BLOW-UP OF SOLUTIONS TO THE PERIODIC GENERALIZED MODIFIED CAMASSA-HOLM EQUATION WITH VARYING LINEAR DISPERSION

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Abstract. Considered herein is the blow-up mechanism to the periodic generalized modified Camassa-Holm equation with varying linear dispersion. The first one is designed for the case when linear dispersion is absent and derive a finite-time blow-up result. The key feature is the ratio between solution and its gradient. The second one handles the general situation when the weak linear dispersion is at present. Fortunately, there exist some conserved quantities that bound the $\|u_x\|_{L^4}$ for the periodic generalized modified Camassa-Holm equation, then the breakdown mechanisms are set up for the general case.

1. Introduction. This paper is to study the following periodic generalized modified Camassa-Holm (gmCH) equation.

$$
\begin{aligned}
&m_t + k_1((u^2 - u_x^2)m_x) + k_2(2u_xm + um_x) + \gamma u_x = 0, \quad x \in \mathbb{S}, t \in [0, T) \\
&m = u - u_{xx}, \\
&u(0, t) = u(1, t).
\end{aligned}
$$

(1.1)

where $u(x, t)$ is a horizontal velocity, which is obtained by applying tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation. The equation (1.1) would be reduced to the modified Camassa-Holm (mCH) equation when $k_1 = 1, k_2 = 0$, and the Camassa-Holm (CH) equation when $k_1 = 0, k_2 = 1$.

We would first like to review some basic integrability properties of the three equations, CH equation, mCH equation and gmCH equation.

The CH equation

$$m_t + um_x + 2mu_x = 0, \quad m = u - u_{xx},$$

was also proposed as a model describing the uni-directional propagation of shallow water waves over a flat bottom \cite{6, 17, 25}. It also models the propagation of axially symmetric waves in hyperelastic rods \cite{20, 21}.

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The CH equation is completely integrable for a large class of initial data, for which it can be solved by the inverse scattering method \[10, 18\]. In contrast to the KdV equation, the CH equation has three remarkable distinctive properties. First, the CH equation is completely integrable, it can describe wave breaking phenomenon: the solution remains bounded while its slope becomes infinite in finite time. The second is the existence of peakons, which are nonanalytic solitary waves that are global weak solutions and interact cleanly like solitons. Indeed, the CH equation has the single peakon \[6\] and the multi-peakon solutions \[27\]. It is significant that the peakons are orbitally stable: the shape is stable under small perturbations \[19, 29\]. These peakons capture a feature of the waves of greatest height for the free-boundary incompressible Euler equations \[11, 15, 40\]. The last one is the variety of interesting geometric formulations of the CH equation \[8, 16, 28, 33\].

Well-posedness and wave breaking of the CH equation were studied in a number of papers. It has been shown \[13, 30, 39\] that the Cauchy problem is locally well-posed for initial data \(u_0 \in H^s(\mathbb{R})\) with \(s > 3/2\). Moreover, if the initial momentum density

\[
m_0(x) = m(0, x) = (1 - \partial_x^2)u_0 = u_0(x) - u_0''(x)
\]

does not change sign, the Cauchy problem admits global solution for certain initial values \[9, 13, 14\], whereas solutions may blow up if their initial momentum density \(m_0\) changes sign \[9, 12, 13, 14\]. After blow-up, the solutions to CH can be continued uniquely as global weak solutions \[4, 5\]. Moreover, the existence of global weak solution was investigated in \[41, 42\].

Like the KdV equation, the CH equation has quadratic nonlinear terms. It is of great interest to find such integrable equations with cubic or higher nonlinear terms. Two CH-type equations with cubic nonlinearities have been proposed: the mCH equation \[44\],

\[
m_t + (u^2 - u_x^2)m_x + \gamma u_x = 0.
\]

(1.2)

and the Novikov equation \[34\]

\[
m_t + u^2 m_x + 3m u_x = 0, \quad m = u - u_{xx}.
\]

Both equations have peaked solitons and can be used to model wave breaking.

The geometric formulation, integrability, local well-posedness, blow-up criteria and wave breaking, existence of peaked solitons (peakons), and the stability of single peakons and periodic peakons to the mCH equation were studied recently in \[7, 24, 26, 31, 36, 37\]. It is shown that even when the initial momentum density \(m_0(x)\) does not change sign, the solutions to the Cauchy problem mCH may blow up in finite time, in contrast to the CH equation, the Degasperis-Procesi (DP) equation \[22, 23, 32\], or Novikov equation. Recently, local-in-space blow up criterion has been presented by Brandolese et al \[11, 2, 3\]. They unify a few of well-known results on wave breaking for the CH equation. This new type criterion is only involving the properties of the initial data in a neighborhood of a single spatial point, which means local perturbation of data around that point does not prevent the singularity formation. The method they used is to analyze two families of Lyapunov functions \(\beta u \pm u_x\) along the characteristics and to estimate the non-local term by the local quantities. Motivated by those idea, we may consider local-in-space blow up criterion for the mCH equation.

The mCH equation (1.2) is formally integrable and can be rewritten as the bi-Hamiltonian form \[35\]. Moreover, the mCH equation exhibits new features, including wave breaking and blowup criteria that do not appear in the original CH
equation. On the other hand, since the mCH equation (1.2) also arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [26], it can be regarded as a Euclidean-invariant counterpart to the KdV equation from the viewpoint of curve flows in Klein geometries [8]. It is worth mentioning that the authors [26] showed the scaling limit equation of mCH equation, when combined with the first-order term $\gamma u_x$, satisfies the short-pulse equation

$$v_{xt} = \frac{1}{3}(v^3)_{xx} + \gamma v,$$

which is a model for the propagation of ultra-short light pulses in silica optical fibres.

As an extension of both CH equation and the mCH equation, and integrable equation with both quadratic and cubic nonlinearities has been introduced by Fokas, which is the gmCH equation.

There is a distinctive feature for gmCH equation is that it is a integrable model for the breakdown of regularity. Moreover, it admits a remarkable variety of the so-called “peakon” solutions, which means wave solutions with a discontinuous derivative at crest. Physically, those peakons reveal some similarity to the well-known Storkes waves of greatest height – the traveling waves of maximum possible amplitude that are solutions to the governing equations for irrotational water waves.

The gmCH equation (1.1) also admits the Lax pair and has the bi-Hamiltonian form

$$m_t = J \frac{\delta H_1}{\delta m} = K \frac{\delta H_2}{\delta m}$$

where

$$J = -k_1 \partial_x m \partial_x^{-1} m \partial_x - \frac{1}{2} k_2 (m \partial_x + \partial_x m) - \frac{1}{2} \gamma \partial_x,$$

and

$$k = \frac{1}{4} (\partial_x^3 - \partial_x)$$

$$H_1[u] = \int_S u m dx, \quad H_2[u] = k_1 I_1 + 2 k_2 I_2 + 2 \gamma \int_S u^2 dx,$$

(1.3)

with

$$I_1 = \int_S (u^4 + 2 u^2 u_x^2 - \frac{1}{3} u_x^4) dx, \quad I_2 = \int_S (u^3 + u u_x^2) dx.$$ 

Unlike the semilinear dispersive system, for instance, the KdV and Schrodinger equations, where in many cases the linear dispersion dominates over the nonlinearity and a contraction principle can be applied even in a very low-regularity regime to obtain well-posedness, and consequently the smoothness of the solution propagates with the help of conservation laws, the equations considered here are all quasilinear with weak linear dispersion, suggesting that well-posedness can only be established in a high regularity regime, and the initial profile can determine the existence time and the regularity of the solution map rather strongly.

From the transport theory, the blow-up criteria assert that singularities are caused by the focusing of characteristics, which involve the information on the gradient $u_x$ and $m$. The dynamics of the blow-up quantity $B = B(u, u_x, m)$ along the characteristics is governed by an equation

$$B'(t) \leq -B^2 + f(u, m)(C^2 u^2 - u_x^2) + \Phi(u, u_x, m) + \Psi_\gamma,$$

(1.4)

where $\Phi(u, u_x, m)$ is the nonlocal part, which usually consists of convolutions against the kernel $G(x) = \frac{\cosh(x-|x|)}{2\sinh(x/2)}$, the fundamental solution of $(1-\partial_x^2)^{-1}$ on $S$, and $\Psi_\gamma$ comes from the weak linear dispersion and is of lower order. In many classical cases there is no linear dispersion, and then $\Psi_\gamma = 0$. Standard approaches seek certain
conservation laws (such as the conservation of the $H^1$-norm of $u$, the persistence of the sign of $m$, etc.) to control the local quantities involving $u, m$ as well as the nonlocal term $\Phi$ by constants. In particular, for the CH and DP equations, the term $f(u, m)(C^2u^2 - u_x^2)$ can be replaced by a function $f(u)$ depending solely on $u$, then the Riccati-type inequality $B' \leq -B^2 + C$, which leads to a finite-time blow-up provided $B$ is sufficiently negative initially. As such approaches make an intensive use of the “global” information of solutions, the blow-up mechanism ignores the local structure of the solution.

Recently Brandolese and Cortez introduced a new type of blow-up criteria in the study of the CH-type equations which highlights how local structure of the solution affects the blow-ups. Their argument relies that the convolution terms are quadratic and positively definite, for gmCH equations, the convolution contains cubic terms which do not have a lower bound in terms of the local terms. For this reason, it is not clear whether a purely local condition on the initial data can generate finite-time blow-ups.

We present two different ideas to investigate the breakdown mechanism of equation (1.1). The first approach deals particularly with the dispersionless situation $\gamma = 0$. We look for some global property, namely the sign persistence of momentum density $m$, to bound the nonlocal term $\Phi$. Thus from (1.4), the blow-up can be deduced by the interplay between $u$ and $u_x$. More precisely, this motivates us carry out a refined analysis of the characteristic dynamics of $M = Cu - u_x$ and $N = Cu + u_x$. This way are not available for $M$ and $N$. But note that an alternative way to show that $C^2u^2 - u_x^2$ is to track the relative ratio $|u_x|/u$, and to prove that the ratio stays sufficiently large. Intuitively, one would expect that fast oscillation causes breakdown of solutions. It turns out that the dynamics of $\frac{u}{u}$ can be put in a rather clean form. However the inhomogeneity of the nonlinearities in the equation makes it difficult to extract a clear ratio condition out of the estimate. By performing a vertical shift of the solution, we are able to make the estimates homogeneous, which in turn provides the desired ratio property.

The other approach we adopt similarity the general situations with weak linear dispersion. There are two difficulties in the approach. Firstly, the convolution part in $\Phi$ contains cubic nonlinearities in $u_x$, and thus it requires some higher-order conservation law. Fortunately, there exist some conserved quantities that bound the $\|u_x\|_{L^4}$ for gmCH equation. Then the cubic terms in the convolution can be control with the $\|u_x\|_{L^4}$ and the $H^1$ estimates of $u$.

The second difficulty is the blow-up quantity for the gmCH equation is $B = (k_1 m + k_2)u_x$, in which both $m$ and $u_x$ can potentially blow-up in finite time. By checking their dynamics individually we find that the dynamic equation for $m$ has a very simple structure $m'(t) = q(m)u_x$, where $q(m)$ is a quadratic polynomial in $m$. Moreover, although the equation for $u'_x$ still contains local terms involving $u_x$, it can be made of a definite sign with the help of the conservation laws. This way $u_x$ will be monotone. With appropriate choice of initial data, the later dynamics of $m$ satisfies $m'(t) \geq q(m)$. Solving this differential inequality one obtains a finite-time blow-up of $m$. Due to the monotonicity of $u_x$ and the proper choice of its initial value, $u_x$ can be made uniformly away from zero. Hence $B$ blow up.

The rest of the paper is organized as follow. Section 2: some preliminary estimates and results are recalled and presented. In Section 3, the blow-up results in the absence of the weak dispersion are illustrated for the periodic gmCH equation.
Section 4 the breakdown mechanisms are set up to the periodic gmCH equation for the general case when the weak linear dispersion is at present.

2. Preliminaries. In this section, we recall some basic results concerning the formation of singularities in the mCH equation.

Denote \( G(x) = \frac{\cosh(x - \lfloor x \rfloor - \frac{1}{2})}{2\sinh(\frac{1}{2})} \), the fundamental solution of \( 1 - \partial_x^2 \) on \( \mathbb{S} \), that is, \( (1 - \partial_x^2)^{-1}f = G * f \), we have the relation

\[
G * f(t,x) = \int_0^1 \frac{\cosh ((x-y) - \lfloor x-y \rfloor - \frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y)dy = \int_0^x \frac{\cosh (x-y - \frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y)dy + \int_x^1 \frac{\cosh (x-y + \frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y)dy.
\]

(2.1)

Similar to the proof of Theorem 6.2 in [38]. We have the following blow-up criterion for the gmCH equation.

Lemma 2.1. Suppose that \( u_0 \in H^s(\mathbb{S}) \) with \( s > \frac{5}{2} \). Then the corresponding solution \( u \) to the initial value problem blow-up in finite time \( T > 0 \) if and only if

\[
\lim_{t \to T^\inf} \inf_{x \in \mathbb{S}} (k_1 m(t,x) + k_2)u_x(t,x) = -\infty.
\]

Lemma 2.2. [43] For every \( f \in H^1(\mathbb{S}) \), we have

\[
\max_{x \in [0,1]} f^2(x) \leq C_1 \int_{\mathbb{S}} (f^2 + \alpha^2 f_x^2)dx,
\]

where

\[
C_1 = \frac{\cosh(\frac{1}{2\alpha})}{2\alpha \sinh(\frac{1}{2\alpha})}.
\]

Moreover \( C_1 \) is the minimum value, so in this sense, \( C_1 \) is the optimal constant which is obtained by the associated Green function.

\[
G = \frac{\cosh(\frac{x}{\alpha} - \lfloor x \rfloor - 1/2)}{2\alpha \sinh(1/2\alpha)}.
\]

Note: When \( \alpha = 1 \), the constant \( C_1 = \frac{e + 1}{2(e - 1)} \) is the optimal constant.

Lemma 2.3. [23] For all \( f \in H^1(\mathbb{S}) \), the following inequality holds

\[
G * (u^2 + \frac{1}{2} f_x^2) \geq \kappa u^2(x),
\]

with

\[
\kappa = \frac{1}{2} + \frac{\text{arctan} (\sinh(1/2))}{2\sinh(1/2) + 2\text{arctan} (\sinh(1/2)) \sinh^2(1/2)} \approx 0.869.
\]

Moreover, \( \kappa \) is the optimal constant obtained by the function

\[
f_0 = \frac{1 + \text{arctan} (\sinh(x - \lfloor x \rfloor - 1/2)) \sinh(x - \lfloor x \rfloor - 1/2)}{1 + \text{arctan} (\sinh(1/2)) \sinh(1/2)}.
\]
Lemma 2.4. [44] \( m_0 \in H^s(\mathbb{S}) \) with \( s \geq 3 \), \( m_0 \geq 0 \) for all \( x \in \mathbb{S} \) and \( \gamma = 0 \). Let \( T > 0 \) be the maximal time of existence of the solution \( m(t, x) \) to the periodic mCH equation with initial data \( m_0 \). Then

\[
|u_x(t, x)| \leq u(t, x).
\]

If \( m_0 \in H^s(\mathbb{S}) \cap L^1(\mathbb{S}) \), then

\[
u(t, x) \leq \frac{\cosh(\frac{1}{2})}{2\sinh(\frac{1}{2})} H_1[u_0].
\]

Moreover,

\[
|u_x(t, x)| \leq \frac{1}{2} H_1[u_0],
\]

for all \((t, x) \in [0, T) \times \mathbb{S} \).

Note: The conclusion in Lemma 2.4 is correct to gmCH equation.

3. Dynamics along the characteristics. In this section we focus on the finite-time blow-up of waves for the periodic gmCH equation. The characteristics associated to the periodic gmCH equation is determined as follows

\[
\begin{cases}
\frac{dq(t, x)}{dt} = (k_1(u^2 - u_x^2) + k_2u)(t, q(t, x)), & x \in \mathbb{S}, t \in [0, T) \\
q(0, x) = x,
\end{cases}
\]

(3.1)

We now compute the dynamics of some important quantities along the characteristic \( q(t, x) \) associated to the gmCH equation. Where \( \ell \) denote the derivative \( \partial_t + |k_1(u^2 - u_x^2) + k_2u| \partial_x \) along the characteristic.

Theorem 3.1. Let \( u_0 \in H^s(\mathbb{S}), s > 3 \). Then \( u(t, q(t, x)), u_x(t, q(t, x)) \) and \( m(t, q(t, x)) \) satisfy the following integro-differential equations

\[
u'(t) = -\gamma G \ast u_x - \frac{2}{3} k_1 u_x^3 - \frac{1}{12 \sinh(\frac{1}{2})} \times (e^{x-\frac{1}{2}} \int_0^x e^{-y} (u + u_y)^3 dy - e^{\frac{1}{2}-x} \int_0^x e^y (u - u_y)^3 dy
\]

\[
+ e^{x+\frac{1}{2}} \int_x^1 e^{-y} (u + u_y)^3 dy - e^{-x-\frac{1}{2}} \int_x^1 e^y (u - u_y)^3 dy)
\]

\[
- k_2 G_x \ast (u^2 + \frac{1}{2} u_x^2).
\]

\[
u_x'(t) = k_1(\frac{1}{3} u^3 - u_x^2) + k_2(u^2 - \frac{1}{2} u_x^2) + \gamma(u - G \ast u) - k_2 G \ast (u^2 + \frac{1}{2} u_x^2)
\]

\[
- \frac{k_1}{12 \sinh(\frac{1}{2})} (e^{x-\frac{1}{2}} \int_0^x e^{-y} (u + u_y)^3 dy + e^{\frac{1}{2}-x} \int_0^x e^y (u - u_y)^3 dy
\]

\[
+ e^{x+\frac{1}{2}} \int_x^1 e^{-y} (u + u_y)^3 dy + e^{-x-\frac{1}{2}} \int_x^1 e^y (u - u_y)^3 dy).
\]

and

\[
m'(t) = -2k_1 u_x m^2 - 2k_2 u_x m - \gamma u_x.
\]
Moreover, we can obtain according (1.1), we have
\[ (1 - \partial_x^2)(u_t + (k_1(u^2 - u_x^2) + k_2 u)_x) \]
\[ = k_1[(-((u^2 - u_x^2)m)_x + u_x(u^2 - u_x^2) - (u_xu(u^2 - u_x^2))_x - 2(u_x^2)m)_x] \]
\[ + k_2[-2uu_x - uu - \partial_x(u^2 + uu_x)] - \gamma u_x \]
\[ = -k_1(2muu_x + 2(u_x^2)m_x) - k_2(2uu_x + uu_x) - \gamma u_x, \]
which implies that
\[ u'(t) = -\gamma G * u_x - k_1 G * ((2muu_x) + 2(u_x^2)m)_x - k_2 G * (2uu_x + uu_x). \] (3.5)

Now using the equation (2.1). In which, it is easy to see that the following equations hold true:
\[ \int_0^x \cosh(x - y - \frac{1}{2}) f(y) dy = \frac{e^{x-\frac{1}{2}}}{2} \int_0^x e^{-y} f(y) dy + \frac{e^{\frac{1}{2} - x}}{2} \int_0^x e^y f(y) dy, \]
\[ \int_x^1 \cosh(x - y - \frac{1}{2}) f(y) dy = \frac{e^{x+\frac{1}{2}}}{2} \int_x^1 e^{-y} f(y) dy + \frac{e^{\frac{1}{2} - x}}{2} \int_x^1 e^y f(y) dy, \]
\[ \int_0^x \sinh(x - y - \frac{1}{2}) f(y) dy = \frac{e^{x-\frac{1}{2}}}{2} \int_0^x e^{-y} f(y) dy - \frac{e^{\frac{1}{2} - x}}{2} \int_0^x e^y f(y) dy, \]
\[ \int_x^1 \sinh(x - y - \frac{1}{2}) f(y) dy = \frac{e^{x+\frac{1}{2}}}{2} \int_x^1 e^{-y} f(y) dy - \frac{e^{\frac{1}{2} - x}}{2} \int_x^1 e^y f(y) dy. \] (3.6)

Moreover, we can obtain
\[ G * ((2muu_x) + 2(u_x^2)m)_x \]
\[ = 2G * (muu_x) + 2G_x(u_x^2)m \]
\[ = \frac{2}{3} u_x^3 + \frac{1}{12 \sinh(\frac{1}{2})} \times (e^{x-\frac{1}{2}} \int_0^x e^{-y} (u + u_y)^3 dy - e^{\frac{1}{2} - x} \int_0^x e^y (u - u_y)^3 dy) \] (3.7)
\[ + e^{x+\frac{1}{2}} \int_x^1 e^{-y} (u + u_y)^3 dy - e^{\frac{1}{2} - x} \int_x^1 e^y (u - u_y)^3 dy \]
and
\[ G * (2uu_x + uu_x) \]
\[ = \int_0^x \frac{\sinh(x - y - \frac{1}{2})}{2 \sinh(\frac{1}{2})} (u_x^3 + \frac{1}{2} u_x^2) dy + \int_x^1 \frac{\sinh(x - y + \frac{1}{2})}{2 \sinh(\frac{1}{2})} (u_x^3 + \frac{1}{2} u_x^2) dy. \] (3.8)

Plugging the above identities (3.7) and (3.8) into (3.5), we obtain that (3.2).

Differentiating (3.5) with respect to x, we have
\[ u'_x = \gamma (u - G * u) - 2k_1 (G_x * (muu_x) + G * (u_x^2)m) \]
\[ - k_2 \left( \frac{1}{2} u_x^2 - u^3 + G * u_x^2 + \frac{1}{2} u_x^2 \right). \] (3.9)

Using the same way, we can obtain the following equation
\[ 2G_x * (muu_x) + 2G * (u_x^2)m = -\frac{1}{3} u^3 + uu_x^2 \]
\[ + \frac{1}{12 \sinh(\frac{1}{2})} (e^{x-\frac{1}{2}} \int_0^x e^{-y} (u + u_y)^3 dy + e^{\frac{1}{2} - x} \int_0^x e^y (u - u_y)^3 dy) \]
\[ + e^{x+\frac{1}{2}} \int_x^1 e^{-y} (u + u_y)^3 dy + e^{\frac{1}{2} - x} \int_x^1 e^y (u - u_y)^3 dy. \] (3.10)
Plugging the identity \((3.10)\) into equation \((3.9)\), we obtain the equation \((3.3)\).

On account of \((1.1)\), we deduce that
\[
m'(t) = m_t + [k_1(u^2 - u_x^2) + k_2 u]m_x
\]
\[
= k_1((u^2 - u_x^2)m_x - ((u^2 - u_x^2)m)_x) + k_2[um_x - 2u_x u - um_x] - \gamma u_x
\]
\[
= -2k_1 u_x m^2 - 2k_2 u_x m - \gamma u_x.
\]

thereby the proof of Theorem 3.1 is complete. \(\square\)

4. Non-sign-changing momentum. In this subsection, we derive some sufficient conditions for the blow-up of the initial-value problem when the parameter \(\gamma = 0\).

Similar to the proof of Lemma 4.3 in \([7]\). We have the following lemma 4.1. It shows that, if \(m_0 = (1 - \partial_x^2)u_0\) does not change sign, then \(m(t, x)\) will not change sign for any \(t \in [0, T)\). This conservative property of the momentum \(m\) will be crucial in the proof of our blow-up result.

**Lemma 4.1.** Let \(u_0 \in H^s(S), s > \frac{5}{2}, \gamma = 0\) and \(T > 0\) be the maximal existence time of the corresponding strong solution \(u\) to \((1.1)\). Then \((1.1)\) has a unique solution \(q \in C^1([0, T) \times S, S)\) such that the map \(q(t, \cdot)\) is an increasing diffeomorphism of \(S\) with
\[
q_x(t, x) = \exp(\int_0^t (2k_1 m + k_2) u_x(s, q(s, x))ds) > 0, \quad \forall (t, x) \in [0, T) \times S. \quad (4.1)
\]

Furthermore, for all \((t, x) \in [0, T) \times S\) it holds that
\[
m(t, q(t, x)) = m_0(x) \exp(-2 \int_0^t (k_1 m + k_2) u_x(s, q(s, x))ds). \quad (4.2)
\]

Now, we give the following result on the blow-up for a non-changing-sign momentum.

**Theorem 4.2.** Let \(\gamma = 0, k_1 > 0, k_2 > 0, u_0 \in H^s(S)\) for \(s > \frac{5}{2}\) and \(m_0 \geq 0\). Suppose there exists a point \(x_1 \in S\) such that
\[
m_0(x_1) > 0, u_{0x}(x_1) < -\frac{1}{\sqrt{2}}(u_0(x_1) + \frac{H_1[u_0]}{2(1 - \kappa)}), \quad (4.3)
\]

Then the corresponding solution \(u(t, x)\) blows up in finite time with an estimate of the blow-up time \(T^*\) as
\[
T^* \leq -\frac{1}{2k_1 m_0(x_1) u_{0x}(x_1)}. \quad (4.4)
\]

**Proof.** We will trace the dynamics along the characteristics emanating from \(x_1\). Denote
\[
\hat{u}(t) = u(t, q(t, x_1)), \hat{u}_x(t) = u_x(t, q(t, x_1)),
\]
\[
\hat{m}(t) = m(t, q(t, x_1)), \hat{M}(t) = (mu_x)(t, q(t, x_1)).
\]

Since we know that \(m_0 \geq 0\), in particular, \(m_0(x_1) > 0\), so from \((4.2)\), we know that \(m(t, x) \geq 0\) and \(\hat{m}(t) > 0\). According Lemma 2.4 we know that \(|u_x(t, x)| \leq u(t, x)\). Therefore \(u_x\) does not blow up. From the blow-up criterion in Lemma 2.1, it suffices to consider the quantity \(M = mu_x\). Using \((3.3)\) and \((4.4)\), a simple calculation then gives the following equation.
\[
M'(t) = -2k_1 m^2 u_x^2 - 2k_2 mu_x^2 - \gamma u_x^2 + k_1(\frac{1}{3} mu^3 - mu_x^2) + k_2 (mu^2 - \frac{1}{2} mu_x^2)
\]
\[ + k_2 (mu^2 - \frac{1}{2} mu_x^2) + \gamma (mu - mG \ast u) - k_2 mG \ast (u^2 + \frac{1}{2} u_x^2) \]

\[ - \frac{k_1 m}{12 \sinh(\frac{1}{2})} \times \left( e^{x-\frac{x}{2}} \int_0^x e^{-y}(u + u_x)^3 dy + e^{\frac{1}{2} - x} \int_0^1 e^y(u - u_x)^3 dy \right) \]

\[ + e^{x+\frac{1}{2}} \int_x^1 e^{-y}(u + u_x)^3 dy + e^{-\frac{1}{2} - x} \int_x^1 e^y(u - u_x)^3 dy \right). \tag{4.5} \]

let \( \gamma = 0. \)

\[
\begin{align*}
\hat{M}'(t) &= -2k_1 \hat{M}^2 + \frac{k_1}{3} \hat{m} \hat{w}(\hat{u}^2 - 3\hat{u}_x^2) \\
&+ k_2 (\hat{m} \hat{w} - \frac{5}{2} \hat{m} \hat{w}_x^2) - k_2 \hat{m} G \ast (u^2 + \frac{1}{2} u_x^2) \\
&- \frac{k_1 \hat{m}}{12 \sinh(\frac{1}{2})} \times \left( e^{x-\frac{1}{2}} \int_0^x e^{-y}(u + u_x)^3 dy + e^{\frac{1}{2} - x} \int_0^1 e^y(u - u_x)^3 dy \right) \\
&+ e^{x+\frac{1}{2}} \int_x^1 e^{-y}(u + u_x)^3 dy + e^{-\frac{1}{2} - x} \int_x^1 e^y(u - u_x)^3 dy \right). \tag{4.6} \end{align*}
\]

Taking account of Lemma 2.3, it is then deduced that

\[
\hat{M}'(t) \leq -2k_1 \hat{M}^2 + \frac{k_1}{3} \hat{m} \hat{w}(\hat{u}^2 - 3\hat{u}_x^2) + k_2 (\hat{m} \hat{w} - \frac{5}{2} \hat{m} \hat{w}_x^2) - k_2 \hat{m} G \ast (u^2 + \frac{1}{2} u_x^2) \]

\[= -2k_1 \hat{M}^2 + \frac{k_1}{3} \hat{m} \hat{w}(\hat{u}^2 - 3\hat{u}_x^2) + k_2 (1 - \hat{\kappa}) \hat{m} \left( \hat{u}^2 - \frac{5}{2(1 - \hat{\kappa})} \hat{u}_x^2 \right). \tag{4.7} \]

Our argument is to find certain conditions on the initial data under which there holds the Riccati-like inequality \( \hat{M}'(t) \leq -C \hat{M}^2. \) Thus from the sign conditions \( \hat{u}(t) > 0 \) and \( \hat{m}(t) > 0, \) we would like to have \( \hat{u}^2 - 3\hat{u}_x^2 \leq 0, \) and this would also imply that \( \hat{u}^2 - \frac{5}{2(1 - \hat{\kappa})} \hat{u}_x^2 \leq 0. \) That is to say, it suffices to recognize finite-time blow up of \( \hat{M}(t) \) if the ratio \( |u_x/u| \) stays big along the characteristics. However, due to the inhomogeneity of the nonlinearities, one can only show that

\[
\left( \frac{\hat{u}_x}{\hat{u}} \right)'(t) \leq D_1 (\hat{u}_x^2 - 2\hat{u}^2_x + D_2 \hat{u}) \]

Thus a large (negative) ratio is not enough to make the right-hand side negative. Therefore, we will track the dynamics of \( \frac{\hat{u}_x}{\hat{u} + a} \) along the characteristics, where \( a \geq 0 \) will be chosen later.

\[
\left( \frac{\hat{u}_x}{\hat{u} + a} \right)'(t) = \frac{u_x'(u + a) - u_x u'}{(u + a)^2}. \tag{4.8} \]

Where

\[
\begin{align*}
u_x'(u + a) - u_x u' &= \frac{k_1}{3} (u^2 - u_x^2)(u^2 - 2u_x^2) + \frac{ak_1}{3} (u^2 - 3u_x^2) + k_2 (u + a)(u^2 - \frac{1}{2} u_x^2) \\
- \frac{k_1 (u - u_x + a) e^{x-\frac{1}{2}}}{12 \sinh(\frac{1}{2})} \int_0^x e^{-y}(u + u_y)^3 dy &- \frac{k_1 (u - u_x + a) e^{\frac{1}{2} - x}}{12 \sinh(\frac{1}{2})} \int_0^1 e^y(u - u_y)^3 dy \\
- \frac{k_1 (u - u_x + a) e^{x+\frac{1}{2}}}{12 \sinh(\frac{1}{2})} \int_x^1 e^{-y}(u + u_y)^3 dy &- \frac{k_1 (u - u_x + a) e^{-\frac{1}{2} - x}}{12 \sinh(\frac{1}{2})} \int_x^1 e^y(u - u_y)^3 dy \\
- k_2 (u + a) G \ast (u^2 + \frac{1}{2} u_x^2) + k_2 u_x G_x \ast (u^2 + \frac{1}{2} u_x^2). \tag{4.9} \end{align*}\]
then
\[
\left( \frac{\hat{u}_x}{\hat{u} + a} \right)'(t) \leq \frac{k_1}{3(\hat{u} + a)^2} (\hat{u}^2 - u_x^2)(\hat{u}^2 - 2\hat{u}_x^2) + \frac{k_2}{(\hat{u} + a)^2} (\hat{u}^2 - \frac{1}{2}u_x^2)(\hat{u} + a) \\
+ \frac{ak_1 \hat{u}}{3(\hat{u} + a)^2} (\hat{u}^2 - 3u_x^2) - \frac{k_2}{(\hat{u} + a)^2} (\hat{u} + a)\kappa u_x^2 + \frac{k_2}{(\hat{u} + a)^2} \hat{u} H_1[u_0] \\
\leq \frac{k_1}{3(\hat{u} + a)^2} (\hat{u}^2 - \hat{u}_x^2)(\hat{u}^2 - 2\hat{u}_x^2) + \frac{ak_1 \hat{u}}{3(\hat{u} + a)^2} (\hat{u}^2 - 3\hat{u}_x^2) \\
+ \frac{k_2(1 - \kappa)}{(\hat{u} + a)^2} \hat{u} \left( \hat{u}^2 - \frac{1}{2(1 - \kappa)} \hat{u}_x^2 \right) \\
+ \frac{k_2a(1 - \kappa)}{(\hat{u} + a)^2} \left( (\hat{u} + \frac{H_1[u_0]}{2a(1 - \kappa)})^2 - \frac{1}{2(1 - \kappa)} \hat{u}_x^2 \right).
\]
\[
(4.10)
\]

So choose \( a = \frac{H_1[u_0]}{2a(1 - \kappa)} \), then from condition (4.3), we see that the right-hand side of the above is negative initially. Hence \( \frac{\hat{u}_x}{\hat{u} + \frac{H_1[u_0]}{2a(1 - \kappa)}}(0) \leq -\frac{1}{\sqrt{2(1 - \kappa)}} \). Therefore
\[
\hat{u}(t) + \frac{1}{\sqrt{2(1 - \kappa)}} \hat{u}_x(t) < 0, \hat{u}(t) + \sqrt{3}\hat{u}_x(t) < 0, \hat{u}(t) + \sqrt{\frac{5}{2(1 - \kappa)}} \hat{u}_x(t) < 0. (4.11)
\]

We also have
\[
\hat{u}(t) - \frac{1}{\sqrt{2(1 - \kappa)}} \hat{u}_x(t) < 0, \hat{u}(t) - \sqrt{3}\hat{u}_x(t) < 0, \hat{u}(t) - \sqrt{\frac{5}{2(1 - \kappa)}} \hat{u}_x(t) < 0. (4.12)
\]

Now plugging (4.11) and (4.12) into (4.7) we have
\[
\hat{M}'(t) \leq -2k_1 \hat{M}^2.
\]

and hence \( \hat{M}(t) \) blows up in finite time with an estimate of the blow-up time \( T^* \) as
\[
T^* \leq -\frac{1}{2k_1 \hat{M}(0)} = -\frac{1}{2k_1 m_0(x_1) u_{0,x}(x_1)}.
\]

This complete the proof of Theorem 4.2. \( \square \)

5. Blow-up for a general momentum. In this section we turn our attention to the general case where \( \gamma \) needs not equal to zero, therefore there is no sign-preservation for the momentum density. Then we need some estimates for later use. Our assumptions are \( k_1 > 0, k_2 \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \). From the conservation law (4.13), we infer that

\[
\|u_x\|^4 = \int_S u_x^4 dx \\
= 3 \int_S (u^4 + 2u^2 u_x^2) dx + \frac{6k_2}{k_1} \int_S (u^3 + u_x^2) dx + \frac{6\gamma}{k_1} \int_S u^2 dx \\
- \frac{3}{k_1} (k_1 \int_S (u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4) dx + 2k_2 \int_S u^3 + u_x^2 dx + 2\gamma \int_S u^2 dx)
\]
Moreover, using Lemma 2.3, we have

\[
\|u\|_{L^\infty(\Omega)} \leq \sqrt{\frac{e+1}{2(e-1)}} H_{1/2}^2[u_0],
\]

we obtain

\[
\|u_\varepsilon\|^4 \leq (\frac{2(e+1)}{e-1}) H_1[u_0] + \sqrt{\frac{18(e+1)}{e-1}} \frac{k_2}{k_1} H_{1/2}^1[u_0] + \frac{6\gamma_1}{k_1} H_1[u_0] - \frac{3}{k_1} H_2[u_0].
\]

Moreover

\[
|e^{x-\frac{1}{2}} \int_0^x e^{-y}(u + u_\varepsilon)^3 dy| \leq |e^{x-\frac{1}{2}} (\|u\|^3 + \|u_\varepsilon\|^3)(1 - e^{-x})| \leq e^{\frac{1}{4}} (\|u\|^3 + \|u_\varepsilon\|^3)
\]

By the Young equality, we have the following equations

\[
\int_S u_\varepsilon^2 dx = \int_S u_\varepsilon^2 u_\varepsilon dx \leq \|u_\varepsilon\|^2 \|u_\varepsilon\|_2^2
\]

\[
\leq \left( \left( \frac{2(e+1)}{e-1} \right) H_1[u_0] + \sqrt{\frac{18(e+1)}{e-1}} \frac{k_2}{k_1} H_{1/2}^1[u_0] + \frac{6\gamma_1}{k_1} \right) H_1[u_0] - \frac{3}{k_1} H_2[u_0] \right)^2 \frac{1}{H_1^2}[u_0]
\]

and

\[
\int_S u^3 dx \leq \|u\|_{L^\infty} \int_S u^2 dx = \sqrt{\frac{e+1}{2(e-1)}} H_{1/2}^2[u_0] \cdot H_1[u_0] = \sqrt{\frac{e+1}{2(e-1)}} H_{1/2}^2[u_0].
\]

Then we substitute the above inequalities into the (5.2), we have

\[
|e^{x-\frac{1}{2}} \int_0^x e^{-y}(u + u_\varepsilon)^3 dy| \leq e^{\frac{1}{4}} \left( \left( \frac{2(e+1)}{e-1} \right) H_1[u_0] + \sqrt{\frac{18(e+1)}{e-1}} \frac{k_2}{k_1} H_{1/2}^1[u_0] + \frac{6\gamma_1}{k_1} \right) H_2[u_0] \]

\[
- \frac{3}{k_1} H_1[u_0] \right)^2 + \frac{1}{\sqrt{2}} H_{1/2}^2[u_0] \right] =: Q.
\]

Using the same way, we have more inequalities

\[
|e^{\frac{1}{2}-x} \int_0^x e^y(u - u_\varepsilon)^3 dy| \leq Q;
\]

\[
|e^{\frac{1}{2}+x} \int_0^x e^{-y}(u + u_\varepsilon)^3 dy| \leq Q;
\]

\[
|e^{-\frac{1}{2}-x} \int_0^x e^y(u - u_\varepsilon)^3 dy| \leq Q;
\]

We can further utilize the conservation of $H_1[u_0]$ to bound

\[
\|G_x \ast u\| = \left| \int_0^x \frac{\sinh(x-y-\frac{1}{2})}{2 \sinh(\frac{1}{2})} u dy + \int_x^1 \frac{\sinh(x-y + \frac{1}{2})}{2 \sinh(\frac{1}{2})} u dy \right|
\]
\[ \leq \frac{1}{2}\|u\|_\infty \leq \frac{1}{2}\sqrt{\frac{e + 1}{2(e - 1)}} H_1^\frac{1}{2}[u_0]. \]

\[ |u - G \ast u| \leq (1 + \cosh(\frac{1}{2})\|u\|_\infty) \leq (1 + \cosh(\frac{1}{2})) \sqrt{\frac{e + 1}{2(e - 1)}} H_1^\frac{1}{2}[u_0]. \quad (5.5) \]

\[ |G \ast (u^2 + \frac{1}{2}u_x^2)| \leq \|G\|_\infty u^2 + \frac{1}{2}\|u_x\|^2 \leq \cosh(\frac{1}{2}) H_1[u_0] \]

\[ |\gamma G \ast u_x| = |\gamma G_x \ast u| \leq |\gamma| \cosh(\frac{1}{2}) \|u\|_\infty \leq |\gamma| \cosh(\frac{1}{2}) \sqrt{\frac{e + 1}{2(e - 1)}} H_1^\frac{1}{2}[u_0] \]

Our blow-up theorem for the periodic gmCH equation with general \( \gamma \) can be stated as follows.

**Theorem 5.1.** Let \( \gamma \in \mathbb{R}, k_1 > 0 \) and \( u_0 \in H^s(\mathbb{S}) \) with \( s > \frac{5}{2} \). Assume that there exists an \( x_2 \in \mathbb{S} \) such that

\[ u_0(x_2) > \max\{0, -\frac{k_2}{2k_1}\}, m_0(x_2) > \max\{\alpha, 0\}, \]

\[ u_{0,x}(x_2) \leq \min\{-(A_1)^\frac{1}{2}, -\sqrt{\frac{A_2}{k_1u_0(x_2) + \frac{k_2}{k_0}}}\}. \quad (5.6) \]

where

\[ A_1 = -(|\gamma| \left[ \frac{3}{4k_1} \sqrt{\frac{e + 1}{2(e - 1)}} H_1^\frac{1}{2}[u_0] + \frac{2Q}{9k_1 \sinh(\frac{1}{2})} + \frac{3k_2}{4k_1} H_1[u_0] \right] \]

and

\[ A_2 = \frac{1}{3} k_1 \left( \frac{e + 1}{2(e - 1)} \right)^\frac{1}{2} H_1^\frac{1}{2}[u_0] + |k_2| \left( \frac{e + 1}{2(e - 1)} \right) H_1[u_0] \]

\[ + |\gamma| \left( 1 + \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} \right) \left( \frac{e + 1}{2(e - 1)} \right)^\frac{1}{2} H_1^\frac{1}{2}[u_0] + k_2 \cosh(\frac{1}{2}) H_1[u_0] + \frac{k_1}{4k_1} \frac{Q}{\sinh(\frac{1}{2})}. \]

with \( Q \) defined in (5.3), and

\[ \alpha = \begin{cases} -\frac{k_2 + \sqrt{\Delta}}{2k_1}, & \text{if } \Delta = (k_2^2 - 2k_1\gamma) > 0, \\ -\frac{k_2}{2k_1}, & \text{if } \Delta = (k_2^2 - 2k_1\gamma) \leq 0. \end{cases} \]

Then the solution \( u(t, x) \) blows up in finite time with an estimate of the blow-up time \( T^* \) as

\[ T^* \leq \begin{cases} -\frac{1}{2u_{0,x}(x_2)^{\sqrt{\Delta}}} \log\left( \frac{m_0(x_2) + \frac{k_2 + \sqrt{\Delta}}{2k_1}}{m_0(x_2) + \frac{k_2 - \sqrt{\Delta}}{2k_1}} \right), & \text{if } \Delta > 0, \\ -\frac{1}{2k_1 u_{0,x}(x_2) + \frac{k_2}{2k_1}}, & \text{if } \Delta = 0, \\ \sqrt{\frac{\Delta}{\alpha}} - \arctan \left( \frac{2k_1 m_0(x_2) + k_2}{\frac{\sqrt{\Delta}}{u_{0,x}(x_2)}} \right), & \text{if } \Delta < 0. \end{cases} \quad (5.7) \]
Proof. Plugging the estimates (5.1)-(5.5) into (3.2) and (3.3), we deduced that
\[ \tilde{u}' = u'(t, q(t, x_2)) \geq -\frac{2}{3} k_1 u_x^2 - |\gamma| \frac{1}{2} \sqrt{\frac{e+1}{2(e-1)}} H_{t_1}^2 [u_0] - \frac{Q}{3 \sinh(\frac{1}{2})} - \frac{k_2}{2} H_1 [u_0]. \] (5.8)

and
\[ \tilde{u}_x'(t) = \tilde{u}_x'(t, q(t, x_2)) \]
\[ \leq -(k_1 u + \frac{1}{2} k_2) u_x^2 + \frac{1}{3} k_1 (\frac{e+1}{2(e-1)})^2 H_{t_1}^2 [u_0] \]
\[ + |k_2| (\frac{e+1}{2(e-1)}) H_1 [u_0] + |\gamma| (1 + \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})}) (\frac{e+1}{2(e-1)}) \frac{1}{2} H_{t_1}^2 [u_0] \]
\[ + k_2 \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} H_1 [u_0] + \frac{k_1 Q}{4 \sinh(\frac{1}{2})} \]
\[ =: A_1 \] (5.9)

Then we know that \( \tilde{u}(t) \) is increasing when
\[ u_x^3 \leq -(|\gamma| \frac{3}{4 k_1} \sqrt{\frac{e+1}{2(e-1)}} H_{t_1}^2 [u_0] + \frac{2Q}{9 k_1 \sinh(\frac{1}{2})} + \frac{3k_2}{4k_1} H_1 [u_0]) := A_1 \]
and the \( \tilde{u}_x(t) \) is decreasing when
\[ (k_1 u + \frac{1}{2} k_2) u_x^2 \geq \frac{1}{3} k_1 (\frac{e+1}{2(e-1)})^2 H_{t_1}^2 [u_0] + |k_2| (\frac{e+1}{2(e-1)}) H_1 [u_0] \]
\[ + |\gamma| (1 + \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})}) (\frac{e+1}{2(e-1)}) \frac{1}{2} H_{t_1}^2 [u_0] + k_2 \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} H_1 [u_0] \]
\[ + \frac{k_1 Q}{4 \sinh(\frac{1}{2})} := A_2 \]

Hence, we accord the assumption on the initial data, we know the over the time of existence of solutions \( \tilde{u}(t) \) is increasing and \( \tilde{u}_x(t) \) is decreasing. Then
\[ \tilde{u}(t) \geq u_0(x_2) > 0, \quad \tilde{u}_x(t) \leq u_0,x(x_2) < 0. \] (5.10)

By the assumption in this theorem on \( m_0(x) \), we know that
\[ 2k_1 m_0^2(x) + 2k_2 m_0(x) + \gamma > 0. \]

Using (5.10), we deduce from Theorem 3.1 that over the time of solution \( m(t) \) is increasing, so
\[ m(t) > m_0(x_2) \]
and
\[ \tilde{m}'(t) := m'(t, q(t, x_2)) = -\tilde{u}_x(t)(2k_1 \tilde{m}^2(t) + 2K_2 \tilde{m}(t) + \gamma) \]
\[ \geq -u_0,x(x)(2k_1 \tilde{m}^2(t) + 2k_2 \tilde{m}(t) + |\gamma|) > 0. \] (5.11)

We show that blow-up must occur in finite time. The proof is divided into three cases.

Case 1. \( \Delta > 0 \). Let \( y_1 \) and \( y_2 \) be the two distinct real roots of the equation \( 2k_1 y^2 + 2k_2 y + \gamma = 0 \), so
\[ y_{1,2} = \frac{-k_2 \pm \sqrt{\Delta}}{2k_1}, \quad y_1 < y_2, \]
\[ 2k_1 y^2 + 2k_2 y + \gamma = 2k_1 (y - y_1)(y - y_2). \]
Integrating inequality over \([0, t]\) gives
\[
\hat{m}(t) \geq \frac{y_2 - y_1 E(t)}{1 - E(t)} \to +\infty,
\]
when \(t \to -\frac{1}{2k_1 u_{0,x}(y_2 - y_1)} \log \left( \frac{m_0(x) - y_1}{m_0(x) - y_2} \right)\), where
\[
E(t) = \frac{m_0(x) - y_2}{m_0(x) - y_1} \exp \left(-2k_1 u_{0,x}(y_2 - y_1)t\right).
\]
Notice that \(\hat{u}_x(t) \leq u_{0,x}(x) < 0\). It is then deduced that in this case
\[
(k_1 m + k_2)u_x(t, q(t, x)) \to -\infty,
\]
as \(t \to -\frac{1}{2k_1 u_{0,x}(y_2 - y_1)} \log \left( \frac{m_0(x) - y_1}{m_0(x) - y_2} \right)\).

**Case 2.** \(\Delta = 0\). In this case the equation \(2k_1 y^2 + 2k_2 y + \gamma = 0\) has the unique real root \(y = -\frac{k_2}{2k_1}\), and so
\[
2k_1 y^2 + 2k_2 y + \gamma = 2k_1 \left( y + \frac{k_2}{2k_1} \right)^2.
\]
Integrating inequality in the interval \([0, t]\) gives
\[
\hat{m}(t) \geq \frac{m_0(x) + \frac{k_2}{2k_1}}{2k_1 u_{0,x}(x)(m_0(x_5) + \frac{k_2}{2k_1}) t + 1} - \frac{k_2}{2k_1} \to +\infty,
\]
as
\[
t \to -\frac{1}{2k_1 u_{0,x}(x)(m_0(x) + \frac{k_2}{2k_1})}.
\]
Then we obtain that
\[
(k_1 m + k_2)u_x(t, q(t, x)) \to -\infty,
\]
as \(t \to -\frac{1}{2k_1 u_{0,x}(x)(m_0(x) + \frac{k_2}{2k_1})}\).

**Case 3.** \(\Delta < 0\). In this case we have
\[
2k_1 y^2 + 2k_2 y + \gamma = 2k_1 \left( y + \frac{k_2}{2k_1} \right)^2 - \frac{\Delta}{2k_1}.
\]
Integrating inequality in the interval \([0, t]\) gives
\[
\hat{m}(t) \geq -\frac{k_2}{2k_1} + \frac{\sqrt{-\Delta}}{2k_1} \tan[-u_{0,x}(x)\sqrt{-\Delta} t]
+ \arctan \frac{2k_1 (m_0(x) + \frac{k_2}{2k_1})}{\sqrt{-\Delta} t} \to +\infty,
\]
as
\[
t \to -\frac{\pi}{2} - \arctan \frac{2k_1 m_0(x) + k_2}{\sqrt{-\Delta}\sqrt{\frac{k_2}{2k_1}}} -u_{0,x}(x)\sqrt{-\Delta}.
\]
Then we obtain that
\[
(k_1 m + k_2)u_x(t, q(t, x)) \to -\infty,
\]
as \(t \to -\frac{\pi}{2} - \arctan \frac{2k_1 m_0(x) + k_2}{\sqrt{-\Delta}\sqrt{\frac{k_2}{2k_1}} - u_{0,x}(x)\sqrt{-\Delta}}\).
Following the proof of Theorem 5.1, we can deal with the case of $\gamma = 0$ and $k_2 < 0$. Indeed, we have the following result, which can not be obtained from Theorem 4.2.

**Corollary 5.2.** Let $k_1 > 0, k_2 < 0, \gamma = 0$ and $u_0 \in H^s(S)$ with $s > \frac{5}{2}$ and $m_0(x) \geq 0, \forall x \in S$. Assume that there exists a point $x_3 \in S$ such that

$$u_0(x_3) > \frac{k_2}{2k_1}, m_0(x_3) > -\frac{k_2}{k_1},$$

$$u_{0,x}(x_3) \leq \min \left\{ -(A_1)^{\frac{1}{2}}, -\sqrt{\frac{A_4}{k_1u_0(x_3) + k_2^2}} \right\},$$

where

$$A_3 = \frac{2Q}{9k_1 \sinh(\frac{1}{2})} + \frac{3k_2}{4k_1} H_1[u_0]$$

and

$$A_4 = \frac{1}{3} k_1 \left( \frac{e + 1}{2(e - 1)} \right)^{\frac{3}{2}} H_1^{\frac{3}{2}}[u_0] + \left| k_2 \right| \left( \frac{e + 1}{2(e - 1)} \right) H_1[u_0]$$

$$+ k_2 \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} H_1[u_0] + \frac{k_1 Q}{4 \sinh(\frac{1}{2})}.$$  

and $Q$ is defined by (5.3). Then the solution $u(t, x)$ blows up in finite time with an estimate of the blow-up time $T^*$ as

$$T^* \leq \frac{1}{2k_2 u_{0,x}(x_3)} \log \frac{m_0(x_3)}{m_0(x_3) + \frac{k_2}{k_1}}.$$

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