Multistep collocation methods for weakly singular Volterra integral equations with application to fractional differential equations

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Abstract We discuss the application of multistep collocation methods to Volterra integral equations which contain a weakly singular kernel \((t - \tau)^{\alpha-1}\) with \(0 < \alpha < 1\). Convergence orders of the methods are determined and their superconvergence is also analyzed. The paper closes with numerical examples and application to fractional differential equations.

Keywords Volterra integral equations · Multistep collocation methods · Convergence · Superconvergence · Fractional differential equations.

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1 Introduction

We study the numerical solution of the weakly singular Volterra integral equations (VIEs)

\[
y(t) = g(t) + \int_0^t (t - \tau)^{\alpha-1} K(t, \tau, y(\tau)) \, d\tau, \quad t \in I = [0, T], \ T < \infty
\]

where \(0 < \alpha < 1\), \(g\) and \(k\) are given smooth functions. If we apply uniform mesh sequences, then the global order of convergence of the collocation approximation will be \(1 - \alpha\), to recover the optimal convergence orders one has to use suitable graded meshes \(t_i = (i/N)^r T, \ i = 0, 1, \ldots, N\) with the graded exponent \(r(\alpha) = \frac{\mu}{1-\alpha}, \ \mu \geq 1 - \alpha\) and \(\mu \geq m [2]\).

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In this work to increase the order of convergence in collocation method, the numerical methods to be applied will be multistep collocation method (MCM). This method for VIEs at first introduced in [7], then the authors extended the method for VIEs in [9] with the aim of increasing the order of classical one-step collocation methods without increasing the computational cost. We refer the reader to a survey on collocation based methods for the numerical solution of VIEs presented in [8]. The interested reader can see [3,6,10] and references therein for more research works in the subject. As a recent literature for solving these kind of equations, Wu in [13] compared the collocation methods on graded meshes with that on uniform meshes and Baratella in [1] derived a Nyström type interpolant of the solution based on Gauss-Radau nodes.

The organization of the paper is as follows: In the next section we will recall MCM from [9] thus making our exposition self-contained. Section 3 contains convergence analysis of the method and also superconvergence of the method considered in section 4. In section 5, the discretized version of MCM will be presented. In section 6, we will construct some methods and in section 7, we indicated that presented MCM can be applied for solving fractional differential equations. Section 8 provides some numerical results. Finally, the last section is devoted to conclusion.

2 Multistep collocation for weakly singular VIEs

Let \( I_h = \{ t_n : 0 = t_0 < t_1 < \ldots < t_N = T \} \) be a partition of the time interval \([0, T]\) with constant stepsize \( h = t_{n+1} - t_n, n = 0, \ldots, N - 1 \). Eq. (1) can be written, by relating it to mesh \( I_h \), as

\[
y(t) = F_n(t) + \Phi_n(t), \quad t \in [t_n, t_{n+1}],
\]

where the lag-term \( F_n \) and the increment-term \( \Phi_n \) are given by

\[
F_n(t) = g(t) + \int_0^t \frac{K(t, \tau, y(\tau))}{(t - \tau)^{1-\alpha}} d\tau, \quad \Phi_n(t) = \int_{t_n}^t \frac{K(t, \tau, y(\tau))}{(t - \tau)^{1-\alpha}} d\tau. \tag{2}
\]

Now, in this method by defining the collocation parameters \( 0 < c_1 < \ldots < c_m \leq 1 \) and the collocation points \( t_{nj} = t_n + c_j h, j = 1, \ldots, m \), we approximate the solution \( y(t) \) of the Eq. (1) via a piecewise polynomial \( u_n(t) \), which in the interval \([t_n, t_{n+1}]\) depends on the approximations \( y_{n-k} \approx y(t_{n-k}) \), \( k = 0, 1, \ldots, r-1 \), computed at \( r \) previous time steps

\[
u_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s)y_{n-k} + \sum_{j=1}^m \psi_j(s)U_{nj}, \quad s \in [0, 1],
\]

\[U_{nj} = u_n(t_{nj}) = u_n(t_n + c_j h),\tag{3}\]

where \( \varphi_k(s) \) and \( \psi_j(s) \) are polynomials of degree \( m + r - 1 \), suitably chosen to satisfy the conditions

\[u_n(t_{n-k}) = y_{n-k}, \quad k = 0, 1, \ldots, r - 1, \quad U_{nj} = u_n(t_{nj}), \quad j = 1, 2, \ldots, m. \tag{4}\]
For any fixed set of collocation parameters $c_1, \ldots, c_m$, these conditions lead to the interpolation problem
\[
\varphi_l(c_j) = 0, \quad \varphi_k(-k) = \delta_{lk}, \quad l, k = 0, 1, \ldots, r - 1, \\
\psi_j(c_i) = \delta_{ij}, \quad \psi_j(-k) = 0, \quad i, j = 1, 2, \ldots, m.
\]

(5)

Remark 1 The construction of the polynomials $\varphi_k$ and $\psi_j$ are obtained by Lagrange interpolation formula. Assuming that $c_i \neq c_j$ and $c_1 \neq 0$, the unique solution of interpolation problem assumes the form
\[
\varphi_k(s) = \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \prod_{i=0, i \neq k}^{r-1} \frac{s + i}{-k + i}, \\
\psi_j(s) = \prod_{i=0}^{r-1} \frac{s + i}{c_j + i} \prod_{i=1, i \neq j}^{m} \frac{s - c_i}{c_j - c_i}.
\]

(6)

Therefore the exact multistep collocation method for solving Eq. (1), which is obtained by imposing the collocation conditions (i.e. the collocation polynomial (3) exactly satisfies the VIE (1) at the collocation points $t_{ni}$) and computing $y_{n+1} = u_n(t_{n+1})$, is defined by
\[
U_{ni} = F_n(t_{ni}) + \Phi_n(t_{ni}), \\
y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1)y_{n-k} + \sum_{j=1}^{m} \psi_j(1)U_{nj},
\]

(7)

where
\[
F_n(t_{ni}) = g(t_{ni}) + \int_0^{t_{ni}} (t_{ni} - \tau)^{-1} K(t_{ni}, \tau, u_n(\tau)) d\tau \\
= g(t_{ni}) + h^{\alpha - 1} \sum_{v=0}^{n-1} \int_0^{1} \left( \frac{t_{ni} - t_v}{h} - \tau \right)^{-1} K(t_{ni}, t_v + \tau h, u_v(t_v + \tau h)) d\tau,
\]

(8)

\[
\Phi_n(t_{ni}) = \int_{t_{ni}}^{t_{ni+1}} (t_{ni} - \tau)^{-1} K(t_{ni}, \tau, u_n(\tau)) d\tau \\
= (c_i h)^{\alpha} \int_0^{1} (1 - \tau)^{-1} K(t_{ni}, t_n + c_i \tau h, u_n(t_n + c_i \tau h)) d\tau,
\]

(9)

$n = r - 1, r, \ldots, N - 1.$
3 Global convergence results

**Theorem 1** Let \( \varepsilon(t) = y(t) - u_n(t) \) be the error of exact collocation method \([8]-[9]\) and \( p = m + r \). Suppose that

(i) the given functions describing the VIE \([7]\) satisfy \( k \in C^p(D \times \mathbb{R}) \), \( g \in C^p(I) \),

(ii) the starting error is \( \|\varepsilon\|_{\infty,[0,t_r]} = O(h^p) \),

(iii) \( \rho(A) < 1 \), where

\[
A = \begin{bmatrix}
0_{r-1,1} & I_{r-1} \\
\varphi_{r-1}(1) & \varphi_{r-2}(1), \ldots, \varphi_{0}(1)
\end{bmatrix},
\]

(10)

and \( \rho \) denotes the spectral radius.

Then

\[
\|\varepsilon\|_{\infty} = O(h^{p+\alpha-1}).
\]

**Proof** Our proof will be by the same way as given in \([9]\) with some differences. We do the proof for the linear case

\[
y(t) = g(t) + \int_0^t (t - \tau)^{\alpha-1} K(t, \tau)y(\tau) \, d\tau, \quad t \in I,
\]

(11)

it can be easily extended to nonlinear case \([2,9]\). The exact collocation error \( \varepsilon(t) = y(t) - u_n(t) \) has the local representation

\[
\varepsilon(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) \varepsilon_{n-k} + \sum_{j=1}^m \psi_j(s) \varepsilon_{nj} + h^p R_{m,r,n}(s), \quad n \geq r,
\]

(12)

where \( \varepsilon_{n-k} = \varepsilon(t_{n-k}) \), \( \varepsilon_{nj} = \varepsilon(t_{nj}) \)

\[
R_{m,r,n}(s) = \int_{-r+1}^1 K_{mr}(s, v) y^{(m+r)}(t_n + vh) \, dv
\]

\[
K_{mr}(s, v) = \frac{1}{(m + r - 1)!} \left\{ (s - v)^{m+r-1} - \sum_{k=0}^{r-1} \varphi_k(s)(-k - v)^{m+r-1} \right. \\
- \left. \sum_{j=1}^m \psi_j(s)(c_j - v)^{m+r-1} \right\}.
\]

The first equation in \([7]\) with \( K(t, \tau, y) = K(t, \tau)y \) leads to

\[
u_n(t_{ni}) = g(t_{ni}) + h \sum_{v=0}^{n-1} \int_0^1 (t_{ni} - t_v - \tau h)^{\alpha-1} K(t_{ni}, t_v + \tau h) u_v(t_v + \tau h) \, d\tau \\
+ h \int_0^{c_i} (c_i h - \tau)^{\alpha-1} K(t_{ni}, t_n + \tau h) u_n(t_n + \tau h) \, d\tau.
\]

(13)
By evaluating (11) for \( t = t_{ni}, \) i.e.

\[
y(t_{ni}) = g(t_{ni}) + h \sum_{v=0}^{n-1} \int_0^1 \left( t_{ni} - t_v - \tau h \right)^{\alpha-1} K(t_{ni}, t_v + \tau h) y(t_v + \tau h) \, d\tau
\]

\[
+ h \int_0^{c_n} (c_n h - \tau)^{\alpha-1} K(t_{ni}, t_n + \tau h) y(t_n + \tau h) \, d\tau.
\]

(14)

and subtracting (13) from (14), we get

\[
\varepsilon_{ni} = h \sum_{v=0}^{n-1} \int_0^1 \left( t_{ni} - t_v - \tau h \right)^{\alpha-1} K(t_{ni}, t_v + \tau h) \varepsilon(t_v + \tau h) \, d\tau
\]

\[
+ h \int_0^{c_n} (c_n h - \tau)^{\alpha-1} K(t_{ni}, t_n + \tau h) \varepsilon(t_n + \tau h) \, d\tau.
\]

(15)

By the hypothesis on the starting error, it follows that

\[
\varepsilon(t_v + \tau h) = h^p q_v(s), \quad v = 0, \ldots, r - 1, \quad \tau \in [0, 1]
\]

(16)

with \( \|q_v\|_{\infty} \leq C_1 \) independent of \( h. \) By Substituting the expressions (12) and (16) in Eq. (15), we obtain

\[
\varepsilon^{(2)}_n - h \tilde{B}_n \varepsilon^{(2)}_n = h \sum_{v=r}^{n-1} \tilde{B}_n^{(v)} \varepsilon^{(1)}_v + h \sum_{v=r}^{n} \tilde{B}_n^{(v)} \varepsilon^{(2)}_v + h^{p+1} \sum_{v=0}^{n} \tilde{\rho}_n^{(v)} , \quad n \geq r,
\]

(17)

where \( \varepsilon^{(1)}_v \in R^r, \varepsilon^{(2)}_v, \tilde{\rho}_n^{(v)} \in R^m, \tilde{B}_n^{(v)} \in R^{m \times r}, \tilde{B}_n, \tilde{B}_n^{(v)} \in R^{m \times m} \) are defined as

\[
\varepsilon^{(1)}_v = [\varepsilon_{v-r+1}, \ldots, \varepsilon_v]^T, \quad \varepsilon^{(2)}_v = [\varepsilon_v, \ldots, \varepsilon_{vm}]^T,
\]

\[
\left( \tilde{B}_n^{(v)} \right)_{ik} = \begin{cases} 
\int_0^1 (t_{ni} - t_v - \tau h)^{\alpha-1} K(t_{ni}, t_v + \tau h) \varphi_k(\tau) \, d\tau, & v = r, r + 1, \ldots, n - 1, \\
\int_0^{c_n} (t_{ni} - t_n - \tau h)^{\alpha-1} K(t_{ni}, t_n + \tau h) \varphi_k(\tau) \, d\tau, & v = n,
\end{cases}
\]

\[
\left( \tilde{B}_n \right)_{ij} = \begin{cases} 
\int_0^1 (t_{ni} - t_n - \tau h)^{\alpha-1} K(t_{ni}, t_n + \tau h) \psi_j(\tau) \, d\tau, 
\end{cases}
\]

\[
\left( \tilde{B}_n^{(v)} \right)_{ij} = \begin{cases} 
\int_0^1 (t_{ni} - t_v - \tau h)^{\alpha-1} K(t_{ni}, t_v + \tau h) \psi_j(\tau) \, d\tau, & v = r, r + 1, \ldots, n - 1, \\
\int_0^{c_n} (t_{ni} - t_n - \tau h)^{\alpha-1} K(t_{ni}, t_n + \tau h) \psi_j(\tau) \, d\tau, & v = n.
\end{cases}
\]

(18)

(19)

Setting \( n = v - 1 \) and \( s = 1 \) in (12), leads to

\[
\varepsilon^{(1)}_v = A \varepsilon^{(1)}_{v-1} + S \varepsilon^{(2)}_{v-1} + h^{p+1} \tilde{\rho}_{v-1,r-1}, \quad l \geq 1,
\]
where \( A \) is given by (11).
\[
S = \begin{bmatrix} 0_{r-1,m} \\
\psi T(1) \end{bmatrix}, \quad \bar{\rho}_{m,r,j} = \begin{bmatrix} 0_{r-1,1} \\
R_{m,r,j}(1) \end{bmatrix}, \quad \psi(1) = [\psi_0(1), \ldots, \psi_m(1)]^T,
\]
and the solution is
\[
\epsilon^{(1)}_v = A^{v-r+1} \epsilon^{(1)}_{r-1} + \sum_{j=r-1}^{v-1} A^{v-j-1} \left( S \epsilon^{(2)}_j + h^p \bar{\rho}_{m,r,j} \right).
\]
substituting this solution in (17), gives
\[
(I - h\bar{B}_n)\epsilon^{(2)}_n = h \sum_{v=r}^{n-1} \bar{B}_n^{(v)} \epsilon^{(2)}_v + h \sum_{v=r}^{n-1} \left( \sum_{j=r+1}^{v} \bar{B}_n^{(j)} A^{v-j-1} S \right) \epsilon^{(2)}_j
\]
\[
+ h \left( \sum_{v=r}^{n-1} \bar{B}_n^{(v)} A^{v-r+1} \right) \epsilon^{(1)}_{r-1} + h \left( \sum_{v=r}^{n} \bar{B}_n^{(v)} A^{v-r} S \right) \epsilon^{(2)}_{r-1}
\]
\[
+ h^{p+1} \sum_{j=r-1}^{n-1} \left( \sum_{v=j+1}^{n} \bar{B}_n^{(v)} A^{v-j-1} \right) \bar{\rho}_{m,r,j} + h^{p+1} \sum_{v=0}^{n} \bar{\rho}_n^{(v)}, \quad n \geq r.
\]
We set
\[
K_{mr} = \max_{s \in [0, 1]} \int_{-r+1}^{1} |K_{mr}(s, v)| dv, \quad M_{mr} = \|g^{(m+r)}\|_{\infty} = \max_{t \in [0, 1]} |g^{(m+r)}(t)|,
\]
\[
\bar{K} = \max_{t, r \in [0, 1]} |K(t, s)|, \quad \|\bar{\rho}_{m,r,j}\|_1 \leq K_{mr} M_{gr} = \beta_{mr},
\]
\[
M_{mr} = \max \left\{ \|\varphi_k\|_{\infty}, \|\psi_j\|_{\infty}, k = 0, \ldots, r - 1, j = 1, \ldots, m \right\},
\]
and apply Lemma 6.2.10 (see [2]), we have
\[
\|\bar{B}_n^{(v)}\|_1 \leq \bar{K} M_{mr} \frac{2^{1-\alpha}}{\alpha} (n-v)^{\alpha-1} h^{\alpha-1}
\]
\[
= \bar{K} M_{mr} \gamma(\alpha) h^{\alpha-1} (n-v)^{\alpha-1} \leq \bar{K} M_{mr} \gamma(\alpha) h^{\alpha-1},
\]
\[
\|\bar{B}_n^{(\alpha)}\|_1 \leq \bar{K} M_{mr} \frac{\alpha}{\alpha} h^{\alpha-1} = \bar{K} M_{mr} \gamma_1(\alpha) h^{\alpha-1},
\]
\[
\|\bar{\rho}_n^{(v)}\|_1 \leq \begin{cases} C_1 \bar{K} \gamma(\alpha), & v = 0, 1, \ldots, r - 1 \\
\beta_{mr} \bar{K} \gamma(\alpha), & v = r, r + 1, \ldots, n - 1, \\
\beta_{mr} \bar{K} \gamma_1(\alpha), & v = n. \end{cases}
\]
Then, from (20) a bound for \( \epsilon^{(2)}_n \) can be found by the same way as described in [9] (Theorem 4.2) that leads to the estimate
\[
\|\epsilon^{(2)}_n\| \leq \Omega_1 h^{p+\alpha-1},
\]
and from (19) a bound for $\epsilon_n^{(1)}$ can be obtained in the form

$$\|\epsilon_n^{(1)}\| \leq \omega_2 h^{p+\alpha-1}.$$  

Note that the coefficients $\omega_1$ and $\omega_2$ depend on the bounds of the matrices in (20). Using the local error representation (12) and the above inequalities together with the expression (10) for the starting errors, complete the proof.

4 Local superconvergence results

The following theorem provides conditions on the collocation parameters to guarantee a local superconvergence in the mesh points.

**Theorem 2** Let us suppose that the hypothesis of Theorem 1 hold with $p = m + r$ and the collocation parameters satisfy $c_m = 1$ and

$$\frac{1}{m + r + 1} - \sum_{k=0}^{r-1} \beta_k (-k)^{m+r} - \sum_{j=1}^{m} \gamma_j (c_j)^{m+r} = 0, \quad m > 1, \quad (21)$$

with

$$\beta_k = \int_0^1 \varphi_k(s) \, ds, \quad \gamma_j = \int_0^1 \psi_j(s) \, ds.$$  

Then

$$\max_{n=0,\ldots,N} |\epsilon(t_n)| = O(h^{m+r+2\alpha-1}).$$

**Proof** Since $u_n(t)$ satisfies the integral equation at the collocation points, we have

$$u_n(t) = g(t) + \int_0^t (t-\tau)^{\alpha-1} K(t,\tau) u_n(\tau) \, d\tau - \delta(t), \quad t \in I$$

with $\delta(t_{ni}) = 0$. Subtracting this from (11) yields

$$\epsilon(t) = \delta(t) + \int_0^t (t-\tau)^{\alpha-1} K(t,\tau) \epsilon(\tau) \, d\tau,$$

with the solution given by

$$\epsilon(t) = \delta(t) + \int_0^t R(t,\tau) \delta(\tau) \, d\tau,$$

where the resolvent kernel $R(t,\tau)$ corresponding to the kernel $(t-\tau)^{\alpha-1} K(t,\tau)$ inherits the weak singularity term $(t-s)^{\alpha-1}$ and has the form $(t-\tau)^{\alpha-1} Q(t,\tau;\alpha)$.
and lie in the space $C^p$(see [2]). The error in a mesh point $t_n$ is then

$$\varepsilon(t_n) = \delta(t_n) + h \sum_{v=0}^{n-1} \int_0^1 R(t_n, tv + sh) \delta(tv + sh) \, ds$$

$$= \delta(t_n) + h \sum_{v=0}^{n-1} \int_0^1 (t_n - tv - sh)^{\alpha-1} Q(t_n, tv + sh; \alpha) \delta(tv + sh) \, ds$$

$$= \delta(t_n) + h^{\alpha} \int_0^1 (1 - s)^{\alpha-1} Q(t_n, t_{n-1} + sh; \alpha) \delta(t_{n-1} + sh) \, ds$$

$$+ h^{\alpha} \sum_{v=0}^{n-2} \int_0^1 (n - v - s)^{\alpha-1} Q(t_n, tv + sh; \alpha) \delta(tv + sh) \, ds \quad (22)$$

The hypothesis $c_m = 1$ assures that $t_n$ is a collocation point for each $n$. Since the defect function vanishes in the collocation points, we have $\delta(t_n) = 0$. By the global convergence result of Theorem 1, an upper bound is obtained for the first integral as

$$h^{m+r+2\alpha-1} Q_0(\alpha)/\alpha = O(h^{m+r+2\alpha-1}),$$

since $\|\delta\|_\infty = O(h^{m+r+\alpha-1})$. To compute the second integral, we use the quadrature formula

$$\int_0^1 (n - v - s)^{\alpha-1} Q(t_n, tv + sh; \alpha) \delta(tv + sh) \, ds$$

$$= \sum_{k=0}^{r-1} \beta_k (n - v - k)^{\alpha-1} Q(t_n, t_{v-k}; \alpha) \delta(t_{v-k})$$

$$+ \sum_{j=1}^{m} \gamma_j (n - v - c_j)^{\alpha-1} Q(t_n, t_{vj}; \alpha) \delta(t_{vj}) + E_{nv}, \quad v = 0, 1, ..., n - 2.$$}

which by construction of the polynomials $\varphi_k$ and $\psi_j$ has at least $m + r - 1$ degree of precision. By imposing condition (21) the degree of precision will be improved to $m + r$, so we have $|E_{nv}| = O(h^{m+r+1})$. Consequently, an upper bound for absolute value of second integral is given by $O(h^{n+r+\alpha})$. This finishes the proof.

5 The discretized multistep collocation method

The integrals occurring in the collocation equations (8) and (9) usually cannot be found analytically, so they need to be approximated by a suitable quadrature formula. Since at the remark 4 we considered $c_1 \neq 0$, we have

$$\frac{t_n + c_i h - tv}{h} - \tau \geq (1 + c_i) - \tau \geq c_i > 0.$$
Consequently, for the integral on the right side of (8) we don’t encounter to any singularity and we can approximate it by a suitable quadrature formula like as

\[ F_n(t_{ni}) = g(t_{ni}) + h^\alpha \sum_{i=0}^{n-1} \int_0^1 \left( \frac{t_{ni} - t_v}{h} - \tau \right)^{\alpha-1} K(t_{ni}, t_v + \tau h, u_v(t_v + \tau h)) d\tau \]

\[ \approx g(t_{ni}) + h^\alpha \sum_{i=0}^{n-1} \sum_{l=0}^{\mu_1} \omega_{1l} \left( \frac{t_{ni} - t_v}{h} - \xi_l \right)^{\alpha-1} K(t_{ni}, t_v + \xi_l h, u_v(t_v + \xi_l h)). \]

(23)

To approximate the integral in (23), we use an appropriate quadrature formula to obtain

\[ \Phi_n(t_{ni}) = (c_i h)^\alpha \int_0^1 (1 - \tau)^{\alpha-1} K(t_{ni}, t_n + c_i \tau h, u_n(t_n + c_i \tau h)) d\tau \]

\[ \approx (c_i h)^\alpha \sum_{l=0}^{\mu_2} \omega_{2l} K(t_{ni}, t_n + c_i \eta_l h, u_n(t_n + c_i \eta_l h)), \]

(24)

where \(\xi_l, \eta_l\) are quadrature nodes and \(\omega_{1l}, \omega_{2l}\) are quadrature weights. Therefore, the discretized multistep collocation polynomial, denoted by \(P_n(t)\), is then of the form

\[ P_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s)y_{n-k} + \sum_{j=1}^m \psi_j(s)\hat{U}_{nj}, \quad s \in [0, 1], \quad n = 0, 1, ..., N - 1, \]

where \(\hat{U}_{nj} = P_n(t_{nj})\) are determined by the solution of the following nonlinear system

\[ \hat{U}_{ni} = g(t_{ni}) + h^\alpha \sum_{i=0}^{n-1} \sum_{l=0}^{\mu_1} \omega_{1l} \left( \frac{t_{ni} - t_v}{h} - \xi_l \right)^{\alpha-1} \]

\[ \times K\left(t_{ni}, t_v + \xi_l h, \sum_{k=0}^{r-1} \varphi_k(\xi_l)y_{n-k} + \sum_{j=1}^m \psi_j(\xi_l)\hat{U}_{vj}\right) \]

\[ + (c_i h)^\alpha \sum_{l=0}^{\mu_2} \omega_{2l} K\left(t_{ni}, t_n + c_i \eta_l h, \sum_{k=0}^{r-1} \varphi_k(c_i \eta_l)y_{n-k} + \sum_{j=1}^m \psi_j(c_i \eta_l)\hat{U}_{nj}\right), \]

\[ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1)y_{n-k} + \sum_{j=1}^m \psi_j(1)\hat{U}_{nj}. \]

(25)

(26)

**Theorem 3** Let \(e(t) = y(t) - P_n(t)\) be the error of the discretized multistep collocation method (23)-(26), and let \(p = m + r\). Suppose that

(i) the given functions describing the VIE (1) satisfy \(k \in C^{(p)}(D \times R)\), \(g \in C^{(p)}(I)\),
(ii) the lag-term and increment-term quadrature formulas (23)-(24) are of order at least $p$,
(iii) the starting error is $\|e\|_{\infty,[a,t_r]} = O(h^p)$,
(v) $\rho(A) < 1$, where $A$ is given by (17).

Then
$$\|e\|_{\infty} = O(h^{p+\alpha-1}).$$

Proof The result is obtained from Theorem 1, the inequality
$$\|e\|_{\infty} \leq \|\epsilon\|_{\infty} + \|u_n - P_n\|_{\infty}$$
and the order of the quadrature formulas.

**Theorem 4** Let us suppose that
- the hypothesis of Theorem (5) hold,
- the collocation parameters satisfy $c_m = 1$ and in (27).

Then
$$\max_{n=0,..,N} |e(t_n)| = O(h^{p+2\alpha-1}).$$

Proof The result follows by Theorem 2, the inequality
$$|e(t_n)| \leq |\epsilon(t_n)| + |u_n(t_n) - P_n(t_n)|$$
and considering hypotheses on the order of quadrature formulas.

6 Constructing some methods

- (Two-step one-parameter method) Let us consider the case $m = 1$, $r = 2$ and denote the collocation parameter by $c$. The polynomials $\varphi_k$, $k = 0, 1$ and $\psi_1$ can be written by (27) as follows
  $$\varphi_0(s) = \frac{(c-s)(s+1)}{c}, \quad \varphi_1(s) = \frac{s(s-c)}{1+c}, \quad \psi_1(s) = \frac{s(s+1)}{c(c+1)}.$$ 
To evaluate the integrals (23) and (24), we must choose a quadrature method with order at least 3 to obtain a one-stage method of order $2 + \alpha$ for any choice of $c \in [\sqrt{2}, 1]$, note that this interval obtained under the condition $\rho(A) < 1$, also the results of Table 2 confirm that out of this interval our method maybe diverge. By choosing $c = 1$, the order of super-convergence will be $2 + 2\alpha$.

- (Two-step two-parameters method) Let us consider the case $m = 2$, $r = 2$ and denote the collocation parameters by $c_1$ and $c_2$. The polynomials $\varphi_k$, $k = 0, 1$ and $\psi_j$, $j = 1, 2$ can be written by (29) as follows
  $$\varphi_0(s) = \frac{(c_1-s)(c_2-s)(s+1)}{c_1c_2}, \quad \varphi_1(s) = \frac{(c_1-s)(s-c_2)s}{(1+c_1)(1+c_2)}, \quad \psi_1(s) = \frac{s(s+1)(s-c_2)}{c_2(c_2+1)(c_2-c_1)}, \quad \psi_1(s) = \frac{s(s+1)(s-c_1)}{c_1(c_1+1)(c_1-c_2)}.$$
as the same as former case to evaluate integrals (23) and (24) we must choose a quadrature method with order at least 4 to obtain a two-stage method of order $3 + \alpha$ for any choice of $c_1$ and $c_2$. By choosing $c_2 = 1$ and imposing condition (21) on $c_1$ to obtain $c_1 = \frac{8}{15}$ the order of superconvergence will be $3 + 2\alpha$. Note that however choosing collocation parameters $c_1$ and $c_2$ is arbitrary, according to the condition $\rho(A) < 1$ we must choose them carefully, Figure 1 (right side) shows that in some cases the value of $\rho(A)$ maybe become so large.

7 Multistep collocation method for fractional differential equations

Fractional differential equation of order $n - 1 < \alpha < n$ can be considered as a special case of integral equation (1), so we will be able to apply mentioned method for these kind of equations. To this end, consider the nonlinear fractional differential equation

\begin{equation}
(CD^\alpha y)(t) = f [t, y(t)], \quad t \in [0, T], \ n - 1 < \alpha < n, \ n \in \mathbb{N}
\end{equation}

\begin{equation}
y^{(k)}(0) = a_k \quad (a_k \in \mathbb{R}, \ k = 0, 1, ..., n - 1),
\end{equation}

involving the Caputo fractional derivative on a finite interval $[0, T], T \in \mathbb{R}$, defined by

\begin{equation}
(CD^\alpha f)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+n+1}} ds, \quad n - 1 < \alpha < n, \ n = \lfloor \alpha \rfloor + 1.
\end{equation}

By applying fractional integral of order $\alpha$ on both side of Eq. (22), we get

\begin{equation}
y(t) = \sum_{j=0}^{n-1} \frac{a_j}{j!} t^j + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f [\tau, y(\tau)] d\tau}{(t-\tau)^{1-\alpha}}
\end{equation}
\[ y(t) = \sum_{j=0}^{n-1} \frac{a_j}{j!} t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} (t - \tau)^{\lfloor \alpha \rfloor} f(\tau, y(\tau)) d\tau \quad (29) \]

which is special case of integral equation (1) with
\[ g(t) = \sum_{j=0}^{n-1} \frac{a_j}{j!} t^j, \quad K(t, \tau, y(\tau)) = \frac{1}{\Gamma(\alpha)} (t - \tau)^{\lfloor \alpha \rfloor} f(\tau, y(\tau)). \]

8 Numerical experiments

In this section to confirm the order that obtained in theorems 1 and 2, we will present the following four test problems.

Example 1 Linear Volterra internal equation
\[ y(t) = \frac{1}{\sqrt{1 + t}} + \frac{1}{8} \pi - \frac{1}{4} \sin^{-1} \left( \frac{1-t}{1+t} \right) - \frac{1}{4} \int_0^t (t - \tau)^{-\frac{1}{2}} y(\tau) d\tau, \quad t \in [0, 1], \]
with the exact solution \( y(t) = \frac{1}{\sqrt{1 + t}} \).

Example 2 Linear fractional differential equation
\[ (C D^{\frac{1}{2}} y)(t) = \sqrt{\pi} \text{erf} \sqrt{\pi} t y(t), \quad y(0) = -1, \quad t \in [0, 1], \]
with the exact solution \( y(t) = -e^{\pi t} \).

Example 3 Nonlinear Volterra integral equation
\[ y(t) = \arctan(t) + \frac{4}{3} t^{\frac{3}{2}} - \int_0^t (t - \tau)^{-\frac{1}{2}} \tan(y(\tau)) d\tau, \quad t \in [0, 1], \]
with the exact solution \( y(t) = \arctan(t) \).

Example 4 Nonlinear Volterra integral equation
\[ y(t) = t^{\frac{5}{2}} \left( 1 - \frac{9}{10} t^{\frac{5}{2}} \right) + \int_0^t (t - \tau)^{-\frac{1}{2}} y^3(\tau) d\tau, \quad t \in [0, 1], \]
with the exact solution \( y(t) = t^{\frac{5}{2}} \).

The first problem that is used by Linz and Cameron [4, 12] solved her e with the first constructed multistep collocation method with \( c = 1 \). In the second and forth problems, although the related functions don’t fulfill all adequate conditions required by Theorem 1 the results are satisfying. The third nonlinear problem is solved with the second constructed multistep collocation method with the following choices of the collocation abscissas: \{ \( c_1 = 0.7, \ c_2 = 1 \) \} and \{ \( c_1 = \frac{8}{15}, \ c_2 = 1 \) \}. Also we mention that the starting values have been
Multistep collocation methods for weakly singular Volterra integral equations

13
taken from the known exact solutions and also we can obtain them by a one-
step classical collocation method of the same order of the presented method.
The columns labeled by Order give the estimated order of convergence via:
Order = \log_2 \left( \frac{\text{Error}(n^2)}{\text{Error}(n)} \right).

To approximate the integrals in the discretized multistep collocation method
we have used Simpson and the proposed method by [5] based on the piecewise
cubic interpolation as follows
\[
\int_0^{t_n} (1 - \tau)^{\alpha-1} y(\tau) d\tau = \frac{h^\alpha}{(\alpha + 3)(\alpha + 2)(\alpha + 1)\alpha} \sum_{i=0}^{n} W_{2,i} y(t_i)
+ \frac{h^{\alpha+1}}{(\alpha + 3)(\alpha + 2)(\alpha + 1)\alpha} \sum_{i=0}^{n} \tilde{W}_{2,i} y'(t_i) + O(h^4)
\]
where the coefficients are given by
\[
W_{2,0} = -6(n - 1)^{2+\alpha}(1 + 2n + \alpha) + n^\alpha(12n^3 - 6n^2(3 + \alpha)
+ (1 + \alpha)(2 + \alpha)(3 + \alpha)),
\]
\[
W_{2,n} = 6(1 + \alpha),
\]
\[
W_{2,i} = 6 \left( 4(n - i)^{3+\alpha} + (n - i - 1)^{2+\alpha}(-1 + 2i - 2n - \alpha)
+ (1 - i + n)^{2+\alpha}(1 + 2i - 2n + \alpha) \right), \quad i = 1, 2, ..., n - 1,
\]
\[
\tilde{W}_{2,0} = -2(n - 1)^{2+\alpha}(3n + \alpha) + n^{1+\alpha}(6n^2 - 4n(3 + \alpha) + (2 + \alpha)(3 + \alpha)),
\]
\[
\tilde{W}_{2,n} = -2\alpha,
\]
\[
\tilde{W}_{2,i} = 2(n - i - 1)^{\alpha+2}(3i - 3n - \alpha) - 2(n - i + 1)^{\alpha+2}(3i - 3n + \alpha)
- (n - i)^{\alpha+2}(24 + 8\alpha), \quad i = 1, 2, ..., n - 1.
\]

9 Conclusion

This work has been concerned with the application of multistep collocation
methods for obtaining numerical solution of the nonlinear Volterra integral
equation which has a weakly singular kernel. The main aim of the paper was
increasing the order of classical one-step collocation methods without increasing
the computational cost. In the proposed method approximation of solution
in each interval depended on the approximations of some values that computed
at previous time steps. In this investigation we saw the order of global conver-
gence that obtained in [9] was special case of that obtained in this paper, but
there were not any relation between the superconvergence orders. In addition,
we found that, unlike standard collocation method which choosing collocation
parameters were arbitrary, here we should choose them carefully.

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   integral equations, Journal of Computational and Applied Mathematics, 237, 542-555
   (2013)
Table 1 The error norms $\max|y(t_n) - u_n(t_n)|$ for example 1

| $N$ | Error norm | Order | $N$ | Error norm | Order |
|-----|-------------|-------|-----|-------------|-------|
| 4   | 1.7517E-04  |       | 32  | 2.9821E-07  | 3.1291|
| 8   | 2.2695E-05  | 32    | 64  | 3.6525E-08  | 3.0293|
| 16  | 2.6090E-06  | 64    | 128 | 4.6123E-09  | 2.9853|

Table 2 The error norms $\max|y(t_n) - u_n(t_n)|$ for example 1 with $n = 16$ and different values of collocation parameter $c$.

| $c$   | $N$ | Error norm | Order | $N$ | Error norm | Order |
|-------|-----|-------------|-------|-----|-------------|-------|
| 0.1   | 8   | 1.4688e+13  | 2.8198e+07 | 5.4298e+04 | 208.6841 | 1.2801 |
| 0.2   | 16  | 2.8741e-02  | 2.7808 | 6.3912e-05 | 2.9690 |
| 0.3   | 32  | 3.8543e-03  | 2.8985 | 8.2542e-06 | 2.9529 |

Table 3 The error norms $\max|y(t_n) - u_n(t_n)|$ for example 2

| $N$ | Error norm | Order | $N$ | Error norm | Order |
|-----|-------------|-------|-----|-------------|-------|
| 8   | 1.9751E-01  | 64    | 6.0042E-04 | 2.9453|
| 16  | 2.8741E-02  | 2.7808 | 6.3912E-05 | 2.9690 |
| 32  | 3.8543E-03  | 2.8985 | 8.2542E-06 | 2.9529 |

Table 4 The error norms $\max|y(t_n) - u_n(t_n)|$ for example 3

| $N$ | Error norm | Order | $N$ | Error norm | Order (Superconvergence) |
|-----|-------------|-------|-----|-------------|--------------------------|
| 4   | 1.0498e-05  |       | 4   | 2.4245e-05  |                          |
| 8   | 7.2074e-07  | 3.8645 | 8   | 1.9166e-06  | 3.6611                   |
| 16  | 4.6194e-08  | 3.9637 | 16  | 1.0547e-07  | 4.1836                   |
| 32  | 3.2521e-09  | 3.8283 | 32  | 5.1758e-09  | 4.3489                   |
| 64  | 2.1392e-10  | 3.9262 | 64  | 2.2973e-07  | 4.4938                   |

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Table 5 The error norms $\max|y(t_n) - u_n(t_n)|$ for example 4

| $N$ | Error norm | Order | $N$ | Error norm | Order |
|-----|-------------|-------|-----|-------------|-------|
| 4   | 2.6380E-1   |       | 32  | 1.5082E-03  | 1.9409|
| 8   | 1.6507E-2   | 0.6763| 64  | 3.7079E-04  | 2.0242|
| 16  | 5.7913E-3   | 1.5112| 128 | 9.0677E-05  | 2.0318|
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