THE FRASER-LI CONJECTURE AND THE LIOUVILLE TYPE BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we provide a sufficient condition for a curve on a surface in \( \mathbb{R}^3 \) to be given by an orthogonal intersection with a sphere. This result makes it possible to express the boundary condition entirely in terms of the Weierstrass data without integration when dealing with free boundary minimal surfaces in a ball. Moreover, we show that solving the Fraser-Li conjecture in the case of real analytic boundaries is equivalent to proving that a solution of the Liouville type boundary value problem in an annulus has radial symmetry. This suggests a new PDE theoretic approach to the Fraser-Li conjecture.

1. INTRODUCTION

A topic of free boundary minimal surfaces has been a very active field of research. One of the most widely accepted conjectures is the following.

**Conjecture 1.1** (Fraser and Li, \([3]\)). The critical catenoid is the only embedded free boundary minimal annulus in \( \mathbb{R}^3 \), up to rigid motions.

The above open question deals with the free boundary analog of the Lawson conjecture in which the Clifford torus is the only embedded minimal torus in \( \mathbb{S}^3 \). Lawson’s conjecture was proved in \([1]\) by applying a maximum principle to a two-point function obtained from the geometric observation of the inner and outer spheres of a surface in \( \mathbb{S}^3 \). It is tempting to check whether a similar method holds, but two boundary components of the surface make it difficult to apply a maximum principle type method to the Fraser-Li conjecture.

Instead, another useful tool, called the *Weierstrass representation formula*, is used in the classical minimal surface theory:

\[
\text{Re} \int \left[ \frac{1}{2} (1 - g^2) \omega, \frac{i}{2} (1 + g^2) \omega, g \omega \right],
\]

where \( g \) is a meromorphic function and \( \omega \) is a holomorphic one-form on a Riemann surface. Since the Weierstrass representation formula is presented in the integral form, it is difficult to translate all the information related to free boundary into the data \( g \) and \( \omega \).

Meanwhile, two facts on boundary curves can be obtained from the free boundary condition. More generally, let \( \Sigma \) be a surface in \( \mathbb{R}^3 \) that meets a sphere orthogonally.
along a curve $\Gamma$. As implied by the Terquem-Joachimsthal theorem \cite{6}, $\Gamma$ is a curvature line on $\Sigma$. Moreover, as the conormal vector of $\Sigma$ along the curve coincides with the unit normal vector to the sphere, geodesic curvature along $\Gamma$ computed on $\Sigma$ is identical with normal curvature on the sphere equal to 1. It turns out that the converse is also true. Indeed, we gained the below observation as the two conditions are so powerful:

**Proposition 1.2** (Proposition 3.3 in Section 3). Let $\Gamma$ be a compact real analytic curve on a surface $\Sigma$ in $\mathbb{R}^3$. Suppose that $\Gamma$ contains only finitely many umbilic points of $\Sigma$. If $\Gamma$ is a line of curvature and has constant geodesic curvature $c$ on $\Sigma$, then there exists a sphere $S$ of radius $\frac{1}{|c|}$ (if $c = 0$, it means a plane) where $\Sigma$ intersects $S$ orthogonally along $\Gamma$. When $\Gamma$ is a piecewise real analytic curve, the same result can be obtained once a sphere is replaced by a union of spheres.

This proposition generalizes the well-known fact that if a surface contains a principal geodesic, then it meets the plane containing the geodesic perpendicularly. Since curvature lines on a nonplanar minimal surface are real analytic and contain possibly a finite number of umbilic points, the proposition is applied to both a curve on the interior of a minimal surface and real analytic boundaries. It should be noted that the conditions in the proposition can be expressed in terms of the Weierstrass data $(g, \omega)$ without undergoing a process of integration. In this way, the proposition solves difficulties associated with using the representation formula for free boundary minimal surfaces in a ball.

After applying the above proposition and the Weierstrass representation formula, we found a close relationship between the Fraser-Li conjecture and the Liouville equation. More specifically, we demonstrated that solving the Fraser-Li conjecture in real analytic boundaries is equivalent to proving that the solution of the following Liouville equation, whenever it exists, shows a radial symmetry for arbitrary $R > 1$, $0 < \epsilon < 1$, and a nonzero real constant $C_0$:

$$
\begin{align*}
\Delta v + 2C_0^2 e^v &= 0 \text{ in } A(1 - \epsilon, R + \epsilon), \\
\frac{\partial v}{\partial n} &= 2e^{-\frac{1}{2}v} - 2 \text{ if } |z| = 1, \\
\frac{\partial v}{\partial n} &= 2\frac{1}{R}e^{-\frac{1}{2}v} + \frac{2}{R} \text{ if } |z| = R, \\
2 \int_0^{2\pi} \int_1^R \frac{1}{r} e^{-v} dr d\theta &= \int_0^{2\pi} e^{-\frac{1}{2}v(1, \theta)} d\theta + \int_0^{2\pi} Re^{-\frac{1}{2}v(R, \theta)} d\theta.
\end{align*}
$$

The main idea is to construct a free boundary minimal annulus in a ball from the solution of the equation above:

**Theorem 1.3** (Theorem 4.1 in Section 4). Let $v$ be a solution of $E[R, \epsilon, C_0]$. There exist a minimal immersion $X : A(1 - \epsilon, R + \epsilon) \to \mathbb{R}^3$ and a unit sphere where they intersect orthogonally along level curves $|z| = 1$ and $|z| = R$. Moreover, the solution $v$ satisfies $v = \log \frac{1}{|z|^2 \lambda^2}$, where $\lambda$ is a metric factor induced by $X$.

This equivalence enables us to approach the Fraser-Li conjecture by using PDE theoretic methods. It can be remarked that Jiménez \cite{5} solved the Liouville equation in an annulus to classify constant curvature annuli. Although it was successful to deal with the boundary in \cite{5} by considering the Schwarzian derivative of a
meromorphic function (for instance, $h$ in (4.1)), the free boundary condition does not work well with this method. Moreover, if we translate the conjecture into a problem of the Gauss map by using the Weierstrass data, it is easy to see that the boundary condition depends not only on a geometric property, but on a specific parametrization. In this regard, only using the holomorphic function theory is a difficult approach. The equivalence suggests a new research direction to the Fraser-Li conjecture.

The paper is organized as follows. In Section 2, the Weierstrass representation and the well-known fact on a characterization of spherical space curves are reviewed. In Section 3, we prove that necessary conditions obtained from the orthogonal intersection with a sphere actually become the sufficient condition for the existence of a sphere that meets a surface orthogonally. In the final section, the equivalence between the Fraser-Li conjecture and the Liouville equation is addressed.

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2. Preliminaries

2.1. Weierstrass representation. We recall the Weierstrass representation of a minimal surface in $\mathbb{R}^3$. Since coordinate functions are harmonic, it can be expressed as a conformal harmonic immersion from a Riemann surface $C$ into $\mathbb{R}^3$:

$$\text{Re} \int \left[ \frac{1}{2}(1 - g^2)\omega, i \frac{1}{2}(1 + g^2)\omega, g\omega \right],$$

where $g$ is a meromorphic function and $\omega$ is a holomorphic one-form on $C$ such that $g$ has order $n$ pole at $p \in C$ if and only if $\omega$ has order $2n$ zero at $p \in C$. Note that $g$ corresponds to the Gauss map and $(g, \omega)$ is called a Weierstrass data.

The induced metric is

$$ds^2 = \frac{1}{4}(1 + |g|^2)^2|\omega|^2$$

and the second fundamental form is given by

$$\text{Re}\{dg \cdot \omega\}.$$

Note that to obtain a well-defined immersion from $C$, it should satisfy the period condition:

$$\text{Re} \int_{\delta} \left[ \frac{1}{2}(1 - g^2)\omega, i \frac{1}{2}(1 + g^2)\omega, g\omega \right] = 0$$

for every closed curves $\delta$ on $C$. Otherwise, the Weierstrass data gives rise to a periodic minimal surface.
2.2. Characterization of spherical space curves. We may recall an elementary fact on spherical curves, which is one of the main ingredients of the paper. Since a curve in $\mathbb{R}^3$ is determined (up to a rigid motion) by the curvature and torsion, it is possible to characterize spherical space curves in terms of them as follows.

Let $\alpha = \alpha(t) : I \rightarrow \mathbb{R}^3$ be an arclength parametrized curve such that $\tau(t) \neq 0$ and $\kappa'(t) \neq 0$ for all $t \in I$. Here, $\kappa$ is the curvature and $\tau$ is the torsion. Then $\alpha(I) \subset S^2(R)$ if and only if

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\kappa}'\right)^2 \left(\frac{1}{\tau}\right)^2 = R^2. \quad (2.1)$$

Moreover, if $\alpha(I) \subset S^2(R)$, then the unit normal vector of the sphere can be expressed as

$$-\frac{1}{R\kappa}n + \frac{\kappa'}{R\kappa^2\tau}b, \quad (2.2)$$

where $n$ and $b$ are normal and binormal vector of $\alpha$, respectively.

3. Sufficient condition to meet a sphere orthogonally

In this section, we discuss a sufficient condition for a surface to meet a sphere orthogonally. We first prove two lemmas that describe the local nature of orthogonal intersections. Then by combining two lemmas and some global arguments, we prove the main result.

**Lemma 3.1.** Let $\Gamma$ be a line of curvature on a surface $\Sigma$ with the constant geodesic curvature $c \neq 0$. Moreover, assume that $\Gamma$ has a non-vanishing torsion as a curve in $\mathbb{R}^3$ and does not contain any umbilic points. Then there exists a sphere of radius $\frac{1}{|c|}$ such that it intersects $\Sigma$ orthogonally along $\Gamma$.

**Proof.** Let $t$ be the unit tangent vector of $\Gamma$. Let us denote the unit conormal vector along $\Gamma$ as $\nu$ and the unit normal vector of $\Sigma$ as $N$ such that $\{N, t, \nu\}$ is positively oriented in $\mathbb{R}^3$. Also we write the unit normal and binormal vectors of $\Gamma$ by $n$ and $b$, respectively. Here, $b$ is given by $t \wedge n$.

As $\{N, \nu\}$ and $\{b, n\}$ form oriented orthonormal bases for the normal plane of the curve, we may write

$$\left\{ \begin{array}{l}
    b = \cos \theta N + \sin \theta \nu \\
    n = -\sin \theta N + \cos \theta \nu
  \end{array} \right. \quad (3.1)$$

for some function $\theta$ defined on $\Gamma$.

Let $\kappa$ and $\tau$ be the curvature and torsion of $\Gamma$ as a space curve. Since $\Gamma$ is a line of curvature on $\Sigma$,

$$\langle \nabla_t N, \nu \rangle = -\langle N, \nabla_t \nu \rangle = 0 \quad (3.2)$$

and we obtain from (3.1) that

$$\langle \nabla_t N, n \rangle = \cos \theta \langle \nabla_t N, \nu \rangle = 0,$n
$$\langle \nabla_t \nu, n \rangle = -\sin \theta \langle \nabla_t \nu, N \rangle = 0.$$
Here, $\nabla$ means the Riemannian connection in $\mathbb{R}^3$. Then it follows from the Frenet-Serret formulas and (\ref{3.1}) that
\[
-\tau = \langle \nabla_t b, n \rangle \\
= \langle \nabla_t (\cos \theta N + \sin \theta \nu), n \rangle \\
= (D_t \theta) \langle -\sin \theta N + \cos \theta \nu, n \rangle + \cos \theta \langle \nabla_t N, n \rangle + \sin \theta \langle \nabla_t \nu, n \rangle \\
= D_t \theta,
\]
where $D_t$ denotes the directional derivative. Hence $\tau = -D_t \theta$.

Again by (\ref{3.1}) and the Frenet-Serret formulas, the geodesic curvature of $\Gamma$ computed on $\Sigma$ is given by
\[
c^2 = \langle \nabla_t t, \nu \rangle^2 = \langle \kappa n, \nu \rangle^2 = \kappa^2 \cos^2 \theta. \tag{3.3}
\]
Since $\Gamma$ does not contain umbilic points and $c \neq 0$, we have $\kappa \neq 0$, $\cos \theta \neq 0$, $\sin \theta \neq 0$.

The constancy of the geodesic curvature along $\Gamma$ implies that
\[
0 = D_t (\kappa \cos \theta) = (D_t \kappa) \cos \theta - \kappa \sin \theta (D_t \theta) = (D_t \kappa) \cos \theta + \tau \kappa \sin \theta,
\]
where we used $\tau = -D_t \theta$ in the last step. Therefore we obtain
\[
\frac{D_t \kappa}{\tau \kappa} = -\tan \theta \neq 0. \tag{3.4}
\]

Now we compute
\[
\frac{1}{\kappa^2} + \left( D_t \left( \frac{1}{\kappa} \right) \right)^2 \frac{1}{\tau^2} = \frac{1}{\kappa^2} \left( 1 + \frac{(D_t \kappa)^2}{\tau^2 \kappa^2} \right) \\
= \frac{1}{\kappa^2} (1 + \tan^2 \theta) \\
= \frac{1}{c^2}
\]
so that the characterization result in Section 2 shows that $\Gamma$ lies on a sphere of radius $\frac{1}{|c|}$. Moreover, we deduce from
\[
\left\langle -\frac{1}{\kappa} n + \frac{D_t \kappa}{\kappa^2 \tau} b, N \right\rangle = -\frac{1}{\kappa} \langle n, N \rangle + \frac{1}{\kappa} \cdot \frac{D_t \kappa}{\tau \kappa} \langle b, N \rangle \\
= \frac{1}{\kappa} \sin \theta - \frac{1}{\kappa} \tan \theta \cdot \cos \theta \\
= 0
\]
that the sphere intersects $\Sigma$ orthogonally. \qed

Next, we have the following lemma for a vanishing torsion case:

**Lemma 3.2.** Let $\Gamma$ be a line of curvature on a surface $\Sigma$ with the constant geodesic curvature $c \neq 0$, which does not contain any umbilic points. If the torsion of $\Gamma$ is identically zero, then $\Gamma$ is a part of a circle. Therefore $\Sigma$ is orthogonal to a sphere of radius $\frac{1}{|c|}$ along $\Gamma$. 
Proof. We may use the same notation as in the proof of Lemma 3.1. Since the torsion is identically zero, $\Gamma$ lies on a plane orthogonal to $b$. Moreover, $\tau = -D_t \theta = 0$ implies that $\Sigma$ has a constant contact angle $\theta$ with the plane along $\Gamma$. Now (3.3) shows that $\Gamma$ has a constant curvature on the plane and the result follows. □

Combining Lemma 3.1 and 3.2, we obtain the proposition:

**Proposition 3.3.** Let $\Gamma$ be a compact real analytic curve on a surface $\Sigma$ in $\mathbb{R}^3$. Suppose that $\Gamma$ contains only finitely many umbilic points of $\Sigma$. If $\Gamma$ is a line of curvature and has constant geodesic curvature $c$ on $\Sigma$, then there exists a sphere $S$ of radius $\frac{1}{|c|}$ (if $c = 0$, it means a plane) where $\Sigma$ intersects $S$ orthogonally along $\Gamma$. When $\Gamma$ is a piecewise real analytic curve, the same result can be obtained once a sphere is replaced by a union of spheres.

Proof. It is well-known that if there exists a principal geodesic on $\Sigma$, then it is a plane curve and $\Sigma$ intersects that plane orthogonally. So we may assume that the geodesic curvature is nonzero, i.e., $c \neq 0$.

We will consider the real analytic case first. Since $\Gamma$ contains finitely many umbilic points of $\Sigma$, it divides into finite pieces by umbilic points: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$. Moreover, by the analyticity, the torsion of $\Gamma$ satisfies either one of the following: it is identically zero, or it vanishes only at finitely many points.

In the first case, we may apply Lemma 3.2 to each $\Gamma_i$ and obtain a sphere $S_i$ of radius $\frac{1}{|c|}$ for each $\Gamma_i$. Then we can conclude that all $S_i$’s must be the same by the continuity of the curvature of $\Gamma$.

For the second case, each $\Gamma_i$ divides into finite pieces by torsion-vanishing points. Then Lemma 3.1 implies that we can find a sphere of radius $\frac{1}{|c|}$ for each piece, and again by the continuity, we complete the proof.

If $\Gamma$ is a piecewise real analytic curvature line, then its singular points can occur at umbilic points. In this case, it is not possible to use the continuity argument at singular points to obtain only one sphere as in the above. Instead, a similar argument shows that there exists a union of spheres that intersects $\Sigma$ orthogonally along $\Gamma$. □

4. The Fraser-Li conjecture and the Liouville equation

In this section, the equivalence between the Fraser-Li conjecture and the Liouville equation will be studied. This equivalence not only suggests a possible new approach to the Fraser-Li conjecture but also provides an interesting boundary value problem of partial differential equations to study.

Let $F : A(1, R) \to \mathbb{R}^3$ be a free boundary minimal annulus in a ball $\mathbb{B}^3$ with real analytic boundaries, where $A(1, R) := \{ z \in \mathbb{C} \mid 1 < |z| < R \}$. The real analyticity implies that $F$ can be extended as a minimal immersion defined in a slightly larger annulus $A(1 - \epsilon, R + \epsilon)$ for some $\epsilon > 0$. By abuse of notation, we may also denote it by $F : A(1 - \epsilon, R + \epsilon) \to \mathbb{R}^3$. 
By using the Hopf differential, one can show that the second fundamental form is given by
\[ |\sigma|^2 = \frac{2C_0^2}{r^4\lambda^4} \]
for some real constant \( C_0 \). From this we have
\[ |\sigma|^2 = 2C_0^2 r^{-4}\lambda^{-4} \]
Here, \( \lambda \) is the metric factor defined by
\[ ds^2 = \lambda^2 |dz|^2 \]
and \( r = |z| \). Then Simons’ identity \( \Delta \log |\sigma|^2 = -2|\sigma|^2 \) (cf. [2], p. 71) gives
\[ \frac{1}{\lambda^2} \Delta \log 2C_0^2 r^{-4}\lambda^{-4} = -\frac{4C_0^2}{r^4\lambda^4}, \]
where we used \( \Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \).

Now let \( v := \log \frac{1}{r^4\lambda^2} \). Since \( \log r \) is a harmonic function, the Simons identity implies that \( v \) satisfies the Liouville equation in \( A(1-\epsilon, R+\epsilon) \):
\[ \Delta v + 2C_0^2 e^v = 0. \]
Moreover, it follows from the free boundary condition that the geodesic curvature of level curves \( r = 1 \) and \( r = R \) are equal to 1, which can be expressed as
\[ -\frac{1}{r\lambda} \left( 1 + \frac{r}{\lambda} \frac{\partial \lambda}{\partial r} \right) = 1 \text{ if } r = 1, \]
\[ -\frac{1}{r\lambda} \left( 1 + \frac{r}{\lambda} \frac{\partial \lambda}{\partial r} \right) = -1 \text{ if } r = R. \]
Hence we obtain
\[ \frac{\partial v}{\partial n} = 2e^{-\frac{v}{2}} - 2 \text{ if } r = 1, \]
\[ \frac{\partial v}{\partial n} = \frac{2}{R^2} e^{-\frac{v}{2}} + \frac{2}{R} \text{ if } r = R, \]
where \( n \) is the inner unit normal vector to \( A(1, R) \).

Combining all things above, we observe that \( v \) gives rise to the solution of the following Liouville type boundary value problem:
\[
\begin{cases}
  \Delta v + 2C_0^2 e^v = 0 \text{ in } A(1-\epsilon, R+\epsilon), \\
  \frac{\partial v}{\partial n} = 2e^{-\frac{v}{2}} - 2 \text{ if } |z| = 1, \\
  \frac{\partial v}{\partial n} = \frac{2}{R^2} e^{-\frac{v}{2}} + \frac{2}{R} \text{ if } |z| = R, \\
  2 \int_0^{2\pi} \int_1^R \frac{1}{r} e^{-v} drd\theta = \int_0^{2\pi} e^{-\frac{v}{2}(1,\theta)} d\theta + \int_0^{2\pi} Re^{-\frac{v}{2}(R,\theta)} d\theta. 
\end{cases} \quad (E[R, \epsilon, C_0])
\]
The last equation comes from \( 2|F(A(1, R))| = |\partial F(A(1, R))| \).

We now prove that the solution of \( E[R, \epsilon, C_0] \) also gives rise to a free boundary minimal annulus in a ball:

**Theorem 4.1.** Let \( v \) be a solution of \( E[R, \epsilon, C_0] \). There exist a minimal immersion \( X : A(1-\epsilon, R+\epsilon) \to \mathbb{R}^3 \) and a unit sphere where they intersect orthogonally along level curves \( |z| = 1 \) and \( |z| = R \). Moreover, the solution \( v \) satisfies \( v = \log \frac{1}{|z|\lambda^2} \), where \( \lambda \) is a metric factor induced by \( X \).
Proof. By applying the same method in [5], we observe that the solution $v$ is given by

$$v = \log \frac{4|h_z|^2}{(1 + C_0^2|h|^2)^2}$$ (4.1)

for some locally univalent meromorphic function $h$ in $A(1 - \epsilon, R + \epsilon)$.

Let us consider a minimal immersion $X : A(1 - \epsilon, R + \epsilon) \to \mathbb{R}^3$ defined by the Weierstrass data $(g, \omega)$, where

$$g = C_0 h, \quad \omega = \frac{1}{z^2 h_z} dz.$$ 

Later we will prove that it has no real period so that $X$ is well-defined.

The induced metric is given by

$$\lambda^2 |dz|^2 = \frac{1}{4} |\omega|^2 (1 + |g|^2)^2 = \frac{(1 + C_0^2|h|^2)^2}{4|h_z|^2|z|^4} |dz|^2,$$

which implies that

$$\lambda^2 = \frac{(1 + C_0^2|h|^2)^2}{4|h_z|^2|z|^4}.$$ 

Hence we obtain

$$\log \frac{1}{|z|^4 \lambda^2} = \log \frac{4|h_z|^2}{(1 + C_0^2|h|^2)^2} = v,$$

and it follows easily from the second and third equations of E[$R, \epsilon, C_0$] that the geodesic curvature of level curves $|z| = 1$ and $|z| = R$ are equal to 1. Moreover, the second fundamental form is

$$\text{Re}\{dg \cdot \omega\} = \text{Re}\left\{\frac{C_0}{z^2} dz^2\right\}$$

so that each level curve of $|z|$ is a line of curvature on the minimal surface.

Now it is possible to apply Proposition 3.3. We first show that the Weierstrass data $(g, \omega)$ does not have a real period. Indeed, if there was a real period, the data gives rise to a periodic minimal surface. Consider a minimal surface corresponding to the triple of the period. Then Proposition 3.3 implies that the level curve $|z| = 1$ lies on a sphere. It is impossible since three points corresponding to the image of $z = 1$ are collinear, but every lines can only intersect a sphere in at most two points. Therefore there is no real period and the immersion $X : A(1 - \epsilon, R + \epsilon) \to \mathbb{R}^3$ is well-defined.

Again by Proposition 3.3 there exist unit spheres $S_{O_1}$, centered at $O_1$, and $S_{O_2}$, centered at $O_2$, such that $S_{O_1}$ and $S_{O_2}$ intersect the minimal surface orthogonally along level curves $|z| = 1$ and $|z| = R$, respectively. As $h$ is locally univalent, there is no umbilic point and the last condition of E[$R, \epsilon, C_0$] gives $2|X(A(1, R))| = |\partial X(A(1, R))|$. Then, by Lemma 4.2 below, we can conclude that $O_1 = O_2$ and we finish the proof. □
Lemma 4.2. Let $\Sigma$ be a minimal annulus in $\mathbb{R}^3$ with $\partial \Sigma = \Gamma_1 \cup \Gamma_2$. Suppose that there exist unit spheres $S_{O_1}$ and $S_{O_2}$, with centers $O_1$ and $O_2$, respectively, such that each $S_{O_k}$ intersects $\Sigma$ orthogonally along $\Gamma_k$. If $2 |\Sigma| = |\partial \Sigma|$ and the boundary does not contain umbilic points, then $O_1 = O_2$.

Proof. Let $\rho := O_2 - O_1$ and let $Y$ be the position vector with the origin at $O_1$. The divergence theorem and the minimality of $\Sigma$ imply that

$$2|\Sigma| = \int_{\Sigma} \text{div} Y \, dA = \int_{\Gamma_1} Y \cdot \nu_1 \, ds + \int_{\Gamma_2} Y \cdot \nu_2 \, ds,$$

where $\nu_k$'s are outward unit conormal vectors. Since the surface and spheres intersect orthogonally, we have $Y \cdot \nu_1 = 1$ and $Y \cdot \nu_2 = 1 + \rho \cdot \nu_2$. Hence

$$2|\Sigma| = |\Gamma_1| + |\Gamma_2| + \rho \cdot \int_{\Gamma_2} \nu_2 \, ds = |\partial \Sigma| + \rho \cdot \text{Flux}(\Sigma)$$

and we obtain $\rho \cdot \text{Flux}(\Sigma) = 0$.

On the other hand, as the torque with respect to $O_1$ is zero at $\Gamma_1$, it should also be zero at $\Gamma_2$:

$$0 = \int_{\Gamma_2} Y \wedge \nu_2 \, ds = \rho \wedge \int_{\Gamma_2} \nu_2 \, ds = \rho \wedge \text{Flux}(\Sigma).$$

Since the boundary does not contain umbilic points, its curvature on each sphere does not change the sign. Therefore $\text{Flux}(\Sigma) \neq 0$ and we have $\rho = 0$. \qed

Since the rotationally symmetric free boundary minimal annulus in a ball is known to be the critical catenoid, it follows from Theorem 4.1 that solving the Fraser-Li conjecture in real analytic boundaries is equivalent to showing that the solution of $E[R, \epsilon, C_0]$ has a radial symmetry, whenever it exists, for arbitrary $R > 1$, $0 < \epsilon < 1$, and a nonzero real constant $C_0$.

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