RINEHART COMPLEXES AND
BATALIN-VILKOVISKY ALGEBRAS

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Introduction

In this note we push further the observation made in [1] that certain Batalin-Vilkovisky algebras are entirely classical objects: Let \((A, L)\) be a Lie-Rinehart algebra such that, as an \(A\)-module, \(L\) is finitely generated and projective of finite constant rank. We will show that the relationship between generators of the Gerstenhaber bracket on \(\Lambda_A L\) and connections on the highest \(A\)-exterior power of \(L\) given in [1] arises from the canonical pairing between the exterior \(A\)-powers of \(L\).

Within the framework of duality for Lie-Rinehart algebras [2], this yields another conceptual explanation of the observation, made already in [1] that, given an exact generator for the Gerstenhaber algebra \(\Lambda_A L\), the chain complex underlying the resulting Batalin-Vilkovisky algebra coincides with the Rinehart complex computing the corresponding Lie-Rinehart homology. Thereafter we will spell out some of the connections of [1] with Koszul’s paper [5]; in particular we will explain the significance of the notions of torsionfree linear connection and divergence in the Lie-Rinehart context. The reader is assumed familiar with our paper [1].

1. Rinehart complexes and Batalin-Vilkovisky algebras

Let \((A, L)\) be a Lie-Rinehart algebra. By Theorem 1 of [1], the formula

\[
    a \circ \alpha = a(D\alpha) - \alpha(a), \quad a \in A, \; \alpha \in L,
\]

establishes a bijective correspondence between right \((A, L)\)-connections \(\circ: A \otimes_R L \to A\) on \(A\) (written \((a, \alpha) \mapsto a \circ \alpha\)) and \(R\)-linear operator \(D\) on \(\Lambda_A L\) generating the Gerstenhaber bracket. Under this correspondence, flat right \((A, L)\)-connections, that is, right \((A, L)\)-module structures on \(A\), correspond to exact operators (i.e. operators

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of square zero). Given a right \((A, L)\)-connection \(\circ: A \otimes_R L \to A\), an explicit formula for the generator \(D\) is given by

\[
D(\vartheta_1 \wedge \ldots \wedge \vartheta_p) = \sum_{i=1}^{p} (-1)^{(i-1)}(1 \circ \vartheta_i) \vartheta_1 \wedge \ldots \hat{\vartheta}_i \ldots \wedge \vartheta_p \\
+ \sum_{1 \leq j < k \leq p} (-1)^{(j+k)} [\vartheta_j, \vartheta_k] \wedge \vartheta_1 \wedge \ldots \hat{\vartheta}_j \ldots \hat{\vartheta}_k \ldots \wedge \vartheta_p
\]

(1.2)

where \(\vartheta_1, \ldots, \vartheta_p \in L\), cf. (1.6) in [1]. Furthermore, by Theorem 2 of [1], given an exact generator \(\partial\) for the Gerstenhaber algebra \(\Lambda_A L\), the chain complex underlying the Batalin-Vilkovisky algebra \((\Lambda_A L, \partial)\) coincides with the Rinehart complex \((A \otimes \mathcal{U}_{(A, L)} K(A, L), d)\) where \(A_0\) refers to \(A\), endowed with the right \((A, L)\)-module structure determined by \(\partial\). In particular, the resulting generator \(\partial\) for the Gerstenhaber bracket then comes down to a standard Koszul-Rinehart operator, and when \(L\) is projective as an \(A\)-module, the Batalin-Vilkovisky algebra \((\Lambda_A L, \partial)\) computes the Lie-Rinehart homology \(H_*(L, A_0)\) of \(L\) with coefficients in the right \((A, L)\)-module \(A_0\).

We now suppose that, as an \(A\)-module, \(L\) is projective of finite constant rank (say) \(n\), so that \(\Lambda^n_A L\) is the highest non-zero exterior power of \(L\) in the category of \(A\)-modules. Write \(\lambda: L \otimes_R \Lambda^n_A L \to \Lambda^n_A L\) for the operation of Lie derivative. For later reference, we recall that, for \(\alpha \in L\) and \(x = \xi_1 \wedge \cdots \wedge \xi_n \in \Lambda^n_A L\) where \(\xi_j \in L\),

\[
\lambda \alpha(x) = \sum_j \xi_1 \wedge \ldots [\alpha, \xi_j] \ldots \wedge \xi_n.
\]

Theorem 3 in [1] says that the formula

\[
(1 \circ \alpha) x = \lambda \alpha (x) - \nabla \alpha (x)
\]

(1.3)

establishes a bijective correspondence between right \((A, L)\)-connections \(\circ: A \otimes_R L \to A\) on \(A\) and \((A, L)\)-connections \(\nabla: L \otimes_R \Lambda^n_A L \to \Lambda^n_A L\) on \(\Lambda^n_A L\) (written \((\alpha, x) \mapsto \nabla \alpha (x)\)) in such a way that right \((A, L)\)-module structures on \(A\) correspond to left \((A, L)\)-module structures on \(\Lambda^n_A L\) (i.e. flat connections).

**Remark 1.4.** A conceptual explanation of the formula (1.3) is this: Given an \(A\)-module \(M\) with an \((A, L)\)-connection \(\nabla: L \otimes_R M \to M\), a slight generalization of [2] (2.8) entails that, given \(\phi \in \text{Hom}_A(\Lambda^n_A L, M)\) and \(\alpha \in L\), the formula \(\phi \circ \alpha = -\lambda^\nabla(\phi)\), where \((\lambda^\nabla(\phi))(x) = \nabla \phi (\phi(x)) - \phi(\lambda \alpha(x)) \ (x \in \Lambda^n_A L)\), endows \(\text{Hom}_A(\Lambda^n_A L, M)\) with a right \((A, L)\)-connection

\[
\circ: \text{Hom}_A(\Lambda^n_A L, M) \otimes_R L \to \text{Hom}_A(\Lambda^n_A L, M).
\]

The formula (1.3) is a special case thereof, for \(M = \Lambda^n_A L\). Within the world of supermanifolds, a version of the association \(M \to \text{Hom}_A(\Lambda^n_A L, M)\) from modules with left connection to modules with right connection may be found in [6].

The above observations entail, cf. the Corollary in Section 2 of [1], that the formula

\[
(D \alpha) x = \lambda \alpha (x) - \nabla \alpha (x), \ x \in \Lambda^n_A L,
\]

(1.5)
yields a bijective correspondence between \((A,L)\)-connections \(\nabla\) on \(\Lambda^n_A L\) and linear operators \(D\) generating the Gerstenhaber bracket on \(\Lambda_A L\) in such a way that flat connections correspond to operators of square zero, that is, to differentials. The existence of such a bijective correspondence is due to Koszul \cite{5} for the special case where \(A\) is the ring of smooth functions and \(L\) the \((\mathbb{R},A)\)-Lie algebra of smooth vector fields on a smooth manifold. The approach in terms of right \((A,L)\)-module structures on \(A\) is more general, though, and yields in particular a conceptual explanation of the Batalin-Vilkovisky algebra \((\Lambda^*_A L,D)\) resulting from a flat connection \(\nabla\) on \(\Lambda^n_A L\) as the Rinehart complex calculating the corresponding Rinehart homology. It is, furthermore, completely formal and hence immediately generalizes to other situations, e.g. graded Lie-Rinehart algebras, cf. \cite{3,4}, vvsheaf versions thereof, Lie algebroids on supermanifolds, etc.

We now give another description of the relationship between \((A,L)\)-connections \(\nabla\) on \(\Lambda^n_A L\) and generators \(D\) for the Gerstenhaber bracket on \(\Lambda_A L\). The canonical pairing

\[
\wedge: \Lambda^*_A L \otimes_A \Lambda^n_A = \Lambda^n_A L
\]

of graded \(A\)-modules is perfect and its adjoint

\[
\phi: \Lambda_A^* L \to \text{Hom}_A(\Lambda^n_A^* L, \Lambda^*_A L) = \text{Alt}_A^{n-*}(L, \Lambda^n_A L)
\]

is an isomorphism of graded \(A\)-modules. Given \(\alpha \in \Lambda_A^* L\), write \(\phi_\alpha \in \text{Alt}_A^{n-*}(L, \Lambda^n_A L)\) for the image of \(\alpha\) under the isomorphism \((1.7)\). Thus, when \(\alpha \in \Lambda^n_A L\), given \(\xi_{p+1}, \ldots, \xi_n \in L\),

\[
\phi_\alpha(\xi_{p+1}, \ldots, \xi_n) = \alpha \wedge \xi_{p+1} \wedge \ldots \wedge \xi_n.
\]

For an \((A,L)\)-connection \(\nabla: M \to \text{Hom}_A(L,M)\) on a left \(A\)-module \(M\), we denote its operator of covariant derivative by

\[
d^\nabla: \text{Alt}_A(L,M) \to \text{Alt}_A(L,M).
\]

The correct degree for an element of \(\text{Alt}_A^{n-p}(L, \Lambda^n_A L)\) is \(p\), so that \((1.7)\) is degree preserving. Henceforth we use the standard sign convention for the differential in a Hom-complex and, more generally, for operators of covariant derivative: Thus, given a connection \(\nabla: L \otimes_R \Lambda_A^* L \to \Lambda_A^* L\), for \(f \in \text{Alt}_A^{n-p}(L, \Lambda^n_A L)\) and \(\xi_p, \ldots, \xi_n \in L\),

\[
(d^\nabla f)(\xi_p, \ldots, \xi_n) = (-1)^{p+1} \sum_{p \leq j \leq n} (-1)^{j-p} \nabla_{\xi_j} (f(\xi_p, \ldots, \hat{\xi}_j, \ldots, \xi_n))
\]

\[+ (-1)^{p+1} \sum_{p \leq j < k \leq n} (-1)^{j-p+1+k-p} f([\xi_j, \xi_k], \xi_p, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots, \xi_n)
\]

\[= \sum_{p \leq j \leq n} (-1)^{j-1} \nabla_{\xi_j} (f(\xi_p, \ldots, \hat{\xi}_j, \ldots, \xi_n))
\]

\[+ (-1)^{p+1} \sum_{p \leq j < k \leq n} (-1)^{j+k} f([\xi_j, \xi_k], \xi_p, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots, \xi_n).
\]

The relationship between generators for the Gerstenhaber bracket and connections on \(\Lambda^n_A L\) is made explicit by the following.
Theorem 1.8. The relationship
\[ v \nu \phi_D(\alpha) = -d^\nabla (\phi_\alpha), \quad \alpha \in \Lambda^n_A L, \]
establishes a bijective correspondence between generators $D$ for the Gerstenhaber bracket on $\Lambda_A L$ and $(A, L)$-connections $\nabla$ on $\Lambda^n_A L$ in such a way that exact generators correspond to left $(A, L)$-module structures $\nabla$, i.e., flat $(A, L)$-connections, on $\Lambda^n_A L$. This correspondence coincides with the one determined by (1.5).

Thus the operator $D$ and the connection $\nabla$ determine each other via (1.8.1).

Proof. Let $D$ be a generator for the Gerstenhaber bracket and $\nabla$ a connection on $\Lambda^n_A L$. First we show that $D$ and $\nabla$ are related by (1.5) if and only if, for $0 \leq p \leq n$, the diagram
\[ \begin{array}{ccc} \Lambda^p_A L & \xrightarrow{\phi} & \operatorname{Alt}^{n-p}_A (L, \Lambda^n_A L) \\ D \downarrow & & \downarrow -d^\nabla \\ \Lambda^{p-1}_A L & \xrightarrow{\phi} & \operatorname{Alt}^{n-(p-1)}_A (L, \Lambda^n_A L) \end{array} \]
is commutative. In view of what was said above, given $\xi_p, \ldots, \xi_n \in L$, for any $\alpha \in \Lambda^n_A L$,
\[
(d^\nabla \phi_\alpha)(\xi_p, \ldots, \xi_n) = \sum_{p \leq j \leq n} (-1)^{j-1} \nabla_{\xi_j} (\phi_\alpha(\xi_p, \ldots, \hat{\xi}_j, \ldots, \xi_n)) \\
+ (-1)^{p+1} \sum_{p \leq j < k \leq n} (-1)^{j+k} \phi_\alpha([\xi_j, \xi_k], \xi_p, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots, \xi_n),
\]
\[= \sum_{p \leq j \leq n} (-1)^{j-1} \nabla_{\xi_j} (\alpha \wedge \xi_p \wedge \ldots \hat{\xi}_j \ldots \xi_n) \\
+ (-1)^{p+1} \sum_{p \leq j < k \leq n} (-1)^{j+k} \alpha \wedge [\xi_j, \xi_k] \wedge \xi_p \wedge \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \xi_n.\]

In particular, given $\vartheta_1, \ldots, \vartheta_p \in L$, for $\alpha = \vartheta_1 \wedge \ldots \wedge \vartheta_p$, we obtain
\[
(d^\nabla \phi_\alpha)(\xi_p, \ldots, \xi_n) = \sum_{p \leq j \leq n} (-1)^{j-1} \nabla_{\xi_j} (\vartheta_1 \wedge \ldots \wedge \vartheta_p \wedge \xi_p \wedge \ldots \hat{\xi}_j \ldots \xi_n) \\
+ (-1)^{p+1} \sum_{p \leq j < k \leq n} (-1)^{j+k} \vartheta_1 \wedge \ldots \vartheta_p \wedge [\xi_j, \xi_k] \wedge \xi_p \wedge \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \xi_n.
\]

On the other hand, in view of (1.2), with reference to the right $(A, L)$-connection $\circ$ on $A$ which is determined by $D$ and which determines $D$ as well,
\[
(\phi_D(\alpha))(\xi_p, \ldots, \xi_n) = (D(\alpha)) \wedge \xi_p \wedge \ldots \wedge \xi_n \\
= \sum_{i=1}^p (-1)^{(j-1)} (1 \circ \vartheta_i) \vartheta_1 \wedge \ldots \hat{\vartheta}_i \ldots \wedge \vartheta_p \wedge \xi_p \wedge \ldots \wedge \xi_n \\
+ \sum_{1 \leq j < k \leq p} (-1)^{(j+k)} [\vartheta_j, \vartheta_k] \wedge \vartheta_1 \wedge \ldots \hat{\vartheta}_j \ldots \hat{\vartheta}_k \ldots \wedge \vartheta_p \wedge \xi_p \wedge \ldots \wedge \xi_n.
\]
Since, as an $A$-module, $L$ is finitely generated and projective, it will suffice to establish the claim for the special case where $(\vartheta_1, \ldots, \vartheta_p) = (\xi_1, \ldots, \xi_p)$. Indeed, locally $L$ will be free as an $A$-module, and we may then take $\xi_1, \ldots, \xi_n$ to be an $A$-basis. Now

\[
(d\nabla \phi_\alpha)(\xi_1, \ldots, \xi_n) = (-1)^{p-1} \nabla_{\xi_p} (\xi_1 \wedge \ldots \wedge \xi_p \wedge \xi_{p+1} \wedge \ldots \wedge \xi_n)
+ (-1)^{p+1} \sum_{p+1 \leq k \leq n} (-1)^{p+k} \xi_1 \wedge \ldots \wedge \xi_p \wedge [\xi_p, \xi_k] \wedge \xi_{p+1} \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n
+ (-1)^{p+1} \sum_{p+1 \leq k \leq n} (-1)^{k} [\xi_p, \xi_k] \wedge \xi_1 \wedge \ldots \wedge \xi_p \wedge \xi_{p+1} \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n
\]

Thus

\[
(-1)^{p-1}(d\nabla \phi_\alpha)(\xi_1, \ldots, \xi_n) = \nabla_{\xi_p} (\xi_1 \wedge \ldots \wedge \xi_p \wedge \xi_{p+1} \wedge \ldots \wedge \xi_n)
+ \sum_{p+1 \leq k \leq n} (-1)^{k} [\xi_p, \xi_k] \wedge \xi_1 \wedge \ldots \wedge \xi_p \wedge \xi_{p+1} \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n
\]

where the range of the last summation has been extended to $j \leq p$ (it was $j \leq p-1$ before); this extension of summation does not add a non-zero term since $[\xi_p, \xi_p] = 0$.

In view of (1.3), for every $\xi \in L$, with the notation $x = \xi_1 \wedge \ldots \wedge \xi_n$ and $\nabla^\circ$ for the $(A, L)$-connection on $\Lambda^n_A L$ corresponding to $D$ via (1.3),

\[
(1 \circ \xi)\xi_1 \wedge \ldots \wedge \xi_n = \lambda_\xi(x) - \nabla^\circ_\xi(x)
= \sum_{1 \leq k \leq n} \xi_1 \wedge \ldots \wedge [\xi, \xi_k] \wedge \ldots \wedge \xi_n - \nabla^\circ_\xi(\xi_1 \wedge \ldots \wedge \xi_n)
= \sum_{1 \leq k \leq n} (-1)^{k-1} [\xi, \xi_k] \wedge \xi_1 \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n - \nabla^\circ_\xi(\xi_1 \wedge \ldots \wedge \xi_n).
\]

Thus we conclude

\[
(-1)^{p-1}(\phi_D(\alpha))(\xi_1, \ldots, \xi_n) = \sum_{1 \leq k \leq n} (-1)^{k-1} [\xi_p, \xi_k] \wedge \xi_1 \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n
- \nabla^\circ_{\xi_p}(\xi_1 \wedge \ldots \wedge \xi_n)
+ \sum_{1 \leq j \leq p} (-1)^{j} [\xi_p, \xi_j] \wedge \xi_1 \wedge \ldots \tilde{\xi}_j \ldots \wedge \xi_{p-1} \wedge \xi_p \wedge \ldots \wedge \xi_n
= \sum_{p+1 \leq k \leq n} (-1)^{k-1} [\xi_p, \xi_k] \wedge \xi_1 \wedge \ldots \tilde{\xi}_k \ldots \wedge \xi_n
- \nabla^\circ_{\xi_p}(\xi_1 \wedge \ldots \wedge \xi_n)
= (-1)^p(d\nabla^\circ \phi_\alpha)(\xi_1, \ldots, \xi_n)
\]
Consequently the diagram (1.8.2) is commutative for $0 \leq p \leq n$ if and only if $\nabla = \nabla^\circ$, that is, if and only if $D$ and $\nabla$ are related by (1.5).

Reading the calculations backwards we see that, given an $(A, L)$-connection $\nabla$ on $\Lambda^n_A L$, the operator $D$ determined by the commutativity of the diagram (1.8.2) is a generator for the Gerstenhaber bracket, and vice versa. This proves the claim. □

**Remark 1.9.** With the notation $\beta = \xi_p \wedge \ldots \wedge \xi_n$, the Gerstenhaber algebra property entails

$$0 = D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^p \alpha \wedge (D\beta) + (-1)^p [\alpha, \beta],$$

and the reasoning in the above proof may somewhat concisely be summarized by the formula

$$0 = \phi_{D\alpha} \wedge \beta + d\nabla \phi_{\alpha}(\beta),$$

so that

$$d\nabla \phi_{\alpha}(\beta) = (-1)^p \alpha \wedge (D\beta) + (-1)^p [\alpha, \beta].$$

### 2. The relationship with linear connections

As before, let $(A, L)$ be a Lie-Rinehart algebra, and maintain the hypothesis that, as an $A$-module, $L$ be projective of finite constant rank $n$. Let $\nabla^L: L \otimes_R L \to L$ be an $(A, L)$-connection on $L$. Given $\alpha \in L$, define a morphism $\Phi^{\nabla^L}_\alpha: L \to L$ of $A$-modules by means of the formula

$$\Phi^{\nabla^L}_\alpha(\xi) = [\alpha, \xi] - \nabla^L_\alpha(\xi), \quad \xi \in L,$$

and denote its trace by $\text{Tr}(\Phi^{\nabla^L}_\alpha) \in A$. For $x \in \Lambda^n_A L$, it is obvious that

$$\text{Tr}(\Phi^{\nabla^L}_\alpha)x = \lambda_\alpha(x) - \nabla_\alpha(x) = (1 \circ \alpha)x = (D\alpha)x;$$

where $\nabla: L \otimes_R \Lambda^n_A L \to \Lambda^n_A L$ is the induced $(A, L)$-connection on $\Lambda^n_A L$. Hence, in view of (1.3) and (1.5) above,

$$\text{Tr}(\Phi^{\nabla^L}_\alpha)x = (1 \circ \alpha)x = (D\alpha)x,$$

where “$\circ$” refers to the corresponding right $(A, L)$-connection on $A$, and $D$ to the corresponding generator of the Gerstenhaber bracket on $\Lambda_A L$. Consequently, for any $\alpha \in L$,

$$\text{Tr}(\Phi^{\nabla^L}_\alpha) = 1 \circ \alpha = D\alpha.$$

We note that in [5] the notion of divergence is used which, given $\alpha \in L$, is written $\text{div}_{\nabla^L}(\alpha) = \text{Tr}(\Phi^{\nabla^L}_\alpha)$ but we will avoid it since it conflicts with other usages of the terminology “divergence”; see what is said below. Given the connection $\nabla^L$, the value $1 \circ \alpha = D\alpha$ depends only on the induced connection $\nabla$ on $\Lambda^n_A L$. The corresponding generator $D$ for the Gerstenhaber bracket on $\Lambda_A L$ depends only on the induced connection $\nabla$ on $\Lambda^n_A L$ as well.

In [5], for the special case where $(A, L)$ is the Lie-Rinehart algebra $(C^\infty(X), \text{Vect}(X))$ of smooth functions and smooth vector fields on a smooth manifold $X$, Koszul explicitly constructs a generator $D$ for the Gerstenhaber bracket
on $\Lambda_A L$ from a linear connection on $X$ which has zero torsion. The zero torsion hypothesis then entails that, for any $\alpha$,

$$\Phi^L_\alpha(\xi) = [\alpha, \xi] - \nabla^L_\alpha(\xi) = -\nabla^L_\xi(\alpha), \quad \xi \in L.$$ 

The description (1.5) of the generator $D$ given above shows that such a generator is determined by its induced connection on $\Lambda^n_A L$ since this description involves only the induced connection on $\Lambda^n_A L$. The significance of the occurrence in [5] of a linear connection on $X$ with zero torsion is that the zero torsion property provides an explicit expression for the corresponding generator $D$ entirely within the language of differential operators which, in turn, is used in [5] to establish the existence of $D$. The description (1.2) given above which involves a little bit more homological algebra shows that the zero torsion hypothesis is superfluous. This circle of ideas is now closed by the following observation where we must perhaps make some assumption of the kind which enables us to apply an appropriate gluing procedure, see below.

**Proposition.** Given any $(A, L)$-connection $\nabla$ on $\Lambda^n_A L$, there is an $(A, L)$-connection on $L$ which induces $\nabla$ and has zero torsion.

**Proof.** Let $\nabla^L$ be an $(A, L)$-connection on $L$ having zero torsion, and let $\nabla$ be the induced $(A, L)$-connection on $\Lambda^n_A L$. An arbitrary $(A, L)$-connection $\nabla$ on $\Lambda_A^n L$ may be written in the form

$$\nabla_\alpha(x) = \nabla_\alpha(x) + (\phi(\alpha))x, \quad x \in \Lambda_A^n L,$$

for some morphism $\phi: L \rightarrow A$ of $A$-modules. Let $\Phi: L \rightarrow \text{End}_A(L)$ be a morphism of $A$-modules such that, for every $\alpha \in L$, $\phi(\alpha) = \text{Tr}(\Phi(\alpha))$ and such that for every $\alpha, \beta \in L$, $\Phi(\alpha)\beta = \Phi(\beta)\alpha$. The existence of $\Phi$ is obvious when $L$ is a (finitely generated) free $A$-module. When $L$ is only projective, the existence of $\Phi$ may be established locally where $L$ is free and then the resulting data may be glued together appropriately. For example, $(A, L)$ could be the algebra of smooth functions on a smooth manifold $W$ and $L$ the space of sections of a Lie algebroid on $W$. With this preparation out of the way, the formula

$$\nabla^L_\alpha = \nabla^L_\alpha + \Phi(\alpha), \quad \alpha \in L,$$

yields an $(A, L)$-connection $\nabla^L$ on $L$ having zero torsion which induces the $(A, L)$-connection $\nabla$ on $\Lambda_A^n L$. $\square$

**Remark 1.** Let $M$ be a free $A$-module of rank one, with basis element $b$. Given an $R$-module endomorphism $\alpha$ of $M$, its divergence $\text{div}_b(\alpha) \in A$ is defined by the identity

$$\alpha(b) = \text{div}_b(\alpha)b \in M.$$ 

In particular, let $\nabla$ be an $(A, L)$-connection on $\Lambda_A^n L$ and, given $\alpha \in L$, consider the operation of generalized Lie derivative

$$\lambda^\nabla_\alpha: \text{Alt}_A^n(L, \Lambda_A^n L) \rightarrow \text{Alt}_A^n(L, \Lambda_A^n L), \quad (\lambda^\nabla_\alpha \phi)(x) = \nabla_\alpha(\phi x) - \phi(\lambda_\alpha x), \quad x \in \Lambda_A^n L,$$
where $\phi \in \text{Alt}^n_A(L, \Lambda^n_A L) = \text{Hom}_A(\Lambda^n_A L, \Lambda^n_A L)$. Its negative, with $\phi = \text{Id}$, yields the right-hand side of (1.3). Now, $\text{Alt}^n_A(L, \Lambda^n_A L)$ is a free $A$-module with basis element $\phi = \text{Id}$. Hence, for $\alpha \in L$,

$$\text{div}_\phi(\lambda^\nabla_\alpha) = \text{Tr}(\Phi^\nabla_\alpha) = 1 \circ \alpha = D\alpha.$$  

**Remark 2.** In supermanifold theory, there is also a notion of integral forms, cf. e. g. [7]. What corresponds to it under our circumstances where $(A, L)$ is a Lie-Rinehart algebra having $L$ finitely generated and projective of finite constant rank is the complex $C_L \otimes_U K(A, L) \cong C_L \otimes_A \Lambda A L$ computing the homology $H_*(L, C_L)$ where $C_L$ is the dualizing module, cf. [2]. When $C_L$ is free as an $A$-module, this complex of integral forms is just a Batalin-Vilkovisky algebra of the kind $(\Lambda A L, \partial)$ described above. We hope to return to these issues at another occasion.

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