A simple construction of Grassmannian polylogarithms

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To Andrey Suslin for his 60th birthday

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1 Introduction and main definitions

Abstract. The classical $n$-logarithm is a multivalued analytic function defined by induction as

$$\text{Li}_n(z) := \int_0^z \text{Li}_{n-1}(t)d\log t, \quad \text{Li}_1(z) = -\log(1-z).$$

In this paper we give a simple explicit construction of the Grassmannian $n$-logarithm, which is a multivalued analytic function on the quotient of the Grassmannian of $n$-dimensional subspaces in $\mathbb{C}^{2n}$ in generic position to the coordinate hyperplanes by the natural action of the torus $(\mathbb{C}^*)^{2n}$. The classical $n$-logarithm appears at certain one dimensional boundary stratum.

We introduce and study Tate iterated integrals, which are homotopy invariant integrals of 1-forms $d\log f_i$ where $f_i$ are rational functions. We give a simple explicit formula for the Tate iterated integral related to the Grassmannian $n$-logarithm.

Another example is the Tate iterated integrals for the multiple polylogarithms on the moduli spaces $\mathcal{M}_{0,n}$, calculated in Section 4.4 of [G2] using the combinatorics of plane trivalent trees decorated by the arguments of the multiple polylogarithms.

It is a pleasure to contribute this paper to the volume dedicated to Andrey Suslin, whose works [Su] and [Su2] played an essential role in the development of the story.
1.1 The Grassmannian polylogarithms and their properies

Configurations and Grassmannians. A configurations of $m$ points of a $G$-set $X$ are the orbits of the group $G$ on $X^m$. Recall the classical dictionary relating configurations of points in projective/vector spaces to Grassmannians.

1. If $X = V_n$ is an $n$-dimensional complex vector space and $G = GL_n(\mathbb{C})$ we have configurations of vectors in $V_n$. Configurations of vectors in isomorphic vector spaces are canonically identified. Such a configuration is generic if any $k \leq n$ vectors are linearly independent.

Denote by $G_n$ the moduli space of generic configurations of $2n$ vectors in an $n$-dimensional vector space. Its complex points are identified with the points of the open part of the Grassmmanian of $n$-dimensional subspaces in the coordinate space $\mathbb{C}^{2n}$ parametrising the subspaces which are in generic position to the coordinate planes. Namely, such a subspace $H \subset \mathbb{C}^{2n}$ provides a configuration of $2n$ vectors in $H^*$ given by the restriction of the coordinate functions.

2. If $X = \mathbb{C}P^{n-1}$ and $G = PGL_n(\mathbb{C})$ we have configurations of points in $\mathbb{C}P^{n-1}$. Such a configuration is generic if any $k \leq n$ of the points generate a $(k-1)$-plane in $\mathbb{C}P^{n-1}$.

Denote by $PG_n$ the moduli space of generic configurations of $2n$ points in $\mathbb{P}^{n-1}$. Its complex points are identified with the orbits of the torus $(\mathbb{C}^*)^{2n}$ on the Grassmannian $G_n^0(\mathbb{C})$. Namely, an $n$-dimensional subspace $H \subset \mathbb{C}^{2n}$ provides a configuration of $2n$ hyperplanes in the projectivisation of $H$ given by intersection with the coordinate hyperplanes. By the projective duality this is the same as a generic configuration of $2n$ points in $\mathbb{C}P^{n-1}$.

Construction of the Grassmannian polylogarithms. The Grassmannian $n$-logarithm is a multivalued analytic function $L_n^G$ on $PG_n(\mathbb{C})$, which we define as the integral of an explicit closed 1-form $\Omega$ on $PG_n(\mathbb{C})$. The 1-form $\Omega$ is defined by using the Aomoto $(n-1)$-logarithms $[\mathcal{A}]$, whose definition we recall now.

The Aomoto $n$-logarithm. A simplex in $\mathbb{C}P^n$ is a collection of $n+1$ hyperplanes $(L_0,...,L_n)$. In particular, a collection of $n+1$ points in generic position determines a simplex with the vertices at these points. A pair of simplices $(L;M)$ in $\mathbb{C}P^n$ is admissible if $L$ and $M$ have no common faces of the same dimension. There is a canonical $n$-form $\omega_L$ in $\mathbb{C}P^n$ with logarithmic poles at the hyperplanes $L_i$. If $z_i = 0$ are homogeneous equations of $L_i$ then

$$\omega_L = d \log(z_1/z_0) \wedge ... \wedge d \log(z_n/z_0).$$

Let $\Delta_M$ be a topological $n$-cycle representing a generator of $H_n(\mathbb{C}P^n,M)$. The Aomoto $n$-logarithm is a multivalued analytic function on configurations of admissible pairs of simplices $(L;M)$ in $\mathbb{C}P^n$ given by

$$\mathcal{A}_n(L;M) := \int_{\Delta_M} \omega_L.$$ 

Examples. 1. Let $(l_1,l_2)$ and $(m_1,m_2)$ be two pairs of distinct points in $\mathbb{C}P^1$. Then

$$\mathcal{A}_1(l_1,l_2;m_1,m_2) := \int_{m_1}^{m_2} d \log \frac{z-l_2}{z-l_1} = \log r(l_1,l_2,m_1,m_2).$$

where $r(x_1,x_2,x_3,x_4)$ is the cross-ratio of four points on the projective line:

$$r(x_1,x_2,x_3,x_4) := \frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}.$$
2. The classical \( n \)-logarithm \( \text{Li}_n(z) \) is given by an \( n \)-dimensional integral

\[
\text{Li}_n(z) = \int_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_n} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}.
\]

Below we always use the following convention about the integration cycles \( \Delta_M \). Given a generic configuration of points \( (x_1, \ldots, x_m) \) in \( \mathbb{CP}^{n-1} \), a compatible system of cycles is the following data. For every two points \( (x, y) \) of the configuration we choose a generic path \( \varphi(x, y) \) connecting them, for every three points \( (x, y, z) \) we choose a generic topological triangle \( \varphi(x, y, z) \) which bounds \( \varphi(x, y) + \varphi(y, z) + \varphi(z, x) \), and so on, so that for every subconfiguration \( (x_i, \ldots, x_k) \), \( k \leq n \) we choose a generic topological simplex \( \varphi(x_i, \ldots, x_k) \), and these choices are compatible with the boundaries. In the definition of the Aomoto polylogarithms we always choose a \( \varphi \)-symplex as the chain \( \Delta_M \).

Let \( V_n \) be an \( n \)-dimensional complex vector space. Choose a volume form \( \omega_n \in \det V_n^* \). Given vectors \( l_1, \ldots, l_n \) in \( V_n \), set

\[
\Delta(l_1, \ldots, l_n) := (l_1 \wedge \ldots \wedge l_n, \omega_n).
\]

Notice that a simplex in a projective space \( \mathbb{P}(V) \) can be defined as either a collection of hyperplanes, or vertices. Below we employ the second point of view, and use vectors \( l_i \in V \) to determine the vertices as the lines spanned by the vectors.

Consider the following multivalued analytic \( 1 \)-form on the Grassmannian \( G_n(\mathbb{C}) \):

\[
\Omega(l_1, ..., l_{2n}) := \text{Alt}_{2n}(A_{n-1}(l_1, ..., l_n; l_{n+1}, ..., l_{2n}) \, d \log \Delta(l_{n+1}, ..., l_{2n})). \tag{1}
\]

It does not depend on the choice of the form \( \omega_n \).

**Theorem 1.1** The \( 1 \)-form \( \Omega(l_1, ..., l_{2n}) \) is closed. It depends only on the configuration of points in \( \mathbb{CP}^{n-1} \) obtained by projection of the vectors \( l_i \).

**Definition 1.2** The Grassmannian \( n \)-logarithm \( L_n^G(l_1, ..., l_{2n}) \) is the skewsymmetrization under the permutations of the vectors \( l_1, ..., l_{2n} \) of the primitive of the \( 1 \)-form (1).

A primitive of the \( 1 \)-form (1) is a multivalued analytic function defined up to a scalar. The scalar vanishes under the skewsymmetrization. So the Grassmannian \( n \)-logarithm is a well defined multivalued analytic function. Thanks to the last claim of Theorem 1.1 we can consider it as a function \( L_n^G(x_1, ..., x_{2n}) \) on configurations of \( 2n \) points in \( \mathbb{CP}^{n-1} \).

**Properties of the Grassmannian \( n \)-logarithm.** Given a configuration of \( m + 1 \) vectors \( (l_0, ..., l_m) \) in \( V_n \), denote by \( (l_0 | l_1, ..., l_m) \) a configuration of vectors obtained by projection of the vectors \( l_1, ..., l_m \) to the quotient of \( V_n \) along the subspace generated by \( l_0 \). We employ a projective version of this construction. Given a configuration of \( m + 1 \) points \( (y_0, y_1, ..., y_m) \) in \( \mathbb{CP}^{n-1} \), denote by \( (y_0 | y_1, ..., y_m) \) a configuration of \( m \) points in \( \mathbb{CP}^{n-2} \) obtained by projection of the points \( y_i \) with the center at the point \( y_0 \).

**Theorem 1.3** The function \( L_n^G(x_1, ..., x_{2n}) \) enjoys the following properties.

1. The \( (2n + 1) \)-term equation. For a generic configuration of \( 2n + 1 \) points \( (x_1, ..., x_{2n+1}) \) in \( \mathbb{CP}^{n-1} \) one has

\[
\sum_{i=1}^{2n+1} (-1)^i L_n^G(x_1, ..., \hat{x}_i, ..., x_{2n+1}) = \text{a constant}.
\]
2. Dual \((2n + 1)\)-term equation. For a generic configuration of points \((y_1, \ldots, y_{2n+1})\) in \(\mathbb{CP}^n\)

\[
\sum_{j=1}^{2n+1} (-1)^j L^G_n(y_j | y_1, \ldots, \tilde{y}_j, \ldots, y_{2n+1}) = \text{a constant.}
\]

Here we assumed that compatible systems of cycles for the configurations of points \((x_1, \ldots, x_{2n+1})\)
and \((y_1, \ldots, y_{2n+1})\) were chosen.

**Example.** For \(n = 2\) we get the Rogers version of the dilogarithm:

\[
L^G_2(x_1, x_2, x_3) = L_2(r(x_1, x_2, x_3)), \quad \text{where } L_2(z) := \text{Li}_2(z) + \frac{1}{2} \log(1 - z) \log(z).
\]

![Figure 1: Special configuration of 8 points in \(\mathbb{P}^3\).](image)

A configuration \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) of points in \(\mathbb{P}^{n-1}\) is called a *special configuration* if \((x_1, \ldots, x_n)\)
form a generic configuration, and for every \(i\) the point \(y_i\) lies on the line \(x_i x_{i+1}\). See an example on
Fig 1. Special configurations are parametrised by one parameter, denoted by \(r(x_1, \ldots, x_n, y_1, \ldots, y_n)\),
see [G4], Section 4.4. For \(n = 2\) it is the cross-ratio. One can show that the restriction of the
function \(L^G_n\) to the special configuration is expressed via the classical \(n\)-logarithm function:

The Grassmannian \(n\)-logarithm is a period of a variation of framed mixed \(\mathbb{Q}\)-Hodge-Tate structures of geometric origin on \(PG_n(\mathbb{C})\). We call it the Grassmannian variation of mixed Tate motives.
We calculate the Tate iterated integral related to the Grassmannian polylogarithm function.

### 1.2 The history and ramifications of the problem.

There are three incarnations of the dilogarithm function:

i) The real valued Rogers dilogarithm \(L_2(x)\) defined on \(\mathbb{RP}^1 - \{0, 1, \infty\}\) by the condition:

\[
dL_2(x) = \frac{1}{2} \left( - \log |1 - x| d \log |x| + \log |x| d \log |1 - x| \right), \quad L_2(-1) = L_2(1/2) = L_2(2) = 0. \quad (2)
\]

Notice that \(\mathbb{RP}^1 - \{0, 1, \infty\}\) is the moduli space of generic configuration of 4 points in \(\mathbb{RP}^1\). The function \(L_2(r(l_1, \ldots, l_4))\) is the unique solution of the differential equation (2) which is skew symmetric under the permutations of the vectors \(l_i\). Its restriction to the interval \((0, 1)\) is given by

\[
L_2(x) = \text{Li}_2(x) + \frac{1}{2} \log(1 - x) \log x, \quad x \in (0, 1).
\]

It satisfies the 5-term relation

\[
\sum_{i=1}^{5} (-1)^k L_2(l_0, \ldots, \hat{l}_k, \ldots, l_4) = -\varepsilon \frac{\pi^2}{6}, \quad \varepsilon = \frac{1}{2} \prod_{0 \leq i < j \leq 4} \text{sgn} \Delta(l_i, l_j).
\]
ii) The multivalued complex analytic dilogarithm function $L_i(z)$, whose properties are best described by the corresponding variation of framed mixed $\mathbb{Q}$-Hodge structures.

iii) The single valued Bloch-Wigner function, defined on $\mathbb{CP}^1$ by

$$L_2(z) := \text{Im} \left( Li_2(z) + \log(1 - z) \log |z| \right).$$

It satisfies the 5-term relation

$$\sum_{i=1}^{5} (-1)^k L_2(l_0, \ldots, \hat{l}_k, \ldots, l_4) = 0.$$

The Bloch-Wigner function is nothing else but the real period of the variation which appears in ii). The motivic nature of the dilogarithm is described by the Bloch-Suslin complex and its relations to the algebraic K-theory, see [Su2].

In accordance to this, there are three directions for a generalization of the dilogarithm function:

i) Gelfand and MacPherson [GM] defined a real valued Grassmannian $2n$-logarithm function on $PG_{2n}(\mathbb{R})$ by constructing its differential. Notice that our construction of the Grassmannian $n$-logarithm also starts from a closed 1-form $\Omega$ on $G_n(\mathbb{C})$. The relationship between these two functions is not clear. It should reflect the relationship between the Chern and Pontryagin classes.

ii) The construction of Hanamura and MacPherson [HM1], [HM2] provides a Grassmannian $n$-logarithm function. The construction is geometric but rather complicated. I do not know how to relate it to the function $L_n^G$. An explicit motivic construction of Grassmannian $n$-logarithm function was given for $n = 3$ in [G] and for $n = 4$ in [G3].

iii) In [G1], see also [G4], we defined a single-valued Grassmannian $n$-logarithm function $L_n^G$. Precise relationship between this function and the multivalued analytic function $L_n^G$ is not known.

The bi-Grassmannian $n$-logarithm cocycles. We denote by $G_p^n$ the Grassmannian of $q$-dimensional subspaces in a coordinate vector space of dimension $p+q$, transversal to the coordinate hyperplanes. The weight $n$ bi-Grassmannian $\mathbb{G}(n)^\bullet$ is given by a collection of Grassmannians $G_p^n$, $p \geq n$, arranged in a form of a truncated bisimplicial variety:

$$\cdots \rightarrow  \cdots \rightarrow G_{n+1}^2 \rightarrow G_{n+1}^3 \rightarrow \cdots$$

Here a horisontal arrow stands for a collection of maps given by the intersection of the subspaces with the coordinate hyperplanes, and the vertical one for projection along the coordinate axes, see [G5]. The bottom line is the semisimplicial weight $n$ Grassmannian $G_n^\bullet$ introduced in [BMS].

The weight $n$ bi-Grassmannian $\mathbb{G}(n)^\bullet$ and the related polylogarithms play a key role in the explicit combinatorial construction of Chern classes suggested in [G5].

Points of the bi-Grassmannian $\mathbb{G}(3)$ with values in a field $F$ form a truncated bisimplicial set. Applying to it the “free abelian group” functor $S \rightarrow \mathbb{Z}[S]$ we get a bi-Grassmannian complex. Its bottom line is the Grassmannian complex, whose homology were studied by Suslin in the fundamental paper [Su].
Each of the three versions of the Grassmannian $n$-logarithm functions should appear as a component of the corresponding bi-Grassmannian $n$-logarithm cocycle, which is a cocycle in the complex calculating cohomology of the bi-Grassmannian with coefficients in certain complex of sheaves. These complexes are:

i) A real analog of the weight $2n$ Deligne complex on $G(n)^\bullet(\mathbb{R})$.

ii) The multivalued analytic weight $n$ Deligne complex on $G(n)^\bullet(\mathbb{C})$ considered in [BMS].

iii) The real weight $n$ Deligne complex on $G(n)^\bullet(\mathbb{C})$ — see, for example, [G1].

Here is what is known about the corresponding cocycles.

i) The real bi-Grassmannian $2n$-logarithm cocycle is the crucial building block in the Gabrielov, Gelfand and Losik [GGL] approach to a combinatorial formula for the $n$-th Pontryagin class. However such a cocycle is available only when $2n = 2$ [GGL], and, mostly, when $2n = 4$ [Yu], [G3].

ii) The existence of a multivalued analytic Grassmannian $n$-logarithm cocycle was conjectured by Beilinson, MacPherson and Schechtman [BMS]. An explicit geometric construction was found in [HM1], [HM2]. A weaker existence theorem was proved in [H]. There is an explicit motivic construction of the bi-Grassmannian $n$-logarithm cocycle for $n = 3$ [G] and $n = 4$ [G3].

iii) A single-valued bi-Grassmannian $n$-logarithm cocycle was defined in [G1], see also [G4]. It has a rather peculiar property: its components assigned to the Grassmannians $G_n^p(\mathbb{C}), m > n$ (i.e. above the bottom row in (3)) are identically zero. This is not expected to hold for the motivic/multivalued analytic bi-Grassmannian $n$-logarithm cocycles for $n > 3$.

The structure of the paper. In Section 2 we recall the scissor congruence groups $A_n(F)$, whose properies reflect the ones of the Aomoto $n$-logarithm. The functional equations of the Grassmannian $n$-logarithm stated in Theorem 1.3 follow immediately from basic properties of the Aomoto $(n − 1)$-logarithm. However Theorem 1.1, and therefore the existence of the function $L^G_n$, is less obvious. It is proved in Section 2.

In Section 3 we introduce Tate iterated integrals. Tate iterated integrals on a complex algebraic variety $X$ are certain (conjecturally all) homotopy invariant iterated integrals of 1-forms $d\log f_i$, where $f_i$ are rational functions on $X$. Denote by $\mathbb{C}(X)$ the field of rational functions on $X$. The length $n$ Tate iterated integrals are uniquely determined by their $\otimes^n$-invariants, given by elements

$$I \in \bigotimes^n \mathbb{C}(X)^* \quad (4)$$

satisfying certain integrability condition of algebraic nature. For $n = 2$ the integrability condition just means that the image of the element $I$ in $K_2(\mathbb{C}(X))$ modulo $K_2(\mathbb{C})$ is zero.

We give a simple definition of geometric variations of mixed $\mathbb{Q}$-Hodge-Tate structures — it does not rely on the theory of motives. We show that Tate iterated integrals are periods of variations of mixed Tate motives, understood as geometric variations of mixed $\mathbb{Q}$-Hodge-Tate structures. Precisely, any geometric variation of framed mixed $\mathbb{Q}$-Hodge-Tate structures on $X$ determines an element (4). We show that the elements obtained this way are exactly the integrable ones.

In Section 4 we define explicitly a Tate iterated integral on the Grassmannian $G_n(\mathbb{C})$ by exibiting its $\otimes^n$-invariant $I_n$. We prove that $I_n$ coincides with a $\otimes^n$-invariant for the form $\Omega$.

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2 Properties of the Grasmannian polylogarithms

2.1 Motivic avatar of the form $\Omega$

The scissors congruence groups $A_n(F)$. They were defined in [BMS], [BGSV]. We use slightly modified groups, adding one more relation – the dual additivity relation.

Let $F$ be a field. The group $A_n(F)$ is generated by the elements

$$\langle l_0, \ldots, l_n; m_0, \ldots, m_n \rangle_{A_n}$$

corresponding to generic configurations of $2(n+1)$ points $(l_0, \ldots, l_n; m_0, \ldots, m_n)$ in $\mathbb{P}^n(F)$. We use a notation $\langle L; M \rangle_{A_n}$ where $L = (l_0, \ldots, l_n)$ and $M = (m_0, \ldots, m_n)$. The relations, which reflect properties of the Aomoto polylogarithms, are the following:

1. **Nondegeneracy.** $\langle L; M \rangle_{A_n} = 0$ if $(l_0, \ldots, l_n)$ or $(m_0, \ldots, m_n)$ belong to a hyperplane.

2. **Skew symmetry.** $\langle \sigma L; M \rangle_{A_n} = (-1)^{|\sigma|} \langle L; \sigma M \rangle_{A_n}$ for any $\sigma \in S_{n+1}$.

3. **Additivity.** For any configuration $(l_0, \ldots, l_{n+1})$

$$\sum_{i=0}^{n+1} (-1)^i \langle l_0, \ldots, \hat{l}_i, \ldots, l_{n+1}; m_0, \ldots, m_n \rangle_{A_n} = 0,$$

and a similar condition for $(m_0, \ldots, m_{n+1})$.

4. **Dual additivity.** For any configuration $(l_0, \ldots, l_{n+1})$

$$\sum_{i=0}^{n+1} (-1)^i \langle l_i|l_0, \ldots, \hat{l}_i, \ldots, l_{n+1}; m_0, \ldots, m_n \rangle_{A_n} = 0,$$

and a similar condition for $(m_0, \ldots, m_{n+1})$.

4. **Projective invariance.** $\langle gL; gM \rangle = \langle L; M \rangle_{A_n}$ for any $g \in PGL_{n+1}(F)$.

The cross-ratio provides a canonical isomorphism

$$a_1 : A_1(F) \to F^*, \quad a_1 : \langle l_0, l_1; m_0, m_1 \rangle_{A_1} \mapsto r(l_0, l_1, m_0, m_1).$$

**Lemma 2.1** The Aomoto polylogarithm function satisfies all the above properties 1)-4).

**Proof.** Follows straight from the definitions. Notice that it is essential to use the coherent system of topological simplices $\varphi$ as representatives of the relative cycles $\Delta_M$.

The coalgebra $A_\bullet(F)$. Set $A_0(F) = \mathbb{Z}$. There is a graded coassociative coalgebra structure on $A_\bullet(F) := \oplus_{n \geq 0} A_n(F)$ with a coproduct $\nu$, see [BMS], [BGSV]. We need only one component of the coproduct:

$$\nu_{n-1,1} : A_n(F) \to A_{n-1}(F) \otimes F^*.$$  

1The coproduct $\nu$ is defined by the same formula as in loc. cit.. The combinatorial formula for the coproduct used in loc. cit. in the Hodge or $l$-adic realizations coincides with (and was motivated by) the general formula for the coproduct of framed objects in mixed categories, [G2], Appendix. Derivation of the former from the latter is a good exercise. A detailed solution of a similar problem for a different kind of scissor congruence groups is given in Theorem 4.8 in [G6]. See Section 4 there for further details.
We employ a formula for $\nu_{n-1,1}$ derived in Proposition 2.3 of [G3], which is much more convenient than the original one for computations and makes the skew-symmetry obvious. Namely, using the notation $\text{Alt}_{3,3}$ for the skewsymmetrization of $(l_0, l_1, l_2)$ as well as $(m_0, m_1, m_2)$, we have\footnote{I am grateful to the referee who pointed out that according to Proposition 2.3 in [G3] the coefficient below is $-1/8$ rather then $1/4$. This leads, up to a sign, to a coefficient $(n!)^2$ in Comparison Theorem 4.2 rather than $2(n!)^2$.}

\begin{equation}
\nu_{1,1}(l_0, l_1, l_2; m_0, m_1, m_2)_{A_2} = \end{equation}

\begin{equation}
-\frac{1}{8} \text{Alt}_{3,3} \left( \Delta(m_0, l_1, l_2) \otimes \langle m_0 | l_1, l_2; m_1, m_2 \rangle_{A_1} + \langle l_0 | l_1, l_2; m_1, m_2 \rangle_{A_1} \otimes \Delta(l_0, m_1, m_2) \right).
\end{equation}

For $n > 2$:

\begin{equation}
\nu_{n-1,1}(l_0, \ldots, l_n; m_0, \ldots, m_n)_{A_{n-1}} =
\end{equation}

\begin{equation}
- \sum_{i,j=0}^{n} (-1)^{i+j} \langle l_i | l_0, \ldots, \hat{l}_i, \ldots, l_n; m_0, \ldots, \hat{m}_j, \ldots, m_n \rangle_{A_{n-1}} \otimes \Delta(l_i, m_0, \ldots, \hat{m}_j, \ldots, m_n).
\end{equation}

It is straightforward to prove that $\nu_{n-1,1}$ is well defined, i.e. kills the relations.

The map $\nu_{n-1,1}$ and the differential of the Aomoto polylogarithm. Let $\mathbb{A}_n$ be the field of rational functions on the space of pairs of simplices in $\mathbb{C}P^n$. There is a natural map

$$A_n \otimes d \log : A_n(\mathbb{A}_n) \otimes A_n^* \longrightarrow \Omega^1_{\text{mv}},$$

$$\langle L, M \rangle \otimes F \longmapsto A_n(L, M) \ d \log(F).$$

where $\Omega^1_{\text{mv}}$ is the space of multivalued analytic 1-forms on the space of pairs of simplices in $\mathbb{C}P^n$.

Lemma 2.2. One has

$$dA_n(l_0, \ldots, l_n; m_0, \ldots, m_n) = A_{n-1} \otimes d \log \circ \nu_{n-1,1}(l_0, \ldots, l_n; m_0, \ldots, m_n)_{A_{n-1}}.$$ 

Proof. This is a very special case of the general formula for the differential of the period of a variation of Hodge-Tate structures, see Lemma 3.6 formula (20).

One can easily prove it directly as follows. We can assume that the vectors $l_0, \ldots, l_n$ form a standard basis. Let us consider a small deformation $m_i(t)$ of the vectors $m_i$, where $0 \leq t \leq \varepsilon$. By the Stokes formula, to calculate the differential of the function $A_n(l_0, \ldots, l_n; m_0(t), \ldots, m_n(t))$ we have to calculate the linear in $\varepsilon$ term of $\sum (-1)^j \int_{M_j(\varepsilon)} \Omega_L$, where $M_j(\varepsilon)$ is the $n$-dimensional body obtained by moving the $j$-th face $(m_0(t), \ldots, \hat{m}_j(t), \ldots, m_n(t))$. One can easily see that this match the $j$-th term in (6). The lemma is proved.

Motivic avatar of the form $\Omega$. Recall the notation $\mathbb{Q}(X)$ for the field of rational functions on a variety $X$ over $\mathbb{Q}$. Consider the following element of

$$\Lambda_{n-1,1}(l_1, \ldots, l_{2n}) \in A_{n-1}(\mathbb{Q}(G_n)) \otimes \mathbb{Q}(G_n)^*.$$ \hspace{1cm} (7)

$$\Lambda_{n-1,1}(l_1, \ldots, l_{2n}) := \text{Alt}_{2n}(l_1, \ldots, l_n; l_{n+1}, \ldots, l_{2n})_{A_n} \otimes \Delta(l_{n+1}, \ldots, l_{2n}).$$ \hspace{1cm} (8)

Lemma 2.3. For any $2n+1$ vectors $(l_1, \ldots, l_{2n+1})$ in generic position in $V_n$ one has

$$\sum_{i=1}^{2n+1} (-1)^i \Lambda_{n-1,1}(l_1, \ldots, \hat{l}_i, \ldots, l_{2n+1}) = 0.$$ 


For any \(2n+1\) vectors \((m_1, ..., m_{2n+1})\) in generic position in \(V_{n+1}\) one has
\[
\sum_{j=1}^{2n+1} (-1)^j \Lambda_{n-1,1}(m_j | m_1, ..., \hat{m}_j, ..., m_{2n+1}) = 0.
\]

**Proof.** The first formula reduces to the statement that
\[
\sum_{i=1}^{n+1} (-1)^i (l_1, ..., \hat{l}_i, ..., l_{n+1}; l_{n+2}, ..., l_{2n+1})_A \otimes \Delta(l_{n+2}, ..., l_{2n+1}) = 0
\]
which follows from the additivity. The second reduces to the dual additivity. The lemma is proved.

2.2 Proof of Theorems 1.1 and 1.3
Below we always work modulo 2-torsion.

We start from the following observations. Let \(A\) be a coassociative coalgebra with the coproduct \(\nu\), and \(A_+\) the kernel of the counit. Let
\[
\tilde{\nu} := \nu - (\text{Id} \otimes 1 + 1 \otimes \text{Id}) : A_+ \longrightarrow A_+^{\otimes 2}
\]
be the restricted coproduct. Then there is a map \(\nu[k] : A_+ \longrightarrow \otimes^k A_+\) given by a composition
\[
A_+ \xrightarrow{\tilde{\nu}} A_+ \otimes A_+ \xrightarrow{\tilde{\nu} \otimes \text{Id}} A_+ \otimes A_+ \otimes A_+ \xrightarrow{\tilde{\nu} \otimes \text{Id}} ... \xrightarrow{\tilde{\nu} \otimes \text{Id}} A_+^{\otimes k}.
\]
The coassociativity of \(A\) implies that one can replace anywhere here \(\tilde{\nu} \otimes \text{Id}\) by \(\text{Id} \otimes \tilde{\nu}\).

In particular, if \(A = \oplus A_n\) is graded by positive integers then we have a map (G2 or G3, p. 26):
\[
\nu[n] : A_n \longrightarrow \otimes^n A_1.
\]

**Proof of Theorem 1.1.** The proof below works equally well for \(\nu_1, 1\). One has
\[
(\nu_{n-2, 1} \otimes \text{Id}) \circ \Lambda_{n-1,1}(l_1, ..., l_n; m_1, ..., m_n) =
\]
\[
(\nu_{n-2, 1} \otimes \text{Id}) \text{Alt}_2n \left( (l_1, ..., l_n; m_1, ..., m_n)_A \otimes \Delta(m_1, ..., m_n) \right) =
\]
\[
-n^2 \cdot \text{Alt}_2n \left( (l_1 | l_2, ..., l_n; m_2, ..., m_n)_A \otimes \Delta(l_1, m_2, ..., m_n) \otimes \Delta(m_1, ..., m_n) \right).
\]
So thanks to Lemma 2.2 we need to prove that
\[
\text{Alt}_2n \left( A_{n-2}(l_1 | l_2, l_3, ..., l_n; m_2, m_3, ..., m_n) d \log \Delta(l_1, m_2, ..., m_n) \wedge d \log \Delta(m_1, ..., m_n) \right) = 0. \quad (10)
\]
We will deduce this from the following Lemma

**Lemma 2.4**
\[
\text{Alt}_2n \left( dA_{n-2}(l_1 | l_2, l_3, ..., l_n; m_2, m_3, ..., m_n) \otimes d \log \Delta(l_1, m_2, ..., m_n) \wedge d \log \Delta(m_1, ..., m_n) \right) = 0.
\]
Lemma 2.4 implies the first claim of Theorem 1.1 by the following argument: Integrating each of the 1-forms \( dA_{n-2}(l_1, l_2, \ldots, l_n; m_2, m_3, \ldots, m_n) \) we recover (10) plus a sum
\[
\sum C_{\alpha_1, \alpha_2} d \log \Delta_{\alpha_1} \wedge d \log \Delta_{\alpha_2},
\]
where \( \alpha_1 = \{l_1, m_2, \ldots, m_n\} \), \( \alpha_2 = \{m_1, \ldots, m_n\} \), and \( C_{\alpha_1, \alpha_2} \) are the integration constants. It is zero since we alternate an expression symmetric in \( (m_{n-1}, m_n) \).

**Proof of Lemma 2.4** Using (9), one has
\[
\langle n, m \rangle = 1 \quad \text{if} \quad n = m, \quad \text{and} \quad 0 \quad \text{otherwise},
\]

(11) is equal to
\[
\text{Alt}_2(n) \left( l_1, l_2 | l_3, \ldots, l_n; m_3, \ldots, m_n \right) \Delta(n, m) \Delta(1, m_2, \ldots, m_n). \tag{12}
\]

We use the following formula (G, Lemma 2.6), valid only modulo 2-torsion \(^3\).
\[
\delta\{r(v_1, v_2, v_3, v_4)\} = \frac{1}{2} \text{Alt}_4 \left( \Delta(v_1, v_2) \wedge \Delta(v_1, v_3) \right). \tag{13}
\]

We say that a single term in formula (13), say \( \Delta(v_1, v_2) \wedge \Delta(v_1, v_3) \), is obtained by choosing \( v_1 \) and forgetting \( v_4 \).

So the product of the last two factors in the expression under the alternation sign in (11) is obtained by choosing \( m_2 \) and forgetting \( l_2 \) in
\[
\delta\{r(m_3, \ldots, m_n | l_1, l_2, m_1, m_2)\}. \tag{14}
\]

1. Due to skewsymmetry, the term obtained by choosing \( m_i \) and forgetting \( l_j \), where \( i = 1, 2 \) and \( j = 1, 2 \), also appears. We use a similar argument in 2-4 below.

2. The term obtained by choosing \( m_2 \) and forgetting \( m_1 \) vanishes. This follows by applying the additivity relation to the configuration
\[
(l_1, l_2 | m_1, l_3, \ldots, l_n; m_3, \ldots, m_n).
\]

Indeed, none of the vectors \( m_1, l_3, \ldots, l_n \) enters to the last three factors (the second row below) of the expression
\[
\text{Alt}_2(n) \left( l_1, l_2 | l_3, \ldots, l_n; m_3, \ldots, m_n \right) \Delta(n, m) \wedge \Delta(l_1, m_2, m_3, \ldots, m_n) \wedge \Delta(l_2, m_2, m_3, \ldots, m_n).
\]

3. The term obtained by choosing \( l_1 \) and forgetting \( l_2 \) vanishes. This follows by applying the dual additivity relation to the configuration
\[
(l_1 | l_3, \ldots, l_n; l_2, m_3, \ldots, m_n).
\]

\(^3\)recall that we work modulo 2-torsion throughout the paper.
Indeed, the dual additivity relation provides us the first of the following two equalities:

\[ \text{Alt}_{2n}(l_1, l_2 | l_3, ..., l_n; m_3, ..., m_n)_{A_{n-2}} \otimes \]
\[ \Delta(l_1, l_2, m_3, ..., m_n) \otimes \Delta(l_1, m_1, m_3, ..., m_n) \land \Delta(l_1, m_2, m_3, ..., m_n) = \]
\[ -\sum_{k=3}^{n} (-1)^k \text{Alt}_{2n}(l_1, m_k | l_3, ..., l_n; l_2, m_3, ..., m_k, ..., m_n)_{A_{n-2}} \otimes \]
\[ \Delta(l_1, l_2, m_3, ..., m_n) \otimes \Delta(l_1, m_1, m_3, ..., m_n) \land \Delta(l_1, m_2, m_3, ..., m_n) = 0. \]

To prove the second equality, notice that the pair \((l_1, m_k), \) where \(k \geq 3,\) enters to every four factors of the last expression symmetrically, and thus the sum vanishes after the alternation.

4. The term obtained by choosing \(l_1\) and forgetting \(m_1\) vanishes. This follows by applying the additivity relation for the configuration

\((l_1, l_2 | m_1, l_3, ..., l_n; m_3, ..., m_n).\)

Indeed, none of the vectors \(m_1, l_3, ..., l_n\) enters to the last three factors (the second row below) of the expression

\[ \text{Alt}_{2n}(l_1, l_2 | l_3, ..., l_n; m_3, ..., m_n)_{A_{n-2}} \otimes \]
\[ \Delta(l_1, l_2, m_3, ..., m_n) \otimes \Delta(l_1, l_2, m_3, ..., m_n) \land \Delta(l_1, m_2, m_3, ..., m_n) \]

Lemma 2.5 and hence Lemma 2.4 and the first claim of Theorem 1.1 are proved.

The form \(\Omega\) does not change if we multiply the vector \(l_{2n}\) by a constant \(a \in \mathbb{C}^*:\)

\[ \Omega(l_1, ..., a l_{2n}) - \Omega(l_1, ..., l_{2n}) = \text{Alt}_{2n-1}(A_{n-1}(l_1, ..., l_n; l_{n+1}, ..., l_{2n})) \otimes d \log a = 0. \]

Indeed, it is easy to prove using Lemma 2.2 that \(\text{Alt}_{2n}(d A_{n-1}(l_1, ..., l_n; m_1, ..., m_n)) = 0.\) This implies the claim, just as above. Theorem 1.1 is proved.

**Conjecture 2.6** \(A_{n-1,1}(l_1, ..., l_{2n})\) does not change if one of the vectors \(l_i\) is multiplied by \(\lambda \in F^*.\) So it depends only on the configurations of \(2n\) points in \(\mathbb{P}^{n-1}\) defined by the vectors \(l_i.\)

**Proof of Theorem 1.3** Applying the map \(A_{n-1} \otimes d \log\) to the element (7) we get the form \(\Omega.\) Therefore the proof follows from Lemma 2.3.

### 3 Tate iterated integrals and variations of mixed Tate motives

**Iterated integrals of smooth 1-forms.** Let \(M\) be a manifold. Let \(\omega_1, ..., \omega_n\) be smooth 1-forms on \(M.\) Then given a path \(\gamma : [0, 1] \rightarrow M\) there is an iterated integral

\[ \int_{\gamma} \omega_1 \circ ... \circ \omega_n := \int_{0 \leq t_1 \leq ... \leq t_n \leq 1} \gamma^* \omega_1(t_1) \land ... \land \gamma^* \omega_n(t_n). \quad (15) \]

Let \((A^*(M), d)\) be the commutative DG algebra of smooth forms on \(M.\) By linearity an element

\[ I \in \bigotimes^n (A^1(M)[1]) := A^1(M)[1] \otimes ... \otimes A^1(M)[1] \]

give rise to an iterated integral \(\int_{\gamma} (I).\)
Homotopy invariant iterated integrals. Denote by \( T(A) \) the tensor algebra of the graded vector space \( A \). The bar complex of the commutative DG algebra \( A^*(M) \) is defined as \( T(A^*(M)[1]) \) equipped with a differential

\[
D : T(A^*(M)[1]) \rightarrow T(A^*(M)[1]).
\]

The differential is the sum of the de Rham differential \( d \) and the maps given by the products of the consecutive factors in the tensor product. A theorem of K.T. Chen \([Ch]\) tells that an iterated integral \( \int_\gamma(I) \) is homotopy invariant, i.e. invariant under deformations of the path \( \gamma \) preserving its endpoints, if and only if \( D(I) = 0 \).

In particular, a collection of closed 1-forms \( \omega_1^{(s)} \) such that for every \( 1 \leq k \leq n-1 \) one has

\[
\sum_s \int_\gamma \omega_1^{(s)} \otimes \ldots \otimes \omega_k^{(s)} \otimes \omega_{k+1}^{(s)} \otimes \ldots \otimes \omega_n^{(s)} = 0 \tag{16}
\]
gives rise to a homotopy invariant integrated integral \( \sum_s \int_\gamma \omega_1^{(s)} \otimes \ldots \otimes \omega_n^{(s)} \).

Tate iterated integrals. Now let \( X \) be a complex algebraic variety. Recall that \( \mathbb{C}(X) \) is the field of rational functions on \( X \). Our goal is to study iterated integrals of 1-forms \( d \log f_i \) where \( f_i \) are rational functions on \( X \). There is a map

\[
d \log : \bigotimes^n \mathbb{C}(X)^* \longrightarrow \Omega^n_{\log}(X), \quad f_1 \otimes \ldots \otimes f_n \longmapsto d \log f_1 \wedge \ldots \wedge d \log f_n.
\]

So given a path \( \gamma : [0,1] \rightarrow X(\mathbb{C}) \) in \( X(\mathbb{C}) \) and

\[
I = f_1(x) \otimes \ldots \otimes f_n(x) \in \bigotimes^n \mathbb{C}(X)^*
\]

there is an iterated integral

\[
\int_\gamma d \log(I) = \int_\gamma d \log f_1 \circ d \log f_2 \circ \ldots \circ d \log f_n.
\]

The forms \( d \log f \) are closed. So condition \([16]\) implies the homotopy invariance of the corresponding iterated integral. There is an algebraic condition on the functions \( f_i \) which implies condition \([16]\), and which is hypothetically equivalent to it. The key point is this. The map \( d \log \) annihilates the Steinberg element \( (1 - f) \otimes f \). Conjecturally the ideal generated by the Steinberg elements and constants is the kernel of the map \( d \log \).

This leads to the following definitions. Let \( F \) be a field. There is a natural projection

\[
\pi : F^* \otimes F^* \longrightarrow K_2(F), \quad a \otimes b \longmapsto \{a, b\}.
\]

For \( 1 \leq k \leq n-1 \) let \( \pi_{k,n} \) be a map obtained by applying \( \pi \) to the \( k \)-th factor \( \otimes^k F^* \) in \( \otimes^n F^* \):

\[
\pi_{k,n} : \bigotimes F^* \longrightarrow \bigotimes F^* \otimes K_2(F) \otimes \bigotimes F^*, \quad \pi_k = \text{Id} \otimes \pi \otimes \text{Id}.
\]

**Definition 3.1** An element \( I \) of \( \otimes^n F^* \) is integrable if \( \pi_{k,n}(I) = 0 \) for every \( 1 \leq k \leq n-1 \).
Definition 3.2 A Tate iterated integral is an iterated integral provided by an integrable element

\[ I \in \bigotimes^n C(X)^*. \]  

(17)

Chen’s theorem immediately implies the following Lemma.

Lemma 3.3 A Tate iterated integral is homotopy invariant.

Let \( \text{div}(I) \subset X \) be the union of divisors of the factors \( f_i \) of an element \( E \). Then the iterated integral provided by (17) is an iterated integral on \( X(\mathbb{C}) - \text{div}(I) \).

Tate iterated integrals and periods of geometric variations of Hodge-Tate structures. Below we use the notion of variations framed Hodge-Tate structures. The key point is that it is a mixed Tate category, so one can use the formalism, see Appendix in [G2]. We also need the notion of the period of a variation of framed Hodge-Tate structures, see Section 4 of [G6].

A variation \( V \) of Hodge-Tate structure has a weight filtration \( W^i \) whose subquotients \( \text{gr}^W_{-2m+1} V = 0 \). A \( (\mathbb{Q}(m-1)_X, \mathbb{Q}(m)_X) \)-framing on \( V \) consists of choosing non-zero maps

\[ \mathbb{Q}(m-1)_X \rightarrow \text{gr}^W_{-2m+2} V, \quad \text{gr}^W_{-2m} V \rightarrow \mathbb{Q}(m)_X. \]

So it gives rise to an element of \( \text{Ext}^1_{\mathbb{Q}^{-\text{MHS}}}(\mathbb{Q}(0)_X, \mathbb{Q}(1)_X) \). There is a natural map

\[ O(X)_\mathbb{Q}^* \rightarrow \text{Ext}^1_{\mathbb{Q}^{-\text{MHS}}}(\mathbb{Q}(0)_X, \mathbb{Q}(1)_X) \]  

(18)

where the Ext group is in the category of variations of \( \mathbb{Q} \)-Hodge-Tate structures on \( X(\mathbb{C}) \).

Definition 3.4 A variation of framed \( \mathbb{Q} \)-Hodge-Tate structures on \( X(\mathbb{C}) \) is geometric if the Ext\(^1\) defined by any \( (\mathbb{Q}(m-1)_X, \mathbb{Q}(m)_X) \)-framing is in the image of map (18).

Theorem 3.5 a) The period of a geometric\(^4\) variation of framed \( \mathbb{Q} \)-Hodge-Tate structures on \( X(\mathbb{C}) \) is given by a Tate iterated integral.

b) Assume that \( X \) is a rational variety. Then a Tate iterated integral on \( X \) is a period of a motivic, that is realized in cohomology of algebraic varieties, variation of framed \( \mathbb{Q} \)-Hodge-Tate structures.

Proof. a) The Tannakian Hopf algebra of the category of geometric variations of framed \( \mathbb{Q} \)-Hodge-Tate structures at the generic point of \( X \) is commutative graded Hopf algebra \( \mathcal{H}_\bullet(\mathbb{C}(X)) \), graded by the non-negative integers. One has (Appendix in [G2])

\[ \mathcal{H}_0(\mathbb{C}(X)) = \mathbb{Q}, \quad \mathcal{H}_1(\mathbb{C}(X)) = \mathbb{C}(X)_{\mathbb{Q}}^*. \]

So there is a map

\[ \nu_{[n]} : \mathcal{H}_n(\mathbb{C}(X)) \rightarrow \otimes^n \mathbb{C}(X)^*. \]

Recall (loc. cit.) that a geometric variation of \( (\mathbb{Q}(0), \mathbb{Q}(n)) \)-framed mixed \( \mathbb{Q} \)-Hodge-Tate structures at the generic point of \( X \) provides us an element

\[ I \in \mathcal{H}_n(\mathbb{C}(X)). \]  

(19)

\(^4\)in the sense of Definition 3.4
Different variations may lead to the same element. Let \( p(\mathcal{V}) \) be the multivalued analytic function at the generic point of \( X(\mathbb{C}) \) given by the period of a framed variation \( \mathcal{V} \). The period functions assigned to variations with the same invariant \( 19 \) are the same.

Therefore there is a map \( p \otimes d \log \) from \( \mathcal{H}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)_Q^* \) to multivalued analytic 1-forms at the generic point of \( X(\mathbb{C}) \). Consider the \((n-1,1)\)-component of the coproduct

\[
\nu_{n-1,1} : \mathcal{H}_n(\mathbb{C}(X)) \to \mathcal{H}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)_Q^*.
\]

**Lemma 3.6** The differential of the period \( p(I) \) of a framed Hodge-Tate variation \( I \) is given by

\[
dp(I) = p \otimes d \log \left( \nu_{n-1,1}(I) \right).
\]

The iterated integral for the period \( p(I) \) is described by the integrable element

\[
\nu_{[n]}(I) \in \bigotimes^n \mathbb{C}(X)_Q^*.
\]

**Proof.** The first claim is Lemma 4.6a) in [G6].
To see the integrability of element \( 20 \) we present the map \( \nu_{[n]} \) as a composition:

\[
\mathcal{H}_n \to \cdots \to \bigotimes^{n-k-1} \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \bigotimes^n \mathcal{H}_1.
\]

Now we use the following argument. The composition below is zero:

\[
\mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_1 = \mathbb{C}_Q^* \otimes \mathbb{C}_Q^* \to K_2(\mathbb{C}).
\]

Then there is a commutative diagram, where the \( \mathbb{Z}[\mathbb{C}] \)-component of the left vertical map is provided by the framed Hodge-Tate structure corresponding to the dilogarithm iterated integral:

\[
\begin{array}{ccc}
\mathcal{H}_2 & \to & \mathcal{H}_1 \otimes \mathcal{H}_1 = \mathbb{C}_Q^* \otimes \mathbb{C}_Q^* \\
\downarrow & & \downarrow \\
\mathcal{H}_2 & \to & \mathbb{C}_Q^* \otimes \mathbb{C}_Q^*
\end{array}
\]

Here the top differential is given by \( \delta \) on \( \mathbb{Z}[\mathbb{C}] \), and by \( a \cdot b \mapsto a \otimes b + b \otimes a \) on \( \mathbb{S}_2 \mathbb{C}_Q^* \). The left arrow is supposed to be surjective, but we do not use it. The claim follows from this.

b) A Tate iterated integral \( \int_\gamma d \log(I) \) assigned to an integrable element \( 17 \) is a period of the framed mixed Hodge structure provided by Beilinson’s construction, see [DG]. Namely, take the mixed Hodge structure \( \mathcal{P}(X; x, y) \) on the pronilpotent torsor of path between the base points \( x, y \). The framing is given by the cohomology class \( d \log(I) \) and the relative homology class provided by a homotopy class of a path \( \gamma \) between \( x \) and \( y \). By the very construction, \( \int_\gamma d \log(I) \) is the period of a variation of framed mixed Hodge structures realized in cohomology of algebraic varieties. Finally, it is Hodge-Tate since \( X \) is rational, and hence its cohomology are of type \((p, p)\).

The theorem is proved.

**Remark.** We do not prove that the variation corresponding to a Tate iterated integral is geometric in the sense of Definition 3.4.

If \( X \) is not rational, in general the variation will not be a variation of Hodge-Tate structures. However we conjecture that the framed variation is equivalent to a framed Hodge-Tate variation:

\footnote{It is enough to work with \( \mathbb{C} \) rather then \( \mathbb{C}(X) \); or just use \( \mathbb{C}(X) \) below.}
Conjecture 3.7  The Tate iterated integral \( \int d\log(1) \) given by \([17]\) is the period of a variation of framed mixed \( \mathbb{Q} \)-Hodge-Tate structures on \((X - \text{div}(I))^2\).

Notice that for the length one and two iterated integrals (i.e. \( n = 1, 2 \) in \([17]\)) this is obvious. So \( n = 3 \) is the first non-trivial case.

4  Tate iterated integral for the Grassmannian polylogarithm

Definition 4.1  An element \( I_n(l_1, \ldots, l_{2n}) \in \bigotimes^n \mathcal{O}(G_n)^* \) is given by the formula

\[
I_n(l_1, \ldots, l_{2n}) := \text{Alt}_{2n} \left( \Delta(l_1, \ldots, l_{n-1}, l_n) \otimes \Delta(l_2, \ldots, l_{n+1}) \otimes \cdots \otimes \Delta(l_n, \ldots, l_{2n-1}) \right). \tag{21}
\]

Comparison Theorem. It relates \( \Lambda_{n-1,1} \) and \( I_n \). Observe that it is sufficient to know \( \nu_{n-1,1} \) in order to compute \( \nu_{[n]} \).

Theorem 4.2  One has

\[
(\nu_{[n-1]} \otimes \text{Id}) \circ \Lambda_{n-1,1}(l_1, \ldots, l_n, m_1, \ldots, m_n) = (-1)^n (n!)^2 I_n(l_1, \ldots, l_n; m_1, \ldots, m_n). \]

Proof. Using (8) and (11), and continuing the same line, we come to the expression

\[
(-1)^{n-3} n^2 \cdot \ldots \cdot 4^2 \cdot \text{Alt}_{2n} \left( \langle l_1, \ldots, l_{n-3}, l_{n-2}, l_{n-1}, l_n; m_{n-2}, m_{n-1}, m_n \rangle A_1 \right) \otimes \\
\text{Alt}_{2n} \left( \langle l_1, \ldots, l_{n-3}, l_{n-2}, l_{n-1}, l_n; m_{n-2}, m_{n-1}, m_n \rangle A_2 \right) \otimes \\
\Delta(l_1, \ldots, l_{n-3}, m_{n-2}, m_{n-1}, m_n) \otimes \cdots \otimes \Delta(m_1, \ldots, m_n).
\]

Taking into account formula (5) for \( \nu_{1,1} \), we get

\[
(-1)^{n-2} n^2 \cdot \ldots \cdot 4^2 \cdot \frac{3^2}{2} \text{Alt}_{2n} \left( \Delta(l_1, \ldots, l_{n-3}, m_{n-2}, l_{n-1}, l_n) \otimes \langle l_1, \ldots, l_{n-3}, m_{n-2} l_{n-1}, l_n; m_{n-1}, m_n \rangle A_1 \right) \otimes \\
\Delta(l_1, \ldots, l_{n-3}, m_{n-2}, m_{n-1}, m_n) \otimes \cdots \otimes \Delta(m_1, m_2, \ldots, m_n) + \\
\langle l_1, \ldots, l_{n-2} l_{n-1}, l_n; m_{n-1}, m_n \rangle A_1 \otimes \Delta(l_1, \ldots, l_{n-2}, m_{n-1}, m_n) \otimes \cdots \otimes \Delta(m_1, m_2, \ldots, m_n). \tag{23}
\]

Using the formula

\[
\langle l_1, \ldots, l_{n-2} l_{n-1}, l_n; m_{n-1}, m_n \rangle A_1 = \frac{\Delta(l_1, \ldots, l_{n-2}, l_{n-1}, m_{n-1}) \Delta(l_1, \ldots, l_{n-2}, l_n; m_{n-2}, m_{n-1}, m_n)}{\Delta(l_1, \ldots, l_{n-2}, l_{n-1}, m_{n}) \Delta(l_1, \ldots, l_{n-2}, l_n; m_{n-1})} \tag{24}
\]

we write the term (22) as follows

\[
(-1)^{n} \frac{1}{2} (n!)^2 \text{Alt}_{2n} \left( \Delta(l_1, \ldots, l_{n-3}, m_{n-2}, l_{n-1}, l_n) \otimes \Delta(l_1, \ldots, l_{n-3}, m_{n-2}, l_{n-1}, l_n) \right) \otimes \\
\Delta(l_1, \ldots, l_{n-3}, m_{n-2}, m_{n-1}, m_n) \otimes \cdots \otimes \Delta(m_1, m_2, \ldots, m_n) = \\
(-1)^{n} \frac{1}{2} (n!)^2 \text{Alt}_{2n} \left( \Delta(l_1, \ldots, l_{n-1}, m_n) \otimes \cdots \otimes \Delta(m_1, m_2, \ldots, m_n) \right). \tag{25}
\]

In the last step we use the fact that each of the permutations \((l_{n-2}, l_{n-1}, l_n) \rightarrow (l_n, l_{n-2}, l_{n-1})\) and \((m_{n-2}, m_{n-1}, m_n) \rightarrow (m_n, m_{n-2}, m_{n-1})\) are even.

It is easy to see using (24) that (23) also equals (25). The theorem is proved.
Theorem 4.3  a) The element $I_n$ is integrable.

b) It leaves on $PG_n$, and satisfies two $(2n+1)$-term relations:
1) For a generic configuration of $2n+1$ vectors $(l_1, ..., l_{2n+1})$ in $V_n$ one has

$$\sum_{i=1}^{2n+1} (-1)^i I_n(l_1, ..., \hat{l}_i, ..., l_{2n+1}) = 0.$$  \hfill (26)

2) For a generic configuration of vectors $(m_1, ..., m_{2n+1})$ in $V_{n+1}$ one has

$$\sum_{j=1}^{2n+1} (-1)^i I_n(m_j|m_1, ..., \hat{m}_j, ..., m_{2n+1}) = 0.$$  \hfill (27)

**Proof.** a) Follows easily from Lemma 2.5 by using Comparison Theorem 4.2.

b) Changing the vector $l_1$ to $al_1$ we get

$$I_n(al_1, ... l_{2n}) - I_n(l_1, ... l_{2n}) = \text{Alt}_{2n}(a \otimes \Delta(l_2, ..., l_{n+1}) \otimes \cdots \otimes \Delta(l_n, ..., l_{2n-1})) = 0.$$  

Indeed, we skewsymmetrize an expression which does not contain the pair of vectors $(l_1, l_{2n})$.

The two relations follow immediately from Comparison Theorem 4.2 and Lemma 2.3.

**Conclusion.** The iterated integral assigned to the element $I_n$ is a multivalued analytic function on $G_n(\mathbb{C}) \times G_n(\mathbb{C})$. By Theorem 4.3 it is the period of a motivic variation of framed Hodge-Tate structures on $G_n(\mathbb{C}) \times G_n(\mathbb{C})$.

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