Cold and Freezing Sets in the Digital Plane

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Abstract

Cold sets and freezing sets belong to the theory of (approximate) fixed points for continuous self-maps on digital images. We study some properties of cold sets for digital images in the digital plane, and we examine some relationships between cold sets and freezing sets.

Key words and phrases: digital topology, digital image, approximate fixed point, freezing set, cold set

MSC2020 classification: 54H30, 54H25

1 Introduction

Digital topology is concerned with exploring topological and geometric properties of digital images as stored in computer memory, i.e., as sets of discrete pixels, usually treated as graphs in which some notion of “nearness” determines adjacency. Pioneering publications in the discipline include [20] [21] [19] [18]. Considerable success has been obtained in showing that digital images and the Euclidean objects they represent are often similar with respect to properties such as connectedness, fundamental group, contractibility, retraction, et al. However, the discrete and usually finite nature of a graph often constricts continuous functions on digital images in ways unmatched by similar limitations for continuous functions on Euclidean objects. Among these restrictions are those associated with cold sets and freezing sets.

Cold sets and freezing sets were introduced in [7] in order to study properties of fixed points and approximate fixed points in digital topology. Subsequent papers [8] [9] developed our understanding of freezing sets. In this paper, we give more attention to cold sets.

Among our results are the following.

• Multiple results concerning points that must belong to a cold set for a given digital image: section 5 and Theorem 6.1

• Results for which cold sets and freezing sets coincide: Theorem 6.3 and Proposition 7.1. These augment a result of [7] saying that for a digital
image that is rigid, i.e., the only continuous self-map homotopic to the identity is the identity [15], cold and freezing are equivalent. In general, cold and freezing are not equivalent [7].

Some of our results concerning when cold and freezing are equivalent show that like freezing (0-cold) sets, cold (1-cold) sets are often found in the boundary of $X$. For $s > 1$, $s$-cold sets may be found in the boundary of $X$, as in Example 8.1, or in the interior of $X$ [7].

2 Preliminaries

Let $N$ denote the set of natural numbers; $\mathbb{N}^* = \{0\} \cup \mathbb{N}$, the set of nonnegative integers; $\mathbb{Z}$, the set of integers; and $\mathbb{R}$, the set of real numbers. \# $X$ will be used for the number of members of a set $X$.

2.1 Adjacencies

Material in this section is largely quoted or paraphrased from [12].

A digital image is a pair $(X, \kappa)$ where $X \subseteq \mathbb{Z}^n$ for some $n$ and $\kappa$ is an adjacency on $X$. Thus, $(X, \kappa)$ is a graph for which $X$ is the vertex set and $\kappa$ determines the edge set. Usually, $X$ is finite, although there are papers that consider infinite $X$. Usually, adjacency reflects some type of “closeness” in $\mathbb{Z}^n$ of the adjacent points. When these “usual” conditions are satisfied, one may consider a subset $Y$ of $\mathbb{Z}^n$ containing $X$ as a model of a black-and-white “real world” image in which the black points (foreground) are represented by the members of $X$ and the white points (background) by members of $Y \setminus \{X\}$.

We write $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when $\kappa$ is understood or when it is unnecessary to mention $\kappa$, to indicate that $x$ and $y$ are $\kappa$-adjacent. Notations $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when $\kappa$ is understood, indicate that $x$ and $y$ are $\kappa$-adjacent or are equal.

The most commonly used adjacencies are the $c_u$ adjacencies, defined as follows. Let $X \subseteq \mathbb{Z}^n$ and let $u \in \mathbb{Z}$, $1 \leq u \leq n$. Then for points

$$x = (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) = y$$

we have $x \leftrightarrow_{c_u} y$ if and only if

- for at most $u$ indices $i$ we have $|x_i - y_i| = 1$, and
- for all indices $j$, $|x_j - y_j| \neq 1$ implies $x_j = y_j$.

The $c_u$-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. E.g.,

- in $\mathbb{Z}$, $c_1$-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, $c_1$-adjacency is 4-adjacency and $c_2$-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, $c_1$-adjacency is 8-adjacency, $c_2$-adjacency is 18-adjacency, and $c_3$-adjacency is 26-adjacency.
In this paper, we mostly use the \( c_1 \) and \( c_2 \) adjacencies in \( \mathbb{Z}^2 \).

Let \( x \in (X, \kappa) \). We use the notations

\[
N(X, x, \kappa) = \{ y \in X \mid y \leftrightarrow_\kappa x \}
\]

and

\[
N^*(X, x, \kappa) = \{ y \in X \mid y \leftrightarroweq_\kappa x \} = N(X, x, \kappa) \cup \{ x \}.
\]

We say \( \{ x_n \}_{n=0}^k \subset (X, \kappa) \) is a \( \kappa \)-path (or a path if \( \kappa \) is understood) from \( x_0 \) to \( x_k \) if \( x_i \leftrightarrow_\kappa x_{i+1} \) for \( i \in \{0, \ldots, k-1\} \), and \( k \) is the length of the path.

A subset \( Y \) of a digital image \( (X, \kappa) \) is \( \kappa \)-connected [21], or connected when \( \kappa \) is understood, if for every pair of points \( a, b \in Y \) there exists a \( \kappa \)-path in \( Y \) from \( a \) to \( b \).

2.2 Digitally continuous functions

Material in this section is largely quoted or paraphrased from [12].

We denote by id or \( \text{id}_X \) the identity \( \text{id}(x) = x \) for all \( x \in X \).

**Definition 2.1.** [21, 2] Let \((X, \kappa)\) and \((Y, \lambda)\) be digital images. A function \( f : X \to Y \) is \((\kappa, \lambda)\)-continuous, or digitally continuous when \( \kappa \) and \( \lambda \) are understood, if for every \( \kappa \)-connected subset \( X' \) of \( X \), \( f(X') \) is a \( \lambda \)-connected subset of \( Y \). If \((X, \kappa) = (Y, \lambda)\), we say a function is \( \kappa \)-continuous to abbreviate \"(\kappa, \kappa)\)-continuous."

**Theorem 2.2.** [2] A function \( f : X \to Y \) between digital images \((X, \kappa)\) and \((Y, \lambda)\) is \((\kappa, \lambda)\)-continuous if and only if for every \( x, y \in X \), if \( x \leftrightarrow_\kappa y \) then \( f(x) \leftrightarroweq_\lambda f(y) \).

A function \( f : (X, \kappa) \to (Y, \lambda) \) is an isomorphism (called a homeomorphism in [1]) if \( f \) is a continuous bijection such that \( f^{-1} \) is continuous.

We use the following notation. For a digital image \((X, \kappa)\),

\[
C(X, \kappa) = \{ f : X \to X \mid f \text{ is continuous} \}.
\]

Given \( f \in C(X, \kappa) \), a point \( x \in X \) is a fixed point of \( f \) if \( f(x) = x \). We denote by \( \text{Fix}(f) \) the set \( \{ x \in X \mid x \text{ is a fixed point of } f \} \). A point \( x \in X \) is an almost fixed point [21, 22] or an approximate fixed point [11] of \( f \) if \( x \leftrightarrow_\kappa f(x) \). Other papers in which approximate fixed points were studied include [3, 4, 5, 6, 10, 17]. The paper [17] has inappropriate citations and unoriginal results; these will be discussed in section 9. However, one of the implications of Theorem 4.4 of that paper is an important and original contribution.

2.3 Freezing and cold sets

Material in this section is largely quoted or paraphrased from [7].

In a Euclidean space, knowledge of the fixed point set of a continuous self-map \( f : X \to X \) often gives little information about \( f|_{X\setminus\text{Fix}(f)} \). By contrast, knowledge of \( \text{Fix}(f) \) for \( f \in C(X, \kappa) \) can tell us much about \( f|_{X\setminus\text{Fix}(f)} \). This motivates the study of freezing and cold sets.
Definition 2.3. [7] Let \((X, \kappa)\) be a digital image. We say \(A \subseteq X\) is a freezing set for \(X\) if given \(g \in C(X, \kappa)\), \(A \subseteq \text{Fix}(g)\) implies \(g = \text{id}_X\). If no proper subset of a freezing set \(A\) is a freezing set for \((X, \kappa)\), then \(A\) is a minimal freezing set.

Definition 2.4. [8] Let \(X \subseteq \mathbb{Z}^n\).

- The boundary of \(X\) with respect to the \(c_i\) adjacency, \(i \in \{1, 2\}\), is \(\text{Bd}_i(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_i} x\}\). \(\text{Bd}_1(X)\) is what is called the boundary of \(X\) in [20]. This paper uses both \(\text{Bd}_1(X)\) and \(\text{Bd}_2(X)\).

- The interior of \(X\) with respect to the \(c_i\) adjacency is \(\text{Int}_i(X) = X \setminus \text{Bd}_i(X)\).

Theorem 2.5. [7] Let \(X \subseteq \mathbb{Z}^n\) be finite. Then for \(1 \leq u \leq n\), \(\text{Bd}_1(X)\) is a freezing set for \((X, c_u)\).

Theorem 2.6. [7] Let \(X = \prod_{i=1}^n [0, m_i] \subseteq \mathbb{Z}^n\). Let \(A = \prod_{i=1}^n \{0, m_i\}\).

- \(\text{Let } Y = \prod_{i=1}^n [a_i, b_i] \subseteq \mathbb{Z}^n\) be such that \(X \subseteq Y\). Let \(f : X \to Y\) be \(c_1\)-continuous. If \(A \subseteq \text{Fix}(f)\), then \(X \subseteq \text{Fix}(f)\).

- \(A\) is a freezing set for \((X, c_1)\); minimal for \(n \in \{1, 2\}\).

Theorem 2.7. [7] Let \(X = \prod_{i=1}^n [0, m_i] \subseteq \mathbb{Z}^n\), where \(m_i > 1\) for all \(i\). Then \(\text{Bd}_1(X)\) is a minimal freezing set for \((X, c_n)\).

In the following, we use the path-length metric \(d\) for connected digital images \((X, \kappa)\), defined [16] as \(d_\kappa(x, y) = \min\{\ell \mid \ell \text{ is the length of a } \kappa\text{-path in } X \text{ from } x \text{ to } y\}\).

If \(X\) is finite and \(\kappa\)-connected, the diameter of \((X, \kappa)\) is \(diam(X, \kappa) = \max\{d_\kappa(x, y) \mid x, y \in X\}\).

Definition 2.8. [7] Given \(s \in \mathbb{N}^*\), we say \(A \subseteq X\) is an \(s\)-cold set for the connected digital image \((X, \kappa)\) if given \(f \in C(X, \kappa)\) such that \(f|_A = \text{id}_A\), then for all \(x \in X\), \(d_\kappa(x, f(x)) \leq s\). If no proper subset of \(A\) is an \(s\)-cold set for \((X, \kappa)\), then \(A\) is minimal. A cold set is a 1-cold set.

Theorem 2.9. [7] Let \((X, \kappa)\) be a connected digital image, let \(A\) be an \(s\)-cold set for \((X, \kappa)\), and let \(F : (X, \kappa) \to (Y, \lambda)\) be an isomorphism. Then \(F(A)\) is an \(s\)-cold set for \((Y, \lambda)\).

Remark 2.10. [7] The following are easily observed.

1. A 0-cold set is a freezing set.
2. If \( A \subset A' \subset X \) and \( A \) is an \( s \)-cold set for \((X, \kappa)\), then \( A' \) is an \( s \)-cold set for \((X, \kappa)\).

3. \( A \) is a cold set (i.e., a 1-cold set) for \((X, \kappa)\) if and only if given \( f \in C(X, \kappa) \) such that \( f|_A = \text{id}_A \), every \( x \in X \) is an approximate fixed point of \( f \).

4. In a finite connected digital image \((X, \kappa)\), every nonempty subset of \( X \) is a \( \text{diam}(X) \)-cold set.

5. If \( s_0 < s_1 \) and \( A \) is an \( s_0 \)-cold set for \((X, \kappa)\), then \( A \) is an \( s_1 \)-cold set for \((X, \kappa)\).

2.4 Digital disks and bounding curves

Material in this section is largely quoted or paraphrased from [8].

Let \( \kappa \in \{c_1, c_2\} \), \( n > 1 \). We say a \( \kappa \)-connected set \( S = \{x_i\}_{i=1}^n \subset \mathbb{Z}^2 \) is a (digital) line segment if the members of \( S \) are collinear.

**Remark 2.11.** [8] A digital line segment must be vertical, horizontal, or have slope of \( \pm 1 \).

We say a segment with slope of \( \pm 1 \) is slanted. An axis-parallel segment is horizontal or vertical.

A (digital) \( \kappa \)-closed curve is a path \( S = \{s_i\}_{i=0}^m \) such that \( s_0 = s_m \), and \( 0 < |i - j| < m \) implies \( s_i \neq s_j \). If, also, \( 0 \leq i < n \) implies

\[
N(S, x_i, \kappa) = \{x_{(i-1) \mod n}, x_{(i+1) \mod n}\}
\]

\( S \) is a (digital) \( \kappa \)-simple closed curve. For a simple closed curve \( S \subset \mathbb{Z}^2 \) we generally assume

- \( m \geq 8 \) if \( \kappa = c_1 \), and
- \( m \geq 4 \) if \( \kappa = c_2 \).

These requirements are necessary for the Jordan Curve Theorem of digital topology, below, as a \( c_1 \)-simple closed curve in \( \mathbb{Z}^2 \) must have at least 8 points to have a nonempty finite complementary \( c_2 \)-component, and a \( c_2 \)-simple closed curve in \( \mathbb{Z}^2 \) must have at least 4 points to have a nonempty finite complementary \( c_1 \)-component. Examples in [20] show why it is desirable to consider \( S \) and \( \mathbb{Z}^2 \setminus S \) with different adjacencies.

**Theorem 2.12.** [20] (Jordan Curve Theorem for digital topology) Let \( \{\kappa, \kappa'\} = \{c_1, c_2\} \). Let \( S \subset \mathbb{Z}^2 \) be a simple closed \( \kappa \)-curve such that \( S \) has at least 8 points if \( \kappa = c_1 \) and such that \( S \) has at least 4 points if \( \kappa = c_2 \). Then \( \mathbb{Z}^2 \setminus S \) has exactly 2 \( \kappa' \)-connected components.

One of the \( \kappa' \)-components of \( \mathbb{Z}^2 \setminus S \) is finite and the other is infinite. This suggests the following.
Definition 2.13. [8] Let $S \subset \mathbb{Z}^2$ be a $c_2$-closed curve such that $\mathbb{Z}^2 \setminus S$ has two $c_1$-components, one finite and the other infinite. The union $D$ of $S$ and the finite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is a (digital) disk. $S$ is a bounding curve of $D$. The finite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is the interior of $S$, denoted $\text{Int}(S)$, and the infinite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is the exterior of $S$, denoted $\text{Ext}(S)$.

Notes:

- If $D$ is a digital disk determined as above by a bounding $c_2$-closed curve $S$, then $(S, c_1)$ can be disconnected. See Figure 1.
- There may be more than one closed curve $S$ bounding a given disk $D$. See Figure 2. When $S$ is understood as a bounding curve of a disk $D$, we use the notations $\text{Int}(S)$ and $\text{Int}(D)$ interchangeably.
- Since we are interested in finding minimal freezing or cold sets and since it turns out we often compute these from bounding curves, we may prefer those of minimal size. A bounding curve $S$ for a disk $D$ is minimal if there is no bounding curve $S'$ for $D$ such that $\#S' < \#S$.
- In particular, a bounding curve need not be contained in $\text{Bd}_1(D)$. E.g., in the disk $D$ shown in Figure 2(i), $(2, 2)$ is a point of the bounding curve; however, all of the points $c_1$-adjacent to $(2, 2)$ are members of $D$, so by Definition 2.14 $(2, 2) \notin \text{Bd}_1(D)$. However, a bounding curve for $D$ must be contained in $\text{Bd}_2(D)$.
- In Definition 2.13, we use $c_2$ adjacency for $S$ and we do not require $S$ to be simple. Figure 2 shows why these seem appropriate.
  - The $c_2$ adjacency allows slanted segments in bounding curves and makes possible a bounding curve in subfigure (ii) with fewer points than the bounding curve in subfigure (i) in which adjacent pairs of the bounding curve are restricted to $c_1$ adjacency.
  - Neither of the bounding curves shown in Figure 2 is a $c_2$-simple closed curve. E.g., non-consecutive points of each of the bounding curves, $(0, 1)$ and $(1, 0)$, are $c_2$-adjacent. The bounding curve shown in Figure 2(ii) is clearly also not a $c_1$-simple closed curve.
- A closed curve that is not simple may be the boundary $\text{Bd}_2$ of a digital image that is not a disk. This is illustrated in Figure 3.

More generally, we have the following.

Definition 2.14. [8] Let $X \subset \mathbb{Z}^2$ be a finite, $c_i$-connected set, $i \in \{1, 2\}$. Suppose there are pairwise disjoint $c_2$-closed curves $S_j \subset X$, $1 \leq j \leq n$, such that

- $X \subset S_1 \cup \text{Int}(S_1)$;
- for $j > 1$, $D_j = S_j \cup \text{Int}(S_j)$ is a digital disk;
Figure 1: The $c_1$-disk $D = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| < 2\}$. The bounding curve $S = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 1\} = D \setminus \{(0, 0)\}$ is not $c_1$-connected.

Figure 2: Two views of $D = [0, 3]^2 \setminus \{(3, 3)\}$, which can be regarded as a $c_1$-disk with either of the closed curves shown in dark as a bounding curve.
(i) The dark line segments show a $c_1$-simple closed curve $S$ that is a bounding curve for $D$. Note the point $(2, 2)$ in the bounding curve shown. By Definition 2.4, $(2, 2) \notin Bd_1(D)$; however, $(2, 2) \in Bd_2(D)$.
(ii) The dark line segments show a $c_2$-closed curve $S$ that is a minimal bounding curve for $D$.

Figure 3: $D = [0, 6]^2 \times [0, 2] \setminus \{(3, 2)\}$ shown with a bounding curve $S$ in dark segments. $D$ is not a disk with either the $c_1$ or the $c_2$ adjacency, since with either of these adjacencies, $\mathbb{Z}^2 \setminus S$ has two bounded components, $\{(1, 1), (2, 1)\}$ and $\{(4, 1), (5, 1)\}$.
Figure 4: $p \in \overline{uv}$ in a bounding curve, with $uv$ slanted. Note $u \not\leftrightarrow c_1, p \not\leftrightarrow c_1, v, p \leftrightarrow c_2 c \not\leftrightarrow c_1, p, \{p, c\} \subset N(\mathbb{Z}^2, c_1, b) \cap N(\mathbb{Z}^2, c_1, d)$. If $X$ is slant-thick at $p$ then $c \in X$. (Not meant to be understood as showing all of $X$.)

- no two of $S_1 \cup Ext(S_1), D_2, \ldots, D_n$ are $c_1$-adjacent or $c_2$-adjacent; and
- we have

$$\mathbb{Z}^2 \setminus X = Ext(S_1) \cup \bigcup_{j=2}^{n} Int(S_j).$$

Then $\{S_j\}_{j=1}^{n}$ is a set of bounding curves of $X$.

Note: As above, a digital image $X \subset \mathbb{Z}^2$ may have more than one set of bounding curves.

2.5 Thickness

A notion of “thickness” in a digital image $X$, introduced in [8], means, roughly speaking, $X$ is “locally” like a disk.

Our definition of thickness depends on a notion of an “interior angle” of a disk. We have the following.

**Definition 2.15.** [8] Let $s_1$ and $s_2$ be sides of a digital disk $X \subset \mathbb{Z}^2$, i.e., maximal digital line segments in a bounding curve $S$ of $X$, such that $s_1 \cap s_2 = \{p\} \subset X$. The interior angle of $X$ at $p$ is the angle formed by $s_1, s_2$, and $Int(S)$.

**Definition 2.16.** Let $X \subset \mathbb{Z}^2$ be a digital disk. Let $S$ be a bounding curve of $X$ and $p \in S$.

- Suppose $p$ is in a maximal slanted segment $\sigma$ of $S$ such that $p$ is not an endpoint of $\sigma$. Then $X$ is slant-thick at $p$ if there exists $c \in X$ such that (see Figure 4)

$$c \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} c,$$

(1)

- Suppose $p$ is the vertex of a $90^\circ$ ($\pi/2$ radians) interior angle $\theta$ of $S$. Then $X$ is $90^\circ$-thick at $p$ if there exists $q \in Int(X)$ such that

  - if $\theta$ has axis-parallel sides then $q \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} q$ (see Figure 5(1));

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Figure 5: (1) \( \angle apb \) is a 90° \((\pi/2 \text{ radians})\) angle of a bounding curve of \( X \) at \( p \in A_1 \), with horizontal and vertical sides. If \( X \) is 90°-thick at \( p \) then \( q \in \text{Int}(X) \). (Not meant to be understood as showing all of \( X \).)

(2) \( \angle apb \) is a 90° \((\pi/2 \text{ radians})\) angle between slanted segments of a bounding curve. If \( X \) is 90°-thick at \( p \) then \( q \in \text{Int}(X) \). (Not meant to be understood as showing all of \( X \).)

Figure 6: \( \angle apq \) is an angle of 135° \((3\pi/4 \text{ radians})\) of a bounding curve of \( X \) at \( p \), with \( ap \cup pq \) a subset of the bounding curve. If \( X \) is 135°-thick at \( p \) then \( b, b' \in X \). (Not meant to be understood as showing all of \( X \).)

\( \bullet \) if \( \theta \) has slanted sides then \( q \leftrightarrow c_1 p \) (see Figure 5(2)).

\( \bullet \) Suppose \( p \) is the vertex of a 135° \((3\pi/4 \text{ radians})\) interior angle \( \theta \) of \( S \). Then \( X \) is 135°-thick at \( p \) if there exist \( b, b' \in X \) such that \( b \) and \( b' \) are in the interior of \( \theta \) and (see Figure 6)

\[ b \leftrightarrow c_2 p \not\leftrightarrow c_1 b \quad \text{and} \quad b' \leftrightarrow c_1 p. \]

**Definition 2.17.** Let \( X \subset \mathbb{Z}^2 \) be a digital disk. We say \( X \) is **thick** if the following are satisfied. For some bounding curve \( S \) of \( X \),

\( \bullet \) for every maximal slanted segment of \( S \), if \( p \in S \) is not an endpoint of \( S \), then \( X \) is slant-thick at \( p \), and

\( \bullet \) for every \( p \) that is the vertex of a 90° \((\pi/2 \text{ radians})\) interior angle \( \theta \) of \( S \), \( X \) is 90°-thick at \( p \), and

\( \bullet \) for every \( p \) that is the vertex of a 135° \((3\pi/4 \text{ radians})\) interior angle \( \theta \) of \( S \), \( X \) is 135°-thick at \( p \).

2.6 Convexity

A set \( X \) in a Euclidean space \( \mathbb{R}^n \) is **convex** if for every pair of distinct points \( x, y \in X \), the line segment \( \overline{xy} \) from \( x \) to \( y \) is contained in \( X \). The **convex hull of**
\(Y \subset \mathbb{R}^n\), denoted \(\text{hull}(Y)\), is the smallest convex subset of \(\mathbb{R}^n\) that contains \(Y\). If \(Y \subset \mathbb{R}^2\) is a finite set, then \(\text{hull}(Y)\) is a single point if \(Y\) is a singleton; a line segment if \(Y\) has at least 2 members and all are collinear; otherwise, \(\text{hull}(Y)\) is a polygonal disk, and the endpoints of the edges of \(\text{hull}(Y)\) are its vertices.

A digital version of convexity can be stated for subsets of the digital plane \(\mathbb{Z}^2\) as follows. A finite set \(Y \subset \mathbb{Z}^2\) is (digitally) convex \(\ref{8}\) if either

- \(Y\) is a single point, or
- \(Y\) is a digital line segment, or
- \(Y\) is a digital disk with a bounding curve \(S\) such that the endpoints of the maximal line segments of \(S\) are the vertices of \(\text{hull}(Y) \subset \mathbb{R}^2\).

**Remark 2.18.** \(\ref{8}\) Let \((X, \kappa)\) be a digital disk in \(\mathbb{Z}^2\), \(\kappa \in \{c_1, c_2\}\). Let \(s_1\) and \(s_2\) be sides of \(X\) such that \(s_1 \cap s_2 = \{p\} \subset X\). Then the interior angle of \(X\) at \(p\) is well defined.

**Remark 2.19.** \(\ref{8}\) It follows from Remark 2.11 that every interior angle measures as a multiple of 45° (\(\pi/4\) radians). For a convex disk, an interior angle must be 45° (\(\pi/4\) radians), 90° (\(\pi/2\) radians), or 135° (\(3\pi/4\) radians).

### 3 Tools for determining fixed point sets

The following assertions will be useful in determining fixed point and freezing sets.

**Proposition 3.1.** (Corollary 8.4 of \(\ref{12}\)) Let \((X, \kappa)\) be a digital image and \(f \in C(X, \kappa)\). Suppose \(x, x' \in \text{Fix}(f)\) are such that there is a unique shortest \(\kappa\)-path \(P\) in \(X\) from \(x\) to \(x'\). Then \(P \subseteq \text{Fix}(f)\).

Lemma 3.2 below,

... can be interpreted to say that in a \(c_u\)-adjacency, a continuous function that moves a point \(p\) also moves a point that is “behind” \(p\). E.g., in \(\mathbb{Z}^2\), if \(q\) and \(q'\) are \(c_1\)- or \(c_2\)-adjacent with \(q\) left, right, above, or below \(q'\), and a continuous function \(f\) moves \(q\) to the left, right, higher, or lower, respectively, then \(f\) also moves \(q'\) to the left, right, higher, or lower, respectively \(\ref{7}\).

**Lemma 3.2.** \(\ref{7}\) Let \((X, c_u) \subset \mathbb{Z}^n\) be a digital image, \(1 \leq u \leq n\). Let \(q, q' \in X\) be such that \(q \leftrightarrow_{c_u} q'\). Let \(f \in C(X, c_u)\).

1. If \(p_i(f(q)) > p_i(q) > p_i(q')\) then \(p_i(f(q')) > p_i(q')\).
2. If \(p_i(f(q)) < p_i(q) < p_i(q')\) then \(p_i(f(q')) < p_i(q')\).

**Remark 3.3.** \(\ref{7}\) If \(X \subset \mathbb{Z}^2\) is finite, then a set of bounding curves for \(X\) is a freezing set for \((X, c_i), i \in \{1, 2\}\).
In particular, we have:

**Theorem 3.4.** Let $D$ be a digital disk in $\mathbb{Z}^2$. Let $S$ be a bounding curve for $D$. Then $S$ is a freezing set for $(D, c_1)$ and for $(D, c_2)$.

The next two results form a dual pair.

**Theorem 3.5.** [8] Let $X$ be a thick convex disk with a bounding curve $S$. Let $A_1$ be the set of points $x \in S$ such that $x$ is an endpoint of a maximal axis-parallel edge of $S$. Let $A_2$ be the union of slanted line segments in $S$. Then $A = A_1 \cup A_2$ is a minimal freezing set for $(X, c_1)$.

**Theorem 3.6.** [8] Let $X$ be a thick convex disk with a minimal bounding curve $S$. Let $B_1$ be the set of points $x \in S$ such that $x$ is an endpoint of a maximal slanted edge in $S$. Let $B_2$ be the union of maximal axis-parallel line segments in $S$. Let $B = B_1 \cup B_2$. Then $B$ is a minimal freezing set for $(X, c_2)$.

### 4 General result

**Theorem 4.1.** Let $A$ be a cold set for the connected digital image $(X, \kappa)$. Assume $\#X > 2$. Let $p \in X$ such that $\#N(X, p, \kappa) = 1$. Then $p \in A$.

*Proof.* By hypothesis, there exists $p' \in X$ such that $\{p'\} = N(X, p, \kappa)$. Since $X$ is connected and has more than 2 points, there exists $q \in N(X, \kappa, p') \setminus \{p\}$. Suppose $p \notin A$. Then the function $f : X \to X$ given by

$$f(x) = \begin{cases} q & \text{if } x = p; \\ x & \text{if } x \neq p, \end{cases}$$

is a member of $C(X, \kappa)$; this follows from the observation that

$$x \leftrightarrow p \Rightarrow x = p' \Rightarrow f(x) = f(p') = p' \leftrightarrow q = f(p).$$

Clearly $f|_A = \text{id}_A$ and $d_\kappa(p, f(p)) = d_\kappa(p, q) = 2$. The latter contradicts the assumption that $A$ is a cold set. The assertion follows. 

### 5 Results for vertices of boundary angles

In this section, we state results concerning whether the vertex of an interior angle formed by sides of a bounding curve must belong to a cold set.

#### 5.1 $45^\circ$ ($\pi/4$ radians)

**Proposition 5.1.** Let $X \subset \mathbb{Z}^2$. Let $S$ be a member of a set of minimal bounding curves for $X$. Let $a \in S$ be the vertex of an interior angle, with measure $45^\circ$ ($\pi/4$ radians), formed by edges $E_1$ and $E_2$ of $S$. Let $A$ be a freezing set or a cold set for $(X, c_1)$. Then $a \in A$. 
Figure 7: Illustration of the function $g$ of Proposition 5.3. All points of $X$ other than $p_0$ are fixed points of $f$. Notice the point marked $p_0$ is moved 2 units by $g$. (Not to be understood as showing the entire image $X$.) The only $c_1$-neighbor of $p_0$ in $X$ is $(0,1)$, a fixed point of $g$ and a $c_1$-neighbor of $g(p_0)$, so $g \in C(X,c_1)$. The points $(-1,1), (0,1), \text{ and } (1,1)$ are the $c_2$-neighbors of $p_0$, are fixed points of $g$, and are $c_2$-neighbors of $g(p_0)$, so $g \in C(X,c_2)$.

Proof. By Theorem 2.9, there is no loss of generality in assuming $a = (0,0)$, the points $(x,y)$ of $E_1$ satisfy $y = x \geq 0$, and the points of $E_2$ satisfy $x \geq 0 = y$. The function $f : X \to X$ given by

$$f(x) = \begin{cases} (1,1) & \text{if } x = a; \\ x & \text{if } x \neq a, \end{cases}$$

is easily seen (see Figure 4) to belong to $C(X,c_1)$. Further, if $a \notin A$ then $f|_A = id_A$ and $d_{c_2}(a,f(a)) = 2$, the latter contrary to assumption if $A$ is either a freezing set or a cold set for $(X,c_1)$. The assertion follows.

Proposition 5.2. Let $X \subset \mathbb{Z}^2$. Let $S$ be a member of a set of minimal bounding curves for $X$. Let $a \in S$ be the vertex of an interior angle, with measure $45^\circ$ ($\pi/4$ radians), formed by edges $E_1$ and $E_2$ of $S$. Let $p \in E_1$, $p \leftrightarrow c_2 a$. Let $X$ be slant-thick at $p$. Let $A$ be a freezing set or a cold set for $(X,c_2)$. Then $a \in A$.

Proof. By Theorem 2.9 there is no loss of generality in assuming $a = (0,0)$, the points $(x,y)$ of $E_1$ satisfy $y = x \geq 0$, and the points of $E_2$ satisfy $x \geq 0 = y$.

Since $X$ is slant-thick at $p$, $c = (2,0) \in X$ (see Figure 4). Consider the function $f : X \to X$ given by

$$f(x) = \begin{cases} c & \text{if } x = a; \\ x & \text{if } x \neq a, \end{cases}$$

It is easily seen that $f \in C(X,c_2)$. Also, we have that $f|_A = id_A$, and $d_{c_2}(a,f(a)) = 2$, so assuming $a \notin A$ is contrary to the assumption that $A$ is a freezing or cold set. The assertion follows.
5.2 \(90^\circ (\pi/2 \text{ radians})\)

**Proposition 5.3.** Let \(X \subset \mathbb{Z}^2\). Let \(S\) be a minimal bounding curve for \(X\). Let \(p_0\) be the vertex of an interior angle of \(S\), formed by slanted edges \(E_1\) and \(E_2\) of \(S\), of measure \(90^\circ (\pi/2 \text{ radians})\). Let \(A\) be any of a freezing set for \((X, c_1)\), a cold set for \((X, c_1)\), a freezing set for \((X, c_2)\), or a cold set for \((X, c_2)\). Let \(X\) be \(90^\circ\)-thick at \(p_0\). Then \(p_0 \in A\).

*Proof.* By Theorem 2.9, there is no loss of generality in assuming \(p_0 = (0, 0)\), points of \(E_1\) satisfy \(y = x \geq 0\), and points of \(E_2\) satisfy \(y = -x \leq 0\).

Since \(X\) is \(90^\circ\)-thick at \(p_0\), \(q = (2, 0) \in X\) (see Figure 7). Suppose \(p_0 \notin A\). Consider the function \(g : X \to X\) given by

\[
g(x) = \begin{cases} q & \text{if } x = p_0; \\ x & \text{if } x \neq p. \end{cases}
\]

It is easily seen that \(g \in C(X, c_1)\) and \(g \in C(X, c_2)\). If \(p \notin A\) then \(f|A = \text{id}_A\) and \(d_c(p_0, g(p_0)) = 2\) for \(i \in \{1, 2\}\), contrary to the assumption that \(A\) is a freezing or cold set. Therefore we must have \(p_0 \in A\). 

**Proposition 5.4.** Let \(X \subset \mathbb{Z}^2\). Let \(S\) be a bounding curve for \(X\). Let \(p\) be the vertex of an interior angle of \(S\) formed by axis-parallel edges \(E_1\) and \(E_2\) of \(S\), of measure \(90^\circ (\pi/2 \text{ radians})\). Let \(X\) be \(90^\circ\)-thick at \(p\). Let \(A\) be a cold set for \((X, c_1)\). Then \(p \in A\).

*Proof.* By Theorem 2.9 we may assume \(p = (0, 0)\), points of \(E_1\) satisfy \(x \geq 0\), \(y = 0\), and points of \(E_2\) satisfy \(x = 0\), \(y \leq 0\). Since \(X\) is \(90^\circ\)-thick at \(p\), \(q = (1, -1) \in X\) (see Figure 5(1)). Let \(A\) be a cold set for \((X, c_1)\). Suppose \(p \notin A\). Let \(f : X \to X\) be the function given by

\[
f(x) = \begin{cases} q & \text{if } x = p; \\ x & \text{if } x \neq p. \end{cases}
\]

It is easily seen that \(f \in C(X, c_1)\), \(f|A = \text{id}_A\). However, \(d_{c_1}(p, f(p)) = 2\), contrary to the assumption that \(A\) is cold for \((X, c_1)\). Therefore, we must have \(p \in A\).

We do not obtain a similar conclusion if \(c_2\) is substituted for \(c_1\) in the hypotheses of Proposition 5.4 as shown in the following example.

**Example 5.5.** Let \(X = [0, 2]\). Then \(p_0 = (0, 0)\) is the vertex of an interior angle of \(90^\circ (\pi/2 \text{ radians})\) with axis-parallel sides, and \(X\) is \(90^\circ\)-thick at \(p_0\), but \(p_0\) is not a member of every cold set for \((X, c_2)\).

*Proof.* Let \(A = X \setminus \{p_0\}\). Let \(g \in C(X, c_2)\) such that \(g|A = \text{id}_A\). Then continuity implies

\[g(p_0) \in N^*(X, (1, 0), c_2) \cap N^*(X, (0, 1), c_2) = \{p_0, (1, 1)\} \subset N^*(X, p_0, c_2)\]

Therefore, \(A\) is a cold set for \((X, c_2)\).
5.3 $135^\circ (3\pi/4 \text{ radians})$

**Proposition 5.6.** Let $X \subset \mathbb{Z}^2$ have a $135^\circ (3\pi/4 \text{ radians})$ interior angle at $p_0$. Suppose $X$ is $135^\circ$-thick at $p_0$. Then for every cold set $A$ for $(X, c_1)$, $p_0 \in A$.

**Proof.** By Theorem 2.9, we may assume $p_0 = (0, 0)$, points $(x, y) \in E_1$ satisfy $x \geq 0 = y$, and points $(x, y) \in E_2$ satisfy $y = -x \geq 0$.

Suppose $p_0 \notin A$. Since $X$ is $135^\circ$-thick at $p_0$, $b = (1, 1) \in X$ (see Figure 6). The function $f : X \to X$ given by

$$f(x) = \begin{cases} b & \text{if } x = p_0; \\ x & \text{if } x \neq p_0, \end{cases}$$

is a member of $C(X, c_1)$, since $N(X, p_0, c_1) = \{(0, 1), (1, 0)\}$ and

$$(0, 1) = f(0, 1) \leftrightarrow_{c_1} f(p_0) \leftrightarrow_{c_1} (1, 0) = f(1, 0).$$

Also, $f|_A = \text{id}_A$. However, $d_{c_1}(p_0, f(p_0)) = 2$, contrary to the assumption that $A$ is cold. The contradiction yields the assertion. \qed

If we replace $c_1$ with $c_2$ in Theorem 5.6, we do not obtain a similar conclusion, as shown in the following example.

**Example 5.7.** Let $X = ([0, 2] \times \{0\}) \cup ([-1, 1] \times \{1\}) \cup ([0, 2] \times \{2\})$ (see Figure 8). Then $(0, 0)$ is the vertex of an interior angle in $X$ measuring $135^\circ (3\pi/4 \text{ radians})$, $X$ is $135^\circ$-thick at $(0, 0)$, and $A = X \setminus \{(0, 0)\}$ is a cold set for $(X, c_2)$.

**Proof.** Let $f \in C(X, c_2)$ such that $f|_A = \text{id}_A$. By continuity, we must have

$$f(0, 0) \in N^*(X, f(-1, 1), c_2) \cap N^*(X, f(1, 0), c_2) =$$

$$N^*(X, (-1, 1), c_2) \cap N^*(X, (1, 0), c_2) = \{(0, 0), (0, 1)\} \subset N^*(X, (0, 0), c_2).$$

The assertion follows. \qed
Figure 9: A digital image \( X \) used for Example 5.9.

If \( f \in C(X, c_1) \), \( p = (2, 2) \), \( f(1, 2) = (1, 2) \), and \( f(3, 2) = (3, 2) \), then we must have \( f(p) = p \). Thus \( X \backslash \{p\} \) is a freezing set, hence cold set, for \((X, c_1)\).

However, we have the following.

**Proposition 5.8.** Let \( X \) be a digital disk in \( \mathbb{Z}^2 \) that is \( 135^\circ \)-thick at \( p \), where \( p \) is the vertex of an interior angle of \( X \) formed by edges \( E_1 \) and \( E_2 \) of a minimal bounding curve \( S \) for \( X \). Let \( A \) be a freezing set for \((X, c_2)\). Then \( p \in A \).

**Proof.** By Theorem 2.9, we may assume \( p = (0, 0) \), points \((x, y)\) \( \in E_1 \) satisfy \( x \geq 0 = y \), and points \((x, y)\) \( \in E_2 \) satisfy \( y = -x \geq 0 \).

Suppose there is a freezing set \( A \) for \((X, c_2)\) such that \( p \notin A \). Since \( X \) is \( 135^\circ \)-thick at \( p \), \( b' = (0, 1) \in N(X, p, c_2) \) (see Figure 6). Then the function \( f : X \to X \) given by

\[
f(x) = \begin{cases} 
  b' & \text{if } x = p; \\
  x & \text{if } x \neq p,
\end{cases}
\]

is easily seen to belong to \( C(X, c_2) \), with \( f|_A = id_A \) and \( d_{c_2}(p, f(p)) = 1 \), contrary to the assumption that \( A \) is freezing. The assertion follows.

5.4 \( 225^\circ \) (5\( \pi/4 \) radians)

The following example shows that a vertex of a \( 225^\circ \) (5\( \pi/4 \) radians) angle need not be a member of a given cold set.

**Example 5.9.** Let \( X = \{(0, 0), (0, 1), (1, 1)\} \cup [0, 4] \times [2, 4] \) (see Figure 9). Let \( p = (2, 2) \). Note \( p \) is a member of a bounding curve of \( X \) and is a vertex at which the interior angle is \( 225^\circ \) (5\( \pi/4 \) radians). If \( A = X \backslash \{p\} \), then \( A \) is a freezing set, hence a cold set, for both \((X, c_1)\) and \((X, c_2)\).

**Proof.** Let \( f \in C(X, c_1) \) be such that \( f|_A = id_A \). Then

\[
f(p) \in N^*(X, (1, 2), c_1) \cap N^*(X, (3, 2), c_1) = \{p\}.
\]
Thus \( f = \text{id}_A \), so \( A \) is a freezing set, hence a cold set for \((X, c_1)\).

Let \( f \in C(X, c_2) \) be such that \( f|_A = \text{id}_A \). Then

\[
f(p) \in N^*(X, (1, 1), c_2) \cap N^*(X, (3, 3), c_2) = \{p\}.
\]

Thus \( f = \text{id}_A \), so \( A \) is a freezing set, hence a cold set for \((X, c_2)\).

\[\square\]

### 5.5 270° (3\(\pi/2\) radians)

The following examples show that the vertex of an interior angle that measures 270° (3\(\pi/2\) radians) need not belong to a given freezing, hence cold, set for its digital image when either the \(c_1\) or the \(c_2\) adjacency is used.

**Example 5.10.** Let \( X = ([0, 2]_\mathbb{Z} \times [0, 2]_\mathbb{Z}) \cup ([2, 4]_\mathbb{Z} \times [0, 3]_\mathbb{Z}) \) (see Figure 10). A minimal freezing set, and therefore a cold set, for \((X, c_1)\) is

\[
A = \{(0, 0), (4, 0), (4, 3), (2, 3), (0, 2)\} \quad [9].
\]

A freezing set, and therefore a cold set, for \((X, c_2)\) is, by Theorem 3.6

\[
B = \{(0, i)\}_{i=0}^2 \cup \{(j, 0)\}_{j=0}^4 \cup \{(4, k)\}_{k=0}^3 \cup \{(1, 2), (2, 3), (3, 3)\}.
\]

The point \( p = (2, 2) \), at which \( X \) has an internal angle of 270° (3\(\pi/2\) radians), is not a member of \( A \), nor of \( B \). Note \( p \) is also not a member of the minimal bounding curve of \( X \), which bypasses \( p \) by using the diagonal path \( \{(1, 2), (2, 3)\} \).

In general, a vertex of a bounding curve of an image \( Y \in \mathbb{Z}^2 \) at which the interior angle is 270° (3\(\pi/2\) radians), is not a member of the minimal bounding curve of \( Y \).

**Example 5.11.** Let \( X = [0, 4]_\mathbb{Z} \setminus \{(1, 0), (2, 0), (2, 1), (3, 0)\} \) (see Figure 11). The point \( p = (2, 2) \) is the vertex of an interior angle of 270° (3\(\pi/2\) radians) with slanted sides, and does not belong to every freezing, hence cold, set for \((X, c_1)\) or for \((X, c_2)\).

**Proof.** Let \( A = X \setminus \{p\} \). We will show \( A \) is a freezing set, hence a cold set, for both \((X, c_1)\) and \((X, c_2)\).

Let \( f \in C(X, c_1) \) be such that \( f|_A = \text{id}_A \). Then

\[
f(p) \in N^*(X, f(1, 2), c_1) \cap N^*(X, f(2, 3), c_1) \cap N^*(X, f(3, 2), c_1) =
N^*(X, (1, 2), c_1) \cap N^*(X, (2, 3), c_1) \cap N^*(X, (3, 2), c_1) = \{p\}.
\]

Thus \( f = \text{id}_X \), so \( A \) is a freezing set, hence a cold set, for \((X, c_1)\).

Let \( f \in C(X, c_2) \) be such that \( f|_A = \text{id}_A \). Then

\[
f(p) \in N^*(X, f(1, 1), c_2) \cap N^*(X, f(2, 3), c_2) \cap N^*(X, f(3, 1), c_2) =
N^*(X, (1, 1), c_2) \cap N^*(X, (2, 3), c_2) \cap N^*(X, (3, 1), c_2) = \{p\}.
\]

Thus \( f = \text{id}_X \), so \( A \) is a freezing set, hence a cold set, for \((X, c_2)\). \[\square\]
Figure 10: The digital image $X$ of Example 5.10. Points of a cold set $A$ for $(X, c_1)$ are marked “a”. Notes:
1) The point $p = (2, 2)$, at which $X$ has an internal angle of $270^\circ$ ($3\pi/2$ radians), is not a member of $A$.
2) The point $p$ does not belong to the minimal bounding curve, since the $c_1$-path $\{(1, 2), p, (2, 3)\}$ of the $c_1$-bounding curve can be replaced by the $c_2$-path $\{(1, 2), (2, 3)\}$ to obtain the minimal bounding curve.

Figure 11: The digital image $X$ of Example 5.11. The point $p = (2, 2)$ is the vertex of an interior angle of $270^\circ$ ($3\pi/2$ radians).
Figure 12: The interior angle with sides from $p$ to $(2,0)$ and from $p$ to $(4,0)$ measures $315^\circ$ ($7\pi/4$ radians). Note $p$ is an interior point of the digital image shown.

5.6 $315^\circ$ ($7\pi/4$ radians)

Example 5.12. If $X \subset \mathbb{Z}^2$ is a thick convex digital disk and $p \in X$ is the vertex of an angle in $X$ of measure $315^\circ$ ($7\pi/4$ radians), then $p \in \text{Int}(S)$ for any bounding curve $S$ of $X$. There are cold sets for both the $c_1$ and the $c_2$ adjacencies that do not contain $p$.

Proof. Figure 12 shows how in a thick convex digital disk $X$ with $p \in X$ as the vertex of an angle of $315^\circ$ ($7\pi/4$ radians), $p$ must belong to $\text{Int}(S)$ for any bounding curve $S$ of $X$. By Theorems 3.5 and 3.6, $X$ has freezing sets, hence cold sets, for both the $c_1$ and the $c_2$ adjacencies, that do not contain $p$. $\Box$

6 Results for $c_1$ adjacency in $\mathbb{Z}^2$

In this section, we obtain results for cold sets of digital images $X \subset \mathbb{Z}^2$ with respect to the $c_1$ adjacency.

Theorem 6.1. Let $X$ be a thick convex digital disk in $\mathbb{Z}^2$. Let $S$ be a minimal bounding curve for $X$. Let $p_0$ be a vertex of $\text{hull}(X)$. Let $A$ be a cold set for $(X,c_1)$. Then $p_0 \in A$.

Proof. Since $X$ is convex, the interior angle of $S$ at $p_0$ must be $45^\circ$ ($\pi/4$ radians), $90^\circ$ ($\pi/2$ radians), or $135^\circ$ ($3\pi/4$ radians). The assertion follows from Propositions 5.1, 5.3, 5.4, and 5.6. $\Box$

Proposition 6.2. Let $X$ be a thick digital disk in $\mathbb{Z}^2$ with bounding curve $S$. Let $\sigma$ be a slanted edge of $S$. Let $p \in \sigma$ such that $p$ is not an endpoint of $\sigma$. Let $A$ be a cold set for $(X,c_1)$. Then $p \in A$.

Proof. By choice of $p$, there exists $q \in \text{Int}(X)$ such that $p \leftrightarrow_{c_2} q$ and $p \not\leftrightarrow_{c_1} q$. We must have $p \in A$, for otherwise the function $f : X \rightarrow X$ given by

$$f(x) = \begin{cases} q & \text{if } x = p; \\ x & \text{if } x \neq p, \end{cases}$$
Figure 13: Example of the $c_1$-continuous function $f$ of Proposition 6.2. A point $p$ of a slanted edge is marked, as is the point $f(p)$. All other points of the image $X$ are fixed points of $f$. Note that $d_{c_1}(p, f(p)) = 2$.

(see Figure 13) belongs to $C(X, c_1)$, $f|_A = \text{id}_A$, and $d_{c_1}(p, f(p)) = 2$, contrary to the assumption that $A$ is cold. □

**Theorem 6.3.** Let $X$ be a thick convex disk in $\mathbb{Z}^2$. Let $A \subset X$. Then $A$ is a cold set for $(X, c_1)$ if and only if $A$ is a freezing set for $(X, c_1)$.

**Proof.** Let $A$ be a cold set for $(X, c_1)$. Let $S$ be a minimal bounding curve for $X$. From Theorem 6.1 we know that

the endpoints of edges of $S$ belong to $A$. (2)

It follows from Proposition 6.2 that

every slanted edge of $S$ is a subset of $A$. (3)

It follows from (2), (3), and Theorem 3.6 that $A$ is a freezing set for $(X, c_1)$.

The converse follows from Remark 2.10(1),(5). □

7 Results for $c_2$ adjacency in $\mathbb{Z}^2$

In this section, we obtain results for cold sets of digital images $X \subset \mathbb{Z}^2$ with respect to the $c_2$ adjacency.

**Proposition 7.1.** Let $X$ be a 4-sided thick digital disk in $\mathbb{Z}^2$, all sides of which are slanted. Let $A$ be the set of endpoints of the edges of $X$. Then $A$ is a minimal cold set for $(X, c_2)$.

**Proof.** By Theorem 3.6, $A$ is a freezing set, hence a cold set for $(X, c_2)$. It follows from Proposition 6.3 that $A$ is minimal as a cold set. □
Proposition 7.2. [7] Let \( m, n \in \mathbb{N} \). Let \( X = [0, m] \times [0, n] \). Let \( A \subset Bd_{1}(X) \) be such that no pair of \( c_{1} \)-adjacent members of \( Bd_{1}(X) \) belong to \( Bd_{1}(X) \setminus A \). Then \( A \) is a cold set for \((X, c_{2})\). Further, for all \( f \in C(X, c_{2}) \), if \( f|_{A} = \text{id}_{A} \) then \( f|_{\text{Int}(X)} = \text{id}|_{\text{Int}(X)} \).

We extend Proposition 7.2 as follows.

Theorem 7.3. Let \( X \) be a thick disk in \( \mathbb{Z}^{2} \) with bounding curve \( S \) made up of axis-parallel segments. Let \( A \subset S \) be such that
\[
\text{no pair of } c_{1} \text{-adjacent members of } S \text{ belong to } S \setminus A. \tag{4}
\]
Then \( A \) is a cold set for \((X, c_{2})\). Further, for all \( f \in C(X, c_{2}) \) such that \( f|_{A} = \text{id}_{A} \), we have \( f|_{\text{Int}(X)} = \text{id}|_{\text{Int}(X)} \).

Proof. Let \( f \in C(X, c_{2}) \) be such that \( f|_{A} = \text{id}_{A} \). Let \( x \in X \). We must show \( x \) is an approximate fixed point of \( f \), i.e., that \( x \leftrightarrow_{c_{2}} f(x) \). We consider the following cases.

- If \( x \in A \) then \( x = f(x) \).
- If \( x \in S \setminus A \) then since \( S \) has only axis-parallel segments, by (4), there are distinct \( y_{0}, y_{1} \in A \) such that \( y_{0} \leftrightarrow_{c_{1}} x \leftrightarrow_{c_{1}} y_{1} \). Therefore, either \( y_{0}, x, \) and \( y_{1} \) are collinear or these points form a right angle at \( x \). In either case, the continuity of \( f \) implies
  \[
y_{0} = f(y_{0}) \leftrightarrow_{c_{2}} f(x) \leftrightarrow_{c_{2}} f(y_{1}) = y_{1}.
\]
  Thus, \( f(x) \in N(X, y_{0}, c_{2}) \cap N(X, y_{1}, c_{2}) \), which implies \( x \leftrightarrow_{c_{2}} f(x) \).

- If \( x \in \text{Int}(X) \) then we create a \( c_{2} \)-path \( P = \{ q_{i} \}_{i=0}^{n} \) through \( x \) such that \( q_{i} \leftrightarrow_{c_{2}} q_{i+1} \) and \( p_{1}(q_{i+1}) = p_{1}(q_{i}) + 1 \) for \( i \in \{0, \ldots, n-1\} \), and \( \{q_{0}, q_{n}\} \subset A \). This is done as follows. Let \( L \) be a minimal horizontal line segment with the properties of containing \( x \) and having endpoints in \( S \).
  
  - If the endpoints of \( L \) are both members of \( A \), take \( P = L \) with \( q_{0}, q_{n} \) as these endpoints.
  - (See Figure 14(i).) If an endpoint of the horizontal through \( x \) is in a vertical segment of \( S \) and not in \( A \), by (4) we can replace the endpoint with one of the adjacent members of \( S \).
  - (See Figure 14(ii).) If an endpoint of the horizontal through \( x \) meets \( S \setminus A \) at the vertex of a right angle, by (4) we can replace the endpoint with the adjacent member of the incident vertical side of \( S \).

In all cases, the resulting path \( P \) has endpoints in \( A \) and is monotone increasing, from left to right, in the first coordinate. By Lemma 3.2, \( p_{1}(f(x)) = p_{1}(x) \).

Similarly, \( p_{2}(f(x)) = p_{2}(x) \). Thus, \( f(x) = x \).

Thus we have shown that for all \( x \in X, f(x) \leftrightarrow_{c_{2}} x \); and \( f|_{\text{Int}(X)} = \text{id}|_{\text{Int}(X)} \). \( \square \)
Figure 14: A digital disk $X$ with axis-parallel boundary segments to illustrate Theorem 7.3. Points of the set $A$ are labeled “a”. No $c_1$-adjacent pair of members of the bounding curve $S$ belong to $S \setminus A$. Given $x \in \text{Int}(X)$, we want to find a $c_2$-path between members of $A$ through $x$ that is monotone increasing in the first coordinate. This is easy if a horizontal segment in $X$ through $x$ has endpoints in $A$. Otherwise:

(i) If an endpoint of the horizontal through $x$ is in a vertical segment of $S$ and not in $A$, replace the endpoint with one of the adjacent members of $S$. E.g., with $x = (3, 3)$, replace $(4, 3)$ with $(4, 2)$. This yields the $c_2$-path $\{(i, 3)\}_{i=0}^3 \cup \{(4, 2)\}$.

(ii) If an endpoint of the horizontal through $x$ meets $S \setminus A$ at the vertex of a right angle, replace the endpoint with the adjacent member of the incident vertical side of $S$. E.g., with $x = (3, 2)$, replace $(2, 2)$ with $(2, 1)$. This yields the $c_2$-path $\{(2, 1), (3, 2), (4, 2)\}$. 
Figure 15: $X = [-n, n]_2$ (shown for $n = 3$) for Example 8.1. Pixels (other than the origin) labeled $i$ belong to triangle $T_i$. The origin belongs to $\bigcap_{i=1}^8 T_i$; other members of $D$ (the union of the two diagonals) each belong to two distinct $T_i$.

8 More on the choice of adjacency

Example 8.1 below shows the importance of the adjacency used, since by Theorems 2.6 and 2.9, for the same sets $X$ and $A$, with the $c_1$ adjacency, $A$ is a freezing set.

Example 8.1. Let $X = [-n, n]_2$ for $n \geq 1$. Let

$$A = \{(-n, -n), (-n, n), (n, -n), (n, n)\}.$$  

Then, for $(X, c_2)$, $A$ is an $n$-cold set and not an $(n - 1)$-cold set.

Proof. Let $f \in C(X, c_2)$ such that $f|_A = \text{id}_A$. By Proposition 5.1 the diagonals

$$D = \{(x, y) \in X \mid y = \pm x\} \subset \text{Fix}(f). \quad (5)$$

Consider the digital triangle of points,

$$T_1 = \{(x, y) \in X \mid 0 \leq x \leq n, 0 \leq y \leq x\}.$$  

(see Figure 15).

Let $q = (u, v) \in T_1$. We will show that

$$|u - p_1(f(q))| \leq n. \quad (6)$$
Suppose otherwise. Then \( u - p_1(f(q)) > n \). Let \( q_1 = (n, v) \). By the \( c_1 \)-continuity of \( f \), it follows from Lemma 3.2 that \( p_1(f(q_1)) < 0 \). Therefore, \( d_{c_2}(f(q_1), f(n, n)) = d_{c_2}(f(q_1), (n, n)) > n \), although \( d_{c_2}(q_1, (n, n)) = n - v \leq n \). This is a contradiction, since \( f \in C(X, c_2) \). Thus \( (\text{9}) \) is established.

Next, we show

\[
p_2(f(u, v)) = v. \tag{7}
\]

This follows from Lemma 3.2 since \( p_2(f(u, v)) < v \) would imply \( p_2(f(u, u)) < u \), contrary to \( (u, u) \in \text{Fix}(f) \); and \( p_2(f(u, v)) > v \) would imply \( p_2(f(u, -u)) > -u \), contrary to \( (u, -u) \in \text{Fix}(f) \).

From \( (6) \) and \( (7) \), it follows that \( d_{c_2}(q, f(q)) \leq n \).

Similarly, \( d_{c_2}(q, f(q)) \leq n \) for \( q \) a member of each of the following subsets of \( X \).

\[
T_2 = \{(x, y) \in X \mid 0 \leq x \leq y \leq n \},
\]

\[
T_3 = \{(x, y) \in X \mid -n \leq x \leq 0, -x \leq y \leq n \},
\]

\[
T_4 = \{(x, y) \in X \mid -n \leq x \leq 0, y \leq -x \leq n \},
\]

\[
T_5 = \{(x, y) \in X \mid -n \leq x \leq y \leq 0 \},
\]

\[
T_6 = \{(x, y) \in X \mid -n \leq y \leq x \leq 0 \},
\]

\[
T_7 = \{(x, y) \in X \mid 0 \leq x \leq n, -n \leq y \leq -x \},
\]

\[
T_8 = \{(x, y) \in X \mid 0 \leq x \leq n, -x \leq y \leq 0 \}.
\]

Since \( X = \bigcup_{i=1}^8 T_i \), we have \( d_{c_2}(q, f(q)) \leq n \) for all \( q \in X \). It follows that \( A \) is an \( n \)-cold set for \((X, c_2)\).

To see \( A \) is not an \((n-1)\)-cold set for \((X, c_2)\), let \( g : X \to D \subset X \) be defined for \( q = (u, v) \) by

\[
g(q) = \begin{cases} (u, |u|) & \text{if } q \in T_2 \cup T_3; \\ q & \text{otherwise} \end{cases}
\]

(see Figure 10). Roughly, \( g \) projects \( T_2 \cup T_3 \) vertically to \( D \) and leaves all other points of \( X \) fixed. It is easy to see that \( g \in C(X, c_2) \), \( g|_A = \text{id}_A \), and

\[
d_{c_2}((0, n), g(0, n)) = d_{c_2}((0, n), (0, 0)) = n.
\]

Thus \( A \) is not an \((n-1)\)-cold set for \((X, c_2)\). \( \square \)

9 Remarks on [17]

Much of this section is quoted or paraphrased from a comment posted to the website researchgate.com on the paper [17], by Kang and Han. The paper has a result that is original, correct and correctly proven, and interesting; however, the paper is greatly flawed. Some papers cited in [17] should not be rewarded with automated incremented citation counts, so papers cited in this section but not otherwise cited in the current paper are listed at the end of this section rather
Figure 16: \(X = [-n, n]^2\) (shown for \(n = 3\)) for Example 8.1. Illustration of the function \(g\). Members of \(D\) are shown dotted. Each \(q \in X \setminus \text{Fix}(g)\) is shown joined to \(g(q) \in D\) by a line segment.

than among the references at the end of this paper. Papers cited elsewhere in the current paper are referenced as elsewhere in the current paper.

The paper [17] misguides readers through much of the literature of the almost fixed point property, also known as the approximate fixed point property (AFPP), for digital images. The AFPP generalizes the familiar fixed point property (FPP).

On page 7216, we find “… any digital space \((X, k)\) on \(\mathbb{Z}^n\) does not have the FPP for (digitally) \(k\)-continuous maps [21] (for more details, see [H19, H20]).” This statement is obviously false if \(X\) has a single point; it is true when \(X\) has more than one point (indeed, this case is noted on page 7218), the proof appearing in [11], a paper not cited in [17]. While [21] gives examples of digital images \((X, k)\) that lack the FPP, it says nothing like the general statement above attributed to it by [17]. Further, Han’s papers [H19, H20] contribute nothing to our knowledge of this assertion. Variants on these errors appear in the last paragraph of page 7218 of [17], where we also find Han’s paper [H17] falsely credited as a source for the FPP assertion above.

Also on page 7216, we find “… Banach contraction principle and a Cauchy sequence for complete metric spaces, we have also studied this issue…” followed by citation of several papers. However, Cauchy sequences for digital metric spaces were shown in Han’s own paper [H16] to be trivial, and the Banach contraction principle was shown in [BxSt] to be trivial for digital images when the most natural metrics, including all \(\ell_p\) metrics, are used. Indeed, most of the
“contributions” for digital metric spaces in papers cited by [17] were shown in [BxSt] to be either trivial or incorrect.

Example 3.2(1) of [17] has no originality, being a case of Theorem 3.3 of [21]. Example 3.2(2) of [17] has no originality in either of its parts. Part (1-1) is implied by Theorem 3.5 of [5]. Both parts, (1-1) and (1-2), are implied by Theorem 4.8 of [6].

The adjacency given at Definition 4.1 of [17], attributed to Han’s paper [16], is the normal product adjacency. It was in the literature for decades before [16] appeared; see, e.g., [Ber].

Example 4.1 of [17] should have been derived as an immediate consequence of Theorem 3.3 of [21] and Theorem 4.2 of [11].

The only significant original contribution of [17] is its Theorem 4.4. The assertion that the normal product of \((X, k_1)\) and \((Y, k_2)\) has the AFPP if the factors have the AFPP, is correctly proven. However, the implication that the factors have the AFPP if the product with the normal product adjacency has the AFPP, follows immediately and more simply from the fact that retractions preserve the AFPP [11]. Also, it should be noted that Theorem 4.4 generalizes Theorem 4.5 of [6].

Theorem 4.9 of [17] is unoriginal. See Theorem 4.8 of [6].

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10 Further remarks

We have studied properties of cold sets for digital images \((X, \kappa)\) in the digital plane. In sections 4 and 5, we have considered essential members of cold sets for \((X, \kappa)\). In sections 6 and 7, for the \(c_1\) and \(c_2\) adjacencies, respectively, we have derived cold sets for thick digital disks \(X\), with particular attention to convex disks. In section 8, we showed that the same sets \(X \subset \mathbb{Z}^2, A \subset X\), can have very different cold-set properties depending on whether the adjacency considered is \(c_1\) or \(c_2\).

The suggestions of an anonymous reviewer are acknowledged with gratitude.

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