A fractal-like structure for the fractional quantum Hall effect

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Abstract

We have pursued in the literature a fractal-like structure for the fractional quantum Hall effect-FQHE which consider the Hausdorff dimension associated with the quantum mechanics paths and the spin of the particles or quasiparticles termed fractons. These objects carry rational or irrational values of spin and satisfy a fractal distribution function associated with a fractal von Neumann entropy. We show that our approach offers a rationale for all FQHE data including possible filling factors suggested by some authors. Our formulation is free of any empirical formula and this characteristic appears as a foundational insight for this FQHE-phenomenon. The connection between a geometrical parameter, the Hausdorff dimension \( h \), associated with the quantum paths and the spin \( s \) of particles, \( h = 2 - 2s, 0 < s < \frac{1}{2} \), is a physical analogous to the fractal dimension formula, \( \Delta(\Gamma) = 2 - H \), of the graph of the functions in the context of the fractal geometry, where \( H \) is known as Hölder exponent, with \( 0 < H < 1 \). We emphasize that our fractal approach to the fractional spin particles gives us a new perspective for charge-flux systems defined in two-dimensional multiply connected space.

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I. INTRODUCTION

The Fractal Geometry of Nature [1] is manifest in the context of the fractional quantum Hall effect [2] through a theoretical formulation introduced by us in the literature [3–10]. Another viewpoint about the fractal approach to the FQHE was discussed in [11]. We have defined universal classes \( h \) of particles or quasiparticles, termed fractons, which satisfy a fractal distribution function associated with a fractal von Neumann entropy. Such objects are charge-flux systems defined in two-dimensional multiply connected space and they carry rational or irrational values of spin. We have considered the Hausdorff dimension \( h \), defined within the interval \( 1 < h < 2 \) and associated with the fractal curves of the quantum mechanics particles. We have found an expression which relates the fractal dimension \( h \) and the spin \( s \) of the particles, \( h = 2 - 2s, \ 0 < s < \frac{1}{2} \). This result has a mathematical analogous, in the branch of the fractal geometry, to the fractal dimension formula of the graph of the functions and given by: \( \Delta(\Gamma) = 2 - H \), where \( H \) is known as Hölder exponent, with \( 0 < H < 1 \) [12]. The fractal properties of the quantum paths can be extracted from the propagators of the particles in the momentum space \(^1\) [3,13]. Our formula, when we consider the spin-statistics relation \( \nu = 2s \), is written as \( h = 2 - \nu, \ 0 < \nu < 1 \). In this way, a fractal spectrum was defined taking into account a mirror symmetry:

\[
\begin{align*}
    h - 1 &= 1 - \nu, \quad 0 < \nu < 1; \\
    h - 1 &= \nu - 1, \quad 1 < \nu < 2; \\
    h - 1 &= 3 - \nu, \quad 2 < \nu < 3; \\
    h - 1 &= \nu - 3, \quad 3 < \nu < 4; \quad \text{etc.}
\end{align*}
\]  

(1)

Now, given the statistical weight for these classes of fractons [3]

\[
W[h, n] = \frac{[G + (nG - 1)(h - 1)]!}{[nG]![G + (nG - 1)(h - 1) - nG]!}
\]

(2)

and from the condition of the entropy be a maximum, we obtain the fractal distribution function

\[
n[h] = \frac{1}{\mathcal{Y}[\xi] - h}.
\]

(3)

The function \( \mathcal{Y}[\xi] \) satisfies the equation

\[
\xi = \left\{ \mathcal{Y}[\xi] - 1 \right\}^{h-1} \left\{ \mathcal{Y}[\xi] - 2 \right\}^{2-h},
\]

(4)

with \( \xi = \exp \left\{ (\epsilon - \mu)/kT \right\} \). We understand the fractal distribution function as a quantum-geometrical description of the statistical laws of nature, since the quantum path is a fractal curve and this reflects the Heisenberg uncertainty principle.

We can obtain for any class its distribution function considering the Eqs.(3,4). For example, the universal class \( h = \frac{3}{2} \) with distinct values of spin \( \left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h=\frac{3}{2}} \) has a specific fractal distribution

\(^1\)Another viewpoint about this discussion can be obtained via a gamma representation of the statistical weight given bellow [25].
This result coincides with another one of the literature of fractional spin particles for the statistical parameter $\nu = \frac{1}{2}$ [14], however our interpretation is completely distinct\(^2\). We also have

$$n\left[\frac{3}{2}\right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}.$$  \hfill (5)

\(^2\)In particular, the authors in [14] never considered the possibility of to define universal class of particles for charge-flux systems defined in two-dimensional multiply connected space. We have here, sets of particles (fractons) with rational or irrational values of spin, satisfying a specific fractal distribution function. This idea works in the same way that fermions constitute a universal class of particles obeying the Fermi-Dirac distribution function. We emphasize that, in this fractal approach to the fractional spin particles, fractons realize the spin-statistics connection. On the other hand, the Hausdorff dimension associated to the quantum paths of particles with any value of spin can be obtained promptly.
Now, as we can check, each universal class $h$ of particles, within the interval of definition has its entropy defined by the Eq.(10). Thus, for fractons of the self-dual class $\left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \right\}_{h=\frac{1}{4}}$, we obtain

$$S_G\left[\frac{3}{2}\right] = K \left\{ (2 + n) \ln \sqrt{\frac{2+n}{2n}} - (2 - n) \ln \sqrt{\frac{2-n}{2n}} \right\}. \tag{11}$$

We have also introduced the topological concept of fractal index, which is associated with each class. As we saw, $h$ is a geometrical parameter related to the quantum paths of the particles and so, we define [5]

$$i_f[h] = \frac{6}{\pi^2} \int_{\infty(T=0)}^{1(T=\infty)} \frac{d\xi}{\xi} \ln \{\Theta[Y(\xi)]\}. \tag{12}$$

For the interval of the definition $1 \leq h \leq 2$, there exists the correspondence $0.5 \leq i_f[h] \leq 1$, which signalizes a connection between fractons and quasiparticles of the conformal field theories, in accordance with the unitary $c < 1$ representations of the central charge. For $\nu$ even it is defined by

$$c[\nu] = i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right] \tag{13}$$

and for $\nu$ odd it is defined by

$$c[\nu] = 2 \times i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \tag{14}$$

where $i_f[h, \nu]$ means the fractal index of the universal class $h$ which contains the particles with distinct values of spin. In another way, the central charge $c[\nu]$ can be obtained using the Rogers dilogarithm function, i.e.

$$c[\nu] = \frac{L[x^\nu]}{L[1]}, \tag{15}$$

with $x^\nu = 1 - x$, $\nu = 0, 1, 2, 3, etc.$ and

$$L[x] = -\frac{1}{2} \int_0^x \left\{ \frac{\ln(1-y)}{y} + \frac{\ln y}{1-y} \right\} dy, \ 0 < x < 1. \tag{16}$$

Thus, we have established a connection between fractal geometry and number theory, given that the dilogarithm function appears in this context, besides another branches of mathematics [15].

Such ideas can be applied in the context of the FQHE. This phenomenon is characterized by the filling factor parameter $f$, and for each value of $f$ we have the quantization of Hall resistance and a superconducting state along the longitudinal direction of a planar system of electrons, which are manifested by semiconductor doped materials, i.e., heterojunctions under intense perpendicular magnetic fields and lower temperatures [2].

4
The parameter $f$ is defined by $f = N\phi_0$, where $N$ is the electron number, $\phi_0$ is the quantum unit of flux and $\phi$ is the flux of the external magnetic field throughout the sample. The spin-statistics relation is given by $\nu = 2s = 2\phi/\phi_0$, where $\phi$ is the flux associated with the charge-flux system which defines the fracton $(\hbar, \nu)$. According to our approach there is a correspondence between $f$ and $\nu$, numerically $f = \nu$. This way, we verify that the filling factors experimentally observed appear into the classes $h$ and from the definition of duality between the equivalence classes, we note that the FQHE occurs in pairs of these dual topological quantum numbers.

II. FRACTAL STRUCTURE FOR THE FQHE DATA

In [10] we have considered recent experimental results and checked the validity of our approach. On the one hand, some papers have suggested other values for the filling factors. In this Letter, we show that these possible values and their duals can be obtained by our approach. On the one hand, some papers have suggested other values for the filling factors. On the other hand, the suggestion in [16] about the fractal-like structure to a deeper understanding of FQHE echoed just with our ideas advanced in the literature [3–10].

Let us consider the filling factors suggested in [16–18] and the FQHE data [11,19–23] together. Taking into account the fractal spectrum and the duality symmetry, we can write the sequence:

$$
\begin{align*}
4 & \rightarrow 19 \rightarrow 34 \rightarrow 15 \rightarrow 26 \rightarrow 11 \rightarrow 18 \rightarrow 25 \rightarrow 7 \rightarrow 24 \rightarrow 17 \rightarrow 10 \rightarrow 23 \\
1 & \rightarrow 5 \rightarrow 9 \rightarrow 4 \rightarrow 7 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 2 \rightarrow 7 \rightarrow 5 \rightarrow 3 \rightarrow 7 \\
13 & \rightarrow 29 \rightarrow 16 \rightarrow 3 \rightarrow 14 \rightarrow 25 \rightarrow 11 \rightarrow 19 \rightarrow 8 \rightarrow 13 \rightarrow 18 \rightarrow 23 \rightarrow 28 \\
4 & \rightarrow 9 \rightarrow 5 \rightarrow 1 \rightarrow 5 \rightarrow 9 \rightarrow 4 \rightarrow 7 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 9 \rightarrow 11 \\
33 & \rightarrow 5 \rightarrow 32 \rightarrow 27 \rightarrow 22 \rightarrow 17 \rightarrow 12 \rightarrow 7 \rightarrow 16 \rightarrow 9 \rightarrow 20 \rightarrow 11 \rightarrow 2 \\
13 & \rightarrow 2 \rightarrow 13 \rightarrow 11 \rightarrow 9 \rightarrow 7 \rightarrow 5 \rightarrow 3 \rightarrow 7 \rightarrow 4 \rightarrow 9 \rightarrow 5 \rightarrow 1 \\
15 & \rightarrow 43 \rightarrow 28 \rightarrow 13 \rightarrow 11 \rightarrow 9 \rightarrow 16 \rightarrow 7 \rightarrow 19 \rightarrow 12 \rightarrow 17 \rightarrow 5 \rightarrow 8 \\
8 & \rightarrow 23 \rightarrow 15 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 9 \rightarrow 4 \rightarrow 11 \rightarrow 7 \rightarrow 10 \rightarrow 3 \rightarrow 5 \\
11 & \rightarrow 14 \rightarrow 17 \rightarrow 20 \rightarrow 3 \rightarrow 19 \rightarrow 16 \rightarrow 13 \rightarrow 10 \rightarrow 7 \rightarrow 4 \rightarrow 13 \rightarrow 9 \\
7 & \rightarrow 9 \rightarrow 11 \rightarrow 13 \rightarrow 2 \rightarrow 13 \rightarrow 11 \rightarrow 9 \rightarrow 7 \rightarrow 5 \rightarrow 3 \rightarrow 10 \rightarrow 7 \\
14 & \rightarrow 5 \rightarrow 11 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 17 \rightarrow 26 \rightarrow 9 \rightarrow 1 \rightarrow 8 \rightarrow 15 \rightarrow 7 \\
11 & \rightarrow 4 \rightarrow 9 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 15 \rightarrow 23 \rightarrow 8 \rightarrow 1 \rightarrow 9 \rightarrow 17 \rightarrow 8 \\
13 & \rightarrow 6 \rightarrow 11 \rightarrow 16 \rightarrow 5 \rightarrow 19 \rightarrow 14 \rightarrow 9 \rightarrow 22 \rightarrow 13 \rightarrow 17 \rightarrow 21 \rightarrow 25 \\
15 & \rightarrow 7 \rightarrow 13 \rightarrow 19 \rightarrow 6 \rightarrow 23 \rightarrow 17 \rightarrow 11 \rightarrow 27 \rightarrow 16 \rightarrow 21 \rightarrow 26 \rightarrow 31 \\
4 & \rightarrow 23 \rightarrow 19 \rightarrow 15 \rightarrow 11 \rightarrow 18 \rightarrow 7 \rightarrow 10 \rightarrow 13 \rightarrow 16 \rightarrow 19 \rightarrow 3 \rightarrow 17 \\
5 & \rightarrow 29 \rightarrow 24 \rightarrow 19 \rightarrow 14 \rightarrow 23 \rightarrow 9 \rightarrow 13 \rightarrow 17 \rightarrow 21 \rightarrow 25 \rightarrow 4 \rightarrow 23 \\
14 & \rightarrow 11 \rightarrow 8 \rightarrow 5 \rightarrow 12 \rightarrow 7 \rightarrow 16 \rightarrow 9 \rightarrow 20 \rightarrow 11 \rightarrow 24 \rightarrow 13 \rightarrow 2 \\
19 & \rightarrow 15 \rightarrow 11 \rightarrow 7 \rightarrow 17 \rightarrow 10 \rightarrow 23 \rightarrow 13 \rightarrow 29 \rightarrow 16 \rightarrow 35 \rightarrow 19 \rightarrow 3 \\
11 & \rightarrow 20 \rightarrow 9 \rightarrow 16 \rightarrow 7 \rightarrow 12 \rightarrow 17 \rightarrow 5 \rightarrow 18 \rightarrow 13 \rightarrow 8 \rightarrow 3 \rightarrow 16 \\
17 & \rightarrow 31 \rightarrow 14 \rightarrow 25 \rightarrow 11 \rightarrow 19 \rightarrow 27 \rightarrow 8 \rightarrow 29 \rightarrow 21 \rightarrow 13 \rightarrow 5 \rightarrow 27 \\
29 & \rightarrow 13 \rightarrow 23 \rightarrow 10 \rightarrow 17 \rightarrow 24 \rightarrow 7 \rightarrow 25 \rightarrow 18 \rightarrow 11 \rightarrow 4 \rightarrow 21 \rightarrow 38 \\
49 & \rightarrow 22 \rightarrow 39 \rightarrow 17 \rightarrow 29 \rightarrow 41 \rightarrow 12 \rightarrow 43 \rightarrow 31 \rightarrow 19 \rightarrow 7 \rightarrow 37 \rightarrow 67
\end{align*}

(17)
Thus, we identify dual pairs of filling factors observed

$$(\nu, \bar{\nu}) = \left( \frac{1}{3}, \frac{2}{3} \right), \left( \frac{5}{3}, \frac{4}{3} \right), \left( \frac{1}{5}, \frac{4}{5} \right), \left( \frac{2}{7}, \frac{5}{7} \right), \left( \frac{2}{5}, \frac{3}{5} \right), \left( \frac{7}{3}, \frac{8}{3} \right), \left( \frac{3}{7}, \frac{4}{7} \right), \left( \frac{4}{9}, \frac{5}{9} \right), \left( \frac{8}{7}, \frac{7}{7} \right), \left( \frac{6}{11}, \frac{7}{11} \right), \left( \frac{5}{6}, \frac{6}{7} \right), \left( \frac{10}{11}, \frac{11}{11} \right), \left( \frac{4}{7}, \frac{7}{7} \right), \left( \frac{13}{11}, \frac{15}{11} \right), \left( \frac{14}{5}, \frac{11}{5} \right), \left( \frac{9}{4}, \frac{4}{4} \right), \left( \frac{19}{7}, \frac{16}{7} \right) \text{ etc.} $$

and other ones to be verified as

$$(\nu, \bar{\nu}) = \left( \frac{2}{11}, \frac{9}{11} \right), \left( \frac{3}{19}, \frac{16}{19} \right), \left( \frac{3}{17}, \frac{14}{17} \right), \left( \frac{2}{13}, \frac{11}{13} \right), \left( \frac{1}{7}, \frac{6}{7} \right), \left( \frac{2}{15}, \frac{13}{15} \right), \left( \frac{5}{13}, \frac{8}{13} \right), \left( \frac{4}{9}, \frac{9}{9} \right), \left( \frac{5}{17}, \frac{12}{17} \right), \left( \frac{26}{23}, \frac{23}{23} \right), \left( \frac{19}{16}, \frac{16}{16} \right), \left( \frac{34}{29}, \frac{29}{29} \right) \text{ etc.} $$

Here, we observe that our approach, in terms of equivalence classes for the filling factors, embodies the structure of the modular group as discussed in the literature [3,24] and the quantum Hall transitions satisfy some properties related with the Farey sequences of rational numbers. The transitions allowed are those generated by the condition $|p_2q_1 - p_1q_2| = 1$, with $\nu_1 = \frac{p_1}{q_1}$ and $\nu_2 = \frac{p_2}{q_2}$. Thus, we verify distinct possible transitions for the fractional quantum Hall effect. We define families of universality classes to them, for example, consider the group I and II:

**Group I**

\[
\begin{align*}
\left\{ \frac{1}{5}, \frac{9}{5}, \frac{11}{5}, \frac{19}{5} \right\}_{h=\frac{3}{8}} & \rightarrow \left\{ \frac{2}{9}, \frac{16}{9}, \frac{20}{9}, \frac{34}{9} \right\}_{h=\frac{3}{16}} \rightarrow \left\{ \frac{1}{4}, \frac{7}{4}, \frac{9}{4}, \frac{15}{4} \right\}_{h=\frac{1}{8}} \\
\left\{ \frac{2}{7}, \frac{12}{7}, \frac{16}{7}, \frac{26}{7} \right\}_{h=\frac{3}{12}} & \rightarrow \left\{ \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3} \right\}_{h=\frac{3}{5}} \rightarrow \left\{ \frac{2}{5}, \frac{8}{5}, \frac{12}{5}, \frac{18}{5} \right\}_{h=\frac{3}{5}}
\end{align*}
\]
\[
\begin{equation}
\begin{array}{c}
\begin{array}{c}
\{3, 11, 17, 25, \ldots\} \quad h=\frac{4}{7} \\
\{3, 7, 13, 17, \ldots\} \quad h=\frac{5}{9} \\
\{3, 5, 11, 13, \ldots\} \quad h=\frac{5}{7}
\end{array}
\rightarrow
\begin{array}{c}
\{1, 3, 5, 7, \ldots\} \quad h=\frac{4}{7} \\
\{2, 4, 8, 10, \ldots\} \quad h=\frac{5}{9} \\
\{7, 11, 25, 29, \ldots\} \quad h=\frac{5}{7}
\end{array}
\rightarrow
\begin{array}{c}
\{4, 10, 18, 24, \ldots\} \quad h=\frac{5}{9} \\
\{5, 9, 19, 23, \ldots\} \quad h=\frac{5}{7} \\
\{7, 19, 33, 45, \ldots\} \quad h=\frac{7}{11}
\end{array}
\end{array}
\end{equation}
\]

\[\text{Group II}\]

\[
\begin{equation}
\begin{array}{c}
\begin{array}{c}
\{4, 6, 14, 16, \ldots\} \quad h=\frac{5}{9} \\
\{7, 9, 15, 19, \ldots\} \quad h=\frac{7}{11} \\
\{5, 7, 9, 15, \ldots\} \quad h=\frac{7}{11}
\end{array}
\rightarrow
\begin{array}{c}
\{1, 3, 5, 7, \ldots\} \quad h=\frac{4}{7} \\
\{2, 4, 8, 10, \ldots\} \quad h=\frac{5}{9} \\
\{7, 11, 25, 29, \ldots\} \quad h=\frac{5}{7}
\end{array}
\rightarrow
\begin{array}{c}
\{3, 5, 11, 13, \ldots\} \quad h=\frac{5}{7} \\
\{3, 7, 13, 17, \ldots\} \quad h=\frac{7}{11} \\
\{5, 9, 19, 23, \ldots\} \quad h=\frac{5}{7}
\end{array}
\rightarrow
\begin{array}{c}
\{4, 10, 18, 24, \ldots\} \quad h=\frac{5}{9} \\
\{5, 13, 23, 31, \ldots\} \quad h=\frac{11}{16} \\
\{9, 9, 9, 9, \ldots\} \quad h=\frac{11}{16}
\end{array}
\end{array}
\end{equation}
\]

\[\text{III. CONCLUSIONS}\]

We have verified that our approach to the FQHE reproduces all experimental data and can predicting the occurrence of this phenomenon for other filling factors. According to our formulation, the topological character of these quantum numbers is related with the Hausdorff dimension of the quantum paths of fractons. For that, we have obtained a physical analogous to the mathematical one formula of the Hausdorff dimension associated with fractal curves. The FQHE occurs in pairs of dual topological quantum numbers filling factors. The foundational basis of our theoretical formulation is free of any empirical formula and this characteristic constitutes the great difference between our insight and others of the literature. Besides this, we have obtained other results, such as [3–10]: a relation between the fractal parameter and the Rogers dilogarithm function, through the concept of fractal index, which is defined in terms of the partition function associated with each universal class of particles; a connection between the fractal parameter \( h \) and the Farey sequences of rational numbers. Farey series \( F_n \) of order \( n \) is the increasing sequence of irreducible fractions in the range 0 – 1 whose denominators do not exceed \( n \). We have the following

\textbf{Theorem} [8]: The elements of the Farey series \( F_n \) of the order \( n \), belong to the fractal sets, whose Hausdorff dimensions are the second fractions of the fractal sets. The Hausdorff
dimension has values within the interval \(1 < h < 2\) and these ones are associated with fractal curves.

Along this discussion we have established a connection between the FQHE and some concepts of the fractal geometry. We believe that our formulation sheds some light on that phenomenon. In another direction, a quantum computing in terms of fractons can be also defined and in this way, via the concept of entanglement fractal von Neumann entropy, we have observed that different Hall states have equivalent entanglement content [25]. This is so because these Hall states belong to the same universality class of the quantum Hall transitions which are labelled just by the Hausdorff dimension. Also, we noted that some properties of the FQHE are independent of dynamical aspects or others details of the system: certain peculiarities of the FQHE can be extracted considering global properties as the modular group and the fractal dimension, which are implicit in all our theoretical formulation. Finally, the comments in [16] about a possible fractal-like structure to a deeper understanding of the FQHE, maybe can start with our ideas.
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