On the Supergravity Gauge theory Correspondence and the Matrix Model

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Abstract

We review the assumptions and the logic underlying the derivation of DLCQ Matrix models. In particular we try to clarify what remains valid at finite $N$, the role of the non-renormalization theorems and higher order terms in the supergravity expansion. The relation to Maldacena’s conjecture is also discussed. In particular the compactification of the Matrix model on $T_3$ is compared to the $AdS_5 \times S_5 \, \mathcal{N} = 4$ super Yang-Mills duality, and the different role of the branes in the two cases is pointed out.
1 Introduction

There appear to be two conjectures on the relation between gauge theory and gravity. One is the Matrix model [1] which was originally proposed as a microscopic theory whose low-energy limit is 11 dimensional supergravity. The other is the more recent conjecture on the relation between gauge theory and supergravity [2], [3], [4], [5] whose clearest manifestation is in the correspondence between $\mathcal{N} = 4$ $SU(N)$ four dimensional Yang-Mills theory and supergravity (string theory?) on a $AdS_5 \times S_5$ background. The Matrix model can also be compactified and in particular on a three torus, it is supposed to be represented by the same Yang-Mills theory. One of the purposes of this investigation is to elucidate the connection between the two conjectures [1]. The other purpose is to understand why finite $N$ calculations work at least in certain cases.

In the next section we will review the arguments given in [7], [8] for obtaining the Matrix model. In the course of the discussion we will try to be careful about the logic of these arguments by distinguishing between what is actually derived and that which is still conjecture. In particular by expanding on arguments given in [9] we will try to explain precisely what the connection to supergravity should be. We will also comment on exactly what is achieved by the recently proven non-renormalization theorems for the model in relation to the connection between gauge theory and gravity.

In the third section we will discuss the correspondence between the higher order terms in the supergravity expansion and the non-renormalization theorem. We will point out that the latter imposes certain regularities in the supergravity terms and we will also identify the supergravity terms from which certain non-diagonal terms (in the terminology of [10]) in the Matrix model expansion arise. In the third section we will briefly review the recent work [3], [4], [5] on the gauge theory/gravity connection. In particular we will

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1Recently there have been two papers by S. Hyun [6] on this issue. While there is some overlap between the present work and those papers our conclusions are somewhat different especially with regard to the interpretation of the Matrix model on the three torus and the corresponding AdS picture.
compare and contrast this with the Matrix model conjecture. The natural place for this is clearly the \( AdS_5 \times S_5 \) supergravity/string theory, \( \mathcal{N} = 4 \) four dimensional Yang-Mills correspondence. In particular we will argue that although in the interpretation of this connection given in [5] the gauge theory is located at the boundary of the space-time, in the Matrix model the whole space is supposed to be the moduli space of the gauge theory. In fact there is a singularity at the origin which is to be interpreted as a break down of the moduli space approximation and is to be replaced by the non-Abelian quantum dynamics. Alternatively from the supergravity point of view one may regard the singularity as being resolved by the branes which are sitting there.

2 On the Matrix model

We begin by summarizing the arguments of Seiberg [8] which suggest a connection to D0 quantum mechanics of the Discrete Light Cone Quantization (DLCQ) (i.e. the quantization of the theory compactified on a null circle) of M-theory.

a) A microscopic Lorentz invariant M-theory should include a framework for calculating scattering amplitudes of the fundamental degrees of freedom (the supergravitons ?). At low energies these amplitudes should yield 11 dimensional supergravity. (This is exactly what happens in string theory. There is a Lorentz covariant formulation, which yields by general arguments on the consistent coupling of spin two fields, the 10 D supergravity low energy effective action. The challenge in M theory is to find the analog of this.)

b) Given a theory satisfying a) its compactification on a null circle will yield scattering amplitudes which at low energies become those of 11 D supergravity compactified on a null circle.

c) The theory compactified on a null circle (of radius \( R \)) is related by an infinite boost to the theory compactified on a space-like circle. The study of states in DLCQ M theory (with Planck length \( l_P \) and finite values of light cone energy \( P_+ \)) is most conveniently done in terms of a \( \tilde{M} \) theory compactified on a space-like circle with vanishing radius \( R_+ \).
and a vanishing Planck length $\tilde{l}_P$ such that

$$
\frac{R_s}{l_p^2} = \frac{R}{l_p^2}, \quad \frac{\tilde{R}_i}{l_P} = \frac{R_i}{l_P}
$$

(2.1)

where the right hand sides are fixed.

d) This limit of $\tilde{M}$ theory is equivalent to string theory in a certain regime. Namely one where

$$
l_s \to 0; \quad g_{YM}^2 \equiv \frac{1}{l_m^2} = \frac{g_s}{l_s^2} = \frac{R_s^3}{l_P^6} \text{ fixed}, \quad \frac{R_i}{l_s^2} = \frac{R_i}{l_m l_P} \equiv U_i \text{ fixed}.\quad (2.2)
$$

In the above we have introduced the string scale $l_s$ and string coupling $g_s$ which are related to the $\tilde{M}$ quantites by

$$
\tilde{l}_P = g_s^{1/3} l_s, \quad R_s = l_s g_s \quad (2.3)
$$

This limit is often referred to as the DKPS limit and we will use this name for it. Note that the radius of the null circle $R$ has no physical significance and we may conveniently set $R = l_P$ so that the length scale set by the gauge theory may be identified with the Planck length, $l_P = l_m$.

e) String theory in the regime defined in d) is given by D0-brane quantum mechanics; i.e. $U(N)$ quantum mechanics with 16 supercharges where $N$ is the number of D0-branes and this corresponds to the sector with $P_+ = N/R$ in the original $M$ theory.

In the above list a) is clearly influenced by what happens in string theory and b) is certainly very plausible. c) on the other hand involves an infinite boost and thus may be problematic but for the purposes of this paper we will assume that it is meaningful. d) involves a hidden assumption that is normally not made explicit. The relation between $M$ theory and string theory is established only at the level of the effective actions. What is assumed here is that this relation holds also at the microscopic level. However this is a standard and plausible assumption that we will not question here.

\footnote{Strictly speaking we should consider these as quantities with tildes since they are related to $\tilde{M}$ theory rather than to $M$ theory, but since we are not going to discuss the space like compactification or the $M$ theory it is not essential to make the distinction.}
The real problem is e). The (perturbative) string action (i.e. the sigma model action) is not defined in this limit (2.2). In fact all D-brane actions are also ill-defined in the limit (since the tensions become infinite) except for the D0-brane action. If one took the open string representation of the latter, it becomes the quantum mechanics action

\[ S_{QM} = -\frac{1}{4g_{YM}^2} \int W_1 \text{tr}(D_\alpha X_i D^\alpha X_i + \frac{1}{4}[X_i, X_j]^2) + \text{fermion terms.} \]  

(2.4)
since the higher order terms in \( \alpha' \) disappear. Here the \( X_i \) are the ten dimensional gauge fields which in this case are to be interpreted as operators governing the position fluctuations of the branes.

However it is the closed string representation of this action that is directly related to the Kaluza-Klein reduction of the 11 D graviton. (see for example [13]). This action for a D0-brane in a background field given by the metric \( g \) and RR field \( C \) is

\[ S_{sugra} = -\frac{1}{g l_s} \int dt e^{-\phi} \sqrt{\det g} + \frac{1}{l_s} \int C \]  

(2.5)
One would then expect a relation of the form

\[ \int dX' e^{iS_{QM}[U+X']} = \lim_{DKPS} e^{iS_{sugra}[U]} . \]  

(2.6)
between these two when \( g \) and \( C \) are due to a cluster of D0-branes and \( S_{sugra} \) is the supergravity representation of the probe brane action when it is a distance \( U = r/l_s \) (in units with mass dimension!) from the cluster and moving with velocity \( \dot{U} = v/l_s^2 \).

It is precisely relations of this sort that must be established if the gauge theory gravity connection implied by the arguments of [1], [7], [8] is to be proven. The problem is that the supergravity form of the action is meaningful when a massless closed string representation is valid i.e. when \( r/l_s > 1 \), whereas the DKPS limit takes us to \( r/l_s = l_s U \rightarrow 0 \).

The supergravity solution corresponding to N zero branes is given by [12]

\[ ds_{10}^2 = -H_0^{-1/2} dt^2 + H_0^{1/2} dx^i dx^i \]
\[ e^{-\phi} = H_0^{-3/4}, \quad C_t = H_0^{-1} - 1. \]  

(2.7)
where \( H = 1 + h, \ h = \frac{Nc_0 g T}{r} \) and \( c_0 \) is a known constant whose value is irrelevant for our purposes. If we lift this solution to 11 dimensions using the standard formulae (see for
example \[13\]) then we get

\[
\begin{align*}
   ds^2_{11} &= -(1-h)dt^2 - 2hdx^{11} dt + (1+h)dx^{11}^2 + dx^{i2} \\
   &= 2d\tau dx^- + hdx^{-2} + dx^{i2} \\
   &= e^{-2\phi/3}d\bar{s}^2_{10} + e^{4\phi/3}(dx^- + \bar{C}_\tau d\tau)^2 \\
\end{align*}
\]

(2.8)

where in the last equation,

\[
\begin{align*}
   \bar{d}s^2_{10} &= -h^{-1/2}d\tau^2 + h^{1/2}dx^{i2}, \\
   e^{-2\phi/3} &= h^{-1/2}, \quad \bar{C}_\tau = h^{-1}. \\
\end{align*}
\]

(2.9)

In particular the ten dimensional metric above is just the (asymptotically) light like compactified Aichelburg-Sexl \[7\] metric which can be rewritten as.

\[
\begin{align*}
   ds^2_{10} &= l_s^2(-\bar{h}^{-1/2}d\tau^2 + \bar{h}^{1/2}(dU^2 + U^2d\Omega^2_8)), \\
\end{align*}
\]

(2.10)

where

\[
\bar{h} = l_s^4h = \frac{c_0N g_s^2 M}{U^2}. \\
\]

(2.11)

The argument above was given in essence in \[10\] and elaborated on in \[16\].

On the other hand let us consider again the 10 dimensional metric (2.7) and take the limit (2.2). This limit also leads to the light-like compactified M-theory metric (2.10) except that we now have \(\tau \to t\). Thus we might expect that this fact on the supergravity side of the D0-brane metric is reproduced by the gauge theory on the D0-brane in the same limit. In other words what we should expect is (2.6). However as mentioned earlier the problem is that this limit gives us a region of string theory which takes us to substring scales where supergravity is not expected to be valid. Thus it is far from obvious that all graviton scattering amplitudes should be reproduced by the Matrix model.

Let us now review the argument of \[9\] in the light of the above discussion. The idea is to explain the agreement of the calculation of \[13\], \[10\] by using string theory as the interpolating theory connecting supergravity and gauge theory. In the above mentioned references the gauge theory effective action was calculated in a background corresponding to a situation in which one brane is separated from the rest by a distance \(r\) and moving.
with some velocity $v$. In terms of the variables in the gauge theory this means that a variable $U = r/l_s^2$ and $\dot{U} = v/l_s^2$ have acquired expectation values. In the limit $l_s \to 0$ with $U$ fixed, since $r \to 0$, the physical separation of the branes are below the string scale and are best described by the gauge theory. Using dimensional analysis the perturbative expansion is given by \[10\]

$$
C_{I,L}(N) g_{YM}^{2(l_s^2 - 2)} \frac{\dot{U}^I}{U^{3l_s^2 + 2(I - 2)}} = C_{I,L}(N) \frac{U^2}{g_{YM}^2} \left( \frac{g_{YM}^2 \dot{U}^2}{U^7} \right)^L \left( \frac{\dot{U}}{U^2} \right)^{I-2l_s^2-2}.
$$

(2.12)

Before we go onto discuss the argument further it is important to stress the meaning of the recently proven non-renormalization theorem\[17\] in this context. Firstly it is clear purely from the dimensional analysis that the numerical coefficient of a given $\frac{\dot{U}^I}{U^7}$ term can get a contribution only from the $L = (N - 2(I - 2))/3$ loop level. In particular this means that $\dot{U}^4/U^7$ term only gets a contribution from one loop and that the $\dot{U}^6/U^{14}$ from two loops. There is no question of renormalization of these numerical coefficients and so the agreement of these with supergravity cannot possibly be affected by going to strong coupling. Thus the non-renormalization theorem is irrelevant for the purpose of explaining this numerical agreement with supergravity. What it does tell us is that the only power of $U$ which comes with the $\dot{U}^4$ term is $U^{-7}$ and that the only one which comes with $\dot{U}^6$ is $U^{-14}$. The relation of this fact to supergravity will be discussed in the next section. The numerical agreement with supergravity still needs to be explained and this is precisely what was done in \[9\].

The corresponding open string perturbation expansion is obtained by replacing the coefficients $C_{I,L}(N)$ by functions $C_{I,L}(N,l_sU)$ and it was argued in \[9\] that $C_{I,L}(N,0) = C_{I,L}(N) \[9\]$. On the other hand for $l_sU = \frac{r}{l_s}$ greater than some critical value (say 1) the physics can be described by closed string fields. In this region one typically writes the effective action in a power series in $l_s^2$ but one may expect it to be convergent giving some effective action functional $S[g, \phi, C, l_s]$ ($C$ stands for the RR field). Now in this closed string formalism a D0-brane is represented by the action \[2.3\].

\footnote{This fact is true only for configurations such as the one being considered with some unbroken supersymmetry, see \[8\].}
In the configuration that we are considering the closed string fields have the solutions given in (2.7) to lowest order in $l^2_s$. Suppose now the solution to the exact effective action $S$ is known. This solution when plugged into (2.5) will have an expansion of the same form as (2.12) but with the coefficients $C_{I,L}(N)$ replaced by functions $C_{I,L}^{SG}(N, l_s U)$. These functions (since they are obtained from the exact action functional for closed string fields) would be analytic continuations of the corresponding power series obtained from the $\alpha'$ expansion. Thus they must be the same as $C_{I,L}(N, l_s U)$ in the region $l_s U < 1$ and in particular at $l_s U = 0$. However it turns out that the exact value of the so-called diagonal coefficients $C_{2L+2,L}(N) = C_{2L+2,L}^{SG}(N, 0)$ can be calculated simply from the leading term of the closed string expansion. To see this we first need to plug in the leading order supergravity solution into (2.5) and then take the limit $l_s \to 0$. This gives the (finite!) result\footnote{This was first observed in \cite{18}}

$$-\frac{1}{g^2_{YM}} k^{-1}(\sqrt{1 - k\hat{U}^2} - 1), \quad (2.13)$$

where $k \equiv \frac{c g^2_{YM} N}{\hat{U}^2}$ with $c$ a known constant. Now the important point is that one expects the DKPS limit of the full $\alpha'$ expansion to go over into the light like compactification of the corresponding low energy M-theory expansion (this is now a quantum M-theory expansion). But purely on dimensional grounds none of the higher derivative terms in the expansion can contribute to correcting the numerical coefficients of the “diagonal terms” which occur in the expansion of (2.13) (see \cite{10} and the discussion in the next section). Thus the analytically continued value of the diagonal functions $C_{2L+2,L}(N, l_s U)$ at the origin $l_s = 0$ are given by the leading supergravity values obtained from (2.13). This argument then explains why the supergravity calculation agrees with the loop expansion calculation in gauge theory.

Now the above argument did not actually use large $N$. This is just as well since the calculations of \cite{13}, \cite{10} were done for $N = 2$ but they still agreed with supergravity. The reason is that regardless of the value of $N$ only the leading term in the supergravity expansion contributes to the diagonal ($I = 2L + 2$) terms. Thus one does not need a
suppression of the higher powers of $R$. However there are other comparisons between the
gauge theory calculations and supergravity which involve at least two scales where finite
$N$ calculations disagree with supergravity. The classic case is the calculation of Dine and
Rajaraman [19]. In this case the argument used above does not apply directly (though
there may be a generalization of it). The reason is that in the above discussion we have
used the limit (2.2) of the probe action in a background solution of supergravity corre-
sponding to a cluster of coincident D0-branes which can be lifted to eleven dimensions
and identified with the Aichelburg-Sexl metric (averaged over the light like circle). In
the more complicated case of [19] (and also the cases considered in [20],[21]) there is no
corresponding argument whence one can regard the scattering of three gravitons to three
in terms of the action of one probe. However if recent work [22] which contradict [19] is
correct, (see also [23],[24]) there is possibly a more general argument than the one given
above that shows agreement between the finite $N$ Matrix model and arbitrary supergrav-
ity processes in a background with one light like compactified circle. On the other hand
there are processes [21] where the finite $N$ argument is definitely violated but agreement
is obtained at large $N$. This does not necessarily mean that only the large $N$ result of
the Matrix model is reliable. What it does mean is that both bound state effects and
higher order supergravity terms must be taken into account when such comparisons are
being made. The simple dimensional arguments that enabled us to conclude that only
the leading order supergravity term contributes to the diagonal terms for instance may
not be valid. In fact as we shall see in the next section agreement of even the one loop
Matrix model calculation with supergravity for the two graviton to two graviton case
requires taking into account the higher derivative terms in the supergravity side. Thus
one should not in general expect agreement with just the contributions from the Einstein
term.
3 On the non-renormalization theorem and supergravity

In order to get some perspective on this issue\footnote{I would like to acknowledge the collaboration of E. Keski-Vakkuri and P. Kraus in this section.} it is necessary to recall some history. In the BFSS paper it was stated after their observation (based on the calculation of \cite{11}) that the $v^4/r^7$ term\footnote{For convenience in comparing with standard results in the literature we have reverted back to the standard notation where $\dot{U} \rightarrow v, U \rightarrow r$.} in the Matrix model agreed with the 11D supergravity calculation of two graviton scattering at zero momentum transfer, that a non-renormalization theorem was needed in order to protect this agreement. Since there was no discussion of $R^4$ and higher derivative terms on the supergravity side the point they were making presumably was that since on the supergravity side the calculation gave only the term $v^4/r^7$ at order $v^4$ (i.e. that there are no other powers of $1/r$) this should be the only contribution in the Matrix model as well. The situation is much more complicated however, since first of all the Matrix model (or string theory) one loop calculation has an infinite number of non-vanishing terms. Thus even for agreement with the one loop Matrix model calculation one needs on the supergravity side (an infinite number of) higher derivative terms. In fact we may reverse the logic that led to the above quoted statement from BFSS and ask what restrictions the non-renormalization theorems have on the supergravity expansion.

As pointed out in \cite{25}, comparison with type II strings implies that the M-theory low energy expansion has (very schematically) the following form,

$$S \sim \sum_{r=0}^{\infty} \int_p l_3^{3r-9} \int R^{r,3r+1}. \quad (3.1)$$

The inverted commas are a reminder of the fact that in general there may be covariant derivatives as well as Riemann tensors so that the counting is in powers of squared derivatives. The first term here is the Einstein term. The second term is the by now
well-known $R^4$ derivative term,\[ t^{\mu_1 \ldots \mu_8} t^{\nu_1 \ldots \nu_8} R_{\mu_1 \nu_1} \cdots R_{\mu_7 \nu_8}, \tag{3.2}\]

Where $t$ is a rank eight tensor constructed out of the metric. It is important to note that at the eight derivative level there are no covariant derivative terms in the action.

First let us note that the structure of this series is exactly what is required for agreement with the Matrix model expansion. This is simply because the expansion is in integer powers of $l_p^3$ and therefore fits in with the expansion in $g_{YM} \equiv \frac{1}{l_m}$ since $l_m$ is to be identified with the Planck length. The contribution of the Einstein term was discussed above and it gives exactly the diagonal $I = 2L + 2$ terms in the Matrix model expansion.

The comparison with the Matrix model, of contributions from this $R^4$ term, was made in [27] (see also [28]) where the basic technique for going beyond the Einstein term was developed. Let us first briefly review their method. Write the metric as

$$ds^2 = (\eta_{\mu\nu} + \Delta_{\mu\nu})dx^\mu dx^\nu \tag{3.3}$$

where

$$\Delta_{\mu\nu} = h_{--} \delta_\mu^- \delta_\nu^- - \kappa f_{\mu\nu} \tag{3.4}$$

The first term on the right hand side is the Aichelberg-Sexl metric which is an exact solution to the string effective action (3.1) (see [11] and references therein). The second term is a small perturbation due to the probe. Thus we assume that $f << 1$ so that the metric does not change significantly. Substituting in (3.1) we keep only the quadratic terms. It is important to note that the linear terms vanish since $f = 0$ gives the Aichelburg-Sexl metric which is an exact solution to the quantum corrected equations of motion. Now for small enough $f$ we can choose the transverse traceless gauge for $f$ so that in particular ($\mu = +, -, i, \tau = x^+/2$ as in section two) only $f_{ij} \neq 0$. The contribution from the $R^4$ term is of the form (using the $SO(9) \times SO(1,1)$ symmetry of the configuration and the fact that $h$ depends only on $r = \sqrt{(x^i)^2}$) is schematically of the form $\partial_+ f \partial_+ f \partial_+ h_- \partial_+ h_-$.

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7See [25] for the original references to this.
8This seems to have been first observed in [26].
where the subscript \( \perp \) denotes transverse components. Thus we have the equation of motion,

\[
(-\partial_+ \partial_+ - \partial_+^2 + h \partial_+^2) f_{ij} + b \partial_+^4 f_{ij} \partial_+^2 h_{\perp} \partial_+^2 h_{\perp} = 0
\] (3.5)

Writing \( f \sim e^{ixp} \) we have, solving iteratively for the Routhian,\(^9\)

\[
L' = L - p_- x^- = p_+ \dot{x}^i + p_r = \frac{p_+}{h} (1 - \sqrt{1 - h_{\perp} v_{\perp}^2}) + \Delta L'
\] (3.6)

The first term here is the exact solution to the Einstein term alone and corresponds to
the diagonal terms of the Matrix model expansion as discussed in the previous section. In the case considered here we have from (3.5) the result,

\[
\Delta L' \sim \frac{p_+^4}{p_-} (\partial_+^2 h_{\perp})^2 = \frac{N^3 N_s^2 v_{\perp}^8}{R^8 r^{18}} + \ldots.
\] (3.7)

In the last step we’ve used the formulae \( p_r \sim \frac{p_-^2}{p_-} \sim p_- v_-^2 \) which are valid to leading order in \( h \) and \( p_- = \frac{N}{R} \). This term is not ruled out by the non-renormalization theorem (which only restricts the \( v^4 \) and \( v^6 \) terms). However its \( N \) depends disagrees with the naive perturbative \( N \) dependence which must go like \( N R N_s^2 \). We will find more such disagreements later and we assume that such disagreements are to be expected since bound state effects will almost certainly affect the \( N \) dependence of the perturbation series.\(^9\)

It is actually easy to see that these \( R^4 \) terms will not contribute to renormalizing
the \( v^4 \) or \( v^6 \) terms. This is because, as can be seen from (3.5), in order to maintain the \( \text{SO}(1,1) \) invariance the term must have four powers of \( \partial_+ \) and this leads to at least eight
powers of \( v \). It should be stressed that the form (3.5) obtains, because of the absence of covariant derivative terms in the \( R^4 \) term.

At this point one might wonder from whence the infinite number of non-vanishing
one-loop terms on the gauge theory side namely terms like \( v^8 / r^{15} \) etc.\(^11\) come. This term

\(^9\)This is the correct object to compute in order to compare with the gauge theory calculation as
argued in \[10\].

\(^{10}\)We wish to thank S. Sethi for discussions on this.

\(^{11}\)The coefficient of the \( v^8 / r^{11} \) vanishes in the one loop calculation and this can be explained by the
non-renormalization theorem \[17\].
clearly does not arise from the $R^4$ term so it has to come from a $R^7$ term or higher order term. It is easy to see that this term cannot come from a pure (i.e. with no covariant derivatives) term. In fact it comes from a 14 derivative term of the form $R\nabla^2 R \nabla^6 R$. This leads to a term of the form

$$\partial_\perp f_{ij} \partial_\perp \partial_\perp f_{ij} \partial_\perp h_{--} \sim \frac{N^5 N v^8}{R^7 r^{15}}$$

(3.8)

Thus we establish that in order to agree even with the one loop Matrix model result the $R^7$ expression must have covariant derivative terms (unlike the $R^4$ term). The fact that such terms must exist starting at the 14 derivative level means that there is no simple argument on the supergravity side that would correspond to the Matrix model non-renormalization theorem. To put it another way the non-renormalization theorem on the Matrix model side implies that on the supergravity side certain types of terms involving covariant derivatives are not allowed. For instance a 14 derivative term of the form $R\nabla^{10} R$ gives a term $\partial_+ f_{ik} \partial_+ f_{jk} \partial_\perp \partial_\perp h_{--}$ and this would give a contribution proportional to $v^4/r^{19}$ and hence if the Matrix model supergravity correspondence is valid, must vanish by the non-renormalization theorem [17]. Similarly a term of the form $R\nabla^8 R^2$ gives a contribution $v^6/r^{17}$ and must also be absent. In general it appears that all terms of the form $R\nabla^{6r-2} R$ and $R\nabla^{6r-4} R^2$ must be absent in order to have agreement with the Matrix model non-renormalization theorem.

4 The Matrix model and supergravity on $AdS_5 \times S_5$

Now let us try to generalize the arguments of the first part of section 2 to the case of Matrix models on torii.

The supergravity solution for an (extremal) Dp-brane is given by

$$ds^2 = H_p^{-1/2}(-dt^2 + \sum_{i=1}^{p} (dx^i)^2) + H_p^{1/2}(dr^2 + r^2 d\Omega^2_{8-p}).$$

(4.1)

for the metric with the dilaton and the RR field taking the values

$$e^{-2\phi} = g^{-2} H_p^{\nu/2}, \quad C_{0..p} = (H_p^{-1} - 1).$$

(4.2)
In the above
\[ H = 1 + \frac{Ng d_p l_s^{7-p}}{r^{7-p}}, \quad (4.3) \]
with \( \bar{d}_p \) a known \( p \) dependent constant and \( N \) the number of p-branes and \( g \) is the string coupling. In the weak coupling limit the Dp-brane is described by some non-Abelian version of the Born-Infeld action whose exact form is currently unknown. However one can take the limit \[ \alpha' \to 0, \text{with } g_{Y,M}^2 = (2\pi)^{p-2} g_s (\alpha')^{\frac{p-3}{2}} \text{fixed}. \quad (4.4) \]

Note that in this limit the gauge field \( A \) on the p-brane as well as the transverse position operator \( U \) (the 9-p dimensional scalar field on the brane which is really the transverse components of the 10 dimensional gauge field) are kept fixed. The effective dimensionless coupling constant of the gauge theory is \( g_{\text{eff}} \simeq Ng_{Y,M}^2 U^{p-3} \) and the theory is strongly coupled in the infra-red for \( p < 3 \) and is weakly coupled in the infrared for \( p > 3 \) while at \( p = 3 \) we have \( N = 4 \) super Yang-Mills which is a conformal field theory.

The same scaling may be done in the supergravity solution and gives
\[
\frac{ds^2}{l_s^2} = \frac{U^{7-p}}{g_{Y,M} \sqrt{d_p N}} (-dt^2 + \sum_{i=1}^{p} (dx^i)^2) + \frac{g_{Y,M} \sqrt{(2\pi)^{p-2} d_p N}}{U^{7-p}} dU^2 \\
+ g_{Y,M} \sqrt{(d_p N U^{p-3})} d\Omega_{8-p}^2 \quad (4.5)
\]
where \( d_p = (2\pi)^{p-2} \bar{d}_p \). These solutions are supposed to be valid if one can ignore both string loop effects and \( \alpha' \) corrections. As discussed in the second paper of \[ \text{this} \] this is possible if the following conditions are satisfied,
\[
\alpha' R \sim \frac{1}{g_{\text{eff}}} \ll 1, \quad e^{\phi} \sim \frac{g_{\text{eff}}^{7-p}}{N} \ll 1, \quad g_{\text{eff}}^2 = Ng_{Y,M}^2 U^{p-3} \quad (4.6)
\]

For the case \( p = 3 \) this metric becomes that of \( AdS_5 \times S_5 \). From such arguments (and the agreements that have been shown to exist between calculations in black hole physics and gauge theory such as those in \[ \text{this} \]) Maldacena conjectured that gauge theory the large \( N \) limit is dual in some sense to supergravity in the above background. Also including the \( O(1/g_{Y,M}^2 N) \) corrections to the strong coupling expansion in the gauge theory should
be equivalent to including the string corrections on the above supergravity background, while string loop corrections are governed by $g_{YM}^2$. Actually in this case it has been argued that there are no string correction to this background. [29] so one may even work with small $g_{YM}^2 N$.

Let us now review the Matrix model argument for relating gauge theory and gravity after compactifying on a p-torus. One starts with the $p = 0$ (D0-brane) case of the earlier discussion (see section 2). The limit one takes is the same as (4.4) for $p = 0$. As we reviewed in section 2 the theory thus obtained is then interpreted as a microscopic model of M-theory on a light like circle. Now while the limit for $p = 0$ is the same as the one taken by Maldacena [3] eqn(4.4) the interpretation in the other cases is somewhat different. On the one hand the higher dimensional branes in M-theory are supposed to be obtained as condensates of the D0-branes. Secondly the matrix theory description of M-theory compactified on a p-torus is obtained by T-dualizing the D0-brane theory [30], [1]. Let us compare the latter procedure with the above discussion of duality.

Under compactification on a p-torus (with radii $r_i$) and T-dualization,

$$r_i \rightarrow \sigma_i = \frac{l_s^2}{r_i}; \quad g \rightarrow g^{(p)} = g \frac{\prod_{i=1}^{p} l_s}{r_i} = \frac{l_s^{3-p}}{l_m^{3}} \prod \sigma_i. \quad (4.7)$$

where we have put $g_{YM}^2 \equiv 1/l_m^3$. It is important to observe that the limit $\lambda_s \rightarrow 0$ in the compactified Matrix model means in addition to (2.2) that we keep the radii of the dual torus $\sigma_i$ fixed. (This corresponds to holding $U = r/l_s^2 f i x e d$). Doing this Matrix model rescaling in the supergravity solutions we get the following:

$$H_p = 1 + \frac{N d_p \prod \sigma_i}{l_s l_m^3} \frac{1}{U^{7-p}} \rightarrow \frac{N d_p \prod \sigma_i}{l_s l_m^3} \frac{1}{X^{7-p}}. \quad (4.8)$$

where $X = l_m^3 U$. Rescaling the metric $ds^2 \rightarrow \frac{l_s^2}{l_m^2} ds^2$ we have

$$ds^2 \rightarrow \frac{X^{7-p}}{R^{7-p}} (-dt^2 + \sum_{\alpha=1}^{p} (dx^\alpha)^2) + \frac{R^{7-p}}{X^{7-p}} (dX^2 + X^2 d\Omega_{8-p}^2). \quad (4.9)$$

where

$$R^{p-7} = \frac{l_s^{3-p}}{N d_p \prod \sigma_i} \quad (4.10)$$
It is instructive to compare this in the case \( p = 3 \) to the \( AdS_5 \times S_5 \) case considered in \([3]\). For this case the above becomes (rewriting \( X \to U \) in order to conform to notation that seems to have become standard for AdS spaces),

\[
ds \to \frac{U^2}{R^2} (-dt^2 + \sum_{i=1}^{p} (dx^i)^2) + \frac{R^2}{U^2} (dU^2 + U^2 d\Omega_{8-p}^2).
\] (4.11)

This metric is locally the same as the metric (for the case \( p = 3 \)) in (4.5) but it is not the same globally. The reason is that in this Matrix model case one has actually divided out by a discrete symmetry which is a subgroup of the (apparent) translation isometry (under \( x^\alpha \to x^\alpha + a^\alpha \alpha = 0, i \)) of the above metric. However the actual (freely acting) isometry group of \( AdS_5 \) is \( SO(4,2) \). The translation isometry has a fixed point at \( U = 0 \). To see this let us it is only necessary to observe that the above coordinates of the AdS metric are ill-defined at \( U = 0 \). The \( AdS_{p+1} \) space is defined as the hyperboloid

\[-UV + (X^\alpha)^2 = -R^2.\] (4.12)

embedded in a \( p + 2 \)-dimensional space with metric

\[ds^2 = -dUdV + (dX^\alpha)^2.\] (4.13)

The metric in the form (4.11) is obtained by eliminating \( V \) and defining the coordinates \( x^\alpha = \frac{X^\alpha R}{U} \). The translation symmetry of the \( x^\alpha \) clearly have a fixed point at \( U = 0 \) and hence when dividing by a discrete subgroup of this symmetry in order to get a 3-torus one gets a singularity at \( U = 0 \). Thus the space-time metric that is related to the Matrix model on \( T_3 \) is not \( AdS_5 \times S_5 \) which is a smooth space but a space which locally looks like it away from \( U = 0 \), but has a singularity at \( U = 0 \).

However this singularity is just the point at which the moduli space approximation of the gauge theory breaks down. The singularity must in fact be replaced by full quantum non-abelian description. In contrast to the situation in the non-orbifolded case here it is unclear whether there is a holographic interpretation. The holographic interpretation in the case of \( AdS_5 \times S_5 \) comes from the ansatz of \([5]\) (see also \([4]\) for a slightly different interpretation) according to which the \( N = 4 \) superconformal field theory sits on the
boundary of the AdS space and the correlation functions of the former are obtained from the bulk supergravity by using the relation (in a Euclidean signature)

$$\int [dA] e^{-S_{\text{CFT}}[\phi_0, A]} = e^{-S[\phi]}.$$  \hfill (4.14)

The functional integral is over all gauge theory variables and $\phi$ is a classical fluctuation around the background AdS space which has boundary value $\phi_0$. The left hand side of this equation is the generating functional for connected correlation functions and $\phi_0$ is an external source which uniquely determines the bulk value $\phi$. Thus the theory in the bulk is uniquely determined by the theory on the boundary giving a holographic picture of bulk physics. It should also be noted that since the space has no singularity there is no need to have branes anywhere in the space.

By contrast in the Matrix model case the equation which replaces (4.14) is the analog of (2.6)

$$\int dX' e^{-S_{\text{MM}}[U + X']} = \lim_{\text{DKPS}} e^{-S_{\text{Sugra}}[U]}.$$ \hfill (4.15)

where (in the present case) $S_{\text{MM}}$ is the same gauge theory except it is now on a three torus and the right hand side is the supergravity representation of the probe D3 brane in the background space given by (4.11) which is singular at the origin. The latter is effectively to be replaced by the branes (i.e. the Matrix model). Clearly it is not straightforward to give this a holographic interpretation.

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References

[1] T. Banks, W. Fischler, S. Shenker, L. Susskind, Phys. Rev. D55 (1997) 5112; hep-th/9610043.

[2] I. Klebanov, Nucl. Phys. B496 (1997) 231: I. Klebanov and S. Gubser, Phys. Lett B413 (1997) 41, hep-th/9708005.

[3] J. Maldacena, “The Large N limit of Superconformal Field Theories And Supergravity”, hep-th/9711200. N. Itzhaki, J.Maldacena, J. Sonnenshein, S. Yankielowicz, “Supergravity and The Large N Limit of Theories With Sixteen Supercharges” hep-th/9802042.

[4] S. Gubser, I. Klebanov and A. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory”, hep-th/9802109.

[5] E. Witten, “Anti-de Sitter Space And Holography”. hep-th/9802150.

[6] S. Hyun, “The Background Geometry of DLCQ Supergravity”, hep-th/9802026. “Background geometry of DLCQ M theory on a p-torus and holography”, hep-th/9805136.

[7] A. Sen, “D0 Branes on $T^n$ and Matrix Theory” hep-th/9709220.

[8] N. Seiberg, Phys. Rev. Lett. 79 (1997) 3577, hep-th/9710009.

[9] S. de Alwis, ‘Phys.Lett. B423 (1998) 59, hep-th/9710219.

[10] K. Becker, M. Becker, J. Polchinski, A. Tseytlin, Phys.Rev. D56 (1997) 3174, hep-th/9706072.
[11] M. Douglas, D. Kabat, Pouliot and S. Shenker, Nucl. Phys. B485 (1997) 85, hep-th/9608024.

[12] G. Horowitz and A. Strominger, Nucl. Phys. B360, (1991) 197.

[13] P. Townsend, “Four Lectures on M-theory” hep-th/9612121.

[14] P. Aichelburg and R. Sexl, Gen. Rel. Grav. 2 (1971) 303.

[15] M. Becker and K. Becker, Nucl.Phys. B506 (1997) 48, hep-th/9705091.

[16] E. Keski-Vakkuri and P. Kraus, Nucl.Phys. B518 (1998) 212, hep-th/9709122.

[17] S. Paban, S. Sethi, M. Stern, “Constraints From Extended Supersymmetry in Quantum Mechanics”, hep-th/9805018. “Supersymmetry and Higher Derivative Terms in the Effective Action of Yang-Mills Theories”, hep-th/9806028.

[18] J. Maldacena, “Branes probing black holes”, hep-th/9709099.

[19] M. Dine and A. Rajaraman, “Multigraviton Scattering in the Matrix Model”, hep-th/9710174.

[20] M. Douglas, H. Ooguri and S. Shenker, Phys.Lett. B402 (1997) 36, hep-th/9702203

[21] D. Kabat and W. Taylor, “Spherical membranes in Matrix theory” hep-th/9711078. “Linearized supergravity from Matrix theory”, hep-th/9712183.

[22] Y. Okawa and T. Yoneya, “Multi-Body Interactions of D-Particles in Supergravity and Matrix Theory” hep-th/9806108.

[23] M. Fabbrichesi, G. Ferretti and R. Iengo, “Supergravity and matrix theory do not disagree on multi-graviton scattering” hep-th/9806018. R. Echols and J. Gray, “Comment on multigraviton scattering in the Matrix model” hep-th/9806109. J. McCarthy, L. Susskind, and A. Wilkins, “Large N and the Dine-Rajaraman problem” hep-th/9806136.
[24] W. Taylor IV and M. Van Raamsdonk, “Three-graviton scattering in Matrix theory revisited”, hep-th/9806066.

[25] J. Russo and A. Tseytlin, Nucl.Phys. B508 (1997) 245, hep-th/9707134.

[26] V. Balasubramanium, R. Gopakumar and F. Larson, “Gauge Theory, Geometry and the Large N Limit”, hep-th/9712077.

[27] E. Keski-Vakkuri and P. Kraus, “Short Distance Contributions to Graviton-Graviton Scattering: Matrix Theory versus Supergravity” hep-th/9712013.

[28] K. Becker and M. Becker, Phys.Rev. D57 (1998) 6464, hep-th/9712238.

[29] R. Kallosh and A. Rajaraman, “Vacua of M-theory and string theory” hep-th/9805041.

[30] W. Taylor, Phys.Lett. B394 (1997) 283, hep-th/9811042.