Linear idempotents in Matsuo algebras

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Abstract

Matsuo algebras are an algebraic incarnation of 3-transposition groups with a parameter \(\alpha\), where idempotents take the role of the transpositions. We show that a large class of idempotents in Matsuo algebras satisfy the Seress property, making these nonassociative algebras well-behaved analogously to associative algebras, Jordan algebras and vertex (operator) algebras. We calculate eigenvalues in the Matsuo algebra of \(\text{Sym}(n)\) for any \(\alpha\), generalising some vertex algebra results for which \(\alpha = \frac{1}{4}\). Finally, in the Matsuo algebra of the root system \(D_n\), we show \(n - 3\) conjugacy classes of involutions coming from the Weyl group are in natural bijection with idempotents in the algebra via their fusion rules.

Idempotents play a distinguished role in algebras. In matrix algebras and generally in associative algebras, idempotents are projections onto subspaces, with eigenvalues 1 and 0. In nonassociative algebras, the situation is more subtle. In for example the classical theory of Jordan algebras, structural theorems depend on the existence of idempotents, which now admit eigenvalues 1, 0 and \(\frac{1}{2}\); a key result is that the product of a \(\phi\)-eigenvector with a \(\psi\)-eigenvector is a sum of \(\phi \ast \psi\)-eigenvectors, according to the fusion rules \(\Phi(\alpha)\) of Table 2 with \(\alpha = \frac{1}{2}\). In some ways, idempotents are also analogous to \(sl_2\)-subalgebras of Lie algebras.

These ideas are captured by axial algebras, which are algebras generated by idempotents satisfying some fusion rules \(\Phi\). An important source of axial algebras are the weight-2 subalgebras of a special class of vertex (operator) algebras, where the fusion rules come from the representation theory of the Virasoro algebra. The most famous instance is the Griess algebra—the weight-2 subalgebra of the vertex algebra \(V\)—whose automorphism group is the Monster. This algebra is generated by idempotents with eigenvalues 1, 0, \(\frac{1}{2}, \frac{1}{2}\) satisfying the Ising fusion rules. These fusion rules are \(\mathbb{Z}/2\)-graded, so each such idempotent induces an involution, in the conjugacy class \((2A)\). These involutions generate the entire group and have pairwise products of order at most 6, whence the Monster is a 6-transposition group.

The fusion rules \(\Phi(\alpha)\) are simpler but also \(\mathbb{Z}/2\)-graded. Using the grading, in [HRS14] it is shown that the involutions induced from \(\Phi(\alpha)\)-idempotents generate a 3-transposition group \(G\) if and only if those idempotents generate a Matsuo algebra \(M_\alpha(G)\), and this is always the case when \(\alpha \neq 0, \frac{1}{2}, 1\). Here \(G\) is the Fischer space, a graph, of \(G\).

In this text we investigate further algebraic properties of such \(M_\alpha(G)\). For a 3-transposition group \(G\), its Matsuo algebra \(M_\alpha(G)\) may be thought of as an alternative to its ordinary group algebra \(\mathbb{F}G\); the theory of group algebras, for example relating central primitive idempotents to irreducible characters, has already been fruitfully well-developed [P79]. The key to our approach is that linear idempotents provide a direct link between group-theoretic properties of \(G\) and structural properties of the algebra \(M_\alpha(G)\). Our results, outlined below, are a first step of the same programme for arbitrary idempotents in axial algebras.

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1 when \(\alpha = \frac{1}{2}\), the Jordan algebras with associated 3-transposition groups are classified in [DMR15]
Section 1 recalls definitions—in particular, the Seress property of the fusion rules of an idempotent $e$ is that $e$ is globally associative with its $1,0$-eigenspaces—and preliminary results. Section 2 presents Hypothesis 2.3 on 3-transposition groups, asking that maximal 3-transposition subgroups act transitively on the transpositions they do not contain, and Theorem 2.2 showing that this holds in large classes of examples. In Section 3, we introduce Definition 3.7, the linear idempotents: idempotents which are identities of parabolic subalgebras, that is, come from 3-transposition subgroups, closed under differences $e - f$ when $f$ is in the 1-eigenspace of $e$. We then prove that the following weakening of associativity holds:

**Theorem (3.8).** Linear idempotents in $M_\alpha(G)$ are Seress (over a suitable field) if the 3-transposition $G$ group of $G$ satisfies Hypothesis 2.3.

The Seress property is well-known to hold for all idempotents in Jordan algebras by the Peirce decomposition and multiplication of eigenspaces. Similarly, it holds for all idempotents in weight-2 subalgebras of vertex algebras via the fusion rules of the Virasoro algebra and application of [M96], Lemma 5.1. However, c.f. [M03] Proposition 3.3.8 and [DMR15], these are far from including all Matsuo or axial algebras; to our knowledge, this paper is the first which handles a general class of idempotents in these nonassociative algebras.

A particular application of Theorem 3.8 is that we can find those tori, i.e., maximal associative subalgebras, which arise from chains of parabolic subgroups via identity elements of chains of parabolic subalgebras. The search for tori was initiated in [CR15], and goes back to classical work on the Monster, including [MN93] and framed vertex algebras, although we believe they still merit further investigation.

Specialising to the case of the Matsuo algebra of the symmetric group $\text{Sym}(n)$ of degree $n$, in Section 4, we achieve

**Theorem (4.7).** The eigenvalues of primitive linear idempotents in $M_\alpha(A_n^\pm)$ are determined.

When $\alpha = \frac{1}{4}$, by Theorem 1.8, due to [DLMN96], the specialisation of Theorem 4.7 in Lemma 4.8 determines the highest weights of the Virasoro algebra occurring at weight 2 in the lattice vertex algebra of the root system $\sqrt{2}A_n$, giving a proof of the results of [Y01] that generalises to other situations.

Finally in Section 5 we consider $D_n$, where we observe some new bijections between involutions in the group and idempotents in the algebra, via their fusion rules.

**Theorem (5.3).** In $\text{Aut}(W(D_n)/Z(W(D_n)))$, $n - 3$ conjugacy classes of involutions are realised as Miyamoto involutions of linear idempotents in $M_\alpha(D_n)$.

Note that, as $A_n^\pm = D_{n+1}$, in these last two sections we are considering the same object from two (combinatorially) different points of view.

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# Preliminaries

**1.1 Definition.** A 3-transposition group $(G, D)$ consists of a group $G$ generated by a set of involutions $D \subseteq G$ such that

i. $D$ is closed under $G$-conjugation, and ii. for any $c, d \in D$, $|cd| \leq 3$. 

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For further material on 3-transposition groups, including their classification, we refer to [A97] or [H93].

Recall that any 3-transposition group \((G, D)\) is uniquely characterised, up to center, by a

1.2 Definition. A Fischer space \(G\) is a graph whose lines are sets of vertices of size 3 such that, if \(\ell_1, \ell_2\) are two distinct intersecting lines,

i. \(|\ell_1 \cap \ell_2| = 1\) and

ii. their points span a subspace isomorphic to \(P_2^\vee\) or \(P_3\) in Figure 1.

![Figure 1: The dual affine plane \(P_2^\vee\) and the affine plane \(P_3\)](image)

Namely, for \((G, D)\) a 3-transposition group its Fischer space \(G\) has point set \(D\) and lines \(\{c,d,e\}\) for any \(c,d,e \in D\) such that \(\langle c, d, e \rangle \cong \text{Sym}(3)\).

Some interesting Fischer spaces are Weyl groups \(W(X_n)\) of root systems \(X_n\) (for the latter two topics, we refer to [C05]); Recall that the Weyl group of \(A_n\) is \(\text{Sym}(n+1)\) and the Weyl group of \(D_n\) is \(2^n : \text{Sym}(n+1)\). The set of reflections \(D\) coming from the roots make \((W(X_n), D)\) a 3-transposition group in each case. When \(X_n\) is simply-laced, as \(A_n, D_n\) are, we use \(X_n\) to denote the Fischer space of its Weyl group.

For distinct points \(x, y\) in a Fischer space \(G\), we write \(x \sim y\) if there exists a line in \(G\) containing both \(x\) and \(y\), and \(x \not\sim y\) otherwise. If \(x \sim y\), the line \(\ell\) containing \(x\) and \(y\) is unique by i. of Definition 1.2. We write \(x \land y\) for \(\ell = \{x, y, x \land y\}\).

In [M03], Matsuo introduced an algebra on Fischer spaces:

1.3 Definition. The Matsuo algebra \(M_\alpha(G)_R\) of the Fischer space \(G\) over the ring \(R\) containing \(\frac{\alpha}{2}\) is the free \(R\)-module spanned by the points of \(G\) together with the bilinear multiplication, for \(x, y\) points of \(G\),

\[
xy = \begin{cases} 
  x & \text{if } x = y, \\
  0 & \text{if } x \not\sim y, \\
  \frac{\alpha}{2}(x + y - x \land y) & \text{if } x \sim y.
\end{cases}
\]
We view $G$ as embedded in $M_{\alpha}(G)_{R}$, so that $x \in G$ is an idempotent, that is, $xx = x$. The following definitions come with a view towards the idempotents in Matsuo algebras.

Suppose that $A$ is an algebra over a ring $R$. For arbitrary $x \in A$, write $\text{ad}(x)$ for the adjoint map in $\text{End}(A)$ that is left-multiplication: $\text{ad}(x): y \rightarrow xy$. The eigenvalues, eigenvectors and eigenspaces of $x$ are the eigenvalues, eigenvectors and eigenspaces of $\text{ad}(x)$. The element $x$ is also said to be diagonalisable if $\text{ad}(x)$ is diagonalisable as a matrix, that is, there exists a basis of $A$ consisting of eigenvectors of $x$. We write, for $\phi \in R$,

$$A_{\phi} = \{ y \in A \mid xy = \phi y \}$$

(2)

for the $\phi$-eigenspace of $x$, and extend the notation so that, for $\Phi \subseteq R$ a set,

$$A_{\Phi} = \bigoplus_{\phi \in \Phi} A_{\phi}$$

(3)

including $A_{\emptyset} = 0$.

1.4 Definition. Fusion rules are a set $\Phi \subseteq R$ together with a symmetric map $\star : \Phi \times \Phi \rightarrow 2^{\Phi}$. A diagonalisable idempotent $x \in A$ is a $\Phi$-axis if all of its eigenvalues lie in $\Phi$ and

$$A_{\phi}A_{\psi} \subseteq A_{\phi \star \psi} = \bigoplus_{\chi \in \phi \star \psi} A_{\chi}$$

(4)

that is, the product $yz$ of a $\phi$-eigenvector with a $\psi$-eigenvector is in the span of $\chi$-eigenvectors with $\chi \in \phi \star \psi$.

1.5 Definition. Fusion rules $\Phi$ containing $0, 1$ are Seress if $1 \star \phi \subseteq \{ \phi \} \supseteq 0 \star \phi$ for all $\phi \in \Phi$. In particular, this means $1 \star 0 = \emptyset$.

We observe that an idempotent in an associative algebra has eigenvalues 1, 0 and its eigenvectors multiply according to $1 \star 1 = \{ 1 \}, 0 \star 0 = \{ 0 \}$ and $1 \star 0 = \emptyset$, so it satisfies Seress fusion rules.

1.6 Lemma. An element $e \in A$ has fusion rules $\Phi$ which are Seress if and only if $e$ associates with its 1, 0-eigenspace $A_{\{1,0\}}^{e}$.

Proof. Suppose that $e \in A$ a $\Phi$-axis. Let $x, z \in A$ be arbitrary. By linearity, we may take $x \in A_{\phi}^{e}$. Then $ex = \phi x$ and in particular $(ex)z = \phi xz$.

Observe that $xz \in A_{\phi}^{e}$ for any $x \in A_{\phi}^{e}, z \in A_{\{1,0\}}^{e}$, for any $\phi \in \Phi$, if and only if $\Phi$ is Seress. Furthermore $xz \in A_{\phi}^{e}$ if and only if $e(xz) = \phi xz$, that is, $e(xz) = (ex)z$. □

| $\cdot$ | 1 | 0 | $\alpha$ |
|-------|---|---|---------|
| 1     | $\{1\}$ | $\emptyset$ | $\{\alpha\}$ |
| 0     | $\{0\}$ | $\{\alpha\}$ |        |
| $\alpha$ | | $\{1,0\}$ |        |

Table 2: Jordan fusion rules $\Phi(\alpha)$

The Jordan fusion rules of Table 2 take a primary role in this work. It is not difficult to see that they are Seress, and
1.7 Lemma ([HRS14], Theorem 6.2). Any point $x \in \mathcal{G} \subseteq M_{\alpha}(\mathcal{G})$ is a $\Phi(\alpha)$-axis. □

We finally note

1.8 Theorem ([DLMN96] Theorem 3.1). The Matsuo algebra $M_{1/4}^{1/2}(\mathcal{X}_n^\pm)$ (c.f. Definition 4.1), modulo its radical (c.f. (24)), is the weight-2 subalgebra of a vertex algebra $V_+^{\pm2}$ when $\mathcal{X}_n$ is the Fischer space of a simply-laced root system $X_n$, that is, $A_n, n \in \mathbb{N}, D_n, n \geq 4$, or $E_6, E_7, E_8$. The radical of $M_{1/4}^{1/2}(A_n^\pm)$ is 0 for all $n$. □

1.9 Theorem (Perron-Frobenius, [GR01] Theorem 8.8.1). For an irreducible matrix $A$ over $\mathbb{C}$, there exists a real positive eigenvalue $\rho$ of $A$ such that $|\lambda| \leq |\rho|$ for all eigenvalues $\lambda$ of $A$, and the $\rho$-eigenspace of $A$ is 1-dimensional. □

Recall that the adjacency matrix of a connected graph is irreducible, and if $A$ is $k$-regular (that is, the neighbourhood of every point has size $k$) then $\rho = k$ in Theorem 1.9.

2 3-transposition groups

Suppose that $(G, D)$ is a 3-transposition group. A subgroup $H \subseteq G$ is parabolic (also called a $D$-subgroup) if $H$ is generated by $H \cap D$. A maximal parabolic subgroup is a parabolic subgroup $H$ maximal by inclusion among parabolic subgroups. We call $(G, D)$ connected if $D$ is a single conjugacy class, that is, $D = d^G$ for any $d \in D$. When $(G, D)$ is connected, so is the Fischer space $\mathcal{G}$ of $(G, D)$, and there exists a constant $k_{G} \in \mathbb{N}$ such that for any $x \in \mathcal{G}$, $|x^\sim| = |\{y \in \mathcal{G} \mid x \sim y\}| = k_{G}$, that is, as a graph $\mathcal{G}$ is $k_{G}$-regular.

The boundary graph $\mathcal{G}/\mathcal{H}$ of an embedding $\mathcal{H} \subseteq \mathcal{G}$ of Fischer spaces is the graph with point set

$$\mathcal{H}^\sim = \{x \in \mathcal{G} \mid x \notin \mathcal{H}, x \sim y \text{ for some } y \in \mathcal{H}\}$$

and lines $\{x, y\}$ for all $x, y \in \mathcal{G} \setminus \mathcal{H}$ such that $x \wedge y \in \mathcal{H}$.

2.1 Definition. An embedding $\mathcal{H} \subseteq \mathcal{G}$ of Fischer spaces is very regular if $\mathcal{H}$ is a maximal subspace in $\mathcal{G}$, and $\mathcal{H}, \mathcal{G}, \mathcal{G}/\mathcal{H}$ are connected.

A parabolic subgroup $H$ of a 3-transposition group $(G, D)$ is very regular if it induces a very regular embedding of Fischer spaces $\mathcal{H} \subseteq \mathcal{G}$.

For a very regular embedding $\mathcal{H} \subseteq \mathcal{G}$ it follows that $\mathcal{G}/\mathcal{H}$ is $k_{G}^\mathcal{H}$-regular for some $k_{G}^\mathcal{H} \in \mathbb{N}$.

We conjecture that an arbitrary maximal connected parabolic subgroup $H$ of a connected 3-transposition group $(G, D)$ is very regular; this holds for many known examples, as shown in the following Theorem 2.2.

Recall that $A_n$ is the affine extension of $A_{n-1}$. For a Weyl group $W(X_n)$, the transpositions are the conjugacy class of reflections of roots in $X_n$. To define the group $G = W_k(A_n)$ for $k = 2, 3$, let $V$ be the vector space $F_{n+1}^k$ with basis $\{v_0, \ldots, v_n\}$ and $\hat{G}$ the semidirect product of $V$ with $\text{Sym}(n + 1)$ using the permutation action on the given basis. Then $G$ is the quotient $\hat{G}/\langle v_0 + \cdots + v_n \rangle$, and the transpositions in $G$ are the image of the conjugacy class of $(1, 2)^{\text{Sym}(n+1)}$. By $3^n : 2$ we mean the elementary abelian group $3^n$ extended by an inverting involution, unless otherwise indicated. In all groups of shape $3^n : 2$, the transpositions are the unique class of involutions.

2.2 Theorem. The connected maximal parabolic subgroups $H$ of $(G, D)$ induce very regular Fischer spaces $\mathcal{H} \subseteq \mathcal{G}$ when $(G, D)$ is, for any $n \in \mathbb{N}$, the Weyl group of $A_n, D_n, E_6, E_7, E_8$, or $W_k(A_n)$ for $k = 2, 3$, or $3^n : 2$, or M. Hall’s $3^{10} : 2$. 

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Proof. Suppose that $\mathcal{H}, \mathcal{G}$ are connected. We now show that the group-theoretic condition that

$$D = (H \cap D) \cup d^H$$

for any $d \in D \setminus H$ implies that $\mathcal{G}/\mathcal{H}$ is connected: namely, the conjugation action of $H$ on $d$ is afforded by its generators $H \cap D$, and two elements $d, d' \in D$ are $H$-conjugate if and only if the corresponding points $d, d' \in \mathcal{G}$ can be path-connected in $\mathcal{G}$ by lines nontrivially intersecting $\mathcal{H}$. Thus, the proof of this theorem is reduced to showing that (6) holds in each case.

Recall that the Weyl group $(G, D)$ of $A_n$ is $G = \text{Sym}(n+1), D = (1, 2)^G$. Let $E \subseteq D$ and $S \subseteq \{1, \ldots, n+1\}$ be the support of $E$, that is, the smallest subset $S$ of $\{1, \ldots, n+1\}$ such that any transposition $e \in E$ is of the form $(s_1, s_2)$ for some $s_1, s_2 \in S$. Then partition $S$ into orbits $S_1, \ldots, S_n$ of $\langle E \rangle$. Observe that $\langle E \rangle \cong \text{Sym}(|S_1|) \times \cdots \times \text{Sym}(|S_n|)$ and therefore $E$ does not satisfy the hypothesis of connectedness unless $S = S_1$ is a single orbit. Furthermore if $|S|$ is less than $n$ then $H$ is not maximal. Therefore a connected maximal parabolic subgroup $H$ of $G$ has support $\{1, \ldots, j-1, j+1, \ldots, n+1\}$ for some $j$ and $H \cong \text{Sym}(n)$. In these cases let $d = (1, j), \text{or } d = (1, 2)$ if $j = 1$, so that $d \in D \setminus (D \cap H)$. We see that $D = (H \cap D) \cup d^H$.

As $W(D_n) \cong W_2(\hat{A}_{n-1})$ by [H93], we cover it below as part of $W_k(\hat{A}_{n-1})$.

The cases for $W(E_n), n = 6, 7, 8$, were checked in [MAGMA] with the computational assistance of Raul Moragues Moncho.

Suppose that $(G, D)$ comes from $W_k(\hat{A}_n)$ when $k = 2, 3$ and $n \geq 3$. There are two possibilities for a parabolic subgroup $H$ such that $H \cap D$ is a single conjugacy class: either $H$ is isomorphic to $\text{Sym}(n)$ or to $W_k(\hat{A}_{n-1})$. We use a representation of $G$ as a matrix group. Let

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-1}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}, \ldots, \quad g_{n-1} = I_{n-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (1),$$

$$g_{n+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{r-3} \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} I_{n+1} \\ 1 \end{pmatrix}$$

over $\mathbb{F}_k$.

Then $G \cong \langle g_1, \ldots, g_{n+1} \rangle/\langle h \rangle$ and $D$ is the set of conjugates of $\{g_i\}_{1 \leq i \leq n+1}$. We also set $\hat{G} = \langle g_1, \ldots, g_{n+1} \rangle$ and $\hat{D}$ the set of conjugates of $\{g_i\}_{1 \leq i \leq n+1}$. Now $H \cong W_k(\hat{A}_{n-1})$ if and only if, up to conjugation, $H = \hat{H}/\langle h \rangle$ for $\hat{H} = \langle g_1, \ldots, g_{n-1}, g_{n+1} \rangle$. Then it is clear that in $\hat{G}$, $\hat{D} = (\hat{H} \cap \hat{D}) \cup g_n^H$. The same property descends to the quotient, so that $D = (H \cap D) \cup (g_n(n))^H$. This shows that $\mathcal{G}/\mathcal{H}$ is connected, so $\mathcal{H} \subseteq \mathcal{G}$ is very regular. The other possibility is that $H = \langle g_1, \ldots, g_n \rangle/\langle n \rangle$. In this case, when $k = 2$ we see that $W_2(\hat{A}_{n-1}) \cong W(D_n)$. We can observe that in general in $\hat{G}$, the orbit of $g_{n+1}$ under the action of $\hat{H} = (g_1, \ldots, g_n)$ has size $\frac{1}{2}n(n+1)$ if $k = 2$ and $n(n+1)$ if $k = 3$, so that $\hat{H}$ is transitive on the transpositions in $\hat{G}$ outside $\hat{H}$. This again holds in the quotient $H$.

When $G = 3^n : 2$, there is only one conjugacy class $D$ of involutions. Observe that any subset of involutions of $G$ generates a subgroup $H \cong 3^m : 2$ for some $m$. Then $H$ is maximal if $m = n - 1$. In this case, if $t, s$ are two transpositions in $D \setminus H$, then $\langle t, s \rangle \cap H = \{t^s\}$ as $t^s \not\in H$ would contradict maximality, so $\mathcal{G}/\mathcal{H}$ is connected. This shows that it is also regular by transitivity.

That the statement holds for M. Hall’s $G \cong 3^{10} : 2$ was checked in [GAP] using the presentation

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^3 = (ac)^3 = (ad)^3 = (bc)^3 = (bd)^3 = (cd)^3 = (b^c d)^3 = (a^b c)^3 = (a^b d)^3 = (a^c d)^3 = (a^b c d)^3 = (a^b c d)^3 = 1 \rangle,$$
given in [H93], Proposition 2.9.

By inductive application of Theorem 2.2, we have that the hypothesis

2.3 Hypothesis. For any parabolic subgroups $K, H$ of $(G, D)$ such that $K$ is a maximal parabolic subgroup of $H$, $K$ is very regular in $H$.

holds for many examples of $(G, D)$.

3 Idempotents

Let $\mathbb{F}$ be a field containing $\frac{1}{2}$ and $\alpha$. In this section, we investigate an important class of idempotents, coming from identity elements of parabolic subalgebras, and show that they are well-behaved.

3.1 Lemma. If $G$ is a connected Fischer space and $\alpha \neq -\frac{2}{k_G}$ then $A = M_\alpha(G)\mathbb{F}$ is unital, with

$$\text{id}_G = \frac{1}{1 + \frac{2}{k_G}} \sum_{x \in G} x. \quad (9)$$

Proof. We show that, for $x \in G$,

$$x \sum_{y \in G} y = (1 + \frac{1}{2} \alpha k_G)x. \quad (10)$$

Observe that $G = \{x\} \cup x^\sim \cup x^{\not{\sim}}$, for $x^\sim = \{y \in H \mid x \sim y\}$ and $x^{\not{\sim}} = \{y \in H \mid x \not{\sim} y\}$; also, $|x^\sim| = k_G$. Then

$$x \sum_{y \in G} y = xx + x \sum_{y \in x^\sim} y + x \sum_{y \in x^{\not{\sim}}} y = x + \frac{\alpha}{2} \sum_{y \in x^\sim} (x + y - x \land y) + 0 = (1 + \frac{1}{2} \alpha k_G)x, \quad (11)$$

where the last equality follows since, as $y$ ranges over $x^\sim$, so does $x \land y$: that is, $\{x \land y \mid y \in x^\sim\} = x^\sim$ and $\sum_{y \in x^\sim} (y - x \land y) = \sum_{y \in x^\sim} y - \sum_{y \in x^\sim} x \land y = 0$.

If $1 + \frac{2}{k_G}$ is nonzero, that is, $\alpha \neq -\frac{2}{k_G}$, it follows that $\text{id}_G$ of (9) is the identity of $A$. \qed

This result, and its sequels, admits generalisation to nonconnected Fischer spaces $G$. If $G = G_1 \cup \cdots \cup G_n$ is a partition into pairwise disconnected Fischer spaces such that each $M_\alpha(G_i)$ is unital, then $\text{id}_G = \sum_i \text{id}_{G_i}$.

Write $\text{Spec}(M)$ for the eigenvalues of a matrix $M$, and by extension $\text{Spec}(G)$ for the eigenvalues of the adjacency matrix of a graph $G$. Also recall the Minkowski addition, and difference, of sets: $X \pm Y = \{x \pm y \mid x \in X, y \in Y\}$. For $x \in \mathbb{F}$ and $Y \subseteq \mathbb{F}$, we will write $x - Y = \{x\} - Y$. Now we can describe the eigenvalues of $\text{id}_H$.

3.2 Lemma. Suppose that $H \subseteq G$ is very regular and $\alpha \neq -\frac{2}{k_H}$. Then $\text{id}_H$ in $A = M_\alpha(G)\mathbb{F}$ acts diagonalisably on the subspaces spanned by

$H$ with eigenvalue 1, and

$H \cup H^\sim$ with further eigenvalues $\frac{\alpha}{2 + \alpha k_H} (k_G^H - \text{Spec}(G/H)).$

Proof. By Lemma 3.1, $(H)\mathbb{F}$ is a subspace of the 1-eigenspace of $\text{id}_H$.

Take $y \in H^\sim$, where $H^\sim$ is defined in (5). Then $y \not{\in} H$ and

$$\text{id}_H y = \frac{1}{1 + \frac{2}{k_H} \alpha} \sum_{x \in H \cap y^\sim} \frac{\alpha}{2} (x + y - x \land y). \quad (12)$$
If $x \in \mathcal{H}$ and $y \in x^\sim \setminus \mathcal{H}$ then $y, x \wedge y \in \mathcal{H}^\sim$, so that $\text{id}_\mathcal{H}$ fixes the subspace spanned by $\mathcal{H} \cup \mathcal{H}^\sim$. Furthermore, as $k_\mathcal{H}^\mathcal{H} = |y^\sim \cap \mathcal{H}|$,

$$\text{id}_\mathcal{H} y = \frac{\alpha k_\mathcal{H}^\mathcal{H}}{2 + \alpha k_\mathcal{H}} y + \frac{\alpha}{2 + \alpha k_\mathcal{H}} \sum_{x \in y^\sim \cap \mathcal{H}} (x - x \wedge y). \quad (13)$$

Observe that $x \in \mathcal{H}$ and $x \wedge y \in \mathcal{H}^\sim$ (for, if $x \wedge y \not\in \mathcal{H}$, then as $\mathcal{H}$ is a subspace we would have $x \wedge (x \wedge y) = y \in \mathcal{H}$). Now suppose that $e \in (\mathcal{H} \cup \mathcal{H}^\sim)_F$ is an eigenvector for $\text{id}_\mathcal{H}$. Write $e_\sim$ for the projection of $e$ to $(\mathcal{H}^\sim)_F$ and $e_0 = e - e_\sim$. Then

$$\text{id}_\mathcal{H} e = e_0 + \text{id}_\mathcal{H} e_\sim = \lambda e$$

for some $\lambda$, and, using (13), the projection of $\text{id}_\mathcal{H} e_\sim$ to $(\mathcal{H}^\sim)_F$ is

$$\frac{\alpha}{2 + \alpha k_\mathcal{H}} (k_\mathcal{H}^\mathcal{H} I_{|\mathcal{H}|} + \text{ad}(G/\mathcal{H})) e_\sim = \lambda e_\sim,$$

where $\text{ad}(G/\mathcal{H}) y = \sum_{x \in y^\sim \cap \mathcal{H}} x \wedge y$ is extended $F$-linearly to $(\mathcal{H}^\sim)_F$. Therefore if $e_\sim \not= 0$, then $\lambda$ is an eigenvalue of

$$\frac{\alpha}{2 + \alpha k_\mathcal{H}} (k_\mathcal{H}^\mathcal{H} I_{|\mathcal{H}|} - \text{ad}(G/\mathcal{H})). \quad (16)$$

Therefore $\lambda$ is in $\frac{\alpha}{2 + \alpha k_\mathcal{H}} (k_\mathcal{H}^\mathcal{H} - \text{Spec}(G/\mathcal{H}))$. By comparing dimensions, the eigenspaces of $\text{id}_\mathcal{H}$ span $(\mathcal{H} \cup \mathcal{H}^\sim)_F$, so $\text{id}_\mathcal{H}$ is diagonalisable. \hfill $\square$

This enables us to show that (the adjoint operator of) $\text{id}_\mathcal{H}$ is diagonalisable:

**3.3 Lemma.** Suppose that $\mathcal{H} \subseteq G$ and $\mathcal{H}$ is very regular in any parabolic subspace $\mathcal{G}' \subseteq G$ in which $\mathcal{H}$ is maximal. If $\alpha \not= -\frac{2}{k_\mathcal{G}'},$ then $\text{id}_\mathcal{H}$ is diagonalisable in $M_\alpha(G)$.

*Proof.* By Lemma 3.1, the subalgebra of $M_\alpha(G)$ spanned by $\mathcal{H}$ has an identity $\text{id}_\mathcal{H}$.

Let $x \in \mathcal{G} \setminus \mathcal{H}$ be arbitrary and set $\mathcal{G}' = \langle x, \mathcal{H} \rangle$. If $x \not\in \mathcal{H}^\sim$, then $\text{id}_\mathcal{H} x = 0$; otherwise, $\text{id}_\mathcal{H}$ acts on $\mathcal{G}'$ diagonalisably by Lemma 3.2. Now $\mathcal{G} \setminus \mathcal{H}$ can be partitioned in $\mathcal{G}'_1, \mathcal{G}'_2, \ldots, \mathcal{G}'_r$ and $\mathcal{H}^\sim$ where each $\mathcal{G}'_i$ is a subgraph of $\mathcal{G}$ in which $\mathcal{H}$ is maximal. That $\mathcal{G}'_i \cap \mathcal{G}'_j = \mathcal{H}$ if $i \not= j$ follows from the fact that, if $y \in \mathcal{G}'_i \cap \mathcal{G}'_j \setminus \mathcal{H}$ then $\mathcal{G}'_i = \langle \mathcal{H}, y \rangle = \mathcal{G}'_j$ by maximality, so the $\mathcal{G}'_i$ have pairwise trivial intersection. Thus $\text{id}_\mathcal{H}$ acts diagonally on a basis of $M_\alpha(G)$. \hfill $\square$

We can also classify the 1- and 0-eigenspaces of any $\text{id}_\mathcal{H}$.

**3.4 Lemma.** Suppose that $\mathcal{H} \subseteq G$ is very regular. If $\alpha$ is an indeterminate over $F$ and $\Lambda = M_\alpha(G)_F(\alpha)$, then the 1-eigenspace of $\text{id}_\mathcal{H}$ is $\langle \mathcal{H} \rangle_F(\alpha)$, the 0-eigenspace is 1-dimensional, and these are the only eigenvalues of $\text{id}_\mathcal{H}$ contained in $F \subseteq F(\alpha)$. If $\alpha \in F$, the 0-eigenspaces of $\text{id}_\mathcal{H}$ in $M_\alpha(G)_F$ are the same if $\alpha \not= \frac{2}{k_\mathcal{G}' - k_\mathcal{H} - \lambda}$ for any $\lambda \in \text{Spec}(\text{ad}(G/\mathcal{H}))$.

*Proof.* The eigenvalues of $\text{id}_\mathcal{H}$ on $\mathcal{G}$ are classified by Lemma 3.2, showing that if $\alpha$ is an indeterminate, then 1 and 0 are the only eigenvalues in $F \subseteq F(\alpha)$. The eigenvalues of $\text{id}_\mathcal{H}$ are 1 and $\frac{\alpha}{2 + \alpha k_\mathcal{H}} (k_\mathcal{G}' - \lambda)$, for $\lambda \in \text{Spec}(G/\mathcal{H})$. Evidently $\mathcal{H} \subseteq A_1^{\text{id}_\mathcal{H}}$. By Theorem 1.9, the $k_\mathcal{H}$-eigenspace of $\text{ad}(G/\mathcal{H})$ is 1-dimensional, so when $\alpha \not= 0$ the 0-eigenspace of $\text{id}_\mathcal{H}$ is also 1-dimensional. It only remains to consider other 1-eigenvectors. The only solution to the equation, if $\alpha \not= k_\mathcal{G}'$, \hfill $\square$

$$\alpha = 2/(k_\mathcal{G}' - k_\mathcal{H} - \lambda).$$

(17)
We say that an element $x \in A$ is Serres if it acts diagonally and the fusion rules $\Phi$ satisfied by its eigenspaces are Seres as in Definition 1.5. In particular, this applies to $\text{id}_H$ in certain cases:

**3.5 Lemma.** If $H \subseteq G$ are very regular Fischer spaces and $\alpha \neq -\frac{2}{k_H}, \frac{2}{k_H} - k_H - \lambda$ for any $\lambda \in \text{Spec}(\text{ad}(G/H))$, then $\text{id}_H$ is Serres in $M_\alpha(G)$. Furthermore $\text{id}_H$ is Serres in $M_\alpha(G)$ if $H \subseteq G$ is very regular whenever $H \subseteq G'$ is maximal and $G' \subseteq G$, and $\alpha \neq -\frac{2}{k_H} - k_H - \lambda$ for $\lambda \in \text{Spec}(\text{ad}(G'/H))$.

**Proof.** Lemma 3.2 showed that $\text{id}_H$ acts diagonally. We use the classification of 1- and 0-eigenvectors of Lemma 3.4 to prove that $1 \star \phi \subseteq \{\phi\}$ for all eigenvalues $\phi$ of $\text{id}_H$ in $A = M_\alpha(G)$ (which in particular implies $1 \star 0 = \emptyset$). We first take the case when $H \subseteq G$ is very regular.

Since under our hypotheses the 1-eigenspace of $\text{id}_H$ is spanned by $H$, which is closed under multiplication, we already have that $1 \star 1 = \{1\}$.

We will use four facts for the sequel. Firstly, observe that

$$y^{\tau}(x) = \begin{cases} x \land y & \text{if } x \sim y, \\ y & \text{otherwise.} \end{cases} \quad (18)$$

Secondly, for any $h \in H$, $a \in A$, by application of (18),

$$ha = \frac{\alpha}{2}(\lambda_h h + a - a^{\tau(h)}) \text{ for some } \lambda_h \in \mathbb{F}. \quad (19)$$

Thirdly, if $t \in \text{Aut}(A) \subseteq \text{End}(A)$ fixes $a \in A$, then $t$ centralises $\text{ad}(a) \in \text{End}(A)$ and the eigenspaces $A^\alpha_\phi$ of $a$. Fourthly, if $h \in H$ then, as $H$ is closed under $\land$, $\tau(h)$ permutes the points of $H$ and therefore fixes $\text{id}_H$.

To show that $1 \star \phi = \{\phi\}$ for $\phi \neq 1$, suppose that $h \in H$ and $y$ is a $\phi$-eigenvector of $\text{id}_H$ in $A$. Set $y = y_0 + y_\sim$, for $y_\sim \in \langle H^\sim \rangle_{\mathbb{F}}$ the projection of $y$ onto the subspace spanned by $H^\sim$ and $y_0 = y - y_\sim$. Now as $y$ is a $\phi$-eigenvector for $\text{id}_H$, $\text{id}_H y = \phi y$ is again a $\phi$-eigenvector. On the other hand, using Lemma 3.1 and (19),

$$\phi y = \text{id}_H y = \frac{1}{1 + \frac{\alpha}{2}k_H} \sum_{h \in H} hy = \frac{\alpha}{2 + \alpha k_H} \sum_{h \in H} (\lambda_h h + y - y^{\tau(h)}). \quad (20)$$

Noting that $y^{\tau(h)} \in (A^1_\phi)^{\tau(h)} = A^1_\phi$, by rearranging terms we have an expression for $\sum_{h \in H} \lambda_h h$ in terms of $\phi$-eigenvectors. On the other hand, any $h \in H$ is a 1-eigenvector and $A^1_\phi \cap A^1_{\text{id}_H} = 0$, so that $\sum_{h \in H} \lambda_h h = 0$. As the points $h \in H$ are linearly independent, this means $\lambda_h = 0$ for all $h \in H$. Therefore $hy = \frac{\alpha}{2}(y - y^{\tau(h)}) \in A^1_\phi$, so $1 \star \phi = \{\phi\}$.

To show that $1 \star 0 = \emptyset$, observe that the 0-eigenspace of $\text{id}_H$ is 1-dimensional by Lemma 3.4, and fixed by any automorphism $t$ fixing $\text{id}_H$. In particular, $\tau(h)$ fixes $y \in A^0_{\text{id}_H}$, so by the previous paragraph, $hy = \frac{\alpha}{2}(y - y) = 0$.

Therefore a 0-eigenvector $z$ of $\text{id}_H$ in $M_\alpha(G)$ is also a 0-eigenvector of any $h \in H$. By Lemma 1.6, for any $x \in A$ we have $h(xz) = (hx)z$. As $\text{id}_H$ is a linear combination of $h \in H$, we conclude $\text{id}_H(xz) = (\text{id}_H x)z$. Thus $\text{id}_H$ and $z$ associate, and using the other direction of Lemma 1.6 this implies that $0 \star \phi = \{\phi\}$ for all $\phi \neq 1$.

We now tackle the general case of connected $H$ in some $G$ such that $H \subseteq G'$ is very regular in every $G' \subseteq G$ for which $H \subseteq G'$ is maximal. The 1-eigenspace of $\text{id}_H$ in $M_\alpha(G)$ is still spanned by $H$ and, by the same argument as that in the proof of Lemma 3.3, any $\phi$-eigenvector can be decomposed into a sum of $\phi$-eigenvectors lying in $M_\alpha(G')$ for $H \subseteq G'$ very regular, unless $\phi = 0$, in which case the 0-eigenspace also includes $H^\sim$. Therefore the fusion rules $1 \star \phi = \{\phi\}$ for $\phi \neq 0$ are satisfied.
Suppose that $z \in \mathcal{H}^\alpha$; then for $h \in \mathcal{H}$, $h \not\sim z$ so $hz = 0$. This shows that $1 \ast 0 = \emptyset$ in $M_n(\mathbb{G})$.

To show that $0 \ast \phi = \{\phi\}$ in $M_n(\mathbb{G})$, we repeat our observation that the 0-eigenvectors of $\text{id}_H$ are 0-eigenvectors of $h \in \mathcal{H}$, which are Seress, so that by linearity $\text{id}_H$ associates with its 0-eigenspace and, using Lemma 1.6, therefore $0 \ast \phi = \{\phi\}$ for all $\phi \neq 1$.

\[ \square \]

3.6 Lemma. Suppose that $e$, $f$ are idempotents and $f \in A_1^e$. Then

i. $e - f$ is an idempotent;

ii. if further $f$ is Seress and $e$, $f$ are diagonalisable, then $e - f$ is diagonalisable;

iii. if further $e$ is Seress and $A_{(1,0)}^{e-f} \subseteq A_{(1,0)}^e \cap A_{(1,0)}^f$, then $e - f$ is Seress.

Proof. i. This is

\[(e - f)(e - f) = ee - 2ef + ff = e - 2f + f = e - f. \tag{21}\]

ii. As $f$ is Seress, it associates with its 1-eigenspace, in particular, with $e$. Therefore $(ef)f = e(xf)$ for all $x \in A$; equivalently, $\text{ad}(e)\text{ad}(f) = \text{ad}(f)\text{ad}(e) \in \text{End}(A)$, so that $[\text{ad}(e), \text{ad}(f)] = 0$. Thus $\text{ad}(e), \text{ad}(f)$ are two commuting diagonalisable matrices, so they are simultaneously diagonalisable and their difference $\text{ad}(e) - \text{ad}(f) = \text{ad}(e - f)$ is diagonalisable.

iii. If $A_{(1,0)}^{e-f} \subseteq A_{(1,0)}^e \cap A_{(1,0)}^f$, then any 1-eigenvector $x$ of $e - f$ is a 1-eigenvector of $e$ and a 0-eigenvector of $f$, and a 0-eigenvector $z$ of $e - f$ is a 0-eigenvector of $e$ and $f$. Since both $e$ and $f$ associate with their 1, 0-eigenspaces, $e - f$ associates with $A_{(1,0)}^e \cap A_{(1,0)}^f$. Therefore $e - f$ associates with $A_{(1,1,0)}^{e-f}$, and by Lemma 1.6, $e - f$ is Seress.

\[ \square \]

Observe that the hypotheses of iii. hold if, whenever $\lambda, \mu$ are eigenvalues of $e$, $f$ respectively with $\lambda - \mu \in \{1, 0\}$, then $\mu = 0$. This is key to our last definition and theorem.

3.7 Definition. Let $\mathcal{G}$ be a Fischer space and $A = M_\alpha(\mathcal{G})_F$ its Matsuo algebra. Write $L_0$ for the set of identity elements of parabolic subalgebras. The set $L$ of linear idempotents of $A$ is the minimal set containing $L_0$ such that, for all $e, f \in L$ with $f \in A_1^e$, also $e - f \in L$.

3.8 Theorem. Suppose that $(G, D)$ is a 3-transposition group satisfying Hypothesis 2.3 and set $A_1 = M_\alpha(G)_F$. The linear idempotents in $A_1(\alpha)$ are Seress when $\alpha$ is indeterminate over $\mathbb{F}$.

Proof. By Lemmas 3.3 and 3.5, the identities of parabolic subalgebras are diagonalisable and Seress when $\alpha$ is an indeterminate (as this rules out any coincidences of eigenvalues such as $\alpha = -\frac{2}{n\lambda - k_n^2} \frac{2}{k_n^2 - k_n^2 - \lambda}$). Suppose that $e, f \in L$, the linear idempotents of $A$, and that $f \in A_1^e$. Then it follows by ii. of Lemma 3.6 that $e - f$ is diagonalisable with eigenvalues $\text{Spec}(e) - \text{Spec}(f)$. Therefore, to show that iii. of Lemma 3.6 holds for an arbitrary linear idempotent $e$, which can be written as a sum $e = \sum_{i=1}^{n}(\lambda_{i}^{(1)})^{i+1} \text{id}_i$ for $\text{id}_i$ the identity of a parabolic subalgebra, we need to consider when sums $\sum_{i=1}^{n}(\lambda_{i}^{(1)})^{i+1} \lambda_i$ of eigenvalues $\lambda_i$ of $\text{id}_i$ can equal 1 or 0.

Contributions of eigenvalue 0 coming from a constituent term $\text{id}_i$ can be neglected. For a simultaneous eigenvector, as the 1-eigenspaces satisfy the inclusions $A_1^i \subseteq A_1^{i+1}$, only the first $m$ consecutive idempotents may take eigenvalue 1 for some $m \leq n$. As the sum is alternating, these contributions cancel to either 1 or 0. Therefore observe that an eigenvalue of $e$ is

\[ \lambda = \alpha \sum_{i=1}^{n} \frac{\mu_i}{2 + \alpha k_i} \quad \text{or} \quad 1 - \lambda, \tag{22} \]

where $\mu_i, k_i \in \mathbb{Z}$; here, $\text{id}_i$ is the identity of a subalgebra $A_1^{id_i}$ with Fischer space $\mathcal{H}_i \subseteq \mathcal{G}$ which is $k_i$-regular, and $\mu_i = (\lambda^{(1)})^{1+i} (k_{\mathcal{H}_i}^{(1)} - \lambda)$ for some very regular embedding $\mathcal{H}_i \subseteq \mathcal{G}' \subseteq \mathcal{G}$ and
\( \lambda \in \text{Spec}(\text{ad}(\mathcal{G}/\mathcal{H}_r)) \). We solve for \( \lambda \) or \( 1 - \lambda \) equal to 1,0; without loss of generality, we need only to find when \( \lambda = 1,0 \).

Comparing degrees of \( \alpha \) in the expression for the numerator and denominator, we see that the denominator has a constant term, whereas the term of lowest degree in the numerator has degree 1. Therefore they cannot be equal, so that the expression cannot evaluate to \( \lambda = 1 \).

The other possibility is that \( \lambda = 0 \), hence \( \sum \frac{\mu_i}{2 + \alpha k_i} = 0 \), which we now rule out. The denominators \( 2 + \alpha k_i \) are all different, as the maximal, or Perron-Frobenius, eigenvalues \( k_i, k_i + 1 \) of graphs \( \mathcal{H}_i \subseteq \mathcal{H}_{i+1} \) satisfy \( k_i < k_{i+1} \) when \( \mathcal{H}_i \) is strictly smaller than \( \mathcal{H}_{i+1} \), which must be the case as \( \text{id}_i \neq \text{id}_{i+1} \). Now the collection \( \{ \frac{\mu_i}{2 + \alpha k_i} \}_{1 \leq i \leq n} \subseteq F(\alpha) \) is linearly independent over \( F \), as \( \sum_{i=1}^{n} \frac{\mu_i}{2 + \alpha k_i} = 0 \) if and only if \( \sum_{i=1}^{n} \mu_i \prod_{j \neq i} (2 + \alpha k_j) = 0 \), and by specialising \( \alpha \mapsto -\frac{2}{k_i} \), since there are no repeated factors we see \( \mu_i = 0 \) in this sum.

Thus indeed \( A_{1,0}^f \subseteq A_{1,0}^f \cap A_{1,0}^f \), so \( e = f \) is Seress by application of iii. of Lemma 3.6.

\[ \square \]

4 \hspace{1em} \textbf{Eigenvalues in} \( M_\alpha(A_n) \)

For the results of this section we first present a graph construction.

\[ \textbf{4.1 Definition.} \text{ Suppose that } \mathcal{G} \text{ is a Fischer space. Its double graph } \mathcal{G}^\pm \text{ is the graph with point set } \{ x^+, x^- \mid x \in \mathcal{G} \} \text{ and lines } \{ x^\varepsilon, y^\eta, (x \wedge y)^\varepsilon\eta \} \text{ for any } x \sim y \text{ in } \mathcal{G}, \varepsilon, \eta \in \{ +, - \}. \]

\[ \textbf{4.2 Lemma.} \text{ The double graph of } A_n \text{ is } D_{n+1}. \]

\[ \text{Proof.} \text{ Suppose that } \{ x_1, \ldots, x_m \} \text{ are the points in } A_n, \text{ inducing transpositions } \{ t_1, \ldots, t_{n+1} \} \text{ in the Weyl group } W(A_n). \text{ Then there are } s_1, \ldots, s_n \text{ transpositions among them satisfying the Coxeter presentation for } W(A_n) \text{ in Figure 3.} \]

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\circ \end{array} \]

\[ s_1 \begin{array}{c}
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\circ \end{array} \begin{array}{c}
\circ \end{array} \begin{array}{c}
\circ \end{array} s_n \]

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figures/3.png}
\caption{Coxeter presentation for \( W(A_n) \)}
\end{figure}

Let \( x_1^+, \ldots, x_m^+, x_1^-, \ldots, x_m^- \) be the points of \( A_{n+1}^\pm \) and \( t_\varepsilon \) the transposition \( \tau(x_\varepsilon) \) of \( x_\varepsilon \) in the permutation representation. Then it follows that \( S = \{ t_1^-, t_1^+, t_2^+, t_2^-, \ldots, t_n^+ \} \), transpositions induced from the points of \( A_n^\pm \), satisfies the Coxeter presentation for \( W(D_{n+1}) \) in Figure 4.

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\[ t_1^+ \begin{array}{c}
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\end{array} t_2^+ \begin{array}{c}
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\end{array} t_4^+ \]

\[ t_1^- \begin{array}{c}
\circ
\end{array} t_2^- \begin{array}{c}
\circ
\end{array} t_3^- \begin{array}{c}
\circ
\end{array} t_4^- \]

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figures/4.png}
\caption{Coxeter presentation for \( W(D_{n+1}) \)}
\end{figure}

Moreover, \( S \) generates \( G = W(A_n^\pm) \), so \( G \) is a quotient of \( W(D_{n+1}) \). In fact a counting argument shows that \( G = W(D_{n+1}) \), since \( W(D_{n+1}) \) has \( n(n+1) \) transpositions and \( W(A_n) \) has \( 2 \cdot \frac{1}{2} n(n+1) \), the same number. The corresponding points \( x_1^+, x_1^+, x_2^+, \ldots, x_n^+ \) generate \( A_n^\pm \), therefore \( D_{n+1} \cong A_n^\pm \).

\[ \square \]

\[ \textbf{4.3 Lemma.} \text{ The boundary graph } A_n/A_{n-1} \text{ is } K_n, \text{ the complete graph on } n \text{ points.} \]
4.2 Lemma. We record the eigenvalues, where superscripts indicate multiplicities.

Proof. Recall that the group \( W(A_n) \) generated by Miyamoto involutions of points \( x \in A_n \) is the symmetric group \( \text{Sym}(n+1) \) on \( n+1 \) letters. Taking the embedding \( H = \text{Sym}(n) \subseteq \text{Sym}(n+1) = G \) that corresponds to \( A_{n-1} \subseteq A_n \) gives that \( H \) has support \( \{1, \ldots , n\} \) and \( G \) has support \( \{1, \ldots , n+1\} \) in the standard permutation realisation of \( G \). Then if \( s,t \in G \setminus H \) are transpositions, they each move two letters in \( \{1, \ldots , n+1\} \). If \( s \) moves two letters in \( \{1, \ldots , n\} \) then \( s \in H \), so \( s \) moves \( n+1 \); the same goes for \( t \). We can therefore write \( s = (i,i+n+1) \) and \( t = (j,j+n+1) \) for \( 1 \leq i,j \leq n \). Then \( st = (i,j) \) lies in \( H \). This shows that the points \( x,y \in A_n \) corresponding to \( s,t \) satisfy \( x \wedge y \in A_{n-1} \). As \( s,t \) were arbitrary, any two points in \( A_n/A_{n-1} \) are connected.

4.4 Lemma. The double graph \( (G/H)^\pm \) of \( G/H \), for \( G,H \) linear 3-graphs, is \( G^\pm/H^\pm \).

Proof. The naive bijection works: take \( x^e \in (G/H)^\pm \). Then \( x \in G/H \) and is uniquely identified with a point \( x' \in G \setminus H \), for which there exists \( y' \in G \setminus H \) with \( x' \wedge y' \in H \). Now \( x^e, y^e \in G^\pm \setminus H^\pm \) and \( x^e \wedge y^e \in H^e \subseteq H^\pm \), so \( x^e \in G^\pm/H^\pm \). Therefore \( (G/H)^\pm \) has the same cardinality as \( G^\pm/H^\pm \). Indeed identifying \( y' \in G \setminus H \) in the above argument with \( y \in G/H \) shows that this bijection also preserves lines \( x \sim y \), so that we have an isomorphism of graphs.

4.5 Lemma. If \( G \) is a Fischer space containing no isolated points, then \( G^\pm/G \) is isomorphic to \( G \).

Proof. Let \( x^-, y^- \in G^\pm \setminus G^+ \) be arbitrary. Then \( x^- \sim y^- \) if and only if \( x \sim y \) by definition, and if so, then \( x^- \wedge y^- = (x \wedge y)^- = (x \wedge y)^+ \in G^+ \). Furthermore since \( G \) contains no isolated points, every \( x^- \in G^- \) is connected to at least one other point \( y^- \in G^- \). Therefore the point set of \( \mathcal{X} = G^\pm/G^+ \) is \( G^- \), and \( \mathcal{X} \) has lines \( \{x^-,y^-\} \) exactly when \( \{x,y,x \wedge y\} \) is a line in \( G \). Thus the incidence relations of the points are the same (although note that the lines are not the same, as they have differing cardinalities).

We now give results for specific graphs.

4.6 Lemma. We record the eigenvalues, where superscripts indicate multiplicities,

\[
\begin{align*}
\text{Spec}(\text{ad}(K_n)) &= \{(n-1)^1, -1^{n-1}\}, \quad k_{A_n}^{A_{n+1}}(\alpha) = n - 1, \\
\text{Spec}(\text{ad}(A_1)) &= \{0^1\}, \quad \text{Spec}(\text{ad}(A_2)) = \{2^1, -2^1\}, \\
\text{Spec}(\text{ad}(A_{n \geq 3})) &= \{(2n-2)^1, (n-3)^n, -2(n+1)(n-2)/2\}, \\
\text{Spec}(\text{ad}(D_{n \geq 4})) &= \{(4n-8)^1, (2n-8)n^{-1}, -4(n-3)/2, 0(n-1)/2\}.
\end{align*}
\]

Proof. These facts are folklore; we used unpublished work of Hall and Spectorov for details. For \( D_n \), we can also deduce the values using Lemma 4.2 from those for \( A_n \).

For application to vertex algebras, we need to calculate central charges. Suppose that \( A = M_n(G) \) is a Matsuo algebra and that \( c \in \mathbb{F} \). Then, for \( x,y \in G \), by [M03]

\[
(x,y) = \begin{cases} 
2c & \text{if } x = y, \\
0 & \text{if } x \neq y, \\
c \alpha & \text{if } x \sim y 
\end{cases}
\]

defines a bilinear form on \( A \). The central charge \( cc(e) \) of an idempotent \( e \in A \) is \( \frac{1}{2}(e,e) \). This matches the scaling of the form and the definition of central charge in Theorem 1.8 and [M96].

Fix embeddings \( A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \) and set, in \( M_n^+(A_0^\pm) \),

\[
\text{id}_i = \text{id}_{A_i}, \quad \tilde{id}_i = \text{id}_{A_i^\pm}, \quad e_i = \text{id}_i - \text{id}_{i-1}, \quad \hat{e}_i = \tilde{id}_i - \text{id}_i.
\]
4.7 Theorem. In \( A = M^n_\alpha(A^n_\alpha) \), for \( 4 \leq i < n \), for

\[
\eta_\alpha(i) = \frac{\alpha(i + 1)}{2 + 2\alpha(i - 1)}, \quad \tilde{\eta}_\alpha(i) = \frac{\alpha i}{1 + \alpha(i - 1)}, \tag{26}
\]

we have

\[
\text{Spec}(\epsilon_i) = \{ 1, 0, \eta_\alpha(i), 1 - \eta_\alpha(i - 1), \eta_\alpha(i) - \eta_\alpha(i - 1), \tilde{\eta}_\alpha(i) - \eta_\alpha(i - 1), \tilde{\eta}_\alpha(i) - \tilde{\eta}_\alpha(i - 1) \}, \tag{27}
\]

\[
\text{Spec}(\tilde{\epsilon}_i) = \{ 1, 0, 1 - \eta_\alpha(i - 1), 1 - \tilde{\eta}_\alpha(i - 1) \}, \tag{28}
\]

\[
cc^c_\alpha(\epsilon_i) = \frac{c}{2(1 + \alpha(i - 1))(1 + \alpha(i - 2))}, \tag{29}
\]

\[
cc^c_\alpha(\tilde{\epsilon}_i) = \frac{c}{2(1 + 2\alpha(i + 1))(1 + \alpha(i + 1))}. \tag{30}
\]

Proof. It follows from from Lemma 3.2, and substitutions from Lemma 4.6, that the eigenvalues of \( \text{id}_{A_i} \) in \( A \) are

\[
\text{Spec}(\text{id}_{A_0}) = \{ 0 \}, \quad \text{Spec}(\text{id}_{A_{i=1,2}}) = \{ 1, 0, \eta_\alpha(i) \}, \tag{31}
\]

\[
\text{Spec}(\text{id}_{A_{i=3}}) = \{ 1, 0, \eta_\alpha(i), \tilde{\eta}_\alpha(i) \}.
\]

By observations on inclusions of eigenspaces and the fact that, for commuting matrices \( x, y \),

\[
\text{Spec}(x - y) = \text{Spec}(x) - \text{Spec}(y),
\]

we deduce the spectrum of \( \epsilon_i \) and \( \tilde{\epsilon}_i \). Namely, denote \( \text{id}_{A_1}^{\epsilon_1} \)

by \( A_1^\alpha \); then \( A_1^{i-1} \subseteq A_1^i \) is clear, \( A_0 \subseteq A_0^{i-1} \) implies that an eigenvalue \( 0 - \phi \) is only realised for \( \phi = 0 \), and \( A_1^i \subseteq A_1^{i-1} \).

\[\square\]

In view of Theorem 1.8, we calculate the specialisation of Theorem 4.7 for \( \alpha = \frac{1}{4}, c = \frac{1}{2} \) in Lemma 4.8. With respect to the lattice vertex algebra of \( \sqrt{2}A_0 \), we find \( \epsilon_i \) induces a Virasoro algebra of central charge \( c_i \), and \( \tilde{\epsilon}_i \) is the conformal vector of a \( W \)-algebra of central charge \( \frac{2i}{i+3} \). The notation \( h_i^s \) indicates the highest weights of the Virasoro algebra at central charge \( c_i \), as per [M96] or [Y01].

4.8 Lemma. The specialisation for \( \alpha = \frac{1}{4}, c = \frac{1}{2} \) of Theorem 4.7 is,

\[
cc^{1/2}_{1/4}(\epsilon_i) = 1 - \frac{6}{(i + 2)(i + 3)} = c_i, \quad cc^{1/2}_{1/4}(\tilde{\epsilon}_i) = \frac{2i}{i + 3}, \tag{32}
\]

\[
0 = h_{1,1}^i, \tag{33}
\]

\[
\eta_{1/4}(i) = \frac{1}{2} \frac{i + 1}{i + 3} = \frac{1}{2} h_{1,3}^i, \tag{34}
\]

\[
1 - \eta_{1/4}(i - 1) = \frac{1}{2} \frac{i + 4}{i + 2} = \frac{1}{2} h_{3,1}^i, \tag{35}
\]

\[
\eta_{1/4}(i) - \eta_{1/4}(i - 1) = \frac{1}{2} \frac{i - 1}{(i + 2)(i + 3)} = \frac{1}{2} h_{3,3}^i, \tag{36}
\]

\[
\tilde{\eta}_{1/4}(i) - \eta_{1/4}(i - 1) = \frac{1}{2} \frac{i(i - 1)}{(i + 2)(i + 3)} = \frac{1}{2} h_{3,5}^i, \tag{37}
\]

\[
\tilde{\eta}_{1/4}(i) - \tilde{\eta}_{1/4}(i - 1) = \frac{3}{2} \frac{i(i - 1)}{(i + 2)(i + 3)} = \frac{1}{2} h_{5,5}^i. \tag{38}
\]

Proof. By direct evaluation, we see

\[
\eta_{1/4}(i) = \frac{1}{2} \frac{i + 1}{i + 3}, \quad \tilde{\eta}_{1/4}(i) = \frac{i}{i + 3} \tag{39}
\]
and the results are then straightforward manipulations.

5 Involutions and $D_n$

5.1 Lemma. Suppose that $H \subseteq G$ satisfies the hypotheses of Lemma 3.5, and that $x, y \in G$ are collinear. If $x, y \in H^\perp$, then $x \wedge y \in H^\perp$.

Proof. Suppose that $x, y \in H^\perp$. Then $x, y$ are 0-eigenvectors for $id_H$ in $A = M_\alpha(G)_{\mathbb{F}(\alpha)}$, an indeterminate; our calculations will take place in $A$. Since $id_H$ is Seress, $xy$ is again a 0-eigenvector of $id_H$. As $xy = \frac{\alpha}{2}(x + y - x \wedge y)$, $x \wedge y$ must also be a 0-eigenvector. The 0-eigenvectors of $id_H$ are classified in $G'$ for any $G' \subseteq G$ such that $H \subseteq G'$ is very regular, by Lemma 3.4, so that either $x \wedge y \in H^\perp$ or $x \wedge y \in H^\sim$ and there exists $H \subseteq G' \ni x \wedge y$. In this latter case, the only 0-eigenvector of $id_H$ in the span of $G'$ has full support in $G'$, so that $G' = H \cup \{x \wedge y\}$, contradicting that $G'$ is connected. Therefore $x \wedge y \in H^\perp$. □

5.2 Lemma. The fusion rules of $id_{D_i}$ and

$$ f_i = id_{D_i} - id_{D_{i-1}} $$

in $M_\alpha(D_m)$, $3 \leq i \leq m$, are $\mathbb{Z}/2$-graded.

Proof. The eigenvectors of $x = id_{D_i}$ are $\{0, \eta_\alpha(i), \eta'_\alpha(i)\}$. We will show that $\Phi_+ \cup \Phi_0 = \{0, \eta_\alpha(i)\} \cup \{\eta'_\alpha(i)\}$ is a $\mathbb{Z}/2$-graded partition of the fusion rules. We first observe that the $\eta'_\alpha(i)$-eigenvectors are of the form $x^+ - x^-$ for $x \in \mathbb{A}_i^\perp \subseteq \mathbb{A}_m$ using the identification $D_m = \mathbb{A}_m^\perp$ from Lemma 4.2. We can verify by direct computation that $id_{D_i}(x^+ - x^-) = \eta'_\alpha(i)(x^+ - x^-)$. Furthermore note that the quotient graph of $D_m$ by $\{x^+ - x^- \mid x \in \mathbb{A}_{m-1}\}$ is exactly $\mathbb{A}_{m-1}^\perp/\mathbb{A}_{m-1} \cong \mathbb{A}_{m-1}$ (see Lemma 4.5), and the image of $id_{D_i}$ under this map is a scalar multiple of $id_{A_{i-1}}$. Every vector which is annihilated in the quotient is a $\eta_\alpha(i)$-eigenvector, so in particular no $\eta_\alpha(i)$-eigenvector is mapped to 0. As $id_{A_{i-1}}$ has only 3 distinct eigenvalues in $M_\alpha(A_{m-1})$ by Lemma 3.2, and the image of $id_{D_i}$ are again 1, 0-eigenvectors, it follows that the $\eta_\alpha(i)$-eigenspace of $id_{D_i}$ is mapped to the $\eta_\alpha(i-1)$-eigenspace of $id_{A_{i-1}}$ and the $\eta'_\alpha(i)$-eigenspace is completely annihilated, so that all $\eta'_\alpha(i)$-eigenvectors lie in the span of $\{x^+ - x^- \mid x \in \mathbb{A}_i^\perp\}$.

Let $t = \tau(id_{D_i})$ be the map

$$ x \mapsto \begin{cases} x^+ & \text{if } x \in \mathbb{A}_i \cup \mathbb{A}_i^\perp, \\ x^- & \text{if } x \in \mathbb{A}_i^\sim. \end{cases} $$

(41)

Observe that $t$ inverts the $\eta'_\alpha(i)$-eigenspace of $id_{D_i}$ and fixes the other eigenspaces. By showing that $t$ is an automorphism of $A = M_\alpha(G)$, together with Lemma 3.5, we show that the fusion rules of $id_{D_i}$ are a subset of Table 5, which is $\mathbb{Z}/2$-graded.

Again identify $D_m$ as $\mathbb{A}_{m-1}^\perp$. Let $\varepsilon, \eta \in \{+, -\}$ and $x, y \in \mathbb{A}_{m-1} \subseteq \mathbb{A}_{m-1}^\perp$. We will consider the product $\wedge$ on collinear points $x^\varepsilon, y^\eta$ from the subspaces $D_i, D_i^\sim$ and $D_i^\perp$.

If $x^\varepsilon, y^\eta \in D_i$, then $x^\varepsilon \wedge y^\eta \in D_i$, since $D_i$ is a closed subspace. If $x^\varepsilon, y^\eta \in D_i^\perp$ then $x^\varepsilon \wedge y^\eta \in D_i^\perp$ by Lemma 5.1. If $x^\varepsilon \in D_i^\sim, y^\eta \in D_i$, then $x^\varepsilon \wedge y^\eta \in D_i^\sim$, as $y \sim (x \wedge y)$ rules out $x^\varepsilon \wedge y^\eta \in D_i^\perp$ and $x^\varepsilon \wedge y^\eta \in D_i$ would force $x^\varepsilon \in D_i$, a contradiction. If $x^\varepsilon \in D_i^\perp, y^\eta \in D_i^\perp$ then $x^\varepsilon \wedge y^\eta \in D_i^\sim$, as $y \sim (x \wedge y)$ rules out $x^\varepsilon \wedge y^\eta \in D_i^\sim$ and $x^\varepsilon \wedge y^\eta \in D_i$ would force $x^\varepsilon \in D_i^\perp$, a contradiction.

Finally, suppose that $x^\varepsilon, y^\eta \in D_i^\sim$. We show that $x^\varepsilon \wedge y^\eta \in D_i \cup D_i^\perp$. It is sufficient to show that for $x, y \in A_{i-1}$ in $A_m$ we have $x \wedge y \in A_{i-1} \cup A_{i-1}^\perp$. Suppose that the points of $A_{i-1}$ are labelled by transpositions in $\text{Sym}(i)$ with support $\{1, \ldots, i\}$ inside $\text{Sym}(m+1)$ with support $\{1, \ldots, m+1\}$. Then $x, y$ are labelled $(i_x, j_x), (i_y, j_y)$ respectively with $i_x, i_y \in \{1, \ldots, i\}$ and
\[ \tau \text{ on } D \text{ acts as e of e by } \tau. \]

**5.3 Theorem.** Let \( \tau \) be the Miyamoto involution \( \tau(\text{id}_{D_i}) \in \text{Aut}(M_\alpha(D_m)) \) of \( \text{id}_{D_m} \) for some embedding \( D_i \subseteq D_m, i \geq 3 \). Then \( \tau \) has an action on the Fischer space \( D_m \) and on \( W(D_m)/Z(W(D_m)) \), \( \tau \) acts by permuting collinear points (see \( 18 \)). Therefore \( \tau(\text{id}_{D_i}) \) is not in the conjugacy class of any Miyamoto involution \( \tau(x) \) for \( x \in D_m \), which are the transpositions in \( W(D_m) \). It also follows that \( \tau_i \) acts as \(-1\) on a subspace of dimension \( |A_{i-1}^\sim| \). As \( |A_{i-1}^\sim| \neq |A_{j-1}^\sim| \) for \( i \neq j \), and conjugation preserves the dimensions of eigenspaces, we have that \( \tau_i, \tau_j \) cannot be conjugate for \( i \neq j \).

Recall that, if \( t \) is an automorphism of an algebra \( A \) and \( e \in A \) is a \( \Phi \)-axis for some \( \Phi \), then \( e^t \) is again a \( \Phi \)-axis. Furthermore, when \( \Phi \) is \( \mathbb{Z}/2 \)-graded and \( \tau(e) \) is the Miyamoto involution of \( e \), we have \( \tau(e^t) = \tau(e)^t \). Therefore \( \tau(x^{\tau(\text{id}_{D_i})}) = \tau(x)^{\tau(\text{id}_{D_i})} \), so that the action of \( \tau(\text{id}_{D_i}) \) on \( D_m \) induces an action on \( \{ \tau(x) \mid x \in D_m \} \) and the subgroup of \( \text{Aut}(M_\alpha(D_m)) \) it generates.

By [A97] and [HRS14], the Miyamoto involutions of \( M_\alpha(D_m) \), corresponding to involutions of points in the Fischer space \( D_m \), generate \( W(D_m)/Z(W(D_m)) \).

| \( \star \) | 1 | 0 | \( \eta_\alpha(i) \) | \( \eta'_\alpha(i) \) |
|---|---|---|---|---|
| 1 | \{1\} | \{\} | \{\eta_\alpha(i)\} | \{\eta'_\alpha(i)\} |
| 0 | \{0\} | \{\} | \{\eta_\alpha(i)\} | \{\eta'_\alpha(i)\} |

Table 5: Fusion rules \( \Phi \) of \( \text{id}_{D_i} \),

\( j_x, j_y \in \{i + 1, \ldots, m + 1\} \). That \( x \sim y \) implies that either \( i_x = i_y \) or \( j_x = j_y \). Thus \( x \wedge y \) is labelled \( (j_x, j_y) \) or \( (i_x, i_y) \) respectively, and hence \( x \wedge y \in A_{i-1} \cup A_{j-1}^\sim \).

To show that \( t \) is an automorphism of \( M_\alpha(\mathcal{G}) \), by linearity it suffices to show that for any \( x^\varepsilon, y^\eta \in \mathcal{G} \) we have

\[
(x^\varepsilon y^\eta)^\dagger = (x^\varepsilon y^\eta)^t.
\]

When \( x \not\sim y \), both sides are seen to be 0. By a case-by-case analysis for \( x^\varepsilon, y^\eta \) coming from the subspaces \( D_i, D_j^\sim \) and \( D_i^\sim \), using our information on \( \wedge \) calculated previously, we see that (42) is satisfied in all cases, for example, when \( x^\varepsilon, y^\eta \in D_i^\sim \),

\[
(x^\varepsilon y^\eta)^\dagger = x^{-\varepsilon} y^{-\eta} = \frac{\alpha}{2}(x^{-\varepsilon} + y^{-\eta} - x^{-\varepsilon} \wedge y^{-\eta}),
\]

\[
(x^\varepsilon y^\eta)^t = \frac{\alpha}{2}(x^\varepsilon + y^\eta - x^\varepsilon \wedge y^\eta),
\]
and as \( x^{-\varepsilon} \wedge y^{-\eta} = x^\varepsilon \wedge y^\eta \), we have the desired equality.

Therefore \( t \) is an automorphism, and is the Miyamoto involution of \( \text{id}_{D_i} \).

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Suppose that $t = \tau_i^g$ for some $g \in G$. Let $P$ be the set of points not fixed by $\tau_i$ on $D_m$. The embedding $D_i \subseteq D_m$ such that $\tau_i = \tau(\text{id}_{D_i})$ is the unique embedding of $D_i \subseteq D_m$ such that, if $D_m = A_{m-1}^\pm$ and $A_i-1 = D_i \cap A_{m-1}$, then $P = (A_{i-1}^\pm)^\circ$. Thus we can recover $D_i \subseteq D_m$ and $\text{id}_i$ from the set $P$. Now any element $g \in G$ has an action on the transpositions of $G$, which we have identified with points of the Fischer space $D_m$. Hence $t = \tau_i^g$ acts on $D_m$ by fixing all points except $P^g$. The points $P^g$ uniquely identify an embedding $D_i^g \subseteq D_m$ and hence $\text{id}_{D_i^g} = \text{id}_i^g$, so that $t = \tau(\text{id}_i^g)$ as required.

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