ON STRONG SOLUTIONS OF ITÔ’S EQUATIONS WITH
\( A \in W^1_d \) AND \( B \in L_d \)

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Abstract. We consider Itô uniformly nondegenerate equations with
time independent coefficients, the diffusion coefficient in \( W^1_{d,\text{loc}} \), and
the drift in \( L_d \). We prove the unique strong solvability for any starting
point and prove that as a function of the starting point the solutions are
Hölder continuous with any exponent \(< 1\). We also prove that if we are
given a sequence of coefficients converging in an appropriate sense to the
original ones, then the solutions of approximating equations converge to
the solution of the original one.

1. Introduction

Let \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space of points \( x = (x^1, \ldots, x^d) \)
with \( d \geq 3 \). Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, let \( \{\mathcal{F}_t\} \)
be an increasing filtration of \( \sigma\)-fields \( \mathcal{F}_t \subset \mathcal{F} \), that are complete. Let \( w_t \)
be a \( d_1\)-dimensional Wiener process relative to \( \{\mathcal{F}_t\} \), where \( d_1 \geq d \).

Assume that on \( \mathbb{R}^d \) we are given \( \mathbb{R}^d \)-valued Borel functions \( b, \sigma^k = (\sigma^k) \),
\( k = 1, \ldots, d_1 \). We are going to fix \( x_0 \in \mathbb{R}^d \) and investigate the equation

\[
  x_t = x_0 + \int_0^t \sigma^k(x_s) \, dw^k_s + \int_0^t b(x_s) \, ds,
\]

where and everywhere below the summation over repeated indices is under-
stood.

We are interested in the so-called strong solutions, that is solutions such
that, for each \( t \geq 0 \), \( x_t \) is \( \mathcal{F}_t^w \)-measurable, where \( \mathcal{F}_t^w \) is the completion of
\( \sigma(w_s : s \leq t) \). We present sufficient conditions for the equation to have a
strong solution and also for the solution to be unique (strong uniqueness).
A very reach literature on the weak uniqueness problem for (1.1) is beyond
the scope of this article.

After the classical work by Itô showing that there exists a unique strong
solution of (1.1) if \( \sigma^k \) and \( b \) are Lipschitz continuous (may also depend on
time and \( \omega \)), many efforts were applied to relax these Lipschitz conditions.
In case \( d = d_1 = 1 \) T. Yamada and S. Watanabe [26] relaxed the Lipschitz
condition on \( \sigma \) to the Hölder \((1/2)\)-condition (and even slightly weaker
condition) and kept \( b \) Lipschitz (slightly less restrictive). Much attention

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was paid to equations with continuous coefficients satisfying the so-called
monotonicity conditions (see, for instance, [8] and the references therein).

T. Yamada and S. Watanabe [26] also put forward a very strong theorem, basically, saying that strong uniqueness implies the existence of strong
solutions. Unlike the present paper, the majority of papers on the subject
after that time are using their theorem. S. Nakao ([18]) proved the strong
solvability in time homogeneous case if \( d = d_1 = 1 \) and \( \sigma \) is bounded away
from zero and infinity and is locally of bounded variation. He also assumed
that \( b \) is bounded, but from his arguments it is clear that the summability
of \(|b|\) suffices. In this respect his result shows that our results are also true
if \( d = 1 \). However, the general case that \( d = 2 \) is quite open.

A. Veretennikov seems to be the first who in [24] not only proved the
existence of strong solutions in the time inhomogeneous multidimensional
case when \( b \) is bounded, but also considered the case of \( \sigma^k \) in Sobolev class,
namely, \( \sigma^k_2 \in L_{2d,\text{loc}} \). He used A. Zvonkin’s method (see [29]) of transforming
the equation in such a way that the drift term disappears. X. Zhang in [27]
considered time inhomogeneous equations under some conditions which for
the time homogeneous case (our case) become \( \sigma^k x, b \in L^p \) with \( p > d \). For
more detailed information on the time inhomogeneous case we refer the
reader to [27], [28], and the references therein.

Even the case when the \( \sigma^k \)'s are constant and the process is nondegenerate
attracted very much attention especially in the time inhomogeneous setting.
We discuss some of the results in the particular case of \( b \) independent of \( t \).
M. Röckner and the author in [15] proved, among other things, the existence
of strong solutions when \( b \in L^p \) with \( p > d \). If \( b \) is bounded A. Shaposhnikov
([21], [22]) proved the so called path-by-path uniqueness, which, basically,
means that for almost any trajectory \( w \) there is only one solution (adapted
or not). This result was already announced by A. Davie before with a very
entangled proof which left many doubtful.

In a fundamental work by L. Beck, F. Flandoli, M. Gubinelli, and M. Mau-
relli ([3]) the authors investigate such equations from the points of view of
Itô stochastic equations, stochastic transport equations, and stochastic con-
tinuity equations. Their article contains an enormous amount of information
and a vast references list. We compare only those of their results which have
counterparts in the present article. In what concerns our situation they re-
tain (\( \sigma^k \) constant and the process is nondegenerate) \( b \in L_{p,\text{loc}} \) with \( p > d \)
or \( p = d \) but \( \|b\|_{L^p} \) to be sufficiently small and they prove strong solvability
and strong uniqueness (actually, path-by-path-uniqueness which is stronger)
only for almost all starting points \( x \). We assume that \( \sigma^k_2, b \in L_d \) and for
uniformly nondegenerate and bounded \( \sigma^k \) prove that, for any \( x \), equation
(1.1) has a unique strong solution.

Our approach is absolutely different from all articles mentioned above and
all articles which one can find in their references. We do not use Yamada-
Watanabe theorem or transformations of the noise. Instead, our method is
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based on an analytic criterion for the existence of strong solutions which first appeared in [25].

Simple examples of equations for which we prove the existence of unique strong solutions are

$$dx_t = (2 + I_{x_t \neq 0}) \zeta(x_t) \sin(\ln |x_t|) \, dw_t, \quad dx_t = dw_t + \zeta(x_t) (|x_t| \ln |x_t|)^{-1} \, dt,$$

where $\zeta$ is any smooth function vanishing for $|x| > 1/2$ satisfying $|\zeta| \leq 1$ and $l$ is any vector in $\mathbb{R}^d$. Observe that in the first equation the diffusion coefficient is discontinuous at the origin.

We conclude the introduction by some notation. We set $u_x = Du$ to be the gradient of $u$, $u_{xx}$ to be the matrix of its second-order derivatives,

$$D_{x^i} = D_i u = u_{x^i} = \frac{\partial}{\partial x^i} u, \quad u_{x^i \eta^j} = D_{x^i} \eta^j u = D_{x^i} D_{\eta^j} u,$$

$$\partial_t u = \frac{\partial}{\partial t} u, \quad u(\xi) = \xi^i u_{x^i}.$$

If $\sigma(x) = (\sigma^i(x))$ is vector-valued (column-vector), by $\sigma_x$ we mean the matrix whose $ij$th element is $\sigma^j_{x^i}$. If $c$ is a matrix (in particular, vector), we set $|c|^2 = \text{tr} c^* c$ (if $c$ is complex-valued).

For $p \in [1, \infty]$ by $L^p$ we mean the space of Borel (perhaps complex-vector- or matrix-valued) functions on $\mathbb{R}^d$ with finite norm given by

$$\|f\|_{L^p} = \int_{\mathbb{R}^d} |f(x)|^p \, dx.$$

By $W^{2}_p$ we mean the space of Borel functions $u$ on $\mathbb{R}^d$ whose Sobolev derivatives $u_x$ and $u_{xx}$ exist and $u, u_x, u_{xx} \in L^p$. The norm in $W^{2}_p$ is given by

$$\|u\|_{W^{2}_p} = \|u_{xx}\|_{L^p} + \|u\|_{L^p}.$$

Similarly $W^{1}_p$ is defined. As usual, we write $f \in L_{p, \text{loc}}$ if $f \zeta \in L^p$ for any $\zeta \in C^\infty_0(= C^\infty_0(\mathbb{R}^d))$. Similarly $W^{1}_{p, \text{loc}}$ are defined.

If a Borel $\Gamma \subset \mathbb{R}^d$, by $|\Gamma|$ we mean its Lebesgue measure. Finally,

$$B_R(x) = \{ y \in \mathbb{R}^d : |x - y| < R \}, \quad B_R = B_R(0).$$

2. MAIN RESULTS

Set $a^{ij} = \sigma^{ik} \sigma^{jk}$, $a = (a^{ij})$. Fix numbers $\delta \in (0, 1)$ and $\|b\|, \|\sigma^k_x\| \in (0, \infty)$.

**Assumption 2.1.** We have

$$\delta^{-1} |\lambda|^2 \geq a^{ij}(x) \lambda^i \lambda^j \geq \delta |\lambda|^2 \quad (2.1)$$

for all $\lambda, x \in \mathbb{R}^d$. Also

$$\|b\|_{L^d} \leq \|b\|.$$

**Assumption 2.2.** For any $k$ we have $\sigma^k_x \in W^{1}_{d, \text{loc}}$ and

$$\|\sigma^k_x\|_{L^d} \leq \|\sigma^k_x\|.$$


Theorem 2.3. Under the above assumptions, for any \( x_0 \in \mathbb{R}^d \), equation (1.1) has a strong solution \( x_t \). If \( y_t \) is also a solution of (1.1), then with probability one \( x_t = y_t \) for all \( t \).

Theorem 2.4. Under the above assumptions suppose that we are also given sequences \( \sigma^k(n), b(n), n = 1, 2, ..., k = 1, ..., d_1 \), of functions having the same meaning as \( \sigma, b \) and satisfying Assumptions 2.1 and 2.2 with the same \( \delta, \|b\| \) and \( \|\sigma_x^k\| \). Assume that \( b(n) \to b \) and \( \sigma^k_x(n) \to \sigma^k_x \) in \( L_d \) as \( n \to \infty \) and we are given a sequence \( x(n) \to x_0 \). Finally, let \( \sigma^k(n) \to \sigma^k \) (a.e.) as \( n \to \infty \). Then for any \( m, T \in (0, \infty) \)
\[
\lim_{n \to \infty} E \sup_{t \leq T} |x_t(n, x(n)) - x_t|^m = 0,
\]
where \( x_t(n, x(n)) \) are the solutions of (1.1) in which \( x_0, \sigma^k, \) and \( b \) are replaced by \( x(n), \sigma^k(n) \), and \( b(n) \), respectively.

Theorem 2.5. Under the above assumptions, there is a function \( x_t(x) = x_t(\omega, x) \) which for \( x = x_0 \) is a solution of (1.1) and for each \( \alpha < 1 \) and \( \omega \) is \( \alpha \)-Hölder continuous with respect to \( x \) and \( \alpha/2 \)-Hölder continuous with respect to \( t \) on each set \( [0, T] \times \bar{B}_R, T, R \in (0, \infty) \).

Remark 2.6. The main emphasis of the article is to treat the case that \( b \in L_d \). It is known (see, for instance, [2] [12]) that, even if \( d_1 = d \) and \( (\sigma^k) \) is a unit matrix, there are cases when \( b \in L_{d-\epsilon} \) for any \( \epsilon \in (0, 1) \), but not for \( \epsilon = 0 \), and there are no solutions of (1.1).

However, our results are new also if \( b \) is bounded or \( b \equiv 0 \). In that case the arguments are not so technically involved and allow any \( d \geq 1 \) rather than \( d \geq 3 \). In Remark 5.10 we show an example with \( b \equiv 0 \) and \( \sigma^k \in L_{d-\epsilon} \) for any \( \epsilon \in (0, 1) \), but not for \( \epsilon = 0 \), when there are no strong solutions. In this regard Assumption 2.2 seems to be optimal.

The rest of the article is organized as follows. As we mentioned above our main tool is an analytic criterion for the existence of strong solutions. To derive it we develop necessary facts from the theory of semigroups generated by elliptic operators in Section 3. Then in Section 4 we relate the semigroup from Section 3 to the semigroup of the corresponding Markov diffusion process. In Section 5 we derive our analytic criterion. Section 6 is devoted to some estimates of the series involved in the criterion when \( \sigma^k \) and \( b \) are smooth. In Sections 7, 8, and 9 we prove Theorems 2.3, 2.4, and 2.5, respectively.

3. An analytic semigroup

In this section Assumption 2.1 is supposed to be satisfied but Assumption 2.2 is replaced with a weaker Assumption 3.5 which comes after some discussion.

Introduce the uniformly elliptic operators
\[
Lu(x) = (1/2)u^{ij}(x)u_{x^i x^j}(x) + b^i(x)u_{x^i}(x),
\]
\[ L_0 u(x) = (1/2) a^{ij}(x) u_{x_i x_j}(x) \]

acting on functions given on \( \mathbb{R}^d \).

Remark 3.2 similar to (3.1) where implies that in Lemma 3.1 one can replace (3.1) with

\[ u \] this estimate \( \varepsilon_i D \)

Denote

\[ a^\#_r = \sup_{x \in \mathbb{R}^d} \sup_{\rho < r} \text{osc} (a, B_\rho(x)) . \]

Here is a consequence of Theorem 1 of [6]. We are dealing with complex-valued functions and denote \( \mathbb{R}^{d+1} = \{ (x^0, x^1, \ldots, x^d) : x^k \in \mathbb{R} \} \).

Lemma 3.1. For any \( p \in (1, \infty) \) and \( \varepsilon \in (0, 1] \) there exists \( \theta_0 = \theta_0(d, \delta, \varepsilon, p) \) such that, if there is \( r_0 > 0 \) for which \( a^\#_r \leq \theta_0 \), then there exist \( \lambda_0 \geq 1, N_0 \), depending only on \( d, \delta, \varepsilon, p, r_0 \), such that, for any \( u \in W^2_p \) and \( \lambda \geq \lambda_0 \),

\[
\sum_{r,s=0}^{d} \| D_{rs} u \|_{L^p(\mathbb{R}^{d+1})} + \lambda \| u \|_{L^p(\mathbb{R}^{d+1})} \leq N_0 \| L_0 u \pm \varepsilon D_0^2 u - (1 \pm \varepsilon) \lambda u \|_{L^p(\mathbb{R}^{d+1})}. \tag{3.1}
\]

Proof. As is easy to see, Theorem 1 of [6] is applicable to the operator

\[ M u := (1 \pm \varepsilon)^{-1} (L_0 u \pm \varepsilon D_0^2 u) \]

and it yields an estimate for

\[ u \in W^{1,2}_p((-\infty, 0) \times \mathbb{R}^{d+1}) = \{ u \in L^p((-\infty, 0) \times \mathbb{R}^{d+1}) : \partial_t u, \]

\[ u_x, u_{xx} \in L^p((-\infty, 0) \times \mathbb{R}^{d+1}) \}

similar to (3.1) where \( \mathbb{R}^{d+1} \) is replaced with \( (-\infty, 0) \times \mathbb{R}^{d+1} \) and \( L_0 u \pm \varepsilon D_0^2 u - (1 \pm \varepsilon) \lambda u \) is replaced with \( M u - \partial u / \partial t - \lambda u \). By substituting in this estimate \( u(x)e^t \), we get (3.1) and the lemma is proved.

Remark 3.2. Without introducing the new coordinate, Theorem 1 of [6] implies that in Lemma 3.1 one can replace (3.1) with

\[
\sum_{r,s=1}^{d} \| D_{rs} u \|_{L^p} + \lambda \| u \|_{L^p} \leq N_0 \| L_0 u - \lambda u \|_{L^p} \tag{3.2}
\]

valid for any \( u \in W^2_p \) and \( \lambda \geq \lambda_0(d, \delta, p) \) as long as \( a^\#_r \leq \theta_0(d, \delta, p) \). Since \( \| L_0 u - \lambda u \|_{L^p} \leq \| L_0 u - \mu u \|_{L^p} + |\lambda - \mu| \| u \|_{L^p} \), estimate (3.2) easily implies that in the same situation

\[
\sum_{r,s=1}^{d} \| D_{rs} u \|_{L^p} + |\mu| \| u \|_{L^p} \leq 2N_0 \| L_0 u - \mu u \|_{L^p} \tag{3.3}
\]

as long as \( |\mu| \geq \lambda_0 \) and \( |\Im \mu| \leq 2\varepsilon_0 \Re \mu \), where \( \varepsilon_0 = \varepsilon_0(d, \delta, p, r_0) > 0 \).
Lemma 3.3. For any \( p \in (1, \infty) \) there exists \( \theta_0 = \theta_0(d, \delta, p) \) such that, if there is \( r_0 > 0 \) for which \( a_{r_0}^\# \leq \theta_0 \), then there exist \( \lambda_0 \geq 1, N_0, \) depending only on \( d, \delta, p, r_0 \), such that, for any \( u \in W_p^2 \) and complex \( \lambda \) such that \( \Re \lambda \geq \lambda_0 \),

\[
\sum_{r,s=1}^d \|D_{rs}u\|_{L_p} + |\lambda||u||_{L_p} \leq N_0\|L_0u - \lambda u\|_{L_p}. \tag{3.4}
\]

Proof. We use an idea from [1]. Take a nonnegative \( \zeta \in C_0^\infty(\mathbb{R}) \) such that \( \zeta^p \) has unit integral, \( u \in W_p^2 \), and plug into (3.1) the function \( u(x)e^{ix\zeta(x)} \) and \( \varepsilon = \varepsilon_0 \). Then we get for \( \lambda \geq \lambda_0 \) and \( \mu \in \mathbb{R} \) that

\[
\sum_{r,s=1}^d \|D_{rs}u\|_{L_p} + (\lambda + \mu^2)||u||_{L_p} - N(1 + |\mu|)||u||_{L_p}
\leq N\|L_0u - [(1 \pm \varepsilon_0i)\lambda \pm \varepsilon_0\mu^2]u||_{L_p} + N(1 + |\mu|)||u||_{L_p}. \tag{3.5}
\]

Now take \( \hat{\lambda} \) such that \( \Re \hat{\lambda} \geq \lambda_0 \). If \( |\Im \hat{\lambda}| \leq 2\varepsilon_0\Re \hat{\lambda} \), we have (3.4) with \( \hat{\lambda} \) in place of \( \lambda \) thanks to (3.3).

If \( \Re \hat{\lambda} \geq 2\varepsilon_0\Re \hat{\lambda} \) set \( \lambda = \Re \hat{\lambda}, \varepsilon_0\mu^2 = \Im \hat{\lambda} - \varepsilon_0\lambda \). Then

\[|\hat{\lambda}|^2 \leq ((2\varepsilon_0)^{-2} + 1)(\Im \hat{\lambda})^2, \quad \mu^2 \leq \varepsilon_0^{-1}\Im \hat{\lambda} \leq \varepsilon_0^{-1}|\hat{\lambda}|, \quad \lambda + \mu^2 = \varepsilon_0^{-1}\Im \hat{\lambda}\]

and (3.5) with upper signs yields

\[
\sum_{r,s=1}^d \|D_{rs}u\|_{L_p} + |\hat{\lambda}||u||_{L_p} \leq N\|L_0u - \hat{\lambda}u\|_{L_p} + N(1 + |\hat{\lambda}|^{1/2})||u||_{L_p}.
\]

By increasing \( \lambda_0 \) we absorb the last term on the right into the left-hand side for \( \Re \hat{\lambda} \geq \lambda_0 \) and we come to (3.4) with \( \hat{\lambda} \) in place of \( \lambda \) if \( \Im \hat{\lambda} \geq 0 \). The case of \( \Im \hat{\lambda} \leq 0 \) is treated by using (3.5) with lower signs. The lemma is proved.

The argument in the second part of Remark 3.2 also allows us to deduce from Lemma 3.3 the following.

Lemma 3.4. Lemma 3.3 holds true if we replace the restriction \( \Re \lambda \geq \lambda_0 \) in it with \( \lambda \in \Gamma \), where \( \Gamma = \{ \Re \lambda \geq \lambda_0 \} \cup \{ \varepsilon_0|\Im \lambda| \geq -\Re \lambda + \mu_0 \} \), with \( \varepsilon_0 > 0 \) and \( \mu_0 > 0 \) which depend only on \( d, \delta, p, r_0 \).

In the rest of the section we impose the following.

Assumption 3.5 \((p, r_0)\). We have \( a_{r_0}^\# \leq \theta_0(d, \delta, p) \), where \( \theta_0 \) is taken in a way to accommodate Lemmas 3.3 and 3.4.

Remark 3.6. It is well known that if \( a_x \in L_d \), then \( a_{r_0}^\# \to 0 \) as \( r_0 \downarrow 0 \). Therefore, Assumption 3.5 is weaker than Assumption 2.2.

On the basis of Lemma 3.4 we can repeat what was done in [14] and obtain the first part of the following result about the full operator \( L \).
Theorem 3.7. Let $p \in (1,d)$. Then under Assumptions 2.1 and 3.5 there exist $\lambda_0 \geq 1, N_0$, depending only on $d, \delta, p, r_0$, and $\nu_b$ (introduced below), such that, for any $u \in W^2_p$ and $\lambda \in \Gamma$,

$$
\sum_{r,s=1}^{d} \|D_{rs}u\|_{L^p} + |\lambda|\|u\|_{L^p} \leq N\|Lu - \lambda u\|_{L^p},
$$

(3.6)

where $\nu_b$ is defined by the condition

$$
N_1\|bI| \geq \nu_b \|b\|_{L^d} \leq 1
$$

with a constant $N_1 = N_1(d, \delta, r_0, p)$. Furthermore, for any $\lambda \in \Gamma$ and $f \in L^p$ there is a unique $u \in W^2_p$ such that $\lambda u - Lu = f$.

The “existence” part of this theorem, as usual, is proved by the method of continuity.

Denote by $R_\lambda f$ the solution $u$ from Theorem 3.7. Then the fact that the norm of $R_\lambda$ as an operator in $L^p$ decreases as $N/|\lambda|$ for $\lambda \in \Gamma$ allows us to use the well-known construction introduced by Hille ([4]). We use the following facts which the reader can find, for instance, in [19]. For complex $t$ in the sector $S := \{\Re t \geq \varepsilon_0\}$ set

$$
\hat{T}_t = \frac{1}{2\pi i} \int_{\partial \Omega} e^{tz} R_z \, dz,
$$

(3.7)

where the integral is taken in a counter clockwise direction. Below in this section

$$
p \in (1,d).
$$

Theorem 3.8. (i) Formula (3.7) defines $\hat{T}_t$ in $S$ as an analytic semigroup of bounded operators in $L^p$ with norms bounded by a constant, depending only on $\varepsilon, d, \delta, r_0, p, \nu_b$, as long as $t \in \{\varepsilon t < \varepsilon_0 \Re t\}$ for any given $\varepsilon < \varepsilon_0$;

(ii) The infinitesimal generator of this semigroup is $L$ with domain $W^2_p$;

(iii) For $g \in W^2_p$ the function $\hat{T}_tg(x)$ is a unique solution of the problem

$$
\partial_t u(t, x) = Lu(t, x), \quad t > 0, \quad \lim_{t \to 0} u(t, \cdot) - g \in L^p
$$

in the class of $u$ such that $u(t, \cdot) \in W^2_p$ and (strong $L^p$-derivative) $\partial_t u(t, \cdot) \in L^p$ for each $t > 0$;

(iv) For any $T \in (0, \infty)$ there is a constant $N$, depending only on $T, d, \delta, p, r_0, \nu_b$, such that for each $t \in (0, T]$ and $f \in L^p$

$$
\|\hat{T}_t f\|_{W^2_p} \leq \frac{N}{t} \|f\|_{L^p}, \quad \|D\hat{T}_t f\|_{L^p} \leq \frac{N}{\sqrt{t}} \|f\|_{L^p}.
$$

(3.8)

Actually, the second estimate in (3.8) is not to be found explicitly in [19] but it follows by interpolation from the first one and the fact that $\|\hat{T}_t f\|_{L^p} \leq N\|f\|_{L^p}$.

We will also need a stability result before which we make the following.
Therefore, it also goes to zero and the theorem is proved.

Concerning the second

Moreover, by embedding theorems, if \( u \in W^2_p \), then for any \( x \in \mathbb{R}^d \)

\[
|u(x)| \leq N\left(\|u_{xx}\|_{L^p} + \|u\|_{L^p}\right),
\]

where \( N = N(d, p) \). By substituting here \( u(cx) \) in place of \( u(x) \) and taking minimum of the right-hand side with respect to \( c > 0 \) we come to the well-known estimate

\[
|u(x)| \leq N\|u_{xx}\|_{L^p}^{d/(2p)}\|u\|_{L^p}^{1-(d/(2p))}.
\]

Applying this and (3.8) yields that for \( t \leq T \) and any \( x \in \mathbb{R}^d \)

\[
|\hat{T}_tf(x)| \leq \frac{N}{t^{d/(2p)}}\|f\|_{L^p},
\]

where \( N \) depends only on \( T, d, \delta, p, r_0 \), and \( v_0 \).

**Theorem 3.10.** Let \( d > p > d/2 \) and let \( a_n, b_n, n = 1, 2, ..., \) have the same meaning as \( a, b \), respectively. Suppose that, for each \( n \), they satisfy Assumptions 2.1 (with the same \( \delta, \|b\| \)) and 3.5 (\( p, r_0 \)) (with the same \( \theta_0 \). Assume that \( a_n \to a \) (a.e.) and \( b_n \to b \) in \( L^d \) as \( n \to \infty \). Denote by \( \hat{T}_t^n \) the semigroups constructed on the basis of (3.7) when \( R_z \) is replaced with \( R_z^n \) that is the inverse operator to \( z - L_n \), where \( L_n = (1/2)a^nD_{ij} + b^n_iD_i \). Then for any \( t > 0 \) and \( f \in L^p \) we have \( \hat{T}_t^n f \to \hat{T}_tf \) in \( W^2_p \) and, hence, uniformly on \( \mathbb{R}^d \) as \( n \to \infty \).

Proof. It suffices to prove the convergence in \( W^2_p \) and formula (3.7) and estimate (3.6) and the dominated convergence theorem show that it suffices to prove that \( \|R^n_z f - R_z f\|_{W^2_p} \to 0 \) for \( z \in \Gamma \). In light of (3.6)

\[
\|R^n_z f - R_z f\|_{W^2_p} \leq N\|(z - L^n)(R^n_z f - R_z f)\|_{L^p} = N\|(L - L^n)R_z f\|_{L^p} \\
\leq N\|a^n - a\|\|(R_z f)_{xx}\|_{L^p} + N\|b^n - b\|\|(R_z f)_{x}\|_{L^p},
\]

where the constants \( N \) are independent of \( n \). In the last sum the first term tends to zero by the dominated convergence theorem. Concerning the second one, observe that by the Hölder and Sobolev inequalities

\[
\|b^n - b\|\|(R_z f)_{x}\|_{L^p} \leq \|b^n - b\|_{L^d}\|(R_z f)_{x}\|_{L^{pd/(d-p)}} \\
\leq N(d, p)\|b^n - b\|_{L^d}\|(R_z f)_{xx}\|_{L^p}.
\]

Therefore, it also goes to zero and the theorem is proved.
4. Relation of $\hat{T}_t$ to a diffusion process

Fix $p \in [d_0, d]$, where $d_0 = d_0(d, \delta) \in (d/2, d)$ is taken from [12], and suppose that Assumptions 2.1 and 3.5 $(p, r_0)$ are satisfied.

Define $\Omega = \mathcal{C}'((0, \infty), \mathbb{R}^d)$ and for $\omega = \omega_\cdot \in \Omega$ define $x_t(\omega) = \omega_t$. Also set $\mathcal{M}_t = N_t = \sigma(x_s : s \leq t)$. Let $X = (x_t, \infty, \mathcal{M}_t, P_x)$ be a Markov process corresponding to the operator $L$ constructed in [12] (we need only Assumptions 2.1 for that). We know from [12] that, for each $x_0 \in \mathbb{R}^d$, with $P_{x_0}$-probability one

$$x_t = x_0 + \int_0^t \sqrt{a(x_s)} dB_s + \int_0^t b(x_s) ds,$$

where $B_t$ is a $d$-dimensional Wiener process on $\Omega$ relative to $\mathcal{N}_t^{x_0}$ that are the completions of $\mathcal{N}_t$ with respect to $P_{x_0}$.

Lemma 4.1. There is an extension of the probability space $(\Omega, N^{x_0}_\infty, P_{x_0})$ that carries a $d_1$-dimensional Wiener process $w_t$ such that the above $x_t$ satisfies (1.1).

Proof. Enlarge the probability space $(\Omega, N^{x_0}_\infty, P_{x_0})$ in such a way that it will carry a $d_1$-dimensional Wiener process $\hat{B}_t$ the first $d$ coordinate of which coincide with those of $B_t$. Then introduce $\hat{\sigma}^k = a^{1/2} \sigma^k$ and observe that, for each $x$, the vectors $\xi_i(x) = (\hat{\sigma}^{i1}(x), ..., \hat{\sigma}^{id_1}(x))$, $i = 1, ..., d$, are orthogonal to each other and have unit length. By using the Gram-Schmidt procedure it is not hard to complement them in such a way that $\xi_i(x)$, $i = 1, ..., d_1$, are orthogonal to each other, have unit length, and are Borel with respect to $x$. In that case the matrix $Q(x)$, having as rows the $\xi_i(x)$'s, is orthogonal and

$$w_t := \int_0^t Q^*(x_s) \ d\hat{B}_s$$

is a $d_1$-dimensional Wiener process. After that it only remains to note that $\hat{\sigma}^k Q^{rk} = e_r I_{r \leq d}$, where $e_r$ is the $r$th basis vector, so that

$$\sigma^k(x_s) \ dw^k_s = a^{1/2}(x_s) \sigma^k(x_s) \ Q^{rk}(x_s) \ d\hat{B}^r_s = a^{1/2}(x_s) \ dB_s.$$ 

The lemma is proved.

We know from [14] that under Assumption 3.5 $(p, r_0)$ solutions of (1.1) are weakly unique and therefore talking about the properties of solutions of (1.1) we may use some results from [13] about the process $X$.

The following result regarding $X$ is taken from [13].

Lemma 4.2. Denote

$$T_t f(x) = E_x f(x_t).$$

Then (Theorem 4.8 of [13]) for any $q \geq d_0$ there are constants $N$ and $\mu > 0$, depending only on $d, q, \delta$, and $\|b\|$, such that for any Borel nonnegative $f$ given on $\mathbb{R}^d$ and $t > 0$ we have

$$T_t f(0) \leq N \ t^{-d/(2q)} \|\Phi(t)\|_{L_q},$$

(4.2)
where $\Phi_t(x) = \exp(-\mu|x|/\sqrt{t})$. Furthermore (Corollary 4.9 of [13]), for $q \geq d_0$ such that $q > d/2 + 1$ there exists a constant $N = N(q, d, \delta, \|b\|)$ such that for any $T \in (0, \infty)$ and nonnegative Borel $f(t, x)$ given on $[0, T] \times \mathbb{R}^d$ we have

$$E_0 \int_0^T f(t, x_t) dt \leq NT^{(q-1)/q-d/(2q)}\|\Phi_T f\|_{L_\infty([0, T] \times \mathbb{R}^d)}. \tag{4.3}$$

Finally (Lemma 6.4 of [13] and (4.2)), $T_t f(x)$ is a continuous (even locally Hölder continuous) function of $(t, x) \in (0, \infty) \times \mathbb{R}^d$ if $f \in L_q$ with $q \geq d_0$.

Note that if $u \in W_q^{1,2}([0, T] \times \mathbb{R}^d)$ and $q > d/2 + 1$, then $u$ has a modification which is bounded and continuous on $[0, T] \times \mathbb{R}^d$. Therefore, talking about $u$ of class $W_q^{1,2}([0, T] \times \mathbb{R}^d)$ we will always mean this modification.

**Theorem 4.3** (Itô’s formula). Let $q \geq d_0$ and $q > d/2 + 1$, and let $u \in W_q^{1,2}([0, T] \times \mathbb{R}^d)$. Then with probability one for all $t \in [0, T]$ we have

$$u(t, x_t) = u(0, x_0) + \int_0^t (\partial_t + L) u(s, x_s) ds + \int_0^t \sigma^i_k D_i u(s, x_s) dw^k_s, \tag{4.4}$$

where the stochastic integral is a square integrable martingale on $[0, T]$ (and $x_t$ is a solution of (1.1)).

This theorem is proved by using (4.3) in the same way as Theorem 1.3 of [12] is proved on the basis of Theorem 2.6 of [12].

Recall that $p \in [d_0, d)$ and $d_0 \in (d/2, d)$, so that there are values of $p > d/2 + 1$ since $d \geq 3$.

**Theorem 4.4.** Let $p > d/2 + 1$, $T \in (0, \infty)$, and $f \in L_p \cap L_{2p}$. Then

(i) For each $t > 0$ and $x \in \mathbb{R}^d$, we have $T_t f(x) = T_t f(x)$;

(ii) For each $t > 0$, for solutions of (1.1), with probability one we have

$$f(x_t) = T_t f(x_0) + \int_0^t \sigma^i_k D_i T_{t-s} f(x_s) dw^k_s, \tag{4.5}$$

where $\sigma^i_k D_i T_{t-s} f(x) = (\sigma^i_k D_i T_{t-s} f)(x)$ and similar notation is also used below;

(iii) For each $t > 0$ and $x \in \mathbb{R}^d$

$$T_t f^2(x) = (T_t f(x))^2 + \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^i_k D_i T_{t-s} f \right)^2 \right](x) ds. \tag{4.6}$$

Proof. If $f \in W_p^2$, then $u(s, x) := \hat{T}_{t-s} f(x)$, $s \leq t$, satisfies the condition of Theorem 4.3 and we get (4.5) with $\hat{T}$ in place of $T$, by that theorem. By taking the expectations of both sides we get that $T_t f(x_0) = \hat{T}_t f(x_0)$. This holds for any $x_0$ and yields (4.5) as is. By taking the expectations of the squares of both sides of (4.5) we obtain (4.6). Thus, all assertions of the theorem are true if $f \in W_p^2$. 


Assertion (i) holds for any \( f \in L^p_p \), which is seen from the fact that according to embedding theorems and (4.2) both \( \hat{T}_t f(x) \) and \( T_t f(x) \) are bounded linear functionals on \( L^p_p \) and \( W^2_p \) is dense in \( L^p_p \).

Then, as \( f^n \in W^2_p \) tend to \( f \) in \( L^p_p \cap L^2_p \), \( T_{t-s} f^n \to T_{t-s} f \) in \( W^2_p \) for \( s < t \) (see (3.8)). By embedding theorems \( (p \geq d/2) DT_{t-s} f^n \to DT_{t-s} f \) in \( L^2_p \) and in light of (4.2)

\[
T_s \left[ \sum_i \sigma^{ik} D_i T_{t-s} f^n \right]^2 (x) \to T_s \left[ \sum_i \sigma^{ik} D_i T_{t-s} f \right]^2 (x)
\]
for any \( 0 < s < t \) and \( x \in \mathbb{R}^d \). Furthermore, \( (f^n)^2 \to f^2 \) in \( L^p_p \) and, due to (4.2), \( T_t (f^n)^2 (x) \to T_t f^2 (x) \). It follows by Fatou’s lemma (and (4.6)) that

\[
T_t f^2 (x) \geq (T_t f(x))^2 + \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^{ik} D_i T_{t-s} f \right)^2 \right] (x) \, ds. \quad (4.7)
\]

Hence, the right-hand side of (4.5) is well defined. Furthermore,

\[
E \left| \int_0^t \sigma^{ik} D_i T_{t-s} f(x_s) \, dw^k_s - \int_0^t \sigma^{ik} D_i T_{t-s} f^n(x_s) \, dw^k_s \right|^2
\]

\[
= \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^{ik} D_i (f - f^n) \right)^2 \right] (x_0) \, ds
\]

\[
\leq T_t (f - f^n)^2 (x_0) - (T_t (f - f^n)(x_0))^2,
\]

where the inequality is due to (4.7). The last expression tends to zero, which allows us to get (4.5) by passing to the limit in its version with \( f^n \) in place of \( f \). After that (4.6) follows as above. The theorem is proved.

**Remark 4.5.** In light of Theorem 4.4 (i) estimate (4.2) is weaker in what concerns the restriction on \( q \) than (3.9). However, (4.2) is proved for just measurable \( \sigma^k \).

5. **A criterion for strong solutions of Itô’s equations**

In this section

\[
p \in (d_0, d), \quad p > d/2 + 1
\]

and we suppose that Assumptions 2.1 and 3.5 \((p, r_0)\) are satisfied.

Recall the setting from the beginning of the article. We are given a complete probability space \((\Omega, \mathcal{F}, \mathcal{P})\) with an increasing filtration of \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F} \), that are complete. We are also given a \(d_1\)-dimensional Wiener process \(w_t\) relative to \(\mathcal{F}_t\). Finally, for an \(x_0 \in \mathbb{R}^d\) we are given a solution \(x_t\) of (1.1). We know from Lemma 4.1 that such a situation is quite realistic and we also know that the solution is weakly unique. In particular, it has the same distributions as the process \(x_t\) from Section 4 has relative to \(\mathcal{P}_{x_0}\).

For further discussion we need the following, in which \(\mathcal{P}\) is the \(\sigma\)-field of predictable sets and \(\mathcal{B}(0, \infty)\) is the Borel \(\sigma\)-field in \((0, \infty)\).
Lemma 5.1. Assume that for \( s, r \in (0, \infty), \omega \in \Omega \) we are given a real-valued function \( g(s, r) = g(s, r, \omega) \), \( s \in (0, \infty) \), \( (r, \omega) \in (0, \infty) \times \Omega \) which is measurable with respect to \( \mathcal{B}(0, \infty) \otimes \mathcal{P} \) and such that for each \( s \)

\[
E \int_0^\infty g^2(s, r) \, dr < \infty.
\]

Then there is a function \( m_{s,t} = m(s, t, \omega) \) on \([0, \infty) \times (0, \infty) \times \Omega\) measurable with respect to \( \mathcal{B}(0, \infty) \otimes \mathcal{P} \), continuous in \( t \) for each \((s, \omega)\) and such that for each \( s \) it is martingale starting from zero and, moreover, for each \( s \) (a.s.)

\[
m_{s,t} = \int_0^t g(s, r) \, dw_r. \tag{5.1}
\]

Proof. Introduce

\[
\Omega_s = \{ \omega : \int_0^\infty g^2(s, r) \, dr < \infty \}, \quad \hat{g}(s, r) = I_{\Omega_s} g(s, r),
\]

\[
B_t(s) = \int_0^t \hat{g}^2(s, r) \, dr.
\]

Observe that \( P(\Omega_s) = 1 \) so that \( \Omega_s \in \mathcal{F}_0 \). Also \( B_\infty(s) < \infty \) for any \( s \) and \( \omega \).

By Lemma 2.6 of [10] there exists a function \( m_{s,t} \) on \([0, \infty) \times (0, \infty) \times \Omega\) with the properties described in the statement of the lemma but satisfying (5.1) with \( \hat{g} \) in place of \( g \). Since \( P(\Omega_s) = 1 \) the integrals of \( \hat{g} \) and \( g \) coincide with probability one and the lemma is proved.

Remark 5.2. As we have noted if \( f \in L_p \), then, for any \( t > 0 \), we have \( T_t f \in W^2_p \), and hence \( (p > d/2) \), \( DT_t f \in L_p \cap L^2_p \). Therefore, we can apply

Theorem 4.4 and write that for any \( s < t \) (a.s.)

\[
\sigma^{ik} D^k T_{t-s} f(x_s) = T_s(\sigma^{ik} D^k T_{t-s} f)(x_0)
\]

\[
+ \int_0^s \sigma^{im} D^j T_{s-r} (\sigma^{ik} D^k T_{t-s} f)(x_r) \, dw_r^m. \tag{5.2}
\]

After that we want to substitute the result into (4.5) to get

\[
f(x_t) = T_t f(x_0) + \int_0^t T_s(\sigma^{ik} D^k T_{t-s} f)(x_0) \, dw_s^k
\]

\[
+ \int_0^t \left( \int_0^s \sigma^{im} D^j T_{s-r} (\sigma^{ik} D^k T_{t-s} f)(x_r) \, dw_r^m \right) dw_s^k. \tag{5.3}
\]

The formal objection to do that is that we should know that the integral in (5.2) is, for instance, predictable as a function of \((\omega, s)\) and this may not happen if we allow any version of the stochastic integral to be taken for each \( s \). However, set \( h^k(s, x) = I_{s < t} \sigma^{ik} D^k T_{t-s} f(x) \) and consider

\[
I^k(s, u) = \int_0^u I_{r < s} \sigma^{im} D^j T_{s-r} h^k(s, \cdot)(x_r) \, dw_r^m. \tag{5.4}
\]
This is the sum over \( m \) of stochastic integrals and

\[
E \int_0^\infty I_{r<s} \left| \sum_j \sigma^{jm} D_j T_{s-r} h^k(s, \cdot)(x_r) \right|^2 \, dr
\]

\[
= E \int_0^s \left| \sum_j \sigma^{jm} D_j T_{s-r} h^k(s, \cdot)(x_r) \right|^2 \, dr \leq T_s \left( (h^k(s, \cdot))^2 \right)(x_0),
\]

where the inequality is due to (4.6). It follows from Lemma 5.1 that \( I(s, u) = I(s, u, \omega) \) has a version which we denote again \( I(s, u) \), that is continuous in \( u \) for each \( s, \omega \) and measurable with respect to \( \mathcal{B}(0, \infty) \otimes \mathcal{P} \). Then \( I^k(s, s) \) is predictable and we take this modification of the right-hand side of (5.4) in the right-hand side of (5.3) thus justifying (5.3).

Then we want to repeat this procedure. Introduce

\[
Q^k_t f(x) = \sigma^{ik}(x) D_i T_t f(x).
\]

In this notation (4.5) and (5.3) become, respectively,

\[
f(x_t) = T_t f(x_0) + \int_0^t Q^{k_1}_{t-t_1} f(x_{t_1}) \, dw_{t_1}^{k_1};
\]

\[
f(x_t) = T_t f(x_0) + \int_0^t T_{t_1} Q^{k_1}_{t-t_1} f(x_0) \, dw_{t_1}^{k_1}
\]

\[
+ \int_0^t \left( \int_0^{t_1} Q^{k_2}_{t_1-t_2} Q^{k_1}_{t-t_1} f(x_{t_2}) \, dw_{t_2}^{k_2} \right) dw_{t_1}^{k_1}.
\]

By induction we obtain that for any \( n \geq 1 \) (a.s.) for all \( t \geq 0 \) (\( t_0 = t \))

\[
f(x_t) = T_t f(x_0) + \sum_{m=1}^n \int_{t > t_1 > \ldots > t_m} T_{t_m} Q^{k_m}_{t_m-t_{m-1}} \ldots Q^{k_2}_{t_2-t_1} f(x_0) \, dw_{t_m}^{k_m} \ldots dw_{t_1}^{k_1}
\]

\[
+ \int_{t > t_1 > \ldots > t_{n+1}} Q^{k_{n+1}}_{t_{n+1}-t_n} \ldots Q^{k_1}_{t_1-t_1} f(x_{t_{n+1}}) \, dw_{t_{n+1}}^{k_{n+1}} \ldots dw_{t_1}^{k_1},
\]

where by the expressions like

\[
\int_{t > t_1 > \ldots > t_m} \cdots \, dw_{t_m}^{k_m} \cdots dw_{t_1}^{k_1}
\]

we mean

\[
\int_0^t dw_{t_1}^{k_1} \int_0^{t_1} dw_{t_2}^{k_2} \ldots \int_0^{t_{m-1}} \cdots \, dw_{t_m}^{k_m}.
\]

By taking expectations of the squares of the sides in (5.6) we conclude that

\[
T_t f^2(x_0) = (T_t f(x_0))^2
\]

\[
+ \sum_{m=1}^n \int_{t > t_1 > \ldots > t_m} \left[ T_{t_m} Q^{k_m}_{t_m-t_{m-1}} \ldots Q^{k_1}_{t_1-t_1} f(x_0) \right]^2 \, dt_m \cdots dt_1
\]

\[
+ \int_{t > t_1 > \ldots > t_{n+1}} \sum_{k_1, \ldots, k_{n+1}} T_{t_{n+1}} \left[ Q^{k_n}_{t_{n+1}-t_n} \ldots Q^{k_1}_{t_1-t_1} f \right]^2(x_0) \, dt_{n+1} \cdots dt_1.
\]
In particular, the sequence of
\[ \int_{t>1 > \ldots > t_n} \sum_{k_1, \ldots, k_n} T_{t_n} \left[ Q_{n-1-t_n}^{k_n} \ldots Q_{t-t_1}^{k_1} f \right]^2 (x_0) \, dt_n \ldots dt_1 \]
is decreasing.

**Remark 5.3.** It turns out that proving directly that each term in the right-hand side of (5.7) is finite presents significant difficulties. However, observe that, due to (3.8) and (4.2), for \( p \in (d_0, d) \) and \( f \in L_p \) we have
\[ |T_m Q_{m-1-t_m}^{k_m} \ldots Q_{t-t_1}^{k_1} f(x)| \leq \frac{N}{t_m^{d/(2p)} (t_{m-1} - t_m)^{1/2} \ldots (t - t_1)^{1/2}} \| f \|_{L_p}, \]
where \( N \) depends only on \( m, d, \delta, \| b \|, \) and \( \nu_0 \). Furthermore,
\[ \int_{t>1 > \ldots > t_m} \sum_{k_1, \ldots, k_m} T_{t_m} Q_{m-1-t_m}^{k_m} \ldots Q_{t-t_1}^{k_1} f(x_0) \, dw_{t_m}^{k_m} \ldots dw_{t_1}^{k_1} < \infty. \]

Recall that \( F_t^w \) is the completion of \( \sigma(w_s : s \leq t) \). Remark 5.2 allows us to repeat literally some arguments in [25] and leads to the following results.

**Theorem 5.4.** Let \( f \in L_p \cap L_{2p}, t > 0. \) Then
\[ E(f(x_t) \mid F_t^w) = T_t f(x_0) \]
\[ + \sum_{m=1}^{\infty} \int_{t>1 > \ldots > t_m} T_{t_m} Q_{m-1-t_m}^{k_m} \ldots Q_{t-t_1}^{k_1} f(x_0) \, dw_{t_m}^{k_m} \ldots dw_{t_1}^{k_1}, \]
where the series converges in the mean square sense.

For \( n \geq 1, t > 0, \) and \( s_1, \ldots, s_n > 0 \) define
\[ Q_{s_1, \ldots, s_n} f(x) = \sum_{k_1, \ldots, k_n} \left[ Q_{s_n}^{k_n} \ldots Q_{s_1}^{k_1} f \right]^2 (x). \quad (5.8) \]

**Theorem 5.5.** Let \( f \in L_p \cap L_{2p}, t_0 > 0. \) Then \( f(x_{t_0}) \) is \( F_{t_0}^w \)-measurable iff
\[ \lim_{n \to \infty} \int_{t_0>1 > \ldots > t_n} T_{t_n} Q_{n-1-t_n, \ldots, 0-t_1} f(x_0) \, dt_n \ldots dt_1 = 0. \quad (5.9) \]
Furthermore, under either of the above equivalent conditions
\[ f(x_t) = T_t f(x_0) \]
\[ + \sum_{m=1}^{\infty} \int_{t>1 > \ldots > t_m} T_{t_m} Q_{m-1-t_m}^{k_m} \ldots Q_{t-t_1}^{k_1} f(x_0) \, dw_{t_m}^{k_m} \ldots dw_{t_1}^{k_1}. \quad (5.10) \]

**Theorem 5.6.** If equation (1.1) has two solutions which are not indistinguishable, then it does not have any strong solution. In particular, if (1.1) has at least one strong solution, then the solution is unique.

**Theorem 5.7.** If equation (1.1) has a strong solution on one probability space then it has a strong solution on any other probability space carrying a \( d_1 \)-dimensional Wiener process.
Remark 5.8. By making the change of variables \( t_k = s_k + \ldots + s_n, \ k = 1, \ldots, n \) we rewrite (5.9) as

\[
\lim_{n \to \infty} \int_{S_n(t_0)} T_{s_n} Q_{s_{n-1}, \ldots, s_1, t_0 - (s_1 + \ldots + s_n)} f(x_0) \, ds_n \cdot \ldots \cdot ds_1 = 0, \quad (5.11)
\]

where \( S_n(t_0) = \{(s_1, \ldots, s_n) : s_k \geq 0, s_1 + \ldots + s_n < t_0\} \)

The sequence under the limit sign in (5.11), call it \( u_n(t_0) \), is decreasing for any \( t_0 \) (and \( x \)). Therefore, its limit will be zero for almost any \( t \) if

\[
\lim_{n \to \infty} \int_0^\infty u_n(t_0)e^{-\nu t_0} \, dt_0 = 0,
\]

where \( \nu > 0 \) is any number. In that case, actually, the limit of \( u_n(t_0) \) is zero for all \( t_0 \), since, in light of Theorem 5.5, \( f(x_0) \) is \( \mathcal{F}^u_{t_0} \)-measurable for almost all \( t_0 \), and by continuity, for all \( t_0 \). In this way after simple change of variables we come to the following.

**Theorem 5.9.** Let \( f \in L_p \cap L_{2p} \). Then \( f(x_t) \) is \( \mathcal{F}^u_t \)-measurable for any \( t > 0 \) if there exists a \( \nu > 0 \) such that

\[
\int_0^\infty e^{-\nu s_n} T_{s_n} \left( \int_{R^+_p} e^{-\nu(s_1 + + \ldots + s_0)} Q_{s_{n-1}, \ldots, s_0} f(x) \, ds_{n-1} \cdot \ldots \cdot ds_1 \right) (x_0) \, ds_n \to 0
\]

as \( n \to \infty \), where \( R^+_p = (0, \infty)^n \), which in light of (4.2) holds for any \( x_0 \) if

\[
\left\| \int_{R^+_p} e^{-\nu(s_1 + + \ldots + s_0)} Q_{s_{n-1}, \ldots, s_0} f(x) \, ds_{n-1} \cdot \ldots \cdot ds_1 \right\|_{L_p} \to 0. \quad (5.13)
\]

We are going to prove that (5.13) holds under Assumptions 2.1 and 2.2 by showing that the series composed of the left-hand sides of (5.13) converges.

**Remark 5.10.** The criterion (5.9) is proved under Assumptions 2.1 and 3.5 (\( p, r_0 \)), assumptions, which involve the \( \sigma^k \)'s only implicitly and it turns out that for some choice of the \( \sigma^k \)'s (5.9) may hold and for another fail to hold. To illustrate this we take \( b \equiv 0 \). In that case the restriction \( p < d \) disappears along with \( d \geq 3 \) (which is a consequence of \( p < d \) and \( p > d/2 + 1 \)). Then we take \( d_1 = d = 2 \) and following [16] set \( \sigma^1(x) = x/|x|, \ \sigma^2(x) = x^*/|x| \), where \( x^* = (-x^2, x^1) \) for \( x \neq 0 \), \( \sigma^{1k}(0) = \delta^{ik} \). Then \( \sigma^j(x) = \delta^j \), equation (1.1) has a solution for any \( x_0 \) (see, for instance, Lemma 4.1), and each solution is a Wiener process starting from \( x_0 \). For \( x_0 \neq 0 \) the solutions are strong and, hence, (5.9) holds, because the solution never reaches the origin, the point where \( \sigma^k \) are not smooth. However, for \( x_0 = 0 \) there are no strong solutions, because, as is easy to see, rotation in \( x^1 x^2 \) coordinates by any angle brings any solution it to another solution of the same equation. Therefore, for \( x_0 = 0 \) equation (5.9) does not hold.

Also observe that in this example \( \sigma^k \in W^1_{d-2-\varepsilon, \text{loc}} \) for any \( \varepsilon \in (0, 1) \) but not for \( \varepsilon = 0 \). One can construct similar examples for \( d \geq 3 \) starting from
the following with \( d = 3, d_1 = 9, \) and \( \sigma^k \)'s that are the \( k \)th columns of the matrix

\[
\frac{1}{|x|} \begin{pmatrix}
  x^1 & x^2 & x^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & x^1 & x^2 & x^3 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & x^1 & x^2 & x^3
\end{pmatrix}.
\]

Again \( a^{ij} = \delta^{ij} \), \( \sigma^k \in W^{1,\infty}_{d-\varepsilon, \text{loc}} \) for any \( \varepsilon \in (0,1) \) but not for \( \varepsilon = 0 \), and, if \( x_t \) is a solution of (1.1) with \( x_0 = 0 \), then \(-x_t\) is also a solution of (1.1) with \( x_0 = 0 \).

### 6. Some Estimates in the Case of \( C^\infty \) Coefficients

We suppose that \( \sigma^k, b \) satisfy Assumption 2.1 and are infinitely differentiable with each derivative bounded.

Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, let \( \{\mathcal{F}_t\} \) be an increasing filtration of \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \), that are complete. Let \( w_t \) be a \( d_1 \)-dimensional Wiener process relative to \( \mathcal{F}_t \). We also assume that there is a \( (d + 1) \)-independent \( d \)-dimensional Wiener, relative to \( \{\mathcal{F}_t\} \), process \( B_t^{(0)}, \ldots, B_t^{(d)} \) independent of \( w_t \). Take \( x, \eta \in \mathbb{R}^d \), a nonnegative bounded infinitely differentiable \( K_0 \) with each derivative bounded given on \( \mathbb{R}^d \), and consider the following system

\[
x_t = x + \int_0^t \sigma^k(x_s) \, dw^k_s + \int_0^t b(x_s) \, ds, \quad (6.1)
\]

\[
\eta_t = \eta + \int_0^t \sigma^k(\eta_s) \, dw^k_s + \int_0^t b(\eta_s)(x_s) \, ds \\
+ \int_0^t K_0(x_s) \, dB^{(0)}_s + \int_0^t K_0(x_s) \eta_s^k \, dB^{(k)}_s. \quad (6.2)
\]

As is well known, (6.1) has a unique solution which we denote by \( x_t(x) \). By substituting it into (6.2) we see that the coefficients of (6.2) grow linearly in \( \eta \) and hence (6.2) also has a unique solution which we denote by \( \eta_t(x, \eta) \). By the way observe that equation (6.2) is linear with respect to \( \eta_t \). Therefore \( \eta_t(x, \eta) \) is an affine function of \( \eta \). For the uniformity of notation we set \( x_t(x, \eta) = x_t(x) \). It is also well known (see, for instance, Sections 2.7 and 2.8 of [7]) that, as a function of \( x \) and \( (x, \eta) \), the processes \( x_t(x) \) and \( \eta_t(x, \eta) \) are infinitely differentiable in an appropriate sense (specified below), their derivatives satisfy the equations which are obtained by formal differentiation of (6.1) and (6.2), respectively, and, for any \( n \geq 0, T \in (0, \infty), \ell_k, \xi_k \in \mathbb{R}^d, k = 1, \ldots, n \) (if \( n \geq 1 \)), \( x, \eta \in \mathbb{R}^d \), and \( q \geq 1 \),

\[
E \sup_{t \leq T} \left| \prod_{k=1}^n (lb) D_{(\ell_k, \xi_k)} (x_t, \eta_t)(x, \eta) \right|^q \leq N(1 + |\eta|^m), \quad (6.3)
\]

where \( N \) is a certain constant independent of \( (x, \eta) \), \( m = m(n, q) \), and, for instance, by \((lb) D_{(\ell, \xi)} \eta_t(x, \eta)\) we mean a process \( \zeta_t \) such that, for any \( q \geq 1 \)
and \( S \in (0, \infty) \)
\[
\lim_{\varepsilon \downarrow 0} E \sup_{t \leq S} \left| \xi_t - \varepsilon^{-1} \left( \eta_t(x + \varepsilon t, \eta + \varepsilon \xi) - \eta_t(x, \eta) \right) \right|^q = 0.
\]

**Lemma 6.1.** Let \( f(x, \eta) \) be infinitely differentiable and such that each of its derivatives grows as \(|x| + |\eta| \to \infty\) not faster than polynomially. Then the function \( u(t, x, \eta) := Ef((x_t, \eta_t)(x, \eta)) \) is infinitely differentiable in \((x, \eta)\) and each of its derivatives is continuous in \( t \) and is by absolute value bounded on each finite time interval by a constant times \((1 + |x| + |\eta|)^m\) for some \( m \).

Furthermore, \( u(t, x, \eta) \) is continuously differentiable in \( t \) and for \( t \geq 0 \) and \((x, \eta) \in \mathbb{R}^{2d} \)
\[
\partial_t u(t, x, \eta) = \left( \frac{1}{2} \right) \sigma^{ik} \sigma^{jk}(x) u_{x^i x^j}(t, x, \eta) + \sigma^{ik} \sigma^{jk}_{(\eta)}(x) u_{x^i \eta^j}(t, x, \eta)
\]
\[
+ \left( \frac{1}{2} \right) \sigma^{ik}_{(\eta)} \sigma^{jk}(x) u_{\eta^i \eta^j}(t, x, \eta) + \left( \frac{1}{2} \right) K_0^2(x)(1 + |\eta|^2) \delta^{ij} u_{\eta^i \eta^j}(t, x, \eta)
\]
\[
+ b_i u_{x^i}(t, x, \eta) + b_{i(\eta)}(x) u_{\eta^i}(t, x, \eta) =: \tilde{L}(x, \eta) u(t, x, \eta). \quad (6.4)
\]

The first assertion of this lemma follows easily from what is said before it. Then the fact that (6.4) holds follows from the Markov property of \((x_t, \eta_t)\) and from the first assertion. The claimed property of \( \partial_t u \) follows from (6.4).

**Lemma 6.2.** Let \( \eta \in \mathbb{R}^d \) and \( \xi_t(x, \eta) = (lb)D_\eta x_t(x) \). Then

(i) \( \xi_t(x, \eta) \) satisfies (6.2) with \( K_0 \equiv 0 \).

(ii) For any \( t \)
\[
\xi_t(x, \eta) = E(\eta_t(x, \eta) \mid \mathcal{F}_t^w). \quad (6.5)
\]

(iii) If \( f(x) \) is infinitely differentiable with bounded derivatives, then
\[
Ef_{(\eta_t(x, \eta))}(x_t(x)) = Ef(x_t(x)) \quad (6.6)
\]

Proof. Assertion (i) is well known (see, for instance, [7]). The right-hand side of (6.5) satisfies (6.2) with \( K_0 \equiv 0 \) owing to the linearity of \( g(\eta) \) in \( \eta \) and independence of \( B \) and \( w \). Therefore, due to uniqueness, assertion (ii) follows from (i). Assertion (iii) follows from (ii) and the fact that (see, for instance, [7])
\[
(Ef(x_t(x)))_{(\eta)} = Ef_{(\xi_t(x, \eta))}(x_t(x)).
\]

The lemma is proved.

Now follows one of the most important computations. The idea behind it is the following. If we formally differentiate both parts of (5.10) in the direction \( \eta \) and then take the expectations of the squares of both sides, then we obtain an equality in (6.7) below if we also replace on the left \( \eta_t(x, \eta) \) by \( \xi_t(x, \eta) \). Then the inequality follows from Lemma 6.2.

**Lemma 6.3.** Let \( x, \eta \in \mathbb{R}^d \) and let \( f(x) \) be infinitely differentiable with bounded derivatives. Then for any \( t \in (0, \infty) \) \((t_0 = t)\)
\[
Ef_{(\eta_t(x, \eta))}(x_t(x))^2 \geq \left[ (T_t f(x))_{(\eta)} \right]^2
\]

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+ \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_n} \int_{t_0 > \cdots > t_n} \left( T_{t_n} Q_{t_{n-1} - t_n}^{k_n} \cdots Q_{t_{t_1} - t_n}^{k_1} f(x) \right) \eta \ dt_n \cdots dt_1. \quad (6.7)

Proof. Introduce the notation \( \tilde{T}_t u(x, \eta) = Eu(\{(x, \eta t) \}(x, \eta)) \). Then, similarly to (4.5), for smooth bounded \( u(x, \eta) \) by dropping for simplicity the arguments \( x \) and \( \eta \) in \( x_s(x, \eta) \), we get

\[
 u(x_t, \eta_t) = \tilde{T}_t u(x, \eta) + \int_0^t K_0 D_{\eta t} \tilde{T}_{t-t_1} u(x_{t_1}, \eta_{t_1}) \left(dB^{(0)}_{t_1} + \eta_{t_1}^k dB^{(k)}_{t_1} \right)
 + \int_0^t \left[ \sigma_{\eta t}^{ik}(x_{t_1}) D_{\eta t} \tilde{T}_{t-t_1} u(x_{t_1}, \eta_{t_1}) + \sigma_{\eta t}^{ik}(x_{t_1}) D_{\eta t} \tilde{T}_{t-t_1} u(x_{t_1}, \eta_{t_1}) \right] dw_{t_1}^k.
\]

It follows that

\[
 Eu^2(x_t, \eta_t) \geq \left( \tilde{T}_t u(x, \eta) \right)^2
 + \sum_k \int_0^t \left[ \sigma_{\eta t}^{ik}(x_{t_1}) D_{\eta t} \tilde{T}_{t-t_1} u(x_{t_1}, \eta_{t_1}) + \sigma_{\eta t}^{ik}(x_{t_1}) D_{\eta t} \tilde{T}_{t-t_1} u(x_{t_1}, \eta_{t_1}) \right]^2 dt_1. \quad (6.8)
\]

By using Fatou’s lemma and estimates like (6.3) one easily carries (6.8) over to smooth \( u(x, \eta) \) whose derivatives have no more than polynomial growth as \( \|x\| + \|\eta\| \to \infty \). In particular, one can apply (6.8) to \( u(x, \eta) = f(\eta t)(x) \). Then, after noting that in light of (6.6) in that case

\[
\sigma_{\eta t}^{ik}(x) D_{\eta t} \tilde{T}_{t-t_1} u(x, \eta) + \sigma_{\eta t}^{ik}(x) D_{\eta t} \tilde{T}_{t-t_1} u(x, \eta)
 = \sigma_{\eta t}^{ik}(x) D_{\eta t} \tilde{T}_{t-t_1} f(x) \eta + \sigma_{\eta t}^{ik}(x) D_{\eta t} \tilde{T}_{t-t_1} f(x) \eta
 = \sigma_{\eta t}^{ik}(x) D_{\eta t} \tilde{T}_{t-t_1} f(x) \eta = (Q_{t-t_1}^k f(x)) \eta,
\]

we obtain

\[
 Eu^2(\eta_t(x)) \geq \left( \tilde{T}_t f(x) \eta \right)^2 + \sum_{k_1} \int_0^t E[\{(Q_{t-t_1}^k f(\eta_t(x))) \eta \}]^2 dt_1.
\]

By applying this formula to \( Q_{t-t_1}^k f \) in place of \( f \) we get

\[
 Eu^2(\eta_t(x)) \geq \left( \tilde{T}_t f(x) \eta \right)^2 + \sum_{k_1} \int_0^t E[\{(T_{t_1} Q_{t-t_1}^k f(\eta_t(x))) \eta \}]^2 dt_1
 + \sum_{k_1, k_2} \int_0^t dt_1 \int_0^{t_1} E[\{(Q_{t_1-t_2}^k Q_{t-t_1}^k f(\eta_{t_2}) \eta_{t_2}) \eta \}]^2 dt_2.
\]

Using induction yields that for any \( n \geq 1 \ (t_0 = t) \)

\[
 Eu^2(\eta_t(x)) \geq \left( \tilde{T}_t u(x) \eta \right)^2
 + \sum_{m=1}^n \sum_{k_1, \ldots, k_m} \int_{t_0 > \cdots > t_m} \left[ (T_{t_m} Q_{t_{m-1} - t_m}^{k_m} \cdots Q_{t_{t_1} - t_n}^{k_1} f(x) \eta) \right]^2 dt_m \cdots dt_1
\]
In particular, it follows that

This yields (6.7) and proves the lemma.

Next, we want to estimate the left-hand side of (6.7) which according to Lemma 6.1 satisfies (6.4).

In the future we will be interested in estimating not only the left-hand side of (6.7) but a slightly more general quantity. Therefore, we take an infinitely differentiable \( f(x, \eta) \geq 0 \) such that for an \( m > 0 \) and a constant \( N \)

\[
(|f| + |f_x| + |f_{\eta}| + |f_{xx}| + |f_{x\eta}|)(x, \eta) \leq N(1 + |\eta|)^m
\]

for all \( x, \eta \) and such that \( f(x, \eta) = 0 \) for all \( \eta \) if \( |x| \geq R \) for some \( R > 0 \). Then denote \( u(t, x, \eta) = T_t f(x, \eta) \). According to what was said before Lemma 6.2 and in that lemma, if we denote \( (l_t, \xi_t) = (lb)D(t, \xi)(x_t, \eta_t)(x, \eta) \), then

\[
u(l_t, \xi_t)(t, x, \eta) = E\nu(l_t, \xi_t)(x_t, \eta_t)(x, \eta).
\]

In particular, it follows that

\[
|\nu(l_t, \xi_t)(t, x, \eta)| \leq P^{1/2}(|x_t(x)| \leq R)(E|\nu(l_t, \xi_t)(x_t, \eta_t)(x, \eta)|^2)^{1/2}.
\]

Here the second factor on the right is estimated by using (6.3). The first factor is less than \( 2 \exp(-\mu \text{dist}^2(x, B_R)/t) \) by Theorem 2.10 of [11], where \( \mu > 0 \) depends only on \( d, \delta, \) and \( \|b\| \). Similar estimates are available for the second-order derivatives of \( u(t, x, \eta) \). More precisely, observe that there exist constants \( \mu > 0, \kappa = \kappa(m) \), and a function \( M(t) \) bounded on each time interval \([0, T]\) such that for all \( t, x, \eta \) we have

\[
|u(t, x, \eta)| + |u_x(t, x, \eta)| + |u_\eta(t, x, \eta)| + |u_{xx}(t, x, \eta)| + |u_{x\eta}(t, x, \eta)| + |u_{\eta\eta}(t, x, \eta)| \leq M(t)e^{-\mu|x|}(1 + |\eta|^2)^\kappa.
\]

This justifies the integration we perform below.

Introduce

\[
h = (1 + |\eta|^2)^{-\kappa - d}
\]

and observe that for a constant \( N = N(d, \kappa) \) we have

\[
|\eta|h \leq Nh, \quad (1 + |\eta|^2)|h_{\eta\eta}| \leq h.
\]

**Theorem 6.4.** Let \( q \geq 2 \) and suppose that the above \( u \geq 0 \). Then there is a constant \( N_0 \geq 1 \), depending only on \( d, d_1, \delta, m, \) and \( q \), such that for any \( \lambda \geq 1 \) satisfying

\[
N_0 \left( \sum_k \|\sigma^k f|\sigma^k| > \lambda\|L_d + \|bI|\|L_d \right) \leq 1
\]

there exists a constant \( N \), depending only on \( \lambda, \|\sigma^k\|, d, \delta, m, \) and \( q \), and there is a function \( K \) \( \geq 0 \) such that for \( t \geq 0 \)

\[
\int \mathbb{R}^{2d} h(\eta)u^q(t, x, \eta) dx d\eta \leq e^{Nt} \int \mathbb{R}^{2d} h(\eta)f^q(x, \eta) dx d\eta.
\]
The proof of this theorem proceeds as usual by multiplying (6.4) by $h(\eta)u^{q-1}(t, x, \eta)$ and integrating by parts over $[0, t] \times \mathbb{R}^d$. The integral of the left-hand side is

$$q^{-1} \int_{\mathbb{R}^d} h(\eta)u^q(t, x, \eta) \, dx \, d\eta - q^{-1} \int_{\mathbb{R}^d} h(\eta)f^q(x, \eta) \, dx \, d\eta.$$ 

Therefore, in light of Gronwall’s inequality, to prove the theorem it suffices to prove the following estimate.

**Lemma 6.5.** Let $q \geq 2$ and $\kappa \geq 0$. Then there is a constant $N_0 \geq 1$, depending only on $d, d_1, \delta, \kappa,$ and $q,$ such that for any $\lambda \geq 1$ satisfying

$$N_0 \left( \sum_k \| \sigma_k^k I_{|\sigma_k^k|>\lambda} \|_{L_d} \right) \leq 1 \tag{6.12}$$

there exists a constant $N,$ depending only on $\lambda, \| \sigma_k \|, d, d_1, \delta, \kappa,$ and $q,$ and there is a function $K_0$ such that for any a smooth function $v(x, \eta) \geq 0$ (independent of $t$), for which condition (6.9) is satisfied with $v$ in place of $u$ and some $M,$ we have

$$\int_{\mathbb{R}^d} h(\eta)v^{q-1}(x, \eta)\tilde{L}v(x, \eta) \, dx \, d\eta \leq N \int_{\mathbb{R}^d} h(\eta)v^q(x, \eta) \, dx \, d\eta. \tag{6.13}$$

Proof. For simplicity of notation we drop the arguments $x, \eta.$ We also write $U \sim V$ if their integrals over $\mathbb{R}^d$ coincide, and $U < V$ if the integral of $U$ is less than or equal to that of $V.$ Below the constants called $N,$ sometimes with indices, depend only on $d, d_1, \delta, \kappa,$ and $q$ unless specifically noted otherwise.

Set $w = v^{q/2}$ and note simple formulas:

$$v^{q-1}v_x = (2/q)vw_x, \quad v^{q-2}v_{xx} = (4/q^2)w_xw_{xx}.$$ 

Then denote by $\tilde{L}_1$ the sum of the first-order terms in $\tilde{L}$ and observe that integrating by parts shows that

$$hv^{q-1}v_{ij}v_{ij} \sim -(1/q)h_{\eta \eta}v_i^j v^q - (1/q)hh_{ij}^i v^q$$

$$\sim (2/q)\eta^k h_{ij} b^i w_{w,x} + (2/q)hb^i w_{w,x}.$$ 

Hence,

$$hv^{q-1}\tilde{L}_1v \sim (2/q)\eta^k h_{ij} b^i w_{w,x} + (4/q)hb^i w_{w,x}.$$ 

We take a number $\lambda \geq 1$ and write $b = \hat{b} + \tilde{b},$ where $\hat{b} = bI_{|b|>\lambda}.$ Following [23] we observe that by the Hölder and Sobolev inequalities ($d \geq 3$)

$$\int_{\mathbb{R}^d} |\hat{b}w_{w,x}| \, dx \leq \left( \int_{\mathbb{R}^d} |w_x|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |\hat{b}|^2 |w|^2 \, dx \right)^{1/2}$$

$$\leq \left( \int_{\mathbb{R}^d} |w_x|^2 \, dx \right)^{1/2} \| \hat{b} \|_{L_d} \| w \|_{L_{2d/(d-2)}} \leq N\| \hat{b} \|_{L_d} \int_{\mathbb{R}^d} |w_x|^2 \, dx, \tag{6.14}$$

where $N$ depends only on $d.$ Since $|\eta| |h_{\eta}| \leq N(\kappa, d)h,$ it follows that

$$\eta^k h_{\eta \eta} \hat{b}^i w_{w,x} \sim N\| \hat{b} \|_{L_d} h |w_x|^2.$$
Similarly, \( (4/q)h^q \hat{b}^{i} w_{x,i} \ll N\|\hat{b}\|_{L_d} h|w_x|^2 \). We estimate the remaining terms in \( hv^q - 1 \tilde{L}_1 v \) roughly like
\[
|\hat{b}^{i} w_{x,i}| \leq \lambda |w| |w_x| \leq \varepsilon |w_x|^2 + \varepsilon^{-1} \lambda^2 |w|^2
\]
and conclude that, for any \( \varepsilon > 0 \),
\[
hv^q - 1 \tilde{L}_1 v \ll (N\|\hat{b}\|_{L_d} + \varepsilon) h|w_x|^2 + N\varepsilon^{-1} \lambda^2 h|w|^2. \tag{6.15}
\]

Starting to deal with the second order derivatives note that
\[
hv^q - 1 (1/2) \sigma^j \theta^k \eta_{x,i} \sim -(q-1/2) v^{q-2} h \sigma^j \eta_{x,i} \sigma^k \eta_{x,i}
\]
\[
-(1/2) h \left[ \sigma_x^j \theta^k \eta_{x,i} + \sigma_x^k \theta^j \eta_{x,i} \right] v^{q-1} \eta_{x,i} = -(2q-2) / q^2 h \sigma_x^j \eta_{x,i} \sigma_x^k w_{x,i}
\]
\[
-(1/2) h \left[ \sigma_x^j \theta^k \eta_{x,i} + \sigma_x^k \theta^j \eta_{x,i} \right] \eta_{x,i} \leq -(1/2) h \sigma_x^j \eta_{x,i} \sigma_x^k w_{x,i}
\]
\[
+h \left[ \sigma_x^j \theta^k \eta_{x,i} + \sigma_x^k \theta^j \eta_{x,i} \right] w_{x,i},
\]
where the inequality (to simplify the writing) is due to the fact that \( q \geq 2 \). In this inequality the first term on the right is dominated in the sense of \( \prec \) by
\[
-(1/2) \delta h|w_x|^2
\]
(see (2.1)). The remaining term contains \( w_{x,i} \) and we treat it like above writing \( \sigma_x^k = \hat{\sigma}^k + \hat{\sigma}^k \), where \( \hat{\sigma}^k = \sigma_x^k I_{\sigma_x^k > \lambda} \). Then we get
\[
hv^q - 1 (1/2) \sigma^j \theta^k \eta_{x,i} \prec N \lambda^2 \varepsilon^{-1} h|w|^2
\]
\[
- \left[ (1/2) \delta - N \sum_k \| \hat{\sigma}^k \|_{L_d} - \varepsilon \right] h|w_x|^2. \tag{6.16}
\]

Next,
\[
hv^q - 1 \sigma^j \theta^k \eta_{x,i} \sim -(q-1) h \sigma_x^j \eta_{x,i} \sigma_x^k \eta_{x,i}
\]
\[
- \eta^{q-1} \sigma_x^j \left[ h_{\eta \eta} \sigma_x^j \theta^k + h_{\eta \eta} \sigma_x^k \theta^j \right] = -(4q-4)/q^2 h \sigma_x^j \eta_{x,i} \sigma_x^k \eta_{x,i}
\]
\[
- (2/q) \sigma_x^j \sigma_x^k \eta_{x,i}. \]

We estimate the first term on the right roughly using
\[
|\sigma_x^j \sigma_x^k \eta_{x,i}| \ll \varepsilon |w_x|^2 + N \varepsilon^{-1} |\eta| \sum_k |\sigma_x^k|^2 |\eta|^2.
\]

The second term contains \( w_{x,i} \) and allows the same handling as before. Therefore
\[
hv^q - 1 \sigma^j \theta^k \eta_{x,i} \ll \left( \varepsilon + N \sum_k \| \hat{\sigma}^k \|_{L_d} \right) h|w_x|^2
\]
\[
+ N \varepsilon^{-1} h|\eta| \sum_k |\sigma_x^k|^2 |\eta|^2 + N \lambda \varepsilon^{-1} h|w|^2. \tag{6.17}
\]

The last term in \( hv^q - 1 \tilde{L}_1 v \) containing \( \sigma \) is
\[
hv^q - 1 (1/2) \sigma_x^j \theta^k \eta_{x,i} \sim -(q-1/2) h \sigma_x^j \eta_{x,i} \sigma_x^k \eta_{x,i}
\]
\[
-(1/2) v^{q-1} \sigma_x^j \eta_{x,i} \left[ h_{\eta \eta} \sigma_x^j + h_{\sigma_x^j \sigma_x^k} \right] -(1/(2q)) h \sigma_x^j \eta_{x,i} \sigma_x^k \eta_{x,i}.
\]
\[ \langle N h (|\eta|^2 |w_\eta|^2 + w^2) \sum_k |\sigma_x^k|^2 + I, \]

where

\[ I = -(1/(2q)) h(w^2)_{\eta'} \sigma_x^{jk} \sigma_{(\eta)}^{jk} \]

\[ \sim (1/(2q)) w \sigma_x^{jk} [h_{\eta'} \sigma_{(\eta)}^{jk} + h \sigma_{x}^{jk}] \prec N h \sum_k |\sigma_x^k|^2 w^2. \]

To estimate the last term we basically use the derivation of (6.14). We have

\[ \int_{\mathbb{R}^d} |\hat{\sigma}^k|^2 w^2 \, dx \leq \|\hat{\sigma}^k\|_{L_d}^2 \|w\|^2_{L_{2d/(d-2)}} \leq N \|\hat{\sigma}^k\|_{L_d}^2 \int_{\mathbb{R}^d} |w_x|^2 \, dx. \]  

(6.18)

Below we show how to choose the constant \( N_0 \) in the condition (6.12) under which our assertion is true. But observe that with any such choice \( \|\hat{\sigma}^k\|_{L_d}^2 \leq 1 \) and therefore, (just to keep some uniformity in our estimates) (6.18) implies that

\[ \int_{\mathbb{R}^d} |\hat{\sigma}^k|^2 w^2 \, dx \leq N \|\hat{\sigma}^k\|_{L_d} \int_{\mathbb{R}^d} |w_x|^2 \, dx. \]  

(6.19)

Hence,

\[ I \prec N h \lambda^2 w^2 + N h \sum_k \|\hat{\sigma}^k\|_{L_d} |w_x|^2 \]

and

\[ h v^{\alpha-1} (1/2) \sigma_{(\eta)}^{jk} \sigma_{\eta'}^{jk} \prec N h |\eta|^2 |w_\eta|^2 \sum_k |\sigma_x^k|^2 \]

\[ + N h \lambda^2 w^2 + N h \sum_k \|\hat{\sigma}^k\|_{L_d} |w_x|^2 \]  

(6.20)

Finally,

\[ h v^{\alpha-1} (1/2) K_0^2 (1 + |\eta|^2) \delta^{ij} v_{\eta' \eta'} \prec -((2q - 2)/q^2) h K_0^2 (1 + |\eta|^2) |w_\eta|^2 \]

\[ - (2/q) K_0^2 (h (1 + |\eta|^2)) v_{\eta' \eta'} w \eta' \]

\[ \sim -((2q - 2)/q^2) h K_0^2 (1 + |\eta|^2) |w_\eta|^2 + (1/q) w^2 K_0^2 \delta^{ij} (h (1 + |\eta|^2)) v_{\eta' \eta'} \]

\[ \leq - (1/q) h K_0^2 (1 + |\eta|^2) |w_\eta|^2 + N w^2 K_0^2 h. \]  

(6.21)

By combining (6.15), (6.16), (6.17), (6.20), and (6.21), we see that for any \( \varepsilon \in (0, 1] \)

\[ h v^{\alpha-1} \tilde{L} v \prec N \varepsilon^{-1} \lambda^2 h |w|^2 \]

\[ - [(1/q) \delta - N_1 \left( \sum_k \|\hat{\sigma}^k\|_{L_d} + \|\hat{b}\|_{L_d} \right) - N_2 \varepsilon] h |w_x|^2 \]

\[ + |w_\eta|^2 h (1 + |\eta|^2) \left[ N_3 \varepsilon^{-1} \sum_k |\sigma_x^k|^2 - (1/q) K_0^2 \right] + N_4 w^2 K_0^2 h. \]  

(6.22)

Take and fix \( \varepsilon \) so that \( N_2 \varepsilon \leq \delta/(2q) \). After that set

\[ K_0^2 = 1 + N_3 q \varepsilon^{-1} \sum_k |\sigma_x^k|^2 \]
(1 is added to guarantee the smoothness of $K_0$) and observe that similarly to (6.18) and (6.19)

$$N_4 w^2 K_0^2 h = N_4 w^2 h + N w^2 \sum_k |\sigma^k_x|^2 < N_5 h \sum_k ||\dot{\sigma}^k||_{L_4} |w_x|^2 + N h \lambda^2 w^2.$$  

Then (6.22) becomes

$$hv^{\frac{q-1}{2}} L v < N \lambda^2 h |w|^2 - \left[ \frac{1}{(2q)} \lambda - (N_1 + N_5) \left( \sum_k ||\dot{\sigma}^k||_{L_4} + ||\dot{b}||_{L_4} \right) \right] |w_x|^2.$$  

We can certainly believe that $N_1 \geq 1$, take $N_0$ in (6.12) to be equal to $(2q/\delta)(N_1 + N_5) (\geq 1)$, and conclude

$$hv^{\frac{q-1}{2}} L v < N \lambda^2 h |w|^2.$$  

The lemma is proved.

We finish the section with an approximation result.

**Lemma 6.6.** Let Assumptions 2.1 and 2.2 be satisfied. Then there are sequences $\sigma^k(n), b(n), n = 1, 2, \ldots, k = 1, \ldots, d_1$, of infinitely differentiable functions with each derivative bounded having the same meanings as $\sigma^k, b$ in the beginning of the article, satisfying Assumptions 2.1 and 2.2 with $\delta/2$ in place of $\delta$ and the same $||b||$ and $||\sigma^k_x||$ for sufficiently large $n$, and such that $\sigma^k(n) \to \sigma^k$ as $n \to \infty$ (a.e.) and $\sigma^k_x(n), b(n) \to \sigma^k_x, b$ in $L_d$ as $n \to \infty$.

Proof. Take a nonnegative $\zeta \in C_0^\infty$ with unit integral and support in $B_1$ and set $\zeta_n(x) = n^d \zeta(nx), u(n, x) = u(x) * \zeta(nx)$, then the well-known properties of convolutions imply all stated properties apart from what concerns (2.1).

Denote by $\sigma$ the $d \times d_1$-matrix whose columns are the $\sigma^k$’s and note that

$$|\sigma^*(n, x) \lambda| \leq \zeta(nx)|\sigma^*(x) \lambda| \leq \delta^{-1/2} |\lambda|.$$  

Therefore we need only prove that for sufficiently large $n$

$$|\sigma^*(n, x) \lambda| \geq |\lambda| \delta^{-1/2}/\sqrt{2}. \tag{6.23}$$  

For any $y$ we have

$$|\sigma^*(n, x) \lambda| \geq |\sigma^*(y) \lambda| - |(\sigma^*(n, x) - \sigma^*(y)) \lambda| \geq |\lambda| \delta^{1/2} - |(\sigma^*(n, x) - \sigma^*(y)) \lambda|$$

$$\geq |\lambda| (\delta^{1/2} - |(\sigma^*(n, x) - \sigma^*(y))|)$$

Furthermore,

$$\int_{R^d} |(\sigma^*(n, x) - \sigma^*(x - y))| \zeta_n(y) dy$$

$$\leq \int_{R^d} \int_{R^d} |(\sigma^*(x - z) - \sigma^*(x + y))| \zeta_n(y) \zeta_n(z) dy dz$$

$$\leq N(d) \max \zeta^2 \text{osc } \sigma, B_{1/n}(x).$$

The latter tends to zero uniformly with respect to $x$ since $\sigma_x \in L_d$ (cf. Remark 3.6). This certainly proves the lemma.
7. Proof of Theorem 2.3

According to Theorems 5.6 and 5.7 it suffices to prove that at least one of the solutions of (1.1) is strong. We will be dealing with the solution from Lemma 4.1.

Let \( f \in C_0^\infty \). First we deal with smooth coefficients and develop necessary estimates. By Lemma 6.3 and Theorem 6.4 for \( t \geq 0 \) and \( q \geq 2 \) we have

\[
\int_{\mathbb{R}^d} h(\eta)u^q(t, x, \eta) \, d\eta \leq Ne^{Nt},
\]

where (and below) \( N \) depends only on \( f, d, d_1, \delta, m = m(f), q, \) and \( \lambda \) defined by \( 6.10 \) and

\[
\begin{align*}
  &u(t, x, \eta) = \int_{\mathbb{R}^d} \sum_{k_1, \ldots, k_n} \int_{t > t_1 > \ldots > t_n} \left[ (T_{t_n}Q_{t_{n-1}-t_n} \cdots Q_{t_{1}-t_1}f(x))_\eta \right]^2 \, dt_n \cdots dt_1. \\
\end{align*}
\]

Obviously, \( u(t, x, \eta) \) is a quadratic function of \( \eta \). Hence, (7.1) implies that, for any \( R \in (0, \infty) \)

\[
\int_{|\eta| \leq R} \sup \left| \eta \right| \leq \frac{R}{N} u^q(t, x, \eta) \, dx \leq Ne^{Nt} R^{2q}.
\]

Observe that in notation (5.5) and (5.8)

\[
\sum_k u(t, x, \sigma^k) = \sum_{n=1}^{\infty} \int_{t > t_1 > \ldots > t_n} Q_{t_n, t_{n-1}-t_n, \ldots, t_1-t_1} f(x) \, dt_n \cdots dt_1
\]

\[
= \sum_{n=1}^{\infty} \int_{S_n(t)} Q_{s_n, s_{n-1}-s_n, \ldots, s_1-t_1-t_1} f(x) \, ds_n \cdots ds_1 =: \sum_{n=1}^{\infty} I_n(t, x)
\]

\((S_n(t)\) is introduced in Remark 5.8). Next, for \( \nu > 0 \) by Jensen’s inequality

\[
\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\nu t} I_n(t, x) \, dt \right)^q \, dx \leq \nu^{1-q} \int_0^\infty \int_{\mathbb{R}^d} e^{-\nu t} \left( \sum_{n=1}^{\infty} I_n(t, x) \, dx \right) \, dt
\]

\[
\leq \nu^{1-q} \int_0^\infty \int_{\mathbb{R}^d} \left( \sum_k u(t, x, \sigma^k) \right)^q \, dx \, dt,
\]

which thanks to (7.2) implies that for appropriate \( \nu \), depending only on \( f, d, d_1, \delta, q, \) and \( \lambda \),

\[
\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\nu t} I_n(t, x) \, dt \right)^q \, dx \leq N,
\]

where \( N \) depends only on \( f, d, d_1, \delta, q, \) and \( \lambda \).

Estimate (7.3) has been derived only for infinitely differentiable \( \sigma \) and \( b \). However, using smooth approximations (Lemma 6.6), Theorem 3.10, and Fatou’s lemma prove (7.3) also in our general case. Indeed, although the constant \( N \) in (7.3) for each approximation depends on \( \lambda \), satisfying (6.10)
for the approximating $\sigma^K$ and $b$, it can be taken the same as long as the approximations are sufficiently close in $L_d$ to the original $\sigma^K$ and $b$.

Finally, by observing that

$$\int_0^\infty e^{-\nu t}I_n(t, x) \, dt = \int_{\mathbb{R}^+} e^{-\nu(s_0+\ldots+s_n)} Q_{s_n,\ldots,s_0}f(x) \, ds_n \cdot \ldots \cdot ds_0,$$

referring to Theorem 5.9, and taking $q = p$, we conclude that $f(x_t)$ is $\mathcal{F}_t^\nu$-measurable. The arbitrariness of $f$ and $t$ finishes the proof.

8. Proof of Theorem 2.4

Take a bounded smooth function $f$ with compact support. By Theorems 2.3 and 5.5 for any $t$

$$f(x_t(n, x(n))) = T_t(n)f(x(n))$$

$$+ \sum_{m=1}^{\infty} \int_{t > t_1 > \ldots > t_m} T_{t_m}(n)Q_{t_{m-1}-t_m}(n) \cdot \ldots \cdot Q_{t-t_1}(n)f(x(n)) \, dw_{t_m} \cdot \ldots \cdot dw_{t_1},$$

(8.1)

where $T_t(n)$ and $Q^K_t(n)$ are the operators corresponding to $\sigma^K(n)$, $b(n)$. First we prove that $E|f(x_t(n, x(n))) - f(x_t)|^2 \to 0$ as $n \to \infty$. Since $Ef^2(x_t(n, x(n))) \to Ef^2(x_t)$ (see Theorem 3.10), it suffices to prove that $f(x_t(n, x(n))) \to f(x_t)$ weakly in $L_2(\Omega, \mathcal{F}_t^\nu, P)$. Furthermore, according to [5] the linear combinations of the multiple Itô integrals of the type

$$\int_{t > t_1 > \ldots > t_m} \phi(t_1, \ldots, t_m) \, dw_{t_m} \cdot \ldots \cdot dw_{t_1},$$

where $m$ is arbitrary and $\phi$ is an arbitrary bounded (nonrandom) Borel function, are dense in $L_2(\Omega, \mathcal{F}_t^\nu, P)$. Therefore, it suffices to prove that for all such $m$ and $\phi$

$$Ef(x_t(n, x(n))) \int_{t > t_1 > \ldots > t_m} \phi(t_1, \ldots, t_m) \, dw_{t_m} \cdot \ldots \cdot dw_{t_1}$$

$$\to Ef(x_t) \int_{t > t_1 > \ldots > t_m} \phi(t_1, \ldots, t_m) \, dw_{t_m} \cdot \ldots \cdot dw_{t_1}.$$
Next, observe that for any \( T \in (0, \infty) \) and bounded smooth \( \mathbb{R}^d \)-valued \( \tilde{b} \) with compact support

\[
I := \lim_{n \to \infty} E \sup_{t \leq T} \left| \int_0^t b(n, x_s(n, x(n))) \, ds - \int_0^t b(x_s) \, ds \right|
\]

\[
\leq \lim_{n \to \infty} E \int_0^T \left| b(n, x_s(n, x(n))) - b(x_s) \right| \, ds
\]

\[
\leq \lim_{n \to \infty} E \int_0^T \left| b(n, x_s(n, x(n))) - \tilde{b}(x_s(n, x(n))) \right| \, ds
\]

\[
+ \lim_{n \to \infty} \int_0^T E \left| \tilde{b}(x_s(n, x(n))) - \tilde{b}(x_s) \right| \, ds
\]

\[
+ \lim_{n \to \infty} E \int_0^T \left| \tilde{b}(x_s) - b(x_s) \right| \, ds.
\]

Here the middle term vanishes by the first part of the proof. Owing to Lemma 4.2 the two remaining terms are majorated by

\[
N \left( \lim_{n \to \infty} \| b_n - \tilde{b} \|_{L_d} + \| b - \tilde{b} \|_{L_d} \right) = N \| b - \tilde{b} \|_{L_d},
\]

that can be made arbitrarily small by an appropriate choice of \( \tilde{b} \). Hence, \( I = 0 \).

Similarly,

\[
J := \lim_{n \to \infty} \left( E \sup_{t \leq T} \left| \int_0^t \sigma^k(n, x_s(n, x(n))) \, dw^k_s - \int_0^t \sigma^k(x_s) \, dw^k_s \right| \right)^2
\]

\[
\leq N \lim_{n \to \infty} \sum_k E \int_0^T \left| \sigma^k(n, x_s(n, x(n))) - \sigma^k(x_s) \right|^2 \, ds
\]

\[
\leq N \lim_{n \to \infty} \sum_k E \int_0^T \left| \sigma^k(n, x_s(n, x(n))) - \sigma^k(x_s) \right| \, ds
\]

\[
\leq N \sum_k \| \Phi_T(\sigma^k - \hat{\sigma}^k) \|_{L_d},
\]

where \( \hat{\sigma}^k \) are smooth functions with compact support. It follows that \( J = 0 \) and together with \( I = 0 \) this implies that

\[
\lim_{n \to \infty} E \sup_{t \leq T} |x_t(n, x(n)) - x_t| = 0 \quad (8.2)
\]

By Corollary 1.2 of [11] for any \( m \geq 0 \)

\[
E \sup_{t \leq T} |x_t(n, x(n)) - x(n)|^{2m} + E \sup_{t \leq T} |x_t - x_0|^{2m} \leq N(m, d, \delta, \| \bar{b} \|) T^m
\]

and this along with (8.2) yields the result. The theorem is proved.
9. Proof of Theorem 2.5

First we assume that $\sigma^k$ and $b$ are infinitely differentiable with each derivative bounded. In that case, as it is known since [2] (see also [17]) that one can define $x_t(x)$ in such a way that it becomes differentiable in $x$ for all $(\omega, t)$ and the derivative of $x_t$ in the direction of $\eta$ satisfies the same equation as $\xi_t(x, \eta)$ from Lemma 6.2, for which (6.5) holds. In particular, for any even $\kappa \geq 2$, and $f$ with compact support $((\cdot, \cdot)$ is the scalar product in $\mathbb{R}^d$)

$$E((Df)(x_t(x)), \eta(x, \eta))^\kappa \geq E((Df)(x_t(x)), \xi_t(x, \eta))^\kappa =: v(t, x, \eta).$$

By Theorem 6.4, with $q = 2$ there, there is a constant $m = m(\kappa)$ such that for any $\lambda > 0$ satisfying (6.10) there exists a constant $N$, depending only on $\lambda, d, \delta, m$, such that for $t \geq 0$

$$\int_{\mathbb{R}^d} h(\eta) v^2(t, x, \eta) \, dx d\eta \leq e^{Nt} \int_{\mathbb{R}^d} h(\eta) |f(\eta)(x)|^{2\kappa} \, dx d\eta = N(d, \kappa) e^{Nt} \int_{\mathbb{R}^d} |f(x)|^{2\kappa} \, dx =: M_t. \quad (9.1)$$

Next, for any $R \in (0, \infty)$

$$E \int_{B_R} |D(f(x_t(x)))|^\kappa \, dx = E \int_{B_R} \int_{\mathbb{R}^d} (f(x_t(x)) \eta(t, x, \eta) \, d\eta \, dx$$

$$= N \int_{B_R} \int_{\mathbb{R}^d} v(t, x, \eta) h(\eta) \, d\eta.$$  

By using (9.1) and Hölder’s inequality we obtain that

$$E \int_{B_R} |D(f(x_t(x)))|^\kappa \, dx \leq N(d, \kappa) R^{d/2} M_t^{1/2}.$$

By Morrey’s theorem (see, for instance, Theorem 10.2.1 of [9]) this implies that for any $\kappa > d$

$$E \sup_{x, y \in B_R} \frac{|f(x_t(x)) - f(x_t(y))|^{\kappa}}{|x - y|^{\kappa-d}} \leq N(d, \kappa) R^{d/2} M_t^{1/2}. \quad (9.2)$$

Note that (9.2) is certainly applicable to vector-valued $f$. Fix $\rho \geq 2R$ and take a smooth $f$ with support in $B_4\rho$ such that $f(x) = x$ for $|x| \leq 2\rho$ and $|f_x| \leq 2$. Then $M_t \leq N(T, d, \delta, \lambda, \kappa) \rho^d$ and for $x, y \in B_R$ and $t \leq T$

$$E|x_t(x) - x_t(y)|^\kappa \leq N \rho^{d/2} |x - y|^{\kappa-d} + N(\kappa) E(|x_t(x)|^\kappa + |x_t(x)|^\kappa) I_{|x_t(x)|^\kappa + |x_t(y)|^\kappa \geq 2\rho},$$

where $N$ depends only on $R, T, \lambda, d, \delta$, and $\kappa$. We estimate the first term on the right by using Hölder’s inequality and Theorem 2.10 of [11] and find that it is dominated by

$$N(\kappa) P^{1/2}(|x_t(x)| + |x_t(y)| \geq 2\rho) \left( (E|x_t(x)|^{2\kappa})^{1/2} + (E|x_t(x)|^{2\kappa})^{1/2} \right) \leq N e^{-\mu \rho^2},$$
where $N$ depends only on $R, T, \lambda, d, \delta$, and $\kappa$ and $\mu > 0$ depends only on $T, d, \delta$, and $\|b\|$. Thus, for $\rho \geq 2R$

$$E|\xi_t(x) - \xi_t(y)|^\kappa \leq Ne^{-\mu \rho^2} + N\rho^{d/2}|x - y|^\kappa - d.$$  

By taking here $\kappa > 2d$ and $\mu \rho^2 = -\ln |x - y|^{\kappa - 2d}$ we find that

$$E|\xi_t(x) - \xi_t(y)|^\kappa \leq N|x - y|^{\kappa - 2d},$$  \hspace{1cm} (9.3)

where $N$ depends only on $R, T, \lambda, d, \delta, \|b\|$, and $\kappa$, provided that $-\ln |x - y|^{\kappa - 2d} \geq 2R/\mu$. However, if $-\ln |x - y|^{\kappa - 2d} \leq 2R/\mu$, (9.3) is obvious.

Estimate (9.3) so far is proved only for infinitely differentiable coefficients, but usual approximations, Theorem 2.4, and Fatou’s lemma allow us to obtain (9.3) in our general case where, naturally, by $\xi_t(x)$ we mean the strong solution of (1.1) with $x_0 = x$.

In the general case we also have by Corollary 1.2 of [11] that for any $q \geq 1$

$$E|\xi_t(x) - \xi_s(x)|^q \leq N|t - s|^{q/2},$$  \hspace{1cm} (9.4)

where $N = N(d, \delta, \|b\|, q)$.

Now the arbitrariness of $\kappa$ and $q$ leads to the claimed result by a version of Kolmogorov’s theorem which can be found, for instance, in [20] or simply derived by using $|\xi_t(x) - \xi_s(y)| \leq |\xi_t(x) - \xi_s(x)| + |\xi_s(x) - \xi_s(y)|$. The theorem is proved.

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