Principal Fairness: 
Removing Bias via Projections

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Abstract

Reducing hidden bias in the data and ensuring fairness in algorithmic data analysis has recently received significant attention. We complement several recent papers in this line of research by introducing a general method to reduce bias in the data through random projections in a “fair” subspace.

We apply this method to densest subgraph and $k$-means. For densest subgraph, our approach based on fair projections allows to recover both theoretically and empirically an almost optimal, fair, dense subgraph hidden in the input data. We also show that, under the small set expansion hypothesis, approximating this problem beyond a factor of 2 is NP-hard and we show a polynomial time algorithm with a matching approximation bound. We further apply our method to $k$-means. In a previous paper, Chierichetti et al. [NIPS 2017] showed that problems such as $k$-means can be approximated up to a constant factor while ensuring that none of two protected class (e.g., gender, ethnicity) is disparately impacted.

We show that fair projections generalize the concept of fairlet introduced by Chierichetti et al. to any number of protected attributes and improve empirically the quality of the resulting clustering. We also present the first constant-factor approximation for an arbitrary number of protected attributes thus settling an open problem recently addressed in several works [8, 9, 24, 31].

1 Introduction

Fairness of machine learning and algorithmic methods has recently received considerable attention by the research community. Although algorithms certainly lack biases that humans might exhibit, if said biases are already present in the input data set, an oblivious algorithm might optimize towards reinforcing them, thus producing discriminatory results. For example, analysis of a statistical method commonly used in the US criminal justice system to assess recidivism risk, highlighted its bias against African–American defendants [5]. An important step in addressing this issue was recently taken by Feldman et al. [15], who proposed a definition of algorithmic fairness based on the legal notion of disparate impact. Given a protected attribute $A$ (e.g. gender, ethnicity), we define the conditional odds with respect to a dependent variable $Y$ as $\frac{P[Y|A=1]}{P[Y|A=0]}$. Informally speaking, if this ratio is close to 1, the protected attribute has no bearing on the decision of the algorithm, whereas if the ratio is close to 0 or tends to $\infty$, the protected attribute is heavily correlated with the algorithm’s choices.
The general goal of algorithmic fairness is to minimize the effects of the aforementioned biases, at the same time retaining the objectives of quality and efficiency of classic algorithms analysis. Unfortunately, constraining a problem towards notions of fairness often makes it intractable or significantly more difficult.

In this work we jointly study the problem of de-biasing input data, at the same time designing near optimal algorithms that enforce fairness constraints for multiple private attributes. At a high level, we pursue these goals by regarding bias in input data as a form of noise that needs to be filtered out before the algorithm is applied.

The recent history of algorithmic data analysis provides several examples of algebraic and probabilistic methods (random projections, low-dimensional approximations, random perturbations) whose goal is to make data more amenable to the application of algorithmic techniques. This naturally raises the following, general question:

**Question 1.** What simple pre-processing methods can be applied to input data, so that subsequent application of an algorithm produces fair(er) solutions?

Chierichetti et al. [13] recently made an important contribution towards addressing the issue of tractability of algorithmic fairness by giving a general framework to address disparate-impact–based fairness in metric spaces. Similar strategies are unlikely to be successful for clustering problems based on graph cuts such as spectral clustering, as in these cases we are working with non-metric spaces. Therefore, a second question arises:

**Question 2.** What other natural partitioning problems admit fair algorithms with respect to the disparity of impact measure?

While the case of a protected binary attribute considered by Chierichetti et al. [13] captures important attributes such as sex, other important ones such as ethnicity or gender admit multiple potential values. Unfortunately, extension of their approach to multi-color classes is not straightforward, as it entails computing high dimensional matchings, which are NP-hard problems. This raises a third question:

**Question 3.** Can disparity of impact be minimized for multiple color classes?

**Contribution and approach.** We make progress on the three questions above, by adopting an algebraic perspective for the disparity-of-impact measure. In more detail:

We introduce the fair transform, a technique whose main underlying idea is treating bias in the data as a form of noise and applying spectral de-noising techniques. In practice, the fair transform can be defined in terms of a projection onto a suitable “fair” space and, as such, it can be be used as a preprocessing tool by virtually any algorithm relying on an algebraic formalization of a problem of interest. To exemplify the effect of the fairness transformation on the input data, Figure 1 presents plots obtained from Amazon books on US politics. On the left, we observe the books ordered according to their corresponding entries in the main eigenvector of the adjacency matrix of the co-purchase graph. Books are also colored according to political orientation. We can observe that, whereas liberal books are well distributed, conservative ones are clustered. On the right we observe the results after applying the transformation. Notice that now conservative books are also well distributed along the principal component. Our technique can be regarded as a way to enforce “soft” fairness constraints to a problem of interest, thus possibly mitigating tractability issues arising when hard fairness constraints are imposed.

We showcase application of the fair transform by investigating fair variants of the densest subgraph and $k$-means clustering problems. The former has application in ensuring diversity in recommender systems and the latter is a fundamental building block in machine-learning applications. Analysis of these two problems highlights interesting properties of the fair transform mentioned above. For densest subgraph, application of the fair transform apparently allows us to provably overcome the general intractability of the problem when certain assumptions hold, a result that is further supported by empirical evidence. For $k$-means, applying the fair transform is similar to the popular method of projecting onto the subspace spanned by the first $k$ singular vectors, with the difference that instead of removing excess variance, we remove “excess unfairness.” The state-of-the-art way for computing fair $k$-mean solutions is based on the concept of fairlets [13]. In Section 3, we show how

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1 As will become clear in Section 3, the main eigenvector of the adjacency matrix can leveraged to compute a dense subgraph.
applying $k$-means after the fair transform generalizes the concept of fairlets, while being also usable for problems where the concept of fairlets is inapplicable.

We also study approximation algorithms for these problems. For the fair densest subgraph problem, it is straightforward to show that the problem is as difficult to approximate as the densest at most $k$-subgraph problem. On the positive side, we show that if the ground set is fair (i.e., the number of nodes in a graph with color 0 and color 1 in the graph is equal), we can give an optimal 2-approximation for the problem. For fair $k$-means, we further extend the approximation algorithm designed by Chierichetti et al. [13] to an arbitrary number of colors, while preserving approximation up to constant factors. Our results hold for the $k$-means and the $k$-median problem, complementing recent results that either only applied to $k$-center [9, 24, 31] or only yielded bicriteria approximations [8, 9].

1.1 Related work

Algorithmic Fairness and Fair Clustering Fairness in algorithms has recently received considerable attention, see [21, 34, 38, 40] and references therein. Disparate impact was first proposed by Feldman et al. [15]. It has since been used by Zafar et al. [39] and Noriega-Campero et al. [29] for classification and Celis et al. [11, 12] for voting and ranking problems. Fair clustering as a problem was first proposed by Chierichetti et al. [13]. They showed that for two protected classes, fair clustering for various objectives such as $k$-median, $k$-center, and (implicitly, though unstated) $k$-means can be approximated as well as the unconstrained variants of the problem (up to constant factors). Building upon their work, Backurs et al. [7] and Schmidt et al. [32] considered this problem for large data sets. The main open problem left in their work is whether the approximability can be extended for multiple color classes. Here, the $k$-center problem has received the most attention [9, 24, 31], with the current state of the art being a 5 approximation. Conversely, prior to our work, for $k$-means with multiple protected classes, only a PTAS for constant $k$ [32] and bicriteria approximation algorithms were known [8, 9].

Densest Subgraph The densest subgraph problem consists in finding a set of nodes $S$ such that the ratio between the number of edges of the induced subgraph and $|S|$ is maximized. Identifying dense subgraphs is a key primitive in a number of applications; see [16, 18, 35]. The problem can be solved optimally in polynomial time [19]. The fair densest subgraph problem is highly related to the densest subgraph problem limited to at most $k$ nodes, which cannot be approximated up to a factor of $n^{1/\log\log n}$ for some $c > 0$ assuming the exponential time hypothesis [25] and for which state-of-the-art methods yield an $O(n^{1/4+\varepsilon})$ approximation [10]. For the densest subgraph of cardinality at least $k$, there exists an optimal 2-approximation [4, 23, 26].

1.2 Preliminaries

We consider graphs $G(V, E, w)$, where $V$ is the set of $n$ nodes, $E \subseteq V \times V$ is the set of edges, and $w : E \rightarrow \mathbb{R}_{\geq 0}$ is a weight function. We denote the (weighted) adjacency matrix of $G$ by $A$. For a subset of edges $E' \subseteq E$, we denote $w(E') = \sum_{e \in E'} w(e)$. We only consider undirected graphs, that is, $w(u, v) = w(v, u)$ for all $u, v \in V$. If the image of $w$ is $\{0, 1\}$, we call $G$ unweighted and omit $w$ from the notation. We do not consider graphs with self-loops, that is, $w(v, v) = 0$ for all $v \in V$.
We call $G$ is $d$-regular, if $\sum_{e \in [v] \times V} w(e) = d$ for every node $v$. For any subset of nodes $S \subset V$, we write $\overline{S} := V \setminus S$. We denote the induced subgraph containing the edges $E_S := E \cap (S \times S)$ as $G_S(S, E_S, w)$. We further denote the induced cut of $S$ by $E_{S, \overline{S}} := E \cap (S \times \overline{S})$. The weight of the cut $E_{S, \overline{S}}$ is defined as $cut(S, \overline{S}) := \sum_{e \in E_{S, \overline{S}}} w(e)$. For a $d$-regular weighted graph, the expansion $\Phi(S)$ of $S \subset V$ is defined as $\Phi(S) := \frac{cut(S, \overline{S})}{d \min(|S|, |\overline{S}|)}$.

Additionally, we are given a coloring $c : V \rightarrow [\ell]$ of $V$, where $[\ell] := \{1, 2, \ldots, \ell\}$. If $\ell = 2$, for simplicity we denote the colors red and blue and we use Red := $\{v \in V \mid c(v) = \text{red}\}$ and Blue := $\{v \in V \mid c(v) = \text{blue}\}$ to refer to nodes of the respective color. A set $S \subset V$ is called fair if $|S \cap \{v \in V \mid c(v) = 1\}| = |S \cap \{v \in V \mid c(v) = 2\}| = \cdots = |S \cap \{v \in V \mid c(v) = \ell\}|$. A graph is called fair if $V$ is fair.

Finally, the $p$-norm of a vector $x \in \mathbb{R}^n$ is $\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$. Let $A$ be an $n \times n$ matrix. The Frobenius norm is defined as $\|A\|_F = \sqrt{\sum_{i,j} A_{i,j}^2}$. The matrix $AA^\dagger$ (resp. $A^\dagger A$) is defined as the orthogonal projection onto the columnspace (resp. rowspace) of $A$. An eigenvector $v$ of an $n \times n$ matrix $A$ satisfies $Av = \lambda v$, where $\lambda$ is the eigenvalue associated with $v$.

## 2 Fair Projections

To get an intuitive grasp, consider the simple case where $V$ may be partitioned into two sets of colors Red and Blue. Let $f = \{-1, 1\}^n$ be the vector denoting class membership, where $f_i = -1$ if node $i \in \text{Red}$ and $f_i = 1$ if $i \in \text{Blue}$. Next, suppose that the output of any algorithm under consideration is represented by an indicator vector $x \in \{0, 1\}^n$. We observe that $x$ has disparate impact 1 if and only if $f^T x = 0$. We call this the fairness constraint wrt Red and Blue. Notice that this implies $f f^T x = 0$. In other words, $x$ is drawn from the null space of $f$ or, formally, $x$ is a linear combinations of the orthogonal basis of $I - ff^T$.

This simple example generalizes to multiple colors as follows.

**Definition 1.** Let $M$ be an $n \times d$ matrix and let $c : [n] \rightarrow [\ell]$ be a coloring of the rows of $M$. Define the $n \times (\ell - 1)$ matrix $F_{i,j} := \begin{cases} 1 & \text{if } c(v_i) = 1 \\ -1 & \text{if } c(v_i) = j + 1. \\ 0 & \text{otherwise} \end{cases}$

We call $(I - F F^\dagger)M$ the fair subspace of $M$. Any orthogonal projection $YY^T \in \text{span}(I - F F^\dagger)$ is called a fair projection.

While it might appear that colors are not treated identically according to the above definition, this indeed is not the case, since matrix $F$ has rank $\ell - 1$ (assuming every color is present) if and only if the fairness constraint is satisfied for all pairs of colors.

Especially for large data sets, the number of colors $\ell$ may become large as well. In this case, computing a fair subspace may lead to a considerable overhead and naive methods running in time $O(n\ell^2)$ will not scale. The following theorem shows that computation of the fair transform can indeed be done efficiently, the key ingredient being the fast Haar transform [20].

**Theorem 1** (Fast Fair Transform). Let $M$ be an $n \times d$ matrix with a coloring $c : [n] \rightarrow [\ell]$ of the rows, where $\ell$ is a power of 2. Then the fair subspace of $M$ can be computed in time $O(n\ell^2 \log\ell)$.

In the remainder, we provide the theoretical framework for the application of the fair transform to the recovery of dense fair subgraphs, as well as empirical evidence of its potential benefits for both the densest subgraph and the $k$-means problems, in realistic scenarios based on real datasets.

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\(^2\) $A^\dagger$ is the pseudo-inverse of $A$. Its further properties are not relevant for this paper.

\(^3\) Note that, while the above definition corresponds to the notion of fairness we consider in this paper, it easily generalizes to settings in which we want to enforce a given proportion of samples from each class. This corresponds to setting $F_{i,j} = \alpha$ for a suitable $\alpha > 0$, if $c(v_i) = j + 1$. 
3 Two illustrative applications of Fair Projections

In general, enforcing fairness can make a problem intractable, even when it originally wasn’t. The fair transform can be regarded as a way to mitigate this issue, by enforcing soft fairness constraints to virtually any problem that is amenable to an algebraic formulation. In this section, we briefly discuss two exemplifying applications, namely densest subgraph and $k$-means clustering. We emphasize that the method itself is more general and can be applied to a number of problems such as spectral clustering, graph bisection, and max cut.

3.1 Densest Fair Subgraph

The densest-subgraph problem consists of finding a subgraph induced by a set of nodes $S \subset V$ with maximum average (potentially weighted) degree. The density of $S$ is thus $D_S := \frac{2|E_S|}{|S|^2}$, where $E_S := E \cap (S \times S)$. Using the indicator vector

$$x_i := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else} \end{cases},$$

we have $D_S = \frac{2x^T A x}{x^T x}$. If $x$ is fair, it is drawn from the fair subspace of $A$. Therefore, for any fair subgraph with indicator $S$, we have

$$\frac{2x^T A x}{x^T x} = \frac{2x^T (I - FF^\top) A (I - FF^\top) x}{x^T x}.$$

Conversely, for any indicator vector $x \notin \text{span}(I - FF^\top)$, the objective value only decreases.

We note that by relaxing $x$ to be an arbitrary vector, the above expression is maximized by the first eigenvector of $A$. Indeed, Kannan and Vinay [22] established a relationship between the first eigenvector of the graph and an approximately densest subgraph. Similar ideas are also implicit in the work of McSherry [28] and Vu [36]. Under certain regularity conditions (which we also observe in our experiments), we show that fair projections similarly allow approximate recovery of a hidden fair subgraph.

**Theorem 2.** We are given a graph $G$ with a 2-coloring of the nodes. Assume the spectrum of $G$’s adjacency matrix satisfies $\lambda_1 \geq 4\lambda_2$.

Assume further that $G$ contains a fair subset $S$ such that:

1. the subgraph $G_S$ induced by $S$ is almost regular, that is, for every $i \in S$, the (weighted) degree of $i$ in $G_S$ belongs to the interval $[(1 - \epsilon)d, (1 + \epsilon)d]$ for some $\epsilon$ and $d$.

2. $d \geq (1 - \theta) d_{\text{max}}$ where $d_{\text{max}}$ is the maximum (weighted) degree of vertices in $G$.

In this case, it is possible to recover all but $16(\epsilon + \theta)|S|$ of the vertices in $S$ in polynomial time.

Intuitively, the result above states that, if the underlying network $G$ is an almost-regular expander containing a dense, fair subset, we can approximately retrieve it in polynomial time. Succinctly, this follows because, under these assumptions, the indicator vector of $S$ forms a small angle with the main eigenvector of $(I - ff^T) A (I - ff^T)$. To prove this theorem, we first introduce a number of lemmas that characterize the spectrum of $A$, and $I - ff^T$. We then proceed to relate the spectrum with the indicator vector of the densest fair subgraph.

We will use the following notations. Denote by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ the eigenvalues of a matrix $A$. and by $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_n$, those of $(I - ff^T) A (I - ff^T)$. Further, denote by $v_i$ and $\hat{v}_i$ the $i$-th eigenvector of $A$ and $(I - ff^T) A (I - ff^T)$ respectively. Note that $\lambda_1 \leq d_{\text{max}}$, with the latter the maximum degree of $G$. For a subset $C \subset V$, we denote by $\chi$ its normalized indicator vector, where $C$ is understood from context. Namely, $\chi_i = 1/\sqrt{|C|}$ if $i \in C$, $\chi_i = 0$ otherwise. We also denote by $v$ the main eigenvector of the matrix $(I - ff^T) A (I - ff^T)$.

The following result states a simple, yet useful property of the eigenvalues of $(I - ff^T) A (I - ff^T)$.

**Lemma 1.** We have $(I - ff^T) \hat{v}_i = \hat{v}_i$ whenever $\hat{\lambda}_i \neq 0$.\footnote{That is, $G$ is an expander.}
Proof. If \( \hat{\lambda}_1 \neq 0 \), we have:
\[
(I - f f^T)A(I - f f^T)\hat{v}_i = \hat{\lambda}_i \hat{v}_i.
\]
Since \((I - f f^T)\) is a projection matrix, if we pre-multiply both members of the above equation by \((I - f f^T)\) we have:
\[
(I - f f^T)A(I - f f^T)\hat{v}_i = \hat{\lambda}_i (I - f f^T)\hat{v}_i.
\]
Subtracting the first equation from the second and recalling that \( \hat{\lambda}_1 \neq 0 \) immediately yields the thesis. \( \square \)

Note that Lemma 1 immediately implies the following

**Corollary 1.** For every non-trivial eigenvalue \( \hat{\lambda}_i \) the following holds for the corresponding eigenvector:
\[
\hat{\lambda}_i = \hat{v}_i^T (I - f f^T)A(I - f f^T)\hat{v}_i = \hat{v}_i^T A\hat{v}_i.
\]

We will also need the following technical lemma:

**Lemma 2.** Assume the spectrum of \( A \) satisfies the condition \( \lambda_1 \geq 4\lambda_2 \). Then \( \hat{\lambda}_2 \leq \frac{3}{4} \lambda_1 \).

**Proof.** We begin by expressing \( \hat{v}_2 \) as linear combination of the eigenvectors of \( A \), namely, \( \hat{v}_2 = \sum_{i=1}^{n} \beta_i v_i \). Note that, since \( A \) is symmetric, considering its orthogonal decomposition we have:
\[
\hat{\lambda}_2 = \hat{v}_2^T (I - f f^T)A(I - f f^T)\hat{v}_2 = \hat{v}_2^T A\hat{v}_2 = \sum_{i=1}^{n} \lambda_i \beta_i^2.
\]

Next, assume that \( \beta^2 = 1 - \gamma \). We next prove that \( \gamma \) cannot be “too small”. To this purpose, note that the last equation immediately implies that \( \hat{\lambda}_2 \geq \lambda_1 (1 - \gamma) \). We next show that \( \hat{v}_1 \)'s projection onto \( v_1 \) has length at most \( \gamma \). To prove this, let \( \psi_{xy} \) denote the angular distance between vectors \( x \) and \( y \). From the triangle inequality we have:
\[
\psi_{\hat{v}_1, \hat{v}_2} = \frac{\pi}{2} \leq \psi_{v_1, v_2} = \psi_{v_1, \hat{v}_1} + \psi_{\hat{v}_1, \hat{v}_2},
\]
from which \( \psi_{\hat{v}_1, v_1} \geq \frac{\pi}{2} - \psi_{v_1, \hat{v}_2} \) follows. This in turn implies:
\[
< \hat{v}_1, v_1 > = \cos(\psi_{\hat{v}_1, v_1}) \leq \cos \left( \frac{\pi}{2} - \psi_{v_1, \hat{v}_2} \right) = \sin(\psi_{\hat{v}_1, \hat{v}_2}) \leq \gamma.
\]

On the other hand, if \( \hat{v}_1 = \sum_{i=1}^{n} \gamma_i v_i \), we have:
\[
\hat{\lambda}_1 = \hat{v}_1^T \hat{v}_1 = \sum_{i=1}^{n} \lambda_i \gamma_i^2 \leq \lambda_1 \gamma^2 + \sum_{i=2}^{n} \lambda_i \gamma_i^2 \leq \lambda_1 \gamma^2 + (1 - \gamma^2) \lambda_2 \leq \lambda_1 \gamma^2 + \lambda_2.
\]

Since \( \hat{\lambda}_2 \leq \hat{\lambda}_2 \) by definition and since \( \hat{\lambda}_2 \geq (1 - \gamma) \lambda_1 \) this implies:
\[
\gamma^2 \lambda_1 + \lambda_2 \geq \lambda_1 (1 - \gamma).
\]
Solving with respect to \( \gamma \) yields \( \gamma \geq \frac{1}{2} \), whenever \( \lambda_1 \geq 4\lambda_2 \). Finally:
\[
\hat{\lambda}_2 = \sum_{i=1}^{n} \lambda_i \beta_i \leq \frac{\lambda_1}{2} + \sum_{i=2}^{n} \lambda_i \beta_i^2 \leq \frac{\lambda_1}{2} + \lambda_2 \leq \frac{3}{4} \lambda_1,
\]
since \( \lambda_2 \leq 4\lambda_1 \). This concludes the proof. \( \square \)

The next result states that the indicator vector of the fair densest subgraph is close to \( \hat{v}_1 \):

**Lemma 3.** Assume the subgraph spanned by \( C \) is \( \epsilon \) regular. Assume further that \( d \geq (1 - \theta) d_{\text{max}} \) and that \( \lambda_1 \geq 4\lambda_2 \). Then:
\[
\| \chi - \hat{v}_1 \|^2 \leq 4(\epsilon + \theta).
\]

\(^3\)Since we are interested in \( \beta_i^2 \), we assume without loss of generality that \( \psi_{v_1, \hat{v}_2} < \frac{\pi}{2} \).
Proof. In the remainder of the proof we let \( m = |C| \) for simplicity. Consider the vector \( \chi \) and note that \( \chi^T f = 0 \) by definition, which implies \((I - ff^T)\chi = \chi\). On the other hand, \( \chi^T \chi = m \) (by definition), so that we have:

\[
\chi^T(I - ff^T)A(I - ff^T)\chi = \chi^TA\chi = \frac{\sum_{i \in C} d_i}{m} \geq (1 - \epsilon)d,
\]

where \( d_i \) denotes \( i \)'s degree in the subgraph induced by \( S \). Next, we decompose \( \chi \) along its components respectively parallel and orthogonal to \( \tilde{v}_1 \), namely, \( \chi = \alpha v + z \). Again we have:

\[
\begin{align*}
\chi^T(I - ff^T)A(I - ff^T)\chi &= (\alpha v + z)^T(I - ff^T)A(I - ff^T)(\alpha v + z) \\
&= \alpha^2 \hat{\lambda}_1 + z^T(I - ff^T)A(I - ff^T)z \\
&\leq \alpha^2 \hat{\lambda}_1 + \hat{\lambda}_2 \|ar{z}\|^2 \\
&\leq \alpha^2 \hat{\lambda}_1 + (1 - \alpha^2)\hat{\lambda}_2
\end{align*}
\]

Putting together (1) and (2) yields:

\[
\alpha^2 \geq \frac{(1 - \epsilon)d - \hat{\lambda}_2}{\hat{\lambda}_1 - \hat{\lambda}_2},
\]

which in turn gives the thesis if we recall that \( \|\chi\| = 1 \) and that \( v \) and \( z \) are orthogonal by definition.

We further note the following:

\[
\hat{\lambda}_1 = \max_{\|w\|=1} w^T(I - ff^T)A(I - ff^T)w \leq \|(I - ff^T)A\| \leq \|A\| = \lambda_1,
\]

where the third equality follows since \( I - ff^T \) is a projection matrix and its norm is 1. As a consequence,

\[
\begin{align*}
\|\chi - v\|^2 &\leq 1 - \frac{(1 - \epsilon)d - \hat{\lambda}_2}{\hat{\lambda}_1 - \hat{\lambda}_2} \leq 1 - \frac{(1 - \epsilon)(1 - \theta)d_{\text{max}} - \hat{\lambda}_2}{\hat{\lambda}_1 - \hat{\lambda}_2} \\
&\leq 1 - \frac{(1 - \epsilon)(1 - \theta)\lambda_1 - \hat{\lambda}_2}{\hat{\lambda}_1 - \hat{\lambda}_2} < 1 - \frac{\lambda_1 - \hat{\lambda}_2 - (\epsilon + \theta)\lambda_1}{\lambda_1 - \hat{\lambda}_2} \\
&= \frac{(\epsilon + \theta)\lambda_1}{\lambda_1 - \hat{\lambda}_2} \leq 4(\epsilon + \theta).
\end{align*}
\]

Here, the second inequality follows from our hypotheses on \( d \), the third inequality follows since the main eigenvalue of an adjacency matrix is upper-bounded by the maximum degree of the underlying graph, while the last inequality follows from Lemma 2.

Corollary 2. Under the hypotheses of Lemma 2, the number of vertices \( i \) for which \( |v_1(i) - \chi(i)| > \frac{\epsilon}{2\sqrt{m}} \) is at most \( 16m(\epsilon + \theta) \).

Proof. Assume there are \( x \) such vertices. This implies \( \|v - \chi\|^2 \geq \frac{\epsilon^2}{4m} \). On the other hand, the upper bound following from Lemma 3 immediately implies \( x \leq 16m(\epsilon + \theta) \).

The algorithm Assume first we know \( m \). In this case the algorithm is as follows:

1. Find the main eigenvector \( v_1 \) of \((I - ff^T)A(I - ff^T)\).
2. For every \( i \): accept node \( i \) if \( v_1(i) \geq 1/2\sqrt{m} \), reject node \( i \) otherwise.

Corollary 2 immediately implies that we keep all but \( 16m(\epsilon + \theta) \) nodes belonging to the fair densest subgraph.

Unfortunately, we do not know \( m \). To overcome this issue, we instead run the following algorithm:

1. Find the main eigenvector \( v_1 \) of \((I - ff^T)A(I - ff^T)\).
2. For every value of \( s \): build a solution in which every node \( i \) is accepted if \( v_1(i) \geq 1/2\sqrt{s} \) and rejected otherwise.
3. Keep the solution corresponding to the value of \( s \) such that: i) the unbalance between red and blue nodes is at most \( 16s(\epsilon + \theta) \); ii) the graph is densest.

Corollary 2 ensures that such a solution has to exist and it is not worse than the solution obtained choosing \( s = m \). This concludes the proof of Theorem 2.
3.2 Fair k-Means

As a second example, we outline application of the fair transform to k-means. Recall that the k-means problem is defined as finding a set \( C \) of \( k \) centers, such that the sum of the squared distances between any point of \( A \) to its closest center (i.e., \( \sum_{x \in A} \min_{c \in C} ||x - c||^2 \)) is minimized. The fair k-means problem additionally requires every cluster to be fair.

To apply the fair transform, we consider a spectral formulation of k-means. This is a consequence of the following, well-known decomposition of the k-means objective, which is central in showing that k-means clustering can be expressed as a constrained matrix factorization problem.

**Fact 1.** Given a point set \( A \) in \( d \) dimensions, define the centroid \( \mu(A) := \frac{1}{|A|} \sum_{x \in A} x \) and let \( c \) be an arbitrary point in \( \mathbb{R}^d \). Then the following identities hold:

\[
\begin{align*}
\sum_{x \in A} ||x - c||^2 &= \sum_{x \in A} ||x - \mu(A)||^2 + |A| \cdot ||\mu(A) - c||^2 \\
\sum_{x \in A} \sum_{y \in A} ||x - y||^2 &= 2|A| \cdot \sum_{x \in A} ||x - \mu(A)||^2.
\end{align*}
\]

This observation implies that given a partition of \( A \) into \( k \) clusters, the optimal centers are the centroids of the partition. The mapping of the points of \( A \) to these centroids can be expressed algebraically. We define the \( n \times k \) clustering matrix \( X \) (using the shorthand rank \( k \) c.m. \( X \)) with entries

\[
X_{i,j} = \begin{cases} 
\frac{1}{\sqrt{|C_j|}} & \text{if } A_i \in C_j, \\
0 & \text{otherwise}
\end{cases}
\]

\( X \) has pairwise orthogonal columns and \( XX^T A \) maps the \( i \)th row of \( A \) to the centroid of cluster \( C_j \). Then, we can express the k-means objective as

\[
\min_{\text{rank } k \text{ c.m. } X} \| A - XX^T A \|_F^2.
\]

We note that if we lift the constraint that \( X \) is a clustering matrix and instead require \( X \) only to be orthogonal, the problem is known as PCA, or equivalently the low rank subspace approximation problem. We call \( X \) fair if all clusters encoded by it are fair. In this case \( X \) is in the span of \((I - FF^T)\).

Hence, by the Pythagorean theorem, the successive application of \( XX^T(I - FF^T) \) to \( A \) is still an orthogonal projection. Then for fair cluster matrices \( X \), Equation (3) becomes

\[
\| A - XX^T A \|_F^2 = \| (I - FF^T) A - XX^T(I - FF^T) A \|_F^2 + \| A - (I - FF^T) A \|_F^2.
\]

The first term is the objective value (variance) after the application of the fair transform. Thus, by applying the fair transform, we reduce the variance by \( \| FF^T A \|_F^2 \). We emphasize that if \( X \) does not encode a fair clustering matrix then this identity does not hold. In this case \( \| A - XX^T A \|_F^2 \) is strictly less than \( \| (I - FF^T) A - XX^T(I - FF^T) A \|_F^2 + \| FF^T A \|_F^2 \), hence running an algorithm on the data matrix \((I - FF^T) A\) penalizes unfair solutions while not impacting fair ones.

3.3 Connection to Fairlets

The fairlet approach studied in [7][31][32] is based on the observation that a min-cost perfect matching (with respect to some suitable distance function) is a lower bound on the value of any optimal fair clustering. Given 2 color classes consisting of \( n \) points, the matching itself is nothing but an optimal fair \( n \)-clustering for a point set of size \( 2n \). In the following, we argue why (for the k-means problem) the fairlet algorithm is special case of fair projections.

First, we observe that the min-cost perfect matching problem associated with k-means in Euclidean spaces consists in finding a matching \( M = \{F_1, \ldots, F_n\} \) such that

\[
\sum_{j=1}^{n} \sum_{(p,q) \in F_j} \| p - q \|^2
\]

is minimized, where the \( F_j \) ∈ \( \text{Red} \times \text{Blue} \) are the fairlets in some fixed but arbitrary ordering. Now, consider the fair \( 2n \times n \) clustering matrix \( Y_{i,j} = \frac{1}{\sqrt{2}} \) if \( A_i \in F_j \). Then because of Fact [1], the term in
Equation 5 is equal to
\[ \sum_{j=1}^{n} \sum_{(p,q) \in F_j} \|p - q\|^2 = 2\|A - YY^T A\|_F^2. \]

The clustering through fairlets algorithm now merely solves the problem
\[ \min_{\text{rank } k \text{ c.m. } X} \|YY^T A - XX^T YY^T A\|_F^2 + \|A - YY^T A\|_F^2, \]

which is a special case of Equation 4 in the sense that $YY^T$ is an orthogonal projection from the span of $(I - FF^\top)$. We have $\|A - YY^T A\|_F^2 \geq \|FF^\top A\|_F^2$ because $(I - YY^T) FF^\top = FF^\top$. Therefore the amount of "bias" removed by computing the fair subspace may be (much) more than what could be achieved via the fairlets. The fairlet algorithm [7,13,32] benefits from fair projections just like any other clustering does via Equation 4. We study this relationship in the experiments detailed in Section 5.

4 Fair Approximation Algorithms

The “soft” fairness of the fair transform is in our view one of its most important features. Nevertheless, in some cases it might be important to assess the 
price of fairness, by comparing the quality of fair solutions to that of solutions for the original, unconstrained problem. We therefore complement our algorithmic treatment of fairness with a number of approximation algorithms.

Densest Fair Subgraph It is straightforward to show that the densest fair subgraph problem contains the densest at most $k$ subgraph problem as a special case. Conversely, we can show that any algorithm for the densest at most $k$ subgraph problem can be used for the densest fair subgraph problem.

**Theorem 3.** The densest fair subgraph problem is at least as hard as the densest at most $k$ subgraph problem. Moreover, any $\alpha$-approximation algorithm for the densest at most $k$ subgraph problem is an $8\alpha$ approximation algorithm for the densest fair subgraph problem.

**Proof.** We first show hardness, then approximation. The densest at-most-$k$ subgraph problem asks to find a set of nodes $S$ with $|S| \leq k$, such that the density $D_S := \frac{|E_S|}{|S|}$ is maximized. We can reduce the problem to densest fair subgraph as follows. Consider an arbitrary graph $G(V,E)$. We consider $V$ to be colored red. Add $k$ blue nodes with no edges. Then the density of the fair densest subgraph is, up to a multiplicative factor of exactly $\frac{1}{2}$, equal to the density of the densest at most $2k$ subgraph.

For the approximation result, we denote by $OPT$ the density of the densest fair subgraph. Further, we consider the density of a densest at most $k$ subgraph $OPT_k$. For $k = \min(|Red \cap V|, |Blue \cap V|)$, observe that $OPT \leq OPT_k \leq 4 \cdot OPT_{k/2}$. We now use an algorithm to compute an approximate densest at most $k/2$ subgraph, and subsequently we add nodes of the underrepresented color until the subgraph is fair. The initial density of approximate densest at most $k/2$ subgraph is (by assumption) at least $\frac{1}{4} \cdot OPT_{k/2} \geq \frac{1}{8}\cdot OPT$. After adding at most $k/2$ additional nodes to this subgraph, the density decreases by at most a factor of 2.

If the graph itself is fair, the problem becomes significantly easier. Here we can show a 2-approximation algorithm. The algorithm is simple: First compute a densest subgraph and then add as many nodes of the underrepresented color class until the subgraph is fair. This result showcases that in terms of the densest subgraph objective, policies such as affirmative action do not significantly decrease the quality of the solution. Moreover, the approximation factor is optimal, assuming that the small set expansion hypothesis holds.

**Theorem 4.** Given a fair graph, the densest fair subgraph problem admits a 2-approximation algorithm.

**Proof.** We refer to the set $S_1$ of densest unconstrained subgraph, which implies $D_{S_1} > OPT$. Let $S_2$ be the final output of our algorithm. For $S_2$, we observe that $|S_2| \leq |S_1| + |S_1 \cap Blue| - |S_1 \cap Red| \leq 2 \cdot |S_1|$, hence

\[ D_{S_2} = \frac{w(E_{S_2})}{|S_2|} \geq \frac{w(E_{S_1})}{2|S_1|} \geq \frac{OPT}{2}. \]
Theorem 5. Assuming the Small Set Expansion Hypothesis \([30]\), computing a \(2 + \varepsilon\) approximation for the densest fair subgraph problem in a fair graph is \(NP\)-hard for any constant \(\varepsilon > 0\).

Proof. Our hardness proof is based on the Small Set Expansion Hypothesis (see Conjecture \([1]\) below) formulated by \([30]\). Given a \(d\)-regular weighted graph \(G\) and two constants \(\delta, \eta \in (0, 1)\), the Small Set Expansion problem \(SSE(\delta, \eta)\) asks to distinguish between the following two cases:

Completeness There exists a set of nodes \(S \subset V\) of size \(\delta \cdot |V|\) such that \(\Phi(S) \leq \eta\).

Soundness For every set of nodes set of nodes \(S \subset V\) of size \(\delta \cdot |V|\), \(\Phi(S) \geq 1 - \eta\).

Conjecture 1 (SSEH). For every \(\eta > 0\) there exists a \(\delta := \delta(\eta) > 0\) such that \(SSE(\eta, \delta)\) is \(NP\)-hard.

Without loss of generality, we will assume that \(\delta \leq 1/2\), otherwise we simply consider \(\Phi(S)\) instead of \(\Phi(S)\). We say that a cut \((S, \overline{S})\) is fair, if \(S\) and \(\overline{S}\) are fair.

For any set \(S \subset V\) of size \(s := \delta \cdot |V|\), we have \(w(E_S) := d \cdot s - \Phi(S) \cdot d \cdot s\). We construct a colored graph \(G'(V', E', w')\) by considering all nodes of \(G\) to be colored red, and by adding \(|V|\) blue nodes. Of these nodes, we select an arbitrary but fixed subset of \(\delta \cdot |V|\) blue nodes that we denote by \(B\). Each edge in \(E_B\) is weighted uniformly by \(t := \frac{2 \cdot d}{s-1}\). The remaining edges are weighted with 0.

Reformulating SSE, we assume that distinguishing between the following two cases is \(NP\)-hard.

Completeness If there exists some \(S \subset V\) of size \(s\) with \(\Phi(S) \leq \eta\), then \(w(E_S) \geq (1 - \eta) \cdot d \cdot s\). Then the density of the fair subgraph induced by \(S \cup B\) of size \(2s\) satisfies

\[
D_{S\cup B} = \frac{w(E_S) + w(B)}{2|S|} \geq \frac{(1 - \eta) \cdot d \cdot s + t \cdot \binom{s}{2}}{2s} = \frac{(1 - \eta) \cdot d + t \cdot \frac{s-1}{2}}{2} \geq (1 - \eta) \cdot d.
\]

Soundness If for all \(S \subset V\) of size \(s\), \(\Phi(S) \geq 1 - \eta\), then

\[
w(E_S) \leq \eta \cdot d \cdot s.
\]

Denote the size of the densest fair subgraph \(C\) by \(k\). Further, let \(C_{red} = C \cap Red\). We will distinguish between four basic cases: (1) \(k < 2\mu \cdot s\), (2) \(2\mu \cdot s \leq k < 2 \cdot s\), (3) \(2 \cdot s \leq k < \frac{2}{\mu} s\), and (4) \(\frac{2}{\mu} s \leq k\), where \(\mu > 0\) is a small constant specified later. For cases (1) and (3), we can show \(D_{C_{red} \cup B_k} \leq (1 + 2\mu) \cdot \frac{d}{2}\) and for the other two cases we can show \(D_{C_{red} \cup B_k} \leq \left(1 + \frac{2\eta}{\mu}\right) \cdot \frac{d}{2}\).

First, let \(k < 2\mu \cdot s\) and again let \(B_k\) be an arbitrary subset of \(B\) of size \(k\). Then

\[
D_{C_{red}\cup B_k} \leq \frac{d}{2} + \frac{d \cdot 2\mu}{2} \leq (1 + 2\mu) \cdot \frac{d}{2}.
\]

Now, let \(2\mu \cdot s \leq k < 2 \cdot s\). We have

\[
D_{C_{red}\cup B_k} \leq \frac{\eta \cdot d}{\mu} + \frac{d}{2} \leq \left(1 + \frac{2\eta}{\mu}\right) \cdot \frac{d}{2}.
\]

Now, let \(2 \cdot s \leq k \leq \frac{2}{\mu} \cdot s\). We will first show that

\[
w(C) \leq \frac{2}{\mu} \cdot \eta \cdot d \cdot k.
\]
For the sake of contradiction, assume that this is not the case. The argument revolves around double counting \( w(C) \). There exist \( \binom{k}{2} \) subsets of size \( s \) of \( C \). Observe that for any such subset \( S' \) has weight \( w(S') \leq \eta \cdot d \cdot s \) and hence

\[
\sum_{S' \subset C \land |S'|=s} w(S') \leq \eta \cdot d \cdot s \cdot \binom{k}{s}.
\]

At the same time, every (possibly 0 valued) edge appears in \( \binom{k-2}{s-2} \) of these subsets. Hence

\[
\sum_{S' \subset C \land |S'|=s} w(S') = w(C) \cdot \binom{k-2}{s-2} > \frac{2}{\mu} \cdot \eta \cdot d \cdot k \cdot \binom{k-2}{s-2}.
\]

Combining both equations, we have

\[
\frac{2}{\mu} < \frac{k \cdot (k-1) s}{s \cdot (s-1) k} \leq \frac{2}{\mu},
\]

which is a contradiction.

Consider now the density of any fair cut containing \( C \cup B_k \), where \( B_k \) contains \( B \) and \( k-s \) further arbitrary blue nodes. Using Equation 10, we have

\[
D_{C \cup B_k} \leq \frac{2n \cdot d \cdot k + t \cdot \binom{s}{2}}{2 \cdot k} = \frac{\eta \cdot d \cdot s}{k} + \frac{d \cdot s}{2 \cdot k} \leq \left( 1 + \frac{2\eta}{\mu} \right) \cdot \frac{d}{2}.
\]  

(11)

Finally, consider the case \( k > \frac{2}{\mu} s \). Then the density of any fair cut containing \( C \cup B_k \), where \( B_k \) contains \( B \) and \( k-s \) further arbitrary blue nodes, is

\[
D_{C \cup B_k} \leq \frac{d \cdot k + t \cdot \binom{s}{2}}{2 \cdot k} = \frac{d + d \cdot s}{2 \cdot k} \leq (1 + 2\mu) \cdot \frac{d}{2}.
\]  

(12)

We note that bounds from Equations 9 and 11 and Equations 8 and 12 are identical. For \( \varepsilon < \frac{1}{4} \), we set \( \mu = \frac{s}{2} \), \( \eta \leq \frac{s}{3} \cdot \varepsilon^2 \). Then the ratio between the terms \( \delta \) and the densities are at least \( 2 - \varepsilon \). Therefore, approximating the densest fair subgraph problem beyond a factor of 2 solves the \( SSE(\eta, \delta) \) problem.

**Fair k-Clustering** Similar to Equation 3, we can formulate \( k \)-clustering objectives such as \( k \)-median and \( k \)-center algebraically. For an \( n \times d \) matrix \( A \), define the cascaded \( (p, q) \) norm \( \|A\|_{p,q} := \sqrt[p]{\sum_{i=1}^{n} \left( \sum_{j=1}^{d} |A_{ij}|^q \right)^{p/q}} \), that is, we first compute the \( q \)-norms of the rows of \( A \), and then compute the \( p \) norm of the resulting vector. It is perhaps instructive to note that \( \|A\|_{2,2} \) is identical to \( \|A\|_F \). Then the \( (k, p, q) \) clustering problem consists of computing an \( n \times d \) matrix \( C \) with at most \( k \) distinct rows minimizing \( \|A - C\|_{p,q} \). For example, \((k,1,1)\) clustering is the \( k \)-median problem in Hamming space, \((k,2,2)\) clustering is Euclidean \( k \)-means, and \((k,\infty,2)\) clustering is Euclidean \( k \)-center.

For two colors, the (suitably computed) fairlets of Chierichetti et al. [13] yield constant factor approximations for all \((k, p, q)\)-clustering objectives. However, for multiple colors, computing fairlets entails solving high dimensional min-cost perfect matching problems. Our main insight to bypass this issue is that a 2-approximate fairlet can always be computed for any \((k, p, q)\) clustering problem, as the cascaded \((p, q)\) norm of \( A \) is, indeed, a norm. By elaborating on this observation we are able to prove the following result:

**Theorem 6.** Given an \( \alpha \)-approximation of the unconstrained \((k, p, q)\)-clustering problem, there exists a \((\alpha + 2)\)-approximation for the fair \((k, p, q)\)-clustering problem with an arbitrary number of colors. For fair \( k \)-means with an arbitrary number of colors, the approximation factor is \( 2\alpha + 4 \).

We will use the following approximate triangle inequality that holds for squared Euclidean spaces.

**Lemma 4 (Weak Triangle Inequality).** Let \( p \geq 0 \) and \( 1/2 > \varepsilon > 0 \). For any \( a, b, c \in \mathbb{R} \), we have \((a-b)^2 \leq (1+\varepsilon)(a-c)^2 + (c-b)^2(1+1/\varepsilon)\).
We remark that this also implies an efficient fair clustering algorithm for $k$ centers. Indeed, by Lemma 4, we obtain from applying the algorithm to cluster points from $A$ we can compute $A(i)$ by simply evaluating a min-cost perfect matching with respect to the squared Euclidean distances between all pairs $A(j)$ and $A(j')$ for $j, j' \in \ell$ and selecting $A(i)$, such that the sum of min-cost perfect matchings is minimized. Note that, at the end of this step, each point of $A(i)$ is matched to exactly $k$ points, each belonging to a different $A(i)$, Clearly, any $k$ clustering of the points in $A(i)$ will induce a clustering of $A$ that is fair by construction.

We remark that this also implies an efficient fair clustering algorithm for $k$-means and $k$-median in any metric space. This follows by the fact that any finite $n$ point metric can be embedded into $\| \cdot \|_\infty$ with $O(n \log n)$ dimensions; see, for instance, Matousek [27]. Constraining input points to be centers loses an additional (additive) factor 2 in the approximation and makes the underlying metric finite. This allows us to apply Matousek’s embedding and then use any algorithm for the $(k, p, \infty)$ clustering problem.

5 Experimental Analysis

In this section, we provide an experimental analysis, showcasing the effect of our approach on two scenarios of potential interest. We selected the two scenarios to highlight the suitability of fair projections to tackle problems arising in various application domains.

We first use fair densest subgraph to provide diverse recommendations. Our goal is to find a diverse recommendation set of high quality (which we evaluated in terms of density).

For $k$-means, our main goal is to determine the tradeoff between cost and fairness of the computed solution, as well as the effect of fair projections on the underlying algorithms.

Given two color classes, we define the balance as $\min \left( \frac{|\text{Red}|}{|\text{Blue}|}, \frac{|\text{Blue}|}{|\text{Red}|} \right)$. For example, if a candidate subgraph (or cluster) contains 10 nodes of color Red and 1 node of color Blue, the balance is $1/10$. 

Proof. Throughout this proof, we denote by $OPT$ the cost of an optimal fair $(k, p, q)$ clustering.

Assume that we have $\ell$ colors consisting of $n$ points each, with each point belonging to $\mathbb{R}^d$. We first observe that the cost of an optimal fair $n$ clustering $\mathcal{L} = \{L_1, \ldots, L_n\}$ is a lower bound on the cost of the optimal fair $k$ clustering. We denote the point with color $j$ of cluster $L_i$ by $p_{i,j}$. Further, denote by $A$ the $n\times\ell \times d$ matrix such that row $A_{n,j-1}+i = p_{i,j}$ and let $C(L)$ denote the $n \times d$ matrix such that the row $C(L_i)$ contains the center of cluster $L_i$. Finally let $C(L)$ be $n \times \ell \times d$ matrix obtained by stacking $\ell$ copies of $C(L)$. We then have cost$(A, \mathcal{L}) = \left( \sum_{i=1}^n \sum_{j=1}^\ell (\|A_{n,j-1}+i - C(L_i)\|_q) \right)^{1/p} = \|A - C(\mathcal{L})\|_{p,q}$. Next, denote by $A(i)$ the $n \times d$ matrix whose rows are all points of color $j$ (i.e., $A(j)$ corresponds to rows $A_{n,j-1}+1, \ldots, A_{n,j}$ of $A$). Now consider the cost of clustering to the best color $A(i)$, i.e. the matrix minimizing the expression

$$\min_{i \in \ell} \left( \left\| \begin{bmatrix} A(1) \\ A(2) \\ \vdots \\ A(\ell) \end{bmatrix} - \begin{bmatrix} A(i) \\ A(i) \\ \vdots \\ A(i) \end{bmatrix} \right\|_{p,q} \right),$$

where inequality follows by application of triangle inequality, for every possible value of $i$. Finally, observe that minimization wrt $i$ implies that the value on the right hand side is at most $2\text{cost}(A, \mathcal{L})$. Moreover, the very same definition of the optimal $A(i)$ (left hand side of the inequality above) suggests that we can compute $A(i)$ by simply evaluating a min-cost perfect matching with respect to the squared Euclidean distances between all pairs $A(j)$ and $A(j')$ for $j, j' \in \ell$ and selecting $A(i)$, such that the sum of min-cost perfect matchings is minimized. Note that, at the end of this step, each point of $A(i)$ is matched to exactly $k$ points, each belonging to a different $A(i)$. Clearly, any $k$ clustering of the points in $A(i)$ will induce a clustering of $A$ that is fair by construction.
For a clustering with \(k\) clusters, we consider the following geometric mean over the balance of the clusters (though we report balance in terms of arithmetic mean and minimum as well). Our reason for choosing the geometric mean, as opposed to arithmetic mean or minimum, is that the former may be too optimistic and the latter may be too pessimistic.

### 5.1 Datasets

**Densest Subgraph.** We used data from the Amazon product co-purchasing graph [37]. Nodes of this graph represent products in the Amazon catalog and unweighted edges represent pairs of commonly co-purchased products. We processed a total number of 1581 graphs each containing between 100 and 3000 nodes. Each graph used in our experiments is a connected component extracted from the Amazon product co-purchasing graph, where all nodes belong to the same Amazon product category. Moreover, we partitioned the nodes of our graphs into two classes based on the output of the spectral clustering algorithm described in [33] executed on the graph itself, and next, we considered as a protected attribute of a node the cluster it belongs to.

We also use the POLBOOKS data set [3]. We solve the fair densest subgraph problem to identify a set of books on US politics to be recommended to users. This dataset has been already analyzed in the introduction as an example of applications of fair projections.

Nodes represent books on US politics included in the Amazon catalog. An edge between two books exists if both books are frequently co-purchased by the same buyers, as indicated by the “customers who bought this book also bought these other books” feature on Amazon. Each book is further labeled by its political stance, possible stances being “liberal”, “neutral”, and “conservative”. For our experiments, we removed “neutral” nodes from the graph obtaining 92 nodes in total, 49 of which were associated with a “conservative” worldview, 43 with a “liberal” worldview. The final graph had 362 edges in total.

Given a selection of political books with different stances from Amazon catalog, we aim to find a subset of books with many co-purchases, that equally represents Democrat and Republican viewpoints.

**K-Means.** The ATHLETES data set [11] used for \(k\)-means contains bio data on Olympic athletes and medal results from Athens 1896 to Rio 2016. The selected features are age, height, and weight, and the protected attribute is sex; 1000 records are sampled from the subset of athletes who won at least one medal. From a quick and simple descriptive analysis of the data, it is clear that height and weight are strictly related to sex.

The other data set used for fair \(k\)-means clustering is ADULTS [2] also used by Chierichetti et al. [13]. It contains 1994 US census records about registered individuals including age, education, marital status, occupation, ethnicity, sex, hours worked per week, native country, and others. Here, the numerical attributes chosen to represent points in the Euclidean space are age, education, and hours worked per week (education is encoded by a numerical scale: the higher the number, the higher the education level). The number of records sampled is set to 1000, but analysis was also carried out on a greater number of points (6000 or 9000). The aim is to cluster these points so to balance ethnicity. This selected protected attribute actually had five possible values (White, Black, Asian-Pac-Islander, Amer-Indian-Eskimo, Other), but it was reduced to a binary class using the labels White or Black, which were more numerous than the other classes.

### 5.2 Description of Algorithms

**Densest Subgraph.** For densest subgraph, we evaluated the optimal approximation algorithm, as well as a number of faithful implementations of the spectral rounding algorithm used to prove Theorem 2.

**Single-Sweep (SS).** Single-Sweep is an eigenvector rounding method [22,28]. We compute the main eigenvector of the adjacency matrix of the graph and then perform eigenvector rounding based on a linear sweep over the eigenvector entries ordered according to four different criteria: i) non-increasing; ii) non-decreasing; iii) non-increasing absolute values; iv) non decreasing absolute values. Each eigenvector rounding criteria provides at most \(n\) solutions, one for each size of the subgraph. Not all these solutions have a fair balance of 1. Among all the subgraphs, the ones with maximum density and a fair balance of 1 are considered for the final solution.
**Paired Sweep (PS).** Paired-Sweep is a modification of Single-Sweep in which the fairness constraint is always satisfied in each graph produced by the rounding algorithm. This is done by selecting at each step the pair of entries with different discriminant attribute value that appear next in the order.

**2-DFSG Approximation.** The optimal 2-approximation algorithms based on Goldberg’s optimal algorithm for densest subgraph used to prove Theorem 4.

**Fair Single Sweep (FSS).** It is the execution of SS after the fair projections have been applied. It is a variant of the rounding algorithm used in the proof of Theorem 2.

**Fair Paired Sweep (FPS).** It is the execution of PS after the fair projections have been applied.

**k-Means** We further considered the following algorithms, both on the original data set $A$, and on the fair subspace $(I − FF^\top)A$. All variants of the algorithms were tested for all values of $k$ in \{2, ..., 20\}.

**k-means++.** As a baseline algorithm, we used $k$-means++ [6]. It consists of iteratively sampling a center proportionate to the squared Euclidean distances of the previously picked centers. This process is repeated until $k$ centers have been added.

**CKLV.** We implemented the fairlet algorithm by Chierichetti et al. [13], algorithm by first computing a min-cost perfect matching wrt. squared Euclidean distances, followed by running $k$-means++ on the resulting fairlets.

### 5.3 Results

**Densest Fair Subgraph.** The plot on the right shows the distributions of the normalized density, over the entire set of instances, of the fair subgraphs retrieved by different algorithms. The normalization, computed to make solutions for different instances comparable, is done by scaling the maximum density one can obtain for the corresponding graph running [19]. The algorithm that turns out to be the best at retrieving dense fair subgraphs, both in terms of quality and robustness, is a spectral rounding algorithm run on $(I − FF^\top)A(I − FF^\top)$. The FPS algorithm has the highest median density and the smaller deviations from the median. Moreover, it is worth highlighting that the spectral rounding algorithms run on $(I − FF^\top)A(I − FF^\top)$ (FSS and FPS) respectively outperform their counterparts (SS and PS) run on $A$.

We first provide evidence of the statistical significance of the median differences of the normalized density distributions of fair algorithms FSS and FPS. In Table 1 we report $p$-values at level 0.05. These data, in combination with the boxplots of section 5.3, allow us to conclude that the quality of the fair solutions given by FSS and FPS is statistically significant better than the non fair algorithms SS, PS an 2-DFSG that we use for comparison. For FSS and 2-DFSG the medians are not statistically significant different.

| Algorithms | FSS     | FPS     |
|------------|---------|---------|
| SS         | 0.0035  | 7.8465e-234 |
| PS         | 2.2521e-46 | 2.2989e-20 |
| 2-DFSG     | 0.7221  | 1.1957e-300 |

Table 1

In the boxplots of section 5.3 we also observe that the distributions related to SS, FSS and 2-DFSG are skewed toward lower density values. One reason of this fact is that we assign density 0 to retrieved solutions that do not have a fair balance equal to 1.
Figure 2: Results on fair instances.

We report in Table 2 the percentage of instances each algorithm is not able to solve, i.e., for which it does not return a fair solution. We also report in Figure 2 the results obtained on instances for which each algorithm gives a fair solution. The number of these instances is 913 out of 1581.

| Algorithm | % of unfair solutions |
|-----------|-----------------------|
| SS        | 28.59                 |
| FSS       | 35.80                 |
| PS        | 0                     |
| FPS       | 0                     |
| 2-DFSG    | 38.96                 |

Table 2

As we observe in Figure 2, FPS is still the algorithm reporting fair subgraphs with highest density. FPS is also more robust than the others, since 100% of its fair solutions, excluding few outliers, have a normalized density greater than 0.6. On the other hand, SS and FSS algorithms report density distributions skewed towards 0 even if we remove all the unfair solutions from the analysis.

The quality of the application of the algorithms for densest subgraphs to the POLBOOKS data set are illustrated in Table 3. All randomized algorithms were repeated 100 times, with averages being reported in the results.

Our experiments show that the fair algorithms FS and FPS that implement the spectral recovery scheme of Theorem 2 in conjunction with fair projections have the best performances. In particular, for a fairness balance of 1, the algorithms based on fair projections outperform the 2-DFSG approximation algorithm and the linear sweep algorithm SS. However, we also observe that the PS algorithm performs in this case as well as the algorithms for densest subgraph based on fair projections.

\textbf{k-Means} We first observe that the data set does not seem to admit a fair clustering that is close to the best unconstrained clustering; assuming that \textit{k}-means++ is a close approximation to the optimum, the cost of the clustering output by \textit{k}-means++ with fair projections is a lower bound for the cost of an optimal fair clustering. Second, the fair projections passively improve the fairness of the resulting

| Algorithm | density / balance |
|-----------|-------------------|
| 2-DFSG    | 3.57 / 1          |
| SS        | 3.08 / 1          |
| PS        | 4.48 / 1          |
| FS        | 4.48 / 1          |
| FPS       | 4.48 / 1          |

Table 3
clustering, even without specifically optimizing towards it. Lastly, fair projections improve the quality of the CKLV algorithm, while preserving fairness across all clusters.

First, we compared the balance achieved by running fair $k$-means before the fair projection (i.e. on the original data matrix $A$) and after fair projection (i.e. on the data matrix $(I - FF^T)A$). Recall that the balance for a single cluster $C_i$ is defined $B(C_i) = \min \left(\frac{|\text{RED} \cap C_i|}{|\text{BLUE} \cap C_i|}, \frac{|\text{BLUE} \cap C_i|}{|\text{RED} \cap C_i|}\right)$ For a $k$-clustering $C = \{C_1, \ldots, C_k\}$, we extend the notion of balance of $C$ by defining the aggregate balance $B_C = M_p(B(C_1), \ldots, B(C_k))$, where $M_p$ is the power mean $(\frac{1}{n} \sum X_i^p)$. Since we typically want to highlight imbalance, good choices for $p$ are 1 (the arithmetic mean), 0 (the geometric mean) and $-\infty$ (the minimum). Comparisons of these balance values for both ADULTS and ATHLETES are given in Figures 4, 5, and 6.
The \( k \)-means fair clustering algorithms are compared on the ADULTS and the ATHLETES datasets. On the ADULTS data set fair projections had no effect. Before projecting, the Frobenius norm of \( A \) was 1871.88, and the fair projection removed only 31.84 units.

Less than one percentile of the mass was removed. This resulted in essentially identical fairlets and clusters being computed compared to the CKLV algorithm run on \( A \). Nevertheless, the best unfair solution was still cheaper, see Figure 7. In particular, for increasing \( k \), the fair clustering no longer improved with respect to the \( k \)-means cost, whereas the unconstrained solution kept on improving up to the maximum number of chosen clusters 20. The balance, accordingly of the \( k \)-means++ algorithm run on both \( A \) and \((I - FF^\dagger)A\) was virtually the same for all measures.

The ATHLETES data set paints a different picture. Here, we reported a Frobenius norm of roughly 607.22, of which the fair projections removed 311.65 units, i.e. almost 50% of the mass. On the ATHLETES data set, the solution computed by \( k \)-means++ was cheaper than the CKLV algorithm by a significant margin. Unsurprisingly, the former solution was also extremely unfair, achieving an aggregate balance of around 0.2 for most values of \( k \). When applying the fair projections, we observed a significant decrease in cost for for the CKLV algorithm (by nearly 20%), while still computing a fair solution. When running \( k \)-means++ on the fair subspace, the cost significantly decreased wrt to the fair solutions (by roughly 40% for both CKLV and CKLV run in conjunction with fair projections). Compared to unconstrained \( k \)-means++, the solution was roughly twice as expensively, while also being more than three times as balanced, see Figure 8. The balance of \( k \)-means++ when run on \((I - FF^\dagger)A\) also noticeably improved compared to unconstrained \( k \)-means++

6 Discussion and Outlook

In this paper we proposed a spectral technique to enforce soft fairness constraints in problems of interest in machine learning and data mining applications. Our technique virtually applies to any algorithmic technique that relies on an algebraic formulation of a problem of interest. We show-cased application of the fair transform, by studying fair variants of basic clustering problems on graphs.
After showing that fairness constraints make problems computationally harder, we provided a general approach based on the application of fair projections to the original input data.

Our approach can be seen as a fast preprocessing step on the dataset, which tends to remove unfair solutions from the search space of the underlying problem. We analytically showed that our approach used in conjunction with standard spectral methods is able to uncover fair solutions if these are hidden in the input data, and our experiments indicate that it is effective in practice as well.

For example, when applied to identification of fair, dense subgraphs in the Amazon political books’ dataset, a standard densest subgraph algorithm would typically recover a mono-colored subgraph, something not surprising given political polarization reflected in the structure of this co-purchase network. On the other hand, application of fair projections afforded recovery of dense, yet fair subgraphs, reflecting representative subsets of books expressing alternative viewpoints on related topics, a result that suggests potential use as a diversification tool in recommender systems. At a more technical level, we observed that, despite identifying a subgraph with promising density and balance, the quality of the normal eigenvector rounding deteriorated as soon as we attempted to find fairer solutions. In contrast, when applying fair projections, the fair eigenvector recovered far denser subgraphs at higher balance rates.

For \( k \)-means clustering, results generally showed that, especially as the number of protected attributes increases, fair projections can have a significant impact on fairness of the resulting clustering. While the bias removed by the projection may be small compared to the overall variance in the data set, fair projections nevertheless had a large effect on cluster composition, especially for the ATHLETES data set (to a lesser extent for ADULTS data set, when more than two protected attributes were considered).

We interpret these results as suggesting that (1) an approximation algorithm is likely not the correct way to attack this problem, and (2) fair projections are a relatively inexpensive tool to improve fairness of recovery and learning algorithms. In general, we believe the different impact of fair projections itself could potentially be leveraged as a measure of the extent to which bias in a dataset discriminates with respect to a given objective. This might be useful, since fairness constraints might greatly increase cost of a solution, despite little disparate impact being present.

Our work opens up several directions for future research. A first direction is the extension of our work to other applications that have seen the successful approach of spectral methods, as for example recommender systems and text mining. We also plan to extend our work to optimize clustering problems under different objective functions and levels of fairness.

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\(^7\) We emphasize that we use the term “discriminate” without moral judgment: For instance, given biological data with medical applications like the ATHLETES data set, separation along sex may be necessary and desirable.
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A Missing proofs

A.1 Proof of Theorem 1

We use the following, simple characterization of $FF^\top$ in terms of Haar-matrices. Recall that the $\ell$ by $\ell$ Haar-matrix is recursively defined as $H^\ell = \begin{bmatrix} H^{\ell/2} \otimes [1,1] \\ H^{\ell/2} \otimes [1,-1] \end{bmatrix}$, where the Kronecker product $A \otimes B$ is defined as $A \otimes B := \begin{bmatrix} A_{1,1}B & A_{1,2}B & \cdots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \cdots & A_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}B & A_{n,2}B & \cdots & A_{n,n}B \end{bmatrix}$. Further, the Hadamard product between two matrices $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{n \times m}$ is defined as $(A \circ B)_{i,j} = A_{i,j}B_{i,j}$.

Lemma 5. Let $M$ be a $n \times d$ matrix with a coloring $c : [n] \rightarrow [\ell]$ of the rows, where $\ell$ is a power of 2. Define the $n \times \ell$ orthogonal matrix $F$ with columns
\begin{align*}
F_{:,j} &= \frac{1}{\sqrt{|i \in [n] \mid c(i) = c(j)|}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \circ H^\ell_{c(j)}.
\end{align*}

Then $I - FF^\top$ induces the diverse subspace of $M$.

Proof. The columns of $F$ are orthogonal as the Haar matrix itself is orthogonal. We further note that the number of non-zero entries in a column of $F$ exactly corresponds to $\sum_{i=1}^{\ell} \mathbb{1}_{H_{c(i)+1,j}} \neq 0$, i.e. the columns are also normalized. What remains to be shown is that the kernel of $F$ spans the fair subspace. We prove this by induction. We first note that for $H_1 := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ the second row exactly corresponds to $F$ in the case of only two colors. Assume now that all fairness conditions for $2^{k-1}$ colors are in the span of the $2^{k-1}$-dimensional Haar-matrix $H_{2^{k-1}}$. When considering $2^{k}$ colors, these conditions are still obeyed between the pairs of colors $(j, j+1)$ for odd $j \in [2^k]$ due to the Kronecker product $I_{2^{k-1}} \otimes [1,-1]$. Let us now ”fuse” these colors, i.e. we consider $(j, j+1)$ to be a single color, leaving us with $2^{k-1}$ colors. Since the kernel of the bottom $2^{k-1} - 1$ rows of $H_{2^{k-1}}$ spans the fair subspace of $2^{k-1}$ colors, the kernel of the bottom $2^{k-1} - 1$ rows of the Kronecker product $H_{2^{k-1}} \otimes [1,1]$ contains the fair subspace of all fused colors. 

We use this lemma as follows. We modify $M$ such that the the rows are ordered by color, i.e. $M_1, \ldots, M_{k'}$ are colored 1, $M_{k'+1}, \ldots, M_{k''}$ are colored 2, etc. Then we ”fuse” the rows of equal color such that $M'_1 = \sum_{c(j)=i} M_{c(i)}$, obtaining an $\ell$ by $n$ matrix in the process. Observing $F^\top M + \frac{1}{\sqrt{n}} M^\top = H_{k'}M'$, and that $\frac{1}{\sqrt{n}} M^\top$ requires $O(n \cdot d)$ time, the total running time is dominated by the generation of $M'$ which runs in time $O(n \cdot d)$ and applying the fast Haar transform on each of the $d$ columns of $M'$ running in time $O(d \cdot \ell)$ for $\ell$-dimensional vectors [20].