On source coupling and the teleparallel equivalent to GR

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Abstract

Alternatives to the usual general relativity (GR) Riemannian framework include Riemann-Cartan and teleparallel geometry. The “teleparallel equivalent of GR” (TEGR, aka GR||) has certain virtues, however there have been allegations of serious source coupling limitations. Now it is quite straightforward to show that the coupled dynamical field equations of Einstein’s GR with any source can be accurately represented in terms of any other connection, in particular teleparallel geometry. Using an argument similar to one used long ago to show the “effective equivalence” between GR and the Einstein-Cartan theory, we construct the teleparallel action which is equivalent to a given Riemannian one; thereby finding the “effectively equivalent” coupling principle for all sources, including spinors. No auxiliary field is required. Can one decide which is the real “physical” geometry? Invoking the minimal coupling principle may give a unique answer.

1 Background

The world has geometry, but which kind?\textsuperscript{[1]} In general, geometry includes both a metric and a connection. They determine three tensors: the non-metricity $Q$, torsion $T$ and curvature $R$. Here we consider only the vanishing $Q$ cases: Riemann-Cartan ($R \neq 0 \neq T$), Riemannian ($R \neq 0 = T$) and teleparallel ($R = 0 \neq T$).

Aldrovandi and Pereira\textsuperscript{[2]} have emphasized that spaces do not have curvature and torsion, rather it is connections that have curvature and torsion. Since any two connections differ by a tensor, it is straightforward to change from one connection to any other. In this way we can take the field equations

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of, for example, the Poincaré Gauge theory and represent them in terms of the Riemannian connection. The price is that one needs to add to the Riemannian geometry an extra tensor field (e.g., the contortion, or torsion). And, moreover, the extra field couples in a characteristic “non-minimal” fashion. As long as one allows an extra tensor field with its associated special type of non-minimal coupling, one can transcribe any physical equations from Riemann-Cartan to Riemannian to teleparallel geometry and vice versa. For a complete “equivalence” we also want the conserved quantities, like energy-momentum and angular momentum, hence we should also find the equivalent action. The effectively equivalent action also generally needs extra fields and non-minimal coupling.

There is a teleparallel equivalent to GR (TEGR, aka GR\|). The simplest description is in terms of orthonormal-teleparallel (OT) frames. It is easily constructed: given the GR metric, simply choose any orthonormal frame field and declare it to be parallel (i.e., take the connection coefficients to vanish in this frame).

It has been alleged that GR\| has serious source coupling limitations, e.g. it has been said that only scalar matter fields or gauge fields are allowed as sources, whereas matter carrying spin cannot be consistently coupled; some argue that torsion does not couple to electromagnetism, others have a different opinion.

For the special case of GR and GR\| we can use an argument similar to that used long ago to establish the effective equivalence of GR and the Einstein-Cartan (EC) theory. As in that case, we find that the GR\| effective equivalent field equations and action also do not require an extra field. Nevertheless we get a complete equivalence for all sources, including, in particular, spinors.

2 Riemannian equations re-expressed in teleparallel form

The Riemannian GR equations for any source follow from an action of the form:

\[ L = e e^a_{\mu} e^b_{\nu} R^a_{\mu \nu} (\Gamma) + T^a_{\mu \nu} \tau^a_{\mu \nu} + L(\phi, e, \partial \phi + \Gamma \phi) \]

by variation with respect to the frame \( e^a_{\mu} \), the connection \( \Gamma \), the source field \( \phi \) and the multiplier \( \tau \) (which enforces the vanishing of torsion for \( \Gamma \)).
These standard GR dynamical equations can be represented in terms of any other connection, e.g. Riemann-Cartan with torsion, simply by a transformation involving an extra field: \( \Gamma^a_{b\mu} = \bar{\Gamma}^a_{b\mu} + K^a_{b\mu} \), here \( K^a_{b\mu} \) is the contorsion tensor (linearly related to the torsion). Minimally coupled Riemannian equations become non-minimally coupled to the contorsion. In particular we can also force the new connection \( \bar{\Gamma} \) to be teleparallel, so that \( \bar{R}^a_{\mu\nu} = 0 \).

Within each of the dynamic equations we make the above substitution. In this way, all of these Riemannian equations of motion can be transcribed into teleparallel form. But the form of the coupling has become non-minimal.

What about the conserved quantities? Can we find an action that gives the new equations?

### 3 Equivalent Teleparallel Action

For complete equivalence we need an equivalent teleparallel action. It can be obtained by the same substitution \( \Gamma^a_{b\mu} = \bar{\Gamma}^a_{b\mu} + K^a_{b\mu} \). Expanding the Hilbert scalar curvature Lagrangian of GR gives \( \bar{R} + \bar{D}K + K^2 \). We can enforce vanishing teleparallel curvature with a Lagrange multiplier field. After removing a total divergence the effectively equivalent action is of the form

\[
L_{||} = -\frac{1}{4} T^{\alpha\mu\nu} T_{\alpha\mu\nu} - \frac{1}{2} T^{\alpha\beta\nu} T_{\alpha\beta\nu} + T^\alpha_{\alpha\mu} T^\beta_{\beta\mu} + R^{ab}_{\mu\nu} \lambda_{ab}^{\mu\nu} + L(\phi, e, \bar{\nabla}\phi, K),
\]

(2)

to be varied with respect to the source \( \phi \), the frame \( e^a_\mu \), the teleparallel connection \( \bar{\Gamma} \), and the multiplier \( \lambda \). In short:

\[
L_{GR} \xrightarrow{\delta} \text{Riemannian form of equations} \quad \downarrow \quad \text{Riemannian form of equations} \quad \downarrow
\]

\[
L_{||} \xrightarrow{\delta} \text{teleparallel form of equations}
\]

where \( \downarrow \) is \( \Gamma \rightarrow \bar{\Gamma} + K \). As in the EC case studied earlier, (unlike the corresponding situation for the Poincaré Gauge theory or Metric-Affine gravity) we do not need any extra field. We obtain a complete equivalence for all sources — including spinors.
4 Conclusion

There is a one-to-one correspondence between the two representations. No physical experiment can distinguish between them; no observation can decide whether spacetime has curvature without torsion or torsion with vanishing curvature. GR and GR$_{||}$ are completely equivalent for all sources. TEGR truly is a good name.

Can we determine the true “physical” geometry? Is it Riemannian or teleparallel? This may be possible with the aid of the minimal coupling principle. Sources which are minimally coupled with respect to the Riemannian geometry will not generally be minimally coupled with respect to the teleparallel geometry, and vice versa. We could hope that observations would yield results which are consistent with one of these geometries and the associated minimal coupling.

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