Pinched Gluon Vertex Operator in Super Worldline Formalism

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Abstract

We reformulate, using super worldline formalism, the pinched gluon vertex operator proposed by Strassler. The pinched vertex operator turns out to be the product of two gluon vertex operators with the insertion of $\delta$-function which makes the super distances between them zero. Thus the pinch procedures turn out to be nothing but the insertions of $\delta$-function. Applying our formulation to two-loop diagrams which are the QED correction to gluon scatterings via a single spinor loop, with the QED charge $e$ being replaced by the strong coupling $g$, we show various formulae on pinched $N$-point functions.

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1 Introduction

The worldline formalism of field theory has become an important concept in the relevance to the Bern-Kosower rules, which provide a simple reorganization of one-loop Feynman amplitudes in Yang-Mills gauge theory (see for a review). Roughly speaking, we have two advantages in this formalism. In the first place, one is a technical merit to get rid of tedious and extravagant Feynman rule calculation. For example, 5-point gluon scattering amplitude was calculated through the Bern-Kosower rules. The simplified calculation method seems to be promising to obtain further complicated results such as graviton scatterings and multi-loop generalization. Although the worldline formalism only generates the effective action (the sum of 1PI graphs) and not the full scattering amplitudes, it certainly plays an important role to make a connection between particle and string theories. Another one is a clarification of underlying physical or mathematical structure and concept, for example, worldline supersymmetry, which enables us to express gluon amplitudes as correlation functions of worldline superfields. These interesting properties are strongly related to string and conformal field theories and thus provide a phenomenological motivation of studying lower dimensional field theories. None of these points can be understood through standard Feynman rule calculation.

There are already many one-loop studies of worldline formalism. However the results of this approach concerning two-loop theories have not been so much accumulated as those of one-loop theories. Only simple theories, $\phi^3$-theory, spinor QED, $\phi^4$-theory and scalar QED, were investigated. Ref. enables us to calculate, using super-worldline techniques, photon scattering amplitudes via single spinor loop with one (or more) internal photon insertion(s). Various types of multi-loop worldline Green function are obtained from the viewpoints of string and particle theories. The analysis of $\phi^4$-type interaction of seems to suggest us a clue how the four-point vertices of Yang-Mills theory should be handled within the realm of super-worldline formulation. The scalar $\phi^3$-theory with an internal color symmetry was also analyzed, using bosonic string theory, as a preliminary for Yang-Mills theory. If we successfully combine all these significant results, we can expect a new development toward Bern-Kosower-like rules for QCD multi-loop scattering theory. This is the present scope beyond the one-loop studies.
In this paper, we consider, combining two works, two-loop gluon scatterings via a single spinor/gluon loop with one internal gluon insertion. One is Strassler’s work which formulated gluon scatterings in one-loop Yang-Mills theory [3]. In particular, he showed that the contribution of gluon self-interaction to the one-loop amplitudes can be evaluated by means of inserting a pinched gluon vertex operator. Another one is ref. [10], in which Schmidt and Schubert studied the photon scatterings in two-loop QED using super-worldline formulation (Thus, we focus only on the same type of diagrams studied by Schmidt-Schubert). The problem of multi-loop formulation for the gluon scatterings is to find out a method how to evaluate pinched contributions. In the one-loop case, it is enough to insert the pinched vertex operators into a fundamental loop expression which corresponds to the un-pinched function. However, in the higher loop cases, this simple situation becomes complicated because one of external lines at a pinched vertex should participate in the internal gluon propagator part. For this reason, we have to reformulate Strassler’s pinch prescription into more suitable form to be applicable beyond one-loop order. To this end, the super-worldline formalism is very useful.

The contents of this paper are as follows. First in sect.2, we briefly review how the pinched gluon vertex operator works in Strassler’s one-loop argument, which is not organized into the super-worldline formalism. A special care about a \( \delta \)-function must be taken into account there. In sect.3, we reorganize the statements of sect.2 introducing a worldline superfield. The pinched gluon vertex operator turns out to be exactly the product of two gluon vertex operators which are joined by a \( \delta \)-function at the same vertex on a super-worldline. In this formulation, there is no need to give the \( \delta \)-function the special treatment remarked in sect.2. Then in sect.4, applying the super-worldline pinch prescription of sect.3 to two-loop \( N \)-point diagrams which are the QED correction to gluon scatterings via a single spinor loop with the QED charge \( e \) being replaced by the strong coupling \( g \), we derive several vanishing formulae on the main parts (vertex position integrals) of pinched \( N \)-point functions. In sects.5 and 6, we calculate pinched two- and three-point functions. In sect. 7, we discuss a connection of our pinched functions between one-loop and two-loop cases. In sect. 8, we comment on the gluon loop case in short. Conclusion is sect.9.
2 Notes on Strassler’s method

It is worth reviewing Strassler’s pinch method \cite{6} in order to clearly understand the difference between his method and ours, which will be explained later. We will be mainly concerned with spinor (with bare mass \( m \)) loop case, since gluon loop case is essentially parallel to the spinor case.

The proper \( N \)-point functions of gluon scatterings with single spinor loop in the worldline approach to the Bern-Kosower rules are written in the following integral \cite{6}

\[
\Gamma_N = -\frac{1}{2} \text{Tr} \int_0^\infty \frac{dT}{T} e^{-m^2T} \hat{\Gamma}_N. \tag{2.1}
\]

The value of \( \hat{\Gamma}_N \) is given by the expectation value of \( N \) gluon vertex operators

\[
\hat{\Gamma}_N = \oint [dx][d\psi] e^{-S_0} \prod_{n=1}^N V_n, \quad V_n \equiv V(k_n, \epsilon_n), \tag{2.2}
\]

where \( x(\tau) \) and \( \psi(\tau) \) are one-dimensional bosonic and fermionic fields, and the gluon vertex operator is

\[
V(k, \epsilon) = (-igT^a_n) \int_0^\tau d\tau' (\epsilon_\mu \dot{x}^\mu + 2i\psi^\mu \epsilon_\mu \psi_{\nu} k_\nu) \exp[ik_\mu x^\mu]. \tag{2.3}
\]

The worldline action \( S_0 \), of which form depends on the particle moving around the loop, is

\[
S_0 = \int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi \cdot \dot{\psi} \right), \tag{2.4}
\]

where \( \dot{x}^2 = \dot{x}^\mu \dot{x}_\mu \), and \( \dot{x} = dx/d\tau \) (\( A \cdot B \) means \( A^\mu B_\mu \), and \( \mu \) runs over 1, 2, \ldots \( D \)). Note also that the minus sign and \( \frac{1}{2} \) of the pre-factor in eq.(2.1) count the statistics and degrees of freedom of the particle moving around the loop. For example, in the cases of complex boson/ghost loop, the pre-factor must be \( 2 \times \frac{1}{2} \) and \( -2 \times \frac{1}{2} \) (Gluon loop case is assigned to \( \frac{1}{2} \) exceptionally).

In Strassler’s method, the closed path integrals on \( x \) and \( \psi \) in \( \hat{\Gamma}_N \) are performed introducing additional Grassmann integration variables:

\[
\hat{\Gamma}_N = \mathcal{N} \prod_{n=1}^N \int_0^T d\tau_n d\theta_n d\bar{\theta}_n \exp[-\int d\tau d\tau' \left\{ \frac{1}{2} J^\mu(\tau) G_B(\tau, \tau') J_\mu(\tau') + \frac{1}{4} \eta^\mu(\tau) G_F(\tau, \tau') \eta_\mu(\tau') \right\}], \tag{2.5}
\]

where

\[
G_B(\tau_j, \tau_i) = |\tau_j - \tau_i| - \frac{(\tau_j - \tau_i)^2}{T} \equiv G_{Bji}^j, \tag{2.6}
\]

\[
G_F(\tau_j, \tau_i) = \text{sign}(\tau_j - \tau_i) \equiv G_{Fji}^j, \tag{2.7}
\]
\[ N = \oint [dx][d\psi]e^{-S_0} = 2D \left( \frac{1}{4\pi T} \right)^\frac{D}{2} \tag{2.8} \]

and

\[ J^\mu(\tau) = \sum_{n=1}^N \delta(\tau - \tau_n)(\overline{\theta}_n e^\mu_n \partial_\tau + i k^\mu_n), \tag{2.9} \]

\[ \eta^\mu(\tau) = \sum_{n=1}^N \delta(\tau - \tau_n)\sqrt{2(\theta^\mu_n + i \overline{\theta}_n k^\mu_n)}. \tag{2.10} \]

On the other hand, a pinched function is obtained through the replacement of a pair of two vertex operators \( V_jV_i \) by \( O_{ji} \):

\[ \hat{\Gamma}_N(j,i) \equiv \oint [dx][d\psi]e^{-S_0} N \prod_{n \neq i,j} V_n O_{ji}, \tag{2.11} \]

\[ O_{ji} = (-igT^{a_j})(-igT^{a_i}) \int_0^T d\tau_i d\tau_j \delta(\tau_i - \tau_j)2\epsilon_j \cdot \psi \epsilon_i \cdot e^{(k_i+k_j)x}. \tag{2.12} \]

Note that the pinched vertex operator \( O_{ji} \) does never look like \( V_jV_i \) at this stage. In addition, equality in (2.11) should be understood under an appropriately ordered color factor (The positions of \( T^{a_j} \) and \( T^{a_i} \) are not clear in the above formal expression). For simplicity, let us ignore it for the moment. A clear explanation will be given in the next section along the super-worldline context. Now, similarly as done in (2.5), we have the path-integrated form of (2.11)

\[ \hat{\Gamma}_N(j,i) = N \prod_{n=1}^N \int_0^T d\tau d\theta d\overline{\theta}\delta(\tau_i - \tau_j)(k_i \cdot k_j G^ji_F)^{-1} \times \exp\left[ -\overline{\theta}_i \overline{\theta}_j k_i \cdot k_j G^ji_F - \int d\tau d\tau' \frac{1}{2} \overline{J}^\mu(\tau)G_B(\tau,\tau')\overline{J}_\mu(\tau') \right. \]

\[ + \left. \frac{1}{4} \overline{\eta}^\mu(\tau)G_F(\tau,\tau')\overline{\eta}_\mu(\tau') \right], \tag{2.13} \]

where

\[ \overline{J}^\mu(\tau) = \sum_{n \neq i,j}^N \delta(\tau - \tau_n)(\overline{\theta}_n e^\mu_n \partial_\tau + i k^\mu_n) + \sum_{n=i,j}^N \delta(\tau - \tau_n)ik^\mu_n, \tag{2.14} \]

\[ \overline{\eta}^\mu(\tau) = \sum_{n \neq i,j}^N \delta(\tau - \tau_n)\sqrt{2}(\theta^\mu_n + i \overline{\theta}_n k^\mu_n) + \sum_{n=i,j}^N \delta(\tau - \tau_n)\sqrt{2}\theta^\mu_n. \tag{2.15} \]

Note that the \( \overline{J}G_B\overline{J} \) term includes none of \( \overline{\theta}_i, \overline{\theta}_j, \theta_i \) and \( \theta_j \). \( \overline{\eta}G_F\overline{\eta} \) does not include \( \overline{\theta}_i \) and \( \overline{\theta}_j \). Hence, as seen from (2.13), the result of Grassmann integrals of the exponential part is exactly proportional to \( k_i \cdot k_j G^ji_F \). Obviously, such terms can be alternatively extracted from
the un-pinched exponent \( \exp[-\frac{1}{2}JG_B J - \frac{1}{4}J G F J] \) in (2.3), in terms of just picking up the terms proportional to \( k_i \cdot k_j G_{ji}^F \). This means that we have to throw irrelevant terms away by hand from (2.5). Otherwise, we have to perform \( 2N \) Grassmann integrals of (2.13). Also note that the \( \delta \)-function originally included in the pinch vertex operator \( O_{ji} \) should be performed after \((G_{ji}^F)^{-1}\) cancels \( G_{ji}^F \)'s coming up from the exponent in (2.13). After all, we must not integrate \( \delta(\tau_j - \tau_i) \) until we finish two procedures (i) \( 2N \) Grassmann integrals, (ii) cancellation of \((G_{ji}^F)^{-1}\).

As mentioned in \([6]\), the pinch contribution of two-point function \( \hat{\Gamma}_2(2,1) \) is zero. The simplest example of non-vanishing pinch is the three point function \( \hat{\Gamma}_3(2,1) \)

\[
\hat{\Gamma}_3(2,1) = -iN \int d\tau_3 d\tau_2 (k_3 \cdot \epsilon_1 \epsilon_2 - k_3 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_1)(G_{ji}^F)^{-1} e^{-k_3^2 G_{ji}^F}.
\] (2.16)

One can directly verify, through checking (2.16), how the above attention should be paid in calculation.

3 Super worldline formulation

Let us reformulate the above pinch formulation using super-worldline technique \([10]\). We will see in this section that the super-worldline formulation does not need the careful treatment about the \( \delta \)-function remarked in previous section. This time, we do not have to introduce the artificial \( 2N \) Grassmann integrals, and the \( \delta \)-function can be integrated away immediately. These improvements simplify the calculation. Furthermore, differently from (2.12), \( O_{ji} \) reasonably looks like \( V_j V_i \) in our formulation.

With the following super worldline notation

\[
X(\tau, \theta) = x(\tau) + \sqrt{2}\psi(\tau), \quad D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \tau},
\] (3.1)

the gluon vertex operator (2.3) and the spinor loop action (2.4) can be re-expressed \([10]\)

\[
S_0 = -\int_0^T d\tau d\theta \frac{1}{4} XD^3 X,
\] (3.2)

\[
V_n = -(i g T^{an}) \int_0^T d\tau d\theta e_n \cdot DX \exp[i k_n \cdot X],
\] (3.3)

and the un-pinched \( N \)-point function defined in (2.2) reads

\[
\hat{\Gamma}_N = \oint [dX] e^{-S_0} \prod_{n=1}^N V_n.
\] (3.4)
Differently from the argument of sect. 2, we apply the Wick contraction to the evaluation of this \( N \)-point function using the super-worldline Green function \[10\]

\[
<X_1^\mu X_2^\nu> = \mathcal{N}^{-1} \int [dX] e^{-S_0} X_1^\mu X_2^\nu = -g^{\mu\nu} G(1, 2),
\]

where

\[
G(1, 2) = G_B^{12} + \theta_1 \theta_2 G_F^{12}.
\]

Now, the pinched gluon vertex operator \[2.12\] turns out to be

\[
\mathcal{O}_{ji} = (-igT^a_j)(-igT^a_i) \int_0^T d\tau_i d\tau_j d\theta_i d\theta_j \delta(\tau_i - \tau_j) \epsilon_i \cdot DX_i \epsilon_j \cdot DX_j e^{ik_j \cdot X_i + ik_i \cdot X_j},
\]

where \( X_i \) means \( X(\tau_i, \theta_i) \). It is now clear that \( \mathcal{O}_{ji} \) is created by the product of two vertex operators \( V_j V_i \). There exists the following equality (denoted by \( \sim \)) between \( V_j V_i \) and \( \mathcal{O}_{ji} \) at the level of integrand,

\[
\mathcal{O}_{ji} \sim V_j \theta_j \theta_i \delta(\tau_j - \tau_i) V_i.
\]

This means that the insertion of \( \mathcal{O}_{ji} \) is equivalent to the insertion of \( \theta_j \theta_i \delta(\tau_j - \tau_i) \) between \( V_j \) and \( V_i \) (or \( DX_j \) and \( DX_i \)). Note that \( \theta_j \theta_i \delta(\tau_j - \tau_i) \) is the \( \delta \)-function which makes the super-distances \( \tau_j - \tau_i + \theta_j \theta_i \) zero, and this \( \delta \)-function coincides with a part of a supersymmetric step function \[13\]. In the following, we consider a fixed color ordering, and the ordering of \( V_n \) (\( n = 1, 2, \cdots N \)) should be fixed before inserting the pinch \( \delta \)-function \( \theta_j \theta_i \delta(\tau_j - \tau_i) \). For example, we choose \( V_N, V_{N-1}, \cdots, V_1 \). Eq.\[2.11\] then becomes

\[
\hat{\Gamma}_N(j, i) = \int [dX] e^{-S_0} V_N \cdots V_j \cdots V_1 \theta_i \theta_j \delta(\tau_j - \tau_i).
\]

If one wants to obtain also \( \hat{\Gamma}_N(i, j) \), it is enough to insert \( \theta_i \theta_j \delta(\tau_i - \tau_j) \) and to reverse the ordering between \( T^{a_j} \) and \( T^{a_i} \). The sign of inserted \( \delta \)-function is then reversed, and the sum of these pinched functions is naturally accompanied by the structure constant produced by the commutator between \( T^{a_j} \) and \( T^{a_i} \).

Hereafter, we will ignore this kind of color trace structure, since this is just a matter of counting factors, and it can be easily recovered after getting main expression (integral parts) of \( \hat{\Gamma}_N(j, i) \). By reason of this, keep in mind that the vertex operators \( V_n \) will be dealt as if commuting quantities.
Let us look how the \( \delta \)-function insertion, which is equivalent to the insertion of super pinch vertex operator \( (3.7) \), works in the two-point function case. The result must be zero as seen in sect.2. First, write down \( \hat{\Gamma}_2 \) using Wick contractions

\[
\hat{\Gamma}_2 = -N \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \epsilon_1^\mu \epsilon_2^\nu < DX_\mu^1 DX_\nu^2 e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} >
\]

\[
= -N \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \epsilon_1^\mu \epsilon_2^\nu < e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} >
\]

\[
\times [D_1 D_2 < X_1^\mu X_2^\nu > - D_1 < X_1^\mu k_2 \cdot X_2 > D_2 < X_2^\nu k_1 \cdot X_1 >],
\]

where \( \dot{G}_B \) and \( \ddot{G}_B \) mean the first and second derivatives w.r.t. the first argument of \( G_B \). Following the method explained above, the pinched two-point function \( \hat{\Gamma}_2(2,1) \) can be obtained by inserting \( \theta_2 \theta_1 \delta(\tau_2 - \tau_1) \) into eq. \((3.10)\). We then immediately see

\[
\hat{\Gamma}_2(2,1) = \int [dX] e^{-S_0} V_1 \theta_2 \theta_1 \delta(\tau_2 - \tau_1) = 0,
\]

while getting the un-pinched function as well

\[
\hat{\Gamma}_2 = N \int d\tau_1 d\tau_2 \epsilon_1^k \epsilon_2^k \dot{G}_B e^{\epsilon_1 \cdot \epsilon_2 \ddot{G}_B} + \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 (\dot{G}_B^2) - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 (\dot{G}_B^2)
\]

which coincides with equation \((3.28)\) of ref. \([3]\). It is clear that the insertion of \( \theta_2 \theta_1 \) removes irrelevant terms from \((3.10)\) and that \( \delta(\tau_2 - \tau_1) \) can be integrated right at the moment when it is inserted. (A \( \delta \)-function multiplied by a sign function, i.e. \( G_B^{12} \delta(\tau_2 - \tau_1) \), does not have a well-defined value when \( \tau_1 = \tau_2 \). However, if one considers the \( \delta \)-function to be even, then the result follows.) These are the different points from the method of sect.2 and make calculation simple.

Next, let us check whether our method works in a non-trivial example, say \( \hat{\Gamma}_3(2,1) \). Writing down the un-pinched three-point function

\[
\hat{\Gamma}_3 = N \int d\tau_1 d\tau_2 d\tau_3 d\theta_1 d\theta_2 d\theta_3 \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho < DX_\mu^3 DX_\nu^2 DX_\rho^1 \prod_{j=1}^{3} e^{ik_j \cdot X_j} >
\]
\[ K^{\mu\nu} = \mathcal{D}_3 \mathcal{D}_2 < X_3^{\mu} X_2^{\nu} > \mathcal{D}_1 < X_1^{\mu} \sum_{j=1}^{3} ik_j \cdot X_j > \]
\[ - \mathcal{D}_3 \mathcal{D}_1 < X_3^{\mu} X_1^{\nu} > \mathcal{D}_2 < X_2^{\nu} \sum_{j=1}^{3} ik_j \cdot X_j > \]
\[ + \mathcal{D}_2 \mathcal{D}_1 < X_2^{\nu} X_1^{\mu} > \mathcal{D}_3 < X_3^{\mu} \sum_{j=1}^{3} ik_j \cdot X_j >, \]

where we insert \( \theta_2 \theta_1 \delta(\tau_2 - \tau_1) \) according to the integrand equality \( \hat{\Gamma}_3(2, 1) \sim \hat{\Gamma}_3 \theta_2 \theta_1 \delta(\tau_2 - \tau_1) \). Obviously, the third term of \( K^{\mu\nu} \) vanishes after this pinch procedure (see eq.(3.11)). Applying the formulae (3.11) and (3.12) to \( K^{\mu\nu} \) and integrating w.r.t. \( \tau_1 \) and all \( \theta_i \), we arrive at
\[ \hat{\Gamma}_3(2, 1) = \int [dX] e^{-S_0 V_3 V_2 V_1 \theta_2 \theta_1 \delta(\tau_2 - \tau_1)} \]
\[ = \mathcal{N} \int d\tau_3 d\tau_2 d\tau_1 e^{\frac{\rho}{2}} \sum_{i<j} k_i \cdot k_j G(i, j) [K^{\mu\nu}], \] (3.16)

This coincides with (2.16).

We can calculate pinched 4-point functions in the same way. There are two types: the single-pinch type
\[ \hat{\Gamma}_4(2, 1) = \mathcal{N} \int d\tau_4 d\tau_3 d\tau_1 \exp[k_4 \cdot k_3 G_B^{43} + k_4 \cdot (k_1 + k_2) G_B^{41} + k_3 \cdot (k_1 + k_2) G_B^{31}] \]
\[ \times \left[ G_F^{43} G_F^{31} G_F^{41} (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot k_3 \epsilon_1 \cdot k_4 - \epsilon_4 \cdot \epsilon_3 \epsilon_2 \cdot k_4 \epsilon_1 \cdot k_3 + \epsilon_4 \cdot \epsilon_2 \epsilon_3 \cdot k_4 \epsilon_1 \cdot k_3 \right. \]
\[ - \epsilon_4 \cdot \epsilon_1 \epsilon_3 \cdot k_4 \epsilon_2 \cdot k_3 - \epsilon_3 \cdot \epsilon_2 \epsilon_4 \cdot k_3 \epsilon_1 \cdot k_4 + \epsilon_3 \cdot \epsilon_1 \epsilon_4 \cdot k_3 \epsilon_2 \cdot k_4 \]
\[ + \hat{G}_B^{31} (G_F^{41})^2 \{ \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot (k_1 + k_2) \epsilon_1 \cdot k_3 - \epsilon_4 \cdot \epsilon_1 \epsilon_3 \cdot (k_1 + k_2) \epsilon_2 \cdot k_4 \} \]
\[ + \hat{G}_B^{41} (G_F^{31})^2 \{ \epsilon_1 \cdot \epsilon_2 \epsilon_4 \cdot (k_1 + k_2) \epsilon_1 \cdot k_3 - \epsilon_3 \cdot \epsilon_1 \epsilon_4 \cdot (k_1 + k_2) \epsilon_2 \cdot k_3 \}, \] (3.19)

and the double-pinch type
\[ \hat{\Gamma}_4(4, 3|2, 1) \equiv \int [dX] e^{-S_0 V_3 V_2 V_1 \theta_4 \theta_3 \theta_2 \theta_1 \delta(\tau_4 - \tau_3) \delta(\tau_2 - \tau_1)} \]
\[ = \mathcal{N} \int d\tau_3 d\tau_1 \exp[(k_3 + k_4) \cdot (k_1 + k_2) G_B^{31}] \]
\[ \times (\epsilon_4 \cdot \epsilon_1 \epsilon_3 \cdot \epsilon_2 - \epsilon_4 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_1)(G_F^{31})^2. \] (3.20)

Here is one remark. If we consider scalar vertex operators discarding polarization vectors \( V = \int d\tau d\theta e^{ikX} \), where super field expression may be kept because fermionic integral becomes
just a normalization, we can recover a four-point function of $\phi^4$-theory from (3.20),

$$\hat{\Gamma}_4 = \mathcal{N} \int d\tau_3 d\tau_1 \exp[(k_3 + k_4) \cdot (k_1 + k_2) G_{B}^{\delta 1}]. \quad (3.21)$$

4 Two-loop pinch formulas

The super worldline formulation of multi-loop $N$-photon amplitudes are discussed in [10] for the set of diagrams that $p$ photon propagators are inserted into the spinor loop. For this type of $(p+1)$-loop $N$-point function, we have only to evaluate $(N + 2p)$-point correlation function. Here, we confine ourselves to the two-loop case $p = 1$ because we do not have to make our situation complex.

First, let us recall that those two-loop $N$-point functions in QED are proportional to the following integral [10]

$$\hat{\Gamma}^{(1)}_N \equiv \int \frac{dT}{T} e^{-m^2 T} \int d\bar{T} (4\pi \bar{T})^{-\frac{D}{2} - 1} \hat{\Gamma}^{(1)}_N, \quad (4.1)$$

where an appropriate pre-factor associated to the theory is again dropped. $\hat{\Gamma}^{(1)}_N$ is given by the integrals of $(N+2)$-point correlator

$$\hat{\Gamma}^{(1)}_N = \int d\tau_a d\tau_b d\theta_a d\theta_b N_1 < DX_b \cdot DX_a \prod_{n=1}^{N} V_n >_{(1)}, \quad (4.2)$$

where the path integral normalization $N_1$ is

$$N_1 = \oint [dX] e^{-S_0} \exp \left[ -\frac{(X_a - X_b)^2}{4T}\right], \quad = 2^D (4\pi T)^{-D/2} (1 + \frac{1}{T} G(a,b))^{-D/2}. \quad (4.3)$$

With the use of Wick contraction, the correlator $< \cdots >_{(1)}$ can be decomposed into two-point correlators, i.e., two-loop worldline Green functions,

$$< X_1^\mu X_2^\nu >_{(1)} = N_1^{-1} \oint [dX] e^{-S_0} \exp \left[ -\frac{(X_a - X_b)^2}{4T}\right] X_1^\mu X_2^\nu \quad = -g^\mu\nu G^{(1)}(1,2). \quad (4.4)$$

These equations are derived by connecting a pair of external photon lines through one internal photon propagator. After Wick contracting, the $N$-point function $\hat{\Gamma}^{(1)}_N$ generally takes the following form

$$\epsilon_1^\mu \cdots \epsilon_N^\nu K_{\mu_1 \cdots \mu_N}^{(1)} \exp \left[ \frac{1}{2} \sum_{i,j=1}^{N} k_i \cdot k_j G^{(1)}(i,j) \right], \quad (4.5)$$
where \( K^{(1)}_{\mu...\nu} \) consists of possible contraction terms among \( X \) fields, such as (3.17). A pinched \( N \)-point function can be obtained through inserting a pinch \( \delta \)-function into \( \hat{\Gamma}^{(1)}_N \).

Before considering pinch situation, we have to remark on the ambiguity problem of multi-loop worldline Green functions. For example, two-loop worldline Green functions are known as the following two forms \[9\], \[10\]

\[
\tilde{G}^{(1)}(x,y) = G(x,y) + \frac{1}{2} \frac{(G(x,a) - G(x,b))(G(y,a) - G(y,b))}{T + G(a,b)},
\]

or

\[
G^{(1)}(x,y) = G(x,y) - \frac{1}{4} \frac{(G(x,a) - G(x,b) - G(a,y) + G(b,y))^2}{T + G(a,b)}.
\]

The first form was used in deriving the two-loop QED \( \beta \)-function \[10\]. The latter form can be derived from a pinching \((\alpha' \to 0)\) limit of closed string theory \[13\]. Both are related by the relation

\[
G^{(1)}(x,y) = \tilde{G}^{(1)}(x,y) - \frac{1}{2} \tilde{G}^{(1)}(x,x) - \frac{1}{2} \tilde{G}^{(1)}(y,y).
\]

This modification is harmless to the exponential part of (4.5), i.e.

\[
\exp\left[\frac{1}{2} \sum_{i,j} k_i \cdot k_j G^{(1)}(i,j)\right] = \exp\left[\frac{1}{2} \sum_{i,j} k_i \cdot k_j \tilde{G}^{(1)}(i,j)\right],
\]

because of momentum conservation concerning external legs. However, it is not clear to the Wick contraction parts \( K^{(1)}_{\mu...\nu} \), whether or not both (4.6) and (4.7) give the same results mutually.

Similarly to the one-loop case, pinched contributions can be evaluated by means of replacing a pair of \( V_j V_i \) with the pinched vertex operator \( O_{ji} \), namely, by insertion of \( \theta_j \theta_i \delta(\tau_j - \tau_i) \) between \( \mathcal{D}X_j \) and \( \mathcal{D}X_i \). In accordance with a color ordering, we again fix the ordering of all \( \mathcal{D}X_n \) \((n = a, b, 1, 2, \ldots N)\) before we insert the pinched vertex operator i.e. the \( \delta \)-function \( \theta_j \theta_i \delta(\tau_j - \tau_i) \). The ordering we choose here is \( a, 1, 2, \ldots N, b \) from right to left. Thus (for \( j > i \)),

\[
\hat{\Gamma}^{(1)}_N(j, i) = \int d\tau_a d\tau_b d\theta_a d\theta_b N_1 g_{\mu\nu} < \mathcal{D}X^\mu_b \prod_{n=1}^N V_n \mathcal{D}X^\nu_a >_{(1)} \theta_j \theta_i \delta(\tau_j - \tau_i)
\]

\[
\sim \hat{\Gamma}^{(1)}_N \theta_j \theta_i \delta(\tau_j - \tau_i).
\]

Note that the pinch of both edges of internal gluon line becomes zero irrespectively of the number of external gluon legs

\[
\hat{\Gamma}^{(1)}_N(b, a) = 0,
\]
in which $\theta_b \theta_a$ should be inserted between $DX_b$ and $DX_a$. This can be proved by direct calculation for each choice of $G^{(1)}$ or $\tilde{G}^{(1)}$ ($N = 0, 1$ cases are checked), however the following proof is simple for generic $N$. Interchanging the integration variables carrying $a$ and $b$ in

$$\hat{\Gamma}_N^{(1)}(b, a) = \int d\tau_a d\tau_b d\theta_a d\theta_b N_1 < DX_b \cdot DX_a \prod_{n=1}^{N} V_n >_{(1)} \theta_b \theta_a \delta(\tau_a - \tau_b), \quad (4.12)$$

and anti-commuting $DX_a$ with $DX_b$, we see RHS of the above equation becomes $-\hat{\Gamma}_N^{(1)}(b, a)$. Therefore $\hat{\Gamma}_N^{(1)}(b, a) = 0$. In particular, $\hat{\Gamma}_0^{(1)}(b, a) = 0$ is naturally understood if we notice two facts that the zero-point function $\hat{\Gamma}_0^{(1)}$ originally corresponds to the one-loop two-point function $\hat{\Gamma}_2$ and that the one-loop pinched two-point function $\hat{\Gamma}_2(2, 1)$ is zero. More general correspondence between two-loop $N$-point functions and one-loop $(N + 2)$-point functions will be discussed in later section.

In the same way as the proof of (4.11), we obtain another pinch formula between internal and external gluon lines

$$\hat{\Gamma}_N^{(1)}(b, n) + \hat{\Gamma}_N^{(1)}(n, a) = 0. \quad (4.13)$$

This formula reduces to (4.11) if we put $n = a$ or $b$.

We put one remark. When directly checking (4.13), there is a subtle difference in calculation between $G^{(1)}$ and $\tilde{G}^{(1)}$ (although the final result is independent of this choice). For example, consider $\Gamma_1^{(1)}$. On the one hand for $\tilde{G}^{(1)}$, $\hat{\Gamma}_1^{(1)}(b, 1) \neq 0$ by itself. It is cancelled by

$$\hat{\Gamma}_1^{(1)}(1, a) = \frac{1}{2} i \epsilon \cdot k (D - 1) \int d\tau_a d\tau_b N_1^B (G_F)^2 \frac{G^{ab}}{T + G^{ab}}, \quad (4.14)$$

where $N_1^B$ consists of only bosonic Green function

$$N_1^B = 2^D (4\pi T)^{-D/2} (1 + \frac{1}{T} G^{ab})^{-D/2}. \quad (4.15)$$

On the other hand for $G^{(1)}$,

$$\hat{\Gamma}_1^{(1)}(b, 1) = \hat{\Gamma}_1^{(1)}(1, a) = 0. \quad (4.16)$$

Note that the interchange symmetry of integration variables $a \leftrightarrow b$ makes the proofs of (4.11) and (4.13) simple. For this reason, their explicit check of choice-independence of $G^{(1)}$ has not been needed. In the same way, we can show the similar formulae for arbitrary number of pinch pairs $(n_i, m_i \neq a, b)$

$$\hat{\Gamma}_N^{(1)}(b, a|n_1, m_1| \cdots |n_k, m_k) = 0, \quad (4.17)$$

$$\hat{\Gamma}_N^{(1)}(b, n|n_1, m_1| \cdots |n_k, m_k) + \hat{\Gamma}_N^{(1)}(n, a|n_1, m_1| \cdots |n_k, m_k) = 0, \quad (4.18)$$

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where the multi-pinch functions are defined by
\[
\hat{\Gamma}_N^{(1)}(n_1, m_1 | \cdots | n_k, m_k) \sim \hat{\Gamma}_N^{(1)} \theta_{n_1} \theta_{m_1} \cdots \theta_{n_k} \theta_{m_k} \prod_{i=1}^{k} \delta(\tau_{n_i} - \tau_{m_i}).
\] (4.19)

In general cases, there is no such useful interchange symmetry. We must perform and compare direct calculation in each choice of \(G^{(1)}\). Eq.(4.18) is due to the abelian approximation (the color structure at the \(a\)- and \(b\)- vertices will destroy the relation), and it is still unclear whether or not the equation comes from gauge invariance.

These pinch formulae, (4.11), (4.13), (4.17) and (4.18), tell us that (sum of) all the internal ‘gluon’ pinch contributions may be ignored, with the exception of the diagrams where two edges of an internal ‘gluon’ are pinched to distinct external gluon lines like
\[
\hat{\Gamma}_N^{(1)}(b, n|m,a|n_1,m_1|\cdots |n_k,m_k).
\] (4.20)

Hence in our particular situation of QED correction, we have only to gather external gluon pinch functions \(\hat{\Gamma}_N^{(1)}(n,m|\cdots)\) as well as the types of (4.20). In the following sections, we calculate 2- and 3-point functions where double-pinch functions of the type (4.20) appear.

### 5 Pinched two-point functions

In this section, we verify, calculating pinched two-point functions, the consistency between \(\hat{G}^{(1)}\) and \(G^{(1)}\). As mentioned at the end of sect.4, the candidates of non-vanishing pinched contributions are the following single-pinch function
\[
\hat{\Gamma}_2^{(1)}(2,1) = \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 N_1 g_{\mu\nu} \langle DX_b^\mu V_2^\nu DX_a^\mu \rangle^{(1)} \theta_2 \theta_1 \delta(\tau_2 - \tau_1),
\] (5.1)

and the double-pinch function of the type (4.20)
\[
\hat{\Gamma}_2^{(1)}(b,2|1,a) = \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 g_{\mu\nu} N_1 \langle DX_b^\mu V_2^\nu DX_a^\mu \rangle^{(1)} \theta_2 \theta_1 \theta_1 \delta(\tau_a - \tau_1) \delta(\tau_b - \tau_2).
\] (5.2)

Although \(V_1\) and \(V_2\) commute with each other (up to color factors), this does not mean the interchange symmetry of integration variables \(1 \leftrightarrow 2\). Hence there is no simple analysis compared to (4.11) and (4.13) where the interchange symmetry \(a \leftrightarrow b\) was useful for their proofs.

We perform direct calculations. First, we write down the un-pinched function
\[
\hat{\Gamma}_2^{(1)} = \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \epsilon_{1} \epsilon_{2} \epsilon_{X_1} \epsilon_{X_2} N_1 \langle DX_b \cdot DX_a DX_2^\nu DX_1^\mu \rangle^{(1)} \cdot (5.3)
\]
Using the Wick contraction, the sixfold correlator takes the following form
\[
< DX_b \cdot DX_a DX_1^{\mu} e^{i k_1 \cdot X_1} e^{i k_2 \cdot X_2} >_{(1)} = K^{(1)}_{\mu\nu} \exp[k_1 \cdot k_2 G^{(1)}(1, 2)],
\]
and \( \epsilon^\mu_1 \epsilon^\nu_2 K^{(1)}_{\mu\nu} \) is given by the following 15 terms
\[
ed^\mu_1 \epsilon^\nu_2 K^{(1)}_{\mu\nu} = \epsilon_1 \cdot \epsilon_2 [D \dot{G}^{(1)}_{ab} \dot{G}^{(1)}_{12} - \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{12} + \dot{G}^{(1)}_{a2} \dot{G}^{(1)}_{b1}
+ \epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 [- \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{12} \dot{G}^{(1)}_{b2} + \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{12} \dot{G}^{(1)}_{b2}]
+ \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 [\dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b2} \dot{G}^{(1)}_{12} - \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b2} \dot{G}^{(1)}_{12} + D \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b1} \dot{G}^{(1)}_{12}
- \dot{G}^{(1)}_{21} \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b2} + \dot{G}^{(1)}_{21} \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b2}]
+ \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 [- \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b1} \dot{G}^{(1)}_{22} + \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b1} \dot{G}^{(1)}_{22} - D \dot{G}^{(1)}_{a1} \dot{G}^{(1)}_{b1} \dot{G}^{(1)}_{22}
+ \dot{G}^{(1)}_{11} \dot{G}^{(1)}_{a2} \dot{G}^{(1)}_{b2} - \dot{G}^{(1)}_{11} \dot{G}^{(1)}_{a2} \dot{G}^{(1)}_{b2}],
\]
where
\[
\dot{G}^{(1)}_{ij} = D_i G^{(1)}(i, j), \quad \dot{G}^{(1)}_{ij} = D_i D_j G^{(1)}(i, j).
\]
To estimate (5.1) and (5.2), we insert \( \theta_2 \theta_1 \) at least, and we thereby discard the terms proportional to \( \theta_1 \) or \( \theta_2 \) contained in the above 15 terms. Then pick up the terms proportional to 1 for \( \tilde{\Gamma}^{(1)}_2(b, 2|1, a) \), and similarly those proportional to \( \theta_a \theta_b \) for \( \tilde{\Gamma}^{(1)}_2(2, 1) \). Note that the exponent \( k_1 \cdot k_2 \dot{G}^{(1)}_{12} \) does not contribute to these pinched functions, but it does to \( \tilde{\Gamma}^{(1)}_2(1, a) \). Note also that \( \dot{G}^{(1)}_{ij} \) does not depend on the choice whether \( G^{(1)} \) or \( \dot{G}^{(1)} \).

Now, let us consider \( \tilde{\Gamma}(b, 2|1, a) \). The 4-15th terms in (5.5) are of the form \( \dot{G}^{(1)} \dot{G}^{(1)} \dot{G}^{(1)} \), and these are made of the form \( (\theta_a + \theta_b)(1 + \theta_a \theta_b)(\theta_a + \theta_b) \), whichever \( G^{(1)} \) we use. These terms are hence zero for \( \tilde{\Gamma}^{(1)}_2(b, 2|1, a) \). Since the remaining first three terms are composed of only \( \dot{G}^{(1)} \), the result is independent of the choice of \( G^{(1)} \)
\[
\tilde{\Gamma}^{(1)}_2(b, 2|1, a) = (D - 1) \epsilon_1 \cdot \epsilon_2 \int d\tau_1 d\tau_2 e^{k_1 \cdot k_2 G^{(1)}_{(1, 2)}(1, 2)} (G^{12})^2 N_1^B |_{a \rightarrow 1, b \rightarrow 2}.
\]

Next, let us show that
\[
\tilde{\Gamma}^{(1)}_2(2, 1) = 0.
\]
In the case of \( G^{(1)} \), taking the limit \( 1 \rightarrow 2 \) in (5.3), it is easy to see this equality because of the properties
\[
\lim_{\tau_1 \rightarrow \tau_2} \theta_1 \theta_2 \dot{G}^{(1)}(1, 2) = 0, \quad \lim_{\tau_1 \rightarrow \tau_2} \dot{G}^{(1)}(1, 2) = 0,
\]
which mean $\hat{G}^{(1)}(j, i)$ and $\hat{G}^{(1)}(j, i)$ can be dropped in the pinch situation $j \rightarrow i$. In the case of $\hat{G}^{(1)}$, the first five terms in (6.1) vanish from the same reason, because $\hat{G}^{(1)}$ is independent of the choice of $G^{(1)}$. The 6, 7, 8, 9 and 10th terms cancel the 11, 15, 13, 12 and 14th terms respectively. Therefore all 15 terms vanish, and the proof ends.

6 Three-point function

For 3-point functions, we have to start with the following 8-fold correlator

$$\hat{\Gamma}^{(1)}_3 = \int d\tau_a d\tau_b d\tau_1 d\tau_2 d\tau_3 d\theta_a d\theta_b d\theta_1 d\theta_2 d\theta_3 N_1$$

$$< DX_a \cdot DX_b \cdot DX_1 \cdot DX_2 \cdot DX_3 \cdot e^{ik_1 \cdot X_1} e^{ik_2 \cdot X_2} e^{ik_3 \cdot X_3} > (1) g_{\sigma \delta} \epsilon_{\mu \nu} \epsilon_{\rho \sigma}.$$ $$(6.1)$$

Let us consider the pinch situation $2 \rightarrow 1$. In this section, we only discuss the $G^{(1)}$ case which makes equations simpler owing to (5.10). Although the Wick contraction creates 68 terms, these are reduced to 30 terms by taking the pinch limit $2 \rightarrow 1$. Further, using the interchange symmetry of $a \leftrightarrow b$, the number of terms becomes 15. Among them, the following 6 terms turn out to be independent

$$I_1 = \hat{G}^{(1)}_{ab} \hat{G}^{(1)}_{13} \hat{G}^{(1)}_{13}, \quad I_2 = \hat{G}^{(1)}_{a1} \hat{G}^{(1)}_{b3} \hat{G}^{(1)}_{13}, \quad (6.2)$$

$$I_3 = \hat{G}^{(1)}_{a1} \hat{G}^{(1)}_{b3} \hat{G}^{(1)}_{b1}, \quad I_4 = \hat{G}^{(1)}_{a1} \hat{G}^{(1)}_{b3} \hat{G}^{(1)}_{b3}, \quad (6.3)$$

$$I_5 = \hat{G}^{(1)}_{b1} \hat{G}^{(1)}_{a1} \hat{G}^{(1)}_{13} \hat{G}^{(1)}_{13}, \quad I_6 = \hat{G}^{(1)}_{a1} \hat{G}^{(1)}_{b1} \hat{G}^{(1)}_{13} \hat{G}^{(1)}_{b3}. \quad (6.4)$$

After all, using $k_3 = -(k_1 + k_2)$, the pinched function $\hat{\Gamma}^{(1)}_3(2, 1)$ is expressed as

$$\hat{\Gamma}^{(1)}_3(2, 1) = 2i \int d\tau_a d\tau_b d\tau_1 d\tau_2 d\tau_3 d\theta_a d\theta_b d\theta_1 d\theta_2 d\theta_3$$

$$N_1 \theta_2 \theta_1 \delta(\tau_2 - \tau_1) \exp[\sum_{i<j}^3 k_i \cdot k_j G^{(1)}(i, j)]$$

$$\times [(\frac{D}{2} I_1 - I_2 + I_3 - I_4 - k_1 \cdot k_2 I_5)(\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_3 - \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_3)$$

$$+ I_6 \epsilon_1 \cdot k_3(\epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 + \epsilon_2 \cdot k_2 \epsilon_3 \cdot k_1)$$

$$- I_6 \epsilon_2 \cdot k_3(\epsilon_1 \cdot k_1 \epsilon_3 \cdot k_2 + \epsilon_1 \cdot k_2 \epsilon_3 \cdot k_1)]. \quad (6.5)$$

Let us consider the integrations on $\tau_2$ and all $\theta_i$. To this end, pick up only $\theta_a \theta_b \theta_3$ terms from
the integrand $\mathcal{N}_1 I_i \exp \left[ \sum_{i<j} k_i \cdot k_j G^{(1)}(i, j) \right]$, and note
\[
G^{(1)}(1, 3) = G_B^{(1)}(1, 3) + \frac{1}{2} g_{ab}(1, 3)(\theta_a \theta_3 G_F^{a3} - \theta_b \theta_3 G_F^{b3}) + \mathcal{O}(\theta_1, \theta_2),
\]
where
\[
g_{ab}(i, j) = \frac{C_B^{ia} - C_B^{ib} - C_B^{aj} + C_B^{bj}}{T + G_B^{ab}}.
\]
This, with $\theta_a$ and $\theta_b$ terms of $I_i$, actually contributes to $\theta_a \theta_b \theta_3$ terms which are proportional to $k_3^2$. Also, a $\theta_a \theta_b$ term from $\mathcal{N}_1$ contributes with $\theta_3$ terms of $I_i$. Then integrating $\theta_a$, $\theta_b$ and $\theta_3$,
\[
J_i = \int d\theta_3 d\theta_b d\theta_a I_i \exp \left[ -k_3^2 G_B^{(1)}(1, 3) \right],
\]
we obtain the following values of $J_i$,
\[
J_1 = (G_F^{13})^2 \left( \tilde{G}_B^{ab} + \frac{1}{2} \frac{(\tilde{G}_B^{ab})^2 - (G_F^{ab})^2}{T + G_B^{ab}} \right) - \frac{D (G_F^{ab})^2 (G_F^{13})^2}{2 T + G_B^{ab}}
+ \frac{1}{4} k_3^2 g_{ab}(1, 3)(G_F^{1b} G_F^{a3} - G_F^{a1} G_F^{b3}) G_B^{ab} G_F^{13},
\]
\[
J_2 + J_4 - J_3 = \frac{1}{2} (G_F^{13})^2 \left( \tilde{G}_B^{ab} + \frac{1}{2} \frac{(\tilde{G}_B^{ab})^2 - (G_F^{ab})^2}{T + G_B^{ab}} \right) - \frac{D (G_F^{ab})^2 (G_F^{13})^2}{2 T + G_B^{ab}}
+ \frac{1}{2} k_3^2 G_F^{ab} G_F^{13} (\tilde{G}_B^{ab} - \tilde{G}_B^{b3}) g_{ab}(1, 3)
+ \frac{1}{4} k_3^2 g_{ab}(1, 3)(G_F^{ab} G_F^{b3} - G_B^{ab} G_F^{a3}) G_F^{13},
\]
\[
J_5 = -\frac{1}{4} (G_F^{13})^2 G_F^{ab} \left( G_F^{ab} + G_F^{1b} - G_B^{ab} - G_F^{a1} \right)
- \frac{1}{4} G_F^{13} (G_F^{ab})^2 \left( G_F^{ab} + G_F^{1b} - G_B^{ab} - G_F^{a1} \right) \frac{T + G_B^{ab}}{2},
\]
\[
J_6 = \frac{1}{4} G_F^{ab} \left( G_B^{ab} + G_F^{1b} - G_B^{ab} - G_F^{b1} \right) g_{ab}(1, 3)
\left( G_F^{1b} \left\{ G_F^{a1} + \frac{1}{2} (\tilde{G}_B^{a1} - \tilde{G}_B^{b3}) g_{ab}(3, 1) \right\} + G_F^{13} G_F^{b3} \right)
\left( G_F^{a1} \left\{ G_B^{ab} + G_B^{b1} + G_B^{a1} \right\} \frac{T + G_B^{ab}}{2} \right)
\times \left( G_F^{13} G_F^{a1} + G_F^{1b} \left\{ -\tilde{G}_B^{b1} + \frac{1}{2} (\tilde{G}_B^{a1} - \tilde{G}_B^{b3}) g_{ab}(3, 1) \right\} \right),
\]
which should be finally integrated in the expression
\[
\hat{\Gamma}^{(1)}_3(2, 1) = -2i \int d\tau_a d\tau_3 d\tau_1 d\tau_3 \exp [k_3 \cdot (k_1 + k_2) G_B^{(1)}(\tau_1, \tau_3)] \mathcal{N}_1^R
\]
\[
\times \left( \frac{D}{2} J_1 - J_2 + J_3 - J_4 - k_1 \cdot k_2 J_5 \right) \left( \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot k_3 - \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_3 \right) \\
+ J_6 \epsilon_1 \cdot k_3 (\epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 + \epsilon_2 \cdot k_2 \epsilon_3 \cdot k_1) \\
- J_6 \epsilon_2 \cdot k_3 (\epsilon_1 \cdot k_1 \epsilon_3 \cdot k_2 + \epsilon_1 \cdot k_2 \epsilon_3 \cdot k_1) \right]. \tag{6.13}
\]

Similarly, the following double-pinched function can be calculated
\[
\tilde{\Gamma}_3^{(1)}(b, 2|1, a) \sim \tilde{\Gamma}_3^{(1)} b \theta_2 \theta_1 \theta_3 \delta(\tau_b - \tau_2) \delta(\tau_1 - \tau_a). \tag{6.14}
\]

The pinch \(\delta\)-functions \(a \to 1\) and \(b \to 2\) remove 34 terms (among the 68 terms). In addition, the following formula
\[
\mathcal{D}_i G^{(1)}(i, j) = 0 + \mathcal{O}(\theta_1, \theta_2, \theta_a, \theta_b) \quad i, j \neq 3 \tag{6.15}
\]
removes further 24 terms. Gathering remaining 10 terms, we obtain
\[
\tilde{\Gamma}_3^{(1)}(b, 2|1, a) = -i \int d\tau_1 d\tau_2 d\tau_3 N_1^B \exp\left[ \sum_{i<j} k_j \cdot k_i G^{(1)}_B(\tau_j, \tau_i) \right]_{a \to 1, b \to 2} \\
\times \left[ (D - 1) \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_1 (G_{12}^{12})^2 \left\{ -\dot{G}^{31}_B + \frac{1}{2}(\dot{G}^{31}_B - \dot{G}^{32}_B)g_{ab}(3, 1) \right\} \right. \\
+ (D - 1) \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_2 (G_{12}^{12})^2 \left\{ -\dot{G}^{32}_B + \frac{1}{2}(\dot{G}^{31}_B - \dot{G}^{32}_B)g_{ab}(3, 2) \right\} \\
\left. + (D - 2) G_{12}^{12} G_{23}^{12} G_{31}^{12} (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot k_3 - \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_3) \right]. \tag{6.16}
\]

7 Pinched \(N\)-point functions

In previous sections, we have calculated one-, two- and three-point functions directly, \(i.e.\) according to the definition (4.10). This kind of calculations becomes more complicated as increasing the number of external legs. However, there must be a simple method to evaluate two-loop \(N\)-point functions, if we start from known one-loop \((N + 2)\)-point functions. In this section, let us consider this possibility for a while.

The un-pinched two-loop function eq.(4.2) can be written in a similar form as the one-loop formula (3.14)
\[
\hat{\Gamma}_N^{(1)} = \oint [dX] e^{-S_1} V_b V_a \prod_{n=1}^N V_n \Big|_{k_a = k_b = 0, \epsilon_a = \epsilon_b = 0} \tag{7.1}
\]
where
\[
S_1 = -\frac{(X_a - X_b)^2}{4T} - \frac{1}{4} \int_0^T d\tau d\theta X D^3 X \tag{7.2}
\]
This suggests that we have only to (i) evaluate one-loop \((N+2)\)-point function \(<V_b V_a \prod V_n>\), (ii) then substitute one-loop quantities by two-loop ones

\[
\mathcal{N} \rightarrow \mathcal{N}_1, \quad G_B(\tau_1, \tau_2) \rightarrow G_B^{(1)}(\tau_1, \tau_2), \quad (7.3)
\]

with

\[
\epsilon^\mu_a \epsilon^\nu_b \rightarrow g^{\mu \nu}, \quad k_a, k_b \rightarrow 0, \quad (7.4)
\]

where \(a\) and \(b\) are supposed to be the leg labels which will be joined by a propagator insertion to make another loop. The rules (7.4) coincide with those shown in [11]. However, note that \(G_B^{(1)}\) should be the combination of \(G_B\) defined not in (4.6) but in (4.7), in accordance with the fact \(G_B(\tau_i, \tau_i) = 0\).

We can easily check the validity of the above replacements up to between one-loop pinched 4-point functions and two-loop pinched 2-point functions. The first simple example is between the one-loop \(\hat{\Gamma}_3(2, 1)\) and two-loop \(\hat{\Gamma}^{(1)}_1(1, a)\) functions. Choosing the joining label set \((a, b)\) to be \((3, 1)\) in eq.(3.18) and setting \(k_3 = k_1 = 0, \epsilon^\mu_3 \epsilon^\nu_1 = g^{\mu \nu}\), we see that

\[
\hat{\Gamma}_3(2, 1) \rightarrow 0, \quad (7.5)
\]

which coincides with eq.(4.16) i.e. \(\hat{\Gamma}^{(1)}_1(1, a)\). Similarly, if we choose \((a, b) = (1, 2)\) in eq.(3.19) and \((a, b) = (3, 4)\) in eq.(3.19), we verify the following map relations respectively

\[
\hat{\Gamma}_4(2, 1) \rightarrow 0 = \hat{\Gamma}^{(1)}_2(b, a), \quad (7.6)
\]

\[
\hat{\Gamma}_4(2, 1) \rightarrow 0 = \hat{\Gamma}^{(1)}_2(2, 1). \quad (7.7)
\]

For a non-zero example, put \((a, b) = (4, 1)\) in eq.(3.20). Then we recover eq.(5.7) from the one-loop 4-point function

\[
\hat{\Gamma}_4(4, 3|2, 1) \rightarrow (D - 1) \epsilon_3 \cdot \epsilon_2 \int d\tau_3 d\tau_2 \mathcal{N}^B_1 (G^{32}_F)^2 \exp[k_3 \cdot k_2 G_B^{(1)}(\tau_3, \tau_2)],
\]

\[
= \hat{\Gamma}^{(1)}_2(b, 3|2, a), \quad (7.8)
\]

where \(\tau_a\) and \(\tau_b\) in \(\mathcal{N}^B_1\) must be replaced by \(\tau_2\) and \(\tau_3\).

It is nontrivial whether (7.3) is further valid for the two-loop pinched 3-point functions obtained in previous section, since several contributions from \(G^{(1)}\) and \(\mathcal{N}_1\) exist as remarked there — although the un-pinched function (7.1) clearly support the replacement (7.3). For
the \((m + 1)\)-loop cases of Schmidt-Schubert’s type [9], [10], we can easily generalize these
replacements, using \(m\) copies of (7.4) and introducing \(N_m\) and \(G_B^{(m)}\) in (7.3) (\(N_m\) and \(G_B^{(m)}\)
are given in eq.(28) of [9], however note that the \(G_B^{(m)}\) here should be the one which satisfies
\(G_B^{(m)}(x, x) = 0\).

8 Gluon loop

Our pinch technique can be easily applied to gluon loop cases as well, once the un-pinched
part of \(N\)-point function \(\hat{\Gamma}^{(1)}_N\) for gluon loop is written in a form of super-worldline correlator.
Let us recall the difference between spinor and gluon loops at the one-loop level. In the gluon
loop case, we have to change
\[
S_0 \rightarrow \tilde{S}_0 = \frac{1}{4} x^2 + \psi_\mp \dot{\psi}_\mp + C(\psi_\mp \dot{\psi}_\mp - 1),
\]
(8.1)
\[
\oint [dX] \rightarrow \lim_{C \to \infty} \frac{1}{2} \sum_{p=0}^{1} (-)^{p+1} \oint_{(p)} [dX],
\]
(8.2)
where \(p = 0\) (\(p = 1\)) denotes (anti-)periodic worldline fermions, which are defined by \(\psi = \psi_\mp + \psi_\pm\) and satisfy \(\{\psi^\mu_\pm, \psi^\nu_\pm\} = g^{\mu\nu}, \{\psi_\pm, \psi_\pm\} = 0\). The above action inevitably changes the
fermionic worldline Green function \(G_F\) into
\[
G_F^{(1)}(\tau_1, \tau_2) = 2 \text{sign}(\tau_1 - \tau_2) e^{-CT/2 \cosh(\frac{CT}{2} - C|\tau_1 - \tau_2|)},
\]
(8.3)
\[
G_F^{(0)}(\tau_1, \tau_2) = 2 \text{sign}(\tau_1 - \tau_2) e^{-CT/2 \sinh(\frac{CT}{2} - C|\tau_1 - \tau_2|)}.
\]
(8.4)

Taking account of these modifications, ‘gluon’ two-loop functions may be written as
\[
\hat{\Gamma}^{(1)}_N = \int d\tau_a d\tau_b d\theta_a d\theta_b \lim_{C \to \infty} \sum_{p=0}^{1} (-)^{p+1} \mathcal{N}_1^{(p)} < \mathcal{D}X_b \cdot \mathcal{D}X_a \prod_{n=1}^{N} V_n >^{p}_{(1)},
\]
(8.5)
where
\[
\mathcal{N}_1^{(p)} = \oint_{(p)} [dX] e^{-\tilde{S}_0} \exp[-\frac{(X_a - X_b)^2}{T}] = Z_p(4\pi T)^{-\frac{D}{2}} (1 + \frac{1}{T} G(a, b))^{-\frac{D}{2}},
\]
(8.6)
\[
Z_p = e^{CT}(1 + (-)^{p+1} e^{-CT})^4,
\]
(8.7)
and
\[
<X_1 X_2>^{p}_{(1)} = \frac{1}{\mathcal{N}_1^{(p)}} \oint_{(p)} [dX] e^{-\tilde{S}_0} \exp[-\frac{(X_a - X_b)^2}{T}] X_1 X_2.
\]
(8.8)
Note that $G(i, j)$ is now modified

$$G(i, j) = G_B(\tau_i, \tau_j) + 2 \theta_1 \theta_2 G_F^{(p)}(\tau_1, \tau_2). \quad (8.9)$$

Similarly as the spinor loop case, pinched functions of (8.5) can be obtained through inserting pinch $\delta$-functions $\theta_j \theta_i \delta(\tau_j - \tau_i)$ etc. Now, the simplest example is the two-point function

$$\hat{\Gamma}_2^{(1)}(b, 2|1, a) = (D-1)\epsilon_1 \cdot \epsilon_2 \int d\tau_1 d\tau_2 e^{k_1 \cdot k_2 G_B^{(1)}(1, 2)} \lim_{C \to \infty} \frac{1}{2} (-)^{p+1} \mathcal{N}^{[2]} G_F^{(p)}(\tau_1, \tau_2)^2$$

$$= 8(D-1)\epsilon_1 \cdot \epsilon_2 \int d\tau_1 d\tau_2 e^{k_1 \cdot k_2 G_B^{(1)(1,2)}(4\pi T) \frac{1}{4\pi T} (1 + \frac{1}{T} G_B^{12})^{\frac{\tau_2 - \tau_1}{2}}}. \quad (8.10)$$

Other pinched functions can be obtained from the pinched functions $\hat{\Gamma}_N^{(1)}$ of spinor loop through the replacement rule which was developed in one-loop studies [3]. Namely, replace the single $G_F$ chain appeared in the $\hat{\Gamma}_N^{(1)}$ of spinor loop

$$\prod_{k=1}^{d} G_F^{i_{k+1}, i_k} = \begin{cases} 
-2^d & \text{if } \tau_{max} \tau_{id} > \tau_{id-1} > \cdots > \tau_{i_2} > \tau_{i_1} = \tau_{min} \\
-(-2)^d & \text{if } \tau_{max} \tau_{id} > \tau_{i_1} > \tau_{i_2} > \cdots > \tau_{id-1} = \tau_{min} \\
-8 & \text{if } d = 2 \\
0 & \text{otherwise.} 
\end{cases} \quad (8.11)$$

Of course, this rule recovers the second line of eq. (8.10). As seen from the above argument, the gluon loop case is almost same as the spinor loop case. An important point is that every pinched function can be easily evaluated by inserting a possible number of the $\delta$-function $\theta_j \theta_i \delta(\tau_j - \tau_i)$ as long as the (un-pinched) $N$-point function is expressed in a form of super-worldline correlator. This enables us to calculate any pinch between external and internal gluon lines.

9 Conclusion

We have, using the super worldline formalism, reformulated Strassler’s method to evaluate pinched $N$-point functions associated to the quadratic terms of field strength in non-abelian gauge theory. It is much convenient that the pinched gluon vertex operator has been written in the elegant superfield expression which is given by the product of two gluon vertex operators where the pinch $\delta$-function is inserted. Owing to this result, we have only to insert the
pinch $\delta$-function $\theta_j \theta_i \delta(\tau_j - \tau_i)$ into the un-pinched $N$-point function $\hat{\Gamma}_N$, in order to obtain the pinched function $\hat{\Gamma}_N^{(j,i)}$. After that, one can put an appropriate color factor. This method makes calculation simpler and more transparent than the original pinch method of Strassler. Furthermore, our method can be applied straightforwardly to two-loop diagrams (although the present cases are nothing but the QED corrections), and we have derived various formulae on the two-loop pinched $N$-point functions. In particular, the pinch between internal and external gluon has become calculable in terms of the $\delta$-function insertion.

Of course, there exist several unsolved problems as well. First of all, the direct proof of coincidence of our two-loop $N$-point functions with Feynman rule results is difficult and non-trivial at the present expressions, since our expressions of $N$-point functions are extremely different from standard ones. For example, it is hard to guess the origin of two-loop Green function $G_B^{(1)}$ in the corresponding Feynman rule calculation. A reasonable comparison could be done after all (or some of) integrations w.r.t. $\tau$ variables, like done in [10] — where they performed every $\tau$ integration using the Fock-Schwinger gauge and extracted a relevant divergent constant to the QED $\beta$-function. This kind of analysis would seem to be the only one that we can do at the present level of our techniques.

Second one is the following. We have considered the equivalence between two different Green functions in the situations of pinched one- and two-point functions, whereas it is still unclear in the un-pinched function cases. In these pinched situations, all the $\tau$-integrations were easy because of the $\delta$-function insertions, which reduce the number of non-trivial integrations on $\tau$ variables. The equivalence in un-pinched functions would become clear after performing all of integrations on $\tau_a$, $\tau_b$ and $\tau_n$, although we did not analyze the cases. Otherwise, there should be found a vanishing integral formula to compensate the difference of Green functions.

Thirdly, we have not considered any insertions of vertex operators into the inserted ‘gluon’ propagator line. This kind of vertex operator insertions occurs also in more general situation like in Schmidt-Schubert type multi-loop diagrams [1], [10]. In scalar $\phi^3$-theory, worldline Green functions for such vertex insertions are already found in [12], [13]. As easily expected from the scalar theory analysis, these multi-loop cases have further three types of worldline Green functions, and the $N$-point function exponent of these types should be given by the sum of these possible vertex insertions. Our pinch formalism presented here could be basically applied
to these complicated multi-loop diagrams. However, having various types of worldline Green functions, we do not know which one should be used in Wick contractions when evaluating non-exponent parts $K_{\mu\nu...\rho}$. Without solving this problem, we will not be able to go ahead for further complicated multi-loop diagram analyses.

Finally, another consistency check should be pursued from the string theoretical approach. String theories offer us a reasonable input to particle theory through the infinite limit of string tension. Since a particle diagram corresponds to a corner of moduli space, we can start from a universal expression of world-sheet Green function \cite{16}. This might solve one of the above problems. Also, it is interesting to note that the pinch prescription presented here is very similar to the one developed in superstring theory \cite{14}.

Although we have not yet arrived at any familiar result obtained by the standard (Feynman rule) calculation, we believe that our pinch formulation and formulae will be useful for further understanding and development of Bern-Kosower-like rules for multi-loop scattering amplitudes in Yang-Mills theory.

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**Note added:** In this paper, we concentrated on the photon propagator insertion. After submitting this work, an exact path integral expression of gluon propagator has been proposed in \cite{17}. Together with that, one can perform some calculations of pinch contribution from the internal gluon line.

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