EVENT-TRIGGERED GAIN SCHEDULING OF REACTION-DIFFUSION PDES
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Abstract. This paper deals with the problem of boundary stabilization of 1D reaction-diffusion PDEs with a time- and space-varying reaction coefficient. The boundary control design relies on the backstepping approach. The gains of the boundary control are scheduled under two suitable event-triggered mechanisms. More precisely, gains are computed/updated on events according to two state-dependent event-triggering conditions: static-based and dynamic-based conditions, under which, the Zeno behavior is avoided and well-posedness as well as exponential stability of the closed-loop system are guaranteed. Numerical simulations are presented to illustrate the results.

Key words. reaction-diffusion systems, backstepping control design, event-triggered sampling, gain scheduling.

1. Introduction. Control design of complex systems modeled by partial differential equations (PDEs) has become a central research area. The two traditional ways to act on those complex systems are the in-domain control and boundary control. For boundary control, the backstepping method has been used as standard tool for designing feedback laws. It has initially emerged to deal with 1D reaction-diffusion parabolic PDEs in [1], [32] and since then, the method has been employed to deal with the boundary stabilization of broader classes of PDEs (for an overview see [19] and [24]). One of the most remarkable features of the backstepping approach is the possibility to obtain closed-form analytical solutions for the kernels of the underlying integral Volterra transformation. Having explicit expressions for the kernels and for controllers makes implementation simpler and more precise. For instance, for reaction diffusion systems with constant parameters, closed-form solutions for the kernels has been obtained in terms of special functions such as the modified Bessel function [32]. When having a time-varying reaction coefficient, a closed-form solution can be obtained through power series for exponential stabilization [33], or, in the context of fixed-time stability, closed-form of time-varying kernels can be obtained using special functions as in [7].

Nevertheless, for general reaction diffusion PDEs, obtaining closed-form solutions for the kernels is in general very hard. For this class of PDEs with time-and space varying coefficients, the problem of boundary stabilization has been very challenging. Time-and space varying reaction coefficients come into play in some applications such as in trajectory planning and multi-agent systems (see e.g. [25, 8]), to mention a few. In general, the resulting kernel-PDE is of the form of hyperbolic spatial operator and first order derivative with respect to time since the kernel of the Volterra transformation has to be time-varying. This brings additional source of complexity to the problem that requires a careful well-posedness analysis and numerical methods for the solvability. The solution of the kernel, indeed, is needed to be found numerically by e.g. the method of integral operators or the so called method of successive approximations (which traces back to the seminal work [2]). In this line, some contributions

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have rigorously handled the well-posedness of time-varying kernels solutions where the reaction term is time and space dependent as in [25, 38] and some efficient algorithms to better handle the solvability of kernels have been proposed e.g. in [14].

More recent contributions focus on coupled parabolic PDEs [26], with space varying reaction coefficients [37, 3] and time- and space- varying coefficients [18] all of which able to handle more challenging issues related to the solvability, suitable choices of target systems, coupling matrices structure and well-posedness issues in general.

In this paper, we aim at stabilizing scalar 1D reaction–diffusion PDEs with a time- and space- varying reaction coefficient from a different perspective. Our approach combines some ideas from hybrid systems, specifically from the framework of state-dependent switching laws, sampled-data and event-triggered sampling/control strategies. For an overview of the literature on sampled-data, event-triggered and switching strategies, we refer to e.g. [13, 21, 35, 12, 10, 29, 15, 22] for finite-dimensional systems and to [23, 9, 16, 31, 11],[36, 30, 20, 4, 5, 28] for some classes of infinite dimensional systems.

Having said that, the main ideas in this paper state that instead of handling a time-varying kernel capturing the time- and space- varying coefficient, we use a simpler kernel capturing only the spatial variation of the reaction coefficient. This is possible on the basis that the reaction coefficient is sampled in time, thus the kernel-PDE reduces to a form involving space-varying coefficient only between two successive sampling times. In order to determine the time instants, we introduce event-triggered mechanisms that form an increasing sequence of triggering times (or time of events). At those event times, kernels are computed/updated in aperiodic fashion and only when needed. In other words, kernels for the control are scheduled according to some event triggering condition (state-dependent law). Doing so, the approach constitutes a kind of a gain scheduling strategy suggesting then the adopted name for our approach: event-triggered gain scheduling.

Sampling in time the reaction coefficient introduces an error (called error when sampling) that is reflected in the target system after transformation. This requires the study of well-posedness issues, ISS properties ([17]) and the exponential stability of the closed loop system when the control gains are scheduled according to the event-triggered mechanisms. In this paper we propose two strategies: the first one relies on a static triggering condition which takes into account the effect of the error when sampling after transformation and the current state of the closed-loop system. The second strategy relies on a dynamic triggering condition which makes uses of a dynamic variable that can be seen as a filtered version of the static triggering condition. Moreover, under the two proposed strategies, the avoidance of the so called Zeno phenomenon is proved. Hence, we can guarantee the well-posedness as well as the exponential stability of the closed-loop system.

The paper is organized as follows. In Section 2, we introduce the class of reaction-diffusion parabolic systems, the control design which includes the introduction of the event-triggered strategies for gain scheduling and the notion of existence and uniqueness of solutions. Section 3 provides the main results which include the avoidance of the Zeno phenomenon, the well-posedness of the closed-loop system and the exponential stability result. Section 4 provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section 5. The Appendix contains the proof of an auxiliary result.

Notation. $\mathbb{R}_+$ will denote the set of nonnegative real numbers. Let $S \subseteq \mathbb{R}^n$ be an open set and let $A \subseteq \mathbb{R}^n$ be a set that satisfies $S \subseteq A \subseteq \bar{S}$. By $C^0(A;\Omega)$,
we denote the class of continuous functions on $A$, which take values in $\Omega \subseteq \mathbb{R}$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A$, which takes values in $\Omega$ and has continuous derivatives of order $k$. In other words, the functions of class $C^k(A; \Omega)$ are the functions which have continuous derivatives of order $k$ in $S = \text{int}(A)$ that can be continued continuously to all points in $\partial S \cap A$. $L^2(0, 1)$ denotes the equivalence class of Lebesgue measurable functions $f, g : [0, 1] \to \mathbb{R}$ for which $\|f\| = \left(\int_0^1 |f(x)|^2 \, dx\right)^{1/2} < \infty$ and with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$.

Let $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ be given. $u[t]$ denotes the profile of $u$ at certain $t \geq 0$, i.e. $u[t](x) = u(t, x)$, for all $x \in [0, 1]$. For an interval $I \subseteq \mathbb{R}_+$, the space $\mathcal{C}^0(I; L^2(0, 1))$ is the space of continuous mappings $I \ni t \mapsto u[t] \in L^2(0, 1)$. $H^2(0, 1)$ denotes the Sobolev space of functions $f \in L^2(0, 1)$ with square integrable (weak) first and second-order derivatives $f'(\cdot), f''(\cdot) \in L^2(0, 1)$.

2. Problem description and control design. Consider the following scalar reaction-diffusion system with time- and space-varying reaction coefficient:

\begin{align}
(2.1) & \quad u_t(t, x) = \varepsilon u_{xx}(t, x) + \lambda(t, x) u(t, x) \\
(2.2) & \quad u_x(t, 0) = qu(t, 0) \\
(2.3) & \quad u(t, 1) = U(t), \quad \text{or} \quad u_x(t, 1) = U(t)
\end{align}

and initial condition:

\begin{equation}
(2.4) \quad u(0, x) = u_0(x)
\end{equation}

where $\varepsilon > 0$, $q \in (-\infty, +\infty)$ (the case $q = +\infty$ is interpreted as the Dirichlet case) and $\lambda \in \mathcal{C}^0(\mathbb{R}_+ \times [0, 1])$ with $\lambda(t) \in \mathcal{C}^1([0, 1])$. $u : [0, \infty) \times [0, 1] \to \mathbb{R}$ is the system state and $U(t) \in \mathbb{R}$ is the control input. We assume that $\lambda \in \mathcal{C}^0(\mathbb{R}_+ \times [0, 1])$ is bounded, i.e. there exists a constant $\bar{\lambda} > 0$ such that

\begin{equation}
(2.5) \quad |\lambda(t, x)| \leq \bar{\lambda}, \quad \forall t \geq 0, \quad x \in [0, 1]
\end{equation}

Moreover, we assume the following:

Assumption 2.1. There exists a constant $\varphi > 0$ such that the following inequality holds:

\begin{equation}
(2.6) \quad |\lambda(t, x) - \lambda(s, x)| \leq \varphi |t - s|, \quad \forall x \in [0, 1], \quad t, s \geq 0
\end{equation}

Assumption 2.1 means that the reaction coefficient is Lipschitz with respect to time with Lipschitz constant $\varphi$. The constant $\varphi$ is a quantity that depends on the rate of change of the reaction coefficient: a high rate of change of the reaction coefficient implies a large value for $\varphi$. All subsequent results are also valid (with some modifications) if Assumption 2.1 is replaced by the less demanding assumption of Hölder continuity instead of Lipschitz continuity, i.e., if we replace the right hand side of (2.6) by $\varphi |t - s|^a$, where $a \in (0, 1)$ is a constant. However, for simplicity we will restrict the presentation of the results to the Lipschitz case.

In what follows, we do not consider system (2.1)-(2.3) as a time-varying system but we consider system (2.1)-(2.3) as a time-invariant system with two inputs: the control input $U(t)$ and the time-varying distributed parameter $\lambda[t]$. This is very important for theoretical reasons: since system (2.1)-(2.3) is not a time-varying system we can always assume that the initial time is zero. Therefore, the proposed event-triggered control scheme may be seen as a feedforward control scheme that compensates the effect of the distributed disturbance input $\lambda[t]$. 

2.1. Backstepping control design. Our aim is the global exponential stabilization of the system (2.1)-(2.3) at zero using boundary control. To that end, we follow the backstepping approach which makes uses of an invertible Volterra transformation to map the system into a target system simpler to handle and with desired stability properties. Since the reaction coefficient is both time and space varying, typically, the kernels of the transformation have to be chosen to depend on time. This brings an additional source of complexity since the resulting kernel PDE equation contains a time-derivative of the kernel and involves the time and space varying coefficient. Overall, the problem is much harder to solve but has been the object of extensive investigation since the seminal work [2]. Numerical strategies such as the method of successive approximation have been widely employed (see e.g. [25] and [38]).

Our approach takes a different direction. We avoid solving a kernel-PDE hyperbolic spatial operator and first order derivative with respect to time capturing the reaction coefficient (dependent on both time and space). We use simpler kernels for the control under which we are still able to stabilize exponentially the closed-loop system. This brings some degree of robustness to the controller. Inspired by event-triggered control strategies (in both finite and infinite-dimensional settings), the key idea of our approach is to schedule the kernel gain at a certain increasing sequence \( \{t_j\}_{j \in \mathbb{N}} \) \( t_0 = 0 \) which will be characterized later on.

Let \( \{t_j\}_{j \in \mathbb{N}} \) be an increasing sequence of times with \( t_0 = 0 \) and define:

\[
\begin{align*}
\text{(2.7)} \quad b_j(x) &:= \lambda(t_j, x), \quad \text{for} \quad x \in [0, 1] \\
\text{(2.8)} \quad e_j(t, x) &:= \lambda(t, x) - b_j(x), \quad \text{for} \quad t \in [t_j, t_{j+1}), \quad x \in [0, 1]
\end{align*}
\]

which is the sampled version of the reaction coefficient \( \lambda(t, x) \). We define also the error when sampling:

\[
\begin{align*}
\text{(2.9)} \quad u_t(t, x) &:= \varepsilon u_{xx}(t, x) + b_j(x)u(t, x) + e_j(t, x)u(t, x) \\
\text{(2.10)} \quad u_x(t, 0) &:= qu(t, 0) \\
\text{(2.11)} \quad u(t, 1) &:= U(t), \quad \text{or} \quad u_x(t, 1) = U(t)
\end{align*}
\]

The backstepping boundary control design is performed by transforming (2.9)-(2.11) into a target system which will reflect of the error when sampling \( e_j(t, x) \) (2.8). Therefore, consider the following invertible Volterra transformation, for \( j \geq 0 \),

\[
\begin{align*}
\text{(2.12)} \quad w_j(t, x) &:= u(t, x) - \int_0^x K_j(x, y)u(t, y)dy := (K_ju[t])(x) \\
\text{whose inverse is given by} \\
\text{(2.13)} \quad u(t, x) &:= w_j(t, x) + \int_0^x L_j(x, y)w(t, y)dy := (L_jw_j[t])(x)
\end{align*}
\]

with kernels \( K_j, L_j \in C^2(\mathcal{T}) \) evolving in a triangular domain given by \( \mathcal{T} = \{(x, y) : 0 \leq y < x \leq 1\} \) and satisfying [32, Theorem 2]:

\[
\begin{align*}
\text{(2.14)} \quad K_{j,xx}(x, y) - K_{j,yy}(x, y) &= \frac{b_j(y) + c}{\varepsilon} K_j(x, y)
\end{align*}
\]
\begin{align}
(2.15) \quad K_jy(x, 0) &= qK_j(x, 0) \\
(2.16) \quad K_j(x, x) &= -\frac{1}{2\varepsilon} \int_0^x (b_j(s) + c) ds \\
(2.17) \quad L_{j,xx}(x, y) - L_{j,yy}(x, y) = -\frac{(b_j(x) + c)}{\varepsilon} L_j(x, y) \\
(2.18) \quad L_{jy}(x, 0) &= qL_j(x, 0) \\
(2.19) \quad L_j(x, x) &= -\frac{1}{2\varepsilon} \int_0^x (b_j(s) + c) ds
\end{align}

Under (2.12), (2.14)-(2.16) and selecting the control \( U(t) \) to satisfy
\begin{equation}
(2.20) \quad U(t) = \int_0^1 K_j(1, y) u(t, y) dy, \quad t \in (t_j, t_{j+1})
\end{equation}
for Dirichlet actuation or by
\begin{equation}
(2.21) \quad U(t) = K_j(1, 1) u(t, 1) + \int_0^1 K_{j,x}(1, y) u(t, y) dy, \quad t \in (t_j, t_{j+1})
\end{equation}
for Neumann actuation, the transformed system, for all \( j \geq 0, t \in (t_j, t_{j+1}) \), is as follows:
\begin{align}
(2.22) \quad w_{j,t}(x, t) &= \varepsilon w_{j,xx}(x, t) - cw_j(t, x) + (K_j f_j(t))(x) \\
(2.23) \quad w_{jx}(t, 0) &= qw_j(t, 0) \\
(2.24) \quad w_j(t, 1) &= 0, \quad \text{or} \quad w_{jx}(t, 1) = 0
\end{align}
where
\begin{equation}
(2.25) \quad f_j(t, x) := e_j(t, x) u(t, x)
\end{equation}
and \( c \) is a design parameter which is chosen as \( c \geq \varepsilon \tilde{q}^2 \) (for Dirichlet actuation) or \( c \geq \varepsilon \tilde{q}^2 + \varepsilon/2 \) (for Neumann actuation) where \( \tilde{q} = \max\{0, -q\} \) (see [32]).

Moreover, the following estimate holds, for all \( j \geq 0 \)
\begin{equation}
(2.26) \quad \max \left\{ |K_j(x, y)|, |L_j(x, y)| \right\} \leq M \exp(2Mx), \quad \text{for} \quad (x, y) \in T
\end{equation}
where \( M := \frac{\lambda + c}{\varepsilon} \).

Definitions (2.12), (2.13) imply the following estimates, for all \( j \geq 0 \):
\begin{align}
(2.27) \quad \|\mathcal{K}_j u[t]\| &\leq K_j \|u[t]\| \\
(2.28) \quad \|\mathcal{L}_j w_j[t]\| &\leq \tilde{L}_j \|w[t]\|
\end{align}
where \( K_j \) and \( \tilde{L}_j \) are defined, respectively by \( K_j := 1 + \left( \int_0^1 (\int_0^x |K_j(x, y)|^2 dy) dx \right)^{1/2} \) and \( \tilde{L}_j := 1 + \left( \int_0^1 (\int_0^x |L_j(x, y)|^2 dy) dx \right)^{1/2} \).

In addition, inequality (2.26) implies the existence of a constant \( G > 0 \) such that:
\begin{equation}
(2.29) \quad \max \left\{ K_j, \tilde{L}_j \right\} \leq G, \quad \text{for} \quad j \geq 0
\end{equation}
where

\[
G := 1 + \sqrt{\frac{\lambda + c}{4\varepsilon}} \left( \exp \left( \frac{4(\lambda + c)}{\varepsilon} \right) - 1 \right)
\]

which is independent of \( j \).

**2.2. Well-posedness analysis.** In order to study well-posedness issues for (2.22)-(2.24) and in turn for (2.9)-(2.11) (by virtue of the bounded invertibility the backstepping transformations (2.12)-(2.13)) under any event-triggered gain scheduling implementation, we need first a more general result (see Theorem 2.2 below) which establishes the notion of solution for the following reaction-diffusion system:

\[
w_t(t, x) = \varepsilon w_{xx}(t, x) - cw(t, x) + (F(t)w(t))(x), \quad x \in (0, 1)
\]

\[
a_0w(t, 0) + b_0w_x(t, 0) = a_1w(t, 1) + b_1w_x(t, 1) = 0
\]

where \( \varepsilon > 0, c, a_0, b_0, a_1, b_1 \) are constants with \( a_0^2 + b_0^2 > 0, a_1^2 + b_1^2 > 0 \) and for each \( t \geq 0 \) the operator \( F(t) : L^2(0, 1) \to L^2(0, 1) \) is a linear bounded operator for which there exist constants \( \Omega_1, \Omega_2 > 0 \) such that

\[
\|F(t)\| \leq \Omega_1, \quad \text{for all } t \geq 0
\]

\[
\|F(t) - F(s)\| \leq \Omega_2|t - s|, \quad \text{for all } t, s \geq 0
\]

**Theorem 2.2.** For every initial condition \( w_0 \in L^2(0, 1) \) and \( T > 0 \), the initial-boundary value problem (2.31) with

\[
w(0) = w_0
\]

has a unique solution \( w \in C^0([0, T]; L^2(0, 1)) \cap C^1((0, T); L^2(0, 1)) \) with \( w[t] \in D \) for all \( t \in (0, T) \), where

\[
D := \{ f \in H^2(0, 1) : a_0f(0) + b_0f'(0) = a_1f(1) + b_1f'(1) = 0 \}
\]

that satisfies (2.34) and (2.31) for all \( t \in (0, T) \).

**Proof.** See Appendix A. \( \square \)

We are in position to specialize the well-posedness result to the system (2.22)-(2.24) so as we can construct the solution by the step method. To do so, let us take in (2.34) \( a_0 = q, b_0 = -1 \) for \( q < +\infty; a_0 = 1, b_0 = 0 \) for \( q = +\infty; a_1 = 1, b_1 = 0 \) for Dirichlet actuation and \( a_1 = 0, b_1 = 1 \) for Neumann actuation. In addition, it suffices to observe that the operator \( (K_j f_j(t)))(x) \) in (2.22) has the form of \((F(t)w[t])(x)\), i.e,

\[
(F(t)w[t])(x) = (K_j(e_j[t]L_jw_j[t]))(x), \quad t \in [t_j, t_{j+1}]
\]

which is indeed the case by virtue of (2.8), (2.12), (2.13) and (2.25). Moreover, due to (2.6) in Assumption 2.1, the operator \( F(t) \) satisfies (2.32)-(2.33). Extending continuously the operator \( F(t) \) defined by (2.35) for \( t \geq t_{j+1} \) and using Theorem 2.2 we obtain the following proposition:
Proposition 2.3. For every initial data $w_j[t_j] = (K_j u[t_j])(x) \in L^2(0,1)$, there exists a unique function $w_j \in C^0 \left( (t_j, t_{j+1}); L^2(0,1) \right) \cap C^1 \left( (t_j, t_{j+1}); L^2(0,1) \right)$ with $w_j[t] \in D$ for $t \in (t_j, t_{j+1})$ that satisfies (2.22)-(2.24) for all $t \in (t_j, t_{j+1})$ where $D \subset H^2([0,1])$ is the set of functions $f : [0,1] \to \mathbb{R}$ for which one has $f'(0) = qf(0)$ and $f(1) = 0$ for the case of Dirichlet actuation or $f'(1) = 0$ for the case of Neumann actuation.

Consequently, by the bounded invertibility of the backstepping transformation, for every $u_0 \in L^2(0,1)$, we can construct a solution $u \in C^0 \left( [0,\lim_{j \to \infty}(t_j)]; L^2(0,1) \right)$ with $u[t] \in H^2(0,1)$ for $t \in (0,\lim_{j \to \infty}(t_j))$ and $u \in C^1(\bar{I}; L^2(0,1))$ where $\bar{I} = [0,\lim_{j \to \infty}(t_j)) \setminus \{t_j : j = 0, 1, 2, \ldots\}$ which also satisfies (2.9)-(2.11) for all $t \in \bar{I}$.

2.3. Event-triggered gain scheduling. Let us consider the following Sturm-Liouville operator $B : D \to L^2(0,1)$ defined by

\begin{equation}
(Bh)(x) = -\varepsilon \frac{d^2 h}{dx^2}(x)
\end{equation}

for all $h \in D$ and $x \in (0,1)$ where $D \subset H^2([0,1])$ is the set of functions $h : [0,1] \to \mathbb{R}$ for which one has $h'(0) = qh(0)$ and $h(1) = 0$ for the case of Dirichlet actuation or $h'(1) = 0$ for the case of Neumann actuation.

We denote $\mu_1 < \mu_2 < \ldots < \mu_n < \ldots$ with $\lim_{n \to \infty}(\mu_n) = +\infty$ and $\phi_n(x) \in C^2([0,1],\mathbb{R})$, $(n = 1, 2, \ldots)$ the eigenvalues and the eigenfunctions, respectively, of the operator $B$.

2.3.1. Event-triggered gain scheduling with a static triggering condition. We introduce the first event-triggering strategy (or mechanism) for gain scheduling considered in this paper. The triggering condition is a state-dependent law and determines the time instants at which the reaction coefficient has to be sampled and thereby when the kernel computation/updating has to be done.

Definition 2.4 (Definition of the static event-triggered mechanism for gain scheduling). Let $K_j$ be the kernel satisfying (2.14)-(2.16) and let $f_j(t,x)$ be given by (2.25), $j \geq 0$. Let $\mu_1$ be the principal eigenvalue of the Sturm-Liouville operator $B$ (2.36). Let $R \in (0,1)$ be a design parameter and define

\begin{equation}
\mu := c + \varepsilon \mu_1
\end{equation}

The static event-triggered gain scheduler is defined as follows:

The times of events $t_j \geq 0$ with $t_0 = 0$ form a finite or countable set of times which is determined by the following rules for some $j \geq 0$:

a) if $\{t > t_j : \left\langle (K_j u[t]), (K_j f_j[t]) \right\rangle > \mu R \Vert(K_j u[t])\Vert^2 \} \neq \emptyset$ then the set of the times of the events is $\{t_0, \ldots, t_j\}$.

b) if $\{t > t_j : \left\langle (K_j u[t]), (K_j f_j[t]) \right\rangle > \mu R \Vert(K_j u[t])\Vert^2 \} \neq \emptyset$, then the next event time is given by:

\begin{equation}
t_{j+1} := \inf \{ t > t_j : \left\langle (K_j u[t]), (K_j f_j[t]) \right\rangle > \mu R \Vert(K_j u[t])\Vert^2 \}
\end{equation}

where $u[t]$ denotes the solution of (2.1)-(2.3) with (2.20) or (2.21) for $t \geq t_j$. 

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2.3.2. Event-triggered gain scheduling with a dynamic triggering condition. Inspired by [10] and [5], we introduce the second event-triggering mechanism for gain scheduling in this paper. It involves a dynamic variable which can be viewed as a filtered value of the static triggering condition in (2.38). With this strategy we expect to reduce updating times for the kernel scheduling and obtain larger inter-execution times.

Definition 2.5 (Definition of the dynamic event-triggered mechanism for gain scheduling). Let \( K_j \) be the kernel satisfying (2.14)-(2.16), \( f_j(t, x) \) be given by (2.25), \( j \geq 0 \) and \( \mu \) be given by (2.37). Let \( R \in (0, 1) \), \( \eta \geq 2\mu(1 - R) \) and \( \theta > 0 \) be design parameters.

The dynamic event-triggered gain scheduler is defined as follows:

The times of events \( t_j \geq 0 \) with \( t_0 = 0 \) form a finite or countable set of times which is determined by the following rules for some \( j \geq 0 \):

a) if \( \{ t > t_j : \langle (K_j u[t]), (K_j f_j[t]) \rangle - \mu R\| (K_j u[t])\|^2 > \frac{1}{\theta} m(t) \} = \emptyset \) then the set of the times of the events is \( \{ t_0, ..., t_j \} \).

b) if \( \{ t > t_j : \langle (K_j u[t]), (K_j f_j[t]) \rangle - \mu R\| (K_j u[t])\|^2 > \frac{1}{\theta} m(t) \} \neq \emptyset \), then the next event time is given by:

\[
(2.39) \quad t_{j+1} := \inf \{ t > t_j : \langle (K_j u[t]), (K_j f_j[t]) \rangle - \mu R\| (K_j u[t])\|^2 > \frac{1}{\theta} m(t) \}
\]

where \( u[t] \) denotes the solution of (2.1)-(2.3) with (2.20) or (2.21) for \( t > t_j \) and \( m \) satisfies the ordinary differential equation

\[
(2.40) \quad \dot{m}(t) = -\eta m(t) + \left( \mu R\| (K_j u[t])\|^2 - \langle (K_j u[t]), (K_j f_j[t]) \rangle \right), \quad \text{for} \quad t \geq t_j
\]

and we set \( m(t_j) = 0 \).

Remark 2.6. Let us remark that the static event-triggered strategy has only one design parameter (i.e. \( R \in (0, 1) \)) whereas the dynamic event-triggered strategy has three additional design parameters, namely, \( R \) (as in the static case), \( \eta \) and \( \theta \). Essentially, \( \eta \) adjusts the convergence rate of the filter (2.40) that can be characterized as \( \eta \geq 2\mu(1 - R) \). The parameter \( \theta \), on the other hand, can be selected to contribute to reduce updating times for the kernel scheduling and obtain larger inter-execution times.

The following result guarantees that the dynamic variable \( m(t) \) remains always positive between two successive triggering times. This fact is going to be helpful in the stability analysis of the closed-loop system.

Lemma 2.7. Under the definition of the event-triggered gain scheduling with dynamic trigger condition (2.39)-(2.40), it holds, for \( t \in [t_j, t_{j+1}) \), \( j \geq 0 \), that

\[
\frac{1}{\theta} m(t) + \mu R\| (K_j u[t])\|^2 - \langle (K_j u[t]), (K_j f_j[t]) \rangle \geq 0 \quad \text{and} \quad m(t) \geq 0.
\]

Proof. From definition of the the event-triggered gain scheduling with dynamic triggering condition (2.39)-(2.40), events are triggered to guarantee, for \( t \in [t_j, t_{j+1}) \), \( j \geq 0 \), that

\[
\frac{1}{\theta} m(t) + \mu R\| (K_j u[t])\|^2 - \langle (K_j u[t]), (K_j f_j[t]) \rangle \geq 0. \quad \text{This inequality in}
\]
conjunction with (2.40) yields:

$$\dot{m}(t) \geq - (\eta + \frac{1}{\theta}) m(t)$$

for which the Comparison principle can be used to guarantee $m(t) \geq 0$, for all $t \in [t_j, t_{j+1})$, $j \geq 0$ and provided that $m(t_j) \geq 0$.

Lemma 2.7 guarantees that $\frac{1}{\theta} m(t_{j+1}) + \mu R \|((K_j u(t_{j+1})\|^2 - \langle (K_j u(t_{j+1}), (K_j f_j(t_{j+1})) \rangle \geq 0$ and that $m(t_{j+1}) \geq 0$ when $t_{j+1} < +\infty$ (by continuity).

**Proposition 2.8.** If the time of the next event generated by (2.38) is finite, then the time of the next event generated by the dynamic event triggered mechanism (2.39)-(2.40) is strictly larger than the time of the next event generated by the static event triggered mechanism (2.38).

**Proof.** Without loss of generality we may assume that $j = 0$ (and consequently $t_0 = 0$). Notice that if $u(0) = 0$ then both the static and dynamic event triggering conditions give $t_1 = +\infty$. By assumption, the time of the next event generated by the static strategy is finite; therefore it holds that $u(0)$ is not zero. Consequently, $K_0 u(0)$ is not zero.

Let $t_1$ be the time of the next event generated by the static event triggered mechanism and let $T$ be the time of the next event generated by the dynamic one. We show next that $T > t_1$ by contradiction. Assume that $T \leq t_1$. Define

$$q(t) := \mu R \|K_0 u(t)\|^2 - \langle (K_0 u(t), (K_0 f_0(t)) \rangle$$

Then we have by virtue of (2.38), (2.42) $q(t) \geq 0$ for all $t \in [0, t_1]$ and by virtue of (2.39), (2.42) $q(T) = -\frac{1}{\theta} m(T)$, implying that $m(T) \leq 0$. Since $m(0) = 0$ and $\dot{m}(t) = -\eta m(t) + q(t)$ for all $t \in [0, T]$, we have

$$m(t) = \int_0^t \exp(-\eta(t-s))q(s)ds, \quad \text{for all } t \in [0, T]$$

Since $q(t) \geq 0$ for all $t \in [0, T]$ we get $m(T) \geq 0$ and thus we conclude that $m(T) = 0$. By continuity of $q(t)$ and the fact that $q(t) \geq 0$ for all $t \in [0, T]$, the integral $\int_0^T \exp(-\eta(T-s))q(s)ds$ is zero only if $q(t)$ is identically zero on $[0, T]$. However, that is not possible since $f_0(0) = 0$ (recall (2.7), (2.8) and (2.25)) and since $q(0) = \mu R \|K_0 u(0)\|^2 - \langle (K_0 u(0), (K_0 f_0(0)) \rangle = \mu R \|K_0 u(0)\|^2 > 0$. Thus, it must hold that $T > t_1$.

3. Analysis of the closed-loop system and main results. In this section we present our main results: the avoidance for the Zeno behavior \(^1\), the well-posedness and the exponential stability of the closed-loop system under boundary controller whose gains are scheduled according to the two event-triggered strategies.

3.1. Avoidance of the Zeno phenomenon.

3.1.1. Event-triggered gain scheduling with a static triggering condition.

\(^1\)We recall, the Zeno phenomenon means infinite triggering times in a finite-time interval. In practice, Zeno phenomenon would represent infeasible implementation into digital platforms since one would require to sample infinitely fast.
Proposition 3.1. Under (2.38), there exists a minimal dwell-time between two triggering times, i.e. there exists a constant \( \tau > 0 \) (independent of the initial condition \( u_0 \)) such that \( t_{j+1} - t_j \geq \tau \), for all \( j \geq 0 \). More specifically, \( \tau \) satisfies:

\[
\tau = \frac{1}{\varphi} \frac{\mu R}{G^2}
\]

with \( \mu = c + \varepsilon \mu_1 \) (recall (2.37) with \( \mu_1 \) being the principal eigenvalue of the Sturm-Liouville operator \( B \) (2.36)), \( R \in (0, 1) \) being the design parameter involved in the event-triggering condition (2.38), \( \varphi \) as in Assumption 2.1 and \( G \) given by (2.30).

Proof. Assume that an event occurs at \( t = t_{j+1} \). Then, from and using (2.12), continuity of all mappings involved with respect to time and the Cauchy-Schwarz inequality, the following more conservative estimate holds:

\[
\|w(t_{j+1})\| \|(K_j f_j(t_{j+1}))\| \geq \mu R \|w(t_{j+1})\|^2
\]

Using (2.13), (2.25), (2.27)-(2.30) we get from (3.2):

\[
G^2 \|w(t_{j+1})\|^2 \|e_j(t_{j+1})\|_{\infty} \geq \mu R \|w(t_{j+1})\|^2
\]

Therefore,

\[
G^2 \|e_j(t_{j+1})\|_{\infty} \geq \mu R
\]

By virtue of Assumption 2.1 in conjunction with (2.7) and (2.8), we obtain for all \( j \geq 0 \)

\[
G^2 \varphi(t_{j+1} - t_j) \geq \mu R
\]

from which we can deduce (using definition (3.1))

\[
t_{j+1} - t_j \geq \tau
\]

being \( \tau \) the minimal dwell-time (independent on initial conditions).

Proposition 3.1 allows us to conclude that \( \lim(t_j) = +\infty \) and thereby we can apply Proposition 2.3 to get the following result on the existence of solutions of the closed-loop system (2.1)-(2.4) with control (2.20), under the static event-triggered gain scheduler (2.38).

Corollary 3.2. For every initial condition \( u_0 \in L^2(0, 1) \), there exists a unique mapping \( u \in C^0(\mathbb{R}_+; L^2(0, 1)) \cap C^1(I; L^2(0, 1)) \), \( u[t] \in H^2(0, 1) \) for \( t > 0 \) that satisfies (2.1)-(2.4) with control (2.20) or (2.21), under the static event-triggered gain scheduler (2.38) for all \( t \in I \) and \( I = \mathbb{R}_+ \setminus \{t_j \geq 0, j = 0, 1, 2, \ldots \} \).

Proof. It is an immediate consequence of Proposition 2.3 and Proposition 3.1. Indeed, the solution is constructed (by the step method) iteratively between successive triggering times.

Remark 3.3. The minimal dwell-time depends on the rate of change of the reaction coefficient. A higher rate of change of the reaction coefficient (i.e., large \( \varphi \)) would give a smaller minimal dwell-time (i.e., more frequent event triggering). This is expected since a high rate of change of the reaction coefficient requires a more frequent update of the control law.
3.1.2. Event-triggered gain scheduling with a dynamic triggering condition. By virtue of Proposition 2.8, the inter-execution time for the dynamic event triggered mechanism (2.39)-(2.40) always exceeds the inter-execution time for the static event triggered mechanism (2.38). Due to Proposition 3.1, the Zeno phenomenon is immediately excluded.

Therefore, as in the static case, we can also conclude that \( \lim(t_j) = +\infty \) and thereby we can apply Proposition 2.3 to get the following result on the existence of solutions of the closed-loop system (2.1)-(2.4) with control (2.20) or (2.21), under the dynamic event-triggered gain scheduler (2.39)-(2.40).

**Corollary 3.4.** For every initial condition \( u_0 \in L^2(0,1) \), there exists a unique mapping \( u \in C^0(\mathbb{R}_+; L^2(0,1)) \cap C^1(I; L^2(0,1)) \), \( u[t] \in H^2(0,1) \) for \( t > 0 \) that satisfies (2.1)-(2.4) with control (2.20) or (2.21), under the dynamic event-triggered gain scheduler (2.39)-(2.40) for all \( t \in I \) and \( I = \mathbb{R}_+ \{ t_j \geq 0, j = 0, 1, 2, \ldots \} \).

3.2. Exponential stability analysis. We present next the stability results under our two event-triggered gain scheduling strategies.

3.2.1. Event-triggered gain scheduling with a static triggering condition.

**Theorem 3.5.** Under Assumption 2.1, if the following condition is fulfilled,

\[
\varphi < \frac{\mu^2 R (1 - R)}{G^2 \ln(G)},
\]

where \( \varphi, G, \mu, R \) are defined, respectively, in (2.6), (2.30), (2.37), (2.38); then, the closed-loop system (2.1)-(2.4) with control (2.20) or (2.21), under the static event-triggered gain scheduler (2.38), is globally exponentially stable. More specifically, there exists a constant \( \sigma > 0 \) such that:

\[
\|u[t]\| \leq G \exp(-\sigma t)\|u[0]\|, \text{ for all } t \geq 0
\]

**Proof.** By using the variational characterization of eigenvalues (see [34, Section 11.4]) in conjunction with (2.22)-(2.24),(2.25), the following estimate holds for all \( t \in (t_j, t_{j+1}) \):

\[
\frac{d}{dt} \left( \frac{1}{2} \|w_j[t]\|^2 \right) \leq -\mu \|w_j[t]\|^2 + \langle w_j[t], (K_j f_j[t]) \rangle
\]

where \( \mu = c + \varepsilon \mu_1 \) (recall (2.37) \( \mu_1 \) being the principal eigenvalue of the Sturm-Liouville operator (2.36)). We can rewrite (3.9) as follows:

\[
\frac{d}{dt} \left( \frac{1}{2} \|w_j[t]\|^2 \right) \leq -\mu(1 - R) \|w_j[t]\|^2 - \mu R \|w_j[t]\|^2 + \langle w_j[t], (K_j f_j[t]) \rangle
\]

where \( R \in (0,1) \) is the parameter involved in (2.38).

Therefore, from the definition of the static event-triggered gain scheduler, events are triggered to guarantee, \( \langle w_j[t], (K_j f_j[t]) \rangle \leq \mu R \|w[t]\|^2 \), for all \( t \in (t_j, t_{j+1}) \). Then, we obtain for all \( t \in [t_j, t_{j+1}) \):

\[
\|w_j[t]\|^2 \leq \exp(-2\mu(1 - R)(t - t_j)) \|w_j[t_j]\|^2
\]
Using (2.12), (2.13), (2.27)-(2.30) and (3.11), we get:

\[ \|u[t]\|^2 \leq G^2 \exp(-2\mu(1 - R)(t - t_j))\|u[t_j]\|^2 \]  \hspace{1cm} (3.12)

for all \( t \in [t_j, t_{j+1}) \). Since \( u \in C^0(\mathbb{R}_+; L^2(0, 1)) \), it follows that (3.12) holds for \( t = t_{j+1} \) as well, i.e.

\[ \|u[t_{j+1}]\|^2 \leq G^2 \exp(-2\mu(1 - R)(t_{j+1} - t_j))\|u[t_j]\|^2 \]  \hspace{1cm} (3.13)

Now, for all \( t \geq 0 \), an estimate of \( \|u[t]\| \) in terms of \( \|u[0]\| \) can be derived recursively, by using (3.6) and the fact that there have been \( j \) events and that \( j \tau \) units of time have (at least) been passed until \( t \). To that end, we can apply induction on \( j \) and prove that, for all \( j \geq 0 \),

\[ \|u[t_j]\|^2 \leq (G^j) \exp(-2\mu(1 - R)t_j)\|u[0]\|^2 \]  \hspace{1cm} (3.14)

and that \( t_j \geq j \tau \). Let \( j \geq 0 \) be given (arbitrary) and \( t \in [t_j, t_{j+1} \rangle \) (arbitrary). We obtain from (3.12),(3.13) and (3.14):

\[ \|u[t]\|^2 \leq (G^2)^{j+1} \exp(-2\mu(1 - R)t)\|u[0]\|^2 \]  \hspace{1cm} (3.15)

Moreover, since \( j \leq \frac{t}{\tau} \), it holds:

\[ \|u[t]\|^2 \leq G^2 \exp\left(-\left(2\mu(1 - R) - \frac{2ln(\bar{G})}{\varphi}\right)t\right)\|u[0]\|^2 \]  \hspace{1cm} (3.16)

In light of condition (3.7) in conjunction with (3.6) where \( \tau = \frac{1}{\nu} \frac{\varphi R}{G^2} \), we finally obtain:

\[ \|u[t]\| \leq G \exp(-\sigma t)\|u[0]\|, \text{ for all } t \geq 0 \]  \hspace{1cm} (3.17)

with \( \sigma = \frac{\mu^2 R(1 - R) - \varphi G^2 \ln(G)}{\mu R} > 0 \). This concludes the proof.

\textbf{Remark 3.6.} Notice that \( \mu^2 R(1 - R) - \varphi G^2 \ln(G) > 0 \) holds true provided that \( \varphi \) is sufficiently small (this corresponds to the case where \( \lambda(t, x) \) is slowly time-varying coefficient). In addition, it is worth remarking that we can select \( R = \frac{1}{2} \) in order to maximize the allowable upper bound \( \varphi \). Nevertheless, different values of \( R \) may be used in practice since the obtained estimates are conservative. The proof of Theorem 2 provides a (conservative) explicit estimate of the convergence rate \( \sigma > 0 \). The obtained estimate shows that the smaller \( \varphi \) is (i.e., the slower the change of the reaction coefficient), the higher the convergence rate is.

### 3.2.2. Event-triggered gain scheduling with a dynamic triggering condition.

**Theorem 3.7.** If condition (3.7) of Theorem 3.5 holds, then, the closed-loop system (2.1)-(2.4) with control (2.20) or (2.21), under dynamic event-triggered gain scheduler (2.39)-(2.40) is globally exponentially stable. More specifically, there exists a constant \( \sigma > 0 \) such that:

\[ \|u[t]\| \leq G \exp(-\sigma t)\|u[0]\|, \text{ for all } t \geq 0 \]  \hspace{1cm} (3.18)

\textbf{Proof.} An estimate of the time-derivative of the following function \( W(t) := \frac{1}{2}\|w_j[t]\|^2_2 + m(t) \) along the solutions (2.22)-(2.24),(2.25),(2.40) is given by:

\[ \dot{W}(t) \leq -\mu\|w_j[t]\|^2_2 + \left<w_j[t], (K_j f_j[t])\right> - \eta m(t) - \left<w_j[t], (K_j f_j[t])\right> + \mu R\|w_j[t]\|^2_2 \]  \hspace{1cm} (3.19)
which can be rewritten as follows:

\[ W(t) \leq -\mu(1 - R)(\|w_j[t]\|^2 + 2m(t)) - m(t)(\eta - 2\mu(1 - R)) \tag{3.20} \]

By Lemma 2.7, we guarantee that \(m(t) \geq 0\) and since \(\eta \geq 2\mu(1 - R)\) (recall Definition 2.5), thus we get:

\[ W(t) \leq -2\mu(1 - R)W(t) \tag{3.21} \]

Therefore, we obtain for \(t \in [t_j, t_{j+1})\):

\[ \frac{1}{2}\|w_j[t]\|^2 + m(t) \leq \exp(-2\mu(1 - R)(t - t_j)) \left( \frac{1}{2}\|w_j[t_j]\|^2 + m(t_j) \right) \tag{3.22} \]

Notice that \(\frac{1}{2}\|w_j[t]\|^2 \leq \frac{1}{2}\|w_j[t]\|^2 + m(t)\) and that by Definition 2.5, \(m(t_j) = 0\). Therefore, from (3.22) we have, for all \(t \in [t_j, t_{j+1})\):

\[ \|w_j[t]\|^2 \leq \exp(-2\mu(1 - R)(t - t_j))\|w_j[t_j]\|^2 \tag{3.23} \]

The remaining part of the proof follows the same reasoning as the proof of Theorem 3.5 (see from (3.12)). This concludes the proof.

\[ \text{Remark 3.8.} \text{ The function } W(t) \text{ is monotonically decreasing (see (3.21)), for all } t \in [t_j, t_{j+1}). \text{ However, the function } \frac{1}{2}\|w_j[t]\|^2 \text{ may not be monotonically decreasing on that interval. The design parameter } \theta \text{ involved in the dynamic event-triggering condition (2.39)-(2.40) and also discussed in Remark 2.6, allows to limit the potential increase of } \frac{1}{2}\|w_j[t]\|^2. \text{ Indeed, since events are triggered to guarantee } \frac{1}{2}m(t) + \mu R|||w_j[u(t)]||| - \langle (K_j u[t]), (K_j f_j[t]) \rangle \geq 0, \text{ it holds that:} \]

\[ \frac{d}{dt} \left( \frac{1}{2}\|w_j[t]\|^2 \right) \leq -\mu(1 - R)||w_j[t]\|^2 + \frac{1}{\theta}m(t) \]

Notice that the larger the value of \(\theta\), the more limited the increase. We approach then to the case as we were dealing with the static event-triggered gain scheduler.

4. Numerical simulations. We illustrate the results by considering (2.1)-(2.4) with \(\varepsilon = 1\), \(q = +\infty\) and initial condition \(u_0(x) = 2(x - x^2)\). For numerical simulations, the state of the system has been discretized by divided differences on a uniform grid with the step \(h = 0.02\) for the space variable. The discretization with respect to time was done using the implicit Euler scheme with step size \(\Delta t = h^2\). We run simulations on a frame of 2s. We choose the time- and space-varying coefficient \(\lambda(t, x)\) to have a simple form as \(\lambda(t, x) = \lambda_c + \lambda_t(t) + \lambda_x(x)\). More specifically:

\[ \lambda(t, x) = 10 + \frac{50}{\cosh^2(5(t - 1))} + 7 \cos(5\pi t) + \frac{50}{\cosh^2(5x)}, \quad t > 0, \quad x \in [0, 1] \tag{4.1} \]

which has a profile depicted in Figure 1. We stabilize the closed-loop system (2.1)-(2.4) under Dirichlet actuation with boundary control (2.20) whose kernel gains satisfy (2.14)-(2.16) and are scheduled according to the two event-triggered mechanisms we introduced in Definition 2.4 (static-based triggering condition) and Definition 2.5 (dynamic-based triggering condition).

The parameters of the triggering conditions are set \(R = 0.15\), \(\mu = c + \varepsilon \pi^2 = \pi^2\) with \(c = 0\). In addition, \(\eta = 16.7\) and \(\theta = 0.15\).
Fig. 1. Profile of the time- and space- varying reaction coefficient $\lambda(t, x) = 10 + \frac{50}{\cosh(5(t_j-1))} + 7 \cos(5\pi t_j) + \frac{50}{\cosh(5x)}$ for the reaction-diffusion system (2.1)-(2.4).

Fig. 2. Sampled version of the profile of the time- and space- varying reaction coefficient (4.1), i.e. $\lambda(t_j, x)$ for all $\{t_j\}_{j \in \mathbb{N}}$ according to the static event-triggered gain scheduler (2.38) (depicted on the left) and the dynamic event-triggered gain scheduler (2.39)-(2.40) (depicted on the right).

From (2.7), $b_j(x) = \lambda(t_j, x) = \tilde{\lambda}_j + \frac{50}{\cosh(5x)}$, where $\tilde{\lambda}_j := 10 + \frac{50}{\cosh(5(t_j-1))} + 7 \cos(5\pi t_j)$ with $\{t_j\}_{j \in \mathbb{N}}$ according to (2.38) (static) or (2.39)-(2.40) (dynamic). In either cases, kernels $K_j$ satisfying (2.14)-(2.16), for all $t \in [t_j, t_{j+1})$, admit a closed-form solution which is given as follows [32, Section VIII.E]:

$$K_j(x, y) = -\tilde{\lambda}_j y \frac{I_1 \left( \sqrt{\tilde{\lambda}_j (x^2 - y^2)} \right)}{\sqrt{\tilde{\lambda}_j (x^2 - y^2)}} - 5 \tanh(5y) I_0 \left( \sqrt{\tilde{\lambda}_j (x^2 - y^2)} \right)$$

where $I_m(\cdot), m \in \mathbb{Z}$ is a modified Bessel function of the first kind of order $m$. Figure 2 shows the event-triggered sampled version of the profile of the time- and space- varying reaction coefficient (4.1) for all $t \in [t_j, t_{j+1}), j \geq 0$, according to the static event-triggered gain scheduler (2.38) (depicted on the left) and the dynamic event-triggered gain scheduler (2.39)-(2.40) (depicted on the right). Hence, the kernel updating is done on events and aperiodically. One of the main features of this approach is that the kernel of the control does not need to be computed using the method of successive approximations to solve a PDE kernel which involves a time- and space- varying...
coefficient. As motivated throughout the paper, it suffices to schedule the kernel in a suitable way and only when needed while using a simpler kernel (in some cases admitting closed-form solution).

Figures 3 and 4 show the time-evolution of the $L^2$ norm of the closed-loop system (2.1)-(2.4), (4.1) and the time-evolution of the boundary control (2.20), respectively, under static event-triggered gain scheduler (2.38) (red dashed line) and dynamic event-triggered gain scheduler (2.39)-(2.40) (blue line). For both figures, on the right, there are zooms of the two curves to illustrate the difference. It can be observed that under the two strategies, the behavior is similar with same theoretical guarantees. Finally, we run simulations for 100 different initial conditions given by

\[ u_0(x) = \sqrt{2/n} \sin(\sqrt{n}\pi x) + \sqrt{n}(x - x^2), \text{ for } n = 1, \ldots, 100 \text{ on a frame of } 2s. \]

We compare the static event triggered mechanism with respect to the dynamic one while computing the inter-execution times between two triggering times. We compare several cases by tuning different parameters. For all cases, $\eta$ is selected as $\eta = 2\mu(1-R)$. The mean value of the numbers of events generated under the two strategies is reported in Table 1. The mean value and coefficient of variation (ratio between the standard deviation and the mean value) of inter-execution times for both approaches are reported in Tables 2 and 3, respectively. In addition, Figure 5 shows the density of the inter-execution times (axis in logarithmic scale). The red bars correspond to the inter-execution times under the static event triggered mechanism (2.38); whereas the blue bars correspond to the dynamic event triggered mechanism (2.39)-(2.40) resulting in larger inter-execution times. Therefore, it can be asserted that, as expected, with the dynamic triggering condition one obtains larger inter-execution times and we can reduce the number of events rendering the strategy slightly less conservative. In general, dynamic event-triggered strategies may offer benefits with respect to static strategies as in the framework of even-triggered control (in finite and infinite dimensional settings).

5. Conclusion. In this paper, we have addressed the problem of exponential stabilization of a reaction-diffusion PDE with time- and space- varying reaction coefficient. The boundary control design relies on the backstepping method and the gains are computed/updated on events according to two event-triggered gain scheduling schemes. Two event-triggered strategies are prosed for gain scheduling: static and
dynamic. The latter involves a dynamic variable that can be viewed as the filtered value of the static one. It has been observed that under this strategy it is possible to reduce the number of events for the gain scheduling. We show that under the two proposed event-triggered gain scheduling schemes Zeno behavior is avoided, which allows to prove well-posedness as well as the exponential stability of the closed-loop system.

Our approach can be seen as an efficient way of kernel computation as it is scheduled aperiodically, when needed and relying on the current state information of
Table 1

|                  | Static ET | Dynamic ET (θ = 100) | Dynamic ET (θ = 1) | Dynamic ET (θ = 0.015) |
|------------------|-----------|----------------------|-------------------|------------------------|
| R = 0.15, η = 16.7 | 39.93     | 37.08                | 29.8              | 17.01                  |
| R = 0.5 η = 9.86  | 18.58     | 17.6                 | 16.02             | 8.99                   |

Table 2

|                  | Static ET | Dynamic ET (θ = 100) | Dynamic ET (θ = 1) | Dynamic ET (θ = 0.015) |
|------------------|-----------|----------------------|-------------------|------------------------|
| R = 0.15, η = 16.7 | 0.0460    | 0.0354               | 0.0374            | 0.0546                 |
| R = 0.5 η = 9.86  | 0.0738    | 0.0521               | 0.0582            | 0.1112                 |

the closed-loop system. This work constitutes an effort towards the “robustification” of boundary controllers designed under backstepping method.

In future work, we expect to combine this result with event-triggered control strategies for boundary controlled reaction-diffusion PDEs systems recently introduced in [6] (which deals with constant reaction coefficient only). The results in this paper may suggest that the triggering times for gain scheduling may be synchronized with the time instants for control updating. The control is going to be piecewise constant and not piecewise continuous as in the present work. This would represent a more realistic way of actuation on the PDE system towards digital realizations.

Appendix A. Proof of Theorem 2.2.

Proof. It suffices to show that there exists k > 0 such that for each w0 ∈ L2(0, 1) and T > 0 the initial value problem

\[\begin{align*}
\dot{y}[t] + (A + kI)y[t] &= F(t)y[t] \\
y[0] &= w_0
\end{align*}\]  
(A.1)

has a unique classical solution on [0, T] in the sense described in [27] where A : D → L2(0, 1) is the Sturm-Liouville operator defined by the following formula for every f ∈ D :

\[(Af)(x) = -\varepsilon f''(x) + cf(x), \text { for } x \in (0, 1)\]  
(A.2)

Notice that any solution y[t] of (A.1) provides a solution of the initial-boundary value problem (2.31) with (2.34) by means of the formula w[t] = exp(kt)y[t] and any solution w[t] of the initial boundary value problem (2.31) with (2.34) provides a solution of the initial value problem (A.1) by means of the formula y[t] = exp(-kt)w[t].

Since -A is the infinitesimal generator of a C₀ semigroup $S(t), t ≥ 0$ on $L^2(0, 1)$ and since for each $t ≥ 0$ the operator $F(t) : L^2(0, 1) → L^2(0, 1)$ is a linear bounded operator for which there exist constants $Ω_1, Ω_2 > 0$ such that (2.32) and (2.33) hold, it
Coefficient of variation of inter-execution times for static event-triggered gain scheduler (2.38) and for dynamic event-triggered gains scheduler (2.39)-(2.40).

|                  | Static ET | Dynamic ET ($\theta = 100$) | Dynamic ET ($\theta = 1$) | Dynamic ET ($\theta = 0.015$) |
|------------------|-----------|-----------------------------|---------------------------|-------------------------------|
| $R = 0.15$, $\eta = 16.7$ | 1.9814    | 2.110                       | 2.6251                    | 2.3457                        |
| $R = 0.5$, $\eta = 9.86$     | 2.093     | 2.693                       | 2.8905                    | 2.0965                        |

follows from Theorem 1.2 on page 184 in [27] that there exists a unique mild solution $y \in C^0\left([0, T]; L^2(0, 1)\right)$ of the initial value problem (A.1), i.e.,

\[
(A.3) \quad y[t] = \exp(-kt)S(t)w_0 + \int_0^t \exp(-k(t-s))S(t-s)\mathcal{F}(s)y[s]ds, \text{ for all } t \in [0, T]
\]

Theorem 2.2 on page 4 in [27] implies the existence of constants $M, \omega > 0$ such that the estimate $\|S(t)\| \leq M \exp(\omega t)$ holds for all $t \geq 0$. Exploiting the previous estimate in conjunction with (A.3) and (2.32), we get for all $t \in [0, T]$:

\[
(A.4) \quad \|y[t]\| \leq M \exp(-(k - \omega)t)\|w_0\| + \frac{Mc}{k - \omega} \max_{0 \leq s \leq T}(\|y[s]\|), \text{ for all } t \in [0, T]
\]

Selecting $k > 0$ so that $\frac{Mc}{k - \omega} < 1$, estimate (A.4) implies the following estimate:

\[
(A.5) \quad \max_{0 \leq s \leq T}(\|y[s]\|) \leq \left(1 - \frac{Mc}{k - \omega}\right)^{-1} M \|w_0\|
\]

Notice that the mild solution $y \in C^0\left([0, T]; L^2(0, 1)\right)$ of the initial value problem (A.1) is a mild solution of the inhomogeneous initial value problem

\[
(A.6) \quad \dot{y}[t] + (A + kI)y[t] = g(t) \\
\quad y[0] = w_0
\]

with $g(t) = \mathcal{F}(t)y[t]$ for $t \in [0, T]$. Inequalities (A.5) and (2.32) imply that $g \in L^p\left([0, T]; L^2(0, 1)\right)$ for every $p \in [1, +\infty)$. Since $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ on $L^2(0, 1)$, it follows from Theorem 3.1 on page 110 in [27] that for every $p \in (1, +\infty)$, the mapping $t \rightarrow y[t]$ is locally Hölder continuous on $(0, T]$ with exponent $\frac{p-1}{p}$. Using (2.32), (2.33) and the fact that $g(t) = \mathcal{F}(t)y[t]$ for $t \in [0, T]$, it follows that for every $p \in (1, +\infty)$, the mapping $t \rightarrow g[t]$ is locally Hölder continuous on $(0, T]$ with exponent $\frac{p-1}{p}$. By virtue of Corollary 3.3 on page 113 in [27] we conclude that $y[t]$ is the unique classical solution of (A.1).

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