The ponderomotive force due to the intrinsic spin in extended fluid and kinetic models

Martin Stefan†, Jens Zamanian‡, Gert Brodin,† Amar Prasad Misra§ and Mattias Marklund¶

Department of Physics, Umeå University, SE–901 87 Umeå, Sweden
(Dated: 02 December, 2010)

In this paper we calculate the contribution to the ponderomotive force in a plasma from the electron spin using a recently developed model. The spin-fluid model used in the present paper contains spin-velocity correlations, in contrast to previous models used for the same purpose. Is its then found that previous terms for the spin-ponderomotive force are recovered, but also that additional terms appear. Furthermore, the results due to the spin-velocity correlations are confirmed using the spin-kinetic theory. The significance of our results is discussed.

PACS numbers: 52.25.Dg, 52.25.Xz, 52.35.Hr, 52.35.Mw

I. INTRODUCTION

The ponderomotive force plays a crucial role in the nonlinear dynamics of plasmas. Phenomena induced by the ponderomotive force include, e.g. wakefield generation [1, 2], soliton formation, self-focusing and wave collapse [3]. The classical expression for the ponderomotive force in a magnetized plasma first derived by Karpman et al. [4] was recently generalized to account for quantum mechanical effects, in particular due to the electron spin [5]. The physics of quantum plasmas (for recent reviews, see e.g. [6, 7]) has recently received much interest due to applications in, e.g. quantum wells [8], plasmonics [9] and spintronics [10]. In particular, the dynamical effects due to the intrinsic spin of the electron has been investigated using both fluid [11–13] and semiclassical kinetic approaches [14–18]. In Ref. [5] the ponderomotive force related to the magnetic dipole moment of the electrons (due to the spin) was shown to induce a spin-polarized plasma, i.e. the spin-up and spin-down states of the electrons are separated. This effect was shown to be pronounced also for a plasma of relatively modest density [5], and the expressions has recently been applied to astrophysical plasmas [19, 20]. Furthermore, the spin-polarization, in turn, was shown to induce cubically nonlinear terms that may influence the high-frequency dynamics [5].

When the spin contribution of electrons to the ponderomotive force was calculated in Ref. [5], a relative simple fluid model was used. The model included the magnetic dipole force, spin-precession as well as the magnetization current due to the spin, but not the spin-velocity correlations. However, in a recent work [21] a fluid model of spin that includes spin-velocity correlations was shown to capture the spin effects of kinetic theory much more accurately, although the comparison between fluid and kinetic theory in Ref. [21] was limited to linear phenomena.

In the present paper, we aim to improve the calculation made in Ref. [5] to capture the effects of the spin-velocity tensor using the newly developed four-equation hierarchy [21] for spin-fluid dynamics. Furthermore, in order to validate this model, we compare our results with the full kinetic treatment from the equations derived in Ref. [22]. The set-up is that of a weakly nonlinear electromagnetic pulse propagating parallel to an external magnetic field. It is then confirmed that a spin-ponderomotive term of the same kind as in previous calculations exists (together with the classical ponderomotive force), but it is also found that the spin-velocity correlations induce an additional term. This new term, due to the spin-velocity correlations, gives rise to a force in the same directions for all particles, regardless of the spin-states (up or down) relative to the external magnetic field. This is in contrast to the previously found term [5] which acted in opposite directions for spin-up and down states. Furthermore, the new term has higher order resonances at the spin precession frequency, as compared to the previous term [5]. The comparisons with kinetic theory give a perfect agreement in the low-temperature limit. This comparison also includes a calculation of the low-frequency magnetization induced by the spin-ponderomotive force. The main purpose of the present paper is to validate the newly developed model against kinetic theory also in the regime of nonlinear perturbations. However, we also point out that the newly found contribution to the spin-ponderomotive force is likely to be important for the nonlinear evolution in quantum plasmas.

II. FLUID DESCRIPTION

In Ref. [21] (see also Ref. [23]) a fluid moment hierarchy was derived from a quantum kinetic theory with spin, extending previous hydrodynamical models for particles with spin-1/2. The time evolution equations are given by

\[ \partial_t n + \partial_i (nv_i) = 0 \]  \hspace{1cm} (1)

\[ m (\partial_t + v_j \partial_j) v_i = q (E_i + \epsilon_{ijk} B_k) + \frac{1}{n} \partial_j P_{ij} + \frac{2\mu}{\hbar} S_j \partial_j B_j \]  \hspace{1cm} (2)

\[ (\partial_t + v_j \partial_j) S_i = - \frac{2\mu}{\hbar} E_{ijk} B_k S_j - \frac{1}{nm} \partial_j \Sigma_{ij} \]  \hspace{1cm} (3)
\[(\partial_t + v_k \partial_k)P_{ij} = -P_{ik} \partial_i u_j - P_{jk} \partial_k u_i - P_{ij} \partial_k u_k + \frac{q}{m} \varepsilon_{lmn} P_{jm} B_n + \frac{q}{m} \varepsilon_{mn} P_{im} B_n \]

\[+ \frac{2 \mu}{\hbar} \Sigma \partial_i B_k + \frac{2 \mu}{\hbar} \Sigma \partial_i B_k - \partial_i Q_{ijk} \]

where \(n\) is the particle number density, \(v\) is the fluid velocity, \(q\) is the particle charge, \(m\) is the mass, \(P_{ij}\) is the pressure tensor, \(\Sigma_i\) is the spin density which is defined to have length \(\hbar/2\) for a coherent spin state, and \(Q_{ijk}\) is the heat flux tensor. The magnetic moment of the particle is given by \(\mu = g q h/4m\), where \(g\) is the \(g\)-factor of the particle in question (\(g = 2.0023\) for electrons). Summation over repeated indices \(i, j, k\cdots = 1, 2, 3\) is understood and we have used the notation \(\partial_i = \partial/\partial x_i\) and \(\varepsilon_{ijk}\) is the antisymmetric Levi-Civita tensor. The tensor \(\Sigma_{ij}\) is the spin-velocity tensor which correlates spin and velocity and captures, in a macroscopic description, how different forces act on different parts of the spin probability distribution. In Ref. [21] inclusion of \(\Sigma_{ij}\) was shown to reproduce some of the more subtle effects predicted by kinetic theory in the linear regime. Note that quantum mechanical effects associated with particle dispersion (e.g., the Bohm potential) could have been included in the above model. However, since such effects will not influence the ponderomotive force of transverse waves propagating along a magnetic field [5], which is our problem of consideration, we do not include such terms here.

Next, we consider circularly polarized electromagnetic waves propagating parallel to an external magnetic field, \(B_0 = B_0 \hat{z}\), and use the following ansatz

\[E = \frac{1}{2} \left[ \vec{E}(z,t) e^{i(kz-\omega t)} + \vec{E}^*(z,t) e^{-i(kz-\omega t)} \right], \]

\[B = \frac{1}{2} \left[ \vec{B}(z,t) e^{i(kz-\omega t)} + \vec{B}^*(z,t) e^{-i(kz-\omega t)} \right], \]

where the amplitudes are assumed to vary much slower than the exponential factors, and the star denotes complex conjugates. Since the basic wave modes propagating parallel to \(B_0\) are either left- or right-circularly polarized, we have \(\vec{E}, \vec{B} \propto \hat{\mathbf{x}} \pm i \hat{\mathbf{y}}\). Furthermore, all perturbations are small, such that weakly nonlinear perturbation theory is applicable. The equilibrium density and spin density will be denoted by \(n_0\) and \(S_0\), respectively.

### A. Spin-Ponderomotive Force

When thermal effects are small, the classical (superscript cl) low-frequency (subscript (lf)) ponderomotive force corresponding to the ansatz (5) (with transverse fields) is

\[F_{\text{cl}}^{\text{lf}} = -\frac{q^2}{2m(\omega \mp \omega_e)} \left[ \partial_t \mp \frac{k \omega_e}{\omega(\omega \mp \omega_e)} \right] |E_\pm|^2, \]

where \(\omega_e \equiv qB_0/m\) is the electron cyclotron frequency [24]. (We will here limit ourselves to electrons, although all results in Sec. II can straightforwardly be generalized to any particle species fulfilling Eqs. (1)–(5). Equation (7) was first derived by Karpman et al [4], and has been verified by many subsequent authors. On the other hand, the spin-dependent part of the ponderomotive force was recently calculated by Brodin et al [5], but starting from a model without the spin-velocity tensor. Now, we follow their outline to calculate the spin contribution to the ponderomotive force, but taking also the parts originating from \(\Sigma_{ij}\) into account.

To find the spin dependent part of the ponderomotive force, we first linearize the Eq. (5) and note that the only components of the spin-velocity vector that have a driving term are \(\Sigma_{13}\) and \(\Sigma_{23}\) and thus we define \(\Sigma_{\pm} = \Sigma_{13} \pm i \Sigma_{23}\), \(B_+ = B_x \pm i B_y\) and \(S_\pm = S_x \pm i S_y\). Neglecting the slow derivatives, i.e., derivatives acting on the amplitudes, we get the linear solution to Eq. (5) given by

\[\Sigma_\pm = -\frac{\hbar \mu n_0 k}{4(\omega \mp \omega_g)} B_\pm, \]

where \(\omega_g \equiv 2 \mu B_0/h\) is the spin-precession frequency (which is close to the cyclotron frequency for electrons, as \(|g| = 2.0023\)). Iterating by plugging this back into Eq. (5) we find the correction due to the slow derivatives as

\[\Sigma_\pm = -\frac{\hbar \mu n_0 k}{4(\omega \mp \omega_g)} \left( k - \frac{ik}{\omega \mp \omega_g} \partial_t - i \partial_t \right) B_\pm. \]

This last step could also be done by simply making the substitutions \(\omega \rightarrow \omega + i \partial_t\) and \(k \rightarrow k - i \partial_t\) in Eq. (8) and expanding to lowest order in the slow derivatives. From Faraday’s law the electric and magnetic fields are similarly related by

\[B_\pm = \pm i \frac{k}{\omega} E_\pm \pm \frac{1}{\omega} \frac{\partial E_\pm}{\partial t} \mp \frac{k}{\omega} \frac{\partial E_\pm}{\partial t}. \]

The lowest order approximation, \(B_\pm = \pm i (k/\omega) E_\pm\), can be used to switch between magnetic and electric fields in, e.g., the right-hand side of Eq. (7). Repeating the above steps for Eq. (5) and using Eq. (9) we can express the spin variable as

\[S_\pm = \mp \frac{\mu S_0}{\hbar(\omega \mp \omega_g)} B_\pm + \frac{\hbar \mu}{4m(\omega \mp \omega_g)^2} \left[ -k^2 + i \left( \pm \frac{4mS_0}{\hbar^2} + \frac{2k^2}{\omega \mp \omega_g} \right) \partial_t + 2ik \partial_x \right] B_\pm. \]

The spin-ponderomotive force density is then given by the last term of Eq. (11) and can be written as

\[F_{\text{clf}} = \frac{\mu}{2\hbar} \left( S_\pm \partial_z B_\pm + S_\mp \partial_z B_\pm \right). \]
the slow derivative expansion. Thus, using Eq. \(11\) to the first order in slow derivatives we obtain

\[
F_{zlf} = \mp \frac{\mu^2 \Sigma_{0}}{2 \hbar^2 (\omega + \omega_{x})^2} \left[ \partial_t - \frac{k}{\omega + \omega_{x}} \partial_z \right] |B_\pm|^2 + \frac{\mu^2 k^2}{8 m (\omega + \omega_{x})^2} \left[ \partial_t + \frac{2k}{\omega + \omega_{x}} \partial_z \right]|B_\pm|^2. \tag{13}
\]

Comparing this force with the result of Ref. \([5]\) we see that the first term (proportional to \(S_0\)) corresponds exactly to their result [after a correction of factor 2 in Eq. (13) of Ref. \([5]\)] has been made. The second term comes in due to the inclusion of the spin-velocity tensor (which was neglected in Ref. \([5]\)). It has an extra factor \(\hbar k^2 / 2 m (\omega + \omega_{x})\) compared to the first term and could in a sense be viewed as a higher-order quantum correction. This term might, however, dominate over the first term in some cases where the zeroth order magnetization is small. For thermal equilibrium we have \(S_0 = (\hbar / 2) \tanh (\mu B_0/k_B T)\) where \(\tanh (\mu B_0/k_B T) \simeq \mu B_0/k_B T\) for a moderate magnetic field strength, and thus both the terms of the spin-ponderomotive force are quadratic in \(h\). In this case, the ratio of the first to second term scales as \(\sim m \omega_{x} (\omega - \omega_{x}) / k_B T k^2\), which could be both smaller or larger than unity. On the other hand, for a higher magnetic field strength for which \(\tanh (\mu B_0/k_B T) \sim 1\), the same ratio is scaled as \(\sim m (\omega - \omega_{x}) / \hbar k^2\), which can also be either smaller or larger than the unity. Thus, depending on the parameter regimes we consider for a specific problem, both the classical and the spin-ponderomotive force may either be comparable or even dominate over each other \([12, 20]\). As for example, for a wave frequency close to the electron-cyclotron frequency, the spin contribution to the ponderomotive force can, indeed, dominate over the classical one when \(\hbar k^2 / m \omega \gg 1\) \([5, 12, 20]\).

A slightly different perspective was taken in, e.g. Ref. \([5]\). There a two-fluid model for electrons was used, where spin-up and down states were treated as different species, and thus the two species have \(S_0 = \pm (\hbar / 2)\). As a consequence, the force on each separate species [due to the first term of Eq. \((13)\)] is typically stronger [since the diminishing factor \(\mu B_0/k_B T\) disappears from the first terms, the ratio between the two types of terms are now \(\hbar k^2 / 2 m (\omega + \omega_{x})\)], but on the other hand, the forces on the two spin-populations act in opposite directions due to the terms proportional to \(S_0\). Since the physics is different depending on whether the terms induce spin-polarization or not, both types of terms may be needed for an accurate description even in the cases when one of the terms is much larger than the other. As argued in Ref. \([21]\), the two-fluid model captures some of the effects of a kinetic theory, but it is less needed when the spin-velocity correlations of the \(\Sigma_{ij}\) tensor is included in the model. We will return to this issue of one-fluid versus two-fluid models in a little more detail below. As a final note of this subsection we point out that for slowly modulated waves as considered here we may in many cases use the approximate relation \(\partial_t \simeq -v_g \partial_z\) (where \(v_g\) is the group velocity) to simplify Eqs. \((9)\) and \((13)\), although more accurate expressions might be needed in case the terms from the spatial and temporal derivative tend to cancel.

### B. Low-frequency magnetization

The slowly modulated wave also gives rise to a nonlinearly induced low-frequency magnetization. To further investigate the fluid model we next calculate the low frequency magnetization defined as

\[
M_{lf} = \mu \left( n_0 S_{lf} + n_i S_0 \right). \tag{14}
\]

We are particularly interested in the case where the low-frequency magnetization is induced in the absence of prior magnetization, and hence we take \(S_0 = 0\) for simplicity. As \(S_0 = (\hbar / 2) \tanh (\mu B_0/k_B T)\) in a thermal equilibrium background, this may be a useful approximation also from a practical perspective, provided \(\mu B_0/k_B T \ll 1\).

Applying Eq. \((3)\), considering the low-frequency time scale and keeping terms up to the second order in amplitude, we obtain

\[
\partial_t S_{lf} + \frac{\mu}{2 \hbar} (i B_\pm S_{lf}^* + \text{c.c.}) \delta_{ij} - \frac{1}{n_{om}} \partial_j \Sigma_{ij}^{lf}, \tag{15}
\]

where c.c. denotes the complex conjugate. The first term can be calculated from Eq. \((11)\), and we note that only terms involving slow derivatives survive when adding the complex conjugates. As for the second term, only the \(\Sigma_{ij}^{lf}\) components need to be calculated. Thus, out of these three, the only component with a nonzero driving term is

\[
\partial_t S_{lf}^{33} = \pm \frac{\mu}{2 \hbar} \Sigma_{33} B_\pm^* + \text{c.c.} \tag{16}
\]

where +c.c. means adding the complex conjugate of the right-hand side. Combining the above formulas with the linear results from the previous section, Eqs. \((9)\) and \((11)\) we obtain

\[
\partial_t^2 M_{lf} = - \frac{\mu^3 n_0}{4 m h^2 (\omega + \omega_{x})^2} \left[ \partial_t - \frac{2k}{\omega + \omega_{x}} \partial_z \right] \left( \partial_z + \frac{k}{\omega + \omega_{x}} \partial_t \right)|B_\pm|^2. \tag{17}
\]

### C. Magnetization in spin two-fluid models

In Ref. \([5]\) a similar problem was studied, where the first term of Eq. \((13)\) induced opposite density perturbations for particles with initial spin-up (i.e. with \(S_0 = +\hbar / 2\)) and for particles with initial spin-down (with \(S_0 = -\hbar / 2\)). As pointed out above, it is straightforward to include such a formalism within the present theory, we just consider two electrons fluids which are identical in all aspects, except that the unperturbed spin in the z-direction is +\(\hbar / 2\) or -\(\hbar / 2\). The expression Eq. \((13)\) is then the same, except that we substitute \(S_0 = \pm (h/2)\) where \(+(-)\) stands for up (down) species, rather than the thermodynamic equilibrium expression \(S_0 = (h/2) \tanh (\mu B_0/k_B T)\) of the one-fluid theory.

Let us now calculate the two-fluid version corresponding to Eq. \((17)\), which is still evaluated when \(\mu B_0/k_B T \ll 1\) such that \(S_0\) of the one-fluid theory is negligible. This corresponds
to equal densities of the initial spin-up and down-populations in the two-fluid model. Somewhat surprisingly, we obtain exact agreement with Eq. (17) using such a two-fluid theory. However, contributions to the magnetization that come from the spin-velocity correlations in the one-fluid model is, to some extent, replaced by contributions that come from density perturbations of the two-species in the two-fluid model. Specifically, we note that our current model (as we recall, includes a modified ponderomotive force due to the spin-velocity tensor) agrees with the density perturbations of the spin-up and spin-down populations of the less elaborate model used in Ref. [5]. Furthermore, we stress that the additional spin-ponderomotive force term does not contribute to the spin-polarization. Nevertheless, the corresponding magnetization found here can be much lower in certain cases, as compared to the model without spin-velocity correlations. In particular, this holds for the specific case of an unmagnetized plasma \((B_0 = 0)\), and a weakly dispersive driver, i.e. from Eq. (17) we find that for \(\omega_c \gg \omega\), \(B_0 = 0\) and zero dispersion (i.e. \(v_e = \omega/k\)), the right-hand side of Eq. (17) vanishes. This may seem surprising, as intuitively one would expect that a net density difference between the up- and down-species (as found in a two-fluid model both with and without spin-velocity correlations) implies a net magnetization. However, the reason of why this does not give rise to a finite magnetization for the given assumptions specified above, is that the density difference is compensated by a rotation of the spin direction, which also is of second order in the transverse wave field amplitudes.

In a two-fluid model [where the unperturbed spin of the species are \(S_0 = \pm \hbar/2\)] the magnetization for each species is then given by Eq. (14) and the total low-frequency magnetization is obtained as

\[
M_{\text{nl}} = M_{\text{nl}l} + M_{\text{nl}u} = \frac{2\mu_B}{\hbar} \int_0^\infty \left[ n_{\text{nl}l} - n_{\text{nl}u} \right] |S_0|, \quad (18)
\]

with \(S_0 = \hbar/2\). For the assumptions specified above where the induced magnetization vanishes, we accordingly have \((S_{\text{nl}l} + S_{\text{nl}u}) = -(n_{\text{nl}l} - n_{\text{nl}u}) \hbar/2n_0\).

### III. KINETIC APPROACH

In order to further investigate the fluid model of Eqs. (1)-\(\text{4a}\), we now consider the same situations as above but using a kinetic model. The evolution of the scalar distribution for a spin particle is, in the long scalelength limit, given by [22]

\[
\partial_t f + v \cdot \nabla f + \frac{q}{m} (E + v \times B) \cdot \nabla f + \left( \frac{\mu}{m} \right) (B \cdot s + B \cdot \nabla) \cdot \nabla f + \frac{2\mu}{\hbar} (s \times B) \cdot \nabla f = 0. \quad (19)
\]

Here, the distribution function \(f = f(x, v, s, t)\) contains a spin variable \(s\), which is defined to lie on the unit sphere. This can be directly related to the fluid model, with the macroscopic spin vector given by \(S(x, t) = \langle h/2 \rangle \int d\Omega 2s f(x, v, s, t)\), where the integration element is \(d\Omega = d^3v d^2s\) and the two-dimensional spin integration is carried out over the unit sphere. For a more detailed description we refer to Ref. [22]. However, we point out that the spin part of the distribution function is more general than a semi-classical treatment, such as that made in, e.g. Refs. [14, 15], where the evolution equation for the distribution function has the same structure as for a classical magnetic dipole moment.

In order to calculate the weakly nonlinear low-frequency response to an incoming transverse wave packet we make the ansatz

\[
f(x, v, s, t) = f_0(v^2, \theta) + f_{\text{lf}}(z, t, v, \Theta),
\]

\[
+ \frac{1}{2} \left[ f_1(z, t, v, s) e^{ikz+it0} + f_1^*(z, t, v, s) e^{-ikz+it0} \right], \quad (20)
\]

where \(f_0\) is the background distribution, \(f_{\text{lf}}\) is a low-frequency part due to quadratic nonlinearities and \(f_1\) is a slowly modulated high-frequency wave. The background distribution will be taken to be of the form

\[
f_0 = \frac{n_0}{2\pi v_T^2} e^{-v^2/v_T^2} \left[ 1 + \tanh \left( \frac{\mu B_0}{k_B T} \cos \Theta \right) \right], \quad (21)
\]

where \(n_0\) is the equilibrium density, the thermal velocity \(v_T\) is defined as \(v_T = \sqrt{2k_BT/m}\). This is the generalization of thermal equilibrium for the scalar distribution function, applicable for low or moderate densities, which contains the quantum effects on the angular spin distribution. The fully quantum mechanical background distribution (applicable in the regime of high densities and/or strong magnetic fields) is presented in Ref. [22]. The spin part of the distribution function above can also be written as \(f_1(1 + \cos \Theta, \Theta) + f_1(1 + \cos \Theta)\), where \(f_1 \propto \exp(-\mu_B B_0/k_BT)\) and \(f_1 \propto \exp(\mu_B B_0/k_BT)\). Thus, we see that the part proportional to \(\cos \Theta\) scales as \((f_1 - f_0)/(f_1 + f_0)\), which gives the factor \(\tan(\mu_B B_0/k_BT)\) in the second term of Eq. (21). It is then more clearly seen that the distribution function is separated into two parts, one for each spin population. This can be considered as a basis for derivation of a two-fluid model [21], but we will not pursue this further here. Note also that a semi classical reasoning would suggest us to take the distribution function as \(f_0 \propto \exp[-E/(k_BT)]\), where \(E = mv^2/2 - \mu_B \cos \Theta\), which differs from the quantum mechanical angular distribution used here. We will assume that the temperature is sufficiently low for \(v_T \ll \omega/k\) such that we can neglect thermal effects in the final result.

The ansatz for the incoming high-frequency field in Eq. (6) is the same as before. The aim is then to find an equation for the low-frequency part of the distribution function. From such an equation we can then calculate the low-frequency response in the current density and magnetization, and compare with the results from the previous section. The high-frequency perturbation of the distribution function is \(f_1 = f_1(\cdot)\) for left-hand (right-hand) circularly polarized waves. The expressions for \(f_{\text{lf}}\) found from Eq. (19) to linear order can be easily com-
pute from Eq. (9) in Ref. [25]

\[ f_\pm = \frac{-i e^{\pm i \phi}}{\omega - kv_c + \omega_e} \frac{q}{2m} E_{\pm} \partial_{v_z} f_0 + \frac{\omega - kv_c + \omega_e}{\omega - kv_c + \omega_e} \frac{\mu}{2m} \left[ k B_{\pm} \left( \sin \theta \partial_{v_z} f_0 + \cos \theta \partial_{v_r} f_0 \right) + \frac{2m}{\hbar} B_{\pm} \partial_{\theta} f_0 \right]. \]  

Next, allowing for slow modulations and solving the equation to first order in \( \partial_t/k, \partial_t/\omega \), we note that the zero-order solution applies after making the substitution \( \omega \rightarrow \omega + i \partial_t \) and \( k \rightarrow k - i \partial_t \) in Eq. (21), and then expanding to first order in the slow derivatives. Inserting the ansatz above into the evolution equation and consider the slow-time scale and keeping only up to quadratic nonlinearities we obtain the equation

\[
(\partial_t + v_z \partial_z) f_{1f} + \frac{q}{m} E_{1f} \partial_{v_z} f_0 = - \frac{q}{4m} \left( E + v \times B \right) + \frac{\mu}{4m} (ik + \partial_z) (s \cdot B + B \cdot \nabla) \hat{z} \cdot \nabla f_1^s + \frac{\mu}{2\hbar} \vec{B} \cdot \nabla f_1^s + \text{c.c.} \tag{23}
\]

Here we have also added a low-frequency electric field in the \( z \)-direction, \( E_{1f} \), which has \( f_{1f} \) as source. Equations (22) and (23) now constitute a basis for calculating the nonlinear response in the current density and magnetization.

### A. Ponderomotive Force

Inserting the first order solution \( f_1 \) in Eq. (23), multiplying by \( q v_z \) and integrating over \( d\Omega = d^3v d^2s \) we can derive an equation for the current density \( J_z \). We will neglect Landau damping associated with the particle resonances. Furthermore, since we have assumed a low-temperature (\( v_T \ll \omega/k \)) we may expand the denominators in \( f_1 \) to the lowest order in \( v_z \). All the integrals can then be evaluated.

As an intermediate step, after integrating Eq. (23) we obtain

\[
\partial_t J_{1f} + q v_z \int d\Omega \left( v_{||}^2 f_{1f} - \frac{q^2 n_0}{m} E_{1f} \right) = \text{nonlinear terms}. \tag{24}
\]

The second term on the left-hand side of Eq. (24) is a thermal correction to the low-temperature limit, which we will neglect. Using Ampere’s law for the first term on the left-hand side, we can then obtain a closed equation for the time derivative of the low-frequency electric field in terms of the incoming wave. After some algebra we find

\[
\left[ \partial_t^2 + \omega_p^2 \right] E_{1f} = C_\pm + D_\pm,
\]

where \( \omega_p = \sqrt{n_0 q^2 / m e \hbar} \) is the plasma frequency and the pond-

deromotive source terms on the right-hand side are given by

\[
C_\pm = -\frac{q \omega_0^2 \omega}{8mk^2(\omega + \omega_c)} \left( \partial_t + \frac{\partial t}{\omega + k - \omega_c} \right) \left| B_\pm \right|^2 \tag{25}
\]

\[
D_\pm = -\frac{q \omega_0^2 h^2 k^2}{16m^2(\omega + \omega_c)^2} \left( \partial_t + \frac{2k}{\omega + \omega_c} \partial_t \right) \left| B_\pm \right|^2 + \frac{q \omega_0^2 S_0}{4m^2(\omega + \omega_c)} \left( \partial_t - \frac{k}{\omega + \omega_c} \partial_t \right) \left| B_\pm \right|^2. \tag{26}
\]

Note that the term \( C_\pm \) is due to the classical ponderomotive force stemming from the magnetic part of the Lorentz force, and \( D_\pm \) is due to the spin effects. Taking into account the normalization factor \( n_0/q \) (which appears when an evolution equation for \( E_{1f} \) is derived) we have perfect agreement with the fluid theory in the previous section.

### B. Magnetization

We can also obtain the low-frequency response in the magnetization. This is done by multiplying Eq. (23) by \( 3 \mu \hbar s_z \) and integrating over the velocity and the spin. Thus, we obtain

\[
\partial_t M_{1f} + 3 \mu \hbar s_z \int d\Omega \partial_t f_{1f} = \text{nonlinear terms}. \tag{27}
\]

In order to be able to evaluate the second term, we take the time derivative of Eq. (24) and once again use Eq. (23) to evaluate the resulting term \( 3 \mu \hbar \partial_t f_{1f} \). We still neglect all the thermal contributions including the particle resonances. To simplify the problem further, and compare with the fluid result, we also neglect the zeroth order magnetization which corresponds to neglecting the factor proportional to \( S_0 = (\hbar/2) \tanh(\mu B_0/k_B T) \) in Eq. (21). Also, to obtain an evolution equation for \( M_{1f} \), one takes the time derivative of Eq. (27), substitute (23) for \( \partial_t f_{1f} \), apply Eq. (22) and keep only the low-frequency source terms (those proportional to \( \left| B_\pm \right|^2 \) or \( \left| E_{1f} \right|^2 \), and carry out the integrals over \( d\Omega \), omitting thermal corrections. After lengthy calculations and evaluating the corresponding integrals we eventually obtain an exact agreement with Eq. (27). This is an important verification that the truncation of the moment hierarchy used to close the fluid equations is valid in the low-temperature limit, also when nonlinearities are present.

### IV. SUMMARY AND CONCLUSIONS

In this paper, we have calculated the ponderomotive force due to the electron spin property in both a recently derived fluid model [21] and in the kinetic model (see Eq. (83) in Ref. [22] that is the basis for the fluid theory. The kinetic result has been evaluated in the low-temperature limit, in which case we obtain a complete agreement with the fluid result. The fluid theory considered here extends a simpler fluid model (e.g. Ref. [5]) by including the spin-velocity correlations. As a result, the spin part of the ponderomotive force found here has a contribution that is additional to that recently derived...
in Ref. \[5\]. The previously derived spin force contained a term proportional to the unperturbed spin \( S_0 \) (the first term in \[13\]) but not the second term in Eq. \[13\] that is independent of \( S_0 \). Nevertheless, Ref. \[5\] found that the spin-ponderomotive force could be important even when the unperturbed magnetization (and hence the net value of \( S_0 \)) is zero, such as in an unmagnetized plasma. The reason was that a two-fluid model of electrons was used, where the spin-up and spin-down states were exposed to spin-ponderomotive forces pointing in opposite directions, which induced a spin-polarized plasma. It is straightforward to include such an approach also within Eqs. \[1\]-\[5\]. Furthermore, within a nonlinear perturbation scheme, the division of the electrons into spin-up- and spin-down-populations can be put on a firm basis in the more complete kinetic theory.

However, provided the more subtle effects of spin-velocity correlations are taken into account, our results here, as well as those in Ref. \[21\], suggest that a two-fluid spin model (which may capture certain effects of the microscopic spread in the spin-probability distribution) is not needed. This results from the fact that the spin-velocity tensor seems to capture the physics of spin-polarization, as well as additional effects of the spin. On the other hand, it is still too early to exclude the possibility that treating spin-up and -down populations as different species captures new physics compared to a one-fluid model. In particular, a difference between the one- and two-fluid models may reveal itself when cubically nonlinear calculations are carried out.

One of the main conclusions of this paper is that Eqs. \[1\]-\[5\] agree completely with kinetic theory when thermal effects are negligible, whereas the same would not be true when the spin-velocity tensor is omitted. This comparison holds not only for the spin-ponderomotive force, but also for the induced low-frequency magnetization, that has been calculated both from the fluid and the kinetic theory. It should be noted that the previous comparison between the current spin fluid model and the spin kinetic theory was limited to linearized theory. The previous expressions for the ponderomotive force were shown to be significant in plasmas of relatively modest density and also when the plasma was unmagnetized, due to the induced spin-polarization \[5\]. The additional term in the expression for the spin ponderomotive force found here may be at least as important, particularly when the wave frequency is close to \( \omega_{qz} \), due to the higher-order of the resonance in this new term. A detailed evaluation of the effect of the spin-ponderomotive expression found here is a project for future work.