Deterministic Brownian Motion:
The Effects of Perturbing a Dynamical System by a Chaotic
Semi-Dynamical System

Michael C. Mackey* and Marta Tyran-Kamińska †
March 22, 2022

Abstract

Here we review and extend central limit theorems for highly chaotic but deterministic semi-
dynamical discrete time systems. We then apply these results show how Brownian motion-like
results are recovered, and how an Ornstein-Uhlenbeck process results within a totally determin-
istic framework. These results illustrate that the contamination of experimental data by “noise”
may, under certain circumstances, be alternately interpreted as the signature of an underlying
chaotic process.

Contents

1 Introduction 2

2 Semi-dynamical systems 5
  2.1 Density evolution operators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.2 Probabilistic and ergodic properties of density evolution . . . . . . . . . . . . . . . . 12
  2.3 Brownian motion from deterministic perturbations . . . . . . . . . . . . . . . . . . . . . 16
      2.3.1 Central limit theorems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
      2.3.2 FCLT for noninvertible maps . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
  2.4 Weak convergence criteria . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31

3 Analysis 33
  3.1 Weak convergence of \(v(t_n)\) and \(v_n\) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35
  3.2 The linear case in one dimension . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
      3.2.1 Behaviour of the velocity variable . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
      3.2.2 Behaviour of the position variable . . . . . . . . . . . . . . . . . . . . . . . . . . 38

4 Identifying the Limiting Velocity Distribution 41
  4.1 Dyadic map . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42
  4.2 Graphical illustration of the velocity density evolution with dyadic map perturbations 45
  4.3 r-dyadic map . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 45

*e-mail: mackey@cnld.mcgill.ca, Departments of Physiology, Physics & Mathematics and Centre for Nonlinear
Dynamics, McGill University, 3655 Promenade Sir William Osler, Montreal, QC, CANADA, H3G 1Y6
†Corresponding author, email: mtyran@us.edu.pl, Institute of Mathematics, Silesian University, ul. Bankowa 14,
40-007 Katowice, POLAND
1 Introduction

Almost anyone who has ever looked through a microscope at a drop of water has been intrigued by the seemingly erratic and unpredictable movement of small particles suspended in the water, e.g. dust or pollen particles. This phenomena, noticed shortly after the invention of the microscope by many individuals, now carries the name of “Brownian motion” after the English botanist Robert Brown who wrote about his observations in 1828. Almost three-quarters of a century later, Einstein (1905) gave a theoretical (and essentially molecular) treatment of this macroscopic motion that predicted the phenomenology of Brownian motion. (A very nice English translation of this, and other, work of Einstein’s on Brownian motion can be found in Fürth (1956).) The contribution of Einstein led to the development of much of the field of stochastic processes, and to the notion that Brownian movement is due to the summated effect of a very large number of tiny impulsive forces delivered to the macroscopic particle being observed. This was also one of the most definitive arguments of the time for an atomistic picture of the microscopic world.

Other ingenious experimentalists used this conceptual idea to explore the macroscopic effects of microscopic influences. One of the more interesting is due to Kappler (1931), who devised an experiment in which a small mirror was suspended by a quartz fiber (c.f Mazo (2002) for an analysis of this experimental setup). Any rotational movement of the mirror would tend to be counterbalanced by a restoring torsional force due to the quartz fiber. The position of the mirror was monitored by shining a light on it and recording the reflected image some distance away (so small changes in the rotational position of the mirror were magnified). Air molecules striking the mirror caused a transient deflection that could thus be monitored, and the frequency of these collisions was controlled by changing the air pressure. Figure 1.1, taken from Kappler (1931), shows two sets of data taken using this arrangement and offers a vivid depiction of the macroscopic effects of microscopic influences.

In trying to understand theoretically the basis for complicated and irreversible experimental observations, a number of physicists have supplemented the reversible laws of physics with various hypotheses about the irregularity of the physical world. One of the first of these, and arguably one of the most well known, is the so-called “molecular chaos” hypothesis of Boltzmann (1995). This hypothesis, which postulated a lack of correlation between the movement of molecules in a small collision volume, allowed the derivation of the Boltzmann equation from the Liouville equation and led to the celebrated H theorem. The origin of the loss of correlations was never specified. In an effort to understand the nature of turbulent flow, Ruelle (1978, 1979, 1980) postulated a type of mixing dynamics to be necessary. More recently, several authors have made chaotic hypotheses about the nature of dynamics at the microscopic level. The most prominent of these is Gallavotti (1999), and virtually the entire book of Dorfman (1999) is predicated on the implicit assumption that microscopic dynamics have a chaotic (loosely defined, but usually taken to be mixing) nature. All of these hypotheses have been made in spite of the fact that none of the microscopic dynamics that we write down in physics actually display such properties.

Others have taken this suggestion (chaotic hypothesis) quite seriously, and attempted an experimental confirmation. Figure 1.2 shows a portion of the data, taken from Gaspard et al. (1998), that was obtained in an examination of a microscopic system for the presence of chaotic behavior.
Figure 1.1: In the upper panel is shown a recording of the movement of the mirror in the Kappler (1931) experiment over a period of about 30 minutes at atmospheric pressure (760 mm Hg). The bottom panel shows the same experiment at a pressure of $4 \times 10^{-3}$ mm Hg. Both figures are from Kappler (1931). See the text for more detail.
Figure 1.2: The data shown here, taken from Gaspard et al. (1998), show the position of a 2.5 μm Brownian particle in water over a 300 second period with a sampling interval of \( \frac{1}{60} \) sec (see Gaspard et al. (1998); Briggs et al. (2001) for the experimental details). The inset figure shows the power spectrum, which displays a typical decay (for Brownian motion) with \( \omega^{-2} \).

Their data analysis showed a positive lower bound on the sum of Lyapunov exponents of the system composed of a macroscopic Brownian particle and the surrounding fluid. From their analysis, they argued that the Brownian motion was due to (or the signature of) deterministic microscopic chaos. However, Briggs et al. (2001) were more cautious in their interpretation, and Mazo (2002, Chapter 18) has explored the possible interpretations of experiments like these in some detail.

If true, the existence of deterministic chaos (whatever that means) would be an intriguing possibility since, if generated by a non-invertible dynamics (semi-dynamical system), it could serve as an explanation of a host of unresolved problems in the sciences. Most notably, it could serve as an explanation for the manifest irreversibility of our physical and biological world in the face of physical laws that fail to encompass irreversibility without the most incredulous of assumptions. In particular, it would clarify the foundations of irreversible statistical mechanics, e.g. the operation of the second law of thermodynamics (Dorfman, 1999; Gallavotti, 1999; Mackey, 1989, 1992; Schulman, 1997), and the implications of the second law for the physical and biological sciences.

In this paper, we have a rather more modest goal. We address a different facet of this chaotic hypothesis by studying how and when the characteristics of Brownian motion can be reproduced by deterministic systems. To motivate this, in Figures 1.3 through 1.6 we show the position \( (x) \)
and velocity \((v)\) of a particle of mass \(m\) whose dynamics are described by

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{mv}{dt} &= -\Gamma v + F(t).
\end{align*}
\tag{1.1, 1.2}
\]

In Equations 1.1 and 1.2, \(F\) is a fluctuating “force” consisting of a sequence of delta-function like impulses given by

\[
F(t) = m\kappa \sum_{n=0}^{\infty} \xi(t)\delta(t - n\tau),
\tag{1.3}
\]

and \(\xi\) is a “highly chaotic” (exact, see Section 2) deterministic variable generated by \(\xi_{t+\tau} = T(\xi_t)\) where \(T\) is the hat map on \([-1,1]\) defined by:

\[
T(y) = \begin{cases} 
2 \left( y + \frac{1}{2} \right) & \text{for } y \in [-1,0) \\
2 \left( \frac{1}{2} - y \right) & \text{for } y \in [0,1).
\end{cases}
\tag{1.4}
\]

In this paper we examine the behavior of systems described by equations like (1.1) through (1.4) and establish, analytically, the eventual limiting behavior of ensembles. In particular, we address the question of how Brownian-like motion can arise from a purely deterministic dynamics. We do this by studying the dynamics from a statistical, or ergodic theory, standpoint.

The outline of the paper is as follows. Section 2 gives required background and new material including the definitions of a hierarchy of chaotic behaviours (ergodic, mixing, and exact) with a discussion of their different behaviours in terms of the evolution of densities under the action of transfer operators such as the Frobenius-Perron operators. We then go on to treat Central Limit theorems and Functional Central Limit Theorems for non-invertible dynamical systems. Section 3 returns to the specific problem that Equations 1.1 through 1.4 illustrate. We show how the particle velocity distribution may converge and how the particle position may become asymptotically Gaussian, but for a more general class of maps than given by (1.4). In Section 4 we illustrate the application of the results from Section 3 using a specific chaotic map (the dyadic map, Equation 1.4) to act as a surrogate noise source. Section 5 considers the question when one can obtain Gaussian processes by studying appropriate scaling limits of the velocity and position variables, and the convergence of the velocity process to an Ornstein-Uhlenbeck process as the interval \(\tau\) between chaotic perturbations approaches 0. The paper concludes with a brief discussion in Section 6. The Appendix collects and extends general central limit theorems from probability theory that are used in the main results of the paper.

2 Semi-dynamical systems

We are going to examine the behavior illustrated in Section 1 using techniques from ergodic theory, and closely related concepts from probability theory, applied to the dynamics of semi-dynamical (non-invertible) systems. In this section we collect together the necessary machinery to do so. Much of this background material can be found in Lasota and Mackey (1994).

2.1 Density evolution operators

Let \((Y_1, B_1, \nu_1)\) and \((Y_2, B_2, \nu_2)\) be two \(\sigma\)-finite measure spaces and let the transformation \(T : Y_1 \to Y_2\) be measurable, \(i.e.\) \(T^{-1}(B_2) \subseteq B_1\) where \(T^{-1}(B_2) = \{T^{-1}(A) : A \in B_2\}\). Then we say that
Figure 1.3: The top panel shows the simulated position of a particle obeying Equations 1.1 through 1.4 using Equation 3.28, while the bottom panel shows the velocity of the same particle computed with Equation 3.21. The parameters used were: $\gamma = \Gamma/m = 10$, $\kappa = 1$, and $\tau = -\frac{1}{10} \ln(9 \times 10^{-4}) \simeq 0.932$ so $\lambda \equiv e^{-\gamma \tau} = 9 \times 10^{-4}$. The initial condition on the hat map given by Equation 1.4 was $y_0 = 0.12562568$. 
Figure 1.4: As in Figure 1.3 except that \( \tau = \frac{1}{10} \ln(2) \simeq 0.069 \) so \( \lambda = \frac{1}{2} \).
Figure 1.5: As in Figure 1.4 except that $\tau = -\frac{1}{10} \ln(0.9) \simeq 0.011$ so $\lambda = 0.9$. 
Figure 1.6: As in Figure 1.3 with the parameters of Figure 1.5 and an initial condition on the hat map (1.4) of $y_0 = 0.1678549321$. 
\( T \) is nonsingular (with respect to \( \nu_1 \) and \( \nu_2 \)) if \( \nu_1(T^{-1}(A)) = 0 \) for all \( A \in \mathcal{B}_2 \) with \( \nu_2(A) = 0 \). Associated with the transformation \( T \) we have the Koopman operator \( U_T \) defined by

\[
U_T g = g \circ T
\]

for every measurable function \( g : Y_2 \to \mathbb{R} \). We define the transfer operator \( P_T : L^1(Y_1, \mathcal{B}_1, \nu_1) \to L^1(Y_2, \mathcal{B}_2, \nu_2) \) as follows. For any \( f \in L^1(Y_1, \mathcal{B}_1, \nu_1) \), there is a unique element \( P_T f \) in \( L^1(Y_2, \mathcal{B}_2, \nu_2) \) such that

\[
\int_A P_T f(y) \nu_2(dy) = \int_{T^{-1}(A)} f(y) \nu_1(dy).
\]

Equation (2.1) simply gives an implicit relation between an initial density of states \( f \) and that density after the action of the map \( T \), i.e. \( P_T f \). The Koopman operator \( U_T : L^\infty(Y_2, \mathcal{B}_2, \nu_2) \to L^\infty(Y_1, \mathcal{B}_1, \nu_1) \) and the transfer operator \( P_T \) are adjoint, so

\[
\int_{Y_2} g(y) P_T f(y) \nu_2(dy) = \int_{Y_1} f(y) U_T g(y) \nu_1(dy)
\]

for \( g \in L^\infty(Y_2, \mathcal{B}_2, \nu_2) \), \( f \in L^1(Y_1, \mathcal{B}_1, \nu_1) \).

In some special cases Equation (2.1) allows us to obtain an explicit form for \( P_T \). Let \( Y_2 = \mathbb{R} \), \( \mathcal{B}_2 = \mathcal{B}(\mathbb{R}) \) be the Borel \( \sigma \)-algebra, and \( \nu_2 \) be the Lebesgue measure. Let \( Y_1 \) be an interval \([a, b] \) on the real line \( \mathbb{R} \), \( \mathcal{B}_1 = [a, b] \cap \mathcal{B}(\mathbb{R}) \) and \( \nu_1 \) be the Lebesgue measure restricted to \([a, b] \). We will simply write \( L^1([a, b]) \) when the underlying measure is the Lebesgue measure.

The transformation \( T : [a, b] \to \mathbb{R} \) is called piecewise monotonic if

(i) there is a partition \( a = a_0 < a_1 < \ldots < a_l = b \) of \([a, b] \) such that for each integer \( i = 1, \ldots, l \) the restriction of \( T \) to \((a_{i-1}, a_i) \) has a \( C^1 \) extension to \([a_{i-1}, a_i] \) and

(ii) \(|T'(x)| > 0 \) for \( x \in (a_{i-1}, a_i), i = 1, \ldots, l \).

If a transformation \( T : [a, b] \to \mathbb{R} \) is piecewise monotonic, then for \( f \in L^1([a, b]) \) we have

\[
P_T f(y) = \sum_{i=1}^{l} \frac{f(T^{-1}_{(i)}(y))}{|T'(T^{-1}_{(i)}(y))|} 1_{T([a_{i-1}, a_i])}(y),
\]

where \( T^{-1}_{(i)} \) is the inverse function for the restriction of \( T \) to \((a_{i-1}, a_i) \). Note that we have equivalently

\[
P_T f(y) = \sum_{x \in T^{-1}(\{y\})} \frac{f(x)}{|T'(x)|}.
\]

Of course these formulas hold almost everywhere with respect to the Lebesgue measure.

Let \((Y, \mathcal{B})\) be a measurable space and let \( T : Y \to Y \) be a measurable transformation. The definition of the transfer operator for \( T \) depends on a given \( \sigma \)-finite measure on \( \mathcal{B} \), which in turn gives rise to different operators for different underlying measures on \( \mathcal{B} \). If \( \nu \) is a probability measure on \( \mathcal{B} \) which is invariant for \( T \), i.e. \( \nu(T^{-1}(A)) = \nu(A) \) for all \( A \in \mathcal{B} \), then \( T \) is nonsingular. The transfer operator \( P_T : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu) \) is well defined and when we want to emphasize that the underlying measure \( \nu \) in the transfer operator is invariant under the transformation \( T \) we will write \( P_{T, \nu} \). The Koopman operator \( U_T \) is also well defined for \( f \in L^1(Y, \mathcal{B}, \nu) \) and is an isometry of \( L^1(Y, \mathcal{B}, \nu) \) into \( L^1(Y, \mathcal{B}, \nu) \), i.e. \( \|U_T f\|_1 = \|f\|_1 \) for all \( f \in L^1(Y, \mathcal{B}, \nu) \). The following relation holds between the operators \( U_T, P_{T, \nu} : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu) \)

\[
P_{T, \nu} U_T f = f \quad \text{and} \quad U_T P_{T, \nu} f = E(f|T^{-1}(\mathcal{B}))
\]
for \( f \in L^1(Y, \mathcal{B}, \nu) \), where \( E(\cdot | T^{-1}(B)) : L^1(Y, \mathcal{B}, \nu) \rightarrow L^1(Y, T^{-1}(B), \nu) \) denotes the operator of conditional expectation (see Appendix). Both of these equations are based on the following change of variables [Billingsley, 1995, Theorem 16.13]: \( f \in L^1(Y, \mathcal{B}, \nu) \) if and only if \( f \circ T \in L^1(Y, \mathcal{B}, \nu) \), in which case the following holds

\[
\int_{T^{-1}(A)} f \circ T(y)\nu(dy) = \int_A f(y)\nu(dy), \quad A \in \mathcal{B}.
\]  

(2.4)

If the measure \( \nu \) is finite, we have \( L^p(Y, \mathcal{B}, \nu) \subset L^1(Y, \mathcal{B}, \nu) \) for \( p \geq 1 \). The operator \( U_T : L^p(Y, \mathcal{B}, \nu) \rightarrow L^p(Y, \mathcal{B}, \nu) \) is also an isometry in this case. Note that if the conditional expectation operator \( E(\cdot | T^{-1}(B)) : L^1(Y, \mathcal{B}, \nu) \rightarrow L^1(Y, \mathcal{B}, \nu) \) is restricted to \( L^2(Y, \mathcal{B}, \nu) \), then this is the orthogonal projection of \( L^2(Y, \mathcal{B}, \nu) \) onto \( L^2(Y, T^{-1}(B), \nu) \).

One can also consider any \( \sigma \)-finite measure \( m \) on \( \mathcal{B} \) with respect to which \( T \) is nonsingular and the corresponding transfer operator \( P_T : L^1(Y, \mathcal{B}, m) \rightarrow L^1(Y, \mathcal{B}, m) \). To be specific, let \( Y \) be a Borel subset of \( \mathbb{R}^k \) with Lebesque measure \( m \) and \( \mathcal{B} = \mathcal{B}(Y) \) be the \( \sigma \)-algebra of Borel subsets of \( Y \). Throughout this paper \( m \) will denote Lebesque measure and \( L^1(Y) \) will denote \( L^1(Y, \mathcal{B}, m) \). The transfer operator \( P_T : L^1(Y) \rightarrow L^1(Y) \) is usually known as the [Frobenius-Perron operator](#). A measure \( \nu \) (on \( Y \)) is said to have a density \( g \), if \( \nu(A) = \int_A g(y)\nu(dy) \) for all \( A \in \mathcal{B} \), where \( g \in L^1(Y) \) is nonnegative and \( \int_Y g(y)\nu(dy) = 1 \). A measure \( \nu \) is called absolutely continuous if it has a density. If the Frobenius-Perron operator \( P_T \) has a nontrivial fixed point in \( L^1(Y) \), *i.e.* the equation \( P_T f = f \) has a nonzero solution in \( L^1(Y) \), then the transformation \( T \) has an absolutely continuous invariant measure \( \nu \), its density \( g_\ast \) is a fixed point of \( P_T \), and we call \( g_\ast \) an invariant density under the transformation \( T \). The following relation holds between the operators \( P_T \) and \( P_{T, \nu} \nolimits 

\[
P_T(f g_\ast) = g_\ast P_{T, \nu} f \quad \text{for} \quad f \in L^1(Y, \mathcal{B}, \nu).
\]  

(2.5)

In particular, if the density \( g_\ast \) is strictly positive, *i.e.* \( g_\ast(y) > 0 \) a.e. for \( y \in Y \), then the measures \( m \) and \( \nu \) are equivalent and we also have

\[
P_T(f) = g_\ast P_{T, \nu} \left( \frac{f}{g_\ast} \right) \quad \text{for} \quad f \in L^1(Y).
\]

The notion of a piecewise monotonic transformation on an interval can be extended to “piecewise smooth” transformations \( T : Y \rightarrow Y \) with \( Y \subset \mathbb{R}^k \). Therefore if \( T \) has, for example, a finitely many inverse branches and the Jacobian matrix \( DT(x) \) of \( T \) at \( x \) exists and \( \det DT(x) \neq 0 \) for almost every \( x \), then the Frobenius-Perron operator is given by

\[
P_T f(y) = \sum_{x \in T^{-1}(\{y\})} \frac{f(x)}{\left| \det DT(x) \right|} \quad \text{a.e.}
\]

for \( f \in L^1(Y) \). If \( T \) is invertible then we have \( P_T f(y) = f(T^{-1}(y))| \det DT^{-1}(y) | \).

Finally, we briefly mention Ruelle’s transfer operator. Let \( Y \) be a compact metric space, \( T : Y \rightarrow Y \) be a continuous map such that \( T^{-1}(\{y\}) \) is finite for each \( y \in Y \) and let \( \phi : Y \rightarrow \mathbb{R} \) be a function (typically continuous or Hölder continuous). The so called Ruelle operator \( L_\phi \) acts on functions rather than on \( L^1(Y) \) elements and is defined by

\[
L_\phi f(y) = \sum_{x \in T^{-1}(\{y\})} e^{\psi(x)} f(x)
\]

for every \( y \in Y \). The function \( \psi \) is a so called potential. If we take \( \psi(x) = -\log | \det DT(x) | \), provided it makes sense, then we arrive at the representation for the Frobenius-Perron operator.
2.2 Probabilistic and ergodic properties of density evolution

Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space and let \(T : Y \to Y\) be a measurable transformation preserving the measure \(\nu\). We can discuss the ergodic properties of \(T\) in terms of the convergence behavior of its transfer operator \(P_{T,\nu} : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)\). To this end, we note that the transformation \(T\) is

(i) **Ergodic** (with respect to \(\nu\)) if and only if every invariant set \(A \in \mathcal{B}\) is such that \(\nu(A) = 0\) or \(\nu(Y \setminus A) = 0\). This is equivalent to: \(T\) is ergodic (with respect to \(\nu\)) if and only if for each \(f \in L^1(Y, \mathcal{B}, \nu)\) the sequence \(\dfrac{1}{n} \sum_{k=0}^{n-1} P_{T,\nu}^k f\) is weakly convergent in \(L^1(Y, \mathcal{B}, \nu)\) to \(\int f(y) \nu(dy)\), i.e. for all \(g \in L^\infty(Y, \mathcal{B}, \nu)\)

\[
\lim_{n \to \infty} \int \dfrac{1}{n} \sum_{k=0}^{n-1} P_{T,\nu}^k f(y) g(y) \nu(dy) = \int f(y) \nu(dy) \int g(y) \nu(dy);
\]

(ii) **Mixing** (with respect to \(\nu\)) if and only if

\[
\lim_{n \to \infty} \nu(\{A \cap T^{-n}(B)\}) = \nu(A) \nu(B) \quad \text{for} \quad A, B \in \mathcal{B}.
\]

Mixing is equivalent to: For each \(f \in L^1(Y, \mathcal{B}, \nu)\) the sequence \(P_{T,\nu}^n f\) is weakly convergent in \(L^1(Y, \mathcal{B}, \nu)\) to \(\int f(y) \nu(dy)\), i.e.

\[
\lim_{n \to \infty} \int P_{T,\nu}^n f(y) g(y) \nu(dy) = \int f(y) \nu(dy) \int g(y) \nu(dy) \quad \text{for} \quad g \in L^\infty(Y, \mathcal{B}, \nu).
\]

(iii) **Exact** (with respect to \(\nu\)) if and only if

\[
\lim_{n \to \infty} \nu(T^n(A)) = 1 \quad \text{for} \quad A \in \mathcal{B} \quad \text{with} \quad T(A) \in \mathcal{B}, \quad \nu(A) > 0.
\]

Exactness is equivalent to: For each \(f \in L^1(Y, \mathcal{B}, \nu)\) the sequence \(P_{T,\nu}^n f\) is strongly convergent in \(L^1(Y, \mathcal{B}, \nu)\) to \(\int f(y) \nu(dy)\), i.e.

\[
\lim_{n \to \infty} \int |P_{T,\nu}^n f(y) - \int f(y) \nu(dy)| \nu(dy) = 0.
\]

The characterization of the ergodic properties of transformations through the properties of the evolution of densities requires that we know an invariant measure \(\nu\) for \(T\). Examples of ergodic, mixing, and exact transformations are given in the following.

**Example 1** The transformation on \([0, 1]\)

\[T(y) = y + \phi \pmod{1},\]

known as rotation on the circle, is ergodic with respect to the Lebesgue measure when \(\phi\) is irrational. The associated Frobenius-Perron operator is given by

\[P_T f(y) = f(y - \phi).\]
Example 2. The baker map on $[0, 1] \times [0, 1]$

$$T(y, z) = \begin{cases} 
(2y, \frac{1}{2}z) & 0 \leq z \leq \frac{1}{2} \\
(2y - 1, \frac{1}{2} + \frac{1}{2}z) & \frac{1}{2} < z \leq 1 
\end{cases}$$

is mixing with respect to the Lebesgue measure. The Frobenius-Perron operator is given by

$$P_T f(y, z) = \begin{cases} 
f(\frac{1}{2}y, 2z) & 0 \leq z \leq \frac{1}{2} \\
f(\frac{1}{2} + \frac{1}{2}y, 2z - 1) & \frac{1}{2} < z \leq 1 
\end{cases}$$

Example 3. The hat map on $[-1, 1]$ defined by Equation 1.4 is exact with respect to the Lebesgue measure, and has a Frobenius-Perron operator given by

$$P_T f(y) = \frac{1}{2} \left[ f \left( \frac{1}{2}y - \frac{1}{2} \right) + f \left( \frac{1}{2} - \frac{1}{2}y \right) \right].$$

Example 4. A class of piecewise linear transformations on $[0, 1]$ are given by

$$T_N(y) = \begin{cases} 
N \left( y - \frac{2n}{N} \right) & \text{for } y \in \left[ \frac{2n}{N}, \frac{2n+1}{N} \right) \\
N \left( \frac{2n+2}{N} - y \right) & \text{for } y \in \left( \frac{2n+1}{N}, \frac{2n+2}{N} \right], 
\end{cases} \quad (2.6)$$

where $n = 0, 1, \ldots, [(N - 1)/2]$ and $[z]$ denotes the integer part of $z$. For $N \geq 2$, these piecewise linear maps generalize the hat map, are exact with respect to the Lebesgue measure, and have the invariant density

$$g_*(y) = 1_{[0,1]}(y). \quad (2.7)$$

Example 5. The Chebyshev maps (Adler and Rivlin (1964)) on $[-1, 1]$ studied by Beck and Roepstorff (1987), Beck (1996) and Hilgers and Beck (1999) are given by

$$S_N(y) = \cos(N \arccos y), \quad N = 0, 1, \ldots \quad (2.8)$$

with $S_0(y) = 1$ and $S_1(y) = y$. They are conjugate to the transformation of Example 4, and satisfy the recurrence relation $S_{N+1}(y) = 2yS_N(y) - S_{N-1}(y)$. For $N \geq 2$ they are exact with respect to the measure with the density

$$g_*(y) = \frac{1}{\pi \sqrt{1 - y^2}}.$$

For $N = 2$ the Frobenius-Perron operator is given by

$$P_{S_2} f(y) = \frac{1}{2\sqrt{2y + 2}} \left[ f \left( \frac{1}{2}y + \frac{1}{2} \right) + f \left( -\frac{1}{2}y + \frac{1}{2} \right) \right]$$

and the transfer operator by

$$\mathcal{P}_{S_2, \nu} f(y) = \frac{1}{2} \left[ f \left( \frac{1}{2}y + \frac{1}{2} \right) + f \left( -\frac{1}{2}y + \frac{1}{2} \right) \right].$$

The construction of a transfer operator for a conjugate map is presented in the next theorem from Lasota and Mackey (1994).
Theorem 1 (Lasota and Mackey (1994, Theorem 6.5.2)) Let \( T : [0,1] \to [0,1] \) be a measurable and nonsingular (with respect to the Lebesgue measure) transformation. Let \( \nu : \mathcal{B}([a,b]) \to [0,\infty) \) be a probability measure with a strictly positive density \( g_* \), that is \( g_*(y) > 0 \) a.e. Let a second transformation \( S : [a,b] \to [a,b] \) be given by \( S = G^{-1} \circ T \circ G \), where

\[
G(x) = \int_a^x g_*(y) \, dy, \quad a \leq x \leq b.
\]

Then the transfer operator \( \mathcal{P}_{S,\nu} \) is given by

\[
\mathcal{P}_{S,\nu} f = U_G P_T U_G^{-1} f, \quad \text{for } f \in L^1([a,b], \mathcal{B}([a,b]), \nu), \tag{2.9}
\]

where \( U_G, U_G^{-1} \) are Koopman operators for \( G \) and \( G^{-1} \), respectively, and \( P_T \) is the Frobenius-Perron operator for \( T \). As a consequence, \( \nu \) is invariant for \( S \) if and only if the Lebesgue measure is invariant for \( T \).

Example 6 The dyadic map on \([-1,1]\) is given by

\[
T(y) = \begin{cases} 
2y + 1, & y \in [-1,0] \\
2y - 1, & y \in (0,1),
\end{cases}
\tag{2.10}
\]

and has the uniform invariant density

\[
g_*(y) = \frac{1}{2} 1_{[-1,1]}(y).
\]

Like the hat map, it is exact with respect to the normalized Lebesgue measure on \([-1,1]\). It has a Frobenius-Perron operator given by

\[
P_T f(y) = \frac{1}{2} \left[ f\left(\frac{1}{2}y - \frac{1}{2}\right) + f\left(\frac{1}{2}y + \frac{1}{2}\right)\right].
\]

Example 7 Alexander and Yorke (1984) defined a generalized baker transformation (also known as a fat/skiny baker transformation) \( S_\beta : [-1,1] \times [-1,1] \to [-1,1] \times [-1,1] \) by

\[
S_\beta(x,y) = (\beta x + (1-\beta)h(y), T(y))
\]

where \( 0 < \beta < 1 \), \( T \) is the dyadic map on \([-1,1]\), and

\[
h(y) = \begin{cases} 
1, & y \geq 0, \\
-1, & y < 0.
\end{cases}
\]

For every \( \beta \in (0,1) \) the transformation \( S_\beta \) has an invariant probability measure on \([-1,1] \times [-1,1]\) and is mixing. The invariant measure is the product of a so called infinitely convolved Bernoulli measure (see Section 3) and the normalized Lebesgue measure on \([-1,1]\). If \( \beta = \frac{1}{2} \), the transformation \( S_\beta \) is conjugated through a linear transform of the plane to the baker map of Example 2. If \( \beta < \frac{1}{2} \), the transformation \( S_\beta \) does not have an invariant density (with respect to the planar Lebesgue measure).

Example 8 The continued fraction map

\[
T(y) = \frac{1}{y} \mod 1 \quad y \in (0,1) \tag{2.11}
\]
has an invariant density
\[ g_\ast(y) = \frac{1}{(1 + y) \ln 2} \quad (2.12) \]
and is exact. The Frobenius-Perron operator is given by
\[ P_T f(y) = \sum_{k=1}^{\infty} \frac{1}{(y + k)^2} f \left( \frac{1}{y + k} \right) \]
and the transfer operator by
\[ P_{T,\nu} f(y) = \sum_{k=1}^{\infty} \frac{y + 1}{(y + k)(y + k + 1)} f \left( \frac{1}{y + k} \right). \]

Example 9 The quadratic map is given by
\[ T_\beta(y) = 1 - \beta y^2, \quad y \in [-1, 1] \]
where \(0 < \beta \leq 2\). It is known that there exists a positive Lebesgue measure set of parameter values \(\beta\) such that the map \(T_\beta\) has an absolutely continuous (with respect to Lebesgue measure) invariant measure \(\nu_\beta\) (Jakobson [1981] and Benedics and Carleson [1985]). Let \(\alpha > 0\) be a very small number and let
\[ \Delta_\epsilon = \{ \beta \in [2 - \epsilon, 2] : |T_\beta^n(0)| \geq e^{-\alpha n} \text{ and } |(T_\beta^n)'(T_\beta(0))| \geq (1.9)^n \forall n \geq 0 \} \]
for \(\epsilon > 0\). Young [1992] proved that for sufficiently small \(\epsilon\) and for every \(\beta \in \Delta_\epsilon\) the transformation \(T_\beta\) is exact with respect to \(\nu_\beta\) and this measure is supported on \([T_\beta^2(0), T_\beta(0)]\).

Example 10 The Manneville-Pomeau map \(T_\beta : [0, 1] \to [0, 1]\) is given by
\[ T_\beta(y) = y + y^{1+\beta} \mod 1, \]
where \(\beta \in (0, 1)\). The map has an absolutely continuous invariant probability measure \(\nu_\beta\) with density satisfying
\[ \frac{c_1}{y^\beta} \leq g_\ast(y) \leq \frac{c_2}{y^\beta} \]
for some constants \(c_2 \geq c_1 > 0\) (cf. Thaler [1980]), and is exact.

Finally, we discuss the notion of Sinai-Ruelle-Bowen measure or SRB measure of \(T\) which was first conceived in the setting of Axiom A diffeomorphisms on compact Riemannian manifolds. This notion varies from author to author (Alexander and Yorke [1984], Eckmann and Ruelle [1985], Tsujii [1995], Young [2002], Hunt et al., 2002). Let \((Y, \rho)\) be a compact metric space with a reference measure \(m\), e.g. a compact subset of \(\mathbb{R}^k\) and \(m\) the Lebesgue or a compact Riemannian manifold and \(m\) the Riemannian measure on \(Y\). If \(T : Y \to Y\) is a continuous map, then by the Bogolyubov-Krylov theorem there always exists at least one invariant probability measure for \(T\). When there is more than one measure, the question arises which invariant measure is “interesting”, and has led to attempts to give a good definition of “physically” relevant invariant measures. Though this seems to be a rather vague and poorly defined concept, loosely speaking one would expect that one criteria for a physically relevant invariant measure would be whether or not it was observable in the context of some laboratory or numerical experiment.
An invariant measure \( \nu \) for \( T \) is called a natural or physical measure if there is a positive Lebesgue measure set \( Y_0 \subset Y \) such that for every \( y \in Y_0 \) and for every continuous observable \( f : Y \to \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(y)) = \int f(z) \nu(dz). \tag{2.13}
\]

In other words the average of \( f \) along the trajectory of \( T \) starting in \( Y_0 \) is equal to the average of \( f \) over the space \( Y \). We can also say that for each \( y \in Y_0 \) the measures \( \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(y)} \) are weakly convergent to \( \nu \) (see the next section for this notation).

Observe that if \( \nu \) is ergodic then from the individual Birkhoff ergodic theorem it follows that for every \( f \in L^1(Y, B, \nu) \) Condition 2.13 holds for almost all \( y \in Y \), i.e. except for a subset of \( Y \) of \( \nu \) measure zero. Thus, if \( T \) has an ergodic absolutely continuous invariant measure \( \nu \) with density \( g \) then every continuous function \( f \) is integrable with respect \( \nu \) and Condition 2.13 holds for almost every point from the set \( \{ y \in Y : g(y) > 0 \} \), i.e. except for a subset of Lebesgue measure zero. Therefore such \( \nu \) is a physical measure for \( T \). Not only absolutely continuous measures are physical measures. Consider, for example, the generalized baker transformation \( S_\beta \) of Example 7.\footnote{Alexander and Yorke (1984) showed that there is a unique physical measure \( \nu_\beta \) for each \( \beta \in (0, 1) \). This measure is mixing and hence ergodic. Although for \( \beta > \frac{1}{2} \) the transformation expands areas, the measure \( \nu_\beta \) might not be absolutely continuous for certain values of the parameter \( \beta \) (e.g. \( \beta = \frac{-1 + \sqrt{5}}{2} \)) in which case the Birkhoff ergodic theorem only implies Condition 2.13 on a zero Lebesgue measure set. Therefore a completely different argument was needed in the proof of the physical property.}

In the context of smooth invertible maps having an Axiom A attractor the existence of a unique physical measure on the attractor was first proved for Anosov diffeomorphisms by \textbf{Sinai} (1972) and later generalized by \textbf{Ruelle} (1976) and \textbf{Bowen} (1975). Roughly speaking, these are maps having uniformly expanding and contracting directions and their physical invariant measures have densities with respect to the Lebesgue measure in the expanding directions (being usually singular in the contracting directions). This property lead then to the characterization of a Sinai-Ruelle-Bowen measure. In a recent attempt to go beyond maps having an Axiom A attractor, \textbf{Young} (2002) additionally requires that \( T \) has a positive Lyapunov exponent a.e. The precise definition strongly relies on the smoothness and invertibility of the map \( T \). Note that the generalized baker transformation \( S_\beta \) has Lyapunov exponents equal to \( \ln 2 \) and \( \ln \beta \) and the measure \( \nu_\beta \) is absolutely continuous along all vertical directions.

### 2.3 Brownian motion from deterministic perturbations

#### 2.3.1 Central limit theorems

We follow the terminology of \textbf{Billingsley} (1968). If \( \zeta \) is a measurable mapping from a probability space \( (\Omega, \mathcal{F}, \Pr) \) into a measurable space \( (Z, \mathcal{A}) \), we call \( \zeta \) a \( Z \)-valued random variable. The distribution of \( \zeta \) is the normalized measure \( \mu = \Pr \circ \zeta^{-1} \) on \((Z, \mathcal{A})\), i.e.

\[
\mu(A) = \Pr(\zeta^{-1}(A)) = \Pr\{\omega : \xi(\omega) \in A\} = \Pr\{\xi \in A\}, \quad A \in \mathcal{A}.
\]
If \( Z = \mathbb{R}^k \), we also have the associated distribution function of \( \zeta \) or \( \mu \), defined by
\[
F(x) = \mu\{y : y \leq x\} = \Pr\{\zeta \leq x\}, \quad x \in \mathbb{R}^k,
\]
where \( \{y : y \leq x\} = \{y : y_i \leq x_i, i = 1, \ldots, k\} \) for \( x = (x_1, \ldots, x_k) \). The random variables \( \zeta \) and \( \xi \) are, by definition, \((\text{statistically})\) independent if
\[
\Pr\{\zeta \in A, \xi \in B\} = \Pr\{\zeta \in A\} \Pr\{\xi \in B\},
\]
i.e. the distribution of the pair \((\zeta, \xi)\) is the product of the distribution of \( \zeta \) with that of \( \xi \).

Let \((Z, \rho)\) be a metric space and \( \mathcal{B}(Z) \) be the \( \sigma \)-algebra of Borel subsets of \( Z \). A sequence \((\mu_n)\) of normalized measures on \((Z, \mathcal{B}(Z))\) is said to converge weakly to a normalized measure \( \mu \) if
\[
\lim_{n \to \infty} \int_Z f(z) \mu_n(dz) = \int_Z f(z) \mu(dz)
\]
for every continuous bounded function \( f : Z \to \mathbb{R} \). Note that the integrals \( \int_Z f(z) \mu(dz) \) completely determine \( \mu \), thus the sequence \((\mu_n)\) cannot converge weakly to two different limits. Note also that weak convergence depends only on the topology of \( Z \), not on the specific metric that generates it; thus two equivalent metrics give rise to the same notion of weak convergence. If we have a family \( \{\mu_{\tau} : \tau \geq 0\} \) of normalized measures instead of a sequence, we can also speak of weak convergence of \( \mu_{\tau} \) to \( \mu \) when \( \tau \) goes to \( \infty \) or some finite value \( \tau_0 \) in a continuous manner. This then means that \( \mu_{\tau} \) converges weakly to \( \mu \) as \( \tau \to \tau_0 \) if and only if \( \mu_{\tau_n} \) converges weakly to \( \mu \) for each sequence \((\tau_n)\) such that \( \tau_n \to \tau_0 \) as \( n \to \infty \).

If \( Z = \mathbb{R}^k \) and \( F \) and \( F_n \) are, respectively, the distribution functions of \( \mu \) and \( \mu_n \), then \((\mu_n)\) converges weakly to \( \mu \) if and only if
\[
\lim_{n \to \infty} F_n(z) = F(z) \quad \text{at continuity points } z \text{ of } F.
\]
The characteristic function \( \varphi_\mu \) of a normalized measure \( \mu \) on \( \mathbb{R}^k \) is defined by
\[
\varphi_\mu(r) = \int \exp \left( i < r, z > \right) \mu(dz),
\]
where \( i = \sqrt{-1} \) and \(< r, z > = \sum_{j=1}^k r_j z_j \) denotes the inner product in \( \mathbb{R}^k \). The continuity theorem \cite{Billingsley1968} gives us the following: \((\mu_n)\) converges weakly to \( \mu \) if and only if
\[
\lim_{n \to \infty} \varphi_{\mu_n}(r) = \varphi_\mu(r) \quad \text{for each } r \in \mathbb{R}^k.
\]
A sequence \( \zeta_n \) of \( Z \)-valued random variables converges \emph{in distribution}, or \emph{weakly}, to a normalized measure \( \mu \) on \((Z, \mathcal{B}(Z))\), if the corresponding distributions of \( \zeta_n \) converge weakly to \( \mu \). This is denoted by
\[
\zeta_n \xrightarrow{d} \mu.
\]
If \( \mu \) is the distribution of a random variable \( \zeta \), we write \( \zeta_n \xrightarrow{d} \zeta \). Note that the underlying probability spaces for the random variables \( \zeta, \zeta_1, \zeta_2, \ldots \) may be all distinct.

A sequence \( \zeta_n \) of \( Z \)-valued random variables converges \emph{in probability} to a \( Z \)-valued random variable \( \zeta \) if
\[
\lim_{n \to \infty} \Pr(\rho(\zeta_n, \zeta) > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.
\]
This is denoted by
\[
\zeta_n \xrightarrow{P} \zeta.
\]
Here all the random variables are defined on the same probability space. Note that if Condition 2.14 holds then
\[
\lim_{n \to \infty} \Pr_0(\rho(\zeta_n, \zeta) > \varepsilon) = 0 \text{ for all } \varepsilon > 0
\]
for every probability measure \( \Pr_0 \) on \( (\Omega, \mathcal{F}) \) which is absolutely continuous with respect to \( \Pr \). In other words convergence in probability is preserved by an absolutely continuous change of measure. We will also frequently use the following result from Billingsley (1968, Theorem 4.1): If \((Z, \rho)\) is a separable metric space, and \( \tilde{\zeta}_n \to^d \zeta \) and \( \rho(\zeta_n, \tilde{\zeta}_n) \to^P 0 \), then \( \zeta_n \to^d \zeta \). \( \tag{2.15} \)

We will write \( N(0, \sigma^2) \) for either a real-valued random variable which is Gaussian distributed with mean 0 and variance \( \sigma^2 \), or the measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) with density
\[
\frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{x^2}{2\sigma^2} \right).
\tag{2.16}
\]
Since \( \sigma N(0, 1) = N(0, \sigma^2) \) when \( \sigma > 0 \), we can always write \( \sigma N(0, 1) \) for \( \sigma \geq 0 \), which in the case \( \sigma = 0 \) reduces to 0. The characteristic function of \( N(0, 1) \) is of the form \( \phi(r) = \exp \left( -\frac{1}{2} r^2 \right) \), \( r \in \mathbb{R} \).

Let \( (\zeta_j)_{j \geq 1} \) be a sequence of real-valued zero-mean random variables with finite variance. If there is \( \sigma > 0 \) such that
\[
\frac{\sum_{j=1}^{n} \zeta_j}{\sqrt{n}} \to^d \sigma N(0, 1), \tag{2.17}
\]
then \( (\zeta_j)_{j \geq 1} \) is said to satisfy the \textbf{central limit theorem (CLT)}. Note that if the random variables \( \zeta_j \) are defined on the same probability space \( (\Omega, \mathcal{F}, \Pr) \), we have the equivalent formulation of (2.17) in the case of \( \sigma > 0 \)
\[
\lim_{n \to \infty} \Pr \left\{ \omega \in \Omega : \frac{\sum_{j=1}^{n} \zeta_j(\omega)}{\sigma \sqrt{n}} \leq z \right\} = \Phi(z), \quad z \in \mathbb{R},
\]
where
\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp \left( -\frac{1}{2} t^2 \right) \, dt \tag{2.18}
\]
is the standard Gaussian distribution function. In the case of \( \sigma = 0 \)
\[
\frac{\sum_{j=1}^{n} \zeta_j}{\sqrt{n}} \to^P 0.
\]

Let \( \{w(t), t \in [0, \infty)\} \) be a standard Wiener process (Brownian motion), i.e. \( \{w(t), t \in [0, \infty)\} \) is a family of real-valued random variables on some probability space \( (\Omega, \mathcal{F}, \Pr) \) satisfying the following properties:

(i) the process starts at zero: \( w(0) = 0 \) a.e;
(ii) for \( 0 \leq t_1 < t_2 \) the random variable \( w(t_2) - w(t_1) \) is Gaussian distributed with mean 0 and variance \( t_2 - t_1 \);
(iii) for times \( t_1 < t_2 < \ldots < t_n \) the increments \( w(t_2) - w(t_1), \ldots, w(t_n) - w(t_{n-1}) \) are independent random variables.
Existence of the Wiener process \( \{ w(t), t \in [0, 1] \} \) is equivalent to the existence of the Wiener measure \( W \) on the space \( C[0, 1] \) of continuous functions on \( [0, 1] \) with uniform convergence, in a topology which makes \( C[0, 1] \) a complete separable metric space. Then, simply, \( W \) is the distribution of a random variable \( W : \Omega \to C[0, 1] \) defined by \( W(\omega) : t \mapsto w(t)(\omega) \).

Let \( D[0, 1] \) be the space of right continuous real valued functions on \( [0, 1] \) with left-hand limits. We endow \( D[0, 1] \) with the Skorohod topology which is defined by the metric

\[
\rho_S(\psi, \tilde{\psi}) = \inf_{s \in S} \left( \sup_{t \in [0,1]} |\psi(t) - \tilde{\psi}(s(t))| + \sup_{t \in [0,1]} |t - s(t)| \right), \quad \psi, \tilde{\psi} \in D[0,1],
\]

where \( S \) is the family of strictly increasing, continuous mappings \( s \) of \( [0, 1] \) onto itself such that \( s(0) = 0 \) and \( s(1) = 1 \) [Billingsley, 1968, Section 14]. The metric space \( (D[0,1], \rho_S) \) is separable and is not complete, but there is an equivalent metric on \( D[0,1] \) which turns \( D[0,1] \) with the Skorohod topology into a complete separable metric space. Since the Skorohod topology and the uniform topology on \( C[0,1] \) coincide, \( W \) can be considered as a measure on \( D[0,1] \).

A stronger result than the CLT is a weak invariance principle, also called a functional central limit theorem (FCLT). Let \( (\zeta_j)_{j \geq 1} \) be a sequence of real-valued zero-mean random variables with finite variance. Let \( \sigma > 0 \) and define the process \( \{ \psi_n(t), t \in [0,1] \} \) by

\[
\psi_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \zeta_j \quad \text{for} \quad t \in [0,1],
\]

(where the sum from 1 to 0 is set equal to 0). Note that \( \psi_n \) is a right continuous step function, a random variable of \( D[0,1] \) and \( \psi_n(0) = 0 \). If

\[
\psi_n \xrightarrow{d} w
\]

(here the convergence in distribution is in \( D[0,1] \)), then \( (\zeta_j)_{j \geq 1} \) is said to satisfy the FCLT.

If for every \( k \geq 1 \) and every vector \( (t_1, \ldots, t_k) \in [0,1]^k \) with \( t_1 < \ldots < t_k \) the joint distribution of the vector \( (\psi_n(t_1), \ldots, \psi_n(t_k)) \) converges to the joint distribution of \( (w(t_1), \ldots, w(t_k)) \), then we say that the finite dimensional distributions of \( \psi_n \) converge to those of \( w \). For one dimensional distribution this convergence is equivalent to the central limit theorem.

The convergence of all finite-dimensional distributions of \( \psi_n \) to those of \( w \) is not sufficient to conclude that \( \psi_n \xrightarrow{d} w \) in \( D[0,1] \). According to Theorems 15.1 and 15.5 of Billingsley [1968], if, additionally, for each positive \( \epsilon \)

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{|t-s| \leq \delta} \Pr \left( \sup_{|t-s| \leq \delta} |\psi_n(s) - \psi_n(t)| > \epsilon \right) = 0,
\]

then \( \psi_n \) converges in distribution to the Wiener process \( w \).

The term functional central limit theorem comes from the mapping theorem [Billingsley, 1968, Theorem 5.1], according to which for any functional \( f : D[0,1] \to \mathbb{R} \), measurable and continuous on a set of Wiener measure 1, the distribution of \( f(\psi_n) \) converges weakly to the distribution of \( f(w) \). This applies in particular to the functional \( f(\psi) = \sup_{0 \leq s \leq 1} \psi(s) \). Instead of real-valued functionals one can also consider mappings with values in a metric space. For example, this theorem applies for any \( f : D[0,1] \to D[0,1] \) of the form \( f(\phi)(t) = \sup_{s \leq t} \phi(s) \) or \( f(\phi)(t) = \int_0^t \phi(s)ds \).

19
2.3.2 FCLT for noninvertible maps

How can we obtain, and in what sense, Brownian-like motion from a (semi) dynamical system? This question is intimately connected with Central Limit Theorems (CLT) for non-invertible systems and various invariance principles.

Many CLT results and invariance principles for maps have been proved, see e.g. the survey by Denker (1983). These results extend back over some decades, including contributions by Ratner (1973), Bovarsky and Scarowsky (1979), Wong (1979), Keller (1980), Jabłoński and Malczak (1983a), Jabłoński (1991), Liverani (1996) and Viana (1997).

First, however, remember that if we have a time series \( y(j) \) and a bounded integrable function \( h : X \to R \), then the correlation of \( h \) is defined as

\[
R_h(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} h(y(j+n))h(y(j)).
\]

If the time series is generated by a measurable transformation \( T : Y \to Y \) operating on a normalized measure space \( (Y, B, \nu) \), and if further \( \nu \) is invariant under \( T \) and \( T \) is ergodic, then we can rewrite the correlation as

\[
R_h(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} h(y(j+n))h(y(j)) = \int h(y)h(T^n(y))\nu(dy).
\]

The average \( \langle h \rangle \) is just

\[
\langle h \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} h(y(j)) = \int h(y)\nu(dy).
\]

Let \( (Y, B, \nu) \) be a normalized measure space, and \( T : Y \to Y \) be a measurable transformation such that \( \nu \) is \( T \)-invariant. \( (Y, B, \nu) \) will serve as our probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Let \( h \in L^2(Y, B, \nu) \) be such that \( \int h(y)\nu(dy) = 0 \). The random variables \( \zeta_j = h \circ T^j \), \( j \geq 1 \) are real-valued, have zero-mean and finite variance equal to \( ||h||^2_2 = \int h^2(y)\nu(dy) \). Thus the terminology from Section 2.3.1 applies. The explicit formulae for the Frobenius-Perron operators in Section 2.2 show that the equations \( P_T h = 0 \) or \( P_{T,\nu} h = 0 \) can be easily solved. For instance, in Example 3 every function \( h \) which is odd is a solution of these equations. In particular, considering \( h \) with \( P_{T,\nu} h = \text{turns out to be very fruitful. Statistical properties of the sequence} \ (h \circ T^j)_{j \geq 0} \text{are summarized in the following}

**Theorem 2 (CLT) Let} \ (Y, B, \nu) \text{be a normalized measure space and} \ T : Y \to Y \text{be ergodic with respect to} \ \nu. \text{If} \ h \in L^2(Y, B, \nu) \ \text{is such that} \ P_{T,\nu} h = 0, \text{then}

(i) \ \int h(y)\nu(dy) = 0 \text{ and} \ \int h(y)h(T^n(y))\nu(dy) = 0 \text{ for all} \ n \geq 1.

(ii) \ \frac{\sum_{j=0}^{n-1} h \circ T^j}{\sqrt{n}} \to^d \sigma N(0,1) \text{ and} \ \sigma = ||h||_2.

(iii) \text{If} \ \sigma > 0 \text{ then} \ (h \circ T^j)_{j \geq 0} \text{satisfies the CLT and FCLT.}

(iv) \text{If} \ h \in L^\infty(Y, B, \nu) \text{ and} \ \sigma > 0 \text{ then all moments of} \ \frac{\sum_{j=0}^{n-1} h \circ T^j}{\sqrt{n}} \text{converge to the corresponding moments of} \ \sigma N(0,1), \text{i.e. for each} \ k \geq 1 \text{ we have}

\[
\lim_{n \to \infty} \int \left( \frac{\sum_{j=0}^{n-1} h(T^j(y))}{\sqrt{n}} \right)^{2k} \nu(dy) = \frac{(2k)!\sigma^k}{k!2^k}.
\]
\[
\lim_{n \to \infty} \int \left( \frac{\sum_{j=0}^{n-1} h(T^j(y))}{\sqrt{n}} \right)^{2k-1} \nu(dy) = 0.
\]

**Proof.** First note that the transfer operator \( P_{T, \nu} \) preserves the integral, i.e \( \int P_{T, \nu} h(y) \nu(dy) = \int h(y) \nu(dy) \). Hence \( \int h(y) \nu(dy) = 0 \). Now let \( n \geq 1 \). Since \( \nu \) is a finite measure, the Koopman and transfer operators are adjoint on the space \( L^2(Y, \mathcal{B}, \nu) \). This implies

\[
\int h(y) h(T^n(y)) \nu(dy) = \int h(y) U^n h(y) \nu(dy) = \int P_{T, \nu} h(y) U^{n-1} h(y) \nu(dy) = 0
\]

and completes the proof of (i). Part (ii) follows from Lemma 1 and Theorem 12 in Appendix, since for each \( n \geq 1 \) we have

\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h \circ T^j = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} h \circ T^{n-j}.
\]

If \( \sigma > 0 \), then the CLT is a consequence of part (ii), while the FCLT follows from Lemma 2 and 3 in Appendix. Finally, the existence and convergence of moments follow from Theorem 5.3 and 5.4 of Billingsley (1968) and from Lemma 1 in Appendix.

**Remark 1** Note that if \( T \) is invertible then the equation \( P_{T, \nu}(h) = 0 \) has only a zero solution, so the theorem does not apply.

We now address the question of solvability of the equation \( P_{T, \nu}(h) = 0 \).

**Proposition 1** Let \( (Y, \mathcal{B}, \nu) \) be a normalized measure space and \( T : Y \to Y \) be a measurable map preserving the measure \( \nu \). Let \( Y = Y_1 \cup Y_2 \) with \( Y_1, Y_2 \in \mathcal{B} \) and \( \nu(Y_1 \cap Y_2) = 0 \) and let a bijective map \( \varphi : Y_1 \to Y_2 \) be such that both \( \varphi \) and \( \varphi^{-1} \) are measurable and preserve the measure \( \nu \). Assume that for every \( A \in \mathcal{B} \) there is \( B \in \mathcal{B} \) such that \( B \subseteq Y_1 \) and

\[
T^{-1}(A) = B \cup \varphi(B).
\]

If \( h(y) + h(\varphi(y)) = 0 \) for almost every \( y \in Y_1 \), then \( P_{T, \nu} h = 0 \).

**Proof.** From Condition 2.20 it follows that

\[
\int_{T^{-1}(A)} h(y) \nu(dy) = \int_{B} h(y) \nu(dy) + \int_{\varphi(B)} h(y) \nu(dy).
\]

Since

\[
\int_{\varphi(B)} h(y) \nu(dy) = \int_{\varphi(B)} h(\varphi^{-1}(\varphi(y))) \nu(dy),
\]

the last integral is equal to

\[
\int_{B} h(\varphi(y)) (\nu \circ \varphi)(dy) = \int_{B} h(\varphi(y)) \nu(dy)
\]

by the change of variables applied to \( \varphi^{-1} \) and finally by the invariance of \( \nu \) for \( \varphi \). This, with the definition of \( P_{T, \nu} \), completes the proof.
Remark 2 The above proposition can be easily generalized. For example, we can have

\[ T^{-1}(A) = B \cup \varphi_1(B) \cup \varphi_2(B) \]

with the sets \( B, \varphi_1(B), \varphi_2(B) \) pairwise disjoint.

Note that if \( Y \) is an interval then it is enough to check Condition [2.20] for intervals of the form \([a, b)\).

Example 11 For an even transformation \( T \) on \([-1, 1]\) with an even invariant density we can take \( Y_1 = [-1, 0] \) and \( \varphi(y) = -y \). In this case \( P_{T, \nu}h = 0 \) for every odd function on \([-1, 1]\). In particular, this applies to the tent map and to the Chebyshev maps \( S_N \) of Example 5 with \( N \) even. We also have \( P_{S_N, \nu}h = 0 \) when \( h(y) = y \) and \( S_N \) is the Chebyshev map with \( N \) odd. Indeed, first observe that by Theorem 4 we have \( P_{S_N, \nu}h = 0 \) if \( T_Nf = 0 \) where \( T_N \) is given by \( 2.6 \) and \( f(y) = \cos(\pi y) \) for \( y \in [0, 1] \) and then note that \( P_{T, \nu}h = 0 \) follows because the expression

\[ f(y) + \sum_{n=1}^{(N-1)/2} \left( f\left( \frac{2n}{N} + y \right) + f\left( \frac{2n}{N} - y \right) \right) \]

reduces to

\[ \cos(\pi y)(1 + 2 \sum_{n=1}^{(N-1)/2} \cos\left( \frac{2n\pi}{N} \right) ) \]

which is equal to 0. For the dyadic map, we can take \( \varphi(y) = y + 1 \). Then any function satisfying \( h(y) + h(y + 1) = 0 \) gives a solution to \( P_{T, \nu}h = 0 \).

The next example shows that the assumption of ergodicity in Theorem 2 is in a sense essential.

Example 12 Let \( T : [0, 1] \to [0, 1] \) be defined by

\[ T(y) = \begin{cases} 2y, & y \in [0, \frac{1}{2}) \\ 2y - \frac{1}{2}, & y \in [\frac{1}{2}, \frac{3}{4}) \\ 2y, & y \in [\frac{3}{4}, 1] \\ -1, & y \in [\frac{1}{2}, \frac{3}{4}) \\ 1, & y \in [\frac{3}{4}, 1] \end{cases} \]

The Frobenius-Perron operator is given by

\[ P_Tf(y) = \frac{1}{2} f\left( \frac{1}{2} y \right) 1_{[0, \frac{1}{2})}(y) + \frac{1}{2} f\left( \frac{1}{2} y + \frac{1}{4} \right) 1_{[\frac{1}{2}, \frac{3}{4})}(y) + \frac{1}{2} f\left( \frac{1}{2} y + \frac{1}{2} \right) 1_{[\frac{3}{4}, 1]}(y) \]

Observe that the Lebesgue measure on \((0, 1], B([0, 1]))\) is invariant for \( T \) and that \( T \) is not ergodic since \( T^{-1}([0, \frac{1}{2})] = [0, \frac{1}{2}) \) and \( T^{-1}([\frac{3}{4}, 1]) = [\frac{1}{2}, 1] \). Consider the following functions

\[
\begin{align*}
    h_1(y) &= \begin{cases} 
        1, & y \in [0, \frac{1}{2}) \\
        -1, & y \in [\frac{1}{2}, \frac{3}{4}), \\
        1, & y \in [\frac{3}{4}, 1] 
    \end{cases} \\
    h_2(y) &= \begin{cases} 
        1, & y \in [0, \frac{1}{2}) \\
        -1, & y \in [\frac{1}{2}, \frac{3}{4}), \\
        -2, & y \in [\frac{3}{4}, \frac{5}{4}) \\
        2, & y \in [\frac{5}{4}, 1] 
    \end{cases}
\end{align*}
\]

We see at once that \( P_T h_1 = P_T h_2 = 0 \), \( P_T h_1^2 = h_1^2 \), and \( P_T h_2^2 = h_2^2 \). It is immediate that for every \( y \in [0, 1] \) we have \( h_1^2(T(y)) = h_1^2(y) = 1 \) and \( h_2^2(T(y)) = h_2^2(y) \). Lemma 4 and Theorem 13 in Appendix show that

\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h_1 \circ T^j \to^d N(0, 1),
\]

22
while
\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h_2 \circ T^j \to^d \zeta,
\]
where \(\zeta\) has the characteristic function of the form
\[
\phi_\zeta(r) = \int_0^1 \exp \left( -\frac{r^2}{2} h_2^2(y) \right) dy.
\]
The density of \(\zeta\) is equal to
\[
\frac{1}{2} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right) + \frac{1}{2} \frac{1}{\sqrt{8\pi}} \exp\left( -\frac{x^2}{8} \right), \quad x \in \mathbb{R}.
\]
Consequently, the sequence \((h_2 \circ T^j)_{j \geq 0}\) does not satisfy the CLT.

We now discuss the problem of changing the underlying probability space for the sequence \((h \circ T^j)_{j \geq 0}\). The random variables \(h \circ T^j\) in Theorem 2 are defined on the probability space \((Y, \mathcal{B}, \nu)\). Since \(\nu\) is invariant for \(T\), they have the same distribution and constitute a stationary sequence. We shall show that the result (iii) of Theorem 2 remains true if the transformation \(T\) is exact and \(h \circ T^j\) are random variables on \((Y, \mathcal{B}, \nu_0)\) where \(\nu_0\) is an arbitrary normalized measure absolutely continuous with respect to \(\nu\). In other words, we can consider random variables \(h(T^j(\xi_0))\) with \(\xi_0\) distributed according to \(\nu_0\). Now these random variables are not identically distributed and constitute a non-stationary sequence. For example, consider the hat map \(T(y) = 1 - 2|y|\) on \([-1,1]\) and \(h(y) = y\). Then Theorem 2 applies if \(\xi_0\) is uniformly distributed. We will show that we can also consider \(\xi_0\) having a density with respect to the normalized Lebesgue measure on \([-1,1]\).

**Theorem 3** Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space and \(T : Y \to Y\) be exact with respect to \(\nu\). Let \(h \in L^2(Y, \mathcal{B}, \nu)\) be such that \(\mathcal{P}_{T,\nu} h = 0\) and \(\sigma = ||h||_2 > 0\). If \(\xi_0\) is distributed according to a normalized measure \(\nu_0\) on \((Y, \mathcal{B})\) which is absolutely continuous with respect to \(\nu\), then \((h \circ T^j(\xi_0))_{j \geq 0}\) satisfies both the CLT and FCLT.

**Proof.** Let \(g_0\) be the density of the measure \(\nu_0\) with respect to \(\nu\). On the probability space \((Y, \mathcal{B})\) define the random variables \(\zeta_n\) by
\[
\zeta_n(y) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{n-1} h(T^j(y)), \quad y \in Y.
\]
To prove the CLT we shall use the continuity theorem (Billingsley, 1968, Theorem 7.6), and to do so we must show that
\[
\lim_{n \to \infty} \int \exp(ir\zeta_n(y))g_0(y)\nu(dy) = \exp\left( -\frac{r^2}{2} \right), \quad r \in \mathbb{R}.
\]
Fix \(\epsilon > 0\). Since \(T\) is exact, there exists \(m \geq 1\) such that
\[
\int |\mathcal{P}_{T,\nu}^m g_0(y) - 1|\nu(dy) \leq \epsilon.
\]
Define
\[
\tilde{\zeta}_n(y) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{n-m-1} h(T^j(y)), \quad n > m
\]
and observe that
\[ \zeta_n = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{m-1} h \circ T^j + \tilde{\zeta}_n \circ T^m \]
for sufficiently large \( n \). Since for every \( y \) the sequence \( \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{m-1} h(T^j(y)) \) converges to 0 as \( n \to \infty \), we obtain
\[ \lim_{n \to \infty} \left| \int \exp(ir\zeta_n(y))g_0(y)\nu(dy) - \int \exp(ir\tilde{\zeta}_n(T^m(y)))g_0(y)\nu(dy) \right| = 0. \]
The equality \( \exp(ir\tilde{\zeta}_n \circ T^m) = U_T^m(\exp(ir\tilde{\zeta}_n)) \) implies that
\[ \int \exp(ir\tilde{\zeta}_n(T^m(y)))g_0(y)\nu(dy) = \int \exp(ir\tilde{\zeta}_n(y))P_{T,\nu}^m g_0(y)\nu(dy), \]
since the operators \( U_T \) and \( P_{T,\nu} \) are adjoint. This gives
\[ \left| \int \exp(ir\tilde{\zeta}_n(T^m(y)))g_0(y)\nu(dy) - \int \exp(ir\tilde{\zeta}_n(y))\nu(dy) \right| \leq \int |P_{T,\nu}^m g_0(y) - 1|\nu(dy) \leq \epsilon. \]
From Theorem 2 and the continuity theorem (Billingsley 1968, Theorem 7.6), it follows that
\[ \lim_{n \to \infty} \int \exp(ir\tilde{\zeta}_n(y))\nu(dy) = \exp\left(-\frac{r^2}{2}\right). \]
Consequently,
\[ \limsup_{n \to \infty} \left| \int \exp(ir\zeta_n(y))g_0(y)\nu(dy) - \exp\left(-\frac{r^2}{2}\right) \right| \leq \epsilon, \]
which leads to the desired conclusion, as \( \epsilon \) was arbitrary.

Similar arguments as above, the multidimensional version of the continuity theorem, and Lemma 2 in Appendix allow us to show that the finite dimensional distributions of
\[ \psi_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]-1} h(T^j(\xi_0)) \]
converge to the finite dimensional distributions of the Wiener process \( w \). By Lemma 8 in Appendix Condition 2.19 holds with \( \Pr = \nu \). Since \( \nu_0 \) is absolutely continuous with respect to \( \nu \), it is easily seen that this Condition also holds with \( \Pr = \nu_0 \), which completes the proof.

Does the CLT still hold when \( h \) does not satisfy the equation \( P_{T,\nu}h = 0 \)? The answer to this question is positive provided that \( h \) can be written as a sum of two functions in which one satisfies the assumptions of Theorem 2 while the other is irrelevant for the Central Limit Theorem to hold. This idea goes back to Gordin (1963).

**Theorem 4** Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \( T : Y \to Y \) be ergodic with respect to \( \nu \), and \( h \in L^2(Y, \mathcal{B}, \nu) \) be such that \( \int h(y)\nu(dy) = 0 \). If there exists \( \tilde{h} \in L^2(Y, \mathcal{B}, \nu) \) such that \( P_{T,\nu}\tilde{h} = 0 \) and the sequence \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \) is convergent in \( L^2(Y, \mathcal{B}, \nu) \) to 0, then
\[ \lim_{n \to \infty} \left| \frac{\sum_{j=0}^{n-1} h \circ T^j}{n} \right|^2 = \left| \tilde{h} \right|^2 \] (2.21)
Theorem 5

Let sequence is convergent in $L$, i.e. $||\nu||_2 \rightarrow^d ||\tilde{h}||_2 N(0, 1)$.

Moreover, if the series $\sum_{j=1}^{\infty} \int h(y)h(T^j(y))\nu(dy)$ is convergent, then

$$||\tilde{h}||_2^2 = \int h^2(y)\nu(dy) + 2 \sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy).$$

(2.22)

Proof. We have

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h \circ T^j \rightarrow^d ||\tilde{h}||_2 N(0, 1).$$

and therefore Equation (2.21) holds. Since the sequence $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j$ is convergent to 0 in $L^2(Y, B, \nu)$, it is also convergent to 0 in probability. Equation (2.23) of Theorem 5 applied to $\tilde{h}$ and property (2.15) complete the proof of the first part.

It remains to show that

$$\lim_{n \rightarrow \infty} \frac{||\sum_{j=0}^{n-1} h \circ T^j||_2^2}{n} = \int h^2(y)\nu(dy) + 2 \sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy).$$

Since $\nu$ is $T$-invariant, we have

$$\frac{1}{n} \int \left( \sum_{j=0}^{n-1} h(T^j(y)) \right)^2 \nu(dy) = ||h||_2^2 + 2 \sum_{j=1}^{n-1} \int h(y)h(T^j(y))\nu(dy),$$

but the sequence $(\sum_{j=1}^{n} \int h(y)h(T^j(y))\nu(dy))_{n \geq 1}$ is convergent to $\sum_{j=1}^{\infty} \int h(y)h(T^j(y))\nu(dy)$, which completes the proof.

Remark 3 Note that in the above proof of the CLT we only used the weaker condition that the sequence $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j$ is convergent to 0 in probability. The stronger assumption that this sequence is convergent in $L^2(Y, B, \nu)$ was used to derive Equation (2.21). Note also that all of the computations are useless when $||\tilde{h}||_2 = 0$ and the most interesting situation is when $\tilde{h}$ is nontrivial, i.e. $||\tilde{h}||_2 > 0$.

Strengthening the assumptions of the last theorem leads to the functional central limit theorem.

Theorem 5 Let $(Y, B, \nu)$ be a normalized measure space, $T : Y \rightarrow Y$ be ergodic with respect to $\nu$, and $h \in L^2(Y, B, \nu)$ be such that $\int h(y)\nu(dy) = 0$. If there exists a nontrivial $\tilde{h} \in L^2(Y, B, \nu)$ such that $P_{T} \tilde{h} = 0$ and the sequence $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j$ is $\nu$-a.e. convergent to 0, then $(h \circ T^j)_{j \geq 0}$ satisfies the FCLT.
Proof. Since every sequence convergent \( \nu \) almost everywhere is convergent in probability, the CLT follows by the preceding Remark and Theorem 4. To derive the FCLT define

\[
\tilde{\psi}_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]-1} \tilde{h} \circ T^j \quad \text{and} \quad \psi_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j, \quad t \in [0, 1],
\]

where \( \sigma = ||\tilde{h}||_2 \). Then by (iii) of Theorem 2 we have

\[
\tilde{\psi}_n \to^d \nu.
\]

By property (2.15) it remains to show that

\[
\rho_S(\psi_n, \tilde{\psi}_n) \to^p 0.
\]

To this end observe that

\[
\rho_S(\psi_n, \tilde{\psi}_n) \leq \sup_{0 \leq t \leq 1} |\psi_n(t) - \tilde{\psi}_n(t)| \leq \frac{1}{\sigma \sqrt{n}} \max_{1 \leq k \leq n} \sum_{j=0}^{k-1} (h - \tilde{h}) \circ T^j.
\]

Since the sequence \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \) is \( \nu \)-a.e. convergent to 0, the same holds for the sequence \( \frac{1}{\sigma \sqrt{n}} \max_{1 \leq k \leq n} |\sum_{j=0}^{k-1} (h - \tilde{h}) \circ T^j| \) by an elementary analysis, which completes the proof.

Remark 4 With the settings and notation of Theorem 4 and respectively Theorem 5, if \( T \) is exact, then the same conclusions hold for the sequence \( (h \circ T^j(\xi_0))_{j \geq 0} \) provided that \( \xi_0 \) is distributed according to a normalized measure \( \nu_0 \) on \((Y, \nu)\) which is absolutely continuous with respect to \( \nu \). Indeed, since convergence to zero in probability is preserved by an absolutely continuous change of measure, we can apply the above arguments again, with Theorem 2 replaced by Theorem 3.

One situation when all assumptions of the two preceding theorems are met is described in the following

Theorem 6 Let \((Y, B, \nu)\) be a normalized measure space, \( T : Y \to Y \) be ergodic with respect to \( \nu \), and \( h \in L^2(Y, B, \nu) \). Suppose that the series

\[
\sum_{n=0}^{\infty} \mathcal{P}_{T, \nu}^n h
\]

is convergent in \( L^2(Y, B, \nu) \). Define \( f = \sum_{n=1}^{\infty} \mathcal{P}_{T, \nu}^n h \) and \( \tilde{h} = h + f - f \circ T \).

Then \( \tilde{h} \in L^2(Y, B, \nu) \), \( \mathcal{P}_{T, \nu}\tilde{h} = 0 \), \( \int h(y) \nu(dy) = 0 \), and the sequence \( \left( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \right)_{n \geq 1} \) is convergent to 0 both in \( L^2(Y, B, \nu) \) and \( \nu \)-a.e.

In particular, \( ||\tilde{h}||_2 = 0 \) if and only if \( h = f \circ T - f \) for some \( f \in L^2(Y, B, \nu) \).

Proof. Since \( \mathcal{P}_{T, \nu}(h + f) = f \), we have by Equation 2.3

\[
\mathcal{P}_{T, \nu}\tilde{h} = \mathcal{P}_{T, \nu}(h + f) - \mathcal{P}_{T, \nu}(f \circ T) = f - \mathcal{P}_{T, \nu}U_T f = 0.
\]

Thus it remains to study the behavior of the sequence \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \), which with our notations reduces to \( \frac{1}{\sqrt{n}}(f \circ T^n - f) \). This sequence is obviously convergent to 0 in \( L^2(Y, B, \nu) \) because \( ||f \circ T^n - f||_2 \leq 2||f||_2 \). It is also \( \nu \)-a.e. convergent to 0 which follows from the Borel-Cantelli lemma and the fact that for every \( \epsilon > 0 \) the series \( \sum_{n=1}^{\infty} \nu(f^2 \circ T^n \geq n \epsilon) = \sum_{n=1}^{\infty} \nu(f^2 \geq n \epsilon) \) is convergent as \( f \in L^2(Y, B, \nu) \), which completes the proof.

Summarizing our considerations for general \( h \) we arrive at the following sufficient conditions for the CLT and FCLT to hold.
**Corollary 1** Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \(T : Y \to Y\) be ergodic with respect to \(\nu\), and \(h \in L^2(Y, \mathcal{B}, \nu)\). If
\[
\sum_{n=0}^{\infty} ||P_{n,T,\nu}^n h||_2 < \infty,
\]
then \(\sigma \geq 0\) given by
\[
\sigma^2 = \int h^2(y)\nu(dy) + 2\sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy)
\]
is finite and \((h \circ T^j)_{j\geq 0}\) satisfies the CLT and FCLT provided that \(\sigma > 0\).

**Proof.** Since the operators \(P_{n,T,\nu}\) and \(U_T\) are adjoint on the space \(L^2(Y, \mathcal{B}, \nu)\), we have
\[
\int h(y)h(T^n(y))\nu(dy) = \int P_{n,T,\nu}^n h(y)h(y)\nu(dy).
\]
Thus
\[
\left| \int h(y)h(T^n(y))\nu(dy) \right| \leq \int |P_{n,T,\nu}^n h(y)h(y)|\nu(dy) \leq ||P_{n,T,\nu}^n h||_2 ||h||_2
\]
by Schwartz’s inequality. Hence
\[
\sum_{n=1}^{\infty} \int |h(y)h(T^n(y))|d\nu \leq ||h||_2 \sum_{n=1}^{\infty} ||P_{n,T,\nu}^n h||_2,
\]
which shows that the series \(\sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy)\) is convergent. Since assumption 2.24 implies that the series \(\sum_{n=0}^{\infty} P_{n,T,\nu}^n h\) is absolutely convergent in \(L^2(Y, \mathcal{B}, \nu)\), the assertions follow from Theorems 4, 5, and 6.

**Remark 5** Note that if Condition 2.24 holds then
\[
\lim_{n \to \infty} ||P_{n,T,\nu}^n h||_2 = 0.
\]
Since \(\nu\) is finite, we have
\[
\lim_{n \to \infty} ||P_{n,T,\nu}^n h||_1 = 0.
\]
Therefore the validity of Condition 2.24 on a dense subset of \(\{h \in L^1(Y, \mathcal{B}, \nu) : \int h(y)\nu(dy) = 0\}\) implies that \(T\) is exact.

Assume that \(Y\) is an interval \([a, b]\) in \(\mathbf{R}\) for some \(a, b\). Recall that a function \(h : [a, b] \to \mathbf{R}\) is said to be of bounded variation if
\[
\sqrt[b]{a} \int h = \sup \sum_{i=1}^{n} |h(y_{i-1}) - h(y_i)| < \infty,
\]
where the supremum is taken over all finite partitions, \(a = y_0 < y_1 < \ldots < y_n = b, n \geq 1, \text{ of } Y\).

Let \(V([0, 1])\) denote the space of all integrable functions with bounded variation over \([0, 1]\) such that \(\int_0^1 h(y)dy = 0\). We have
\[
|h(y)| \leq \sqrt[1]{0 \int h} \text{ for } h \in V([0, 1]), y \in [0, 1].
\]

\[
(2.25)
\]
Example 13 For the continued fraction map there exists a positive constant \( c < 1 \) such that for every function \( h \) of bounded variation over \([0, 1]\) we have (Iosifescu, 1992, Corollary, p. 904)

\[
\int_0^1 P_{T,\nu}^n h \leq c^n \int_0^1 h \quad \text{for all } n \geq 1,
\]

where \( \nu \) is Gauss’s measure with density \( g_* \) as in Example 8. From this and Condition 2.25 it follows that for every \( h \in V([0, 1]) \)

\[
\|P_{T,\nu}^n h\|_2 \leq \sup_{y \in [0, 1]} |P_{T,\nu}^n h(y)| \leq c^n \int_0^1 h.
\]

Consequently, Condition 2.24 is satisfied and Corollary 1 applies.

By definition, the Frobenius-Perron operator is a linear operator from \( L^1([0, 1]) \) to \( L^1([0, 1]) \), but for sufficiently smooth piecewise monotonic maps it can be defined as a pointwise map of \( V([0, 1]) \) into \( V([0, 1]) \). Since functions of bounded variation have only countably many points of discontinuity, redefining \( P_T \) at those points does not change its \( L^1 \) properties. If, moreover, one is able to give an estimate for the iterates of \( P_T \) in the bounded variation norm

\[
\|P_T^n h\|_{BV} = \int_0^1 h + \int_0^1 |h(y)| dy,
\]

then obviously one is able to estimate the norm of \( P_T^n f \) in all \( L^p([0, 1]) \) spaces. In many cases there exist \( c_1, c_2 > 0 \) and \( r \in (0, 1) \) such that

\[
\|P_T^n h\|_{BV} \leq c_1 r^n (\int_0^1 h + c_2 \|h\|_1), \quad h \in V([0, 1]).
\] (2.26)

We now describe two classes of chaotic maps for which one can easily show that Condition 2.24 holds for every \( h \in V([0, 1]) \). Consider a transformation \( T : [0, 1] \to [0, 1] \) having the following properties

(i) there is a partition \( 0 = a_0 < a_1 < \ldots < a_l = 1 \) of \([0, 1]\) such that for each integer \( i = 1, \ldots, l \) the restriction of \( T \) to \( [a_{i-1}, a_i) \) is continuous and convex,

(ii) \( T(a_{i-1}) = 0 \) and \( T'(a_{i-1}) > 0 \),

(iii) \( T'(0) > 1 \).

For such transformation the Frobenius-Perron operator has a unique fixed point \( g_* \), where \( g_* \) is of bounded variation and a decreasing function of \( y \) (Lasota and Mackey, 1994). Moreover it is bounded from below when, for example, \( T([0, a_1]) = [0, 1] \). It is known (Jabłoński and Malecak, 1983) that the estimate in Equation 2.26 holds for the Frobenius-Perron operator \( P_T \) for transformations with these three properties. Suppose that \( g_*(y) > 0 \) for a.a. \( y \in [0, 1] \). Since \( \|P_T\|_\infty \leq \|f\|_\infty \) for all \( f \in L^\infty([0, 1], \mathcal{B}, \nu) \), we have for all \( h \in f \in L^\infty([0, 1], \mathcal{B}, \nu) \)

\[
\|P_T^n h\|_2 \leq \|h\|_\infty^{1/2} ||P_T^n h||_1^{1/2}
\]

(2.27)
If $h$ is of bounded variation with $\int h(y)g_*(y)dy = 0$, then $hg_* \in V([0, 1])$ and $\|P^n_{T,\nu}h\|_1 = \|P^n_T(hg_*)\|_1$. Thus
\[
\|P^n_{T,\nu}h\|_2 = O(n^{1/2})
\]
by Equation 2.26 and Corollary 1 applies.

Let a transformation $T : [0, 1] \rightarrow [0, 1]$ be piecewise monotonic, the function $\frac{1}{|T'(y)|}$ be of bounded variation over $[0, 1]$ and $\inf_{y \in [0, 1]} |T'(y)| > 1$. For such transformations the Frobenius-Perron operator has a fixed point $g_*$ and $g_*$ is of bounded variation (Lasota and Mackey, 1994). Suppose that $P_T$ has a unique invariant density $g_*$ which is strictly positive. Then the transformation $T$ is ergodic and $g_*$ is bounded from below. There exists $k$ such that $T^k$ is exact, the estimate in Equation 2.26 is valid for the Frobenius-Perron operator $P_{T^k}$ corresponding to $T^k$, and the following holds (Jabłoński et al., 1985): There exists $c_1 > 0$ and $r \in (0, 1)$ such that
\[
|P^n_{T^k}f(y)| \leq c_1r^n \left( \int_0^1 f + \int_0^1 |f(y)|dy \right), \quad y \in [0, 1], \ f \in V([0, 1]);
\]
Hence Corollary 1 applies to $T^k$ and every $h$ of bounded variation with $\int h(y)g_*(y)dy = 0$. One can relax the assumption that $g_*$ is strictly positive and have instead $g_* \geq c$ for a.e. $y \in Y_* = \{y \in [0, 1] : g_*(y) > 0\}$. Then the above estimate is valid for $y \in Y_*$ and $f$ with $\text{supp} f = \{y \in [0, 1] : f(y) \neq 0\} \subset Y_*$ and Corollary 1 still applies to $T^k$.

We now describe how to obtain the conclusions of Corollary 1 for conjugated maps. If the Lebesgue measure on $[0, 1]$ is invariant with respect to $T$, then Theorem 1 offers the following.

**Corollary 2** Let $T : [0, 1] \rightarrow [0, 1]$ be a transformation for which the Lebesgue measure on $[0, 1]$ is invariant and for which Equation 2.26 holds. Let $g_*$ be a positive function and $S : [a, b] \rightarrow [a, b]$ be given by $S = G^{-1} \circ T \circ G$, where
\[
G(x) = \int_a^x g_*(y)dy, \quad a \leq x \leq b.
\]
If $h : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation with $\int h(y)g_*(y)dy = 0$, then
\[
\sum_{n=0}^{\infty} \|P^n_{S,\nu}h\|_2 < \infty,
\]
where $\| \cdot \|_2$ denotes the norm in $L^2([a, b], \mathcal{B}([a, b]), \nu)$ and $\nu$ is the measure with density $g_*$. 

**Proof.** By Theorem 1 we have
\[
P_{S,\nu}f = U_GP_TU_{G^{-1}}f, \quad \text{for } f \in L^1([a, b], \mathcal{B}([a, b]), \nu).
\]
The operator $U_G : L^2([0, 1]) \rightarrow L^2([a, b], \mathcal{B}([a, b]), \nu)$ is an isometry, thus
\[
\|P^n_{S,\nu}h\|_2 = \|P^n_TU_{G^{-1}}h\|_{L^2([0, 1])}.
\]
Since $G$ is increasing, $G^{-1}$ is a function of bounded variation, as a result $U_{G^{-1}}h = h \circ G^{-1}$ is of bounded variation over $[0, 1]$, which completes the proof.

Finally we discuss the case of quadratic maps. We follow the formulation in Viana (1997). Consider the quadratic map $T_\beta$, $\beta \in (0, 2)$, of Example 1 and assume that for the critical point $c = 0$ there are constants $\lambda_c > 1$ and $\alpha > 0$ such that $\lambda_c > e^{2\alpha}$ and
Theorem 7 (Tyran-Kamińska (2004)) Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \(T : Y \to Y\) be ergodic with respect to \(\nu\), and \(h \in L^2(Y, \mathcal{B}, \nu)\) be such that \(\int h(y) \nu(dy) = 0\). Suppose that
\[
\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \left\| \sum_{k=0}^{n-1} \mathcal{P}_{T,\nu}^k h \right\|_2 < \infty. \tag{2.28}
\]
Then there exists \(\tilde{h} \in L^2(Y, \mathcal{B}, \nu)\) such that \(\mathcal{P}_{T,\nu} \tilde{h} = 0\) and the sequence \(\left( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \right)_{n \geq 1}\) is convergent in \(L^2(Y, \mathcal{B}, \nu)\) to zero and if
\[
\left\| \sum_{k=0}^{n-1} \mathcal{P}_{T,\nu}^k h \right\|_2 = O(n^\alpha) \quad \text{with} \quad \alpha < \frac{1}{2}
\]
then \(\left( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \right)_{n \geq 1}\) is convergent \(\nu\)-a.e. to 0.

Now we give a simple result that derives CLT and FCLT from a decay of correlations.

Corollary 3 (Tyran-Kamińska (2004)) Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \(T : Y \to Y\) be ergodic with respect to \(\nu\), and let \(h \in L^\infty(Y, \mathcal{B}, \nu)\) be such that \(\int h(y) \nu(dy) = 0\). If there are \(\alpha > 1\) and \(c > 0\) such that
\[
\left| \int h(y) g(T^n(y)) \nu(dy) \right| \leq \frac{c}{n^\alpha} ||g||_\infty \tag{2.29}
\]
for all \(g \in L^\infty(Y, \mathcal{B}, \nu)\) and \(n \geq 1\), then \(\sigma \geq 0\) given by
\[
\sigma^2 = \int h^2(y) \nu(dy) + 2 \sum_{n=1}^{\infty} \int h(y) h(T^n(y)) \nu(dy)
\]
is finite and \((h \circ T^j)_{j \geq 0}\) satisfies the CLT and FCLT provided that \(\sigma > 0\).

Only recently the FCLT was established by Pollicott and Sharp (2002) for maps such as the Manneville-Pomeau map of Example 10 and for Hölder continuous functions \(h\) with \(\int h(y) \nu(dy) = 0\) under the hypothesis that \(0 < \beta < \frac{1}{2}\). When \(0 < \beta < \frac{1}{2}\) the CLT was proved by Young (1999), where it was shown that in this case condition 2.29 holds. Thus our Corollary 3 gives both the CLT and the FCLT for maps satisfying the following:
(i) \( T(0) = 0, T'(0) = 1, T \) is increasing and piecewise \( C^2 \) and onto \([0, 1]\),

(ii) \( \inf_{\epsilon \leq y \leq 1} |T'(y)| > 1 \) for every \( \epsilon > 0 \),

(iii) \( \lim_{y \to 0} y^{1-\beta}T''(y) \neq 0. \)

as the “tower method” of Young (1999) gives us the estimate in Equation 2.29 with \( \alpha = \frac{1}{\beta} - 1 \) for all Hölder continuous \( h \) and \( g \in L^\infty([0, 1], \mathcal{B}, \nu) \) with the constant \( c \) dependent only on \( h \).

2.4 Weak convergence criteria

Let \((X, | \cdot |)\) be a phase space which is either \( \mathbb{R}^k \) or a separable Banach space, and denote by \( \mathcal{M}_1 \) the space of all probability measures defined on the \( \sigma \)-algebra \( \mathcal{B}(X) \) of Borel subsets of \( X \). For a real-valued measurable bounded function \( f \), and \( \mu \in \mathcal{M}_1 \), we introduce the scalar product notation

\[
\langle f, \mu \rangle = \int_X f(x) \mu(dx).
\]

One way to characterize weak convergence in \( \mathcal{M}_1 \) is to use the Fortet-Mourier metric in \( \mathcal{M}_1 \), which is defined by

\[
d_{FM}(\mu_1, \mu_2) = \sup \{|\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle| : f \in \mathcal{F}_{FM}\} \text{ for } \mu_1, \mu_2 \in \mathcal{M}_1,
\]

where

\[
\mathcal{F}_{FM} = \{f : X \to \mathbb{R} : \sup_{x \in X} |f(x)| \leq 1, |f|_L \leq 1\}
\]

and \( |f|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \). This defines a complete metric on \( \mathcal{M}_1 \), and we have \( \mu_n \to \mu \) weakly if and only if \( d_{FM}(\mu_n, \mu) \to 0 \) (cf. Dudley (1989) [Chapter 3]).

We further introduce a distance on \( \mathcal{M}_1 \) by

\[
d(\mu_1, \mu_2) = \sup \{|\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle| : |f|_L \leq 1\} \text{ for } \mu_1, \mu_2 \in \mathcal{M}_1.
\]

This quantity is always defined, but for some measures it may be infinite. It is easy to check that the function \( d \) is finite for elements of the set

\[
\mathcal{M}_1^1 = \{\mu \in \mathcal{M}_1 : \int_X |x| \mu(dx) < \infty\},
\]

and defines a metric on this set. Moreover, \( \mathcal{M}_1^1 \) is a dense subset of \((\mathcal{M}_1, d_{FM})\) and

\[
d_{FM}(\mu_1, \mu_2) \leq d(\mu_1, \mu_2).
\]

Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space and let \( R_n : X \times Y \to X \) be a measurable transformation for each \( n \in \mathbb{N} \). We associate with each transformation \( R_n \) an operator \( P_n : \mathcal{M}_1 \to \mathcal{M}_1 \) defined by

\[
P_n \mu(A) = \int_X \int_Y 1_A(R_n(x,y)) \nu(dy) \mu(dx)
\]

for \( \mu \in \mathcal{M}_1 \), where

\[
1_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \not\in A 
\end{cases}
\]
is the indicator function of a set $A$. Write
\[ U_n f(x) = \int_Y f(R_n(x,y))\nu(dy) \]
for measurable functions $f : X \to \mathbb{R}$, for which the integral is defined. The operators $U_n$ and $P_n$ satisfy the identity $<U_n f, \mu> = <f, P_n \mu>$. Note that if $\mu = \delta_x$, where $\delta_x$ is the point measure at $x$ defined by
\[ \delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A, \end{cases} \]
then $U_n f(x) = <f, P_n \delta_x>$.

**Remark 6** Note that if $R_n(x,y)$ does not depend on $y$, then $U_n f = U_{R_n} f$ where $U_{R_n}$ is the Koopman operator corresponding to $R_n : X \to X$. The following relation holds between the Frobenius-Perron operator $P_{R_n}$ on $L^1(X,\mathcal{B}(X),m)$ and the operator $P_n$: If $\mu$ has a density $f$ with respect to $m$, then $P_{R_n} f$ is a density of $P_n \mu$.

On the other hand if $R_n(x,y)$ does not depend on $x$, then $U_n f$ is equal to $\int U_{R_n} f(y)\nu(dy)$, where $U_{R_n}$ is the Koopman operator corresponding to $R_n : Y \to X$. The operator $P_n$ has the same value $\nu \circ R_n^{-1}$ for every $\mu \in \mathcal{M}_1$.

Assume that for each $n \in \mathbb{N}$ the transformation $R_n : X \times Y \to X$ satisfies the following conditions:

(A1) There exists a measurable function $L_n : Y \to \mathbb{R}_+$ such that
\[ |R_n(x,y) - R_n(\bar{x},y)| \leq L_n(y)|x - \bar{x}| \quad \text{for} \quad x, \bar{x} \in X, \ y \in Y. \]

(A2) The series
\[ \sum_{n=1}^{\infty} \int_Y |R_n(0,T(y)) - R_{n+1}(0,y)|\nu(dy) \]

is convergent, where $T : Y \to Y$ is a transformation preserving the measure $\nu$.

(A3) The integral $\int_Y |R_n(0,y)|\nu(dy)$ is finite for at least one $n$.

**Proposition 2** Let the transformations $R_n$ satisfy conditions (A1)-(A3). If
\[ \lim_{n \to \infty} \int_Y L_n(y)\nu(dy) = 0, \]
then there exists a unique measure $\mu_* \in \mathcal{M}_1$ such that $(P_n \mu)$ converges weakly to $\mu_*$ for each measure $\mu \in \mathcal{M}_1$.

**Proof.** Assumptions (A2) and (A3) imply that $P_n(\mathcal{M}_1^1) \subset \mathcal{M}_1^1$. By the definition of the metric $d$ we have
\[ d(P_n \delta_0, P_{n+1} \delta_0) = \sup\{|U_n f(0) - U_{n+1} f(0)| : |f|_L \leq 1\}. \]

Since the transformation $T$ preserves the measure $\nu$, we can write
\[ U_n f(0) = \int_Y f(R_n(0,y))\nu(dy) = \int_Y f(R_n(0,T(y))\nu(dy) \]
for any $f$ with $|f|_L \leq 1$. Hence

$$|U_n f(0) - U_{n+1} f(0)| \leq \int_Y |R_n(0, T(y)) - R_{n+1}(0, y)| \nu(dy).$$

Consequently

$$d(P_n \delta_0, P_{n+1} \delta_0) \leq \int_Y |R_n(0, T(y)) - R_{n+1}(0, y)| \nu(dy),$$

and

$$d_{FM}(P_n \delta_0, P_{n+1} \delta_0) \leq d(P_n \delta_0, P_{n+1} \delta_0).$$

From Condition (A2), the sequence $(P_n \delta_0)$ is a Cauchy sequence. Since the space $(\mathcal{M}_1, d_{FM})$ is complete, $(P_n \delta_0)$ is weakly convergent to a $\mu_* \in \mathcal{M}_1$. From (A1) it follows that

$$d(P_n \mu_1, P_n \mu_2) \leq \int_Y L_n(y) \nu(dy) d(\mu_1, \mu_2).$$

Hence $(P_n \mu)$ is weakly convergent for each $\mu \in \mathcal{M}_1^1$ and has the limit $\mu_*$. Since, for sufficiently large $n$, each operator $P_n$ satisfies

$$d_{FM}(P_n \mu_1, P_n \mu_2) \leq d_{FM}(\mu_1, \mu_2)$$

and the set $\mathcal{M}_1^1$ is dense in $(\mathcal{M}_1, d_{FM})$, the proof is complete.

### 3 Analysis

We now return to the original problem posed in Section 1. We consider the position ($x$) and velocity ($v$) of a dynamical system defined by

$$\frac{dx(t)}{dt} = v(t), \quad (3.1)$$

$$\frac{dv(t)}{dt} = b(v(t)) + \eta(t), \quad (3.2)$$

with a perturbation $\eta$ in the velocity. We assume that $\eta(t)$ consists of a series of delta-function-like perturbations that occur at times $t_0, t_1, t_2, \cdots$. These perturbations have an amplitude $h(\xi(t))$, and $\eta(t)$ takes the explicit form

$$\eta(t) = \kappa \sum_{n=0}^{\infty} h(\xi(t)) \delta(t - t_n). \quad (3.3)$$

We assume that $\xi$ is generated by a dynamical system that at least has an invariant measure for the results of Section 3.1 to hold, or is at least ergodic for the Central Limit Theorem to hold as in Section 3.2.

In practice, we will illustrate our results assuming that $\xi$ is the trace of a highly chaotic semidynamical system that is, indeed, even **exact** in the sense of Lasota and Mackey [1994] (c.f. Section 2). $\xi$ could, for example, be generated by the differential delay equation

$$\delta \frac{d\xi}{dt} = -\xi(t) + T(\xi(t - 1)), \quad (3.4)$$

33
where the nonlinearity $T$ has the appropriate properties to generate chaotic solutions (Mackey and Glass (1977); an der Heiden and Mackey (1982)). The parameter $\delta$ controls the time scale for these oscillations, and in the limit as $\delta \to 0$ we can approximate the behavior of the solutions through a map of the form

$$\xi_{n+1} = T(\xi_n).$$

Thus, we can think of the map $T$ as being generated by the sampling of a chaotic continuous time signal $\xi(t)$ as, for example, by the taking of a Poincaré section of a semi-dynamical system operating in a high dimensional phase space.

Let $(Y, B, \nu)$ be a normalized measure space. Let $b : \mathbb{R}^k \to \mathbb{R}^k$, $h : Y \to \mathbb{R}^k$ be measurable transformations, and let $(t_n)_{n \geq 0}$ be an increasing sequence of real numbers. Assume that $\xi : \mathbb{R}_+ \times Y \to Y$ is such that $\xi(t_{n+1}) = T(\xi(t_n))$ for $n \geq 0$, where $T : Y \to Y$ is a measurable transformation preserving the measure $\nu$. Combining (3.2) with (3.3) we have

$$\frac{dv(t)}{dt} = b(v(t)) + \kappa \sum_{n=0}^{\infty} h(\xi(t)) \delta(t - t_n).$$

We say that $v(t)$, $t \geq t_0$, is a solution of Equation 3.6 if, for each $n \geq 0$, $v(t)$ is a solution of the Cauchy problem

$$\begin{cases}
\frac{dv(t)}{dt} = b(v(t)), & t \in (t_n, t_{n+1}) \\
v(t_n) = v(t_n^-) + \kappa h(\xi(t_n)),
\end{cases}$$

(3.7)

where $v_0$ is an arbitrary point of $\mathbb{R}^k$ and $v(t_n^-) = \lim_{t \to t_n^-} v(t)$ for $n \geq 1$.

Let $\pi : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}^k$ be the semigroup generated by the Cauchy problem (if $b$ is a Lipschitz map then $\pi$ is well defined)

$$\begin{cases}
\frac{d\tilde{v}(t)}{dt} = b(\tilde{v}(t)), & t > 0 \\
\tilde{v}(0) = \tilde{v}_0,
\end{cases}$$

(3.8)

i.e. for every $\tilde{v}_0 \in \mathbb{R}^k$ the unique solution of (3.8) is given by $\tilde{v}(t) = \pi(t, \tilde{v}_0)$ for $t \geq 0$. As a result, the solution of (3.6) is given by

$$v(t) = \pi(t - t_n, v(t_n)), \quad \text{for } t \in [t_n, t_{n+1}), n \geq 0.$$  

(3.9)

After integration, for $t \in [t_n, t_{n+1})$ we have

$$x(t) - x(t_n) = \int_{t_n}^{t} v(s) ds = \int_{t_n}^{t} \pi(s - t_n, v(t_n)) ds = \int_{0}^{t-t_n} \pi(s, v(t_n)) ds.$$  

Consequently, the solutions of Equations 3.1 and 3.2 are given by

$$\begin{align*}
x(t) &= x(t_n) + \int_{0}^{t-t_n} \pi(s, v(t_n)) ds, \\
v(t) &= \pi(t - t_n, v(t_n)), \quad \text{for } t \in [t_n, t_{n+1}), n \geq 0.
\end{align*}$$

(3.10)

Observe that $x(t)$ is continuous in $t$, while $v(t)$ is only right continuous, with left-hand limits, and $v(t_n) = \lim_{t \to t_n^+} v(t)$. 

34
We are interested in the variables \( v(t_n), v_n := v(t_n^-), \xi_n := \xi(t_n), \) and \( x_n := x(t_n) \) which appear in the definition of the solution \( v(t) \) and \( x(t) \). We have

\[
\begin{align*}
v(t_n) &= v_n + \kappa h(\xi_n), \\
v_{n+1} &= \pi(t_{n+1} - t_n, v(t_n)), \\
\xi_{n+1} &= T(\xi_n), \\
x_{n+1} &= x_n + \int_0^{t_{n+1} - t_n} \pi(s, v(t_n)) ds.
\end{align*}
\]

(3.11) (3.12) (3.13) (3.14)

We are going to examine the dynamics of these variables from a statistical point of view.

Suppose that \( v_0 \) has a distribution \( \mu \), \( \xi_0 \) has a distribution \( \nu \), and that the random variables are independent. What can we say about the long-term behavior of the distribution of the random variables \( v(t_n) \) or \( v_n \)?

### 3.1 Weak convergence of \( v(t_n) \) and \( v_n \).

To simplify the presentation and easily use Proposition 2 of Section 2.4, assume that the differences \( t_{n+1} - t_n \) do not depend on \( n \), and that \( t_n = n\tau \) for \( n \geq 0 \). Define \( \Lambda : \mathbb{R}^k \to \mathbb{R}^k \) by

\[
\Lambda(v) = \pi(\tau, v), \quad v \in \mathbb{R}^k,
\]

(3.15)

where \( \pi \) describes the solutions of the unperturbed system as defined by Equation 3.8. In particular, adding chaotic deterministic perturbations to any exponentially stable system produces a stochastically stable system as stated in the following

**Corollary 4** Let \( \Lambda : \mathbb{R}^k \to \mathbb{R}^k \) be a Lipschitz map with a Lipschitz constant \( \lambda \in (0, 1) \). Let \( T : Y \to Y \) be a transformation preserving the measure \( \nu \), and \( h : Y \to \mathbb{R}^k \) be such that \( \int_Y |h(y)| |\nu(dy)| < \infty \).

Assume that the random variables \( v_0 \) and \( \xi_0 \) are independent and that \( \xi_0 \) has a distribution \( \nu \). Then \( v(n\tau) \) converges in distribution to a probability measure \( \mu_* \) on \( \mathbb{R}^k \) and \( \mu_* \) is independent of the distribution of the initial random variable \( v_0 \). Moreover, \( v_n \) converges in distribution to the probability measure \( \mu_* \circ \Lambda^{-1} \).

**Proof.** From Equations 3.11 and 3.12 it follows that

\[
v((n + 1)\tau) = \Lambda(v(n\tau)) + \kappa h(\xi_{n+1}), \quad n \geq 0.
\]

Define the transformation \( R_n : \mathbb{R}^k \times Y \to \mathbb{R}^k \) recursively:

\[
\begin{align*}
R_0(v, y) &= v + \kappa h(y), \\
R_{n+1}(v, y) &= \Lambda(R_n(v, y)) + \kappa h(T^{n+1}(y)), \quad v \in \mathbb{R}^k, y \in Y, n \geq 0.
\end{align*}
\]

(3.16)

Then \( v(n\tau) = R_n(v_0, \xi_0) \). One can easily check by induction that all assumptions of Proposition 2 are satisfied. Thus \( v(n\tau) \) converges in distribution to a unique probability measure \( \mu_* \) on \( \mathbb{R}^k \).

Since \( v_{n+1} = \Lambda(v(n\tau)) \) and \( \Lambda \) is a continuous transformation, it follows from the definition of weak convergence that the distribution of \( v_{n+1} \) converges weakly to \( \mu_* \circ \Lambda^{-1} \).

We call the measure \( \mu_* \) the limiting measure for \( v(n\tau) \). Note that \( \mu_* \) may depend on \( \nu \).

**Remark 7** Although this Corollary shows that there is a unique limiting measure, we cannot conclude in general that this measure has a density absolutely continuous with respect to the Lebesgue measure. See Example 7 and Remark 12.
3.2 The linear case in one dimension

We now consider Equation \(3.2\) when \(b(v) = -\gamma v \) and \(\gamma \geq 0\). In this situation, we are considering a frictional force linear in the velocity, so Equations \(3.1\) and \(3.2\) become

\[
\frac{dx(t)}{dt} = v(t), \\
\frac{dv(t)}{dt} = -\gamma v(t) + \kappa \sum_{n=0}^{\infty} h(\xi(t))\delta(t - t_n).
\] (3.17)

To make the computations of the previous section completely transparent, multiply Equation \(3.17\) by the integrating factor \(\exp(\gamma t)\), rearrange, and integrate from \((t_n - \epsilon)\) to \((t_{n+1} - \epsilon)\), where \(0 < \epsilon < \min_{n \geq 0}(t_{n+1} - t_n)\), to give

\[
v(t_{n+1} - \epsilon)e^{\gamma (t_{n+1} - \epsilon)} - v(t_n - \epsilon)e^{\gamma (t_n - \epsilon)} = \kappa e^{\gamma (t_n - \epsilon)}h(\xi(t_n - \epsilon))
\] (3.18)

Taking the \(\lim_{\epsilon \to 0}\) in Equation \(3.18\) and remembering that \(v(t_n) = v_n\) and \(\xi(t_n) = \xi_n\), we have

\[
v_{n+1} = \lambda_n v_n + \kappa \lambda_n h(\xi_n),
\] (3.19)

where \(\tau_n \equiv t_{n+1} - t_n\) and \(0 \leq \lambda_n \equiv e^{-\gamma \tau_n} < 1\).

We simplify this formulation by taking \(t_{n+1} - t_n \equiv \tau > 0\) so the perturbations are assumed to be arriving periodically. As a consequence, \(\lambda_n \equiv \lambda\) with

\[
\lambda = e^{-\gamma \tau}.
\] (3.20)

Then, Equation \(3.19\) becomes

\[
v_{n+1} = \lambda v_n + \kappa \lambda h(\xi_n).
\] (3.21)

This result can also be arrived at from other assumptions \(^1\).

3.2.1 Behaviour of the velocity variable

For a given initial \(v_0\) we have, by induction,

\[
v_n = \lambda^n v_0 + \kappa \lambda \sum_{j=0}^{n-1} \lambda^{n-1-j} h(\xi_j)
\]

\[
= \lambda^n v_0 + \kappa \lambda \sum_{j=0}^{n-1} \lambda^{n-1-j} h(T^j(\xi_0)).
\] (3.24)

\(^1\)Alternately but, as it will turn out, equivalently, we can think of the perturbations as constantly applied. In this case we write an Euler approximation to the derivative in Equation \(3.17\) so with an integration step size of \(\tau\) we have

\[
v(t + \tau) = (1 - \gamma \tau)v(t) + \tau \kappa h(\xi(t)).
\] (3.22)

Measuring time in units of \(\tau\) so \(t_{n+1} = t_n + \tau\) we then can write this in the alternate equivalent form

\[
v_{n+1} = \lambda v_n + \kappa_1 h(\xi_n),
\] (3.23)

where, now, \(\lambda = 1 - \gamma \tau\) and \(\kappa_1 = \kappa \tau \lambda^{-1}\). Again, by induction we obtain Equation \(3.24\).
We now calculate the asymptotic behaviour of the variance of \( v_n \) when \( \xi_0 \) is distributed according to \( \nu \), the invariant measure for \( T \). Assume for simplicity that \( v_0 = 0 \), set \( \sigma^2 = \int h^2(y)\nu(dy) \) and assume that \( \int h(y)h(T^n(y))\nu(dy) = 0 \) for \( n \geq 1 \). Then we have

\[
\int v_n^2\nu(dy) = \kappa^2 \int \left( \sum_{j=0}^{n-1} \lambda^{n-j}h(T^j(y)) \right)^2 \nu(dy).
\]

Since the sequence \( h \circ T^j \) is uncorrelated by our assumption,

\[
\int \left( \sum_{j=0}^{n-1} \lambda^{n-j}h(T^j(y)) \right)^2 \nu(dy) = \sum_{j=0}^{n-1} \lambda^{2n-2j} \int h(T^j(y))^2\nu(dy) = \frac{1 - \lambda^{2n}}{1 - \lambda^2 \sigma^2}.
\]

Thus

\[
\int v_n^2\nu(dy) = \kappa^2 \sigma^2 \left( \frac{1 - \lambda^{2n}}{1 - \lambda^2} \right). \quad (3.25)
\]

Since \( b(v) = -\gamma v \), we have \( \pi(t,v) = e^{-\gamma t} v \). Equation 3.21 in conjunction with Equations 3.12 and 3.11 leads to \( v_{n+1} = \lambda v(n\tau) \) where \( v(n\tau) = v_n + \kappa h(\xi_n) \). Thus

\[
v(n\tau) = \lambda^n v_0 + \kappa \sum_{j=0}^{n} \lambda^{n-j}h(T^j(\xi_0)) \quad (3.26)
\]

**Case 1:** If \( \lambda < 1 \) and \( \xi_0 \) is distributed according to \( \nu \), then by Corollary 4 there exists a unique limiting measure \( \mu_* \) for \( v(n\tau) \) provided that \( h \) is integrable with respect to \( \nu \). The sequence \( v_n \) also converges in distribution. Since the function \( \Lambda \) defined by Equation 3.15 is linear, \( \Lambda(v) = \lambda v \), both sequences \( v(n\tau) \) and \( v_n \) are either convergent or divergent in distribution.

What can happen if the random variable \( \xi_0 \) in Equation 3.26 is distributed according to a different measure?

**Proposition 3** Let \( \lambda < 1 \), the transformation \( T \) be exact with respect to \( \nu \), and \( h \in L^1(Y,B,\nu) \). If the random variable \( \xi_0 \) is distributed according to a normalized measure \( \nu_0 \) on \( (Y,B) \) which is absolutely continuous with respect to \( \nu \), then

\[
\kappa \sum_{j=0}^{n} \lambda^{n-j}h(T^j(\xi_0)) \rightarrow^d \mu_* \quad (3.27)
\]

and \( \mu_* \) does not depend on \( \nu_0 \).

**Proof.** Recall that by the continuity theorem

\[
\kappa \sum_{j=0}^{n} \lambda^{n-j}h(T^j(\xi_0)) \rightarrow^d \mu_*
\]

if and only if for every \( r \in \mathbb{R} \)

\[
\lim_{n \rightarrow \infty} \int_Y \exp(ir\kappa \sum_{j=0}^{n} \lambda^{n-j}h(T^j(y)))\nu_0(dy) = \int_{\mathbb{R}} \exp(irx)\mu_*(dx).
\]
We know that the last Equation is true when \( \nu_0 = \nu \). Since for every \( m \geq 1 \) the sequence
\[
\kappa \sum_{j=0}^{m-1} \lambda^{n-j} h(T^j(y))
\]
is convergent to 0 as \( n \to \infty \) and
\[
\sum_{j=0}^{n} \lambda^{n-j} h \circ T^j = \sum_{j=0}^{n} \lambda^{n-m-j} h \circ T^j \circ T^m,
\]
an analysis similar to that in the proof of Theorem completes the demonstration.

**Case 2:** If \( \lambda = 1 \) we have \( v_n = v_0 + \kappa \sum_{j=0}^{n-1} h(T^j(\xi_0)) \). Since \( v_0 \) and \( \xi_0 \) are independent random variables, \( v_0 \) and \( \kappa \sum_{j=0}^{n-1} h(T^j(\xi_0)) \) are also independent. Hence \( v_n \) converges if and only if \( \kappa \sum_{j=0}^{n-1} h(T^j(\xi_0)) \) does. Moreover, if there is a limiting measure \( \mu^* \) for \( v_n \), then the sequence \( v(n\tau) \) converges in distribution, say to \( \nu^* \), and \( \mu^* \) is a convolution of the distribution of \( v_0 \) and \( \nu^* \). As a result, \( \mu^* \) depends on the distribution of \( v_0 \). However, if the map \( T \) and function \( h \) satisfy the FCLT, then
\[
\frac{\sum_{j=0}^{n-1} h(T^j(\xi_0))}{\sqrt{n}} \to^d N(0,1).
\]
Hence \( \sum_{j=0}^{n-1} h(T^j(\xi_0)) \) is not convergent in distribution since the density is spread on the entire real line.

### 3.2.2 Behaviour of the position variable

For the position variable we have, for \( t \in [n\tau, (n+1)\tau) \),
\[
x(t) - x(n\tau) = \int_{n\tau}^{t} v(s) ds = \int_{n\tau}^{t} e^{-\gamma(s-n\tau)} v(n\tau) ds = \frac{1 - e^{-\gamma(t-n\tau)}}{\gamma} v(n\tau).
\]

With \( x(n\tau) = x_n \) we have
\[
x_{n+1} - x_n = \frac{1 - e^{-\gamma\tau}}{\gamma} v(n\tau) = \frac{1 - \lambda}{\gamma} v(n\tau).
\]
(3.28)

Summing from 0 to \( n \) gives
\[
x_{n+1} = x_0 + \frac{1 - \lambda}{\gamma} \sum_{j=0}^{n} v(j\tau).
\]
(3.29)

From this and Equation 3.26 we obtain
\[
x_{n+1} = x_0 + \frac{1 - \lambda}{\gamma} \sum_{j=0}^{n} \left( \lambda^j v_0 + \kappa \sum_{i=0}^{j} \lambda^{j-i} h(T^i(\xi_0)) \right)
\]
\[
= x_0 + \frac{(1 - \lambda^{n+1})}{\gamma} v_0 + \frac{(1 - \lambda)}{\gamma} \sum_{j=0}^{n} \sum_{i=0}^{j} \lambda^{j-i} h(T^i(\xi_0)).
\]

38
Changing the order of summation in the last term gives

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} \lambda^{j-i} h(T^i(\xi_0)) = \sum_{i=0}^{n} \sum_{j=i}^{n} \lambda^{j-i} h(T^i(\xi_0)) = \sum_{i=0}^{n} \frac{1 - \lambda^{n-i+1}}{1 - \lambda} h(T^i(\xi_0)) = \frac{1}{1 - \lambda} \sum_{i=0}^{n} h(T^i(\xi_0)) - \frac{\lambda}{1 - \lambda} \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0)).
\]

Consequently

\[
x_{n+1} = x_0 + \frac{(1 - \lambda^{n+1})}{\gamma} v_0 + \frac{\kappa}{\gamma} \sum_{i=0}^{n} h(T^i(\xi_0)) - \frac{\lambda \kappa}{\gamma} \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0)).
\]

In conjunction with Equation 3.30 this gives

\[
x_n = x_0 + \frac{(1 - \lambda^n)}{\gamma} v_0 + \frac{\kappa}{\gamma} \sum_{i=0}^{n} h(T^i(\xi_0)) - \frac{\lambda \kappa}{\gamma} \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0)).
\] (3.30)

Next we calculate the asymptotic behavior of the variance of \(x_n\). Assume as before that \(x_0 = v_0 = 0\), and that \(\int h(y) h(T^j(y)) \nu(dy) = 0\) and \(\sigma^2 = \int h^2(y) \nu(dy)\). We have

\[
\left( \sum_{i=0}^{n} h(T^i(\xi_0)) - \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0)) \right)^2 = \left( \sum_{i=0}^{n} h(T^i(\xi_0)) \right)^2 + \left( \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0)) \right)^2 - 2 \sum_{i=0}^{n} h(T^i(\xi_0)) \sum_{i=0}^{n} \lambda^{n-i} h(T^i(\xi_0))
\]

Since, by assumption, the sequence \(h \circ T^i\) is again uncorrelated we have

\[
\int \left( \sum_{i=0}^{n} h(T^i(y)) \right)^2 \nu(dy) = \sum_{i=0}^{n} \sum_{j=0}^{n} \int h(T^i(y)) h(T^j(y)) \nu(dy) = \sum_{i=0}^{n} \int h(T^i(y))^2 \nu(dy) = (n + 1) \sigma^2.
\]

Analogous to the computation for the velocity variance

\[
\int \left( \sum_{i=0}^{n} \lambda^{n-i} h(T^i(y)) \right)^2 \nu(dy) = \sum_{i=0}^{n} \lambda^{2n-2i} \int h(T^i(y))^2 \nu(dy) = \frac{1 - \lambda^{2n+2}}{1 - \lambda^2} \sigma^2
\]

and

\[
\int \sum_{i=0}^{n} h(T^i(y)) \sum_{j=0}^{n} \lambda^{n-j} h(T^j(y)) \nu(dy) = \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda^{n-j} \int h(T^i(y)) h(T^j(y)) \nu(dy) = \sum_{i=0}^{n} \lambda^{n-i} \int h(T^i(y))^2 \nu(dy) = \frac{1 - \lambda^{n+1}}{1 - \lambda} \sigma^2.
\]

Consequently, if \(x_0 = v_0 = 0\) then

\[
\int x_n^2 \nu(dy) = \frac{\kappa^2 \sigma^2}{\gamma^2} \left( n + 1 + \frac{1 - \lambda^{2n+2}}{1 - \lambda^2} - \frac{1 - \lambda^{n+1}}{1 - \lambda} \right).
\] (3.31)
**Theorem 8** Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \(T : Y \to Y\) be a measurable map such that \(T\) preserves the measure \(\nu\), let \(\sigma > 0\) be a constant, and \(h \in L^2(Y, \mathcal{B}, \nu)\) be such that \(\int h(y)\nu(dy) = 0\). Then
\[
\sum_{i=0}^{n} h(T^i(\xi_0)) \sqrt{n} \to^d N(0, \sigma^2)
\] (3.32)
if and only if
\[
\frac{x_n}{\sqrt{n}} \to^d N \left( 0, \frac{\kappa^2 \gamma^2}{\gamma^2} \right).
\] (3.33)

**Proof.** Assume that Condition (3.32) holds. From Equation (3.30) we obtain
\[
\frac{x_n}{\sqrt{n}} = \frac{x_0}{\sqrt{n}} + \frac{1 - \lambda^n}{\gamma} \frac{v_0}{\sqrt{n}} + \frac{k}{\gamma \sqrt{n}} \sum_{i=0}^{n} h(T^i(\xi_0)) - \frac{\kappa}{\gamma} \sum_{i=0}^{n} \lambda^n - i h(T^i(\xi_0)).
\]
By assumption
\[
\frac{\kappa}{\gamma \sqrt{n}} \sum_{i=0}^{n} h(T^i(\xi_0)) \to^d \frac{\kappa}{\gamma} N(0, \sigma^2).
\]
Thus the result will follow when we show that the remaining terms are convergent in probability to zero. The first term
\[
\frac{x_0}{\sqrt{n}} + \frac{1 - \lambda^n}{\gamma} \frac{v_0}{\sqrt{n}}
\]
is convergent to zero a.e. hence in probability. The sequence \(\sum_{i=0}^{n} \lambda^n - i h(T^i(\xi_0))\) is convergent in distribution and \(\frac{k}{\gamma \sqrt{n}} \to 0\) as \(n \to \infty\). Consequently, the sequence
\[
\frac{k}{\gamma \sqrt{n}} \sum_{i=0}^{n} \lambda^n - i h(T^i(\xi_0))
\]
is convergent in probability to zero which completes the proof. The proof of the converse is analogous.

**Remark 8** Observe that if the transformation \(T\) is exact and \(\xi_0\) is distributed according to a measure absolutely continuous with respect to \(\nu\), then the conclusion of Theorem 8 still holds.

Theorem 8 generalizes the results of Chew and Ting (2002). In Section 2.3.2 we have discussed when Condition (3.32) holds for a given ergodic transformation.

**Remark 9** Note that if we multiply Gaussian distributed random variable \(N(0,1)\) by a positive constant \(c\), then it becomes Gaussian distributed \(N(0,c^2)\) with variance \(c^2\). Thus if we multiply both sides of Equation (3.33) by \(\frac{1}{\sqrt{\tau}}\), we obtain
\[
\frac{x(n\tau)}{\sqrt{n\tau}} \to^d N(0, \frac{\kappa^2 \sigma^2}{\tau \gamma^2}).
\]
So if \(\kappa = \sqrt{m_\gamma}\), as in Chew and Ting (2002), then
\[
\frac{\kappa^2 \sigma^2}{\tau \gamma^2} = \frac{m \sigma^2}{\gamma}.
\]
4 Identifying the Limiting Velocity Distribution

Let $Y \subset \mathbb{R}$ be an interval, $B = B(Y)$, and $T : Y \to Y$ be a transformation preserving a normalized measure $\nu$ on $(Y, B(Y))$. Recall from Section 3.1 that $\mu_*$ is the limiting measure for the sequence of random variables $(v(n\tau))$ starting from $v_0 \equiv 0$, i.e.

$$v(n\tau) = \kappa \sum_{i=0}^{n} \lambda^{n-i} h(\xi_i),$$

where $h : Y \to \mathbb{R}$ is a given integrable function, $0 < \lambda < 1$, $\xi_i = T^i(\xi_0)$, and $\xi_0$ is distributed according to $\nu$.

**Proposition 4** Let $Y = [a, b]$ and $h : Y \to \mathbb{R}$ be a bounded function. Then the limiting measure $\mu_*$ has moments of all order given by

$$\int x^k \mu_*(dx) = \lim_{n \to \infty} \int v(n\tau)^k \nu(dy) \quad (4.1)$$

and the characteristic function of $\mu_*$ is of the form

$$\phi_*(r) = \sum_{k=0}^{\infty} \frac{(ir)^k}{k!} \int x^k \mu_*(dx), \quad r \in \mathbb{R}.$$ 

Moreover, $\mu_*([-\frac{\kappa c}{1-\lambda}, \frac{\kappa c}{1-\lambda}]) = 1$, where $c = \sup_{y \in Y} |h(y)|$.

**Proof.** Since $h$ is bounded, we have

$$|v(n\tau)|^k \leq \left( \frac{\kappa c}{1-\lambda} \right)^k, \quad n, k \geq 0.$$ 

The existence and convergence of moments now follow from Theorem 5.3 and 5.4 of Billingsley (1968). Since $v(n\tau)$ has all its values in the interval $[-\frac{\kappa c}{1-\lambda}, \frac{\kappa c}{1-\lambda}]$, we obtain $\mu_n([-\frac{\kappa c}{1-\lambda}, \frac{\kappa c}{1-\lambda}]) = 1$ where $\mu_n$ is the distribution of $v(n\tau)$. Convergence in distribution (Billingsley, 1968, Theorem 2.1) implies that

$$\limsup_{n \to \infty} \mu_n(F) \leq \mu_*(F)$$

for all closed sets. Therefore $\mu_*([-\frac{\kappa c}{1-\lambda}, \frac{\kappa c}{1-\lambda}]) = 1$. The formula for the characteristic function is a consequence of the other statements, and the proof is complete.

**Remark 10** Note that if the characteristic function of $\mu_*$ is integrable, then $\mu_*$ has a continuous and bounded density which is given by

$$f_*(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixr)\phi_*(r)dr, \quad x \in \mathbb{R}.$$ 

On the other hand if $\mu_*$ has a density then $\phi_*(r) \to 0$ as $|r| \to \infty$.

Note also that if a density exists then it must be zero outside the interval $[-\frac{\kappa c}{1-\lambda}, \frac{\kappa c}{1-\lambda}]$. 

41
Let $Y = [a, b]$ be an interval and $h(y) = y$ for $y \in Y$. Thus $h$ is bounded and for this choice of $h$ all moments of the corresponding limiting distribution $\mu_*$ exists by Proposition 4. However, the calculation might be quite tedious. We are going to determine the measure $\mu_*$ for a specific example of the transformation $T$ by using a different method.

Let $h_n : Y \to \mathbb{R}$ be defined by

$$h_n(y) = \sum_{i=0}^{n} \lambda^{n-i}T_i(y), \quad y \in Y, n \geq 0. \quad (4.2)$$

Then $v(n\tau) = \kappa h_n(\xi_0)$ and $v_n = \kappa \lambda h_{n-1}(\xi_0)$. Thus knowing the limiting distribution for these sequences is equivalent to knowing the limiting distribution for $h_n(\xi_0)$.

### 4.1 Dyadic map

To give a concrete example for which much of the preceding considerations can be completely illustrated, consider the generalized dyadic map defined by Equation 2.10:

$$T(y) = \begin{cases} 2y + 1, & y \in [-1, 0] \\ 2y - 1, & y \in (0, 1] \end{cases}.$$  

**Proposition 5** Let $\xi, \xi_0$ be random variables uniformly distributed on $[-1, 1]$. Let $(\alpha_k)$ be a sequence of independent random variables taking values drawn from $\{-1, 1\}$ with equal probability. Assume that $\xi$ is statistically independent of the sequence $(\alpha_k)$. Then for every $\lambda \in (0, 1)$

$$h_n(\xi_0) \to d \frac{1}{2 - \lambda} \left( \xi + \sum_{k=0}^{\infty} \lambda^k \alpha_{k+1} \right). \quad (4.3)$$

**Proof.** The random variable

$$\xi_0 = \sum_{k=1}^{\infty} \frac{\alpha_k}{2^k}$$

is uniformly distributed on $[-1, 1]$. It is easily seen that for the dyadic map

$$\xi_i = T^i(\xi_0) = \sum_{k=1}^{\infty} \frac{\alpha_{k+i}}{2^k} \text{ for } i \geq 1. \quad (4.4)$$

Using this representation we obtain

$$\sum_{i=0}^{n-1} \lambda^{n-1-i} \xi_i = \sum_{i=0}^{n-1} \lambda^{n-1-i} \sum_{k=1}^{\infty} \frac{\alpha_{k+i}}{2^k} + \sum_{i=0}^{n-1} \lambda^{n-1-i} \sum_{k=n-i+1}^{\infty} \frac{\alpha_{k+i}}{2^k}.$$  

Changing the order of summation leads to

$$\sum_{k=1}^{n} \left( \sum_{i=1}^{k} \frac{\lambda^{n-1-k+i}}{2^i} \right) \alpha_k = \sum_{i=0}^{n-1} \lambda^{n-1-i} \sum_{k=1}^{\infty} \frac{\alpha_{k+n}}{2^k} = \frac{1}{2 - \lambda} \sum_{k=1}^{n} \lambda^{n-k} \left( 1 - \left( \frac{\lambda}{2} \right)^k \right) \alpha_k + \frac{1 - (\lambda/2)^n}{2 - \lambda} \xi_n.$$  

42
Consequently
\[ \sum_{i=0}^{n-1} \lambda^{n-1-i} \xi_i = \frac{1}{2 - \lambda} \sum_{k=1}^{n} \lambda^{n-k} \alpha_k + \frac{1}{2 - \lambda} \xi_n - \lambda^n w_n, \]
where
\[ w_n = \frac{1}{2 - \lambda} \left[ \sum_{k=1}^{n} \left( \frac{1}{2} \right)^k \alpha_k + \left( \frac{1}{2} \right)^n \xi_n \right]. \]
This gives
\[ h_{n-1}(\xi_0) + \lambda^n w_n = \frac{1}{2 - \lambda} \left( \sum_{k=1}^{n} \lambda^{n-k} \alpha_k + \xi_n \right). \] (4.5)

Note that for every \( n \) we have \(|w_n| \leq 2\). Therefore \( \lambda^n w_n \) is a.s. convergent to 0 as \( n \to \infty \). Since \( h_n(\xi_0) \) converges in distribution, say to \( \mu_* \), we have \( h_{n-1}(\xi_0) + \lambda^n w_n \to^d \mu_* \) and the random variables on the right-hand side of Equation (4.5) converge in distribution to \( \mu_* \). Since the random variables \( \alpha_k \) are independent, the random variables \( \sum_{k=1}^{n} \lambda^{n-k} \alpha_k \) and \( \xi_n \) are also independent for every \( n \). The same is true for \( \sum_{k=0}^{n-1} \lambda^k \alpha_{k+1} \) and \( \xi \). Moreover, \( \xi_n + \sum_{k=1}^{n} \lambda^{n-k} \alpha_k \) and \( \xi + \sum_{k=0}^{n-1} \lambda^k \alpha_{k+1} \) have identical distributions. Thus
\[ \frac{1}{2 - \lambda} \left( \xi + \sum_{k=0}^{n-1} \lambda^k \alpha_{k+1} \right) \to^d \mu_* \]
on the other hand \( \sum_{k=0}^{n-1} \lambda^k \alpha_{k+1} \to \sum_{k=0}^{\infty} \lambda^k \alpha_{k+1} \) almost surely as \( n \to \infty \), but this implies convergence in distribution. The proof is complete.

Before stating our next result, we review some of the known properties of the random variable which appears in Equation (4.5). For every \( \lambda \in (0, 1) \) let
\[ \zeta_\lambda = \sum_{k=0}^{\infty} \lambda^k \alpha_{k+1}, \] (4.6)
and let \( g_\lambda \) be the distribution function of \( \zeta_\lambda \), \( g_\lambda(x) = \Pr\{\zeta_\lambda \leq x\} \) for \( x \in \mathbb{R} \). Explicit expressions for \( g_\lambda \) are, in general, not known. The measure induced by the distribution \( g_\lambda \) is called an infinitely convolved Bernoulli measure (see Peres et al. (2000) for the historical background and recent advances).

It is known (Jessen and Wintner, 1935) that \( g_\lambda \) is continuous and it is either absolutely continuous or singular. Recall that \( x \) is a point of increase of \( g_\lambda \) if \( g_\lambda(x - \epsilon) < g_\lambda(x + \epsilon) \) for all \( \epsilon > 0 \). The set of points of increase of \( g_\lambda \) (Kershner and Wintner, 1935) is either the interval \([-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]\) when \( \lambda \geq \frac{1}{2} \) or a Cantor set \( K_\lambda \) of zero Lebesgue measure contained in this interval when \( \lambda < \frac{1}{2} \), \( g_\lambda \) is always singular for \( \lambda \in (0, \frac{1}{2}) \) and the Cantor set \( K_\lambda \) satisfies \( K_\lambda = (\lambda K_\lambda + 1) \cup (\lambda K_\lambda - 1) \) and \( \frac{1}{1-\lambda}, -\frac{1}{1-\lambda} \in K_\lambda \). Wintner (1935) noted that \( g_\lambda \) has the uniform density \( \rho_\lambda(x) = \frac{1}{4}|1-2x|(x) \) for \( \lambda = \frac{1}{2} \) and that it is absolutely continuous for the \( k \)th roots of \( \frac{1}{2} \). Thus it was suspected that \( g_\lambda \) is absolutely continuous for all \( \lambda \in (\frac{1}{2}, 1) \). However, Erdős (1939a) showed that \( g_\lambda \) is singular for \( \lambda = \frac{\sqrt{5} - 1}{2} \) and for the reciprocal of the so called Pisot numbers in \((1, 2)\). Later Erdős (1939b) showed that there is a \( \beta < 1 \) such that for almost all \( \lambda \in (\beta, 1) \) the measure \( g_\lambda \) is absolutely continuous. Only recently, Solomyak (1993) showed that \( \beta = \frac{1}{2} \).

**Proposition 6** For every \( \lambda \in (0, 1) \) the density \( f^*_\lambda \) of the limiting measure \( \mu^*_\lambda \) of \( v(\nu \tau) \) satisfies
\[ f^*_\lambda(v) = 0 \] if and only if \( |v| \geq \frac{\kappa}{1-\lambda}. \)
Moreover, on the interval \((\frac{-\kappa}{1-\lambda}, \frac{\kappa}{1-\lambda})\) we have

\[
f^*_\lambda(v) = \begin{cases} 
\frac{2-\lambda}{2\kappa} \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v + 1 \right), & -\frac{\kappa}{1-\lambda} < v < -\frac{\kappa(2-\lambda)(1-\lambda)}{(2-\lambda)(1-\lambda)} \\
\frac{2-\lambda}{2\kappa} \left[ \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v + 1 \right) - \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v - 1 \right) \right], & |v| \leq \frac{\kappa(2-\lambda)(1-\lambda)}{(2-\lambda)(1-\lambda)} < v < \frac{\kappa}{1-\lambda} \\
\frac{2-\lambda}{2\kappa} \left[ 1 - \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v - 1 \right) \right], & \frac{\kappa(2-\lambda)(1-\lambda)}{(2-\lambda)(1-\lambda)} < v < \frac{\kappa}{1-\lambda},
\end{cases}
\]

where \(\varrho_\lambda\) is the distribution function of \(\zeta_\lambda\) defined by Equation 4.6.

**Proof.** Recall that we have \(v(n\tau) = h_n(\xi_0)\). By Proposition 2, the sequence \((2-\lambda)h_n(\xi_0)\) converges in distribution to \(\xi + \zeta_\lambda\) and the random variables \(\xi\) and \(\zeta_\lambda\) are statistically independent. Since \(\xi\) has the uniform density on \([-1,1]\) and \(\varrho_\lambda\) is continuous, the density of \(\xi + \zeta_\lambda\) is given by

\[
\int_{-\infty}^{\infty} \frac{1}{2} [1-x, 1] (x-z) d\rho_\lambda(z) = \int_{-\infty}^{\infty} \frac{1}{2} [1-x, 1] (z) d\rho_\lambda(z) = \int_{-x}^{x+1} \frac{1}{2} d\rho_\lambda(z) = \frac{1}{2} (\rho_\lambda(x+1) - \rho_\lambda(x-1)).
\]

Since \(v(n\tau)\) converges in distribution to \(\frac{\kappa}{2-\lambda}(\xi + \zeta_\lambda)\), it follows that \(\mu^*_\lambda\) is the distribution of \(\frac{\kappa}{2-\lambda}(\xi + \zeta_\lambda)\). Thus \(\mu^*_\lambda\) has a density given by

\[
f^*_\lambda(v) = 2-\lambda \frac{2-\lambda}{2\kappa} \left( \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v + 1 \right) \right) - \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v - 1 \right), \quad v \in \mathbb{R}.
\]

Consequently, \(f^*_\lambda(v) = 0\) if and only if \(\varrho_\lambda \left( \frac{2-\lambda}{\kappa} v + 1 \right) = \varrho_\lambda \left( \frac{2-\lambda}{\kappa} v - 1 \right).\)

Since \(\varrho_\lambda\) is nondecreasing, it must be constant outside the set of points on which it is increasing, which is contained in the interval \([-\frac{\kappa}{1-\lambda}, \frac{\kappa}{1-\lambda}]\). Hence \(\varrho_\lambda(x) = 0\) for \(x \leq -\frac{\kappa}{1-\lambda}\) and \(\varrho_\lambda(x) = 1\) for \(x \geq \frac{1}{1-\lambda}\). Therefore if \(|v| \geq \frac{\kappa}{1-\lambda}\), then \(f^*_\lambda(v) = 0\). If \(\lambda \geq \frac{1}{2}\) the function \(\varrho_\lambda\) is increasing on the interval \([-\frac{\kappa}{1-\lambda}, \frac{\kappa}{1-\lambda}]\), thus \(f^*_\lambda\) is positive on \([-\frac{\kappa}{1-\lambda}, \frac{\kappa}{1-\lambda}]\). Now let \(\lambda < \frac{1}{2}\). Since \(\frac{1}{1-\lambda}, -\frac{1}{1-\lambda} \in K_\lambda\), we also have \(\frac{1-2\lambda}{1-\lambda}, \frac{1-2\lambda}{1-\lambda} \in K_\lambda\) and they divide the interval \([-\frac{1}{1-\lambda}, \frac{\kappa}{1-\lambda}]\) into three intervals of length 2, \(2\frac{1-2\lambda}{1-\lambda}\), and 2 respectively. Since the middle interval has length less than 2 and the distance between the points \(2\frac{1-2\lambda}{\kappa} v + 1\) and \(2\frac{1-2\lambda}{\kappa} v - 1\) is always 2, the result follows in this case as well.

**Remark 11** If \(\lambda = \frac{3}{2}\) then \(\varrho_\lambda(x) = \frac{1}{2}(x+2), 2-\lambda = \frac{3}{2}\) and the density is equal to

\[
f^*_\lambda(v) = \begin{cases} 
\frac{9}{32\kappa}(v+2), & -2\kappa < v < -\frac{2\kappa}{3} \\
\frac{3}{8\kappa}, & |v| \leq \frac{2\kappa}{3} \\
\frac{9}{32\kappa}(2-v), & \frac{2\kappa}{3} < v < 2\kappa
\end{cases}
\]

**Remark 12** The invariant measure for the baker transformation \(S_\beta\) of Example 7 is the product of the distribution of \((1-\lambda)\zeta_\lambda\) and the normalized Lebesgue measure. Thus, in this example the limiting measure for \(v_n\) is the distribution of \((1-\lambda)\zeta_\lambda\), which may be either singular or absolutely continuous.
4.2 Graphical illustration of the velocity density evolution with dyadic map perturbations

What is the probability density function of \( h_n(\xi_0) \) defined by Equation 4.2 when \( \xi_0 \) is distributed according to \( \nu \)? For many maps, including the dyadic map example being considered here, this can be calculated analytically which is the subject of this section.

Let \( Y \) be an interval and let \( \nu \) have a density \( g_* \) with respect to Lebesgue measure. Then the distribution of \( h_n(\xi_0) \) is given by

\[
\Pr\{h_n(\xi_0) \in A\} = \Pr\{\xi_0 \in h_n^{-1}(A)\} = \int_{h_n^{-1}(A)} g_*(y)dy, \quad A \in \mathcal{B}(\mathbb{R}).
\]

To obtain the density of \( h_n(\xi_0) \) with respect to the Lebesgue measure, one has to write the last integral as \( \int_A g_n(x)dx \) for some nonnegative function \( g_n \). If the map \( h_n : Y \rightarrow \mathbb{R} \) is nonsingular with respect to Lebesgue measure, then the Frobenius-Perron operator \( P_{h_n} : L^1([-1, 1]) \rightarrow L^1(\mathbb{R}) \) for \( h_n \) exists and \( g_n = P_{h_n}g_* \).

Let \( Y = [-1, 1] \) and let \( T \) be the dyadic map. Remember that \( P_T \) has uniform invariant density \( g_*(y) = \frac{1}{2}1_{[-1, 1]}(y) \). Since \( h_n \) is a linear function on each interval \( (\frac{k}{r}, \frac{k+1}{r}] \) with constant derivative, say \( h'_n \), we have

\[
g_n(v) = P_{h_n}g_*(v) = \frac{1}{2h'_n} \sum_{k=-2^n}^{2^n-1} 1_{h_n((\frac{k}{r}, \frac{k+1}{r}]})(v), \quad v \in \mathbb{R}. \tag{4.8}
\]

The derivative \( h'_n \) satisfies the recurrence equation \( h'_n = \lambda^{n-1} + 2h'_{n-1} \), \( n \geq 1 \) and \( h'_0 = 0 \) and is equal to \( \frac{(\lambda - 3)2^n + \lambda^n}{\lambda - 2} \) for each \( n \).

In Figure 4.1 we show the evolution of the velocity densities \( g_n \) when \( T \) is the dyadic map for two different values of \( \lambda \). For \( \lambda = \frac{1}{2} \) (left hand panels) the density rapidly (by \( n = 8 \)) approaches the analytic form given in Equation 4.7. On the right hand side, for \( \lambda = 0.8 \) the velocity densities have, by \( n = 8 \), approached a Gaussian-like form but supported on a finite interval. In both cases the support of the limiting densities is in agreement with Proposition 6. Figure 4.2 shows \( g_8(v) \) for six different values of \( \lambda \).

4.3 r-dyadic map

Let \( r \geq 2 \) be an integer. Consider the \( r \)-dyadic transformation on the interval \([0, 1]\)

\[
T(y) = ry \pmod{1}, \quad y \in [0, 1].
\]

The proof of Proposition 5 carries over to this transformation when \( (\alpha_k) \) is a sequence of independent random variables taking values in \( \{0, 1, \ldots, r - 1\} \) with equal probabilities, i.e. \( \Pr(\alpha_k = i) = \frac{1}{r} \), \( i = 0, 1, \ldots, r - 1 \), and \( \xi \) is a random variable uniformly distributed on \([0, 1]\) and independent of the sequence of random variables \( (\alpha_k) \). Then the limiting measure for \( v_n \) is the distribution of the random variable

\[
\frac{\kappa}{r - \lambda} \left( \xi + \sum_{k=0}^{\infty} \lambda^k \alpha_{k+1} \right).
\]
Figure 4.1: This figure shows the successive evolution of the densities $g_n$, $n = 2, \ldots, 8$ of the velocity under perturbations from the dyadic map. In the left hand series of densities, $\lambda = \frac{1}{2}$, while on the right $\lambda = 0.8$. The densities $g_n$ when $\lambda = \frac{1}{2}$ rapidly approach the limiting analytic form $f_{\frac{1}{2}}$ given in Equation 4.7. In both cases, $\kappa = 1$. 
Figure 4.2: This figure illustrates the form of the density \( g_8(v) \) for perturbations coming from the dyadic map, as computed from Equation 4.8 and various values of \( \lambda \) as indicated, with \( \kappa = 1 \). In every case the initial velocity density was the uniform invariant density of the dyadic map.

5 Gaussian Behaviour in the limit \( \tau \to 0 \).

In a series of papers Beck and Roepstorff (1987), Beck (1990b), Beck (1990a), Beck (1996) and Hilgers and Beck (1999), motivated by questions related to alternative interpretations of Brownian motion, have numerically examined the dynamic character of the iterates of dynamical systems of the form

\[
\begin{align*}
    u_{n+1} &= \lambda u_n + \sqrt{\tau} y_n, & \lambda \equiv e^{-\gamma \tau} \\
    y_{n+1} &= T(y_n),
\end{align*}
\]

(5.1)

(5.2)

in which \( T \) is a ‘chaotic’ mapping and \( \tau \) is a small temporal scaling parameter. They refer to these systems as linear Langevin systems, and point out that they arise from the following

\[
\dot{u} = -\gamma u + \sqrt{\tau} \sum_{i=1}^{\infty} y_{i-1} \delta(t - i\tau).
\]

Integrating this equation one obtains Equation 5.1 with \( u_n = u(n\tau) \).

For situations in which the map \( T \) is selected from the class of Chebyshev maps [c.f. Equation 2.8 and Adler and Rivlin (1964)], Hilgers and Beck (2001) have provided abundant numerical evidence that the density of the distribution of the sequence of iterates \( \{u_i\} \) for \( N \) quite large, may be approximately normal, or Gaussian, as \( \lambda \to 1 \), and Hilgers and Beck (1999) have provided some of the same type of numerical evidence for perturbations coming from the dyadic and hat maps. Our results provide the analytic basis for these observations.

In this section we consider and answer the question when one can obtain Gaussian processes by studying appropriate scaling limits of the velocity and position variables. We first recall what is meant by a Gaussian process.
An \( \mathbb{R} \)-valued stochastic process \( \{ \zeta(t); t \in (0, \infty) \} \) is called Gaussian if, for every integer \( l \geq 1 \) and real numbers \( 0 < t_1 < t_2 < \ldots < t_l < \infty \) the random vector \( (\zeta(t_1), \ldots, \zeta(t_l)) \) has a joint normal distribution or equivalently, for all \( d_j \in \mathbb{R}, j = 1, \ldots, k \), the random variable \( \sum_{j=1}^{l} d_j \zeta(t_j) \) is Gaussian. The finite dimensional distributions of a Gaussian process are completely determined by its first moment \( m(t) = E(\zeta(t)) \) and its covariance function

\[
K_{\zeta}(t, s) = E(\zeta(t) - m(t))(\zeta(s) - m(s)), \quad s, t > 0.
\]

If \( m(t) \equiv 0, t > 0 \) we say that \( \zeta \) is a zero-mean Gaussian process. The initial random variable \( \zeta(0) \) can be either identically equal to zero or can be any other random variable independent of the process \( \{ \zeta(t); t \in (0, \infty) \} \).

Now we recall the Ornstein-Uhlenbeck theory of Brownian motion for a free particle. The Ornstein-Uhlenbeck velocity process is a solution of the stochastic differential equation

\[
dV(t) = -\gamma V(t)dt + \sigma_0 dw(t),
\]

where \( w \) is a standard Wiener process, and the solution of this equation is

\[
V(t) = e^{-\gamma t}V(0) + \sigma_0 \int_0^t e^{-\gamma(t-s)}dw(s).
\]

In other words, \( V \) is an Ornstein-Uhlenbeck velocity process if \( \zeta \) defined by \( \zeta(t) = V(t) - e^{-\gamma t}V(0), t \geq 0 \), is a zero-mean Gaussian process with covariance function

\[
K_{\zeta}(t, s) = \frac{\sigma_0^2}{2\gamma}(e^{2\gamma \min(t,s)} - 1)e^{-\gamma(t+s)}.
\]

If the initial random variable \( V(0) \) has a normal distribution with mean zero and variance \( \frac{\sigma_0^2}{2\gamma} \), then \( V \) itself is a stationary, zero-mean Gaussian process with covariance function

\[
K_V(t, s) = \frac{\sigma_0^2}{2\gamma}e^{-\gamma|t-s|}.
\]

Let \( X(t) \) denote the position of a Brownian particle at time \( t \). Then

\[
X(t) = X(0) + \int_0^t V(s)ds.
\]

In other words, \( X \) is an Ornstein-Uhlenbeck position process if \( \eta \) defined by \( \eta(t) = X(t) - X(0) - \frac{1}{\gamma}e^{-\gamma t}V(0) \) is a zero-mean Gaussian process with covariance function

\[
K_{\eta}(t, s) = \frac{\sigma_0^2}{2\gamma^3} \left( 2\gamma \min(t, s) - 2 + 2e^{-\gamma t} + 2e^{-\gamma s} - e^{-\gamma|t-s|} - e^{-\gamma(t+s)} \right).
\]

In particular the variance of \( \eta(t) \) is equal to \( \frac{\sigma_0^2}{2\gamma}(2\gamma t - 3 + 4e^{-\gamma t} - e^{-2\gamma t}) \).

Let \( h \in L^2(Y, \mathcal{B}, \nu) \) be such that \( \int h(y)\nu(dy) = 0 \). Assume that \( x_0, v_0, \) and \( \xi_0 \) are independent random variables on \( (Y, \mathcal{B}, \nu) \) and \( \xi_0 \) is distributed according to \( \nu \). The solution of Equation 3.17 is of the form

\[
v(t) = e^{-\gamma(t-n\tau)}v(n\tau), \quad t \in [n\tau, (n+1)\tau), \quad n \geq 0.
\]
We indicate the dependence of $x(t)$ and $v(t)$ on $\tau$ by writing $x_\tau(t)$ and $v_\tau(t)$ respectively. Let $n = \left\lfloor \frac{t}{\tau} \right\rfloor$, where the notation $\lfloor \cdot \rfloor$ indicates the integer value of the argument, for $t \in [n\tau, (n + 1)\tau)$, substitute $\lambda = e^{-\gamma\tau}$, and use Equation 5.26 to obtain

$$v_\tau(t) = e^{-\gamma t}v_0 + \kappa e^{-\gamma t} \sum_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} e^{\gamma j\tau} h(T^j(\xi_0)), \quad t \geq 0,$$

and Equation 5.30 to obtain

$$x_\tau(t) = x_0 + \frac{1 - e^{-\gamma t}}{\gamma} v_0 + \frac{\kappa}{\gamma} \left( \sum_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} h(T^j(\xi_0)) - e^{-\gamma t} \sum_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} e^{\gamma j\tau} h(T^j(\xi_0)) \right).$$

Observe that the first moment of $v_\tau(t)$ is equal to

$$\int v_\tau(t) \nu(dy) = e^{-\gamma t} \int v_0(y) \nu(dy),$$

since the random variables $h(T^j(\xi_0))$ have a first moment equal to 0. Assume for simplicity that $\int h(y)h(T^j(y))\nu(dy) = 0$ for $j \geq 1$ and set $\sigma^2 = \int h^2(y)\nu(dy)$. Since the random variables $v_0$ and $h(T^j(\xi_0))$ are independent, the second moment of $v_\tau(t)$ takes the form

$$\int v_\tau(t)^2 \nu(dy) = e^{-2\gamma t} \int v_0^2(y) \nu(dy) + \kappa^2 e^{-2\gamma t} \sum_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} e^{2\gamma j\tau} \int h^2(T^j(y))\nu(dy)$$

$$= e^{-2\gamma t} \int v_0^2(y) \nu(dy) + \sigma^2 \kappa^2 e^{-2\gamma t} \frac{1 - e^{2\gamma(\lfloor \frac{t}{\tau} \rfloor + 1)}}{1 - e^{2\gamma t}}.$$

If $\sigma$ and $\gamma$ do not depend on $\tau$, we have

$$\lim_{\tau \to 0} \sigma^2 \kappa^2 e^{-2\gamma t} \left( e^{2\gamma(\lfloor \frac{t}{\tau} \rfloor + 1)} - 1 \right) \frac{\tau}{e^{2\gamma t} - 1} = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}).$$

Hence the limit of the second moment of $v_\tau(t)$ as $\tau \to 0$ is finite and positive if and only if $\kappa$ depends on $\tau$ in such a way that

$$\lim_{\tau \to 0} \frac{\kappa^2}{\tau}$$

is finite and positive.

Beck and Roepstorff (1987) take $\kappa_\tau = \sqrt{\tau}$ from the outset, and claim that in the limit $\tau \to 0$ the process $v_\tau(t)$ converges to the Ornstein-Uhlenbeck velocity process when the sequence $(h \circ T^j)$ has a so called $\phi$-mixing property$^2$ on the probability space $(Y, B, \nu)$. In fact, the following result can be proved.

$^2$A sequence of random variables $\{\xi_j : j \geq 0\}$ is called $\phi$-mixing if

$$\lim_{n \to \infty} \sup_{A \in \mathcal{F}_1^k, b \in \mathcal{F}_k^{k+n}, k \geq 1} \left\{ \left| \Pr(A \cap B) - \Pr(A) \Pr(B) \right| / \Pr(A) \right\} = 0$$

where $\mathcal{F}_1^k$ and $\mathcal{F}_k^{k+n}$ denote the $\sigma$-algebra generated by the random variables $\xi_1, \ldots, \xi_k$ and $\xi_{k+n}, \xi_{k+n+1}, \ldots$ respectively.
Theorem 9  Let \( (Y, \mathcal{B}, \nu) \) be a normalized measure space and \( T : Y \to Y \) be ergodic with respect to \( \nu \). Let \( h \in L^2(Y, \mathcal{B}, \nu) \) be such that \( \sum_{n=0}^{\infty} \|\mathcal{P}_{T,\nu}^n h\|_2 < \infty \) and let
\[
\sigma = \left( \int h(y)^2 \nu(dy) + 2 \sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy) \right)^{1/2}
\]
be positive. Assume that \( \gamma > 0 \) and
\[
\lim_{\tau \to 0} \frac{\kappa^2_{\tau}}{\tau} = 1.
\]
Then for each \( v_0 \) the finite dimensional distributions of the velocity process \( v_\tau \) given by Equation \ref{v_\tau} converge weakly as \( \tau \to 0 \) to the finite dimensional distributions of the Ornstein-Uhlenbeck velocity process \( V \) for which \( V(0) = v_0 \) and \( \sigma_0 = \sigma \).

Proof.  By Theorem \ref{lem:asymptotic} we have \( \mathcal{P}_{T,\nu}(\tilde{h}) = 0 \) where \( \tilde{h} = h + f \circ T \) and \( f = \sum_{n=1}^{\infty} \mathcal{P}_{T,\nu}^n h \). For \( t \geq 0 \) and \( \tau > 0 \) define
\[
\zeta_{\tau}(t) = \kappa_{\tau} e^{-\gamma t} \sum_{j=0}^{\lfloor \frac{j}{\tau} \rfloor} e^{\gamma j} \tilde{h} \circ T^j, \quad \tilde{\zeta}_{\tau}(t) = \kappa_{\tau} e^{-\gamma t} \sum_{j=0}^{\lfloor \frac{j}{\tau} \rfloor} e^{\gamma j} (f \circ T^j + f \circ T^j).
\]
Then
\[
v_{\tau}(t) = e^{-\gamma t} v_0 + \zeta_{\tau}(t) + \tilde{\zeta}_{\tau}(t).
\]
Observe that
\[
\tilde{\zeta}_{\tau}(t) = \kappa_{\tau} e^{-\gamma t} (e^{\gamma \lfloor \frac{j}{\tau} \rfloor} f \circ T^{\lfloor \frac{j}{\tau} \rfloor} + f) + \kappa_{\tau} e^{-\gamma t} (e^{\gamma \tau} - 1) \sum_{j=1}^{\lfloor \frac{j}{\tau} \rfloor} e^{\gamma j} f \circ T^j.
\]
Hence
\[
\|\tilde{\zeta}_{\tau}(t)\|_2 \leq 2 \kappa_{\tau} \|e^{-\gamma t} (e^{\gamma \lfloor \frac{j}{\tau} \rfloor} + 1)\|_2, \quad t \geq 0, \quad \tau > 0.
\]
This and Lemma \ref{lem:asymptotic} imply that the finite dimensional distributions of \( v_{\tau}(t) - e^{-\gamma t} v_0 \) converge weakly to the corresponding finite dimensional distributions of a zero mean Gaussian process \( \zeta \) with \( \zeta(0) = 0 \) and the covariance function \( K_{\zeta}(t, s) \) given by Equation \ref{K_\zeta} where \( \sigma_0^2 = \|\tilde{h}\|_2^2 \), which completes the proof.

For the corresponding process we have the following.

Theorem 10  Under the assumptions of Theorem \ref{lem:asymptotic} let \( V(0) = v_0 \). Then for each \( x_0 \) the finite dimensional distributions of the position process \( x_{\tau} \) given by Equation \ref{position_process} converge weakly as \( \tau \to 0 \) to the finite dimensional distributions of the Ornstein-Uhlenbeck position process \( X \) for which \( X(0) = x_0 \).

Proof.  This follows from Lemma \ref{lem:asymptotic} similarly as the preceding theorem follows from Lemma \ref{lem:asymptotic}.

Example 14  Let us apply Theorem \ref{lem:asymptotic} to a transformation \( T : [-1, 1] \to [-1, 1] \) and \( h(y) = y \). We have \( \mathcal{P}_{T,\nu} h = 0 \) when \( T \) is the hat map, Equation \ref{hat_map}. Then \( \sigma^2 = 1/3 \). Thus all assumptions of Theorem \ref{lem:asymptotic} are satisfied. When \( T \) is one of the Chebyshev maps \ref{chebyshev_map} \( S_N \) we also have \( \mathcal{P}_{T,\nu} h = 0 \) by Example \ref{chebyshev_example} and \( \sigma^2 = 1/2 \). For the dyadic map \ref{dyadic_map}, the series \( \sum_{n=1}^{\infty} \mathcal{P}_{T,\nu}^n h \) is absolutely convergent in \( L^2([-1, 1], \mathcal{B}([-1, 1]), \nu) \) and is equal to \( h \). This implies that \( \sigma = \|2h - h \circ T\|_2 \), thus in this case \( \sigma = 1 \), which can be easily calculated. Thus all of the numerical examples and studies of Beck and co-workers cited above are covered by this example.
6 Discussion

In this paper we were motivated by the strong statistical properties of discrete dynamical systems to consider when Brownian motion like behaviour could emerge in a simple toy system. To do this, we have reviewed and significantly extended a class of central limit theorems for discrete time maps. These new results, presented primarily in Sections 2.3 and 2.4, were then applied in Section 3 to the Langevin-like equations

\[
\frac{dx(t)}{dt} = v(t), \quad \frac{dv(t)}{dt} = -\gamma v(t) + \kappa \eta(t)
\]

in which the underlying noise \(\eta(t)\) need not be a Gaussian noise but may be substituted by

\[
\eta(t) = \sum_{n=0}^{\infty} h(\xi(t)) \delta(t - n\tau)
\]

with a highly irregular deterministic function \(\xi(n\tau)\). When the variables \(h(\xi(n\tau))\) are uncorrelated Gaussian distributed (thus in fact independent) random variables then the limiting distribution of \(v(n\tau)\) is Gaussian. This is need not be the case for the deterministic noise produced by perturbations derived from highly chaotic semi-dynamical systems. However (Section 5), in the limit \(\tau \to 0\) both types of noise produce the same stochastic process in this limit, the Ornstein-Uhlenbeck process. Finally, in Section 4 we have illustrated all of our results of Section 3 for the specific case of perturbations derived from the exact dyadic map.

The significance of these considerations is rather broad. It is the norm in experimental observations that any experimental variable that is recorded will be “contaminated” by “noise”. Sometimes the distribution of this noise is approximately Gaussian, sometimes not. The considerations here illustrate quite specifically that the origins of the noise observed experimentally need not be due to the operation of a random process (random in the sense that there is no underlying physical law allowing one to predict exactly the future of the process based on the past). Rather, the results we present strongly suggest, as an alternative, that the fluctuations observed experimentally might well be the signature of an underlying deterministically chaotic process.

Acknowledgments

This work was supported by MITACS (Canada) and the Natural Sciences and Engineering Research Council (NSERC grant OGP-0036920, Canada). The first draft of this paper was written while the second author was visiting McGill University, whose hospitality and support are gratefully acknowledged. We thank Radosław Kamiński for providing the computations on which Figure 1.3 was based.

A Appendix: Limit Theorems for Dependent Random Variables

This Appendix reviews known general central limit theorems from probability theory which we can use directly in the context of noninvertible dynamical systems. With their help we then prove several results which we have used in the preceding Sections.
Consider random variables arranged in a double array
\[
\begin{align*}
\zeta_{1,1}, \zeta_{1,2}, \ldots, \zeta_{1,k_1} \\
\zeta_{2,1}, \zeta_{2,2}, \ldots, \zeta_{2,k_2} \\
\vdots \\
\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,k_n} \\
\vdots
\end{align*}
\] (1.1)
with \(k_n \to \infty\) as \(n \to \infty\). We shall give conditions which imply that the row sums converge in distribution to a Gaussian random variable \(\sigma N(0, 1)\), that is
\[
\sum_{i=1}^{k_n} \zeta_{n,i} \to^d \sigma N(0, 1).
\] (1.2)
We shall require the Lindeberg condition
\[
\lim_{n \to \infty} \sum_{i=1}^{k_n} E(\zeta_{n,i}^2 1\{|\zeta_{n,i}| > \epsilon\}) = 0 \text{ for every } \epsilon > 0.
\] (1.3)

If independence in each row is allowed, then we have the classical Lindeberg-Feller theorem.

**Theorem 11** \(\text{(Chung (2001, Theorem 7.2.1))}\) Let the random variables \(\zeta_{n,1}, \ldots, \zeta_{n,k_n}\) be independent for each \(n\). Assume that \(E(\zeta_{n,i}) = 0\) and \(\sum_{i=1}^{k_n} E(\zeta_{n,i}^2) = 1, \ n \geq 1\). Then the Lindeberg condition holds if and only if
\[
\sum_{i=1}^{k_n} \zeta_{n,i} \to^d N(0, 1),
\]
and
\[
\text{for all } \delta > 0, \ \max_{1 \leq i \leq k_n} \Pr\{|\zeta_{n,i}| > \delta\} \to 0.
\] (1.4)
If one considers a map \(T\) on a probability space \((Y, B, \nu)\) which preserves the measure \(\nu\), and defines \(\zeta_{n,i}\) to be \(\frac{1}{\sqrt{n}} f \circ T^{i-1}\) for \(i = 1, \ldots, n\) with \(f\) measurable, then Condition 1.4 holds. This is because
\[
\Pr\{|\zeta_{n,i}| > \delta\} = \nu(\{y \in Y : |f(T^{i-1}(y))| > \delta \sqrt{n}\}),
\]
and \(\{y \in Y : |f(T^{i-1}(y))| > \delta \sqrt{n}\} = T^{-i+1}(\{y \in Y : |f(y)| > \delta \sqrt{n}\})\). Thus by the invariance of \(\nu\) this leads to
\[
\max_{1 \leq i \leq k_n} \Pr\{|\zeta_{n,i}| > \delta\} = \nu(\{y \in Y : |f(y)| > \delta \sqrt{n}\}) \to 0.
\]
Similarly, if one takes a square integrable \(f\), then the Lindeberg condition 1.3 is satisfied. Indeed, we have
\[
E(|\zeta_{n,i}|^2 1\{|\zeta_{n,i}| > \epsilon\}) = \frac{1}{\sqrt{n}} \int_{\{z : |f(T^{i-1}(z))| \geq \sqrt{n}\}} f^2(T^{i-1}(y)) \nu(dy)
\]
and by the change of variables applied to \(T^{i-1}\) this reduces to
\[
\frac{1}{n} \int_{\{|f| \geq \sqrt{n}\}} f^2(y) \nu(dy).
\]
Hence
\[ \sum_{i=1}^{n} E(|\zeta_{n,i}|^2 1_{\{|\zeta_{n,i}|>\epsilon\}}) = \int_{\{|f|>\sqrt{n}\epsilon\}} f^2(y)\nu(dy), \]
which converges to 0 by the Dominated Convergence Theorem under our assumption that \( f^2 \) is integrable.

Since our random variables are dependent, we cannot apply the above theorem. Instead, we use the notion of martingale differences for which there is a natural generalization of Theorem 11. Moreover, additional assumptions are needed, as one can easily check that if \( T \) is the identity map, then
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f \circ T^{i-1} = \frac{n}{\sqrt{n}} f \]
which cannot be convergent to a Gaussian random variable.

We first recall the definition of conditional expectation. Let a probability space \((\Omega, \mathcal{F}, \Pr)\) be given and let \( \mathcal{G} \) be a sub-\(\sigma\)-algebra of \( \mathcal{F} \). For \( \zeta \in L^1(\Omega, \mathcal{F}, \Pr) \) there exists a random variable \( E(\zeta | \mathcal{G}) \), called the conditional expected value of \( \zeta \) given \( \mathcal{G} \), having the following properties: it is \( \mathcal{G} \)-measurable, integrable and satisfies the equation
\[ \int_A E(\zeta | \mathcal{G})(\omega) \Pr(d\omega) = \int_A \zeta(\omega) \Pr(d\omega), \quad A \in \mathcal{G}. \]
The existence and uniqueness of \( E(\zeta | \mathcal{G}) \) for a given \( \zeta \) follows from the Radon-Nikodym theorem. The transformation \( \zeta \mapsto E(\zeta | \mathcal{G}) \) is a linear operator between the spaces \( L^1(\Omega, \mathcal{F}, \Pr) \) and \( L^1(\Omega, \mathcal{G}, \Pr) \), so sometimes it is called an operator of conditional expectation.

Let \( \{\zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\} \) be a family of random variables defined on a probability space \((\Omega, \mathcal{F}, \Pr)\). For each \( n \geq 1 \), let a family \( \{\mathcal{F}_{n,i} : i \geq 0\} \) of sub-\(\sigma\)-algebras of \( \mathcal{F} \) be given. Consider the following set of conditions
\begin{enumerate}[(i)]  
  \item \( E(\zeta_{n,i}) = 0 \) and \( E(\zeta_{n,i}^2) < \infty \),  
  \item \( \mathcal{F}_{n,i-1} \subseteq \mathcal{F}_{n,i} \),  
  \item \( \zeta_{n,i} \) is \( \mathcal{F}_{n,i} \) measurable,  
  \item \( E(\zeta_{n,i} | \mathcal{F}_{n,i-1}) = 0 \) for each \( 1 \leq i \leq k_n, n \geq 1 \).
\end{enumerate}
A family \( \{\mathcal{F}_{n,i}, \zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\} \) satisfying conditions (i)-(iv) is called a (square integrable) martingale differences array.

The next theorem is from Billingsley (1995).

**Theorem 12** (Billingsley (1995, Theorem 35.12)) Let \( \{\zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\} \) be a martingale difference array satisfying the Lindeberg condition \( i.3 \). If
\[ \sum_{i=1}^{k_n} E(\zeta_{n,i}^2 | \mathcal{F}_{n,i-1}) \rightarrow^p \sigma^2, \tag{1.5} \]
where \( \sigma \) is a nonnegative constant, then
\[ \sum_{i=1}^{k_n} \zeta_{n,i} \rightarrow^d \sigma N(0,1). \tag{1.6} \]
If the limit in Condition 1.5 is a random variable instead of the constant $\sigma^2$, we obtain convergence to mixtures of normal distributions Eagleson (1975, Corollary p. 561).

**Theorem 13** Let $\{\zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\}$ be a martingale difference array satisfying the Lindeberg condition 1.3. If there exists an $F_{\infty} = \bigcap_{n=1}^{\infty} F_{n,0}$-measurable, a.s. positive and finite random variable $\eta$ such that

$$\sum_{i=1}^{k_n} E(\zeta_{n,i}^2 | F_{n,i-1}) \to P \eta,$$

then $\sum_{i=1}^{k_n} \zeta_{n,i}$ is convergent in distribution to a measure whose characteristic function is $\varphi(r) = E(\exp(-\frac{1}{2}r^2\eta))$.

The above result shows that to obtain a normal distribution in the limit a specific normalization is needed and we have the following

**Theorem 14** (Gaenssler and Joos (1992, Theorem 3.6)) Let $\{\zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\}$ be a martingale difference array and let $\eta$ be a real-valued random variable such that $Pr(0 < \eta < \infty) = 1$. Suppose that

$$\lim_{n \to \infty} E(\max_{1 \leq i \leq k_n} |\zeta_{n,i}|) = 0$$

(1.8)

and

$$\sum_{i=1}^{k_n} \zeta_{n,i}^2 \to P \eta.$$  

(1.9)

If $\eta$ is $F_{n,i}$-measurable for each $n$ and for each $1 \leq i \leq k_n$, then

$$\frac{\sum_{i=1}^{k_n} \zeta_{n,i}}{\sqrt{\sum_{i=1}^{k_n} \zeta_{n,i}^2}} \to d N(0,1).$$

**Remark 13** If $\eta$ in Condition 1.9 is constant, $\eta = \sigma^2$, then the conclusion of Theorem 14 is equivalent to

$$\sum_{i=1}^{k_n} \zeta_{n,i} \to d \sigma N(0,1).$$

Note also that Condition 1.8 is implied by the Lindeberg condition.

The next result gives conditions for moment convergence in Theorem 12.

**Theorem 15** (Hall (1978, Theorem) Teicher (1988, Theorem 3)) Let $\{\zeta_{n,i} : 1 \leq i \leq k_n, n \geq 1\}$ be a martingale difference array with $\sum_{i=1}^{k_n} E(\zeta_{n,i}^2) = \sigma^2$ where $\sigma > 0$. Suppose that for $p > 1$

$$\sum_{i=1}^{k_n} E|\zeta_{n,i}|^{2p} \to 0 \quad \text{and} \quad E\left| \sum_{i=1}^{k_n} E(\zeta_{n,i}^2 | F_{n,i-1}) - \sigma^2 \right|^p \to 0.$$  

(1.10)

Then

$$\lim_{n \to \infty} E|\sum_{i=1}^{k_n} \zeta_{n,i}|^{2p} = E|N(0,\sigma^2)|^{2p}.$$  

54
Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space and \(T : Y \to Y\) be a measurable map such that \(T\) preserves the measure \(\nu\). Recall from Section 2 the relation between the transfer operator \(\mathcal{P}_{T,\nu}\), the Koopman operator and the operator of conditional expectation which gives

\[
\mathcal{P}_{T,\nu} \circ U_T f = f \quad \text{and} \quad U_T \circ \mathcal{P}_{T,\nu} f = E(f | T^{-1}(\mathcal{B})), \quad f \in L^1(Y, \mathcal{B}, \nu).
\]

**Lemma 1** Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, and \(T : Y \to Y\) be a measurable map such that \(T\) preserves the measure \(\nu\). Let \(\{c_{n,i} : 1 \leq i \leq k_n, n \geq 1\}\) be a family of real numbers and \(h \in L^2(Y, \mathcal{B}, \nu)\). Suppose that \(\mathcal{P}_{T,\nu} h = 0\). Then

\[
\zeta_{n,i} = c_{n,i} h \circ T^{k_n-i}, \quad 1 \leq i \leq k_n, \quad \zeta_{n,i} = 0, \quad i > k_n,
\]

with

\[
\mathcal{F}_{n,i} = T^{-k_n+i}(\mathcal{B}), \quad 0 \leq i \leq k_n, \quad \text{and} \quad \mathcal{F}_{n,i} = \mathcal{B}, \quad i > k_n, \quad n \geq 1
\]

is a martingale difference array and if \(c_{n,i} = \frac{1}{\sqrt{k_n}}, \quad 1 \leq i \leq k_n\) then the following hold

(i) Lindeberg condition \[135\]

(ii) Conditions \[146\] and \[144\] with \(\eta = E(h^2|\mathcal{I})\) where \(\mathcal{I}\) is the \(\sigma\)-algebra of all \(T\)-invariant sets;

(iii) Condition \[136\] with \(\sigma^2 = \int h^2 d\nu\) provided that \(T\) is ergodic;

(iv) Condition \[144\] for every \(p > 1\) provided that \(T\) is ergodic and \(h \in L^\infty(Y, \mathcal{B}, \nu)\).

**Proof.** To check conditions (ii), and (iii) of the definition of a martingale difference array, observe that \(T^{-j-1}(\mathcal{B}) \subset T^{-j}(\mathcal{B})\) and \(h \circ T^j\) is \(T^{-j}(\mathcal{B})\) measurable. The Koopman and transfer operators for the iterated map \(T^j\) are just the \(j^{th}\) iterates of the operators \(U_T\) and \(\mathcal{P}_{T,\nu}\). From this and Equation \[2\] we have \(\mathcal{P}_{T,\nu} U_T h = h\) and

\[
E(h \circ T^j | T^{-j-1}(\mathcal{B})) = U_T^{j+1} \mathcal{P}_{T,\nu}^j (h \circ T^j) = U_T^{j+1} \mathcal{P}_{T,\nu} h.
\]

Since \(\mathcal{P}_{T,\nu} h = 0\), we see that \(E(h \circ T^j | T^{-j-1}(\mathcal{B})) = 0\) for \(j \geq 0\) which proves condition (iv).

The Lindeberg condition reduces, through a change of variables, to

\[
\sum_{i=1}^{k_n} E(\zeta_{n,i}^2 I_{\{|\zeta_{n,i}| > \epsilon\}}) = \int_{\{|h| \geq \sqrt{\epsilon} k_n\}} h^2 \nu(dy),
\]

but \(h^2\) is integrable and the Lindeberg condition follows. To obtain Condition \[144\] use Equation \[2\], change the order of summation

\[
\sum_{i=1}^{k_n} E(\zeta_{n,i}^2 | \mathcal{F}_{n,i-1}) = \frac{1}{k_n} \sum_{i=1}^{k_n} E(h^2 \circ T^{k_n-i} | T^{-k_n+i-1}(\mathcal{B}))
\]

\[
= \frac{1}{k_n} \sum_{i=1}^{k_n} U_T^{k_n-i+1} \mathcal{P}_{T,\nu}^{k_n-i+1} U_T^{k_n-i}(h^2)
\]

\[
= \frac{1}{k_n} \sum_{i=1}^{k_n} U_T^{k_n-i+1} \mathcal{P}_{T,\nu}(h^2) = \frac{1}{k_n} \sum_{i=0}^{k_n-1} U_T^{i+1} \mathcal{P}_{T,\nu}(h^2)
\]

and apply Birkhoff’s ergodic theorem to the integrable function \(U_T \mathcal{P}_{T,\nu}(h^2)\) to conclude that this sequence is convergent to \(E(U_T \mathcal{P}_{T,\nu}(h^2)|\mathcal{I})\) almost everywhere (with respect to \(\nu\), and consequently
in probability. Since $U_T\mathcal{P}_T u(h^2) = E(h^2|T^{-1}(B))$ and $I \subseteq T^{-1}(B)$, we have $E(U_T\mathcal{P}_T u(h^2)|I) = E(h^2|I)$. Similarly, Condition 1.2 follows from the Birkhoff ergodic theorem. In addition, if $T$ is ergodic, then $\eta$ is constant a.e. and is equal to $\int h^2 d\nu$. Since $h^2 \in L^p(Y, \mathcal{B}, \nu)$ and $p > 1$, we have

$$\sum_{i=1}^{k_n} E|\zeta_{n,i}|^{2p} = \frac{1}{n^p} \int h^{2p}(y)\nu(dy) \rightarrow 0$$

and $U_T\mathcal{P}_T u(h^2) \in L^p(Y, \mathcal{B}, \nu)$. By the ergodic theorem in $L^p$ spaces we get

$$\sum_{i=1}^{k_n} E(\zeta_{n,i}^2|F_{n,i-1}) = \frac{1}{k_n} \sum_{i=0}^{k_n-1} U_{T_i}^i U_T\mathcal{P}_T u(h^2) \rightarrow \int U_T\mathcal{P}_T u(h^2)(y)\nu(dy)$$

in $L^p(Y, \mathcal{B}, \nu)$, but $\int U_T\mathcal{P}_T u(h^2)(y)\nu(dy) = \int h^2(y)\nu(dy)$ and the proof is complete.

We now turn to the FCLT for $(\psi(T))^2$. Similarly, Condition 1.9 follows from the Birkhoff ergodic theorem. In addition, if $h$ be ergodic with respect to $\nu$, $\nu$ be a normalized measure space and $T : Y \rightarrow Y$ be ergodic with respect to $\nu$. Let $h \in L^2(Y, \mathcal{B}, \nu)$ and $\sigma = ||h||_2 > 0$. We define a random function

$$\psi_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{i=0}^{[nt]-1} h \circ T^i \text{ for } t \in [0, 1]$$

(where the sum from 0 to $-1$ is set to be 0). Note that $\psi_n$ is a right continuous step function, a random variable of $D[0, 1]$ and $\psi_n(0) = 0$.

**Lemma 2** If $\mathcal{P}_T u h = 0$, then the finite dimensional distributions of $\psi_n$ converge to those of the Wiener process $w$.

**Proof.** To show that the finite dimensional distributions of $\psi_n$ converge to the corresponding finite dimensional distributions of $w$ we use the Cramer-Wold technique. If the $c_1, \ldots, c_k$ are arbitrary real numbers and $t_0 = 0 < t_1 < \ldots < t_k \leq 1$, we put

$$\zeta_{n,i} = \begin{cases} \frac{c_j}{\sigma \sqrt{n}} h \circ T^{m-i}, & n - [nt_j] < i \leq n - [nt_{j-1}], \ j = 1, \ldots, k \\ 0, & 1 \leq i \leq n - [nt_k] \text{ and } t_k < 1 \\ 0, & i > n \end{cases}$$

Observe that

$$\sum_{i=1}^{n} \zeta_{n,i} = \sum_{j=1}^{k} c_j(\psi_n(t_j) - \psi_n(t_{j-1})).$$

By Lemma 1 $\zeta_{n,i}$ is a martingale differences array and we will verify the conditions of Theorem 1.2. For the Lindeberg condition note that

$$\sum_{i=1}^{n} E(\zeta_{n,i}^21_{\{\zeta_{n,i} > c\}}) = \sum_{j=1}^{k} \frac{c_j^2}{\sigma^2 n} (\lfloor nt_j \rfloor - [nt_{j-1}]) E(h^21_{\{|h| > c\sqrt{\sigma^2 n_j}^{-1} \}})$$

and as a finite sum of sequences converging to 0 it is convergent to 0. For Condition 1.5 observe that

$$\sum_{i=1}^{n} E(\zeta_{n,i}^2|T^{-n+i-1}(B)) = \sum_{j=1}^{k} \frac{c_j^2}{\sigma^2 n} \sum_{i=\lfloor nt_{j-1} \rfloor + 1}^{\lfloor nt_j \rfloor} U_{T_i}^i U_T h^2 \rightarrow F \sum_{j=1}^{k} c_j^2 (t_j - t_{j-1})$$

56
by the ergodicity of $T$, and the fact that $\sigma^2 = \int h^2 d\nu$. Therefore, by Theorem 12,

$$\sum_{i=1}^{n} \zeta_{n,i} \to^d \left( \sum_{j=1}^{k} c_j^2 (t_j - t_{j-1}) \right) N(0, 1).$$

Thus $\sum_{j=1}^{k} c_j (\psi_n(t_j) - \psi_n(t_{j-1}))$ converges to the Gaussian distributed random variable with mean 0 and variance $\sum_{j=1}^{k} c_j^2 (t_j - t_{j-1})$, but this is the distribution of $\sum_{j=1}^{k} c_j (w(t_j) - w(t_{j-1}))$ which completes the proof.

**Lemma 3** If $\mathcal{P}_{T,\nu,h} = 0$, then Condition 2.1 holds for each positive $\epsilon$.

**Proof.** Define a martingale difference array

$$\zeta_{n,i} = \begin{cases} \frac{1}{\sigma \sqrt{n}} h \circ T^{n-i}, & 1 \leq i \leq n \\ 0, & i > n, \; n \geq 1 \end{cases}$$

Let also $\zeta_{n,0} = 0$ and $\bar{\psi}_n(t) = \sum_{i=0}^{\lfloor nt \rfloor} \zeta_{n,i}$, $t \in [0,1], \; n \geq 1$. We have

$$\psi_n(t) = \bar{\psi}_n(1) - \bar{\psi}_n(1-t).$$

We first observe that

$$\sup_{|t-s| \leq \delta} |\psi_n(s) - \psi_n(t)| \leq \sup_{|t-s| \leq \delta} |\bar{\psi}_n(s) - \bar{\psi}_n(t)| \leq 4 \sup_{k \leq (k+1)\delta} |\bar{\psi}_n(t) - \bar{\psi}_n(k\delta)|.$$ 

This gives

$$\nu\left( \sup_{|t-s| \leq \delta} |\psi_n(s) - \psi_n(t)| > \epsilon \right) \leq \sum_{k<1} \nu\left( \sup_{k\delta < t \leq (k+1)\delta} |\bar{\psi}_n(t) - \bar{\psi}_n(k\delta)| > \frac{\epsilon}{4} \right).$$

Now applying Lemma 2 and arguments similar to those of [Brown, 1971, pp. 64-65], one can complete the proof. 

For the next results we need the following

**Lemma 4** Let $(z_i)_{i \geq 1}$ be a sequence of real numbers such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} z_i = z.$$ 

Then

$$\lim_{n \to \infty} \frac{1}{\sum_{i=1}^{k_n} a_i} = \frac{z}{\sum_{i=1}^{k_n} a_i}$$

for every sequence of integers $k_n \geq 1$ and every sequence of real numbers $a_n$ satisfying

$$\lim_{n \to \infty} k_n = \infty, \; \lim_{n \to \infty} a_n = 1, \; \text{and} \; \lim_{n \to \infty} a_n^{k_n} \neq 1 \quad (1.11)$$

and either $a_n > 1$ or $0 < a_n < 1$ for all $n \geq 1$. 

57
We shall apply Theorem 14. We have
\[ c_i b_i = c_m \sum_{i=k}^m b_i - \sum_{i=k}^{m-1} (c_{i+1} - c_i) \sum_{j=k}^i b_i \]
can be used to write
\[ \sum_{i=1}^{k_n} a_n^i z_i - z \sum_{i=1}^{k_n} a_n^i = - \sum_{i=1}^{k_n-1} (a_n^{i+1} - a_n^i)(\sum_{j=1}^i z_j - iz) + a_n^{k_n}(\sum_{i=1}^{k_n} z_i - k_n z). \]

Fix \( \epsilon > 0 \) and let \( n_0 \) be such that
\[ \left| \frac{\sum_{i=1}^m z_i}{m} - z \right| \leq \epsilon \quad \text{for} \quad m \geq n_0. \]

Suppose that \( a_n > 1 \) for all \( n \geq 1 \). The other case is proved analogously. Combining these yields, for \( k_n > n_0 \),
\[ \left| \sum_{i=1}^{k_n} a_n^i z_i - z \sum_{i=1}^{k_n} a_n^i \right| \leq \frac{\sum_{i=1}^{n_0-1} (a_n^{i+1} - a_n^i) |\sum_{j=1}^i z_j - iz|}{\sum_{i=1}^{k_n} a_n^i} + \epsilon \frac{\sum_{i=n_0}^{k_n-1} (a_n^{i+1} - a_n^i) i + a_n^{k_n} k_n}{\sum_{i=1}^{k_n} a_n^i}. \]

Letting \( n \to \infty \) we see that the first term on the right goes to zero, while the second term goes to \( \epsilon \) times a constant not depending on \( \epsilon \), which completes the proof.

**Theorem 16** Let \((Y, B, \nu)\) be a normalized measure space and \( T : Y \to Y \) be ergodic with respect to \( \nu \). Let \((k_n), (a_n)\) be sequences satisfying Condition 1.1. Let \( c \in \mathbb{R} \) and let \((c_n)\) be a sequence of real numbers such that
\[ \lim_{n \to \infty} k_n c_n^2 = c^2. \]
If \( h \in L^2(Y, B, \nu) \) is such that \( \mathbb{P}_{T, \nu} h = 0 \), then
\[ c_n \sum_{i=1}^{k_n} a_n^i h \circ T^i \to^d \sigma N(0, 1), \]
where \( \sigma = \sqrt{\frac{c^2(a^2 - 1)}{\ln a^2} ||h||^2} \) and \( a = \lim_{n \to \infty} a_n^{k_n} \).

**Proof.** From Lemma 1 it follows that \( \zeta_{n,i} = c_n a_n^{k_n+1-i} h \circ T^{k_n+1-i} \) is a martingale difference array and that
\[ \lim_{n \to \infty} \frac{1}{\sqrt{k_n}} E(\max_{1 \leq j \leq k_n} |h \circ T^j|) = 0. \]
We shall apply Theorem 14. We have
\[ \max_{1 \leq i \leq k_n} |\zeta_{n,i}| \leq |c_n| \max(a_n^{k_n}, 1) \max_{1 \leq i \leq k_n} |h \circ T^i|, \]
so
\[ E(\max_{1 \leq i \leq k_n} |\zeta_{n,i}|) \leq \sqrt{k_n} |c_n| \max(a_n^{k_n}, 1) E(\frac{1}{\sqrt{k_n}} \max_{0 \leq i \leq k_n} |h \circ T^i|). \]
Letting $n \to \infty$ we see that Condition 1.8 holds. To verify Condition 1.9 note that
\[
\sum_{i=1}^{k_n} c_{n,i}^2 = c_n^2 \sum_{i=1}^{k_n} a_{n,i}^2 h^2 \circ T^i = c_n^2 \left( \sum_{i=1}^{k_n} a_{n,i}^2 \right) \sum_{i=1}^{k_n} a_{n,i}^2 h^2 \circ T^i.
\]
Therefore Birkhoff’s ergodic theorem, Lemma 4, and the fact that
\[
\lim_{n \to \infty} c_n^2 \sum_{i=1}^{k_n} a_{n,i}^2 = \frac{c^2(a^2 - 1)}{\ln a^2}
\]
complete the proof.

**Corollary 5** Under the assumptions of Theorem 16, if for $h \in L^2(Y, \mathcal{B}, \nu)$ the series $\sum_{n=0}^{\infty} P^n_{T,\nu} h$ is convergent in $L^2(Y, \mathcal{B}, \nu)$, then
\[
c_n \sum_{i=1}^{k_n} a_{n,i}^i h \circ T^i \to^d \sigma N(0,1),
\]
where $\sigma = \sqrt{\frac{c^2(a^2 - 1)}{\ln a^2}} ||h + f - f \circ T||_2$ and $f = \sum_{n=1}^{\infty} P^n_{T,\nu} h$.

**Proof.** Theorem 6 implies $P_{T,\nu}(h + f - f \circ T) = 0$. Thus
\[
c_n \sum_{i=1}^{k_n} a_{n,i}^i (h + f - f \circ T) \circ T^i \to^d \sigma N(0,1)
\]
by Theorem 16. Therefore it remains to prove that
\[
c_n \sum_{i=1}^{k_n} a_{n,i}^i (f \circ T - f) \circ T^i \to^P 0.
\]
(1.12)

Observe that the left-hand side of Equation 1.12 is equal to
\[
c_n (a_n^{k_n} f \circ T^{k_n+1} - f \circ T) + c_n (a_n^{-1} - 1) \sum_{i=1}^{k_n} a_{n,i}^i f \circ T^i.
\]
Since $c_n \to 0$ as $n \to \infty$, the first term converges in probability to 0. From Lemma 4 and Birkhoff’s ergodic theorem it follows that
\[
\frac{\sum_{i=1}^{k_n} a_{n,i}^i f \circ T^i}{\sum_{i=1}^{k_n} a_{n,i}^i} \to^P \int f(y) \nu(dy).
\]
Therefore the sequence
\[
c_n (a_n^{-1} - 1) \sum_{i=1}^{k_n} a_{n,i}^i f \circ T^i = c_n (1 - a_n^{k_n}) \sum_{i=1}^{k_n} a_{n,i}^i f \circ T^i
\]
is also convergent in probability to 0, which completes the proof.
Remark 14 Note that we can conclude from Theorem 10 that
\[ c_n \sum_{i=m}^{k_n} a_i^n h \circ T^i \to^d \sigma N(0,1), \]
where \( m \geq 0 \) is any fixed integer, because \( c_n \) goes to zero and \( a_n \) to 1 as \( n \to \infty \), so the difference
\[ c_n \sum_{i=1}^{k_n} a_i^n h \circ T^i - c_n \sum_{i=m}^{k_n} a_i^n h \circ T^i, \]
which is either equal to \( c_n h \) or \( c_n \sum_{i=1}^{m-1} a_i^n h \circ T^i \), converges in probability to zero.

Lemma 5 Let \((Y, \mathcal{B}, \nu)\) be a normalized measure space, \( T : Y \to Y \) be ergodic with respect to \( \nu \), and \( \gamma \neq 0 \) be a constant. Let \( \kappa_\tau, \tau > 0 \), be such that
\[ \lim_{\tau \to 0} \frac{\kappa_\tau^2}{\tau} = 1. \]
If \( h \in L^2(Y, \mathcal{B}, \nu) \) is such that \( \mathbb{P}_{T^\tau, \nu} h = 0 \), then the finite dimensional distributions of the process \( \zeta_\tau \) defined by
\[ \zeta_\tau(t) = \kappa_\tau e^{-\gamma t} \sum_{j=0}^{[\frac{t}{\tau}]} e^{\gamma \tau j} h \circ T^j, \quad t \geq 0, \quad \tau > 0 \]
converge weakly as \( \tau \to 0 \) to the corresponding finite dimensional distributions of the zero-mean Gaussian process \( \zeta \) for which \( \zeta(0) = 0 \) and
\[ E\zeta(t)\zeta(s) = \frac{||h||_2^2}{2\gamma} (e^{2\gamma \min(t,s)} - 1)e^{-\gamma(t+s)}, \quad t, s > 0. \]

Proof. To prove the convergence of the finite dimensional distributions of \( \zeta_\tau \) to the corresponding finite dimensional distributions of the Gaussian process \( \zeta \), it is enough to prove that for any \( l \geq 1 \), real numbers \( 0 < t_1 < \ldots < t_l < \infty \) and \( d_1, \ldots, d_l \) the distribution of \( \sum_{j=1}^l d_j \zeta_\tau(t_j) \) is Gaussian and that \( \sum_{j=1}^l d_j \zeta_\tau(t_j) \) converges in distribution as \( \tau \to 0 \) to \( \sum_{j=1}^l d_j \zeta(t_j) \).

We consider first the case of \( l = 1 \). It follows from Theorem 10 that for \( t > 0 \) the distribution of \( \zeta_\tau(t) \) converges weakly as \( \tau \to 0 \) to a Gaussian random variable. To see this let \( \tau_n \) be a sequence going to zero as \( n \to \infty \). Take \( k_n = \left[ \frac{t}{\tau_n} \right], a_n = e^{\gamma \tau_n}, c_n = \kappa_{\tau_n} e^{-\gamma t} \), and observe that
\[ \lim_{n \to \infty} k_n = \infty, \quad \lim_{n \to \infty} a_n = 1, \quad \lim_{n \to \infty} e_n^n = e^{\gamma t} \quad \text{and} \quad \lim_{n \to \infty} k_n e_n^n = t e^{-2\gamma t}, \]
and \( \frac{t e^{-2\gamma t}}{\ln e^{2\gamma t}} = \frac{1 - e^{-2\gamma t}}{2\gamma} \). The theorem then implies that
\[ \kappa_{\tau_n} e^{-\gamma t} \sum_{j=0}^{[\frac{t}{\tau_n}]} e^{\gamma \tau_n j} h(T^j(\xi_0)) \to N \left( 0, \frac{\sigma^2_0}{2\gamma} (1 - e^{-2\gamma t}) \right) \]
where \( \sigma^2_0 = \int h^2(y) \nu(dy) \) and \( \xi_0 \) is distributed according to \( \nu \). Consequently \( \zeta_\tau(t) \to^d \zeta(t) \) as \( \tau \to 0 \), where \( \zeta(t) \) is a Gaussian distributed random variable with mean 0 and variance given by
\[ \frac{||h||_2^2}{2\gamma} (1 - e^{-2\gamma t}), \quad t > 0. \]
Note that $\zeta_\tau(0) = \kappa_\tau h$. Since $\lim_{\tau \to 0} \kappa_\tau = 0$, we also have $\zeta_\tau(0) \to 0$ as $\tau \to 0$.

We next consider the case of $l = 2$. The case of arbitrary $l$ is deduced analogously from Theorem 14. Let $t_1 < t_2$ and $d_1, d_2$ be given. Let $\tau_n$ be a sequence going to zero as $n \to \infty$. Set $k_{n,1} = \left[\frac{t_1}{\tau_n}\right]$, $k_{n,2} = \left[\frac{t_2}{\tau_n}\right]$, $k_n = k_{n,2} + 1$, and observe that $k_{n,1} < k_{n,2}$ for all $n$ sufficiently large. Define

$$\eta_{n,j} = \begin{cases} 
2e^{-\gamma t_2} \kappa_{\tau_n} e^{\gamma t_n (k_n - j)} h \circ T^{k_n - j}, & 0 < j \leq k_{n,2} - k_{n,1}, \\
(2e^{-\gamma t_2} + d_1 e^{-\gamma t_1}) \kappa_{\tau_n} e^{\gamma t_n (k_n - j)} h \circ T^{k_n - j}, & k_{n,2} - k_{n,1} < j \leq k_n, \\
0, & \text{otherwise}.
\end{cases}$$

Then we have

$$d_1 \zeta_{\tau_n}(t_1) + d_2 \zeta_{\tau_n}(t_2) = \sum_{j=1}^{k_n} \eta_{n,j}.$$ 

Observe that

$$\sum_{j=1}^{k_n} \eta_{n,j}^2 = d_2^2 e^{-2\gamma t_2} \kappa_{\tau_n}^2 \sum_{j=0}^{k_{n,2}} e^{2\gamma t_n j} h^2 \circ T^j + (2d_2 d_1 e^{-\gamma (t_2 + t_1)} + d_1^2 e^{-2\gamma t_1}) \kappa_{\tau_n}^2 \sum_{j=0}^{k_{n,1}} e^{2\gamma t_n j} h^2 \circ T^j.$$ 

As in the proof of Theorem 16, we check that Theorem 14 applies to $\eta_{n,j} : 1 \leq j \leq k_n, n \geq 1$ and conclude that

$$d_1 \zeta_{\tau_n}(t_1) + d_2 \zeta_{\tau_n}(t_2) \to^d \sigma N(0, 1)$$

where $\sigma^2 = \frac{|h|_2^2}{2\gamma} (d_2^2 (1 - e^{-2\gamma t_2}) + 2d_2 d_1 e^{-\gamma (t_2 + t_1)} (e^{2\gamma t_1} - 1) + d_1^2 (1 - e^{-2\gamma t_1}))$. Since $\sigma N(0, 1)$ is the distribution of $d_1 \zeta(t_1) + d_2 \zeta(t_2)$ and $E(\zeta(t_1) \zeta(t_2)) = \frac{|h|_2^2}{2\gamma} e^{-\gamma (t_2 + t_1)} (e^{2\gamma t_1} - 1)$, the proof of the lemma is complete.

**Lemma 6** Let $(Y, \mathcal{B}, \nu)$ be a normalized measure space, $T : Y \to Y$ be ergodic with respect to $\nu$, and $\gamma \neq 0$ be a constant. Let $\kappa_\tau, \tau > 0$, be such that

$$\lim_{\tau \to 0} \frac{\kappa_\tau^2}{\tau} = 1.$$ 

If $h \in L^2(Y, \mathcal{B}, \nu)$ is such that $\mathcal{P}_{T, \nu} h = 0$, then the finite dimensional distributions of the process $\eta_\tau$ defined by

$$\eta_\tau(t) = \kappa_\tau \left[ \frac{t}{\gamma} \right] \sum_{j=0}^{\left[ \frac{t}{\tau} \right]} (1 - e^{\gamma (\tau j - t)}) h \circ T^j, \quad t \geq 0, \quad \tau > 0$$

converge weakly as $\tau \to 0$ to the corresponding finite dimensional distributions of the zero-mean Gaussian process $\eta$ for which $\eta(0) = 0$ and

$$E(\eta(t) \eta(s)) = \frac{|h|_2^2}{\gamma^3} (2\gamma \min(t, s) - 2 + 2e^{-\gamma t} + 2e^{-\gamma s} - e^{-\gamma |t-s|} - e^{-\gamma (t+s)})$$

for $t, s > 0$.

The lemma follows from Theorem 14 in a similar fashion as the preceding lemma.
References

Adler, R. and Rivlin, T. (1964). Ergodic and mixing properties of Chebyshev polynomials. *Proc. Amer. Math. Soc.*, 15:794–796.

Alexander, J. C. and Yorke, J. A. (1984). Fat baker’s transformations. *Erg. Theory Dyn. Syst.*, 4:1–23.

an der Heiden, U. and Mackey, M. C. (1982). The dynamics of production and destruction: Analytic insight into complex behaviour. *J. Math. Biol.*, 16:75–101.

Beck, C. (1990a). Brownian motion from deterministic dynamics. *Physica A*, 169:324–336.

Beck, C. (1990b). Ergodic properties of a kicked damped particle. *Commun. Math. Phys.*, 130:51–60.

Beck, C. (1996). Dynamical systems of Langevin type. *Physica A*, 233:419–440.

Beck, C. and Roepstorff, G. (1987). From dynamical systems to the Langevin equation. *Physica A*, 145:1–14.

Benedics, M. and Carleson, L. (1985). On iterations of $1 - ax^2$ on $(-1, 1)$. *Ann. Math.*, 122:1–25.

Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley & Sons, New York.

Billingsley, P. (1995). *Probability and Measure*. John Wiley & Sons, New York.

Boltzmann, L. (1995). *Lectures on Gas Theory*. Dover, Mineola, N.Y.

Bowen, R. (1975). Equilibrium states and the ergodic theory of Anosov diffeomorphisms. volume 470 of *Springer Lecture Notes in Math.*, New York.

Boyarsky, A. and Scarowsky, M. (1979). On a class of transformations which have unique absolutely continuous invariant measures. *Trans. Amer. Math. Soc.*, 255:243–262.

Briggs, M., Sengers, J., Francis, M., Gaspard, P., Gammon, R., Dorfman, J., and Calabrese, R. (2001). Tracking a colloidal particle for the measurement of dynamic entropies. *Physica A*, 296:42–59.

Brown, B. M. (1971). Martingale central limit theorems. *Ann. Math. Statist.*, 42:59–66.

Chew, L. and Ting, C. (2002). Microscopic chaos and Gaussian diffusion processes. *Physica A*, 307:275–296.

Chung, K. L. (2001). *A course in probability theory*. Academic Press, 3 edition.

Denker, M. (1989). The central limit theorem for dynamical systems. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 33–62. PWN, Warsaw.

Dorfman, J. (1999). *An Introduction to Chaos in Nonequilibrium Statistical Mechanics*, volume 14 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, Cambridge, New York.

Dudley, R. M. (1989). *Real Analysis and Probability*. Wadsworth, Belmont, California.
Eagleson, G. K. (1975). Martingale convergence to mixtures of infinitely divisible laws. *Ann. Probab.*, 3:557–562.

Eckmann, J.-P. and Ruelle, D. (1985). Ergodic theory of chaos and strange attractors. *Rev. Modern Phys.*, 57(3, part 1):617–656.

Einstein, A. (1905). Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Ann. d. Physik*, 17:549–560.

Erdös, P. (1939a). On a family of symmetric Bernoulli convolutions. *Amer. J. Math.*, 61:974–976.

Erdös, P. (1939b). On the smoothness properties of a family of Bernoulli convolutions. *Amer. J. Math.*, 62:180–186.

Fürth, R., editor (1956). *Investigations on the Theory of the Brownian Movement*, New York. Dover.

Gaenssler, P. and Joos, K. (1992). Another view on martingale central limit theorems. *Stochastic Processes Appl.*, 40(2):181–197.

Gallavotti, G. (1999). *Statistical Mechanics: A Short Treatise*. Springer Verlag, Berlin, New York.

Gaspard, P., Briggs, M., Francis, M., Sengers, J., Gammon, R., Dorfman, J., and Calabrese, R. (1998). Experimental evidence for microscopic chaos. *Nature*, 394:865–868.

Gordin, M. I. (1969). The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, 188:739–741.

Hall, P. (1978). The convergence of moments in the martingale central limit theorem. *Z. Wahrsch. Verw. Gebiete*, 44(3):253–260.

Hilgers, A. and Beck, C. (1999). Approach to Gaussian stochastic behavior for systems driven by deterministic chaotic forces. *Phys. Rev. E.*, 60:5385–5393.

Hilgers, A. and Beck, C. (2001). Higher-order correlations of Tchebyscheff maps. *Physica D*, 156:1–18.

Hunt, B. R., Kennedy, J. A., Li, T.-Y., and Nusse, H. E. (2002). SLYRB measures: natural invariant measures for chaotic systems. *Phys. D*, 170(1):50–71.

Iosifescu, M. (1992). A very simple proof of a generalization of the Gauss-Kuzmin-Levy theorem on continued fractions, and related questions. *Rev. Roumaine Math. Pures Appl.*, 37:901–914.

Jabłoński, M. (1991). A central limit theorem for processes generated by a family of transformations. *Dissertationes Mathematicae*, 307:1–62.

Jabłoński, M., Kowalski, Z., and Malczak, J. (1985). The rate of convergence of iterates of the Frobenius-Perron operator for Lasota-Yorke transformations. *Univ. Jagel. Acta Math.*, 25:189–193.

Jabłoński, M. and Malczak, J. (1983a). A central limit theorem for piecewise convex mappings of the unit interval. *Tôhoku Math. J.*, 35:173–180.
Jabłoński, M. and Malczak, J. (1983b). The rate of convergence of iterates of the Frobenius-Perron operator for piecewise convex transformations of the unit interval. *Bull. Pol. Ac.: Math.*, 31:249–254.

Jakobson, M. (1981). Absolutely continuous invariant measure for one-parameter families of one-dimensional maps. *Commun. Math. Phys.*, 81:39–88.

Jessen, B. and Wintner, A. (1935). Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.*, 38:48–88.

Kappler, E. (1931). Versuche zur Messung der Avogadro-Loschmidtschen Zeit aus der Brownschen bewegung einer Drehwaage. *Ann. Physik*, 11:233–256.

Keller, G. (1980). Un théorème de la limite centrale pour une classe de transformations monotones par morceaux. *C.R. Acad. Sc. Paris*, 291:155–158.

Kershner, R. and Wintner, A. (1935). On symmetric Bernoulli convolutions. *Amer. J. Math.*, 57:541–548.

Lasota, A. and Mackey, M. C. (1994). *Chaos, Fractals and Noise: Stochastic Aspects of Dynamics*. Springer-Verlag, Berlin, New York, Heidelberg.

Liverani, C. (1996). Central limit theorem for deterministic systems. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 56–75. Longman, Harlow.

Mackey, M. C. (1989). The dynamic origin of increasing entropy. *Rev. Mod. Phys.*, 61:981–1016.

Mackey, M. C. (1992). *Time’s Arrow: The Origins of Thermodynamic Behaviour*. Springer-Verlag, Berlin, New York, Heidelberg.

Mackey, M. C. and Glass, L. (1977). Oscillation and chaos in physiological control systems. *Science*, 197:287–289.

Mazo, R. (2002). *Brownian Motion: Fluctuations, Dynamics, and Applications*. Claredon Press, Oxford.

Peres, Y., Schlag, W., and Solomyak, B. (2000). Sixty years of Bernoulli convolutions. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, Progr. Probab. 46, pages 39–65, Basel. Birkhäuser.

Pollicott, M. and Sharp, R. (2002). Invariance principles for interval maps with an indifferent fixed point. *Comm. Math. Phys.*, 229:337–346.

Ratner, M. (1973). The central limit theorem for geodesic flows on n-dimensional manifolds of negative cuorvature. *Israel J. Math.*, 16:181–197.

Ruelle, D. (1976). A measure associated with Axiom A attractors. *Am. J. Math.*, 98:619–654.

Ruelle, D. (1978). Sensitive dependence on initial condition and turbulent behavior of dynamical systems. *Ann. N.Y. Acad. Sci.*, 316:408–416.

Ruelle, D. (1979). Microscopic fluctuations and turbulence. *Phys. Let.*, 72A:81–82.
Ruelle, D. (1980). Measures describing a turbulent flow. *Ann. N.Y. Acad. Sci.*, 357:1–9.

Schulman, L. S. (1997). *Time’s Arrows and Quantum Measurement*. Cambridge University Press, Cambridge.

Sinai, Y. G. (1972). Gibbs measure in ergodic theory. *Russian Math. Surveys*, 27:21–69.

Solomyak, B. (1995). On the random series $\sum \pm \lambda^n$ (an Erdős problem). *Annals of Math.*, 142:611–625.

Teicher, H. (1988). Distribution and moment convergence of martingales. *Probab. Theory Related Fields*, 79(2):303–316.

Thaler, M. (1980). Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37:303–314.

Tsujii, M. (1996). On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps. *Comm. Math. Phys.*, 177(1):1–11.

Tyran-Kamińska, M. (2004). An invariance principle for maps with polynomial decay of correlations. *Preprint*.

Viana, M. (1997). Stochastic dynamics of deterministic systems. *Col. Bras. de Matemática*, 21:197.

Wintner, A. (1935). On convergent Poisson convolutions. *Amer. J. Math.*, 57:827–838.

Wong, S. (1979). A central limit theorem for piecewise monotonic mappings of the unit interval. *Ann. Prob.*, 7:500–514.

Young, L.-S. (1992). Decay of correlations for certain quadratic maps. *Commun. Math. Phys.*, 146:123–138.

Young, L.-S. (1999). Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188.

Young, L.-S. (2002). What are SRB measures, and which dynamical systems have them? *J. Statist. Phys.*, 108:733–754.