On the global log canonical threshold of Fano complete intersections

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We show that the global log canonical threshold of generic Fano complete intersections of index 1 and codimension $k$ in $\mathbb{P}^{M+k}$ is equal to 1 if $M \geq 3k + 4$ and the highest degree of defining equations is at least 8. This improves the earlier result where the inequality $M \geq 4k + 1$ was required, so the class of Fano complete intersections covered by our theorem is considerably larger. The theorem implies, in particular, that the Fano complete intersections satisfying our assumptions admit a Kähler-Einstein metric. We also show the existence of Kähler-Einstein metrics for a new finite set of families of Fano complete intersections.

Bibliography: 9 titles.

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1. The canonical and log canonical thresholds. Consider a smooth Fano variety $X$, such that $\text{Pic} X = \mathbb{Z}K_X$, $D \sim -nK_X$ an effective divisor, $n \geq 1$.

Definition 1. The pair $(X, \alpha D)$, where $\alpha \in \mathbb{Q}$, is not canonical (respectively, not log canonical), if there exists a prime divisor $E$ over $X$ such that the inequality

$$\alpha \text{ ord}_E D > a(E)$$

(respectively, $\alpha \text{ ord}_E D > a(E) + 1$) is satisfied, where $a(E) = a(X, E)$ is the discrepancy of $E$ with respect to the model $X$.

Explicitly, this means that there are a birational morphism $\varphi: \tilde{X} \to X$, where $\tilde{X}$ is a smooth projective variety, and a prime $\varphi$-exceptional divisor $E \subset \tilde{X}$ such that

$$\alpha \text{ ord}_E \varphi^* D > a(E)$$

(respectively, $\alpha \text{ ord}_E \varphi^* D > a(E) + 1$).

Definition 2. The global canonical (respectively, log canonical) threshold of the
variety $X$ is defined by the equality
\[ \text{ct}(X) = \sup\{\lambda \in \mathbb{Q}_+ \mid \text{the pair } (X, \frac{\lambda}{n}D) \text{ is canonical for all } D \in |-nK_X|\}, \]
(respectively,
\[ \text{lct}(X) = \sup\{\lambda \in \mathbb{Q}_+ \mid \text{the pair } (X, \frac{\lambda}{n}D) \text{ is log canonical for all } D \in |-nK_X|\}). \]
(Note that as $D \sim -nK_X$, the integer $n \geq 1$ depends on the effective divisor $D$.)

The importance of canonical and log canonical thresholds comes from their applications to complex differential geometry and to certain problems of higher-dimensional birational geometry.

In [1, 2, 9] the following fact was shown.

**Theorem 1.** Assume that the inequality
\[ \text{lct}(X) > \frac{\dim X}{\dim X + 1} \]
holds. Then on the variety $X$ there exists a Kähler-Einstein metric.

The log canonical threshold is important in this differential-geometric context because it indicates the non-triviality of certain multiplier ideal sheaves. In their analytic interpretation, these multiplier ideal sheaves in turn measure the failure of a priori estimates sufficient to solve a Monge-Ampère equation for a Kähler-Einstein metric.

**Definition 3.** The mobile canonical threshold of a variety $X$, which is denoted by the symbol $\text{mct}(X)$, is the supremum of such $\lambda \in \mathbb{Q}_+$ that the pair $(X, \frac{\lambda}{n}D)$ is canonical for a generic divisor $D \in \Sigma$ of any mobile linear system $\Sigma \subset |-nK_X|$ (that is to say, any system $\Sigma$ with no fixed components).

In [5] the following fact was shown.

**Theorem 2.** Assume that primitive Fano varieties $F_1, \ldots, F_K$, $K \geq 2$, satisfy the conditions $\text{lct}(F_i) = \text{mct}(F_i) = 1$. Then their direct product
\[ V = F_1 \times \cdots \times F_K \]
is a birationally superrigid variety. In particular,

(i) Every structure of a rationally connected fiber space on the variety $V$ is given by a projection onto a direct factor. More precisely, let $\beta: V^\sharp \to S^\sharp$ be a rationally connected fiber space and $\chi: V \dashrightarrow V^\sharp$ a birational map. Then there exists a subset of indices
\[ I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, K\} \]
and a birational map
\[ \alpha: F_I = \prod_{i \in I} F_i \dashrightarrow S^\sharp, \]
such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\chi} & V^2 \\
\downarrow \pi_I & & \downarrow \beta \\
F_I & \xrightarrow{\alpha} & S^2
\end{array}
\]

commutes, that is, \(\beta \circ \chi = \alpha \circ \pi_I\), where

\[\pi_I: \prod_{i=1}^{K} F_i \to \prod_{i \in I} F_i\]

is the natural projection onto a direct factor.

(ii) Let \(V^2\) be a variety with \(\mathbb{Q}\)-factorial terminal singularities, satisfying the condition

\[\dim_{\mathbb{Q}}(\text{Pic} V^2 \otimes \mathbb{Q}) \leq K,\]

and \(\chi: V \dashrightarrow V^2\) a birational map. Then \(\chi\) is a (biregular) isomorphism.

(iii) The groups of birational and biregular self-maps of the variety \(V\) coincide:

\[\text{Bir } V = \text{Aut } V.\]

In particular, the group \(\text{Bir } V\) is finite.

(iv) The variety \(V\) admits no structures of a fibration into rationally connected varieties of dimension strictly smaller than \(\min\{\dim F_i\}\). In particular, \(V\) admits no structures of a conic bundle or a fibration into rational surfaces.

(v) The variety \(V\) is non-rational.

For the precise definition of birational (super)rigidity, a discussion of its properties and examples of birationally (super)rigid varieties, see [8].

2. Fano complete intersections. Fix an integer \(k \geq 2\). Consider an arbitrary system \((d_1, \ldots, d_k)\) of positive integers, satisfying the condition \(d_k \geq \cdots \geq d_1 \geq 2\). Set \(M = d_1 + \cdots + d_k - k\). Fix the complex projective space \(\mathbb{P} = \mathbb{P}^{M+k}\) and consider the family \(\mathcal{F}(d_1, \ldots, d_k)\) of non-singular complete intersections \(V\) of the type \(d_1 \cdot \cdots \cdot d_k\) in \(\mathbb{P}\):

\[V = F_1 \cap \cdots \cap F_k \subset \mathbb{P},\]

Here \(F_i \subset \mathbb{P}\) is a hypersurface of degree \(d_i\), and \(\text{codim } V = k\).

The following two theorems collect the known information about the global canonical and log canonical thresholds of Fano complete intersections as above.

**Theorem 3.** Assume that \(M \geq 4k + 1\) and \(d_k \geq 8\). Then for a generic (in the sense of Zariski topology on the space \(\mathcal{F}(d_1, \ldots, d_k)\)) variety \(V \in \mathcal{F}(d_1, \ldots, d_k)\) the equality \(\text{ct}(V) = 1\) holds.

**Proof:** see [9] Section 3].

Thus under the assumptions of Theorem 3 on \(V\) exists a Kähler-Einstein metric. Besides, since \(\lct(V) \geq \text{ct}(V)\) and \(\mct(V) \geq \text{ct}(V)\), the variety \(V\) satisfies the
assumptions of Theorem 2 and for that reason can be used as a direct factor in birationally superrigid Fano direct products.

**Theorem 4.** Assume that $M \geq 4k+1$ and any of the following conditions holds:

(i) $d_k = d_{k-1} = 7$ and $M \leq 47$,
(ii) $d_k = 7$, $d_{k-1} \leq 6$ and $M \leq 19$.
(iii) $k = 2$, $d_1 = d_2 = 6$, $M = 10$.

Then the canonical threshold $\text{ct}(V)$ (and hence also the log canonical threshold $\text{lct}(V)$) of a generic variety $V \in \mathcal{F}(d_1, \ldots, d_k)$ satisfies the inequality

$$\text{ct}(V) > \frac{M}{M+1}.$$  

**Proof:** see [7].

Therefore, on a general variety $V \in \mathcal{F}(d_1, \ldots, d_k)$, satisfying one of the conditions listed in Theorem 4 there exists a Kähler-Einstein metric.

The aim of the present note is to improve the claims of Theorems 3 and 4, extending them to a larger class of Fano complete intersections of index 1. We will show the following two facts.

**Theorem 5.** Assume that $M \geq 3k+4$ and $d_k \geq 8$. Then for a generic (in the sense of Zariski topology on the space $\mathcal{F}(d_1, \ldots, d_k)$) variety $V \in \mathcal{F}(d_1, \ldots, d_k)$ the equality $\text{ct}(V) = 1$ holds.

**Theorem 6.** Assume that $M \geq 3k+4$ and any of the two following conditions holds:

(i) $d_k = d_{k-1} = 7$ and $M \leq 47$,
(ii) $d_k = 7$, $d_{k-1} \leq 6$ and $M \leq 19$.
(iii) $k = 2$, $d_1 = d_2 = 6$, $M = 10$.

Then the canonical threshold $\text{ct}(V)$ (and hence also the log canonical threshold $\text{lct}(V)$) of a generic variety $V \in \mathcal{F}(d_1, \ldots, d_k)$ satisfies the inequality

$$\text{ct}(V) > \frac{M}{M+1}.$$  

**Remark 1.** Theorem 5 covers a considerably larger class of Fano complete intersections than Theorem 3. The same is true for the part (i) of Theorems 6 and 4. For the part (iii), Theorem 6 gives nothing new compared to Theorem 4. As for the part (ii), Theorem 6 gives the existence of the Kähler-Einstein metric for the following 7 families of Fano complete intersections that do not fit into the assumptions of Theorem 4, all of them in $\mathbb{P}^{24}$:

$$\mathcal{F}(2, 5, 5, 5, 7), \quad \mathcal{F}(2, 4, 5, 6, 7), \quad \mathcal{F}(2, 3, 6, 6, 7), \quad \mathcal{F}(3, 3, 5, 6, 7),$$
$$\mathcal{F}(3, 4, 5, 5, 7), \quad \mathcal{F}(3, 4, 4, 6, 7), \quad \mathcal{F}(4, 4, 4, 5, 7).$$

**3. The conditions of general position.** Now we will explain what we mean by the genericity of a Fano complete intersection $V \in \mathcal{F}(d_1, \ldots, d_k)$. Fix any point
o ∈ V and let (z₁, ..., z_{M+k}) be a system of affine coordinates on ℙ with the origin at the point o,

\[ f_i = q_{i,1} + \cdots + q_{i,d_i} \]

the equation of the hypersurface \( F_i \) with respect to (non-homogeneous) coordinates \( z_* \), decomposed into homogeneous in \( z_* \) components \( q_{i,j} \), \( \deg q_{i,j} = j \). Since \( V \) is a non-singular variety, the system of linear equations

\[ q_{1,1} = \cdots = q_{k,1} = 0 \]

defines a linear subspace \( T_o V \subset ℂ^{M+k} \) of codimension \( k \), the tangent space to the variety \( V \) at the point \( o \). We define a finite set of pairs \( I \subset ℤ_+ \times ℤ_+ \) in the following way: if \( d_{k-1} = d_k \), then we set

\[
I = \{ (i,j) \mid 1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \notin \{ (k,d_k), (k-1,d_{k-1}) \} \},
\]

and if \( d_{k-1} \leq d_k - 1 \), then we set

\[
I = \{ (i,j) \mid 1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \notin \{ (k,d_k), (k,d_k-1) \} \}.
\]

**Definition 4.** We say that the complete intersection \( V \) is regular at the point \( o \), if for any linear form \( l(z_*) \), not vanishing identically on the subspace \( T_o V \), the set of homogeneous polynomials

\[
\{ l \} \cup \{ q_{i,j} \mid (i,j) \in I \}
\]

forms a regular sequence in \( O_{o,ℙ} \), that is, the system of equations in \( ℂ^{M+k} \)

\[
l = 0, \quad q_{i,j} = 0, \quad (i,j) \in I,
\]

defines a subset of codimension \( \#I+1 \). Finally, we say that the complete intersection \( V \) is regular, if it is regular at every point.

In Sec. 4 below we show the following fact.

**Theorem 7.** For \( M \geq 3k+4 \) there exists a non-empty Zariski open subset

\[
\mathcal{F}_{\text{reg}}(d_1, \ldots, d_k) \subset \mathcal{F}(d_1, \ldots, d_k),
\]

such that any variety \( V \in \mathcal{F}_{\text{reg}}(d_1, \ldots, d_k) \) is regular.

By genericity in Theorems 3-6 we mean the regularity. For that reason, the main results of this paper (Theorems 5,6) are essentially dependent on Theorem 7 which allows us to assume regularity of the complete intersection \( V \).

**4. Proof of the regularity conditions.** Let us show Theorem 7. The proof proceeds in a series of reduction steps and case distinctions, through the estimates (1) - (12).

Let the incidence variety \( V \subset H^0(ℙ^{M+k}, O_{ℙ^{M+k}}(1)) \times ℙ^{M+k} \times \mathcal{F}(d_1, \ldots, d_k) \) consist of tuples \((L,o,F_1,\ldots,F_k)\) such that \( o \in \{ L \equiv F_1 = \cdots = F_k = 0 \} \). Let \( p, q, r \) be the natural projections of \( V \) to \( H^0(ℙ^{M+k}, O_{ℙ^{M+k}}(1)) \), \( ℙ^{M+k} \) resp. \( \mathcal{F}(d_1, \ldots, d_k) \).

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The image of $\mathcal{V}$ under the projection $p \times q$ is the incidence variety

$$\mathcal{L} \subset H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1)) \times \mathbb{P}^{M+k}$$

consisting of pairs $(L, o)$ such that $L(o) = 0$. All the $(p \times q)$-fibers $\mathcal{V}_{L, o} \subset \mathcal{V}$ over points $(L, o) \in \mathcal{L}$ are isomorphic to a product of affine spaces and have codimension $k$ in $\mathcal{F}(d_1, \ldots, d_k)$, as vanishing in $o$ imposes one linear condition on the sections in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_i))$. Hence $\mathcal{V}$ is irreducible.

Bertini’s theorem shows that for a general tuple $(F_1, \ldots, F_k) \in \mathcal{F}(d_1, \ldots, d_k)$ the algebraic subset $\{F_1 = \cdots = F_k = 0\}$ is an $M$-dimensional complete intersection. Hence the general $r$-fiber in $\mathcal{V}$ has dimension $(M + k) + M = 2M + k$.

Let $\mathcal{V}_{\text{sm,nonreg}}$ be the locally Zariski-closed subset of tuples $(L, o, F_1, \ldots, F_k) \in \mathcal{V}$ such that $V = \{F_1 = \cdots = F_k = 0\}$ is a smooth complete intersection of dimension $M$, but the dehomogenisation $l$ of the linear form $L$ in affine coordinates around $o$ and the homogeneous parts $q_{i,j}$ of the dehomogenized $F_i$ do not satisfy the regularity condition of Definition 4. Then the theorem is shown if the projection of the Zariski-closure of $\mathcal{V}_{\text{sm,nonreg}}$ does not cover $\mathcal{F}(d_1, \ldots, d_k)$.

To show this claim we note that every $r$-fiber of $\mathcal{V}_{\text{sm,nonreg}}$ must be at least $k$-dimensional: If the subscheme $V = \{F_1 = \cdots = F_k = 0\} \subset \mathbb{P}^{M+k}$ is smooth of dimension $M$ in $o \in \mathbb{P}^{M+k}$ then we can choose homogeneous coordinates $[Z_0 : \ldots : Z_{M+k}]$ on $\mathbb{P}^{M+k}$ such that $o = [1 : 0 : \ldots : 0]$ and $q_{1,1} = z_{M+1}, \ldots, q_{k,1} = z_{M+k}$ in the affine coordinates $z_1, \ldots, z_{M+k}$ dehomogenized with respect to $Z_0$. Hence there is a $k$-dimensional linear subspace of linear forms $L \in H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1))$ with the same intersection of $\{L = 0\} \subset \mathbb{P}^{M+k}$ and the tangent space $T_o V$, seen as a linear subspace of $\mathbb{P}^{M+k}$. In particular, if the sequence $l, q_{1,1}, \ldots, q_{k,1}, q_{1,2}, \ldots$ is not regular for one such linear form $L$ (dehomogenized to $l$) then the sequence will not be regular for any other linear form in this $k$-dimensional linear subspace. Consequently, we only need to show that

$$\text{codim}_\mathcal{V} \mathcal{V}_{\text{sm,nonreg}} \geq (2M + k) - k + 1 = 2M + 1. \quad (1)$$

The $q$-fibers $\mathcal{V}_o$ over points $o \in \mathbb{P}^{M+k}$ are all isomorphic. Consequently it is enough to show for all the subsets $\mathcal{V}_{\text{sm,nonreg}} \cap \mathcal{V}_o =: \mathcal{V}_{\text{sm,nonreg}}^o$ locally Zariski-closed in $\mathcal{V}_o$ that

$$\text{codim}_{\mathcal{V}_o} \mathcal{V}_{\text{sm,nonreg}}^o \geq 2M + 1. \quad (2)$$

Let $\mathcal{P}_d^N$ denote the vector space of homogeneous polynomials of degree $d$ in the affine coordinates $z_1, \ldots, z_N$, $N \leq M + k$, introduced above. Then

$$\mathcal{V}_o \cong \mathcal{P}_1^{M+k} \times \prod_{i=1}^{k} \prod_{j=1}^{d_i} \mathcal{P}_j^{M+k},$$

by identifying the dehomogenized sections in $H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(j))$ vanishing in $o$ with $\mathcal{P}_j^{M+k}$ and decomposing the dehomogenisation $f_i = q_{i,1} + \cdots + q_{i,d_i}$ of the $F_i$ in the tuple $(L, F_1, \ldots, F_k) \in \mathcal{V}_o$ into homogeneous parts $q_{i,j}$ of degree $j$. 
Consider the projection $s$ of $\mathcal{V}_o$ to $\mathcal{P}^{M+k}_1 × \prod_{i=1}^k(\mathcal{P}^{M+k}_1)$, that is, to the linear form $l$ coming from $L$ and the linear parts $q_{i,1}$ of the dehomogenized $F_i$. The map $s$ is not defined everywhere on $\mathcal{V}_o$ but since $\mathcal{V}_{o,\text{sm},\text{nonreg}}$ consists of tuples $(L, F_1, \ldots, F_k)$ such that $V = \{F_1 = \ldots = F_k = 0\} \subset \mathbb{P}^{M+k}$ is smooth of codimension $k$ in $o$ and $l|_{T_o V} \neq 0$, hence none of the $l$ and $q_{i,1}$ can be $0$, we conclude that $s$ is defined on $\mathcal{V}_{o,\text{sm},\text{nonreg}}$.

Therefore, it is enough to show that the codimension of the isomorphic $s$-fibers in $\mathcal{V}_{o,\text{sm},\text{nonreg}}$ is $\geq 2M + 1$. Choosing the coordinates $z_*$ such that $l = z_M, q_{i,1} = z_{M+1}, \ldots, q_{k,1} = z_{M+k}$, that means to show that the set $U(d_1, \ldots, d_k)$ of tuples of homogeneous polynomials in variables $z_1, \ldots, z_{M-1}$ defined by

$$\{(q_{i,j})_{1 \leq i \leq k, 2 \leq j \leq d_i} \mid (q_{i,j})_{(i,j) \in I, j \neq 1} \text{ is not a regular sequence in } \mathcal{O}_{0,\mathbb{A}^{M-1}}\}$$

and Zariski-closed in $\prod_{i=1}^k \prod_{j = 2}^{d_i} \mathcal{P}^{M-1}_j$ satisfies

$$\text{codim}_{\prod_{i=1}^k \prod_{j = 2}^{d_i} \mathcal{P}^{M-1}_j} U(d_1, \ldots, d_k) \geq 2M + 1.$$  

(3)

In order to obtain this estimate only the degrees of the homogeneous polynomials $q_{i,j}$ matter. So we will discuss the codimension of the set of tuples

$$U := \{(q_i)_{1 \leq i \leq M} \mid (q_i)_{1 \leq i \leq M-2} \text{ is not a regular sequence in } \mathcal{O}_{0,\mathbb{A}^{M-1}}\}$$

in $\prod_{i=1}^M \mathcal{P}^{M-1}_{m_i}$ where $2 \leq m_1 \leq \cdots \leq m_M = d_k$ and we have

$$k_d := \# \{d_i : d_i \geq d, 1 \leq l \leq k\}$$

polynomials $q_i$ of degree $d$. In particular $k = k_2 \geq k_3 \geq \cdots \geq k_{d_k}$ and

$$\sum_{i=1}^M m_i = \sum_{d=2}^{d_k} k_d \cdot d = \sum_{i=1}^k \sum_{d=2}^{d_i} d = \sum_{i=1}^k \frac{d_i (d_i + 1)}{2} - k.$$ 

Let $Z(q_1, \ldots, q_j) \subset \mathbb{A}^{M-1}$ denote the vanishing locus of $q_1, \ldots, q_j$ in $\mathbb{A}^{M-1}$. Since the $q_i$ are homogeneous $Z(q_1, \ldots, q_j)$ is a cone over the origin in $\mathbb{A}^{M-1}$, and we denote its projectivization in $\mathbb{P}^{M-2}$ by $V(q_1, \ldots, q_j)$. In particular, $(q_1, \ldots, q_j)$ is regular in $0 \in \mathbb{A}^{M-1}$ if and only if

$$\text{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_j) = \text{codim}_{\mathbb{A}^{M-2}} Z(q_1, \ldots, q_j) = j,$$

where the codimension is set to be the minimum of the codimensions of the irreducible components.

Consequently, $U$ is covered by locally Zariski-closed subsets $U_j × \prod_{i=j+1}^M \mathcal{P}^{M-1}_{m_i}$, where $1 \leq j \leq M - 2$ and

$$U_j := \{(q_1, \ldots, q_j) \mid \text{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_j) = \text{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_{j-1}) = j - 1\}$$
is a locally Zariski-closed subset of $\prod_{i=1}^{j} P_{m_{i}}^{M-1}$. Thus we need to show
\[
\text{codim}_{\prod_{i=1}^{j} P_{m_{i}}^{M-1} U} = \min_{1 \leq j \leq M-2} \text{codim}_{\prod_{i=1}^{j} P_{m_{i}}^{M-1} U_j} \geq 2M + 1,
\]
and this means to verify
\[
\text{codim}_{\prod_{i=1}^{j} P_{m_{i}}^{M-1} U_j} \geq 2M + 1 \text{ for } 1 \leq j \leq M - 2.
\]

If $m_j = 2$ (that is, $1 \leq j \leq k_2 = k$) then we estimate the codimension of $U_j$ in $\prod_{i=1}^{j} P_{m_{i}}^{M-1}$ using the method of [3]: Choose a general $(j - 2)$-dimensional hyperplane $S \subset \mathbb{P}^{M-2}$. Then the projection $\pi_S : \mathbb{P}^{M-2} \to \mathbb{P}^{M-1-j}$ restricts to a finite morphism on each of the irreducible components of $V(q_1, \ldots, q_{j-1})$ for given $q_1, \ldots, q_{j-1}$ such that $\text{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_{j-1}) = j - 1$. Consequently, polynomials in $\mathbb{P}^{M-1}_{m_{j}} = \mathbb{P}^{M-1}$ obtained as a pullback of a homogeneous quadratic polynomial defined on $\mathbb{P}^{M-1-j}$ do not vanish on $V(q_1, \ldots, q_{j-1})$. The linear space $W_j$ of such pulled-back quadratic polynomials has dimension $\binom{M-1-j+2}{2}$. By construction, $W_j$ intersects the space of polynomials in $\mathbb{P}^{M-1}$ vanishing on one of the irreducible components of $V(q_1, \ldots, q_{j-1})$ only in 0. Therefore,
\[
\text{codim}_{\prod_{i=1}^{j} P_{m_{i}}^{M-1} U_j} \geq \binom{M-j+1}{2} \geq \binom{M-k+1}{2}, \quad j = 1, \ldots, k.
\]

Note that this estimate also holds if $j = 1$. Furthermore, $\binom{M-k+1}{2} \geq 2M + 1$ since $3k + 4 \leq M$. Hence [5] is shown for $j = 1, \ldots, k$.

If $m_j > 2$ (that is, $k + 1 \leq j \leq M - 2$) then we estimate the codimension of $U_j$ in $\prod_{i=1}^{j} P_{m_{i}}^{M-1}$ using the method of [4]: If $(q_1, \ldots, q_j) \in U_j$ choose an irreducible component $B$ of $V(q_1, \ldots, q_j)$. By definition of $U_j$, the codimension of $B$ in $\mathbb{P}^{M-2}$ is $j - 1$. Assume that the codimension of the linear subspace $\langle B \rangle \subset \mathbb{P}^{M-2}$ spanned by $B$ is $b$; that means in particular that $0 \leq b \leq j - 1$.

**Lemma 1.** If $b < j - 1$, then there are indices $1 \leq i_1 < \cdots < i_{j-1-b} \leq j - 1$ such that $B$ is an irreducible component of
\[
V(q_1, \ldots, q_{i_{j-1-b}}) \cap \langle B \rangle.
\]

**Proof.** Since $\text{codim}_{\mathbb{P}^{M-2}} \langle B \rangle = b < j - 1 = \text{codim}_{\mathbb{P}^{M-2}} B$ one of the $q_1, \ldots, q_{j-1}$ must not vanish on $\langle B \rangle$. Let $i_1$ be the smallest index such that $q_{i_1} \mid \langle B \rangle \neq 0$. If $b = j - 2$ one of the irreducible components of $V(q_{i_1}) \cap \langle B \rangle$ must be $B$ and we are done. For $b < j - 2$ choose an irreducible component $R_1$ of $V(q_{i_1}) \cap \langle B \rangle$ containing $B$. Since $\dim R_1 > \dim B$ and $q_i \mid R_1 \equiv 0$ for $i = 1, \ldots, i_1$ we can find an index $i_1 + 1 \leq i_2 \leq j - 1$ such that $q_{i_2}$ does not vanish on $R_1$: Otherwise, $R_1 \subset V(q_1, \ldots, q_{j-1})$, and this is a contradiction to $\text{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_{j-1}) = j - 1$. In the same way we can inductively find $i_3, \ldots, i_{j-1-b}$ such that finally $B$ is an irreducible component of $V(q_1, \ldots, q_{i_{j-1-b}}) \cap \langle B \rangle$. Q.E.D. for the lemma.
The lemma implies that $U_j$ is contained in the union of all locally Zariski-closed subsets $U_j(P; i_1, \ldots, i_{j-1-b}) \subset \prod_{i=1}^{j} P_{m_i}^{M-1}$ of the form
\[
\left\{ (q_1, \ldots, q_j) \mid \begin{array}{l}
\text{thereexistsanirreducible component } B \text{ of } V(q_1, \ldots, q_{j-1-b}) \cap P : \\
\langle B \rangle = P, \codim_P B = j - 1 - b, q_i \equiv 0 \text{ for all } i = 1, \ldots, j
\end{array} \right\},
\]
where $P$ ranges over all codimension $b$ linear subspaces of $\mathbb{P}^{M-2}$, $0 \leq b \leq j - 1$, and the indices $i_1, \ldots, i_{j-1-b}$ range over all increasing sequences $1 \leq i_1 < \cdots < i_{j-1-b} \leq j - 1$. Note that for $b = j - 1$ we just consider the sets
\[
U_j(P) := \left\{ (q_1, \ldots, q_j) \left| P \text{ is an irreducible component of } V(q_1, \ldots, q_{j-1}) \text{ and } q_{ij} \equiv 0 \right\} \subset \prod_{i=1}^{j} P_{m_i}^{M-1}.
\]
Since the dimension of the Grassmann variety $\mathbb{G}(M - 2 - b, M - 2)$ is $b(M - 1 - b)$, estimate (5) will follow if
\[
\text{codim}_{\prod_{i=1}^{j} P_{m_i}^{M-1}} U_j(P; i_1, \ldots, i_{j-1-b}) \geq 2M + 1 + b(M - 1 - b). \tag{7}
\]
holds. By a linear change of coordinates we can assume that
\[
P = \{ z_{M-b} = \cdots = z_{M-1} \}.
\]
Let $q_{i_1}, \ldots, q_{i_{j-1-b}}$ be polynomials in $\prod_{r=1}^{j-1-b} P_{m_r}^{M-1}$ such that an irreducible component $B$ of $V(q_1, \ldots, q_{i_{j-1-b}})$ lies in $P$ with $\codim_P B = j - 1 - b$ and $\langle B \rangle = P$. Then for any degree $m$, products of $m$ linear forms
\[
\prod_{i=1}^{m} (a_{i,1} z_1 + \cdots + a_{i,M-b-1} z_{M-b-1})
\]
do not vanish on $B$. These products span a linear subspace of $P_m^{M-1}$ of dimension $(M - b - 2)m + 1$ intersecting the linear subspace of polynomials vanishing on $B$ only in $0$. We apply these facts to the polynomials $q_i \in P_{m_i}^{M-1}$ with $i \in \{1, \ldots, j\} - \{i_1, \ldots, i_{j-1-b}\}$ and obtain
\[
\text{codim}_{\prod_{i=1}^{j} P_{m_i}^{M-1}} U_j(P; i_1, \ldots, i_{j-1-b}) \geq \left( \sum_{i=1}^{j} m_i \right) (M - b - 2) + 1 \geq \left( \sum_{i=1}^{b} m_i + m_j \right) (M - b - 2) + 1.
\]
So (7) follows from
\[
\left( \sum_{i=1}^{b} m_i + m_j - b \right) (M - b - 2) \geq 2M. \tag{8}
\]
Since $m_i \geq 2$ for $i = 1, \ldots, j - 1$ and we consider the case $m_j \geq 3$ we have

$$\sum_{i=1}^{b} m_i + m_j - b \geq b + 3.$$ 

The polynomial $(b + 3)(M - 2 - b) - 2M = -b^2 + (M - 5)b + M - 6$ quadratic in $b$ increases for $b \leq \frac{M - 5}{2}$ and decreases for $b \geq \frac{M - 5}{2}$. Since $M \geq 3k + 4 \geq 7$, hence $3(M - 2) - 2M \geq 0$, estimate (8) is shown for $b = 0, M - 5$, hence for all $0 \leq b \leq M - 5$. This leaves the cases $b = M - 4, M - 3$.

If $b = M - 4$ then $j = M - 3$ or $M - 2$, and by the assumptions on the degrees $m_i$ we have

$$\sum_{i=1}^{b} m_i + m_j \geq \sum_{i=1}^{M} m_i - 2d_k - (d_k - 1) = \sum_{i=1}^{k-1} \frac{d_i(d_i + 1)}{2} + \frac{(d_k - 3)(d_k - 2)}{2} + 2 - k.$$ 

Similarly, if $b = M - 3$ then $j = M - 2$, and

$$\sum_{i=1}^{b} m_i + m_j \geq \sum_{i=1}^{M} m_i - d_k - d_k = \sum_{i=1}^{k-1} \frac{d_i(d_i + 1)}{2} + \frac{(d_k - 2)(d_k - 1)}{2} + 1 - k.$$ 

This last case is the worst possible situation: $V(q_1, \ldots, q_{M-3})$ is a line in $\mathbb{P}^{M-2}$, and $q_{M-2}$ vanishes on this line.

Solving an optimization problem and using $\sum_{i=1}^{k} d_i = M + k$ we obtain that

$$\sum_{i=1}^{k-1} d_i^2 + (d_k - 3)^2 \geq k \left( \frac{M - 3 + k}{k} \right)^2$$ and $$\sum_{i=1}^{k-1} d_i^2 + (d_k - 2)^2 \geq k \left( \frac{M - 2 + k}{k} \right)^2.$$ 

Consequently, (8) follows for $b = M - 4, M - 3$ if

$$\left[ \frac{(M - 3 + k)^2}{2k} + \frac{M - 3 + k}{2} - k - M + 6 \right] \cdot 2 \geq 2M$$ \hspace{1cm} (9)

and

$$\left[ \frac{(M - 2 + k)^2}{2k} + \frac{M - 2 + k}{2} - k - M + 3 \right] \cdot 1 \geq 2M.$$ \hspace{1cm} (10)

Now (9) is equivalent to

$$k \leq \frac{(M - 3)^2}{M}$$ \hspace{1cm} (11)

and (10) is equivalent to

$$k \leq \frac{(M - 2)^2}{3M - 2}.$$ \hspace{1cm} (12)

Both inequalities are satisfied if $M \geq 3k + 4$. Then $k \geq 1$ implies $M \geq 7$, so no further lower bound on $M$ is needed. Proof of Theorem 7 is complete.
5. Hypertangent divisors. Let us prove Theorem 5. The argument is similar to the proof of Theorem 4 in [6], so we will only sketch the main steps.

**Step 1.** Assume that the pair \((V, D)\) is not canonical for an effective divisor \(D \sim nH\), where \(H \in \text{Pic} V\) is the class of a hyperplane section, \(K_V = -H\). By linearity of all conditions involved in this assumption, the divisor \(D\) can be assumed to be irreducible. For some prime divisor \(E\) over \(V\) the inequality \(\text{ord}_E D > na(E, V)\) holds.

We look at the centre \(B \subset V\) of the divisor \(E\). Since by [6, Proposition 3.6] for any irreducible subvariety \(Y \subset V\) of dimension at least \(k\) (where \(k = \text{codim}(V \subset \mathbb{P})\)) we have \(\text{mult}_Y D \leq n\), we conclude that \(\text{dim } B \leq k - 1\). Let \(o \in B\) be a point of general position, \(\sigma: V^+ \rightarrow V\) its blow up and \(E^+ = \sigma^{-1}(o)\) the exceptional divisor. A standard argument (see, for example, [6, Proposition 2.5]) gives us a hyperplane \(\Delta \subset E^+ \cong \mathbb{P}^{M-1}\) satisfying the inequality

\[
\text{mult}_o D + \text{mult}_\Delta D^+ > 2n,
\]

where \(D^+\) is the strict transform of \(D\) on \(V^+\).

Now let \(T\) be a general hyperplane section of \(V\), containing the point \(o\) and cutting out \(\Delta\) on \(E^+\): that is to say, \(T^+ \cap E^+ = \Delta\). It is easy to see that the effective cycle of the scheme-theoretic intersection \(D_T = (D \circ T)\) is well defined and satisfies the estimate

\[
\text{mult}_o D_T > 2n. \quad (13)
\]

We will consider \(D_T\) as an effective divisor on the complete intersection \(T \subset \mathbb{P}^{M+k-1}\) of codimension \(k\), \(D_T \sim nH_T\).

**Step 2.** By Sec. 3-4, the complete intersection \(T \subset \mathbb{P}^{M+k-1}\) satisfies the regularity condition. Namely, for a system of linear coordinates \(z_1, \ldots, z_{M+k-1}\) with the origin at the point \(o\), the variety \(T\) is given by a system of non-homogeneous polynomial equations:

\[
\bar{f}_i = \bar{q}_{i,1} + \cdots + \bar{q}_{i,d_i},
\]

where the bar means the restriction onto the hyperplane \(\{l = 0\}\) — the hyperplane that cuts out \(T\) on \(V\). Now the set of homogeneous polynomials

\[
\{\bar{q}_{i,j} \mid (i, j) \in I\}
\]

forms a regular sequence in \(O_{o, \mathbb{P}^{M+k-1}}\).

**Step 3.** Now we can apply the technique of hypertangent divisors [8, Chapter 3] to the divisor \(D_T\) on the complete intersection \(T \subset \mathbb{P}^{M+k-1}\) in precisely the same way as it was done in [6, Section 3] or, in more details, in [7, Section 5] and obtain the estimate

\[
\frac{\text{mult}_o D_T}{2n} \leq \max \left\{ 1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1} \right\},
\]
where \(d_+ = d_k\), if \(d_{k-1} = d_k\), and \(d_+ = d_k - 1\), otherwise. It is easy to see that if \(d_k \geq 8\), this gives us the inequality \(\text{mult}_o D_T \leq 2n\), which contradicts the estimate obtained in Step 1. The contradiction completes the proof of Theorem 5.

**Proof of Theorem 6** follows the same lines and repeats the argument given in [7, Section 4] word for word. What is different from the proof of Theorem 5 given above, is the starting point: assuming the inequality

\[ \text{ct}(V) \leq \frac{M}{M+1}, \]

we obtain for any rational number \(\lambda > \frac{M}{M+1}\) an effective divisor \(D \sim nH\) such that the pair \((V, \frac{\lambda}{n}D)\) is not canonical. Now, repeating the proof of Theorem 5, we use the technique of hypertangent divisors to obtain the inequality

\[ 1 < \lambda \max \left\{ 1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1} \right\}, \]

and taking the limit as \(\lambda \to \frac{M}{M+1}\), we conclude that

\[ \max \left\{ 1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1} \right\} \geq \frac{M + 1}{M}. \]

However, it is a trivial check that in the assumptions of Theorem 6 the last inequality cannot be true. Q.E.D. for Theorem 6.

**References**

[1] Demailly J.-P. and Kollár J., Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. de l’École Norm. Sup. 34 (2001), 525-556.

[2] Nadel A., Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. Ann. Math. 132 (1990), 549-596.

[3] Pukhlikov A. V., Birational automorphisms of Fano hypersurfaces, Invent. Math. 134 (1998), no. 2, 401-426.

[4] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. 541 (2001), 55-79.

[5] Pukhlikov A. V., Birational geometry of Fano direct products, Izvestiya: Mathematics, 69 (2005), no. 6, 1225-1255.

[6] Pukhlikov A. V. Birational geometry of algebraic varieties with a pencil of Fano complete intersections, Manuscripta Mathematica. 121 (2006), 491-526.
[7] Pukhlikov A. V., Existence of the Kähler-Einstein metric on certain Fano complete intersections, Math. Notes 88, No. 4, 33-39.

[8] Pukhlikov Aleksandr, Birationally Rigid Varieties. Mathematical Surveys and Monographs 190, AMS, 2013.

[9] Tian G., On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. Invent. Math. 89 (1987), 225-246.

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