Computational quantum-classical boundary of commuting quantum circuits

Keisuke Fujii\textsuperscript{1,2} and Shuhei Tamate\textsuperscript{3}

\textsuperscript{1}The Hakubi Center for Advanced Research, Kyoto University, Yoshida-Ushinomiyacho, Sakyo-ku, Kyoto 606-8302, Japan
\textsuperscript{2}Graduate School of Informatics, Kyoto University, Yoshida Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
\textsuperscript{3}RIKEN Center for Emergent Matter Science, Wako, Saitama 351-0198, Japan

(Dated: June 27, 2014)

It is often said that the transition from quantum to classical worlds is caused by decoherence originated from an interaction between a system of interest and its surrounding environment. Here we establish a computational quantum-classical boundary from the viewpoint of classical simulatability of a quantum system under decoherence. Specifically, we consider the commuting quantum circuits as dynamics of the quantum system. To show intractability of classical simulation above the boundary, we utilize the postselection argument introduced by M. J. Bremner, R. Jozsa, and D. J. Shepherd [Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 465, 1413 (2009).] and crucially strengthen its statement by taking noise effect into account. Classical simulatability below the boundary is shown by taking a projected-entangled-pair-state picture. Not only the separability criteria but also the condition for the entangled pair to become a convex mixture of stabilizer states is developed to show classical simulatability of highly entangling operations. We found that when each qubit is subject to a single-qubit complete-positive-trace-preserving noise, the computational quantum-classical boundary is tightly given by the dephasing rate required for the magic state distillation.

Introduction. — Understanding a boundary between quantum and classical worlds is one of the most important quests in physics. Sometimes it is said that decoherence originated from an interaction with an environment causes the transition from quantum to classical worlds\textsuperscript{1,2}. However, the definition of “quantumness” varies depending on a situation where the system is located and a purpose of its usage. One of the most successful definition would be a violation of the Bell inequality\textsuperscript{3}; if the measurement outcomes of Alice and Bob violate the Bell inequality, the measurement outcomes cannot be expressed by any local hidden variable theory. In this sense, whether or not the system obeys the Bell inequality serves as a quantum-classical boundary. Nonlocality, or more widely, entanglement, in the quantum regime is also utilized as a resource for quantum information processing, especially in a communication scenario\textsuperscript{1,2}.

Is there any other quantum-classical boundary, which would be useful in another scenario? In many experiments, the quantum system of interest is held in a local experimental apparatus, such as a vacuum chamber and a refrigerator. In such a situation, can we decide whether or not the system is quantum in a reasonable sense? In this paper, we establish a quantum-classical boundary from the viewpoint of classical simulatability of a quantum dynamics under decoherence, which we call a computational quantum-classical (CQC) boundary. For this purpose, nonlocality or entanglement would not capture the boundary completely since there are a lot of classically simulatable classes of quantum computation, which can generate highly entangled states\textsuperscript{4,5}. Moreover, highly mixed state quantum computation without any entanglement exhibits nontrivial quantum dynamics\textsuperscript{10,11}. Thus we have to develop a novel criterion, which determines whether or not the system is classically simulatable or not.

Specifically we consider commuting quantum circuits, which is quite simple and less powerful than universal quantum computation but still exhibits nontrivial quantum dynamics\textsuperscript{9,12,13} and can be applied, for example, to a random state generation and a thermalizing algorithm of classical Hamiltonian\textsuperscript{14}. Here we derive a threshold on the noise strength, below which the system has quantumness in the sense that the measurement outcomes cannot be simulated efficiently by any classical computer under some reasonable assumptions. If the noise strength lies above the threshold, the measurement outcomes can be efficiently simulated by a classical computer. Thus the threshold provides a boundary between quantum and classical worlds in a computational sense.

To show intractability of classical simulation below the boundary, we utilize and further extend the postselection argument introduced by Bremner, Jozsa and Shepherd\textsuperscript{13} to the system being subject to rather general decoherence. This extension is crucial for our purpose, since the original postselection argument holds only for an approximation with a multiplicative error, which is hard to archive in actual experimental systems, where noise is introduced inevitably. Classical simulatability above the boundary is shown by taking a projected-entangled-pair-state (PEPS) picture\textsuperscript{15}. Not only the separable criteria\textsuperscript{16,17}, we also develop a criteria for the shared entangled pair to become a convex mixture of stabilizer states. This allows us to show classical simulatability of highly entangling operations. We found that when
each qubit is subject to a single-qubit complete-positive-trace-preserving (CPTP) noise, the CQC boundary is exactly given by the dephasing rate 14.6%, which is a tight boundary of whether or not the magic state distillation succeeds \[13, 19\]. We also discuss how to verify quantumness in the computational sense by a single-shot experimental result under some physical assumptions.

Postselected threshold theorem.— The commuting quantum circuit consists of an input state, dynamics, and measurements. The input state is given as a product state of \(N\) qubits, \(\{0\}, e^{i\theta Z}|+\rangle\)^\(N\), located on vertices of a cubic lattice. The dynamics \(D\) consists of controlled-Z (CZ) gates \(\Lambda(Z)\) acting on all nearest-neighbor two qubits on a cubic lattice (later we will generalize this to arbitrary two-qubit commuting gates). The measurements are done in the \(X\)-basis. By choosing an input state of a qubit to be \(|0\rangle\), the CZ gates acting on the qubit can be effectively canceled. Then the commuting quantum circuits belong to the class IQP \[12, 13\] with a geometrical constrain.

Since adaptive measurements are not allowed, the commuting quantum circuits (or more generally IQP) are less powerful than universal quantum computation. However, if we are allowed to use postselection, we can simulate universal measurement-based quantum computation (MBQC) by choosing the measurement outcomes that do not need any feedforward operation. This implies that the postselected commuting quantum circuits are as powerful as probabilistic polynomial-time computation (PP) by virtue of post-BQP=PP theorem \[20\]. As shown in Ref. \[13\], if the output \(\{m_k\}\) of such a commuting quantum circuit can be efficiently sampled with a multiplicative error \(1 < c < \sqrt{2}\) using a classical randomized algorithm, the polynomial hierarchy (PH) collapses at the third level \[13\]. The collapse of the PH at the third level is relatively more likely than \(P=NP\) but is also highly implausible to occur. Thus classical simulation of the commuting quantum circuits is intractable on a classical computer in this sense.

The above postselection argument has been quite successful, showing classical intractability of the experimentally feasible systems, such as commuting quantum circuits (so-called IQP) \[13\], linear optics (boson sampling) \[21\], and highly-mixed state quantum computation (deterministic quantum computation with one-clean qubit \[10\] \[11\]. However, the above argument holds only for sampling with a multiplicative approximation error, which is experimentally hard to achieve and verify. This is the reason why Aaronson and Arkhipov have also argued the intractability with an additive error under some plausible complexity conjectures \[21\].

Below we will show that intractability of commuting quantum circuits is robust against noise. That is, we can show that classical simulation of the output distributions are still hard even when the commuting quantum circuits are subject to noise. This is shown by a virtual use of fault-tolerant quantum computation. As mentioned before, the postselected commuting quantum circuits can simulate universal measurement-based quantum computation (MBQC). This implies that topologically protected MBQC on a three-dimensional (3D) cluster state can also be simulated by the commuting quantum circuits under postselection \[22, 23\]. Specifically, input states are chosen to be \(|0\rangle, |+\rangle\), and \(e^{i(\pi/8)Z}|+\rangle\) to create the defect, vacuum, and singular-qubit regions, respectively. If the noise level is sufficiently smaller than the threshold value for topologically protected MBQC, classical simulation of such a noisy commuting quantum circuit is also hard. More importantly, we can go further beyond the standard noise threshold by virtue of postselection. Since we are allowed to use postselection, we can execute error detection, without any cost, which discards any possible error events. Since the noise threshold for error detection is much higher than the noise threshold for error correction \[24, 23\], intractability of the commuting quantum circuits is much more robust against noise than the standard universal quantum computation.

We model the noise as a \(k\)-spatially-local (\(k\)-slocal) CPTP map \(\mathcal{N}_j\). Here \(\mathcal{N}_j\) is a super-operator acting on the \(j\)th and its at most \((k−1)\)th nearest neighbor qubits. We are assumed not to know the detail of the noise except that it is spatially local. Nevertheless we can show the following theorem.

**Theorem 1 (Postselected threshold)** Suppose the dynamics \(D\) is followed by arbitrary \(k\)-slocal noise \(\prod_{j=1}^N \mathcal{N}_j\). There is a constant threshold \(\epsilon_{th}\) such that if \(\|\mathcal{N}_j - I\|_\diamond \leq \epsilon_{th}\), then efficient classical simulation of the output of the noisy commuting quantum circuits is impossible unless the PH collapses at the third level. Here \(\|\cdot\|_\diamond\) denotes the diamond norm of the super-operators \[22\].

**Proof:** Suppose an input state is chosen such that the 3D cluster state for topologically protected MBQC is generated \[22, 23\]. Specifically the qubits located at \((e, e, e)\) or \((o, o, o)\) on the cubic lattice are removed by setting their input state to be \(|0\rangle\). Here “e” and “o” denote even and odd integers, respectively. By appropriately choosing the input state to be \(|0\rangle\) on some of the remaining qubits, we can introduce defect regions. The magic state injection can be done by using the input state \(e^{i(\pi/8)Z}|+\rangle\). By the \(X\)-basis measurements and postselection, we can perform topologically protected MBQC. The postselection is utilized not only to choose the measurement outcomes with no feedforward operation but also to discard the erroneous events, where the error syndrome suggests any existence of errors.

Below we will bound the logical error probability by using the argument developed in Ref. \[27\] with modifying it for the postselected case. We first decompose the \(k\)-slocal noise \(\mathcal{N}_j\) into

\[\mathcal{N}_j \rho = I \rho + \mathcal{E}_j \rho,\]
where $\mathcal{I}$ is an identity super-operator and $\mathcal{E}_j$ is a residual $k$-sloC-S super-operator. The noise can be rewritten as

$$
\prod_{j=1}^{N} N_j = \sum_{\eta=0}^{N} \left[ \mathcal{I} \sum_{(j_1, \ldots, j_N)} \prod_{l=1}^{\eta} \mathcal{E}_{j_l} \right],
$$

where the sumation $\sum_{(j_1, \ldots, j_N)}$ is taken over all possible configurations $(j_1, \ldots, j_N)$. Let us consider an error chain $\mathcal{L}$ of length $L$ in the 3D lattice. We need at least $r \equiv \lfloor L/(2k-1) \rfloor$ $k$-sloC-S noises, which have supports on the chain. Thus the probability of such an error is calculated to be

$$
p_{\text{tail}}(\mathcal{L}) = \text{Tr} \left[ \sum_{\eta=r}^{L(2k^2-2k+1)} \left( \mathcal{I} \sum_{(j_1, \ldots, j_N)} \prod_{l=1}^{\eta} \mathcal{E}_{j_l} \right) \rho \right] \leq \sum_{\eta=r}^{L(2k^2-2k+1)} \left( \frac{L(2k^2 - 2k + 1)}{\eta} \right) \epsilon^\eta (1)
$$

where $\epsilon \equiv \max_j ||\mathcal{E}_j||_o$, and we used the properties of the diamond norm $[26]$. The number of error chains of length $L$ in the 3D lattice can be bounded by $N(6/5)5^L$ from the number of 3D self-avoiding walks. Thus the total failure probability is bounded by

$$
N(6/5) \sum_{L=L_d}^{N} \left( 5 \cdot 2^{k^2-2k+1} L^{1/(2k-1)} \right),
$$

where $L_d$ is the minimum size of the defects. The total failure probability decreases exponentially in the defect size $L_d$, if $\epsilon < 1/(5 \cdot 2^{k^2-2k+1} 2^{k-1})$. Since $k$ is a finite constant, there is a constant threshold on $\epsilon$, below which Clifford gates are topologically protected under postselection. Furthermore, if $\epsilon$ is sufficiently smaller than a certain constant value, the magic state distillation for universal quantum computation $[18, 19]$ can also be done under postselection. Accordingly there exists a postselected noise threshold $\epsilon_{th}$, below which the noisy commuting quantum circuits with postselection are as hard as postBQP, and hence no efficient classical simulation exists unless the PH collapses at the third level $[13, ]$.

Since the dynamics consists only of CZ gates of a constant depth, noises introduced by the input states, the commuting gates, and the measurements can also be regarded as a $k$-sloC-S noise as long as they are also local in space.

A tight CQC boundary.— Let us consider the simplest case with $k=1$ to proceed a detailed analysis. The dynamics is subject to a homogeneous single-qubit CPTP map

$$
\mathcal{N}_\rho = \sum_i W_i \rho W_i^*,
$$

where $W_i = \sum_j c_{ij} \sigma_j$ with $\sigma_j$ being the Pauli matrices. Since we are supposed to be blind to the detail of the noise in experiments, we have to transform the CPTP noise into dephasing by using a subprotocol as follows.

In the vacuum and singular-qubit regions, the input state is chosen to be $X^{\xi_j} Z^{\nu_j} e^{i\theta_j} |+\rangle_j$, where $\nu_j \equiv \nu_j \oplus \epsilon_j \partial_j \xi_k$ with $\partial_j$ being neighbors of the $j$th qubit, and $\{\xi_j\}$ and $\{\nu_j\}$ are random binary variables with probability 1/2. The measurement outcomes are reinterpreted as $\bar{m}_i = m_i \oplus \nu_i$. This subprotocol is equivalent to the original commuting quantum circuit where each single-qubit CPTP noise is sandwiched by stochastic Pauli operations $[28]$. Thus the threshold for the topological protection is given by $q = 20\%$. On the other hand, the threshold for the magic state distillation is given tightly by $q = (1 - \sqrt{2}/2)/4 = 0.0146 [18, 19]$. Thus if $q \leq 14.6\%$, classical simulation of such a noisy commuting quantum circuit is impossible. On the other hand, if $q > 14.6\%$, any input state lies inside the octahedron of the Bloch sphere and hence can be written as a convex mixture of the Pauli-basis states. The dynamics consists only of Clifford gates. The measurements are done in the Pauli-basis. Thus the output distribution is classically

$$
\begin{align*}
X^\nu Z^\nu e^{i\theta_j} |+\rangle_j 
&\rightarrow X^\nu Z^\nu |+\rangle_j \\
&\rightarrow X^\nu Z^\nu |+\rangle_j
\end{align*}
$$

with a depolarizing rate $q$.

From Eq. (1), the total failure probability is given by $N(6/5) \sum_{L=L_d}^{N} (5q)^L$. Thus the threshold for the topological protection is given by $q = 20\%$. On the other hand, the threshold for the magic state distillation is given tightly by $q = (1 - \sqrt{2}/2)/4 = 0.0146 [18, 19]$. Thus if $q \leq 14.6\%$, classical simulation of such a noisy commuting quantum circuit is impossible. On the other hand, if $q > 14.6\%$, any input state lies inside the octahedron of the Bloch sphere and hence can be written as a convex mixture of the Pauli-basis states. The dynamics consists only of Clifford gates. The measurements are done in the Pauli-basis. Thus the output distribution is classically
simulatable due to Gottesman-Knill’s theorem. This indicates that the CQC boundary, which divides classically simulatable and not simulatable regions, is tightly given by \( q = 14.6\% \) in the present setup. If we can estimate the dephasing rate \( q \) in an experiment efficiently (later we will show how to do this), the CQC boundary serves as an efficient experimental criterion that the dynamics has quantumness in a computational sense.

Note that in the standard quantum computation, the threshold for Clifford gates are much lower than that for the magic state distillation. Thus the threshold for fault-tolerant universal quantum computation is determined by the threshold for the Clifford gates. This has raised a natural question how powerful the system in the intermediate region is. However, as shown above, if we consider the classical simulatability by using the postselection argument, the threshold, i.e. CQC boundary, is given solely by the distillation threshold of the magic state. This result is quite reasonable since the magic state distillation is an essential ingredient for universal quantum computation.

Next we show how to estimate the dephasing rate \( q \) from a single-shot measurement under some physical assumptions. This and the previous argument allow us to establish a criteria to verify whether or not the dynamics has quantumness in the computational sense.

**Theorem 2 (Single-shot verification)** Suppose the noise is given by homogeneous 1-local noise. Let \( S_u = \bigoplus_{i \in \partial u} \tilde{m}_i \) be a parity of six measurement outcomes on a unit cube \( u \) in the vacuum region of 3D cluster state. We define a set of parities \( \{S_u\} \) such that the number of parities \( |S_u| \) is \( O(N) \), and each parity \( S_u \) does not share measurement outcomes \( \{\tilde{m}_i\} \) with each other. If the spatial average \( \langle S_u \rangle = 1/|S_u| \sum S_u \) is larger than 1/8, such a noisy commuting quantum circuit is guaranteed to be hard for classical simulation with a probability exponentially close to 1 in the system size \( N \).

**Proof:** As mentioned previously, if the \( j \)th input state is chosen to be \( X^j Z^j e^{i \theta_j} X Z \) randomly, the 1-local noise \( \mathcal{N}_j \) can be rewritten as a dephasing \( P_j(q) \) with the probability \( q \equiv \sum_{i=2,3} |c_i|^2 \). The parities \( \{S_u = \pm 1\} \) are independent binary variables with probability \( |1 + S_u(1 - 2q)|^2/2 \). The spatial average of \( S_u \) is calculated to be

\[
\langle S_u \rangle = (1 - 2q)^6.
\]

If \( q = (1 - \sqrt{2}/2)/2 \), this reads 1/8. By virtue of the large deviation principle, if we obtain \( \langle S_u \rangle > 1/8 \) experimentally, the probability that \( q > (1 - \sqrt{2}/2)/2 \) is exponentially small, and hence classical intractability is guaranteed with a probability exponentially close to 1.

The above arguments can be straightforwardly generalized into \( k \)-local CPTP noises.

**CQC boundary for general commuting circuits.**— In the previous argument, we explicitly utilized the fact that the dynamics consists only of CZ gates. Here we generalize the dynamics to two-qubit nearest-neighbor commuting gates

\[
D = \prod_{(ij)} e^{i \theta_{ij} Z_i Z_j},
\]

where \( \theta_{ij} \in [0, \pi/4] \), and \( \prod_{(ij)} \) is taken over all nearest-neighbor two qubits. For simplicity, we assume that noise is intrinsically provided as a dephasing \( P(q) \). The lower bound, i.e. classical intractability, with \( \theta_{ij} = \pi/4 \) is \( q = 14.6\% \), since the previous case is a special case of the present one.

Below we will first derive an upper bound of the CQC boundary showing classically simulatability. We regard the state before the measurement, which we call a quantum output hereafter, as a PEPS. At the center of the site, the input state \( |\alpha_j\rangle \) is located. An entangled pair

\[
|\theta_{ij}\rangle \equiv e^{i \theta_{ij} Z_i Z_j}|+\rangle +
\]

is shared between nearest-neighbor sites as shown in Fig. 2(a). The isometry (projection)

\[
\Pi_{iso}^{(j)} = |0\rangle\langle 0| \otimes 7 + |1\rangle\langle 1| \otimes 7,
\]

defined on each site \( i \) reproduces the quantum output as follows:

\[
|\Psi_{out}\rangle \equiv D \bigotimes_k |\alpha_k\rangle = \mathcal{C} \prod_{k} \Pi_{iso}^{(k)} |\theta_{ij}\rangle \otimes |\alpha_k\rangle,
\]

where \( \mathcal{C} \) is a normalization factor. By denoting \( \rho_{out} \equiv |\Psi_{out}\rangle \langle \Psi_{out}| \) and \( \rho_{ij} = |\theta_{ij}\rangle \langle \theta_{ij}| \), the dephasing can be taken as

\[
\prod_i P_i(q) \rho_{out} = C^2 \prod_i \Pi_{iso}^{(i)} \left( \bigotimes_{(ij)} P_j(q_{ij}) P_{(ij)} \rho_{ij} \bigotimes_{k} \right. \nonumber
\]

\[
\left. \otimes \left( \bigotimes_k P(q_k) |\alpha_k\rangle \langle \alpha_k| \right) \right) \prod_{k} \Pi_{iso}^{(k)} \bigg)^{\dagger},
\]

where \( q_{ij} \) and \( q_k \) are chosen such that

\[
1 - 2q = (1 - 2q_k) \prod_{j \in \delta i} (1 - 2q_{ij}).
\]

By choosing \( q_{ij} = q_{j,i} = q^{(i,j)} \), the dephased entangled pair \( \tilde{\rho}_{ij} \) can be written as

\[
\tilde{\rho}_{ij} = P_r(q^{(i,j)}) P_j(q^{(i,j)}) \rho_{ij}
\]

\[
= \frac{1}{4} [1 + (1 - 2q^{(i,j)}) \cos 2 \theta_{ij} (IX + XIT) - (1 - 2q^{(i,j)}) \sin 2 \theta_{ij} (ZY + YZ) + (1 - 2q^{(i,j)})^2 XX].
\]
The separability criterion for two-qubit mixed state $\rho$ provides the condition

$$(1 - 2q^{(i,j)}) \leq -\sin 2\theta_{ij} + \sqrt{\sin^2 2\theta_{ij} + 1}.$$  

Each site has six nearest-neighbor bonds. If at least two nearest-neighbor bonds per site are made separable for as shown in Fig. 2 (b), the 3D PEPS can be decoupled into quasi one-dimensional entangled states (more precisely matrix product states).

In a classical simulation, we have to take into account the success probability of the projections for the PEPS. Suppose the dephased entangled pair is decomposed into separable states as follows:

$$\tilde{\rho}_{ij} = \sum_k p_{ij}^{(k_{ij})} \tilde{\rho}_i^{(k_i)} \otimes \tilde{\rho}_j^{(k_j)}.$$  

To handle the success probability of projections, we have to sample separable states $\{\rho_{ij}^{(k_{ij})} \equiv \tilde{\rho}_i^{(k_i)} \otimes \tilde{\rho}_j^{(k_j)}\}_{\text{sep}}$ with a posterior probability conditioned on the success of projections $P_{\text{iso}}^{(l)} = |00\rangle\langle 00| + |11\rangle\langle 11|$ on all site $l$:

$$p(\{k_{ij}\}_{\text{sep}}) = \frac{\text{Tr} \left[ \prod_l P_{\text{iso}}^{(l)} \left( \prod_{(ij)_{\text{sep}}} \rho_{ij}^{(k_{ij})} \otimes \rho_l \right) \prod_{(ij)_{\text{sep}}} p_{ij}^{(k_{ij})} \right]}{\text{Tr} \left[ \prod_l P_{\text{iso}}^{(l)} \left( \prod_{(ij)_{\text{sep}}} \tilde{\rho}_{ij} \otimes \rho_l \right) \right]},$$  

where $\{\cdot\}_{\text{sep}}$ and $\langle\cdot\rangle_{\text{sep}}$ are sets with respect to the separable bonds, and $\rho_l$ indicates the remaining entangling bonds and central qubits $\otimes |\alpha_j\rangle$ for the input state. To this end, a separable state $\rho_{ij}^{(k_{ij})}$ is sampled independently for each dephased entangled pair $\tilde{\rho}_{ij}$ according to a posterior probability given that the projections at site $i$ and $j$ succeed:

$$\tilde{\rho}_{ij}^{(k_{ij})} = \frac{\text{Tr} \left[ P_{\text{iso}}^{(i)} P_{\text{iso}}^{(j)} \left( \tilde{\rho}_{ij}^{(k_{ij})} \bigotimes_{j' = \partial i \setminus j} \psi_{i'}^{(j')} \bigotimes_{i' = \partial j \setminus i} \psi_{i'}^{(i')} \right) \prod_{(ij)_{\text{sep}}} |\alpha_i\rangle \langle \alpha_i| \otimes |\alpha_j\rangle \langle \alpha_j| \right] p_{ij}^{(k_{ij})}}{\text{Tr} \left[ P_{\text{iso}}^{(i)} P_{\text{iso}}^{(j)} \left( \tilde{\rho}_{ij} \bigotimes_{j' = \partial i \setminus j} \psi_{i'}^{(j')} \bigotimes_{i' = \partial j \setminus i} \psi_{i'}^{(i')} \right) \prod_{(ij)_{\text{sep}}} |\alpha_i\rangle \langle \alpha_i| \otimes |\alpha_j\rangle \langle \alpha_j| \right]},$$  

(2)

Here if the sampling on bond $(i, j')$ is not yet completed, $\psi_{i'}^{(j')} = \text{Tr}_{j'}[\tilde{\rho}_{i'j'}]$ with $\text{Tr}_{q}[]$ being a partial trace with...
respect to qubit $a$. Otherwise, $\psi_{ij}^{(j')} = \rho_{ij}^{(k,ij)}$ according to the sampling result. Similarly $\psi_{ij}^{(k,ij)} = \mathrm{Tr}_x[p_{ij}]$ or $\psi_{ij}^{(j')} = \rho_{ij}^{(k,ij)}$ depending on whether or not the sampling on bond $(j',j)$ is completed. In other words, the calculation of the posterior probability is done with updating the states on the bonds depending on the sampling results. Since both commuting gate and dephasing operations commute with the isometry, the joint probability distribution for the successful projections on all sites are divided into a product of probabilities of successful projections on each site. This is also the case for the sampled states, since they are separable. By using these facts, as proved in Ref. [17], the sampling according to $\prod_{ij} p_{ij}^{(k,ij)}$ reproduces the distribution $p((k_{ij}))$.

After the sampling, the PEPS is decoupled into multiple quasi one-dimensional entangled states. The probability distributions on the quasi one-dimensional entangled states can be evaluated via the matrix products. Hence the measurement outcomes can be simulated efficiently if

$$1 - 2q \leq \left( -\sin 2\theta_m + \sqrt{\sin^2 2\theta_m + 1} \right)^2,$$

where $\theta_m = \max\{\theta_{ij}\}$ and $q_k = 0$ is taken.

The above argument using the separability criteria cannot reproduce classical simulatability with $\theta_{ij} = \pi/4$, where the quantum output is highly entangled. Next we derive another bound with respect to Gottesman-Knill’s theorem. If

$$(1 - 2q_{ij}) \leq \cos 2\theta_{ij} + \sin 2\theta_{ij} - \sqrt{2 \cos 2\theta_{ij} \sin 2\theta_{ij}}$$

the entangled pair becomes a convex mixture of the stabilizer states. The input state $e^{i\theta_{ij}Z|+\rangle}$ becomes a convex mixture of Pauli-basis states, if $1 - 2q_k \leq 1/(\sin 2\theta_k + \cos 2\theta_k) \geq 1/\sqrt{2}$. Thus if

$$1 - 2q \leq \left( \cos 2\theta_m' + \sin 2\theta_m' - \sqrt{2 \cos 2\theta_m' \sin 2\theta_m'} \right)^6,$$

with $\theta_m' = \max\{\theta_{ij} - \pi/4\}$, the quantum output becomes a convex mixture of stabilizer states, on which the Pauli-basis measurements are efficiently classically simulatable. More precisely, for each bond, we first choose a pure stabilizer state from the convex mixture according to the posterior probability conditioned on the successful projections as mentioned previously. In this case, one of the sampled state is given as an entangled state

$$II - (ZY + YZ) + XX \over 4.$$

This state can be made separable by using the commuting gate $e^{-i(\pi/4)ZZ}$, which commutes with the isometry. Thus even in this case, the joint probability of successful projections on all sites can be divided into probabilities of successful projections on each site. Then, the sampling with the posterior probability can be done appropriately.

The $X$-basis measurement of the $i$th qubit after the isometry (projection) is equivalent to the measurement of an operator $\prod_{a} X_a^{(i)}$ at site $i$ before the isometry. Thus the probability distribution of the output of the commuting circuits is given by the probability distribution for $\prod_{a} X_a^{(i)}$ conditioned on obtaining $+1$ eigenvalues for all parity operators $\{Z_a^{(i)}\}$. Such a probability can be evaluated efficiently by virtue of the Gottesman-Knill’s theorem.

For simplicity, let us assume $\phi = |\pi/4 - \theta_{ij}|$ for all $(i,j)$, that is, all commuting gates have the same entangling power. Then the separable and stabilizer-mixture criteria are shown in Fig. 3. When $\phi = 0.0144$, the dephasing rate $q$ required for classical simulation becomes the highest. In the region $\phi > 0.0144$, the state before the measurements is highly entangled but can be written as a convex mixture of stabilizer states, and hence the measurement outcomes can be efficiently simulated.

Finally we discuss classical intractability, i.e., lower bound of the CQC boundary for the general two-qubit commuting gates with $\phi = |\pi/4 - \theta_{ij}|$ ($\theta_{ij} \in [-\pi/4, \pi/4]$). Note that two different types of two-qubit commuting gates ($\theta_{ij} = \pi/4 \pm \phi$) of the same entangling power are employed. By choosing $\theta_{ij} = \pi/4 \pm \phi$ randomly with probability 1/2, the two-qubit commuting gate can be rewritten as $e^{i(\pi/4)ZZ} \over 2$ (equivalent to $\text{CZ}$ up to a single-qubit rotation) followed by a collective dephasing with

![FIG. 3. A phase diagram of commuting quantum circuits with respect to the rotational angle $\phi = |\pi/4 - \theta_{ij}|$ of the two-qubit commuting gates and the dephasing strength $q$. The solid line indicates the condition for the entangled pair to be a convex mixture of the stabilizer states. The dashed line indicates the separable criterion such that the residual entangled pairs can be treated as matrix product states. Inside the region colored blue, universal fault-tolerant quantum computation can be executed under postselection, which leads classical simulation of it is hard.

The $X$-basis measurement of the $i$th qubit after the isometry (projection) is equivalent to the measurement of an operator $\prod_{a} X_a^{(i)}$ at site $i$ before the isometry. Thus the probability distribution of the output of the commuting circuits is given by the probability distribution for $\prod_{a} X_a^{(i)}$ conditioned on obtaining $+1$ eigenvalues for all parity operators $\{Z_a^{(i)}\}$. Such a probability can be evaluated efficiently by virtue of the Gottesman-Knill’s theorem.

For simplicity, let us assume $\phi = |\pi/4 - \theta_{ij}|$ for all $(i,j)$, that is, all commuting gates have the same entangling power. Then the separable and stabilizer-mixture criteria are shown in Fig. 3. When $\phi = 0.0144$, the dephasing rate $q$ required for classical simulation becomes the highest. In the region $\phi > 0.0144$, the state before the measurements is highly entangled but can be written as a convex mixture of stabilizer states, and hence the measurement outcomes can be efficiently simulated.

Finally we discuss classical intractability, i.e., lower bound of the CQC boundary for the general two-qubit commuting gates with $\phi = |\pi/4 - \theta_{ij}|$ ($\theta_{ij} \in [-\pi/4, \pi/4]$). Note that two different types of two-qubit commuting gates ($\theta_{ij} = \pi/4 \pm \phi$) of the same entangling power are employed. By choosing $\theta_{ij} = \pi/4 \pm \phi$ randomly with probability 1/2, the two-qubit commuting gate can be rewritten as $e^{i(\pi/4)ZZ} \over 2$ (equivalent to $\text{CZ}$ up to a single-qubit rotation) followed by a collective dephasing with
probability $q(\phi) \equiv \sin^2 \phi$:

$$\rho \rightarrow [1 - q(\phi)]\rho + q(\phi)ZZ\rho ZZ.$$  

Topological quantum error corrections are done independently on the primal and dual lattices, respectively. Suppose the primal lattice is utilized to inject magic states and perform universal quantum computation and the dual lattice is utilized to detect errors. If a total of the dephasing rates $q$ and $q(\phi)$ is below the topological threshold $20\%$ (although this is far from tight), that is, 

$$[1 - 2q(\phi)]^6(1 - 2q) \geq 0.6,$$

then the correlated errors are detected and removed on the primal lattice. Besides, if $(1 - 2q) < 1/\sqrt{2}$, magic state distillation succeeds and hence the commuting quantum circuits can simulate universal quantum computation under postselection. The classically intractable region $(q, \phi)$, in which the dynamics cannot be simulated efficiently unless the PH collapses at the third level, is shown in Fig. 3.

Note that while we here randomly choose the angle $\theta_{ij} = \pi/4 \pm \phi$ to depolarize a commuting gate into a correlated dephasing, we can also calculate the intractable region for $\theta_{ij} = \pi/4 - \phi$ by taking $e^{-\phi ZZ}$ as a noise and evaluating its diamond norm.

**Discussion.**— Here we have established the CQC boundary for the commuting quantum circuits under decoherence. The condition for the system to be a convex mixture of the stabilizer states is far from tight and should be further improved. Such a technique required to show classical simulatability will be useful to describe a complex and noisy quantum system efficiently.

On the other hand, the technique to show classical intractability is useful to certify quantumness in an experimentally feasible setup. It will be interesting to study a relation with unconditionally verifiable blind quantum computation [30], where the quantum tasks are verified without any assumption but unfortunately have no error tolerance, meaning that any small error is detected as an evil attack by the quantum server.

The commuting quantum circuits, which we adopted as an experimentally feasible setup, can be readily applicable for a wide range of non-commuting quantum dynamics by using the Trotter-Suzuki expansion and a path integral method.

Finally, it would be interesting to investigate the relationship between the preset CQC boundary and contextuality [31], a nonlocal property of quantum systems, which has been shown to be relevant for universal quantum computation via magic state distillation, recently.

KF is supported by JSPS Grant-in-Aid for Research Activity start-up 25887034. ST is supported by the Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program).

[1] W. H. Zurek, Phys. Today **44**, 2003. arXiv preprint quant-ph/0306072 (2003).
[2] W. H. Zurek, Reviews of Modern Physics **75**, 715 (2003).
[3] J. Bell, Physics **1**, 195200 (1964).
[4] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Physical Review Letters **70**, 1895 (1993).
[5] A. K. Ekert, Physical review letters **67**, 661 (1991).
[6] D. Gottesman, *Stabilizer codes and quantum error correction.*, Ph.D. thesis, California Institute of Technology (1997).
[7] L. G. Valiant, SIAM Journal on Computing **31**, 1229 (2002).
[8] S. Bravyi and R. Raussendorf, Physical Review A **76**, 022304 (2007).
[9] K. Fujii and T. Morimae, arXiv preprint arXiv:1311.2128 (2013).
[10] E. Knill and R. Laflamme, Phys. Rev. Lett. **81**, 5672 (1998).
[11] T. Morimae, K. Fujii and J. F. Fitzsimons, Phys. Rev. Lett. **112**, 130502 (2014).
[12] D. Shepherd and M. J. Bremer, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science **465**, 1413 (2009).
[13] M. J. Bremer, R. Jozsa, and D. J. Shepherd, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science **467**, 459 (2011).
[14] Y. Nakata and M. Murao, arXiv preprint arXiv:1405.6552 (2014).
[15] F. Verstraete and J. I. Cirac, Physical Review A **70**, 060302 (2004).
[16] R. Raussendorf, S. Bravyi, and J. Harrington, Phys. Rev. A **71**, 062313 (2005).
[17] S. D. Barrett, S. D. Bartlett, A. C. Doherty, D. Jennings, and T. Rudolph, Phys. Rev. A **80**, 062328 (2009).
[18] S. Bravyi and A. Kitaev, Physical Review A **71**, 022316 (2005).
[19] B. W. Reichardt, Quantum Information Processing **4**, 251 (2005).
[20] S. Aaronson, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science **461**, 3473 (2005).
[21] S. Aaronson and A. Arkhipov, in *Proceedings of the 43rd annual ACM symposium on Theory of computing* (ACM, 2011) pp. 333–342.
[22] R. Raussendorf, J. Harrington, and K. Goyal, Annals of physics **321**, 2242 (2006).
[23] R. Raussendorf, J. Harrington, and K. Goyal, New Journal of Physics **9**, 199 (2007).
[24] E. Knill, Nature **434**, 39 (2005).
[25] E. Knill, Physical Review A **71**, 042322 (2005).
[26] D. Aharonov, A. Kitaev, and N. Nisan, in *Proceedings of the thirteenth annual ACM symposium on Theory of computing* (ACM, 1998) pp. 20–30.
[27] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Journal of Mathematical Physics **43**, 4452 (2002).
[28] W. Dür, M. Hein, J. I. Cirac, and H.-J. Briegel, Phys. Rev. A **72**, 052326 (2005).
[29] W. K. Wootters, Physical Review Letters **80**, 2245 (1998).
[30] J. F. Fitzsimons and E. Kashefi, arXiv preprint
arXiv:1203.5217 (2012).

[31] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Nature (2014).