Interacting scalar fields in de Sitter space

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Abstract

We investigate the massless $\lambda \phi^4$ theory in de Sitter space. It is unnatural to assume a minimally coupled interacting scalar field, since $\xi = 0$ is not a fixed point of the renormalization group once interactions are included. In fact, the only case where perturbation theory can be trusted is when the field is non-minimally coupled at the minimum of the effective potential. Thus, in perturbation theory, there is no infrared divergence associated with this scalar field.

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1 Introduction

Especially since the advent of the inflationary cosmology,[1] the study of de Sitter space has become more relevant. Inflation aside, the well-known conundrum called the cosmological constant problem,[2] (i.e., why the vacuum energy density is either finely tuned or does not gravitate) is especially intransigent. This unyielding theoretical puzzle challenges our notions of naturalness in field theory in other contexts, such as the hierarchy problem in the Standard Model. Various attempts have been made to identify something that is “wrong” with de Sitter space that would drive it toward Minkowski space. In particular, Ford and Parker [3] pointed out that some two-point functions for a massless, minimally coupled scalar (MMCS) field are infrared divergent. A closely related phenomenon is that the vacuum expectation value of \( \phi^2 \) grows with time \[4\][5][6],

\[
\langle \phi^2 \rangle = \frac{H^3}{4\pi^2} t. \tag{1}
\]

Here we have parameterized the de Sitter metric as

\[
ds^2 = dt^2 - e^{2Ht} d\mathbf{x}^2. \tag{2}
\]

The result, Eq. (1), is surprising in that it is not de Sitter invariant. Because of the infrared divergence, \( \langle \phi^2 \rangle \) is calculated either by introducing a physically well motivated infrared cutoff [4] or by choosing other physically acceptable vacuum states which are free of infrared divergences [3]. Either way the result is Eq. (1) with the possible addition of a constant (whose effect is to shift the origin of the time coordinate) and of terms which decrease exponentially in time [7].

Ford [7] has used this growth in time of \( \langle \phi^2 \rangle \) to argue that an interacting theory which includes a massless, minimally coupled scalar field may, under certain circumstances, cause a decrease in the cosmological constant from a large value to a small value, thus making de Sitter space unstable. We note that this problem with the infrared divergence does not arise for a non-minimally coupled scalar field or for a massive scalar field. In these cases the de Sitter invariant vacuum is free of infrared problems, and \( \langle \phi^2 \rangle \) does not grow with time. This property is crucial to the conclusion of this paper.

In this paper we discuss the one-loop effective potential for massless \( \lambda \phi^4 \) theory in de Sitter space and show that, in all cases where perturbation theory is valid, the field \( \phi \) is not minimally coupled, that is, the coupling to the curvature, \( \xi \), is not equal to zero, at the minimum of the effective potential. Although we only treat the \( \lambda \phi^4 \) theory in detail, the lessons learnt here are applicable to other models. The upshot of our analysis is this – once the field \( \phi \) becomes an interacting field, it cannot be considered minimally coupled and, hence, Eq. (1) does not hold for an interacting scalar field in de Sitter space. Thus in such a situation, there is no infrared problem. We have not considered the feedback on the metric, which may lead to other instabilities [8].
2 The Effective Potential

As the simplest model, consider a massless, non-minimally coupled, \( \lambda \phi^4 \) theory in de Sitter space described by the Lagrangian density,

\[
L = \sqrt{-g} \left[ \frac{1}{2} g^{\mu
u} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{\xi R}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 + c.t. \right]
\]  

In the zeta-function scheme, the one-loop effective potential is [9],

\[
V_{\text{eff}}(\phi) = \frac{\xi R}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{1}{2\Omega} \left[ \zeta'(0) + \zeta(0) \ln \left( \frac{\mu^2}{H^2} \right) \right]
\]

where \( \Omega = \frac{8}{3} \pi^2 \frac{1}{\mu^4}, \ R = 12H^2, \)

\[
\zeta(0) = \frac{\Delta^2}{12} - \frac{\Delta}{24} - \frac{17}{2880}
\]

\[
\zeta'(0) = -\frac{1}{3} \left[ \int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}} U(U - \frac{1}{2})(U - 1) \Psi(U) dU 
+ \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}} U(U - \frac{1}{2})(U - 1) \Psi(U) dU \right] + \frac{\Delta^2}{12} + \frac{\Delta}{72}
\]

(\text{up to an additive constant}) and,

\[
\Delta \equiv \frac{9}{4} - \frac{V''_{\text{tree}}}{H^2} - \frac{9}{4} - \frac{1}{H^2} \left( \xi R + \frac{\lambda}{2} \phi^2 \right) = \frac{9}{4} - 12\xi - \frac{\lambda}{2} \frac{\phi^2}{H^2}.
\]

Here \( \mu \) is the (arbitrary) renormalization scale, on which the renormalized coupling constants \( \xi(\mu), \) and \( \lambda(\mu) \) depend. Depending on the value of \( \Delta, \) the integration is along the line \( \text{Re}(U) = 1/2 \) (when \( \Delta \) is negative) or along the real axis (when \( \Delta \) is positive). The digamma function (defined by \( \Psi(U) = d\Gamma(U)/dU \)) has simple poles at \( U = 0, -1, -2, \ldots \) The pole at \( U = 0 \) is cancelled in the integrand. The integration around the poles at \( U = -1, -2, \ldots \) is performed by deforming the contour under the poles in the complex \( U \) plane. The imaginary part of \( V_{\text{eff}} \) is given by \( i\pi \times \text{the residue of the poles} \).

Now we want to investigate where the minimum of \( V_{\text{eff}} \) occurs and whether it is consistent to have \( \xi = 0 \) at the global minimum of \( V_{\text{eff}} \). In the models that Ford [7] considered, the renormalized value of \( \xi \) was set to zero at some renormalization scale \( \mu, \) and Eq.(4) was used for \( \langle \phi^2 \rangle \). But since the field \( \phi \) is an interacting field in the models considered in Ref. [7], the beta function for \( \xi \) is non-zero, and more importantly, \( \xi = 0 \) is not a fixed point of the renormalization group. Therefore it is important to verify whether one can arrange for

\[1\] For the interpretation of \( \text{Im}\{V_{\text{eff}}\} \), see Weinberg and Wu [10].
the renormalized value of $\xi$ to be zero at the global minimum of the effective potential (the vacuum). What we find is that perturbation theory breakdown in theories where $\xi$ vanishes at the minimum. Also, normalizing the theory at the scale where $\xi$ vanishes, also leads to the breakdown of perturbation theory. We will illustrate in the simplest case of $\lambda \phi^4$, but we expect that this model is prototypical of more general interacting scalar theories, and the results we obtain will be directly relevant for such theories (the interacting scalar model in Ref. [7] for example).

Now let us examine the effective potential in the limits $\phi^2 \gg H^2$ and $\phi^2 \ll H^2$ to see what choices we have to make in these regions for $\mu^2$ so as to obtain a satisfactory perturbation expansion. In the limit $\phi^2 \gg H^2$, the asymptotic behaviour of the digamma functions is logarithmic. In this limit $\zeta'(0)$ tends to,

$$\zeta'(0) \to -\zeta(0) \ln \left( \frac{\lambda}{2} \frac{\phi^2}{H^2} \right)$$

Thus the one loop correction in this limit tends to

$$\to \frac{1}{2\Omega} \zeta(0) \ln \left( \frac{\lambda}{2} \frac{\phi^2}{\mu^2} \right)$$

which is of course the one loop correction in flat space. To keep the log under control in this region we should choose $\mu^2 \approx \phi^2$.

In the region $\phi^2 \ll H^2$, the behaviour of the digamma functions is not logarithmic. To examine the behaviour of $V_{\text{eff}}$ more precisely in this region, it is simpler, and for our purposes sufficient to look at $dV_{\text{eff}}/d\phi \equiv V'_{\text{eff}}$. Differentiating Eq.(4) we obtain,

$$V'_{\text{eff}} = \phi \left[ \xi R + \frac{\lambda}{6} \phi^2 - \frac{\lambda R}{384 \pi^2} \left[ \left( \Delta - \frac{1}{4} \right) \left\{ \Psi \left( \frac{1}{2} + \sqrt{\Delta} \right) - \Psi \left( \frac{1}{2} - \sqrt{\Delta} \right) - 1 - \ln \left( \frac{\mu^2}{H^2} \right) \right\} - \frac{1}{3} \right] \right]$$

For $\phi^2/H^2 \to 0$, $\Delta \to 9/4 - 12 \xi$. So the behaviour of the digamma functions is regular and they tend to a constant value as $\phi^2/H^2 \to 0$ (except for particular values of $\xi$ which we will discuss later). Therefore, for $\phi^2/H^2 \ll 1$, if we choose $\mu^2 = H^2$, there will be no large log in the expression. Hence, the Hubble scale, $H$, acts as an infrared cutoff. Upto small factors, this choice for the infrared cutoff matches with that of Ref.[11] and Ref.[12].

Therefore the perturbation expansion will be improved if we exploit the renormalization group to resum the large logs for $\phi^2 > H^2$. For all $\phi^2 \leq H^2$ we may simply choose to normalize at $\mu^2 = H^2$. 
3 The Renormalization Group Equation

The renormalization group equation satisfied by $V_{\text{eff}}$ is,

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta_\xi \frac{\partial}{\partial \xi} + \beta_\lambda \frac{\partial}{\partial \lambda} - \gamma \phi \frac{\partial}{\partial \phi} \right) V_{\text{eff}}(\phi, \xi, \lambda, \mu) = 0
$$

The beta functions calculated to one-loop using the background field method are,

$$
\beta_\xi = \frac{\lambda}{16\pi^2} \left( \xi - \frac{1}{6} \right) \quad \beta_\lambda = \frac{3\lambda^2}{16\pi^2} \quad \gamma = 0
$$

Note that $\xi = 0$ is not a fixed point, and the conformally coupled value, $\xi = 1/6$, is a fixed point at one loop. What we are primarily interested in is the position of the minimum of $V_{\text{eff}}$. Instead of applying the RG to $V_{\text{eff}}(\phi)$ and then differentiating, it is simpler to apply the RG to $V'_{\text{eff}}(\phi)$ and work with the RG-improved $V'_{\text{eff}}$. Differentiating (9) with respect to $\phi$, and using the fact that $\gamma = 0$ to one-loop, we obtain the RGE satisfied by $V'_{\text{eff}}$.

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta_\xi \frac{\partial}{\partial \xi} + \beta_\lambda \frac{\partial}{\partial \lambda} \right) V'_{\text{eff}}(\phi, \xi, \lambda, \mu) = 0
$$

A solution to this equation is,

$$
V'_{\text{eff}}(\phi, \xi, \lambda, \mu) = V'_{\text{eff}}(\phi, \xi(\phi), \lambda(\phi), \phi)
$$

where the running couplings are given by,

$$
\xi(\phi) = \frac{1}{6} + \frac{\xi(\mu) - \frac{1}{6}}{\left( 1 - \frac{3\lambda(\mu)r}{16\pi^2} \right)^{\frac{1}{2}}} \quad \lambda(\phi) = \frac{\lambda(\mu)}{1 - \frac{3\lambda(\mu)r}{16\pi^2}}
$$

where $t = \frac{1}{2} \ln(\phi^2/\mu^2)$, $\xi(\mu) \equiv \xi$, and $\lambda(\mu) \equiv \lambda$. Thus the RG improved $V'_{\text{eff}}$ is,

$$
V'_{\text{eff}} = \phi \left[ \xi(\phi) R + \frac{\lambda(\phi)}{6} \phi^2 - \frac{\lambda(\phi) R}{384\pi^2} \left[ \left( \Delta(\phi) - \frac{1}{4} \right) \left\{ \Psi \left( \frac{1}{2} + \sqrt{\Delta(\phi)} \right) \right. \right.

\left. \right. + \left. \Psi \left( \frac{1}{2} - \sqrt{\Delta(\phi)} \right) - 1 - \ln \left( \phi^2 / R^2 \right) \right] \right] - \frac{1}{3}
$$

By $\Delta(\phi)$ we mean, use Eq.(5) with the running coupling constants. Since $\gamma = 0$, the full content of the RGE is obtained by setting $\mu = \phi$ in Eq.(5).

The solution we have obtained, Eq.(15), is the solution to the RG equation with beta functions calculated using a mass independent renormalization prescription. Hence, physical
“mass thresholds” are not automatically incorporated and so they have to be separately incorporated. Drawing upon the analysis from the previous section, we see that Eq.(14) should be used only for scales $\phi^2 \geq H^2$. For scales $\phi^2 < H^2$, we simply choose $\mu^2 = H^2$. So the explicit log term goes to zero and the couplings remain frozen at the values $\xi(H)$ and $\lambda(H)$ for all $\phi^2 < H^2$. Therefore for $\phi^2 \geq H^2$, $V_{\text{eff}}'$ is given by Eq.(15), and for $\phi^2 < H^2$, $V_{\text{eff}}'$ is given by,

$$V_{\text{eff}}' = \phi \left[ \xi(H)R + \frac{\lambda(H)R}{6} \phi^2 - \frac{\lambda(H)R}{384\pi^2} \left( (\Delta(H) - \frac{1}{4}) \left\{ \Psi \left( \frac{1}{2} + \sqrt{\Delta(H)} \right) \right\} - \frac{1}{3} \right) \right]$$

(16)

Here, by $\Delta(H)$ we mean, use Eq.(5) with the couplings normalized at $H$. Henceforth by $V_{\text{eff}}'$, we will mean $V_{\text{eff}}'$ given by Eq.(15) and Eq.(16), depending on the value of $\phi^2$. The RG improved $V_{\text{eff}}$ can be obtained by integrating $V_{\text{eff}}'$.

4 The Form of the Effective Potential

The form of $V_{\text{eff}}$ is determined by the parameters: $\xi(\mu)$, $\lambda(\mu)$, and $\mu/H$, or equivalently by $\xi(H)$ and $\lambda(H)$. Depending on the values of these dimensionless coupling constants, $V_{\text{eff}}$ takes on one of the two forms labeled (a) and (b) in Figure 1 (We are of course only dealing with theories in which $\lambda(H)/(16\pi^2) \ll 1$). Note that we have subtracted the appropriate constant so that $V_{\text{eff}}$ is zero at $\phi = 0$. Figure 1(a) shows the symmetric case, and Figure 1(b) shows the spontaneously broken case, that is, when the symmetry is spontaneously broken. It turns out that there is another possibility. For some values of $\xi(H)$, and $\lambda(H)$, perturbation theory breaks down in the neighborhood of the ‘apparent minimum’ of $V_{\text{eff}}$. Therefore in this case we can not identify the minimum of the effective potential using a perturbation calculation.

Before we go further we will describe qualitatively how this breakdown of perturbation theory shows up in the one-loop effective potential. Consider the tree potential with the
running couplings (recall that we keep the couplings frozen for $\phi^2 < H^2$). We will refer to this potential as $V_0$ henceforth. Suppose $V_0$ has a minimum at $\phi = \phi_0$, which could be zero or non-zero. We would expect that as long as $\lambda(\phi_0)/(16\pi^2) \ll 1$, perturbation theory could be trusted, and $V_0$ would only be modified slightly upon the inclusion of the full one loop corrections. But this is not always true. For some range of values of the couplings, one of the poles in $V''_\text{eff}$ (the pole that occurs when the curvature of the tree potential, $V''_\text{tree}(\phi)$, vanishes) significantly affects the structure of the effective potential near this apparent minimum, $\phi_0$. This implies a breakdown of perturbation theory in the neighborhood of $\phi_0$, even though $\lambda$ may be perturbatively small. We note that a divergence at $V''_\text{tree}(\phi) = 0$ also occurs in the flat space effective potential, but only at higher orders [16]. As we mentioned, this situation can arise when the minimum of $V_0$, $\phi_0$, is at the origin or away from the origin. We will discuss both these cases below.

We will treat the symmetric case first. We will show that a necessary condition for the symmetric case is that $\xi(H) > 0$. But $\xi(H)$ can not be too small (we will make this quantitative later), since, if the curvature of the effective potential becomes too flat, the fluctuations become uncontrolled leading to a breakdown in perturbation theory. This is the physical reason for the appearance of the singularity at $V''_\text{tree}(\phi) = 0$ in the perturbatively calculated effective potential.

Note that $V''_\text{eff}$ is zero at $\phi = 0$ except when $\xi$ is such that the pole in $V''_\text{eff}$ occurs at $\phi = 0$. $V''_\text{eff}$ has poles when $1/2 - \sqrt{\Delta} = -1, -2, -3, \ldots$. This translates into poles at (see Eq. (17)), $V''_\text{tree}/H^2 = 0, -4, -10, \ldots, -(n-1)(n+2), \ldots$. Therefore a pole will occur at $\phi = 0$ if $\xi(H) = 0, -1/3, \ldots, -(n-1)(n+2)/12, \ldots$. For these values of $\xi(H)$, perturbation theory breakdown in the neighborhood of the origin. Since in all cases where perturbation theory is valid in the neighborhood of the origin, $V''_\text{eff}(0) = 0$, the origin is an extremum of $V''_\text{eff}$. Now if we can show that $V''_\text{eff}(0) < 0$ for $\xi(H) < 0$, then we will have proven that a necessary condition for the symmetric case is that $\xi(H) > 0$.

In the neighbourhood of the origin, $V''_\text{eff}$ is given by Eq. (16). Differentiating this expression with respect to $\phi$ we obtain,

$$V''_\text{eff} = R\left(\xi - \frac{1}{384\pi^2} \left[\left(\Delta - \frac{1}{4}\right) f - \frac{1}{3}\right]\right) + \frac{\lambda}{2} \phi^2 \left(1 + \frac{\lambda}{16\pi^2} \left[f + \left(\Delta - \frac{1}{4}\right) \frac{\psi'(\frac{1}{2} \pm \sqrt{\Delta}) - \psi'(\frac{1}{2} - \sqrt{\Delta})}{2\sqrt{\Delta}}\right]\right)$$

(17)

where $f$ represents the expression inside the curly brackets in Eq. (16), and all the couplings are normalized at $\mu^2 = H^2$. This expression is regular at $\phi = 0$ except for the values of $\xi$ that we have listed above. For $\xi$ not equal to one of these values, we have,

$$V''_\text{eff}(0) = R \left(\xi + \frac{\lambda}{384\pi^2} \left[12 \left(\xi - \frac{1}{6}\right) |f|_{\phi=0} + \frac{1}{3}\right]\right)$$

(18)

There is no possibility for a Coleman-Weinberg [15] type of situation here.
where,

\[ [f]_{\phi=0} = \Psi \left( \frac{1}{2} + \sqrt{\frac{9}{4} - 12\xi} \right) + \Psi \left( \frac{1}{2} - \sqrt{\frac{9}{4} - 12\xi} \right) - 1 \] (19)

The plot of \( V''_{\text{eff}}(0) \) versus \( \xi(H) \) is shown for \( \lambda(H) = 1 \) in Figure 2. It is evident that for \( \xi(H) < 0, V''_{\text{eff}}(0) < 0 \), except for values of \( \xi(H) \) when perturbation theory is not valid at the origin. Thus a necessary condition for the symmetric case is that \( \xi(H) > 0 \).

As we mentioned earlier, \( \xi(H) \) can not be too small. In order to get an estimate on how large \( \xi(H) \) must be for perturbation theory to be valid, we again look at \( V''_{\text{eff}}(0) \). For perturbation theory to be valid, a sufficient condition is (see Eq. (18)),

\[ \left| \frac{\lambda}{384\pi^2} \left\{ 12 \left( \xi - \frac{1}{6} \right) [f]_{\phi=0} + \frac{1}{3} \right\} \right| \ll \xi \] (20)

For \( \lambda \) of \( O(1) \), this term will get large only near the pole. Therefore we may approximate this expression with just the pole term. We can extract the pole at \( \xi = 0 \) from the function \( \Psi(1/2 - \sqrt{9/4 - 12\xi}) \) using the following property of the digamma function [17],

\[ \Psi(z + 1) = \Psi(z) + \frac{1}{z} \] (21)

Thus we obtain the condition,

\[ \left| \frac{\lambda}{384\pi^2} 12 \left( -\frac{1}{6} \right) \left( -\frac{1}{4\xi} \right) \right| \ll \xi \] (22)

giving,

\[ \xi^2(H) \gg \frac{\lambda(H)}{768\pi^2} \] (23)
Therefore for all $\xi(H) > 0$ that satisfies the above condition, we have the symmetric case.

Now we will treat the spontaneously broken case. At this point we introduce the following notation for brevity.

$$\tilde{\xi}(\phi) = \begin{cases} \xi(\phi) & \text{if } \phi \geq H \\ \xi(H) & \text{if } \phi < H \end{cases}$$

$$\tilde{\lambda}(\phi) = \begin{cases} \lambda(\phi) & \text{if } \phi \geq H \\ \lambda(H) & \text{if } \phi < H \end{cases}$$

Now $V_0$ has minima away from the origin. These minima occur at $\phi = \pm v_0$ given by

$$v_0^2 = \frac{-6\tilde{\xi}(v_0)}{\tilde{\lambda}(v_0)} R = \frac{-72\tilde{\xi}(v_0)}{\tilde{\lambda}(v_0)} H^2.$$

In the cases where perturbation theory can be trusted, the one-loop corrections make only small shifts in the positions of these minima. For some values of the couplings, upon the inclusion of the full one-loop corrections, we find that perturbation theory breaks down in the neighborhood of $v_0$. As we mentioned earlier, this is because the pole in $V''_{\text{eff}}$ that occurs when the curvature of the tree potential vanishes significantly affects the structure of $V_{\text{eff}}$ near this apparent minimum.

To see this in greater detail, let us examine the effective potential in the neighborhood of $\phi = +v_0$. $V''_{\text{eff}}$ has poles at $\sqrt{\tilde{\Delta}(\phi)} = 3/2, 5/2, 7/2, \ldots$. The pole that lies closest to the minimum is at $\sqrt{\tilde{\Delta}(\phi)} = 3/2$, that is, when $V''_{\text{tree}}(\phi) = 0$. We extract the pole term as we did in the previous case. The pole term is,

$$[V'_{\text{eff}}]_{\text{pole term}} = \frac{\tilde{\lambda}(\phi) R \phi}{384\pi^2} \left( \tilde{\Delta}(\phi) - \frac{1}{4} \right) \frac{1}{\frac{3}{2} - \sqrt{\tilde{\Delta}(\phi)}}$$

Now we want to find the range of $\phi$ over which the pole term is dominant. That is, to a good approximation, the range of $\phi$ over which,

$$|[V'_{\text{eff}}]_{\text{pole term}}| \geq |V'_0|$$

where,

$$V'_0 \equiv \phi \left( \tilde{\xi}(\phi) R + \frac{\tilde{\lambda}(\phi)}{3} \phi^2 \right)$$

To determine the limits of the allowed range of $\phi$, we need to solve,

$$|[V'_{\text{eff}}]_{\text{pole term}}| = |V'_0|$$

Approximating all the couplings with their values at the pole, and approximating the L.H.S. with the first term of its Laurent expansion, we obtain,

$$\left| \frac{\tilde{\lambda}(\phi_p) R \phi_p}{64\pi^2} \right| \geq \frac{2}{3} \tilde{\xi}(\phi_p) R \phi_p$$
where $\phi_p$ is the position of the pole given by,

$$\phi_p^2 = -\frac{2\tilde{\xi}(\phi_p)R}{\tilde{\lambda}(\phi_p)} \quad (30)$$

The two solutions to (29) are,

$$\phi^2 ± \phi_p^2 = \phi^2_p ± \frac{R}{256\pi^2|\xi(\phi_p)|} \quad (31)$$

Thus for the dominance of the pole to die out well before $\phi = v_1$, where $v_1$ is the minimum of the full one-loop effective potential, we need,

$$\phi^2_± - \phi^2_p \ll v^2_1 - \phi^2_p \quad (32)$$

That is,

$$\frac{1}{256\pi^2|\xi(\phi_p)|} \ll \frac{v^2_1}{R} - \frac{-2\tilde{\xi}(\phi_p)}{\tilde{\lambda}(\phi_p)} \quad (33)$$

If we make the approximations, $v^2_1 \simeq v^2_0$, $\tilde{\xi}(\phi_p) \simeq \tilde{\xi}(v_0)$, and $\tilde{\lambda}(\phi_p) \simeq \tilde{\lambda}(v_0)$, the above condition becomes,

$$64\tilde{\xi}^2(v_0) \gg \frac{\tilde{\lambda}(v_0)}{16\pi^2} \quad (34)$$

Note that if $v_0 < H$, $\tilde{\xi}(\phi_p) = \tilde{\xi}(v_0) = \xi(H)$, and likewise for $\tilde{\lambda}$. This condition is essentially the same as (23) for the case where $\phi_0 = 0$. As long as (34) is satisfied, the pole does not dominate the form of the effective potential near the minimum, and we can trust the one-loop effective potential near the minimum.

It is interesting to note that the ratio of $\xi^2$ and $\lambda$ appears in the expression for the value of the tree potential at its minimum. If we normalize the couplings at the scale of the minimum, the value of the tree potential at $\phi = v_0$ relative to the value of the potential at $\phi = 0$ is,

$$V_{tree}(\phi = v_0) = -\frac{3}{2} \frac{\tilde{\xi}^2(v_0)R^2}{\tilde{\lambda}(v_0)} \quad (35)$$

If the inequality (34) is satisfied then,

$$V_{tree}(\phi = v_0) \ll -\frac{3R^2}{2048\pi^2} \quad (36)$$

This condition implies that in the cases where perturbation theory is valid, the value of the potential at the minimum is not degenerate or “nearly” degenerate with its value at the origin.

Thus we find that in all cases where perturbation theory is valid, $|\tilde{\xi}|$ (normalized at the appropriate scale as described above) can not be too small. What is too small is determined by $\tilde{\lambda}$ as given by Eq. (23), and Eq. (34) for the two cases. In any case $\tilde{\xi}$ can not be zero at the minimum. Thus in all cases where perturbation theory can be trusted, the field will be non-minimally coupled at the minimum of the effective potential.
5 Conclusion

We have analyzed the one-loop, renormalization-group-improved effective potential for the \( \lambda \phi^4 \) theory in de Sitter space and have characterized the ranges of the parameters that lead to the different forms of the effective potential. The important result we have obtained is that in all cases where perturbation theory can be trusted, the field is necessarily non-minimally coupled at the minimum of the effective potential. Consequently, normalizing the theory at the scale where \( \xi \) vanishes will lead to the breakdown of perturbation theory.

Although we have investigated only the \( \lambda \phi^4 \) theory here, there is one aspect of this theory that is generally true in all theories. It is that once interactions are included, \( \xi = 0 \) is never a fixed point \([14]\). Therefore, although one would have to do a complete analysis of the other theories to determine their precise behavior, the fact that \( \xi = 0 \) is not a fixed point makes the assumption of a minimally coupled scalar field suspect or, at least, unnatural. Thus we conclude that, in perturbation theory, there are no long-distance problems associated with an interacting massless scalar in de Sitter space since the field is necessarily non-minimally coupled.

It is worth reflecting on the fact that gravitons resemble to some degree the massless, minimally-coupled scalar to the extent that the free field equation for the transverse, traceless modes of the metric tensor is the same, so that the graviton propagator suffers the same infrared problems. However, the minimal coupling of these modes is a result of gauge invariance (general coordinate invariance) and is not artificial. In that case, we believe that a de Sitter invariant vacuum does not strictly exist, although the resolution of the infrared problem may not be so serious \([9]\). There remain however other possible mechanisms that feed back on the metric, \([8, 19]\) and it is not yet clear whether the necessary modifications of the de Sitter background in quantum gravity are relevant to the solution of the cosmological constant problem. The point is that, unfortunately, when interactions are taken into account, the scalar case is not a good prototype for quantum gravity.

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\(^3\) See Ref. \([18]\) for a discussion of the modification of the vacuum state that we have in mind.
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