THE GREEN FUNCTION FOR WAVES ON THE 2-REGULAR BETHE LATTICE

KAÏS AMMARI AND GILLES LEBEAU

Abstract. In this paper, we compute an explicit analytic expression for the Green function of the wave operator on the 2-regular lattice called the “Bethe lattice” equipped with its standard metric. In particular, we exhibit a phenomena of abnormal speed of propagation for waves: the effective speed of propagation of energy for large time is \( c^* = 2 \sqrt{2}/3 < 1 \), and there exists a true propagation at any speed \( c < c^* \).

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1. Introduction

A \( z \)-regular Bethe lattice (a particular kind of Cayley graph, introduced by Hans Bethe in 1935), is an infinite connected cycle-free graph where each node is connected to \( z + 1 \) neighbors. The \( z \)-regular Bethe lattice \( \mathcal{B}_z \) is thus a 1-dimensional Riemannian manifold with singularities (the nodes). The canonical metric is the obvious one: each edge is identified with \( [0, 1] \) and has length 1. Moreover, given two points \( p, q \) in \( \mathcal{B}_z \), there exists a unique geodesic connecting \( p \) and \( q \). The number of nodes at a given distance \( k \in \mathbb{N}^* \) of a given node is equal to

\[
N_k = (z + 1)z^{k-1}.
\]

Due to its distinctive topological structure, the statistical mechanics of lattice models on graphs are often exactly solvable (see [3]). In this paper, we are interested in the study of the wave equation on \( \mathcal{B}_z \) with speed 1. We will show that it is effectively exactly solvable. To our knowledge, our explicit formulas are new.

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Let $\mathcal{N}_z$ be the set of nodes and $\mathcal{A}_z$ the set of edges. Then $\mathcal{B}_z \setminus \mathcal{N}_z$ is the disjoint union of the edges $A \in \mathcal{A}_z$, and each edge $A$ is identified with the interval $[0, 1]$. By definition, a wave on $\mathcal{B}_z$, with speed 1, is a collection of distributions $(u_A(x, t))_{A \in \mathcal{A}_z}$ defined on $]0, 1[ \times \mathbb{R}_t$ which satisfy the usual 1-D wave equation on each edge

$$
\frac{\partial^2 u_A}{\partial t^2}(x, t) - \frac{\partial^2 u_A}{\partial x^2}(x, t) = 0, \quad 0 < x < 1, \ t \in \mathbb{R}, \ A \in \mathcal{A}_z
$$

and the Kirchhoff boundary conditions at the nodes $N \in \mathcal{N}_z$ (see [1, 2, 4, 5, 6]). If one denotes by $A_i$ the set of edges adjacent to $N$, this means

$$
u_{A_i}(x = N, t) = u_{A_j}(x = N, t), \quad \forall i, j, \tag{1.2}
$$

$$
\sum_i \partial_n u_{A_i}(x = N, t) = 0. \tag{1.3}
$$

Observe that since the distribution $u_A(x, t)$ satisfies the wave equation (1.1), the traces $u_A(x = N, t), \partial_n u_A(x = N, t)$ are well defined distributions on $\mathbb{R}_t$, and therefore the boundary conditions (1.2) and (1.3) are well defined. It is well known that the associated Cauchy problem to the equations (1.1), (1.2) and (1.3) is well posed.

For simplicity, in all the paper, we restrict ourselves to the case $z = 2$ and we set $\mathcal{B} = \mathcal{B}_2, \mathcal{N}_z = \mathcal{N}$ and $\mathcal{A}_z = \mathcal{A}$.

![Figure 1. Bethe Lattice](image)

Observe that each edge $A$ has two orientations and a middle point $m(A)$, defined by $x = 1/2$ in the identification $A = ]0, 1[$. If $\vec{A}$ is a given orientation, we will denote by $-\vec{A}$ the opposite orientation. We will write $\vec{A} \rightarrow \vec{B}$ if the right end point of $\vec{A}$ is equal to the left end point of $\vec{B}$.

As in Figure 1, we fix a given edge $A$, oriented from left to right. We denote by $W(t, p, \vec{A})$ the unique wave solution of (1.1), (1.2) and (1.3) and such that

$$
u_{A}(x, t) = \delta_{x = 1/2, t} \quad \forall t \in ]-1/2, 1/2[ \tag{1.4}
$$

$$
u_{B}(x, t) = 0 \quad \forall t \in ]-1/2, 1/2[ \text{, } \forall B \neq A$$
It is not difficult to see by finit speed of propagation and elementary properties of a 1-d waves, that the knowledge of $W(t, p, \tilde{A})$ allows to solve the Cauchy problem associated to (1.1), (1.2) and (1.3) for any Cauchy data. Moreover, it is not difficult to verify (see section 2), that $W(t, p, \tilde{A})$ is at any time a finite sum of Dirac masses. For $t \in \mathbb{Z}$, these Dirac masses are located at the middle points of $B$, and they propagate according to one of the two orientations on each edge. Therefore, one has for any $k \in \mathbb{Z}
olinebreak
\begin{equation}
(1.5) \quad W(t, p, \tilde{A}) = \sum_{\tilde{B}, \tilde{B} \in \mathcal{A}} G(k, \tilde{B}, \tilde{A}) \delta_{x=1/2z,t} \quad \forall t \in ]k - 1/2, k + 1/2[ \nolinebreak
\end{equation}

where the $\pm$ sign depends on the choice of the orientation of the edge $B$.

The aim of this paper is to give an explicit formula for the coefficients $G(k, \tilde{B}, \tilde{A})$ with $k \in \mathbb{N}$ (the case $k \in -\mathbb{N}$ follows by time symmetry and reverse orientation). This will be done in section 2, see proposition 3.10. An important quantity, related to the energy propagation, is $E(k, d)$, with $k, d \in \mathbb{N}$. It is defined by

\begin{equation}
(1.6) \quad E(k, d) = \sum_{\tilde{A} \text{fixed}, \text{dist}(m(B), m(A)) = d} |G(k, \tilde{B}, \tilde{A})|^2.
\end{equation}

By finit speed of propagation, one has obviously

$E(k, d) = 0 \quad \forall d > k$.

The main result of this paper is the following theorem.

1.1. **Theorem.** Let $c_* = 2 \sqrt{2}/3 < 1$. For $k \in \mathbb{N}^*$ and $d \in \mathbb{N}$, let $\gamma = d/k$.

1. For $\gamma > c_*$, there exists $\alpha = a(\gamma) > 0$ such that $E(k, d) \in O(e^{-\alpha k})$.
2. For $\gamma \in [0, c_*]$, there exists $b_{j,m} = b_{j,m}(\gamma), a_{j,m} = a_{j,m}(\gamma), j, m \in \{1, 2, 3, 4\}$ s.t.

$E(k, d) = k^{-1} \sum_j \sum_m e^{ia_{j,m}k}(b_{j,m} + O(k^{-1}))^2$.

3. $E(k, 0) \in O(k^{-3})$.

Observe that this theorem is about the behavior of $E(k, d)$ for large values of $k$ (large time). It will be obtained in section 3 as a consequence of the study of a phase integral with large parameter $k$. The precise statement for $d \approx c_* k$ is given in Theorem 4.3; it involves a transition describe by Airy functions.

1.2. **Remark.** We thus find an effective speed of propagation of energy which is at most $c_* = 2 \sqrt{2}/3 < 1$.

Moreover, any speed $c \in [0, c_*]$ appears in the propagation of the energy: this is a kind of "diffusion", but is quite different from the heat or parabolic diffusion, since there is no regularization at all, and the scaling factor between space and time is the same.
2. The propagation of waves at nodes

In this section, we compute the reflection and transmission coefficient for a Dirac mass at a given node.

As in Figure 2, we consider an incoming wave on the left edge identified with \([-1, 0]\) of the form \(u_{\text{in}}(x, t) = \delta_{x=1} \) for \( t < 0 \) small. We identified the two right edges with \([0, 1]\). Then the outgoing wave is equal for \( t > 0 \) small to

\[ u_{\text{out}}(x, t) = (\alpha \delta_{x=-1}, \beta \delta_{x=t}, \beta \delta_{x=0}) \]

where \( \alpha \) is the reflection coefficient and \( \beta \) the transmission coefficient. The continuity condition gives

\[ 1 + \alpha = \beta, \]

and the Kirchhoff law gives

\[ -(1 - \alpha) + 2\beta = 0. \]

This implies

\[ \alpha = -\frac{1}{3}, \quad \beta = \frac{2}{3} \]

2.1. Remark. For the general Bethe lattice \( \mathcal{B}_z \), with \( z = N + 1 \), one find

\[ 1 + \alpha = \beta, \quad -(1 - \alpha) + N\beta = 0, \]

hence

\[ \alpha = -(N - 1)/(N + 1), \quad \beta = 2/(N + 1). \]

Observe that we always have:

\[ \alpha + N\beta = 1, \quad \text{charge conservation}, \]

\[ \alpha^2 + N\beta^2 = 1, \quad \text{energy conservation}. \]
3. The computation of $G(k, \vec{B}, \vec{A})$

Recall from (1.5) that for $k \in \mathbb{N}$, $G(k, \vec{B}, \vec{A})$ denotes the coefficient on the oriented wedge $\vec{B}$ of the Dirac mass (propagating in the direction of $\vec{B}$), for the solution of the wave equation with Cauchy data the Dirac mass at $m(\vec{A})$ (propagating in the direction of $\vec{A}$). We thus have $G(0, \vec{B}, \vec{A}) = \delta_{\vec{B}=\vec{A}}$ and we get from (2.7):

$$G(1, \vec{B}, \vec{A}) = \begin{cases} -\frac{1}{3} & \text{if } \vec{B} = -\vec{A} \\ \frac{2}{3} & \text{if } \vec{A} \rightarrow \vec{B} \text{ and } \vec{B} \neq -\vec{A} \\ 0 & \text{in all other cases} \end{cases}$$

Observe that we have $\vec{A} \rightarrow \vec{B}$ iff $G(1, \vec{B}, \vec{A}) \neq 0$. For $k \in \mathbb{Z}$, $G(k, \vec{B}, \vec{A})$ is an (infinite) matrix indexed by the oriented wedges (with only a finite number of non-zero coefficients). If the Cauchy data at $t = 0$ of the solution $f$ of the wave equation is $X(0) = \sum_{\vec{A}} x_{\vec{A}} \delta_{\vec{A}}$, which means that for all wedge $\vec{A}$ one has:

$$f|_{t=0, s=\vec{A}} = \left(x_{\vec{A}} + x_{-\vec{A}}\right) \delta_{s=\frac{1}{2}}, \quad \frac{\partial f}{\partial t}|_{t=0, s=\vec{A}} = \left(-x_{\vec{A}} + x_{-\vec{A}}\right) \delta'_{s=\frac{1}{2}}$$

then the Cauchy data at any time $t = k \in \mathbb{Z}$ is equal to

$$X(k) = \sum_{\vec{B}} y_{\vec{B}} \delta_{\vec{B}}, \quad y_{\vec{B}} = \sum_{\vec{A}} G(k, \vec{B}, \vec{A}) . x_{\vec{A}}.$$ 

One has the group law

$$G(k + l) = G(k)G(l), \quad \forall k, l \in \mathbb{Z},$$

and thus we get

$$G(k) = G(1)^k.$$ 

We are thus reduce to compute the $k$-th power of the matrix $G(1)$ given by (3.8). We equip the space $X = \sum_{\vec{A}} x_{\vec{A}} \delta_{\vec{A}}$ with the $\ell^2$ norm : $\|X\|^2 = \sum_{\vec{A}} |x_{\vec{A}}|^2$.

3.1. Lemma. The matrix $G(1)$ is unitary.

Proof. For any oriented wedge $\vec{A}$, one has from (3.8) $\|G(1)\delta_{\vec{A}}\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$. For $\vec{A} \neq \vec{B}$, let us verify $\langle G(1)\delta_{\vec{A}}|G(1)\delta_{\vec{B}}\rangle = 0$. One has

$$G(1)(\delta_{\vec{A}}) = -\frac{1}{3} \delta_{\vec{A}} + \frac{2}{3} \delta_{\vec{B}_1} + \frac{2}{3} \delta_{\vec{B}_2},$$

as shown in the diagram.
Thus for $\vec{A} \neq \vec{B}$, $(G(1)\delta_{\vec{A}} G(1)\delta_{\vec{B}}) \neq 0$ implies $\vec{B} = -\vec{B}_1$ or $\vec{B} = -\vec{B}_2$. But one has

$$G(1)\delta_{\vec{A}} = -\frac{1}{3}\delta_{\vec{A}} + \frac{2}{3}\delta_{\vec{A}}' + \frac{2}{3}\delta_{\vec{B}}',$$

thus

$$\left( G(1)\delta_{\vec{A}} G(1)\delta_{\vec{B}} \right) = -\frac{2}{9} + \frac{2}{9} + \frac{4}{9} = 0.$$

□

In order to compute $G(k, \vec{B}, \vec{A})$ we will use a (signed) sum over paths.

3.2. Definition. A path $\gamma$ of length $\ell(\gamma) = k$ connecting $\vec{A}$ to $\vec{B}$ is a $k + 1$-plet

$$\gamma = (\vec{A}, \vec{C}_1, ..., \vec{C}_{k-1}, \vec{B}) = (\vec{C}_0, ..., \vec{C}_k), \quad \vec{A} = \vec{C}_0, ..., \vec{B} = \vec{C}_k$$

with $\vec{C}_j \to \vec{C}_{j+1}$ for $j \in \{0, ..., k-1\}$. One says that $(\vec{C}_j, \vec{C}_{j+1})$ is an inversion if $\vec{C}_{j+1} = -\vec{C}_j$ and we denote by $r(\gamma)$ the number of $j$ such that $(\vec{C}_j, \vec{C}_{j+1})$ is an inversion. We have $0 \leq r(\gamma) \leq k = \ell(\gamma)$.

We denote by $C_{k,r}(\vec{A}, \vec{B})$ the set of paths of length $k \geq 1$, with $r$ inversion, connecting $\vec{A}$ to $\vec{B}$ and we set $C_k(\vec{A}, \vec{B}) = \cup_{r \geq 0} C_{k,r}(\vec{A}, \vec{B})$.

By definition of the product of two matrices, from (3.8) and (3.10) we get with $\alpha = -\frac{1}{3}$ and $\beta = \frac{2}{3}$

$$G(k, \vec{B}, \vec{A}) = \sum_{\gamma \in C_k(\vec{A}, \vec{B})} \alpha^{r(\gamma)} \beta^{k-r(\gamma)}, \quad \forall k \geq 1.$$

Equivalently we get the following formula:

$$(3.11) \quad G(k, \vec{B}, \vec{A}) = \beta^k \sum_{r=0}^{k} \left( \frac{\alpha}{\beta} \right)^r \left| C_{k,r}(\vec{A}, \vec{B}) \right|, \quad \forall k \geq 1.$$

Thus it remains to compute $\left| C_{k,r}(\vec{A}, \vec{B}) \right| = \text{number of paths of length } k \geq 1, \text{ with } r \text{ inversion, connecting } \vec{A} \text{ to } \vec{B}.$

Computation of $\left| C_{k,r}(\vec{A}, \vec{B}) \right|$.

By symmetry, $\left| C_{k,r}(\vec{A}, \vec{B}) \right|$ depends only on the distance $d = \text{dist}(m(A), m(B))$ and on the 4 possible orientations of $\vec{A}$ and $\vec{B}$ with respect to the unique geodesic connecting $m(A)$ to $m(B)$.
We denote by $\Gamma_{k,r}(d,j)$, $j \in \{1, 2, 3, 4\}$ the number of paths of length $k$, with $r$ inversions, connecting $\vec{A}$ to $\vec{B}$ such that $\text{dist}(m(\vec{A}), m(\vec{B})) = d$ and the orientation of $(\vec{A}, \vec{B})$ in position $j \in \{1, 2, 3, 4\}$ like in figure 3.

Observe that in the particular case $d = 0$ one has $\Gamma_{k,r}(0,1) = \Gamma_{k,r}(0,3)$ and $\Gamma_{k,r}(0,2) = \Gamma_{k,r}(0,4)$. We will calculate $|C_{k,r}(\vec{A}, \vec{B})|$ by induction on $k$.

**Initialization**: $k = 1$. Then one has $(d = 1, r = 0)$ or $(d = 0, r = 1)$, and we find

\begin{equation}
\Gamma_{1,0}(1,1) = 1 \\
\Gamma_{1,1}(0,2) = 1 = \Gamma_{1,1}(0,4) \\
\Gamma_{1,r}(d, j) = 0 \text{ in all other cases.}
\end{equation}

**Induction**: $k - 1 \rightarrow k$ with $k \geq 2$ and the convention $\Gamma_{k,r}(d, j) = 0$ for $r < 0$ or $d > k$, or $r > k$.

3.3. **Lemma.** For $k \geq 2$, the following formulas hold true:

(1) For $d \geq 1$

\begin{equation}
\begin{aligned}
\Gamma_{k,r}(d, 1) &= \Gamma_{k-1,r-1}(d, 2) + \Gamma_{k-1,r}(d - 1, 1) + \Gamma_{k-1,r}(d, 2) \\
\Gamma_{k,r}(d, 2) &= \Gamma_{k-1,r-1}(d, 1) + 2\Gamma_{k-1,r}(d + 1, 1) \\
\Gamma_{k,r}(d, 3) &= \Gamma_{k-1,r-1}(d, 4) + 2\Gamma_{k-1,r}(d + 1, 3) \\
\Gamma_{k,r}(d, 4) &= \Gamma_{k-1,r-1}(d, 3) + \Gamma_{k-1,r}(d - 1, 4) + \Gamma_{k-1,r}(d, 3).
\end{aligned}
\end{equation}

(2) For $d = 0$

\begin{equation}
\begin{aligned}
\Gamma_{k,r}(0, 1) &= \Gamma_{k-1,r-1}(0, 2) + 2\Gamma_{k-1,r}(1, 3) \\
\Gamma_{k,r}(0, 2) &= \Gamma_{k-1,r-1}(0, 1) + 2\Gamma_{k-1,r}(1, 2) \\
\Gamma_{k,r}(0, 3) &= \Gamma_{k-1,r-1}(0, 4) + 2\Gamma_{k-1,r}(1, 3) \\
\Gamma_{k,r}(0, 4) &= \Gamma_{k-1,r-1}(0, 3) + 2\Gamma_{k-1,r}(1, 2).
\end{aligned}
\end{equation}

**Proof.** Let us verify the first line of (3.13).

Let $\gamma = (\vec{A}, \vec{C}_0, ..., \vec{C}_{k-1}, \vec{B})$ a path of length $k$ with $r$ inversions and the orientation $\vec{A}, \vec{B}$ in position 1 like in Figure 3. There is 3 possibilities:
Lemma. 3.4. The following Lemma is obvious.

(3.19) \[ F \Theta \]

Then we define the generating functions \[ \Theta \]

Thus we get that the first line of (3.13) holds true. The verification is the same for the others 7 formulas of (3.13), (3.14) and we leave the details to the reader. \( \Box \)

Let us define \( \Gamma_{0,d} \) by

(3.15) \[ \Gamma_{0,0}(0,1) = \Gamma_{0,0}(0,3) = 1, \Gamma_{0,d}(d, j) = 0 \text{ in all other cases.} \]

Observe that this convention for \( k = 0 \) is compatible with formulas (3.12), (3.13) and (3.14). Then we define the generating functions \( F_j, j \in \{1, 2, 3, 4\} \) as the formal series in \( X, Y, Z \)

(3.16) \[ F_j = \sum_{k \geq 0, 0 \leq j, r \geq 0} \Gamma_{k,r}(d, j)X^kY^rZ^d. \]

From formulas (3.12), (3.15), (3.13) and (3.14), we deduce the system of equations:

(3.17) \[
\begin{align*}
F_1 &= 1 + XZF_1 + XYF_2 + X[F_2 - F_3|Z=0] + 2X\partial_Z F_3|Z=0 \\
F_2 &= XYF_1 + \frac{2X}{Z} [F_2 - F_3|Z=0] \\
F_3 &= 1 + XYF_4 + \frac{2X}{Z} [F_3 - F_3|Z=0] \\
F_4 &= XZF_4 + XYF_3 + X[F_3 - F_3|Z=0] + 2X\partial_Z F_2|Z=0.
\end{align*}
\]

The following Lemma is obvious.

3.4. Lemma. Let \( \Theta \) be the operator acting on formal series of \( (X, Y, Z) \):

\[ \Theta(F) = F - \frac{2X}{Z} \left( F - F|Z=0 \right). \]

Then \( \Theta \) commutes with the multiplication by \( X, Y \), one has \( \Theta(1) = 1 \) and the equation

\[ \Theta \left( \sum_{n=0}^{\infty} F_n(X, Y)Z^n \right) = \sum_{n=0}^{\infty} G_n(X, Y)Z^n \]

admits the unique solution \( F \):

(3.18) \[ F = \sum_{n=0}^{\infty} F_n(X, Y)Z^n, \quad F_n = \sum_{p=0}^{\infty} (2X)^p G_{n+p}. \]

From the lines 2 and 3 of (3.17) one has

(3.19) \[
\begin{align*}
F_2 &= \Theta^{-1}(XYF_1) = XY\Theta^{-1}(F_1) \\
F_3 &= \Theta^{-1}(1 + XYF_4) = 1 + XY\Theta^{-1}(F_4).
\end{align*}
\]
Let us define the operators $P$ and $Q$:

$$P(F) = F - XZF - X^2 \left[ Y^2 + Y \right] \Theta^{-1}(F) + X^2Y\Theta^{-1}(F)_{Z=0}$$

$$Q(F) = 2X^2Y \frac{\partial}{\partial Z} \Theta^{-1}(F)_{Z=0}. $$

From (3.17) we get the following Lemma.

3.5. Lemma. $F_1, F_4$ satisfy the system:

$$\begin{bmatrix}
P & -Q \\
-Q & P
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_4
\end{bmatrix} =
\begin{bmatrix}
1 \\
XY
\end{bmatrix}. $$

From (3.11), when the orientation of $(\vec{A}, \vec{B})$ is of type $j \in \{1, 2, 3, 4\}$, and $\text{dist}(m(\vec{A}), m(\vec{B})) = d$, one has

$$\begin{equation}
G(k, \vec{B}, \vec{A}) = \left( \frac{2}{3} \right)^k \frac{1}{k!} \frac{\partial^k}{\partial x^d} \frac{\partial^k}{\partial z^2} F \big|_{X=Z=0, Y=-\frac{1}{2}}.
\end{equation}$$

Observe that $Y$ is just a parameter in the system (3.20). Since we are in the case $Y = \frac{a}{\beta} = -\frac{1}{2}$, we can forget the variable $Y$ by setting $Y = -\frac{1}{2}$; then $F_1, F_4$ are formal series in $X, Z$ and they satisfy:

$$\begin{equation}
\begin{cases}
P = 1 - XZ + \frac{\beta}{4} \Theta^{-1} - \frac{\beta}{2} \Theta^{-1}(\cdot)_{Z=0} \\
Q = -X^2 \partial_z \Theta^{-1}(\cdot)_{Z=0}.
\end{cases}
\end{equation}$$

We are thus reduce to solve the system (3.22). From (3.19), we will get that $F_2$ and $F_3$ are defined by

$$\begin{equation}
\begin{align*}
F_2 &= -\frac{\beta}{2} \Theta^{-1}(F_1) \\
F_3 &= 1 - \frac{\beta}{2} \Theta^{-1}(F_4).
\end{align*}
\end{equation}$$

Resolution of the system (3.22)

Using the change of variables $X = \sqrt{2}x, Z = \sqrt{2}z$, (3.22) becomes

$$\begin{equation}
\begin{cases}
P = 1 - 2xz + \frac{\beta}{2} \Theta^{-1} - x^2 \Theta^{-1}(\cdot)_{Z=0} \\
Q = -x^2 \partial_x \Theta^{-1}(\cdot)_{Z=0}
\end{cases}
\end{equation}$$

with $\Theta(f) = f - \frac{2x}{3} \left(f - f_{z=0} \right)$. With the new unknowns $\tilde{F}_1 = \Theta(f_1), \tilde{F}_4 = \Theta(f_4)$, (3.24) reads, with $r = \frac{\sqrt{2}}{3}, \mu = \mu(x) = \frac{2x}{14 + \frac{\beta}{3}}, \delta = \delta(x) = \frac{x^2}{14 + \frac{\beta}{3}} = \frac{\mu}{2} \mu$

$$\begin{equation}
\begin{cases}
p & -q \\
-q & p
\end{cases}
\begin{bmatrix}
f_1 \\
f_4
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 + 2x^2
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{x^2} \\
-\frac{1}{y^2}
\end{bmatrix}
\end{equation}$$

$$\begin{align*}
p(f) &= 1 - \mu z - \frac{\mu}{2} \right) f + \left( \frac{\mu}{z} - 5\delta \right) f_{z=0} \\
q(f) &= -\sqrt{2} \delta f_{z=0}.
\end{align*}$$

The system (3.25) is an equation on fonctions of $z$, with $x$ as a holomorphic parameter in the complex disc $\{ x \in \mathbb{C}, |x| < r = \frac{\sqrt{2}}{3} \}$. For $x = 0$ one has $p = 1d$ and $q = 0$. 

3.6. Definition. Let $U$ be the open subset of $\mathbb{C}$

$$U = \mathbb{C} \setminus \{ \mu \in \mathbb{C}, \mu^2 \in \left[ \frac{1}{4}, +\infty \right] \}.$$

For $\mu \in U$ we define the holomorphic function $z_-(\mu)$ by

$$z_-(\mu) = \frac{1}{2\mu} \left[ 1 - \sqrt{1 - 4\mu^2} \right],$$

where $\sqrt{1 - 4\mu^2}$ is the principal determination of the square root of $1 - 4\mu^2$ on $U$. One has

$$z_-(0) = 0, \quad z'_-(0) = 1, \quad z_2 - \frac{z_-}{\mu} + 1 = 0.$$

We also define $z_+(\mu) = \frac{1}{z_-(\mu)} = \frac{1}{2\mu} \left[ 1 + \sqrt{1 - 4\mu^2} \right]$. Then $z_+(\mu)$ is meromorphic on $U$ with a simple pole at $\mu = 0$.

Let $V$ be the open subset of $\mathbb{C}$

$$V = \mathbb{C} \setminus \{ \mu \in \mathbb{C}, \mu^2 \in \left[ r^2, +\infty \right] \}.$$

From $\frac{\sqrt{3}}{3} = r < \frac{1}{2}$ one has $V \subset U$. We denote by $\theta_0$ the unique solution of

$$\cos \theta_0 = 2r, \quad \theta_0 \in ]0, \frac{\pi}{2}[.$$

3.7. Lemma. The holomorphic function $x \mapsto \mu(x) = \frac{2x}{1 + 2x^2}$ is a bijection from $D_r = \{ x, |x| < r \}$ onto $V$, and its restriction on the boundary $\partial D_r \setminus \{ \pm ir \}$ satisfies

$$\mu \left( \{ re^{i\theta}, -\pi/2 < \theta < \pi/2 \} \right) = [r, +\infty[, \mu \left( \{ re^{i\theta}, \pi/2 < \theta < 3\pi/2 \} \right) = [-\infty, -r].$$

Proof. For $\mu \neq 0$, the equation $\mu(x) = \mu \iff x^2 - 2x \frac{2}{\mu} + r^2 = 0$ admits 2 solutions $x_+ x_- = r^2$. Thus, when $|x_+| \neq |x_-|$, there is a unique unique solution in $D_r$. In the case $|x_+| = |x_-| = r$ one has $x_+ = re^{i\varphi}, x_- = re^{-i\varphi}$ and $x_+ + x_- = 2r \cos \varphi = \frac{2r^2}{\mu}$, thus $\mu \in \mathbb{R}$ et $|\mu| = \left| \frac{r}{\cos \varphi} \right| \geq r$. Since $\mu(x) = 0 \iff x = 0$, this implies that $x \mapsto \mu(x)$ is a holomorphic bijection from $D_r$ onto $V$. The last two equalities follow from

$$\mu(re^{i\varphi}) = \frac{2re^{i\varphi}}{1 + e^{2i\varphi}} = \frac{r}{\cos \varphi}. \quad \square$$
3.8. Lemma. For all \( \mu \in U = \mathbb{C} \setminus \{ \mu \in \mathbb{C}, \mu^2 \in [\frac{1}{4}, +\infty) \} \) one has \(|z(\mu)| < 1\).

Proof. One has \( z(0) = 0 \) and for \( \mu \neq 0 \), \( z_-(\mu) \) and \( z_+ (\mu) = \frac{1}{z_-(\mu)} \) satisfy the equation
\[
z^2 - \frac{2}{\mu} + 1 = 0.
\]
Thus for \( \mu \) near 0 one has \(|z_-(\mu)| < 1\) and if there exists \( \mu \in U \) such that \(|z_-(\mu)| = 1\) one has also \(|z_+(\mu)| = 1\), thus \( z_-(\mu) = e^{i\varphi}, z_+(\mu) = e^{-i\varphi} \); then \( z_+ + z_- = \frac{1}{\mu} \) gives \( \cos \varphi = \frac{1}{2\mu} \), hence \( \mu \in \mathbb{R} \) and \(|\mu| \geq 1\): this contradicts \( \mu \in U \). \( \square \)

3.9. Lemma. Let \( x \in \mathbb{C} \) close to 0. Let \( p, q \) be the operators defined in (3.25). For \((\alpha, \beta) \in \mathbb{C}^2\), the system
\[
\begin{align*}
p(f) - q(g) &= \alpha \\
-q(f) + p(g) &= \beta
\end{align*}
\]
admits a unique solution
\[
\begin{bmatrix}
f \\
g
\end{bmatrix}
= \left( \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \right)
\]
holomorphic in \(|z| < 1\) and this solution is given by
\[
\begin{align*}
f &= \frac{az_- (\mu)}{\mu} \frac{1}{1 - z z_-(\mu)} \\
g &= \frac{b z_+ (\mu)}{\mu} \frac{1}{1 - z z_-(\mu)}
\end{align*}
\]
where the couple \((a, b) \in \mathbb{C}^2\) is solution of the equation
\[
\begin{bmatrix}
(1 - 5\delta z_0) & \sqrt{2\delta z_0} \\
\sqrt{2\delta z_0} & (1 - 5\delta z_0)
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}.
\]

Proof. When \( x = 0 \), one has \( \mu(x) = \mu = 0, z_-(\mu) = 0, \frac{z_-(\mu)}{\mu} = 1, \delta = 0 \) and \( p = Id, q = 0 \). Then the result is obvious. We may thus assume \( x \neq 0 \), hence \( \mu \neq 0 \). We multiply (3.26) by \( z \) and we get
\[
(z - \mu(z^2 + 1)) f = az - c, \quad a = \alpha + 5\delta f_0 - \sqrt{2} \delta g_1, \quad c = \mu f_0,
\]
\[(z - \mu(z^2 + 1)) g = bz - d, \quad b = \beta + 5\delta g_0 - \sqrt{2}\delta f_1, \quad d = \mu g_0.\]

since \(z - \mu(z^2 + 1) = -\mu(z - z_+)(z - z_-), \) we get \(c = az_- \) et \(d = bz_- \). Thus we find

\[f = \frac{-a}{\mu(z - z_+)} = \frac{a}{\mu(z_+ - z)} = \frac{a}{\mu z_+(1 - \frac{z}{z_+})} = az_- \frac{1}{\mu} 1 - zz_-\]

and \(g = \frac{bz_-}{\mu} \frac{1}{1 - zz_-}. \) Therefore (3.27) holds true, and by evaluation at \(z = 0\) we get

\[f_0 = \frac{az_-}{\mu}, \quad g_0 = \frac{bz_-}{\mu}, \quad f_1 = \frac{az_-^2}{\mu}, \quad g_1 = \frac{bz_-^2}{\mu}.\]

Thus we get that the unknowns \((a, b)\) satisfy the system

\[a = \alpha + 5\delta f_0 - \sqrt{2}\delta g_1 = \alpha + 5\delta \frac{az_-}{\mu} - \sqrt{2}\delta \frac{bz_-^2}{\mu} \]

\[b = \beta + 5\delta g_0 - \sqrt{2}\delta f_1 = \beta + 5\delta \frac{bz_-}{\mu} - \sqrt{2}\delta \frac{az_-^2}{\mu} \]

which is equivalent to (3.28). Finally, since \(|x|\) is small, the invertibility of the matrix (3.28) is obvious. The proof of Lemma 3.9 is complete. \(\square\)

Recall \(\delta = \frac{a}{2}\mu(x), \) \(\mu = \mu(x) = \frac{2x}{1 + \frac{x^2}{\bar{r}^2}}. \) We denote by \((A(x), B(x)) = (a, b)\) the solution of \((3.28)\) with right hand side associated to the equation (3.25) i.e.,

\[
\begin{pmatrix}
\alpha, \\
\beta
\end{pmatrix}
= \left(\frac{1}{1 + \frac{x^2}{\bar{r}^2}, \frac{-x}{\sqrt{2}} \left(1 + \frac{x^2}{\bar{r}^2}\right)}\right).
\]

Then \(A(x)\) and \(B(x)\) are well defined holomorphic functions near \(x = 0.\) The resolution of system (3.28) leads to (with \(z_+ = z_-(\mu(x))\))

\[
A(x) = \frac{1}{1 + \frac{x^2}{\bar{r}^2}} \left\{ \frac{1}{(1 - \frac{5}{2}zx_-)^2 - \frac{x^2}{2}z_-^4} \left[ 1 - \frac{5}{2}zx_- + \frac{x^2}{2}z_-^2 \right] \right\},
\]

\[
B(x) = -\frac{x}{\sqrt{2}} \frac{1}{1 + \frac{x^2}{\bar{r}^2}} \left\{ \frac{1}{(1 - \frac{5}{2}zx_-)^2 - \frac{x^2}{2}z_-^4} \left[ 1 - \frac{5}{2}zx_- + \frac{x^2}{2}z_-^2 \right] \right\}.
\]

Let us define the functions \(g_j(x), j \in \{1, 2, 3, 4\}, \) holomorphic near \(x = 0,\) by the formulas

\[g_1(x) = A(x) \frac{z_-(\mu(x))(1 - 2zx_-(\mu(x)))}{\mu(x)}, \quad g_4(x) = B(x) \frac{z_-(\mu(x))(1 - 2zx_-(\mu(x)))}{\mu(x)}, \]

\[g_2(x) = -\frac{xA(x)}{\sqrt{2}} \frac{z_-(\mu(x))}{\mu(x)}, \quad g_3(x) = -\frac{xB(x)}{\sqrt{2}} \frac{z_-(\mu(x))}{\mu(x)}.\]
From $\tilde{F}_1 = \Theta(f_1)$, $\tilde{F}_4 = \Theta(f_4)$ and (3.23) we get

$$
\tilde{F}_1(x, z) = \frac{g_1(x)}{1 - z_-(\mu(x))}, \\
\tilde{F}_2(x, z) = \frac{g_2(x)}{1 - z_-(\mu(x))}, \\
\tilde{F}_3(x, z) = 1 + \frac{g_3(x)}{1 - z_-(\mu(x))}, \\
\tilde{F}_4(x, z) = \frac{g_4(x)}{1 - z_-(\mu(x))}.
$$

(3.30)

Since $\mu(x) = 2x(1 + x^2/r^2)^{-1}$, we find the following formulas for the functions $g_j$, with $z_- = z_-(\mu(x))$:

$$
\begin{align*}
g_1(x) &= \frac{z_-}{2x} \left( 1 - 2xz_- \right) \left( 1 - \frac{5}{2}xz_- + \frac{x^2}{2}z_-^2 \right) \left( 1 - \frac{5}{2}xz_- \right)^2 - \frac{x^2}{2}z_-^4, \\
g_2(x) &= -\frac{z_-}{2\sqrt{2}} \left( 1 - \frac{5}{2}xz_- + \frac{x^2}{2}z_-^2 \right) \left( 1 - \frac{5}{2}xz_- \right)^2 - \frac{x^2}{2}z_-^4, \\
g_3(x) &= \frac{xz_-}{4} \left( 1 - \frac{5}{2}xz_- + z_-^2 \right) \left( 1 - \frac{5}{2}xz_- \right)^2 - \frac{x^2}{2}z_-^4, \\
g_4(x) &= -\frac{z_-}{2\sqrt{2}} \left( 1 - 2xz_- \right) \left( 1 - \frac{5}{2}xz_- + \frac{x^2}{2}z_-^2 \right) \left( 1 - \frac{5}{2}xz_- \right)^2 - \frac{x^2}{2}z_-^4. 
\end{align*}
$$

(3.31)

When the orientation of $(\tilde{A}, \tilde{B})$ is of type $j \in \{1, 2, 3, 4\}$, we will write

$$
G(k, \tilde{B}, \tilde{A}) = G_j(k, d), \text{ with } d = \text{dist}(m(A), m(B)).
$$

3.10. **Proposition.** For $k \geq 1$, and $j \in \{1, 2, 3, 4\}$, one has

$$
G_j(k, d) = \left( \frac{\sqrt{2}}{3} \right)^k \left( \frac{1}{\sqrt{2}} \right)^d \frac{1}{k!} \partial_x^d \left[ g_j(x)z_-^d(\mu(x)) \right]_{x=0}.
$$

(3.32)

*Proof.* This follows from (3.21), (3.30) and $Z = \sqrt{2}z, X = \sqrt{2}x$. □

We denote also by

$$
\mathcal{E}_j(k, d) = \sum_{(\tilde{A}, \tilde{B}) \text{ type } j, \tilde{A} \text{ fixed}, d(m(A), m(B)) = d} \left| G(k, \tilde{A}, \tilde{B}) \right|^2
$$

the contribution to the energy at time $k$ of all oriented wedges $\tilde{B}$ such that $m(B)$ is at distance $d$ of $m(A)$ and the orientation of $(\tilde{A}, \tilde{B})$ is of type $j$. One has

$$
\mathcal{E}(k, d) = \sum_{j=1,2,3,4} \mathcal{E}_j(k, d).
$$
The number of oriented wedge $\vec{B}$, with $\text{dist}(m(A), m(B)) = d$, and the orientation of $(\vec{A}, \vec{B})$ of a given type $j \in \{1, 2, 3, 4\}$ is equal to $2^d$. Thus, from (3.32), we get
\[
E_j(k, d) = 2^d \left| \frac{\partial}{\partial z^\mu} (g_j(x)z^\mu(\mu(x)))) \right|_{x=0}^2, \quad r = \sqrt{2}/3.
\]

4. Proof of Theorem 1.1

In this section, we study the functions of $d \in \{0, 1, ..., k\}$, $d \mapsto E_j(k, d)$, and we prove Theorem 1.1. We start by the following Lemma.

4.1. Lemma. The functions $g_j(x)$ are holomorphic in the complex disc $D_r = \{x, |x| < r\}$. Their boundary value on $\partial D_r$, $\theta \mapsto g_j(re^{i\theta})$ is analytic except for $\theta$ equal to $\pm \theta_0, \pm (\theta_0 + \pi)$ with $\cos \theta_0 = 2r$. Near the singular points $x_0 \in \{re^{i\theta_0}, re^{-i\theta_0}, -re^{i\theta_0}, -re^{-i\theta_0}\}$ one has $g_j(x) = a_j(x) + (x-x_0)^{1/2}b_j(x)$, with $a_j, b_j$ holomorphic near $x_0$.

Proof. Since we know a priori that \( \sum_{d=0}^{k} E_j(k, d) \leq 1 \), (3.33) implies that the functions \( x \mapsto g_j(x) \) are holomorphic in the complex disc $D_r = \{x, |x| < r = \sqrt{2}/3\}$ (take $d = 0$).

More precisely we get from (3.33) with $g_j(x) = \sum_{n=0}^{\infty} g_{j,n}x^n$:
\[
|g_{j,n}| = \frac{1}{n!} \left| \frac{\partial^n}{\partial x^n} g_j(0) \right| \leq r^{-n} \sqrt{E_j(n, 0)} \leq r^{-n}.
\]

To analyze the singularities of $g_j$ on $\partial D_r$, we use formulas (3.31). From Lemmas 3.7 et 3.8, we know that the function $h(x) = z_-(\mu(x))$ is holomorphic in $D_r$ and $|h(x)| < 1$. Moreover, the function $h$ satisfies $h(0) = 0$, $h'(0) = 1$, and
\[
h^2 - (1 + \frac{x^2}{r^2}) \frac{h}{2x} + 1 = 0.
\]

This shows $h(x) = 0$ iff $x = 0$, and the only possible singularities of $h$ are located at the zeros of the discriminant of this second order equation, which means $1 + \frac{x^2}{r^2} = \pm 4x$ which is equivalent to $x \in \{re^{i\theta_0}, re^{-i\theta_0}, -re^{i\theta_0}, -re^{-i\theta_0}\}$. Moreover, near the singular points $x_0 \in \{re^{i\theta_0}, re^{-i\theta_0}, -re^{i\theta_0}, -re^{-i\theta_0}\}$ one has $h(x) = a(x) + (x-x_0)^{1/2}b(x)$, with $a, b$ holomorphic near $x_0$.

It remains to verify
\[
|x| \leq r \Rightarrow \left( 1 - \frac{5}{2}xz_- \right)^2 - \frac{x^2}{2}z_+^2 \neq 0,
\]
and we already know that (4.35) is true for $|x| < r$. Let us assume $1 - \frac{5}{2}xz_- = \varepsilon \frac{x^2}{2}z_+^2$ with $\varepsilon = \pm 1$. Then one has
\[
\begin{cases}
\frac{\varepsilon x^2}{2}z_+^2 + \frac{5}{2}xz_- - 1 = 0 \\
z_- - \frac{\varepsilon x}{\mu} + 1 = 0.
\end{cases}
\]
We eliminate $z_-$ in the polynomial system \((4.36)\). Then, with $x = ry = \frac{\sqrt{5}}{3}y$ we get that $y$ satisfies the polynomial equation:

\[(4.37)\quad y^4 - 6ey^2 + 8y^2 - 6ey - 9 = (y^2 - 2ey + 3)(y^2 - 4ey - 3) = 0.\]

Therefore, one has ever $\varepsilon y = 1 \pm i\sqrt{2}$ which implies $|x| = r|y| > r$, or $\varepsilon y = 2 \pm \sqrt{7}$, and then $|x| = r|y| \neq r$.

\[\text{Remark.} \quad \text{Observe that in the case} \quad \varepsilon y = 2 - \sqrt{7} \approx -0.646, \text{we have} \quad |x| = r|y| < r, \text{but for this particular value of} \quad x, \text{one verifies} \quad 1 - \frac{2}{3}xz_+ + \frac{2}{3}z_+^2 \neq 0. \text{Since} \quad g_2 \text{is holomorphic near} \quad x, \text{this shows that the value} \quad \varepsilon y = 2 - \sqrt{7} \text{is associated to the root} \quad z_+ = \frac{1}{z_-} \text{in} \quad \text{(4.36)}.\]

The proof of Lemma 4.1 is complete. \hfill \Box

We define $\psi_j(k, d)$, $j \in \{1, 2, 3, 4\}$, such that $E_j(k, d) = |\psi_j(k, d)|^2$, by the formula

\[(4.38)\quad \psi_j(k, d) = r^k \frac{1}{k!} \partial_x^k \left[ g_j(x)z_+^{j} (\mu(x)) \right]_{x=0}.\]

By the proof of Lemma 4.1, we know that the function $h(x) = z_+ (\mu(x))$ is holomorphic in $D_r$, continuous on $\overline{D_r}$, satisfies $|h(x)| \leq 1$, $h(x) = 0$ iff $x = 0$, and its singularities on $\partial D_r$ are at $x \in \{r e^{i \theta_0}, r e^{-i \theta_0}, -r e^{i \theta_0}, -r e^{-i \theta_0}\}$. Therefore, there exists a function $\varphi(\theta)$, holomorphic in $Im(\theta) > 0$, with $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$, continuous for $Im(\theta) \geq 0$, such that $Im(\varphi(\theta)) \geq 0$, with singularities at $\theta \in [\pm \theta_0, \pm (\theta_0 + \pi)]$, such that

\[z_-(\mu r e^{i \theta})) = z_-(\frac{r}{\cos \theta}) = e^{i \varphi(\theta)}.\]

We normalize it by choosing $\varphi(\theta_0) = 0$.

From $z_- = e^{i \varphi}$, $z_+ = e^{-i \varphi}$ and $z_+ + z_- = \frac{1}{\mu} = \frac{\cos \theta}{r}$, we get

\[2r \cos \varphi = \cos \theta.\]

One find the following explicit formulas for the function $\varphi(\theta)$

\[(4.39)\quad \varphi(\theta) = i \text{argch} \left( \frac{\cos \theta}{2r} \right) \quad \text{for} \quad \theta \in [-\theta_0, \theta_0],
\varphi(\theta) = \text{arccos} \left( \frac{\cos \theta}{2r} \right) \quad \text{for} \quad \theta \in [\theta_0, \pi - \theta_0],
\varphi(\theta) = \pi + i \text{argch} \left( \frac{|\cos \theta|}{2r} \right) \quad \text{for} \quad \theta \in [\pi - \theta_0, \pi + \theta_0],
\varphi(\theta) = \pi + \text{arccos} \left( -\frac{\cos \theta}{2r} \right) \quad \text{for} \quad \theta \in [\pi + \theta_0, 2\pi - \theta_0].\]

In particular, we get $\varphi(\theta + \pi) = \varphi(\theta) + \pi$.

The variations of the function $\theta \mapsto \varphi(\theta)$ are resumed in the following picture:

\[\text{Figure 6}\]
By Cauchy formula, one has
\[
\psi_j(k,d) = \frac{r^k}{2\pi i} \int_{|z|=r} \frac{g_j(z) \zeta^{d}}{\zeta^{k+1}} \, d\zeta = \frac{1}{2\pi} \int_{0}^{2\pi} g_j(re^{i\theta}) e^{i(d\phi(\theta) - k\theta)} \, d\theta.
\]

For \(d \in [0,k]\), we define \(\gamma \in [0,1]\) by \(d = \gamma k\). We will apply the phase stationary method with \(k\) as great parameter to the integral
\[
\psi_j(k,d) = \frac{1}{2\pi} \int_{0}^{2\pi} g_j(re^{i\theta}) e^{i(\gamma \phi(\theta) - \theta)} \, d\theta.
\]

For \(\theta \notin [\theta_0, \pi - \theta_0] \cup [\pi + \theta_0, -\theta_0]\), one has \(\text{Im} \phi > 0\). For \(\theta \in [\theta_0, \pi - \theta_0]\), one has \(\phi \in [0, \pi]\) and \(\partial_{\theta} \phi = \frac{\sin \theta}{2r \sin \phi}\). We thus get
\[
\partial_{\theta} \phi = \frac{\sin \theta}{2r(1 - \cos^2 \phi)^{1/2}} = \frac{1}{2r} \frac{\sqrt{1 - 4r^2 z^2}}{\sqrt{1 - z^2}}, \quad z = \cos \theta \in [-1, 1].
\]

Hence, one has \(\partial_{\theta} \phi \geq \frac{1}{2r}\) and equality iff \(\theta = \frac{\pi}{2}\), and \(\partial_{\theta} \phi(\theta_0) = \partial_{\theta} \phi(\pi - \theta_0) = +\infty\). For \(\theta \in [\theta_0, \pi - \theta_0]\), one finds
\[
4r^2 (\sin \phi)^3 \partial_{\theta}^2 \phi = 2r \cos \theta (\sin \phi)^2 - (\sin \theta)^2 \cos \phi = -\cos \theta \left(\frac{1}{2r} - 2r\right).
\]

The phase in (4.41) is equal to \(\gamma \phi(\theta) - \theta\). Thus the phase is stationary iff \(\partial_{\theta} \phi = 1/\gamma\).

**4.3. Theorem.** Let \(c_* = 2^{\sqrt{2}/3} = 2r < 1\) and \(j \in \{1, 2, 3, 4\}\). For \(k \in \mathbb{N}\) and \(d \in \mathbb{N}\), let \(\gamma = d/k\).

1. For \(\gamma > c_*\), there exists \(a = a_j(\gamma) > 0\) such that \(\psi_j(k,d) \in O(e^{-ak})\).
2. For \(\gamma \in ]0, c_*[\), there exist \(b_l = b_{l,j}(\gamma), a_l = a_{l,j}(\gamma)\), such that
\[
\psi_j(k,d) = k^{-1/2} \sum_{l \in \{1,2,3,4\}} e^{i\delta k} \left(b_l + O(k^{-1})\right).
\]
3. For \(\gamma = c_* = \delta\) with \(\delta\) small, there exist two functions \(\zeta(\delta) = \frac{2r}{\rho} \delta + O(\delta^2),\)
\(\tau(\delta) = O(\delta^2),\) where \(\rho = \frac{1}{2}(\frac{1}{4r^2} - 1)\), and for \(l = 1, 2\), symbols of degree 0 in \(k\),
\(a_{l,\pm,j}(\delta,k) \approx \sum_{n \geq 0} k^{-n} a_{l,\pm,n,j}(\delta)\), such that
\[
\psi_j(k,d) = e^{ik\tau(\delta)} \left(A_+ e^{i\delta \zeta(\delta)} A_0 + e^{i\delta \zeta(\delta)} A_-\right).
\]
\(A_+ = e^{i\delta/2} \left(a_{1,+,j}Ai((kp)^{2/3} \zeta) + k^{-3/3} a_{2,+,j}Ai'(((kp)^{2/3} \zeta)\right),\)
where \(Ai(z) = \frac{1}{2\pi} \int e^{t(z/3 + zt)} dt\) is the Airy function.
Proof. \(1\) For \(\gamma > c_*\), the phase has no real critical points, and the result follows by deformation of the integral (4.41) on the contour \(\theta = \mu + i\epsilon, \epsilon > 0, \mu \in [0,2\pi]\) since \(\text{Im}(\gamma \varphi(\theta) - \mu) > 0\) on this contour. Equivalently, we replace in (4.40) the circle \(|\zeta| = r\) by the circle \(|\zeta| = re^{-\epsilon}\).

\(2\) For \(\epsilon \in [0,\epsilon]\), the phase has 4 non degenerate critical points \(\theta_1, \theta_2, \theta_3, \theta_4\), with \(\theta_0 < \theta_1 < \pi/2 < \theta_2 < \pi - \theta_0\) and \(\theta_3 = \pi + \theta_1, \theta_4 = \pi + \theta_2\). This follows from the fact that \(\partial^2 \varphi\) is strictly decreasing from \(+\infty\) to \(1/2\) on \([\theta_0, \pi/2]\) and strictly increasing from \(1/2\) to \(+\infty\) on \([\pi/2, \pi - \theta_0]\). We can then apply the stationary phase theorem, on a complex deformation of the contour leaving the critical points fixed, and avoiding the singularities \(\{\pm \theta_0, \pm (\pi - \theta_0)\}\) of the phase and symbols in (4.41) (since \(\gamma \partial \varphi(\theta) - 1 \gg 0\) near \(\theta_0, \pi - \theta_0\), we use a deformation \(\theta = \mu + i\epsilon, \epsilon > 0\) near the singular points \(\theta_0, \pi - \theta_0\)).

\(3\) \(\gamma = c_* = 2\pi\) is a transition point with two double critical points at \(\theta = \frac{\pi}{2}\) and \(\theta = \frac{3\pi}{2}\). This gives contributions of Airy type. More precisely, let \(\gamma - c_* = \delta\) small and set \(\Phi(\theta) = \gamma \varphi(\theta) - \theta\). One has

\[
\varphi(\pi/2) = \pi/2, \quad \varphi'(\pi/2) = \frac{1}{2r}, \quad \varphi''(\pi/2) = 0, \quad \varphi'''(\pi/2) = \frac{1}{2r} \left( \frac{1}{4r^2} - 1 \right).
\]

Thus for \(r\) small, we find

\[
\Phi(\pi/2 + y) = \frac{\pi}{2} (2r - 1) + \left( \frac{1}{4r^2} - 1 \right) y^3/6 + O(y^4) + \delta \varphi(\pi/2 + y).
\]

Then from classical results on phase integrals with degenerate critical points of order two, we get the term \(A_+\) which is associated to the integral

\[
\int_0^\pi g_j(re^{i\theta}) e^{ik(\gamma \varphi(\theta) - \theta)} d\theta.
\]

Using \(\Phi(\theta + \pi) = \Phi(\theta) + (\gamma - 1)\pi\), the same analysis give the term \(A_-\) which is associated to the integral

\[
\int_{2\pi}^{2\pi} g_j(re^{i\theta}) e^{ik(\gamma \varphi(\theta) - \theta)} d\theta.
\]

The proof of Theorem 4.3 is complete. \(\square\)

Proof. of Theorem 1.1 The points \(1\) and \(2\) of Theorem 1.1 are consequences of points \(1\) and \(2\) of Theorem 4.3. The proof of \(3\) in Theorem 1.1 is easy: for \(\gamma = 0\), i.e \(d = 0\), the result follows from the fact that the \(\psi_j(k,0)\) are the Fourier coefficient of the function \(g_j\), which by Lemma 4.1 has at most \(\sqrt{\gamma}\) type singularities. This leads to \(\psi_j(k,0) \in O(k^{-3/2})\). \(\square\)

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UR Analyse et Contrôle des Edp, UR13ES64, Département de Mathématiques, Faculté des Sciences de Monastir, Université de Monastir, 5019 Monastir, Tunisie

E-mail address: kais.ammari@fsm.rnu.tn

Département de Mathématiques, Université de Nice Sophia-Antipolis, Parc Valrose, 06000 Nice Cedex 02, France

E-mail address: lebeau@unice.fr