Fast Approximations for Rooted Connectivity in Weighted Directed Graphs

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Abstract

We consider approximations for computing minimum weighted cuts in directed graphs. We consider both rooted and global minimum cuts, and both edge-cuts and vertex-cuts. For these problems we give randomized Monte Carlo algorithms that compute a $(1 + \epsilon)$-approximate minimum cut in $\tilde{O}(n^2/\epsilon^2)$ time. These results extend and build on recent work [4] that obtained exact algorithms with similar running times in directed graphs with small integer capacities.

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1 Introduction

Let $G = (V, E)$ be a directed graph with $m$ edges and $n$ vertices. Let $G$ have positive edge weights $w : E \to \mathbb{R}_{>0}$. Recall that $G$ is strongly connected if any vertex can reach any other in the graph. The (global, weighted) edge connectivity is the minimum weight of edges that needs to be removed so that $G$ is no longer strongly connected. The minimum (weight) cut is the corresponding set of edges. Determining the edge connectivity and computing the minimum cut are basic problems in graph algorithms. This work develops a faster randomized algorithm for approximating the minimum weight cut.

The algorithm for edge connectivity is developed alongside for the following related connectivity problems also of basic interest. Let $r \in V$ be a fixed vertex, called the root. The minimum rooted cut (from $r$), also called the minimum $r$-cut, is the minimum weight set of edges whose removal disconnects $r$ from at least one vertex. Global connectivity follows from rooted connectivity by choosing any root arbitrarily, and computing the rooted connectivity in both $G$ and the reversed graph. Most of the algorithmic discussion in this work is focused on rooted connectivity and global connectivity is obtained as a by-product. Rooted connectivity has other connections in combinatorial optimization [11, 33]; for example, Edmonds [7] showed that the $r$-rooted edge connectivity equals the maximum number of arborescences rooted at $r$ that can be packed into the graph. Now suppose instead that the graph has vertex weights $w : V \to \mathbb{R}_{>0}$. The (global) vertex connectivity is the minimum weight of vertices that needs to be removed so that $G$ either is no longer strongly connected or consists of only a single vertex; the minimum vertex cut is the corresponding set of vertices. One can also define rooted vertex connectivity analogously to rooted edge connectivity. This work also develops fast approximation algorithms for rooted and global vertex connectivity.

These connectivity problems are well-studied and we first give an overview of classical results, with particular focus on algorithms for directed graphs, before discussing more recent developments. There is a long line of algorithms for directed edge connectivity [12, 17, 27, 28, 32, 35] (see also [33]), of which we highlight the most pertinent. For general weights, an algorithm by Hao and Orlin [17] finds the minimum (weight) cut in $O(mn \log(n^2/m))$ time. For multigraphs, Gabow [12] gives an $O(m\lambda \log(n^2/m))$ time algorithm for the minimum $r$-cut, where $\lambda$ is the weight of the minimum rooted cut. Directed vertex connectivity likewise has had many algorithms, and many of these running times are parametrized by the weight of the vertex cut [5, 8, 13, 14, 18, 31]. Of those that are not, we highlight the randomized $O(mn \log(n))$ time algorithm of Henzinger, Rao, and Gabow [18] that remains the fastest algorithm in weighted and directed graphs.

Recently there has been a flurry of results for graph algorithms several of which impact directed connectivity. There have been many significant developments for $(s, t)$-flow for both edge- and vertex-capacitated directed graphs. [2, 3, 6, 15, 16, 20, 21, 24–26, 29, 30]. Very recently [2] obtained an $\tilde{O}(m + n^{1.5})$ running time for edge-capacitated $(s, t)$-max flow, generalizing a preceding $\tilde{O}(m + n^{1.5})$ algorithm for vertex capacitated $(s, t)$-max flow [3]. Another recent development is a randomized $\tilde{O}(mk^2)$ time exact algorithm and a randomized $\tilde{O}(mk/\epsilon)$ time $(1 + \epsilon)$-approximation for global vertex connectivity in unweighted directed graphs [10]. These algorithms are based on local algorithms for vertex connectivity and influence the local algorithms that appear in this work. A very recent and independent work of [23] has obtained an $\tilde{O}(mn^{1-1/12+o(1)})$ time algorithm for vertex connectivity in directed and unweighted graphs. (We have not yet had time to digest and make a proper comparison to [23].) The last recent work we discuss is a randomized, $\tilde{O}(n^2U^2)$ time (exact) algorithm for rooted and global edge connectivity in directed graphs with small integer capacities between 1 and $U$ [4]. [4] also gives a $\tilde{O}(knW)$-time exact algorithm for rooted and global vertex capacity with integer weights, where $W$ is the total weight in the graph, and $\kappa$ is the weight of

\footnote{Here and throughout $\tilde{O}(\cdots)$ hides polylogarithmic factors. The algorithms in this work use the $\tilde{O}(m + n^{1.5})$ time algorithms [2, 3] as a subroutine which incur large polylogarithmic factors hidden in the $\tilde{O}(\cdots)$ notation. Consequently we generally do not try to optimize polylogarithmic factors in this article.}
the minimum vertex cut. [4] introduces elementary ideas to sparsify rooted connectivity problems but the crux of the argument needs the capacities to be small. The driving motivation of this article is to overcome the limitations of [4] to small integer capacities and extend the ideas to the weighted setting.

1.1 Results.

The primary results of this work extend the \( \bar{O}(n^2) \) randomized running times of [4] to the weighted setting while allowing for approximation. These algorithms take as input an additional parameter \( \epsilon > 0 \); the goal is to compute a cut whose weight is at most a \((1 + \epsilon)\)-multiplicative factor greater than the minimum cut.

**Edge connectivity.** The first result is for rooted and global edge connectivity.

**Theorem 1.1.** Given a directed graph with polynomially bounded weights, and \( \epsilon \in (0, 1) \), a \((1 + \epsilon)\)-approximate minimum rooted or global edge cut can be computed with high probability in \( \bar{O}(n^2/\epsilon^2) \) randomized time.

It is of theoretical interest to obtain \( o(mn) \) running times due to the longstanding \( O(mn \log(n^2/m)) \) running time of [17] and the connection to flow decompositions. Observe that the above running time for edge connectivity is \( o(mn) \) except for the case where \( m \leq n^{1+o(1)} \). One can modify the algorithm in Theorem 1.1 (leveraging, in particular, the recent \( m^{1.5-1/28} \) time algorithm for \((s, t)\)-flow [15]) to establish a \( o(mn) \) running time, as follows.

**Corollary 1.2.** There exists a constant \( c > 0 \) such that, for all fixed \( \epsilon > 0 \), an \((1 + \epsilon)\)-approximate minimum weight rooted or global cut in a directed graph with polynomially bounded edge weights can be computed with high probability in \( \bar{O}(mn^{1-c}/\epsilon^2) \) randomized time.

**Vertex connectivity.** The other main result is for rooted vertex connectivity. Here, \( \deg^+(v) \) denotes the unweighted out-degree of a vertex \( v \).

**Theorem 1.3.** Let \( \epsilon \in (0, 1) \), let \( G = (V, E) \) be a directed graph with polynomially bounded vertex weights, and let \( r \in V \) be a fixed root. A \((1 + \epsilon)\)-approximate minimum vertex \( r \)-cut can be computed with high probability in \( \bar{O}(m + n(n – \deg^+(r))/\epsilon^2) \) randomized time.

An argument by [18] (with some modifications) implies that the above running time for rooted vertex connectivity, combined with randomly sampling roots by weight, leads to the following running time for global vertex connectivity.

**Corollary 1.4.** For all \( \epsilon \in (0, 1) \), a \((1 + \epsilon)\)-approximate minimum weight global vertex cut in a directed graph with polynomially bounded vertex weights can be computed with high probability in \( \bar{O}(n^2/\epsilon^2) \) expected time.

In the same way as for edge connectivity above, Theorem 1.3 and Corollary 1.4 leads to \( o(mn) \) time approximation algorithms for rooted and global vertex connectivity, as follows.

**Corollary 1.5.** There exists a constant \( c > 0 \) such that, for all fixed \( \epsilon > 0 \), an \((1 + \epsilon)\)-approximate minimum weight rooted or global cut in a directed graph with polynomially bounded edge weights can be computed with high probability in \( O((mn)^{1-c}/\epsilon^2) \) randomized time.
1.2 Key ideas.

The high level approach is inspired by previous work in [4], which was limited to small integer capacities. Let us focus on edge connectivity as the ideas for vertex connectivity are similar. An important idea that emerges from [4] is that there are useful tradeoffs based on the number of vertices in the sink component. Let $k$ denote the number of vertices in the sink component of the minimum $r$-cut, and for simplicity, suppose $k$ is known. [4] observed that if $k$ is small, and the graph has small capacities, then the graph can be sparsified by contracting vertices with in-degree greater than $O(k)$ into the root. Meanwhile, if $k$ is large, then it is easier to sample a vertex from the sink component and then apply $(s,t)$-flow. Balancing these tradeoffs leads to an $\tilde{O}(n^2)$ time exact algorithm for rooted edge connectivity (for small integer capacities). Note that the sampling approach for large $k$ extends to the weighted setting. However the sparsification argument requires the assumption of small capacities. The high-level goal of this work is to extend the ideas from [4] to the weighted setting.

The first step is to take advantage of the approximation error and discretize the edge weights (and also reduce the number of edges) by random sampling. While random sampling is known to preserve cuts in undirected graphs [1, 19, 34], there are no such guarantees in directed graphs. Here we continue the theme of balancing tradeoffs in the size of the sink component, $k$. Rather than sampling as to preserve all cuts, we only sample to preserve the in-cuts of vertex sets of size less than or equal to $k$ of the minimum rooted cut (which can be guessed). This requires paying some overhead in proportion to the target component size. To address large vertex sets for which the random sampling might drastically underestimate the in-cut, we add appropriately weighted auxiliary edges from the root to every vertex. The auxiliary edges make it impossible for large vertex sets to induce the minimum weight rooted cuts, but also have limited impact on small vertex sets. (This approach is inspired in part by the sparsification ideas in [4] and in another part by the pessimistic estimator in [22].) Thereafter one can contract high-degree nodes in the root similar to [4]. The end result is a sparser graph where the sparsity depends on $k$. Moreover, up to scaling, the sparsified graph has integer capacities, and the weight of the minimum $r$-cut becomes proportional to $k$.

In this sparsified graph, depending on the (guessed) size of the sink component, the algorithm pursues one of two options. If $k$ is smaller than (roughly) $\sqrt{n}$, the graph is very sparse and the size of the cut is small, and we run an algorithm based on a new deterministic local cut algorithm that takes advantage of the small value $k$. If $k$ is larger, then we try to sample a vertex from the sink component and run $(s,t)$-flow. In both cases we work in the sparsified graph. Balancing terms between the running times of these two approaches leads to the claimed $\tilde{O}(n^2/\epsilon^2)$ running time.

We highlight that in previous work for small capacities, in the regime where $k$ is small, [4] was able to use Gabow’s algorithm (on a sparsified graph) to find the minimum rooted cut. Here, in the presence of capacities (even after sparsification), Gabow’s algorithm has a larger polynomial dependency on $k$ than desired and one needs new ideas. The local cut approach developed here is inspired by the recent randomized algorithms of [4, 10]. Compared to these previous works, the new local cut algorithm has a better dependency on $k$ and $\epsilon$ and is also deterministic. The local cut algorithm routes flow in the reversed graph from a fixed vertex $t$ to the root, with localized running times depending only on $k$. It takes advantage of auxiliary edges from the root added in the sparsification step to find short augmenting paths. The improved running time comes from a refined analysis based on how many of these auxiliary edges have been saturated. The fact that we are always routing flow to the root also removes the guess work from [4, 10] and makes the algorithm deterministic.2

Organization. The remainder of this article is divided into two sections. Section 2 considers edge connectivity, and proves Theorem 1.1 and Corollary 1.2. Section 3 considers vertex connectivity, and proves Theorem 1.3, Corollary 1.4, and Corollary 1.5.

2That said, the overall algorithm for rooted edge connectivity is still randomized.
2 Rooted edge connectivity

In this section, we design and analyze an $\tilde{O}(n^2/e^2)$-time approximation algorithm for the minimum weight rooted edge cut. We present the algorithm as three main steps. Each of the steps are parameterized by values $\lambda > 0$ and $k \in \mathbb{N}$ that, in principle, are meant to be constant factor estimates for the weight of the minimum rooted cut and the number of vertices in the sink component of the minimum rooted cut, respectively. The first step is a sparsification result that (assuming $\lambda$ and $k$ are accurate) reduces the problem to a rooted graph with roughly $nk$ edges and rooted connectivity roughly $k$ in addition to a few other helpful properties. This sparsification procedure is used by both of the remaining two steps. The second step, preferable for small $k$, approximates the minimum rooted cut in roughly $nk^2$ time, and is based on a new deterministic local cut algorithm that makes essential use of some of the specific properties of the sparsification lemma. The third step, preferable for large $k$, approximates the minimum rooted cut in roughly $n^2 + n^{2.5}/k$ time, via random sampling and $(s,t)$-flow in the sparsified graph. Balancing terms leads to the claimed running time.

Sparsification. Our first lemma sparsifies the graph while preserving the minimum rooted cut. The algorithm is parameterized by a target number of vertices in the sink component and the sparsity of the output graph depends on this input parameter.

Lemma 2.1. Let $G = (V, E)$ be a directed graph with positive edge weights. Let $r \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\lambda > 0$, and $k \in \mathbb{N}$ be given parameters. In randomized linear time, one can compute a randomized directed and edge-weighted graph $G_0 = (V_0, E_0)$, where $V_0 \subseteq V$ and $r \in V_0$, and a scaling factor $\tau > 0$, with the following properties.

(i) $G_0$ has integer edge weights between 1 and $O(k \log(n)/\epsilon^2)$.
(ii) Every vertex $v \in V_0$ has unweighted in-degree at most $O(k \log(n)/\epsilon^2)$ in $G_0$.
(iii) For every $v \in V_0 - r$ there is an edge $(r, v)$ with capacity at least $\Omega(\log(n)/\epsilon)$.
(iv) With high probability, for all $S \subseteq V_0 - r$, the weight of the in-cut induced by $S$ in $G_0$ (up to scaling by $\tau$) is at least the minimum of $(1 - \epsilon)$ times the weight of the induced in-cut in $G$ and $c \lambda$ for any desired constant $c > 1$, and at most $(1 + \epsilon)$ times of the weight of the induced in-cut $G$ plus $\epsilon \lambda |S|/k$.
(v) With high probability, for all $S \subseteq V - r$ such that $|S| \leq k$ and the weight of the induced in-cut is $\leq O(\lambda)$, we have $S \subseteq V_0$.

In particular, if the minimum $r$-cut has weight $\Theta(\lambda)$, and the sink component of a minimum $r$-cut has at most $k$ vertices, then with high probability $G_0$ preserves the minimum $r$-cut up to a $(1 + O(\epsilon))$-multiplicative factor.

Proof. Consider the following randomized algorithm applied to the input graph $G$.

1. Let $\tau = c_\tau \epsilon^2 \lambda/k \log(n)$ and $\Delta = c_\Delta k \log(n)/\epsilon^2$ for a sufficiently small constant $c_\tau > 0$ and a sufficiently large constant $c_\Delta > 0$.
2. Importance sample each edge weight to be a discrete multiple of $\tau$. Drop any edge with weight 0.
3. Add an edge of weight $\epsilon \lambda/2k$ from the root to every vertex.
   
   // Decreasing $c_\tau$ and $\epsilon$ as needed, we assume $\lambda$ and $\epsilon \lambda/2k$ are multiples of $\tau$.
4. Scale down all edge weights by $\tau$ (which makes them integers).
5. Truncate all edge weights to be at most $c_w k \log(n)/\epsilon^2$ for a sufficiently large constant $c_w > 0$ (while maintaining integrality).
6. For any vertex $v$ with unweighted in-degree $\geq \Delta$, contract $v$ into $r$.

Consider the graph $G_0$ obtained by the above steps. Of the claimed properties, (i), (ii), and (iii) follow directly from the construction. The remaining proof is dedicated to proving the high-probability claims in (iv) and (v). We first show that the initial steps (1) to (3) – before rescaling – preserves the weights of the $r$-cuts in the sense of (iv) (without the rescaling). We then analyze the remaining steps which rescale and contract the graph.

For each set $S$, let $f(S)$ denote the weight of the in-cut at $S$ in $G$. Let $g(S)$ denote the randomized weight of the in-cut after step (2). Let $h(S)$ denote the randomized weight of the in-cut of $S$ after adding the auxiliary edges in (3). The first claim analyzes the concentration of $g(S)$ for all sets $S$.

**Claim 1.** With high probability, for all $S \subseteq V$

$$|g(S) - f(S)| \leq \epsilon f(S) + \frac{\epsilon \lambda |S|}{2k}.$$  

The above claim consists of an upper and lower bound on $g(S)$ for all $S$. We first show the lower bound on $g(S)$ holds for all $S$ with high probability. Fix $S \subseteq V$. $g(S)$ is an independent sum with expected value $f(S)$ and where each term in the sum is nonnegative and varies by at most $\tau$. By a variation of standard Chernoff inequalities,

$$\mathbb{P}[g(S) \leq (1 - \epsilon) f(S) - \gamma] \leq e^{-\epsilon \gamma / \tau} = n^{-\gamma k \log(n) / c_\tau \epsilon \lambda},$$

In particular, for $\gamma = \epsilon \lambda |S| / 2k$, the RHS is at most $n^{-c_0 |S|}$ where $c_0$ is a constant under our control (via $c_\tau$). For large enough $c_0$, we can take the union bound over all sets of vertices. This establishes that the lower bounds for $g(S)$ hold for all $S$ with high probability. The upper bounds also hold with high probability by a symmetric argument.

Now we analyze the in-cuts after step (3). Recall that for $S \subseteq V$, $h(S)$ denotes the weight of the in-cut of $S$ after adding the auxiliary edges in (3).

**Claim 2.** With high probability, for all $S \subseteq V - r$, we have

$$(1 - \epsilon) f(S) \leq h(S) \leq (1 + \epsilon) f(S) + \epsilon \lambda |S| / k.$$  

Indeed, we have $h(S) = g(S) + \epsilon \lambda |S| / 2k$ for all $S \subseteq V - r$. The additive term introduced by $h$ offsets the additive error in the lower bound on $g(S)$ in Claim 1. This term also adds to the additive error in the upper bound of Claim 1 for a total of $\epsilon \lambda |S| / k$. Thus in the high probability event of Claim 1, we have the bounds described by Claim 2 for all $S$.

Henceforth, let us assume that the high probability event in Claim 2 holds. (Otherwise the algorithm fails.) Claim 2 implies that, after step (3), the cuts in $G_0$ preserve the weight of the cuts in $G$ in the approximate sense of (iv) (without the scaling). Now, after step (3), all the weights are divisible by $\tau$. After

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3 Here we apply the following bounds (appropriately rescaled) which follow from the same proof as the standard multiplicative Chernoff bound.

Let $X_1, \ldots, X_n \in [0, 1]$ independent random variables. Then for all $\epsilon > 0$ sufficiently small and all $\gamma > 0$,

$$\mathbb{P}[X_1 + \cdots + X_n \leq (1 - \epsilon) \mathbb{E}[X_1 + \cdots + X_n] - \gamma] \leq e^{-\epsilon \gamma}$$

and

$$\mathbb{P}[X_1 + \cdots + X_n \geq (1 + \epsilon) \mathbb{E}[X_1 + \cdots + X_n] + \gamma] \leq e^{-\epsilon \gamma}.$$
scaling down by $\tau$ in step (4), we will continue to preserve the $r$-cuts in the desired sense (up to scaling). Truncating weights in $G_0$ to $O(\lambda/\tau)$ in (5) decreases the weight of some cuts, but to no less than $O(\lambda/\tau)$. The contractions in step (6) only removes some $r$-cuts from consideration and does not effect the weight of any remaining cuts This establishes property (iv).

To show that the contractions in step (6) preserve property (v), let $T$ be the sink component of any $r$-cut of capacity $\leq O(\lambda)$ and with $|T| \leq k$. We claim that any vertex in $T$, has unweighted in-degree at most $\Delta$ in the graph obtained after (4). Indeed, fix any such vertex $v \in T$, and consider the edges going into $v$. At most $k - 1$ of those edges can come from another vertex in $T$, since $T$ has at most $k$ vertices. The remaining edges must be in the in-cut of $T$, and the in-cut of $T$ has at most $O(\lambda/\tau) = O(k \log(n)/\epsilon^2)$ edges (per property (iv)). Thus there are less than $\Delta = O(k \log(n)/\epsilon^2)$ edges incident to $v$. In conclusion, any vertex $v$ with in-degree more than $\Delta$ lies outside $T$ and can be safely contracted into the root. This establishes property (v) and completes the proof.

**Rooted edge connectivity for small sink components.** This section presents an approximation algorithm for rooted vertex connectivity for the particular setting where the sink component is small. In particular, we are given an upper bound $k$ on the number of vertices in the sink component, and want to obtain running times of the form $\text{poly}(k)$. When a similar situation arose previously for small integer capacities in [4], [4] used Gabow’s algorithm which works well for unweighted multigraphs. Here, while Lemma 2.1 produces relatively sparse graphs with integral edge capacities, the edge capacities imply a multigraph with roughly $nk^3$ edges, and Gabow’s algorithm would then take roughly $nk^3$ time. This section develops an alternative approach that reduces the dependency on $k$ to $k^2$, and is inspired by existing local algorithms for (global and rooted) vertex cuts [4, 10]. Compared to [4, 10], the algorithm here is for edge cuts and is designed to take full advantage of the properties of the graph produced by Lemma 2.1. These modifications have some tangible benefits. First, it improves the dependency on $k$ and $\epsilon$. (We estimate that previous approaches lead to an $\tilde{O}(nk^3/\epsilon^3)$ running time.) Second, the local subroutine here is deterministic whereas before they were randomized. Third and last, as suggested by the better running time and the determinism, the version presented here is arguably simpler and more direct than the previous algorithms (for this setting).

**Lemma 2.2.** Let $G = (V, E)$ be a directed graph with positive edge weights $w : V \to \mathbb{R}_{>0}$. Let $r \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\lambda > 0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized linear time Monte Carlo algorithm that, with high probability, produces a deterministic data structure that supports the following query.

For $t \in V$, let $\lambda_{t,k}$ denote the weight of the minimum $(r,t)$-cut such that the sink component has at most $k$ vertices. Given $t \in V$, deterministically in $O(k^3 \log(n)/\epsilon^4)$ time, the data structure either (a) returns the sink component of an $(r,t)$-cut of weight at most $\lambda_{t,k} + c \epsilon$, or (b) declares that $\lambda_{t,k} > \lambda$.

**Proof.** We first apply Lemma 2.1 to $G$ with root $r$ and parameters $\lambda$, $k$, and $c\epsilon$ for a sufficiently small constant $c > 0$. This produces an edge capacitated graph $G_0 = (V_0, E_0)$, where $r \in V_0$ and $V_0 \subseteq V$. We briefly highlight the features of $G_0$ guaranteed by Lemma 2.1 that we leverage. The edge weights in $G_0$ are scaled down so that the weight $\lambda$ in $G$ corresponds to weight $O(k \log(n)/\epsilon^2)$ in $G_0$. The edge weights are integral, with value between 1 and $O(k \log(n)/\epsilon^2)$. Every vertex has unweighted in-degree at most $O(k \log(n)/\epsilon^2)$. Lastly, for every non-root vertex $v \in V_0 - r$, there is an edge from $r$ to $v$ with capacity at least $\Omega(\log(n)/\epsilon)$. With high probability, we have the following guarantees on the cuts of $G_0$. Modulo scaling, every $r$-cut in $G_0$ has weight no less than the minimum of its weight in $G$ and $2\lambda$. Additionally, moduling scaling, the sink component of an $r$-cut in $G$ with capacity at most $\lambda$ and at most $k$ vertices in the sink component is preserved in $V_0$, and the corresponding cut in $G_0$ has weight at most an $c_0(\epsilon \lambda$ additive factor bigger in $G_0$, for any desired constant $c_0 > 0$. In particular we preserve $\lambda_{t,k}$ within the desired approximation factor for all $t$ such that $\lambda_{t,k} \leq \lambda$. Henceforth we assume that the edge cuts are preserved in the sense described above. Otherwise we consider the algorithm to have failed.
We propose a data structure that, given \( t \in V \), will search for a small \((r, t)\)-cut in \( G_0 \) via a customized, edge-capacitated flow algorithm. The search may or may not return the sink component of an \((r, t)\)-cut. If the search does return a sink component, and the corresponding in-cut in \( G_0 \) has weight that, upon rescaling back to the scale of the input graph \( G \), is at most \((1 + \epsilon/2)\lambda\), the data structure returns it. Otherwise the data structure indicates that \( \lambda_{t, k} > \lambda \).

To develop the \((r, t)\)-cut algorithm, let \( G_{rev} \) be the reversed graph of \( G_0 \). In \( G_{rev} \), given \( t \in V \), we run a specialization of the Ford-Fulkerson algorithm [9] with source \( t \) and sink \( r \) that either computes a minimum \((t, r)\)-cut or concludes that the minimum \((t, r)\)-cut is at least \( O(k \log(n)/\epsilon^2) \) after \( O(k \log(n)/\epsilon^2) \) iterations. To briefly review, each iteration in the Ford-Fulkerson algorithm searches for a path from \( t \) to \( r \) in the residual graph of the flow to that point. If such a path is found, then it routes one unit of flow along this path, and updates the residual graph by reversing (one unit capacity) of each edge along the path. After \( \ell \) successful iterations we have a flow of size \( \ell \) and in particular the minimum \((t, r)\)-cut is at least \( \ell \). If, after \( \ell \) iterations, there is no path in the residual graph from \( t \) to \( r \), then the set of vertices reachable from \( t \) gives a minimum \((t, r)\)-cut of size \( \ell \).

Within the Ford-Fulkerson framework, we give a refined analysis that takes advantages of the auxiliary \((v, r)\) edges (for all \( v \neq r \)) that each have capacity at least \( \Omega(\log(n)/\epsilon) \). Call a non-root vertex \( v \) saturated if the auxiliary edge \((v, r)\) is saturated; that is, if \((v, r)\) is not in the residual graph. (A vertex \( v \) is called unsaturated if it is not saturated.) We modify the search for an augmenting path so that whenever we visit an unsaturated \( v \), we automatically complete a path to \( r \) via \((v, r)\). It remains to bound the running time of this search. We first bound the number of saturated vertices.

Claim 1. There are at most \( O(k/\epsilon) \) saturated \( v \)'s.

Indeed, each saturated \( v \) implies \( \Omega(\log(n)/\epsilon) \) units of flow via the edge \((v, r)\). The size of the flow is limited to \( O(k \log(n)/\epsilon^2) \).

The above bound on the number of saturated vertices leads to the following bound on the total number of edges visited in each search.

Claim 2. Every (modified) search for an augmenting path traverses at most \( O\left(k^2/\epsilon^2\right) \) edges.

We first observe that every vertex visited in the search, except the unsaturated vertex terminating the search, is a saturated vertex. By Claim 1, there are at most \( O(k/\epsilon) \) saturated vertices. In turn there are at most \( O(k^2/\epsilon^2) \) edges between saturated vertices. Thus we can traverse at most \( O(k^2/\epsilon^2) \) edges before visiting either an unsaturated vertex or \( r \), as claimed.

Claim 2 implies that each iteration takes \( O(k^2/\epsilon^2) \) time. The algorithm runs for at most \( O(k \log(n)/\epsilon^2) \) iterations before either finding a minimum \((r, t)\)-cut or concluding that the minimum \((r, t)\)-cut in \( G_0 \) is at least \( O(k \log(n)/\epsilon^2) \) (which corresponds to weight \( O(\lambda) \) in \( G \)). The total running time follows.

We now present the overall algorithm for finding \( r \)-cuts with small sink components. The algorithm combines Lemma 2.2 with randomly sampling for a vertex \( t \) in the sink component of the desired \( r \)-cut.

Lemma 2.3. Let \( G = (V, E) \) be a directed graph with positive edge weights \( w : E \to \mathbb{R}_{>0} \). Let \( r \in V \) be a fixed root vertex. Let \( \lambda > 0 \) and \( k > 0 \) be given parameters. There is a randomized algorithm that runs in \( O(m \log k + nk^2 \log^2(n)/\epsilon^4) \) time and has the following guarantee. If there is an \( r \)-cut of capacity at most \( \lambda \) and where the sink component has at most \( k \) vertices, then with high probability, the algorithm returns an \( r \)-cut of capacity at most \( (1 + \epsilon)\lambda \).

Proof. Let \( T^* \) be the sink component of the minimum \( r \)-cut subject to \( |T^*| \leq k \). Suppose the capacity of the in-cut of \( T^* \) is at most \( \lambda \). (Otherwise the algorithm makes no guarantee.)
Suppose we had a factor-2 overestimate \( \ell \) of the number of vertices in \( T^* \); i.e., \( |T^*| \leq \ell \leq 2|T^*| \). We apply Lemma 2.2 with parameter upper bound \( \lambda \) on the size of the cut and \( \ell \) on the number of vertices in the sink component, which produces a deterministic data structure that with high probability is correct for all queries. Let us assume the data structure is correct (and otherwise the algorithm fails). We then randomly sample \( O(n \log(n)/\ell) \) vertices from \( V - r \). For each sampled vertex \( t \), we query the data structure from Lemma 2.2. Observe that if \( t \in T^* \), then the query for \( t \) return an \((r,t)\)-cut with capacity at most \((1 + \epsilon)\lambda\).

With high probability we sample at least one vertex from \( T^* \), with produces the desired \( r \)-cut. By Lemma 2.2, the running time for \( O(n \log(n)/\ell) \) queries is \( O(m + n^2 \log^2(n)/\epsilon^4) \).

While we do not have such an estimate \( \ell \ a \ priori \), we can try all powers of 2 between 1 and \( 2k \). One of these choices of \( \ell \) will be accurate and succeed with high probability. Note that the sum of \( O(n \ell^2 \log^2(n)/\epsilon^4) \) over the range of \( \ell \) is dominated by the maximum \( \ell \). The claimed running time follows.

**Rooted connectivity for large sink components.** The second subroutine we present is better suited for cases where the sink component is very large.

**Lemma 2.4.** Let \( G = (V, E) \) be a directed graph with positive edge weights. Let \( r \in V \) be a fixed root. Let \( \epsilon, k, \lambda > 0 \) be given parameters with \( \epsilon \) sufficiently small. Let \( \lambda^* \) be the minimum weight of all \( r \)-cuts where the sink component has between \( k/2 \) and \( k \) vertices. Then there is a randomized \( \tilde{O}(n^2/\epsilon^2 + n^{2.5}/k) \) time algorithm that has the following guarantee. If \( \lambda^* \leq \lambda \), then with probability, the algorithm returns an \( r \)-cut of capacity at most \( \lambda^* + \epsilon \lambda \).

**Proof.** We assume the graph \( G_0 = (V_0, E_0) \) produced by Lemma 2.1 with parameters \( k, \lambda \), and \( c \) for a sufficiently small constant \( c > 0 \). Let \( T^* \subseteq V - r \) be the sink component of the minimum \( r \)-cut subject to \( k/2 \leq |T^*| \leq k \), and suppose the capacity of \( T^* \) is at most \( \lambda \). We randomly sample \( O(n \log(n)/k) \) sinks \( t \in V_0 \), and for each compute the minimum \((r,t)\) cut. We output the minimum of these cuts.

If \( \lambda^* \leq \lambda \), then Lemma 2.1 asserts that with high probability we have \( T^* \subseteq V_0 \). With high probability, some \( t \) will be drawn from \( T^* \) and the corresponding \((r,t)\)-cut has capacity at most \((1 + \epsilon)\lambda^* + \epsilon \lambda \). We use \( EC(m, n) = \tilde{O}(m + n^{1.5}) \) [2]. By Lemma 2.1, we have \( m = O(n \ell \log(n)/\epsilon^2) \).

**Rooted and global edge connectivity.** Finally we combine the two approaches above for rooted connectivity – Lemma 2.3 for small components, and Lemma 2.4 for large components – in an overall algorithm for rooted edge connectivity and establish Theorem 1.1. We restate Theorem 1.1 for the sake of convenience.

**Theorem 1.1.** Given a directed graph with polynomially bounded weights, and \( \epsilon \in (0, 1) \), a \((1 + \epsilon)\)-approximate minimum rooted or global edge cut can be computed with high probability in \( \tilde{O}(n^2/\epsilon^2) \) randomized time.

**Proof.** We focus on rooted connectivity from which global connectivity follows immediately. As the weights are polynomially bounded, we can guess the rooted connectivity \( \lambda \) and the number of vertices \( k \) in the sink component of the minimum rooted cut up to a multiplicative factor of 2 with polylogarithmic overhead. For each choice of \( k \), we apply the faster of Lemma 2.3 and Lemma 2.4. We balance the \( \tilde{O}(nk^2/\epsilon^4) \) running time of Lemma 2.3 with the \( \tilde{O}(n^2/\epsilon^2 + n^{2.5}/k) \) running time of Lemma 2.4. For \( k = \sqrt{n} \epsilon^{4/3} \), we obtain the claimed running time.

**o(mn)-time approximations.** Recall from the introduction that there is theoretical interest in obtaining a \( o(mn) \) running time, partly to improve on the running time of [17] for exact edge connectivity and partly due to the connection to the flow-decomposition barrier. Theorem 1.1 gives an \( o(mn) \) running time for all but the extremely sparse regime where \( m = n^{1 + o(1)} \). Part of the problem is that the \( \tilde{O}(m + n^{1.5}) \) running time of [2] is not as compelling for extremely sparse graphs. However, the minimum \((s, t)\)-cut can also be obtained in \( \tilde{O}(m^{1.5-\delta}) \) time for \( \delta = 1/128 \) [15]. This is slightly faster than \( \tilde{O}(m + n^{1.5}) \) for \( m = n^{1 + o(1)} \).
Using this second \((s, t)\)-flow algorithm leads to the following slightly improved running time for extremely sparse graphs.

**Lemma 2.5.** Let \( \epsilon \in (0, 1) \), and let \( G = (V, E) \) be a directed graph with polynomially bounded edge weights. Let \( \epsilon > 0 \). Suppose that the minimum \((s, t)\)-cut can be computed in \( \tilde{O}(m^{3/2-\delta}) \) time for a constant \( \delta > 0 \). A \((1 + \epsilon)\)-approximate minimum rooted (or global) cut in \( G \) can be computed with high probability in \( \tilde{O}(nm^{1-2\delta/3}/\epsilon^{4/3}) \) randomized time.

**Proof.** We take the same approach as Theorem 1.1 except we use the \( \tilde{O}(m^{3/2-\delta}) \) running time for \((s, t)\)-flow, and we don’t sparsify the graph first. As before, we guess the weight \( \lambda \) of the minimum \( r \)-cut and the number of vertices \( k \) in the sink component of the minimum \( r \)-cut to within a multiplicative factor of 2 with polylogarithmic overhead. For a fixed choice of \( k \), we run the faster of two options. The first option is to invoke Lemma 2.3 which runs in \( O\left(nk^2 \log^2(n)/\epsilon^4\right) \) randomized time. The second option replaces Lemma 2.4 and is as follows. We sample \( O\left(k \log(n)\right) \) vertices with high probability. For each sampled vertex \( t \), we compute the minimum \((r, t)\)-cut in \( \tilde{O}(m^{3/2-\delta}) \) time. For a correct value of \( k \), with high probability, one of the sampled vertices \( t \) is in the sink component of the minimum cut, and we obtain the minimum \((r, t)\)-cut. The total running time of this combined approach is obtained (up to logarithmic factors) by choosing \( k \) to balance the two running times of \( \tilde{O}(nk^2 \log(n)/\epsilon^4) \) and \( \tilde{O}(nm^{3/2-\delta}/k) \). For \( k = \epsilon^{4/3} m^{1/2-\delta/3} \), we obtain the claimed running time.

Balancing Lemma 2.5 with Theorem 1.1 gives the \( \tilde{O}\left((mn)^{1-\epsilon}/\epsilon^2\right) \) running time, where \( c > 0 \) is a constant, that is claimed in Corollary 1.2.

### 3 Rooted vertex connectivity

In this section we present the approximation algorithm for rooted and global vertex connectivity. Similar to Section 2, the main focus is on rooted connectivity, and the algorithm is presented in three main parts. All three parts are parameterized by values \( \kappa > 0 \) and \( k \in \mathbb{N} \) that, in principle, are meant to be constant factor estimates for the rooted vertex connectivity and the number of vertices in the sink component of the minimum rooted vertex cut. The first part reduces the number of edges to roughly \( nk \) and the rooted connectivity to \( k \) in a graph with integer weights. This sparsification is used in the remaining two parts. The second part gives a roughly \( nk^2 \) time approximation algorithm for the minimum rooted cut. The third part gives a roughly \( n^2 + n^{2.5}/k \) time approximation algorithm for the minimum rooted cut. Balancing term leads to the claimed running time. The rooted connectivity algorithm then leads to a global connectivity algorithm via an argument due to [18] (with some modifications).

**Sparsification.** The first part is a sparsification lemma that preserves rooted vertex cuts where the number of vertices in the sink component is below some given parameter. In the following, we let \( N^+(r \mid G) \) denote the set of out-neighbors of \( r \) in the graph \( G \). We omit \( G \) and simply write \( N^+(r) \) when \( G \) can be inferred from the context.

**Lemma 3.1.** Let \( G = (V, E) \) be a directed graph with positive vertex weights. Let \( r \in V \) be a fixed vertex. Let \( k, \kappa > 0 \) be given parameters. Let \( V' = V \setminus (\{r\} \cup N^+(r)) \). In randomized linear time, one can compute a randomized directed and vertex-weighted graph \( G_0 = (V_0, E_0) \), and a scaling factor \( \tau > 0 \), with the following properties.

(i) \( r \in V_0 \).

(ii) Let \( V'_0 = V_0 \setminus (\{r\} \cup N^+(r \mid G_0)) \). We have \( V'_0 = V' \).
(iii) $G_0$ has integer vertex weights between 0 and $O(k \log(n)/\epsilon^2)$.

(iv) Every vertex $v \in V_0$ has at most $O(k \log(n)/\epsilon^2)$ incoming edges.

(v) Every vertex $v$ with weight 0 has no outgoing edges.

(vi) With high probability, for all $S \subseteq V'$, the weight of the vertex in-cut induced by $S$ in $G_0$ (up to scaling by $\tau$) is at least the minimum of the $(1 - \epsilon)$ times the weight of the induced vertex in-cut in $G$ or $c_k$ (for any desired constant $c > 1$), and at most $(1 + \epsilon)$ times its weight in $G$ plus $\epsilon k |S|/k$.

(vii) With high probability, for all $S \subseteq V'$ such that $|S| \leq k$ and the weight of the induced vertex in-cut is $\leq O(k)$, we have $S \subseteq V'_0$. (That is, $S$ is still the sink component of an $r$-cut in $G_0$.)

In particular, if the minimum vertex $r$-cut has weight $\Theta(k)$, and the sink component of a minimum vertex $r$-cut has at most $k$ vertices, then with high probability $G_0$ preserves the minimum vertex $r$-cut up to a $(1 + O(\epsilon))$-multiplicative factor.

Proof. Consider the following randomized algorithm applied to the input graph $G$.

1. Let $\tau = c_\tau \epsilon^2 k/k \log(n)$ and let $\Delta = c_\Delta k \log(n)/\epsilon^2$, where $c_\tau > 0$ is a sufficiently small constant and $c_\Delta > 0$ is a sufficiently large constant.
2. Important sample each vertex weight to be a discrete multiple of $\tau$.
3. For each vertex $v$, introduce an auxiliary vertex $a_v$ with weight $\epsilon k/2k$. Add edges from the $r$ to $a_v$, and from $a_v$ to $v$.

// Decreasing $c_\tau$ and $\epsilon$ as needed, we assume that $k$ and $\epsilon k/2k$ are multiples of $\tau$.

4. Remove all outgoing edges from any vertex with weight 0.
5. Scale down all vertex weights by $\tau$ (which makes them integers).
6. Truncate all vertex weights to be at most $c_w k \log(n)/\epsilon^2$ for a sufficiently large constant $c_w > 0$.
7. For all $v$ with unweighted in-degree at least $\Delta$, replace all incoming edges to $v$ with a single edge from $r$.

Let $G_0$ be the graph obtained at the end of the algorithm above. Properties (i) through (v) follow directly from the construction. The remaining proof is dedicated to proving the high probability bounds of (vi) and (vii). We first show that the initial steps (1) to (3) – before rescaling – preserves the minimum weight rooted vertex-cut approximately in the sense of (vi) (sans scaling). We then analyze the remaining steps. For each set $S$, let $f(S)$ denote the weight of the vertex in-cut of $S$. Let $g(S)$ denote the randomized weight of the vertex in-cut after step (2). Let $h(S)$ denote the randomized weight of the vertex in-cut after step (3).

Claim 1. With high probability, for all $S \subseteq V$, we have

$$|g(S) - f(S)| \leq \epsilon f(S) + \frac{\epsilon k |S|}{2k}.$$ 

The claim and proof are similar to Claim 1 in the proof of Lemma 2.1. The claim consists of an upper bound and a lower bound on $g(S)$ for all $S$ and we first show the lower bound holds with high probability. Fix any set $S$. $g(S)$ is an independent sum with expected value $f(S)$ and where each term in the sum is nonnegative and varies by at most $\tau$. Concentration bounds (see footnote 3 on page 6) imply that for any $\gamma \geq 0$, we have

$$P[g(S) \leq (1 - \epsilon) f(S) - \gamma] \leq e^{-\epsilon \gamma/\tau} = n^{-\gamma k \log(n)/c_\tau \epsilon k},$$

for a sufficiently large $c_\tau$.
In particular, for $\gamma = \epsilon \kappa |S|/2k$, the RHS is at most $n^{-c_0|S|}$ where $c_0 > 0$ is again a constant under our control (via $c_\gamma$). For sufficiently large $c_0$, we can take the union bound over all sets $S$, establishing the high probability lower bound. The high probability upper bound follows by a symmetric argument.

Now we analyze the vertex $r$-cuts after step (3). Recall that for $S \subseteq V$, $h(S)$ denotes the weight of the in-cut of $S$ after adding auxiliary vertices in step (3).

**Claim 2.** For all $S \subseteq V'$, we have

$$(1 - \epsilon) f(S) \leq h(S) \leq (1 + \epsilon) f(S) + \frac{\epsilon \kappa |S|}{k}.$$ 

This claim and its proof is similar to Claim 2 in Lemma 2.1. We have $h(S) = g(S) + \epsilon \kappa |S|/2k$ for all $S \subseteq V'$. The additive factor of $\epsilon \kappa |S|/2k$ combine with the high-probability additive error in Claim 1 to establish the claim.

We point out that Claim 2 implies that, with high probability after step (3), the weights of all the vertex $r$-cuts are preserved the approximate sense described by (vi) (without the scaling). Henceforth we assume that the high probability event of Claim 2 holds. Now, after step (3), all the weights are integer multiples of $\tau$. We have $\kappa/\tau = O(k \log(n)/\epsilon^2)$. After scaling down by $\tau$ in step (5), we still preserve the $r$-cuts per property (vi). Truncating weights in $G_0$ to $O(\kappa/\tau)$ decreases the weight of some cuts, but to no less than $O(\kappa/\tau)$ (which maps to $O(\kappa)$ when rescaled back to the scale of $G$). Removing the outgoing edges of vertices with weight 0 also has no impact on the weight of any vertex $r$-cut. The final step adding edges from $r$ only eliminates some of the vertex $r$-cuts from consideration and does not impact the weight of the remaining vertex $r$-cuts. This establishes (vi).

It remains to prove (vii) and in particular we must show that it is not impacted by the last step, (7). Recall that step (7) replaces the incoming edges to any vertex $v$ with unweighted in-degree greater than $\Delta = O(k \log(n)/\epsilon^2)$ with a single edge in $r$. In particular, this edge places $v$ in $N^+(r)$ and destroys all $r$-cuts where the sink component contains $v$. Let $T \subseteq V'$ be the sink component of a vertex $r$-cut in $G$ where the capacity of the cut is at most $\kappa$, and $|T| \leq k$. We want to show that all vertices in $T$ have in-degree less than $\Delta$, in which case the extra edges in (7) have no impact on $T$. The vertex in-cut induced by $T$ has weight at most $(1 + 2\epsilon)\kappa/\tau$ in the randomized graph before (7) (per (vi)). Fix any $v \in T$ and consider the edges going into $v$. At most $k - 1$ of those edges can come from another vertex in $T$, since $T$ has at most $k$ vertices. The remaining edges must be from vertices in the vertex in-cut of $T$. Each of these vertices have weight at least 1, and by (vi) the in-cut has weight at most $(1 + 2\epsilon)\kappa/\tau$, so there are at most $O(\kappa/\tau)$ of these vertices. This gives a maximum total of less than $\Delta$ edges incident to $v$, as desired. In conclusion, for any vertex $v$ that is the endpoint to at least $\Delta$ edges, it is safe to replace all of $v$'s incoming edges with a single edge from the root, without violating (vii). This establishes (vii) and completes the proof.

**Rooted vertex connectivity for small sink components.** This section presents an approximation algorithm for rooted vertex connectivity for the particular setting where the sink component of the minimum rooted cut is small. The algorithm is similar to the algorithms presented in Lemma 2.2 and Lemma 2.3 for rooted edge connectivity, and has the same inspirations (from [4, 10]) and motivations. As with Lemma 2.3, the local algorithm presented here is customized to take full advantage of the properties of the graph produced by the sparsification lemma, Lemma 3.1, and is a deterministic algorithm with better dependency on $k$ and $\epsilon$ compared to previous algorithms.

**Lemma 3.2.** Let $G = (V, E)$ be a directed graph with positive vertex weights. Let $r \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\kappa > 0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized linear time Monte Carlo algorithm that, with high probability, produces a deterministic data structure that supports the following query.
For \( t \in V' \) \( \overset{\text{def}}{=} V \setminus (\{r\} \cup N^+(r)) \), let \( \kappa_{t,k} \) denote the weight of the minimum \((r,t)\)-vertex cut such that the sink component has at most \( k \) vertices. Given \( t \in V' \), deterministically in \( O(k^3 \log^2(n)/\epsilon^4) \) time, the data structure either (a) returns the sink component of a minimum \((r,t)\)-vertex cut of weight at most \((1+\epsilon)\kappa_{t,k} \), or (b) declares that \( \kappa_{t,k} > \kappa \).

**Proof.** We first apply Lemma 3.1 to \( G \) with root \( r \) and parameters \( \kappa, k \), and \( c \) \( \epsilon \) for a constant \( c > 0 \) sufficiently small. This produces a vertex capacitated graph \( G_0 = (V_0,E_0) \) with \( V \subset V_0 \). We highlight the features that we leverage. All new vertices (in \( V_0 \setminus V \)) are in \( N^+(r \mid G_0) \); that is, \( V' \) equals \( V_0 \overset{\text{def}}{=} V_0 \setminus (\{r\} \cup N^+(r \mid G_0)) \). Put alternatively, none of the new vertices are in the sink component of any \( r \)-cut. The vertex weights are integers between 0 and \( O(k \log(n)/\epsilon^2) \). Every vertex has unweighted in-degree at most \( O(k \log(n)/\epsilon^2) \). Every vertex with weight 0 has no outgoing edges.

With high probability, we have the following guarantees on the vertex \( r \)-cuts of \( G_0 \). The vertex weights in \( G_0 \) are scaled so that a weight of \( \kappa \) in \( G \) corresponds to weight \( O(k \log(n)/\epsilon^2) \) in \( G_0 \). Modulo scaling, every vertex \( r \)-cut in \( G_0 \) has weight no less than the minimum of its weight in \( G \) and \( 2k \). Additionally, modulo scaling, for every vertex \( r \)-cut in \( G_0 \) with capacity at most \( k \) and at most \( k \) vertices in the sink component, the corresponding vertex cut in \( G_0 \) has weight at most a \( c_0 \) \( \epsilon k \) additive factor larger than in \( G \), for any desired constant \( c_0 > 0 \). We consider the algorithm to fail if the cuts are not preserved in the sense described above.

Given \( t \in V \), the data structure will search for a small \((r,t)\)-cut in \( G_0 \) via a customized, edge-capacitated flow algorithm. This algorithm may or may not return the sink component of \((r,t)\)-cut. If the search does return a sink component, and the corresponding vertex in-cut in \( G_0 \) has weight that, upon rescaling back to the scale of the input graph \( G \), is at most \((1+\epsilon/2)\kappa \), the data structure returns it. Otherwise the data structure indicates that \( \kappa_{t,k} > \kappa \).

Proceeding with the flow algorithm, let \( G_{rev} \) be the reverse of \( G_0 \), and let \( G_{split} \) be the standard “split-graph” of \( G_{rev} \) modeling vertex capacities with edge capacities. We recall that the split graph splits each vertex \( v \) into an auxiliary “in-vertex” \( v^- \) and an auxiliary “out-vertex” \( v^+ \). For each \( v \) there is a new edge \((v^-,v^+)\) with capacity equal to the vertex capacity of \( v \). Each edge \((u,v)\) is replaced with an edge \((u^+,v^-)\) with capacity\(^4 \) equal to the vertex capacity of \( u \). Every \((r,t)\)-vertex cut in \( G_0 \) maps to a \((t^+,r^-)\)-edge cut in \( G_{rev} \) with the same capacity. Any \((t^+,r^-)\)-edge capacitated cut maps to a \((r,t)\)-vertex cut in \( G_0 \) (with negligible overhead in the running time). Now, recall that for each \( v \in V' \), the sparsification procedures introduces an auxiliary path \((r,a_v,r^-)\) where \( a_v \) is was given weight \( \Theta(\epsilon k/k) \). It is convenient to replace the corresponding auxiliary path \((v^+,a_v^+,a_v^-,r^-)\) in \( G_{rev} \) with a single edge \((v^+,r^-)\) with capacity equal to the weight of \( a_v \). This does not effectively the weight of the minimum \((t^+,r^-)\)-edge cut for any \( t \in V' \). This adjustment can be easily made within the allotted preprocessing time.

In this graph, given \( t \in V' \), we run a specialization of the Ford-Fulkerson algorithm [9] that either computes a minimum \((t^+,r^-)\)-cut or concludes that the minimum \((t^+,r^-)\)-cut is at least \( O(k \log(n)/\epsilon^2) \) (which corresponds to \( O(\kappa) \) in \( G \)) after \( O(k \log(n)/\epsilon^2) \) iterations. To briefly review, each iteration in the Ford-Fulkerson algorithm searches for a path from \( t \) to \( r \) in the residual graph. If such a path is found, then it routes one unit of flow along this path, and updates the residual graph by reversing (one unit capacity) of each edge along the path. After \( \ell \) successful iterations we have a flow of size \( \ell \). If, after \( \ell \) iterations, there is no path in the residual graph from \( t^+ \) to \( r^- \), then the set of vertices reachable from \( t \) gives a minimum \((t^+,r^-)\)-cut of size \( \ell \). Observe that updating the residual graph along a \((t^+,r^-)\)-path preserves the weighted in-degree and out-degree of every vertex except \( t^+ \) and \( r^- \). The weighted out-degree of \( t^+ \) decreases by 1 and the weighted in-degree of \( r^- \) changes by 1. Moreover, updating the residual graph along a path increases the unweighted out-degree of any vertex by at most one, since a path contains at most one edge going into any single vertex. Since every vertex initially has unweighted out-degree at most \( O(k \log(n)/\epsilon^2) \) in \( G_{rev} \) (reversing the upper bound on the unweighted in-degrees in \( G_0 \)), and the flow algorithm updates the residual

\(^{4}\) Usually, this edge is set to capacity \( \infty \), but either the weight of \( u \) or the weight of \( v \) are also valid.
We first observe that every vertex visited in the search, except the unsaturated vertex terminating the search, is not in the residual graph. Call an in-vertex $v^-$ saturated if the auxiliary edge $(v^-, v^+)$ is saturated; that is, if $(v^-, v^+)$ is not an unsaturated vertex. We modify the search for an augmenting path to effectively end when we first visit an unsaturated vertex $v^+$ or an unsaturated $v^-$. If we visit an unsaturated $v^-$, then we automatically complete a path to $r^-$ via $v^+$. If we find an unsaturated $v^+$, then we automatically complete a path to $r^-$ via the edge $(v^+, r^-)$. It remains to bound the running time of this search. We first bound the number of saturated $v^+$’s.

**Claim 1.** There are at most $O(k/e)$ saturated $v^+$’s.

Indeed, each saturated $v^+$ implies $O(\log(n)/e)$ units of flow along $(v^+, r^-)$, and the flow is bounded above $O(k \log(n)/e^2)$.

Note that Claim 1 also implies there are at most $O(k/e)$ $v^-$’s such that $v^+$ is saturated. The next claim bounds the total out-degree of saturated $v^-$’s.

**Claim 2.** The sum of out-degrees of saturated $v^-$’s is at most the amount of flow that has been routed to $r^-$. Indeed, the out-degree of a $v^-$ in the residual graph is bounded above by the amount of flow through $(v^-, v^+)$, since initially $(v^-, v^+)$ is the only outgoing edge from $v^-$. Recall that if $v^-$ is saturated, then by definition $v^+$ is unsaturated. As long as $v^+$ is unsaturated, each unit of flow through $(v^-, v^+)$ goes directly to $r^-$ via the edge $(v^+, r^-)$, and can be charged to the total flow.

We now apply the above two claims to bound the total running time for each search, as follows.

**Claim 3.** Every (modified) search for an augmenting path traverses at most $O\left(k^2 \log(n)/e^2\right)$ edges.

We first observe that every vertex visited in the search, except the unsaturated vertex terminating the search, is either (a) a saturated $v^-$, (b) a saturated $v^+$, or (c) an unsaturated $v^-$ such that $v^+$ is saturated. We will upper bound the number of edges traversed in each iteration based on the type of vertex at the initial point of that edge. First, the amount of time spent exploring edges leaving either (b) a saturated $v^+$ or (c) an unsaturated $v^-$ such that $v^+$ is saturated. By Claim 1, there are at most $O(k/e)$ such vertices, and each has out-degree at most $O(k \log(n)/e^2)$. Thus we spend $O(k^2 \log(n)/e^2)$ time traversing such edges. All together, we obtain an upper bound of $O(k^2 \log(n)/e^2)$ total edges per search.

Claim 3 also bounds the running time for each iteration. The algorithm runs for at most $O(k \log(n)/e^2)$ iterations before either finding an $(t^*, r^-)$-cut or concluding that the weight of the minimum $(t^*, r^-)$-cut, rescaled to the input scale of $G$, is at least a constant factor greater than $\kappa$. The total running time follows.

We now present the overall algorithm for finding vertex $r$-cuts with small sink components. The algorithm combines Lemma 3.2 with randomly sampling for a vertex $t$ in the sink component of an approximately minimum $r$-cut.

**Lemma 3.3.** Let $G = (V, E)$ be a directed graph with positive vertex weights. Let $r \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\kappa > 0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized algorithm that runs in $O\left(m + (n - \deg^+(r))k^2 \log^3(n)/\epsilon^4\right)$ time and has the following guarantee. If there is a vertex $r$-cut of capacity at most $\kappa$ and where the sink component has at most $k$ vertices, then with high probability, the algorithm returns a vertex $(r, t)$-cut of capacity at most $(1 + \epsilon)\kappa$.
Proof. Let \( T^* \) be the sink component of the minimum vertex \( r \)-cut subject to \(|T^*| \leq k \). Assume the capacity of the vertex in-cut of \( T^* \) is at most \( \kappa \) (since otherwise the algorithm makes no guarantees). Let \( V' = V \setminus (\{r\} \cup N^+(r)) \) and note that \(|V'| = n - 1 - \deg^+(r)\).

Suppose we had a factor-2 overestimate \( \ell \in ([T^*], 2|T^*|) \) of the number of vertices in \( T^* \). We apply Lemma 3.2 with upper bounds \( \kappa \) on the size of the cut and \( \ell \) on the number of vertices in the sink component, which returns a data structure that, with high probability, is correct for all queries. Let us assume the data structure is correct (and otherwise the algorithm fails). We randomly sample \( O\left( (n - \deg^+(r)) \log(n)/\ell \right) \) vertices from \( V' \). For each sampled vertex \( t \), we query the data structure from Lemma 3.2. Observe that if \( t \in T^* \), then the query for \( t \) returns an \( r \)-cut with capacity at most \((1 + \epsilon)k\). With high probability we sample at least one vertex from \( T^* \), which produces the desired \( r \)-cut. By Lemma 3.2, the total running time to serve all queries is \( O\left( m + (n - \deg^+(r)) \ell^2 \log^3(n)/\epsilon^4 \right) \).

A factor-2 overestimate \( \ell \) can be obtained by enumerating all powers of 2 between 1 and \( 2k \). One of these choices of \( \ell \) will be accurate and produce the minimum \( r \)-cut with high probability. Note that the sum of \( O\left( (n - \deg^+(r)) \ell^2 \log^3(n)/\epsilon^4 \right) \) over this range of \( \ell \) is dominated by the maximum \( \ell \). The claimed running time follows.

Rooted vertex connectivity for large sink components. The third and final part (before the overall algorithm) is an approximation for the rooted vertex cut that is well-suited for large sink components.

Lemma 3.4. Let \( G = (V, E) \) be a directed graph with positive vertex weights. Let \( r \in V \) be a fixed root vertex. Let \( \epsilon \in (0, 1) \), \( \kappa > 0 \), and \( k \in \mathbb{N} \) be given parameters. There is a randomized algorithm that runs in \( \tilde{O}(m + (n - \deg^+(r))\log(n)/k) \) time and has the following guarantee. If there is a vertex \( r \)-cut of capacity at most \( \kappa \) and where the sink component has at most \( k \) vertices, then with high probability, the algorithm returns a vertex \((r, t)\)-cut of capacity at most \((1 + \epsilon)k\).

Proof. Let \( T^* \) be the sink component of the minimum \( r \)-cut subject to \(|T^*| \leq k \). We assume the capacity of the \( r \)-cut induced by \( T^* \) is at most \( \kappa \). (Otherwise the output is not well-defined.) Let \( V' = V \setminus (\{r\} \cup N^+(r)) \) and note that \(|V'| < n - \deg^+(r)\).

We apply Lemma 3.1 to produce the graph \( G_0 \). Lemma 3.1 succeeds with high probability and for the rest of the proof we assume it was successful. (Otherwise the algorithm fails.) We sample \( O\left( (n - \deg^+(r)) \log(n)/k \right) \) vertices \( t \in V' \). For each sampled \( t \), we compute the minimum \((r, t)\)-vertex cut in \( G_0 \). With high probability, some \( t \) will be drawn from the sink component of the true minimum \( r \)-cut, in which case the minimum \((r, t)\)-cut in \( G_0 \) gives an \((1 + \epsilon)\)-approximate \( r \)-cut in \( G \) (by Lemma 3.1). We use \( \text{VC}(m, n) = \tilde{O}(m + n^{1.5}) \) [3]. By Lemma 3.1, we have \( m = O(nk \log(n)/\epsilon^2) \). This gives the total running time.

Approximating the rooted and global vertex connectivity. We now combine the two parameterized approximation algorithms for rooted vertex connectivity to give the following overall algorithm for rooted edge connectivity and establish Theorem 1.3. We restate Theorem 1.3 for the sake of convenience.

Theorem 1.3. Let \( \epsilon \in (0, 1) \), let \( G = (V, E) \) be a directed graph with polynomially bounded vertex weights, and let \( r \in V \) be a fixed root. A \((1 + \epsilon)\)-approximate minimum vertex \( r \)-cut can be computed with high probability in \( \tilde{O}(m + n(n - \deg^+(r))/\epsilon^2) \) randomized time.

Proof. The high-level approach is similar to Theorem 1.1 for edge connectivity – we are balancing two algorithms for rooted vertex connectivity, where one is better suited for small sink components, the second is better suited for large sink components. Both leverage the randomized sparsification lemma. As before, with polylogarithmic overhead, we can assume access to values \( \kappa \) and \( k \) that are within a factor 2 of the weight of the minimum \( r \)-cut and the number of vertices in the sink component of the minimum \( r \)-cut,
respective. For a fixed choice of $k$ and $\kappa$, we run the faster of two randomized algorithms, both of which would succeed with high probability when $k$ and $\kappa$ are (approximately) correct. The first option, given by Lemma 3.3, runs in $\tilde{O}((n - \deg^+(r))k^2/\epsilon^3)$. The second option, given by Lemma 3.4, runs in $\tilde{O}(n(n - \deg^+(r))/\epsilon^2 + n^{1.5}(n - \deg^+(r))/k)$. The overall running time is obtained by choosing $k$ to balance the running times. For $k = \epsilon \sqrt{n}$, we obtain the claimed running time. 

Next we use the algorithm for rooted vertex connectivity to obtain an algorithm for global vertex connectivity and establish Corollary 1.4. Henzinger, Rao, and Gabow [18] showed that running times of the form $(n - \deg^+(r))T$ for rooted connectivity from a root $r$ imply a randomized $nT$ expected time algorithm for global vertex connectivity. Theorem 1.3 gives a $\tilde{O}(m + n(n - \deg^+(r))/\epsilon^2)$ running time, so some modifications have to be made to address the additional $\tilde{O}(m)$ additive factor. We restate Corollary 1.4 for the sake of convenience.

**Corollary 1.4.** For all $\epsilon \in (0, 1)$, a $(1 + \epsilon)$-approximate minimum weight global vertex cut in a directed graph with polynomially bounded vertex weights can be computed with high probability in $\tilde{O}(n^2/\epsilon^2)$ expected time.

**Proof.** Let $T = \tilde{O}(n/\epsilon^2)$. Let $w : V \to \mathbb{R}_{\geq 0}$ denote the vertex weights, and let $W = \sum_{v \in V} w(v)$ be the total weight of the graph. Let $\kappa$ denote the weight of the minimum global vertex cut. The algorithm samples $L = O(W \log(n)/(W - \kappa))$ vertices $r$ in proportion to their weight, and – morally, but not actually – computes the minimum $r$-vertex cut for each sampled vertex $r$ via Theorem 1.3. It returns the smallest cut found.

For the sake of running time, we adjust the algorithm from Theorem 1.3. Recall that for a fixed root $r$, and for each of a logarithmic number of values for $k$ and $\kappa$, the algorithm from Theorem 1.3 applies Lemma 3.1 which reduces the graph to having $\tilde{O}(nk/\epsilon^2)$ edges and rooted connectivity $\tilde{O}(k/\epsilon^2)$. For fixed $k$ and $\kappa$, rather than rerun Lemma 3.1 entirely for each $r$ we sample, we execute most of it just once for all $r$, and make local modifications for each different root $r$. Referring to the algorithm given in the proof of Lemma 3.1, observe that the only step that directly mentions $r$ is step (3), which adds auxiliary vertices between $r$ and each other vertex $v$. We move this step to the very end of the algorithm. (Here the vertex weight of auxiliary vertices is scaled down appropriately.) It is easy to see that the proof of Lemma 3.1 still goes through (with minor rearrangement in the argument). The advantage is that, over all $L$ roots $r$, we now spend a total of $O(m + nL)$ time, rather than $O(mL)$. Thereafter, the rest of the rooted connectivity algorithm takes $\tilde{O}((n - \deg^+(r))T)$ per root $r$. Note that $\tilde{O}((n - \deg^+(r))T)$ dominates the $O(n)$ time required to complete the sparsification for each root.

Consider a single root $r$ sampled from $V$ in proportion to its weight. The expected running time to compute the minimum $r$-cut is

$$E[(n - \deg^+(r))T] = nT - \frac{T}{W} \sum_{v \in V} \deg^+(v)w(v) \overset{(a)}{=} nT - \frac{T}{W} \sum_{v \in V} \sum_{x \in N^+(v)} w(x) \overset{(b)}{\leq} nT \left(1 - \frac{\kappa}{W}\right).$$

Here, in (a), $N^+(v)$ denotes the in-neighborhood of $v$. The equality is obtained by implicitly interchanging sums. (b) is because for each $v$, the sum $\sum_{x \in N^+(v)} w(x)$ is the weighted in-degree of $v$, and at least $\kappa$. Thus the overall expected running time over all the sampled roots is

$$O\left(\frac{W \log(n)}{W - \kappa} E[(n - \deg^+(r))T]\right) = O\left(\left(\frac{W \log(n)}{W - \kappa}\right) \cdot nT \left(1 - \frac{\kappa}{W}\right)\right) = \tilde{O}\left(n^2/\epsilon^2\right).$$

Meanwhile, when we sample $O(W \log(n)/(W - \kappa))$ vertices in proportion to their weight, then with high probability, at least one sampled vertex lies outside the minimum global vertex cut. Such a vertex then leads to the minimum global vertex cut with high probability.
\textbf{o(mn)-time approximations for vertex connectivity.} We now show how to approximate rooted and global vertex connectivity in o(mn) time, just as we did previously for edge connectivity. We point out that the minimum (s, t)-edge cut algorithm of [15] that runs in $\tilde{O}(m^{1.5-\delta})$ time for $\delta = 1/128$ is also an $\tilde{O}(m^{1.5-\delta})$ time algorithm for (s, t)-vertex cut by standard reductions. As the ideas here to develop an o(mn) time approximation for vertex connectivity are the same as for edge connectivity, we restrict ourselves to a sketch.

\begin{lemma}
Let $\varepsilon \in (0, 1)$, and let $G = (V, E)$ be a directed graph with polynomially bounded vertex weights. Suppose that the minimum (s, t)-vertex cut can be computed in $\tilde{O}(m^{3/2-\delta})$ time for a constant $\delta > 0$. For a fixed root $r \in V$, a (1 + $\varepsilon$)-approximate minimum vertex r-cut can be computed with high probability in $\tilde{O}(m + (n - \deg^+(r))m^{1-2\delta}/\varepsilon^{4/3})$ randomized time. A (1 + $\varepsilon$)-approximate minimum global vertex cut can be computed with high probability in $\tilde{O}(m + mm^{1-2\delta}/\varepsilon^{4/3})$ randomized time.
\end{lemma}

\textbf{Proof sketch.} We first consider rooted vertex cuts. We take the same approach as Theorem 1.3 except modifying the algorithm Lemma 3.4 as follows. First, we do not sparsify the graph. Second, we use the $\tilde{O}(m^{3/2-\delta})$ time algorithm for (s, t)-vertex cut instead of $\tilde{O}(m+n^{3/2})$. The result replaces the running time in Lemma 3.4 with $\tilde{O}((n - \deg^+(r))m^{3/2-\delta}/k)$. Choosing $k$ to balance this running time with the $\tilde{O}((n - \deg^+(r))k^2 \log(n)/\varepsilon^4)$ running time of Lemma 3.3 gives the claimed running time.

The running time for global vertex connectivity follows from rooted connectivity in the same way as Corollary 1.4 above.

Balancing Lemma 3.5 with Theorem 1.3 and Corollary 1.4 gives $\tilde{O}((mn)^{1-c}/\varepsilon^2)$ running times for approximating rooted and global vertex cuts, where $c > 0$ is a constant, as claimed in Corollary 1.5.

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