The one-loop form factors in the effective action, and
production of coherent gravitons from the vacuum.

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Abstract

We present the solution of the problem of the $1/\Box, \Box \to 0$, asymptotic terms discovered in the one-loop form factors of the gravitational effective action. Owing to certain constraints among their coefficients, which we establish, these terms cancel in the vacuum stress tensor and do not violate the asymptotic flatness of the expectation value of the metric. They reappear, however, in the Riemann tensor of this metric and stand for a new effect: a radiation of gravitational waves induced by the vacuum stress. This coherent radiation caused by the backreaction adds to the noncoherent radiation caused by the pair creation in the case where the initial state provides the vacuum stress tensor with a quadrupole moment.
1 Introduction

We consider the expectation-value equations for the gravitational field in an in-state \( |\psi \rangle \). The model-independent, or phenomenological approach [4-7] makes it possible to write down the general form of these equations in terms of the form factors in the vacuum action. The form factors are to be calculated from a given dynamical model. However, for obtaining predictions of various models, the expectation-value equations should be first analysed with arbitrary form factors in order to relate the properties of the form factors to the important properties of the solution [4].

This analysis has thus far been limited to the behaviour of the vacuum stress tensor \( T^\mu_\nu_{\text{vac}} \) at infinity [6,7]. It was shown [6] that the requirement of asymptotic flatness of the solution imposes restrictions on the behaviours of the form factors at small values of their arguments. Namely, these behaviours in one (each) of the arguments with the others fixed should be \( w(0) \log(-\Box) + O(1), \Box \to -0 \). The coefficients \( w(0) \) of the \( \log(-\Box), \Box \to -0 \) behaviours (the spectral weights at zero spectral mass) determine the rate of the vacuum radiation through the future null infinity (\( \mathcal{I}^+ \)).

However, at each given order in the curvature, only certain combinations of the form factors must behave in this way. The condition of finiteness of \( T^\mu_\nu_{\text{vac}} \) at \( \mathcal{I}^+ \) which brought to the result above leaves some arbitrariness in the asymptotic behaviours of the individual form factors in the basis decomposition of the action. It is this arbitrariness that allows for the existence of the effect discussed in the present paper.

The present paper deals with the problem which appeared when the field-theoretic form factors were calculated. It is worth noting that, as emphasized in [6], the loop expansion of field theory has a domain of validity. It is near null infinity where the

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1 The gravitational collapse problem was first considered in this setting in refs. [1-3].
results of this expansion are valid and can be used to calculate the energy of the vacuum radiation. The ultraviolet divergent terms which appear in the expectation-value equations when $T_{\text{vac}}^{\mu\nu}$ is expanded in loops are local and vanish at infinity; only nonlocal terms survive, and these are unambiguous. Technically, the loop expanded form factors are reliable in the limit of small $\Box$ arguments up to terms $O(\Box^0), \Box \rightarrow -0$.

The field-theoretic form factors have been calculated in the one-loop approximation for a generic quantum field model in refs. [8-15]. Their asymptotic behaviours at $\Box \rightarrow -0$ up to terms $O(\Box^0)$, which are of interest for the above-mentioned reason, are presented in ref. [15], and these behaviours offer a problem. While the second-order form factors behave as expected: they are $w(0) \log(-\Box) + O(1)$ for $\Box \rightarrow -0$, the third-order form factors contain also the asymptotic terms $1/\Box, \Box \rightarrow -0$ which apparently violate the asymptotic flatness of the solution [6,15].

Since the third-order form factors are functions of three $\Box$ arguments, their behaviours in one of the arguments with the two others fixed cannot be predicted on dimensional grounds. The $1/\Box$ asymptotic terms in the form factors appear as a result of an explicit calculation of loops [12,15]. In the present paper we propose an explanation of this result as well as of the following remarkable fact [12] which one can establish by a direct inspection of the expressions in [15]. The inspection shows that the alarming $1/\Box$ terms appear only in the curvature invariants containing the gravitational field strength and act selectively only on the Ricci curvature. The matter field strengths contained in the commutator and potential curvatures (see [15]) remain unaffected by these terms. Thus the presence of the $1/\Box$ terms breaks the democracy of massless vacuum particles; gravitons appear to be distinguished.

Since the same vacuum action describes also the transition amplitudes between in- and out- states [8], the problem of the $1/\Box$ terms appears also in scattering theory where these terms either signal an infrared divergence of the on-shell amplitudes.
with gravitons or, in the favourable case, stand for some inelastic process. In the language of expectation values, the former case corresponds to a breakdown of the asymptotic flatness. That the situation is not hopeless is seen from the fact that the form factors are not quite the vertices; they are coefficients of the curvatures rather than the field disturbances. As compared to the field disturbances, the curvatures contain extra derivatives which, in the favourable case, may cancel the $1/\Box$ factors in the on-shell amplitudes.

Below we present the solution of the problem as it appears in expectation-value theory. As mentioned above, only certain combinations of the form factors should behave like $\log(-\Box), \Box \to -0$ to ensure finiteness of $T_{\text{vac}}^{\mu \nu}$ at $\mathcal{I}^+$. We show that the $1/\Box$ terms precisely cancel in these combinations leaving indeed the $\log(-\Box)$ behaviour as the leading one. The cancellation occurs owing to certain constraints among the coefficients of the $1/\Box$ terms which we establish by an analysis of the asymptotic flatness and next check with the explicit expressions in [15]. The fulfilment of these constraints is by itself a powerful check on the results in [15] apart from the checks that have already been carried out in [12]. Thus we prove that the $1/\Box, \Box \to -0$ terms discovered in [12,15] do not violate the asymptotic flatness of the solution of the expectation-value equations. The proof is given for a generic quantum field model for which the results in [15] are obtained and which is characterized by a set of field strengths consisting of the Ricci, commutator, and potential curvatures. Such a general proof is possible owing to the above-mentioned fact that the $1/\Box$ operators in the asymptotic expressions for the form factors act only on the Ricci curvature. Therefore, for the consideration of the leading asymptotic terms of the equations at $\mathcal{I}^+$, one does not need to know the variational derivatives of the commutator and potential curvatures with respect to the metric.

Next, we reveal the significance of the $1/\Box, \Box \to -0$ terms in the vacuum form factors. These terms vanish in the energy-momentum tensor but, as we show, they
reappear in the Riemann tensor of the solution and determine its leading \(O(1/r)\) behaviour at \(I^+\). The coefficient of this behaviour is known to give the energy of the outgoing gravitational waves. Thus the \(1/\Box, \Box \to -0\) terms of the vacuum form factors discovered in [12,15] stand for a new effect: a generation of the gravitational waves from the vacuum. This is an effect of the backreaction of the vacuum stress on the metric. All massless particles including gravitons [16] contribute to \(T_{\mu\nu}^{\text{vac}}\) and are radiated through the future null infinity by the quantum mechanism of pair creation. The energy of this component of radiation is determined by the \(\log(-\Box), \Box \to -0\) terms of the vacuum form factors. On top of this, \(T_{\mu\nu}^{\text{vac}}\) as a whole acts as a source of the gravitational field and causes a secondary radiation of gravitons. This component of radiation has the shape of a classical wave but with a quantum amplitude, and its energy is determined by the \(1/\Box, \Box \to -0\) terms of the vacuum form factors.

An important difference between the two cases is that the gravitational-wave component will be nonvanishing only if the initial state has a sufficient asymmetry to provide the vacuum stress with a quadrupole moment whereas the contribution of gravitons in \(T_{\mu\nu}^{\text{vac}}|_{I^+}\) is present even in a spherically symmetric in-state because the out-states in which these gravitons appear at \(I^+\) are squeezed vacuum states rather than coherent states (see [17,18] and references therein).

The paper is organized as follows. In sec. 2 we briefly review the structure of the expectation-value equations and present a new expression for the solution of the Bianchi identities to second order in the curvature. This expression simplifies obtaining the news functions of the gravitational waves. Sec. 3 contains an analysis of contributions to \(T_{\mu\nu}^{\text{vac}}\) at \(I^+\). This analysis is not complete but sufficient for obtaining the asymptotic flatness constraints and for calculating \(T_{\mu\nu}^{\text{vac}}|_{I^+}\) to second order in the curvature. We point out an important distinction of this calculation from the one in two dimensions [3], and also make some first step in giving the expectation-value
equations a closed form. Sec. 4 contains the derivation of the asymptotic flatness constraints and the proof that they are satisfied by the one-loop form factors. In sec. 5 we propose a new method for calculating the energy of the gravitational waves and obtain the contribution of the third-order vacuum form factors to the news functions. Sec. 6 completes the calculation of the vacuum news functions in the lowest nonvanishing approximation. Appendix A contains reference equations pertaining to the behaviour of the asymptotically flat metric at null infinity. Appendix B summarizes the properties of the retarded Green function used in the text.

2 The expectation-value equations in an in-state

In the framework of quantum field theory one starts with the assumption that there exists a quantum state such that the expectation value of the metric in this state is an asymptotically flat gravitational field. Under this assumption one goes to the past null infinity ($I^-$) of the spacetime with the expectation value of the metric and, for all massless fields, builds the Fock space of states (the in-states) based on the standard in-vacuum [19]. The assumed state belongs to this space. The choice of the state determines the initial data at $I^-$ for the field’s expectation values and, generally, affects also the form of their dynamical equations since these equations are nonlocal. There exists an action which produces the expectation-value equations although the procedure by which it does so is not the least-action principle (see below).

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[^2]: After the choice of the state has been made, one checks the original assumption. Since the causality relationship is set by the expectation value of the metric, this is a self-consistent problem even in its original setting. That the consistency check is nontrivial is seen, for example, from the fact that with the massive quantum fields one generally arrives at a contradiction with the asymptotic flatness.
It makes sense to choose the initial state in which the matter quanta form some heavy classically behaved source of the gravitational field, and gravitons are in a coherent state so that, for the mean metric, they form, generally, a classical incoming gravitational wave. Here we consider the case where such a wave is absent. The action for the expectation value of the metric in such a state can be taken as the sum

\[ S = S_{\text{vac}} + S_{\text{source}} \] (2.1)

where \( S_{\text{source}} \) is the action of a source which moves along a classical trajectory in the mean metric, and \( S_{\text{vac}} \) is the action for the gravitational field in the in-vacuum state.

The action \( S_{\text{vac}} \) is to be calculated from a given quantum field model. Within certain approximations (which are not completely unsatisfactory, see above) this calculation is feasible and, in the one-loop approximation, it can be done for a generic field model [8-15]. However, since the gravitational interaction is universal, all particles existing in nature contribute to \( S_{\text{vac}} \), and, at higher loop orders, all details of their interactions matter. Therefore, if \( S_{\text{vac}} \) is to be ultimately calculated from a model (of fields or strings or whatever), then this should be the Model and the Calculation.

In the phenomenological approach of refs. [4-7], the action (2.1) is viewed as an effective action (in the loose sense) for the observable field which should be a part of predictions of any fundamental dynamical theory. Irrespectively of the nature of this theory, one assumes the existence of a functional, the action \( S_{\text{vac}} \), which describes the elastic properties of real vacuum i.e. its response to the introduction of a gravitationally charged source. For \( S_{\text{vac}} \) one writes down the most general expansion in terms of nonlocal invariants of \( N \)th order in the curvature. One has to go explicitly to \( N = 3 \) because third order in the curvature in the action corresponds to second order in the equations, and it has been shown that, at first order in the
curvature, the flux of vacuum energy through $\mathcal{I}^+$ is pure quantum noise [7]. The full bases of nonlocal invariants of second order and third order are built in [5] for a set of field strengths consisting of the Ricci, commutator and potential curvatures:

$$\mathcal{R} = \{ R_{\mu\nu}, \tilde{R}_{\mu\nu}, \tilde{P} \}$$ (2.2)

(for the definitions see [15]). The explanation for the absence of the Riemann tensor from the basis invariants can be found in [9,4,5] but it makes sense to repeat it here since the respective equations will be of use below.

By differentiating and contracting the Bianchi identities, one obtains the equation

$$\Box R^\alpha{}_{\mu\nu} = 2\nabla[\mu \nabla^\alpha R^\nu]_{\beta} - 2\nabla[\mu \nabla^\beta R^\nu]_{\alpha} - 4R^\alpha{}_{\gamma, \sigma} R^\beta{}_{\gamma\nu\sigma} + 2R^\gamma{}_{\nu} R^\alpha{}_{\beta\gamma\nu} - R^\alpha{}_{\gamma\sigma} R^\mu{}_{\nu\gamma\sigma}$$ (2.3)

which can be solved iteratively with respect to the Riemann tensor. In this equation the Ricci tensor plays the role of a source, and the solution is fixed by the initial data for the gravitational field at $\mathcal{I}^-$. The solution with zero data (no incoming gravitational wave) corresponds to the in-vacuum state and is expressed in terms of the retarded Green function (see appendix B):

$$R^\alpha{}_{\mu\nu} = \frac{1}{\Box} \left( 4\nabla[\mu \nabla^\alpha R^\nu]_{\beta} + O[R^2] \right).$$ (2.4a)

Here and below, $1/\Box$ stands for the retarded Green function, and both types of brackets $[\ldots]$ and $<>\ldots$ denote antisymmetrization of the respective indices. Since the Riemann tensor is expressed in this way through the Ricci tensor, the nonlocal invariants with the Riemann tensor in the vacuum action are redundant.

Below we shall confine ourselves to the case where the flux components of the Ricci tensor at $\mathcal{I}^-$ vanish [3]. In this case the derivatives in (2.4a) can be made
external by commuting them with the Green function $1/\Box$. To second order in the Ricci curvature the solution is then of the form

$$R^{\alpha\beta\mu\nu} = \nabla^{[\mu} \nabla^{\alpha < \delta} \left( \frac{4}{\Box} \right) \left[ R^{\nu]\beta >} + \left( \nabla^{[\nu} \frac{1}{\Box} R^{\gamma]\delta} \right) \left( \nabla^{\delta >} \frac{1}{\Box} R_{\gamma\delta} \right) - 2 \left( \nabla^{\gamma} \frac{1}{\Box} R^{\nu]\delta} \right) \left( \nabla^{\delta} \frac{1}{\Box} R^{\beta>] \gamma} \right) \right] + 8g_{\gamma\delta} \left( \frac{1}{\Box} \nabla^{[\alpha} R^{\beta]} R^{\nu]|\gamma| \right) \left( \frac{1}{\Box} \nabla^{[\beta} R^{\nu]|\delta} \right) + 8g_{\gamma\delta} \left( \frac{1}{\Box} \nabla^{[\mu} R^{\nu]|\alpha} \right) \left( \frac{1}{\Box} \nabla^{[\nu} R^{\beta]>\delta} \right) + 8g_{\gamma\delta} \left( \frac{1}{\Box} \nabla^{[\mu} R^{\nu]} R^{[\alpha} \right) \left( \frac{1}{\Box} \nabla^{[\nu} R^{\beta]>\delta} \right) - 2 \left( \nabla^{\gamma} \frac{1}{\Box} R^{\mu\nu} \right) \left( \frac{1}{\Box} \nabla^{[\alpha} R^{\beta]>\gamma} \right) \left( \frac{1}{\Box} R_{\gamma\delta} \right) \right) + O[R^3]$$

(cf. the result in [5,12]). The advantage of making the derivatives external is in the appearance of the terms in the curly brackets in (2.4b) which have no overall $1/\Box$ factor. These terms do not contribute to the leading asymptotic behaviour of the Riemann tensor at $\mathcal{I}^+$. On the other hand, the terms which have the overall $1/\Box$ factor have also the overall derivatives. This facilitates solving the equation for the news functions of the gravitational waves (see sec. 5).

To third order in the curvature the vacuum action is of the form

$$S_{\text{vac}} = S(1) + S(2) + S(3) + O[\Re^4], \quad (2.5)$$

$$S(1) = \frac{1}{16\pi} \int dx g^{1/2} R, \quad (2.6)$$

$$S(2) = \frac{1}{2(4\pi)^2} \int dx g^{1/2} \sum_{i=1}^5 \gamma_i \left( -\Box_2 \right) R_1 R_2(i), \quad (2.7)$$

which can be taken, e.g., as having a compact spatial support. Then, since the energy-momentum tensor of the in-vacuum has no incoming fluxes at $\mathcal{I}^-$, the same will be true also of $R^\mu\nu$ (see eq. (2.18) below). In the general case, the flux components of $R^\mu\nu$ at $\mathcal{I}^-$ cancel in the combination $\nabla^{[\mu} \nabla^{\alpha} R^{\nu]|\beta} > \gamma$ appearing in (2.4a) but the derivatives cannot be commuted with $1/\Box$ for otherwise the action of the retard Green function will become ill-defined.

We follow the notation of ref. [15] but change the overall sign of the action as appropriate for the lorentzian signature of the metric. We use the signature $(-++; +)\ldots$ and the conventions $R^\mu_{\alpha\nu\beta} = \partial_\nu \Gamma^\mu_{\alpha\beta} - \ldots$, $R_{\alpha\beta} = R^\mu_{\alpha\mu\beta}$, $R = g^{\alpha\beta} R_{\alpha\beta}$.  

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where $\mathcal{R}_1 \mathcal{R}_2(i)$ with $i = 1$ to 5, and $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$ with $i = 1$ to 29 are the quadratic and cubic basis invariants listed in [15]. This list is reduced in comparison to the full list in [5] because, in the trace of the heat kernel and hence in the one-loop vacuum action, the invariants linear in the commutator curvature all but one prove to be absent [12,13]. The only one that is present is number 13 in the list of ref. [15] which we use here. In low-dimensional manifolds there exist hidden constraints between nonlocal invariants, reducing the basis. In four dimensions, the second-order basis is unconstrained, and the only constraint which exists among the third-order invariants boils down to the condition that the completely symmetric part of the form factor $\Gamma_{28}$ vanishes identically [5,12]. In the field-theoretic form factors of refs. [12,15] this condition is explicitly implemented.

The commutator and potential curvatures are functions of the metric and matter fields, different in different models, but in any case their contribution to the purely gravitational sector of the action boils down to a modification of the form factors of the basis invariants with the Ricci tensor only. There are only two such in $S(2)$ and ten in $S(3)$. Below, when referring to the purely gravitational form factors, we shall assume that this reduction has already been made. The full set of invariants for gravity and matter is considered here because it is important that the maintenance of asymptotic flatness be proved for the one-loop action in full generality. Apart from this, our main concern in the discussion below is the vacuum action for the metric.

The functions $\gamma_i$ and $\Gamma_i$ in (2.7) and (2.8) are the second-order and third-order form factors. The principal assumption about these and higher-order form factors made in the axiomatic approach is their analyticity which allows one to put them in
the spectral forms [4]. For example, the spectral form used in [6] for the lowest-order form factors is

\[ \gamma(-\Box) = (\Box + \mu^2)^n \int_0^\infty \frac{dm^2}{m^2 - \Box(m^2 + \mu^2)^n} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} (\Box + \mu^2)^k \left( \frac{\partial}{\partial \mu^2} \right)^k \gamma(\mu^2) \]  

(2.10)

where \( w(m^2) \) is the spectral weight

\[ w(m^2) = \frac{1}{2 \pi i} \left[ \gamma(-m^2 - i0) - \gamma(-m^2 + i0) \right] , \]  

(2.11)

\( \mu^2 > 0 \) is an arbitrary parameter on which \( \gamma(-\Box) \) does not depend, and \( n \) is the degree of growth of \( \gamma(-\Box) \) at large \( \Box \) which will presumably be fixed or bounded by the requirement of regularity of the solution (see [1-3]). The requirement of asymptotic flatness of the solution imposes restrictions only on the small-\( \Box \) behaviours of the form factors [6,7]. In the small-\( \Box \) limit, eq. (2.10) reduces to the simple spectral form since the terms modifying this form for \( n > 0 \) vanish in this limit [6]. On the other hand, this form should be generalized to allow for the behaviour \( 1/\Box, |\Box| \to 0 \) of \( \gamma(-\Box) \). It remained unnoticed in paper [6] that the derivation in this paper for the two purely gravitational second-order form factors brings in fact to the following general result:

\[ \gamma_1(-\Box) = -\frac{2a}{\Box} - w_1(0) \log(-\Box) + O(1), \quad -\Box \to 0 \]  

(2.12)

\[ \gamma_2(-\Box) = \frac{a}{\Box} - w_2(0) \log(-\Box) + O(1), \quad -\Box \to 0 \]  

(2.13)

in which there appears an arbitrary constant \( a \). Only the combination

\[ \gamma_1(-\Box) + 2\gamma_2(-\Box) = -\left( w_1(0) + 2w_2(0) \right) \log(-\Box) + O(1), \quad -\Box \to 0 \]  

(2.14)

\[ ^5 \text{Eq. (39) of ref. [6] admits one more solution: } \Box \left[ \gamma_1(-\Box) + 3\gamma_2(-\Box) \right] \to a, -\Box \to 0 \text{ which had been overlooked.} \]
should behave like \( \log(-\Box) \) by the analysis in [6]. As will be seen below, a similar situation takes place for the higher-order form factors.

In the form factors \( \gamma_i \) calculated from field theory, \( a = 0 \) [9,15] since, by dimension, the terms \( 1/\Box \) cannot appear in the loop expansion of the second-order form factors. They appear, however, already in the third-order form factors \( \Gamma_i \) [15]. For the generalized spectral forms of these form factors see [10,12].

With the form factors in the spectral forms, the only nonlocal structure that remains in \( S_{\text{vac}} \) is the inverse operator \( 1/(m^2 - \Box) \). This simplifies the procedure of obtaining the expectation-value equations. When the action \( S_{\text{vac}} \) is varied, the inverse operators are regarded as obeying the variational rule

\[
\delta \frac{1}{m^2 - \Box} = \frac{1}{m^2 - \Box} \delta \frac{1}{m^2 - \Box}, \tag{2.15}
\]

and, after the variation has been completed, all inverse operators are replaced by the retarded Green functions [6]. If

\[
\left. \frac{\delta S_{\text{vac}}}{\delta g_{\mu\nu}(x)} \right|_{\Box \to \Box_{\text{ret}}} \tag{2.16}
\]

is to denote the result of this procedure, and a further notation is introduced to separate the classical term of \( S_{\text{vac}} \):

\[
T_{\text{vac}}^{\mu\nu} \equiv \frac{2}{g^{1/2}} \left. \frac{\delta S_{\text{vac}}}{\delta g_{\mu\nu}} \right|_{\Box \to \Box_{\text{ret}}} - \frac{2}{g^{1/2}} \left. \frac{\delta S(1)}{\delta g_{\mu\nu}} \right|_{\Box \to \Box_{\text{ret}}} \tag{2.17}
\]

then the expectation-value equations corresponding to the action (2.1) are of the form

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi \left( T_{\text{vac}}^{\mu\nu} + T_{\text{source}}^{\mu\nu} \right) \tag{2.18}
\]

where

\[
T_{\text{source}}^{\mu\nu} = \frac{2}{g^{1/2}} \left. \frac{\delta S_{\text{source}}}{\delta g_{\mu\nu}} \right|_{\Box \to \Box_{\text{ret}}} \tag{2.19}
\]

\[6\]For the derivation of this procedure in QFT see [8] and references therein. In the phenomenological approach this set of rules is taken for granted [4].
and $T_{\mu\nu}^{\text{vac}}$ in (2.17) can be interpreted as the energy-momentum tensor of the in-vacuum. These equations are to be solved with zero initial data for the gravitational field at $\mathcal{I}^-$. One arrives at a Cauchy problem [2,3] for nonlocal equations with the retarded kernels which are to be integrated from $\mathcal{I}^-$ to the future until the solution hits a singularity if there remains one. One hopes that it doesn’t.

We do not consider the more general initial data, with a gravitational wave at $\mathcal{I}^-$, because in this case the action should also be calculated more generally. Specifically, the solution of eq. (2.3) can no more be taken in the form (2.4).

3 The structure of $T_{\mu\nu}^{\text{vac}}$ at $\mathcal{I}^+$

It is natural to begin the study of the expectation-value equations with the behaviour of $T_{\mu\nu}^{\text{vac}}$ at null infinity since the nonlocal terms of the equations should be responsible for the effect of the vacuum radiation.

At the future null infinity one has the Bondi-Sachs equation [20,21] (see also appendix A) which is an exact consequence of the expectation-value equations:

\[
\frac{-dM(u)}{du} = \frac{1}{4\pi} \int d^2 \mathcal{S} \left[ \left( \frac{\partial}{\partial u} C_1 \right)^2 + \left( \frac{\partial}{\partial u} C_2 \right)^2 \right] + \int d^2 \mathcal{S} \left( \frac{1}{4} r^2 T_{\mu\nu}^\text{source} \nabla_\mu v \nabla_\nu v + \frac{1}{4} r^2 T_{\nu\rho}^\text{vac} \nabla_\mu v \nabla_\nu v \right) \bigg|_{\mathcal{I}^+},
\]

\[
(\nabla u)^2 = 0, \quad (\nabla u, \nabla r) \bigg|_{\mathcal{I}^+} = -1, \quad (3.2)
\]

\[
(\nabla v)^2 = 0, \quad (\nabla u, \nabla v) \bigg|_{\mathcal{I}^+} = -2. \quad (3.3)
\]

Here $u$ is the retarded time along $\mathcal{I}^+$ with the natural normalization in (3.2), the integrals are over the 2-sphere $\mathcal{S}$ (normalized to have the area $4\pi$) at which the null congruence $u = \text{const.}$ crosses $\mathcal{I}^+$, $r$ is the luminosity distance along the rays of
this congruence, \( M(u) \) is the Bondi mass, and \( \partial C_1/\partial u, \partial C_2/\partial u \) are the Bondi-Sachs news functions of the gravitational waves.

Eq. (3.1) is the conservation law missing in the theory of quantum fields on a fixed gravitational background. In the collapse problem, this is the backreaction equation relating ”the changing mass of the black hole” with the energy of the quanta radiated by this black hole. In full quantum theory, both the ”black hole” and the quantum fields ”on its background” evolve from one and the same initial state, and one is able to answer the question where does the black-hole radiation take its energy from. It takes it ultimately from the energy of the collapsing source \( T^\mu_\nu_{\text{source}} \) which equals the ADM mass of the expectation value of the metric and serves as an initial datum \( M(-\infty) \) for eq. (3.1).

The last term of eq. (3.1) is the flux of the vacuum energy through \( \mathcal{I}^+ \). For it to be finite, the flux component of \( T^\mu_\nu_{\text{vac}} \) should decrease at \( \mathcal{I}^+ \) like \( 1/r^2 \). This is a necessary condition of asymptotic flatness. Below, terms \( O(1/r^3) \) in \( T^\mu_\nu_{\text{vac}} \) will be referred to as vanishing at \( \mathcal{I}^+ \).

When computing \( T^\mu_\nu_{\text{vac}}(x) \) at \( \mathcal{I}^+ \) from the action (2.5), all terms in which the Ricci curvature appears at the observation point \( x \) can be discarded because \( R^\mu_\nu \) decreases at least like \( 1/r^2 \) and will always be multiplied by a decreasing function. Thus, the covariant derivatives \( \nabla \) which appear in the basis invariants \( \mathcal{R}_\mu^\nu(\mathcal{R}) \), etc. need not be varied because the contributions of their variations to \( T^\mu_\nu_{\text{vac}} \) contain the curvature at the observation point. For the same reason, in the expression

\[
\delta R^\gamma_\nu = \frac{1}{2} g^\gamma_\mu \left( \nabla_\mu \nabla^\alpha \delta g_{\nu\alpha} + \nabla_\nu \nabla^\alpha \delta g_{\mu\alpha} - \nabla_\mu \nabla_\nu g^\alpha_\beta \delta g_{\alpha\beta} - \Box \delta g_{\mu\nu} + 2 R^\alpha_\beta g_{\mu\alpha} \delta g_{\nu\beta} \right)
\]

\[
+ \frac{1}{2} R^\alpha_\beta \delta g_{\nu\alpha} - \frac{1}{2} g^\gamma_\alpha R^\alpha_\beta \delta g_{\nu\beta} ,
\]

(3.4)

the terms with the Ricci tensor can be discarded but the term with the Riemann tensor cannot since the Riemann tensor has components decreasing like \( 1/r \). In (3.4), the expression for \( \delta R^\gamma_\nu \) has been brought by commutations to the form used below.
Only the variations of the Ricci tensors and the variations of the form factors in $S_{\text{vac}}$ can give nonvanishing contributions to $T_{\text{vac}}^{\mu\nu}$ at $\mathcal{I}^+$. It is easy to see that the variation of a form factor

$$\Gamma(-\Box_1, -\Box_2, ..., -\Box_N)$$

in the argument $\Box_p$ can contribute at $\mathcal{I}^+$ only if, in this argument, the form factor behaves like $1/\Box_p$, $\Box_p \to -0$. Thus, assuming $a = 0$ in eqs. (2.12), (2.13), as is the case in the field theoretic form factors, one can calculate the variations of $\gamma_i(-\Box)$ in the action (2.7) by using the spectral form (2.10) in which, moreover, the terms appearing at $n > 0$ can be disregarded when the observation point tends to $\mathcal{I}^+$. One finds

$$\int dx g^{1/2} \mathcal{R}_1 \delta \gamma(-\Box) \mathcal{R}_2 = \int dx g^{1/2} \int_0^\infty dm^2 w(m^2) \left( \frac{1}{m^2 - \Box} \mathcal{R}_1 \right) \delta \Box \left( \frac{1}{m^2 - \Box} \mathcal{R}_2 \right).$$

Since, by the result in [6],

$$\left( \frac{1}{(m^2 - \Box_{\text{ret}})} \mathcal{R}(x) \right) \bigg|_{x \to \mathcal{I}^+} \propto r^{-1}(x) \exp(-|\text{const.|}m \sqrt{r(x)}) (1 + \mathcal{O}),$$

$$\mathcal{O} \to 0, r(x) \to \infty, x \to \mathcal{I}^+,$$

and

$$w(0) = \text{finite},$$

one concludes that the contribution of $\delta \gamma(-\Box)$ to $T_{\text{vac}}^{\mu\nu}$ at $\mathcal{I}^+$ is $O(1/r^3)$. This result applies to all form factor of the form (3.5) provided that their behaviours in individual arguments are $O(\log(-\Box))$, $\Box \to -0$. On the other hand, with the behaviour

$$\Gamma(-\Box_1, -\Box_2, -\Box_3, ...) = \frac{1}{\Box_1} F(\Box_2, \Box_3, ...) (1 + \mathcal{O}), \quad \mathcal{O} \to 0, \Box_1 \to -0$$

the variation of $\Box_1$ in the term

$$\int dx g^{1/2} \Gamma(-\Box_1, -\Box_2, -\Box_3, ...) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \ldots$$

(3.10)
of the action is asymptotically of the form

$$- \int dx g^{1/2} \left( \frac{1}{\Box} \mathcal{R}_1 \right) \delta \Box \frac{1}{\Box} \left( F(\Box_2, \Box_3, \ldots) \mathcal{R}_2 \mathcal{R}_3 \ldots \right)$$  \quad (3.11)$$

and gives a nonvanishing contribution to $T_{\mu\nu}^{\text{vac}}$ at $I^+$ proportional to $1/r^2$.

The fact that, in four dimensions, the variations of the second-order form factors do not contribute to the energy flux at infinity makes an important distinction of this case from the case in two dimensions where $\gamma(-\Box) \propto 1/\Box$ and the relevant contribution comes from $\delta \gamma(-\Box)$ [3]. The variations of the third-order form factors $\Gamma_i$ in (2.8) can already contribute to $T_{\mu\nu}^{\text{vac}}$ at $I^+$ owing to the $1/\Box, \Box \to -0$ terms discovered in [15] but, to second order in the curvature (in the equations), only the Ricci tensors in the action $S(3)$ need to be varied.

For all terms of the vacuum action, in which the form factors are of the form (3.5)\footnote{We do not consider here the more general form factors $\Gamma(-\Box_1, \ldots, -\Box_N, -\Box_{1+2}, -\Box_{1+3}, \ldots)$ in which the operator arguments $\Box_{n+m}$ act on products of two curvatures since such form factors appear in $S_{\text{vac}}$ only beginning with $N = 4$ [4,5].}, it is useful to introduce a quantity, the generalized current, defined by varying the action with respect to the Ricci tensor only. The sum of such terms in $S_{\text{vac}}$ can be represented in the form

$$\tilde{S}_{\text{vac}} = \frac{1}{2(4\pi)^2} \sum_N \sum_i S_i(N) \quad (3.12)$$

$$S_i(N) = \int dx g^{1/2} P_1(\nabla_1) \ldots P_N(\nabla_N) \Gamma(-\Box_1, \ldots, -\Box_N) R(x_1) \ldots R(x_N) \quad (3.13)$$

where (3.13) is a contribution of the $N$th order basis invariant number \\

"$i$", and the sum in (3.12) extends over both $i$ and $N$. In (3.13), $R_i$ are the Ricci tensors with mixed indices, and the polynomials in covariant derivatives $P_n(\nabla_n)$ which are generally present in the tensor basis [15,5] act on the respective $R_i(x_n)$ after the action of the operator arguments $\Box_n$ of $\Gamma$. In this representation all operators $\Box_n$ in $\Gamma$ are uniformly defined as applied to a mixed second-rank tensor, and the advantage
of taking it mixed is in the absence of $g^{\mu\nu}$ factors contracting indices in (3.13) which otherwise would need to be varied. These factors are generally present in the polynomials $P(\nabla)$ but, in any case, their variations (as well as the variation of $g^{1/2}$ in the measure) do not contribute to $T_{\mu\nu}^{\text{vac}}$ at $I^+$. The order of operations in (3.13) is different from the one in (2.8) (cf. the explicit expressions in [15]) but, at every order in the curvature, the action can be brought to the form (3.13) by commutations.

Since the argument $\Box_n$ of $\Gamma$ is the first operator acting on $R(x_n)$ in (3.13), it will be the last in the variational derivative of (3.13) with respect to $R(x_n)$. The generalized current $I_\mu^\nu(\xi, x)$ is then defined by the relation

$$\delta R \tilde{S}_{\text{vac}} = \frac{1}{2(4\pi)^2} \int dx g^{1/2} I_\mu^\nu(-\Box, x) \delta R_\mu(x) = \frac{1}{2(4\pi)^2} \int dx g^{1/2} I_\mu^\nu(-\Box, x) \delta R_\mu(x)$$

(3.14)

where the notation $\delta R$ points out that only the Ricci tensors in (3.13) are varied, and the argument $\Box$ of $I_\mu^\nu(-\Box, x)$ is the argument of the form factor $\Gamma$ in (3.13) that acts on the varied Ricci tensor. The $I_\mu^\nu(\xi, x)$ is a tensor function of the spacetime point $x$ and a function of a parameter $\xi$ which in eq. (3.14) gets replaced by the operator $-\Box$. This operator next acts in either of the two ways pointed out in (3.14).

Given the action of the form (3.12), it is easy to calculate $I_\mu^\nu(\xi, x)$ to each given order in the curvature. For the action (2.5) we have

$$I_\beta^\alpha(\xi, x) = I_{2\beta}^\alpha(\xi, x) + I_{3\beta}^\alpha(\xi, x) + O[\mathcal{R}^3]$$

(3.15)

where $I_{2\beta}^\alpha(\xi, x)$ and $I_{3\beta}^\alpha(\xi, x)$ are the contributions of $S(2)$ and $S(3)$ respectively, and

$$I_{2\beta}^\alpha(\xi, x) = 2\gamma_1(\xi)R_\beta^\alpha(x) + 2\gamma_2(\xi)\delta_\beta^\alpha R(x)$$

(3.16)

$$I_{3\beta}^\alpha(\xi, x) = 3 \left[ \delta_\nu^\mu \Gamma_1(\xi, -\Box_1, -\Box_2) R_\mu(x_1) R_\nu^\alpha(x_2) \right]_{x_1 = x_2 = x} + \left[ \nabla_1^\mu \nabla_2^\nu \Gamma_2(\xi, -\Box_1, -\Box_2) R(x_1) R(x_2) \right]_{x_1 = x_2 = x} - 2 \delta_\beta^\alpha \nabla_\nu \left[ \nabla_1^\mu \Gamma_2(\xi, -\Box_1, -\Box_2, x) R_\mu(x_1) R(x_2) \right]_{x_1 = x_2 = x} + \ldots$$

(3.17)
where only the purely gravitational terms are written down. We do not present the latter expression in full but exemplify it with the contributions of two third-order invariants, number 10 and number 22 (see [15]). Eq. (3.17) illustrates the general structure of the current \( I^\mu_\nu(\xi, x) \).

A remarkable property of \( I^\mu_\nu(\xi, x) \) is that the variations of both the Ricci tensors and the form factors in (3.12) are expressed entirely through this quantity. Indeed, in addition to (3.14), we have

\[
\delta \Gamma \tilde{S}_{\text{vac}} = \frac{1}{2(4\pi)^2} \int dx g^{1/2} \delta \Box' \frac{1}{\Box' - \Box''} \left[ I^\mu_\nu(-\Box', x'') - I^\mu_\nu(-\Box'', x'') \right] R^\mu_\nu(x') \quad (3.18)
\]

where the notation \( \delta \Gamma \) points out that only the form factors \( \Gamma \) in (3.13) are varied, and we have used the general formula for a variation of an operator function [12,14]:

\[
\int dx g^{1/2} A(\delta f(\Box)) B = \int dx g^{1/2} \delta \Box_B f_\Box B \frac{f(\Box_A) - f(\Box_B)}{\Box_A - \Box_B} AB \quad (3.19)
\]

It is understood that \( \Box_A \) (or \( \Box_B \)) is the operator \( \Box \) acting on \( A \) (or \( B \)), and similarly in (3.18) \( \Box' \) acts on \( x' \), and \( \Box'' \) on \( x'' \) with subsequently setting \( x' = x'' = x \). The operators \( \delta \Box_B \) and \( \Box_B \), and similarly \( \delta \Box' \) and \( \Box' \) in (3.18), do not commute and act in the indicated order. The identity

\[
\int dx g^{1/2} (\delta \Box'' - \delta \Box') F(x', x'') = \int dx g^{1/2} (-\delta \log g^{1/2})(\Box'' - \Box') F(x', x'') \quad (3.20)
\]

(with an arbitrary two-point tensor \( F(x', x'') \) contracting into a scalar at \( x' = x'' \)) serves to check that varying the left-hand side of the equality

\[
\int dx g^{1/2} f(\Box_B) AB = \int dx g^{1/2} f(\Box_A) AB \quad (3.21)
\]

with the aid of eq. (3.19) gives the same result as varying its right-hand side.

Although the generalized current \( I^\mu_\nu(\xi, x) \), as calculated from the action (2.5), is given in the form of an expansion, it enters the expectation-value equations as a single whole and determines the vacuum stress at null infinity. Indeed, since (3.14)
and (3.18) are the only contributions surviving in (2.16) when the observation point tends to $I^+$, $T_{\mu\nu}^{\text{vac}}$ at $I^+$ is obtained in a closed form. As seen from (3.16), (3.17), the behaviour of $I_{\mu}^{\nu}(\xi, x)$ in $\xi$ includes all the behaviours of the form factors in individual arguments. The $I_{\mu}^{\nu}(\xi, x)$ may, therefore, be of a significance in axiomatic theory. For this current as a function of its parameter argument one must postulate the existence of a spectral form similar to (2.10) but allowing for the spectral weight to have a $\delta(m^2)$ singularity at $m^2 = 0$. Assuming for simplicity $n = 0$ in (2.10) (the modification concerning the large $\Box$ is irrelevant to the present discussion) we set

$$I_{\mu}^{\nu}(\xi, x) = \int_0^\infty \frac{dm^2}{m^2 + \xi} w_{\mu}^{\nu}(m^2, x) ,$$

(3.22)

and then eqs. (3.14) and (3.18) take the form

$$\delta_R \tilde{S}_{\text{vac}} = \frac{1}{2} \frac{1}{(4\pi)^2} \int dx g^{1/2} \int \frac{dm^2 \delta R_{\mu}^{\nu}(x)}{m^2 - \Box} \left[ \frac{1}{m^2 - \Box w_{\mu}^{\nu}(m^2, x)} \right] ,$$

(3.23)

$$\delta_{\Gamma} \tilde{S}_{\text{vac}} = \frac{1}{2} \frac{1}{(4\pi)^2} \int dx g^{1/2} \int \frac{dm^2 \delta \Box R_{\mu}^{\nu}(x)}{m^2 - \Box} \left[ \frac{1}{m^2 - \Box w_{\mu}^{\nu}(m^2, x)} \right] .$$

(3.24)

Along with expression (3.4) and an easily derivable expression for $\delta \Box$ in (3.24) they determine $T_{\mu\nu}^{\text{vac}}$ at $I^+$.

4 The asymptotic flatness constraints

Even with the $1/\xi, \xi \to 0$ behaviour of $I_{\mu}^{\nu}(\xi, x)$, the contribution to $T_{\mu\nu}^{\text{vac}}$ coming from (3.24) is $O(1/r^2)$ at $I^+$. Also the contributions coming from the term with the Riemann tensor and the term with the $\Box$ operator in (3.4) are $O(1/r^2)$. We have, therefore, from (3.14) and (3.4)

$$T_{\mu\nu}^{\text{vac}}(x) = \frac{1}{2(4\pi)^2} \left( \nabla_{\mu} \nabla_{\alpha} I^{\alpha\nu}(- \Box_{\text{ret}}, x) + \nabla_{\nu} \nabla_{\alpha} I^{\alpha\mu}(- \Box_{\text{ret}}, x) \right) .$$

(4.1)
The terms of order $r^{-2}(x)$ in (4.1) are the ones to be retained but it is not our purpose here to calculate $T^\mu\nu_{\text{vac}}$. The point is that the remaining terms in (4.1) should also behave like $O(r^{-2}(x))$ for the asymptotic flatness to be maintained. This behaviour should, moreover, hold at every order in the curvature because even small disturbances of the metric can violate the asymptotic flatness. This does not mean, however, that the function

$$I^{\alpha\mu}(\xi, x) = g^{\alpha\gamma}(x) I^{\beta\gamma}(\xi, x).$$

(4.2)

should behave like $O(r^{-2}(x))$; it suffices that

$$\nabla_\alpha I^{\alpha\mu}(\xi, x) = O(r^{-2}(x)), \ x \to \mathcal{I}^+.$$  

(4.4)

Owing to this fact, the $1/\xi, \xi \to 0$ behaviour of $I^{\alpha\mu}(\xi, x)$ is not completely ruled out but condition (4.4) imposes a constraint on the coefficient of this behaviour.

For implementing this constraint the derivative $\nabla_\alpha$ in (4.4) should be commuted with the operator $\Box_{\text{ret}}$:

$$\nabla_\alpha I^{\alpha\mu}(\xi, x) = \int \frac{dm^2}{m^2 - \Box_{\text{ret}}} \nabla_\alpha w^{\alpha\mu}(m^2, x) +$$

$$+ \frac{1}{m^2 - \Box_{\text{ret}}} \left[ \nabla_\alpha, \Box \right] \frac{1}{m^2 - \Box_{\text{ret}}} w^{\alpha\mu}(m^2, x).$$

(4.5)

Here all operators act to the right on $x$, and

$$w^{\mu\nu}(m^2, x) = g^{[\mu\gamma}(x) w^{\nu\gamma]}(m^2, x).$$

(4.6)

For this calculation to lowest order in the curvature see [6,7]. The cancellation of the $1/\Box$ terms being established in the present paper, this calculation can now be done to second order in the curvature on the basis of the results in [15].
Both terms in (4.5) are generally $O(r^{-1}(x))$ but the commutator term contains an extra power of the curvature. If we denote
\[ w^\mu(m^2, x) \equiv \nabla_\alpha w^{\alpha \mu}(m^2, x) + [\nabla_\alpha, \Box] \frac{1}{m^2 - \Box_{ret}} w^{\alpha \mu}(m^2, x) \quad (4.7) \]
and
\[ I^\mu(\xi, x) \equiv \int_0^\infty \frac{dm^2}{m^2 + \xi} w^\mu(m^2, x) , \quad (4.8) \]
so that
\[ \nabla_\alpha I^{\alpha \mu}(- \Box_{ret}, x) = I^\mu(- \Box_{ret}, x) , \quad (4.9) \]
then, by (4.4) and the result in [6], the behaviour of the function (4.8) should already be
\[ I^\mu(\xi, x) = - w^\mu(0, x) \log \xi + O(1) , \quad \xi \to 0 \quad (4.10) \]
\[ w^\mu(0, x) = \text{finite.} \quad (4.11) \]

The vector current (4.8) is obtained from (3.15) as an expansion:
\[ I^\mu(\xi, x) = I^\mu_2(\xi, x) + I^\mu_3(\xi, x) + O[\mathbb{R}^3] \quad (4.12) \]
where $I^\mu_2$ and $I^\mu_3$ are the contributions of the actions $S(2)$ and $S(3)$. For the contribution of the action $S(2)$ one finds by using eq. (3.16):
\[ I^\mu_2(- \Box_{ret}, x) = \left( \gamma_1(- \Box_{ret}) + 2 \gamma_2(- \Box_{ret}) \right) \nabla^\mu R + 2 \left[ \nabla_\alpha, \gamma_1(- \Box_{ret}) \right] R^{\alpha \mu} + \left( \gamma_1(- \Box_{ret}) \right) \nabla^\mu R \quad (4.13) \]
and
\[ I^\mu_3(\xi, x) = \left( \gamma_1(\xi) + 2 \gamma_2(\xi) \right) \nabla^\mu R(x) + O[\mathbb{R}^2] \quad (4.14) \]
whence the constraint (2.14) for the second-order form factors immediately follows.

Should the constant $a$ in (2.12),(2.13) be nonvanishing, the commutator terms in
(4.13) would contribute to the constraint condition for the third-order form factors. Since, however, \( a = 0 \) as discussed above, these commutator terms are \( O(1/r^2) \) as seen from their spectral forms.

The constraint condition for the third-order form factors is thus of the form

\[
I^\mu_3(-\Box_{\text{ret}}, x) + O[\mathcal{R}^3] = O\left(r^{-2}(x)\right), \quad x \to \mathcal{I}^+
\]  

(4.15)

and

\[
I^\mu_3(\xi, x) = \nabla_\alpha I^{\alpha \mu}_3(\xi, x) + O[\mathcal{R}^3] \tag{4.16}
\]

since, in this case, the commutator of \( \nabla_\alpha \) with \( \xi = -\Box \) contributes to \( O[\mathcal{R}^3] \) already. On the other hand, with the field-theoretic form factors [15] one obtains

\[
I^{\alpha \mu}_3(\xi, x) = \frac{1}{\xi} A^{\alpha \mu}(x) + (\log \xi) B^{\alpha \mu}(x) + O(\xi^0), \tag{4.17}
\]

\[
I^{\mu}_3(\xi, x) = \frac{1}{\xi} A^{\mu}(x) + O(\log \xi), \xi \to 0, \tag{4.18}
\]

\[
A^{\mu}(x) = \nabla_\alpha A^{\alpha \mu}(x) + O[\mathcal{R}^3] \tag{4.19}
\]

where the coefficients \( A^{\mu}(x) \), etc. can be expanded in some vector basis second-order in the curvature:

\[
A^{\mu}(x) = \text{tr} \sum_p A_p(\Box_1, \Box_2) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}^{\mu}(p) + O[\mathcal{R}^3]. \tag{4.20}
\]

Examples of the basis structures in (4.20) are

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}^{\mu}(1) = \nabla^{\mu} R_1 \cdot R_2 \hat{1}, \tag{4.21}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}^{\mu}(2) = \nabla^{\mu} R_1^\alpha \cdot \nabla_\alpha \nabla_\beta R_2 \hat{1}, \tag{4.22}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}^{\mu}(3) = \hat{\mathcal{R}}^\alpha_1 \nabla^{\mu} \hat{\mathcal{R}}_{2\alpha} \hat{1}, \tag{4.23}
\]

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}^{\mu}(4) = \hat{R}_1^{\mu \alpha} \nabla_\alpha \hat{P}_2, \tag{4.24}
\]

etc. where all curvatures of the set (2.2) participate, and the trace in (4.20) refers to the matrices \( \hat{1}, \hat{P} \), etc. (cf. a similar construction of the basis of invariants in
We do not present the basis in (4.20) in full although it is important to have it in full for obtaining the results below. In order that (4.15) hold for any choice of the in-state \( \Psi \), there should be

\[
A^\mu(x) + O[\mathbb{R}^3] = 0
\]

and hence

\[
A_p(\Box_1, \Box_2) = 0.
\] (4.26)

Eqs. (4.26) are the constraints to be satisfied by the coefficients of the \( 1/\Box \) asymptotic behaviours of the third-order form factors. Let us introduce a notation for these coefficients:

\[
\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_1} F^1_i(\Box_2, \Box_3) + O(\log(-\Box_1)), \Box_1 \to 0
\] (4.27)

\[
\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_2} F^2_i(\Box_3, \Box_1) + O(\log(-\Box_2)), \Box_2 \to 0
\] (4.28)

\[
\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = \frac{1}{\Box_3} F^3_i(\Box_1, \Box_2) + O(\log(-\Box_3)), \Box_3 \to 0
\] (4.29)

where there appear functions of two variables \( F^m_i \) with \( m = 1 \) to \( 3 \) and \( i = 1 \) to \( 29 \).

The functions \( A_p \) in (4.20) are certain linear combinations of the \( F^m_i \). By an explicit calculation with the action \( S(3) \) in (2.8) and [15] one can work up these combinations to see if they vanish. The commutator and potential curvatures appear in the basis in (4.20) but the contributions of \( \delta \hat{R}_{\mu\nu} \) and \( \delta \hat{P} \) to (3.14) may, in this calculation, be omitted since all \( F^m_i \) with \( \Box_m \) acting on \( \hat{R}_{\mu\nu} \) or \( \hat{P} \) vanish [12,15]. This makes it possible to carry out the check of asymptotic flatness for a generic quantum field model.

\footnote{With the specifications made in sec. 2, this choice boils down to the choice of \( T^\mu\nu_{\text{source}} \) in eq. (2.18). Variations in \( T^\mu\nu_{\text{source}} \) induce variations in the curvature of the solution. Eq. (4.15) should, therefore, hold for any configuration of the curvature. It is also important for inferring (4.25) that, by the construction of the curvature basis in the action [5], \( I_3^{\mu\nu}(\xi, x) \) can have no total derivative terms of the form \( \Box X^{\alpha\mu}(\xi, x) \) or \( \nabla(\alpha X^{\mu})(\xi, x) \). Hence \( A^{\alpha\mu}(x) \) can have no such terms.}
The results are as follows. Of $3 \times 29$ functions $F^m_i$ only 21 in the table of ref. [15] do not vanish and are not related to each other by the symmetries of the form factors. With the 21 nonvanishing $F^m_i$ the expansion (4.20) gives rise to 14 constraints (4.26) which, by linearly combining them, can be brought to the following form:

$$F^3_{25}(\Box_1, \Box_2) = \frac{1}{2}(\Box_1 - \Box_2)F^3_{25}(\Box_1, \Box_2),$$  \hspace{1cm} (4.30)

$$F^1_{10}(\Box_1, \Box_2) = -\frac{1}{12}(\Box_2 - \Box_1)^2F^3_{25}(\Box_1, \Box_2),$$  \hspace{1cm} (4.31)

$$F^1_{28}(\Box_1, \Box_2) = 2F^3_{27}(\Box_1, \Box_2) + \frac{3}{2}(\Box_2 - \Box_1)F^1_{29}(\Box_1, \Box_2),$$  \hspace{1cm} (4.32)

$$F^3_{24}(\Box_1, \Box_2) = \frac{1}{2}F^1_{25}(\Box_1, \Box_2) - \frac{1}{4}(\Box_1 - \Box_2)F^1_{28}(\Box_2, \Box_1),$$  \hspace{1cm} (4.33)

$$F^3_{22}(\Box_1, \Box_2) = -\frac{1}{2}F^1_{24}(\Box_1, \Box_2) + \frac{1}{8}(\Box_1 + \Box_2)F^1_{28}(\Box_1, \Box_2),$$  \hspace{1cm} (4.34)

$$F^1_{11}(\Box_1, \Box_2) = -\frac{1}{4}(\Box_2 - \Box_1)F^1_{23}(\Box_1, \Box_2),$$  \hspace{1cm} (4.35)

$$F^1_{23}(\Box_1, \Box_2) = -2F^3_{22}(\Box_1, \Box_2) + (\Box_1 - \Box_2)F^1_{27}(\Box_1, \Box_2) + \frac{1}{4}(\Box_1 + \Box_2)F^1_{28}(\Box_2, \Box_1) + \frac{3}{4}(\Box_2 - \Box_1)\Box_1F^1_{29}(\Box_1, \Box_2),$$  \hspace{1cm} (4.36)

$$F^1_{5}(\Box_1, \Box_2) = \frac{1}{32}(\Box_1 + \Box_2)(\Box_1 - \Box_2)^2F^1_{29}(\Box_1, \Box_2) - \frac{1}{24}(\Box_1 + \Box_2)^2F^3_{27}(\Box_1, \Box_2),$$  \hspace{1cm} (4.37)

$$F^1_{22}(\Box_1, \Box_2) = -\frac{1}{4}F^1_{25}(\Box_1, \Box_2) + \frac{3}{8}(\Box_1 - \Box_2)^2F^1_{29}(\Box_1, \Box_2),$$  \hspace{1cm} (4.38)

$$F^1_{5}(\Box_1, \Box_2) = \frac{1}{4}(\Box_1 - \Box_2)^2F^1_{26}(\Box_1, \Box_2),$$  \hspace{1cm} (4.39)

$$F^1_{16}(\Box_1, \Box_2) = (\Box_1 - \Box_2)F^1_{20}(\Box_1, \Box_2),$$  \hspace{1cm} (4.40)

$$F^1_{1}(\Box_1, \Box_2) = \frac{1}{4}(\Box_1 + \Box_2)F^1_{21}(\Box_1, \Box_2),$$  \hspace{1cm} (4.41)

$$F^1_{18}(\Box_1, \Box_2) = \frac{1}{2}F^1_{21}(\Box_1, \Box_2),$$  \hspace{1cm} (4.42)

$$F^1_{19}(\Box_1, \Box_2) = -\frac{1}{4}F^1_{21}(\Box_1, \Box_2).$$  \hspace{1cm} (4.43)

It is now a matter of a direct inspection to check if the $F^m_i$ calculated in [15] satisfy
these constraints. They do!

Relations (4.30) - (4.43) leave only 7 independent nonvanishing \( F^m_i \) for which one can take the functions \( F^1_{21}, F^1_{25}, F^1_{26}, F^1_{27}, F^3_{27}, F^1_{28}, F^1_{29} \). With the exception of \( F^1_{26} \) and \( F^1_{27} \), these functions are symmetric in their \( \Box \) arguments: \( F^1_{25}, F^3_{27} \) and \( F^3_{28} \) are symmetric owing to the respective symmetries of the form factors \( \Gamma_{25}, \Gamma_{27} \) and \( \Gamma_{28} \), and the symmetry of \( F^1_{21}, F^1_{29} \) is a property of the explicit expressions in [15].

By expressing all \( F^m_i \) through the 7 independent ones, one can bring the coefficient \( A^{\mu\nu}(x) \) in (4.17) to the form

\[
A^{\mu\nu}(x) = -\nabla_\alpha \nabla_\beta K^{\mu\nu\alpha\beta}(x) + O[\Re^3] \tag{4.44}
\]

in which the fulfilment of condition (4.25) is manifest, and the function \( K^{\mu\nu\alpha\beta}(x) \) which appears antisymmetrized in (4.44) is of the following form:

\[
K^{\mu\nu\alpha\beta}(x) = 3\text{tr}F^1_{29}(\Box_1, \Box_2)\left\{ 4\nabla_\mu \nabla_\nu R_1^\gamma R_2^\sigma \cdot \nabla_\gamma \nabla_\sigma R_2^{\alpha\beta} + 2(\Box_2 - \Box_1)\left[ \nabla^{(\mu} R_1 \cdot \nabla^{\nu)} R_2^{\alpha\beta} \right] - 2(\Box_2 - \Box_1)g^{\mu\nu}\left[ R_1^{\alpha\beta} \cdot R_2 \right] \right\} 1 + \text{tr}F^3_{28}(\Box_1, \Box_2)\left\{ 4\nabla_\mu R_1^{\gamma\nu} \cdot \nabla_\nu R_2^{\alpha\beta} \right\} 1 + \\
+ \text{tr}F^1_{27}(\Box_1, \Box_2)\left\{ 8\nabla_\mu \nabla_\nu R_1^{\alpha\beta} \cdot R_2 \right\} 1 + 8 \text{tr}F^1_{26}(\Box_1, \Box_2)\left\{ \nabla_\mu \nabla_\nu R_1^{\alpha\beta} \cdot \hat{P}_2 \right\} + \\
+ \text{tr}F^1_{25}(\Box_1, \Box_2)\left\{ 4R_1^{\alpha\beta} \cdot R_2^{\nu\alpha} + 2g^{\mu\nu} R_1^{\nu\alpha} \cdot R_2 \right\} 1 + \\
+ \frac{1}{2}\text{tr}F^1_{23}(\Box_1, \Box_2)\left\{ \hat{R}_1^{\alpha\beta} \cdot R_2 \right\} + \text{tr}F^1_{27}(\Box_1, \Box_2)\left\{ 8\nabla^{(\mu} R_1^{\nu)\gamma} \cdot \nabla_\gamma R_2^{\alpha\beta} + 4\nabla^{(\mu} R_1 \cdot \nabla^{\nu)} R_2^{\alpha\beta} + \\
+ 4g^{\mu\nu}\nabla_\gamma R_1^{\alpha\gamma} \cdot \nabla_\sigma R_2^{\gamma\beta} + 4\nabla^{(\mu} R_1 \cdot \nabla^{\nu)} R_2^{\alpha\beta} + \\
+ 2(\Box_1 - \Box_2)g^{\mu\nu}\left[ R_1 \cdot R_2^{\alpha\beta} \right] + g^{\mu\nu}\nabla_\alpha R_1^\beta \cdot R_2 \right\} 1. \tag{4.45}
\]

The one-loop expressions for the functions \( F \) entering (4.45) are given in [15] but one may conjecture that the 7 independent structures in (4.45) and (4.44) (5 in the case of pure gravity) is the general result independent of models and approximations (although this remains to be checked by repeating the analysis above for the general form of the action). As will be seen below, expression (4.45) gets considerably simplified when inserted in the formula for the news functions.
5 The news functions. Contribution of the third-order form factors

A significance of the $1/\Box$ asymptotic terms in the vacuum form factors is that they contribute to the energy of the outgoing gravitational waves. This energy is the term with the news functions $\partial C_1/\partial u, \partial C_2/\partial u$ in the Bondi-Sachs equation (3.1).

Obtaining the news functions requires solving the dynamical equations already. However, there is a short cut: eq. (2.4). We may use the fact that the news functions appear as a coefficient of the $1/r$ behaviour of the Riemann tensor at null infinity. Indeed, we have (see appendix A)

$$\nabla_{\alpha}v\nabla_{\mu}v\, m_{\beta}m_{\nu} R^{\alpha\beta\mu\nu} \bigg|_{I^+} = -\frac{8}{r} \frac{\partial^2}{\partial u^2} C + O\left(\frac{1}{r^2}\right)$$

(5.1)

where

$$C = C_1 + i C_2,$$

(5.2)

and $m_{\beta}$ is a complex null vector tangent to the 2-sphere $S$:

$$(m, \nabla u) = (m, \nabla v) = (m, m) = 0, \quad (m, m^*) = -2$$

(5.3)

with $\nabla u, \nabla v$ and $S$ in (3.1)-(3.3), and $m^*$ complex conjugate to $m$. The contribution of the outgoing gravitational waves to the mass loss, eq. (3.1), is then

$$\frac{1}{4\pi} \int d^2S \left| \frac{\partial}{\partial u} C \right|^2,$$

(5.4)

and

$$-\frac{\partial^2}{\partial u^2} C = \left\{ \frac{r}{8} \nabla_{\alpha}v\nabla_{\mu}v m_{\beta}m_{\nu} R^{\alpha\beta\mu\nu} \right\} \bigg|_{I^+}.$$  

(5.5)
On the other hand, the Riemann tensor can be calculated with the aid of eq. (2.4). Its $1/r$ behaviour is then obtained as the leading asymptotic behaviour of the retarded Green function. Using (2.4b) we have

$$-\frac{\partial^2}{\partial u^2} C = \left\{ \frac{r}{2} \nabla_\alpha \nabla_\mu \nabla_\nu m_\beta m_\nu \nabla^{[\mu} \nabla^{\nu]} \frac{1}{\Box} \left( R^{\nu\beta} + O[R^2] \right) \right\} \bigg|_{\mathcal{I}^+}$$

(5.6)

where the quadratic terms are to be copied from (2.4b), and the quadratic terms with no overall $1/\Box$ factor - all terms in the curly brackets in (2.4b) - do not contribute.

Expression (5.6) can next be simplified as follows. The derivatives in (5.6) appear projected either as $\nabla_\nu v$ or as $m_\nu \nabla_\nu$. In both cases the projected derivatives can be commuted with the remaining factors of $\nabla v$ and $m$ since the components of $\nabla \nabla v$ and $\nabla m$ in the null tetrad basis are $O(1/r)$ at $\mathcal{I}^+$. The projected derivatives become then acting on some scalar $X$ which behaves like $1/r$ at $\mathcal{I}^+$, and in this case

$$m^\nu \nabla_\nu X \bigg|_{\mathcal{I}^+} = O \left( \frac{1}{r^2} \right) ,$$

(5.7)

$$\nabla^\nu v \nabla_\nu X \bigg|_{\mathcal{I}^+} = -2 \frac{\partial}{\partial u} X + O \left( \frac{1}{r^2} \right)$$

(5.8)

(see appendix A). In this way one obtains

$$\frac{\partial^2}{\partial u^2} C = -\frac{\partial}{\partial u} \left\{ \frac{\partial}{\partial u} \frac{r}{2} m_\beta m_\nu \frac{1}{\Box} \left( R^{\nu\beta} + O[R^2] \right) \right\} \bigg|_{\mathcal{I}^+} .$$

(5.9)

Eq. (5.9) can now be integrated over $u$ from $-\infty$ to a given point of $\mathcal{I}^+$ to obtain the news functions. Since, at $u = -\infty$, $\partial C/\partial u = 0$, there remains to be shown that the expression in the curly brackets in (5.9) also vanishes at $u = -\infty$. This is shown in appendix B. As a result one obtains

$$\frac{\partial}{\partial u} C = -\frac{\partial}{\partial u} \left\{ \frac{r}{2} m_\beta m_\nu \frac{1}{\Box} \left[ R^{\nu\beta} + (\nabla^\nu \frac{1}{\Box} R^{\gamma\delta})(\nabla^\beta \frac{1}{\Box} R_{\gamma\delta}) - 2(\nabla^\gamma \frac{1}{\Box} R^{\mu\delta})(\nabla^\delta \frac{1}{\Box} R^{\beta\gamma}) + O[R^3] \right] \right\} \bigg|_{\mathcal{I}^+} .$$

(5.10)
where for the Ricci tensor one can use the dynamical equations\textsuperscript{10}.

There is, of course, a classical gravitational radiation or, more generally, a radiation induced by the classical source $T_{\text{source}}^{\mu\nu}$ in eq. (2.18). The respective contribution\textsuperscript{11} to the news functions (call it $\partial C_{\text{source}}/\partial u$) is obtained by substituting for $R^{\nu\beta}$ its Einstein value

$$R_{\text{cl}}^{\nu\beta} = 8\pi\left(T_{\text{source}}^{\nu\beta} - \frac{1}{2} g^{\nu\beta} T_{\text{source}}\right)$$

in all terms of (5.10) including $O[R^3]$. The remaining contributions in (5.10) stand for the gravitational radiation induced by the vacuum stress,

$$\frac{\partial}{\partial u} C = \frac{\partial}{\partial u} C_{\text{source}} + \frac{\partial}{\partial u} C_{\text{vac}}.$$  

By (5.10) and the dynamical equations (2.18)\textsuperscript{12},

$$\frac{\partial}{\partial u} C_{\text{vac}} = -\frac{\partial}{\partial u} \left\{4\pi r m_{\beta} m_{\nu} \frac{1}{\Box} \left(T_{\text{vac}}^{\nu\beta} + O[R^2]\right)\right\}_{\mathcal{I}^+}$$

and

$$T_{\text{vac}}^{\mu\nu} = T_{\text{vac}}^{\mu\nu}(2) + T_{\text{vac}}^{\mu\nu}(3) + O[R^3]$$

where $T_{\text{vac}}^{\mu\nu}(2)$ and $T_{\text{vac}}^{\mu\nu}(3)$ are the contributions of the actions $S(2)$ and $S(3)$ in (2.5).

In (5.10) and (5.13) there appears an expression of the form

$$\left\{r m_{\beta} m_{\nu} \frac{1}{\Box} X_{\nu\beta}\right\}_{\mathcal{I}^+}$$

with some tensor $X^{\mu\nu}$, and we have used the fact that a contribution of this form with

$$X^{\mu\nu} = g^{\mu\nu} X$$

\textsuperscript{10} This method can also be applied to the classical problems in the gravitational radiation.

\textsuperscript{11} Even this contribution is not purely classical since the metric to be used is the solution of the expectation-value equations. The same concerns the flux of $T_{\text{source}}^{\mu\nu}$ at $\mathcal{I}^+$ in eq. (3.1). It would be interesting to consider a case where $T_{\text{source}}^{\mu\nu}$ does not induce the gravitational waves but $T_{\text{vac}}^{\mu\nu}$ does.

\textsuperscript{12} Since, at $\mathcal{I}^-$, the flux components of $T_{\text{vac}}^{\mu\nu}$ vanish, eq. (5.13) is valid even without the limitation implied in (2.4b) (see sec. 2).
vanishes by the orthogonality relation in (5.3). Another property of expression (5.13) which will be used below is that a contribution of the form (5.15) with

\[ X^{\mu\nu} = \nabla^{\mu}X^{\nu} + \nabla^{\nu}X^{\mu} \]  

(5.17)
is of a higher order in the curvature:

\[ O[X^{\mu} \times R_\cdot]. \]  

(5.18)

Indeed, in this case,

\[ m_\beta m_\nu \frac{1}{\Box} X^{\nu\beta} = 2 m_\beta m_\nu \nabla^{\nu} \frac{1}{\Box} X^{\beta} + 2 m_\beta m_\nu \frac{1}{\Box} (\nabla^{\nu}, \Box \frac{1}{\Box} X^{\beta}). \]  

(5.19)

By (5.7), the first term of this expression is \( O(1/r^2) \), and the remaining term contains a commutator.

To lowest order in the curvature, only the contribution of the action \( S(2) \) is to be considered. By (3.4) and (3.14),

\[
T_{\nu\nu}^{\mu\nu}(2) = \frac{1}{2(4\pi)^2} \left( \nabla^{\mu} \nabla_{\alpha} I_2^{\alpha\nu} + \nabla^{\nu} \nabla_{\alpha} I_2^{\alpha\mu} - g^{\mu\nu} \nabla_{\beta} \nabla_{\alpha} I_2^{\alpha\beta} - \Box I_2^{\mu\nu} \right) + O[R_2^2] \]  

(5.20)

where

\[ I_2^{\mu\nu} = I_2^{\mu\nu}(-\Box_{ret}, x), \]  

(5.21)

and the explicit form of \( I_2(\xi, x) \) is given in (3.16). The first two terms in (5.20) are of the form (5.17), and, therefore, their contribution to (5.13) is \( O[R_2^2] \). The third term in (5.20) is of the form (5.16), and, therefore, its contribution to (5.13) vanishes. In the remaining term of (5.20), the \( \Box \) operator kills the Green function:

\[
\frac{\partial}{\partial u} C_{\nu\nu} = \frac{\partial}{\partial u} \left\{ \frac{r}{8\pi} m_\beta m_\nu I_2^{\nu\beta}(-\Box_{ret}, x) \right\}_{\mathcal{I}^+} + O[R_2^2], \]  

(5.22)
and this is the reason why at all the asymptotic behaviours of the form factors at small $\Box$ are relevant to the gravitational waves. Since the second-order form factors $\gamma_1, \gamma_2$ behave like $\log(-\Box)$ at small $\Box$, we have

$$I_2^\beta (\Box_{ret}, x) = O\left(\frac{1}{r^2}\right), \quad x \to \mathcal{I}^+$$

(5.23)

and the contribution (5.22) vanishes. Thus the vacuum contribution to the news functions begins with second order in the curvature:

$$\frac{\partial}{\partial u} C_{vac} = O[R^2_{\Box}] \cdot (5.24)$$

At second order in the curvature both $S(2)$ and $S(3)$ contribute to $C_{vac}$. We are presently interested in the contribution of $S(3)$ which we shall denote $C_{vac}(3)$. Since the action $S(3)$ is cubic in the curvature, the accuracy in (5.13) is sufficient for calculating this contribution:

$$\frac{\partial}{\partial u} C_{vac}(3) = -\frac{\partial}{\partial u} \left\{ 4\pi r m_\beta m_\nu \frac{1}{\Box} \left( T_{\nu\beta}^{\nu\beta}(3) + O[R^3_{\Box}] \right) \right\}_{x \to \mathcal{I}^+}. (5.25)$$

To second order in the curvature, $T_{\nu\beta}^{\nu\beta}(3)$ is of the form similar to (5.20):

$$T_{\nu\beta}^{\nu\beta}(3) = \frac{1}{2(4\pi)^2} \left( \Box I_3^{\nu\beta} \right) \cdot (5.26)$$

with

$$I_3^{\nu\beta} = I_3^{\nu\beta} (\Box_{ret}, x). (5.27)$$

Again, the first two terms in (5.26) are of the form (5.17), and their contribution to (5.25) is $O[R^3_{\Box}]$. Again the third term is proportional to the metric, and its contribution vanishes. Again the $\Box$ operator in the remaining term kills the Green function:

$$\frac{\partial}{\partial u} C_{vac}(3) = \frac{\partial}{\partial u} \left\{ r m_\beta m_\nu I_3^{\nu\beta} \left( -\Box_{ret}, x \right) \right\}_{x \to \mathcal{I}^+} + O[R^3_{\Box}], (5.28)$$

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but this time the contribution (5.28) does not vanish because the third-order form factors behave like $1/\Box$ at small $\Box$. By (4.17),

$$I_3^{\nu\beta}(-\Box,x) = -\left(\frac{1}{\Box}\right)A^{\nu\beta}(x) + \log(-\Box)B^{\nu\beta}(x) + O\left(\frac{1}{\Box^3}\right),$$

and only the term with $1/\Box$ survives in (5.28). With the expression for $A^{\nu\beta}(x)$ given in (4.44) the result is

$$\frac{\partial}{\partial u} C_{\text{vac}}(3) = \frac{\partial}{\partial u} \left\{ \frac{r}{8\pi} m_\mu m_\nu \frac{1}{\Box} \nabla_\alpha \nabla_\beta K^{\mu\alpha\nu\beta}(x) + O[R^3_1] \right\}_{I+}. \quad (5.30)$$

Expression (4.45) for $K^{\mu\alpha\nu\beta}$ can now be simplified by using that (i) all terms containing the metric with the indices of $K^{\mu\alpha\nu\beta}$ can be discarded since any contraction among $m_\mu m_\nu \nabla_\alpha \nabla_\beta$ in (5.30) results in a vanishing contribution, and (ii) any derivative $\nabla$ with the indices of $K^{\mu\alpha\nu\beta}$ can be treated as in integration by parts because the respective total derivative contracts with either $m$ or $\nabla$, and its contribution vanishes. As a result, $K^{\mu\alpha\nu\beta}$ in (5.30) can be replaced by the following expression:

$$\tilde{K}^{\mu\alpha\nu\beta}(x) = 12 \text{tr} F_{29}^1(\Box_1, \Box_2) \left[ \nabla^\mu \nabla^\nu R_1^{\gamma\sigma} \cdot \nabla_\gamma \nabla_\sigma R_2^{\alpha\beta} \right] \hat{1} +$$

$$+ 4 \text{tr} F_{28}^3(\Box_1, \Box_2) \left[ \nabla^\mu R_1^{\gamma\nu} \cdot \nabla_\nu R_2^{\beta} \right] \hat{1} + 8 \text{tr} F_{25}^2(\Box_1, \Box_2) \left[ \nabla^\mu R_1^{\nu\gamma} \cdot \nabla_\gamma R_2^{\alpha\beta} \right] \hat{1} +$$

$$+ 4 \text{tr} F_{25}^1(\Box_1, \Box_2) \left[ R_1^{\mu\beta} \cdot R_2^{\nu\alpha} \right] \hat{1} + \text{tr} \left\{ 6(\Box_2 - \Box_1) F_{29}^1(\Box_1, \Box_2) + 8 F_{27}^1(\Box_1, \Box_2) -

- 4 F_{27}^3(\Box_1, \Box_2) \right\} \left[ \nabla^\mu \nabla^\nu R_1^{\alpha\beta} \cdot R_2 \right] \hat{1} + 8 \text{tr} F_{26}^1(\Box_1, \Box_2) \left[ \nabla^\mu \nabla^\nu R_1^{\alpha\beta} \cdot \hat{P}_2 \right] +$$

$$+ \frac{1}{2} \text{tr} F_{21}^1(\Box_1, \Box_2) \left[ \tilde{R}_1^{\mu\alpha} \cdot \tilde{R}_2^{\nu\beta} \right].$$
6 The vacuum news functions in the lowest nonvanishing approximation

To complete the calculation of the vacuum news functions in the lowest nonvanishing approximation we must consider the contribution of the action $S(2)$:

$$\frac{\partial}{\partial u} C_{\text{vac}}(2) = -\frac{\partial}{\partial u} \left\{ 4\pi r m_\beta m_\gamma \left( T_{\text{vac}}^{\beta\gamma}(2) + 2\left( \nabla^\beta \frac{1}{\Box} R_{\mu\nu}^\text{cl} \right) \left( \nabla^\gamma \frac{1}{\Box} T_{\text{vac}}^{\mu\nu}(2) \right) - \left( \nabla^\beta \frac{1}{\Box} R_{\text{cl}}^\mu \right) \left( \nabla^\gamma \frac{1}{\Box} T_{\text{vac}}(2) \right) - 4\left( \nabla_\alpha \frac{1}{\Box} R_{\mu\nu}^\alpha \right) \left( \nabla_\beta \frac{1}{\Box} T_{\text{vac}}^{\mu\nu}(2) \right) + 2\left( \nabla^\gamma \frac{1}{\Box} R_{\text{cl}}^{\beta\mu} \right) \left( \nabla_\mu \frac{1}{\Box} T_{\text{vac}}(2) \right) \right\} \bigg|_{I+} + O(\hbar^2) + O[R^3]. \quad (6.1)$$

Here $R_{\mu\nu}^\text{cl}$ is the notation in (5.11), and in the nonlinear terms of eq. (5.10) we omitted the contributions $T_{\text{vac}} \times T_{\text{vac}} = O(\hbar^2)$ of second order in the Planck constant. As shown above, expression (6.1) is of second order in the curvature; the contribution of first order in this expression vanishes.

The expression (5.20) for $T_{\text{vac}}^{\mu\nu}(2)$ completed with terms $O[R^2]$ is of the form

$$T_{\text{vac}}^{\mu\nu}(2) = \frac{1}{2(4\pi)^2} \left( \nabla^\mu \nabla_\alpha I_2^{\alpha\nu} + \nabla^\nu \nabla_\alpha I_2^{\alpha\mu} - g^{\mu\nu} \nabla_\beta \nabla_\alpha I_2^{\beta\alpha} - \Box I_2^{\mu\nu} + 2R_{\alpha..\beta} I_2^{\alpha\beta} + \frac{1}{2}g^{\mu\nu} R_\alpha I_2^{\alpha\beta} \right) + \Pi^{\mu\nu} \quad (6.2)$$

where

$$\Pi^{\mu\nu} = \frac{2}{g^{1/2}} \frac{\delta, S(2)}{\delta g_{\mu\nu}} \bigg|_{\Box \rightarrow \Box_{\text{ret}}} \quad (6.3)$$

with

$$\delta, S(2) = \frac{1}{2(4\pi)^2} \int dx g^{1/2} \left( R_\nu^{\mu} \delta \gamma_1 (-\Box) R_\nu^\nu + R \delta \gamma_2 (-\Box) R \right) \quad (6.4)$$

is the contribution of the variations of the lowest-order form factors (see eqs. (3.19), (3.24)). When inserting expression (6.2) in the linear term of (6.1), all terms in (6.2)
proportional to $g^{\mu \nu}$ or to the $\Box$ operator can be omitted, and for the contribution of the first two terms in (6.2) one can use eq. (5.19).

In the nonlinear terms of (6.1), the expression (6.2) is needed only up to $O[R^2]$. By using (3.16), we obtain

$$T_{\text{vac}}(2) = \frac{1}{(4\pi)^2} \left[ \nabla^\mu \nabla^\nu \left( \gamma_1(-\Box) + 2\gamma_2(-\Box) \right) R - \Box \gamma_1(-\Box) R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} \Box \left( \gamma_1(-\Box) + 4\gamma_2(-\Box) \right) R \right] + O[R^2].$$

(6.5)

This makes it possible to calculate also the Riemann tensor with accuracy $O[R^2]$:

$$R^{\alpha \beta \mu \nu} = R_{\text{cl}}^{\alpha \beta \mu \nu} - \frac{2}{\pi} \gamma_1(-\square) \nabla^\mu \nabla^{<\alpha} \nabla_{\beta>} R_{\beta <\alpha} + \frac{1}{4\pi} \left( \gamma_1(-\Box) + 2\gamma_2(-\Box) \right) \times$$

$$\times \left( g^{\nu \beta} \nabla_\mu \nabla^\alpha - g^{\nu \alpha} \nabla_\mu \nabla^\beta - g^{\mu \beta} \nabla_\nu \nabla^\alpha + g^{\alpha \mu} \nabla_\nu \nabla^\beta \right) R + O[R^2].$$

(6.6)

Here we used eq. (2.4a) which defines also

$$R_{\text{cl}}^{\alpha \beta \mu \nu} = \frac{1}{\Box} \left( 2\nabla^\mu \nabla^{<\alpha} R_{\text{cl}}^{\beta >} - 2\nabla^\nu \nabla^{<\alpha} R_{\text{cl}}^{\mu >} + O[R^2] \right).$$

(6.7)

Finally, by combining the results above, the following expression is obtained for the contribution $\partial C_{\text{vac}}(2)/\partial u$ to the news functions:

$$\frac{\partial}{\partial u} C_{\text{vac}}(2) = -\frac{\partial}{\partial u} \left\{ \frac{r M_r m_r}{4\pi} \frac{1}{\Box} \left[ \gamma_1(-\Box_2) \left[ 4\nabla_\alpha \frac{1}{\Box} R_{1 \text{cl}}^{\mu \beta} \cdot \nabla_\beta R_{2 \text{cl}}^{\nu \alpha} + 4\nabla^\mu \nabla^\nu \frac{1}{\Box} R_{1 \text{cl}}^{\beta} \cdot R_{2 \text{cl}}^{\alpha \beta} - 2\nabla^\alpha \nabla^\beta \frac{1}{\Box} R_{1 \text{cl}}^{\mu \nu} \cdot R_{1 \text{cl}}^{\alpha \beta} + R_{1 \text{cl}}^{\mu \nu} \cdot R_{2 \text{cl}} \right] + \left( \gamma_1(-\Box_2) + 2\gamma_2(-\Box_2) \right) \left[ \nabla^\mu \frac{1}{\Box} R_{1 \text{cl}}^{\beta} \cdot \nabla_\beta R_{2 \text{cl}} - 2\nabla_\nu \frac{1}{\Box} R_{1 \text{cl}}^{\mu \alpha} \cdot \nabla_\alpha R_{2 \text{cl}} \right] + \frac{1}{2} \frac{1}{\Box} R_{1 \text{cl}}^{\mu \nu} \cdot R_{2 \text{cl}} - \frac{1}{2} R_{1 \text{cl}}^{\mu \nu} \cdot R_{2 \text{cl}} \right] + (4\pi)^2 \Pi^{\mu \nu} \right) + O(h^2) + O[R^3] \right\} \bigg|_{\mathcal{I}^+}.$$

This contribution involves the lowest-order vacuum form factors in the whole range of their dependence on the $\Box$ argument.

The total result is

$$\frac{\partial}{\partial u} C = \frac{\partial}{\partial u} C_{\text{source}} + \frac{\partial}{\partial u} C_{\text{vac}}(2) + \frac{\partial}{\partial u} C_{\text{vac}}(3) + O(h^2) + O[R^3]$$

(6.9)
with $\partial C_{\text{vac}}(2)/\partial u$ in (6.8) and $\partial C_{\text{vac}}(3)/\partial u$ in (5.30). Note that the news functions appear squared in the mass-loss formula (3.1), and the vacuum contribution to $\partial C/\partial u$ begins with second order in the curvature, eq. (5.24). Therefore, in the absence of a classical radiation, the energy of the vacuum gravitational waves is of order $O[h^2 R^4]$ which makes this effect difficult to be noticed in perturbation theory.

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**Appendix A. The asymptotically flat metric at null infinity**

The general asymptotically flat metric in a chart covering $I^+$ is built by considering a congruence of null hypersurfaces $u = \text{const}$. generated by the light rays reaching $I^+$. The generators are labeled by two parameters $\theta, \varphi$ taking values on a 2 - sphere $S$, and the luminosity distance $r$ is used as a parameter along the generators. The metric is then of the form [21]

$$ds^2 = -V du^2 + 2\Psi dudr + r^2 f_{ab}(dx^a - U^a du)(dx^b - U^b du) \quad (A.1)$$

where

\begin{align*}
a, b &= 1, 2; \quad x^1 = \theta, \ x^2 = \varphi, \end{align*}
\[ f_{ab}dx^a dx^b = \frac{1}{2}(e^{2\gamma} + e^{2\delta})d\theta^2 + (e^{\gamma-\delta} - e^{\delta-\gamma})\sin \theta d\theta d\varphi + \frac{1}{2}(e^{-2\gamma} + e^{-2\delta})\sin^2 \theta d\varphi^2 \] (A.2)

and

\[ \frac{1}{\Psi} = (\nabla u, \nabla r) < 0 \] (A.3)
\[ \Psi \bigg|_{\mathcal{I}^+} = -1. \] (A.4)

In this metric, \((\nabla u)^2 = 0\) is to ensure that the hypersurfaces \(u = \text{const.}\) are null, \((\nabla x^a, \nabla u) = 0\) is to ensure that the lines \(u = \text{const.}, x^a = \text{const.}\) are null geodesics, \(\det f_{ab} = \sin^2 \theta\) is to ensure that \(r\) is the luminosity parameter along these geodesics, condition (A.3) is to ensure that this parameter is monotonic, and condition (A.4) is to choose the retarded time \(u\) coincident with the proper time of an observer at large and constant \(r\).

At the limit of \(\mathcal{I}^+(r \to \infty, u = \text{const.}, \theta = \text{const.}, \varphi = \text{const.})\) the metric behaves as follows [21]:

\[ V = 1 - \frac{2\mathcal{M}}{r} + O \left( \frac{1}{r^2} \right), \] (A.5)
\[ \frac{\gamma + \delta}{2} = \frac{C_1}{r} + O \left( \frac{1}{r^2} \right), \] (A.6)
\[ \frac{\gamma - \delta}{2} = \frac{C_2}{r} + O \left( \frac{1}{r^2} \right), \] (A.7)
\[ U^a = \frac{2N^a}{r^2} + O \left( \frac{1}{r^3} \right), \] (A.8)
\[ \Psi = -1 - \frac{2B}{r^2} + O \left( \frac{1}{r^3} \right) \] (A.9)

where \(\mathcal{M}, C_1, C_2, N^a, B\) are functions of \(\theta, \varphi, u\).

The \(C_1, C_2\) differentiated with respect to \(u\) are the Bondi-Sachs news functions [20, 21]. In the gauge (A.1) they stand for the radiation degrees of freedom of the
gravitational field. The Bondi mass $M(u)$ is obtained by averaging the coefficient $\mathcal{M}$ in (A.5) over the unit 2-sphere:

$$M(u) = \frac{1}{4\pi} \int d^2S \mathcal{M} = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \mathcal{M}(\theta, \varphi, u). \quad (A.10)$$

Its limiting value at $u \to -\infty$ is the ADM mass, and the difference

$$M(-\infty) - M(u) = \int_{-\infty}^{u} du \left(-\frac{dM}{du}\right) \quad (A.11)$$

is the energy radiated away through $I^+$ by the instant $u$ of retarded time (see [22] and references therein).

Some components of the Riemann and Ricci tensors calculated in the metric (A.1) - (A.9) at $I^+$ are as follows:\(^\text{13}\):

$$R_{\theta\theta\theta\theta} \bigg|_{I^+} = -r \frac{\partial^2}{\partial u^2} C_1 + O(1), \quad (A.12)$$

$$R_{\theta\theta\phi\phi} \bigg|_{I^+} = -r \sin \theta \frac{\partial^2}{\partial u^2} C_2 + O(1), \quad (A.13)$$

$$R_{\phi\phi\phi\phi} \bigg|_{I^+} = r \sin^2 \theta \frac{\partial^2}{\partial u^2} C_1 + O(1), \quad (A.14)$$

$$R_{rr} \bigg|_{I^+} = -\frac{2}{r^4} \left(\frac{C_1^2 + C_2^2 + 4B}{r^2}\right) + O \left(\frac{1}{r^5}\right), \quad (A.15)$$

$$R_{r\theta} \bigg|_{I^+} = -\frac{1}{r^2} \left[2N^\theta + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} (C_1 \sin^2 \theta) + \frac{\partial}{\partial \varphi} \left(\frac{C_2}{\sin \theta}\right)\right] + O \left(\frac{1}{r^3}\right), \quad (A.16)$$

$$R_{r\varphi} \bigg|_{I^+} = -\frac{1}{r^2} \left[2N^\varphi \sin^2 \theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{C_2}{\sin \theta}\right) - \frac{\partial}{\partial \varphi} C_1\right] + O \left(\frac{1}{r^3}\right), \quad (A.17)$$

$$R_{uw} \bigg|_{I^+} = \frac{1}{r^3} \frac{\partial}{\partial u} \left(\frac{C_1^2 + C_2^2 + 4B}{r^2}\right) + O \left(\frac{1}{r^4}\right), \quad (A.18)$$

\(^{13}\)By our calculations, the equation $R_{rr} = (2/r)(\Psi^{-1} \partial_r \Psi - (\partial_r \gamma)^2)$ (in the present notation) given in [21] for the axisymmetric case is in error. We obtain $R_{rr} = (2/r)\Psi^{-1} \partial_r \Psi - 2(\partial_r \gamma)^2$. It follows from the former equation that, for $R_{\mu\nu}$ of a compact spatial support, $\Psi = -1 + O(1/r^3)$. With our result, $\Psi = -1 + O(1/r^2)$. The significance of the behaviour of $\Psi$ is seen from eqs. (A.15), (A.18).
\[ R_{u\theta} \bigg|_{I^+} = \frac{1}{r} \frac{\partial}{\partial u} \left[ 2N^\theta + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( C_1 \sin^2 \theta \right) + \frac{\partial}{\partial \varphi} \left( \frac{C_2}{\sin \theta} \right) \right] + O \left( \frac{1}{r^2} \right), \quad (A.19) \]

\[ R_{u\varphi} \bigg|_{I^+} = \frac{1}{r} \frac{\partial}{\partial u} \left[ 2N^\varphi \sin^2 \theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( C_2 \sin^2 \theta \right) - \frac{\partial}{\partial \varphi} C_1 \right] + O \left( \frac{1}{r^2} \right), \quad (A.20) \]

\[ R_{uu} \bigg|_{I^+} = -\frac{2}{r^2} \left[ \frac{\partial}{\partial u} M + \left( \frac{\partial}{\partial u} C_1 \right)^2 + \left( \frac{\partial}{\partial u} C_2 \right)^2 + \frac{1}{\sin \theta} \frac{\partial}{\partial u} (\sin \theta \frac{\partial}{\partial u} N^\alpha) \right] + O \left( \frac{1}{r^3} \right). \quad (A.21) \]

When referring to tensors at \( I^+ \) we always mean their projections on the null tetrad \( \nabla u, \nabla v, m, m^* \) introduced in (3.2), (3.3) and (5.3) where \( v \) and \( m_\alpha \) are asymptotically of the form

\[ v \bigg|_{I^+} = 2r + u + O \left( \frac{1}{r} \right), \quad (A.22) \]

\[ m_\alpha \bigg|_{I^+} = r(\nabla_\alpha \theta + i \sin \theta \nabla_\alpha \varphi) + O(1). \quad (A.23) \]

The null-tetrad components of physical quantities are regular at \( I^+ \) i.e. are either finite or decreasing like inverse powers of \( r \). This is true specifically of tensors obtained by the action of the retarded form factors (see below). The null-tetrad vectors may be regarded as covariantly constant at \( I^+ \) since the null-tetrad components of their derivatives are \( O(1/r) \). Thus, up to curvature terms,

\[ \nabla_\mu \nabla_\alpha v \bigg|_{I^+} = \frac{1}{2} \left( m_\mu m_\alpha^* + m_\alpha^* m_\mu \right) + O[R.], \quad (A.24) \]

and the curvature terms are \( O(1/r) \). Eqs. (5.7), (5.8) are obtained by calculating the derivatives projected on the null tetrad in terms of the Bondi-Sachs coordinates. Specifically, eq. (5.7) owes to the fact that \( m^\alpha \bigg|_{I^+} = O(1/r) \) as seen from (A.23) and (A.1).

It follows from the asymptotic expressions above that the null-tetrad components of the curvature tensor decrease at \( I^+ \) like \( 1/r \) or faster. Specifically,

\[ \nabla_\alpha v \nabla_\mu v m_\beta m_\nu R^{\alpha\beta\mu\nu} \bigg|_{I^+} = -\frac{8}{r} \left( \frac{\partial^2}{\partial u^2} C_1 + i \frac{\partial^2}{\partial u^2} C_2 \right) + O \left( \frac{1}{r^2} \right) \quad (A.25) \]
which is eq. (5.1). The energy flux component of the Ricci tensor at \( I^+ \)

\[
R_{\mu\nu} \nabla^\mu u \nabla^\nu v \bigg|_{I^+} = 4 R_{uu} \bigg|_{I^+} \tag{A.26}
\]
is given in eq. (A.21). By averaging (A.21) over the unit 2-sphere one obtains the relation

\[
- \frac{dM(u)}{du} = \frac{1}{4\pi} \int d^2 S \left[ (\frac{\partial}{\partial u} C_1)^2 + (\frac{\partial}{\partial u} C_2)^2 \right] + \frac{1}{8\pi} \int d^2 S \frac{1}{4} r^2 R_{\mu\nu} \nabla^\mu v \nabla^\nu v \bigg|_{I^+} \tag{A.27}
\]

which, after using the dynamical equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{total}^{\mu\nu} \tag{A.28}
\]
becomes the conservation law (3.1). Here \( T_{total}^{\mu\nu} \) is the total energy-momentum tensor which in eq. (2.18) is

\[
T_{total}^{\mu\nu} = T^{\mu\nu}_{source} + T^{\mu\nu}_{vac} \tag{A.29}
\]

**Appendix B. The retarded Green function in the past of \( I^+ \)**

To lowest order in the curvature, the retarded operator \( \frac{1}{\Box} \) acting on an arbitrary tensor source \( X^{\alpha_1...\alpha_k} \) is of the following form:

\[
\frac{1}{\Box} X^{\alpha_1...\alpha_k}(x) = \int_{\text{past of } x} d\bar{x} \bar{g}^{1/2}(\sigma(x, \bar{x})) g^{\alpha_1}_{\bar{\alpha}_1}(x, \bar{x}) ... g^{\alpha_k}_{\bar{\alpha}_k}(x, \bar{x}) X^{\bar{\alpha}_1...\bar{\alpha}_k}(\bar{x}) + O[X \times \mathbb{R}] \tag{B.1}
\]

where \( \sigma(x, \bar{x}) \) is the world function [23], or geodetic interval biscal [24], \( g^{\alpha}_{\bar{\alpha}}(x, \bar{x}) \) is the geodetic parallel displacement bivector [24], and the integration point \( \bar{x} \) is in
the past of the observation point $x$. Here and below, the bar over a symbol means that this symbol refers to the point $\bar{x}$.

It follows from a comparison of the equations defining $g^\alpha_\bar{\alpha}(x, \bar{x})$ with the ones defining $\sigma(x, \bar{x})$ that, up to the curvature terms, the parallel displacement bivector can be calculated as follows:

$$g^\alpha_\bar{\alpha}(x, \bar{x}) = -\nabla^\alpha\nabla_{\bar{\alpha}}\sigma(x, \bar{x}) + O[R.]$$  \hspace{1cm} (B.2)

whence it also follows that

$$\nabla_\mu g^\alpha_\bar{\alpha}(x, \bar{x}) = O[R.]$$ \hspace{1cm} (B.3)

If $\ell^\mu_i$ with $i = 1$ to 4 are the vectors of the null tetrad, then the null-tetrad components of the tensor (B.1) are obtained by calculating the contractions

$$\ell^\mu_i(x)g_{\mu\bar{\mu}}(x, \bar{x}).$$ \hspace{1cm} (B.4)

By using eq. (B.2) and a perturbative expression for the world function, it is easy to see that, when $x$ tends to $\mathcal{I}^+$, and $\bar{x}$ is in a compact domain, the contractions (B.4) remain finite. Hence the null-tetrad components of the tensor (B.1) decrease at $\mathcal{I}^+$ like $O(1/r)$ - the fact assumed in the main text. The expression for the massive retarded Green function is similar to (B.1) [6]. Therefore, generally, the null-tetrad components of tensors obtained by the action of the retarded form factors are regular at $\mathcal{I}^+$.

As seen from (B.1), we are always dealing with some scalar source

$$\mathcal{Y}(\bar{x}) = g^\alpha_{\alpha_1}(x, \bar{x}) \ldots g^\alpha_{\alpha_k}(x, \bar{x})X^{\alpha_1 \ldots \alpha_k}(\bar{x})$$ \hspace{1cm} (B.5)

which may depend parametrically on the observation point but it suffices to consider the action of the Green function on a scalar :

$$-\frac{1}{\Box}X^{\alpha_1 \ldots \alpha_k}(x) = \frac{1}{4\pi} \int_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta(\sigma(x, \bar{x}))\mathcal{Y}(\bar{x}) + O[X \times \mathbb{R}].$$ \hspace{1cm} (B.6)
The integration over the light cone in (B.6) includes subintegrations along the light rays coming from $\mathcal{I}^-$ to the observation point $x$:

$$\int_0^{\infty} d\mu(\rho) Y|_L.$$

(B.7)

Here $L$ is a generator of the past light cone of $x$, $\rho$ is the luminosity parameter along $L$, and the measure in (B.7) is asymptotically of the form

$$d\mu(\rho)|_{\rho \to \infty} = d\rho \cdot \rho.$$

(B.8)

We shall, therefore, assume that the source decreases at $\mathcal{I}^-$ like

$$X^{\alpha_1...\alpha_k}|_{\mathcal{I}^-} = O\left(\frac{1}{r^3}\right).$$

(B.9)

Another important assumption [6] is analyticity of the source in time including the past timelike infinity ($i^-$). The real sources appearing in the expectation-value equations are built out of the curvature, and the condition of analyticity implies in particular that, at $i^-$, the metric becomes asymptotically static. This should be provided by imposing the respective requirement on $T_{\mu \nu}^{\text{source}}$. By analyticity, the limit $r \to \infty$ of the source at $i^-$ coincides with its limit in the past of $\mathcal{I}^-$:

$$\left(X^{\alpha_1...\alpha_k}|_{i^-}\right)|_{r \to \infty} = \left(X^{\alpha_1...\alpha_k}|_{\mathcal{I}^-}\right)|_{v \to -\infty}$$

(B.10)

where $v$ is the advanced time along $\mathcal{I}^-$. Hence, by (B.9),

$$\left(X^{\alpha_1...\alpha_k}|_{i^-}\right)|_{r \to \infty} = O\left(\frac{1}{r^3}\right).$$

(B.11)

For the calculation of the integral (B.6) at $x \to \mathcal{I}^+$ we may use the Bondi-Sachs frame (A.1). For the past of $\mathcal{I}^+$ this is safe even if the metric has closed apparent horizons [2,3] since no one of these will be encountered by the light rays emitted
sufficiently early. To lowest order in the curvature, the world function is then of the form
\[ \sigma(x, \bar{x}) = -(u - \bar{u})(r - \bar{r} + \frac{u - \bar{u}}{2}) + r\bar{r}(1 - \cos \omega) + O[R_\cdot] \quad (B.12) \]
where \(\frac{1}{2}\omega^2\) is the world function on the 2-sphere:
\[ \cos \omega = \cos \theta \cos \bar{\theta} + \sin \theta \sin \bar{\theta} \cos(\varphi - \bar{\varphi}) . \quad (B.13) \]

By solving the equation \(\sigma(x, \bar{x}) = 0\) with respect to \(\bar{u}\) and choosing the solution which corresponds to the past light cone of \(x, \bar{u} = f\), we obtain
\[ -\frac{1}{\Box} X^{\alpha_1 \ldots \alpha_k}(r, \theta, \varphi, u) = \frac{1}{4\pi} \int d^2 \bar{S} \int_0^\infty d\bar{r} \bar{r}^2 \left| \nabla \bar{u}, \nabla \bar{r} \right| \frac{\partial \sigma}{\partial \bar{u}} \bigg|_{\bar{u}=f} + \]
\[ + O[X \times \Re] \quad (B.14) \]
and
\[ -\frac{1}{\Box} X^{\alpha_1 \ldots \alpha_k}(r, \theta, \varphi, u) \bigg|_{\mathcal{I}^+} = \frac{1}{r} Q(\theta, \varphi, u) + O \left( \frac{1}{r^2} \right) \quad (B.15) \]
where
\[ Q(\theta, \varphi, u) = -\frac{1}{4\pi} \int d^2 \bar{S} \int_0^\infty d\bar{r} (\log \bar{r}) \frac{\partial}{\partial \bar{r}} \left[ \bar{r}^3 Y(\bar{r}, \bar{\theta}, \bar{\varphi}, \bar{u}) \bigg|_{\bar{u}=f^*} \right] + O[X \times \Re] , \quad (B.16) \]
\[ f^* = u - \bar{r}(1 - \cos \omega) + O[R_\cdot] , \quad (B.17) \]
and the equation \(\bar{u} = f^*\) is the equation of the limiting light cone of the point \(x\) at \(\mathcal{I}^+\). Here we wrote \(\bar{r}^2 = \bar{r}^3 \partial \log \bar{r} / \partial \bar{r}\) and integrated by parts for being able to consider sources decreasing at \(\mathcal{I}^-\) like \(1/r^3\). At the limit \(u \to -\infty\), the source in (B.16) turns out to be at \(\bar{u} \to -\infty\). By the assumption of analyticity, we then have
\[ Q(\theta, \varphi, u) \bigg|_{u \to -\infty} = Q_0 + \frac{1}{u} Q_1 + \cdots \quad (B.18) \]
with

$$Q_0 = -\frac{1}{4\pi} \int d^2 \mathcal{S} \int_0^\infty \tilde{d}\tilde{r}(\log \tilde{r}) \frac{\partial}{\partial \tilde{r}} [\tilde{r}^3 \mathcal{Y}(\tilde{r}, \tilde{\theta}, \tilde{\varphi}, -\infty)] + O[X \times \mathbb{R}] , \quad (B.19)$$

eqno etc., and the convergence of the integral in (B.19) is now owing to (B.11).

Eqs. (B.15)-(B.19) make manifest the fact which is more general than the approximations made. Namely, the integral (B.1) in the past of $I^+$ involves only the source $X^{\alpha_1...\alpha_k}$ at $i^-$. Under the assumption of analyticity at $i^-$, this integral becomes a static Coulomb potential. One can show that it preserves this form also at two other limits: at the past null infinity and spatial infinity ($i^0$),

$$-\frac{1}{\Box} X^{\alpha_1...\alpha_k} \bigg|_{I^-_0} \rightarrow 0 \quad (B.20)$$

Consider now equation (5.9) for the news functions which is valid under the assumption made in sec. 2 that the flux components of the Ricci tensor at $I^-$ vanish. Since, in this case, $R^{\mu\nu}$ at $I^-$ is $O(1/r^3)$, the integral implied in $(\Box^{-1/2}) R^{\mu\nu}$ converges. Moreover, the derivatives of this integral in the null-tetrad basis behave at $I^-$ like

$$\left(\nabla_\gamma \frac{1}{\Box} R^{\mu\nu}\right) \bigg|_{I^-} = O \left(\frac{1}{r^2}\right) \quad (B.21)$$

(see below). Therefore, the nonlinear additions to $R^{\nu\beta}$ in eq. (5.10) are $O(1/r^4)$ at $I^-$. We then have

$$m_\beta m_\nu \left[ R^{\mu\beta} + (\nabla^\nu \frac{1}{\Box} R^\gamma_\delta) (\nabla^\beta \frac{1}{\Box} R^\gamma_\delta) - 2(\nabla^\gamma \frac{1}{\Box} R^\nu_\delta) (\nabla^\delta \frac{1}{\Box} R^\gamma_\nu) + O[R^3]\right]_{I^+} = \frac{1}{r} q(\theta, \varphi, u) + O \left(\frac{1}{r^2}\right) . \quad (B.22)$$

\[14\] Since, at $I^+$, the flux components of $R^{\mu\nu}$ do not vanish, the integral $(1/\Box) R^{\mu\nu}$ behaves at $I^+$ like $(\log r)/r$. The $(\log r)/r$ asymptotic terms vanish in expression (2.4b) owing to the presence of the antisymmetrized derivatives, and in expressions (5.10), (5.13), (B.22) owing to the contraction with $m_\beta m_\nu$. 

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We need to calculate the limit
\[
\lim_{u \to -\infty} \frac{\partial}{\partial u} q(\theta, \varphi, u)
\]
which serves as an initial datum for eq. (5.9). By analyticity of the metric at \(i^-\),
\[
q(\theta, \varphi, u) \bigg|_{u \to -\infty} = q_0 + \frac{1}{u} q_1 + \cdots,
\]
and the limit (B.23) vanishes owing to the presence of the derivative \(\partial/\partial u\).

Eq. (B.20) can also be used to prove that the solution of the Bianchi identities (2.3) with zero initial data for the gravitational field at \(I^-\) is expressed indeed in terms of the retarded Green function. This is easily seen in expression (2.4b). Since, by (B.20),
\[
-\frac{1}{\Box} \left( R^\nu_\beta + O[R^2] \right) \bigg|_{I^-} = \frac{Q^\nu_\beta}{r} + O \left( \frac{1}{r^2} \right),
\]
and \(Q^\nu_\beta\) does not depend on the advanced time along \(I^-\), the derivatives in (2.4b) enhance the power of \(1/r\). Eq. (B.21) is valid for the same reason.

The proof is more involved in the general case, where the fluxes of \(R^\nu_\mu\) through \(I^-\) are nonvanishing, since one has to address eq. (2.4a). In terms of the null tetrad at \(I^-\)
\[
\nabla[^\mu \nabla <^\alpha R'^\nu_\beta>] \bigg|_{I^-} = \frac{\nabla[^\mu \nabla <^\alpha \nabla^\gamma u > (\nabla^\gamma u \nabla^\sigma u) \nabla^\sigma \nabla^\nu R'^\nu_\beta]}{\nabla u, \nabla v^2 ((\nabla^\gamma u) (\nabla^\gamma u \nabla^\gamma R^\nu_\beta) + O \left( \frac{1}{r^3} \right)).
\]
Since the flux components of \(R^\nu_\beta\) at \(I^-\) are proportional either to \(\nabla^\nu v\) or to \(\nabla_\beta v\), they cancel in (B.26) owing to the antisymmetrizations:
\[
\nabla[^\mu \nabla <^\alpha R'^\nu_\beta>] \bigg|_{I^-} = O \left( \frac{1}{r^3} \right).
\]
Thus the source in (2.4a) satisfies condition (B.9), and we have
\[
R^{\alpha_\beta \mu \nu}_{\nu} \bigg|_{I^-} = -\frac{Q^\alpha_\beta \mu \nu}{r} + O \left( \frac{1}{r^2} \right)
\]
with $Q_0^{0\beta\mu\nu}$ constant along $\mathcal{I}^-$. By (B.19) and (B.5),

$$Q_0^{\alpha\beta\mu\nu} = -\frac{1}{4\pi} \int d^2S \int_0^\infty d\bar{r} (\log \bar{r}) \frac{\partial}{\partial \bar{r}} \left( \bar{r}^3 \bar{\mathcal{Y}} \right)_{i^-} + O[R^2] \, ,$$

(B.29)

$\bar{\mathcal{Y}} = 4g^{[\mu} g^{\nu]} g^{[\alpha} g^{\beta]} \nabla^{\mu} \nabla^{\alpha} R_{\nu\beta}$,  

(B.30)

and, up to $O[R^2]$, $\bar{\mathcal{Y}}$ is a total derivative:

$$\bar{\mathcal{Y}} \bigg|_{i^-} = \bar{g}^{-1/2} \partial_{\bar{r}} \left( \bar{g}^{1/2} Z^\mu \right) \bigg|_{i^-} + O[R^2] \, ,$$

(B.31)

$$Z^{\mu} = 4g^{[\mu} g^{\nu]} g^{[\alpha} g^{\beta]} \nabla^{\mu} R_{\nu\beta} \, ,$$

(B.32)

where use is made of eq. (B.3). Since, in this total derivative, the time derivative vanishes owing to the asymptotic stationarity of the metric at $i^-$, and the angle derivatives vanish in the integral $\int d^2S$, we obtain

$$Q_0^{\alpha\beta\mu\nu} = -\frac{1}{4\pi} \int d^2S \int_0^\infty d\bar{r} (\log \bar{r}) \frac{\partial}{\partial \bar{r}} \left[ \bar{r} \frac{\partial}{\partial \bar{r}} \left( \bar{r}^2 \nabla_{\mu} \bar{Z}^\mu \bigg|_{i^-} \right) \right] + O[R^2] \, .$$

(B.33)

Since the power of decrease of $Z^{\mu} \bigg|_{i^-}$ at $\bar{r} \to \infty$ is at least $O(1/\bar{r}^2)$, we may integrate by parts in (B.33) to remove the integration over $\bar{r}$ completely:

$$Q_0^{\alpha\beta\mu\nu} = \lim_{\bar{r} \to \infty} \frac{1}{4\pi} \int d^2S \left( 4g^{[\mu} g^{\nu]} g^{[\alpha} g^{\beta]} \nabla_{\mu} \nabla_{\nu} R_{\alpha\beta} \right) \left( \bar{r}^2 \nabla_{\alpha} \bar{R}_{\nu\beta} \bigg|_{i^-} \right) + O[R^2] \, .$$

(B.34)

Only the terms in $\bar{R}_{\rho\beta}$ that decrease like $1/\bar{r}^2$ can survive in (B.34). However, the integrand in (B.34) contains one more derivative of $\bar{R}_{\rho\beta}$, and, again, the respective time derivative vanishes at $i^-$. Therefore, this integrand is $O(1/\bar{r})$ (actually $O(1/\bar{r}^2)$, see below), and the limit (B.34) vanishes. The presence of an extra derivative is in fact not essential. By (B.10), the sequence of limits in (B.34) can be replaced by the limit of the same quantity in the past of $\mathcal{I}^-$. It is then seen that the constant (B.34) hangs solely on the flux components of the Ricci tensor in the past of $\mathcal{I}^-$. However, the metric in the past becomes asymptotically static, and for a static metric the
fluxes of $R_{\nu\beta}$ through $I^-$ are absent as seen from the counterparts at $I^-$ of eqs. (A.18) - (A.21). We have, therefore,

$$ (R_{\nu\beta} |_{i^-})|_{r \to \infty} = (R_{\nu\beta} |_{I^-})|_{v \to -\infty} = O \left( \frac{1}{r^3} \right), \quad (B.35) $$

and

$$ Q_0^{\alpha\beta\mu\nu} = 0. \quad (B.36) $$

Hence

$$ R^{\alpha\beta\mu\nu} |_{I^-} = O \left( \frac{1}{r^2} \right) \quad (B.37) $$

which proves that, in the retarded solution, there are no incoming gravitational waves.

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