Abstract Present work contains an elementary method to obtain Jackson and Stechkin type inequalities of approximation by integral functions of finite degree (IFFD) in some non-translation invariant Banach spaces of functions defined on $R := (-\infty, +\infty)$. To do this we employ a transference theorem which produce norm inequalities starting from norm inequalities in $C(R)$, the class of bounded uniformly continuous functions defined on $R$. As a model example we consider the variable exponent Lebesgue spaces. Let $B \subseteq R$ be a measurable set, $p(x) : B \to [1, \infty)$ be a measurable function. For the class of functions $f$ belonging to variable exponent Lebesgue spaces $L^{p(x)}(B)$ we consider difference operator $(I - T_\delta)^r f(\cdot)$ under the condition that $p(x)$ satisfies the Log Hölder continuity condition and $1 \leq \text{ess inf}_{x \in B} p(x), \text{ess sup}_{x \in B} p(x) < \infty$ where $I$ is the identity operator, $r \in \mathbb{N} := \{1, 2, 3, \ldots\}, \delta \geq 0$ and

$$T_\delta f(x) = \frac{1}{\delta} \int_0^\delta f(x + t) dt, \quad x \in R, \quad T_0 \equiv I,$$

is the forward Steklov operator. It is proved that

$$\| (I - T_\delta)^r f \|_{p(\cdot)},$$

is a suitable measure of smoothness for functions in $L^{p(x)}(B)$ where $\| \cdot \|_{p(\cdot)}$ is some norm in $L^{p(x)}(B)$. We obtain main properties of $\| (I - T_\delta)^r f \|_{p(\cdot)}$ in $L^{p(x)}(B)$. We give proof of direct and inverse theorems of approximation by IFFD in $L^{p(x)}(R)$.

Key Words Variable exponent Lebesgue space, One sided Steklov operator, Integral functions of finite degree, Best approximation, Direct theorem, Inverse theorem, Modulus of smoothness, Marchaud inequality, K-functional.

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1. Part 1: Introduction and A Transference Result

Some inequalities of Approximation Theory in a Homogenous Banach Spaces (HBS) can be obtained their uniform-norm counterparts. This information is known for a
long time. (see e.g., [22]). Here, this elegant method is generalized to some non-
translation invariant (with respect to ordinary translation $x \to f(x + a)$) Banach
spaces of functions defined on $\mathbb{R}$. (see Theorem 1.4 below). This is the first part
of this work. After then, we obtain several uniform-norm inequalities on $C(\mathbb{R})$. (see
results of Part 2). In the Part 3, we apply the results of Parts 2 and 3 to obtain
several inequalities of approximation by IFFD in some variable exponent Lebesgue
spaces $L_{p(x)}(\mathbb{R})$. Under some condition on $p(x)$ of $L_{p(x)}(\mathbb{R})$ we obtain main
inequalities of exponential approximation by IFFD such as Jackson-Stechkin-Timan
type estimates and equivalence of $K$-functional with suitable modulus of smoothness (**)
given in abstract for functions of $L_{p(x)}(\mathbb{R})$. Note that many results of approximation
by IFFD can be obtained easily their uniform-norm counterparts in $C(\mathbb{R})$. (See proofs of
Part 3.)

Now we consider Transference Result (TR).

Let $C(\mathbb{R})$ be the class of continuous functions defined on $\mathbb{R}$. For $r \in \mathbb{N}$, we define
$C^r(\mathbb{R})$ consisting of every member $f \in C(\mathbb{R})$ such that the derivative $f^{(k)}$ exists and
is continuous on $\mathbb{R}$ for $k = 1, ..., r$. We set $C^\infty(\mathbb{R}) := \{f \in C^r(\mathbb{R})\}$
for any $r \in \mathbb{N}$. We denote by $C_c(\mathbb{R})$, the collection of real valued continuous
functions on $\mathbb{R}$ and support of $f \equiv \text{spt } f$ is compact set in $\mathbb{R}$. We define
$C^r_c(\mathbb{R}) := C^r_c(\mathbb{R}) \cap C_c(\mathbb{R})$ for $r \in \mathbb{N}$ and $C^\infty_c := C^\infty_c(\mathbb{R}) \cap C_c(\mathbb{R})$.
Let $L_{p(\cdot)}(\mathbb{R})$, $1 \leq p \leq \infty$ be the classical Lebesgue
space of functions on $\mathbb{R}$.

For $i \in \mathbb{N}$, all constants $C_i (\ldots)$ will be some positive number such that they depend on
the parameters given in the brackets. Also constants $C_i (\ldots)$ will not change throughout
this paper.

Let $X$ (and $X'$) be a Banach space (and its dual).

**Definition 1.1.** We collect here conditions required for Banach space $X$, considered
here, as the following three assumptions:

**X1** (a) $C^\infty_c \cap X$ is a dense subset of $X$. (b) $C^\infty_c \cap X'$ is a dense subset of $X'$ with
$X' \neq L^\infty(\mathbb{R})$.

**X2** For any compact set $A \subset \mathbb{R}$, there exists a $C_1(X,A) > 0$ such that
\[
\|f\|_{1,A} \leq C_1(X,A) \|f\|_X, \quad \text{and}
\]

**X3** $\|f\|_X \leq \tilde{C}_1 \sup \{ \int_{\mathbb{R}} |f(x)g(x)| \, dx : G \in X' \cap C^\infty_c \text{ and } \|G\|_{X'} \leq 1 \}$.

First of all we provide some examples of Banach space $X$ satisfying conditions in
Definition 1.1.

**Example 1.2.** (i) Let $X := L_p(\mathbb{R})$ ($1 < p < \infty$) be the classical Lebesgue spaces
on $\mathbb{R}$. Then $X$ satisfy properties (X1)-(X2)-(X3). When $p = 1$, we do not require
(X1)(b) and results holds for $X := L_1(\mathbb{R})$ with minor modifications on constants.

(ii) Let $p \in P^{\log}(\mathbb{R})$ see Definition 3.4 Then variable exponent Lebesgue space $L_{p(\cdot)}$
(see Definition 3.1) satisfy properties (X1)-(X2)-(X3) see Theorem 3.1 below. When
$p(\cdot) \equiv 1$ the same information in (i) need here. It is well known that variable exponent
Lebesgue spaces are not translation invariant.
(iii) All Banach Function Spaces (see Benneth Sharpley [14, Chapter 1]) with property (X1) satisfy conditions in Definition 1.1.

(iv) Lebesgue spaces \( X := L_p ([−π, π], \omega) \) (1 < p < ∞) with Muckenhoupt weights \( \omega \) (see [41]) satisfy conditions in Definition 1.1 (see paper [9]).

We define \( \langle f, g \rangle := \int_R f(x)g(x)dx \) when integral exists.

For an \( f \in X \), we define

\[
F_{f,G}(u) := \int_R f(u+x)|G(x)|dx, \quad u \in R,
\]

where \( G \in X' \cap C_c^\infty \) and \( \|G\|_{X'} \leq 1 \).

Let \( W^r_X, r \in \mathbb{N} \), be the class of functions \( f \in X \) such that derivatives \( f^{(k)} \) exist for \( k = 1, ..., r-1 \), \( f^{(r-1)} \) absolutely continuous and \( f^{(r)} \in X \).

Some properties of the function \( F_f(\cdot) \) is given in the following theorem.

**Theorem 1.3.** Let \( X \) be a Banach space with the properties (X1)-(X2)-(X3). Then,

(a) if \( f \in X \), then, the function \( F_{f,G}(\cdot) \) defined in (1.1) is uniformly continuous on \( R \),

(b) if \( r \in \mathbb{N} \), and \( f \in W^r_X \), then, \( (F_{f,G}(u))^{(k)} \) exists and

\[
(F_{f,G})^{(k)}(u) = F_{f^{(k)},G}(u)
\]

for \( k \in \{1, ..., r\} \).

Main theorem of this part is as follows.

**Theorem 1.4.** Let \( X \) be a Banach space with the properties (X1)-(X2)-(X3) and \( f, g \in X \). If

\[
\|F_{f,G}\|_{C(R)} \leq C_2 \|F_{g,G}\|_{C(R)},
\]

with an absolute constant \( C_2 > 0 \), then, norm inequality

\[
\|f\|_X < 2C_1 (X, sptG) \tilde{C}_1 C_2 \|G\|_\infty \|g\|_X
\]

holds.

1.1. Proof of the results of part 1.

Proof of Theorem 1.3 (a) Since \( C_c^\infty \) is a dense subset of \( X \), we consider functions \( f \in C_c^\infty \). For any \( \varepsilon > 0 \), there exists \( \delta := \delta (\varepsilon) > 0 \) so that

\[
|f(x+u_1) - f(x+u_2)| < \frac{\varepsilon}{1 + |sptG|}
\]

for any \( u_1, u_2 \in \mathbb{R} \) with \( |u_1 - u_2| < \delta \). Then, there holds inequality

\[
|F_{f,G}(u_1) - F_{f,G}(u_2)| \leq \int_R |f(x+u_1) - f(x+u_2)||G(x)|dx
\]
Proof of Theorem 1.4. Let \( \xi > 0 \), then one can find an \( \varepsilon > 0 \) such that for any \( u_1, u_2 \in \mathbb{R} \) with \( |u_1 - u_2| < \delta \). Thus conclusion of Theorem 1.3 follows. For the general case \( f \in X \) there exists an \( g \in C^\infty_c \) so that

\[
\|f - g\|_X < \frac{\xi}{4(1 + |sptG|)} \|G\|_\infty
\]

for any \( \xi > 0 \). Therefore

\[
|F_{f,G}(u_1) - F_{f,G}(u_2)| = |F_{f,G}(u_1) - F_{g,G}(u_1)| + |F_{g,G}(u_1) - F_{g,G}(u_2)| + |F_{g,G}(u_2) - F_{f,G}(u_2)| = |F_{f-g,G}(u_1)| + \frac{\xi}{2} + |F_{g-f,G}(u_2)|
\]

\[
\leq 2(1 + |sptG|)\|G\|_\infty \|f - g\|_X + \frac{\xi}{2} < \xi.
\]

As a result \( F_{f,G} \) is uniformly continuous on \( \mathbb{R} \).

(b) is seen from (a) and (1.1). \( \square \)

Proof of Theorem 1.4. Let \( 0 \leq f, g \in X \). If \( \|g\|_X = 0 \), then, the result is obvious. So we assume that \( \|g\|_X > 0 \). In this case

\[
\|F_{f,G}\|_{C(X)} \leq C_2 \|F_{g,G}\|_{C(X)} = C_2 \left\| \int R g(u+x) |G(x)| \, dx \right\|_{C(r)}
\]

\[
= C_2 \max_{u \in R} \int R g(u+x) |G(x)| \, dx = C_2 \max_{u \in sptG} \int sptG g(u+x) |G(x)| \, dx
\]

\[
= C_2 \max_{u \in sptG} \|G\|_{\infty,sptG} \|g(u\cdot)\|_{1,sptG} = C_2 \|G\|_{\infty} \|g\|_{1,sptG}
\]

\[
\leq C_2 \|G\|_{\infty} C_1(X,sptG) \|g\|_X
\]

by (X2).

On the other hand, for any \( \varepsilon > 0 \) and appropriately chosen \( \tilde{G}_\varepsilon \in X' \cap C^\infty_c \) with

\[
\int R g(x) \left| \tilde{G}_\varepsilon(x) \right| \, dx \geq \frac{1}{C_1} \|g\|_X - \varepsilon, \quad \left\| \tilde{G}_\varepsilon \right\|_{X'} \leq 1,
\]

one can find

\[
\|F_{f,G}\|_{C(X)} \geq |F_{f,G}(0)| \geq \int R f(x) |G(x)| \, dx
\]

\[
= \frac{1}{C_1} \|f\|_X - \varepsilon.
\]

In the last inequality we take as \( \varepsilon \rightarrow 0^+ \) and obtain

\[
\|F_{f,G}\|_{C(X)} \geq \frac{1}{C_1} \|f\|_X.
\]
Combining these inequalities we get
\[
\| f \|_X \leq \tilde{C}_1 \| F_{f,G} \|_{C(R)} \leq \tilde{C}_1 C_2 \| G \|_\infty \| g \|_X .
\]

In the general case \( f, g \in X \) we get
\[
(1.2) \quad \| f \|_X \leq 2 \tilde{C}_1 (X, \text{spt} G) \tilde{C}_1 C_2 \| G \|_\infty \| g \|_X .
\]

**Remark 1.5.** (i) Note that, inequalities type (1.2) will be used frequently in the application Part 3.

(ii) We can see from the Main Theorem 1.4 that uniform norm inequalities are important for applications. In the next part, we will concentrate on uniform norm inequalities on measurable subsets of \( R \).

2. **Part 2: Uniform norm estimates**

In this part, let \( \Omega \subseteq R \) be a measurable set and \( C(\Omega) \) be the collection of functions continuous on \( \Omega \). If \( \Omega \neq R \) and \( f \in C(\Omega) \), we will extend \( f \) to whole \( R \) by " \( f(s) \equiv 0 \) whenever \( s \notin \Omega \)." when necessary. For \( f \in C(\Omega) \) and \( \delta \geq 0 \), we define the modulus of smoothness as
\[
(2.1) \quad \Omega_r(f, \delta) := \|(I - T_\delta)^r f\|_{C(\Omega)}, \quad r \in \mathbb{N},
\]

\[
\Omega_0(f, \cdot) := \|f\|_{C(\Omega)}
\]

with \( T_\delta f \) of (*).

**Lemma 2.1.** Let \( 0 \leq \delta < \infty, r \in \mathbb{N} \) and \( f \in C^r(\Omega) \). Then
\[
(2.2) \quad \frac{d^r}{dx^r} T_\delta f(x) = T_\delta \frac{d^r}{dx^r} f(x) \quad \text{on} \Omega.
\]

The following theorem states the main properties of (2.1).

**Theorem 2.2.** For \( f \in C(\Omega), 0 \leq \delta < \infty, \) and \( r \in \mathbb{N} \), the following properties hold.

1. \( \Omega_r(f, \delta)_{C(\Omega)} \) is non-negative, non-decreasing function of \( \delta \),
2. \( \Omega_r(f, \delta)_{C(\Omega)} \) is subadditive with respect to \( f \),
3. \( \|T_\delta f\|_{C(\Omega)} \leq \|f\|_{C(\Omega)} \),
4. \( \Omega_r(f, \delta)_{C(\Omega)} \leq 2\Omega_{r-1}(f, \delta)_{C(\Omega)} \leq \cdots \leq 2^{r-1}\Omega_1(f, \delta)_{C(\Omega)} \leq 2^r \|f\|_{C(\Omega)}, \quad (***)
5. \( \Omega_r(f, \delta)_{C(\Omega)} \leq 2^{-1}\delta\Omega_{r-1}(f^r, \delta)_{C(\Omega)} \leq \cdots \leq 2^{-r}\delta \|f^{(r)}\|_{C(\Omega)}, \quad \text{if} f \in C^r(\Omega). \)

Let \( X \) be a Banach space with a norm \( \| \cdot \|_X \) and \( r \in \mathbb{N} \). We define Peetre’s \( K \)-functional for the pair \( X \) and \( W^r_X \) as follows:
\[
K_r(f, \delta, X)_X := \inf_{g \in W^r_X} \{ \|f - g\|_X + \delta \|g^{(r)}\|_X \}, \quad \delta > 0.
\]

We set \( T^r_\delta f := (T_\delta f)^r \).
Lemma 2.3. Let $0 \leq \delta < \infty$, $r - 1 \in \mathbb{N}$, and $f \in C^r(\Omega)$ be given. Then
\[
\frac{d^r}{dx^r} T_\delta f (x) = \frac{d}{dx} T_\delta \left(\frac{d^{r-1}}{dx^{r-1}} T_\delta f (x)\right) \quad \text{on } \Omega.
\]

Lemma 2.4. (see e.g. [19, p.177]) Let $\Omega \subseteq \mathbb{R}$ be a measurable set, $\delta > 0$, $f \in C(\Omega)$ and $\tilde{T}_\delta f (\cdot) = f (\cdot + \delta)$. Then, for any $r \in \mathbb{N}$, there holds
\[
\frac{1}{r^r + 2^r} \leq \sup_{|h| \leq \delta} \left\| \left( I - \tilde{T}_h \right)^r f \right\|_{C(\Omega)} \leq 2^r.
\]

Main result of this part is the following theorem.

Theorem 2.5. Let $\Omega \subseteq \mathbb{R}$ be a measurable set, $0 < \delta < \infty$, $f \in C(\Omega)$, $r \in \mathbb{N}$ and $g \in C^2(\Omega)$. Then, the following inequalities
\[
\left\| \frac{d}{dx} T_\delta f (x) \right\|_{C(\Omega)} \leq \frac{2}{\delta} \left\| f \right\|_{C(\Omega)},
\]
\[
\left\| T_\delta f (x) \right\|_{C(\Omega)} \leq \frac{2}{\delta} \left\| \frac{d}{dx} T_\delta f \right\|_{C(\Omega)},
\]
\[
\left\| g (x) - T_\delta g (x) + \frac{\delta}{2} \frac{d}{dx} g (x) \right\|_{C(\Omega)} \leq \frac{\delta^2}{6} \left\| \frac{d^2}{dx^2} g \right\|_{C(\Omega)},
\]
\[
\left( (C_3 (r))^{-1} K_r (f, \delta, C(\Omega)) \right)_{C(\Omega)} \leq \left\| (I - T_\delta)^r f \right\|_{C(\Omega)} \leq 2^r K_r (f, \delta, C(\Omega)),
\]
are hold with $C_3 (1) = 36$, $C_3 (r) = 2^r (r^r + 34^r)$ for $r > 1$.

As a corollary of Theorem 2.5 we can state the following result.

Proposition 2.6. If $0 < h \leq \delta < \infty$ and $f \in C(\Omega)$, then
\[
\left\| (I - T_h) f \right\|_{C(\Omega)} \leq 72 \left\| (I - T_\delta) f \right\|_{C(\Omega)}.
\]

As a corollary of (2.4) and Lemma 2.4 we can write

Corollary 2.7. Let $\Omega \subseteq \mathbb{R}$ be a measurable set, $\delta > 0$, $f \in C(\Omega)$ and $r \in \mathbb{N}$. Then,

(i) there holds
\[
1 + 2^{-r} r^r \leq \frac{\sup_{|h| \leq \delta} \left\| \left( I - \tilde{T}_h \right)^r f \right\|_{C(\Omega)}}{\left\| (I - T_\delta)^r f \right\|_{C(\Omega)}} \leq 2^r C_3 (r),
\]

(ii) for $0 < \delta_1 \leq \delta_2$, there holds
\[
(1 + 2^{-r} r^r) \Omega_r (f, \delta_1)_{C(\Omega)} \leq C_3 (r) 2^r \Omega_r (f, \delta_2)_{C(\Omega)}.
\]

Remark 2.8. From Theorem 23.62 of [58, p.579] we have
\[
\lim_{\delta \searrow 0} \Omega_1 (f, \delta)_{C(\mathbb{R})} = \lim_{\delta \searrow 0} \left\| (I - T_\delta) f \right\|_{C(\mathbb{R})} = 0.
\]
Corollary 2.9. If \( f \in C(\mathbb{R}) \), \( 0 < \delta < \infty \), and \( r \in \mathbb{N} \), then, by (2.6) and (**),
\[
\lim_{\delta \to 0} \Omega_r(f, \delta)_{C(\Omega)} = \lim_{\delta \to 0} \| (I - T_\delta)^r f \|_{C(\Omega)} = 0
\]
holds.

2.1. Exponential Approximation. Consider an entire function \( f(z) \) and put \( M(r) = \max_{|z|=r} |f(z)| \) for \( z = x + iy \). We say that an entire function \( f \) is of exponential type \( \sigma \) if
\[
\limsup_{r \to \infty} \frac{\ln M(r)}{r} \leq \sigma, \quad \sigma < \infty.
\]
Let \( \mathcal{G}_\sigma(X) \) be the subspace of entire function of exponential type \( \sigma \) that belonging to a Banach space \( X \). The quantity
\[
A_\sigma(f)_X := \inf_{g} \{ \| f - g \|_X : g \in \mathcal{G}_\sigma(X) \}
\]
is called the deviation of the function \( f \in X \) from \( \mathcal{G}_\sigma(X) \).

Remark 2.10. Let \( \sigma > 0 \), \( 1 \leq p \leq \infty \), \( f \in L_p(\mathbb{R}) \),
\[
\vartheta(x) := \frac{2 \sin(x/2) \sin(3x/2)}{x^2}
\]
and
\[
J(f, \sigma) = \sigma \int_R f(x-u) \vartheta(\sigma u) du
\]
be the dela Val`ee Poussin operator ([13, definition given in (5.3)]). It is known (see (5.4)-(5.5) of [13]) that, if \( f \in L_p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), then,

(i) \( J(f, \sigma) \in \mathcal{G}_{2\sigma}(L_p(\mathbb{R})) \),

(ii) \( J(g_\sigma, \sigma) = g_\sigma \) for any \( g_\sigma \in \mathcal{G}_\sigma(L_p(\mathbb{R})) \),

(iii) \( \| J(f, \sigma) \|_{L_p(\mathbb{R})} \leq \frac{3}{2} \| f \|_{L_p(\mathbb{R})} \),

(iv) \( J(f, \sigma)^{(r)} = J(f^{(r)}, \sigma) \) for any \( r \in \mathbb{N} \) and \( f \in W^r_{L_p(\mathbb{R})} \);

(v) \( \| J\left(f, \frac{\sigma}{2}\right) - f \|_{L_p(\mathbb{R})} \to 0 \) (as \( \sigma \to \infty \)) and hence
\[
\| J\left(f, \frac{\sigma}{2}\right)^{(k)} - f^{(k)} \|_{L_p(\mathbb{R})} \to 0 \text{ as } \sigma \to \infty,
\]
for \( f \in W^r_{L_p(\mathbb{R})} \) and \( 1 \leq k \leq r \).

Corollary 2.11. Let \( 0 < \sigma < \infty \).

(i) If \( 1 \leq p \leq \infty \), \( f \in L_p(\mathbb{R}) \). Then, using (v) of the last remark, we conclude
\[
\lim_{\sigma \to \infty} A_\sigma(f)_{L_p(\mathbb{R})} = 0.
\]

(ii) Let \( g : \mathbb{R} \to \mathbb{C} \) be bounded on the real axis \( \mathbb{R} \). Then (see [15])
\[
\lim_{\sigma \to \infty} A_\sigma(g)_{C(\mathbb{R})} = 0 \Leftrightarrow g \text{ is uniformly continuous on } \mathbb{R}.
\]
Theorem 2.12. Let \( r \in \mathbb{N}, \sigma > 0, \delta \in (0,1) \) and \( f \in C(R) \). Then, the following Jackson type inequality
\[
(2.8) \quad A_\sigma (f)_{\mathcal{C}(R)} \leq 5\pi 4^{-1} C_3 (r) \Omega_r (f, 1/\sigma)_{\mathcal{C}(R)}, \text{ and }
\]
its weak inverse
\[
(2.9) \quad \Omega_r (f, \delta)_{\mathcal{C}(R)} \leq (1 + 2^{r-1}) 2^{-1} \sigma^r \left( A_0 (f)_{\mathcal{C}(R)} + \int_{1/2}^{1/3} u^{r-1} A_u (f)_{\mathcal{C}(R)} du \right)
\]
are hold.

We set \( |\sigma| := \max \{ n \in \mathbb{Z} : n \leq \sigma \} \).

Theorem 2.13. Let \( r \in \mathbb{N}, f \in X^r_\mathcal{C}(R) \) and \( \sigma > 0 \). Then,

(a) (i) there exists (see [13, Proposition 25]) a \( g_\sigma \in \mathcal{G}_\sigma (C(R)) \) such that
\[
A_\sigma (f)_{\mathcal{C}(R)} \leq \| f - g_\sigma \|_{\mathcal{C}(R)} \leq \frac{5\pi 4^r}{4 \sigma^r} \| f^{(r)} \|_{\mathcal{C}(R)}, \text{ and }
\]
(ii) its weak inverse
\[
\| f^{(k)} \|_{\mathcal{C}(R)} \leq (1 + 2^{2k-1}) 2^{k+2} \pi^k C_3 (k) \sum_{\nu=0}^{\infty} \frac{(\nu + 1)^{r}}{\nu + 1} A_\nu (f)_{\mathcal{C}(R)},
\]
holds whenever \( k = 1, 2, \ldots, r \) and \( \sum_{\nu=0}^{\infty} (\nu + 1)^{r-1} A_\nu (f)_{\mathcal{C}(R)} < \infty \).

(b) (i) the following inequality (see [29, p.397])
\[
A_\sigma (f)_{\mathcal{C}(R)} \leq \frac{(5\pi)^r}{\sigma^r} A_{\sigma^{(r)}} (f^{(r)})_{\mathcal{C}(R)}, \text{ and }
\]
(ii) its weak inverse
\[
A_{\sigma} (f^{(r)})_{\mathcal{C}(R)} \leq \left\| f^{(r)} - (J (f^{(r)}, \frac{\sigma}{2})) \right\|_{\mathcal{C}(R)} \leq
\leq (1 + 2^{2r-1}) 2^{r+2} \pi^r C_3 (r) \left( A_\sigma (f)_{\mathcal{C}(R)} \sum_{k=0}^{[\sigma]} \frac{k^r}{k} + \sum_{\nu=[\sigma]+1}^{\infty} \frac{(\nu + 1)^{r}}{\nu + 1} A_\nu (f)_{\mathcal{C}(R)} \right)
\]
hold when \( \sum_{\nu=0}^{\infty} (\nu + 1)^{r-1} A_\nu (f)_{\mathcal{C}(R)} < \infty \).

Theorem 2.14. Let \( r, k \in \mathbb{N}, 0 < t \leq 1/2, 0 \leq \delta < \infty \) and \( f \in C(R) \). Then

(i) there holds
\[
\Omega_{r+k} (f, \delta)_{\mathcal{C}(R)} \leq 2^k \Omega_r (f, \delta)_{\mathcal{C}(R)}, \text{ and }
\]
(ii) its weak inverse (Marchaud inequality)
\[
\Omega_r (f, t)_{\mathcal{C}(R)} \leq C_4 (r, k) t^r \int_{t}^{1} \frac{\Omega_{r+k} (f, u)_{\mathcal{C}(R)}}{u^{r+1}} du
\]
with \( C_4 (r, k) = 10\pi (1 + 2^{2r-1}) 2^{2r+3k} C_3 (r + k) \).
Theorem 2.15. Let $\sigma > 0$ and $f \in C(R)$. If $\sum_{\nu=0}^{\infty}(\nu + 1)^{k-1}A_{\nu}(f)_{C(R)} < \infty$, holds for some $k \in N$, then,

(i) the following Jackson type inequality for derivatives

$$A_{\sigma}(f)_{C(R)} \leq (5\pi)^{k+1}C_{3}(r)\sigma^{-k}\Omega_{r}(f^{(k)}, \sigma^{-1})_{C(R)},$$

and

(ii) its weak inverse (see Theorem 6.3.4 of [29, p.343])

$$\Omega_{r}(f^{(k)}, \frac{1}{\sigma})_{C(R)} \leq 2^{2k+r+1}\frac{1}{\sigma^{r}}\sum_{\nu=0}^{[\sigma]}\frac{(\nu + 1)^{r+k}}{\nu + 1}A_{\nu}(f)_{C(R)} + \sum_{\nu=[\sigma]+1}^{\infty}\frac{\nu^{k}}{\nu}A_{\nu}(f)_{C(R)}$$

are hold.

2.2. Proofs of the results of part 2.

Proof of Lemma 2.1. For $\delta = 0$ (2.2) is obvious. For $0 < \delta < \infty$ and $r = 1$, one can find

$$\frac{d}{dx}T_{\delta}f(x) = \frac{d}{dx}\left(\frac{1}{\delta}\int_{0}^{\delta} f(x + t) dt\right) = \frac{d}{dx}\left(\frac{1}{\delta}\int_{x}^{x+\delta} f(\tau) d\tau\right) = \frac{1}{\delta}\int_{x}^{x+\delta} \frac{d}{dx} f(\tau) d\tau = T_{\delta}\frac{d}{dx}f(x).$$

For $r > 1$, (2.2) follows from (2.10).

Proof of Theorem 2.2. (1)-(3) is known. (4) is seen from binomial expansion. To prove (5) it is sufficient to note inequality (see [10])

$$\| (I - T_{\delta}) f \|_{C(\Omega)} \leq 2^{-1}\delta \| f' \|_{C(\Omega)}, \ \delta > 0$$

for $f \in C^{1}(\Omega)$. Then

$$\| (I - T_{\delta})^{r} f \|_{C(\Omega)} \leq 2^{-1}\delta \| (I - T_{\delta})^{r-1} f' \|_{C(\Omega)} \leq \cdots \leq 2^{-r}\delta^{r} \| f^{(r)} \|_{C(\Omega)}$$

for $f \in C^{r}(\Omega)$, because

$$[(I - T_{\delta})^{r} f]' = (I - T_{\delta})^{r} f'.$$

Proof of Lemma 2.3. For $r = 2$, by Lemma 2.1

$$\frac{d^2}{dx^2}T_{\delta}f = \frac{d}{dx}\frac{d}{dx}T_{\delta}f = \frac{d}{dx}\frac{d}{dx}T_{\delta}\Psi = \frac{d}{dx}T_{\delta}\frac{d}{dx}\Psi [\Psi := T_{\delta}f]$$

and the result (2.3) follows. For $r = 3$, by Lemma 2.1

$$\frac{d^3}{dx^3}T_{\delta}f = \frac{d}{dx}\frac{d^2}{dx^2}T_{\delta}f = \frac{d}{dx}\frac{d^2}{dx^2}T_{\delta}\Psi = \frac{d}{dx}\frac{d}{dx}T_{\delta}\frac{d}{dx}\Psi = \frac{d}{dx}T_{\delta}\frac{d}{dx}T_{\delta}\Psi.$$
\[
\frac{d}{dx} T_\delta f = \frac{d^2}{dx^2} T_\delta^2 f = \frac{d}{dx} T_\delta \frac{d}{dx} T_\delta f = \frac{d}{dx} T_\delta \frac{d^2}{dx^2} T_\delta^2 f
\]

and \((2.3)\) holds. Let \((2.3)\) holds for \(k \in \mathbb{N}\):

\[
(2.11) \quad \frac{d^k}{dx^k} T_\delta^k f = \frac{d}{dx} T_\delta \frac{d^{k-1}}{dx^{k-1}} T_\delta^{k-1} f.
\]

Then, for \(k + 1\), \((2.11)\) and Lemma 2.1 implies that

\[
\frac{d^{k+1}}{dx^{k+1}} T_\delta^{k+1} f = \frac{d}{dx} T_\delta \frac{d^k}{dx^k} T_\delta^k f = \frac{d}{dx} T_\delta \frac{d^{k-1}}{dx^{k-1}} T_\delta^{k-1} f = \frac{d}{dx} T_\delta \frac{d^k}{dx^k} T_\delta^k f.
\]

\(\square\)

**Proof of Theorem 2.5.** For \(f \in C(\Omega)\) we have

\[
\left\| \frac{d}{dx} T_\delta f (x) \right\|_{C(\Omega)} = \left\| \frac{1}{\delta} \int_0^\delta f(x + t) dt \right\|_{C(\Omega)} =
\]

\[
(2.12) \quad \left\| \frac{1}{\delta} \int_x^{x+\delta} f(\tau) d\tau \right\|_{C(\Omega)} = \left\| \frac{1}{\delta} (f(x+\delta) - f(x)) \right\|_{C(\Omega)} \leq \frac{2}{\delta} \|f\|_{C(\Omega)}.
\]

Inequality \((2.12)\) also implies

\[
\left\| \left(\frac{d}{dx}\right)^2 T_\delta f (x) \right\|_{C(\Omega)} \leq \frac{2}{\delta} \left\| \frac{d}{dx} T_\delta f \right\|_{C(\Omega)}
\]

for \(f \in C(\Omega)\). If \(f \in C^2(\Omega)\) one can get

\[
(2.13) \quad \left\| f(x) - T_\delta f(x) + \frac{\delta}{2} \frac{d}{dx} f(x) \right\|_{C(\Omega)} \leq \frac{\delta^2}{6} \left\| \frac{d^2}{dx^2} f \right\|_{C(\Omega)}.
\]

To obtain \((2.13)\) we will use the Taylor formula

\[f(x + t) = f(x) + t \frac{d}{dx} f(x) + \frac{t^2}{2} \frac{d^2}{dx^2} f(\xi)\]

for some \(\xi \leq [x, x + t]\). Then integrating the last equation with respect to \(t\)

\[
\frac{1}{\delta} \int_0^\delta f(x + t) dt = f(x) + \frac{1}{\delta} \int_0^\delta t dt \frac{d}{dx} f(x) + \frac{1}{2\delta} \int_0^\delta t^2 dt \frac{d^2}{dx^2} f(\xi),
\]

\[
T_\delta f(x) = f(x) + \frac{\delta}{2} \frac{d}{dx} f(x) + \frac{\delta^2}{6} \frac{d^2}{dx^2} f(\xi)
\]

and \((2.13)\) holds.

Now \((2.12)\) and \((2.13)\) imply that

\[
(1/36) K_1 (f, \delta, C(\Omega))_{C(\Omega)} \leq \| (I - T_\delta) f \|_{C(\Omega)} \leq 2K_1 (f, \delta, C(\Omega))_{C(\Omega)}.
\]
Firstly, let us prove the right hand side of (2.14). For any \( g \in C^1 (\Omega) \)
\[
\| f - T_\delta f \|_{C(\Omega)} \leq \| f - g \|_{C(\Omega)} + \| g - T_\delta g \|_{C(\Omega)} + \| T_\delta (g - f) \|_{C(\Omega)}
\leq 2 \| f - g \|_{C(\Omega)} + \frac{\delta}{2} \| g' \|_{C(\Omega)} \leq 2 K_1 (f, \delta, C(\Omega))_{C(\Omega)}.
\]

For the left hand side of inequality (2.14) we need inequalities
\begin{align}
(2.15) \quad \| f - T_\delta^2 f \|_{C(R)} & \leq 2 \| f - T_\delta f \|_{C(R)}, \\
(2.16) \quad \delta \left( \left( \frac{d}{dx} \right)^2 T_\delta^2 f \right)_{C(R)} & \leq 34 \| f - T_\delta f \|_{C(R)}.
\end{align}

First we prove (2.15). Then
\[
\| f - T_\delta^2 f \|_{C(\Omega)} \leq \| f - T_\delta f \|_{C(\Omega)} + \| T_\delta f - T_\delta T_\delta f \|_{C(\Omega)}
\leq 2 \| f - T_\delta f \|_{C(\Omega)}.
\]

Now we consider inequality (2.16). In (2.15) we replace \( f \) by \( T_\delta^2 f \) and obtain
\[
\left\| T_\delta^2 f (x) - T_\delta T_\delta f (x) + \frac{\delta}{2} \frac{d}{dx} T_\delta^2 f (x) \right\|_{C(\Omega)} \leq \frac{\delta^2}{6} \left\| \frac{d^2}{dx^2} T_\delta^2 f \right\|_{C(\Omega)}.
\]

On the other hand, by (2.12),
\[
\left\| \frac{d^2}{dx^2} T_\delta^2 f \right\|_{C(\Omega)} \leq \frac{2}{\delta} \left\| \frac{d}{dx} T_\delta f \right\|_{C(\Omega)}
\leq \frac{2}{\delta} \left\{ \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} + \left\| \frac{d}{dx} (T_\delta f - f) \right\|_{C(\Omega)} \right\}
\leq \frac{2}{\delta} \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} + \frac{4}{\delta^2} \| T_\delta f - f \|_{C(\Omega)}.
\]

Hence,
\[
\frac{\delta}{2} \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} \leq \left\| T_\delta^2 f - T_\delta T_\delta^2 f - \frac{\delta}{2} \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} + \| T_\delta^2 f - T_\delta T_\delta^2 f \|_{C(\Omega)}
\leq \frac{\delta^2}{6} \left\| \frac{d^2}{dx^2} T_\delta^2 f \right\|_{C(\Omega)} + \| T_\delta^2 f - T_\delta T_\delta^2 f \|_{C(\Omega)}
\leq \frac{\delta^2}{6} \left\{ \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} + \frac{2}{\delta} \| T_\delta f - f \|_{C(\Omega)} \right\} + \| T_\delta^2 f - f \|_{C(\Omega)}
\]
\[
+ \| T_\delta (T_\delta^2 f - f) \|_{C(\Omega)} + \| T_\delta f - f \|_{C(\Omega)}.
\]
Then
\[
\frac{\delta}{6} \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} \leq \frac{17}{3} \left\| T_\delta f - f \right\|_{C(\Omega)},
\]
\[
\delta \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)} \leq 34 \left\| T_\delta f - f \right\|_{C(\Omega)}.
\]
To finish proof of the left hand side of inequality (2.4) with \( r = 1 \), we proceed as
\[
K_1 (f, \delta, C(\Omega)) \leq \left\| f - T_\delta^2 f \right\|_{C(\Omega)} + \delta \left\| \frac{d}{dx} T_\delta^2 f \right\|_{C(\Omega)}
\leq 36 \left\| T_\delta f - f \right\|_{C(\Omega)}.
\]
The proof of (2.4) with \( r = 1 \) now completed.

Let \( r > 1 \) be a natural number and we define
\[
g(\cdot) = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} T_\delta^{2r} f(\cdot).
\]
Then,
\[
\left\| f - g \right\|_{C(\Omega)} = \left\| (I - T_\delta^{2r}) f \right\|_{C(\Omega)} \leq (2r)^r \left\| (I - T_\delta)^r f \right\|_{C(\Omega)}.
\]
On the other hand,
\[
\delta^r \left\| \frac{d}{dx^r} T_\delta^{2r} f \right\|_{C(\Omega)} = \delta^{r-1} \delta \left\| \frac{d}{dx} T_\delta^2 \left( \frac{d^{r-1}}{dx^{r-1}} \right) T_\delta^{2r-2} f \right\|_{C(\Omega)}
\leq 34 \delta^{r-1} \left\| (I - T_\delta)^2 \frac{d^{r-1}}{dx^{r-1}} T_\delta^{2r-2} f \right\|_{C(\Omega)}
\leq (34)^2 \delta^{r-2} \left\| (I - T_\delta)^2 \frac{d^{r-2}}{dx^{r-2}} T_\delta^{2r-4} f \right\|_{C(\Omega)}
\leq \cdots \leq (34)^r \left\| (I - T_\delta)^r f \right\|_{C(\Omega)}.
\]
Then
\[
\delta^r \left\| \frac{d}{dx^r} T_\delta^{2r} f \right\|_{C(\Omega)} \leq (34)^r \left\| (I - T_\delta)^r T_\delta^{2r(l-1)} f \right\|_{C(\Omega)}
= (34)^r \left\| T_\delta^{2r(l-1)} (I - T_\delta)^r f \right\|_{C(\Omega)} \leq (34)^r \left\| (I - T_\delta)^r f \right\|_{C(\Omega)}.
\]
Using the last inequality we find
\[
\delta^r \left\| \frac{d}{dx^r} g \right\|_{C(\Omega)} = \delta \left\| \frac{d}{dx^r} \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} T_\delta^{2r} f \right\|_{C(\Omega)}
= \delta \left\| \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \frac{d}{dx^r} T_\delta^{2r} f \right\|_{C(\Omega)}
\leq \sum_{i=1}^{r} \left\| \binom{r}{i} \right\| \delta^r \left\| \frac{d}{dx^r} T_\delta^{2r} f \right\|_{C(\Omega)} \leq 2^r (34)^r \left\| (I - T_\delta)^r f \right\|_{C(\Omega)}.
and
\[ \begin{aligned} K_r(f, \delta, C(\Omega))_{C(\Omega)} & \leq \|f - g\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{C(\Omega)} \\ & \leq 2^r (r^r + (34)^r) \|(I - T_\delta)^r f\|_{C(\Omega)}. \end{aligned} \]

For the opposite direction of the last inequality, when \( g \in W^r_{p(\cdot)} \),
\[ \Omega_r(f, \delta)_{C(\Omega)} \leq 2^r \|f - g\|_{C(\Omega)} + \Omega_r(g, \delta)_{C(\Omega)}, \]
(2.17)
\[ \leq 2^r \|f - g\|_{C(\Omega)} + 2^{-r} \delta^r \|g^{(r)}\|_{C(\Omega)}, \]
and taking infimum on \( g \in W^r_{p(\cdot)} \) in (2.17) we get
\[ \Omega_r(f, \delta)_{C(\Omega)} \leq 2^r K_r(f, \delta, C(\Omega))_{C(\Omega)}. \]
\[ \square \]

Proof of Proposition 2.6. Let \( f \in C(\Omega) \). Then
\[ \| (I - T_h) f \|_{C(\Omega)} \leq 2K_1(f, h, C(\Omega))_{C(\Omega)} \leq 2K_1(f, \delta, C(\Omega))_{C(\Omega)} \leq 72 \| (I - T_\delta) f \|_{C(\Omega)}. \]
\[ \square \]

Proof of Theorem 2.12. (i) We consider Jackson type inequality (2.8). For any \( g \in X^r_{C(R)} \) we have
\[ A_\sigma(f)_{C(R)} \leq A_\sigma(f - g)_{C(R)} + A_\sigma(g)_{C(R)} \leq \|f - g\|_{C(R)} + \frac{5\pi}{4} \frac{d^r}{dx^r} \left\| \frac{d^r}{dx^r} g \right\|_{C(R)}. \]

Taking infimum on \( g \in X^r_{C(R)} \) in the last inequality we have
\[ A_\sigma(f)_{C(R)} \leq \frac{5\pi}{4} K_r\left( f, \frac{1}{\sigma}, C(R) \right)_{C(R)} \leq \frac{5\pi}{4} C_3(r) 4^r \left\| (I - T_{1/\sigma})^r f \right\|_{C(R)}. \]

(ii) We give the proof of inverse estimate (2.9). Let \( \sigma > 0 \) and \( g_\sigma \in G_\sigma(C(R)) \) be the best approximating IFFD of \( f \in C(R) \). Suppose that \( r \in \mathbb{N}, 0 < \delta < 1 \). Then, there exists a \( m \in \mathbb{N} \) such that \( \lfloor 1/\delta \rfloor = 2^{m-1} \). Hence, \( 2^{m-1} \leq 1/\delta < 2^m \). Now we have
\[ \Omega_r(f, \delta)_{C(R)} \leq \Omega_r(f - g_{2^m}, \delta)_{C(R)} + \Omega_r(g_{2^m}, \delta)_{C(R)} \leq 2^r A_{2^m}(f)_{C(R)} + 2^{-r} \delta^r \left\| \frac{d^r}{dx^r} g_{2^m} \right\|_{C(R)}. \]

On the other hand
\[ \left\| \frac{d^r}{dx^r} g_{2^m} \right\|_{C(R)} = \sum_{\gamma=1}^m \left( \frac{d^r}{dx^r} g_{2^\gamma} - \frac{d^r}{dx^r} g_{2^{\gamma-1}} \right) + \left( \frac{d^r}{dx^r} g_{1} - \frac{d^r}{dx^r} g_{0} \right) \right\|_{C(R)} \leq \sum_{\gamma=1}^m 2^{\gamma r} \left\| g_{2^\gamma} - g_{2^{\gamma-1}} \right\|_{C(R)} + \left\| g_1 - g_0 \right\|_{C(R)} \]
inequality, one gets

\[ \leq A_0 (f)_{C(R)} + A_1 (f)_{C(R)} + \sum_{\gamma=1}^{m} 2^{\gamma r} \left( A_{2\gamma^2} (f)_{C(R)} + A_{2\gamma - 1} (f)_{C(R)} \right) \]

\[ \leq A_0 (f)_{C(R)} + 2^{r} A_1 (f)_{C(R)} + 2 \sum_{\gamma=1}^{m} 2^{\gamma r} A_{2\gamma - 1} (f)_{C(R)} \]

\[ \leq 2 \left( A_0 (f)_{C(R)} + \sum_{\gamma=1}^{m} 2^{\gamma r} A_{2\gamma - 1} (f)_{C(R)} \right). \]

Then

\[ \frac{\delta^r}{2^r} \left\| \frac{d^r}{dx^r} g_{2m} \right\|_{C(R)} \leq \frac{2}{2^r} \delta^r \left( A_0 (f)_{C(R)} + \sum_{\gamma=1}^{m} 2^{\gamma r} A_{q\gamma - 1} (f)_{C(R)} \right). \]

Hence

\[ \Omega_r (f, \delta)_{C(R)} \leq \frac{2^{(m+1)r}}{2^m} A_{2m} (f)_{C(R)} + \frac{2}{2^r} \delta^r \left( A_0 (f)_{C(R)} + \sum_{\gamma=1}^{m} 2^{\gamma - 1} A_{2\gamma - 1} (f)_{C(R)} \right) \]

\[ \leq (1 + 2^{2r-1}) 2^{1-r} 2^{2r} \delta^r \left( A_0 (f)_{C(R)} + \sum_{\gamma=1}^{m} 2^{\gamma - 1} \int_{1/2} u r^{-1} A_u (f)_{C(R)} du \right) \]

\[ \leq (1 + 2^{2r-1}) 2^{r-1} \delta^r \left( A_0 (f)_{C(R)} + \int_{1/2}^{1/\delta} u r^{-1} A_u (f)_{C(R)} du \right). \]

□

**Proof of Theorem 2.13.** Results a) (i) and b) (i) are known. Let us consider a) (ii).

Suppose that \( \sum_{\nu=0}^{[\nu+1]/2} A_\nu (f)_{C(R)} < \infty \) and \( k \in \{1, 2, \cdots, r\} \). Then, using Nikolskii inequality, one gets

\[ \| f^{(k)} \|_{C(R)} = \lim_{\sigma \to \infty} \| J \left( f^{(k)}, \frac{\sigma}{2} \right) \|_{C(R)} = \lim_{\sigma \to \infty} \| J \left( f, \frac{\sigma}{2} \right) \|_{C(R)}^{(k)} \]

\[ \leq \pi^k \left[ \sup_{|b| \leq \delta} \left\| (I - \tilde{T}_h)^k \left( J \left( f, \frac{\sigma}{2} \right) \right) \right\|_{C(R)} \right] \]

\[ \leq (1 + 2^{2k-1}) 2^{k+2} \pi^k C_3 (k) \sum_{\nu=0}^{[\nu+1]/2} A_\nu \left( J \left( f, \frac{\sigma}{2} \right) \right)_{C(R)} \]

Note that (ii) b) is follow from (i) b). □
Proof of Theorem 2.14. (i) follows from properties of modulus of smoothness. We consider Marchaud type inequality (ii). Let \( 0 < t < 1/2 \). Assume that \( 2^{m-1} \leq \frac{1}{t} < 2^m \) for some \( m \in \mathbb{N} \). Then

\[
\Omega_t(f, t)_{C(R)} \leq \left( 1 + 2^{2r-1} \right) 2^{1-r} t^r \left( \sum_{\nu=1}^m 2^{\nu r} A_{2^{\nu-1}} (f)_{C(R)} + A_0 (f)_{C(R)} \right)
\]

\[
\leq \frac{5\pi}{2} \left( 1 + 2^{2r-1} \right) 2^{r+2k} C_3 (r + k) t^r \left( A_0 (f)_{C(R)} + \sum_{\nu=1}^m 2^{\nu r} \Omega_{k+r} (f, \frac{1}{2^\nu})_{C(R)} \right)
\]

\[
\leq \frac{5\pi}{2} \left( 1 + 2^{2r-1} \right) 2^{r+3k} C_3 (r + k) t^r \left( \Omega_{k+r} (f, \frac{1}{2})_{C(R)} + \sum_{\nu=1}^m 2^{-\nu+1} \Omega_{k+r} (f, u)_{C(R)} du \right)
\]

\[
\leq \frac{5\pi}{2} \left( 1 + 2^{2r-1} \right) 2^{r+3k} C_3 (r + k) t^r \left( \Omega_{k+r} (f, \frac{1}{2})_{C(R)} + \int_{2^{-\nu}}^{2^{-\nu+1}} \Omega_{k+r} (f, u)_{C(R)} du \right)
\]

\[
\leq 5\pi \left( 1 + 2^{2r-1} \right) 2^{r+3k} C_3 (r + k) t^r \left( \int_{1/2}^1 \Omega_{k+r} (f, u)_{C(R)} \frac{du}{u^{r+1}} \right) + \int_{1}^t \Omega_{k+r} (f, u)_{C(R)} \frac{du}{u^{r+1}}
\]

\[
\leq 10\pi \left( 1 + 2^{2r-1} \right) 2^{r+3k} C_3 (r + k) t^k \int_{1}^t \Omega_{k+r} (f, u)_{C(R)} \frac{du}{u^{r+1}}
\]

\[\square\]

3. Part 3: Applications

In this part we will exhibit some applications of previous parts. Let \( B \subseteq \mathbb{R} \) be a measurable set and \( p(x) : B \rightarrow [1, \infty) \) be a measurable function. We define \( P(B) \) as the class of measurable functions \( p(x) \) satisfying the conditions

\[ 1 \leq p^-_B := \text{ess inf}_{x \in B} p(x), \quad p^+_B := \text{ess sup}_{x \in B} p(x) < \infty. \]

We also set \( p^- := p^-_B \) and \( p^+ := p^+_B \).

Definition 3.1. We define the \( L_{p(\cdot)}(B) \) as the set of all functions \( f : B \rightarrow \mathbb{R} \) such that

\[ \int_B \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy < \infty \]

for some \( \lambda > 0 \). We set \( I_{p(\cdot)} (f) := I_{p(\cdot),B} (f) \). The set of of functions \( L_{p(\cdot)}(B) \), with norm

\[ \| f \|_{p(\cdot),B} := \inf \left\{ \eta > 0 : I_{p(\cdot),B} \left( \frac{f}{\eta} \right) < 1 \right\} \]

is Banach space. We set \( L_{p(\cdot)} := L_{p(\cdot)}(\mathbb{R}) \). The following embedding results are hold.
Theorem 3.2. ([17, Theorem 2.26]) Let \( B \subseteq \mathbb{R} \) be a measurable set. If \( 1 \leq p(x) < p^+_B < \infty \), \( p'(x) = p(x)/(p(x) - 1) \), \( f \in L_{p(\cdot)}(B) \), and \( g \in L_{p'(\cdot)}(B) \), then, Hölder Inequality
\[
\int_B f(x)g(x)dx \leq C_5 \left( p^+_B, p^-_B, B \right) \| f \|_{p(\cdot), B} \| g \|_{p'(\cdot), B}
\]
holds with \( C_5 := C_5 \left( p^+_B, p^-_B, B \right) = 1 + \frac{1}{p_B} - \frac{1}{p'_B} \).

Remark 3.3. By Theorem 2.26 and Remark 2.27 of [17] that
\[ 1 < C_5 \left( p^+_B, p^-_B, B \right) \leq 4 \]
for any nonconstant \( p(\cdot) \).

Let \( G_{\sigma,p(\cdot)} := \mathcal{G}_{\sigma} \left( L_{p(\cdot)} \right) \) be the subspace of integral function \( f \) of exponential type \( \sigma \) that belonging to \( L_{p(\cdot)} \). The quantity
\[
A_\sigma(f)_{p(\cdot)} := \inf_g \left\{ \| f - g \|_{p(\cdot)} : g \in G_{\sigma,p(\cdot)} \right\}
\]
is the deviation of the function \( f \in L_{p(\cdot)} \) from \( \mathcal{G}_\sigma \).

Definition 3.4. For measurable \( B \subseteq \mathbb{R} \), let \( P_{Log}(B) \) be the class of measurable functions \( p(x) \) satisfying the condition \( P(B) \), and there are positive constants \( C_6(p(\cdot)), C_7(p(\cdot)) \) and \( p_\infty > 1 \) such that
\begin{align*}
(3.3) & \quad |p(x) - p(y)| \ln \left( e + 1/|x - y| \right) \leq C_6 < \infty, \\
& \quad \text{for any } x, y \in B, \text{ and} \\
(3.4) & \quad |p(x) - p_\infty| \ln \left( e + |x| \right) \leq C_7 < \infty, \\
& \quad \text{for any } x \in B. \text{ Let } c_{log}(p) := \max \{ C_6, C_7 \}.
\end{align*}

The approximation by entire function of finite degree in the real line was originated in the beginning of twentieth century by Serge Bernstein [16] and became a separate branch of analysis due to the efforts of many mathematicians such as N. Wiener and R. Paley [46], N.I. Ahiezer [4], S.M. Nikolskii [43], I.I. Ibragimov [29], A. F. Timan [53], M. F. Timan [54], R. Taberski [55, 56], F.G. Nasibov [42], V. Yu. Popov [17], A. A. Ligun [44], and others.

Studying function spaces with variable exponent is now an extensively developed field after their applications in elasticity theory [59], fluid mechanics [48, 49], differential operators [21, 49], nonlinear Dirichlet boundary value problems [10], nonstandard growth [59], and variational calculus. See the books [17, 20, 52] for more references. Nowadays many mathematician solved many problems for the approximation of function in these type spaces defined on \( [0, 2\pi] \subset \mathbb{R} \) (see e.g., [7, 8, 27, 30, 31, 34], [1, 2, 3, 11, 12], [5, 6, 9, 13, 15], [21, 25, 26, 28, 32, 33, 36, 37, 38, 43, 50, 51, 57]). In this paper we propose generalized our last results in [10] which we obtained a direct and inverse theorems for approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis \( \mathbb{R} \) with
\[
(3.5) \quad \sup_{0 < h \leq \delta} \| (I - T_h)f \|_{p(\cdot)}
\]
as modulus of continuity \( \Omega_1(f, \delta)_{p(.)} \). Instead of (3.5), here we will use

\[
\|(I - T_\delta)^p f\|_{p(.)}
\]
as modulus smoothness \( \Omega_r(f, \delta)_{p(.)} \) and we obtain stronger Jackson inequality than obtained in [10].

3.1. Translated Steklov Averages and Mollifiers.

**Definition 3.5.** Suppose that \( 0 < \delta < 1 \) and \( \tau \in \mathbb{R} \). We define family of translated Steklov operators \( \{S_{\delta, \tau} f\} \), by

\[
S_{\delta, \tau} f(x) := \Phi_\delta(f)(x + \tau) = \frac{1}{\delta} \int_{x + \tau - \delta/2}^{x + \tau + \delta/2} f(t) \, dt, \quad x \in \mathbb{R}
\]

for locally integrable function \( f \) defined on \( \mathbb{R} \).

Mollifiers in variable exponent Lebesgue spaces is obtained by D. Cruz-Uribe and A. Fiorenza (see [18]).

**Definition 3.6.** Let \( B \subseteq \mathbb{R} \) be an open set, \( \phi \in L_1(B) \) and \( \int_B \phi(t) \, dt = 1 \). For each \( t > 0 \) we define \( \phi_t(x) = \frac{1}{t} \phi(\frac{x}{t}) \). Sequence \( \{\phi_t\} \) will be called approximate identity. A function

\[
\tilde{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|
\]

will be called radial majorant of \( \phi \). If \( \tilde{\phi} \in L_1(B) \), then, sequence \( \{\phi_t\} \) will be called potential-type approximate identity.

**Definition 3.7.** ([20]) Let \( \mathbb{N} := \{1, 2, 3, \cdots \} \) be natural numbers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

(a) A family \( Q \) of measurable sets \( E \subset \mathbb{R} \) is called locally \( N \)-finite \( (N \in \mathbb{N}) \) if

\[
\sum_{E \in Q} \chi_E(x) \leq N
\]

almost everywhere in \( \mathbb{R} \) where \( \chi_U \) is the characteristic function of the set \( U \).

(b) A family \( Q \) of open bounded sets \( U \subset \mathbb{R} \) is locally 1-finite if and only if the sets \( U \in Q \) are pairwise disjoint.

(c) Let \( U \subset \mathbb{R} \) be a measurable set and

\[
A_U f := \frac{1}{|U|} \int_U |f(t)| \, dt.
\]

(d) For a family \( Q \) of open sets \( U \subset \mathbb{R} \) we define averaging operator by

\[
T_Q : L^1_{loc} \rightarrow L^0,
\]

\[
T_Q f(x) := \sum_{U \in Q} \chi_U(x) A_U f, \quad x \in \mathbb{R},
\]

where \( L^0 \) is the set of measurable functions on \( \mathbb{R} \).
For a measurable set $A \subset \mathbb{R}$, symbol $|A|$ will represent the Lebesgue measure of $A$.

**Theorem 3.8.** ([20, Theorem 4.4.8]) Suppose that $p \in P^{\text{Log}}$, and $f \in L^p(\cdot)$. If $Q$ is $1$-finite family of open bounded subsets of $\mathbb{R}$ having Lebesgue measure $1$, then, the averaging operator $T_Q$ is uniformly bounded in $L^p(\cdot)$, namely,

$$
\|T_Qf\|_{p(\cdot)} \leq \frac{2}{\beta} \|f\|_{p(\cdot)}
$$

holds with $\beta := e^{-4c\log(1/p)}$.

**Theorem 3.9.** ([20]) Suppose that $B \subseteq \mathbb{R}$ be an open set, $p \in P^{\text{Log}}(B)$, $f \in L^p(\cdot)(B)$, $\phi$ is a potential-type approximate identity. Then, for any $t > 0$,

$$
\|f \ast \phi_t\|_{p(\cdot),B} \leq C_8 \|f\|_{p(\cdot),B}
$$

and

$$
\lim_{t \to 0} \|f \ast \phi_t - f\|_{p(\cdot),B} = 0
$$

hold with

$$
C_9 := \frac{12}{\beta}, \quad C_8 := \max \left\{ C_9^\frac{1}{p-1}(B), C_9^\frac{1}{p+1}(B) \right\}.
$$

Let $S_c$ be the collection of the simple functions with compact support.

**Theorem 3.10.** ([20, Corollary 4.4.6]) Let $p \in P^{\text{Log}}(\mathbb{R})$. Then

$$
\sup_{g \in C_c^\infty, \|g\|_{p'(\cdot)} \leq 1} \int_{\mathbb{R}} |f(x)G(x)| \, dx \geq \frac{\beta}{24} \|f\|_{p(\cdot)}
$$

for $f \in L^p(\cdot)$ with $\beta$ of Theorem 3.8.

**Theorem 3.11.** Let $p \in P^{\text{Log}}(\mathbb{R})$. Then $L^p(\cdot)$ satisfy properties $(X1)$-$(X2)$-$(X3)$. Further, if $f, g \in L^p(\cdot)$ and

$$
\|F_{f,G}\|_{C(\mathbb{R})} \leq C_2 \|F_{g,G}\|_{C(\mathbb{R})},
$$

with an absolute constant $C_2 > 0$, then, norm inequality

$$
\|f\|_X \leq 96 (1 + |\text{spt}G|) \beta^{-1} \|G\|_\infty C_2 \|g\|_X
$$

holds.

As a corollary of Theorem 3.11 we have

**Theorem 3.12.** Suppose that $p \in P^{\text{Log}}(\mathbb{R})$, $0 < \delta < \infty$ and $\tau \in \mathbb{R}$. Then,

(i) $F_{S_{\delta,\tau}f,G} = S_{\delta,\tau}F_{f,G}$ and

(ii) the family of operators $\{S_{\delta,\tau}f\}$, defined by (3.7), is uniformly bounded (in $\delta$ and $\tau$) in $L^p(\cdot)$, namely,

$$
\|S_{\delta,\tau}f\|_{p(\cdot)} \leq C_{10} \|f\|_{p(\cdot)},
$$

holds with

$$
C_{11} := 96 (1 + |\text{spt}G|) \beta^{-1} \|G\|_\infty.
$$
$C_{10} := \max \left\{ C_{11}^{P_-}, C_{11}^{P_+} \right\}$.

As a corollary of Theorem 3.11 we get

Corollary 3.13. Let $p \in P^{\log}(R)$, $0 < \delta < \infty$, $f \in L_{p(\cdot)}$. If $\tau = \delta/2$ then,

$$S_{\delta,\delta/2} f(x) = \frac{1}{\delta} \int_0^\delta f(x+t) \, dt = T_\delta f(x),$$

$F_{T_\delta f, G} = T_\delta F_{f, G}$ and

$$\|T_\delta f\|_{p(\cdot)} \leq C_{10} \|f\|_{p(\cdot)}.$$

3.2. Modulus of Smoothness. For $p \in P^{\log}(R)$, $f \in L_{p(\cdot)}$, $0 < \delta < \infty$, $r \in \mathbb{N}$, modulus of smoothness $\Omega_r(f, \delta)_{p(\cdot)} = \|(I - T_\delta)^r f\|_{p(\cdot)}$ has property

$$\|(I - T_\delta)^r f\|_{p(\cdot)} \leq (1 + C_{10})^r \|f\|_{p(\cdot)}.$$

Lemma 3.14. Let $p \in P^{\log}(R)$, $r \in \mathbb{N}$, and $0 < \delta < \infty$. Then

$$\|(I - T_\delta)^r f\|_{p(\cdot)} \leq C_{10}^r 2^{-r} \delta^r \|f\|_{p(\cdot)}, \quad f \in W_{L_{p(\cdot)}}^r$$

hold.

We will use notation $K_r(f, \delta, p(\cdot)) := K_r(f, \delta, L_{p(\cdot)})_{L_{p(\cdot)}}$ for $r \in \mathbb{N}$, $p \in P^{\log}(B)$, $\delta > 0$ and $f \in L_{p(\cdot)}(B)$.

As a corollary of Transference Result we can obtain the following lemma.

Lemma 3.15. Let $0 < h \leq \delta < \infty$, $p \in P^{\log}(R)$ and $f \in L_{p(\cdot)}$. Then

$$F_{(I - T_h)f, G} = (I - T_h) F_{f, G} \quad \text{and}$$

$$\|(I - T_h) f\|_{p(\cdot)} \leq 72 \cdot 96 (1 + |spt G|) \beta^{-1} \|G\|_{\infty} \|(I - T_\delta) f\|_{p(\cdot)} \quad (3.9)$$

holds.

In the following theorem we show that $K$-functional $K_r(f, \delta, p(\cdot))_{p(\cdot)}$ and $\Omega_r(f, \delta)_{p(\cdot)}$ are equivalent.

Theorem 3.16. Let $p(\cdot) \in P^{\log}(R)$. If $L_{p(\cdot)}$, then the $K$-functional $K_r(f, \delta, p(\cdot))_{p(\cdot)}$ and the modulus $\Omega_r(f, \delta)_{p(\cdot)}$, are equivalent, namely,

$$\frac{\beta}{24 \cdot 2^r (1 + |spt G|) \|G\|_{\infty}} \leq \frac{K_r(f, \delta, p(\cdot))_{p(\cdot)}}{\Omega_r(f, \delta)_{p(\cdot)}} \quad \text{and}$$

$$\frac{K_r(f, \delta, p(\cdot))_{p(\cdot)}}{\Omega_r(f, \delta)_{p(\cdot)}} \leq \frac{24 \{2^r(34)^r\} \beta}{(1 + |spt G|) \|G\|_{\infty}}.$$

Theorem 3.17. For $p(\cdot) \in P^{\log}(R)$, $f, g \in L_{p(\cdot)}$ and $\delta > 0$, the modulus of smoothness $\Omega_r(f, \delta)_{p(\cdot)}$, has the following properties:

(1) $\Omega_r(f, \delta)_{p(\cdot)}$ is non-negative; non-decreasing function of $\delta$;
(2) For $f, g \in L_{p(\cdot)}$ and $\delta > 0$,
\[ \Omega_r (f + g, \delta)_{p(\cdot)} \leq \Omega_r (f, \delta)_{p(\cdot)} + \Omega_r (g, \delta)_{p(\cdot)}. \]

(3.10) $\Omega_r (f, \delta)_{p(\cdot)} \leq \Omega_r (f + g, \delta)_{p(\cdot)}$.

(3) For $f \in L_{p(\cdot)}$,
\[ \lim_{\delta \to 0} \Omega_r (f, \delta)_{p(\cdot)} = 0. \]

As a corollary of Theorem 3.16

**Corollary 3.18.** Let $p(\cdot) \in P^\text{Log} (R)$. If $\delta, \lambda \in (0, \infty)$, $f \in L_{p(\cdot)}$, then,
\[ \left( \frac{\Omega_r (f, \lambda \delta)_{p(\cdot)}}{(1 + [\lambda])^r \Omega_r (f, \delta)_{p(\cdot)}} \right)^{1/2} \leq (24)^2 2^{2r} \{ r^r + (34)^r \} \beta^{-2} (1 + |sptG|)^2 \| G \|_\infty^2. \]

holds.

**Theorem 3.19.** Let $p(\cdot) \in P^\text{Log} (R)$, $r \in N$, $\sigma > 0$ and $f \in L_{p(\cdot)}$. Then,
\[ \| A_\sigma (f)_{p(\cdot)} \| \leq C \| (I - T_{1/\sigma})^r f \|_{p(\cdot)} \]
with $C = 1800 \pi 8^r \{ (2r)^r + 2^r (34)^r \} \beta^{-2} (1 + |sptG|)^2 \| G \|_\infty^2$.

Now we present the inverse theorem.

**Theorem 3.20.** Let $p(\cdot) \in P^\text{Log} (R)$, $r \in N$, $\delta \in (0, \infty)$ and $f \in L_{p(\cdot)}$. Then,
\[ \Omega_r (f, \delta)_{p(\cdot)} \leq C_{12} \delta^r \left( A_0 (f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{-1} A_{u/2} (f)_{p(\cdot)} du \right) \]
holds with $C_{12} = \frac{12}{\beta} (1 + 2^{2r-1}) 2^r (1 + |sptG|) \| G \|_\infty (1 + 144 (1 + |sptG|) \beta^{-1} \| G \|_\infty)$.

In this section we obtain Marchaud inequality.

**Theorem 3.21.** Let $r, k \in N$, $p \in P^\text{Log} (R)$, $f \in L_{p(\cdot)}$ and $t \in (0, 1/2)$. Then,
\[ \Omega_r (f, t)_{p(\cdot)} \leq \frac{240 \pi}{\beta} (1 + 2^{2r-1}) 2^{2r+3k} C_3 \| sptG \| \| G \|_\infty t^r \int_{1/2}^{1/\delta} \frac{\Omega_{r+k} (f, u)_{p(\cdot)} du}{u^{r+1}} \]
holds.

**Theorem 3.22.** Let $p \in P^\text{Log} (R)$, $r \in N$ and $f \in L_{p(\cdot)}$. If
\[ \sum_{k=0}^{\infty} u^{k-1} A_{\nu/2} (f)_{p(\cdot)} < \infty \]
holds for some $k \in N$, then $f^{(k)} \in L_{p(\cdot)}$ and
\[ \Omega_r \left( f^{(k)}, \frac{1}{\sigma} \right)_{p(\cdot)} \leq C_{13} \left( \frac{1}{\sigma^r} \sum_{\nu=0}^{[\sigma]} (\nu + 1)^{r+k-1} A_{\nu/2} (f)_{p(\cdot)} + \sum_{\nu=\sigma+1}^{\infty} \nu^{k-1} A_{\nu/2} (f)_{p(\cdot)} \right) \]
with $C_{13} = C_{10} 2^{2k+r+2}$.
3.3. Proofs of the results of Part 3.

Proof of Theorem 3.11. Using [39] Theorem 4.1, the set $C^\infty_c$ is a dense subset of $L_{p(\cdot)}$ and (X1) follows. Corollary 3.3.4 of [20] gives (X2). Theorem 3.10 gives (X3). Then, we get
\[ \|f\|_X \leq 96 (1 + |\text{spt}G|) \beta^{-1} \|G\|_\infty C_2 \|g\|_X. \]

Proof of Lemma 3.14. We note that (see [10]) the following inequality
\[ (3.14) \quad \|(I - T_\delta) f\|_{p(\cdot)} \leq 2^{-1} C_{10} \|f\|_{p(\cdot)} , \quad \delta > 0 \]
holds for $f \in L_{p(\cdot)}$. Then
\[ \Omega_r (f, \delta)_{p(\cdot)} = \|(I - T_\delta)^r f\|_{p(\cdot)} \leq \ldots \leq 2^{-r} C_{10}^r \|f^{(r)}\|_{p(\cdot)}, \quad \delta > 0 \]
for $f \in W^r_{L_p(\cdot)}$.

Proof of Theorem 3.10. For any $g \in W^r_{L_p(\cdot)} (\Omega)$ we have $F_g \in C^r (\Omega)$, $F_{(I - T_\delta)^r f, G} = (I - T_\delta)^r F_{f, G}$ and
\[
\|(I - T_\delta)^r f\|_{p(\cdot)} \leq \frac{24}{\beta} \|F_{(I - T_\delta)^r f, G}\|_{C(\Omega)}
\]
\[
= \frac{24}{\beta} \|(I - T_\delta)^r F_{f, G}\|_{C(\Omega)} \leq \frac{24}{\beta} \cdot 2^r K_r (F_f, \delta, C (\Omega))_{C(\Omega)}
\]
\[
\leq \frac{24}{\beta} 2^r \left\{ \|F_f - F_g\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} F_g \right\|_{C(\Omega)} \right\}
\]
\[
= \frac{24}{\beta} 2^r \left\{ \|F_{f - g, G}\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{C(\Omega)} \right\}
\]
\[
\leq \frac{24}{\beta} 2^r (1 + |\text{spt}G|) \|G\|_\infty \left\{ \|f - g\|_{p(\cdot)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{p(\cdot)} \right\}.
\]
Taking infimum and considering definition of $K$-functional one gets
\[
\|(I - T_\delta)^r f\|_{p(\cdot)} \leq \frac{24 \cdot 2^r}{\beta} (1 + |\text{spt}G|) \|G\|_\infty K_r (f, \delta, p (\cdot))_{p(\cdot)}.
\]
Now we consider the opposite direction of the last inequality. For
\[ g (\cdot) = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} T_\delta^{2i} f (\cdot) \]
we have
\[
K_r (f, \delta, p (\cdot))_{p(\cdot)} \leq \|f - g\|_{p(\cdot)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{p(\cdot)}
\]
\[
\leq \frac{24}{\beta} \left\{ \|F_{f - g}\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{C(\Omega)} \right\}.\]
\[\frac{24}{\beta} \left\{ \left\| F_f - F_g \right\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} F_g \right\|_{C(\Omega)} \right\} \]
\[\leq \frac{24}{\beta} \left\{ \left\| (I - T_\delta^r)^r F_f \right\|_{C(\Omega)} + \delta^r \left\| \frac{d^r}{dx^r} \sum_{l=1}^r (-1)^{l-1} \binom{r}{l} T_\delta^{2rl} F_f \right\|_{C(\Omega)} \right\} \]
\[= \frac{24}{\beta} \left\{ \left\| (I - T_\delta^r)^r F_f \right\|_{C(\Omega)} + \sum_{l=1}^r \left\| \binom{r}{l} \delta^r \left\| \frac{d^r}{dx^r} T_\delta^{2rl} F_f \right\|_{C(\Omega)} \right\} \]
\[\leq \frac{24}{\beta} \left\{ (2r)^r \left\| (I - T_\delta^r)^r F_f \right\|_{C(\Omega)} + 2^r (34)^r \left\| (I - T_\delta^r)^r F_f \right\|_{C(\Omega)} \right\} \]
\[= \frac{24}{\beta} \left\{ (2r)^r + 2^r (34)^r \right\} \left\| F_f \right\|_{C(\Omega)} \]
\[\leq \frac{24}{\beta} \left\{ (2r)^r + 2^r (34)^r \right\} \left( 1 + |sptG| \right) \left\| G \right\|_\infty \left\| (I - T_\delta^r)^r f \right\|_{p(\cdot)} . \]

\[\square\]

**Proof of Theorem 3.17**: Properties (1) and (2), by definition of $\Omega_r (f, \delta)_{p(\cdot)}$ and the triangle inequality of $L_{p(\cdot)}$, are clearly valid. By using [23, Theorem 10.1] and [35, Lemma 2], the relation (3.11) is satisfied.

\[\square\]

**Proof of Corollary 3.18**: We have $\Omega_r (f, \lambda \delta)_{p(\cdot)} (1 + |\lambda|)^{-r} \Omega_r (f, \delta)^{-1}_{p(\cdot)} \leq \frac{(24)^2 2^2 r^r + (34)^r}{(1 + |\lambda|)^r} K_r (f, \lambda \delta, p(\cdot))_{p(\cdot)} \leq (24)^2 2^r \left\{ r^r + (34)^r \right\} \beta^{-2} \left( 1 + |sptG| \right)^2 \left\| G \right\|_\infty^2 . \]

\[\square\]

**Proof of Theorem 3.12**: First we obtain

(3.15) $A_{2^\sigma} (f)_{p(\cdot)} \leq \frac{180 \pi 8^r}{\beta^2} \left\{ (2r)^r + 2^r (34)^r \right\} \left( 1 + |sptG| \right)^2 \left\| G \right\|_\infty^2 \left\| (I - T_1/(2\sigma))^r f \right\|_{p(\cdot)}$

and (3.12) follows from (3.15). Using $V_\sigma F_{f,G} = F_{V_\sigma f,G}$ and $V_\sigma g_\sigma = g_\sigma$ we get

$A_{2^\sigma} (f)_{p(\cdot)} \leq \left\| f - V_\sigma f \right\|_{p(\cdot)} \leq \frac{24}{\beta} \left\| F_f - V_\sigma F_f, G \right\|_{C(R)} = \frac{24}{\beta} \left\| F_f - V_\sigma F_f, G \right\|_{C(R)}$

\[\leq \frac{24}{\beta} \left\| F_f, G - g_\sigma + V_\sigma F_f, G \right\|_{C(R)} = \frac{24}{\beta} \left\| F_f, G - g_\sigma + V_\sigma g_\sigma - V_\sigma F_f, G \right\|_{C(R)} \]

\[\leq \frac{24}{\beta} A_{2^\sigma} (F_f, G)_{C(R)} + \frac{24}{\beta} A_{2^\sigma} (F_f, G)_{C(R)} = \frac{60}{\beta} A_{2^\sigma} (F_f, G)_{C(R)} . \]

For any $g \in W_r^r_{C(R)}$

$A_{2^\sigma} (u)_{C(R)} \leq A_{2^\sigma} (u - g)_{C(R)} + A_{2^\sigma} (g)_{C(R)}$

$\leq \left\| u - g \right\|_{C(R)} + \frac{5 \pi 4^r}{4 \sigma^r} \left\| \frac{d^r}{dx^r} g \right\|_{C(R)}$
Proof of Theorem 3.20.

\[ \begin{align*}
\leq & \frac{5\pi 4^r}{4} K_r \left( u, \frac{1}{\sigma}, C(R) \right)_{C(R)} \leq \frac{5\pi 8^r}{4} K_r \left( u, \frac{1}{2\sigma}, C(R) \right)_{C(R)} \\
\leq & 30\pi 8^r \{(2r)^r + 2^r(34)^r\} \beta^{-1} \left(1 + |spt G|\right) \|G\|_\infty \left\| \left(I - T_{\frac{1}{2\sigma}}\right)^r u \right\|_{C(R)}.
\end{align*} \]

Therefore

\[ A_{2\sigma}(f)_{p(\cdot)} \leq \frac{60}{\beta} A_{\sigma}(F_{f,G})_{C(R)} \]

\[ \leq \frac{1800}{\beta} \pi 8^r \{(2r)^r + 2^r(34)^r\} \beta^{-1} \left(1 + |spt G|\right) \|G\|_\infty \left\| \left(I - T_{\frac{1}{2\sigma}}\right)^r F_f \right\|_{C(R)} \]

\[ = \frac{1800}{\beta} \pi 8^r \{(2r)^r + 2^r(34)^r\} \beta^{-1} \left(1 + |spt G|\right) \|G\|_\infty \left\| F_{f,G} \right\|_{C(R)} \]

\[ \leq 1800\pi 8^r \{(2r)^r + 2^r(34)^r\} \beta^{-1} \left(1 + |spt G|\right)^2 \|G\|_\infty^2 \left\| \left(I - T_{1/(2\sigma)}\right)^r f \right\|_{p(\cdot)} \]

\[ \square \]

Proof of Theorem 3.27.

\[ \Omega_r(f,\delta)_{p(\cdot)} = \left\| (I - T_\delta)^r f \right\|_{p(\cdot)} \leq \frac{24}{\beta} \left\| (I - T_\delta)^r F_f \right\|_{C(R)} = \frac{24}{\beta} \left\| (I - T_\delta)^r F_f \right\|_{C(R)} \]

\[ \leq \frac{12}{\beta} \left(1 + 2^{2r-1}\right) 2^r \delta^r \left(A_0(F_f)_{C(R)} + \int_{1/2}^{1/\delta} u^{r-1} A_{u/2}(F_f)_{C(R)} du \right) \]

\[ \leq \frac{12}{\beta} \left(1 + 2^{2r-1}\right) 2^r \left(1 + |spt G|\right) \|G\|_\infty \left(1 + 144 \left(1 + |spt G|\right) \beta^{-1} \|G\|_\infty \right) \times \]

\[ \times \delta^r \left(A_0(f)_{p(\cdot)} + \int_{1/2}^{1/\delta} u^{r-1} A_{u/2}(f)_{p(\cdot)} du \right) \]

because

\[ A_{2\sigma}(F_{f,G})_{C(R)} \leq \left\| F_{f,G} - V_\sigma F_{f,G} \right\|_{C(R)} = \left\| F_{f,G} - V_\sigma f \right\|_{C(R)} \]

\[ \leq (1 + |spt G|) \|G\|_\infty \left\| f - V_\sigma f \right\|_{p(\cdot)} \]

\[ = (1 + |spt G|) \|G\|_\infty \left\| f - g_\sigma + g_\sigma - V_\sigma f \right\|_{p(\cdot)} \]

\[ \leq (1 + |spt G|) \|G\|_\infty \left(\left\| f - g_\sigma \right\|_{p(\cdot)} + \| V_\sigma g_\sigma - V_\sigma f \right\|_{p(\cdot)} \]

\[ \leq (1 + |spt G|) \|G\|_\infty \left(\left\| f - g_\sigma \right\|_{p(\cdot)} + 144 (1 + |spt G|) \beta^{-1} \|G\|_\infty \| g_\sigma - f \right\|_{p(\cdot)} \]

\[ = (1 + |spt G|) \|G\|_\infty \left(1 + 144 (1 + |spt G|) \beta^{-1} \|G\|_\infty \right) A_{\sigma}(f)_{p(\cdot)} . \]

\[ \square \]
Proof of Theorem 3.21 Let $g_\sigma$ be an exponential type entire function of degree $\leq \sigma$, belonging to $L^{p(\cdot)}$, as best approximation of $f \in L^{p(\cdot)}$. Then
\[
\Omega_r(f, t)_{p(\cdot)} = \|(I - T_t)^r f\|_{p(\cdot)} \leq \frac{24}{\beta} \|F(I - T_t)^r f\|_{C(R)} = \frac{24}{\beta} \|(I - T_t)^r F f\|_{C(R)}
\]
\[
\leq \frac{24}{\beta} 10\pi \left(1 + 2^{2r-1}\right) 2^{2r+3k} C_3 (r + k) t^r \int_t^1 \|F(I - T_t)^{r+k} f\|_{C(R)} du
\]
\[
= \frac{24}{\beta} 10\pi \left(1 + 2^{2r-1}\right) 2^{2r+3k} C_3 (r + k) t^r \int_t^1 \|F(I - T_t)^{r+k} f\|_{C(R)} du
\]
\[
\leq \frac{24}{\beta} 10\pi \left(1 + 2^{2r-1}\right) 2^{2r+3k} C_3 (r + k) \left(1 + |spt G|\right) \|G\|_\infty t^r \int_t^1 \frac{\|F(I - T_t)^{r+k} f\|_{p(\cdot)} du}{u^{r+1}}
\]
\[
= \frac{24}{\beta} 10\pi \left(1 + 2^{2r-1}\right) 2^{2r+3k} C_3 (r + k) \left(1 + |spt G|\right) \|G\|_\infty t^r \int_t^1 \Omega_{r+k} (f, t)_{p(\cdot)} du.
\]

Proof of Theorem 3.22 Proof of (8.13) is similar to that of proof of Theorem 3.21.

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