Exact Solution of a Spin-$\frac{1}{2}$ Particle for a Linear Potential

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Abstract

The problem of a spin-$\frac{1}{2}$ particle moving in a linear potential field in two-dimensions is searched to obtain for nonzero energy eigenvalues and the corresponding normalized eigenfunctions. The zero-mode ($E = 0$) eigenfunctions are also studied and it is seen that they are normalizable. The variation of the non-zero eigenfunctions and also eigenvalues are according to the position and potential parameter $\gamma$, respectively, given in the text.

Keywords: Dirac equation, linear potential field, zero-mode eigenfunctions

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I. INTRODUCTION

The potential having a linear form of position and a "Coulomb-like" potential play an important role in various branches of physics. The gluon condensation in high energy physics can be studied with an effective linear potential [1]. The different properties of low-lying baryons have been searched by using a linear potential and the corrections to the mass spectrum have also been calculated one-gluon exchange [2]. It has been suggested that the hadron spectrum can be studied by assuming that the quarks are bounded by a long-range potential plus a short-range potential (a Coulomb-like) arising in gluon-photon exchange diagrams [3]. It is obtained that baryon resonances can be studied by using a gluon-perturbed linear potential. In addition, it has been analytically showed that the static QCD potential can be designed as a sum of a "Coulomb-like" potential and a linear potential by using an approximation based on the renormalon picture [4]. The linear potential arising from an effective vector exchange can be used to calculate fine-structure corrections $\psi/J$ particles and also compute the electromagnetic decay rates between low-lying $s$ and $p$ states [5].

Analytical solutions of Dirac equation for a "Coulomb-like" potential and/or a linear potential or their linear combination are also a subject within the relativistic quantum mechanics [6-8].

In the present work the exact solutions of two-dimensional Dirac equation for a particle moving in a linear potential are obtained by turning it into a second order cylinder differential equation [9]. The nonzero eigenfunctions are written in terms of the functions $1F_1(a; b; z)$ and then they are given in terms of the Hermite polynomials. The nonzero eigenenergies are obtained by using some restrictions on the eigenfunctions. The normalization constants are also computed and the variation of the first three eigenfunctions according to the position are given in plots. The zero-mode eigenenergies ($E = 0$) and the corresponding eigenfunctions are computed from the Dirac equation.

II. ANALYTICAL SOLUTIONS

The (1+1)-dimensional time-independent Dirac equation for a spin-$\frac{1}{2}$ particle with rest mass $m$ moving in a time-independent scalar potential is given

$$\left[ c\vec{\alpha} \cdot \vec{\sigma} + \beta \left( mc^2 + V(x) \right) \right] \Psi(x) = E \Psi(x),$$

(1)
where $\vec{p}$ is the momentum operator and $\alpha$ and $\beta$ are Hermitian square matrices, respectively. $E$ is the energy of particle and $c$ is the velocity of light. Choosing $\alpha$ and $\beta$ as $2 \times 2$ Pauli spin matrices as $\alpha = \sigma_2$ and $\beta = \sigma_2$ and writing the spinor in terms of the upper and lower components as

$$
\Psi(x) = \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix},$

the Dirac equation reads as

$$
(mc^2 + V(x))\phi(x) - hc\frac{\phi(x)}{dx} = E\psi(x),$$

(3a)

$$
(mc^2 + V(x))\psi(x) + hc\frac{\psi(x)}{dx} = E\phi(x),$$

(3b)

Last two equations can be written as two second order differential equations

$$
\frac{d^2\psi(x)}{dx^2} + U_u(x)\psi(x) = \epsilon_u \psi(x),$$

(4a)

$$
\frac{d^2\phi(x)}{dx^2} + U_l(x)\phi(x) = \epsilon_l \phi(x),$$

(4b)

where $u$ stands for "upper" and $l$ for "lower", respectively and

$$
U_u(x) = -\frac{1}{\hbar^2 c^2}V^2(x) - \frac{2m}{\hbar^2}V(x) \pm \frac{1}{\hbar c} \frac{dV(x)}{dx},
$$

(5a)

$$
\epsilon_u = \frac{1}{\hbar^2 c^2}(E^2 - m^2 c^4),
$$

(5b)

It should be noted that Eqs. (4a) and (4b) are not independent because of the eigenvalue $\epsilon_{ul}$. So, we have to look for bound state solutions for $U_{ul}(x)$ having a common energy eigenvalue $\epsilon_{ul}$.

Now let us consider a scalar potential with the following form

$$
V(x) = -V_0 - \gamma x,
$$

(6)

where $\gamma$ is a real parameter. With the help of this equation and using Eq. (5a), we obtain the following from Eq. (4a)

$$
\left\{ \frac{d^2}{dx^2} - (\alpha_1 x^2 + \alpha_2 x + \alpha_3) \right\} \psi(x) = 0,
$$

(7)

where

$$
\alpha_1 = \frac{\gamma^2}{\hbar^2 c^2}; \quad \alpha_2 = \frac{2\gamma}{\hbar^2} \left( \frac{V_0}{c^2} - m \right); \quad \alpha_3 = \frac{V_0}{\hbar^2} \left( \frac{V_0}{c^2} - 2m \right) + \frac{\gamma}{\hbar c} + \frac{1}{\hbar^2 c^2} (E^2 - m^2 c^4),
$$

(8)
Eq. (8) is written in terms of a new variable \( y = (4\alpha_1)^{1/4} x \) as

\[
\left\{ \frac{d^2}{dy^2} - \left( \frac{1}{4} y^2 + \frac{\alpha_2}{(4\alpha_1)^{3/4}} y + \frac{\alpha_3}{(4\alpha_1)^{1/4}} \right) \right\} \psi(y) = 0 ,
\]

(9)

In order to obtain a cylinder differential equation [9] we use a new variable \( z = y + \frac{2\alpha_2}{(4\alpha_1)^{1/2}} \) in last equation and then we obtain

\[
\frac{d^2\psi(z)}{dz^2} - \left( \frac{1}{4} z^2 + A \right) \psi(z) = 0 ,
\]

(10)

where \( A = \frac{\alpha_1}{(4\alpha_1)^{1/2}} - \frac{\alpha_2}{(4\alpha_1)^{1/2}} \). For the convenience, we want to write this equation as following

\[
\left( \frac{d^2}{dz^2} - \frac{1}{4} z^2 \right) \psi(z) = \frac{1}{2} (B + 1) \psi(z) ,
\]

(11)

where \( B = \frac{1}{\hbar c \gamma} (E^2 - m^2 c^4) \).

The general solutions of Eq. (10) are written in terms of a confluent hypergeometric function [9]

\[
\psi(z) \sim e^{-z^2/4} \, _1F_1 \left( \frac{A}{2} + \frac{1}{4}; \frac{1}{2}; \frac{1}{2} z^2 \right) ,
\]

(12a)

\[
\psi(z) \sim z e^{-z^2/4} \, _1F_1 \left( \frac{A}{2} + \frac{1}{4}; \frac{1}{2}; \frac{1}{2} z^2 \right) ,
\]

(12b)

corresponding to "odd" and "even" solutions, respectively. In order to get a finite (physical) solutions we have to write following equality in Eq. (12a)

\[
\frac{A}{2} + \frac{1}{4} = -n \quad (n \in \mathbb{N}) ,
\]

(13)

which gives a quantization condition for the energy eigenvalues. Using the last restriction, we write the eigenfunctions for \( x > 0 \)

\[
\psi(z)_{x>0} = Ne^{-z^2/4} \, _1F_1 \left( -n; \frac{1}{2}; \frac{1}{2} z^2 \right) ,
\]

(14)

where \( N \) is obtained from the normalization. The requirement in Eq. (13) implies into energy eigenvalues

\[
E = \pm \sqrt{2n' \hbar c \gamma + 2m^2 c^4 - \hbar c \gamma} ,
\]

(15)

where \( n' = 2n + \frac{1}{2} \). It is seen that the energy levels are symmetric about \( E = 0 \) and proportional to the potential parameter \( \gamma \).
We could write the eigenfunctions in Eq. (14) in a more suitable form by using the following identity between confluent hypergeometric function and the Laguerre polynomials $L_n^m(x)$ [9]

\[ _1F_1(-n; m + 1; x) = \frac{m!n!}{(m+n)!}L_n^m(x), \]

and the identity between the Laguerre polynomials and the Hermite ones $H_n(x)$

\[ L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{x}), \]

With the help of last two equations, we write the eigenfunctions as

\[ \psi(z)_{x>0} = N\frac{(-1)^n(n-1)!}{(2n-1)!}e^{-z^2/4}H_{2n}\left(\sqrt{\frac{1}{2}}z\right), \]

where the normalization constant is obtained from $\int_{-\infty}^{\infty}(|\psi|^2 + |\phi|^2)dx = 1$.

Following the same steps we collect the differential equations satisfying the eigenfunctions $\psi(x)$ and $\phi(x)$ as

\[ \left(\frac{d^2}{dz^2} - \frac{1}{4}z^2\right)\psi(z)_{x>0} = \frac{1}{2}(B+1)\psi(z)_{x>0}, \]

\[ \left(\frac{d^2}{dz^2} - \frac{1}{4}z^2\right)\phi(z)_{x>0} = \frac{1}{2}(B-1)\phi(z)_{x>0}, \]

and

\[ \left(\frac{d^2}{dz^2} - \frac{1}{4}z^2\right)\psi(z)_{x<0} = \frac{1}{2}(B-1)\psi(z)_{x<0}, \]

\[ \left(\frac{d^2}{dz^2} - \frac{1}{4}z^2\right)\phi(z)_{x<0} = \frac{1}{2}(B+1)\phi(z)_{x<0}, \]

These equations shows that the eigenfunctions could be constructed from Hermite polynomials of order $2n$ and $2n+1$. On the other hand, the continuity condition at $x = 0$ gives the requirement that $H_{2n+1} \sim \text{const.} H_{2n}$ where the ”const.” could be determined from the continuity condition [7].

Here we should to say that the energy eigenvalues for $x < 0$ is little different from the result given in Eq. (15). The quantized energy values for $x < 0$ is given as

\[ E = \pm \sqrt{2n'hc\gamma + 2m'^2c^4 + hc\gamma}, \]

which shows that the energy levels are also symmetric about $E = 0$. The difference between two results obtained from Eqs. (15) and (21) is plotted in Fig. (1). It is seen that the
dependence of the eigenvalues on the parameter $\gamma$ is linear while the contribution of $\gamma$ to the energy values is lower in the case given in Eq. (21) than the ones given in Eq. (15).

In order to get the normalization constant in Eq. (14) we use the following identity for the Hermite polynomials \[9\]

$$
\int_{-\infty}^{+\infty} H_n(x)H_m(x)e^{-x^2}dx = 2^n(n!)\sqrt{\pi}\delta_{mn},
$$

and we obtain as

$$
N = \frac{1}{(n-1)!(1)^n}\sqrt{\frac{(2n-1)!}{n2^{2n-1}\sqrt{\pi}}},
$$

(23)

The dependence of the normalized eigenfunctions on the coordinate $z$ with the help of the last equation could be seen in Fig. (2). All eigenfunctions have the same behavior near the origin and for $z \rightarrow +\infty$ and also finite values. They oscillate between the range where they are well defined.

Finally let us search the zero-mode solutions which can be obtained from Eq. (3). In this case we obtain first-order differential equations and the solutions could be summarized as

$$
\psi(x)_{x>0} \sim e^{h_{x>0}(x)},
$$

(24a)

$$
\phi(x)_{x>0} \sim e^{-h_{x>0}(x)},
$$

(24b)

$$
\psi(x)_{x<0} \sim e^{h_{x<0}(x)},
$$

(24c)

$$
\phi(x)_{x<0} \sim e^{-h_{x<0}(x)},
$$

(24d)

where

$$
h_{x>0}(x) = \frac{\gamma^2}{2\hbar c}x^2 + \frac{1}{\hbar c}(V_0 - mc^2)x,
$$

(25a)

$$
h_{x<0}(x) = -\frac{\gamma^2}{2\hbar c}x^2 + \frac{1}{\hbar c}(V_0 - mc^2)x,
$$

(25b)

These solutions can be normalized by using the following identity \[9\]

$$
\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2-4ac}{4a}\right], \quad (a > 0)
$$

(26)

and then we find the normalization constant $N'$ as

$$
N' = \left[\frac{\pi}{\sqrt{\frac{\gamma^2}{\hbar c}}\exp\left(-\frac{1}{\hbar c\gamma}(mc^2-V_0)^2\right)}\right]^{-1/2},
$$

(27)
We have completely analyzed the problem of a spin-$\frac{1}{2}$ particle subject to a linear potential field. We have computed the energy eigenvalues and the corresponding normalized eigenfunctions by converting the Dirac equation to the second-order cylinder differential equation. We have written the eigenfunctions by using the Laguerre and the Hermite polynomials [7]. We have seen that the upper and lower components of the Dirac spinor are the same except a constant which could be determined from the continuity condition at $x = 0$. We have computed the probability density $|\Psi|^2$ and $|\psi|^2$ and numerically showed their variation with respect to $x$ in figure. We have also searched the zero-mode energy eigenvalues and the corresponding normalized wave functions.
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FIG. 1: Variation of the first three positive energy eigenvalues with the potential parameter $\gamma$ obtained from Eq. (15) (left panel) and from Eq. (21) (right panel) ($m = c = \hbar = 1$).

FIG. 2: Variation of eigenfunctions according to $z$ for $n = 1$ (left upper panel), $n = 2$ (right upper panel) and $n = 3$ (lower panel) ($m = c = \hbar = 1$).