Threshold Corrections in $K3 \times T2$

Heterotic String Compactifications

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Abstract

We consider compactifications of the heterotic string on $K3 \times T2$ so that the resulting theory in $d = 4$ space-time dimensions has $N = 2$ supersymmetry. The gravitational and gauge coupling constants of the low-energy effective theory receive threshold corrections from loops of super-heavy string states. We calculate these corrections for the case when the $K3$-surface is a $\mathbb{Z}_n$ orbifold of a four torus $T4$. The results are used to determine the one-loop prepotential $F_0^{(1)}$ for the vector multiplets and the gravitational coupling $F_1^{(1)}$.

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1. Introduction

Theories in $d = 4$ space-time dimensions with $N = 2$ supersymmetry have proven to possess a rich structure while still being simple enough to be tractable with present methods. In the case of string theory, the gravitational and gauge coupling constants of the low-energy effective supergravity theory have received a lot of interest. Their one-loop dependence on the momentum scale and the moduli is given by

$$\frac{1}{g_{\text{grav}}^2(p^2)} = 24\text{Re} \left( -iS + \frac{1}{16\pi^2} \Delta_{\text{univ}}^{\text{grav}} \right) + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{1}{16\pi^2} \Delta_{\text{grav}}, \quad (1.1)$$

$$\frac{1}{g_{\text{gauge}}^2(p^2)} = \text{Re} \left( -iS + \frac{1}{16\pi^2} \Delta_{\text{univ}}^{\text{gauge}} \right) + \frac{b_{\text{gauge}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{1}{16\pi^2} \Delta_{\text{gauge}},$$

where $S$ is the dilaton and $\Delta_{\text{univ}}$ is a certain universal contribution related to the Green-Schwarz anomaly cancellation term. The coefficients $b_{\text{grav}}$ and $b_{\text{gauge}}$ are related to the one-loop beta-functions and are thus determined by the spectrum of massless $N = 2$ multiplets $[1][2]$:

$$b_{\text{grav}} = 46 + 2(n_H - n_V) \quad (1.2)$$

$$b_{\text{gauge}} = 2\text{Tr}_H(Q^2) - 2\text{Tr}_V(Q^2).$$

Here $n_H$ and $n_V$ denote the number of massless hyper multiplets and vector multiplets respectively, and $Q$ is some generator of the simple factor in the gauge group under consideration. The effect of integrating out super-heavy string states is summarized by the threshold corrections $\Delta_{\text{grav}}$ and $\Delta_{\text{gauge}} [1][2][3][4]$. They are given at one-loop in terms of the fundamental domain integrals

$$\Delta_{\text{grav}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \frac{-i}{\eta^2(\tau)} \text{Tr} \left\{ J_0 e^{i\pi J_0} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} \left( E_2(\tau) - \frac{3}{\pi \tau_2} \right) \right\} - b_{\text{grav}} \right]$$

$$\Delta_{\text{gauge}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \frac{-i}{\eta^2(\tau)} \text{Tr} \left\{ J_0 e^{i\pi J_0} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} \left( Q^2 - \frac{1}{8\pi \tau_2} \right) \right\} - b_{\text{gauge}} \right] \quad (1.3)$$

respectively. The traces are over the Ramond sector of the internal conformal field theory, and $J_0$ denotes the $U(1)$ generator of the $N = 2$ superconformal algebra. The functions $\eta$ and $E_2$ are the Dedekind eta function and the Eisenstein series respectively. Throughout this paper, subscripts 1 and 2 on a complex quantity such as the modular parameter $\tau$ are used to denote its real and imaginary parts respectively. Given these threshold corrections, the one-loop prepotential $\mathcal{F}^{(1)}_0$ for the vector multiplets and the gravitational
coupling $F^{(1)}_1$ can be determined by comparing the string theory expressions (1.1) and (1.3) with equations arising from considerations of holomorphicity and duality transformation properties in $N = 2$ supersymmetric field theories [4][5][6].

We will be considering compactifications of the ten-dimensional heterotic string to four space-time dimensions with $N = 2$ extended space-time supersymmetry. This means that the right-moving internal $\bar{c} = 9$ conformal field theory must decompose into a $\bar{c} = 3$ piece and a $\bar{c} = 6$ piece with $N = 2$ and $N = 4$ world-sheet supersymmetry respectively [7]. The geometrical interpretation is that the six-dimensional internal space is a direct product of a two-torus $T2$ and a $K3$-surface. Suppose the gauge bundle on the $K3$ surface leaves a rank $s$ subgroup of $E_8 \times E_8$ unbroken. Wilson line and toroidal moduli then lead to a Narain moduli space $\mathcal{N}^{s+2,2}$ for vectormultiplets. Here $\mathcal{N}^{s+2,2} = O(\Gamma^{s+2,2})/O(s + 2, 2; R)/K$, where $O(\Gamma^{s+2,2})$ is a discrete automorphism group of the lattice and $K$ is the maximal compact subgroup. Our main object will be to calculate the threshold corrections $\Delta_{\text{gauge}}$ and $\Delta_{\text{grav}}$ as functions of these moduli. Such calculations are feasible essentially because the threshold corrections only depend on the elliptic genus of the $\bar{c} = 6$ theory and therefore can be performed in some convenient limit where this theory becomes free, such as an orbifold limit.

The standard embedding of the spin connection in the gauge group generically breaks $E_8 \times E_8 \to E_7 \times E_8$. The $E_7$ can be completely Higgsed leaving a rank 12 group. The vectormultiplet moduli space is then $\mathcal{N}^{s+2,2}$ with $s = 8$. The calculation of $\Delta_{\text{gauge}}$ and $F^{(1)}_0$ was performed recently in this case for a $Z_2$ orbifold in [8] by generalizing techniques for performing fundamental domain integrals introduced in [9]. Ref. [8] also studied in detail the subspace with unbroken $E_8$ corresponding to $s = 0$. The first gravitational coupling was evaluated using the same methods for a special case $s = 0$ in [10]. The methods used in [8] apply to a wider class of backgrounds than the special cases examined in [8][10], and in this paper we carry out the calculations for a class of $Z_n$ orbifolds. The results of [8][10] left room for generalization even for the standard embedding. In this case one can begin with the group $E_7 \times SU(2) \times E_8$ in the orbifold limit and consider turning on Wilson lines going to a Coulomb branch of $E_7 \times E_8$ but leaving the $SU(2)$ unbroken. We compute the prepotential on the resulting vectormultiplet moduli space $\mathcal{N}^{17,2}$ by computing the threshold correction to the $SU(2)$ coupling. We can then Higgs the $SU(2)$ by turning on $K3$ moduli. From the calculations below we can also find the prepotential and gravitational coupling for other topologies of the gauge group.
It should be noted that these methods apply in principle to an even larger class of backgrounds. For example, suppose that the lattice $\Gamma^{s+2,2} \in N^{s+2,2}$ is $SO(s+2,2)$-related to a lattice of the form $\Gamma^{s+2,2} = \Pi[\tilde{\Gamma} + \delta]$, where $\tilde{\Gamma} \cong \Gamma_{s,0} \oplus \Pi^{2,2}$ for some even rank $(s,0)$ lattice $\Gamma_{s,0}$ and the $\delta$ are orthogonal to $\Pi^{2,2}$. The modular invariant integrands of (1.3) take the form

$$\sum_i \left\{ \sum_{p \in \tilde{\Gamma} + \delta_i} q^\frac{1}{2} p_1^2 \bar{q}^\frac{1}{2} q_2^2 \right\} f_i(q)$$

(1.4)

where $f_i(q)$ form a representation of the modular group of weight $-s/2$. Under such conditions the method of [9] continues to apply with the result that the threshold correction $\Delta$ can be decomposed into four distinct contributions:

$$\Delta = \Delta^{\text{log}} + \Delta^{\text{const}} + \Delta^{\text{rat}} + \Delta^{\text{transc}}.$$  

(1.5)

Here $\Delta^{\text{log}}$ and $\Delta^{\text{const}}$ are related to the Kähler potential of the moduli space in a simple way. $\Delta^{\text{transc}}$ is an interesting sum of polylogarithms of exponentials of the moduli, weighted by the Fourier coefficients $c_i(x)$ of $f_i(q)$. The most subtle term is $\Delta^{\text{rat}}$, which is a rational function of the moduli. In order to obtain ‘nice’ automorphic forms (for example, infinite products with a Weyl vector) in the threshold corrections it is necessary that the rational terms obey certain non-trivial identities. It is far from obvious that this will happen in general, but we show in section four that these identities indeed follow from $N = 2$ supersymmetry. Our main results are the formulas (4.1) - (4.6) for the running coupling constants and the expressions (4.13) and (4.14) for the prepotential and the gravitational coupling respectively.

The outline of this paper is as follows: In section two, we present the calculation of the threshold corrections in the case of $\mathbb{Z}_n$ orbifolds. In section three, we specialize the results to the fundamental chamber of the moduli space. The one-loop prepotential for the vector multiplets and the gravitational coupling are calculated in section four. The details of the calculation in section two are discussed in Appendix A. In Appendix B, we consider the case of the standard embedding of the spin connection in the gauge group in somewhat more detail.

While this manuscript was being typed, a paper by T. Kawai appeared [15] which has some overlap with the present paper.

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1 In an intriguing set of papers [11] [12] the same methods were applied to obtain interesting product formulae for automorphic forms, such as the product formula for the Siegel automorphic form $\Delta_5$, due to [13]. Unfortunately, the relevance of the expressions in [11] [12] to physical threshold corrections is unclear. Nevertheless, the results have been applied in a very interesting way to black hole physics [14].
2. The threshold corrections for $Z_n$ orbifolds

We consider $K3$-surfaces which can be obtained as $Z_n$ orbifolds of some four-torus $T^4$. Introducing complex coordinates $z^1, z^2$ for $T^4$, we let the group $Z_n$ act as

\[
\begin{align*}
    z^1 &\to e^{2\pi i a/n}z^1 \\
    z^2 &\to e^{-2\pi i a/n}z^2
\end{align*}
\]

for $a \in \mathbb{Z}$ mod $n$. Tori which admit such a $Z_n$ symmetry exist for $n = 2, 3, 4, 6$. As usual, the Hilbert space of the theory decomposes into different twist sectors labeled by $a = 0, \ldots, n - 1$. We will also have to project onto $Z_n$ invariant states in the partition function, which is equivalent to inserting a group element indexed by $b$ in the ‘time’ direction, summing over $b = 0, \ldots, n - 1$ and dividing by $n$.

In the bosonized formulation of the heterotic string, the gauge degrees of freedom are described by sixteen left-moving bosons. Together with the two left-moving and two right-moving bosons from the two-torus $T^2$, these fields take their values on a torus given by an even self-dual lattice $\Gamma^{18,2}$ of the indicated signature. To cancel the space-time anomaly, we let $Z_n$ act by a shift $\frac{a}{n}\gamma$ on the momentum, where $\gamma \in \Gamma^{18,2}$. The contribution to the integrands in (1.3) in the $(a, b)$ sector from these degrees of freedom is then

\[
Z^{\text{torus}}_{a,b}(\tau, \bar{\tau}) = 2e^{-2\pi i \frac{ab}{n^2}\gamma^2} \eta^{-18}(\tau) \sum_{p \in \Gamma^{18,2} + \frac{a}{n}\gamma} e^{2\pi i \frac{b}{n^2} p \cdot \gamma} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2} \left( \frac{(p \cdot Q)^2}{Q \cdot Q} - \frac{1}{4\pi \tau_2} \right)
\]

for $\Delta_{\text{grav}}$ and

\[
\hat{Z}^{\text{torus}}_{a,b}(\tau, \bar{\tau}) = e^{-2\pi i \frac{ab}{n^2}\gamma^2} \eta^{-18}(\tau) \times \sum_{p \in \Gamma^{18,2} + \frac{a}{n}\gamma} e^{2\pi i \frac{b}{n^2} p \cdot \gamma} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2} \left( \frac{(p \cdot Q)^2}{Q \cdot Q} - \frac{1}{4\pi \tau_2} \right)
\]

for $\Delta_{\text{gauge}}$. Notice that there are no right-moving oscillator contributions because of a cancellation between bosons and fermions. It is easy to check that

\[
\begin{align*}
    Z^{\text{torus}}_{a,b+n}(\tau, \bar{\tau}) &= Z^{\text{torus}}_{a,b}(\tau, \bar{\tau}) \\
    \hat{Z}^{\text{torus}}_{a,b+n}(\tau, \bar{\tau}) &= \hat{Z}^{\text{torus}}_{a,b}(\tau, \bar{\tau})
\end{align*}
\]

and that

\[
\begin{align*}
    Z^{\text{torus}}_{a,b}(\tau + 1, \tau + 1) &= \exp 2\pi i \left( \frac{1}{2} \frac{a^2}{n^2}\gamma^2 - \frac{3}{4} \right) Z^{\text{torus}}_{a,a+b}(\tau, \bar{\tau}) \\
    Z^{\text{torus}}_{a,b}(-1/\tau, -1/\bar{\tau}) &= \exp 2\pi i \left( \frac{ab}{n^2}\gamma^2 \right) (i\bar{\tau})^{1/2} Z^{\text{torus}}_{b,-a}(\tau, \bar{\tau}) \\
    \hat{Z}^{\text{torus}}_{a,b}(\tau + 1, \tau + 1) &= \exp 2\pi i \left( \frac{1}{2} \frac{a^2}{n^2}\gamma^2 - \frac{3}{4} \right) \hat{Z}^{\text{torus}}_{a,a+b}(\tau, \bar{\tau}) \\
    \hat{Z}^{\text{torus}}_{a,b}(-1/\tau, -1/\bar{\tau}) &= \exp 2\pi i \left( \frac{ab}{n^2}\gamma^2 \right) (-i\tau)(i\bar{\tau})^{1/2} \hat{Z}^{\text{torus}}_{b,-a}(\tau, \bar{\tau}).
\end{align*}
\]
The modular transformation laws for $\hat{Z}_{a,b}^{\text{torus}}(\tau, \bar{\tau})$ require that the charge $Q$ be purely left-moving. Physically, this means that the unbroken gauge group contains at least an $SU(2)$ factor.

Next, we consider the contributions to the integrands in (1.3) from the degrees of freedom associated with the $K3$ surface. We begin with untwisted sector, where $a = 0$. The contributions from the non-zero modes of the right-moving bosons and fermions cancel against each other, so we only get a factor $16 \sin^2 \pi \frac{b}{n}$ from the fermionic zero-modes. This is the well known holomorphicity of the elliptic genus. The contribution from the left-movers is

$$q^{-\frac{1}{k}} \prod_{j=1}^{\infty} \left( 1 - e^{2\pi i \frac{k}{n} q^j} \right)^{-2} \left( 1 - e^{-2\pi i \frac{k}{n} q^j} \right)^{-2}. \quad (2.6)$$

The contributions in the twisted sectors, where $a \neq 0$, are now determined by the requirement that the integrands

$$- i \frac{\tau_2}{\eta^2(\tau)} \text{Tr} \left\{ J_0 e^{i\pi J_0 q L_0 c/24 \bar{q} \bar{\bar{L}}_0 - \bar{c}/24} \left( E_2(q) - \frac{3}{\pi \tau_2} \right) \right\}$$

$$= \frac{\tau_2}{\eta^2(\tau)} \sum_{a,b} \frac{1}{n} Z_{a,b}^{K3}(\tau) Z_{a,b}^{\text{torus}}(\tau, \bar{\tau}) \left( E_2(q) - \frac{3}{\pi \tau_2} \right) \quad (2.7)$$

and

$$- i \frac{\tau_2}{\eta^2(\tau)} \text{Tr} \left\{ J_0 e^{i\pi J_0 q L_0 c/24 \bar{q} \bar{\bar{L}}_0 - \bar{c}/24} \left( Q^2 - \frac{1}{8\pi \tau_2} \right) \right\}$$

$$= \frac{\tau_2}{\eta^2(\tau)} \sum_{a,b} \frac{1}{n} Z_{a,b}^{K3}(\tau) \hat{Z}_{a,b}^{\text{torus}}(\tau, \bar{\tau}) \quad (2.8)$$

be modular invariants. In general, we can write

$$Z_{a,b}^{K3}(\tau) = k_{a,b} \ q^{-\left( \frac{1}{k} \right)^2} \eta^2(\tau) \Theta_1^{-2} \left( \frac{a}{n} + \frac{b}{n} | \tau \right), \quad (2.9)$$

where $\eta(\tau)$ and $\Theta_1(\nu | \tau)$ are the Dedekind eta function and the Jacobi theta function respectively and $k_{a,b}$ are some constants. From the above, it follows that in the untwisted sector

$$k_{0,b} = 64 \sin^4 \pi \frac{b}{n}. \quad (2.10)$$

The $k_{a,b}$ for $a \neq 0$ are now determined by modular invariance through the relations

$$\frac{k_{a,b}}{k_{a,a+b}} = e^{i\pi \frac{a^2}{n}(2-\gamma^2)}$$

$$\frac{k_{a,b}}{k_{b,-a}} = e^{-2\pi i \frac{ab}{n}(2-\gamma^2)}. \quad (2.11)$$
These relations are only consistent provided that the level matching condition

$$\gamma^2 - 2 \in 2n\mathbb{Z} \quad (2.12)$$

is fulfilled.

The $\Gamma^{18,2}$ lattice is obtained by an $SO(18,2)$ rotation of some standard lattice, which we take to be of the form $II^{16,0} \oplus II^{1,1} \oplus II^{1,1}$. Here $II^{16,0}$ is a sixteen-dimensional, even self dual Euclidean lattice, i.e. either the $E_8 \times E_8$ root lattice or the $\text{Spin}(32)/\mathbb{Z}_2$ weight lattice. The shift $\gamma$ and the charge $Q$ transform as vectors under $SO(18,2)$. In the standard lattice, we take $\gamma$ to be given by a lattice vector $\bar{\gamma}$ in the $II^{16,0}$ factor. The unbroken gauge group is then generated by the root vector $s\bar{r}$ of $II^{16,0}$ which fulfil

$$\frac{1}{n}\bar{r} \cdot \bar{\gamma} \in \mathbb{Z}. \quad (2.13)$$

In these coordinates, the left- and right-moving components of $p \in \Gamma^{18,2} + \frac{a}{n}\gamma$ are given by

$$\frac{1}{2}p_L^2 - \frac{1}{2}p_R^2 = \frac{1}{2}\bar{R} \cdot \bar{R} - m^+n^- + m^0n^0 \quad (2.14)$$

$$\frac{1}{2}p_R^2 = \frac{1}{2}\bar{y} \cdot \bar{y}' - y^+y'^- - y^-y'^+ + \frac{1}{2}n^0(y, y) \quad (2.15)$$

where $\bar{y} \cdot \bar{y}'$ denotes the standard Euclidean product between two sixteen dimensional vectors. The moduli space is a Kähler space with Kähler potential

$$K = -\log(- (y_2, y_2)). \quad (2.16)$$

We see that the requirement that the charge $Q$ be purely left-moving is equivalent

\footnotetext[2]{A bar over a vector, as in $\bar{\gamma}$ indicates the component in the $II^{16,0}$ factor.}
to demanding that $\bar{y} \cdot \bar{Q} = 0$. This means that the $Q$ gauge boson is massless so that the unbroken gauge group is at least $SU(2)$.

The constants $b_{\text{grav}}$ and $b_{\text{gauge}}$ are determined by the requirement that $\Delta_{\text{grav}}$ and $\Delta_{\text{gauge}}$ be finite at a generic point in the moduli space and are thus given by the coefficients of $\tau_2 q^0 \bar{q}^0$ in the integrands. Introducing the Fourier coefficients $c_{a,b}(h)$ and $\tilde{c}_{a,b}(h)$ through

$$e^{-2\pi i \frac{ab}{n^2} \gamma^2} \eta^{-20}(\tau) Z_{a,b}^3(\tau) = \sum_{h \geq -1} c_{a,b}(h) q^h$$

$$e^{-2\pi i \frac{ab}{n^2} \gamma^2} \eta^{-20}(\tau) E_2(\tau) Z_{a,b}^3(\tau) = \sum_{h \geq -1} \tilde{c}_{a,b}(h) q^h. \tag{2.17}$$

we get

$$b_{\text{grav}} = \sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\bar{\gamma}}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) 2$$

$$b_{\text{gauge}} = \sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\bar{\gamma}}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q}. \tag{2.18}$$

The superscript 0 on the summation signs indicate that, for a given $\bar{r}$, the sum is only over $a$ such that $\bar{R} \cdot \bar{y} = 0$ for generic values of the moduli $\bar{y}$ in the moduli space. In terms of the massless spectrum, we get (1.2). The sums in (2.17) should be regarded as sums over the real numbers, with $c_{a,b}(x)$ only taking nonzero values on the discrete spectrum of the model.

To write the final answer for $\Delta_{\text{grav}}$ and $\Delta_{\text{gauge}}$, we make the following definitions: First we introduce a notion of positivity of the vectors $\bar{R} = \bar{r} + \frac{a}{n} \bar{\bar{\gamma}}$. (In the case of $\bar{\bar{\gamma}}$ being a root vector, corresponding to the standard embedding of the spin connection in the gauge group, this can be done by first defining a positivity condition on lattice vectors $\bar{r} \in \mathbb{II}^{16,0}$ such that $\bar{\bar{\gamma}}$ is the smallest positive root. We then define $\bar{R}$ to be positive if $\bar{r} > 0$ or $\bar{r} = 0$ and $a > 0$.) We now say that the triplet $R = (\bar{R}, -k, -l)$ is positive if

$$k > 0 \text{ or }$$

$$k = 0, \ l > 0 \text{ or }$$

$$k = l = 0, \ \bar{R} > 0. \tag{2.19}$$

It is convenient to introduce the functions

$$d(R) \equiv \sum_b \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\bar{\gamma}}} c_{a,b} \left( -\frac{1}{2} (R, R) \right)$$

$$\tilde{d}(R) \equiv \sum_b \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\bar{\gamma}}} \tilde{c}_{a,b} \left( -\frac{1}{2} (R, R) \right). \tag{2.20}$$
We will also use these functions with argument $\bar{R}$ instead of $R$, meaning that $k = l = 0$. We define the product $(R; y)$ as

$$(R; y) = \begin{cases} \bar{R} \cdot \bar{y}_1 + l y_1^- + k y_1^+ + i |\bar{R} \cdot \bar{y}_2 + l y_2^- + k y_2^+| & \text{for } k > 0 \\
\bar{R} \cdot \bar{y} + l y^- - \left[\frac{\bar{R} \cdot y_2}{y_2}\right] y^- & \text{for } k = 0, \bar{R} \geq 0 \\
\bar{R} \cdot \bar{y} + l y^- + \left[\frac{\bar{R} \cdot y_2}{y_2}\right] y^- & \text{for } k = 0, \bar{R} < 0 , \end{cases}$$

(2.21)

where $[x]$ is the greatest integer less than or equal to $x$. Finally, we introduce the functions

$$\text{li}_m(x) = \text{Li}_m(e^{2\pi i x}) = \sum_{p=1}^{\infty} \frac{(e^{2\pi i x})^p}{p^m}$$

(2.22)

$$\mathcal{P}(x) = x^2 \text{li}_2(x) + \frac{1}{2\pi} \text{li}_3(x).$$

The calculation of $\Delta_{\text{grav}}$ and $\Delta_{\text{gauge}}$ is performed in Appendix A. The result is

$$\Delta_{\text{grav}} = \Delta_{\text{log}}^{\text{grav}} + \Delta_{\text{const}}^{\text{grav}} + \Delta_{\text{rat}}^{\text{grav}} + \Delta_{\text{transc}}^{\text{grav}}$$

$$\Delta_{\text{gauge}} = \Delta_{\text{log}}^{\text{gauge}} + \Delta_{\text{const}}^{\text{gauge}} + \Delta_{\text{rat}}^{\text{gauge}} + \Delta_{\text{transc}}^{\text{gauge}},$$

(2.23)

where

$$\Delta_{\text{log}}^{\text{grav}} = b_{\text{grav}} (-\log(-(y_2, y_2)) - K)$$

$$\Delta_{\text{const}}^{\text{grav}} = \frac{96\zeta(3) \chi}{\pi^2(y_2, y_2)}$$

$$\Delta_{\text{rat}}^{\text{grav}} = \sum_{\bar{r}, a} \left( \bar{d}(R) \frac{4 y_2^-}{\pi} \text{Re li}_2 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) \right.$$

$$+ d(\bar{R}) \left( -\frac{\pi(y_2, y_2)}{5 y_2^-} + \frac{24(y_2^-)^3}{\pi^3(y_2, y_2)} \text{Re li}_4 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) \right) \left. \right)$$

$$\Delta_{\text{transc}}^{\text{grav}} = \sum_{R > 0} d(R) 8 \text{Re li}_1 \left( (R; y) \right) + d(R) \frac{48}{\pi(y_2, y_2)} \text{Re } \mathcal{P} \left( (R; y) \right)$$

(2.24)

and

$$\Delta_{\text{log}}^{\text{gauge}} = b_{\text{gauge}} (-\log(-(y_2, y_2)) - K)$$

$$\Delta_{\text{const}}^{\text{gauge}} = \frac{4\zeta(3) \chi}{\pi^2(y_2, y_2)}$$

$$\Delta_{\text{rat}}^{\text{gauge}} = \sum_{\bar{r}, a} d(\bar{R}) \left( \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} \left( \frac{\pi(y_2, y_2)}{6 y_2} + \frac{2 y_2^-}{\pi} \text{Re li}_2 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) \right) \right.$$

$$+ \frac{\pi(y_2, y_2)}{120 y_2^-} + \frac{(y_2^-)^3}{\pi^3(y_2, y_2)} \text{Re li}_4 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) \left. \right)$$

$$\Delta_{\text{transc}}^{\text{gauge}} = \sum_{R > 0} d(R) \left( \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} 4 \text{Re li}_1 \left( (R; y) \right) + \frac{2}{\pi(y_2, y_2)} \text{Re } \mathcal{P} \left( (R; y) \right) \right).$$

(2.25)
Here $\K = 1 - \gamma_E + \log \frac{4\pi}{3\sqrt{3}}$ and 
\[
\chi = \frac{1}{4} \sum_{\tilde{r},\alpha}^\circ d(\tilde{R}).
\] (2.26)

The prime on the sums over $R > 0$ in $\Delta^\text{transc}_{\text{grav}}$ and $\Delta^\text{transc}_{\text{gauge}}$ indicates that terms with $k = l = 0$ and $\tilde{R} \cdot \tilde{y} = 0$ for generic values of the moduli are omitted. These are exactly the terms which are included in the sum with the superscript zero in $\chi$.

3. The fundamental chamber

The above formulas can be simplified in the (generalized) fundamental chamber of the moduli space defined by the conditions
\[
0 < \frac{\tilde{R} \cdot \tilde{y}_2}{y_2} < 1 \quad \text{for } \tilde{R} > 0, \quad \tilde{R} \cdot \tilde{R} \leq 2;
\]
\[
0 < y_2^- < y_2^+.
\] (3.1)

This means that $(\R; y) = (\R, y)$ for all $\R$ such that $-\frac{1}{2}(\R, \R) \geq -1$. Furthermore, using the identities
\[
\text{Re } \text{li}_2(x) = \pi^2 \left( \frac{1}{6} - |x| + x^2 \right)
\]
\[
\text{Re } \text{li}_4(x) = \pi^4 \left( \frac{1}{90} - \frac{1}{3} x^2 + \frac{2}{3} |x|^3 - \frac{1}{3} x^4 \right),
\] (3.2)

which are valid for $-1 \leq x \leq 1$, the rational terms can now be written as
\[
\Delta^\text{rat}_{\text{grav}} = \frac{24\pi}{(y_2, y_2)} \left( (d_{\text{grav}})_{ABC} y_A^2 y_B^2 y_C^2 + \frac{1}{y_2} (\hat{d}_{\text{grav}})_{ABCD} y_A^2 y_B^2 y_C^2 y_D^2 \right)
\]
\[
\Delta^\text{rat}_{\text{gauge}} = \frac{\pi}{(y_2, y_2)} \left( (d_{\text{gauge}})_{ABC} y_A^2 y_B^2 y_C^2 + \frac{1}{y_2} (\hat{d}_{\text{gauge}})_{ABCD} y_A^2 y_B^2 y_C^2 y_D^2 \right),
\] (3.3)

where
\[
(d_{\text{grav}})_{ABC} y_A^2 y_B^2 y_C^2 = \sum_{\tilde{r},\alpha}^\circ d(\tilde{R}) \left( -\frac{1}{6} |\tilde{R} \cdot \tilde{y}_2| y_2^- y_2^2 + \frac{1}{3} |\tilde{R} \cdot \tilde{y}_2| y_2^2 y_2^- 
\right.
\]
\[
- \frac{1}{3} (\tilde{R} \cdot \tilde{y}_2)^2 y_2^1 + \frac{1}{30} y_2^- y_2 y_2^- - \frac{1}{18} y_2^2 y_2^- y_2 
\]
\[
+ d(\tilde{R}) \left( \frac{2}{3} |\tilde{R} \cdot \tilde{y}_2|^3 + \frac{1}{30} y_2^- y_2 y_2^2 - \frac{1}{3} (\tilde{R} \cdot \tilde{y}_2)^2 y_2^- 
\right.
\]
\[
- \frac{1}{30} y_2^+ y_2^+ y_2^- + \frac{1}{90} y_2^- y_2^- y_2^- \right),
\] (3.4)
\[(d_{\text{gauge}})_{ABCD} y_1^A y_2^B y_3^C y_4^D \]
\[= \sum_{\bar{r},a} d(\bar{R}) \left( \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} \right) \left( -2|\bar{R} \cdot \bar{y}_2|\bar{y}_2 \cdot \bar{y}_2 + 4|\bar{R} \cdot \bar{y}_2|y_2^+ y_2^- + \frac{2}{3}y_2^+ \bar{y}_2 y_2^- \right. \\
\[ \quad \quad - 4(\bar{R} \cdot \bar{y}_2)^2 y_2^+ + \frac{1}{3}y_2 \cdot \bar{y}_2 y_2^- - \frac{2}{3}y_2^+ \bar{y}_2 y_2^- - \frac{2}{3}y_2^+ y_2^- \bar{y}_2 \right) \quad (3.5) \\
\[ \left. + \frac{2}{3}|\bar{R} \cdot \bar{y}_2|^3 - \frac{1}{30}y_2 \cdot \bar{y}_2 y_2^+ - \frac{1}{3}(\bar{R} \cdot \bar{y}_2)^2 y_2^- + \frac{1}{30}y_2^+ y_2^+ y_2^- \right. \\
\[ \left. + \frac{1}{90}y_2^- \bar{y}_2 y_2^- \right), \]

\[(\hat{d}_{\text{grav}})_{ABCD} y_1^A y_2^B y_3^C y_4^D \]
\[= \sum_{\bar{r},a} d(\bar{R}) \left( \frac{1}{6}(\bar{R} \cdot \bar{y}_2)^2 \bar{y}_2 \cdot \bar{y}_2 + d(\bar{R}) \left( - \frac{1}{120}(\bar{y}_2 \cdot \bar{y}_2)^2 - \frac{1}{3} (\bar{R} \cdot \bar{y}_2)^4 \right) \right) \quad (3.6) \]

and

\[(d_{\text{gauge}})_{ABCD} y_1^A y_2^B y_3^C y_4^D \]
\[= \sum_{\bar{r},a} d(\bar{R}) \left( \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} \right) \left( 2(\bar{R} \cdot \bar{y}_2)^2 \bar{y}_2 \cdot \bar{y}_2 - \frac{1}{6}(\bar{y}_2 \cdot \bar{y}_2)^2 \right) \\
\[ \quad + \frac{1}{120}(\bar{y}_2 \cdot \bar{y}_2)^2 - \frac{1}{3} (\bar{R} \cdot \bar{y}_2)^4 \right). \quad (3.7) \]

In fact, as we will see in the next section, these tensors obey the following identities on the subspace where $\bar{y} \cdot \bar{Q} = 0$:

\[(\hat{d}_{\text{grav}})_{ABCD} y_1^A y_2^B y_3^C y_4^D = 0 \]
\[(d_{\text{gauge}})_{ABCD} y_1^A y_2^B y_3^C y_4^D = 0 \quad (3.8)\]

and

\[(d_{\text{grav}} - d_{\text{gauge}})_{ABC} y_1^A y_2^B y_3^C y_4^D = 0 \mod (y_2, y_2) \tilde{d}_A y_2^A \quad (3.9)\]

for some real $\tilde{d}_A$. The last equation obviously implies that the tensor $(d_{\text{gauge}})_{ABCD} y_1^A y_2^B y_3^C y_4^D$ is independent of the factor of the gauge group under consideration up to terms of the form $(y_2, y_2) \tilde{d}_A y_2^A$. As we show in the next section, it follows from $N = 2$ supersymmetry that these conditions are obeyed in general, although this is far from obvious in our calculation. In Appendix B, we check that they are indeed fulfilled in the case of the standard embedding of the spin connection in the gauge group.
We have thus found that, in the fundamental chamber and on the subspace where $\bar{y} \cdot \bar{Q} = 0$, the threshold corrections are

$$
\Delta_{\text{grav}} = b_{\text{grav}} \left( - \log \left( - (y_2, y_2) \right) - K \right) + \frac{96 \zeta(3)}{\pi^2 (y_2, y_2)}
+
\frac{24\pi}{(y_2, y_2)} (d_{gauge})_{AB} y_2^A y_2^B y_2^C
+
\sum_{R>0}^\prime \left( d(R) 8 \Re \text{li}_1 (\left( R, y \right)) + d(R) \frac{48}{\pi (y_2, y_2)} \Re \mathcal{P} (\left( R, y \right)) \right)
$$

$$
\Delta_{\text{gauge}} = b_{\text{gauge}} \left( - \log \left( - (y_2, y_2) \right) - K \right) + \frac{4 \zeta(3)}{\pi^2 (y_2, y_2)}
+
\frac{\pi}{(y_2, y_2)} (d_{gauge})_{AB} y_2^A y_2^B y_2^C
+
\sum_{R>0}^\prime \left( 4 (\bar{R} \cdot \bar{Q})^2 \Re \text{li}_1 (\left( R, y \right)) + 2 \frac{\pi (y_2, y_2)}{\pi (y_2, y_2)} \Re \mathcal{P} (\left( R, y \right)) \right).
$$

(3.10)

4. The prepotential and the gravitational coupling

In this section, we will compute the prepotential for the vector multiplets $F^{(1)}_0$ and the gravitational coupling $F^{(1)}_1$ and show that the effective (non-Wilsonian) coefficients of the operators $\text{Tr} F^2$ and $R \wedge R^*$ may be written in the form

$$
\frac{1}{g^2_{\text{grav}} (p^2)} = 24 \Im \tilde{S} + \frac{b_{\text{grav}}}{16 \pi^2} \log \frac{M^2}{p^2} - \frac{3}{4 \pi^2} \log ||\Psi_{\text{grav}}||^2
$$

$$
\frac{1}{g^2_{\text{gauge}} (p^2)} = \Im \tilde{S} + \frac{b_{\text{gauge}}}{16 \pi^2} \log \frac{M^2}{p^2} - \frac{1}{(s+4)4 \pi^2} \log ||\Psi_{\text{gauge}}||^2,
$$

(4.1)

where $\tilde{S}$ is the pseudo-invariant dilaton defined below and the holomorphic functions $\Psi_{\text{grav}}$ and $\Psi_{\text{gauge}}$ are automorphic forms of the $T$-duality group. In particular, the invariant norm squares are

$$
||\Psi_{\text{grav}}||^2 = \left( - (y_2, y_2) \right)^{w_{\text{grav}}} ||\Psi_{\text{grav}}||^2
$$

$$
||\Psi_{\text{gauge}}||^2 = \left( - (y_2, y_2) \right)^{w_{\text{gauge}}} ||\Psi_{\text{gauge}}||^2
$$

(4.2)

with the weights

$$
w_{\text{grav}} = \frac{1}{12} b_{\text{grav}}
$$

$$
w_{\text{gauge}} = \frac{s+4}{4} b_{\text{gauge}}.
$$

(4.3)

---

1. $\tilde{S}$ differs from the dilaton $S$ by the addition of a holomorphic function such that $\tilde{S}$ transforms with a real shift under duality transformations. Note that this is slightly different from the invariant dilaton of [5].
Furthermore, these holomorphic functions may be written in the form
\[ \Psi_{\text{grav}} = e^{2\pi i (\rho_{\text{grav}})} y^A \prod_{R > 0} \left( 1 - e^{2\pi i (R, y)} \right)^{l_{\text{grav}}(R)} \]
\[ \Psi_{\text{gauge}} = e^{2\pi i (\rho_{\text{gauge}})} y^A \prod_{R > 0} \left( 1 - e^{2\pi i (R, y)} \right)^{l_{\text{gauge}}(R)} , \]
where the Weyl vectors and exponents are given by
\[ (\rho_{\text{grav}})_A = \frac{576}{s + 4} (d_{\text{grav}})^B A \]
\[ (\rho_{\text{gauge}})_A = 3 (d_{\text{gauge}})^B A \]
and
\[ l_{\text{grav}}(R) = \frac{96}{s + 4} d(R) (R, R) - 8 \tilde{d}(R) \]
\[ l_{\text{gauge}}(R) = d(R) \left( \frac{1}{2} (R, R) - \frac{s + 4}{2} (\bar{R} \cdot \bar{Q})^2 \right) \]
respectively.

The proof is based on a comparison of the string theory expressions (1.1) for the gauge and gravitational coupling constants with the following field theoretical expressions [4][5]:
\[ \frac{1}{g_{\text{grav}}^2 (p^2)} = \text{Re} \left( \log F_{\text{heterotic}}^{1} + \frac{b_{\text{grav}}}{16\pi^2} \left( \log \frac{M_{\text{Planck}}^2}{p^2} + K \right) \right) \]
\[ \frac{1}{g_{\text{gauge}}^2 (p^2)} = \text{Re} \left( -i\tilde{S} - \frac{1}{2(s + 4)\pi^2} \log \Psi_{\text{gauge}} + \frac{b_{\text{gauge}}}{16\pi^2} \left( \log \frac{M_{\text{Planck}}^2}{p^2} + K \right) \right) , \]
where the Kähler potential is
\[ K = -\log \text{Re} (-iS) - \log \left( -(y_2, y_2) \right) + \text{const} \]
and the Planck and string scales are related as \( M_{\text{Planck}}^2 = M_{\text{string}}^2 \text{Re} (-iS) \). The functions \( \log F_{\text{heterotic}}^{1} \) and \( \Psi_{\text{gauge}} \) are holomorphic and transform under duality in such a way that \( g_{\text{grav}}^{-2} (p^2) \) and \( g_{\text{gauge}}^{-2} (p^2) \) are invariant.

We start with the equations for \( g_{\text{gauge}}^{-2} (p^2) \). From the above, it follows that
\[ \frac{1}{2(s + 4)\pi^2} \text{Re} \log \Psi_{\text{gauge}} + \frac{b_{\text{gauge}}}{16\pi^2} \log \left( -(y_2, y_2) \right) \]
should be invariant under the duality group. We take this quantity to equal the manifestly invariant \( \frac{1}{16\pi^2} \left( \frac{1}{s + 4} \nabla^2 - 1 \right) \Delta_{\text{gauge}} \), where
\[ \nabla^2 = -2(y_2, y_2) \left( \eta^{AB} - \frac{2}{(y_2, y_2)} y_2^A y_2^B \right) \partial_A \partial_{\bar{B}} \]
is the Laplacian for the metric following from the Kahler potential (2.15) restricted to the subspace where $\bar{y} \cdot \bar{Q} = 0$. (The tensor $\eta^{AB}$ is the inverse of $\eta_{AB}$ defined through $\eta_{AB} y^A y'^B = (y, y')$; note that the indices $A$ and $B$ take 17 different values on this subspace.) It is now a straightforward computation to verify the above expression for $\Psi_{\text{gauge}}$. This is in fact the unique solution with the correct singularity structure [8]. Note that if the quartic tensors (3.6) and (3.7) in the rational terms had not vanished, this procedure to construct a holomorphic function with the right transformation properties would not have worked. Comparing the two expressions for $g^{-2}_{\text{gauge}}(p^2)$ and using

$$\frac{1}{16\pi^2} \Delta^{\text{univ}} = \frac{1}{-(y_2, y_2)} \text{Re} \left( \mathcal{F}_{0}^{(1)} - iy_2^A \frac{\partial}{\partial y^A} \mathcal{F}_{0}^{(1)} \right)$$

we get the following differential equation for the vector multiplet prepotential $\mathcal{F}_{0}^{(1)}$:

$$\text{Re} \left[ -\frac{1}{s + 4} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \mathcal{F}_{0}^{(1)} + \frac{1}{(y_2, y_2)} \left( \mathcal{F}_{0}^{(1)} - iy_2^A \frac{\partial}{\partial y^A} \mathcal{F}_{0}^{(1)} \right) \right] = \frac{1}{16\pi^2} \Delta_\text{gauge} + \frac{1}{2(s + 4)\pi^2} \text{Re} \log \Psi_{\text{gauge}} + \frac{b_{\text{gauge}}}{16\pi^2} \log((y_2, y_2)) + \text{const.}$$

By direct substitution, one checks that a solution is

$$\mathcal{F}_{0}^{(1)} = \frac{-i}{32\pi} (d_{\text{gauge}})_{ABC} y^A y^B y^C + \frac{\zeta(3)\chi}{4\pi^4} + \frac{1}{16\pi^4} \sum_{R > 0} d'(R) \text{Li}_3((R, y)).$$

Note that this form of the prepotential as a sum of a cubic tensor, a constant and an infinite sum of polylogarithms is what one expects when the heterotic theory has a type II dual. On the type II side, the coefficients of the cubic tensor, the constant $\chi$ and the coefficients of the polylogarithms are then interpreted as the intersection form, the Euler characteristic and the number of rational curves of the Calabi-Yau space respectively.

In fact, it is not difficult to see that (4.12) would not be satisfied by any holomorphic $\mathcal{F}_{0}^{(1)}$ and $\Psi_{\text{gauge}}$ if the quartic tensor (3.7) had not vanished. (After multiplication by $(y_2, y_2)$, all other terms in (4.12) can be written as a sum of holomorphic functions in the moduli $y^A$ times at most quadratic polynomials in their complex conjugates $y^*A$ plus the complex conjugate terms. The term in $\Delta_\text{gauge}$ involving the quartic tensor is not of this form and must therefore vanish.) Furthermore, the prepotential should be independent
of the factor in the gauge group that we are considering up to physically irrelevant terms of the form \((y_2, y_2)\tilde{d}_A y^A\) for some real \(\tilde{d}_A\). (Indeed, one can show that all solutions to (4.12) are physically equivalent in this sense \([3]\).) Together with analogous arguments for the gravitational corrections, these observations show that the conditions (3.8) and (3.9) indeed follow from \(N = 2\) supersymmetry.

We now turn to the equations for \(g^{-2}_{\text{grav}}(p^2)\). Comparing the two expressions, we get

\[
\log F_{\text{heterotic}}^1 = 24(-iS) + \log F_1^{(1)} \quad \text{with}
\]

\[
\log F_1^{(1)} = \frac{1}{2\pi^2} \sum_{R > 0} \tilde{d}(R) \text{li}_1((R, y))
\]

modulo terms linear in \(y\). Here we have used the condition (3.9). Finally, we should check that \(F_{\text{heterotic}}^1\) transforms in the correct way. Writing \(\log F_{\text{heterotic}}^1 = 24 \left(-i\tilde{S} - \frac{1}{16\pi^2} \log \Psi_{\text{grav}}\right)\), this is equivalent to

\[
\frac{1}{16\pi^2} \text{Re} \log \Psi_{\text{grav}} + \frac{b_{\text{grav}}}{16\pi^2} \log \left(-(y_2, y_2)\right)
\]

being invariant. It is straightforward to check that this equals the manifestly invariant quantity \(\frac{1}{16\pi^2} \left(\frac{1}{s+4} \nabla^2 - 1\right) \Delta_{\text{grav}}\), and one verifies the expression for \(\Psi_{\text{grav}}\).

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Appendix A. Poisson resummation and the orbit method

In this appendix, we will prove that the threshold corrections can be written as in (2.23) with (2.24) and (2.25), following the methods introduced in appendix B of [3] and further developed in [8].
The first step is to perform a Poisson resummation of the sums over \( m^+ \) and \( m^0 \):\[
\sum_{\vec{r}} \sum_{m^+,m^-} q^{\frac{1}{2}p_L} q^{\frac{1}{2}p_R} = \sum_{\vec{r}} \sum_A \frac{-(y_2,y_2)}{2\tau_2 y_2^2} q^{\frac{1}{2} \vec{R} \cdot \vec{R}} \exp G. \tag{A.1}
\]
Here
\[
A = \begin{pmatrix} n^- & \hat{j} \\ \hat{n}^0 & p \end{pmatrix}
\tag{A.2}
\]
is summed over all integer \( 2 \times 2 \) matrices and
\[
G = \frac{\pi(y_2,y_2)}{2(y_2)^2 \tau_2} |A|^2 - 2\pi i y^+ \det A + \frac{\pi}{y_2} \left( \vec{R} \cdot \vec{q} \vec{A} - \vec{R} \cdot \vec{y}^* \vec{A} \right)
- \frac{\pi}{2y_2} n^0 \left( \vec{y} \cdot \vec{q} \vec{A} - \vec{y}^* \cdot \vec{y}^* \vec{A} \right) + \frac{i\pi y_2 \cdot \vec{y}_2}{(y_2)^2} (n^- + n^0 y^-) A,
\tag{A.3}
\]
with \( A = (1 \ y^-) A \left( \begin{array}{c} \tau \\ 1 \end{array} \right) \) and \( \vec{A} = (1 \ y^-) A \left( \begin{array}{c} \tau \\ 1 \end{array} \right) \). We can now write
\[
\Delta_{\text{grav}} = \int \sum_{\vec{r}} \sum_{a,b} \sum_{A} \sum_{h} \frac{1}{n} e^{2\pi i \frac{\vec{R} \cdot \vec{Q}}{4\pi \tau_2} - \frac{3c_{a,b}(h)}{\pi \tau_2} \left( \vec{c}_{a,b}(h) - \frac{3c_{a,b}(h)}{\pi \tau_2} \right)}
\times \frac{-(y_2,y_2)}{2y_2^2} q^{h+\frac{1}{2} \vec{R} \cdot \vec{R}} \exp G - \tau_2 b_{\text{grav}}
\]
\[
\Delta_{\text{gauge}} = \int \sum_{\vec{r}} \sum_{a,b} \sum_{A} \sum_{h} \frac{1}{n} e^{2\pi i \frac{\vec{R} \cdot \vec{Q}}{4\pi \tau_2} - \frac{1}{4\pi \tau_2}} \left( \frac{(\vec{R} \cdot \vec{Q})^2}{Q \cdot Q} - \frac{1}{4\pi \tau_2} \right)
\times \frac{-(y_2,y_2)}{2y_2^2} q^{h+\frac{1}{2} \vec{R} \cdot \vec{R}} \exp G - \tau_2 b_{\text{gauge}}
\tag{A.4}
\]
with the Fourier coefficients \( c_{a,b}(h) \) and \( \vec{c}_{a,b}(h) \) defined in (2.17).

A key property is the behavior of the sum over \( A \) under modular transformations: The contributions from the matrix \( A' = AV^{-1} \) for \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \) is given by the contribution from the matrix \( A \) if we change \( \tau \) to \( \tau' = \frac{a\tau + b}{c\tau + d} \). We can therefore apply the method of orbits \[3\]: Rather than integrating the contributions of all integer \( 2 \times 2 \) matrices over the fundamental region, we pick a representative matrix \( A \) from each orbit of \( SL(2,\mathbb{Z}) \) and integrate over the image of the fundamental region under all \( SL(2,\mathbb{Z}) \) transformations \( V \) that yield distinct matrices \( AV^{-1} \). In this way we get
\[
\Delta_{\text{grav}} = \Delta_{\text{grav}}^{\text{zero}} + \Delta_{\text{grav}}^{\text{reg}} + \Delta_{\text{grav}}^{\text{deg}},
\]
\[
\Delta_{\text{gauge}} = \Delta_{\text{gauge}}^{\text{zero}} + \Delta_{\text{gauge}}^{\text{reg}} + \Delta_{\text{gauge}}^{\text{deg}} \tag{A.5}
\]
where the three terms correspond to contributions from the zero matrix, matrices with non-zero determinant and non-zero matrices with zero determinant respectively, and the $b$-terms are included in the last term.

To get the contribution from the zero orbit, we take $A = 0$ and integrate over the fundamental region $\mathcal{F} = \{\tau \mid -1/2 < \tau_1 < 1/2, \tau_2 > 0, |\tau| > 1\}$. We thus get

$$
\Delta_{\text{zero}}^{\text{grav}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} J_{\text{grav}} \quad \text{and} \quad \Delta_{\text{zero}}^{\text{gauge}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} J_{\text{gauge}},
$$

where

$$
J_{\text{grav}} = \sum_{\tilde{r}} \sum_{a,b} \sum_{h} \frac{1}{n} e^{2\pi i \frac{a}{n} R \cdot \tilde{R}} \left( \tilde{c}_{a,b}(h) - \frac{3c_{a,b}(h)}{\pi \tau_2} \right) 2 \left( y_2, y_2 \right) - \frac{y_2}{2y_2} Q \cdot Q - \frac{1}{4\pi \tau_2} - \frac{y_2}{2y_2} Q \cdot Q.
$$

It follows from modular invariance that $J_{\text{grav}}$ and $J_{\text{gauge}}$ can be written as linear combinations of $(E_2(\tau) - \frac{3}{\pi \tau_2}) E_4(\tau) E_6(\tau) \eta^{-24}(\tau)$, $J(\tau)$ and 1 with coefficients that can be determined by considering the coefficients of $q^{-1}$, $q^0 \tau_2^{-1}$ and $q^0$:

$$
J_{\text{grav}} = \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{a}{n} R \cdot \tilde{R}} c_{a,b}(h) \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) - \left( y_2, y_2 \right) - \frac{y_2}{2y_2} - \frac{7}{20} + \left( \frac{\tilde{R} \cdot \tilde{Q}}{Q \cdot Q} \right)^2
\times \left( -\frac{1}{120} \right) (E_2(\tau) - \frac{3}{\pi \tau_2}) E_4(\tau) E_6(\tau) \eta^{-24}(\tau)
$$

$$
J_{\text{gauge}} = \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{a}{n} R \cdot \tilde{R}} c_{a,b}(h) \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) - \left( y_2, y_2 \right) - \frac{y_2}{2y_2} - \frac{7}{20} + \left( \frac{\tilde{R} \cdot \tilde{Q}}{Q \cdot Q} \right)^2
\times \left( -\frac{1}{2880} (E_2(\tau) - \frac{3}{\pi \tau_2}) E_4(\tau) E_6(\tau) \eta^{-24}(\tau) + \frac{1}{2880} J(\tau) \right)
$$

The formula

$$
\int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \left( E_2 - \frac{3}{\pi \tau_2} \right)^k W = \frac{\pi}{3(k + 1)} E_2^{(k+1)} W \bigg|_{q^0},
$$

which is valid for an arbitrary meromorphic modular form $W$ of weight $-2k$ [17], now gives

$$
\Delta_{\text{zero}}^{\text{grav}} = \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{a}{n} R \cdot \tilde{R}} c_{a,b}(h) \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) - \left( y_2, y_2 \right) - \frac{y_2}{2y_2} \left( \frac{2\pi}{5} \right)
$$

$$
\Delta_{\text{zero}}^{\text{gauge}} = \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{a}{n} R \cdot \tilde{R}} c_{a,b}(h) \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) - \left( y_2, y_2 \right) - \frac{y_2}{2y_2} \left( \frac{\pi}{3} \right) \left( \frac{\tilde{R} \cdot \tilde{Q}}{Q \cdot Q} - \frac{1}{20} \right).
$$

(A.8)
For the regular orbits, we take \( A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \) with \( 0 \leq j < k \) and \( p \neq 0 \) so that

\[
G = \frac{\pi (y_2 \cdot y_2)}{2(y_2^2)^2} \left| k\tau + j + py^- \right|^2 - 2\pi iy^+ kp \\
+ \frac{\pi}{y_2^3} \left( R \cdot \bar{y}(k\tau + j + py^-) - \bar{R} \cdot \bar{y^*}(k\tau + j + py^-) \right) \\
+ \frac{i\pi}{2(y_2^2)} \bar{y}_2 \cdot \bar{y}_2(k\tau + j + py^-)
\]

(A.10)

and integrate over the double cover of the upper half-plane \( \mathcal{H} = \{ \tau \mid \tau_2 > 0 \} \). We thus get

\[
\Delta_{\text{grav}}^{\text{reg}} = \sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \bar{r} \cdot \bar{R}} 2 \left( \hat{S}_{1/2} - \frac{3}{\pi} S_{3/2} \right) \\
\Delta_{\text{gauge}}^{\text{reg}} = \sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \bar{r} \cdot \bar{R}} \left( \frac{(\bar{R} \cdot \bar{Q})^2}{\bar{Q} \cdot Q} S_{1/2} - \frac{1}{4\pi} S_{3/2} \right),
\]

(A.11)

where

\[
\hat{S}_{1/2} = 2 \int_{\mathcal{H}} \frac{d^2 \tau}{\tau_2^2} \sum_{k>0} \sum_{0 \leq j < k} \sum_{p \neq 0} \sum_{h} \tilde{c}_{a,b}(h) \frac{-(y_2 \cdot y_2)}{2y_2^3} q^{h+1/2} \bar{R} \cdot \bar{R} \exp G \\
S_{1/2} = 2 \int_{\mathcal{H}} \frac{d^2 \tau}{\tau_2^2} \sum_{k>0} \sum_{0 \leq j < k} \sum_{p \neq 0} \sum_{h} c_{a,b}(h) \frac{-(y_2 \cdot y_2)}{2y_2^3} q^{h+1/2} \bar{R} \cdot \bar{R} \exp G \\
S_{3/2} = 2 \int_{\mathcal{H}} \frac{d^2 \tau}{\tau_2^2} \sum_{k>0} \sum_{0 \leq j < k} \sum_{p \neq 0} \sum_{h} c_{a,b}(h) \frac{-(y_2 \cdot y_2)}{2y_2^3} q^{h+1/2} \bar{R} \cdot \bar{R} \exp G.
\]

(A.12)

After performing the Gaussian integral over \( \tau_1 \), the only \( j \)-dependence of these quantities is through the factor \( \exp \left( -2\pi i(h + \frac{1}{2} \bar{R} \cdot \bar{R}) \frac{j}{k} \right) \). It follows from the invariance under \( \tau \rightarrow \tau + 1 \) that we only need to consider the case where \( h + \frac{1}{2} \bar{R} \cdot \bar{R} \) is integral, and the sum over \( j \) then amounts to replacing \( h \) by \( kl - \frac{1}{2} \bar{R} \cdot \bar{R} \) and summing over all integers \( l \). The \( \tau_2 \) integrals give rise to the Bessel functions \( K_{1/2} \) and \( K_{3/2} \), which may be expressed in terms of elementary functions. Finally, the sums over \( p \) may be expressed in terms of the functions \( \text{li}_1 \) and \( P \).
defined in (2.22). In this way we get

\[
\begin{align*}
\tilde{S}_{1/2} = & \text{Re} \sum_{k > 0} \sum_{l \in \mathbb{Z}} 4\tilde{c}_{a,b} \left( kl - \frac{1}{2} \check{R} \cdot \check{R} \right) \\
& \times \text{li}_1 \left( \check{R} \cdot \tilde{y}_1 + l\tilde{y}_1^- + k\tilde{y}_1^+ + i |\check{R} \cdot \tilde{y}_2 + l\tilde{y}_2^- + k\tilde{y}_2^+ | \right)
\end{align*}
\]

\[S_{1/2} = \text{Re} \sum_{k > 0} \sum_{l \in \mathbb{Z}} 4c_{a,b} \left( kl - \frac{1}{2} \check{R} \cdot \check{R} \right) \]

\[\times \text{li}_1 \left( \check{R} \cdot \tilde{y}_1 + l\tilde{y}_1^- + k\tilde{y}_1^+ + i |\check{R} \cdot \tilde{y}_2 + l\tilde{y}_2^- + k\tilde{y}_2^+ | \right)
\]  \hspace{1cm} (A.13)

\[S_{3/2} = \text{Re} \sum_{k > 0} \sum_{l \in \mathbb{Z}} \frac{8}{(y_2, y_2)} c_{a,b} \left( kl - \frac{1}{2} \check{R} \cdot \check{R} \right) \]

\[\times \mathcal{P} \left( \check{R} \cdot \tilde{y}_1 + l\tilde{y}_1^- + k\tilde{y}_1^+ + i |\check{R} \cdot \tilde{y}_2 + l\tilde{y}_2^- + k\tilde{y}_2^+ | \right). \]

For the degenerate orbits, we take \( A = \left( \begin{array}{cc} 0 & j \\ 0 & p \end{array} \right) \) with \((j, p) \neq (0, 0)\) so that

\[
\mathcal{G} = \frac{\pi(y_2, y_2)}{2(y_2)^2\tau_2} |j + py^-|^2 + \frac{\pi}{y_2} \left( \check{R} \cdot \tilde{y}(j + py^-) - \check{R} \cdot \tilde{y}^*(j + py^-) \right)
\]  \hspace{1cm} (A.14)

and integrate over the strip \( S = \{ \tau | -1/2 < \tau_1 < 1/2, \tau_2 > 0 \} \). The integral over \( \tau_1 \) amounts to putting \( h = -\frac{1}{2} \check{R} \cdot \check{R} \) and we are left with

\[
\begin{align*}
\Delta_{\text{grav}}^{\text{deg}} &= \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) 2U^0 \\
&+ \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) 2U' \\
&+ \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) \left( -\frac{6}{\pi} \right) V
\end{align*}
\]

\[
\begin{align*}
\Delta_{\text{gauge}}^{\text{deg}} &= \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) \frac{(\check{R} \cdot \check{Q})^2}{\check{Q} \cdot \check{Q}} U^0 \\
&+ \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) \frac{(\check{R} \cdot \check{Q})^2}{\check{Q} \cdot \check{Q}} U' \\
&+ \sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \tilde{R} \cdot \tilde{\gamma}_{a,b}} \left( -\frac{1}{2} \check{R} \cdot \check{R} \right) \left( -\frac{1}{4\pi} \right) V,
\end{align*}
\]  \hspace{1cm} (A.15)
where

\[ U^0 = \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \sum_{(j,p) \neq (0,0)} \frac{-(y_2, y_2)}{2y_2^2} \exp \left( \frac{\pi(y_2, y_2)}{2y_2^2} |j + py|^2 \right) - \int_F \frac{d^2 \tau}{\tau_2} \]

\[ U' = \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \sum_{(j,p) \neq (0,0)} \frac{-(y_2, y_2)}{2y_2^2} \exp G \]

\[ V = \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \sum_{(j,p) \neq (0,0)} \frac{-(y_2, y_2)}{2y_2^2} \exp G. \]

(A.16)

The sums in the first two terms in \( \Delta_{\text{grav}} \) and \( \Delta_{\text{gauge}} \) are over \( a \) such that \( \bar{R} \cdot \bar{y} = 0 \) and \( \bar{R} \cdot \bar{y} \neq 0 \) for generic values of \( \bar{y} \) respectively, whereas the sum in the third term is unrestricted. The contributions from \( b_{\text{grav}} \) and \( b_{\text{gauge}} \) have been included in the first term. The calculation of \( U^0 \) has been performed in [9]. To calculate \( U' \) and \( V \), one first integrates over \( \tau_2 \). The sum over \( j \) for \( p = 0 \) then directly gives rise to \( \text{li}_2 \) and \( \text{li}_4 \) functions, whereas for \( p \neq 0 \) we can perform the \( j \)-sum by means of a Sommerfeld-Watson transformation and get \( \text{li}_1 \) and \( \mathcal{P} \) functions. In this manner one finds

\[ U^0 = \frac{\pi}{3} - \log \left( -(y_2, y_2) \right) - K + 4 \sum_{l \geq 0} \text{Re } \text{li}_1 (y^- l) \]

\[ U' = \frac{2y_2^-}{\pi} \text{Re } \text{li}_2 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) + 4 \sum_{l \geq 0} \text{Re } \text{li}_1 \left( \bar{R} \cdot \bar{y} + l y^- - \left[ \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right] y^- \right) \]

\[ V = \frac{-4(y_2^-)^3}{\pi^2(y_2, y_2)} \text{Re } \text{li}_4 \left( \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right) \]

\[ - \frac{8}{(y_2, y_2)} \sum_{l \geq 0} \text{Re } \mathcal{P} \left( \bar{R} \cdot \bar{y} + l y^- - \left[ \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right] y^- \right), \]

(A.17)

for an arbitrary function \( f \).

where \([x]\) denotes the largest integer less than or equal to \( x \).

In order to compare with (2.23), we separate the \( \mathcal{P} \) terms with \( l = 0 \) in \( \Delta_{\text{deg}} \) and \( \Delta_{\text{deg}} \) according to whether \( \bar{R} \cdot \bar{y} = 0 \) or \( \bar{R} \cdot \bar{y} \neq 0 \) for generic value of the moduli and note that \( \mathcal{P}(0) = \frac{\zeta(3)}{2\pi} \). We also separate the \( \text{li}_1 \) and \( \mathcal{P} \) terms according to whether \( \bar{R} \) is negative or not and use that

\[ \sum_{l \geq 0} f \left( \bar{R} \cdot \bar{y} + l y^- - \left[ \frac{\bar{R} \cdot \bar{y}_2}{y_2^-} \right] y^- \right) = \begin{cases} \sum_{l \geq 0} f \left( (R; y) \right) & \text{for } \bar{R} \geq 0; \\ \sum_{l > 0} f \left( (R; y) \right) & \text{for } \bar{R} < 0; \end{cases} \]

(A.18)

for an arbitrary function \( f \).
Appendix B. The standard embedding

In this appendix, we will consider the case of the standard embedding of the spin connection in the gauge group in somewhat more detail. This means that the shift vector $\bar{\gamma}$ is a root vector of the $E_8 \times E_8$ lattice, i.e. $\bar{\gamma} \cdot \bar{\gamma} = 2$, which we take to belong to the first $E_8$ factor. This is consistent with the level matching condition (2.12) for $n = 2, 3, 4, 6$, and the constants $k_{a,b}$ are given in the table below:

| $\frac{b}{n}$ | $\frac{0}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ | $\frac{3}{12}$ | $\frac{4}{12}$ | $\frac{5}{12}$ | $\frac{6}{12}$ | $\frac{7}{12}$ | $\frac{8}{12}$ | $\frac{9}{12}$ | $\frac{10}{12}$ | $\frac{11}{12}$ |
|---------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\frac{a}{n}$ | $\frac{0}{12}$ | 0              | 4              | 16             | 36             | 64             | 36             | 16             | 4              |
| $\frac{1}{12}$ |               | 4              | 4              | 4              | 4              | 4              | 4              | 4              |
| $\frac{2}{12}$ |               | 16             | 16             | 16             | 16             |                |                |                |
| $\frac{3}{12}$ |               | 36             | 4              | 36             | 4              | 36             | 4              |                |
| $\frac{4}{12}$ |               | 64             | 4              | 16             | 4              | 64             | 4              | 16             | 4              |
| $\frac{5}{12}$ |               |                |                |                |                |                |                |                |
| $\frac{6}{12}$ |               | 36             | 4              | 36             | 4              | 36             | 4              |                |
| $\frac{7}{12}$ |               | 16             | 16             | 16             | 16             |                |                |
| $\frac{8}{12}$ |               | 4              | 4              | 4              | 4              | 4              | 4              |
| $\frac{9}{12}$ |               |                |                |                |                |                |                |
| $\frac{10}{12}$ |              |                |                |                |                |                |                |
| $\frac{11}{12}$ |             |                |                |                |                |                |                |

The unbroken gauge group is generated by the root vectors $\bar{b}$ of the $E_8 \times E_8$ lattice which fulfil $\frac{1}{n} \bar{b} \cdot \bar{\gamma} \in \mathbb{Z}$. For $n = 3, 4, 6$, this means that $\bar{b}$ is one of the $240 + 126$ roots which are orthogonal to $\bar{\gamma}$, and the gauge group in this case is $U(1) \times E_7 \times E_8 \times U(1)^4$. For $n = 2$ there is the additional possibility that $\bar{b} = \pm \bar{\gamma}$, and the gauge group is enhanced to $SU(2) \times E_7 \times E_8 \times U(1)^4$. The first $U(1)$ factor for $n = 3, 4, 6$ or the $SU(2)$ factor for $n = 2$ arise only in the orbifold limit, so for a smooth $K3$-surface the gauge group is $E_7 \times E_8 \times U(1)^4$. This can then be broken down further by turning on the Wilson line moduli $\bar{y}$. To calculate the threshold correction to the gauge coupling constant, there must be at least an unbroken $SU(2)$ factor generated by some $\bar{Q}$, which is either a root vector in the first $E_8$ factor such that $\bar{\gamma} \cdot \bar{Q} = 0$ or an arbitrary root vector in the second $E_8$ factor.

In fact, these four orbifold models only differ by being at different points in the moduli space of the $K3$-surface. Since these moduli belong to $N = 2$ hypermultiplets, which do
not mix with the vector multiplet moduli, the threshold corrections should be independent of \( n \). One must be careful at this point, since the answers are in fact different when \( \bar{y} \cdot \bar{\gamma} \neq 0 \). The reason is that in this case, we turn on the Wilson modulus corresponding to the extra \( U(1) \) or \( SU(2) \) factor which only exists in the orbifold limit, thus freezing the hypermultiplet moduli at that particular point in the moduli space of the \( K3 \)-surface and we are exploring inequivalent branches of the total moduli space. It is worthwhile checking that we do get equivalent answers for different \( n \) when \( \bar{y} \cdot \bar{\gamma} = 0 \). To this end, we note that the threshold corrections are of the form

\[
\sum_{a,b} \frac{1}{n} e^{2\pi i \frac{k}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} + kl \right) g \left( \bar{R} \cdot \bar{y}, ky^{+}, ly^{-} \right) = \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{k}{n} \bar{R} \cdot \bar{\gamma}} q^{\frac{1}{2} \bar{R} \cdot \bar{R} \eta} e^{-2\pi i \frac{ab}{n^2} \eta^2} \eta^{-20} Z_{a,b}^{K3}(\tau) \quad (B.1)
\]

for some function \( g \). Furthermore the quantity \( \bar{R} \cdot \bar{y} \) is invariant under shifts of \( \bar{r} \) by \( \bar{\gamma} \).

Altogether, this means that the equivalence would follow from the \( n \)-independence of the functions

\[
\sum_{a,b} \frac{1}{n} e^{2\pi i \frac{k}{n} (R+j\bar{\gamma}) \cdot \bar{\gamma}} q^{\frac{1}{2} (R+j\bar{\gamma}) \cdot (R+j\bar{\gamma})} e^{-2\pi i \frac{ab}{n^2} \eta^2} \eta^{-20} Z_{a,b}^{K3}(\tau) \quad (B.2)
\]

for any \( \bar{r} \). Performing the sum over \( j \) by means of the Jacobi triple product identity, using the form (2.9) of \( Z_{a,b}^{K3}(\tau) \) and dropping some manifestly \( n \)-independent factors, we get

\[
\sum_{a,b} \frac{1}{n} k_{a,b} \left( e^{2\pi i \frac{k}{n} q^{\frac{1}{2}} (\bar{r} - \bar{\gamma})} \right)^{(\bar{r} \cdot \bar{\gamma} + 1)} \left( 1 - q^{\frac{k}{n}} e^{2\pi i \frac{k}{n}} \right)^{-2} \prod_{j=1}^{\infty} \left( 1 - q^{j+\frac{k}{n}} e^{2\pi i \frac{k}{n}} \right)^{-2} \left( 1 - q^{j-\frac{k}{n}} e^{-2\pi i \frac{k}{n}} \right)^{-2} \left( 1 + q^{2j-1+\bar{r} \cdot \bar{\gamma} + 2 \frac{k}{n}} e^{4\pi i \frac{k}{n}} \right) \left( 1 + q^{2j-1-\bar{r} \cdot \bar{\gamma} - 2 \frac{k}{n}} e^{-4\pi i \frac{k}{n}} \right). \quad (B.3)
\]

By expanding out the first few terms, one may check that, for integer \( \bar{r} \cdot \bar{\gamma} \), we indeed get the same result for \( n = 2, 3, 4, 6 \), although this is not manifest in (B.3).

For \( n = 2 \), it is now straightforward to calculate

\[
\sum_{a,b} \frac{1}{n} e^{2\pi i \frac{k}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) = 128 \quad (B.4)
\]

\[
\sum_{a,b} \frac{1}{n} e^{2\pi i \frac{k}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) = -64
\]
for $\bar{r} = a = 0$ and
\[
\sum_b \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) = \sum_b \frac{1}{n} e^{2\pi i \frac{b}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right)
\]
\[
= \begin{cases} 
8 & \text{for } a = 0, \bar{r} \cdot \bar{r} = 2, \bar{r} \cdot \bar{\gamma} = 0 \\
-8 & \text{for } a = 0, \bar{r} \cdot \bar{r} = 2, \bar{r} \cdot \bar{\gamma} = \pm 1 \\
8 & \text{for } a = 0, \bar{r} = \pm \bar{\gamma} \\
-256 & \text{for } a = 1, \bar{r} = 0 \text{ or } a = 1, \bar{r} = -\bar{\gamma} \\
-64 & \text{for } a = 1, \bar{r} \cdot \bar{r} = 2, \bar{r} \cdot \bar{\gamma} = -1 \\
0 & \text{otherwise.}
\end{cases}
\]
(B.5)

The constant $\chi$ receives contributions from $\bar{r}$ and $a$ such that $\bar{R}$ is parallel to $\gamma$, i.e.
\[
\chi = \frac{1}{4} \left( 128 + 8 + 8 - 256 - 256 \right) = -92.
\]
(B.6)

This agrees with the expected result $2 \times (19 - 65) = -92$ from 19 vector multiplets and $20 + 45 = 65$ hypermultiplet moduli for the $K3$-surface and the gauge bundle.

To calculate the rational terms in the threshold corrections, we also need some information about the $E_8 \times E_8$ root lattice. Let $\bar{v}$ be an arbitrary vector in the vector space of the $E_8$ root lattice. Using the rotational symmetry to fourth order of this lattice [18], we can calculate the following sums over root vectors $\bar{b}$ with the indicated inner products with $\bar{\gamma}$:
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 0} 1 = 126
\]
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 1} 1 = 56
\]
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 0} \left( \bar{b} \cdot \bar{v} \right)^2 = 36 \bar{v} \cdot \bar{v} - 18 (\bar{v} \cdot \bar{\gamma})^2
\]
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 1} \left( \bar{b} \cdot \bar{v} \right)^2 = 12 \bar{v} \cdot \bar{v} + 8 (\bar{v} \cdot \bar{\gamma})^2
\]
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 0} \left( \bar{b} \cdot \bar{v} \right)^4 = 24 (\bar{v} \cdot \bar{v})^2 - 24 \bar{v} \cdot \bar{v} (\bar{v} \cdot \bar{\gamma})^2 + 6 (\bar{v} \cdot \bar{\gamma})^4
\]
\[
\sum_{\bar{b} \cdot \bar{\gamma} = 1} \left( \bar{b} \cdot \bar{v} \right)^4 = 6 (\bar{v} \cdot \bar{v})^2 + 12 \bar{v} \cdot \bar{v} (\bar{v} \cdot \bar{\gamma})^2 - 4 (\bar{v} \cdot \bar{\gamma})^4.
\]
(B.7)
We may now calculate (Again using $n = 2$ for simplicity, although the result would be the same with $n = 3, 4, 5$ by the above argument):

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) = -1920
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} \bar{c}_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) = -2112. \tag{B.8}
\]

In the following formulas, we have decomposed $\bar{y}_2 = \bar{y}_1^1 + \bar{y}_2^2$, where the terms refer to the first and second $E_8$ factor. We also introduce the sets $r_0^1$ and $r_1^1$ of positive roots of the first $E_8$ factor whose inner product with $\bar{\gamma}$ equals 0 and 1 and the set $r^2$ of positive roots of the second $E_8$ factor. The symbol $c_{a,b}$ stands for either of the functions $c_{a,b}$ or $\bar{c}_{a,b}$.

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) |\bar{R} \cdot \bar{y}_2| = 16 \sum_{\bar{r} \in r_0^1} \bar{r} \cdot \bar{y}_2^1 - 160 \sum_{\bar{r} \in r_1^1} \bar{r} \cdot \bar{y}_2^1 + 16 \sum_{\bar{r} \in r^2} \bar{r} \cdot \bar{y}_2^2
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} \bar{c}_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) (\bar{R} \cdot \bar{y}_2)^2 = -672 \bar{y}_2^1 \cdot \bar{y}_2^1 + 480 \bar{y}_2^2 \cdot \bar{y}_2^2 \tag{B.9}
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) |\bar{R} \cdot \bar{y}_2|^3 = 16 \sum_{\bar{r} \in r_0^1} (\bar{r} \cdot \bar{y}_2^1)^3 - 160 \sum_{\bar{r} \in r_1^1} (\bar{r} \cdot \bar{y}_2^1)^3 + 16 \sum_{\bar{r} \in r^2} (\bar{r} \cdot \bar{y}_2^2)^3
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} \bar{c}_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) (\bar{R} \cdot \bar{y}_2)^4 = -288(\bar{y}_2^1 \cdot \bar{y}_2^1)^2 + 288(\bar{y}_2^2 \cdot \bar{y}_2^2)^2.
\]

The sums involving the charge $\bar{Q}$ depend on which $E_8$ factor $\bar{Q}$ belongs to. In the case where $\bar{Q}$ belongs to the first $E_8$ factor, we must have $\bar{Q} \cdot \bar{\gamma} = \bar{y}_1 \cdot \bar{Q} = \bar{y}_1 \cdot \bar{\gamma} = 0$. We then get

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} c_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} = -672
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} \bar{c}_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} |\bar{R} \cdot \bar{y}_2| = 16 \sum_{\bar{r} \in r_0^1} \frac{(\bar{r} \cdot \bar{Q})^2}{Q \cdot Q} \bar{r} \cdot \bar{y}_2^1
\]

\[
- 160 \sum_{\bar{r} \in r_1^1} \frac{(\bar{r} \cdot \bar{Q})^2}{Q \cdot Q} \bar{r} \cdot \bar{y}_2^1 \tag{B.10}
\]

\[
\sum_{\bar{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{1}{n} \bar{R} \cdot \bar{\gamma}} \bar{c}_{a,b} \left( -\frac{1}{2} \bar{R} \cdot \bar{R} \right) \frac{(\bar{R} \cdot \bar{Q})^2}{Q \cdot Q} (\bar{R} \cdot \bar{y}_2)^2 = -96 \bar{y}_2^1 \cdot \bar{y}_2^1.
\]
In the case where $\bar{Q}$ belongs to the second $E_8$ factor, we must have $\bar{y}^1 \cdot \bar{Q} = \bar{y}^2 \cdot \bar{\gamma} = 0$. The sums are then

$$\sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{r}{n}} \tilde{R} \cdot \tilde{c}_{a,b} \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) \frac{(\tilde{R} \cdot \tilde{Q})^2}{Q \cdot Q} = 480$$

$$\sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{r}{n}} \tilde{R} \cdot \tilde{c}_{a,b} \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) \frac{(\tilde{R} \cdot \tilde{Q})^2}{Q \cdot Q} |\tilde{R} \cdot \tilde{y}_2| = 16 \sum_{\tilde{r} \in r^2} \frac{(\tilde{r} \cdot \tilde{Q})^2}{Q \cdot Q} \tilde{r} \cdot \tilde{y}_2^2 \quad (B.11)$$

$$\sum_{\tilde{r}} \sum_{a,b} \frac{1}{n} e^{2\pi i \frac{r}{n}} \tilde{R} \cdot \tilde{c}_{a,b} \left( -\frac{1}{2} \tilde{R} \cdot \tilde{R} \right) \frac{(\tilde{R} \cdot \tilde{Q})^2}{Q \cdot Q} (\tilde{R} \cdot \tilde{y}_2)^2 = 96 \tilde{y}_2^2 \cdot \tilde{y}_2^2.$$  

With this information, one may calculate the rational terms in the threshold corrections explicitly and in particular verify that the conditions (3.8) and (3.9) are indeed obeyed in the case of the standard embedding.
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