Exponential Stability of Solutions to Stochastic Differential Equations Driven by $G$-Lévy Process

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Abstract In this paper, BDG-type inequality for $G$-stochastic calculus with respect to $G$-Lévy process is obtained and solutions of stochastic differential equations driven by $G$-Lévy process under non-Lipschitz condition are constructed. Moreover, we establish the mean square exponential stability and quasi sure exponential stability of the solutions by means of $G$-Lyapunov function method. An example is presented to illustrate the efficiency of the obtained results.

Key words $G$-Lévy Process; Non-Lipschitz; Exponential stability.

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1 Introduction

In recent years much effort has been made to develop the theory of sublinear expectations connected with the volatility uncertainty and so-called $G$-Brownian motion. $G$-Brownian motion was introduced by Shige Peng in [10] as a way to incorporate the unknown volatility into financial models. Its theory is tightly associated with the uncertainty problems involving an undominated family of probability measures. Soon other connections have been discovered, not only in the field of financial mathematics, but also in the theory of path-dependent partial differential equations or backward stochastic differential equations. Thus $G$-Brownian motion and connected $G$-expectation are attractive mathematical objects. We refer the reader to Gao [3], Denis et al. [2], Soner [24], Bai et al. [1], Li et al. [7], Peng [11,13,15,16] and the references therein.

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Returning however to the original problem of volatility uncertainty in the financial models, one feels that $G$-Brownian motion is not sufficient to model the financial world, as both $G$- and the standard Brownian motion share the same property, which makes them often unsuitable for modelling, namely the continuity of paths. Therefore, it is natural that Hu and Peng [6] introduced the process with jumps, which they called $G$-Lévy process. Then Ren [22] introduced the representation of the sublinear expectation as an upper-expectation. In [18], the author concentrated on establishing the integration theory for $G$-Lévy process with finite activity, introduced the integral w.r.t the jump measure associated with the pure jump $G$-Lévy process and gave the Itô formula for general $G$-Itô Lévy process.

In [4], the author proved the BDG inequality for $G$-stochastic calculus with respect to $G$-Brownian motion. In this article, we will prove the BDG-type inequality for $G$-stochastic calculus with respect to $G$-Lévy Process, which will be used in section 3.

In [20] and [1] the authors considered the stochastic differential equations driven by $G$-Brownian motion, where the coefficients do not satisfy the Lipschitz condition. Motivated by the aforementioned works, in section 3, the following stochastic differential equations driven by $G$-Lévy process (GSDEs) is studied:

$$
\begin{align*}
    dY_t &= b(t, Y_t)dt + h_{ij}(t, Y_t)dB^i_t + \sigma_i(t, Y_t)dB_t^i + \int_{\mathbb{R}^d} K(t, Y_t, z) L(dt, dz), \\
    Y_{t_0} &= Y_0,
\end{align*}
$$

(1.1)

where $Y_0$ is the initial value with $\mathbb{E}[|Y_0|^2] < \infty$, $((B^i_t)_{t \geq 0})$ is the mutual variation process of the $d$-dimension $G$-Brownian motion $(B_t)_{t \geq 0}$, $L(\cdot, \cdot)$ is a Poisson random measure associated with the $G$-Lévy process $X$. The coefficients $b(\cdot, x), h_{ij}(\cdot, x), \sigma_i(\cdot, x)$ are in the space $M^2_G(0, T; \mathbb{R}^n)$, $K(\cdot, x, \cdot) \in H^2_G([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ (these spaces will be defined in section 2). If $b, h, \sigma$ and $K$ satisfy Lipschitz conditions, Paczka [13] established the existence and uniqueness of solution for the stochastic differential equations (1.1) in the space $M^2_G(0, T; \mathbb{R}^n)$. However, many coefficients do not satisfy the Lipschitz condition. Therefore, the extension to non-Lipschitz conditions is necessary.

Stability of stochastic differential equations has been well studied by many authors. In particular, by employing Lyapunov function method, Liu [8] obtained the comparison principles of $p$-th moment exponential stability for impulsive stochastic differential equations. For more details on this topic, one can see Peng and Jia [17], Shen and Sun [23], Wu and Han et al. [25], Wu and Sun [24], Wu and Yan et al. [27] and the references therein. Very recently, Hu et al. [5] investigated the sufficient conditions for $p$-th moment stability of solutions to stochastic differential equations driven by $G$-Brownian motion by means of a very special Lyapunov function. Zhang and Chen [28], Fei and Fei [3] respectively established the sufficient conditions for the exponential stability and quasi sure exponential stability for a kind of special stochastic differential equations driven by $G$-Brownian motion. Then Ren et al. [21] established the $p$-th moment exponential stability and quasi sure exponential stability of solutions to impulsive stochastic differential equations driven by $G$-Brownian motion.

To our best knowledge, there is no work reported on the mean square exponential stability and quasi sure exponential stability of solutions to GSDEs driven by $G$-Lévy process. So, this should be developed. Motivated by the aforementioned works, the second part of this paper aims...
to establish the mean square exponential stability and quasi sure exponential stability of solutions to GSDEs driven by $G$-Lévy process by means of the $G$-Lyapunov function method.

The rest of this paper is organized as follows. In section 2, we introduce some preliminaries and give the proof of BDG-type inequality for $G$-stochastic calculus with respect to $G$-jump measure. In section 3, the solution of (1.1) is constructed. Section 4 is devoted to proving the mean square exponential stability and quasi sure exponential stability of solutions to GSDEs driven by $G$-Lévy process by means of the $G$-Lyapunov function method. In the last section, an example is given to illustrate the effectiveness of the obtained results.

## 2 Preliminaries

In this section, we introduce some notations and preliminary results in $G$-framework which are needed in the following section. More details can be found in [6, 9, 14, 18, 19, 22].

**Definition 2.1.** Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$. Moreover, if $X_i \in \mathcal{H}, i = 1, 2, ..., d$, then $\varphi(X_1, ..., X_d) \in \mathcal{H}$ for all $\varphi \in C_{b, lip}(\mathbb{R}^d)$, where $C_{b, lip}(\mathbb{R}^d)$ is the space of all bounded real-valued Lipschitz continuous functions. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

1. **Monotonicity:** $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.
2. **Constant preserving:** $\mathbb{E}[C] = C$ for $C \in \mathbb{R}$.
3. **Sub-additivity:** $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
4. **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \mathbb{E})$. We often call $Y = (Y_1, ..., Y_d), Y_i \in \mathcal{H}$ a $d$-dimensional random vector in $(\Omega, \mathcal{H}, \mathbb{E})$.

**Definition 2.2.** In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a $n$-dimensional random vector $Y = (Y_1, ..., Y_n)$ is said to be independent from an $m$-dimensional random vector $X = (X_1, ..., X_m)$ if for each $\varphi \in C_{b, lip}(\mathbb{R}^{m+n})$,

$$
\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].
$$

**Definition 2.3.** Let $X_1, X_2$ be two $n$-dimensional random vectors defined on a sublinear expectation space $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if

$$
\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{b, lip}(\mathbb{R}^n).
$$

$X$ is said to be an independent copy of $X$, if $X$ is identically distributed with $X$ and independent of $X$.

**Definition 2.4.** ($G$-Lévy process). Let $X = (X_t)_{t \geq 0}$ be a $d$-dimensional càdlàg process on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We say that $X$ is a Lévy process if:

1. $X_0 = 0,$
(ii) for each $s, t \geq 0$ the increment $X_{t+s} - X_s$ is independent of $(X_{t_1}, \cdots, X_{t_n})$ for every $n \in \mathbb{N}$ and every partition $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq s$, 

(iii) the distribution of the increment $X_{t+s} - X_s$, $s, t \geq 0$ is stationary, i.e. does not depend on $s$.

Moreover, we say that a Lévy process $X$ is a $G$-Lévy process, if it satisfies additionally following conditions

(iv) there is a $2d$-dimensional Lévy process $(X^c_t, X^d_t)_{t \geq 0}$ such that for each $t \geq 0$, $X_t = X^c_t + X^d_t$,

(v) process $X^c_t$ and $X^d_t$ satisfy the following conditions

$$\lim_{t \downarrow 0} \mathbb{E}[|X^c_t|^3]t^{-1} = 0; \quad \mathbb{E}[|X^d_t|] < C_t \text{ for all } t \geq 0.$$ 

Remark 2.1. The condition (v) implies that $X^c$ is a $d$-dimensional generalized $G$-Brownian motion, whereas the jump part $X^d$ is of finite variation. (See [6] for details.)

Hu and Peng [6] noticed in their paper that each $G$-Lévy process might be characterized by a non-local operator $G_X$.

Theorem 2.1. ( [6]) Let $X$ be a $G$-Lévy process in $\mathbb{R}^d$. For every $f \in C^2_b(\mathbb{R}^d)$ such that $f(0) = 0$ we put

$$G_X[f(\cdot)] := \lim_{\delta \downarrow 0} \mathbb{E}[f(X_\delta)]\delta^{-1}.$$

The above limit exist. Moreover, $G_X$ has the following Lévy-Khintchine representation

$$G_X[f(\cdot)] = \sup_{(v,p,Q) \in \mathcal{U}} \{ \int_{\mathbb{R}^d} f(z)v(dz) + \langle Df(0), p \rangle + \frac{1}{2} \text{tr}[D^2f(0)QQ^T] \},$$

where $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$, $\mathcal{U}$ is a subset $\mathcal{U} \subset \mathcal{V} \times \mathbb{R}^d \times \mathcal{Q}$ and $\mathcal{V}$ is a set of all Borel measures on $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$. $\mathcal{Q}$ is a set of all $d$-dimensional positive definite symmetric matrices in $\mathbb{S}^d$ ($\mathbb{S}^d$ is the space of all $d \times d$-dimensional symmetric matrices) such that

$$\sup_{(v,p,Q) \in \mathcal{U}} \{ \int_{\mathbb{R}^d_0} |z|v(dz) + |p| + \text{tr}[QQ^T] \} < \infty. \quad (2.1)$$

Theorem 2.2. ( [6]) Let $X$ be a $d$-dimensional $G$-Lévy process. For each $\phi \in C_{b,\text{lip}}(\mathbb{R}^d)$, define $u(t, x) := \mathbb{E}[\phi(x + X_t)]$. Then $u$ is the unique viscosity solution of the following integro-PDE

$$0 = \partial_t u(t, x) - G_X[u(t, x + \cdot) - u(t, x)]$$

$$= \partial_t u(t, x) - \sup_{(v,p,Q) \in \mathcal{U}} \{ \int_{\mathbb{R}^d_0} [u(t, x + z) - u(t, x)]v(dz) + \langle Du(t, x), p \rangle + \frac{1}{2} \text{tr}[D^2u(t, x)QQ^T] \}, \quad (2.2)$$

with initial condition $u(0, x) = \phi(x)$.

Theorem 2.3. Let $\mathcal{U}$ satisfy (2.1). Consider the canonical space $\Omega := \mathcal{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ of all càdlàg functions taking values in $\mathbb{R}^d$ equipped with the Skorohod topology. Then there exists a sublinear expectations $\mathbb{E}$ on $\mathcal{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ such that the canonical process $(X_t)_{t \geq 0}$ is a $G$-Lévy process satisfying Lévy-Khintchine representation with the same set $\mathcal{U}$. 


The proof of above Theorem might be found in (Theorem 38 and 40 in [6]). We will give however the construction of $\hat{E}$, as it is important to understand it. We denote $\Omega_T := \{w \wedge T : w \in \Omega\}$. Put

$$Lip(\Omega_T) := \{\xi \in L^0(\Omega_T) : \xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}), \phi \in C_{b, lip}(R^{d \times n}), 0 \leq t_1 \leq \cdots \leq t_n \leq T\},$$

where $X_t(w) = w_t$ is the canonical process on the space $D_0(R^+, R^d)$ and $L^0(\Omega)$ is the space of all random variables, which are measurable to the filtration generated by the canonical process. We also set

$$Lip(\Omega) := \bigcup_{T=1}^{\infty} Lip(\Omega_T).$$

Firstly, consider the random variable $\xi = \phi(X_{t+s} - X_s), \phi \in C_{b, lip}(R^d)$. We define

$$\hat{E}[\xi] := u(s, 0),$$

where $u$ is a unique viscosity solution of integro-PDE (2.2) with the initial condition $u(0, x) = \phi(x)$. For general

$$\xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}), \phi \in C_{b, lip}(R^{d \times n})$$

we set $\hat{E}[\xi] := \phi_n$, where $\phi_n$ is obtained via the following iterated procedure

$$\phi_1(x_1, ..., x_{n-1}) = \hat{E}[\phi(x_1, ..., x_{n-1}, X_{t_n} - X_{t_{n-1}})],$$

$$\phi_2(x_1, ..., x_{n-2}) = \hat{E}[\phi_1(x_1, ..., x_{n-2}, X_{t_{n-1}} - X_{t_{n-2}})],$$

$$\vdots$$

$$\phi_{n-1}(x_1) = \hat{E}[\phi_{n-1}(x_1, X_{t_2} - X_{t_1})],$$

$$\phi_n = \hat{E}[\phi_n(X_{t_1})].$$

Lastly, we extend definition of $\hat{E}$ on the completion of $Lip(\Omega_T)$ (respectively $Lip(\Omega)$) under the norm $\|\cdot\|_p = \hat{E}[|\cdot|^p], p \geq 1$. We denote such a completion by $L^p_G(\Omega_T)$ (or resp. $L^p_G(\Omega)$).

Let $G_B$ denote the set of all Borel function $g : R^d \rightarrow R^d$ such that $g(0) = 0$. Assume that for all Lévy measure $\mu$ and $v \in V$ there exist a function $g_v \in G_B$ such that $v(B) = \mu(g_v^{-1}(B)), \forall B \in B(R_0^d)$. Then we can consider a different parametrizing set in the Lévy-Khintchine formula. Namely using

$$\tilde{U} := \{(g_v, p, Q) \in G_B \times R^d \times Q : (v, p, Q) \in U\}.$$
associated with that Lévy process. Define $N_t = \int_{R^d} x N(t, dx)$, which is finite $P_0$-a.s. as we assume that $\mu$ integrates $|x|$. Moreover, in the finite activity case, i.e. $\lambda = \sup_{v \in V} v(R^d_0) < \infty$, we define the Poisson process $M$ with intensity $\lambda$ by putting $M_t = N(t, R^d_0)$. We also define the filtration generated by $W$ and $N$:

$$F_t := \sigma(W_s, N_s : 0 \leq s \leq t) \vee N; \quad N := \{ A \in \Omega : P_0(A) = 0 \}; \quad F := (F_t)_{t \geq 0}.$$  

**Theorem 2.4.** ([13]) Introduce a set of integrands $A^U_{t,T}, 0 \leq t \leq T$, associated with $U$ as $s$ set of all processes $\theta = (\theta^d, \theta^1, \theta^2)$ defined on $[t,T]$ satisfying the following properties:

1. $(\theta^1, \theta^2)$ is $F$-adapted process and $\theta^d$ is $F$-predictable random field on $[t,T] \times R^d$.
2. For $P_0$-a.a. $w \in \Omega$ and a.e. $s \in [t,T]$ we have that $(\theta^d(s,\cdot)(w), \theta^1_c(w), \theta^2_c(w)) \in U$.
3. $\theta$ satisfies the following integrability condition

$$E^{P_0}[\int_t^T (|\theta^1| + |\theta^2_c|^2 + \int_{R^d} |\theta^d(s,z)|\mu(dz))ds] < \infty.$$  

For $\theta \in A_{0,\infty}^U$ denote the following Lévy-Itô integral as

$$B^{\theta}_{t,T} = \int_t^T \theta^1_s ds + \int_t^T \theta^2_s dw + \int_t^T \int_{R^d} \theta^d(s,z)N(ds,dz).$$  

For every $\xi \in \phi(X_{t_1}, X_{t_2} - X_{t_1}, X_{t_n} - X_{t_{n-1}}) \in Lip(\Omega_T)$, then $\hat{E}[\xi] = \sup_{\theta \in A_{0,\infty}^U} E^{P_0}[\phi(B^{\theta}_{t_1}, B^{\theta}_{t_2}, \ldots, B^{\theta}_{t_n})]$. Let $\xi \in L^1(\Omega)$, we can represent the sublinear expectation in the following way

$$\hat{E}[\xi] = \sup_{\theta \in A_{0,\infty}^U} E^{P_0}[\xi],$$  

where $P^\theta := P_0 \circ (B^{0,\theta})^{-1}, \theta \in A_{0,\infty}^U$. We also denote $\mathcal{B} := \{ P^\theta : \theta \in A_{0,\infty}^U \}$.  

**Definition 2.5.** We define the capacity $c$ associated with $\hat{E}$ by putting

$$c(A) := \sup_{P \in \mathcal{B}} P(A), \quad A \in \mathcal{B}(\Omega).$$  

We will say that a set $A \in \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. We say that a property holds quasi-surely (q.s.) if it holds outside a polar set.

**Lemma 2.1.** Let $X \in L^1(\Omega_T)$ and for some $p > 0$, $\hat{E}[|X|^p] < \infty$. Then, for each $M > 0$,

$$c(|X| > M) \leq \frac{\hat{E}[|X|^p]}{M^p}.$$  

Assume that $G$-Lévy process $X$ has finite activity, i.e.

$$\lambda := \sup_{v \in V} v(R^d_0) < \infty.$$  

Without loss of generality we will also assume that $\lambda = 1$ and that also $\mu(R^d_0) = 1$. Let $X_{u-}$ denote the left limit of $X$ at point $u$, $\Delta X_u = X_u - X_{u-}$, then we can define a Poisson random measure $L(ds,dz)$ associated with the $G$-Lévy process $X$ by putting

$$L([s,t], A) = \sum_{s < u \leq t} 1_A(\Delta X_u), \quad q.s.
Lemma 2.2. For every \( s < t < \infty \) and \( A \in B(R^d_0) \). The random measure is well-defined and may be used to define the pathwise integral.

Let \( H^S_0([0, T] \times R^d_0) \) be a space of all elementary random fields on \([0, T] \times R^d_0\) of the form

\[
K(t, z)(w) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} F_{k,l}(w)\mathbb{I}_{[t_k, t_{k+1})}(r)\psi_l(z), \quad n, m \in \mathbb{N},
\]

where \( 0 \leq t_1 < \ldots < t_n \leq T \) is the partition of \([0, T]\), \( \{\psi_l\}_{l=1}^m \subset C_{b, lip}(R^d) \) are functions with disjoint supports s.t. \( \psi_l(0) = 0 \) and \( F_{k,l} = \phi_{k,l}(X_{t_1}, \ldots, X_{t_{k+1}} - X_{t_k}) \), \( \phi_{k,l} \in C_{b, lip}(R^{d \times k}) \). We introduce the norm on this space

\[
\|K\|_{H^S_0([0, T] \times R^d_0)}^p := \mathbb{E}\left[ \int_0^T \sup_{v \in \mathcal{V}} \int_{R^d_0} |K(t, z)|^p v(dz)dt \right], \quad p = 1, 2.
\]

**Definition 2.6.** Let \( 0 \leq s < t \leq T \). The Itô integral of \( K \in H^S_0([0, T] \times R^d_0) \) w.r.t. jump measure \( L \) is defined as

\[
\int_s^t \int_{R^d_0} K(r, z)L(dr, dz) := \sum_{s < r \leq t} K(r, \Delta X_r), \quad \text{q.s.}
\]

**Lemma 2.2.** For every \( K \in H^S_0([0, T] \times R^d_0) \), we have that \( \int_0^T \int_{R^d_0} K(r, z)L(dr, dz) \) is an element of \( L^p_0(\Omega_T) \).

Let \( H^P_0([0, T] \times R^d_0) \) denote the topological completion of \( H^S_0([0, T] \times R^d_0) \) under the norm \( \| \cdot \|_{H^P_0([0, T] \times R^d_0)} \), \( p = 1, 2 \). Then Itô integral can be continuously extended to the whole space \( H^P_0([0, T] \times R^d_0), p = 1, 2 \). Moreover, the extended integral takes value in \( L^p_0(\Omega_T) \), \( p = 1, 2 \).

We now give the following BDG-type inequality for the integral defined above.

**Lemma 2.3.** For \( K(r, z) \in H^P_0([0, T] \times R^d_0) \), set \( Y_t := \int_0^t \int_{R^d_0} K(r, z)L(dr, dz) \). Then there exists a càdlàg modification \( \tilde{Y}_t \) of \( Y_t \) for all \( t \in [0, T] \) such that

\[
\mathbb{E}[\sup_{0 \leq t \leq T} |\tilde{Y}_t|^2] \leq C_T \mathbb{E}\left[ \int_0^T \sup_{v \in \mathcal{V}} \int_{R^d_0} K^2(r, z)v(dz)dr \right],
\]

where \( C_T > 0 \) is a constant depend on \( T \).

**Proof.** Firstly, let us consider the case:

\[
K(r, z)(w) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} F_{k,l}(w)\mathbb{I}_{[t_k, t_{k+1})}(r)\psi_l(z) \in H^S_0([0, T] \times R^d_0).
\]

For this case, the proof is similar to the theorem 27 in [18]. However, for completeness, we prove it as follows. By the definition of the Itô integral and Theorem 2.4 we have

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left( \int_0^t \int_{R^d_0} K(r, z)L(dr, dz) \right)^2 \right] = \sup_{\theta \in A_{0,T}^d} E^{P_{\theta}}\left[ \sup_{0 \leq t \leq T} ( \sum_{0 \leq r \leq t} K(r, \Delta X_r))^2 \right]
\]

\[
= \sup_{\theta \in A_{0,T}^d} E^{P_{\theta}}\left[ \sup_{0 \leq t \leq T} ( \sum_{0 \leq r \leq t} \sum_{k=1}^{n-1} \sum_{l=1}^{m} \phi_{k,l}(X_{t_1}, \ldots, X_{t_{k+1}} - X_{t_k})\mathbb{I}_{[t_k, t_{k+1})}(r)\psi_l(\Delta X_r))^2 \right]
\]
Then we can rewrite (2.4) as

\[ \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}^\theta \int_{[t_k, t_{k+1} \wedge t]} \psi \left( \frac{\theta^4(r, \Delta N_t)}{\theta^4(r, z)} \right)^2 \right) \right] = \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m \theta^4(r, z) \mu(dz)dr \right)^2 \right], \tag{2.5} \]

where \( F_{k,l}^\theta := \phi_{k,l}(B_{t_1 \wedge t}, \ldots, B_{t_{k+1} \wedge t}) \) and \( N_t \) is the Poisson process in Theorem 2.4. Define a predictable process \( K^\theta(r, z) \) as

\[ K^\theta(r, z) := \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}^\theta \int_{[t_k, t_{k+1} \wedge t]} \psi \left( \frac{\theta^4(r, \Delta N_t)}{\theta^4(r, z)} \right)^2. \]

Then we can rewrite (2.4) as

\[ \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}^\theta \int_{[t_k, t_{k+1} \wedge t]} \psi \left( \frac{\theta^4(r, \Delta N_t)}{\theta^4(r, z)} \right)^2 \right) \right] = \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m \theta^4(r, z) \mu(dz)dr \right)^2 \right], \tag{2.6} \]

where \( N(dr, dz) \) and \( \tilde{N}(dr, dz) \) are respectively the Poisson random measure and the compensated Poisson measure associated with the Lévy process with the Lévy triplet \((0,0,\mu)\). Using the standard BDG inequality and Hölder inequality we get:

\[ \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}^\theta \int_{[t_k, t_{k+1} \wedge t]} \psi \left( \frac{\theta^4(r, \Delta N_t)}{\theta^4(r, z)} \right)^2 \right) \right] \leq 2 \mathbb{E}_t \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m \theta^4(r, z) \mu(dz)dr \right)^2 \right]. \]

where \( \tilde{C}_T \) is a constant depend on \( T \) and \( C_T = 2(T + \tilde{C}_T) \). Note that the intervals \([t_k, t_{k+1}]\) are mutually disjoint, hence

\[ \mathbb{E}_0 \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}^\theta \int_{[t_k, t_{k+1} \wedge t]} \psi \left( \frac{\theta^4(r, \Delta N_t)}{\theta^4(r, z)} \right)^2 \right) \right] \leq C_T \mathbb{E}_0 \left[ \sum_{k=1}^{n-1} \sum_{l=1}^m \int_{t_k}^{t_{k+1}} \mathbb{E}_0 \left[ \left( \int_0^t \int_{R^d} K^\theta(r, z)L(dr, dz) \right)^2 \right] \right] \]

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\[\begin{align*}
&= C_T \sup_{\theta \in \mathcal{A}_T^*} \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} E^{P_0}[\sum_{l=1}^{m} \phi_{k,l}(B^{0,\theta}_{t_l}, \ldots, B^{t_{k-1},\theta}_{t_l})] \int_{R_0^d} \psi_t^2(\theta^l(r, z)) \mu(dz) dr.
\end{align*}\] (2.7)

By the assumptions on the process \(\theta^l\), we know that for a.a. \(w\) and a.e. \(r\) function \(z \to \theta^l(r, z)(w)\) is equal to \(g_t\) for \(v \in \mathcal{V}\). Hence we can transform (2.7) to get

\[\begin{align*}
&\mathbb{E}[\sup_{0 \leq t \leq T} \left( \int_{R_0^d} K(r, z) L(dr, dz) \right)^2] \\
&\leq C_T \sup_{\theta \in \mathcal{A}_T^*} \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} E^{P_0}[\sum_{l=1}^{m} \phi_{k,l}(B^{0,\theta}_{t_l}, \ldots, B^{t_{k-1},\theta}_{t_l})] \int_{R_0^d} \psi_t^2(z) \mu(dz) dr \\
&\leq C_T \sup_{\theta \in \mathcal{A}_T^*} E^{P_0}[\int_{0}^{T} \sup_{v \in \mathcal{V}} \int_{R_0^d} \sup_{k,m} \phi_{k,l}^2(B^{0,\theta}_{t_l}, \ldots, B^{t_{k-1},\theta}_{t_l}) \int_{R_0^d} \psi_t^2(z) \mu(dz) dr] \\
&= C_T \mathbb{E}[\sup_{v \in \mathcal{V}} \int_{R_0^d} K^2(r, z) \mu(dz) dr].
\end{align*}\] (2.8)

For general \(K(r, z) \in H^2([0, T] \times R_0^d)\), choose \(\{K^n, n \geq 1\} \subset H^S([0, T] \times R_0^d)\) such that

\[\|K - K^n\|_{H^S([0, T] \times R_0^d)} \to 0 \quad \text{as} \quad n \to \infty.\]

Set \(Y_t^n = \int_{0}^{T} K^n(r, z) L(dr, dz)\). Then as \(n, m \to \infty\),

\[\mathbb{E}[\sup_{0 \leq t \leq T} (Y_t^n - Y_t^m)^2] \leq C_T \|K^n - K^m\|_{H^2([0, T] \times R_0^d)}^2 \to 0\]

and so there exist a subsequence \(\{Y_t^{n_k}, k \geq 1\}\) such that for any \(k \geq 1\),

\[\left(\mathbb{E}[\sup_{0 \leq t \leq T} (Y_t^{n_k+1} - Y_t^{n_k})^2]\right)^{\frac{1}{2}} \leq \frac{1}{2^k}.\]

Then

\[\left(\mathbb{E}[\sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} (Y_t^{n_k+1} - Y_t^{n_k})^2]\right)^{\frac{1}{2}} = \sup_{\theta \in \mathcal{A}_T^*} \left( E^{P_0} \left( \sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} (Y_t^{n_k+1} - Y_t^{n_k})^2 \right) \right)^{\frac{1}{2}} \leq 1,\]

which implies

\[\sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} |Y_t^{n_k+1} - Y_t^{n_k}| < \infty, \quad \text{q.s.}\]

Set \(\tilde{Y}_t = Y_t^{n_1} + \sum_{k=1}^{\infty} (Y_t^{n_{k+1}} - Y_t^{n_k})\), then \(\tilde{Y}_t\) is q.s. defined on \(\Omega\) for all \(t \in [0, T]\) and for q.s. \(w, t \to \tilde{Y}_t(w)\) is càdlàg. Moreover, \(\mathbb{E}[\sup_{0 \leq t \leq T} (\tilde{Y}_t^2)] < \infty\), and

\[\left(\mathbb{E}[\sup_{0 \leq t \leq T} |\tilde{Y}_t|^2]\right)^{\frac{1}{2}} \leq \left(\mathbb{E}[\sum_{l=0}^{\infty} \sup_{0 \leq t \leq T} |Y_t^{n_{l+1}} - Y_t^{n_l}|^2]\right)^{\frac{1}{2}}\]
Lemma 2.4. For \( p \geq 2 \), \( \eta \in M^p_G(0, T) \) Then

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} | \int_0^s \eta_t dB_t |^p \right] \leq C_p T^{\frac{p}{2} - 1} \int_0^T \mathbb{E}[|\eta_t|^p] dt,
\]

where \( C_p \) is a constant only dependent on \( p \).
Lemma 2.5. For \( p \geq 1, \eta \in M_p^G(0, T) \). Then there exists a constant \( C_p > 0 \) such that
\[
\mathbb{E}[\sup_{0 \leq u \leq T} \int_0^u \eta_t d\langle B \rangle_t |^p] \leq C_p T^{p-1} \int_0^T \mathbb{E}[\eta_t]^p dt.
\]

Let \( M_p^G([0, T]; R^n) \) and \( H_p^G([0, T] \times R^n_0; R^n) \) be the space of \( n \)-dimension stochastic process with each element belong to \( M_p^G(0, T) \) and \( H_p^G([0, T] \times R^n_0) \) respectively.

3 SDEs Driven by G-Lévy Process

In this section, we consider the solution of the following \( n \)-dimension GSDEs:
\[
\begin{cases}
    dY_t = b(t, Y_t)dt + h_{ij}(t, Y_t)d(B^i, B^j)_t + \sigma_i(t, Y_t)dB^i_t + \int_{R^n_0} K(t, Y_t, z) L(dt, dz), \\
    Y_0 = Y_0,
\end{cases}
\]
(3.1)

where \( b(\cdot, x), h_{ij}(\cdot, x), \sigma_i(\cdot, x) \in M^2_G([0, T]; R^n), K(\cdot, x, \cdot) \in H^2_G([0, T] \times R^n_0; R^n) \) for each \( x \in R^n \), \( Y_0 \in R^n \) is the initial value with \( \mathbb{E}|Y_0|^2 < \infty \), \( (\langle B^i, B^j \rangle_t)_{t \geq t_0} \) is the mutual variation process of the \( d \)-dimension G-Brownian motion \( (B_t)_{t \geq t_0} \).

Here and in the rest of this paper we use the Einstein convention, i.e., the above repeated indices of \( i \) and \( j \) within one term imply the summation form 1 to \( d \), i.e.,

\[
\int_0^t h_{ij}(s, Y_s)d\langle B^i, B^j \rangle_s := \sum_{i,j=1}^d \int_0^t h_{ij}(s, Y_s)d\langle B^i, B^j \rangle_s,
\]

\[
\int_0^t \sigma_i(s, Y_s)dB^i_s := \sum_{i=1}^d \int_0^t \sigma_i(s, Y_s)dB^i_s.
\]

Theorem 3.1. Suppose that

(a) there exists a function \( H(t, u) : R_+ \times R_+ \rightarrow R_+ \) such that

(a1) for fixed \( t, H(t, u) \) is continuous nondecreasing with respect to \( u \),

(a2) for \( 0 \leq t_0 < t \leq T \) and \( X_t \in L^2_G(\Omega_t) \),

\[
b(t, X_t), h_{ij}(t, X_t), \sigma_i(t, X_t) \in M^2_G((0, T]; R^n), K(t, X_t, z) \in H^2_G([0, T] \times R^n_0; R^n)
\]

and

\[
\mathbb{E}|b(t, X_t)|^2 + \mathbb{E}|h_{ij}(t, X_t)|^2 + \mathbb{E}|\sigma_i(t, X_t)|^2 + \mathbb{E}[\sup_{v \in \mathbb{V}} \int_{R^n_0} |K(t, X_t, z)|^2 v(dz)]
\]

\[
\leq H(t, \mathbb{E}|X_t|^2),
\]

(3.2)

(a3) for any \( M > 0 \), the differential equation
\[
\frac{du}{dt} = MH(t, u)
\]

has a global solution \( u_t \) for any initial value \( u_{t_0} \);
(b) there exist a function $F(t,u) : R_+ \times R_+ \rightarrow R_+$ such that

(b1) for fixed $t$, $F(t,u)$ is continuous nondecreasing in $u$ and $F(t,0) = 0$,

(b2) for $t_0 < t \leq T$ and $X_t, Y_t \in L^2_G(\Omega)$,

\[
\mathbb{E}[\|b(t,X_t) - b(t,Y_t)\|^2] + \mathbb{E}[h_{ij}(t,X_t) - h_{ij}(t,Y_t)]^2 + \mathbb{E}[\sigma_i(t,X_t) - \sigma_i(t,Y_t)]^2 + \mathbb{E}[\sup_{r \leq t} \|X_r - Y_r\|^2] \leq F(t, \sup_{r \leq t} |X_r|)
\]

(b3) for any constant $M > 0$, if a non-negative function $\varphi_t$ satisfies

\[
\varphi_t \leq M \int_{t_0}^t F(s, \varphi_s) ds
\]

for all $t > t_0$, then $\varphi_t = 0$.

Then (3.1) has a unique càdlàg solution $Y_t \in L^2_G(\Omega)$ for $t_0 < t \leq T$.

**Proof** Let $Y^n_0 := Y_0$ and for $n \in \mathbb{N}$,

\[
Y^n_t := Y_0 + \int_{t_0}^t b(s,Y_s^{n-1}) ds + \int_{t_0}^t h_{ij}(s,Y_s^{n-1}) d(B^i,B^j)_s + \int_{t_0}^t \sigma_i(s,Y_s^{n-1}) dB^i_s + \int_{t_0}^t \int_{\mathbb{R}^d} K(s,Y_s^{n-1},z)L(ds,dz).
\]

First of all, we show that for $t_0 < t \leq T$ and $n \in \mathbb{N}$,

\[
Y^n_t \in L^2_G(\Omega) \quad \text{and} \quad \mathbb{E}[\sup_{r \leq t} |Y_r^n|^2] \leq u_t \leq u_T,
\]

where $u_t$ is the solution of differential equation in (a3) satisfies

\[
u_t = C_1(T)\mathbb{E}[|Y_0|^2] + C_1(T) \int_{t_0}^t H(s,\nu_s) ds
\]

and

\[
C_1(T) := 5(1 + T + C_2 T + C_2 + C_T).
\]

Suppose $Y_t^{n-1} \in L^2_G(\Omega)$ and $\mathbb{E}[\sup_{r \leq t} |Y_r^{n-1}|^2] \leq u_t$, which together with the definition of the $G$-stochastic integral and (a2) yield $Y^n_t \in L^2_G(\Omega)$.

Secondly, by $C_r$-inequality, Lemma 2.3.2-5, Hölder inequality and (a1)-(a2), we get

\[
\mathbb{E}[\sup_{r \leq t} |Y_t^n|^2] \leq 5[\mathbb{E}[|Y_0|^2] + \mathbb{E}[\sup_{t_0 \leq r \leq t} |b(s,Y_s^{n-1})|^2] + \mathbb{E}[\sup_{t_0 \leq r \leq t} |h_{ij}(s,Y_s^{n-1})|^2] + \mathbb{E}[\sup_{t_0 \leq r \leq t} |\sigma_i(s,Y_s^{n-1})||2B^i_s|^2] + \mathbb{E}[\sup_{t_0 \leq r \leq t} |K(s,Y_s^{n-1},z)L(ds,dz)|^2] \leq 5[\mathbb{E}[|Y_0|^2] + t \int_{t_0}^t \mathbb{E}[|b(s,Y_s^{n-1})|^2] ds + C_2 t \int_{t_0}^t \mathbb{E}[|h_{ij}(s,Y_s^{n-1})|^2] ds + C_2 t \int_{t_0}^t \mathbb{E}[|\sigma_i(s,Y_s^{n-1})|^2] ds + C_T \int_{t_0}^t \mathbb{E}[\sup_{r \leq t} |K(s,Y_r^{n-1},z)|^2|v(dz)| ds]
\]

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By (b3), we obtain that
\[ \xi \leq C_1(T)\mathbb{E}[|Y_0|^2] + \int_{t_0}^t H(s, \mathbb{E}[\sup_{r \leq s}|Y_r^{m-1}|^2])ds \]
\[ \leq C_1(T)\mathbb{E}[|Y_0|^2] + \int_{t_0}^t H(s, u_s)ds \leq u_t \tag{3.6} \]
for all \( t \leq T \). By the induction method, (3.5) is proved.

Next, by the same deduction as above, we have
\[
\mathbb{E}[\sup_{r \leq t}|Y^n_r - Y_r^{m-1}|^2] \leq 4\mathbb{E}[\sup_{r \leq t}\int_{t_0}^r (b(s, Y_s^{n-1}) - b(s, Y_s^{m-1}))ds]^2 \\
+ \mathbb{E}[\sup_{r \leq t}\int_{t_0}^r (h_{ij}(s, Y_s^{n-1}) - h_{ij}(s, Y_s^{m-1}))d(B^i, B^j)_s] + \mathbb{E}[\sup_{r \leq t}\int_{t_0}^r (\sigma_i(s, Y_s^{n-1}) - \sigma_i(s, Y_s^{m-1}))dB^i_s] \\
+ \mathbb{E}[\sup_{r \leq t}\int_{t_0}^r (K(s, Y_s^{n-1}, z) - K(s, Y_s^{m-1}, z)L(ds, dz)]^2 \\
\leq C_2(T)\int_{t_0}^t \mathbb{E}[|b(s, Y_s^{n-1}) - b(s, Y_s^{m-1})|^2]ds \\
+ \int_{t_0}^t \mathbb{E}[|h_{ij}(s, Y_s^{n-1}) - h_{ij}(s, Y_s^{m-1})|^2]ds + \int_{t_0}^t \mathbb{E}[|\sigma_i(s, Y_s^{n-1}) - \sigma_i(s, Y_s^{m-1})|^2]ds \\
+ \int_{t_0}^t \mathbb{E}[\sup_{v \in V} \int_{R^d} |K(s, Y_s^{n-1}, z) - K(s, Y_s^{m-1}, z)|^2 v(dz)]ds \tag{3.7} \\
\leq C_2(T)\int_{t_0}^t F(s, \mathbb{E}[\sup_{r \leq s}|Y_r^{n-1} - Y_r^{m-1}|^2])ds,
\]
where \( C_2(T) = 4(T + C_1T + C_2 + C_T) \).

Let
\[ \xi_t = \limsup_{n,m \to \infty} \mathbb{E}[\sup_{r \leq t}|Y_r^{n-1} - Y_r^{m-1}|^2] \]
It follows from the Fatou lemma and (b1) that
\[ \xi_t \leq C_2(T)\int_{t_0}^t F(s, \xi_s)ds. \]
By (b3), we obtain that \( \xi_t = 0 \), i.e.
\[ \limsup_{n,m \to \infty} \mathbb{E}[\sup_{r \leq t}|Y_r^{n-1} - Y_r^{m-1}|^2] = 0. \]
Then there exists a subsequence \( Y_{t_k}^{n_k} \) such that for any \( k \geq 1 \),
\[ (\mathbb{E}[\sup_{r \leq t}|Y_{t_{k+1}}^{n_{k+1}} - Y_{t_k}^{n_k}|^2])^{\frac{1}{2}} \leq \frac{1}{2^k}. \]
Thus
\[
(\mathbb{E}\left[\sup_{r \leq t}|Y_{t_{k+1}}^{n_{k+1}} - Y_{t_k}^{n_k}|^2\right])^{\frac{1}{2}} = \sup_{\theta \in A_{t_{k+1}}^R} (E_{t_{k+1}}^{P\theta} \sup_{r \leq t}|Y_{r}^{n_{k+1}} - Y_{r}^{n_k}|^2)^{\frac{1}{2}} \\
\leq \sup_{\theta \in A_{t_{k+1}}^R} \sum_{k=1}^{\infty} (E_{t_{k+1}}^{P\theta} \sup_{r \leq t}|Y_{r}^{n_{k+1}} - Y_{r}^{n_k}|^2)^{\frac{1}{2}} \leq \sum_{k=1}^{\infty} (\mathbb{E}[\sup_{r \leq t}|Y_{r}^{n_{k+1}} - Y_{r}^{n_k}|^2])^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{1}{2^k},
\]
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which implies
\[ \sum_{k=1}^{\infty} \sup_{r \leq t} |Y_{r,n}^{k+1} - Y_{r,n}^{k}| < \infty \text{ q.s.} \]

Set \( Y_t = Y_t^{n_1} + \sum_{k=1}^{\infty} (Y_{r,n}^{k+1} - Y_{r,n}^{k}) \), then \( Y_t \) is q.s. defined on \( \Omega \) for all \( t \in [0, T] \) and càdlàg. Moreover, \( (\hat{E} \sup_{r \leq t} |Y_{r}|^2)^{1/2} < \infty \), and

\[
\begin{align*}
\left( \hat{E} \sup_{r \leq t} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2 \right)^{1/2} & \leq \left( \hat{E} \left( \sum_{i=k}^{\infty} |Y_{r,n}^{i+1} - Y_{r,n}^{i}|^2 \right)^2 \right)^{1/2} \\
& = \sup_{\theta \in \mathcal{A}^{l}_{\mathcal{F}}} \left( E^{\mathcal{F}} \left( \sum_{i=k}^{\infty} |Y_{r,n}^{i+1} - Y_{r,n}^{i}|^2 \right)^{1/2} \right) \leq \sum_{i=k}^{\infty} \left( \hat{E} \sup_{r \leq t} |Y_{r,n}^{i+1} - Y_{r,n}^{i}|^2 \right)^{1/2}.
\end{align*}
\]

(3.9)

Letting \( k \to \infty \) and taking limits on both sides of the above inequality, we get

\[ \lim_{k \to \infty} \hat{E} \sup_{r \leq t} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2 = 0. \]

Then by the Hölder inequality, (b2) and Lemma 2.3-2.5, it holds that

\[ \hat{E} \sup_{r \leq t} \left| \int_{t_0}^{r} b(s,Y_{s}^{n})ds - \int_{t_0}^{r} b(s,Y_{s})ds \right|^2 \leq C_2(T) \int_{t_0}^{t} F(s, \hat{E} \sup_{r \leq s} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2)ds, \]

\[ \hat{E} \sup_{r \leq t} \left| \int_{t_0}^{r} h_{ij}(s,Y_{s}^{n})d(B_{s}^{i},B_{s}^{j}) - \int_{t_0}^{r} b(s,Y_{s})d(B_{s}^{i},B_{s}^{j}) \right|^2 \leq C_2(T) \int_{t_0}^{t} F(s, \hat{E} \sup_{r \leq s} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2)ds, \]

\[ \hat{E} \sup_{r \leq t} \left| \int_{t_0}^{r} \sigma_i(s,Y_{s}^{n})dB_{s}^{i} - \int_{t_0}^{r} \sigma_i(s,Y_{s})dB_{s}^{i} \right|^2 \leq C_2(T) \int_{t_0}^{t} F(s, \hat{E} \sup_{r \leq s} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2)ds, \]

and

\[ \hat{E} \sup_{r \leq t} \left| \int_{t_0}^{r} K(s,Y_{s}^{n},z)L(ds,dz) - \int_{t_0}^{r} K(s,Y_{s},z)L(ds,dz) \right|^2 \]

\[ \leq C_2(T) \int_{t_0}^{t} F(s, \hat{E} \sup_{r \leq s} |Y_{r,n}^{k} - Y_{r}^{\sharp}|^2)ds. \]

Taking limits on both sides of (3.4) in \( L^2_{\mathcal{F}}(\Omega) \), we obtain that \( Y \) satisfies (3.1).

Next, let \( Y \) and \( Y' \) be both solutions of (1.1), then by the same way as above, we obtain that

\[ \hat{E} \sup_{r \leq t} |Y_{r} - Y_{r}^{\sharp}|^2 \leq C_2(T) \int_{t_0}^{t} F(s, \hat{E} \sup_{r \leq s} |Y_{r} - Y_{r}^{\sharp}|^2)ds \]

for all \( t \leq T \). We can apply (b3) deduce that \( \hat{E} \sup_{r \leq t} |Y_{r} - Y_{r}^{\sharp}|^2 = 0 \), which implies that \( Y_t = Y_t' \), \( t_0 < t \leq T \) q.s.. Thus the proof is completed.
4 Exponential stability of the solutions

In this section, we consider exponential stability of the following $n$-dimension GSDEs:

$$
\begin{aligned}
dY_t &= b(t, Y_t)dt + h_{ij}(t, Y_t)d(B^i, B^j)_t + \sigma_i(t, Y_t)dB^i_t + \int_{\mathbb{R}_+^d} K(t, Y_t, z)L(dt, dz), \\
Y_{t_0} &= Y_0
\end{aligned}
$$

(4.1)

where $b, h_{ij}, \sigma_i \in M_n^2([0, T]; \mathbb{R}^n)$, $K \in H^2_{G}([0, T] \times \mathbb{R}_+^d; \mathbb{R}^n)$, $Y_0 \in \mathbb{R}^n$ is the initial value with $\hat{E}|Y_0|^2 < \infty$, $(\langle B^i, B^j \rangle_t)_{t \geq t_0}$ is the mutual variation process of the $d$-dimension $G$-Brownian motion $(B_t)_{t \geq t_0}$. We assume the functions $b, h_{ij}, \sigma_j$ and $K$ satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \geq t_0$. For the purpose of stability in this paper, we also assume that $b(t, 0) = 0, h_{ij}(0) = 0, \sigma_i(t, 0) = 0, K(t, 0, z) = 0$. Thus, the system (4.1) has a trivial solution.

**Definition 4.1.** The trivial solution of the system (4.1) is said to be

1. mean square exponential stable if for any initial $Y_0$, the solution $Y_t$ satisfies that

$$
\hat{E}|Y_t|^2 \leq C\hat{E}|Y_0|^2 e^{-\lambda(t-t_0)},
$$

where $\lambda$ and $C$ are positive constants independent of $t_0$.

2. quasi sure exponentially stable if the solution $Y_t$ satisfies that

$$
\limsup_{t \to \infty} \frac{1}{t} \ln |Y_t| \leq -\lambda, \quad \text{q.s.,}
$$

for any initial data $Y_0$ and $\lambda > 0$.

**Definition 4.2.** The function $V$ is said to belong to the class $v_0$, if $V(t, Y) \in C^{1,2}([t_0, +\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, i.e., $V_t, V_Y, V_{YY}$ are continuous on $[t_0, +\infty) \times \mathbb{R}^n$, and $V_{YY}$ satisfy local Lipschitz condition, where

$$
V_i(t, Y) := \frac{\partial V(t, Y)}{\partial t}, \quad V_Y(t, Y) := \left( \frac{\partial V(t, Y)}{\partial Y_1}, \frac{\partial V(t, Y)}{\partial Y_2}, \ldots, \frac{\partial V(t, Y)}{\partial Y_n} \right)
$$

and

$$
V_{YY}(t, Y) := \left( \frac{\partial^2 V(t, Y)}{\partial Y_i \partial Y_j} \right)_{i,j=1}^{n \times n}.
$$

**Definition 4.3.** For each $V \in v_0$, we define an operator $L$ by

$$
\begin{aligned}
LV(t, Y_t) &:= V_t(t, Y_t) + \langle V_Y(t, Y_t), b(t, Y_t) \rangle \\
&+ \sup_{Q \in \mathcal{Q}} \text{tr}\left[ (\langle V_Y(t, Y_t), h(t, Y_t) \rangle + \frac{1}{2} \langle V_{YY}(t, Y_t) \sigma(t, Y_t), \sigma(t, Y_t) \rangle)QQ^T \right] \\
&+ \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_+^d} (V(t, Y_{t-} + K(t, Y_t, z)) - V(t, Y_{t-}))v(dz),
\end{aligned}
$$

(4.2)

where $\langle V_Y(t, Y_t), h(t, Y_t) \rangle + \langle V_{YY}(t, Y) \sigma(t, Y_t), \sigma(t, Y_t) \rangle$ is the symmetric matrix in $\mathbb{S}^d$, with the form

$$
\begin{aligned}
\langle V_Y(t, Y_t), h(t, Y_t) \rangle + \langle V_{YY}(t, Y_t) \sigma(t, Y_t), \sigma(t, Y_t) \rangle &= \left[ \langle V_Y(t, Y_t), h_{ij}(t, Y_t) \rangle + \langle V_{YY}(t, Y_t) \sigma_i(t, Y_t), \sigma_j(t, Y_t) \rangle \right]_{i,j=1}^{d}.
\end{aligned}
$$
Let $Y_t$ be a solution of (4.1), for convention, we use the following notations in the sequel

$$M_t^{\lambda} := \int_{s}^{t} e^{\lambda r}[(V_Y(r,Y_r), h_{ij}(r,Y_r)) + \frac{1}{2}(V_{YY}(r,Y_r)\sigma_i(r,Y_r), \sigma_j(r,Y_r))]d(B^i, B^j)_r$$

$$- \int_{s}^{t} e^{\lambda r} \sup_{Q \in \mathbb{Q}} \text{tr}[(V_Y(t,Y_t), h(t,Y_t)) + \frac{1}{2}(V_{YY}(r,Y_r)\sigma(r,Y_r), \sigma(r,Y_r))]QQ^T]dr,$$

$$P_t^{\lambda} = \int_{s}^{t} \int_{R_0^d} e^{\lambda r}[V(r,Y_{r-} + K(r,Y_r,z)) - V(r,Y_{r-})]L(dr,dz)$$

$$- \int_{s}^{t} \sup_{v \in V} \int_{R_0^d} e^{\lambda r}[V(r,Y_{r-} + K(r,Y_r,z)) - V(r,Y_{r-})]v(dz)dr. \quad (4.3)$$

From Theorem 2.2 in Peng [14], $\{M_t^{\lambda}\}_{t \geq s}$ is a $G$-martingale. From Theorem 13 in [19], $P_t^{\lambda}$ is also a $G$-martingale.

We are now in a position to propose the mean square exponentially stability for the system (4.1).

**Theorem 4.1.** Assume that there exist a $V \in v_0$, constants $C_4 > C_3 > 0$ and $\lambda > 0$ such that

(c) $C_3|Y|^2 \leq V(t,Y) \leq C_4|Y|^2$ for all $t \geq t_0, Y \in \mathbb{R}^n$,

(d) $LV(t,Y_t) \leq -\lambda V(t,Y_t)$.

Then, the trivial solution of system (4.1) is mean square exponentially stable.

**Proof** For $t \in [t_0,T]$, applying the G-Itô formula (Theorem 32 in [18]) to $e^{\lambda r}V(t,Y_t)$, we obtain

$$d(e^{\lambda r}V(t,Y_t)) = e^{\lambda r}[LV(t,Y_t) + V_t(t,Y_t) + \langle V_Y(t,Y_t), b(t,Y_t) \rangle]dt$$

$$+ e^{\lambda r}(V_Y(t,Y_t), T_{ij}(t,Y_t))dB^i_t + e^{\lambda r}(V_Y(t,Y_t), h_{ij}(t,Y_t))d(B^i, B^j)_t$$

$$+ \int_{R_0^d} e^{\lambda r}[V(t,Y_{r-} + K(t,Y_r,z)) - V(t,Y_{r-})]L(dt,dz). \quad (4.4)$$

Thus, we have

$$e^{\lambda r}V(t,Y_t) = e^{\lambda t_0}V(t_0,Y_0) + \int_{t_0}^{t} e^{\lambda r}[LV(r,Y_r) + LV_r(t,Y_t)]dr + \int_{t_0}^{t} e^{\lambda r}[V_Y(r,Y_r), \sigma_j(r,Y_r)]dB^j_t$$

$$+ M_t^{\lambda t_0} + P_t^{\lambda t_0}. \quad (4.5)$$

Since the last three terms are $G$-martingale, then take expectation on the two sides, we get

$$\mathbb{E}[e^{\lambda r}V(t,Y_t)] \leq \mathbb{E}[e^{\lambda t_0}V(t_0,Y_0)] + \mathbb{E}[\int_{t_0}^{t} e^{\lambda r}[LV(r,Y_r) + LV_r(t,Y_t)]dr]. \quad (4.6)$$

From condition (d), we have

$$\mathbb{E}[e^{\lambda r}V(t,Y_t)] \leq \mathbb{E}[e^{\lambda t_0}V(t_0,Y_0)]. \quad (4.7)$$

Since $V(t,Y) \leq C_4|Y|^2$, it holds that $\mathbb{E}[e^{\lambda t_0}V(t_0,Y_0)] \leq C_4\mathbb{E}[|Y_0|^2]$ and

$$\mathbb{E}[e^{\lambda r}V(t,Y_t)] \leq C_4\mathbb{E}[|Y_0|^2]e^{\lambda t_0},$$
Theorem 4.2. Assume that there exist a $V \in v_0$, constants $C_4 > C_3 > 0$ and $\lambda > 0$ such that

(c) $C_3|Y|^2 \leq V(t, Y) \leq C_4|Y|^2$ for all $t \geq t_0, Y \in \mathbb{R}^n$,

(d1) $LV(t, Y) \leq (-\lambda + \lambda_1(t))V(t, Y)$, where $\lambda_1 : [t_0 + \infty) \rightarrow \mathbb{R}$ is a continuous function such that $\int_{t_0}^{t} \lambda_1^+(s)ds < \infty$.

Then, the trivial solution of system (4.1) is mean square exponentially stable.

Proof Since $\int_{t_0}^{t} \lambda_1^+(s)ds < \infty$, it follows that there exists a positive constant $M_1$, such that $\int_{t_0}^{t} \lambda_1^+(s)ds < M_1$. With the same discussions as in Theorem 4.1, applying the G-Itô formula to $e^{\lambda t} V(t, Y_t)$, we obtain

$$
\hat{E}[e^{\lambda t} V(t, Y_t)] \leq \hat{E}[e^{\lambda t_0} V(t_0, Y_0)] + \hat{E}[\int_{t_0}^{t} e^{\lambda r} [LV(r, Y_r) + LV_r(t, Y_r)]dr].
$$

From condition (d1), we have

$$
\hat{E}[e^{\lambda t} V(t, Y_t)] \leq \hat{E}[e^{\lambda t_0} V(t_0, Y_0)] + \hat{E}[\int_{t_0}^{t} e^{\lambda r} \lambda_1(r)V(r, Y_r)dr]
$$

$$
\leq \hat{E}[e^{\lambda t_0} V(t_0, Y_0)] + \int_{t_0}^{t} \lambda_1^+(r) \hat{E}[e^{\lambda r} V(r, Y_r)]dr.
$$

By the Gronwall inequality and condition (c), we have

$$
\hat{E}[e^{\lambda t} V(t, Y_t)] \leq \hat{E}[e^{\lambda t_0} V(t_0, Y_0)] e^{\int_{t_0}^{t} \lambda_1^+(r)dr}
$$

$$
\leq C_4 \hat{E}[|Y_0|^2] e^{\int_{t_0}^{t} \lambda_1^+(r)dr} \leq M_1 C_4 \hat{E}[|Y_0|^2] e^{\lambda t},
$$

so

$$
\hat{E}|Y_t|^2 \leq \frac{\hat{E}[V(t, Y_t)]}{C_3} \leq \frac{M_1 C_4}{C_3} \hat{E}[|Y_0|^2] e^{-\lambda(t-t_0)}.\tag{4.11}
$$

The following theorem shows that the solution of system (4.1) is quasi sure exponentially stable under some additional conditions.

Theorem 4.3. Assume that there exist a $V \in v_0$, positive constants $C_3, C_4, \lambda$ and $\alpha$ such that

(c) $C_3|Y|^2 \leq V(t, Y) \leq C_4|Y|^2$ for all $t \geq t_0, Y \in \mathbb{R}^n$,

(d1) $LV(t, Y) \leq (-\lambda + \lambda_1(t))V(t, Y)$, where $\lambda_1 : [t_0 + \infty) \rightarrow \mathbb{R}$ is a continuous function such that $\int_{t_0}^{t} \lambda_1^+(s)ds < \infty$.

(e) $\hat{E}[b(t, Y_t)^2 + h_{ij}(t, Y_t)^2] + |\sigma_i(t, Y_t)|^2 + \sup_{v \in \mathcal{V}} \int_{R^d} |K(t, Y_t, z)|^2 v(dz) < \alpha \hat{E}[|Y_t|^2].$

Then, the trivial solution of system (4.1) is quasi sure exponentially stable.

Proof The conditions of Theorem 4.3 imply that all the conditions of Theorem 4.2 hold, so the solution of system (4.1) is mean square exponentially stable. Therefore, there exist a positive constant $M_2$ such that

$$
\hat{E}[|Y_t|^2] \leq M_2 e^{-\lambda(t-t_0)}.\tag{4.12}
$$
In fact,

\[ Y_{t+s} = Y_t + \int_t^{t+s} b(r, Y_r) dr + \int_t^{t+s} h_{ij}(r, Y_r) d(B^i, B^j)_r \\
+ \int_t^{t+s} \sigma_i(r, Y_r) dB^i_r + \int_t^{t+s} \int_{R^d_0} K(r, Y_r, z) L(dr, dz). \]  \hspace{1cm} (4.13)

By \(C_t\)-inequality, it holds that

\[ |Y_{t+s}|^2 \leq 5|Y(t)|^2 + \int_t^{t+s} |b(r, Y_r)|^2 dr + \int_t^{t+s} h_{ij}(r, Y_r) d(B^i, B^j)_r^2 \\
+ \int_t^{t+s} |\sigma_i(r, Y_r)|^2 dB^i_r^2 + \int_t^{t+s} \int_{R^d_0} K(r, Y_r, z) L(dr, dz)^2]. \]  \hspace{1cm} (4.14)

Furthermore, we obtain

\[ \hat{E}[\sup_{0 \leq s \leq \tau} |Y_{t+s}|^2] \leq 5\hat{E}[|Y_t|^2] + \hat{E}\int_t^{t+\tau} |b(r, Y_r)|^2 dr + \hat{E}[\sup_{0 \leq s \leq \tau} |h_{ij}(r, Y_r)|^2 dB^i_r dB^j_r^2] \\
+ \hat{E}[\sup_{0 \leq s \leq \tau} |\sigma_i(r, Y_r)|^2 dB^i_r^2] + \hat{E}[\sup_{0 \leq s \leq \tau} \int_t^{t+s} \int_{R^d_0} K(r, Y_r, z) L(dr, dz)^2]], \]  \hspace{1cm} (4.15)

where \(\tau\) is a positive constant.

By the Hölder inequality, (e) and (4.12), we have

\[ \hat{E}\int_t^{t+\tau} |b(r, Y_r)|^2 dr \leq \tau \int_t^{t+\tau} \hat{E}|Y_r|^2 dr \\
\leq \tau \int_t^{t+\tau} \alpha \hat{E}|Y_r|^2 dr \leq \frac{\alpha M_2}{\lambda} e^{-\lambda (t-t_0)}. \]  \hspace{1cm} (4.16)

By Lemma 2.5, (e) and (4.12), we have

\[ \hat{E}[\sup_{0 \leq s \leq \tau} |h_{ij}(r, Y_r)|^2 dB^i_r dB^j_r^2] \leq C_2^2 \int_t^{t+\tau} \hat{E}|h_{ij}(r, Y_r)|^2 dr \\
\leq C_2^2 \alpha \tau \hat{E}|Y_r|^2 dr \leq \frac{C_2^2 \alpha \tau M_2}{\lambda} e^{-\lambda (t-t_0)}. \]  \hspace{1cm} (4.17)

By Lemma 2.4, (e) and (4.12), we have

\[ \hat{E}[\sup_{0 \leq s \leq \tau} |\sigma_i(r, Y_r)|^2 dB^i_r^2] \leq C_2 \int_t^{t+\tau} \hat{E}|\sigma_i(r, Y_r)|^2 dr \\
\leq C_2 \alpha \tau \hat{E}|Y_r|^2 dr \leq \frac{C_2 \alpha M_2}{\lambda} e^{-\lambda (t-t_0)}. \]  \hspace{1cm} (4.18)

Similarly, by Lemma 2.3, we have

\[ \hat{E}[\sup_{0 \leq s \leq \tau} \int_t^{t+s} \int_{R^d_0} K(r, Y_r, z) L(dr, dz)^2] \leq C_\tau \int_t^{t+\tau} \hat{E}[\sup_{v \in V} \int_{R^d_0} |K(r, Y_r, z)|^2 v(dz)] dr \\
\leq C_\tau \alpha \int_t^{t+\tau} \hat{E}|Y_r|^2 dr \leq \frac{C_\tau \alpha M_2}{\lambda} e^{-\lambda (t-t_0)}. \]  \hspace{1cm} (4.19)
Substituting (4.16)-(4.19) into (4.15), we get
\[
\hat{E}[ \sup_{0 \leq s \leq \tau} |Y_{t+s}|^2] \leq M_3 e^{-\lambda t},
\]
(4.20)
where \(M_3 > 0\) is a constant. Then for \(n = 1, 2, \ldots\), it follows that
\[
\hat{E}[ \sup_{n\tau \leq t \leq (n+1)\tau} |Y_t|^2] \leq M_3 e^{-\lambda n\tau}.
\]
(4.21)
Hence, for an arbitrary \(\varepsilon \in (0, \lambda)\) and \(n \in \mathbb{N}\), from Lemma 2.1, we derive that
\[
c(w) \sup_{n\tau \leq t \leq (n+1)\tau} |Y_t|^2 \leq e^{-\frac{(\lambda-\varepsilon)n\tau}{2}}, \quad \text{q.s.}
\]
(4.22)
Taking \(\limsup\) in (4.22) leads to quasi-surely exponential estimate, that is,
\[
\limsup_{t \to \infty} \frac{\ln |Y_t|}{t} \leq -\frac{(\lambda-\varepsilon)}{2} \quad \text{q.s.}
\]
(4.23)
Letting \(\varepsilon \to 0\), we obtain the desired result.

5 An example

In this section, an example is given to illustrate the effectiveness of the obtained results in section 4.

We consider the following one dimensional G-stochastic differential equations
\[
\begin{aligned}
\frac{dY_t}{t} &= -2Y_t dt - \left(\frac{\sin^2 t}{2(1+t^2)} Y_t d(B_t - B_t) + (1 + \frac{\sin|t|}{\sqrt{1+t^2}}) Y_t dB_t + \int_{R_0} R_t R(z) L(dt, dz), \\
Y_0 &= Y_0
\end{aligned}
\]
(5.1)
where \(B_t\) is a one dimensional G-Brownian motion. Let \(b(t, Y_t) = -2Y_t, \ h(t, Y(t)) = -\frac{\sin^2 t}{2(1+t^2)} Y_t, \)
\(\sigma(t, Y_t) = (1 + \frac{\sin|t|}{\sqrt{1+t^2}}) Y_t, \ K(t, Y(t), z) = R(z) Y_t, \ V(t, Y) = |Y|^2\) and the function \(R(z)\) is assumed to satisfy
\[
\sup_{v \in V} \int_{R_0} |R(z)|^2 v(dz) < k
\]
and
\[
\sup_{v \in V} \int_{R_0} [(1 + R(z))^2 - 1] v(dz) =: l < 3.
\]
Then we have
\[ V_Y(t,Y)b(t,Y_t) = -4Y^2, \]
\[ V_Y(t,Y)h(t,Y_t) = -\frac{\sin^2 t}{1+t^2}Y^2, \]
\[ V_Y(t,Y)\sigma^2(t,Y_t) = 2(1 + \frac{\sin t}{\sqrt{1+t^2}})^2Y^2, \]
\[ LV(t,Y) = -4Y^2 - \frac{\sin^2 t}{1+t^2}Y^2 + (1 + \frac{\sin t}{\sqrt{1+t^2}})^2Y^2 + \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} [(1 + R(z))^2 - 1]Y^2v(dz) \]
\[ \leq -3Y^2 + \frac{2|\sin t|}{\sqrt{1+t^2}}Y^2 + \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} [(1 + R(z))^2 - 1]v(dz)Y^2 \]
\[ \leq (-3 + t)Y^2 + \frac{2|\sin t|}{\sqrt{1+t^2}}Y^2. \quad (5.2) \]

Let \( \lambda_1(t) = \frac{2|\sin t|}{\sqrt{1+t^2}} \), we can see \( \int_0^\infty \lambda_1^2(s)ds < \infty \). From Theorem 4.2, we easily know that the solution of system (5.1) is exponentially stable in mean square. Moreover
\[ |b(t,Y(t))|^2 = 4|Y|^2, \quad |h(t,Y(t))|^2 \leq |Y|^2, \quad |\sigma(t,Y(t))|^2 \leq 4|Y|^2, \]
\[ |\sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} K(t,Y,z)v(dz)|^2 \leq k|Y|^2. \]

So, we have
\[ \hat{E}[|b(t,Y_t)|^2 + |h(t,Y_t)|^2 + |\sigma(t,Y_t)|^2 + \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} |K(t,Y,z)|^2v(dz)] < (9 + k)\hat{E}[|Y_t|^2]. \]

From Theorem 4.3, letting \( \alpha = 9 + k \), we easily know that the solution of system (5.1) is quasi sure exponentially stable.

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