OSTROWSKI TYPE INEQUALITIES FOR $s$–LOGARITHMICALLY CONVEX FUNCTIONS IN THE SECOND SENSE WITH APPLICATIONS

MEVLÜT TUNÇ AND AHMET OCAK AKDEMIR

ABSTRACT. In this paper, we establish some new Ostrowski type inequalities for $s$–logarithmically convex functions. Some applications of our results to P.D.F.'s and in numerical integration are given.

1. INTRODUCTION

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, the interior of the interval $I$, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [2]):

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right].$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1.1) see ([2]-[6]) and the references therein.

Let us recall some known definitions and results which we will use in this paper.

The function $f : I \subset \mathbb{R} \to [0, \infty)$ is said to be log–convex or multiplicatively convex if log $t$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality (See [7], p.7):

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)}.$$

In [8], authors introduced the class of $s$–logarithmically convex functions in the first sense as the following:

Definition 1. A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be $s$–logarithmically convex in the first sense if

$$f(\alpha x + \beta y) \leq [f(x)]^{\alpha s} [f(y)]^{\beta s}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $\alpha^s + \beta^s = 1$.

In [9], authors introduced the class of $s$–logarithmically convex functions in the second sense as the following:

Definition 2. A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be $s$–logarithmically convex in the second sense if

$$f(tx + (1-t)y) \leq [f(x)]^{ts} [f(y)]^{(1-t)s}$$

2000 Mathematics Subject Classification. 26D10, 26A15, 26A16, 26A51.

Key words and phrases. Hadamard’s inequality, $s$–geometrically convex functions.
for some $s \in (0, 1]$, where $x, y \in I$ and $t \in [0, 1]$.

Clearly, when taking $s = 1$ in Definition 1.2 or Definition 1.3 then $f$ becomes the standard logarithmically convex function on $I$.

The main purpose of this paper is to establish some new Ostrowski’s type inequalities for $s-$logarithmically convex functions. We also give some applications to P.D.F.’s and to midpoint formula.

2. THE NEW RESULTS

In order to prove our main results, we will use following Lemma which was used by Alomari and Darus (see [3]):

Lemma 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, be a differentiable mapping on $I$ where $a, b \in I$, with $a < b$. Let $f' \in L[a, b]$, then the following equality holds:

$$f(x) - \frac{1}{b - a} \int_{a}^{b} f(u) du = (b - a) \int_{0}^{1} p(t) f'(ta + (1 - t)b) dt$$

for each $t \in [0, 1]$, where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b - x}{b - a}\right] \\ t - 1, & t \in \left(\frac{b - x}{b - a}, 1\right] \end{cases}$$

for all $x \in [a, b]$.

Theorem 1. Let $I \supseteq [0, \infty)$ be an open interval and $f : I \rightarrow (0, \infty)$ is differentiable mapping on $I$. If $f' \in L[a, b]$ and $|f'|$ is $s-$logarithmically convex functions in the first sense on $[a, b]$, $a, b \in I$, with $a < b$, for some fixed $s \in (0, 1]$ then the following inequality holds:

$$(2.1) \quad \left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(u) du \right| \leq |f'(b)| \Psi(\tau, s, a, b)$$

where

$$\Psi(\tau, s, a, b) = \begin{cases} (b - a) \left[ \frac{\tau^s(\frac{b - x}{b - a}) (2s(\frac{b - x}{b - a}) - 1) + 1}{2s^2 \ln \tau} \right], & \tau < 1 \\ \frac{(a-x)^2 + (b-x)^2}{2(b-a)} & \tau = 1 \end{cases} \quad , \tau = \frac{|f'(a)|}{|f'(b)|}.$$
Proof. By Lemma 1 and since 
\[ |f'| \text{ is } s-\text{logarithmically convex function in the first sense on } [a, b], \]
we have
\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(u)du \right| 
\leq (b - a) \left[ \int_{0}^{\frac{b - a}{s - 2}} t |f'(ta + (1 - t)b)| dt + \int_{\frac{b - a}{s - 2}}^{1} (1 - t) |f'(ta + (1 - t)b)| dt \right]
\leq (b - a) \left[ \int_{0}^{\frac{b - a}{s - 2}} t |f'(a)|^{s-t} |f'(b)|^{1-t'} dt + \int_{\frac{b - a}{s - 2}}^{1} (1 - t) |f'(a)|^{s-t} |f'(b)|^{1-t'} dt \right].
\]
\[
\text{If } \left| \frac{f'(a)}{|f'(b)|} \right| = 1, \text{ it easy to see that}
\]
\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(u)du \right| \leq |f'(b)| \left[ \frac{(a - x)^2 + (b - x)^2}{2(b - a)} \right].
\]
\[
\text{If } \left| \frac{f'(a)}{|f'(b)|} \right| < 1, \text{ then } \left( \frac{|f'(a)|}{|f'(b)|} \right)^{s-t} \leq \left( \frac{|f'(a)|}{|f'(b)|} \right)^{s-t}, \text{ we have}
\]
\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(u)du \right| \leq (b - a) |f'(b)| \left[ \frac{\left( \frac{|f'(a)|}{|f'(b)|} \right)^{s-t} \left( \frac{2s(b - a)}{|f'(b)|} - 1 \right) + 1}{2s^2 \ln \left( \frac{|f'(a)|}{|f'(b)|} \right)} \right]
\]
\[
+ \left( \frac{|f'(a)|}{|f'(b)|} \right)^{s-t} \left( \frac{|f'(a)|}{|f'(b)|} \right)^{s-t} \left( \frac{2s(b - a)}{|f'(b)|} - 1 \right) \right].
\]
This completes the proof. \(\square\)

**Corollary 1.** If we choose \( |f'(x)| \leq M \) in (2.7), we obtain the inequality:
\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(u)du \right| \leq M\Psi(r, s, a, b)
\]
inequality holds:

\[
\begin{cases}
(b-a) \left[ M^\tau \left( \frac{2\tau - s}{2\tau - a} \right) (2s - b) - 1 \right] + M^\tau \left( \frac{2\tau - s}{2\tau - a} \right) (2b - a) - 1 \right] , & M < 1 \\
\frac{(a-x)^2 + (b-x)^2}{2(b-a)} , & M = 1
\end{cases}
\]

Corollary 2. If we choose \( x = \frac{a+b}{2} \) in (2.1), we obtain the inequality:

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq |f'(b)| \cdot \Psi (\tau, s, a, b)
\]

where

\[
\Psi (\tau, s, a, b) = \begin{cases}
(b-a) \left[ \frac{\tau \left( \frac{2\tau - s}{2\tau - a} \right) (s(b-a) - 1) + \frac{\tau (2\tau - s)}{2\tau - a} (a-b) - 1 \right] , & \tau < 1 \\
\frac{b-a}{4} , & \tau = 1
\end{cases}
\]

and

\[
\tau = \frac{|f'(a)|}{|f'(b)|}.
\]

Corollary 3. If we choose \( s = 1 \) in (2.1), we obtain the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq |f'(b)| \cdot \Psi (\tau, 1, a, b)
\]

where

\[
\Psi (\tau, 1, a, b) = \begin{cases}
(b-a) \left[ \frac{\tau \left( \frac{2\tau - s}{2\tau - a} \right) (2s - a) - 1 \right] + \frac{\tau (2\tau - s)}{2\tau - a} (a-b) - 1 \right] , & \tau < 1 \\
\frac{(a-x)^2 + (b-x)^2}{2(b-a)} , & \tau = 1
\end{cases}
\]

and

\[
\tau = \frac{|f'(a)|}{|f'(b)|}.
\]

Theorem 2. Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable mapping on \( I \). If \( f' \in L[a, b] \) and \( |f'|^p \) is \( s \)-logarithmically convex functions in the first sense on \( [a, b] \), \( a, b \in I \), with \( a < b \), for some fixed \( s \in (0, 1] \), then the following inequality holds:

\[
(2.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{|f'(b)|}{(p+1)^{\frac{1}{p}}} \cdot \Psi (\tau, s, a, b)
\]
Proof. From Lemma 1 and by using the Hölder integral inequality, we have

\[
\Psi(\tau, s, a, b) = \begin{cases} 
\frac{1}{(b-a)^p} \left[ (b-x)^{\frac{p+1}{p}} \left( \frac{e^{s\left(\frac{b-x}{b-a}\right)}}{s q \ln \tau} \right)^{\frac{1}{p}} + (x-a)^{\frac{p+1}{p}} \left( \frac{e^{s\left(\frac{a-x}{b-a}\right)}}{s q \ln \tau} \right)^{\frac{1}{p}} \right], & \tau < 1 \\
\frac{(b-x)^2 + (x-a)^2}{b-a}, & \tau = 1
\end{cases}
\]

and

\[ \tau = \frac{|f'(a)|}{|f'(b)|}. \]

for \( q > 1 \) and \( p^{-1} + q^{-1} = 1. \)

Proof. From Lemma 1 and by using the Hölder integral inequality, we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \left( \int_0^t t^p \, dt \right)^{\frac{1}{p}} \left( \int_0^t \left| f'(ta + (1-t)b) \right|^q \, dt \right)^{\frac{1}{q}} + \left( \int_{\frac{b-a}{a}}^1 (1-t)^p \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{b-a}{a}}^1 \left| f'(ta + (1-t)b) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

Since \(|f'|\) is \( s \)--logarithmically convex function in the first sense, we can write

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) |f'(b)| \left( \int_0^t t^p \, dt \right)^{\frac{1}{p}} \left( \int_0^t \left( \frac{|f'(a)|}{|f'(b)|} \right)^q t^r \, dt \right)^{\frac{1}{q}} + \left( \int_{\frac{b-a}{a}}^1 (1-t)^p \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{b-a}{a}}^1 \left( \frac{|f'(a)|}{|f'(b)|} \right)^q t^r \, dt \right)^{\frac{1}{q}}.
\]

If \( \frac{|f'(a)|}{|f'(b)|} = 1 \), then we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{|f'(b)|}{(p+1)^{\frac{1}{p}}} \left[ (b-x)^2 + (x-a)^2 \right].
\]
On the other hand, if \( \frac{|f'(a)|}{|f'(b)|} < 1 \), then 
\[
\left( \frac{|f'(a)|}{|f'(b)|} \right)^{qt} \leq \left( \frac{|f'(a)|}{|f'(b)|} \right)^{sqt},
\]
thereby
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right|
\leq (b-a) |f'(b)| \left[ \left( \int_0^{\frac{1}{p}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{sqt}} \left( \frac{|f'(a)|}{|f'(b)|} \right)^{sqt} dt \right)^{\frac{1}{sqt}} + \left( \int_0^{\frac{1}{p}} (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{sqt}} \left( \frac{|f'(a)|}{|f'(b)|} \right)^{sqt} dt \right)^{\frac{1}{sqt}} \right].
\]
By computing the above integrals the proof is completed. \( \square \)

**Corollary 4.** If we choose \(|f'(x)| \leq M\) in (2.2), we obtain the inequality:
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{(p+1)^{\frac{1}{p}}} \Psi(\tau, s, a, b)
\]
where
\[
\Psi(\tau, s, a, b) = \left\{ \begin{array}{ll}
\frac{1}{(b-a)^p} \left[ (b-x)^{\frac{p+1}{p}} \left( M^{\frac{a}{sq in M}} - 1 \right) + (x-a)^{\frac{p+1}{p}} \left( M^{\frac{a}{sq in M}} - 1 \right) \right] , & M < 1 \\
\frac{(b-a)^2 + (x-a)^2}{b-a} , & M = 1
\end{array} \right.
\]

**Corollary 5.** If we choose \( x = \frac{a+b}{2} \) in (2.2), we obtain the inequality:
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{|f'(b)|}{|f'(b)|} \right) \Psi(\tau, s, a, b)
\]
where
\[
\Psi(\tau, s, a, b) = \left\{ \begin{array}{ll}
\left( \frac{\tau x s}{sq in \tau} \right)^{\frac{1}{\tau}} + \left( \frac{\tau x s}{sq in \tau} \right)^{\frac{1}{\tau}} , & \tau < 1 \\
\frac{1}{2} , & \tau = 1
\end{array} \right.
\]
and
\[
\tau = \frac{\left| f'(a) \right|}{|f'(b)|}.
\]
3. APPLICATIONS FOR P.D.F’s

Let \( X \) be a random variable taking values in the finite interval \([a, b]\), with the probability density function \( f : [a, b] \rightarrow [0, 1] \) with the cumulative distribution function \( F(x) = \Pr(X \leq x) = \int_a^b f(t) dt \).

**Theorem 3.** Under the assumptions of Theorem 1, we have the inequality:

\[
\left| \Pr(X \leq x) - \frac{1}{b-a} (b - E(x)) \right| \leq |f'(b)| \Psi (\tau, s, a, b)
\]

where \( E(x) \) is the expectation of \( X \) and \( \Psi (\tau, s, a, b) \) as defined in Theorem 1.

**Proof.** The proof follows immediately from the fact that:

\[
E(x) = \int_a^b t F(t) = b - \int_a^b F(t) dt.
\]

\( \square \)

**Theorem 4.** Under the assumptions of Theorem 2, we have the inequality:

\[
\left| \Pr(X \leq x) - \frac{1}{b-a} (b - E(x)) \right| \leq \frac{|f'(b)|}{(p+1)^3} \Psi (\tau, s, a, b)
\]

where \( E(x) \) is the expectation of \( X \) and \( \Psi (\tau, s, a, b) \) as defined in Theorem 2.

**Proof.** Likewise the proof of the previous theorem, by using the fact that:

\[
E(x) = \int_a^b t F(t) = b - \int_a^b F(t) dt
\]

the proof is completed. \( \square \)

4. APPLICATIONS IN NUMERICAL INTEGRATION

Let \( d \) be a division of the closed interval \([a, b]\), i.e., \( d : a = x_0 < x_1 < ... < x_{n-1} < x_n = b \), and consider the midpoint formula

\[
M(f, d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_i + x_{i+1}}{2} \right).
\]

It is well known that if the mapping \( f : [a, b] \rightarrow \mathbb{R} \), is differentiable such that \( f''(x) \) exists on \((a, b)\) and \( K = \sup_{x \in (a, b)} |f''(x)| < \infty \), then

\[
I = \int_a^b f(x) dx = M(f, d) + E_M(f, d),
\]

where the approximation error \( E_M(f, d) \) of the integral \( I \) by the midpoint formula \( M(f, d) \) satisfies

\[
|E_M(f, d)| \leq \frac{K}{24} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.
\]

Now, it is time to give some new estimates for the remainder term \( E_M(f, d) \) in terms of the first derivative of \( s \)-logarithmically convex functions.
Proposition 1. Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable mapping on $I$. If $f' \in L[a, b]$ and $|f'|$ is $s$-logarithmically convex functions in the first sense on $[a, b]$, $a, b \in I$, with $a < b$, for some fixed $s \in (0, 1)$, then the following inequality holds for every division $d$ of $[a, b]$: 

$$|E_M(f, d)| \leq \sum_{i=0}^{n-1} |f'(x_{i+1})| \Psi(\tau, s, x_i, x_{i+1})$$

where 

$$\Psi(\tau, s, x_i, x_{i+1}) = \begin{cases} 
(x_{i+1} - x_i) \left[ \frac{(x_{i+1} - x_i)}{\tau} \frac{2^s}{\ln \tau} + \frac{2^s}{\ln \tau} \right], & \tau < 1 \\
\frac{x_{i+1} - x_i}{4}, & \tau = 1 
\end{cases}$$

and 

$$\tau = \frac{|f'(x_i)|}{|f'(x_{i+1})|}.$$ 

Proof. By applying Corollary 2 on the sub-intervals $[x_i, x_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ of the division $d$, we have 

$$\left| f \left( \frac{x_i + x_{i+1}}{2} \right) - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(u) du \right| \leq |f'(x_{i+1})| \Psi(\tau, s, x_i, x_{i+1}).$$

By summing over $i$ from 0 to $n - 1$, it is easy to see that 

$$\left| M(f, d) - \int_a^b f(x) dx \right| \leq \sum_{i=0}^{n-1} |f'(x_{i+1})| \Psi(\tau, s, x_i, x_{i+1})$$

which completes the proof. \qed 

Proposition 2. Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable mapping on $I$. If $f' \in L[a, b]$ and $|f'|$ is $s$-logarithmically convex functions in the first sense on $[a, b]$, $a, b \in I$, with $a < b$, for some fixed $s \in (0, 1)$, then the following inequality holds for every division $d$ of $[a, b]$: 

$$|E_M(f, d)| \leq \sum_{i=0}^{n-1} \left( \frac{x_{i+1} - x_i}{2} \right)^{q} \left( \frac{|f'(x_{i+1})|}{(p + 1)^{q}} \right) \Psi(\tau, s, x_i, x_{i+1})$$

where 

$$\Psi(\tau, s, x_i, x_{i+1}) = \begin{cases} 
\left( \frac{(x_{i+1} - x_i)}{\tau} \frac{1}{\ln \tau} - 1 \right)^{\frac{1}{q}}, & \tau < 1 \\
\frac{1}{q}, & \tau = 1 
\end{cases}$$

and 

$$\tau = \frac{|f'(x_i)|}{|f'(x_{i+1})|}.$$ 

for $q > 1$ and $p^{-1} + q^{-1} = 1$.

Proof. The proof of the result is similar to the proof of the Proposition 1, by applying Corollary 5. \qed
ON SOME INEQUALITIES OF OSTROWSKI TYPE

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KILIS 7 Aralık University, Faculty of Science and Arts, Department of Mathematics, Kilis, 79000, Turkey.

E-mail address: mevluttunc@kilis.edu.tr

AĞrı İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, AĞrı, Turkey.

E-mail address: ahmetakdemir@agri.edu.tr