Phase Space Structure of
Non-Abelian Chern-Simons Particles

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Abstract

We investigate the classical phase space structure of \( N \, SU(n+1) \) non-Abelian Chern-Simons (NACS) particles by first constructing the product space of associated \( SU(n+1) \) bundle with \( \mathbb{CP}^n \) as the fiber. We calculate the Poisson bracket using the symplectic structure on the associated bundle and find that the minimal substitution in the presence of external gauge fields is equivalent to the modification of symplectic structure by the addition of field strength two form. Then, we take a direct product of the associated bundle by the space of all connections and choose a specific connection by the condition of vanishing momentum map corresponding to the gauge transformation, thus recovering the quantum mechanical model of NACS particles in Ref. [1].

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I. INTRODUCTION

Exotic statistics in two spatial dimensions has attracted much attention recently. It has relevance to various areas of physics and in particular could be realized in condensed matter physics. Particles exhibiting exotic statistics and carrying anomalous spin, called anyons, can be described as particles carrying both charge and magnetic flux and a simple model for them can be constructed by coupling their charges minimally with the Abelian Chern-Simons gauge field. The notion of anyon can be generalized by introducing a non-Abelian gauge group, i.e., particles which carry non-Abelian charges and interact with each other through the non-Abelian Chern-Simons theory. One can also construct the classical action for these non-Abelian Chern-Simons (NACS) particles by introducing isospin degree of freedom and coupling isospin charge with the non-Abelian Chern-Simons gauge fields. The resulting quantum mechanical model of $SU(2)$ NACS particles indicates that they also carry anomalous spins and satisfy the non-Abelian braid statistics. In Ref. [11], a model of $SU(n + 1)$ NACS particles with arbitrary $n \geq 1$ has been achieved by considering the isospin degree of freedom defined on $\mathbb{CP}^n$ manifold. It can be generalized to arbitrary group with invariant nonsingular metric. Also, equivalent second-quantized description of NACS particles has been formulated.

In this paper we investigate the model of Ref. [11] from a more geometrical point of view and explore the symplectic structure of the phase space of the NACS particles. In order to do this, we first identify the phase space of isospin particles in the presence of external gauge fields. Naively, one could try to construct the phase space by introducing the tangent bundle $T^*G$ of group manifold $G = SU(n + 1)$ for the isospin degree of freedom and taking direct product of this by the canonical phase space. Then, to account for the external gauge field, one could consider associated bundle with $T^*G$ as the fiber. Although this procedure works eventually, one has to go through some extra reduction procedure to achieve the goal. More economical way has been known for some time and it is to use associated bundle with coadjoint orbit as the fiber. It can be shown that the symplectic structure defined
on this associated bundle with a given connection is automatically gauge invariant and can be used to write down the equations of motions of a classical particle in the presence of gauge fields. The resulting equations of motion are the well-known Wong’s equations [17].

To make a connection with the quantum mechanical model of NACS particles, we have to work with a specific connection. This connection, for example, was determined as a solution of the Gauss constraints in the holomorphic gauge in Ref. [1,11]. In this more geometrical approach, this can be achieved by introducing the space of all connections and take a direct product of this by the associated bundle constructed previously. Then, gauge transformation can be defined on the product space and the corresponding momentum map can be constructed. The condition of vanishing of this momentum map yields the Gauss law constraints which can be solved as before.

In this paper, we elaborate on the above procedure using the associated bundle with \( \mathbb{C}P^n \) as the fiber. In Sec. 2, we start with a brief review on symplectic geometry and momentum map which are essential ingredients of phase space structure. In Sec. 3, we prove that the associated bundle is isomorphic to direct product of canonical tangent bundle by \( \mathbb{C}P^n \) manifold and evaluate the Poisson bracket between the dynamical variables using the symplectic structure. We show that the minimal substitution is equivalent to the modification of symplectic structure by the addition of field strength two form. In Sec. 4, we construct the momentum map corresponding to the gauge transformation and use the solution of Gauss constraints in holomorphic gauge to recover the model of Ref. [1,11].

II. MOMENTUM MAP FOR \( \mathbb{C}P^N \)

We start with a brief summary of symplectic geometry [18]. Consider a phase space \( \mathcal{O} \) with symplectic structure \( \Omega_{AB} \). Using the symplectic structure, we define Poisson bracket [19,20]:

\[
\{ F, H \} = \Omega^{AB} \partial_A F \partial_B H
\]  

(2.1)
where $F, H \in C^\infty(\mathcal{O})$ and $\Omega^{AB}$ is the inverse matrix of $\Omega_{AB}$. We define for a given $F$, the corresponding vector field $\zeta_F$ by

$$\zeta_F] \Omega + dF = 0. \quad (2.2)$$

The vector field $\zeta_F$ preserves the symplectic structure $\mathcal{L}_{\zeta_F} \Omega = 0$ and $\zeta_F$ defines one-parameter family of canonical transformation of $\mathcal{O}$. $\zeta_F$ are called Hamiltonian vector field generated by $F$ and the set of all Hamiltonian vector fields on $\mathcal{O}$ is denoted by $HV(\mathcal{O})$. One can show that the Poisson bracket defines a Lie algebra homomorphism of $C^\infty(\mathcal{O})$ onto $HV(\mathcal{O})$:

$$[\zeta_F, \zeta_H] = \zeta_{\{F,H\}} \quad (2.3)$$

Let us assume that $\mathcal{O}$ is a Hamiltonian $G$-space. This means that $\mathcal{O}$ is a symplectic manifold and the $G$ group action $L_g : \mathcal{O} \to \mathcal{O}$ defined by $f^g = g f (g \in G, f \in \mathcal{O})$ is a symplectomorphism:

$$L_g^* \Omega = \Omega. \quad (2.4)$$

The case in which $\mathcal{O}$ is the coadjoint orbit $K$ of group $G$ is most interesting. The mapping $\sigma : \mathcal{G} \to HV(\mathcal{O})$ given by $\sigma(\zeta)f = d/dt|_{t=0}(\exp t \zeta) \circ f$ where $\zeta \in \mathcal{G}$ is a Lie algebra homomorphism: $\sigma([\zeta_1, \zeta_2]) = [\sigma(\zeta_1), \sigma(\zeta_2)]$. We assume that there is a lifting $\tilde{\sigma} : \mathcal{G} \to C^\infty(\mathcal{O})$ such that the Lie algebra structure is given by the Poisson bracket on $\mathcal{O}$: $\tilde{\sigma}([\zeta_1, \zeta_2]) = \{\tilde{\sigma}(\zeta_1), \tilde{\sigma}(\zeta_2)\}$. Thus, to each $\zeta \in \mathcal{G}$, we get a function $\tilde{\sigma}(\zeta) \equiv F_\zeta$ and a Hamiltonian vector fields $\zeta_F$ of $\mathcal{O}$ so that

$$\zeta_F] \Omega + dF_\zeta = 0. \quad (2.5)$$

We define the momentum mapping $Q : \mathcal{O} \to \mathcal{G}^*$ where

$$\langle \zeta, Q(f) \rangle = F_\zeta(f), \quad f \in \mathcal{O}. \quad (2.6)$$

Here $\mathcal{G}^*$ denotes the dual algebra of $\mathcal{G}$ and $\langle , \rangle$ denotes the pairing between $\mathcal{G}^*$ and $\mathcal{G}$.
The momentum map for $SU(n+1)$ action on $\mathbb{CP}^n$ can be calculated according to the above definition. Let $u_0, u_1, \cdots, u_n$ be coordinates on $\mathbb{C}^{n+1}$ and consider the $SU(n+1)$ group action on $\mathbb{C}^{n+1}$ given by

$$u^g = gu, g = \exp(-i\theta^a T^a) \in SU(n+1)$$

where $u^T = (u_0, \cdots, u_n)$ and the $T^a$'s are the generators of the Lie algebra $su(n+1)$:

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

We use the normalization $Tr(T^a T^b) = (1/2)\delta_{ab}$. Now we introduce the coordinates $\xi_p = u_p/u_0(p,q = 1, \cdots, n)$ on the open set $U_0 = (u_0 \neq 0)$ in $\mathbb{CP}^n$.

To calculate the vector field generated by the generator $T^a$ on $\mathbb{CP}^n$, we note that

$$\zeta_p = \frac{d}{dt} \bigg|_{t=0} \left( \exp\{-itT^a\} \right)_{ps} u_s (\exp\{-itT^a\})_{os} u_s (s, t = 0, 1, \cdots, n)$$

$$= -i \left[ (T^a)_{po} + (T^a)_{pq} \xi_q - (T^a)_{00} \xi_p - (T^a)_{0q} \xi_q \xi_p \right].$$

Hence the vector field generated by $T^a$ is given by

$$\zeta_a = -i \left[ (T^a)_{po} + (T^a)_{pq} \xi_q - (T^a)_{00} \xi_p - (T^a)_{0q} \xi_q \xi_p \right] \frac{\partial}{\partial \xi_p} + (c.c).$$

The above vector field is in fact a Hamiltonian vector field since the unitary group $SU(n+1)$ acts transitively on $\mathbb{CP}^n$ and leaves the following symplectic two form $\Omega$ on $\mathbb{CP}^n$ invariant:

$$\Omega = 2iJ \left[ \frac{d\bar{\xi} \wedge d\xi}{1 + |\xi|^2} - \frac{\langle \xi d\bar{\xi} \rangle \wedge (\bar{\xi} d\xi)}{(1 + |\xi|^2)^2} \right], \quad |\xi|^2 = \sum_p |\xi_p|^2.$$  (2.11)

Thus the momentum map function $Q^a \equiv F_{T^a}$ on $\mathbb{CP}^n$ associated with the Hamiltonian vector field $\zeta_a$ defined by Eq. (2.5) satisfy the following:

$$dQ^a = 2J(d\xi \cdot \frac{\partial}{\partial \xi} + d\bar{\xi} \cdot \frac{\partial}{\partial \bar{\xi}}) \left[ \frac{(T^a)_{00} + (T^a)_{0q} \xi_q + \bar{\xi}_p (T^a)_{po} + \xi_p (T^a)_{pq} \xi_q}{1 + |\xi|^2} \right].$$

One can show that $Q^a$ can be written compactly as follows (up to a constant which we neglect):
\[ Q^a = 2J \sum_{s,t=0}^N \bar{u}_s (T^a)_{st} u_t \bigg|_{u_0 = \frac{1}{\sqrt{1 + |\xi|^2}}, \bar{u}_p = u_0 \bar{\xi}_p}. \] (2.13)

For example, for \( SU(2) \) group with Pauli matrices \( T^a = \sigma^a / 2 \), we have

\[ Q^1 = \frac{J(\xi + \bar{\xi})}{1 + |\xi|^2}, \quad Q^2 = i \frac{J(\xi - \bar{\xi})}{1 + |\xi|^2}, \quad Q^3 = \frac{J(1 - |\xi|^2)}{1 + |\xi|^2} \] (2.14)

In terms of stereographical projection \( \xi = \tan(\theta/2)e^{i\phi} \), they are just the ordinary angular momentum:

\[ Q^1 = J \sin \theta \cos \phi, \quad Q^2 = J \sin \theta \sin \phi, \quad Q^3 = J \cos \theta \] (2.15)

To calculate the Poisson brackets for the momentum map function \( Q^a \) defined by Eq.(2.1), we must use the symplectic structure on \( \mathbb{C}P^n \). To do that, we introduce the notation \( \xi^A = (\bar{\xi}_p, \xi_q) \) and write the symplectic two form as \( \Omega = \frac{1}{2} \Omega_{AB} d\xi^A \wedge d\xi^B \). The Poisson bracket defined by Eq.(2.11) with the use of Eq.(2.11) have the following expression:

\[ \{F, H\} = -i \sum_{p,q} g^{pq} \left( \frac{\partial F}{\partial \bar{\xi}_p} \frac{\partial H}{\partial \xi_q} - \frac{\partial F}{\partial \xi_q} \frac{\partial H}{\partial \bar{\xi}_p} \right), \] (2.16)

where \( g^{pq} \) is the inverse of the Fubini-Study metric given by

\[ g^{pq} = \frac{1}{2J} (1 + |\xi|^2)(\delta_{pq} + \bar{\xi}_p \xi_q). \] (2.17)

A simple calculation yields the fundamental commutators as follows

\[ \{\bar{\xi}_p, \xi_q\} = -i \frac{1}{2J} (1 + |\xi|^2)(\delta_{pq} + \bar{\xi}_p \xi_q), \] (2.18)

\[ \{\xi_p, \xi_q\} = \{\bar{\xi}_p, \bar{\xi}_q\} = 0, \]

and shows that the momentum map functions \( Q^a \) satisfy \( su(n + 1) \) algebra

\[ \{Q^a, Q^b\} = -f^{abc} Q^c. \] (2.19)
Consider two dimensional configuration space $M$ which could be in general arbitrary Riemann surfaces. For simplicity, we assume that $M$ is a plane. (We present one particle case and later extend to $N$ particles in a straightforward manner.) Let $\tilde{P} \rightarrow M$ be a principal $G$ bundle over $M$. Let $X = T^*M$ and $P$ be the pull-back of the bundle $\tilde{P}$ by the projection $\pi' : T^*M \rightarrow M$. Then $P$ is a principal $G$ bundle over $X$. Because $P$ is a principal $G$ bundle and $G$ acts on $\mathcal{O}$, we can form the associated bundle $\mathcal{P} \equiv P \times_G \mathcal{O}$. $\mathcal{P}$ is the phase space of an isospin particle under the influence of external gauge field. Sternberg showed [15] that given a connection $\Theta$ on $P$, there exists a unique symplectic structure defined on $\mathcal{P}$. We will explicitly calculate this symplectic structure for $\mathcal{O} = \mathbb{C}P^n$.

We present the essence of Sternberg’s results. Let $\zeta_P$ denote vector field on $P$ generated by the one parameter group consisting of right multiplication by $\exp(-t\zeta) : \zeta_P(p) = d/dt|_{t=0} p \circ \exp(-t\zeta), p \in P$. Let $R_g$ denote the right multiplication $R_g(p) = pg^{-1}$. The connection $\Theta$ on $P$ is defined as a $G$-valued differential one form which satisfies the following two conditions:

$$R_g^*\Theta = Ad_g\Theta, \quad \Theta(\zeta_P) = \zeta$$

(3.1)

for all $\zeta \in \mathcal{G}$. The infinitesimal version of first condition is $L_{\zeta_P} \Theta = ad_\zeta \Theta$. Let $U_g$ denote the action of $g \in G$ on $P \times \mathcal{O}$ by $U_g(p, f) = (pg^{-1}, gf)$.

By definition, the associated bundle $\mathcal{P}$ is the quotient space of $P \times \mathcal{O}$ under the action of $U_g$. Now define the real valued differential one form $\langle \Theta, Q \rangle$ on $P \times \mathcal{O}$. Then one can easily show that $\langle \Theta, Q \rangle$ is invariant under the action of $U_g : U_g^*\langle \Theta, Q \rangle = \langle \Theta, Q \rangle$. However, this does not imply that $\langle \Theta, Q \rangle$ is well defined on the quotient space $\mathcal{P}$. One can show this by proving that $d\langle \Theta, Q \rangle$ is not defined on $\mathcal{P}$. A simple calculation using $L_{\zeta_P} \Theta = ad_\zeta \Theta$ indeed shows that

$$\zeta_U d\langle \Theta, Q \rangle = -\pi^*(\mathcal{O} \Omega).$$

(3.2)

Here $\zeta_U$ denotes the vector field on $P \times \mathcal{O}$ corresponding to the action $U$ on $P \times \mathcal{O}$ so that
\( \zeta_U = \zeta_P + \zeta_O \) and \( \pi \) is the projection map \( \pi : P \times O \to O \). The above equation implies that \( d\langle \Theta, Q \rangle \) is not naturally defined on the quotient space \( P \) because it does not vanish when evaluated on vectors, one of which is along the direction of projection \( P \times O \to P \).

To have expression for two form which descends on \( P \), consider \( d\langle \Theta, Q \rangle + \pi^*\Omega \). Then, we have

\[
\zeta_U \| (d\langle \Theta, Q \rangle + \pi^*\Omega) = 0.
\]  

(3.3)

by using Eq.(3.2). One can also easily check that \( d\langle \Theta, Q \rangle + \pi^*\Omega \) is invariat under the \( U_g \) action. These suggest that there exist a unique form \( \Omega_\Theta \) on \( P \) such that

\[
d\langle \Theta, Q \rangle + \pi^*\Omega = \bar{\pi}^*\Omega_\Theta
\]  

(3.4)

where \( \bar{\pi} \) is the projection map \( \bar{\pi} : P \times O \to P \). Since \( \bar{\pi}^* \) is one-to-one and \( d\langle \Theta, Q \rangle + \pi^*\Omega \) is closed, the two form \( \Omega_\Theta \) is closed:\( d\Omega_\Theta = 0 \). Also, \( \Omega_\Theta \) is nondegenerate in the case when \( X = T^*M \) and \( P \) is the pull-back of the bundle \( \tilde{P} \) and the connection is the pull-back of a connection defined on \( \tilde{P} \) [15]. Denoting \( \tilde{\omega} \) for the pull-back of \( \omega \) which is the canonical symplectic structure defined on \( X \) to \( P \) via projection onto \( X \), we have symplectic structure on \( P \) as

\[
\Omega_T = \tilde{\omega} + \Omega_\Theta
\]  

(3.5)

When \( M \) is a plane, \( T^*M \) is contractible and every associated bundle is trivial. So we have

\[
P = P \times_G O = T^*M \times O
\]  

(3.6)

In fact, the above holds for arbitrary Riemann surfaces \( M \). This can be seen from the following simple homotopy arguement. It is well known that the set of equivalence classes of principal G-bundles \( K_G(T^*M) \) is isomorphic to \( [T^*M, BG] \) where \([T^*M, BG]\) is the set of homotopy classes of maps from \( T^*M \) to the classifying space \( BG \) of \( G \). Consider the homotopy exact sequence of universal principal G-bundle
\[\pi_i(G) \to \pi_i(EG) \to \pi_i(BG) \to \pi_{i-1}(G) \to \pi_{i-1}(EG).\]  

(3.7)

Since \(EG\) is contractible, \(\pi_i(EG) = \pi_{i-1}(EG) = 0\) and we have \(\pi_i(BG) = \pi_{i-1}(G)\). By the fact [22] that

\[
\pi_k(SU(n+1)) = \begin{cases} 0 & \{k \leq 2\} \\ \mathbb{Z} & \{k = 3\}, \end{cases}
\]

(3.8)

we have

\[
\pi_k(BG) = \begin{cases} 0 & \{k \leq 3\} \\ \mathbb{Z} & \{k = 4\}. \end{cases}
\]

(3.9)

Consider the reduced singular cohomology group \(\tilde{H}^k(T^*M, \pi_k(BG))\). Since \(T^*M\) is homotopy equivalent to \(M\), \(\tilde{H}^k(T^*M, \pi_k(BG)) = \tilde{H}^k(M, \pi_k(BG))\). Since \(M\) is a real two dimensional manifold, \(\tilde{H}^k(M, \pi_k(BG)) = 0\) for \(k \geq 3\). If \(k = 1, \pi_1(BG) = 0\) and \(\tilde{H}^1(M, \pi_1(BG)) = 0\). Thus there exists a surjective map [23] from \(\tilde{H}^2(T^*M, \pi_2(BG)) = \tilde{H}^2(M, \pi_2(BG))\) to \([T^*M, BG]\). Since for \(k = 2, \pi_2(BG) = 0\) and \(\tilde{H}^2(M, \pi_2(BG)) = 0\), this implies that every principal fiber bundle over \(M\) is trivial and we have \(P = P \times_G \mathcal{O} = T^*M \times \mathcal{O}\). Hence we have \(\tilde{\omega} = \omega\) and

\[\Omega_T = \omega + \sigma^*(d\langle \Theta, Q \rangle + \pi^*\Omega)\]

\[= \omega + d(A^a Q^a) + \Omega\]

(3.10)

where \(\sigma\) is the cross section \(P \times_G \mathcal{O} \to P \times \mathcal{O}\) and we used \(\sigma^*\Theta = A\), the gauge field one form on \(M\). Notice that \(\omega + \Omega\) is not gauge invariant. We must have Sternberg’s two form \(d\langle \Theta, Q \rangle\) to achieve the gauge invariance. Physically, this term describes the interaction between isospin charge and the external gauge fields. Now, we calculate the symplectic structure on \(P \times_G \mathcal{O}\). We start from the two form on \(P \times_G \mathcal{O} = T^*M \times \mathcal{O}\) given by

\[\Omega_T = dp_i \wedge dq^i + d(A^a_i Q^a dq^i) + \Omega\]

(3.11)

where the \(\Omega\) is given by the expression Eq.(2.11). To achieve the notational simplifications, we introduce \(\eta^I = (p_i, q^j)\) and \(x^M = (\xi^A, \eta^I)\). Then we can write \(\Omega_T = \frac{1}{2} \Omega_{MN} dx^M \wedge dx^N\) where the matrix \(\Omega_{MN}\) can be expressed as
\(\Omega_{MN} = \begin{pmatrix} \Omega_{AB} & A_j^a(\partial Q^a/\partial \xi^A) \\ -A_i^a(\partial Q^a/\partial \xi^B) & \omega_{IJ} \end{pmatrix}. \) \hspace{1cm} (3.12)

Here, \(A_i^a = (0, A_i^a)\) and \(\omega_{IJ}\) is given by

\[\omega_{IJ} = \begin{pmatrix} 0 & I \\ -I & A_a^a \end{pmatrix}. \] \hspace{1cm} (3.13)

A short calculation gives the following inverse matrix \(\Omega^{MN}\):

\[\Omega^{MN} = \begin{pmatrix} \Omega^{AB} & -F^{KJ}\Omega^{AC} A_K^a(\partial Q^a/\partial \xi^C) \\ F^{KI}\Omega^{BD} A_K^a(\partial Q^a/\partial \xi^D) & F^{IJ} \end{pmatrix}. \] \hspace{1cm} (3.14)

where \(F^{IJ}\) is the inverse matrix of \(F_{IJ} = \omega_{IJ} - f^{abc} A^a_i A^b_j Q^c\). Using Eq.(3.13), we find the matrix \(F^{IJ}\):

\[F^{IJ} = \begin{pmatrix} F^{a}_{ij} Q^a -I \\ I & 0 \end{pmatrix}. \] \hspace{1cm} (3.15)

where \(F^{a}_{ij} \equiv \partial_j A^a_i - \partial_i A^a_j - f^{abc} A^b_i A^c_j\) is the Yang-Mills field strength.

The Poisson bracket on \(P \times G \mathcal{O}\) is defined by the use of inverse matrix \(\Omega^{MN}\) as before

\[\{F, H\} = \Omega^{MN} \frac{\partial F}{\partial x^M} \frac{\partial H}{\partial x^N}. \] \hspace{1cm} (3.16)

The calculation is greatly simplified by the use of Eq.(2.19) and we find the following Poisson bracket among the dynamical variables

\[\{Q^a, p_i\} = -f^{abc} A^b_i Q^c, \quad \{Q^a, q^i\} = 0\] \hspace{1cm} (3.17)

\[\{p_i, p_j\} = F^{a}_{ij} Q^a, \quad \{p_i, q^j\} = -\delta^j_i, \quad \{q^i, q^j\} = 0.\]

The above relations are in accordance with the minimal substitution

\[p_i \to P_i = p_i - A_i^a Q^a. \] \hspace{1cm} (3.18)

In terms of canonical momentum \(P_i\), we have, among others,
\[ \{Q^a, P_i\} = 0 \quad \{P_i, P_j\} = 0 \quad \{P_i, q^j\} = -\delta^j_i. \quad (3.19) \]

Thus, one can work in \((p_i, q^j, Q^a)\) coordinates using the symplectic structure given by Eqs.(3.14) and (3.15) or with \((P_i, q^j, Q^a)\) using the canonical symplectic structure without mixing between \(P_i\) and \(Q^a\). The two procedures are equivalent [13]. Consider, for example, the free Hamiltonian \(H = (1/2m)p^2\) with symplectic structure given by Eqs.(3.14) and (3.15). The Hamiltonian equations of motion

\[ \dot{x}^M = \Omega^{MN} \frac{\partial H}{\partial x^N} \quad (3.20) \]

reproduces the well known Wong’s equations

\[ m\ddot{q}_i = F^a_{ij}Q^a\dot{q}^j \quad \dot{Q}^a = -f^{abc}A^b_i\dot{q}^iQ^c, \quad (3.21) \]

which describes the dynamics of an isospin particle in external gauge fields \(A^a_i\). Minimal substitution implies that alternatively, we can work with

\[ H = \frac{1}{2m}(P_i - A^a_iQ^a)^2, \quad (3.22) \]

with canonical symplectic structure Eq.(3.19). Obviously, we get the same equations of motions. The above procedures can be generalized to a system of \(N\) particles in an obvious manner. We will consider from now on a system of \(N\) particles and denote the particle index by \(\alpha, \beta, \cdots = 1, \cdots, N\).

**IV. REDUCED PHASE SPACE AND CONNECTION**

The analysis we have done so far holds for most part in describing isospin particles under the influence of arbitrary external gauge fields. In this section, we attempt to determine a specific connection which is essential in describing the quantum mechanics of fractional spin and braid statistics. To do that, we first consider phase space \(P_T \equiv \prod_\alpha \mathcal{P}^\alpha \times \mathcal{A} = \prod_\alpha T^*_\alpha M^\alpha \times O^\alpha \times \mathcal{A}\) where \(\mathcal{A}\) is the space of all connections. We define the gauge transformation of \(f \in \mathcal{O}\) and \(A \equiv A^a_iT^a_i\dot{q}^i \in \mathcal{A}\) by
\[ f^g_\alpha = g f_\alpha, \quad A^g = g^{-1} A g + i g^{-1} d g \] (4.1)

for \( g \in G = SU(n+1) \). The momentum map for \( G \) action on \( \prod_{\alpha} O^\alpha = \prod_{\alpha} C P^{\alpha n} \) is

\[ Q = \sum_{\alpha} Q^\alpha a T^\alpha \delta(\mathbf{x} - q_\alpha) dx^1 \wedge dx^2 \] with \( Q^\alpha \) given by Eq.\((2.13)\). To describe the momentum map for \( G \) action on \( A \), consider the symplectic two form on \( A \):

\[ \Omega_A(a,b) = \kappa \int_M Tr(a \wedge b) \] (4.2)

where \( a, b \in \mathcal{G} \) are tangent vectors at \( A \in \mathcal{A} \) and one form on \( M \). \( \kappa \) is the coefficient of the Chern-Simons gauge theory. The above form is invariant under the \( G \) action (4.1) and the corresponding momentum map is given by the curvature two form \( \kappa F \),

\[ F = dA + i A \wedge A \equiv (1/2) F^a_{ij} T^a dx^i \wedge dx^j \] [24]. To see this first define the momentum map function \( \mu_\zeta \) on \( \mathcal{A} \) by

\[ \mu_\zeta(A) \equiv \langle \zeta, \mu_A \rangle = \int_M Tr(\mu_A \wedge \zeta) \] (4.3)

where \( \zeta \in \mathcal{G} \) are zero forms on \( M \). The above definition suggests that the momentum map is two form on \( M \). The vector field generated by \( \zeta \) is given by the gauge transformations \( d_A \zeta \) and from the definition of momentum map Eq.(2.5), we have for arbitrary \( a \)

\[ \langle a, d\mu_\zeta(A) \rangle = d_A \zeta \big\langle \Omega_A(a) = \kappa \int_M Tr(d_A \zeta \wedge a) = -\kappa \int_M Tr(\zeta \wedge d_A a) \] (4.4)

where we integrated by parts in the last step. We see that the last term is the gauge transformation of \( F_\zeta \) along the direction of \( a \) and can be written as \( \kappa \langle a, dF_\zeta(A) \rangle \) [24]. So we have \( \mu_A = \kappa F \). Hence the total momentum map is given by \( \Phi = Q + \mu_A = Q + \kappa F \). We can form the quotient space \( \Phi^{-1}(0)/G \) which is the Marsden-Weinstein reduction [18] and this moduli space is the reduced phase space of the NACS particles. Note that the \( \Phi = 0 \) can be written as

\[ \Phi^a = \frac{\kappa}{2} \epsilon^{ij} F^a_{ij}(\mathbf{x}) + \sum_{\alpha} Q^\alpha \delta(\mathbf{x} - q_\alpha) = 0, \] (4.5)

which is the Gauss constraints in Chern-Simons theory coupled with point sources.

Rather than probing the geometry of the reduced phase space, we attempt to determine a specific connection as the solution of the above Gauss constraints which can be solved
explicitly in two gauge conditions. The first one is the axial gauge \[25\] in which, for example, we set \( A^a_1 = 0 \). The remaining \( A^a_2 \) field becomes highly singular with strings attached to each source and we do not adopt this solution. The less singular solutions can be obtained by performing the analytic continuation of the gauge fields. Introducing complex coordinates, 
\[ z = x + iy, \bar{z} = x - iy, z_\alpha = q_\alpha^1 + iq_\alpha^2, \bar{z}_\alpha = q_\alpha^1 - iq_\alpha^2, A^a_\alpha = \frac{1}{2}(A^a_1 + iA^a_2), A^a_{\bar{\alpha}} = \frac{1}{2}(A^a_1 - iA^a_2), \]
analytic continuation means that \( A^a_\alpha \) and \( A^a_{\bar{\alpha}} \) are treated as independent variables. We recall \[26\] that choosing a gauge condition corresponds to reduction to the space \( \Phi^{-1}(0)/G \) by the choice of representatives in \( \Phi^{-1}(0) \) of all orbits. In the analytic continuation, the representatives are fixed in the complexified gauge orbit space and this is consistent with the coherent state quantization method \[27\]. We choose \( A^a_{\bar{\alpha}} = 0 \) as a gauge fixing condition in this space which was called holomorphic gauge in ref \[1\]. The solution of the Gauss constraints

\[ \Phi^a(z) = -\kappa \partial_\bar{z} A^a_\bar{z} + \sum_\alpha Q^a_\alpha \delta(z - z_\alpha) = 0, \]  
\[ (4.6) \]
in holomorphic gauge turns out to be \[1\]

\[ A^a_\bar{z}(z, \bar{z}) = i \frac{\kappa}{2\pi} \sum_\alpha Q^a_\alpha \frac{1}{z - z_\alpha} + P(z), \]  
\[ (4.7) \]
where \( P(z) \) is an arbitrary holomorphic polynomial in \( z \). The further choice of \( P(z) = 0 \) resulted in the quantum mechanical model which provides a unified framework for fractional spin, braid statistics and Knizhnik-Zamolodchikov equation \[28\]. In other words, essential ingredients in the quantum mechanical model of Ref. \[1\] consist of classical phase which is \( \prod_\alpha T^*M^\alpha \times \mathcal{O}^\alpha \) and a specific connection in Eq.(4.7) with \( P(z) = 0 \). This connection known as Knizhnik-Zamolodchikov (KZ) connection in the literature plays an important role in establishing the relation of Chern-Simons theory with conformal field theory and quantum group \[29,30\]. Substituting the above connection into the \( N \) particle Hamiltonian where Hamiltonian of each particle is given by Eq.(3.22), we obtain (replacing \( P_i \rightarrow p_i \))

\[ H = \sum_\alpha \frac{2}{m_\alpha} p^2_\alpha \left( p^2_\alpha - \frac{i}{2\pi \kappa} \sum_\beta \frac{Q^a_\alpha Q^a_\beta}{z_\alpha - z_\beta} \right), \]
\[ (4.8) \]
Quantum mechanically, the dynamics of the NACS particles are governed by the operator version \( \hat{H} \) of the Hamiltonian Eq. (4.8)

\[
\hat{H} = -\sum_{\alpha} \frac{1}{m_{\alpha}} (\nabla_{z_{\alpha}} \nabla_{z_{\alpha}} + \nabla_{z_{\alpha}} \nabla_{\bar{z}_{\alpha}})
\]

\[
\nabla_{z_{\alpha}} = \frac{\partial}{\partial z_{\alpha}} + \frac{1}{2\pi \kappa} \left( \sum_{\beta \neq \alpha} \hat{Q}_{\alpha}^{a} \hat{Q}_{\beta}^{a} \frac{1}{z_{\alpha} - z_{\beta}} + \hat{Q}_{\alpha}^{2} a_{z}(z_{\alpha}) \right) \tag{4.9}
\]

\[
\nabla_{\bar{z}_{\alpha}} = \frac{\partial}{\partial \bar{z}_{\alpha}}
\]

where \( a_{z}(z_{\alpha}) = \lim_{z \to z_{\alpha}} 1/(z - z_{\alpha}) \) and the isospin operators \( \hat{Q}^{a} \)'s satisfy the \( SU(N + 1) \) algebra, \([\hat{Q}^{a}_{\alpha}, \hat{Q}^{b}_{\beta}] = i f^{abc} \hat{Q}^{c}_{\alpha} \delta_{\alpha \beta} \) upon quantizing the classical algebra Eq. (2.19). The second term and the third term in \( \nabla_{z_{\alpha}} \) are responsible for the non-Abelian statistics and the anomalous spins of the NACS particles respectively. The detailed analysis of the above quantum mechanical model has been performed in Ref. [1,31].

V. CONCLUSION

In this paper, we have investigated the phase space structure of \( N \) NACS particles from a geometrical point of view. We first considered the product space of \( N \) associated \( SU(n + 1) \) bundle with \( \text{CP}^{n} \) as the fiber. The momentum map which is the essential ingredient necessary to probe the phase space structure and to construct the gauge invariant symplectic two form was obtained for \( \text{CP}^{n} \). An interesting consequence of a simple homotopy argument was that the associated bundle is equivalent to the direct product of canonical cotangent bundle by the \( \text{CP}^{n} \) manifold for arbitrary two-dimensional Riemann surfaces. The explicit evaluation of the symplectic structure showed that the minimal substitution of canonical momentum in the external non-Abelian gauge fields is equivalent to the modification of canonical symplectic structure with the addition of field strength two form as in the Abelian case. Then, we introduced the space of all Chern-Simons connections and adopted the procedure of symplectic reduction in order to obtain the reduced phase space of NACS.
particles. A specific connection was determined by the condition of vanishing momentum map with the holomorphic gauge choice and this produced the well-known KZ connection. This connection endows NACS particles with the non-Abelian magnetic flux and introduces topological interaction between them. The canonical quantization for two particle system using this connection was performed [31] in detail. The geometric quantization of such a model is another interesting possibility and will be reported elsewhere.

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