Analysis of thin film flow of generalized Maxwell fluid confronting withdrawal and drainage on non-isothermal cylindrical surfaces

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Abstract
This investigation is concerned with the study of thin film flow of a generalized Maxwell fluid along with slip conditions, confronting withdrawal and drainage on non-isothermal cylindrical surfaces. The governing equations have been formulated from the continuity equation, momentum equation, and energy equation. Analytical solutions for the velocity field, volume flow rate, average film velocity, tangential stress, and temperature are obtained in series form through the Binomial expansion technique in both withdrawal and drainage cases. The well-known solutions for a Newtonian fluid are regained as a particular case of our acquired general solutions in all flow cases. In addition, solutions for the power-law fluid model, executing alike motion, can be recovered as a limiting case of our acquired general solutions. The influence of different dimensionless parameters on all physical quantities (i.e. velocity, volume flow rate, average film velocity, tangential stress, and temperature profile) is examined and discussed graphically for both generalized Maxwell and Newtonian fluids.

Keywords
Thin film flow, generalized Maxwell fluid, binomial series method, withdrawal and drainage, non-isothermal cylindrical surfaces

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Introduction
In recent time, the non-Newtonian fluids have gained astonishing interest by researchers and engineers in various branches of science and technology. Such interest is initiated by practical applications of these fluids in bioengineering, chemical industries, and material processing such as slurry fuels, colloidal and suspension solutions, exotic lubricants, extrusion of polymer fluids, and many more. Fluids belonging to this group including drilling mud, cement, shampoo, grease, ketchup, sludge, granular suspension, paints, aqueous foams, food products, plastics, paper pulp, and several others illustrate the characteristics of non-Newtonian fluids which are pretty different when equated with the linearly viscous fluids. These fluids are elucidated by a non-linear association between stress and the rate of deformation tensors, and hence, there is not a solo constitutive model which can forecast all the prominent features of non-Newtonian fluids because of their...
assorted physical structures. Generally, non-Newtonian fluid models are classified as (a) the rate type, (b) the differential type, and (c) the integral type. But the most prominent among them is the first one which has gained stunning importance in the field of research, especially in applied mathematics, engineering, and industry-related research problems. An imperative class of the rate type model of non-Newtonian fluids, that have the properties of both viscosity and elasticity, is viscoelastic fluids. The study of viscoelastic fluids has many applications to industrial processes, including the cooling of metallic plates in a bath and food stuff. The first rate type model of viscoelastic fluids which is still extensively used to approximate the response of some first rate type model of viscoelastic fluids which is still extensively used to approximate the response of some first rate type and (c) the integral type. But the most prominent among them is the first one which has gained stunning importance in the field of research, especially in applied mathematics, engineering, and industry-related research problems. An imperative class of the rate type model of non-Newtonian fluids, that have the properties of both viscosity and elasticity, is viscoelastic fluids. The study of viscoelastic fluids has many applications to industrial processes, including the cooling of metallic plates in a bath and food stuff. The first rate type model of viscoelastic fluids which is still extensively used to approximate the response of some dilute polymeric solutions is Maxwell fluids. This is a methodical thermodynamic technical fields. These applications comprise power technology, it has gained considerable interest in the time. Due to its important applications in industrial manufacturing of CDs and plastic sheets, painting, rinsing, bearings, chemical and nuclear reactor design, machining and technology, so an understanding of their practical applications, such as coating of photographic films, enameling, hot tinning, lubrication of moving machine parts as in lubricating gears and bearings, chemical and nuclear reactor design, manufacturing of CDs and plastic sheets, painting, rinsing, dip coating, spreading of sauce on food, and pickling. A typical thin film flow comprises a span of fluid partly bounded by a solid surface, whereas the other surface is freely interrelated to another fluid, generally a gas and, most frequently, air in applications. For the formulation of these films, three basic conditions such as gravitational force, surface tension, and centrifugal force are required. A simple and clear example is the flow of a thin raindrop down a windowpane under the action of gravity. With the research advancement, thin film flow of non-Newtonian fluids has received significant approach by many intellectual workers and researchers due to their fruitful applications in the vicinities of physical and biological sciences, particularly with the progress of polymer, petroleum, and other kinds of pulp industries. Landau and Lifshitz studied the drainage thin film flow phenomena of Newtonian fluids. Over the past few decades, significant attempts have been made by scientists for the establishment of numerical solutions and analytical algorithms of thin film flows. Hayat and colleagues analyzed the thin film flow by considering differential type fluids in varieties of articles. They studied the influence of numerous physical parameters on flow fields. Siddiqui et al. investigated the flow of thin films of third- and fourth-grade fluids down an inclined plane and vertical cylinder by means of the homotopy perturbation method. They also studied the flow behavior of non-Newtonian fluid films over a vertical moving belt. At present, the study of non-Newtonian fluid flow along with slip boundary conditions has become spirited due to the extensive usage of such fluids in polymer melt, power engineering, petroleum production, and food engineering. Farooq et al. explored the withdrawal and drainage on a vertical cylinder for generalized second-grade fluid with slip conditions. Gul and colleagues probed the lifting and drainage for magnetohydrodynamic (MHD) thin film flow of third-grade fluid using constant and variable viscosities.

The purpose of this article is to venture further into the establishment of a generalized Maxwell fluid. More precisely, this article claims to examine the impact of slip condition on thin film flow of a generalized Maxwell fluid over a vertical upward moving cylinder and down a stationary vertical cylinder. Series solutions in both cases are acquired by using efficient methodical techniques, namely, binomial series method. Analytical solutions for velocity field, volume flow rate, average film velocity, tangential stress, and temperature are obtained in both withdrawal and drainage cases. The equivalent solutions for Newtonian fluid are also achieved as a particular case of our general solutions by taking the flow behavior index is equal to zero. Moreover, the influence of various physical parameters on velocity field, volume flow rate, average film velocity, tangential stress, and temperature is discussed and presented graphically in both cases under the state of generalized Maxwell and Newtonian fluids.

Governing equations

The main equations governing the motion of an incompressible generalized Maxwell fluid on non-isothermal cylindrical surfaces are

\begin{equation}
\nabla \cdot \mathbf{v} = 0
\end{equation}

\begin{equation}
\rho \frac{D \mathbf{v}}{Dt} + \nabla \rho = \rho \mathbf{b} + \nabla \mathbf{S}
\end{equation}

\begin{equation}
\rho \frac{D T}{Dt} = \alpha \nabla^2 T + \mathbf{S} \cdot \mathbf{L}
\end{equation}

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \) is the material time derivative, \( \rho \) is the fluid density, \( \mathbf{S} \) is the extra stress tensor, \( \mathbf{b} \) is the body force, \( \mathbf{v} \) is the velocity vector, \( T \) is the temperature,
\( \varepsilon \) is the thermal conductivity, and \( \varphi \) is the specific heat constant.\(^{37,39}\)

The Cauchy stress tensor \( \tau \) for a generalized Maxwell fluid is defined as\(^{37}\)

\[
\tau = -pI + S = \lambda \left[ \dot{S} - SL - SL^T \right] = \mu_{\text{eff}} A_1 \quad (4)
\]

where \(-pI\) symbolizes the indeterminate spherical stress due to the constraint of incompressibility, \( \lambda \) is the relaxation time, \( A_1 = L + LL^T \) is the first Rivlin–Ericksen tensor,\(^{43}\) \( L = \nabla v \) is the velocity gradient, the superposed dot indicates material time derivative, the superscript \( T \) designates the transpose operation, and \( \mu_{\text{eff}} \) is the effective viscosity which for a generalized Maxwell fluid is defined as\(^{23,37}\)

\[
\mu_{\text{eff}} = \beta \left[ \frac{1}{2} \frac{\text{tr} A_1}{\lambda} \right]^{q} \quad (5)
\]

where \( \beta \) is the flow consistency index, \( q \) is the flow behavior index with \(-1 < q < 1\). The model characterized by constitutive equation (4) contains the significant consequences such that for \( q < 0 \), the fluid is shear thinning (pseudoplastic); for \( q > 0 \), the fluid is shear thickening (dilatant); and for \( q = 0 \), the classical Maxwell fluid is retrieved. Furthermore, for \( \lambda \rightarrow 0 \), the generalized Maxwell fluid model will be reduced to the power-law fluid model.\(^{17,22}\)

**Problem formulation for withdrawal case**

Consider a container filled with an incompressible generalized Maxwell fluid. An infinite vertical cylinder of radius \( R \) is lifted upward through this container with constant speed \( u_0 \). As the cylinder moves, a thin layer of the fluid of constant thickness \( X \) adheres to the cylindrical surface. This film has a tendency to drain down the cylinder under the influence of gravity. We use cylindrical coordinates system such that the \( z \)-axis is alongside the axis of the cylinder in the upward direction and \( r \)-axis is normal to it as shown in Figure 1.\(^{22,37,39}\) Suppose that the atmospheric pressure \( p = \text{constant} \) ⇒ \( \partial_z p = 0 \), then equation (9) becomes

\[
\frac{\beta}{r} D_r \left[ r(D_r u)(D_r u)^q \right] - \rho g - \partial_z p = 0 \quad (9)
\]

Using equations (4) and (6), equation (3) becomes\(^{35,37}\)

\[
\varepsilon D_r^2 T + \frac{\varepsilon}{r} D_r T + \beta \left[ (D_r u)^2 (D_r u)^q \right] = 0 \quad (11)
\]

Boundary conditions related with equations (10) and (11) are\(^{35,39}\)

\[
\text{Free space boundary conditions : } \quad S_{rz} = 0, \quad D_r T = 0 \quad \text{at } r = D \quad (12)
\]

\[
\text{Slip boundary conditions : } \quad u = u_0 - \alpha S_{rz}, \quad T = T_0 \quad \text{at } r = R \quad (13)
\]

where \( \alpha \) is the slip coefficient, \( D = R + X \) and \( S_{rz} = \beta (D_r u)^q + 1 \)
By integration of equation (10) with respect to “r” and consuming the boundary conditions given in equation (12), we have \(^{37,39}\)

\[
D_u = (1 - 1) \frac{\rho g}{2B} \left[ \frac{D^2 - r^2}{r} \right] \frac{1}{r} \quad (15)
\]

In series form, using binomial expansion, equation (15) becomes

\[
D_u = (1 - 1) \frac{\rho g}{2B} \sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) (1)^k r^{2k - \frac{1}{q + 1} + D^{-2k + \frac{2}{q + 1}})
\]

Equation (11) in view of equation (16) becomes

\[
\varepsilon \left[ \frac{D_u T}{k} + \frac{1}{D_u T} \right] = -k(1)^{\sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) (1)^k r^{2k - \frac{1}{q + 1} + D^{-2k + \frac{2}{q + 1}})}
\]

\[
(17)
\]

**Solutions for withdrawal case**

Solutions for the generalized Maxwell fluid and for Newtonian fluid are presented here in this section.

**Solutions for generalized Maxwell fluid**

For \(q \neq 0\), the following expressions are obtained for the generalized Maxwell fluid.

**Velocity field.** Integrating equation (16) with respect to “r” and inserting the given boundary conditions, the expression for velocity field is obtained as

\[
u = u_0 + \frac{\alpha \rho g}{2R} \left[ \frac{D^2 - r^2}{r} \right] + (1)^{\sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) (1)^k r^{2k - \frac{1}{q + 1} + D^{-2k + \frac{2}{q + 1}})}
\]

\[
(18)
\]

**Volume flow rate.** Volume flow rate \(Q\) of the fluid passing through the surface of the cylinder is defined as\(^{37,39}\)

\[
Q = \int ru(r)drd\theta, \quad R \leq r \leq D, \quad 0 \leq \theta \leq 2\pi
\]

\[
(19)
\]

Using equation (18), we acquire

\[
Q = \pi (D^2 - R^2) \left[ u_0 + \frac{\alpha \rho g}{2R} (D^2 - R^2) \right]
\]

\[
+ 2 \pi (1)^{\sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) (-1)^k r^{2k - \frac{1}{q + 1} + D^{-2k + \frac{2}{q + 1}})}
\]

\[
\left[ \frac{D^{2k + \frac{2}{q + 1}} - R^{2k + \frac{2}{q + 1}}}{2k + \frac{q}{q + 1}} \right] D^{2k - \frac{2}{q + 1}} \ln \left( \frac{r}{R} \right)
\]

\[
(19)
\]

**Average film velocity.** Average film velocity \(U\) is defined as\(^{37,39}\)

\[
U = \frac{Q}{\pi (D^2 - R^2)}
\]

Inserting equation (20) into equation (21), the average film velocity is

\[
U = u_0 + \frac{\alpha \rho g}{2R} \left[ D^2 - R^2 \right] + \pi \left[ \frac{1}{2} \right] \sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) (-1)^k r^{2k + \frac{2}{q + 1} + D^{-2k + \frac{2}{q + 1}}}
\]

\[
\left[ \frac{D^{2k + \frac{2}{q + 1}} - R^{2k + \frac{2}{q + 1}}}{2k + \frac{q}{q + 1}} \right] D^{2k - \frac{2}{q + 1}} \left[ (D^2 - R^2) - \frac{R^{2k + \frac{2}{q + 1}}}{2} \right]
\]

\[
(22)
\]

**Tangential stress.** The tangential stress on cylindrical surface is given in equation (14). Using equation (4), we have

\[
S_{\tau} = -\frac{\rho g}{2} \left[ \frac{D^2 - r^2}{r} \right]
\]

\[
(23)
\]

Consuming the slip boundary conditions given in equation (13), the tangential stress employed by the fluid on cylindrical surface in withdrawal case is

\[
S_{\tau} \big|_{r = R} = -\frac{\rho g}{2} \left[ \frac{D^2 - R^2}{R} \right]
\]

\[
(24)
\]

**Temperature profile.** Solution of equation (17) for temperature profile by using boundary conditions defined in equations (12) and (13) is

\[
T = T_0 + \frac{\beta}{\varepsilon} (1)^{\sum_{k = 0}^{\infty} \left( \frac{1}{k + 1} \right) \sum_{k = 0}^{\infty} \left( \frac{q + 2}{q + 1} \right) (-1)^k r^{2k + \frac{2}{q + 1} + D^{-2k + \frac{2}{q + 1}}}}
\]

\[
\left[ \frac{R^{2k + \frac{2}{q + 1}} - r^{2k + \frac{2}{q + 1}} + D^{2k + \frac{2}{q + 1}} \ln \left( \frac{r}{R} \right)}{2} \right]
\]

\[
(25)
\]
Solutions for Newtonian fluid

When \( q = \lambda = 0 \), the generalized Maxwell fluid diminishes to a Newtonian fluid which is the particular case of our obtained solutions for withdrawal case.

### Velocity field

Putting \( q = 0 \) in equation (15) and consuming the given boundary condition, the expression for velocity field is

\[
u = u_0 - \frac{pg}{4\beta} \left[ 2D^2 \ln \left( \frac{D}{R} \right) + \left( R^2 - r^2 \right) - 2\alpha \beta \left( \frac{D^2 - R^2}{R} \right) \right]
\]

(26)

### Volume flow rate

Volume flow rate \( Q \) for Newtonian fluid by using equation (26) in equation (19) is

\[
Q = - \frac{\pi pg}{2\beta} \left[ D^4 \ln \left( \frac{D}{R} \right) + \pi (D^2 - R^2) - u_0 + \frac{pg}{8\beta} \left( 2D^2 + \left[ 1 + \frac{4\alpha \beta}{R} \right] (D^2 - R^2) \right) \right]
\]

(27)

### Average film velocity

By using equation (27) into equation (21), the average film velocity \( U \) is

\[
U = u_0 - \frac{pg}{8\beta} \left\{ \frac{4D^4}{D^2 - R^2} \ln \left( \frac{D}{R} \right) - 2D^2 - \left[ 1 + \frac{4\alpha \beta}{R} \right] (D^2 - R^2) \right\}
\]

(28)

### Tangential stress

Tangential stress for Newtonian fluid is the same as it is for generalized Maxwell fluid, which is given by equation (24).

### Temperature profile

Putting \( q = 0 \) and consuming boundary conditions given in equations (12) and (13), the solution of equation (11) for temperature profile is

\[
T = T_0 - \frac{\beta}{\varepsilon} \left( \frac{pg}{8\beta} \right)^2 \left[ 4D^4 (4\ln D - 3) \ln \left( \frac{R}{r} \right) + 8D^4 \left\{ \left[ \ln \left( \frac{R}{r} \right) \right]^2 - \left[ \ln R \right]^2 \right\} + (r^4 - R^4) + 8D^2 (R^2 - r^2) \right]
\]

(29)

Solutions for withdrawal case in dimensionless form

Introducing the following dimensionless parameters

\[
\tilde{r} = \frac{r}{R}, \quad \tilde{u} = \frac{u}{u_0}, \quad \tilde{H} = \frac{D}{R}, \quad \tilde{T} = \frac{T - T_0}{T_1 - T_0}, \quad \tilde{\beta} = \frac{\beta(u_0)^q}{R^q}, \quad S_i = \frac{pgR^2}{u_0 \mu_{\text{eff}}}, \quad B_r = \frac{u_0^2 \mu_{\text{eff}}}{\varepsilon(T_1 - T_0)}, \quad \tilde{S}_{rz} = \frac{1}{\beta} \left[ \frac{R}{u_0} \right]^{q+1} S_{rz}
\]

(30)

in differential equations (10) and (11) and in the boundary conditions given by equations (12) and (13). The new transformed differential equations along with boundary conditions in dimensionless form after dropping the “hats” for simplicity are

\[
\frac{1}{r} D_r \left[ r(D_r u)(D_r u)^q \right] - S_i = 0 \Rightarrow D_r \left[ r(D_r u)^q + 1 \right] = rS_i
\]

(31)

\[
D_r^2 T + \frac{1}{r} D_r T + B_r \left[ (D_r u)^2 (D_r u)^q \right] = 0
\]

(32)

Free space boundary condition :

\[
D_r u = 0 \text{ and } D_r T = 0 \text{ at } r = H
\]

(33)

Slip boundary condition :

\[
U = 1 - \frac{\alpha}{u_0} A(D_r u)^q + 1 \text{ and } T = 0 \text{ at } r = 1
\]

(34)

where \( S_i \) is the Stokes number, \( B_r \) is the Brinkman number, \( T_1 \) is the reference temperature, \( A = \beta(u_0/R)^q + 1 \), and \( H \) is the fluid parameter.

By integration of equation (31) with respect to “\( r \)” and consuming boundary conditions defined in equation (33), we have

\[
D_r u = (-1)^{\frac{q+2}{q+1}} \left( \frac{S_i}{2} \right)^{\frac{q+1}{q+2}} \left[ \frac{H^2 - r^2}{r} \right]^{\frac{q+2}{q+1}}
\]

(35)

In the above expression, signs of \( D_r u \) and \( r \) are always opposite to each other, since velocity of the fluid decreases as \( r \) increases. In series form, using binomial expansion, equation (35) is written as

\[
D_r u = (-1)^{\frac{q+2}{q+1}} \left( \frac{S_i}{2} \right)^{\frac{q+1}{q+2}} \sum_{k=0}^{\infty} \left( \frac{1}{k+1} \right) (-1)^k 2^{2k+1} r^{2+2k} H^{-2k} + \frac{2^{2k+4} r^{2k+4} H^{-2k+4}}{q+1}
\]

(36)

Equation (32) in view of equation (36) yields

\[
D_r^2 T + \frac{1}{r} D_r T = -B_r (-1)^{\frac{q+2}{q+1}} \left( \frac{S_i}{2} \right)^{\frac{q+1}{q+2}} \sum_{k=0}^{\infty} \left( \frac{q+2}{q+1} \right) (-1)^k 2^{2k+2} r^{2k+2} H^{-2k} + \frac{2^{2k+4} r^{2k+4} H^{-2k+4}}{q+1}
\]

(37)

Inserting equation (35) into equation (34), we have
\[ u = 1 + \frac{\alpha S}{2\mu_0} A \left[H^2 - 1 \right] \text{ and } T = 0 \text{ at } r = 1 \] (38)

**Solutions for generalized Maxwell fluid**

For \( q \neq 0 \), the following expressions are obtained for generalized Maxwell fluid.

**Velocity field.** Integrating equation (36) with respect to “\( r \)” and inserting boundary conditions given in equation (38), the expression for velocity field is acquired as

\[
u = 1 + \frac{\alpha S}{2\mu_0} A \left[H^2 - 1 \right] + (-1)^{\frac{r}{q+1}} \left\{ \frac{S}{2} \right\} \sum_{k=0}^{\infty} \left( \frac{q+1}{k} \right) \left( -1 \right)^k \frac{r^{2k+q+1}}{2k + \frac{q}{q+1}} H^{2k+q+1} \] (39)

**Volume flow rate.** Volume flow rate \( Q \) of the fluid passing through cylindrical surface is defined as

\[
Q = \int r u(r) dr d\theta, \quad 1 \leq r \leq H, \quad 0 \leq \theta \leq 2\pi
\] (40)

Inserting equation (39) into equation (40), the volume flow rate is obtained as

\[
Q = \pi (H^2 - 1) \left[ 1 + \frac{\alpha S}{2\mu_0} A (H^2 - 1) \right] + 2\pi(-1)^{\frac{r}{q+1}} \left\{ \frac{S}{2} \right\} \sum_{k=0}^{\infty} \left( \frac{q+1}{k} \right) \left( -1 \right)^k \frac{H^{2k+q+1}}{2k + \frac{q}{q+1}} \times \left[ \frac{H^{2k+q+1} - 1}{2k + \frac{q}{q+1}} - H^2 - 1 \right]
\] (41)

**Average film velocity.** Average film velocity \( U \) is given by Ullah \(^3\)

\[
U = \frac{Q}{\pi (H^2 - 1)}
\] (42)

Inserting equation (41) into equation (42), the average film velocity \( U \) for withdrawal case is achieved as

\[
U = 1 + \frac{\alpha S}{2\mu_0} A \left[H^2 - 1 \right] + 2\pi(-1)^{\frac{r}{q+1}} \left\{ \frac{S}{2} \right\} \sum_{k=0}^{\infty} \left( \frac{q+1}{k} \right) \left( -1 \right)^k \frac{H^{2k+q+1}}{2k + \frac{q}{q+1}} \left[ \frac{H^{2k+q+1} - 1}{2k + \frac{q}{q+1}} - H^2 - 1 \right]
\] (43)

**Tangential stress.** Tangential stress employed by the fluid over the cylindrical surface is

\[
S_{rz} = (D_r u)^{q+1}
\] (44)

Using equation (35) into equation (44), tangential stress is

\[
S_{rz}\big|_{r=1} = - \frac{S[H^2 - 1]}{2}
\] (45)

**Temperature profile.** Solving equation (37) using boundary conditions given in equations (33) and (34), expression for temperature profile is

\[
T = B_r(-1)^{\frac{r}{q+1}} \left\{ \frac{S}{2} \right\} \sum_{k=0}^{\infty} \left( \frac{q+1}{k} \right) \left( -1 \right)^k \frac{H^{2k+q+1}}{2k + \frac{q}{q+1}} \left[ H^{2k+q+1} \ln r + \frac{1 - r^{2k+q+1}}{2k + \frac{q}{q+1}} \right]
\] (46)

**Solutions for Newtonian fluid**

When \( q = \lambda = 0 \), the generalized Maxwell fluid diminishes to Newtonian fluid which is the particular case of our general solutions. The exact solutions for Newtonian fluid in dimensionless form are presented below.

**Velocity field.** Putting \( q = 0 \) and consuming boundary conditions given in equation (38), solution of equation (35) for velocity field is

\[
u = 1 + \frac{S}{2} \left[ \frac{\alpha \beta}{R} \left( H^2 - 1 \right) - H^2 \ln r + \frac{r^2 - 1}{2} \right]
\] (47)

**Volume flow rate.** Volume flow rate \( Q \) is obtained from equation (40) by using equation (47), which is

\[
Q = \pi (H^2 - 1) \left[ 1 + \frac{S}{4} \left\{ \frac{2\alpha \beta}{R} \left( H^2 - 1 \right) - 1 \right\} \right] - \frac{\pi S}{8} \left[ 4H^4 (\ln H) - (H^2 - 1)(3H^2 + 1) \right]
\] (48)

**Average film velocity.** By using equation (48) into equation (42), the average film velocity \( U \) is

\[
U = 1 + \frac{S}{8} \left[ \frac{4\alpha \beta}{R} \left( H^2 - 1 \right) - \frac{4H^4 (\ln H)}{H^2 - 1} + 3H^2 - 1 \right]
\] (49)
Tangential stress. Using equation (35) into equation (44) and then solving for $q = 0$, we obtain tangential stress for Newtonian fluid which is similar as it is calculated for generalized Maxwell fluid given by equation (45).

Temperature profile. Putting $q = 0$ and consuming boundary conditions given by equations (33) and (34), the solution of equation (32) for temperature profile is

\[
T = \frac{-B_z S_z^2}{64} \left[ 4H^2(3 + 2\ln r - 4\ln H)\ln r + 8H^2(1 - r^2) + r^4 - 1 \right]
\]  

Problem formulation for drainage case

Consider a steady, uniform thin film of generalized Maxwell fluid draining down over a stationary vertical cylinder. In drainage case, $z$-axis is in the downward direction and the motion of fluid is due to the gravity as shown in Figure 2. The geometry and assumptions for drainage case are alike as in the preceding case, but the boundary conditions of velocity are changed. Therefore, equations (10) and (11) turn into the following form:

\[
\beta \frac{D_r}{r} [r (D_r u) (D_r u)] \beta + \rho g = 0
\]

\[
\Rightarrow D_r \left[ r (D_r u)^{q+1} \right] = -\frac{\rho g}{\beta}
\]

Boundary conditions connected with equations (51) and (52) are

Free space boundary condition:

\[
S_{rz} = 0, \ D_r T = 0 \text{ at } r = D
\]

Slip boundary condition:

\[
u = -\alpha S_{rz}, \ T = T_0 \text{ at } r = R
\]

where $D$ and $S_{rz}$ are given in equation (14). By integration of equation (51) with respect to “$r$” and then using equations (14) and (53), we have

\[
(D_r u)^{q+1} = \frac{\rho g}{2\beta} \left[ \frac{D_r^2 - r^2}{r} \right]^{\frac{q+1}{q+2}}
\]

In series form, using binomial expansion, equation (55) becomes

\[
D_r u = \left\{ \frac{\rho g}{2\beta} \right\}^{\frac{q+1}{q+2}} \sum_{k=0}^{\infty} \left( \frac{1}{q+1} \right) \left( -1 \right)^k r^{2k-\frac{q}{q+1}} D_r D_r^{q+2k+\frac{q+4}{q+2}}
\]

Equation (52) in view of equation (56) yields

\[
\epsilon D_r^2 T + \frac{\rho g}{2\beta} D_r T = -\beta \left\{ \frac{\rho g}{2\beta} \right\}^{\frac{q+1}{q+2}} \sum_{k=0}^{\infty} \left( \frac{q+2}{q+1} \right) \left( -1 \right)^k r^{2k-\frac{q}{q+1}} D_r D_r^{q-2k+\frac{q+4}{q+2}}
\]

Solutions for drainage case

Solutions for generalized Maxwell fluid and also for Newtonian fluid are presented here in this section.

Solutions for generalized Maxwell fluid

For $q \neq 0$, the following expressions are obtained for generalized Maxwell fluid.

Velocity field. Solving equation (56) using the given boundary conditions, the expression for velocity field in drainage case is acquired as
\[ u = -\frac{\alpha \rho g}{2} \left[ \frac{D^2 - R^2}{R} \right] + \left( \frac{\rho g}{2\beta} \right)^{\frac{1}{q+1}} \sum_{k=0}^{\infty} \left( \frac{1}{q+1} \right)^k \]

\[ \left( -1 \right)^k \frac{R^{2k + \frac{q}{q+1}} - R^2}{2(2k + \frac{q}{q+1})} \left[ 2k + \frac{q}{q+1} \right] \]

**Volume flow rate.** Volume flow rate \( Q \) in drainage case is obtained by using equation (58) into equation (19) as

\[ Q = -\frac{\pi \alpha \rho g}{2R} \left[ D^2 - R^2 \right]^2 + 2\pi \left( \frac{\rho g}{2\beta} \right)^{\frac{1}{q+1}} \sum_{k=0}^{\infty} \left( \frac{1}{q+1} \right)^k \]

\[ \left( -1 \right)^k \frac{D^{2k + \frac{q}{q+1}} - R^{2k + \frac{q}{q+1}}}{2(2k + \frac{q}{q+1})} \left[ 2k + \frac{q}{q+1} \right] \left( \frac{D^2 - R^2}{R} \right) \]

**Average film velocity.** Expression for average film velocity \( U \) is attained by using equation (59) into equation (21) as under

\[ U = -\frac{\alpha \rho g}{2R} \left[ D^2 - R^2 \right] + \frac{1}{2} \left( \frac{\rho g}{2\beta} \right)^{\frac{1}{q+1}} \sum_{k=0}^{\infty} \left( \frac{1}{q+1} \right)^k \]

\[ \left[ \frac{D^{2k + \frac{q}{q+1}} - R^{2k + \frac{q}{q+1}}}{2(2k + \frac{q}{q+1})} \left( \frac{D^2 - R^2}{R} \right) \right] \]

**Tangential stress.** Tangential stress exerted on cylindrical surface in this case is

\[ S_{zz} = \frac{\rho g}{2} \left[ \frac{D^2 - r^2}{r} \right] \Rightarrow S_{zz}|_{r - R} = \left( \frac{\rho g}{2} \right)^{\frac{1}{q+1}} \left[ \frac{D^2 - R^2}{R} \right] \]

**Temperature profile.** Solution of equation (57) for temperature profile, when the boundary conditions defined by equations (53) and (54) are used, is

\[ T = T_0 + \frac{\beta}{\nu} \left( \frac{\rho g}{2\beta} \right)^{\frac{1}{q+1}} \sum_{k=0}^{\infty} \left( \frac{1}{q+1} \right)^k \left( -1 \right)^k \frac{D^{2k + \frac{q}{q+1}}}{2k + \frac{q}{q+1}} \]

\[ \frac{R^{2k + \frac{q}{q+1}} - r^{2k + \frac{q}{q+1}}}{2k + \frac{q}{q+1}} + D^{2k + \frac{q}{q+1}} \ln \left( \frac{r}{R} \right) \]

**Solutions for Newtonian fluid**

When \( q = \lambda = 0 \), the generalized Maxwell fluid diminishes to a Newtonian fluid which is the particular case of our obtained solutions for drainage case.

**Velocity field.** Putting \( q = 0 \) and consuming boundary conditions defined in equation (53) and (54), solution of equation (51) for velocity field is

\[ u = \frac{\rho g}{4\beta} \left[ 2D^2 \ln \left( \frac{r}{R} \right) + \left( R^2 - r^2 \right) - 2\alpha \beta \left( \frac{D^2 - R^2}{R} \right) \right] \]

**Volume flow rate.** The volume flow rate \( Q \) for Newtonian fluid in drainage case is

\[ Q = \frac{\pi \rho g}{2\beta} \left[ D^4 \ln \left( \frac{D}{R} \right) \right] - \frac{\pi \rho g}{8\beta} \left( D^2 - R^2 \right) \]

\[ 2D^2 + \left\{ 1 + \frac{4\alpha \beta}{R} \right\} \left( D^2 - R^2 \right) \]

**Average film velocity.** By utilizing equation (64) in equation (21), the average film velocity \( U \) is

\[ U = \frac{\rho g}{8\beta} \left[ 4D^4 \ln \left( \frac{D}{R} \right) - 2D^2 - \left\{ 1 + \frac{4\alpha \beta}{R} \right\} (D^2 - R^2) \right] \]

**Tangential stress.** Tangential stress for Newtonian fluid is the same as it is for generalized Maxwell fluid, which is given by equation (61).

**Temperature profile.** Putting \( q = 0 \) and consuming boundary conditions specified in equations (53) and (54), the solution of equation (52) for temperature profile is

\[ T = T_0 - \frac{\beta}{\nu} \left( \frac{\rho g}{2\beta} \right)^{\frac{1}{q+1}} \left[ 4D^4 (4\ln D - 3) \ln \left( \frac{R}{r} \right) \right] + 8D^4 \left\{ (\ln r)^2 - (\ln R)^2 \right\} + (r^4 - R^4) + 8D^2 (R^2 - r^2) \]

**Solutions for drainage case in dimensionless form**

Inserting the non-dimensional parameters defined in equation (30) into equations (51), (52), (53) and (54). After dropping “hats” for simplicity, the new converted
dimensionless equations together with boundary conditions are

\[ \frac{1}{r} D_r [r (D_r u) (D_r u)^q] + S_t = 0 \Rightarrow D_r \left[ (D_r u)^q + 1 \right] = -r S_t \]

(67)

\[ D_r^2 T + \frac{1}{r} D_r T + B_r \left[ (D_r u)^2 (D_r u)^q \right] = 0 \]

(68)

\[ \Rightarrow D_r^2 T + \frac{1}{r} D_r T = -B_r (D_r u)^q + 2 \]

Free space boundary condition :

\[ D_r u = 0 \text{ and } D_r T = 0 \text{ at } r = H \]

(69)

Slip boundary condition :

\[ u = -\frac{\alpha S_t}{2 u_0} A (D_r u)^q + 1 \text{ and } T = 0 \text{ at } r = 1 \]

(70)

By integration of equation (67) with respect to “r” and using the given boundary conditions, we acquire

\[ D_r u = \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \left[ \frac{H^2 - r^2}{r} \right]^\frac{1}{q + 1} \]

(71)

In series form, using binomial expansion, equation (71) is written as

\[ D_r u = \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{r^{q + 1}}{k!} \right) (-1)^k r^{2k + \frac{q + 1}{q + 1}} H^{2k + \frac{q + 1}{q + 1}} \]

(72)

Using equation (72), equation (68) yields

\[ D_r^2 T + \frac{1}{r} D_r T = -B_r \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{q + 2}{k + 1} \right) (-1)^k r^{2k + \frac{q + 2}{q + 1}} H^{2k + \frac{q + 2}{q + 1}} \]

(73)

Equation (70) in view of equation (71) gives

\[ u = -\frac{\alpha S_t}{2 u_0} A \left[ H^2 - 1 \right] \text{ and } T = 0 \text{ at } r = 1 \]

(74)

**Solutions for generalized Maxwell fluid**

For \( q \neq 0 \), the following expressions are obtained for generalized Maxwell fluid in drainage case.

**Velocity field.** Solving equation (72) using boundary conditions defined in equation (74), the expression for velocity field is

\[ u = -\frac{\alpha S_t}{2 u_0} A \left[ H^2 - 1 \right] + \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{q + 1}{k} \right) \]

\[ \frac{(-1)^k r^{2k + \frac{q + 1}{q + 1}} H^{2k + \frac{q + 1}{q + 1}}}{2k + \frac{q}{q + 1}} \]

(75)

**Volume flow rate.** Using equation (75) into equation (40), the volume flow rate \( Q \) in drainage case is

\[ Q = -\frac{\pi \alpha S_t}{2 u_0} A \left[ H^2 - 1 \right]^2 + 2 \pi \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{q + 1}{k} \right) \]

\[ \frac{(-1)^k H^{2k + \frac{q + 2}{q + 1}} - 1}{2k + \frac{q}{q + 1}} \frac{H^{2k + \frac{q + 2}{q + 1}} - 1}{2} \]

(76)

**Average film velocity.** Profile for average film velocity \( U \) is attained by using equation (76) into equation (42) as under

\[ U = -\frac{\alpha S_t}{2 u_0} A \left[ H^2 - 1 \right] + 2 \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{q + 1}{k} \right) \]

\[ \frac{(-1)^k H^{2k + \frac{q + 2}{q + 1}} - 1}{2k + \frac{q}{q + 1}} \frac{H^{2k + \frac{q + 2}{q + 1}} - 1}{2} \]

(77)

**Tangential stress.** Tangential stress exerted on cylindrical surface in the drainage case is

\[ S_{rz} = \frac{S_t}{2} \frac{H^2 - r^2}{r} \Rightarrow S_{rz}|_{r = 1} = \frac{S_t}{2} \left[ H^2 - 1 \right] \]

(78)

**Temperature profile.** Solution of equation (73) for temperature profile, when the boundary conditions specified in equations (69) and (74) are used, is

\[ T = B_r \left\{ \frac{S_t}{2} \right\}^{\frac{1}{q + 1}} \sum_{k=0}^{\infty} \left( \frac{q + 2}{k + 1} \right) \]

\[ \frac{(-1)^k H^{2k + \frac{q + 2}{q + 1}} - 1}{2k + \frac{q}{q + 1}} \frac{H^{2k + \frac{q + 2}{q + 1}} - 1}{2k + \frac{q}{q + 1}} \]

(79)

**Solutions for Newtonian fluid**

When \( q = \lambda = 0 \), the generalized Maxwell fluid diminishes to a Newtonian fluid which is the particular case of our general solutions obtained for drainage case.
Velocity field. Putting \( q = 0 \) and consuming boundary conditions defined by equations (69) and (74), solution of equation (67) for velocity field is

\[
\mathbf{u} = \frac{S_t}{4} \left[ 2H^2 \ln(r) - r^2 + 1 - \frac{2a\beta}{R} (H^2 - 1) \right]
\]  

(80)

Volume flow rate. Volume flow rate \( Q \) in this case is obtained by using equation (80) in equation (40) as

\[
Q = \frac{\pi S_t}{8} \left[ 4H^4 \ln H - (H^2 - 1) \left\{ 2H^2 + \left( 1 + \frac{4a\beta}{R} \right) (H^2 - 1) \right\} \right]
\]  

(81)

Average film velocity. By utilizing equation (81) into equation (42), the average film velocity \( U \) is

\[
U = \frac{S_t}{8} \left[ \frac{4H^4 \ln H}{H^2 - 1} - 2H^2 - \left\{ 1 + \frac{4a\beta}{R} \right\} (H^2 - 1) \right]
\]  

(82)

Tangential stress. Tangential stress for Newtonian fluid is same as it is for generalized Maxwell fluid given by equation (78).

Temperature profile. Putting \( q = 0 \) and consuming boundary conditions specified by equations (69) and (70), solution of the differential equation (68) for temperature profile is

\[
T = -\frac{B_s S_t^2}{64}
\left[ 4H^4 \left\{ 3 + 2\ln r - 4 \ln H \right\} \ln r + 8H^2 (1 - r^2) + r^4 - 1 \right]
\]  

(83)

Results and discussion

In this article, we analyzed the thin film flow of generalized Maxwell fluid over a vertical cylinder under the impact of slip and free space boundary conditions. The problem has been modeled for withdrawal and drainage of velocity field and temperature profile. Furthermore, series solutions for volume flow rate, average film velocity, and tangential stress have been predicted by using the binomial series technique. The geometry of problem has been shown in Figures 1 and 2 for withdrawal and drainage, respectively. The influences of dimensionless model parameters, for instance, flow behavior index \( q \), Stocks number \( S_t \), Brinkman number \( B_r \), and fluid parameter \( H \), have been investigated on velocity, volume flow rate, average film velocity, tangential stress, and temperature for the proposed problem. The impacts of these pertinent parameters have been reflected in Figures 3–24 which are plotted for real solutions.

It has been investigated that, when \( S_t \) increases, velocity decreases in withdrawal case, either the fluid is dilatant, pseudoplastic, or Newtonian, and this effect is shown graphically by Figures 3(a), 4(a), and 5(a), while Figures 3(b), 4(b), and 5(b) show that the magnitude of velocity increases in drainage case, respectively, for the above stated fluid types.

Figures 6–8 show the effect of \( H \) on velocity field for both cases. We diagnosed that velocity decreases when the value of \( H \) increases in withdrawal case, either the fluid is shear thickening (dilatant), shear thinning (pseudoplastic), or Newtonian as this impact is given by Figures 6(a), 7(a), and 8(a), respectively, and in the drainage case, velocity rises with the rise in \( H \) as shown by Figures 6(b), 7(b), and 8(b), respectively, for the above stated fluid types.

Figure 9(a) shows that velocity declines as the fluid is becoming thicker in withdrawal case, and it rises in the drainage case, as shown by Figure 9(b).
Figures 10–12 display the influence of Stock’s number $S_r$ on the volume flow rate $Q$ for modeled problem. Figures 10(a), 11(a), and 12(a) indicate that with the increase in $S_r$ (Stock’s number), the volume flow rate decreases in withdrawal case and it increases in the drainage case as shown by Figures 10(b), 11(b), and 12(b).

Figures 14–16 display the influence of Stock’s number on average film velocity. Figures 14(a), 15(a), and 16(a) indicate that with the increase of Stock’s number, the fluid becomes thicker in withdrawal case and it becomes thinner in the drainage case as shown by Figures 14(b), 15(b), and 16(b).

Figure 13(a) shows that the volume flow rate rises as the fluid becomes thicker in withdrawal case and it declines in the drainage case, as shown by Figure 13(b). Figures 14–16 display the influence of Stock’s number on average film velocity. Figures 14(a), 15(a), and 16(a) indicate that with the increase of Stock’s number,
average film velocity decreases in withdrawal case and it increases in the drainage case as shown by Figures 14(b), 15(b), and 16(b).

Figures 17 and 18 present the effects of $H$ and $q$ on average film velocity. With the increase in $H$, average film velocity decreases as shown by Figure 17(a) in withdrawal case, whereas in the drainage case, average film velocity increases and is presented by Figure 17(b).

We examined from Figure 18 that average film velocity increases as the fluid is becoming shear thickening in withdrawal case and vice versa in the drainage case.

Figures 19 and 20 show the impacts of parameters $S_r$ and $H$, respectively, on tangential stress. Figures 19(a) and 20(a) for withdrawal case indicate that tangential
stress on the cylindrical surface decreases as $S_t$ and $H$ increased, and Figures 19(b) and 20(b) show that tangential stress rises with the rise in $S_t$ and $H$ for drainage case.

Figures 21–23 indicate that temperature increases by rising the values of $S_t$, $B_t$, and $H$ for both withdrawal and drainage cases, either the fluid is dilatant or pseudoplastic. Moreover, temperature increases as the fluid is becoming thicker in both withdrawal and drainage cases, and this effect is shown by Figure 22.

**Conclusion**

The important findings of this article are listed below.
The theoretical study of steady uniform thin film flow of incompressible generalized Maxwell fluid over vertical cylinder under the influence of non-isothermal effects for withdrawal and drainage cases is presented.

The problem is formulated in terms of differential equations along with appropriate initial and boundary conditions.

Exact methodical solutions are reported for velocity, volume flow rate, average film velocity,
tangential stress, and temperature for both cases by applying binomial series technique which is the most efficient, effective, and reliable technique for the solutions of such problems.

- Solutions for the Newtonian fluid are recovered as a particular case of our obtained solutions for generalized Maxwell fluid.
- Tangential stress on cylindrical surface is similar for Maxwell and Newtonian fluids in withdrawal case. Moreover, the effect of tangential stress also remains invariable in drainage case for either type of the fluid.
- Solutions for temperature profile in both cases remain the same for either type of the fluid.
The impacts of different dimensionless parameters on velocity field, volume flow rate, average film velocity, tangential stress, and temperature profile are examined and discussed graphically for both generalized Maxwell and Newtonian fluids.

The temperature of the fluid rises with a rise in the values of pertinent parameters used in the planned problem. This effect shows that there is a direct relation between these parameters and temperature profile. On the other hand, the magnitude of velocity decreases by rising the

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values of parameters in withdrawal case, while it increases in drainage case.

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