Semiclassical Effects Induced by
Aharonov-Bohm Interaction Between a
Cosmic String and a Scalar Field

M. E. X. Guimarães

Department of Physics and Astronomy
University of Wales, College of Cardiff
PO Box 913, Cardiff CF2 3YB, UK

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Abstract

In the context of the vacuum polarization effect, we consider the
backreaction of the energy-momentum tensor of a charged scalar field
on the background metric of a cosmic string carrying a magnetic flux
Φ. Working within the semiclassical approach to the Einstein eqs. we find the first-order (in $\hbar$) metric associated to the magnetic flux cosmic string. We show that the contribution to the vacuum polarization effect coming from the Aharonov-Bohm interaction is larger than the one coming from the non-trivial gravitational interaction.

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Introduction

In General Relativity, a static, straight axially symmetric cosmic string is described by the metric \[ ds^2 = -dt^2 + dz^2 + d\rho^2 + B^2 \rho^2 d\varphi^2 \tag{1} \] in cylindrical coordinates $(t, z, \rho, \varphi)$ such that $\rho \geq 0$ and $0 \leq \varphi < 2\pi$. The constant $B$ is related to the linear mass density $\mu$ of the string\footnote{Throughout this paper, we work in the system of units in which $G = c = 1$ and $\hbar \sim 2.612 \times 10^{-33} \text{ cm}^2$.} $B = 1 - 4\mu$. For GUT strings, $\mu$ is of order $\mu \sim 10^{22} \text{ g/cm}$.

$B \sim 2.612 \times 10^{-66} \text{ cm}^2$. 
Metric (1) is locally but not globally flat. The presence of the string leads to an azimuthal deficit angle $\Delta = 8\pi \mu$ and, as a result, this spacetime has a conical singularity [2]. One of the most interesting features of this spacetime is that fields and particles are sensitive to its global structure and physical effects may arise due solely to the global properties of this metric. One of these effects - the vacuum polarization - has been extensively studied in the literature [3, 4] and can be understood as an analog to the Casimir effect [5] in which the conducting planes here form an angle equal to the deficit angle $\Delta$. Then, a scalar field placed in the spacetime outside the string has its vacuum state polarized due to non-trivial periodicity conditions on the azimuthal angle $\varphi$ [3]. In papers of reference [4] a more general situation has been considered in which the cosmic string carrying a magnetic flux $\Phi$ interacts with a charged scalar field placed in the metric (1). In this case, the vacuum polarization arises not only via non-trivial gravitational interaction (i.e, the global conical structure) but also via Aharonov-Bohm (AB) interaction: The scalar field acquires an additional phase shift proportional to the magnetic flux in spite of the fact that it is placed in a region where there is no magnetic field [6]. This situation is a realization in Cosmology of the original AB effect. In papers [4] the non-vanishing vacuum expectation values (VEV)
of the energy-momentum tensor for the scalar field in the fixed background (1) were determined. However, if we want to compute the contribution of both the magnetic flux and the scalar field on the original metric of the cosmic string, this non-vanishing energy-momentum tensor must be taken into account to determine a more realistic spacetime metric associated with the magnetic flux cosmic string. This is the purpose of the present letter.

Throughout this paper we will work in the so-called semiclassical approach to the Einstein eqs. $G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle$ and we will treat this problem using the perturbative approach as in Hiscock’s paper [7]. In this approach, the first-order (in $\hbar$) $\langle T_{\mu\nu} \rangle$ is treated as a matter perturbation of the zeroth-order metric (1) and can be used to compute the first-order metric perturbation associated to it by solving the linearized Einstein’s eqs. about the zeroth-order metric. In the present case, there will be contributions from both the non-trivial gravitational and the AB interactions. We find the gravitational force associated with the backreaction of the $\langle T_{\mu\nu} \rangle$ and the first-order corrections to the deficit angle. Our main result is that the AB is the leading interaction between the magnetic flux cosmic string and the charged scalar field and dominates over the gravitational interaction. That is, the sign of both the gravitational force and the deficit angle is determined by the AB
interaction.

Semiclassical Effects Induced by the Backreaction of the $\langle T_{\mu\nu} \rangle$

The $\langle T_{\mu\nu} \rangle$ for a massless, charged scalar field in the geometry (1) is [4]

$$\langle T_{\mu\nu} \rangle = \frac{\hbar}{\rho^4} \left[ \omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma) \right] \text{diag}(1,1,1,-3) + 4 \left( \xi - \frac{1}{6} \right) \frac{\hbar}{\rho^4} \omega_2(\gamma) \text{diag}(1,1,-1/2,3/2), \quad (2)$$

where the constants $\omega_2(\gamma)$ and $\omega_4(\gamma)$ are given by the following expressions

$$\omega_2(\gamma) = -\frac{1}{8\pi^2} \left[ \frac{1}{3} - \frac{1}{2B^2} [4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] \right]$$

$$\omega_4(\gamma) = -\frac{1}{720\pi^2} \left[ 11 - \frac{15}{B^2} [4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] ight] + \frac{15}{8B^4} [16(\gamma - \frac{1}{2})^4 - 8(\gamma - \frac{1}{2})^2 + \frac{7}{15}]], \quad (3)$$

valid only\footnote{This \textit{mathematical} restriction arises from successive integrations to obtain the $\langle T_{\mu\nu} \rangle$. However, it does not correspond to a \textit{physical} restriction because strings of cosmological interest are of order $\mu \sim 10^{-6}$ (recall $B = 1 - 4\mu$). For more details, see, for instance, Guimarães and Linet [4].} when $B > 1/2$ and $\gamma$ is the fractional part of $\{\frac{\Phi}{\Phi_0}\}$, $\Phi_0$ being the quantum flux $\Phi_0 = 2\pi\hbar/e$ and lies in the domain $0 \leq \gamma < 1$. Let us now
define the dimensionless quantities

\[ A(\gamma) \equiv \omega_4(\gamma) - \frac{1}{3}\omega_2(\gamma) \]

\[ B(\gamma) \equiv 4(\xi - \frac{1}{6})\omega_2(\gamma), \] (4)

in such a way that the components of \( \langle T_{\mu\nu} \rangle \) can now be rewritten as

\[ \langle T_{tt} \rangle = \langle T_{zz} \rangle = \bar{h}\rho^4 \left[ A(\gamma) + B(\gamma) \right], \]

\[ \langle T_{\rho\rho} \rangle = \frac{\bar{h}}{\rho^4} \left[ A(\gamma) - \frac{1}{2}B(\gamma) \right], \]

\[ \langle T_{\phi\phi} \rangle = -3\frac{\bar{h}}{\rho^4} \left[ A(\gamma) - \frac{1}{2}B(\gamma) \right]. \] (5)

The \( \langle T_{\mu\nu} \rangle \) above is linear in \( \bar{h} \) and its dimensionality is \([L]^{-2}\). We can now attempt to solve the semiclassical Einstein’s equations \( G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle \) at linearized level to obtain the first-order metric perturbation associated to the backreaction of the \( \langle T_{\mu\nu} \rangle \) (5). We follow here the same approach as Hiscock in paper [7] and we will later compare our results (in which the magnetic flux is present) with his results (in which there is no magnetic flux).

Following Hiscock’s procedure [7], we set a static, cylindrically symmetric metric in the general form

\[ ds^2 = e^{2\Phi(\rho)}(-dt^2 + dz^2 + d\rho^2) + e^{2\Psi(\rho)}d\varphi^2, \] (6)
where $\Phi$ and $\Psi$ are functions of $\rho$ only; and we expand this metric about the background metric

$$
\Phi = \phi_0 + \phi \quad \text{and} \quad \Psi = \psi_0 + \psi \quad (7)
$$

where, for metric (1), we have $\phi_0 = 0$ and $\psi_0 = \ln(B\rho)$. Therefore, we obtain the linearized Einstein’s equations with source (5)

$$
\psi'' + \phi'' + \frac{2}{\rho} \psi' = 8\pi [A(\gamma) + B(\gamma)] h\rho^{-4}
$$

$$
\frac{2}{\rho} \phi' = 8\pi [A(\gamma) - \frac{1}{2} B(\gamma)] h\rho^{-4}
$$

$$
2\phi'' = -24\pi [A(\gamma) - \frac{1}{2} B(\gamma)] h\rho^{-4}. \quad (8)
$$

The general solutions for eqs. (8) can be easily found

$$
\phi = -2\pi [A(\gamma) - \frac{1}{2} B(\gamma)] h\rho^{-2} + C,
$$

$$
\psi = 10\pi [A(\gamma) + \frac{1}{10} B(\gamma)] h\rho^{-2} + D\rho^{-1} + E. \quad (9)
$$

The exterior metric (corrected at first-order in $h$) of the magnetic flux cosmic string is then obtained

$$
ds^2 = \left[1 - 4\pi \frac{h}{\rho^2} [A(\gamma) - \frac{1}{2} B(\gamma)]\right] (dt^2 + dz^2 + d\rho^2)
$$

$$
+ \left[1 + 20\pi \frac{h}{\rho^2} [A(\gamma) + \frac{1}{10} B(\gamma)]\right] (1 - 4\mu)^2 \rho^2 d\phi^2.
$$
The semiclassical approach is legitimated so long as the first-order perturbations are small compared to one:

$$| \hbar \rho^{-2}[A + \frac{1}{10}B] | \ll 10^{-2},$$

which means that $\rho \gg 10[\hbar(A + \frac{1}{10}B)]^{1/2}$. The rhs is approximately equal to $\approx (\hbar \mu)^{1/2}$. Since the radius of a physical cosmic string is approximately of the same order\[4], this means that $\rho \gg \rho_s$. That is, the semiclassical approach is valid everywhere outside the cosmic string.

Now, if we want to describe the string in a coordinate system such that the radial coordinate measures the proper radius from the string, we make the change of variables

$$r = \rho + 2\pi \frac{\hbar}{\rho}[A(\gamma) - \frac{1}{2}B(\gamma)],$$

such that $g_{rr} = 1$ and therefore the exterior metric becomes

$$ds^2 = \left[1 - 4\pi \frac{\hbar}{r^2}[A(\gamma) - \frac{1}{2}B(\gamma)]\right] (-dt^2 + dz^2) + dr^2 + (1 - 4\mu)^2 r^2 \left[1 + 16\pi \frac{\hbar}{r^2}[A(\gamma) + \frac{1}{4}B(\gamma)]\right] d\varphi^2. \quad (10)$$

\[3\]Of the order of the Compton wavelength of the Higgs bosons involved in the phase transitions leading to the formation of the cosmic string, $\sim 10^{-30}$ cm.

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The geometry of the \((r, \varphi)\)-space is no longer flat, but asymptotically approaches the zeroth-order metric (1). The first consequence is the appearance of a non-vanishing gravitational force on a massive test particle. If we set \(g_{00} = -[1 + 2\Phi]\) where \(\Phi\) is the Newtonian potential, we have

\[
f^r = -4\pi \bar{h} r^3 [A(\gamma) - \frac{1}{2} B(\gamma)].
\]

Using the definitions (4), we obtain a general expression for the gravitational force

\[
f^r = -4\pi \frac{\bar{h}}{r^3} [\omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma) - 2(\xi - \frac{1}{6}) \omega_2(\gamma)].
\]

(11)

For the particular values \(\xi = 0\) (minimal coupling) and \(\xi = 1/6\) (conformal parameter), the general expression (11) of the gravitational force takes the form

\[
f^r = -4\pi \frac{\bar{h}}{r^3} \omega_4(\gamma)
\]

(12)

and

\[
f^r = -4\pi \frac{\bar{h}}{r^3} [\omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma)],
\]

(13)

respectively.

We can also obtain the first-order corrections to the deficit angle. Let us define it as \(\Delta \varphi = 2\pi - C/r\), where \(C\) is the circumference of a circle centered
around the string at a fixed proper radius $r$ from it. Therefore, for metric (10), the deficit angle has the following expression (after using definitions (4))

$$\Delta \varphi = 8 \pi \mu - (1 - 4 \mu)16 \pi^2 \frac{\hbar}{r^2} \left[ \omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma) + (\xi - 1/6) \omega_2(\gamma) \right].$$

For the particular values $\xi = 0$ and $\xi = 1/6$ of the coupling parameter, this general expression reduces to

$$\Delta \varphi = 8 \pi \mu - (1 - 4 \mu)16 \pi^2 \frac{\hbar}{r^2} \left[ \omega_4(\gamma) - \frac{1}{2} \omega_2(\gamma) \right], \quad (14)$$

and

$$\Delta \varphi = 8 \pi \mu - (1 - 4 \mu)16 \pi^2 \frac{\hbar}{r^2} \left[ \omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma) \right], \quad (15)$$

respectively.

**Comparison with the Case $\gamma = 0$ and Concluding Remarks**

Now we are able to make some conclusions about the results obtained in the previous section. Let us first consider the gravitational force (11). In the case where there is no magnetic flux ($\gamma = 0$) the gravitational force is always *attractive* for both minimal ($\xi = 0$) and conformal ($\xi = 1/6$) couplings.
However, this behaviour changes when the magnetic flux is present (and $\gamma$ lies in the domain $0 < \gamma < 1$). Indeed, it is easy to see from (12) and (13) that the gravitational force is repulsive for both minimal and conformal couplings.

Considering now the deficit angle, again the behaviour changes if the magnetic flux is present or not. When it is absent, the deficit angle increases as $r \to 0$ for minimally coupled scalar field and decreases as $r \to 0$ for conformally coupled field. When the magnetic flux is present, and $\gamma$ lies in the domain $0 < \gamma < 1$, the result is different: The deficit angle decreases as $r \to 0$ for minimally coupled scalar field and increases as $r \to 0$ for conformally coupled field.

It seems, thus, clear from these analysis that the sign of both the gravitational force and the deficit angle is determined by the AB interaction between the magnetic flux cosmic string and the charged scalar field. We can, then, conclude that the contribution coming from the AB interaction dominates over the one coming from the non-trivial gravitational interaction. It is interesting to remark that this result agrees with previous statement by Alford and Wilczek (and further by de Souza Gerbert) although in a different context: In these papers the authors show that the cross section coming
from AB scattering of fermions in presence of a cosmic string is much larger than the one coming from the gravitational scattering.

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References

[1] J. R. Gott III, Ap. J. 288, 422 (1985); W. A. Hiscock, Phys. Rev. D 31, 3288 (1985); B. Linet, Gen. Rel. Grav. 17, 1109 (1985).

[2] A. Vilenkin, Phys. Rev. D 23, 852 (1981).

[3] T. M. Helliwell and D. A. Konkowski, Phys. Rev. D 34, 1918 (1986); B. Linet, Phys. Rev. D 35, 536 (1987); A. G. Smith, The Formation and Evolution of Cosmic Strings, eds. G. W. Gibbons, S. W. Hawking and T. Vaschaspati (Cambridge: Cambridge Univ. Press) p 680 (1990).

[4] J. S. Dowker, Phys. Review D 36, 3095 (1987); Phys. Review D 36, 3742 (1987); V. P. Frolov and E. M. Serebrianyi, Phys.Review D 35,
3779 (1987); M. E. X. Guimarães and B. Linet, Comm. Math. Phys. 165, 297 (1994).

[5] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge: Cambridge Univ. Press) (1982).

[6] E. M. Serebrianyi, Theor. Math. Phys. 52, 51 (1982); M. Bordag, Ann. Phys. 206, 257 (1991).

[7] W. A. Hiscock, Phys. Lett. B188, 317 (1987).

[8] M. G. Alford and F. Wilczek, Phys. Rev. Lett. 62, 1071 (1989); Ph. de Souza Gerbert, Phys. Rev. D 40, 1346 (1989).