lambda \) is a map of degree two, having four ramification points. The value of \( \lambda \) at each of these ramification points is a non-constant (algebraic) function of \( m \).

**Proof.** This fact could be checked by a (nasty) computation, but we avoid this. Let us fix an \( m \) so that \( E'_m \) is a smooth cubic lying on the plane \( \Pi'_m \). In order to compute the degree of the map \( \lambda \) restricted to \( E'_m \), we have to compute the number of points in \( E'_m \) where it takes a generic value. Now, by our opening construction, a ‘value’ of the map \( \lambda \) is represented by a line passing through \( p_m := \Pi'_m \cap L = (1 + m : 1 - m : 1 + m : 1 - m) \in E'_m \); so, since such a line generically intersects \( E'_m \) in two more points, the degree of \( \lambda \) is 2. The ramified values of \( \lambda \) are the tangent lines passing through \( p_m \), apart from the tangent line at \( p_m \) (unless \( p_m \) is a flexus).

We want to prove that these values are not constant as \( m \) varies. Of course, in order to compare values of \( \lambda \) for different values of \( m \) we have to identify a value of \( \lambda \) with the corresponding plane, not with a line on the \( \mu \)-plane \( \Pi'_m \). More precisely, given the plane \( \Pi'_m \) and a point \( p \in E'_m \), \( \lambda(p) \) will be the \( \lambda \)-plane (i.e. in the family \( \Pi_3 \)) generated by \( p \) and \( L \), i.e. the \( \lambda \)-plane generated by the line connecting \( p \) with \( p_m \) and the line \( L \). It then makes sense to compare two values \( \lambda(p), \lambda(q) \) even if \( p \) and \( q \) do not lie on a same \( \mu \)-plane.

Now, if \( \lambda \) is a ramified value for \( \lambda \) were fixed, say equal to \( l \), the \( \lambda \)-plane \( \Pi_3 \) would contain a tangent line to \( E'_m \), passing through \( p_m \), for all values of \( m \in \Pi_1 \). To see that this does not happen, consider the degenerate cases \( E'_0 \) and \( E'_\infty \). These cubic curves degenerate into three lines, of which only one passes through \( p_m \). For \( E'_0 \) this line has an equation \( x - w = y - z = 0 \), while for \( E'_\infty \) it is the line of equation \( y + z = x + w = 0 \). These two lines are not coplanar, concluding the proof of the lemma.

\[\square\]

We can now conclude the proof of the theorem. By this lemma, the set of branch points of \( \lambda \) restricted to \( E'_m \) may intersect \( \mathcal{R} \) at most for finitely many \( m \in \Pi_1 \). Choose an \( m \in Q - T' \) such that this does not happen. Now fix a morphism \( \varphi : Z \to \mathbb{P}_1 \) as before. The intersection \( \varphi(Z(Q)) \cap \lambda(E'_m(Q)) \) is the image in \( \Pi_1 \) of the rational points in \( W(Q) \), where \( W \to \Pi_1 \) is obtained as a fiber product of \( \varphi : Z \to \mathbb{P}_1 \) and \( \lambda : E'_m \to \mathbb{P}_1 \). Since \( \varphi \) has degree \( > 1 \), \( \varphi \) must ramify above some point of \( \Pi_1 \) (actually above at least two points) and by our choice these points are not branch points for \( \lambda \). Hence the map \( W \to E'_m \) must also ramify somewhere, for each irreducible component of \( W \), which implies that each component of \( W \) has genus \( \geq 2 \), so contains only finitely many rational points by Faltings’ theorem.

This proves that \( \lambda(E'_m(Q)) \cap \varphi(Y(Q)) \) is finite for each morphism \( \varphi \) of the given finite set, hence \( \lambda(E'_m(Q)) \cap T \) is finite.

Summing up, in all cases we obtain a contradiction, proving finally the sought result.

**Remark 3.4.** (i) Inspection shows that for the proof Falting’s theorem may be replaced by Mumford’s earlier estimate for rational points on curves of genus \( > 1 \).
(ii) The given argument applies for more general K3 surfaces, in particular also for the intermediate K3 surface giving rise to the example treated in the next subsection (see Remark 3.15(iii) for a few words on this). We have preferred to stick to the special case of the Fermat surface for the sake of simplicity.

(iii) As already mentioned, we do not know whether this surface $F$ has the WAP or at least the WWAP; this is unlikely to hold if the only rational points in $F$ come from the above construction.

3.2. An Enriques surface without the Hilbert Property and the proof of Theorem 1.3

We shall now present the details of the example announced in the introduction, of a surface without the Hilbert Property, with a Zariski-dense set of rational points and admitting no non-constant maps to abelian varieties, hence proving Theorem 1.3.

This surface is a so-called Enriques surface, defined as a smooth surface with vanishing irregularity and geometric genus, and vanishing $2K$; no such surface can be rational, e.g. because of this last fact. Such surfaces can have a Zariski-dense set of rational points; some explicit examples can be found e.g. in [19]. Actually, Bogomolov and Tschinkel [4] proved that after a finite extension of their field of definition, every Enriques surface has a Zariski-dense set of rational points (potential density of rational points).

To obtain our example, we start from the automorphism $\sigma$ of $F$ defined by

$$\sigma(x : y : z : w) = (x : ty : -iz : -w).$$

It has order 4 and no fixed points in $F$. Its square is $(x : y : z : w) \mapsto (x : -y : z : -w)$ and has the eight fixed points $(1 : 0 : 0 : i^m)$, $(0 : 1 : i^m : 0)$ for $0 \leq m < 4$. Let $F'$ be the quotient of $F$ by the subgroup generated by $\sigma^2$ (see [20], III.12, for quotients of a variety by a finite group). It turns out that a smooth model of $F'$ is also a K3 surface, on which $\sigma$ acts (as an automorphism of order 2) without fixed points. In fact, a fixed point of $\sigma$ on $F'$ would correspond to an orbit $p, \sigma^i(p)$ on $F$, fixed by $\sigma$; this would imply either $\sigma(p) = p$ or $\sigma(p) = \sigma^2(p)$, so either $p$ or $\sigma(p)$ would be fixed by $\sigma$ on $F$, which is not the case. So $F'$ is an unramified cover of its quotient by the subgroup of order 2 generated by $\sigma$. A smooth model of this quotient is the desired Enriques surface.

We omit a discussion of smooth models and also on the quotient varieties so obtained (which is not in fact necessary for our assertions) and only say that a surface birational to the Enriques, call it $E$, may be defined in $\mathbb{P}_3$ by the equation appearing in Theorem 1.3 i.e.

$$E : x_0x_2^4 + x_1x_3^4 = x_0^2x_1^4 + x_0^3x_1^2,$$

(with singular points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$) whereas $F'$ is birational to the cover $\mathcal{F}$ of $E$ given in $\mathbb{P}_4$ by the further equation $x_0x_1 = x_2^2$.

Note the dominant rational map from $F$ to $E$ defined by

$$x_0 = x'^4, x_1 = y'^4, x_2 = xy'^2z, x_3 = x'^2yw,$$

where the four expressions are invariant by $\sigma$. Through the rational points that we have observed on $F$, this yields a Zariski-dense set of rational points on $E$.

We continue by proving Theorem 1.3 where we could use Theorem 1.6 through the above variant of the Chevalley-Weil Theorem; but it is easier to argue directly, on showing that any rational point on $E$ lifts either to $F$ or to the similar cover $F^-$ defined by the equation $x_0x_1 = -x_2^2$.

But before this, let us observe, independently of the above interpretation of $E, F$ as related to quotients of $F$, that these covers are indeed irreducible. For this, we may use the functions $u_i := x_i/x_0$ and the equation $u_2 = u_1(u_1^2 + u_1 - u_3^2)$ for $E$, and the further equation $u_3^2 = \pm u_1$ for $F, F^-$ resp. (exhibiting our surfaces as covers of the $(u_1, u_3)$-plane). Since none of the factors on the right side of the first equation is a square in $\mathbb{C}(u_1, u_3)$, it follows that the equations yield linearly disjoint extensions of $\mathbb{C}(u_1, u_3)$, proving what we need.

Let now $(a_0 : a_1 : a_2 : a_3) \in E(\mathbb{Q})$, where $a_i$ are coprime integers. It suffices to show that one among $\pm a_0a_1$ is a square, where we can assume that $a_0a_1 \neq 0$. For this, let $p$ be a prime number and let $p^{e_i}|a_i$, so $e_i$ are integers $\geq 0$. If the $p$-adic order of $a_0a_1$ is not
even, then \(e_0 + e_1\) is odd. But then the numbers \(e_0 + 4e_2, e_1 + 4e_3, 2e_0 + 3e_1, 3e_0 + 2e_1\) are pairwise distinct, because they are congruent modulo 4 resp. to \(e_0, e_1, e_1 + 2, e_0 + 2\), and hence actually pairwise incongruent modulo 4. However this is impossible because these numbers are the \(p\)-adic orders of the four terms appearing in the equation defining \(E\).

Hence \(a_0a_1\) has even \(p\)-adic order at every prime, proving our contention.\

Of course, this shows that \(E\) has not the Hilbert Property, as stated.

**Remark 3.5.** (i) Exactly the same proof shows that for every discrete valuation \(\nu\) of a field \(K\), and for each point \(P \in E(K)\), the valuation of \((x_i/x_0)(P)\) is even. If we apply this with \(K\) equal to the function field of \(E\), and observe that \(x_1/x_0\) is not a square in that function field, we obtain that every smooth model of \(E\) admits an irreducible unramified double cover, namely the one corresponding to taking the square root of \(x_1/x_0\). Also the quotient variety described above (i.e. \(F'/(\sigma >)\)) is not simply connected, because \(\sigma\) has no fixed points on \(F'\). On the contrary, as already said, this singular model is (topologically, so algebraically) simply connected.

(ii) The argument shows easily that \(E\) has not the Hilbert Property over any number field \(k\): again, we can use the refined Chevalley-Weil Theorem (applying in practice Theorem 1.6) or argue directly, using that elements having even valuation at all primes are squares up to finitely many factors (depending on \(k\)).

(iii) It is possible to prove the HP for the above surface \(F' = F/\sigma^2\); we outline the argument. Using the HP for \(F\) (i.e. Theorem 1.4) the crucial issue is to produce rational points on \(F'\) not coming from covers isomorphic to \(F\) (over \(\overline{\mathbb{Q}}\)). In turn this amounts to find ‘sufficiently many’ points on \(F\) defined over a quadratic field, on which \(\sigma^2\) acts as a conjugation. Our formulae translate this problem into rational points for twisted models of \(F\) defined by \(x^2 + dy^4 = d^2z^4 + w^4\), where \(d\) is a nonzero integer, not a square. Given then finitely many covers of \(F'\) of degree \(> 1\), it is a matter of routine to pick a suitable definite \(d\) (depending on the covers) and repeat the above proof pattern of Theorem 1.4 for this new surface, concluding the argument.

### 3.3. A surface with the Hilbert Property but without the Weak Approximation Property and the proof of Theorem 1.7.

In this subsection we are going to exhibit an example of a surface \(X/\mathbb{Q}\) with the HP but without the WAP, proving in particular Theorem 1.7.

The negation of WAP is due in fact to Swinnerton-Dyer (who used the real valuation). We define \(X\) as a cubic in \(\mathbb{P}_3\), with coordinates \((t : x : y : z)\), by

\[
X : t(x^2 + y^2) = (4z - 7)(z^2 - 2t^2).
\]

It is singular precisely at the points \((0 : 1 : \pm i : 0)\). We observe at once that \(X\) becomes rational over \(\mathbb{Q}(i)\) (where the quadratic form \(x^2 + y^2\) is equivalent to \(xy\)). In particular, it is simply connected and has the WAP and the HP over \(\mathbb{Q}(i)\).

Actually, as stated in Theorem 1.7 \(X\) has the HP even over \(\mathbb{Q}\). To see this, proving then half of the theorem, we argue somewhat similarly to the case of the Fermat surface \(F\) of Theorem 1.3. Again, though this is simpler than before, we do not see any direct deduction of the HP from general known results.

Confining to the affine subset \(t \neq 0\), we divide by \(t\) and, setting \(x/t = \xi, y/t = \mu, z/t = \lambda\), we find a family of conics in coordinates \(\xi, \mu\):

\[
C_\lambda : \xi^2 + \mu^2 = (4\lambda - 7)(\lambda^2 - 2).
\]

and a family of elliptic curves in coordinates \(\lambda, \xi:\)

\[
E_\mu : \xi^2 = (4\lambda - 7)(\lambda^2 - 2) - \mu^2.
\]

---

20The argument yields directly that \(E(\mathbb{Q})\) lifts to \(F(\mathbb{Q}(i))\), which is analogue to the standard Chevalley-Weil theorem. The present device, on replacing one cover by two covers to maintain the ground field, is similar to the above version of Chevalley-Weil.

21We have preferred to use this model of a hypersurface of \(\mathbb{P}_3\) both for simplicity and also because it illustrates some subtleties related to the fundamental group of different models.

22For given \(\mu\), we may see this as a curve in the plane \((\lambda, \xi)\) and take as origin the point at infinity; in our setting this corresponds to the point \((0 : 0 : 1 : 0)\) on \(X\), though this is immaterial for us.
Now, for instance, we find the (smooth) rational point \( x = y = t = 1, z = 2 \) on \( X \). As remarked in [54], one may obtain infinitely many rational points by intersecting \( X \) with the tangent plane at the rational point, which would even show that \( X \) is unirational over \( \mathbb{Q} \); however here we shall proceed differently. Before this, we pause for a remark:

**Remark 3.6.** Certainly the rational points coming from a single unirational parametrization would not be sufficient to confirm the HP: in fact, since this surface is not rational (over \( \mathbb{Q} \)), these points would all come, by the very definition, from a cover of degree \( > 1 \), namely the rational cover which exhibits unirationality. Hence a proof of the HP requires to produce other rational points.

To go ahead, observe that the said rational point yields the rational point \( \xi = 1, \lambda = 2 \) on the elliptic curve \( E_1 \) which may be checked to be non torsion.

Then we find infinitely many rational points \( (\xi_n, \lambda_n) \in E_1(\mathbb{Q}) \) and rational points \( (1 : \xi_n : 1 : \lambda_n) \in X(\mathbb{Q}) \). Now, this also yields that the conic \( C_{\lambda_n} \) has the rational point \( (\xi_n, 1) \) and thus infinitely many rational points, proving that \( X(\mathbb{Q}) \) is Zariski-dense in \( X \).

To prove more, i.e. the HP, we proceed similarly to the case of the Fermat surface (however things shall be simpler now).

Base change by the map \( \mu : C_2 \to \mathbb{P}_1 \) yields a section \( s \) to the surface: i.e. for \( p = (a, b) \in C_2 \) we set \( s(p) = (2, a) \in E_b \). This is not identically torsion, because e.g. the map \( \mu \) is branched above \( \mu = \pm \sqrt{2} \), whereas the bad reduction of the family \( E_\mu \) occurs on a linearly disjoint field. Therefore a corollary of Silverman’s theorem implies that \( s(p) \) may be torsion on \( E_b \) only for finitely many rational points \( p = (a, b) \in C_2(\mathbb{Q}) \). In particular, for all but finitely many such rational values \( b \), we have that \( E_b(\mathbb{Q}) \) is infinite.

Note that for rational \( b \) a rational point \( q = (u, v) \in E_b(\mathbb{Q}) \) produces a rational point \( (v, b) \) on \( C_u \), and therefore \( C_u \) becomes birational to \( \mathbb{P}_1 \) over \( \mathbb{Q} \) and has ‘many’ rational points.

For later reference, let us denote by \( A \) the set of rational numbers \( l \) such that \( C_l \) has a rational point; the given argument says that

\[
l \in A \text{ whenever } l = \lambda(q) \text{ for a } q \in E_b(\mathbb{Q}), \text{ some } b \in \mu(C_2(\mathbb{Q})).
\]

Suppose now that \( X \) has not the Hilbert Property (over \( \mathbb{Q} \)), and let \( \pi : Y \to X \) be one of the covers of \( X \) which occur in this violation of HP, of degree \( > 1 \), defined over \( \mathbb{Q} \). We distinguish between two types of such covers.

- **First type:** this occurs when \( Y \) remains generically irreducible above \( C_\lambda \) (for \( \lambda \in \mathbb{C} \)).

  Then \( Y \) may be reducible above \( C_l \) only for finitely many \( l \in \mathbb{C} \). For each of the remaining \( l \in A \), the image \( \pi(Y(\mathbb{Q})) \) by definition covers at most a thin set in \( C_l(\mathbb{Q}) \).

- **Second type:** this occurs when \( Y \) is generically reducible above \( C_\lambda \) (i.e. reducible for generic values of \( \lambda \) in \( \mathbb{C} \)). Now, for a special value \( l \in \mathbb{Q} \), either \( Y \) is irreducible over \( \mathbb{Q} \) above \( C_l \) (but it is certainly reducible over \( \overline{\mathbb{Q}} \) since it is of second type) or not. In the first sub-case, being reducible over \( \overline{\mathbb{Q}} \) but irreducible over \( \mathbb{Q} \), it may have only finitely many rational points (above \( C_l \)). The second sub-case may happen (taking into account simultaneously all finitely many covers of this second type) only for a thin set \( T \) of values \( l \in \mathbb{Q} \), i.e. the set of these rational \( l \) is covered by images \( \varphi(Z) \), for finitely many morphisms \( \varphi : Z \to \mathbb{P}_1 \) of degree \( > 1 \).

The danger now is that \( T \) contains all the \( l \in \mathbb{Q} \) such that \( C_l \) has a rational point (i.e. that \( T \) contains \( A \)). In particular, in view of the above remarks in this ‘bad’ case \( T \) would contain all values of \( \lambda \) on the rational points of a curve \( E_b \), where \( b \) can be any element of \( \mu(C_2(\mathbb{Q})) \).

To exclude this, we note that the set of branch points of \( \lambda \) on \( E_u \) consists of \( \infty \) and the roots of the polynomial \((4\lambda - 7)(\lambda^2 - 2) - u^2 \). Now, each given complex number \( c \) can be such a root only for finitely many \( u \in \mathbb{C} \). But then we can find a \( b \) in the infinite set \( \mu(C_2(\mathbb{Q})) \) such that the set of branch points of \( \lambda : E_b \to \mathbb{P}_1 \) intersects the set of branch points of \( \varphi : Z \to \mathbb{P}_1 \) at most at infinity, and this for all of the \( \varphi \) in question.

Under this condition, (each component of) the pullback cover \( \lambda^*(Z) \) of \( E_b \) must be ramified, hence of genus \( > 1 \) and can have only finitely many rational points. Therefore for such numbers \( b \in \mathbb{Q} \) we find that \( \lambda(E_b(\mathbb{Q})) \cap T \) is finite, whence we may pick an.
4.1. Appendix 1: On the Hilbert Property for Kummer surfaces. In this Appendix we consider Kummer surfaces over \( \mathbb{Q} \), i.e. quotients \( X = A/\pm I \) of an abelian surface \( A \) defined over \( \mathbb{Q} \); we assume that \( A \) (and thus \( X \)) has a Zariski-dense set of rational points.

We shall sketch a proof that

There is a Zariski-dense set of rational points of \( X \) which may not be lifted to \( A \),

providing a piece of the Hilbert property for \( X \). We remark that the full Hilbert property for \( X \) would follow from this fact combined with the Vojta’s conjectures; this second part could presumably be dealt with unconditionally by the methods of [11], where we hope that someone (we?) will undertake the task to fill the details in.

The statement is invariant by isogeny, and thus we may split the proof into the two basic cases when \( A \) is either the product of two elliptic curves or \( A \) is simple; in turn, in this second case \( A \) is known to be isogenous to the Jacobian of a curve of genus 2, where we limit here ourselves to the case when everything is defined over \( \mathbb{Q} \). We assume that \( A \) has a Zariski-dense set of rational points.

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\( l \in \lambda(E_b(\mathbb{Q})) - T \). For such an \( l \) the covers of the second type as well absorb at most a thin subset of the rational points of \( C \), proving finally what we need.

It remains to check that \( X \) has not the WAP. Now, this is done in [34], where it is actually proved that there are no rational points in the real component of \( X(\mathbb{R}) \) defined by \( |z/t| \leq \sqrt{2} \). This itself excludes the WAP already for the real place of \( \mathbb{Q} \). In the next Remark we shall give an argument similar to the one in [34], for excluding the WAP for a similar surface, for which the rational points are dense in the real ones.

**Remark 3.7.** (i) The slightly different surface given by the equation

\[
(3) \quad t(x^2 + y^2) = (2z - 7t)(z^2 - 2t^2)
\]

has rational points in both connected components of the real points (e.g. \((1 : 2 : 1 : 1)\) in the component \(|z/t| \leq \sqrt{2}\) and \((1 : 13 : 1 : 6)\) in the other one). Then it is not too difficult to prove that the rational points are dense in the real points. However again the WAP fails: for instance there are no rational points \((a : b : c : d)\) in the second component, with coprime integer coordinates and such that \(a > 0, a \equiv d \equiv 1 \pmod{4}\), \(b \equiv 0, c \equiv 1 \pmod{2}\) (conditions which define a certain 2-adic neighbourhood of \((1 : 2 : 1 : 1)\)). For otherwise we would have \(2d - 7a \equiv -1 \pmod{4}\) and \(2d - 7a\) would be a positive integer dividing \(a(b^2 + c^2)\). Let then \(p\) be a prime \(\equiv 3 \pmod{4}\) and dividing \(2d - 7a\) to an odd power \(h\) (which would exist). Now, \(p\) divides \(b^2 + c^2\) to an even power, which is positive if and only if it divides both \(b, c\). It follows that if \(p\) would not divide \(a\) then \(p\) would have to divide \(d^2 - 2a^2\) as well, hence also \(41a^2\), a contradiction. Therefore \(p||a\), where \(l > 0\), and hence \(p\) divides \(d\), and can not divide both \(b, c\), hence it does not divide \(b^2 + c^2\).

We conclude that \(l = h + m\), where \(p^n||d^2 - 2a^2\) and it easily follows that \(p^n||d\) and \(m = 2h, l = 3h\). But then, repeating the argument for all such \(p\) we would conclude that \(a \equiv 2d - 7a \pmod{4}\), a contradiction.

(ii) These last surfaces are unirational, and thus expected to have the WWAP (according to a conjecture of Colliot-Thélène mentioned above). Actually equations (2) and (3) define so-called Châtelet surfaces; for these surfaces, WWAP was proved by Colliot-Thélène, Sansuc and Swinnerton-Dyer in [17]; the case of cubic Châtelet surface appeared also in [35]. For completeness, in the Appendix 2 below we sketch a proof, for the surface of the theorem, of a property very similar (albeit weaker) than the WWAP; it seems that those arguments should give the full WWAP.

In any case, we remark again that we do not know whether the WWAP is generally implied by the HP.

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4. Appendices

4.1. Appendix 1: On the Hilbert Property for Kummer surfaces. In this Appendix we consider Kummer surfaces over \( \mathbb{Q} \), i.e. quotients \( X = A/\pm I \) of an abelian surface \( A \) defined over \( \mathbb{Q} \); we assume that \( A \) (and thus \( X \)) has a Zariski-dense set of rational points.

We shall sketch a proof that

There is a Zariski-dense set of rational points of \( X \) which may not be lifted to \( A \),

providing a piece of the Hilbert property for \( X \). We remark that the full Hilbert property for \( X \) would follow from this fact combined with the Vojta’s conjectures; this second part could presumably be dealt with unconditionally by the methods of [11], where we hope that someone (we?) will undertake the task to fill the details in.

The statement is invariant by isogeny, and thus we may split the proof into the two basic cases when \( A \) is either the product of two elliptic curves or \( A \) is simple; in turn, in this second case \( A \) is known to be isogenous to the Jacobian of a curve of genus 2, where we limit here ourselves to the case when everything is defined over \( \mathbb{Q} \). We assume that \( A \) has a Zariski-dense set of rational points.

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*These reductions rely on the fact that a smooth model of \( X \) is known to be simply connected, and all its ramified covers not factoring via the canonical map \( A \to X \) come from ramified covers of \( A \).*
First case: $A = E_1 \times E_2$. Here we suppose that the elliptic curve $E_i$ is given by Weierstrass equation $y^2 = f_i(x)$, for cubic polynomials $f_1, f_2$ without multiple roots. Then $X$ is given birationally by the equation

$$f_2(x_2) = w^2 f_1(x_1).$$

Then, the above statement on points in $X(\mathbb{Q})$ which do not come from $A(\mathbb{Q})$ translates into the following assertion:

If both curves $y^2 = f_i(x)$ have infinitely many rational points, there exists a Zariski-dense set of rational solutions of the displayed equation such that $f_2(x_2)$ is not a square.

For this we consider the 2-dimensional family $Z$ of curves $Z_{u_1,u_2} : f_1(u_1)f_2(x_2) = f_2(u_2)f_1(x_1)$ parametrized by $(u_1,u_2) \in \mathbb{A}^2$; we have given an affine equation, but we implicitly consider the projective closure with respect to the plane $(x_1,x_2)$.

The generic member has genus 1 and there is a section given by $(x_1, x_2) = (u_1,u_2)$. This section can be taken as origin, giving $Z$ the structure of an elliptic curve $E$ defined over $\mathbb{Q}(u_1,u_2)$.

We can obtain a second section by the ‘tangent method’, i.e. intersecting the cubic $Z_{u_1,u_2}$ with the tangent line at $(u_1,u_2)$. Now, the difference of the sections gives a point on $E$, defined over $\mathbb{Q}(u_1,u_2)$. It may be checked that this point is not identically torsion.

Then by the results of Silverman, specialising first $u_1 = u_1$, for a rational point $(a_1,b_1) \in E_1(\mathbb{Q})$, then $u_2 = u_2$, for a rational point $(a_2,b_2) \in E_2(\mathbb{Q})$, of large enough height in both cases, we obtain a non-torsion rational point on the elliptic curve $b_1^2f_2(x_2) = b_2^2f_1(x_1)$. Multiples of this point are dense of the curve $Z_{a_1,a_2}$ and on varying $a_1,a_2$ this clearly yields the required density of rational points on our surface.

Second case: $A$ is the Jacobian of a curve $H$ of genus 2 defined over $\mathbb{Q}$. We let $y^2 = f(x)$ be an equation for $H$, where we may take $f \in \mathbb{Q}[x]$ to be a polynomial of degree 5 without multiple roots. We may suppose $A$ to be simple, for otherwise we fall in the previous case after isogeny.

There is one point $\infty$ at infinity on a smooth model of $H$ (i.e. the unique pole of $x$) and we may embed $H$ in $\mathbb{A}$ on defining, for $p \in H$, $[p]$ =class of the divisor $(p) = (\infty)$. Then $A$ is the set of sums $[p] + [q]$, for $p, q \in H$ (where the sums $[p] + [p']$ yield $0$, $p \mapsto p'$ being the canonical involution on $H$, because $p, p'$ are the zeros of the function $x - x(p)$).

We have an infinity of quadratic points on $H$ obtained by putting $x = a \in \mathbb{Q}$ and taking $y$ as a square root of $f(a)$ (omitting the cases when $f(a)$ is a square, a thin and actually finite set).

For $p$ one such quadratic point observe that $\phi(p) := 2[p] \in A$ is a point defined over a quadratic extension of $\mathbb{Q}$ and satisfies $\phi(p)^2 + \phi(p) = 0$, where $\sigma$ is the nontrivial automorphism of the quadratic field $\mathbb{Q}(\sqrt{f(a)})$; indeed. $\phi(p)^2 = \phi(p') = \phi(p') = -\phi(p)$, because of the above observation $[p] + [p'] = 0$. We have already observed in [22,11] that these points yield rational points on $X$ which do not come from rational points on $A$.

To find a Zariski-dense set of such points, observe that $n\phi(p)$, for $n \in \mathbb{Z}$ denoting multiplication in $J$, are points with the same property as $\phi(p)$ (relative to the same quadratic field). Certainly the points $\phi(p)$ cannot be all torsion on $A$, and taking their images in $X$ yields rational points of $X$, and provides the required dense set.

Remark 4.1. We observe that the multiples $n\phi(p)$, when written in the shape $[u] + [v]$, for $u, v \in H$, correspond generally to points such that $u^a = v'$ and such that, putting $u = (a,b)$, we have $a, b$ defined over our quadratic field but such that $a \notin \mathbb{Q}$.

Also, we remark that the same methods may be applied over any number field.

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30A priori the torsion order would be absolutely bounded, because a rational torsion point over $\mathbb{Q}(u_1,u_2)$ cannot have ‘too large’ order, due to the Galois theory of Fricke and Weber, no need to use the deeper arithmetical results here. But we do not even need to use this, or to perform computations: indeed, $Z_{u_1,u_2}$ depends only on $w := f_2(u_2)/f_1(u_1)$, whereas we may vary both $u_1,u_2$; then we would obtain a continuous family of points of finite order on a same elliptic curve, which is impossible.
4.2. Appendix 2: WWAP for a cubic surface. In this Appendix we shall sketch a proof of a form of the WWAP for the surface $X$ of §3.3 for a proof of a more general result see [7]. Namely, we shall consider only the following property: for a large enough $q$, for every finite set $S$ of primes $\ell > q$, and every set $\xi_{\ell} \in \mathbb{F}_{\ell}$, $\ell \in S$, there exists a rational point $x \in X(\mathbb{Q})$ such that $x \equiv \xi_{\ell} \pmod{\ell}$ for all $\ell \in S$.

We note that this property, a priori weaker than WWAP, is sufficient to imply the HP (by the same arguments given e.g. in [29]). And actually even less would suffice; for instance we could restrict the $\xi_{\ell}$ to lie outside a prescribed proper subvariety of $X$. In fact, we shall give detail only for this last even weaker property, and only indicate how one can remove this restriction.

In any case, in particular, the argument shall give as a byproduct an independent proof that $X/\mathbb{Q}$ has the HP.

Before going ahead, we let $k$ be any ground field of definition for $X$ (possibly char $k > 0$), and recall the above notation, in particular the rational maps $\lambda, \mu : X \dashrightarrow \mathbb{P}_1$, which fiber $X$ by conics $C_\lambda$ and elliptic curves $E_\mu$.

The affine conic $C : x^2 + y^2 = 1$, isomorphic to the algebraic group $SO_2$, acts simply transitively on $C_\lambda$ as $(\xi, \mu) \mapsto (\xi x - \mu y, \xi y + \mu x)$, which also extends to yield a group of automorphisms of $X$. For $g \in C$, we shall denote by $g \mapsto g(q)$ this action, for $q \in X$.

Corresponding to the elliptic curves $E_\mu$, we have multiplication maps $[m] : E_\mu \to E_\mu$, and we shall denote again by $[m]$ the rational map $[m] : X \dashrightarrow X$ obtained on viewing $X$ as an elliptic curve over $k(\mu)$.

By combining these maps, we can obtain a lot of rational self-maps on $X$ which are dominant and defined over the prime subfield of $k$. In particular, we can produce rational points starting from a given one. For the application we are discussing, it turns out that it suffices to use the dominant rational maps (of degree 4)

$$\sigma_g := [2] \circ g : X \dashrightarrow X, \quad g \in C.$$ 

We note that composing maps from these families, and looking at the orbits of a given point, we can exhibit $X$ as a unirational variety over the prime field.

We denote by $X_0$ the affine open subset $t \neq 0$ on $X$, with coordinates $\xi, \mu, \lambda$ and start from the point $p_0 = (1, 1, 2) \in X_0$. By acting with $\sigma_g$ we obtain the following points: if $g^* := g(p_0) = (a, b) \in C_2$, $b \neq 0$, then

$$\sigma_g(p_0) = (a, s, r), \quad r = 9\left(\frac{1}{16} - \frac{1}{4}\right), \quad s = -\frac{6}{b}(r - 2) - b.$$ 

Now, if $h \in C$, we obtain points $h(\sigma_g(p_0)) = (h(a, s), r)$. Putting $h(a, s) =: (u, v)$, supposing $v \neq 0$ and duplicating, we obtain, on denoting $f(z) = (4z - 7)(z^2 - 2)$,

$$\lambda(\sigma_h \sigma_g(p_0)) = \frac{7}{4} + \frac{f'(r)^2}{16v^2} - 2r.$$ 

Let now as above $k$ be a ground field, and let us choose $z_0 \in k$; we try to equate the last displayed value to $z_0$ (with $v \neq 0$), which amounts to

$$(4) \quad 4v^2(4z_0 + 8r - 7) = f'(r)^2.$$

We view this equation as defining (for given $z_0$) a curve $\Gamma$ in $C_2 \times C$, with coordinates $g^* \times h = (a, b) \times h := (a, b) \times (a, \beta)$. Note for future reference that in these coordinates we have $v = a\beta + s\alpha$, whereas $r$ is given above in terms of $b$.

Suppose for the moment that $z_0 \in \mathbb{F}_{\ell}$ and that $\Gamma$ is absolutely irreducible (over $\mathbb{F}_{\ell}$). Then, this curve would have absolutely bounded genus, so for $\ell$ large enough Weil’s Riemann-hypothesis for curves over finite fields would imply the existence of (several) points $g_{0}^* \times h_{0} \in \Gamma(\mathbb{F}_{\ell})$, i.e. points in $(C_2 \times C)(\mathbb{F}_{\ell})$ solving the equation (41), and we may also require that $v \neq 0$.

But now, since $C_2, C$ are rational and smooth over $\mathbb{Q}$, and thus have the WAP, we may lift $g_{0}^*, h_{0}$ to points in $C(\mathbb{Q})$, integral at $\ell$; composing then the corresponding maps $\sigma$, starting from $p_0$, we would obtain a rational point $w := (\xi_{0}, \mu_{0}, \lambda_{0}) \in X_{0}(\mathbb{Q})$, integral at $\ell$ and such that the reduction modulo $\ell$ of $\lambda_{0} := \lambda(w)$ is defined and equal to $z_0$. 


Finally, take a point $\tilde{x} \in X(\mathbb{F}_\ell)$ with $\lambda(\tilde{x}) = z_0$. Now, using once more the transitive action of $C$ on the conic $C_{\lambda_0}$ and the WAP for $C$, we can deform $w$ to a rational point on $C_{\lambda_0}$, with reduction $\tilde{x}$ modulo $\ell$.

We observe that use of the WAP for $C$ (in practice the Chinese theorem) allows to deal simultaneously with any finite set of primes $\ell$, provided they are large enough.

To conclude we have then to investigate the possible values $z_0 \in k$ making the above curve $\Gamma$ reducible over $k$ (where for the present purpose we take $k = \mathbb{F}_\ell$). For this task, in the above formulae we take $g^*, h$ as generic points of $C_{\lambda}/k$, $C/k$ related by the equation \eqref{eq:4}.

We have a rational map $\psi : C_2 \times C \to \mathbb{P}^2_1$, $(a, b) \times (\alpha, \beta) \mapsto (r, v)$, where $r$ is given by the above formula and $v = a\beta + s\alpha$. It is easy to check that $\psi$ has degree 8. Indeed, $[k(C_2) : k(b^2)] = 4$ and $v$ has degree 2 as a map on $C$, for fixed $a, b$.

Denoting $w := b^2$, the extension $k(C_2)/k(r)$ is generated by $\sqrt{w}, \sqrt{2 - w}$, whereas $k(C_2 \times C) = k(C_2)(v, \alpha)$. Now, a minimal equation for $\alpha$ is $(a\alpha)^2 + (v - s\alpha)^2 = a^2$, i.e. $(a^2 + s^2)\alpha^2 - 2v\alpha + 2\alpha^2 - a^2 = 0$, with discriminant $4(v^2s^2 - (v^2 - a^2)(a^2 + s^2)) = 4a^2(a^2 + s^2 - v^2) = 4a^2(\sqrt{f(r)} - v^2)$.

Then we find that $k(C_2 \times C) = k(r, v)(\sqrt{w}, \sqrt{2 - w}, \sqrt{\sqrt{f(r)} - v^2})$.

Now, the equation \eqref{eq:4} defines an irreducible curve $\Delta$ in the $(r, v)$-plane, and we wish to check whether its lift $\Gamma$ to $C_2 \times C$ remains irreducible.

If $\Lambda := w(4z_0 + 8r - 7) = (4z_0 - 25)w + 72$, then the equation for $\Gamma$ reads $4w^2\Lambda = w f'(r)^2$. Hence above $\Delta$ the curve may split only if $\Lambda$ becomes a square in the field $k(\sqrt{w}, \sqrt{2 - w}, \sqrt{\sqrt{f(r)} - v^2})$, where $v$ is obtained using the equation for $\Delta$, i.e. the last square root is to be replaced by $\sqrt{4\Lambda f(r) - w f'(r)^2}/\Lambda$. Note also that $r = 9(w^{-1} - 4^{-1})$.

This may happen only if there is a multiplicative combination of $\Lambda, w, 2 - w, (4\Lambda f(r) - w f'(r)^2)/\Lambda$ equal to a square in $k(w)$, with odd exponent of $\Lambda$. In turn, this implies that either $\Lambda$ or $w^2(4\Lambda f(r) - w f'(r)^2)$ has a root $w = 0$ or $w = 2$, or is constant. The first cases yield $4z_0 = -11$, whereas $\Lambda$ constant occurs only for $4z_0 = 25$. As to the last mentioned case, this is easily checked to be impossible.

So, finally we have proved the following

**Theorem 4.2.** There is a (computable) number $l_0$ such that if $L$ is a finite set of primes $\ell > l_0$ and $\xi_\ell$ are points in the open subset of $X_0$ defined by $\lambda \neq 25/4, -11/4$, then there exists a rational point $x \in X_0(\mathbb{Q}_\ell)$ such that $x \equiv \xi_\ell \pmod{\ell}$ for all $\ell \in L$.

**Remark 4.3.** To get rid of the restrictions on the $\lambda$-coordinate (i.e. the above $z_0$), several methods are available. For instance we could start with a different $p_0$, or use further maps similar to the $\sigma_\ell$ (possibly obtained by multiplication by integers $> 2$).

It seems also quite feasible to carry out a similar proof for approximations in $X(\mathbb{Q}_\ell)$ with arbitrary accuracy, instead of merely modulo $\ell$, then achieving the full WWAP. Since this would not add any further idea, and since the result is proved in a different way (and greater generality) in \cite{7}, we do not pursue this task in the present paper.

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