On Dynamic Pricing with Covariates

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Abstract

We consider the dynamic pricing problem with covariates under a generalized linear demand model: a seller can dynamically adjust the price of a product over a horizon of $T$ time periods, and at each time period $t$, the demand of the product is jointly determined by the price and an observable covariate vector $x_t \in \mathbb{R}^d$ through an unknown generalized linear model. Most of the existing literature assumes the covariate vectors $x_t$'s are independently and identically distributed (i.i.d.); the few papers that relax this assumption either sacrifice model generality or yield sub-optimal regret bounds. In this paper we show that a simple pricing algorithm has an $O(\sqrt{dT \log T})$ regret upper bound without assuming any statistical structure on the covariates $x_t$ (which can even be arbitrarily chosen). The upper bound on the regret matches the lower bound (even under the i.i.d. assumption) up to logarithmic factors. Our paper thus shows that (i) the i.i.d. assumption is not necessary for obtaining low regret, and (ii) the regret bound can be independent of the (inverse) minimum eigenvalue of the covariance matrix of the $x_t$'s, a quantity present in previous bounds. Furthermore, we discuss a condition under which a better regret is achievable and how a Thompson sampling algorithm can be applied to give an efficient computation of the prices.

1 Introduction

In this paper we consider the problem of dynamically pricing a product over time when additional covariate information is available. In the literature this is also referred to as “feature-based dynamic pricing”, “personalized dynamic pricing”, or “dynamic pricing with side information”. Over time periods $t = 1, \ldots, T$ the seller observes covariates $x_t \in \mathbb{R}^d$ in each period, sets a price and then observes the realized demand. The covariates generally are a set of features that characterize the pricing environment at both a macro level (market or environment) and a micro level (customer-specific). The seller’s objective is to maximize revenue, and the demand depends on the price and the covariates through a parametric function. The seller needs to balance the learning of the demand function with the earning of the revenue, the classic exploration-exploitation tradeoff. The performance of a pricing policy or algorithm is measured by the gap between the expected cumulative revenue obtained by the policy and that of an optimal policy which knows the demand function a priori.

In this paper, we study a setting where the covariates $x_t$ are arbitrarily (with no i.i.d. assumption, even adversarially) generated and the demand depends on the covariates and the price through a generalized linear model. The generalized linear model includes the two mainstream demand models, the binary demand model and the linear demand model as special cases. Both have been covered in the literature extensively, generally under an i.i.d. assumption on the covariates. The i.i.d. assumption can be restrictive in many situations in practice, say when there is serial correlation or peer-effects in the data. We elaborate on these limitations further in §2.1.3.

Table 1 summarizes existing works on the problem of dynamic pricing with covariates along with the assumptions, demand model, and the derived regret bound. For the regret bound, we focus on the
Table 1: Summary of Existing Results: The notation $\tilde{O}(\cdot)$ omits the logarithmic factors, $d$ is the dimension of the covariates and $T$ is the horizon length. The column “Covariates” describes the assumption on the generation of the covariates. In column “Demand Model”, $\beta, \gamma \in \mathbb{R}^d$ and $b \in \mathbb{R}$, which are defaulted as fixed but unknown parameters without further comment in “Key assumptions”, and $\epsilon$ is the unobserved error term. The column “Key assumptions” summarizes key assumptions for the corresponding paper.

| Paper                        | Regret Bound | Covariates | Demand Model          | Key Assumptions                  |
|------------------------------|--------------|------------|-----------------------|----------------------------------|
| Qiang and Bayati (2016)      | $O(d \log T)$| Martingale | $x^\top \beta + bp + \epsilon$ | With known incumbent price       |
| Cohen et al. (2020)          | $O(d^2 \log T)$ | —         | $1 \{x^\top \beta \geq p\}$ | —                                |
| Cohen et al. (2020)          | $O(d^{9/4} \log T)$ | —         | $1 \{x^\top \beta + \epsilon \geq p\}$ | Sub-Gaussian noise                |
| Mao et al. (2018)            | $O(T^{\frac{2}{21}})$ | —         | $1 \{f(x) \geq p\}$ | Log-concave noise with \(f(x)\) is Lipschitz |
| Javanmard (2017)             | $O(\sqrt{T})$ | —         | $1 \{x^\top \beta + \epsilon \geq p\}$ | Log-concave noise with \(\beta\) changing |
| Javanmard and Nazerzadeh (2019) | $O(d \log T)$ | i.i.d.     | $1 \{x^\top \beta + \epsilon \geq bp\}$ | Log-concave noise and sparsity   |
| Javanmard and Nazerzadeh (2019) | $O(\log d \sqrt{T})$ | i.i.d.     | $1 \{f(x) \geq p\}$ | Strictly log-concave noise       |
| Luo et al. (2021)            | $O(dT^{\frac{2}{7}})$ | —         | $1 \{x^\top \beta + \epsilon \geq bp\}$ | Log-concave noise with unknown distribution |
| Xu and Wang (2021)           | $O(d \log T)$ | —         | $1 \{f(x) \geq p\}$ | Sub-Gaussian noise and sparsity  |
| Ban and Keskin (2021)        | $O(dv \sqrt{T})$ | i.i.d.     | $g(x^\top \beta + x^\top \gamma \cdot p) + \epsilon$ | Sub-Gaussian noise and sparsity  |
| Ours                         | $O(d^{2} \log^2 T)$ | —         | $g(x^\top \beta + bp) + \epsilon$ | Sub-Gaussian noise and known price coefficient $b$ |
| Ours                         | $O(dv \sqrt{T})$ | —         | $g(x^\top \beta + x^\top \gamma \cdot p) + \epsilon$ | Sub-Gaussian noise                |

On a high-level, the existing literature on dynamic pricing with covariates can be grouped into three categories:

- Allow arbitrary covariates but assume both the knowledge of the price coefficient and the parameters of the distribution of demand shock. In §2.1 we discuss modeling issues associated with this assumption.
- Assume i.i.d. covariates and allow an unknown price coefficient and an unknown parameter of the distribution of demand shock.
- Allow arbitrary covariates with unknown price coefficient and parameters of the error distribution, but able to derive only a sub-optimal regret bound.

Our work (the first that we know of) makes no assumptions on how the covariates are generated and yet achieves an optimal order of regret.

2 Literature Review

As we mentioned earlier, there are two main models considered in the literature, linear and binary. Ban and Keskin (2021) consider the generalized linear demand model which includes both as special cases, extending the covariate-free case of the generalized linear demand model in (Broder and Rusmevichientong, 2012). Our paper considers this generalized linear demand model, but our algorithm and analysis differs from that of Ban and Keskin (2021).
Qiang and Bayati (2016) impose a martingale-type condition on the covariates which is similar to the i.i.d. assumption because both are to ensure the minimum eigenvalue of the sample covariance matrix is bounded away from zero. In Subsection 2.1.3, we discuss the technical and practical advantages of removing the i.i.d. assumption. In addition, our first algorithm is very simple, essentially an upper confidence bound (UCB) algorithm. Our second algorithm is based on Thompson sampling which has also been explored for the dynamic pricing problem (Ferreira et al., 2018; Bastani et al., 2021). Ferreira et al. (2018) consider the price-based network revenue management problem where there are finitely many allowable prices and a number of resource constraints.

We note that Javanmard and Nazerzadeh (2019) and Ban and Keskin (2021) pose their problem in a high-dimensional setting, with a sparsity assumption, and perform a variable selection subroutine in their pricing algorithm via $\ell_1$ regularization. To this end, we believe the i.i.d. assumption and the dependence of the bounds on the minimum eigenvalue in both the papers are necessary to handle the high-dimensional setting, in contrast to our low-dimensional one where we can remove those assumptions.

For more on the history and origin of the dynamic pricing problem, especially the covariate-free case, we refer the readers to the review paper (Den Boer, 2015).

2.1 Modeling issues

In this section we discuss some modeling issues and assumptions in the dynamic pricing literature.

2.1.1 Coefficient of price and the distribution of the random shock

We first point out an issue with assumptions on the stochastic quasi-linear demand model, the so-called binary demand model. The model is as follows. At time $t$ the demand is

$$D_t = \begin{cases} 
1, & \text{if } x_t^\top \beta^* + \xi_t \geq p_t \\
0, & \text{if } x_t^\top \beta^* + \xi_t < p_t
\end{cases}$$

where $x_t \in \mathbb{R}^d$ denotes the covariate vector, $\beta^* \in \mathbb{R}^d$ is the linear coefficient vector, $p_t$ is the price, and $\xi_t$ models an unobserved utility shock. Under this model, $x_t^\top \beta^* + \xi_t$ represents the customer’s utility where the term $x_t^\top \beta^*$ captures the part of the utility explained by the covariates $x_t$.

An assumption often made is (i) the coefficient of $p_t$ is fixed at 1 and (ii) $\xi_t$’s are i.i.d. and follow a distribution with known parameters. The vector $\beta^*$ then is assumed unknown that we wish to learn from observations over time. This is roughly the setting used in (Javanmard, 2017; Cohen et al., 2020; Xu and Wang, 2021) and part of (Javanmard and Nazerzadeh, 2019).

There are two issues in general with this setting:

1. Estimation of the parameters of the distribution generating $\xi_t$ cannot be separated from the estimation of the $\beta^*$. So it raises the question of how one arrives at this knowledge of the parameters of the distribution of $\xi_t$’s.

2. Somewhat related to the above point – say the seller starts collecting new data covariates $y_t \in \mathbb{R}^m$ in addition to $x_t$. The additional covariates $y_t$ can potentially improve the prediction of customer utility and thus reduce the variance of the demand shock $\xi_t$. Assuming the variance is known will then be problematic.

2.1.2 Parametric distribution of the error term

Assume the distribution of $\xi_t$ belongs to a parametric family of distributions with some unknown parameter(s). Say $\xi_t$ has zero mean and is scale-invariant with variance $\sigma^{-2}$ for some unknown $\sigma > 0$. 

Scaling appropriately, model (1) is then equivalent to the following binary demand model where the distribution of \( \xi_t \) is known but the price coefficient \( \sigma \) is unknown,

\[
D_t = \begin{cases} 
1, & \text{if } x_t^\top \beta^* + \xi_t \geq \sigma p_t \\
0, & \text{if } x_t^\top \beta^* + \xi_t < \sigma p_t 
\end{cases}
\]  

(2)

where \( \xi_t \) follows some known distribution with mean zero and variance one.

So in (2) we can assume either (i) known price coefficient (say fixed to 1) and unknown variance, or (ii) unknown price coefficient and known variance (say 1). We note that it is unnecessary to assume both parts unknown because the thresholding conditions in (1) and (2) are scale-invariant.

Remark 1. In the literature, Javanmard and Nazerzadeh (2019) and Luo et al. (2021) consider the model (2). Specifically, Javanmard and Nazerzadeh (2019) focus on the high-dimensional setting and impose the i.i.d. assumption on the covariates; Luo et al. (2021) allow a non-parametric structure for the shock distribution (more general than (2)) and achieve an \( \tilde{O}(dT^{2/3}) \) regret. Xu and Wang (2021) also mention the model (2) and leave the achievability of \( \sqrt{T} \) regret as an open question. The generalized linear model replaces the price coefficient \( \sigma \) in (2) with a linear function of the covariates \( x_t \). Thus it can be viewed as a further generalization of the model (2). The rationale is that the extent to which the covariates \( x_t \) explain the customer utility can be dependent not only on some unknown constant \( \sigma \) but also on \( x_t \), and the dependency is unknown. Our result resolves the open question in (Xu and Wang, 2021).

Linear Demand Model: Another prevalent demand model (den Boer and Zwart (2014); Keskin and Zeevi (2014, 2018); Qiang and Bayati (2016); Javanmard and Nazerzadeh (2019); Ban and Keskin (2021); Bastani et al. (2021)) is the linear demand model, where demand

\[
D_t = x_t^\top \beta^* + x_t^\top \gamma^* \cdot p_t + \epsilon_t
\]  

(3)

and \( \beta^* \) and \( \gamma^* \) are unknown parameter vectors. The demand shock \( \epsilon_t \)'s are mean-zero random variables, and it makes no essential difference to adapt the distribution of \( \epsilon_t \) to the history up to time \( t \).

Under the linear demand model, the distribution of \( \epsilon_t \) makes no difference to the optimal pricing strategy as long as it is mean-zero and sub-Gaussian. This explains why papers on the linear demand model, unlike the case of the binary demand model’s unobserved utility shock \( \xi \), do not assume knowledge of the distribution of \( \epsilon_t \).

2.1.3 Case for making no assumptions on the covariate distributions

The covariates are typically customer information or features of the environment and are almost always exogenous and not in the control of the firm. Making assumptions on their distribution is restrictive for the following reasons:

- Network/Peer effects: A wide range of products are influenced by network or peer effects (Seiler et al., 2017; Goolsbee and Klenow, 2002; Bailey et al., 2019; Nasr and Elshar, 2018). Baardman et al. (2020) show that demand prediction can be improved by incorporating such effects. Thus, the customers’ features are more likely to exhibit short-term dependencies.

- Seasonality and life-cycle of product: Seasonality, day-of-week and time-of-day patterns create serial correlation in the covariates (Neale and Willems, 2009). Product life-cycle effects (for tech or fashion items) also come into play, so the distribution of the covariates changes over the life-cycle
of the product as the customer segment mix may be different at different stages of the product life-cycle.

- Competitors: Competing products and the action of the competitors influence demand (Armstrong et al., 2005). However, the effect from competitors’ actions is complicated and the covariates of the environment cannot be identically distributed.

The remainder of the paper is organized as follows. Section 3 gives the formal definition of the problem with assumptions and performance measure. Section 4 designs a UCB-type pricing algorithm with a recipe when quasi-MLE problem cannot be solved in closed-form. Section 5 purposes an efficient pricing algorithm based on Thompson sampling to alleviate the potential computation issue in UCB pricing algorithm, and also gives a sufficient condition for improving the regret bound from $O(\sqrt{T})$ to $O(\log T)$. Finally, section 6 provides empirical simulations and some discussion with future directions.

3 Model and Performance Measure

We consider the generalized linear demand model

$$D_t = g(x_t^\top \beta^* + x_t^\top \gamma^* \cdot p_t) + \epsilon_t \quad \forall t = 1, 2, \ldots, T,$$

where $\beta^*, \gamma^* \in \mathbb{R}^d$ are true unknown parameters, $g(\cdot)$ is a known function, $x_t$ is an observable customer covariate vector, and $\epsilon_t$ is an unobservable and idiosyncratic demand shock in period $t$. Here, by observable we mean that the seller knows $x_t$ before setting the price $p_t$. We let $X$ denote the domain of $x_t$ and $\theta^* := (\beta^*; \gamma^*)$ as the concatenated parameter vector.

We present the expected revenue function as

$$r(p; \theta, x) := p \cdot g(x_t^\top \beta + x_t^\top \gamma \cdot p),$$

and denote the optimal expected revenue function as

$$r^*(\theta, x) := \max_{p \geq 0} p \cdot g(x_t^\top \beta + x_t^\top \gamma \cdot p),$$

with the optimal pricing function

$$p^*(\theta, x) = \arg \max_{p \geq 0} p \cdot g(x_t^\top \beta + x_t^\top \gamma \cdot p).$$

Throughout the paper, we assume $p^*(\theta, x)$ can be efficiently computed for any given parameter $\theta$ and covariates $x$.

Assumption 1 (Boundedness). We assume

(a) There exists $\bar{\theta} > 0$ such that $\Theta = \{ \theta \in \mathbb{R}^{2d} : \| \theta \|_2 \leq \bar{\theta} \}$ and $\theta^* \in \Theta$.

(b) The seller is allowed with a price range $[\underline{p}, \bar{p}]$ under all possible $x$ and $\theta^*$. We assume that $p^*(\theta, x)$ is in the interior of the feasible set $[\underline{p}, \bar{p}]$ for all $\theta \in \Theta$ and $x \in X$.

(c) For all $x \in X$ and $p \in [\underline{p}, \bar{p}]$, $\|(x, px)\|_2 \leq 1$.

The boundedness assumption in above is standard in the dynamic pricing literature and it is also well-grounded in a practical application context. We note that if the raw covariates do not satisfy the last part, one can always perform some normalization for the covariates to meet the condition.
Assumption 2 (Properties of \( g(\cdot) \)). Assume \( g(\cdot) \) is strictly increasing and differentiable, with bounded derivative over its domain. Specifically, there exist constants \( \underline{g}, \bar{g} \in \mathbb{R} \) such that \( 0 < \underline{g} \leq g(z) \leq \bar{g} < \infty \) for all \( z = x^\top \beta + x^\top \gamma \cdot p \) where \( x \in \mathcal{X}, \theta \in \Theta, \) and \( p \in [\underline{p}, \bar{p}] \).

Assumption 3 (Demand Shock). Assume \( \{\epsilon_t, t = 1, 2, \ldots\} \) form a \( \sigma^2 \)-sub-Gaussian martingale difference, i.e.,
\[
\mathbb{E}[\epsilon_t | \mathcal{H}_{t-1}] = 0 \quad \text{and} \quad \log \left( \mathbb{E} \left[ \exp(s\epsilon_t) | \mathcal{H}_{t-1} \right] \right) \leq \frac{\sigma^2 s^2}{2}
\]
for all \( s \in \mathbb{R} \), where \( \mathcal{H}_t := \sigma(p_1, \ldots, p_t, \epsilon_1, \ldots, \epsilon_t, x_1, \ldots, x_t, x_{t+1}) \) and \( \mathcal{H}_0 = \sigma(0, \Omega) \). Moreover, we assume \( \sigma^2 \) is known a priori.

Assumption 3 on the error term covers common distributions such as Normal and random variables with bounded support. The sub-Gaussian parameter \( \sigma^2 \) is an upper bound proxy for the true variance of the random variable, which can be easily obtained for a bounded random variable. We note that the filtration definition includes the covariates at time \( t + 1 \). This small change allows the demand shock \( \epsilon_t \) to be dependent on the covariates \( x_t \), and gives us more modeling flexibility.

In the following we show how the general model (4) recovers the binary demand model (2) and the linear demand model (3) as special cases.

Example 1 (Binary Demand Model). As noted in the previous section, the binary demand model (2) (also (1)) is considered by a number of papers as in Table 1 (denoted by binary model in the “Demand Model” column). We first restrict the first dimension of \( x_t \) to be always 1. To recover (2), we can then set the function \( g(\cdot) \) in (4) to be the cumulative distribution function of \( -\xi_t, \gamma^* = (-\sigma, 0, \ldots, 0)^\top \), and
\[
\epsilon_t = \begin{cases} 
1 - g \left( x_t^\top \beta - \sigma p_t \right) & \text{w.p.} \ g \left( x_t^\top \beta - \sigma p_t \right), \\
- g \left( x_t^\top \beta - \sigma p_t \right) & \text{w.p.} \ 1 - g \left( x_t^\top \beta - \sigma p_t \right).
\end{cases}
\]
Thus it becomes the binary demand model (2). The parameter \( \sigma \) is an unknown parameter that represents the price coefficient or describes the variance of the utility shock in (2); the sub-Gaussian parameter \( \sigma^2 \) can be easily chosen as 1/4 from the boundedness of \( \epsilon_t \).

Example 2 (Linear Demand Model). The linear demand model (3) can be easily recovered from (4) by letting the function \( g(\cdot) \) be an identity function. As in Example 1, we restrict the first dimension of \( x_t \) to be always 1. Then, the model in (Qiang and Bayati, 2016) can be recovered by setting \( \gamma^* = (b, 0, \ldots, 0) \); the covariate-free linear demand model in (den Boer and Zwart, 2014; Keskin and Zeevi, 2014) can be recovered by setting \( \beta^* = (a, 0, \ldots, 0) \) and \( \gamma^* = (b, 0, \ldots, 0) \).

Performance Measure. Now, we define regret as the performance measure for the problem. Specifically,
\[
\text{Reg}_T^\pi(x_1, \ldots, x_T) = \sum_{t=1}^T r^*(\theta^*, x_t) - \mathbb{E} \left[ \sum_{t=1}^T r_t \right]
\]
where \( \theta^* \) denotes the true parameter vector, \( \pi \) denotes the policy/algorithm, \( r_t \) is the revenue obtained at time \( t \) under \( \pi \), and the expectation is taken with respect to the demand shock \( \epsilon_t \)’s. The benchmark oracle (the first summation in above) is defined based on the optimal revenue function \( r^*(\theta, x) \). It assumes the knowledge of the true \( \theta^* \) but does not observe the realization of the demand shock \( \epsilon_t \) when setting the price. In defining the regret, we allow the covariates \( x_t \)’s to be arbitrarily generated. Therefore, no expectation is taken for \( x_t \)’s in the regret definition, and we seek for a worst-case regret upper bound over all possible \( x_t \)’s. For the case when \( x_t \)’s are i.i.d., the regret definition involves one more layer of expectation on \( x_t \)’s. Thus our regret bound is stronger and can directly translate into a regret bound for the i.i.d. case.
4 UCB-Based Pricing

In this section, we introduce our first generic algorithm for the problem of dynamic pricing with arbitrary covariates. Since we do not assume the knowledge of the distribution $\epsilon$’s, we adopt the quasi-maximum likelihood estimation (MLE) to learn the parameter $\theta$. We first introduce some analytical results for the quasi-MLE problem and then present our algorithm.

4.1 Regularized Quasi-Maximum Likelihood Estimation

For the dynamic pricing problem, we define the *misspecified likelihood function* for the $t$-th observation as follows

$$ l_t(\theta) := -\int_{D_t} g(z_t^\top \theta) \frac{1}{h(u)} (u - D_t) du, \quad (5) $$

where $\theta = (\beta, \gamma)$ encapsulates the parameters, $z_t = (x_t, p_t, x_t)$ is a column vector by concatenating the covariates, and $h(u) = g'(g^{-1}(u))$ for $u \in \mathbb{R}$. To gain some intuition into $l_t(\cdot)$, we can treat it as the weighted mean of the gap between observation $D_t$ and the estimator $g(z_t^\top \theta)$: by noting $1/h(u) = (g^{-1})'(u)$, it can be interpreted as a weight which captures the first-order information (derivative) of $\theta$ revealed at point $u$. Then the function $l_t$ as a first-order Taylor expansion proxy of $g^{-1}(\cdot)$ for the true likelihood function aims to find a $\theta$ that minimizes the weighted gap between $g(z_t^\top \theta)$ and $D_t$. This can be seen when one maximizes $l_t$ over $\theta$, it achieves its maximum when $g(z_t^\top \theta) = D_t$ (if possible). The likelihood function $l_t$ takes the same form as the one in (Ban and Keskin, 2021), but our analysis differs from the analysis therein and is more aligned with the analyses in the bandits literature (Abbasi-Yadkori et al., 2011; Filippi et al., 2010). We also refer readers to (Gill, 2001; Heyde, 2008) for more details.

Based on $l_t$, we define the regularized quasi-likelihood estimator with parameter $\lambda$ as:

$$ \hat{\theta}_t := \arg\max_{\theta \in \Theta} -\lambda \|\theta\|_2^2 + \sum_{\tau=1}^t l_\tau(\theta), \quad (6) $$

where $g$ denotes the lower bound of $g'(\cdot)$ as defined in Assumption 2. The estimator $\hat{\theta}_t$ will be used throughout the paper. The motivation for the regularization term is to overcome the singularity caused by the arbitrary covariates and to ensure a curvature for the likelihood function.

Now we analyze the property of the estimator. The gradient and Hessian of $l_t$ are

$$ \nabla l_t(\theta) = \frac{g'(z_t^\top \theta) \cdot z_t}{h(g(z_t^\top \theta))} \left(D_t - g(z_t^\top \theta)\right) = \xi_t(\theta) z_t, \quad (7) $$

$$ \nabla^2 l_t(\theta) = -\eta_t(\theta) z_t z_t^\top, \quad (8) $$

where $\xi_t(\theta) := D_t - g(z_t^\top \theta)$ and $\eta_t(\theta) := g'(z_t^\top \theta)$. The concise form of the gradient and Hessian justifies the choice of $h(u)$ in (5).

The following lemma states that under a non-anticipatory pricing policy/algorithm, the sequence of $\{\xi_t(\theta^*)\}_{t=1}^T$ is a martingale difference sequence adapted to history observations with zero-mean $\sigma^2$-sub-Gaussian increments.

**Lemma 1.** For $t = 1, \ldots, T$, we have

$$ \mathbb{E} [\xi_t(\theta^*) | \mathcal{H}_{t-1}] = 0. $$

In addition, $\xi_t(\theta^*) | \mathcal{H}_{t-1}$ is $\sigma^2$-sub-Gaussian.

**Proof.** Note that

$$ \mathbb{E} [\xi_t(\theta^*) | \mathcal{H}_{t-1}] = \mathbb{E} [\epsilon_t | \mathcal{H}_{t-1}] = 0. $$
Both the last part and the sub-Gaussianity come from Assumption 3.

Let the (cumulative) score function

$$S_t := \sum_{\tau=1}^{t} \frac{\xi_{\tau}(\theta^*)}{\sigma} z_\tau,$$

and define the (cumulative) design matrix

$$M_t := \lambda I_{2d} + \sum_{\tau=1}^{t} z_\tau z_\tau^\top$$

where $I_{2d}$ is a $2d$-dimensional identity matrix.

Define the $M$-norm of a vector $z$,

$$\|z\|_M := \sqrt{z^\top M z}$$

for a positive definite matrix $M$. We use $\det M$ to denote the determinant of the matrix $M$.

The following theorem measures $S_t$’s deviation in terms of the metric induced by $M_t$. It can be easily proved by an application of the martingale maximal inequality on the sequence of $\xi_t(\theta^*)$.

**Theorem 1** (Theorem 20.4, Lattimore and Szepesvári (2020)). For any regularization parameter $\lambda > 0$ and $\delta \in (0, 1)$,

$$\mathbb{P} \left( \exists t \in \{1, \ldots, T\} : \|S_t\|_{M_t}^2 \geq 2 \log \left( \frac{1}{\delta} \right) + \log \left( \frac{\det M_t}{\lambda^{2d}} \right) \right) \leq \delta.$$

An implication of the theorem is to produce the following bound on the estimation error of $\hat{\theta}_t$. The proof is a combination of the curvature analysis of the objective function (6) and the above theorem.

Proposition 1 is a key ingredient in removing the statistical assumption on the covariates $x_t$. Specifically, when we impose some i.i.d. assumption on $x_t$ and assume the minimum eigenvalue of its covariance matrix is bounded away from zero, then we can upper bound the estimation error of $\theta^*$ in the Euclidean norm. When we do not impose such assumptions, the following proposition tells that we can still obtain an estimation error bound by measuring the distance according to the sampled design matrix $M_t$.

As noted earlier, the technique is often seen in the linear bandits literature where we usually assume the features associated with the actions arrive in an arbitrary manner (Abbasi-Yadkori et al., 2011; Filippi et al., 2010).

**Proposition 1.** For any regularization parameter $\lambda > 0$, the following bound holds

$$\mathbb{P} \left( \exists t \in \{1, \ldots, T\} : \|\hat{\theta}_t - \theta^*\|_{M_t} \geq 2\sqrt{\lambda} \theta + \frac{2\sigma}{\sqrt{2}} \sqrt{2 \log \left( \frac{1}{\delta} \right) + \log \left( \frac{\det M_t}{\lambda^{2d}} \right)} \right) \leq \delta.$$

for any $\delta \in (0, 1)$.

The proof of Proposition 1 is deferred to the Appendix and is based on the Taylor expansion for the objective function (6); the first-order term will be characterized by Theorem 1, while the second-order term is controlled jointly by the matrix $M_t$ and the properties of the function $g$ given in Assumption 2.

We choose $\delta = \frac{1}{T}$ in Proposition 1 and define the function

$$\alpha(M) := 2\sqrt{\lambda} \theta + \frac{2\sigma}{\sqrt{2}} \sqrt{2 \log T + \log \left( \frac{\det M}{\lambda^{2d}} \right)}.$$
Then we obtain the following corollary which gives us a small-probability confidence bound for the estimator \( \hat{\theta}_t \). The first part of the corollary prescribes the confidence interval in our following algorithms, and the second part provides a uniform upper bound of \( \alpha(M_t)'s \). Specifically, the first part is obtained by plugging \( \delta = \frac{1}{T} \) into Proposition 1. And given that \( \|z_t\|_2^2 \leq 1 \) by assumption, we can apply Lemma 19.4 of Lattimore and Szepesvári (2020) (purely algebraic analysis with no stochasticity) and obtain the second part.

**Corollary 1.** For all \( \lambda > 0 \),

\[
P \left( \exists t \in \{1, ..., T\} : \left\| \hat{\theta}_t - \theta^* \right\|_{M_t} \geq \alpha(M_t) \right) \leq \frac{1}{T}.
\]

Moreover, for all \( t = 1, ..., T \),

\[
\alpha(M_t) \leq \bar{\alpha} := 2\sqrt{X\theta} + \frac{2\theta}{d} \sqrt{2 \log(T) + 2d \log \left( \frac{2d\lambda + T}{2d\lambda} \right)}.
\]

Now we complete the discussion of the quasi-MLE and we proceed to the algorithm and regret analysis in the following.

### 4.2 Algorithm and Regret Analysis

Algorithm 1 describes a UCB-based pricing algorithm. At each time \( t \), the algorithm first constructs an estimator and the corresponding confidence bound based on the quasi-MLE introduced earlier. Then the algorithm finds the most “optimistic” parameter within the confidence bound and sets the price pretending this parameter to be the true. It is a standard exemplification of the idea of upper confidence bound (UCB), also known as the principle of optimism in the face of uncertainty.

**Algorithm 1 UCB Pricing**

**Input:** Regularization parameter \( \lambda \).

**for** \( t = 1, ..., T \) **do**

1. Compute the estimators \( \hat{\theta}_{t-1} \) by (6) and its confidence interval
   \[
   \Theta_t := \left\{ \theta \in \Theta : \left\| \theta - \hat{\theta}_{t-1} \right\|_{M_{t-1}} \leq \alpha(M_{t-1}) \right\}.
   \]

2. Observe feature \( x_t \) and choose the UCB parameter which maximizes the expected revenue:
   \[
   \theta_t := (\beta_t, \gamma_t) = \arg \max_{\theta \in \Theta_t} r^* (\theta, x_t)
   \]
   \[\text{(9)}\]

3. Set the price by
   \[
   p_t = p^* (\theta_t, x_t)
   \]

**end for**

While many existing dynamic pricing algorithms more or less utilize the special model structure for designing algorithms, Algorithm 1 is very simple yet generally applicable. In this light, we hope the algorithm and its analysis can work as a prototype for future study on this topic. Theorem 2 states the regret bound of Algorithm 1. For dimension \( d \) and horizon \( T \), it meets the lower bound of the problem (See (Ban and Keskin, 2021)) up to logarithmic factors. Compared to the existing bounds (Qiang and Bayati, 2016; Javanmard and Nazerzadeh, 2019; Ban and Keskin, 2021), our bound does not involve the term \( \lambda_{\min}^{-1} \) where \( \lambda_{\min} \) represents the minimum eigenvalue of the covariance matrix of \( X_t \).
For other parameters like $\bar{p}$ and $\bar{\theta}$, we are unclear about whether their dependencies are optimal. They also appear in the existing regret bounds under the i.i.d. setting (Javanmard and Nazerzadeh, 2019; Ban and Keskin, 2021). Under the linear demand model, the parameter $\bar{\theta}$ is of the same magnitude with the demand $D_t$ so it can be treated as a constant. But for binary linear demand model, whether the parameter $\bar{\theta}$ has some implicit dependency on $d$ is contingent on the distribution of the utility shock in (2).

For Theorem 2, the proof idea is very intuitive. As the algorithm represents the confidence interval based on the matrix $M_{t-1}$, the current observation $x_t$ will either induce a small single step regret or reduce the confidence interval significantly. Then we upper bound the regret of the algorithm to a summation sequence involving the covariates $x_t$’s and matrices $M_t$’s and then employ the elliptical potential lemma (as follows) to conclude the proof.

**Lemma 2** (Elliptical Potential Lemma, (Lai and Wei, 1982)). For any constant $\lambda \geq 1$ and sequence of $\{x_t\}_{t \geq 1}$ with $\|x_t\|_2 \leq 1$ for all $t \geq 1$ and $x_t \in \mathbb{R}^d$, define the sequence of covariance matrices:

$$
\Sigma_0 := \lambda I_d, \quad \Sigma_t := \lambda I_d + \sum_{\tau=1}^{t} x_{\tau}x_{\tau}^\top \quad \forall t \geq 1,
$$

where $I_d$ is the identity matrix with dimension $d$. Then for any $T \geq 1$, the following inequality holds

$$
\sum_{t=1}^{T} \|x_t\|_{\Sigma_{t-1}}^2 \leq 2d \log \left( \frac{\lambda d + T}{\lambda d} \right).
$$

**Theorem 2.** Under Assumptions 1, 2 and 3, with any sequence $\{x_t\}_{t=1,...,T}$, if we choose the regularization parameter $\lambda = 1$, the regret of Algorithm 1 is upper bounded by

$$
4\bar{p}\bar{g}\bar{\alpha} \sqrt{T d \log \left( \frac{2d + T}{2d} \right)} + \bar{p} = O(d\sqrt{T \log T})
$$

where $\bar{\alpha} = 2\theta + \frac{2\sigma_g}{t} \sqrt{2\log T + 2d \log \left( \frac{2d + T}{2d} \right)}$ defined in Corollary 1 represents an upper bound for the confidence volume.

### 4.3 Discussion

Analysis-wise, the regret derivation of Algorithm 1 largely mimics the analysis of bandits problem with a generalized linear dependence (Filippi et al., 2010). The dynamic pricing with covariates problem differs in that (i) the action space becomes infinite and changes over time; (ii) there is a slight misalignment between the objective (reward) and the observation (demand). Specifically, the pricing decision at time $t$, if viewed as an action for the bandits problem, is a line segment prescribed jointly by the covariate $x_t$ and the allowable price $p_t$. The analysis here draws a connection between the dynamic pricing and the bandits problem, and also provides an alternative route for the existing analyses on the dynamic pricing with covariates problem (See (Qiang and Bayati, 2016; Ban and Keskin, 2021; Zhu and Zheng, 2020) among others). As noted earlier, this new analysis relaxes the i.i.d. assumption and removes the dependency on the inverse minimum eigenvalue of the covariance matrix.

As for the computational aspect, the dynamic pricing problem has one extra layer of optimization problem (9) than a finite-arm bandits problem. Specifically, Algorithm 1 has two optimization problems to solve: (i) the quasi-MLE problem (6); (ii) the UCB optimization problem (9). In the following, we discuss these two problems separately.
Quasi-MLE problem (6)

In general, the quasi-MLE problem cannot be solved in closed-form. In practice, the problem (6) has to be solved by some optimization algorithm, and thus the solution output from the optimization algorithm might not be the exact optimal solution. We show that the pricing algorithm can also be adapted to the case of an approximate optimal solution. Algorithm 2 describes such an adaption where at each time $t$, an approximate solution $\hat{\theta}_t$ is used to construct the confidence interval. The approximation gap $\Delta_t$ is formally defined as follows and it can be obtained by monitoring the dual optimization problem,

$$
\Delta_t := \sum_{\tau=1}^{t} l_{\tau}(\hat{\theta}_t) - \lambda g\|\hat{\theta}_t\|_2^2 - \sum_{\tau=1}^{t} l_{\tau}(\hat{\theta}_t) + \lambda g\|\hat{\theta}_t\|_2^2.
$$

To account for this approximation gap, the confidence bound needs to be enlarged accordingly such that it covers the true parameter $\theta^*$.

**Algorithm 2**

**UCB Pricing with Approximation**

**Input:** Regularization parameter $\lambda$.

**for** $t = 1, ..., T$ **do**

- Compute the estimators $\hat{\theta}_{t-1}$ by (6) with approximation gap $\Delta_{t-1}$ and its confidence interval

  $$
  \Theta_t := \left\{ \theta \in \Theta : \|\theta - \hat{\theta}_{t-1}\|_{M_{t-1}} \leq \alpha(M_{t-1}) + \frac{\sqrt{2}}{2} \Delta_{t-1} \right\}.
  $$

- Observe feature $x_t$ and choose the UCB parameter which maximizes the expected revenue:

  $$
  \theta_t := (\beta_t, \gamma_t) = \arg\max_{\theta \in \Theta} r^*(\theta, x_t)
  $$

- Set the price by

  $$
  p_t = p^*(\theta_t, x_t)
  $$

**end for**

**Theorem 3.** Under Assumptions 1, 2 and 3, with any sequence $\{x_t\}_{t=1,...,T}$, if we choose the regularization parameter $\lambda = 1$, the regret of Algorithm 2 is upper bounded by

$$
4\bar{p}\bar{g} \left( \alpha \sqrt{Td \log \left( \frac{2d + T}{2d} \right)} + \frac{\sqrt{2} \sum_{t=1}^{T} \Delta_{t-1} E\left[\|z_t\|_{M_{t-1}}]\right)}{2 \sqrt{2}} \right) + \bar{p}.
$$

Moreover, if $\Delta_{t-1} \leq \bar{\Delta}$ for all $t = 1, ..., T$, the regret upper bound becomes

$$
4\bar{p}\bar{g} \left( \alpha + \frac{\sqrt{2} \Delta_{t-1}}{2} \right) \sqrt{Td \log \left( \frac{2d + T}{2d} \right)} + \bar{p}.
$$

Theorem 3 provides a regret upper bound for Algorithm 2. Given that the objective function (6) is unnormalized, the uniform bound $\Delta$ can be achieved through linear steps of gradient descent or stochastic gradient descent. This explains the application of online optimization techniques for the dynamic pricing problem in (Javanmard, 2017; Xu and Wang, 2021). The proof idea of Theorem 3 is almost identical to that of Theorem 2 by using the following lemma captures the distance between the approximate solution $\hat{\theta}_t$ and the optimal solution $\hat{\theta}_t$. The lemma can be implied from the optimality condition.

**Lemma 3.** Recall that $\hat{\theta}_t$ is the optimal solution to the optimization problem (6). For any $\theta \in \Theta$, we
have
\[ \sum_{\tau=1}^{t} l_{\tau}(\hat{\theta}_{\tau}) - \lambda g\|\hat{\theta}_{\tau}\|^2 - \sum_{\tau=1}^{t} l_{\tau}(\theta) + \lambda g\|\theta\|^2 \geq \frac{1}{2\lambda g}\|\hat{\theta}_{\tau} - \theta\|_{M_{\tau}}^2. \]

The lemma justifies the choice of the confidence bound in Algorithm 2 and it ensures that the confidence bound covers the true parameter with high probability.

**UCB optimization problem (10)**

We note that in general the optimization subroutine (10) may be non-convex and hard to solve. In fact, computation efficiency is a common issue for UCB algorithms. For example, the action selection step can be NP-hard for the linear bandit problem (Dani et al., 2008). The following two examples show that the subroutine could be solved efficiently under some special cases.

**Example 3** (Covariate-free linear demand function). Consider the demand \( D_t = a + bp_t + \epsilon_t \), where \( a > 0, b < 0 \) are unknown real-valued parameters. Then (10) becomes an optimization problem with concave objective function \(-\frac{a}{2b}\) subject to a quadratic constraint.

**Example 4** (Binary demand observation with constant price coefficient). Consider the binary demand model (2) where the parameters \( \beta \) and \( \sigma \) are unknown. For each fixed value of \( \sigma \), it is easy to verify that the revenue function is increasing with respect to \( x^\top \beta \); hence the optimization problem (10) reduces to quadratically constrained linear program (QCLP). Then we can discretize the domain of \( \sigma \) into a number of possible candidates, and solve a QCLP for each candidate. Lastly, we pick the pair of \( \beta \) and \( \sigma \) that outputs the largest objective value.

While these specific examples rely on the structure of the problem, one general solution to compute the UCB optimization problem is through Monte Carlo method. Note that for each time \( t \), the confidence interval is an ellipsoid, so random sampling from the confidence interval can be done efficiently.

Specifically, we randomly generated \( K \) samples from the uniform distribution over the confidence interval and we denote the samples as \( \tilde{\theta}_{k} \)’s. Then the optimization subroutine is solved by

\[ \theta_t = \arg \max_{k=1,...,K} r^*(\tilde{\theta}_{k}, x_t). \]

In the numerical experiments, we will try out different values for \( K \) and examine the effect of sample size on the algorithm performance.

## 5 Towards More Efficient Computation and Better Regret

### 5.1 Thompson Sampling

Now we apply the method of Thompson sampling as a more efficient implementation of the algorithms in the previous section. Previous work (Ferreira et al., 2016) has discussed the possibility of applying Thompson sampling for revenue management where the authors studied on the covariate-free case and analyzed the constrained dynamic pricing/revenue management problem. Our result focuses on the handling of covariates and analyzes the problem under a frequentist rather than Bayesian setting.

In Algorithm 3, we first compute the quasi-MLE estimator \( \hat{\theta}_{t-1} \) just as the previous algorithms. Then the algorithm samples a multivariate Gaussian vector \( \eta_t \) and uses \( \eta_t \) to generate a randomized estimator \( \tilde{\theta}_{t-1} \). This sampling step can be viewed as an efficient substitute of the UCB optimization step (10) in Algorithm 1. Intuitively, the design matrix \( M_{t-1} \) represents the confidence bound for the
current estimation of $\theta^*$ based on the past observations. In the algorithm, the matrix $M_{t-1}$ twists the Gaussian vector $\eta_t$ so as to encourage a random exploration in the direction that the current estimator is less confident. Then the algorithm pretends the sampled parameter $\hat{\theta}_{t-1}$ as true and uses it to set the price $p_t$. The algorithm design and analysis largely follow the analysis of Thompson sampling for linear bandits by Abeille and Lazaric (2017).

Algorithm 3 Thompson Sampling with Covariates

Input: Regularization parameter $\lambda$.

for $t = 1, \ldots, T$ do

Compute the estimator $\hat{\theta}_{t-1}$ by (6) and observe covariates $x_t$.

Sample $\eta_t \sim \mathcal{N}(0, I_{2d})$ and compute the parameter

$$\tilde{\theta}_{t-1} := \hat{\theta}_{t-1} + \alpha(M_{t-1})M_{t-1}^{-1/2}\eta_t.$$ 

Set the price by

$$p_t = \arg\max_{r \in [\underline{p}, \bar{p}]} \mathbb{E}_{x_t}(r(p, \tilde{\theta}_{t-1}, x_t)).$$

end for

Assumption 4 (Properties of $g(\cdot)$). Let

$$\tilde{\Theta} := \{\theta \in \mathbb{R}^{2d} : \|\theta - \hat{\theta}\|_2 \leq 2\tilde{\alpha}\sqrt{d\log(4dT^2)} \text{ for some } \hat{\theta} \in \Theta\}$$

where $\Theta$ is defined in Assumption 1. We assume $g(z)$ is strictly increasing, differentiable, convex and there exist constants $\underline{g}, \bar{g} \in \mathbb{R}$ such that $0 < \underline{g} \leq g'(z) \leq \bar{g} < \infty$ for all $z = x^\top \beta + x^\top \gamma \cdot p$ where $x \in \mathcal{X}$, $\theta = (\beta, \gamma) \in \tilde{\Theta}$, and $p \in [\underline{p}, \bar{p}]$.

Assumption 4 is a stronger version of Assumption 2 on the properties of $g$. Specifically, the domain of parameter $\theta$ is enlarged from $\Theta$ to $\tilde{\Theta}$ so as to cover (with high probability) the randomized sampled parameters $\tilde{\theta}_t$'s in the algorithm. Assumption 4 basically requires that the properties of $g$ in Assumption 2 hold on this larger domain of $\tilde{\Theta}$. We remark that the constants $\underline{g}$ and $\bar{g}$ are always one under the linear demand model; but for the binary demand model, these two constants may change after the domain is enlarged.

Theorem 4. Under Assumption 1, 3, 4, with any sequence $\{x_t\}_{t=1}^{T}$, if we choose the regularization parameter $\lambda = 1$, the regret of Algorithm 3 can be bounded by

$$8(8\sqrt{e\pi} + 1)\underline{p}\bar{d}\bar{g}\sqrt{T\log(4dT^2)}\log\left(\frac{2d + T}{2d}\right) + 2\bar{p} = O\left((d\log T)^{3/2}\sqrt{T}\right).$$

Compared to Algorithm 1, there is an extra factor of $\sqrt{d}$ in the regret bound here. This extra factor is also inevitable for the existing analyses of Thompson sampling algorithms on the linear bandits problem (Agrawal and Goyal, 2013; Abeille and Lazaric, 2017). The proof of Theorem 4 largely follows the derivation in (Abeille and Lazaric, 2017). Some special care needs to be taken with respect to the price optimization step in the Algorithm 3. The price optimization restricts the optimal price to an interval of $[\underline{p}, \bar{p}]$, and thus some projection into this interval may sometimes be required for the unconstrained optimal price. The projection prevents a direct application of the gradient-based single step regret bound in (Abeille and Lazaric, 2017), but a similar argument can be made using the Lipschitzness of the function $g$ and the boundedness of the covariates and the parameters. We defer the detailed proof to Section B.
resolves the computational efficiency of the UCB optimization (2019). In particular, the price coefficient can be estimated through small perturbed pricing experiments. In general, the algorithm and analysis presented in the following are mainly for technical illustration purposes: (i) to identify a difference between the dynamic pricing problem and the bandits problem; (ii) to explain the achievability of \( O(\log T) \) regret dependency in the literature.

Assumption 5 (Known \( \gamma^* \) and smoothness). Assume \( \gamma^* \) in (4) is known. In addition, assume there exists a constant \( C \) such that the optimal expected revenue function satisfies

\[
|r^*(\theta^*, x) - r(p^*(\theta^*; x); \theta^*, x)| \leq C(x^\top \beta^* - x^\top \beta)^2,
\]

for all \( \theta, \theta^* \in \Theta \) with \( \theta = (\beta; \gamma^*) \) and \( \theta^* = (\beta^*; \gamma^*) \).
To interpret the condition in the assumption, we recall that \(p^*(\theta, x)\) denotes the optimal price under the parameter \(\theta\). So the left-hand-side represents the revenue loss caused by using a wrong parameter \(\theta\) for pricing, while the right-hand-side is quadratic in terms of the linear estimation error. One sufficient condition for the assumption is that \(r(p; \theta^*, x)\) is continuously twice differentiable with respect to \(p\) for all possible \(\theta^*\) and \(x\), and \(p^*(\theta, x)\) is Lipschitz in \(x^\top \beta\). Essentially, this condition does not impose extra restriction upon Assumptions 1, 2, and 3; in other words, almost all the demand models that satisfy the previous assumptions also meet this condition under the knowledge of \(\gamma^*\). For example, this condition can be met by the binary demand model (2) with a log-concave unknown noise (Javanmard and Nazerzadeh, 2019) and by the linear demand model (3). It is also analogous to the “well separation” condition in the covariate-free case (Broder and Rusmevichientong, 2012).

To proceed with the algorithm description, we first slightly revise the MLE estimator in Section 4.1 for known \(\gamma^*\). Specifically, we redefine the misspecified likelihood function for the case of known \(\gamma^*\) as

\[
\tilde{l}_t(\beta) := -\int_{D_t} g(x_t^\top \beta + x_t^\top \gamma^* p_t) \frac{1}{h(u)}(u - D_t)du,
\]

where the function \(h(u)\) is the same as in Section 4.1. Then the estimator becomes

\[
\hat{\beta}_t := \arg \max_{\beta \in \Theta} -\lambda \|\beta\|_2^2 + \sum_{\tau=1}^t \tilde{l}_\tau(\beta),
\]

where \(\Theta_\beta\) denotes the subspace \(\{\beta : (\beta, \gamma^*) \in \Theta\}\). Compared to the previous case of unknown \(\gamma^*\), the only change made here is to plug in the known value of \(\gamma^*\) and to restrict the attention to estimating the unknown \(\beta^*\). Accordingly, we revise the definition of the (cumulative) design matrix as

\[
\tilde{M}_t := \lambda I_d + \sum_{\tau=1}^t x_\tau x_\tau^\top
\]

with \(I_d\) as an identity matrix of dimension \(d\).

The following result is parallel to Corollary 1. We omit the proof as it is the same as the previous case of unknown \(\gamma^*\) except for some minor notation changes.

**Proposition 2.** For all \(\lambda > 0\),

\[
P\left(\exists t \in \{1, ..., T\} : \left\|\hat{\beta}_t - \beta^*\right\|_{\tilde{M}_t} \geq \alpha(\tilde{M}_t)\right) \leq \frac{1}{T}.
\]

Moreover, for all \(t = 1, ..., T\),

\[
\alpha(\tilde{M}_t) \leq \tilde{\alpha} := 2\sqrt{\tilde{\theta}} + \frac{2\tilde{\sigma}}{d} \sqrt{2 \log(T) + 2d \log \left(\frac{d\lambda + T}{\lambda d}\right)}.
\]

Algorithm 5 describes a certainty-equivalent pricing policy. At each time step, the algorithm performs a regularized quasi-MLE to obtain the estimator \(\hat{\beta}_{t-1}\). Then it assumes \(\hat{\beta}_{t-1}\) to be the true parameter and finds the corresponding optimal price.

**Theorem 6.** Under Assumptions 1, 2, 3, 5 and with any sequence \(\{x_t\}_{t=1,...,T}\), if we choose the regularization parameter \(\lambda = 1\), the regret of Algorithm 5 is upper bounded by

\[
2C\tilde{\alpha}^2 d \log \left(\frac{d + T}{d}\right) + \bar{p} = O(d^2 \log^2 T)
\]
Algorithm 5 Certainty-Equivalent Pricing

**Input:** Regularization parameter $\lambda$.

for $t = 1, ..., T$ do
  Compute the estimator $\hat{\beta}_{t-1}$ by (11), observe feature $x_t$ and set the price by
  $$p_t = p^* \left( \hat{\theta}_{t-1}, x_t \right)$$
  where $\hat{\theta}_{t-1} = (\hat{\beta}_{t-1}; \gamma^*)$
end for

where $\hat{\theta} = \sqrt{\frac{d}{2}} a^2 + \frac{2a}{2b} a^2 \sqrt{2 \log T + d \log \left( \frac{d + T}{d} \right)}$ is defined in Proposition 2 and $C$ is defined in Assumption 5.

Theorem 6 provides a regret upper bound for Algorithm 5. We remark that it is unnecessary for Algorithm 5 to compute the estimator at every time step. Javanmard and Nazerzadeh (2019) solve an $L_1$ regularized linear regression on geometric time intervals, and the scheme can also be applied to Algorithm 5 with the same order of regret bound. The frequent or infrequent estimation scheme makes no analytical difference and the choice mainly accounts for computation consideration. Xu and Wang (2021) study the special case of the binary demand model with unit price coefficient (i.e., known $\gamma^*$), and they derive the same order of regret bound as Theorem 6 under arbitrary covariates using online Newton’s method. The intuition is that the convergence rate of Newton’s method is on the same order with the MLE estimator, so the corresponding output can be viewed as an approximate MLE estimator at each time step, and the approximation will not deteriorate the regret performance. This is aligned with Theorem 3 and Theorem 5 where an approximate quasi-MLE estimator is used.

In general, many existing the $O(\sqrt{T})$ regret bounds (Broder and Rusmevichientong, 2012; Javanmard, 2017; Javanmard and Nazerzadeh, 2019; Xu and Wang, 2021) fall into this paradigm of

known price coefficient + certainty-equivalent policy.

Intuitively, when the price coefficient is known, the price $p_t$ will not interfere the learning of $\beta^*$. Thus there is no need to do price exploration like UCB or TS, and the regret purely reflects the cumulative learning rate of $\beta^*$. This disentanglement of pricing decisions from parameter estimation makes the setting of known $\gamma^*$ analogous to the “full information” setting in online learning literature. In contrast, when the price coefficient is unknown, the pricing decisions will affect the learning rate of $\gamma^*$, thus the setting of unknown $\gamma^*$ is more aligned with the “partial information” setting such as the bandits problem.

Interpreting the result under a linear demand model.

We use a linear demand model to further illustrate the contrast between $O(\sqrt{T})$ and $O(\log T)$ regret dependency. Consider the demand follows

$$D_t = a + bp_t + \epsilon_t$$

for some $a > 0$ and $b < 0$. At time $t$, the seller sets the price by $p_t = -\frac{a}{2b}$ with for some estimators $\hat{a}_t$ and $\hat{b}_t$ which could be from either Algorithm 1 (optimistic estimators) or Algorithm 5 (CE estimators). Then the single step regret can be expressed by a function of the true parameters and the estimators,

$$\text{Reg}_t = r^*((a, b)) - r(p_t; (a, b)) = -2b \left( \frac{\hat{a}_t}{2b_t} - \frac{a}{2b} \right)^2 = -\frac{(\hat{a}_t b_t - \hat{b}_t a)^2}{2bb_t^2} \quad (12)$$
• When the price coefficient is known, \( \hat{b}_t = b \). The equality becomes
\[
\text{Reg}_t = -\frac{1}{2b}(\hat{a}_t - a)^2.
\]

• When the price coefficient is unknown, we cannot do more than a first-order Taylor expansion when we want to upper bound Reg, by the estimation error, i.e.,
\[
\text{Reg}_t \leq c(|\hat{a}_t - a| + |\hat{b}_t - b|)
\]
for some \( c > 0 \).

For this example, whether the price coefficient \( b \) is known determines the space in which we view the right-hand-side of (12) as a function of \( \hat{a}_t \) and \( \hat{b}_t \). Intuitively, suppose that the estimation error is on the order of \( \sqrt{1/t} \) (the intuition is precise when the covariates are i.i.d.). Then the right-hand-side will recover two different regret bounds under the two settings.

Generally, we remark that the first-order bound like (13) under a proper norm is always the first step for the analysis of UCB and TS algorithms, including our analysis for the dynamic pricing problem. For linear bandits problem, the LinUCB algorithm (Chu et al., 2011; Abbasi-Yadkori et al., 2011) can directly obtain this first-order bound and under TS algorithms, it can be obtained by some Bayesian arguments (Russo and Van Roy, 2014) or by maintaining a constant probability of choosing an optimistic action with anti-concentration sampling (Abeille and Lazaric, 2017).

6 Numerical Experiments and Conclusion

6.1 Numerical Experiments

We consider the linear demand model (3) for three groups of numerical experiments:

(a) Covariates \( x_t \) are i.i.d. generated throughout all time periods.

(b) The horizon is split into two phases with equal length: In first phase, the first half of the covariates (dimension 1 to \( d/2 \)) as a sub-vector are i.i.d. generated over time while the second half (dimension \( d/2 + 1 \) to \( d \)) are all zero; in the second phase, the first half of the covariates (dimension 1 to \( d/2 \)) are all zero while the second half (dimension \( d/2 + 1 \) to \( d \)) as a sub-vector are i.i.d. generated over time.

(c) The horizon is split into six phases with equal length: In first three phases, at Phase \( m = 1, 2, 3 \), only \( x_t \)’s \( m \)-th third covariates are non-zero and i.i.d. generated over time (just like (b) but with three groups of covariates). For Phase \( m = 4, 5, 6 \), it repeats the generation mechanism of Phase \( m - 3 \), respectively.

Moreover, we set the allowable price range as \([0, 1, 5]\). For each simulation trial, the parameter \( \beta^* \) is generated by a uniform distribution over \( \frac{1}{\sqrt{d}}[1, 2]^d \) and \( \gamma^* \) is generated by a uniform distribution over \( -\frac{1}{\sqrt{d}}[0, 1]^d \). The dimension \( d \) will be tested for different values as shown in the figures below. For experiments (a), (b) and (c), the covariates (if non-zero) is always generated i.i.d. from a uniform distribution over \( \frac{1}{\sqrt{d}}[0, 1] \). The normalizing factor \( \frac{1}{\sqrt{d}} \) ensures the demand always stays on the same magnitude for different \( d \).
For benchmark purpose, we adapt the covariate-free constrained iterated least square (CILS) algorithm (Keskin and Zeevi, 2014) for the covariate setting. Specifically, the price is set by

\[ p_t = \begin{cases} \bar{p}_{t-1} + \text{sgn}(\delta_t)\kappa t^{-\frac{1}{4}}, & \text{if } |\delta_t| < \kappa t^{-\frac{1}{4}} \\ p^*(\hat{\theta}_t), & \text{otherwise} \end{cases} \]

where \( \hat{\theta}_t \) is the least square estimator for the unknown parameters, \( \bar{p}_{t-1} \) is the average of the prices over the period 1 to \( t-1 \), and \( \delta_t = p^*(\hat{\theta}_t) - \bar{p}_{t-1} \). The intuition is that if the tentative price \( p^*(\hat{\theta}_t) \) stays too close to the history average, we will introduce a small perturbation as price experimentation to encourage the parameter learning. The parameter \( \kappa \) is a hyper-parameter and after a moderate tuning, we choose \( \kappa = \frac{d}{10} \) in our experiments. For Algorithm 1, we also set the confidence set by

\[ \Theta_t = \left\{ \theta \in \Theta : \left| \hat{\theta}_{t-1} - \theta \right|^2 \leq \frac{d}{10} \right\} \]

For Algorithm 3, we choose the sampled parameter by

\[ \tilde{\theta}_t = \hat{\theta}_t + \frac{\sqrt{d}}{25} M_{t-1}^{-1/2} \eta_t. \]

For both UCB and TS algorithms, we choose the regularization parameter \( \lambda = 1 \). We set the horizon \( T = 1500 \) and plot the cumulative gap between the online revenue and the optimal revenue. The curve is plotted based on an average over 100 simulation trials. We use covariate CILS to denote the benchmark algorithm.

![Graphs showing cumulative regret for different algorithms and choices of \( d \) for experiments (a), (b), and (c)].

Figure 1: Experiment (a): i.i.d. covariates

Figure 2: Experiment (b): two phases with different distributions

Figure 1 shows Experiment (a) where the covariates are i.i.d. generated over time. We can see that all the algorithms exhibit sublinear curve and their performances are quite comparable to each other. Figure 2 shows Experiment (b) where there are two phases with covariates from different distributions. We notice that the CILS algorithm performs well for the first phase but fails to learn the parameter
Figure 3: Experiment (c): six phases with repeating patterns during the second phase. Algorithm 1 performs stably well under all numbers of Monte Carlo samples. At the beginning of Phase 2, the curves of Algorithm 1 and Algorithm 3 grow quickly but then they all flatten. Figure 3 shows Experiment (c) where there are six phases and the later three phases repeat the distributions of the first three phases, respectively. We observe that both Algorithm 1 and Algorithm 3 have successfully learned all the parameters over the first three phases, so the corresponding curves do not grow significantly over the later three phases. Besides, we also note that the performance of Algorithm 1 is quite insensitive with respect to the number of samples $K$ in the Monte Carlo method across all the experiments.

6.2 Discussion and Future Directions

In this paper, we consider the dynamic pricing problem under a generalized linear demand model where the demand is dependent on the price and covariates. We develop general-purposed algorithms and derive regret bounds under the settings of unknown price coefficient and known price coefficient. We conclude our discussion with a few possible future work directions.

- **Demand model beyond Lipschitzness and unimodality.**
  All the existing works discussed in this paper assume some Lipschitzness on the demand function. For the generalized linear demand model, this is exemplified in Assumption 2. Though the structure may be well justified in practice, the regret’s dependency on the Lipschitz constant, as well as some other constants, is often overlooked in the regret analysis. A more complete characterization of the regret in terms of all the model parameters will bring more insights and it deserves more future efforts. *den Boer and Keskin (2020)* consider a piece-wise linear (discontinuous) demand function and the model therein might point to a direction for non-Lipschitzness. Besides, the consideration of a multimodal revenue function (*Wang et al., 2021*) in dynamic pricing under the arbitrary covariates is not addressed in this work.

- **Non-parametric distribution of $\epsilon_t$ under the binary demand model.**
  In Section 2.1, we discuss the modeling issue of assuming the price coefficient to be 1. The binary demand model (2) partially resolves the problem but still imposes a parametric structure for the distribution. The question is what if the parametric distribution family is misspecified and what effect the misspecification will bring to the algorithm performance. *Luo et al. (2021)* consider a fully non-parametric distribution for $\epsilon_t$ and obtain a regret on the order of $T^{2/3}$. The performance deterioration might be the paid price for not knowing the parameterized structure of $\epsilon_t$. It remains unclear whether the dependency on $T$ can be further improved under a non-parametric distribution on $\epsilon_t$.

- **Online model selection.**
The existing works on dynamic pricing often assume the demand function belongs to a family of functions. The question is how to choose the proper function family and what if the true demand function lies outside the family. Besbes and Zeevi (2015) consider the covariate-free case and illustrate the robustness of a linear demand model. With the presence of covariates, Javanmard and Nazerzadeh (2019) and Ban and Keskin (2021) both consider the high-dimensional setting where the variable selection can be viewed as a model selection procedure. More generally, the question remains open whether the problem of dynamic pricing with covariates exhibits a similar formulation as the online model selection problem (Chatterji et al., 2020; Foster et al., 2019) where the true demand model may belong to a large family of models and the algorithm’s regret is adaptively determined by the some complexity measure of the true demand model.

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A  Proofs for Section 4

A1 Proof of Proposition 1

Proof. We perform a second-order Taylor’s expansion for the objective function of regularized quasi-MLE (6) around the true parameter $\theta^*$. Let

$$Q_t(\theta) := \sum_{\tau=1}^{t} l_{\tau}(\theta).$$

We have

$$Q_t(\theta^*) - \lambda g\|\theta^*\|_2^2 - Q_t(\theta) + \lambda g\|\theta\|_2^2$$

$$= - \langle \nabla Q_t(\theta^*) - \lambda g\theta^*, \theta - \theta^* \rangle - \frac{1}{2} \langle \theta - \theta^*, (\nabla^2 Q_t(\theta^*) - \lambda gI_{2d}) (\theta - \theta^*) \rangle$$

for some $\theta'$ on the line segment between $\theta$ and $\theta^*$.

By the optimality of $\hat{\theta}_t$,

$$Q_t(\theta^*) - \lambda g\|\theta^*\|_2^2 \leq Q_t(\hat{\theta}_t) - \lambda g\|\hat{\theta}_t\|_2^2.$$

Then from (14), we have

$$\langle \nabla Q_t(\theta^*) - \lambda g\theta^*, \hat{\theta}_t - \theta^* \rangle + \frac{1}{2} \langle \hat{\theta}_t - \theta^*, (\nabla^2 Q_t(\theta^*) - \lambda gI_{2d}) (\hat{\theta}_t - \theta^*) \rangle \geq 0,$$
for some $\hat{\theta}$ on the line segment between $\theta$ and $\theta^*$.

From Assumption 2,

$$-\nabla^2 Q_t(\hat{\theta}) = \sum_{\tau=1}^{t} g' \left( z_{\tau}^T \hat{\theta} \right) z_{\tau} z_{\tau}^T \geq \frac{g}{2} \sum_{\tau=1}^{t} z_{\tau} z_{\tau}^T \quad \forall \hat{\theta} \in \Theta.$$  

Further, with Holder’s inequality and (15),

$$\|\nabla Q_t(\theta^*) - \lambda g \theta^*\|_{M_t^{-1}} \geq \left\langle \hat{\theta}_t - \theta^* \right\rangle_{M_t}$$

almost surely. Consequently,

$$\|\nabla Q_t(\theta^*) - \lambda g \theta^*\|_{M_t^{-1}} \geq \frac{1}{2} \|\hat{\theta}_t - \theta^*\|_{M_t} \quad a.s. \quad (16)$$

Recall that

$$S_t = \sum_{\tau=1}^{t} \frac{\xi_\tau(\theta^*)}{\sigma} z_{\tau} = \frac{1}{\sigma} \nabla Q_t(\theta^*),$$

which implies

$$\frac{1}{2\sigma^2} \|\hat{\theta}_t - \theta^*\|_{M_t} \leq \frac{1}{\sigma} \left\| -\nabla Q_t(\theta^*) + \lambda g \theta^* \right\|_{M_t^{-1}}$$

Here the first line comes from (16), the second line comes from the definition of $S_t$, the third lines comes from the norm inequality, and the last line is from the fact that $\lambda M_t^{-1} \leq I_{2d}$. Thus, we complete the proof from combining Theorem (1) with $\|\theta^*\|_2 \leq \bar{\theta}$.

A2 Proof of Theorem 2

**Proof.** We define the “good event” as $\mathcal{E} = \{\theta^* \in \Theta_t \text{ for } t = 1, ..., T\}$. From Corollary 1, we know

$$\mathbb{P}(\mathcal{E}) \geq 1 - 1/T. \quad (17)$$

At time $t$, under the event $\mathcal{E}$, the choice of $\theta_t$ in Algorithm 1 ensures

$$r^*(\theta_t, x_t) \geq r^*(\theta^*, x_t). \quad (18)$$
Thus, under the event $E$, the single period regret can be bounded by

$$\text{Reg}_t := r^*(\theta_t, x_t) - r(p_t; \theta^*, x_t)$$

$$\leq p_t \cdot (g(x_t^T \beta_t + x_t^T \gamma_t \cdot p_t) - g(x_t^T \beta^* + x_t^T \gamma^* \cdot p_t))$$

$$\leq \bar{p}g(z_t^T (\theta_t - \theta^*))$$

$$\leq \bar{p}g\|z_t\|_{M^{-1}_{t-1}} \|\theta_t - \theta^*\|_{M_{t-1}}$$

$$\leq \bar{p}g\|z_t\|_{M^{-1}_{t-1}} (\|\theta_t - \hat{\theta}_{t-1}\|_{M_{t-1}} + \|\hat{\theta}_{t-1} - \theta^*\|_{M_{t-1}})$$

$$\leq 2\bar{p}g\alpha\|z_t\|_{M^{-1}_{t-1}}$$

where the functions $r^*$ and $r$ are introduced in the Section 3. Here the first line is from (18), the third line is from Assumption 1, the fourth line is from Holder’s inequality, and the last inequality is by Corollary 1 under the event $E$.

Thus, the total expected regret (the expectation is with respect to the randomness of demand shocks) can be bounded by

$$\text{Reg}_{\pi_1}(x_1, ..., x_T) = \sum_{t=1}^{T} \mathbb{E}[\text{Reg}_t \cdot 1_E] + \mathbb{E}[\text{Reg}_t \cdot 1_{E^c}]$$

$$\leq 2\sum_{t=1}^{T} \bar{p}g\alpha\|z_t\|_{M^{-1}_{t-1}} + \mathbb{P}(E^c) \cdot \bar{p}T$$

$$\leq 2\bar{p}g\alpha \sqrt{T \sum_{t=1}^{T} \|z_t\|^2_{M^{-1}_{t-1}} + \mathbb{P}(E^c) \cdot \bar{p}T}$$

$$\leq 4\bar{p}g\alpha \sqrt{T d \log \left( \frac{2d\lambda + T}{2d\lambda} \right) + \bar{p}}$$

where $\pi_1$ denotes the pricing policy specified by Algorithm 1 and $E^c$ denotes the complement of the event $E$. The second inequality is by Holder’s inequality and the last inequality is because (17) and Lemma 2.

**A3 Proof of Lemma 3**

*Proof.* Recall that the regularized quasi-MLE at time $t$ is defined as

$$Q_t(\theta) - \lambda g\|\theta\|^2$$

with Hessian matrix

$$\sum_{t=1}^{T} -g'(z_t^T \theta) z_t z_t^T - \lambda gI_{2d}$$

which is negative definite by the assumption that $g'(z_t^T \theta) > 0$ (in the feasible domain of related parameters). Thus, the regularized quasi-MLE is concave in $\theta$. We can then perform a second-order Taylor’s
expansion around the optimal solution $\hat{\theta}_t$ in $\Theta$ with any point $\theta \in \Theta$,

$$Q_t(\hat{\theta}_t) - \lambda g_\theta \|\hat{\theta}_t\|^2_2 - Q_t(\theta) + \lambda g_\theta \|\theta\|^2_2$$

$$= -\left\langle \nabla Q_t(\hat{\theta}_t) - \lambda g_\theta \hat{\theta}_t, \theta - \hat{\theta}_t \right\rangle - \frac{1}{2} \left\langle \hat{\theta}_t - \theta, (\nabla^2 Q_t(\theta') - \lambda g_{I_2d})(\hat{\theta}_t - \theta) \right\rangle$$

$$\geq -\frac{1}{2} \left\langle \hat{\theta}_t - \theta, (\nabla^2 Q_t(\theta') - \lambda g_{I_2d})(\hat{\theta}_t - \theta) \right\rangle$$

$$\geq \frac{1}{2} g_\theta \|\hat{\theta}_t - \theta\|^2_{M_t},$$

where $\theta' \in \Theta$ is a point between $\theta$ and $\hat{\theta}_t$, the first inequality is by the concavity and the optimality of $\hat{\theta}_t$ in a compact set $\Theta$, and the second inequality is by $-\nabla^2 Q_t(\theta') \succeq q \sum_{t=1}^T z_t z_t^\top$. \hfill \Box

## B Proofs for Section 5

We first introduce some requirement for the sampling distribution.

**Definition 1** (Sampling distribution (Abelle and Lazaric, 2017)). A distribution $D^{TS}$ is suitable for Thompson sampling if it is a multivariate distribution on $\mathbb{R}^{2d}$ absolutely continuous with respect to the Lebesgue measure which satisfies the following properties:

1. *(anti-concentration)* there exists a positive probability $q$ such that for any $u \in \mathbb{R}^{2d}$ with $\|u\|_2 = 1$, 

   $$\mathbb{P}_{\eta \sim D^{TS}}(u^\top \eta \geq 1) \geq q,$$

2. *(concentration)* there exist constants $c, c'$ such that $\forall \delta \in (0,1)$,

   $$\mathbb{P}_{\eta \sim D^{TS}}(\|\eta\|_2 \leq \sqrt{2cd \log \frac{c'd^2d}{\delta}}) \geq 1 - \delta.$$

As shown in (Abelle and Lazaric, 2017), the Gaussian distribution $\eta \sim \mathcal{N}(0, I_{2d})$ that we use in Algorithm 3 satisfies the above definition with $c = c' = 2$ and $q = \frac{1}{\sqrt{2\pi}}$. An alternative choice is $\eta \sim U([0, \sqrt{2d}])^{2d}$, uniform distribution on domain $[0, \sqrt{2d}]^{2d}$, and then $c = 1$, $c' = \frac{\sqrt{2d}}{2}$ and $q = \frac{1}{\sqrt{16\sqrt{2}\pi}}$.

When the sampling distribution of $\eta$ in Algorithm 3 satisfies the Definition 1, it has the following property. Specifically, the following lemma tells that the sampled parameter $\hat{\theta}_{t-1}$ will be “optimistic” with respect to some convex function $f$ for a constant probability. By “optimistic”, we mean the objective value under this parameter $\hat{\theta}_{t-1}$ is no smaller than that under $\theta^*$.

**Lemma 4** (Lemma 3 of (Abelle and Lazaric, 2017)). For the sampled parameter in Algorithm 3 $\hat{\theta}_{t-1} = \hat{\theta}_{t-1} + \alpha(M_{t-1})M_{t-1}^{-1}/2 \eta$ with $\eta \sim \mathcal{N}(0, I_{2d})$, then for any convex function $f(\theta)$ in $\theta$ and any $t \geq 1$,

$$\mathbb{P}\left( f(\hat{\theta}_{t-1}) \geq f(\theta^*) \bigg| \mathcal{H}_{t-1}, \theta^* \in \Theta_{t-1} \right) \geq \frac{1}{8\sqrt{e\pi}}$$

where $\Theta_t$ is as defined in Algorithm 1.

Using this lemma, we proceed with the proof of Theorem 4.

**B1 Proof of Theorem 4**

*Proof*. Let $\kappa(M) := 2\sqrt{d \log(4dT^2)} \alpha(M)$. From Corollary 1, we know $\kappa(M) \leq 2\alpha \sqrt{d \log(4dT^2)}$, which justifies the choice of $\Theta$ in Assumption 4. Compared to the definition of $\alpha(M)$ in the UCB case, the
extra factor $2 \sqrt{d \log (4dT^2)}$ in $\kappa(M)$ aims to account for the dispersion caused by the sampling in the Thompson sampling algorithm. Specifically, $\alpha(M_t)$ describes the volume of confidence bound for $\hat{\theta}_t$ in the UCB algorithm, while $\kappa(M_t)$ describes the volume (with high probability) of possible sampled parameter $\tilde{\theta}_t$. Define

$$\hat{\Theta}_{t-1} := \{ \theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_{t-1}\|_{M_{t-1}} \leq \kappa(M_{t-1}) \}.$$ 

Now, we define the good event for Thompson sampling as

$$\mathcal{E} = \{ \theta^* \in \Theta_t, \tilde{\theta}_t \in \hat{\Theta}_t \text{ for } t = 0, ..., T - 1 \}$$

where $\Theta_t$ is as defined in Algorithm 1.

Then by Definition 1 and Corollary 1, we have

$$P(\mathcal{E}) \geq 1 - \frac{2}{T}.$$ 

Since the sampled parameter $\tilde{\theta}_t$ may be out of the original parameter set $\Theta$, we revise the definition of optimal objective function by

$$\tilde{r}^*(\theta, x) := \max_{p \in [\bar{p}, \bar{p}]} r(p; \theta, x).$$

By Assumption 4, the function $g(\cdot)$ is convex, so $\tilde{r}^*(\theta, x)$ is also convex in $\theta$ by the preservation of convexity under linear transformation and maximization.

Now under event $\mathcal{E}$, the regret can be decomposed into

$$\text{Reg}_{T}^{\pi_3} \cdot 1_{\mathcal{E}} = \sum_{t=1}^{T} (\tilde{r}^*(\theta^*, x_t) - \tilde{r}^*(\tilde{\theta}_{t-1}, x_t)) \cdot 1_{\mathcal{E}}$$

$$= \sum_{t=1}^{T} \left( \tilde{r}^*(\theta^*, x_t) - \tilde{r}^*(\tilde{\theta}_{t-1}, x_t) + \tilde{r}^*(\tilde{\theta}_{t-1}, x_t) - r(p_t; \theta^*, x_t) \right) \cdot 1_{\mathcal{E}}$$

where $\pi_3$ denotes the pricing policy specified by Algorithm 3.

We denote

$$\text{Reg}_{\pi}^{(1)} := \left( \tilde{r}^*(\theta^*, x_t) - \tilde{r}^*(\tilde{\theta}_{t-1}, x_t) \right) \cdot 1_{\mathcal{E}},$$

$$\text{Reg}_{\pi}^{(2)} := \left( \tilde{r}^*(\tilde{\theta}_{t-1}, x_t) - r(p_t; \theta^*, x_t) \right) \cdot 1_{\mathcal{E}}.$$ 

With the same approach as the proof of Theorem 2, we have

$$\mathbb{E} \left[ \sum_{t=1}^{T} \text{Reg}_{\pi}^{(2)} \right] \leq 8 \bar{p} \bar{g} d \sqrt{T \log (4dT^2) \log \left( \frac{2d + T}{2d} \right)}.$$ 

Now we focus on analyzing $\text{Reg}_{\pi}^{(1)}$. Define

$$\Theta_{t-1}^{\text{OPT}} = \{ \theta \in \hat{\Theta}_{t-1} : \tilde{r}^*(\theta, x_t) \geq \tilde{r}^*(\theta^*, x_t) \}.$$ 

The set contains parameters for which the corresponding optimal objective is larger than the optimal
objective value under the true parameter. Then for any $\bar{\theta} \in \Theta_t^{OPT}$, we have

$$
\mathbb{E} \left[ \text{Reg}^{(1)}_{t} | \mathcal{H}_{t-1} \right] \leq \mathbb{E} \left[ \left( \tilde{r}^* (\tilde{\theta}; x_t) - \hat{r}^*(\tilde{\theta}_{t-1}; x_t) \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \right] \\
= \mathbb{E} \left[ \left( \tilde{r}^* (\tilde{\theta}; x_t) - p_t \cdot g(x_t^T \tilde{\beta}_{t-1} + x_t^T \tilde{\gamma}_{t-1} \cdot p_t) \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \right] \\
\leq \mathbb{E} \left[ \tilde{p}^* (\tilde{\theta}, x_t) \cdot \left( g(x_t^T \tilde{\beta} + x_t^T \tilde{\gamma} \cdot \tilde{p}^* (\tilde{\theta}, x_t)) - g(x_t^T \tilde{\beta}_{t-1} + x_t^T \tilde{\gamma}_{t-1} \cdot \tilde{p}^* (\tilde{\theta}, x_t)) \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \right] \\
\leq \tilde{p} g \mathbb{E} \left( \| \tilde{z}_t (\tilde{\theta})^T (\tilde{\theta} - \tilde{\theta}_{t-1}) \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \\
\leq 4 \tilde{p} g \sqrt{\log (4d T^2)} \mathbb{E} \left( \| \tilde{z}_t \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1},
$$

where $\tilde{p}^* (\tilde{\theta}, x) = \arg \max_{p \in [0,1]} r(p; \tilde{\theta}, x)$ and $\tilde{z}_t (\tilde{\theta}) = (x_t, \tilde{p}^* (\tilde{\theta}, x_t))$. Here the third line is from the optimality of $p_t$, the fourth line is from Assumptions 1 and 4, and the last line comes from the definition of $\mathcal{E}$ together with Holder’s inequality.

Let $\tilde{\theta}_{t-1}$ be an independent copy of $\tilde{\theta}_{t-1}$ following the same distribution. Then we have

$$
\mathbb{E} \left[ \text{Reg}^{(1)}_{t} | \mathcal{H}_{t-1} \right] \leq 4 \tilde{p} g \sqrt{\log (4d T^2)} \mathbb{E} \left( \| \tilde{z}_t (\tilde{\theta}_t) \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \\
= 4 \tilde{p} g \sqrt{\log (4d T^2)} \mathbb{E} \left( \| \tilde{z}_t (\tilde{\theta}_t) \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \cdot P^{-1} \left( \tilde{\theta}_{t-1} \in \Theta_t^{OPT} | \mathcal{H}_{t-1}, \tilde{\theta}^* \in \Theta_{t-1} \right) \\
\leq 4 \tilde{p} g \sqrt{\log (4d T^2)} \mathbb{E} \left( \| \tilde{z}_t (\tilde{\theta}_t) \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1} \cdot 8 \sqrt{\pi} \\
= 32 \tilde{p} g \sqrt{\log (4d T^2)} \mathbb{E} \left( \| \tilde{z}_t \|_{M_{t-1}} \right) \cdot I_{\mathcal{E}} \mid \mathcal{H}_{t-1},
$$

where $z_t = (x_t, p_t, x_t)$ and $p_t$ is the price used in the algorithm. Here the first line comes by replacing $\tilde{\theta}$ in (19) with a randomized parameter of $\tilde{\theta}_{t-1}$ restricted to the set $\Theta_t^{OPT}$. The second line comes from the property of conditional expectation. The third line removes the indicator functions. The fourth line applies Lemma 4 by setting the function $f$ as $\hat{r}^*(\cdot, x_t)$. The last line comes from the definition of $\tilde{z}_t (\tilde{\theta}$) and $\tilde{z}_t$.

Finally, by using Lemma 2, we can conclude

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \text{Reg}^{(1)}_{t} \right] \leq 64 \tilde{p} g \sqrt{\log (4d T^2)} \sqrt{T d \log \left( \frac{2d + T}{2d} \right)}.
$$

Combining the two parts of regret with the additional loss caused by the event $\tilde{\mathcal{E}}$ (bounded by $2\tilde{p}$), we complete the proof.

\begin{flushright}
$\Box$
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**B2 Proof of Theorem 6**

**Proof.** Revise the definition of the “good event” as

$$
\tilde{\mathcal{E}} := \left\{ \left\| \beta_t - \beta^* \right\|_{M_t} \leq \alpha (M_t) \text{ for } t = 0, \ldots, T - 1 \right\},
$$

27
From Proposition 2, we know 
\[ \mathbb{P}(\mathcal{E}) \geq 1 - 1/T. \]

Under the event \( \mathcal{E} \), the single period regret can be bounded by 
\[
\text{Reg}_t = r^*(\theta^*, x_t) - r(p_t; \theta^*, x_t) 
\leq C \left| x_t^\top \beta^* - x_t^\top \hat{\beta}_{t-1} \right|^2
\leq C \|x_t\|_{\bar{M}_{t-1}}^2 \|\hat{\beta}_{t-1} - \beta^*\|_{\bar{M}_{t-1}}^2
\leq C \bar{\alpha}'^2 \|x_t\|_{\bar{M}_{t-1}}^2.
\]

Here the second line is from Assumption 5, the third line is from Holder’s inequality, and the last inequality is by Proposition 2 under the event \( \mathcal{E} \).

Thus, the total expected regret (the expectation is with respect to the randomness of demand shocks) can be bounded by 
\[
\text{Reg}^{\pi_{CE}}_T(x_1, ..., x_T) = \sum_{t=1}^{T} \mathbb{E} [\text{Reg}_t \cdot 1_{[\mathcal{E}]}] + \mathbb{E} [\text{Reg}_t \cdot 1_{[\mathcal{E}^c]}]
\leq \sum_{t=1}^{T} C \bar{\alpha}'^2 \|x_t\|_{\bar{M}_{t-1}}^2 + \mathbb{P}(\mathcal{E}^c) \cdot \bar{p}T
\leq 2C \bar{\alpha}'^2 d \log \left( \frac{d\lambda + T}{d\lambda} \right) + \bar{p}
\]

where \( \pi_{CE} \) denotes Algorithm 5 and \( \mathcal{E}^c \) denotes the complement of the event \( \mathcal{E} \). The last inequality is because of Lemma 2. \( \square \)