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ON THETA FUNCTIONS OF BINARY QUADRATIC FORMS WITH CONGRUENCE CONDITION

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Abstract. We study modular properties of theta functions of binary quadratic forms with congruence condition and compute their values at arbitrary cusps.

1. Introduction

Let \( A \) be a positive definite symmetric matrix of size \( r \) and \( P(x) \) a spherical function of degree \( \nu \). For a column vector \( x \), we denote \( ^1xAx \) by \( A[x] \). We take \( h \in \mathbb{Z}^r \) and a positive integer \( N \) such that

\[
NA^{-1} \in M_r(\mathbb{Z}), \quad (1.1)
\]
\[
Ah \in N\mathbb{Z}^r. \quad (1.2)
\]

Further, we always assume that

the diagonal entries of \( A \) are even. \quad (1.3)

For \( N \) satisfying (1.1), we denote the set of \( h \) satisfying (1.2) by

\[
\mathcal{H}_{A,N} = \{ h \in \mathbb{Z}^r / N\mathbb{Z}^r \mid Ah \equiv 0 \pmod{N} \}. \quad (1.4)
\]

The theta function we consider in this paper is defined by

\[
\theta(z; h, A, N, P) = \sum_{m \in \mathbb{Z}^r, m \equiv h \pmod{N}} P(m) e\left(\frac{A[m]}{2N^2}z\right),
\]

with \( e(z) = \exp(2\pi iz) \). Shimura [5] showed that \( \theta(z; h, A, N, P) \) converges absolutely and uniformly on any compact subset of the upper half-plane \( \mathfrak{H} \) and is holomorphic on \( \mathfrak{H} \). He also showed that the theta function is a modular form on a certain congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) of weight \( k = r/2 + \nu \) and is a cusp form if \( \nu \geq 1 \) with some additional condition, hence, then the weight is greater than one.

On the other hand, Kida and Namura showed in [2] that if the Galois group of a number field over \( \mathbb{Q} \) is isoclinic to the dihedral group of order 8 and admits odd representation, then the inverse Mellin transform of the Artin \( L \)-function gives a newform of weight one which is a linear combination of this kind of theta function of binary quadratic forms. Therefore it is important to study how to construct cusp forms of weight one from the theta functions with \( r = 2 \) and \( \nu = 0 \). The aim of this paper is to carry out this construction.

In the following we therefore concentrate on theta series of positive definite binary quadratic forms defined by the following.

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Definition 1.1. For a positive definite $A \in M_2(\mathbb{Z})$ and $N$ satisfying (1.1) and $h \in \mathcal{H}_{A,N}$, we define

$$\theta(z; h, A, N) = \sum_{m \in \mathbb{Z}^2 \atop m \equiv h \pmod{N}} e\left(\frac{A[m]}{2N^2}z\right),$$

where the sum is taken over $m \in \mathbb{Z}^2$ satisfying the congruence condition $m \equiv h \pmod{N}$.

In the following section, we construct modular forms on the congruence subgroup $\Gamma_0(M)$ for some $M$ by linear combinations of the theta functions (1.5). In order to construct cusp forms, we compute the Fourier expansions at arbitrary cusps in Section 3. In Section 4, we construct cusp forms using the result in Section 3. Several explicit examples will be given in the section.

Throughout this paper we will use the following standard notation as in [1]. We denote by $M_k(\Gamma)$ the vector space of modular forms of weight $k$ with respect to a congruence subgroup $\Gamma$ and by $M_k(M, \chi)$ the $\chi$-eigen subspace of $M_k(\Gamma_1(M))$. We also denote by $S_k(\Gamma)$ and $S_k(M, \chi)$ the subspaces of cusp forms. For a subgroup $H$ of $(\mathbb{Z}/m\mathbb{Z})^\times$, we denote by $H^\perp$ the set of the Dirichlet characters modulo $m$ vanishing on $H$.

2. Construction of modular forms on $\Gamma_0(M)$

In this section, we construct modular forms on the congruence subgroup $\Gamma_0(M)$ for some $M$ by linear combinations of the theta functions (1.5). The main theorem in this section is Theorem 2.7.

Before starting the construction, we prove some preliminary classification results of theta functions. The following lemma describes the set $\mathcal{H}_{A,N}$ of the congruence conditions defined by (1.4) explicitly.

Lemma 2.1. Let $A$ be a positive definite integral symmetric matrix of size two and

$$UAV = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

the Smith normal form of $A$, where $U, V \in \text{GL}_2(\mathbb{Z})$ and $0 < d_1 \mid d_2$. For a positive integer $N$ satisfying (1.1), we define $n_i$ by $d_in_i = N$ for $i = 1, 2$. Let $v_1, v_2$ be the column vectors of $V$. Then $(n_1v_1, n_2v_2)$ is a basis for $\mathcal{H}_{A,N}$. In particular, we have $\#\mathcal{H}_{A,N} = \det A$.

Proof. We compute

$$\mathcal{H}_{A,N} = \left\{ h \in \mathbb{Z}^2/N\mathbb{Z}^2 \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = V^{-1}h, \quad UAV \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv 0 \pmod{N} \right\}$$

$$= \left\{ h \in \mathbb{Z}^2/N\mathbb{Z}^2 \mid h = V \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} d_1x_1 \\ d_2x_2 \end{bmatrix} \equiv 0 \pmod{N} \right\}$$

$$= (n_1v_1, n_2v_2).$$
The last statement follows from \( \# \mathcal{H}_{A, N} = \# \langle n_1 v_1, n_2 v_2 \rangle = d_1 d_2 = |\det A| \). This completes the proof.

Two different congruence conditions in \( \mathcal{H}_{A, N} \) may define the same theta function. The following proposition shows exactly when this happens.

**Proposition 2.2.** Let \( O(A) = \{ P \in \text{GL}_2(\mathbb{Z}) \mid ^t P A P = A \} \) be the orthogonal group of \( A \). Then we have \( \theta(z; h, A, N) = \theta(z; h', A, N) \) if and only if there exists \( P \in O(A) \) such that \( h' = P h \).

**Proof.** If \( h \) and \( h' \) satisfy \( h' = P h \) for some \( P \in O(A) \), then we obtain \( \theta(z; h', A, N) = \theta(z; h, ^t P A P, N) \). Conversely, suppose that \( \theta(z; h, A, N) = \theta(z; h', A, N) \). Then for any \( g \in \mathbb{Z}^2 \) with \( g \equiv h \mod N \), there exists \( g' \equiv h' \mod N \) such that \( A[g] = A[g'] \). This implies that there is \( P \in O(A) \) such that \( h' \equiv P h \mod N \).

The structure of \( O(A) \) is given by the following proposition.

**Proposition 2.3.** Let \( K \) be the imaginary quadratic field \( \mathbb{Q}(\sqrt{-\det A}) \). The orthogonal group \( O(A) \) fits into the split exact sequence

\[
1 \longrightarrow \mathcal{O}^{\times}_K \xrightarrow{\varphi} O(A) \xrightarrow{\det} \{ \pm 1 \},
\]

where \( \varphi \) is induced by the regular representation of \( \mathcal{O}_K \).

**Proof.** It is clear that

\[
1 \longrightarrow O(A) \cap \text{SL}_2(\mathbb{Z}) \longrightarrow O(A) \xrightarrow{\det} \{ \pm 1 \}
\]

is an exact sequence. Thus it remains to prove that \( \mathcal{O}^{\times}_K \xrightarrow{\varphi} O(A) \cap \text{SL}_2(\mathbb{Z}) \) is an isomorphism.

We may assume that

\[
A = k \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix},
\]

\( \gcd(a, b, c) = 1 \), and \( Q(x, y) = ax^2 + bxy + cy^2 \) is a reduced form, since, if two forms are equivalent, then the orthogonal groups are conjugate. Let \( D \) be the discriminant of \( K \). Since \( Q \) is reduced, an easy computation shows that

\[
O(A) \cap \text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\},
\]

\( D < -4 \),

\[
D = -4,
\]

\[
D = -3.
\]

If we take the standard integral basis of \( \mathcal{O}_K \), then the image of the regular representation coincides with the above group.

Now we proceed to construct modular forms on \( \Gamma_1(M) \) by theta functions. For this purpose, we need the modular transformation property of theta function \( \theta(z; h, A, N) \) studied by Shimura in [5].
THEOREM 2.4. (Shimura) Let
\[ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(\delta N), \]
where
\[ \delta = \begin{cases} 1, & \text{if the diagonal elements of } NA^{-1} \text{ are even,} \\ 2, & \text{otherwise}. \end{cases} \tag{2.1} \]
The theta series \( \theta(z; h, A, N) \) in (1.5) satisfies the transformation law
\[ \theta(z; h, A, N) \mid \gamma = e \left( \frac{abA[h]}{2N^2} \right) \psi_{-\det A} \theta(ah; A, N), \tag{2.2} \]
where
\[ \psi_{-\det A} = \left( \frac{-\det A}{.} \right) \]
denotes the Kronecker symbol.

**Proof.** This is a special case of [5, Proposition 2.1]. \( \square \)

Let \( \Gamma \) be a congruence subgroup contained in \( \Gamma_0(\delta N) \). If the root of unity \( e(abA[h]/2N^2) \) in (2.2) equals 1 for any
\[ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \]
then \( \theta(z; h, A, N) \) is a modular form on \( \Gamma \). In the following proposition, we find such congruence subgroups.

For a positive divisor \( s \) of \( N \), we define the subset \( \mathcal{H}_{A,N}(s) \) of \( \mathcal{H}_{A,N} \) by
\[ \mathcal{H}_{A,N}(s) = \{ h \in \mathcal{H}_{A,N} : 2N^2 \mid sA[h] \text{ and } v_p(2N^2) = v_p(sA[h]) \forall p \mid s \}, \tag{2.3} \]
where \( p \) is a prime dividing \( s \) and \( v_p \) is the \( p \)-adic valuation. For \( s = 1 \), the set \( \mathcal{H}_{A,N}(s) \) is defined only by the first condition. It is easy to see that
\[ \mathcal{H}_{A,N} = \bigsqcup_{s \mid N, s > 0} \mathcal{H}_{A,N}(s) \] (disjoint union).

**PROPOSITION 2.5.** Let \( h \in \mathcal{H}_{A,N}(s) \). We have
\[ \theta(z; h, A, N) \in M_1(\Gamma_1(\delta N) \cap \Gamma^1(s)) \tag{2.4} \]
and
\[ \theta(sz; h, A, N) \in M_1(\Gamma_1(\delta sN)), \tag{2.5} \]
where
\[ \Gamma^1(s) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \mod s \right\} \tag{2.6} \]
and \( \delta \) is defined by (2.1).
Proof. If \( h \in \mathcal{H}_{A,N}(s) \), then \( e(abA[h]/2N^2) = 1 \) for any \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(\delta N) \cap \Gamma^1(s) \).

This proves the assertion (2.4). To prove (2.5), let

\[
\gamma_s = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}.
\]

For all \( \gamma \in \Gamma_1(\delta s N) \), we have \( \gamma_s \gamma \gamma_s^{-1} \in \Gamma_1(\delta N) \cap \Gamma^1(s) \). Since \( \theta(sz; h, A, N) = \theta(z; h, A, N) |_{\gamma_s} \), the assertion (2.5) follows immediately from (2.4). Here, we denote, as usual, by

\[
f(z) |_{\gamma} = (cz + d)^{-1} f(\gamma z) \quad \text{for} \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}).
\]

We can construct modular forms on the congruence subgroup \( \Gamma_0(\delta s N) \) by means of linear combinations of the theta functions \( \theta(sz; h, A, N) \) in (2.5).

Definition 2.6. For \( h \in \mathcal{H}_{A,N} \), let

\[
U(h) = \{ u \in (\mathbb{Z}/N\mathbb{Z})^\times \mid \text{there is } P \in O(A) \text{ such that } Ph = uh \} \quad (2.7)
\]

be the set of eigenvalues of elements of \( O(A) \) whose eigenvector is \( h \).

Moreover, for \( h \in \mathcal{H}_{A,N}(s) \), and a Dirichlet character \( \chi \) modulo \( N \) with \( \chi \in U(h)^\perp \), we define

\[
\Theta(z; h, A, N, \chi) = \sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / U(h)} \chi(u) \theta(sz; uh, A, N). \quad (2.8)
\]

If

\[
h = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

then we have \( U(h) = (\mathbb{Z}/N\mathbb{Z})^\times \) and \( \Theta(z; h, A, N, \chi) = \theta(sz; h, A, N) \).

The purpose of this section is to prove the following theorem.

Theorem 2.7. The theta series \( \Theta(z; h, A, N, \chi) \) of (2.8) is a modular form of weight one and character \( \chi \psi_{-\det A} \) on \( \Gamma_0(\delta s N) \), i.e.

\[
\Theta(z; h, A, N, \chi) \in M_1(\delta s N, \chi \psi_{-\det A}). \quad (2.9)
\]

Proof. For \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(\delta s N) \), we compute by using (2.2) and (2.5):

\[
\Theta(z; h, A, N, \chi) |_{\gamma} = \sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / U(h)} \chi(u) \psi_{-\det A}(d) \theta(sz; uh, A, N).
\]
This completes the proof of Theorem 2.7. □

**Remark 2.8.** Recall that the projection \( \pi_\chi : M_1(1_{sN}) \to M_1(sN, \chi) \) maps each \( f(z) \in M_1(1_{sN}) \) to
\[
\pi_\chi(f(z)) = \frac{1}{\phi(sN)} \sum_{d \in (\mathbb{Z}/sN\mathbb{Z})^\times} \chi^{-1}(d) f(\gamma_d z),
\]
where \( \phi \) is the Euler totient function and
\[
\gamma_d = \begin{bmatrix} * & * \\
* & d \end{bmatrix} \in \Gamma_0(sN).
\]
We can express \( \Theta(z; h, A, N, \chi \psi^{-1}_{\det A}) \) in terms of \( \pi_\chi \) as follows:
\[
\Theta(z; h, A, N, \chi \psi^{-1}_{\det A}) = \frac{1}{\#U(h)} \pi_\chi(\theta(sz; h, A, N)).
\]

### 3. Fourier expansions at cusps

In this section, we compute the constant term of the Fourier expansion of \( \theta(sz; h, A, N) \) at the cusps using the following quadratic Gauss sums defined by Springer in [6].

Let \( G \) be a finite abelian group. A quadratic character on \( G \) is a map \( T : G \to \mathbb{C}^\times \) such that \( f_T(x, y) = T(x + y)T(x)^{-1}T(y)^{-1} \) is a bicharacter. We set
\[
N_T = \{ l \in G | f_T(l, G) = 1 \}, \quad R_T = \ker T|_{N_T},
\]
and define the Gauss sum of \( T \) on \( G \) by
\[
S(T, G) = \sum_{x \in G} T(x).
\]
We shall use the Gauss sum of the following quadratic character
\[
T_h(l) = e\left(\frac{sm(NA[l] + 2hA)}{2nN}\right)
\]
on \( G = \mathbb{Z}^2/n\mathbb{Z}^2 \) with
\[
f_T(l_1, l_2) = e\left(\frac{sm^1l_1A_l_2}{n}\right).
\]
Moreover, we define
\[
\Phi_{A,N}(h, \frac{m}{n}) = e\left(\frac{smA[h]}{2nN^2}\right) S(T_h, G).
\]
Substituting (3.1) and (3.2) in (3.3) and using \( A[h + NL] = A[h] + 2N^2hA + N^2A[l] \), we obtain the formula
\[
\Phi_{A,N}(h, \frac{m}{n}) = \sum_{l \in G} e\left(\frac{smA[h + NL]}{2nN^2}\right).
\]
Although $T_h(l)$ is not well defined for $h \in \mathbb{Z}^2/N \mathbb{Z}^2$, this formula implies that $\Phi_{A,N}(h, m/n)$ is obviously well defined for $h \in \mathbb{Z}^2/N \mathbb{Z}^2$.

The following is the main theorem in this section.

**Theorem 3.1.** Let $A = k A_0$ be a positive definite symmetric matrix of size two, where

$$A_0 = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

with $\gcd(a, b, c) = 1$ and $a, k \in \mathbb{Z}_{>0}$. For

$$\gamma = \begin{bmatrix} m & * \\ n & * \end{bmatrix} \in \SL_2(\mathbb{Z}),$$

let $\theta(sz; h, A, N)|_1 \gamma = \sum_{k=0}^{\infty} a_k q^k$ be the Fourier expansion at the cusp $m/n \in \mathbb{Q} \setminus \mathbb{Z}$. The following assertions describe the constant term of the Fourier expansion step by step.

(i) We have

$$a_0 = \frac{-i}{sn \sqrt{\det A}} \Phi_{A,N} \left( h, \frac{m}{n} \right). \quad (3.5)$$

(ii) Assume that $b$ is odd and $N = k \det A_0$ and $h \in \mathcal{H}_{A,N}(1)$ and $n \mid N$. If $N_{T_h} \neq R_{T_h}$, then $\Phi_{A,N}(h, m/n) = 0$. If $N_{T_h} = R_{T_h}$, then $(\mathbb{Z}^2/n \mathbb{Z}^2)/N_{T_h}$ is a cyclic group. Let $g$ be a generator of the cyclic group. Then we have

$$\Phi_{A,N} \left( h, \frac{m}{n} \right) = e \left( \frac{mA[h]}{2nN^2} \right) e \left( \frac{-mA[g] \alpha^2}{2n} \right) \Phi_{A,N} \left( 0, \frac{m}{n} \right), \quad (3.6)$$

where $\alpha \in \mathbb{Z}^2/n \mathbb{Z}$ is chosen so that

$$\left( \frac{1}{N} hA \right)^g \equiv A[g] \alpha \pmod{n}$$

holds.

(iii) Assume that $N_{T_h} = R_{T_h}$ in (ii). Let $g$ be a generator of the cyclic group $(\mathbb{Z}^2/n \mathbb{Z}^2)/N_{T_h}$. If we set $d = \gcd(k, n)$, $n' = n/d$, $L = mA[g]/2d$, $L' = L/\gcd(L, n')$, and $n'' = n'/\gcd(L, n')$, then we have

$$\Phi_{A,N} \left( 0, \frac{m}{n} \right) = \#N_{T_0} \times \gcd(L, n') \left( \frac{L'}{n''} \right) \epsilon_{n''} \sqrt{n''},$$

where $\epsilon_{n''} = 1$ or $\sqrt{-1}$ according as $n'' \equiv 1$ or 3 mod 4.

**Proof of Theorem 3.1(i).** We use the following three transformation formulas:

$$\theta(z + 1; h, A, N) = e \left( \frac{A[h]}{2N^2} \right) \theta(z; h, A, N), \quad (3.7)$$

$$\theta(z; h, A, N) = \sum_{g \equiv h \mod N} \sum_{g \mod c N} \theta(cz; g, cA, cN), \quad (3.8)$$

$$\theta \left( \frac{-1}{z}; h, A, N \right) = \frac{-iz}{\sqrt{\det A}} \sum_{k \mod N} \sum_{Ak \equiv 0 \mod N} e \left( \frac{ikAh}{N^2} \right) \theta(z; k, A, N). \quad (3.9)$$
The first two formulas are obvious by definition. The formula (3.9) is proved in [4, Lemma 4.9.1].

For

\[
\gamma = \begin{bmatrix} m & \mu \\ n & \nu \end{bmatrix} \in \text{SL}_2(\mathbb{Z}),
\]

we have \( n \gamma(z) = m - (nz + \nu)^{-1} \). By applying the formulas (3.8), (3.7), (3.9) and (3.7) in order, we obtain

\[
\begin{align*}
\theta(sz; h, A, N)_{1} |_{\gamma} &= \theta(z; sh, sA, sN)_{1} |_{\gamma} \\
&= (nz + \nu)^{-1} \sum_{g \equiv h \mod N} \theta(n \gamma(z); sg, snA, snN) \\
&= (nz + \nu)^{-1} \sum_{g \equiv h \mod N} e\left(\frac{smA[g]}{2nN^2}\right) \theta(- (nz + \nu)^{-1}; sg, snA, snN) \\
&= \frac{-i}{s \sqrt{\det A}} \sum_{g \equiv h \mod N} e\left(\frac{smA[g]}{2nN^2}\right) \sum_{Ak \equiv 0 \mod N} \sum_{keZ/nN^2} e\left(\frac{1}{nN^2} \theta(nz + \nu; k, snA, snN)\right) \\
&= \frac{-i}{s \sqrt{\det A}} \sum_{Ak \equiv 0 \mod N} \sum_{g \equiv h \mod N} e\left(\frac{smA[g]}{2nN^2}\right) e\left(\frac{2s^t kAg + \nu A[k]}{2snN^2}\right) \theta(nz; k, snA, snN).
\end{align*}
\]

Since the constant term of \( \theta(sz; h, A, N)_{1} |_{\gamma} \) appears only at

\[
k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

and it equals 1, we obtain (3.5) by using (3.4).

**Proof of Theorem 3.1(ii).** By Definition 1.1, we have

\[
\theta(sz; h, A, N) = \theta(z; sh, sA, sN).
\]

(3.10)

Because we may replace \( h, A, N \) by \( sh, sA, sN \), respectively, it suffices to show the formula (3.6) for \( h \in \mathcal{H}_{A,N}(1) \).

Springer showed in [6, Lemma 1.5] that

\[
S(T, G) = \begin{cases} 
0 & (N_T \neq R_T), \\
\#N_T \times S(T, G/N_T) & (N_T = R_T),
\end{cases}
\]

(3.11)

where \( T \) is the quadratic character on \( G/N_T \) induced by \( T \). Therefore, if \( N_{Th} \neq R_{Th} \), then \( \Phi_{A,N}(h, m/n) = 0 \).

We now assume that \( N_{Th} = R_{Th} \). Since \( m/n \) is a cusp, we may assume that \( n | N \). To compute \( S(T_{Th}, (Z^2/nZ^2)/N_{Th}) \) in (3.11), we describe \( N_{Th} \) explicitly. Since \( b \) is odd and thus
so is $\det A_0$, the Smith normal form of $A$ is

\[ UAV = k \begin{bmatrix} 1 & 0 \\ 0 & \det A_0 \end{bmatrix}. \]

Therefore using $N = k \det A_0$, we have

\[ NT_h = \{ l \in (\mathbb{Z}^2/n\mathbb{Z}^2) \mid A l \in n\mathbb{Z}^2 \} = \left\{ l \in (\mathbb{Z}^2/n\mathbb{Z}^2) \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Vl, \begin{bmatrix} kx_1 \\ Nx_2 \end{bmatrix} \in n\mathbb{Z}^2 \right\}. \]

Setting $d = \gcd(k, n)$, $n' = n/d$, we see that $NT_h \cong n'\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-module because $n$ divides $N$ by our assumption. Therefore, we have $(\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h \cong \mathbb{Z}/n'\mathbb{Z}$.

Let $g$ be a generator of the cyclic group $(\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h$. We obtain

\[ S(T_h, (\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h) = \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left(\frac{mA[g]x^2/2d}{n'}\right) e\left(\frac{m^2(h/N)Agx}{n}\right). \]

Since this sum does not depend on the choice of $x \in \mathbb{Z}/n'\mathbb{Z}$, we have

\[ S(T_h, (\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h) = \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left(\frac{Lx^2 + Mx}{n'}\right), \]

where $L = mA[g]/2d$ and $M = m^2(h/N)Ag/d$ are integers.

Next, we show $\gcd(L, n') \mid \gcd(M, n')$. We write

\[ L' = L/\gcd(L, n'), \quad n'' = n'/\gcd(L, n') \]

and we have

\[ \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left(\frac{Lx^2 + Mx}{n'}\right) = \sum_{y_1 \in \mathbb{Z}/n''\mathbb{Z}} e\left(\frac{L'y_1^2}{n''}\right) \sum_{y_2 \in \mathbb{Z}/\gcd(L, n')\mathbb{Z}} e\left(\frac{M(n''y_2 + y_1)}{n'}\right) = \sum_{y_1 \in \mathbb{Z}/n''\mathbb{Z}} e\left(\frac{L'y_1^2}{n''}\right) e\left(\frac{My_1}{n'}\right) \sum_{y_2 \in \mathbb{Z}/\gcd(L, n')\mathbb{Z}} e\left(\frac{My_2}{\gcd(L, n')}\right). \]

From the assumption $NT_h = R_{T_h}$, the above sum is not zero by (3.11). This implies $\gcd(L, n') \mid M$ and thus $\gcd(L, n') \mid \gcd(M, n')$. Therefore, we obtain

\[ S(T_h, (\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h) = \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left(\frac{L'x^2 + M'x}{n''}\right), \]

where $M' = M/\gcd(L, n')$. Since $\gcd(L, n') \mid M$, there exists $\alpha \in \mathbb{Z}/n''\mathbb{Z}$ such that $M' = 2L'\alpha \mod n''$. Consequently, we have

\[ S(T_h, (\mathbb{Z}^2/n\mathbb{Z}^2)/NT_h) = e\left(\frac{-L'\alpha^2}{n''}\right) \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left(\frac{L'(x + \alpha)^2}{n''}\right) \]
From the definition, we can easily deduce
\[ \frac{\Phi_1}{\phi_1} q \]
By Theorem 2.7, we have
\[ \zeta \]
Proof of Theorem 3.1(iii). By (3.3) and (3.11), we have
\[ \Phi_{A,N}(0, \frac{m}{n}) = \# N_{T_0} \times S(T_0, (\mathbb{Z}^2/n\mathbb{Z})^2)/N_{T_0}. \]
We rewrite this using (3.11) and (3.3) and the formula (3.6) follows.

We compute
\[ S(T_0, (\mathbb{Z}^2/n\mathbb{Z})^2)/N_{T_0} = \sum_{x \in \mathbb{Z}/n'\mathbb{Z}} e\left( \frac{L'x^2}{n''} \right) \]
\[ = \text{gcd}(L, n') \sum_{x \in \mathbb{Z}/n''\mathbb{Z}} e\left( \frac{L'x^2}{n''} \right) \]
\[ = \text{gcd}(L, n') \left( \frac{L'}{n''} \right) \epsilon_{n''} \sqrt{n''}, \]
where \( \epsilon_{n''} = 1 \) or \( \sqrt{-1} \) according as \( n'' \equiv 1 \) or 3 mod 4. The last equality follows from the well-known property of the classical Gauss sum.

We remark that Kitaoka studied the case of
\[ h = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
in [3] and obtained the formula for that case.

Example 3.2. Let
\[ A = 9 \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \ N = 63, \ h_1 = 21 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ h_2 = 21 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
By Theorem 2.7, we have
\[ \Theta(z; h_1, A, N, 1) = \theta(z, h_1, A, N) \in M_1(63, \psi_{-7}), \]
\[ \Theta(z; h_2, A, N, 1) = \theta(z, h_2, A, N) \in M_1(63, \psi_{-7}). \]
We compute the constant term of the Fourier expansion of \( \theta(z; h_i, A, N) \) at the cusp \( m/n \in \mathbb{Q} \setminus \mathbb{Z} \) for \( i = 1, 2 \) using Theorem 3.1. The cusps of \( \Gamma_0(N) \) are \( \infty, 0, \frac{1}{2}, \frac{2}{7}, \frac{1}{7}, \frac{1}{27}, \frac{1}{27}, \frac{2}{27} \). For all the cusps \( m/n \in \mathbb{Q} \setminus \mathbb{Z} \), we have to compute \( \Phi_{A,N}(h_i, m/n) \) defined by (3.3) for \( i = 1, 2 \). From the definition, we can easily deduce \( \Phi_{A,N}(h_1, \frac{1}{3}) = 9\zeta_3 \) and \( \Phi_{A,N}(h_1, \frac{1}{3}) = 9\zeta_3 \), where \( \zeta_3 \) is a primitive third root of unity. The equation \( \Phi_{A,N}(h_i, \frac{1}{3}) = 0 \) follows from \( N_0 \neq R_0 \). Using Theorem 3.1, we get \( \Phi_{A,N}(h_1, \frac{1}{3}) = \Phi_{A,N}(0, \frac{1}{3}) = 7\sqrt{-7} \) and \( \Phi_{A,N}(h_1, \frac{1}{27}) = \zeta_3 \Phi_{A,N}(0, \frac{1}{27}) = -63\zeta_3 \sqrt{-7} \). Table 1 gives the details in the case \( h = h_1 \).
Theorem 2.7.

Example 3.2.

Table 1. Values at cusps of $\theta(z; h_1, A, N)$.

| Cusp                  | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{7}$ | $\frac{1}{9}$ | $\frac{1}{21}$ | $\frac{2}{21}$ |
|-----------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $N_{T_{h_1}}$         | $\mathbb{Z}^2 / \mathbb{Z}^2$ | $\mathbb{Z}^2 / \mathbb{Z}^2$ | $\langle 1 \rangle$ | $\mathbb{Z}^2 / \mathbb{Z}^2$ | $\langle 0 \rangle$, $\langle 1 \rangle$ | $\langle 0 \rangle$, $\langle 1 \rangle$ |
| $\# N_{T_{h_1}}$     | 9            | 9            | 7            | 81           | 63           | 63           |
| $R_{T_{h_1}}$         | $\mathbb{Z}^2 / \mathbb{Z}^2$ | $\mathbb{Z}^2 / \mathbb{Z}^2$ | $\langle 1 \rangle$ | $\langle 0 \rangle$, $\langle 1 \rangle$ | $\langle 0 \rangle$, $\langle 1 \rangle$ | $\langle 0 \rangle$, $\langle 1 \rangle$ |
| $(\mathbb{Z}^2 / \mathbb{Z}^2) / N_{T_{h_1}}$ | {1}          | {1}          | $\langle 0 \rangle$ | $\langle 1 \rangle$ | $\langle 0 \rangle$ | $\langle 1 \rangle$ |
| $(g, \alpha)$         | (0, 0)       | (0, 0)       | $\left( \frac{0}{1}, \frac{3}{1} \right)$ | $\left( \frac{0}{1}, \frac{3}{1} \right)$ | $\left( \frac{0}{1}, \frac{3}{1} \right)$ |
| $(d, L, n')$          | (3, 0, 1)    | (3, 0, 1)    | (1, 18, 7)   | (3, 6, 7)    | (3, 12, 7)   |
| $(\gcd(L, n'), L', n'')$ | (1, 0, 1)  | (1, 0, 1)    | (1, 18, 7)   | (1, 6, 7)    | (1, 12, 7)   |
| $\Phi_{A,N}(0, m/n)$  | 9            | 9            | $7\sqrt{-7}$ | $-63\sqrt{-7}$ | $-63\sqrt{-7}$ |
| $e(mA[h_1]/2nN^2)e(-mA[g]a^2/2n)$ | $\zeta_3$  | $\zeta_3^2$ | 1            | $\zeta_3$  | $\zeta_3^2$  |
| $\Phi_{A,N}(h_1, m/n)$ | $9\zeta_3$ | $9\zeta_3^2$ | $7\sqrt{-7}$ | 0            | $-63\sqrt{-7}\zeta_3$ | $-63\sqrt{-7}\zeta_3^2$ |
4. Construction of cusp forms

In this section, we use the following corollary of Theorem 3.1 to construct cusp forms by linear combinations of the theta functions \( \theta(sz; h, A, N) \) whose values at arbitrary cusps are zero.

**Corollary 4.1.** Let \( h \in \mathcal{H}_{A,N}(s) \) and \( \chi \in U(h) \). Then \( \Theta(z; h, A, N, \chi) \) defined by (2.6) is a cusp form, i.e. \( \Theta(z; h, A, N, \chi) \in S_1(\delta s N, \chi \psi_{-\det A}) \) if and only if \( h, A, N, \chi \) satisfy the conditions

(i) \( h \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \),

(ii) \( \chi \neq 1 \),

(iii) for every cusp \( m/n \in \mathbb{Q} \setminus \mathbb{Z} \) of \( \Gamma_0(\delta N) \),

\[
\sum_{u \in (\mathbb{Z}/N\mathbb{Z})^* / U(h)} \chi(u) \Phi_{A,N}(uh, m/n) = 0. \tag{4.1}
\]

**Proof.** The theta series \( \Theta(z; h, A, N, \chi) \) is a cusp form if and only if the constant term of the Fourier expansion of \( \Theta(z; h, A, N, \chi) \) at every cusp of \( \Gamma_0(\delta N) \) is zero. Let

\[
\theta(sz; h, A, N) = \sum_{n=0}^{\infty} a_n q^n \quad \text{and} \quad \theta(sz; h, A, N) |_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sum_{n=0}^{\infty} b_n q^n
\]

be the Fourier expansions at the cusps \( \infty \) and 0, respectively. By (1.5) and (3.9), we have

\[
a_0 = \begin{cases} 
0, & h \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
1, & h = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{cases}, \quad b_0 = \frac{-i}{s \sqrt{\det A}}. \tag{4.2}
\]

We see that \( b_0 \) does not depend on \( h \). Therefore, this proposition follows from (4.2) and Theorem 3.1(i). \( \square \)

**Example 4.2.** Let

\[
A = 9 \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad N = 63, \quad h_1 = 21 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h_2 = 21 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

We shall show

\[
\theta(z; h_1, A, N) - \theta(z; h_2, A, N) \in S_1(63, \psi_{-\gamma}).
\]

We have \( \Phi_{A,N}(h_1, m/n) = \Phi_{A,N}(h_2, m/n) \) for any cusps \( m/n \in \mathbb{Q} \setminus \mathbb{Z} \) by Example 3.2. We see that the constant terms of the Fourier expansions of \( \theta(z; h_1, A, N) \) and \( \theta(z; h_2, A, N) \) at the cusps \( \infty \) and 0 are equal by (4.2). Since the constant term of the Fourier expansion of \( \theta(z; h_1, A, N) - \theta(z; h_2, A, N) \) at any cusps of \( \Gamma_0(\delta N) \) is zero, thus we obtain the result.

The following theorem shows that the condition of Corollary 4.1(iii) is satisfied if we choose an appropriate Dirichlet character \( \chi \).
THEOREM 4.3. Let $h \in \mathcal{H}_{A,N}(1)$ be non-zero and $\chi \in U(\mathcal{h})$. We assume that the diagonal elements of $NA^{-1}$ are even. Let $\phi$ be the map defined by $\phi(n) = n^2$ on $(\mathbb{Z}/N\mathbb{Z})^\times$. If $\chi \notin (\ker \phi)^\perp$, then $\Theta(z; h, A, N, \chi) \in S_1(N, \chi \psi_{-\det A})$.

Proof. By Corollary 4.1, it is sufficient to show that

$$
\sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / U(h)} \chi(u) \Phi_{A,N}(uh, \frac{m}{n}) = 0 \tag{4.3}
$$

for any cusp $m/n \in \mathbb{Q} \setminus \mathbb{Z}$. Since $m/n$ is a cusp of $\Gamma_0(N)$, we may assume $n | N$. We compute

$$
\sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / U(h)} \chi(u) \Phi_{A,N}(uh, \frac{m}{n})
= \sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / U(h)} \chi(u) \sum_{\substack{g \equiv h \mod N \\ g \in \mathbb{Z}/nN\mathbb{Z}^2}} e\left(\frac{mA[ug]}{2nN^2}\right)
= \frac{1}{\#U(h)} \sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(u) \sum_{\substack{g \equiv h \mod N \\ g \in \mathbb{Z}/nN\mathbb{Z}^2}} e\left(\frac{mu^2A[g]}{2nN^2}\right)
= \frac{1}{\#U(h)} \sum_{v \in \ker \phi} \chi(v) \sum_{u \in (\mathbb{Z}/N\mathbb{Z})^\times / \ker \phi} \chi(u) \sum_{\substack{g \equiv h \mod N \\ g \in \mathbb{Z}/nN\mathbb{Z}^2}} e\left(\frac{mu^2A[g]}{2nN^2}\right).
$$

The last equality can be seen by $v^2 \equiv 1 \mod n$. By the assumption $\chi \notin (\ker \phi)^\perp$, we have $\sum_{v \in \ker \phi} \chi(v) = 0$. This completes the proof.

Example 4.4. Let

$$
A = 15 \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}, \quad N = 15^2, \quad h = 15 \begin{bmatrix} 1 \\ 13 \end{bmatrix} \in \mathcal{H}_{A,N}(1).
$$

We construct a cusp form using Theorem 4.3. There exists a Dirichlet character $\chi$ modulo $N$ such that $\chi(U(h)) = \{1\}$ and $\chi(7) = \sqrt{-1}$. Since $\chi \notin (\ker \phi)^\perp$, we have

$$
\Theta(z; h, A, N, \chi) = \theta \left( z; 15 \begin{bmatrix} 1 \\ 13 \end{bmatrix}, A, N \right) + \sqrt{-1} \theta \left( z; 15 \begin{bmatrix} 7 \\ 1 \end{bmatrix}, A, N \right) - \theta \left( z; 15 \begin{bmatrix} 11 \\ 8 \end{bmatrix}, A, N \right) - \sqrt{-1} \theta \left( z; 15 \begin{bmatrix} 2 \\ 11 \end{bmatrix}, A, N \right)
\in S_1(225, \chi \psi_{-15})
$$

by Theorem 4.3.
In [2, Section 7.3], it is shown that $\Theta(z; h, A, N, \chi)$ coincides with the inverse Mellin transformation of the Artin $L$-function of a certain ray class field of the quadratic field $\mathbb{Q}(\sqrt{-15})$. Therefore, it is a newform in $S_1(15^2, \chi \psi_{-15})$.

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