Spiked separable covariance matrices and principal components

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Abstract

We study a class of separable sample covariance matrices of the form \( \tilde{Q}_1 := \tilde{A}^{1/2} \tilde{B} X^* \tilde{A}^{1/2} \). Here \( \tilde{A} \) and \( \tilde{B} \) are positive definite matrices whose spectrums consist of bulk spectrums plus several spikes, i.e. larger eigenvalues that are separated from the bulks. Conceptually, we call \( \tilde{Q}_1 \) a spiked separable covariance matrix model. On the one hand, this model includes the spiked covariance matrix as a special case with \( \tilde{B} = I \). On the other hand, it allows for more general correlations of datasets. In particular, for spatio-temporal dataset, \( \tilde{A} \) and \( \tilde{B} \) represent the spatial and temporal correlations, respectively.

In this paper, we study the outlier eigenvalues and eigenvectors, i.e. the principal components, of the spiked separable covariance model \( \tilde{Q}_1 \). We prove the convergence of the outlier eigenvalues \( \tilde{\lambda}_i \) and the generalized components (i.e. \( \langle v, \tilde{\xi}_i \rangle \) for any deterministic vector \( v \)) of the outlier eigenvectors \( \tilde{\xi}_i \), with optimal convergence rates. Moreover, we also prove the delocalization of the non-outlier eigenvectors. We state our results in full generality, in the sense that they also hold near the so-called BBP transition and for degenerate outliers. Our results highlight both the similarity and difference between the spiked separable covariance matrix model and the spiked covariance model matrix in [8]. In particular, we show that the spikes of both \( \tilde{A} \) and \( \tilde{B} \) will cause outliers of the eigenvalue spectrum, and the eigenvectors can help to select the outliers that correspond to the spikes of \( \tilde{A} \) (or \( \tilde{B} \)).

Contents

1 Introduction 2
2 Definitions and main results 4
3 Statistical estimation for spiked separable covariance matrices 14
4 Basic tools 20
5 Outlier eigenvalues 32
6 Outlier eigenvectors 44
7 Non-outlier eigenvectors 56

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1 Introduction

High-dimensional data obtained at space-time points has been increasingly employed in various scientific fields, such as geophysical and environmental sciences [32, 36], wireless communications [28, 52, 54], medical imaging [49] and financial economics [11, 12, 59]. The structural assumption of separability is a popular assumption in the analysis of spatio-temporal data. Although this assumption does not allow for space-time interactions in the covariance matrix, in many real data applications (for instance the study of Irish wind speed [24]), the covariance matrix can be well approximated using separable covariance matrices by solving a nearest Kronecker product for a space-time covariance matrix problem (NKPST) [23].

Consider a \(p \times n\) data matrix \(Y\) of the form
\[
Y = \hat{A}^{1/2}X\hat{B}^{1/2},
\]
where \(X = (x_{ij})\) is a \(p \times n\) random matrix with independent entries such that \(E x_{ij} = 0\) and \(E|x_{ij}|^2 = n^{-1}\), and \(\hat{A}\) and \(\hat{B}\) are respectively \(p \times p\) and \(n \times n\) deterministic positive-definite matrices. We say \(Y\) has a separable covariance structure because the joint spatio-temporal covariance of \(Y\), viewed as an \((np)\)-dimensional vector consisting of the columns of \(Y\) stacked on top of one another, is given by a separable form \(\hat{A} \otimes \hat{B}\), where \(\otimes\) denotes the Kronecker product. This model has different names and meanings in different fields. For example, in wireless communications [28, 52, 54], especially for the multiple-input-multiple-output (MIMO) systems, the \(\hat{A}\) and \(\hat{B}\) represent the covariances between the receiver antennas and between the transmitter antennas, respectively. Also, \(Y\) is called the doubly-heteroscedastic noise in [35] for matrix denoising and the separable idiosyncratic part in factor model [41]. However, as a convention, in this paper we always say that the row indices of \(Y\) correspond to spatial locations while the column indices correspond to time points. Moreover, we shall call \(\hat{A}\) and \(\hat{B}\) as spatial and temporal covariance matrices, respectively. In this paper, we are mainly interested in the so-called separable sample covariance matrix \(\hat{Q} = YY^*\) for the above separable data model \(Y\).

One special case of the separable covariance matrix is the classic sample covariance matrix with \(\hat{B} = I_n\), which has been a central object of study in multivariate statistics. In the null case with \(\hat{A} = I_p\), it is well-known that the empirical spectral distribution (ESD) of \(\hat{Q}\) converges to the celebrated Marchenko-Pastur (MP) law [37], whose rightmost edge \(\lambda_r\) gives the asymptotic location of the largest eigenvalue. Later on the convergence result of the ESD is extended to various settings with general positive definite covariance matrices \(\hat{A}\): we refer the readers to the monograph [3] and the review paper [46]. For the extremal eigenvalues, it was proved in a series of papers that under an \(n^{2/3}\) scaling, the distribution of the largest eigenvalue \(\lambda_1\) around \(\lambda_r\) converges to the Tracy-Widom distribution [50, 51]. This result is commonly referred to as the edge universality, in the sense that it is independent of the detailed distribution of the entries of \(X\). The Tracy-Widom distribution of the extremal eigenvalue was first proved in [20] for sample covariance matrices with \(\hat{A} = I_p\) and Gaussian \(X\) (i.e. the entries of \(X\) are i.i.d. Gaussian), whereas the edge universality for \(X\) with generally distributed entries was proved later in [15]. When \(\hat{A}\) is a general non-scalar matrix, the Tracy-Widom distribution for the extremal eigenvalues was first proved for the case with i.i.d. Gaussian \(X\) in [17, 40]; the edge universality was later proved under various moment assumptions on the entries \(x_{ij}\) [5, 13, 30, 54]. Finally, for the (non-outlier) sample eigenvectors, the completely delocalization [30, 43], quantum unique ergodicity [8], distribution of the eigenvector components [13] and convergence of eigenvector empirical spectral distribution [54] have been constructed.

In the statistical study of sample covariance matrices, a popular model is the Johnstone’s spiked covariance matrix model [20]. In this model, a few spikes, i.e., eigenvalues detached from the bulk eigenvalue spectrum, are added to \(\hat{A}\). Since the seminal work of Baik, Ben Arous and Péché [4], it is now well-understood that the extremal eigenvalues undergo a so-called BBP transition along with the change of the strength of the spikes. Roughly speaking, there is a critical value such that the following properties hold: if the strength...
of the spike is smaller than the critical value, then the extremal eigenvalue of the spiked sample covariance matrix will stick to the right endpoint of the bulk eigenvalue spectrum (and hence is not an outlier), and the corresponding sample eigenvector will be delocalized; otherwise, if the strength of the spike is larger than the critical value, then the associated eigenvalue will jump out of the bulk eigenvalue spectrum, and the outlier sample eigenvector will be concentrated on a cone with axis parallel to the population eigenvector with an (almost) deterministic aperture. For an extensive overview of such results, we refer the reader to [8, 11, 15].

One purpose of this paper is to generalize some important results for sample and spiked covariance matrices to the more general separable and spiked separable covariance matrices. The convergence of the ESD of separable covariance matrices to a limiting law were shown in [47, 53, 60]. The edge universality and delocalization of eigenvectors have been proved by the second author [58] for separable covariance matrices without spikes on \( \hat{A} \) and \( \hat{B} \). The convergence of VESD of separable covariance matrices was proved in [57], which is an extension of the result in [55]. Then the main goal of this paper is to study the outlier eigenvalues and eigenvectors of separable covariance matrices with spikes on both the spatial and temporal covariance matrices \( \hat{A} \) and \( \hat{B} \), which we shall refer to as the spiked separable covariance matrices. The precise definition is given in Section 2.4. Roughly speaking, to define a general spiked separable covariance model, we first introduce the reference matrix \( Q_1 = A^{1/2}XBX^*A^{1/2} \), where \( A \) and \( B \) are covariance matrices without spikes. Then the spiked covariance matrices \( \hat{A} \) and \( \hat{B} \) are finite rank perturbations of \( A \) and \( B \), respectively; see the model in (2.7) and (2.8).

In this paper, we derive precise large deviation estimates on the outlier eigenvalues and the generalized components of the outlier eigenvectors. In particular, our results give both the first order limits and the (almost) optimal rates of convergence of the relevant quantities. We now briefly describe our results. Let \( \hat{A} = \sum_{i=1}^{r+1} \tilde{\sigma}_i^a v_i^a(v_i^a)^* \) and \( \hat{B} = \sum_{i=1}^{r+1} \tilde{\sigma}_i^b v_i^b(v_i^b)^* \) be the eigendecomposition of \( \hat{A} \) and \( \hat{B} \), respectively, where we label the eigenvalues in descending order. We assume that the spiked eigenvalues are \( \{\tilde{\sigma}_i^a\}_{i=1}^{s} \) and \( \{\tilde{\sigma}_i^b\}_{i=1}^{b} \), where \( r \) and \( s \) are some fixed integers. Then there exist a threshold \( \ell_a \) (or \( \ell_b \)) such that \( \tilde{\sigma}_i^a \) (or \( \tilde{\sigma}_i^b \)) gives rise to outliers of \( \hat{Q}_1 \) if and only if \( \tilde{\sigma}_i^a > \ell_a \) (or \( \tilde{\sigma}_i^b > \ell_b \)). Moreover, the outlier lies around a fixed location determined by the spike \( \tilde{\sigma}_i^a \) (or \( \tilde{\sigma}_i^b \)); see Theorem 2.12. If \( \tilde{\sigma}_i^a - \ell_a \gg n^{-1/3} \) or \( \tilde{\sigma}_i^b - \ell_b \gg n^{-1/3} \), i.e. the spike is supercritical, then the outlier will be well-separated from the bulk spectrum (which is also the eigenvalue spectrum of the reference matrix \( Q_1 \)) and can be detected readily. As in [4], a BBP transition occurs at the scale \( n^{-1/3} \). For \( 0 < \tilde{\sigma}_i^a - \ell_a \ll n^{-1/3} \) or \( 0 < \tilde{\sigma}_i^b - \ell_b \ll n^{-1/3} \), i.e. the spike is subcritical, the corresponding “outlier” cannot be distinguished from the bulk spectrum and will instead stick to the right-most edge of the bulk spectrum up to some random fluctuation of order \( O(n^{-2/3}) \). The rest of the non-outlier eigenvalues will stick to the eigenvalues of \( Q_1 \); see Theorem 2.13. Next for the sample eigenvector of \( \hat{Q}_1 \), that is associated with the outlier caused by the spike \( \tilde{\sigma}_i^a \), we show that it is concentrated on a cone with axis parallel to the population eigenvector \( v_i^a \) with an explicit aperture determined by \( \tilde{\sigma}_i^a \). On the other hand, the sample eigenvector of \( \hat{Q}_1 \), that is associated with the spike \( \tilde{\sigma}_i^b \) is delocalized. Similar results hold for the right singular vectors of \( Y \), i.e. the eigenvectors of \( \hat{Q}_2 := B^{1/2}X^*AXB^{1/2} \), by switching the roles of \( \hat{A} \) and \( \hat{B} \). Finally, for the non-spiked singular vectors, we proved that they are delocalized. We point out that our results are in the same spirit as the ones for deformed Wigner matrix [29], deformed rectangular matrix [6, 12] and spiked covariance matrices [8, 11, 15]. Also we would like to remark that we prove these results in full generality, even near the BBP transition point and for the cases with degenerate or near-degenerate outliers.

The information from sample singular vectors is very important in the estimation of spiked separable covariance matrices. For example, one important parameter to estimate is the number of spikes. For spiked separable covariance matrices, the outliers have two different origins from either \( \hat{A} \) or \( \hat{B} \). Hence we need to estimate the number of spikes for each of them. In the literature of spiked covariance matrices [14], the number of spikes is estimated using statistic constructed from eigenvalues only. However, this only gives an estimation of the total number of spikes. To distinguish the two types of spikes, we also need to utilize the information from singular vectors. This will be discussed in detail in Section 3.
Before concluding the introduction, we summarize the main contributions of our work.

(i) We introduce the spiked separable covariance matrix model; see (2.12). It allows more general covariance structure and is suitable for the spatio-temporal data analysis with possible spikes in both space and time.

(ii) For both supercritical and subcritical spikes, we obtain the first order limits of the corresponding outliers and the generalized components of the associated eigenvectors. Moreover, our results provide a precise rate of convergence, which we believe to be optimal up to some \( n^c \) factor. These results are presented in Theorem 2.12 and Theorem 2.19.

(iii) We prove large deviation bounds for the non-outlier eigenvalues and eigenvectors. In particular, we prove that the non-outlier eigenvalues will stick with those of the reference matrix. Moreover, the non-outlier eigenvectors near the spectrum edge will be biased in the direction of the population eigenvectors of the subcritical spikes. These results are presented in Theorem 2.13 and Theorem 2.19.

(iv) We address two important issues in the estimation of spiked separable covariance matrices. First, we provide statistics to estimate the number of spikes for \( \hat{A} \) and \( \hat{B} \). In particular, we will show that the eigenvectors are important for us to separate the outliers from the spikes of \( \hat{A} \) and those from the spikes of \( \hat{B} \). Second, we obtain the optimal shrinkage for the eigenvalues, which is adaptive to the data matrix only. These are discussed in Section 6.

Finally, to have a complete description of the principal components, we still need to consider the second order asymptotics, i.e. the limiting distribution of the outlier eigenvalues and eigenvectors. This will be the subject of future research.

This paper is organized as follows. In Section 2 we define the spiked separable covariance matrix and state the main results. In Section 6 we address two important issues regarding the statistical estimation of the proposed spiked separable covariance matrices. In Section 3 we collect some basic tools for the proofs of the main results. The results on the eigenvalues are proved in Section 5, and the results on the outlier and non-outlier eigenvectors are proved in Section 6 and Section 7 respectively.

Conventions. The fundamental large parameter is \( n \) and we always assume that \( p \) is comparable to and depends on \( n \). We use \( C \) to denote a generic large positive constant, whose value may change from one line to the next. Similarly, we use \( \varepsilon, \tau, c, \) etc. to denote generic small positive constants. If a constant depend on a quantity \( a \), we use \( C(a) \) or \( C_a \) to indicate this dependence. For two quantities \( a_n \) and \( b_n \) depending on \( n \), the notation \( a_n = O(b_n) \) means that \( |a_n| \leq C|b_n| \) for some constant \( C > 0 \), and \( a_n = o(b_n) \) means that \( |a_n| \leq c_n|b_n| \) for some positive sequence \( c_n \downarrow 0 \) as \( n \to \infty \). We also use the notations \( a_n \leq b_n \) if \( a_n = O(b_n) \), and \( a_n \sim b_n \) if \( a_n = O(b_n) \) and \( b_n = O(a_n) \). For a matrix \( A \), we use \( \|A\| := \|A\|_{\infty} \) to denote the operator norm; for a vector \( v = (v_i)_{i=1}^n, \|v\| = \|v\|_2 \) stands for the Euclidean norm. For a matrix \( A \) and a number \( a > 0 \), we write \( A = O(a) \) if \( \|A\| = O(a) \). In this paper, we often write an identity matrix of any dimension as \( I \) or 1 without causing any confusions.

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2 Definitions and main results

2.1 Spiked separable covariance matrices

We consider a class of separable sample covariance matrices of the form \( Q_1 := A^{1/2}XBX^*A^{1/2} \), where \( A \) and \( B \) are deterministic non-negative definite symmetric (or Hermitian) matrices. Note that \( A \) and \( B \) are not
necessarily diagonal. We assume that $X = (x_{ij})$ is a $p \times n$ random matrix, where the entries $x_{ij}$, $1 \leq i \leq p$, $1 \leq j \leq n$, are real or complex independent random variables satisfying

$$
\mathbb{E} x_{ij} = 0, \quad \mathbb{E}|x_{ij}|^2 = n^{-1}.
$$

(2.1)

For definiteness, in this paper we focus on the real case, i.e. the random variables $x_{ij}$ are real. However, we remark that our proof can be applied to the complex case after minor modifications if we assume in addition that $\text{Re} x_{ij}$ and $\text{Im} x_{ij}$ are independent centered random variables with variance $(2n)^{-1}$. We also assume that the random variables $x_{ij}$ have arbitrarily high moments, in the sense that for any fixed $k \in \mathbb{N}$, there is a constant $\mu_k > 0$ such that

$$
\max_{i,j} (\mathbb{E}|x_{ij}|^k)^{1/k} \leq \mu_k n^{-1/2},
$$

(2.2)

for all $n$. We will also use the $n \times n$ matrix $Q_2 := B^{1/2}X^*AXB^{1/2}$. We denote the eigenvalues of $Q_1$ and $Q_2$ in descending order by $\lambda_1(Q_1) \geq \ldots \geq \lambda_p(Q_1)$ and $\lambda_1(Q_2) \geq \ldots \geq \lambda_n(Q_2)$. Since $Q_1$ and $Q_2$ share the same nonzero eigenvalues, we will simply write $\lambda_j$, $1 \leq j \leq p \leq n$, to denote the $j$-th eigenvalue of both $Q_1$ and $Q_2$ without causing any confusion.

We shall consider the high-dimensional setting in this paper. More precisely, we assume that there exists a constant $0 < \tau < 1$ such that the aspect ratio $d_n := n/p$ satisfies

$$
\tau \leq d_n \leq \tau^{-1} \quad \text{for all } n.
$$

(2.3)

We assume that $A$ and $B$ have eigendecompositions

$$
A = V^a\Sigma^a(V^a)^*, \quad B = V^b\Sigma^b(V^b)^*, \quad \Sigma^a = \text{diag}(\sigma_1^a, \ldots, \sigma_p^a), \quad \Sigma^b = \text{diag}(\sigma_1^b, \ldots, \sigma_n^b),
$$

(2.4)

where

$$
V^a = (\mathbf{v}_1, \ldots, \mathbf{v}_p), \quad V^b = (\mathbf{v}_1, \ldots, \mathbf{v}_n).
$$

We denote the empirical spectral distributions (ESD) of $A$ and $B$ by

$$
\pi_A = \pi_A^{(p)} := \frac{1}{p} \sum_{i=1}^p \delta_{\sigma_i^a}, \quad \pi_B = \pi_B^{(n)} := \frac{1}{n} \sum_{i=1}^n \delta_{\sigma_i^b}.
$$

(2.5)

We assume that there exists a small constant $0 < \tau < 1$ such that for all $n$ large enough,

$$
\max\{\sigma_1^a, \sigma_1^b\} \leq \tau^{-1}, \quad \max\left\{\pi_A^{(p)}([0, \tau]), \pi_B^{(n)}([0, \tau])\right\} \leq 1 - \tau.
$$

(2.6)

Note the first condition means that the operator norms of $A$ and $B$ are bounded by $\tau^{-1}$, and the second condition means that the spectrums of $A$ and $B$ cannot concentrate at zero.

In this paper, we study spiked separable sample covariance matrices, which can be realized through a low rank perturbation of the non-spiked version. We shall assume that $Q_1$ is a separable sample covariance matrix without spikes (see Assumption 2.5 below). To add spikes, we follow the setup in [11] and assume that there exist some fixed integers $r, s \in \mathbb{N}$ and constants $d_{ij}$, $1 \leq i \leq r$, and $d_{\mu}$, $1 \leq \mu \leq s$, such that

$$
\tilde{A} = V^a\tilde{\Sigma}^a(V^a)^*, \quad \tilde{B} = V^b\tilde{\Sigma}^b(V^b)^*, \quad \tilde{\Sigma}^a = \text{diag}(\tilde{\sigma}_1^a, \ldots, \tilde{\sigma}_p^a), \quad \tilde{\Sigma}^b = \text{diag}(\tilde{\sigma}_1^b, \ldots, \tilde{\sigma}_n^b),
$$

(2.7)

where

$$
\tilde{\sigma}_i^a = \begin{cases} \sigma_i^a(1 + d_{ij}), & 1 \leq i \leq r, \\ \sigma_i^a, & \text{otherwise}, \end{cases} \quad \tilde{\sigma}_\mu^b = \begin{cases} \sigma_\mu^b(1 + d_\mu), & 1 \leq \mu \leq s, \\ \sigma_\mu^b, & \text{otherwise}. \end{cases}
$$

(2.8)
Without loss of generality, we assume that we have reordered indices such that
\[ \sigma_1^a \geq \sigma_2^a \geq \ldots \geq \sigma_p^a \geq 0, \quad \sigma_1^b \geq \sigma_2^b \geq \ldots \geq \sigma_n^b \geq 0. \] (2.9)

Moreover, we assume that
\[ \max\{\sigma_1^a, \sigma_1^b\} \leq \tau^{-1}. \] (2.10)

With (2.7) and (2.8), we can write
\[ \tilde{A} = A \left( 1 + V_a^a D^a (V_a^a)^* \right), \quad \tilde{B} = B \left( 1 + V_a^b D^b (V_a^b)^* \right). \] (2.11)

where
\[ D^a = \text{diag}(d_1^a, \ldots, d_p^a), \quad V_a^a = (v_1^a, \ldots, v_r^a), \quad D^b = \text{diag}(d_1^b, \ldots, d_q^b), \quad V_a^b = (v_1^b, \ldots, v_q^b). \]

Then we define the spiked separable sample covariance matrices as
\[ \tilde{Q}_1 = \tilde{A}^{1/2} X \tilde{B}^{1/2} \tilde{A}^{1/2}, \quad \tilde{Q}_2 = \tilde{B}^{1/2} X^* \tilde{A}^{1/2} \tilde{B}^{1/2}. \] (2.12)

We summarize our basic assumptions here for future reference.

**Assumption 2.1.** We assume that \( X \) is a \( p \times n \) random matrix with real entries satisfying (2.1) and (2.2), \( A \) and \( B \) are deterministic non-negative definite symmetric matrices satisfying (2.3) and (2.6), \( \tilde{A} \) and \( \tilde{B} \) are deterministic non-negative definite symmetric matrices satisfying (2.11), (2.8), (2.9) and (2.10), and \( d_n \) satisfies (2.3).

### 2.2 Resolvents and limiting laws

In this paper, we study the eigenvalue statistics of \( Q_{1,2} \) and \( \tilde{Q}_{1,2} \) through their resolvents (or Green’s functions). Throughout the paper, we shall denote the upper half complex plane and the right half real line by
\[ \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} z > 0 \}, \quad \mathbb{R}_+ := [0, \infty). \]

**Definition 2.2** (Resolvents). For \( z = E + \imath \eta \in \mathbb{C}_+ \), we define the following resolvents as
\[ \mathcal{G}_{1,2}(X, z) := (Q_{1,2}(X) - z)^{-1}, \quad \tilde{\mathcal{G}}_{1,2}(X, z) := \left( \tilde{Q}_{1,2}(X) - z \right)^{-1}. \] (2.13)

We denote the ESD \( \rho^{(p)} \) of \( Q_1 \) and its Stieltjes transform as
\[ \rho = \rho^{(p)} := \frac{1}{p} \sum_{i=1}^{p} \delta_{\lambda_i(Q_1)}, \quad m(z) = m^{(n)}(z) := \int \frac{1}{x - z} \rho^{(p)}(dx) = \frac{1}{p} \text{Tr} \mathcal{G}_1(z). \] (2.14)

We also introduce the following quantities:
\[ m_1(z) = m_1^{(n)}(z) := \frac{1}{n} \text{Tr} (A \mathcal{G}_1(z)), \quad m_2(z) = m_2^{(n)}(z) := \frac{1}{n} \text{Tr} (B \tilde{\mathcal{G}}_2(z)). \] (2.15)

It was shown in [47] that if \( d_n \to d \in (0, \infty) \) and \( \pi_A^{(p)}, \pi_B^{(n)} \) converge to certain probability distributions, then almost surely \( \rho^{(p)} \) converges to a deterministic distribution \( \rho_\infty \). We now describe it through its Stieltjes transform
\[ m_\infty(z) := \int_R \frac{\rho_\infty(dx)}{x - z}, \quad z \in \mathbb{C}_+. \]
For any finite \( n, p = nd_n \) and \( z \in \mathbb{C}_+ \), we define \((m_{1c}^{(n)}(z), m_{2c}^{(n)}(z)) \in \mathbb{C}_+^2 \) as the unique solution to the system of self-consistent equations

\[
m_{1c}^{(n)}(z) = d_n \int \frac{x}{-z + 1 + x m_{2c}^{(n)}(z)} \pi_A^{(p)}(dx), \quad m_{2c}^{(n)}(z) = \int \frac{x}{-z + 1 + x m_{1c}^{(n)}(z)} \pi_B^{(n)}(dx).
\] (2.16)

Then we define

\[
m_c(z) = m_c^{(n)}(z) := \int \frac{1}{-z + 1 + x m_{2c}^{(n)}(z)} \pi_A^{(p)}(dx).
\] (2.17)

It is easy to verify that \( m_c(z) \in \mathbb{C}_+ \) for \( z \in \mathbb{C}_+ \). Letting \( \eta \downarrow 0 \), we can obtain a probability measure \( \rho_c^{(n)} \) with the inverse formula

\[
\rho_c^{(n)}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_c^{(n)}(E + i\eta).
\] (2.18)

If \( d_n \to d \in (0, \infty) \) and \( \pi_A^{(p)}, \pi_B^{(n)} \) converge to certain probability distributions, then \( m_c^{(n)} \) also converges and we have

\[
m_c(z) := \lim_{n \to \infty} m_c^{(n)}(z), \quad z \in \mathbb{C}_+.
\]

Letting \( \eta \downarrow 0 \), we can recover the asymptotic eigenvalue density \( \rho_c \) with

\[
\rho_c(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_c(E + i\eta).
\] (2.19)

It is also easy to see that \( \rho_c \) is the weak limit of \( \rho_c^{(n)} \).

The above definitions of \( m_c^{(n)}, \rho_c^{(n)}, m_c, \rho_c \) make sense due to the following theorem. Throughout the rest of this paper, we often omit the super-indices \( (p) \) and \( (n) \) from our notations for simplicity.

**Theorem 2.3** (Existence, uniqueness, and continuous density). For any \( z \in \mathbb{C}_+ \), there exists a unique solution \((m_{1c}, m_{2c}) \in \mathbb{C}_+^2 \) to the systems of equations in (2.16). The function \( m_c \) in (2.17) is the Stieltjes transform of a probability measure \( \mu_c \) supported on \( \mathbb{R}_+ \). Moreover, \( \mu_c \) has a continuous derivative \( \rho_c(x) \) on \((0, \infty)\).

**Proof.** See [60, Theorem 1.2.1], [25, Theorem 2.4] and [10, Theorem 3.1]. \( \square \)

From (2.16), it is easy to see that if we define the function

\[
f(z, m) := -m + \int \frac{x}{-z + xd_n} \frac{1}{\pi+im} \pi_A(dt) \pi_B(dx),
\] (2.20)

then \( m_{2c}(z) \) can be characterized as the unique solution to the equation \( f(z, m) = 0 \) that satisfies \( \text{Im} m > 0 \) for \( z \in \mathbb{C}_+ \), and \( m_{1c}(z) \) can be defined using the first equation in (2.16). Moreover, \( m_{1,2c}(z) \) are the Stieltjes transforms of densities \( \rho_{1,2c} \):

\[
\rho_{1,2c}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_{1,2c}(E + i\eta).
\]

Then we have the following result.

**Lemma 2.4.** The densities \( \rho_c \) and \( \rho_{1,2c} \) all have the same support on \((0, \infty)\), which is a union of intervals:

\[
\text{supp} \rho_c \cap (0, \infty) = \text{supp} \rho_{1,2c} \cap (0, \infty) = \bigcup_{k=1}^L [a_{2k}, a_{2k-1}] \cap (0, \infty),
\] (2.21)
Moreover, such an outlier is around a deterministic location $g$ where matrices $M(x,m)$. Meanwhile, we shall use the words outlier eigenvalues and eigenvectors for those of the separable covariance this paper, we use the words spikes and spiked eigenvectors for those of the population matrices respectively. Note that the inverse functions exist because In this section, we state the main results on the eigenvalues and eigenvectors of 
\[ f(x,m) = 0, \quad \text{and} \quad \frac{\partial f}{\partial m}(x,m) = 0. \] (2.22) Finally, we have $a_1 = O(1)$, $m_{1c}(a_1) \in ((-\tilde{\sigma}^a_i)^{-1},0)$ and $m_{2c}(a_1) \in ((-\tilde{\sigma}^b_{\mu})^{-1},0)$.

**Proof.** See Section 3 of [10].

We shall call $a_k$ the spectral edges. In particular, we focus on the rightmost edge $\lambda_r := a_1$. Now we make the following assumption. It guarantees a regular square-root behavior of the spectral densities $\rho_{1,2c}$ near $\lambda_r$ and rules out the existence of spikes.

**Assumption 2.5.** There exists a constant $\tau > 0$ such that
\[ 1 + m_{1c}(\lambda_r)\sigma^b_i \geq \tau, \quad 1 + m_{2c}(\lambda_r)\sigma^a_i \geq \tau. \] (2.23)

Under this assumption, we have the following lemma.

**Lemma 2.6** (Lemma 2.6 of [58]). Under the assumptions (2.24), (2.26) and (2.28), there exist constants $a_{1,2} > 0$ such that
\[ \rho_{1,2c}(\lambda_r - x) = a_{1,2}x^{1/2} + O(x), \quad x \downarrow 0, \] (2.24) and
\[ m_{1,2c}(z) = m_{1,2c}(\lambda_r) + \pi a_{1,2}(z - \lambda_r)^{1/2} + O(|z - \lambda_r|), \quad z \to \lambda_r, \quad \text{Im } z \geq 0. \] (2.25) The estimates (2.24) and (2.25) also hold for $\rho_\epsilon$ and $m_\epsilon$ with different constants.

### 2.3 Main results

In this section, we state the main results on the eigenvalues and eigenvectors of $\tilde{Q}_1$ and $\tilde{Q}_2$. Throughout this paper, we use the words spikes and spiked eigenvectors for those of the population matrices $\tilde{A}$ and $\tilde{B}$. Meanwhile, we shall use the words outlier eigenvalues and eigenvectors for those of the separable covariance matrices $\tilde{Q}_1$ and $\tilde{Q}_2$.

We will see that a spike $\tilde{\sigma}^a_i$, $1 \leq i \leq r$, or $\tilde{\sigma}^b_{\mu}$, $1 \leq \mu \leq s$, causes an outlier eigenvalue beyond $\lambda_r$, if
\[ \tilde{\sigma}^a_i > -m_{2c}^{-1}(\lambda_r) \quad \text{or} \quad \tilde{\sigma}^b_{\mu} > -m_{1c}^{-1}(\lambda_r). \] (2.26)

Moreover, such an outlier is around a deterministic location
\[ \theta_1(\tilde{\sigma}^a_i) := g_{1c}(-e^{-1}) \quad \text{and} \quad \theta_2(\tilde{\sigma}^b_{\mu}) := g_{2c}(-e^{-1}), \] (2.27)
where $g_{1c}$ and $g_{2c}$ are the inverse functions of $\rho_{1c} : (\lambda_r, \infty) \to (m_{1c}(\lambda_r), 0)$ and $m_{2c} : (\lambda_r, x) \to (m_{2c}(\lambda_r), 0)$, respectively. Note that the inverse functions exist because
\[ m_{1,2c}(x) = \int_0^\lambda \frac{\rho_{1,2c}(t)}{t - x} dt, \quad x > \lambda_r, \] (2.28) are monotonically increasing functions of $x$ for $x > \lambda_r$. 

8
Assumption 2.7. We assume that (2.29) holds for all $1 \leq i \leq r$ and $1 \leq \mu \leq s$. Otherwise, if (2.26) fails for some $\tilde{\sigma}_i^a$ or $\tilde{\sigma}_\mu^b$, we can simply redefine it as the unperturbed version $\sigma_i^a$ or $\sigma_\mu^b$. Moreover, we define the integers $0 \leq r^+ \leq r$ and $0 \leq s^+ \leq s$ such that

$$\tilde{\sigma}_i^a \geq -m_{2c}(\lambda_r) + n^{-1/3} \text{ if and only if } 1 \leq i \leq r^+,$$

and

$$\tilde{\sigma}_\mu^b \geq -m_{1c}(\lambda_r) + n^{-1/3} \text{ if and only if } 1 \leq \mu \leq s^+.$$

The lower bound $n^{-1/3}$ is chosen for definiteness, and it can be replaced with any $n$-dependent parameter $\psi$ such that $\psi \sim n^{-1/3}$.

Remark 2.8. A spike $\tilde{\sigma}_i^a$ or $\tilde{\sigma}_\mu^b$ that does not satisfy (2.29) or (2.30) will give an outlier that lies within an $O(n^{-2/3})$ neighborhood of the rightmost edge $\lambda_r$. It is essentially indistinguishable from the extremal eigenvalue of $Q_1$, which has typical fluctuation of order $n^{-2/3}$ around $\lambda_r$. Hence in (2.29) and (2.30), we simply choose the “real” spikes of $\tilde{A}$ and $\tilde{B}$.

We will use the following notion of stochastic domination, which was first introduced in [19] and subsequently used in many works on random matrix theory, such as [7, 8, 9, 20, 21, 30]. It simplifies the presentation of the results and their proofs by systematizing statements of the form “$\xi$ is bounded by $\zeta$ with high probability up to a small power of $n$”.

Definition 2.9 (Stochastic domination). (i) Let

$$\xi = \left(\xi^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)}\right), \quad \zeta = \left(\zeta^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)}\right)$$

be two families of nonnegative random variables, where $U^{(n)}$ is a possibly $n$-dependent parameter set. We say $\xi$ is stochastically dominated by $\zeta$, uniformly in $u$, if for any fixed (small) $\varepsilon > 0$ and (large) $D > 0$,

$$\sup_{u \in U^{(n)}} \mathbb{P}\left(\xi^{(n)}(u) > n^\varepsilon \zeta^{(n)}(u)\right) \leq n^{-D}$$

for large enough $n \geq n_0(\varepsilon, D)$, and we shall use the notation $\xi \prec \zeta$. Throughout this paper, the stochastic domination will always be uniform in all parameters that are not explicitly fixed (such as matrix indices, and $z$ that takes values in some compact set). Note that $n_0(\varepsilon, D)$ may depend on quantities that are explicitly constant, such as $\tau$ in Assumption 2.1 and (2.33). If for some complex family $\xi$ we have $|\xi| \prec \zeta$, then we will also write $\xi \prec \zeta$ or $\xi = O_{\prec}(\zeta)$.

(ii) We extend the definition of $O_{\prec}(\cdot)$ to matrices in the weak operator norm sense as follows. Let $A$ be a family of random matrices and $\zeta$ be a family of nonnegative random variables. Then $A = O_{\prec}(\zeta)$ means that $|\langle v, Aw \rangle| \prec |\langle v, \zeta \rangle|_2 |\langle w, \zeta \rangle|_2$ uniformly in any deterministic vectors $v$ and $w$. Here and throughout the following, whenever we say “uniformly in any deterministic vectors”, we mean that “uniformly in any deterministic vectors belonging to a set of cardinality $n^O(1)$”.

(iii) We say an event $\Xi$ holds with high probability if for any constant $D > 0$, $\mathbb{P}(\Xi) \geq 1 - n^{-D}$ for large enough $n$.

The following lemma collects basic properties of stochastic domination $\prec$, which will be used tacitly in the proof.

Lemma 2.10 (Lemma 3.2 in [7]). Let $\xi$ and $\zeta$ be families of nonnegative random variables.

(i) Suppose that $\xi(u, v) \prec \zeta(u, v)$ uniformly in $u \in U$ and $v \in V$. If $|V| \leq n^C$ for some constant $C$, then $\sum_{v \in V} \xi(u, v) \prec \sum_{v \in V} \zeta(u, v)$ uniformly in $u$.

(ii) If $\xi_1(u) \prec \zeta_1(u)$ and $\xi_2(u) \prec \zeta_2(u)$ uniformly in $u \in U$, then $\xi_1(u)\xi_2(u) \prec \zeta_1(u)\zeta_2(u)$ uniformly in $u$.

(iii) Suppose that $\Psi(u) \geq n^{-C}$ is deterministic and $\xi(u)$ satisfies $\mathbb{E}\xi(u)^2 \leq n^C$ for all $u$. Then if $\xi(u) \prec \Psi(u)$ uniformly in $u$, we have $\mathbb{E}\xi(u) \prec \Psi(u)$ uniformly in $u$. 

9
Moreover, we define the nontrivial eigenvalues of $r_1$.

**Definition 2.11.** We define the labelling functions $\alpha : \{1, \ldots, p\} \rightarrow \mathbb{N}$ and $\beta : \{1, \ldots, n\} \rightarrow \mathbb{N}$ as follows. For any $1 \leq i \leq r$, we assign to it a label $\alpha(i) \in \{1, \ldots, r + s\}$ if $\theta_1(\bar{\sigma}_i^q)$ is the $\alpha(i)$-th largest element in $(\theta_1(\bar{\sigma}_i^q))_{i=1}^s \cup (\theta_2(\bar{\sigma}_i^b))_{\mu=1}^s$. We also assign to any $1 \leq \mu \leq s$ a label $\beta(\mu) \in \{1, \ldots, r + s\}$ in a similar way. Moreover, we define $\alpha(i) = i + s$ if $i > r$ and $\beta(\mu) = \mu + r$ if $\mu > s$. We define the following sets of outlier indices:

$$\mathcal{O} := \{\alpha(i) : 1 \leq i \leq r\} \cup \{\beta(\mu) : 1 \leq \mu \leq s\}, \quad \mathcal{O}^+ := \{\alpha(i) : 1 \leq i \leq r^+\} \cup \{\beta(\mu) : 1 \leq \mu \leq s^+\}.$$ 

We first state the results on the locations of the outliers and the first few non-outlier eigenvalues. Denote the nontrivial eigenvalues of $\hat{Q}_{1,2}$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p,n}$. For $1 \leq i \leq r$ and $1 \leq \mu \leq s$, we define

$$\Delta_1(\bar{\sigma}_i^q) := (\bar{\sigma}_i^q + m_{2c}(\lambda_r))^{1/2}, \quad \Delta_2(\bar{\sigma}_\mu^b) := (\bar{\sigma}_\mu^b + m_{4c}(\lambda_r))^{1/2}.$$  

**Theorem 2.12.** Suppose that Assumptions 2.1, 2.5 and 2.7 hold. Then we have

$$|\lambda_{\alpha(i)} - \theta_1(\bar{\sigma}_i^q)| < n^{-1/2} \Delta_1(\bar{\sigma}_i^q), \quad 1 \leq i \leq r^+,$$

and

$$|\lambda_{\beta(\mu)} - \theta_2(\bar{\sigma}_\mu^b)| < n^{-1/2} \Delta_2(\bar{\sigma}_\mu^b), \quad 1 \leq \mu \leq s^+.$$  

Furthermore, for any fixed $\varpi > r + s$, we have

$$|\hat{\lambda}_i - \lambda_r| < n^{-2/3}, \quad \text{for } i \notin \mathcal{O}^+ \text{ and } i \leq \varpi.$$  

The above theorem gives the large deviation bounds for the locations of the outlier and the first few extremal non-outlier eigenvalues. Note that Theorem 2.12 shows that the fluctuation of the outlier changes from the order $n^{-1/2} \Delta_1(\bar{\sigma}_i^q)$ to $n^{-2/3}$ when $\Delta_1(\bar{\sigma}_i^q)$ or $\Delta_2(\bar{\sigma}_\mu^b)$ crosses the scale $n^{-1/6}$. This implies the occurrence of the BBP transition [3]. In a future work, we will show that under certain assumptions, the outlier eigenvalues are normally distributed, whereas the extremal non-outlier eigenvalues follow the Tracy-Widom law.

Next, we study the non-outlier eigenvalues of $\hat{Q}_1$. We prove that the eigenvalues of $\hat{Q}_1$ for $i > r^+ + s^+$ are governed by *eigenvalue sticking*, which states that the non-outlier eigenvalues of $\hat{Q}_1$ “stick” with high probability to the eigenvalues of the reference matrix $Q_1$. Recall that we denote the eigenvalues of $Q_1$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p,n}$.

**Theorem 2.13.** Suppose that Assumptions 2.1, 2.5 and 2.7 hold. We define

$$\alpha_+ := \min \left\{ \min_i |\bar{\sigma}_i^q + m_{2c}(\lambda_r)|, \min_\mu |\bar{\sigma}_\mu^b + m_{4c}(\lambda_r)| \right\}.$$  

Fix any sufficiently small constant $\tau > 0$. We have that for $1 \leq i \leq \tau n$,

$$|\hat{\lambda}_{i+r+s} - \lambda_i| < \frac{1}{n \alpha_+} + n^{-3/4} + 2^{1/3} n^{-5/6}. \quad (2.36)$$

If either (a) the third moments of the entries of $X$ vanish in the sense that

$$\mathbb{E} x_{ij}^3 = 0, \quad 1 \leq i \leq p, \quad 1 \leq j \leq n, \quad (2.37)$$

or (b) either $A$ or $B$ is diagonal, then we have the stronger estimate

$$|\hat{\lambda}_{i+r+s} - \lambda_i| < \frac{1}{n \alpha_+}, \quad 1 \leq i \leq \tau n. \quad (2.38)$$
Theorem 2.13 establishes the large deviation bounds for the non-outlier eigenvalues of $\tilde{Q}_1$ with respect to the eigenvalues of $Q_1$. In particular, when $\alpha_+ \gg n^{-1/3}$, the right-hand side of (2.36) or (2.38) is much smaller than $n^{-2/3}$ for $i = O(1)$. In fact it was proved in [8] that the limiting joint distribution of the first few eigenvalues $\{\lambda_i\}_{1 \leq i \leq k}$ of $Q_1$ is universal under an $n^{2/3}$ scaling for any fixed $k \in \mathbb{N}$. Together with (2.36), this implies that the limiting distribution of the largest non-outlier eigenvalues of $\tilde{Q}_1$ is also universal under an $n^{2/3}$ scaling as long as $\alpha_+ \gg n^{-1/3}$. In a future paper, we will prove that $[n^{2/3}(\lambda_1 - \lambda_\nu)]_{1 \leq i \leq k}$ converges to the Tracy-Widom law, which immediately implies that the largest non-outlier eigenvalues of $\tilde{Q}_1$ also satisfy the Tracy-Widom law.

Remark 2.14. The Theorem 2.12 and 2.13 can be combined to potentially estimate the spikes of $\tilde{A}$ and $\tilde{B}$ if $A$ and $B$ are low-rank perturbations of identity matrices. By Theorem 2.12, the spike $\tilde{\sigma}_i$ or $\tilde{\sigma}_\mu$ can be effectively estimated using $-m_{2c}^{-1}(\tilde{\lambda}_{\alpha(i)})$ or $-m_{\nu}^{-1}(\tilde{\lambda}_\beta(\mu))$. Although calculating $m_{1c}$ and $m_{2c}$ needs the knowledge of the spectrums of $A$ and $B$, we will see from the averaged local law (c.f. Theorem 4.7) that $m_{1c}$ and $m_{2c}$ can be well approximated using the eigenvalues of $\tilde{Q}_1$ and $\tilde{Q}_2$ only. We record such result in Theorem 3.5.

On the other hand, for the non-spiked eigenvalues, to our best knowledge there does not exist any literature on the estimation of the spectrums of general $A$ and $B$ using the eigenvalues of $Q_1$ and $Q_2$ only. However, for sample covariance matrices with $B = I$, the spectrum of $A$ can be estimated using the eigenvalues of $A^{1/2}XX^*A^{1/2}$ by solving a convex optimization problem involving the self-consistent equation for $m_{2c}$ in [18, 31]. In the future work, we will try to generalize their results to the separable covariance matrices with more general $B$. Note that although we cannot observe the eigenvalues of $Q_1$, Theorem 2.13 implies that the non-outlier eigenvalues of $\tilde{Q}_1$ are close to those of $Q_1$.

We now state the results for the eigenvectors of $\tilde{Q}_1$ and $\tilde{Q}_2$. We denote the eigenvectors of $\tilde{Q}_1$ by $\tilde{\xi}_k$, $1 \leq k \leq p$, and the eigenvectors of $\tilde{Q}_2$ by $\tilde{\zeta}_k$, $1 \leq \mu \leq n$. To remove the arbitrariness in the definitions of eigenvectors, we shall consider instead the products of generalized components

$$\langle v, \tilde{\xi}_k \rangle \langle \tilde{\xi}_k, w \rangle, \quad \langle v', \tilde{\zeta}_k \rangle \langle \tilde{\zeta}_k, w' \rangle,$$

where $v, w, v'$ and $w'$ are some given deterministic vectors. Note that these products characterize the eigenvectors $\tilde{\xi}_k$ and $\tilde{\zeta}_k$ completely up to the ambiguity of a phase. More generally, if we consider degenerate or near-degenerate outliers, then only eigenspace matters. Hence as in [8], we shall consider the generalized components $\langle v, P_S w \rangle$ of the random projection

$$P_S := \sum_{k \leq S} \tilde{\xi}_k \tilde{\xi}_k^*,$$

for $S \subset \mathbb{C}^+$.

In particular, in the non-degenerate case $S = \{k\}$, the generalized components of $P_S$ are the products of the generalized components of $\tilde{\xi}_k$.

For $1 \leq i \leq r^+$, $1 \leq j \leq p$ and $1 \leq \nu \leq n$, we define

$$\delta_{\alpha(i), \alpha(j)}^a := |\tilde{\sigma}_j^a - \tilde{\sigma}_i^a|, \quad \delta_{\alpha(i), \beta(\nu)}^a := |\tilde{\sigma}_\nu^a + m_{1c}^{-1}(\tilde{\theta}_1(\tilde{\sigma}_i^a))|. \quad (2.39)$$

Similarly, for $1 \leq \mu \leq s^+$, $1 \leq j \leq p$ and $1 \leq \nu \leq n$, we define

$$\delta_{\beta(\mu), \alpha(j)}^b := |\tilde{\sigma}_j^b + m_{2c}^{-1}(\tilde{\theta}_2(\tilde{\sigma}_\nu^b))|, \quad \delta_{\beta(\mu), \beta(\nu)}^b := |\tilde{\sigma}_\nu^b - \tilde{\sigma}_\mu^b|. \quad (2.40)$$

Given any $S \subset \mathbb{C}^+$, if $a \in S$, then we define

$$\delta_a (S) := \begin{cases} 
\left( \min_{k, \alpha(k) \notin S} \delta_{\alpha(k), \alpha(k)}^a \right) \wedge \left( \min_{\mu, \beta(\mu) \notin S} \delta_{\beta(\mu), \beta(\mu)}^b \right), & \text{if } a = \alpha(i) \in S \\
\left( \min_{k, \alpha(k) \notin S} \delta_{\alpha(k), \alpha(k)}^b \right) \wedge \left( \min_{\mu, \beta(\mu) \notin S} \delta_{\beta(\mu), \beta(\mu)}^a \right), & \text{if } a = \beta(\mu) \in S 
\end{cases}.$$
Then for any deterministic vector \( v \in \mathbb{C}^p \), we have that

\[
|\langle v, P_S v \rangle - \langle v, Z_S v \rangle| < \sum_{i : \alpha(i) \in S} \frac{|v_i|^2}{n^{1/2} \Delta_i(\tilde{\sigma}_i^2)} + \sum_{j=1}^p \frac{|v_j|^2}{n \delta_{\alpha(j)}(S)}^2 + \langle v, Z_S v \rangle \left( \sum_{\alpha(j) \in S} \frac{|v_j|^2}{n \delta_{\alpha(j)}(S)}^2 \right)^{1/2}.
\] (2.42)

**Remark 2.16.** For any deterministic vectors \( v, w \in \mathbb{C}^p \), we can state Theorem 2.15 for more general quantities of the form \( \langle v, Z_S w \rangle \) using the polarization identity.

The index set \( S \) in Theorem 2.15 can be chosen according to user’s goal. We now consider two typical cases to illustrate the idea.

**Example 2.17 (Non-degenerate case).** If all the outliers are well-separated, then we can choose \( S = \{ \alpha(i) \} \) or \( S = \{ \beta(\mu) \} \). For example, suppose \( S = \{ \alpha(i) \} \) and \( v = v^\alpha \). Then we get from (2.42) that

\[
|\langle v^\alpha, \hat{\xi}_i^\alpha \rangle|^2 = \frac{1}{\sigma_i^2} \frac{g_{2c}^2(\tilde{\sigma}_i^2 - \tilde{\sigma}_i^2)^{-1}}{g_{2c}(-\tilde{\sigma}_i^2 - \tilde{\sigma}_i^2)^{-1}} + O\left( \frac{1}{n^{1/2}(\tilde{\sigma}_i^2 + m_{2c}(\lambda_r))^1/2} + \frac{1}{n \delta_{\alpha(i)}} \right), \quad \delta_{\alpha(i)} := \delta_{\alpha(i)}(\{ \alpha(i) \}).
\] (2.43)

Note that \( \hat{\xi}_i \) is concentrated on a cone with axis parallel to \( v^\alpha_i \) if the error term is much smaller than the first item, which is of order

\[
\frac{1}{\sigma_i^2} \frac{g_{2c}^2(\tilde{\sigma}_i^2 - \tilde{\sigma}_i^2)^{-1}}{g_{2c}(-\tilde{\sigma}_i^2 - \tilde{\sigma}_i^2)^{-1}} \sim \tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r)
\]

by the estimates (2.37) below. This leads to the following conditions

\[
\tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r) \gg n^{-1/3}, \quad \delta_{\alpha(i)} \gg (\tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r))^{-1/2} n^{-1/2}.
\] (2.44)

The first condition means that \( \tilde{\lambda}_{\alpha(i)} \) is truly an outlier (c.f. Theorem 2.12), whereas the second condition is an non-overlapping condition. In fact, by (2.32) \( \hat{\lambda}_i \) fluctuates around \( \theta_1(\tilde{\sigma}_i^2) \) on the scale of order \( (\tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r))^{1/2} n^{-1/2} \). Therefore, \( \hat{\lambda}_i \) is well-separated from the other outlier eigenvalues if

\[
\min_{(\alpha(i)) \in \Omega \setminus \{ \alpha(i) \}} |\theta_1(\tilde{\sigma}_i^2) - \theta_1(\tilde{\beta}_j^2)| \wedge \min_{\beta(\mu) \in \Omega} |\theta_1(\tilde{\sigma}_i^2) - \theta_2(\tilde{\beta}_j^2)| \gg (\tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r))^{1/2} n^{-1/2}.
\] (2.45)

With (1.17) below and (2.39), it is easy to show that the left-hand side of (2.44) is of order \( (\tilde{\sigma}_i^2 + m_{2c}^{-1}(\lambda_r))^{1/2} \delta_{\alpha(i)} \). This gives the second condition of (2.44).

For degenerate or near-degenerate outliers, their indices should be included in the same set \( S \). We now consider an example with multiple outliers that share exactly the same classical location.
Example 2.18 (Degenerate case). Suppose that we have an $|S|$-fold degenerate outlier, i.e., for some $\theta_0 > \lambda_r$,

$$\theta_1(\hat{\sigma}^a_i) = \theta_1(\hat{\sigma}^a_j) = \theta_2(\hat{\sigma}^b_j) = \theta_0, \quad \text{for all } \alpha(i), \alpha(j), \beta(\mu), \beta(\nu) \in S.$$  

Suppose the outlier $\theta_0$ is well-separated from both the bulk and the other outliers (i.e., with distances of order 1), then by (2.42), we have that

$$P_S = \sum_{\alpha(i) \in S} \frac{1}{n!} \frac{g_{2c}(-\hat{\sigma}^a_i)^{-1}}{\lambda_i} \psi^a_i(v_i^a)^* + \mathcal{E},$$

where $\mathcal{E}$ is an error that is delocalized in the basis of $v_i^a$, i.e., $\langle v_i^a, \mathcal{E} v_j^a \rangle < n^{-1/2}$. This can be regarded as a generalized cone concentration for the subspace spanned by $\{\tilde{\xi}_a\}_{a \in S}$.

Then we state the delocalization results on the non-outlier eigenvectors when $i \notin \mathcal{O}^+$. Denote

$$\eta_i := n^{-3/4} + n^{-5/6} \nu_1; \quad \kappa_i := \nu^{-2/3} n^{-2/3}.$$  

**Theorem 2.19.** Suppose that Assumptions 2.4, 2.5 and 2.7 hold. Fix any sufficiently small constant $\tau > 0$. For $\alpha(i) \notin \mathcal{O}^+$, $i \leq \tau p$ and any deterministic vector $v \in \mathbb{R}^p$, we have

$$|\langle v, \tilde{\xi}_{\alpha(i)} \rangle|^2 \leq \sum_{j=1}^p |v_j|^2 \frac{n^{-1} + \eta_i \sqrt{\kappa_i}}{\lambda_j^2 + m_{2c}^{-1}(\lambda_j)^2 + \kappa_i}. \quad (2.46)$$

If we have (a) (2.37) holds, or (b) either $A$ or $B$ is diagonal, then the following stronger estimate holds:

$$|\langle v, \tilde{\xi}_{\alpha(i)} \rangle|^2 \leq \sum_{j=1}^p \frac{|v_j|^2}{n} \frac{n^{-1} + \eta_i \sqrt{\kappa_i}}{\lambda_j^2 + m_{2c}^{-1}(\lambda_j)^2 + \kappa_i}. \quad (2.47)$$

**Remark 2.20.** Note that for $i \leq n^{1/4}$, we have $\eta_i \sqrt{\kappa_i} \leq C n^{-1}$. Hence (2.46) becomes the stronger estimate (2.47) for the non-eigenvalues with indices $i \leq n^{1/4}$.

Note that if $\hat{\sigma}^a_j - m_{2c}^{-1}(\lambda_j) \geq 1$, i.e. $\hat{\sigma}^a_j$ is well separated from the threshold, then $\tilde{\xi}_{\alpha(i)}$ is completely delocalized in the direction of $v_j^a$ for all $i \notin \mathcal{O}^+$ and $i \leq n^{1/4}$. We then consider the outliers that is close to the threshold.

**Example 2.21.** Suppose that $i \leq C$, i.e. $\hat{\lambda}_i$ is near the edge. Then (2.47) gives

$$|\langle v_j^a, \tilde{\xi}_{\alpha(i)} \rangle|^2 \leq \frac{1}{n(|\hat{\sigma}^a_j + m_{2c}^{-1}(\lambda_j)|^2 + n^{-2/3})}. \quad (2.48)$$

Therefore, the delocalization bound for the generalized component $|\langle v_j^a, \tilde{\xi}_{\alpha(i)} \rangle|$ changes from the optimal order $n^{-1/2}$ to $n^{-1/6}$ as $\hat{\sigma}^a_j$ approaches the transition point $m_{2c}^{-1}(\lambda_j)$. This shows that the non-outlier eigenvectors near the edge are biased in the direction of $v_j^a$ provided that $\hat{\sigma}^a_j$ is near the transition point $m_{2c}^{-1}(\lambda_j)$. In particular, for $|\hat{\sigma}^a_j + m_{2c}^{-1}(\lambda_j)| \leq n^{-1/3}$, we have that

$$|\langle v_j^a, \tilde{\xi}_{\alpha(i)} \rangle|^2 \leq \frac{1}{n(|\hat{\sigma}^a_j + m_{2c}^{-1}(\lambda_j)|^2). \quad (2.49)$$

In the literature, the $\hat{\sigma}^a_j$ in this case is called a weak spike in statistics [27] or subcritical spike in probability [8]. Thus (2.49) shows that the non-outlier eigenvectors still retain information about the weak spikes of $\hat{\lambda}_i$ in contrast to the non-outlier eigenvalues as seen from (2.46).
The Theorems \([2.12\ 2.13\ 2.15\ \text{and } 2.19]\) give the first order limits and convergent rates of the principal eigenvalues and eigenvectors of \(Q_1\). The second order asymptotics of the outlier eigenvalues and eigenvectors will be studied in another paper.

Note that for separable covariance matrices, \(\hat{A}^{1/2}X\hat{B}^{1/2}\) and \(\hat{B}^{1/2}X^*\hat{A}^{1/2}\) take exactly the same form. Hence by exchanging the roles of \((\hat{A}, X)\) and \((\hat{B}, X^*)\), one can immediately obtain from Theorems \([2.15\ \text{and } 2.19]\) the similar results for the eigenvectors \(\hat{c}_k\). For reader’s convenience, we state them in the following two theorems. Denote
\[
P_S' := \sum_{k \in S} \hat{c}_k\hat{c}_k^*, \quad \text{for } S \subset \mathbb{O}^+.
\]

**Theorem 2.22.** Suppose that Assumptions \([2.1\ 2.5\ \text{and } 2.7]\) hold. Fix any \(S \subset \mathbb{O}^+\), we define the following deterministic positive quadratic form
\[
\langle w, Z'_S w \rangle := \sum_{\mu, \beta(\mu) \in S} \frac{1}{\delta_\mu^p} g_1(\delta_\mu^b)^{-1}|w_\mu|^2, \quad \text{for } w \in \mathbb{C}^n, \quad w_\mu := \langle v_\mu, w \rangle.
\]
Then for any deterministic vector \(w \in \mathbb{C}^p\), we have that
\[
|\langle w, P'_S w \rangle - \langle w, Z'_S w \rangle| < \sum_{\mu, \beta(\mu) \in S} \frac{|w_\mu|^2}{n^{1/2} \Delta_2(\delta_\mu^p)} + \sum_{\mu = 1}^n \frac{|w_\mu|^2}{n \delta_\beta(\mu)(S)^2} + \langle w, Z'_S w \rangle^{1/2} \left( \sum_{\beta(\mu) \notin S} \frac{|w_\mu|^2}{n \delta_\beta(\mu)(S)^2} \right)^{1/2}.
\]

**Theorem 2.23.** Suppose that Assumptions \([2.4\ 2.5\ \text{and } 2.7]\) hold. Fix any sufficiently small constant \(\tau > 0\). For \(\beta(\mu) \notin \mathbb{O}^+, \mu \leq \tau n\) and any deterministic vector \(w \in \mathbb{R}^n\), we have
\[
|\langle w, \tilde{c}_\beta(\mu) \rangle|^2 < \sum_{\nu = 1}^n |w_\nu|^2 \frac{n^{-1} + \eta_\beta \sqrt{m_\mu}}{\delta_\nu^b + m_1c(\lambda_\nu)^2 + \kappa_\mu}.
\]
If we have (a) \([2.37]\) holds, or (b) either \(A\) or \(B\) is diagonal, then we have the stronger estimate
\[
|\langle w, \tilde{c}_\beta(\mu) \rangle|^2 < \sum_{\nu = 1}^n \frac{|w_\nu|^2}{n ((\delta_\nu^b + m_1c(\lambda_\nu)^2 + \kappa_\mu)}.
\]

## 3 Statistical estimation for spiked separable covariance matrices

In this section, we consider the estimation of \(\hat{A}\) and \(\hat{B}\) from the data matrix \(\hat{A}^{1/2}X\hat{B}^{1/2}\). In particular, we address two fundamental issues:

1. estimating the number of spikes in \(\hat{A}\) and \(\hat{B}\);
2. adaptive optimal shrinkage of the eigenvalues of \(\hat{A}\) and \(\hat{B}\).

To ease our discussion, till the end of this section, we will replace Assumption \([2.4]\) with the following stronger super-critical condition. It is commonly used in the statistical literature, for instance \([6\ 15\ 16\ 38]\).

**Assumption 3.1.** For some fixed constant \(\tau > 0\), we assume that there are \(r\) spikes for \(\hat{A}\) and \(s\) spikes for \(\hat{B}\), which satisfy
\[
\tilde{\sigma}_i^a + m_2c(\lambda_i) > \tau, \quad 1 \leq i \leq r; \quad \text{and} \quad \tilde{\sigma}_\mu^b + m_1c(\lambda_\mu) > \tau, \quad 1 \leq \mu \leq s.
\]
For simplicity of presentation, we will also assume the following non-overlapping condition.

**Assumption 3.2.** For some fixed constant \( \tau > 0 \), we have

\[
\min_{1 \leq j \leq r} \delta^a_{\lambda_1(\alpha), \lambda_1(\beta)} \wedge \min_{1 \leq \mu \leq s} \delta^b_{\lambda_1(\beta), \lambda_1(\mu)} \geq \tau, \quad 1 \leq i \leq r,
\]

and

\[
\min_{1 \leq \nu \leq s} \delta^b_{\lambda_1(\beta), \lambda_1(\nu)} \wedge \min_{1 \leq i \leq r} \delta^a_{\lambda_1(\alpha), \lambda_1(\mu)} \geq \tau, \quad 1 \leq \mu \leq s.
\]

### 3.1 Estimating the number of spikes

The number of spikes has important meaning in practice. For instance, it represents the number of factors in factor model [11, 12] and number of signals in signal processing [59]. Such a problem has been studied for spiked covariance matrix, see e.g. [44]. In this section, we extend the discussion to the more general spiked separable model [2, 12].

Different from the spiked covariance matrix model, we have two sources of spikes from either \( \hat{A} \) or \( \hat{B} \). For spiked covariance matrices, the statistic only involves sample eigenvalues. However, as we have seen from Theorem 2.13, the sample eigenvectors only contain information of the total number of spikes, i.e. \( r + s \). One way to deal with this issue is to use the information from the sample eigenvectors and apply Theorem 2.15.

For an illustration, we refer the reader to Figure 1. In the following discussion, we assume that the population eigenvectors of \( \hat{A} \) and \( \hat{B} \) are known. For the more general case where such information is unavailable, we will study it somewhere else (see also Remark 3.3).

We provide our statistic and start with a heuristic discussion. Note that under Assumptions 3.1 and 3.2, we get from Theorems 2.12 and 2.15 that

\[
\tilde{\lambda}_{\alpha(i)} = \theta_1(\tilde{\sigma}^a_1) + O_\prec(n^{-1/2}), \quad |\langle \mathbf{v}^a_i, \mathbf{\tilde{\xi}}_k \rangle|^2 = 1(k = \alpha(i)) \frac{1}{\theta_1(\tilde{\sigma}^a_1)} + O_\prec(n^{-1/2}), \quad 1 \leq i \leq r.
\]

Hence, if all the spiked eigenvalues are well-separated, the ratio between \( \tilde{\lambda}_{\alpha(i)} \) and \( \tilde{\lambda}_{\alpha(i+1)} \) are strictly greater than 1. However, for the non-outlier eigenvalues, these ratios will converge to 1 at a rate \( O_\prec(n^{-2/3}) \) by Theorem 2.13. Moreover, the (cosine of) the angles \(|\langle \mathbf{v}^a_i, \mathbf{\tilde{\xi}}_k \rangle|\) is of order \( O_\prec(n^{-1/2}) \) except when \( k = \alpha(i) \), in which case we have that \(|\langle \mathbf{v}^a_i, \mathbf{\tilde{\xi}}_k \rangle|\) is larger than a constant. Therefore, the ratios between consecutive eigenvalues and the angles will be used as our statistics.

Formally, for a given threshold \( \omega > 0 \) and a properly chosen constant \( c > 0 \), we define the statistic \( q \) by

\[
q = q(\omega) := \arg \min_{1 \leq i \leq c(n \wedge n)} \left\{ \frac{\tilde{\lambda}_{i+1}}{\tilde{\lambda}_{i+2}} - 1 \leq \omega \right\}, \quad (3.1)
\]

and \( q_{a,b} = q_{a,b}(\omega) \) by

\[
q_a(\omega) := \arg \min_{1 \leq i \leq c(n \wedge n)} \left\{ \max_{1 \leq k \leq c(n \wedge n)} \left| \langle \mathbf{v}^a_{i+1}, \mathbf{\tilde{\xi}}_k \rangle \right|^2 \leq \omega \right\}, \quad q_b(\omega) := \arg \min_{1 \leq \mu \leq c(n \wedge n)} \left\{ \max_{1 \leq \nu \leq c(n \wedge n)} \left| \langle \mathbf{v}^b_{\mu+1}, \mathbf{\tilde{\xi}}_{\nu} \rangle \right|^2 \leq \omega \right\}.
\]

As discussed above, \( q \) is used to estimate the total number of spikes, whereas \( q_a \) and \( q_b \) are used to estimate the number of spikes for \( \hat{A} \) and \( \hat{B} \), respectively. With Theorems 2.12, 2.13, 2.15, 2.19, 2.22 and 2.23 it is easy to show that they are consistent estimators for carefully chosen threshold \( \omega \). Denote the event \( \Omega = \Omega(\omega) \) by

\[
\Omega := \{ q = r + s, q_a = r, q_b = s \}.
\]
Eigenvalues for Cases I and II. Here $p = 150, n = 200$.

(a) Eigenvalues for Cases I and II.

(b) Eigenvectors of $\tilde{Q}^1$: $|\langle v_a^p, \xi_{\alpha(1)} \rangle|^2$.

(c) Eigenvectors of $\tilde{Q}^2$: $|\langle v_b^p, \xi_{\beta(1)} \rangle|^2$.

Figure 1: Eigenvalues and eigenvectors for spiked separable covariance matrices. We consider two different settings:

Case I

$\tilde{\Sigma}^a = \text{diag}(5, 1, \cdots, 1), \quad \tilde{\Sigma}^b = \text{diag}(5, 1, \cdots, 1),$

and

Case II

$\tilde{\Sigma}^a = \text{diag}(3, 2, 1, \cdots, 1), \quad \tilde{\Sigma}^b = \text{diag}(1, 1, \cdots, 1).$

Figure (a) shows that there are two spikes in both cases. However, from Figure (b) and Figure (c), we can see that there are two parts of spikes in Case I, but only one part in Case II as expected. It shows the necessity to take into consideration the information from the eigenvectors.

Theorem 3.3. Suppose that the Assumptions 2.1, 2.5, 3.1 and 3.2 hold. Then if $\omega$ satisfies that for some constant $\varepsilon > 0$,

$\omega \rightarrow 0, \quad \omega n^{2/3 - \varepsilon} \rightarrow \infty,$

then we have that $\Omega$ holds with high probability for large enough $n$. 

16
Proof. This theorem is an easy consequence of Theorems 2.12, 2.13, 2.15, 2.19, 2.22 and 2.23.

For the practical implementation, we employ a resampling procedure to choose the threshold $\omega$ using a reference matrix. Such procedure has been used in estimating the number of spikes for spiked covariance matrix [44]. In the null case with $r = s = 0$, we conclude from the edge universality result [58 Theorem 2.7] that the edge eigenvalues of the non-spiked $Q_{1,2}$ fluctuate on the scale $n^{-2/3}$. Since the edge eigenvalues of Wishart matrix (i.e., $XX^*$ with the entries of $X$ being i.i.d. Gaussian) satisfy the Tracy-Widom distribution up to an $n^{-2/3}$ rescaling, the edge eigenvalue ratios of $Q_{1,2}$ should be close to those of the Wishart matrix. More precisely, we can use Wishart matrix as the reference matrix and take the following steps to choose $\omega$.

**Step (i):** Generate a sequence of $N$ (say $N = 10^4$), $p \times n$ Wishart matrices $X_iX_i^*$ and the associated sequence of statistics $\{T_i\}_{i=1}^N$,

$$T_i := \max_{1 \leq k \leq c(p \wedge n)} \left\{ \frac{\lambda_k^{(i)}}{\lambda_{k+1}^{(i)}} \right\},$$

where $\{\lambda_k^{(i)}\}_{k=1}^{p \wedge n}$ are the eigenvalues of $X_iX_i^*$ arranged in descending order.

**Step (ii):** Given the nominal level $\varepsilon$ (say $\varepsilon = 0.05$), we choose $\omega$ such that

$$\frac{\# \{ T_i \leq 1 + \omega \}}{N} \geq 1 - \varepsilon.$$

We next use some Monte-Carlo simulations to verify the validity of our statistics. Consider the setting

$$\tilde{A} = \text{diag}(4,1,\cdots,1), \quad \tilde{B} = \text{diag}(x+2,x,\cdots,1), \quad x \geq 1.$$

In Figure 3, we consider the estimation of the number of spikes of $\tilde{B}$ and analyze the frequency (over $10^4$ simulations) of misestimation as a function of the value of $x$ under different combinations of $p$ and $n$. We make use of the statistic $q_b$ and choose $\omega$ according to the above steps (i) and (ii). We can see that our estimator performs quite well for $x$ above some threshold.

![Figure 2: Frequency of misestimation for different values of $x$.](image)
3.2 Adaptive optimal shrinkage for spiked separable covariance matrices

In most of the real applications, we have no a priori information on the true eigenvectors of $\hat{A}$ or $\hat{B}$. Then the natural choice for us is to use the sample eigenvectors $\{\hat{\xi}_i\}_{1 \leq i \leq p}$ and $\{\hat{\zeta}_\mu\}_{1 \leq \mu \leq n}$. Consider Johnstone’s spiked covariance model \cite{JS01} with $A = I_p$ and $B = I_n$. Suppose we know the number of spikes $r + s$. Then we want to estimate

$$\hat{A} = \sum_{i=1}^{r+s} \sigma^2_i \hat{\xi}_i^*(\hat{\xi}_i)^* + \sum_{i=r+1}^{p} \hat{\xi}_i^*(\hat{\xi}_i)^*, \quad \hat{B} = \sum_{\mu=1}^{s} \sigma^2_\mu \hat{\zeta}_\mu^*(\hat{\zeta}_\mu)^* + \sum_{\mu=s+1}^{n} \hat{\zeta}_\mu^*(\hat{\zeta}_\mu)^*,$$

using the estimators

$$\hat{A} = \sum_{i=1}^{r+s} g_a(\hat{\lambda}_i) \hat{\xi}_i^* \hat{\xi}_i^* + \sum_{i=r+s+1}^{p} \hat{\xi}_i^* \hat{\xi}_i^*, \quad \hat{B} = \sum_{\mu=1}^{s} g_b(\hat{\lambda}_\mu) \hat{\zeta}_\mu^* \hat{\zeta}_\mu^* + \sum_{\mu=s+1}^{n} \hat{\zeta}_\mu^* \hat{\zeta}_\mu^*, \quad (3.3)$$

where $g^a(\cdot)$ and $g^b(\cdot)$ are some shrinkage functions characterized by the minimizers of certain loss functions:

$$\hat{A} := \arg\min_A \mathcal{L}_a(A, \hat{A}), \quad \hat{B} := \arg\min_B \mathcal{L}_b(B, \hat{B}).$$

In \cite{JS01}, the authors consider this problem for spiked covariance matrices for a variety of loss functions assuming that $r, s$ are known. In this section, we study this problem for spiked separable covariance matrices using the Frobenius norm as the loss functional. We will also prove the optimal convergent rate for such estimators. The other loss functions as discussed in \cite{JS01} can be studied in a similar way.

We shall only consider $g_a(\lambda_i)$, while $g_b(\lambda_\mu)$ can be handled with the same argument by symmetry. We calculate that

$$\|\hat{A} - \hat{A}\|_F^2 = \|T\|_F^2, \quad T := \sum_{i=1}^{r+s} \left( (g(\hat{\lambda}_i) - 1)\hat{\xi}_i^* \hat{\xi}_i^* - (\hat{\sigma}_i^a - 1)\hat{\psi}_i^a(\hat{\psi}_i^a)^* \right). \quad (3.4)$$

We expand $T$ to get

$$\|T\|_F^2 = \sum_{i=1}^{r+s} \left[ (g_a(\hat{\lambda}_i) - 1)^2 + (\hat{\sigma}_i^a - 1)^2 - 2(\hat{\psi}_i^a, \hat{\xi}_i^*)^2(g_a(\hat{\lambda}_i) - 1)(\hat{\sigma}_i^a - 1) \right] - 2 \sum_{i \neq j} (g_a(\hat{\lambda}_i) - 1)(\hat{\sigma}_i^a - 1)(\hat{\psi}_i^a, \hat{\xi}_j^*).$$

Therefore, (3.4) is minimized if

$$g_a(\hat{\lambda}_i) = 1 + \sum_{j=1}^{r+s} (\hat{\sigma}_j^a - 1)(\hat{\psi}_j^a, \hat{\xi}_i^*).$$

Under Assumptions \ref{assumption} and \ref{assumption2} by Theorems \ref{thm1} and \ref{thm2} we find that for $\hat{\sigma}_k^a := d_k^a + 1,$

$$g_a(\hat{\lambda}_i) = 1(i = \alpha(k) \text{ for some } k = 1, \ldots, r) \frac{d_k^a}{d_k^a} g_k(\frac{-d_k^a}{d_k^a}) + O_{\alpha}(n^{-1/2}).$$

Under the setting with $A = I_p$ and $B = I_n$, $m_2(z)$ is the Stieltjes transform of the standard Marchenko-Pastur (MP) law. Then it is known that $g_2_\alpha$ is given by [30 Section 2.2]

$$g_2_\alpha(x) = \frac{-1}{x} + \frac{1}{x} + 1,$$

where recall that $d_n = p/n$. Therefore, we can calculate that

$$g_a(\hat{\lambda}_i) = \frac{(d_k^a)^2 - d_n}{(d_k^a)^2 + d_n} + O_{\alpha}(n^{-1/2}), \quad i = \alpha(k).$$
For $d_k^2$, we can use Theorem 2.2 to get that $d_k^2 = -m_{2c}^{-1}(\lambda_i) - 1 + O_\prec(n^{-1/2})$ for $i = \alpha(k)$. We have the following explicit form for $m_{2c}$ (see e.g. (4.10) of [12]):

$$m_{2c}(x) = \frac{d_n - 1 - x + \sqrt{(x - \lambda_+)(x - \lambda_-)}}{2x}, \quad \lambda_\pm = (1 \pm d_n^{1/2})^2,$$

when $x > \lambda_r := \lambda_+$. Thus we can define the following shrinkage function

$$\hat{g}_a(\hat{\lambda}_i) = 1(i = \alpha(k)) \text{ for } k = 1, \cdots, r, \quad \frac{(\hat{d}_k^2 - d_n)}{d_n + d_m}, \quad \hat{d}_k^2 = -m_{2c}^{-1}(\hat{\lambda}_{\alpha(k)}) - 1,$$

which satisfies that

$$\hat{g}_a(\hat{\lambda}_i) = 1(i = \alpha(k)) \text{ for } k = 1, \cdots, r, \quad \frac{(\hat{d}_k^2 - d_n)}{d_n + d_m}, \quad \hat{d}_k^2 = -m_{2c}^{-1}(\hat{\lambda}_{\alpha(k)}) - 1,$$

Remark 3.4. Note that the definition of the shrinkage function depends on a priori knowledge of the indices of the outliers caused by the spikes of $\hat{A}$, which may not be available in applications. Moreover, the methods in Section 3.1 cannot be used since we have no information on the eigenvectors of $\hat{A}$ and $\hat{B}$. However, this is still possible by exploring the “cone condition”, see Example 2.17, i.e. we can project the left and right outlier-singular vectors onto some suitably chosen directions and take average over many samples. To have a rigorous theory, it is necessary to establish the second order asymptotics of the outlier eigenvectors. Both of these topics will be explored elsewhere.

Now we present the results of some Monte-Carlo simulations designed to illustrate the finite-sample properties of the shrinkage estimator $\hat{A}$. We study the improvement of $\hat{A}$ over the separable covariance matrix $\hat{Q}_1$, which also uses the sample eigenvectors. Denote $\hat{A}$ as in [33] with $\hat{g}_a(\hat{\lambda}_i)$ replaced by $\hat{g}_a(\hat{\lambda}_i)$. We report the Percentage Relative Improvement in Average Loss (PRIAL) [33] Section 1.3] for $\hat{A}$:

$$\text{PRIAL} := 100 \times \left\{ 1 - \frac{E\|\hat{A} - \hat{\hat{A}}\|_F^2}{E\|\hat{Q}_1 - \hat{\hat{A}}\|_F^2} \right\},$$

where $E(\cdot)$ denotes the average over $10^4$ Monte-Carlo simulations. We can see that our estimators perform better than sample separable covariance matrix even for “not so large” matrix dimensions.

Figure 3: PRIAL against matrix dimension $n$. We consider the setting $\hat{A} = \text{diag}(8, 5, 1, \cdots, 1)$ and $\hat{B} = \text{diag}(3, 1, \cdots, 1)$.
Before concluding this section, we provide a useful result for the estimation of spikes. By Theorem 2.12, we need to know the form of $m_{2c}$ in order to estimate the spikes of $\tilde{A}$. However, thanks to the anisotropic local law in Theorem 4.10 below, it is possible to use an adaptive estimator for the spikes of $\tilde{A}$ based only on the data matrices $Q_2$ if $\tilde{B}$ is a small-rank perturbation of the identity matrix. We define

$$\tilde{\sigma}_i^a := -\frac{1}{n} \sum_{\nu=r+s+1}^{n} \frac{1}{\lambda_{\nu}(Q_2) - \lambda_{\alpha(i)}}^{-1}, \quad 1 \leq i \leq r + s.$$  

Similarly, if $A$ is a small-rank perturbation of the identity matrix, then we have the following estimator for the spikes of $\tilde{B}$:

$$\tilde{\sigma}_\mu^b := -\frac{1}{n} \sum_{k=r+s+1}^{n} \frac{1}{\lambda_k(Q_1) - \lambda_{\beta(\mu)}}^{-1}, \quad 1 \leq \mu \leq r + s.$$  

We claim the following result, whose proof will be presented in Section 4.2 after we state Theorem 4.10.

**Theorem 3.5.** Suppose that the Assumptions 2.1, 2.5 and 3.1 hold. Suppose $\tilde{B} = I_n + M_n$, where $M_n$ is a matrix of rank $l_n$. Then we have that for $1 \leq i \leq r$

$$\tilde{\sigma}_i^a = \tilde{\sigma}_i^a + O_r(l_n n^{-1/2}). \quad (3.5)$$  

Similarly, if $\tilde{A}$ is an $l_n$-rank perturbation of the identity matrix, then for $1 \leq \mu \leq s$,

$$\tilde{\sigma}_\mu^b = \tilde{\sigma}_\mu^b + O_r(l_n n^{-1/2}). \quad (3.6)$$  

Finally, we use some Monte-Carlo simulations to illustrate the accuracy of the above estimators. We set $\tilde{A} = \text{diag}(\tilde{\sigma}^a, 1, \cdots, 1), \quad \tilde{B} = \text{diag}(3, 1, \cdots, 1)$.  

In Table 1 we give the estimation of $\tilde{\sigma}^a$ using $\tilde{\sigma}^a$ for various combinations of $p$ and $n$. Each value is recorded by taking an average over 2,000 simulations. We find that our estimator is quite accurate even for a small sample size.

| $\tilde{\sigma}^a/(p, n)$ | (100, 200) | (200, 400) | (300, 400) | (400, 300) | (500, 400) |
|--------------------------|------------|------------|------------|------------|------------|
| 4                        | 3.67       | 3.58       | 3.83       | 4.61       | 4.43       |
| 5                        | 4.78       | 4.65       | 4.84       | 5.49       | 5.37       |
| 8                        | 7.75       | 7.62       | 7.86       | 8.47       | 8.33       |
| 10                       | 9.83       | 9.65       | 9.88       | 10.51      | 10.37      |
| 15                       | 14.95      | 14.86      | 14.93      | 15.56      | 15.42      |

Table 1: The value of $\tilde{\sigma}^a$. We record the average of $\tilde{\sigma}^a$ over 2,000 simulations.

4 Basic tools

In this section, we collect some tools that will be used in the proof of Theorems 2.12, 2.13, 2.15 and 2.19. We first record the following lemma for matrix perturbation, which follows from a simple algebraic calculation.
Lemma 4.1 (Woodbury matrix identity). For $A, S, B, T$ of conformable dimensions, we have

$$(A + SBT)^{-1} = A^{-1} - A^{-1}S(B^{-1} + T^{-1}S)^{-1}TA^{-1}. \tag{4.1}$$

as long as all the operations are legitimate. As a special case, we have the following Hua’s identity:

$$A - A(A + B)^{-1}A = B - B(A + B)^{-1}B \tag{4.2}$$

if $A + B$ is non-singular.

We also need the following eigenvalue interlacing result for our model \footnote{Recall that the eigenvalues of $Q_1$ and $Q_1$ are denoted by $\{\lambda_i\}$ and $\{\lambda_i\}$, respectively. Then we have

$$\lambda_i \in [\lambda_i, \lambda_{i-r+1}], \tag{4.2}$$

where we adopt the convention that $\lambda_i = \infty$ if $i < 1$ and $\lambda_i = 0$ if $i > p$.}

Proof. We first consider the rank one deformation with $r = 1$ and $s = 0$: $\tilde{A} = (1 + d^a v^a(v^a)^*)A$ with $d^a > 0$ and $v^a$ being an eigenvector of $A$. Then we have

$$\tilde{G}_1 = \frac{1}{P^{1/2}A^{1/2}XBX^*A^{1/2}P^{1/2} - z} = P^{-1/2} \left[ G_1^{-1} + v^a \frac{zd^a}{d^a + 1} (v^a)^* \right]^{-1} P^{-1/2}, \tag{4.3}$$

where $P := 1 + d^a v^a(v^a)^*$. Then applying Lemma 4.1 to (4.3), we obtain that

$$(\tilde{G}_1)_{v^a v^a} = \left( \frac{(G_1)_{v^a v^a}}{d^a + 1} - \frac{(G_1)^{2}_{v^a v^a}}{d^a + 1} \right) \frac{z}{(d^a)^{-1} + 1 + z(G_1)_{v^a v^a}}. \tag{4.4}$$

Thus we get

$$\frac{1}{(\tilde{G}_1)_{v^a v^a}} = \frac{1 + d^a}{(G_1)_{v^a v^a}} + zd^a. \tag{4.5}$$

We denote the eigenvectors of $Q_1$ and $\tilde{Q}_1$ by $\{\xi_k\}_{k=1}^p$ and $\{\tilde{\xi}_k\}_{k=1}^p$, respectively. Then writing (4.5) in spectral decomposition gives

$$(d^a + 1) \left( \sum_k \frac{|(v^a, \xi_k)^2|}{\lambda_k - z} \right)^{-1} = \left( \sum_k \frac{|(v^a, \tilde{\xi}_k)^2|}{\lambda_k - z} \right)^{-1} - zd^a. \tag{4.6}$$

By adding a small perturbation to $Q_1$, we may assume without loss of generality that (i) $\lambda_1, \ldots, \lambda_p$ are all positive and distinct, and (ii) all $\langle v^a, \xi_k \rangle$ and $\langle v^a, \tilde{\xi}_k \rangle$ are nonzero. Note that since eigenvalues and eigenvectors depend continuously on the matrix entries, we can remove the arbitrarily small perturbation and obtain the corresponding result for the original matrices $Q_1$ and $\tilde{Q}_1$. Moreover, it is always possible to choose such perturbation. For example, we can add a matrix $\varepsilon H$, where the entries of $H$ are bounded and have absolutely continuous densities. Then (i) and (ii) hold with probability 1 for any $\varepsilon > 0$. Thus there must exist a realization of $H$ such that (i) and (ii) hold for $Q_1 + \varepsilon H$ and $\tilde{Q}_1 + \varepsilon H$.

By (i) and (ii), the left-hand side of (4.6) defines a function of $z \in (0, \infty)$ with $(p - 1)$ poles and $p$ zeros. The function is smooth and decreasing away from the singularities, and its zeros are $\lambda_1, \ldots, \lambda_p$. Now using
the fact that $z$ is an eigenvalue of $\tilde{Q}_1$ if and only if the left-hand side of (4.6) is equal to $-zd^n < 0$, we obtain the interlacing property (4.2) for $r = 1$ and $s = 0$.

Next, for the case $r = 0$ and $s = 1$, we conclude the proof easily by applying (4.12) to $\tilde{Q}_2$ and using the fact that $\tilde{Q}_2$ have the same nonzero eigenvalues as $\tilde{Q}_1$. Note that the above arguments are purely deterministic. They work for any non-negative definite matrix $A^{1/2}XBX^*A^{1/2}$ and any rank one deformation of the form $\tilde{A} = A(1 + d^a v^a(v^a)^*)$ or $\tilde{B} = B(1 + d^b v^b(v^b)^*)$, where

$$\tilde{A} = A \left( 1 + d^a v^a(v^a)^* \right) \quad \text{or} \quad \tilde{B} = B \left( 1 + d^b v^b(v^b)^* \right),$$

with $d^a > 0$, $d^b > 0$, and $v^a$ and $v^b$ being eigenvectors of $A$ and $B$, respectively. Then the general case (4.12) with any finite $r, s = O(1)$ follows from a simple induction. 

4.1 Properties of limiting laws

For any constants $\varsigma_1, \varsigma_2 > 0$, we denote a domain of the spectral parameter $z$ as

$$S(\varsigma_1, \varsigma_2) := \{ z = E + i\eta : \lambda_r - \varsigma_1 \leq E \leq \varsigma_2 \lambda_r, \ 0 < \eta \leq 1 \}. \quad (4.7)$$

For $z = E + i\eta$, we define the distance to the rightmost edge as

$$\kappa := \kappa_E := |E - \lambda_r|. \quad (4.8)$$

Then we have the following lemma, which summarizes some basic properties of $m_{1,2c}$ and $\rho_{1,2c}$.

**Lemma 4.3.** Suppose Assumptions (2.7) and (2.9) hold. Then there exists sufficiently small constant $\varsigma_1 > 0$ such that the following estimates hold:

(i) 

$$\rho_{1,2c}(x) \sim \sqrt{\lambda_r - x}, \quad \text{for } x \in [\lambda_r - 2\varsigma_1, \lambda_r]; \quad (4.9)$$

(ii) for $z = E + i\eta \in S(\varsigma_1, \varsigma_2)$,

$$|m_{1,2c}(z)| \sim 1, \quad \text{Im} m_{1,2c}(z) \sim \begin{cases} \frac{\sqrt{\kappa + \eta}}{\sqrt{\kappa + \eta}}, & \text{if } E \geq \lambda_r, \\ \sqrt{\frac{\kappa + \eta}{\kappa + \eta}}, & \text{if } E \leq \lambda_r, \end{cases}, \quad (4.10)$$

and

$$|\text{Re} m_{1,2c}(z) - m_{1,2c}(\lambda_r)| \sim \begin{cases} \frac{\sqrt{\kappa + \eta}}{\sqrt{\kappa + \eta} + \kappa}, & \text{if } E \geq \lambda_r, \\ \sqrt{\frac{\kappa + \eta}{\kappa + \eta} + \kappa}, & \text{if } E \leq \lambda_r, \end{cases}; \quad (4.11)$$

(iii) there exists constant $\tau' > 0$ such that

$$\min_1 |1 + m_{1c}(z)\sigma^h_1| \geq \tau', \quad \min_1 |1 + m_{2c}(z)\sigma^h_1| \geq \tau', \quad (4.12)$$

for any $z \in S(\varsigma_1, \varsigma_2)$.

The above estimates (i)-(iii) also hold for $z$ on the real axis, i.e., $z \in S(\varsigma_1, \varsigma_2)$. Finally, the estimates (4.9)-(4.11) also hold for $\rho_{c}$ and $m_{c}$.

**Proof.** The estimates (4.9), (4.10) and (4.12) have been proved in [58, Lemma 3.4]. The estimate (4.11) follows directly from (4.25).
Lemma 4.4. Suppose that Assumptions 2.1 and 2.5 hold. For \( \sigma_1 \geq -m_{1c}^{-1}(\lambda_r) \) and \( \sigma_2 \geq -m_{2c}^{-1}(\lambda_r) \), we have
\[
\theta_1(\sigma_2) - \lambda_r = g_{2c}(\sigma_2^{-1}) - \lambda_r \sim (\sigma_2 + m_{2c}^{-1}(\lambda_r))^2, \quad \theta_2(\sigma_1) - \lambda_r = g_{1c}(\sigma_1^{-1}) - \lambda_r \sim (\sigma_1 + m_{1c}^{-1}(\lambda_r))^2. \tag{4.13}
\]
For \( x > \lambda_r \) and \( m_{1,2} > m_{1,2c}(\lambda_r) \), we have
\[
m'_{2c}(x) \sim \kappa_x^{-1/2}, \quad m'_{1c}(x) \sim \kappa_x^{-1/2}, \tag{4.14}
g'_{2c}(m_2) \sim (m_2 - m_{2c}(\lambda_r)), \quad g'_{1c}(m_1) \sim (m_1 - m_{1c}(\lambda_r)). \tag{4.15}
\]
Moreover, the above estimates imply that
\[
m'_{2c}(\theta_1(\sigma_2)) \sim (\sigma_2 + m_{2c}^{-1}(\lambda_r))^{-1}, \quad m'_{1c}(\theta_2(\sigma_1)) \sim (\sigma_1 + m_{1c}^{-1}(\lambda_r))^{-1}, \tag{4.16}
g'_{2c}(-\sigma_2^{-1}) \sim \sigma_2 + m_{2c}^{-1}(\lambda_r), \quad g'_{1c}(-\sigma_1^{-1}) \sim \sigma_1 + m_{1c}^{-1}(\lambda_r). \tag{4.17}
\]
Proof. With the definitions \ref{eq:2.24} and \ref{eq:2.25}, we can obtain that
\[
-\sigma_2^{-1} = m_{2c}(\theta_1(\sigma_2)) = m_{2c}(\lambda_r) + \pi a_2 \sqrt{\theta_1(\sigma_2) - \lambda_r} + O(|\theta_1(\sigma_2) - \lambda_r|),
\]
if \( \theta_1(\sigma_2) - \lambda_r \ll \zeta_1 \) for some sufficiently small constant \( 0 < \zeta_1 < 1 \), and
\[
-\sigma_2^{-1} = m_{2c}(\theta_1(\sigma_2)) \gg m_{2c}(\lambda_r + \zeta_1) = m_{2c}(\lambda_r) + \pi a_2 \sqrt{\zeta_1} + O(\zeta_1),
\]
if \( \theta_1(\sigma_2) - \lambda_r \gg \zeta_1 \), where in the second inequality we use the fact that \( m_{2c}(x) \) is monotone increasing when \( x > \lambda_r \). The above two estimates imply the first estimate in \ref{eq:4.13}. The second estimate in \ref{eq:4.13} can be proved in the same way.

Differentiating the equation \( f(z,m) = 0 \) in \ref{eq:2.20} with respect to \( m \), we can get that \( z'(m_r) = 0 \) and \( z''(m_r) = -c^2 f(\lambda_r, m_r)/\partial_z f(\lambda_r, m_r) \), where \( m_r := m_{2c}(\lambda_r) \). It was proved in \cite{58} Lemma 2.6 that \( z''(m_r) \sim 1 \) under the assumptions \ref{eq:2.6} and \ref{eq:2.23}. Moreover, using implicit differentiation of the equation \( f(z,m) = 0 \) and \ref{eq:4.12}, it is easy to show that \( z^{(3)}(m) = O(1) \) if \( m_r - c \leq m \leq 0 \) for some sufficiently small constant \( c > 0 \). Hence we conclude that
\[
z'(m) = O(|m - m_r|), \quad \text{for} \quad m_r - c \leq m \leq 0. \tag{4.18}
\]
This implies the first estimate in \ref{eq:4.13}. Since \( m_{2c} \) is the inverse function of \( g_{2c} \), we get from the inverse function theorem that
\[
m'_{2c}(x) = \frac{1}{g'_{2c}(m_{2c}(x))} \sim (m_{2c}(x) - m_{2c}(\lambda_r))^{-1} \sim \kappa_x^{-1/2},
\]
where we used \ref{eq:2.24} in the last step. This implies the first estimate in \ref{eq:4.14}. Now taking \( x = \theta_1(\sigma_2) \) and \( m_2 = -\sigma_2^{-1} \) in the first estimates in \ref{eq:4.14} and \ref{eq:4.15}, respectively, and using \ref{eq:4.13}, we obtain the first estimates in \ref{eq:4.10} and \ref{eq:4.17}.

Exchanging the roles of \( (A, m_{1c}, g_{1c}) \) and \( (B, m_{2c}, g_{2c}) \), one can prove the second estimates in \ref{eq:4.13}-\ref{eq:4.17} in the same way.

In the proof, it is important to extend the real functions \( g_{1c} \) and \( g_{2c} \) to the complex plane. The following lemma can be proved with a simple complex analytical argument.
Lemma 4.5. Suppose the assumptions of Lemma 4.4 hold. Then for any constant \( \zeta > 0 \), there exist constants \( \tau_0, \tau_1, \tau_2 > 0 \) such that the following statements hold.

(i) \( m_{1c} \) and \( m_{2c} \) are holomorphic homeomorphisms on the spectral domain

\[ D(\tau_0, \zeta) := \{ z = E + i\eta : \lambda_r < E < \zeta, \ -\tau_0 < \eta < \tau_0 \} \]

As a consequence, the inverse functions of \( m_{1c} \) and \( m_{2c} \) exist and we again denote them by \( g_{1c} \) and \( g_{2c} \), respectively.

(ii) We have \( D_1(\tau_1, \zeta) \subset m_{1c}(D(\tau_0, \zeta)) \) and \( D_2(\tau_2, \zeta) \subset m_{2c}(D(\tau_0, \zeta)) \), where

\[ D_1(\tau_1, \zeta) := \{ \xi = E + i\eta : m_{1c}(\lambda_r) < E < m_{1c}(\zeta), \ -\tau_1 < \eta < \tau_1 \} \]

and

\[ D_2(\tau_2, \zeta) := \{ \xi = E + i\eta : m_{2c}(\lambda_r) < E < m_{2c}(\zeta), \ -\tau_2 < \eta < \tau_2 \} \]

In other words, \( g_{1c} \) and \( g_{2c} \) are holomorphic homeomorphisms on \( D_1(\tau_1, \zeta) \) and \( D_2(\tau_2, \zeta) \), respectively.

(iii) For \( z \in D(\tau_0, \zeta) \), we have

\begin{equation}
|m_{1c}(z) - m_{1c}(\lambda_r)| \sim |z - \lambda_r|^{1/2}, \quad |m_{2c}(z) - m_{2c}(\lambda_r)| \sim |z - \lambda_r|^{1/2},
\end{equation}

and

\begin{equation}
|m'_{1c}(z)| \sim |z - \lambda_r|^{-1/2}, \quad |m'_{2c}(z)| \sim |z - \lambda_r|^{-1/2}.
\end{equation}

(iv) For \( \xi \in D_1(\tau_1, \zeta) \) and \( \zeta \in D_2(\tau_2, \zeta) \), we have

\begin{equation}
|g_{1c}(\xi) - \lambda_r| \sim |\xi - m_{1c}(\lambda_r)|^2, \quad |g_{2c}(\zeta) - \lambda_r| \sim |\zeta - m_{2c}(\lambda_r)|^2,
\end{equation}

and

\begin{equation}
|g'_{1c}(\xi)| \sim |\xi - m_{1c}(\lambda_r)|, \quad |g'_{2c}(\zeta)| \sim |\zeta - m_{2c}(\lambda_r)|.
\end{equation}

(v) For \( z_1, z_2 \in D(\tau_0, \zeta) \), \( \xi_1, \xi_2 \in D_1(\tau_1, \zeta) \) and \( \zeta_1, \zeta_2 \in D_2(\tau_2, \zeta) \), we have

\begin{equation}
|m_{1c}(z_1) - m_{1c}(z_2)| \sim |m_{2c}(z_1) - m_{2c}(z_2)| \sim \frac{|z_1 - z_2|}{\max_{i=1,2} |z_i - \lambda_r|^{1/2}},
\end{equation}

and

\begin{equation}
|g_{1c}(\xi_1) - g_{1c}(\xi_2)| \sim |\xi_1 - \xi_2| \cdot \max_{i=1,2} |\xi_i - m_{1c}(\lambda_r)|, \quad |g_{2c}(\zeta_1) - g_{2c}(\zeta_2)| \sim |\zeta_1 - \zeta_2| \cdot \max_{i=1,2} |\zeta_i - m_{2c}(\lambda_r)|.
\end{equation}

Proof. For the proof, we choose a sufficiently small constant \( \omega > 0 \) such that (2.25) can be applied to \( z \in D_{\omega} := \{ z = E + i\eta : 0 < E - \lambda_r < 2\omega, -\omega < \eta < \omega \} \). We also define the spectral domain \( \bar{D}_{\omega} := \{ z = E + i\eta : 0 < E - \lambda_r < \omega, -\omega < \eta < \omega \} \). Then the constants \( \tau_0, \tau_1, \tau_2 > 0 \) will be chosen such that they are much smaller than \( \omega \). Without loss of generality, we only prove the relevant statements for \( m_{2c} \) and \( g_{2c} \).

Note that \( m_{2c} \) is holomorphic on \( \mathbb{C} \setminus [0, \lambda_r] \). By (2.25), we see that \( m_{2c} \) is a holomorphic homeomorphism for \( z \in D_{\omega} \) as long as \( \omega \) is sufficiently small. Moreover, we have

\begin{equation}
g_{2c}(\xi) = \frac{1}{\pi i a_2^2} (\xi - m_{2c}(\lambda_r))^2 + O((\xi - m_{2c}(\lambda_r))^3), \quad \xi \in m_{2c}(D_{\omega}).
\end{equation}
On the other hand, with (4.20) it is easy to see that there exists a constant $c_{\omega', \zeta} > 0$ such that $m_{2c}(x) \geq c_{\omega', \zeta}$ for all $\lambda + \omega < x < \zeta$. Then combining the implicit function theorem, analytic continuation and a compactness argument, we can conclude statement (i). The statement (ii) follows immediately from that

$$\text{Im} m_{2c}(E + i\eta) = \eta \int_0^{\lambda_r} \frac{\rho_{2c}(x)dx}{(x - E)^2 + \eta^2} \geq \eta.$$  

The estimates in (iii) and (iv) can be proved using (2.25), (4.25), and implicit differentiation of the equation $f(z, m) = 0$ as in the proof for Lemma 4.4. We leave the details to the reader. Finally, notice that (4.24) follows directly from (1.24) together with (1.24). Thus it only remains to prove (4.23).

The upper bound in (4.23) is given by (4.20). We only need to show the lower bound. Without loss of generality, we assume that $|z_1 - \lambda_r| \geq |z_2 - \lambda_r|$. We consider the following three cases: (i) $z_1, z_2 \in D_{\omega}$; (ii) $z_1, z_2 \in D(\tau_0, \zeta) \setminus D_\omega$; (iii) $z_1 \in D(\tau_0, \zeta) \setminus D_\omega$ and $z_2 \in D(\tau_0, \zeta) \cap \bar{D}_\omega$.

In case (i), first suppose that $|z_1 - z_2| \leq |z_1 - \lambda_r|/2$. Then (4.23) follows from the mean value theorem by using (4.20) and the fact that $|\xi - \lambda_r| \sim |z_1 - \lambda_r|$ for any $\xi$ on the line between $z_1$ and $z_2$. Now for $|z_1 - z_2| \geq |z_1 - \lambda_r|/2$, then by (4.25) we get

$$|m_{2c}(z_1) - m_{2c}(z_2)| \geq \pi a_2 \left( \sqrt{|z_1 - \lambda_r|} - \sqrt{|z_2 - \lambda_r|} \right) + C|z_1 - \lambda_r| \geq c \frac{|z_1 - z_2|}{|z_1 - \lambda_r|^{1/2}}$$

as long as we take $\omega$ to be sufficiently small.

In case (ii), by mean value theorem and (4.20), we have

$$|m_{2c}(z_1) - m_{2c}(z_2)| \sim |z_1 - z_2| \sim \frac{|z_1 - z_2|}{|z_1 - \lambda_r|^{1/2}}.$$  

Finally, in case (iii), we have

$$|m_{2c}(z_1) - m_{2c}(z_2)| \geq |\text{Re} m_{2c}(z_1) - \text{Re} m_{2c}(z_2)|.$$  

(4.26)

Denote $z = E_1 + i\eta_1$ and $z = E_2 + i\eta_2$. Then applying (2.25) to $m_{2c}(z_2)$ and the Stieltjes transform formula to $m_{2c}(z_1)$, we obtain that

$$|\text{Re} m_{2c}(z_1) - m_{2c}(E_1)| \leq C\sqrt{\eta_1}, \quad |\text{Re} m_{2c}(z_2) - m_{2c}(E_2)| \leq C\omega \eta_2.$$  

Together with (4.20), we get that

$$|m_{2c}(z_1) - m_{2c}(z_2)| \geq |m_{2c}(E_1) - m_{2c}(E_2)| - C\sqrt{\eta_1} - C\omega \eta_2 \geq c_\omega$$

as long as we take $\tau_0$ to be small enough. Here we used that $|m_{2c}(E_1) - m_{2c}(E_2)| \sim 1$ since $m_{2c}(x)$ is strictly decreasing.

Combining the above three cases, we get the lower bound in (4.23).

Remark 4.6. As a corollary of (4.20) and (4.23), we see that the following approximate isometry properties hold:

$$|g_{1c}(m_{2c}(z_1)) - g_{1c}(m_{2c}(z_2))| \sim |z_1 - z_2|, \quad |g_{2c}(m_{1c}(z_1)) - g_{2c}(m_{1c}(z_2))| \sim |z_1 - z_2|,$$

(4.27)

and

$$|m_{1c}(g_{2c}(z_1)) - m_{1c}(g_{2c}(z_2))| \sim |\xi_1 - \xi_2|, \quad |m_{2c}(g_{1c}(z_1)) - m_{2c}(g_{1c}(z_2))| \sim |\xi_1 - \xi_2|,$$

(4.28)

for $z_1, z_2 \in D(\tau, \zeta)$, $\xi_1, \xi_2 \in D_1(\tau, \zeta)$ and $\zeta_1, \zeta_2 \in D_2(\tau, \zeta)$ for sufficiently small constant $\tau > 0$.  

25
4.2 Local law

We first introduce a convenient self-adjoint linearization trick, which has been proved to be useful in studying the local laws of random matrices of the Gram type \[11\] [30] [55] [58]. We define the following \((p+n) \times (p+n)\) self-adjoint block matrix, which is a linear function of \(X\):

\[
H = H(X, z) := z^{1/2} \begin{pmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+.
\] (4.29)

where \(z^{1/2}\) is taken to be the branch cut with positive imaginary part. Then we define its resolvent (Green’s function) as

\[
G = G(X, z) := (H(X, z) - z)^{-1}.
\] (4.30)

By Schur complement formula, we can verify that (recall (4.13))

\[
G(z) = \begin{pmatrix} G_1 & z^{-1/2}G_1Y \\ z^{-1/2}Y^*G_1 & G_2 \end{pmatrix} = \begin{pmatrix} G_1 & z^{-1/2}G_1Y \\ z^{-1/2}Y^* & G_2 \end{pmatrix},
\] (4.31)

where \(Y := A^{1/2}XB^{1/2}\). Thus a control of \(G\) yields directly a control of the resolvents \(G_{1,2}\). Similarly, we can define \(\tilde{H}\) and \(\tilde{G}\) by replacing \(A\) and \(B\) with \(\tilde{A}\) and \(\tilde{B}\).

For simplicity of notations, we define the index sets

\[
\mathcal{I}_1 := \{1, \ldots, p\}, \quad \mathcal{I}_2 := \{p+1, \ldots, p+n\}, \quad \mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2.
\]

Then we label the indices of the matrices according to

\[
X = (X_{ij} : i \in \mathcal{I}_1, \mu \in \mathcal{I}_2), \quad \Lambda = (\lambda_{ij} : i, j \in \mathcal{I}_1), \quad B = (B_{\mu \nu} : \mu, \nu \in \mathcal{I}_2).
\]

In the rest of this paper, we will consistently use the latin letters \(i, j \in \mathcal{I}_1\) and greek letters \(\mu, \nu \in \mathcal{I}_2\). Note that for the index \(1 \leq \mu \leq n\) used in previous sections, it can be translated into an index in \(\mathcal{I}_2\) by taking \(\mu \to \mu + p\).

Next we introduce the spectral decomposition of \(G\). Let

\[
A^{1/2}XB^{1/2} = \sum_{k=1}^{p+n} \sqrt{\lambda_k} \xi_k^* \xi_k^*,
\]

be a singular value decomposition of \(A^{1/2}XB^{1/2}\), where

\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{p+n} \geq 0 = \lambda_{p+n+1} = \ldots = \lambda_{p+n}
\]

are the eigenvalues of \(\tilde{G}_1\), \(\{\xi_k\}_{k=1}^p\) and \(\{\xi_k\}_{k=1}^n\) are the left and right singular vectors of \(A^{1/2}XB^{1/2}\), respectively. Then using (4.13), we can get that for \(i, j \in \mathcal{I}_1\) and \(\mu, \nu \in \mathcal{I}_2\),

\[
G_{ij} = \sum_{k=1}^p \frac{\xi_k(i)\xi_k^*(j)}{\lambda_k - z}, \quad G_{\mu\nu} = \sum_{k=1}^n \frac{\xi_k(\mu)\xi_k^*(\nu)}{\lambda_k - z},
\] (4.32)

\[
G_{\mu i} = z^{-1/2} \sum_{k=1}^{p+n} \sqrt{\lambda_k} \xi_k(i)\xi_k^*(\mu) \frac{1}{\lambda_k - z}, \quad G_{\mu i} = z^{-1/2} \sum_{k=1}^{p+n} \sqrt{\lambda_k} \xi_k(\mu)\xi_k^*(i) \frac{1}{\lambda_k - z}.
\] (4.33)

We define the deterministic limit \(\Pi\) of the resolvent \(G\) in (4.30) as

\[
\Pi(z) := \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix}, \quad \Pi_1 := -z^{-1}(1 + m_{2c}(z)A)^{-1}, \quad \Pi_2 := -z^{-1}(1 + m_{1c}(z)B)^{-1}.
\] (4.34)
Define the control parameter \( z \) for

\[
\frac{1}{n} \text{Tr } \Pi_1 = m_c, \quad \frac{1}{n} \text{Tr } (A \Pi_1) = m_{1c}, \quad \frac{1}{n} \text{Tr } (B \Pi_2) = m_{2c}.
\]  

(4.35)

Define the control parameter

\[
\Psi(z) := \sqrt{\frac{\text{Im } m_{2c}(z)}{n\eta}} + \frac{1}{n\eta}
\]

(4.36)

Note that by (4.10) and (4.12), we have

\[
\text{constant } \epsilon
\]

Lemma 4.3. Then for any fixed \( \epsilon \) and \( \text{Theorem 4.7} \),

\[
\text{uniformly in } \mathcal{P}, \mathcal{S}
\]

as in (2.14).

Moreover, outside of the spectrum we have the following stronger estimate

\[
|m(z) - m_c(z)| + |m_1(z) - m_{1c}(z)| + |m_2(z) - m_{2c}(z)| < \frac{1}{n(k + \eta)} + \frac{1}{(n\eta)^2 \sqrt{k + \eta}},
\]

(4.42)

uniformly in \( z \in \tilde{S}(\xi_1, \xi_2, \epsilon) \cap \{ z = E + i\eta : E \geq \lambda, n\eta \sqrt{k + \eta} \geq n^\epsilon \} \), where \( k \) is defined in (4.8).

(3) If we have (a) (4.37) holds, or (b) either \( A \) or \( B \) is diagonal, then the estimate (4.40) - (4.42) hold for \( z \in S_0(\xi_1, \xi_2, \epsilon) \).

The above estimates are uniform in \( z \) and any set of deterministic vectors of cardinality \( N \mathcal{O}(1) \).

Proof. See Theorem 3.6 of [58].

As a corollary of the averaged local law, the so-called eigenvalue rigidity holds for \( \mathcal{Q}_1 \). We first define the classical locations of eigenvalues.

27
Definition 4.8 (Classical locations of eigenvalues). The classical location $\gamma_j$ of the $j$-th eigenvalue of $Q_1$ is defined as

$$\gamma_j := \sup_x \left\{ \int_x^{+\infty} \rho_c(x) dx > \frac{j - 1}{n} \right\}. \quad (4.43)$$

In particular, we have $\gamma_1 = \lambda_r$.

Note that for any fixed $E \leq \lambda_r$, $\Psi^2(E + inj)$ is monotonically decreasing with respect to $\eta$. Hence there is a unique $\eta(E)$ such that $n^{1/2}\Psi^2(E + i\eta(E)) = 1$. Note that by (4.10) and (4.36), we have

$$\eta(E) = O(n^{-3/4} + n^{-1/2} \sqrt{\kappa_r}), \quad \text{for} \quad E \leq \lambda_r. \quad (4.44)$$

For $E > \lambda_r$, we define $\eta(E) := \eta(\lambda_r) = O(n^{-3/4})$.

Theorem 4.9 (Rigidity of eigenvalues). Suppose that (4.41) and (4.42) hold. Then we have the following estimates for any fixed constant $0 < \zeta < \zeta_1$.

(1) For any $E \geq \lambda_r - \zeta$, we have

$$|n(E) - n_c(E)| < n^{-1} + (\eta(E))^{3/2} + \eta(E)\sqrt{\kappa_E}, \quad \text{where} \quad n(E) := \frac{1}{N} \# \{ \lambda_j \geq E \}, \quad n_c(E) := \int_E^{+\infty} \rho_{2c}(x) dx. \quad (4.45)$$

(2) For any $j$ such that $\lambda_r - \zeta \leq \gamma_j \leq \lambda_r$, we have for any fixed $\varepsilon > 0$,

$$|\lambda_j - \gamma_j| < n^{-2/3} j^{-1/3} + \eta(\gamma_j), \quad (4.47)$$

where $\eta(\gamma_j) = O(n^{-3/4} + n^{-5/6} j^{1/3})$.

(3) If we have (a) (2.37) holds, or (b) either $A$ or $B$ is diagonal, then

$$|\lambda_j - \gamma_j| < n^{-2/3} j^{-1/3}. \quad (4.48)$$

Proof. The bounds (4.45) and (4.48) were proved in Theorem 3.8 of [58]. With (4.45), we follow the proof of Theorem 2.13 in [22] to get that

$$|\lambda_j - \gamma_j| < n^{-2/3} \left[ j^{-1/3} + 1 \left( j \leq n^\varepsilon \left( 1 + n^{3/2}(\gamma_j) \right) \right) \right] + n^{2/3} \eta^2(\gamma_j) j^{-2/3} + \eta(\gamma_j). \quad (4.49)$$

With (4.44) and $\kappa_{\gamma_j} \sim (j/n)^{2/3}$, it is easy to show that

$$n\eta^{3/2}(\gamma_j) \leq n^{-1/8} + j^{1/2} n^{-1/4}, \quad n^{2/3} \eta^2(\gamma_j) j^{-2/3} \leq n^{-2/3} j^{-1/3}.$$

Together with (4.49), we get (4.47) since $\varepsilon$ can be arbitrarily small. \hfill $\square$

Away from the support of $\rho_c$, i.e. for $\text{Re} z > \lambda_r$, the anisotropic local law can be strengthened as follows.

Theorem 4.10 (Anisotropic local law outside of the spectrum). Suppose the Assumptions 2.1 and 2.5 hold. Fix any $\varepsilon > 0$. Then for any

$$z \in S_{out}(\zeta_2, \varepsilon) := \left\{ E + in : \lambda_r + n^{-2/3+\varepsilon} \leq E \leq \zeta_2 \lambda_r, \eta \in [0, 1] \right\}, \quad (4.50)$$

and any deterministic unit vectors $u, v \in \mathbb{C}^2$, we have the anisotropic local law

$$|\langle u, G(X, z)v \rangle - \langle u, \Pi(z)v \rangle| < \sqrt{\frac{\text{Im} m_{2c}(z)}{m_1}} n^{-1/2} (\kappa + \eta)^{-1/4}. \quad (4.51)$$
**Proof.** The second step of (4.51) follows from (4.10). Moreover, for \( \eta \geq \eta_0 := n^{-1/2} \kappa^{1/4} \) and \( \kappa \geq n^{-2/3 + \varepsilon} \), it is easy to verify that
\[
n^{1/2} \Theta^2(z) \leq n^{-1/6}, \quad \frac{1}{n \eta} \leq \frac{\text{Im} m_2(z)}{n \eta}.
\]
Then by (4.20), we have that (4.51) holds for \( z \in S_{\text{out}}(\varepsilon_2, \varepsilon) \) with \( \eta \geq \eta_0 \). Hence it remains to prove that for \( z \in S_{\text{out}}(\varepsilon_2, \varepsilon) \) with \( 0 \leq \eta \leq \eta_0 \), we have
\[
|\langle \mathbf{v}, G(X, z) \mathbf{v} \rangle - \langle \mathbf{v}, \Pi(z) \mathbf{v} \rangle| < n^{-1/2} \kappa^{-1/4}, \quad (4.52)
\]
for any deterministic unit vector \( \mathbf{v} \in \mathbb{C}^{p+n} \). Note that (4.52) implies (4.51) by polarization identity.

Now fix any \( z = E + i \eta \in S_{\text{out}}(\varepsilon_2, \varepsilon) \) with \( \eta \leq \eta_0 \). We denote \( z_0 := E + i \eta_0 \). With (4.52) at \( z_0 \), it suffices to prove that
\[
\langle \mathbf{v}, (\Pi(z) - \Pi(z_0)) \mathbf{v} \rangle < n^{-1/2} \kappa^{-1/4}, \quad (4.53)
\]
and
\[
\langle \mathbf{v}, (G(z) - G(z_0)) \mathbf{v} \rangle < n^{-1/2} \kappa^{-1/4}. \quad (4.54)
\]
With (4.12), to prove (4.53) it is enough to show that
\[
|m_{1c}(z) - m_{1c}(z_0)| + |m_{2c}(z) - m_{2c}(z_0)| < n^{-1/2} \kappa^{-1/4}. \quad (4.55)
\]
Using (4.23), we obtain that
\[
|m_{1c}(z) - m_{1c}(z_0)| \leq \frac{\eta - \eta_0}{|z - z_0|^{1/2}} \leq n^{-1/2} \kappa^{-1/4}.
\]
We can deal with the \( m_{2c} \) term in the same way. This finishes the proof of (4.53).

For (4.54), we write \( \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \) and use (4.32)-(4.33). The upper left block gives that
\[
|\langle \mathbf{v}_1, (G(z) - G(z_0)) \mathbf{v}_1 \rangle| \leq \sum_{k=1}^{p} \frac{\eta_0 |\langle \mathbf{v}_1, \xi_k \rangle|^2}{[(E - \lambda_k)^2 + \eta^2]^{1/2} [(E - \lambda_k)^2 + \eta_0^2]^{1/2}}. \quad (4.56)
\]
Here and throughout the rest of this paper, we will always identify vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) with their embeddings \( \begin{pmatrix} \mathbf{v}_1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ \mathbf{v}_2 \end{pmatrix} \), respectively. By (4.37), we have for any \( k \), \( E - \lambda_k \geq E - \lambda_1 \approx \eta_0 \) with high probability. Using the notations in (4.3), we can bound (4.56) by
\[
|\langle \mathbf{v}_1, (G(z) - G(z_0)) \mathbf{v}_1 \rangle| \leq \sum_{k=1}^{p} \frac{\eta_0 |\langle \mathbf{v}_1, \xi_k \rangle|^2}{(E - \lambda_k)^2 + \eta_0^2} = \text{Im} \mathcal{G}_{\mathbf{v}_1} \mathbf{v}_1(z_0) < n^{-1/2} \kappa^{-1/4} + \text{Im} \Pi_{\mathbf{v}_1} \mathbf{v}_1(z_0) < n^{-1/2} \kappa^{-1/4},
\]
where in the third step we used (4.40), and in the last step we used (4.34), (4.12) and (4.10) to get
\[
\text{Im} \Pi_{\mathbf{v}_1} \mathbf{v}_1(z_0) < \frac{\eta_0}{\sqrt{\kappa + \eta_0}} < n^{-1/2} \kappa^{-1/4}.
\]
Similarly, for the upper right block we have
\[
|\langle \mathbf{v}_1, (G(z) - G(z_0)) \mathbf{v}_2 \rangle| < \left| 1 - (z - z_0)^{-1/2} \right| |\langle \mathbf{v}_1, G(z) \mathbf{v}_2 \rangle| + \sum_{k=1}^{p+n} \frac{\eta_0 |\langle \mathbf{v}_1, \xi_k \rangle \cdot (\xi_k, \mathbf{v}_2) |}{|\lambda_k - z||\lambda_k - z_0|}
\]
\[
\begin{align*}
&< \eta_0 + \sum_{k=1}^{n^2} \eta_0 \left( \frac{1}{|\lambda_k - \gamma_k|^2} + \sum_{k=1}^{n^2} \frac{\eta_0}{|\lambda_k - \gamma_k|^2} \right) = \eta_0 + \text{Im} G_{v_1 v_1}(z_0) + \text{Im} G_{v_2 v_2}(z_0) < n^{-1/2} \kappa^{-1/4}.
\end{align*}
\]

The lower left and lower right blocks can be handled in the same way. This proves (4.54), which completes the proof.

The anisotropic local law (4.40) implies the following delocalization properties of eigenvectors.

**Lemma 4.11** (Isotropic delocalization of eigenvectors). Suppose (4.40) and (4.47) hold. Then we have the following estimates for any fixed constant \(0 < \xi < \xi_1\).

1. For any deterministic unit vectors \(u \in \mathbb{C}^2\) and \(v \in \mathbb{C}^2\), we have

\[
|\langle u, \xi_k \rangle|^2 + |\langle v, \xi_k \rangle|^2 < n^{-1} + \eta_l(\gamma_k) \left( \frac{k}{n} \right)^{1/3},
\]

for all \(k\) such that \(\lambda_r - \xi \leq \gamma_k \leq \lambda_r\), where \(\eta_l(\gamma_k) = O(n^{-3/4} + n^{-5/6} k^{1/3})\).

2. For any integers \(k_1 < k_2\) satisfying

\[
\lambda_r - \xi \leq \gamma_{k_1}, \gamma_{k_2} \leq \lambda_r, \quad n^{-1/2} + n^{-1/6} k_1^{1/3} < k_2^{2/3} - k_1^{2/3},
\]

we have

\[
\frac{1}{k_2 - k_1} \sum_{k=k_1}^{k_2} \left[ |\langle u, \xi_k \rangle|^2 + |\langle v, \xi_k \rangle|^2 \right] < n^{-1}.
\]

3. If we have (a) (2.37) holds, or (b) either \(A\) or \(B\) is diagonal, then we have

\[
\max_{k, \lambda_r - \xi \leq \gamma_k \leq \lambda_r} \left\{ |\langle u, \xi_k \rangle|^2 + |\langle v, \xi_k \rangle|^2 \right\} < n^{-1}.
\]

**Proof.** By (4.47) and (4.48), it is easy to show that \(|\lambda_k - \gamma_k|^2 \gamma_k \ll 1\) with high probability, which gives that \(\eta_l(\lambda_k) = \eta_l(\gamma_k)(1 + o(1))\) with high probability. In particular, \(z_k := \lambda_k + \text{int}(\eta_l(\gamma_k)) \in \bar{S}(\xi_1, \xi_2, \xi)\) with high probability for every \(k\) such that \(\lambda_r - \xi \leq \gamma_k \leq \lambda_r\). Then using the spectral decomposition (4.32), we get

\[
\sum_{k=1}^{n^2} \frac{n^2 \eta_l(\gamma_k) |\langle v, \xi_k \rangle|^2}{(\lambda_k - E)^2 + n^2 \eta_l^2(\gamma_k)} = \text{Im} \langle v, G(z_0) v \rangle.
\]

Choosing \(E = \lambda_k\) in (4.61) and using (4.40), we obtain that

\[
|\langle v, \xi_k \rangle|^2 \leq C n^2 \eta_l(\gamma_k) \text{Im} \langle v, G(z_0) v \rangle < n^2 \eta_l(\gamma_k) \left[ \text{Im} m_{2e}(\lambda_k + \text{int}(\eta_l(\gamma_k)) \right] + \frac{1}{n^{1+\xi_1 \eta_l(\gamma_k)}}. \]

With (4.10), (4.47) and \(\kappa_{\gamma_k} \sim (k/n)^{2/3}\), we can bound that

\[
\text{Im} m_{2e}(\lambda_k + \text{int}(\eta_l(\gamma_k)) \leq \left( \frac{k}{n} \right)^{2/3} + n^{-2/3} k^{-1/3} + n^2 \eta_l(\gamma_k) \right)^{1/2} \leq C \left( \frac{k}{n} \right)^{1/3}.
\]

Plugging it into (4.62), we obtain that

\[
|\langle v, \xi_k \rangle|^2 < n^{-1} + n^2 \eta_l(\gamma_k) \left( \frac{k}{n} \right)^{1/3}.
\]

30
Since \( \epsilon \) is arbitrary, we get (4.57) for \( |\langle \mathbf{v} , \xi_k \rangle |^2 \). In a similar way, we can prove (4.57) for \( |\langle \mathbf{u} , \xi_k \rangle |^2 \). The proof for (4.60) is the same, except that we can take \( z_k := \lambda_k + in^{-1+\epsilon} \in S_0(\xi_1, \xi_2, \epsilon) \) in this case.

In order to proved (4.59), we set
\[
\begin{align*}
    z &= E + \eta, \quad E := \gamma k_1, \quad \eta := n\epsilon(\gamma k_1 - \gamma k_2) \sim n\epsilon \left( \frac{k_2}{n} \right)^{2/3} - \left( \frac{k_1}{n} \right)^{2/3}.
\end{align*}
\]

Under the condition (4.58), we know that
\[
\sum_{k = k_1}^{k_2} |\langle \mathbf{u} , \xi_k \rangle |^2 < \sum_{k = k_1}^{k_2} \frac{n^2}{(\lambda_k - E)^2 + \eta^2} < \eta |\langle \mathbf{u} , \Im G(z) \mathbf{u} \rangle | < \eta \left( \frac{1}{n\eta} + \Im m_{2c}(z) \right)
\]
where in the first step we used (4.47) to conclude that with high probability,
\[
|\lambda_k - E| < \eta(E) + \left( \frac{k}{n} \right)^{2/3} - \left( \frac{k_1}{n} \right)^{2/3} < \eta,
\]
in the third step the local law (4.49) and \( \Im \langle \mathbf{u} , \Pi \mathbf{u} \rangle = O(\Im m_{2c}(z)) \), in the fourth step (4.10) and \( \kappa \sim (k_1/n)^{1/3} \), and in the last step the elementary inequalities
\[
k_2^{2/3} - k_1^{2/3} \leq (k_2 - k_1)^{2/3}, \quad k_2^{2/3} - k_1^{2/3} \leq k_1^{-1/3}(k_2 - k_1).
\]
Similarly, we can bound the \( \sum_{k = k_1}^{k_2} |\langle \mathbf{v} , \xi_k \rangle |^2 \) term. Since \( \epsilon > 0 \) is arbitrary, we conclude (4.59).

Before concluding this section, we give the proof of Theorem 3.5.

**Proof of Theorem 3.5** By Theorem 2.12 under Assumption 3.1 we have that
\[
\tilde{\lambda}_\alpha(i) = g_{2c}(-\tilde{\sigma}_i^\alpha)^{-1} + O_\prec(n^{-1/2}).
\]
Moreover, this shows that \( \tilde{\lambda}_\alpha(i) - \lambda \geq 1 \) with high probability. Together with (4.16) and Theorem 4.10 we obtain from (4.64) that
\[
\tilde{\sigma}_i^\alpha = -m_{2c}^{-1}(\tilde{\lambda}_\alpha(i)) + O_\prec(n^{-1/2}) = -m_{2c}^{-1}(\tilde{\lambda}_\alpha(i)) + O_\prec(n^{-1/2}).
\]
Since \( \tilde{B} \) is an \( l_n \)-rank perturbation of the identity matrix, with Theorem 4.10 and (2.15), we obtain that
\[
m_{2c}(\tilde{\lambda}_\alpha(i)) = \frac{1}{n} \text{Tr} \mathcal{G}_2(\tilde{\lambda}_\alpha(i)) + O_\prec(l_n n^{-1/2}).
\]
Finally, using Theorem 2.13 and the fact that \( |\tilde{\lambda}_\nu - \tilde{\lambda}_\alpha(i)| \geq 1 \) with high probability for all \( \nu \gg r + s + 1 \), we obtain that
\[
\frac{1}{n} \text{Tr} \mathcal{G}_2(\tilde{\lambda}_\alpha(i)) = \frac{1}{n} \sum_{\nu = r + s + 1}^{n} \frac{1}{\tilde{\lambda}_\nu(\mathcal{G}_2) - \tilde{\lambda}_\alpha(i)} + O_\prec(n^{-1}).
\]
Combining (4.63)-(4.67), we conclude (4.50). The estimate (4.50) can be proved in the same way.
5 Outlier eigenvalues

In this section, we prove Theorem 2.12 and Theorem 2.13. The argument is an extension of the ones in [8, Section 4] and [29, Section 6]. The proof consists of the following three steps.

(i) We first find the permissible regions which contain all the eigenvalues of \( \hat{Q}_1 \) with high probability.

(ii) Then we apply a counting argument to a special case (such as a case where the Assumption 3.2 holds), and show that each connected component of the permissible region contains the right number of eigenvalues of \( \hat{Q}_1 \).

(iii) Finally we use a continuity argument to extend the result in (ii) to the general case using the gaps in the permissible regions.

Our proof is more complicated than the ones in [8, Section 4] and [29, Section 6], since we need to keep track of two types of outliers from the spikes of either \( \hat{A} \) or \( \hat{B} \).

5.1 Outlier locations

As in (4.29), we introduce the following linearization of the spiked separable covariance matrices \( \hat{Q}_{1,2} \):

\[
\tilde{H}(X, z) = z^{1/2} \left( \begin{array}{cc} 0 & \tilde{A}^{1/2}X\tilde{B}^{1/2} \\ \tilde{B}^{1/2}X^*\tilde{A}^{1/2} & 0 \end{array} \right), \quad z \in \mathbb{C}_+ \cup \mathbb{R}.
\]

Note that the non-zero eigenvalues of \( z^{-1/2}\tilde{H} \) is given by

\[
\pm \sqrt{\lambda_1(\hat{Q}_1)}, \pm \sqrt{\lambda_2(\hat{Q}_1)}, \cdots, \pm \sqrt{\lambda_{p+n}(\hat{Q}_1)}.
\]

Hence it is easy to see that \( x > 0 \) is an eigenvalue of \( \hat{Q}_1 \) if and only if

\[
\det \left( \tilde{H}(X, x) - x \right) = 0. \tag{5.1}
\]

With the notations in (2.11), we can write

\[
\tilde{H}(X, z) = PH(X, z)P, \quad P = \begin{pmatrix} (1 + V_o^a D^a (V_o^a)^*)^{1/2} & 0 \\ 0 & (1 + V_o^b D^b (V_o^b)^*)^{1/2} \end{pmatrix}. \tag{5.2}
\]

We introduce the \((p+n) \times (r+s)\) matrix \( U \) and the \((r+s) \times (r+s)\) diagonal matrix \( D \) as

\[
U = \begin{pmatrix} V_o^a & 0 \\ 0 & V_o^b \end{pmatrix}, \quad D = \begin{pmatrix} (D^a(D^a + 1))^{-1} & 0 \\ 0 & D^b(D^b + 1))^{-1} \end{pmatrix}. \tag{5.3}
\]

The next lemma gives the master equation for the locations of the outlier eigenvalues.

**Lemma 5.1.** If \( x \neq 0 \) is not an eigenvalue of \( Q_1 \), then it is an eigenvalue of \( \hat{Q}_1 \) if and only if

\[
\det \left( D^{-1} + xU^*G(x)U \right) = 0. \tag{5.4}
\]
Proof. Since $P$ is always invertible, by (5.1) $x \neq 0$ is an eigenvalue of $\tilde{H} = PHP$ if and only if
\[
0 = \det(\tilde{H} - x) = \det(\tilde{H} - x) = \det(P^2) \det(G(x)) \det(1 + xG(x)(1 - P^{-2})) \\
= \det(P^2) \det(G(x)) \det(1 + xG(x)U^*U^*) = \det(P^2) \det(G(x)) \det(1 + xU^*G(x)U).
\]
where in the second step we used $\det(1 + AB) = \det(1 + BA)$. The claim then follows.

Heuristically, by (5.4), (4.51) and (4.34), an outlier location $x > \lambda_r$ almost satisfies the equation
\[
\det(D^{-1} + xU^*\Pi(x)U) = 0 \Rightarrow \prod_{i=1}^r \left( \frac{d_i^a + 1}{1 + m_{2c}(x)\sigma_i^a} \right) \prod_{\mu=1}^s \left( \frac{d_\mu^b + 1}{1 + m_{1c}(x)\sigma_\mu^b} \right) = 0.
\]
Since $(1 + m_{2c}(x)\sigma_i^a)^{-1}$ is a monotonically decreasing function in $x$ for $x > \lambda_r$, hence the equation
\[
1 + (d_i^a)^{-1} - (1 + m_{2c}(x)\sigma_i^a)^{-1} = 0
\]
has a solution on the right of $\lambda_r$ if and only if
\[
\frac{d_i^a + 1}{1 + m_{2c}(\lambda_r)\sigma_i^a} < \frac{1}{1 + m_{2c}(\lambda_r)\sigma_i^a} \Leftrightarrow \tilde{a}_i^a > -m_{2c}^{-1}(\lambda_r).
\]
We can do a similar calculation for $\tilde{a}_\mu^b$. This explains the conditions in (5.29).

Proof of Theorem 2.13 By Theorem 4.7, Theorem 4.9 and Theorem 4.10, for any fixed $\varepsilon > 0$ we can choose a high-probability event $\Xi$ in which the following estimates hold:
\[
1(\Xi) \|U^*(G(z) - \Pi(z))U\| \leq n^{\varepsilon/2} \Psi(z), \quad \text{for } z \in S(s_1, s_2, \varepsilon); \tag{5.5}
\]
\[
1(\Xi) \|U^*(G(z) - \Pi(z))U\| \leq n^{-1/2 + \varepsilon/2} \kappa^{-1/4}, \quad \text{for } z \in S_{out}(s_2, \varepsilon), \tag{5.6}
\]
\[
1(\Xi) |\lambda_i(Q_1) - \lambda_{\lambda_r}| \leq n^{-2/3 + \varepsilon}, \quad \text{for } 1 \leq i \leq \varpi. \tag{5.7}
\]
We remark that the randomness of $X$ only comes into play to ensure that $\Xi$ holds with high probability. The rest of the proof is restricted to $\Xi$ only, and will be entirely deterministic.

For any fixed constant $\varepsilon > 0$, we define the index sets
\[
\mathcal{O}^{(a)}(\varepsilon) := \left\{ i : \tilde{a}_i^a + m_{2c}^{-1}(\lambda_r) \geq n^{-1/3 + \varepsilon} \right\}, \quad \mathcal{O}^{(b)}(\varepsilon) := \left\{ \mu : \mu \leq \mu_\varepsilon \right\},
\]
where
\[
\mu_\varepsilon := \sup \left\{ 1 \leq \mu - p \leq s^+ : \theta_2(\tilde{a}_\mu^b) \geq \inf_{i \in \mathcal{O}^{(a)}} \theta_1(\tilde{a}_i^a) \right\}.
\]
Notice that we have
\[
\sup_{\mu \notin \mathcal{O}^{(b)}} (\tilde{a}_\mu^b + m_{1c}^{-1}(\lambda_r)) \leq n^{-1/3 + \varepsilon}, \quad \inf_{\mu \in \mathcal{O}^{(b)}} (\tilde{a}_\mu^b + m_{1c}^{-1}(\lambda_r)) \geq n^{-1/3 + \varepsilon}.
\]
Here we have defined the set of indices such that
\[
\sup_{i \notin \mathcal{O}^{(a)}} \theta_1(\tilde{a}_i^a) \leq \inf_{\mu \in \mathcal{O}^{(b)}} \theta_2(\tilde{a}_\mu^b), \quad \sup_{\mu \notin \mathcal{O}^{(b)}} \theta_2(\tilde{a}_\mu^b) \leq \inf_{i \in \mathcal{O}^{(a)}} \theta_1(\tilde{a}_i^a).
\]
This will simplify the labelling of indices: we can label the largest outliers of the $\bar{Q}_1$ according to the indices $i \in \mathcal{O}^{(a)}_\epsilon$ and $\mu \in \mathcal{O}^{(b)}_\epsilon$—the other spikes will only give smaller outliers.

One can see that to prove (5.32)-(5.34), it suffices to prove that for arbitrarily small constant $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$1(\Xi) \left| \bar{\lambda}_{\alpha(i)} - \theta_1(\bar{\sigma}_i^a) \right| \leq C n^{-1/2+2\varepsilon} \Delta_1(\bar{\sigma}_i^a), \quad 1(\Xi) \left| \bar{\lambda}_{\beta(\mu)} - \theta_2(\bar{\sigma}_\mu^b) \right| \leq C n^{-1/2+2\varepsilon} \Delta_2(\bar{\sigma}_\mu^b),$$

(5.9)

for all $i \in \mathcal{O}^{(a)}_\epsilon$ and $\mu \in \mathcal{O}^{(b)}_\epsilon$, and

$$\left| \bar{\lambda}_{\alpha(i)} - \lambda_v \right| \leq C n^{-2/3+12\varepsilon}, \quad \left| \bar{\lambda}_{\beta(\mu)} - \lambda_v \right| \leq C n^{-2/3+12\varepsilon},$$

(5.10)

for all $i \in \{1, \ldots, r\} \setminus \mathcal{O}^{(a)}_\epsilon$ and $\mu \in \{p + 1, \ldots, p + s\} \setminus \mathcal{O}^{(b)}_\epsilon$.

**Step 1:** Our first step is to show that on $\Xi$, there exist no eigenvalues outside the neighborhoods of the classical outlier locations $\theta_1(\bar{\sigma}_i^a)$ and $\theta_2(\bar{\sigma}_\mu^b)$. For each $1 \leq i \leq r^+$, we define the permissible interval

$$I^{(a)}_i = I^{(a)}(D^a, D^b) := \left[ \theta_1(\bar{\sigma}_i^a) - n^{-1/2+\varepsilon} \Delta_1(\bar{\sigma}_i^a), \theta_1(\bar{\sigma}_i^a) + n^{-1/2+\varepsilon} \Delta_1(\bar{\sigma}_i^a) \right].$$

Similarly for each $1 \leq \mu - p \leq s^+$, we define the permissible interval

$$I^{(b)}_\mu = I^{(b)}(D^a, D^b) := \left[ \theta_2(\bar{\sigma}_\mu^b) - n^{-1/2+\varepsilon} \Delta_2(\bar{\sigma}_\mu^b), \theta_2(\bar{\sigma}_\mu^b) + n^{-1/2+\varepsilon} \Delta_2(\bar{\sigma}_\mu^b) \right].$$

We then define

$$I = I(D^a, D^b) := I_0 \cup \left( \bigcup_{i \in \mathcal{O}^{(a)}_\epsilon} I^{(a)}_i \right) \cup \left( \bigcup_{\mu \in \mathcal{O}^{(b)}_\epsilon} I^{(b)}_\mu \right), \quad I_0 := \left[ 0, \lambda_v + n^{-2/3+3\varepsilon} \right].$$

(5.11)

We claim the following result.

**Lemma 5.2.** The complement of $I(D^a, D^b)$ contains no eigenvalues of $\bar{Q}_1$.

**Proof.** By (5.11), (5.7) and (5.6), we see that $x \notin I_0$ is an eigenvalue of $\bar{Q}_1$ if and only if

$$1(\Xi)(D^{-1} + x U^* G(x) U) = 1(\Xi) \left( \tilde{D}^{-1} + x U^* \tilde{\Pi}(x) U + O(\kappa \tilde{\epsilon} n^{-1/2+\varepsilon/2}) \right),$$

(5.12)

is singular. To prove the claim, it suffices to show that if $x \notin I$, then

$$\min_{1 \leq i \leq r} \left| \frac{d_i^n + 1}{d_i^n} - \frac{1}{1 + m_2c(x) \sigma_i^n} \right|, \quad \min_{1 \leq \mu - p \leq s} \left| \frac{d_\mu^n + 1}{d_\mu^n} - \frac{1}{1 + m_1c(x) \sigma_\mu^n} \right| \geq \kappa_x^{-1/4} n^{-1/2+\varepsilon/2},$$

(5.13)

If (5.13) holds, then the smallest singular value of $(D^{-1} + x U^* \Pi(x) U)$ is much larger than $\kappa_x^{-1/4} n^{-1/2+\varepsilon/2}$, and the matrices in (5.12) has to be non-singular. Note that for $x > \lambda_v$, we have

$$\left| \frac{d_i^n + 1}{d_i^n} - \frac{1}{1 + m_2c(x) \sigma_i^n} \right| = \left| \frac{1}{1 + m_2c(x) \theta_1(\bar{\sigma}_i^a) \sigma_i^n} - \frac{1}{1 + m_2c(x) \sigma_i^n} \right| = O \left( \left| m_2c(x) - m_2c(\theta_1(\bar{\sigma}_i^a)) \right| \right),$$

where we used (4.12) in the last step.

For any $1 \leq i \leq r$, we claim that

$$\left| x - \theta_1(\bar{\sigma}_i^a) \right| \geq n^{-1/2+\varepsilon} \Delta_1(\bar{\sigma}_i^a) \quad \text{for all } x \notin I.$$

(5.14)
In fact, (5.13) is true for \( i \in O_z^{(a)} \) by definition. For \( i \notin O_z^{(a)} \), we have \( \tilde{\sigma}_i^a + m_{2c}^{-1}(\lambda_r) \leq n^{-1/3+\varepsilon} \) and by (4.13),
\[
\theta_1(\tilde{\sigma}_i(a)) - \lambda_r \leq n^{-2/3+3\varepsilon} \ll n^{-2/3+3\varepsilon}.
\]

Now to prove (5.13), we first assume that there exists a constant \( c > 0 \) such that \( \theta_1(\tilde{\sigma}_i^a) \notin \{x - c\kappa_x, x + c\kappa_x\} \).
Then since \( m_{2c} \) is monotonically increasing on \( (\lambda_r, +\infty) \), we have that
\[
|m_{2c}(x) - m_{2c}(\theta_1(\tilde{\sigma}_i^a))| \geq |m_{2c}(x) - m_{2c}(x \pm c\kappa_x)| \sim \kappa_x^{1/2} \gg n^{-1/2+2\varepsilon} \kappa_x^{-1/4},
\]
where we used (4.14) in the second step, and \( \kappa_x \gg n^{-2/3+3\varepsilon} \) for \( x \notin I_0 \) in the last step. On the other hand, suppose \( \theta_1(\tilde{\sigma}_i^a) \in \{x - c\kappa_x, x + c\kappa_x\} \) such that \( \theta_1(\tilde{\sigma}_i^a) - \lambda_r \sim \kappa_x \). With (4.13) and \( \tilde{\sigma}_i^a + m_{2c}^{-1}(\lambda_r) \geq n^{-1/3+\varepsilon} \), it is easy to show that
\[
\theta_1(\tilde{\sigma}_i^a) - \lambda_r \sim \Delta_1(\tilde{\sigma}_i^a)^4 \gg n^{-1/2+\varepsilon} \Delta_1(\tilde{\sigma}_i^a).
\]
Together with (4.10), we have that
\[
|m_{2c}(\xi)| \sim |m_{2c}(\theta_1(\tilde{\sigma}_i^a))| \sim (\tilde{\sigma}_i^a + m_{2c}^{-1}(\lambda_r))^{-1} = [\Delta_1(\tilde{\sigma}_i^a)]^{-2}
\]
for \( \xi \in I_i^{(a)} \). Since \( m_{2c} \) is monotonically increasing on \( (\lambda_r, +\infty) \), we get that for \( x \notin I_i^{(a)} \),
\[
|m_{2c}(x) - m_{2c}(\theta_1(\tilde{\sigma}_i^a))| \geq |m_{2c}(\theta_1(\tilde{\sigma}_i^a)) \pm n^{-1/2+\varepsilon} \Delta_1(\tilde{\sigma}_i^a) - m_{2c}(\theta_1(\tilde{\sigma}_i^a))| \sim n^{-1/2+\varepsilon} |\Delta_1(\tilde{\sigma}_i^a)|^{-1}
\approx n^{-1/2+\varepsilon} \theta_1(\tilde{\sigma}_i^a) - \lambda_r^{-1/4} \sim n^{-1/2+\varepsilon} \kappa_x^{-1/4} \gg n^{-1/2+2\varepsilon} \kappa_x^{-1/4},
\]
where we used (4.13) in the third step. The \( d_\mu \) term can be handled in the same way. This proves (5.13). \( \square \)

Step 2: In this step we will show that each \( I_i^{(a)}, i \in O_c^{(a)}, \) or \( I_i^{(b)}, \mu \in O_c^{(b)}, \) contains the right number of eigenvalues of \( \hat{Q}_1 \), under a special case; see (5.10) below. For simplicity, we relabel the indices in \( O_c^{(a)} \cup O_c^{(b)} \) as \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{r_x} \), and call them spikes. Moreover, we assume that they correspond to classical locations of outliers as \( x_1, \ldots, x_{r_x} \) (some of them are determined by \( \theta_1 \), while others are given by \( \theta_2 \)), such that
\[
x_1 \geq x_2 \geq \cdots \geq x_{r_x}.
\]
The corresponding permissible intervals \( I_i^{(a)} \) and \( I_i^{(b)} \) are relabelled as \( I_i, 1 \leq i \leq r_x \). In this step, we consider a special configuration \( x = x(0) := (x_1, x_2, \ldots, x_{r_x}) \) of the outliers that is independent of \( n \) and satisfies
\[
x_1 \geq x_2 \geq \cdots \geq x_{r_x} > \lambda_r.
\]

In this step, we claim that each \( I_i(x), 1 \leq i \leq r_x \), contains precisely one eigenvalue of \( \hat{Q}_1 \). Fix any \( 1 \leq i \leq r_x \) and pick up a small \( n \)-independent positively oriented closed contour \( C \subset C[0, \lambda_r] \) that encloses \( x_i \) but no other point of the set \( \{x_i\}_{i=1}^{r_x} \). Define two functions
\[
h(z) := \det(D^{-1} + zU^{-1}G(z)U), \quad l(z) := \det(D^{-1} + zU^{-1}H(z)U).
\]
The functions \( h, l \) are holomorphic on and inside \( C \) when \( n \) is sufficiently large by (5.7). Moreover, by the construction of \( C \), the function \( l \) has precisely one zero inside \( C \) at \( x_i \). By (5.6), we have
\[
\min_{z \in C} |l(z)| \geq 1, \quad |h(z) - l(z)| = \mathcal{O}(n^{-1/2+\varepsilon}/2).
\]
The claim then follows from Rouché’s theorem.

Step 3: In order to extend the results in Step 2 to arbitrary \( n \)-dependent configuration \( x_n \), we shall employ a continuity argument as in [29] Section 6.5. We first choose an \( n \)-independent \( x(0) \) that satisfies (5.10). We then choose a continuous (\( n \)-dependent) path of the eigenvalues of \( D^a \) and \( D^b \), which gives a continuous path of the configurations \( (x(t)) : 0 \leq t \leq 1 \) that connects \( x(0) \) and \( x(1) = x_n \). Correspondingly, we have a continuous path of eigenvalues \( \{\tilde{\lambda}_i(t)\}_{i=1}^{r_x} \). We require that \( x(t) \) satisfies the following properties.
(i) For all $t \in [0, 1]$, the eigenvalues of $D^a(t)$ and $D^b(t)$ are all non-negative.

(ii) For all $t \in [0, 1]$, the number $r_\varepsilon$ of $\varepsilon$-spikes is unchanged and we denote them by $\tilde{\sigma}_1(t), \ldots, \tilde{\sigma}_{r_\varepsilon}(t)$. Moreover, we always have the following order of the outliers: $x_1(t) \geq x_2(t) \geq \cdots \geq x_{r_\varepsilon}(t)$.

(iii) For all $t \in [0, 1]$, we denote the permissible intervals as $I_i(t)$. If $I_i(1) \cap I_j(1) = \emptyset$ for $1 \leq i < j \leq r_\varepsilon$, then $I_i(t) \cap I_j(t) = \emptyset$ for all $t \in [0, 1]$. The interval $I_0$ in (5.11) is unchanged along the path.

It is easy to see that such a path $x(t)$ exists. With a bootstrap argument along the path $x(t)$, we can prove the following lemma.

**Lemma 5.3.** On the event $\Xi$, the estimate (5.9) holds for the configuration $x(1)$.

**Proof.** Along the path, we denote the corresponding separable covariance matrices as $\tilde{\mathcal{Q}}_i(t)$, with eigenvalues $\{\tilde{\lambda}_i(t)\}$. We define $I(t) := I_0 \cup (\cup_{1 \leq i \leq r_\varepsilon} I_i(t))$. Combining Step 1 and Step 2 above, we obtain that on $\Xi$,

\[ \tilde{\lambda}_i(0) \in I_i(0), \quad 1 \leq i \leq r_\varepsilon, \quad \text{and} \quad \tilde{\lambda}_i(0) \in I_0, \quad i \geq r_\varepsilon. \tag{5.17} \]

To apply a continuity argument, recall that we have shown that all the eigenvalues of $\tilde{\mathcal{Q}}_i(t)$ lie in $I(t)$ for all $t \in [0, 1]$. Moreover, since $t \mapsto \tilde{\mathcal{Q}}_i(t)$ is continuous, we find that $\tilde{\lambda}_i(t)$ is continuous in $t \in [0, 1]$ for all $i$. During the proof, we shall call $i \in \{1, \ldots, r_\varepsilon\}$ a type-$a$ index if $\tilde{\sigma}_i = \tilde{\sigma}_k^a$ for some $k$. Otherwise, we shall call $i$ a type-$b$ index. Note that if the $r_\varepsilon$ intervals are disjoint when $t = 1$, then they are disjoint for all $t \in [0, 1]$ by property (iii) of the path. Together with (5.17) and the continuity of $\tilde{\lambda}_i(t)$, we conclude that

\[ \tilde{\lambda}_i(t) \in I_i(t), \quad 1 \leq i \leq r_\varepsilon, \]

for all $t \in [0, 1]$.

Now we consider the general case where some of the intervals are not disjoint. Let $\mathcal{B}$ denote the finest partition of $\{1, \ldots, r_\varepsilon\}$ such that $i$ and $j$ belong to the same block of $\mathcal{B}$ if $I_i(1) \cap I_j(1) \neq \emptyset$. Denote by $B_i$ the block of $\mathcal{B}$ that contains $i$. Note that elements of $B_i$ are sequences of consecutive integers. We now pick any $1 \leq i \leq r_\varepsilon$ and let $j \in B_i$ such that it is not the smallest index in $B_i$. Our first task is to estimate $x_{j-1}(1) - x_j(1)$. We claim that there exists a constant $C > 0$ such that

\[ x_j - x_{j-1} \leq Cn^{-1/2 + \varepsilon} \Delta(\tilde{\sigma}_j), \quad \Delta(\tilde{\sigma}_j) := \begin{cases} \Delta_1(\tilde{\sigma}_j), & \text{if } j \text{ is of type-}a \\ \Delta_2(\tilde{\sigma}_j), & \text{if } j \text{ is of type-}b \end{cases}. \tag{5.18} \]

Without loss of generality, we assume that $j$ is a type-$a$ index. Let $\xi \geq \tilde{\sigma}_j$ be a value such that $\theta_1(\xi) = g_{2\varepsilon}(-\xi^{-1}) \in I_j(1)$. Then we have

\[ \min_{\xi \in [\tilde{\sigma}_j, \xi]} g_{2\varepsilon}(-\xi^{-1})(\tilde{\sigma}_j^{-1} - \xi^{-1}) \leq g_{2\varepsilon}(-\xi^{-1}) - g_{2\varepsilon}(-\tilde{\sigma}_j^{-1}) \leq Cn^{-1/2 + \varepsilon} \Delta_1(\tilde{\sigma}_j). \]

By (1.17), this implies that

\[ \xi - \tilde{\sigma}_j \leq n^{-1/2 + \varepsilon}(\tilde{\sigma}_j + m_{2\varepsilon}^{-1}(\lambda_r))^{-1/2}. \]

Thus we get that

\[ \Delta_1(\xi) = \Delta_1(\tilde{\sigma}_j) \left(1 + \frac{\xi - \tilde{\sigma}_j}{\tilde{\sigma}_j + m_{2\varepsilon}^{-1}(\lambda_r)}\right)^{1/2} \leq \Delta_1(\tilde{\sigma}_j) \left(1 + n^{-1/2 + \varepsilon}(\tilde{\sigma}_j + m_{2\varepsilon}^{-1}(\lambda_r))^{-3/2}\right) \leq \Delta_1(\tilde{\sigma}_j)(1 + o(1)), \]

36
where in the last step we used that that $\tilde{\sigma}_j \in \mathcal{O}_\varepsilon^{(a)}$ defined in (5.8). With the same arguments, we can also prove that for $\xi \leq \tilde{\sigma}_{j-1}$,

\[
\begin{align*}
\Delta_1(\tilde{\sigma}_{j-1}) &\leq \Delta_1(\xi)(1 + o(1)), & \text{if } \tilde{\sigma}_{j-1} \text{ is of type-a and } \theta_1(\xi) \in I_{j-1}(1), \\
\Delta_2(\tilde{\sigma}_{j-1}) &\leq \Delta_2(\xi)(1 + o(1)), & \text{if } \tilde{\sigma}_{j-1} \text{ is of type-b and } \theta_2(\xi) \in I_{j-1}(1).
\end{align*}
\]

Now we pick $x \in I_j(1) \cap I_{j-1}(1)$, and denote $\xi_1 := -m_{2c}^{-1}(x)$ and $\xi_2 := -m_{1c}^{-1}(x)$. Note that we have $x = \theta_1(\xi_1) = \theta_2(\xi_2)$, and

\[
\Delta_1(\xi_1) = (m_{2c}^{-1}(\lambda_r) - m_{2c}^{-1}(x))^{1/2} \sim \kappa_x^{1/4} - (m_{1c}^{-1}(\lambda_r) - m_{1c}^{-1}(x))^{1/2} = \Delta_2(\xi_2),
\]

where we used (4.14) in the second and third steps. Then if $(j - 1)$ is of type-a, we have

\[
\Delta_1(\tilde{\sigma}_{j-1}) \leq \Delta_1(\xi_1)(1 + o(1)) \leq \Delta_1(\tilde{\sigma}_j)(1 + o(1)).
\]

If $(j - 1)$ is of type-b, then using (5.19) we can obtain that

\[
\Delta_2(\tilde{\sigma}_{j-1}) \leq \Delta_2(\xi_2)(1 + o(1)) \leq \Delta_1(\tilde{\sigma}_j)(1 + o(1)).
\]

This proves the claim (5.18).

Repeating the estimate (5.18) for all the remaining $j \in B_i$, since $|B_i|$ is trivially bounded by $r + s$, we obtain that

\[
\text{diam} \left( \bigcup_{j \in B_i} I_j(1) \right) \leq C n^{-1/2+\varepsilon} \Delta(\tilde{\sigma}_{\text{max}}(j; j \in B_i)) \leq C n^{-1/2+\varepsilon} \Delta(\tilde{\sigma}_i).
\]

On the other hand, since $i \in \mathcal{O}_{4\varepsilon}^{(a)} \cup \mathcal{O}_{4\varepsilon}^{(b)}$, by (4.15) we have that

\[
\theta_1(\tilde{\sigma}_i) - \lambda_r - \text{diam} \left( \bigcup_{j \in B_i} I_j(1) \right) \geq c \Delta(\tilde{\sigma}_i)^4 - C n^{-1/2+\varepsilon} \Delta(\tilde{\sigma}_i) \gg n^{-2/3+3\varepsilon}.
\]

Hence there is a gap between the right of $I_0$ and the left of $\bigcup_{j \in B_i} I_j(1)$. Then by (5.17), property (iii) of the path and the continuity of the eigenvalues along the path, we obtain that

\[
\tilde{\lambda}_i(t) \in \bigcup_{j \in B_i} I_j(t), \quad 0 \leq t \leq r_4 \varepsilon,
\]

for all $t \in [0, 1]$. This proves (5.9) by (5.20).

---

**Step 4:** Finally, we consider the non-outlier eigenvalues, i.e. eigenvalues corresponding to $i \notin \mathcal{O}_{4\varepsilon}^{(a)} \cup \mathcal{O}_{4\varepsilon}^{(b)}$. First, we fix a configuration $x(0)$ satisfying (5.16). By Step 2, (5.17) and Lemma 4.2 we have

\[
\tilde{\lambda}_i(0) \in I_0, \quad \text{and} \quad \tilde{\lambda}_i(0) \geq \lambda_r - n^{-2/3+3\varepsilon}.
\]

The above two estimates give that $|\tilde{\lambda}_i(0) - \lambda_r| \leq n^{-2/3+3\varepsilon}$. Next we employ a similar continuity argument as in Step 3. For $t \in [0, 1]$, by (5.7) and Lemma 4.2, we always have that

\[
\lambda_i(t) \geq \lambda_r - n^{-2/3+\varepsilon}, \quad \text{if } t \geq r + s + 1.
\]

As in the proof of Lemma 5.3, if $I_0$ is disjoint from the other $I_j$’s, then by the continuity of $\tilde{\lambda}_i(t)$ and Lemma 5.2, we can conclude that $\tilde{\lambda}_i(t) \in I_0(t)$ for all $t \in [0, 1]$. Otherwise, we again consider the partition $B$ as in
the proof of Lemma \ref{lem:5.3} and let $B_0$ be the block of $B$ that contains $i$. With the same arguments as in the proof of Lemma \ref{lem:5.3} we can prove that
\[ \text{I}_0(1) \cup \left( \bigcup_{j \in B_0} I_j(1) \right) \subset [0, \lambda_r + 2n^{-2/3+3\varepsilon}]. \]

Then using (5.22), (5.23) and the continuity of the eigenvalues along the path, we obtain that
\[ |\tilde{\lambda}_i(t) - \lambda_r| \leq 2n^{-2/3+3\varepsilon}, \quad r_\varepsilon < i \leq r + s, \]
for all $t \in [0,1]$. Obviously, we can apply the same arguments to $r_\varepsilon < i \leq r + s$ by replacing $I_0(1)$ with $[0, \lambda_r + n^{-2/3+12\varepsilon}]$, and hence conclude (5.10). This finishes the proof of Theorem \ref{thm:2.12}.

5.2 Eigenvalue Sticking

In this section, we prove the eigenvalue sticking result, i.e. Theorem \ref{thm:2.13}. By Theorem \ref{thm:2.12}, Theorem \ref{thm:4.7}, Theorem \ref{thm:4.9}, Theorem \ref{thm:4.10} and Lemma \ref{lem:4.11} for any fixed $\varepsilon > 0$, we can choose the high-probability event $\Xi$ in which (5.5)-(5.7) and the following estimates hold:
\[ 1(\Xi) |\tilde{\lambda}_i - \lambda_r| \leq n^{-2/3+\varepsilon/2}, \quad \text{for } r_+ + s_+ + 1 \leq i \leq \varpi, \tag{5.24} \]
for some fixed large integer $\varpi \geq r + s$;
\[ 1(\Xi) |\lambda_i - \gamma_i| \leq i^{-1/3}n^{-2/3+\varepsilon/2} + n^{\varepsilon/2} \eta_i(\gamma_i), \quad \text{for } i \leq \tau p; \tag{5.25} \]
for $k_1 < k_2$ satisfying
\[ n^{-1/2} + n^{-1/6} k_1^{1/3} \leq n^{\varepsilon/20} k_2^{-1/3}(k_2 - k_1), \tag{5.26} \]
we have
\[ 1(\Xi) \frac{1}{k_1 - k_2} \sum_{k=k_1}^{k_2} \left[ |\langle u, \xi_k \rangle|^2 + |\langle v, \xi_k \rangle|^2 \right] \leq n^{-1+\varepsilon/20} \tag{5.27} \]
for $u, v$ in some given set of deterministic vectors of cardinality $n^{O(1)}$. Again the randomness of $X$ only comes into play to ensure that $\Xi$ holds with high probability. The rest of the proof is restricted to $\Xi$ only, and will be entirely deterministic.

Our strategy is similar to the one described at the beginning of Section 5. We first find the permissible region. For any $i$, we define the set
\[ \Omega_i := \left\{ x \in [\lambda_{i-r-s-1}, \lambda_r + c_0 n^{-2/3+2\varepsilon}] : \text{dist} \left( x, \text{Spec}(Q_1) \right) > n^{-1+\varepsilon} \alpha_+^{-1} + \eta(x) \right\}, \tag{5.28} \]
where $\text{Spec}(Q_1)$ stands for the spectrum of $Q_1$ and $c_0 > 0$ is some small constant.

Lemma 5.4. For $\alpha_+ \geq n^{-1/3+\varepsilon}$ and $i \leq n^{1-2\varepsilon} \alpha_+^3$, there exists a constant $c_0 > 0$ such that the set $\Omega_i$ contains no eigenvalue of $\tilde{Q}_1$.

Proof. In the proof, we always use the following parameters
\[ \eta_x := n^{-1+\varepsilon} \alpha_+^{-1} + \eta(x), \quad z_x = x + i\eta_x. \tag{5.29} \]
Suppose \( x \in \Omega_i \). We now apply a similar argument as in (5.13). We first claim that for any \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) and \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) with \( u_1, v_1 \in \mathbb{C}^{d_1} \) and \( u_2, v_2 \in \mathbb{C}^{d_2} \), we have

\[
|G_{u,v}(x) - G_{u,v}(\lambda)| \leq \sum_{i=1}^{2} |\text{Im} \ G_{u_i,u}(x) + \text{Im} \ G_{v_i,v}(\lambda)|, \quad x \in \Omega_i.
\]

(5.30)

As in the proof for Theorem 4.10, we identify vectors \( u_i \) and \( v_i \) with their natural embeddings in \( \mathbb{C}^{d} \). We now prove (5.30) using (5.32) and (5.33). For the terms with \( G_{u_1,v_2}(\cdot) \), we have

\[
|G_{u_1,v_2}(x) - G_{u_1,v_2}(\lambda)| \leq \sum_{k=1}^{p+n} \eta_k \left| \langle u_1, \xi_k \rangle \langle \xi_k, v_2 \rangle \right| \frac{\eta_k}{\lambda_k - x - i(\lambda_k - x)}
\]

where in the second step we used the definition of \( \Omega_i \). For the rest of the cases with \( G_{u_1,v_1}(\cdot), G_{u_2,v_1}(\cdot) \) and \( G_{u_3,v_2}(\cdot) \), the proof is similar.

Now using (4.10), (5.3) and (5.30), we obtain that

\[
\det(D^{-1} + xU^*G(x)U) = \det(D^{-1} + x\eta + z_\nu^*G(z_\nu)U + xU^*G(x) - G(z_\nu)U - i\eta \ U^*G(z_\nu)U)
\]

\[
= \det \left( D^{-1} + x\eta + z_\nu^*G(z_\nu)U + O \left( \eta + n^{\epsilon/2} \Psi(z_\nu) + \text{Im} \ m_2(z_\nu) \right) \right)
\]

\[
= \det \left( D^{-1} + x\eta + z_\nu^*G(z_\nu)U + O \left( n^{\epsilon/2} \text{Im} \ m_2(z_\nu) + \eta^{\epsilon/2} \eta \right) \right),
\]

(5.31)

where in the second step we also used that

\[
\max \left\{ \max_{1 \leq i \leq p} \text{Im} \ \Omega_i \nu_i \nu_i^*, \max_{p+1 \leq \mu \leq n} \text{Im} \ \Omega_i^\mu \nu^*_\mu \right\} \sim \text{Im} \ m_2(z_\nu)
\]

due to (4.12), and in the last step we used

\[
\Psi(z_\nu) \leq \text{Im} \ m_2(z_\nu) + (n\eta)^{-1}.
\]

Therefore, by Lemma 5.1 we conclude that \( x \) is not an eigenvalue of \( \tilde{Q}_1 \) if

\[
\min \left\{ \min_{1 \leq i \leq r} \frac{d_i^a + 1}{d_i^b} - \frac{1}{1 + m_2(z_\nu) \sigma^a_j}, \min_{1 \leq \mu \leq p} \frac{d_\mu^b + 1}{d_\mu^a} - \frac{1}{1 + m_1(z_\nu) \sigma^b_\mu} \right\} \geq n^{\epsilon/2} \text{Im} \ m_2(z_\nu) + \frac{n^{\epsilon/2}}{\eta \nu_\nu}.
\]

(5.32)

For any \( 1 \leq j \leq r \), we have

\[
\frac{d_j^a + 1}{d_j^b} - \frac{1}{1 + m_2(z_\nu) \sigma^a_j} = \frac{1 + m_2(z_\nu) \sigma^a_j}{d_j^b(1 + m_2(z_\nu) \sigma^a_j)}.
\]

(5.33)

Since \( i \leq n^{1-2\epsilon} \alpha_1^3 \), by (5.25) we have

\[
-c_0 n^{-2/3+2\epsilon} \leq \lambda_r - x \leq \left( \frac{i}{n} \right)^{2/3} + i^{-1/3} n^{-2/3+\epsilon/2} + n^{\epsilon/2} \eta \gamma_i \leq n^{-4\epsilon/3} \alpha_1^3, \quad x \in \Omega_i.
\]

(5.34)
where we also used \( \gamma_i \sim (i/n)^{2/3} \), \( \alpha_+ \geq n^{-1/3+\varepsilon} \) and \((1.44)\). Then by \((2.25)\), we have

\[
|m_2c(x) - m_2c(\lambda_r)| \leq C n^{-2\varepsilon/3} \alpha_+ \ll \alpha_+, \quad x \in \Omega_i \cap \{ x : x \leq \lambda_r \}.
\]

and

\[
|m_2c(x) - m_2c(\lambda_r)| \leq C \sqrt{c_0 n^{-1/3+\varepsilon}} \leq C \sqrt{c_0} \alpha_+ \ll \alpha_+, \quad x \in \Omega_i \cap \{ x : x > \lambda_r \}
\]

for some constant \( C \) independent of \( c_0 \). Plugging the above two estimates into \((5.33)\) and using \( |\tilde{\sigma}^0_j + m_2c^{-1}(\lambda_r)| \geq \alpha_+ \) by \((2.35)\), we obtain that

\[
\left| \frac{d^0_j + 1}{d^0_j} - \frac{1}{1 + m_2c(x) \sigma^0_j} \right| \gtrsim \alpha_+
\]

as long as \( c_0 \) is sufficiently small. On the other hand, using \((1.10),\) \((5.29)\) and \((5.34)\), we can verify that

\[
n^{\varepsilon/2} \text{Im} m_2c(z_x) + \frac{n^{\varepsilon/2}}{n^2 \eta_x} \leq n^{\varepsilon/2} \sqrt{\kappa_x + \eta_x} + \frac{n^{\varepsilon/2}}{n^2 \eta_x} \ll \alpha_+, \quad x \in \Omega_i \cap \{ x : x \leq \lambda_r \},
\]

and

\[
n^{\varepsilon/2} \text{Im} m_2c(z_x) + \frac{n^{\varepsilon/2}}{n^2 \eta_x} \leq n^{\varepsilon/2} \frac{\eta_x}{\sqrt{\kappa_x + \eta_x}} + \frac{n^{\varepsilon/2}}{n^2 \eta_x} \ll \alpha_+, \quad x \in \Omega_i \cap \{ x : x > \lambda_r \}.
\]

This proves \((5.32)\) (where the \( d^0_j \) terms can be handled in the same way). \( \square \)

Now we perform a counting argument for a special case. More precisely, we have the following lemma. We postpone its proof until we finish the proof of Theorem \(2.13\).

**Lemma 5.5.** We fix a configuration \( x \equiv x(0) := (x_1, x_2, \cdots, x_{r+s}) \) of the outliers that is independent of \( n \) and satisfies

\[
x_1 > x_2 > \cdots > x_{r+s} > \lambda_r.
\]

For \( \alpha_+ \geq n^{-1/3+2\varepsilon} \) and \( i \leq n^{-1-4\varepsilon} \alpha_+^3 \), we have that

\[
|\lambda_{i+r+s} - \lambda_i| \leq n^{-1+2\varepsilon} \alpha_+^{-1} + n^{3\varepsilon} \eta_i(\gamma_i).
\]

**Proof of Theorem 2.13** First, we consider the case with \( \alpha_+ \geq n^{-1/3+2\varepsilon} \) and \( i \leq n^{-1-4\varepsilon} \alpha_+^3 \). We shall apply a similar continuity argument as in Step 4 of the proof of Theorem 2.12 in Section 5.1. We define

\[
\tilde{I}_0 := \left\{ x \in [0, \lambda_r + c_0 n^{-2/3+2\varepsilon}] : \text{dist}(x, \text{Spec}(Q_1)) \leq n^{-1+\varepsilon} \alpha_+^{-1} + n^{2\varepsilon} \eta_i(x) \right\}.
\]

Note that \( \tilde{I}_0 \) is a union of connected intervals. We again define a continuous path of configurations \( x(t) \) such that \( x(0) \) satisfies \((5.33)\) and \( x(1) \) is the configuration we are interested in.

Note that by interlacing, Lemma \(1.2\) we have

\[
\lambda_i \leq \tilde{\lambda}_i(t) \leq \lambda_{i-r-s}.
\]

By Lemma \(5.5\) and Lemma \(5.4\) we know

\[
|\tilde{\lambda}_{i+r+s}(0) - \lambda_i| \leq n^{-1+2\varepsilon} \alpha_+^{-1} + n^{3\varepsilon} \eta_i(\gamma_i),
\]

and

\[
\text{dist} \left( \tilde{\lambda}_{i+r+s}(t), \text{Spec}(Q_1) \right) \leq n^{-1+\varepsilon} \alpha_+^{-1} + n^{3\varepsilon/2} \eta_i(\gamma_i),
\]

(5.38)
where we used that
\[ \eta(\tilde{\lambda}_{i+r+s}(t)) \ll n^{3/2} \eta(\gamma_i) \] (5.39)
since \( \tilde{\lambda}_{i+r+s}(t) \) satisfies (5.37) and \( \lambda_i \) satisfies (5.25). In addition, by continuity of the eigenvalues, we know that \( \tilde{\lambda}_{i+r+s}(t) \) is in the same connected component of \( \tilde{I}_0 \) as \( \tilde{\lambda}_{i+r+s}(0) \). Let \( B_i \) be the set of \( 1 \leq j \leq p \) such that \( \lambda_i \) and \( \lambda_j \) are in the same connected component of \( \tilde{I}_0 \). Then we conclude that
\[
\tilde{\lambda}_{i+r+s}(t) \in \bigcup_{j \in B_i+r+s-j \leq r+s} \left[ \lambda_j - (n^{-1+2\varepsilon} \alpha_+^{-1} + n^{3\varepsilon} \eta(\gamma_i)), \lambda_j + (n^{-1+2\varepsilon} \alpha_+^{-1} + n^{3\varepsilon} \eta(\gamma_i)) \right]
\subset \bigcup_{j \in B_i+r+s-j \leq r+s} \left[ \lambda_j - (n^{-1+2\varepsilon} \alpha_+^{-1} + n^{4\varepsilon} \eta(\gamma_i)), \lambda_j + (n^{-1+2\varepsilon} \alpha_+^{-1} + n^{4\varepsilon} \eta(\gamma_i)) \right]
\]
where we again used estimates that are similar to (5.39). This gives that
\[ \left| \tilde{\lambda}_{i+r+s}(t) - \lambda_i \right| \leq 2(r+s) \left( n^{-1+2\varepsilon} \alpha_+^{-1} + n^{4\varepsilon} \eta(\gamma_i) \right) \] (5.40)
when \( \alpha_+ \gg n^{-1/3+2\varepsilon} \) and \( \iota \leq n^{-4\varepsilon} \alpha_+^{-3} \).

Finally we consider the cases: \( \alpha_+ < n^{-1/3+2\varepsilon} \), or \( i > n^{-4\varepsilon} \alpha_+^{-3} \). Suppose first that \( \alpha_+ < n^{-1/3+2\varepsilon} \), then by (5.25) and Lemma 4.2 we obtain that
\[ |\tilde{\lambda}_{i+r+s} - \lambda_i| \leq (r+s) \left( n^{-1/3} n^{-2/3+\varepsilon/2} + n^{\varepsilon/2} \eta(\gamma_i) \right) \leq (r+s) \left( n^{-1+3\varepsilon} \alpha_+^{-1} + n^{\varepsilon/2} \eta(\gamma_i) \right). \]
Similarly, if \( i > n^{-4\varepsilon} \alpha_+^{-3} \), we have
\[ |\tilde{\lambda}_{i+r+s} - \lambda_i| \leq (r+s) \left( n^{-1/3} n^{-2/3+\varepsilon/2} + n^{\varepsilon/2} \eta(\gamma_i) \right) \leq (r+s) \left( n^{-1+2\varepsilon} \alpha_+^{-1} + n^{\varepsilon/2} \eta(\gamma_i) \right). \]
Together with (5.40), we finish the proof of (2.38) using (4.14), since \( \varepsilon > 0 \) can be arbitrarily small.

For (2.38), the proof is exactly the same, except that we can set \( \eta(E) = n^{-1} \) by using the stronger anisotropic local law (4.40) for \( \varepsilon \in S_0(s_1, s_2, \varepsilon) \) and the stronger rigidity estimate (4.48).

The strategy for the proof of Lemma 5.3 is an extension of the one for the proof of [20, Proposition 6.8]. We remark that in [20], the results are only obtained for the eigenvalues near the edge with \( \iota \ll (\log n)^C \log \log n \), for some constant \( C > 0 \). We will prove that the same results hold further into the bulk.

**Proof of Lemma 5.3.** If \( j \) is large enough such that \( j^{-1/3} n^{-2/3} \leq \delta n^{3\varepsilon/2} \eta(\gamma_j) \) for some constant \( \delta > 0 \), then we immediately obtain (5.38) by (5.25) and interlacing, Lemma 4.2. Hence in the following proof, we always assume that the indices in the proof satisfy
\[ n^{3\varepsilon/2} \eta(\gamma_j) \leq C_1 j^{-1/3} n^{-2/3} \Rightarrow n^{-1/12} + n^{-1/6} j^{1/3} \leq C_2 n^{-3\varepsilon/2} j^{-1/3} \] (5.41)
for some constant \( C_1, C_2 > 0 \).

In the first step, we group together the eigenvalues that are close to each other. More precisely, let \( \mathcal{A} = \{ A_k \} \) be the finest partition of \( \{ 1, \ldots, p \} \) such that \( i < j \) belong to the same block of \( \mathcal{A} \) if
\[ |\lambda_i - \lambda_j| \leq \delta(j) := n^{-1+7\varepsilon/6} \alpha_+^{-1} + n^{7\varepsilon/6} \eta(\gamma_j). \]
Note that each block \( A_k \) of \( \mathcal{A} \) consists of a sequence of consecutive integers. We order the blocks in the descending order, i.e., if \( k < l \) then \( \lambda_{i_k} > \lambda_{i_l} \) for all \( i_k \in A_k \) and \( i_l \in A_l \).

We first derive a bound on the sizes of the blocks near the edge with \( i \leq \alpha_* \), where we denote \( \alpha_* = \max\{ n^{-4\varepsilon} \alpha_+^{-3}, j_0 \} \), where \( j_0 \) is the largest integer such that the second estimate in (5.38) holds. We define
\( k^* \) such that \( \alpha^* \in A_{k^*} \). For any \( k \leq k^* \), we take \( i < j \) such that \( i \) and \( j \) both belong to the block \( A_{k} \). Then by (5.25) and Lemma 4.2, we find that for some constants \( c, C > 0 \),
\[
c \left[ \frac{(j/n)^2}{3} - \left(\frac{i}{n}\right)^{2/3} \right] - C \left( i^{-1/3} n^{-2/3+\varepsilon/2} + n^{\varepsilon/2} \eta(\gamma) \right) \leq \lambda_i - \lambda_j \leq C (j-i) \left( n^{-1+7\varepsilon/6} \alpha^{-1} + n^{7\varepsilon/6} \eta(\gamma_j) \right).
\]

Now using (4.44), (4.63), (5.41) and \( j \leq \alpha^* \), we obtain that
\[
n^{-2/3} j^{-1/3} (j-i) \leq C i^{-1/3} n^{-2/3+\varepsilon/2} \Rightarrow j - i \leq C (j/i)^{1/3} n^{\varepsilon/2}.
\]

We claim that
\[
|A_k| \leq C n^{3\varepsilon/4} \quad \text{for } k = 1, \cdots, k^*.
\]

and for any given \( i_k \in A_k \),
\[
|\lambda_i - \gamma_{i_k}| \leq i^{-1/3} n^{-2/3+\varepsilon} \quad \text{for all } i \in A_k.
\]

We denote
\[
m_k := \max_{i \in A_k} i, \quad l_k := \min_{i \in A_k} i.
\]

If \( i \in A_k \) satisfies \( i \geq m_k/2 \), then (5.42) gives that \( m_k - i \leq C n^{7\varepsilon/2} \). Using (4.63), we get that
\[
|\gamma_i - \gamma_{m_k}| \leq C n^{7\varepsilon/2} i^{-1/3} n^{-2/3}.
\]

On the other hand, if \( i \in A_k \) satisfies \( i \leq m_k/2 \), then (5.42) gives that \( m_k - i \leq C n^{3\varepsilon/4} \). Thus we get
\[
|\gamma_i - \gamma_{m_k}| \leq \left| \gamma_1 - \gamma_{m_k} \right| \leq C n^{-2/3+\varepsilon/2} \leq C i^{-1/3} n^{-2/3+7\varepsilon/4}.
\]

Together with (5.25) and (5.41), we obtain that
\[
|\lambda_i - \gamma_{i_k}| \leq |\lambda_i - \gamma_i| + |\gamma_i - \gamma_{m_k}| + |\gamma_{m_k} - \gamma_{i_k}| \leq i^{-1/3} n^{-2/3+\varepsilon/2} + n^{\varepsilon/2} \eta(\gamma_i) + C n^{3\varepsilon/4} i^{-1/3} n^{-2/3} \leq i^{-1/3} n^{-2/3+\varepsilon}.
\]

Combining the two cases, we obtain (5.43) and (5.44).

We are now ready to give the main argument. For any \( 1 \leq k \leq k^* \), we denote
\[
a^k := \min_{i \in A_k} \lambda_i = \lambda_{m_k}, \quad b^k := \max_{i \in A_k} \lambda_i = \lambda_{i_k}.
\]

We introduce a continuous path as
\[
x^k_1 = (1-t) \left( a^k - \delta(m_k)/3 \right) + t \left( b^k + \delta(l_k)/3 \right), \quad t \in [0,1].
\]

Note that \( x^k_0 = a^k - \delta(m_k)/3 \) and \( x^k_1 = b^k + \delta(l_k)/3 \). The interval \([x^k_0, x^k_1]\) contains precisely the eigenvalues of \( Q_1 \) that are in \( A_k \), and the endpoint \( x^k_0 \) (or \( x^k_1 \)) is at a distance at least of the orders \( \delta(m_k)/3 \) (or \( \delta(l_k)/3 \)) from any eigenvalue of \( Q_1 \).

In order to avoid problems with exceptional events, we add some randomness to \( D^a \) and \( D^b \). Recall that their eigenvalues satisfy (5.35). Let \( \Delta \) be an \((r+s) \times (r+s)\) Hermitian random matrix, which only has nonzero entries in the upper left \( r \times r \) block and the lower right \( s \times s \) block. Moreover, we assume the upper triangular entries of \( \Delta \) are independent and have an absolutely continuous law supported in the unit disk.

Following the notations in (5.2) and (5.3), for any \( \omega > 0 \), we define \( D^{a,\omega} \) and \( D^{b,\omega} \) such that
\[
(\tilde{D}^{\omega})^{-1} := D^{-1} + \omega \Delta.
\]
Correspondingly, we define $\tilde{Q}_i^{\omega}$ and
\[
\tilde{H}^\omega = P^\omega H P^{\omega}, \quad P = \begin{pmatrix}
(1 + V_\omega^2 D^{\omega}(V_\omega^{\ast})^*)^{1/2} & 0 \\
0 & (1 + V_\omega^2 D^{\omega}(V_\omega^{\ast})^*)^{1/2}
\end{pmatrix}.
\]

We shall take $\omega$ to be sufficiently small, say $\omega \leq ce^{-n}$ for some $c \to 0$. From now on, we use “almost surely” to mean almost surely with respect to the randomness of $\Delta$.

By Lemma 4.2, we find that there are at most $\omega$ Proposition 5.6.

Next, we will use the standard interlacing argument to show that $\tilde{Q}_1$ has at most $|A_k|$ eigenvalues in $[x_0^k, x_1^k]$. By Lemma 5.1, we know that $\tilde{Q}_1$ has exactly $|A_1|$ eigenvalues of $\tilde{Q}_1$ in $[x_0^k, \infty)$. Hence, by Theorem 2.12 and (5.40), there are exactly $|A_1|$ eigenvalues of $\tilde{Q}_1$ in $[x_0^k, x_1^k]$. Repeating this argument, we can show that $\tilde{Q}_1$ has at least $|A_k|$ eigenvalues in $[x_0^k, x_1^k]$ for all $k = 2, \ldots, k^*$. Moreover, by (5.43), we find that for any $i \in A_k$,
\[
\sup \left\{ |x - \lambda_i| : i \in A_k, x \in [x_0^k, x_1^k] \right\} \leq C_\gamma^{3\varepsilon/4} \left( n^{-1+7\varepsilon/6} \alpha_{\gamma}^{1} + n^{7\varepsilon/6} \eta(\gamma_{mk}) \right) \leq n^{-2+2\varepsilon} \alpha_{\gamma}^{1} + n^{2\varepsilon} \eta(\gamma_{mk}).
\]

Together with $\eta(\gamma_{mk}) \leq n^2 \eta(\gamma_i)$, we conclude the proof of Lemma 5.5.

The proof of Proposition 5.6 is very similar to the argument in [29, Section 6.4]. We only prove the part that is different from the proof there, and omit the rest of the details.

**Proof of Proposition 5.6.** For $x \notin \text{spec}(\tilde{Q}_1)$, we define
\[
M^\omega(x) := D^{\omega} + xU^\ast G(x)U.
\]

By Lemma 5.1, we know that $x \in \text{Spec}(\tilde{Q}_1) \setminus \text{Spec}(\tilde{Q}_1)$ if and only if $M^\omega(x)$ is singular.

We split $G$ into $P_{A_k}G + P_{A_k^c}G$ according to whether $i \in A_k$ or $i \notin A_k$ in the spectral decompositions (4.32) and (4.33). For example, the upper left blocks of $P_{A_k}G$ and $P_{A_k^c}G$ are defined as
\[
P_{A_k}G_{ij}(x) := \sum_{i \in A_k} \xi(i) \xi^\ast(j) \lambda_i - x, \quad P_{A_k^c}G_{ij}(x) := \sum_{i \notin A_k} \xi(i) \xi^\ast(j) \lambda_i - x.
\]

Similarly, we can define the other three blocks of $P_{A_k}G$ and $P_{A_k^c}G$. Let $x \in [x_0^k, x_1^k]$. Then given any deterministic vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, similar to (5.30) we have
\[
|P_{A_k^c}G_{\mathbf{u}, \mathbf{v}}(x) - P_{A_k}G_{\mathbf{u}, \mathbf{v}}(x)| \lesssim \sum_{s=1}^2 \left| \text{Im} \left( G_{\mathbf{u}, \mathbf{u}}(x) + \text{Im} \left( G_{\mathbf{v}, \mathbf{v}}(x) \right) \right) \right|
\]
(5.47)
For example, for the terms with $G_{uv_2}(\cdot)$, we have

$$|P_{A_k}G_{uv_2}(z_x) - P_{A_k}G_{uv_2}(x)| \leq \eta_x |G_{uv_2}(z_x)| + \sum_{\ell \notin A_k} \sqrt{\eta_x} |\langle u_1, \xi_l \rangle | \frac{\eta_x}{(\lambda_l - x - i\eta_x)(\lambda_l - x)}.$$

where in the second step we used that $|x - \lambda_l| \geq \eta_x$ for any $x \in [x_0^k, x_1^k]$ and $\ell \notin A_k$. For the rest of the cases with $G_{uv_1}(\cdot)$, $G_{uv_1}(\cdot)$ and $G_{uv_2}(\cdot)$, the proof of (5.47) is similar. Moreover, we claim that

$$|P_{A_k}G_{uv}(z_x)| \leq n^{-\varepsilon/3}.$$  \hspace{1cm} (5.48)

For example, we have

$$\left| \sum_{\ell \in A_k} \left\langle \frac{|u_1, \xi_l \rangle \langle \xi_l, v_1 \rangle}{\lambda_l - x} \right| \leq Cn^{3z/4} \eta_x^{-1} n^{-1+\varepsilon/10} \ll n^{-\varepsilon/3},$$  \hspace{1cm} (5.49)

where we used (5.43) and (5.27) (the condition (5.26) can be verified using (5.41)). For the rest of the cases with $G_{uv_2}(\cdot)$, $G_{uv_1}(\cdot)$ and $G_{uv_2}(\cdot)$, the proof of (5.48) is similar. Then by a discussion similar to (5.31), we have

$$M^\omega(x) = x U^* P_{A_k} G(x) U + x U^* (P_{A_k} G(x) - P_{A_k} G(z_x)) U + (z_x + (x - z_x)) U^* G(z_x) U$$

$$- x U^* P_{A_k} G(z_x) U + D^{-1} + \omega \Delta$$

$$= x U^* P_{A_k} G(x) U + D^{-1} + \omega \Delta + z_x U^* (z_x) U + R_0(x)$$

$$= x U^* P_{A_k} G(x) U + D^{-1} + \omega \Delta + \lambda_\varepsilon U^* (\lambda_\varepsilon) U + R(x),$$  \hspace{1cm} (5.50)

where

$$R_0(x) = O \left( \eta_x + n^{\varepsilon/2} \Psi(z_x) + \text{Im} m_{2c}(z_x) + n^{-\varepsilon/3} \right) = O \left( n^{-\varepsilon/3} \right)$$

and

$$R(x) = R_0(x) + O(\sqrt{\eta_x} + \eta_x) = O \left( n^{-\varepsilon/3} \right).$$

Moreover, $R(x)$ is real (since all the other terms in (5.50) are real), continuous in $x$ on the extended real line $\mathbb{R}$, and independent of $\Delta$.

The rest of the proof follows from a continuity argument, which is exactly the same as the proof in Section 6.4 between (6.27) and (6.28). We remark that the small $\omega \Delta$ is used only in this proof to avoid some problems with exceptional events. We omit the details. This finishes the proof of Proposition 5.3. \hspace{1cm} $\square$

6 Outlier eigenvectors

In this section, we study the outlier eigenvectors. More precisely, we prove Theorem 2.15 under the following stronger assumption.

**Assumption 6.1.** For some fixed small constant $\tau > 0$, we assume that for $\alpha(i) \in S$ and $\beta(\mu) \in S$,

$$\tilde{\sigma}_i^\alpha + m_{2c}^{-1}(\lambda_\varepsilon) \geq n^{-1/3+\tau}, \quad \tilde{\sigma}_p^\beta + m_{1c}^{-1}(\lambda_\varepsilon) \geq n^{-1/3+\tau}. $$  \hspace{1cm} (6.1)

The necessary argument to remove this assumption will be given in Section 7 after we finish the proof of Theorem 2.10, since we need the delocalization bounds in Theorem 2.10. Thus the main goal of this section is to prove the following weaker proposition.
Proposition 6.2. Suppose the assumptions in Theorem 2.15 hold. Then under Assumption 6.1 we have
\[
\left| \langle v_i^a, P_S v_j^a \rangle - \delta_{ij} \mathbf{1}(\alpha(i) \in S) \frac{1}{\Delta_i^a} g_{2c} \left( - (\tilde{\alpha}_i^a)^{-1} \right) \right| < \frac{1}{n^{1/2}} \sqrt{\Delta_1(\tilde{\alpha}_i^a) \Delta_1(\tilde{\alpha}_j^a)}
\]
\[
+ \frac{1}{n} \left( \frac{1}{\delta_{\alpha(i)}(S)} + \frac{1}{\delta_{\alpha(j)}(S)} \right) \left( \frac{1}{\Delta_1(\tilde{\alpha}_i^a)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_j^a)^2} \right) + \frac{1}{n^{1/2} \delta_{\alpha(i)}(S)} (i \leftrightarrow j),
\]
where \((i \leftrightarrow j)\) denotes the same terms but with \(i\) and \(j\) interchanged.

It is easy to check that, under Assumption 6.1, Theorem 2.15 follows from Proposition 6.2 by using the Cauchy-Schwarz inequality and the fact that
\[
\frac{1}{\Delta_i^a} g_{2c} \left( - (\tilde{\alpha}_i^a)^{-1} \right) \sim \Delta_1(\tilde{\alpha}_i^a)^2, \quad 1 \leq i \leq r,
\]
since \(g_{2c}(-\tilde{\alpha}_i^{-1}) \sim 1, \tilde{\alpha}_i^a \sim 1\) and \(g_{2c}(-\tilde{\alpha}_j^{-1}) \sim \Delta_1(\tilde{\alpha}_j^a)^2\) by (4.17).

The rest of this section is devoted to proving Proposition 6.2. Our strategy is an extension of the one in [3] Section 5. But there is additional complication in our case, because we need to simultaneously handle the outliers caused by the spikes of \(B\).

6.1 Non-overlapping condition

We first prove Proposition 6.2 under the following additional non-overlapping condition. We will remove it later in Section 6.2.

Assumption 6.3. For some fixed small constant \(\tilde{\tau} > 0\), we assume that for all \(\alpha(i) \in S\) and \(\beta(\mu) \in S\),
\[
\delta_{\alpha(i)}(S) \geq \left[ \Delta_1(\tilde{\alpha}_i^a) \right]^{-1} n^{-1/2 + \tilde{\tau}}, \quad \text{and} \quad \delta_{\beta(\mu)}(S) \geq \left[ \Delta_2(\tilde{\alpha}_\mu^b) \right]^{-1} n^{-1/2 + \tilde{\tau}}.
\]

Remark 6.4. This condition is actually a generalization of the second condition in (2.14). Note that for \(1 \leq i \leq r^+, \) using (4.13), (4.16) and (4.17), we have
\[
\delta_{\alpha(i), \alpha(j)}^a = |\tilde{\alpha}_i^a - \tilde{\alpha}_j^a| \sim \frac{|\theta_1(\tilde{\alpha}_i^a) - \theta_1(\tilde{\alpha}_j^a)|}{[\Delta_1(\tilde{\alpha}_i^a)]^2}, \quad 1 \leq j \leq r^+,
\]
and
\[
\delta_{\alpha(i), \beta(\nu)}^a = |\tilde{\alpha}_i^b + m^{-1}_c(\theta(\tilde{\alpha}_i^a))| \sim \frac{|m_1c(\theta(\tilde{\alpha}_i^a) - m_1c(\theta(\tilde{\alpha}_i^a)))|}{[\Delta_1(\tilde{\alpha}_i^a)]^2}.
\]

Thus under Assumption 6.3 we have that for \(\alpha(i) \in S\),
\[
\Delta_1(\tilde{\alpha}_i^a) n^{-1/2 + \tilde{\tau}} \lesssim \begin{cases} |\theta_1(\tilde{\alpha}_i^a) - \theta_1(\tilde{\alpha}_j^a)|, & \text{if } \alpha(j) \notin S, \\ |\theta_1(\tilde{\alpha}_i^a) - \theta_2(\tilde{\alpha}_j^a)|, & \text{if } \beta(\nu) \notin S. \end{cases}
\]

With a similar arguments for \(\beta(\mu) \in S\), we conclude that the eigenvalues with indices in \(S\) do not overlap with any other eigenvalues by (2.52) and (2.53).

The main estimate for outlier eigenvectors under the non-overlapping assumption is included in the following proposition.
Proposition 6.5. Suppose the assumptions in Proposition 6.2 hold. Then under Assumption 6.3, we have that for all \(i, j = 1, \cdots, p\),
\[
\left| \langle v_i^{(a)}, P_S v_j^{(a)} \rangle - \delta_{ij} 1(\alpha(i) \in S) \frac{1}{s_i} g_{2c}(\cdot) \left( \frac{1}{s_i} \right) \right| < \frac{1}{n} \left( \frac{1}{\delta_{\alpha(i)}(S)} + \frac{1}{\Delta_1(\delta^a_i)} \right)^{1/2} + \frac{1}{n^{1/2}} \delta_{\alpha(i)}(S) \Delta_1(\delta^a_i) + (i \rightarrow j). \tag{6.3}
\]
where \(\delta^a_i\) is defined in (3.39).

Remark 6.6. Note that (6.3) is slightly stronger than (6.2) since we always have \(\delta^a_{\alpha(i)}(S) \leq \delta^a_{\alpha(i), \alpha(j)}\) for \(\alpha(i) \in S\) and \(\alpha(j) \notin S\). This is necessary in order to remove the non-overlapping Assumption 6.3 later.

The rest of this subsection is devoted to proving Proposition 6.5. Suppose that Assumptions 6.1 and 6.3 hold. Let \(\omega < \tau/2\) and \(0 < \varepsilon < \min\{\tau, \tilde{\tau}\}/10\) be small positive constants to be chosen later. By Theorem 4.9, Theorem 1.10, and Theorem 2.12, we can choose a high-probability event \(\Xi_1 = \Xi_1(\varepsilon, \omega, \tau, \tilde{\tau})\) in which the following estimates hold.

(i) For all \(z \in S_{\text{out}}(\varepsilon_2, \varepsilon) := \{E + \eta: \lambda_{c, r} + n^{-2/3+\omega} \leq E \leq \omega + n - 1, \eta \in [0, 1]\}\),
we have the anisotropic local law
\[
1(\Xi_1) \| U^a(G(z) - \Pi(z)) U \| \leq n^{-1/2+\varepsilon}(\kappa + \eta)^{-1/4}. \tag{6.5}
\]
(ii) For all \(1 \leq i \leq r^+ + 1 \leq \mu - p \leq s^+\), we have
\[
1(\Xi_1) \left| \lambda_{\alpha(i)} - \theta_1(\delta^a_i) \right| \leq n^{-1/2+\varepsilon} \Delta_1(\delta^a_i), \quad 1(\Xi_1) \left| \lambda_{\beta(\mu)} - \theta_2(\delta^a_i) \right| \leq n^{-1/2+\varepsilon} \Delta_2(\delta^a_i). \tag{6.6}
\]
(iii) For any fixed integer \(\bar{\omega} > r + s\) and all \(r^+ + s^+ < \bar{\omega} \leq \omega\), we have
\[
1(\Xi_1) \left| \lambda_1 - \lambda_{r^+} \right| \leq n^{-2/3+\varepsilon}. \tag{6.7}
\]
As in the proof in Section 3, the randomness of \(X\) only comes into play to ensure that \(\Xi_1\) holds with high probability. The rest of the proof is restricted to the event \(\Xi_1\) only, and will be entirely deterministic.

Given any \(1 \leq i \leq r^+\), our first step is to give a contour integral representation of the generalized components \(\langle v_i^{(a)}, P_S v_j^{(a)} \rangle\) using resolvents. We define the radius
\[
\rho^a_i = c_i \left[ \delta_{\alpha(i)}(S) \cap (\delta^a_i + m_{2c}(\lambda_r)) \right], \quad \alpha(i) \in S, \tag{6.8}
\]
and
\[
\rho^a_\mu = c_\mu \left[ \delta_{\beta(\mu)}(S) \cap (\delta^a_\mu + m_{1c}(\lambda_r)) \right], \quad \beta(\mu) \in S, \tag{6.9}
\]
for some sufficiently small constants \(0 < c_i, c_\mu < 1\). Define the contour \(\Gamma := \partial C\) as the boundary of the union of open discs
\[
C := \bigcup_{\alpha(i) \in S} C_i \cup \bigcup_{\beta(\mu) \in S} C_\mu, \quad C_i := B_{\rho^a_i}(-\delta^a_i), \quad C_\mu := B_{\rho^a_\mu}(m_{2c}(\theta_2(\delta^a_\mu))), \tag{6.10}
\]
where \(B_r(x)\) denotes an open disc of radius \(r\) around \(x\). By choosing sufficiently small \(c_i\) and \(c_\mu\), we can assume that \(C \subset D_2(\tau_3, \varepsilon)\) in Lemma 4.5. In the following lemma, we shall show that: (i) \(g_{2c}(C)\) is a subset of the parameter set in (6.4) so that we can use the estimate (6.5); (ii) \(\hat{c} g_{2c}(C) = g_{2c}(\Gamma)\) only encloses the outliers with indices in \(S\).
Lemma 6.7. Suppose that Assumptions 6.1 and 6.3 hold true. Then the set \( g_{2c}(\mathbb{C}) \) lies in the parameter set \( S_{\text{out}}(\xi^2, \varepsilon) \) in (6.4) as long as the \( \gamma_i \)'s and \( c_\gamma \)'s are sufficiently small. Moreover, we have \( \{ \lambda_i \} \in S \subset g_{2c}(\mathbb{C}) \) and all the other eigenvalues lie in the complement of \( g_{2c}(\mathbb{C}) \).

Proof. Our proof is similar to the one for [8, Lemma 5.4 and 5.5]. We first show that each \( g_{2c}(C_i) \) is a subset of \( S_{\text{out}}(\xi^2, \varepsilon) \). By (4.21), it is easy to see that \( |g_{2c}(\zeta)| \leq \omega^{-1} \) for all \( \zeta \in \mathbb{C} \) as long as \( \omega \) is sufficiently small. For the lower bound on \( \text{Re} g_{2c}(\zeta) \), we claim that for any constant \( \tilde{C} > 0 \) and sufficiently small constant \( 0 < \tilde{c}_0 < 1 \), there exists a constant \( \tilde{c}_1 \equiv \tilde{c}_1(\tilde{c}_0, \tilde{C}) \) such that

\[
\text{Re} g_{2c}(\zeta) \geq \lambda_r + \tilde{c}_1(\text{Re} \xi - m_{2c}(\lambda_r))^2, \tag{6.11}
\]

for \( \text{Re} \zeta \geq m_{2c}(\lambda_r) \), \( |\text{Im} \zeta| \leq \tilde{c}_0(\text{Re} \zeta - m_{2c}(\lambda_r)) \), and \( |\zeta| \leq \tilde{C} \). In fact, if \( 0 \leq \text{Re} \xi - m_{2c}(\lambda_r) \leq c_0 \) for some sufficiently small constant \( c_0 > 0 \), then (6.11) follows from (2.25) that

\[
\text{Re} g_{2c}(\zeta) - \lambda_r \sim \text{Re}(\xi - m_{2c}(\lambda_r))^2 \sim (\text{Re} \xi - m_{2c}(\lambda_r))^2, \quad \text{for } |\text{Im} \zeta| \leq \tilde{c}_0(\text{Re} \xi - m_{2c}(\lambda_r)).
\]

On the other hand, if \( \text{Re} \xi - m_{2c}(\lambda_r) \geq c_0 \), then using (4.22) we get

\[
\text{Re} g_{2c}(\zeta) - \lambda_r \geq g_{2c}(\text{Re} \xi) - \lambda_r - C|\zeta - m_{2c}(\lambda_r)| |\text{Im} \zeta| \geq c,
\]

for some constants \( C > 0 \) and \( c = c(c_0, \tilde{c}_0, \tilde{C}, C) > 0 \) as long as \( \tilde{c}_0 \) is small enough. The claim (6.11) then follows by first choosing a sufficiently small constant \( \tilde{c}_0 \) and then choosing an appropriate constant \( \tilde{c}_1 \).

Now as long as \( c_0 \) is sufficiently small, we conclude that \( g_{2c}(C_i) \subset S_{\text{out}}(\xi^2, \varepsilon) \) using (6.11),

\[
\text{Re} \zeta - m_{2c}(\lambda_r) \geq \left( \frac{1}{\tilde{\sigma}^a(\tilde{\sigma}^b - m_{2c}(\lambda_r))} - c_1 \right) (\tilde{\sigma}^a + m_{2c}(\lambda_r)), \quad \text{Im} \zeta \leq c_1 (\tilde{\sigma}^a + m_{2c}(\lambda_r)),
\]

and \( (\tilde{\sigma}^a + m_{2c}(\lambda_r)) \geq n^{-1+3\tau} \). Similarly, for \( \zeta \in \mathbb{C}_\mu \), using (4.23) and (4.13) we have

\[
\text{Re} \zeta - m_{2c}(\lambda_r) \geq m_{2c}(\theta_2(\tilde{\sigma}^a)) - m_{2c}(\lambda_r) - c_1 (\tilde{\sigma}^b + m_{2c}(\lambda_r)) \geq \tilde{c}_2 (\tilde{\sigma}^b + m_{2c}(\lambda_r)),
\]

and

\[
\text{Im} \zeta \leq \tilde{C}_2 c_\mu (\tilde{\sigma}^a + m_{2c}(\lambda_r)) \]

for some constants \( \tilde{c}_2, \tilde{C}_2 > 0 \) that are independent of \( c_\mu \). Then using (6.11) and (6.1), we obtain that \( g_{2c}(C_\mu) \subset S_{\text{out}}(\xi^2, \varepsilon) \) as long as \( c_0 \) is sufficiently small. This finishes the proof of the first statement.

Next, we prove the second statement. If suffices to show that:

(i) \( \tilde{\lambda}_\alpha(i) \in g_{2c}(C_i) \) and \( \tilde{\lambda}_\beta(\mu) \in g_{2c}(C_\mu) \) for all \( \alpha(i) \in S \) and \( \beta(\mu) \in S \);

(ii) all the other eigenvalues \( \tilde{\lambda}_j \) satisfies \( \tilde{\lambda}_j \notin g_{2c}(C_i) \) and \( \tilde{\lambda}_j \notin g_{2c}(C_\mu) \) for all \( \alpha(i) \in S \) and \( \beta(\mu) \in S \).

To prove (i), we notice that under Assumptions 6.1 and 6.3

\[
\rho^a_i \geq \left[ \Delta_1(\tilde{\sigma}^a) \right]^{-1} n^{-1/2+2\tau}, \quad \rho^b_\mu \geq \left[ \Delta_2(\tilde{\sigma}^b_\mu) \right]^{-1} n^{-1/2+2\tau},
\]

where we recall that \( \varepsilon < \min\{\tau, \tilde{\tau}\}/10 \). Together with (4.17), we get that

\[
|g_{2c} \left( \left( \tilde{\sigma}^a - \rho^a_i \right)^{-1} - \left( \tilde{\sigma}^a - \rho^a_i \right)^{-1} \right) \right| \geq \Delta_1(\tilde{\sigma}^a)n^{-1/2+2\tau}, \quad \alpha(i) \in S,
\]

and

\[
|g_{2c} \left( \left( \tilde{\sigma}^b - \rho^b_\mu \right)^{-1} - \left( \tilde{\sigma}^b - \rho^b_\mu \right)^{-1} \right) \right| \geq \Delta_2(\tilde{\sigma}^b_\mu)n^{-1/2+2\tau}, \quad \beta(\mu) \in S.
\]

Then we conclude (i) using (6.6). In order to prove (ii), we consider the two cases: (1) \( j \in O^+ \setminus S \); (2) \( j \notin O^+ \). In case (1), the claim follows from Assumption 6.3; (6.10) and (4.17); see Remark 6.3. In case (2), the claim follows from the first statement and (4.17). This concludes the proof.
For the proof of Proposition 6.5 we shall use a contour integral representation of $\mathcal{P}_S$. As in (4.32) and (4.33), we have the following spectral decompositions for $\tilde{G}$:

$$\tilde{G}_{ij} = \sum_{k=1}^{p} \frac{\tilde{G}_k(i)\tilde{G}_k(j)}{\lambda_k - z}, \quad \tilde{G}_{\mu
u} = \sum_{k=1}^{n} \frac{\tilde{G}_k(\mu)\tilde{G}_k(\nu)}{\lambda_k - z},$$

(6.12)

By (6.12), Lemma 6.7 and Cauchy’s integral formula, we have

$$\langle \nu_i, \mathcal{P}_S \nu_j \rangle = -\frac{1}{2\pi i} \int_{g_{2\varepsilon}(\Gamma)} \langle \nu_i, \tilde{G}(z)\nu_j \rangle dz,$$

(6.13)

where $\nu_{i/j}$ is the natural embedding of $\nu_{i/j}$ in $\mathbb{C}^T$. We next provide a representation for $\langle \nu_i, \tilde{G}(z)\nu_j \rangle$ for $1 \leq i, j \leq r$. Using (5.2) and the Woodbury matrix identity in Lemma 4.1, we obtain that

$$U^* \tilde{G}(z)U = U^* P^{-1} (H - z + (1 - P^{-2})^{-1} P^{-1} U = U^* P^{-1} (G(z) + zUDU^*)^{-1} P^{-1} U$$

$$= U^* P^{-1} \left( \frac{G(z) - zG(z)U}{D - 1 + zUGU^*} \right) P^{-1} U$$

$$= \tilde{D}^{1/2} \left[ U^*G(z)U - zU^*G(z)U \right] \frac{1}{D^{-1} + zUGU^*} U^*G(z)U \tilde{D}^{1/2},$$

(6.14)

where

$$\tilde{D} := \begin{pmatrix} (1 + D^a)^{-1} & 0 \\ 0 & (1 + D^b)^{-1} \end{pmatrix}.$$ 

With (6.13) and (6.14), we now finish the proof of Proposition 6.5.

**Proof of Proposition 6.5** We denote $\mathcal{E}(z) = zU^*(\Pi(z) - G(z))U$. Then we can write

$$zU^*G(z)U = zU^*\Pi(z)U - \mathcal{E}(z).$$

We now perform a resolvent expansion for the denominator in (6.14) as

$$\frac{1}{D^{-1} + zU^*G(z)U} = \frac{1}{D^{-1} + zU^*\Pi(z)U} + \frac{1}{D^{-1} + zU^*G(z)U} \mathcal{E} \frac{1}{D^{-1} + zU^*\Pi(z)U}$$

$$+ \frac{1}{D^{-1} + zU^*\Pi(z)U} \mathcal{E} \frac{1}{D^{-1} + zU^*G(z)U} \mathcal{E} \frac{1}{D^{-1} + zU^*\Pi(z)U}.$$ 

(6.15)

Inserting it into (6.13) and using that $\Gamma$ does not enclose any pole of $G$ by (6.7), we obtain that

$$\langle \nu_i, \mathcal{P}_S \nu_j \rangle = \sqrt{(1 + d^a_i)(1 + d^a_j)} \frac{1}{d^a_i d^a_j} (s_0 + s_1 + s_2),$$

where $s_0, s_1$ and $s_2$ are defined as

$$s_0 = \frac{\delta_{ij}}{2\pi i} \int_{g_{2\varepsilon}(\Gamma)} \frac{1}{(d^a_i)^{-1} + 1 - (1 + m_2c(z)\sigma^2)^{-1}} \frac{dz}{z},$$

48
Next we consider three different cases. First suppose that $C$ where $\alpha$ for $r$ the same bound. Next we suppose that $\alpha$. Then we get from (6.18) that $|s_1| \leq C \Delta_1 (\tilde{\sigma}_r^{\alpha}) n^{-1/2} D^{-1/2}$. (6.20)

where we used (6.19) in the last step. If $\tilde{\sigma}_r^\alpha = \tilde{\sigma}_j^\alpha$, then a simple application of the residue’s theorem gives the same bound. Next we suppose that $\alpha(i) \in S$ and $\alpha(j) \notin S$. Then we get from (6.18) that

$$|s_1| \leq C \frac{|h_{ij}(-\tilde{\sigma}_r^{\alpha})|}{|\tilde{\sigma}_r^\alpha - \tilde{\sigma}_j^\alpha|} \leq \frac{C \Delta_1 (\tilde{\sigma}_r^{\alpha}) n^{-1/2} D^{-1/2}}{\delta_{\alpha(i), \alpha(j)}}.$$

(6.21)
We have a similar estimate if \( \alpha(i) \notin S \) and \( \alpha(j) \in S \). Finally, if \( \alpha(i) \notin S \) and \( \alpha(j) \notin S \), we have \( s_1 = 0 \) by Cauchy’s residue theorem.

It remains to estimate the second order error \( s_2 \). We decompose the contour into

\[
\Gamma = \bigcup_{\alpha(i) \in S} \Gamma_i \cup \bigcup_{\beta(\mu) \in S} \Gamma_{\mu}, \quad \Gamma_i := \Gamma \cap \partial C_i, \quad \Gamma_{\mu} := \Gamma \cap \partial C_{\mu}.
\] (6.22)

We have the following basic estimates on each of these components.

**Lemma 6.8.** For any \( \alpha(i) \in S, \, 1 \leq j \leq r, \, 1 \leq \nu - p \leq s \) and \( \zeta \in \partial C_i \), we have

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \sim \rho_i^q + \delta_{\alpha(i),\alpha(j)},
\] (6.23)

and

\[
|m_{1c}(g_{2c}(\zeta)) + (\tilde{\beta}_j^q)^{-1}| \sim \rho_i^h + \delta_{\alpha(i),\beta(\nu)}. \] (6.24)

For any \( \beta(\mu) \in S, \, 1 \leq j \leq r, \, 1 \leq \mu - p \leq s \) and \( \zeta \in \partial C_{\mu} \), we have

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \sim \rho_\mu^h + \delta_{\beta(\mu),\alpha(j)},
\] (6.25)

and

\[
|m_{1c}(g_{2c}(\zeta)) + (\tilde{\beta}_j^q)^{-1}| \sim \rho_\mu^h + \delta_{\beta(\mu),\beta(\nu)}. \] (6.26)

**Proof.** The proof is similar to but a little more complicated than the one for [5, Lemma 5.6]. The upper bound in (6.23) follows from the triangle inequality:

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \leq \rho_i^q + |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \leq \rho_i^q + \delta_{\alpha(i),\alpha(j)}.
\]

We only needs to prove a lower bound. For \( \alpha(j) \notin S \), by Assumptions 6.1 and 6.3 we trivially have

\[
|(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \geq 2\rho_i^q,
\]

from which we obtain that

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \geq |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| - \rho_i^q \geq \rho_i^q + |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}|.
\]

Next we consider the case \( \alpha(j) \in S \). Define \( \delta := |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| - \rho_i^q - \rho_j^q \). First suppose that \( C_0 \delta > |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \) for some constant \( C_0 > 1 \). It then follows that \( \rho_i^q + \rho_j^q \leq \frac{C_0 - 1}{C_0} |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \).

As a consequence, we obtain that

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \geq |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| - \rho_i^q \geq \frac{1}{C_0} |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \geq \rho_i^q + \delta_{\alpha(i),\alpha(j)}.
\]

Suppose now that \( C_0 \delta \leq |(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \). Then we have

\[
|(\tilde{\beta}_j^q)^{-1} - (\tilde{\beta}_j^q)^{-1}| \leq \frac{C_0}{C_0 - 1} (\rho_i^q + \rho_j^q).
\]

We claim that for large enough constant \( C_0 > 0 \), there exists a constant \( \tilde{\epsilon}(c_i, c_j, C_0) > 0 \) such that

\[
\tilde{\epsilon}^{-1} \rho_i^q \leq \rho_j^q \leq \tilde{\epsilon} \rho_i^q.
\] (6.27)

If (6.27) holds, then we have

\[
|\zeta + (\tilde{\beta}_j^q)^{-1}| \geq \rho_j^q \geq \rho_i^q + \rho_j^q \geq \rho_i^q + \delta_{\alpha(i),\alpha(j)}.
\]
This concludes (6.23).

It remains to prove (6.27). Recall the definitions of $\rho_i^a$ in (6.8). We consider the following three cases.

(i) If $\rho_i^a = c_i \delta_{\alpha(1), \alpha(k)}$ for some $k$ such that $\alpha(k) \not\in S$, then we have

$$\frac{\rho_i^a}{c_j} \leq \delta_{\alpha(j), \alpha(k)} \leq \delta_{\alpha(1), \alpha(k)} + |\bar{\delta}_i^a - \bar{\delta}_j^a| \leq \frac{\rho_i^a}{c_i} + \frac{C_0 \bar{\delta}_i^a \bar{\delta}_j^a}{C_0 - 1} (\rho_i^a + \rho_j^a). \quad (6.28)$$

Thus as long as $c_j$ and $C_0$ is chosen such that $\frac{\rho_i^a}{c_j} \leq \frac{C_0 \bar{\delta}_i^a \bar{\delta}_j^a}{C_0 - 1}$, we can obtain the upper bound in (6.27). (ii) If $\rho_i^a = c_i (\bar{\delta}_i^a + m_2c(\lambda_r))$, the proof is the same as for case (i). (iii) If $\rho_i^a = c_i \delta_{\alpha(1), \beta(\nu)}$ for some $\nu$ such that $\beta(\nu) \not\in S$, then there exists a constant $C > 0$ independent of $c_i, c_j, C_0$ such that

$$\frac{\rho_i^a}{c_j} \leq \frac{|m_1c1(\bar{\delta}_i^a) - m_2c(\lambda_r)|}{\frac{C_0}{c_i} + C |\bar{\delta}_i^a - \bar{\delta}_j^a|} \leq \frac{\rho_i^a}{c_i} + \frac{CC_0 \bar{\delta}_i^a \bar{\delta}_j^a}{C_0 - 1} (\rho_i^a + \rho_j^a),$$

where in the second step we used (4.28). Again we obtain the upper bound in (6.27) by choosing appropriate $c_j$ and $C_0$. Finally, the lower bound in (6.27) follows immediately by switching the roles of $i$ and $j$.

The proof for (6.24), (6.25) and (6.26) is similar; the only difference is that we need to use the approximate isometry properties in (4.27) and (4.28).

Now we finish the estimate of $s_2$. First with (6.5), (4.21) and (4.22), we can estimate that

$$|s_2| \leq C \int \frac{n^{-1+2\varepsilon}}{m_1(\bar{\delta}_i^a)} \frac{|g_2^a(\zeta)|}{|\bar{\delta}_i^a - \bar{\delta}_j^a|^{1/2}} \left( |\mathcal{D}^{-1} + g_2^a(\zeta)U^*G(g_2^a(\zeta))U^{-1}| \right) |d\zeta|,$$

where

$$\mathfrak{d}(\zeta) := \left( \min_{1 \leq l \leq r} |(d^l_j)^{-1} + 1 - (1 + \zeta \sigma_j^{l-1})^{-1}| \right) \wedge \left( \min_{1 \leq l \leq p \leq s} |(d^l_p)^{-1} + 1 - (1 + m_1c(g_2^a(\zeta))\sigma_j^{p-1})^{-1}| \right),$$

and we can bound $||\mathcal{E}(g_2^a(\zeta))||$ using (6.5), (6.21) and the Hilbert-Schmidt norm as

$$||\mathcal{E}(g_2^a(\zeta))|| \leq C \sqrt{r} s (\zeta - m_2c(\lambda_r))^{-1/2} n^{-1/2 + \varepsilon}. \quad (6.30)$$

For $\mathfrak{d}(\zeta)$, we have

$$\frac{d^j_j + 1}{d^j_j} - \frac{1}{1 + \zeta \sigma_j^j} = \frac{1 + \bar{\delta}_j^a \zeta}{d^j_j \sigma_j^j (\zeta + (\sigma_j^j)^{-1})}, \quad 1 \leq j \leq r, \quad (6.31)$$

and

$$\frac{d^p_p + 1}{d^p_p} - \frac{1}{1 + m_1c(g_2^a(\zeta))\sigma_j^p} = \frac{1 + \bar{\delta}_j^a m_1c(g_2^a(\zeta))}{d^p_p \sigma_j^p (m_1c(g_2^a(\zeta)) + (\sigma_j^p)^{-1})}, \quad 1 \leq p \leq s. \quad (6.32)$$

Note that we have $|\zeta + (\sigma_j^j)^{-1}| \sim 1$ and $m_1c(g_2^a(\zeta)) + (\sigma_j^p)^{-1} \sim 1$ by (4.12). On the other hand, we can use Lemma 6.8 to bound the numerators from below. Thus we obtain that

$$||\mathcal{E}(g_2^a(\zeta))|| \leq \left( \bar{\delta}_i^a + m_2c(\lambda_r) \right)^{-1/2} n^{-1/2+\varepsilon} \leq \begin{cases} \rho_i^a \leq \mathfrak{d}(\zeta), & \text{for } \zeta \in \Gamma_i, \\ \rho_i^a \leq \mathfrak{d}(\zeta), & \text{for } \zeta \in \Gamma_j, \end{cases}$$

51
where we used Assumption 6.30 and \(6.33\). Thus we have
\[
\frac{1}{\mathfrak{S}(\zeta) - \|\mathcal{L}(g_{2r}(\zeta))\|} \leq \begin{cases} 
(\rho^a_i)^{-1}, & \text{for } \zeta \in \Gamma_i, \\
(\rho^b_{\beta})^{-1}, & \text{for } \zeta \in \Gamma_{\mu}.
\end{cases}
\] (6.33)

Decomposing the integral contour in (6.29) as in (6.22), using (6.33) and Lemma 6.8 and recalling that the length of \(\Gamma_i\) (or \(\Gamma_{\mu}\)) is at most \(2\pi \rho^a_i\) (or \(2\pi \rho^b_{\mu}\)), we get that
\[
|s_2| \leq C \sum_{(\alpha) \in S} \frac{n^{-1+2\varepsilon}}{(\rho^a_k + \delta^a_{\alpha(k),\alpha(i)})^2} + C \sum_{\beta(\mu) \in S} \frac{n^{-1+2\varepsilon}}{(\rho^b_{\mu} + \delta^b_{\beta(\mu),\alpha(i)})^2}.
\] (6.34)

Finally, we estimate the RHS of (6.34) using Cauchy-Schwarz inequality. For \(\alpha(i) \notin S\), we have
\[
\sum_{\alpha(k) \in S} \frac{1}{(\rho^a_k + \delta^a_{\alpha(k),\alpha(i)})^2} + \sum_{\beta(\mu) \in S} \frac{1}{(\rho^b_{\mu} + \delta^b_{\beta(\mu),\alpha(i)})^2} \leq C \frac{1}{\delta^a_{\alpha(i)}(S)^2} + C \frac{1}{\delta^b_{\beta(\mu)}(S)^2}.
\]
For \(\alpha(i) \in S\), we have \(\rho^a_k + \delta^a_{\alpha(k),\alpha(i)} \geq \rho^a_i\) for \(\alpha(k) \in S\), and \(\rho^b_{\mu} + \delta^b_{\beta(\mu),\alpha(i)} \geq \rho^b_{\mu}\) for \(\beta(\mu) \in S\) (which follow from arguments that are similar to the first two inequalities in (6.29)). Then we have for some constant \(C > 0\)
\[
\sum_{\alpha(k) \in S} \frac{1}{(\rho^a_k + \delta^a_{\alpha(k),\alpha(i)})^2} + \sum_{\beta(\mu) \in S} \frac{1}{(\rho^b_{\mu} + \delta^b_{\beta(\mu),\alpha(i)})^2} \leq C \frac{1}{\delta^a_{\alpha(i)}(S)^2} + C \frac{1}{\Delta_1(\tilde{\sigma}^a_i)^2}.
\]
Plugging the above two estimates into (6.34), we get that
\[
|s_3| \leq C n^{-1+2\varepsilon} \left( \frac{1}{\delta^a_{\alpha(i)}(S)} + \frac{1}{\Delta_1(\tilde{\sigma}^a_i)^2} \right) \left( \frac{1}{\delta^b_{\alpha(j)}(S)} + \frac{1}{\Delta_1(\tilde{\sigma}^b_j)^2} \right).
\] (6.35)

Combining (6.10), (6.20), (6.21) and (6.35), we obtain (6.3) for \(1 \leq i, j \leq r\) since \(\varepsilon\) can be arbitrarily small.

We can easily extend the above arguments to the general case. For any \(i, j \in \{1, \cdots, p\}\), we define \(\mathcal{R} := \{1, \cdots, r\} \cup \{i, j\}\). Then we define a perturbed model with (recall (2.11))
\[
\widehat{A} = A \left(1 + \tilde{V}_o \tilde{D}^a (\tilde{V}_o^a)^*\right), \quad \widehat{D}^a = \text{diag}(d^a_k)_{k \in \mathcal{R}}, \quad V^a_o = (v^a_k)_{k \in \mathcal{R}},
\]
where
\[
d^a_k := \begin{cases} 
d^a_k, & \text{if } 1 \leq k \leq r \\
\tilde{\varepsilon}, & \text{if } k \in \mathcal{R} \text{ and } k > r.
\end{cases}
\]

Then all the previous proof goes through for the perturbed model as long as we replace the \(U\) and \(D\) in (6.3) with
\[
\widehat{U} = \begin{pmatrix} 
\tilde{V}_o & 0 \\
0 & V^a_o
\end{pmatrix}, \quad \widehat{D} = \begin{pmatrix} 
\tilde{D}^a (\tilde{D}^a + 1)^{-1} & 0 \\
0 & D^b (D^b + 1)^{-1}
\end{pmatrix}.
\] (6.36)

Note that in the proof, only the upper bound on the \(d^a_k\)'s were used. Moreover, the proof does not depend on the fact that \(\tilde{\sigma}^a_i\) or \(\tilde{\sigma}^b_j\) satisfy (2.26) (we only need the indices in \(S\) to satisfy Assumptions 6.1 and 6.3). Finally, taking \(\tilde{\varepsilon} \downarrow 0\) and using continuity, we get (6.3) for general \(i, j \in \{1, \cdots, p\}\).
6.2 Removing the non-overlapping condition

In this subsection, we prove Proposition 6.2 by removing the non-overlapping Assumption 6.3 in Proposition 6.5. In fact once we have Proposition 6.5, the proof of Proposition 6.2 is relatively more standard and is almost the same as the one in [8, Section 5.2]. Hence we will refrain from writing down all the details.

Proof of Proposition 6.2. Recall the constants $\tau$ in Assumption 6.1 and $\tilde{\tau}$ in Assumption 6.3. Let $\tilde{\tau} < \tau / 4$. We define the index set (recall (5.8))

$$O_{\tau / 2}^+ := \left\{ \alpha(i) : i \in O_{\tau / 2}^{(a)} \right\} \cup \left\{ \beta(\mu) : \mu \in O_{\tau / 2}^{(b)} \right\} .$$

For simplicity, we denote

$$\delta_{a,i} := \delta_{a,i} ; \quad \text{if } a = \alpha(i)$$

$$\delta_{a,i} := \delta_{a,i} ; \quad \text{if } a = \beta(\mu)$$

for any $a \in O^+$. We say that $a \neq b \in O_{\tau / 2}^+$ overlap if

$$\delta_{a,b} \wedge \delta_{b,a} \leq \left\{ \begin{array}{ll}
[\Delta_i(\tilde{\tau})]^{-1} n^{-1/2 + \tilde{\tau}}, & \text{if } a = \alpha(i) \\
[\Delta_j(\tilde{\tau})]^{-1} n^{-1/2 + \tilde{\tau}}, & \text{if } a = \beta(\mu),
\end{array} \right.$$

or $\delta_{a,b} \wedge \delta_{b,a} \leq \left\{ \begin{array}{ll}
[\Delta_i(\tilde{\tau})]^{-1} n^{-1/2 + \tilde{\tau}}, & \text{if } b = \alpha(j) \\
[\Delta_j(\tilde{\tau})]^{-1} n^{-1/2 + \tilde{\tau}}, & \text{if } b = \beta(\nu).
\end{array} \right.$

Definition 6.9. For $S$ satisfying Assumption 6.1, we define sets $L_1(S) \subset S \subset L_2(S)$ such that $L_1(S)$ is the largest subset of $S$ that do not overlap with its complement, and $L_2(S)$ is the smallest subset of $O_{\tau / 2}^+$ that do not overlap with its complement.

It is easy to see that $L_1(S)$ and $L_2(S)$ exist and are unique. For an illustration of these two sets, we refer the reader to Fig. 4 of [8]. The main reason for defining these two sets is that Proposition 6.5 now holds for $(\tau / 2, L_1(S))$ or $(\tau / 2, L_2(S))$.

Now we are ready to prove (6.2). As discussed at the end of Section 6.1, without loss of generality, we can assume that $1 \leq i, j \leq r$. There are four cases to consider.

Case (a): $\alpha(i) = \alpha(j) \notin S$. If $\alpha(i) \notin L_2(S)$, then using $r + s = O(1)$ we see that $\delta_{\alpha(i)}(L_2(S))$. Then Proposition 6.5 gives that

$$\langle v_i^a, P_S v_i^a \rangle \leq \langle v_i^a, P_{L_2(S)} v_i^a \rangle \leq \frac{1}{n \delta_{\alpha(i)}(L_2(S))^2} \leq \frac{1}{n \delta_{\alpha(i)}(L_2(S))^2} .$$

If $\alpha(i) \in L_2(S)$, an easy argument gives that

$$\delta_{\alpha(i)}(S) \leq C \Delta_1(\tilde{\tau})^{-1} n^{-1/2 + \tilde{\tau}} \leq C \delta_{\alpha(i)}(L_2(S)) .$$

Then Proposition 6.5 gives that

$$\langle v_i^a, P_S v_i^a \rangle \leq \langle v_i^a, P_{L_2(S)} v_i^a \rangle \leq \frac{1}{\sigma_i} \frac{1}{n^{1/2} \Delta_1(\tilde{\tau})^{-1}} + \frac{1}{n \delta_{\alpha(i)}(L_2(S))^2} \leq \frac{C n^{2/3}}{n \delta_{\alpha(i)}(S)^2} ,$$

where we also used (4.17) in the third step.

Case (b): $\alpha(i) = \alpha(j) \in S$. We first consider the case $\alpha(i) \in L_1(S)$. We can write

$$\langle v_i^a, P_S v_i^a \rangle = \langle v_i^a, P_{L_1(S)} v_i^a \rangle + \langle v_i^a, P_{S,L_1(S)} v_i^a \rangle .$$

(6.40)
Using Proposition 6.5 and the fact that \( \delta_{\alpha(i)}(S) \sim \delta_{\alpha(i)}(L_1(S)) \), we can calculate the first term as

\[
\frac{1}{\delta_{\alpha(i)}(S)^2} \left( \frac{1}{\Delta_1(\tilde{\alpha}_i)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^4} \right).
\]

For the second term in (6.40), it suffices to assume that \( S \neq \emptyset \) (otherwise it is equal to zero). Then we observe that \( \delta_{\alpha(i)}(S) \sim \delta_{\alpha(i)}(S \setminus L_1(S)) \). Applying (6.37) with \( S \) replaced by \( S \setminus L_1(S) \), we obtain that

\[
\langle v_i^a, P_{S \setminus L_1(S)} v_i^a \rangle < \frac{1}{n \delta_{\alpha(i)}(S)^2}.
\]

Next, for the case \( \alpha(i) \notin L_1(S) \), it is easy to show that (6.38) holds, and as in (6.39), we get

\[
\frac{1}{\delta_{\alpha(i)}(S)^2} \left( \frac{1}{\Delta_1(\tilde{\alpha}_i)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^4} \right) < \frac{n^{2\tau}}{n \delta_{\alpha(i)}(S)^2}.
\]

(6.43)

Combining (6.37), (6.39), and (6.41)-(6.43), we conclude that

\[
\langle v_i^a, P_S v_i^a \rangle - 1 \frac{1}{\delta_{\alpha(i)}(S)^2} \left( \frac{1}{\Delta_1(\tilde{\alpha}_i)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^4} \right) < \frac{n^{2\tau}}{n \delta_{\alpha(i)}(S)^2} + \frac{1}{\delta_{\alpha(i)}(S)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^2}.
\]

(6.44)

This concludes (6.2) for the i = j case since \( \tau \) can be chosen arbitrarily small.

**Case (c):** \( i \neq j \) and \( \alpha(i) \notin S \) or \( \alpha(j) \notin S \). Using (6.44) and the basic estimate

\[
\langle v_i^a, P_S v_j^a \rangle \leq \langle v_i^a, P_{L_1(S)} v_j^a \rangle \langle v_j^a, P_S v_j^a \rangle,
\]

we find that in this case, (6.2) holds with an additional \( n^{2\tau} \) factor multiplying the RHS.

**Case (d):** \( i \neq j \) and \( \alpha(i), \alpha(j) \in S \). Our goal is to prove that

\[
\langle v_i^a, P_S v_j^a \rangle \leq \frac{n^{2\tau}}{n \delta_{\alpha(i)}(S)^2} + \frac{1}{n \delta_{\alpha(j)}(S)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^2}.
\]

(6.46)

We again split \( P_S \) into

\[
\langle v_i^a, P_S v_j^a \rangle = \langle v_i^a, P_{L_1(S)} v_j^a \rangle + \langle v_i^a, P_{S \setminus L_1(S)} v_j^a \rangle.
\]

(6.47)

There are four cases: (i) \( \alpha(i), \alpha(j) \in L_1(S) \); (ii) \( \alpha(i) \in L_1(S) \) and \( \alpha(j) \notin L_1(S) \); (iii) \( \alpha(i) \notin L_1(S) \) and \( \alpha(j) \in L_1(S) \); (iv) \( \alpha(i), \alpha(j) \notin L_1(S) \). In case (i), we can bound the first term in (6.47) using Proposition 6.3 and the estimates that \( \delta_{\alpha(i)}(S) \sim \delta_{\alpha(i)}(L_1(S)) \) and \( \delta_{\alpha(j)}(S) \sim \delta_{\alpha(j)}(L_1(S)) \). The second term in (6.47) can be bounded as in case (c) above (with \( S \) replaced by \( S \setminus L_1(S) \)) together with the estimates \( \delta_{\alpha(i)}(S) \leq C \delta_{\alpha(i)}(S \setminus L_1(S)) \) and \( \delta_{\alpha(j)}(S) \leq C \delta_{\alpha(j)}(S \setminus L_1(S)) \).

In case (ii), we have

\[
\delta_{\alpha(i)}(S) \sim \delta_{\alpha(i)}(L_1(S)), \quad \delta_{\alpha(j)}(S) \leq C \delta_{\alpha(j)}(S \setminus L_1(S)), \quad \delta_{\alpha(i)}(S) \leq \Delta_1(\tilde{\alpha}_i)^{-1} n^{-1/2+\tau} \leq C \delta_{\alpha(j)}(L_1(S)).
\]

Then with Proposition 6.5, we can bound the first term in (6.47) as

\[
\langle v_i^a, P_{L_1(S)} v_j^a \rangle \leq \frac{1}{n \delta_{\alpha(i)}(S)^2} + \frac{1}{n \delta_{\alpha(j)}(S)^2} + \frac{1}{\Delta_1(\tilde{\alpha}_i)^2}.
\]

54
where the last term can be bounded as
\[
\frac{\Delta_1(\tilde{\sigma}_n^\alpha)}{n^{1/2}\delta_{\alpha(i),\alpha(j)}} < \frac{1}{n^{1/2}\Delta_1(\tilde{\sigma}_n^\alpha)\Delta_1(\tilde{\sigma}_n^\eta)} + \frac{n^{\tilde{\tau}}}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)}
\]

For its proof, we refer the reader to (5.38)-(5.40) in [8]. The second term in (6.47) can be bounded as in case (c) above (with \(S\) replaced by \(S\mid L_1(S)\)) together with the estimates
\[
\delta_{\alpha(i)}(S) \sim \delta_{\alpha(i)}(S\mid L_1(S)), \quad \delta_{\alpha(i)}(S) \leq C\delta_{\alpha(i),\alpha(j)}, \quad \delta_{\alpha(j)}(S) \leq C\delta_{\alpha(j)}(S\mid L_1(S)) \leq C\left[\Delta_1(\tilde{\sigma}_n^\alpha)^{-1}n^{-1/2+\tilde{\tau}}\right].
\]

In this case, we get that
\[
\frac{|\langle \psi_i^\alpha, \mathcal{P}_{S\mid L_1(S)} \psi_j^\alpha \rangle|}{\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)} < \frac{n^{2\tilde{\tau}}}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)} + \frac{n^{2\tilde{\tau}}\Delta_1(\tilde{\sigma}_n^\alpha)}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)}
\]

Combining the above estimates, we prove (6.46) for case (ii). The case (iii) can be handled in the same way by interchanging \(i\) and \(j\).

Finally, we deal with case (iv). For the first term in (6.47), we use Proposition 6.5 and the estimates
\[
\delta_{\alpha(i)}(S) \leq C\delta_{\alpha(i)}(L_1(S)), \quad \delta_{\alpha(j)}(S) \leq C\delta_{\alpha(j)}(L_1(S)),
\]

to get that
\[
|\langle \psi_i^\alpha, \mathcal{P}_{L_1(S)} \psi_j^\alpha \rangle| < \frac{1}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)}.
\]

For the second term in (6.47), we use the estimate
\[
\delta_{\alpha(j)}(S) \leq C\delta_{\alpha(j)}(S\mid L_1(S)) \leq C\left[\Delta_1(\tilde{\sigma}_n^\alpha)^{-1}n^{-1/2+\tilde{\tau}}\right]
\]
and case (b) to get that
\[
\langle \psi_i^\alpha, \mathcal{P}_{S\mid L_1(S)} \psi_j^\alpha \rangle < \Delta_1(\tilde{\sigma}_n^\alpha)^2 + \frac{1}{n^{1/2}\Delta_1(\tilde{\sigma}_n^\alpha)} + \frac{n^{2\tilde{\tau}}\Delta_1(\tilde{\sigma}_n^\alpha)}{n\delta_{\alpha(i)}(S\mid L_1(S))} + \frac{1}{n\delta_{\alpha(i)}(S\mid L_1(S))} \leq Cn^{2\tilde{\tau}}
\]
A similar estimate holds for \(\langle \psi_i^\alpha, \mathcal{P}_S \psi_j^\alpha \rangle\). Then we conclude that
\[
\frac{|\langle \psi_i^\alpha, \mathcal{P}_S \psi_j^\alpha \rangle|}{\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)} \leq \frac{1}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)} = \frac{n^{2\tilde{\tau}}}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)}.
\]

This proves (6.46) for case (iv), and hence concludes the proof for case (d).

Combining cases (c) and (d), we conclude (6.2) for the \(i \neq j\) case since \(\tilde{\tau}\) can be chosen arbitrarily small. This finishes the proof of Proposition 6.2 together with (6.44).
7 Non-outlier eigenvectors

In this section, we first prove Theorem 2.19 which will then be used to complete the proof of Theorem 2.15. In other words, we will remove Assumption 6.1 in Proposition 6.2.

Our first goal of this section is to prove the following proposition, from which Theorem 2.19 follows.

Proposition 7.1. Fix a constant \( \tilde{\tau} \in (0, 1/3) \). For \( \alpha(i) \notin \mathcal{O}^+ \) and \( i \leq \tau p \), where \( \tau > 0 \) is given in Theorem 2.15, we have

\[
|\langle \mathbf{v}_j, \tilde{\xi}_{\alpha(i)} \rangle|^2 < \frac{n^{-1} + \eta(i) \sqrt{\kappa_{\gamma_i}}}{|\tilde{\sigma}_j^a + m_{2c}^{-1}(\lambda_i)|^2 + \kappa_{\gamma_i}}.
\]

where we recall the definitions (1.38) and (1.44). Moreover, if \( \alpha(i) \in \mathcal{O}^+ \) satisfies

\[
\tilde{\sigma}_j^a + m_{2c}^{-1}(\lambda_i) \leq n^{-1/3 + \tilde{\tau}},
\]

then we have

\[
|\langle \mathbf{v}_j, \tilde{\xi}_{\alpha(i)} \rangle|^2 < n^{4\tau} \left( \frac{n^{-1} + \eta(i) \sqrt{\kappa_{\gamma_i}}}{|\tilde{\sigma}_j^a + m_{2c}^{-1}(\lambda_i)|^2 + \kappa_{\gamma_i}} \right).
\]

Proof. By Theorem 2.12, Theorem 2.13, Theorem 4.1, Theorem 4.9, Theorem 4.10 and Lemma 4.11, for any fixed \( \varepsilon > 0 \), we can choose a high-probability event \( \Xi_2 \) in which (4.44) and (7.4) hold for \( i \leq \tau p \).

In fact, (7.4) follows from (7.5) and (7.6) combined with the interlacing, Lemma 4.12.

Now we fix an \( \alpha(i) \notin \mathcal{O}^+ \) or \( \alpha(i) \in \mathcal{O}^+ \) satisfying (7.5), and some \( 1 \leq j \leq \tau p \). As discussed at the end of Section 6.1, we may define \( \mathcal{R} := \{1, \ldots, r\} \cup \{j\} \) and can assume without loss of generality that \( \tilde{\sigma}_j^a \) also has a nonzero perturbation \( d_j^a \) (even though it may not cause any outlier). For simplicity, we still use the unperturbed notations and denote \( \hat{A} \) as \( A \).

We choose a specific spectral parameter as \( z_i = \tilde{\lambda}_i + i\eta_i \). Here \( \eta_i := \tilde{\eta}_i \vee n^{\varepsilon} \eta(i) \), where \( \tilde{\eta}_i \) is defined as the solution of

\[
\text{Im} m_{2c}(\tilde{\lambda}_i + i\tilde{\eta}_i) = n^{-1} + 6\varepsilon \tilde{\eta}_i^{-1}.
\]

In fact, the solution exists and is unique since \( \eta \text{ Im} m_{c}(\tilde{\lambda}_i + i\eta) \) is a strictly monotonically increasing function of \( \eta \). With (4.10), it is easy to check that

\[
\tilde{\eta}_i \approx \begin{cases} 
\frac{n^{6\varepsilon}}{n \sqrt{\kappa_{\gamma_i} + n^{-2/3+2\varepsilon}}}, & \text{if } \tilde{\lambda}_i \leq \lambda_r + n^{-2/3+4\varepsilon} \\
n^{-1/2+3\varepsilon} \kappa_{\gamma_i}^{1/4}, & \text{if } \tilde{\lambda}_i \geq \lambda_r + n^{-2/3+4\varepsilon}.
\end{cases}
\]

Moreover, with (4.44) and (7.4), we obtain that

\[
\kappa_{\gamma_i} \lesssim \sqrt{n} \kappa_{\gamma_i} + i^{-1/3} n^{-2/3+2\varepsilon/2} + n^{-3/4+\varepsilon/2} + n^{-1/2+\varepsilon/2} \sqrt{\kappa_{\gamma_i}} \lesssim n^{-\varepsilon/2} \kappa_{\gamma_i}.
\]

and

\[
\eta(\tilde{\lambda}_i) \lesssim n^{-3/4} + n^{-1/2} \left( \kappa_{\gamma_i} + i^{-1/3} n^{-2/3+2\varepsilon/2} + n^{-3/4+\varepsilon/2} + n^{-1/2+\varepsilon/2} \sqrt{\kappa_{\gamma_i}} \right)^{1/2} \lesssim \eta(i).
\]

56
Thus as in (6.33), we see that \( z_i \in \tilde{S}(\varsigma_1, \varsigma_2, \varepsilon) \) and (5.5) can be applied at \( z_i \). We consider two cases: (i) \( \bar{\eta}_i \geq n^\varepsilon \eta(\gamma_i) \) and (ii) \( \bar{\eta}_i < n^\varepsilon \eta(\gamma_i) \). In case (i), (5.5) gives that

\[
\| U^*(G(z_i) - \Pi(z_i)) U \| \leq n^{\varepsilon/2} \Psi(\bar{\lambda}_i + i\bar{\eta}_i) \leq \frac{n^{4\varepsilon}}{n\bar{\eta}_i} = n^{-2\varepsilon} \text{Im} m_{2c}(z_i). \tag{7.8}
\]

In case (ii), with (7.4) and (6.6), we see that

\[
\text{In particular, together with (7.4) and (6.6), we have that}
\]

\[
\text{By (7.8) and (7.10), we have that}
\]

\[
\text{where we used the resolvent expansion in (6.15) and abbreviated}
\]

\[
\text{Together with (6.10), we get that}
\]

\[
\| U^*(G(z_i) - \Pi(z_i)) U \| \leq n^{\varepsilon/2} \Psi(\bar{\lambda}_i + i\eta(\gamma_i)) \leq \sqrt{\frac{\sqrt{\kappa}}{n\eta(\gamma_i)}} + \frac{1}{n^{1+\varepsilon/2}\eta(\gamma_i)} \leq n^{-\varepsilon} \sqrt{\kappa} \leq n^{-\varepsilon} \text{Im} m_{2c}(z_i). \tag{7.10}
\]

After these preparations, we are ready to give the proof. As in (4.62), with the spectral decomposition (6.12), we have the following bound

\[
\left| \langle \nu_j^a, \tilde{G}_i \nu_j^b \rangle \right| \leq \eta_i \text{Im}(\nu_j^a, \tilde{G}(z_i) \nu_j^b). \tag{7.11}
\]

Applying (4.11) to (6.14), we obtain another identity

\[
U^* \tilde{G}(z) U = z^{-1} \tilde{D}^{1/2} \left( D^{-1} - D^{-1} \frac{1}{D^{-1} + z U^* G(z) U} D^{-1} \right) \tilde{D}^{1/2}. \tag{7.12}
\]

In particular, we have

\[
z \langle \nu_j^a, \tilde{G}(z) \nu_j^b \rangle = \frac{1}{d^2_j} \left( 1 + d^2_j \right) \left( \frac{1}{D^{-1} + z U^* G(z) U} \right)_{jj} \Phi_j(z) + \Phi_j^2(z) \left( \mathcal{E}(z) + \mathcal{E}(z) \frac{1}{D^{-1} + z U^* G(z) U} \mathcal{E}(z) \right)_{jj}, \tag{7.13}
\]

where we used the resolvent expansion in (6.15) and abbreviated

\[
\Phi_j(z) := \frac{1}{(d^m_j)^{-1} + 1 - (1 + m_{2c}(z) \sigma_j^a)^{-1}}.
\]

By (7.8) and (7.11), we have that

\[
\min_j \left| (d^m_j)^{-1} + 1 - \frac{1}{1 + m_{2c}(z_i) \sigma_j^a} \right| \geq \text{Im} m_{2c}(z_i) \gg \| \mathcal{E}(z_i) \|.
\]

Thus as in (6.33), we conclude that

\[
\left| \frac{1}{D^{-1} + z U^* G(z_i) U} \right| \leq \frac{C}{\text{Im} m_{2c}(z_i)} \ll \| \mathcal{E}(z_i) \|^{-1}.
\]

57
Inserting it into (7.13) and using (6.31), we obtain that

\[ z\langle \psi_j^a, \tilde{G}(z_i) \psi_j^a \rangle = -(1 + m_{2c}(z_i)\tilde{\sigma}_j^a)^{-1} + O\left( \frac{||\mathcal{E}(z_i)||}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2} \right). \tag{7.14} \]

The next lemma provides a lower bound for \((1 + m_{2c}(z)\tilde{\sigma}_j^a)^{-1}\). Its proof is the same as the one for (6.10) in [2], where the only input is Lemma 4.3.

**Lemma 7.2.** For any fixed \(\delta \in [0, 1/3 - \varepsilon]\), there exists a constant \(c > 0\) such that

\[ |1 + m_{2c}(z_i)\tilde{\sigma}_j^a| \geq c \left[ n^{-2\varepsilon}|\tilde{\sigma}_j^a|^2 + m_{2c}^{-1}(\lambda_r) + \text{Im} m_{2c}(z_i) \right] \]

holds whenever \(\tilde{\lambda}_i \in [0, \theta_1(-m_{2c}^{-1}(\lambda_r) + n^{-1/3+\varepsilon})]\).

Now we fix the \(\delta\) in Lemma 7.2. By (7.11) and (7.14), we have that

\[ \left| \langle \psi_j^a, \tilde{G}(z_i) \rangle \right|^2 \leq -\frac{\eta_i^2}{|z_i|^2} \text{Re} \left( 1 + m_{2c}(z_i)\tilde{\sigma}_j^a \right)^{-1} - \frac{\eta_i^2}{|z_i|^2} \text{Im} \left( 1 + m_{2c}(z_i)\tilde{\sigma}_j^a \right)^{-1} + \frac{C\eta_i \|\mathcal{E}(z_i)\|}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2}. \tag{7.15} \]

We next estimate the terms in (7.15) one by one. First, \(|z_i| \sim 1\) by (7.4) and hence we have

\[ -\frac{\eta_i^2}{|z_i|^2} \text{Re} \left( 1 + m_{2c}(z_i)\tilde{\sigma}_j^a \right)^{-1} \leq \frac{C\eta_i^2}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|} \leq \frac{C\eta_i^2}{\text{Im} m_{2c}(z_i)}. \tag{7.16} \]

where we used Lemma 7.2 in the second step. If \(\tilde{\eta}_i \geq n^6 \eta_i(\gamma_i)\), then with (7.6) we get

\[ \tag{7.10} \leq C\eta_i^2 n^{-1-6\varepsilon} \leq C n^{-1+6\varepsilon+3\delta}, \]

where we used that \(\tilde{\eta}_i \leq n^{-2/3+4\varepsilon+\delta}\), as follows from (7.6), (7.4) and (6.6). If \(\tilde{\eta}_i < n^6 \eta_i(\gamma_i)\), by (7.10) and (7.6) we get

\[ \tag{7.10} \leq \frac{C\eta_i^2}{\sqrt{\eta_i}} \leq \eta_i^2 \eta_i(\gamma_i) \sqrt{\eta_i}. \]

Similarly, for the second item of (7.15), we have

\[ -\frac{\eta_i^2}{|z_i|^2} \text{Im} \left( 1 + m_{2c}(z_i)\tilde{\sigma}_j^a \right)^{-1} \leq \frac{C\eta_i \text{Im} m_{2c}(z_i)}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2} \leq \begin{cases} \frac{C\eta_i n^{-1+6\varepsilon}}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2}, & \text{if } \tilde{\eta}_i \geq n^6 \eta_i(\gamma_i) \\ \frac{C\eta_i n^{-1+6\varepsilon}}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2} \eta_i(\gamma_i) \sqrt{\eta_i}, & \text{if } \tilde{\eta}_i < n^6 \eta_i(\gamma_i). \end{cases} \]

Finally, the third term of (7.15) can be estimated using (7.8) and (7.10) by

\[ \frac{C\eta_i \|\mathcal{E}(z_i)\|}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2} \leq \begin{cases} \frac{C\eta_i n^{-1+6\varepsilon}}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2}, & \text{if } \tilde{\eta}_i \geq n^6 \eta_i(\gamma_i) \\ \frac{C\eta_i \eta_i(\gamma_i) \sqrt{\eta_i}}{|1 + m_{2c}(z_i)\tilde{\sigma}_j^a|^2}, & \text{if } \tilde{\eta}_i < n^6 \eta_i(\gamma_i). \end{cases} \]
Combining all the above estimates, we conclude that

\[
|\langle v_j^a, \tilde{\xi}_i \rangle|^2 \leq Cn^{-1+6\varepsilon+3\delta} + \frac{Cn^{-1+6\varepsilon} + Cn^2 \eta(\gamma_i) \sqrt{\kappa_{\gamma_i}}}{1 + m_{2c}(z) \tilde{\sigma}_j^2}|^2.
\]  

(7.17)

We still need to estimate the denominator of (7.17) from below using Lemma 7.2, which requires a lower bound on $\text{Im} m_{2c}(z_i)$. For $\alpha(i) \notin \mathcal{O}^+$, with (6.10), (7.6) and (7.8) we find that $m_{2c}(z_i) \gtrsim \sqrt{\kappa_{\gamma_i}}$. Together with (7.17), this concludes the proof of (7.8) by choosing $\delta = 0$ in Lemma 7.2, since $\varepsilon$ can be arbitrarily small. On the other hand, when $\alpha(i) \in \mathcal{O}^+$ such that (7.2) holds, with (6.6) and (7.6) we can verify that

\[
\tilde{\lambda}_i \leq \theta_1(-m_{2c}^{-1}(\lambda_r) + n^{-1/3+\varepsilon}),
\]

and

\[
\text{Im} m_{2e}(z_i) \gtrsim n^{-1/3+2\varepsilon}. \tag{7.19}
\]

We can therefore conclude the proof of (7.3) with (7.17) by letting $\delta = \tilde{\tau}$ in Lemma 7.2.

Proof of Theorem 2.19. The estimate (2.36) is an easy corollary of (7.1) using (4.44). If we have (a) (2.37) holds, or (b) either $A$ or $B$ is diagonal, then we can remove the $\eta_i$ term and prove the stronger estimate (2.47) by using the stronger versions of Theorem 2.13 (Theorem 2.7 and Theorem 1.9).

Finally, we can prove Theorem 2.13 without the Assumption 6.1.

Proof of Theorem 2.13. As remarked below Proposition 6.2, it suffices to prove that (6.2) holds for $S \in \mathcal{O}^+$, where for all $\alpha(i) \in S$ and $\beta(\mu) \in S$,

\[
\tilde{\sigma}_i^a + m_{2c}^{-1}(\lambda_r) \gtrsim n^{-1/3}, \quad \tilde{\sigma}_i^b + m_{1c}^{-1}(\lambda_r) \gtrsim n^{-1/3}.
\]  

(7.18)

Then Theorem 2.13 immediately follows by using the Cauchy-Schwarz inequality.

Fix a constant $\varepsilon > 0$. Note that it is easy to check by contradiction that there exists some $x_0 \in [1, r+s]$ satisfying the following gap property: for all $k$ such that $\tilde{\sigma}_k^a \leq -m_{2c}^{-1}(\lambda_r) + x_0 n^{-1/3+\varepsilon}$, we have $\tilde{\sigma}_k^a \leq -m_{2c}^{-1}(\lambda_r) + (x_0 + 1)n^{-1/3+\varepsilon}$. Following the idea in [5, Section 6.2], for such $x_0$, we split $S = S_0 \cup S_1$ such that $\tilde{\sigma}_k^a \leq -m_{2c}^{-1}(\lambda_r) + x_0 n^{-1/3+\varepsilon}$ for $\alpha(k) \in S_0$, and $\tilde{\sigma}_k^a \geq -m_{2c}^{-1}(\lambda_r) + (x_0 + 1)n^{-1/3+\varepsilon}$ for $\alpha(k) \in S_1$. Without loss of generality, we assume that $S_0 \neq \emptyset$, since otherwise the claim already follows from Proposition 6.2.

There are totally six cases: (a) $\alpha(i), \alpha(j) \in S_0$; (b) $\alpha(i) \in S_0$ and $\alpha(j) \in S_1$; (c) $\alpha(i) \in S_0$ and $\alpha(j) \notin S$; (d) $\alpha(i), \alpha(j) \in S_1$; (e) $\alpha(i) \in S_1$ and $\alpha(j) \notin S$; (f) $\alpha(i), \alpha(j) \notin S$.

Case (a): $\alpha(i), \alpha(j) \in S_0$. We have the splitting

\[
\langle v_i^a, \mathcal{P}_S v_j^a \rangle = \langle v_i^a, \mathcal{P}_S v_j^a \rangle + \langle v_i^a, \mathcal{P}_S v_j^a \rangle.
\]  

(7.19)

Applying (6.45) and (7.3) to the first term, and Proposition 6.2 to the second term, we get that

\[
\langle v_i^a, \mathcal{P}_S v_j^a \rangle - \delta_{ij} \frac{1}{\tilde{\sigma}_i^a \tilde{\sigma}_j^b} \left| g_{ij}^a(\tilde{\sigma}_i^a, \tilde{\sigma}_j^b) \right| \leq \delta_{ij} \Delta_1(\tilde{\sigma}_i^a)^2 + \frac{n^{4\varepsilon}}{n \Delta_1(\tilde{\sigma}_i^a)^2 \Delta_1(\tilde{\sigma}_j^b)^2} + \frac{n \delta_{i(i)}(S_1) \delta_{j(i)}(S_1)}{n \Delta_1(\tilde{\sigma}_i^a)^2 \Delta_1(\tilde{\sigma}_j^b)^2},
\]

where we used that $\eta(\gamma_k) \sqrt{\kappa_{\gamma_k}} \leq n^{-1}$ for $k \leq n^{1/4}$ in the first step, and $\Delta_1(\tilde{\sigma}_i^a)^2 \leq n^{-1/3+\varepsilon} \leq \delta_{i(i)}(S_1)$ in the second step.
Case (b): $\alpha(i) \in S_0$ and $\alpha(j) \in S_1$. First suppose that Assumption 6.3 holds for some constant $0 < \eta < \varepsilon$. Applying Cauchy-Schwarz and Proposition 7.1 to the first term in (7.19), and applying 6.3 to the second term in (7.19), we get that

$$|\langle \nu, P \nu \rangle| < n^{4c} \frac{\Delta_1(\tilde{\sigma}^o_j)^2}{n\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n\delta_{\alpha(i)}(S_1)} \left( \frac{1}{\delta_{\alpha(j)}(S_1)} + \frac{1}{\Delta_1(\tilde{\sigma}_j)^2} \right) + \frac{\Delta_1(\tilde{\sigma}_j)^2}{n^{1/2}\delta_{\alpha(i),\alpha(j)}}$$

where we used $\delta_{\alpha(i)}(S_1) \geq \Delta_1(\tilde{\sigma}_j)^2$, $\delta_{\alpha(j)}(S_1) \geq |\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$, and $\Delta_1(\tilde{\sigma}_j)^2 \geq \Delta_1(\tilde{\sigma}_j)^2$. This concludes the proof of case (b) if the non-overlapping Assumption 6.3 holds. Otherwise, the argument is similar to the one in Section 6.4 by using the set $L_1(S_1)$, and we ignore the details.

Cases (c), (e) and (f): We use the splitting (7.19), where we will apply 6.3 to the first term, and Proposition 7.1 to the second term. Note that in all cases, we have $\delta_{\alpha(j)}(S) \leq \delta_{\alpha(j)}(S_1)$ and $\delta_{\alpha(j)}(S) \leq n^c|\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$. In case (c) with $\alpha(i) \in S_0$ and $\alpha(j) \notin S$, we obtain that

$$|\langle \nu, P \nu \rangle| < n^{4c} \frac{\Delta_1(\tilde{\sigma}^o_j)^2}{n\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n\delta_{\alpha(i)}(S_1)\delta_{\alpha(j)}(S_1)} \leq \frac{n^{5c}}{n\Delta_1(\tilde{\sigma}^o_j)^2 \delta_{\alpha(j)}(S)}$$

where we used $\delta_{\alpha(i)}(S_1) \geq |\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ in the second step. In case (e) with $\alpha(i) \in S_1$ and $\alpha(j) \notin S$, we have

$$|\langle \nu, P \nu \rangle| < n^{4c} \frac{\Delta_1(\tilde{\sigma}^o_j)^2}{n\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n\delta_{\alpha(i)}(S_1)} \left( \frac{1}{\delta_{\alpha(j)}(S_1)} + \frac{1}{\Delta_1(\tilde{\sigma}_j)^2} \right) + \frac{\Delta_1(\tilde{\sigma}_j)^2}{n^{1/2}\delta_{\alpha(i)}(S)}$$

where we used $\delta_{\alpha(i)}(S_1) \geq |\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ and $\delta_{\alpha(i)}(S)$ in the second step. In case (f) with $\alpha(i), \alpha(j) \notin S$, we obtain that

$$|\langle \nu, P \nu \rangle| < n^{4c} \frac{\Delta_1(\tilde{\sigma}^o_j)^2}{n\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n\delta_{\alpha(i)}(S_1)\delta_{\alpha(j)}(S_1)} \leq \frac{n^{6c}}{n\delta_{\alpha(i)}(S)\delta_{\alpha(j)}(S)}$$

where we used $\delta_{\alpha(i)}(S_1) \geq |\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ and $\delta_{\alpha(i)}(S) \leq n^c|\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ in the second step.

Case (d): $\alpha(i), \alpha(j) \in S_1$. Again using (7.19), Proposition 7.1 and Proposition 6.2 we get that

$$|\langle \nu, P \nu \rangle| - \delta_{\alpha(i)} \frac{1}{\delta_{2c}(\tilde{\sigma}_j^o)^{-1}} < n^{4c} \frac{\Delta_1(\tilde{\sigma}^o_j)^2}{n\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n^{1/2}\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n} \left( \frac{1}{\delta_{\alpha(j)}(S_1)} + \frac{1}{\Delta_1(\tilde{\sigma}_j)^2} \right)$$

$$< \frac{1}{n^{1/2}\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{n^{4c}}{n^{1/2}\Delta_1(\tilde{\sigma}^o_j)^2 \Delta_1(\tilde{\sigma}_j)^2} + \frac{1}{n} \left( \frac{1}{\delta_{\alpha(j)}(S_1)} + \frac{1}{\Delta_1(\tilde{\sigma}_j)^2} \right)$$

where we used $\delta_{\alpha(i)}(S_1) \geq |\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ and $\delta_{\alpha(i)}(S_1) \geq n^c|\tilde{\sigma}_j^o + m_2^{-1}(\lambda_r)|$ in the second step.

Combining all the above six cases, we conclude that even without the Assumption 6.3 the estimate 6.2 still holds with an additional factor $n^{6c}$ multiplying with the RHS. Since $\varepsilon$ can be arbitrarily small, we conclude the proof.
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