A Strehl Version of Fourth Franel Sequence

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Abstract

We give a combinatorial identity related to the Franel numbers involving the sum of fourth power of binomial coefficients. Furthermore, investigating in J. Mikić’s proof of the first Strehl Identity, we provide a combinatorial proof of this identity using the double counting argument.

Keywords: Franel number, Strehl Identity, double counting argument.

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1. Introduction

It is well known that
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{1}
\]

Also, the sums of power two of binomial coefficients is given by Vandermonde Identity
\[
\beta_n := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \tag{2}
\]

it is established that the sequence \((\beta_n)_{n \in \mathbb{N}}\) satisfy the following recurrence relation (see [2, Corollary 3.1])

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\begin{align*}
    n\beta_n &= 2(2n-1)\beta_{n-1} \text{ where } \beta_0 = 1. \quad (3)
\end{align*}

In 1894, Franel, [8], introduced the sequences \((f_n)_{n \in \mathbb{N}}\) as
\begin{align*}
    f_n := \sum_{k=0}^{n} \binom{n}{k}^3,
\end{align*}
where he provided that \((f_n)_{n \in \mathbb{N}}\) verified the following recurrence relation
\begin{align*}
    (n+1)^2 f_{n+1} = (7n^2 + 7n + 2) f_n + 8n^2 f_{n-1} \text{ where } f_0 = 1, f_1 = 2. \quad (5)
\end{align*}

Later, in 1895, Franel, [9], shows that the sequence
\begin{align*}
    \varphi_n := \sum_{k=0}^{n} \binom{n}{k}^4,
\end{align*}
satisfies a three-term recurrence formula as follow
\begin{align*}
    (n+1)^3 \varphi_{n+1} = 2(2n+1)(3n^2 + 3n + 1) \varphi_n + 4n(4n-1)(4n+1) \varphi_{n-1},
\end{align*}
where \(\varphi_0 = 1, \varphi_1 = 2\) (for more details about the recurrence of sums that involving power of binomial coefficient one can see [4]). In 1905, MacMahon, [10], investigated the master Theorem where he found the following identity
\begin{align*}
    \sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} \binom{n+k}{k} (xy)^k (x+y)^{n-2k}. \quad (6)
\end{align*}

Recently, in 1993, through applying the Chu-Vandermonde convolution [1], Strehl, [12], obtained the following identity
\begin{align*}
    f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}. \quad (8)
\end{align*}
by simplifying $\binom{2k}{n} \binom{n}{k}$ on Strehl Identity, we get

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2n-2k}{n-k}, \quad (9)$$

in which Gould, [7], derived it using Carlitz formula [3]. Furthermore, Zhao, [13], presents a combinatorial proof of the equivalence between Formula (7) (for $x = y = 1$) and Identity (9), employing free 2-Motzkin paths. Also, J. Mikić et al. [11], established a combinatorial proof of Identity (8), using double counting argument.

The purpose of this paper is to prove a new extension of Franel number $\varphi_n$. Furthermore, we develop the J. Mikić’s argument, [11], of the first Strehl Identity to prove Identity (13), below combinatorially.

The paper is structured as follow; in Section 2, we present our result, then in Section 3, we provide the combinatorial proof of Identity (13), below using double counting argument.

2. Main Theorem

Let us start by following lemma.

Lemma 1 ([1, 6]). Let $n, m$ be positive integers and $x$ be complex, we have

$$\binom{x}{n} \binom{n}{m} = \binom{x}{m} \binom{x-m}{n-m}, \quad (10)$$

$$\sum_{k=0}^{n} \binom{x}{k} \binom{m}{n-k} = \binom{x+m}{n}, \quad (11)$$

$$\sum_{k=0}^{n} \binom{n}{i} \binom{m}{n-k} \binom{x+n-k}{n+m} = \binom{x}{n} \binom{x}{m}. \quad (12)$$
Remark 1. The Identity (12), is exactly the Riordan Identity see [6, (6.45)].

Theorem 1. Let $n$ be positive integer, the following identity holds true
\[ \varphi_n = \sum_{k=0}^{n} \binom{n}{k}^2 \left( \frac{2k}{n} \right) \left( \frac{2n-k}{n} \right). \]  
(13)

Proof. Let $s$ be left hand side of (13). From (11), we have
\[ \left( \frac{2k}{n} \right) = \sum_{i=0}^{n} \binom{k}{i} \left( \frac{k}{n} \right) \left( \frac{n-i}{n} \right), \]
then
\[ s = \sum_{k=0}^{n} \binom{n}{k}^2 \left( \frac{2n-k}{n} \right) \sum_{i=0}^{n} \binom{k}{i} \left( \frac{k}{n} \right) \left( \frac{n-i}{n} \right); \]
\[ = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{n}{n-i} \right) \sum_{k=i}^{n} \binom{n-i}{k-i} \left( \frac{i}{k+i-n} \right) \left( \frac{2n-k}{n} \right); \]
\[ = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{n}{n-i} \right) \sum_{k=i}^{n} \binom{n-i}{k-i} \left( \frac{i}{n-k} \right) \left( \frac{2n-k}{n} \right), \]
set $k-i=j$, we get
\[ s = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{n}{n-i} \right) \sum_{j=0}^{n-i} \binom{n-i}{j} \left( \frac{i}{n-i-j} \right) \left( \frac{2n-i-j}{n} \right); \]
finally, we apply the Riordan Identity (12), where get to result. \( \square \)

Remark 2. Follow the same steps of precedent proof we get this general form for all complex $x$
\[ \sum_{k=0}^{n} \binom{n}{k}^2 \left( \frac{x}{k} \right) \left( \frac{x}{n-k} \right) = \sum_{k=0}^{n} \left( \frac{x}{n-k} \right) \left( \frac{x+k}{k} \right) \left( \frac{2n-k}{n} \right) \left( \frac{n}{k} \right). \]  
(14)

As a consequence we get for $x = 1/2$ and $x = -1/2$, the following identities
\[ \sum_{k=0}^{n} \binom{n}{k}^2 \left( \frac{2k}{k} \right) \left( \frac{2(n-k)}{n-k} \right) \frac{1}{(2k-1) (2(n-k)-1)} \]  
\[ = \sum_{k=0}^{n} \left( \frac{2k}{k} \right) \left( \frac{2(n-k)}{n-k} \right) \left( \frac{n}{k} \right) \left( \frac{n}{2(n-k)-1} \right) \]  
\[ \frac{(2k+1) (-1)^{n-k+1}}{2(n-k)-1}, \]  
(15)
\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{2(n-k)}{n} \binom{n}{k} (-1)^k, \]

where (16), is Domb numbers \[5\]. Furthermore, set \( x = -n - 1 \), we get
\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2n-k}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-k}{n} \binom{2(n-k)}{n} . \] (17)

3. Combinatorial Proof

Before staring the combinatorial proof of Identity (13), we need to remember some fundamental concepts.

- Let \( n \) be non negative integer, \([n]\) denotes the set \( \{1, 2, \ldots, n\} \), \([0]\) denotes the empty set \( \emptyset \).
- Let \( A \) be finite set, we note by \(|A|\) the cardinal of \( A \).

Now we are already to start our combinatorial proof in two steps. Let \( n \in \mathbb{N} \) and \( Y \) be the set define as follow
\[ Y := \{(A, B, C) | A, B \subset [2n], C \subset [3n] \setminus [n], |A| = |B| = |C| = n, A \subset B \Delta [n], B \setminus [n] \subset C\} . \]

First step. Suppose \( k = |B \setminus [n]| \), it is clear that \( 0 \leq k \leq n \), so we can choose those elements from the set \([2n] \setminus [n]\) in \( \binom{n}{k} \) ways. The remain elements \( n - k \) belong to \( B \cap [n] \), so we can choose them in \( \binom{n}{n-k} \) ways (see (11)). Since, the \( k \) elements of \( B \setminus [n] \) must be already in \( C \), then the remain elements \( n - k \) can be choose from remain elements of the set \(([3n] \setminus [n]) \setminus (B \setminus [n])\) in \( \binom{2n-k}{n-k} \) ways.
Since, $|B \Delta [n]| = 2k$, then we have $\binom{2k}{n}$ ways to choose the elements of the set $A$ (see [11]). We conclude that

$$|Y| = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} \binom{2k}{n} \binom{2n-k}{n-k}. \quad (18)$$

Second Step. Observe that

$$Y = \{(A, B, C) \mid A, B \subset [2n], C \subset [3n] \setminus [n], |A| = |B| = |C| = n, A \setminus [n] \subset B, B \setminus [n] \subset C, A \cap B \cap C \cap [n] = \emptyset \}.$$ 

Let us start by counting the number of elements of $A$. Suppose $k = |A \cap [n]|$. Obviously, $0 \leq k \leq n$, we choose those elements from the set $[n]$ in $\binom{n}{k}$ ways, the remain $n - k$ elements belong to $A \setminus [n]$ and we choose them from the set $[2n] \setminus [n]$ in $\binom{n}{n-k}$ ways. In addition, these elements are in the set $B$, so the remain $k$ elements can be choose from $[2n] \setminus A$ in $\binom{n}{k}$ ways (for more details see [11]). Clearly $A \setminus [n] \subset B \setminus [n]$, then the $n - k$ elements of $A \setminus [n]$ must already be in $C$. The remain elements of $C$ can be choose from the set $[3n] \setminus [2n]$ in $\binom{n}{k}$ ways. We conclude that

$$|Y| = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} \binom{n}{n-k} \binom{n}{k}. \quad (19)$$

From Identity (18) and Identity (19), the proof of (13) is done.

**Remark 3.** Motivated by J. Mikić’s observation about generalization of Identity (8), see [11, Remark 1], we generalize the set $Y$ as follow

$$Y := \{(A, B, C) \mid A, B \subset [m], C \subset [m+n] \setminus [n], |A| = |B| = |C| = n,$$

$$\quad , A \subset B \Delta [n], B \setminus [n] \subset C \}.$$
where by double counting argument we get

\[
\min(m-n,n) \sum_{k=0}^{\min(m-n,n)} \binom{m-n}{k} \binom{n}{n-k} \binom{2k}{n} \binom{m-k}{n-k} = \min(m-n,n) \sum_{k=0}^{\min(m-n,n)} \binom{n}{k} \binom{m-n}{n-k} \binom{m-n}{k} \binom{n}{k}.
\]  

(20)

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