An analogue of a formula for Chebotarev Densities

Biao Wang

Department of Mathematics
University at Buffalo, The State University of New York
Buffalo, NY 14260, USA
bwang32@buffalo.edu

In this short note, we show an analogue of Dawsey’s formula on Chebotarev densities for finite Galois extensions of $\mathbb{Q}$ with respect to the Riemann zeta function $\zeta(ms)$ for any integer $m \geq 2$. Her formula may be viewed as the limit version of ours as $m \to \infty$.

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1. Introduction and statement of results

Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\text{Re } s > 1$ be the Riemann zeta function and let $\mu(n)$ be the Möbius function defined by $\mu(n) = (-1)^k$ if $n$ is the product of $k$ distinct primes and is zero otherwise. It is well-known (e.g., [5, (4.5)]) that the prime number theorem is equivalent to the assertion that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \quad (1.1)$$

or equivalently,

$$-\sum_{n=2}^{\infty} \frac{\mu(n)}{n} = 1. \quad (1.2)$$

Let $p(n)$ be the smallest prime divisor of $n$ and let $\varphi$ be the Euler totient function. Let $k \geq 1$, $\ell$ be integers and $(\ell, k) = 1$. In 1977, Alladi [2] proved that

$$-\sum_{\substack{n \geq 2 \\ p(n) \equiv \ell (\text{mod } k)}} \frac{\mu(n)}{n} = \frac{1}{\varphi(k)}. \quad (1.3)$$

In 2017, Dawsey [4] generalized formula (1.3) to the setting of Chebotarev densities for finite Galois extensions of $\mathbb{Q}$. That is, for any conjugacy class $C$ in the Galois group $G = \text{Gal}(K/\mathbb{Q})$ of a finite Galois extension $K$ of $\mathbb{Q}$, we have

$$-\sum_{\substack{n \geq 2 \\ [K(n):\mathbb{Q}] = C}} \frac{\mu(n)}{n} = \frac{|C|}{|G|}. \quad (1.4)$$

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where
\[
\left[ \frac{K}{\mathbb{Q}} \right]_p := \left\{ \left[ \frac{K}{\mathbb{Q}} \right]_p : p \subseteq \mathcal{O}_K \text{ and } p \mid p \right\}
\]
for unramified prime \( p \) and \( \left[ \frac{K}{\mathbb{Q}} \right]_p \) is the Artin symbol for Frobenius map. Here \( \mathcal{O}_K \) denotes the ring of integers in \( K \) and \( p \) denotes a prime ideal in \( \mathcal{O}_K \).

Alladi’s result (1.3) is the special case of (1.4) when \( K = \mathbb{Q}(\zeta_k) \) and \( C \) is the conjugacy class of \( \ell \), where \( \zeta_k \) is a primitive \( k \)-th root of unity.

In this note, we give an analogue of Alladi’s and Dawsey’s results relating to \( \zeta(ms) \) for any integer \( m \geq 2 \). Let \( \lambda_m(n) \) be the function defined as the coefficient of term \( \frac{1}{n^s} \) in the Dirichlet series expansion of \( \zeta(ms) \zeta(s) \) for \( \text{Re } s > 1 \). That is,
\[
\sum_{n=1}^{\infty} \frac{\lambda_m(n)}{n^s} = \frac{\zeta(ms)}{\zeta(s)} \quad (1.5)
\]
for \( \text{Re } s > 1 \). When \( m = 2 \), \( \lambda_2(n) = (-1)^{\Omega(n)} \) is the Liouville function (e.g., [7, Theorem 300]), where \( \Omega(n) = \sum_{p\mid n} \alpha \). Hence \( \lambda_m(n) \) is a generalization of Liouville function. In section 2 we will see that \( \lambda_m(n) = \sum_{d \mid n} \mu \left( \frac{n}{d^m} \right) \) and the prime number theorem is equivalent to the assertion that
\[
\sum_{n=1}^{\infty} \frac{\lambda_m(n)}{n} = 0. \quad (1.6)
\]

Analogous to Alladi’s formula (1.3), for \( (\ell, k) = 1 \) we have that
\[
- \sum_{\substack{n \geq 2 \\text{mod } k \\mid \ell \\text{mod } k}} \frac{\lambda_m(n)}{n} = \frac{1}{\varphi(k)}. \quad (1.7)
\]
As [4], Eq. (1.7) can be thought of as a special case in the following main theorem.

**Theorem 1.1.** Let \( K \) be a finite Galois extension of \( \mathbb{Q} \) with Galois group \( G = \text{Gal}(K/\mathbb{Q}) \). Then for any conjugacy class \( C \subseteq G \), we have
\[
- \sum_{\left[ \frac{K}{\mathbb{Q}} \right]_p = C} \frac{\lambda_m(n)}{n} = \frac{|C|}{|G|}. \quad (1.8)
\]

**Remark 1.2.** Since \( \lim_{m \to \infty} \zeta(ms) = 1 \) for \( s > 1 \), we have \( \lim_{m \to \infty} \lambda_m(n) = \mu(n) \). Hence Alladi’s and Dawsey’s results may be viewed as the limit version of (1.7) and (1.8), respectively.

**Remark 1.3.** In 2019, Sweeting and Woo [9] generalized (1.4) to finite Galois extensions of number fields. One may also generalize (1.8) to number fields.

For the proof of Theorem (1.1), we shall use a prime divisor function \( P_m(n) \) which will be defined in section 3 to estimate the difference between the partial sums of (1.4) and (1.8). As a result, \( P_m(n) \) is very close to the largest prime divisor function \( P(n) \) and satisfies Alladi’s duality property. Then we apply Dawsey’s result in [4].
2. Some properties of $\lambda_m(n)$

In this section, we mainly introduce the relation between $\lambda_m$ and $\mu$ and prove the prime number theorem with respect to $\lambda_m$.

Lemma 2.1. Let $m \geq 2$ be a fixed integer. For the $\lambda_m$ defined by (1.3), we have

(1) $\lambda_m$ is a multiplicative function.
(2) $\lambda_m(n) = \sum_{d|m \mid n} \mu\left(\frac{n}{d}\right)$.
(3) For any integer $n \geq 1$, we can write it as $n = k^m \cdot l$ for $k,l \geq 1$ and $l$ is $m$-th power-free (i.e., it has no $m$-th power divisor except 1). Then $\lambda_m(n) = \mu(l)$.
(4) $\mu(n) = \mu^2(n)\lambda_m(n)$ for all integer $n \geq 1$.

Proof. Set

$$a(n) := \begin{cases} 1, & \text{if } n = d^m \text{ for some integer } d \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then $a(n)$ is multiplicative and $\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(ms)$ for $\text{Re } s > 1$.

(1) It is well known (e.g. [6, Corollary 11.3]) that $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ for $\text{Re } s > 1$.

By (1.5), the definition of $\lambda_m(n)$, for $\text{Re } s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\lambda_m(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \quad (2.2)$$

It follows that $\lambda_m = a \ast \mu$ is the Dirichlet convolution of $a$ and $\mu$, which are both multiplicative functions. Hence $\lambda_m$ is multiplicative.

(2) Since $\lambda_m = a \ast \mu$, we have

$$\lambda_m(n) = \sum_{d|n} a(d) \mu\left(\frac{n}{d}\right). \quad (2.3)$$

Plugging (2.1) into (2.3), we get the part (2).

(3) Since $\lambda_m$ is multiplicative, it suffices to consider the prime powers. Suppose $n = p^{\alpha}$, $\alpha \geq 1$. Write $\alpha$ as $\alpha = m\beta + r$ with integers $\beta \geq 0$ and $0 \leq r < m$.

Then $p^\alpha = (p^\beta)^m \cdot p^r$ and we can use part (2) to compute $\lambda_m(p^\alpha)$ as follows:

$$\lambda_m(p^\alpha) = \sum_{d|m|p^\alpha} \mu\left(\frac{p^\alpha}{d}\right) = \sum_{j=0}^{\beta} \mu\left(\frac{p^\alpha}{p^{jm}}\right) = \sum_{j=0}^{\beta} \mu(p^{m(\beta-j)+r}) = \mu(p^r).$$

(4) By part (3), $\lambda_m(n) = \mu(n)$ if $n$ is square-free. Then part (4) follows immediately by the fact that $\mu$ is supported on square-free numbers. \(\square\)

Remark 2.2. Due to Lemma 2.1(2), analogous to the Möbius function $\mu(n)$, the Riemann hypothesis is equivalent to the estimate that for all $\epsilon > 0$ we have

$$\sum_{n \leq x} \lambda_m(n) = O(x^{\frac{1}{2}+\epsilon}) \quad (2.4)$$
where the implied constant depends on \( \epsilon \), see [3, Theorem 4.16, 4.18].

**Remark 2.3.** Sarnaks conjecture with respect to \( \mu \) is equivalent to Sarnaks conjecture with respect to \( \lambda_m \) due to Lemma 2.1(2) and (4), see [6, Corollary 11.25].

**Lemma 2.4.** The prime number theorem is equivalent to the assertion that

\[
\sum_{n=1}^\infty \frac{\lambda_m(n)}{n} = 0. \tag{2.5}
\]

**Proof.** Let

\[
A(x) := \sum_{n \leq x} \frac{\mu(n)}{n}.
\]

Then by [2, (2.24)], we have

\[
A(x) = O \left( \exp(-c(\log x)^{\frac{1}{4}}) \right) \tag{2.6}
\]

for some constant \( c > 0 \). Here we note that the aftermentioned \( c \) is always a positive constant that may vary according to the context.

By Lemma 2.1(2), we have

\[
\sum_{n \leq x} \frac{\lambda_m(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d^m = n} \mu(e) = \sum_{d^m \leq x^\frac{1}{m}} \frac{1}{d^m} A \left( \frac{x}{d^m} \right) = \sum_{d^m \leq x^\frac{1}{m}} \frac{1}{d^m} A \left( \frac{x}{d^m} \right) + \sum_{x^{\frac{1}{m}} < d^m \leq x} \frac{1}{d^m} A \left( \frac{x}{d^m} \right) \tag{2.7}
\]

For the first sum, note that \( \frac{x}{d^m} \geq x^{\frac{1}{2}} \) and \( \sum_{d=1}^\infty \frac{1}{d^m} \leq \zeta(2) < \infty \). So

\[
\sum_{d^m \leq x^{\frac{1}{m}}} \frac{1}{d^m} A \left( \frac{x}{d^m} \right) = \sum_{d^m \leq x^{\frac{1}{m}}} \frac{1}{d^m} \cdot O \left( \exp(-c(\log x)^{\frac{1}{4}}) \right) = O \left( \exp(-c(\log x)^{\frac{1}{4}}) \right). \tag{2.8}
\]

For the second sum, we have

\[
\sum_{x^{\frac{1}{m}} < d^m \leq x} \frac{1}{d^m} A \left( \frac{x}{d^m} \right) = O \left( \sum_{x^{\frac{1}{m}} < d^m \leq x} \frac{1}{d^m} \right) = O \left( x^{-\frac{m-1}{m}} \right) \tag{2.9}
\]

Combining (2.7), (2.8) and (2.9) together, we get an estimate for the partial sum

\[
\sum_{n \leq x} \frac{\lambda_m(n)}{n} = O \left( \exp(-c(\log x)^{\frac{1}{2}}) \right) \tag{2.10}
\]

and Eq. (2.5) follows immediately, as \( x \to \infty \). \( \square \)
3. Duality of prime factors

Lemma 3.1 (Duality Lemma). For any arithmetic function \( f(n) \) with \( f(1) = 0 \), we have

\[
\sum_{d \mid n} \lambda_m(d) f(p(d)) = -f(P_m(n)) \tag{3.1}
\]

where \( p(1) = 1 \) and \( P_m(n) \) is the largest prime factor of \( n \) of order \( \not\equiv 0 \pmod{m} \) and is 1 if \( n \) is a perfect \( m \)-th power.

Proof. Let \( a(n) \) be the function defined by (2.1). By (1.5), we have \( \zeta(ms) = \zeta(s) \sum_{n=1}^\infty \frac{\lambda_m(n)}{n^s} \), which implies that \( a(n) = \sum_{d \mid n} \lambda_m(d) \). Note that \( \lambda_m(n) \) is a multiplicative function. Following [2], for \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), \( p_1 < \cdots < p_r \), we have

\[
\sum_{d \mid n} \lambda_m(d) f(p(d)) = \lambda_m(1)f(1) + \sum_{j=1}^r f(p_j) \sum_{d \mid n, p(d) = p_j} \lambda_m(d)
\]

\[
= \sum_{j=1}^r f(p_j) \sum_{k=1}^{\alpha_j} \sum_{e \mid d_{j+1}} \lambda_m(p_j^k e)
\]

\[
= \sum_{j=1}^r f(p_j) \left( \sum_{k=1}^{\alpha_j} \lambda_m(p_j^k) \right) \sum_{e \mid d_{j+1}} \lambda_m(e)
\]

\[
= \sum_{j=1}^r f(p_j) \left( a(p_j^{\alpha_j}) - 1 \right) a(d_{j+1}) \tag{3.2}
\]

where \( d_i = p_{j_i}^{\alpha_i} p_{j_i+1}^{\alpha_{i+1}} \cdots p_{r_i}^{\alpha_{r_i}} \) for \( 1 \leq j \leq r \) and \( d_{r+1} = 1 \).

Let \( j_0 \) be the largest index \( j \) such that \( m \not\mid \alpha_j \). Then \( a(p_j^{\alpha_j}) = 1 \) for \( j > j_0 \), \( a(d_{j+1}) = 1 \) for \( j \geq j_0 \) and \( a(d_{j+1}) = 0 \) for \( j < j_0 \). The sum (3.2) turns out to be \( -f(p_{j_0}) \) and (3.1) follows. \( \square \)

Remark 3.2. Similarly, one can prove that for \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), \( p_1 < \cdots < p_r \),

\[
\sum_{d \mid n} \lambda_m(d) f(P_m(d)) = -\sum_{j=1}^{j_0} f(p_j)d_j(n)
\]

where \( d_j(n) = \sum_{m \mid \alpha_1, \ldots, m \mid \alpha_{j-1}, \quad d^m \mid p_j^{\alpha_j} p_{j+1}^{\alpha_{j+1}} \cdots p_r^{\alpha_r}} 1 \) and \( j_0 \) is the first index \( j \) such that \( m \not\mid \alpha_j \).
4. Proof of Theorem 1.1

Theorem 4.1 ([8, Theorem (1.7)]). Let $P(n)$ be the largest prime divisor of $n$. Then for $r > -1$,

$$
\sum_{\substack{n \leq x \\ P(n) \neq P(n)} } \frac{1}{P(n)^r} = x \exp\left\{ -(2r+2)\frac{x}{\log x} \left[ 1 + g_r(x) + O\left( \left( \frac{\log x}{\log(2) x} \right)^3 \right) \right] \right\}
$$

where $\log^{(k)} x = \log(\log^{(k-1)} x)$ is the $k$-fold iterated natural logarithm of $x$ and

$$
g_r(x) = \frac{\log^{(3)} x + \log(1+r) - 2 - \log 2}{2 \log^{(2)} x} \left( 1 + \frac{2}{\log^{(2)} x} \right) - \frac{\log^{(3)} x + \log(1+r) - 2}{8 (\log^{(2)} x)^2}.
$$

Corollary 4.2. There exists some constant $C_m$ such that

$$
\sum_{\substack{n \leq x \\ P_m(n) \neq P(n)} } 1 = O(x \exp(-c(\log x \log^{(2)} x)^{\frac{1}{2}}))
$$

and

$$
\sum_{\substack{n \leq x \\ P_m(n) \neq P(n)} } \frac{1}{n} = C_m + O(\exp(-c(\log x \log^{(2)} x)^{\frac{1}{2}})),
$$

where $c > 0$ is a positive constant.

Proof. Equation (4.2) follows by the case $r = 0$ in Theorem 4.1.

Put $e(x) = \sum_{\substack{n \leq x \\ P_m(n) \neq P(n)} } 1$. Then (4.3) can be deduced by (4.2) as follows

$$
\sum_{\substack{n \leq x \\ P_m(n) \neq P(n)} } \frac{1}{n} = \int_1^x \frac{e(t)}{t} dt = e(t) \bigg|_1^x + \int_1^x \frac{c(t) dt}{t^2} = C_m - \int_x^{\infty} \frac{c(t) dt}{t^2} + \frac{e(x)}{x},
$$

where $C_m = \int_1^{\infty} \frac{e(t) dt}{t^2}$.

Remark 4.3. Due to this corollary, $P_m(n)$ inherits a lot of properties of $P(n)$. For example, one can get a version of Theorem 4.1 for $P_m(n)$. Another example we would like to mention is that $P_m(n)$ is equi-distributed (mod $k$) for $k \geq 2$ by Theorem 1 in [2].

Now we prove the Theorem 1.1 by showing the following theorem.

Theorem 4.4. Under the notations and assumptions of Theorem 1.1 we have

$$
- \sum_{\substack{2 \leq n \leq x \\ \left( \frac{k}{P(n)} \right) = 1}} \frac{\lambda_n(n)}{n} = \frac{|C|}{|G|} + O\left( \exp(-c(\log x)^{\frac{1}{2}}) \right),
$$
where \( c \) is a positive constant.

**Proof.** Here we follow the ideas in the proof 2 of [2, Theorem 4] and the proof of [4, Theorem 1].

Let \( f(n) \) be an arithmetic function defined by

\[
    f(n) = \begin{cases} 
        1, & \text{if } \left[ \frac{K/Q}{p} \right] = C, n = p > 1; \\
        0, & \text{otherwise.}
    \end{cases}
\]

Then

\[
    \sum_{2 \leq n \leq x, \left[ \frac{K/Q}{p(n)} \right] = C} \frac{\lambda_m(n)}{n} = \sum_{n \leq x} \frac{\lambda_m(n)f(p(n))}{n}.
\]

As [2, (2.35)], by Möbius inversion formula and Duality Lemma 3.1 we have

\[
    \sum_{n \leq x \frac{1}{2}} \frac{\lambda_m(n)f(p(n))}{n} = - \sum_{n \leq x} \frac{1}{n} \sum_{d|n} \mu(n/d)f(P_m(d)) = - \sum_{n \in \mathfrak{N}} \mu(n) \cdot \frac{f(P_m(n))}{n} - \sum_{n \leq x} \frac{f(P_m(n))}{n} \sum_{d \leq x} \frac{\mu(d)}{d}.
\]

It follows that the difference between the partial sums on \( \lambda_m \) and \( \mu \) is

\[
    \sum_{2 \leq n \leq x, \left[ \frac{K/Q}{p(n)} \right] = C} \frac{\lambda_m(n)}{n} = \sum_{n \in \mathfrak{N}} \frac{\mu(n)}{n} \sum_{d \leq x} \frac{f(P_m(d)) - f(P(d))}{d} = S_1 + S_2 \tag{4.5}
\]

For \( S_2 \), by [2, (2.24)]

\[
    \sum_{n \leq x \frac{1}{2}} \frac{\mu(n)}{n} = O\left( \exp\left( -c\left(\log x\right)^{\frac{1}{2}} \right) \right) \tag{4.6}
\]

we get that

\[
    \sum_{x \frac{1}{2} < d \leq x} \frac{\mu(d)}{d} = O\left( \exp\left( -c\left(\log \frac{x}{n}\right)^{\frac{1}{2}} \right) \right) \tag{4.7}
\]

This implies that

\[
    S_2 = O\left( \sum_{n \leq x \frac{1}{2}} \frac{1}{n} \exp\left( -c\left(\log \frac{x}{n}\right)^{\frac{1}{2}} \right) = O\left( \exp(-c(\log x)^{\frac{1}{2}}) \right) \right) \tag{4.8}
\]
For $S_1$, by (4.3) in Corollary 4.2
\[
\sum_{d \leq x} f(P_m(d)) - f(P(n)) = C_m + O\left(\exp\left(-c(\log\frac{x}{n})^{1/2}\right)\right). \tag{4.10}
\]
Similar to (4.9) and by (4.7) again, we get that
\[
S_1 = -C_m \sum_{n \leq x} \frac{\mu(n)}{n} + O\left(\exp\left(-c(\log x)^{1/2}\right)\right) = O\left(\exp\left(-c(\log x)^{1/2}\right)\right). \tag{4.11}
\]
Thus, (4.4) follows by combining (4.6), (4.9), (4.11) and [4, (10)] together. \hfill \Box

**Remark 4.5.** Similar to the proof of Theorem 4.4 one can also prove the analogues of formula (1.7) and (1.8) for functions \((-1)\omega(n)\) and \((-1)A(n)\), where \(\omega(n) = \sum_{p \mid n} \alpha_p\) is the prime divisor counting function and \(A(n) = \sum_{p \mid n} \alpha_p\) is the additive prime divisor function which was introduced by Alladi and Erdős [1] in 1977. This is mainly due to the Duality Lemma 3.1 with respect to \((-1)\omega(n)\) and \((-1)A(n)\) holds for the numbers \(n\) satisfying \(P(n)\mid n\) and \(P(n) \geq 3\).

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