The $C$-Numerical Range for Schatten-Class Operators

Gunther Dirr$^a$ and Frederik vom Ende$^b$

$^a$Department of Mathematics, University of Würzburg, 97074 Würzburg, Germany
$^b$Department of Chemistry, TU Munich, 85747 Garching, Germany

ABSTRACT

We generalize the $C$-numerical range $W_C(T)$ from trace-class to Schatten-class operators, i.e. to $C \in \mathcal{B}^p(\mathcal{H})$ and $T \in \mathcal{B}^q(\mathcal{H})$ with $1/p + 1/q = 1$, and show that its closure is always star-shaped with respect to the origin. For $q \in (1, \infty]$, this is equivalent to saying that the closure of the image of the unitary orbit of $T \in \mathcal{B}^q(\mathcal{H})$ under any continous linear functional $L \in (\mathcal{B}^q(\mathcal{H}))'$ is star-shaped with respect to the origin. For $q = 1$, one has star-shapedness with respect to $\overline{\text{tr}(T)W_e(L)}$, where $W_e(L)$ denotes the essential range of $L$.

Moreover, the closure of $W_C(T)$ is convex if $C$ or $T$ is normal with collinear eigenvalues. If $C$ and $T$ are both normal, then the $C$-spectrum of $T$ is a subset of the $C$-numerical range, which itself is a subset of the closure of the convex hull of the $C$-spectrum. This closure coincides with the closure of the $C$-numerical range if, in addition, the eigenvalues of $C$ or $T$ are collinear.

KEYWORDS

$C$-numerical range; $C$-spectrum; Schatten-class operators

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1. Introduction

This is a follow-up paper of [1]. There, we studied the $C$-numerical range $W_C(T)$ of $T$ generalized to trace-class operators $C$ and bounded operators $T$ acting on some infinite-dimensional separable complex Hilbert space $\mathcal{H}$, i.e.

$$W_C(T) = \{ \text{tr}(CU^\dagger T) \mid U \in \mathcal{B}(\mathcal{H}) \text{ unitary} \},$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. In this setting, however, symmetry in $C$ and $T$ compared to the matrix case is lost in the sense that by construction the mapping $(C,T) \mapsto W_C(T)$ is no longer defined on a symmetric domain. Probably, the most natural symmetric domain where $\text{tr}(CT)$ is still well-defined is the set $\mathcal{B}^2(\mathcal{H})$ of all Hilbert-Schmidt operators. Thus a natural question to ask is whether the known results about convexity, star-shapedness and the $C$-spectrum carry over to Hilbert-Schmidt operators.

While analyzing this problem, it rapidly becomes evident that one can easily go one step further by considering operators $C$ and $T$ which belong to conjugate Schatten-classes, as the set $\mathcal{B}^p(\mathcal{H})$ of all $p$-Schatten-class operators constitutes a two-sided ideal ideal.
in the $C^*$-algebra $\mathcal{B}(\mathcal{H})$ for all $p \in [1, \infty]$. Starting from the symmetry requirement, our line-of-thought will arrive in a quite natural way at the outlined Schatten-class setting of the $C$-numerical range of $T$.

The paper is organized as follows: After a preliminary section collecting notation and basic results on Schatten-class operators, we present our main results in Section 3. We show that the closure of $W_C(T)$ for conjugate Schatten-class operators $C$ and $T$ is always star-shaped with respect to the origin. We reformulate this result in terms of the image of the unitary orbit of $T \in \mathcal{B}^q(\mathcal{H})$ under any continuous linear functional $L \in (\mathcal{B}^q(\mathcal{H}))'$. Moreover, we prove that the closure of $W_C(T)$ is convex if either $C$ or $T$ is normal with collinear eigenvalues. Finally, we introduce the $C$-spectrum of $T$ and derive some inclusion and convexity results, which are well known for matrices, under the assumption that both Schatten-class operators $C$ and $T$ are normal.

2. Notation and Preliminaries

Unless stated otherwise, here and henceforth $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary infinite-dimensional complex Hilbert spaces while $\mathcal{H}$ and $\mathcal{G}$ are reserved for infinite-dimensional separable complex Hilbert spaces (for short i.s.c. Hilbert spaces). Moreover, $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{B}^p(\mathcal{X}, \mathcal{Y})$ denote the set of all bounded, compact and $p$-th Schatten-class operators between $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Scalar products are conjugate linear in the first argument and linear in the second one. For an arbitrary subset $S \subset \mathbb{C}$, the notations $\overline{S}$ and $\text{conv}(S)$ stand for its closure and convex hull, respectively. Finally, given $p, q \in [1, \infty]$, we say $p$ and $q$ are conjugate if $\frac{1}{p} + \frac{1}{q} = 1$.

2.1. Infinite-dimensional Hilbert Spaces and the Trace Class

For a comprehensive introduction to infinite-dimensional Hilbert spaces and Schatten-class operators, we refer to, e.g., [2] and [3]. Here, we recall only some basic results which will be use frequently throughout this paper.

Let $(e_i)_{i \in I}$ be any orthonormal basis of $\mathcal{X}$ and let $x \in \mathcal{X}$. Then one has Parseval’s identity

$$\sum_{i \in I} |\langle e_i, x \rangle|^2 = \|x\|^2$$

which reduces to Bessel’s inequality

$$\sum_{j \in J} |\langle f_j, x \rangle|^2 \leq \|x\|^2$$

if $(f_j)_{j \in J}$ is any orthonormal system in $\mathcal{X}$ instead of an orthonormal basis.

Lemma 2.1 (Schmidt decomposition). For each $C \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$, there exists a decreasing null sequence $(s_n(C))_{n \in \mathbb{N}}$ in $[0, \infty)$ as well as orthonormal systems $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$
and \((g_n)_{n \in \mathbb{N}}\) in \(\mathcal{Y}\) such that
\[
C = \sum_{n=1}^{\infty} s_n(C) \langle f_n, \cdot \rangle g_n,
\]
where the series converges in the operator norm.

As the singular numbers \((s_n(C))_{n \in \mathbb{N}}\) in Lemma 2.1 are uniquely determined by \(C\), the \(p\)-th Schatten-class \(B^p(\mathcal{X}, \mathcal{Y})\) is defined by
\[
B^p(\mathcal{X}, \mathcal{Y}) := \left\{ C \in \mathcal{K}(\mathcal{X}, \mathcal{Y}) \mid \sum_{n=1}^{\infty} s_n(C)^p < \infty \right\}
\]
for \(p \in [1, \infty)\). The Schatten-\(p\)-norm
\[
\nu_p(C) := \left( \sum_{n=1}^{\infty} s_n(C)^p \right)^{1/p}
\]
turns \(B^p(\mathcal{X}, \mathcal{Y})\) into a Banach space. Moreover, for \(p = \infty\), we identify \(B^\infty(\mathcal{X}, \mathcal{Y})\) with the set of all compact operators \(\mathcal{K}(\mathcal{X}, \mathcal{Y})\) equipped with the norm
\[
\nu_\infty(C) := \sup_{n \in \mathbb{N}} s_n(C) = s_1(C).
\]
Note that \(\nu_\infty(C)\) coincides with the ordinary operator norm \(\|C\|\). Hence \(B^\infty(\mathcal{X}, \mathcal{Y})\) constitutes a closed subspace of \(B(\mathcal{X}, \mathcal{Y})\) and thus a Banach space, too. The following results can be found in [4, Corollary XI.9.4 & Lemma XI.9.9].

**Lemma 2.2.** (a) Let \(p \in [1, \infty]\). Then, for all \(S, T \in B(\mathcal{X})\) and \(C \in B^p(\mathcal{X})\), one has
\[
\nu_p(SCT) \leq \|S\| \nu_p(C) \|T\|.
\]
(b) Let \(1 \leq p \leq q \leq \infty\). Then \(B^p(\mathcal{X}, \mathcal{Y}) \subseteq B^q(\mathcal{X}, \mathcal{Y})\) and \(\nu_p(C) \geq \nu_q(C)\) for all \(C \in B^p(\mathcal{X}, \mathcal{Y})\).

Note that due to (a), all Schatten-classes \(B^p(\mathcal{X})\) constitute – just like the compact operators – a two-sided ideal in the \(C^*\)-algebra of all bounded operators \(B(\mathcal{X})\).

**Lemma 2.3.** Let \(T \in \mathcal{K}(\mathcal{X})\) and \((e_k)_{k \in \mathbb{N}}\) be any orthonormal system in \(\mathcal{X}\). Then
(a) \(\sum_{k=1}^{n} |\langle e_k, Te_k \rangle| \leq \sum_{k=1}^{n} s_k(T)\) for all \(n \in \mathbb{N}\) and
(b) \(\lim_{k \to \infty} \langle e_k, Te_k \rangle = 0\).

**Proof.** (a) Consider a Schmidt decomposition \(\sum_{m=1}^{\infty} s_m(T) \langle f_m, \cdot \rangle g_m\) of \(T\). Then
\[
\sum_{k=1}^{n} |\langle e_k, Te_k \rangle| \leq \sum_{m=1}^{\infty} s_m(T) \left( \sum_{k=1}^{n} |\langle e_k, f_m \rangle| |\langle g_m, e_k \rangle| \right) =: \lambda_m.
\]

3
Note that by Cauchy-Schwarz and Bessel’s inequality one has
\[ \lambda_m \leq \left( \sum_{k=1}^{n} |\langle e_k, f_m \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |\langle g_m, e_k \rangle|^2 \right)^{1/2} \leq 1 \]
for all \( m \in \mathbb{N} \). On the other hand, Cauchy-Schwarz and Bessel’s inequality also imply
\[ \sum_{m=1}^{\infty} \lambda_m \leq \sum_{k=1}^{n} \left( \sum_{m=1}^{\infty} |\langle e_k, f_m \rangle|^2 \right)^{1/2} \left( \sum_{m=1}^{\infty} |\langle g_m, e_k \rangle|^2 \right)^{1/2} \leq \sum_{k=1}^{n} \|e_k\|^2 = n. \]
Hence an upper bound of \( \sum_{m=1}^{\infty} s_m(T) \lambda_m \) is given by choosing \( \lambda_1 = \ldots = \lambda_n = 1 \) and \( \lambda_j = 0 \) whenever \( j > n \), since \( s_1(T) \geq s_2(T) \geq \ldots \) by construction. This shows the desired inequality. A proof of (b) can be found, e.g., in [3, Lemma 16.17].

Now for any \( C \in \mathcal{B}^1(\mathcal{X}) \), the trace of \( C \) is defined via
\[ \text{tr}(C) := \sum_{i \in I} \langle f_i, Cf_i \rangle, \tag{1} \]
where \( (f_i)_{i \in I} \) can be any orthonormal basis of \( \mathcal{X} \). The trace is well-defined as one can show that the right-hand side of (1) does not depend on the choice of \( (f_i)_{i \in I} \). Important properties are the following, cf. [4, Lemma XI.9.14].

**Lemma 2.4.** Let \( C \in \mathcal{B}^p(\mathcal{X}) \) and \( T \in \mathcal{B}^q(\mathcal{X}) \) with \( p, q \in [1, \infty] \) conjugate. Then one has \( CT, TC \in \mathcal{B}^1(\mathcal{X}) \) with
\[ \text{tr}(CT) = \text{tr}(TC) \]
\[ |\text{tr}(CT)| \leq \nu_p(C) \nu_q(T). \tag{2} \]

Note that the space of so called Hilbert-Schmidt operators \( \mathcal{B}^2(\mathcal{X}) \) turns into a Hilbert space under the scalar product \( \langle C, T \rangle := \text{tr}(C^\dagger T) \) [3, Prop. 16.22].

### 2.2. Set Convergence

In order to transfer results about convexity and star-shapedness of the \( C \)-numerical range of matrices to Schatten-class operators, we need some basic facts about set convergence. We will use the Hausdorff metric on compact subsets (of \( \mathbb{C} \)) and the associated notion of convergence, see, e.g., [3].

The distance between \( z \in \mathbb{C} \) and any non-empty compact subset \( A \subseteq \mathbb{C} \) is defined by
\[ d(z, A) := \min_{w \in A} d(z, w) = \min_{w \in A} |z - w|. \tag{3} \]
Based on [3], the **Hausdorff metric** \( \Delta \) on the set of all non-empty compact subsets of \( \mathbb{C} \) is given by
\[ \Delta(A, B) := \max \left\{ \max_{z \in A} d(z, B), \max_{z \in B} d(z, A) \right\}. \]
The following result is proven in [1, Lemma 2.5].

**Lemma 2.5.** Let \((A_n)_{n \in \mathbb{N}}\) and \((B_n)_{n \in \mathbb{N}}\) be bounded sequences of non-empty compact subsets of \(\mathbb{C}\) such that \(\lim_{n \to \infty} A_n = A\), \(\lim_{n \to \infty} B_n = B\) and let \((z_n)_{n \in \mathbb{N}}\) be any sequence of complex numbers with \(\lim_{n \to \infty} z_n = z\). Then the following statements hold.

(a) If \(A_n \subseteq B_n\) for all \(n \in \mathbb{N}\), then \(A \subseteq B\).

(b) The sequence \((\text{conv}(A_n))_{n \in \mathbb{N}}\) of compact subsets converges to \(\text{conv}(A)\), i.e.

\[
\lim_{n \to \infty} \text{conv}(A_n) = \text{conv}(A).
\]

(c) If \(A_n\) is convex for all \(n \in \mathbb{N}\), then \(A\) is convex.

(d) If \(A_n\) is star-shaped with respect to \(z_n\) for all \(n \in \mathbb{N}\), then \(A\) is star-shaped with respect to \(z\).

### 3. Results

Let \(\mathcal{H}\) denote an arbitrary infinite-dimensional separable complex (i.s.c.) Hilbert space. Our goal will be to carry over the characterizations of the geometry of the \(C\)-numerical range \(W_C(T)\), like star-shapedness or convexity, from the trace class \([1]\) to conjugate Schatten-class operators on \(\mathcal{H}\).

**Definition 3.1.** Let \(p, q \in [1, \infty]\) be conjugate. Then for \(C \in B^p(\mathcal{H})\) and \(T \in B^q(\mathcal{H})\), we define the \(C\)-numerical range of \(T\) to be

\[
W_C(T) := \{\text{tr}(CU^TU) \mid U \in B(\mathcal{H}) \text{ unitary}\}.
\]

Note that the trace \(\text{tr}(CU^TU)\) is well-defined due to Lemma 2.2 and 2.4.

Moreover, throughout this paper we need some mechanism to associate bounded operators on \(\mathcal{H}\) with matrices. In doing so, let \((e_n)_{n \in \mathbb{N}}\) be some orthonormal basis of \(\mathcal{H}\) and let \((\hat{e}_i)_{i=1}^n\) be the standard basis of \(\mathbb{C}^n\). For any \(n \in \mathbb{N}\) we define

\[
\Gamma_n : \mathbb{C}^n \to \mathcal{H}, \quad \hat{e}_i \mapsto \Gamma_n(\hat{e}_i) := e_i
\]

and its linear extension to all of \(\mathbb{C}^n\). Next, let

\[
[\cdot]_n : B(\mathcal{H}) \to \mathbb{C}^{n \times n}, \quad A \mapsto [A]_n := \Gamma_n^* A \Gamma_n
\]

be the operator which “cuts out” the upper \(n \times n\) block of (the matrix representation of) \(A\) with respect to \((e_n)_{n \in \mathbb{N}}\).

#### 3.1. Star-Shapedness

Our strategy is to transfer well-known properties of the finite-dimensional \([C]_n\)-numerical range of \([T]_n\) to \(W_C(T)\) via the convergence results of Lemma 2.5.
Lemma 3.2. Let \( p \in [1, \infty] \), \( C \in \mathcal{B}^p(\mathcal{H}) \) and \((S_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{B}(\mathcal{H}) \) which converges strongly to \( S \in \mathcal{B}(\mathcal{H}) \). Then one has \( S_nC \to SC \), \( CS_n^\dagger \to CS^\dagger \), and \( S_nCS_n^\dagger \to SCS^\dagger \) for \( n \to \infty \) with respect to the norm \( \nu_p \).

Proof. The cases \( p = 1 \) and \( p = \infty \) are proven in [1, Lemma 3.2]. As the proof for \( p \in (1, \infty) \) is essentially the same, we sketch only the major differences. First, choose \( K \in \mathbb{N} \) such that

\[
\sum_{k=K+1}^{\infty} |s_k(C)|^p < \frac{\varepsilon^p}{(3K)^p},
\]

where \( \kappa > 0 \) satisfies \( \|S\| \leq \kappa \) and \( \|S_n\| \leq \kappa \) for all \( n \in \mathbb{N} \). The existence of the constant \( \kappa > 0 \) is guaranteed by the uniform boundedness principle. Then decompose \( C = \sum_{k=1}^{\infty} s_k(C)\langle e_k, \cdot \rangle f_k \) into \( C = C_1 + C_2 \) with \( C_1 := \sum_{k=1}^{K} s_k(C)\langle e_k, \cdot \rangle f_k \) finite-rank. By Lemma 2.2 one has

\[
\nu_p(SC - S_nC) \leq \nu_p(SC_1 - S_nC_1) + \|S\|\nu_p(C_2) + \|S_n\|\nu_p(C_2) < \nu_p(SC_1 - S_nC_1) + \frac{2\varepsilon}{3}.
\]

Thus, what remains is to choose \( N \in \mathbb{N} \) such that \( \nu_p(SC_1 - S_nC_1) < \varepsilon/3 \) for all \( n \geq N \). Starting from

\[
\nu_p(SC_1 - S_nC_1) \leq \sum_{k=1}^{K} s_k(C)\nu_p(\langle e_k, \cdot \rangle(Sf_k - S_nf_k)) = \sum_{k=1}^{K} s_k(C)\|Sf_k - S_nf_k\|,
\]

the strong convergence of \((S_n)_{n \in \mathbb{N}}\) yields \( N \in \mathbb{N} \) such that

\[
\|Sf_k - S_nf_k\| < \frac{\varepsilon}{3\sum_{k=1}^{K} s_k(C)}
\]

for \( k = 1, \ldots, K \) and all \( n \geq N \). This shows \( \nu_p(SC - S_nC) \to 0 \) as \( n \to \infty \). All other assertions are an immediate consequence of \( \nu_p(A) = \nu_p(A^\dagger) \) for all \( A \in \mathcal{B}^p(\mathcal{H}) \) and

\[
\nu_p(SCS^\dagger - S_nCS_n^\dagger) \leq \|S\|\nu_p(CS^\dagger - CS_n^\dagger) + \nu_p(SC - S_nC)\|S_n\|
\]

\[
\leq \kappa(\nu_p(CS^\dagger - CS_n^\dagger) + \nu_p(SC - S_nC)) \quad \square
\]

Lemma 3.3. Let \( C \in \mathcal{B}^p(\mathcal{H}) \) and \( T \in \mathcal{B}^q(\mathcal{H}) \) with \( p, q \in [1, \infty] \) conjugate and let \((S_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{B}(\mathcal{H}) \) which converges strongly to \( S \in \mathcal{B}(\mathcal{H}) \). Then

\[
\lim_{n \to \infty} \text{tr}(CS_n^\dagger TS_n) = \text{tr}(CS^\dagger TS).
\]

Furthermore, the sequence of linear functionals \((\text{tr}(CS_n^\dagger(\cdot)S_n))_{n \in \mathbb{N}}\) converges uniformly to \( \text{tr}(CS^\dagger(\cdot)S) \) on \( \nu_q \)-bounded subsets of \( \mathcal{B}^q(\mathcal{H}) \), while the sequence \((\text{tr}((\cdot)S_n^\dagger TS_n))_{n \in \mathbb{N}}\) converges uniformly to \( \text{tr}((\cdot)S^\dagger TS) \) on \( \nu_p \)-bounded subsets of \( \mathcal{B}^p(\mathcal{H}) \).
Proof. The statement is a simple consequence of \([\square]\) and Lemma \([\square]\) as

\[
|\text{tr}(CS^\dagger TS) - \text{tr}(CS_n^\dagger T S_n)| = |\text{tr}((SCS^\dagger - S_nCS_n^\dagger)T)| \\
\leq \nu_p(SCS^\dagger - S_nCS_n^\dagger)\nu_q(T) \to 0 \quad \text{as } n \to \infty.
\]

Theorem 3.4. Let \(C \in \mathcal{B}^p(\mathcal{H}), \ T \in \mathcal{B}^q(\mathcal{H})\) with \(p, q \in [1, \infty]\) conjugate be given. Furthermore, let \((e_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}\) be arbitrary orthonormal bases of \(\mathcal{H}\). Then

\[
\lim_{n \to \infty} W[C]_n ((T)^g_{2n}) = W_C(T)
\]

where \([\cdot]_k^e\) and \([\cdot]_k^g\) are the maps given by \([\square]\) with respect to \((e_n)_{n \in \mathbb{N}}\) and \((g_n)_{n \in \mathbb{N}}\), respectively.

Proof. The proof for \(p = 1\) and \(q = \infty\) (or vice versa) given in \([\square]\) Thm. 3.1] can be adjusted to the case \(p, q \in (1, \infty)\) by minimal modifications. \(\square\)

Before proceeding with the star-shapedness of \(W_C(T)\), we need the following auxiliary result to characterize the star-center later on.

Lemma 3.5. Let \(C \in \mathcal{B}^p(\mathcal{H})\) with \(p \in (1, \infty]\) and let \(q \in [1, \infty)\) such that \(p, q\) are conjugate. Furthermore, let \((e_n)_{n \in \mathbb{N}}\) be any orthonormal system in \(\mathcal{H}\). Then

\[
\lim_{n \to \infty} \frac{1}{n^{1/q}} \sum_{k=1}^{n} \langle e_k, Ce_k \rangle = 0.
\]

Proof. First, let \(p = \infty\), so \(q = 1\). As \(C\) is compact, by Lemma \([\square]\) (b), one has \(\lim_{k \to \infty} \langle e_k, Ce_k \rangle = 0\) and thus the sequence of arithmetic means converges to zero as well. Next, let \(p \in (1, \infty)\) and \(\varepsilon > 0\). Moreover, we assume w.l.o.g. \(C \neq 0\) so \(s_1(C) = \|C\| \neq 0\). As \(C \in \mathcal{B}^p(\mathcal{H})\), one can choose \(N_1 \in \mathbb{N}\) such that

\[
\sum_{k=N_1+1}^{\infty} s_k(C)^p < \frac{\varepsilon^p}{2^p}
\]

and moreover \(N_2 \in \mathbb{N}\) such that

\[
\frac{1}{n^{1/q}} < \frac{\varepsilon}{2 \sum_{k=1}^{N_1} s_k(C)}
\]

for all \(n \geq N_2\). Then, for any \(n \geq N := \max\{N_1 + 1, N_2\}\), by Lemma \([\square]\) and Hölder’s
inequality we obtain

\[
\left| \frac{1}{n^{1/q}} \sum_{k=1}^{n} \langle e_k, C e_k \rangle \right| \leq \frac{1}{n^{1/q}} \sum_{k=1}^{N_1} s_k(C) + \frac{1}{n^{1/q}} \sum_{k=N_1+1}^{n} s_k(C) \\
\leq \frac{1}{n^{1/q}} \sum_{k=1}^{N_1} s_k(C) + \left( \sum_{k=N_1+1}^{n} \frac{1}{n} \right)^{1/q} \left( \sum_{k=N_1+1}^{n} \frac{1}{n} \right)^{1/q} \\
< \frac{\varepsilon}{2} + \left( \sum_{k=N_1+1}^{\infty} \frac{1}{n} \right)^{1/p} \left( \frac{n - N_1}{n} \right)^{1/q} \leq \varepsilon .
\]

This concludes the proof. \(\square\)

Now, our main result of this section reads as follows.

**Theorem 3.6.** Let \(C \in \mathcal{B}^p(\mathcal{H})\) and \(T \in \mathcal{B}^q(\mathcal{H})\) with \(p, q \in [1, \infty]\) conjugate. Then \(W_C(T)\) is star-shaped with respect to the origin.

**Proof.** Let \((e_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}\) be arbitrary orthonormal bases of \(\mathcal{H}\). For \(n \in \mathbb{N}\), it is readily verified that

\[
\frac{\text{tr}([C]_n^g) \text{tr}([T]_n^g)}{2n} = \frac{\text{tr}([C]_n^e) \text{tr}([T]_n^g)}{2n^{1/q} (2n)^{1/p}} \\
= \left( \frac{1}{(2n)^{1/q}} \sum_{j=1}^{2n} \langle e_j, C e_j \rangle \right) \left( \frac{1}{(2n)^{1/p}} \sum_{j=1}^{2n} \langle g_j, T g_j \rangle \right).
\]

Both factors converge and, by Lemma 3.5, at least one of them goes to 0 as \(n \to \infty\). Moreover, \(W_{[C]_n^g}([T]_n^g)\) is star-shaped with respect to \((\text{tr}([C]_n^g) \text{tr}([T]_n^g))/(2n)\) for all \(n \in \mathbb{N}\), cf. [6, Thm. 4]. Thus Lemma 2.5 (d) and Theorem 3.4 imply that \(W_C(T)\) is star-shaped with respect to 0 \(\in \mathbb{C}\), i.e. with respect to the origin. \(\square\)

**Remark 1.** The limit case \(p = 1\) and \(q = \infty\) returns the known star-shapedness result in the case of trace-class [1, Thm. 3.3] because the essential numerical range satisfies \(W_e(T) = \{0\}\) if (and only if) \(T\) is compact [7, Thm. 34.2].

In analogy to the essential numerical range of a bounded linear operator as characterized in, e.g., [7, Thm. 34.9], we introduce the **essential range** of a bounded linear functional \(L \in (\mathcal{B}^q(\mathcal{H}))'\) via

\[
W_e(L) := \left\{ \lim_{n \to \infty} L((f_n, \cdot) f_n) \mid (f_n)_{n \in \mathbb{N}} \text{ ONS of } \mathcal{H} \right\} \subset \mathbb{C}.
\]

By the canonical isomorphism \(A \mapsto \text{tr}(A \cdot)\) one has \((\mathcal{B}^1(\mathcal{H}))' \simeq \mathcal{B}(\mathcal{H})\) and \((\mathcal{B}^q(\mathcal{H}))' \simeq \mathcal{B}^p(\mathcal{H})\) for \(q \in (1, \infty]\) with \(p, q\) conjugate, refer to [3, Thm. V.15] and [4, Prop. 16.26]. Thus for \(q \in [1, \infty]\), to each \(L \in (\mathcal{B}^q(\mathcal{H}))'\) we can associate a unique bounded linear operator \(C \in \mathcal{B}(\mathcal{H})\) if \(q = 1\) and \(C \in \mathcal{B}^p(\mathcal{H})\) if \(q \in (1, \infty]\), such that

\[
W_e(L) = W_e(C).
\]
This shows that \( W_e(L) \) is non-empty, compact and convex and, in particular, \( W_e(L) = \{0\} \) for \( q \in (1, \infty) \), cf. \cite{3}, Thm. 34.2. With the above terminology one has the following straightforward conclusion.

**Corollary 3.7.** (a) Let \( q \in (1, \infty] \) and \( T \in \mathcal{B}^q(\mathcal{H}) \) be given. The closure of the image of the unitary orbit of \( T \) under any bounded linear functional \( L \in (\mathcal{B}^q(\mathcal{H}))' \), i.e. the closure of

\[
L(\mathcal{O}_U(T)) := \{ L(U^*TU) \mid U \in \mathcal{B}(\mathcal{H}) \text{ unitary} \},
\]

is star-shaped with respect to the origin.

(b) Let \( q = 1 \) and \( T \in \mathcal{B}^1(\mathcal{H}) \) be given. The closure of the image of the unitary orbit of \( T \) under any bounded linear functional \( L \in (\mathcal{B}^1(\mathcal{H}))' \) is star-shaped with respect to \( \text{tr}(T)W_e(L) \), i.e. all \( z \in \text{tr}(T)W_e(L) \) are possible star centers.

**Proof.** (a) Let \( q \in (1, \infty] \) with conjugate \( p \in [1, \infty) \). Then, as seen above, \( \mathcal{B}^q(\mathcal{H}) \simeq (\mathcal{B}^q(\mathcal{H}))' \) by means of the canonical map \( A \mapsto \text{tr}(A \cdot) \). Now, \( L(\mathcal{O}_U(T)) = W_C(T) \) for some unique \( C \in \mathcal{B}^p(\mathcal{H}) \) and thus, by Theorem 3.6, the closure of this set is star-shaped with respect to 0 ∈ \( \mathbb{C} \).

(b) For \( q = 1 \), again as seen above one has \( (\mathcal{B}^1(\mathcal{H}))' \simeq \mathcal{B}(\mathcal{H}) \) and thus \( L = \text{tr}(B \cdot) \) for some \( B \in \mathcal{B}(\mathcal{H}) \). Hence, \( L(\mathcal{O}_U(T)) \) equals \( W_T(B) \), cf. \cite{1}, Defi. 3.1, and therefore is star-shaped with respect to \( \text{tr}(T)W_e(B) = \text{tr}(T)W_e(L) \), refer to \cite{5} and \cite{1}, Thm. 3.3. \( \square \)

### 3.2. Convexity and the C-Spectrum

Convexity is definitely one of the most beautiful properties in the context of numerical ranges. A useful tool in order to characterize convexity of the C-numerical range is the C-spectrum, which was first introduced for matrices in \cite{9} and was generalized to infinite dimensions (more precisely, to trace-class operators) in \cite{1}. Consequently, the next step is to transfer this concept and some of the known results to the Schatten-class setting.

In order to define the C-spectrum, we first have to fix the term *eigenvalue sequence* of a compact operator \( T \in \mathcal{K}(\mathcal{H}) \). In general, it is obtained by arranging the (necessarily countably many) non-zero eigenvalues in decreasing order with respect to their absolute values and each eigenvalue is repeated as many times as its algebraic multiplicity.\(^1\) If only finitely many non-vanishing eigenvalues exist, the sequence is filled up with zeros, see \cite{3}, Ch. 15]. For our purposes, we have to pass to a slightly modified eigenvalue sequence as follows:

- If the range of \( T \) is infinite-dimensional and the kernel of \( T \) is finite-dimensional, then put \( \dim(\ker T) \) zeros at the beginning of the eigenvalue sequence of \( T \).
- If the range and the kernel of \( T \) are infinite-dimensional, mix infinitely many zeros into the eigenvalue sequence of \( T \).\(^2\)

\(^1\)By \cite{3}, Prop. 15.12, every non-zero element \( \lambda \in \sigma(T) \) of the spectrum of \( T \) is an eigenvalue of \( T \) and has a well-defined finite algebraic multiplicity \( \nu_\lambda(T) \), e.g., \( \nu_\lambda(T) := \dim \ker(T - \lambda)^n \), where \( n_0 \in \mathbb{N} \) is the smallest natural number such that \( \ker(T - \lambda)^n = \ker(T - \lambda)^{n+1} \).

\(^2\)Since in Definition 3.8 arbitrary permutations will be applied to the modified eigenvalue sequence, we do not need to specify this mixing procedure further, cf. also \cite{1}, Lemma 3.6.
• If the range of \( T \) is finite-dimensional leave the eigenvalue sequence of \( T \) unchanged.

Note that compact normal operators have a spectral decomposition of the form

\[
T = \sum_{n=1}^{\infty} \tau_n \langle f_n, \cdot \rangle f_n
\]

where \( (f_n)_{n \in \mathbb{N}} \) is an orthonormal basis of \( \mathcal{H} \) and \( (\tau_n)_{n \in \mathbb{N}} \) denotes the modified eigenvalue sequence of \( T \), cf. [2, Thm. VIII.4.6]. Hence it is evident that for arbitrary \( p \in [1, \infty) \), the absolute values of the non-vanishing eigenvalues and the singular values of a normal \( T \in \mathcal{B}^p(\mathcal{H}) \) coincide and thus

\[
\nu_p(T) = \left( \sum_{n=1}^{\infty} |\tau_n|^p \right)^{1/p} < \infty.
\]

**Definition 3.8 (C-spectrum).** Let \( p, q \in [1, \infty] \) be conjugate. Then, for \( C \in \mathcal{B}^p(\mathcal{H}) \) with modified eigenvalue sequence \( (\gamma_n)_{n \in \mathbb{N}} \) and \( T \in \mathcal{B}^q(\mathcal{H}) \) with modified eigenvalue sequence \( (\tau_n)_{n \in \mathbb{N}} \), we define the C-spectrum of \( T \) to be

\[
P_C(T) := \left\{ \sum_{n=1}^{\infty} \gamma_n \tau_{\sigma(n)} \mid \sigma : \mathbb{N} \to \mathbb{N} \text{ is permutation} \right\}.
\]

Due to Hölder’s inequality and the standard estimate \( \sum_{n=1}^{\infty} |\gamma_n(A)|^p \leq \sum_{n=1}^{\infty} s_n(A)^p \), cf. [3, Prop. 16.31], one has

\[
\sum_{n=1}^{\infty} |\gamma_n \tau_{\sigma(n)}| \leq \left( \sum_{n=1}^{\infty} s_n(C)^p \right)^{1/p} \left( \sum_{n=1}^{\infty} s_n(T)^q \right)^{1/q} = \nu_p(C)\nu_q(T).
\]

Thus, the series \( \sum_{n=1}^{\infty} \gamma_n \tau_{\sigma(n)} \) in the definition of \( P_C(T) \) are well-defined and bounded by \( \nu_p(C)\nu_q(T) \).

A comprehensive survey on basic results regarding the C-spectrum of a matrix can be found in [10, Ch. 6]. Below, in Theorem 3.10 we generalize some well-known inclusion relations between the C-numerical range and the C-spectrum of matrices to Schatten-class operators. Prior to this, however, we have to derive an approximation result similar to Theorem 3.4.

**Theorem 3.9.** Let \( C \in \mathcal{B}^p(\mathcal{H}) \) and \( T \in \mathcal{B}^q(\mathcal{H}) \) both be normal with \( p, q \in [1, \infty] \) conjugate. Then

\[
\lim_{n \to \infty} P_{C^n}[T^g_n] = P_C(T).
\]

Here, \( [\cdot]_k^c \) and \( [\cdot]_k^g \) are the maps given by (4) with respect to the orthonormal bases \( (e_n)_{n \in \mathbb{N}} \) and \( (g_n)_{n \in \mathbb{N}} \) of \( \mathcal{H} \) which diagonalize \( C \) and \( T \), respectively.

**Proof.** A proof for \( p = 1, q = \infty \) (or vice versa) is given in [11, Thm. 3.6] and can be adjusted to \( p, q \in (1, \infty) \) by minimal modifications. \( \square \)

Now our main result of this section reads as follows.
Theorem 3.10. Let \( C \in \mathcal{B}^p(\mathcal{H}) \) and \( T \in \mathcal{B}^q(\mathcal{H}) \) with \( p, q \in [1, \infty) \) conjugate. Then the following statements hold.

(a) If either \( C \) or \( T \) is normal with collinear eigenvalues, then \( W_C(T) \) is convex.

(b) If \( C \) and \( T \) both are normal, then
\[
P_C(T) \subseteq W_C(T) \subseteq \text{conv}(P_C(T)).
\]

(c) If \( C \) and \( T \) both are normal and the eigenvalues of \( C \) or \( T \) are collinear, then
\[
W_C(T) = \text{conv}(P_C(T)).
\]

Proof. (a) W.l.o.g. let \( C \) be normal with collinear eigenvalues. There exists an orthonormal basis \( (e_n)_{n \in \mathbb{N}} \) of \( \mathcal{H} \) such that \( C = \sum_{n=1}^{\infty} \gamma_n \langle e_n, \cdot \rangle e_n \). Since \( \gamma_n \to 0 \) as \( n \to \infty \), due to the collinearity assumption there exists \( \phi \in [0, 2\pi) \) such that \( e^{i\phi}C \) is hermitian. Thus, by Theorem 3.4, one has
\[
W_C(T) = W_{e^{i\phi}C}(e^{-i\phi}T) = \lim_{n \to \infty} W_{[e^{i\phi}C]_{2n}}([e^{-i\phi}T]_{2n}).
\]
As \( [e^{i\phi}C]_{2n} \in \mathbb{C}^{2n \times 2n} \) is obviously hermitian for all \( n \in \mathbb{N} \), it follows that \( W_{[e^{i\phi}C]_{2n}}([e^{-i\phi}T]_{2n}) \) is convex for \( n \in \mathbb{N} \), cf. [11]. Hence Lemma 2.5 (c) yields the desired result.

(b) The statement can be proven completely analogously to [1, Thm. 3.4 – second inclusion].

(c) Finally, applying the closure and the convex hull to (b) yields \( \text{conv}(P_C(T)) = \text{conv}(W_C(T)) = W_C(T) \), where the last equality holds because of (a).

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