Abstract. We classify the point objects in the derived category $D(X)$ of a torsor under an abelian variety over a field of characteristic 0.

1. Introduction

Let $k$ be a field of characteristic 0 and let $X/k$ be a torsor under an abelian variety $A/k$ of dimension $d$. Let $A^t$ denote the dual abelian variety of $A$ and let $D(X)$ denote the bounded derived category of coherent sheaves on $X$.

Recall (see for example [9]) that if $k$ is algebraically closed then a vector bundle $E$ on $X$ is called semi-homogeneous if for every $a \in A(k)$ there exists a line bundle $L_a$ on $X$ such that $t^*_{a}E \simeq E \otimes L_a$.

For general fields $k$ we call a vector bundle $E$ semi-homogeneous if its base change $E_{\bar{k}}$ to $X_{\bar{k}}$ is semi-homogeneous, where $\bar{k}$ is an algebraic closure of $k$.

The main result of this article is the following:

**Theorem 1.1.** Let $F \in D(X)$ be an object satisfying the following:

(i) $\text{Ext}^i(F, F) = 0$ for $i < 0$.
(ii) The $k$-vector space $\text{Ext}^0(F, F)$ has dimension 1 and $\text{Ext}^1(F, F)$ has dimension $\leq d$.

Then

$$F \simeq i_* \mathcal{E}|r$$

where $r$ is an integer, $i : Z \hookrightarrow X$ is a torsor under a sub-abelian variety $H \subset A$, and $\mathcal{E}$ is a geometrically simple semi-homogeneous (with respect to the $H$-action) vector bundle on $Z$.

Conversely, if $i : Z \hookrightarrow X$ is a torsor under a sub-abelian variety $H \subset A$ and $\mathcal{E}$ is a geometrically simple semi-homogeneous vector bundle on $Z$ then $F := i_* \mathcal{E}$ satisfies conditions (i) and (ii).

**Remark 1.2.** If $k$ is algebraically closed then there is a classification of simple semi-homogeneous vector bundles on an abelian variety $A$ [9, 5.6 and 5.8]. They are all of the form $\pi_* \mathcal{L}'$, where $\pi : A' \to A$ is an isogeny for which the induced map

$$\lambda : A' \to A^t, \quad a' \mapsto t^*_{a'} \mathcal{L}' \otimes \mathcal{L}'^{-1}$$

is injective on $\text{Ker}(\pi)$. This description will be important in our arguments and we revisit some of the arguments proving this in a more general setting in sections 2-4.

**Remark 1.3.** The assumption that $k$ has characteristic 0 is used in exactly one place in the proof (see 6.1). The rest of the argument does not use this assumption.
Remark 1.4. After completion of the initial draft of this article we were made aware of the close connection between this work and that of Polishchuk in [12, 13, 14]. In fact, the methods proving 1.1 also yield a classification of “Lagrangian pairs” studied in loc. cit. We explain this in section 7.

Example 1.5. Let \( A \) be an abelian variety over \( k \) and let \( \mathcal{L} \) be a line bundle on \( A \). Let \( \Phi : D(A) \to D(A^t) \) be the equivalence provided by the Poincaré bundle. Then \( \Phi(\mathcal{L}) \) is a point object on \( A^t \) which is described in [6, 5.1] as follows. Let \( \lambda : A \to A^t, \ a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1} \) be the map defined by \( \mathcal{L} \), let \( C \subset A \) be the connected component of the identity in \( \text{Ker}(\lambda) \), and let \( q : A \to B := A/C \) be the induced quotient. By a theorem of Kempf [10, Appendix, Theorem 1] we can write the line bundle \( \mathcal{L} \) as

\[
\mathcal{L} \simeq q^*\mathcal{M} \otimes \mathcal{S}
\]

for a non-degenerate line bundle \( \mathcal{M} \) on \( B \) and a line bundle \( \mathcal{S} \) algebraically equivalent to 0. Gulbrandsen then shows that

\[
\Phi(\mathcal{L}) = t_{q^*}^*(q^*_t\Phi_B(\mathcal{M})),
\]

where \( \Phi_B : D(B) \to D(B^t) \) is the Fourier transform for \( B \) and \( q^t : B^t \to A^t \) is the dual of \( B \). Now since \( \mathcal{M} \) is non-degenerate the transform \( \Phi_B(\mathcal{M}) \) is a shift of a vector bundle on \( B^t \) [12, 11.11] and we see that \( \Phi(\mathcal{L}) \) has the form described in 1.1.

1.6. Idea of the proof. The basic idea of the proof of 1.1 is to study the action of \( A \times A^t \) on the derived category \( D(X) \) provided by Rouquier’s work [15]. If \( F \) satisfies conditions (i) and (ii) in 1.1 then we show that the stabilizer \( S \subset A \times A^t \), suitably defined, of \( F \) under this action has dimension equal to \( \dim(A) \). This gives \( F \) the structure of a “semi-homogeneous complex” under the action of \( S \) in the sense that for \( (a, [\mathcal{L}]) \in S \) we have

\[
t_a^*F \simeq F \otimes \mathcal{L}.
\]

The rest of the proof then reduces to understanding the structure of such complexes.

Sections 2-4 are devoted to developing the necessary theory of semi-homogeneous complexes and proving some technical results that will be needed in the proof. In the case of vector bundles, a reference for the material in this section is [9]. Then in section 5 we collect some basic results about moduli of point objects in derived categories which enables us to make precise the meaning of the stabilizer \( S \). Finally in section 6 we put the technical ingredients together and complete the proof of 1.1. As noted above we also include a discussion in section 7 of the relationship with Polishchuk’s work on Lagrangian pairs.

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2. Homogeneous complexes

Let \( A \) be an abelian variety over an algebraically closed field \( k \).

**Definition 2.1.** A *homogeneous complex* on \( A \) is a nonzero object \( K \in D(A) \) such that for every \( a \in A(k) \) we have
\[
t_a^* K \simeq K,
\]
where \( t_a : A \to A \) is translation by \( a \).

In the case when \( K \) is a sheaf we obtain the usual notion of a homogeneous vector bundle [9].

**2.2.** Note that if \( K \) is a homogeneous complex then each cohomology sheaf \( H^i(K) \) is a homogeneous vector bundle on \( A \). Let
\[
\Phi : D(A) \to D(A')
\]
be the equivalence of derived categories given by the Poincaré bundle. Then \( \Phi(H^i(K)) \) is set-theoretically supported on a finite set of points of \( A' \) [9, 4.19]. It follows that we have a decomposition
\[
\Phi(K) = \bigoplus_{\mathcal{L} \in A'(k)} K_{\mathcal{L}},
\]
where \( K_{\mathcal{L}} \) is a complex on \( A' \) supported at \( \mathcal{L} \in A'(k) \). Applying the inverse transform we have
\[
K \simeq \bigoplus_{\mathcal{L} \in A'(k)} K_{\mathcal{L}},
\]
where
\[
K_{\mathcal{L}} := \Phi^{-1}(K_{\mathcal{L}}).
\]
Now observe that \( K_{\mathcal{L}} \) admits a finite filtration whose successive quotients are of the form \( \kappa(\mathcal{L})[s] \) for various integers \( s \), and therefore we can endow the complex \( K_{\mathcal{L}} \) with the structure of a filtered complex whose graded pieces are of the form \( \mathcal{L}[s] \) (note that this structure is not canonical). Setting
\[
U_{\mathcal{L}} := K_{\mathcal{L}} \otimes \mathcal{L}^{-1}
\]
we obtain a decomposition
\[
K \simeq \bigoplus_{\mathcal{L} \in A'(k)} \mathcal{L} \otimes U_{\mathcal{L}},
\]
where \( U_{\mathcal{L}} \) admits a filtration whose successive quotients are of the form \( \mathcal{O}_A[s] \).

**Lemma 2.3.** For \( \mathcal{L} \neq \mathcal{L}' \) we have
\[
\text{Ext}^*(\mathcal{L} \otimes U_{\mathcal{L}}, \mathcal{L}' \otimes U_{\mathcal{L}'}) = 0,
\]
and therefore
\[
\text{Ext}^*(K, K) \simeq \bigoplus_{\mathcal{L} \in A'(k)} \text{Ext}^*(\mathcal{L} \otimes U_{\mathcal{L}}, \mathcal{L} \otimes U_{\mathcal{L}}) \simeq \bigoplus_{\mathcal{L} \in A'(k)} \text{Ext}^*(U_{\mathcal{L}}, U_{\mathcal{L}}).
\]

**Proof.** To prove the vanishing (2.3.1) note that since \( U_{\mathcal{L}} \) and \( U_{\mathcal{L}'} \) admit filtrations whose successive quotients are of the form \( \mathcal{O}_A[s] \) it suffices to show that
\[
\text{Ext}^*(\mathcal{L}, \mathcal{L}') \simeq H^*(A, \mathcal{L}^{-1} \otimes \mathcal{L}') = 0,
\]
which follows from the fact that \( \mathcal{L}^{-1} \otimes \mathcal{L}' \) is a nontrivial line bundle algebraically equivalent to 0. \( \square \)
Lemma 2.4. Let $K$ be a homogeneous complex on $A$.

(i) Each $U_L$ in the decomposition above is, up to a shift, a sheaf if and only if $\text{Ext}^i(K, K) = 0$ for $i < 0$.

(ii) If $\text{Ext}^i(K, K) = 0$ for $i < 0$ and $\text{Ext}^0(K, K) = k$ then $K$ is a line bundle, up to a shift.

Proof. The “only if” part of statement (i) is immediate. To see the “if” direction we proceed as follows. With notation as in (2.2.2) we have

$$\text{Ext}^*(U_L, U_L) \cong \text{Ext}^*_{A^t}(K^t_L, K^t_L).$$

Assume that $K^t_L$ is concentrated in more than one degree, and let $a$ (resp. $b$) be the degree of the top (resp. bottom) cohomology of $K^t_L$. So $a > b$ and $H^a(K^t_L)$ and $H^b(K^t_L)$ are finite length modules supported at the point $[L] \in A^t$. Since these are finite length modules over the same point there exists a nonzero morphism

$$\varphi : H^a(K^t_L) \to H^b(K^t_L).$$

Such a morphism defines a nonzero morphism

$$K^t_L \to K^t_L[b - a],$$

or equivalently a nonzero element of

$$\text{Ext}^{b-a}(K^t_L, K^t_L)$$

contradicting the vanishing of the negative Ext-groups. It follows that $K^t_L$ is supported in a single degree. By [9, 4.12] it follows that $K_L$, and therefore also $U_L$, is also supported in a single degree proving (i).

To see (ii) note that if $K^t_L$ has length $> 1$ then there exists non-scalar endomorphisms of $K^t_L$ and therefore also non-scalar endomorphisms of $K_L$. Therefore each $K^t_L$ is, up to a shift, the skyscraper sheaf associated to the point $[L] \in A^t$, and $K_L \simeq L$, up to a shift. Furthermore, since $\text{Ext}^0(K, K) = k$ there is only one factor in the decomposition (2.2.3) proving (ii). $\square$

3. Semi-homogeneous complexes

3.1. Let $k$ be an algebraically closed field and fix a diagram of abelian varieties over $k$

$$S \xrightarrow{u} A^t \xrightarrow{v} A,$$

where $u$ is surjective.

Definition 3.2. A complex $K \in D(A)$ is semi-homogeneous with respect to (3.1.1) if for every $s \in S(k)$ we have

$$t^*_u(s)K \simeq K \otimes L_{v(s)},$$

where $L_{v(s)}$ is the line bundle corresponding to $v(s)$.

To study semi-homogeneous complexes we will consider additional geometric structure on (3.1.1) as follows (a priori such additional data may not exist but see 3.5 below).
**Data 3.3.** (i) A commutative diagram

\[ S \xrightarrow{u'} A' \xrightarrow{\pi} A \]

where $\pi$ is an isogeny and $\alpha' : A' \rightarrow A^n$ is a morphism of abelian varieties.

(ii) A line bundle $\mathcal{N}'$ on $A'$ such that the map $\alpha'$ is equal to the map $\lambda_{\mathcal{N}'} : A' \rightarrow A^n$, $a' \mapsto t^*_{a'} \mathcal{N}' \otimes \mathcal{N}'^{-1}$.

**3.4.** Let $K$ be a nonzero semi-homogeneous complex with respect to (3.1.1). Then for every integer $i$ the sheaf $\mathcal{H}^i(K)$ is a semi-homogeneous sheaf and therefore isomorphic to a vector bundle, since the semi-homogeneous condition implies that the maximal open in $A$ over which $\mathcal{H}^i(K)$ is locally free is translation invariant. Fix an integer $i$ for which $\mathcal{H}^i(K)$ is nonzero, let $r$ be its rank, and let $\mathcal{M}$ denote the determinant of $\mathcal{H}^i(K)$. Taking determinants of the isomorphism (3.2.1) we then obtain that for every $s \in S(k)$ we have

\[ t^*_{u(s)}\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{L}_{v(s)}^r. \]

In other words, if

\[ \lambda_{\mathcal{M}} : A \rightarrow A^t, \ a \mapsto t^*_{a} \mathcal{M} \otimes \mathcal{M}^{-1} \]

denotes the homomorphism defined by $\mathcal{M}$ then the diagram

\[ S \xrightarrow{u} A^t \]

\[ A \xrightarrow{\lambda_{\mathcal{M}}} A^t \]

commutes, where $[r] : A^t \rightarrow A^t$ denotes multiplication by $r$.

**Lemma 3.5.** If there exists a non-zero complex $K$ semi-homogeneous with respect to (3.1.1) then there exists data as in 3.3.

**Proof.** We proceed with notation as in 3.4. To obtain the diagram (3.3.1) in (i), let $A'$ be the connected component of the identity in

\[ A \times_{\lambda_{\mathcal{M}}, A^t, [r]} A^t, \]

let $\alpha : A' \rightarrow A^t$ (resp. $\pi$) be the map given by the second (resp. first) projection, and let $u'$ be the map induced by the commutative square (3.4.1). Setting $\alpha' := \pi \circ \alpha$ we then get the desired commutative diagram.

For the existence of $\mathcal{N}'$ in 3.3 (ii) note that the composition

\[ A' \xrightarrow{\alpha'} A^n \xrightarrow{[r]} A^n \]

is equal to the map $\lambda_{\mathcal{M}'}$, where $\mathcal{M}' := \pi^* \mathcal{M}$. Indeed the precomposition of this map with the surjective $u' : S \rightarrow A'$ is the map (using the commutativity of (3.4.1) and (3.3.1))

\[ [r] \circ \pi^* \circ v = \pi^* \circ [r] \circ v = \lambda_{\mathcal{M}} \circ u. \]
It follows that $A'[r] \subset \text{Ker}(\lambda_{\alpha'})$, and therefore by [11, Theorem 3 on p. 214] the line bundle $\mathcal{M}'$ has an $r$-th root $\mathcal{N}'$. The two maps $\lambda_{\alpha'}, \alpha' : A' \to A''$ have equal compositions with $[r] : A'' \to A''$, and therefore must be equal. 

**Lemma 3.6.** Fix (3.1.1) and data (3.3) and let $K$ be a semi-homogeneous complex with respect to (3.1.1). 

(i) For any $a' \in A'(k)$ we have 

$$t^*_a \pi^* K \simeq \pi^* K \otimes \mathcal{L}_{\alpha'(a')}.$$ 

(ii) There is a decomposition 

$$(3.6.1) \quad \pi^* K \simeq \bigoplus_{d \in A'(k)} U_d \otimes \mathcal{L} \otimes \mathcal{N}',$$ 

where $U_d$ is a homogeneous complex admitting a filtration whose successive quotients are of the form $A'[[s]]$ for various integers $s$.

**Proof.** To see (i), note first of all that $u'$ is surjective since the composition $\pi \circ u' = u$ is surjective by assumption. Given $a' \in A'(k)$ with $\pi(a') = a$ choose $s \in S(k)$ with $u'(s) = a'$. Since $K$ is semi-homogeneous with respect to (3.3.1) we then have 

$$t^*_a K \simeq K \otimes \mathcal{L}_{\psi(s)}.$$ 

Applying $\pi^*$ and noting that $\pi^* t^*_a K \simeq t^*_a \pi^* K$ and $\pi^* \mathcal{L}_{\psi(s)} \simeq \mathcal{L}_{\pi^* \psi(s)}$ we find that 

$$t^*_a \pi^* K \simeq \pi^* K \otimes \mathcal{L}_{\pi^* \psi(s)}.$$ 

Statement (i) then follows from the commutativity of the left triangle in (3.3.1) which implies that 

$$\mathcal{L}_{\pi^* \psi(s)} \simeq \mathcal{L}_{\psi'(a')}.$$ 

Finally (ii) follows from (i) by considering the decomposition (2.2.3) for the homogeneous complex 

$$\pi^* K \otimes \mathcal{N}'^-$$ 

on $A'$.

For data (3.3) we can consider the degree of the isogeny $\pi$, an integer $\geq 1$.

**Lemma 3.7.** For data (3.3) with $\pi$ of minimal degree the group scheme 

$$(3.7.1) \quad I := \text{Ker}(\pi) \cap \text{Ker}(\alpha')$$

is 0.

**Proof.** Assume to the contrary that the group scheme $I$ is nontrivial. We show that there exists data (3.3) with $\pi$ of smaller degree. Since $\alpha'$ is defined by a line bundle there is a skew-symmetric pairing (see for example [11, p. 205]) 

$$e : \text{Ker}(\alpha') \times \text{Ker}(\alpha') \to \mathbb{G}_m.$$ 

There is a nonzero subgroup scheme $I' \subset I$ such that the restriction of $e$ to $I'$ is identically zero. This is clear if $I(k) \neq 0$ from the skew-symmetry of $e$: simply choose a nonzero element $x \in I(k)$ and let $I'$ denote the cyclic subgroup generated by $x$. If $I$ is a local group scheme note that $I$ contains a subgroup scheme $I' \subset I$ of order the characteristic $p$ of $k$. Indeed $I[p]$ is
nonzero since \( I \) is local, and is contained in the group scheme \( A'[p] \) whose simple constituents are given by \( \mathbb{Z}/(p) \), \( \mu_p \), and \( \alpha_p \). It then follows from \[ \text{Lemma 1, p. 207} \] that the restriction of \( e \) to this \( I' \) is zero. See also \[ \text{Proof of Lemma 3, p. 216} \] for a similar argument.

Let \( A'' \) denote \( A'/I' \). By \[ \text{Theorem 2, p. 213} \] the line bundle \( \mathcal{N}' \) descends to a line bundle \( \mathcal{N}'' \) on \( A'' \). Let \( \alpha'' = \lambda_{\mathcal{N}''} : A'' \to A'' \) denote the induced homomorphism. Setting \( u'' : S \to A'' \) equal to \( u' \) composed with the projection \( q : A' \to A'' \) we then obtain a commutative diagram

\[ S \xrightarrow{u'} A' \xrightarrow{q} A'' \xrightarrow{\pi''} A \]

Now the compositions of the two maps
\[ \pi'' \circ v, \alpha'' \circ q \circ u' : S \to A'' \]

with \( q' \) are equal, and therefore these two maps are also equal. Denoting \( q \circ u' \) by \( u'' \) we then obtain a commutative diagram

\[ S \xrightarrow{u''} A'' \xrightarrow{\pi''} A \]

and \( \deg(\pi'') < \deg(\pi) \). \qed

**Theorem 3.8.** Fix data \([3.3]\) with \( \deg(\pi) \) minimal, and let \( K \) be a nonzero semi-homogeneous complex with respect to \((3.1.1)\). Assume further that \( \text{Ker}(\pi) \) is an étale group scheme. Then there exists a homogeneous complex \( H \in D(A') \) and a line bundle \( \mathcal{N}' \) on \( A' \) such that \( K \simeq \pi_*(H \otimes \mathcal{N}') \).

**Remark 3.9.** The assumption on \( \text{Ker}(\pi) \) holds for example if the ground field \( k \) has characteristic 0.

Before starting the proof it is convenient to fix some additional notation.

**3.10.** Let \( G \) denote the quotient
\[ A''/\alpha'(\text{Ker}(\pi)). \]

If \( K \) is a semi-homogeneous complex on \( A \) with respect to \((3.1.1)\) we then have a decomposition \((3.6.1)\) of \( \pi^*K \). Let
\[ \tilde{\Sigma}(K) \subset A''(k) \]
be the set of those \( L \) for which \( U_L \) is nonzero, and let
\[ \Sigma(K) \subset G(k) \]
be the image of \( \tilde{\Sigma}(K) \).
Lemma 3.11. The set $\tilde{\Sigma}(K)$ is the preimage under the projection map $A^n(k) \to G(k)$ of $\Sigma(K)$.

Proof. It suffices to show that $\tilde{\Sigma}(K)$ is stable under translation by $\text{Ker}(\pi)$. To see this note that for $b \in \text{Ker}(\pi)(k)$ we have an isomorphism 
$$t_b^* \pi^* K = \bigoplus_{\mathcal{L}} t_b^* (U_{\mathcal{L}} \otimes \mathcal{L} \otimes \mathcal{N}') \to \bigoplus_{\mathcal{L}} U_{\mathcal{L}} \otimes \mathcal{L} \otimes \mathcal{N}'$$
giving the natural $\text{Ker}(\pi)$-linearization on $\pi^* K$. Now
$$t_b^*(U_{\mathcal{L}} \otimes \mathcal{L} \otimes \mathcal{N}') \simeq (U_{\mathcal{L}} \otimes \mathcal{L} \otimes t_b^* \mathcal{N}' \otimes \mathcal{N}'^{-1}) \otimes \mathcal{N}' \simeq U_{\mathcal{L}} \otimes \mathcal{L} \otimes \mathcal{L}_{\alpha'(b)} \otimes \mathcal{N}'$$
so using [2.3] we find that $\tilde{\Sigma}(K)$ is stable under translation by $\alpha'(b)$. □

Proof of 3.8. Since $\alpha'$ is injective on $\text{Ker}(\pi)$ by [3.7] we can write $\tilde{\Sigma}(K)$ as a disjoint union of $\text{Ker}(\pi)$-orbits, say 
$$\tilde{\Sigma}(K) = \Sigma(K) \times \text{Ker}(\pi).$$
Let $H$ denote the homogeneous complex 
$$H := \bigoplus_{\mathcal{L} \in \Sigma(K)} U_{\mathcal{L}} \otimes \mathcal{L}$$
so 
$$\pi^* K \simeq \bigoplus_{b \in \text{Ker}(\pi)} t_b^*(H \otimes \mathcal{N}')$$
with $\text{Ker}(\pi)$ acting by permuting the factors. Projecting onto the factor corresponding to $b = 0$ we get a map 
$$q : \pi^* K \to H \otimes \mathcal{N}'$$
such that the map (3.11.1) is described as the sum
$$\sum_{b \in \text{Ker}(\pi)} t_b^*(q) : \pi^* K \to \bigoplus_{b \in \text{Ker}(\pi)} t_b^*(H \otimes \mathcal{N}').$$
By adjunction the map $q$ gives a map 
$$\phi : K \to \pi_* (H \otimes \mathcal{N}').$$
This map $\phi$ is an isomorphism. Indeed to verify this it suffices to show that it is an isomorphism after applying $\pi^*$ where we recover the isomorphism (3.11.1). □

4. Further results about semi-homogeneous complexes

4.1. In this section we record some additional technical results we will need for our later arguments. We continue with the notation of the previous section. That is, we fix a diagram (3.1.1) as well as additional data 3.3. We will assume that $\pi : A' \to A$ has minimal degree, so that by [3.7] the composition 
$$\text{Ker}(\pi) \hookrightarrow A' \overset{\alpha'}{\rightarrow} A^n$$
is injective. As in the previous section let $G$ denote the quotient 
$$A^n / \alpha'(\text{Ker}(\pi)).$$
So for a complex $K$ semi-homogeneous with respect to (3.1.1) we have subsets (see 3.10)

\[ \tilde{\Sigma}(K) \subset A^t(k), \quad \Sigma(K) \subset G(k). \]

**Remark 4.2.** If $K$ is a vector bundle semi-homogeneous with respect to (3.1.1) then from (3.6.1) the vector bundle $\pi^*K$ is graded by $\tilde{\Sigma}(K)$. This grading is not preserved by the $\text{Ker}(\pi)$-linearization of $\pi^*K$, but the coarser grading by $\Sigma(K)$ is preserved. By descent theory we therefore obtain a decomposition

\[ K = \bigoplus_{\sigma \in \Sigma(K)} K_{\sigma}. \]

We generalize this to complexes in 4.6 below.

**Lemma 4.3.** (i) Let $K, K' \in D(A)$ be semi-homogeneous complexes with respect to (3.1.1) such that $\Sigma(K) \cap \Sigma(K') = \emptyset$. Then

\[ \text{Ext}^*(K, K') = 0. \]

(ii) Let $K, K'$ be semi-homogeneous vector bundles with respect to (3.1.1) for which $\Sigma(K) \cap \Sigma(K') \neq \emptyset$. Then

\[ \text{Hom}(K, K') \neq 0. \]

**Proof.** For (i) fix a representative in $A^t(k)$ for each element of $\Sigma(K)$ and $\Sigma(K')$. By 3.8 we can then write

\[ K = \pi_* \left( \bigoplus_{\sigma \in \Sigma(K)} U_{\sigma} \otimes \mathcal{L}_{\sigma} \otimes N' \right), \quad K' = \pi_* \left( \bigoplus_{\sigma' \in \Sigma(K')} U_{\sigma'} \otimes \mathcal{L}_{\sigma'} \otimes N' \right) \]

with the $U_{\sigma}$ and $U_{\sigma'}$ as in 3.6 (ii). By adjunction we then have

\[ \text{Ext}^*_A(K, K') \simeq \bigoplus_{b \in \text{Ker}(\pi)} \bigoplus_{(\sigma, \sigma')} \otimes_{\Sigma(K) \times \Sigma(K')} \otimes_{\Sigma(K') \times \Sigma(K')} \text{Ext}^*_A(U_{\sigma} \otimes \mathcal{L}_{\sigma} \otimes \mathcal{L}_{\sigma'}(b) \otimes N', U_{\sigma'} \otimes \mathcal{L}_{\sigma'} \otimes N') \]

\[ \simeq \bigoplus_{b \in \text{Ker}(\pi)} \bigoplus_{(\sigma, \sigma')} \otimes_{\Sigma(K) \times \Sigma(K')} \otimes_{\Sigma(K') \times \Sigma(K')} \text{Ext}^*_A(U_{\sigma} \otimes \mathcal{L}_{\sigma} \otimes \mathcal{L}_{\sigma'}(b), U_{\sigma'} \otimes \mathcal{L}_{\sigma'}). \]

From this and using the filtrations on the $U_{\sigma}$'s we see that it suffices to show that if $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on $A'$ corresponding to distinct points of $A^t(k)$ then

\[ \text{Ext}^*(\mathcal{L}, \mathcal{L}') = 0, \]

which is standard (see for example [4, 7.19]).

Similarly statement (ii) reduces to showing that if $\mathcal{E}$ and $\mathcal{E}'$ are unipotent vector bundles on $A'$ then $\text{Hom}(\mathcal{E}, \mathcal{E}') \neq 0$, which is immediate. \qed

**4.4.** Consider a closed immersion $i : A \hookrightarrow B$ of abelian varieties and let $K$ be a semi-homogeneous complex on $A$ with respect to (3.1.1). Since $i$ is a homomorphism we have for every $a \in A(k)$ an isomorphism

\[ t^*_a i^* i_* \simeq i^* i_* t^*_a. \]

From this it follows that $i^* i_* K$ is again a semi-homogeneous complex with respect to (3.1.1).

**Lemma 4.5.** Let $K$ be a semi-homogeneous vector bundle on $A$. Then $\Sigma(i^* i_* K) = \Sigma(K)$. 
Proof. Note that by the projection formula we have

\[ i_s^* i_s K \simeq (i_s \mathcal{O}_A) \otimes \mathcal{L}_B K \simeq i_*(i^* i_s \mathcal{O}_A) \otimes \mathcal{L}_B i_s K \simeq i_*(i^* \mathcal{O}_A) \otimes K. \]

Let \( \overline{B} \) denote the quotient \( B/A \), let \( g : B \to \overline{B} \) be the quotient map, and let \( s : \text{Spec}(k) \to \overline{B} \) be the zero section. Then

\[ i_s \mathcal{O}_A \simeq g^* s \mathcal{O}_{\text{Spec}(k)}, \]

and we find that

\[ i_s \mathcal{H}^s(i^* i_s K) \simeq H^s(k \otimes \mathcal{L}_s k) \otimes_k i_s K. \]

It follows that we have

\[ \mathcal{H}^s(i^* i_s K) \simeq W_s \otimes_k K \]

for vector spaces \( W_s \), not all of which are zero. In particular,

\[ \Sigma(i^* i_s K) = \bigcup_s \Sigma(\mathcal{H}^s(i^* i_s K)) = \Sigma(K). \]

\[ \square \]

Lemma 4.6. Let \( F \in D(B) \) be a complex on \( B \) such that for every \( s \in S(k) \) we have

\[ t^*_u(s) F \simeq F \otimes \mathcal{R}_s \]

in \( D(B) \), for a line bundle \( \mathcal{R}_s \) on \( B \) lifting \( \mathcal{L}_{v(s)} \), and such that for all \( s \) we have \( \mathcal{H}^s(F) \simeq i_* \mathcal{G}_s \) for a vector bundle \( \mathcal{G}_s \) on \( A \) semi-homogeneous with respect to \( (3.1.1) \). Then

\[ F \simeq \bigoplus_{\sigma \in \Sigma(F)} F_{\sigma}, \]

where

\[ \Sigma(F) := \bigcup_s \Sigma(\mathcal{G}_s) \subset G(k) \]

and \( F_{\sigma} \) has the property that \( \mathcal{H}^s(F_{\sigma}) = i_* \mathcal{G}_{s,\sigma}. \)

Proof. Set \( F_{\geq s} := \tau_{\geq s} F \). We prove by descending induction on \( s \) that the lemma holds for \( F_{\geq s} \). The base case is trivial since \( F_{\geq s} = 0 \) for \( s \) sufficiently large. For the inductive step we assume the result holds for \( s \) and prove it for \( s - 1 \). Considering the distinguished triangle

\[
\begin{array}{ccc}
  i_* \mathcal{G}_{s-1}[-s+1] & \rightarrow & F_{\geq s} - 1 \\
  & \simeq \downarrow & \simeq \downarrow \\
  & \oplus_{\sigma} F_{\geq s,\sigma} & \oplus_{\sigma} i_* \mathcal{G}_{s-1,\sigma}[-s+2]
\end{array}
\]

we see that to prove the inductive step it suffices to show that for \( \sigma \neq \sigma' \) we have

\[ \text{Ext}^*(F_{\geq s,\sigma}, i_* \mathcal{G}_{s-1,\sigma'}) = 0. \]

Considering the canonical filtration on \( F_{\geq s,\sigma} \) we see that for this in turn it suffices to show that for \( s \neq t \)

\[ \text{Ext}^t_B(i_* \mathcal{G}_{t,\sigma}, i_* \mathcal{G}_{s,\sigma'}) \simeq \text{Ext}^*_A(i^* i_* \mathcal{G}_{t,\sigma}, \mathcal{G}_{s,\sigma'}) = 0. \]

This follows from \( 4.3 \) (i) and \( 4.3 \). \[ \square \]

Proposition 4.7. Let \( F \in D(B) \) be a complex satisfying the following:

(i) For every \( a \in A(k) \) with lift \( s \in S(k) \) we have \( t^*_u F \simeq F \otimes \mathcal{R}_s \) for a line bundle \( \mathcal{R}_s \)

on \( B \) restricting to the line bundle \( \mathcal{L}_{v(s)} \) in \( 3.2 \).

(ii) The complex \( F \) is set-theoretically supported on \( A \).
(iii) \( \text{End}(F) = k \).
(iv) \( \text{Ext}^i(F, F) = 0 \) for \( i < 0 \).

Then there exists a line bundle \( \mathcal{L} \) defining the map \( \alpha' \) such that \( F \simeq i_* \pi_* \mathcal{L}[s] \) for an integer \( s \).

**Proof.** Note first of all that conditions (i)-(iii) imply that each cohomology sheaf \( \mathcal{H}^s(F) \) is of the form \( i_* G_s \) for a vector bundle \( G_s \) on \( A \) which is semi-homogeneous with respect to \( (3.1.1) \). To see this it suffices to show that each \( \mathcal{H}^s(F) \) is scheme-theoretically supported on \( A \). Consider the quotient \( B/A \), let \( U = \text{Spec}(R) \subset B/A \) be an affine neighborhood of 0, and let \( B_U \) denote the preimage of \( U \). Since \( F \) is set-theoretically supported on \( A \) the sheaf \( \mathcal{H}^s(F) \) is the pushforward of its restriction to \( B_U \). If \( m \subset R \) denotes the maximal ideal corresponding to the origin then it suffices to show that \( m \) annihilates \( \mathcal{H}^s(F)|_{B_U} \). This follows from noting that the action of \( R \) on \( \mathcal{H}^s(F)|_{B_U} \) factors through the map \( R \to \text{Ext}^0_{B_U}(F|_{B_U}, F|_{B_U}) \simeq \text{Ext}^0(F, F) = k \).

Observe also that in light of (4.6) and (iii) we have \( \Sigma(F) = \{ \sigma \} \) for a single element \( \sigma \in A'(k)/\alpha'(%Ker(\pi)) \).

Next we show that condition (iv) implies that \( F \) is, up to a shift, a sheaf. Suppose to contrary that \( F \) is concentrated in more than one degree and let \( n \) (resp. \( m \)) be the top (resp. bottom) nonzero degree so \( n > m \). Then
\[
\text{Ext}^{n-m}(F, F) \simeq \text{Ext}^0(i_* G_n, i_* G_m),
\]
and the natural map
\[
\text{Ext}^0(i_* G_n, i_* G_m) \to \text{Ext}^0(i_* G_n, i_* G_m)
\]
is injective since a map \( G_n \to G_m \) is determined by its pushforward. Now since \( \Sigma(F) \) is a point we must have \( \Sigma(G_n) = \Sigma(G_m) \) so by (4.3) (ii) there exists a nonzero map \( G_n \to G_m \), a contradiction.

Thus after possibly applying a shift, \( F \) is a sheaf and by (3.8) we have
\[
F \simeq i_* \pi_* (U \otimes \mathcal{L}_\sigma \otimes N')
\]
for a unipotent vector bundle \( U \). This vector bundle \( U \) must be of rank 1, for otherwise it has nonscalar endomorphisms contradicting (iii). We conclude that \( F \) has the desired form. \( \Box \)

### 5. Moduli of weak point complexes

#### 5.1. Let \( X \) be a smooth projective geometrically connected scheme over a field \( k \) of dimension \( d \), and let \( D(X) \) be its bounded derived category of coherent sheaves.

In the literature one finds various possible notions of point objects in \( D(X) \) (see [2, 2.1], [5, 5.1]). To avoid confusion we introduce the following definition which captures the necessary properties for \( 1.1 \).
**Definition 5.2.** A *weak point object* is a complex $F \in D(X)$ satisfying the following conditions:

(i) $\text{Ext}^i(F, F) = 0$ for $i < 0$.

(ii) $\text{Ext}^0(F, F) = k$.

(iii) $\text{Ext}^1(F, F)$ has dimension $\leq d$.

**5.3.** For a $k$-scheme $S$ we define a *relative weak point object* on $X_S$ to be an $S$-perfect complex $F \in D(X_S)$ such that for every geometric point $\bar{s} \to S$ the induced object $F_{\bar{s}} \in D(X_{\kappa(\bar{s})})$ is a weak point object.

Let $\mathcal{P}$ be the fibered category over the category of $k$-schemes which to any $S$ associates the groupoid of relative weak point objects in $D(X_S)$.

**Lemma 5.4.** The fibered category $\mathcal{P}$ is an algebraic stack locally of finite type over $k$.

**Proof.** Let $\mathcal{D}$ be the fibered category which to any scheme $S$ associates the groupoid of objects $K \in D(X_S)$ such that for all geometric points $\bar{s} \to S$ we have $\text{Ext}^i(K_{\bar{s}}, K_{\bar{s}}) = 0$ for $i < 0$. By [7, Theorem on p. 176] the fibered category $\mathcal{D}$ is an algebraic stack locally of finite type over $k$. We claim that the natural inclusion $\mathcal{P} \hookrightarrow \mathcal{D}$ is represented by locally closed immersions. Indeed the substack of $\mathcal{D}$ satisfying (iii) is open by semi-continuity and within this substack condition (ii) is represented by locally closed immersions by [16, Tag 0BDL].

**Remark 5.5.** Note that $\mathcal{P}$ is a $\mathbb{G}_m$-gerbe over an algebraic space, which we will denote by $P$.

**5.6.** For any object $F \in D(X)$ there is an induced map $\kappa_F : HH^1(X) \to \text{Ext}^1(F, F)$ defined as follows, where $HH^1(X)$ denotes the first Hochschild cohomology of $X$. By definition we have $HH^1(X) = \text{Hom}_{X \times X}(\Delta_X \ast O_X, \Delta_X \ast O_X[1])$, and therefore an element $\alpha \in HH^1(X)$ defines a morphism of functors $\alpha : \text{id}_{D(X)} \to \text{id}_{D(X)}[1]$. Evaluating on $F$ we get an element $\alpha(F) : F \to F[1]$, or equivalently an element of $\text{Ext}^1(F, F)$. The map $\kappa_F$ sends $\alpha$ to $\alpha(F)$.

**Remark 5.7.** Note that if $x \in X(k)$ is a $k$-point with associated skyscraper sheaf $O_x$ then we have $\text{Ext}^1(O_x, O_x) \simeq T_X(x)$ and in characteristic 0 the map $\kappa_{O_x}$ gets identified, via the HKR isomorphism [17], with a map $H^1(X, O_X) \oplus H^0(X, T_X) \to T_X(x)$. 
This map is simply the projection to $H^0(X, T_X)$, followed by the natural map $H^0(X, T_X) \to T_X(x)$.

5.8. For $F \in \mathcal{P}(k)$ the map $\kappa_F$ has the following geometric interpretation. Following [8, 5.4] let $\mathcal{R}_X^0$ denote the fibered category over $k$ whose fiber over a scheme $S$ is the groupoid of objects $Q \in D((X \times X)_S)$ satisfying the "Rouquier condition". So $\mathcal{R}_X^0$ is a $\mathbf{G}_m$-gerbe over

$$R_X^0 := \text{Aut}_X^0 \times \text{Pic}_X^0,$$

and such a complex $Q$ is given by $\Gamma_S^*L$, where $\sigma$ is an automorphism of $X$ and $L$ is a line bundle on $X$ such that the pair $(\sigma, L)$ defines a point of $\text{Aut}_X^0 \times \text{Pic}_X^0$.

The given object $F$ defines a morphism of stacks

$$(5.8.1) \quad \mathcal{R}_X^0 \to \mathcal{D}, \quad (\sigma, \mathcal{L}) \mapsto (\sigma^*F) \otimes \mathcal{L},$$

where $\mathcal{D}$ is as in the proof of 5.4. The tangent space at the origin of $\mathcal{R}_X^0$ corresponds to isomorphism classes of deformations of $\Delta_\times \mathcal{O}_X$ in $D(X \times X)$, which is given by

$$HH^1(X) = \text{Ext}^1(\Delta_\times \mathcal{O}_X, \Delta_\times \mathcal{O}_X),$$

and the tangent space to $[F] \in \mathcal{D}(k)$ is given by

$$\text{Ext}^1(F, F).$$

The map $\kappa_F$ is induced by (5.8.1) by passing to the tangent spaces.

Note also that since $\mathcal{R}_X^0$ is a $\mathbf{G}_m$-gerbe over $R_X^0$ we can also view $HH^1(X)$ as the tangent space of $R_X^0$ at the origin.

6. Proof of 1.1

It suffices to prove the theorem after making a base change to an algebraic closure of $k$, so without loss of generality we may assume that $k$ is algebraically closed.

We first show that if $F$ satisfies conditions (i) and (ii) then $F$ has the form indicated in 1.1.

6.1. Let $S_X \subset R_X^0 = A \times A^t$ be the stabilizer of the point $[F] \in P$ (recall that $P$ is the coarse space of the $\mathbf{G}_m$-gerbe $\mathcal{P}$). The tangent space of $S_X$ at the origin is then the kernel of $\kappa_F$. Since $\text{Ext}^1(F, F)$ has dimension $\leq d$ by assumption, it follows that the dimension of $S_X$ is equal to some integer $g \geq d$ (here we use the fact that $k$ has characteristic 0). Let $S_X^0 \subset S_X$ be the connected component of the identity, and let

$$S_X' \subset A$$

be the image of $S_X^0$ under the first projection and set

$$K := \text{Ker}(S_X^0 \to S_X').$$

So $S_X'$ is an abelian subvariety of $A$ consisting of those points $a \in A$ for which $t_a^*F \simeq F \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$ algebraically equivalent to 0 (the reason for considering the space $P$ is to make this loose statement precise). Let $g'$ denote the dimension of $S_X'$. 

6.2. Let $Z \subset X$ be the set-theoretic support of $F$, viewed as a subscheme with the reduced structure. Since $Z$ is invariant under $S'_X$ every irreducible component of $Z$ has dimension $\geq g'$.

Let $Z' \subset Z$ be an irreducible component of dimension $d'$ and let $\tilde{Z} \to Z'$ be a surjective morphism with $\tilde{Z}$ smooth and projective over $k$ of dimension $d'$ (such a $\tilde{Z}$ exists using the theory of alterations \[3\]). If $\tilde{T}$ denotes the Albanese torsor of $\tilde{Z}$ then since $X$ is its own Albanese torsor the map $\tilde{Z} \to X$ extends to a map

$$\rho : \tilde{T} \to X.$$

Since $Z'$ has dimension $d'$ it follows that this map has image of dimension $\geq d'$. Taking duals we find that the image of

$$A' = \text{Pic}^0_X \to \text{Pic}^0_{\tilde{T}}$$

has dimension $\geq d'$.

Now observe that the connected component of the identity $K^0$ of $K \subset A'$ is in the kernel of this map. Indeed let $\tilde{F}$ be the pullback of $F$ to $\tilde{Z}$. Then for some integer $i$ the sheaf $\mathcal{H}^i(\tilde{F})$ has generic rank $r_i \neq 0$. Now if $\mathcal{L}$ is a line bundle on $X$ corresponding to a point of $K^0$ then we have

$$\tilde{F} \otimes \mathcal{L}|_{\tilde{Z}} \simeq \tilde{F}.$$

Taking determinants we find that

$$\det(\mathcal{H}^i(\tilde{F})) \otimes \mathcal{L}^{\otimes r_i}|_{\tilde{Z}} \simeq \det(\mathcal{H}^i(\tilde{F})),$$

and therefore $\mathcal{L}|_{\tilde{Z}}$ is a torsion line bundle. Since $K^0$ is connected it follows that $\mathcal{L}|_{\tilde{Z}}$ is trivial.

We conclude that the dimension of $K$ is $\leq d - d'$. Therefore

$$g = g' + \dim(K) \leq d' + (d - d') = d$$

with equality if and only $g' = d'$ and $\dim(K) = d - d'$. Since $g \geq d$ we conclude that $d' = g'$ and that each irreducible component of $Z$ is a $S'_X$-orbit. Since $F$ is a point sheaf the support $Z$ is connected and we conclude that $Z$ is, in fact, a torsor under $S'_X$.

6.3. We now apply the results of the previous sections with \[(3.1.1)\] the diagram

$$\begin{array}{ccc}
S_X^0 & \xrightarrow{u} & S^n \\
\downarrow & & \downarrow \\
S' & & S'.
\end{array}$$

We then conclude from \[4.7\] that $F$ has the desired form.

6.4. For the converse statement in \[1.1\] proceed as follows. Let $h$ be the dimension of $Z$ so that $A/H$ has dimension $g - h$. We have

$$\text{Ext}^*_X(i_*\mathcal{E}, i_*\mathcal{E}) \simeq \text{Ext}^*_Z(i^*i_*\mathcal{E}, \mathcal{E}) \simeq H^*(Z, (i^*i_*\mathcal{E})^\vee \otimes \mathcal{E}).$$

This implies that

$$\text{Ext}^0_X(i_*\mathcal{E}, i_*\mathcal{E}) \simeq H^0(Z, \mathcal{E}^\vee \otimes \mathcal{E}) \simeq k,$$

since $\mathcal{E}$ is geometrically simple. To calculate $\text{Ext}^1$, note that by the proof of \[4.5\] we have a distinguished triangle

$$\mathcal{E}^\vee \otimes \mathcal{E} \to \tau_{\leq 1}(i^*i_*\mathcal{E})^\vee \otimes \mathcal{E} \to (\mathcal{E}^\vee \otimes \mathcal{E})^{\oplus (g-h)}[-1] \to \mathcal{E}^\vee \otimes \mathcal{E}[1].$$
It follows that we an exact sequence
\[ \text{Ext}^1_Z(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^1_X(i_\ast \mathcal{E}, i_\ast \mathcal{E}) \rightarrow \text{Ext}^0_Z(\mathcal{E}, \mathcal{E}) \oplus (g-h). \]

Now by [9, 5.8] the dimension of \( \text{Ext}^1_Z(\mathcal{E}, \mathcal{E}) \) is \( h \) and since \( \mathcal{E} \) is simple the dimension of \( \text{Ext}^0(\mathcal{E}, \mathcal{E}) \) is 1. We conclude that the dimension of \( \text{Ext}^1_X(i_\ast \mathcal{E}, i_\ast \mathcal{E}) \) is less than or equal to \( g \) as desired.

This completes the proof of 1.1. □

**Example 6.5.** Let \( k \) be an algebraically closed field, let \( i : A \hookrightarrow B \) be a closed immersion of abelian varieties over \( k \), and let \( \mathcal{E} \) be a simple semi-homogeneous vector bundle on \( A \). For \( F = i_\ast \mathcal{E} \) we can then describe the group scheme \( S_X \) in the above discussion as follows (a consequence of 1.1 is that this example is, in fact, general). By [9, 5.6 and 5.8] (or use 3.8) there is an isogeny \( \pi : A' \rightarrow A \) and a line bundle \( N' \) on \( A' \) such that \( \alpha' : \lambda_{\pi^*} : A' \rightarrow A'' \) restricts to an injection on \( \text{Ker}(\pi) \) and such that \( \mathcal{E} \simeq \pi^* N' \). The abelian variety \( S_X \) can then be described as follows.

The \( k \)-points of \( S_X \) are pairs \( (b, [\mathcal{L}]) \in B(k) \times B^t(k) \) such that there exists an isomorphism \( t_b^!(F) \simeq F \otimes \mathcal{L} \). Such an isomorphism necessarily preserves the scheme-theoretic support of \( F \), which implies that \( b \in A(k) \) and that we have an isomorphism \( \sigma : t_b^! \mathcal{E} \simeq \mathcal{E} \otimes i^* \mathcal{L} \) on \( A \).

Now we have \( \mathcal{E} \otimes i^* \mathcal{L} \simeq (\pi_\ast N') \otimes i^* \mathcal{L} \simeq \pi_\ast (N' \otimes \mathcal{L}|_{A'}) \).

By adjunction the map \( \sigma \) corresponds to a map over \( A' \)
\[ \pi_\ast t_b^! \mathcal{E} \simeq \oplus_{b' \in \pi^{-1}(b)} t_{b'}^! N' \rightarrow N' \otimes \mathcal{L}|_{A'}. \]

From this we conclude that there exists a unique element \( b' \in \pi^{-1}(b) \) such that \([\mathcal{L}]_{A'} = \alpha'(b')\) in \( A''(k) \). From this we conclude that there is a cartesian diagram
\[
\begin{array}{ccc}
S_X & \rightarrow & A' \\
\downarrow & & \downarrow (\text{id}_{A'}, \alpha') \\
A' \times B^t & \rightarrow & A' \times A''.
\end{array}
\]

with the map to \( B \times B^t \) given by the composition
\[
S_X \rightarrow A' \times B^t \xrightarrow{i_\ast \times \pi \times \text{id}_{B^t}} B \times B^t.
\]

Equivalently, we have a cartesian diagram
\[
\begin{array}{ccc}
S_X & \rightarrow & A' \\
\downarrow & & \downarrow \alpha' \\
B^t & \rightarrow & A''.
\end{array}
\]

From this description it follows that \( S_X \) is connected. Indeed dualizing the exact sequence
\[ 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0 \]
we get an exact sequence (using that \( B \rightarrow B/A \) has geometrically connected fibers)
\[ 0 \rightarrow (B/A)^t \rightarrow B^t \rightarrow A^t \rightarrow 0. \]
We therefore get an exact sequence

\[ 0 \to (B/A)^t \to S_X \to A^t \times_{A^t, \alpha'} A' \to 0. \]

Now \( A^t \to A'^t \) is the cover corresponding by duality to \( \text{Ker}(\pi) \hookrightarrow A' \). More concretely, if we write \( \mathcal{P}_{A', \alpha'} \) for the restriction of the Poincaré bundle on \( A' \times A'^t \) to \( \{ a' \} \times A'^t \) then \( A^t \) is described, as a scheme over \( A'^t \), by

\[ \bigoplus_{a' \in \text{Ker}(\pi)} \mathcal{P}_{A', a'}. \]

with algebra structure given by the biextension structure on \( \mathcal{P}_{A'} \). It follows from this that the pullback of this cover along \( \alpha' \) is the cover of \( A' \) given by

\[ \bigoplus_{a' \in \text{Ker}(\pi)} t^* a' \mathcal{N}' \otimes \mathcal{N}'^{-1}. \]

That is, \( A^t \times_{A^t, \alpha'} A' \to A' \) is the cover given by the composition

\[ \text{Ker}(\pi) \longrightarrow A' \xrightarrow{\alpha'} A'^t. \]

Since this map is injective by assumption it follows that \( A^t \times_{A^t, \alpha'} A' \), and therefore also \( S_X \), is connected. From this it also follows that we have an exact sequence

\[ 0 \to A'^t/\alpha'(\text{Ker}(\pi)) \to S_X^t \to (B/A) \to 0. \]

7. Lagrangians

As noted in the introduction our main result 1.1 is closely related to Polishchuk’s work in [13, 14]. In this section we discuss this connection in more detail.

Let \( k \) be an algebraically closed field of characteristic 0.

7.1. Recall that if \( S \) is an abelian variety over \( k \) and \( \mathcal{N} \) is a line bundle on \( S \) then there is an associated biextension \( \Lambda(\mathcal{N}) \) [1, VII] (see also [12, II.10.3]). This biextension is obtained from the Poincaré bundle \( \mathcal{P} \) on \( S \times S'^t \), endowed with its canonical structure of a biextension, by pullback along the map \( \text{id}_{S \times S'} : S \times S' \to S \times S'^t \).

If \( A/k \) is an abelian variety then a Lagrangian pair in \( A \times A'^t \) [13, 2.2.2] consists of the following data:

(i) An abelian subvariety \( i : S \subset A \times A^t \) of the same dimension as \( A \).

(ii) A line bundle \( \mathcal{N} \) on \( S \) together with an isomorphism \( \Lambda(\mathcal{N}) \simeq i^* \mathcal{B}_A \) of biextensions, where \( \mathcal{B}_A \) is the biextension over \( (A \times A^t) \times (A \times A^t) \) obtained by pulling back the Poincaré bundle along the projection to the first and fourth factor.

By [13, 2.4.5] (applied with \( (Y, \alpha) = (\{0\} \times A^t, \Theta(\{0\} \times A^t)) \)) and [13, 2.3.2] a Lagrangian pair \( (S, \mathcal{N}) \) gives rise to a coherent sheaf \( S_{(S, \mathcal{N})} \) on \( A \) equipped with an isomorphism over \( A \times S \)

\[ m^* S_{(S, \mathcal{N})} \otimes \text{pr}_{13} \mathcal{B}^{-1}_A \otimes \text{pr}_2 \mathcal{N}^{-1} \simeq \text{pr}_1^* S_{(S, \mathcal{N})}, \]

where \( m : A \times S \to A \) is the composition of the projection map \( A \times S \to A \times A \) and the addition map, \( \text{pr}_{13} \mathcal{B}_A \) is the pullback of the Poincaré bundle along the first and third projections \( A \times S \to A \times A^t \), and \( \text{pr}_2 \) is given by the projection \( A \times S \to S \) followed by \( u : S \to A \).
Remark 7.2. The formula (7.1.1) differs from the one in [13, 2.4] since we are taking inverses of the line bundles, so what we denote by $S_{(S,N)}$ corresponds to the dual of the objects in [13]. This sign convention will be useful in the proofs below.

Lemma 7.3. There exists a subabelian variety $H \subset A$ and an $H$-torsor $i: Z \hookrightarrow A$ such that $S_{(S,N)} \cong i^*\mathcal{E}$ for a simple semi-homogeneous vector bundle $\mathcal{E}$ on $Z$. In particular, $S_{(S,N)}$ satisfies conditions (i) and (ii) in 1.1.

Proof. Let $\mathcal{D}$ be as in the proof of 5.4. Again by [16, Tag 0BDL] there is a locally closed substack $\mathcal{D}' \subset \mathcal{D}$ classifying objects $K \in D(A)$ with $\text{End}(K) = k$, and $\mathcal{D}'$ is a $\mathbb{G}_m$-gerbe over an algebraic space $D'$. The object $S_{(S,N)}$ defines an object in $\mathcal{D}'(k)$ since it is “endosimple” in the terminology of [13]. We can therefore consider the stabilizer $S_X \subset A \times A'$ of the induced point of $D'$, as in 6.1. By [13, 2.4.10] this stabilizer equals $S$ and therefore has dimension equal to the dimension of $A$. Applying the argument of section 6, noting that condition (ii) is used in the argument to conclude that the dimension of $S_X$ is $\geq \dim(A)$, we then obtain the lemma.

Theorem 7.4. The map of sets

$$
\{\text{Lagrangian pairs } (S,N)\} \to \{\text{coherent sheaves } \mathcal{F} \text{ satisfying (i) and (ii) in 1.1}\}
$$

is a bijection.

Proof. Starting with a coherent sheaf $\mathcal{F}$ satisfying (i) and (ii) in 1.1 we construct the Lagrangian pair $(S,N)$ mapping to it as follows.

Let $Z \subset A$ be the support of $\mathcal{F}$, so $Z$ is a torsor under an abelian variety $i: H \subset A$, which is a quotient $S_X \twoheadrightarrow H$ of the subgroup scheme $S_X \subset A \times A'$ defined as in 6.1. Let $m: A \times S_X \to A$ be the action map given by

$$(a, a', [L]) \mapsto a + a',$$

and let $\mathcal{B}$ be the line bundle on $A \times S_X$ given by pulling back the Poincaré bundle on $A \times A'$ along $\text{pr}_{13}: A \times S_X \to A \times A'$.

To prove the theorem it suffices to prove the following claim, the argument for which occupies the remainder of the proof.

Claim 7.5. There exists a unique line bundle $\mathcal{N}$ on $S_X$ for which $(S_X, \mathcal{N})$ is a Lagrangian pair and such that there exists an isomorphism of sheaves on $A \times S_X$ as in 7.1.1 with $S_{(S_X, \mathcal{N})}$ replaced by $\mathcal{F}$.

If $\mathcal{N}_1$ and $\mathcal{N}_2$ are two invertible sheaves on $S_X$ for which $(S_X, \mathcal{N}_i)$ $(i = 1, 2)$ are both Lagrangian pairs then the biextension $\Lambda(\mathcal{N}_1 \otimes \mathcal{N}_2^{-1})$ is trivial and therefore $\mathcal{N}_2 \simeq \mathcal{N}_1 \otimes \mathcal{R}$ for a translation invariant line bundle $\mathcal{R}$ on $S_X$ (recall that $\Lambda(\mathcal{N})$ is the biextension over $S_X \times S_X$ given by $m^*\mathcal{N} \otimes \text{pr}_{1}^{*}\mathcal{N}^{-1} \otimes \text{pr}_{2}^{*}\mathcal{N}^{-1}$). From the formula (7.1.1) for both $\mathcal{N}_1$ and $\mathcal{N}_2$ we get an isomorphism $\text{pr}_{1}^{*}\mathcal{F} \simeq \text{pr}_{1}^{*}\mathcal{F} \otimes \text{pr}_{2}^{*}\mathcal{R}$, which defines a nonzero map

$$
\text{pr}_{2}^{*}\mathcal{R} \to \mathcal{R}\text{Hom}(\text{pr}_{1}^{*}\mathcal{F}, \text{pr}_{1}^{*}\mathcal{F}).
$$
By adjunction and using property (i) in \[1\] we get a nonzero map 
$$\mathcal{R} \to \text{pr}_2^* \mathcal{R}^0 \mathcal{H}om(\text{pr}_1^* \mathcal{F}, \text{pr}_1^* \mathcal{F}) \simeq \mathcal{O}_{\mathcal{S}}.$$ 
Since \(\mathcal{R}\) is algebraically equivalent to 0 we conclude that this map is an isomorphism and that \(\mathcal{R}\) is trivial. This proves the uniqueness part of the claim.

For the existence part, note that there exists an isogeny \(\pi : H' \to H\), a line bundle \(\mathcal{N}'\) on \(H'\) for which the map \(\lambda_{\mathcal{N}'} : H' \to H''\) is injective on \(\text{Ker}(\pi)\), and a point \(a \in A(k)\) such that \(\mathcal{F} = t_a^* i_* \pi_* \mathcal{N}'\).

By the discussion in \[6.5\] we have \(\mathcal{S}_X = A^t \times_{H, \alpha'} H'\). Now consider the commutative diagram 
$$
\begin{array}{ccc}
H' \times S_X & \xrightarrow{pr_1} & H' \\
\downarrow{id} & \downarrow{id \times \alpha'} & \downarrow{id \times \alpha'} \\
H' \times A^t & \xrightarrow{id \times i} & H' \times H'' \\
\downarrow{A \times A^t} & \downarrow{A \times A^t} & \downarrow{A \times A^t} \\
A \times A^t,
\end{array}
$$
where the square is cartesian. Now we have an isomorphism of biextensions \(\mathcal{P}_A|_{H' \times A^t} \simeq (1 \times i')^* \mathcal{P}_H\), so we get an isomorphism of biextensions \(\text{pr}_1^* \mathcal{P}_A \simeq \text{pr}_1^* \Lambda(\mathcal{N}')\) over \(H' \times S_X\). \(\text{pr}_1^* \mathcal{P}_A \simeq \text{pr}_1^* \Lambda(\mathcal{N}')\) Therefore if \(\mathcal{N}\) denotes the pullback of \(\mathcal{N}'\) along the projection \(S_X \to H'\) then \((S_X, \mathcal{N})\) is a Lagrangian pair. Furthermore, writing out this isomorphism we get 
$$
\text{pr}_1^* \mathcal{P}_A \simeq m^* \mathcal{N}' \otimes \text{pr}_1^* \mathcal{N}'^{-1} \otimes \text{pr}_2^* \mathcal{N}'^{-1},
$$
which rearranges to 
$$
m^* \mathcal{N}' \otimes \text{pr}_1^* \mathcal{P}_A^{-1} \otimes \text{pr}_2^* \mathcal{N}'^{-1} \simeq \text{pr}_1^* \mathcal{N}'.
$$

Pushing forward to \(A \times S_X\) we see that the claim holds for \(i_* \pi_* \mathcal{N}'\).

To handle the case of \(\mathcal{F} = t_a^* i_* \pi_* \mathcal{N}'\) for some \(a \in A\) consider the commutative diagram 
$$
\begin{array}{ccc}
A \times S & \xrightarrow{m} & A \\
\downarrow{t_a \times \text{id}} & & \downarrow{t_a} \\
A \times S & \xrightarrow{m} & A,
\end{array}
$$
and note that using the biextension structure on \(\mathcal{P}_A\) we have 
$$
(t_a \times \text{id})^* \text{pr}_1^* \mathcal{P}_A \simeq \text{pr}_1^* \mathcal{P}_A \otimes \text{pr}_2^* \mathcal{P}_{A,a},
$$
where \(\mathcal{P}_{A,a}\) is the line bundle on \(A^t\) obtained by restriction \(\mathcal{P}_A\) to \(\{a\} \times A^t\). From this it follows that taking \(\mathcal{N}\) equal to the pullback of \(\mathcal{N}'\) tensored with the pullback of \(\mathcal{P}_{A,a}^{-1}\) verifies the claim for \(t_a^* i_* \pi_* \mathcal{N}'\).

This completes the proof of the claim and \[7.4\].

\begin{flushright}
\(\Box\)
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