VANISHING LINES IN GENERALIZED ADAMS SPECTRAL SEQUENCES ARE GENERIC

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Abstract. We show that in the generalized Adams spectral sequence, the presence of a vanishing line of fixed slope at some $E_r$-term is a generic property.

1. Introduction

Let $E$ be a nice ring spectrum, let $X$ be a spectrum, and consider the $E$-based Adams spectral sequence converging to $\pi_\ast X$. In this note, we prove that, for any number $m$, the property that the spectral sequence has a vanishing line of slope $m$ at some term of the spectral sequence is generic.

Definition 1.1. We say that a spectrum $X$ is $E$-complete if the inverse limit of the Adams tower for $X$ is contractible. (See Section 2 for a definition of the Adams tower.) A property $P$ of $E$-complete spectra is said to be generic if

- whenever $Y$ is $E$-complete and $Y$ satisfies $P$, then so does any retract of $Y$; and
- if $X \to Y \to Z$ is a cofibration of $E$-complete spectra and two of $X$, $Y$, and $Z$ satisfy $P$, then so does the third.

In other words, a property is generic if the full subcategory of all $E$-complete spectra satisfying it is thick.

Given a connective spectrum $W$, we write $|W|$ for its connectivity.

We assume that our ring spectrum $E$ satisfies the standard assumptions for convergence of the $E$-based Adams spectral sequence—in other words, the assumptions necessary for Theorem 15.1(iii) in [Ada74, Part III]; see also Assumptions 2.2.5(a)–(c) and (e) in [Rav86].

Theorem 1.2. Let $E$ be a ring spectrum as above, and consider the $E$-based Adams spectral sequence $E_r^{s,t}(X) \Rightarrow \pi_\ast(X)$. Fix a number $m$. The following properties of an $E$-complete spectrum $X$ are each generic:

(i) There exist numbers $r$ and $b$ so that for all $s$ and $t$ with $s \geq m(t-s) + b$, we have $E_r^{s,t}(X) = 0$.

(ii) There exist numbers $r$ and $b$ so that for all finite spectra $W$ with $|W| = w$ and for all $s$ and $t$ with $s \geq m(t-s-w) + b$, we have $E_r^{s,t}(X \wedge W) = 0$.

Remark 1.3. (a) One usually draws Adams spectral sequences $E_r^{s,t}$ with $s$ on the vertical axis and $t-s$ on the horizontal; in terms of these coordinates, the properties say that $E_r^{s,t}$ is zero above a line of slope $m$.

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(b) Assuming that $X$ is $E$-complete ensures that the spectral sequence converges, which we need to prove the theorem. We do not need to identify the $E_2$-term of the spectral sequence, so we do not need to know that $E$ is a flat ring spectrum, for example.

We also mention one or two possible applications of the theorem. Since there is a classification of the thick subcategories of the category of finite spectra (see [Hop87], [HS], [Rav92]), then if one is dealing with finite spectra $X$, one may be able to identify all spectra with vanishing line of a given slope. For example, in the classical mod 2 Adams spectral sequence, since the mod 2 Moore spectrum has a vanishing line of slope $\frac{1}{2}$ at the $E_2$-term, then the mod 2 Moore spectrum, and indeed any type 1 spectrum, has a vanishing line of slope $\frac{1}{2}$ at some $E_r$-term. Similarly, any type $n$ spectrum has a vanishing line of slope $\frac{1}{2^n}$ at some $E_r$-term of the classical mod 2 Adams spectral sequence. Theorem 1.2 gives no control over the term $r$ or the intercept $b$ of the vanishing line.

Since the proof is formal, this theorem also applies in other stable homotopy categories. The second author has used this result in an appropriate category of modules over the Steenrod algebra to prove a version of Quillen stratification for the cohomology of the Steenrod algebra. See [Pala, Palb] for details.

2. Proof of Theorem 1.2

The difficulty in proving a result like Theorem 1.2 is that the $E_r$-term of an Adams spectral sequence does not have nice exactness properties if $r \geq 3$—a cofibration of spectra does not lead to a long exact sequence of $E_r$-terms, for instance. So we prove the theorem by showing that the purported generic conditions are equivalent to other conditions on composites of maps in the Adams tower, and then we show that those other conditions are generic.

We start by describing the standard construction of the Adams spectral sequence, as found in [Ada74, III.15], [Rav86, 2.2], and any number of other places. Given a ring spectrum $E$, we let $F_sX = E \wedge^s S^0 \to E$. For any integer $s \geq 0$, we let

\[ F_sX = E \wedge^s X, \]
\[ K_sX = E \wedge^s \wedge X. \]

We use these to construct the following diagram of cofibrations, which we call the Adams tower for $X$:

\[ X \quad F_0X \xleftarrow{g} \quad F_1X \xleftarrow{g} \quad F_2X \xleftarrow{g} \quad \ldots \]
\[ \quad K_0X \quad K_1X \quad K_2X \]

This construction satisfies the definition of an “$E_r$-Adams resolution” for $X$, as given in [Rav86, 2.2.1]—see [Rav86, 2.2.9]. Note also that $F_sX = X \wedge F_sS^0$, and the same holds for $K_sX$—the Adams tower is functorial and exact.

Given the Adams tower for $X$, if we apply $\pi_*$, we get an exact couple and hence a spectral sequence. This is called the $E$-based Adams spectral sequence.
Each condition depends on a pair of numbers $r$, Fix numbers.

Lemma 2.2. $s$ ghost map (zero on homotopy) whenever map whenever $s$.

D\textsuperscript{\textbullet,d} indicate the bidegrees of the maps):

Unfolding this exact couple leads to the following exact sequence:

This leads to the following $r$th derived exact couple, where $D^{s,t}_r$ is the image of $g^{-1}_s$, and the map $D^{s+1,t+1}_r \to D^{s,t}_r$ is the restriction of $g_s$:

Unfolding this exact couple leads to the following exact sequence:

(2.1) $\ldots \to E^{s,t+1}_r \to D^{s+1,t+1}_r \to D^{s,t}_r \to E^{s+r-r-1}_r \to \ldots$

Fix a number $m$. With respect to the $E$-based Adams spectral sequence $E^{s*,*(-)}_s$, we have the following conditions on a spectrum $X$:

(1) There exist numbers $r$ and $b$ so that for all $s$ and $t$ with $s \geq m(t - s) + b$, the map $g^{-1}_s: \pi_{t-s}(F_{s+r-1}X) \to \pi_{t-s}(F_sX)$ is zero. (In other words, $D^{s,t}_r(X) = 0$.)

(2) There exist numbers $r$ and $b$ so that for all $s$ and $t$ with $s \geq m(t - s) + b$, we have $E^{s,t}_r(X) = 0$.

(3) There exist numbers $r$ and $b$ so that for all finite spectra $W$ with $|DW| = -w$ and for all $s$ with $s \geq mw + b$, then the composite $W \to F_{s+r-1}X \to F_sX$ is null. (Here, $DW$ denotes the Spanier-Whitehead dual of $W$.)

(4) There exist numbers $r$ and $b$ so that for all finite spectra $W$ with $|W| = w$ and for all $s$ and $t$ with $s \geq m(t - s - w) + b$, we have $E^{s,t}_r(X \wedge W) = 0$.

Each condition depends on a pair of numbers $r$ and $b$, and we write $\text{(1)}_{r,b}$ to mean that condition (1) holds with the numbers specified, and so forth.

Notice that if $m = 0$, then condition (3) says that $F_{s+r-1}X \to F_sX$ is a phantom map whenever $s \geq b$. If $m = 0$, then condition (1) says that $F_{s+r-1}X \to F_sX$ is a ghost map (zero on homotopy) whenever $s \geq b$.

Lemma 2.2. Fix numbers $m$, $r$, and $b$. We have the following implications:

(a) If $r \geq -m$, then $\text{(1)}_{r,b} \Rightarrow \text{(2)}_{r,b+r-1}$. If $r < -m$, then $\text{(1)}_{r,b} \Rightarrow \text{(2)}_{r,b-m}$.
(b) If \( r \geq 1 - m \), then (2) \( r,b \Rightarrow (1),r,b-m \). If \( r < 1 - m \), then (2) \( r,b \Rightarrow (1),r,b-r+1 \).

(c) If \( r \geq -m \), then (3) \( r,b \Rightarrow (4),r,b+r-1 \). If \( r < -m \), then (3) \( r,b \Rightarrow (4),r,b-m \).

(d) If \( r \geq 1 - m \), then (4) \( r,b \Rightarrow (3),r,b-m \). If \( r < 1 - m \), then (4) \( r,b \Rightarrow (3),r,b-r+1 \).

(Obviously, (3) \( r,b \Rightarrow (1),r,b \) and (4) \( r,b \Rightarrow (2),r,b \), but we do not need these facts.)

**Proof.** As above, we write \( g \) for the map \( F_{s+1}X \to F_sX \) and \( g_s \) for the map \( D_{s,t}^{r+1,t+1} \to D_{s,t}^{r,t+1} \), so that \( D_{s,t}^{r,t} \) is the image of \( g_{s,t}^{-1} : \pi_{t-s}F_{s+r-1}X \to \pi_{t-s}F_sX \).

(a): Assume that if \( s \geq m(t-s) + b \), then
\[
g_{s,t}^{-1} : \pi_{t-s}(F_{s+r-1}X) \to \pi_{t-s}(F_sX)
\]
is zero; i.e., \( D_{s,t}^{r,t} = 0 \). In the case \( r \geq -m \), if \( s \geq m(t-s) + b \), then \( s + r \geq m(t-r-1) - (s+r) + b \); so we see that \( D_{s,t}^{r+1,t+r-1} = 0 \). By the long exact sequence (2.1), we conclude that \( D_{s,t}^{r+1,t+r-1} = 0 \) when \( s \geq m(t-s) + b \). Reindexing, we find that \( E_p^{r,t} = 0 \) when \( p \geq m(q-p) + b + r - 1 \); i.e., condition (2) \( r,b+r-1 \) holds. The case \( r < -m \) is similar; in this case, the long exact sequence implies that \( E_s^{r,t+1} = 0 \).

(b): Assume that \( r \geq 1 - m \). If \( E_p^{r,t}(X) = 0 \) whenever \( s \geq m(t-s) + b \), then \( E_s^{r+1,t+r-1}(X) = 0 \) when \( s \geq m(t-s) + b \). So by the exact sequence (2.1), we see that \( D_{s,t}^{r+1,t} \to D_{s,t}^{r,t-1} \) is an isomorphism under the same condition. This map is induced by \( g_s : \pi_{t-s-1}F_{s+1}X \to \pi_{t-s-1}F_sX \), so we conclude that when \( s \geq m(t-s) + b \), we have
\[
\lim_{q} g_{t-s-1}F_qX = D_{s,t}^{r,t-1},
\]
\[
\lim_{q} F_{t-s-1}F_qX = 0.
\]
But by convergence of the spectral sequence, we know that \( \lim_{q} \pi_{t-s-1}F_qX = 0 \), so \( D_{s,t}^{r,t-1} = \text{im} g_{s,t}^{r-1} = 0 \). Reindexing gives \( D_{p,q}^{r,t} = 0 \) when \( p \geq m(q+1-p) + b \); i.e., (2) \( r,b \) implies (1) \( r,b+m \).

If \( r < 1 - m \), then a similar argument shows that \( D_{s,t}^{r+1,t+r-1} = 0 \).

Parts (c) and (d) are similar. \( \square \)

It is easy to prove Theorem 1.3, once we have the lemma.

**Proof of Theorem 1.3.** The proofs of the genericity of the two statements are similar, so we only prove that condition (i) is generic.

We know by Lemma 1.2 that condition (i) is equivalent, up to a reindexing, to

(*) There exist numbers \( r \) and \( b \) so that for all \( s \) and \( t \) with \( s \geq m(t-s) + b \), the map \( g_{r}^{-1} : F_{s+r-1}X \to F_sX \) is zero on \( \pi_{t-s} \).

We show that this condition is generic. Since the Adams tower is functorial, if \( Y \) is a retract of \( X \), then the Adams tower for \( Y \) is a retract of the Adams tower for \( X \). So if \( F_{s+r-1}X \to F_sX \) is zero on \( \pi_{t-s} \), then so is \( F_{s+r-1}Y \to F_sY \). (Given
\[ S^{t-s} \rightarrow F_{s+r-1}Y, \text{ then consider} \]
\[
\begin{array}{ccc}
S^{t-s} & \longrightarrow & F_{s+r-1}Y \\
& \downarrow i & \downarrow i \\
F_{s+r-1}X & \longrightarrow & F_sX \\
& \downarrow j & \downarrow j \\
F_{s+r-1}Y & \longrightarrow & F_sY
\end{array}
\]
Since \( \pi_{t-s}F_{s+r-1}X \rightarrow \pi_{t-s}F_sX \) is 0, then the map \( S^{t-s} \rightarrow F_sX \) is null. But \( S^{t-s} \rightarrow F_sY \) factors through this map, and hence is also null.

Given a cofibration sequence \( X \rightarrow Y \rightarrow Z \) in which \( X \) and \( Z \) satisfy conditions \((*)_{r,b}\) and \((*)_{r',b'}\), respectively, we show that \( Y \) satisfies \((*)_{r+r'-1, \max(b, b'-r+1)}\).

Consider the following commutative diagram, in which the rows are cofibrations:
\[
\begin{array}{ccc}
F_{s+r+r'-2}X & \longrightarrow & F_{s+r+r'-2}Y \\
& \downarrow & \downarrow \\
F_{s+r-1}X & \longrightarrow & F_{s+r-1}Y \\
& \downarrow & \downarrow \\
F_sX & \longrightarrow & F_sY
\end{array}
\]
\[
\begin{array}{ccc}
& \alpha & \downarrow \beta \\
& & \downarrow \\
F_{s+r-1}X & \longrightarrow & F_{s+r-1}Y \\
& & \downarrow \\
F_sX & \longrightarrow & F_sY
\end{array}
\]
\[
\begin{array}{ccc}
& \gamma & \downarrow \delta \\
& & \downarrow \\
F_{s+r-1}X & \longrightarrow & F_{s+r-1}Z \\
& & \downarrow \\
F_sX & \longrightarrow & F_sZ
\end{array}
\]

We assume that \( s \geq m(t-s) + \max(b, b'-r+1) \), so that we have
\[
s \geq m(t-s) + b,
\]
\[
s + r - 1 \geq m(t-s) + b'.
\]

If we map \( S^{t-s} \) into this diagram, then since \( \pi_{t-s}\beta = 0 \), any map
\[
S^{t-s} \rightarrow F_{s+r+r'-2}Y \xrightarrow{\alpha} F_{s+r-1}Y
\]
factors through \( F_{s+r-1}X \). Since \( \pi_{t-s}\gamma = 0 \), though, then the composite
\[
S^{t-s} \rightarrow F_{s+r+r'-2}Y \xrightarrow{\alpha} F_{s+r-1}Y \xrightarrow{\delta} F_sY
\]
is null.

This shows that condition \((*)\), and hence condition (i), is generic.

The same proof, in the case \( m = 0 \), also shows the following (using the language of [Chr97]).

**Corollary 2.3.** If \( I \) is an ideal of maps that is part of a projective class, then the following property is generic for \( E \)-complete spectra \( X \):

- There exist numbers \( r \) and \( b \) so that for all \( s \geq b \), the composite
  \[
g^{r-1}: F_{s+r-1}X \rightarrow F_sX
  \]
is in \( I \).
REFERENCES

[Ada74] J. F. Adams, Stable homotopy and generalised homology, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics.

[Chr97] J. D. Christensen, Ideals in triangulated categories: Phantoms, ghosts, and skeleta, Ph.D. thesis, Mass. Inst. of Tech., 1997.

[Hop87] M. J. Hopkins, Global methods in homotopy theory, Proceedings of the Durham Symposium on Homotopy Theory (J. D. S. Jones and E. Rees, eds.), 1987, LMS Lecture Note Series 117, pp. 73–96.

[HS] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory II, preprint.

[Pal9] J. H. Palmieri, Quillen stratification for the Steenrod algebra, preprint.

[Palb] J. H. Palmieri, Stable homotopy over the Steenrod algebra, preprint.

[Rav86] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press, 1986.

[Rav92] D. C. Ravenel, Nilpotence and periodicity in stable homotopy theory, Annals of Mathematics Studies, vol. 128, Princeton University Press, 1992.

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