The Evolution of the MLLA Parton Spectra.

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Abstract

The evolution with energy scale of the partonic logarithmic scaled energy spectra is investigated in the framework of the modified leading logarithmic approximation (MLLA). The behaviour of the higher order moments is compared to a number of analytic predictions and $e^+e^-$ data.

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1 Introduction

Quantum Chromodynamics (QCD) currently does not allow direct calculations of the hadronic final state observables. To make predictions of the final state it is necessary to model the transition from partons to hadrons. One approach that has been successful in describing the general features of the inclusive energy spectra, in both $e^+e^-$ annihilation and deep inelastic scattering experiments, is the modified leading log approximation (MLLA) using local parton hadron duality (LPHD) as the method for relating the MLLA partonic predictions to the hadronic observables.

The production of hadrons in hard scattering processes is controlled by the underlying partonic behaviour. At small values of momentum fraction (of the outgoing parton from the original hard scatter) this parton behaviour, often referred to as a parton shower, is dominated by gluon bremsstrahlung. The branching processes $q \rightarrow qg$ and $g \rightarrow gg$ (double logarithmic processes) in addition to $g \rightarrow q\bar{q}$ (single logarithmic process) give rise to this parton shower. MLLA accounts for both these double and single logarithmic effects in the evolution equations. The perturbative properties of partonic distributions have been calculated in the framework of the MLLA. They are governed by two free parameters: a running strong coupling, defined by a QCD scale $\Lambda$, and an energy cut-off, $Q_0$, below which the parton evolution is truncated.

Using LPHD, the non-perturbative effects of particle distributions are reduced to a simple factor of normalisation that relates the hadronic distributions to the partonic ones. At large enough energies, away from the influence of $Q_0$, this ‘hadronisation’ factor should be independent of the energy scale at which the spectra are being calculated.

Various approaches, detailed below, have been taken to calculate the single particle spectra and their moments within the MLLA framework. In this paper these theoretical approaches are compared as a function of energy scale, $Q$. Substantial difference are found in the predictions of the higher moments. The theoretical results, in particular the higher order moments, are also compared with $e^+e^-$ data over a range of centre of mass energies, $Q = E_{CM}/2$. It is found that the general characteristics are well described but that the theory is not in accord with all features of the data if it is assumed that $\Lambda$, $Q_0$ and the LPHD normalisation are energy independent. The studies reported here avoid the need to extrapolate into regions not experimentally measured.

2 Single Particle Spectra

Given a high energy parton which fragments via secondary partons into a jet of hadrons, the MLLA evolution equation allows the secondary parton spectra for the logarithmic scaled energy, $\xi$, to be calculated. The variable $\xi$ is defined as $\ln(Q/E) \equiv \ln(1/x)$, where $Q$ is the energy of the original parton and $E$ is the energy of the secondary parton. The cut-off, $Q_0$, limits the parton energy to $E \geq k_T \geq Q_0$, where $k_T$ is the transverse energy of the decay products in the jet evolution. In order to reconstruct the $\xi$ distributions an inverse Mellin transformation is performed

$$
\frac{1}{N} \frac{dn_h}{d\xi} \propto D(\xi, Y, \lambda) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} D(\omega, Y, \lambda)
$$

where the integral runs parallel to the imaginary axis on the right of all singularities in the complex $\omega$-plane, $Y = \ln(Q/Q_0)$ and $\lambda = \ln(Q_0/A)$.

The Mellin-transformed distributions, $D(\omega, Y, \lambda)$, can be expressed in terms of confluent hypergeometric functions, $\Phi$, as

$$
D(\omega, Y, \lambda) = \frac{\Gamma(A)\Gamma(B+1)}{\Gamma(A+B+1)} \Phi(-A + B + 1, B + 2; -t_1) \Phi(A - B, 1 - B; t_2)
$$

$$
+ \left(\frac{\omega}{t_1}\right)^B \Phi(-A, -B; -t_1) \Phi(A, B + 1; t_2),
$$

where

$$
t_1 = \omega(Y + \lambda) \quad t_2 = \omega \lambda
$$

and $A$ and $B$ are defined as

$$
A = 4N_c/b\omega, \quad B = a/b,
$$

$$
\text{(3)}
$$

$$
\text{(4)}
$$
where \( N_c \) is the number of colours, \( a = 11N_c/3 + 2nf/3N_c^2 \), \( nf \) is the number of flavours and \( b = 11N_c/3 + 2nf/3 \). Throughout this paper the number of flavours is assumed to be 3. Equation (1) is then calculated using a numerical integration in the complex \( \omega \)-plane.

It is convenient to investigate the MLLA spectra in terms of moments. The cumulant moments of the \( \xi \) distribution can be written as:

\[
K_m = \left( -\frac{\partial}{\partial \omega} \right)^m D(\omega, Y, \lambda) \bigg|_{\omega=0}.
\]

(5)

The analytic form of these cumulant moments have been calculated in [4] for the first four moments. This allows the normalised moment, \( \langle \xi^m \rangle \), to be calculated and hence the dispersion (\( \sigma \)), skewness (\( s \)) and the kurtosis (\( k \)) to be constructed, see for example [5]

\[
\langle \xi \rangle = K_1
\]

(6)

\[
\sigma^2 = K_2 = \langle \xi^2 \rangle - \langle \xi \rangle^2
\]

(7)

\[
s = K_3/\sigma^3 = \frac{\langle \xi^3 \rangle - 3\langle \xi^2 \rangle \langle \xi \rangle + 2\langle \xi \rangle^3}{\sigma^3}
\]

(8)

\[
k = K_4/\sigma^4 = \frac{\langle \xi^4 \rangle - 4\langle \xi^3 \rangle \langle \xi \rangle - 3\langle \xi^2 \rangle^2 + 12\langle \xi^2 \rangle \langle \xi \rangle^2 - 6\langle \xi \rangle^4}{\sigma^4}
\]

(9)

The so-called limiting spectrum is the case when \( Q_0 = \Lambda \) i.e. \( \lambda = 0 \). For this case, Fong and Webber [6] have also calculated the behaviour of the moments of the \( \xi \) spectra with energy scale, \( Q \). They point out that the spectra can be represented close to the maximum of the distribution by a distorted Gaussian of the form:

\[
D(\xi, Y) \propto \exp \left[ \frac{1}{8} k - \frac{1}{2} s\delta - \frac{1}{4} (2 + k)\delta^2 + \frac{1}{6} s\delta^3 + \frac{1}{24} k\delta^4 \right]
\]

(10)

where \( \delta = (\xi - l)/\sigma \) and \( l \) is the mean. This expression, when fitted to the spectrum, allows the moments to be determined when the full spectrum is either unmeasured or uncalculable.

The changes to the parton spectra for quark, \( q \), and gluon, \( g \), jets are Next-to-MLLA effects which result in additional terms in the integral shown in Eq. (11). These corrected spectra (at \( Q_0 = \Lambda \)) for the quark and gluon jet can be related to the limiting spectra [3] as follows

\[
\tilde{D}_{q,g} = \left[ 1 + \Delta_{q,g} \left( \frac{\partial}{\partial l} + \frac{\partial}{\partial Y} \right) \right] \tilde{D}^{lim}(l, Y) = \tilde{D}^{lim}(l + \Delta_{q,g}, Y + \Delta_{q,g}),
\]

(11)

with

\[
\Delta_g = -\frac{1}{3} nf \frac{N_c}{2N_c^2}, \quad \Delta_q = \Delta_0 + \Delta_g, \quad \Delta_0 = \frac{a - 3N_c}{4N_c}.
\]

The effect on the moments is that the \( \sigma \), skewness and kurtosis for the quark/gluon distribution would be approximately the same as the limiting distribution at an effective energy \( Y + \Delta_{q,g} \). The mean is the same as that of the limiting spectra at this effective energy but shifted in value by \( -\Delta_{q,g} \). These shifts are small with the limiting spectra being between the distribution for the quark and gluon jets. It should be noted that the predictions of Fong and Webber [3] for the relative shifts between quark and gluon jets are slightly different from those discussed above.

In the MLLA approach, the partons are assumed massless so the scaled energy and momentum spectra are identical. Experimentally the scaled momentum distribution is usually measured and as the observed hadrons are massive the equivalence of the two spectra no longer holds. In [3] the assumption is made that the cut-off \( Q_0 \) can be related to the masses of hadrons. This allows the logarithmic scaled momentum distribution, \( \xi_p \), to be written as

\[
\frac{1}{N} \frac{dn_h}{d\xi_p} \propto \frac{p_h}{E_h} D(\xi_p, Y),
\]

(12)
where
\[ \xi = \log \frac{Q}{\sqrt{Q^2 e^{-2\xi p_e} + Q_0^2}}. \]
the energy of a hadron with a momentum \( p_h \) is \( E_h = \sqrt{p_h^2 + Q_0^2} \). Limiting momentum spectra based on massless partons and massive partons will be referred to as MLLA-0 and MLLA-M spectra, respectively.

### 3 Behaviour of Theoretical Spectra

In this section the general characteristics of the spectra are discussed in order to illustrate their behaviour as a function of the three variables \( Q \), \( \lambda \), and \( Q_0 \).

Figure 3(a) shows the limiting energy spectra for three energies \( Q = 183.2 \) (LEP II), 91.2 (LEP) and 14.82 GeV (TASSO) with a \( \Lambda \) value of 250 MeV. As the energy scale increases, the parton multiplicity (the area under the curve) grows and the peak position shifts to the right i.e. the parton scaled energy spectra is softer. The cut-off on the right hand side of the plot corresponds to the truncation in the parton energy spectra at \( Q_0 = \Lambda \).

Figure 3(b) shows the scaled energy spectra at fixed energy scale \( Q = 91.2 \) GeV and \( \Lambda = 250 \) MeV but at two different values for the cut-off of the parton evolution, \( Q_0 = \Lambda \) and \( 2\Lambda \). (The curves are cropped at \( \xi = 1.0 \) as below this value the numerical integration of the truncated spectra is unstable.) Truncating the cascade at higher values of \( Q_0 \) leads to a lower parton multiplicity with a harder energy spectra.

Figure 3(c), again, shows the limiting energy spectra but at a fixed energy scale, \( Q = 91.22 \) GeV, with values of \( \Lambda = 50, 250 \) and 400 MeV. As \( \Lambda \) decreases, more partons are produced and their energy spectra is softer. In the case of the limiting spectra, this behaviour is dominated by the fact that an increase in \( \Lambda \) is accompanied by an increase in the cut-off \( Q_0 \).

Figure 3(d) shows the limiting scaled momentum spectra for the MLLA-0 (\( E_h = p_h \)) and MLLA-M case (\( E_h \neq p_h \)) at an \( E_{CM} = 91.2 \) GeV. The MLLA-0 spectra displays the usual truncation at large \( \xi \) associated with the cut-off \( Q_0 \), but the MLLA-M case does not. This is due to the fact the calculation is now regulated by the cut-off entering as a mass term in the expression of \( \xi_p \) (see Eq. 12) As the momentum \( \rightarrow Q \), i.e. \( \xi_p \rightarrow 0.0 \), the introduction of a mass term has no effect on the MLLA calculation. As \( \xi_p \) increases (the momentum becomes smaller) the mass term begins to play an increasingly important rôle. From these arguments it can be seen that as \( Q \) decreases mass term has a more significant influence over a larger (fractional) range of the \( \xi_p \) spectra.

Figure 4 shows the evolution of the mean, \( \sigma \), skewness and kurtosis of the limiting energy spectra with \( \Lambda = 250 \) MeV as a function of \( Q \). Three different approaches to calculating the moments have been investigated: (i) the analytic expression of Fong and Webber for gluon jets (dashed line) and quark jets (dash-dotted line); (ii) the analytic calculation of Dokshitzer et al. (dotted line); (iii) fitting a distorted Gaussian (Eq. 11) over \( \pm 1 \sigma \) around the mean value (full line). The range of the fit is motivated by the phenomenological scope of validity. All approaches exhibit the same trends, in that all the cumulants increase as \( Q \) increase. The values of skewness and kurtosis are negative, tending towards zero as \( Q \) increases, i.e. the spectra are becoming more like a pure Gaussian. The mean and the \( \sigma \) exhibit a very similar dependence on \( Q \) for all three approaches, though with different offsets. The skewness and kurtosis exhibit a different dependence on \( Q \), but as \( Q \) approaches the asymptotic limit the predictions are converging.

The differences between quark and gluon jets for the predictions of Fong and Webber, in Fig. 4, are small. A feature of the data is the marked difference between the analytic calculations of Dokshitzer et al. and the fit to the limiting spectra. Calculating the moments over the full range of the spectra gives the same result as the analytic calculations, not too surprisingly since both calculations use the same assumptions. Investigating a limited range of the spectra, as is done during the fit, produces marked differences, highlighting the sensitivities of these higher-order moments to the behaviour of the tails as well as the incorrect form of the distorted Gaussian away from the peak position. The difference between the analytic calculation of Fong and Webber and that of Dokshitzer et al. could well be due to the differences in the predictions at large \( x_p \), where the theoretical assumption made in both calculations are no longer valid.

Figures 4(b) and 4(c) imply that \( \Lambda \) influences the position of the maximum and the width of the distribution of \( \xi \). The effect of the relation between \( Q_0 \) and \( \Lambda \) is more complex; setting \( Q_0 = 2\Lambda \) lowers the position of the maximum for a given \( \Lambda \). The maximum can be returned to the original position by reducing \( \Lambda \) but with a consequential increase of the width of the distribution relative to that found for the limiting spectrum. The effect of introducing mass terms into the momentum spectra is to increase the tail of the spectra at high \( \xi \), thus increasing skewness. As \( Q \) increases, the influence of the mass term becomes less important.
4 Comparison with data

The differential $\xi_p$ distributions for $e^+e^-$ data and the models are shown in Figure \[3\]. The limiting spectra were calculated with $\Lambda = 250 \text{ MeV}$; this value was chosen to give the correct peak position of the spectra for an $E_{CM} (= 2Q)$ at the mass of the $Z^0$, $M_z$. For each energy, the theoretical spectra are normalised to the same maximum height as the data.

The theoretical spectra follow the general trend of the data for both for both MLLA-0 and MLLA-M spectra; in detail there are discrepancies. At low energies, for the MLLA-M case, the skewness is larger (i.e. more positive) than that observed for the data; at high energies, the limiting spectra have a smaller width than the data. Also shown are the MLLA energy spectra for $Q_0 = 2\Lambda$. Here $\Lambda = 50 \text{ MeV}$ is required to get the correct position of the maximum for $E_{CM} = M_z$. The position of the maximum has approximately the correct energy dependence; however, the width of the theoretical spectrum is consistently greater than that of the data.

To study the energy evolution of the MLLA theoretical spectra and the $e^+e^-$ data, fits around the peak position, using the distorted Gaussian (Eq. [10]) have been made to determine the moments of the $\xi_p$ distributions. In view of the statistical limitations, the range of the fit for the data was about three units around the peak (see Figure \[3\]). The distorted Gaussian describes the data well over the fitted region with a $\chi^2$/dof of 1.0 or better. To permit direct comparison with the data, a similar fit range was used for the theoretical spectra. The distorted Gaussian gives a good description of the MLLA-M spectra. The description is less good for the MLLA-0 spectra but for all energies the Gaussian represents the model to better than 1%.

The moments from the fits are shown in Figure \[4\] as solid lines for fits to MLLA-0 spectra and as dashed lines for fits to the MLLA-M spectra. The data points are from our fit to $e^+e^-$ data \[9, 10\].

It may be concluded from Figure \[4\] that the MLLA-0 model gives a good description of the $e^+e^-$ data for the mean and skewness. There are however discrepancies in $\sigma$ and kurtosis as is also evident in Figure \[5\] $\sigma$ is smaller, and the model is more platykurtic than the data. The MLLA-M model gives a poorer description of all variables at low $E_{CM}$ but approaches both the MLLA-0 predictions and the data at high $E_{CM}$.

This is contrary to the conclusions reported by Lupia and Ochs \[11\]. In that paper the moments of the experimental distributions corrected for the mass effects were compared to the predictions for the limiting spectra and found to be in good agreement. It should be noted that in \[11\] an extrapolation was made into regions where no experimental measurement exists to allow the moments to be calculated. In addition, the agreement of the MLLA limiting spectra with the mass corrected experimental $\xi$ distribution is poor. The spectra at low $\xi$ are well described but the region around the peak and at higher $\xi$ is generally not well reproduced. The studies reported here avoid the need to extrapolate into regions not experimentally measured.

LPHD postulates that a constant factor relates the MLLA predicted spectrum to the experimental spectrum, independent of the $E_{CM}$. In the spectra discussed above, the maxima of the MLLA predictions have been scaled to the data. This scale factor, for the MLLA-0 model, is shown in Figure \[6\] as a function of $E_{CM}$. It falls from $\approx 1.4$ at the lowest $E_{CM}$ to $\approx 1.2$ at the highest. A similar result is obtained for an area normalisation over approximately three units of $\xi_p$ for the MLLA-M case. Thus our observations disagree with the LPHD postulate of a constant scale factor. This suggests that at the currently accessible experimental energies the theoretical spectra are still influenced by the $Q_0$ cut-off.

Similar comparisons of $e^+e^-$ data with MLLA predictions have been made by the DEPHI \[12\] and OPAL \[10\] collaborations. CDF has examined jets produced in $p\bar{p}$ interactions \[13\]. In contrast to the analysis presented here, no explicit analysis of the higher moments of the $\xi$ distribution has been made. However, where the analyses can be compared, the results are in agreement with those presented here. In particular, $\Lambda$ is close to 250MeV, the shape of the MLLA predicted spectrum is close to that of the data but disagrees in detail, and the value of the LPHD constant changes with energy.

5 Conclusions

The MLLA predictions of the logarithmic scaled energy spectra, and in particular their moments, have been investigated as function of energy scale. Various theoretical approaches have been compared and differences discussed. We note that care should be taken in the comparisons to ensure that a consistent approach is maintained; in particular attention should be paid to the range of application in $\xi$. There is a large discrepancy in the skewness and kurtosis at low $Q$ but the various predictions converge as the asymptotic limit is approached.
The theoretical results are compared to measurements taken in $e^+e^-$ annihilation experiments. The calculations are generally in good agreement with the data. It is observed that the limiting spectra is preferred over the predictions of the truncated cascade ($Q_0 \neq \Lambda$). The introduction of the mass term has a large effect at all but the highest $E_{CM}$. The data is consistently broader than the limiting spectra over the energy range studied here.

The normalisation factor of LPHD between the data and the MLLA theoretical predictions is not constant. Contrary to expectation it displays a dependence on $E_{CM}$, decreasing as $E_{CM}$ increases, thus suggesting a residual influence of the $Q_0$ cut-off.

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Figure 1: The logarithmic scaled energy (a)-(c) and scaled momentum (d) spectra of partons as calculated in the MLLA formalism. Unless otherwise stated values of $\Lambda = 250$ MeV and $Q_0 = \Lambda$ (the so-called limiting spectra) at $Q = 91.2/2$ GeV were used to calculate the curves. (a) The three curves are predictions at three different energy scales, $Q = 183.0/2, 91.2/2$ and $14.8/2$ GeV represented by the full, dashed and dotted lines respectively. (b) The two curves are predictions for two different values of $Q_0 = \Lambda$ and $2\Lambda$ represented by the full and dashed lines respectively. (c) The three curves are predictions for three different values of $\Lambda = 50, 250$ and $400$ MeV represented by the full, dashed and dotted lines respectively. (d) The logarithmic scaled momentum spectra of partons as calculated in the MLLA formalism. The two curves are predictions for a massless (full line) and massive assumption (dashed line).
Figure 2: The evolution of the mean, $\sigma$, skewness and kurtosis as a function of $Q$ for $\Lambda = 250$ MeV. The dashed and dash-dotted lines are the analytic calculation of Fong and Webber for the gluon and quark respectively, the dotted line from the analytic calculation of Dokshitzer et al., and the full line from fitting a distorted Gaussian around $\pm 1\sigma$ of the mean of the MLLA-0 spectra.
Figure 3: $e^+e^-$ data compared with MLLA-0 spectra (dashed histogram), MLLA-M spectra (solid histogram) and MLLA $Q_0 = 2\Lambda$ with $\Lambda = 50$ MeV (thin solid line). The thick solid line is from the fit of a distorted Gaussian that is used to determine the moments for the data.
Figure 4: Moments as a function of $E_{CM} (= 2Q)$. The points with error bars are from fits of a distorted Gaussian to $e^+e^-$ data; circles: TASSO; square: TOPAZ; triangles: OPAL. The solid lines are moments from a fit of a distorted Gaussian to the MLLA-0 spectrum. The dashed lines are moments from a fit of a distorted Gaussian to a MLLA-M spectrum.
Figure 5: Scale factor as a function of $E_{CM} (= 2Q)$. The full symbols are from normalisation to the same maximum height for data and MLLA-0 spectra. The open symbols (displaced by 1 GeV for clarity) are from a normalisation of the areas of data and MLLA-M spectra; circles: TASSO; square: TOPAZ; triangles: OPAL. The line is a linear fit of the height-based scale factor to log($E_{CM}$).