OPE and a low-energy theorem in QCD-like theories

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Abstract: We verify, both perturbatively and nonperturbatively asymptotically in the ultraviolet (UV), a special case of a low-energy theorem of the NSVZ type in QCD-like theories, recently derived in Phys. Rev. D 95 (2017) 054010, that relates the logarithmic derivative with respect to the gauge coupling, or the logarithmic derivative with respect to the renormalization-group (RG) invariant scale, of an \(n\)-point correlator of local operators in one side to an \(n+1\)-point correlator with the insertion of \(\text{Tr} F^2\) at zero momentum in the other side. Our computation involves the operator product expansion (OPE) of the scalar glueball operator, \(\text{Tr} F^2\), in massless QCD, both perturbatively and in its RG improved form, by means of which we extract both the perturbative divergences and the nonperturbative UV asymptotics in both sides. We also discuss the role of the contact terms in the OPE, both finite and divergent, discovered some years ago in JHEP 1212 (2012) 119, in relation to the low-energy theorem. Besides, working the other way around by assuming the low-energy theorem for any 2-point correlator of a multiplicatively renormalizable operator, we compute the corresponding lower-order perturbative OPE and nonperturbative asymptotics. The low-energy theorem has a number of applications: to the renormalization in asymptotically free QCD-like theories, both perturbatively and nonperturbatively in the large-\(N\) ’t Hooft and Veneziano expansions, and to the way the open/closed string duality may or may not be realized in the would-be solution by canonical string theories for QCD-like theories, both perturbatively and in the ’t Hooft large-\(N\) expansion. Our computations will also enter further developments based on the low-energy theorem.
One of the aims of this paper is to verify, both perturbatively and nonperturbatively asymptotically in the ultraviolet (UV), a special case of a recently derived low-energy theorem [1] of the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) type [2] in SU(N) QCD-like gauge theories.
It relates the logarithmic derivative with respect to the 't Hooft gauge coupling, $g^2 = g_M^2 N$, of an $n$-point correlator of local operators, $\mathcal{O}_k$, to an $n + 1$-point correlator with the insertion of $\text{Tr} \mathcal{F}^2$ at zero momentum [1]:

$$\frac{\partial \langle O_1 \cdots O_n \rangle}{\partial \log g} = \frac{N}{g} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \text{Tr} \mathcal{F}^2(x) \rangle - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \langle \text{Tr} \mathcal{F}^2(x) \rangle d^4 x \quad (1.1)$$

where the Wilsonian normalization of the YM action is chosen (Sec. 2) and the operators, $\text{Tr} \mathcal{F}^2$ and $\mathcal{O}_k$, are $g$ independent (Sec. 2).

It admits another version [1], where the logarithmic derivative with respect to the gauge coupling is replaced by the logarithmic derivative with respect to the renormalization-group (RG) invariant scale, $\Lambda_{QCD}$, in an asymptotically free QCD-like theory (Sec. 2.2):

$$\frac{\partial \langle O_1 \cdots O_n \rangle}{\partial \log \Lambda_{QCD}} = -\frac{N\beta(g)}{g^3} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \text{Tr} \mathcal{F}^2(x) \rangle - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \langle \text{Tr} \mathcal{F}^2(x) \rangle d^4 x \quad (1.2)$$

After rescaling the gauge field in the functional integral by a factor of $\frac{g}{\sqrt{N}}$, the low-energy theorem admits a trivially equivalent canonical version with the canonical normalization of the YM action – Eq. (2.6) in Subsec. 2.3 – in terms of operators, $\text{Tr} \mathcal{F}^2$ and $\mathcal{O}_k$, defined by the very same rescaling, satisfying $\frac{g^2}{\sqrt{N}} \text{Tr} \mathcal{F}^2 = \text{Tr} \mathcal{F}^2$ and $(\frac{g}{\sqrt{N}})^c \mathcal{O}_k = \mathcal{O}_k$ for some $c_k$. The canonically normalized operators, $\text{Tr} \mathcal{F}^2$ and $\mathcal{O}_k$, depend now on $g$ (Subsec. 2.3).

The canonical version of Eq. (1.1), which is most suitable for perturbative computations, – Eq. (2.6) with $n = 2$ and $\mathcal{O}_k = \text{Tr} \mathcal{F}^2$ – has been employed to analyze the renormalization properties of QCD-like theories perturbatively to one loop in [3] and the way the open/closed string duality [4, 5] may be actually implemented [3] in string theories realizing perturbatively [5] QCD-like theories.

The second version – Eq. (1.2) with $\mathcal{O}_k = \text{Tr} \mathcal{F}^2$ – has been employed to compute the nonperturbative counterterms [1] in the large-$N$ 't Hooft [6] and Veneziano expansions [7] of QCD-like theories.

Moreover, it has entered crucially a no-go theorem [3] that the nonperturbative renormalization in the 't Hooft large-$N$ QCD S matrix is incompatible with the open/closed string duality of a would-be canonical string solution, which therefore does not exist [3]. By a canonical string solution we mean [3] a perturbative expansion in the string coupling, $g_s \sim \frac{1}{\sqrt{N}}$, for the string S matrix that matches the topology of the 't Hooft large-$N$ expansion and computes the large-$N$ QCD S matrix by means of an auxiliary 2d conformal field theory living on the string world-sheet with fixed topology [6–9]. A noncanonical way-out to the no-go theorem has been suggested in [3, 10]. Relatedly, the large-$N$ Veneziano expansion [7] has been briefly discussed in [10].

Because of the importance of the low-energy theorem for the above subjects, a deeper understanding and an explicit evaluation of both sides in Eqs. (1.1) and (1.2) are most interesting.
In fact, the divergent parts in both sides of the canonical version of Eq. (1.1) – Eq. (2.6) for \( n = 2 \) and \( O_k = \text{Tr} F^2 \) – have already been computed to one loop in perturbation theory in [3], thus partially verifying a special case of the low-energy theorem perturbatively. Yet, we include (Sec. 3) in the aforementioned one-loop computation the finite contact term that arises from the operator product expansion (OPE) of \( \text{Tr} F^2 \) (Eq. (1.3)) in the canonical version of Eq. (1.1) that has been skipped in [3].

Moreover, we extend the one-loop perturbative computation in [3] to two loops in Subsec. 3.2, including as well the divergent contact term in the OPE of \( \text{Tr} F^2 \) (Eq. (1.3)) discovered some years ago to two loops in [11], whose renormalization has been recently discussed in [17].

Besides, we compute in Sec. 4 the nonperturbative universal, i.e., renormalization-scheme independent, UV asymptotics\(^1\) in the coordinate representation in both sides of Eq. (1.2) for \( n = 2 \) and \( \mathcal{O}_k = \text{Tr} \mathcal{F}^2 \).

Our nonperturbative computation furnishes a detailed derivation and an improvement to the next to leading logs of the crucial – for the no-go theorem in [3] – nonperturbative leading-log asymptotic estimate in the coordinate representation in [3], and provides another check of the low-energy theorem.

In order to perform the computations, we employ firstly the perturbative OPE worked out in [11, 12, 17] that is recalled in App. B. A previous perturbative lower-order computation appeared in [13]. We pass from the perturbative OPE in the momentum representation [11, 12, 17] in Apps. B.1 and B.3 to the OPE in the coordinate representation by the Fourier transform in Apps. B.2 and B.4. From it we get the normalization for the nonperturbative universal RG-improved UV asymptotics of the OPE coefficients in the coordinate representation derived a priori in App. A, which includes the leading and next to leading logs.

Moreover, by making the OPE coefficients RG invariant by a suitable rescaling of the operators, we verify in App. B their nonperturbative asymptotics by rewriting the perturbative computation, originally expressed in terms of \( g(\mu) \) and large logs [11, 12, 17], in terms of the running coupling, \( g(x) \), to the given perturbative order.

Previous related results about the universal asymptotics of the OPE both in the momentum and coordinate representation have been obtained in [14, 15] and, only about the leading logs, in the momentum representation in [16] and in the coordinate representation in [3].

Besides, working the other way around by assuming the low-energy theorem for any 2-point correlator, \( \langle O \mathcal{O} \rangle \), of a multiplicatively renormalizable operator, \( \mathcal{O} \), we compute the corresponding lower-order perturbative OPE and nonperturbative UV asymptotics in Sec. 5.

\(^1\)We define in App. A what we mean by the universal UV asymptotics.
Our computations will also enter further developments that assume the low-energy theorem.

Now, we describe the organization of the paper and the details of the computations.

In Sec. 2 we recall the proof of various versions of the low-energy theorem. The version that involves the canonically normalized YM action – Eq. (2.6) in Subsec. 2.3 – is suitable for its perturbative check. The version that involves $\Lambda_{QCD}$ – Eq. (1.2) in Subsec. 2.2 – enters the nonperturbative UV asymptotics of the low-energy theorem in Sec. 4.

Firstly, we aim to verify Eq. (2.6) in the coordinate representation for $n = 2$ and $O_k = \text{Tr} F^2$ perturbatively in Sec. 3.

For brevity we define $F^2(x) \equiv 2 \text{Tr} F^2(x)$. While the 2-point correlator for $F^2$ in the lhs of Eq. (2.6) is available perturbatively [11, 12, 17], a direct computation of the 3-point correlator in the rhs is not available. Therefore, we exploit the OPE for $F^2(x) F^2(0)$ in massless QCD (i.e., QCD with massless quarks):

$$F^2(x) F^2(0) = C_0^{(S)}(x) I + C_1^{(S)}(x) F^2(0) + \cdots \quad (1.3)$$

$C_0^{(S)}(x)$ is the coefficient of the identity operator, $I$, $C_1^{(S)}(x)$ is the coefficient of the operator $F^2$ itself, and the dots stand for operators whose contribution to the 3-point correlator in Eq. (2.6) turns out to be subleading as $x \to 0$ [18].

Thus, our computation cannot be exact, but it is affected by finite ambiguities due both to the incomplete OPE and to regularizing in the infrared (IR) the integral in the rhs of Eq. (2.6) in perturbation theory (Sec. 3). Yet, our computation suffices to verify the perturbative universal divergent parts in both sides of Eq. (2.6) (Sec. 3) and, as we will see, the nonperturbative UV asymptotics in both sides of Eq. (1.2) (Sec. 4).

For multiplicatively renormalized $F^2$ in massless QCD, $C_0^{(S)}$ has been computed in the $\overline{\text{MS}}$ scheme to three loops, both in the momentum and coordinate representation, in [11]:

$$C_0^{(S)}(x) = \frac{N_c^2 - 1}{x^8} \frac{48}{\pi^4} \left( 1 + g^2(\mu)(A_{0,1}^{(S)} + 2\beta_0 \log(x^2 \mu^2)) 
+ g^4(\mu)(A_{0,2}^{(S)} + A_{0,3}^{(S)} \log(x^2 \mu^2) + 3\beta_0^2 \log^2(x^2 \mu^2)) \right) 
+ \Delta^2 \delta^{(4)}(x) \frac{N_c^2 - 1}{4\pi^2} \left( 1 + \log\left(\frac{\Lambda^2}{\mu^2}\right) + g^2(\mu) \left( A_{0,4}^{(S)} + A_{0,5}^{(S)} \log\left(\frac{\Lambda^2}{\mu^2}\right) 
- \beta_0 \log^2\left(\frac{\Lambda^2}{\mu^2}\right) \right) 
+ g^4(\mu) \left( A_{0,6}^{(S)} + A_{0,7}^{(S)} \log\left(\frac{\Lambda^2}{\mu^2}\right) + A_{0,8}^{(S)} \log^2\left(\frac{\Lambda^2}{\mu^2}\right) 
+ \beta_0^2 \log^3\left(\frac{\Lambda^2}{\mu^2}\right) \right) \right) \right) \quad (1.4)$$
$C_1^{(S)}$ has been computed in the $\overline{MS}$ scheme, in the momentum representation to two loops in [11] and to three loops in [12]. We perform in App. B.4 its Fourier transform in the coordinate representation:

$$C_1^{(S)}(x) = \frac{4\beta_0}{\pi^2} \log^2(\mu) \left( 1 + g^2(\mu)(A_{1,1}^{(S)} + 2\beta_0 \log(x^2\mu^2)) + g^4(\mu)(A_{1,2}^{(S)} + A_{1,3}^{(S)} \log(x^2\mu^2) + 3\beta_0^2 \log^2(x^2\mu^2)) \right) + \delta^{(4)}(x) \left( 4 + g^2(\mu)A_{1,3}^{(S)} + g^4(\mu)A_{1,5}^{(S)} + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) \right) + g^6(\mu) \left( A_{1,6}^{(S)} + 8\beta_2 \log(\frac{\Lambda^2}{\mu^2}) - 4\beta_0\beta_1 \log^2(\frac{\Lambda^2}{\mu^2}) \right)$$

(1.5)

where:

$$\begin{align*}
\beta_0 &= \frac{1}{(4\pi)^2} \left( \frac{11}{3} - \frac{2}{3} \frac{N_f}{N} \right) \\
\beta_1 &= \frac{1}{(4\pi)^4} \left( \frac{34}{3} - \frac{13}{3} \frac{N_f}{N} + \frac{N_f}{N^3} \right) \\
\beta_2 &= \frac{1}{(4\pi)^6} \left( \frac{2857}{54} - \frac{1709}{54} \frac{N_f}{N} + \frac{56}{27} \frac{N_f^2}{N^2} + \frac{187}{36} \frac{N_f}{N^3} - \frac{11}{18} \frac{N_f^2}{N^4} + \frac{N_f}{4N^5} \right)
\end{align*}$$

(1.6), (1.7), (1.8)

are the first-three coefficients of the QCD beta function, $\frac{\partial g}{\partial \log \Lambda} = \beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 \cdots$, and $A_{0,j}^{(S)}$, $A_{1,j}^{(S)}$ are finite coefficients computed in [11, 12].

$\delta^{(4)}(x)$ and $\Delta^2 \delta^{(4)}(x)$ are contact terms in the coordinate representation, i.e., distributions supported at coinciding points. They arise from polynomials in the momentum representation. Interestingly, both finite and divergent contact terms occur in the perturbative OPE [11, 17]. The divergent contact terms require further additive renormalizations [17] with respect to the multiplicative renormalization of $F^2$ due to its anomalous dimension.

By exploiting the OPE in Eq. (1.3), Eq. (2.6) reads:

$$4C_0^{(S)}(z) + 2g^2 \frac{\partial C_0^{(S)}(z)}{\partial g^2} \sim C_0^{(S)}(z) \int C_1^{(S)}(x) d^4x$$

(1.9)

We test Eq. (1.9) in perturbation theory by extracting the divergent parts in both sides from the unrenormalized OPE both to one loop in Subsec. 3.1 and two loops in Subsec. 3.2. Indeed, the low-energy theorem is derived for the bare coupling in Eqs. (2.2) and (2.6). The perturbative unrenormalized coefficients are obtained from the renormalized ones by setting $\mu = \Lambda$ in Eqs. (1.4) and (1.5).

The contact terms in $C_0^{(S)}$ are, in fact, inessential to verify the low-energy theorem, since they can be skipped by choosing $z \neq 0$ in Eq. (1.9). Instead, the contact terms in $C_1^{(S)}$ cannot be skipped, because $C_1^{(S)}$ is integrated over the whole space-time.
We demonstrate that the finite contact term to the order of $g^0$ in $C_1^{(S)}$ in the rhs of Eq. (1.9) is crucial to match the first term in the lhs perturbatively \(^2\) (Sec. 3). Thus, somehow surprisingly, the rationale for its occurrence is the low-energy theorem.

We observe that the log-divergent contact term in $C_1^{(S)}$ to the order of $g^4$ \([11, 12, 17]\) mixes in Eq. (1.9) with the scheme-dependent two-loop divergences due to the anomalous dimension of $F^2$ (Sec. 3). Since the latter divergences to the order of $g^4$ are affected by the aforementioned finite ambiguities to the order of $g^2$, presently we cannot argue on the basis of the low-energy theorem about the renormalization of the two-loop divergent contact term in $C_1^{(S)}$ perturbatively. We have anyway computed the constant finite parts to two loops in both sides of Eq. (1.9), with the IR subtraction point of the integral in the rhs specified below, for future employment as well.

Yet, if we skip the divergent contact term in $C_1^{(S)}$, but not the finite one, the low-energy theorem is verified nonperturbatively for the renormalized correlators asymptotically in the UV in Sec. 4. This is compatible with the previous result in \([17]\), obtained independently of the low-energy theorem, that the divergent contact term in $C_1^{(S)}$ should be renormalized to zero.

We discover in Subsec. 3.1 the prescription for the IR subtraction point of the integral in the rhs of Eq. (1.9) that also allows us to reproduce the finite small-$z$ dependence in the lhs in perturbation theory to one loop.

We apply the very same subtraction prescription to compute the nonperturbative UV asymptotics in both sides of Eq. (4.3), verifying their asymptotic agreement in the UV.

Our computation involves the nonperturbative UV asymptotics of the OPE coefficients in the coordinate representation, which we establish a priori, up to a constant overall normalization, by means of the Callan-Symanzik equation in App. A, following standard methods worked out in \([14, 15]\). Then, we employ the perturbative results in App. B to fix the overall normalization as well.

Alternatively, following \([14]\) we verify in App. B.3 and B.6 the nonperturbative UV asymptotics of the RG-invariant coefficients, $\left(\frac{\beta(g)}{g}\right)^2 C_1^{(S)}$ and $-\frac{\beta(g)}{g} C_1^{(S)}$, previously derived a priori in App. A, by means of a change of the perturbative renormalization scheme that allows us to rewrite the perturbative results, originally expressed in terms of $g(\mu)$ and large logs \([11, 12, 17]\), in terms of the running coupling, $g(x)$, to the given perturbative order.

Finally, by inverting the logic of the computation, we derive from the low-energy theorem for any 2-point correlator, $\langle \mathcal{O} \mathcal{O} \rangle$, of a multiplicatively renormalizable oper-

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\(^2\)In \([3]\) both the aforementioned contact term and the first term in the lhs of Eq. (1.9) have been skipped.
operator, $\mathcal{O}$, the lower-order perturbative OPE and the nonperturbative UV asymptotics in Sec. 5.

Since the rhs of the low-energy theorem contains an operator insertion at zero momentum, it is somehow surprising that the low-energy theorem is asymptotically verified nonperturbatively in the UV.

Yet, our RG-improved computations in Secs. 4 and 5 seem to show a posteriori that the integral on space-time in rhs of the low-energy theorem, given the aforementioned specific choice of the IR subtraction point dictated by one-loop perturbation theory, is in fact dominated by the UV asymptotics of the integrand. This is compatible with the universal belief that, nonperturbatively, the IR of the integrand in the rhs of Eqs. (1.9) and (5.1) is exponentially suppressed because of the glueball mass gap [3].

2 The low-energy theorem

2.1 Low-energy theorem in terms of the Wilsonian coupling

For completeness we report the proof of the low-energy theorem in QCD-like theories according to [1]. Given a set of local operators, $\mathcal{O}_k$, and the Wilsonian normalization of the action, by deriving:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{\int \mathcal{O}_1 \cdots \mathcal{O}_n e^{-\frac{N}{g^2} \int \text{Tr} \mathcal{F}^2(x) d^4x + \cdots}}{e^{-\frac{N}{g^2} \int \text{Tr} \mathcal{F}^2(x) d^4x + \cdots}} \quad (2.1)$$

with respect to $-\frac{1}{g^2}$, we obtain:

$$\frac{\partial \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle}{\partial \log g} = \frac{N}{g^2} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \text{Tr} \mathcal{F}^2(x) \rangle - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \langle \text{Tr} \mathcal{F}^2(x) \rangle d^4x \quad (2.2)$$

where the trace, Tr, is in the fundamental representation, $\text{Tr} \mathcal{F}^2 = \text{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})$, the sum over repeated indices is understood, and $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. From the derivation it is clear that all the operators $- \mathcal{O}_k$, $\text{Tr} \mathcal{F}^2$, and in the dots -- are chosen to be $g$ independent. It also is clear that $g$ in Eqs. (2.1) and (2.2) is the bare coupling.

2.2 Low-energy theorem in terms of $\Lambda_{QCD}$

A second version [1] of the low-energy theorem holds in an asymptotically free QCD-like theory:

$$\frac{\partial \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle}{\partial \log \Lambda_{QCD}} = -\frac{N \beta(g)}{g^2} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \text{Tr} \mathcal{F}^2(x) \rangle - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \langle \text{Tr} \mathcal{F}^2(x) \rangle d^4x \quad (2.3)$$

as it follows by employing the chain rule, $\frac{\partial}{\partial \log g} = \frac{\partial \Lambda_{QCD}}{\partial \log g} \frac{\partial}{\partial \Lambda_{QCD}}$, the defining relation:

$$\left( \frac{\partial}{\partial \log \Lambda} + \beta(g) \frac{\partial}{\partial g} \right) \Lambda_{QCD} = 0 \quad (2.4)$$
and the identity:
\[
\frac{\partial \Lambda_{QCD}}{\partial \log \Lambda} = \Lambda_{QCD}
\] (2.5)
since \(\Lambda_{QCD} = e^{\log \Lambda} f(g)\) for a function \(f(g)\).

### 2.3 Low-energy theorem in terms of the canonical coupling

In order to verify the low-energy theorem in perturbation theory it is convenient to employ the canonical normalization of the action [3]. Thus, we rescale the gauge fields in the functional integral by a factor of \(g \sqrt{N}\). Of course, the rescaling does not affect the expectation value of the operators. As a consequence, \(g^2 N \text{Tr} F^2 = \text{Tr} F^2\) and \((\frac{g}{\sqrt{N}})^c_k O_k = O_k\) for some \(c_k\), where now \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \frac{g}{\sqrt{N}} [A_\mu, A_\nu]\) and \(O_k\) are \(g\) dependent. After the rescaling, Eq. (2.2) reads:

\[
\left( \sum_{k=1}^{k=n} c_k \right) \langle O_1 \ldots O_n \rangle + \frac{\partial \langle O_1 \ldots O_n \rangle}{\partial \log g} = \int \langle O_1 \ldots O_n \text{Tr} F^2(x) \rangle - \langle O_1 \ldots O_n \rangle \langle \text{Tr} F^2(x) \rangle d^4 x
\] (2.6)

Similarly, Eq. (2.3) becomes:

\[
-\frac{\beta(g)}{g} \left( \sum_{k=1}^{k=n} c_k \right) \langle O_1 \ldots O_n \rangle + \frac{\partial \langle O_1 \ldots O_n \rangle}{\partial \log \Lambda_{QCD}} = -\frac{\beta(g)}{g} \int \langle O_1 \ldots O_n \text{Tr} F^2(x) \rangle - \langle O_1 \ldots O_n \rangle \langle \text{Tr} F^2(x) \rangle d^4 x \] (2.7)

### 3 Low-energy theorem for \(\langle F^2(z)F^2(0) \rangle\) in perturbation theory

Perturbatively, for \(n = 2\) and \(O = F^2\), Eq. (2.6) becomes:

\[
4 \langle F^2(z)F^2(0) \rangle + 2g^2 \frac{\partial \langle F^2(z)F^2(0) \rangle}{\partial g^2} = \frac{1}{2} \int \langle F^2(z)F^2(0)F^2(x) \rangle d^4 x
\] (3.1)

since the condensate, \(\langle F^2 \rangle\), vanishes identically in dimensional regularization to every order in perturbation theory.

We verify Eq. (3.1) in perturbation theory for the unrenormalized operator, \(F^2\), [3]. We choose \(z \neq 0\) in order to skip the inessential contact terms in the lhs of Eq. (3.1). The lhs is in general log divergent [3] to one loop because of the anomalous-dimension coefficient, \(\gamma_0 = 2\beta_0\), of \(F^2\), where:

\[
\gamma(g) = -\frac{\partial \log Z}{\partial \log \mu} = -\gamma_0 g^2 - \gamma_1 g^4 + \cdots
\] (3.2)
By a standard argument reported in [14] the anomalous dimension of $F^2$ is related to the beta function:

$$\gamma_{F^2}(g) = g \frac{\partial}{\partial g} \left( \frac{\beta(g)}{g} \right)$$  \hspace{1cm} (3.3)$$

It follows:

$$\gamma_{F^2}(g) = -2\beta_0 g^2 - 4\beta_1 g^4 + \cdots$$  \hspace{1cm} (3.4)$$

Only $\gamma_0$ is scheme independent. We will construct explicitly in App. B.3 the scheme where $\gamma_1 = 4\beta_1$ following [14]. It turns out to be the scheme where the constant finite parts of $C_0^{(S)}(z)$ for $z \neq 0$ vanish to two loops.

Hence, also the rhs in Eq. (3.1) must be divergent, and the divergence can be evaluated by means of the OPE in Eq. (1.3) [3]. Thus, it suffices to evaluate asymptotically the 3-point correlator, $\langle F^2(z) F^2(0) F^2(x) \rangle$, by fixing $z$ while $x$ may be close either to $z$ or $0$.

For $x$ close to 0 we get:

$$\langle F^2(z) F^2(0) F^2(x) \rangle = C_1^{(S)}(x) \langle F^2(z) F^2(0) \rangle + \cdots$$  \hspace{1cm} (3.5)$$

Therefore, as far as the divergent parts are concerned, the low-energy theorem reads:

$$4C_0^{(S)}(z) + 2g^2 \frac{\partial C_0^{(S)}(z)}{\partial g^2} \sim C_0^{(S)}(z) \int C_1^{(S)}(x) d^4 x$$  \hspace{1cm} (3.6)$$

where a factor of 2 has been included in the rhs to take into account that $x$ can be close either to $z$ or $0$ [3].

To evaluate Eq. (3.6) we employ in the rhs the perturbative version of $C_1^{(S)}$. After extracting from $C_1^{(S)}$ the lowest-order contact term:

$$C_1^{(S)}(x) = 4\delta^4(x) + C_1^{(S)'}(x)$$  \hspace{1cm} (3.7)$$

Eq. (3.6) simplifies significantly:

$$2g^2 \frac{\partial C_0^{(S)}(z)}{\partial g^2} \sim C_0^{(S)}(z) \int C_1^{(S)'}(x) d^4 x$$  \hspace{1cm} (3.8)$$

Interestingly, the lowest-order contact term is crucial to satisfy Eq. (3.6) to the order of $g^0$. In [3] both this contact term and the compensating first term in the lhs of Eq. (3.6) have been skipped (Sec. 3.1).

### 3.1 One loop

To verify Eq. (3.8) to one loop [3], we employ the corresponding unrenormalized one-loop OPE coefficients in the coordinate representation. They are obtained simply setting $\mu = \Lambda$ in the renormalized ones:

$$C_0^{(S)}(z) = \frac{N^2 - 1}{z^8} \frac{48}{\pi^4} \left( 1 + g^2(\Lambda) \left( A_{0,1}^{(S)} + 2\beta_0 \log(\Lambda^2/\mu^2) + 2\beta_0 \log(z^2 \mu^2) \right) \right)$$  \hspace{1cm} (3.9)$$

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In Eq. (3.9) we have skipped the inessential contact terms in $C_0^{(S)}(z)$ by choosing $z \neq 0$. Hence, the divergent part of Eq. (3.8) reads to one loop:

$$N^2 - 1 \frac{4}{z^8} \frac{\pi^4 g^2(\Lambda)}{4 \beta_0} \log(\Lambda^2) = \frac{N^2 - 1}{z^8} \frac{4 \beta_0}{\pi^4} \log(\Lambda^2) \int \frac{1}{x^4} d^4 x$$

The integral in the rhs is both UV and IR divergent [3]. Incidentally, this divergence plays a key role for the compatibility of the open/closed string duality with perturbation theory in massless QCD [3].

We regularize the integral by restricting to the domain $D^\Lambda = \{ x^2 : 1/\Lambda^2 \leq x^2 \leq 1/\mu^2 \}$. Hence, performing the integral in polar coordinates, we get:

$$\int_{D^\Lambda} \frac{1}{x^4} d^4 x = \pi^2 \log(\frac{\Lambda^2}{\mu^2})$$

that implies Eq. (3.11) according to [3].

Our key observation is that we can also match the $z$ dependence of the finite part in the lhs of Eq. (3.8), provided that we suitably modify the integration domain in the IR, $D^\mu = \{ x^2 : 1/\mu^2 \leq x^2 \leq z^2 \}$. By including the constant finite parts for future employment, we get to one loop:

$$N^2 - 1 \frac{4}{z^8} \frac{\pi^4 g^2(\Lambda)}{4 \beta_0} \left( 2 A^{(S)}_{0,1} + 4 \beta_0 \log(\Lambda^2) \right)$$

$$\sim \frac{N^2 - 1}{z^8} \frac{4 \beta_0}{\pi^4} \left( A^{(S)}_{1,1} + \frac{4 \beta_0}{\pi^2} \int_{D^\Lambda} \frac{1}{x^4} d^4 x \right) g^2(\Lambda)$$

This prescription plays a key role for getting the correct nonperturbative RG-improved UV asymptotics in the rhs of Eq. (4.3). Of course, any prescription for the IR cutoff in the rhs leads, already to one loop, to an ambiguity for the constant finite parts to the order of $g^2$ in the rhs. Presently, we cannot resolve this finite ambiguity in the framework of our computation that is either based in this Sec. on the perturbative OPE or on its universal nonperturbative asymptotics in Sec. 4.

### 3.2 Two loops

The one-loop $O(g^2)$ UV divergence computed in Subsec. 3.1 is universal, i.e., it depends only on the first coefficient of the anomalous dimension. The $O(g^4)$ quadratic-log divergence in the lhs of Eq. (3.8) is universal as well, because it is essentially the square of the one-loop divergence.

Instead, the $O(g^4)$ log divergence is scheme dependent, and therefore depends on the constant finite parts to the order of $g^2$. Thus, we verify perturbatively Eq. (3.8) by limiting ourselves to the universal divergences.
However, we compute as well the scheme-dependent finite and log-divergent parts in both sides of Eq. (3.8) to the order of $g^4$ for future employment. We evaluate the rhs of Eq. 3.8 to two loops:

\[
\frac{N^2 - 1}{\pi^4} g^4(A) \left( \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} \left( A^{(s)}_{1,1} + 2\beta_0 \log(x^2 \Lambda^2) \right) d^4 x + A^{(s)}_{1,5} \right) + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) + \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} d^4 x \\
+ \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) A^{(s)}_{1,4}
\]

(3.14)

It reads to the order of $g^4$:

\[
\frac{N^2 - 1}{\pi^4} g^4(A) \left( \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} \left( A^{(s)}_{1,1} + 2\beta_0 \log(x^2 \Lambda^2) \right) d^4 x + A^{(s)}_{1,5} \right) + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) + \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} d^4 x \\
+ \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) A^{(s)}_{1,4}
\]

(3.15)

Hence, we evaluate Eq. 3.8 to the order of $g^4$:

\[
\frac{N^2 - 1}{\pi^4} g^4(A) \left( 4A^{(s)}_{0,2} + 4A^{(s)}_{0,3} \log(z^2 \Lambda^2) + 12\beta_0^2 \log^2(z^2 \Lambda^2) \right)
\]

\[
\sim \frac{N^2 - 1}{\pi^4} g^4(A) \left( \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} \left( A^{(s)}_{1,1} + 2\beta_0 \log(x^2 \Lambda^2) \right) d^4 x + A^{(s)}_{1,5} \right) + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) + \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) \int_{D_\Lambda} \frac{4\beta_0}{\pi^2 x^4} d^4 x \\
+ \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) A^{(s)}_{1,4}
\]

\[
= \frac{N^2 - 1}{\pi^4} g^4(A) \left( 4\beta_0 A^{(s)}_{1,1} \log(z^2 \Lambda^2) + 4\beta_0^2 \log^2(z^2 \Lambda^2) + A^{(s)}_{1,5} \right) + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) + \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) 4\beta_0 \log(z^2 \Lambda^2) \\
+ \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} + 2\beta_0 \log(z^2 \Lambda^2) \right) A^{(s)}_{1,4}
\]

\[
= \frac{N^2 - 1}{\pi^4} g^4(A) \left( 4\beta_0 A^{(s)}_{1,1} \log(z^2 \Lambda^2) + 4\beta_0^2 \log^2(z^2 \Lambda^2) + A^{(s)}_{1,5} \right) + 4\beta_1 \log(\frac{\Lambda^2}{\mu^2}) + \frac{N^2 - 1}{\pi^4} g^4(A) \left( A^{(s)}_{0,1} A^{(s)}_{1,4} + 8\beta_0^2 \log^2(z^2 \Lambda^2) \\
+ 2\beta_0 A^{(s)}_{1,4} \log(z^2 \Lambda^2) + 4\beta_0 A^{(s)}_{0,1} \log(z^2 \Lambda^2) \right)
\]
\[
N^2 - \frac{1}{\pi^4} g^4(\Lambda) \left( A_1^{(S)} + A_0^{(S)} A_1^{(S)} + 4\beta_0 (A_1^{(S)} + A_0^{(S)} + \frac{1}{2} A_1^{(S)}) \log(z^2 \Lambda^2) \\
+ 4\beta_1 \log\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{12\beta_0^2}{48} \log^2(z^2 \Lambda^2)\right)
\]

(3.16)

Therefore, the universal quadratic-log divergences in both sides of Eq. (3.8) agree.

4 Nonperturbative UV asymptotics of the low-energy theorem for \(\langle F^2(z)F^2(0)\rangle\)

We compute now the nonperturbative UV asymptotics of the low-energy theorem for \(\langle F^2(z)F^2(0)\rangle\) within the universal leading and next to leading logarithmic accuracy, by means of the nonperturbative UV asymptotics of the renormalized OPE coefficients in App. A, and of their perturbative normalization in App. B.

It is convenient to introduce the RG-invariant coefficients \(\frac{\beta(g)}{g} C_0^{(S)}(z)\) and \(-\frac{\beta(g)}{g} C_1^{(S)}(x)\) associated to the OPE of the RG-invariant operator \(\frac{\beta(g)}{g} F^2\). The change of normalization does not affect the universal asymptotics but for the overall normalization, yet it is specifically convenient for the perturbative computation in Apps. B.3 and B.6. Their universal UV asymptotics reads:

\[
\left(\frac{\beta(g)}{g}\right)^2 C_0^{(S)}(z) \sim \frac{N^2 - 148}{\pi^4} g^4(z) \frac{1}{\log^2\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right)} \left(1 - 2\beta_1 \frac{\log \log\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right)}{\beta_0 \log\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right)}\right)
\]

(4.1)

and:

\[
-\frac{\beta(g)}{g} C_1^{(S)}(x) \sim 4\beta_0 \frac{1}{\pi^2} g^4(x) \log^2\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right) \left(1 - 2\beta_1 \frac{\log \log\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right)}{\beta_0 \log\left(\frac{1}{x^2 \Lambda_{QCD}^2}\right)}\right)
\]

(4.2)

where the second asymptotic equalities follow from Eq. (A.13).

It is convenient to employ the version of the low-energy theorem that involves \(\Lambda_{QCD}\) and the canonical normalization of the action in Eq. (2.7). We skip the finite contact term in \(C_1^{(S)}\), the compensating term in the lhs of Eq. (2.7), and the divergent contact terms in \(C_1^{(S)}\) that, according to [17], should be renormalized to zero.

Then, for the renormalized correlators, it should hold nonperturbatively and asymptotically:

\[
\left(\frac{\beta(g)}{g}\right)^2 2\Lambda_{QCD}^2 \frac{\partial C_0^{(S)}(z)}{\partial \Lambda_{QCD}^2}
\]
\[ - \left( \frac{\beta(g)}{g} \right)^2 C_0^{(S)}(z) \int_{D_{\frac{1}{z}}^\Lambda} \frac{\beta(g)}{g} C_1^{(S)'}(x) d^4 x \] (4.3)

Firstly, we compute the lhs of Eq. (4.3):

\[
\left( \frac{\beta(g)}{g} \right)^2 C_0^{(S)}(z) \partial C_0^{(S)}(z) \\sim \frac{N^2 - 1.48}{\pi^4 z^8} \frac{4}{\log^4 \left( \frac{1}{z^2 \Lambda_{QCD}^2} \right)} \left( 1 - 3 \frac{\beta_1}{\beta_0^2} \log \log \left( \frac{1}{z^2 \Lambda_{QCD}^2} \right) \right) \]

\[ \sim \frac{N^2 - 1.48}{\pi^4 z^8} 4 \beta_0^3 g^6(z) \] (4.4)

In the rhs of Eq. (4.3) the crucial step is the integration of \(- \frac{\beta(g)}{g} C_1^{(S)}\). According to the prescription in Subsec. 3.1, we have restricted the integral to the domain \( D_{\frac{1}{z}} = \{ x^2 : \frac{1}{\Lambda^2} \leq x^2 \leq z^2 \} \).

But now, after the RG resummation, the integral is UV convergent because of the AF, and therefore we can remove the UV cutoff. Thus, we may extend the integration to the new domain \( D_{\frac{1}{z}}^\infty = \{ x^2 : 0 \leq x^2 \leq z^2 \} \).

Incidentally, the nonperturbative UV finiteness of the integral in Eq. (4.3), as opposed to the UV divergence of the integral in Eq. (3.11) in perturbation theory, plays a key role for the no-go theorem in [3]. We get:

\[ - \int_{D_{\frac{1}{z}}^\infty} \frac{\beta(g)}{g} C_1^{(S)'}(x) d^4 x \]

\[ \sim \int_{D_{\frac{1}{z}}^\infty} \frac{4}{\pi^2 x^4 \log^2 \left( \frac{1}{z^2 \Lambda_{QCD}^2} \right)} \left( 1 - 2 \frac{\beta_1}{\beta_0^2} \log \log \left( \frac{1}{z^2 \Lambda_{QCD}^2} \right) \right) d^4 x \]

\[ \sim 4 \beta_0^3 g^6(z) \] (4.5)

that substituted in the rhs of Eq. (4.3) implies:

\[ - \left( \frac{\beta(g)}{g} \right)^2 C_0^{(S)}(z) \int_{D_{\frac{1}{z}}^\infty} \frac{\beta(g)}{g} C_1^{(S)'}(x) d^4 x \]

\[ \sim \frac{N^2 - 1.48}{\pi^4 z^8} 4 \beta_0^4 g^4(z) 4 \beta_0^3 g^6(z) \]

\[ \sim \frac{N^2 - 1.48}{\pi^4 z^8} 4 \beta_0^3 g^6(z) \] (4.6)

that actually matches Eq. (4.4).
Just as an aside, the integral in Eq. (4.5) is computed in polar coordinates by means of the obvious change of variables and by integrating by parts:

$$\int_{D^N} \frac{1}{\pi^2 x^4} \log^2(\frac{1}{2z^2 \Lambda_{QCD}^2}) \left(1 - 2 \frac{\beta_1}{\beta_0} \log(\frac{1}{2z^2 \Lambda_{QCD}^2})\right) d^4x$$

$$= \int_0^{|x|} \frac{2}{\log^2(\frac{1}{|x|^2 \Lambda_{QCD}^2})} \left(1 - 2 \frac{\beta_1}{\beta_0} \log(\frac{1}{|x|^2 \Lambda_{QCD}^2})\right) \frac{d|x|}{|x|}$$

$$= \frac{1}{\log(\frac{1}{2z^2 \Lambda_{QCD}^2})} \left(1 - \frac{\beta_1}{\beta_0} \log(\frac{1}{2z^2 \Lambda_{QCD}^2})\right)$$

with $|x| = \sqrt{x^2}$.

5 Lower-order perturbative OPE and nonperturbative UV asymptotics from the low-energy theorem for $\langle O(z)O(0) \rangle$

By exploiting the OPE in Eqs. (A.1) and (A.2), the low-energy theorem for any 2-point correlator, $\langle O(z)O(0) \rangle$, of a canonically normalized operator, $O$, reads:

$$2cC_0^{(O)}(z) + 2g^2 \frac{\partial C_0^{(O)}(z)}{\partial g^2} \sim C_0^{(O)}(z) \int C_0^{(O,F^2)}(x) d^4x$$

(5.1)

with $c$ the exponent, defined in Sec. 2.3, of the canonical rescaling of the operator $O$. For a multiplicatively renormalizable operator of spin $s$, the unrenormalized $C_0^{(O)}$, up to a finite term on the order of $g^2$, reads to one loop for $z \neq 0$:

$$C_0^{(O)}(z) = A \frac{P^{(s)}(z)}{z^{2D}} \left(1 + g^2(\Lambda) \gamma_0^{(O)} \log(z^2 \Lambda^2)\right)$$

(5.2)

with $P^{(s)}(z)$ the spin projector in the coordinate representation in the conformal limit, $A$ a constant normalization factor, $D$ the canonical dimension, and $\gamma_0^{(O)}$ the first coefficient of the anomalous dimension.

The first term in the lhs of Eq. (5.1) implies that a finite contact term should occur for $c \neq 0$ to the order of $g^0$:

$$C_0^{(O,F^2)}(x) = 2c \delta^4(x) + C_0^{(O,F^2)'}(x)$$

(5.3)

By skipping the contact term in the rhs and the compensating term in the lhs of Eq. (5.1), to the order of $g^2$ the low-energy theorem implies:

$$\frac{2A \gamma_0^{(O)} P^{(s)}(z)}{z^{2D}} g^2(\Lambda) \log(z^2 \Lambda^2) = A \frac{P^{(s)}(z)}{z^{2D}} \int_{D^4} C_0^{(O,F^2)'}(x) d^4x$$

(5.4)
This fixes $C_{O}^{(O,F^{2})}$ in massless QCD to the lower orders in terms of the first coefficient of the anomalous dimension, $\gamma^{(O)}_{0}$, and of the exponent, $c$, of the canonical rescaling:

$$C_{O}^{(O,F^{2})}(x) = g^{2}(\Lambda) \frac{2\gamma^{(O)}_{0}}{x^{4}\pi^{2}} + 2c\delta^{4}(x) \quad (5.5)$$

Indeed, the $x$ dependence in Eq. (5.5) is just dimensional analysis. Thus, $C_{O}^{(O,F^{2})}$ necessarily starts to the order of $g^{2}$ in perturbation theory because of the derivative with respect to $g$ in the lhs of Eq. (5.1). It follows from Eq. (A.20):

$$C_{O}^{(O,F^{2})}(x) \sim \frac{2\gamma^{(O)}_{0}}{x^{4}\pi^{2}} g^{2}(x) \left(\frac{g(x)}{g(\mu)}\right)^{2} \quad (5.6)$$

Then, the nonperturbative UV asymptotics of the low-energy theorem reads:

$$2\Lambda_{QCD}^{2} \frac{\partial C_{0}^{(O)}}{\partial\Lambda_{QCD}^{2}} \sim \frac{\beta(g)}{g} C_{0}^{(O)}(z) \int_{D_{\Lambda}^{\infty}} C_{O}^{(O,F^{2})}(x) d^{4}x \quad (5.7)$$

with:

$$C_{0}^{(O)}(z) \sim \frac{P^{(s)}(z)}{z^{2D}} \left(\frac{g(z)}{g(\mu)}\right)^{\gamma^{(O)}_{0}/\beta_{0}} \quad (5.8)$$

The lhs of Eq. (5.7) reads:

$$2\Lambda_{QCD}^{2} \frac{\partial C_{0}^{(O)}}{\partial\Lambda_{QCD}^{2}} \sim \frac{2\gamma^{(O)}_{0}}{z^{2D}} \frac{A P^{(s)}(z)}{(g^{2}(\mu))^{\gamma^{(O)}_{0}/\beta_{0}}^{1+2\gamma^{(O)}_{0}/\beta_{0}} \log^{1+2\gamma^{(O)}_{0}/\beta_{0}}(\frac{1}{z^{2}\Lambda_{QCD}^{2}})} \left(1 - (1 + \frac{\gamma^{(O)}_{0}}{\beta_{0}}) \frac{\beta_{0}}{\beta_{0}^{2}} \log(\frac{1}{z^{2}\Lambda_{QCD}^{2}})\right)$$

$$\sim A P^{(s)}(z) \frac{2\gamma^{(O)}_{0}}{z^{2D}} \left(\frac{g(z)}{g(\mu)}\right)^{\gamma^{(O)}_{0}/\beta_{0}} g^{2}(z) \quad (5.9)$$

The integral in the rhs of Eq. (5.7) is UV convergent exactly as in Sec. 4, and the integration domain can be extended to $D_{\Lambda}^{\infty}$:

$$-\frac{\beta(g)}{g} \int_{D_{\Lambda}^{\infty}} C_{O}^{(O,F^{2})}(x) d^{4}x \quad (5.10)$$
Therefore, the rhs of Eq. (5.7) reads:

\[-\frac{\beta(g)}{g} C_0^{(O)}(z) \int_{D^\infty} C_0^{(O,F^2)}(x) d^4 x \sim \frac{A P(s)(z) g(z)}{g(\mu)} \frac{\gamma_0^{(O)}}{\pi} 2 \gamma_0^{(O)} g^2(z) \]  

(5.12)

that actually matches Eq. (5.9).

A  Nonperturbative UV asymptotics of the OPE

According to the RG, the nonperturbative UV asymptotics of the renormalized OPE coefficients, \(C_0, C_1\) and \(C_O^{(O,O_1)}\), for multiplicatively renormalizable operators, \(O\) and \(O_1\):

\[O(x) O(0) \sim C_0^{(O)}(x) 1 + C_1^{(O)}(x) O_1(0) + \cdots\]  

(A.1)

\[O(0) O_1(x) \sim C_O^{(O,O_1)}(x) O(0) + \cdots\]  

(A.2)

follows from the associated Callan-Symanzik equations in the coordinate representation, which guarantees the absence of contact terms for \(x \neq 0\) and, consequently, of additive renormalizations [14, 15, 18]:

\[\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2D + 2 \gamma_0(g) \right) C_0^{(O)}(x) = 0 \]  

(A.3)

\[\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2D - D_1 + 2 \gamma_0(g) - \gamma_{0_1}(g) \right) C_1^{(O)}(x) = 0 \]  

(A.4)

\[\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + D_1 + \gamma_{0_1}(g) \right) C_O^{(O,O_1)}(x) = 0 \]  

(A.5)

with \(D, \gamma_0(g)\) and \(D_1, \gamma_{0_1}(g)\) the canonical and anomalous dimension of the operators \(O\) and \(O_1\) respectively.

The general solutions [14, 15, 18] are:

\[C_0^{(O)}(x) = \frac{1}{x^{2D}} g_0^{(O)}(g(x)) Z^{(O)2}(x, g(\mu))\]  

(A.6)

\[C_1^{(O)}(x) = \frac{1}{x^{2D-D_1}} g_1^{(O)}(g(x)) Z^{(O)2}(x, g(\mu)) Z^{(O_1)-1}(x, g(\mu))\]  

(A.7)

\[C_O^{(O,O_1)}(x) = \frac{1}{x^{D_1}} g_O^{(O,O_1)}(g(x)) Z^{(O_1)}(x, g(\mu))\]  

(A.8)
which are expressed in terms of the RG-invariant functions, $G^{(O)}_0$, $G^{(O)}_1$ and $G^{(O,O)}_O$, of $g(x)$ only, and of the renormalized multiplicative factors, $Z^{(O)}$:

$$Z^{(O)}(x \mu, g(\mu)) = \exp \int_{g(\mu)}^{g(x)} \frac{\gamma^{(O)}(g)}{\beta(g)} dg$$  \hspace{1cm} (A.9)

determined by the anomalous dimension, $\gamma^{(O)}(g)$:

$$\gamma^{(O)}(g) = -\frac{\partial \log Z^{(O)}}{\partial \log \mu} = -\gamma^{(O)}_0 g^2 - \gamma^{(O)}_1 g^4 + \cdots$$  \hspace{1cm} (A.10)

and the beta function, $\beta(g)$:

$$\beta(g) = \frac{\partial g}{\partial \log \mu} = -\beta_0 g^3 - \beta_1 g^5 + \cdots$$  \hspace{1cm} (A.11)

that can be computed in perturbation theory.

The asymptotic expansion of $Z^{(O)}$ for $x \to 0$ follows from Eq. (A.9):

$$Z^{(O)}(x \mu, g(\mu)) \sim \left(\frac{g(x)}{g(\mu)}\right)^{\frac{(O)}{\beta_0}} \exp \left(\frac{\gamma^{(O)}_0 \beta_0 - \gamma^{(O)}_1 \beta_1}{2 \beta_0^2} (g^2(x) - g^2(\mu)) + \cdots\right)$$  \hspace{1cm} (A.12)

where the dots represent a series in powers of $g^2(x)$ and $g^2(\mu)$. By an abuse of notation we have set $g(x) \equiv g(x \Lambda_{QCD})$ and $g(\mu) \equiv g(\mu^{-1} \Lambda_{QCD})$, where within the universal – i.e., renormalization-scheme independent – leading and next to leading asymptotic accuracy:

$$g^2(x \Lambda_{QCD}) \sim \frac{1}{\beta_0 \log(\frac{1}{x^2 \Lambda_{QCD}})} \left(1 - \frac{\beta_1}{\beta_0^2} \log \log(\frac{1}{x^2 \Lambda_{QCD}})\right)$$  \hspace{1cm} (A.13)

Indeed, in Eq. (A.13) we may substitute to $\Lambda_{QCD}$ any finite scale, $\mu$, without changing the universal asymptotics. Thus:

$$Z^{(O)}(x \mu, g(\mu)) \sim \left(\frac{g(x)}{g(\mu)}\right)^{\frac{(O)}{\beta_0}} Z^{(O)}(g(\mu))$$  \hspace{1cm} (A.14)

where the constant factor, $Z^{(O)}(g(\mu))$, is the limit of the exponential in Eq. (A.12) as $g(x) \to 0$. For brevity, in writing the universal UV asymptotics of $Z^{(O)}$, we skip systematically the factor of $Z^{(O)}(g(\mu))$, which is on the order of $1 + O(g^2(\mu))$.

Hence, the universal UV asymptotics of the OPE coefficients [14, 15, 18] is:

$$C^{(O)}_0(x) \sim \frac{1}{z^{2D}} \left(\frac{g(x)}{g(\mu)}\right)^{\frac{2 \gamma^{(O)}_0}{\beta_0}}$$  \hspace{1cm} (A.15)
$C_1^{(O)}(x) \sim \frac{1}{x^{2D-D_1}} g^{2l}(x) \left( \frac{g(x)}{g(\mu)} \right)^{2^{(O)} - \gamma^{(O_1)}_{0}}$  \hspace{1cm} (A.16)

$C_0^{(O,O_1)}(x) \sim \frac{1}{x^{D_1}} g^{2k}(x) \left( \frac{g(x)}{g(\mu)} \right)^{\gamma^{(O_1)}_{0}}$  \hspace{1cm} (A.17)

for some integer $l$ and $k$, up to constant normalization factors that can be fixed by perturbation theory.

The RG-invariant factors, $g^{2l}(x)$ and $g^{2k}(x)$, which arise from the asymptotics of $G_i^{(O)}(g(x))$ and $G_i^{(O,O_1)}(g(x))$ respectively, account for the possibility that the 3-point correlator, $\langle O(x)O(0)O_1 \rangle$, vanishes to some perturbative order.

Indeed, we have shown in Sec. 5 that $\langle O(z)O(0)F^2(x) \rangle$ necessarily starts to the order of $g^2$ in perturbation theory up to contact terms as a consequence of the low-energy theorem, and thus actually vanishes to the lowest order up to contact terms.

Instead, the 2-point correlator, $C_0^{(O)}(x)$, of a nontrivial Hermitian operator, $O$, is necessarily nonvanishing [14, 15, 18] in the conformal limit of a unitary massless QCD-like theory.

It follows from from Eqs. (3.9):

$C_0^{(S)}(x) \sim \frac{N_c^2 - 1}{\pi^8 x^8} \left( \frac{g(x)}{g(\mu)} \right)^4$  \hspace{1cm} (A.18)

from Eq. (3.10):

$C_1^{(S)}(x) \sim \frac{4\beta_0}{\pi^2 x^4} g^2(x) \left( \frac{g(x)}{g(\mu)} \right)^2$  \hspace{1cm} (A.19)

and from Eq. (5.5):

$C_0^{(O,F^2)}(x) \sim \frac{2^{(O)}_{0}}{\pi^2 x^4} g^2(x) \left( \frac{g(x)}{g(\mu)} \right)^2$  \hspace{1cm} (A.20)

B Perturbative OPE

B.1 Perturbative $C_0^{(S)}$ in the momentum representation

For the reader convenience, we report the relative normalization of the OPE coefficients computed in [11, 12, 17] for the multiplicatively renormalized operator, $F^2$, in the $\overline{MS}$ scheme:

$F^2(z)F^2(0) \sim 16C_{0CZ}^{(S)}(z)I - 4C_{1CZ}^{(S)}(z)F^2(0)$  \hspace{1cm} (B.1)

with respect to our conventions:

$F^2(z)F^2(0) \sim C_0^{(S)}(z)I + C_1^{(S)}(z)F^2(0)$  \hspace{1cm} (B.2)
Thus:

\[
C_0^{(S)} = 16 C_{0_{CZ}}^{(S)} \tag{B.3}
\]
\[
C_1^{(S)} = -4 C_{1_{CZ}}^{(S)} \tag{B.4}
\]

\(C_{0_{CZ}}^{(S)}\) is given in [17] to three loops in the momentum representation:

\[
C_{0_{CZ}}^{(S)} = \frac{N^2 - 1}{16 \pi^2} p^4 \left\{ -\frac{\log \left( \frac{F}{p^2} \right)}{4} + \frac{1}{4} + a_s \left( \frac{11}{48} C_A \log^2 \left( \frac{p^2}{\mu^2} \right) - \frac{73C_A \log \left( \frac{F}{p^2} \right)}{48} - \frac{3C_A \zeta_3}{4} + \frac{485 C_A}{192} \right) \\
- \frac{1}{12} \log^2 \left( \frac{p^2}{\mu^2} \right) N_f T_F + \frac{7}{12} \log \left( \frac{p^2}{\mu^2} \right) N_f T_F - \frac{17N_f T_F}{16} \right) \\
+ a_s^2 \left( -\frac{121}{576} C_A^2 \log^3 \left( \frac{p^2}{\mu^2} \right) + \frac{313}{128} C_A^2 \log^2 \left( \frac{p^2}{\mu^2} \right) + \frac{55}{32} C_A^2 \log \left( \frac{p^2}{\mu^2} \right) \zeta_3 \\
- \frac{37631 C_A^2 \log \left( \frac{p^2}{\mu^2} \right)}{3456} - \frac{2059}{288} C_A^2 \zeta_3 + \frac{11}{64} C_A^2 \zeta_4 + \frac{25}{16} \frac{C_A^2 \zeta_5 + \frac{707201 C_A^2}{41472}}{16} \right) \\
+ \frac{11}{72} C_A \log \left( \frac{p^2}{\mu^2} \right) N_f T_F - \frac{85}{48} C_A \log \left( \frac{p^2}{\mu^2} \right) N_f T_F + \frac{1}{8} C_A \log \left( \frac{p^2}{\mu^2} \right) N_f T_F \zeta_3 \\
+ \frac{6665}{864} C_A \log \left( \frac{p^2}{\mu^2} \right) N_f T_F + \frac{169}{144} C_A N_f T_F \zeta_3 - \frac{7}{16} C_A N_f T_F \zeta_4 \\
- \frac{7847}{648} C_A N_f T_F - \frac{1}{8} C_f \log \left( \frac{p^2}{\mu^2} \right) N_f T_F - \frac{3}{4} C_f \log \left( \frac{p^2}{\mu^2} \right) N_f T_F \zeta_3 \\
+ \frac{131}{96} C_f \log \left( \frac{p^2}{\mu^2} \right) N_f T_F + \frac{41}{24} C_f N_f T_F \zeta_3 + \frac{3}{8} C_f N_f T_F \zeta_4 \\
- \frac{5281 C_f N_f T_F}{1728} - \frac{1}{36} \log \left( \frac{p^2}{\mu^2} \right) N_f^2 T_F^2 + \frac{7}{24} \log \left( \frac{p^2}{\mu^2} \right) \zeta_3 \\
- \frac{127}{108} \log \left( \frac{p^2}{\mu^2} \right) N_f^2 T_F^2 + \frac{4715 N_f^2 T_F^2}{2592} \right) \right\} \tag{B.5}
\]

with \(C_A = N\), \(C_F = \frac{N^2 - 1}{2N}\), \(T_F = \frac{1}{2}\) and \(a_s = \frac{g\sqrt{\alpha(M)}}{4\pi^2}\).

Hence, multiplying Eq. (B.5) by 16, and expressing \(a_s\) in terms of the 't Hooft coupling, \(g\), we get \(C_0^{(S)}\) in the momentum representation:

\[
C_0^{(S)}(p) = \frac{N^2 - 1}{4\pi^2} p^4 \left( 1 - \log \left( \frac{p^2}{\mu^2} \right) + g^2(\mu) \left( B_{0,1}^{(S)} - B_{0,2}^{(S)} \log \left( \frac{p^2}{\mu^2} \right) + \beta_0 \log^2 \left( \frac{p^2}{\mu^2} \right) \right) \\
+ g^4(\mu) \left( B_{0,3}^{(S)} - B_{0,4}^{(S)} \log \left( \frac{p^2}{\mu^2} \right) + B_{0,5}^{(S)} \log^2 \left( \frac{p^2}{\mu^2} \right) + \beta_0^2 \log^3 \left( \frac{p^2}{\mu^2} \right) \right) \right) \\
+ \frac{N^2 - 1}{4\pi^2} p^4 \left( \log \left( \frac{\Lambda^2}{\mu^2} \right) + g^2(\mu) \left( B_{0,6}^{(S)} \log \left( \frac{\Lambda^2}{\mu^2} \right) - \beta_0 \log^2 \left( \frac{\Lambda^2}{\mu^2} \right) \right) \right)
\]
+\phi'(\mu) \left( B_{0,7}^{(S)} \log(\frac{A^2}{\mu^2}) + B_{0,8}^{(S)} \log^2(\frac{A^2}{\mu^2}) + \beta_0^2 \log^3(\frac{A^2}{\mu^2}) \right) \right) \quad (B.6)

with:

\begin{align*}
B_{0,1}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{485}{12} - 12\zeta_3 - \frac{17N_f}{2N} \right) \\
B_{0,2}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{73}{3} - 14N_f - \frac{3}{3N} \right) \\
B_{0,3}^{(S)} &= \frac{1}{(4\pi)^4} \left( 11\zeta_4 + 100\zeta_5 - \frac{4118\zeta_3}{9} + \frac{707201}{648} + \frac{584N_f\zeta_3}{9N} - \frac{141395N_f}{324N} + \frac{4715N_f^2}{162N^2} - \frac{6N_f\zeta_4}{N^3} - \frac{82N_f\zeta_3}{3N^3} + \frac{5281N_f}{108N^3} \right) \\
B_{0,4}^{(S)} &= \frac{1}{(4\pi)^4} \left( -110\zeta_3 + \frac{37631}{54} - 4\zeta_3 \frac{N_f}{N} - \frac{6665N_f}{27} + 24\zeta_3 \frac{N^2}{2N^2} - \frac{1}{2N^2} \frac{1}{N} \right) \\
B_{0,5}^{(S)} &= \frac{1}{(4\pi)^4} \left( \frac{313}{2} - \frac{170N_f}{3} - \frac{4}{2N^2} \frac{N^2 - 1}{N} + \frac{14N_f^2}{3N^2} \right) \\
B_{0,6}^{(S)} &= \frac{1}{(4\pi)^4} \left( \frac{17}{2} - 5N_f \right) \\
B_{0,7}^{(S)} &= \frac{1}{(4\pi)^4} \left( \frac{22\zeta_3}{3} + \frac{22351}{324} - \frac{28\zeta_3 N_f}{3N} - \frac{1598N_f}{81N} + 8\zeta_3 \frac{N^2 - 1}{2N^2} - \frac{107N^2 - 1}{2N^2} \frac{N_f}{N} + \frac{49N_f^2}{81N^2} \right) \\
B_{0,8}^{(S)} &= \frac{1}{(4\pi)^4} \left( -\frac{833}{18} + \frac{146N_f}{9N} + \frac{8N^2 - 1}{3} \frac{N_f}{2N^2} - \frac{10N_f^2}{9N^2} \right)
\end{align*}

In Eq. (B.6) we have also included the contact terms computed in [17].

**B.2 Perturbative \( C_0^{(S)} \) in the coordinate representation**

To get \( C_0^{(S)} \) in the coordinate representation, we Fourier transform employing the relations [14]:

\[ \int p^4 \log(\frac{p^2}{\mu^2}) e^{ipx} \frac{d^4p}{(2\pi)^4} = -\frac{2^6\gamma_3}{\pi^2x^6} \quad (B.15) \]

\[ \int p^4 \log^2(\frac{p^2}{\mu^2}) e^{ipx} \frac{d^4p}{(2\pi)^4} = \frac{2^7\gamma_3}{\pi^2x^6} \left( \log(\mu^2x^2) - \frac{10}{3} + 2\gamma_E - \log 4 \right) \quad (B.16) \]

\[ \int p^4 \log^3(\frac{p^2}{\mu^2}) e^{ipx} \frac{d^4p}{(2\pi)^4} = \frac{2^6\gamma_3}{\pi^2x^6} (-3(\log 4 - \log(\mu^2x^2))^2 \]
\begin{align*}
 \delta^\beta - \Delta_{\theta}^\beta + \delta^\beta + \theta_{\theta}^\beta \log(\mu^2 x^2) - 12 \gamma_E^2 \left( \frac{51}{2} + 40 \gamma_E - (20 - 12 \gamma_E) \log 4 \right) & \quad \text{(B.17)}
\end{align*}

It follows \( C_0^{(S)} \) in the coordinate representation:

\begin{align*}
 C_0^{(S)}(x) &= \frac{N^2 - 1}{\pi^2} \left( 1 + g^2(\mu)(A_{0,1} + 2 \beta_0 \log(x^2 \mu^2)) \right. \\
 & \quad + g^4(\mu)(A_{0,2} + A_{0,3} \log(x^2 \mu^2) + 3 \beta_0^2 \log^2(x^2 \mu^2)) \\
 & \quad + \Delta \delta^{(4)}(x) \frac{N^2 - 1}{4 \pi^2} \left( 1 + \log(\frac{\Lambda^2}{\mu^2}) + g^2(\mu) \left( A_{0,4} + A_{0,5} \log(\frac{\Lambda^2}{\mu^2}) \right) \\
 & \quad - \beta_0 \log^2(\frac{\Lambda^2}{\mu^2}) \right) + g^4(\mu) \left( A_{0,6} + A_{0,7} \log(\frac{\Lambda^2}{\mu^2}) + A_{0,8} \log^2(\frac{\Lambda^2}{\mu^2}) \\
 & \quad + \beta_0^2 \log^3(\frac{\Lambda^2}{\mu^2}) \right) \right) & \quad \text{(B.18)}
\end{align*}

with \( A_{0,1}, A_{0,2}, A_{0,3} \) scheme-dependent:

\begin{align*}
 A_{0,1}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{132 \gamma_E - 1 - 132 \log 2}{9} - \frac{2 N_f (12 \gamma_E + 1 - 12 \log 2)}{9 N} \right) & \quad \text{(B.19)} \\
 A_{0,2}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{-98 + 4356 \log^2 2 - 8712 \gamma_E \log 2 - 2382 \log 2}{27} \\
 & \quad + \frac{2970 \zeta_3 + 4356 \gamma_E^2 + 2382 \gamma_E}{N_f (-2112 \log 2 - 6366 \gamma_E \log 2)} - \frac{432 \zeta_3 + 3168 \gamma_E^2 + 2112 \gamma_E + 121 + 3168 \log^2 2}{54 N} \\
 & \quad + \frac{2 N^2 (72 \gamma_E^2 + 12 \gamma_E - 13 + 72 \log^2 2 - 144 \gamma_E \log 2 - 12 \log 2)}{27 N^2} \\
 & \quad + \frac{N_f (-24 \zeta_3 + 16 \gamma_E + 17 - 16 \log 2)}{2 N^3} \right) & \quad \text{(B.20)} \\
 A_{0,3}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{1452 \gamma_E + 397 - 1452 \log 2}{9} - \frac{88 N_f (6 \gamma_E + 2 - 6 \log 2)}{9 N} \\
 & \quad + \frac{4 N_f^2 (12 \gamma_E + 1 - 12 \log 2)}{9 N^2} + \frac{4 N_f}{N^3} \right) & \quad \text{(B.21)}
\end{align*}

and \( A_{0,4}, A_{0,5}, A_{0,6}, A_{0,7}, A_{0,8} \) coefficients of the contact terms:

\begin{align*}
 A_{0,4}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{485}{12} - 12 \zeta_3 - \frac{17 N_f}{2 N} \right) & \quad \text{(B.22)} \\
 A_{0,3}^{(S)} &= \frac{1}{(4\pi)^2} \left( \frac{17}{2} - \frac{5 N_f}{3 N} \right) & \quad \text{(B.23)}
\end{align*}
\[ A_{0,6}^{(S)} = \frac{1}{(4\pi)^4} \left( 11\zeta_4 + 100\zeta_5 - \frac{4118\zeta_3}{9} + \frac{707201}{648} + \frac{584N_f\zeta_3}{9N} \right) \]
\[ - \frac{141395N_f}{324N} + \frac{4715N_f^2}{162N^2} - \frac{6N_f\zeta_4}{N^3} - \frac{82N_f\zeta_3}{3N^3} + \frac{5281N_f}{108N^3} \]  \hspace{1cm} (B.24)
\[ A_{0,7}^{(S)} = \frac{1}{(4\pi)^4} \left( \frac{22\zeta_3}{3} + \frac{22351}{324} - \frac{28\zeta_3 N_f}{3N} - \frac{1598N_f}{81N} + \frac{8\zeta_3 N^2 - 1}{2N^2} \right) \]
\[ - \frac{107 N^2 - 1 N_f}{9} + \frac{49 N_f^2}{2N^2} \] \hspace{1cm} (B.25)
\[ A_{0,8}^{(S)} = \frac{1}{(4\pi)^4} \left( - \frac{833}{18} + \frac{146 N_f}{9N} + \frac{8 N^2 - 1 N_f}{3} - \frac{10 N_f^2}{9N^2} \right) \] \hspace{1cm} (B.26)

### B.3 Verifying the UV asymptotics of \( C_0^{(S)} \) by a change of renormalization scheme in perturbation theory

We skip the contact terms, and we multiply Eq. (B.18) by \( g^4 \) that is equivalent to consider the OPE of \( g^2 F^2 \). Then, following [14] we change the renormalization scheme by redefining the coupling constant:

\[ g_{ab}^2(\mu) = g^2(\mu)(1 + a g^2(\mu) + b g^4(\mu)) \]  \hspace{1cm} (B.27)

with \( a \) and \( b \) such that the constant finite parts \(^3\) of \( C_0^{(S)} \) vanish to two loops. Eq. (B.18) becomes:

\[ g_{ab}^4(\mu) C_0^{(S)}(x) = \frac{N^2 - 1}{\pi^4 x^8} g_{ab}^4(\mu) \left( 1 + g_{ab}^2(\mu) (A_{0,1}^{(S)} - 2a + 2b \log(x^2 \mu^2)) \right) \]
\[ + g_{ab}^4(\mu) (A_{0,2}^{(S)} + 5a^2 - 2b - 3a A_{0,1}^{(S)}) \]
\[ + (A_{0,3}^{(S)} + \frac{a(2N_f - 11N)}{8\pi^2N}) \log(x^2 \mu^2) + 3b^2 \log^2(x^2 \mu^2) \] \hspace{1cm} (B.28)

with \( a \) and \( b \):

\[ a = \frac{A_{0,1}^{(S)}}{2} \] \hspace{1cm} (B.29)
\[ b = \frac{4A_{0,2}^{(S)} - A_{0,1}^{(S)} \delta^2}{8} \] \hspace{1cm} (B.30)

Remarkably, the coefficient of the log term to the order \( g_{ab}^4 \) is [14] now:

\[ A_{0,3}^{(S)} + \frac{a(2N_f - 11N)}{8\pi^2N} \]
\[ = A_{0,3}^{(S)} + \frac{A_{0,1}^{(S)} (2N_f - 11N)}{16\pi^2N} \]
\[ = \frac{1}{(4\pi)^4} \left( \frac{1452 \gamma_E + 397 - 1452 \log 2}{9} - \frac{88N_f(6\gamma_E + 2 - 6 \log 2)}{9N} \right) \]

\(^3\)We define the divergent parts as the terms that, after setting \( \mu = \Lambda \), become divergent as \( \Lambda \to \infty \). The constant finite parts are the remaining constant terms.
In order to verify Eq. (3.4) this cannot be done immediately, because the coefficient of the term \( g^{12}\gamma_{E} + 1 - 12 \log 2 \) in Eq. (A.18) reads to two loops 

\[
N^{3} - 13 N^{2} \frac{1}{N} + 3 \frac{N_{f}}{2} = 4 \beta_{1} \tag{B.31}
\]

Thus, \( \gamma_{1}^{(F^{2})} = 4 \beta_{1} \) in this scheme, according to Eq. (3.4). It follows:

\[
g^{ab}(\mu)C_{0}^{(S)}(x) = \frac{N^{2} - 48 g^{4}\gamma_{E}^{2}}{1 + 4 \beta_{1} g^{2}\mu\log(x^{2} \mu^{2})} (1 + 4 \beta_{1} \mu\log(x^{2} \mu^{2}))
\]

In order to verify Eq. (A.18) we should express Eq. (B.32) in terms of \( g(x) \), which reads to two loops [14]:

\[
g^{2}(x) = g^{2}(\mu) \left( 1 + g^{2}(\mu) \beta_{0} \log(x^{2} \mu^{2}) + g^{4}(\mu) \beta_{1} \log(x^{2} \mu^{2}) \right)
\]

Hence:

\[
g^{4}(x) = g^{4}(\mu) \left( 1 + 2 g^{4}(\mu) \beta_{0} \log(x^{2} \mu^{2}) + g^{4}(\mu) \beta_{1} \log(x^{2} \mu^{2}) \right)
\]

This cannot be done immediately, because the coefficient of the term \( g^{4}(\mu) \log(x^{2} \mu^{2}) \) in Eq. (B.32) is \( 4 \beta_{1} \) instead of \( 2 \beta_{1} \) in Eq. (B.34). Following [14] we insert the identity:

\[
1 = \frac{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x)}{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu)} \tag{B.35}
\]

in Eq. (B.32) to the relevant order:

\[
g^{ab}(\mu)C_{0}^{(S)}(x) = \frac{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x)}{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x)} \frac{N^{2} - 148 g^{4}(\mu)}{x^{8}}
\]

\[
\sim \frac{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu)}{1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu) + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(x)} \frac{N^{2} - 148 g^{4}(\mu)}{x^{8}}
\]

where in the first line we have expanded to the order of \( g^{4} \):

\[
1 + 2 \frac{\beta_{1}}{\beta_{0}} g^{2}(\mu) = 1 - 2 \beta_{1} g^{4}\mu \log(x^{2} \mu^{2}) \tag{B.37}
\]
and in the third line we have expanded to the order of $g^2$:

$$
1 + \frac{2\beta_0 g^2_{ab}(x)}{1 + 2\frac{\beta_0}{\beta_0} g^2_{ab}(\mu)} = 1 + 2\frac{\beta}{\beta_0} g^2_{ab}(x) - 2\frac{\beta_1}{\beta_0} g^2_{ab}(\mu)
$$

(B.38)

The factor of \big(1 + \frac{2\beta_0 g^2_{ab}(x) - 2\frac{\beta_0}{\beta_0} g^2_{ab}(\mu)}{\big)} in the last line of Eq. (B.36) arises according to Eq. (A.12). Therefore, Eq. (B.36) agrees with Eq. (A.18).

We may remove the scale-dependent term in Eq. (B.36), $-\frac{2\beta_0}{\beta_0} g^2_{ab}(\mu)$, multiplying Eq. (B.36) by $\beta_0^2 (1 + \beta_0^2 g^2_{ab}(\mu))^2$ that makes $g^2_{ab}(\mu)C_0^{(S)}$ RG invariant [14]. Hence, we get the relevant perturbative order:

$$
\left(\frac{\beta(g)}{g}\right)^2 C_0^{(S)}(x) = \frac{N^2 - 148\beta_0^2}{\pi^4} x^8 g^4_{ab}(x) \left(1 + 2\frac{\beta_1}{\beta_0} g^2_{ab}(x)\right)
$$

(B.39)

according to the nonperturbative asymptotics in Eq. (4.1).

**B.4 Perturbative $C_1^{(S)}$ in the momentum representation**

$C_1^{(S)}_{1CZ}$ has been given in [11, 12] to three loops in the momentum representation in the $\overline{MS}$ scheme:

$$
C_1^{(S)}_{1CZ} = -1 + a_s \left\{ \frac{49C_A}{36} + \frac{5N_f T_F}{9} - \log\left(\frac{p^2}{\mu^2}\right) \left(\frac{N_f T_F}{3} - \frac{11C_A}{12}\right) \right\}
$$

\[ + a_s^2 \left\{ \frac{33C_A^2 C_3}{8} - \frac{11509 C_A^2}{1296} + \frac{3}{2} C_A N_f T_F \zeta_3 + \frac{3095 C_A N_f T_F}{648} - 3 C_F N_f T_F \zeta_3 \right. \]

\[ + \frac{13 C_F N_f T_F}{4} + \frac{25 N_f^2 T_F^2}{81} - \log\left(\frac{p^2}{\mu^2}\right) \left(-\frac{1151 C_A^2}{216} + \frac{97 C_A N_f T_F}{27} + C_F N_f T_F \right) \]

\[ - \frac{10 N_f^2 T_F^2}{27} \log\left(\frac{p^2}{\mu^2}\right) \left(-\frac{121 C_A^2}{144} + \frac{11 C_A N_f T_F}{18} - \frac{N_f^2 T_F^2}{9}\right) \]

\[ + \log\left(\frac{N_f^2 T_F^2}{\mu^2}\right) \left(-\frac{17 C_A^2}{24} + \frac{5 C_A N_f T_F}{12} + \frac{C_F N_f T_F}{4}\right) \] 

\[ + a_s^3 \left\{ \frac{53155 C_A^3 C_3}{144} - \frac{55 C_A^4 C_5}{8} - \frac{977563 C_A^3}{186624} - \frac{263 C_A^2 N_f T_F \zeta_3}{144} \right. \]

\[ - 5 C_A^2 N_f T_F \zeta_3 + \frac{31104}{1299295} C_A^2 N_f T_F \zeta_3 + \frac{3341}{16} C_A C_F N_f T_F \zeta_3 - \frac{15}{2} C_A C_F N_f T_F \zeta_3 \]

\[ + \frac{35707 C_A C_F N_f T_F}{15552} - \frac{121}{36} C_A N_f^2 T_F^2 \zeta_3 - \frac{116773 C_A N_f^2 T_F^2}{15552} - 9 C_F N_f T_F \zeta_3 \]

\[ + 15 C_F N_f T_F \zeta_5 - \frac{45}{16} C_F N_f T_F + 13 \frac{C_F N_f^2 T_F^2 \zeta_3}{288} \]

\[ - \log\left(\frac{p^2}{\mu^2}\right) \left(\frac{363 C_A^3 C_3}{32} - \frac{360325 C_A^3}{10368} + \frac{55757 C_A^2 N_f T_F}{1728} - \frac{33}{4} C_A C_F N_f T_F \zeta_3 \right. \]

\[ + \frac{2527}{192} C_A C_F N_f T_F - \frac{3}{2} C_A N_f^2 T_F^2 \zeta_3 \]

\[ - \frac{2057}{288} C_A N_f^2 T_F^2 - \frac{9}{32} C_f N_f T_F \]

\[ + \frac{2057}{2304} C_A^2 N_f T_F \zeta_3 - \frac{2057}{288} C_A N_f^2 T_F^2 \zeta_3 - \frac{9}{32} C_f N_f T_F \]
\[+3C_F N_f T_F^2 \zeta_3 - \frac{209}{48} C_F N_f^2 T_F^2 \left( \frac{25N^3 T_F^3}{81} \right) + \log^2 \left( \frac{p^2}{\mu^2} \right) \left( -\frac{1793 C_A^3}{216} \right) + \frac{273}{32} C_A^2 N_f T_F + \frac{55}{32} C_A C_F N_f T_F - \frac{181}{72} C_A N_f^2 T_F - \frac{5}{8} C_F N_f^2 T_F^2 + \frac{5N^3 T_F^3}{27} \]
\[-\log^3 \left( \frac{p^2}{\mu^2} \right) \left( -\frac{1331 C_A^3}{1728} + \frac{121}{144} C_A^2 N_f T_F - \frac{11}{36} C_A N_f^2 T_F + \frac{N^3 T_F^3}{27} \right) \log \left( \frac{p^2}{\mu^2} \right) \left( \frac{1451 C_A^2 N_f T_F}{864} - \frac{2857 C_A^3}{1728} + \frac{205 C_A C_F N_f T_F}{288} - \frac{79 C_A N_f^2 T_F}{432} \right) \]
\[-\frac{11 C_A C_F N_f T_F}{16} + \frac{11 C_A N_f^2 T_F^2}{72} + \log^2 \left( \frac{p^2}{\mu^2} \right) \left( -\frac{89 C_A^2 N_f T_F}{144} + \frac{187 C_A^3}{288} \right) \frac{5 C_A N_f^2 T_F^2}{36} + \frac{C_F N_f^2 T_F^2}{12} \right \}

(B.40)

It follows \( C_1^{(S)} \) in the momentum representation:

\[
C_1^{(S)}(\mu) = 4 + g^2(\mu) \left( B_{1,1}^{(S)} + \beta_0 \log \left( \frac{p^2}{\mu^2} \right) \right) + g^4(\mu) \left( B_{1,2}^{(S)} + B_{1,3}^{(S)} \log \left( \frac{p^2}{\mu^2} \right) \right) - \beta_0^2 \log^2 \left( \frac{p^2}{\mu^2} \right) + 4 \beta_1 \log \left( \frac{p^2}{\mu^2} \right) \log \left( \frac{p^2}{\mu^2} \right) + g^6(\mu) \left( B_{1,4}^{(S)} + B_{1,5}^{(S)} \log \left( \frac{p^2}{\mu^2} \right) \right) + B_{1,6}^{(S)} \log^2 \left( \frac{p^2}{\mu^2} \right) - \beta_3^2 \log^3 \left( \frac{p^2}{\mu^2} \right) + 8 \beta_2 \log \left( \frac{p^2}{\mu^2} \right) - 4 \beta_0 \beta_1 \log^2 \left( \frac{p^2}{\mu^2} \right)
\]

with:

\[
B_{1,1}^{(S)} = \frac{1}{(4\pi)^2} \left( \frac{196}{9} - \frac{40N_f}{9N} \right)
\]

(B.42)

\[
B_{1,2}^{(S)} = \frac{1}{(4\pi)^4} \left( \frac{264\zeta_3}{2} + \frac{46036}{81} - \frac{12380 N_f}{81} - \frac{48 \zeta_3 N_f}{N} + \frac{8 \zeta_3 N_f^2 - 1 N_f}{2N^2 N} \right)
\]

(B.43)

\[
B_{1,3}^{(S)} = \frac{1}{(4\pi)^4} \left( \frac{9208}{27} + \frac{3104 N_f}{27 N} + \frac{16 N_f^2 - 1 N_f}{27 N^2} \right)
\]

(B.44)

\[
B_{1,4}^{(S)} = \frac{1}{(4\pi)^6} \left( \frac{85040 \zeta_3}{9} - \frac{1760 \zeta_5}{9} - \frac{9775633}{729} - \frac{16612 N_f \zeta_3}{9N} - \frac{640 N_f \zeta_5}{N} + \frac{3518939 N_f}{486 N} - \frac{64 N_f^2 \zeta_3}{9 N^2} - \frac{181546 N_f^2}{243 N^2} + \frac{1900 N_f \zeta_3}{N^3} - \frac{480 N_f \zeta_5}{N^3} - \frac{32467 N_f}{18 N^3} \right)
\]

(B.45)

\[
B_{1,5}^{(S)} = \frac{1}{(4\pi)^6} \left( \frac{134014 N_f}{9 N^2} + \frac{5368 N_f}{9 N^2} - \frac{800 N_f^3}{81 N^3} - \frac{528 N_f \zeta_3}{9 N^3} - \frac{2473 N_f \zeta_5}{3 N^4} + \frac{96 N_f^2 \zeta_3}{3 N^4} - \frac{418 N_f^2 \zeta_5}{3 N^4} + \frac{9 N_f^3}{N^5} \right)
\]

(B.46)
As demonstrated in [17], the divergent contact terms in Eq. (B.41) are expressed in terms of the coefficients of the QCD beta function:

\[
\beta_0 = \frac{1}{(4\pi)^2} \left( \frac{11}{3} - \frac{2N_f}{3N_f} \right) \tag{B.48}
\]

\[
\beta_1 = \frac{1}{(4\pi)^4} \left( \frac{34}{3} - \frac{13N_f}{3N_f} + \frac{N_f}{N_f^3} \right) \tag{B.49}
\]

\[
\beta_2 = \frac{1}{(4\pi)^6} \left( \frac{2857}{54} - \frac{1709N_f}{54} + \frac{56N_f^2}{27N_f^2} + \frac{187N_f^2}{36N_f^3} - \frac{11N_f^2}{18N_f^4} + \frac{N_f}{4N_f^5} \right) \tag{B.50}
\]

### B.5 Perturbative \(C_1^{(S)}\) in the coordinate representation

The Fourier transform of Eq. (B.40) is:

\[
C_1^{(S)}(x) = \frac{4\beta_0}{\pi x^4} \gamma^2(\mu) \left( 1 + g^2(\mu)(A_{1,1}^{(S)} + 2\beta_0 \log(x^2\mu^2)) + g^4(\mu)(A_{1,2}^{(S)} + A_{1,3}^{(S)} \log(x^2\mu^2)) \right) + \delta^{(4)}(x) \left( 4 + g^2(\mu)(A_{1,4}^{(S)} + A_{1,5}^{(S)} \log(x^2\mu^2)) \right) + \delta^{(2)} \left( \frac{A_2}{\mu^2} \right) + g^4(\mu) \left( A_{1,6}^{(S)} + 8\beta_2 \log(\frac{A_2}{\mu^2}) \right) - 4\beta_0 \beta_1 \log^2(\frac{A_2}{\mu^2}) \tag{B.51}
\]

with \(A_{1,1}^{(S)}, A_{1,2}^{(S)}, A_{1,3}^{(S)}\) scheme dependent:

\[
A_{1,1}^{(S)} = \frac{4}{\beta_0} \left( \frac{1}{(4\pi)^4} \left( \frac{363\gamma_E + 394 - 363 \log 2}{27} - \frac{N_f(132\gamma_E + 155 - 132 \log 2)}{27N_f} \right) \right) \tag{B.52}
\]

\[
A_{1,2}^{(S)} = \frac{16}{\beta_0} \left( \frac{1}{(4\pi)^6} \left( \frac{-117612\zeta_3 + 95832\gamma_E^2 + 248424\gamma_E + 188197 + 95832 \log 2}{2592} - \frac{N_f(-7128\zeta_3 + 17424\gamma_E^2 + 47484\gamma_E)}{864N_f} \right) \right) \tag{B.53}
\]
\[ A_{1,3}^{(S)} = \frac{16}{\beta_0 (4\pi)^6} \left( \frac{11(726\gamma_E + 941 - 726 \log 2)}{216} - \frac{N_f(968\gamma_E + 1319 - 968 \log 2)}{48N} ight) 
+ \frac{11N_f^2(72\gamma_E + 75 - 48 \log(2) - 24 \log 2)}{2} 
+ \frac{-96\gamma_E N_f^3 - 32N_f^3 + 96N_f^3 \log 2 + 1485N_f}{432N^3} - \frac{5N_f^2}{8N^4} \] (B.54)

and \( A_{1,4}^{(S)}, A_{1,5}^{(S)}, A_{1,6}^{(S)} \) coefficients of the contact terms:

\[ A_{1,4}^{(S)} = \frac{1}{(4\pi)^2} \left( \frac{196}{9} - \frac{40N_f}{9N} \right) \] (B.55)

\[ A_{1,5}^{(S)} = \frac{1}{(4\pi)^4} \left( -\frac{264\zeta_3}{2} + \frac{46036}{81} - \frac{12380N_f}{81N} - 48\zeta_3 \frac{N_f}{N} + \frac{8}{3} \zeta_3 \frac{N^2 - 1N_f}{2N^2} \right) 
- \frac{104N^2 - 1N_f^2}{2N^2} \frac{400N^2}{81N^2} \] (B.56)

\[ A_{1,6}^{(S)} = \frac{1}{(4\pi)^6} \left( \frac{85040\zeta_3}{9} - 1760\zeta_5 - \frac{9775633}{729} - \frac{16612N_f\zeta_3}{9N} - \frac{640N_f\zeta_5}{N} \right) 
+ \frac{3518939N_f}{486N} \frac{N}{9N^2} - \frac{181546N_f^2}{243N^2} + \frac{1900N_f\zeta_3}{N^3} \frac{480N_f\zeta_5}{N^3} \frac{32467N_f}{18N^3} \] 
+ \frac{4000N^3}{729N^3} \frac{52N^2\zeta_3}{N^4} + \frac{2399N^2}{9N^4} - \frac{288N\zeta_3}{N^5} + \frac{480N\zeta_5}{N^5} - \frac{90N_f}{N^5} \] (B.57)

\section*{B.6 Verifying the UV asymptotics of \( C_1^{(S)} \) by a change of renormalization scheme in perturbation theory}

As for \( C_0^{(S)} \), we skip the contact terms, and we multiply Eq. (B.51) by \( g^2 \) that is equivalent to consider the OPE of \( g^2 F^2 \). Following [14] we change the renormalization scheme by redefining the coupling constant:

\[ g^2_{cd}(\mu) = g^2(\mu)(1 + cg^2(\mu) + dg^4(\mu)) \] (B.58)

Then, Eq. (B.51) becomes:

\[ g^2_{cd}(\mu)C_1^{(S)}(x) = \frac{4\beta_0}{\pi^2x^4}g^4_{cd}(\mu) \left( 1 + g^2_{cd}(\mu)(A_{1,1}^{(S)} - 2\beta_0 \log(x^2 \mu^2)) 
+ g^4_{cd}(\mu)(5c^2 - 2d - 3cA_{1,1}^{(S)} + A_{1,2}^{(S)}) 
+ (A_{1,3}^{(S)} - 6\beta_0) \log(x^2 \mu^2) + 3\beta_0 \log^2(x^2 \mu^2) \right) \] (B.59)

We fix \( c \) and \( d \) by requiring that the finite parts vanish to two loops in the new renormalization scheme:

\[ c = \frac{A_{1,1}^{(S)}}{2} \] (B.60)

\[ d = \frac{4A_{1,2}^{(S)} - A_{1,1}^{(S)}^2}{8} \] (B.61)
Remarkably, as for $c^{(S)}_0$, in this renormalization scheme the coefficient of the two-loop log divergence is proportional to the second coefficient of the QCD beta function:

\[ A_{1,3}^{(S)} = -6c \beta_0 \]

\[ = A_{1,3}^{(S)} - 3A_{1,1}^{(S)} \beta_0 \]

\[ = \frac{16}{\beta_0 (4\pi)^6} \left( \frac{11(72\gamma_E + 941 - 726 \log 2)}{216} - \frac{N_f(968\gamma_E + 1319 - 968 \log 2)}{48N} \right. \]

\[ + \frac{11N_f^2(72\gamma_E + 75 - 48 \log(2) - 24 \log 2)}{216N} \]

\[ + \left. \frac{-96\gamma_E N_f^3 - 32N_f^3 + 96N_f^2 \log 2 + 1485N_f}{432N^3} - \frac{5N_f^2}{8N^4} \right) \]

\[ - \frac{12}{(4\pi)^4} \left( \frac{363\gamma_E + 394 - 363 \log 2}{27} - \frac{N_f(132\gamma_E + 155 - 132 \log 2)}{27N} \right) \]

\[ + \frac{4N_f^2(3\gamma_E + 1 - 3 \log 2)}{27N^2} + \frac{N_f}{N^2} \]

\[ = \frac{34N^3 - 13N^2 N_f + 3N_f}{256\pi^4 N^3} = 3\beta_1 \] (B.62)

Eq. (B.59) reads now:

\[ g_{cd}(\mu)C_i^{(S)}(x) = \frac{4\beta_0}{\pi^2 x^4} g_{cd}(\mu) \left( 1 + 2\beta_0 g_{cd}(\mu) \right) \log(x^2 \mu^2) + g_{cd}(\mu)(3\beta_1 \log(x^2 \mu^2) + 3\beta_0^2 \log^2(x^2 \mu^2)) \]

(B.63)

In order to verify Eq. (A.19) we insert the identity:

\[ 1 = \frac{1 + \frac{\beta_1}{\beta_0} g_{cd}(\mu)}{1 + \frac{\beta_1}{\beta_0} g_{cd}(x)} \]

(B.64)

in Eq. (B.63) to the relevant perturbative order:

\[ g_{cd}(\mu)C_i^{(S)}(x) \sim \frac{1 + \frac{\beta_1}{\beta_0} g_{cd}(\mu)}{1 + \frac{\beta_1}{\beta_0} g_{cd}(x)} \]

\[ \frac{4\beta_0}{\pi^2 x^4} g_{cd}(x) \left( 1 + \frac{\beta_1}{\beta_0} g_{cd}(x) - \frac{\beta_1}{\beta_0} g_{cd}(\mu) \right) \]

(B.65)

where in the first line we have expanded to the order of $g^4$:

\[ \frac{1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)}{1 + \frac{\beta_1}{\beta_0} g_{ab}(x)} = 1 - \beta_1 g^4_{ab}(\mu) \log(x^2 \mu^2) \]

(B.66)
and in the third line we have expanded to the order of $g^2$:

$$\frac{1 + \frac{\beta_1}{\beta_0} g_{ab}(x)}{1 + 2\frac{\beta_1}{\beta_0} g_{ab}(\mu)} = 1 + \frac{\beta_1}{\beta_0} g_{ab}(x) - 2\frac{\beta_1}{\beta_0} g_{ab}(\mu)$$  \hspace{1cm} (B.67)

The factor of $\left(1 + \frac{\beta_1}{\beta_0} g_{cd}(x) - \frac{\beta_1}{\beta_0} g_{cd}(\mu)\right)$ in the last line of Eq. (B.65) arises according to Eqs. (A.12). Therefore, Eq. (B.65) agrees with Eq. (A.19).

We may remove the scale-dependent term in the last line of Eq. (B.65), $-\frac{\beta_1}{\beta_0} g_{cd}(\mu)$, multiplying Eq. (B.65) by $\frac{\beta_1}{\beta_0} \left(1 + \frac{\beta_1}{\beta_0} g_{cd}(\mu)\right)$ that makes $g_{cd}^2 C_1^{(S)}$ RG invariant. Hence, we get to the relevant perturbative order:

$$\frac{\beta(g)}{g} C_1^{(S)}(x) \sim \frac{4\beta_0^2}{\pi^2 x^4} g_{cd}^4(x) \left(1 + \frac{\beta_1}{\beta_0} g_{cd}(x)\right) \sim \frac{4\beta_0^2}{\pi^2 x^4} g_{cd}^4(x)$$  \hspace{1cm} (B.68)

according to the nonperturbative asymptotics in Eq. (4.2).

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