Quantum Enhanced Precision Estimation of Transmission with Bright Squeezed Light

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Squeezed light enables measurements with sensitivity beyond the quantum noise limit (QNL) for optical techniques such as spectroscopy, gravitational wave detection, magnetometry and imaging. Precision of a measurement — as quantified by the variance of repeated estimates — has also been enhanced beyond the QNL using squeezed light. However, sub-QNL sensitivity is not sufficient to achieve sub-QNL precision. Furthermore, demonstrations of sub-QNL precision in estimating transmission have been limited to picowatts of probe power. Here we demonstrate simultaneous enhancement of precision and sensitivity to beyond the QNL for estimating modulated transmission with a squeezed amplitude probe of 0.2 mW average (25 W peak) power, which is 8 orders of magnitude above the power limitations of previous sub-QNL precision measurements of transmission. Our approach enables measurements that compete with the optical powers of current classical techniques, but have both improved precision and sensitivity beyond the classical limit.

Optical measurements are fundamentally limited by quantum fluctuations in the probe. The Poisson distributed photon number $n$ of coherent light — often used as a probe in classical experiments — results in shot-noise, which represents the quantum noise limit (QNL) in the precision of parameter estimation with classical resources [1]. Because the QNL scales with $\sim 1/\sqrt{n}$, longer measurements and higher intensity can increase precision. We may also increase precision with more interaction between probe and sample via multiple passes [2, 3] or optimising sample length [4]. However, there can often exist restrictions on the total optical exposure, the measurement time and sample properties [5]. By using non-classical light, the fluctuations in the probe can be reduced below the QNL, thus providing ‘sub-shot-noise’ precision per photon [6]. The QNL defines the best precision achievable without the use of quantum correlations for a given apparatus and average photon number [7]. This is distinguished from the standard quantum limit (SQL), which defines a measurement-independent limit to the precision that may be achieved using a minimum uncertainty state of a given average photon number, without quantum resources [8]. Because squeezed light can offer significant reduction in noise below the QNL [9], and can be generated with arbitrary intensity using coherent laser light [10], it offers a practical approach for enhancing optical techniques beyond classical limitations.

Precision in measuring a parameter can be quantified by the inverse of the variance of corresponding measurement outcomes, and is bounded by the Fisher information according to the Cramér-Rao bound [11]. By contrast, the sensitivity of a measurement is the smallest possible signal that may be observed [12], and thus depends only on the signal-to-noise ratio (SNR). Photonic (definite photon number) quantum metrology uses photon counting to observe quantum correlations between modes to reduce the noise of a measurement [13–15]. Here one can attain a quantum advantage in both precision and sensitivity, by increasing the Fisher information and reducing the optical noise floor. However, due to limitations in both the maximum photon flux and detector saturation power, the probe powers achievable are in practice $O(10^6)$ photons detected per second (pW) [13, 14], which limits use to only cases that are reliant on ultra low intensities. Homodyne detection of squeezed vacuum has been used to estimate phase with sub-QNL precision [16, 17]. This is possible because measurement is performed away from low frequency technical noise, in a shot-noise limited bandwidth where squeezing reduces vacuum noise. However, as with photonic quantum metrology, strategies using squeezed vacuum for sub-QNL precision have also been restricted in maximum optical probe power.

Measurements using high power squeezed light can reach sub-QNL sensitivity in detection of phase modulation [6] and amplitude modulation (AM) [18]. This is because modulation introduces AC components in the detected signal, which can be made to coincide with a shot-noise-limited detector bandwidth, while squeezed light reduces the optical noise relative to the signal. The sensitivity of any frequency domain measurement of an optical signal in a shot-noise limited bandwidth may be improved by such techniques and this has been demonstrated in a range of applications (e.g. [7, 19–30]). However, enhancing sensitivity is not a sufficient condition to enhance precision. When bright optical probes are used, the noise in the bright field often dominates over the vacuum noise, which prohibits the use of squeezed light for reaching precision beyond the QNL. For squeezed light to provide a precision improvement in such a measurement, the variance of the measured signal must be limited by optical shot-noise. Here, we fulfil this condition and use bright amplitude squeezed light to measure modulated transmission with precision beyond the QNL.

The parameter estimated in this work is the modulation index $\delta_m = (P - P')/P$, where $P$ and $P'$ are the maximum and minimum output power transmitted through a modulated loss (Fig. 1(e)) [1]. For modulation frequency $\Omega$, sinusoidal AM generates optical sidebands at $\pm \Omega$ from the carrier frequency. Upon photodetection, this leads to a single electronic side-
band in the spectral noise power at frequency $\Omega$ that contains information about $\delta_m$. Figure 1(a-c) illustrate the behaviour of the spectral noise power of an initial laser input (a), where the noise characteristics at $\Omega$ approximate that of a coherent state $|\alpha\rangle$ and so quantum noise dominates the variance of intensity. The light is subsequently squeezed in amplitude (b) and then modulated in amplitude (c). The insets illustrate the ideal evolution of the state at $\pm\Omega$ for an initial coherent state $|\alpha\rangle$. The final state is amplitude squeezed with an average photon number of $\langle n(\pm\Omega)\rangle = \delta_m|\alpha|^2/2$.

We derive an estimator for $\delta_m$ from the SNR of direct photodetection, similar to [18] which uses homodyne detection. For direct photodetection of AM in a shot-noise limited bandwidth around $\Omega$, the SNR is given by

$$\delta_{SNR} = \frac{\langle p_s \rangle}{\langle p_n \rangle}$$

where $\langle p_s \rangle$ is the average signal component of the generated electronic power at $\Omega$, and $\langle p_n \rangle$ is the average electronic power generated from optical noise. In the limit of weak AM ($\delta_m \ll 1$) loss due to AM has a negligible effect on the squeezing parameter $\Phi$ and the average optical power on the modulator output. Therefore, average measured photocurrent is expressed as $i_0 = q\eta P(1 - (\delta_m/2))/\hbar\omega \approx q\eta P/\hbar\omega$, with electron charge $q$, photodiode efficiency $\eta$, reduced Planck constant $\hbar$ and carrier angular frequency $\omega$. We then obtain

$$\delta_{SNR} = \frac{\langle p_s \rangle}{\langle p_n \rangle} \approx \frac{\delta_m i_0}{4q\Phi B}.$$  

(see Appendix Section 1), where $B$ is the frequency resolution bandwidth (RBW) of the noise spectrum, and corresponds to the inverse of the integration time over which the spectrum is measured. From Eq. (1), we define the estimator

$$\hat{\delta}_m = \sqrt{\frac{4q\Phi B\delta_{SNR}}{i_0}},$$

(2)

where

$$\delta_{SNR} = \frac{p_0 - p_N}{p_N - p_E}, \quad \text{and} \quad i_0 = \frac{q\eta(P)}{\hbar \omega}. \quad (3)$$

$p_0$, $p_N$ and $p_E$ are the spectral noise powers of the electronic sideband, the optical noise floor and the electronic noise floor respectively. $\langle P \rangle$ is the average optical power output from the modulator, and both $\langle P \rangle$ and $\langle N \rangle$ may be pre-calibrated with high precision. The dependence of $\delta_m$ on the optical noise is then contained in the measurement of $p_0$.

For an input resistance of $R$ to the measuring device (e.g. spectrum analyser or oscilloscope), we can define the power of the electronic sideband as

$$p_0 = 2R|\hat{i}_\Omega|^2,$$

(4)

where $\hat{i}_\Omega$ is the photocurrent in the frequency bin centered on $\Omega$. By considering power fluctuations due to quantum optical noise, low frequency classical optical noise, and electronic noise, we find

$$\text{Var}(p_0) = \frac{\langle p_0^2 \rangle - \langle p_0 \rangle^2}{\langle p_0 \rangle^2} \approx \frac{R^2}{M} \left[ 2q^2\delta_m^2 i_0^4 \Phi B + 4q^2\delta_m^4 i_0^4 \text{Var}[\Re[H]] \right]$$

(5)
We find that for higher RBWs, squeezing provides sub-QNL precision (dashed lines), with all other parameters fixed. We find Ω limited at the RBW for a typical laser source which is quantum noise propagation. We find the number of photons detected in the measurement time as unbiased. Therefore, light. The variance of F can be obtained from standard error analysis with [33] 

\[ F(\delta_{SNR}) = \frac{1}{\text{Var}(\hat{\delta}_{SNR})} = \left[ \left( \frac{\partial(\hat{\delta}_{SNR})}{\partial(\hat{\delta}_{m})} \right)^2 \text{Var}(\hat{\delta}_{m}) \right]^{-1} \]  

(6)

F(\delta_{m}) can be obtained from F(\delta_{SNR}) by using [33]

\[ F(\delta_{m}) = \left( \frac{\partial(\delta_{SNR})}{\partial(\delta_{m})} \right)^2 F(\delta_{SNR}) \]  

(7)

We find that Var[\Re(N)] contributes negligibly to F(\delta_{m}), and from Eq. (2-7), this leads to

\[ F(\delta_{m}) \approx M \left[ \frac{2\Phi B}{\lambda_0} + 4\delta_m^2 \text{Var}(\Re(\hat{H})) \right]^{-1} \]  

(8)

The quantum advantage is then the ratio Q(\delta_{m}) between the values of F(\delta_{m}) for squeezed (Φ < 1) and coherent (Φ = 1) light. The variance of \δ_{m} can be obtained by standard error propagation. We find

\[ \text{Var}(\hat{\delta}_{m}) = \left( \frac{\partial(\hat{\delta}_{m})}{\partial(\hat{\delta}_{m})} \right)^2 \text{Var}(\hat{\delta}_{m}) = \frac{1}{F(\delta_{m})} \]  

(9)

Therefore, \delta_{m} is an efficient estimator. We also find that, in the limit of weak AM, \langle \delta_{m} \rangle = \delta_{m}, meaning our estimator is unbiased.

The Fisher information per detected photon may be defined as \( F'(\delta_{m}) = F(\delta_{m}) / \langle N \rangle \), where \( \langle N \rangle = i_0 / qB \) is the number of photons detected in the measurement time \( B^{-1} \). Figure 1(d) illustrates the dependence of \( F'(\delta_{m}) \) on the RBW for a typical laser source which is quantum noise limited at Ω (solid black line) and various levels of squeezing (dashed lines), with all other parameters fixed. We find that for higher RBWs, squeezing provides sub-QNL precision in estimating \delta_{m}. This can be seen from Eq. (8), since for \( 2\Phi B / i_0 \gg 4\delta_m^2 \text{Var}(\Re(\hat{H})) \), quantum noise limits the precision of the measurement, and we find \( Q(\delta_{m}) \rightarrow Q_{opt} \), where

\[ Q_{opt} = \frac{1}{\Phi} \]  

(10)

Because all information on \delta_{m} is contained at modulation frequency Ω, this model suggests a practically achievable quantum advantage per detected photon.

For the measurement, we built a source of amplitude squeezed light, based on [34]. 100 fs pulses with central wavelength \( \lambda_0 = 740 \) nm from a Spectra Physics Mai Tai Ti:Sapphire laser are coupled into an asymmetric Sagnac interferometer, with a 90:10 splitting ratio beamsplitter (BS) (Fig. 1(e)). 14 m of photonic crystal fibre (PCF) provides a strong \( \chi(3) \) nonlinearity in the interferometer. The fibre samples used were originally fabricated for photon pair generation work [35]. As the brighter 90° reflected pulses propagate through the PCF, they undergo self-phase modulation and become quadrature squeezed [10, 34]. These pulses interfere with the weaker (10%) counter-propagating pulses transmitted initially at the BS — these provide a coherent displacement in phase space. This leads to amplitude squeezing on the output of the interferometer [34]. The chosen central wavelength of \( \lambda_0 = 740 \) nm is close to the 730 nm zero-dispersion wavelength of the PCF, in order to minimise the spectral broadening, which enables optimal interference at the 90:10 BS. The zero-dispersion wavelength of the PCF may be tailored by the fibre structure, making this approach applicable to a large range of wavelengths. The average optical power of the output state is 0.2 mW, which equates to 25 W of peak power. The amplitude squeezed light passes through a Thorlabs EO-AM-NR-C1 electro-optic modulator (EOM), that modulates polarisation. A subsequent polarising beamsplitter (PBS) translates the polarisation modulation into a weak AM of depth \delta_{m}, and generates optical sidebands at distance ±Ω from the carrier frequency. The resulting state is measured with direct detection, by collecting all the light at photodiode PD1. We calibrate the shot-noise level using the balanced subtraction photocurrent of PD1 and PD2. The balanced amplified photodetector used is a Thorlabs PDB440A(-AC). The spectral photocurrent is analysed with a Rohde & Schwarz FPC1000 spectrum analyser.

Figure 2(a) shows the relative noise power traces of amplitude modulated squeezed light (−1.2 dB) and antisqueezed light (2.7 dB) produced by the setup. The RBW is \( B = 10 \) kHz, which is considerably wider than the linewidth of the optical sidebands, measured to be < 1 Hz. The frequency separation of trace points in Fig. 2(a) is smaller than the RBW since the trace is a result of multiple samples within each RBW interval. This measurement demonstrates enhanced sensitivity detection of AM due to amplitude squeezing, as shown in [18]. The antisqueezed data in Fig. 2(a) is corrected for the difference in optical power required to generate anti-squeezing and squeezing, by subtracting the difference in the respective shot-noise levels from this trace. The electronic noise has also been subtracted from each trace.

From Eq. (9) we know that Var(\delta_{m}) is proportional to Φ and inversely proportional to \langle P \rangle. However, the profile of squeezing with optical power is such that change in power is negligible across the maximum observed squeezing range [−1.6, 2.7] dB in our setup, so here Var(\delta_{m}) scales linearly...
with $\Phi$. By fitting measured $\text{Var}(\delta_m)$ to a line, we infer measured quantum advantage using

$$Q(\delta_m) = \frac{\text{Var}(\delta_m)}{\text{Var}(\delta_m)}^{QNL},$$

where $\text{Var}(\delta_m)$ is the variance of estimates of $\delta_m$, for coherent and squeezed light respectively.

Figure 2(b) shows measured $Q(\delta_m)$ for a range of squeezing $\Phi$ with fixed RBW $B = 100$ kHz. The value of $\delta_m$ was taken from 50 samples of $\delta_m$. Due to the high RBW, quantum noise limits the variance of $\delta_m$, and the measurement saturates the optimal quantum bound $Q_{opt}$ given by Eq. (10) (red curve). A quantum advantage of $Q(\delta_m) = 1.44 \pm 0.09$ is observed with $-1.6$ dB of squeezing, in agreement with $Q_{opt} = 1.45$.

By repeating this for a range of $B$ and a fixed $-1.3$ dB of average squeezing, we plot in Fig. 2(c) the dependence of $Q(\delta_m)$ on RBW. The red curve is a theoretical fitting calculated using Eq. (8), with $\text{Var}(|\Psi|^2)$ as a fitting parameter, which gives $\text{Var}(|\Psi|^2) = 7 \pm 1 \times 10^{-6}$. We observe sub-QNL precision down to $B = 100$ Hz. The maximum quantum advantage observed here is $Q(\delta_m) = 1.34 \pm 0.07$, which again closely agrees with the optimal $Q_{opt} = 1.35$ for the average squeezing parameter of $\Phi = 0.74$. In Fig. 2(b) and Fig. 2(c), the error bars were derived from standard propagation of error calculations, with the data averaged over 236 variance measurements. Reducing optical loss to measure higher squeezing [9] enables greater enhancement in precision. Accounting for detection efficiency (we measure $\eta_d = 0.84$) and coupling efficiency between the interferometer output and the detector ($\eta_{opt} = 0.81$), we infer our maximum measured squeezing value of $-1.6$ dB corresponds to $-2.6$ dB of generated amplitude squeezing.

We have demonstrated quantum enhanced precision parameter estimation with bright squeezed amplitude light. Our model shows the degree of precision enhancement is dictated by the amount of squeezing, the RBW and the classical noise on the generated sidebands. This exemplifies that for measurements of high power optical signals, sub-shot-noise sensitivity does not alone imply sub-shot-noise precision. We verify our model with experiment, reporting up to a $44\%$ quantum advantage in precision in the estimation of the modulation index, per detected photon. This general demonstration motivates applications in areas such as spectroscopy [36] and imaging [37], where precision may determine the performance of the measurement, which can be improved by using squeezed light. We use amplitude squeezed light of $0.2$ mW average optical power ($25$ W peak power) as a probe. This power is comparable to the photon dose required to induce a photophobic response in living cells [39], therefore indicating this technique’s relevance to biological measurements.

Supporting data is available on request.

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1. Calculation of the Signal-to-Noise Ratio

Here we derive the expected value of the electronic power $p_\Omega$ in the $\pm B/2$ frequency interval centered on the modulation frequency $\Omega$, generated by the current $i(t)$. We may write $p_\Omega$ as

$$p_\Omega = 2R|\hat{i}_\Omega|^2,$$

where $R$ is the input resistance, $\hat{i}_\Omega = \int_\Omega^{\Omega+B/2} i(\nu)d\nu$ is the photocurrent in the frequency bin centered on $\Omega$, using the notation $\int_\Omega^{\Omega+B/2} \equiv \int_{\Omega-B/2}^{\Omega}$, and the factor of 2 comes from the integration over positive and negative frequencies. It is important to note that this differs from the standard definition of the power in a frequency band, $p_\Omega = 2R \int_\Omega^{\Omega+B/2} |i(\nu)|^2d\nu$ [1]. The reason for the definition used in Eq. (12) is that measuring devices such as spectrum analysers and oscilloscopes are fundamentally voltage detectors, and therefore the displayed power level is computed from the voltage measured in a given frequency bin. This means that the integration of the photocurrent density effectively occurs before taking the absolute square. While the average value $\langle p_\Omega \rangle$ does not significantly differ between these definitions, Var$(p_\Omega)$ does. To keep the model consistent with the measurements obtained by our spectrum analyser, we will use Eq. (12) as a definition for $p_\Omega$.

The amplitude of the optical field before modulation is applied may be written as $\hat{A}_0(t) = [1 + \zeta(t)]\alpha_0 e^{i\theta} + \hat{a}(t)$, where $\theta$ is the phase of the classical field, $\hat{a}(t)$ describes the quantum amplitude fluctuations and $\zeta(t)$ is a stochastic noise function which corresponds to the low frequency classical noise of the laser. We have assumed a continuous-wave amplitude $\alpha_0$ for simplicity. The amplitude of the detected light after modulation may then be written as

$$\hat{A}(t) = (\Psi_0 + \Psi_m \cos(2\pi \Omega t))([1 + \zeta(t)]\alpha e^{i\theta} + \hat{a}(t)),$$

where $\Psi_0$ and $\Psi_m$ are related to the modulation index by $\Psi_0 = 1 - \delta_m/2$ and $\Psi_m = \delta_m/2$, and the detection efficiency $\eta$ is modelled as an additional loss before detection, such that $\alpha = \sqrt{\eta}\alpha_0$. By making the assumption of large amplitude $\alpha \gg 1$ and small modulation $\Psi_m \ll 1$, we can approximate $\hat{A}(t)$ as

$$\hat{A}(t) \approx (\Psi_0 + \Psi_m \cos(2\pi \Omega t))[1 + \zeta(t)]\alpha e^{i\theta} + \hat{a}(t) \equiv |\alpha(t)|e^{i\theta} + \hat{a}(t),$$

where the effect of amplitude modulation on the quantum noise term has been neglected. The current at time $t$ may then be written as

$$\hat{i}(t) = q \left( \hat{A}(t)^\dagger \hat{A}(t) + n_e(t) \right) = q \left( |\alpha(t)|^2 + \sqrt{2}|\alpha(t)|\hat{x}_\theta(t) + \hat{a}(t)^\dagger \hat{a}(t) + n_e(t) \right),$$

where we have defined the quadrature operator $\hat{x}_\theta(t) = \frac{1}{\sqrt{2}}[\hat{a}(t)e^{-i\theta} + \hat{a}(t)^\dagger e^{i\theta}]$ and the electronic noise term $n_e(t)$ corresponds to the number of electrons generated independently of the optical field. The component of the photocurrent at frequency $\nu$ is then given by

$$\hat{i}(\nu) = q \left[ I(\nu) + \sqrt{2} \int \alpha(\mu)\hat{x}_\theta(\nu - \mu)d\mu + \int \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu)d\mu \right],$$

where $\int \equiv \int_{-\infty}^{\infty}$, and we have defined the unitary Fourier transforms

$$I(\nu) = \int (|\alpha(t)|^2 + n_e(t)) e^{-2\pi i\nu t}dt$$

and

$$\hat{x}_\theta(\nu) = \int \hat{x}_\theta(t)e^{-2\pi i\nu t}dt = \frac{1}{\sqrt{2}}[\hat{a}(\nu)^\dagger e^{i\theta} + \hat{a}(\nu)e^{-i\theta}].$$

We also define $\hat{a}(\nu)$ as the squeezed vacuum operator [40]

$$\hat{a}(\nu) = \hat{b}(\nu) \cosh r(\nu) - e^{2i(\nu)}\hat{b}(\nu)^\dagger \sinh r(\nu),$$

and

$$\hat{b}(\nu) = \int \hat{b}(\mu)e^{-2\pi i\nu \mu}d\mu.$$
where $\hat{b}(\nu)$ and $\hat{b}^\dagger(\nu)$ are the bosonic creation and annihilation operators. The squeezing is defined such that $r(\nu) = r$ and $\theta(\nu) = \theta$ within the frequency bandwidth $-\Lambda/2 \leq \nu \leq \Lambda/2$ (where $\Lambda/2 > \Omega$) and $r(\nu) = 0$ outside of this frequency range. The $2\theta$ phase factor then orients the squeezing in the amplitude direction. By defining $I_\Omega = \int \Omega I(\nu) d\nu$ we can write $p_\Omega$ as

$$
p_\Omega = 2q^2 R \left[ |I_\Omega|^2 + \sqrt{2} I^*_\Omega \int \Omega \int \alpha(\mu) \hat{x}_\theta(\nu - \mu) d\mu d\nu + I^*_\Omega \int \Omega \int \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu - \mu) d\mu d\nu \\
+ \sqrt{2} \int \Omega \int \int \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger d\mu d\nu + 2 \int \Omega \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger d\mu d\nu d\nu d\sigma \\
+ \sqrt{2} \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\nu d\sigma + I_\Omega \int \Omega \int \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu \\
+ \sqrt{2} \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\nu d\sigma \\
+ \int \Omega \int \int \int \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\nu d\sigma \right],
$$

where $(\bullet)^*$ denotes the complex conjugate. From Eq. (14), we can find the frequency dependence of the classical field amplitude:

$$
\alpha(\nu) = \int |\alpha(t)|e^{-2\pi t\nu} dt = \alpha \left[ \Psi_0(\delta(\nu + h(\nu)) + \frac{\Psi_m}{2} (\delta(\nu - \Omega) + \delta(\nu + \Omega) + h(\nu - \Omega) + h(\nu + \Omega)) \right],
$$

where $h(\nu) = \int \zeta(t)e^{-2\pi t\nu} dt$, and since classical noise is only observed at low frequencies ($\lesssim 2$ MHz), we can write for example $h(\Omega) = 0$. Equation (14) allows us to define $I(\nu)$ as

$$
I(\nu) = \alpha^2 \left[ \Psi_0^2 \Psi_m^2 \delta(\nu + 2h(\nu) + \int h(\mu)h(\nu - \mu)d\mu + \Psi_0 \Psi_m \left( \delta(\Omega - \nu) + \delta(\Omega + \nu) \right) \\
+ 2h(\nu - \Omega) + 2h(\nu + \Omega) + \int h(\mu)h(\nu - \mu - \Omega)d\mu + \int h(\mu)h(\nu - \mu + \Omega)d\mu \right) + \\
+ \Psi_m^2 \left( \delta(\nu - 2\Omega) + \delta(\nu + 2\Omega) + 2\delta(\nu) + 2h(\nu - 2\Omega) + 2h(\nu + 2\Omega) + 4h(\nu) + 2h(\nu - \Omega) + 2h(\nu + \Omega) \right) \\
+ \int h(\mu)h(\nu - \mu - 2\Omega)d\mu + \int h(\mu)h(\nu - \mu + 2\Omega)d\mu + 2 \left( \int h(\mu)h(\nu - \mu)d\mu \right) \right) + n_c(\nu).
$$

We can then find the expectation $\langle p_\Omega \rangle$ with respect to the random variables $h(\nu)$, $n_c(\nu)$ and $\hat{a}(\nu)$. Since these variables are independent, the expectation value may be defined as $\langle \bullet \rangle \equiv \langle \langle 0 | \bullet | 0 \rangle \rangle_{h(\nu), n_c(\nu)}$. To calculate this, we first compute

$$
\langle p_\Omega \rangle = 2q^2 R \left[ \int \Omega^2 + \Phi \int \Omega \int \int \alpha(\mu)^* \alpha(\mu + \nu - \nu) d\mu d\nu + 4 \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger d\mu d\nu d\sigma + 2 \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\sigma \right] + \int \Omega \int \int \int n_c(\nu)d\nu d\sigma,
$$

where $\Re[\bullet]$ signifies the real part. Then, by evaluating the quantum part of the expectation value, we obtain the result:

$$
\langle p_\Omega \rangle = 2q^2 R \left[ 4 \left( \int \Omega^2 \right) + \Phi \int \Omega \int \int \alpha(\mu)^* \alpha(\mu + \nu - \nu) d\mu d\nu + 4 \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger d\mu d\nu d\sigma + 2 \int \Omega \int \int \int \alpha(\mu)^* \alpha(\mu) \hat{x}_\theta(\nu - \mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\sigma \right] + \int \Omega \int \int \int n_c(\nu)d\nu d\sigma.
$$
\[ +4\Psi_0^2 \Psi_m^2 \left( \int_0^\Omega h(\nu - \Omega) d\nu \right)^2 + 4\Psi_0^2 \Psi_m^2 \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \Omega)^* h(\mu) h(\nu - \Omega)] \rangle d\mu d\nu \]

\[ + \Psi_0^2 \Psi_m^2 \left( \int_0^\Omega \int_0^\Omega (h(\mu) h(\nu - \Omega) d\mu d\nu \right)^2 \right) + \alpha^2 \left( 2\Psi_0 \Psi_m \int_0^\Omega \langle \mathcal{R}[n_c(\nu)] \rangle d\nu \right.

\[ + 4\Psi_0 \Psi_m \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \Omega)^* n_c(\nu)] \rangle d\nu d\sigma + 2\Psi_0 \Psi_m \int_0^\Omega \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\mu) h(\nu - \Omega)^* n_c(\nu)] \rangle d\mu d\nu d\sigma \]

\[ + \left( \Psi_0^2 + \frac{\Psi_m^2}{2} \right) \Phi B + (2\Psi_0^2 + \Psi_m^2) \Phi \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \sigma)] \rangle d\nu d\sigma \right) \)

\[ + \frac{\Phi^2}{8} + \frac{1}{8\Phi^2} - \frac{1}{4} \right]. \tag{24} \]

for the squeezing parameter \( \Phi = e^{-2\sigma} \). In Eq. (24), we have neglected terms involving the expectation value of the product of an odd number of creation or annihilation operators, and terms outside the domain of \( h(\nu) \). By observing that \( \alpha \gg 1 \), \( \langle |\int h(\nu) d\nu| \rangle \ll 1 \) and \( \delta_m \ll 1 \) for \( I_0 \approx \tilde{q} \alpha^2 \) we find

\[ \langle p_\Omega \rangle \approx R \left( \frac{\tilde{q}^2 \delta_m^2}{4} + 2q_i \Phi B + 2q^2 \left( \left| \int_0^\Omega n_c(\nu) d\nu \right|^2 \right) \right). \tag{25} \]

Similarly, at a small frequency interval \( \Delta \nu \) from \( \Omega \), we find that the electronic power of the optical noise floor and electronic noise floor are respectively

\[ \langle p_N \rangle \approx R \left( 2q_i \Phi B + 2q^2 \left( \left| \int_0^\Omega n_c(\nu) d\nu \right|^2 \right) \right) \quad \text{and} \quad \langle p_E \rangle = R \left( 2q^2 \left( \left| \int_0^\Omega n_c(\nu) d\nu \right|^2 \right) \right). \tag{26} \]

We then find that the optical signal-to-noise ratio \( \delta_{\text{SNR}} \) may be expressed as

\[ \delta_{\text{SNR}} = \frac{\langle p_\Omega \rangle - \langle p_N \rangle}{\langle p_N \rangle - \langle p_E \rangle} \approx \frac{I_0 \delta_m^2}{4q \Phi B}. \tag{27} \]

2. Calculation of the Variance of the Sideband Power

In order to calculate the variance \( \text{Var}(p_\Omega) = \langle p_\Omega^2 \rangle - \langle p_\Omega \rangle^2 \), an expression for \( \langle p_\Omega \rangle^2 \) may be evaluated directly from Eq. (24) to give

\[ \langle p_\Omega \rangle^2 = 4q^4 R^2 \left[ (|I_\Omega|^2)^2 + (2\Psi_0^2 + \Psi_m^2)(|I_\Omega|^2)^2 \right] \alpha^2 \left( \Phi B + 2\Phi \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \sigma)] \rangle d\nu d\sigma \]

\[ + \Phi \int_0^\Omega \int_0^\Omega \left( h(\nu) h(\mu) h(\nu - \sigma) \right) d\mu d\nu d\sigma \right) + (|I_\Omega|^2)^2 \Lambda B \left( \frac{\Phi^2}{4} + \frac{1}{4\Phi^2} - \frac{1}{2} \right)

\[ + \left( \Psi_0^2 + \frac{\Psi_m^2}{2} \right) \Phi B + (2\Psi_0^2 + \Psi_m^2) \Phi \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \sigma)] \rangle d\nu d\sigma \]

\[ + 2\Phi^2 B \int_0^\Omega \int_0^\Omega \left( h(\nu) h(\mu + \nu - \sigma) \right) d\mu d\nu d\sigma + 4\Phi^2 \int_0^\Omega \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \sigma)] \rangle \langle \mathcal{R}[h(\bar{\nu} - \bar{\sigma})] \rangle d\nu d\sigma d\bar{\nu} d\bar{\sigma} \]

\[ + 4\Phi^2 \int_0^\Omega \int_0^\Omega \int_0^\Omega \int_0^\Omega \langle \mathcal{R}[h(\nu - \sigma)] \rangle \langle h(\mu) h(\mu + \nu - \sigma) \rangle d\mu d\nu d\sigma d\bar{\nu} d\bar{\sigma} \]

\[ + \Phi^2 \int_0^\Omega \int_0^\Omega \int_0^\Omega \int_0^\Omega \langle h(\mu) h(\nu - \sigma) \rangle \langle h(\bar{\mu}) h(\bar{\nu} - \bar{\sigma}) \rangle d\mu d\nu d\sigma d\bar{\nu} d\bar{\sigma} \]
\[ \langle \hat{p}^2 \rangle = 4q^2 R^2 \left( |I_\Omega|^4 + 4|I_\Omega|^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{1\Omega}^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{2\Omega} \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 4|I_{1\Omega}|^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 3|I_{1\Omega}|^2 \int_\Omega \int \int \int \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger d\mu d\nu d\varphi + |I_{1\Omega}|^2 \int_\Omega \int \int \int \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{2\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) + 4 \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2 \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) + 4q^2 R^2 \left( |I_\Omega|^4 + 4|I_\Omega|^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{1\Omega}^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{2\Omega} \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 4|I_{1\Omega}|^2 \int_\Omega \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 3|I_{1\Omega}|^2 \int_\Omega \int \int \int \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger d\mu d\nu d\varphi + |I_{1\Omega}|^2 \int_\Omega \int \int \int \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger \hat{a}(-\mu)^\dagger \hat{a}(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{2\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu) \hat{x}_\phi(\nu - \mu) d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2I_{1\Omega} \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) + 4 \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi + 2 \int_\Omega \int \int \int \int \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \alpha(\mu)^* \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger \hat{x}_\phi(\nu - \mu)^\dagger d\mu d\nu d\varphi \right) \right). \] (28)
\[ + 2 \int_\Omega \int_\Omega \int_\Omega \left\{ \int \int \int \left( \alpha(\mu) \ast \alpha(\pi) \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right) \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

To explicitly evaluate \( \langle p^2 \rangle \) in the following, we calculate the integrals in Eq. (29) separately, using Eq. (21) and the commutation relations of the boson operators with the expectation value taken on the vacuum state. Terms 2-5 of Eq. (29) respectively give:

\[ \left\langle 4 |I| \right\rangle \left\{ \left\langle I \right\rangle \ast \alpha(\pi) \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

\[ \left( 2 \Psi_3^2 + \Psi_m^2 \right) \alpha^2 \Phi \left( \left\langle I \right\rangle \ast \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

\[ \left( 2 \Psi_3^2 \right) \left( 2 \Psi_m^2 \right)^2 \alpha^2 \Phi \left( \left\langle I \right\rangle \ast \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

\[ \left( 2 \Psi_3^2 \right) \left( 2 \Psi_m^2 \right)^2 \alpha^2 \Phi \left( \left\langle I \right\rangle \ast \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

and

\[ \left( 4 \Psi_3^2 \right) \left( 4 \Psi_m^2 \right)^2 \alpha^2 \Phi \left( \left\langle I \right\rangle \ast \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)

\[ \left( 4 \Psi_3^2 \right) \left( 4 \Psi_m^2 \right)^2 \alpha^2 \Phi \left( \left\langle I \right\rangle \ast \hat{a}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \eta(\nu - \pi) \hat{\eta}(\nu - \pi) \hat{a}(\nu - \mu) \hat{\eta}(\nu - \pi) \hat{\eta}(\nu - \pi) \right\} \right\} \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right) \cdot \right)
The summation of terms 6 and 7 of Eq. (29) gives

\[
3|I_\Omega|^2 \int_\Omega \int \int \int \hat{a}(\nu - \mu)\hat{a}(-\mu)\hat{a}(\bar{\nu})\hat{a}(-\bar{\mu})d\mu d\bar{\nu} d\bar{\mu} d\nu \\
+ \left| I_\Omega \right|^2 \int_\Omega \int \int \int \hat{a}(-\mu)\hat{a}(\nu - \mu)\hat{a}(\bar{\nu})\hat{a}(-\bar{\mu})d\mu d\bar{\nu} d\bar{\mu} d\nu \right) = |I_\Omega|^2 \Lambda B \left( \frac{\Phi^2}{2} + \frac{1}{2\Phi^2} - 1 \right). \tag{34}
\]

It is also possible to combine terms 8-15 of Eq. (29) as follows:

\[
2 \int_\Omega \int \int \int \left[ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{\xi}_\theta(\bar{\nu}) \hat{a}(-\bar{\mu})\hat{a}(\nu - \mu) \hat{a}(\bar{\nu}) \hat{a}(-\mu) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu})\hat{a}(\nu - \mu) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \right] d\mu d\bar{\nu} d\bar{\mu} d\nu \bar{\nu} d\bar{\mu} d\nu \bar{\nu} d\bar{\mu} d\nu \\
= (4\Phi^2 - 2) \int_\Omega \int \int \int \left[ \langle \Im[I_\Omega]\alpha(\mu)\alpha(\mu + \nu - \bar{\nu})\rangle d\mu d\nu d\bar{\nu} d\bar{\mu} \right] \\
= \Psi_0 \Psi_m \alpha^2 (4\Phi^2 - 2) \left( \langle \Im[I_\Omega]\rangle B^2 + 2 \int_\Omega \int \int \int \langle \Im[I_\Omega]\Im[h(\nu + \nu - \bar{\nu} - \Omega)]\rangle d\nu d\nu d\bar{\nu} d\bar{\mu} \right) \\
+ \int_\Omega \int \int \int \langle \Im[I_\Omega]\Im[h(\mu - \Omega + \nu + \nu - \bar{\nu})]\rangle d\nu d\nu d\bar{\nu} d\bar{\mu} \right). \tag{35}
\]

Similarly, we find that terms 16-19 of Eq. (29) simplify as

\[
2 \int_\Omega \int \int \int \int \left[ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{\xi}_\theta(\bar{\nu}) \hat{a}(-\bar{\mu})\hat{a}(\nu - \mu) \hat{a}(\bar{\nu}) \hat{a}(-\nu) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu})\hat{a}(\nu - \mu) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \\
+ \langle I_\Omega^* \alpha(\mu)\alpha(\bar{\mu})\hat{\xi}_\theta(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(\nu - \mu)\hat{a}(\bar{\nu}) \hat{a}(-\bar{\mu}) \rangle \right] d\mu d\bar{\nu} d\bar{\mu} d\nu \bar{\nu} d\bar{\mu} d\nu \bar{\nu} d\bar{\mu} d\nu \\
= \Phi^2 \int_\Omega \int \int \int \langle I_\Omega^* \alpha(\mu)\alpha(\mu + \nu - \bar{\nu})\rangle d\mu d\nu d\bar{\nu} d\bar{\mu} \\
+ \Phi^2 \int_\Omega \int \int \int \langle I_\Omega^* \alpha(\mu)\alpha(\mu + \nu - \bar{\nu})\rangle d\mu d\nu d\bar{\nu} d\bar{\mu} \\
= 2\Psi_0 \Psi_m \alpha^2 \Phi^2 \left( \langle \Im[I_\Omega]\rangle B^2 + 2 \int_\Omega \int \int \int \langle \Im[I_\Omega]\Im[h(\nu + \nu - \bar{\nu} - \Omega)]\rangle d\nu d\nu d\bar{\nu} d\bar{\mu} \right) \\
+ \int_\Omega \int \int \int \langle \Im[I_\Omega]\Im[h(\mu - \Omega + \nu + \nu - \bar{\nu})]\rangle d\nu d\nu d\bar{\nu} d\bar{\mu} \right). \tag{36}
\]

We can write term 20 of Eq. (29) as
In the expansion of Eq. (37), many terms vanish due to both the restricted domain of \( h(\nu) \) and the action of creation and annihilation operators on the vacuum. This allows Eq. (37) to be significantly simplified, leading to the result:

\[
4 \int \Omega \int \Omega \int \Omega \int \Omega \int \int \int (\alpha(\mu)^* \alpha(\bar{p}) \alpha(\bar{p})^* \alpha(\bar{p})^*) \hat{x}_\theta(\nu - \mu)^\dagger \hat{x}_\theta(\bar{p} - \bar{p})^\dagger \hat{x}_\theta(\bar{p} - \bar{p}) \hat{x}_\theta(\bar{p} - \bar{p})) d\mu d\nu d\bar{p} d\bar{p} d\bar{p} d\bar{p}
\]

\[
= 4 \left( \left( \int \Omega \int \Omega \int \Omega \int \Omega \int \int \int (\alpha(\mu)^* \alpha(\bar{p}) \alpha(\bar{p})^* \alpha(\bar{p})^*) \hat{x}_\theta(\nu - \mu)^\dagger \hat{x}_\theta(\bar{p} - \bar{p})^\dagger \hat{x}_\theta(\bar{p} - \bar{p}) \hat{x}_\theta(\bar{p} - \bar{p})) d\mu d\nu d\bar{p} d\bar{p} d\bar{p} d\bar{p} \right)^4 \right)
\]

(37)

The summation of terms 21-24 of Eq. (29) can be written as

\[
2 \int \Omega \int \Omega \int \Omega \int \Omega \int \int \int \left[ (\alpha(\mu)^* \alpha(\bar{p}) \hat{x}_\theta(\nu - \mu)^\dagger \hat{x}_\theta(\bar{p} - \bar{p}) \hat{a}(\bar{p} - \bar{p})^\dagger \hat{a}(\bar{p} - \bar{p}) \hat{a}(\bar{p} - \bar{p})) d\mu d\nu d\bar{p} d\bar{p} d\bar{p} d\bar{p}
\]

\[
= \left( \frac{5 \Phi^3}{2} - 2 \Phi + \frac{1}{2 \Phi} \right) \int \Omega \int \Omega \int \Omega \int \Omega \int \int \int (\alpha(\mu)^* \alpha(\mu + \nu + \bar{p} - \bar{p})) d\mu d\nu d\bar{p} d\bar{p} d\bar{p} d\bar{p}
\]

(38)
Similarly, we can combine terms 25 and 26 of Eq. (29) to give

\[
2 \int \int \int \int \int \left[ \langle \alpha(\mu)^* \alpha(\nu) \rangle \hat{x}_\theta(\nu - \mu) \hat{a}(\bar{\nu} - \bar{\mu}) \hat{a}(\bar{\nu} - \bar{\mu}) \right] d\mu d\nu d\sigma d\theta d\bar{\theta}
\]
\[
+ \langle \alpha(\mu) \alpha(\bar{\nu}) \rangle \hat{x}_\theta(\nu - \bar{\mu}) \hat{a}(\bar{\nu} - \bar{\mu}) \hat{a}(\bar{\nu} - \bar{\mu}) \right] d\mu d\nu d\sigma d\theta d\bar{\theta}
\]
\[
= (2\Psi_\omega^2 + \Psi_m^2) \alpha^2 \left( \Phi^3 - \Phi \right) \left( \frac{1}{2} B^3 + \int \int \int \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu} - \bar{\nu})] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\]
\[
+ \frac{1}{2} \int \int \int \int \int \langle \mathcal{R}[\mathcal{R}[h(\mu + \nu - \bar{\nu} - \bar{\nu})]] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right).
\]

(40)

The final term of Eq. (29) gives the result:

\[
\int \int \int \int \langle \hat{a}(\nu - \mu) \hat{a}(\nu - \mu) \hat{a}(\bar{\nu} - \bar{\mu}) \hat{a}(\bar{\nu} - \bar{\mu}) \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\]
\[
= B^3 \Lambda \left( \frac{7\Phi^4}{32} - \frac{3\Phi^2}{8} + \frac{7}{32\Phi^4} - \frac{3}{8\Phi^2} + \frac{5}{16} \right).
\]

(41)

By combining all the terms calculated for Eq. (29), we obtain the result for \( \langle p_\Omega^2 \rangle \):

\[
\langle p_\Omega^2 \rangle = 4\lambda^2 R^2 \left[ \langle |I_\Omega|^2 \rangle + (4\Psi_m^2 + 2\Psi_m^2) \alpha^2 \Phi \left( \langle |I_\Omega|^2 \rangle B + 2 \int \int \langle |I_\Omega|^2 \mathcal{R}[h(\nu - \nu)] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \int \int \langle |I_\Omega|^2 h(\mu)^* h(\nu - \bar{\nu}) \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right] + \frac{\Psi_m^2}{2} \alpha^2 \left( \langle \mathcal{R}[I_\Omega^2] \rangle B + 2 \int \int \langle \mathcal{R}[I_\Omega^2 h(\nu + \nu - \bar{\nu})] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \int \int \langle \mathcal{R}[I_\Omega^2 h(\mu - \Omega) h(\nu + \nu - \bar{\nu})] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ 2\Psi_\omega \Psi_m \alpha^2 \Phi^2 \left( \langle \mathcal{R}[I_\Omega] \rangle B + 2 \int \int \langle \mathcal{R}[I_\Omega h(\nu + \nu - \bar{\nu})] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \int \int \langle \mathcal{R}[I_\Omega^2 h(\mu - \Omega) h(\nu + \nu - \bar{\nu})] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ (4\Psi_m^4 + 4\Psi_m^2 \Psi_m^2 + \Psi_m^4) \alpha^2 \Phi^2 \left( \frac{1}{2} B^2 + 2B \int \int \langle \mathcal{R}[h(\nu - \nu)] \rangle d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \int \int \langle \mathcal{R}[h(\nu - \nu)] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ B \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu})] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \frac{1}{2} \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu})] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ \Psi_m^4 \alpha^2 \Phi^2 \left( \frac{1}{2} B^2 + 2B \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu})] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ 2 \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu})] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
+ 2 \int \int \langle \mathcal{R}[h(\nu + \nu - \bar{\nu})] d\mu d\nu d\sigma d\theta d\bar{\theta}
\right)
\]
\[
= \frac{7\Phi^4}{32} - \frac{3\Phi^2}{8} + \frac{7}{32\Phi^4} - \frac{3}{8\Phi^2} + \frac{5}{16}.
\]
\begin{align*}
+ \frac{1}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \langle h(\mu - \Omega)^{*} h(\nu - \Omega) h(\nu + \mu - \Omega)^{*} h(\nu' + \mu - \Omega) \rangle d\mu d\nu d\nu' d\mu' \\
+ (2\Psi_{0}^{2} + \Psi_{m}^{2}) \alpha^{2} \left( \frac{5\Phi^{2}}{2} - 2\Phi + \frac{1}{2}\Phi \right) \left( \frac{1}{2} B^{3} + \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \langle \Re[h(\nu + \nu - \Omega)] \rangle d\mu d\nu d\nu' d\mu' \\
\quad + \frac{1}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \langle h(\mu + \nu - \Omega - \Omega)^{*} h(\mu + \nu - \Omega - \Omega) \rangle d\mu d\nu d\nu' d\mu' \\
+ (2\Psi_{0}^{2} + \Psi_{m}^{2}) \alpha^{2} (\Phi^{3} - \Phi) \left( \frac{1}{2} B^{3} + \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \langle \Re[h(\nu + \nu - \Omega)] \rangle d\mu d\nu d\nu' d\mu' \\
\quad + \frac{1}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \langle \Re[h(\mu + \nu - \Omega - \Omega)] \rangle d\mu d\nu d\nu' d\mu' \\
\quad + B^{3} \Lambda \left( \frac{7\Phi^{4}}{32} - 3\Phi^{2} + \frac{7}{32\Phi^{2}} - \frac{3}{8\Phi^{2}} + \frac{5}{16} \right) \right). (42)
\end{align*}

In the expression for \( \text{Var}(p_{\Omega}) = \langle p_{\Omega}^{2} \rangle - \langle p_{\Omega} \rangle^{2} \), there is significant cancellation between \( \langle p_{\Omega}^{2} \rangle \) and \( \langle p_{\Omega} \rangle^{2} \), and by taking the leading remaining terms we find that

\begin{align*}
\text{Var}(p_{\Omega}) &= \langle p_{\Omega}^{2} \rangle - \langle p_{\Omega} \rangle^{2} \approx 4q^{4} R^{2} \left[ (|I_{\Omega}|^{4}) - (|I_{\Omega}|^{2})^{2} + 2(|I_{\Omega}|^{2})\Psi_{0}^{2} \alpha^{2} \Phi B \right] \\
&\approx 4q^{4} R^{2} \left[ 16\Psi_{0}^{4} \Psi_{m}^{4} \alpha^{8} \left( \int_{\Omega} \int_{\Omega} \langle \Re[h(\nu - \Omega)] \Re[h(\nu - \Omega)] \rangle d\nu d\sigma - \int_{\Omega} \int_{\Omega} \langle \Re[h(\nu - \Omega)] \Re[h(\nu - \Omega)] \rangle d\nu d\sigma \right) \\
\quad + 4\Psi_{0}^{2} \Psi_{m}^{2} \alpha^{4} \left( \int_{\Omega} \int_{\Omega} \langle \Re[n_{c}(\nu)] \Re[n_{c}(\sigma)] \rangle d\nu d\sigma - \int_{\Omega} \int_{\Omega} \langle \Re[n_{c}(\nu)] \Re[n_{c}(\sigma)] \rangle d\nu d\sigma \right) + 2\Phi B \Psi_{0}^{4} \Psi_{m}^{4} \alpha^{6} \right]. (43)
\end{align*}

By associating \( \mathcal{H} = \int_{\Omega} h(\nu - \Omega) d\nu = \int_{-B/2}^{B/2} h(\nu) d\nu \) as the relative amplitude of the classical optical noise in the DC component, \( \mathcal{N} = \int_{\Omega} n_{c}(\nu) d\nu \) as the amplitude of the electronic noise in the \( \pm B/2 \) frequency interval around \( \Omega \), and substituting \( \Psi_{0} \approx 1 \), \( \Psi_{m} = \delta_{m}/2 \) and \( i_{0} \approx q\eta\alpha_{0}^{2} \), we find that for \( M \) spectral averages,

\begin{align*}
\text{Var}(p_{\Omega}) &\approx \frac{R^{2}}{M} \left[ 2q\delta_{m}^{2} i_{0}^{3} \Phi B + 4\delta_{m}^{4} i_{0}^{4} \text{Var}(|\Re[\mathcal{H}]|) + 4q^{2}\delta_{m}^{2} i_{0}^{2} \text{Var}(|\Re[\mathcal{N}]|) \right]. (44)
\end{align*}

From Eq. (44), an improvement in precision beyond the quantum noise limit may be obtained in the case that squeezing (\( \Phi < 1 \)) provides a significant reduction in \( \text{Var}(p_{\Omega}) \).