Quantum Kinetic Theory of BEC Lattice Gas: Boltzmann Equations from 2PI-CTP Effective Action

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We continue our earlier work [Ana Maria Rey, B. L. Hu, Esteban Calzetta, Albert Roura and Charles W. Clark, Phys. Rev. A 69, 033610 (2004)] on the nonequilibrium dynamics of a Bose Einstein condensate (BEC) selectively loaded into every third site of a one-dimensional optical lattice. From the two-particle irreducible (2PI) closed-time-path (CTP) effective action for the Bose-Hubbard Hamiltonian, we show how to obtain the Kadanoff-Baym equations of quantum kinetic theory. Using the quasiparticle approximation, we show that the local equilibrium solutions of these equations reproduce the second-order corrections to the self-energy originally derived by Beliaev. This work paves the way for the use of effective action methods in the derivation of quantum kinetic theory of many atom systems.

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I. INTRODUCTION

In many respects, the dynamics of cold atoms in optical lattices resemble those of electrons in crystals. Cold-atom systems exhibit many favorable attributes for studying quantum many-body dynamics, such as the absence of defects in the optical lattice, and the high degree of experimental control over all relevant parameters [1, 2]. In particular, by varying the depth of the optical lattice, the superfluid-insulator phase transition can be induced. For weakly-confining optical lattices, the system has macroscopic quantum coherence, and interesting matter wave interference phenomena induced by the periodicity of the lattice have been demonstrated in experiments [3, 4, 5, 6]. For tightly-confining lattices, the matter-wave coherence is lost, and the system undergoes a transition to the Mott-insulator phase [7]. This regime has become also experimentally accessible [8, 9, 10]. Outside the weakly interacting regime, standard mean field techniques are inapplicable to describe the evolution of the system, and alternative methods are required.

Motivated by a recent patterned loading experiment [1], we previously adopted a functional effective action approach capable of dealing with non equilibrium situations that require a treatment beyond mean field theory ([11], hereafter I). We applied the CTP functional formalism [12] and the two-particle irreducible (2PI) effective action [13] to the Bose-Hubbard Hamiltonian, and derived equations of motion. This method enabled us to go beyond the Hartree-Fock-Bogoliubov (HFB) approximation [14, 15, 16] and to incorporate nonlinear and non-Markovian aspects of quantum dynamics, which underlie dissipation and fluctuation phenomena.

In its pristine form the 2PI-CTP equations of motion for the mean field and the two-point correlation function are complicated nonlocal nonlinear equations, which defy even numerical solutions for realistic experimental systems with many lattice sites. It is obvious that to get more physical insight we need ways to simplify this full theory. In this paper we continue this investigation with the goal of showing how to formulate a quantum kinetic theory [17, 18] by way of the 2PI-CTP formalism. For earlier work addressing this problem in quantum field theory, see [19]-[25]. There exist an extensive literature on quantum kinetic theory, many addressing BEC dynamics with condensate-noncondensate interactions [14, 26]-[39]. Those relevant to our present discussions are [14, 20, 22].

Towards this goal, we ask the question when quantum kinetic theory is a reasonable attainable limit of the more complete theory based on the 2PI-CTP effective action. Physically, a kinetic theory regime exists when the system dynamics has a clear separation of two time (or length) scales, one pertaining to the macroscopic scale describing the kinetic motion such as the mean free time and the other to the microscopic scale such as the duration of collision event. Alternatively, when perturbations induce disturbances of wavelength longer than the thermal wavelengths and frequencies much lower than characteristic excitation frequencies, standard kinetic theories may give a reasonable description of the system’s dynamics. This is the case for weakly interacting gases confined by a slowly-varying external potential. For quantum systems, when the quantum features of the many-body system act effectively only on the microscopic scale (e.g., when one can use a quasiparticle type of approximation), quantum kinetic theory can provide an adequate description. It fails when such a two-time separation does not exist, such as in strongly correlated systems or systems with macroscopic quantum coherence [1].

The organization of this paper with a brief of our findings is as follows. In Sec.II we summarize our prior results for the

1 We have in mind systems whose quantum coherence or correlation or entanglement extends to macroscopic dimensions. Examples are coherence tunneling phenomena [40], quantum properties of microelectro-mechanical systems [41, 42] and of course, BEC, which certainly has macroscopic quantum coherence. The impossibility of a two-time separation refers only to the condensate state alone. The interaction between the condensate and the non-condensate atoms can under general conditions allow a two-time separation and a kinetic theory description, as is the topic of our present discussion and much prior work.
HFB and second order equations of motion \[1\] and express them with lightened notation in a more compact form. In Sec. \[\text{III}\] we discuss how a quantum kinetic theory can be derived from a quantum theory of interacting particles. We first discuss this issue under more general conditions, where a two-time separation may not exist. A kinetic theory is obtained from the full hierarchy of correlation functions by truncation of higher order correlations and the imposition of causal factorizable conditions. We use the nPl-effective action to illustrate this conceptual framework. In Sec. \[\text{IV}\] we focus on situations where there is a two-time separation in the system dynamics. We delineate the physical conditions and show the procedures in deriving quantum kinetic equations from the 2PI-CTP equations of motion. Then we introduce further simplifications and discuss how to derive the familiar Boltzmann equations. In Sec.\[\text{V}\] we study how these kinetic equations ad- plifications and discuss how to derive the familiar Boltzmann 2PI-CTP equations of motion. Then we introduce further sim- ilifications and discuss how to derive the familiar Boltzmann 2PI equations yield to the same second-order damping rates their with the useful equations obtained from our earlier investigation \[1\]. We will refer to the numbering of equations therein with a prefix I.

In terms of these fields the classical action takes the form

\[
S[a_i^\ast, a_i] = \int \! dt \sum_i \left[ \frac{1}{2} h a^\ast_i \dot{a}_i + \int \! dt \sum_i \frac{U}{2} a^\ast_i a_i + \frac{1}{2} V_i a^\ast_i a_i \right],
\]

(2)

To compactify our notation we introduce \(a^b_i (b = 1, 2)\) defined by \(a_i = a_i^1, a^\ast_i = a_i^2\). In contrast to I, where we set \(V_i(t) = 0\), here we allow the presence of an external potential \(V_i\) in \(S[a_i^\ast, a_i]\). In the derivation of Boltzmann equations we will assume that \(V_i\) is a slowly-varying function in position and time and treat it as an external perturbation. In terms of these fields the classical action takes the form

\[
S[a_i] = \int \! dt \sum_i \frac{1}{2} h a^\ast_i \dot{a}_i + \int \! dt \sum_i \frac{1}{2} V_i a^\ast_i a_i + \frac{1}{2} V_i a^\ast_i a_i 
\]

(3)

where \(N\) is the number of fields, which is two in this case, and summation over repeated field indices \(a_i\) is implied. \(h_{ab}\) and \(\sigma_{ab}\) are matrices defined as

\[
h_{ab} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(4)

In terms of the familiar Pauli matrices, \(\sigma_{ab} = \sigma_x\) and \(h_{ab} = -\sigma_y\). We define the following index lowering convention

\[
X_a = \sigma_{ab} X^b.
\]

(5)

After second quantization the fields \(a_i^\ast\) are promoted to operators. We denote the mean field or the expectation value of the field operator by \(\langle \hat{a}_i^\ast \rangle\) (called \(\hat{\Phi}_i\) in Paper I) and the expectation value of the fluctuation field \(\hat{\phi}_i^\ast\) by \(G_{ij}^{ab}(t, t')\). Physically, \(\langle \hat{a}_i^\ast \rangle\) is the condensate population and the two point functions \(G_{ij}^{ab}(t, t')\) determines the quantum fluctuations around the mean field:

\[
G_{ij}^{ab}(t, t') = \langle \hat{a}_i^\ast(t) \hat{a}_j(t') \rangle.
\]

(6)

The brackets denote taking the expectation value with respect to the density matrix and \(T_C\) denotes time ordering along a contour \(C\) in the complex plane.

In order to describe the non-equilibrium dynamics we specify the contour of integration to be the Schwinger-Keldysh
\footnotesize

contour \cite{12} along the real-time axis or closed time path (CTP) contour. Using the CTP contour, the two-point functions are decomposed as

\[ G_{ij}^{ab}(t, t') = \theta_{ctp}(t, t') G_{ij}^{ab>}(t, t') + \theta_{ctp}(t', t) G_{ij}^{abc}(t, t'), \tag{7} \]

where

\[ G_{ij}^{ab>}(t, t') = \langle \tilde{\phi}_i^a(t) \tilde{\phi}_j^b(t') \rangle, \tag{8} \]

\[ G_{ij}^{abc}(t, t') = \langle \tilde{\phi}_i^b(t') \tilde{\phi}_j^c(t) \rangle, \tag{9} \]

with \( \theta_{ctp}(t - t') \) being the CTP complex contour ordered theta function defined in Eq. (1.30).

All correlation functions of the quantum theory can be obtained from the two particle irreducible (2PI) effective action \( \Gamma[z, G] \). In Ref. \cite{11} we showed \( \Gamma[z, G] \) is given by:

\[ \Gamma[z, G] = S[z] + \frac{i\hbar}{2} Tr \ln G^{-1} + \frac{i\hbar}{2} Tr D^{-1}(z) G + \Gamma_2[z, G] + \text{const}, \tag{10} \]

where \( iD^{-1}(z) \) is the classical inverse propagator given by

\[ iD_{ij}^{ab}(t, t')^{-1} = \frac{\delta S[z]}{\delta z_i^a(t) \delta z_j^b(t')} \tag{11} \]

\[ = \left( \delta_{ij} \hbar \delta_{ab} \partial_t + J(\delta_{i+1,j} + \delta_{i-1,j}) \sigma_{ab} \right) \delta(t - t') - \frac{1}{\mathcal{N}} \left( 2z_{ia}(t) z_{ib}(t) + \sigma_{ab} z_i^c(t) z_{ic}(t) \right) \delta_{ij} \delta(t - t'), \]

and \( \Gamma_2[z, G] \) consists of all two-particle irreducible vacuum graphs in the theory (the diagrams that do not become disconnected by cutting two propagator lines) with propagators set equal to \( G \) and vertices determined by the interaction terms in \( S[z + \varphi] \).

The dynamical equations of motion for the mean field \( z_i^a(t) \) and the propagators \( G_{ij}^{ab}(t, t') \) are found by solving the equations \( \frac{\delta S[z, G]}{\delta z_i^a(t)} = 0 \) and \( \frac{\delta S[z, G]}{\delta G_{ij}^{ab}(t, t')} = 0 \). They were given in (I. 24) and (I.26) respectively.

The action \( \Gamma \) including the full diagrammatic series for \( \Gamma_2 \) gives the full dynamics. It is of course not feasible to obtain an exact expression for \( \Gamma_2 \) in a closed form. Various approximations for the full 2PI effective action can be obtained by truncating the diagrammatic expansion for \( \Gamma_2 \). The ones relevant for this paper are the HFB approximation and the full second order approximation. The HFB approximation corresponds to a truncation of \( \Gamma_2 \) retaining only the first order diagram in \( U \) which is \( z \) independent, i.e. keeping only the double-bubble diagram (Fig. 1 in paper I). The full second order approximation corresponds to a truncation retaining also diagrams of second order in \( U \) (the basket-ball and the setting-sun).

Hereafter, to lighten the notation, we introduce a more compact set of symbols for the physical quantities than was used in Paper I, which contains more details:

\[ z(t_i) = \begin{pmatrix} \langle \hat{a}_i^\dagger(t) \rangle \\ \langle \hat{a}_i(t) \rangle \end{pmatrix} = \begin{pmatrix} z(t_i) \\ z^*(t_i) \end{pmatrix} \tag{12} \]

\[ iH(t, t') = z_i^a(t) z_{jb}(t') \tag{13} \]

\[ i\sigma(t, t') = g_{ij}^a(t, t'), \tag{14} \]

\[ ig(t, t') = G_{ij}^{\sigma_a}(t, t'), \tag{15} \]

\[ g(t, t') = G_{ij}^{\sigma_a}(t, t') = G_{ji}^{\sigma_a}(t, t'). \tag{16} \]

The notation \( t_i \) means that the function must be evaluated at the time \( t \) and at the lattice site \( i \).

\section*{B. The HFB and full second order equations of motion}

The equations of motion derived from the 2PI-CTP effective action (I-24) and (I-26) have terms which can be grouped as the single particle, the HFB and the second order contributions, as follows:

\[ \sum_k \int dt'' \left( D_{o}^{-1}(t_i, t_k') - S^{HFB}(t_i, t_k') \right) H(t_k', t_j') = \sum_k \int dt'' S(t_i, t_k') H(t_k', t_j'), \tag{17} \]

\[ \sum_k \int dt'' \left( D_{o}^{-1}(t_i, t_k') - \Sigma^{HFB}(t_i, t_k') \right) g(t_k', t_j') = \sum_k \int dt'' \Sigma(t_i, t_k') g(t_k', t_j') - \delta_{ij} \delta_C(t - t'), \tag{18} \]

where \( \sigma_z \) is the Pauli matrix:

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20} \]
\( S^{HFB} \) and \( \Sigma^{HFB} \) are the HFB self-energies of \( H \) and \( g \) respectively, and \( S \) and \( \Sigma \) are the remaining parts of the self-energies of \( H \) and \( g \), which we will assume are given by the second order corrections.

Using Eq. (I-43) it can be shown that \( S^{HFB} \) and \( \Sigma^{HFB} \) are given by:

\[
\Sigma^{HFB}(t_i, t'_j) \equiv \frac{U}{N}(Tr(H(t_i, t'_j) + g(t_i, t'_j))) I + 2 \left( H(t_i, t'_j) + g(t_i, t'_j) \right) \delta(t - t') \delta_{ij},
\]

\[
S^{HFB}(t_i, t'_j) \equiv \frac{U}{N}(Tr(H(t_i, t'_j) + g(t_i, t'_j))) I + 2g(t_i, t'_j) \delta(t - t') \delta_{ij}.
\]

where \( I \) is the identity matrix.

In paper I, we used the CTP contour of integration (which is also usually called “in-in” contour) to evaluate the second order contribution. Use of the CTP formalism was important there, because it provided the technical means to formulate our initial value problem in a completely causal manner, removing the Feynman boundary conditions on the Green’s function used in the conventional ”in-out” formalism \([17]\). In this work we are more interested in deriving kinetic equations which are especially devised to study relaxation of systems close to equilibrium. With this purpose in mind, as done by Kadanoff and Baym \([17, 18]\), it is better to set the initial conditions in the far past. We follow them hereafter and use the CTP contour, but instead of setting the initial time to zero, as was done in paper I, we choose it to be \(-\infty\). The equations of motion we obtain in this way agree with the equations of motion \([17, 18]\) and are given by:

\[
\sum_k \int_{-\infty}^{\infty} dt'' \left( D_{\omega}^{-1}(t_i, t''_k) - S^{HFB}(t_i, t''_k) \right) H(t''_k, t'_j) - \int_{-\infty}^{t} dt'' \gamma(t_i, t''_k) H(t''_k, t'_j) = 0, \tag{23}
\]

\[
\sum_k \int_{-\infty}^{\infty} dt'' H(t_i, t''_k) \left( D_{\omega}^{-1}(t''_k, t'_j) - S^{HFB}(t''_k, t'_j) \right) + \int_{-\infty}^{t} dt'' H(t_i, t''_k) \gamma(t''_k, t'_j) = 0, \tag{24}
\]

\[
\sum_k \int_{-\infty}^{\infty} dt'' \left( D_{\omega}^{-1}(t_i, t''_k) - \Sigma^{HFB}(t_i, t''_k) \right) g^{(\Sigma)}(t''_k, t'_j) = \tag{25}
\]

\[
\sum_k \int_{-\infty}^{t} dt'' \Gamma(t_i, t''_k) g^{(\Sigma)}(t''_k, t'_j) - \sum_k \int_{-\infty}^{t'} dt'' \Sigma^{(\Sigma)}(t_i, t''_k) A(t''_k, t'_j), \tag{26}
\]

\[
\sum_k \int_{-\infty}^{\infty} dt'' g^{(\Sigma)}(t_i, t''_k) \left( D_{\omega}^{-1}(t''_k, t'_j) - \Sigma^{HFB}(t''_k, t'_j) \right) = \tag{26}
\]

\[
\sum_k \int_{-\infty}^{t} dt'' A(t_i, t''_k) \Sigma^{(\Sigma)}(t''_k, t'_j) - \sum_k \int_{-\infty}^{t'} dt'' g^{(\Sigma)}(t_i, t''_k) \Gamma(t''_k, t'_j). \tag{26}
\]

In the above equations, Eq. (26) is the hermitian conjugate of Eq. (25), Eq. (24) is the hermitian conjugate of Eq. (23) and we have introduced the spectral functions

\[
\gamma(t_i, t'_j) \equiv \left( S^{\Sigma}(t_i, t'_j) - S^{\Sigma^{(\Sigma)}}(t_i, t'_j) \right), \tag{27}
\]

\[
\Gamma(t_i, t''_k) \equiv \left( \Sigma^{\Sigma}(t_i, t''_k) - \Sigma^{\Sigma^{(\Sigma)}}(t_i, t''_k) \right), \tag{28}
\]

\[
A(t_i, t''_k) \equiv \left( g^{\Sigma}(t_i, t''_k) - g^{\Sigma^{(\Sigma)}}(t_i, t''_k) \right). \tag{29}
\]

Notice that \( A(t_i, t''_k), \Gamma(t_i, t''_k), \gamma(t_i, t''_k) \) are just the spectral functions defined in (I. 36) multiplied by a minus sign. In paper I we denoted these by a subscript \((\rho)\). Here, for ease of comparisons with the literature, we have changed to the notation of Kadanoff and Baym \([17]\), \( \gamma, \Gamma \) and \( A \). We will show later that \( \gamma \) and \( \Gamma \) contain information about the condensate and noncondensate particle decay rates respectively.

If we use the full second order expansion, (I.60) and (I.61), \( S^{(\Sigma)} \) and \( \Sigma^{(\Sigma)} \) are given by
\[
S^{(2)}(t_i, t'_j) = -\frac{1}{2} \left( \frac{2U}{N} \right)^2 \left( g^{(2)}(t_i, t'_j) \text{Tr} \left( g^{(2)}(t_i, t'_j) g^{(2)}(t'_j, t_i) \right) + 2g^{(2)}(t_i, t'_j) g^{(2)}(t'_j, t_i) g^{(2)}(t_i, t'_j) \right),
\]
\[
\Sigma^{(2)}(t_i, t'_j) = -\frac{1}{2} \left( \frac{2U}{N} \right)^2 \left\{ H(t_i, t'_j) \text{Tr} \left( g^{(2)}(t_i, t'_j) g^{(2)}(t'_j, t_i) \right) + 2H(t_i, t'_j) g^{(2)}(t'_j, t_i) g^{(2)}(t_i, t'_j) 
+ 2g^{(2)}(t_i, t'_j) \left( H(t'_j, t_i) g^{(2)}(t_i, t'_j) + g^{(2)}(t'_j, t_i) H(t'_j, t_i) \right) g^{(2)}(t_i, t'_j) \right\}.
\]

It is convenient to decompose the above equations in their matrix components. To do that we introduce the definitions
\[
g^{>}(t_i, t'_j) = -i \left( \bar{\rho}_{ij}(t, t') \frac{m_{ij}(t, t')}{\rho_{ji}(t, t')} \right),
\]
\[
g^{<}(t_i, t'_j) = -i \left( \rho_{ij}(t, t') \frac{m_{ij}(t, t')}{\bar{\rho}_{ji}(t, t')} \right).
\]

At equal times, the quantities \(\bar{\rho}_{ij}\) and \(\rho_{ij}\) are related by the bosonic commutation relations. Using Eqs. 32 and 33 into the self-energy equations we get
\[
\Sigma_{11}^{HFB}(t_i, t'_j) = \frac{2U}{N} \left( \frac{2|z_i|^2 + \rho_{ii} + \bar{\rho}_{ii}}{z_i^2 + m_{ii}^*} \right) \delta(t - t') \delta_{ij},
\]
\[
\Sigma_{12}^{HFB}(t_i, t'_j) = \frac{2U}{N} \left( \frac{|z_i|^2 + \rho_{ii} + \bar{\rho}_{ii}}{|z_i|^2 + \rho_{ii} + \bar{\rho}_{ii}} \right) \delta(t - t') \delta_{ij}.
\]

The above expressions for the self-energy, which contain two-particle irreducible diagrams up to second order in the interaction strength, agree exactly with those used in Refs. 28, 30. In Refs. 28, 30 the authors used these equa-
tions as the starting point of a quantum kinetic theory before applying the Markovian approximation. It is important to mention that in contrast to other self-energy approximations that may lead to equations of motion that do not satisfy conservation laws, the 2PI effective action formalism is a “Φ-derivable” approximation and therefore all the equations of motion derived from it are guaranteed to be conserving. Moreover, as we showed in paper I, a truncation up to second order in the interaction strength is not appropriate to describe far-from-equilibrium dynamics outside the weak coupling regime. Away from the weak coupling regime, the 2PI effective action can be a powerful tool. For example a $1/N$ expansion of the 2PI effective action has been shown to provide a practicable controlled nonperturbative description of far-from equilibrium dynamics without the small coupling restriction.

III. FROM QUANTUM THEORY OF INTERACTING PARTICLES TO QUANTUM KINETIC THEORY

From previous sections it can be observed that the equations of motion obtained from the 2PI effective action are quite involved: nonlinear and nonlocal integro-differential equations, not readily solvable in closed form. To progress further we need to introduce approximations based on physical considerations. It is easier to proceed if one can observe and justify a separation of time scales in the relevant physical processes in question, i.e., one related to quantum processes which are usually microscopic in scale (note quantum entanglement and correlation of the system can extend to much greater scales, meso or even macro) and one related to the kinetic or transport properties, which is usually macroscopic in scale. However, this assumption of a scale separation, may not be valid in mesoscopic processes (as in strongly correlated systems) or macroscopic quantum coherence effects (see footnote 1). For those situations where a separation of macroscopic and microscopic time scales which would permit an effective kinetic theory description does not exist, one can adopt the effectively open system framework quantified by the nPI-CTP effective action and the hierarchy of equations it generates. We begin with a discussion of the latter situation which is more demanding and general. We describe the conceptual pathway for the construction of quantum kinetic theory from the nPI effective action. Though somewhat theoretical and formally oriented, it may be of some use, as this is the first point of contact with quantum kinetic theory from the effective action approach, in the atomic and molecular physics (AMO) context. For more details on this subject see [19, 20, 21, 22], where our discussions in the following section are based on.

A. Quantum kinetic theory from (nPI) effective action

It may be useful to begin by defining what we mean by a quantum kinetic theory. It contains, but supersedes, the quantum version of Boltzmann’s theory. Formally it refers to the theory based on the hierarchy of coupled equations for the (relativistic) Wigner function and its higher-correlation analogs, which are obtained by a Fourier transform of the relative coordinates in the Schwinger-Dyson equations for the correlation functions, or alternatively, in the master effective action (defined as the nPI effective action when $n \to \infty$, we are dealing with $n = 2$ here) whose variation yields the Schwinger-Dyson equations. This is a quantum analogue of the BBGKY hierarchy, expressed in a representation convenient for distinguishing between microscopic (quantum field-theoretic) and macroscopic (transport and relaxation) phenomena. As such, it does not require near-equilibrium conditions, and in fact, is applicable for a rather general moment expansion of the initial density matrix.

To understand how quantum kinetic theory is derived from an nPI effective action and how it relates to the familiar Boltzmann’s theory, it is perhaps helpful to examine the relation between this theory in its full generality and an effective Boltzmann description of relaxation phenomena for the one-particle distribution function of quasiparticles. In nonequilibrium statistical mechanics, as is well known, the act of truncating the classical BBGKY hierarchy does not in itself lead to irreversibility and an H-theorem. One must further perform a type of coarse graining of the truncated, coupled equations for n-particle distribution functions. For example, if one truncates the hierarchy to include only the one-particle and two-particle distribution functions, it is the subsequent assumption that the two-particle distribution function at some initial time factorizes in terms of a product of single-particle distribution functions (which is at the heart of the molecular chaos hypothesis where the colliding particles are initially independent, but correlated after a collision ) what leads to the (irreversible) Boltzmann equation. The assumption that the two-particle distribution function factorizes is an example of a type of coarse graining called slaving of the two-particle distribution function to the single-particle distribution function, in the language of [20]. The situation in quantum kinetic field theory is completely analogous. One may choose to work with a truncation of the hierarchy of the Wigner function and its higher correlation analogs, or one may instead perform a slaving of, for example, the Wigner-transformed four-point function, which leads (within the context of perturbation theory) directly to the (relativistic) Boltzmann equation and the usual H-theorem. Typically this slaving of the higher correlation function(s) involves imposing causal boundary conditions to obtain a particular solution for the higher correlation function(s) in terms of the lower order correlation functions.

The truncation and subsequent slaving of the hierarchy within quantum kinetic field theory can be carried out at any desired order, as dictated by the initial conditions and relevant interactions. As with any coarse graining procedure, in implementing the slaving of a higher correlation/distribution function to lower correlation/distribution

2 It should be pointed out that in order to identify the Wigner function with a distribution function for quasiparticles, one must show that the density matrix has decohered, and this is neither guaranteed nor required by the existence of a separation of macroscopic and microscopic time scales.
functions, one is going over from a closed system to an effectively open system, the hallmarks of which are the emergence of dissipation [19] and noise/fluctuations [20]. This fact has led some to search for stochastic generalizations of the Boltzmann equation [52], motivated by the fact that systems in thermal equilibrium always manifest fluctuations, as embodied in the fluctuation-dissipation relation. (A derivation of the stochastic Boltzmann equation from quantum field theory can be found in [20].)

The essential point about the process of slaving of higher correlation (or distribution) functions is that it is a step which is independent of the assumption of macroscopic and microscopic time scales. In fact, a completely analogous procedure exists at the level of the Schwinger-Dyson equations (i.e., without Wigner transformation) for correlation functions in an interacting quantum field theory [20]. Recall that the Schwinger-Dyson equations are, in the context of nonequilibrium field theory in the Schwinger-Keldysh or closed-time-path (CTP) formulation, an infinite chain of coupled dynamical equations for all order correlation functions of the quantum field. The importance of the closed-time-path formalism in nonequilibrium situations is that it ensures that the equations are causal and that the correlation functions are “in-in” expectation values in the appropriate initial state or density matrix. As with the BBGKY hierarchy in nonequilibrium statistical mechanics, the common strategy is to truncate the hierarchy of correlation functions at some finite order. A general procedure has been presented for obtaining coupled equations for correlation functions at any order \( l \) in the correlation hierarchy, which involves a truncation of the master effective action at a finite order in the loop expansion [20].

By working with an \( l \) loop-order truncation of the master effective action, one obtains a closed, time-reversal invariant set of coupled equations for the first \( l + 1 \) correlation functions, \( z = C_1, g = C_2, C_3, \ldots, C_{l+1} \). In general, the equation of motion for the highest order correlation function will be linear, and thus can be formally solved using Green’s function methods. The existence of a unique solution depends on supplying causal boundary conditions. When the resulting solution for the highest correlation function is then back-substituted into the evolution equations for the other lower-order correlation functions, the resulting dynamics becomes non-time-reversal invariant, and generically dissipative. As with the slaving of the higher-order Wigner-transformed correlation functions in quantum kinetic field theory, we have then gone over from a closed system (the truncated equations for correlation functions) to an effectively open system. In addition to dissipation, one expects that an effectively open system will manifest noise/fluctuations (an example of slaving the four-point function, one expects that an effectively open system will manifest fluctuations, as embodied in the fluctuation-dissipation relation. (A derivation of the stochastic Boltzmann equation from quantum field theory can be found in [20].))


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While it is certainly not the only coarse-graining scheme which could be applied to an interacting quantum field, the slaving of higher correlation functions to lower-order correlation functions within a particular truncation of the correlation hierarchy, as a particular coarse graining method, has several important benefits. First, it can be implemented in a truly nonperturbative fashion, where the variance of the mean field can be on the order of the “classical” mass (defined as the second order derivative of the effective potential in the equation of motion for the mean field, which provides the natural time scale of the system dynamics). This necessitates a nonperturbative resummation of daisy graphs (the leading contributions in a large \( \lambda' \) expansion) [13], which can be incorporated in the truncation/slaving of the correlation hierarchy in a natural way. Second, the truncation of the correlation hierarchy accords with our intuition that the degrees of freedom readily accessible to physical measurements are often limited to the mean field and two-point function.

### IV. SYSTEMS WHOSE DYNAMICS ADMIT TWO-TIME SEPARATION

An alternative (actually more common and easier) route to reach a kinetic theory description from \( n \)-body quantum dynamics becomes available when there is a clear separation of two time scales in the system dynamics. This is the usual text book treatment of kinetic theory we are familiar with. The two different scales in the system are the time (or length) scale separation between the duration of a collision event (or scattering length) and the inverse collision rate (or the mean free path). For quantum processes, in the weakly interacting regime, we expect there is also a separation between the kinetic scale of \( n \) particles (expressed in the center of mass coordinate) and the quantum scale (expressed in the momentum corresponding to the Fourier transform of the relative coordinates between two particles), which describes how quantum processes (such as radiative corrections) change the particles’ mass-energy and momenta. Using these approximations it is possible to recast the full quantum dynamics into the simpler forms of two coupled equations which constitute quantum kinetic theory, the Boltzmann equation governing the distribution functions and what is known as the gap equation for the modified dispersion relation.

For a three dimensional uniform Bose gas the duration of a collision event \( \tau_0 \) is given by the time that a particle with average velocity \( v \) spends in the interaction region measured by the range of the two-particle interaction potential. This range for a repulsive potential is typically given by the \( s \)-wave scattering length and thus \( \tau_0 \approx a_s/v \). On the other hand, the inverse collision rate \( \tau_c \) or time between successive collisions is approximately given by \( \tau_c \approx (na_s^2v)^{-1} \), where \( n \) is the particle density. The required separation of time scales, \( \tau_c \gg \tau_0 \) implies the inequality \( na_s^3 \ll 1 \) or in other words, the

---

3 At late times in the thermalization stage, when the quantum field is near equilibrium, an effective kinetic description may be justified, but will likely require resummation of hard thermal loops (see, e.g., [53]). Under such circumstances, even the evaluation of transport coefficients is nontrivial for high temperatures [23, 53].
necessary condition required for the validity of a scale separation is that the system must be in the dilute weakly interacting regime. For atoms in optical lattices the dilute weakly interacting conditions required for the scale separation is fulfilled if the average repulsive interaction energy \( Un \), where \( n \) is the mean number of particles per lattice site, is much smaller than \( J \), the quantum kinetic energy needed to correlate two atoms at adjacent lattice sites, or \( Un/J \ll 1 \).

Perhaps an intuitive way to understand the scale separation is the following. At equilibrium the correlation functions describing a homogeneous system are translationally invariant and stationary. If the system is disturbed from equilibrium, collisions among particles would break both invariances. However, as long as the the interaction energy per particle is smaller than the typical kinetic energy per particle, inter-particle collisions are few and far between. In this case the quantum-mechanical entanglement between collision partners decays faster than the time required for the next collision to take place, particles can be considered as free between collisions and approximate time and space translational invariance holds.

### A. Coarse-graining procedure

To make the scale separation, for BEC systems at hand, it is best to perform first a gauge transformation which makes it easier to identify (and coarse-grain away) the fast variations induced by the rapid change of the condensate phase. Following Ref. [18] we introduce the gauge transformation

\[
\begin{align*}
\tilde{z}(t_i) &= e^{i\theta(t_i)} \sqrt{n_0(t_i)}, \\
\tilde{g}^{(\sigma)}(t_i, t_j') &= e^{i\theta(t_i)\sigma} \tilde{g}^{(\sigma)}(t_i, t_j') e^{-i\theta(t_j')\sigma},
\end{align*}
\]

where \( \sqrt{n_0(t_i)} \) and \( \theta(t_i) \) are real. The equations of motion are invariant under the phase transformation if we replace \( D_o^{-1} \) by \( \tilde{D}_o^{-1} \):

\[ \tilde{D}_o^{-1}(t_i, t_j') = \left( \hbar \delta_{ij} (i\sigma z - \partial_t \theta(t_i)) - \delta_{ij} V_i + J \left( e^{i\sigma z \Delta\theta(t_i+1/2)\delta_{i+1,j}} + e^{-i\sigma z \Delta\theta(t_i-1/2)\delta_{i-1,j}} \right) \delta(t-t') \right), \]

where we have introduced the definition \( \Delta\theta(t_{i+1/2}) = \theta(t_{i+1}) - \theta(t_i) \). As shown in Ref. [16], in the context of the discrete Bose-Hubbard model, it is convenient to map the unitary gauge transformation to the so called phase-twist of the Hamiltonian. The twisted Hamiltonian exhibits additional phase factors, \( e^{\pm i\Delta\theta} \) in the hopping term, which are known as the Peierls phase factors.

The scale separation is performed by introducing the variables:

\[
R = (i + j)/2, \quad T = (t + t')/2,
\]

\[
r = (i - j), \quad \tau = (t - t'),
\]

For a translationally invariant system at equilibrium, the condensate density \( n_0(t_i) \) is position and time independent and the propagators \( g^{(\sigma)}(t_i, t_j') \) only depend on the relative coordinates variables \( r \) and \( \tau \) and are highly peaked about their zeros. If the system is disturbed by small perturbations, such as an external potential \( V(t_i) \) which varies slowly in space and time, we expect for systems with scale separation, that the gauge-transformed propagators, \( \tilde{g}^{(\sigma)}(t_i, t_j') \), acquire a slowly varying dependence on the center of mass coordinates \( R \) and \( T \) but still to be peaked around the zeros of \( r \) and \( \tau \). We emphasize that the gauge transformed, not the original variables, are the ones that are expected to be slowly varying. The reason is that even if the perturbation is slowly varying, the phase \( \theta(t_i) \) can be a rapidly varying function and it can induce strong variations in the condensate amplitude and in the propagators.

Before going further, it is important to discuss the issue that by defining the spatial center of mass coordinates at points that strictly speaking are not lattice sites points we might be introducing un-physical degrees of freedom. We stress though that this is not the case for system with scale separation. Under the slowly varying approximation the un-physical degrees of freedom are excluded, since the functions evaluated at the \( R \) points may be thought of as the average over neighboring physical lattice sites.

We proceed now to describe the coarse-graining procedure that uses the slowly varying property of the propagators in the center of mass variables to simplify the equations of motion.

If the phase twist applied to the system is small \( \Delta\theta \ll \pi \), the Peierls phase factors can be written as, \( e^{i\Delta\theta} = 1 + i\Delta\theta - \frac{1}{2!} \Delta\theta^2 \). In this case, the phase factors can be physically connected to the imposition of an acceleration on the lattice and the energy change resulting from the phase twist can be attributed to the kinetic energy of the superflow generated by the acceleration. Under this picture in the context of the Bose-Hubbard model the quantity \( \Delta\theta \) can be also connected, as is the gradient of the phase in non lattice systems, to the superfluid velocity:

\[
h v_s(t_{i+1/2}) = 2J \Delta\theta(t_{i+1/2}) a_i.
\]

with \( a_i \) the lattice spacing.

If the disturbances introduced by the perturbation are small, the superfluid velocity is expected to be a slowly varying function in space and time and to a good approximation its second order variations can be ignored, i.e. \( \Delta v_s(t) = 2[v_s(t_{i+1/2}) - v_s(t_{i+1/2})] \).
Again, the quantity \( v_s(t_i) \) may be thought of as the average over neighboring lattice sites: \( [v_s(t_{i+1/2}) + v_s(t_{i-1/2})]/2 \). Using the small angle and slowly varying dependence of the superfluid velocity, the propagator \( \tilde{D}_o^{-1} \) can thus be written in terms of the superfluid velocity as:

\[
\tilde{D}_o^{-1}(t_i, t'_j) \approx \left( \delta_{ij} t_{\sigma_z} \partial_t - \hbar \partial_t \theta(t_i) - V(t_i) - J \tau^2 \right) J \left( 1 + \frac{i}{2} \sigma_z \Delta \tau_s(t) \right) [\delta_{i+1j} + \delta_{i-1j}] \delta(t - t')
\]

(54)

where we have introduced the dimensionless superfluid velocity \( \tau_s(t_i) \equiv \frac{\hbar v_s(t_i)}{2 M} \).

At equilibrium, the time derivative of the phase is related to the chemical potential. Extending this identification to the nonequilibrium system we define the chemical potential as

\[
\mu(t_i) = -\hbar \partial_t \theta(t_i) - J \tau^2 \tau_s(t_i) - V(t_i), \quad (55)
\]

If we make a change of variables \( (t_i) \rightarrow (R + (r/2), T + (t/2)) \) in the one-point functions \( n_o(t_i), \mu(t_i), v_s(t_i) \) and \( V(t_i) \) and use the the slowly varying dependence of the functions on the center of mass coordinates, to a good approximation the functions can be treated as continuous functions and second order variations in \( R \) and \( T \) can be neglected. Thus, they can be written as:

\[
\begin{align*}
n_o(t_i) &= n_o(R + (r/2), T + (t/2)) = n_o(R, T) + \frac{r}{2} \partial_R n_o(R, T) + \frac{t}{2} \partial_T n_o(R, T), \\
\mu(t_i) &= \mu(R + (r/2), T + (t/2)) = \mu(R, T) + \frac{r}{2} \partial_R \mu(R, T) + \frac{t}{2} \partial_T \mu(R, T), \\
v_s(t_i) &= v_s(R + (r/2), T + (t/2)) = v_s(R, T) + \frac{r}{2} \partial_R v_s(R, T) + \frac{t}{2} \partial_T v_s(R, T), \\
V(t_i) &= V(R + (r/2), T + (t/2)) = V(R, T) + \frac{r}{2} \partial_R V(R, T) + \frac{t}{2} \partial_T V(R, T).
\end{align*}
\]

(59)

Using Eq. (56) in Eq. (61) we get:

\[
H(R, q; T, \omega) = 2 \pi M \left( I + \sigma_z \right) n_o(R, T) \delta(\omega) \delta_{q0}. \quad (62)
\]

In Eq. (62), the quantity \( n_o(R, T) \) is just related to the condensate density of atoms at the space time point \( (Ra, T) \). In Eq. (62), the upper diagonal component of the two-point function \( g^{(2)}(R, q; T, \omega) \) corresponds to the well known Wigner distribution function \( [47] \). It can be interpreted as the density of noncondensed particles with quasimomentum \( q \) and energy \( \hbar \omega \) at the position \( Ra \) and time \( T \). On the other hand, \( g_{11}^{(1)}(R, q; T, \omega) \) essentially the density of states available to a particle that is added to the system at \( (Ra, T) \) with quasimomentum \( q \) and energy \( \hbar \omega \). As opposed to a normal system, the presence of the condensate gives nonzero values to the off-diagonal terms of the functions \( g^{(2)}_{12}(R, q; T, \omega) \). We refer to them as the anomalous contributions to the respective two point functions.
B. Generalized Boltzmann equations

The generalized Boltzmann equations can be obtained as the Fourier transform of the equations of motion for the case in which the variations in R and T are very small: in particular when the inverse propagator \( D_o^{-1} \) and the self energies vary very little as \( R_{01} \) is changed by a characteristic excitation wavelength or \( T \) is changed by an inverse excitation energy.

If we neglect the second order variation in \( T \) and \( R \), as explained above, the equations of motion Eq. (23) to (26) can be approximated by:

\[
\left( D_o^{-1} - \Re S + \frac{i}{2} \Gamma \right) g^{(\Re)} - g^{(\Re)} \Re g \left( \Re g - \frac{i}{2} A \right) = -\frac{i}{2} \left[ D_o^{-1}, g^{(\Re)} \right] + \frac{i}{2} \left[ \Re S, g^{(\Re)} \right] + \frac{i}{2} \left[ \Re g^{(\Re)}, \Re g \right] \\
+ \frac{1}{4} \left[ \Gamma, g^{(\Re)} \right] - \frac{1}{4} \left[ \Re g^{(\Re)}, A \right], \tag{63}
\]

\[
g^{(\Re)} \left( D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right) - \left( \Re g - \frac{i}{2} A \right) \Re \Sigma^{(\Re)} = -\frac{i}{2} \left[ g^{(\Re)}, D_o^{-1} \right] + \frac{i}{2} \left[ g^{(\Re)}, \Re \Sigma \right] + \frac{i}{2} \left[ \Re g, \Re \Sigma \right] \\
- \frac{1}{4} \left[ g^{(\Re)}, \Gamma \right] + \frac{1}{4} \left[ A, \Re \Sigma^{(\Re)} \right], \tag{64}
\]

with

\[
D_o^{-1}(qR; T, \omega) \equiv (\sigma_z (\hbar \omega - \tau_s(R, T)2J \sin(qa_l)) + (2J \cos(qa_l) + \mu(R, T)) I. \tag{67}
\]

In Eqs. (63, 66) all the quantities depend on \((qR; T, \omega)\).

In the equations we have also introduced the following functions:

\[
\Re S(R, q; T, \omega) = S^{HF}(R, q; T, \omega) + \Re S^B(R, q; T, \omega), \tag{68}
\]

\[
\Re \Sigma(R, q; T, \omega) = \Sigma^{HF}(R, q; T, \omega) + \Re \Sigma^B(R, q; T, \omega), \tag{69}
\]

\[
\Re S^B(R, q; T, \omega) = P \int \frac{d\omega'}{2\pi} \frac{\gamma(R, q; T, \omega)}{\omega' - \omega}, \tag{70}
\]

\[
\Re \Sigma^B(R, q; T, \omega) = P \int \frac{d\omega'}{2\pi} \frac{\Gamma(qR; T, \omega')}{\omega' - \omega}, \tag{71}
\]

\[
\Re g(R, q; T, \omega) = P \int \frac{d\omega'}{2\pi} \frac{A(R, q; T, \omega')}{\omega' - \omega}. \tag{72}
\]

with \(P\) denoting the Cauchy principal value and \(\gamma(R, q; T, \omega), \Gamma(R, q; T, \omega), S^{HF}(R, q; T, \omega), \Sigma^{HF}(R, q; T, \omega)\) and \(A(R, q; T, \omega)\) understood as Fourier transforms of the functions \(\gamma(t_i, t'_j), \Gamma(t_i, t'_j), S^{HF}(t_i, t'_j), \Sigma^{HF}(t_i, t'_j)\) and \(A(t_i, t'_j)\) respectively.

To approximate the discretized equations by the continuous differential equations we have also used the slowly varying dependence of the quantities on \( R \) and \( T \). The brackets in Eqs. (63, 66) denote the generalized Poisson brackets defined as:

\[
[A, B] = \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial T} - \frac{\partial A}{\partial T} \frac{\partial B}{\partial \omega} + \partial_r A \partial_q B - \partial_q A \partial_r B. \tag{73}
\]

Notice that even though the continuous limit has been taken at the kinetic scale, the discreteness introduced by the lattice, crucial for a correct description of the physics, is taken into account at the quantum scale, as can be seen in Eq. (67) where the free propagator has a trigonometric dependence on the quasimomentum \( q \), characteristic of lattice-type systems. If the disturbances in the system are small enough that only long wavelength modes are excited, \( qa \ll 1 \), the excitations only see the lower quarter of the band. In this case the free propagator reduces to

\[
D_o^{-1}(q < 1/a_l R; T, \omega) \approx \sigma_z (\hbar \omega - \nu_s(R, T)p) + \left(2J - \frac{p^2}{2m^*} + \mu(R, T)\right) I. \tag{74}
\]

which is like the free propagator for a non-lattice system and the role of the lattice is just to introduce an effective mass \( m^* \). Here \( p = \hbar q \) and \( m^* = \hbar^2/(2a_l^2J) \).
If we define the statistical functions (which carry a superscript \(F\) in Paper I, Eq. (1.35)) as:

\[
F(R, q; T, \omega) = \frac{g^> (R, q; T, \omega) + g^< (R, q; T, \omega)}{2}, \quad (75)
\]

\[
\Pi(R, q; T, \omega) = \frac{\Sigma^> (R, q; T, \omega) + \Sigma^< (R, q; T, \omega)}{2}. \quad (76)
\]

Eqs. (63), (64) and (77) - (80) are our passage to the Boltzmann equations. They describe the state of the gas at a given time. Different from the HFB equations they include collisional integrals for binary interactions. Eq. (65) and Eq. (66), can be rewritten in terms of statistical and spectral functions as:

\[
\left( D_o^{-1} - \Re \Sigma + \frac{i}{2} \Gamma \right) F - \Pi \left( \Re g + \frac{i}{2} A \right) = - \frac{i}{2} \left\{ \left[ D_o^{-1} - \Re \Sigma + \frac{i}{2} \Gamma, F \right] - \left[ \Pi, \Re g + \frac{i}{2} A \right] \right\}, \quad (77)
\]

\[
F \left( D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right) - \left( \Re g - \frac{i}{2} A \right) \Pi = - \frac{i}{2} \left\{ \left[ F, D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right] - \left[ \Re g - \frac{i}{2} A, \Pi \right] \right\}, \quad (78)
\]

\[
(D_o^{-1} - \Re \Sigma) A - \Re g - \Gamma g = - \frac{i}{2} \left\{ \left[ A, D_o^{-1} - \Re \Sigma \right] - [\Re g, \Gamma] \right\}. \quad (79)
\]

\[
A(D_o^{-1} - \Re \Sigma) - \Re g \Gamma = - \frac{i}{2} \left\{ \left[ A, D_o^{-1} - \Re \Sigma \right] - [\Re g, \Gamma] \right\}. \quad (80)
\]

Eqs. (63), (64) and (77) - (80) are our passage to the Boltzmann equations. They describe the state of the gas at a given time. Different from the HFB equations they include collisional integrals for binary interactions.

C. Ordinary Boltzmann equations

To progress further we can introduce more simplifications based on physical considerations. The ordinary Boltzmann equation emerges from the approximation in which the self energies that appear on the left side of Eqs. (63), (64) and (77) - (80) are handled differently from those which appear on the right. These two appearance of the self-energy play a different physical role in the description of the dynamics (17). The self energies on the right hand side describe the dynamical effects of collisions, i.e., how the collisions transfer particles from one energy-momenta configuration to another. On the other hand, the self energies on the left describe the quantum kinetic effects due to interactions, i.e. how interaction effects change the energy momentum dispersion relations from that of free particles to a more complicated spectrum. Because these two effects are physically distinct, we can treat the left and the right hand sides in a different way.

In the derivation of the ordinary Boltzmann equations, one completely neglects all the kinetic effects in the second order self energies (the dependence on \(T\) and \(R\) in the second order self-energy terms on the right hand side) and retain dynamical effects (\(T\) and \(R\) dependence on the left hand side). In this way, we get the familiar Boltzmann equations which describe the particles as free particles in between collisions with a modified energy-momentum dispersion relation. It is a reasonable assumption in dilute weakly interacting gases in which the duration of a collision is very short compared to the essentially interaction-free dynamics between isolated collisions. Neglecting kinetic effects in the second order self energies, Eqs. (63), (64) and (77) - (80) can be approximated to

\[
\left( D_o^{-1} - \Re S + \frac{i}{2} \Gamma \right) H = - \frac{i}{2} \left[ D_o^{-1} - S^{HFB}, H \right], \quad (81)
\]

\[
H \left( D_o^{-1} - \Re S - \frac{i}{2} \Gamma \right) = - \frac{i}{2} \left[ H, D_o^{-1} - S^{HFB} \right], \quad (82)
\]

\[
\left( D_o^{-1} - \Re \Sigma + \frac{i}{2} \Gamma \right) F - \Pi \left( \Re g + \frac{i}{2} A \right) = - \frac{i}{2} \left[ D_o^{-1} - \Sigma^{HFB}, F \right], \quad (83)
\]

\[
(D_o^{-1} - \Re \Sigma) A - \Re g \Gamma = - \frac{i}{2} \left[ A, D_o^{-1} - \Sigma^{HFB} \right], \quad (84)
\]

\[
F \left( D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right) - \left( \Re g - \frac{i}{2} A \right) \Pi = - \frac{i}{2} \left[ F, D_o^{-1} - \Sigma^{HFB} \right], \quad (85)
\]

\[
A \left( D_o^{-1} - \Re \Sigma \right) - \Re g \Gamma = - \frac{i}{2} \left[ A, D_o^{-1} - \Sigma^{HFB} \right]. \quad (86)
\]
If we take the trace of the sum and the difference of each one of the above equations with its hermitian conjugate, they can be simplified to:

\[
\begin{align*}
Tr \left\{ (D_o^{-1} - \Re \Sigma) H \right\} &= 0, \quad (87) \\
Tr \left\{ (D_o^{-1} - \Re \Sigma) F - \Pi \Re g \right\} &= 0, \quad (88) \\
Tr \left\{ (D_o^{-1} - \Re \Sigma) A - \Gamma \Re g \right\} &= 0, \quad (89) \\
Tr \left[ D_o^{-1} - S^{HFB}, H \right] &= -\text{Tr}(\gamma H), \quad (90) \\
Tr \left[ D_o^{-1} - \Sigma^{HFB}, F \right] &= -\text{Tr}(\Gamma F - \Pi A), \quad (91) \\
Tr \left[ D_o^{-1} - \Sigma^{HFB}, A \right] &= 0. \quad (92)
\end{align*}
\]

Moreover, if we define the operator \( \gamma \equiv M_{12} + M_{21}^* \) and apply it again to the sum and the difference of each one of the equations (87) to (99) with its transpose we also get:

\[
\begin{align*}
\text{Re} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) H \right\} \right) &= \frac{1}{2} \text{Im} \left( \gamma \left[ D_o^{-1} - S^{HFB}, H \right] + \gamma(\gamma H) \right), \quad (93) \\
\text{Re} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) F - \Pi \Re g \right\} \right) &= \frac{1}{2} \text{Im} \left( \gamma \left[ D_o^{-1} - \Sigma^{HFB}, F \right] + \gamma(\Gamma F - \Pi A) \right), \quad (94) \\
\text{Re} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) A - \Gamma \Re g \right\} \right) &= \frac{1}{2} \text{Im} \left( \gamma \left[ D_o^{-1} - \Sigma^{HFB}, A \right] \right), \quad (95) \\
\text{Im} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) H \right\} \right) &= -\frac{1}{2} \text{Re} \left( \gamma \left[ D_o^{-1} - S^{HFB}, H \right] + \gamma(\gamma H) \right), \quad (96) \\
\text{Im} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) F - \Pi \Re g \right\} \right) &= -\frac{1}{2} \text{Re} \left( \gamma \left[ D_o^{-1} - \Sigma^{HFB}, F \right] + \gamma(\Gamma F - \Pi A) \right), \quad (97) \\
\text{Im} \left( \gamma \left\{ (D_o^{-1} - \Re \Sigma) A - \Gamma \Re g \right\} \right) &= -\frac{1}{2} \text{Re} \left( \gamma \left[ D_o^{-1} - \Sigma^{HFB}, A \right] \right). \quad (98)
\end{align*}
\]

with \( \text{Re} \) and \( \text{Im} \) denoting the real and imaginary parts. To close the set of equations, we need an equation of motion for the superfluid velocity which can be found from the definitions Eq. (55) and Eq. (56) to be:

\[
\frac{\partial \sigma_s(R, T)}{\partial T} = -\frac{\partial}{\partial \sigma_s} \left( (\mu(R, T) + V(R, T)) + J_{\sigma_s}^2(R, T) \right). \quad (99)
\]

Eqs. (87) and (99) form a closed set of equations that describe the state of the gas at a given time. Equations (87), (88) and (99) are usually called the gap equations. They describe the quantum properties of a gas which is evolving according to Boltzmann-type equations (90) and (91). Under the derived formalism Eqs. (87) to (99) form a coupled set of equations which replace the original dynamics. The equations have to be solved self consistently for any analysis.

V. EQUILIBRIUM PROPERTIES FOR A HOMOGENEOUS SYSTEM

There are two situations in which we expect an equilibrium solution to come from the Boltzmann equations. Firstly when the system has never been disturbed and it remains in its equilibrium state. Secondly when the system has had sufficient time to relax after an applied perturbation. In this section we will show how the second order nonequilibrium Boltzmann Equations lead, in these special cases to the linear equilibrium solutions obtained from the HFB approximation upgraded with second order corrections in \( U \).

At equilibrium, in the absence of any external potential, the functions \( g^{\Re} \) and \( H \) are completely independent of \( R \) and \( T \). In this case the generalized Poisson-bracket terms are zero and Eqs. (89) and (95) imply that:

\[
A \left( D_o^{-1} - \Re \Sigma \right) - (\Re g) \Gamma = 0. \quad (100)
\]

Because \( \Re g(q, \omega) \) is determined by \( A(q, \omega) \) as indicated in Eq. (72), Eq. (100) is satisfied when \( A(q, \omega) \) is given by

\[
-iA(q, \omega) = \left[ D_o^{-1} - \Re \Sigma + \frac{i}{2} \Gamma \right]^{-1} - \left[ D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right]^{-1} \quad (101)
\]

and the function \( \Re g(q, \omega) \) given by

\[
\Re g(q, \omega) = P \int \frac{d\omega'}{2\pi} \frac{A(q, \omega')}{\omega - \omega'} \quad (102)
\]

\[
= \frac{1}{2} \left\{ \left[ D_o^{-1} - \Re \Sigma + \frac{i}{2} \Gamma \right]^{-1} + \left[ D_o^{-1} - \Re \Sigma - \frac{i}{2} \Gamma \right]^{-1} \right\}.
\]
From Eqs. (103), (106), (101) and (106) we also get, at equilibrium, the conditions
\[ \gamma = 0, \quad (103) \]
\[ \Gamma F - \Pi A = 0. \quad (104) \]
Eqs. (103) and (104) are just the mathematical statement of detailed balance. They represent the physical condition that at equilibrium the net rate of change of the density of particles with momentum \( q \) and energy \( \omega \) is zero. Since it is always possible to write
\[ F(q, \omega) = \left( n_q(\omega) + \frac{1}{2} \right) A(q, \omega), \quad (105) \]
Eq. (104) can only be satisfied if
\[ \Pi(q, \omega) = \left( n_q(\omega) + \frac{1}{2} \right) \Gamma(q, \omega), \quad (106) \]
is satisfied. Detailed study of the structure of the self-energy indicates that \( n_q(\omega) \) is related to the Bose-Einstein thermal distribution, \( n_q(\omega) = \frac{1}{\exp(\beta \omega - 1) + 1} \), with \( \beta \) interpreted as the local inverse temperature in energy units. \( \Pi(q, \omega) \) is identified with the quasiparticle energies \( \epsilon \). In refs. \( [17, 18] \), the authors prove that the only translational invariant solution is the thermal.

Since \( H \) contains delta functions in momentum and energy at equilibrium, we get from Eq. (87):
\[ \mu = -2J + \Re S_{11}(0, 0) + \Re S_{12}(0, 0). \quad (107) \]

### A. Quasiparticle formalism

In the noninteracting case the diagonal terms of \( A(q, \omega) \) are just delta functions with peaks at values of \( \hbar \omega \) that match the possible energy difference which results from adding a single particle with quasimomentum \( q \) to the system. In the many body system the energy spectrum is sufficiently complex so that the diagonal elements of \( A(q, \omega) \) are not delta functions but instead continuous functions of \( \omega \). However, there are always sharp peaks in \( A \). These sharp peaks represent the coherent and long lived excitations which behave like weakly interacting particles. These excitations are called quasiparticles. From Eq. (101) it is possible to see that the quasiparticle decay rate is determined by \( \Gamma \). The quasiparticle approximation is obtained by considering \( \Gamma \) very small for small values of \( \omega \). This assumption implies that \( D^{-1} \equiv D^{-1}_{\text{HF}} = \frac{1}{2} \Sigma + i \Gamma \) is essentially real with only an infinitesimal imaginary part. The zeros of \( D^{-1} \) about which \( A \) is very sharply peaked are identified with the quasiparticle energies \( \hbar \omega_q \).

Using the assumption of \( D \) being small \( \Gamma \), and the identity
\[ \lim_{\epsilon \to 0} \frac{1}{\omega - \omega' + i \epsilon} = \frac{1}{\omega - \omega'} - i \pi \delta(\omega - \omega'), \quad (108) \]
it is possible to write the matrix components of \( D^{-1} \) as:
\[ D^{-1}(q, \omega) = \hbar \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \mathcal{L}_{qq}(q, \omega) & \mathcal{M}_{q-q}(q, \omega) \\ \mathcal{M}^*_{q-q}(-q, -\omega) & \mathcal{L}^*_{qq}(-q, -\omega) \end{pmatrix}, \quad (109) \]
with
\[ \mathcal{L}_{qq}(\omega) = -2J \cos q\delta - \mu + \Sigma_{11}^{HF}(q, \omega) + \frac{d\omega'}{2\pi} \Gamma_{11}(q, \omega') \quad (110) \]
\[ \mathcal{M}_{q-q}(\omega) = \Sigma_{12}^{HF}(q, \omega) + \frac{d\omega'}{2\pi} \Gamma_{12}(q, \omega') \quad (111) \]

The quasiparticle amplitudes \( u_q \) and \( v_q \) are the solutions to the eigenvalue problem
\[ \begin{pmatrix} \mathcal{L}_{qq}(q, \omega) & \mathcal{M}_{q-q}(q, \omega) \\ \mathcal{M}^*_{q-q}(-q, -\omega) & \mathcal{L}^*_{qq}(-q, -\omega) \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \hbar \omega_q \begin{pmatrix} u_q \\ -v_q \end{pmatrix}, \quad (112) \]
and satisfy the normalization condition \( |u_q|^2 - |v_q|^2 = 1 \). In the absence of vortices it is always possible to find an ensemble in which the amplitudes \( (u_q, v_q) \) are purely real and \( u_q = u_{-q}, \ v_q = v_{-q} \). In terms of the quasiparticle amplitudes, the matrix elements of the spectral function \( A \), Eq. (101), are given by:
\[ A_{11}(q, \omega) = -2J \left[ \frac{u^2_q}{\omega - \omega_q + i0^+} - \frac{v^2_q}{\omega - \omega_q - i0^+} \right] \]
\[ = 2\pi \left[ u^2_q \delta(\omega - \omega_q) - v^2_q \delta(\omega + \omega_q) \right], \quad (113) \]
\[ A_{12}(q, \omega) = 2J \left[ \frac{u_q v_q}{\omega - \omega_q + i0^+} - \frac{v_q u_q}{\omega - \omega_q - i0^+} \right] \]
\[ = -2\pi u_q v_q \delta(\omega - \omega_q) - \delta(\omega + \omega_q), \quad (114) \]
\[ A_{22}(q, \omega) = A_{11}(q, -\omega), \quad (115) \]
\[ A_{21}(q, \omega) = A_{12}^*(q, \omega). \quad (116) \]

Finally, using the definitions of \( F \) and \( A \), we can express the matrix components: \( \rho_q(\omega), \tilde{\rho}_q(\omega) \) and \( m_q(\omega) \) defined as the Fourier transform of \( \rho_{ij}, \tilde{\rho}_{ij} \) and \( m_{ij} \) respectively (see Eqs. (32) and (35)) in terms of quasiparticle amplitudes:
\[ \rho_q(\omega) = 2\pi \left[ u^2_q n_q(\omega) \delta(\omega - \omega_q) \right. \]
\[ + v^2_q (1 + n_q(\omega)) \delta(\omega + \omega_q) \],
\[ \tilde{\rho}_q(\omega) = 2\pi \left[ u^2_q (1 + n_q(\omega)) \delta(\omega - \omega_q) \right. \]
\[ + v^2_q n_q(\omega) \delta(\omega + \omega_q) \],
\[ m_q(\omega) = 2\pi u_q v_q \left[ n_q(\omega) \delta(\omega - \omega_q) \right. \]
\[ - \left. (1 + n_q(\omega)) \delta(\omega + \omega_q) \right]. \quad (119) \]

### B. HFB approximation

Under the HFB approximation the matrix \( \Re \Sigma \) and \( \Re S \) are just given by \( \Sigma_{HF} \) and \( S_{HF} \). In terms of the quasiparticle amplitudes and setting \( N = 2 \), they can be written as:
\[
\Sigma^{HFB} = U \begin{pmatrix}
2(n_o + \tilde{n}) & n_o + \tilde{m} \\
n_o + m & 2(n_o + \tilde{n})
\end{pmatrix},
\]
(120)

\[
S^{HFB} = U \begin{pmatrix}
n_o + 2\tilde{n} & \tilde{m} \\
m & n_o + 2\tilde{n}
\end{pmatrix}.
\]
(121)

with

\[
\tilde{n} = \frac{1}{M} \sum_q \left[ (1 + n_q(\omega_q))v_q^2 + u_q^2 n_q \right],
\]
(122)

\[
\tilde{m} = \frac{1}{M} \sum_q u_q v_q (2n_q(\omega_q) + 1).
\]
(123)

In the HFB approximation, Eq. (112) and Eq. (107) then yield:

\[
\left( -2J \cos(qa_l) - \mu + 2U(n_o + \tilde{n}) \right) U(n_o + \tilde{m}) \left( U(n_o + \tilde{m}) -2J \cos(qa_l) - \mu + 2U(n_o + \tilde{n}) \right) \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \hbar \omega_q \begin{pmatrix} u_q \\ -v_q \end{pmatrix},
\]
(124)

\[
\mu = -2J + Un_o + 2U\tilde{n} + U\tilde{m}.
\]
(125)

As a final step, to fix the total density to \( n \), the constraint

\[
n = n_o + \tilde{n},
\]
(126)

has to be satisfied.

For a given density and temperature Eqs. (122)-(126) form a closed set of equations. At zero temperature, they reduce to the HFB equations derived in [16] using the quadratic approximation.

The Hugenholtz-Pines theorem states that a homogeneous system at equilibrium has to fulfill

\[
L_{qq}(0,0) - M_{q-q}(0,0) = 0
\]
(127)

The above equation implies that the energy spectrum of a Bose gas is gapless, i.e. there is an excitation with an energy that tends to zero in the limit of zero momentum. Mathematically the theorem implies that the two-point propagator \( g(q, \omega) \) has a pole at \( q = \omega = 0 \). Physically it reflects the fact that small rotations of the phase of the condensate wave function cost little energy (Goldstone mode of the broken symmetry). The Hugenholtz-Pines theorem is a consequence of the invariance of the mean field and the two point propagators under a phase transformation.

The HFB approximation violates the Hugenholtz-Pines theorem:

\[
L_{qq}(0,0) - M_{q-q}(0,0) = -2U\tilde{m} \neq 0
\]
(128)

One way to solve the gap problem is to set the anomalous term \( \tilde{m} \) to zero in HFB equations. This procedure is known as HFB-Popov approximation. The HFB-Popov equations were first introduced by Popov [59], and at equilibrium they are consider a better approximation than the HFB equations because they yield a gapless spectrum. Nevertheless the HFB-Popov equations are not conserving and therefore they are not appropriate to describe dynamical evolution.

C. Second-order and Beliaev approximations

When second order terms are taken into account the matrices \( L_{qq} \) and \( M_{q-q} \) become energy dependent. For simplicity we restrict the calculations to the zero temperature case when \( n_q = 0 \). In terms of the quasiparticle amplitudes the contributions to the self-energy at second order are given by
\[ M_{q,q}(q, \omega) = U n_o + \bar{U} \bar{n} + \]
\[ \frac{2U^2}{\hbar M} \sum_k \left( \frac{2A_k B_{q-k} + 2C_k A_p + 2C_k B_{q-k} + 3C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]
\[ = \frac{2U^2}{\hbar M} \sum_{k,p} \left( \frac{2A_k B_{q-k} + 2C_k A_p + 2C_k B_{q-k} + 3C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]
\[ \mu = -2J + U n_o + 2U \bar{n} - 2U \frac{2^2}{\hbar M} \sum_{k,p} \left( \frac{2A_k B_{q-k} + 2C_k A_p}{\omega_k + \omega_p + \omega_{q-k}} \right) \]
\[ \Delta M_{q,q}(q, \omega) = \frac{U^2}{\hbar M} n_o \sum_k \left( \frac{2A_k B_{q-k} + 2C_k A_p + 2C_k B_{q-k} + 3C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]
\[ = \frac{2U^2}{\hbar M} \sum_{k,p} \left( \frac{2A_k B_{q-k} + 2C_k A_p + 2C_k B_{q-k} + 3C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]
\[ \Delta L_{q,q}(q, \omega) = -U \bar{n} + \frac{2^2}{\hbar M} \sum_{k,p} \left( \frac{A_k A_{q-k} + 2A_k B_{q-k} + 4C_k A_{q-k} + 2C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]
\[ = \frac{2U^2}{\hbar M} \sum_{k,p} \left( \frac{A_k A_{q-k} + 2A_k B_{q-k} + 4C_k A_{q-k} + 2C_k C_{q-k}}{\omega - \omega_k - \omega_{q-k}} \right) \]

where the quantities \( A, B \) and \( C \) are defined as
\[ A_k = u_k^2, \quad B_k = v_k^2, \quad C_k = -u_k v_k. \]
If second order terms are included in the theory, they change the quasiparticle spectrum not only by shifting the quasiparticle energies but also by making them complex. The imaginary part that the quasiparticle energies acquire comes from the poles of the second order terms and it is associated with a damping rate. The physical meaning is that when the energy denominator in the second order terms vanishes a process where a quasiparticle decays into two of lower energy is energetically allowed. This kind of damping mechanism is known as Beliaev damping and was calculated by Beliaev in the case of a uniform Bose superfluid [43]. In the remainder of this section we calculate the zero temperature Beliaev damping coefficient for atoms in optical lattices using the tight-binding second order Beliaev approximation, Eqs. (137)-(138). We follow the same ideas used by Beliaev to study the uniform system.

1. Perturbative treatment

As the starting point we assume that the net effect of second order plus HFB terms is to introduce small corrections to the Bogoliubov-de Gennes (BdG) self energies \( \varepsilon_q \). In this case instead of solving the equations in a self consistent way we can replace the BdG quasiparticle energies and amplitudes in the HFB and second order self-energy corrections to calculate the shift they introduce in the spectrum.

The quasiparticle energies and amplitudes in the BdG approximation are given by [16]:

\[
\begin{align*}
\varepsilon_q^{(0)} &= \sqrt{\varepsilon_q^2 + 2Un_o} \\
A_q^{(0)} &= u_q^{(0)} e^{i\theta_q^{(0)}} \\
B_q^{(0)} &= v_q^{(0)} e^{-i\theta_q^{(0)}} \\
\eta_q^{(0)} &= n_o \frac{1}{\hbar \omega_i^{(0)}} \\
\eta_q^{(0)} &= n_o \frac{1}{\hbar \omega_i^{(0)}} \\
\end{align*}
\]

\[
\begin{align*}
\Delta L_{q\rightarrow q'}(0,0) - \Delta M_{q\rightarrow q'}(0,0) &= -2U\eta^{(0)} + \frac{2U^2n_o}{\hbar M} \sum_k \left( A_k^{(0)} A_{-k}^{(0)} + B_k^{(0)} B_{-k}^{(0)} - 2C_k^{(0)} C_{-k}^{(0)} \right) \\
&= -2U\eta^{(0)} + \frac{2U^2n_o}{\hbar M} \sum_k \frac{(u_k^{(0)})^2 - (v_k^{(0)})^2}{-2\omega_k^{(0)}} \\
&= 2U\frac{1}{M} \sum_k \frac{Un_o}{2\hbar \omega_k^{(0)}} - \frac{2U^2n_o}{\hbar M} \sum_k \frac{1}{-2\omega_k^{(0)}} = 0.
\end{align*}
\]

with \( n \) the total density, \( n = N/M \).

As shown in the last section, the HFB approximation has the problem that it has a gap in the excitation spectrum and therefore violates Pines-Hugenholtz theorem. However, as shown by Beliaev [43], when second order Beliaev contributions are included the theory becomes gapless. This can be seen from Eqs. (137) and (138):

HFB-Popov [13, 14]:

Beliaev damping

If we include HFB and second order corrections, the quasiparticle energy shifts are given to first order in \( \lambda \) by

\[
\begin{align*}
\Delta L_{q\rightarrow q'}(0,0) - \Delta M_{q\rightarrow q'}(0,0) &= -2U\eta^{(0)} + \frac{2U^2n_o}{\hbar M} \sum_k \left( A_k^{(0)} A_{-k}^{(0)} + B_k^{(0)} B_{-k}^{(0)} - 2C_k^{(0)} C_{-k}^{(0)} \right) \\
&= -2U\eta^{(0)} + \frac{2U^2n_o}{\hbar M} \sum_k \frac{(u_k^{(0)})^2 - (v_k^{(0)})^2}{-2\omega_k^{(0)}} \\
&= 2U\frac{1}{M} \sum_k \frac{Un_o}{2\hbar \omega_k^{(0)}} - \frac{2U^2n_o}{\hbar M} \sum_k \frac{1}{-2\omega_k^{(0)}} = 0.
\end{align*}
\]
Eq. (148) and (149) reduce to the Beliaev uniform gas matrix

If we replace不方便读取 form:

After some algebra, Eq. (146) can be written in the more con-

where the matrices

For a translational-invariant system at equilibrium, all quan-

Because Eq. (150) was found treating the second order correc-

Therefore, in the regime \( Un/J > 1 \,^5\) the interaction term dominates for all quasimomenta and the quasiparticle amplitudes and energies can be expanded as:

\[
\begin{align*}
\delta E_q + i\gamma_q &= U\bar{m}_q^0(u_q^0 - v_q^0)^2 + \frac{4U^2}{\hbar M n_o^0} \sum_k \left( \frac{B_{k,q-k}^2}{\omega_q^0 - \omega_k^0 - \omega_q^{(0)} + i\epsilon} - \frac{\bar{B}_{k,q-k}^2}{\omega_q^0 + \omega_k^0 + \omega_q^{(0)} - i\epsilon} \right),
\end{align*}
\]

where the matrices \(B_{k,q-k}\) and \(\bar{B}_{k,q-k}\) are defined as

\[
\begin{align*}
B_{k,q-k} &= u_q^0(u_k^0 v_{q-k}^0 - v_k^0 v_{q-k}^0) - v_q^0(v_k^0 v_{q-k}^0 - u_k^0 v_q^0 - v_k^0 v_{q-k}^0 - v_q^0 v_q^0), \\
\bar{B}_{k,q-k} &= v_q^0(v_k^0 v_{q-k}^0 - u_k^0 v_{q-k}^0 - u_q^0 v_q^0 - u_k^0 v_{q-k}^0 - v_q^0 v_q^0).
\end{align*}
\]

\(\alpha_k = \eta \left( \frac{\epsilon_k}{J} \right)^{1/4}, \quad \eta \equiv \left( \frac{J}{2n_o^0 U} \right)^{1/4}\)

In the very weakly interacting regime \( Un_o/J \leq 1 \), the approximations used to derive Eqs. (151) to (153) are still valid if the quasimomentum of the excitation involved in the decay process is small, \( qa_l \ll \sqrt{n_o U/J} \).

If one substitutes Eqs. (151) to (153) for the quasiparticle amplitudes in Eq. (150) and makes use of the energy conser-

\(^5\) Notice that for large filling factors \( n \), the parameter \( Un/J \) can be bigger than one but the system can be still far away from the Mott insulator critical point.
viation condition, which is approximately given by

$$\alpha_q^2 - \alpha_k^2 - \alpha_{q-k}^2 = \frac{1}{2}(\alpha_q^6 + \alpha_k^6 - \alpha_{q-k}^6), \quad (156)$$

one gets the following expression for the damping coefficient:

$$\gamma_q = \frac{9\pi}{8M} \frac{J}{n_i^{(0)}} \sum_k \sqrt{\frac{\xi q^2 \xi q - k}{J^3}} \delta (\tau_q - \tau_k - \tau_{q-k}), \quad (157)$$

with $\tau_q$ the dimensionless quasiparticle energies given by

$$\tau_q = \frac{\hbar \omega_{(0)} q}{2n_i^{(0)} U \eta^2}.$$  When the number of lattice sites is large, to a good approximation the discrete sum can be replaced by an integral $1/M \sum_k \to a_l/2\pi \int_{0}^{\pi/a_l} dk$.

For the one dimensional system, we find that the only value of $k$ at which the energy constraint is satisfied is when $k = q$. This value of $k$ leads to a zero damping coefficient and therefore in the one dimensional system the quasiparticles become totally stable against their decay into two of lower energy. In this case higher order decay processes have to be considered. However, the absence of Beliaev damping in one dimensional lattices is not a particular characteristic of the lattice dispersion relation. If the damping coefficient is calculated using the one dimensional uniform Bose gas dispersion relation, it is also found to be zero.

The extension of the expression for the Beliaev damping coefficient to higher dimensional lattice systems can be done straightforwardly. One just has to replace the single particle dispersion relation $\xi_k$ in Eq. (157) by the one in the specific dimension. If we assume a separable square lattice in $d$ dimensions, with the same tunneling matrix energy $J$, and lattice constant $a_l$, in all different directions we get:

$$\gamma_q^{(d)} = \frac{9\pi}{8M} \frac{J}{n_i^{(0)}} \sum_k \sqrt{\frac{\xi q^2 \xi q - k}{J^3}} \delta (\tau_q - \tau_k - \tau_{q-k}), \quad (158)$$

with the definitions $\alpha_k = 4 \sum_{i=1}^{d} \sin^2 (\frac{k_i a_l}{2}), \ h \omega_{(0)} k \simeq \frac{2n_i^{(0)} U (\alpha_k^2 + \frac{1}{4} \alpha_{q-k}^2), \alpha_k = \eta (\frac{q}{a_l})^{1/4}}{2n_i^{(0)} U \eta^2}.$

An analytic expression for the damping coefficient can be easily obtained when the excitations involved in the decay process have long wave number: $qa_l \ll 1$. In this parameter regime for the particular case of a three dimensional lattice the integral yields:

$$\gamma_q^{(d=3)} \approx \frac{9}{32\pi} \frac{J a_l^3}{n_i^{(0)}} \int dk d\theta \sin(\theta) k^2 \sqrt{\frac{\xi q^2 \xi q - k}{J^3}} \delta (qa_l - ka_l - a_l \sqrt{p^2 + q^2 - 2pq \cos \theta}) \quad \text{Eq. 159}$$

with $m^* = \hbar^2/(2Ja_l^2)$ the effective mass. In the long wavelength limit, or phonon regime, the damping coefficient in the lattice reduces to the well known result first obtained by Beliaev in the phonon regime, with the mass replaced by an effective mass.

Outside the phonon regime, the analytic evaluation of the integral is more complicated because of the energy conservation constraint. In the uniform gas case, which has a simpler quasiparticle spectrum, it has been shown that there is a finite threshold momentum $q^*$ such that the decay of an excitation is impossible if $q > q^*$ [60]. We expect that the trigonometric dependence on the quasimomentum of the quasiparticle dispersion relation in lattice-type systems makes the energy conservation constraint even harder to fulfill. In Ref. [61] the authors calculated the finite temperature Landau damping coefficient in a one dimensional optical lattice and showed the disappearance of Landau damping when $Un_o/J > 6$.

VI. CONCLUSIONS

In this work we continued our previous studies of the dynamics of bosonic atoms confined in optical potentials. Here, starting from the 2PI-CTP equations of motion, derived in Paper I from the Bose-Hubbard Hamiltonian, we show how the complicated nonlocal, non-Markovian integro-differential equations can be simplified and reduced to the standard kinetic theory equations. Specifically, by using a two-time separation condition, valid in dilute weakly interacting systems not very far away from equilibrium, we recast the full quantum dynamics into two coupled sets of equations: the first set of Boltzmann equations governing the distribution functions and a second set of gap equations describing the modified dispersion relation. We conclude here with three remarks on some general features of this problem and our approach.

First, a remark on quantum kinetic theory in discrete versus continuous systems: Even though we work with a lattice gas system described by the Bose Hubbard Hamiltonian, the assumption that the propagators are slowly varying in the center of mass coordinates permits one to map the discrete tight
binding equations into a set of continuous differential equations in the center of mass coordinates. For this reason the dynamical equations of motion we derived for discrete systems look very similar to previous kinetic equations derived for continuous systems. On the other hand, to include all the relevant dynamical effects introduced by the lattice, we kept the discrete character of the tight binding Hamiltonian in the equations for the relative coordinates, as manifested in the gap equations which exhibit a dispersion relation different from the homogeneous Bose gas system.

Second, the last section of this work was dedicated to a study of quantum equilibrium solutions. By using the quasiparticle approximation, we recovered from the kinetic equations the linear HFB corrections to the self-energy plus second order corrections. We showed how by neglecting the condensate-independent second order terms in the self-energy, one obtains a tight-binding version of the well known Beliaev equations. We used these equations to derive expressions for the zero temperature Beliaev damping coefficient in lattice systems in certain parameter regimes. In particular, we showed that for long wavelength excitations, the damping coefficient in a three dimensional lattice reduces to the one calculated for a uniform Bose gas in the phonon regime, but with the mass replaced by the effective mass induced by the lattice.

A final remark on the purpose of this work. It is not meant to be a mere academic exercise in our demonstration of how Boltzmann like equations are obtained from the effective action and equilibrium solutions can be obtained from the full quantal solutions. In making explicit the simplifying assumptions en route starting from first principles, it allow us to realize the limitations and the applicability of a kinetic theory formulation for describing the quantum dynamics of many-body lattice systems. It serves to identify the range of validity and the parameter regimes where the underlying assumptions leading to these simplified kinetic equations can become unreliable. We view this effort as having both theoretical and practical significance in seeking a proper description of such systems and better understanding of its behavior – theoretical in scrutinizing the practicing kinetic theories in existence, and practical in providing the correct parameters for comparison with experiments.

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