OPTIMAL REGULARITY RESULTS FOR THE ONE-DIMENSIONAL PRESCRIBED CURVATURE EQUATION VIA THE STRONG MAXIMUM PRINCIPLE

JULIÁN LÓPEZ-GÓMEZ AND PIERPAOLO OMARI

ABSTRACT: A refined version of the strong maximum principle is proven for a class of second order ordinary differential equations with possibly discontinuous non-monotone nonlinearities. Then, exploiting this tool, some optimal regularity results recently established by López-Gómez and Omari, in [15–17], for the bounded variation solutions of non-autonomous quasilinear equations driven by the one-dimensional curvature operator, are substantially improved by admitting general prescribed curvatures and by incorporating general boundary conditions. The new approach developed here yields a new, deeper, interpretation of the assumptions introduced in our previous papers, simultaneously clarifying their meaning and making fully transparent their connection with the strong maximum principle.

1. INTRODUCTION

The aim of this work is twofold. First we prove an extended version of the strong maximum principle for a general class of second order ordinary differential equations

$$v'' = g(t, v, v'),$$

in the absence of any assumption of continuity or monotonicity, which is a crucial feature at the light of our subsequent applications. Then, based on this form of the strong maximum principle, we provide some optimal regularity results for the bounded variation solutions, positive and nodal, of the non-autonomous curvature equation

$$-\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = f(x, u),$$

where $f$ is an arbitrary function prescribing the curvature of $u$. These findings substantially generalize some previous regularity results we established in [14–17] for the positive bounded variation solutions of

$$f(x, s) = h(x) k(s),$$

under homogeneous Neumann boundary conditions, in the special case where

The reader should be aware that, unlike in these papers, here we are imposing neither the structural assumption (1.3), nor the Neumann boundary conditions, nor any requirement on the sign on the solutions. As a consequence, the analysis carried out here allows us, through a completely different technical device, to extend most of the results of [14–17] to more general classes of equations and boundary value problems. Furthermore, in this paper we are going to provide with a new interpretation of the assumptions used
in our previous works, clarifying their meaning and establishing some deep, though previously hidden, connections with the strong maximum principle.

This paper is organized as follows. In Section 2 we establish as Theorem 2.1 the version of the strong maximum principle for equation (1.1) used throughout this paper. Since no kind of continuity or monotonicity, even in the state variable $s$, is imposed on the right hand side $g = g(t, s, \xi)$ of equation (1.1), as well as on the comparison function $G' = G'(s)$ introduced in condition (G) of Theorem 2.1 this form of the strong maximum principle provides a completion and a sharpening of its counterparts in [34] or [53]; its proof being also more delicate than in the classical situations.

Section 3 contains the main regularity results of this work. The first one, stated as Theorem 3.1 shows that a bounded variation solution $u$ of the curvature equation (1.2) can lose its regularity only at the endpoints, but never at the interior points, of any interval where the function $f(\cdot, u(\cdot))$ has a definite sign; yet, $u$ can be singular at an interior point of its domain if such a point separates two adjacent intervals where $f(\cdot, u(\cdot))$ changes sign. This result sharpens [13 Prop. 3.6], where the solutions were assumed, in addition, to satisfy homogeneous Neumann boundary conditions. Although our proof of Theorem 3.1 is a re-elaboration of the proof of [14 Prop. 3.6], for the sake of completeness, we are giving complete technical details here.

Our next two results, Theorems 3.2 and 3.3, establish the complete regularity of the bounded variation solutions $u$ of (1.2). Precisely, Theorem 3.2 guarantees the regularity at the endpoints of any interval where the sign of $f(\cdot, u(\cdot))$ is constant, by imposing at these points a suitable control, expressed by any of the conditions (j)–(jjjj), on the decay rate to zero of $f(\cdot, u(\cdot))$ Theorem 3.3 instead, guarantees the regularity of $u$ at any interior point, $z$, separating two adjacent interval where $f(\cdot, u(\cdot))$ changes sign, by imposing a similar decay property to $f(\cdot, u(\cdot))$ either on the left, or on the right, of $z$, as expressed by the conditions (h) or (hh).

Such kind of decay controls were introduced by the authors in [15] for discussing the regularity of the positive bounded variation solutions of the Neumann problem associated with (1.2) in the very special case when $f$ can be decomposed as in (1.3), namely, $f(x, s) = h(x) k(s)$, where $k$ is a function having a superlinear potential at infinity. They were later used by the authors in [16] to treat the case where the potential of $k$ is asymptotically linear, and in [17] to deal with the case of sublinear potentials. Here, no specific restriction on the asymptotic behavior of $f(\cdot, s)$ with respect to $s$ is imposed. From [15–17] we also know that these assumptions on the decay rate of $f(\cdot, u(\cdot))$ are optimal, in the sense that, if they fail at some point, the derivative $u'$ might blow-up there, and the solution $u$ might even develop a jump discontinuity.

The proof of Theorems 3.2 and 3.3 presented here is completely new and it relies on the use of the strong maximum principle as expressed by Theorem 2.1. Our approach, besides being far more general and versatile, displays the following striking fact. It turns out that the precise condition yielding the regularity of a solution $u$ of (1.2), through a control on the decay rate to zero of $f(\cdot, u(\cdot))$ at some point $z$, is precisely the assumption required by Theorem 2.1 so that the strong maximum principle holds for the differential equation

\[
\left(\frac{v'}{\sqrt{1 + (v')^2}}\right)' = f(z + v, t) \iff v'' = f(z + v, t)\left(1 + (v')^2\right)^{\frac{1}{2}}, \tag{1.4}
\]

satisfied by the shift $v = w - z$ of a local inverse $w$ of $u$. Note that, as $f$ is not assumed to satisfy any regularity condition, the right hand side of (1.4), that is, the function

\[
g(t, s, \xi) := f(z + s, t)\left(1 + \xi^2\right)^{\frac{1}{2}},
\]

may be discontinuous, besides in $t$, in the state variable $s$ as well. Note that this could happen for $g$ even if $f$ were a Carathéodory function.

Essentially, we establish that the validity of the strong maximum principle for equation (1.4) yields the regularity for the solutions of (1.2). As a consequence, the bounded variation solutions of (1.2) can develop singularities only when the conclusions of the strong maximum principle fail for (1.4). This appears to be
a quite remarkable achievement that illuminates and clarify the apparently exotic conditions introduced in [15].

Finally, Section 4 is devoted to the application of the theory developed in Section 3 to establish the regularity, up to the boundary, of the solutions of a number of non-autonomous one-dimensional prescribed curvature equations supplemented with several types of boundary conditions, such as Dirichlet, Neumann, Robin, or even periodic boundary conditions. These and other similar statements, that can be deduced from Theorems 3.1, 3.2, 3.3 complete or extend, as far as regularity is concerned, several existence and multiplicity results previously obtained in [3, 14, 20].

2. A variant of the strong maximum principle

The main result of this section is the following version of the strong maximum principle for second order ordinary differential equations with possibly discontinuous non-monotone right hand sides.

**Theorem 2.1.** Let \( g : (\alpha, \omega) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a given function and let \( v \in W_{\text{loc}}^{2,1}(\alpha, \omega) \cap W^{1,1}(\alpha, \omega) \) be a non-trivial non-negative solution of the differential equation

\[
v''(t) = g(t, v(t), v'(t)) \quad \text{for almost all } t \in (\alpha, \omega).
\]

Assume that:

(G) there exist a constant \( \varepsilon > 0 \) and an absolutely continuous function \( G : [0, \varepsilon] \to \mathbb{R} \) such that

\[
0 \leq g(t, v(t), v'(t)) \leq G'(v(t)) \quad \text{for almost all } t \in (\alpha, \omega) \text{ for which } 0 < v(t) \leq \varepsilon \text{ and } |v'(t)| \leq \varepsilon,
\]

and either

\[
G(s) = 0 \quad \text{for all } s \in [0, \varepsilon],
\]

or

\[
G(s) > 0 \quad \text{for all } s \in [0, \varepsilon] \quad \text{and} \quad \int_{0}^{\varepsilon} \frac{1}{\sqrt{G(s)}} ds = +\infty.
\]

Then, \( v \) is strongly positive, in the sense that the following properties hold true:

(i) \( v(t) > 0 \) for all \( t \in (\alpha, \omega) \),

(ii) \( v'(\alpha^+) > 0 \) if \( v(\alpha) = 0 \) and \( v'(\alpha^+) \) exists,

(iii) \( v'(\omega^-) < 0 \) if \( v(\omega) = 0 \) and \( v'(\omega^-) \) exists.

**Remark 2.1.** It is worth observing that no kind of continuity, or monotonicity, is imposed either on the function \( g \) appearing at the right hand side of equation (2.1), or on the comparison function \( G' \). Indeed, the function \( G' \) considered in assumption (G) is only Lebesgue integrable and does not satisfy any monotonicity condition. These features are crucial for proving our new findings in Section 3.

In this respect, assumption (G) is independent of the conditions required by the classical Vázquez strong maximum principle in [34], where \( G' \) is assumed to be continuous and increasing, as well as of the conditions imposed by Pucci and Serrin in [33, Ch. 5], where \( G' \) is again supposed to be increasing.

As already pointed out in [33, 34], the condition (2.4) is sharp, because if it fails, dead core solutions, i.e., non-negative solutions vanishing on sets of positive measure, may occur. The necessity of this type of conditions goes back to Benilan, Brezis and Crandall [3]. Note that condition (G) entails \( G(0) = 0 \).

**Remark 2.2.** Under condition (G), there are some positive constants, \( k \), that are strict supersolutions of the differential operator

\[
\mathcal{L}v := -v'' + G'(v), \quad v \in W_{\text{loc}}^{2,1}(\alpha, \omega) \cap W^{1,1}(\alpha, \omega),
\]

under Dirichlet boundary conditions in \( (\alpha, \omega) \). Thus, \( \mathcal{L} \) satisfies similar assumptions as those of Theorem 10 in Chapter 2 of Protter and Weinberger [32]. From such a perspective, Theorem 2.1 can be viewed as a sharp one-dimensional nonlinear counterpart of Corollary 2.1 of López-Gómez [13] (going back to [11] and [12]). Indeed, one might re-state Theorem 2.1 by simply saying that, under condition (G), any non-trivial non-negative solution of \( \mathcal{L}v \geq 0 \) in \( W_{\text{loc}}^{2,1}(\alpha, \omega) \cap W^{1,1}(\alpha, \omega) \) must be strongly positive.
This bi-association suggests the validity of Theorem 2.1 even in a multidimensional context, for general operators of the form $L + G'(v)$ where $L$ is a linear second order uniformly elliptic operator in $\Omega$ whose principal eigenvalue, under Dirichlet boundary conditions in $\Omega$, is positive.

The proof of Theorem 2.1 exploits the following maximum principle for first order differential inequalities.

**Lemma 2.1.** Assume that $H : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function satisfying the condition:

(H) there exists a constant $\varepsilon > 0$ such that $H(s) > 0$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{H(s)} \, ds = +\infty = \int_{0}^{\varepsilon} \frac{1}{H(s)} \, ds. \quad (2.5)$$

Then, any non-trivial solution $v \in W^{1,1}(\alpha, \omega)$ of the differential inequality

$$|v'(t)| \leq H(v(t)) \quad \text{for almost all } t \in (\alpha, \omega) \quad (2.6)$$

is either strictly positive, or strictly negative, i.e., either $\min v > 0$, or $\max v < 0$.

**Remark 2.3.** Assumption (H) is the classical Osgood condition, introduced in [31] to guarantee the uniqueness of the solution for the Cauchy problem associated with first order ordinary differential equations.

The proof of Lemma 2.1 is elementary when $H$ is continuous. As here we are only imposing measurability, our proof is far more delicate.

**Proof of Lemma 2.1** Let $v \in W^{1,1}(\alpha, \omega)$ be a non-trivial solution of (2.6). We claim that $\min v > 0$ if $\max v > 0$. It is apparent that, substituting $-v$ for $v$, we can infer that $\max v < 0$ if $\min v < 0$. Arguing by contradiction, we suppose that

$$\max v > 0 \quad \text{and} \quad \min v \leq 0.$$ 

As $v$ is continuous and $\max v > 0$, there exist a point $t_0 \in [\alpha, \omega]$ and two constants $\delta > 0$ and $\eta \in (0, \varepsilon]$ such that $v(t_0) = 0$ and either

$$v((t_0, t_0 + \delta)] = (0, \eta], \quad \text{or} \quad v([t_0 - \delta, t_0)) = (0, \eta].$$

Assume that the former case occurs, the latter one being treated similarly. As $H$ is measurable and $H(s) > 0$ for all $s \in (0, \eta]$, the function $\frac{1}{H} : (0, \eta] \to \mathbb{R}$ is well defined and it is measurable too. Moreover, being absolutely continuous, $v$ satisfies the Lusin’s $N$-property, i.e., it maps sets of null measure to sets of null measure. Thus, the functions

$$\frac{1}{H} : (0, \eta] \to \mathbb{R} \quad \text{and} \quad v : (t_0, t_0 + \delta] \to (0, \eta]$$

fulfill the assumptions of [10, Thm.2]. Consequently, the functions

$$\frac{|v'|}{H \circ v} : (t_0, t_0 + \delta] \to \mathbb{R} \quad \text{and} \quad \frac{N_v(\cdot, (t_0, t_0 + \delta])}{H(\cdot)} : (0, \eta] \to \mathbb{R}$$

are measurable, where $N_v(s, (t_0, t_0 + \delta])$ denotes the Banach indicatrix of $v$ in the interval $(t_0, t_0 + \delta]$, i.e.,

$$N_v(s, (t_0, t_0 + \delta]) = \mathcal{H}^0 (v^{-1}(s) \cap (t_0, t_0 + \delta]),$$

with $\mathcal{H}^0$ the counting measure. Moreover, by (2.6), we have that

$$\frac{|v'(t)|}{H(v(t))} \leq 1 \quad \text{for almost all } t \in (t_0, t_0 + \delta].$$

Hence, integrating yields

$$\int_{t_0}^{t_0 + \delta} \frac{|v'(t)|}{H(v(t))} \, dt \leq \delta. \quad (2.7)$$
Furthermore, by the formula of change of variables established in [10 Thm.2], we find that
\[ \int_{t_0}^{t_0+\delta} \frac{|v'(t)|}{H(v(t))} \, dt = \int_{t_0}^{\eta} \frac{N_\epsilon(s,x_\epsilon,x_\delta)}{H(s)} \, ds. \] (2.8)
As \( v((t_0,t_0+\delta]) = (0,\eta] \), we have that
\[ N_\epsilon(s,(t_0,t_0+\delta]) \geq 1 \quad \text{for all } s \in (0,\eta]. \]
Hence, by (2.8) and (2.7), we get
\[ \int_0^{\eta} \frac{1}{H(s)} \, ds \leq \int_0^{\eta} \frac{N_\epsilon(s,x_\epsilon,x_\delta)}{H(s)} \, ds = \int_{t_0}^{t_0+\delta} \frac{|v'(t)|}{H(v(t))} \, dt \leq \delta, \]
which is impossible, because (2.5) entails that
\[ \int_0^{\eta} \frac{1}{H(s)} \, ds = +\infty \quad \text{for all } \eta \in (0,\epsilon]. \]
This concludes the proof of Lemma 2.1 \( \square \)

Similarly as the proof of Lemma 2.1, also the proof of Theorem 2.1 would be a bit simpler if, in addition, \( g \) and \( G' \) are continuous and \( v \in C^2[\alpha,\omega] \).

**Proof of Theorem 2.1.** Let \( v \in W^{2,1}_{\text{loc}}(\alpha,\omega) \cap W^{1,1}(\alpha,\omega) \) be a non-trivial non-negative solution of (2.1). Arguing by contradiction, we suppose that there is a point \( t_0 \in [\alpha,\omega] \) such that
\[ \min v = v(t_0) = 0. \] (2.9)
Obviously, we must have \( v'(t_0) = 0 \), if \( t_0 \in (\alpha,\omega) \). We further assume that \( v'(t_0) = v'(\alpha^+) = 0 \), if \( t_0 = \alpha \), or respectively \( v'(t_0) = v'(\omega^-) = 0 \), if \( t_0 = \omega \), provided that \( v'(\alpha^+) \), or \( v'(\omega^-) \), exists. As \( \max v > 0 \), we can also suppose, possibly for a different choice of \( t_0 \), that there exists a constant \( \delta > 0 \) such that
\[ 0 < v(t) \leq \varepsilon \quad \text{and} \quad |v'(t)| \leq \varepsilon, \]
either for all \( t \in (t_0,t_0+\delta) \), or for all \( t \in [t_0+\delta,t_0] \), where \( \varepsilon \) is the constant appearing in condition (G). Assume that the former case occurs. The proof, being similar in the latter one, will be omitted here. From (2.1) and (2.2) it follows that \( v''(t) \geq 0 \) for almost all \( t \in (t_0,t_0+\delta] \). Moreover, since
\[ v'(t) - v'(s) = \int_s^t v''(r) \, dr \quad \text{for every } s, t \in (t_0,t_0+\delta], \]
letting \( s \to t_0 \), we infer that
\[ v'(t) = \int_{t_0}^t v''(r) \, dr \quad \text{for every } t \in (t_0,t_0+\delta]. \]
Thus, \( v' \) is absolutely continuous and increasing in \([t_0,t_0+\delta] \). Moreover, as \( v(t) > 0 \) for all \( t \in (t_0,t_0+\delta] \) and \( v(t_0) = 0 = v'(t_0) \), we conclude that
\[ v'(t) > 0 \quad \text{for every } t \in (t_0,t_0+\delta]. \] (2.10)
From (2.2), we have that
\[ g(s,v(s),v'(s)) \leq G'(v(s)) \quad \text{for almost all } s \in [t_0,t_0+\delta], \]
and hence, by (2.1),
\[ v''(s) \leq G'(v(s)) \quad \text{for almost all } s \in [t_0,t_0+\delta]. \] (2.11)
Consequently, multiplying (2.11) by \( v'(s) \), it follows from (2.10) and (2.11) that
\[ v'(s)v''(s) \leq G'(v(s))v'(s) \quad \text{for almost all } s \in [t_0,t_0+\delta]. \] (2.12)
Since \( v \) is absolutely continuous and strictly increasing in \([t_0,t_0+\delta] \) and \( G \) is absolutely continuous in \([0,v(t_0+\delta)] \), the composition \( G \circ v \) is absolutely continuous in \([t_0,t_0+\delta] \). Clearly, \((v')^2 \) is absolutely
continuous in \([t_0, t_0 + \delta]\) too. Therefore, for every \(t \in (t_0, t_0 + \delta]\), integrating \((2.12)\) in \([t_0, t]\) and applying the formula of change of variables yields
\[
\frac{1}{2}(v'(t))^2 = \int_{t_0}^{t} v'(s)v''(s)\,ds \leq \int_{t_0}^{t} G'(v(s))v'(s)\,ds = \int_{v(t_0)}^{v(t)} G'(s)\,ds = G(v(t)),
\]
(2.13)
because \(v'(t_0) = 0\) and \(G(v(t_0)) = G(0) = 0\). From \((2.13)\) it follows that
\[|v'(t)| \leq \sqrt{2G(v(t))} \quad \text{for every} \quad t \in [t_0, t_0 + \delta].\]

In case \((2.9)\) holds, we find that
\[v'(t) = 0 \quad \text{for all} \quad t \in (t_0, t_0 + \delta],\]
thus contradicting \((2.9)\). Whereas, if \((2.4)\) holds, then the function \(H = \sqrt{2G}\) satisfies the assumptions of Lemma \(2.1\) which implies that
\[v(t) > 0 \quad \text{for all} \quad t \in [t_0, t_0 + \delta],\]
thus contradicting \((2.9)\). This concludes the proof of Theorem \(2.1\).

\[\square\]

3. Optimal Regularity Results for the Prescribed Curvature Equation

In this section we discuss the regularity properties of the bounded variation solutions of the one-dimensional non-autonomous prescribed curvature equation
\[
-\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = f(x, u), \quad a < x < b,
\]
(3.1)
where \(f: (a, b) \times \mathbb{R} \to \mathbb{R}\) is any given function. We begin by recalling the notion of bounded variation solution of equation \((3.1)\). To this end, for any \(v \in BV(a, b)\), we denote by \(Dv = D^a v \, dx + D^s v\) the Lebesgue–Nikodym decomposition, with respect to the Lebesgue measure \(dx\) in \(\mathbb{R}\), of the Radon measure \(Dv\) in its absolutely continuous part \(D^a v \, dx\), with density function \(D^a v\), and its singular part \(D^s v\). Further, \(\frac{D^s v}{|D^s v|}\) stands for the density function of \(D^s v\) with respect to its absolute variation \(|D^s v|\). Finally, for every \(x_0 \in [a, b]\), \(v(x_0^+)\) denotes the right trace of \(v\) at \(x_0\) and, for every \(x_0 \in (a, b]\), \(v(x_0^-)\) denotes the left trace of \(v\) at \(x_0\). We refer, e.g., to \([1]\) for additional details on these concepts.

**Definition 3.1.** A function \(u \in BV(a, b)\) is a bounded variation solution of \((3.1)\) if \(f(\cdot, u(\cdot)) \in L^1(a, b)\) and
\[
\int_a^b \frac{D^a u(x)D^a \phi(x)}{\sqrt{1 + (D^a u(x))^2}}\,dx + \int_a^b \frac{D^s u}{|D^s u|}(x)D^s \phi = \int_a^b f(x, u(x))\phi(x)\,dx
\]
(3.2)
for all \(\phi \in BV(a, b)\) such that \(|D^s \phi|\) is absolutely continuous with respect to \(|D^s u|\) and \(\phi(a^+) = \phi(b^-) = 0\).

**Remark 3.1.** It is proven in \([2]\) that \(u \in BV(a, b)\) is a bounded variation solution of \((3.1)\) if and only if it minimizes \(BV(a, b)\) the functional
\[
\mathcal{I}(v) := \int_a^b \sqrt{1 + |Dv|^2} + |v(a^+) - u(a^+)| + |v(b^-) - u(b^-)| - \int_a^b f(x, u(x))v(x)\,dx,
\]
(3.3)
where
\[
\int_a^b \sqrt{1 + |Dv|^2} = \int_a^b \sqrt{1 + |D^a v(x)|^2}\,dx + \int_a^b |D^s v|.
\]

The next regularity result establishes that a bounded variation solution \(u\) of \((3.1)\) can lose its regularity at the endpoints, but never at the interior points, of the intervals where the function \(f(\cdot, u(\cdot))\) has a definite sign; whereas, \(u\) can be singular at an interior point of its domain if such a point separates two adjacent intervals where \(f(\cdot, u(\cdot))\) changes sign. In both cases, the derivative \(u'\) blows up, but, in the latter one, \(u\) can further exhibit a jump discontinuity. Although Theorem \(3.1\) is a substantial refinement of \([14\), Prop. 3.6], which was limited to the solutions of equation \((3.1)\) satisfying the Neumann boundary conditions...
Step 1. To prove the theorem, it suffices to show that the Assertions (i) and (iii) hold true.

Testing (3.2) against functions φ hold.

Let u be a bounded variation solution of equation (3.1). Then, the following statements hold.

(i) If \( f(x, u(x)) \geq 0 \) for almost all \( x \in (a, b) \), then u is concave and either \( u \in W^{2,1}(a, b) \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(b^-) = -\infty \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(a^+) = +\infty \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(b^-) = +\infty \). In all cases, \( u \) satisfies equation (3.1) for almost all \( x \in (a, b) \).

(ii) If \( f(x, u(x)) \leq 0 \) for almost all \( x \in (a, b) \), then u is convex and either \( u \in W^{2,1}(a, b) \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(b^-) = +\infty \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(a^+) = -\infty \), or \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \) and \( u'(b^-) = -\infty \). In all cases, \( u \) satisfies equation (3.1) for almost all \( x \in (a, b) \).

(iii) If there is \( c \in (a, b) \) such that \( f(x, u(x)) \geq 0 \) for almost all \( x \in (a, c) \) and \( f(x, u(x)) \leq 0 \) for almost all \( x \in (c, b) \), then \( u_{\lfloor[a,c]} \) is concave, \( u_{\lfloor(c,b]} \) is convex, and either \( u \in W^{2,1}_{loc}(a, b) \cap W^{1,1}(a, b) \), or \( u \in W^{2,1}_{loc}(a, c) \cap W^{1,1}(a, c), u_{\lfloor(a,c)} \in W^{2,1}_{loc}(c, b) \cap W^{1,1}(c, b), u(c^-) \geq u(c^+) \), and \( u'(c^-) = -\infty = u'(c^+) \). Moreover, in case \( u(c^-) > u(c^+) \), we have that
\[
D^a u = (u(c^+) - u(c^-)) \delta_c,
\]
where \( \delta_c \) stands for the Dirac measure concentrated at \( c \). In any circumstances, \( u \) satisfies equation (3.1) for almost all \( x \in (a, b) \).

Remark 3.2. In cases (ii) and (iii), the behavior of \( u \) at the endpoints \( a \) and \( b \) follows exactly the same patterns as described in (i) and (ii).

Proof. Let \( u \) be a bounded variation solution of equation (3.1), and consider the decomposition of the measure \( Du \),
\[
Du = D^a u \, dx + D^+ u \, dx + D^0 u \, dx + D^- u \, dx + D^c u,
\]
in its absolutely continuous part, \( D^a u \, dx \), its jump part, \( D^+ u \, dx \), and its Cantor part, \( D^c u \), as well as the induced decomposition of the function \( u \),
\[
u = u^a + u^+ + u^-,
\]
where
\[
D^a u \, dx = D^0 u \, dx, \quad Du^+ = D^1 u, \quad Du^- = D^c u,
\]
and
\[
u^a(a) = u(a^-), \quad u^+(a) = 0, \quad u^-(a) = 0.
\]
Hereafter, for convenience, we write
\[
v := u^a \in W^{1,1}(a, b), \quad h := f(X, u(X)) \in L^1(a, b), \quad \psi(s) := \frac{s}{\sqrt{1 + s^2}}, \text{ for all } s \in \mathbb{R}.
\]
To prove the theorem, it suffices to show that the Assertions (i) and (iii) hold true.

Proof of Assertion (i). The proof is divided into three steps.

Step 1. If \( u^a \in W^{2,1}_{loc}(a, b) \), and it is concave in \( (a, b) \), then
\[
\int_a^b \psi(u'(x)) \phi'(x) \, dx = \int_a^b h(x) \phi(x) \, dx.
\]
Thus, \( \psi(v') \in W^{1,1}(a,b) \) and

\[
-(\psi(v'(x)))' = h(x) \quad \text{for almost all } x \in (a,b).
\]  

(3.5)

As \( h(x) \geq 0 \) for almost all \( x \in (a,b) \), \( \psi(v') \) is decreasing in \( (a,b) \). Since, in addition, \( \psi(v') \) is continuous and \( v' \in L^1(a,b) \) is finite almost everywhere, we must have

\[
|\psi(v'(x))| < 1 \quad \text{for all } x \in (a,b).
\]  

(3.6)

This implies that

\[
v'_{\mid(a,b)} = \psi^{-1}(\psi(v')_{\mid(a,b)}) \in W^{1,1}_{\text{loc}}(a,b)
\]

and it is decreasing. Therefore, \( v \in W^{2,1}_{\text{loc}}(a,b) \) and it is concave.

**Step 2:** \( u^j = 0 \).

Arguing by contradiction, we assume that \( u \) has a jump point at \( z \in (a,b) \), and consider the test function

\[
\phi(x) := \begin{cases} 
\frac{z - x}{z - a} & \text{if } x \in [a,z], \\
0 & \text{if } x \in (z,b].
\end{cases}
\]

Clearly, we have that

\[
D^s \phi = -\delta_z,
\]

where \( \delta_z \) is the Dirac measure concentrated at \( z \). Since \( \phi(a) = \phi(b) = 0 \), and \( |D^s \phi| = \delta_z \) is absolutely continuous with respect to \( |D^s u| \) and its unique atom is \( z \), it follows from (3.2) that

\[
\int_a^b \psi(D^s u(x)) D^s \phi(x) \, dx - \int_a^b h(x) \phi(x) \, dx = -\int_a^b \frac{D^s u}{|D^s u|}(x) D^s \phi
\]

\[
= \int_a^b D^s \frac{u}{|D^s u|}(x) \, \delta_z = D^s \frac{u}{|D^s u|}(z).
\]

On the other hand, integrating by parts, it follows from (3.4) that

\[
\int_a^b \psi(D^s u(x)) D^s \phi(x) \, dx - \int_a^b h(x) \phi(x) \, dx = \left[ \int_a^z \psi(D^s u(x)) D^s \phi(x) \, dx - \int_a^z h(x) \phi(x) \, dx \right]
\]

\[
= \left[ \int_a^z \psi(v'(x)) \phi'(x) \, dx - \int_a^z b(x) \phi(x) \, dx \right]
\]

\[
= \psi(v'(z)) \phi(z) - \int_a^z (\psi(v'(x))') \phi(x) \, dx - \int_a^z h(x) \phi(x) \, dx
\]

\[
= \psi(v'(z)) \phi(z) = \psi(v'(z)).
\]

Hence, it follows that

\[
\psi(v'(z)) = \frac{D^s u}{|D^s u|}(z).
\]

(3.7)

Consequently, as the polar decomposition of measures (see, e.g., [1] Cor. 1.29) guarantees that

\[
\left| \frac{D^s u}{|D^s u|}(z) \right| = 1,
\]

we find from (3.7) that

\[
|\psi(v'(z))| = 1,
\]

which contradicts (3.6). Therefore, we conclude that \( u^j = 0 \).

**Step 3:** \( u^c = 0 \).

From the two previous steps, we already know that \( u = u^a + u^c \) in \( (a,b) \) and, hence, \( u \) is continuous. Moreover, \( u \) can be extended by continuity onto \( [a,b] \). Let us prove that \( u \) is concave in \([a,b] \). On the contrary, assume that there exists an interval \([c,d]\) \( \subseteq [a,b] \) such that

\[
u(x) < u(c) + \frac{u(d) - u(c)}{d - c}(x - c) \quad \text{for all } x \in (c,d),
\]

This leads to a contradiction. Therefore, we conclude that \( u^c = 0 \).
and consider the function \( w \in BV(0,1) \) defined by
\[
w(x) = \begin{cases} 
  u(c) + \frac{u(d) - u(c)}{d - c}(x - c) & \text{if } x \in [c, d], \\
  u(x) & \text{elsewhere}.
\end{cases}
\]

It is clear that
\[
\int_0^1 \sqrt{1 + |Dw|^2} < \int_0^1 \sqrt{1 + |Du|^2}
\]
and, since \( w(x) > u(x) \) in \( (c, d) \),
\[
\int_0^1 hw \, dx \geq \int_0^1 hu \, dx.
\]
Thus, we get
\[
\mathcal{I}(w) = \int_0^1 \sqrt{1 + |Dw|^2} - \int_0^1 hw \, dx < \int_0^1 \sqrt{1 + |Du|^2} - \int_0^1 hu \, dx = \mathcal{I}(u),
\]
which contradicts the fact that \( u \) is a global minimizer of the functional \( \mathcal{I} \) defined by (3.3). Therefore, \( u \) being concave, it is locally Lipschitz and, hence, \( u'' = 0 \).

Since we have proved that \( u = u^a \), the remaining conclusions stated in (i) follow directly from the properties of \( u^a \) established in Step 1. This ends the proof of the Assertion (i).

Proof of Assertion (iii). Thanks to (i) and (ii) we know that \( u^{(a,c)} \in W^{2,1}_{\text{loc}}(a, c) \cap W^{1,1}_{\text{loc}}(a, c) \) is concave and \( u^{(c,b)} \in W^{2,1}_{\text{loc}}(c, b) \cap W^{1,1}_{\text{loc}}(c, b) \) is convex, and \( u'' = 0 \). Thus, as \( u = v + w \), we have that, at the point \( c \), either \( u \) is continuous, or it exhibits a jump discontinuity. Arguing as in Step 1, we also see that \( \psi(v') \in W^{1,1}_{\text{loc}}(a, b) \), (3.3) holds, and
\[
|\psi(v'(x))| < 1 \quad \text{for all } x \in (a, b) \setminus \{c\}.
\]
If \( |\psi(v'(c))| < 1 \), then
\[
|\psi(v'(x))| < 1 \quad \text{for all } x \in (a, b),
\]
and hence the argument used in Step 1 yields \( v \in W^{2,1}_{\text{loc}}(a, b) \). Moreover, arguing as in Step 2 shows that \( u^1 = 0 \) and therefore \( u = v \). This proves that \( u \in W^{2,1}_{\text{loc}}(a, b) \cap W^{1,1}_{\text{loc}}(a, b) \).

Suppose now that \( |\psi(v'(c))| = 1 \). Then, we have that \( \psi(v'(c)) = -1 \), or equivalently \( v'(c) = -\infty \), because \( v_{(a,c)} \) is concave and \( v_{(c,b)} \) is convex. If \( u \) is continuous at \( c \), it follows that \( u = v \) and, in particular, \( u'(c) = -\infty \). On the contrary, if \( u \) has a jump discontinuity at \( c \), then it follows that \( u(c^-) > u(c^+) \), as the argument performed in Step 2 of the proof of Assertion (i) yields
\[
\frac{D^+ u}{|D^+ u|}(c) = \frac{D^1 u}{|D^1 u|}(c) = \psi(v'(c)) = -1
\]
and thus
\[
D^1 u(c) = -|u(c^-) - u(c^+)| \delta_c.
\]
Finally, as \( u_{(a,c)} = v_{(a,c)} \) and \( u_{(c,b)} = v_{(c,b)} + u(c^+) - u(c^-) \), we have that
\[
u'(c^-) = u'(c^+) = v'(c) = -\infty.
\]
This concludes the proof. \( \square \)

Essentially, Theorem 3.1 shows that a bounded variation solution of equation (3.1) can only loose its regularity at the endpoints, but not at the interior points, of any interval where the right hand side of the equation has a definite sign. Our next two results, Theorem 3.2 and 3.3, establish, on the contrary, the complete regularity of a bounded variation solution of (3.1). Precisely, Theorem 3.2 guarantees the regularity at the endpoints of any interval where the sign of the right hand side of the equation is constant, by placing at these points a suitable control on its decay rate to zero, while Theorem 3.3 guarantees, instead, the regularity of a bounded variation solution at any interior point separating two adjacent intervals where the right hand side changes sign, provided that a similar constraint on its decay rate to
zero is imposed either on the left, or on the right, of that point. Although this type of constraints were already introduced in some previous papers of ours, [15–17], here we will deliver a completely novel proof of these regularity results based on the strong maximum principle as expressed by Theorem 2.1. This approach shows that the condition that we impose to earn regularity is precisely the assumption required by Theorem 2.1 so that the strong maximum principle holds for a certain equation satisfied by a shift of a local inverse of the considered solution.

**Theorem 3.2.** Let \( u \) be a bounded variation solution of (3.1). Then, the following assertions hold.

(j) If \( f(x, u(x)) \geq 0 \) for almost all \( x \in (a, b) \) and there exist \( \delta > 0 \) and \( \mu \in L^1(a, a + \delta) \) such that

- \( f(x, u(x)) \leq \mu(x) \) for almost all \( x \in (a, a + \delta) \),
- \( M(x) := \int_a^x \mu(t) \, dt > 0 \) for all \( x \in (a, a + \delta) \), and \( \int_a^{a+\delta} \frac{1}{\sqrt{M(x)}} \, dx = +\infty \),

then \( u \in W^{1,1}_{\text{loc}}(a, b) \cap W^{1,1}(a, b) \).

(jj) If \( f(x, u(x)) \geq 0 \) for almost all \( x \in (a, b) \) and there exist \( \delta > 0 \) and \( \mu \in L^1(b - \delta, b) \) such that

- \( f(x, u(x)) \leq \mu(x) \) for almost all \( x \in (b - \delta, b) \),
- \( M(x) := \int_x^b \mu(t) \, dt > 0 \) for all \( x \in [b - \delta, b) \), and \( \int_{b-\delta}^{b} \frac{1}{\sqrt{M(x)}} \, dx = +\infty \),

then \( u \in W^{1,1}_{\text{loc}}(a, b) \cap W^{1,1}(a, b) \).

(jii) If \( f(x, u(x)) \leq 0 \) for almost all \( x \in (a, b) \) and there exist \( \delta > 0 \) and \( \nu \in L^1(a, a + \delta) \) such that

- \( f(x, u(x)) \geq \nu(x) \) for almost all \( x \in (a, a + \delta) \),
- \( N(x) := \int_a^x \nu(t) \, dt < 0 \) for all \( x \in (a, a + \delta) \), and \( \int_a^{a+\delta} \frac{1}{\sqrt{-N(x)}} \, dx = +\infty \),

then \( u \in W^{1,1}_{\text{loc}}(a, b) \cap W^{1,1}(a, b) \).

(iii) If \( f(x, u(x)) \leq 0 \) for almost all \( x \in (a, b) \) and there exist \( \delta > 0 \) and \( \nu \in L^1(b - \delta, b) \) such that

- \( f(x, u(x)) \geq \nu(x) \) for almost all \( x \in (b - \delta, b) \),
- \( N(x) := \int_x^b \nu(t) \, dt < 0 \) for all \( x \in [b - \delta, b) \), and \( \int_{b-\delta}^{b} \frac{1}{\sqrt{-N(x)}} \, dx = +\infty \),

then \( u \in W^{1,1}_{\text{loc}}(a, b) \cap W^{1,1}(a, b) \).

**Proof.** Let \( u \) be a bounded variation solution of equation (3.1). We will prove the validity of Assertion (j). As this proof can be easily adapted to complete the proofs of the remaining assertions, these are omitted here because repetitive. By Theorem 3.1(i), we already know that \( u \) is concave, \( u \in W^{2,1}_{\text{loc}}(a, b) \cap W^{1,1}(a, b) \), and it satisfies equation (3.1) for almost all \( x \in (a, b) \). Arguing by contradiction, we suppose that

\[ u'(a^+) = +\infty. \]

Then, there exists \( c \in (a, a + \delta) \) such that \( u'(x) > 0 \) for all \( x \in [a, c] \). Thus, setting \( \alpha = u(a) \) and \( \omega = u(c) \), we have that \( u : [a, c] \to [\alpha, \omega] \) is invertible and the inverse function \( w : [\alpha, \omega] \to [a, c] \) is continuously differentiable, with derivative

\[ w'(t) = \begin{cases} \frac{1}{u'(w(t))} & \text{if } t \in (\alpha, \omega), \\ 0 & \text{if } t = \alpha. \end{cases} \]  

(3.8)

By the concavity of \( u \), \( u' \) is decreasing and hence \( w' \) is increasing. Thus, (3.8) implies that

\[ 0 < w'(t) \leq w'(\omega) = (u'(c))^{-2} \quad \text{for all } t \in (\alpha, \omega). \]  

(3.9)

As \( u' : (a, c) \to [u'(c), +\infty) \) is locally absolutely continuous, because \( u \in W^{2,1}_{\text{loc}}(a, b) \), and \( w : (\alpha, \omega) \to (a, c) \) is a diffeomorphism of class \( C^1 \), the composition

\[ u' \circ w : (\alpha, \omega) \to [u'(c), +\infty) \]
is locally absolutely continuous. Hence, the reciprocal function
\[
\frac{1}{u' \circ w} : (\alpha, \omega] \to \left(0, \frac{1}{u'(c)}\right)
\]
is locally absolutely continuous too. Consequently, from (5.8) we can infer that \(w' \in W^{1,1}_{\text{loc}}(\alpha, \omega]\), with derivative
\[
w''(t) = -u''(w(t)) \frac{1}{\left(u'(w(t))^2 u'(w(t))\right)^\frac{3}{2}} = -u''(w(t))(w'(t))^3 \quad \text{for almost all } t \in (\alpha, \omega].
\] (3.10)
Since the function \(u \in W^{2,1}_{\text{loc}}(a, c] \cap W^{1,1}(a, c)\) solves the singular problem
\[
\begin{cases}
-u''(x) = f(x, u(x))(1 + (u'(x))^2)^\frac{3}{2} & \text{for almost all } x \in (a, c), \\
u(a) = \alpha, \quad u'(a^+) = +\infty,
\end{cases}
\]
and it satisfies the Lusin's \(N\) property, it follows from (3.8) and (3.10) that the inverse function \(w \in W^{2,1}_{\text{loc}}(\alpha, \omega] \cap C^1[\alpha, \omega]\) is a solution of
\[
\begin{cases}
w''(t) = f(w(t), t)((w'(t))^2 + 1)^\frac{3}{2} & \text{for almost all } t \in (\alpha, \omega), \\
w(\alpha) = \alpha, \quad w'(\alpha) = 0.
\end{cases}
\]
Therefore, if we define \(g : (\alpha, \omega) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) by
\[
g(t, s, \xi) := f(a + s, t)(1 + \xi^2)^\frac{3}{2},
\]
and \(v \in W^{2,1}_{\text{loc}}(\alpha, \omega) \cap C^1[\alpha, \omega]\) by
\[
v := w - a,
\]
the function \(v\) is a non-trivial non-negative solution of the initial value problem
\[
\begin{cases}
v''(t) = g(t, v(t), v'(t)) & \text{for almost all } t \in (\alpha, \omega), \\
v(\alpha) = 0, \quad v'(\alpha) = 0.
\end{cases}
\]
Finally, let us introduce the function \(G : [0, \delta] \to \mathbb{R}\) by setting
\[
G(s) := M(a + s)(1 + (u'(c))^{-2})^\frac{3}{2},
\]
where \(M\) has been defined in (j). From the assumptions in (j) and the definitions of the functions \(g, v, w\) and \(G\), we can infer from (3.9) that
\[
g(t, v(t), v'(t)) = f(a + v(t), t)(1 + (w'(t))^2)^\frac{3}{2}
\leq \mu(a + v(t))(1 + (u'(c))^{-2})^\frac{3}{2} = G'(v(t)) \quad \text{for almost all } t \in (\alpha, \omega),
\]
and
\[
G(s) > 0 \text{ for all } s \in (0, \delta) \quad \text{and} \quad \int_0^\delta \frac{1}{\sqrt{G(s)}} \, ds = +\infty.
\]
Consequently, the conditions (2.2) and (2.4) of (G) are both fulfilled. Thus, thanks to Theorem 2.1 it follows that \(v'(\alpha) > 0\), which is a contradiction.

The following result establishes the regularity of the solutions at the interior points.

**Theorem 3.3.** Let \(u\) be a bounded variation solution of equation (3.1). Then, the following statements hold.

(h) If there is \(c \in (a, b)\) such that \(f(x, u(x)) \geq 0\) for almost all \(x \in (a, c)\) and \(f(x, u(x)) \leq 0\) for almost all \(x \in (c, b)\) and either there exist \(\delta > 0\) and \(\mu \in L^1(c - \delta, c)\) such that
- \(f(x, u(x)) \leq \mu(x)\) for almost all \(x \in (c - \delta, c)\),
- \(M(x) := \int_x^c \mu(t) \, dt > 0\) for all \(x \in [c - \delta, c]\), and \(\int_{c-\delta}^c \frac{1}{\sqrt{M(x)}} \, dx = +\infty\),
or there exist \( \delta > 0 \) and \( \nu \in L^1(c,c+\delta) \) such that
- \( f(x,u(x)) \geq \nu(x) \) for almost all \( x \in (c,c+\delta), \)
- \( N(x) := \int_{\delta}^{c+\delta} \nu(t) \, dt < 0 \) for all \( x \in (c,c+\delta), \) and \( \int_{\delta}^{c} \frac{1}{\sqrt{-N(x)}} \, dx = +\infty, \)
then \( u \in W^{2,1}_{\text{loc}}(a,b) \cap W^{1,1}(a,b). \)

(hh) If there is \( c \in (a,b) \) such that \( f(x,u(x)) \leq 0 \) for almost all \( x \in (a,c) \) and \( f(x,u(x)) \geq 0 \) for almost all \( x \in (c,b) \) and either there exist \( \delta > 0 \) and \( \nu \in L^1(c-\delta,c) \) such that
- \( f(x,u(x)) \geq \nu(x) \) for almost all \( x \in (c-\delta,c), \)
- \( N(x) := \int_{c-\delta}^{c} \nu(t) \, dt < 0 \) for all \( x \in (c-\delta,c), \) and \( \int_{c-\delta}^{c} \frac{1}{\sqrt{-N(x)}} \, dx = +\infty, \)
or there exist \( \delta > 0 \) and \( \mu \in L^1(c,c+\delta) \) such that
- \( f(x,u(x)) \leq \mu(x) \) for almost all \( x \in (c,c+\delta), \)
- \( M(x) := \int_{c}^{c+\delta} \mu(t) \, dt > 0 \) for all \( x \in (c,c+\delta), \) and \( \int_{c}^{c+\delta} \frac{1}{\sqrt{M(x)}} \, dx = +\infty, \)
then \( u \in W^{2,1}_{\text{loc}}(a,b) \cap W^{1,1}(a,b). \)

**Proof.** Let \( u \) be a bounded variation solution of equation (4.1), and, suppose, e.g., that the first alternative of the assumption (h) holds, i.e., there exist \( \delta > 0 \) and \( \mu \in L^1(c-\delta,c) \) such that

\[
f(x,u(x)) \leq \mu(x) \quad \text{for almost all } x \in (c-\delta,c), \quad M(x) := \int_{c-\delta}^{c} \mu(t) \, dt > 0 \quad \text{for all } x \in (c-\delta,c),
\]

and

\[
\int_{c-\delta}^{c} \frac{1}{\sqrt{M(x)}} \, dx = +\infty.
\]

By Theorem 3.1 we already know that either \( u \in W^{2,1}_{\text{loc}}(a,b) \cap W^{1,1}(a,b), \) or

\[
u'(c^-) = -\infty = u'(c^+).
\]

On the other hand, from Theorem 3.2(jj) it follows that \( u \in W^{2,1}_{\text{loc}}(a,c]. \) This rules out \( u'(c^-) = -\infty. \)

Therefore, it follows that \( u \in W^{2,1}_{\text{loc}}(a,b) \cap W^{1,1}(a,b), \) concluding the proof in this case. As the rest of the proof proceeds similarly, we omit the technical details. \(\square\)

4. **Applications to BVPs for the Prescribed Curvature Equation**

In this section we apply the results of Section 3 for establishing the regularity, up to the boundary, of the solutions of non-autonomous one-dimensional prescribed curvature equations subject to possibly non-homogeneous Dirichlet, or Neumann, or Robin boundary conditions, or else periodic boundary conditions. The case of mixed boundary conditions can be also dealt with, in an obvious way, by combining these results and, hence, it is omitted here. In our exposition, we restrict ourselves to discuss the regularity properties of a solution \( u \) when \( f(.,u(.)) \) changes sign at most once. Other more general statements can be easily inferred, in a similar way, from Theorems 3.1, 3.2 and 3.3.

4.1. **The Dirichlet problem.** Let us consider the problem

\[
\begin{align*}
\left( -\frac{u'}{\sqrt{1+(u')^2}} \right)' &= f(x,u), \quad 0 < x < 1, \\
u(0) &= \kappa_0, \quad u(1) = \kappa_1.
\end{align*}
\]

where \( f : (0,1) \times \mathbb{R} \to \mathbb{R} \) and \( \kappa_0, \kappa_1 \in \mathbb{R} \) are given.
Definition 4.1. A function \( u \in BV(0,1) \) is a bounded variation solution of the Dirichlet problem \( (4.1) \) if \( f(\cdot, u(\cdot)) \in L^1(0,1) \) and

\[
\int_0^1 \frac{D^s u(x) D^s \phi(x)}{\sqrt{1 + (D^s u(x))^2}} \, dx + \int_0^1 \frac{D^s u}{|D^s u|} (x) D^s \phi + \text{sgn}(u(0^+) - \kappa_0) \phi(0^+) + \text{sgn}(u(1^-) - \kappa_1) \phi(1^-) = \int_0^1 f(x, u(x)) \phi(x) \, dx
\]

for all \( \phi \in BV(0,1) \) such that \( |D^s \phi| \) is absolutely continuous with respect to \( |D^s u| \), \( \phi(0^+) = 0 \) if \( u(0^+) = \kappa_0 \), and \( \phi(1^-) = 0 \) if \( u(1^-) = \kappa_1 \).

We state below two sample regularity results for problem \( (4.1) \), assuming that either the right hand side of the equation in \( (4.1) \) does not change sign in \((0,1)\), e.g., it is non-negative, or it changes sign exactly once.

Theorem 4.1. Let \( u \) be a bounded variation solution of \( (4.1) \). Assume that

(k) \( f(x, u(x)) \geq 0 \) for almost all \( x \in (0,1) \),

(kk) there exist \( \delta > 0 \) and \( \mu_0 \in L^1(0,\delta) \) such that:
- \( f(x, u(x)) \leq \mu_0(x) \) for almost all \( x \in (0,\delta) \),
- \( M_0(x) := \int_0^\delta \mu_0(t) \, dt > 0 \) for all \( x \in (0,\delta) \), and \( \int_0^\delta \frac{1}{\sqrt{M_0(x)}} \, dx = +\infty \),

(kkk) there exist \( \delta > 0 \) and \( \mu_1 \in L^1(1-\delta,1) \) such that:
- \( f(x, u(x)) \leq \mu_1(x) \) for almost all \( x \in (1-\delta,1) \),
- \( M_1(x) := \int_x^1 \mu_1(t) \, dt > 0 \) for all \( x \in [1-\delta,1) \), and \( \int_{1-\delta}^1 \frac{1}{\sqrt{M_1(x)}} \, dx = +\infty \).

Then, \( u \in W^{2,1}(0,1) \) and it satisfies both the differential equation almost everywhere in \((0,1)\) and the boundary conditions.

Proof. Theorems 3.1 and 3.2 guarantee that \( u \in W^{2,1}(0,1) \). Testing \( (4.2) \) against functions \( \phi \in W^{1,1}(0,1) \) such that \( \phi(0) = \phi(1) = 0 \) and integrating by parts, we infer that \( u \) satisfies the equation almost everywhere in \((0,1)\). To show that \( u \) fulfills the Dirichlet boundary conditions, we argue by contradiction. Suppose, e.g., that \( u(1) \neq \kappa_1 \), and plug in \( (4.2) \) a test function \( \phi \in W^{1,1}(0,1) \) such that \( \phi(0) = 0 \) and \( \phi(1) \neq 0 \). Integrating by parts in \( (4.2) \) yields

\[
\frac{u'(1) \phi(1)}{\sqrt{1 + (u'(1))^2}} + \text{sgn}(u(1) - \kappa_1) \phi(1) = 0
\]

and hence

\[
1 = |\text{sgn}(u(1) - \kappa_1)| = \frac{|u'(1)|}{\sqrt{1 + (u'(1))^2}}
\]

which is impossible, because \( u \in C^1[0,1] \). Therefore, \( u(1) = \kappa_1 \). Similarly, one can prove that \( u(0) = \kappa_0 \). This ends the proof.

Combining the proof of Theorem 4.1 with Theorem 3.3 yields the next result.

Theorem 4.2. Let \( u \) be a bounded variation solution of \( (4.1) \). Assume that

(i) there is \( z \in (0,1) \) such that \( f(x, u(x)) \geq 0 \) for almost all \( x \in (0,z) \) and \( f(x, u(x)) \leq 0 \) for almost all \( x \in (z,1) \),

(ii) there exist \( \delta > 0 \) and \( \mu_0 \in L^1(0,\delta) \) such that
- \( f(x, u(x)) \leq \mu_0(x) \) for almost all \( x \in (0,\delta) \),
- \( M_0(x) := \int_0^\delta \mu_0(t) \, dt > 0 \) for all \( x \in (0,\delta) \), and \( \int_0^\delta \frac{1}{\sqrt{M_0(x)}} \, dx = +\infty \),

(iii) there exist \( \delta > 0 \) and \( \mu_1 \in L^1(1-\delta,1) \) such that
- \( f(x, u(x)) \geq \mu_1(x) \) for almost all \( x \in (1-\delta,1) \),
- \( M_1(x) := \int_x^1 \mu_1(t) \, dt > 0 \) for all \( x \in [1-\delta,1) \), and \( \int_{1-\delta}^1 \frac{1}{\sqrt{M_1(x)}} \, dx = +\infty \).

Then, \( u \in W^{2,1}(0,1) \) and it satisfies both the differential equation almost everywhere in \((0,1)\) and the boundary conditions.
There is $z$ almost everywhere in $(0, 1)$ and the boundary conditions.

Let $u$ be a bounded variation solution of (4.1).

Then, $u \in W^{2,1}(0,1)$ and it satisfies both the differential equation almost everywhere in $(0,1)$ and the boundary conditions.

Example 4.1. A simple situation where the assumptions of Theorems 4.1 and 4.2 are fulfilled is the special case where 
\[ f(x,s) = h(x)k(s), \]
with $h \in L^1(0,1)$ and $k \in C^0(\mathbb{R})$. Suppose, for instance, that $u$ is a bounded variation solution of (4.1) and that $k$ has a definite sign, e.g., $k(s) \geq 0$ for all $s \in \text{Range } u$, and $h$ changes sign once in $(0,1)$, e.g., there is $z \in (0,1)$ such that $h(x) > 0$ for almost all $x \in (0,z)$ and $h(x) < 0$ for almost all $x \in (z,1)$.

Then, according to Theorem 4.2, the regularity of $u$ is granted provided
\[
\int_0^x \left( \int_0^t h(t) \, dt \right)^{-\frac{1}{2}} \, dx = +\infty, \quad \int_1^x \left( \int_0^t h(t) \, dt \right)^{-\frac{1}{2}} \, dx = +\infty,
\]
and
\[
either \int_0^x \left( \int_0^t h(t) \, dt \right)^{-\frac{1}{2}} \, dx = +\infty, \quad or \quad \int_x^1 \left( \int_0^t h(t) \, dt \right)^{-\frac{1}{2}} \, dx = +\infty.
\]

This and other similar regularity results, which can be easily deduced from Theorems 3.1, 3.2 and 3.3, complete the previous regularity results of [21, 22, 26, 30] for (4.1).

4.2. The Neumann problem. Let us consider the problem
\[
\begin{cases}
-\left( \frac{u'}{\sqrt{1+(u')^2}} \right)' = f(x,u), & 0 < x < 1, \\
\frac{u'(0)}{\sqrt{1+(u'(0))^2}} = k_0, & u'(1) \frac{1}{\sqrt{1+(u'(1))^2}} = k_1,
\end{cases}
\]
(4.3)

where $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ and $k_0, k_1 \in [-1,1]$ are given.

Definition 4.2. A function $u \in BV(0,1)$ is a bounded variation solution of the Neumann problem (4.3) if $f(\cdot,u(\cdot)) \in L^1(0,1)$ and
\[
\int_0^1 \frac{u'(x)d'(x)}{\sqrt{1+(u'(x))^2}} \, dx + \int_0^1 \frac{D^s u(x)D^s \phi - \kappa_0 \phi(0^+) - \kappa_1 \phi(1^-)}{|D^s u|} \, dx = \int_0^1 f(x,u(x))\phi(x) \, dx
\]
(4.4)

for all $\phi \in BV(0,1)$ such that $|D^s \phi|$ is absolutely continuous with respect to $|D^s u|$.

Here, we restrict ourselves to state an illustrative regularity result for (4.3), under the assumption that $f(\cdot,u(\cdot))$ changes sign once in $(0,1)$. Note that no assumptions are imposed on $f(\cdot,u(\cdot))$ at the boundary points, the regularity being guaranteed by requiring that $k_0, k_1 \in (-1,1)$, which is optimal.

Theorem 4.3. Let $u$ be a bounded variation solution of (4.3). Suppose that conditions (I) and (III) of Theorem 4.2 hold and $k_0, k_1 \in (-1,1)$. Then, $u \in W^{2,1}(0,1)$ and it satisfies both the differential equation almost everywhere in $(0,1)$ and the boundary conditions.
Proof. Theorems 3.1 and 3.3 guarantee that $u \in W^{2,1}_{loc}(0,1) \cap W^{1,1}(0,1)$. We also know that

$$u' \sqrt{1 + (u')^2} : [0,1] \to [-1, 1]$$

is continuous. Plugging in (4.4) a function $\phi \in W^{1,1}(0,1)$ such that $\phi(0) = \phi(1) = 0$ and integrating by parts in $(-1, 1)$, it becomes apparent that $u$ satisfies the differential equation almost everywhere in $(0, 1)$. Moreover, choosing a test function $\phi \in W^{1,1}(0,1)$ such that $\phi(0) = 0$ and $\phi(1) \neq 0$, and integrating by parts, we find that

$$\frac{u'(1^-) \phi(1)}{\sqrt{1 + (u'(1^-))^2}} - \kappa_1 \phi(1) = 0$$

and hence

$$\frac{u'(1^-)}{\sqrt{1 + (u'(1^-))^2}} = \kappa_1.$$ 

Similarly, one can verify that

$$\frac{u'(0^+)}{\sqrt{1 + (u'(0^+))^2}} = \kappa_0.$$ 

Therefore, $u$ satisfies the conormal boundary conditions. Moreover, since $\kappa_0, \kappa_1 \in (-1, 1)$, we find that $u \in C^1[0,1]$. Finally, the sign properties of $f(\cdot, u(\cdot))$ and, hence, of $u''$, imply that

$$\int_0^1 |u''(x)| \, dx = u'(0) - 2u'(z) + u'(1) \in \mathbb{R} \quad (4.5)$$

and thus $u \in W^{2,1}(0,1)$. This ends the proof.

Theorem 4.3 allows us to establish the regularity of the solutions of (4.3) whose existence was proven in our preceding papers [19, 20] (see also [14–17]).

4.3. The Robin problem. Let us consider the problem

$$\begin{cases} - \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = f(x, u), & 0 < x < 1, \\ u'(0) \sqrt{1 + (u'(0))^2} + \lambda_0 u(0) = \kappa_0, & u'(1) \sqrt{1 + (u'(1))^2} + \lambda_1 u(1) = \kappa_1, \end{cases} \quad (4.6)$$

where $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ and $\lambda_0, \lambda_1 \in \mathbb{R} \setminus \{0\}$, $\kappa_0, \kappa_1 \in \mathbb{R}$ are given.

Definition 4.3. A function $u \in BV(0,1)$ is a bounded variation solution of the Robin problem (4.6) if $f(\cdot, u(\cdot)) \in L^1((0,1)$ and

$$\int_0^1 \frac{u'(x)\phi'(x)}{\sqrt{1 + (u'(x))^2}} \, dx + \int_0^1 \frac{D^+ u}{|D^+ u|}(x) D^+ \phi + (\kappa_0 - \lambda_0 u(0^+))\phi(0^+)$$

$$+ (\lambda_1 u(1^-) - \kappa_1) \phi(1^-) = \int_0^1 f(x, u(x))\phi(x) \, dx,$$

for all $\phi \in BV(0,1)$ such that $|D^+ \phi|$ is absolutely continuous with respect to $|D^+ u|$.

As above, we just state here an illustrative regularity result for problem (4.6) under the assumption that the right hand side of the differential equation in (4.7) changes sign once in $(0, 1)$.

Theorem 4.4. Let $u$ be a bounded variation solution of (4.6). Suppose that conditions (I), (II), (III) and (lll) of Theorem 4.2 hold. Then, $u \in W^{2,1}(0,1)$ and it satisfies both the differential equation almost everywhere in $(0, 1)$ and the boundary conditions.
where $f$ is a function. Definition 4.4. Therefore, $u$ satisfies the Robin boundary conditions. This concludes the proof. \hfill \Box

4.4. The periodic problem. Let us consider the problem

$$
\begin{align*}
-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' &= f(x,u), \quad 0 < x < 1, \\
u(0) &= u(1), \quad u'(0) = u'(1),
\end{align*}
$$

(4.8)

where $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is given.

Definition 4.4. A function $u \in BV(0,1)$ is a bounded variation solution of problem (4.8) if $f(\cdot,u(\cdot)) \in L^1(0,1)$ and

$$
\int_0^1 \frac{D^n u(x) D^n \phi(x)}{\sqrt{1+(u')^2}} \, dx + \int_0^1 \frac{D^n u(x)}{|D^n u|} D^n \phi(x) \, \text{sgn}(u(1^-) - u(0^+))(\phi(1^-) - \phi(0^+))
$$

$$
= \int_0^1 f(x,u(x))\phi(x) \, dx \quad (4.9)
$$

for all $\phi \in BV(0,1)$ such that $|D^n \phi|$ is absolutely continuous with respect to $|D^n u|$ and $\phi(0^+) = \phi(1^-)$ if $u(0^+) = u(1^-)$.

Also in this case we can state a regularity result for problem (4.8) assuming that $f(\cdot,u(\cdot))$ changes sign once in $(0,1)$.

Theorem 4.5. Let $u$ be a bounded variation solution of (4.8). Suppose that conditions (i), (iii), and either (ii), or (iii), of Theorem 4.2 hold. Then, $u \in W^{2,1}(0,1)$ and it satisfies both the differential equation almost everywhere in $(0,1)$ and the boundary conditions.

Proof. Assume that conditions (i), (iii) and, e.g., (iii) of Theorem 4.2 hold. Then, by Theorems 3.1, 3.2 and 3.3, $u \in W^{2,1}_{\text{loc}}(0,1]$ and, in particular, $u \in C^1_{\text{loc}}(0,1)$. Moreover, the function

$$
\frac{u'}{\sqrt{1+(u')^2}} : [0,1] \to [-1,1]
$$

is continuous. Testing (4.9) with functions $\phi \in W^{1,1}(0,1)$ such that $\phi(0) = \phi(1) = 0$ and integrating by parts in $(0,1)$, it is easily seen that $u$ satisfies the differential equation almost everywhere in $(0,1)$. Next, we choose a test function $\phi \in W^{1,1}(0,1)$ such that $\phi(0) = \phi(1) \neq 0$. Then, integrating by parts yields

$$
\frac{u'(0^+) \phi(0)}{\sqrt{1+(u'(0^+))^2}} - \frac{u'(1) \phi(1)}{\sqrt{1+(u'(1))^2}} = 0,
$$
and hence
\[ \frac{u'(0^+)}{\sqrt{1 + (u'(0))^2}} = \frac{u'(1)}{\sqrt{1 + (u'(1))^2}} \in (-1, 1). \]
This implies that \( u'(0^+) = u'(1) \in \mathbb{R} \). In particular, we infer that \( u \in C^1[0, 1] \). The sign properties of \( f(\cdot, u(\cdot)) \) and, hence, of \( u' \) yields (1.5) and hence we can conclude that \( u \in W^{2,1}(0, 1) \). Finally, arguing by contradiction, suppose that \( u(0) \neq u(1) \), and pick a test function \( \phi \in W^{1,1}(0, 1) \) such that \( \phi(1) - \phi(0) = 1 \).
Integrating by parts, it follows that
\[ \frac{u'(1)\phi(1)}{\sqrt{1 + (u'(1))^2}} - \frac{u'(0)\phi(0)}{\sqrt{1 + (u'(0))^2}} + \text{sgn}(u(1) - u(0)) = 0 \]
and hence, as \( u'(0) = u'(1) \),
\[ \frac{|u'(1)|}{\sqrt{1 + (u'(1))^2}} = \frac{|u'(0)|}{\sqrt{1 + (u'(0))^2}} |\phi(1) - \phi(0)| = |\text{sgn}(u(1) - u(0))| = 1, \]
which is impossible, because \( u \in C^1[0, 1] \). Therefore, we can conclude that \( u(0) = u(1) \). This ends the proof. \( \square \)

As far as concerns regularity, Theorem 1.5 complements and completes the existence results obtained in [9, 24, 25, 27, 28].

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