BIFURCATION ANALYSIS OF THE THREE-DIMENSIONAL HÉNON MAP

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Abstract. In this paper, we consider the dynamics of a generalized three-dimensional Hénon map. Necessary and sufficient conditions on the existence and stability of the fixed points of this system are established. By applying the center manifold theorem and bifurcation theory, we show that the system has the fold bifurcation, flip bifurcation, and Neimark-Sacker bifurcation under certain conditions. Numerical simulations are presented to not only show the consistence between examples and our theoretical analysis, but also exhibit complexity and interesting dynamical behaviors, including period-10, -13, -14, -16, -17, -20, and -34 orbits, quasi-periodic orbits, chaotic behaviors which appear and disappear suddenly, coexisting chaotic attractors. These results demonstrate relatively rich dynamical behaviors of the three-dimensional Hénon map.

1. Introduction. The well known Hénon map with the constant Jacobian determinant was proposed by Hénon [10] as a simplified model to the Poincaré section of the Lorenz system [14]. This map is a two-dimensional discrete-time system with a single quadratic non-linearity and exhibits chaotic behavior for the parametric values $a = 1.4$ and $b = 0.3$. Since the late 1970’s, the Hénon map has been extensively studied because of its generality and wide applications. For example, Marotto [16] and Michael [17] analyzed the bifurcation and chaos of the Hénon map with respect to the certain parameters. Curry [4] presented some numerical experiments on the existence of two distinct strange attractors for some parametric values of the Hénon map by using the characteristic exponent, frequency spectrum and a theorem of Smale [22]. The detailed bifurcation diagrams of the Hénon map were described by Mira [18]. By introducing the positive and negative iterative mappings, Luo and Guo [15] investigated the bifurcation and stability of periodic solutions and chaotic layers. Subsequently, different types of the generalized Hénon map were proposed.

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Dullin and Meiss [5] discussed the dynamics of a single, generalized Hénon map. Baier and Klein [2] considered a generalized Hénon map with a single quadratic nonlinearity, where the dimension of this map can be higher than 2.

The $n$-dimensional generalized Hénon map was presented by Richter [20]. In the past decades, considerable attention has been dedicated to the three-dimensional systems due to its significant role in the study of dynamical properties for the high-dimensional systems [1, 2, 8, 11, 13, 25, 24, 6]. For instance, an interesting three-dimensional generalization of the Hénon map was introduced in [11]. Maximum hyperchaos in the generalized Hénon map containing a single quadratic term was investigated in [2]. It was shown in [8] that the three-dimensional Hénon-like map possesses the wild Lorenz-type strange attractors. The existence of strange pseudohyperbolic attractors for the three-dimensional Hénon map was presented in [7].

As we know, many nonlinear systems have special dynamical behaviors with the values of parameters given in the certain intervals or regions. As the values of parameters vary, qualitative structure of orbits and dynamical properties of systems, such as bifurcations and chaotic phenomena, may change dramatically [9]. One of interesting problems on nonlinear dynamical systems is to find how the dynamical behaviors and properties of orbits change and evolve as the values of parameters vary [9].

The aim of this paper is to investigate a three-dimensional Hénon map. At this stage we restrict our attention to the existence of the fold bifurcation, flip bifurcation, and Neimark-Sacker bifurcation by making use of the center manifold theorem [3, 9, 23] and bifurcation theory [9, 12, 23]. Numerical simulations on bifurcation diagrams, Lyapunov exponents and phase portraits are undertaken which agree well with our analytical results. In addition, after analyzing Lyapunov exponents of the system, we demonstrate the orbits of period-10, -13, -14, -16, -17, -20, and -34, attracting invariant cycles, quasi-periodic orbits, coexisting chaotic attractors, period-doubling bifurcation from period-10 leading to chaos, cascades of period-doubling bifurcation in orbits of period-2, -4, -8, ···, and nice chaotic behaviors which appear and disappear suddenly.

The rest of this paper is organized as follows. In Section 2, we introduce the generalized three-dimensional Hénon map and discuss the existence and local stability of the fixed points of the associated system. In Section 3, we explore sufficient conditions of the existence of codimension one bifurcations, including the fold bifurcation, flip bifurcation and Neimark-Sacker bifurcation. In Section 4, numerical simulations are presented to illustrate complexity and dynamical behaviors of our system. Section 5 is a brief conclusion.

2. Three-dimensional Hénon map. Consider the three-dimensional Hénon map [11]:

$$
\begin{align*}
    x_{n+1} &= 1 + y_n - ax_n^2, \\
    y_{n+1} &= bx_n + z_n, \\
    z_{n+1} &= -bx_n,
\end{align*}
$$

where $a, b \in \mathbb{R}$ and $a, b \neq 0$. Let $\bar{u} = ax, \bar{v} = \frac{a}{b} y$ and $\bar{w} = \frac{a}{b} z$. It reduces to

$$
\begin{align*}
    \bar{u}_{n+1} &= a - \bar{u}_n^2 + b\bar{v}_n, \\
    \bar{v}_{n+1} &= \bar{u}_n + \bar{w}_n, \\
    \bar{w}_{n+1} &= -\bar{u}_n.
\end{align*}
$$
For simplicity, we still use \( x, y \) and \( z \) instead of \( \tilde{u}, \tilde{v}, \tilde{w} \), respectively. Then system (1) becomes
\[
\begin{align*}
    x_{n+1} &= a - x_n^2 + by_n, \\
    y_{n+1} &= x_n + z_n, \\
    z_{n+1} &= -x_n.
\end{align*}
\]
It is easy to see that when \( a > -\frac{1}{4}, \) system (2) has two fixed points:
\[ P_1(x_1, 0, -x_1) \quad \text{and} \quad P_2(x_2, 0, -x_2), \]
where \( x_1 = \frac{-1 + \sqrt{1 + 4a}}{2} \) and \( x_2 = \frac{-1 - \sqrt{1 + 4a}}{2} \). When \( a \) tends to \(-\frac{1}{4}\), these two fixed points approach to the point \( P_0(-\frac{1}{4}, 0, \frac{1}{2}) \).

The Jacobian matrix \( J_i \) of system (2) at the fixed point \( P_i \) \((i = 1, 2)\) is given by
\[ J_i = \left( \begin{array}{ccc} -2x_i & b & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{array} \right), \]
and the characteristic equation of the Jacobian matrix \( J_i \) is
\[ \lambda^3 + 2x_i\lambda^2 - b\lambda + b = 0. \]
Denote \( \lambda = \frac{r+1}{r-1} \). The above equation (3) changes to
\[ c_0r^3 + c_1r^2 + c_2r + c_3 = 0, \]
where \( c_0 = 1 + 2x_i, \ c_1 = 3 - 2b + 2x_i, \ c_2 = 3 + 4b - 2x_i, \) and \( c_3 = 1 - 2b - 2x_i. \)
Obviously, \( |\lambda| < 1 \) is equivalent to \( \text{Re}(r) < 0 \). Let
\[ \Delta_1 = c_1, \quad \Delta_2 = \begin{vmatrix} c_1 & c_0 \\ c_3 & c_2 \end{vmatrix} = c_1c_2 - c_0c_3 \quad \text{and} \quad \Delta_3 = \begin{vmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{vmatrix} = c_3\Delta_2. \]
According to the Routh-Hurwitz theorem, \( \text{Re}(r) < 0 \) if and only if there holds
\[ c_j > 0 \ (j = 0, 1, 2, 3) \quad \text{and} \quad \Delta_k > 0 \ (k = 1, 2, 3). \]
That is,
\[ \begin{align*}
    &c_0 > 0, \ c_1 > 0, \ c_2 > 0, \ c_3 > 0, \\
    &\Delta_2 > 0,
\end{align*} \]
which is equivalent to
\[ \begin{align*}
    &x_i > -\frac{1}{2}, \ x_i > b - \frac{3}{2}, \ x_i < 2b + \frac{3}{2}, \ x_i < \frac{1}{2} - b, \\
    &2bx_i - b^2 + b + 1 > 0.
\end{align*} \]
Since \( x_2 = \frac{-1 - \sqrt{1 + 4a}}{2} < -\frac{1}{2} \), the fixed point \( P_2(x_2, 0, -x_2) \) is always unstable.

For the stability of the fixed point \( P_1(x_1, 0, -x_1) \). Substituting \( x_1 = \frac{-1 + \sqrt{1 + 4a}}{2} \) into (1), we get
\[ \begin{align*}
    &-1 < b < -\frac{3}{2}, \\
    &\frac{-1}{\sqrt{1 + 4a}} < \frac{b^2 - 1}{b}, \quad \text{or} \quad \frac{-1}{\sqrt{1 + 4a}} < 2(1 - b).
\end{align*} \]
Notice that the existence of the fixed point \( P_1 \) is assured under the condition \( a > -\frac{1}{4}. \)

The fixed point \( P_1 \) is stable if and only if the following conditions are satisfied
\[ \begin{align*}
    &-1 < b < -\frac{1}{3}, \\
    &-\frac{3}{4} < a < \left(\frac{1}{2b} - \frac{1}{2}\right)^2 - 1, \quad \text{or} \quad -\frac{3}{4} < a < (1 - b)^2 - \frac{1}{4}.
\end{align*} \]
Based on the above analysis, we have the following lemma:
Lemma 2.1. The following statements are true.

(1) The fixed point $P_2(x_2, 0, -x_2)$ is unstable.

(2) The fixed point $P_1(x_1, 0, -x_1)$ is stable if and only if one of the following conditions holds:
   
   (i) either $-1 < b < -\frac{1}{3}$ and $a_0 < a < a_h$, or
   
   (ii) $-\frac{1}{3} < b < 1$ and $a_0 < a < a^*$,

   where $a_0 = -\frac{1}{4}$, $a_h = \left(\frac{b^2 - 1}{2b}\right)^2 - \frac{1}{4}$, and $a^* = (1 - b)^2 - \frac{1}{4}$.

3. Codimension one bifurcations. In this section, we investigate the fold bifurcation, flip bifurcation and Neimark-Sacker bifurcation at the fixed point of system (2). We choose the coefficient $a$ as a bifurcation parameter for analyzing the fold bifurcation, flip bifurcation and Neimark-Sacker bifurcation by means of the center manifold theorem [3, 9, 23] and bifurcation theory [9, 12, 23].

3.1. Fold bifurcation. Consider the fold bifurcation at the unique fixed point $P_0$ of system (2), i.e., $a = a_0 = -\frac{1}{4}$. The eigenvalues of the Jacobian matrix at $P_0$ are $\lambda_1 = 1$ and $\lambda_{2,3} = \pm \sqrt{b}$. Assume $b \neq \pm 1$. It gives $|\lambda_{2,3}| \neq 1$.

Let

\[ \bar{x} = x + \frac{1}{2}, \quad \bar{y} = y, \quad \bar{z} = z - \frac{1}{2} \quad \text{and} \quad \bar{a} = a + \frac{1}{4}. \]

We translate the fixed point $P_0$ to the origin and take $\bar{a}$ as a new dependent variable, then system (2) becomes

\[
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z} \\
\bar{a}
\end{pmatrix} \mapsto \begin{pmatrix} 1 & b & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{a} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} f_1(\bar{x}, \bar{y}, \bar{z}, \bar{a}),
\]

where $f_1(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = -\bar{x}^2$.

For $b > 0$, we find an invertible matrix by making use of corresponding eigenvectors of the $4 \times 4$ matrix in the map (5):

\[
T_1 = \begin{pmatrix}
-1 & -1 & \sqrt{b} & -\sqrt{b} \\
0 & -1 & -1 & \frac{1}{\sqrt{b}} \\
1 & 0 & 1 & 1 \\
0 & b - 1 & 0 & 0
\end{pmatrix}.
\]

Use the transformation of $(\bar{x}, \bar{y}, \bar{z}, \bar{a})^T = T_1 (u, v, w, \mu)^T$. The map (5) can be rewritten as

\[
\begin{pmatrix}
u \\
w \\
w \\
\mu
\end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{b} & 0 \\ 0 & 0 & 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \begin{pmatrix} \frac{1}{b-1} \\ 0 \\ 0 \\ \frac{1}{\sqrt{b}} \end{pmatrix} g_1(u, v, w, \mu),
\]

where $g_1(u, v, w, \mu) = -(u - v + \sqrt{b}w - \sqrt{b}\mu)^2$.

By the center manifold theorem [9], we know that there exists a center manifold $W^c(0)$ around the origin, which can be expressed as follows

\[ W^c(0) = \{(u, v, w, \mu) | w = h_1(u, v), \mu = h_2(u, v), h_i(0, 0) = 0, \] \[ Dh_i(0, 0) = 0, \quad i = 1, 2, \] \]
where
\[ h_1(u, v) = \alpha_1 u^2 + \beta_1 uv + \gamma_1 v^2 + O \left( (|u| + |v|)^3 \right), \]
and
\[ h_2(u, v) = \alpha_2 u^2 + \beta_2 uv + \gamma_2 v^2 + O \left( (|u| + |v|)^3 \right). \]

Since the center manifold must satisfy
\[
\begin{align*}
\{ & h_1 \left( u + v + \frac{g_1(u,v,h_1(u,v),h_2(u,v))}{b-1}, v \right) + \sqrt{b}h_1(u, v) - \frac{g_1(u,v,h_1(u,v),h_2(u,v))}{2(1+\sqrt{b})} = 0, \\
& h_2 \left( u + v + \frac{g_1(u,v,h_1(u,v),h_2(u,v))}{b-1}, v \right) - \sqrt{b}h_2(u, v) - \frac{g_1(u,v,h_1(u,v),h_2(u,v))}{2(1-\sqrt{b})} = 0,
\end{align*}
\]

By comparing the corresponding coefficients of the above two equations, we have
\[
\begin{align*}
\alpha_1 &= -\frac{1}{(1+\sqrt{b})^2}, \quad \beta_1 = -\frac{\sqrt{b}}{(1+\sqrt{b})}, \quad \gamma_1 = -\frac{b-\sqrt{b}}{2(1+\sqrt{b})}, \\
\alpha_2 &= -\frac{1}{(\sqrt{b}-1)^2}, \quad \beta_2 = -\frac{\sqrt{b}}{(\sqrt{b}-1)}, \quad \gamma_2 = -\frac{b+\sqrt{b}}{2(\sqrt{b}-1)}.
\end{align*}
\]

Consider the map on the center manifold given by
\[
F_1 : u \mapsto u + v + \frac{1}{1-b} \left( u^2 + 2uv + v^2 \right) + O \left( (|u| + |v|)^3 \right). \tag{7}
\]

It is easy to see that
\[
F_1(0, 0) = 0, \quad F_{1u}(0, 0) = 1, \quad F_{1v}(0, 0) = 1 \quad \text{and} \quad F_{1uu}(0, 0) = \frac{2}{1-b} \neq 0.
\]

Hence, the fixed point \((u, v) = (0, 0)\) is a fold bifurcation point for the map \((7)\).

For \(b = 0\), let
\[
T_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

By using the transformation of \((\bar{x}, \bar{y}, \bar{z}, \bar{a})^T = T_1(u, v, w, \mu)^T\), the map \((5)\) becomes
\[
\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tilde{g}_1(u, v, w, \mu), \tag{8}
\]

where \(\tilde{g}_1(u, v, w, \mu) = -(v - u)^2\).

To simplify the map \((8)\) on the center manifold, we assume that the center manifold is of the form
\[
W^c(0) = \{(u, v, w, \mu) \mid w = \tilde{h}_1(u, v), \quad \mu = \tilde{h}_2(u, v), \\
\tilde{h}_1(0, 0) = 0, \quad D\tilde{h}_1(0, 0) = 0, \quad i = 1, 2\},
\]

where
\[
\tilde{h}_1(u, v) = \tilde{\alpha}_1 u^2 + \tilde{\beta}_1 uv + \tilde{\gamma}_1 v^2 + O \left( (|u| + |v|)^3 \right),
\]
and
\[
\tilde{h}_2(u, v) = \tilde{\alpha}_2 u^2 + \tilde{\beta}_2 uv + \tilde{\gamma}_2 v^2 + O \left( (|u| + |v|)^3 \right).
\]

Using an approximate computation on the center manifold and equating the corresponding coefficients, we get
\[
\begin{align*}
\tilde{\alpha}_1 &= -1, \quad \tilde{\beta}_1 = -2, \quad \tilde{\gamma}_1 = -1, \\
\tilde{\alpha}_2 &= -1, \quad \tilde{\beta}_2 = 0, \quad \tilde{\gamma}_2 = 0.
\end{align*}
\]
Thus, the map restricted to the center manifold is given by
\[ \tilde{F}_1 : u \mapsto u - v + (v - u)^2 + O \left( (|u| + |v|)^3 \right). \]  
(9)

From the map (9), we find
\[ \tilde{F}_1(0, 0) = 0, \quad \tilde{F}_{1u}(0, 0) = 1, \quad \tilde{F}_{1v}(0, 0) = -1, \quad \text{and} \quad \tilde{F}_{1uv}(0, 0) = -2 < 0. \]
This implies that the fixed point \((u, v) = (0, 0)\) is a fold bifurcation point for the map (9).

For \(b = -s^2 < 0\) (\(s\) is a nonzero real number), let
\[ T_1 = \begin{pmatrix} -1 & -1 & -s & 0 \\ 0 & -1 & -\frac{1}{s} & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -(1 + s^2) & 0 & 0 \end{pmatrix}. \]
By using the transformation of \((\bar{x}, \bar{y}, \bar{z}, \bar{a})^T = T_1(u, v, w, \mu)^T\), the map (5) becomes
\[\begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -s \\ 0 & 0 & s & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \frac{1}{1 + s^2} \begin{pmatrix} -1 \\ 0 \\ -s \\ 1 \end{pmatrix} \tilde{g}_1(u, v, w, \mu), \]  
(10)
where \(\tilde{g}_1(u, v, w, \mu) = -(u - v - sw)^2\).

Applying the center manifold theorem again, we know that there exists a center manifold for (10) which can be written as
\[ W^c(0) = \{(u, v, w, \mu) \mid w = \tilde{h}_1(u, v), \quad \mu = \tilde{h}_2(u, v), \quad \tilde{h}_1(0, 0) = 0, \quad D\tilde{h}_1(0, 0) = 0, \quad i = 1, 2, \} \]
where
\[ \tilde{h}_1(u, v) = \tilde{\alpha}_1 u^2 + \tilde{\beta}_1 uv + \tilde{\gamma}_1 v^2 + O \left( (|u| + |v|)^3 \right), \]
and
\[ \tilde{h}_2(u, v) = \tilde{\alpha}_2 u^2 + \tilde{\beta}_2 uv + \tilde{\gamma}_2 v^2 + O \left( (|u| + |v|)^3 \right). \]

After substitution, by a straightforward computation on equating the corresponding coefficients, it gives
\[\begin{cases} \tilde{\alpha}_1 = \frac{2s}{(1 + s^2)^2}, & \tilde{\beta}_1 = \frac{-2s(1 - 3s^2)}{(1 + s^2)^3}, & \tilde{\gamma}_1 = \frac{s - 10s^3 + 5s^5}{(1 + s^2)^4}, \\ \tilde{\alpha}_2 = \frac{-1 + s^2}{(1 + s^2)^2}, & \tilde{\beta}_2 = \frac{-2s^3(3 - s^2)}{(1 + s^2)^3}, & \tilde{\gamma}_2 = \frac{s^3(5 - 10s^3 + s^5)}{(1 + s^2)^4}. \end{cases}\]
Thus, the map restricted to the center manifold is given by
\[ \tilde{F}_1 : u \mapsto u + v + \frac{1}{1 + s^2}(u + v)^2 + O \left( (|u| + |v|)^3 \right). \]  
(11)

Since
\[ \tilde{F}_1(0, 0) = 0, \quad \tilde{F}_{1u}(0, 0) = 1, \quad \tilde{F}_{1v}(0, 0) = 1, \quad \text{and} \quad \tilde{F}_{1uv}(0, 0) = \frac{2}{1 + s^2} > 0, \]
the fixed point \((u, v) = (0, 0)\) is a fold bifurcation point of the map (11).

Based on the above discussions, we obtain the following results immediately.

**Theorem 3.1.** System (3) undergoes a fold bifurcation at the fixed point \(P_0\) if the conditions \(b \neq \pm 1\) and \(a = a_0\) hold. Moreover, two fixed points bifurcate from \(P_0\) for \(a > a_0\), coalesce as the fixed point \(P_0\) at \(a = a_0\), and disappear for \(a < a_0\).
3.2. Flip bifurcation. In this subsection, we consider the flip bifurcation occurring at the fixed point \( P_1 \) of system \( \left[ \right] \). When \( a = a^* \), the fixed point is \( P_1 = (\bar{x}, \bar{y}, \bar{z}) \), where \( \bar{x} = \frac{1}{2} - b, \bar{y} = 0, \) and \( \bar{z} = -\bar{x} \). The eigenvalues of the associated Jacobian matrix at the fixed point \( P_1 \) are \( \lambda_1 = -1 \) and \( \lambda_{2,3} = b \pm c \), where \( c = \sqrt{b^2 - b} \). The condition \( |\lambda_{2,3}| \neq 1 \) means \( b \neq -\frac{1}{3}, 1 \). Hence, we assume that \( b \neq -\frac{1}{3} \) and \( b \neq 1 \) in the following discussions.

Let
\[
\bar{x} = x - \hat{x}, \quad \bar{y} = y - \hat{y}, \quad \bar{z} = z - \hat{z} \quad \text{and} \quad \bar{a} = a - a^*.
\]

We translate the fixed point \( P_1 \) into the origin and take \( \bar{a} \) as a variable, then system \( \left[ \right] \) can be rewritten as
\[
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z} \\
\bar{a}
\end{pmatrix} \mapsto \begin{pmatrix}
2b - 1 & b & 0 & 1 \\
1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z} \\
\bar{a}
\end{pmatrix} + 
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} f_2(\bar{x}, \bar{y}, \bar{z}, \bar{a}),
\]
where \( f_2(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = -\bar{x}^2 \).

In the case of \( b \neq 0 \), we get an invertible matrix by making use of corresponding eigenvectors of the \( 4 \times 4 \) matrix in the map \( \left[ \right] \):
\[
T_2 = \begin{pmatrix}
1 & -\frac{1}{2(-1+b)} & -b + c & -b - c \\
-2 & 0 & \frac{1}{-b+c(1-b+c)} & \frac{1}{b} \\
1 & \frac{1}{2(-1+b)} & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]

Use the transformation \( (\bar{x}, \bar{y}, \bar{z}, \bar{a})^T = T_2 (u, v, w, \mu)^T \). Then the map \( \left[ \right] \) becomes
\[
\begin{pmatrix}
u \\
w \\
\mu
\end{pmatrix} \mapsto \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & b - c & 0 \\
0 & 0 & 0 & b + c
\end{pmatrix}
\begin{pmatrix}
u \\
w \\
\mu
\end{pmatrix} + 
\frac{1}{2(1+3b)c} \begin{pmatrix}
2c \\
0 \\
2b - c \\
-2b - c
\end{pmatrix} g_2(u, v, w, \mu),
\]
where
\[
g_2(u, v, w, \mu) = -\left[ u + \frac{1}{2(1-b)} v + (c-b)w - (c+b)\mu \right]^2.
\]

Similarly, we reduce the map \( \left[ \right] \) on the center manifold, which can be represented as
\[
W^c(0) = \{(u, v, w, \mu) \mid w = h_3(u, v), \mu = h_4(u, v), h_4(0, 0) = 0, \quad Dh_i(0, 0) = 0, \quad i = 3, 4\},
\]
for \( u \) and \( v \) being sufficiently small. Assume that the center manifold is of the form:
\[
h_3(u, v) = \alpha_3 u^2 + \beta_3 uv + \gamma_3 v^2 + O \left( (|u| + |v|)^3 \right),
\]
and
\[
h_4(u, v) = \alpha_4 u^2 + \beta_4 uv + \gamma_4 v^2 + O \left( (|u| + |v|)^3 \right).
\]
Then, the map (13) restricted to the center manifold can be expressed as

\[ h_3 \left( -u + \frac{g_2(u, v, h_3(u, v), h_4(u, v))}{1 + 3b}, v \right) - (b - c)h_3(u, v) - (2b - c)g_2(u, v, h_3(u, v), h_4(u, v)) = 0, \]

\[ h_4 \left( -u + \frac{g_2(u, v, h_3(u, v), h_4(u, v))}{1 + 3b}, v \right) - (b + c)h_4(u, v) + (2b + c)g_2(u, v, h_3(u, v), h_4(u, v)) = 0. \]

So the center manifold must satisfy

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Use the transformation \( \bar{w} \) rewritten as

\[ \tilde{F} - \tilde{G} \]

In the case of \( \tilde{F} \), we take the invertible matrix

\[ \alpha_3 = \frac{b - c}{2(-1 + b + 2bc + c^2)}, \quad \beta_3 = -\frac{b - c}{2(-1 + b)(1 + b - c)c^2}, \quad \gamma_3 = -\frac{b - c}{8(-1 + b)^2(-1 + b - c)(1 + b - c)c^2}; \]

\[ \alpha_4 = \frac{b + 2c}{2(-1 + b + 2bc + c^2)}, \quad \beta_4 = \frac{b + 2c}{2(-1 + b)(1 + b - c)c^2}, \quad \gamma_4 = -\frac{b + 2c}{8(-1 + b)^2(-1 + b + c)(1 + b + c)c^2}. \]

Then, the map (13) restricted to the center manifold can be expressed as

\[ u \mapsto F_2(u, v), \quad (14) \]

where

\[ F_2(u, v) = -u - \frac{1}{1 + 3b}u^2 - \frac{1}{(1 - b)(1 + 3b)}uv - \frac{1}{4(1 + 3b)(b - 1)^2}v^2 \]

\[ + \frac{4b}{(1 - b)(1 + 3b)^2}u^3 + \frac{2b(-3 + 7b)}{(b - 1)^2(1 + 3b)^3}u^2v - \frac{b(-3 + 7b)}{(-1 - 2b + 3b^2)^3}uv^2 \]

\[ + \frac{b}{2(b - 1)^3(1 + 3b)^2}v^3 + O \left((|u| + |v|)^4\right). \]

In order for the map (14) to undergo a flip bifurcation, we require that two discriminatory quantities \( k_1 \) and \( k_2 \) are not zero, where

\[ k_1 = \left[ \frac{\partial F_2}{\partial v} \left( \frac{\partial^2 F_2}{\partial u^2} + 2 \frac{\partial^2 F_2}{\partial u \partial v} \right) \right]_{(0,0)} = -\frac{2}{(1 - b)(1 + 3b)} \neq 0, \]

and

\[ k_2 = \left[ \frac{1}{2} \left( \frac{\partial^2 F_2}{\partial u^2} \right)^2 + \frac{1}{3} \frac{\partial^3 F_2}{\partial u^3} \right]_{(0,0)} = \frac{2}{(1 - b)(1 + 3b)} \neq 0. \]

In the case of \( b = 0 \), we take the invertible matrix

\[ T_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}. \]

Use the transformation \( (\bar{x}, \bar{y}, \bar{z}, \bar{a})^T = T_2(u, v, w, \mu)^T \). Then the map (12) can be rewritten as

\[ \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \mu \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} \tilde{g}_2(u, v, w, \mu), \quad (15) \]

where \( \tilde{g}_2(u, v, w, \mu) = -(u + v)^2 \).
We reduce the map (15) on a center manifold which can be expressed as
\[ W^c(0) = \{(u, v, w, \mu) \mid u = \tilde{h}_3(u, v), \; v = \tilde{h}_4(u, v), \; \tilde{h}_i(0, 0) = 0, \; D\tilde{h}_i(0, 0) = 0, \; i = 3, 4\}. \]

Assume that \( \tilde{h}_3(u, v) \) and \( \tilde{h}_4(u, v) \) are of the following forms, respectively,
\[ \tilde{h}_3(u, v) = \tilde{\alpha}_3 u^2 + \tilde{\beta}_3 uv + \tilde{\gamma}_3 v^2 + O \left( (|u| + |v|)^3 \right), \]
and
\[ \tilde{h}_4(u, v) = \tilde{\alpha}_4 u^2 + \tilde{\beta}_4 uv + \tilde{\gamma}_4 v^2 + O \left( (|u| + |v|)^3 \right). \]

By a direct computation on the center manifold, we find
\[ \left\{ \begin{array}{l} \tilde{\alpha}_3 = -1, \; \tilde{\beta}_3 = 6, \; \tilde{\gamma}_3 = -1, \\ \tilde{\alpha}_4 = 1, \; \tilde{\beta}_4 = -2, \; \tilde{\gamma}_4 = 1. \end{array} \right. \]

Thus, the map (15) restricted to the center manifold is given by
\[ \tilde{F}_2 : u \mapsto -u - (u^2 + 2uv + v^2) + O \left( (|u| + |v|)^3 \right). \quad (16) \]

Note that
\[ k_1 \equiv \left| \frac{\partial F_2}{\partial v} \cdot \frac{\partial^2 F_2}{\partial u^2} + 2 \frac{\partial^2 F_2}{\partial u \partial v} \right|_{(0,0)} = -4 \neq 0, \]
and
\[ k_2 \equiv \left| \frac{1}{2} \left( \frac{\partial^2 F_2}{\partial u^2} \right)^2 + \frac{1}{3} \left( \frac{\partial^3 F_2}{\partial u^3} \right) \right|_{(0,0)} = 2 > 0. \]

Thus, the fixed point \((u, v) = (0, 0)\) is a flip bifurcation point for the map (16).

In addition, if \(-\frac{1}{3} < b < 1\), then \(|\lambda_{2,3}| < 1\) and \(k_2 > 0\). We summarize our discussions into the following results.

**Theorem 3.2.** The system (2) undergoes a flip bifurcation at the fixed point \(P_1\) when \(b \neq -\frac{1}{3}, 1\) and \(a = a^*\). Moreover, if \(-\frac{1}{3} < b < 1\) \((b < -\frac{1}{3} \text{ or } b > 1)\) holds, then the period-2 orbits that bifurcate from the fixed point \(P_1\) are stable (unstable).

### 3.3. Neimark-Sacker bifurcation

Now, we consider the Neimark-Sacker bifurcation of system (2) at the fixed point \(P_1\). Let \(a = a_h\), where \(a_h\) is given in Lemma 2.1. If \(\Delta = -(3b^2 - 2b - 1) < 0\), the characteristic equation (3) at the point \(P_1\) has a pair of complex conjugate roots \(\lambda_{1,2} = \frac{1 + b \pm i\sqrt{3b^2 - 2b - 1}}{2b} = \rho \pm i\omega\) and a real root \(\lambda_3 = -b\) with \(|\lambda_{1,2}| = 1\) and \(d = \frac{d|\lambda_{1,2}(a_h)|}{da} > 0\). The fixed point turns into \(P_1 = (\tilde{x}, \tilde{y}, \tilde{z})\), where \(\tilde{x} = \frac{\rho^2 - b - 1}{2b}, \; \tilde{y} = 0,\) and \(\tilde{z} = -\tilde{x}\).

In addition, for a Neimark-Sacker bifurcation, it requires that
\[ \lambda_{1,2}^n(a_h) \neq 1 \quad (n = 1, 2, 3, 4) \quad \text{and} \quad |\lambda_3| = |b| \neq 1, \]
which implies that
\[ b \neq -\frac{1}{3}, -\frac{1}{2}, -1, 1. \]

Note that \(\Delta = -(3b^2 - 2b - 1) < 0\) is equivalent to \(b < -\frac{1}{3}\) or \(b > 1\). Hence, we consider the case of \(b < -\frac{1}{3}\) \((b \neq -1, -\frac{1}{2})\) or \(b > 1\) in this subsection.
Let \( \bar{x} = x - \bar{x}, \bar{y} = y - \bar{y}, \) and \( \bar{z} = z - \bar{z}. \) Then the fixed point \((\bar{x}, \bar{y}, \bar{z})\) can be translated to the origin, and system (2) becomes

\[
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix}
= \begin{pmatrix}
\frac{-b^2+b\beta+1}{b} & b & 0 \\
1 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix}
+ \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
f(\bar{x}, \bar{y}, \bar{z})
\end{pmatrix}, \tag{17}
\]

where \( f(\bar{x}, \bar{y}, \bar{z}) = -\bar{x}^2. \) Using the corresponding eigenvectors of the \( 3 \times 3 \) matrix in the map (17), we get an invertible matrix:

\[
T_3 = \begin{pmatrix}
-\sqrt{3b^2 - 2b - 1} & \frac{-1+b}{2b} & b \\
-\sqrt{3b^2 - 2b - 1} & \frac{1+b}{2b} & -\frac{1+b}{b} \\
\frac{3b+1}{b} & \frac{2}{2b} & \frac{1}{1}
\end{pmatrix}.
\]

By making use of the transformation \((\bar{x}, \bar{y}, \bar{z})^T = T_3(u, v, w)^T\), the map (17) becomes

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\rightarrow \begin{pmatrix}
\rho & -\omega & 0 \\
\omega & \rho & 0 \\
0 & 0 & -b
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
+ \frac{b}{2 + b + b^2}
\begin{pmatrix}
-\frac{3b+1}{b} \\
\frac{2}{2b} \\
\frac{1}{1}
\end{pmatrix}
g(u, v, w), \tag{18}
\]

where \( g(u, v, w) = -(-\omega u - \rho v + bw)^2. \)

We simplify the map (18) on the center manifold by assuming that the manifold is of the form

\[W^c(0) = \{(u, v, w)|w = h(u, v), h(0, 0) = 0, Dh(0, 0) = 0\}.\]

Let

\[h(u, v) = \alpha u^2 + \beta uv + \gamma v^2 + O(|u| + |v|)^3).\]

Substituting it into (18) yields

\[
h \left[ \rho u - \omega v - \frac{b(3+b)g(u, v, h(u, v))}{(2+b+b^2)(\sqrt{3b^2-2b-1})}, \omega u + \rho v - \frac{bg(u, v, h(u, v))}{(2+b+b^2)} \right] + bh(u, v) - \frac{bg(u, v, h(u, v))}{2 + b + b^2} = 0.
\]

Equating the corresponding coefficients to zero gives

\[
\alpha = -\frac{b(-1+b)^2(1+3b)}{4(2+9b+9b^2+6b^3+4b^4+b^5+b^6)}, \quad \beta = -\frac{b(1+b)\sqrt{3b^2-2b-1}}{2(2+7b+2b^2+4b^3+b^5)}, \quad \gamma = -\frac{b(3+11b+b^2+b^3)}{4(2+9b+9b^2+6b^3+4b^4+b^5+b^6)}.
\]

Thus, the map (18) restricted to the center manifold is given as follows

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}
\rightarrow \begin{pmatrix}
\rho & -\omega \\
\omega & \rho
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
+ \begin{pmatrix}
f^1(u, v) \\
g^1(u, v)
\end{pmatrix}, \tag{19}
\]
where

\[
f^1(u, v) = \frac{(-1 + b)(3 + b)(1 + 3b)u^2}{4b(2 + b + b^2)\sqrt{3b^2 - 2b - 1}} + \frac{(3 + b)(1 + b)^2v^2}{4b(2 + b + b^2)\sqrt{3b^2 - 2b - 1}} + \frac{(3 + b)(1 + b)uv}{2b(2 + b + b^2)} + \frac{(-1 + b)^2b^2(3 + b)(1 + 3b)u^3}{4(2 + b + b^2)^2(1 + 4b + 2b^2 + b^4)} + \frac{b^2(-1 + b)(3 + b)(1 + 3b)^2u^2v}{4(2 + b + b^2)^2(1 + 3b - b^2 + b^3)\sqrt{3b^2 - 2b - 1}} + \frac{b^2(3 + b)(1 + 3b)(5 + 2b + b^2)uv^2}{4(2 + b + b^2)^2(1 + 4b + 2b^2 + b^4)} + O\left((|u| + |v|)^4\right),
\]

and

\[
g^1(u, v) = \frac{(-1 + b)(1 + 3b)u^2}{4b(2 + b + b^2)} + \frac{(1 + b)\sqrt{3b^2 - 2b - 1}}{2b(2 + b + b^2)}uv + \frac{(1 + b)^2}{4b(2 + b + b^2)}v^2 + \frac{(-1 + b)^2b^2(1 + 3b)\sqrt{3b^2 - 2b - 1}}{4(1 + b)(2 + b + b^2)^2(1 + 3b - b^2 + b^3)}u^3 + \frac{b^2(1 + 3b)(5 + 2b + b^2)\sqrt{3b^2 - 2b - 1}}{4(1 + b)(2 + b + b^2)^2(1 + 3b - b^2 + b^3)}uv^2 + \frac{(-1 + b)^2b^2(1 + 3b)}{4(2 + b + b^2)^2(1 + 3b - b^2 + b^3)}u^2v + \frac{b^2(3 + 11b + b^2 + b^3)}{4(2 + b + b^2)^2(1 + 3b - b^2 + b^3)}v^3 + O\left((|u| + |v|)^4\right).
\]

In order for the map \([9]\) to undergo a Neimark-Sacker bifurcation, we require that the following discriminatory quantity \(l_1\) is not equal to zero \([9] [23] [21]\):

\[
l_1 = -\frac{1}{2} \frac{1 - 2\lambda}{1 - \lambda} \frac{\lambda \xi_{20}}{\xi_{11}} - \frac{1}{2} \xi_{11}^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}),
\]

where the sign of \(l_1\) determines the stability of the invariant circle, and

\[
\xi_{20} = \frac{1}{8} \left[(f^1_{uu} - f^1_{vv} + 2g^1_{uv}) + i\left(g^1_{uu} - g^1_{vv} - 2f^1_{uv}\right)\right],
\]

\[
\xi_{11} = \frac{1}{4} \left[(f^1_{uu} + f^1_{vv}) + i\left(g^1_{uu} + g^1_{vv}\right)\right],
\]

\[
\xi_{02} = \frac{1}{8} \left[(f^1_{uu} - f^1_{vv} - 2g^1_{uv}) + i\left(g^1_{uu} - g^1_{vv} + 2f^1_{uv}\right)\right],
\]

\[
\xi_{21} = \frac{1}{16} \left[(f^1_{uuu} + f^1_{vvv} + g^1_{uvu} + g^1_{vvu}) + i\left(g^1_{uuu} + g^1_{vvv} - f^1_{uvu} - f^1_{vvv}\right)\right].
\]

By a direct calculation, we get

\[
l_1 = \frac{b^2(1 + 2b - 2b^2 + b^3)}{2(b^2 - 1)(2 + b + b^2)(1 + 3b - b^2 + b^3)}.
\]
Note that if $-1 < b < 1$, then $|\lambda_3| < 1$. By the fact that $b < -\frac{1}{3} (b \neq -1, -\frac{1}{2})$, it follows that $l_1 < 0$ for $-1 < b < -\frac{1}{3}$ and $b \neq -\frac{1}{2}$. Consequently, we summarize our result as the following theorem.

**Theorem 3.3.** Suppose that $b < -\frac{1}{3} (b \neq -1, -\frac{1}{2})$ or $b > 1$. Then the system (2) undergoes a Neimark-Sacker bifurcation at the fixed point $P_1$ for $a = a_h$. Moreover, if $-1 < b < -\frac{1}{3}$ and $b \neq -\frac{1}{2}$, then the bifurcation is supercritical and an attracting invariant closed curve bifurcates from the fixed point for $a > a_h$.

In Figure 1, it shows the stability region and bifurcation region of system (2) in the parametric space.

![Figure 1](image)

**Figure 1.** The stability region and bifurcation region of system (2) in the $(b,a)$-plane.

4. **Numerical simulations.** In this section, numerical simulations are provided, including bifurcation diagrams, maximum Lyapunov exponents and phase portraits, to illustrate consistency of our theoretical results, and to show the rich complex dynamics of system (2).

4.1. **Numerical simulations of stability and codimension one bifurcations.** Based on Lemma 2.1 and Theorem 3.1-3.3, we consider the bifurcation parameters in the following three cases.

**Case 1.** Let $b = -0.6$, and $a \in (-0.3, 0.4)$ be the bifurcation parameter. Then $a^* = 2.31$ and $a_h \sim 0.0344$. From Figure 2 (A), we observe that if $a < a_0$, there is no fixed point; if $a = a_0$, there is only one fixed point $P_0$; if $a > a_0$, there are two fixed points $P_1(x_1, 0, -x_1)$ and $P_2(x_2, 0, -x_2)$ bifurcating from $P_0$. It shows that the number of the fixed points varies as the value of the parameter $a$ changes. We also find that $x_2$ is unstable and $x_1$ is stable for $a \in (a_0, a_h)$ in Figure 2 (A). This agrees well with Lemma 2.1 and Theorem 3.1. Furthermore, we see that the fixed point $P_1$ loses its stability at $a_h \sim 0.0344$ and an attracting invariant closed curve bifurcates from the fixed point $P_1$ for $a > a_h$. This is in good agreement with Theorem 3.3.

**Case 2.** Let $b = 0.4$, and $a \in (-0.4, 1)$ be the bifurcation parameter. Then $a^* = 0.11$ and $P_1(0.1, 0, -0.1)$. By a straightforward computation, the eigenvalues of the corresponding Jacobian matrix at the fixed point $P_1$ are $\lambda_1 = -1$ and $\lambda_{2,3} = 0.4 \pm 0.4898979i$, with $k_1 = -1.5151515$ and $k_2 = 1.5151515$. On the basis of Lemma
BIFURCATION ANALYSIS OF THE THREE-DIMENSIONAL HÉNON MAP

FIGURE 2. (A)-(B) bifurcation diagrams of system (2) in the (a, x) plane: (A) \(b = -0.6\), and (B) \(b = 0.4\); (C) bifurcation diagram of system (2) in the (b, x) plane with \(b \in (-0.8, 0.8)\) and \(a = 0.2\). Here, the fold bifurcation, flip bifurcation and Neimark-Sacker bifurcation are labeled as “SN”, “PD” and “NS”, respectively.

From Figures 2(A)-(B) and Theorem 3.2, we know that the fixed point \(P_1\) is stable for \(a \in (a_0, a^*)\) and loses its stability when \(a > a^*\). The flip bifurcation occurs at \(a = a^*\) and the period-2 orbits that bifurcate from \(P_1\) are stable. All these phenomena are clearly presented in Figure 2(B).

**Case 3.** Let \(a = 0.2\), and \(b \in (-0.8, 0.8)\) be the bifurcation parameter. According to Lemma 2.1, the fixed point \(P_1\) is stable for \(b \in (-0.5333, 0.3292)\). From Theorem 3.2, the flip bifurcation occurs at \(P_1\) for \(b \sim 0.3292\). From Theorem 3.3, the supercritical Neimark-Sacker bifurcation occurs at \(P_1\) for \(b \sim -0.5333\). All these phenomena are clearly presented in Figure 2(C) which agree well with Lemma 2.1 and Theorem 3.1-3.3.

4.2. More numerical simulations of system (2). In this subsection, we show that new complex dynamical behaviors change as the parameters of system (2) vary by using the bifurcation diagrams, maximum Lyapunov exponents and phase portraits.

The bifurcation diagrams in the three-dimensional \((a, b, x)\) space are shown in Figure 3. For the bifurcation diagrams in the two-dimensional space, we consider the bifurcation parameters in the following three cases:

(i) changing \(a\) in the range \(0 \leq a \leq 0.4\), and fixing \(b = -0.6\);
(ii) changing $a$ in the range $0.1 \leq a \leq 0.9$, and fixing $b = 0.4$; and (iii) changing $b$ in the range $-0.8 \leq b \leq 0.8$, and fixing $a = 0.23$.

**Figure 3.** Bifurcation diagrams of system (2) in the three-dimensional $(a, b, x)$ space.

**Figure 4.** (A) bifurcation diagram of system (2) in the $(a, x)$ plane $(a \in (0, 0.4))$ for $b = -0.6$; (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for $a \in (0.22, 0.32)$. 
Figure 5. (A)-(H) phase portraits for various values of $a$ corresponding to Figure 4 (A).
Case (i). The bifurcation diagram of system (2) in the \((a,x)\) plane for \(b = -0.6\) and \(0 \leq a \leq 0.4\) with the initial value \((0.03429, 0, -0.03429)\) is demonstrated in Figure 4 (A). The maximum Lyapunov exponents corresponding to Figure 4 (A) are shown in Figure 4 (B). The local amplified bifurcation diagram of Figure 4 (A) for \(a \in (0.22, 0.32)\) is presented in Figure 4 (C). From Figures 4 (A)-(C), we observe that there is a stable fixed point for \(a \in (0, 0.0344)\) and the fixed point loses its stability at \(a \sim 0.0344\). The Neimark-Sacker bifurcation occurs at \(a \sim 0.0344\) and an attracting invariant cycle appears for \(a > 0.0344\). Then, we can see the process of the period-doubling bifurcation which is from the period-10 orbits to chaos and these chaotic regions are interspersed with several periodic windows. For example, we can observe orbits of period-17 for \(a = 0.294\), period-34 for \(a = 0.31\), and period-14 for \(a = 0.361\). Phase portraits for various values of \(a\) are shown in Figure 5 which clearly depicts how a smooth invariant curve bifurcates from the stable fixed point and an invariant curve to chaotic attractors. From Figure 5, we observe that there are an attracting invariant circle, quasi-periodic orbits which are routines to chaos, period-20 orbits, ten-coexisting chaotic attractors, and chaotic sets. In particular, Figures 6 (A)-(C) show chaotic attractors for \(a = 0.385\) in the \((x,y)\) plane, the \((x,z)\) plane, and the \((y,z)\) plane, respectively.

Furthermore, from Figure 4 (B), one can see that the maximum Lyapunov exponents are negative for the parameter \(a \in (0, 0.0344)\) when the fixed point is stable. For \(a \in (0.0344, 0.23)\), the maximum Lyapunov exponents are in the neighborhood.
of zero, which correspond to quasi-period orbits or coexistence of chaos and quasi-period orbits. For $a \in (0.23, 0.4)$, the majority of maximum Lyapunov exponents are positive with a few being negative, which shows that there exist stable fixed points or stable period windows in the chaotic region.

For case (ii). The bifurcation diagram of system (2) in the $(a, x)$ plane for $b = 0.4$ and $0.1 \leq a \leq 1$ with the initial value $(-0.101, 0.101, -0.101)$ is shown in Figure

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{(A) bifurcation diagram of system (2) in the $(a, x)$ plane $(a \in (0, 1))$ for $b = 0.4$; (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for $a \in (0.81, 0.85)$; (D) maximum Lyapunov exponent corresponding to (C); (E)-(F) chaotic attractors for $a = 0.835$ and $a = 0.8445$, respectively.}
\end{figure}
Figure 8. (A) bifurcation diagram of system (2) in the \((b, x)\) plane with \(b \in (-0.8, 0.8)\) and \(a = 0.23\); (B) maximum Lyapunov exponent corresponding to (A); (C) the local amplified bifurcation diagram of (A) for \(b \in (-0.75, -0.5)\); (D) maximum Lyapunov exponent corresponding to (C); (E) the local amplified bifurcation diagram of (A) for \(b \in (0.64, 0.7)\); (F) maximum Lyapunov exponent corresponding to (E).

(A) and the maximum Lyapunov exponents corresponding to Figure 7 (A) are shown in Figure 7 (B). Figure 7 (C) shows the local amplified bifurcation diagram and Figure 7 (D) shows the maximum Lyapunov exponents corresponding to Figure 7 (C). Figure 7 clearly depicts the period-doubling bifurcation being a routine to chaos. From Figures 7 (A) and (C), we can see that the fixed point is stable for
Figure 9. In (A)-(C), phase portraits corresponding to Figure 8 (C): (A) $b = -0.72$, (B) $b = -0.6$, and (C) $b = -0.53$. In (D)-(F), phase portraits corresponding to Figure 8 (E): (D) $b = 0.66$, (E) $b = 0.675$, and (F) $b = 0.691$.

$a < 0.11$, loses its stability at $a = 0.11$, and then a cascade of period-2, -4, -8, \ldots orbits emerge. From Figures 8 (B) and (D), we observe that there exist positive Lyapunov exponents when $a$ exceeds 0.11, which indicates that the chaotic sets arise. This means that the system tends to a stable state at the very beginning, then circulates along period-doubling orbits, and finally follows irregular chaotic sets with some period-2 orbits. For example, the chaotic behavior suddenly disappears for $a \sim 0.8446$, and then the chaotic behavior suddenly appears for $a \sim 0.8447$ and converges to period-2 orbits for $a \sim 0.845$. Phase portraits of four-coexisting chaotic
attractors for \( a = 0.835 \) and a chaotic attractor for \( a = 0.8445 \) are illustrated in Figures 7(E) and (F), respectively.

**Case (iii).** The bifurcation diagram of system (2) in the \((b, x)\) plane for \( a = 0.23 \) is displayed in Figure 8(A). The maximum Lyapunov exponents corresponding to Figure 8(A) are calculated and shown in Figure 8(B), which indicates the existence of chaotic regions and periodic orbits as the parameter \( b \) varies. Figures 8(C) and (E) present the local amplifications of Figure 8(A) and the corresponding maximum Lyapunov exponents are shown in Figures 8(D) and (F), respectively. From Figures 8(D) and (F), we can see when \( b < -0.59 \) or \( b > 0.67 \), some Lyapunov exponents are greater than 0 and some are smaller than 0, so there exist the stable fixed points or stable periodic windows in the chaotic region. Figure 8 clearly depicts that there are period-14, -10, -13, -16 windows and invariant cycles in the chaotic regions, and two onsets of chaos at \( b \approx -0.74 \) and \( b \approx 0.67 \), respectively. From Figures 8(C) and (E), the following phenomena are observed: the period-windows within the chaotic regions and the period-doubling bifurcations from period-2 orbits to chaos. Figures 9(A) and (B) show two nice chaotic attractors at \( b = -0.72 \) and \( b = -0.6 \), respectively. Figure 9(C) shows an invariant circle for \( b = -0.53 \). Furthermore, Figures 9(D)-(F) exhibit the processes from period-doubling bifurcations to chaos corresponding to Figure 8(E), in which two invariant circles at \( b = 0.66 \) are presented in Figure 9(D), eight-coexisting chaotic attractors at \( b = 0.675 \) are demonstrated in Figure 9(E), and a chaotic attractor at \( b = 0.691 \) is shown in Figure 9(F).

5. **Conclusions.** In this study, we investigated the complex dynamical behaviors of a generalized three-dimensional Hénon map. It is shown that system (2) can undergo the fold bifurcation, flip bifurcation and Neimark-Sacker bifurcation by applying the center manifold theorem and bifurcation theory. Moreover, system (2) displays some interesting dynamical behaviors, including invariant circle, orbits of period-10, -13, -14, -16, -17, -20, and -34, quasi-periodic orbits, the new nice types of four-, eight-, and ten-coexisting chaotic attractors, period-doubling bifurcation from period-10 leading to chaos, cascades of period-doubling bifurcation in orbits of period-2, -4, -8, · · · , and chaotic sets. These results show far richer dynamics of the generalized three-dimensional Hénon map, and dynamical properties are different from that of the standard Hénon map. These results can provide us fundamental and useful information to further better understand the dynamic complexity of the higher-dimensional Hénon map arising in different scientific fields.

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