Global existence for the two-component Camassa-Holm system
and the modified two-component Camassa-Holm system

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Abstract

The present work is mainly concerned with global existence for the two-component
Camassa-Holm system and the modified two-component Camassa-Holm system. By dis-
covering new conservative quantities of these systems, we prove several new global existence
results for these two-component shallow water systems.

2000 Mathematics Subject Classification: 35G25, 35L05
Keywords: A two-component Camassa-Holm system, a modified two-component Camassa-
Holm system, global existence.

1 Introduction

Recently, the following integrable two-component Camassa-Holm shallow water system
(CH2)

\[
\begin{align*}
  m_t + 2u_x m + um_x + \sigma \rho \rho_x &= 0, & t > 0, & x \in \mathbb{R}, \\
  \rho_t + (\rho u)_x &= 0, & t > 0, & x \in \mathbb{R}, \\
  m(0, x) &= m_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) &= \rho_0(x), & x \in \mathbb{R},
\end{align*}
\]

(1.1)

where \( m = u - u_{xx}, \sigma = \pm 1, \) has been studied extensively. This system appears originally in [17]. The system (1.1) with \( \sigma = -1 \) was first proposed by Chen et al. in [1] and Falqui in [5]. In
2008, Constantin and Ivanov [2] derived the system (1.1) with \( \sigma = 1 \) in the context of shallow
water theory, where \( u(t, x) \) describes the horizontal velocity of the fluid and \( \rho(t, x) \) is related to
the horizontal deviation of the surface from equilibrium with the boundary assumptions \( u \to 0 \)
and \( \rho \to 1 \) as \( |x| \to \infty \).

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When $\sigma = -1$, the system (1.1) corresponds to the situation in which the gravity acceleration points upwards [2] and its mathematical properties have been studied further in many works [1, 4, 5, 18]. It can be identified with the first negative flow of the AKNS hierarchy and has peakon and multi-kink solutions and possesses the bi-Hamiltonian structure [1, 5]. Moreover, the system (1.1) has been presented as an extended $N = 2$ supersymmetric Camassa-Holm equation recently [18]. In [4], the authors established local well-posedness, derived precise blow-up scenarios for strong solutions and presented two blow-up results for strong solutions to the system (1.1) with $\sigma = -1$. However, the global existence for strong solutions to the system (1.1) with $\sigma = -1$ has not been shown yet. In this paper, we will present two new global existence results for the system (1.1) with $\sigma = -1$.

When $\sigma = 1$, the system (1.1) corresponds to the situation in which the gravity acceleration points downwards [2] and its mathematical properties have been studied further in many works [2, 6, 8, 9, 10, 12]. It has been proven that the system is locally well-posed in $H^s \times H^{s-1}$, $s > \frac{3}{2}$ [2, 6, 8]. The authors proved that the system has strong solutions which blow up in finite time, and also has strong solutions which exist globally in time by use of a Lyapunov functional method [2, 6, 9]. Moreover, smooth traveling waves and infinite propagation speed of the system (1.1) with $\sigma = 1$ have been studied in [16] and [12], respectively.

In 2009, Holm et al. [13] introduced the following modified two-component Camassa-Holm system (MCH2)

\[
\begin{align*}
    m_t + um_x + 2mu_x &= -g\rho \bar{\rho}_x, & t > 0, x \in \mathbb{R}, \\
    \rho_t + (\rho u)_x &= 0, & t > 0, x \in \mathbb{R}, \\
    m(0, x) &= m_0(x), & x \in \mathbb{R}, \\
    \rho(0, x) &= \rho_0(x), & x \in \mathbb{R},
\end{align*}
\]

where $(m_0, \rho_0)$ is a given initial profile, $m = u - u_{xx}$ and $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, $g$ is a positive constant. For convenience, we let $g = 1$ in this paper. The system (1.2) is a modified version of the system (1.1). It is written in terms of velocity $u$ and locally averaged density $\bar{\rho}$ (or depth, in the shallow-water interpretation) and $\bar{\rho}_0$ is taken to be constant. Obviously, it depends on not only the locally averaged density $\bar{\rho}$ but also on the pointwise density $\rho$.

The Cauchy problem of MCH2 (1.2) has been studied in many works [7, 11, 13, 14]. It has been shown that this system is locally well-posed on the line [7] and on the circle [14]. Moreover, the authors presented several blow-up results for it [7, 11, 14]. However, the global existence of strong solutions to the system (1.2) has not been discussed so far. The structure of the system (1.2) makes the method used for the system (1.1) with $\sigma = 1$ invalid. Based on a useful conservation quantity, we will present two new global existence results for strong solutions to the system (1.2).

The aim of this paper is to study global existence for the system (1.1) with $\sigma = \pm 1$ and the system (1.2). The remainder of the paper is organized as follows. In Section 2, we present several new global existence results of strong solutions to the system (1.1) with $\sigma = \pm 1$. In Section 3, we present two new global existence results of strong solutions to the system (1.2).

**Notation** Given a Banach space $Z$, we denote its norm by $\| \cdot \|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations if there is no ambiguity. We denote by $\ast$ the convolution.
2 Global existence for CH2

Note that the conservation laws of CH2 with $\sigma = -1$ and $\sigma = 1$ are different. We divide our discussions into two parts.

2.1 Global existence for CH2 with $\sigma = -1$

Lemma 2.1[3, 4] Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2},$ there exists a maximal $T = T\|z_0\|_{H^s \times H^{s-1}} > 0,$ and a unique solution $z = (u, \rho)$ to the system (1.1) such that

$$z = z(\cdot, z_0) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$$

is continuous. Furthermore, the maximal existence time $T$ may be chosen independent of $s.$

Lemma 2.2[3] Let $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2},$ and let $T$ be the maximal existence time of the solution $z = (u, \rho)$ to the system (1.1) with the initial data $z_0.$ Then for all $t \in [0, T),$ we have

$$\int_{\mathbb{R}} (u^2(t, x) + u_x^2(t, x) - \rho^2(t, x))dx = \int_{\mathbb{R}} (u_0^2(x) + u_x^2(0, x) - \rho_0^2(x))dx.$$

Similar to the proof of Theorem 4.2 in [19], we have the following precise blow-up scenario.

Lemma 2.3 Let $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2},$ and let $T$ be the maximal existence time of the solution $z = (u, \rho)$ to the system (1.1) with the initial data $z_0.$ Then the corresponding solution $z$ blows up in finite time if and only if

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(t, x)\} = -\infty.$$ 

Consider now the following initial value problem

$$\begin{cases} 
q_t = u(t, q), & t \in [0, T), \\
q(0, x) = x, & x \in \mathbb{R}. 
\end{cases} \tag{2.1}$$

Lemma 2.4[3] Let $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2},$ and let $T > 0$ be the maximal existence time of the corresponding solution $z = (u, \rho)$ to the system (1.1). Then the problem (2.1) has a unique solution $q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}).$ Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t, x) = exp \left( \int_0^t u_x(s, q(s, x))ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$ 

Next, we will give two important lemmas to the proof of main results.

Lemma 2.5 Let $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2},$ and let $T$ be the maximal existence time of the solution $z = (u, \rho)$ to the system (1.1) with the initial data $z_0.$ Then we have

$$\exp \left( \int_0^t \frac{(u_{t, x}^2 + 2u_{x,x}^2)}{m(t, q(t, x))}ds \right)m(t, q(t, x))q_x^2(t, x) = m_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{2.2}$$
Proof. Applying Lemma 2.1 and a simple density argument, it suffices to consider the case \( s = 3 \). Differentiating the left-hand side of (2.2) with respect to \( t \), in view of the relation (2.1) and the first equation of the system (1.1), we obtain

\[
\partial_t \left( e^{f_0 t} \frac{(pp_x)(s,q(x),x)}{m(s,q(x),x)} ds \right) = m(t,q(t,x))q_x^2(t,x)
\]

By the first equation in (1.1), we have

\[
\begin{align*}
\text{This completes the proof of the lemma.} & \quad \square
\end{align*}
\]

Lemma 2.6 Let \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2} \) and \( z = (u, \rho) \) is the corresponding solution to the system (1.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0,xx}) \in L^1 \), then

\[
\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} m_0(x)dx = \int_{\mathbb{R}} m(t,x)dx.
\]

Proof. As in the proof of Lemma 2.5 it suffices to prove the above lemma for \( s = 3 \). Note that if \( p(x) := \frac{1}{2} e^{-|x|}, x \in \mathbb{R} \), then \((1 - \partial_x^2)^{-1} f = p \ast f \) for all \( f \in L^2 \). Since \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) and \( u_0 = (1 - \partial_x^2)^{-1} m_0 = p \ast m_0 \), by Young’s inequality we have

\[
\|u_0\|_{L^1} = \|p \ast m_0\|_{L^1} \leq \|p\|_{L^1} \|m_0\|_{L^1} \leq \|m_0\|_{L^1}.
\]

By the first equation in (1.1), we have

\[
u_t + uu_x = -\partial_x p \ast (u^2 + \frac{1}{2} u_x^2 - \frac{1}{2} \rho^2).
\]

It follows that

\[
\frac{d}{dt} \int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_t(t,x)dx = \int_{\mathbb{R}} (-uu_x - \partial_x p \ast (u^2 + \frac{1}{2} u_x^2 - \frac{1}{2} \rho^2))dx = 0
\]

Thus \( \int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx \). Moreover,

\[
\int_{\mathbb{R}} m(t,x)dx = \int_{\mathbb{R}} (u - u_{xx})dx = \int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} (u_0 - u_{0,xx})(x)dx = \int_{\mathbb{R}} m_0(x)dx.
\]

This completes the proof of the lemma. \( \square \)

Then, we will give two global existence results to CH2 with \( \sigma = -1 \).

Theorem 2.1 Let \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u, \rho) \) to the system (1.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) and \( m_0 = u_0 - u_{0,xx} \) does not change sign on \( \mathbb{R} \), then \( T = +\infty \). Moreover, \( \|u_x(t, \cdot)\|_{L^\infty} \leq \|m_0\|_{L^1} \).
for all $t \in [0, +\infty)$.

**Proof.** Again we assume $s = 3$ to prove this theorem. Firstly, we assume that $m_0 \geq 0$ on $\mathbb{R}$. Then, Lemma 2.5 implies $m(t, x) \geq 0$ for all $(t, x) \in [0, T] \times \mathbb{R}$. By $u = (1 - \partial_x^2)^{-1} m = p * m$, we have $u(t, x) \geq 0$. By Lemma 2.6, we get

$$-u_x(t, x) + \int_{-\infty}^{x} u(t, \xi) d\xi = \int_{-\infty}^{x} (u - u_{\xi\xi})(t, \xi) d\xi = \int_{-\infty}^{x} m(t, \xi) d\xi$$

$$\leq \int_{-\infty}^{+\infty} m(t, x) dx = \int_{\mathbb{R}} m_0(x) dx = \|m_0\|_{L^1}.$$

Thus, $u_x(t, x) \geq -\|m_0\|_{L^1}$. It follows from Lemma 2.3 that $T = +\infty$. On the other hand,

$$u_x(t, x) - \int_{-\infty}^{x} u(t, \xi) d\xi = -\int_{-\infty}^{x} m(t, \xi) d\xi \leq 0.$$

It follows that

$$u_x(t, x) \leq \int_{-\infty}^{x} u(t, \xi) d\xi \leq \int_{\mathbb{R}} u(t, x) dx = \int_{\mathbb{R}} m_0(x) dx = \|m_0\|_{L^1}.$$

Therefore, $\|u_x(\cdot, \cdot)\|_{L^\infty} \leq \|m_0\|_{L^1}$. In the case when $m_0 \leq 0$ on $\mathbb{R}$, one can repeat the above proof to get the desired result. $\square$

**Theorem 2.2** Let $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2}$, and let $T$ be the maximal existence time of the corresponding solution $z = (u, \rho)$ to the system (1.1) with the initial data $z_0$. If there exists $x_0 \in \mathbb{R}$ such that

$$\begin{cases}
  m_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 \text{ if } x \leq x_0, \\
  m_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 \text{ if } x \geq x_0
\end{cases}$$

and $m_0(x) \in L^1$. Then the solution $z$ exists globally in time.

**Proof.** We only assume $s = 3$ to prove the above theorem. Since $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t, x) > 0$ for all $(t, x) \in [0, T] \times \mathbb{R}$, it follows from the assumptions of the theorem that

$$\begin{cases}
  m(t, x) \leq 0 \text{ if } x \leq q(t, x_0), \\
  m(t, x) \geq 0 \text{ if } x \geq q(t, x_0)
\end{cases}$$

and $m(t, q(t, x_0)) = 0$. Firstly, we claim that

$$\int_{-\infty}^{\frac{q(t, x_0)}{2}} u(t, x) dx \geq -\|m_0\|_{L^1} \quad \text{ and } \quad \int_{\frac{q(t, x_0)}{2}}^{+\infty} u(t, x) dx \leq \|m_0\|_{L^1}.$$
By the first equation in system (1.1), we have \( m_t = -3u_t + 2u_{tx} + uu_{xx} + \rho \). Thus
\[
\frac{d}{dt} \int_{-\infty}^{q(t,x_0)} m(t,x)dx = \int_{-\infty}^{q(t,x_0)} m_t(t,x)dx + m(t,q(t,x_0))q_t(x_0)
\]
\[
= \int_{-\infty}^{q(t,x_0)} m_t(t,x)dx = \int_{-\infty}^{q(t,x_0)} (-3u_t + 2u_{xx} + uu_{xx} + \rho)dx
\]
\[
= \left(-\frac{3}{2}u^2 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{1}{2}\rho^2\right)_{-\infty}^{q(t,x_0)}
\]
\[
= -\frac{3}{2}u^2(t,q(t,x_0)) + \frac{1}{2}u_x^2(t,q(t,x_0)) + (uu_{xx})(t,q(t,x_0)) + \frac{1}{2}\rho^2(t,q(t,x_0))
\]
\[
\geq -\frac{1}{2}(u^2(t,q(t,x_0)) - u_x^2(t,q(t,x_0)))
\]
here we used \(0 = m(t,q(t,x_0)) = u(t,q(t,x_0)) - u_{xx}(t,q(t,x_0)) \). Next, we will claim that \(u^2(t,q(t,x_0)) - u_x^2(t,q(t,x_0)) \leq 0\). Since \(u = p \ast m\),
\[
u(t,x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^\xi m(t,\xi)d\xi + \frac{e^x}{2} \int_{x}^{+\infty} e^{-\xi} m(t,\xi)d\xi
\]
and
\[
u_x(t,x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^\xi m(t,\xi)d\xi + \frac{e^x}{2} \int_{x}^{+\infty} e^{-\xi} m(t,\xi)d\xi.
\]
From this two equations, we obtain
\[
u(t,x) + \nu_x(t,x) = e^x \int_{x}^{+\infty} e^{-\xi} m(t,\xi)d\xi
\]
\[
u(t,x) - \nu_x(t,x) = e^{-x} \int_{x}^{+\infty} e^\xi m(t,\xi)d\xi.
\]
From (2.3)-(2.5), we deduce that
\[
u^2(t,q(t,x_0)) - \nu_x^2(t,q(t,x_0)) = \int_{q(t,x_0)}^{+\infty} e^{-x} m(t,x)dx \int_{-\infty}^{q(t,x_0)} e^\xi m(t,x)dx \leq 0.
\]
Thus
\[
\frac{d}{dt} \int_{-\infty}^{q(t,x_0)} m(t,x)dx \geq 0.
\]
It follows that
\[
\int_{-\infty}^{q(t,x_0)} m(t,x)dx \geq \int_{-\infty}^{x_0} m_0(x)dx = -\int_{-\infty}^{x_0} |m_0(x)|dx \geq -\|m_0\|_{L^1}.
\]
Note that
\[
\int_{-\infty}^{q(t,x_0)} u(t,x)dx = \int_{-\infty}^{q(t,x_0)} m(t,x)dx + u_x(t,q(t,x_0)).
\]
we obtain

\[ u(t, q(t, x_0)) = -\frac{e^{-q(t, x_0)}}{2} \int_{-\infty}^{q(t, x_0)} e^\xi m(t, \xi) d\xi + \frac{e^{q(t, x_0)}}{2} \int_{q(t, x_0)}^{\infty} e^{-\xi} m(t, \xi) d\xi \geq 0, \]

\[ \int_{-\infty}^{q(t, x_0)} u(t, x) dx \geq \int_{-\infty}^{q(t, x_0)} m(t, x) dx \geq -\|m_0\|_{L^1}. \]

A similar proof implies

\[ \int_{q(t, x_0)}^{\infty} u(t, x) dx \leq \|m_0\|_{L^1}. \]

If \( x \leq q(t, x_0) \), then \( m(t, x) \leq 0 \) and \( u(t, x) = (p \ast m)(t, x) \leq 0 \). In view of

\[ u_x(t, x) - \int_{-\infty}^{x} u(t, \xi) d\xi = - \int_{-\infty}^{x} u(t, \xi) d\xi \geq 0, \]

we have

\[ u_x(t, x) \geq \int_{-\infty}^{x} u(t, \xi) d\xi \geq \int_{-\infty}^{q(t, x_0)} u(t, x) dx \geq -\|m_0\|_{L^1}. \]

If \( x \geq q(t, x_0) \), then \( m(t, x) \geq 0 \) and \( u(t, x) = (p \ast m)(t, x) \geq 0 \). In view of

\[ u_x(t, x) + \int_{x}^{\infty} u(t, \xi) d\xi = \int_{x}^{\infty} m(t, \xi) d\xi \geq 0, \]

we obtain

\[ u_x(t, x) \geq - \int_{x}^{\infty} u(t, \xi) d\xi \geq - \int_{q(t, x_0)}^{\infty} u(t, x) dx \geq -\|m_0\|_{L^1}. \]

Therefore, \( u_x(t, x) \geq -\|m_0\|_{L^1} \) for all \( (t, x) \in [0, T) \times \mathbb{R} \). Lemma 2.3 implies \( T = +\infty \). \( \square \)

**Remark 2.1** For two-component Camassa-Holm shallow water systems, the conservation quantity about \( m = u - u_x \) (see Lemma 2.5) is discovered for the first time. From the proofs of Theorems 2.1-2.2, we find that Lemma 2.5 is a crucial lemma in the proof of the global existence of strong solutions.

**Remark 2.2** In the past few years, there were a lot of global existence results for various two-component shallow water systems. However, almost all of the results were proved by the same Lyapunov functional method, which was proposed by Constantin and Ivanov in [2]. That is, let \( M(t, x) = u_x(t, q(t, x)) \) and \( \alpha(t, x) = \rho(t, q(t, x)) \). By the assumption \( \rho_0 \) does not change sign on \( \mathbb{R} \) and the fact \( \rho(t, x) \) has the same sign with \( \rho_0 \), consider the following Lyapunov functional

\[ w(t, x) = \alpha(t, x) \alpha(0, x) + \frac{\alpha(0, x)}{\alpha(t, x)} (1 + M^2). \]

If the structure of system is suitable, then a direct computation implies the following crucial equality

\[ \frac{\partial w}{\partial t}(t, x) = 2 \frac{\alpha(0, x)}{\alpha(t, x)} M(t, x)(f(t, x) - \frac{1}{2}), \quad (2.6) \]

where \( f(t, x) \) is bounded. Blow-up scenario and (2.6) ensure the corresponding \( T = +\infty \) [6, 9]. However, the structure of some two-component shallow water system is unsuitable, for example
the system (1.1) with \( \sigma = -1 \). Since one can not get the crucial equality (2.6), this Lyapunov functional method is invalid.

**Remark 2.3** The discovery of Lemma 2.5 gives a new passage to obtain the global existence of strong solutions for two-component shallow water systems. Moreover, Theorems 2.1-2.2 give an affirmative answer to global existence for strong solutions to the system (1.1) with \( \sigma = -1 \), which is an open problem that has not been solved before.

### 2.2 Global existence for CH2 with \( \sigma = 1 \)

**Lemma 2.7** \[6\] Suppose that \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \). Then there exists \( T = T(\| z_0 \|_{H^s \times H^{s-1}}) > 0 \), and a unique solution \( z = (u, \rho - 1) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}) \) of (1.1) with \( z(0, x) = z_0 \). Moreover, the solution \( z \) depends continuously on the initial data \( z_0 \) and the maximal existence time \( T \) is independent of \( s \).

**Lemma 2.8** \[6\] Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = (u, \rho - 1) \) to the system (1.1). Then we have

\[
\int_{\mathbb{R}} (u^2 + u_x^2 + (\rho - 1)^2)dx = \int_{\mathbb{R}} (u_0^2 + u_0^2 + (\rho_0 - 1)^2)dx, \quad \forall \ t \in [0, T]. \tag{2.7}
\]

Moreover, we have

\[
\|u(t, \cdot)\|_{L^\infty}^2 \leq \frac{1}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2), \quad \forall \ t \in [0, T).
\]

**Lemma 2.9** \[6\] Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u, \rho - 1) \) to the system (1.1) with the initial data \( z_0 \). Then the corresponding solution \( z \) blows up in finite time if and only if

\[
\lim \inf_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(t, x)\} = -\infty.
\]

Similar to the proof of Lemma 2.5, we have the following crucial lemma.

**Lemma 2.10** Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u, \rho - 1) \) to the system (1.1) with the initial data \( z_0 \). Then we have

\[
e^{\int_0^t \frac{(\rho_0 - 1)(\rho_0 - 1)(q(t, x))}{m(t, q(t, x))}dt} m(t, q(t, x))q_x^2(t, x) = m_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{2.8}
\]

**Theorem 2.3** Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u, \rho) \) to the system (1.1) with the initial data \( z_0 \). If \( m_0 = u_0 - u_{0,xx} \) does not change sign on \( \mathbb{R} \), then the system (1.1) has a global strong solution

\[
z = z(\cdot, z_0) \in C([0, \infty); H^s \times H^{s-1}) \cap C^1([0, \infty); H^{s-1} \times H^{s-2}).
\]

Moreover, \( |u_x(t, x)| \leq \|u(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \) for all \( (t, x) \in [0, +\infty) \times \mathbb{R} \).
Proof. Applying Lemma 2.7 and a simple density argument, we only need to show that
the above theorem holds for some \( s > \frac{3}{2} \). Here we assume \( s = 3 \) to prove the above theorem.
Firstly, we assume that \( m_0 \geq 0 \) on \( \mathbb{R} \). Then, Lemma 2.10 implies \( m(t, x) \geq 0 \) for all \((t, x) \in [0, T) \times \mathbb{R}\). By \( u = (1 - \partial_x^2)^{-1}m = p \ast m \), we have
\[
    u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi
\]
and
\[
    u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi.
\]
From these two equations, we obtain
\[
    u(t, x) + u_x(t, x) = e^x \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi = e^{-x} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi
\]
(2.9)
and
\[
    u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi.
\]
(2.10)
It follows from (2.9)-(2.10) and \( m(t, x) \geq 0 \) for all \((t, x) \in [0, T) \times \mathbb{R}\) that
\[
    |u_x(t, x)| \leq u(t, x) \leq \|u(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
This inequality and Lemma 2.9 imply \( T = \infty \). Moreover, \( |u_x(t, x)| \leq \frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \) for all \((t, x) \in [0, T) \times \mathbb{R}\). In the case when \( m_0 \leq 0 \) on \( \mathbb{R} \), one can repeat the above proof to get the desired result.

Theorem 2.4 Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u, \rho) \) to the system (1.1) with the initial data \( z_0 \). If there exists \( x_0 \in \mathbb{R} \) such that
\[
    \left\{ \begin{array}{l}
    m_0(x) \leq 0 \quad \text{if} \quad x \leq x_0, \\
    m_0(x) \geq 0 \quad \text{if} \quad x > x_0.
    \end{array} \right.
\]
Then the corresponding solution \( z \) exists globally in time. Moreover,
\[
    u_x(t, x) \geq -\frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}}
\]
for all \((t, x) \in [0, +\infty) \times \mathbb{R}\).

Proof. We only assume \( s = 3 \) to prove the above theorem. Since \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_x(t, x) > 0 \) for all \((t, x) \in [0, T) \times \mathbb{R}\), it follows from the assumptions of the theorem that
\[
    \left\{ \begin{array}{l}
    m(t, x) \leq 0 \quad \text{if} \quad x \leq q(t, x_0), \\
    m(t, x) \geq 0 \quad \text{if} \quad x \geq q(t, x_0),
    \end{array} \right. \quad \text{(2.11)}
\]
and \( m(t, q(t, x_0)) = 0 \).
When \( x \leq q(t, x_0) \), by (2.10)-(2.11), we have \( u_x(t, x) \geq u(t, x) \); when \( x \geq q(t, x_0) \), by (2.9) and (2.11), we have \( u_x(t, x) \geq -u(t, x) \). Therefore,

\[
-u_x(t, x) \leq |u(t, x)| \leq \|u(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}}
\]
on \( \mathbb{R} \) for all \( t \in [0, T) \). Thus, \( u_x(t, x) \geq -\frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \). Applying Lemma 2.9, this completes the proof of Theorem 2.4. \( \square \)

**Remark 2.4** Since the system (1.1) with \( \sigma = 1 \) has a better conservation law (2.7), it enables us to eliminate the additional assumptions \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) in Theorems 2.1-2.2. Thus, the global existence results in Theorems 2.3-2.4 are better than those in Theorems 2.1-2.2.

**Remark 2.5** Comparing with the corresponding global existence results in [2, 6, 9], where the assumption that \( \rho_0 \) does not change sign on \( \mathbb{R} \) is required, we consider the global existence of strong solutions for two-component shallow water systems from a different perspective. Therefore, the obtained results are new and distinct.

**Remark 2.6** By Theorem 4.1 in [6], we have if there exists \( x_0 \in \mathbb{R} \) such that \( \rho_0(x_0) = 0 \) and \( u_0'(x_0) < -\frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \), then \( T \) is finite. In view of Theorems 2.3-2.4 in present paper, we get if \( m_0 \) satisfies the conditions in Theorems 2.3-2.4, then \( u_0'(x) \geq -\frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \) for all \( x \in \mathbb{R} \). This implies that our results do not conflict with previous blow-up results.

**Remark 2.7** Next, we will display some results to describe the structure characteristics of strong solutions of the system (1.1) with \( \sigma = 1 \). Let \( z_0 = (u_0, \rho_0 - 1) \in H^s \times H^{s-1}, s > \frac{3}{2} \) and let \( T \) be the maximal existence time of the corresponding solution \( z = (u, \rho) \). Then we have

| conditions on \( \rho_0 \) | conditions on \( u_0 \) or \( m_0 = u_0 - u_{0,xx} \) | results on \( T \) |
|------------------------|----------------------------------|---------------|
| \( \rho_0(x) \neq 0 \) for all \( x \in \mathbb{R} \) | \( u_0'(x) < -\frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2)^{\frac{1}{2}} \) | \( T = \infty \) |
| \( \exists x_0 \in \mathbb{R}, \text{ such that } \rho_0(x_0) = 0 \) | \( m_0 \) does not change sign on \( \mathbb{R} \) or \( \exists x_0 \in \mathbb{R} \) such that \( \begin{cases} m_0(x) \leq 0 & \text{if } x \leq x_0, \\ m_0(x) \geq 0 & \text{if } x \geq x_0. \end{cases} \) | \( T = \infty \) |

### 3 Global existence for MCH2

In this section, we will study global existence of the modified two-component Camassa-Holm system (1.2).

We first introduce the equivalent system of the system (1.2). With \( m = u - u_{xx}, \rho = \gamma - \gamma_{xx} \),
and $\gamma = \bar{\rho} - \rho_0$, we can rewrite the system (1.2) as follows:

$$
\begin{cases}
    m_t + m_x u + 2mu_x = -\rho \gamma, & t > 0, \ x \in \mathbb{R}, \\
    \rho_t + (\rho u)_x = 0, & t > 0, \ x \in \mathbb{R}, \\
    m(0, x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{R}, \\
    \rho(0, x) = \gamma_0 - \gamma_{0,xx}, & x \in \mathbb{R}.
\end{cases}
\tag{3.1}
$$

Note that if $p(x) := \frac{1}{2} e^{-|x|}, \ x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p \ast f$ for all $f \in L^2$, $p \ast m = u$ and $p \ast \rho = \gamma$.

Firstly, we give the following useful lemmas that will be used in the sequel.

**Lemma 3.1** Given $z_0 = (u_0, \gamma_0) \in H^s \times H^s, s > \frac{3}{2}$, there exists a maximal $T = T(||z_0||_{H^s \times H^s}) > 0$, and a unique solution $z = (u, \gamma)$ to the system (3.1) such that

$$
z = z(\cdot, z_0) \in C([0, T); H^s \times H^s) \cap C^1([0, T); H^{s-1} \times H^{s-1}).
$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$
z_0 \mapsto z(\cdot, z_0) : H^s \times H^s \rightarrow C([0, T); H^s \times H^s) \cap C^1([0, T); H^{s-1} \times H^{s-1})
$$

is continuous. Furthermore, the maximal time $T$ may be chosen independent of $s$.

**Lemma 3.2** Let $z_0 \in H^s \times H^s, s > \frac{3}{2}$ and let $T > 0$ be the maximal existence time of the corresponding solution $z = (u, \gamma)$ to the system (3.1). Then we have

$$
E(t) = \int_{\mathbb{R}} (u^2 + u_x^2 + \gamma^2 + \gamma_x^2) dx = \int_{\mathbb{R}} (u_0^2 + u_{0,x}^2 + \gamma_0^2 + \gamma_{0,x}^2) dx.
\tag{3.2}
$$

We know that if $z \in H^s \times H^s, s > \frac{3}{2}$, then by the above lemma we have

$$
\|u(t, \cdot)\|_{L^\infty}^2 + \|\gamma(t, \cdot)\|_{L^\infty}^2 \leq \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{2} \|\gamma\|_{H^1}^2 \tag{3.3}
$$

for all $t \in [0, T)$.

**Lemma 3.3** Let $z_0 = \begin{pmatrix} u_0 \\ \gamma_0 \end{pmatrix} \in H^s \times H^s, s > \frac{3}{2}$ be given and assume that $T$ is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \gamma \end{pmatrix}$ of the system (3.1) with the initial data $z_0$. Then the corresponding solution blows up in finite time if and only if

$$
\limsup_{t \to T} \|u_x(t, \cdot)\|_{L^\infty} = +\infty.
$$

**Lemma 3.4** Let $z_0 = \begin{pmatrix} u_0 \\ \gamma_0 \end{pmatrix} \in H^s \times H^s, s > \frac{3}{2}$, and let $T$ be the maximal existence time
of the solution $z = \begin{pmatrix} u \\ \gamma \end{pmatrix}$ to the system (3.1) with the initial $z_0$. Then the corresponding solution blows up in finite time if and only if
\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(t,x)\} = -\infty.
\]

Consider now the following initial value problem
\[
\begin{aligned}
&\begin{cases}
q_t = u(t,q), & t \in [0,T), \\
q(0,x) = x, & x \in \mathbb{R},
\end{cases} \\
&\text{where $u$ denotes the first component of the solution $z$ to the system (3.1). Applying classical results in the theory of ordinary differential equations, one can obtain the following results on $q$.}
\end{aligned}
\]

Lemma 3.5 Let $u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}), s > \frac{3}{2}$. Then the problem (3.4) has a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with
\[
q_x(t,x) = \exp \left( \int_0^t u_x(s,q(s,x))ds \right) > 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.
\]

Next, we will give a crucial result.

Lemma 3.6 Let $z_0 = \begin{pmatrix} u_0 \\ \gamma_0 \end{pmatrix} \in H^s \times H^s, s > \frac{3}{2}$ be given and assume that $T$ is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \gamma \end{pmatrix}$ of the system (3.1) with the initial data $z_0$. Then we have
\[
e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot m(t,q(t,x))q_x^2(t,x) = m_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{R}.
\]  

(3.5)

Proof. Differentiating the left-hand side of (3.5) with respect to $t$, in view of the relations (3.4) and the first equation in the system (3.1), we obtain
\[
\begin{aligned}
&\partial_t \left( e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot m(t,q(t,x))q_x^2(t,x) \right) \\
&= e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot \frac{(\rho \gamma_x)(t,q(t,x))}{m(t,q(t,x))} \cdot m(t,q(t,x))q_x^2(t,x) \\
&+ e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot (m_t + m_xu)q_x^2(t,x) + e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot m(t,q(t,x)) \cdot 2u_xq_x^2(t,x) \\
&= e^\int_0^t \frac{(\rho \gamma_x)(s,q(s,x))}{m(s,q(s,x))}ds \cdot q_x^2(t,x)(\rho \gamma_x + m_t + m_xu + 2mu_x) = 0
\end{aligned}
\]

This completes the proof of the lemma.

Theorem 3.1 Let $z_0 = \begin{pmatrix} u_0 \\ \gamma_0 \end{pmatrix} \in H^s \times H^s, s > \frac{3}{2}$, and let $T$ be the maximal time of
the solution \( z = \begin{pmatrix} u \\ \gamma \end{pmatrix} \) to the system (3.1) with the initial data \( z_0 \). If \( m_0 = u_0 - u_{0,xx} \) does not change sign on \( \mathbb{R} \), then the system (3.1) has a global strong solution
\[
z = z(\cdot, z_0) \in C([0, \infty); H^s \times H^s) \cap C^1([0, \infty); H^{s-1} \times H^{s-1}).
\]
Moreover, \( |u_x(t, x)| \leq \frac{\sqrt{2}}{2} \|z_0\|_{H^1 \times H^1} \) for all \((t, x) \in [0, +\infty) \times \mathbb{R}\).

**Proof.** Applying Lemma 3.1 and a simple density argument, we only need to show that the above theorem holds for some \( s > \frac{3}{2} \). Here we assume \( s = 3 \) to prove the above theorem.

Firstly, we assume that \( m_0 \geq 0 \) on \( \mathbb{R} \). Then, Lemmas 3.5-3.6 imply \( m(t, x) \geq 0 \) for all \((t, x) \in [0, T) \times \mathbb{R} \). By \( u = (1 - \partial_x^2)^{-1} m = p * m \), we have
\[
u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi
\]
and
\[
u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi.
\]
From these two equations, we obtain
\[
u(t, x) + \nu_x(t, x) = e^{x} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi \tag{3.6}
\]
and
\[
u(t, x) - \nu_x(t, x) = e^{-x} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi. \tag{3.7}
\]
By (3.3), (3.6)-(3.7) and \( m(t, x) \geq 0 \) for all \((t, x) \in [0, T) \times \mathbb{R} \), we have
\[
|\nu_x(t, x)| \leq \nu(t, x) \leq \|\nu(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|z_0\|_{H^1 \times H^1}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
This inequality and Lemma 3.3 imply \( T = \infty \). Moreover, \( |\nu_x(t, x)| \leq \frac{\sqrt{2}}{2} \|z_0\|_{H^1 \times H^1} \) for all \((t, x) \in [0, +\infty) \times \mathbb{R} \). In the case when \( m_0 \leq 0 \) on \( \mathbb{R} \), one can repeat the above proof to get the desired result.

**Theorem 3.2** Let \( z_0 = \begin{pmatrix} u_0 \\ \gamma_0 \end{pmatrix} \in H^s \times H^s, s > \frac{3}{2} \), and let \( T \) be the maximal time of the solution \( z = \begin{pmatrix} u \\ \gamma \end{pmatrix} \) to the system (3.1) with the initial data \( z_0 \). If there exists \( x_0 \in \mathbb{R} \) such that
\[
\begin{cases}
m_0(x) \leq 0 & \text{if } x \leq x_0, \\
m_0(x) \geq 0 & \text{if } x \geq x_0.
\end{cases}
\]
Then the corresponding solution \( z \) exists globally in time.

**Proof.** We only assume \( s = 3 \) to prove the above theorem. Since \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_x(t, x) > 0 \) for all \((t, x) \in [0, T) \times \mathbb{R} \), it follows from the assumptions of the theorem that
\[
\begin{cases}
m(t, x) \leq 0 & \text{if } x \leq q(t, x_0), \\
m(t, x) \geq 0 & \text{if } x \geq q(t, x_0)
\end{cases} \tag{3.8}
\]
and \( m(t,q(t,x_0)) = 0 \).

When \( x \leq q(t,x_0) \), by (3.7)-(3.8), we have \( u_x(t,x) \geq u(t,x) \); when \( x \geq q(t,x_0) \), by (3.6) and (3.8), we have \( u_x(t,x) \geq -u(t,x) \). Therefore,

\[
-u_x(t,x) \leq |u(t,x)| \leq \|u(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|z_0\|_{H^1\times H^1}
\]
on \( \mathbb{R} \) for all \( t \in [0,T) \). Thus, \( u_x(t,x) \geq -\frac{\sqrt{2}}{2} \|z_0\|_{H^1\times H^1} \). Applying Lemma 3.4, this completes the proof of Theorem 3.2. \( \square \)

**Remark 3.1** We used the Lyapunov functional method to obtain a global existence result for the modified two-component Camassa-Holm system (3.1), but unfortunately we find that the result is wrong [15]. The reason is that we can not get the critical equality (2.6) about the system (3.1). However, by using the new conservative quantity in Lemma 3.6, we give two new and distinct global existence results in Theorems 3.1-3.2.

**Remark 3.2** By synthesizing the previous results [7, 11], we give the following form to describe the structure characteristics of strong solutions of the system (3.1). Let \( z_0 = (u_0, \gamma_0) \in H^s \times H^s, s > \frac{3}{2} \), and \( T \) be the maximal time of the solution \( z = (u, \gamma) \) to the system (3.1) with the initial data \( z_0 \). Then we have

| Conditions on \( \rho_0 \) | Conditions on \( u_0 \) or \( m_0 = u_0 - u_{0,xx} \) | Results on \( T \) |
|--------------------------|-----------------------------|-----------------|
| \( \exists x_0 \in \mathbb{R} \) such that \( u_0'(x_0) < -(\|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2)^{\frac{1}{2}} \) | \( m_0(x_0) = 0 \), \( \int_{-\infty}^{x_0} e^{\xi} m_0(\xi) d\xi \geq 0 \) and \( \int_{x_0}^{\infty} e^{-\xi} m_0(\xi) d\xi \leq 0 \) | \( T < \infty \) |
| \( \exists x_0 \in \mathbb{R}, \text{ such that } \rho_0(x_0) = 0 \) and \( \begin{cases} \rho_0(x) \geq 0 & \text{if } x < x_0, \\ \rho_0(x) \leq 0 & \text{if } x > x_0. \end{cases} \) | \( m_0(x) \) does not change sign on \( \mathbb{R} \) or \( \exists x_0 \in \mathbb{R} \) such that \( \begin{cases} m_0(x) \leq 0 & \text{if } x \leq x_0, \\ m_0(x) \geq 0 & \text{if } x \geq x_0. \end{cases} \) | \( T = \infty \) |

**Acknowledgments** The first author was partially supported by NNSFC (No. 11326161), Doctoral Fund of ZZULI (No. 2013BSJJ052) and the key projects of Science and Technology Research of the Henan Education Department (No.14A110011). The second author was partially supported by NNSFC (No. 11271382), RFDP (No. 20120171110014), and the key project of Sun Yat-sen University.

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