Bouncing open universes embeddable in a distorted
Randall-Sundrum brane scenario

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(March 24, 2022)

Abstract

In reference [1] a four-dimensional effective theory of gravity embeddable in a five-dimensional ”distorted" Randall-Sundrum brane scenario was derived. The present paper is aimed at the application of such a theory to describe physics in an open Friedmann-Robertson-Walker (Weyl-symmetric) universe. It is shown that regular bouncing universes arise for a given range of the free parameter of the theory.

One of the main ingredients that enter the basis of the unification scheme of the fundamental interactions is that multiple dimensions are required. In fact, modern field theory suggests spacetime to be 4 + n-dimensional (n = 6 for string theory and n = 7 for supergravity). It is usually assumed that the additional dimensions may be compactified down to size of the order the Planck length or below so they are not observable. However, recent developments suggest that some of the additional dimensions may have compactification radii larger than the Planck length without conflict with observations [2]. Besides, it is possible

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that gravity remains confined to a four-dimensional slice of the bulk spacetime leading to extra-dimensions being even infinite in extent [3].

In reference [1] a four-dimensional effective theory of gravity embeddable in a five-dimensional "distorted" Randall-Sundrum brane scenario was derived. The fifth extra dimension was identified with the dilaton field $\psi$. We started with the Randall-Sundrum scenario improved in Ref. [4] and then we "distorted" it by relaxing some requirements as, for instance, orbifold symmetry and Poincare invariance. Then we elaborated on this "distorted" five-dimensional brane scenario by studying the effective geometry induced on the four-dimensional manifold. It resulted in a Weyl-integrable geometry that may be seen as a distorted Riemann geometry. Then we postulated a four-dimensional theory of gravity that could be embedded in the higher-symmetric structure, i.e., a theory sharing some of the basic properties of the "distorted" brane set-up worked out previously: the theory should be built over Weyl-integrable geometry (then it should be Weyl-symmetric), the matter degrees of freedom should be coupled to the metric induced on the "visible" brane of the higher-dimensional scenario instead of the four-dimensional "Planck" metric, there should be some "signal" from the extra-dimension, etc. Respecting the cosmological issue, the singularity problem was treated for the case of flat Friedmann-Robertson-Walker universes. Now we apply this theory for the description of physics in an open Friedmann-Robertson-Walker (Weyl-symmetric) universe.

We start with an action similar to that used in [1]:

$$S = \int d^4x \sqrt{-g} e^\psi (R - \omega (\nabla \psi)^2 + 16\pi e^\psi L_m),$$  \hspace{1cm} (1)

where $\psi$ is the dilaton field (linked with the extra dimension), $R$ is the curvature scalar of the metric $g_{ab}$, $\omega$ is a free parameter\footnote{It is related with the free parameter in Ref. [1] through $\omega = \frac{1}{4k^2} - \frac{3}{2}$} and $L_m$ is the Lagrangian of the ordinary matter fields. The change of the sign before $\psi$ in the exponents, with respect to Ref. [1], does not
affect the physics. The variational principle gives

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi e^{-\psi}T_{ab} + \omega [\psi, a \psi, b - \frac{1}{2}g_{ab}(\nabla \psi)^2] + [-\psi]_{a b} + \psi, a \psi, b + g_{ab} \Box \psi - g_{ab}(\nabla \psi)^2, \quad (2)$$

$$\Box \psi = 0 \quad (3)$$

where $T_{ab} = 2 \sqrt{-g} \delta(\sqrt{-g}L_m) \delta g_{ab}$, $\Box \psi = g^{mn} \psi_{||mn}$, and the bar denotes covariant differentiation in a metric sense, i.e., defined through the Christoffel symbols of the metric $g_{ab}$. Recall that the theory is built over Weyl-integrable geometry so the affine connection does not coincide with the Christoffel symbols. From this equation one sees that Newton’s constant $G = e^{-\psi}$.

For an open Friedmann-Robertson-Walker (FRW) universe filled with a barotropic perfect fluid, the above field equations can be written explicitly in the following way

$$\frac{\dot{a}}{a}^2 - \frac{1}{a^2} = -\frac{M}{e^{\left(\frac{3\gamma}{2} + 1\right)\psi} a^{3\gamma}} + \frac{\omega \psi^2}{6} + \frac{\dot{a}}{a} \psi, \quad (4)$$

$$\ddot{\psi} + 3 \frac{\dot{a}}{a} \dot{\psi} + \dot{\psi}^2 = 0 \quad (5)$$

$M$ is an arbitrary integration constant and the barotropic index $\gamma$ is in the range $0 < \gamma < 2$. While deriving eq.(4) we have taken into account that the ordinary matter energy density is given by $\mu = \frac{3}{8\pi} \frac{M}{e^{\left(\frac{3\gamma}{2}\right)\psi} a^{3\gamma}}$. After integrating once the wave equation for the scalar field $\psi$, we obtain:

$$e^{\psi} = \pm \frac{\sqrt{N}}{2a^3}. \quad (6)$$

where $N$ is another integration constant.

Substitution of eq. (6) into Eq. (4) gives:

$$\dot{a}^2 - 1 = -\frac{M}{e^{\left(\frac{3\gamma}{2} + 1\right)\psi} a^{3\gamma}} + \frac{N\omega}{24e^{2\psi} a^4} \pm \frac{\sqrt{N} \dot{a}}{2e^\psi a^2}, \quad (7)$$
The curvature scalar for an open FRW universe is found to be

\[ R = \frac{3M(4 - 3\gamma)}{e^{\left(\frac{3}{2} + 1\right)\psi} a^{3\gamma}} - \frac{N\omega}{4e^{2\psi} a^6} \]  

(8)

The high complexity of the system of equations (6)-(7) and of eq. (8) leads us to the use of the conformal transformation technique [5]. Well worth to notice that the conformal theory will be Einstein’s canonical general relativity.

We shall use the Raychaudhuri equation [6]. For a congruence of fluid lines without vorticity and shear, with the time-like tangent vector \( \hat{k}^a = \delta^a_0 \), it can be written in the conformal frame as:

\[ \dot{\theta} = -\dot{R}_{00} - \frac{1}{3} \dot{\theta}^2, \]  

(9)

where the overdot means derivative with respect to the transformed proper time \( \tau \) and \( \dot{\theta} \) is the volume expansion. In eq.(9) we took the reversed sense of time \(-\infty \leq \tau \leq 0\), i.e. \( \dot{a} \) runs from infinity to zero. This equation can be finally written as:

\[ \dot{\theta} = -\frac{3}{a^2} - \frac{9\gamma M}{2a^{3\gamma}} - \frac{3}{2} \frac{(\omega + \frac{2}{3}) N}{a^6}. \]  

(10)

From it one sees that all terms in the right-hand side induce contraction and hence a spacetime singularity is expected to occur (the global singularity at \( \dot{a} = 0 \)).

The evolution of the volume expansion in the original frame, given in terms of the conformal scale factor, can be easily found from eq.(10) if we realize that \( \theta = e^{\frac{\psi}{2}}(\dot{\theta} - \frac{3}{2}\dot{\psi}) \).

We find that

\[ \left(\frac{d\theta}{dt}\right)^\pm = 3e^{\psi\pm} \left\{ -a^4 - \frac{3}{2}\gamma Ma^{3(2-\gamma)} - \left(\frac{\omega}{2} + 1\right)N \pm \frac{\sqrt{N}}{2} \sqrt{a^4 + Ma^{3(2-\gamma)} + \left(\frac{\omega}{6} + \frac{1}{4}\right)N} \right\}, \]  

(11)

where \( t \) is the proper time in the original frame. It is related with \( \tau \) through \( dt = e^{-\frac{1}{2}\psi} d\tau \).

The ‘+’ and ‘-’ signs in eq.(11) correspond to two possible branches of the solution. From (11) one sees that for the ‘+’ branch of the solution, the last term in brackets induces expansion.
We are interested now in the limiting case $\hat{a} \ll 1$, since the singularity in the transformed frame is found at $\hat{a} = 0$. In this case for $\omega = -\frac{3}{2}$ eq.(11) can be written as:

$$\left(\frac{d\theta}{dt}\right)^+ \approx \pm \frac{3\sqrt{NM}e^{\hat{\phi}_0}}{2\hat{a}^{\frac{1}{4}(\gamma+2)}} \exp\left[\mp \frac{2}{3} \sqrt{\frac{N}{M}} \hat{a}^{\frac{1}{2}(2-\gamma)}\right],$$  \hspace{1cm} (12)

for $\gamma > \frac{2}{3}$. $\hat{\phi}_0$ is some integration constant. For $\gamma = \frac{2}{3}$ we obtain:

$$\left(\frac{d\theta}{dt}\right)^+ \approx \pm \frac{3\sqrt{N(M+1)}e^{\hat{\phi}_0}}{2\hat{a}^{4}} \exp\left[\mp \frac{1}{2} \sqrt{\frac{N}{M+1}} \hat{a}^{-2}\right],$$  \hspace{1cm} (13)

while for $\gamma < \frac{2}{3}$:

$$\left(\frac{d\theta}{dt}\right)^+ \approx \pm \frac{3\sqrt{Ne^{\hat{\phi}_0}}}{2\hat{a}^{4}} \exp\left[\mp \sqrt{\frac{N}{2}} \hat{a}^{-2}\right].$$  \hspace{1cm} (14)

For $\omega > -\frac{3}{2}$, in the limit $\hat{a} \ll 1$, eq.(11) can be written in the following way:

$$\left(\frac{d\theta}{dt}\right)^+ \approx \frac{N e^{\hat{\phi}_0}}{6\hat{a}^{\frac{1}{2}}} \frac{1}{\sqrt{\omega + \frac{1}{2}}} (-3\omega - \frac{9}{2} \pm 2\sqrt{6\omega + 9} - \frac{3}{2}),$$  \hspace{1cm} (15)

for $0 < \gamma < 2$.

A careful analysis of eq.(11) shows that, for big $\hat{a}$ the first three terms in brackets prevail over the last one and, consequently, contraction is favored until $\hat{a}$ becomes sufficiently small ($\hat{a} \ll 1$). In this case, when $\omega$ is in the range $-\frac{3}{2} \leq \omega \leq -\frac{4}{3}$, in the '+' branch of the solution, there are not enough conditions for further contraction and the formation of the global singularity is not allowed. A cosmological wormhole is obtained instead. For further analysis of what happen in this case, we need to write down the relevant magnitudes and relationships, in the original frame, in terms of the transformed scale factor. We shall be interested in the behaviour of these magnitudes and relationships for small $\hat{a} \ll 1$ (when the condition for further contraction ceases to hold), in the '+' branch of the solution. In this case the scale factor is found to be

$$a^+ \approx e^{-\frac{1}{4}\hat{\phi}_0} \hat{a}^{\frac{1}{2}} \sqrt{\frac{\omega}{\omega + \frac{1}{2}}},$$  \hspace{1cm} (16)
The Ricci scalar is

$$ R^+ \approx \frac{3}{2} N e^{\phi_0} \hat{a} \sqrt{\omega + \frac{3}{2}}^{-6}, \quad (17) $$

while for the proper time $t$ we have that

$$ t^+ \approx \frac{2 e^{-\frac{1}{2} \phi_0} \hat{a}^{3-\frac{1}{2} \sqrt{\frac{6}{\omega + \frac{3}{2}}}}}{\sqrt{(\omega + \frac{3}{2}) N (6 - \sqrt{\frac{6}{\omega + \frac{3}{2}}})}} \quad (18) $$

for $\omega \neq -\frac{4}{3}$ and

$$ t^+ \approx \frac{e^{-\frac{1}{2} \phi_0}}{\sqrt{(\omega + \frac{3}{2}) N 6} \ln \hat{a}}, \quad (19) $$

for $\omega = -\frac{4}{3}$. Hence, if we choose the '+' branch of the solution, for $\omega \geq -\frac{4}{3}$, $R^+$ is bounded for $\hat{a} \to 0$. In this limit $a^+ \to +\infty$ while $t^+ \to -\infty$. A similar analysis shows that, for big $\hat{a}$, $\hat{a} \to \infty \Rightarrow a^\pm \to \infty$ and $t^\pm \to +\infty$. For intermediate values in the range $0 < \hat{a} < \infty$ the curvature scalar $R^+$ is well behaved and bounded. The scale factor $a$ is a minimum at some intermediate time $t_\ast$. Hence in the original frame, if we choose the '+' branch of the solution, the following picture takes place. If we restrict $\omega$ to fit into the range $-\frac{3}{2} \leq \omega \leq -\frac{4}{3}$ and for $0 < \gamma < 2$, then, the universe evolves from the infinite past $t = -\infty$ when he had an infinite size, through a bounce at some intermediate $t_\ast$ when he reached a minimum size $a_\ast$, into the infinite future $t = +\infty$ when he will reach again an infinite size.

We now elaborate on $\psi$ evolution. Combining equations (6) and (16), and the fact that for $\hat{a} \ll 1$, we find that $\tau \sim \hat{a}^3$ in the conformal frame, we obtain

$$ \psi = \psi_0 + \frac{1}{3} \sqrt{\frac{6}{\omega + \frac{3}{2}}} \ln \tau \quad (20) $$

where $\psi_0$ is an arbitrary constant.

Extrapolating the validity of above equation to arbitrary times, we could conclude that the dilaton (the curved extra-coordinate in our distorted RSL scenario) evolves, in this case, from $\psi^+ \sim -\infty$ at $\tau^+ \sim 0$ into $\psi^+ \sim \infty$ at $t^+ \sim +\infty$. As illustrations to this behaviour...
we shall study the particular cases with $\omega = -\frac{3}{2}$ for dust-filled and radiation-filled universes since, in these very particular situations exact analytic solutions can be easily found.

For a radiation-filled universe ($\gamma = \frac{4}{3}$) the equation conformal to (4), with $\omega = -\frac{3}{2}$ can be written as:

$$\dot{a} = \sqrt{M \dot{a}^{-2}} + 1,$$

and, after integration we obtain for the transformed scale factor

$$\dot{a} = \sqrt{\tau^2 - M}.$$  \hspace{1cm} (22)

The proper time $\tau$ is constrained to the range $|\tau| \geq \sqrt{M}$ or $\sqrt{M} \leq \tau \leq +\infty$ (the case $-\infty \leq \tau \leq -\sqrt{M}$ corresponds to the time reversed solution). The scale factor, in our frame, is then found to be

$$a^{\pm} = \frac{\sqrt{\tau^2 - M}}{\sqrt{\phi_0}} \exp[\pm \frac{1}{2} \frac{\sqrt{N}}{M} \frac{\tau}{\sqrt{\tau^2 - M}}],$$

while the curvature scalar:

$$R^{\pm} = 3 \frac{N \phi_0}{2} \exp[\frac{\pm \sqrt{N}}{M} \frac{\tau}{\sqrt{\tau^2 - M}}].$$

The relationship between the proper time $\tau$ measured in the transformed frame and the one in our initial frame (for the '+1' branch of the solution that is the case of interest) is given by

$$t^+ = -\frac{\tau}{\sqrt{M \phi_0}} \exp[\frac{\sqrt{N}}{2M} \frac{\tau}{\sqrt{\tau^2 - M}}].$$

A careful analysis of eq.(23) shows that $a^+$ is a minimum at some time that is a root of the algebraic equation $\tau^4 - M\tau^2 - \frac{N}{4} = 0$. The curvature singularity occurring in the conformal frame at $\tau = \sqrt{M}$, is removed in our initial frame, where $R^+$ is bounded and well behaved for all times in the range $\sqrt{M} \leq \tau \leq +\infty$ ($-\infty \leq t \leq +\infty$).
Combining equations (6) and (23) we get

\[ \psi = \psi_0 - \frac{\sqrt{N}}{M} \frac{\tau}{\sqrt{\tau^2 - M}}. \]  

(26)

We see that for \( \tau \to \sqrt{M}, \psi \to -\infty \) and for \( \tau \to \infty, \psi \to \psi_0 \), so the "visible" brane starts from an infinite negative separation and asymptotically tends to a finite separation at the infinite future. This way, brane stabilization occurs. In this case conformal (Weyl) symmetry is contiously broken and standard general relativity over Riemann geometry is recovered asymptotically in the future.

For a dust-filled universe \((\gamma = 1)\) the transformed scale factor can be given the form:

\[ \hat{a} = \frac{4M}{\eta^2 - 4}, \]  

(27)

where the time variable \( \eta \) has been introduced through \( d\tau = \frac{\hat{a}^2}{M} d\eta \) and is constrained to the range \( 2 \leq \eta \leq +\infty \) (the case \(-\infty \leq \eta \leq -2\) corresponds to the time reversed solution). In our initial frame we have that

\[ a^\pm(\eta) = \frac{4M}{\sqrt{\phi_0}} \exp[\mp \frac{\sqrt{N}}{24M^2} \eta(\eta^2 - 12)] \frac{1}{\eta^2 - 4}, \]  

(28)

and the relationship between the proper time \( t \) and \( \eta \) is given by the following expression:

\[ t^\pm = \frac{16M}{\sqrt{\phi_0}} \int d\eta \frac{\exp[\mp \frac{\sqrt{N}}{24M^2} \eta(\eta^2 - 12)]}{(\eta^2 - 4)^2}. \]  

(29)

The curvature singularity occurring in the conformal frame at time \( \eta = 2 \) is removed again in our initial representation of the theory. The '+ branch scale factor \( a^+ \) is a minimum at some \( \eta_* \) that is a root of the algebraic equation \( \eta^4 - 8\eta^2 + \frac{16M^2}{\sqrt{N}} \eta + 16 = 0. \)

For the dilaton we have

\[ \psi = \psi_0 + \frac{\sqrt{N}}{12M^2} \eta(\eta^2 - 12) \]  

(30)
We see that for $\eta \to 2, \psi \to \psi_0 - \frac{4\sqrt{3}}{3M^2}$ and for $\eta \to \infty, \psi \to \infty$, so the "visible" brane starts from an arbitrary finite negative separation and tends to an infinite separation at the infinite future. This way, brane unstabilizes and conformal (Weyl) symmetry is recovered in the future.

Summing up, from all above results we could conclude that in this scenario an open FRW evolves from a radiation dominated universe in a visible brane infinitely separated (in the negative $\psi$ direction) from the Planck brane to a dust universe again infinitely separated from the reference (Planck) brane, but now in the positive $\psi$ direction, while in reference [1], the flat universe goes from an infinite separation in the distant past to a finite separation in the future. So, for open universes no brane stabilization occurs, and a more dynamic picture respect to the results obtained for flat universes takes place. In this case conformal (Weyl) symmetry was continuously broken and standard general relativity over Riemann geometry was recovered asymptotically, while the inverse picture will occur in the future. Above results also tell us that both flat and open universes, are free of the cosmological singularity in our initial formulation, in the same region of the parameter space ($-\frac{4}{3} \leq \omega \leq -\frac{2}{3}, 0 < \gamma < 2$). The fact that our non-singular branch is the '+' one, instead of the '-' branch in reference [1] is simply due to the change of the sign before $\psi$ in the exponents of our action.

Finally we shall remark the fact that the cosmological singularity is removed, in our initial frame, only for a given range of the parameter $\omega$. It can be taken just as a restriction on the values this parameter can take. A physical consideration why we chose the '+' branch (the non-singular branch) instead of the '-' branch is based on the following analysis. We shall note that in our frame, $e^{-\psi}$ plays the role of an effective gravitational constant $G$. For the '-' branch $G$ runs from zero to an infinite value, i.e. gravity becomes stronger as the universe evolves and, in the infinite future it dominates over the other interactions, that is in contradiction with the usual picture. On the contrary, for the '+' branch, $G$ runs from an infinite value to zero and hence gravitational effects are weakened as the universe evolves, as required.
REFERENCES

[1] I. Quiros, R. Bonal and R. Cardenas, gr-qc/0104036.

[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429, 263(1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436, 257(1998).

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690(1999).

[4] J. Lykken and L. Randall, JHEP 0006, 014(2000).

[5] V. Faraoni, Phys. Lett. A 245, 26(1998); V. Faraoni, E. Gunzig and P. Nardone, Fund. Cosmic Phys. 20, 121 (1999).

[6] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, 84 (Cambridge University Press, Cambridge, 1973).