Heat Kernel for Fractional Diffusion Operators with Perturbations

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Abstract

Let $L$ be an elliptic differential operator on a complete connected Riemannian manifold $M$ such that the associated heat kernel has two-sided Gaussian bounds as well as a Gaussian type gradient estimate. Let $L^{(\alpha)}$ be the $\alpha$-stable subordination of $L$ for $\alpha \in (1,2)$. We found some classes $K_{\gamma,\beta}^{(\alpha)}$ of time-space functions containing the Kato class, such that for any measurable $b: [0, \infty) \times M \to TM$ and $c: [0, \infty) \times M \to M$ with $|b|, c \in K_{1,1}^{(\alpha)}$, the operator

$$L^{(\alpha)}_{b,c}(t, x) := L^{(\alpha)}(x) + \langle b(t, x), \nabla \cdot \rangle + c(t, x), \quad (t, x) \in [0, \infty) \times M$$

has a unique heat kernel $p^{(\alpha)}_{b,c}(t, x; s, y)$, $0 \leq s < t, x, y \in M$, which is jointly continuous and satisfies

$$\frac{t-s}{C\{\rho(x, y) \vee (t-s)^{\frac{1}{\alpha}}\}^{d+\alpha}} \leq p^{(\alpha)}_{b,c}(t, x; s, y) \leq \frac{C(t-s)}{\{\rho(x, y) \vee (t-s)^{\frac{1}{\alpha}}\}^{d+\alpha}},$$

$$|\nabla_x p^{(\alpha)}_{b,c}(t, x; s, y)| \leq \frac{C(t-s)^{\frac{d-1}{\alpha}}}{\rho(x, y) \vee (t-s)^{\frac{1}{\alpha}}} \{\rho(x, y) \vee (t-s)^{\frac{1}{\alpha}}\}^{d+\alpha}, \quad 0 \leq s < t, \ x, y \in M$$

for some constant $C > 1$, where $\rho$ is the Riemannian distance. The estimate of $\nabla_y p^{(\alpha)}_{b,c}$ and the Hölder continuity of $\nabla_x p^{(\alpha)}_{b,c}$ are also considered. The resulting estimates of the gradient and its Hölder continuity are new even in the standard case where $L = \Delta$ on $\mathbb{R}^d$ and $b, c$ are time-independent.

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1 Introduction

In \cite{13}, the two-sided Gaussian bounds were confirmed for the heat kernel of the time-dependent second order differential operator \( \text{div}(A\nabla) + B \cdot \nabla \) on \( \mathbb{R}^d \), where \( A : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) is uniformly elliptic and uniformly Hölder continuous, and \( B : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is in a class of singular functions. In the same spirit, the sharp heat kernel bounds have been presented in \cite{1} for fractional Laplacian with perturbations. More precisely, let \( \Delta^{(\alpha)} := \Delta^{\frac{a}{2}} \) be the fractional Laplacian on \( \mathbb{R}^d \) for \( \alpha \in (1, 2) \), and let \( b : \mathbb{R}^d \to \mathbb{R}^d \) be in the Kato class \( \mathcal{K}^{\alpha-1} \), i.e.

\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \varepsilon} \frac{|b(y)|}{|x-y|^{d+1-\alpha}} dy = 0,
\]

or equivalently,

\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|b(y)| (\varepsilon \wedge |x-y|^\alpha)}{|x-y|^{d+1}} dy = 0.
\]

Then the heat kernel \( p^{(\alpha)}_b(t, x, y) \) of \( \Delta^{(\alpha)} + \langle b, \nabla \cdot \rangle \) satisfies

\[
(1.1) \quad \frac{t}{C((|x-y| \vee t^{\frac{1}{2}})^{d+\alpha}}} \leq p^{(\alpha)}_b(t, x, y) \leq \frac{Ct}{((|x-y| \vee t^{\frac{1}{2}})^{d+\alpha}}
\]

for some constant \( C > 1 \). Recently, this result was extended in \cite{3} to the Dirichlet heat kernel for the fractional Laplacian with perturbations. The aim of this paper is to derive sharp heat kernel bounds for more general fractional diffusion operators with time-dependent perturbations, and to derive gradient estimates of the heat kernel which are new even in the framework of \cite{1}.

Let \( M \) be a \( d \)-dimensional connected complete Riemannian manifold with Riemannian distance \( \rho \). Let \( L \) be an elliptic differential operator on \( M \) generating a (sub-)Markov semigroup \( P_t \). Then \( P_t \) is a \( C^0 \)-contraction semigroup on the Banach space \( C^b(M) \) equipped with the uniform norm \( \| \cdot \|_{\infty} \), and \( \mathcal{D}(L) \supset C^2_0(M) \), where \( (L, \mathcal{D}(L)) \) is the infinitesimal generator of \( P_t \) on \( C^b(M) \).

Throughout the paper, we assume that \( P_t \) has a density \( p(t, x, y) \) w.r.t. a reference measure \( \mu \) on \( M \) such that

\[
(1.2) \quad \frac{\exp[-\frac{\rho(x,y)^2}{t}]}{C(\mu(M) \land t)^{\frac{d}{2}}} \leq p(t, x, y) \leq \frac{C \exp[-\frac{\rho(x,y)^2}{ct}]}{(\mu(M) \land t)^{\frac{d}{2}}}, \quad t > 0, x, y \in M
\]

and

\[
(1.3) \quad |\nabla_x p(t, x, y)| \leq \frac{C \exp[-\frac{\rho(x,y)^2}{ct}]}{\sqrt{t} (\mu(M) \land t)^{\frac{d}{2}}}, \quad t > 0, x, y \in M
\]

hold for some constant \( C > 1 \), where \( \nabla_x \) stands for the gradient operator w.r.t. variable \( x \). Consider the \( \alpha \)-stable subordination of \( P_t \):

\[
P_t^{(\alpha)} := \int_0^\infty P_s \mu_t^{(\alpha)}(ds), \quad t \geq 0,
\]
where $\mu_t^{(\alpha)}$ is a probability measure on $[0, \infty)$ with Laplace transform
\[ \int_0^\infty e^{-\lambda s} \mu_t^{(\alpha)}(ds) = e^{-t\lambda^\alpha}, \quad \lambda \geq 0. \]

Then $P_t^{(\alpha)}$ is a $C_0$-contraction semigroup on $C_b(M)$. Let $L^{(\alpha)}$ be the infinitesimal generator of $P_t^{(\alpha)}$. Then $\mathcal{D}(L^{(\alpha)}) \supset \mathcal{D}(L) \supset C^2_0(M)$, see e.g. [8 Proposition 12.5]. By (1.2), the density $p^{(\alpha)}(t, x, y)$ of $P_t^{(\alpha)}$ w.r.t. $\mu$ satisfies (see Proposition 2.1 below)
\begin{equation}
(1.4) \quad C^{-1} \xi^{(\alpha)}(t, \rho(x, y)) \leq p^{(\alpha)}(t, x, y) \leq C \xi^{(\alpha)}(t, \rho(x, y)), \quad t > 0, x, y \in M
\end{equation}
for some constant $C > 1$ and
\begin{equation}
(1.5) \quad \xi^{(\alpha)}(t, r) := \frac{t^\alpha}{\mu(M)(r t^{\frac{1}{\alpha}})} + \frac{t}{(r t^{\frac{1}{\alpha}})^{d+\alpha}}, \quad t > 0, r \geq 0.
\end{equation}

Now, to make time-dependent first- and zero-order perturbations of $L^{(\alpha)}$, let $b : [0, \infty) \times M \to TM$ and $c : [0, \infty) \times M \to \mathbb{R}$ be measurable. Consider
\[ L_{b, c}^{(\alpha)}(t, x) := L^{(\alpha)}(x) + \langle b(t, x), \nabla_x \rangle + c(t, x), \quad (t, x) \in [0, \infty) \times M. \]

To construct the heat kernel of this operator, we restrict $|b|$ and $c$ in certain classes of functionals as in [13]. To introduce these classes, a function $f$ on $[0, \infty) \times M$ will be automatically extended to $\mathbb{R} \times M$ by letting $f(s, \cdot) = 0$ for $s < 0$. For $\gamma, \beta \geq 0$, define
\[ K_{\alpha, f}^{\gamma, \beta}(\varepsilon) = \sup_{(t, x) \in [0, \infty) \times M} \left\{ \varepsilon^\frac{\beta}{\alpha} \int_0^\varepsilon \int_M \frac{\xi^{(\alpha)}(s, \rho(x, y)) |f(t \pm s, y)|}{s^{\frac{\alpha}{\beta}}}(x - s)^{\frac{\alpha}{\beta}} \mu(dy)ds \right\}, \quad \varepsilon > 0. \]

**Definition 1.1.** For $\gamma, \beta \geq 0$, let
\[ \mathcal{K}_\alpha^{\gamma, \beta} = \mathcal{K}_\alpha^{\gamma, \beta}(\mu) = \left\{ f \in \mathcal{D}(\mathbb{R} \times M) : \lim_{\varepsilon \downarrow 0} K_{\alpha, f}^{\gamma, \beta}(\varepsilon) = 0 \right\}, \]
where $\mathcal{D}(\mathbb{R} \times M)$ is the set of all measurable functions on $\mathbb{R} \times M$.

It is easy to see that $\mathcal{K}_\alpha^{\gamma, \beta}$ is decreasing in both $\gamma$ and $\beta$. According to Proposition 2.1 below, $\mathcal{K}_\alpha^{1, \beta}(\mu) \supset \mathcal{K}_d^{\alpha-1}(\mu)$ for any $\beta \in [1, \alpha)$, where $\mathcal{K}_d^{\alpha-1}(\mu)$ is the Kato class on $M$ consisting of measurable functions $f$ with
\begin{equation}
(1.6) \quad 1_{\{\mu(M) < \infty\}} \mu(|f|) < \infty, \quad \lim_{\varepsilon \downarrow 0} \sup_{x \in M} \int_M \frac{|f(y)|(\varepsilon \wedge \rho(x, y)^\alpha)}{\rho(x, y)^{d+1}} \mu(dy) = 0.
\end{equation}

When $M = \mathbb{R}^d$ and $\mu(dy) = dy$, this class reduces back to the class $\mathcal{K}_d^{\alpha-1}$ in [1] as mentioned above. See also Proposition 2.2 for explicit subclass of $\mathcal{K}_\alpha^{\gamma, \beta}$ in the time-space functional space.
To introduce the heat kernel of $L_{b,c}^{(α)}$, let us look at the heat equation

\begin{equation}
\begin{cases}
\partial_t u(t, s, \cdot) = L_{b,c}^{(α)}(t, \cdot)u(t, s, \cdot), \quad t > s, \\
u(s, s, \cdot) = \varphi,
\end{cases}
\end{equation}

where $s \geq 0$ and $\varphi \in C_b(M)$. Recall that $u$ is called a mild solution of this equation, if it satisfies

$$u(t, s, x) = P_{t-s}^{(α)}\varphi(x) + \int_s^t \int M P_{t-r}^{(α)}\{ (b(r, \cdot), \nabla u(r, s, \cdot)) + c(r, \cdot)u(r, s, \cdot) \} dr, \quad t \geq s.$$ 

Therefore, it is natural to construct the fundamental solution to the heat equation by solving the integral equation

\begin{equation}
p_{b,c}^{(α)}(t, x; s, y) = p^{(α)}(t-s, x, y) + \int_s^t \int_M p^{(α)}(t-r, x, z) \cdot \{ (b(r, z), \nabla z p_{b,c}^{(α)}(r, z; s, y)) + c(r, z)p_{b,c}^{(α)}(r, z; s, y) \} \mu(dz)
\end{equation}

for $t > s \geq 0, x, y \in M$, so that the mild solution to (1.7) can be formulated as

$$u(t, s, \cdot) = P_{t-s}^{b,c,α} \varphi := \int_M p_{b,c}^{(α)}(t-\cdot, s, y) \varphi(y) \mu(dy).$$

We remark that following the argument of [1,3], the heat kernel of $\Delta^{(α)} + \langle b, \nabla \cdot \rangle$ on $\mathbb{R}^d$ with time-free $b$ was constructed in [1] by solving the dual equation

\begin{equation}
p_{b}^{(α)}(t, x, y) = p^{(α)}(t-x, y) + \int_0^t \int_{\mathbb{R}^d} p_{b}^{(α)}(t-r, x, z) \langle b(z), \nabla z p^{(α)}(r, z, y) \rangle dz,
\end{equation}

where $p^{(α)}$ is the heat kernel for the $\alpha$-stable operator $\Delta^{(α)}$. The advantage of (1.9) is that it does not involve the derivative of the unknown heat kernel, and hence easier to solve. On the other hand, the good point of (1.8) is that from which one can easily derive the gradient estimate and confirm the infinitesimal generator of the solution.

The following three theorems are the main results of the paper.

**Theorem 1.1.** Assume [1,2], [1,3] and let $\alpha \in (1, 2)$. If $|b|, c \in \mathbb{K}_{\alpha}^{1,1}$, then (1.8) has a unique solution $p_{b,c}^{(α)}(t, x; s, y)$ such that for all $t - s \in (0, 1], x, y \in M$,

\begin{equation}
C^{-1} \xi^{(α)}(t-s, \rho(x, y)) \leq p_{b,c}^{(α)}(t, x; s, y) \leq C \xi^{(α)}(t-s, \rho(x, y)),
\end{equation}

and

\begin{equation}
| \nabla x p_{b,c}^{(α)}(t, x; s, y) | \leq \frac{C \xi^{(α)}(t-s, \rho(x, y))}{(t-s)^{\frac{1}{α}}}
\end{equation}

hold for some constant $C > 0$. Moreover, $p_{b,c}^{(α)}$ is continuous and satisfies the following two assertions:
(1) For any \(0 \leq s < r < t\) and \(x, y \in M\),
\[
p^{(\alpha)}_{b,c}(t, x; s, y) = \int_M p^{(\alpha)}_{b,c}(t, x; r, z)p^{(\alpha)}_{b,c}(r, z; s, y)\mu(dz);
\]

(2) If \(\mu\) has a \(C^1\)-density w.r.t. the volume measure, \(b \in C([0, \infty); L^1_{\text{loc}}(M \to TM; \mu))\), \(c \in C([0, \infty); L^1_{\text{loc}}(M \to \mathbb{R}; \mu))\), then for any \(\varphi, \psi \in C^2_0(M)\),
\[
\lim_{t \downarrow s} \frac{1}{t-s} \int_M \psi(P^{b,c}_{t,s}\varphi - \varphi) d\mu = \int_M \psi L^{(\alpha)}_{b,c}(s, \cdot)\varphi d\mu, \quad s \geq 0.
\]

We remark that Theorem 1.1 not only generalizes the main result in [1] for solution to \(\text{(1.3)}\), but also provide the new gradient estimate \(\text{(1.11)}\). The next result says that under a Hessian upper bound condition on \(p^{(\alpha)}_{b,c}\). For \(x \neq x'\), let \(\gamma^{x,x'} : [0, 1] \to M\) be the minimal geodesic from \(x\) to \(x'\), which might be non-unique if \(x\) is in the cut-locus of \(x'\). Define
\[
\eta(t; x, x'; y) = p^{(\alpha)}(t, x, y) + p^{(\alpha)}(t, x', y) + \int_0^1 p^{(\alpha)}(t, \gamma^{x,x'}_\theta, y) d\theta.
\]

**Theorem 1.2.** Assume that \(\text{(1.2)}, \text{(1.3)}\) and
\[
|\nabla_x^2 p(t, x, y)| \leq \frac{C \exp[-\frac{\rho(x,y)^2}{C^2}]}{t(\mu(M) \wedge t)^{\frac{1}{2}}}, \quad t > 0, x, y \in M
\]
hold for some constant \(C > 1\). Let \(\alpha \in (1, 2)\). If \(|b|, c \in \mathbb{K}^{\beta} \alpha\beta\) for some \(\beta \in (1, \alpha)\), then there exists a constant \(C' > 0\) such that
\[
|\nabla_x p^{(\alpha)}_{b,c}(t, x; s, y) - /_{x' \to x} \nabla_{x'} p^{(\alpha)}_{b,c}(t, x'; s, y)| \leq \frac{C' p(x, x')^{\beta-1}\eta(t-s; x, x'; y)}{(1 \wedge (t-s))^{\frac{1}{2}}}
\]
holds for all \(0 \leq s < t\) and \(x, x', y \in M\), where \(/_{x' \to x}\) denotes the parallel transport along the geodesic \(\gamma^{x,x'}\).

Finally, we consider the derivative estimate of \(p^{(\alpha)}_{b,c}\) w.r.t. the variable “\(y\).”

**Theorem 1.3.** In addition to the assumptions of Theorem \(\text{[1.1]}\) we also assume that \(p^{(\alpha)}(t, x, y) = p^{(\alpha)}(t, y, x)\) and \(\text{div}_\mu b \in \mathbb{K}^{1,1} \alpha\) exists, where \(\text{div}_\mu b(s, \cdot)\) is the unique (if exists) element in \(L^1_{\text{loc}}(M \to \mathbb{R}; \mu)\) such that
\[
\int_M \langle \text{div}_\mu b(s, \cdot), f \rangle d\mu = \int_M \langle b(s, \cdot), \nabla f \rangle d\mu, \quad f \in C^1_0(M).
\]
Then
\[
|\nabla_y p^{(\alpha)}_{b,c}(t, x; s, y)| \leq \frac{C \xi^{(\alpha)}(t-s, \rho(x, y))}{(t-s)^{\frac{1}{2}}}
\]
hold for some constant \(C > 0\).
The remainder of the paper is organized as follows. We present in Section 2 some estimates on $p^{(a)}(t,x,y)$ and characterization of the class $K_{\alpha}^{\gamma,\beta}$, then prove the above theorems in Section 3. Finally, some examples are presented in Section 4 to illustrate the above three theorems.

2 Some preliminaries

In this section we aim to characterize the class $K_{\alpha}^{\gamma,\beta}$ and to present some estimates on $p^{(a)}$ which will be used in the proofs of Theorems 1.1 and 1.2.

Proposition 2.1. $K_{\alpha}^{1,\beta}(\mu) \supset K_{\alpha}^{\gamma,\beta}(\mu)$ holds for $\beta \in [1,\alpha)$, where $K_{\alpha}^{\gamma,\beta}(\mu)$ is fixed by (1.6).

Proof. Since

$$\int_{0}^{\xi} \left\{ (s^{1-\frac{1}{\alpha}}r^{-(d+\alpha)}) \wedge s^{-\frac{d+1}{\alpha}} \right\} ds \leq C \{ r^{\alpha-d-1} \wedge (\varepsilon r^{-(d-1)}) \}, \quad \varepsilon, r > 0$$

holds for some constant $C > 0$, it is easy to see that

$$
\varepsilon^{\frac{\alpha}{\gamma}} \int_{M}^{M} |f(y)| \xi^{(\alpha)}(s, \rho(x,y)) s^{-\frac{1}{\alpha}} (\varepsilon - s)^{-\frac{\alpha}{\gamma}} \mu(dy) ds
\leq C_{1} \varepsilon^{\frac{\alpha}{\gamma}} \mu(|f|) 1_{\{\mu(M) < \infty\}} + C_{1} \int_{M} |f(y)| \mu(dy) \int_{0}^{\xi} \left\{ (s^{1-\frac{1}{\alpha}}r^{-(d+\alpha)}) \wedge s^{-\frac{d+1}{\alpha}} \right\} ds
\leq C_{1} \varepsilon^{\frac{\alpha}{\gamma}} \mu(|f|) 1_{\{\mu(M) < \infty\}} + C_{2} \int_{M} |f(y)| \rho^{(\alpha)}(x,y)^{\alpha} \rho(x,y)^{d+1} \mu(dy)
$$

holds for some constants $C_{1}, C_{2} > 0$. Similarly, the same estimate holds for

$$
\varepsilon^{\frac{\alpha}{\gamma}} \int_{M}^{M} |f(y)| \xi^{(\alpha)}(s, \rho(x,y)) s^{-\frac{1}{\alpha}} (\varepsilon - s)^{-\frac{\alpha}{\gamma}} \mu(dy) ds.
$$

Therefore, the proof is finished. \qed

In the next result, we present a lower bound of $K_{\alpha}^{\gamma,\beta}(\mu)$ in the class of time-space functions.

Proposition 2.2. Assume that

(2.1) $\mu(B(x,s)) \leq Cs^{d}, \quad s \geq 0, x \in M$

holds for some constant $C > 0$. Let $\alpha \in (0,2)$, $\gamma, \beta \in [0,\alpha)$ and $p, q \in [1, \infty]$. If

(2.2) $\frac{d}{p} + \frac{\alpha}{q} < \alpha - \gamma, \quad q > \frac{\alpha}{\alpha - \beta},$

then

$$L^{q}(\mathbb{R}; L^{p}(M, \mu)) \subset K_{\alpha}^{\gamma,\beta}(\mu).$$
Proof. Using Hölder’s inequality, it is enough to prove
\[(2.3) \limsup_{t \downarrow 0} I(x,t) = 0,\]
where
\[I(x,t) := t^{\frac{\beta q^*}{\alpha}} \int_0^t \left( \int_M \xi^{(\alpha)}(s, \rho(x,y)) p^* \mu(dy) \right)^{\frac{p^*}{p}} \left( t - s \right)^{-\frac{\beta q^*}{\alpha}} ds\]
for \(q^* = \frac{q}{q-1}\) and \(p^* = \frac{p}{p-1}\). By the definition of \(\xi^{(\alpha)}\), there exists a constant \(C_1 > 0\) such that
\[
\int_M \xi^{(\alpha)}(s, \rho(x,y)) p^* \mu(dy) \\
\leq C_1 \left\{ 1 + \int_{B(x,\frac{1}{s})} s^{-\frac{dp^*}{\alpha}} \mu(dy) + s^{p^*} \int_{B(x,\frac{1}{s})^c} \rho(x,y)^{-\beta q^*} \mu(dy) \right\} \\
= C_1 \left\{ 1 + J_1(x,s) + J_2(x,s) \right\}, \ s \in (0,1].
\]
By (2.1), there exists a constant \(C > 0\) such that \(1 + J_1(x,s) \leq Cs^\frac{d-dp^*}{\alpha}\) and
\[
J_2(x,s) = s^{p^*} \sum_{n=0}^{\infty} \int_{B(x,2^{n+\frac{1}{s}}) - B(x,2^n \frac{1}{s})} \rho(x,y)^{-\beta q^*} \mu(dy) \\
\leq s^{p^*} \sum_{n=0}^{\infty} 2^{-n(d+\alpha)p^*} \mu(B(x,2^{n+\frac{1}{s}})) \\
\leq Cs^\frac{d-dp^*}{\alpha} \sum_{n=0}^{\infty} 2^{-n(d+\alpha)p^*} 2^{(n+1)d} s^{\frac{d}{\alpha}} = Cs^\frac{d-dp^*}{\alpha}
\]
hold for \(s \in (0,1]\). Thus,
\[
I(x,t) \leq C_2 t^{\frac{\beta q^*}{\alpha}} \int_0^t \int_0^s \frac{\frac{\beta q^*}{\alpha}}{\alpha} (t - s)^{-\frac{\beta q^*}{\alpha}} ds = C_2 t^{1+\theta} \int_0^1 s^\theta (1-s)^{-\frac{\beta q^*}{\alpha}} ds \leq C_3 t^{1+\theta},
\]
holds for some constants \(C_2, C_3 > 0\), and all \(t \in (0,1]\), where \(\theta = \frac{dq^*}{ap^*} - \frac{dq^*}{\alpha} - \frac{2q^*}{\alpha} > -1\) and \(\frac{2q^*}{\alpha} < 1\) by (2.2). Then (2.3) holds.

Next, we consider estimates on \(p^{(\alpha)}(t,x,y)\).

**Proposition 2.3.** Assume (1.2).

1. (1.4) holds for some constant \(C > 1\).

2. If there exist constants \(C_1, C_2 > 0\) and a natural number \(k \geq 1\) such that
   \[(2.4) \left| \nabla_x p(t,x,y) \right| \leq \frac{C_1 \exp\left[-\frac{C_2 \rho(x,y)^2}{t}\right]}{t^{\frac{1}{2} \mu(M) \wedge t^{\frac{1}{2}}}}, \ t > 0, x, y \in M, 0 \leq i \leq k,
   \]
then

\begin{equation}
|\nabla_x^k p^{(\alpha)}(t, x, y)| \leq \frac{C_{\xi}^{(\alpha)}(t, \rho(x, y))}{t^{\frac{k}{2}}}, \quad t > 0, x, y \in M
\end{equation}

holds for some constant \( C > 0 \).

(3) If (2.4) holds for \( k = 1, 2 \), then for any \( \beta \in (1, \alpha) \) there exists a constant \( C > 0 \) such that

\[ |\nabla_x p^{(\alpha)}(t, x, y) - f_{x \rightarrow x} \nabla_x p^{(\alpha)}(t, x', y)| \leq C t^{-\frac{d}{2}} \rho(x, x')^{\beta - 1} \eta(t; x, x'; y), \]

where \( \eta \) is in \((1.12)\).

Proof. (1) According to the proof of [2, Theorem 3.1], for any \( \lambda > 0 \) and \( m \geq 0 \) there exists a constant \( C_1 > 1 \) such that

\begin{equation}
\frac{t}{C_1(r \vee t^{\frac{1}{2}})^{m+\alpha}} \leq \int_0^\infty \frac{\exp[-\frac{\lambda r^2}{s}]}{s^{\frac{m}{2}}} \mu_t^{(\alpha)}(ds) \leq \frac{C_1 t}{(r \vee t^{\frac{1}{2}})^{m+\alpha}}
\end{equation}

holds for any \( r, t > 0 \). Combining this with the second inequality in (1.2) we obtain

\[ p^{(\alpha)}(t, x, y) \leq C \int_0^\infty \frac{\exp[-\frac{\rho(x, y)t}{C_s}]}{s^{\frac{d}{2}}} \mu_t^{(\alpha)}(ds) + \frac{C}{\mu(M)} \int_0^\infty \exp[-\frac{\rho(x, y)t}{C_s}] \mu_t^{(\alpha)}(ds) \]

\[ \leq \frac{C't}{(\rho(x, y) \vee t^{\frac{1}{2}})^{(d+\alpha)}} + \frac{C't}{(\rho(x, y) \vee t^{\frac{1}{2}})^{\alpha}} = C' \xi^{(\alpha)}(t, \rho(x, y)) \]

for some constant \( C' > 1 \). On the other hand, noting that

\[ \frac{1}{(\mu(M) \wedge t)^{\frac{d}{2}}} \geq \frac{1}{2} \left( \frac{1}{s^{\frac{d}{2}}} + \frac{1_{\{\mu(M) < \infty\}}}{\mu(M)^{\frac{d}{2}}} \right), \]

we obtain the desired lower bound estimate by using the first inequality in (1.2).

(2) It is well known that (cf. (14) in [2])

\begin{equation}
\mu_t^{(\alpha)}(ds) \leq C_0 t s^{-\frac{d+\alpha}{2}} \exp[-ts^{-\frac{d}{2}}] ds
\end{equation}

holds for some constant \( C_0 > 0 \). Then (2.4) yields that

\[ \sup_{x, y} \sup_{0 \leq i \leq k} \left| \nabla_x^{i} p^{(\cdot), x, y} \right| \in L^1(\mu_t^{(\alpha)}), \]

so that by the dominated convergence theorem we obtain

\[ |\nabla_x^k p^{(\alpha)}(t, x, y)| \]

\[ \leq C_0 C_1 t \int_0^\infty \left\{ s^{-\frac{d+\alpha+k+2}{2}} + 1_{\{\mu(M) < \infty\}} s^{-\frac{\alpha+k+2}{2}} \right\} \exp \left[ -\frac{t}{s^{\frac{d}{2}}} - \frac{C_2 \rho(x, y)^2}{s} \right] ds \]

\[ \leq C_0 C_1 (I_1 \wedge I_2), \]

8
Moreover, in this case (1.5) implies
\[
I_1 := t \int_0^\infty \left\{ s^{-\frac{d+\alpha+k+2}{2}} + 1_{\mu(M) < \infty} s^{-\frac{\alpha+k+2}{2}} \right\} \exp \left[ -\frac{C_2 \rho(x, y)^2}{s} \right] ds \\
\leq C_3 t \left\{ \rho(x, y)^{-(d+\alpha+k)} \int_0^\infty r^{-\frac{d+\alpha+k-2}{2}} e^{-r} dr + 1_{\mu(M) < \infty} \rho(x, y)^{-(\alpha+k)} \int_0^\infty r^{-\frac{\alpha+k-2}{2}} e^{-r} dr \right\} \\
\leq C_4 t \left\{ \rho(x, y)^{-(d+\alpha+k)} + 1_{\mu(M) < \infty} \rho(x, y)^{-(\alpha+k)} \right\},
\]
and
\[
I_2 := t \int_0^\infty \left\{ s^{-\frac{d+\alpha+k+2}{2}} + 1_{\mu(M) < \infty} s^{-\frac{\alpha+k+2}{2}} \right\} \exp \left[ -\frac{t}{s^\frac{1}{2}} \right] ds \\
\leq C_3 \left\{ t^{-\frac{d+k}{\alpha}} \int_0^\infty r^{-\frac{d+k}{\alpha}} e^{-r} dr + 1_{\mu(M) < \infty} t^{-\frac{k}{\alpha}} \int_0^\infty r^{-\frac{k}{\alpha}} e^{-r} dr \right\} \\
\leq C_4 \left\{ t^{-\frac{d+k}{\alpha}} + 1_{\mu(M) < \infty} t^{-\frac{k}{\alpha}} \right\}
\]
for some constants $C_3, C_4 > 0$. Therefore,
\[
|\nabla^k p^{(\alpha)}(t, x, y)| \\
\leq C_4 \min \left\{ t^{(\alpha)}(\rho(x, y)) \right\} \leq C_4 \left\{ t^{-\frac{d+k}{\alpha}} + 1_{\mu(M) < \infty} t^{-\frac{k}{\alpha}} \right\}.
\]
From this we complete the proof by considering the following two cases respectively.

(i) If $\rho(x, y) \leq t^{\frac{1}{\alpha}}$ then (2.8) implies
\[
|\nabla^k p^{(\alpha)}(t, x, y)| \leq C_4 \left\{ t^{-\frac{d+k}{\alpha}} + 1_{\mu(M) < \infty} t^{-\frac{k}{\alpha}} \right\}.
\]
Moreover, in this case (1.5) implies
\[
\xi^{(\alpha)}(t, \rho(x, y)) \geq \frac{1}{\mu(M)} + t^{-\frac{d}{\alpha}}.
\]
Therefore, (2.2) holds for some constant $C$.

(ii) If $\rho(x, y) \geq t^{\frac{1}{\alpha}}$, then from (2.8) and (1.5) we obtain
\[
|\nabla^k p^{(\alpha)}(t, x, y)| \leq C_4 \left\{ t^{(\alpha)}(\rho(x, y)) \right\} \\
\leq \frac{C}{t^{\frac{1}{\alpha}}} \xi^{(\alpha)}(t, \rho(x, y))
\]
for some constant $C > 0$.

(3) Since (2.2) implies (2.5) for $k = 1, 2$, we have
\[
|\nabla x p^{(\alpha)}(t, x, y) - /x' \rightarrow x \nabla x p^{(\alpha)}(t, x', y)| \leq \rho(x, x') \int_0^1 |\nabla^2 p^{(\alpha)}(t, \gamma^x_{\theta}, y)| d\theta \\
\leq Ct^{-\frac{2}{\alpha}} \rho(x, x') \int_0^1 | p^{(\alpha)}(t, \gamma^x_{\theta}, y)| d\theta.
\]
Hence, by (2.5) for \( k = 1 \) and Young’s inequality, we obtain
\[
\left| \nabla_x p^{(a)}(t, x, y) - /x' \nabla_x p^{(a)}(t, x', y) \right| \\
\leq \left( \left| \nabla_x p^{(a)}(t, x, y) \right|^{2-\beta} + \left| \nabla_x p^{(a)}(t, x', y) \right|^{2-\beta} \right) \\
\times \left| \nabla_x B^{(a)}(t, x, y) - /x' \nabla_x p^{(a)}(t, x', y) \right|^{\beta-1} \\
\leq C_3 t^{-\frac{\beta}{\alpha}} \rho(x, x')^{\beta-1} \left\{ p^{(a)}(t, x, y) + p^{(a)}(t, x', y) \right\}^{2-\beta} \\
\times \left( \int_{0}^{1} |p^{(a)}(t, \gamma^{x,x'}_{\theta}, y)|d\theta \right)^{\beta-1} \\
\leq C_4 t^{-\frac{\beta}{\alpha}} \rho(x, x')^{\beta-1} \eta(t; x, x'; y)
\]
for some constants \( C_3, C_4 > 0 \). Then the proof is finished. 

Finally, we present below a (3P)-inequality as in [1, Theorem 4].

**Proposition 2.4.** There exists a constant \( C > 0 \) such that
\begin{equation}
\xi^{(a)}(t, r) \wedge \xi^{(a)}(s, u) \leq C \xi^{(a)}(t + s, r + u), \quad s, t, r, u > 0.
\end{equation}

Consequently, there exists a constant \( C > 0 \) such that for \( s, t > 0, x, y, z \in M \),
\begin{equation}
p^{(a)}(t, x, z) \leq C p^{(a)}(s + t, x, y)\left( p^{(a)}(t, x, z) + p^{(a)}(s, z, y) \right).
\end{equation}

**Proof.** (1) According to the proof of [1, Theorem 4], for any \( m \geq 0 \) the function \( \xi_m(t, r) := t^{-\frac{\alpha}{\beta}} \wedge (tr^{-(m+\alpha)}) \) satisfies
\begin{equation}
\xi_m(t, r) \wedge \xi_m(s, u) \leq 2^{\frac{6m}{\alpha}} \xi_m(t + s, r + u), \quad t, r, s, u > 0.
\end{equation}

So, it suffices to prove (2.9) for \( \mu(M) < \infty \). In this case the proof of (2.9) can be finished by considering the following two situations.

(i) If either \( r \vee t, s \vee u \geq 1 \) or \( r \vee t, s \vee u < 1 \), then for \( m = d \) or \( m = 0 \) respectively one derives from (2.11) that
\[
\xi^{(a)}(t, r) \wedge \xi^{(a)}(s, u) \leq \left( 1 + \frac{1}{\mu(M)} \right) \{ \xi_m(t, r) \wedge \xi_m(s, u) \} \\
\leq 2^{\frac{6m}{\alpha}} \frac{(1 + \mu(M))}{\mu(M)} \xi_m(t + s, r + u) \leq 2^{\frac{6m}{\alpha}} \frac{(1 + \mu(M))^{2}}{\mu(M)} \xi^{(a)}(t + s, r + u).
\]

(ii) If e.g. \( r \vee t \leq 1 \) but \( s \wedge u > 1 \), then
\[
\xi^{(a)}(t, r) \wedge \xi^{(a)}(s, u) \leq \xi^{(a)}(s, u) \leq \frac{s(1 + \mu(M))}{\mu(M) \{ u \wedge s^{1/\alpha} \}} \\
\leq \frac{(1 + \mu(M))(t + s)}{\mu(M) \{ u^{\alpha} \wedge s \}} \leq 2^{\alpha} (1 + \mu(M)) \xi^{(a)}(r + u) \wedge (t + s) \\
\leq 2^{\alpha} (1 + \mu(M)) \xi^{(a)}(t + s, r + u).
\]
Lemma 3.1. Then there exists a constant \( c_{13,1} \). For \( t > s \) to construct the solution of (1.8), we make use of the argument of Picard iteration as in 3 Proofs of Theorems 1.1, 1.2 and 1.3 and noting that \( \rho(x, z) + \rho(z, y) \geq \rho(x, y) \), we prove (2.10) for some (different) constant \( C > 0 \).

3 Proofs of Theorems 1.1, 1.2 and 1.3

To construct the solution of (1.8), we make use of the argument of Picard iteration as in 3 Proofs of Theorems 1.1, 1.2 and 1.3. For \( t > s \geq 0 \) and \( x, y \in M \), let \( p_0(t, x; s, y) = p^{(a)}(t - s, x, y) \) and

\[
p_n(t, x; s, y) = p^{(a)}(t - s, x, y) + \int_s^t \int_M p^{(a)}(t - r, x, z) \cdot \left\{ b(r, z), \nabla_z p_{n-1}(r, z; s, y) + c(r, z) p_{n-1}(r, z; s, y) \right\} \mu(dz)dr
\]

for \( n \geq 1 \). Moreover, let \( \Theta_0(t, x; s, y) := p^{(a)}(t - s, x, y) \) and

\[
\Theta_n(t, x; s, y) := p_n(t, x; s, y) - p_{n-1}(t, x; s, y), \quad n \geq 1.
\]

It is clear that

\[
\Theta_n(t, x; s, y) = \int_s^t \int_M p^{(a)}(t - r, x, z) \langle b(r, z), \nabla_z \Theta_{n-1}(r, z; s, y) \rangle \mu(dz)dr
\]

\[
\quad + \int_s^t \int_M p^{(a)}(t - r, x, z) c(r, z) \Theta_{n-1}(r, z; s, y) \mu(dz)dr.
\]

Lemma 3.1. Assume (1.2), (1.3) and let \( |b|, c \in K^{1,1}_\alpha \). Let

\[
\ell(r) = \sup_{s \in (0, r]} \{ K^{1,1}_{\alpha, |b|}(\varepsilon) + K^{1,1}_{\alpha,c}(\varepsilon) \}, \quad r > 0.
\]

Then there exists a constant \( c_0 > 0 \) such that for any \( n \geq 0 \), \( p_n \) (hence \( \Theta_n \)) is well defined and

\[
|\nabla_x \Theta_n(t, x; s, y)| \leq \{c_0 \ell(t - s)\}^n (t - s)^{-\frac{m}{2}} p^{(a)}(t - s, x, y),
\]

\[
|\Theta_n(t, x; s, y)| \leq \{c_0 \ell(t - s)\}^n p^{(a)}(t - s, x, y), \quad t > s \geq 0, x, y \in M.
\]

Proof. According to Propositions 2.3 and 2.4, we may take a constant \( C \geq 1 \) such that (1.4), (2.5) for \( k = 1 \), and (2.10) hold. Take \( c_0 = 4C \). Then the assertion holds for \( n = 0 \). Assume it holds for \( n \leq m \) for some \( m \geq 0 \), then it is easy to see from (1.4) and \( |b|, c \in K^{1,1}_\alpha \) that \( p_{m+1} \) is well defined. It remains to prove (3.3) for \( n = m + 1 \). For any
holds. Moreover, since all $\Theta_n$ and assertion (2). Let $t > 0$ such that $(1.8)$ has a unique solution satisfying $(1.10)$ and $(1.11)$, and to verify assertion (1) with $t - s \leq t_0$ and assertion (2). Let $t_0 > 0$ be such that $c_0 \ell(t_0) \leq \frac{1}{3}$, where $\ell$ is defined in Lemma 3.1.

(a) Construction of the solution. Define

$$p^{(\alpha)}_{b,c}(t, x; s, y) = \sum_{n=0}^{\infty} \Theta_n(t, x; s, y), \quad t - s \in (0, t_0], x, y \in M.$$ 

By Lemma 3.1, this series, as well as $\sum_{n=0}^{\infty} \nabla_x \Theta_n(t, x; s, y)$, converge uniformly on $\{(t, s, y) : t - s \in (0, t_0], x, y \in M\}$. Then (1.11) holds. By letting $n \to \infty$ in (3.1), we see that (1.8) holds. Moreover, since all $\Theta_n$ are jointly continuous, so is $p^{(\alpha)}_{b,c}$. Finally, by Lemma 3.1
we have

\[ |p_{b,c}^{(a)}(t, s, y) - p^{(a)}(t - s, x, y)| \leq \sum_{n=1}^{\infty} |\Theta_n(t, s, y)| \]

\[ \leq \frac{c_0\ell(t-s)}{1 - c_0\ell(t-s)} p^{(a)}(t-s, x, y) \leq \frac{1}{2} p^{(a)}(t-s, x, y). \]

Then (1.10) follows from (1.4) ensured by Proposition 2.3(1).

(b) Uniqueness. Let \( \tilde{p}_{b,c}^{(a)} \) be another solution to (1.8) satisfying (1.11). Then the induction argument in the proof of Lemma 3.1 implies that \( \Theta := p_{b,c}^{(a)} - \tilde{p}_{b,c}^{(a)} \) satisfies

\[ |\Theta(t, s, y)| \leq (c_0\ell(t-s))^n p^{(a)}(t-s, x, y) \]

for all \( n \geq 0 \), so that letting \( n \to \infty \) we derive \( \Theta(t, s, y) = 0 \) for \( t-s \leq t_0 \). Thus, the solution is unique.

(c) For (1) it suffices to prove that for any \( \varphi \in C_0^\infty(M) \),

\[ P_{t,s}^{b,c} \varphi(x) = P_{t,r}^{b,c} P_{r,s}^{b,c} \varphi(x), \quad s \leq r \leq t, \]

where

\[ P_{t,s}^{b,c} \varphi(x) := \int_M \tilde{p}_{b,c}^{(a)}(t, s, y) \varphi(y) \mu(dy). \]

Set

\[ g_{r,s}(x) = \langle b(r, x), \nabla P_{r,s}^{b,c} \varphi(x) \rangle + c(r, x) P_{r,s}^{b,c} \varphi(x), \quad r > s. \]

By (1.8) we have

\[ P_{t,s}^{b,c} \varphi(x) = P_{t-s}^{(a)} \varphi(x) + \int_s^t P_{t-r}^{(a)} g_{r,s}^{(a)}(x) dr' \]

\[ = P_{t-r}^{(a)} P_{r-s}^{(a)} \varphi(x) + \int_r^t P_{t-r}^{(a)} P_{r-s}^{(a)} g_{r,s}^{(a)}(x) dr' + \int_r^t P_{t-r}^{(a)} g_{r,s}^{(a)}(x) dr \]

\[ = P_{t-r}^{(a)} P_{r,s}^{b,c} \varphi(x) + \int_r^t P_{t-r}^{(a)} \left( \langle b(r', \cdot), \nabla P_{r,s}^{b,c} \varphi \rangle \right)(x) dr' \]

\[ + \int_r^t P_{t-r}^{(a)} \left( c(r', \cdot) P_{r,s}^{b,c} \varphi \right)(x) dr'. \]

On the other hand,

\[ P_{t,r}^{b,c} P_{r,s}^{b,c} \varphi(x) = P_{t-r}^{(a)} P_{r,s}^{b,c} \varphi(x) + \int_r^t P_{t-r}^{(a)} \left( \langle b(r', \cdot), \nabla (P_{r,s}^{b,c} P_{r,s}^{b,c} \varphi) \rangle \right)(x) dr' \]

\[ + \int_r^t P_{t-r}^{(a)} \left( c(r', \cdot) P_{r,s}^{b,c} P_{r,s}^{b,c} \varphi \right)(x) dr'. \]

By the uniqueness as observed in (b), we obtain (3.4).
(d) Finally, we prove (2). Let \( \varphi, \psi \in C_0^2(M) \). By (1.8), we have

\[
\frac{P_{t,s}^{b,c} \varphi - \varphi}{t - s} = L_{b,c}^{(a)}(s) \varphi
\]

\[
= \left( \frac{P_{t,s}^{(a)} \varphi - \varphi}{t - s} - L_{b,c}^{(a)} \varphi \right) + \frac{1}{t - s} \int_s^t \langle b(r, \cdot) - b(s, \cdot), \nabla P_{r,s}^{b,c} \varphi \rangle dr
\]

(3.5)

\[
+ \frac{1}{t - s} \int_s^t \left\{ (c(r, \cdot) - c(s, \cdot)) P_{r,s}^{b,c} \varphi + c(s, \cdot) (P_{r,s}^{b,c} \varphi - \varphi) \right\} dr
\]

\[
+ \frac{1}{t - s} \int_s^t \langle \nabla P_{r,s}^{b,c} \varphi - \nabla \varphi \rangle dr
\]

=: I_0(t, s) + I_1(t, s) + I_2(T, s) + I_3(t, s).

Since \( \varphi \in C_0^2(M) \subset \mathcal{D}(L^{(a)}) \), we have

(3.6)

\[
\lim_{t \downarrow s} \| I_0(t, s) \|_\infty = 0, \quad s \geq 0.
\]

Fix \( s \geq 0 \) and set

\[
\dot{u}(t, x) := P_{t,s}^{b,c} \varphi(x), \quad t \geq s.
\]

First of all, by (1.3), (1.10) and the contraction of \( P_t^{(a)} \) we have

(3.7)

\[
\sup_{t \in [s, 1 + s]} \| u(t, \cdot) \|_\infty < \infty.
\]

Next, by (1.3) and Proposition 2.3(2), (2.5) holds for \( k = 1 \). Combining this with (1.4), we obtain

\[
\| \nabla P_t^{(a)} \varphi - \nabla \varphi \|_\infty \leq \int_0^t \| \nabla P_s^{(a)} L^{(a)} \varphi \|_\infty ds
\]

(3.8)

\[
\leq C_1 \| L^{(a)} \varphi \|_\infty \int_0^t s^{-\frac{1}{n}} ds \leq C, \quad t \in [s, 1 + s]
\]

for some constants \( C_1, C > 0 \). Next, let

\[
\Theta_{t,s}^{(n)} \varphi = \int_M \Theta_n(t, \cdot; s, y) \varphi(y) \mu(dy), \quad t \geq s.
\]

By (3.3), (1.4) and noting that \( \| P_r^{(a)} \varphi \|_\infty \leq \| \varphi \|_\infty < \infty, r \geq 0 \), we obtain

\[
\| \nabla \Theta_{t,s}^{(n)} \varphi \|_\infty
\]

\[
\leq C \| \varphi \|_\infty (c_0 \ell(t - s))^{n - 1} \left\| \int_M \int_s^t \xi^{(a)}(t - r, \rho(\cdot, z)) \left( \frac{[b(r, z)]}{(t - r)^2} + [c(r, z)] \right) \mu(dz) dr \right\|_\infty
\]

\[
\leq C \| \varphi \|_\infty (c_0 \ell(t - s))^{n - 1} \ell(t - s) \leq (c_1 \ell(t - s))^n, \quad n \geq 1,
\]
where $c_1 := c_0 + C \| \varphi \|_\infty$. Since $\ell(r) \to 0$ as $r \to 0$, we may find $t_0 \in (0, 1]$ such that $c_1 \ell(t_0) < 1$. Combining this with (3.8) we conclude from the construction of $p_{b,c}^{(\alpha)}$ and the definition of $P_{t,s}^{b,c}$ that $\| \nabla u(t, \cdot) \|_\infty$ is bounded on $[s, s + t]$. This and (3.7) yield

\begin{equation}
\sup_{r \in [s, t_0 + s]} \left\{ \| \nabla P_{r,s}^{b,c} \varphi \|_\infty + \| P_{r,s}^{b,c} \varphi \|_\infty \right\} < \infty.
\end{equation}

Therefore, there exists a constant $C > 0$ such that

\begin{equation}
\| P_{t,s}^{b,c} \varphi - \varphi \|_\infty \leq \| P_{t-s}^{(\alpha)} \varphi - \varphi \|_\infty + \left\| \int_s^t \int_M p^{(\alpha)}(t - r, \cdot, z) |c(r, z)| |u(r, z)| \mu(dz) dr \right\|_\infty \\
+ \left\| \int_s^t \int_M p^{(\alpha)}(t - r, \cdot, z) |b(r, z)| \cdot \| \nabla_z u(r, z) \| \mu(dz) dr \right\|_\infty \\
\leq \| P_{t-s}^{(\alpha)} \varphi - \varphi \|_\infty + C \sup_{r \in [s, t]} \| P_{r,s}^{b,c} \varphi \|_\infty K_{\alpha,0}^0(t - s) \\
+ C \sup_{r \in [s, t]} \| \nabla P_{r,s}^{b,c} \varphi \|_\infty K_{\alpha,|b|}^0(t - s) \to 0,
\end{equation}

as $t \downarrow s$. Hence,

\begin{equation}
\lim_{t \downarrow s} \sup_{r \in [s, t]} \| P_{r,s}^{b,c} \varphi - \varphi \|_\infty = 0.
\end{equation}

Since $b \in C([0, \infty); L^1_{loc}(M \to TM; \mu))$, $c \in C([0, \infty); L^1_{loc}(M \to \mathbb{R}; \mu))$, this and (3.9) yield that

\begin{equation}
\lim_{t \downarrow s} \int_M \psi \{ I_1(t, s) + I_2(t, s) \} d\mu = 0.
\end{equation}

Combining this with (3.5) and (3.6), we need only to prove

\begin{equation}
\lim_{t \downarrow s} \int_M \psi I_3(t, s) d\mu = 0.
\end{equation}

To this end, take $\{ b_n(s, \cdot) \}_{n \geq 1} \subset C^\infty_0(M; TM)$ such that

\begin{equation}
\lim_{n \to \infty} \int_M |\psi| \cdot |b_n(s, \cdot) - b(s, \cdot)| d\mu = 0.
\end{equation}

Since $\mu$ has a $C^1$-density w.r.t. the volume measure, $\text{div}_\mu(\psi b_n(s)) \in C_0(M)$ for $\psi \in C^2_0(M)$. Combining this with (3.9) and (3.10), and by the dominated convergence theorem we conclude that

\begin{equation}
\lim_{t \downarrow s} \left| \int_M \psi I_3(t, s) d\mu \right| \\
\leq \lim_{n \to \infty} \limsup_{t \downarrow s} \frac{1}{t - s} \int_s^t \int_M |b(s, \cdot) - b_n(s, \cdot)| \cdot |\nabla P_{r,s}^{b,c} \varphi - \nabla \varphi| \cdot |\psi| d\mu dr \\
+ \lim_{n \to \infty} \limsup_{t \downarrow s} \frac{1}{t - s} \int_s^t \int_M |\text{div}_\mu(b_n(s, \cdot) \psi)| \cdot |P_{r,s}^{b,c} \varphi - \varphi| d\mu dr = 0.
\end{equation}

Then the proof is complete. \qed
Proof of Theorem 1.2. Due to Theorem 1.1(1), it suffices to prove for $0 \leq s < t$ with $t-s \leq t_0$, where $t_0 > 0$ is fixed in the proof of Theorem 1.1. According to (a) in the proof of Theorem 1.1, we only need to prove

$$
(3.12) \quad \left| \nabla_x \Theta_n(t, x; s, y) - /_{x' \rightarrow x} \nabla_x \Theta_n(t, x'; s, y) \right|
$$

$$
\leq (c_0 \ell_\beta(t-s))^n (t-s)^{-\frac{n}{2}} \rho(x, x')^\beta - 1 \eta(t-s; x, x'; y), n \geq 1,
$$

where

$$
\ell_\beta(r) := \sup_{\varepsilon \in (0, r]} \left\{ K_{\alpha, |b|}^\beta (\varepsilon) + K_{\alpha, c}^\beta (\varepsilon) \right\}, \quad r > 0.
$$

By (1.4), Lemma 3.1 and Proposition 2.3(2),(3), we have

$$
\left| \nabla_x \Theta_n(t, x; s, y) - /_{x' \rightarrow x} \nabla_x \Theta_n(t, x'; s, y) \right|
$$

$$
\leq \int_M \int_M \left( |b(r, z)| \cdot |\nabla_x \Theta_n(r, z; s, y)| + |c(r, z)| \cdot |\Theta_n(r, z; s, y)| \right) \times \left| \nabla_x p^{(\alpha)}(t-r, x, z) - /_{x' \rightarrow x} \nabla_x p^{(\alpha)}(t-r, x', z) \right| \mu(dz) dr
$$

$$
\leq C \left\{ c_0 \ell_\beta(t-s) \right\}^n \rho(x, x')^\beta - 1 \int \int M \left[ \eta(t-r; x, x'; z) p^{(\alpha)}(r-s, z, y)
$$

$$
\times \left( |b(r, z)| (r-s)^{-\frac{n}{2}} + |c(r, z)| (r-t)^{-\frac{n}{2}} \right) \right] \mu(dz) dr.
$$

By the (3P)-inequality, we have

$$
\eta(t-r; x, x'; z) p^{(\alpha)}(r-s, z, y)
$$

$$
\leq p^{(\alpha)}(t-s, x, y) \left\{ p^{(\alpha)}(t-r, x, z) + p^{(\alpha)}(r-s, z, y) \right\} + p^{(\alpha)}(t-s, x', y) \left\{ p^{(\alpha)}(t-r, x', z) + p^{(\alpha)}(r-s, z, y) \right\}
$$

$$
+ \int_0^1 p^{(\alpha)}(t-s, \gamma_{\theta}, x, y) \left\{ p^{(\alpha)}(t-r, \gamma_{\theta}, x', z) + p^{(\alpha)}(r-s, z, y) \right\} d\theta.
$$

Substituting this into the above estimate, we find that

$$
\left| \nabla_x \Theta_n(t, x; s, y) - /_{x' \rightarrow x} \nabla_x \Theta_n(t, x'; s, y) \right|
$$

$$
\leq C \left\{ c_0 \ell_\beta(t-s) \right\}^n \rho(x, x')^\beta - 1 \eta(t-s; x, x'; y) (t-s)^{-\frac{n}{2}}
$$

$$
\times \left( K_{\alpha, |b|}^{1, \beta} (t-s) + K_{\alpha, |b|}^{\beta, 1} (t-s) + K_{\alpha, c}^{0, \beta} (t-s) + K_{\alpha, c}^{\beta, 0} (t-s) \right).$$

This implies (3.12).
Proof of Theorem 1.3. By (1.11) and the existence of $\text{div}_\mu b$, we see that $\text{div}_\mu \{p^{(\alpha)}(t-r,x,\cdot)b(r,\cdot)\}$ exists. Take $\{h_m\}_{m \geq 1} \subset C^\infty_0(M)$ with $0 \leq h_m \leq 1, h_m \uparrow 1, \|\nabla h_m\|_\infty \downarrow 0$. Then, by approximating $\Theta_{n-1}$ with $h_m \Theta_{n-1}$, we obtain from (3.2) that

$$
\Theta_n(t,x; s,y) = \int_t^s \int_M \text{div}_\mu \{p^{(\alpha)}(t-r,x,\cdot)b(r,\cdot)\}(z)\Theta_{n-1}(r,z; s,y)\mu(dz)dr
$$

where

$$
\tilde{c}(r,z) = c(r,z) + \text{div}_\mu b(r,z).
$$

Notice that by the symmetry and (1.3),

$$
|\nabla_y p(t, x, y)| = |\nabla_y p(t, y, x)| \leq C \exp[-\rho(x,y)^2/c_t] \frac{\sqrt{t}}{(\mu(M) \land t)^{\frac{d}{2}}}, \quad t > 0, x, y \in M.
$$

Using the same arguments as in Lemma 3.1, one can prove

$$
|\nabla_y \Theta_n(t, x; s, y)| \leq \{c_0 \tilde{\ell}(t-s)\}^n(t-s)^{-\frac{d}{2}}p^{(\alpha)}(t-s, x, y),
$$

where

$$
\tilde{\ell}(r) := \sup_{\varepsilon \in (0,r]} \{K^{1,1}_{\alpha,\beta}(\varepsilon) + K^{1,1}_{\alpha,\varepsilon}(\varepsilon)\}, \quad r > 0.
$$

As in the proof of Theorem 1.3, $\sum_{n=0}^\infty \nabla_y \Theta_n(t, x; s, y)$ converges uniformly on $\{(t, x; s, y): t-s \in (0, t_0], x, y \in M\}$. Thus (1.14) holds. \hfill \Box

4 Some examples

Example 4.1. Let $L = \Delta + \langle \nabla V, \nabla \cdot \rangle$ for some $V \in C^2(M)$ such that

$$
\text{Ric}(X,X) - \text{Hess}_V(X,X) - \varepsilon \langle X, \nabla V \rangle^2 \geq 0, \quad X \in TM,
$$

(4.1)

$$
\frac{r^d}{C} \leq \mu(B(x,r)) \leq C r^d, \quad r > 0, x \in M
$$

hold for some constants $\varepsilon > 0, C > 1$, where $B(x,r)$ is the geodesic ball at $x$ with radius $r$, and $\mu(dx) = e^{V(x)}\text{vol}(dx)$ for vol the volume measure on $M$. Then $P_t$ is symmetric in
$L^2(\mu)$ and all assertions in Theorems 1.1 hold. In fact, according to [6] (see also [5] when $V = 0$), the condition (1.2) follows from (4.1). Next, according to e.g. [12, Corollary 4.2] (4.2)

$$|\nabla P_t f| \leq \frac{C(P_t f^2)^{\frac{1}{2}}}{\sqrt{t}}, \quad t > 0, f \in \mathcal{B}_0(M)$$

holds for some constant $C > 0$. Letting $f_{t,y}(z) = p(\frac{t}{2}, z, y)$, this implies that

$$|\nabla x p(t, x, y)| = |\nabla P_{\frac{t}{2}} f_{t,y}(x)| \leq \frac{C(P_{\frac{t}{2}} f_{t,y}^2)^{\frac{1}{2}}}{\sqrt{t}} \leq C \sqrt{|f_{t,y}|_\infty p(t, x, y)}.$$

Combining this with (1.2) we prove (1.3).

**Example 4.2.** Let $M$ be compact and $L = \Delta + \langle \nabla V, \nabla \cdot \rangle$ for some $V \in C^2(M)$. Let $\mu(dx) = e^{V(x)}\text{vol}(dx)$. Then all assertions in Theorem 1.2 hold. In this case $(\frac{1}{C} \leq C(r \wedge 1)^d$ holds for some constants $C > 1$ and all $x \in M, r > 0$. So, (1.2) follows from [5] or [6]. Next, since the compactness of $M$ implies that $\text{Ric} - \text{Hess}_V$ is bounded below, [12, Corollary 4.2] implies (4.2) for $t \in (0, 1]$. Thus, as observed in Example 4.1 that (1.3) holds for $t \in (0, 1]$. Again since $M$ is compact, the second assertion in [12, Theorem 4.4] implies that

$$|\nabla x p(t, x, y)| \leq C e^{-\lambda t}, \quad t \geq 1, x, y \in M$$

holds for some constants $C, \lambda > 0$. Therefore, (1.3) holds also for $t \geq 1$ as $\rho$ is bounded.

**Example 4.3.** Let $M = \mathbb{R}^d, \mu(dx) = dx$ and

$$L = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Assume that $a_{ij}$ are bounded and Hölder continuous functions on $\mathbb{R}^d$, and $(a_{ij}) \geq \lambda_0 I_{d \times d}$ holds for some constant $\lambda_0 > 0$. Then all assertions in Theorems 1.1 and 1.2 hold. In fact, (1.2) follows from [9, Theorem A] with $b = 0$, and (1.3) and (1.13) follow from [9, (1.3)] (see also [4], page 229).

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**References**

[1] K. Bogdan, T. Jakubowski, *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators*, Commun. Math. Phys. 271(2007), 179–198.
[2] K. Bogdan, A. Stós A, P. Sztonyk, *Harnack inequality for stable processes on d-sets*, Studia Math. 158(2003), 163–198.

[3] Z. Chen, P. Kim, R. Song, *Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation*, arXiv:1011.3273.

[4] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1975.

[5] P. Li, S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156(1986), 153–201.

[6] Z. Qian, *Gradient estimates and heat kernel estimates*, Proc. R. Soc. Edinburgh A 125(1995), 975–990.

[7] K. I. Sato, *Lévy Processes and Infinite Divisible Distributions*, Cambridge University Press, Cambridge, 1999.

[8] R. L. Schilling, R. Song, Z. Vondraček, *Bernstein Functions*, De Gruyter, Berlin, 2010.

[9] S.-J. Sheu, *Some estimates of the transition density of a nondegenerate diffusion markov processes*, Ann. Probab. 19(1991), 538–561.

[10] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton, N.J., Princeton University Press, 1970.

[11] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, Amsterdam, 1978.

[12] F.-Y. Wang, *On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups*, Probab. Theory Relat. Fields 108(1997), 87–101.

[13] Q. S. Zhang, *Gaussian bounds for the fundamental solutions of \( \nabla(A \nabla u) + B \nabla u - u_t = 0 \)*, Manuscripta Math. 93(1997), 381–390.