Finite width effects in $\phi$ radiative decays

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Abstract
We calculate the decay rates $\phi \to \gamma \pi^0 \pi^0$ and $\gamma \pi^0 \eta$ in very good agreement with the recent accurate experimental data. We also point out the necessity of a $\phi \gamma K^0 \bar{K}^0$ contact vertex beyond the pure $K^+ K^-$ loop model. The decay widths $\phi \to \gamma f_0(980)$ and $\phi \to \gamma a_0(980)$ are also calculated taking into account the finite widths of the scalar resonances $f_0(980)$ and $a_0(980)$. The latter are shown to be essential in order to obtain meaningful results. The resulting decay rates to $\gamma f_0(980)$ and $\gamma a_0(980)$ are in good agreement with the experimental ones without invoking isospin breaking in the couplings of the $f_0(980)$ and $a_0(980)$ resonances to the $K^+ K^-$ and $K^0 \bar{K}^0$ channels, at odds with recent proposals. The derived formula for calculating these $\phi$ radiative decay widths can be also applied in their own experimental analyses in order to obtain more precise results.
1 Introduction

In this article we study the $\phi$ radiative decays to $\gamma$ and two neutral pseudoscalars and the related rates to $\gamma f_0(980)$ and $\gamma a_0(980)$. These decays were first observed experimentally by the CMD-2 and SND collaborations at Novosibirsk by measuring the invariant mass distributions $d\Gamma(\phi \to \gamma \pi^0\pi^0)/dm_{\pi\pi}$ and $d\Gamma(\phi \to \gamma \pi^0\eta)/dm_{\pi\pi}$, respectively, where we indicate by $m_{M^0N^0}$ the invariant mass of a pair of neutral pseudoscalars $M^0$ and $N^0$. On the other hand, high statistics results have been recently reported by the KLOE collaboration at DAΦNE in refs.[3, 4] for the rates $\phi \to \gamma \pi^0\pi^0$ and $\gamma \pi^0\eta$, respectively. These decays offer an additional source of experimental information on the non-trivial and so important low lying scalar mesons with vacuum quantum numbers $0^{++}$. Our theoretical study generalizes the one undertaken in ref.[3] about the $\phi \to \gamma K^0\bar{K}^0$ decay in order to reproduce the precise data of refs.[5, 6] on $\phi \to \gamma \pi^0\pi^0$ and $\gamma \pi^0\eta$ and connects the previous processes with chiral symmetry and unitarity. Ref.[7] was also followed closely by ref.[5] where the aforementioned radiative decays to the final states $\gamma \pi^0\pi^0$ and $\gamma \pi^0\eta$ were also considered. For recent accounts on the application of similar techniques to the scalar meson-meson and meson-baryon dynamics we refer the reader to refs.[9, 10, 11] where many references are presented and discussed. Other studies of $\phi \to \gamma M^0N^0$ decays are refs.[12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

In order to interpret the experimental results from refs.[3, 4] on the $\phi$ decays to $\gamma f_0(980)$ and $\gamma a_0(980)$, ref.[23] has pointed out recently the necessity of considering important isospin breaking corrections in the couplings of the $f_0(980)$ and $a_0(980)$ resonances to the $K^+K^-$ and $K^0\bar{K}^0$ channels. That is, $|g_{K^+\bar{K}^-}|$ and $|g_{K^0\bar{K}^0}|$, where $R$ represents either the $f_0(980)$ or $a_0(980)$ resonances, are no longer equal as required by isospin symmetry but can be actually quite different according to ref.[23]. Special emphasis is given to the rather large decay width $\phi \to \gamma f_0(980)$ together with the deviation from one, by around a factor four, of the ratio $Br(\phi \to \gamma f_0)/Br(\phi \to \gamma a_0)$ as experimentally established in refs.[1, 3, 4, 5]. Nevertheless, in ref.[24] it is clearly shown that possible isospin breaking effects in the quotient $\Gamma(\phi \to \gamma \pi^0\pi^0)/\Gamma(\phi \to \gamma \pi^0\eta)$ cancel to a large extent. As a result, this ratio is sensitive mostly to the isospin eigenstates $a_0(I = 1)$ and $f_0(I = 0)$. Similar conclusions are obtained within the vector meson dominance model of ref.[25] as well. Other references treating the $f_0(980)$ and $a_0(980)$ mixing are [26, 27, 28, 29].

We show that one does not need to abandon isospin symmetry in the couplings of the scalar resonances $f_0(980)$ and $a_0(980)$ to the $K\bar{K}$ channels so as to account for the experimental results on the $\phi \to \gamma R$ decay widths. We emphasize the fact that the threshold of $\gamma R$ is so close to the mass of the $\phi(1020)$ resonance, that meaningful results for the width $\Gamma(\phi \to \gamma R)$ can only result once the finite widths of the $f_0(980)$ and $a_0(980)$ resonances are taken into account. It is important to stress that the decay width $\Gamma(\phi \to \gamma R)$ depends cubically on the small photon momenta and hence small changes in the nominal masses of the $f_0(980)$ or $a_0(980)$ resonances, e.g. as compared to their widths or to the difference between the real parts of the pole positions and the peaks of the scattering amplitudes, imply dramatic variations on the resulting $\phi$ radiative decay widths to $\gamma R$ if the standard two body decay formula were used. In sec.2 we calculate the $\phi$ decays to $\gamma \pi^0\eta$, $\gamma \pi^0\pi^0$ and $\gamma K^0\bar{K}^0$ where we connect with and extend the results already given in refs.[1, 2]. We also establish a new contribution beyond the pure $K^+\bar{K}^-$ loop model of ref.[12] due to a contact $\phi\gamma K^0\bar{K}^0$ vertex. In sec.3, making use of the results of the previous section, we take into account the finite width effects of the $f_0(980)$ and $a_0(980)$ resonances on the decays $\phi \to \gamma R$ and derive the corresponding formulae. We also consider the connection of the obtained formula for $\Gamma(\phi \to \gamma R)$ to the invariant mass distributions $d\Gamma(\phi \to \gamma M^0N^0)/dm_{M^0N^0}$. Finally, our results are discussed in sec.4 and the conclusions are given in sec.5.
\[ \Gamma (\phi \rightarrow \gamma M^0 N^0) \] rates

We are interested in studying the \( \phi \rightarrow \gamma M^0 N^0 \) decays. The final state interactions (FSI) of the two pseudoscalar mesons are assumed to be dominated by their strong S-wave amplitudes which contain the \( f_0(980) \) and \( a_0(980) \) scalar resonances. Then gauge invariance requires, as noted in refs.\[12, 35, 13\] with respect to the related decays \( \phi \rightarrow \gamma R \) as well, that the transition amplitude \( \phi(1020) \rightarrow \gamma M^0 N^0 \) must have the form \[6, 8\]:

\[
\mathcal{M} [\phi(p) \rightarrow \gamma(k)(M^0 N^0(Q))] = H(Q^2) \left[ g^{\alpha\beta}(p \cdot k) - p^\alpha k^\beta \right] \epsilon^\lambda(\gamma)_\alpha \epsilon^{\nu}(\phi)_\beta .
\] (2.1)

The kinematics is indicated in fig.1 and the four-momentum \( Q \) corresponds to the total one of the pair of pseudoscalars \( M^0 N^0 \).

As in refs.\[12, 10, 7\] we consider the \( K^+ K^- \) meson loop contribution to the decay \( \phi \rightarrow \gamma M^0 N^0 \), fig.1, since the \( K^+ K^- \) system strongly couples to both the \( \phi \) and \( R \) resonances. In addition, since the \( K^0 K^0 \) couples with the same strength as the \( K^+ K^- \) state both to the scalar and \( \phi \) resonances, we also allow for a neutral kaon loop contribution, as indicated in fig.1a, where the vertex on the left of the figure is discussed in length below.

To simplify the discussions, we will consider the isospin channels separately, since they do not mix by final state interactions in the isospin limit, so that \( H^I(Q^2) \) is the invariant function in the same isospin channel as the corresponding scalar resonance \( R \) from the kaon loops of fig.1. We now calculate the function \( H^I(Q^2) \) defined such that \( H^I(Q^2) = \mathcal{H}^I(Q^2) \epsilon_{K+K^- \rightarrow M^0 N^0}^R \), with \( \epsilon_{K+K^- \rightarrow M^0 N^0}^R \) the isospin amplitude \(-t^I_{KK \rightarrow MN} C_{M^0 N^0}^I / \sqrt{2}\) where the \( KK \) and \( MN \) states are in the isospin channel \( I \) of the scalar resonance \( R \), namely, \( I = 0 \) for the \( f_0(980) \) and \( I = 1 \) for the \( a_0(980) \), and \(-1/\sqrt{2}\) and \( C_{M^0 N^0}^I \) are Clebsch-Gordan coefficients. For the \( \gamma\pi^0\eta \) final state only the \( I = 1 \) channel takes place and the \( I = 0 \) one is the only contribution to \( \gamma\pi^0\eta^0 \). For the \( \gamma K^0 \bar{K}^0 \) final state both contributions sum up.

In ref.\[6\] we took as interacting Lagrangian,

\[
\mathcal{L} = 2e g_\phi A^\mu \phi K^+ K^- - i (e A_\mu + g_\phi \phi_\mu)(K^- \partial^\mu K^+ - \partial^\mu K^- K^+) ,
\] (2.2)

upon making the \( \phi \) and \( K^+ K^- \) interactions gauge invariant.

![Diagrams](image)

Figure 1: Set of diagrams for the calculation of \( \phi \rightarrow \gamma M^0 N^0 \) through kaon loops.

We now consider the more refined approach of ref.\[30\], where the couplings of the octet of pseudoscalars to the octet of vector resonances are given at order \( p^2 \) by taking into account chiral symmetry in a chiral
power expansion. The couplings of the singlet vector resonance $\omega_1$ cancel at this order exactly. Making use of ideal mixing we can write the $\phi$ resonance as:

$$\phi = -\frac{2}{\sqrt{6}}\omega_8 + \frac{1}{\sqrt{3}}w_1,$$

(2.3)

and thus the couplings of the $\phi$ resonance from ref.[30] are the ones of the $\omega_8$ resonance times $-2/\sqrt{6}$. The tree level matrix element for the contact vertex $\phi \gamma K^+ K^-$, corresponding to the left vertex of fig.3a, $M(\phi \rightarrow \gamma K^+ K^-)_{\text{contact}}$, calculated from the formalism of ref.[30] is:

$$M(\phi \rightarrow \gamma K^+ K^-)_{\text{contact}} = -\frac{\sqrt{2}eM_\phi G_V}{f^2}e(\gamma) \cdot \epsilon(\phi) - \frac{\sqrt{2}e}{f^2 M_\phi} \left( \frac{F_V}{2} - G_V \right) p_\alpha \epsilon(\phi)_\beta \left( k^\alpha \epsilon(\gamma)^\beta - k^\beta \epsilon(\gamma)^\alpha \right),$$

(2.4)

where $f \simeq 93$ MeV is the weak pion decay constant and $G_V$ measures the strength of the transition $\phi \rightarrow K^+ K^-$. The experimental width $\Gamma(\phi \rightarrow K^+ K^-)$ is reproduced with $G_V = 55$ MeV. Let us note that the second structure in the equation above is gauge invariant by itself. On the other hand, vector meson dominance requires $F_V = 2G_V$ [30] and in this limit the coefficient in front of the second term of eq.(2.3) is zero. At the chiral order considered in ref.[31], there is no direct coupling $\phi \gamma K^0 \bar{K}^0$ although at higher orders this is no longer the case and can mimic e.g. $\phi \gamma K^0 \bar{K}^0$ vertices generated through the exchange of vector resonances, see e.g. [16].

We consider first the $K^+ K^-$ loops of fig.4 that are originated from the surviving local term of eq.(2.4), fig.3a, plus the Bremsstrahlung ones, diagrams 1b and c, and fig.3d. We follow the treatment of ref.[7] where it is shown that the T-matrix element $t^{R, I}_{K^+ K^- \rightarrow K^+ K^-}$ of ref.[31] factorizes on-shell in these diagrams. The reason is, as proved in ref.[7], that the off-shell parts of the T-matrices of ref.[31] do not contribute to the coefficient of the term in eq.(2.1) proportional to $p^\alpha k^\beta$, which is the function $H(Q^2)$ as required by gauge invariance.\#2

Motivated by the self-invariant structure of the second term of eq.(2.4), we consider for each isospin channel a general contact $\phi \gamma K \bar{K}$ interaction with the same structure but parameterized by a coupling $\zeta_I$,

$$V_I(\phi \gamma K \bar{K}) = -\frac{\sqrt{2}e \zeta_I}{f^2 M_\phi} p_\alpha \epsilon(\phi)_\beta \left( k^\alpha \epsilon(\gamma)^\beta - k^\beta \epsilon(\gamma)^\alpha \right).$$

(2.5)

The dependence of $\zeta_I$ on the isospin channel implies a possible non-vanishing $\phi \gamma K^0 \bar{K}^0$ local term since the Clebsch-Gordan coefficient for $K^0 \bar{K}^0$ changes sign from $I = 0$ to $I = 1$ while that of $K^+ K^-$ is the same for both isospins. Similar structures for $\gamma \pi^0 \pi^0$ and $\gamma \pi^0 \eta$ are not considered since they are expected to be suppressed by the OZI rule.

We are now in position to evaluate the loop of fig.2 with the contact term $V_I(\phi \gamma K \bar{K})$ of eq.(2.3), in the vertex on the left of the diagram. The strong amplitude $t^{R, I}_{K^+ K^- \rightarrow M^0 N^0}$ of ref.[31] ($t^{R}_{K^0 \bar{K}^0 \rightarrow M^0 N^0}$ is just proportional to the former) factorizes on-shell since the off-shell part, as in the case of the pure strong interacting problem treated in detail in ref.[31], just renormalizes the coupling on the left part of the loop, namely, the $\zeta_I$ factor of eq.(2.3). This is due to the fact that it only gives rise to tadpole like diagrams. Since the $\zeta_I$ factor is a free one this renormalization process is not relevant for our present purposes and is just reabsorbed in the final value of $\zeta_I$. Since both vertices on the diagram of fig.2 factorize with their physical renormalized values, the one on the left does not involve any $K \bar{K}$ momentum, see eq.(2.5), we
are then left with the logarithmic divergent loop integral,
\[ i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_{K^+}^2 + i0^+} \left( (Q - q)^2 - m_K^2 + i0^+ \right). \]  
(2.6)

In ref.\cite{8} this integral was taken the same as the one fixed in ref.\cite{31} for the strong interactions. Nevertheless, one should point out that unitarity and analyticity, with the latter restricted to the presence of the unitarity or right hand cut, allow the presence of an extra subtraction constant, \( \delta G^I \), in the integral of eq.\cite{26} as compared to \( G_{KK} \) of ref.\cite{31}. We show now this point.

Since the vertex of eq.\cite{2.5} is gauge invariant by itself, we consider its contributions separately. Thus, it is characterized by the corresponding invariant function \( \tilde{H}^I \), a contribution to the total one \( H^I \). Let us denote by \( \tilde{H}_\alpha^I \) the invariant function corresponding to the \( \phi \rightarrow \gamma \alpha \) decay where \( \alpha \) indicates any channel that can couple strongly to the \( KK \) channel and with the same well defined isospin. We treat these invariant functions as form factors depending only on \( Q^2 \) since this is the dependence obtained from unitarity loops as those of fig.\ref{fig:2} that we want to analyze.\footnote{Note that the vertex on the right is the full strong T-matrix with its own unitarity cuts.}

Then unitarity requires:
\[ \text{Im} \left[ \tilde{H}_\alpha^I \right] = \sum_\beta (\tilde{H}_\beta^I)^* \rho_\beta \, t_{\beta \rightarrow \alpha} \theta(Q^2 - W_\beta^2), \]  
(2.7)

where \( \rho_\beta \) is the phase space factor of channel \( \beta \), \( \rho_\beta = q_\beta/(8\pi \sqrt{Q^2}) \). In ref.\cite{32, 33} it is shown that when the T-matrix appearing on the right side of the previous equation has no crossed cuts due to crossed channels, as the one of refs.\cite{31, 34}, \( \tilde{H}_\alpha^I \), collected in the vector column \( \tilde{H}^I \), can be represented as:
\[ \tilde{H}^I = [1 + K^I(Q^2) \cdot G(Q^2)]^{-1} \, R_I(Q^2), \]  
(2.8)

where it is important to remark that \( R_I(Q^2) \) is a vector of functions without any cut. In addition, \( T^I(Q^2) = [1 + K^I(Q^2)G(Q^2)]^{-1} \cdot K^I(Q^2) \) with \( T^I(Q^2) \) the strong T-matrix for the S-wave meson-meson scattering with isospin \( I = 0 \) or 1. In ref.\cite{31} \( K^I(Q^2) \) corresponds to the lowest order CHPT meson-meson amplitudes \cite{35, 36} while in ref.\cite{34} together with this contribution one has the local terms calculated from \( O(p^4) \) CHPT, see refs.\cite{31, 34} for explicit formulae. In addition \( G(Q^2) \) is a diagonal matrix of unitarity loops such that for \( I = 0 \), \( G_{11}(Q^2) \) is the loop analogous to eq.\cite{26} but for \( \pi \pi (\alpha = 1) \) and \( G_{22}(Q^2) \) corresponds to the \( KK (\alpha = 2) \) channel. Analogously for \( I = 1 \), \( G_{11}(Q^2) \) corresponds to \( \pi \eta (\alpha = 1) \) and \( G_{22}(Q^2) \) to \( KK (\alpha = 2) \) and is equal to \( G_{22}(Q^2) \) with \( I = 0 \).

On the other hand, because of the aforementioned factorization of the T-matrix of ref.\cite{31}, \( \tilde{H}_\alpha^I \) can be expressed as:
\[ \tilde{H}^I = \lambda_I - T_I \, G^I(Q^2) \, \lambda_I, \]  
(2.9)
with \( \lambda_I \) the vector column of direct couplings to \( \phi\gamma K\bar{K} \) after removing the tensor structures like in the definition of \( \bar{H}' \), that is \((\lambda_I)^T \propto (0, \zeta_I)\). On the other hand, \( G^I_{ii}(Q^2) \) can differ at most from \( G_{ii}(Q^2) \) in a subtraction constant which we express by writing \( G^I = G + \delta G^I \). Notice that because \((\lambda_I)_1 = 0\) only the matrix element of \( \delta G^I \) for the \( K\bar{K} \) channel is relevant, the only one that we will keep in the following. Now, taking into account the explicit expression of \( T \) in terms of \( G \) and \( K \), one has:

\[
\bar{H}' = \left( 1 - [1 + K'G]^{-1} \{ K'G + K'\delta G^I \} \right) \lambda_I = [1 + K'G]^{-1} \lambda_I - [1 + K'G]^{-1} K'\delta G^I \lambda_I ,
\]

and hence \( R_I = (1 - K'\delta G^I) \lambda_I \). Let us note that \( R_I \) has no cuts, as it should, since neither \( K' \), \( \delta G^I \) or \( \lambda_I \) have any cut.

As a result of this digression, we will keep \( \delta G^I \) as free parameters in addition to \( \zeta_I \). This amounts to four free parameters, two for each isospin channel, although in the end we will show that two of them will turn out to be fixed. Although eq. (2.10) is derived by taking into account the off-shell form of the T-matrices of ref. [31], we will also use it for the more involved T-matrices of refs. [34] since the equivalent form given in eq. (2.10) is equally valid for all of them. That is, the column matrix \( R_I(Q^2) \) is fixed to be \((1 - K'\delta G^I) \lambda_I \), eq. (2.10), for both the T-matrices of ref. [31] and ref. [34]. While for the former strong amplitudes this expression for \( R_I \) is a strict derivation due to its specific off-shell parts that enter in the unitarity loops, for the latter ones \( R_I \) is fixed by analogy.

Summing the \( K^+K^- \) loop contribution of figs. 2a, b and c plus the \( K\bar{K} \) loop of fig. 3 \( \bar{H}' \), we can then write:

\[
H^I(Q^2) = \left\{ \frac{\sqrt{2}eM_{\phi}G_V}{4\pi f^2 m^2_{K^+}} I(a,b) - \frac{2e\zeta_I}{f^2 M_{\phi}} G^I_{K\bar{K}}(Q^2) \right\} t_{K^+K^- \rightarrow M^0N^0}^R ,
\]

where for the \( \gamma\pi^0\pi^0 (\pi^0\eta) \) final state only \( I = 0 \) contributes while for \( \gamma K^0\bar{K}^0 \) one has to sum both \( I = 0 \) and 1 contributions together. From this expression it follows that \( H^I(Q^2) = H^I(Q^2)/t_{K^+K^- \rightarrow M^0N^0}^R \) is given by:

\[
H^I(Q^2) = \frac{\sqrt{2}eM_{\phi}G_V}{4\pi^2 f^4 m^2_{K^+}} I(a,b) - \frac{2e\zeta_I}{f^2 M_{\phi}} G^I_{K\bar{K}}(Q^2) .
\]

Including the three-body phase space and taking into account eq. (2.11), the width \( \Gamma(\phi \rightarrow \gamma M^0N^0) \) for \( M^0N^0 = \pi^0\pi^0 \) or \( \pi^0\eta \) can be expressed as:

\[
\Gamma(\phi \rightarrow \gamma M^0N^0) = \int d\sqrt{Q^2} \frac{\alpha |k|^2 |p_M|}{6\pi^2 f^4} \sum_R \left( \frac{M_{\phi}G_V}{4\pi^2 m^2_{K^+}} I(a,b) - \frac{\sqrt{2}\zeta_I}{M_{\phi}} G^I_{K\bar{K}}(Q^2) \right) t_{K^+K^- \rightarrow M^0N^0}^R \right|^2 ,
\]

where \( |p_M| \) is the momentum of the \( M^0N^0 \) system in their center of mass frame and \( \alpha \) is the electromagnetic fine structure constant. Possible symmetric identity factors 1/2 are omitted in eq. (2.13) since the transition matrix element \( |t_{K^+K^- \rightarrow M^0N^0}^R|^2 \) is taken from refs. [31], [34] and such factors are already included in its normalization. In the previous expression the sum is restricted over those isospin channels included in \( M^0N^0 \). For the \( \gamma K^0\bar{K}^0 \) final state case, \( M^0N^0 = K^0\bar{K}^0 \), one has to add to the term between bars in eq. (2.13) the contribution \((\zeta_0 - \zeta_1)/\sqrt{2}M_{\phi} \) from the direct local terms of eq. (2.1) with \( I = 0 \) and 1.
3 \( \Gamma(\phi \to \gamma R) \) decay widths

As in the previous section, see also refs. [12, 27, 13, 6], gauge invariance requires that the transition amplitude \( \phi(1020) \to \gamma R \) must have the form:

\[
\mathcal{M}[\phi(p) \to \gamma(k)R(Q)] = H^R(Q^2) \left[g^{\alpha\beta}(p \cdot k) - p^\alpha k^\beta\right] \epsilon^\gamma(\gamma)_{\alpha\epsilon^\nu}(\phi)_{\beta}.
\]

where now \( Q \) is the total four-momentum of the resonance \( R \). The function \( H^R(Q^2) \) can be obtained from the expression given above for \( H^I(Q^2) \) replacing \( f^R_{K^+K^-\to M^0N^0} \) by the coupling of the \( K^+K^- \) channel to the \( R \) resonance, \( g^R_{K^+K^-} \). That is, we are extracting the resonance pole contribution from \( t^R_{K^+K^-\to M^0N^0} \). Although not explicitly shown \( g^R_{K^+K^-} \) can depend on the energy, \( Q^2 \), flowing to the resonance \( R \) which is not fixed since its mass is not well defined due to the finite widths of \( R \). Indeed, the latter gives rise to important effects due to the proximity of the mass of the \( \phi(1020) \) resonance to the nominal ones of the \( f_0(980) \) and \( a_0(980) \). Let us consider this in detail.

Since the coupling \( g^R_{K^+K^-} \) appears as a factor in \( H^R(Q^2) \), then we have that \( H^R(Q^2) = H^I(Q^2) g^R_{K^+K^-} \). It is also convenient to define the quantity \( \mathcal{M}' \) so that \( \mathcal{M}(\phi \to \gamma R) = \mathcal{M}' g^R_{K^+K^-} \). Then, the width \( \Gamma(\phi \to \gamma R) \) can be expressed as:

\[
\Gamma(\phi \to \gamma R) = \frac{1}{3} \sum_{\nu=1}^{3} \sum_{\lambda=1}^{2} \int \frac{dk}{(2\pi)^3|k|} \int \frac{dQ^0}{2Q^0} f_R(Q^0) \frac{|\mathcal{M}' g^R_{K^+K^-}|^2}{2M_\phi} (2\pi) \delta(M_\phi - |k| - Q^0).
\]

Of course, because of three-momentum conservation, \( Q = -k \) in the center of mass frame where the \( \phi \) is at rest. We have included energy distributions for the scalar resonances \( f_R(Q^0) \) in order to take care of their finite widths. This is not considered in the literature for the calculation \( \Gamma(\phi \to \gamma R) \) since well defined values for the masses of the scalar resonances \( f_0(980) \) and \( a_0(980) \) are assumed and the standard two body decay formula, eq. (3.10), is then applied, see e.g. [13, 23, 27]. We should note that this is a very unstable procedure since the decay width scales as \( |k|^3 \), where \( |k| \) is the photon three-momentum:

\[
|k| = \frac{M_\phi^2 - Q^2}{2M_\phi}.
\]

Allowing a change of \( m_R \) of just 10 MeV from the central value \( m_R = 980 \) MeV, the quantity \( |k| \) changes by a factor of 1.7 and \( |k|^3 \) by a factor of five. The standard two body decay formula can be obtained from the more general eq. (3.2) by taking \( f_R(Q^0) = \delta(Q^0 - \sqrt{Q^2 + m_R^2}) \).

In order to fix \( f_R(Q^0) \) we invoke unitarity which implies:

\[
2 \text{Imag}[t^R_{K^+K^-\to K^+K^-}] = (K^+K^-|T^\dagger T|K^+K^-) \quad (3.4)
\]

Now, assuming that unitarity is saturated by the exchange of the corresponding scalar resonance \( R \), it follows that,

\[
2 \text{Imag}[t^R_{K^+K^-\to K^+K^-}] = \int \frac{d^4q}{(2\pi)^3|q|^4} f_R(q^0)(2\pi)^4 \delta^4(Q - q) |g^R_{K^+K^-}|^2
\]

\[
= \frac{\pi}{Q^0} f_R(Q^0) |g^R_{K^+K^-}|^2 \quad (3.5)
\]

and then,

\[
\frac{1}{2Q^0} f_R(Q^0) |g^R_{K^+K^-}|^2 = \frac{1}{\pi} \text{Imag}[t^R_{K^+K^-\to K^+K^-}(Q^2)] \quad (3.6)
\]
The previous equation gives our final form for \( f_R(Q^0) \). We show in fig. 3 the combination \( \pi f_R(Q^0)/2Q^0 = \text{Im} \left[ i^{R+K-K} (Q^2) \right] \) for the \( f_0(980) \), left panel, and \( a_0(980) \), right panel. The dashed lines correspond to the matrix elements \( i^{R+K-K} (Q^2) \) given in ref. [31], where the only one free parameter is fixed in this reference by a successful study of the strong \( \pi\pi \), \( KK \) and \( \pi\eta \) S-wave meson-meson interactions. These T-matrices have been used by now in a whole series of studies about the scalar sector stressing the role of unitarity and chiral symmetry in order to understand the scalar dynamics. These studies comprise strong interactions [31], fusion of two photons to \( \pi^0\pi^0 \), \( \pi^+\pi^- \), \( K^+K^- \), \( K^0\bar{K}^0 \) and \( \pi^0\eta \) [38], \( \phi \) radiative decays [6, 8], \( J/\Psi \) decays [32] and pp collisions [9]. In the same figure we also show by the solid lines, \( \text{Im} \left[ i^{R+K-K} (Q^2) \right] \) from the T-matrices of ref. [34] with the use of the Inverse Amplitude Method [39]. We also compare our choice for \( \pi f_R(Q^0)/2Q^0 \) from the T-matrices of refs. [31, 34], with the one that results by taking,

\[
\text{Im} \left[ i^{R+K-K} (Q^2) \right] = -\text{Im} \left[ \frac{(g_{KK}^{R})^2}{D_R(Q^2)} \right],
\]

\[
D_R(Q^2) = Q^2 - m_R^2 - \text{Re} \Pi(Q^2) + \Pi(Q^2),
\]

from ref. [12], since this resonant form was followed in the experimental references [1, 2, 3, 4, 5] to fit their data. The energy distribution resulting from eq. (3.7) is shown in fig. 3 by the thin dotted lines. The free parameters in eqs. (3.7), two masses and four couplings, are fixed to the central values given by the best fits of refs. [3, 4] to \( d\Gamma(\phi \rightarrow \gamma\pi^0\pi^0)/d\mu_{\pi\pi} \) and \( d\Gamma(\phi \rightarrow \gamma\pi^0\eta)/d\mu_{\eta\eta} \), respectively. The \( \Pi_R(Q^2) \) is the sum of two-meson self energies, \( \pi\pi \) and \( KK \) for \( I = 0 \) and \( \pi\eta \) and \( K\bar{K} \) for \( I = 1 \) as in refs. [3, 4]. Explicit formulae for \( \Pi(Q^2) \) can be found in ref. [12]. It is interesting to remark the presence of a long tail in each of the energy distributions \( f_R(Q^0) \) towards low invariant masses. This is a result of keeping the real parts of the unitarity two-meson loops [31, 34] or the real parts in the two-meson self energies in \( D_R(Q^2) \), eq. (3.7) [12].

![Figure 3](image-url)

Figure 3: \( \pi f_R(Q^0)/2Q^0 = \text{Im} \left[ i^{R+K-K} (Q^2) \right] \) for the \( f_0(980) \), left panel, and \( a_0(980) \), right panel. The solid and dashed lines are calculated from the T-matrices of refs. [34, 31], respectively, and the thin dotted ones from the resonant form given in eq. (3.7) following ref. [12] and the experimental analyses [3, 4]. The values of the masses and couplings, necessary to apply eq. (3.7), are determined by the best fits of the latter references.

Then, \( \Gamma(\phi \rightarrow \gamma R) \), eq. (3.2), can be written from eqs. (3.1) and (3.6) as:

\[
\Gamma(\phi \rightarrow \gamma R) = \int |k| |d| |k| \frac{1}{24\pi^2 m_\phi} \text{Im} \left[ i^{R} (Q^2) \right] |\mathcal{H}(Q^2)|^2 (M_\phi^2 - Q^2)^2,
\]

(3.8)
where we have used the shorter notation $t_{22}^R$ instead of $t_{K+K^-\to K^+K^-}^R$. Of course, one should keep in mind that $Q^2$ and $|k|$ are not independent variables but related by energy-momentum conservation, eq. (3.3).

It is also apparent from the previous equation the cubic dependence on the photon three-momentum $|k|$. From eq. (3.8) we can also define the differential decay widths,

$$\frac{d\Gamma(\phi \to \gamma R)}{d|k|} = \frac{M_\phi}{\sqrt{Q^2}} \frac{d\Gamma(\phi \to \gamma R)}{d\sqrt{Q^2}} = \frac{|k|d|k|}{24\pi^2 M_\phi} \text{Imag} \left( t_{22}^R(Q^2) \right) \left| \mathcal{H}^I(Q^2) \right|^2 (M_\phi^2 - Q^2)^2 . \tag{3.9}$$

Note that $\text{Imag} t_{22}^R$ extends from the two pion threshold for the $f_0(980)$ case and from the $\pi\eta$ one for the $a_0(980)$ resonance.

As stated above, the standard two body decay formula for $\phi \to \gamma R$, with a well defined mass $m_R$ for $R$, can be obtained from eq. (3.8) by performing the substitution $f_R(Q^0) = \delta(Q^0 - w_R)$. In this case the decay width is given by:

$$\Gamma(\phi \to \gamma R)_{\text{fixed}} = \frac{|k|^3}{12\pi} |\mathcal{H}^I(m_R^2) g_{K+K^-}^R|^2 . \tag{3.10}$$

Following the previous lines, we can also derive an expression for the decay widths $\Gamma(\phi \to \gamma M^0 N^0)$ by considering explicitly that the strong amplitudes $t_{K+K^-\to M^0N^0}^R$ are dominated by the exchange of the corresponding $R$ resonances. One should keep in mind that the results given in sec. 2 do not make such assumption and are derived directly in terms of the corresponding strong amplitudes $t_{K+K^-\to M^0N^0}^R$.

Let us denote by $q_1$ and $q_2$ the three-momenta of the $M^0$ and $N^0$ particles and by $w_1$ and $w_2$ their corresponding energies. Then, since we are considering the decay $\phi \to \gamma M^0 N^0$ to be mediated by the resonance $R$, if follows that,

$$\Gamma(\phi \to \gamma M^0 N^0) = \frac{1}{3} \sum_{\nu=1}^3 \sum_{\lambda=1}^2 \frac{1}{2M_\phi} \int \frac{dk}{(2\pi)^3 |k|} \left| \frac{\mathcal{M}(\phi \to \gamma R)}{\mathcal{D}_R(Q^2)} \right|^2 \sqrt{Q^2} \times \left\{ \frac{1}{2\sqrt{Q^2}} \int \frac{dp_1}{(2\pi)^3 |p_1|} \frac{2\pi \delta(Q^0 - w_1 - w_2)}{2w_1} |g_{M^0N^0}^R|^2 \right\} , \tag{3.11}$$

where $\mathcal{D}_R(Q^2)$ is any possible inverse propagator for the resonance $R$. Let us note that the term between curly brackets just corresponds to the standard formula for the decay width of a resonance of mass $\sqrt{Q^2}$ to the $M^0N^0$ state with a coupling $g_{M^0N^0}^R$. In addition, we can rewrite,

$$\left| \frac{\mathcal{M}}{\mathcal{D}_R(Q^2)} \right|^2 = \left| \frac{\mathcal{M} g_{K+K^-}^R(Q^2)}{\mathcal{D}_R(Q^2)} \right|^2 = |\mathcal{M}'|^2 |t_{22}^R(Q^2)|^2 . \tag{3.12}$$

On the other hand, within our present assumption that $t_{22}^R(Q^2)$ is dominated by the exchange of the scalar resonance $R$, unitarity implies:

$$\text{Imag}[t_{22}^R(Q^2)] = \frac{1}{2} \sum_{\alpha} t_{22}^R(Q^2) s q_{\alpha} \frac{\theta(Q^2 - W_{\alpha}^2)}{4\pi \sqrt{Q^2}} t_{22}^R(Q^2) = |t_{22}^R(Q^2)|^2 \sum_{\alpha} |g_{\alpha}^R|^2 q_{\alpha} \frac{\theta(Q^2 - W_{\alpha}^2)}{8\pi \sqrt{Q^2}}$$

$$= |t_{22}^R|^2 \sqrt{Q^2} \Gamma_{R,\text{tot}}(Q^2) . \tag{3.13}$$

Where $q_{\alpha}$ is the center of mass three-momentum of the two-body channel $\alpha$, $W_{\alpha}$ is the corresponding threshold energy and $\Gamma_{R,\text{tot}}(Q^2)$ is the total energy of a scalar resonance of mass $\sqrt{Q^2}$ with the same couplings, indicated by $g_{\alpha}^R$, as the $R$ resonance.
As a result of eqs. (3.12), (3.13) we can then write instead of eq. (3.11):

\[
\Gamma (\phi \to \gamma M^0 N^0) = \frac{1}{3} \sum_{\nu=1}^{3} \sum_{\lambda=1}^{2} \frac{1}{2M_\phi} \int \frac{dk}{(2\pi)^3|k|} |M'_\nu|^2 \text{Imag} \left[ t_{22}^R(Q^2) \right] \frac{\Gamma_{R,M^0N^0}(Q^2)}{\Gamma_{R,tot}(Q^2)}
\]

\[
= \int \frac{|k|dk}{24\pi^2M_\phi} |H^I(Q^2)|^2 \text{Imag} \left[ t_{22}^R(Q^2) \right] (M_\phi^2 - Q^2)^2 Br_{M^0N^0}(Q^2),
\]

(3.14)

where \( \Gamma_{R,M^0N^0}(Q^2) \) is the decay width of a scalar resonance with mass \( \sqrt{Q^2} \) and the same coupling \( g_{M^0N^0}^R(Q^2) \) to the \( M^0N^0 \) channel as the \( R \) resonance at that energy. This parameterization can be seen as an alternative to that of ref. [12] and is specially suited for those approaches that directly work with the strong amplitudes, without considering specific forms for the propagators of the scalar resonances which are explicitly needed in the legitimate parameterization of ref. [12].

Consistently with eq. (3.13), the branching ratio \( Br_{M^0N^0}(Q^2) \) of the strong decay \( R(Q^2) \to M^0 N^0 \) is calculated as:

\[
Br_{M^0N^0}(Q^2) = \sum_\alpha \frac{|g_{\alpha}^R|^2 P_{M^0N^0}}{|g_{\alpha}|^2 p_\alpha \theta(Q^2 - W^2)},
\]

(3.15)

where in addition one should include, when necessary, the corresponding \( 1/2 \) factors for the decays to symmetric two particle states. For the sake of completeness we show in table [I] the pole positions, \( \sqrt{s_R} \) and couplings, \( g_{\alpha}^R \), for the \( f_0(980) \) and \( a_0(980) \) resonances from the T-matrices of ref. [31]. The latter are defined by:

\[
|g_{\alpha}^R| = \lim_{s \to s_R} |(s - s_R)t_{\alpha \to \beta}(s)|,
\]

(3.16)

where \( s_R \) is the pole position of the \( R \) resonance in the unphysical sheet where the center of mass three-momentum of the \( M^0 N^0 \) state has negative imaginary part, while that of the \( K\bar{K} \) three-momentum is positive. It is also worth noticing that the couplings \( |g_{\alpha K\bar{K}}^R| \) are different for the \( f_0(980) \) and \( a_0(980) \) resonances like in ref. [19].

Taking from table [I] the values of the couplings and pole positions one calculates from eq. (3.13), \( Br_{f_0}^0 = 0.7 \) and \( Br_{a_0}^0 = 0.63 \) in perfect agreement with the branching ratios given in ref. [31] where the finite widths of the \( f_0(980) \) and \( a_0(980) \) are taken into account for the strong decay rates as well.

|        | \( f_0(980) \) | \( a_0(980) \) |
|--------|----------------|----------------|
| \( \sqrt{s_{f_0}} \) = (992.6 - i 11.8) MeV | \( \sqrt{s_{a_0}} \) = (1009.2 - i 56.1) MeV |
| \( |g_{\pi\pi}^{f_0}| = 1.90 \text{ GeV} \) | \( |g_{\pi\eta}^{a_0}| = 3.54 \text{ GeV} \) |
| \( |g_{K\bar{K}}^{f_0}| = 3.80 \text{ GeV} \) | \( |g_{K\bar{K}}^{a_0}| = 5.20 \text{ GeV} \) |

Table 1: Poles positions, \( \sqrt{s_R} \), and couplings, \( g_{\alpha}^R \), of the \( f_0(980) \) and \( a_0(980) \) resonances from the T-matrices of ref. [31].

Comparing eq. (3.14) with eq. (3.8) for the decay \( \phi \to \gamma R \) one obtains the relation,

\[
\frac{d\Gamma (\phi \to \gamma M^0 N^0)}{d|k|} = \frac{d\Gamma (\phi \to \gamma R)}{d|k|} Br_{M^0N^0}(Q^2),
\]

(3.17)

#4Since the \( I = 0 \) \( \pi\pi \) channel is completely symmetric under the exchange of the pions, the convention of including an extra factor \( 1/\sqrt{2} \) in the definition of this state is followed in ref. [31]. Thus, the \( \pi\pi \) coupling resulting by applying directly eq. (3.14) to the \( I = 0 \) T-matrix of ref. [31] has been multiplied by \( \sqrt{2} \) in table [I].
which, together with eq. (3.9), can be used to analyze in a pure phenomenological way the experimental data of $\phi \to \gamma M^0 N^0$ by parameterizing and fitting $\text{Imag}[t^{R}_{K^+K^-\to K^+K^-}]$, as e.g. in eq. (3.7). Because we are assuming that the decay $\phi \to \gamma M^0 N^0$ is dominated by the exchange of the corresponding $R$ resonance, the previous relation tells us that the differential width $\phi \to \gamma M^0 N^0$ is the same as the one to $\gamma R$ multiplied by the probability that the scalar resonance decays to $M^0 N^0$. The latter is just the branching ratio since the resonance will certainly decay to any channel at asymptotic times. This natural result is indeed the way the experimental values on the $\phi \to \gamma f_0(980)$ and $\phi \to \gamma a_0(980)$ widths are obtained once the rates $\phi \to \gamma f_0(980) \to \gamma \pi^0 \pi^0$ and $\phi \to \gamma a_0(980) \to \gamma \pi^0 \eta$, respectively, are determined by fitting the experimental data [1, 3, 4, 6]. Nevertheless, one should keep in mind that eq. (3.17), when used with the simplified form of $B^{R}_{M^0 N^0}$ given in eq. (3.15), is an approximation since it does not take into account the finite widths of the $f_0(980)$ and $a_0(980)$ resonances in their strong decay rates.

Let us stress that eq. (2.13) is given in terms of the strong $S$-wave $T$-matrices of ref. [31, 34] without any explicit resonance field. Indeed, in ref. [31] the scalar resonances $f_0(980)$ and $a_0(980)$, together with the $\sigma$ meson, are generated dynamically as meson-meson resonances in terms of an interacting kernel given by the lowest order CHPT amplitudes [35, 36] without any explicit resonance field.

Thus, it is an important consistency check whether the invariant mass distribution obtained from eq. (2.13) agrees indeed with the one of eq. (3.17) when substituting in the latter $d\Gamma(\phi \to \gamma R)/|d{k}|$ from eq. (3.7). It is straightforward to demonstrate by relating $\text{Imag}[t^{R}_{K^+K^-\to K^+K^-}]$ and $|t^{R}_{K^+K^-\to M^0 N^0}(Q^2)|^2$ via unitarity, that eq. (2.13) and the combination of eqs. (3.17) and (3.9) yield the same $d\Gamma(\phi \to \gamma M^0 N^0)/|d{k}|$ below the $KK$ threshold. Above it, a specific energy dependent form of $B^{R}_{M^0 N^0}$ should be employed.

Figure 4: Invariant mass distributions $d\Gamma(\phi \to \gamma \pi^0 \pi^0)/dm_{\pi\pi} 10^8$ and $d\Gamma(\phi \to \gamma \pi^0 \eta)/dm_{\pi\eta} 10^8$ from left to right, respectively. The thick lines correspond to a simultaneous fit of the parameters $\zeta_I$ and $\delta G^I$ to the experimental points of $\square$ (empty circles) and $\blacksquare$ (full circles), for the $\gamma \pi^0 \pi^0$ final state, and to the data of $\square$ (empty circles) and $\blacksquare$ (full circles) for the $\gamma \pi^0 \eta$ one. The thin line corresponds to $\zeta_I = -G_V/\sqrt{8}$ and $\delta G^I = 0$, the appropriate values to reproduce the results of ref. [3].

# In ref. [3] an extra and large $\sigma \gamma$ contribution is included in addition to the $f_0(980)\gamma$ one with a destructive interference at low $\pi\pi$ invariant mass. However, this is a controversial result at odds with the conclusions of the previous experimental analysis refs. [3, 4], giving rise to an unconventionally large rate $\phi \to \gamma f_0(980)$ $\blacksquare$, larger than the ones given from any model.
4 Results and discussion

In sec. 2 we have derived the corresponding expressions for the $\phi \to \gamma M^0 N^0$ decay by taking into account the final state interactions due to the strong S-wave meson-meson amplitudes. This is what we denote by the scalar contribution to such decay rates. Nevertheless, for the $\gamma\pi^0\pi^0$ final state the background $\phi \to \pi^0\rho \to \gamma\pi^0\pi^0$ is not negligible, its relative size depending strongly on the energy region, as we show below in fig. 3. In our final results we have included the interference term between the scalar and $\rho\pi^0$ contributions with our own scalar amplitudes and with the vector part calculated as in ref. [22]. For the appropriate formulae for the vector piece we refer to that reference.

We now perform a simultaneous fit, including both the scalar and vector contributions, to the $\phi \to \gamma\pi^0\pi^0$ data of refs. [3, 4] and those of $\phi \to \gamma\pi^0\eta$ of refs. [31, 34]. The fit to the invariant mass distributions $d\Gamma(\phi \to \gamma\pi^0\pi^0)/dm_{\pi\pi}$ and $d\Gamma(\phi \to \gamma\pi^0\eta)/dm_{\pi\eta}$ with the T-matrices of refs. [31, 34] is presented in fig. 4 in the left and right panel, respectively. The solid lines corresponds to the final state interactions calculated from the scalar amplitudes of ref. [34] and the dashed ones to the amplitudes of ref. [31]. The fit reproduces fairly well the experimental invariant mass distributions and the resulting values of the parameters are:

| T-matrix | $\zeta_0 = +164.12$ MeV | $\delta G^0 = 1.46/(4\pi)^2$ |
|----------|------------------------|---------------------------|
| ref. [31] | $\zeta_1 = -165.87$ MeV | $\delta G^1 = 1.36/(4\pi)^2$ |

| T-matrix | $\zeta_0 = +124.99$ MeV | $\delta G^0 = 1.61/(4\pi)^2$ |
|----------|------------------------|---------------------------|
| ref. [34] | $\zeta_1 = -132.26$ MeV | $\delta G^1 = 1.44/(4\pi)^2$ |

Table 2: Values of the parameters from a simultaneous fit to $d\Gamma(\phi \to \gamma\pi^0\pi^0)/dm_{\pi\pi}$ [3, 4] and $d\Gamma(\phi \to \gamma\pi^0\eta)/dm_{\pi\eta}$ [31, 34].

From the previous table of values is clear an obvious symmetry in the results. We have obtained that $\delta G^0 \simeq \delta G^1$ and $\zeta_0 \simeq -\zeta_1$. The $\zeta_I$ parameters represent the direct coupling $\phi\gamma KK$ with isospin $I$, eq. (2.3). Hence the direct coupling for the $K^+K^-$ channel as given by eq. (2.3) and the values in table 2 is proportional to $-(\zeta_1 + \zeta_0)/\sqrt{2} \simeq 0$ and vanishes. On the contrary, the one for $K^0\bar{K}^0$ is $(\zeta_1 - \zeta_0)/\sqrt{2} \simeq \sqrt{2}\zeta_1$. Let us note as well that because $\delta G^0 \simeq \delta G^1$ one has that $G^1 \simeq G^0$. Since the Clebsch-Gordan coefficients for $K^+K^-$ are the same both for $I = 0$ and $I = 1$ its contribution to any process is just proportional to $G^0\zeta_0 + G^1\zeta_1 \simeq 0$ and cancels, as one should expect since the direct coupling $\phi\gamma K^+K^-$ on the left vertex of the diagram of fig. 4 vanishes as we have just seen. Then it is clear the consistency of having obtained within the isospin formalism that $G^1 = G^0$ once $\zeta_0 = -\zeta_1$, otherwise there would have been a mismatch between the results obtained from the physical basis of states and that of isospin. For the $K^0\bar{K}^0$ channels, since the Clebsch-Gordan coefficients change of sign when passing from $I = 0$ to $I = 1$, one has the non-vanishing result $G^1\zeta_1 - G^0\zeta_0 \simeq 2G^1\zeta_1$. Thus our results suggest the existence of a $\phi\gamma K^0\bar{K}^0$ local term of the same type as that of eq. (2.3) but with $\zeta_I$ replaced by $\zeta_I \equiv \sqrt{2}\zeta_1$ and the absence of such terms for $K^+K^-$. The resulting value for $|\zeta_I|$ is very similar to $F_{\rho}$ from $\rho \to e^+e^-$, $F_{\rho} = 154$ MeV or from $\phi \to e^+e^-$, $F_{\rho} = 165$ MeV. The value for $\delta G^I$ is of natural size since it is a number of order one over $16\pi^2$. It is also important to remark that in the reproduction of the lowest energy part of the $d\Gamma(\phi \to \gamma\pi^0\pi^0)/dm_{\pi\pi}$ invariant mass distribution, namely below 600 MeV, the background $\rho\pi^0\pi^0$ plays a significant role, while its negligible above 700 MeV in agreement with ref. [12]. From the integration of the invariant mass distributions we obtain the branching ratios:

$$Br(\phi \to \gamma\pi^0\pi^0) = 1.09 \times 10^{-4},$$

$$Br(\phi \to \gamma\pi^0\eta) = 0.72 \times 10^{-4}.$$  

(4.1)
for the T-matrices of ref.[34]. The values of the previous branching ratios when using the strong amplitudes of ref.[31] are almost the same:

\[
Br(\phi \to \gamma\pi^0\pi^0) = 1.09 \times 10^{-4}, \\
Br(\phi \to \gamma\pi^0\eta) = 0.73 \times 10^{-4}.
\]

For comparison, the precise experimental results from refs.[5,6] are \( Br(\phi \to \gamma\pi^0\pi^0) = (1.09 \pm 0.03_{\text{stat}} \pm 0.05_{\text{sys}}) \times 10^{-4} \) and \( Br(\phi \to \gamma\pi^0\eta) = (0.796 \pm 0.07) \times 10^{-4} \), respectively, in excellent agreement with our results.

In fig.[5] we present a new fit to the same data as before but imposing the constraints \( \delta G^0 = \delta G^1 \) and \( \zeta_0 = -\zeta_1 \) that have emerged in a consistent way from the fit described above. The values obtained are:

\[
\begin{align*}
\text{T-matrix of ref.[31]} & \quad \zeta_0 = -\zeta_1 = +180.83 \text{ MeV}, \quad \delta G_0 = \delta G_1 = 1.42/(4\pi)^2, \\
\text{T-matrix of ref.[34]} & \quad \zeta_0 = -\zeta_1 = +146.42 \text{ MeV}, \quad \delta G_0 = \delta G_1 = 1.54/(4\pi)^2. 
\end{align*}
\]

The quality of the fit is similar to that of fig.[4]. In the same figure we also show by the thin dotted line the \( \rho^0 \pi^0 \) intermediate contribution which as told above turns out to be relevant only in the lowest energy range of the spectrum. Let us note as well that in the experimental analyses at least 6 free parameters [1, 2, 3] are needed to fit the experimental \( \phi \to \gamma\pi^0\eta \) and \( \gamma\pi^0\pi^0 \) invariant mass distributions.

![Figure 5: Invariant mass distributions](image)

Figure 5: Invariant mass distributions \( dBr(\phi \to \gamma\pi^0\pi^0)/dm_{\pi\pi} \times 10^8 \) and \( dBr(\phi \to \gamma\pi^0\eta)/dm_{\pi\eta} \times 10^8 \) from left to right, respectively. The thick lines correspond to a simultaneous fit of the parameters \( \zeta_I \) and \( \delta G^I \) to the experimental points of [3] (empty circles) and [3] (full circles), for the \( \gamma\pi^0\pi^0 \) final state, and to the data of [3] (empty circles) and [3] (full circles) for the \( \gamma\pi^0\eta \) one. The constraints \( \delta G^1 = \delta G^0 \) and \( \zeta_0 = -\zeta_1 \) have been imposed. The thin dotted line corresponds to the \( \rho\pi^0 \) background.

Once the \( \zeta_I \) and \( \delta G^I \) are fixed, table [3] and eq. (4.3), we show in fig.[5], from left to right, the distributions \( dBr(\phi \to \gamma f_0(980))/dm_{f_0} \times 10^8 \) and \( dBr(\phi \to \gamma a_0(980))/dm_{a_0} \times 10^8 \), respectively, calculated from eq. (3.9) and the strong amplitudes of ref.[31, 34]. As in figs.[4] the solid lines corresponds to ref.[34] and the dashed ones to ref.[31]. We have denoted by \( m = \sqrt{Q^2} \) the invariant mass of the \( f_0(980) \) and \( a_0(980) \) resonances. It can be surprising that the tails of the invariant mass distributions extend well below the prominent peaks of the \( f_0(980) \) and \( a_0(980) \) resonances towards rather low invariant masses. This is due to the cubic dependence on the photon three-momentum \( |k| = (M_\phi^2 - Q^2)/2M_\phi \) which largely enhances the low energy part of the invariant mass distributions. Indeed, if we fix \( |k| \) to the value corresponding to some nominal mass of the resonances, e.g. \( m_{f_0} \approx 986 \text{ MeV} \), then one obtains very peaked distributions around the masses of the \( f_0(980) \) and \( a_0(980) \) without any tail towards low energies. Hence, it follows
that although the resonance structure is clear and very prominent the low energy components of the
energy distributions $f_R(Q^0)$, see fig. 3, cannot be neglected because of the enhancement due to the cubic
dependence on the photon three-momentum, fig. 4.

Figure 6: Invariant mass distributions $d\Gamma(\phi \to \gamma R)/d\sqrt{Q^2}$ from left to right, respectively. The solid and dashed lines are calculated from the T-matrices of the
refs. [34, 31], respectively.

We integrate now the invariant mass distributions $d\Gamma(\phi \to \gamma R)/d\sqrt{Q^2}$ from the corresponding thresholds up to $Q^2 = M_0^2$, eq.(2.12), and we obtain the values shown in table 3.

| $Br(\phi \to \gamma f_0(980))$ | T-matrix ref. [31] | T-matrix ref. [34] |
|-------------------------------|--------------------|--------------------|
| $(980)) = 2$                  | 3.19               | 3.11               |
| $(980)) = 0.73$               | 0.73               | 0.73               |
| $(980))$ 10$^4$               | 4.37               | 4.26               |

Table 3: $Br(\phi \to \gamma f_0(980))$, $Br(\phi \to \gamma a_0(980))$ and the quotient between both rates is then 2.72.

In this table the energy distributions $f_R(Q^0)$, eq.(3.6), are determined from the T-matrices of refs. [34, 31]. For $\zeta_I = -G_V/\sqrt{8}$ and $\delta G^I = 0$, which corresponds to the results of ref. [3], one has $Br(\phi \to \gamma f_0(980)) = 2.34 \times 10^{-4}$, $Br(\phi \to \gamma a_0(980)) = 0.86 \times 10^{-4}$ and the quotient between both rates is then 2.72.

The SND collaboration, refs. [3] and [4], reports the branching ratios $Br(\phi \to \gamma f_0(980)) = (3.5 \pm 0.3 \pm 0.3\% )10^{-4}$ and $Br(\phi \to \gamma a_0(980)) = (0.88 \pm 0.17) \times 10^{-4}$, respectively. The CMD-2 collaboration, ref. [4], reports $Br(\phi \to \gamma f_0(980)) = (2.90 \pm 0.21 \pm 1.54) \times 10^{-4}$. Taking into account the latter value for $Br(\phi \to \gamma f_0(980))$, as in ref. [11] since it arises from a combined fit to the $\gamma^0 \pi^0$ and $\gamma^\pm \pi^\mp$ data [30] and the one of ref. [4] for $Br(\phi \to \gamma a_0(980))$, one has $Br(\phi \to \gamma f_0(980))/Br(\phi \to \gamma a_0(980)) \simeq 3.3 \pm 2.0$. These numbers are based in a parameterization of the experimental event distributions of $\pi^0 \pi^0$ and $\pi^0 \eta$ leaving the masses and couplings of the $f_0(980)$ and $a_0(980)$ as free parameters. Once these are determined by the fitting procedure, the branching ratios to $\gamma R$ are obtained by dividing the resulting $Br(\phi \to \gamma f_0(980))$ by the corresponding $Br_R^{M_0 N_0}$. There is some model dependence on the

#In addition, the error in the result $Br(\phi \to \gamma f_0(980)) = (2.90 \pm 0.21 \pm 1.54) \times 10^{-4}$ of ref. [4] is enlarged in this reference so that the rate $\phi \to \gamma f_0$, determined by performing a narrow pole fit, is also compatible within errors.
“experimental” numbers because of the specific forms of the resonance propagators. Nevertheless, we see remarkable agreement between these experimental results and our calculations of table 3. In ref. 3 the KLOE collaboration reports the result $Br(\phi \to \gamma f_0(980)) = (4.47 \pm 0.21) \times 10^{-4}$. This value is obtained by including a very specific destructive interference in the low energy region between the $f_0(980)\gamma$ and a $\sigma\gamma$ contribution, both of them very large at such energies. The $Br(\phi \to \gamma a_0(980))$ is much clearer and very well established due to the absence of any significant background contributions to $\phi \to \gamma\pi^0\eta$ as established both theoretically and experimentally, refs. [19, 17, 22, 4, 6]. The KLOE collaboration [6] determines the number $Br(\phi \to \gamma a_0(980)) = (0.74 \pm 0.07) \times 10^{-4}$ from the very precise measured rate $Br(\phi \to \gamma\pi^0\eta) = (0.796 \pm 0.07) \times 10^{-4}$. Our calculations for both rates presented in eq. (4.1) and in table 3 are in perfect agreement with these determinations since we described very accurately the corresponding invariant mass distribution as noted in figs. 3 and 4.

| Table 4: $Br(\phi \to \gamma f_0(980))$ and $Br(\phi \to \gamma a_0(980))$ branching ratios where the energy distribution $f_R(Q^0)$ is fixed from eq. (3.7) and then eq. (2.12) is used. The results reported in the experimental references [3, 4] are also given. In the second row of numbers, the function $H^I(Q^2)$, eq. (2.12) is used with $\zeta_I = 0$, the choice that corresponds to refs. [3, 4]. |
| --- |
| $Br(\phi \to \gamma f_0(980))$ & $Br(\phi \to \gamma a_0(980))$ |
| reported & $Br(\phi \to \gamma f_0(980))$ & $Br(\phi \to \gamma f_0(980))$ & $Br(\phi \to \gamma a_0(980))$ & $Br(\phi \to \gamma a_0(980))$ |
| eq. (2.12), $\zeta_I = 0$ & 3.21 & 3.51 & 4.6 & 0.88 |
| 2.90 & 2.05 & 4.6 & 0.88 |

We now apply eq. (2.12) to the energy distributions $f_R(Q^0)$ coming from eq. (3.7), where the free parameters, the masses of the $f_0(980)$, $a_0(980)$ and four couplings, take the values of several fits of refs. [1, 3, 4] used to obtain their experimental numbers of the rates $\phi \to \gamma R$. The calculation is performed with the function $H^I(Q^2)$ calculated as in refs. [3, 4], that is, with $\zeta_I = 0$ in eq. (2.12). This corresponds to the second row of numerical results of table 3. In this way, the previous decay widths are calculated without the shortcut to use a fixed value for the branching ratio of the resonance $R$ to the lightest decay channel in all the energy interval up to $M_\phi$, as in the present experimental analyses [1, 3, 4, 3, 1].

In ref. [3] the quoted $Br(\phi \to \gamma f_0(980)) = (3.5 \pm 0.3^{+1.3}_{-0.5}) \times 10^{-4}$ comes by multiplying by three the measured $Br(\phi \to \gamma\pi^0\pi^0)$ with $Br_{\pi^0\pi^0}(Q^2) = 1/3$, see eq. (3.17). In table 3 the quoted value from ref. [3] is the one obtained with a fit to $\phi \to \gamma\pi^0\pi^0$ invariant mass distribution including as well a background from $\rho\pi^0$, which amounts at most to 15%. The error is correspondingly enlarged so as to make both results compatible. For the $a_0(980)$ case the only number that is reported, both in table 3 and ref. [1], is the same as the measured $Br(\phi \to \gamma\pi^0\eta)$ assuming $Br_{\pi^0\eta}(Q^2) = 1$. The fit 1 for the CMD-2 collaboration includes both the $\gamma\pi^0\pi^0$ and $\gamma\pi^+\pi^-$ final states. The CMD-2 second fit includes only the $\gamma\pi^0\pi^0$ final state. We see a general agreement between the reported and calculated numbers from eqs. (3.7), (3.8) and (2.12), given in the first and second rows of numerical results, respectively, although the latter tend to be somewhat larger, particularly for the reported values of the CMD-2 collaboration.

It has been recently claimed in ref. [23] that in order to interpret the experimental results of refs. [3, 4] for the rates $\phi \to \gamma R$ one needs to include sizeable isospin violating effects in the couplings of the $f_0(980)$ and $a_0(980)$ resonances to the $K^+K^-$ and $K^0\bar{K}^0$ channels, so that $|g^R_{K^+K^-}| \neq |g^R_{K^0\bar{K}^0}|$ and the difference is claimed to be as large as a 30%. Particular emphasis is given to the necessity to deviate
from isospin symmetry in order to understand the quotient between the branching ratios of the $\phi$ to $\gamma f_0(980)$ and $\gamma a_0(980)$ that, although with a large experimental uncertainty, as shown above, has a central value of around three instead of one. The much more precise results of refs.\[5, 6\], when taking $Br(\phi \to \gamma f_0(980)) = 3 Br(\phi \to \gamma^0 \pi^0)$ and $Br(\phi \to \gamma a_0(980)) = Br(\phi \to \gamma^0 \eta)$, imply a value $4.1 \pm 0.2$ for this ratio, very close to our results of table 3. Indeed in our approach the calculated $Br(\phi \to \gamma f_0(980))$ from eq.(2.12) and the one obtained by multiplying by three the $Br(\phi \to \gamma^0 \pi^0)$ differ in less than a 6%. Our study clearly shows that one can achieve a good agreement with the experimental data without any deviation from isospin symmetry in the couplings of the $f_0(980)$ and $a_0(980)$ resonances to the $K^+ K^-$ and $K^0 \bar{K}^0$ channels, although a contact interaction term $\phi \gamma K^0 \bar{K}^0$, beyond the pure $K^+ K^-$ loop model of ref.\[12\], has to be included as described above. It is also clear that one should abandon in the study of the $\phi \to \gamma R$ decays the standard two body decay formula, eq.(3.10), used in refs.\[23, 25\], due to the proximity of the threshold of the final state to $M_{\phi}$ and the cubic dependence on $|k|$. As a result, the effects of the finite widths of the scalar resonances, and their associated energy distributions $f_R(Q^0)$, must be included from the very beginning. For instance, had we used the values for the $f_0(980)$ and $a_0(980)$ masses and the $g_{K^+ K^-}$ couplings given in table 1 we would have obtained $Br(\phi \to \gamma f_0) = 1.17 \times 10^{-4}$ and $Br(\phi \to \gamma a_0) = 0.58 \times 10^{-5}$. The latter value is so small due to the rather high pole mass of the $a_0(980)$. Let us stress that this pole mass is clearly different to the value where the S-wave $I=1$ T-matrix elements peak, around 986 MeV, which indeed changes from one matrix element to the other. Hence, it is rather artificial to decide which is the value of $m_R$ to be used for the calculation of the rate $\phi \to \gamma R$ in an extraordinary sensitive two-body standard decay formula to the chosen $m_R$ value.

It is worth mentioning that we have reproduced the numerical results of refs.\[5, 6\], $\zeta_I = 0$, and those of ref.\[8\], $\zeta_I = -G_V/\sqrt{8}$, $\delta G_I = 0$. Nevertheless, the $I(a, b)$ function in ref.\[7\] was evaluated with the mass of the $K^0$ although the mass of the $K^+$ should have been used since it corresponds to a $K^+ K^-$ loop. This kinematical violation gives rise to non-negligible corrections due to the proximity of the $K^0 \bar{K}^0$ threshold to the mass of the $\phi(1020)$. When this is taken into account, one has $Br(\phi \to \gamma K^0 \bar{K}^0) = 3.0 \times 10^{-8}$ instead of $5 \times 10^{-8}$ as given in ref.\[7\] where the T-matrices of ref.\[31\] were used. We now evaluate the previous branching ratio with our present formalism and with the values of the $\delta G_I$ and $\zeta_I$ as given in table 2. For the strong amplitudes of ref.\[31\] we obtain $Br(\phi \to \gamma K^0 \bar{K}^0) = 3.7 \times 10^{-8}$ and with the T-matrices of ref.\[34\] one has $Br(\phi \to \gamma K^0 \bar{K}^0) = 6.43 \times 10^{-9}$. The tree level contact interaction $\phi \gamma K^0 \bar{K}^0$ and its iteration through final state interactions interfere destructively and tend to cancel each other or even they give rise to a negative interference with the pure $K^+ K^-$ kaon loop contribution of ref.\[4\].

5 Conclusions

In this article we have shown how the experimental results for the decay widths $\Gamma(\phi \to \gamma f_0(980))$ and $\Gamma(\phi \to \gamma a_0(980))$ from refs.\[1, 3, 4\] can be described without abandoning isospin symmetry in the calculation of the S-wave strong T-matrix elements \[31, 34\]. Nevertheless, in our final results we have calculated the integral $I(a, b)$ in terms of the $K^+$ mass instead of the average isospin mass. This kinematical isospin violating fact amounts to effects of around a 10% in the rate $\phi \to \gamma f_0(980)$ and around a 20% in the width to $\gamma a_0(980)$. The $\gamma K^0 \bar{K}^0$ branching ratio is much more sensitive to these kinematical effects due to the so much reduced available phase space. The same situation would have arisen in the decays $\phi \to \gamma R$ as well if we had used the standard two body decay formula eq.(3.10), with well defined masses $m_R$, instead of having taken care of the finite width effects of the $f_0(980)$ and $a_0(980)$ resonances. These results are quite opposite to what has been claimed in ref.\[23\] regarding the necessity of including large isospin violating effects in the couplings of the $f_0(980)$ and $a_0(980)$ scalar resonances.
to the $K^+K^-$ and $K^0\bar{K}^0$ channels due to the proximity of the $K\bar{K}$ threshold to the nominal masses of the $f_0(980)$ and $a_0(980)$ resonances. We have also stressed that one should abandon the standard two body decay formula, with well defined masses for the $f_0(980)$ and $a_0(980)$ resonances, in the calculation of the rates $\phi \to \gamma f_0(980)$ and $\phi \to \gamma a_0(980)$. Instead, finite energy distributions, $f_R(Q^2)$, have to be considered from the very beginning because of the dramatic changes in the resulting decay widths under small changes of the $f_0(980)$ and $a_0(980)$ masses as compared to their widths or as compared to the difference between the pole masses and the energy of the peaks in the S-wave $I=0$ and 1 T-matrices.

It is also worth remarking that the formula derived in sec.3 for calculating the decay widths $\phi \to \gamma R$, eq.(3.8), together with eq.(3.17), can be also applied in the experimental analyses of the rates $\phi \to \gamma R$ as an alternative to the rightful one followed in refs.[1, 3, 4, 5, 6] from ref.[12]. In this way, one does not use, even as an intermediate step, the bare theoretical concept $\Gamma(\phi \to \gamma R; \sqrt{Q^2})$ and incorporates the contribution to $H^I(Q^2)$ proportional to $\zeta_I$ in eq.(2.12).

In order to describe the data we have included, beyond the $K^+K^-$ loop model of ref.[12], self-gauge invariant vertices $V_I(\phi \gamma K\bar{K})$ with the $K\bar{K}$ pair in the isospin channel $I$. As a result of the fit to the experimental data of refs.[3, 5, 6, 8] a non-vanishing $V(\phi \gamma K^0\bar{K}^0)$ local term has emerged while the corresponding $V(\phi \gamma K^+K^-)$ tend to vanish. It is also shown that the former plays an important role in order to reproduce the experimental data. We have also included this contribution to the $\phi \to \gamma K^0\bar{K}^0$ decay although its effects do not spoil here the conclusions of ref.[7], where only the $K^+K^-$ loop contribution is included, that this branching ratio is negligible small and does not offer any significant background to study CP violation in a $\phi$-factory like DAΦNE.

It has been repeatedly stated that the study of the $\phi(1020)$ radiative decays to $\gamma R$ constitutes an important test to unveil the nature of the $f_0(980)$ and $a_0(980)$ resonances by comparing the resulting experimental data with the models and approaches present in the literature. Indeed, we see from eq.(2.12) that the study of these decays constitutes an alternative source of experimental information on the $f_0(980)$ resonance since it is sensitive to $\text{Imag}[t_{22}^R]$ which is not directly measured in $\pi\pi$ scattering data. For the $a_0(980)$ resonance the experimental data is much more scarce than for the $f_0(980)$ and hence having new precise data, as that of ref.[6], is of foremost importance. Our simultaneous study of the new and accurate data of $[3]$ on $\phi \to \gamma \pi^0\eta$ together with that of $\phi \to \gamma \pi^0\pi^0$ $[3, 5]$ has given a coherent reproduction of these data in terms of only two free parameters. This constitutes a step forward in the experimental verification of the strong $T$-matrices of refs.[34, 31] already successfully tested in refs.[31, 38, 32, 34].

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