Abstract

We survey the area of strongly regular graphs satisfying the 4-vertex condition and find several new families. We describe a switching operation on collinearity graphs of polar spaces that produces cospectral graphs. The obtained graphs satisfy the 4-vertex condition if the original graph belongs to a symplectic polar space.

Keywords 4-vertex condition · Strongly regular graph · Symplectic polar space

Mathematics Subject Classification 05E30

1 Introduction

In this note we look at graphs with high combinatorial regularity, where this regularity is not an obvious consequence of properties of their group of automorphisms.

A graph $\Gamma$ is said to satisfy the $t$-vertex condition if, for all triples $(T, x_0, y_0)$ consisting of a $t$-vertex graph $T$ together with two distinct distinguished vertices $x_0, y_0$ of $T$, and all pairs of distinct vertices $x, y$ of $\Gamma$, the number of isomorphic copies of $T$ in $\Gamma$, where the isomorphism maps $x_0$ to $x$ and $y_0$ to $y$, does not depend on the choice of the pair $x, y$ but only on whether $x, y$ are adjacent or nonadjacent.

This concept was introduced by Hestenes and Higman [14] (who refer to the unpublished Sims [32]) in order to study rank 3 graphs. Clearly, a rank 3 graph satisfies the $t$-vertex condition for all $t$. If the graph $\Gamma$ satisfies the $t$-vertex condition, where $\Gamma$ has $v$ vertices and $3 \leq t \leq v$, then $\Gamma$ also satisfies the $(t-1)$-vertex condition. A graph satisfies the 3-vertex condition if and only if it is strongly regular (or complete.
or edgeless). It satisfies the $v$-vertex condition if and only if it is rank 3. Thus, we get a hierarchy of conditions of increasing strength between strongly regular and rank 3.

The present paper will focus almost exclusively on the case $t = 4$. A simple criterion for the 4-vertex condition is given in Proposition 2.1. Previously not many graphs were known that satisfy the 4-vertex condition without being rank 3. Here we survey the known examples and give several new constructions. One of our constructions proceeds by switching symplectic graphs (see Sect. 7). As a consequence we find

**Theorem 1.1** For $v \geq 4$ there are at least $\lfloor v^{1/6} \rfloor!$ strongly regular graphs of order at most $v$ satisfying the 4-vertex condition.

It follows that among all non-isomorphic strongly regular graphs of order at most $v$ that satisfy the 4-vertex condition the fraction that is determined by their spectrum goes to 0 when $v$ goes to infinity.

### 2 The 4-Vertex Condition

A graph of order $v$ is called strongly regular with parameters $(v, k, \lambda, \mu)$ if it is neither complete nor edgeless, each vertex has degree $k$, any two adjacent vertices have exactly $\lambda$ common neighbors, and any two non-adjacent vertices have exactly $\mu$ common neighbors.

A graph with vertex set $V$ has rank $r$ if its automorphism group is transitive on $V$ and has exactly $r$ orbits on $V \times V$. Rank 3 graphs are strongly regular.

If $x$ is a vertex of the graph $\Gamma$, then the local graph $\Gamma(x)$ of $\Gamma$ at $x$ is the induced subgraph in $\Gamma$ on the neighborhood of $x$. We say that $\Gamma$ is locally P when all local graphs of $\Gamma$ have property P. If $\Gamma$ is strongly regular, then its 1st subconstituent (at a vertex $x$) is the local graph at $x$, while its 2nd subconstituent (at $x$) is the induced subgraph on the non-neighborhood of $x$. If $xy$ is an edge (resp. nonedge) in $\Gamma$, then the subgraph induced on $\Gamma(x) \cap \Gamma(y)$ is called a $\lambda$-graph (resp. $\mu$-graph).

See [4] for further information about strongly regular graphs.

Details on the parameters of graphs satisfying the 4-vertex condition are given in [14]. In particular, we have the following simple criterion for the 4-vertex condition:

**Proposition 2.1** (Sims [32]) A strongly regular graph $\Gamma$ with parameters $(v, k, \lambda, \mu)$ satisfies the 4-vertex condition, with parameters $(\alpha, \beta)$, if and only if the number of edges in $\Gamma(x) \cap \Gamma(y)$ is $\alpha$ (resp. $\beta$) whenever the vertices $x, y$ are adjacent (resp. nonadjacent). In this case, $k((\lambda^2) - \alpha) = \beta(v - k - 1)$.

The equality here follows by counting 4-cliques minus an edge.

It immediately follows that the collinearity graph of a generalized quadrangle (cf. [28]) or partial quadrangle (cf. [7]) satisfies the 4-vertex condition (with $\alpha = \lambda^2 / 2$ and $\beta = 0$). The same holds for a graph $\Gamma$ with $\lambda \leq 1$.

If $\Gamma$ is locally strongly regular, say with local parameters $(v', k', \lambda', \mu')$ (where clearly $v' = k$ and $k' = \lambda$), then $\Gamma(x) \cap \Gamma(y)$ has valency $\lambda'$ (resp. $\mu'$) when $x \sim y$ (resp. $x \not\sim y$) so that $\Gamma$ satisfies the 4-vertex condition with $\alpha = \lambda\lambda'/2$ and $\beta = \mu\mu'/2$. 

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2.1 A Few Rank 4 Examples

Below we give a small table with the parameters of some edge-transitive rank 4 graphs satisfying the 4-vertex condition. Except for the example with group $HJ.2$ due to Reichard [30], these do not seem to have been noticed in print.

| $v$  | $k$  | $\lambda$ | $\mu$ | $\lambda'$ | $\mu'$ | $\alpha$ | $\beta$ | Group       | Name          | Refs. |
|------|------|-----------|-------|------------|--------|--------|--------|-------------|---------------|-------|
| 144  | 55   | 22        | 20    | –          | 9      | 87     | 90     | $M_{12}.2$  |               |       |
| 280  | 36   | 8         | 4     | –          | 2      | 1      | 4      | $HJ.2$      |               | [30]  |
| 300  | 104  | 28        | 40    | –          | 8      | 78     | 160    | $PGO_5(5)$  | $N_5^-(5)$    | Sect. 6 |
| 325  | 144  | 68        | 60    | –          | 30     | 1153   | 900    | $PGO_5(5)$  | $N_5^+(5)$    | Sect. 6 |
| 512  | 196  | 60        | 84    | 14         | 20     | 420    | 840    | $2^9.\Gamma L_3(8)$ | Dual hyperoval | Sect. 4 |
| 729  | 112  | 1        | 20    | 0          | 0      | 0      | 0      | $3^6.2.L_3(4).2$ | Games graph   | [3]   |
| 1120 | 729  | 468       | 486   | 297        | 306    | 69498  | 74358  | $PSp_6(3).2$ | disj. t.i. planes | Sect. 5 |
| 1849 | 462  | 131       | 110   | –          | –      | 2980   | 1845   | $43^2.(42\times D_{22})$ | power diff. set | Sect. 3.6 |

The numbers $\lambda'$, $\mu'$ give the valency of the $\lambda$- and $\mu$-graphs in case these are regular (and then $\alpha = \lambda\lambda'/2$ and $\beta = \mu\mu'/2$).

The examples on 144 and 729 vertices also satisfy the 5-vertex condition.

2.2 Strongly Regular Graphs with Strongly Regular Subconstituents

As we saw, graphs that are locally strongly regular satisfy the 4-vertex condition. Sometimes it follows that also the 2nd subconstituents must be strongly regular.

**Lemma 2.2** Suppose that a strongly regular graph with parameters $(v, k, \lambda, \mu) = (4t^2, 2t^2-\varepsilon t, t^2-\varepsilon t, t^2-\varepsilon t)$ (where $\varepsilon = \pm 1$) has first subconstituents that are strongly regular with parameters $(v', k', \lambda', \mu') = (2t^2-\varepsilon t, t^2-\varepsilon t, t^2-\varepsilon t)$. Then its second subconstituents are strongly regular with parameters $(v'', k'', \lambda'', \mu'') = (2t^2 + \varepsilon t - 1, t^2, t^2, \frac{1}{2}t^2, \frac{1}{2}t^2)$.

More generally, the spectrum of the 2nd subconstituent at any vertex of a strongly regular graph follows from that of the 1st subconstituent—see [8], Theorem 5.1.

Call the three parameter sets in the above lemma $A(\varepsilon t)$, $B(\varepsilon t)$, and $C(\varepsilon t)$, respectively. They occur again in Sect. 3.3. The parameter sets $A(t)$ and $A(-t)$ are known as (negative) Latin square parameters $LS_t(2t)$ (resp. $NL_t(2t)$). The complementary graphs have parameters $LS_{t+1}(2t)$ (resp. $NL_{t+1}(2t)$).

Cameron et al. [8] studied the situation of a primitive strongly regular graph such that, for some vertex, both subconstituents are strongly regular, and found that such a graph either has a vanishing Krein parameter $q_{11}^1$ or $q_{22}^2$, or has Latin square or negative Latin square parameters. They conjectured that every non-grid example of the latter has parameters as in the above lemma or has a complement with these parameters.
3 Survey of the Known Examples and Results

3.1 Complements

A graph satisfies the $t$-vertex condition if and only if its complement does.

3.2 Generalized Quadrangles

Higman [13] observed that the collinearity graphs of generalized quadrangles satisfy the 4-vertex condition (and there are many examples that are not rank 3, cf. [23]).

More generally the 4-vertex condition holds for partial quadrangles. For example, the Hill graph with parameters $(v, k, \lambda, \mu) = (4096, 234, 2, 14)$ (derived from the cap constructed in [15]) has a rank 10 group and satisfies the 4-vertex condition with $\alpha = 1, \beta = 0$.

Reichard [31] showed that the collinearity graphs of generalized quadrangles satisfy the 5-vertex condition, and that the collinearity graphs of generalized quadrangles $GQ(s, s^2)$ satisfy the 7-vertex condition.

More generally the 5-vertex condition holds for partial quadrangles.

3.3 Binary Vector Spaces with a Quadratic Form

The first non-rank-3 graph satisfying the 5-vertex condition was constructed by A. V. Ivanov [21]: a strongly regular graph $\Gamma_0$ whose subconstituents $\Gamma_1, \Gamma_2$ satisfy the 4-vertex condition. The parameters are as follows.

|   | $v$  | $k$  | $\lambda$ | $\mu$ | $\alpha$ | $\beta$ | $|G|$                      | Remarks                  |
|---|------|------|------------|-------|---------|---------|---------------------------|--------------------------|
| $\Gamma_0$ | 256  | 120  | 56         | 56    | 784     | 672     | $2^{20} \cdot 3^2 \cdot 5 \cdot 7$ | Rank 4: $1 + 120 + 120 + 15$ |
| $\Gamma_1$ | 120  | 56   | 28         | 24    | 216     | 144     | $2^{12} \cdot 3^2 \cdot 5 \cdot 7$ | Rank 4: $1 + 56 + 56 + 7$  |
| $\Gamma_2$ | 135  | 64   | 28         | 32    | 168     | 192     | $2^{12} \cdot 3^2 \cdot 5 \cdot 7$ | Intransitive: $120 + 15$  |

In [6] an infinite family of graphs $\Gamma^{(m)} (m \geq 1)$ is constructed by taking as vertex set $\mathbb{F}_2^{2m}$, where vectors are adjacent when the line joining them meets the hyperplane at infinity in a fixed hyperbolic quadric minus a maximal t.i. subspace. The graphs $\Gamma^{(m)}$ have parameters $A(2^{m-1})$ (see Sect. 2.2). They have a rank 4 group (for $m \geq 4$) and satisfy the 4-vertex condition.

The local graphs $\Delta^{(m)}$ are strongly regular with parameters $B(2^{m-1})$. They have a rank 4 group (for $m \geq 4$) and satisfy the 4-vertex condition.

By Lemma 2.2 also the 2nd subconstituents $E^{(m)}$ are strongly regular, with parameters $C(2^{m-1})$.

We checked by computer that the graph $\Gamma^{(4)}$ is isomorphic to the above $\Gamma_0$.

In [30] it is shown that the graphs $\Gamma^{(m)}$ satisfy the 5-vertex condition.

In [29] it is shown that the graphs $\Gamma^{(m)}$ are triplewise 5-regular, a.k.a. $(3,5)$-regular, where $(s, t)$-regularity is the analog of the $t$-vertex condition where $s$ instead of two
vertices are distinguished. It follows that the 2nd subconstituents $E^{(m)}$ of the graphs $\Gamma^{(m)}$ also satisfy the 4-vertex condition.

In [22], two infinite families of graphs are constructed. One is the above $\Gamma^{(m)}$. The second family has members $\Sigma^{(m)}$ with vertex set $F_2^{2m}$, where vectors are adjacent when the line joining them hits the hyperplane at infinity either in a fixed elliptic quadric minus a maximal t.i. subspace $S$ or in $S^\perp \setminus S$. The graphs $\Sigma^{(m)}$ have parameters $A(−2^m−1)$, have rank 5 (for $m \geq 5$), and satisfy the 4-vertex condition.

Let $\Gamma(V, X)$ be the graph on a vector space $V$ where two vectors are adjacent precisely when the joining line hits the subset $X$ of the hyperplane $PV$ at infinity. Since $\Gamma(V, X)$ is strongly regular if and only if $X$ is a 2-character set ([11]), that is, if and only if $|X \cap H|$ takes only two distinct values when $H$ runs through the hyperplanes of $PV$, the set $(Q \setminus S) \cup (S^\perp \setminus S)$ is a 2-character set when $Q$ is an elliptic quadric, and $S$ a maximal t.i. subspace.

Let $V$ be a vector space over $\mathbb{F}_2$. Then the local graph of $\Gamma(V, X)$ is the collinearity graph of the partial linear space with point set $X$ and whose lines are the projective lines (of size 3) contained in $X$.

The local graphs $T^{(m)}$ are strongly regular with parameters $B(−2^m−1)$. They are intransitive (for $m \geq 5$).

It follows from Lemma 2.2 that also the 2nd subconstituents $\Upsilon^{(m)}$ are strongly regular, with parameters $C(−2^m−1)$. There is a tower of graphs here: If $\Upsilon$ is the 2nd subconstituent of $\Sigma^{(m)}$ at a vertex $x$, and $s \in S$, then the local graph of $\Upsilon$ at its vertex $x + s$ is isomorphic to $\Sigma^{(m−1)}$. (For a proof, see Appendix A.)

In [22] it is conjectured that the graphs $\Sigma^{(m)}$ satisfy the 5-vertex condition, and that the graphs $T^{(m)}$ and $\Upsilon^{(m)}$ satisfy the 4-vertex condition. The former was proved in [30]. The latter is proved in Appendix A. In [29] it is announced that $\Sigma^{(m)}$ is even $(3, 5)$-regular, but we are not aware of a proof in print.

### 3.4 Block Graphs of Steiner Triple Systems

Higman [13] investigated for which $v$-point Steiner triple systems the block graph satisfies the 4-vertex condition. He found that either the system is a projective space $\text{PG}(m, 2)$ or $v$ is one of 9, 13, 25. In [25] the cases 13 and 25 are ruled out, so that the only other example is the affine plane $\text{AG}(2, 3)$. The examples are rank 3.

### 3.5 Smallest Example

In [26] it is shown that the smallest non-rank-3 strongly regular graphs satisfying the 4-vertex condition have $v = 36$ vertices. There are three examples. All have $(v, k, \lambda, \mu) = (36, 14, 4, 6)$ and $\alpha = 0, \beta = 4$.

### 3.6 Cyclotomic Examples

Given $(q, e, J)$, where $e | (q − 1)/2$ and $J$ is a set of nonnegative integers, and a fixed primitive element $\eta$ of $\mathbb{F}_q$, consider the cyclotomic graph with vertex set $\mathbb{F}_q^*$. 
where two elements are adjacent when their difference is in \( D = \{ \eta^i e^j | 0 \leq i < (q - 1)/e, \ j \in J \} \). In some cases this yields a strongly regular graph that satisfies the 4-vertex condition. We give a few examples. The examples on 11 2 and 23 2 vertices are due to Klin and Pech [27].

In all cases \( q = p^f \) where \( p \) is semiprimitive mod \( e \) (that is, \( e \mid (p^i + 1) \) for some \( i \)), so that the parameters of the strongly regular graph can be found in [4, Thm. 7.3.2].

### 4 Graphs From Hyperovals

In [17], Huang et al. constructed various families of graphs. The complement of one of them can be described as follows ( [2]). For \( q = 2^m \), take \( \mathbb{F}_q^3 \) as the vertex set of \( \Gamma \). Let \( \pi \) be the plane at infinity of \( \mathbb{F}_q^3 \). Let \( H^* \) be a dual hyperoval of \( \pi \) (that is, a set of \( q + 2 \) lines, no three on a point). The plane \( \pi \) is partitioned into two parts, \( \frac{1}{2}(q + 1)(q + 2) \) points on two lines of \( H^* \) and \( \frac{1}{2}q(q - 1) \) exterior points on no line of \( H^* \). Two vertices of \( \Gamma \) are adjacent when the line joining them hits \( \pi \) in one of the exterior points. Then \( \Gamma \) is strongly regular and has parameters

\[
(v, k, \lambda, \mu) = (q^3, \frac{1}{2}q(q - 1)^2, \frac{1}{4}q(q - 2)(q - 3), \frac{1}{4}q(q - 1)(q - 2)).
\]

Its local graphs are strongly regular with parameters

\[
\left( \frac{1}{2}q(q - 1)^2, \frac{1}{4}q(q - 2)(q - 3), \frac{1}{8}q(q^2 - 9q + 22), \frac{1}{8}q(q - 3)(q - 4) \right).
\]

Hence, as noted in Sect. 2, \( \Gamma \) satisfies the 4-vertex condition. If \( m = 3 \), then \( \Gamma \) has rank 4.

### 5 Disjoint t.i. Planes in Symplectic 6-Space

Let \( V \) be a 6-dimensional vector space over \( \mathbb{F}_q \), provided with a nondegenerate symplectic form. Let \( \Gamma \) be the graph with as vertices the totally isotropic planes, adjacent when disjoint.

**Proposition 5.1** The graph \( \Gamma \) is strongly regular, with parameters \( v = (q^3 + 1)(q^2 + 1)(q + 1), k = q^6, \lambda = q^2(q^3 - 1)(q - 1), \mu = (q - 1)q^5 \). If \( q \) is even, then \( \Gamma \) is rank

\[
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\]
3, otherwise rank 4. Its local graph $\Delta$ is strongly regular with parameters $v' = k, k' = \lambda, \lambda' = \mu - q^2(q - 2)$ and $\mu' = q^2(q - 1)(q^3 - q^2 - 1)$. It follows that $\Gamma$ satisfies the 4-vertex condition.

For convenience, we give the parameters of $\tilde{\Delta}$, the complement of $\Delta$:
\[
\bar{v} = q^6, \bar{k} = (q^2 + 1)(q^3 - 1), \bar{\lambda} = q^4 + q^3 - q^2 - 2, \bar{\mu} = q^4 + q^2.
\]

**Proof** The dual polar graph $\Sigma$ belonging to $\text{Sp}_6(q)$ is distance-regular of diameter 3 and has eigenvalue $-1$. It follows that its distance-3 graph $\Gamma$ is strongly regular (see [5], Prop. 4.2.17). More generally, the distance 1-or-2 graph of the symplectic dual polar space $\text{Sp}_{2m}(q)$ is distance-regular (cf. [5], Prop. 9.4.10). For $m = 3$ it is the complement of $\Gamma$.

For any vertex $x$, the subgraph induced by $\Sigma$ on $\Sigma_3(x)$ is isomorphic to the symmetric bilinear forms graph on $\mathbb{F}_q^3$ (see [5], Prop. 9.5.10). If $q$ is odd, then distance $j$ ($j = 0, 1, 2, 3$) in $\Sigma_3(x)$ corresponds to $\text{rk}(f - g) = j$ in the symmetric bilinear forms graph and hence to distance $[(j + 1)/2]$ in the quadratic forms graph (see [5], Sect. 9.6). It follows that $\Delta$ is the complement of the quadratic forms graph, and has parameters as claimed.

If $q$ is even, then $\Gamma$ is rank 3 (by triality, it is the complement of the $O_6^-(q)$ polar graph), and $\Delta$ is the complement of the rank 3 graph $V O_6^+(q)$, with parameters as claimed. \qed

### 6 Nonsingular Points Joined by a Tangent

Let $V$ be a vector space of dimension $2m + 1$ over $\mathbb{F}_q$ with $q$ odd, and let $Q$ be a nondegenerate quadratic form on $V$. We also use $Q$ as the symbol for the set of singular projective points.

The projective space $PV$ has $(q^{2m+1} - 1)/(q - 1)$ points, $(q^{2m} - 1)/(q - 1)$ singular, and $q^{2m}$ nonsingular. The nonsingular points come in two types: there are $q^m(q^m + \varepsilon)$ points of type $\varepsilon$ (where $\varepsilon = \pm 1$), with $\varepsilon = +1$ (resp. $-1$) for points $x$ for which $x^\perp$, the hyperplane of points orthogonal to $x$, is hyperbolic (resp. elliptic).

Consider the graph $N\text{O}_{2m+1}^\varepsilon(q)$ that has as vertex set the set of nonsingular points of type $\varepsilon$, where two points are adjacent when the joining line is a tangent.

**Proposition 6.1** (Wilbrink [34], cf. [3]) Let $m \geq 2$. The graph $N\text{O}_{2m+1}^\varepsilon(q)$ is strongly regular with parameters $v = \frac{1}{2}q^m(q^m + \varepsilon), k = (q^{m-1} + \varepsilon)(q^m - \varepsilon), \lambda = 2(q^{2m-2} - 1) + \varepsilon q^{m-1}(q - 1), \mu = 2q^{m-1}(q^{m-1} + \varepsilon)$.

For $m = 1, \varepsilon = -1$ the graph is edgeless. For $m = 1, \varepsilon = 1$ we have the triangular graph $T(q + 1)$. Wilbrink also handled the case of even $q$. We give an explicit proof here; for a different and more general proof see [1].

**Proof** The neighbors of a vertex $x$ lie on the tangents joining $x$ with a singular point of $x^\perp$, and $x^\perp$ has $(q^{m-1} + \varepsilon)(q^m - \varepsilon)/(q - 1)$ singular points. This gives the value of $k$.

A common neighbor $z$ of two adjacent vertices $x, y$ lies on the line $xy$ (and there are $q - 2$ choices) or on some other tangent $T$ on $x$. In the latter case the plane $\langle x, y, z \rangle$
meets $Q$ in a conic or double line. If it is a conic, then $z$ is uniquely determined on $T$ by the fact that $yz$ is the tangent on $y$ other than $xy$. If it is a double line, then each nonsingular point of $T \setminus \{x\}$ is suitable. Let $p$ be the singular point on $xy$. Then $\langle p, x \rangle^⊥/\langle p \rangle$ is a nondegenerate $(2m - 2)$-space of type $\epsilon$, and has $a = (q^{m-2} + \epsilon)(q^{m-1} - \epsilon)/(q - 1)$ singular points. It follows that $xy$ is in a plane that hit $Q$ in a double line, and in $q^{2m-2}$ planes that hit $Q$ in a conic. Consequently, $\lambda = q - 2 + q^{2m-2} + (q - 1)qa$, as desired.

A common neighbor $z$ of two nonadjacent vertices $x, y$ determines a nondegenerate plane $\pi = \langle x, y, z \rangle$ in which $xz$ and $yz$ are tangents, so that $x, y, z$ are exterior points. Now $x, y$ are on two tangents each, and $\pi$ contains 4 common neighbors of $x, y$. If $Q$ is a quadratic form on a $(2m + 1)$-space, then a point $p$ is exterior if and only if $(\pm 1)^m \det(Q) p$ is a nonzero square. In order to have $p$ exterior in $\pi$ but a $\epsilon$-point in $V$, the $(2m - 2)$-space $\pi^⊥$ must be an $\epsilon$-subspace of the $(2m - 1)$-space $\{x, y\}^⊥$. Since there are $b = \frac{1}{2}q^{m-1}(q^{m-1} + \epsilon)$ such $\epsilon$-subspaces, we find $\mu = 4b$, as desired. 

The automorphism group $\text{PGO}_{2m+1}(q)$ of the graph contains $\text{PGO}_{2m+1}(q)$. The latter has $(q + 3)/2$ orbits on pairs of vertices [1]. Hence, the graph has rank $(q + 3)/2$ if $q$ is prime.

For $m = 2, \epsilon = -1$, this is the collinearity graph of a semi-partial geometry found by Metz. Its lines have size $s + 1 = q$ and there are $t + 1 = q^2 + 1$ lines on each point. Each point outside a line has either 0 or 2 neighbors on the line. See Debroey [9], voorbeeld 1.1.3d, and Debroey–Thas [10], example 1.4d, and Hirschfeld–Thas [16], p. 268, and Brouwer-van Lint [3], Sect. 7A, and Brouwer-Van Maldeghem Sect. 8.7, example (ix).

For $m = 2, \epsilon = +1$ this is the collinearity graph of a geometry with $t + 1 = (q + 1)^2$ lines of size $s + 1 = q$ on each point, such that each point outside a line has 0, 2, or $q$ neighbors on the line ( [3], Sect. 7B).

We shall prove that these graphs satisfy the 4-vertex condition. First a lemma.

**Lemma 6.2** Let $S$ be a solid such that $Q|_S$ is nondegenerate. Let $x, y, z$ be distinct nonsingular points of the same type $\epsilon$ such that $\langle z, x \rangle$ and $\langle z, y \rangle$ are tangents and $\langle x, y \rangle$ is nondegenerate. Put $\pi = \langle x, y, z \rangle$. Then there are either 0 or 2 nonsingular points $w \in S \setminus \pi$ of type $\epsilon$ such that $\langle x, w \rangle, \langle y, w \rangle,$ and $\langle z, w \rangle$ are tangents. For $x, y, z$ given, the number of $w$ only depends on the type of $S$. It equals 2 if and only if the nonzero number $2\left(\frac{B(z,z)}{B(x,z)}B(x,y) - 1\right) \det(Q|_S)$ is a square.

**Proof** Replace $x$ by $\frac{B(z,z)}{B(x,z)} x$ and $y$ by $\frac{B(z,z)}{B(y,z)} y$. Then $B(x, z) = B(z, z) = B(y, z)$. Put $x_0 = x - z, y_0 = y - z, w_0 = w - z$, then $B(x_0, z) = B(y_0, z) = B(w_0, z) = 0$. Since the lines $\langle z, x \rangle, \langle z, y \rangle,$ and $\langle z, w \rangle$ are tangents, the points $x_0, y_0, z_0$ are singular, that is, $Q(x_0) = Q(y_0) = Q(w_0) = 0$. The line $\langle x, w \rangle$ is a tangent, so $Q(x + tw) = 0$ has a unique solution $t$. Now

$$Q(x + tw) = Q(z + x_0 + t(z + w_0)) = Q((1 + t)z + x_0 + tw_0) = (1 + t)^2 Q(z) + Q(x_0 + tw_0) = (1 + t)^2 Q(z) + tB(x_0, w_0).$$
It follows that \( (2 + \frac{B(x_0, w_0)}{Q(z)})^2 = 4 \), that is \( \frac{B(x_0, w_0)}{Q(z)} \in \{0, -4\} \).

As \( Q \big|_S \) is nondegenerate, \( z^\perp \cap S \) is a nondegenerate plane. If \( B(x_0, w_0) = 0 \), then \( \langle x_0, w_0 \rangle \) is a totally singular line in this plane, impossible. Hence, \( B(x_0, w_0) = -4Q(z) \). Similarly, \( B(y_0, w_0) = -4Q(z) \).

In the plane \( z^\perp \cap S \), let \( u \) be the point of intersection of the tangents through the points \( x_0 \) and \( y_0 \) and write \( w_0 = ax_0 + by_0 + cu \). Then \( B(x_0, u) = B(y_0, u) = 0 \) and \(-4Q(z) = B(x_0, w_0) = B(x_0, ax_0 + by_0 + cu) = bB(x_0, y_0) \). Similarly, \(-4Q(z) = B(y_0, w_0) = aB(x_0, y_0) \), so that \( a = b = \frac{-4Q(z)}{B(x_0, y_0)} \), independent of \( w \). Also,

\[
0 = Q(w_0) = Q(ax_0 + by_0 + cu) = abB(x_0, y_0) + c^2Q(u) = \frac{16Q(z)^2}{B(x_0, y_0)} + c^2Q(u).
\]

If \(-B(x_0, y_0)Q(u)\) is a square, then we have two solutions for \( c \) (so also \( w_0 \) and, therefore, \( w \)) and otherwise none. Since \( u \) is an exterior point in the plane \( \sigma = z^\perp \cap S \), the number \(-Q(u) \det Q \big|_{\sigma} \) is a square. Also, \( \det Q \big|_S = Q(z) \det Q \big|_{\sigma} \) and \( B(x, y) = B(x_0, y_0) + B(z, z) \).

**Proposition 6.3** The graph \( NO_{2m+1}^\varepsilon(q) \) satisfies the 4-vertex condition.

**Proof** By Proposition 2.1 it suffices to check for \( x \neq y \) that the number of edges in \( \Gamma(x) \cap \Gamma(y) \) does not depend on the choice of the points \( x, y \), but only on whether \( x, y \) are adjacent or not.

Since \( \text{Aut } \Gamma \) is edge-transitive, we only need to check \( \Gamma(x) \cap \Gamma(y) \) for \( x \sim y \).

Claim: this subgraph \( \Gamma(x) \cap \Gamma(y) \) is regular of valency \( 4q^{2m-3} + 3\varepsilon q^{m-1} - 4\varepsilon q^{m-2} - 1 \). In other words, this is the value of \( \mu \) in the local graph (which is regular, but not strongly regular).

If \( x \sim z \sim y, x \sim y \), then \( \pi = \langle x, y, z \rangle \) is a nondegenerate plane in which the common neighbors of \( x \) and \( y \), form a 4-cycle, so that \( x, y, z \) have two common neighbors in \( \pi \), say \( a \) and \( b \).

The plane \( \pi \) lies in \((q^{2m-3} - \varepsilon q^{m-2})/2 \) solids of type \( O^-(4, q) \), equally many solids of type \( O^+(4, q) \), and \((q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon)/(q - 1) \) degenerate solids.

If \( S \) is a degenerate solid through \( \pi \) with apex \( p \), we see that \( w \in S \setminus \pi \) is in \( \Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \) if and only if \( p \) is projected from \( p \) onto an element of \( \{a, b, z\} \) in \( \pi \). Hence, \( |\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \setminus S \setminus \pi| = 3(q - 1) \). Hence, the total number of choices for \( w \) equals \( 3(q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon) \).

Now let \( S \) be a nondegenerate solid on \( \pi \), and let \( p = S \cap \pi^\perp \). By Lemma 6.2, the number of \( w \) in \( S \) is 0 or 2, depending on the determinant of \( Q \) restricted to \( S \). Since \( \pi^\perp \) contains equally many points \( p \) with \( Q(p) \) a square as with \( Q(p) \) a non-square, the total number of choices for \( w \) equals the number of choices for \( p \) which is \( q^{2m-3} - \varepsilon q^{m-2} \).

So the induced subgraph on \( \Gamma(x) \cap \Gamma(y) \) has valency \( 2 + 3(q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon) + (q^{2m-3} - \varepsilon q^{m-2}) = 4q^{2m-3} + 3\varepsilon q^{m-1} - 4\varepsilon q^{m-2} - 1 \). \( \square \)

**7 Polar Switching**

A **polar space** is a partial linear space such that for each line \( L \) any point outside \( L \) is collinear to either all or precisely one of the points of \( L \). A **singular subspace** is
a line-closed set of points, any two of which are collinear. The polar space is called nondegenerate when no point is collinear to all points. Finite nondegenerate polar spaces are the sets of totally isotropic (t.i.) or totally singular (t.s.) points and lines in a vector space over a finite field provided with a suitable symplectic, quadratic or hermitian form. The rank of the polar space is the (vector space) dimension of its maximal singular subspaces.

Let $P$ be a nondegenerate polar space of rank $d \geq 3$ in a vector space $V$ over $\mathbb{F}_q$. Its collinearity graph $\Gamma_0$ is strongly regular and satisfies the 4-vertex condition (since it is rank 3). We shall construct cospectral graphs that satisfy the 4-vertex condition (but are not rank 3) by a switching construction. Let $x^\perp$ be the set of points collinear with $x$ (including $x$ itself).

Suppose $U$ is a maximal singular subspace of $P$ (i.e., a maximal clique in $\Gamma_0$), and let $H_1, H_2$ be two hyperplanes of $U$. We can redefine adjacency and make the points $x$ with $x^\perp \cap U = H_1$ or $H_2$ adjacent to the points in $H_2$ or $H_1$, respectively, and leave all other adjacencies unchanged. This is an example of WQH-switching (Wang, Qiu & Hu [33], cf. [20]) and yields a graph cospectral with $\Gamma_0$. One can repeat this interchange of hyperplanes and get arbitrary permutations of all hyperplanes. We generalize this, even allowing different designs on $U$.

7.1 Construction

Let $P$ be the point set of $P$, and let the subset $U$ be (the set of points of) a totally isotropic $d$-space. Let $D$ be a symmetric design with the same parameters as the symmetric design of points and hyperplanes of $\text{PG}(d-1, q)$, so its parameters are $2-(q^{d-1} - 1, q^{d-1} - 1, q^{d-2} - 1)$. Let $\varphi$ be a bijection from the set $H$ of hyperplanes of $U$ to the blocks of $D$. We assume that the points of $U$ are also the points of $D$.

Following ideas in [12, 24] we define a graph $\Gamma_{\varphi}$ on the vertex set of $\Gamma_0$ as follows:

1. Vertices in $U$ are pairwise adjacent.
2. Distinct vertices $x, y \notin U$ are adjacent if $x \in y^\perp$.
3. Vertices $x \in U, y \notin U$ are adjacent if $x \in (y^\perp \cap U)^\varphi$.

Clearly, $\Gamma_{\varphi} = \Gamma_0$ if we take the hyperplanes of $U$ for the blocks of $D$ and $\varphi$ as the identity.

Theorem 7.1 The graph $\Gamma_{\varphi}$ is strongly regular with the same parameters as the classical graph $\Gamma_0$.

Proof Let $x$ and $y$ be any two vertices. We show that the number of common neighbors $z$ of $x, y$ in $\Gamma_{\varphi}$ does not depend on $\varphi$ (but depends on whether $x, y$ are equal, adjacent or nonadjacent in $\Gamma_{\varphi}$).

If $x, y \in U$, then any $z \in U$ is a common neighbor. The number of $z \in P \setminus U$ such that $x, y \in (z^\perp \cap U)^\varphi$ does not depend on $\varphi$: each hyperplane $H$ of $U$ such that $x, y \in H^\varphi$ contributes $|H^\perp \setminus U|$ such $z$.

Suppose that $x, y \notin U$. Then we are counting the $z$ in $(x^\perp \cap U)^\varphi \cap (y^\perp \cap U)^\varphi$, and also the $z$ in $(x^\perp \cap y^\perp) \setminus U$. The numbers of such $z$ does not depend on $\varphi$. 

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The remainder of the proof concerns the case \( x \in U, y \notin U \). If \( z \in U \) then the requirements are \( z \neq x \) and \( z \in (y^\perp \cap U)^\varphi \). The number of such \( z \) does not depend on \( \varphi \).

So we need to count the \( z \notin U \). First set \( I := y^\perp \cap U \), so \( Y := \langle y, I \rangle \) is totally isotropic. If \( z \in Y \) then \( I^\varphi = (z^\perp \cap U)^\varphi \), and \( x, z \) are adjacent if and only if \( x, y \) are adjacent. The number of such \( z \) is independent of \( \varphi \).

It remains to count the \( z \in y^\perp \setminus Y \) such that \( x \in (z^\perp \cap U)^\varphi \); here \( z^\perp \cap U \neq I \) as \( z \notin Y \). Let \( H \neq I \) be a hyperplane of \( U \) such that \( x \in H^\varphi \). The number of \( H \) does not depend on \( \varphi \) (note that \( x \in I^\varphi \) if and only if \( x, y \) are adjacent in \( \Gamma_\varphi \)). We show that the number of \( z \) in \( y^\perp \setminus Y \) with \( z^\perp \cap U \neq H \) does not depend on \( \varphi \) or \( H \). Using bars to project \( (H \cap I)^\perp \) into the nondegenerate rank 2 polar space \( (H \cap I)^\perp / (H \cap I) \), we see totally isotropic lines \( \bar{U} \) and \( \bar{Y} \) meeting at a point \( \bar{I} \), and a nondegenerate 2-space \( \langle \bar{y}, \bar{H} \rangle \); the number of \( \bar{z} \in \langle \bar{y}, \bar{H} \rangle^\perp \setminus \bar{I} \) does not depend on \( \varphi \) or \( H \), so neither does the number of required \( z \).

\[ \square \]

7.2 Isomorphisms

7.2.1 Emptying Bijections \( \varphi \)

Call a vertex \( e \in U \) emptying for \( \varphi \) if \( \bigcap \{ H \mid H \in \mathcal{H}, e \in H^\varphi \} = \emptyset \). Call \( \varphi \) emptying if the subspace \( U \) is spanned by emptying vertices.

Call a vertex \( f \in U \) dually emptying for \( \varphi \) if \( \bigcap \{ H^\varphi \mid f \in H \in \mathcal{H} \} = \emptyset \). Call \( \varphi \) dually emptying if the subspace \( U \) is spanned by dually emptying vertices.

If \( a \) is not emptying, then \( \bigcap \{ H \mid H \in \mathcal{H}, a \in H^\varphi \} = \{ b \} \) for some vertex \( b \). If \( b \) is not dually emptying, then \( \bigcap \{ H^\varphi \mid b \in H \in \mathcal{H} \} = \{ a \} \) for some vertex \( a \). This establishes a 1-1 correspondence between not emptying vertices \( a \) and not dually emptying vertices \( b \).

**Proposition 7.2** If a permutation \( \varphi \) of \( \mathcal{H} \) is not dually emptying, then it is in \( \mathrm{PGL}(U) \).

**Proof** Let \( E \) denote the set of emptying vertices of \( U \), and put \( A = U \setminus E \). Let \( F \) denote the set of dually emptying vertices of \( U \), and put \( B = U \setminus F \). Let \( \psi : B \rightarrow A \) be the 1-1 correspondence found above. We show that if \( L \) is a line in \( U \) with \( |L \cap B| \geq q \), then \( L \subseteq B \) and \( L^\varphi \) is a line.

Indeed, let \( b, b' \in L \cap B \) and set \( M = \langle b^\psi, b'^\psi \rangle \). Then \( L \subseteq H \) is equivalent to \( M \subseteq H^\varphi \) so that \( (L \cap B)^\psi = M \cap A \). If all points of \( L \) are in \( B \) with a single exception \( w \), then all points of \( M \) are in \( A \) with a single exception \( v \), and all hyperplanes \( H \) with \( w \in H \) satisfy \( v \in H^\varphi \) (since every line meets every hyperplane), and \( v = w^\psi \), that is, \( w \) was no exception.

If \( \varphi \) is not dually emptying, then there exists a hyperplane \( H \) such that \( U \setminus H \subseteq B \). By the above this implies \( B = U \) and \( \psi \) is in \( \mathrm{PGL}(U) \) and induces \( \varphi \) on the set \( \mathcal{H} \). \( \square \)

7.2.2 Large Cliques

We use the presence of maximal cliques of various sizes to study the structure of the graphs \( \Gamma_\varphi \) when \( \varphi \) is a permutation.
Abbreviate the size $q_d^{i-1}$ of an $i$-space with $m_i$, so that maximal singular subspaces have size $m_d$. Since $m_d$ is the Delsarte-Hoffman upper bound for the size of cliques in $\Gamma_\varphi$, each vertex outside a clique of this size is adjacent to precisely $m_d-1$ vertices inside, cf. [4, Proposition 1.1.7].

**Lemma 7.3** Let $d \geq 3$.

(i) If $M \neq U$ is a maximal singular subspace of $\mathbf{P}$, then $C = (M \setminus U) \cup \bigcap\{H^\varphi \mid M \cap U \subseteq H \in \mathcal{H}\}$ is a maximal clique in $\Gamma_\varphi$ of size at least $q^{d-2}(q + 1)$ (and $C \setminus U = M \setminus U$).

(ii) If $C \neq U$ is a maximal clique in $\Gamma_\varphi$ of size at least $q^{d-2}(q + 1)$, then $M = \langle C \setminus U \rangle$ is a maximal singular subspace of $\mathbf{P}$. If, moreover, $|C| = m_d$, then $M \setminus U = C \setminus U$.

**Proof**

(i) Let $M$ be a maximal singular subspace other than $U$. Then $C = (M \setminus U) \cup \bigcap\{H^\varphi \mid M \cap U \subseteq H \in \mathcal{H}\}$ is the largest clique in $\Gamma_\varphi$ containing $M \setminus U$. (Indeed, the set of hyperplanes of $U$ of the form $m^\perp \cap U$ where $m \in M \setminus U$ equals the set of hyperplanes containing $M \cap U$, so $C$ is a clique. No further point outside $U \cup C$ can be adjacent to all of $C$, since $|M \setminus U| > m_d-1$.) If $|C \cap U| = d - 1$, then $|C| = |M| = m_d$. If $|C \cap U| \leq d - 2$, then $|C| \geq |M \setminus U| > m_d - m_d - 2 = q^{d-2}(q + 1)$.

(ii) Let $C \neq U$ be a maximal clique of size at least $q^{d-2}(q + 1)$. If $|C \setminus U| \leq m_d-1$, then $|C \cap U| \geq q^{d-2}(q + 1) - m_d - 1 > m_d - 2$. The set $C \cap U$ is the intersection of sets $H^\varphi$, each of size $m_d - 1$, and any two distinct such sets meet in $m_d-2$ points. It follows that no two different $H$ occur, that is, $H = c^\perp \cap U$ is independent of the choice of $c \in C \setminus U$. Now $C$ is contained in, and hence equals, $H^\varphi \cup (C \setminus U)$, and $|C \setminus U| = m_d - m_d - 1 > m_d - 1$, a contradiction.

If $S$ is a clique in $\Gamma_0$, then also $\langle S \rangle$ is a clique in $\Gamma_0$. In particular, $\langle C \setminus U \rangle$ is a singular subspace. It is maximal since $|\langle C \setminus U \rangle| > m_d-1$.

If $|C| = m_d$, then each vertex outside $C$ is adjacent to precisely $m_d-1$ vertices inside. Hence no point outside $C \cup U$ can be adjacent to all of $C \setminus U$. \hfill \Box

**Lemma 7.4** If the permutation $\varphi$ is dually emptying, then $U$ is uniquely determined within the graph $\Gamma_\varphi$.

**Proof**

The subspace $U$ is a clique of size $m_d$ in $\Gamma_\varphi$, with the two properties

(i) in the subgraph induced on its complement $P \setminus U$ all maximal cliques $N$ have size $m_d - m_i$ (where $m_i = |\langle N \cap U \rangle|$ for some $i$, $0 \leq i \leq d - 1$, and

(ii) the number of maximal cliques of size $m_d$ disjoint from $U$ equals the number of maximal singular subspaces disjoint from any given one.

Let $E \neq U$ be a clique of $\Gamma_\varphi$ of size $m_d$ with the same two properties. First we use (i) to see that $E \cap U$ must be a hyperplane in $U$.

Since $E$ is a maximal clique, and $\varphi$ is a permutation, $E \cap U$ is an intersection of hyperplanes and hence a subspace of $U$. By hypothesis, we can find a dually emptying point $f$ of $U$ not in $E$. If $g \in f^\perp \cap (E \setminus U)$ ($g$ will exist unless $f^\perp \cap E = U \cap E$) and $M$ is a maximal singular subspace containing $f$ and $g$, and meeting $U$ in $\{f\}$, then $C = M \setminus \{f\}$ is a maximal clique in $\Gamma_\varphi$ of size $m_d - 1$. And $N = C \setminus E$ is a maximal clique in $P \setminus E$ of size $m_d - m_i - 1$ in case $|M \cap E| = m_i$. (Note that $C \setminus U = M \setminus U$.)
Why is $N$ maximal? No point can be added since $|N| > m_{d-1}$, unless $q = 2$ and $|N| = |M \cap E| = m_{d-1}$. In that case, no point outside $U$ can be added since $(N) = M$. And no point inside $U$ can be added since $N$ determines all hyperplanes on $f$, and $f$ is dually emptying.

Since $M \cap E \neq \emptyset$, we have $1 \leq i \leq d - 1$, and $m_d - m_i - 1$ is not of the form $m_d - m_h$, violating (i). Therefore, $f^\perp \cap E = U \cap E$, so that $H = \langle E \setminus U \rangle \cap U$ and $H^\varphi = E \cap U$ are hyperplanes.

Now we use (ii) to arrive at a contradiction.

We claim that if a maximal clique $F$ of size $m_d$ is disjoint from $E$, then $\langle F \setminus U \rangle$ is disjoint from $\langle E \setminus U \rangle$. Suppose not. Since $\langle E \setminus U \rangle \setminus U = E \setminus U$ and $\langle F \setminus U \rangle \setminus U = F \setminus U$ by Lemma 7.3(ii), a common vertex must lie in $U$. If $\langle F \setminus U \rangle$ meets $U$ in $m_e$ vertices with $e \geq 2$, then $F$ meets $U$ in a subspace of dimension $e$, that would meet $H^\varphi$, impossible. So, $\langle F \setminus U \rangle$ meets $U$ in a singleton $\{f\}$ on the hyperplane $H$. As $F$ has size $m_d$, $f$ is not dually emptying, so $\bigcap \{H^\varphi \mid f \in H\} = \{f'\}$ for some point $f'$. Now $f' \in E \cap F$, a contradiction. This shows our claim.

By the claim and Lemma 7.3, we have an injection from the set of maximal cliques of size $m_d$ disjoint from $E$ into the set of maximal singular subspaces disjoint from $\langle E \setminus U \rangle$. Since $E$ satisfies (ii), both sets have the same size, so the injection is also a surjection.

On the other hand, since $\varphi$ is dually emptying, there is a dually emptying point $o$ in $U \setminus H$. This $o$ lies in a maximal singular subspace $O$ disjoint from $\langle E \setminus U \rangle$, and this $O$ is not in the image of the surjection. Contradiction. \hfill \Box

**Lemma 7.5** Let $P$ be a nondegenerate polar space with point set $P$, and $U$ a maximal totally isotropic subspace. Let $h : P \setminus U \to P \setminus U$ be a bijection preserving collinearity. Then $h$ can be uniquely extended to an automorphism $h'$ of $P$.

**Proof** Indeed, we can extend $h$ as follows. For $u \in U$, let $R$ be a maximal t.i. subspace with $U \cap R = \{u\}$. Then $R \setminus \{u\}$ is a subspace of $L$ of size $|U| - 1$ and is mapped by $h$ to a similar subspace $S$. In $P$ this subspace is contained in a unique maximal t.i. subspace $\langle S \rangle$ ($= S^\perp$) and we can define $h'(u) = v$ when $\langle S \rangle \setminus S = \{v\}$.

This is well-defined: if $R'$ is a maximal t.i. subspace with $U \cap R' = \{u\}$ and $R$, $R'$ meet in codimension 1, and $h$ maps $R' \setminus \{u\}$ to $S'$, then $\langle S \cap S' \rangle = (S \cap S') \cup \{v\}$. Since the graph on such subspaces $R$, adjacent when they meet in codimension 1, is connected, $v$ is well-defined.

This preserves orthogonality: if $u \in x^\perp$, then there is a maximal t.i. subspace $R$ containing $u$, $x$ with $R \cap U = \{u\}$. Now $h(u) = v$ lies in the t.i. subspace $\langle h(R \setminus \{u\}) \rangle$ which also contains $h(x)$. \hfill \Box

**Proposition 7.6** Let $P$ be a nondegenerate polar space and $U$ a maximal t.i. subspace. Let $\varphi$ and $\chi$ be permutations of $H$ such that $\Gamma_\varphi$ is isomorphic to $\Gamma_\chi$. Then $\varphi$ and $\chi$ are in the same $P\Gamma L(U)$-double coset in $\text{Sym}(H)$.

**Proof** If $\varphi \in P\Gamma L(U)$, then $\Gamma_\varphi$ is isomorphic to $\Gamma_0$ and its group of automorphisms is transitive on the set of maximal singular subspaces. If $\varphi \not\in P\Gamma L(U)$, then according to Lemma 7.4 and Proposition 7.2 the maximal singular subspace $U$ can be recognized in $\Gamma_\varphi$, and hence $\Gamma_\varphi$ is not isomorphic to $\Gamma_0$. Since by assumption $\Gamma_\varphi$ and $\Gamma_\chi$ are
isomorphic, either both or neither are isomorphic to $\Gamma_0$. In the former case both $\varphi$ and $\chi$ are in $\mathrm{PGL}(U)$ and the claim holds. Assume in the following that $\varphi$ and $\chi$ are not in $\mathrm{PGL}(U)$.

We have the set $P$, the point set of $\mathbf{P}$, with three structures defined on it. The polar space structure $\mathbf{P}$, with relation $\perp$, and the two graph structures $\Gamma_\varphi$ and $\Gamma_\chi$. We translate what it means for $\Gamma_\varphi$ and $\Gamma_\chi$ to be isomorphic in terms of the polar space.

Let $g : \Gamma_\varphi \to \Gamma_\chi$ be an isomorphism. By Lemma 7.4, it sends $U$ to itself.

The number of common neighbors of a triple of points in $U$ equals $\lambda - 1$ for collinear triples and is smaller for noncollinear triples. It follows that $g$ preserves projective lines in $U$, and hence induces a permutation $\bar{g}$ of $\mathcal{H}$ that is in $\mathrm{PGL}(U)$.

Let $h$ denote the restriction of $g$ to $P \setminus U$. Then $h$ preserves collinearity (since we have $\{x, y, z\}^\perp \cap (P \setminus U)$ for a triple of pairwise adjacent points $x, y, z$ of $P \setminus U$ if and only if $x, y, z$ are collinear). By Lemma 7.5, $h$ can be uniquely extended to an automorphism $h'$ of $\mathbf{P}$.

Let $\bar{h}$ be the permutation of $\mathcal{H}$ induced by $h'$. Then $\bar{h} \in \mathrm{PGL}(U)$.

For $x \in U$ and $y \notin U$, if $x$ and $y$ are adjacent in $\Gamma_\varphi$, then $x^g$ and $y^g$ are adjacent in $\Gamma_\chi$. This says that $x \in (y^\perp \cap U)^\varphi$ implies that $x^g \in (y^\perp \cap U)^\chi$; $g$ maps the points of any hyperplane of $U$ to the points of another hyperplane. Then $(y^\perp \cap U)^\varphi g = (y^\perp \cap U)^\chi = (y^\perp \cap U)^{\bar{h}\chi}$, so that $\varphi g = \bar{h}\chi$. \hfill \Box

**Theorem 7.7** Let $d \geq 3$. There are at least $q^{d-2}!$ pairwise nonisomorphic strongly regular graphs having the same parameters as the collinearity graph $\Gamma_0$ of the polar space $\mathbf{P}$.

**Proof** Let $q = p^e$, where $p$ is prime. Then $|\mathrm{PGL}(U)| < eq^{d^2}$. In view of Proposition 7.6, we have obtained at least $m_d!/|\mathrm{PGL}(U)|^2 > q^{d-2}!$ pairwise nonisomorphic strongly regular graphs unless $(d, q) = (3, 2)$. For $(d, q) = (3, 2)$, we have four $\mathrm{PGL}(U)$-double cosets in $\mathrm{Sym}(\mathcal{H})$. \hfill \Box

Similar estimates would follow if one generalized Lemma 7.4 to show that $U$ is uniquely determined in $\mathbf{P}$ for arbitrary designs $\mathbf{D}$ (that is, for $\varphi$ that are not permutations). The blocks of $\mathbf{D}$ are then found as $\{\Gamma_\varphi(x) \cap U \mid x \in P \setminus U\}$. In [24, Corollary 3.2] it is shown that for $d \geq 4$ there are at least $q^{d-2}!$ choices for $\mathbf{D}$. Hence, one would obtain the same estimate as in Theorem 7.7 for $d \geq 4$.

### 7.3 Switched Symplectic Graphs with 4-Vertex Condition

We show that in the symplectic case the graphs $\Gamma_\varphi$ satisfy the 4-vertex condition. Let $\mathbf{P}$ be $\mathrm{Sp}_{2d}(q)$, and let $V$ be a $2d$-dimensional vector space over $\mathbb{F}_q$, provided with a nondegenerate symplectic form.

The parameters of $\Gamma_0$ are $v = (q^{2d} - 1)/(q - 1)$, $k = q(q^{2d-2} - 1)/(q - 1)$, $v - k - 1 = q^{2d-1}$, $\lambda = q^2(q^{2d-4} - 1)/(q - 1) + q - 1$, $\mu = (q^{2d-2} - 1)/(q - 1)$, $0 = \frac{1}{2}q^{2d-1}(q^{2d-4} - 1)/(q - 1)$, and those of $\Gamma_\varphi$ will turn out to be the same.

**Proposition 7.8** The graph $\Gamma_\varphi$ satisfies the 4-vertex condition.
Proof Let \( x, y \) be two vertices of \( \Gamma_\varphi \). We show that the number of edges in \( \Gamma_\varphi(x) \cap \Gamma_\varphi(y) \) is independent of \( \varphi \), and only depends on whether \( x, y \) are adjacent or nonadjacent. Since \( \Gamma_0 \) satisfies the 4-vertex condition, \( \Gamma_\varphi \) does too.

Count edges \( ab \) in \( \Gamma_\varphi(x) \cap \Gamma_\varphi(y) \). The vertices \( x, y, a, b \) are pairwise adjacent, except that \( x \) and \( y \) need not be adjacent. We distinguish several cases depending on which of \( x, y, a, b \) are in \( U \). Each of the separate counts will be independent of \( \varphi \). If \( x \notin U \) then let \( X = x \perp \cap U \). If \( y \notin U \) then let \( Y = y \perp \cap U \).

Case \( x, y, a, b \notin U \). In this case adjacencies and counts do not involve \( \varphi \).

Case \( a, b \in \Gamma_0 \). Here \( a, b \) must be chosen distinct from \( x, y \) in case \( x, y \notin U \), or distinct from \( x \) and in \( Y_\varphi \) in case \( x \in U \), \( y \notin U \) (and the count depends on whether \( x \sim y \)), or in \( X_\varphi \cap Y_\varphi \) in case \( x, y \notin U \) (and the count depends on whether \( X = Y \)).

In all cases the count is independent of \( \varphi \).

Case \( x, y, a, b \in \Gamma_0 \). For each hyperplane \( H \) such that \( x, y \in H_\varphi \) we count the \( b \in H_x \cap U \) and the \( a \in H_y \) distinct from \( x, y \).

Case \( x, y \in U \), \( a, b \notin U \). For any two hyperplanes \( H, H' \) of \( U \) with \( x, y \in H_\varphi \cap H_\varphi \) count adjacent \( a, b \) with \( a \in H_x \cap Y \) and \( b \in H_y \perp \cap U \). (The counts will depend on whether \( H = H' \), but not on \( \varphi \).

Case \( x, a \in U \), \( y, b \notin U \). For each hyperplane \( H \) with \( x \in H_\varphi \), count the \( a \in H_\varphi \cap Y_\varphi \) distinct from \( x \), and \( b \in H_y \) adjacent to \( y \). (Here \( H = Y \) occurs when \( x \sim y \). The counts for \( H \neq Y \) do not depend on \( H \).

Case \( x \in U \), \( y, a, b \notin U \). For any two hyperplanes \( H, H' \) with \( x \in H_\varphi \cap H_\varphi \), count edges \( ab \) with \( a \in H_x \) and \( b \in H_y \) in \( y^\perp \cup (U \cup \{y\}) \). (Here \( H = Y \) or \( H' = Y \) occur when \( x \sim y \). The counts for \( H, H' \neq Y \) do not depend on the hyperplanes chosen but only on whether \( H = Y \) or \( H' = Y \) or \( H = H' \).

Finally the least trivial case.

Case \( a \in U \), \( x, y, b \notin U \). Count \( a, H, b \) with \( a \in H \cap X \) and \( H \) a hyperplane of \( U \) on \( a \) and \( b \in \{x, y, H\} \). The count for \( a \) depends on whether \( X = Y \), that for \( b \) depends on whether \( H = X \) or \( H = Y \) or \( H \supseteq X \cap Y \), but does not otherwise depend on the choice of \( H \).

Since all counts were independent of \( \varphi \), this proves our proposition.

By Theorem 7.7, this shows that there are many strongly regular graphs which satisfy the 4-vertex condition. But we still have to show the simplified version of this statement given in the introduction as Theorem 1.1.

Proof of Theorem 1.1. Note that here \( v \) refers to a nonnegative integer as in Theorem 1.1 and no longer is the number of vertices in \( \Gamma_\varphi \).

Apply Theorem 7.7 for \( d = 3 \) to find at least \( q! \) strongly regular graphs satisfying the 4-vertex condition on \( \tilde{v} \) vertices, for \( \tilde{v} = \frac{q^3 - 1}{q^2 - 1} \). Given \( v \), there is a prime \( q \) between \( v^{1/6} \) and \( 2v^{1/6} \) by Bertrand’s postulate. Now \( \tilde{v} < 2q^5 < 64v^{5/6} < v \) for \( v > 2^{36} \).

Checking the prime powers \( q \) for \( 7 \leq q \leq 64 \) one sees that there is a \( q \) with \( \tilde{v} \leq v \leq q^6 \) for \( v \geq 19608 \). One easily verifies the assertion for \( v < 19608 \) using rank 3 graphs.

Further graphs with the same parameters satisfy the 4-vertex condition. Additional examples can be obtained by repeated WQH-switching, see §7.4 and [20], and there are more examples among the graphs constructed in [18]. We have not tried (much) to
determine precisely which graphs in [18] do satisfy the 4-vertex condition. Similarly, we do not know when WQH-switching preserves the 4-vertex condition.

7.4 Small Examples

7.4.1 Examples on 63 Vertices

In [19] a large number of strongly regular graphs are found by applying GM-switching to the $\text{Sp}_6(2)$ polar graph. Among these are 280 non-rank-3 strongly regular graphs with $(v, k, \lambda, \mu) = (63, 30, 13, 15)$ satisfying the 4-vertex condition. All have $\alpha = 30$ and $\beta = 45$. Three of these are among the $\Gamma_\varphi$ constructed above.

We list for each occurring group size the number of examples found.

| $|G|$ | 4 | 8 | 16 | 32 | 48 | 64 | 96 | 128 | 192 | 256 | 384 | 512 | 768 | 1344 | 1536 | 4608 |
|-----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-------|-------|------|
| #   | 3  | 16 | 76 | 62 | 1  | 60 | 2  | 30  | 5   | 12  | 3   | 2   | 1    | 3     | 1     |

None of these examples has a transitive group. We list the orbit lengths in the seven cases with fewer than six orbits.

| $|G|$ | 768 | 768 | 1344 | 1536 | 1536 (twice) | 4608 |
|-----|-----|-----|-------|-------|--------------|------|
| orbits | 3+12+48 | 1+6+24+32 | 7+56   | 1+6+24+32 | 3+4+8+48   | 3+12+48 |

7.4.2 Permutations of Hyperplanes

Let $\mathbf{P}$ be $\text{Sp}_{2,d}(q)$, and let $\varphi$ be a permutation of the set $\mathcal{H}$ of hyperplanes of $U$. For $(d, q) = (3, 2), (3, 3), (4, 2)$, the number of double cosets of $\Gamma L(d, q)$ in $\text{Sym}(\mathcal{H})$ is 4, 252, and 3374, respectively, and these are the numbers of non-isomorphic graphs $\Gamma_\varphi$. In each case, exactly one has rank 3. None of the others has a transitive group (since $U$ can be recognized). The pointwise stabiliser of $U$ in $\text{Aut}(\Gamma_0)$ has size $N = q^{d+1} (q-1)$ and is always contained in $\text{Aut}(\Gamma_\varphi)$. Hence, $N$ divides $|\text{Aut}(\Gamma_\varphi)|$.

Case $(d, q) = (3, 3)$. Here $N = 1458$. We list the group sizes for the 251 graphs $\Gamma_\varphi$ other than $\Gamma_0$.

| $|G|/N$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 39 | 54 | 72 | 144 |
|-------|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|------|
| #     | 172 | 26 | 29 | 6  | 3  | 2  | 2   | 2   | 1   | 1   | 3   | 1   | 2    | 1    |

We list the orbit lengths in the five cases with fewer than six orbits.

| $|G|/N$ | 39 (thrice) | 72 | 144 |
|-------|-------------|----|-----|
| orbits | 13+351      | 1+12+108+243 | 1+12+108+243 |

Case $(d, q) = (4, 2)$. Here $N = 1024$. We list the group sizes for the 3373 graphs $\Gamma_\varphi$ other than $\Gamma_0$.  

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We list the orbit lengths in the eight cases with fewer than six orbits.

| $|G|/N$ | 12 | 18 | 24 | 56 (twice) |
|-------|-----|-----|-----|------------|
| orbits | $3+12+48+192$ | $6+9+96+144$ | $3+12+48+192$ | $1+14+112+128$ |
| $|G|/N$ | 60 | 288 | 1344 |
| orbits | $15+240$ | $3+12+48+192$ | $7+8+16+224$ |

7.4.3 Other Polar Spaces

We made the same exhaustive investigation of all permutations $\varphi$ for the other choices of $P$ in the cases $(d, q) \in \{(3, 2), (3, 3), (4, 2)\}$. The only non-rank-3 examples satisfying the 4-vertex condition occur for $O_7(3)$. Here we obtain 252 graphs in total, of which one is rank 3, and three more satisfy the 4-vertex condition. They all have two orbits (of sizes 13+351) and an automorphism group of size 56862. All other graphs $\Gamma_\varphi$ obtained from $O_7(3)$ have more than two orbits.

One might wonder whether a graph $\Gamma_\varphi$ from $O_{2d+1}(q)$ satisfies the 4-vertex condition if and only if it has at most two orbits. And whether a non-rank-3 graph $\Gamma_\varphi$ can only satisfy the 4-vertex condition if $P$ is $Sp_{2d}(q)$ or $O_{2d+1}(q)$.

7.4.4 Other Designs

There are four 2-$(15, 7, 3)$ designs $D$ other than that of the hyperplanes of $PG(3, 2)$. We investigated the case where $(d, q) = (4, 2)$ and $P$ is $Sp_2(8)$, so that the resulting examples satisfy the 4-vertex condition. We generated several hundred thousand graphs $\Gamma_\varphi$ for each of these designs. None of these graphs occurs for two different designs. We believe our enumeration to be complete.
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Appendix: Details on Ivanov’s Graphs

In Sect. 3.3 we discussed the graphs $\Gamma^{(m)}$ from [6] and $\Sigma^{(m)}$ from [22]. Here we give some more detail on the latter.

For $m \geq 2$, consider $V = \mathbb{P}^{2m}_2$ provided with the elliptic quadratic form $q(x) = x_1^2 + x_2^2 + x_1x_2 + x_3x_4 + \ldots + x_{2m-1}x_{2m}$. Identify the set of projective points (1-spaces) in $V$ with $V^* = V \setminus \{0\}$. Let $Q = \{x \in V^* \mid q(x) = 0\}$ and let $S$ be the maximal t.s. subspace given by $S = \{x \in V^* \mid x_1 = x_2 = 0 \text{ and } x_{2i-1} = 0 \text{ (}2 \leq i \leq m\}\}$. Then $S^\perp = \{x \in V^* \mid x_{2i-1} = 0 \text{ (}2 \leq i \leq m\}\}$. The graph $\Sigma^{(m)}$ has $V$ as vertex set, where two distinct vertices $v, w$ are adjacent when $v - w \in (Q \cup S^\perp) \setminus S$. Let $T^{(m)}$ and $\Upsilon^{(m)}$ be the induced subgraphs on the neighbors (nonneighbors) of the vertex 0. Put $R = V^* \setminus (Q \cup S^\perp)$.

**Proposition**  (i) For $m \leq 4$, the graphs $\Sigma^{(m)}$ are rank 3, and are isomorphic to the complement of $V O_{2m-2}^{(2)}$.

(ii) For $m \geq 5$, the automorphism group of $T^{(m)}$ has two vertex orbits $S^\perp \setminus S$ and $Q \setminus S$, of sizes $3 \cdot 2^{m-1}$ and $2^{2m-1} - 2^m$, respectively. For $2 \leq m \leq 4$, the group is rank 3, and the graph is the complement of $NO_{2m-2}^{(2)}$.

(iii) For $m \geq 5$, the automorphism group of $\Upsilon^{(m)}$ has two vertex orbits $S$ and $R$ of sizes $2^{m-1} - 1$ and $2^{2m-1} - 2^m$, respectively. For $3 \leq m \leq 4$, the group is rank 3, and the graph is the complement of $O_{2m-2}^{(2)}$.

(iv) The $\lambda$- and $\mu$-graphs in $\Upsilon^{(m)}$ and the $\mu$-graphs in $T^{(m)}$ are all regular of valency $2m - 2 (2m - 2 + 1)$. In particular, $\Upsilon^{(m)}$ satisfies the 4-vertex condition.

(v) The $\lambda$-graphs in $T^{(m)}$ have vertices of valencies in $0, 2^{2m-4} - 2^m, 2^{2m-4}, 2^{2m-3} - 2^m$. Edges not in a line contained in $Q$ have $\lambda$-graphs with a single isolated vertex and $\lambda - 1$ vertices of valency $2^{2m-4}$. For edges in a line contained in $Q$ the $\lambda$-graphs have a single vertex with valency $2^{2m-3} - 2^m$, and $2^{2m-3} - 1$ vertices with valency $2^{2m-4} - 2^m$, and the remaining $2^{2m-3} + 2^{m-3}$ vertices have valency $2^{2m-4}$. In particular, $T^{(m)}$ satisfies the 4-vertex condition, with $\alpha = 2^{2m-5} (2^{2m-3} + 2^{m-2} - 1)$ and $\beta = \frac{1}{2} \mu \mu' = 2^{2m-4} (2^{m-2} + 1)^2$.

(vi) The local graph of $\Upsilon^{(m)}$ at a vertex $s \in S$ is isomorphic to $\Sigma^{(m-1)}$.

**Proof** (i)–(iii) This is clear, and can also be found in [22]. (iv)–(v) (the part about $T^{(m)}$):

Let $(v, w) = q(v + w) - q(v) - q(w)$ be the symmetric bilinear form belonging to $q$. Let $X = (Q \cup S^\perp) \setminus S$. Then $T^{(m)}$ is the graph with vertex set $X$, where two vertices $x, y$ are adjacent when the projective line $\{x, y, x + y\}$ they span is contained in $X$. If at least one of $x, y$ is in $S^\perp \setminus S$, then this is equivalent to $(x, y) = 1$. If both are in $Q \setminus S$, then this is equivalent to $((x, y) = 0 \text{ and } x + y \notin S)$ or $((x, y) = 1 \text{ and } x + y \in S^\perp \setminus S)$.

Let $x, y, z$ be pairwise adjacent vertices. The valency $c$ of $z$ in the $\lambda$-graph $\lambda(x, y)$ is the number of common neighbors of $x, y, z$. Distinguish several cases.

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If $z = x + y$, then if $x, y, z \in Q$ we find $c = |\{x, y\} \cap (Q \setminus S)| - 3 = 2^{2m-3} - 2^m$. If $z = x + y$ and at least one of $x$, $y$, $z$ lies in $S^\perp$, then $c = 0$.

Now let $z \neq x + y$. The claims are true for $m \leq 4$. Let $m \geq 5$ and use induction on $m$. Choose coordinates so that $x, y, z$ have final coordinates 00 and let $x', y', z'$ be these points without the final two coordinates. If they have $c'$ common neighbors $w'$ in $T^{(m-1)}$, then we find $2c'$ common neighbors $w = (w', 0, \ast)$. Moreover (since $x, y, z$ are linearly independent), we find $2^{2m-5}$ common neighbors $(w', 1, q'(w'))$ in $Q$, where $w'$ runs through all vectors with the desired inner products with $x', y', z'$. Altogether $c = 2c' + 2^{2m-5}$, as claimed.

For the $\mu$-graphs the argument is similar and simpler: by the definition of adjacency three dependent vertices are pairwise adjacent, so that the case $z = x + y$ does not occur here.

(iv) (the part about $\Upsilon^{(m)}$): Let $Y = V^* \setminus X$. Then $\Upsilon^{(m)}$ is the graph with vertex set $Y$, where two vertices $x, y$ are adjacent when the projective line $\{x, y, x + y\}$ they span is not contained in $Y$. The same argument as before yields the valencies of the $\lambda$- and $\mu$-graphs.

(vi) Consider the graph $\Sigma^{(m)}$. The nonneighbors $z$ of 0 that are neighbors of $s$ are the vertices of the form $z = s + b$ with $z \in S \cup R$ and $b \in (Q \cup S^\perp) \setminus S$. It follows that $s + z \in Q \setminus s^\perp$. Let $s = (0 \ldots 01)$, then $Q \setminus s^\perp$ can be identified with $W = \mathbb{P}_2^{2m-2}$ via $w \mapsto i(w) = (w, 1, \tilde{q}(w))$ for $w \in \mathbb{P}_2^{2m-2}$ and $\tilde{q}(w)$ determined by $q(i(w)) = 0$. The local graph of $\Upsilon$ at $s$ can be identified with the graph with vertices $w$, where $w, w'$ are adjacent when the line joining $i(w), i(w')$ has third point $(w + w', 0, \ast) \in (Q \cup S^\perp) \setminus S$, that is, the line joining $w, w'$ has third point $w'' = w + w'$ satisfying $w'' \notin T$ and $(\tilde{q}(w'')) = 0$ or $w'' \in T^\perp$ where $T = \{w \in W \mid w_1 = w_2 = w_3 = w_5 = \ldots = w_{2m-3} = 0\}$. But this is $\Sigma^{(m-1)}$. □

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