Zero Energy Bound States in Three–Particle Systems

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Abstract

We consider a 3–body system in $\mathbb{R}^3$ with non–positive potentials and non–negative essential spectrum. Under certain requirements on the fall off of pair potentials it is proved that if at least one pair of particles has a zero energy resonance then a square integrable zero energy ground state of three particles does not exist. This complements the analysis in [1], where it was demonstrated that square integrable zero energy ground states are possible given that in all two–body subsystems there is no negative energy bound states and no zero energy resonances. As a corollary it is proved that one can tune the coupling constants of pair potentials so that for any given $R, \epsilon > 0$: (a) the bottom of the essential spectrum is at zero; (b) there is a negative energy ground state $\psi(\xi)$, where $\int |\psi(\xi)|^2 = 1$; (c) $\int_{|\xi| \leq R} |\psi(\xi)|^2 < \epsilon$. 

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I. INTRODUCTION

In [1] under certain restrictions on pair potentials it was proved that the 3–body system, which is at the 3–body coupling constant threshold, has a square integrable state at zero energy if none of the 2–body subsystems is bound or has a zero energy resonance. Naturally, a question can be raised whether the condition on the absence of 2–body zero energy resonances is essential. Here we show that indeed it is. Namely, in this paper we prove that the 3–body ground state at zero energy can only be a resonance and not a $L^2$ state if at least one pair of particles has a resonance at zero energy. The method of proof is inspired by [2, 3]. In the last section we demonstrate that there do exist 3–body systems, where (a) each 2–body subsystem is unbound; (b) one 2–body subsystem is at the coupling constant threshold; (c) the 3–body system has a resonance at zero energy and bound states with the energy less or equal to zero. System like this can be constructed through appropriate tuning of the coupling constants.

II. A ZERO ENERGY RESONANCE IN A 2–BODY SYSTEM

In this section we shall use the method of [2] to prove a result similar to Lemma 2.2 in [3]. Let us consider the Hamiltonian of 2 particles in $\mathbb{R}^3$

$$h_{12}(\varepsilon) := -\Delta_x - (1 + \varepsilon)V_{12}(\alpha x),$$

(1)

where $\varepsilon > 0$ is a parameter and $\alpha = \hbar/\sqrt{2\mu_{12}}$ is a constant, which depends on the reduced mass $\mu_{12}$ [1]. Additionally, we require

R1 $0 \leq V_{12}(\alpha x) \leq b_1 e^{-b_2|x|}$, where $b_{1,2} > 0$ are constants.

R2 $h_{12}(0) \geq 0$ and $\sigma_{pp}(h_{12}(\varepsilon)) \cap (-\infty, 0) \neq \emptyset$ for $\varepsilon > 0$.

The requirement R2 means that $h_{12}(0)$ has a resonance at zero energy, that is, negative energy bound states emerge iff the coupling constant is incremented by an arbitrary amount (in terminology of [2] the system is at the coupling constant threshold).

The following integral operator appears in the Birman–Schwinger principle [2, 5]

$$L(k) := \sqrt{V_{12}} \left(-\Delta_x + k^2\right)^{-1} \sqrt{V_{12}}.$$
$L(k)$ is analytic for $\text{Re } k > 0$. Due to R1 one can use the integral representation and analytically continue $L(k)$ into the interior of the circle on the complex plane, which has its center at $k = 0$ and the radius $|b_2|$. The analytic continuation is denoted as $\tilde{L}(k) = \sum_n \tilde{L}_n k^n$, where $\tilde{L}_n$ are Hilbert-Schmidt operators.

Remark. In Sec. 2 in [2] (page 255) Klaus and Simon consider only finite range potentials. In this case $L(k)$ can be analytically continued into the whole complex plane. As the authors mention in Sec. 9 the case of potentials with an exponential fall off requires only a minor change: the analytic continuation takes place in a circle $|k| < b_2$.

Under requirements R1-2 the operator $L(0) = \tilde{L}(0)$ is Hilbert-Schmidt and its maximal eigenvalue is equal to one

$$L(0)\phi_0 = \phi_0.$$  \hspace{1cm} (3)

$L(0)$ is positivity–preserving, hence, the maximal eigenvalue is non–degenerate and $\phi_0 \geq 0$. We choose the normalization, where $\|\phi_0\| = 1$.

By the standard Kato–Rellich perturbation theory there exists $\rho > 0$ such that for $|k| \leq \rho$

$$\tilde{L}(k)\phi(k) = \mu(k)\phi(k),$$  \hspace{1cm} (4)

where $\mu(k), \phi(k)$ are analytic, $\mu(0) = 1$, $\phi(0) = \phi_0$ and the eigenvalue $\mu(k)$ is non–degenerate. By Theorem 2.2 in [2]

$$\mu(k) = 1 - ak + O(k^2),$$  \hspace{1cm} (5)

where

$$a = (\phi_0, (V_{12})^{1/2})^2/(4\pi) > 0.$$  \hspace{1cm} (6)

The orthonormal projection operators

$$\mathbb{P}(k) := (\phi(k), \cdot)\phi(k) = (\phi_0, \cdot)\phi_0 + O(k),$$  \hspace{1cm} (7)

$$\mathbb{Q}(k) := 1 - \mathbb{P}(k)$$  \hspace{1cm} (8)

are analytic for $|k| < \rho$ as well. Our aim is to analyze the following operator function on $k \in (0, \infty)$

$$W(k) = [1 - L(k)]^{-1}.$$  \hspace{1cm} (9)

By the Birmann–Schwinger principle $\|L(k)\| < 1$ for $k > 0$, which makes $W(k)$ well–defined.
Lemma 1. There exists $\rho_0 > 0$ such that for $k \in (0, \rho_0)$

$$W(k) = \frac{P_0}{ak} + \mathbf{Z}(k),$$

(10)

where $P_0 := (\phi_0, \cdot)\phi_0$ and $\sup_{k \in (0, \rho_0)} \|\mathbf{Z}(k)\| < \infty$.

Proof. $\bar{L}(k) = L(k)$ when $k \in (0, \rho)$. We get from (9)

$$W(k) = [1 - L(k)]^{-1} = [1 - L(k)]^{-1}P(k) + [1 - L(k)]^{-1}Q(k)$$

(11)

$$= [1 - \mu(k)P(k)]^{-1}P(k) + [1 - Q(k)L(k)]^{-1}Q(k)$$

(12)

$$= \frac{1}{1 - \mu(k)}P(k) + \mathbf{Z}(k),$$

(13)

where

$$\mathbf{Z}(k) := [1 - Q(k)L(k)]^{-1}Q(k).$$

(14)

Note that $\sup_{k \in (0, \rho)} \|Q(k)L(k)\| < 1$ because the eigenvalue $\mu(k)$ remains isolated in this range. Thus $\mathbf{Z}(k) = O(1)$. Using (5), (6) and (7) proves the lemma.

Remark. The singularity of $W(k)$ near $k = 0$ has been analyzed in [3] (Lemma 2.2 in [3]), see also [4]. The decomposition (10) differs in the sense that $\mathbf{Z}(k)$ is uniformly bounded in the vicinity of $k = 0$. The price we have to pay for that is the requirement R1 on the exponential fall off of $V_{12}$.

III. MAIN RESULT

Let us consider the Schrödinger operator for three particles in $\mathbb{R}^3$

$$H = H_0 - V_{12}(r_1 - r_2) - V_{13}(r_1 - r_3) - V_{23}(r_2 - r_3),$$

(15)

where $r_i$ are particle position vectors and $H_0$ is the kinetic energy operator with the center of mass removed. Apart from R1-2 we shall need the following additional requirement

R3 $V_{13}, V_{23} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $V_{13}, V_{23} \geq 0$ and $V_{23} \neq 0$.

Here we shall prove

**Theorem 1.** Suppose $H$ defined in (15) satisfies R1-3. Suppose additionally that $H \geq 0$ and $H\psi_0 = 0$, where $\psi_0 \in D(H_0)$. Then $\psi_0 = 0$.  

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We defer the proof to the end of this section. Our next aim would be to derive the inequality (33)-(35).

Like in [1] we use the Jacobi coordinates $x = \sqrt{2\mu_{12}/\hbar}(r_2 - r_1)$ and $y = \sqrt{2M_{12}/\hbar}(r_3 - m_1/(m_1 + m_2)r_1 - m_2/(m_1 + m_2)r_2)$, where $\mu_{ij} = m_i m_j/(m_i + m_j)$ and $M_{ij} = (m_i + m_j)m_l/(m_i + m_j + m_l)$ are reduced masses (the indices $i, j, l$ are all different). The full set of coordinates in $\mathbb{R}^6$ is labeled by $\xi$. In the Jacobi coordinates the kinetic energy operator takes the form

$$H_0 = -\Delta_x - \Delta_y.$$  \hfill (16)

Following the notation in [1] $\mathcal{F}_{12}$ denotes the partial Fourier transform in $L^2(\mathbb{R}^6)$

$$\hat{f}(x, p_y) = \mathcal{F}_{12} f(x, y) = \frac{1}{(2\pi)^{3/2}} \int d^3 y \ e^{-ip_y \cdot y} f(x, y).$$  \hfill (17)

(Here and always a hat over a function denotes its Fourier transform in $y$-coordinates). We shall need the following trivial technical lemmas.

Lemma 2. Suppose an operator $A$ is positivity preserving and $\|A\| < 1$. Then $(1 - A)^{-1}$ is bounded and positivity preserving.

Proof. A simple expansion of $(1 - A)^{-1}$ into von Neumann series. \hfill $\Box$

Lemma 3. Suppose $g(y) \in L^2 \cap L^1(\mathbb{R}^3)$ and $g(y) \geq 0, g \neq 0$. Then

$$\lim_{z \to +0} \int_{|p_y| \leq \epsilon_0} d^3 p_y \frac{\hat{g}^2}{(p_y^2 + z^2)^{3/2}} = \infty$$  \hfill (18)

for all $\epsilon_0 > 0$.

Proof. Let us set

$$J_\epsilon(z) = \int_{|p_y| < \epsilon} d^3 p_y \frac{1}{(p_y^2 + z^2)^{3/2}} \left| \int d^3 y e^{ip \cdot y} g(y) \right|^2.$$  \hfill (19)

We have

$$J_\epsilon(z) \geq \int_{|p_y| < \epsilon} d^3 p_y \frac{1}{(p_y^2 + z^2)^{3/2}} \left| \int d^3 y \ g (y) \cos (p_y \cdot y) \right|^2.$$  \hfill (20)

Let us fix $r$ so that

$$\int_{|y| > r} d^3 y g(y) = \frac{1}{4} \|g\|_1$$  \hfill (21)

Setting $\epsilon = \min[\epsilon_0, \pi/(3r)]$ we get

$$\cos (p_y \cdot y) \geq \frac{1}{2} \quad \text{if} \quad |p_y| \leq \epsilon, |y| \leq r.$$  \hfill (22)
Substituting (22) and (21) into (20) we get

\[ J_\epsilon (z) \geq \frac{\|g\|_1^2}{64} \int_{|p_y|<\epsilon} d^3p_y \frac{1}{(p_y^2 + z^2)^{3/2}}. \]  

(23)

The integral in (23) logarithmically diverges for \( z \to +0. \)

**Remark.** Lemma 2 may hold for \( g(y) \in L^2(\mathbb{R}^3) \) but we could not prove this.

So let us assume that there is a bound state \( \psi_0 \in D(H_0) \) at zero energy, where \( \psi_0 > 0 \) because it is the ground state \([5]\). Then we would have

\[ H_0 \psi_0 = V_{12} \psi_0 + V_{13} \psi_0 + V_{23} \psi_0, \]  

(24)

Adding the term \( z^2 \psi_0 \) (where here and further \( z > 0 \)) and acting with an inverse operator on both sides of (24) gives

\[ \psi_0 = [H_0 + z^2]^{-1} V_{12} \psi_0 + [H_0 + z^2]^{-1} V_{13} \psi_0 + [H_0 + z^2]^{-1} V_{23} \psi_0 \]  

(25)

\[ + z^2 [H_0 + z^2]^{-1} \psi_0. \]  

(26)

From now we let \( z \) vary in the interval \((0, \rho_0/2)\), where \( \rho_0 \) was defined in Lemma [1]. Because the operator \([H_0 + z^2]^{-1}\) is positivity preserving \([5]\) we obtain the inequality

\[ \psi_0 \geq [H_0 + z^2]^{-1} \sqrt{V_{12}} \sqrt{\sqrt{V_{12}} \psi_0} \]  

(27)

Now let us focus on the term \( \sqrt{V_{12}} \psi_0 \). Using (25) we get

\[ \left[ 1 - \sqrt{V_{12}} (H_0 + z^2)^{-1} \sqrt{V_{12}} \right] \sqrt{V_{12}} \psi_0 = \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{13} \psi_0 \]  

(28)

\[ + \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{23} \psi_0 + z^2 \sqrt{V_{12}} [H_0 + z^2]^{-1} \psi_0 \]  

(29)

And by Lemma [2]

\[ \sqrt{V_{12}} \psi_0 \geq \left[ 1 - \sqrt{V_{12}} (H_0 + z^2)^{-1} \sqrt{V_{12}} \right]^{-1} \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{23} \psi_0. \]  

(30)

It is technically convenient to cut off the wave function \( \psi_0 \) by introducing

\[ \psi_1 := \psi_0 (\xi) \chi (\xi : |\xi| \leq b), \]  

(31)

where clearly \( \psi_1 \in L^2 \cap L^1(\mathbb{R}^6) \) and \( b > 0 \) is fixed so that \( \|V_{23} \psi_1\| \neq 0 \) (which is always possible since \( V_{23} \neq 0 \)).
Applying again Lemma 2 we get out of (30)
\[ \sqrt{V_{12}} \psi_0 \geq \left[ 1 - \sqrt{V_{12}}(H_0 + z^2)^{-1} \sqrt{V_{12}} \right]^{-1} \sqrt{V_{12}} [H_0 + 1]^{-1} V_{23} \psi_1. \] (32)

Substituting (32) into (27) gives that for all \( z \in (0, \rho_0/2) \)
\[ \psi_0 \geq f(z) \geq 0, \] (33)

where
\[ f(z) = [H_0 + z^2]^{-1} \sqrt{V_{12}} \left[ 1 - \sqrt{V_{12}}(H_0 + z^2)^{-1} \sqrt{V_{12}} \right]^{-1} \]
\[ \times \sqrt{V_{12}} [H_0 + 1]^{-1} V_{23} \psi_1. \] (34)

Our aim would be to prove that \( \lim_{z \to +0} \|f(z)\| = \infty \), which would be in contradiction with (33). Let us define
\[ \Phi(x, y) := [H_0 + 1]^{-1} V_{23} \psi_1, \] (36)
\[ g(y) := \int dx \, \Phi(x, y) \sqrt{V_{12} (\alpha x)} \phi_0(x), \] (37)

where \( \phi_0 \) is defined in Sec. II.

**Lemma 4.** \( g(y) \in L^1 \cap L^2(\mathbb{R}^3) \) and \( g(y) \neq 0 \).

**Proof.** Following [2] let us denote by \( G_0(\xi - \xi', 1) \) the integral kernel of \( [H_0 + 1]^{-1} \). We need a rough upper bound on \( G_0(\xi, 1) \). Using the formula on p. 262 in [2] we get
\[ (4\pi)^3 |\xi|^4 e^{3|\xi|^2/2} G_0(\xi, 1) = \int_0^\infty t^{-3} e^{3|\xi|^2/2} e^{-t|\xi|^2} e^{-1/(4t)} dt \]
\[ \leq \int_0^\infty t^{-3} e^{-3/(16t)} dt = \frac{256}{9} \] (38)

Hence,
\[ G_0(\xi, 1) \leq \frac{4}{9\pi |\xi|^4} e^{-\frac{|\xi|^2}{2}}. \] (40)

Using \( \|\sqrt{V_{12}} \phi_0\|_\infty < \infty \) we get \( g(y) \in L^1 \cap L^2(\mathbb{R}^3) \) if \( \Phi \in L^1 \cap L^2(\mathbb{R}^6) \). Because \( \Phi \in L^2(\mathbb{R}^6) \) to prove \( \Phi \in L^1(\mathbb{R}^6) \) it suffices to show that \( \chi_{|\xi| \leq 2b} \Phi(\xi) \in L^1(\mathbb{R}^6) \), where \( b \) was defined after Eq. (31). This follows from (40)
\[ \chi_{|\xi| \geq 2b} \Phi(\xi) \leq \frac{4}{9\pi (|\xi| - b)^4} e^{-\frac{|\xi|^2}{2}} V_{23} \Psi_1(\xi'), \] (41)
\[ \leq \chi_{|\xi| \geq 2b} \frac{4}{9\pi (|\xi| - b)^4} e^{-\frac{|\xi|-b/2}{2}} V_{23} \Psi_1 \|_1 \in L^1(\mathbb{R}^6) \] (42)

That \( g \neq 0 \) follows from the inequality \( \Phi(x, y) > 0 \). \( \square \)
Applying \( \mathcal{F}_{12} \) to \((34)\)–\((35)\) we get
\[
\hat{f}(z) = [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \left[ 1 - \sqrt{V_{12}} \left( -\Delta_x + p_y^2 + z^2 \right)^{-1} \sqrt{V_{12}} \right]^{-1} \sqrt{V_{12}} [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \psi_0.
\] (43)
(44)
By Lemma 1 for \( |p_y| < \rho_0 / 2 \) and \( z < \rho_0 / 2 \)
\[
\left[ 1 - \sqrt{V_{12}} \left( -\Delta_x + p_y^2 + z^2 \right)^{-1} \sqrt{V_{12}} \right]^{-1} = \frac{\|p_y\|}{a \sqrt{p_y^2 + z^2}} + \mathcal{Z} \left( \sqrt{p_y^2 + z^2} \right).
\] (45)
From now on \( z \in (0, \rho_0 / 2) \). Notating shortly \( \chi_0(p_y) := \chi_{\{p_y | p_y < \rho_0 / 2\}} \) gives us
\[
\chi_0(p_y) \hat{f}(z) = \hat{f}_1(z) + \hat{f}_2(z),
\] (46)
where
\[
\hat{f}_1(z) = \chi_0(p_y) \frac{g(p_y)}{\sqrt{p_y^2 + z^2}} [-\Delta_x + p_y^2 + z^2]^{-1} \left( \sqrt{V_{12}} \varphi(x) \right),
\] (47)
\[
\hat{f}_2(z) = \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \mathcal{Z} \left( \sqrt{p_y^2 + z^2} \right)
\]
\[
\sqrt{V_{12}} [-\Delta_x + p_y^2 + 1]^{-1} \left( \mathcal{F}_{12} \mathcal{F}_{12}^{-1} \right) \psi_0.
\] (48)
(49)
The next lemma follows from the results of [1].

**Lemma 5.** \( \sup_{z \in (0, \rho_0 / 2)} \| f_2(z) \| < \infty \)

**Proof.** Let us rewrite \((48)\)–\((49)\) in the form
\[
f_2(z) = \mathcal{A}(z) \mathcal{B}(z) \mathcal{C}(z) \psi_0,
\] (50)
where
\[
\mathcal{A}(z) = \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \left[ 1 + \varphi(p_y) + z \right],
\] (51)
\[
\mathcal{B}(z) = \chi_0(p_y) \mathcal{Z} \left( \sqrt{p_y^2 + z^2} \right),
\] (52)
\[
\mathcal{C}(z) = \chi_0(p_y) \sqrt{V_{12}} [-\Delta_x + p_y^2 + 1]^{-1} \left[ 1 + \varphi(p_y) + z \right]^{-1} \left( \mathcal{F}_{12} \mathcal{F}_{12}^{-1} \right),
\] (53)
and \( \varphi(p_y) \) is defined as in Eq. (22) in [1]. Note that by \((10)\) \( \mathcal{Z} \left( \sqrt{p_y^2 + z^2} \right) \) is a difference of two operators each of which commutes with the operator of multiplication by \( 1 + \varphi(p_y) + z \).
That \( \sup_{z \in (0, \rho_0 / 2)} \| \mathcal{A}(z) \|, \| \mathcal{C}(z) \| < \infty \) follows directly from the proofs of Lemmas 6,8 in [1].
\( \sup_{z \in (0, \rho_0 / 2)} \| \mathcal{B}(z) \| < \infty \) follows from Lemma 1.

\( \square \)
The last Lemma needed for the proof of Theorem 1 is

**Lemma 6.** \( \lim_{z \to 0} \| f_1(z) \| = \infty. \)

**Proof.** We get

\[
\| \hat{f}_1(z) \|^2 = \frac{1}{4\pi^2} \int_{|p_y| \leq \rho_0/2} dp_y \frac{|\hat{g}(p_y)|^2}{p_y^2 + z^2} \int dx \int dx' \int dx'' e^{-\sqrt{p_y^2 + z^2}|x - x'|} \times \frac{e^{-\sqrt{p_y^2 + z^2}|x - x''|}}{|x - x''|} \left( \sqrt{V_{12}(\alpha x')} \varphi(x') \right) \left( \sqrt{V_{12}(\alpha x'')} \varphi(x'') \right).
\]

(54)

(55)

The are constants \( R_0, C_0 > 0 \) such that

\[
\int d^3 x' e^{-\delta|x - x'|} \sqrt{V_{12}(\alpha x')} \varphi_0(x') \geq C_0 \frac{e^{-2\delta|x|}}{|x|} \chi \{ x \mid |x| \geq R_0 \}
\]

for all \( \delta > 0 \). Indeed, the following inequality holds for all \( R_0 > 0 \)

\[
\chi \{ x \mid |x| \geq R_0 \} \frac{e^{-\delta|x - x'|}}{|x - x'|} \chi \{ x \mid |x'| \leq R_0 \} \geq \frac{e^{-2\delta|x|}}{2|x|} \chi \{ x \mid |x| \geq R_0 \}.
\]

(56)

(57)

Substituting (57) into the lhs of (56) we obtain (56), where

\[
C_0 = \frac{1}{2} \int_{|x'| \leq R_0} d^3 x' \sqrt{V_{12}(\alpha x')} \varphi_0(x')
\]

(58)

and one can always choose \( R_0 \) so that \( C_0 > 0 \). Using (56) we get

\[
\| \hat{f}_1(z) \|^2 \geq c \int_{|p_y| \leq \rho_0} dp_y \frac{|\hat{g}(p_y)|^2}{(p_y^2 + z^2)^{3/2}},
\]

(59)

where \( c > 0 \) is a constant. Now the result follows from Lemmas 3, 4.

\( \square \)

The proof of Theorem 1 is now trivial.

**Proof of Theorem 1.** A bound state at threshold should it exist must satisfy inequality (33) for all \( z \in (0, \rho_0/2) \). Thus \( \| f(z) \| \) and, hence, \( \| \chi_0 \hat{f}(z) \| \) are uniformly bounded for \( z \in (0, \rho_0/2) \). By (45) and Lemmas 3, 4, this leads to a contradiction.

\( \square \)

**IV. EXAMPLE**

Suppose that R2 is fulfilled and let us introduce the coupling constants \( \Theta, \Lambda > 0 \) in the following manner

\[
H(\Theta, \Lambda) = [-\Delta_x - V_{12}] - \Delta_y - \Theta V_{13} - \Lambda V_{23}.
\]

(60)
For simplicity, let us require that \( V_{ik} \geq 0 \) and \( V_{ik} \in C^\infty_0(\mathbb{R}^3) \). Let \( \Theta_{cr}, \Lambda_{cr} \) denote the 2-body coupling constant thresholds for particle pairs 1,3 and 2,3 respectively. On one hand, using a variational argument it is easy to show that there exists \( \epsilon > 0 \) such that \( H(\Theta, \Lambda) > 0 \) if \( \Theta, \Lambda \in [0, \epsilon] \) (that is in this range \( H(\Theta, \Lambda) \) has neither negative energy bound states nor a zero energy resonance) \[7, 8\]. On the other hand, by the Efimov effect the negative spectrum of \( H(\Theta_{cr}, \Lambda) \) is not empty for \( \Lambda \in (0, \Lambda_{cr}) \) \[3, 4\]. So let us fix \( \Lambda = \epsilon \) and let \( \Theta \) vary in the range \( [\epsilon, \Theta_{cr}] \). The energy of the ground state \( E_{gr}(\Theta) = \inf \sigma(H(\Theta, \epsilon)) \) is a continuous function of \( \Theta \). \( E_{gr}(\Theta) \) decreases monotonically at the points where \( E_{gr}(\Theta) < 0 \). Because \( E_{gr}(\epsilon) = 0 \) there must exist \( \Theta_0 \in (\epsilon, \Theta_{cr}) \) such that \( E_{gr}(\Theta) < 0 \) for \( \Theta \in (\Theta_0, \Theta_{cr}) \) and \( E_{gr}(\Theta_0) = 0 \).

Summarizing, \( H(\Theta_0, \epsilon) \) is at the 3-body coupling constant threshold. By Theorem \[1\] \( H(\Theta_0, \epsilon) \) has a zero energy resonance but not a zero energy bound state. If \( \psi_{gr}(\Theta, \xi) \in L^2(\mathbb{R}^6) \) is a wave function of the ground state defined on the interval \( (\Theta_0, \Theta_{cr}) \) then for \( \Theta \rightarrow \Theta_0 + 0 \) the wave function must totally spread (see Sec. 2 in \[1\]). Which means that for any \( R > 0 \)

\[
\lim_{\Theta \rightarrow \Theta_0 + 0} \int_{|\xi| < R} |\psi_{gr}(\Theta, \xi)|^2 d\xi \rightarrow 0.
\] (61)

Note also that if the particles 1,2 would have a ground state at the energy \( e_{12} < 0 \) then the ground state of the 3 body system at the energy \( e_{12} \) cannot be bound, it can only be a resonance \[3\].

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[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, vol. 4, Academic Press/New York (1978)

[6] T. Kato, *Perturbation Theory for Linear Operators*, Springer–Verlag/Berlin Heidelberg (1995)

[7] J.–M. Richard and S. Fleck, Phys. Rev. Lett. 73, 1464 (1994)

[8] D. K. Gridnev and J. S. Vaagen, Phys. Rev. C61, 054304 (2000)

[9] M. Klaus and B. Simon, Comm. Math. Phys. 78, 153 (1980)