Tracing light propagation to the intrinsic accuracy of spacetime geometry

Mariateresa Crosta
INAF, Astronomical Observatory of Turin, via Osservatorio 20, I-10025 Pino Torinese (TO), Italy
E-mail: crosta@oato.inaf.it

Received 7 July 2011, in final form 16 September 2011
Published 16 November 2011
Online at stacks.iop.org/CQG/28/235013

Abstract
Advancement in astronomical observations requires codification of light propagation and of the processes of its physical measurement at a high level of accuracy. This could unveil a new window of several subtle relativistic effects suffered by light while propagating. Indeed, light modeling and its subsequent detection should be conceived in a fully relativistic context, in order to interpret the outcome of the observing process in accordance with the geometrical environment affecting light propagation itself and the precepts of measurement. This paper deals with the complexity of such a topic by showing how the geometrical framework of RAMOD, a relativistic model initially developed for astrometric observations in the visible, constitutes an appropriate environment for back-tracing photons. Through gauging the energy content of a given gravitationally bound system, the geometrical aspects that match the required accuracy of present and future observational capabilities are evidenced. Then, by comparing different formulations of the null geodesic, their domain of validity within the given geometrical scheme is refined. Finally, by proving its ability in retrieving recent literature cases, RAMOD is promoted as a measurement-based general relativistic method for any present and future advancement in the light-tracing problem.

PACS numbers: 00.04, 02.40.−k, 04.20.−q, 95.30Sf, 95.85.−e, 95.10Jk

1. Introduction

The treatment of light propagation in time-dependent gravitational fields, the fabric of our surrounding universe, is extremely important for astrophysics, encompassing issues from fundamental astronomy to cosmology. Attaining very accurate measurements could allow us to observe a new range of subtle physical effects (see, for example, [1–7] and references therein). Undoubtedly, light tracing must be conceived in a general relativistic framework. Today, general relativity (GR) is the theory in which geometry and physics are joined together.
in order to explain how gravity works and the trajectory of a photon is traced by solving the null geodesic in a curved spacetime. At the same time, the detection process usually takes place in a geometrical environment generated by an $n$-body distribution as it is that of our Solar System.

Nowadays, a few approaches exist that model light propagation in a relativistic context. Among them, the post-Newtonian (pN) and the post-Minkowskian (pM) approximations are those mainly used ([8–12] and references therein). These approximation methods rely upon the idea of pN being as close as possible to the Newtonian theory (everywhere-weak gravitational field, slow sources motion, absolute time, absolute space, auxiliary Euclidean metric and instantaneous potential), while pM to the Minkowskian interpretation of special relativity (weakness of the gravitational field, absolute spacetime, auxiliary Minkowskian metric and retarded potential) [13]. In particular, Kopeikin and Shäfer [9], using the pM approximation, solved Einstein’s equations in the linear regime by expressing the perturbed part of the metric tensor in terms of retarded Lienard–Weichert potentials. Later, Kopeikin and Mashhoon [10] included all the relativistic effects related to the gravitomagnetic field produced by the translational velocity/spin-dependent metric terms. In these works, a special technique of integration of the equation of light propagation is applied once the null geodesic is rewritten as a function of two independent parameters [14].

In this context, RAMOD, the approach discussed here, plays a distinct role. RAMOD is conceived to solve the inverse ray-tracing problem in a general relativistic framework, i.e. not constrained by a priori approximations, according to the precepts of measurement in GR. Therefore, it uses a 3+1 characterization of spacetime in order to measure physical phenomena along the proper time and on the rest-space of a set of fiducial observers. Actually, RAMOD is a family of models of increasing intrinsic accuracy all based on a curved geometry of spacetime and on the small curvature limit [15, 16]. Differently from the other approaches, RAMOD’s full solution requires the integration of a set of coupled nonlinear differential equations, called ‘master equations’. The unknown of these equations is the local line-of-sight as measured by the fiducial observer at the point of observation. Moreover, in order to trace back the light trajectory to the initial position of the emitting source according to Cauchy’s problem, the boundary condition is usually fixed by the required physical measurement. As a matter of fact, RAMOD aims to take advantage of a full solution that naturally entangles all the relativistic contributions suffered by the photon along its trajectory due to the intervening gravitational fields. Any specific effect one is interested to explore can be independently deduced from our formalism as a ‘branch’ output, once we adapt the model to a required specific case and a chosen coordinate system. Proof of this is given in [22], where, within the context of the Gaia mission (ESA, [20]), a first comparison between RAMOD and GREM (Gaia RElativistic Model, [11]) was carried out via the extrapolation of the aberrational term in the local light direction, i.e. at the observer. A semi-analytical solution was found by de Felice and Preti [18] and an analytical one [21] in the case of the parametrized-pN Schwarzschild metric [19, 1].

This work intends to bring further insight toward the physical consistency that one needs to keep in order to solve the light-tracing problem in a situation of increasingly accurate detections. In fact, one should check if the measurement is local or not with respect to the curvature generated by the gravitational field where observations take place [24]. In other words, one has to scrutinize if and to what extent a measurement can be considered local with respect to the geometry, i.e. the ‘vorticity’, as it is discussed in this paper (see section 2). The calculations show that when the vorticity term is needed the light trajectory cannot be laid out on a unique rest-space of simultaneity from the observer to the star, wherever the latter could be located. Without vorticity RAMOD allows a parametrization of the light trajectory (section 3) and sets the level of reciprocal consistency with the existing approaches (section 4).
Throughout the paper, the following notations are used: (i) \( \not P \not l \) stands for the scalar product with respect to the metric \( g_{\alpha\beta} \) and regular bold indicates four-vectors (e.g. \( \mathbf{u} \)); (ii) \( x^\alpha \) designate generic four-coordinates, whereas \( \xi^\alpha \) the four-coordinates in the case of a null shift factor (any tilde used in the text refers to these coordinates); (iii) a dot applied over coordinates means derivative with respect to coordinate time; (iv) partial derivatives with respect to coordinates are usually indicated by a comma (instead, the symbol \( \partial \) is adopted to avoid confusion in cases of more complicated formulas); (v) covariant derivatives are indicated with \( \nabla \); (vi) finally, \( c \) indicates the speed of light in vacuum.

2. From the null geodesic to the master equations

Let us consider a weakly relativistic metric

\[
g_{\alpha\beta} = n_{\alpha\beta} + h_{\alpha\beta} + O(h^2),
\]

where \( n_{\alpha\beta} \) is the flat Minkowskian metric in Cartesian coordinates and the \( h_{\alpha\beta} \)'s are small in the sense that \( |h_{\alpha\beta}| \ll 1 \), i.e. the background geometry is sufficiently small to neglect nonlinear terms. Let us consider a gravitationally bound system. From the virial theorem all forms of energy density within the system must not exceed the maximum amount of the gravitational potential in it, say, \( U \). So, the energy balance requires that \( |h_{\alpha\beta}| \ll U/c^2 \sim v^2/c^2 \), where \( v \) is the characteristic relative velocity within the system. Since the latter is weakly relativistic, the \( h_{\alpha\beta} \)'s are at least of the order of \( (v/c)^2 \) and the level of accuracy, to which it is expected to extend the calculations, is fixed by the order of the small quantity \( \epsilon \sim (v/c) \). In practice, the perturbation tensor \( h_{\alpha\beta} \) contributes with even terms in \( \epsilon \) to \( g_{00} \) and \( g_{ij} \) (lowest order \( \epsilon^2 \)) and with odd terms in \( \epsilon \) to \( g_{\alpha\beta} \) (lowest order \( \epsilon^3 \), \([19, 23]\)); its spatial variations are of the order of \( |h_{\alpha\beta}| \), while its time variation is of the order of \( |\epsilon h_{\alpha\beta}| \). Clearly, the metric form (1) is preserved under infinitesimal coordinate transformations of order \( h \).

In order to describe the spacetime evolution of the system, let us introduce a family of physical observers \( \mathbf{u} \), i.e. a time-like congruence of curves \( C_{\mathbf{u}} \), and consider their vorticity, which measures how a world-line of an observer rotates around a neighboring one. The vorticity \( \omega(\mathbf{u})_{\alpha\beta} \) associated with \( C_{\mathbf{u}} \), with tangent vector field \( \mathbf{u} \), results in (see the definition in chapters 8–9 in \([23]\), page 41 of \([24]\) and appendix A)

\[
\omega(\mathbf{u})_{\alpha\beta} = P(\mathbf{u})_{\mu\rho} P(\mathbf{u})_{\nu\sigma} \nabla u_{\rho\sigma} = O (h_{00}),
\]

where square brackets mean anti-symmetrization and \( P(\mathbf{u})_{\mu\rho} \) is the operator which projects orthogonally to \( \mathbf{u} \). From the Frobenius theorem, if an open set of spacetime manifold admits a vorticity-free congruence of lines, then it can be foliated \([23]\). In practice, the foliation is a mathematical tool which allows us to deal with measurements in a curved spacetime, i.e. to identify a ‘space’ and a ‘time’ relative to any given observer \([24]\).

The requirement of a weakly relativistic background geometry allows us to assume that, locally and only locally, the vorticity is free up to \( \epsilon^3 \), being proportional to the \( g_{00} \) term of the metric, the lowest order established by equation (2). Moreover, at the order of \( \epsilon^3 \), the time dependence of the background metric cannot be ignored any longer. The spacetime still admits a 3+1 splitting with a coordinate representation, say \( (\tau, x^i) \), where each slice \( S(\tau) \) is defined by the hypersurfaces \( \tau = \text{constant} \) and the spatial coordinates \( x^i \) vary along the normals \( \mathbf{u} \) according to the shift law (A.3). Per contra any observer \( \mathbf{u} \) can be considered at rest with

1 For a typical velocity \( \sim 30 \text{ km s}^{-1} \), \( (v/c)^2 \sim 1 \text{ milliarcsec} \).

2 This means that the length scale of the curvature is everywhere small compared to the typical size of the system; this also implies that \( v^2/c^2 \ll 1 \).
respect to the coordinates \( \xi' \) only locally, and for this reason \( \xi' \) is called the \textit{local barycentric observer}, as identified by (A.4) [16].

Let \( \ell \) be the tangent vector field to the null geodesic \( \Upsilon_{\ell} \) that satisfies the following equations:

\[
k^\alpha k_\alpha = 0, \tag{3}\]

\[
\frac{dk^\alpha}{d\lambda} + \Gamma^\alpha_{\rho\sigma} k^\rho k^\sigma = 0; \tag{4}\]

here, \( \lambda \) is a real parameter on \( \Upsilon_{\ell} \) and \( \Gamma^\alpha_{\rho\sigma} \) are the connection coefficients of the given metric. Since at the order of \( \epsilon^3 \) it is not possible to define a rest-space of the barycentric observer that covers the entire spacetime, any local observer \( \xi' \) intersected by the null ray will \textit{measure} the light signal along a spatial direction \( \xi' \) in his rest-space given by \( \bar{\xi}' = P(\xi') k^\rho k^\rho \). It is convenient to parametrize the null curve \( \Upsilon_{\ell} \) with the parameter \( \sigma(\lambda) \equiv \sigma(\xi', \lambda, \tau) \) which marks the proper time of \( \xi' \) that the light trajectory crosses at each \( \tau \). Hence, being \( d\sigma = -(u_\alpha k^\alpha) d\lambda \), we define the new tangent vector field as

\[
\bar{\xi}' = \frac{d\xi'}{d\sigma} = -\frac{\bar{u}^\alpha}{(u^\alpha k^\alpha)}. \tag{5}\]

In the same way, we denote \( \tilde{k}^\alpha \equiv -k^\alpha/(u_\beta k^\beta) \), so that \( \tilde{k}^\alpha = \bar{\xi}' + u^\alpha \) which implies \( \bar{\xi}' \tilde{\xi}' = 1 \).

From (4) the differential equation satisfied by the vector field \( \ell \) is (see appendix B)

\[
\frac{d\bar{\ell}^\alpha}{d\sigma} = -\frac{d\bar{u}^\alpha}{d\sigma} + (\bar{\xi}' + u^\alpha)(\bar{\xi}' \tilde{\xi}' \nabla_\beta u_\beta + \bar{\xi}' \tilde{\xi}' \nabla_\beta u_\beta) - \Gamma^\alpha_{\rho\sigma}(\bar{\xi}' \tilde{\xi}' u^\rho + \bar{\xi}' \tilde{\xi}' u^\rho), \tag{6}\]

which after some algebra, being \( h_{0_\beta} \neq 0, h_{0_0,0} \neq 0, \) and \( h_{i,0} \neq 0 \), results in

\[
\frac{d\bar{\ell}^0}{d\sigma} - \bar{\xi}' \tilde{\xi}' h_{0,0,i} = \frac{1}{2} h_{0,0,0} = 0 \tag{7}\]

\[
\frac{d\bar{\ell}^i}{d\sigma} = -\frac{1}{2} \bar{\xi}' \tilde{\xi}' h_{0,0,i} + \bar{\xi}' \tilde{\xi}' h_{0,i} + \frac{1}{2} \bar{\xi}' \tilde{\xi}' h_{0,i} \tag{8}\]

these equations are named ‘RAMOD4 master equations’ [16]. At this stage we do not need to make the \( h \) terms explicit. Note that there is a differential equation also for the \( \bar{\ell}^0 \) component, which represents an opportunity to better decipher light propagation in future developments. Moreover, each \( h_{0,0,0} \) component is redundant at the order of \( \epsilon^3 \) and did not appear in [16]; we recompute the master equations and keep it in order to compare to the different expressions for the null geodesic in [11, 10], as it will become clear in section 4.

Now, let us consider equation (4) at the \( \epsilon^2 \) regime when coordinates are preserved along the curves, say, \( \bar{\xi}' \) of the Killing congruence \( C_\alpha \), as given in [15]. Even if the metric form (1) holds up to infinitesimal coordinate transformations, the condition of a null shift factor is preserved only to the order \( \epsilon^2 \) as reported in appendix A. We will refer to this as the ‘static case’, or ‘static spacetime’, i.e. a stationary spacetime in which a time-like Killing vector field has vanishing vorticity, or equivalently (by the Frobenius theorem) hypersurface orthogonal [23, 19]. In this case, the parameter \( \sigma \) on \( \bar{\xi}' \) such that \( \bar{u}^\alpha = d\xi^\alpha/d\sigma \) (equation (A.4)) is the proper time of the physical observers who transport the spatial coordinates, from one slide to

\[3 \text{ This definition has nothing to do with the use of the term ‘master equation’ in classical or quantum physics; in our context it represents a set of first-order nonlinear differential equation describing evolution of the spatial light direction components according to the prescribed geometry.} \]

\[4 \text{ } \]
another, without shift. Any hypersurface, at each different coordinate time \( \tau \), can be considered the rest space everywhere of the observer \( \tilde{u} \) and the geometry that each photon feels is, then, identified with metric (1) where \( \tilde{g}_{\mu \nu} = 0 \) (see equation A.5). In these circumstances we can define a one-parameter local diffeomorphism [23]:

\[
\phi_{\Delta \sigma} \equiv \phi_{(\sigma(\xi(\tau_0)) - \sigma(\xi(\tau)))} : \mathcal{G}_k \cap S(\tau) \rightarrow S(\tau_0),
\]

which maps each point of the null geodesic \( \mathcal{G}_k \) with the point on the slice at the time of observation, say \( S(\tau) \) [15, 25]. The mapped curve \( \tilde{\mathcal{G}} \) in \( S(\tau_0) \) that is the image of \( \mathcal{G}_k \) under \( \phi_{\Delta \sigma} \) has a tangent vector [23]:

\[
\dot{\tilde{\mathcal{G}}} = (\phi_{\Delta \sigma} \circ \kappa)^\alpha = \frac{\partial \xi^\alpha(\sigma(\tau))}{\partial \xi^\beta(\sigma(\tau))} \kappa^\beta \equiv \ell^\alpha
\]

and it coincides with the projection on the rest-space of the observer \( \tilde{u} \) in any point of \( \tilde{\mathcal{G}} \) on \( S(\tau_0) \), namely with \( \ell^\alpha = P(\tilde{u})_{\rho}^\alpha k^\rho \). Hence, the curve \( \tilde{\mathcal{G}} \) is the spatial projection of the null geodesic on \( S(\tau_0) \) and is naturally parametrized by \( \lambda \). Then, from equation (10) by using (A.4) (where \( \tilde{u}^\alpha \tilde{u}_\alpha = -1 \)) it follows again that \( \ell^\alpha = k^\alpha + \tilde{u}^\alpha (\tilde{u}_\beta k^\beta) \). From equation (A.4) and setting \( \alpha = 0 \), we deduce \( \ell^0 = 0 \); moreover, as expected, \( \ell^\alpha \) is space-like since it is projected on \( S(\tau_0) \) and lies everywhere on it. After these considerations the geodesic equation (6) is still valid, but it is better to rewrite it explicitly in terms of the expansion \( \Theta(\tilde{u})_{\rho \sigma} = P(\tilde{u})_\rho^\beta P(\tilde{u})_\sigma^\alpha \nabla_\alpha \tilde{u}_\beta \) of \( C_\lambda \) as

\[
\frac{d\tilde{\ell}^\alpha}{d\sigma} = -\frac{d\tilde{u}^\alpha}{d\sigma} + (\tilde{\ell}^\alpha + \tilde{u}^\alpha) (\tilde{\ell}^\nu \nabla_\nu \tilde{u}_\beta + \Theta_{\rho \sigma} \tilde{\ell}^\rho \tilde{\ell}^\sigma) - \Gamma^\alpha_{\rho \sigma} (\tilde{\ell}^\rho + \tilde{u}^\rho) (\tilde{\ell}^\sigma + \tilde{u}^\sigma),
\]

just because the expansion vanishes identically (appendix C). If \( \alpha = 0 \), equation (11) leads to \( d\tilde{\ell}^0/d\sigma = 0 \) assuring that the condition \( \tilde{\ell}^0 = 0 \) holds true all along the curve \( \tilde{\mathcal{G}} \); if \( \alpha = k \) equation (11) gives the following set of differential equations (terms like \( \partial_\sigma h_{0i} \) are null because \( g_{00} = 0 \) and \( \partial_\sigma h_{ii} = 0 \)):

\[
\frac{d\tilde{\ell}^i}{d\sigma} = -\tilde{\ell}^i \left( \frac{1}{2} \tilde{\ell}^0 h_{00,i} \right) - \delta^{ks} \left( h_{s,i} - \frac{1}{2} h_{i,s} \right) \tilde{\ell}^s \tilde{\ell}^j + \frac{1}{2} \delta^{ks} k_{00,s}.
\]

Equations (12) determine light propagation in the static case, and are called ‘RAMOD3 master equations’ [15].

For the sake of clarity, let us further remark on the role of the master equations. The main purpose of the RAMOD approach is to express the null geodesic through all the physical quantities entering the process of measurement without any approximations, in order to entangle all the possible interactions of light with the background geometry. Keeping the geometrical aspects of the problem guarantees the consistency of the measured physical effects with the intrinsic accuracy of spacetime. In this regard, the local line-of-sight \( \mathcal{L} \), the main unknown of the master equations, represents locally what the observer measures of the incoming photons in his/her gravitational environment. At the time of observation, \( \tilde{\mathcal{L}} \) provides the boundary condition for uniquely solving the light path [17].

3. Parametrized mapped trajectories

Let us fix the origin of the coordinates \((\tau, \xi^i)\) at the center of mass (CM) of the matter distribution of the \(n\)-body system, and choose its spatial coordinates as belonging to the

4 The null geodesics crosses each slice \( S(\tau) \) at a point with coordinates \( \xi^i = \xi^i(\sigma(\tau)) \); but this point also belongs to the unique normal to \( S(\tau) \), crossing it with a value of the parameter \( \sigma = \sigma(\xi^i(\lambda), \tau) = \sigma_{i(\xi^i, \tau)}(\tau) \). To shorten the notations, in what follows we denote \( \sigma_{i(\xi^i, \tau)}(\tau) \) as \( \sigma \).
congruence of curves $C_\Lambda$. $^5$ Also, let us assume that the mapped spatial photon trajectory $\hat{\gamma}$ on the slice $S(t_0)$ belongs to a normal neighborhood of CM. Then by definition, there exists a unique geodesic (in any case a space-like geodesic) $\gamma'$ connecting CM to any point $P$ of $\hat{\gamma}$. Denoting with $\sigma(\tau, \xi)$ the parameter along $\hat{\gamma}$, with $\Lambda$ the parameter along each of the curves $\gamma'$ stemming from CM and such that at CM $\Lambda_{\text{CM}} = 0$, and with $\psi$ the homeomorphism defining the chart containing $P$ in the normal neighborhood of CM, we have that $\hat{\gamma} = \tilde{\gamma}(\sigma) = \psi^{-1}(\xi(\sigma))$ [23]. The coordinate $\xi'$ can be expressed in the function of $\sigma$ or $\Lambda$ as $\xi' = \psi(P) = [\psi \circ \tilde{\gamma}(\sigma)]' = (\psi \circ \gamma'((\Lambda)))'$; applying the map $\gamma'^{-1} \circ \psi^{-1}$ to both members, we obtain $\Lambda = (\gamma'^{-1} \circ \psi^{-1}) \circ (\psi \circ \tilde{\gamma}(\sigma)) = \Lambda(\xi'(\sigma))$, namely the $\Lambda$ implicit dependence on $\sigma$ at the point $P$ with coordinates $\xi'$. Now, defining the unit tangent to the curve $\gamma'$ as $\xi'_{\text{CM}} \equiv d\xi'/d\Lambda$, we write the length of $\gamma'$ from CM to $P$ as

$$L = \int_0^\Lambda (\xi'(\Lambda))^{1/2} d\Lambda' = \Lambda(\sigma).$$

The point of closest approach of the photon to CM will correspond to the minimum of $L$. Thus, differentiating (13) with respect to $\sigma$, we obtain $(dL/d\sigma)_P = (\xi')_{P} \tilde{E}$, and, at the point of closest approach with parameter $\sigma$, $(\xi', \tilde{E})_P = 0$.

Let us set $\Lambda(\xi(\sigma)) \equiv \xi$ as the impact parameter which represents the distance from $P(\sigma)$ to CM; this is a constant which labels each mapped photon trajectory on any given $S(t_0)$. Recalling that $\xi(\Lambda(\sigma)) = \xi'(\sigma)$ at any point $P$ of intersection of the curves $\gamma'$ with $\gamma$, at CM, where $\xi'_{\text{CM}} = 0$ and $\Lambda_{\text{CM}} = 0$, we have, to the first order in $\Lambda$,

$$\xi'(\Lambda + \delta \Lambda) = \xi'(\Lambda_{\text{CM}}) + \left(\frac{d\xi'}{d\Lambda}\right)_{\Lambda_{\text{CM}}} \delta \Lambda + \cdots = (\xi'_{\text{CM}})_{(\Lambda_{\text{CM}} - P)} \delta \Lambda + \cdots,$$

where $(\xi'_{\text{CM}})_{(\Lambda_{\text{CM}} - P)}$ refers to the tangent vector at the origin of the unique spacelike geodesic connecting CM, the origin, to the point $P$. From (14) we clearly have

$$\xi'(\Lambda) = (\xi'_{\text{CM}})_{(\Lambda_{\text{CM}} - P)} \int_0^\Lambda d\Lambda = (\xi'_{\text{CM}})_{(\Lambda_{\text{CM}} - P)} \Lambda.$$

Let us introduce one more parameter on $\hat{\gamma}$, namely

$$\hat{\tau} \equiv \sigma - \hat{\tau},$$

which varies as a function of the different values of the parameter $\sigma$ with respect to $\hat{\tau}$ defined at the point of closest approach. At a point $P$ on $\hat{\gamma}$ with parameter $\hat{\tau} + \delta \hat{\tau}$ it is also

$$\xi'(\hat{\tau} + \delta \hat{\tau}) = \xi'(\hat{\tau}) + \left(\frac{d\xi'}{d\tau}\right)_{\hat{\tau}} \delta \hat{\tau} + \cdots.$$  

But from (15), at $\Lambda(\hat{\tau})$, $\xi'(\Lambda(\hat{\tau}))$ is equivalent to $(\xi'_{\text{CM}})_{(\Lambda_{\text{CM}} - P)} \tilde{E} \equiv \hat{E}$; hence, at any point $P$ on $\hat{\gamma}$, we finally obtain

$$\xi'(\sigma) = \hat{E} + \int_0^\tau \hat{E} \, d\hat{\tau}.$$

i.e. the coordinate path of the photon on $S(t_0)$ as function of the two parameters $\xi'$ and $\hat{\tau}$. Formula (18) parametrizes the mapped trajectory and it contains the local spatial direction in integral form; in this sense it generalizes the parametrization of [10] (and references therein), where, instead, the Euclidean scalar product is used (i.e. $\hat{E} \hat{E} = \delta_{ij} \hat{E}^i \hat{E}^j$). Hence, if the Euclidean

$^5$ One can require that the world line of the CM belongs to this congruence, while the world lines of the bodies would differ from the curves of the congruence by an amount which depends on the local spatial velocity relative to the CM; however, at the $\epsilon^2$ order, i.e. neglecting the relative motion of the gravity sources, one can assume that also their world lines belong to that congruence.
scalar product is applied, equation (18) is equivalent to that in [10]. Finally, by calculating the square modulus of (18) we obtain
\[ r^2 = \xi^i \xi_i = \bar{\xi}^2 + \bar{\xi}^2. \] (19)
This result represents an additional comparison between RAMOD and pN/pM approaches.

The reader should bear in mind, however, that as far as RAMOD is concerned, equation (18) has its validity only up to $\epsilon^2$ when it can be traced onto the hypersurface where the photon’s trajectory is mapped.

4. Null geodesics in comparison

The quantity $\bar{\ell}$ is the unitary four-vector representing the local line-of-sight of the photon as measured by the local observer $u$; it represents a physical quantity. Therefore, the scope of this section is twofold: first, demonstrate that substituting $\bar{\ell}$ with its coordinate counterpart in equation (8) of RAMOD4 is equivalent to recovering the geodesic equation in the first pM regime adopted in [9, 10] and [11]; second, as far as the RAMOD framework is concerned, equation (12) of RAMOD3 is already sufficient to recover the parametrized one, i.e. equation (37), in [10] (which is also equivalent to equation (19) in [9]).

To prove these statements let us use the definition of $\bar{\ell}$; since it is
\[ \bar{\ell} = -\frac{k^i}{u^0 k^0} \approx -\frac{k^i}{u^0 \left[-1 + h_{00} + 0_0(k^i/k^0)\right]}, \]
with $u^0 = (-g_{00})^{-1/2}$ (from equation (A.1) and $k^i/k^0 = \text{d}x^i/\text{d}v^0$, the spatial coordinate components of $\bar{\ell}$ result in (in what follows we assume $c = 1$)
\[ \bar{\ell} = \dot{x}^i \left(1 + \frac{1}{2} h_{00} + h_{00} \dot{x}^0\right) + \mathcal{O}(h^2). \] (20)
From (20), neglecting all of the contributions nonlinear in $h$, since $\text{d} \sigma/\text{d} t = (-g_{00})^{-1/2}$ (appendix A), the term $\text{d} \bar{\ell}/\text{d} \sigma)$ in equation (8) becomes
\[ \frac{\text{d} \bar{\ell}}{\text{d} \sigma} \approx \dot{x}^i + \dot{x}^i \left(\frac{1}{2} h_{00} \dot{x}^i + \frac{1}{2} h_{00,i} \dot{x}^j + h_{00,j} \dot{x}^i + h_{00,i} \dot{x}^j\right), \] (21)
while the rest of the terms transform in:
\[ -\frac{1}{2} \dot{x}^i \dot{x}^j h_{ij,0} + \dot{x}^i h_{00,i} (h_{00,i} - \frac{1}{2} h_{ij,i}) + \frac{1}{2} \dot{x}^i \dot{x}^j h_{00,i} \]
\[ + \dot{x}^i (h_{00,i} + h_{00,i} - h_{00,i}) - \frac{1}{2} h_{00,k} - \dot{x}^i h_{00,i} + h_{00,i}. \] (22)
Therefore, equating expressions (21) and (22), equation (8) transforms in
\[ \dot{x}^i \approx \frac{1}{2} \left(h_{00,i} - h_{00,i} - \frac{1}{2} h_{00,0} \dot{x}^2 - h_{00,0} \dot{x}^2 - (h_{00,i} - h_{00,i}) \dot{x}^i - h_{00,i} \dot{x}^i \right) \]
\[ - \left(h_{00,i} - \frac{1}{2} h_{00,i} \right) \dot{x}^i \dot{x}^j + \left(h_{00,i} - h_{00,i} \right) \dot{x}^i \dot{x}^j, \] (23)
which is the same as the first pM approximation of the null geodesic (in coordinate form) as given, for example, by equation (40) in [9]. Note that this last result was expected, since both equations, (8) of RAMOD4 and (40) in [9], are deduced from the null geodesic (4) in a weak field regime.

Once such an equivalence is obtained, one could adopt the same parametrization as those utilized, for instance, in [10] (equation (36)), or in [26] (equation (7)) in order to find a solution

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6 We remind that the term $\dot{x}^i$ is of the order of $h$ because of the same order of the geodesic equation itself.
of the master equation in the RAMOD framework applicable to all of the physical cases already considered in the literature in this context\textsuperscript{7}. On the other hand, as reported in section 3, this kind of parametrization in RAMOD is possible only in a static spacetime where null geodesics can be entirely mapped into the slice of simultaneity at the time of observation, at the $\varepsilon^2$ level of accuracy in the $\tilde{h}$ linear regime. Thus, consistently with the previous reasoning, we need to check if the RAMOD3 master equation can be transformed into equation (36) of [10]. Let us adopt then the parametrization (18) in the case of constant light direction, say $\tilde{\epsilon}_i^0$; moreover, let us introduce the impact parameter $\xi$ as done in [14], by defining as $P(\tilde{\epsilon}_i^j) = \delta^j_i - \tilde{\epsilon}^j_i$, the operator which projects orthogonally to the light direction. The equivalence of the two parametrizations implies the following change of coordinates:

$$\frac{\tilde{\epsilon}^k}{\xi^0} = \tilde{\tau},$$

(24)

where $\Xi^k$ represents a correction to the coordinates of the same order of the perturbation term $|\tilde{h}|$ of the metric, and $d\tilde{\tau} = d\sigma$. From (24), the partial derivatives should also change according to the following rule:

$$\frac{\partial}{\partial \xi^i} = \frac{\partial \tilde{\epsilon}^i}{\partial \xi^0} \frac{\partial}{\partial \tilde{\tau}} + \frac{\partial \tilde{\epsilon}^0}{\partial \xi^i} \frac{\partial}{\partial \tilde{\tau}} + \Xi^i.$$  

(25)

and the components of the metric $h_{\alpha\beta}$ transform as

$$h_{00} = \tilde{h}_{00} - 2\tilde{\epsilon}_0^i \tilde{h}_{0i} + \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \tilde{h}_{ij} + O(h^2),$$

(26)

$$h_{0i} = P_i^j h_{0j} - \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \tilde{h}_{ij} + O(h^2),$$

(27)

$$h_{ij} = P_i^p P_j^q \tilde{h}_{pq} + O(h^2).$$

(28)

By applying approximation (20) and keeping in mind that, according to the one-parameter local diffeomorphism (9), we are not allowed to neglect the derivative of the metric with respect to $\sigma$ (or the new parameter $\tilde{\tau}$) all along the mapped photon trajectory, the RAMOD3 master equation can be recast as

$$\tilde{\epsilon}^k + \xi^k \left(\frac{1}{2} \frac{d h_{00}}{d \sigma}\right) \approx \frac{1}{2} h_{00,ij} - \frac{1}{2} h_{0i,1} \xi^j - \left(h_{ki,j} - \frac{1}{2} h_{ij,k}\right) \xi^i \xi^j,$$

(29)

which is equivalent, with $\tilde{\epsilon}_0^k$ constant, to

$$\frac{d^2 \Xi^k}{d \tilde{\tau}^2} \approx \frac{1}{2} P_i^k h_{00,i} - \frac{1}{2} \frac{d h_{00}}{d \sigma} \tilde{\epsilon}_0^i - \frac{1}{2} P_i^j h_{00,i} \tilde{\epsilon}_0^j - \left(P_i^j h_{k,i} - \frac{1}{2} P_i^k h_{j,k}\right) \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j.$$  

(30)

Also, considering that $d h_{00}/d\tilde{\tau} = P_i^j h_{00,i} \tilde{\epsilon}_0^j + h_{00,1} \xi^0$, we can eliminate the projected term along the corresponding coordinate direction and, after substituting the new metric terms, find

$$\frac{d^2 \Xi^k}{d \tilde{\tau}^2} \approx \frac{1}{2} \left(\hat{h}_{00} - 2 \tilde{\epsilon}_0^i \hat{h}_{0i} + \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \hat{h}_{ij}\right) \tilde{\epsilon}_0^i - \frac{1}{2} \left(\tilde{\epsilon}_0^i \left(\hat{h}_{00} - 2 \tilde{\epsilon}_0^i \hat{h}_{0i} + \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \hat{h}_{ij}\right) + \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \hat{h}_{pq}\right) \xi^i \xi^j.$$  

(31)

This equation is not yet comparable to its analog in [10], but we have to consider the fact that the off-diagonal terms such as $\tilde{g}_{0i}$ in the static case are null, so from (27)

$$- \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \hat{h}_{0p} = - \hat{h}_{0i} \tilde{\epsilon}_0^i \hat{h}_{0i} - \tilde{\epsilon}_0^i \tilde{\epsilon}_0^j \hat{h}_{pq}.$$  

(32)

\textsuperscript{7} These parameters are (i) $\tau$, defined as the Euclidean scalar product between the light ray vector and its unperturbed trajectory (equation (13) in [10], for example), treated as a non-affine parameter, and (ii) $\xi = \tilde{\epsilon}^i \tilde{\epsilon}^i$, the constant impact parameter of the unperturbed trajectory of the light ray.
Once replaced the right-hand side of equation (32) in equation (31), we immediately obtain
\[
\frac{d^2 \xi^k}{d\tau^2} \approx \frac{1}{2} \left( \dot{h}_{00} - 2\dot{r}_0 \ddot{r}_0 + \dddot{r}_0 \dot{r}_0 \right) - \frac{1}{2} \left( \dot{h}_{0k} \dot{r}_k + \dot{r}_0 \ddot{r}_0 - \frac{1}{2} \dot{r}_0 \dddot{r}_0 \dddot{r}_0 \right) \dot{r}_k + \frac{1}{2} \dddot{r}^\alpha \dddot{r}_\alpha \dddot{r}_k \right) \right),
\]
which is precisely equation (37) [10]. Actually, at this stage, the authors of [10] introduce the isotropic four-dimensional vector \( k^\alpha = (-1, k') \) so to have
\[
\frac{d^2 \xi^k}{d\tau^2} \approx \frac{1}{2} k^\alpha k^\beta \dot{h}_{\alpha\beta,\dot{r}} - \frac{1}{2} \dddot{r}_0 k^\alpha \dddot{r}_\alpha - \frac{1}{2} \frac{1}{2} k^\alpha k^\beta k^\gamma \dot{h}_{\gamma,\dot{r}} \right). \]

Briefly, equation (33) can be considered as a second-order differential equation for the perturbation \( \xi^k \) in the coordinates \( \xi^a \), valid in the domain where the \( \epsilon^2 \) accuracy holds, i.e. the domain of applicability of the RAMOD3-like master equations. However, we remark that the integration of the null geodesic in [10] intends to consider the gravitomagnetic effects. In RAMOD the metric coefficients \( h_{\alpha\beta} \) depend on the retarded distance \( r_{(a)} \) as discussed in [15] and [16]. This means that one has to compute the spatial coordinate distance \( r_{(a)} \) from the points on the photon trajectory to the \( a \)th gravity source at the appropriate retarded time and up to the required accuracy. Let us term \( x'(\bar{\sigma}) \) the spatial coordinates along the spatial path of a generic field source, \( \bar{\sigma} \) being the parameter along its spacetime trajectory:
\[
r = |\xi^i(\sigma(\tau)) - x'_i(\bar{\sigma}(\tau'))| \quad (i = 1, 2, 3),
\]
where \( \tau' \) is the retarded time: \( \tau' = \tau - r \). Since all the functions here are smooth and differentiable, we can expand the coordinates of the gravitating bodies in Taylor’s series around their positions at time \( \tau \) which results in the following expansion of the right-hand side of (35) as derived in paper [16]:
\[
\xi^i(\sigma(\tau)) - x'_i(\bar{\sigma}(\tau')) = \xi^i(\sigma(\tau)) - x'_i(\bar{\sigma}(\tau)) + \int_{\tau'}^\tau (u_\mu \dddot{u}_\mu)^2 e^{-\psi} \dddot{\gamma}_\mu d\tau + \cdots,
\]
where \( \dddot{\gamma}_\mu \) is the spatial velocity of the source in the rest frame of the barycenter. Hence, if we wish our model be accurate to \( \epsilon^3 \), it suffices that the retarded distance \( r \) contributes to the gravitational potentials, which we remind are at the lowest of order \( \epsilon^2 \), with terms of the order of \( \epsilon \). Instead, to the order of \( \epsilon^3 \) (static geometry), the contribution of the relative velocities of the gravitating sources can be neglected. Indeed, in the static case one can choose to further expand the retarded distance in order to keep the terms depending on the source’s velocity up to the desired accuracy. Obviously the gravitational field does not vary, since the terms \( g_{\alpha\beta} \) are null and time derivatives of the metric are at lowest of order \( \epsilon^3 \); therefore, the effects due to the bodies’ velocity cannot be related to a dynamical change of spacetime, at least up to the scale where the vorticity can be neglected. Actually, the positions of the bodies at different values of the parameter \( \bar{\sigma}(\tau) \) can be recorded as subsequent snapshots onto the mapped trajectories and deduced as ‘postponed’ corrections in the reconstruction of the photon’s path. Future work will be devoted to show explicitly these corrections and their relation with the other approaches.

5. Conclusions

Modeling light propagation is intrinsically connected to the identification of the geometry where photons naturally move. The results reported here represent an improvement in the understanding of how to handle the null geodesic according to the accuracy required by observations and geometry, namely the physics of GR. The different conception of RAMOD provides a method to exploit high accurate observations to their full extent, as it could be the case for the astrometric data coming from the ESA mission Gaia [20], possibly a new beginning in the field of relativistic astrometry. By implementing coordinate expressions straightforward,
there is the risk of neglecting the role of curvature (in this case the vorticity) in setting up the geometry of the physics under consideration, and in the interpretation of the observables. In RAMOD, light propagation and its detection is different with or without the vorticity term, this being related to the evolution of the congruence of curves that foliates the spacetime and allows the measurement of physical phenomena in that spacetime. The vorticity term cannot be neglected at the order of $\epsilon^3$: ignoring it locally is valid only in a small neighborhood compared to the scale of vorticity itself. Within the scale of the Solar System, for example, there is no slice which extends from the observer up to the photon emitting star; then the RAMOD4 treatment would be necessary. This, in turn, explains why RAMOD3 is sufficient to recover the parametrization used in [9, 10], namely when the geometry allows a rest-space for an observer everywhere. Formula (18) has its validity in the $\epsilon^2$ regime and it generalizes the parametrization used in [9, 10] and, consistently, the RAMOD3 master equations, once converted into a coordinate form, recover the analytical linearized case discussed by Kopeikin and others. Then, without vorticity, RAMOD sets the level of reciprocal consistency with these approaches; instead, RAMOD4, i.e. the case of a dynamical spacetime, fully preserves the active content of gravity. Its master equations are not contemplated in other approaches: equation (8) alone, containing spatial components only, reproduces, once transformed into a coordinate form, the equation of propagation of photons in the first pM approximation, as demonstrated in this paper. Equation (7) is unique with its time component $\bar{\ell}^0$ that could be the clue for uncovering new effects.

In conclusion, by recovering results already presented in the literature, this paper illustrates the full potential of the RAMOD construct and proves its consistency, at the proper level of approximations, with existing approaches.

Acknowledgments

This work is supported by ASI contract I/058/10/0. I would like to thank my collaborators, D Bini, F de Felice, M G Lattanzi, and A Vecchiato for their valuable comments.

Appendix A. Setting up the geometry

Given metric (1), a space-like foliation implies that a unitary one-form $u_a$ exists which is everywhere proportional to the gradient of a smooth and differentiable function $\tau(t, x')$, i.e. $u = -(\tau, a) e^\psi d\sigma$ [23], $e^\psi$ being the normalization factor. From equation (2) one deduces that the congruence $C_u$ is vorticity-free only up to $\epsilon^2$ order. In order to understand the geometrical meaning of the one-form $u_a$, let us adapt the coordinate system to the spatial hypersurfaces $\tau = \text{constant}$ such that $x^0 = \tau(t, x')$ and $x^i = x^i(t, x')$. The new normals are

$$u_a(x^0, x') = -\delta_a^0 e^\psi, \quad u^a(x^0, x') = -g^a\psi e^\psi \frac{dx^a}{d\sigma}, \quad \text{(A.1)}$$

where $\sigma$ is the parameter on the normals $u$, and $\psi = \psi(x^0, x')$ is a function that makes $u^a u_a = -1$. From the unitary condition and (A.1), it is $e^\psi = d\sigma/dt = (-g^00)^{-1/2}$; this means that the parameter $\sigma$ runs uniformly with the coordinate time $\tau$ and depends on the metric term $g_00$. But the congruence $C_u$, however, does not preserve the spatial coordinates $x^i$. In fact, denoting $N^i = -u^i/u^0$ and $N = 1/u^0$, respectively, lapse and shift functions (or

8 Actually, it can be proved that spatial coordinates are Lie-transported along $u$.}
factors, for similar definitions see [19, 27]), the line element can be cast into the form
\[ ds^2 = -(Ndx^0)^2 + g_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0). \]  
(A.2)
Thus, choosing at time \( \tau_1 \), say, an arbitrary event \( P \), labeled by the coordinates \( (\tau_1, x^i_1) \), the unique normal through that point will intersect the slice \( S(\tau_1 + \Delta \tau) \) at a point \( P' \) with spatial coordinates that are shifted from the initial ones by the amount
\[ \Delta x^i = \int_0^{\Delta \tau} N^i(\tau') d\tau'. \]  
(A.3)
In order to vanish the shift factor a new spatial coordinate transformation has to be applied, i.e. \( \xi^i = x^i + N^i \Delta x^0 \) and \( \xi^0 = x^0 = \tau \). Then, the vector field tangent to the congruence \( C_\xi \), which transports the spatial coordinate, results in
\[ \tilde{u}^\alpha(\xi^0, \xi^i) = \frac{d\xi^\alpha}{d\sigma} = e^\phi \delta_0^\alpha \]  
(A.4)
Note that, as expected from (2), it is also
\[ \tilde{g}^{00} = g^{00} \]  
(A.5)
\[ \tilde{g}_{0i} = \frac{\partial \xi^i}{\partial \tau} g^{00} + \frac{\partial \xi^i}{\partial \xi^j} g_{0j} = 0. \]

Appendix B. Formula (6)
From \( k^\alpha = l^\alpha - (u_\beta k^\beta) u^\alpha \) the geodesic equation transforms into
\[ \frac{dl^\alpha}{d\lambda} - (u_\beta k^\beta) \frac{dl^\alpha}{d\lambda} = \alpha^\alpha u^\alpha + \Gamma^\alpha_{\beta \gamma} [l^\beta (l^\gamma + l^\gamma u^\beta + (u_\gamma k^\gamma)^2 u^\beta) + (u_\gamma k^\gamma)^2 u^\beta u^\gamma] = 0, \]
where \( k^\beta \nabla_\gamma u_\beta = l^\beta \nabla_\gamma u_\beta - (u_\gamma k^\gamma)^2 \nabla_\gamma u_\beta \). Let us change the affine parameter \( \lambda \) of the geodesic into the parameter \( \sigma \):
\[ \frac{dl^\alpha}{d\lambda} = \frac{d[l-(u_\beta k^\beta) u^\alpha]}{d\sigma} \frac{d\sigma}{d\lambda} = (u_\beta k^\beta)^2 \frac{d\tilde{\sigma}}{d\sigma} - (u_\beta k^\beta)^2 \tilde{\sigma} u^\alpha \tilde{u}^\beta. \]
Then, substituting the last equation into the previous one and dividing each member by \( (u_\beta k^\beta)^2 \), we find the geodesic equation (6).

Appendix C. The case of null expansion
Let us express the expansion in the linear regime (actually up to terms of order of \( O(\hat{h}^2) \), see equation 10.12 in [27], and chapters 8–9 in [23]) as
\[ \Theta(\tilde{u})_{\rho\sigma} \approx P(\tilde{u})_{\rho}^{\beta} P(\tilde{u})_{\sigma}^{\alpha} \nabla_\alpha \tilde{u}_\beta \]  
(C.1)
where round brackets means symmetrization.
Replacing in \( \Gamma^\alpha_{\beta \gamma} \) the metric coefficients, we obtain [25]
\[ \Theta(\tilde{u})_{\rho\sigma} = \frac{1}{2} e^\phi \delta_{\rho\sigma} + e^{\phi} g_{0\rho} \delta_{0\sigma} + \frac{1}{2} e^{\phi} g_{0\rho} g_{0\sigma} \delta_{00} + \frac{1}{2} e^{\phi} g_{0\rho} g_{0\sigma} \delta_{00} \].
At the order of $\epsilon^2$ it is easy to see that for $\rho = i$ and $\sigma = j$ the last equation becomes
\[ \Theta(\tilde{u})_{ij} = \frac{1}{2} \epsilon^2 \delta_{0ij} = 0; \] (C.2)
for $\rho = 0$ and $\sigma = i$,
\[ \Theta(\tilde{u})_{0i} = \frac{1}{2} \epsilon^2 \delta_{00i} (1 + \epsilon^2 g_{0i}) = 0 \implies \delta_{00i} = 0, \] (C.3)
as expected; finally, for $\rho = 0$ and $\sigma = 0$,
\[ \Theta(\tilde{u})_{00} = \frac{1}{2} \epsilon^2 \delta_{000} (1 + g_{00})^2 = 0. \] (C.4)

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