Decoherence induced by fluctuating boundaries

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Abstract

The effects of fluctuating boundaries on a superposition state of a quantum particle in a box is studied. We consider a model in one space dimension in which the initial state is a coherent superposition of two energy eigenstates. The locations of the walls of the box are assumed to undergo small fluctuation with a Gaussian probability distribution. The spatial probability density of the particle contains an interference term, which is found to decay in time due to the boundary fluctuations. At late times, this term vanishes and the quantum coherence is lost. The system is now described by a density matrix rather than a pure quantum state. This model gives a simple illustration of how environment-induced decoherence can take place in quantum systems. It can also serve as an analog model for the effects of spacetime geometry fluctuations on quantum systems.

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I. INTRODUCTION

The theoretical study of quantum systems is usually performed by assuming that for all practical purposes they can be considered as isolated. However, realistic physical systems are never isolated. Instead, they are always immersed in an environment that continuously interacts with them [1]. As a result of such interactions, quantum superpositions tend to be suppressed and are not usually found at the macroscopic level. Exceptions occur in the phenomenon of superfluidity, and possibly in gravitational wave detection technology, where quantum effects can arise on macroscopic scales [2, 3]. The mechanism behind the loss of coherence between the components states in a quantum superposition has been called environment-induced decoherence (decoherence for short) [4–6], and it has been accepted as the mechanism responsible to the emergence of the classical world from the quantum physics [7, 8]. Decoherence has been subjected to intense investigation in recent decades and plays important roles in several areas of physics, including quantum field theory, condensed matter, quantum optics, quantum-chromo-dynamics and cosmology (for reviews, see Refs. [9–12] and references therein).

Imposition of fluctuating boundary conditions on quantum fields in certain physical configurations was proposed in Ref. [13]. The motivation was to address the problem of infinite energy density appearing when fixed boundaries are assumed [14]. It was found that boundary fluctuations can lead to finite energy densities and a natural description of surface energy densities.

In this paper, we study a simple model of a particle described by a superposition of two quantum states inside a potential well in one space dimension. Fluctuating boundaries are introduced as a mechanism modelling environment interactions. The averaged probability density is derived and its time evolution is investigated. It is shown that the net effect of the fluctuations is to suppress the interference terms between the component states of the particle wave function. As time goes on, the resulting probability density reduces to the case corresponding to a statistical mixture of states. The results indicate that this model can be understood as an example of decoherence. The effects of the fluctuating boundaries are similar to the effects of fluctuating or time dependent electromagnetic fields on quantum charged particles [15].

It is well known that boundary conditions in flat spacetime can model certain aspects of the effects of spacetime geometry or gravity on quantum fields. For example, the vacuum energy of quantum fields in an Einstein universe can be derived as a Casimir energy, using the same techniques used to find Casimir energies in cavities in flat spacetime [16]. This suggests that fluctuating boundaries might be used as analog models for the effects of fluctuations of the gravitational field or of the background spacetime geometry. We will not pursue this idea in detail in the present paper, but leave it as a topic for future study.

The simple quantum mechanical system of a particle in a superposition of states of a potential well is developed in the following section. In Sec. III, we model the interaction between the system and its environment by means of fluctuating boundaries, described by a normal probability distribution. The averaged probability density of the particle is derived and its time evolution is discussed. A numerical example is presented at the end of this section. Final remarks are outlined in Sec. IV, including a short discussion on the extension of the model to quantum systems in three space dimensions and to the case of electromagnetic waves in a cavity.
II. THE MODEL

We begin with a non-relativistic particle of mass \( m \) confined in an infinite potential well whose width is denoted by \( a \). For the sake of simplicity we restrict our analysis to the one-dimensional case. Suppose that the normalized state of the particle in an arbitrary time \( t \) is given by a superposition of the first two available states as

\[
\psi(x,t) = \frac{1}{\sqrt{2}} [\psi_1(x,t) + \psi_2(x,t)].
\]  

(1)

The eigenfunctions \( \psi_n (n = 1, 2) \) are independent solutions of the Schrödinger equation under the conditions that these functions vanish on the boundaries at \( x = 0 \) and \( x = a \), yielding

\[
\psi_n(x,t) = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) e^{-i\omega_n t},
\]  

(2)

with \( \omega_n = n^2\pi^2\hbar/2ma^2 \). Each one of these eigenfunctions describes a stationary state, as the corresponding probability density \( |\psi_n|^2 \) is time-independent. However, time evolution occurs when the particle is governed by a linear superposition of \( \psi_n(x,t) \), such as that given in Eq. (1). The probability density can be obtained as

\[
|\psi(x,t)|^2 = \frac{1}{2} (|\psi_1|^2 + |\psi_2|^2) + |\psi_1||\psi_2| \cos \omega t,
\]  

(3)

where we define \( \omega = \omega_2 - \omega_1 \), which represents the corresponding Bohr angular frequency of the system. As we see, the time evolution of \( |\psi(x,t)|^2 \) is exclusively governed by the interference term between \( \psi_1 \) and \( \psi_2 \). In fact, \( |\psi(x,t)|^2 \) oscillates between its maximum \((|\psi_1| + |\psi_2|)^2/2 \) and minimum \((|\psi_1| - |\psi_2|)^2/2 \) values, as depicted in the down inset frame in Fig. 1 for a particular numerical model.

III. PROBABILITY DENSITY WITH FLUCTUATING BOUNDARIES

Now we wish to investigate the behavior of this quantum system when interaction with the environment takes place. In order to model the interaction between the system and its environment, we allow the positions of the physical boundaries to fluctuate under the influence of an external noise. This can simply be implemented by allowing the width parameter \( a \) to undergo fluctuations around a mean value \( \bar{a} \). We set \( a = \bar{a}(1 + \varepsilon) \), where the dimensionless parameter \( \varepsilon \) is described by a Gaussian distribution as

\[
f(\varepsilon) = \sqrt{\frac{\theta}{\pi}} \exp \left( -\theta\varepsilon^2 \right),
\]  

(4)

with \( \theta \) related to the width \( \sigma \) of the distribution by means of \( \sigma^2 = 1/2\theta \). The mean value of an arbitrary function \( G(\varepsilon) \) over \( \varepsilon \) is a linear operation defined by

\[
\langle G \rangle = \int_{-\infty}^{\infty} G(\varepsilon) f(\varepsilon) d\varepsilon.
\]  

(5)

Notice that \( \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 = \sigma^2 \), the mean squared fluctuation of \( \varepsilon \).
There does not seem to be a meaning to averaging the wave function \( \psi(x,t) \), as it is not directly observable. (See, however, Ref. [18] for a discussion of the possibility of measuring a wave function in the context of a weak measurements approach.) Here we are interested in studying averaged values of observable quantities. Particularly, the modulus squared of the particle wave function, Eq. (3), represents the probability density associated with the position the particle. The average over width fluctuations of this quantity can be calculated by using Eqs. (3) and (5), yielding

\[
\langle |\psi|^2 \rangle = \frac{1}{2} \left( \langle |\psi_1|^2 \rangle + \langle |\psi_2|^2 \rangle \right) + \langle |\psi_1||\psi_2| \cos \omega t \rangle .
\]

Here the angular bracket refers to the average over positions of the boundaries, which was defined in Eq. (5). In what follows, each term appearing in the above equation will be considered separately.

The two first terms appearing in Eq. (6) can be expressed, using Eqs. (2) and (5), as

\[
\langle |\psi_n|^2 \rangle = \sqrt{\frac{4\theta}{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\theta \varepsilon^2}}{a(1 + \varepsilon)} \sin^2 \left[ \frac{n\pi x}{\bar{a}(1 + \varepsilon)} \right] d\varepsilon ,
\]

where \( n = 1, 2 \). We assume a narrow distribution, \( \sigma \ll 1 \), which means that only small values of \( \varepsilon \) contribute in Eq. (7). First use the identity \( \sin^2 u = \frac{1}{2} \left[ 1 - \text{Re}(e^{2iu}) \right] \). Next Taylor expand to first order in \( \varepsilon \) inside the exponential, but only to zeroth order otherwise. The integrals may be performed using the identity

\[
\int_{-\infty}^{\infty} e^{\pm iZ\varepsilon - \theta \varepsilon^2} d\varepsilon = \sqrt{\frac{\pi}{\theta}} e^{-Z^2/4\theta} .
\]

The result is

\[
\langle |\psi_n|^2 \rangle \approx \frac{1}{\bar{a}} - \frac{1}{\bar{a}} \cos \left( \frac{2n\pi x}{\bar{a}} \right) \exp \left( \frac{-2n^2\pi^2\sigma^2 x^2}{\bar{a}^2} \right) .
\]

Further, as the particle is confined \((0 \leq x \leq \bar{a})\), and the small \( \sigma \) approximation, \( \pi \sigma x/\bar{a} \ll 1 \), is assumed, we have

\[
\langle |\psi_n|^2 \rangle \approx \frac{2}{\bar{a}} \sin^2 \left( \frac{n\pi x}{\bar{a}} \right) = |\psi_n|^2 .
\]

Thus, the probability density associated with the energy eigenstates does not change significantly when boundary fluctuations are introduced. In other words, if the particle is initially in an energy eigenstate state \( \psi_n(x,t) \), it will remain in this state and its probability density will not undergo appreciable time evolution when small boundary fluctuations are present.

Next, we consider the last term in Eq. (6), which describes the interference effects occurring in the system. From Eqs. (2) and (5), we obtain that

\[
\langle |\psi_1||\psi_2| \cos \omega t \rangle = \frac{1}{4\bar{a}} \sqrt{\frac{\theta}{\pi}} \left( A_1^+ + A_{-1}^+ - A_3^+ - A_{3-}^+ + A_1^- + A_{-1}^- - A_3^- - A_{3-}^- \right) ,
\]

where \( A_{q\pm} \) is defined by

\[
A_{q\pm} = \int_{-\infty}^{\infty} \frac{1}{1 + \varepsilon} \exp \left[ \frac{q i\pi x}{\bar{a}(1 + \varepsilon)} \pm \frac{i\omega t}{1 + \varepsilon} - \theta \varepsilon^2 \right] d\varepsilon ,
\]
and \( \bar{\omega} = \omega_2(a) - \omega_1(a) = 3\hbar \pi^2 / 2ma^2 \). Proceeding as before, assuming \( \sigma \ll 1 \) and using Eq. (8), we obtain

\[
A_q^\pm \approx \sqrt{\frac{\pi}{\theta}} \exp \left[ \frac{iq\pi}{a} \left( x \pm \frac{a\bar{\omega}t}{q\pi} \right) \right] 
\times \exp \left[ \frac{-q^2\pi^2}{4\theta a^2} \left( x \pm \frac{2a\bar{\omega}t}{q\pi} \right)^2 \right].
\] (13)

After a finite time, \( t \gtrsim 1/\bar{\omega} \), the \( x \) dependence in the real exponential becomes unimportant, and \( A_q^\pm \) can be approximated by

\[
A_q^\pm \approx \sqrt{\frac{\pi}{\theta}} \exp \left[ \frac{iq\pi}{a} \left( x \pm \frac{a\bar{\omega}t}{q\pi} \right) \right] e^{-\Gamma t^2},
\] (14)

where \( \Gamma = 2\bar{\omega}^2 \sigma^2 \). Using this result in Eq. (11), we obtain

\[
\langle |\psi_1| |\psi_2| \cos \omega t \rangle = \frac{1}{a} \left[ \cos \left( \frac{\pi x}{a} \right) - \cos \left( \frac{3\pi x}{a} \right) \right] 
\times \cos(\bar{\omega}t) e^{-\Gamma t^2}.
\] (15)

As one can see this term describes oscillations modulated by a factor which decays exponentially in squared time. The time scale for the onset of this decay is

\[
t_o = \frac{1}{\bar{\omega}}.
\] (16)

In the case of an electron in a potential well with \( \bar{a} \approx 1\text{Å} \), this time is of order \( t_o \approx 10^{-17} \text{s} \). However, once the decay begins, the characteristic decay time is

\[
t_d = \frac{1}{\sqrt{\Gamma}} = \frac{t_o}{\sqrt{2} \sigma},
\] (17)

which is longer by a factor of about \( 1/\sigma \).

Combining the results in Eqs. (10) and (15) with Eq. (6), we find that the average \( \langle |\psi(x,t)|^2 \rangle \) becomes

\[
\langle |\psi|^2 \rangle \approx \frac{1}{2} \left( |\psi_1|^2 + |\psi_2|^2 \right) 
+ \frac{1}{a} \left[ \cos \left( \frac{\pi x}{a} \right) - \cos \left( \frac{3\pi x}{a} \right) \right] \cos(\bar{\omega}t)e^{-\Gamma t^2}.
\] (18)

As time passes, the last term in the above equation falls to zero for \( t \gg t_d \). Thus, the net effect of the fluctuations is to kill the interference term. Neglecting the last term in Eq. (18) we obtain that

\[
\langle |\psi|^2 \rangle = \frac{1}{2} \left( |\psi_1|^2 + |\psi_2|^2 \right),
\] (19)

which corresponds to a weighted sum of probabilities as occurs when a statistical mixture of states is considered.
The magnitude of the parameter $\sigma$ depends upon the source of the boundary fluctuations. However, it is useful to make a simple model in which the boundaries are massive particles in a harmonic potential well. The ground state of a quantum harmonic oscillator is described by a Gaussian with characteristic width

$$\Delta x = \sqrt{\frac{\hbar}{2M\omega_0}}, \tag{20}$$

where $M$ is the mass of the boundary, and $\omega_0$ is the angular frequency of the harmonic oscillator. Plausible choices for these parameters might be a nuclear mass and the vibrational frequency of a nucleus in a molecule or a crystal lattice:

$$\Delta x \approx 0.01 \text{ Å} \left(\frac{30 \text{ amu}}{M}\right)^{\frac{1}{2}} \left(\frac{10^{15} \text{s}^{-1}}{\omega_0}\right)^{\frac{1}{2}}. \tag{21}$$

With these choices, and assuming $\bar{a} \approx 1\text{ Å}$, we have $\sigma \approx 10^{-2}$.

The behavior described by Eq. (18) can be illustrated by a numerical example. For instance, let us study the time evolution of the averaged probability density when fluctuating boundaries are considered in a particular model with $x/\bar{a} = 0.7$ and $\sigma = 0.01$. In this case Eq. (6) can be integrated numerically. The result is depicted in Fig. 1. Alternatively we could have used the approximate solution given by Eq. (18), which leads to an identical graph, confirming the approximation used in obtaining Eq. (18). As the figure shows clearly, the probability density $\langle |\psi|^2 \rangle$ oscillates around the mean value $(\langle |\psi_1|^2 \rangle + \langle |\psi_2|^2 \rangle)/2$, converging to this value when $t \gg t_d$. As anticipated, the net effect of the fluctuating boundaries of the potential well is to kill the interference effect between the two state components $\psi_1$ and $\psi_2$.

FIG. 1: (color online). The figure shows the behavior of the probability density associated with the state defined by Eq. (1) when fluctuating boundaries are assumed. As time goes on, interference effects between the component states of $\psi(x, t)$ are suppressed. For fixed boundaries, no suppression of interference is found, as shown in the down inset frame. We set $x/\bar{a} = 0.7$ and $\sigma = 0.01$. 

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Finally, when no fluctuations are present, the usual stationary solution holds, as illustrated by the down inset frame in Fig. 1.

In order to have an estimate, consider again the case of an electron in a potential well with $\bar{a} \sim 1\text{Å}$. As shown in Fig. 1 the oscillations in $\langle |\psi|^2 \rangle$ are completely suppressed when $\tilde{\omega}t = 200$, that is, $10^{-14}$ seconds after the boundary fluctuations are turned on.

IV. DISCUSSION

We have studied a simple one dimensional model of a quantum particle confined by fluctuating boundaries. A coherent superposition of energy eigenstates is quickly converted into a statistical mixture, with the interference term being damped in time as $e^{-\Gamma t^2}$. This seems to be a simple example of decoherence in quantum systems. This behavior can be understood as a loss of phase coherence. A quantum particle in a box can be viewed as undergoing repeated reflections from the box walls. The uncertainty in position of the walls leads to uncertainty in the phase of the reflected wave, which accumulates in time.

Although we considered only one space dimension for simplicity, the same features are to be expected for a quantum particle in a three dimensional box with fluctuating walls. The crucial feature is the appearance of an oscillating interference term, such as that in Eq. (6), whose frequency depends upon the position of the boundaries, which will also be the case in three dimensions. It is not necessary that all of the boundaries undergo fluctuations. Even if only one boundary is subject to fluctuations, while all others are fixed, this will introduce a dependence on the fluctuation parameter in the time dependence of the interference term. When averages over the fluctuations are taken, a decaying exponential in squared time will result, just as in Eq. (15).

The same comments apply to standing electromagnetic waves in a cavity with fluctuating boundaries. We could average the energy density of such waves over the position of the boundary, and obtain essentially the same results as for a quantum particle. If a single normal mode of the cavity is excited, then the energy density will not be significantly altered by small fluctuations, as is the case for energy eigenstates of a quantum particle. However, a coherent superposition of normal modes will have an interference term in its energy density which will decay in time in the presence of boundary fluctuations. Note that this effect has nothing to do with the damping in the electric field oscillations due to power losses in a cavity. Rather it reflects a loss of the definite phase relation between the superposed modes.

In summary, the effects of boundaries with fluctuating positions is a simple model for decoherence in a quantum system. The effects of fluctuating boundaries can model external influences such as fluctuating electromagnetic or gravitational fields.
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