A NEW TYPE OF COUPLED FIXED POINT THEOREM IN PARTIALLY ORDERED COMPLETE METRIC SPACE

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Abstract. In this paper, we introduce a new type of coupled fixed point theorem in partially ordered complete metric space. We give an example to support of our result.

1. Introduction and Preliminaries

Fixed point theory in recent has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. The first result in this direction was obtained by Ran and Reurings [7]. They presented some applications of results of matrix equations. In [4], Nieto and Lopez extended the result of Ran and Reurings [5], for non decreasing mappings and applied their result to get a unique solution for a first order differential equation. While Agrawal et al. [1] and O’Regan and Petrutel [6] studied some results for generalized contractions in ordered metric spaces. Bhaskar and Lakshmikantham [3] obtained some coupled fixed point results for mixed monotone operators $F : X \times X \to X$ which satisfy a certain contractive type condition, where $X$ is a partially ordered metric space. They established three kinds of coupled fixed point results: 1) existence theorems; 2) existence and uniqueness theorem; and 3) theorems that ensure the equality of the coupled fixed point components. Also, they applied results to the study of existence and uniqueness of solution for a periodic boundary value problem. After, Berinde [2] extended the coupled fixed point theorems for mixed monotone operators $F : X \times X \to X$ obtained in Bhaskar and Lakshmikantham [3] by significantly weakening the involved contractive condition.

In order to state the main result in this paper, we need the following notions.

Definition 1. Let $(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

for $(x, y), (u, v) \in X \times X$, $(u, v) \leq (x, y) \iff x \geq u, y \leq v.$

We say that a mapping $F : X \times X \to X$ has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in $x$ and is monotone increasing in $y$, that is, for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)

and

$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 2. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$F(x, y) = x$ and $F(y, x) = y$.

Theorem 1. [3] Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous

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mapping having the mixed monotone property on \( X \). Assume that there exists a constant \( k \in [0,1) \) with

\[
d(F(x,y), F(u,v)) \leq k \left[ d(x,u) + d(y,v) \right], \quad \forall x \geq u, \ y \leq v.
\]  

(1.1)

if there exist \( x_0, y_0 \in X \) such that

\[
x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0),
\]

then there exist \( x, y \in X \) such that

\[
x = F(x, y) \quad \text{and} \quad y = F(y, x).
\]

In \cite{3} Bhaskar and Lakshmikantham also established some uniqueness results for coupled fixed points, as well as existence of fixed points of \( F \) (\( x \) is a fixed point of \( F \) if \( F(x, x) = x \)).

Inspired by above works, we derive new coupled fixed point theorems for mapping having the mixed monotone property \( F : X \times X \rightarrow X \) in partially ordered metric space and we give an example to support our result.

2. Main Results

Theorem 2. Let \( (X, \leq) \) be a partially ordered set and suppose there is a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Let \( F : X \times X \rightarrow X \) be a continuous mapping having the mixed monotone property on \( X \). Assume that \( F \) satisfies the following condition:

\[
d(F(x,y), F(u,v)) \leq \delta(x,y,u,v) [d(x,u) + d(y,v)].
\]

(2.1)

where

\[
\delta(x,y,u,v) = \frac{d(x,F(u,v)) + d(y,F(v,u)) + d(u,F(x,y)) + d(v,F(y,x))}{1 + 2 [d(x,F(x,y)) + d(y,F(y,x)) + d(u,F(u,v)) + d(v,F(v,u))]},
\]

for all \( x, y, u, v \in X \) with \( x \geq u \) and \( y \leq v \).

If there exist \( x_0, y_0 \in X \) such that

\[
x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0),
\]

then

a) \( F \) has at least a coupled fixed point there exist \( (x, y) \in X \) such that

\[
x = F(x, y) \quad \text{and} \quad y = F(y, x).
\]

b) if \( (x, y), (u, v) \) are two distinct coupled fixed points of \( F \), then \( d(x, u) + d(y, v) \geq \frac{1}{4} \).

Proof. a) Consider the two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that,

\[
x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n)
\]

(2.2)

for all \( n = 0, 1, 2, \ldots \).

Now, we claim that \( \{x_n\} \) is nondecreasing and \( \{y_n\} \) is nonincreasing i.e.,

\[
x_n \leq x_{n+1} \quad \text{and} \quad y_n \geq y_{n+1}
\]

(2.3)

for all \( n = 0, 1, 2, \ldots \). From statement of theorem, we know that \( x_0, y_0 \in X \) with

\[
x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)
\]

(2.4)

By using the mixed monotone property of \( F \), we write

\[
x_1 = F(x_0, y_0) \quad \text{and} \quad y_1 = F(y_0, x_0).
\]

(2.5)

Therefore \( x_0 \leq x_1 \) and \( y_0 \geq y_1 \). That is, the inequality (2.3) is true for \( n = 0 \).

Assume \( x_n \leq x_{n+1} \) and \( y_n \geq y_{n+1} \) for some \( n \). Now we shall prove that (2.3) is true for \( n + 1 \).

Indeed, from (2.4) and the mixed monotone property of \( F \), we have

\[
x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(y_n, x_{n+1}) \geq F(x_n, y_n) = x_{n+1}
\]

(2.6)
and

\[ y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = y_{n+1}. \]

Hence, by induction, \( x_n \leq x_{n+1} \) and \( y_n \geq y_{n+1} \) for all \( n \).

Since (2.1), \( x_{n-1} \leq x_n \) and \( y_{n-1} \geq y_n \), we have

\[
\begin{align*}
\lim_{n \to \infty} \left[ d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq \left( \frac{d(x_n, F(x_{n-1}, y_{n-1})) + d(y_n, F(y_{n-1}, x_{n-1})) + d(x_{n-1}, F(x_n, y_n))}{1 + 2[d(x_n, F(x_{n-1}, y_{n-1})) + d(y_n, F(y_{n-1}, x_{n-1})) + d(x_{n-1}, F(x_n, y_n)) + d(y_{n-1}, F(y_{n-1}, x_{n-1}))]} \right)
\right] \end{align*}
\]

This implies

\[ d(x_{n+1}, x_n) \leq \left( \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{1 + 2[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \right]. \]

Similarly, from (2.1), \( y_{n-1} \geq y_n \) and \( x_{n-1} \leq x_n \), we obtain

\[ d(y_{n+1}, y_n) \leq \left( \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{1 + 2[d(y_n, y_{n+1}) + d(x_n, x_{n+1}) + d(y_{n-1}, y_n) + d(x_{n-1}, x_n)]} \right]. \]

From these inequalities (2.6) and (2.7), we get

\[ d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq 2 \left( \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{1 + 2[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \right). \]

Now, let

\[ \beta_n = 2 \left( \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{1 + 2[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \right). \]

Then

\[ d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \beta_n [d(x_n, x_{n-1}) + d(y_n, y_{n-1})] \leq \beta_n \beta_{n-1} [d(x_{n-1}, x_{n-2}) + d(y_{n-1}, y_{n-2})] \leq \cdots \leq \beta_n \beta_{n-1} \cdots \beta_1 [d(x_1, x_0) + d(y_1, y_0)] \]

Observe that \( (\beta_n) \) is nonincreasing, with positive terms. So, \( \beta_n \beta_{n-1} \cdots \beta_1 \leq \beta_1^n \) and \( \beta_1^n \to 0 \). It follows that

\[ \lim_{n \to \infty} (\beta_n \beta_{n-1} \cdots \beta_1) = 0. \]

Hence, this implies that

\[ \lim_{n \to \infty} [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = 0. \]

From this limit, we have

\[ \lim_{n \to \infty} d(x_{n+1}, x_n) = \lim_{n \to \infty} d(y_{n+1}, y_n) = 0. \]
We claim that \( \{x_n\} \) and \( \{y_n\} \) are a Cauchy sequence in \( X \). Let \( n < m \). Then, from the triangle inequality and (2.6)-(2.8), we have
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)
\]
\[
\leq \frac{\beta^0}{2} [d(x_1, x_0) + d(y_1, y_0)] + \frac{\beta^{m+1}}{2} [d(x_1, x_0) + d(y_1, y_0)]
\]
\[
+ \ldots + \frac{\beta^{m-1}}{2} [d(x_1, x_0) + d(y_1, y_0)]
\]
\[
= \frac{\beta^n}{2} \left( 1 - \frac{\beta^{m-n}}{1 - \beta_1} \right) [d(x_1, x_0) + d(y_1, y_0)]
\]
\[
\leq \frac{\beta^n}{2} \left( 1 - \frac{\beta}{1 - \beta_1} \right) [d(x_1, x_0) + d(y_1, y_0)]
\]
\[
= \frac{\beta^n}{2} [d(x_1, x_0) + d(y_1, y_0)]
\]
and
\[
d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)
\]
\[
\leq \frac{\beta^n}{2} [d(y_1, y_0) + d(x_1, x_0)] + \frac{\beta^{m+1}}{2} [d(y_1, y_0) + d(x_1, x_0)]
\]
\[
+ \ldots + \frac{\beta^{m-1}}{2} [d(y_1, y_0) + d(x_1, x_0)]
\]
\[
\leq \frac{\beta^n}{2} [d(y_1, y_0) + d(x_1, x_0)]
\]
By adding these two inequalities, we obtain
\[
d(x_n, x_m) + d(y_n, y_m) \leq \beta^n [d(x_1, x_0) + d(y_1, y_0)].
\]
This implies that
\[
\lim_{n,m \to \infty} [d(x_n, x_m) + d(y_n, y_m)] = 0.
\]
So, \( \{x_n\} \) and \( \{y_n\} \) are indeed a Cauchy sequence in the complete metric space \( X \) and hence, convergent: there exist \( x, y \in X \) such that
\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y.
\]
Taking limit both sides in (2.2) and using continuity of \( F \), we get
\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) = F \left( \lim_{n \to \infty} (x_{n-1}, y_{n-1}) \right) = F(x, y)
\]
and
\[
y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) = F \left( \lim_{n \to \infty} (y_{n-1}, x_{n-1}) \right) = F(y, x).
\]
Therefore,
\[
x = F(x, y) \quad \text{and} \quad y = F(y, x),
\]
that is, \((x, y)\) is a coupled fixed point of \( F \).

b) If there exist two distinct coupled fixed points \((x, y), (u, v)\) of \( F \), then
\[
d(x, u) + d(y, v) = d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))
\]
\[
\leq [d(x, F(u, v)) + d(y, F(v, u)) + d(u, F(x, y))]
\]
\[
+ [d(v, F(y, x))] [d(x, u) + d(y, v)]
\]
\[
+ [d(y, F(v, u)) + d(x, F(u, v)) + d(v, F(y, x))]
\]
\[
+ [d(u, F(x, y))] [d(x, u) + d(y, v)]
\]
\[
= [d(x, u) + d(y, v)] [4d(x, u) + 4d(y, v)]
\]
\[
= 4[d(x, u) + d(y, v)]^2.
\]
Therefore, we obtain that \( d(x, u) + d(y, v) \geq \frac{1}{4} \). \( \square \)
Now, we will give the following example for such type of mappings which satisfy (2.1).

**Example 1.** Let \( X = \{0, 1\} \) and \( x \leq y \iff x, y \in \{0, 1\} \) and \( x \leq y \) where "\( \leq \)" be usual ordering then \((X, \leq)\) be a partially ordered set. Let \( d : X \times X \to [0, \infty) \) be defined by
\[
d(0, 1) = 2, \quad d(0, 0) = d(1, 1) = 0,
\]
\[
d(a, b) = d(b, a), \forall a, b \in X.
\]

Then \((X, d)\) is a complete metric space.

We define \( F : X \times X \to X \) as
\[
F(0, 0) = 0, \quad F(0, 1) = 0, \quad F(1, 0) = 1, \quad F(1, 1) = 1.
\]

Then \( F \) is continuous and has the mixed monotone property. It is obvious that \((0, 0), (1, 0), (0, 1)\) and \((1, 1)\) are the coupled fixed points of \( F \). If we take \( x = 1, y = 0, u = 0 \) and \( v = 0 \) then we have
\[
d(F(x, y), F(u, v)) = d(F(1, 0), F(0, 0)) = d(1, 0) = 2.
\]

Also, for same value of \( x, y, u \) and \( v \), we obtain
\[
k \cdot \frac{1}{2} [d(x, u) + d(y, v)] = k \cdot \frac{1}{2} [d(1, 0) + d(0, 0)] = k.
\]

Thus the condition \( d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \) where \( \forall x \geq u, y \leq v \) of Theorem [1] is not true for any \( k \in [0, 1) \). Thus we can not use Theorem [1] for the mapping \( F \). On the other hand we will show that the mapping \( F \) satisfies the condition of Theorem [2].

Now for \( x = 1, y = 0, u = 0, v = 0 \) or \( x = 1, y = 0, u = 1, v = 1 \), we have following possibilities for values of \((x, y)\) and \((u, v)\) such that \( x \geq u \) and \( y \leq v \).

Case 1: If we take \((x, y) = (u, v) = r \) where \( r = (0, 0) \) or \((1, 1)\) or \((1, 0)\) or \((0, 1)\), then \( d(F(x, y), F(u, v)) = 0 \). Thus, the inequality (2.7) holds.

Case 2: If we take \((x, y) = (0, 0), (u, v) = (0, 1) \) or \((x, y) = (0, 0), (u, v) = (0, 1) \), then \( d(F(x, y), F(u, v)) = 0 \). Thus, the inequality (2.7) holds.

Case 3: If we take \((x, y) = (1, 0) \) and \((u, v) = (0, 0) \), then
\[
d(F(x, y), F(u, v)) = d(F(1, 0), F(0, 0)) = d(1, 0) = 2,
\]
and
\[
d(x, F(u, v)) + d(y, F(v, u)) + d(u, F(x, y)) + d(v, F(y, x)) = [d(x, u) + d(y, v)]
\]
\[
= d(1, F(0, 0)) + d(0, F(0, 0)) + d(0, F(1, 0)) + d(0, F(0, 0))
\]
\[
= d(1, 0) + d(0, 0) + d(0, 1) + d(0, 0)
\]
\[
= 8.
\]

Or, taking \((x, y) = (1, 0), (u, v) = (0, 1)\), we have
\[
d(F(x, y), F(u, v)) = d(F(1, 0), F(0, 1)) = d(1, 0) = 2,
\]
and
\[
d(x, F(u, v)) + d(y, F(v, u)) + d(u, F(x, y)) + d(v, F(y, x)) = [d(x, u) + d(y, v)]
\]
\[
= d(1, F(0, 1)) + d(0, F(0, 1)) + d(0, F(1, 0)) + d(1, F(0, 1))
\]
\[
= d(1, 0) + d(0, 1) + d(0, 1) + d(1, 0)
\]
\[
= 32.
\]

Therefore, the inequality (2.7) holds.
Case 4: If we take \((x, y) = (1, 0)\) and \((u, v) = (1, 1)\), then
\[
d(F(x, y), F(u, v)) = d(F(1, 0), F(1, 1)) = d(1, 1) = 0,
\]
that is, the inequality (2.1) holds.

Thus all the conditions of Theorem 2 are satisfied. Also, \(F\) has four distinct coupled fixed points \((0, 0), (1, 0), (0, 1)\) and \((1, 1)\) in \(X\) and \(d(x, u) + d(y, v) \ge \frac{1}{2}\) where \((x, y), (u, v)\) are two distinct coupled fixed points of \(F\).

**Remark 1.** The example (2) does not satisfy the conditions of Theorem 1. That is, we can introduce a value because (2.9) might be greater than \(\frac{1}{2}\) and has not introduced an upper bound. If \(d(x, u) + d(y, v) < \frac{1}{4}\) for every \(x, y \in X\), then we have
\[
d(x, F(u, v)) + d(y, F(v, u)) + d(u, F(x, y)) + d(v, F(y, x))
\]
\[
= 2d(x, u) + 2d(y, v) + d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))
\]
\[
< \frac{1}{2} + 2d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))
\]
\[
= \frac{1}{2}(1 + 2[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))]).
\]

It means that
\[
\left(\frac{d(x, F(u, v)) + d(y, F(v, u)) + d(u, F(x, y)) + d(v, F(y, x))}{1 + 2[d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))]}\right) < \frac{1}{2}
\]
which is a special case of the following Theorem 3. Therefore, when \((X, d)\) is a complete metric space such that, for all \(x, y \in X\), \(d(x, u) + d(y, v) \ge \frac{1}{2}\), the above Theorem is valuable because (2.11) might be greater than \(\frac{1}{2}\).

**Remark 2.** The example (2) does not satisfy the conditions of Theorem 1. That is, we cannot say \(F\) has a coupled fixed point in \(X\) or not. But, we can see that \(F\) has a coupled fixed point in \(X\) from Theorem 2. In other words the Theorem 2 is a generalization of Theorem 1.

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