Five Easy Pieces: The Dynamics of Quarks in Strongly Coupled Plasmas

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Abstract: We revisit the analysis of the drag a massive quark experiences and the wake it creates at a temperature \( T \) while moving through a plasma using a gravity dual that captures the renormalisation group runnings in the dual gauge theory. Our gravity dual has a black hole and seven branes embedded via Ouyang embedding, but the geometry is a deformation of the usual conifold metric. In particular the gravity dual has squashed two spheres, and a small resolution at the IR. Using this background we show that the drag of a massive quark receives corrections that are proportional to powers of \( \log T \) when compared with the drag computed using AdS/QCD correspondence. The massive quarks map to fundamental strings in the dual gravity theory. We use the perturbation produced by these strings to compute the wake and compare with the results obtained using AdS/QCD correspondence. We also study the shear viscosity in the theory with running couplings, analyze the viscosity to entropy ratio and compare the result with the bound derived from AdS backgrounds. In the presence of higher order curvature square corrections from the back-reactions of the embedded D7 branes, we argue the possibility of the entropy to viscosity bound being violated. Finally, we show that our set-up could in-principle allow us to study a family of gauge theories at the boundary by cutting off the dual geometry respectively at various points in the radial direction. All these gauge theories can have well defined UV completions, and more interestingly, we demonstrate that any thermodynamical quantities derived from these theories would be completely independent of the cut-off scale and only depend on the temperature at which we define these theories. Such a result would justify the holographic renormalisabilities of these theories which we, in turn, also demonstrate. We give physical interpretations of these results and compare them with more realistic scenarios.
1. Introduction

There is no question that understanding the behavior of many-body QCD in the strong coupling regime is a hard problem to solve, but this needs to be done. The wealth of intriguing experimental data obtained at the Relativistic Heavy Ion Collider (RHIC) has made this situation abundantly clear. One of the main goals of the RHIC program is the creation and the analysis of the quark gluon plasma, a
new phase of matter predicted by lattice QCD. The goal of this paper is to discuss several quantities that can be related to observables measured at RHIC, or to be measured at the LHC. Calculations are done in a regime where the gauge sector is nonperturbative, using techniques borrowed from string theory.

From the early days of the RHIC experiments, the appearance of a strong elliptic hydrodynamic flow was taken as a consequence of early thermalization (before 1 fm/c), and indicative of QGP formation \[\text{[1]}\]. The elliptic flow, defined as the second harmonic component of the momentum distribution, develops when the system undergoing the hydrodynamic expansion has an elliptical shape with different short and long axes. This difference in the spatial shape causes difference in the pressure-gradient, which in turn causes the particles to accelerate more in the short axis direction. The anisotropy in the acceleration then causes the final momentum distribution to be anisotropic. The efficiency of this process of generating the momentum space anisotropy from the spatial anisotropy, however, depends on the size of the shear viscosity $\eta$: a quantity that will be discussed here in some detail. Another experimental observable that has been linked to the formation of a plasma of quarks and gluons is the amount of energy lost by a fast parton travelling through this hot and dense medium. This phenomenon has also been dubbed jet quenching. Interestingly, there is a theoretical link between the concept of a small shear viscosity and that of a large jet quenching \[\text{[4]}\]. The hard partons that travel through the strongly interacting plasma may also leave a wake behind, owing to the medium’s response to the source which is the hard jet. We shall also discuss this in this paper. Somewhat related, the amount of energy lost by a heavy quark has been of great interest as well: the drag force can be related to the properties of the strongly interacting medium. Interestingly, the kinematics of bound states with a heavy quark can be observed through semileptonic decay channels. We compute the drag force experienced by a quark, as it looses energy to the surrounding medium. A brief introduction to some of these topics follows, before the general organization of the paper is outlined.

1.1 Shear viscosity and the viscosity to entropy bound

The shear viscosity represents the strength of the collective interaction between the two laminally flowing layers. Roughly speaking, large shear viscosity means faster mixing of the particles in two neighboring laminas. Somewhat counter-intuitively, the strength of this collective interaction is actually smaller when the microscopic interaction is stronger. This is because the rate of mixing is controlled by the mean free path. When the mean free path is small compared to the typical size of the flow velocity variation, two laminas with different flow velocities cannot easily mix since the exchange of particles is limited to the small volume near the interface of the two laminas: Most particles in the fluid just flows along as if there is no other layers nearby. On the other hand, if the mean free path is comparable to the typical
size of the flow velocity variation, then mixing between different layers can proceed relatively quickly.

When the elliptic flow develops, the fluid has anisotropic fluid velocity distribution. Since the shear viscosity controls the mixing, a large shear viscosity can quickly wash out these difference in the local fluid variables. Therefore, one can say that the smaller the shear viscosity, the stronger the elliptic flow.

In the weakly coupled QCD, the ratio $\eta/s$ is parametrically large since it is of the order $1/\alpha_s^2 \ln(\alpha_s)$. If one is to believe that the QGP created in relativistic heavy ion collisions is in the weak coupling regime, one would then expect the elliptic flow to be small. One of the big surprises from the RHIC experiments is that this expectation is almost maximally violated. It turned out that the ideal hydrodynamics where the value of $\eta$ is just set to 0 consistently describes the elliptic flow at RHIC pointing to a strongly interacting plasma well above the phase transition temperature. In fact, the QGP created at RHIC behaves like the most perfect fluid ever observed.

The analytic tools available to a theorist to tackle this issue, however, have been rather limited. Perturbation theory is obviously not valid in the strong coupling regime and the lattice QCD study has been so far limited to the Euclidean space where extraction of the dynamic quantities such as the perfect-fluidity can be difficult.

This situation changed dramatically when Policastro, Son and Starinets discovered that Maldacena conjecture can be used to calculate the shear viscosity of a certain strongly coupled quantum field exactly. More intriguingly, it turned out that in their solution the shear viscosity and the entropy density ratio, $\eta/s$, had a minimum value at $\frac{\hbar}{4\pi k_B} \equiv \frac{1}{4\pi}$ where $k_B$ is the Boltzmann constant. The authors then made a conjecture that this is indeed the lower bound for any quantum field theory which has a gravity dual. Whether $1/4\pi$ is a true bound or not have been debated many times in the literature. But in all similar calculations, $\eta/s$ remains to be $O(1/4\pi)$. The question is thus: What can we learn from this about strongly coupled QCD?

In this paper, we use the non-AdS/QCD theory to investigate this question using a modified Klebanov-Strassler construction. In the IR limit, this theory eventually leads to a confining $SU(M)$ theory with a large $M$ and $N_f$ flavors\footnote{An alternative possibility is to get an approximately conformal theory with $N_f$ flavors under cascade. This can also be realized in our set-up but we will ignore this possibility.}. This is, of course not a full QCD, but it shares many features with the real world QCD including the appearance of the renormalisation scale\footnote{Note however that by QCD we will always mean a theory that resembles at IR large $N$ QCD (but UV strongly coupled and almost conformal) throughout the text because it has a large number of colors. For finite $N$ there is no known gravity dual. We will however need to keep finite number of fundamental flavors because of certain constraints that will become clearer as we go along.}. There is no gravity-dual construction of QCD yet but our study may shed some light on how the strongly coupled QCD should behave.
Before we go into the details of our calculations, it is instructional to consider the physical meaning of the shear viscosity and its behavior and also why it makes sense to talk about the ratio $\eta/s$ as a measure of its strength.

The definitions of the shear viscosity $\eta$ and the bulk viscosity $\zeta$ are given by the following constitutive equation
\begin{equation}
\langle \delta T_{ij} \rangle = -\frac{\eta}{\varepsilon + P} \left( \nabla_i \langle T^0_j \rangle + \nabla_j \langle T^0_i \rangle - \frac{2}{3} \delta_{ij} \nabla_l \langle T^{0l} \rangle \right) - \frac{\zeta}{\varepsilon + P} \delta_{ij} \nabla_l \langle T^{0l} \rangle \tag{1.1}
\end{equation}
where $\delta T^{ij}$ is the deviation from the ideal fluid stress tensor in the fluid rest frame and $\varepsilon$ and $P$ are the local energy density and the pressure, respectively. Upon using the thermodynamic identity $Ts = \varepsilon + P$ where $s$ is the entropy density, the two coefficients can be also written as $\eta/Ts$ and $\zeta/Ts$. Since the temperature is the only relevant energy scale in the highly relativistic fluid, one can easily see that the importance of the viscous terms depends on the size of the dimensionless ratios $\eta/s$ and $\zeta/s$.

To see what the shear viscosity signifies, consider situation where the fluid is flowing in the $z$ direction but the speed of the flow varies in the $x$ direction (c.f. figure 1). That is, we have $\langle T^{0x} \rangle = \langle T^{0y} \rangle = 0$ everywhere, but $\langle T^{0z}(x) \rangle$ does not have to vanish everywhere. Recall that the stress part, $T^{ij}$, of the stress-energy tensor has the interpretation of the $i$-th component of the current for the conserved momentum density $T^{0j}$.

If the value of the conserved density $\langle T^{0z}(x) \rangle$ at two different points are different, then there must be a net current between these two points. Now consider a $z$-$y$ plane at a fixed $x$ and think about the amount of microscopic current across this plane (c.f. figure 1). A particle crossing this plane from above in figure 1 has the average $v_x$ according to the thermal distribution and average $p_z$ according to the flow velocity above the plane. A particle crossing the plane from the below share the same (but opposite sign) $v_x$ but the average $p_z$ in this case is proportional to the flow velocity below the plane. The net flow of $T^{0z}$ through this $z$-$y$ plane is then
\begin{equation}
T_{xz}(x) \approx T^{0z}(x - l) \langle |v_z| \rangle_{\text{th}} - T^{0z}(x + l) \langle |v_z| \rangle_{\text{th}} \tag{1.2}
\end{equation}
Now for this expression to hold, the particles must not suffer a collision while attempting to cross the plane. Hence the point of origin for the border-crossing particles must not exceed the size of the mean free path. Hence, Taylor expanding the above expression to the first order and setting $l$ to be the mean free path $\lambda$ yields
\begin{equation}
T_{xz}(x) \sim -\lambda(\varepsilon + P) \langle |v_z| \rangle_{\text{th}} \partial_x u_z(x) \tag{1.3}
\end{equation}
Here we used the fact that in case of local equilibrium $T^{0i}_{\text{eq}} = (\varepsilon + P) u^0 u^j$. We also chose the frame where $u_z(x) = 0$. Comparing with the constitutive equation, one sees that
\begin{equation}
\eta \sim \lambda(\varepsilon + P) \langle v \rangle_{\text{th}} \tag{1.4}
\end{equation}
where we have substituted $\langle |v_z| \rangle$ with the average speed $\langle v \rangle$ for this rough estimate and discarded all $O(1)$ constants. Upon using the thermodynamic identity $Ts = \varepsilon + P$, this becomes

$$\frac{\eta}{s} \sim \lambda T\langle v \rangle_{th}$$

(1.5)

From the discussion above, it is clear that the shear viscosity controls the rate of the momentum diffusion in the transverse direction to the flow. In fact, using the constitutive equation, it can be easily shown that the momentum densities satisfy a diffusion equation with the diffusion constant give by

$$D = \frac{\eta}{\varepsilon + P} = \frac{\eta}{Ts} \sim \lambda \langle v \rangle_{th}$$

(1.6)

From Eq.(1.5), we can also argue the existence of the lower bound following [10]. The mean free path $\lambda$ is defined as the average distance between two collisions. This length scale cannot become arbitrarily small due to the uncertainty principle $\Delta x \Delta p \geq 1/2$. The distance between two collisions must be at least longer than the Compton wavelength $1/m$ and the de Broglie wavelength $1/p$. Since the factor $T\langle v \rangle_{th}$ in Eq.(1.5) can be though of as the typical momentum scale in the medium, the $\eta/s$ ratio must satisfy

$$\frac{\eta}{s} \geq B_{low}$$

(1.7)

where $B_{low}$ is a non-zero $O(1)$ constant. Therefore, the fact that $\eta/s$ is bounded from below by an $O(1)$ constant is a rather robust consequence of quantum mechanics as long as the entropy is not completely dominated by large chemical potential; cf. Ref. [24]. What the value of $B_{low}$ is and whether the strongly coupled QCD has the $\eta/s$ ratio close to this bound, of course, is the issue that we are discussing here.

In the theory we are using in this paper it will turn out that $\eta/s$ is smaller than the $1/4\pi$ bound of [3]. Of course it was argued earlier in [7] that if one incorporates

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**Figure 1:** Difference of flows for a particle moving above and below the x-plane.
corrections (where $R$ is the curvature tensor) then the bound is automatically violated in the AdS space. What we find here is that if one goes beyond the AdS case by incorporating RG flows in the gauge theory the bound is violated from the curvature square corrections but the non-trivial RG flow do contribute to this also. We will discuss this in full details in sections 3.4 and 3.5 where we will argue that the required curvature squared corrections can be achieved by adding appropriate number of D7 branes to the background.

There are two main ingredients in our calculation of the $\eta/s$ ratio. The calculation of the shear viscosity in a quantum field theory relies on the Kubo formula

$$\eta = \frac{1}{20} \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{\omega} \int dt dx e^{i \omega t - i k \cdot x} \langle [\hat{\pi}_{ij}(t, x), \hat{\pi}^{ij}(0)] \rangle_{\text{eq}}$$

(1.8)

where $\hat{\pi}_{ij}$ is the spin-2 part (traceless part) of the spatial stress tensor and the average is taken in the fluid rest frame. This Kubo formula relies only on the infrared behavior in the linear response theory. Hence, it is valid for an arbitrarily strong coupling limit.

For the entropy, we may use the Beckenstein-Hawking entropy as originally done by Policastro, Son and Starinets [4]. Alternatively, we can directly calculate the entropy by using the thermodynamic identity $Ts = \varepsilon + P$. When local equilibrium is reached, the energy density and the pressure can easily be obtained once $T^{\mu\nu}$ is known. In the fluid rest frame, we have

$$\langle T^{00}(x) \rangle_{\text{eq}} = \varepsilon(x) \quad \text{and} \quad \frac{1}{3} \langle T^i_i(x) \rangle_{\text{eq}} = P(x)$$

(1.9)

so that the product of entropy density and the equilibrium temperature is

$$Ts = \langle T^{00} \rangle_{\text{eq}} + \frac{1}{3} \langle T^i_i \rangle_{\text{eq}}$$

(1.10)

We expect that the above result matches order by order in $g_s N_f$ with the Bekenstein-Hawking result where $g_s$ is the string coupling and $N_f$ is the number of flavors (we will discuss this in some details in section 3.5). Once we combine these ingredients with the RG flow and curvature squared corrections the $\eta/s$ ratio becomes smaller than the known $1/4\pi$ bound. How small the ratio becomes seems to depend on the details of the model as well as on the UV degrees of freedom in the corresponding dual gauge theory, although the latter dependence is exponentially suppressed.

### 1.2 Background geometry and holographic renormalisation

Our aim in this paper is to study aspects of large $N$ thermal QCD using the dual gravity picture. Of course, as we mentioned earlier, there is no known dual gravitational background to thermal QCD. What we know is a gravitational background dual to a theory with a RG flow that confines in the far IR and has infinite interacting degrees of freedom at the far UV. At zero temperature and in the absence
of fundamental flavors, this is the Klebanov-Tseytlin background \cite{29} with zero deformation parameter. At high temperature and in the presence of $N_f$ fundamental (and $M$ bi-fundamental) flavors the background is much more complicated that, as far as we know, has not been studied before (see \cite{29} for a different model that incorporates flavors but doesn’t have a good UV behavior). In sections 3.1 and 3.3 we show that, to lowest order in $g_s N_f$ and $g_s M^2/N$, there is some analytic control on the background, meaning that we can derive analytic expressions for the metric and fluxes to the lowest order in $g_s N_f$ and $g_s M^2/N$. To higher orders in $g_s N_f$ and $g_s M^2/N$ one can only derive the expressions for the metric and fluxes numerically. We clarify many previously ignored subtleties in the literature, namely the existence of small resolution and the effect of this on the fluxes.

Once we have infinite interacting degrees of freedom in the far UV we naturally face the question of holographic renormalisation. For standard AdS background this has been demonstrated beautifully in the series of papers \cite{13, 14, 15, 16, 17}. For the Klebanov-Strassler background without fundamental flavor, an equivalent treatment has been shown to apply in \cite{18}. In section 3.3 we show that, modulo some subtleties, such a treatment of holographic renormalisation can almost be extended to theories with fundamental flavors also. The subtleties have to do with the existence of non-trivial powers of log $r$ over and above the $1/r$ suppressions in various expressions (here $r$ is the radial coordinate that determines the energy scale of the gauge theory). In the limit of small $g_s N_f$ and $g_s M^2/N$ we argue that once the highest integer power of $r$ has been regularised, the theory is naturally holographically renormalised. On the other hand once $g_s N_f$ and $g_s M^2/N$ are large, we lose all control on our order-by-order expansions and new methods need to be devised to holographically renormalise the theory.

1.3 Organization of the paper

The paper is organized as follows. In section 2 we summarize our main results. This section is meant for readers who would like to know our results without going through the details of the derivations. The summary section also involves a discussion of the subtleties of background associated with RG flows etc. In the presence of non-trivial RG flows and corresponding Seiberg dualities the interpretation of gauge/gravity dualities here is not so straightforward as in the AdS/CFT case. We point this out in some details.

Section 3 is the main section of our paper. As promised in the title, we perform five easy computations in thermal QCD. In sections 3.1 and 3.3 we give a detailed derivations of our background. At zero temperature and in the presence of fundamental and bi-fundamental flavors the gravity dual is given by Ouyang \cite{20}. However inserting a black hole in the Ouyang background to generate a non-zero temperature in the gauge theory changes everything. We can no longer argue that the fluxes, warp factor etc would remain unchanged. Even the internal manifold cannot remain
a simple conifold any more. All the internal spheres would get squashed, and at $r = 0$ there could be both resolution as well as deformation of the two and three cycles respectively. In section 3.1 we present our results to $O(g_s N_f, g_s M^2/N)$, and in section 3.3 we give more detailed derivations and extend this to higher orders in $g_s N_f$ and $g_s M^2/N$. In the limit where the deformation parameter is small, we show that to $O(g_s N_f, g_s M^2/N)$ we can analytically derive the background taking a resolved conifold background. The resolution parameter depends on $g_s N_f, g_s M^2/N$ as well as on the horizon radius $r_h$.

Our background also has D7 branes that give rise to fundamental flavors in the gauge theory. These D7 branes are embedded via Ouyang embedding (3.8) and therefore the strings with one ends on the D7 branes and the other ends falling through the horizon would be associated with $N_f$ thermal quarks. In section 3.2 we evaluate the mass and drag of the quarks in this background. We point out there how our results differ slightly from the analyses done using AdS/QCD techniques.

In section 3.3 we give a more complete picture of the system. Although the section is dedicated to evaluating the wake of the quarks in a thermal medium, we study three related topics here. The first one is already mentioned above: we give a detailed derivation of the background geometry that fills up the gaps left in the discussions of section 3.1. We then give a detailed derivation of the holographic renormalisability of our theory in the limit of small $g_s N_f$ and $g_s M^2/N$. Finally all these analyses are combined to evaluate the finite energy-momentum-tensor of the background plus the quark strings. From there we evaluate the wake of the quarks by removing the energy-momentum-tensor contribution of the quark strings. Due to the complicated nature of the background we could evaluate certain formal quantities in this section without going into numerical details. In the appendices we provide a toy example with only diagonal metric perturbations, where a more direct analysis could be performed.\(^3\)

Another upshot of this section is the realization that we could study infinite number of gauge theories by cutting off the geometry at various $r = r_c$ and UV completing them by inserting “UV caps” at various $r_c$. From gauge theory side this is like inserting correct relevant, marginal or irrelevant operators; and from the gravity side this is like inserting non-trivial geometries from $r = r_c$ to $r = \infty$. We show that any thermal quantities evaluated in these gauge theories are completely independent of the cut-off scale $r = r_c$ and only depend on the temperature (the dependences on the far UV degrees of freedom are exponentially suppressed), justifying the holographic renormalisabilities of these theories.

The cut-off independence is also apparent in the last two sections, sections 3.4 and 3.5 where we study viscosities and the ratios of the viscosities by their corresponding entropy densities. For both these cases there are no cut-off dependences

\(^3\)See for example Appendices A, B and C. Appendix A is not directly related to the main calculations of our paper, but gives the corresponding example for the AdS case.
but the ratios of the viscosities by their corresponding entropy densities are always smaller than the celebrated $1/4\pi$ bound \cite{6}.

Finally in section 4 we present our conclusions. Throughout the paper, we allude to various future directions that will be dealt in the sequel to this paper \cite{55}.

2. Summary of the results

Before we go into the full analysis, let us summarize the main results of our paper. This summary is meant for readers who would want to see the main conclusions without going through our detailed calculations.

In this paper, as the title suggests, we have done five concrete calculations. They are, in order of appearance:

- Construction of the gravity dual of a thermal gauge theory with running coupling constant and fundamental flavors.
- Mass and drag of the quark in the gauge theory from the gravity dual.
- Wake left by the quark when it moves in the QGP medium, again from the gravity dual.
- Shear viscosity of the QGP medium; and finally
- Viscosity by entropy bound for the quark from our dual picture.

Let us now give some brief descriptions of each of the five calculations. First is the gravity dual. Recall that most of the recent analysis have relied heavily on using AdS/CFT correspondence to study the above phenomena. However in this paper we would like to study the gauge theory with non-trivial RG flow, in the hope that we can capture some aspects of the strongly coupled QCD. Although our gravity dual will be far from realistic QCD, we will try to address certain generic issues that could lie in a class of gauge theories that are at least IR confining.

At high temperature the situation is a little subtle, but still IR dynamics of QCD could not be extracted from the AdS-Black Hole (AdS-BH) picture. Here, as one may recall, a black hole is inserted in the AdS picture to account for a non-zero temperature in the gauge theory \cite{11}. To do a better job we need another model that can capture the RG flow in the gauge theory.

The model that comes to mind immediately is the so called Klebanov-Strassler warped conifold construction \cite{12} \textsuperscript{4}. The advantage of this construction is that the gravity dual — which is a warped deformed conifold with three form type IIB fluxes — captures the RG flow of the gauge theory. The KS gauge theory is confining in the far IR but is not asymptotically free. The UV of the theory is a more complicated cascading gauge theory. The other two duals \cite{25} and \cite{26} do not have a good four-dimensional UV description. Additionally, all these constructions are for zero temperature gauge theories.

\textsuperscript{4}A somewhat equivalent constructions given around the same time are \cite{23}, \cite{20}.
The other issue with the KS picture is that the quarks therein are all in the bi-fundamental representations of the two possible UV gauge groups; and they eventually cascade away in the far IR. So what we need is a dual gravity theory that allows fundamental quarks at high temperature.

Before inserting fundamental quark at high temperature let us point out that the situation now, even at zero temperature (and without fundamental flavors), is much more subtle than the AdS case studied earlier. Due to renormalisation group flow in the gauge theory side, we need to ask precisely what aspects of the dual gravity picture captures the dynamics of the strongly coupled gauge theory. The key question to ask here is whether the weakly coupled gravity dual sees the cascading Seiberg dualities or it only sees a smooth RG flow in the gauge theory side. Since the answer to this question lies in the heart of the matter, we will spend some time elaborating this. We would like to caution the readers that this point has been misunderstood in most of the literature and the only article, in our opinion, that has been able to fully explain the subtleties is the one by Strassler [27]. What follows below is an elaboration of these subtleties.

To start off let us ask what \( \mathcal{N} = 4 \) AdS/CFT duality tells us? Recall that due to the tight constraint from supersymmetry all quantum corrections in \( \mathcal{N} = 4 \) theory is cancelled out, leaving us with only classical amplitudes. This in particular means that if we choose any gauge coupling in the theory, it stays there, with possible small finite shifts, under any RG flow (which here means going from UV to IR). For example if we choose the gauge coupling to be very strong, the theory will remain at strong coupling from UV to IR. On the other hand once the gauge theory is at strong coupling we could analyze the theory from weakly coupled supergravity description on AdS space. However when the gauge theory is at very weak coupling there exists no weakly coupled gravity description, and the story there is captured by the full string theory on AdS space\(^5\).

The above conclusions also mean that every value of the coupling is a RG fixed point in the gauge theory. There is no coupling flow and therefore the system is simple without any inherent subtleties. What happens now if we introduce a non-trivial RG flow in the theory? There are three cases to consider:

- The RG flows lead to a non-trivial fixed point (or isolated fixed points) in the theory.
- The RG flows lead to a non-trivial surface of fixed points in the theory; and
- The RG flows lead to no fixed point(s) or fixed surface(s) in the theory.

The first case, as far as we know, leads to no known gravity dual so will ignore this case. The second case is more interesting. We know of one example where non-trivial RG flows in the theory lead to a fixed RG surface. This is the so-called Klebanov-Witten model [42]. There are three couplings in the theory \((g_1, g_2, h)\) corresponding

\(^5\)Alternative one could think that the weakly coupled \( \mathcal{N} = 4 \) gauge theory is captured by string theory in twistor space. This is basically the key essence of [28].
to the two gauge couplings and the coupling associated to the quartic superpotential of the theory respectively. The gauge group is $SU(N) \times SU(N)$ and the three beta functions are:

$$\beta_{g_1} = -\frac{g_1^3 N}{16\pi^2} \left( 1 + \frac{2\gamma_0}{1 - \frac{g_1^2 N}{8\pi^2}} \right), \quad \beta_{g_2} = -\frac{g_2^3 N}{16\pi^2} \left( 1 + \frac{2\gamma_0}{1 - \frac{g_2^2 N}{8\pi^2}} \right), \quad \beta_\eta = \eta (1 + 2\gamma_0) \quad (2.1)$$

where $\eta = h\mu$ is the dimensionless coupling of the theory for any energy scale $\mu$. As discussed in details in [42, 27] all the fields in the theory have the same anomalous dimension $\gamma_0(g_1, g_2, h)$, and therefore the three beta functions vanish exactly when:

$$\gamma_0(g_1, g_2, h) = -\frac{1}{2} \quad (2.2)$$

which is one equation for the three couplings. Therefore the fixed points in this theory form a two-dimensional surface in the three-dimensional space of couplings (see figure 2 for details). This also means that the typical RG flow in the theory will take the following simple form as illustrated in figure 3 below. Since the sign of the two beta functions are negative any arbitrary flow in the coupling constant space will bring us to the fixed point surface. This surface is IR stable. Notice also that at the boundary of the Klebanov-Witten fixed point surface the gauge theory is weakly coupled and so the gravity description is very strongly coupled. To get the weakly coupled supergravity description, we need to go towards the centre of the fixed point surface. The $AdS \times T^{1,1}$ description is valid here. Note also that on the surface the bi-fundamentals scalars all attain anomalous dimensions of $-\frac{1}{2}$, as obvious from the beta functions given earlier.

The story changes quite a bit once we deform away from the AdS background. At zero temperature and in the absence of fundamental flavors, this is of course

![Figure 2: The two-dimensional RG surface in the Klebanov-Witten theory.](image)
the Klebanov-Strassler model \cite{12}. The gauge theory dual of the Klebanov-Strassler model is more complicated now because of the non-trivial RG flows of the two couplings lead to cascades of Seiberg dualities in the theory. Although the UV of the gauge theory has infinite degrees of freedom, the theory is holographically renormalisable\textsuperscript{6} \cite{18} (see also \cite{19}).

To see how the story differs from the AdS case discussed above, let us take the far UV theory to be $SU(kM) \times SU(kM - M)$ with gauge couplings $g_k, g_{k-1}$ for the two gauge groups respectively and $\eta$ to be the other dimensionless coupling defined above. The three beta function now are \cite{12, 27}:

\begin{align*}
\beta_k &= -\frac{g_k^3 kM}{16\pi^2} \left[ \frac{(1 + 2\gamma_0) + \frac{2}{k}(1 - \gamma_0)}{1 - \frac{g_k^2 kM}{8\pi^2}} \right], \quad \beta_\eta = \eta (1 + 2\gamma_0) \quad (2.3) \\
\beta_{k-1} &= -\frac{g_{k-1}^3 (k-1)M}{16\pi^2} \left[ \frac{(1 + 2\gamma_0) - \frac{2}{k-1}(1 - \gamma_0)}{1 - \frac{g_{k-1}^2 (k-1)M}{8\pi^2}} \right] \quad (2.4)
\end{align*}

where we see now that they differ from (2.1) by $O(1/k)$ factors. This also means that there will be no point in the coupling constant space where all the three beta functions could vanish exactly although there would be numerous points where all the beta functions could be very small. Two questions arise immediately:

- Since there are infinite number of gauge theories involved here, what is the gravity description of the theory?
- At the point where the theory has a weakly coupled gravity description, is the gauge theory described by a smooth RG flow or the description involves “choppy” Seiberg dualities?

\textsuperscript{6}We will discuss later that the theory with fundamental flavors is also holographically renormalisable.
Since the answer to these two questions are rather involved, we will go in small steps. It is clear that the geometrical description that involves cascade of Seiberg dualities cannot have a weakly coupled gravity description. The RG flows in the theory have been succinctly presented in [27] so we will be brief here. The RG flows at the boundary of the Klebanov-Witten fixed point surface can be illustrated by figure 4. Observe that the flow takes us from one surface to another because we are moving from one set of gauge theory descriptions convenient at a certain scale to another set of descriptions convenient at a different scale. For the RG

![Figure 4: The RG flow at the boundary of the Klebanov-Witten wall for the Klebanov-Strassler model. The big dots are the Seiberg fixed points of the theory. The vertical distances between the planes don’t signify anything here.](image)

flows illustrated in figure 4 there is no weakly coupled gravity description available. Therefore, contrary to popular belief, the usual cascade of Seiberg dualities do not have a supergravity description on a deformed conifold. In fact there is no simple gravity description that could capture the choppy RG flows in the dual gauge theory! So then where would our gravity calculations fit in? In other words, the analysis that we do in the gravity side captures what aspects of the gauge theory? Clearly since we fail to capture the cascades of Seiberg dualities using our weakly coupled gravity picture, do we then see any interesting aspects of the gauge theory now?

Things are not really that bad once we realise that the strongly coupled gauge theory description or equivalently the weakly coupled gravity description becomes better as we go away from the boundary of the Klebanov-Witten fixed points surface. As we go towards the centre of the surface, the RG flows lose their choppy nature and tend to become smooth. This is illustrated in figure 5. Thus in the gauge theory
Figure 5: The RG flow as we go inside the boundary of the RG fixed point surface. Observe that the gauge theory becomes strongly coupled at the centre of the surface and the RG flow loses its sharp edges and subsequently becomes very smooth.

side we can summarize our situation by three points:

- The theory lies at the boundary of the RG surface and jumps from one Seiberg fixed point to another\(^7\). This is the “usual” cascading and the surface corresponds to the two-dimensional surface of fixed points (for the corresponding Klebanov-Witten theory) in a three-dimensional space of coupling constants.
- The theory lies close to the boundary of the surface but never quite touches the fixed points. So this flow is parallel to the boundary.
- The theory hovers at the centre of the two-dimensional surface and has a smooth RG flow from UV to IR. The degrees of freedom of this theory changes continuously from UV to IR (and is a well defined QFT).

Similarly in the gravity side we can also summarise the situation by three “dual” points:

- For the first case the gauge theory is weakly coupled and therefore there is only a strongly coupled gravity description. This amounts to saying that the tree-level dual is full string theory on a warped deformed conifold (which also means that we cannot say anything from the gravity side other than some protected quantities!).
- For the second case a somewhat similar statement can be made. There is no weakly coupled gravity description available.

\(^7\)As we observed above, all the three beta functions do not vanish simultaneously. Therefore the fixed points are not the absolute Klebanov-Witten type fixed points, rather they have a small tilt given by \(O(1/k)\) corrections \((2.3)\).
For the third case there is a well defined weakly coupled gravity description. This is supergravity defined on warped deformed conifold and tells us clearly that in the dual QFT the colors should decrease logarithmically (although its a little tricky to say how many colors we have).

![Diagram]

Our calculations are done for this regime where there is only a smooth RG flow with a weakly coupled gravity description.

Cascade of Seiberg dualities here, captured by strongly coupled gravity.

**Figure 6:** The complete RG flows depicting the regime of validity of our calculations. This figure is taken from [27]. The yellow wavy line is the branch cut, and corresponds to the various planes in the earlier figures.

All of our above discussions imply that the regime of smooth RG flow in the gauge theory side can be captured by weakly coupled supergravity description. But what supergravity description are we looking for here? We want our supergravity description to capture the RG flow in a gauge theory with fundamental flavor and at a non-zero temperature. In other words we want the sugra description for a thermal gauge theory with fundamental flavors.

The first problem of having fundamental quark may be solved by inserting $N_f$ D7 branes in the KS geometry. However this is subtle as we discuss in sec (3.1). What we know so far is how to insert coincident D7 branes in the Klebanov-Tseytlin background [29]. This is the Ouyang background [20] that has all the type IIB fluxes.
switched on, including the axio-dilaton. Our aim here is to insert a black hole in this geometry and study the resulting gravity dual.

Inserting a black hole in the Ouyang background changes everything. We can no longer argue that the fluxes, warp factor etc would remain unchanged. Even the internal manifold cannot remain a simple conifold any more. All the internal spheres would get squashed, and at \( r = 0 \) there could be both resolution as well as deformation of the two and three cycles respectively. Our metric then should look like (see the definitions of the coordinates in sec (3.1)):

\[
ds^2 = \frac{1}{\sqrt{h}} \left[ - g_1(r) dt^2 + dx^2 + dy^2 + dz^2 \right] + \sqrt{h} \left[ g_2(r)^{-1} dr^2 + d\mathcal{M}_5^2 \right]
\]

where we have been able to work out \( h(r) \), \( g_i(r) \) and \( d\mathcal{M}_5^2 \) only in the limit where the resolution factor of the internal conifold is a constant but the deformation factor is zero\(^8\). Our results are:

- The warp factor \( h(r) \) is given by (3.33).
- The black hole factors \( g_i(r) \) are given by (3.13).
- The internal manifold is given by a warped resolved conifold.

In certain limits along with \( g_1 = g_2 = g \), that we discuss in the text, we can show precisely how our metric differs from the AdS case (see footnote 37 for definitions):

\[
ds^2 = \frac{r^2}{L^2} \left( -g dt^2 + dx^i dx_i \right) + \frac{L^2}{gr^2} dr^2 + L^2 d\mathcal{M}_5^2 - \left( A \log r + B \log^2 r \right) \left[ \frac{r^2}{L^2} \left( -g dt^2 + dx^i dx_i \right) - \frac{L^2}{gr^2} dr^2 + L^2 d\mathcal{M}_5^2 \right]
\]

where the first line is the standard AdS\(_5 \times T^{1,1}\) space with radius \( L \equiv (4\pi g_s N)^{1/4} \), and the second line is the deformation of the AdS as well as the internal spaces. Precisely because of this deformation, as we mentioned above, even the fluxes change drastically from the ones given by Ouyang. Our fluxes capture the fact that the two-cycles in the internal warped resolved conifold are slightly squashed. The three-form RR flux is (the coordinates \((\theta_i, \phi_i, \psi)\) parametrise \( \mathcal{M}_5 \) in (2.3)):

\[
\tilde{F}_3 = 2MA_1 \left( 1 + \frac{3g_s N f}{2\pi} \log r \right) e_\psi \wedge \frac{1}{2} \left( \sin \theta_1 \ d\theta_1 \wedge d\phi_1 - B_1 \sin \theta_2 \ d\theta_2 \wedge d\phi_2 \right) \\
- \frac{3g_s M N f}{4\pi} A_2 \frac{dr}{r} \wedge e_\psi \wedge \left( \cot \frac{\theta_2}{2} \sin \theta_2 \ d\phi_2 - B_2 \cot \frac{\theta_1}{2} \sin \theta_1 \ d\phi_1 \right) \\
- \frac{3g_s M N f}{8\pi} A_3 \sin \theta_1 \sin \theta_2 \left( \cot \frac{\theta_2}{2} \ d\theta_1 + B_3 \cot \frac{\theta_1}{2} \ d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2
\]

\(^8\)Such details won’t matter when we study only the metric. But fluxes will carry the information of background resolution.
and the three form NS-NS flux is:

\[ H_3 = 6g_s A_4 M \left( 1 + \frac{9g_s N_f}{4\pi} \log r + \frac{g_s N_f}{2\pi} \log \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \frac{dr}{r} \wedge \left( \frac{1}{2} \sin \theta_1 d\theta_1 \wedge d\phi_1 ight) \]

\[ - B_4 \sin \theta_2 d\theta_2 \wedge d\phi_2 \] \[ + \frac{3g_s^2 MN_f}{8\pi} A_5 \left( \frac{dr}{r} \wedge e_\psi - \frac{1}{2} d\psi \right) \]

\[ \wedge \left( \cot \frac{\theta_2}{2} d\theta_2 - B_5 \cot \frac{\theta_1}{2} d\theta_1 \right) \] (2.8)

where we differ from Ouyang analysis precisely by the asymmetry factors \( A_i, B_i \). These asymmetry factors incorporate all order corrections and are given in (3.83) (where we present only the first order terms). Finally, the axio-dilaton and the five form are given by (3.31) and (3.84) respectively. These are thus our first set of results related to the gravity dual on which we base all our subsequent calculations. Due to the complicated nature of our background, we have only worked things out to order \( O(g_s N_f, g_s^2 MN_f) \) at small \( r \), and provide some conjectural solution for large \( r \) from the full F-theory embedding of our solution. In fact for most part of this paper our calculations are valid in the following limits of \( g_s, M, N_f \) and \( N \):

\[ \left( g_s, g_s N_f, g_s^2 MN_f, g_s^2 M^2 N \right) \rightarrow 0, \quad (g_s N, g_s M) \rightarrow \infty \] (2.9)

The precise way some of these go to zero or infinity is discussed towards the end of section 3.1. Note that large \( N_f \) limit is probably not realisable here because of the underlying F-theory constraints [30].

Our second set of calculations are related to the mass \( M \) and drag \( \nu \) of a quark from the gravity dual (2.6). The quark is identified in the dual theory with the string whose one end is connected to the D7 branes and the other hand falls in the horizon of the black hole. The drag of the quark is precisely the rate at which the string loses its momentum and energy to the black hole. This is given by\(^9\):

\[ \nu = \frac{T_0}{m L^2} \sqrt{1 + \frac{3g_s M^2}{2\pi N}} \log \left[ \frac{\mathcal{T}}{(1-v^2)^{1/4}} \right] \left( 1 + \frac{3g_s N_f}{2\pi} \left\{ \log \left[ \frac{\mathcal{T}}{(1-v^2)^{1/4}} \right] + \frac{1}{2} \right\} \right) \] (2.10)

where we see that we differ from the AdS result by the \( \log \mathcal{T} \) and \( \log (1-v^2) \) corrections where \( \mathcal{T} = r_h \), the horizon radius, is the characteristic temperature and \( v \) is the velocity of the string (also the quark). These corrections aren’t very big

\(^9\)Note that we will be working in the limit where \( \hbar = c = k_B = M_p = \alpha' \equiv 1 \). Thus all the relevant QCD observables would appear dimensionless throughout. Besides these choices of scales we will also impose \( r_0 \equiv 1 \), where \( r_0 \) is the distance between the tip of the seven brane and the black hole horizon. This will be discussed more in section 3.2.
and occurs precisely because of the logarithmic running in our theory. The other coefficients appearing in (2.10) are defined in section (3.2).

In the above calculations, we had taken the string stretching between D7 branes and the black hole to be a probe in the deformed AdS background. This means that the back reactions are completely neglected. This cannot be the full picture, because a moving string should create some \textit{disturbance} in the surrounding media. This is the premise of our \textbf{third} set of calculations. Under some reasonable assumptions (that we elaborate in sec. (3.3) of the text) we can use this back reaction to calculate the \textit{wake} of the quark in the gauge theory from our dual gravity background. This analysis is simple to state but involves not only a detailed series of manipulations that include, among other things, writing an action and then regularising and renormalising it to extract finite gauge theory variables, but also something much more elaborate and interesting. The interesting part is that not only we can do our wake analysis (or in principle all other thermodynamical quantities) for the dual geometry that we elaborated above, but also on the geometry that is \textit{cut-off} at \(r = r_c\) and a non-trivial UV cap (or UV geometry) attached from \(r = r_c\) to \(r = \infty\). Such a procedure actually creates a new UV completed gauge theory at the boundary that is \textit{different} from the original theory (that we called the \textit{parent} cascading theory). This new theory and the original “parent” one differs by certain operators that we can define carefully at the cut-off. Thus the properties of the new theory are not the universal properties of the parent cascading theory. The parent cascading theory has infinite number of degrees of freedom at the far UV. Our new theory should also have very large degrees of freedom at the UV. We can call these degrees of freedom at the boundary (i.e at \(r = \infty\)) as \(N_{uv}\), such that when \(N_{uv} = e^{-n}, \epsilon \to 0, n \gg 1\) we are studying the parent cascading theory, and the other boundary theories have \(n \geq 1\) (see \textbf{figure 10} for details). An elaborate discussion of this is given in section 3.3 which we would refer the readers for details. In fact the result of studying these gauge theories are both remarkable and instructive. The remarkable thing is that no matter what thermodynamical quantities we want to extract from these theories, the final results are \textit{completely independent} of the cut-off \(r = r_c\) that we imposed to derive these theories. The results only depend on the characteristic temperature \(T\) (that we fix once and for all) and on the UV degrees of freedom via \(e^{-N_{uv}}\). Since \(N_{uv}\) is always infinite for us the results are only sensitive to \(T\). Our third set of calculations related to the wake demonstrates this in full details. No matter how involved are the UV descriptions, or the procedure to holographically renormalise these theories, the final answers for the energy-momentum tensors for the gauge theories are remarkably clean and can be stated as\textsuperscript{10}:

\[
T_{mn}^{\text{medium+quark}} = \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha,\beta} \left\{ (H_{mn}^{\alpha} + H_{mn}^{[\alpha]})(s_{n}^{(4)[\beta]} - 4(K_{mn}^{[\alpha]} + K_{mn}^{[\alpha]})s_{n}^{(4)[\beta]} \right\}
\]

\textsuperscript{10}Repeated indices of the form \(a^nb^n, a^nb_{nn}\) are summed over \(n\), but \(a_nb_n, (ab)^n\) etc may not be.
\[ (\alpha_n \beta_{nn} + \sum_{j=0}^{\infty} \hat{b}^{(\alpha)}_{n(j)} \tilde{J}_n \delta_{nm} e^{-jN_{uv}} + \mathcal{O}(Te^{-N_{uv}})) \]

(2.11)

from where the wake of the quark can be computed using the relation (3.64) which equivalently means we subtract the energy momentum tensors of the quark from (2.11) above. All the functions appearing above are described in section (3.3), with \( \alpha \) denoting the effects of the flavors. Here \((\hat{b}^{(\alpha)}_{n(j)}, N_{uv})\) together specify the full boundary theory for a specific UV complete theory (for a discussion on the back reactions, see after (3.95)). Note that the result has no dependence on \( r = r_c \). In the limit where \( N_{uv} = \epsilon^{-n} \), \( n \gg 1 \) we reproduce the result for the parent cascading theory, and for \( N_{uv} \to \epsilon^{-n}, n \geq 1 \) we have the UV dependences specified above.

The instructive thing about our result is the way we differ from the AdS case. The energy momentum tensor of the system from AdS/CFT case is the first part of the above formula given by \( s^{(4)[0]}_{mn} \) (see (3.115) for details). The rest is the deformation from the AdS result (we have to remove some constant pieces from (2.11) to account for the stress tensor properly). This way we can again see how different UV completed gauge theories can change some of these calculations. Of course all these theories do inherit some of the universal properties of the parent cascading theory.

Our fourth set of calculations is related to studying the shear viscosity with and without curvature squared corrections. Without curvature squared corrections, but involving the RG flow, the result for shear viscosity \( \eta \) is easy to state:

\[ \eta = \frac{T^3 L^2}{2g_s^2 G_N} \left[ 1 + \sum_{k=1}^{\infty} \frac{\alpha_k e^{-4kN_{uv}}}{4\pi + \frac{1}{\pi} \log^2 (1 - T^4 e^{-4N_{uv}})} \right] \]

(2.12)

where \( \alpha_k \) are functions of \( T \) that are given in section 3.4. Note again that the result is independent of the cut-off at \( r = r_c \). In the limit where \( N_{uv} = \infty \) the shear viscosity has a simple form. This is related to the AdS result also. On the other hand the viscosity to the entropy ratio, which is our final set of calculations taken in the limit where we switch on all the ingredients i.e the RG flows, curvature squared corrections as well as the contributions from the UV caps, has the following form:

\[ \frac{\eta}{s} = \left[ \frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4kN_{uv}}}{4\pi + \frac{1}{\pi} \log^2 (1 - T^4 e^{-4N_{uv}})} \right] \]

\[ - \frac{c_3 \kappa}{3L^2 (1 - T^4 e^{-4N_{uv}})^{3/2}} \left[ B_0 (4\pi^2 - \log^2 C_o) + 4\pi A_o \log C_o \right] \]

(2.13)

where \( (A_o, B_o, C_o) \) are given in (3.229) and \( c_3 \) is given in (3.163). We see that the ratio is again independent of any cut-off; and in the limit \( N_{uv} \to \epsilon^{-n} \) and \( c_3 \to 0 \) (2.13) the bound is exactly saturated. However in the limit \( c_3 \neq 0 \) we have a violation of the bound. The plot of the behavior of \( \eta/s \) is given in sec (3.5).
3. Dynamics of quarks in strongly coupled plasmas

This section contains the five elements alluded to in the title of this paper. It is now time to construct a more detailed scenario wherein certain aspects of QCD calculations could be performed. As we discussed in the introduction and in the summary above, most of the previous analysis relied on Anti-deSitte (AdS) spaces whose dual is a conformal field theory (CFT) with no running couplings. Many of the recent works in this field have focussed on AdS/CFT correspondence to study certain behaviors of QCD. Our aim in this section would be to use non-AdS backgrounds to study properties of QCD.

3.1 Construction of the gravity dual

It turns out if we embed D branes in certain geometric background, the gauge theory that lives on the D branes may become confining and exhibit logarithmic running coupling. One of the most popular background to achieve this property is the Klebanov-Strassler background (see also [25, 26]). In Klebanov-Strassler (KS) model, N D3 branes are placed at the tip of a six dimensional conifold and M D5 branes are placed in such a way that they wrap a 2-cycle on the conifold base $T_{1,1}$. The D3 and unwrapped part of the D5 branes extend in four spacetime directions orthogonal to the conifold. The $SU(N) \times SU(N+M)$ gauge theory living on these branes contain matter fields that transform as bifundamental color representation $(N, \bar{N} + \bar{M})_c$ and $(\bar{N}, N + M)_c$ of the group $SU(N) \times SU(N + M)$. Under a Renormalisation Group flow the gauge group cascades down to $SU(M)$ group in IR. The RG flow and fixed points surface for this model have been shown earlier. In the regime where we have a smooth RG flow, the theory confines in the far IR. Thus there is a small regime of the theory that gives us a weakly coupled gravity dual to a confining $SU(M)$ theory. Of course since the UV behavior is not asymptotically free$^{11}$, KS model doesn’t give us the full gravity dual for QCD. However it does come close in giving us at least a dual model that has a running coupling constant. Similar behavior can also be argued for [25] and [26], although the UV behaviors of [25] and [26] are six-dimensional theories and may develop baryonic branches [32]. On the other hand, in far IR, one can show that many BPS quantities have one-to-one correspondences [33].

The original KS model (as well as [25, 26]) do not have quarks in the fundamental representation of the gauge group. To introduce fundamental quarks, we need D7 branes in the gravity dual. However this is subtle because the full global solution that incorporate back reactions of the D7 branes on the KS background has not yet been computed. What we have computed is the local metric that incorporates the deformations of the seven branes when these branes are moved far away from the

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$^{11}$Plus it has infinite number of degrees of freedom.
regime of interest (see [34, 35] for more details). This is given by
\[ ds^2 = h_1 [dz + a_1 \, dx + a_2 \, dy]^2 + h_2 \, [dy^2 + d\tilde{\theta}_2^2] + h_4 \, [dx^2 + h_3 \, d\tilde{\theta}_1^2] + \]
\[ + h_5 \sin \tilde{\psi} \, [dx \, d\tilde{\theta}_2 + dy \, d\tilde{\theta}_1] + h_5 \cos \tilde{\psi} \, [d\tilde{\theta}_1 d\tilde{\theta}_2 - dx \, dy] \]
\[ B_{NS} = b_{x\tilde{\theta}_1} \, dx \wedge d\tilde{\theta}_1 + b_{y\tilde{\theta}_2} \, dy \wedge d\tilde{\theta}_2, \quad B_{RR} = -2A \, dy \wedge dz, \quad \phi = \tilde{\phi} = 0 \quad (3.1) \]

where \((x, y, z, \tilde{\theta}_1, \tilde{\psi}, \tilde{r})\) are the small local regime around the point \(\langle \phi_i \rangle, \langle \psi \rangle, \langle \theta_i \rangle, r_0\) given in the following way:
\[ \psi = \langle \psi \rangle + \frac{2z}{\sqrt{\gamma_0 \sqrt{h_0}}}, \quad \phi_2 = \langle \phi_2 \rangle + \frac{2y}{\sqrt{(\gamma_0 + 4a^2) \sqrt{h_0}} \sin \langle \theta_2 \rangle} \]
\[ \phi_1 = \langle \phi_1 \rangle + \frac{2x}{\sqrt{\gamma_0 \sqrt{h_0}} \sin \langle \theta_1 \rangle}, \quad r = r_0 + \frac{\tilde{r}}{\sqrt{\gamma_0 \sqrt{h_0}}} \]
\[ \theta_1 = \langle \theta_1 \rangle + \frac{2\tilde{\theta}_1}{\sqrt{\gamma_0 \sqrt{h_0}}}, \quad \theta_2 = \langle \theta_2 \rangle + \frac{2\tilde{\theta}_2}{\sqrt{(\gamma_0 + 4a^2) \sqrt{h_0}}} \quad (3.2) \]

where \(\gamma_0(r_0)\) and \(h_0(r_0)\) are some constant functions of \(r_0\) (see [34, 35] for details); and the un-tilded coordinates are used to write the standard metric of the conifold in the following way:
\[ ds^2 = dr^2 + r^2 \left( \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i \, d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2)^2 \right) \quad (3.3) \]

There are a few key issues that we want to point out regarding the metric (3.1):

- First, since the seven branes are kept far away the axion-dilaton vanish for the background locally. Globally there will be non-zero axion-dilaton.
- Secondly, observe that the two spheres (parametrised originally by \((\tilde{\phi}_i, \tilde{\theta}_i)\)) are replaced by two-tori locally. This issue has already been explained in details in [34]. Furthermore the two tori do not appear with the same coefficients. In fact there is a squashing factor associated with the two tori. We believe that globally such squashing factor should also show up for the two spheres.
- Thirdly, the local back reactions on the metric due to fluxes and seven branes would modify the warp factors once we go to the full global scenario.
- Finally, the above local (and the subsequent global) picture is dual only to zero temperature gauge theory. What we need for our purpose is high temperature gauge theory. This would mean that the global extension of the above background (3.1) should contain a black hole whose horizon size should correspond to the temperature in the dual gauge theory at far UV. Since the number of effective degrees of freedom are changing as we go from UV to IR, the entropy and temperature will depend crucially on what UV degrees of freedom we are considering at a given cutoff. This is a subtle issue and we will discuss this carefully a little later.
Thus putting a black hole in the global metric is non-trivial because of all the above considerations. However we can formally write the metric in the following way (although we will use Minkowski coordinates throughout, unless mentioned otherwise, one may write everything in Euclidean coordinates also):

$$ds^2 = \frac{1}{\sqrt{h}} \left(-g_1 dt^2 + dx^2 + dy^2 + dz^2\right) + \sqrt{h} \left[g_2^{-1} dr^2 + r^2 dM_5^2\right]$$  \hspace{1cm} (3.4)

where \((x, y, z)\) not to be confused with \((\mathbf{x}, \mathbf{y}, \mathbf{z})\) discussed above, \(g_i\) are functions\(^\text{12}\) that determine the presence of the black hole, \(h\) is the 10d warp factor that could be a function of all the internal coordinates and \(dM_5^2\) is given by:

$$dM_5^2 = h_1(d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2)^2 + h_2(d\theta_1^2 + \sin^2 \theta_1 \, d\phi_1^2) +$$

$$+ h_3(d\theta_2^2 + \sin^2 \theta_2 \, d\phi_2^2) + h_5 \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2) +$$

$$+ h_5 \sin \psi (\sin \theta_1 \, d\theta_2 d\phi_1 - \sin \theta_2 \, d\theta_1 d\phi_2)$$  \hspace{1cm} (3.5)

with \(h_i\) being the six-dimensional warp factors. One advantage of writing the background in the above form is that it includes all possible deformations in the presence of seven branes and fluxes. The difficulty however is that the equations for the warp factors \(h_i\) are coupled higher order differential equations that do not have simple analytical solutions. The original KS solution is in the limit

$$h_3 = g_i = 1, \quad h_i = \text{fixed}$$  \hspace{1cm} (3.6)

but has no seven branes. In the presence of seven branes we can do slightly better in the limit:

$$h_5 = 0, \quad h_3 = 1, \quad h_4 - h_2 = a, \quad g_i = 1$$  \hspace{1cm} (3.7)

which puts a seven brane in a resolved conifold background with \(a = \text{constant}\) using the so-called Ouyang embedding [20]. A black hole could be inserted in this background by switching on non-trivial \(g_i\). However a naive choice of fluxes in (3.7) is known to break supersymmetry [22].

Our next choice would then be to go to the limit where the resolution parameter \(a\) in (3.7) is vanishing. This is the Ouyang background [20] with \(g_i(r) = 1\) i.e with seven branes but no black holes. Even in the absence of black holes, supersymmetry is an issue here as was pointed out in [22]. Supersymmetry is spontaneously broken but could be restored by switching on appropriate gauge fluxes on the seven branes [22, 23]. The seven branes are embedded via the following equation (see also [20, 22]):

$$r^2 \exp \left[\frac{i(\psi - \phi_1 - \phi_2)}{2}\right] \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu$$  \hspace{1cm} (3.8)

\(^{12}\)They would in general be functions of \((r, \theta)\). We will discuss this later.
where $\mu$ is a complex quantity. In the limit where $\mu \to 0$, the seven branes are oriented along the two branches:

$$\begin{align*}
\text{Branch 1 :} & \quad \theta_1 = 0, \quad \phi_1 = 0 \\
\text{Branch 2 :} & \quad \theta_2 = 0, \quad \phi_2 = 0
\end{align*}$$

From above it is easy to see that the seven branes in Branch 1 wrap a four cycle $(\theta_2, \phi_2)$ and $(\psi, r)$ in the internal space and is stretched along the spacetime directions $(t, x, y, z)$. Similarly in Branch 2 the seven branes would wrap a four-cycle $(\theta_1, \phi_1, r, \psi)$.

From the above discussion, one might get a little concerned by the fact that the seven branes wrap a non-compact four cycle in the internal space. This would suggest a violation of the Gauss’ law as the axion charges of the seven branes have no place to escape. For the time being this apparent paradox can be resolved by allowing the seven brane to wrap a topologically trivial cycle so that it would end abruptly at some $r = r_0$ when the embedding is (3.8). This is similar to the seven brane configuration of [41]. We will discuss a better embedding later.

Once the above issues are resolved, we can insert a black hole in the modified Ouyang background (3.7) by switching on appropriate $g_i(r)$. Clearly we do not expect $h_i$ to remain constant anymore. We also expect $M$ and $N_f$ to be given by some $M_{\text{eff}}$ and $N_{f\text{eff}}$ respectively. Our first approximation would then be to take the following ansatze for the $h_i, M_{\text{eff}}$ and $N_{f\text{eff}}$:

$$\begin{align*}
h_1 &= \frac{1}{9} + O(g_s), \quad h_2 = h_4 = \frac{1}{6} + O(g_s), \quad h_3 = 1 + O(g_s) \\
M_{\text{eff}} &= M + \sum_{m \geq n} a_{mn}(g_sN_f)^m(g_sM)^n, \quad N_{f\text{eff}} = N_f + \sum_{m \geq n} b_{mn}(g_sN_f)^m(g_sM)^n
\end{align*}$$

with $a_{mn}, b_{mn}$ could in principle be functions of the CY coordinates. Note that we have made $m \geq n$ in the above expansions because the precise limits for which our series would be valid are:

$$\left(g_s, g_sN_f, g_s^2MN_f, \frac{g_sM^2}{N}\right) \to 0, \quad (g_sN, g_sM) \to \infty$$

These limits of the variables (which we will concentrate on from now on), bring us closer to the Ouyang solution with very little squashing of the two spheres. This also means that the warp factor $h$ in (3.4) can be written as:

$$h = \frac{L^4}{\sqrt{r^4}} \left[1 + \frac{3g_sM_{\text{eff}}^2}{2\pi N} \log r \left\{1 + \frac{3g_sN_{f\text{eff}}}{2\pi} \left(\log r + \frac{1}{2}\right) + \frac{g_sN_{f\text{eff}}}{4\pi} \log \left(\frac{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}}{2}\right)\right\}\right]$$

with $dM_5$ in (3.5) can be approximated by the angular part of the conifold metric (3.3). We will however modify this further soon to get the result for large $r$. 

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Question now is what choices of \( g \) are we allowed to take to insert the black hole in this geometry? It turns out that to solve EOMs the \( g \)'s have to be functions of \((r, \theta_1, \theta_2)\). Our ansatze therefore would be:

\[
g_1(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}(g_s^2 M N_f), \quad g_2(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}(g_s^2 M N_f) (3.13)
\]

where \( r_h \) is the horizon, and the \((\theta_1, \theta_2)\) dependences come from the \( \mathcal{O}(g_s^2 M N_f) \) corrections. We also expect \( a = a(r_h) + \mathcal{O}(g_s^2 M N_f) \) as the full resolution parameter. Therefore in order to extract temperature from the geometry, we look at the metric in (3.4) in the near horizon limit \( r \to r_h \) and take a five dimensional slice obtained by setting \( \theta_i = \pi, \phi_i = 0, \psi = 0 \). To be exact, we really need to start from the ten dimensional supergravity action and then integrate out the internal directions to obtain a five dimensional effective action. Minimization of that five dimensional effective action will give the five dimensional effective metric. Here we approximate this effective metric by taking the slice

\[
\theta_1 = \theta_2 = \pi, \quad \phi_i = 0, \quad \psi = 0 (3.14)
\]

Such a choice can be justified by observing that the flavor D7 branes are all along this slice (see details below)\(^\text{13}\).

Now, looking at the \( r, t \) direction of the metric and by change of variable, under the assumption that \( g_1(r, \pi, \pi) \approx g_2(r, \pi, \pi) \equiv g(r) \), we can define \( \rho^2 \) as:

\[
\rho^2 = \frac{4 \sqrt{h(r_h)} g(r)}{[g'(r_h)]^2} (3.15)
\]

so that the near horizon limit of five dimensional effective metric takes the following Rindler form:

\[
ds^2 = -\rho^2 \frac{g'(r_h)^2}{4 h(r_h) g(r_c)} dt_c^2 + d\rho^2 (3.16)
\]

where prime denotes differentiation with respect to \( r \) and we only wrote the \( r, t \) part of the metric in terms of new variable \( \rho \) and \( t_c \equiv \sqrt{g(r_c)} t \). The reason behind rescaling time at fixed \( r_c \) is that with this time coordinate \( t_c \), the five dimensional metric induces a four dimensional Minkowski metric at every \( r_c \).

\(^{13}\)Note also that all the analysis presented in the appendices are done without taking such a slice. What we find that taking a slice doesn’t give us results very different from the full analysis. It only makes our calculations a little easier to handle. As an example we can quote (3.173) or (3.205) where the entropy is computed without taking the slice. We can see that the result differs from the one with slice by an \( \mathcal{O}(g_s N_f) \) term, so is very small. Similar statements can be made for the case with dilaton also. The dilaton profile doesn’t vary too much throughout our regime of interest (i.e small \( r \)). So taking a constant dilaton is meaningful here. The full analysis is underway where we plan to take the effects of dilaton and curvatures carefully.
Now the temperature observed by the field theory with time coordinate \( t_c \) can be extracted by writing the metric in (3.16) in the following form
\[
ds^2 = -4\pi^2 T_c^2 \rho^2 \, dt_c^2 + d\rho^2
\]
(3.17)
Thus comparing (3.16) and (3.17), we obtain the temperature \( T_c \) as:
\[
T_c = \frac{g'(r_h)}{4\pi \sqrt{h(r_h) g(r_c)}}
\]
(3.18)
In the limit where we have \( g_1 = g_2 = g = 1 - \frac{r^4}{r_c^4} \), we can easily compute the corresponding temperature using the above formula (3.18). This is given by:
\[
T_c = \frac{r_h}{\pi L^2} + \frac{r_h^5}{2\pi L^2 r_c^4} + \sum_{m,n,p} c_{mnp} \frac{r_h^m \log^n r_h}{r_c^p} \equiv T_b + \mathcal{O}(1/r_c)
\]
(3.19)
where \( L \) is defined earlier, \( c_{mnp} \) is in general functions of \( (g_s, M, N, N_f) \), and \( T_b > T_{\text{deconf}} \) (where \( T_{\text{deconf}} \) is the deconfinement temperature) is the temperature at \( r_c \to \infty \) i.e
\[
T_b \equiv T_{\text{boundary}} = \frac{g'(r_h)}{4\pi \sqrt{h(r_h)}} \quad \Rightarrow \quad r_h \equiv F(T_b) \equiv \mathcal{T}
\]
(3.20)
where \( F(T_b) \) is the inverse transform of the above expression once we know the black hole factor \( g(r_h) \) as well as the warp factor \( h(r_h) \) to all orders in \( g_s N_f, g_s M \). The corresponding temperature function \( \mathcal{T} \) will then be the characteristic temperature of the boundary theory which is fixed once and for all. This temperature function is a scale unique to our cascading theory and will be assumed to be greater than the deconfinement temperature henceforth.

For the attentive readers, something about equations (3.18) and (3.19) may strike as odd. There seems to be an \( \mathcal{O}(1/r_c) \) dependences. What does it mean for a physical variable in our theory to have an \( \mathcal{O}(1/r_c) \) i.e the cutoff dependence? Shouldn’t we make \( r_c \to \infty \) to get a result that is independent of the cutoff? In other words does it make sense to have an explicit cutoff dependence in the physical variables?

Of course in the standard Wilsonian renormalisation we do get results that depend on the explicit UV cutoff. In the limit where we make \( r_c \to \infty \) we reproduce exactly the right temperature for the boundary theory. The fact that the temperature does not have the cutoff dependence going as positive powers of \( r_c \) should be pleasing: this has to do with the holographic renormalisability of the theory (that we elaborate in details later). However the fact that there is some cutoff dependence should signal something quite different from what we have in the AdS/QCD case.

To see what is different here, let us pause for a while and ask what would the presence of a black hole signify in the dual gauge theory. In the standard AdS/QCD
case we only needed to look at the boundary of the AdS Black-Hole (AdS-BH) geometry to study the properties of the dual thermal QCD. Here the situation differs crucially because we can study the weakly coupled gravity description not only at $r \to \infty$ but also at any arbitrary $r = r_c$. What does it mean to study the theory at $r = r_c$ and not $r \to \infty$? Of course, as we commented before, we can always make $r_c \to \infty$ to get the far UV results, but cutting off the theory at $r = r_c$ means we are putting a Wilsonian cutoff in the theory. Such a procedure would make sense if and only if we can carefully describe the degrees of freedom at $r = r_c$. Furthermore—and this is one of the most crucial points—since the theory with flavor is holographically renormalisable (as we will show soon) all the cutoff dependences will come as $O(1/r_c)$. Such a state of affairs lead to two interesting conclusions, one of which is obvious and the other not so obvious. The obvious point is that as long as we put our cutoff at high energy, the low energy dynamics remain completely unaffected by our choice of the cutoff. The not-so-obvious point is that the shear-viscosity etc that we will discuss in detail soon for geometries that are cut-off at $r = r_c$ will in fact be independent of the cut-off once we study them from boundary point of view! This conclusion is surprising because we will not be making $r_c \to \infty$ to study the boundary theory, rather we will add non-trivial UV “cap” to the geometry from $r = r_c$ to $r = \infty$.

So the point that we want to emphasise here is that once we introduce a cutoff at $r = r_c$ we are in principle introducing non-trivial high energy degrees of freedom that will in general take us away from the usual cascading dynamics of the parent theory! Adding such non-trivial degrees of freedom at $r = r_c$ is equivalent to adding a UV cap to the geometry. Thus it all depends on what we do at $r_c$ i.e which boundary conditions we choose there. Therefore they are not universal properties of the cascading theory. Universal properties arise only as $r_c \to \infty$ i.e at the boundary. For example we may think of attaching $\mathcal{N} = 2, 4$ degrees of freedom at $r = r_c$ (somewhat along the lines of [37]) alongwith the remnants of the D7 brane degrees of freedom to have the full F-theory picture (in fact our UV completions

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14 We thank Ofer Aharony for emphasising this point, and clarifying other details about the cutoff dependences.

15 Alternatively this means that we are not adding any UV cap because the geometry can be thought of as being “cut-off” at $r_c = \infty$. One issue here is the connection to the work of [38]. The question is can we write a boundary theory at $r = r_c$ itself instead of going to the actual $r = \infty$ boundary? As shown in [38] this is possible in AdS case because the theory on the surface of a ball in AdS space doesn’t have to be a local quantum field theory. The non-local behavior in such a “boundary” theory is completely captured by the Wilsonian effective action at the so called boundary. As discussed in the text earlier, this is tricky in the Klebanov-Strassler model precisely because there is no unique strongly coupled gauge theory here. Again, as we saw in our RG flow pictures, when the dual gravity theory is weakly coupled we have a smooth RG flow in the gauge theory side, but the theory at any given scale can be given by infinite number of representative theories none of which completely capture the full dynamics. Thus it makes more sense to define the theories at $r = \infty$ boundary only and not at any generic $r = r_c$. 

should always have $24 - N_f$ seven branes attached to the UV caps\textsuperscript{16}. Clearly adding such degrees of freedom will give us a new theory that differs from the UV degrees of freedom of the parent cascading theory. In particular if $r_c$ is finite then this procedure can give us an almost free UV theory with confining IR dynamics. Also since at any given scale there are an infinite number of possible gauge theory descriptions available with different UV degrees of freedom, we can specify a particular set of degrees of freedom at $r = r_c$ to define the UV of this theory. Once this is specified the RG flow will take us to the IR. A sketch of the situation is depicted in figure 7 below where the gravity description captures the smooth RG flow. We can view the dotted

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{An oversimplified picture of the gravity description of the various gauge theories involved. The onion-ring model is only a caricature of a much more involved picture that we gave earlier. The dotted lines emanating from various points (i.e various gauge theories) specify the distinct UV completions of these gauge theories. Once we specify the UV degrees of freedom at various $r = r_c$ the RG flows of each theories should be assumed to bring us to $r = r_c$ points on the smooth RG flow of the original cascading theory. Note that at any point on the gauge theory side (i.e the vertical axis) there are infinite number of possible gauge-theory descriptions available (the horizontal axis).}
\end{figure}

lines coming out from any $r = r_c$ and going to infinity (in the gauge theory side) as describing the UV degrees of freedom for a theory at $r = r_c$. Thus in principle there is a possibility of defining infinite number of theories here by cutting off the geometry at various $r = r_c$ and then “filling” the $r = r_c$ to $r = \infty$ spaces by UV caps that specify the UV degrees of freedom of these gauge theories (of course we

\textsuperscript{16}The fact that there are infinite F-theory backgrounds gluing to our IR solutions will be discussed a little later.
expect small changes in the IR geometries every time we attach UV caps, although the far IR geometries wouldn’t change). Under RG flows these UV theories (that are obviously quite different from the parent cascading theory) meet the RG flow of the cascading theory at various $r = r_c$ (where of course the degrees of freedom match but not the whole set of marginal, relevant and irrelevant operators). The effective degrees of freedom at $r = r_c$ are:

$$N_{\text{eff}} = N + \frac{3g_s M^2}{2\pi} \log r_c + \frac{9g_s^2 M^2 N_f}{4\pi^2} \log^2 r_c \quad (3.21)$$

Therefore $r_c$ dependences in the physical result for our case should be replaced by the effective number of degrees of freedom at that scale, i.e by:

$$r_c = \exp \left[ \frac{\pi}{3g_s} \sqrt{\frac{1}{N_f^2}} - \frac{4(N - N_{\text{eff}})}{M^2 N_f} - \frac{\pi}{3g_s N_f} \right]$$

$$\approx \exp [\alpha + \beta N_{\text{eff}}] \quad (3.22)$$

which in turn means that the right UV degrees of freedom should flow to this value at $r = r_c^{17}$. We have also defined $\alpha = \frac{\pi}{3g_s M} \left( \frac{\sqrt{M^2 - 4NN_f^2}}{N_f} - 1 \right)$ and $\beta = \frac{2\pi}{3g_s M \sqrt{M^2 - 4NN_f}}$ in the limit where $N_{\text{eff}}$ is small compared to the original colors and bi-fundamental flavors in the theory i.e $N > M > N_{\text{eff}} > N_f$ precisely. In general then$^{18}$

$$r_c \equiv e^{N_{\text{eff}}} \quad (3.23)$$

with $N_{\text{eff}}(\Lambda_c)$ being the effective degrees of freedom at a given scale $\Lambda_c$. Thus to summarise:

- Making the cutoff to infinity i.e $r_c \to \infty$ gives us the property of the original cascading theory with a particular UV behavior and a smooth RG flow to IR.
- Defining the theory with a cutoff at $r = r_c$ means that we specify the UV degrees of freedom of a particular gauge theory at that scale or equivalently introduce a UV cap in the geometry so that we are always defining the theory at the boundary$^{19}$.

This theory also has a smooth RG flow but the UV behavior (shown by dotted lines)

$^{17}$Even more interestingly, the final results for any physical quantities from these theories will only depend on the boundary degrees of freedom and not on $N_{\text{eff}}$. The $N_{\text{eff}}$ degrees of freedom are only intermediate. However there are special cases where the boundary degrees of freedom could be close to $N_{\text{eff}}$. We will discuss them later.

$^{18}$There is another flavor-dependent degrees of freedom that one could define in this background. We will define this via $r_{c(\alpha)} = e^{N_{\text{eff}}[1 - \epsilon..]}$, where the variables are defined in (3.35) and (3.36).

$^{19}$It should be clear that unless we specify clearly the UV degrees of freedom, we cannot pinpoint the gauge theory there. The parent cascading theory allows an infinite number of gauge theory description at any given scale. At the point where we have smooth RG flow, none of these descriptions capture the full picture there. Thus to emphasise again, the gauge theory or the UV behavior that we want to specify at that scale is distinct from the UV of the cascading theory as it could have finite (but large) number of degrees of freedom.
is quite different from the parent cascading theory. Only at certain IR scale (i.e. \( r = r_c \) from the gravity side) do these theories match (wrt the degrees of freedom and certain set of marginal and relevant operators).

Once this is settled, let us now introduce fundamental matters in our geometry. As we discussed above, the seven branes introduce fundamental matter with \( N_f \) flavors. Since the above background is a deformation of both KS and Ouyang backgrounds, we will refer the resulting geometry as Ouyang-Klebanov-Strassler (OKS) background. With the black hole, this will henceforth be called as OKS-BH background.

In the OKS-BH background, to order \( \mathcal{O}(g_s N_f) \) and small \( r \), let us start by considering the warp factor choice (3.7) but keeping \( g_i(r) \) as (3.13) instead of 1. The background RR three and five-form fluxes can be succinctly written as

\[
H_3 = dr \wedge e_\psi \wedge (c_1 d\theta_1 + c_2 d\theta_2) + dr \wedge (c_3 \sin \theta_1 d\theta_1 \wedge d\phi_1 - c_4 \sin \theta_2 d\theta_2 \wedge d\phi_2) + \left( \frac{r^2 + 6a^2}{2r} c_1 \sin \theta_2 d\phi_2 - \frac{r}{2} c_2 \sin \theta_1 d\phi_1 \right) \wedge d\theta_1 \wedge d\theta_2,
\]

\[
\tilde{F}_3 = -\frac{1}{g_s} dr \wedge e_\psi \wedge (c_1 \sin \theta_1 d\phi_1 + c_2 \sin \theta_2 d\phi_2) + \frac{1}{g_s} e_\psi \wedge (c_5 \sin \theta_1 d\theta_1 \wedge d\phi_1 - c_6 \sin \theta_2 d\theta_2 \wedge d\phi_2) - \frac{1}{g_s} \sin \theta_1 \sin \theta_2 \left( \frac{r}{2} c_2 d\theta_1 - \frac{r^2 + 6a^2}{2r} c_1 d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2. \tag{3.24}
\]

where \( H_3 \) is closed and \( \tilde{F}_3 \equiv F_3 - C_0 H_3 \), \( C_0 \) being the ten dimensional axion. The derivations of the coefficients appearing in (3.24) are rather involved, and their dependences on the resolution factor etc. will be described in section 3.3\textsuperscript{21}. For the present purpose, let us just quote the results:

\[
c_1 = \frac{g_s^2 M N_f}{4\pi r(r^2 + 6a^2)^2} \left( 72a^4 - 3r^4 + 56a^2 r^2 \log r + a^2 r^2 \log(r^2 + 9a^2) \right) \cot \frac{\theta_1}{2}.
\]

\textsuperscript{20}As we mentioned earlier, at large \( r \) we expect all \( c_i \) to be finite. This means that \( c_i \) should become functions of inverse powers of \( r \) at large \( r \). This is expected from F-theory considerations and would help us remove the Landau pole.

\textsuperscript{21}It is instructive to note that the background EOMs cannot be trivially worked out by solving SUGRA EOMs with fluxes and seven branes sources. This is because, even if we know the energy momentum tensors for the fluxes, the energy momentum tensors for \( N_f \) coincident seven branes are not known in the literature! The difficulty lies in finding a non-abelian Born-Infeld action for \( N_f \) seven branes on a curved background. As far as we are aware of, this problem has remained unsolved till now. So in the absence of such direct approach, we use an alternative method to derive the EOMs. This method uses the ISD (imaginary self-duality) properties of the background fluxes and fields. Details on this have already appeared in [20,22], so we would refer the readers there for a complete analysis. Our present analysis is however more involved than [20,22] because we have a black hole and no supersymmetry. We do however find that even for this scenario, one could find consistent solutions to EOMs using similar arguments. For example see equations (3.71) to equation (3.84) for more detailed derivations.
and discuss later. Finally, this allows us to write the NS 2-form potential:

\[ c_2 = \frac{3g_s^2 MN_f}{4\pi r^3} \left( r^2 - 9a^2 \log(r^2 + 9a^2) \right) \cot \frac{\theta_2}{2} \]

\[ c_3 = \frac{3g_s M r}{r^2 + 9a^2} + \frac{g_s^2 MN_f}{8\pi r (r^2 + 9a^2)} \left[ -36a^2 - 36r^2 \log a + 34r^2 \log r \\
+ (10r^2 + 81a^2) \log(r^2 + 9a^2) + 12r^2 \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right] \]

\[ c_4 = \frac{3g_s M r (r^2 + 6a^2)}{\kappa r^3} + \frac{g_s^2 MN_f}{8\pi \kappa r^3} \left[ 18a^2 - 36(r^2 + 6a^2) \log a + (34r^2 + 36a^2) \log r \\
+ (10r^2 + 63a^2) \log(r^2 + 9a^2) + (12r^2 + 72a^2) \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right] \]

\[ c_5 = g_s M + \frac{g_s^2 MN_f}{24\pi (r^2 + 6a^2)} \left[ 18a^2 - 36(r^2 + 6a^2) \log a + 8(2r^2 - 9a^2) \log r \\
+ (10r^2 + 63a^2) \log(r^2 + 9a^2) \right] \]

\[ c_6 = g_s M + \frac{g_s^2 MN_f}{24\pi r^2} \left[ -36a^2 - 36r^2 \log a + 16r^2 \log r + (10r^2 + 81a^2) \log(r^2 + 9a^2) \right] \]

with \( \kappa = \frac{r^2 + 9a^2}{r^2 + 6a^2} \). All the above coefficients have further corrections that we will discuss later. Finally, this allows us to write the NS 2-form potential:

\[ B_2 = \left( b_1(r) \cot \frac{\theta_1}{2} \frac{d\theta_1}{d\theta_1} + b_2(r) \cot \frac{\theta_2}{2} \frac{d\theta_2}{d\theta_2} \right) \wedge e_\psi \]

\[ + \left[ \frac{3g_s^2 MN_f}{4\pi} \left( 1 + \log(r^2 + 9a^2) \right) \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_3(r) \right] \sin \theta_1 \frac{d\theta_1}{d\phi_1} \\
- \left[ \frac{g_s^2 MN_f}{12\pi r^2} \left( -36a^2 + 9r^2 + 16r^2 \log r + r^2 \log(r^2 + 9a^2) \right) \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_4(r) \right] \]

\[ \times \sin \theta_2 \frac{d\theta_2}{d\phi_2} \]

with the \( r \)-dependent functions

\[ b_1(r) = \frac{g_s^2 MN_f}{24\pi (r^2 + 6a^2)} \left( 18a^2 + (16r^2 - 72a^2) \log r + (r^2 + 9a^2) \log(r^2 + 9a^2) \right) \]

\[ b_2(r) = -\frac{3g_s^2 MN_f}{8\pi r^2} (r^2 + 9a^2) \log(r^2 + 9a^2) \]

(3.27)

and \( b_3(r) \) and \( b_4(r) \) are given by the first order differential equations

\[ b_3(r) = \frac{3g_s M r}{r^2 + 9a^2} + \frac{g_s^2 MN_f}{8\pi r (r^2 + 9a^2)} \left[ -36a^2 - 36a^2 \log a + 34r^2 \log r \\
+ (10r^2 + 81a^2) \log(r^2 + 9a^2) \right] \]

\[ b_4(r) = -\frac{3g_s M (r^2 + 6a^2)}{\kappa r^3} - \frac{g_s^2 MN_f}{8\pi \kappa r^3} \left[ 18a^2 - 36(r^2 + 6a^2) \log a \\
+ (34r^2 + 36a^2) \log r + (10r^2 + 63a^2) \log(r^2 + 9a^2) \right] \]

(3.29)
where \( M \) denote the remnant of the number of fractional three branes (in the gauge theory side) and \( N_f \) denote the number of flavors or seven branes in the dual gravity side. Therefore once we know \( B_2 \) and the string coupling \( e^{-\Phi} \) then it is easy to determine the two couplings at the UV of our dual gauge theory (see also \[42\]):

\[
\begin{align*}
\frac{8\pi^2}{g_1^2} &= e^{-\Phi} \left[ 1 + \frac{1}{2\pi} \left( \int_{S^2} B_2 \right) \right] \\
\frac{8\pi^2}{g_2^2} &= e^{-\Phi} \left[ \frac{1}{2} + \frac{1}{2\pi} \left( \int_{S^2} B_2 \right) \right]
\end{align*}
\]

(3.30)

The string coupling can be determined easily from the monodromy around the seven brane to be:

\[
e^{-\Phi} = \frac{1}{g_s} + \frac{N_f}{8\pi} \log \left( r^6 + 9a^2r^4 \right) - \frac{N_f}{2\pi} \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)
\]

(3.31)

which immediately gives us:

\[
\frac{\partial}{\partial \log \Lambda} \left[ \frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} \right] = -\frac{N_f}{8} \left( \frac{6r^6 + 36a^2r^4}{r^6 + 9a^2r^4} \right)
\]

\[
\frac{\partial}{\partial \log \Lambda} \left[ \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} \right] = 3M \left( 1 + \frac{3g_sN_f}{4\pi} \log (r^2 + 9a^2) + \ldots \right)
\]

(3.32)

which to the leading order is consistent with the Shifman-Vainshtein \( \beta \)-function \[43\]22. The behavior at subleading order will tell us how the color changes as we cascade down from UV to IR. For example, in the presence of \( N_f \) flavors the \( SU(N + M) \) gauge group has \( 2N + N_f \) effective flavors. Under RG flow, Seiberg duality will tell us that the weakly coupled gauge group will become \( SU(N - M + N_f) \times SU(N) \). This also means that the cascade will slow down quite a bit as we approach the IR and therefore the end point of the cascade could either be a conformal theory or a confining theory. As pointed out also in \[20\] if in the end of the cascade \( N \) decreases to zero with finite \( M \) left over, we would have \( SU(M) \) SYM with \( N_f \) flavors in the IR. Extending this to the centre of the RG fixed points surface will allow us to analyse this using weakly coupled supergravity. This is of course the theory we are aiming for. Finally, the five–form flux is as usual given by

\[
\hat{F}_5 = (1 + *_{10}) (dh^{-1} \wedge d^4x).
\]

(3.33)

with \( h \) being the warp factor described above.

Before we end this section we want to point out few subtleties about the background. First as we mentioned before, to maintain Gauss law constraint, we need to

---

22Note that the LHS of the equations (3.32) involve only gauge theory variables whereas the RHS of the equations involve gravity variables. In particular \( \Lambda \) should be identified with the radial coordinate \( r \) of the geometry. Thus the equations (3.32) capture the essence of gauge/gravity duality here.
embed our model in the full F-theory setup. This means that the background configuration that we presented above should be understood as an F-theory on a four-fold where all but \(N_f\) of the seven branes have been moved to infinity. This means that \(N_f < 24\), and the four-fold is a non-trivial torus fibration over a resolved conifold base. Such a four-fold has already been constructed in [34, 22] and so we can direct the readers to those papers for details. What is interesting here is that due to F-theory embedding, and with modified three-form fluxes, we expect the background to not have any naked singularities or Landau poles that are often associated with these backgrounds (see for example [39], [40] for some details). To \(O(g_s N_f)\) our result that we presented may indicate the presence of log \(r\) type singularities. Additionally at:

\[
 r \approx \exp \left( \frac{4\pi}{3g_s N_f} \right), \quad \text{for} \quad a << g_s N_f
\]

we might think that the dilaton is blowing up on the given slice (3.14) leading to some kind of naked singularity. However note that we are seeing such behavior because we have evaluated the background locally near the seven branes, and up to \(O(g_s N_f)\).

Clearly the metric, dilaton and the fluxes have to have a good behavior at infinity to be a F-theory solution. One way to show this would be to rewrite the warp factor (3.12) completely in terms of power series in \(r\) in the following way:

\[
 h = \frac{L_4^4}{r^{4-\epsilon_1}} + \frac{L_4^4}{r^{4-2\epsilon_2}} - \frac{2L_4^4}{r^{4-\epsilon_2}} + \frac{L_4^4}{r^{4-\epsilon_2^2/2}} \equiv \sum_{\alpha=1}^{4} \frac{L_4^4}{r^4(\alpha)}
\]

where \(\epsilon_i, r(\alpha)\) etc are defined as:

\[
 \epsilon_1 = \frac{3g_s M^2}{2\pi N} + \frac{g_s^2 M^2 N_f}{8\pi^2 N} + \frac{3g_s^2 M^2 N_f}{8\pi N} \log \left( \frac{\theta_1^2}{2} \frac{\theta_2^2}{2} \right), \quad \epsilon_2 = \frac{g_s^2 M}{\pi} \sqrt{\frac{2N_f}{N}}
\]

\[
 r(\alpha) = r^{1-\epsilon(\alpha)}, \quad \epsilon(1) = \frac{\epsilon_1}{4}, \quad \epsilon(2) = \frac{\epsilon_2}{2}, \quad \epsilon(3) = \frac{\epsilon_2}{4}, \quad \epsilon(4) = \frac{\epsilon_2}{8}
\]

\[
 r(\pm \alpha) = r^{1+\epsilon(\alpha)}, \quad L(1) = L(2) = L(4) = L^4, \quad L(3) = -2L^4
\]

which makes sense because we can make \(\epsilon_i\) to be very small. The angles \(\theta_i\) take fixed values on the given slice (3.14). Note that the choice of \(\epsilon_i\) doesn’t require us to have \(g_s N_f\) small (although we consider it here). In fact we can have all \((N, M, N_f)\) large but \(\epsilon_i\) small. A simple way to achieve this would be to have the following scaling behaviors of \((g_s, N, M, N_f)\):

\[
 g_s \to \epsilon^\alpha, \quad M \to \epsilon^{-\beta}, \quad N_f \to \epsilon^{-\kappa}, \quad N \to \epsilon^{-\gamma}
\]

where \(\epsilon \to 0\) is the tunable parameter. Therefore all we require to achieve that is to allow:

\[
 \alpha + \gamma > 2\beta + \kappa, \quad \alpha > \kappa, \quad \gamma > \alpha
\]
where the last inequality can keep \( g_s N_f \) small. Thus \( g_s N, g_s M \) are very large, but \( g_s, \frac{g_s M^2}{N} \), \( g_s N_f \) are all very small to justify our expansions (and the choice of supergravity background)\(^\text{23}\).

The warp factor (3.35) has a good behavior at infinity and reproduces the \( \mathcal{O}(g_s N_f) \) result locally. Our conjecture then would be the complete form of the warp factor at large \( r \) will be given by sum over \( \alpha \) as in (3.36) but now \( \alpha \) can take values \( 1 \leq \alpha \leq \infty \). We will use this conjecture to justify the holographic renormalisability of our boundary theory. We will also discuss the case when \( \epsilon(\alpha) \) becomes an integer at large \( r \).

Similarly one would also expect the dilaton to behave in an identical way. Asymptotically the dilaton should go to a constant. So on the given slice (3.14), and near one of the seven brane, we expect:

\[
e^{-\Phi} = \frac{1}{2g_s} \left[ \frac{1}{r^{\epsilon_a}} - \frac{3\epsilon_a a^2}{2r^2} + \text{constant} \right], \quad \epsilon_a = \frac{3g_s N_f}{4\pi} (3.39)\]

where the constant could be determined from the full F-theory picture. Somewhat similar discussion has been given in \( \text{[40]} \). Our conjecture for the large \( r \) behavior stems from the finiteness of F-theory. In fact this also gives us an argument to realise the possibility of infinite F-theory backgrounds gluing to our IR solutions. In the presence of \( 24 - N_f \) seven branes there are infinite possible ways by which we can adjust the positions of the seven branes. For every possible configurations of seven branes there would be non-trivial background axio-dilaton \( \tau \), realised from the degree eight and degree twelve polynomials \( f \) and \( g \) respectively that appear in the Weierstrass equation (see \( \text{[30]} \)), via:

\[
j(\tau) = \frac{55926f^3}{4f^3 + 27g^2} \propto \prod_{i=1}^{8} (z - a_i)^3 \prod_{j=1}^{24} (z - b_j) (3.40)\]

where \( z \) is a complex coordinate orthogonal to the seven branes, and \( a_i \neq b_j \) in general. Since \( a_i, b_j \) take continuous values, there are infinite possible configurations of \( \tau \) here (modulo \( SL(2, \mathbb{Z}) \) transformations). For certain choices of \( a_i, b_j, j(\tau) \) is a constant implying that there are possible configurations of seven branes that give rise to zero axio-dilaton (see the second and the third references of \( \text{[30]} \)). Thus for a generic configuration of seven branes we expect axio-dilaton \( \tau \) to behave as \( \tau = \sum_i C_i \frac{C_i}{\epsilon(\alpha)} \), where \( \{C_i\} \) take a particular set of values for a given configuration of seven branes. Plugging these values of axio-dilaton in sugra equations of motion

\(^\text{23}\)For example we can have \( g_s \) going to zero as \( g_s \rightarrow e^{5/2} \) and \( (N, M, N_f) \) going to infinities as \( (e^{-8}, e^{-3}, e^{-1}) \) respectively. This means \( (g_s N, g_s M) \) go to infinities as \( (e^{-11/2}, e^{-1/2}) \) respectively, and \( (g_s N_f, g_s^2 MN_f, g_s M^2/N) \) go to zero as \( (e^{3/2}, e, e^{9/2}) \) respectively. This is one limit where we can have well defined UV completed gauge theories. Note however that for the kind of background that we have been studying one cannot make \( N_f \) large because of the underlying F-theory constraints \( \text{[31]} \). Since we only require \( g_s N_f \) small, large or small \( N_f \) choices do not change any of our results.
along with a similar configuration of fluxes, we can easily argue the existence of infinite configurations of warp factors of the form (3.35), justifying the IR changes from UV caps. A particular configuration of F-theory background will glue to our IR solution to give us the required UV completion. In fact the configuration with constant coupling (like the last two references of [31]) will give rise to AdS completions of our IR backgrounds! More details will be presented elsewhere.

One caveat is that a full analysis incorporating non-perturbative effects still needs to be performed to justify the whole scenario. However we expect this to be very involved because at large \( r \) we have to consider not only the effects of all the \((p, q)\) seven branes (as discussed above) but also the back-reactions from fluxes etc. We will address this in the sequel [55]. Therefore with this assumption, all the \( \log r \) dependences of the fluxes should also be replaced by inverse powers of \( r \) at large \( r \). We will however, for the purpose of concrete calculations, only work to \( O(g_s N_f) \) locally for many representative examples unless mentioned otherwise. The singularities appearing in these examples at large \( r \) should then be considered as an artifact of the order at which we do the analysis. This will also be clear from the holographic renormalisabilities of these theories.

### 3.2 Quark mass and drag coefficient

We now embed a string in OKS-BH background with one end attached to one of the D7 branes and other end going into the black hole (see figures 8 and 9 below).

If \( X^i(\sigma, \tau) \) is a map from world sheet coordinates \( \sigma, \tau \) to 10 dimensional space time, then string action or fundamental string Born Infeld action is (see for example [31]):

\[
S_{\text{string}} = T_0 \int d\sigma d\tau \left[ \sqrt{-\det(f_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi)} + \frac{1}{2} \epsilon^{ab} B_{ab} + J(\phi) + \partial X^m \partial X^n \Theta \Gamma_m \Gamma^{abc \ldots} \Gamma_n \Theta F_{abc \ldots} + O(\Theta^4) \right] = \int d^{10} x \mathcal{L}_{\text{string}}(x)
\]

---

24More precisely, the non-trivial geometry of the UV will capture the effects of the operators defined at the cut-off; and the F-theory seven-branes will capture the effects of the ultra massive fundamental quarks that help us remove the Landau poles.

25We thank Ingo Kirsch and Diana Vaman for discussion on this issue, and for pointing out the reference [10].

26One might wonder whether the procedure of holographic renormalisability should work for the full F-theory picture. It is easy to see why this should proceed without any complication: F-theory background simply gives us the non-perturbative completion of a given IIB background. These non-perturbative objects are the seven branes and they contribute to the bulk lagrangian as some matter multiplets. As long as the effects of the matter multiplets are not too large (as discussed after (3.97) and before (3.98)) the procedure of holographic renormalisability should proceed as in the usual supergravity case. However if we incorporate these multiplets they still don’t change anything because the procedure of holographic renormalisability requires derivative interactions that come exclusively from \( \sqrt{-G} R \) term of the lagrangian. All the fluxes etc contribute to polynomial interactions, as discussed around (3.117). Of course the effective values of the metric fluctuations are affected by the background seven branes.
The fundamental flavors come from the wrapped D7 branes whereas the bi-fundamental flavors come from the wrapped D5 branes. To introduce temperature all we need to do is to Euclideanise and identify the time coordinate in the gauge theory.

The dual gravity picture for the high temperature gauge theory. The branes (except the D7 branes) are replaced by fluxes and the quark string has one end on the D7 branes and the other end going into the black hole. The conifold is replaced by a non-trivial geometry that we discussed in the previous section.

\[ \mathcal{L}_{\text{string}}(x) = T_0 \int d\sigma d\tau \left[ \sqrt{-\det(f_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi)} + \frac{1}{2} \epsilon^{ab} B_{ab} + J(\phi) + \partial X^m \partial X^n \tilde{\Theta} \Gamma_m \Gamma^{abc \ldots} \Gamma_n \Theta F_{abc \ldots} + \mathcal{O}(\Theta^4) \right] \delta^{10}(X - x) \]  

where \( J(\phi) \) is the additional coupling of the dilaton \( \phi \) to the string world-sheet, \( T_0 \) is the string tension, \( X^n \) are the ten bosonic coordinates, \( \Theta \) is a 32 component spinor, \( F_{abc \ldots} = [dC]_{abc \ldots} \) with \( C_{abc \ldots} \) being the background RR form potentials, and \( f_{\alpha\beta} \) is world sheet metric, given by the standard pull-back of the spacetime metric on the world-sheet:

\[ f = \begin{pmatrix} \dot{X} \cdot \dot{X} & \dot{X} \cdot X' & \dot{X}' \cdot X' \\ \dot{X} \cdot X' & X' \cdot X' & X' \cdot X' \\ \dot{X}' \cdot X' & X' \cdot X' & X' \cdot X' \end{pmatrix} = \begin{pmatrix} \dot{X}^2 - \eta^0 \frac{\eta^0}{\sqrt{\eta^1 \eta^2 + \sqrt{\eta^1}}} \\ \dot{X} \cdot \frac{\eta^0}{\sqrt{\eta^1 \eta^2 + \sqrt{\eta^1}}} \\ \frac{\eta^0}{\sqrt{\eta^1 \eta^2 + \sqrt{\eta^1}}} \end{pmatrix} \]

with \( \gamma = 0, 1, \eta^0 = \tau, \eta^1 = \sigma \) and \( B_{ab} \) is the pull-back of the NS two form field. In the ensuing analysis we will keep both \( B_{ab} \) as well as \( \partial_a \phi \) zero. The former will be imposed (see discussion after (3.75)). The latter case will be addressed soon. The interesting thing however is to do with the background RR forms. Note that the
RR forms always couple to the 32 component spinor. Therefore once we switch-off the fermionic parts in (3.41), the fundamental string is completely unaffected by the background RR forms. Thus in the ensuing analysis, for the mass and drag of the quark, we can safely ignore the RR fields. We have also defined:

$$\det f = -G_{ij} \dot{X}^i X'^j + (G_{ij} X'^i \dot{X}^j)(G_{kl} \dot{X}^k \dot{X}^l)$$

where $G_{ij}$ is more generic than the background metric, and could involve the back reaction of the fundamental string on the geometry. The analysis is very similar to the AdS case discussed in [44] so we will be brief in the following. However, since our background involves running couplings, the results will differ from the ones of [44].

At this point it might be interesting to point out an equivalent classical calculation to evaluate the effect of a drag force on a point charge particle moving through a media. One can consider the point charge to be the endpoint of the fundamental string on the D7 brane. The mass of the point charge is given by the length of the string $l$ as $m = T_0 l$ where $T_0$ is the tension of the string. In the presence of a constant electric flux $F_0$ one can show that the total drag force $F_i$ on the point charge in the dual gauge theory is given by:

$$F_i = -q F_0 (e^{-t/\tau} - 1)$$

where $\tau$ is the so-called collision time. This can be easily determined from the mean free path of the collision. Viewing the end points of the open strings on the D7 brane as a gas of charged particles, we can study the number of collisions at any given point in the gas of particles at a particular temperature, and from there evaluate the collision length between them. The number of collisions $Z$ happening per second per unit volume at a point $\hat{r}$ (with $\hat{r}$ not to be confused with the radial coordinate $r$ in the gravity picture) in the gas can be written as:

$$Z = \int d^3 p_1 \int d^3 p_2 \sigma |v_1 - v_2| g(\hat{r}, v_1, t) g(\hat{r}, v_2, t)$$

where $\hat{r}, v$ are vectors and $g(\hat{r}, v, t)$ is the distribution function. The quantity $\sigma$ is the collision cross-section, and is a constant for our purpose. The distribution function $g(\hat{r}, v, t)$ can be anything generic, and only becomes Maxwell-Boltzmann when the gas attains equilibrium. If there are $n$ strings per unit volume of the gas, one can show that the mean free path $l_m$ in the dual theory is given by the following expression:

$$l_m = \frac{n}{2Z} v_p$$

\[27\]This is of course the familiar statement that the RR fields do not couple in a simple way to the fundamental string.
where $v_p$ is the most probable velocity which differs from the average velocity $\langle v \rangle$ by a numerical constant.

Finally when the system attains equilibrium the distribution function, as discussed above, becomes MB. In that case there are no $\hat{r}, t$ dependences in $g(\hat{r}, \mathbf{v}, t)$, and we have

$$g(\hat{r}, \mathbf{v}, t) \equiv g(\mathbf{p}) = \frac{n}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT}$$ (3.47)

where $m$ is the mass of a single particle in the gas, and $k$ the Boltzmann constant. Using this distribution it is straightforward to show that the mean free path $l_m$ is independent of temperature and is given by:

$$l_m = \sqrt{\frac{\pi}{8}} \cdot \frac{1}{n\sigma}$$ (3.48)

from which the collision time $\tau$ can be determined to be $\tau = \frac{l_m}{\langle v \rangle}$. Plugging this in (3.44) gives us the drag force on a particle in the dual theory.

The above analysis is clean and simple, but doesn't completely give us the full answer. This is because, the above analysis ignores higher order quantum corrections on the collision process. Such corrections are in fact captured by the classical back reactions of the underlying geometry in the gravity side\textsuperscript{28}. The important fact that the D7 brane is wrapped on a curved manifold, changes much of the above analysis.

To see the effect of the background geometry on the process, let us evaluate using the technique of \cite{44}.

To begin, we need the embedding of the D7 brane in our set-up. This has already appeared above as (3.8). We can use the embedding equation to determine $r_0$, the distance upto which the D7 ends in the throat. The embedding equation (3.8) implies:

$$r = \left(\frac{|\mu|^2}{\sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}}\right)^{\frac{1}{4}}$$ (3.49)

from where, by minimising $r$ with respect to the angular coordinates $\theta_i$, we obtain $\theta_1 = \theta_2 = \pi$ and from here one can see that $r_0 = |\mu|^{2/3} \equiv 1$.\textsuperscript{29}

As we said before, a fundamental quark will be a string starting from $r_0$ on the D7 brane to the horizon $r_h$ of the black hole. For simplicity of the calculation, we

\textsuperscript{28}This is of course pure supergravity analysis valid for gauge theory at very strong coupling. There would be stringy corrections over and above this result once we go closer to the boundary of the RG surface. This will additionally introduce $O(g_s)$ corrections to the existing result. For the time being we will ignore these effects as we will always be in the regime shown in figure 6.

\textsuperscript{29}This is done to make $\frac{r}{r_0} = r$ dimensionless. Note that this is another scale in our problem, which we identify to 1. As alluded to before, this will help us to make most of our QCD variables dimensionless.
will then restrict to the case when

\[ X^0 = t, \quad X^4 = r, \quad X^1 = x(\sigma, \tau), \quad X^k = 0 \quad (k = 2, 3, 5, 6, 7) \]

\[ (X^8, X^9) = (\theta_1, \theta_2) = \pi, \quad \Theta = \Bar{\Theta} = 0 \tag{3.50} \]

and we choose parametrization \( \tau = t, \sigma = r \) also known as the static gauge. Thus we are only considering the case when the string extends in the \( r \) direction, does not interact with the RR fields, and moves in the \( x \) direction of our manifold. More general string profile, while being computationally challenging, does not introduce any new physics and hence our simplification is a reasonable one.

Before moving further, let us consider two points. First is the effect of the black hole on the shape of the D7 brane. We expect due to gravitational effects the D7 brane will sag towards the black hole and eventually the string would come very close to the horizon. In fact, as is well known, putting a point charge on the D-brane tends to create a long thin tube on the D-brane that in general extends to infinity. The end point of the string being a source of point charge should show similar effects (see [54] for a discussion of a somewhat similar scenario)\(^{30}\). In this paper we will ignore this effect altogether and discuss it in details in the sequel to this paper [55].

The second point is to see how the background varying dilaton effects the string. Near the local region around the string, we expect the following behavior of the dilaton:

\[ \phi \approx \log g_s + \frac{3g_s N_f}{4\pi} \log r + \frac{9g_s N_f}{8\pi} \frac{a^2}{r^2} \tag{3.51} \]

where we could insert \( \mu \) to make \( r \) dimensionless. In the limit where the resolution parameter \( a \) goes to zero, it is easy to see that near \( r_h \) and \( r_0 \equiv 1 \), the dilaton behaves respectively as (inserting the scale \( \mu \)):

\[ \phi \approx \log g_s + \frac{3g_s N_f}{4\pi} \log \frac{T}{|\mu|^{2/3}} \]

\[ \phi \approx \log g_s - \frac{g_s N_f}{4\pi} \log |\mu| \tag{3.52} \]

which means that near the seven brane the dilaton behavior is almost a constant although there is a \( \log T \) dependence of the dilaton near the horizon, where \( T \) is defined in (3.20). This behavior is quite different from the AdS case where there is no profile of the dilaton. As we will show later, there will be additional \( \log T \) behavior coming from the warp factor also, which for small \( g_s N_f \) will have dependence on \( \frac{g_s M^2}{N} \). Thus the \( \log T \) dilaton dependence can only contribute to order \( \mathcal{O}(g_s N_f) \) if we are close to the horizon. Away from the horizon and near the seven brane, the

\(^{30}\)Even in the supersymmetric case, putting a point charge on a D-brane tends to create a long tube that extends to infinity. One can then view an open string to lie at the end of the thin tube. This effect is somewhat similar to the one discussed in [50].
dilaton is approximately constant. Therefore to simplify our ensuing analysis, we will take the dilaton to be a constant and only consider the detail implication in the sequel. In the following analysis we want to convince the reader that even with these simplifications: $B_{ab} = J(\phi) = 0$ in (3.41), our system will have interesting new physics. For more details on the dilaton behavior see [57].

Now introducing a fundamental string in the geometry will change the IR geometry. We expect the back reaction to be small, i.e we can specify the back reaction via the perturbed metric $G_{ij} = g_{ij} + \kappa l_{ij}$ where $g_{ij}$ is the original metric and $\kappa l_{ij}$ is the back reaction. With our choice of parametization and string profile, it is easy to verify:

$$-\det f = \frac{g_1(r)}{g_2(r)} \frac{g_1(r)}{h(r, \pi, \pi)} x^2 - g_1(r)^{-1} \dot{x}^2 + \mathcal{O}(\kappa) + \mathcal{O}(\kappa^2)$$

(3.53)

where the warp factor $h(r, \theta_1, \theta_2) = h(r, \pi, \pi)$ with $(a_{mn}, b_{mn}) \approx 0$ in (3.12) henceforth (unless mentioned otherwise); and the back reaction of the string appear as $\mathcal{O}(\kappa)$ effect.

With this, the rest of the analysis is a straightforward extension of [14]. The Euler-Lagrangian equation for $X^1 = x(t, r)$ derived from the action (3.41) and the associated canonical momenta are:

$$\frac{1}{g_2(t)} \left( \frac{\dot{x}}{\sqrt{-\det f}} \right) + \frac{d}{dr} \left( \frac{g_1 x'}{h \sqrt{-\det f}} \right) = 0$$

$$\Pi_0^i = -T_0 G_{ij} \frac{(X \cdot X')(X^j)' - (X')^2(X^j)}{\sqrt{-\det f}}$$

$$\Pi_1^i = -T_0 G_{ij} \frac{(\dot{X} \cdot X')(\dot{X}^j) - (\dot{X})^2(X^j)'}{\sqrt{-\det f}}$$

(3.54)

If we consider a static string configuration, i.e. $x(\sigma, \tau) = b = \text{constant}$, then energy can be interpreted as the thermal mass of the quark in the dual gauge theory. Using the static solution in (3.54), we obtain the mass using $E = - \int d\sigma \Pi_0^i$, as

$$m(\mathcal{T}) = T_0 (r_0 - r_h) = T_0 \left( |\mu|^2/3 - \mathcal{T} \right) \equiv T_0 (1 - \mathcal{T})$$

(3.55)

Now recounting our earlier analysis of the drag force (3.44) we see that the velocity of a particle in the gas (given by the end point of the string) will eventually approach a constant velocity $v$ as

$$V(t) = v \left( 1 - e^{-t/\tau} \right)$$

(3.56)

where $v_i = \frac{q F_{0i}}{m}$ with $m$ derived at a constant temperature. Thus once the string (or the fundamental quark in the dual gauge theory) moves with a constant velocity, we need to apply force constantly to keep it in that state. This gives us the drag force. To obtain the drag coefficient $\nu$, we consider strings moving with constant velocity:

$$x(t, r) = \tilde{x}(r) + vt$$

(3.57)
where $v \equiv v_1$ from (3.56), as after large enough time $V(t) \to v$ and the string moves at a uniform speed. Then from (3.54), noting that $f$ is independent of time, we can solve the equation of motion to get:

$$\bar{x}^2 = \frac{h^2 C^2 v^2}{g_1 g_2} \cdot \frac{g_1 - v^2}{g_1 - h C^2 v^2}$$

(3.58)

where $C$ is a constant of integration that can be determined by demanding that $-\det f$ is always positive. Using the value of $\bar{x}^2$ from (3.58) we can give an explicit expression for the determinant of $f$ as:

$$-\det f = g_1 g_2 \cdot g_1 - v^2 g_1 - h C^2 v^2$$

(3.59)

For $-\det f$ to remain positive for all $r$, we need both numerator and denominator to change sign at same value of $r$. This is the same argument as in [44]. The numerator changes sign at

$$r^2 = \frac{r_h^2}{\sqrt{1 - v^2}} + \mathcal{O}(g_s N_f, g_s M)$$

(3.60)

Requiring that denominator also change sign at that value fixes $C$ to be:

$$C = \frac{r_h^2 L^{-2}}{\sqrt{1 - v^2}} \cdot \frac{1}{\sqrt{1 + \frac{3 g_s M^2}{2 \pi N^2} \log\left[\frac{r_h}{(1-v^2)_{1/4}}\right]\left(1 + \frac{3 g_s N_f}{2 \pi} \left\{\log\left[\frac{r_h}{(1-v^2)_{1/4}}\right]\right\} + \frac{1}{2}\right)}}$$

(3.61)

where $\bar{M}$ and $\bar{N}_f$ differs from $M, N_f$ due to the $\mathcal{O}(g_s N_f, g_s M)$ terms in (3.60). The first part of $C$ is the one derived in [44]. The next part is new. Now the rate at which momentum is lost to the black hole is given by the momentum density at horizon

$$\Pi_1^F(r = r_h) = -T_0 C v$$

(3.62)

while the force quark experiences due to friction with the plasma is $\frac{dp}{dt} = -\nu p$ with $p = mv/\sqrt{1 - v^2}$. To keep the quark moving at constant velocity, an external field $E_i$ does work and the equivalent energy is dumped into the medium [44]. This external field $E_i$ is exactly the flux $F_{0i}$ discussed above. Thus the rate at which a quark dumps energy and momentum into the thermal medium is precisely the rate at which the string loses energy and momentum to the black hole. Thus upto $\mathcal{O}(g_s N_f, g_s M)$ we have $\frac{\nu m v}{\sqrt{1 - v^2}} = -\Pi_1^F(r = r_h)$ and

$$\nu = \frac{T_0 C \sqrt{1 - v^2}}{m}$$

(3.63)

$$= \frac{T_0}{m L^2} \frac{T^2}{\sqrt{1 + \frac{3 g_s M^2}{2 \pi N^2} \log\left[\frac{T}{(1-v^2)_{1/4}}\right]\left(1 + \frac{3 g_s N_f}{2 \pi} \left\{\log\left[\frac{T}{(1-v^2)_{1/4}}\right]\right\} + \frac{1}{2}\right)}}$$

Note that by $\mathcal{O}(g_s N_f, g_s M)$ we will always mean $\mathcal{O}(g_s N_f, g_s^2 N_f M, g_s M^2 / N)$ unless mentioned otherwise.
which should now be compared with the AdS result [44]. In the AdS case the drag coefficient $\nu$ is proportional to $T^2$. For our case, when we incorporate RG flow in the gravity dual, we obtain $\mathcal{O}\left(1/\sqrt{A} \log T + B \log^2 T\right)$ correction to the drag coefficient computed using AdS/CFT correspondence [44] [69].

3.3 Wake created by the moving quark

In the previous section we computed the drag force on the quark. Clearly a moving quark should leave some disturbance in the surrounding media. This disturbance is called the wake of the quark. Thus, in order to compute the wake left behind by a fast moving quark in the Quark Gluon Plasma, we need to compute the stress tensor $T_{pq}$, $p, q = 0, 1, 2, 3$ of the entire system. Our goal therefore would be to compute:\footnote{In the published version this section is slightly abridged by emphasising mostly on the holographic renormalisability of the theory with fundamental flavors and very little on the actual wake.}

\[
T_{pq}^{\text{medium+quark}} - T_{pq}^{\text{quark}}
\]

(3.64)

where the first term is basically the energy-momentum tensor of OKS-BH background plus string i.e $T_{pq}^{\text{background+string}}$ restricted to four-dimensional space-time. Similarly the second term is the energy momentum tensor of the string i.e $T_{pq}^{\text{string}}$ restricted to four-dimensional space-time. This is similar to the analysis done in [45] for the AdS case. For our case the above idea, although very simple to state, will be rather technical because of the underlying RG flow in the dual gauge theory side. Our second goal would then be to see how much we differ from the AdS results once we go from CFT to theories with running coupling constants.

For a strongly coupled QGP, we will apply the gauge/gravity duality to compute $T_{pq}$ of QGP using the supergravity action $S_{\text{total}}$. The supergravity action will be defined as a functional of the perturbation $l_{pq}$ from the string on the background metric $g_{pq}$. Making use of the duality, we expect that the Hilbert space of strongly coupled QCD to be mostly contained in the Hilbert space of low energy weakly coupled \textit{classical} Supergravity i.e. the OKS-BH geometry of (3.4), or alternatively, the full Hilbert space of QCD should be contained in the Hilbert space of \textit{string theory} in the OKS-BH background. However there is a subtlety here. The standard supergravity analysis in this theory will lead to actions that blow up at the boundary (i.e taking $r = r_c \to \infty$). The reason for this is rather simple to state (see also [46]). The UV completion of cascading type theories require \textit{infinite} degrees of freedom – much like string theories. This is of course another reason why the dual of cascading theories are given by string theories. Once we require infinite degrees of freedom at the UV, we no longer expect a finite boundary action from supergravity analysis! What we need is to regularise and renormalise the supergravity boundary action so that finite correlation functions could be extracted. This would also mean that the usual Witten type proposal [47] for the AdS/CFT correspondence can be re-expressed...
in terms of the boundary variables to give us the complete picture. Therefore we can rewrite the ansatze proposed by Witten et al [47] for our OKS-BH geometry to take the following Wilsonian form (also as mentioned in footnote 2, by QCD we will only mean a theory that approximates large $N_{\text{IR}}$ QCD, but with a strongly coupled and almost conformal UV):

$$Z_{\text{QCD}}[\phi_0] \equiv \langle \exp \int_{M^4} \phi_0 O \rangle = Z_{\text{total}}[\phi_0] \equiv \exp(S_{\text{total}}[\phi_0] + S_{\text{GH}} + S_{\text{counterterm}})$$

(3.65)

where $M^4$ is Minkowski manifold, $S_{\text{GH}}$ is the Gibbons-Hawking boundary term [48], $\phi_0$ should be understood as a fluctuation over a given configuration of field, and $S_{\text{counterterm}}$ is the counter-term action added to renormalise the action. Observe that in the usual AdS/CFT case we consider the action at the boundary to map it directly to the dual gauge theory side. For the OKS-BH background, as we discussed above, there are many possibilities of defining different gauge theories at the boundary depending on how we cut-off the geometry and add UV caps. Taking the radial coordinate as setting the energy scale $\Lambda$ the gauge theory side would make sense at that scale once we define the UV degrees of freedom there. This is what we called $N_{\text{eff}}(\Lambda)$ in (3.23). Therefore in general the action at any point $r = r_c$ in the OKS-BH geometry will map to the dual gauge theory with $N_{\text{eff}}(\Lambda)$ degrees of freedom at that energy scale. The properties of this dual gauge theory may not necessarily coincide with the universal properties of the parent cascading theory when both are studied from the boundary. However the RG flow of this theory will eventually catch up with the RG flow of the smooth cascading theory at that scale fixed by our choice $r = r_c$ (see figure 7). For large enough $r_c$ the above correspondence (3.65) should give us finite boundary action. Our procedure then would be to fix the boundary action for large $r_c$ (typically $r_c \to \infty$) by adding the corresponding $S_{\text{GH}}$ and $S_{\text{counterterm}}$, and then extrapolate this to smaller radii$^{33}$. Thus, for computing stress tensor of QCD, we have $O = T^{pq}$ and $\phi_0 = \kappa l_{pq}$. It follows that

$$\langle T^{pq} \rangle = \left. \frac{1}{\kappa} \frac{\delta S}{\delta l_{pq}} \right|_{\kappa l_{pq}=0}$$

(3.66)

$^{33}$Once the boundary theory is holographically renormalised, all $r_c$ dependences would go as $r_c^{-1}$. In the dual gauge theories this would mean that in addition to the universal properties inherited by each of these gauge theories, there would be corrections associated with the UV degrees of freedom that we needed to provide to define these theories at $r = r_c$. These are precisely the corrections that take us away from the parent cascading theory and give us results associated with new gauge theories. Needless to say, such extravagant richness of physical theories are not available to us in the AdS/QCD picture. Note however that even for the AdS/QCD case there might arise situations where we would require $\mathcal{N} = 2$ degrees of freedom to UV complete a $\mathcal{N} = 1$ configuration at IR. The issue of holographic renormalisation is not much affected by this because such a UV completion only affects the internal space (here it may change $T^{1,1}$ geometry to $S^5/Z_2$). For our case the full F-theory completion of the parent cascading theory would also be UV complete.
where $S \equiv S_{\text{total}} + S_{\text{GH}} + S_{\text{counterterm}}$ and $S_{\text{total}}$ is the low energy Type IIB supergravity action in ten dimensions defined in string frame as:

$$S_{\text{total}} = \frac{1}{2\kappa_{10}^2} \left[ \int d^{10}x \ e^{-2\Phi} \sqrt{-G} \left( R - 4\partial_i\Phi\partial^i\Phi - \frac{1}{2}|H_3|^2 \right) - \frac{1}{2} \int d^{10}x \sqrt{-G} |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right] - \frac{1}{2} \int C_4 \wedge H_3 \wedge F_3 \right] + S_{\text{solitons}}$$

$$\equiv S_{\text{SUGRA}} + S_{\text{string}} + S_{D7}$$

(3.67)

where $S_{\text{SUGRA}}$ is the background supergravity action and $S_{\text{solitons}}$ is the action of the solitonic objects in our theory, namely the fundamental string and the D7 brane. Recall that the OKS-BH background is constructed by inserting the D7 brane in the supergravity action with an additional black hole singularity. Therefore we will define

$$S_{\text{OKS-BH}} \equiv S_{\text{SUGRA}} + S_{D7}$$

(3.68)

so that the background solutions that we gave in the previous subsection will correspond to $S_{\text{OKS-BH}}$. Once we introduce an additional fundamental string we expect some of the background values to change. The change in the metric will take the following form:

$$G_{ij} = g_{ij} + \kappa l_{ij}$$

$$l_{ij} \equiv l_{ij}(r, x, y, z, t)$$

(3.69)

where $g_{ij}$ is the OKS-BH metric and $l_{ij}$ ($i, j = 0, \ldots, 9$) denote the perturbation from the moving string source (with $\kappa \to 0$). We also expect the NS two form $B_2$ defined in (3.20) to pick up an additional component along the $(0r)$ direction. The three and five forms RR field strengths defined as:

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3, \quad \tilde{F}_5 = F_5 + \frac{1}{2} B_2 \wedge F_3$$

(3.70)

would change from the values defined earlier because $H_3$ changes. In the absence of the string, $F_3$ is the third form sourced by D5 branes and $F_5$ is the fifth form sourced by D3 branes.

To proceed, let us first figure out the possible changes in $H_3$, the NS three form field strength from (3.24) given earlier. To order $O(g_s N_f)$ locally the background value of $H_3$ in the absence of the fundamental string for a Klebanov-Tseytlin type geometry [50] can be given by the following values of $c_i$ [20, 21, 22, 53, 58]:

$$c_1 = -\frac{3g_s^2 M N_f}{4\pi r} \cot \frac{\theta_1}{2} + O(g_s^2 N^2_f), \quad c_2 = -\frac{3g_s^2 M N_f}{4\pi r} \cot \frac{\theta_2}{2} + O(g_s^2 N^2_f)$$

(3.71)

$$c_3 = c_4 = \frac{3g_s M}{r} \left[ 1 + \frac{g_s N_f}{4\pi} \left\{ 9 \log r + 2 \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right\} \right] + O(g_s^2 N^2_f)$$

(3.72)
where the $\mathcal{O}(g_s^2 N_f^2)$ local corrections are typically of the following form (see also [51, 52, 53, 18]):

$$ \sum_{n \geq m,p} a_{mnp} (g_s M)^m (g_s N_f)^n (\log r)^{p+1} $$

(3.72)

with $a_{mnp}$ are not in general constants. For certain examples studied in [11, 52, 53, 18] without D7 branes, $a_{mnp}$ are functions of the radial coordinate $r$. With the D7 branes these corrections have not been computed.

In addition to the $\mathcal{O}(g_s^2 N_f^2)$ corrections we have another set of corrections already to order $\mathcal{O}(g_s N_f)$ that appear because of our choice of background (3.7). These $\mathcal{O}(g_s N_f)$ corrections are in general difficult to work out if in (3.7) the parameter $a$ is not a constant. When $a$ is a constant then this is a small resolution of the conifold and changes the coefficients $c_i$ of (3.71) in the following way:

$$ \frac{\Delta c_1}{c_1} = 6 a^2 \left( \frac{\log r^3 - 2}{r^2} \right) + \mathcal{O}(a^2 \log a) + \mathcal{O}(g_s^2 N_f^2) $$

$$ \frac{\Delta c_2}{c_2} = -18 a^2 \left( \frac{\log r}{r^2} \right) + \mathcal{O}(a^3) + \mathcal{O}(g_s^2 N_f^2) $$

(3.73)

$$ \frac{\Delta c_3}{c_3} = - \frac{9 a^2}{r^2} - \frac{3 g_s N_f a^2}{4 r^2} \left[ 8 + 9 \log r - \frac{2 r^2}{a^2} \log a \right] + \mathcal{O}(a^2 \log a, g_s^2 N_f^2) $$

$$ \frac{\Delta c_4}{c_4} = \frac{3 a^2}{r^2} \cdot \frac{1 + \frac{g_s N_f}{2 \pi} (3 + \log \sin \frac{\alpha}{2} \sin \frac{\alpha}{2})}{1 + \frac{g_s N_f}{4 \pi} (9 \log r + 2 \log \sin \frac{\alpha}{2} \sin \frac{\alpha}{2})} + \mathcal{O}(a^2 \log a) + \mathcal{O}(g_s^2 N_f^2) $$

Putting everything together we see that the NS three form changes a bit from what is given in [20, 51, 52, 53, 18]. To order $g_s N_f$ the local three form is given by:

$$ \sum \left[ \mathcal{O}(a^2 g_s N_f) + \mathcal{O}(g_s^2 N_f) \right] \wedge d\omega_3 $$

(3.74)

where we only pointed out those $\mathcal{O}(a^2 g_s N_f)$ terms that have similar forms as the terms of $H_3$ with $a = 0$. There are a few more $\mathcal{O}(a^2 g_s N_f)$ terms described by generic
three form \(d\omega_3\) in the OKS-BH geometry, that we do not write here but could be easily worked out. In addition to that we have \(O(g_s^2 N_f^2)\) terms of the form \((3.72)\) that we need to incorporate. The two form \(\omega_2\) is defined above as:

\[
\omega_2 = \frac{1}{2} (\sin \theta_1 \, d\theta_1 \wedge d\phi_1 - \sin \theta_2 \, d\theta_2 \wedge d\phi_2)
\]  

(3.75)

Notice also that the fourth term in \((3.74)\) has a relative sign difference from the second term of the angular forms. Similarly the third term has an unequal distribution of the three form on the base two spheres. In addition to that -- for \(d\omega_3\) given exclusively by the internal three forms -- we can always define a \(B_2\) field that lie only along the angular directions \((\theta_i, \phi_i, \psi)\). In that case the dynamics of a fundamental string will get influenced by the background NS field (so to avoid issues like \([49]\) we will take the slice \((3.14)\) where the pull-back \(B_{ab}\) in \((3.41)\) is zero). Also in this paper we will only consider the case where \(B_2\) can be made to lie in the internal angular directions by appropriate gauge transformations. We should however remind the readers that this is not generic. In the presence of branes and \(H_3\) fluxes such procedure cannot always be done without generating noncommutative theories on the branes.

Let us now determine the RR three form \(\tilde{F}_3\) which is a combination of \(F_3\) and \(H_3\). For a Klebanov-Tseytlin \([50]\) kind of background we expect, in addition to the \(c_i\) defined earlier in \((3.71)\), there would be two more \(c_i\) given by:

\[
c_5 = c_6 = g_s M \left( 1 + \frac{3g_s N_f}{2\pi} \log r \right) + O(g_s^2 N_f^2)
\]  

(3.76)

As mentioned earlier, the above choices of \(c_i\) are still incomplete. For constant small resolution \(a\) we expect \(\Delta c_i\) to be given by:

\[
\frac{\Delta c_5}{c_5} = \frac{9g_s N_f}{4\pi} \cdot \frac{a^2}{r^2} \left( \frac{2 - 3 \log r}{1 + \frac{3g_s N_f}{2\pi} \log r} \right) + O(a^2 \log a) + O(g_s^2 N_f^2)
\]

\[
\frac{\Delta c_6}{c_6} = \frac{9g_s N_f}{4\pi} \cdot \frac{a^2}{r^2} \left( \frac{1 + 3 \log r}{1 + \frac{3g_s N_f}{2\pi} \log r} \right) + O(a^2 \log a) + O(g_s^2 N_f^2)
\]  

(3.77)

Combining everything together, we then get the following value of the background RR three form \(\tilde{F}_3\):

\[
\tilde{F}_3 = 2M \left( 1 + \frac{3g_s N_f}{2\pi} \log r \right) e_\psi \wedge \omega_2 \\
- \frac{3g_s M N_f}{4\pi} \frac{dr}{r} \wedge e_\psi \wedge \left( \cot \frac{\theta_2}{2} \sin \theta_2 \, d\phi_2 - \cot \frac{\theta_1}{2} \sin \theta_1 \, d\phi_1 \right) \\
- \frac{3g_s M N_f}{8\pi} \sin \theta_1 \sin \theta_2 \left( \cot \frac{\theta_2}{2} \, d\theta_1 + \cot \frac{\theta_1}{2} \, d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2 \\
+ \frac{9g_s M N_f}{2\pi} \left( 1 + \frac{3g_s N_f}{2\pi} \log r \right) \frac{a^2}{r^2} e_\psi \wedge \left( (2 - 3 \log r) \omega_2 - \frac{9}{2} \log r \sin \theta_2 \, d\theta_2 \wedge d\phi_2 \right) \\
+ \frac{27g_s M N_f}{2\pi} \cdot \frac{a^2 \log r}{r^3} \frac{dr}{r} \wedge e_\psi \wedge \left( \cot \frac{\theta_2}{2} \sin \theta_2 \, d\phi_2 + \cot \frac{\theta_1}{2} \sin \theta_1 \, d\phi_1 \right)
\]
where as before we considered only the $O(a^2 g^2 N_f)$ terms that are proportional to the existing resolution free terms that appeared in [20, 18]. However our form is more involved than the ones considered earlier even if we ignore the $O(g^2 N_f^2)$ corrections of the form (3.72). The fact that the background has a resolution changes much of the details. Notice also the fact that these corrections cannot be absorbed as a renormalisation of the $a = 0$ terms because the form structures have relative sign differences.

In the presence of a non-trivial dilaton, these three forms combine together to give us $G_3 \equiv \tilde{F}_3 - ie^{-\phi}H_3$. It is instructive to construct $G_3$ for our background because most of the details can be expressed directly in terms of $G_3$. The full expression for $G_3$ is involved, but could be combined succinctly using certain one and three forms to take the following background value:

$$G_3 = 2M \left( 1 + \frac{3g_s N_f}{2\pi} \right) \left( e_\psi - \frac{3idr}{r} \right) \wedge \omega_2$$

$$- \frac{3g_s MN_f}{4\pi} \cot \frac{\theta_2}{2} \left( \sin \theta_2 \sin \frac{1}{2} d\phi_2 - \frac{i}{2} d\theta_2 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi + \frac{1}{2} \sin \theta_1 d\theta_1 \wedge d\phi_1 \right)$$

$$+ \frac{3g_s M N_f}{4\pi} \cot \frac{\theta_1}{2} \left( \sin \theta_1 d\phi_1 - \frac{i}{2} d\theta_1 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi + \frac{1}{2} \sin \theta_2 d\theta_2 \wedge d\phi_2 \right)$$

$$+ \frac{3ig_s^2 M N_f^2}{16\pi^2} \log \left( r^{3/2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \left( \cot \frac{\theta_2}{2} d\theta_2 - \cot \frac{\theta_1}{2} d\theta_1 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi - \frac{1}{2} de_\psi \right)$$

$$+ \frac{im g_s N_f^2}{2\pi^2} \left[ \frac{27}{8} \log r + 3 \log r \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + \frac{1}{2} \log^2 \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right]$$

$$\times \left( \frac{dr}{r} \wedge \omega_2 \right) + 2 \hat{m} \left( 1 + \frac{3g_s N_f}{2\pi} \right) \left( e_\psi \wedge \tilde{\omega}_2 + \frac{3ig_s}{g_s} \frac{dr}{r} \wedge \tilde{\omega}_2 \right)$$

$$+ \frac{3N_f}{4\pi} \cot \frac{\theta_2}{2} \left( \tilde{g}_s \tilde{M} \sin \theta_2 \sin \frac{1}{2} d\phi_2 - \frac{i}{2} \frac{\tilde{M} \tilde{g}_s^2}{g_s} d\theta_2 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi + \frac{1}{2} \log r \sin \theta_1 d\theta_1 \wedge d\phi_1 \right)$$

$$- \frac{3N_f}{4\pi} \cot \frac{\theta_1}{2} \left( \tilde{g}_s \tilde{M} \sin \theta_1 d\phi_1 - \frac{i}{2} \frac{\tilde{M} \tilde{g}_s^2}{g_s} d\theta_1 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi + \frac{1}{2} \log r \sin \theta_2 d\theta_2 \wedge d\phi_2 \right)$$

$$- \frac{3ig_s^2 \tilde{M} N_f^2}{16\pi^2} \log \left( r^{3/2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \left( \cot \frac{\theta_2}{2} d\theta_2 + \cot \frac{\theta_1}{2} d\theta_1 \right) \wedge \left( \frac{dr}{r} \wedge e_\psi - \frac{1}{2} de_\psi \right)$$

$$- \frac{ig_s \tilde{g}_s \tilde{M} N_f^2}{2\pi^2} \left[ \frac{27}{8} \log^2 r + 3 \log r \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + \frac{1}{2} \log^2 \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right]$$

$$\times \left( \frac{dr}{r} \wedge \tilde{\omega}_2 \right) + O(a^2 g_s N_f) + O(g^2 N_f^2) \wedge d\omega_3 \left( 1 - \frac{i}{g_s} \right)$$

(3.79)
where we have used two new two-forms \( \tilde{\omega}_2 \) and \( \hat{\omega}_2 \) defined in terms of \( \omega_2 \) in the following way:
\[
\tilde{\omega}_2 \equiv (2 - 3\log r)\omega_2 - \frac{9}{2} \log r \sin \theta_2 \, d\theta_2 \wedge d\phi_2
\]
\[
\hat{\omega}_2 \equiv \omega_2 + \sin \theta_2 \, d\theta_2 \wedge d\phi_2 \quad (3.80)
\]
These forms help us to express the deformation of \( G_3 \) once we take the resolution etc into account. The other effective \( M \) and \( g_s \) are then defined in terms of \((M, g_s, N_f)\) in the following way\(^{34}\):
\[
\tilde{M} = \frac{9g_sMN_f}{4\pi} \cdot \frac{a^2}{r^2}, \quad \hat{M} = \frac{Ma^2}{4r^3}, \quad \tilde{g}_s = \frac{8\pi}{g_sN_f}, \quad \hat{g}_s = 12g_sr \quad (3.81)
\]
Using these definitions we can see how the back reactions effect the three forms in our setup. Note that the deformations appear in \( G_3 \) almost exactly like the undeformed forms, but there are crucial relative signs and extra \((r, \theta_i)\) dependent factors. However the way we have written the backgrounds is not very productive because of many complicated terms. But there exist an alternative way to rewrite the above background which would tell us exactly how the black hole modifies the original Ouyang setup. This can be presented in the following way:
\[
\tilde{F}_3 = 2MA_1 \left( 1 + \frac{3g_sN_f}{2\pi} \log r \right) e_\psi \wedge \frac{1}{2} (\sin \theta_1 \, d\theta_1 \wedge d\phi_1 - B_1 \sin \theta_2 \, d\theta_2 \wedge d\phi_2)
\]
\[
- \frac{3g_sMN_f}{4\pi} A_2 \frac{dr}{r} \wedge e_\psi \wedge \left( \cot \frac{\theta_2}{2} \sin \theta_2 \, d\phi_2 - B_2 \cot \frac{\theta_1}{2} \sin \theta_1 \, d\phi_1 \right)
\]
\[
- \frac{3g_sMN_f}{8\pi} A_3 \sin \theta_1 \sin \theta_2 \left( \cot \frac{\theta_2}{2} \, d\theta_1 + B_3 \cot \frac{\theta_1}{2} \, d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2 \quad (3.82)
\]
\[
H_3 = 6g_sA_4 \left( 1 + \frac{9g_sN_f}{4\pi} \log r + \frac{g_sN_f}{2\pi} \log \frac{\sin \theta_1}{2} \frac{\sin \theta_2}{2} \right) \frac{dr}{r}
\]
\[
\wedge \frac{1}{2} \left( \sin \theta_1 \, d\theta_1 \wedge d\phi_1 - B_4 \sin \theta_2 \, d\theta_2 \wedge d\phi_2 \right) + \frac{3g_s^2MN_f}{8\pi} A_5 \left( \frac{dr}{r} \wedge e_\psi - \frac{1}{2} de_\psi \right)
\]
\[
\wedge \left( \cot \frac{\theta_2}{2} \, d\theta_2 - B_5 \cot \frac{\theta_1}{2} \, d\theta_1 \right)
\]
where we see that the background is exactly of the form presented in \cite{20} except that there are asymmetry factors \( A_i, B_i \). These asymmetry factors contain all the

\(^{34}\)Since upto \( O(g_sN_f) \) our metric remains similar to the Ouyang metric, we then expect the effective number of fiveform flux to change as \( N_{\text{eff}} = N + \frac{3g_sM^2}{2\pi} \left( \log r + \frac{3g_sN_f}{2\pi} \log^2 r \right) \). This would imply that under radial rescaling \( r \to e^{-2\pi/3g_sM} r \), \( N_{\text{eff}} \) decreases by \( M - N_f \) units, exactly as we discussed earlier (see also \cite{20}).
informations of the black hole etc in our background\textsuperscript{35}. To order $\mathcal{O}(g_s N_f)$ these asymmetry factors are given by:

\begin{align*}
A_1 &= 1 + \frac{9 g_s N_f}{4 \pi} \cdot \frac{a^2}{r^2} \cdot (2 - 3 \log r) + \mathcal{O}(a^2 g_s^2 N_f^2) \\
B_2 &= 1 + \frac{36 a^2 \log r}{r^3 + 18 a^2 r \log r} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
A_2 &= 1 + \frac{18 a^2}{r^2} \cdot \log r + \mathcal{O}(a^2 g_s^2 N_f^2) \\
B_1 &= 1 + \frac{81}{2} \cdot \frac{g_s N_f a^2 \log r}{4 \pi r^2 + 9 g_s N_f a^2 (2 - 3 \log r)} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
A_3 &= 1 - \frac{18 a^2}{r^2} \cdot \log r + \mathcal{O}(a^2 g_s^2 N_f^2) \\
B_3 &= 1 + \frac{36 a^2 \log r}{r^2 - 18 a^2 \log r} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
A_4 &= 1 - \frac{3 a^2}{r^2} + \mathcal{O}(a^2 g_s^2 N_f^2), \quad B_4 = 1 + \frac{3 g_s a^2}{r^2 - 3 a^2} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
A_5 &= 1 + \frac{36 a^2 \log r}{r} + \mathcal{O}(a^2 g_s^2 N_f^2), \quad B_5 = 1 + \frac{72 a^2 \log r}{r + 36 a^2 \log r} + \mathcal{O}(a^2 g_s^2 N_f^2)
\end{align*}

These asymmetry factors tell us that corrections to Ouyang background \textsuperscript{[20]} come from $\mathcal{O}(a^2/r^2)$ onwards. Thus to complete the picture all we now need are the values for the axion $C_0$ and the five form $F_5$. They are given by:

\begin{align*}
C_0 &= \frac{N_f}{4 \pi} (\psi - \phi_1 - \phi_2) \\
F_5 &= \frac{1}{g_s} \left[ d^4 x \wedge dh^{-1} + *(d^4 x \wedge dh^{-1}) \right]
\end{align*}

with the dilaton to be taken as approximately a constant near the D7 brane and $h$ is the ten dimensional warp factor discussed above. Thus combining \textsuperscript{[3.82]} and \textsuperscript{[3.84]} our background can be written almost like the Ouyang background \textsuperscript{[20]} with deviations given by \textsuperscript{[3.83]}.

So far we got the background without taking the back reaction of the string. The back reaction of the string can be computed from its energy momentum tensor. Using the action (3.41) we can obtain the energy-momentum tensor of the string as:

\begin{align*}
T_{\text{string}}^{ij}(x) &= \frac{\delta S_{\text{string}}}{\delta G_{ij}} \\
&= \int d\sigma d\tau \left( \frac{2 \dot{X}^i \cdot X' X'^{ij} X'^{ij} - X'^i X'^{ij} X'^{ij} \dot{X}^2}{2 \sqrt{-\det f}} \right) \delta^{10}(X - x)
\end{align*}

\textsuperscript{35}One can easily see from these asymmetry factors that one of the two spheres is squashed. As we mentioned before, this squashing factor is of order $\mathcal{O}(g_s N_f)$ and therefore could have a perturbative expansion. Note also that although the resolution factor in the metric is hidden behind the horizon of the black hole the effect of this shows up in the fluxes. As far as we know, these details have not been considered previously.
where all the variables have been defined earlier.

To study the back reaction without incorporating the backreactions from other F-theory seven-branes, we need to analyse the geometry close to the string. This will be rather involved because our metric (3.4) with the choices of warp factor (3.12) and the black hole functions (3.13) are complicated. However we can impose two immediate simplifications:

$$h(r, \theta_1, \theta_2) = h(r, \pi, \pi), \quad g_1(r) \approx g_2(r) \equiv g(r) = 1 - \frac{r_h^4}{r^4}$$  \hspace{1cm} (3.86)

which is motivated from the fact that we are close to the string and the $O(g_s^2 MN_f)$ corrections to the black hole factor are subleading (as we saw in the previous section). Henceforth we will stick with these choices throughout our analysis. With this therefore we see that the local metric is given from (3.4) and (3.12) (or (3.35)) as:

$$ds^2 = \frac{r^2}{L^2} \left(1 - A \log r - B \log^2 r \right) \left(-g dt^2 + dx^i dx_i \right) + \frac{L^2}{g} \frac{r^2}{r^2} dr^2 \left(1 + A \log r \\
+ B \log^2 r \right) + (L^2 + AL^2 \log r + BL^2 \log^2 r) d\mathcal{M}_5^2$$  \hspace{1cm} (3.87)

where $A$ and $B$ are defined in terms of $g_s, N_{f}^{\text{eff}}, N$ and $M_{\text{eff}}$ as:

$$A = \frac{3g_s M_{\text{eff}}^2}{4\pi N} \left(1 + \frac{3g_s N_{f}^{\text{eff}}}{4\pi} \right), \quad B = \frac{9g_s^2 M_{\text{eff}}^2 N_{f}^{\text{eff}}}{8\pi^2 N}$$  \hspace{1cm} (3.88)

with $M_{\text{eff}}$ and $N_{f}^{\text{eff}}$ are defined in (3.10); and $L^2 = \sqrt{4\pi g_s N}$ being the usual definition. For supergravity to be valid we require weak string coupling but large $N$ such that:

$$g_s \to 0, \quad L^2 >> 1, \quad \frac{N_{f}}{N} << 1, \quad \frac{M}{N} < 1, \quad N >> 1, \quad g_s M >> 1$$  \hspace{1cm} (3.89)

Using these limits (see also footnote 23 for more precise parametrisation) one can easily show that both $A$ and $B$ can be made very small, and

$$\left(\frac{A}{L^2}, \frac{B}{L^2}\right) << 1, \quad (AL^2, BL^2) << 1$$  \hspace{1cm} (3.90)

The above equation is very useful for us because we can recast our metric using (3.90) to show how much we deform from the AdS black hole metric. Since $L^2 >> (AL^2, BL^2)$ we can rewrite (3.87) as\textsuperscript{36}

$$ds^2 = \frac{r^2}{L^2} \left(-g dt^2 + dx^i dx_i \right) + \frac{L^2}{g r^2} dr^2 + L^2 d\mathcal{M}_5^2$$  \hspace{1cm} (3.91)

$$- \left(A \log r + B \log^2 r \right) \left[\frac{r^2}{L^2} \left(-g dt^2 + dx^i dx_i \right) - \frac{L^2}{g r^2} dr^2 + L^2 d\mathcal{M}_5^2 \right]$$

\textsuperscript{36}Using the parametrisation given in footnote 23, observe that $A \to \epsilon^{9/2}, B \to \epsilon^6$ and $L^2 \to \epsilon^{-11/4}$. Thus $AL^2 \to \epsilon^{7/4}, BL^2 \to \epsilon^{13/4}, \frac{A}{L^2} \to \epsilon^{29/4}$ and $\frac{B}{L^2} \to \epsilon^{35/4}$ for $\epsilon \to 0$. This would justify the limits considered above.
where the first line is the \( AdS_5 \times M_5 \) black hole solution, and the second line is the deformation of the \( AdS_5 \) and the internal geometries. Observe that (a) the internal space and the radial direction are very mildly deformed in our limit for regions close to the string\(^{37}\), and (b) the deformation of the AdS geometry is not another AdS space. These conclusions also imply that we can integrate over the internal coordinates \( \psi, \phi_1, \phi_2, \theta_1, \theta_2 \) in (3.67), to obtain the five dimensional effective action \( S_{\text{eff \ total}} \) in the following way:

\[
S_{\text{eff \ total}} = \frac{1}{8\pi G_N} \int d^5x \sqrt{-G} \left[ e^{-2\phi} R(G) - 2\Lambda(r) + G_{ij} \partial^i \Phi^j - m_{ij} \Phi^i \Phi^j + \ldots \right]
\]

\[+ T_5 \int d^5x \left[ F \wedge *F + \text{scalars} + \sum_{n \geq 2} c_n R^n + \sum_{n \geq 1} b_n \text{tr} \left( R \wedge R \right)^n \right] - S_{\text{string}}^{\text{eff \ string}} = S_{\text{OKS-BH}} - S_{\text{string}}^{\text{eff \ string}} \tag{3.92}
\]

where \( G_N \) is the Newton’s constant, \( T_5 \) is the effective tension, \( \Lambda(r) \) is the cosmological “constant” coming from the contributions of the background fields and \( \Phi^i \) are the scalars that we get by dimensionally reducing IIB supergravity fields over the internal manifold. Some of these scalars come from the metric fluctuations of the internal five-manifold \( g_{mn}(y) \) as:

\[
G^{(10)}_{mn}(x, y) := G_{\mu\nu}(x) \oplus \sum_i \Phi^i(x) \Omega^i_{mn}(y) \oplus g_{mn}(y) \tag{3.93}
\]

where \( (x, y) \) are the five-dimensional spacetime and the internal indices respectively, \( G_{mn}(x, y) \) would incorporate the background geometry (3.91) as well as string contribution (to be discussed below), \( \Omega^i \) are the normalisable \( p \)-form satisfying \( \int \Omega^i \wedge \Omega^j = \delta^{ij} \). Similarly the other scalars (excluding the ones that come directly from the axio-dilaton and the five-form) come from the two two-form fields \( B \) in the following way:

\[
\Phi^i = \int \left( B - \langle B \rangle \right) \wedge \Omega^i \tag{3.94}
\]

where \( \langle B \rangle \) are the background \( B_{NS} \) and \( B_{RR} \) fields that we derived earlier. The rest of the scalars and the gauge fields are from the wrapped D7 brane. We have also included the higher derivative \( R^2 \) and \( \text{tr} \left( R \wedge R \right) \) type terms that come from the D7 brane back reaction and world volume Cherns-Simons terms respectively. These terms will have important implications that we will discuss in the next section. For the time being we want to point out that \( O(R^4) \) terms have been studied recently in

\(^{37}\)This is because the coefficients of \( dr^2 \) and the internal space are proportional to \( L^2 \) which go to infinity as \( e^{-11/4} \), whereas the other parts are proportional to \( L^{-2} \) which go to zero. Clearly then the \( dr^2 \) and the internal space are strongly dominated by \( L^2 \) whereas the other parts can have small log \( r \) deviations. Additionally any UV modifications will not affect the IR geometry.
the context of the $\eta/S$ bound, namely the viscosity over entropy bound in [58]. It was found therein that the $\eta/S$ receive quantum corrections that doesn’t lower the known $\frac{1}{4\pi}$ bound of Kovtun-Son-Starinets [5]. However an $O(R^2)$ term should lower the bound as discussed in [7], although one needs to be careful about two issues: the sign of the $R^2$ term and the relative strength of the $R^4$ and $R^2$ terms\(^{38}\).

Once we have our effective five dimensional action, we can derive linearized Einstein equation using this effective action. At this point we can parametrise the string contribution as $\kappa l_{\mu\nu}$ with $\kappa \to 0$. This means that the perturbations are small, which is a reasonable assumption for our case. Thus the total metric is $G_{\mu\nu} = g_{\mu\nu} + \kappa l_{\mu\nu}$ with $g_{\mu\nu}$ given by (3.91). The equation of motion for $G_{\mu\nu}$ would be:

$$R_{\mu\nu} (g_{\alpha\beta} + \kappa l_{\alpha\beta}) - \frac{1}{2} (g_{\mu\nu} + \kappa l_{\mu\nu}) R (g_{\alpha\beta} + \kappa l_{\alpha\beta}) = T^{\text{string}}_{\mu\nu} + T^{\text{fluxes}}_{\mu\nu} + T^{(p,q)7}_{\mu\nu} \tag{3.95}$$

where the $T^{\text{fluxes}}_{\mu\nu}$ come from the five-form fluxes $F(5)$ (that give rise to the $AdS_5$ part) and the remnant of the $H_{NS}, H_{RR}$ and the axio-dilaton along the radial $r$ direction (that give rise to the deformation of the $AdS_5$ part). In five-dimensional space these fluxes would appear as one-forms $F_{r(i)}$ with $i = 1, ..., 4$. The effect of $T^{(p,q)7}_{\mu\nu}$ will not be substantial if we take it as a probe in this background.

At this point we can approach the problem in two ways. The first way is to assume that the fluxes contribute to the five dimensional cosmological constant $\Lambda(r)$ as given in (3.92). In this set-up we have an effective five-dimensional theory (3.92) and we put a string in this background to study the perturbation in the metric. Here the assumption is that the string do not back-react on the cosmological constant $\Lambda(r)$. This seems to be the general approach in the literature.

The second way is to actually consider the back reaction of the string on the five dimensional cosmological constant. This can be worked out if one considers the effects of all the background fluxes in our theory. The result of such an analysis can be presented in powers of $\kappa$. For our case we are only interested in back reactions that are linear in $\kappa$. To this order the equation of motion satisfied by $l_{\alpha\beta}$ is determined by expanding (3.95) in the following way:

$$\kappa \left( \Delta^{\alpha\beta}_{\mu\nu} - B^{\alpha\beta}_{\mu\nu} - A^{\alpha\beta}_{\mu\nu} \right) l_{\alpha\beta} = T^{\text{string}}_{\mu\nu} \tag{3.96}$$

where $\Delta^{\alpha\beta}_{\mu\nu}$ is an operator whereas $B^{\alpha\beta}_{\mu\nu}$ and $A^{\alpha\beta}_{\mu\nu}$ are functions of $r$, the radial coordinate\(^{39}\). We have been able to determine the form for the operator $\Delta^{\alpha\beta}_{\mu\nu}$ for any generic perturbation $l_{\alpha\beta}$. The resulting equations are rather long and involved; and we give them in the Appendix B. For the functions $A^{\alpha\beta}_{\mu\nu}$ and $B^{\alpha\beta}_{\mu\nu}$ we have worked out a toy example in Appendix C with only diagonal perturbations. For off diagonal perturbations we need to take an inverse of a $5 \times 5$ matrix to determine the

\(^{38}\)We thank Aninda Sinha for pointing this out to us.

\(^{39}\)There will be another contribution from the $(p, q)$ seven branes in the background, although for small $g_s N_f$ these are subleading.
functional form. We shall provide details of this in the following. The variables defined in (3.96) are given as:

$$\Delta^{\alpha\beta}_{\mu\nu} = \left( \frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}} \right) - \frac{1}{2} g_{\mu\nu} \left( \frac{\delta R}{\delta g_{\alpha\beta}} \right) - \frac{1}{2} R \delta_{\mu\alpha} \delta_{\nu\beta}$$

$$A^{\alpha\beta}_{\mu\nu} = \sum_{b,c,d,\ldots} F^{(5)}_{a} F^{(5)}_{b} F^{(5)}_{c} F^{(5)}_{d}$$

$$B^{\alpha\beta}_{\mu\nu} = -\frac{5}{8} \sum_{a,b,c,d,\ldots} F^{(5)}_{a} F^{(5)}_{b} F^{(5)}_{c} F^{(5)}_{d}$$

where we have given the most generic form in (3.97) above. Furthermore, $\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}}$ and $\frac{\delta R}{\delta g_{\alpha\beta}}$ are operators and not functions. As usual $g_{ab}$ is the metric of the OKS-BH background.

At this point note that the first approach of ignoring the back reaction of the string on the cosmological constant $\Lambda(r)$ can only be achieved in the limit where there are no additional metric-derivatives and the corrections to $F^{(5)}$ and $F^{(i)}$ are very small. Otherwise this assumption would clearly fail. In this paper we will work out one concrete example (given in Appendix C) using the first approach only and give the full analysis in the sequel [55]. Therefore in the limit where the background fluxes, including the effects of the D7 brane, are very small the above equation (3.96) can be presented as an operator equation of the following form:

$$\kappa \Delta^{\alpha\beta}_{\mu\nu} l^{\alpha\beta}_{\mu\nu}(x) \approx T^{\text{string}}_{\mu\nu}(x)$$

where $x$ is a generic five-dimensional coordinate. In this final form, the operator $\Delta^{\alpha\beta}_{\mu\nu}$ is a second order differential operator derived from (3.95), with $\mu, \nu, \alpha, \beta = 0, 1, 2, 3, 4$.

Before we solve the above equation, we need to determine the relevant number of free components of $l_{\mu\nu}$. Notice that as $l_{\mu\nu}$ is a symmetric tensor, it has fifteen degrees of freedom. We can use coordinate transformations to fix ten degrees of freedom. As an example consider the coordinate transformation: $x^{a} \rightarrow x^{a} = x^{a} + g_{s} e^{a}$, $l_{\mu\nu}$ will transform as:

$$l_{\mu\nu} \rightarrow l_{\mu\nu} - D_{\mu} e_{\nu} - D_{\nu} e_{\mu}$$

where $D_{\mu}$ is the covariant derivative. Using this we can fix five components, say: $l_{4\mu} = 0, \mu = 0, 1, 2, 3, 4$. As one might have expected for a similar example in electromagnetism, this does not completely fix the gauge. The residual gauge transformation allow us to eliminate another five degrees of freedom and we end up with five physical degrees of freedom i.e. five independent metric perturbation using which all
other components could be expressed. Alternatively one can use certain combinations of the fifteen components to write five independent degrees of freedom for the metric fluctuations.

The above result is easy to demonstrate for the AdS space, as has already been discussed in [45]. For non-AdS spaces this is not so easy to construct. Therefore in the following we will take the complete set of ten components and using them we will determine the triangle operator $\Delta_{\mu\nu}^{\alpha\beta}$ in (3.96). If the ten components that can be labelled as a set:

$$l_n = \{ l_{00}, l_{01}, l_{02}, l_{03}, l_{11}, l_{12}, l_{13}, l_{22}, l_{23}, l_{33} \}$$ (3.100)

then the operator $\Delta_{\mu\nu}^{\alpha\beta}$ in (3.96) gives rise to 77 equations that we present in Appendix B. The warp factor $h$ appearing in these equations can be taken to be $h = h(r, \pi, \pi)$ i.e (3.12) with the slice condition (3.14), because we will analyse fluctuations close to the string.

Before analysing (3.96), we make a coordinate transformation $r = 1/u$ which will be useful in the wake analysis at zeroth order in $g_s$. Since the internal space is now independent of warping or more appropriately, has very mild warping (see (3.91)), the local OKS-BH metric under this transformation will become:

$$ds^2 = -g(u)dt^2 + dx^2 + dy^2 + dz^2 + \sqrt{h(u, \pi, \pi)} \frac{u^4 g(u)}{w^4} du^2$$

$$h(u, \pi, \pi) = u^4 L^4 \left[ 1 - A \log u - B \log^2 u \right], \quad g(u) = 1 - \frac{u^4}{u_h^4}$$

$$r_h = \frac{1}{u_h}, \quad A \approx \frac{3g_sM^2}{4\pi N} \left( 1 + \frac{3g_sN_f}{4\pi} \right), \quad B \approx \frac{9g_s^2N_fM^2}{16\pi^2N}$$ (3.101)

We will now solve (3.96) order by order in $g_sN_f$ and $g_sM^2/N$. At zeroth order in $g_sN_f, g_sM^2/N$, the warp factor becomes $h(u) = L^4 u^4$ and the metric (3.101) reduces to that of AdS$_5$. Hence at zeroth order in $g_sN_f, g_sM^2/N$ our analysis will be similar to the AdS/CFT calculations [45] but with certain crucial differences that we will elaborate below. As (3.96) is a second order non linear partial differential equation, we can solve it by Fourier decomposing $x, y, z, t$ dependence of $l_{\mu\nu}$ and writing it as a Taylor series in $u$. To start off then let us decompose $l_{\mu\nu}$ as:

$$l_{\mu\nu} = l_{\mu\nu}^{[0]} + l_{\mu\nu}^{[1]}$$ (3.102)

where the subscript $[0], [1]$ refer to the zeroth and the first order in $(g_sN_f, g_sM^2/N)$. To express the zeroth order fluctuation $l_{\mu\nu}^{[0]}$ in a Fourier series one has to be careful about the fact that our underlying space is not flat. In a curved space the Fourier series is expressed in terms of the corresponding harmonic forms in the space. These forms satisfy a Klein-Gordon type equation that one could easily determine for a given choice of the background metrics. Although in general the spatial parts of
these forms could be complicated, the temporal part however is always of the form $e^{-i\omega \tau}$ where $\tau$ is the time in the curved space. On the other hand, once we take the radial coordinate very large i.e $r = r_c \rightarrow \infty$, then we can take the harmonic forms on the boundary to be approximately wave like with $\tau$ defined as:

$$\tau = \sqrt{g(u_c)} \ t$$  \hspace{1cm} (3.103)

Of course this is not a generic picture at all points away from the boundary, but will nevertheless suffice for us because we will eventually address the theory on the boundary only. Such a procedure will help us extract the universal features of the cascading theory. For all other theories that we will define by specifying UV degrees of freedom will all inherit some aspects of the universal features (plus additional dependences on the UV degrees of freedom). Therefore in this limit, the zeroth order can be succinctly presented as a Fourier series in the following way:

$$l^{[0]}_{\mu\nu}(t, u, x, y, z) = \sum_{k=0}^{\infty} \int \frac{d^3q}{(2\pi)^4} \sqrt{g(u_c)} \left[ e^{-i(\omega\sqrt{g(u_c)} - q_1 x - q_2 y - q_3 z)} \tilde{s}^{(k)[0]}_{\mu\nu}(\omega, q_1, q_2, q_3) u^k \right]$$  \hspace{1cm} (3.104)

where $\tilde{s}^{(k)[0]}_{\mu\nu}$ are expansion coefficients of the solution $l^{[0]}_{\mu\nu}$. The precise deviations from the wave-like behavior for $l^{[0]}_{\mu\nu}$ will not be very relevant for the present analysis.

Similarly, we can also write the source in Fourier space as:

$$T^{\text{string}}_{\mu\nu} = T^{[0]\text{string}}_{\mu\nu} + T^{[1]\text{string}}_{\mu\nu}$$  \hspace{1cm} (3.105)

where as before, $[0, 1]$ refer to the zeroth and first orders in $(g_s N_f, g_s M^2/N)$ respectively. The zeroth order can then be written as:

$$T^{[0]\text{string}}_{\mu\nu}(t, u, x, y, z) = \int \frac{d^3q}{(2\pi)^4} \sqrt{g(u_c)} \ e^{-i(\omega\sqrt{g(u_c)} - q_1 x - q_2 y - q_3 z)} t^{[0]}_{\mu\nu}(\omega, u, q_1, q_2, q_3)$$  \hspace{1cm} (3.106)

where $t^{[0]}_{\mu\nu}$ are expansion coefficients of source $T^{[0]\text{string}}_{\mu\nu}$ at zeroth order in $g_s N_f, g_s M^2/N$. These coefficients are obtained by using explicit expressions for $T^{\text{string}}_{\mu\nu}(x^\mu)$ that can be extracted from (3.41). One may also note that since $g$ is defined at $r = r_c$, we will henceforth be analysing the theory there with appropriate degrees of freedom to be added later so as to have the full UV description. As mentioned before, this theory will correspond to a certain gauge theory with a RG flow that would inherit some properties of the $r_c \rightarrow \infty$ cascading theory, but in general will be different from the parent cascading theory (see figure 10)\(^{40}\).

\(^{40}\)Note however that even if these theories differ significantly in the UV, they all confine in the far IR at zero temperature. This is illustrated in figure 10 by the existence of $r_{\text{min}}$ for both the gravity duals.
Figure 10: The full ten-dimensional picture of the gravity dual. The figure on the left is the actual dual to the cascading theory with infinite (gauge theory) degrees of freedom (dof) at the boundary. The figure on the right is the one with a cutoff at \( r = r_c \) where we add a finite (but large) number of dof to define the UV of that theory. Once the UV dof are specified we can describe this theory at the boundary. The IR physics changes a bit by inclusion of irrelevant operators. Clearly the gravity dual on the right can capture only some universal properties of the parent theory (on the left). Nevertheless the theory on the right is an interesting theory (with good UV behavior) that can be studied directly from our analysis. The RG flows of both these theories have been discussed earlier. The existence of \( r_{min} \) signify the confining nature of these theories in the far IR at zero temperature. At high temperature we should view the blackhole covering the region near \( r_{min} \).

One may also analyse similarly the metric perturbation at linear order in \( g_s N_f \) and \( g_s M^2/N \) where it is easiest to switch to coordinate \( u = \frac{1}{r_c(1-\zeta)} \), so that the entire manifold (from \( r = 0 \) to \( r = r_c \)) is now described by \( 0 \leq \zeta \leq 1 \) with \( r_c \) arbitrarily large and we can get a meaningful Taylor series expansion of the logarithms and other functions appearing in the equation. As we did for the zeroth order cases, we can decompose the first order in \( g_s N_f, g_s M^2/N \) fluctuations via the following Fourier series:

\[
\ell_{\mu
u}^{[1]}(t, \zeta, x, y, z) = \sum_{k=0}^{\infty} \int \frac{d^3q d\omega}{(2\pi)^4 \sqrt{g(u_c)}} e^{-i(\omega \sqrt{g} t - q_1 x - q_2 y - q_3 z)} \tilde{s}_{\mu\nu}^{(k)[1]}(\omega, q_1, q_2, q_3) \zeta^k
\]

where \( \tilde{s}_{\mu\nu}^{(k)[1]} \) are the corresponding Fourier modes. These Fourier modes will eventually appear in the final equations for \( \ell_{\mu\nu}^{[1]} \).

However, as we discussed above, these modes are not all independent. There are only five independent metric fluctuations which could be written as some linear combinations of the ten components (3.100). The zeroth and first order independent
fluctuations can then be expressed as:

\[
L^{[0]}_{mn} = \sum_{n=1}^{10} a_{mn} l^{[0]}_n = \sum_{k=0}^{\infty} \frac{d^3 q d\omega}{(2\pi)^4 \sqrt{g(u_c)}} \left[ e^{-i(\omega \sqrt{\mathcal{H}} \cdot q) \cdot \tau} s^{[0]}_{mn}(\omega, \tau) u^k \right]
\]

\[
L^{[1]}_{mn} = \sum_{n=1}^{10} b_{mn} l^{[1]}_n = \sum_{k=0}^{\infty} \frac{d^3 q d\omega}{(2\pi)^4 \sqrt{g(u_c)}} \left[ e^{-i(\omega \sqrt{\mathcal{H}} \cdot q) \cdot \tau} s^{[1]}_{mn}(\omega, \tau) \zeta^k \right]
\]

where \(a_{mn}\) and \(b_{mn}\) are functions of all the coordinates in general with \(m\) running from 1 to 5. Notice that the way we are expressing the independent modes is different from the way it is presented in [45]. It should also be clear that

\[
s^{[k]}_{mn} = a_m s^{[k]}_{00} + a_{m2} s^{[k]}_{01} + a_{m3} s^{[k]}_{02} + a_{m4} s^{[k]}_{11} + \ldots
\]

\[
s^{[k]}_{mn} = b_m s^{[k]}_{00} + b_{m2} s^{[k]}_{01} + b_{m3} s^{[k]}_{02} + b_{m4} s^{[k]}_{11} + \ldots
\]

Since there are five independent metric components, we expect that there should also be five independent sources. These sources, coming from the string, should again be some linear combination of the ten possible components. We can write them as \(\tilde{T}^{[0]}_{mn}\) and \(\tilde{T}^{[1]}_{mn}\) to denote the zeroth and first orders in \((g_s N_f, g_s M^2 / N)\) respectively. The above considerations also imply that the operator equation (3.96) will become:

\[
\kappa 5 \sum_{n=1}^{5} \Box_{mn} L_{mn} = \tilde{T}_{mn}
\]

where the form of \(\Box_{mn}\) can be easily derived from (3.96).

Knowing the metric perturbations \(L_{mn}\), we can now compute the effective supergravity action in five dimensions and use this action to compute the four dimensional boundary action. Recall that our space is a deformation of the AdS space (3.91) and from all our previous discussions, we need to be careful when we want to give a boundary description. One of the simplest boundary description that appears naturally in this framework is the one with an infinite degrees of freedom. The other alternative descriptions are defined through the cutoffs imposed at various \(r_c\). All these infinite possible descriptions may have finite (but very large) UV degrees of freedom and flow to \(N_{\text{eff}}\) degrees of freedom at \(r = r_c\) (see (3.21)). Once we specify this we expect that the boundary will capture the UV of the corresponding gauge theory. In the limit where \(N >> M\) one can even compute the spectrum of the operators from the boundary description (see for example [37] for a zero-temperature example without flavor). To describe the boundary action we will in principle take a “functional derivative” of our action with respect to the perturbation \(L_{mn}\) – meaning that we will extract only the linear coefficient of the perturbation \(L_{mn}\) – obtained by a fixing a value for the radial coordinate (\(u=\text{fixed or } \zeta=\text{fixed}\)). Using the boundary action our aim then is to compute (3.64) which is the wake left behind by a fast moving quark.
For comparison with AdS/CFT result, a simple way would be to split the five dimensional effective action into two parts:

\[ S_{\text{total}}^{\text{eff}} = S_{\text{total}}^{\text{eff(AdS+string)}} + S_{\text{total}}^{\text{eff(OKS+run)}} \]  

(3.111)

where the first part is the vanilla AdS with a quark string and the second part is the deformation that captures the running of the gauge theory in the dual picture. The AdS-string part is measured at zeroth order in \( g_s N_f \) and \( g_s M^2 / N \) whereas the OKS-run part is measured at first order in \( g_s N_f \) and \( g_s M^2 / N \). It should also be clear that in the limit \( (g_s N_f, g_s M^2 / N) \rightarrow 0 \), our geometry is \( AdS_5 \times T^{11} \) Black-Hole with back reaction from the string\(^{41}\). More concretely, for the AdS-string part we will use \( L^{[0]}_{\text{mm}} \) as our metric perturbation computed using the zeroth order energy momentum tensor of the string (3.105) whereas for the OKS-run part we will use \( L^{[1]}_{\text{mm}} \) as the metric perturbation coming from the first order energy momentum tensor of the string (3.105). To avoid cluttering of formulae, we will henceforth label:

\[ S_{\text{total}}^{\text{eff(AdS+string)}} \equiv S^{(1)}, \quad S_{\text{total}}^{\text{eff(OKS+run)}} \equiv S^{(2)} \]  

(3.112)

Thus we can represent the wake of the quark by the following formal expressions\(^{42}\):

\[
\langle T^{mn} \rangle_{\text{wake}} = \lim_{u \rightarrow 0} \frac{\delta^{mn}}{\kappa} \cdot \frac{\delta b S_{\text{total}}^{\text{eff}}} {\delta b L_{mn}} - T^{mn}_{\text{quark}} \\
= \lim_{u \rightarrow 0} \frac{\delta^{mn}}{\kappa} \left( \frac{\delta b S^{(1)}} {\delta b L^{[0]}_{\text{mm}}} + \frac{\delta b S^{(2)}} {\delta b L^{[1]}_{\text{mm}}} \right) - T^{mn}_{\text{quark}}
\]  

(3.113)

which would give us the result for the UV of the dual gauge theory because \( \zeta = 0 \) would take us to the boundary of our space\(^{43}\), and the operation \( \frac{\delta b S^{(i)}} {\delta b L^{[i]}_{\text{mm}}} \) extracts the linear coefficient from the boundary action. The energy momentum tensor of the quark can then be written as:

\[
T^{mn}_{\text{quark}}(x, y, z, t) = m(T) U^m U^n \sqrt{1 - v^2} \delta^3(x - \vec{v} \cdot t)
\]  

(3.114)

where \( U^m = (U^0, \vec{v}) \) and \( U^0 \) is the energy of the quark and \( \vec{v} \) is the three-velocity. Now at zeroth order, as mentioned before, the five dimensional effective action that we have is that of \( AdS_5 \) plus back reaction of the string. For an infinitely massive

\(^{41}\)We will discuss the back reaction from the D7 brane in the next section when we compute the shear viscosity. In this section we will only consider the back reaction of the string on the geometry.

\(^{42}\)Although the energy momentum tensor in the following equation may seem diagonal in terms of \( L_{mn} \) variables, they are in fact not diagonal in terms of the \( l_{\mu\nu} \) variables. We simply find it convenient to express \( T^{mn} \) in terms of the five independent metric fluctuations. Furthermore both terms of (3.113) are restricted to four dimensional space-time.

\(^{43}\)To be more precise \( \zeta = 0 \) takes us to \( r = r_c \); and then once we add appropriate UV degrees of freedom we can define our theory at the boundary \( r \rightarrow \infty \).
string i.e. \( u_0 \equiv \frac{1}{r_0} = 0.44 \) just like in [45], using exactly the regularisation procedure of [13], [14], [15] and [16], the stress tensor evaluated at \( u = 0 \) is known to be:

\[
\langle T^{mn} \rangle_{\text{wake(AdS)}} = \left( \lim_{u \to 0} \frac{\delta^{mn}}{\kappa} \cdot \frac{\delta_b S^{(1)}}{\delta_b L_{[0]}^{0}} \right)_{\text{FT}} - \hat{T}_{\text{quark}}^{mn}(\omega, \vec{q})
\]

\[
= \int \frac{d^4q}{(2\pi)^4} c_o \delta^{mn} \delta_{r,q}^{[1](4)}(\omega, \vec{q}) - \hat{T}_{\text{quark}}^{mn}(\omega, \vec{q})
\]

where FT is the Fourier transform and \( \hat{T}_{\text{quark}}^{mn}(\omega, \vec{q}) \) are the quark energy-momentum Fourier modes and the value of \( c_o \) will be discussed later. Note however that for a finite length string i.e. the string stretching from \( u_0 \equiv |\mu|^{-2/3} \equiv 1 \) to \( u_h \) the metric perturbation due to the string goes to zero for \( u < u_0 \). This means that the previous analysis done in [45] will not be valid for our case and we would require to regularise our effective action. This is where it may be more convenient to regularise the system to \( \mathcal{O}(g_s N_f, g_s M^2/N) \) without splitting up the action.

In the following therefore we will give a brief discussion of our regularisation scheme using the warp factor (3.35). We will start by analysing the background string configuration by plugging in the \( l_{\mu\nu} \) in the Einstein as well as the flux terms of the five dimensional action. Once the action is expressed in terms of \( l_{\mu\nu} \) we expect that it will be equivalently rewritten in terms of \( L_{\mu\nu\rho\sigma} \). An example of this is given by [45], and we provide another example in Appendix C. Once the dust settles, the result in the most symmetric form is given by:

\[
\mathcal{S}^{(1)}[\Phi] = \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \int dr \left\{ \frac{1}{2} A_1^{mn}(r, q) \left[ \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) + \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) \right] + B_1^{mn}(r, q) \left[ \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) + \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) \right] + C_1^{mn}(r, q) \left[ \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) + \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) \right] + D_1^{mn}(r, q) \phi_m^{[1]}(r, q) \phi_n^{[1]}(r, -q) + T_1^{mn}(r, q) \phi_m^{[1]}(r, q) + E_1^{mn} \phi_m^{[1]}(r, q) + F_1^{mn} \phi_m^{[1]}(r, q) \right\}
\]

(3.116)

where \( m, n = 1, ..., 5 \), prime denotes differentiation with respect to \( r \),\(^{45}\); the script [1] denote the total background to \( \mathcal{O}(g_s N_f, g_s M^2/N) \); and the explicit expressions for \( A_1^{mn}, B_1^{mn}, C_1^{mn}, F_1^{mn} \) for a specific case are given in Appendix D. We have also defined \( \phi_m^{[1]}(r, q) \) in the following way (with \( q_0 \equiv \omega \sqrt{g(r_c)} \) as before):

\[
\phi_m^{[1]}(r, q) = \int \frac{d^4x}{(2\pi)^4 \sqrt{g(r_c)}} e^{i(q_0 t - q_1 x - q_2 y - q_3 z)} L_{mn}(t, r, x, y, z)
\]

(3.117)

\(^{44}\)Recall that for our case the string ended at a finite distance \( u_0 = |\mu|^{-2/3} \) (see (3.49) for details). Once the string is infinite, we are effectively putting the D7 brane at infinity i.e \( u = u_0 = 0 \).

\(^{45}\)This is for simplicity. Prime could be more generic and denote derivatives wrt \( r \) as well as \( \vec{q} \). We will rectify this below. Notice also that we have shifted to \( r \) variable, but the action could equivalently be expressed by \( u \) variable.
We will see that the effective four dimensional boundary action is independent of $D_1^{mn}$ and $T_1^m$ and hence we do not list their explicit expressions in appendix. Furthermore, note that the derivative terms in (3.116) all come exclusively from $\sqrt{-\mathcal{G}} R$, whereas fluxes contribute powers of $\Phi^{[1]}$ but no derivative interactions (this is also one of the reasons why F-theory seven branes will not change our conclusions provided we ignore the anomalous gravitational couplings (3.156) or (3.92)). In the following we keep upto quadratic orders, and therefore the contributions from the fluxes will appear in $D_1^{mn}$ and $E_1^m$. Taking all these affects into account we compute the coefficients $A_1^{mn}, B_1^{mn}, C_1^{mn}, D_1^{mn}, E_1^m$, and $F_1^m$ for a specific example, whose values are given in Appendix D. However, as we mentioned before, we will be ignoring the back reaction of the string on the fluxes because we expect any additional components of fluxes coming from the string to be very small\textsuperscript{46}. The equation of motion for $\Phi^{[1]}(r, -q)$ is given by:

\[
\frac{1}{2} \left[ A_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) \right]'' - \left[ B_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) \right]' - \frac{1}{2} \left[ C_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) \right]' + D_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) + \frac{1}{2} A_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) - \frac{1}{2} C_1^{mn}(r, q) \Phi^{[1]}_n(r, -q) + T_1^m(r, q) - E_1^m(r, q) + F_1^{mn}(r, q) = 0 \tag{3.118}
\]

The next few steps are rather standard and so we will quote the results. The variation of the action (3.119) can be written in terms of the variations $\delta\Phi^{[1]}_n(r, q)$ and $\delta\Phi^{[1]}_n(r, -q)$ in the following way\textsuperscript{47}:

\[
\delta S^{(1)} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \sqrt{g(r_c)} \int_{r_h}^{r_c} dr \left\{ \left[ (A_1^{mn} \Phi^{[1]}_m)^'' - (2B_1^{mn} \Phi^{[1]}_m)' + C_1^{mn} \Phi^{[1]}_m + 2D_1^{mn} \Phi^{[1]}_m \right] \right.
\]

\[
+ A_1^{mn} \Phi^{[1]}_m - (C_1^{mn} \Phi^{[1]}_m)' \right) \delta\Phi^{[1]}_n + \left[ (A_1^{mn} \Phi^{[1]}_m)' - (2B_1^{mn} \Phi^{[1]}_m)' + C_1^{mn} \Phi^{[1]}_m + 2D_1^{mn} \Phi^{[1]}_m \right] \right.
\]

\[
+ A_1^{mn} \Phi^{[1]}_m - (C_1^{mn} \Phi^{[1]}_m)' \right) \delta\Phi^{[1]}_n + 2 \left( T_1^m - E_1^m + F_1^{mn} \right) \delta\Phi^{[1]}_m
\]

\[
\partial_r \left[ A_1^{mn} \Phi^{[1]}_m \delta\Phi^{[1]}_n - (A_1^{mn} \Phi^{[1]}_m)' \delta\Phi^{[1]}_n + 2B_1^{mn} \Phi^{[1]}_m \delta\Phi^{[1]}_n + C_1^{mn} \Phi_m \delta\Phi^{[1]}_n + 2B_1^{mn} \Phi^{[1]}_m \delta\Phi^{[1]}_n \right.
\]

\[
+ C_1^{mn} \Phi^{[1]}_m \delta\Phi^{[1]}_n + 2E_1^m \delta\Phi^{[1]}_m + 2F_1^m \delta\Phi^{[1]}_m - 2F_1^m \delta\Phi^{[1]}_m + A_1^{mn} \Phi^{[1]}_m \delta\Phi^{[1]}_n - (A_1^{mn} \Phi^{[1]}_m)' \delta\Phi^{[1]}_n \right\} \tag{3.119}
\]

\textsuperscript{46}This is easy to motivate from the fact that the size of our quark string is very small. One end of the string goes into the horizon of the black hole and the other end is on the D7 brane placed at $|\mu|^2/3 \equiv 1$. Thus the $B_{NS}$ sources from the string will be negligible. Furthermore fundamental string will not effect any of the background RR forms, at least in the limit (3.50) that we are considering. So it makes sense to ignore the effects of the string on fluxes. See section 3.2 for more details.

\textsuperscript{47}Henceforth, unless mentioned otherwise, $\Phi^{[1]}_m, \Phi^{[1]}_n$ will always mean $\Phi^{[1]}_m(r, q)$ and $\Phi^{[1]}_n(r, -q)$ respectively. Similar definitions go for the variations $\delta\Phi^{[1]}_m$ and $\delta\Phi^{[1]}_n$. 


which includes the equations of motion as well as the boundary term. We can then write the variation of the action $\delta S^{(1)}$ in the following way:

$$
\delta S^{(1)} = \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \left\{ \int_{r_h}^{r_c} dr \left[ (\text{EOM for } \Phi_n^{[1]} \delta \Phi_n^{[1]} + (\text{EOM for } \Phi_m^{[1]} \delta \Phi_m^{[1]} \right]ight.
$$

$$
+ \frac{1}{2} \left[ (2B_1^{mn} - A_1^{mn}) (\Phi_n^{[1]} \delta \Phi_n^{[1]} + \Phi_m^{[1]} \delta \Phi_m^{[1]} \right) + (C_1^{mn} - A_1^{mn}) (\Phi_n^{[1]} \delta \Phi_n^{[1]} + \Phi_m^{[1]} \delta \Phi_m^{[1]} \right)
$$

$$
+ 2(E_1^{nm} - F_1^{nm}) \delta \Phi_m^{[1]} + A_1^{mn} \Phi_n^{[1]} \delta \Phi_n^{[1]} + A_1^{mn} \delta \Phi_n^{[1]} \delta \Phi_m^{[1]} + 2F_1^{nm} \delta \Phi_n^{[1]} \right]_{\text{boundary}} \right\}
$$

(3.120)

where by an abuse of notation by the “boundary” here, and the next couple of pages (unless mentioned otherwise), we mean that the functions are all measured at $r_h$ and $r_c$ i.e the horizon and the cut-off respectively. It is now easy to see why $D_1^{mn}$, $T_1^{mn}$ and $F_0^{mn}$ etc do not appear in the boundary action. Finally, we need to add another boundary term to (3.120) to cancel of the term proportional to $\delta \Phi_n'$. This is precisely the Gibbons-Hawking term:

$$
\mathcal{K}_1 = -\frac{1}{2} \left( \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \left( A_1^{mn} \Phi_n^{[1]} \Phi_m^{[1]} + A_1^{mn} \Phi_n^{[1]} \Phi_m^{[1]} + 2F_1^{nm} \Phi_n^{[1]} \right) \right)_{\text{boundary}}
$$

(3.121)

Taking the variation of (3.121) $\delta \mathcal{K}_1$ we get terms proportional to $\delta \Phi_n^{[1]}$ as well as $\delta \Phi_m^{[1]}$. Adding $\delta \mathcal{K}_1$ to $\delta S^{(1)}$ we can get rid of all the $\delta \Phi_n^{[1]}$ terms from (3.121). This means we can alternately state that the boundary theory should have the following constraints:

$$
\delta \Phi_m^{[1]}(r_c, q) = \delta \Phi_n^{[1]}(r_c, -q) = 0
$$

(3.122)

Next, we can make a slight modification of the boundary integral (3.119) by specifying the coordinate $r$ to be from $r_h$ to certain $r_{\text{max}} = r_c(1 - \zeta)$ instead of just $r_c$ specified earlier. For different choices of $\zeta$ we can allow different UV completions at the boundary such that their degrees of freedom would match with the parent cascading theory only at $r_c(1 - \zeta)$, while $r_h$ would be associated with the characteristic temperature $T$ of the parent cascading theory.

Note that we have not carefully described the degrees of freedom at the boundary as yet. For large enough $r_c$ we expect large degrees of freedom at the UV because of (3.23). This would mean that the contributions to various gauge theories from these degrees of freedom would go like $e^{-N_c \alpha}$, which would be negligible. Thus unless we cut-off the geometry at $r = r_c$ and add UV caps with specified degrees of freedom we are in principle only describing the parent cascading theory. For this theory of course $N_c^{\text{eff}}$ is infinite at the boundary, which amounts to saying that UV degrees of freedom don’t contribute anything here. We will, however, give a more precise description a little later.

One can impose similar constraints at the horizon also.
With all the above considerations we can present our final result for the boundary action. Putting the equations of motion constraints on \((3.120)\), as well as the derivative constraints \((3.122)\), we can show that the variation \((3.120)\) can come from the following boundary 3+1 dimensional action:

\[
S^{(1)} = \int \frac{d^4q}{(2\pi)^4\sqrt{g(r_{\text{max}})}} \left\{ C_1^{mn}(r, q) - A_1^{mn}(r, q) \right\} \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, q) \\
+ \left[ B_1^{mn}(r, q) - A_1^{mn}(r, q) \right] \left[ \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, q) + \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, q) \right] \\
+ \left( E_1^m - F_1^m \right) \Phi_m^{[1]}(r, q) \right\}_{r \rightarrow r_{h}}^{r_{c}(1-c)} \tag{3.123}
\]

However the above action diverges, as one can easily check from the explicit expressions for \(A_1^{mn}, B_1^{mn}, C_1^{mn}, E_1^m\) and \(F_1^m\) for the specific case worked in Appendix D. Indeed, comparing it to the known AdS results, we observe that from the boundary there are terms proportional to \(r_c^4\) in each of \(C_1^{mn}, A_1^{mn}, E_1^m\) and \(F_1^m\) and proportional to \(r_c^5\) in \(A_1^{mn}, B_1^{mn}\). As \(r_c \rightarrow \infty\) the action diverges so as it stands \(r_c\) cannot completely specify the UV degrees of freedom at the QCD scale \(\Lambda_c\). Thus we need to regularise/renormalise it before taking functional derivative of it. This renormalisation procedure will give us a finite boundary theory from which one could get meaningful results of the dual gauge theory.

Once we express the warp factors in terms of power series in \(r_{(\alpha)}\) the renormalisability procedure becomes much simpler. Note however that this renormalisation is only in classical sense, as the procedure will involve removing the infinities in \((3.123)\) by adding counter-terms to it. Comparing with the known AdS results, and the specific example presented in Appendix C, one can argue that the infinities in \((3.123)\) arise from the following three sources:

1. \(C_1^{mn}(r_c, q) - A_1^{mn}(r_c, q) = \sum_\alpha H_{[\alpha]}^{mn}(q) r_{c(\alpha)}^4 + \text{finite terms}\)

2. \(B_1^{mn}(r_c, q) - A_1^{mn}(r_c, q) = \sum_\alpha K_{[\alpha]}^{mn}(q) r_{c(\alpha)}^5 + \text{finite terms}\)

3. \(E_1^m(r_c, q) - F_1^m(r_c, q) = \sum_\alpha I_{[\alpha]}^m(q) r_{c(\alpha)}^4 + \text{finite terms} \tag{3.124}\)

where in the above expressions we are keeping \(\alpha\) arbitrary so that it can in general take both positive and negative values; and the finite terms above are of the form \(r_{c(\alpha)}^{-n}\) with \(n \geq 1\). Therefore to regularise, first we write the metric perturbation also

---

50 We are a little sloppy here. Of course our theory is large \(N\) QCD like only in the far IR. Nevertheless \(r_c\) should specify the UV degrees of freedom as \((3.23)\) at scale \(\Lambda_c\) for the resulting theory.
as a series in $1/r(\alpha)$:

$$
\Phi^{(1)}_n = \sum_{k=0}^{\infty} \sum_{\alpha} \frac{s_{nm}^{(k)\alpha}}{r^{k}(\alpha)} \theta(r_0 - r)
$$

where the above relation could be easily derived using (3.117), (3.108) taking the background warp factor correctly; and $r_0 = |\mu|^{2/3}$ is given by (3.49).

Plugging in (3.123) and (3.124) in (3.123) we can easily extract the divergent parts of the action (3.123). Thus the counter-terms are given by:

$$
S^{(1)}_{\text{counter}} = \int \frac{d^4q}{(2\pi)^4} \sqrt{g(r_{\text{max}})} \sum_{\alpha,\beta,\gamma} \left\{ H^{mn}_{[\alpha]} \theta(r_0 - r) \left[ s_{mm}^{(0)\beta} s_{nn}^{(0)\gamma} r^4(\alpha) + \left(s_{mm}^{(1)\beta} s_{nn}^{(0)\gamma} r^3(\alpha) + r^2(\alpha) \right) \right] 
+ s_{mn}^{(0)\beta} s_{nn}^{(1)\gamma} r^3(\alpha) + \right. 
+ \left( s_{mn}^{(2)\beta} s_{nn}^{(0)\gamma} r^2(\alpha_2) \right) + \left( s_{mn}^{(3)\beta} s_{nn}^{(0)\gamma} r(\alpha_2) \right) + s_{mn}^{(0)\beta} s_{nn}^{(2)\gamma} r^2(\alpha_2) + s_{mn}^{(0)\beta} s_{nn}^{(3)\gamma} r^2(\alpha_2) + s_{mn}^{(0)\beta} s_{nn}^{(4)\gamma} r^2(\alpha_2) + s_{mn}^{(0)\beta} s_{nn}^{(5)\gamma} r^2(\alpha_2) + 
+ K^{mn}_{[\alpha]} \theta(r_0 - r) \left[ r^2(\alpha_{10}) + r^2(\alpha_{11}) + r^2(\alpha_{12}) + r^2(\alpha_{13}) + r^2(\alpha_{14}) + r^2(\alpha_{15}) + r^2(\alpha_{16}) + r^2(\alpha_{17}) + r^2(\alpha_{18}) + r^2(\alpha_{19}) + r^2(\alpha_{20}) + 
+ I^m_{[\alpha]} \theta(r_0 - r) \left[ r^4(\alpha) + r^3(\alpha) + r^2(\alpha) + r^1(\alpha) \right] \right) \right\}
$$

(3.126)

with an equal set of terms with $r(-\alpha)$. In the above expression $r(\alpha) \equiv r^{1-\epsilon(\alpha)}$; and as before, the integrand is defined at the horizon $r_h$ and the cutoff $r_c(1-\zeta)$; with the string stretching between $r_h$ and $r_0$. The other variables namely, $s_{mn}^{(k)\beta}, H^{mn}_{[\alpha]}, K^{mn}_{[\alpha]}$ and $I^m_{[\alpha]}$ are independent of $r$ but functions of $q^i$. For one specific case their values are given in Appendix C. Finally the $\epsilon(\alpha_i)$ can be defined by the following procedure. Let's start with the expression:

$$
H^{mn}_{[\alpha]} s_{mm}^{(a)\beta} s_{nn}^{(b)\gamma} r^p_{(\alpha)} \equiv H^{mn}_{[\alpha]} s_{mm}^{(a)\beta} s_{nn}^{(b)\gamma} \frac{r^p_{(\alpha)}}{r^p_{(\beta)} r^p_{(\gamma)}}
$$

$$
K^{mn}_{[\alpha]} s_{nn}^{(c)\beta} s_{mm}^{(d)\gamma} r^q_{(\alpha)} \equiv K^{mn}_{[\alpha]} s_{nn}^{(c)\beta} s_{mm}^{(d)\gamma} \frac{r^q_{(\alpha)}}{r^q_{(\beta)} r^q_{(\gamma)}}
$$

(3.127)

from where one can easily infer:

$$
p = 4 - a - b, \quad \epsilon(\alpha_k) = \frac{4\epsilon(\alpha) - a\epsilon(\beta) - b\epsilon(\gamma)}{4 - a - b}
$$

$$
q = 5 - c - d, \quad \epsilon(\alpha_k) = \frac{5\epsilon(\alpha) - c\epsilon(\beta) - d\epsilon(\gamma)}{5 - c - d}
$$

(3.128)

Using this procedure we can determine all the $r(\alpha)$ in the counterterm expression (3.126).
At this point the analysis of the theory falls into two possible classes.

- The first class is to analyse the theory right at the usual boundary where \( r_c \to \infty \). This is the standard picture where there are infinite degrees of freedom at the boundary, and the theory has a smooth RG flow from UV to IR till it confines (at least from the weakly coupled gravity dual).

- The second class is to analyse the theory by specifying the degrees of freedom at generic \( r_c \) and then defining the theories at the boundary. All these theories would meet the cascading theory at certain scales under RG flows. The gravity duals of these theories are the usual deformed conifold geometries cutoff at various \( r_c \) with appropriate UV caps added (of course for \( r < r_c \) the geometries change accordingly).

The former is more relevant for the pure AdS/CFT case whereas the latter is more relevant for the present case.

For the pure AdS/CFT case without flavors \( \epsilon_{(\alpha_i)} = 0 \) (so that the subscript \( \alpha_i \)'s can be ignored from all variables), we can subtract the counter-terms (3.126) from the action (3.123) to get the following renormalised action:

\[
S^{\text{ren}} = S^{(1)} - S^{\text{counter}} = \int \frac{d^4q}{(2\pi)^4} \theta(r_0 - r) \left[ H^{mn} \left( s^{(4)}_{mm} s^{(0)}_{nn} + s^{(3)}_{mm} s^{(1)}_{nn} + s^{(2)}_{mm} s^{(2)}_{nn} + s^{(1)}_{mm} s^{(3)}_{nn} \right) \right.

\left. + s^{(0)}_{mm} s^{(4)}_{nn} - K^{mn} \left( 4s^{(0)}_{mm} s^{(4)}_{nn} + 3s^{(1)}_{mm} s^{(3)}_{nn} + 4s^{(2)}_{mm} s^{(2)}_{nn} + s^{(3)}_{mm} s^{(1)}_{nn} \right) \right.

\left. + 4s^{(0)}_{nn} s^{(4)}_{mm} + 3s^{(1)}_{nn} s^{(3)}_{mm} + s^{(3)}_{nn} s^{(1)}_{mm} \right] + I^m s^{(4)}_{mm} \right)
\]

(3.129)

where we have made all the \( \mathcal{O}(1/r_c) \) terms vanishing, and in the limit \( r_h \) small the small shifts to \( s^{(j)}_{nn} \) given by \( s^{(3)}_{nn} + \mathcal{O}(r_h^0) \) can also be ignored. Furthermore for more generic case where primes in (3.116) denote derivatives wrt \( r \) as well as \( \vec{q} \), we can reinterpret the renormalised action (3.123) as the following new action:

\[
S^{\text{ren}} = \int \frac{d^4q}{(2\pi)^4} \left[ Z^{mn} \Phi_m(q) \Phi_n(-q) + U^{mn} \left( \Phi_m(q) \Phi'_n(-q) + \Phi'_m(q) \Phi_n(-q) \right) \right.

\left. + Y^m \Phi_m(q) + Y^n \Phi_n(-q) + V^m \Phi'_m(q) + V^n \Phi'_n(-q) + X \right]
\]

(3.130)

where \( \Phi_m \) are now only functions of \( \pm q \) with prime denoting derivatives wrt \( \vec{q} \), and we isolated all the \( r \) dependences so that \( X, Y, Z, U, V \) could be functions of \( r \) and \( \vec{q} \). We can determine their functional form by comparing (3.130) with (3.129). For us however the most relevant part is the energy momentum tensors which we could determine from (3.130) by finding the coefficients \( Y^m \) and \( Y^n \). One can easily show

\[51\] In both cases of course we need to add appropriate number of seven branes to get the finite F-theory picture. The holographic renormalisation procedure remains unchanged and the far IR physics remains unaltered. The UV caps affect mostly geometries close to \( r_c \), as expected.
that, up to a possible additive constant, \( Y^m, Y^n \) are given by:

\[
Y^m = H^{mn} s^{(4)}_{nm} - 4K^{mn} s^{(4)}_{mn}, \quad Y^n = H^{mn} s^{(4)}_{nm} - 4K^{mn} s^{(4)}_{mm}
\]

from where the wake of the quark can be shown to be exactly given by \( S_1 \) mentioned earlier. We also note that \( c_0 = H^{mn} \theta(r_0 - r) \).

Now let us come to second class of theories wherein we take any arbitrary \( r = r_c \), with appropriate UV degrees of freedom such that they have good boundary descriptions satisfying all the necessary constraints. For these cases, once we subtract the counter-terms \((3.126)\), the renormalised action (specified by \( r \)) takes the following form:

\[
S^{(1)}_{\text{ren}} = S^{(1)} - S^{(1)}_{\text{counter}}
\]

\[
= \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha,\beta,\gamma} \theta(r_0 - r) \left\{ H^{mn}_{[\alpha]} \left( s_{nm}^{(4)} \right) r^{4\epsilon(\beta) + 4\epsilon(\alpha) + 2\epsilon(\gamma)} + \left( s_{mn}^{(3)} \right) r^{3\epsilon(\beta) + \epsilon(\alpha) + \epsilon(\gamma)} + \left( s_{mm}^{(0)} \right) r^{4\epsilon(\beta) + 4\epsilon(\alpha) + 2\epsilon(\gamma)} \right\}
\]

\[
defined at the cut-off and the horizon radii as usual. Notice now the appearance of \( r^{m\epsilon(\alpha) + n\epsilon(\beta) + p\epsilon(\gamma)} \) factors. One can easily show that:

\[
\frac{1}{2} \left[ r^{m\epsilon(\alpha) + n\epsilon(\beta) + p\epsilon(\gamma)} + r^{-m\epsilon(\alpha) - n\epsilon(\beta) - p\epsilon(\gamma)} \right] = 1 + O[\epsilon(\alpha, \beta, \gamma)]^2
\]

Since the warp factor \( h \) is defined only for small values of \( g_sN_f, g_sM^2/N, g_s^2N_fM^2/N \) we don’t know the background (and hence the warp factor) for finite values of these quantities. Therefore for our case we can put \( O[\epsilon(\alpha, \beta, \gamma)]^2 \) to zero so that the value in \((3.133)\) is identically 1. For finite values of these quantities both the warp factor and the background would change drastically and so new analysis need to be performed to holographically renormalise the theory. Our conjecture would be that once we know the background for finite values of \( g_sN_f, g_sM^2/N \), the terms like \((3.132)\) would come out automatically renormalised by choice of our counterterms. We will discuss this in some details later. More elaborate exposition will be given in the sequel \([53]\).

Once this is settled, the renormalised action at the cut-off radius \( r_c \) would only go as powers of \( r^{-1}_c(\alpha) \). Thus we can express the total action as:

\[
S^{(1)}_{\text{ren}} = \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha,\beta} \left\{ \left( \sum_{j=0}^{\infty} \frac{\tilde{c}^{(\alpha)}_{mn(j)}}{r_j^{(\alpha)}} \right) \tilde{G}^{mn} \Phi_m \Phi_n + \left( \sum_{j=0}^{\infty} \frac{\tilde{c}^{(\alpha)}_{mn(j)}}{r_j^{(\alpha)}} \right) \tilde{M}^{mn} (\Phi_m \Phi_n)
\]
\[ + \Phi' \Phi_n \right) + H_{(a)}^{mn} \left[ s^{(4)(\beta)}_{sm} \Phi_m + \frac{s^{(4)(\beta)}_{mn}}{\Phi_n} \right] + K_{(a)}^{mn} \left[ - 4s^{(4)(\beta)}_{nm} \Phi_m - 4s^{(4)(\beta)}_{mm} \Phi_n + \frac{s^{(5)(\beta)}_{nm}}{\Phi_m} \right. \\
+ \frac{s^{(5)(\beta)}_{mm}}{\Phi_n} \right] + \left( \sum_{j=0}^{\infty} \frac{\tilde{b}_{m(j)}}{r_{(a)}} \right) \tilde{J}_m \Phi_m + X[r_{(a)}] \left[ 1 - \frac{r_h^4}{r_c^4(1 - \zeta)^4} \right]^{-\frac{1}{2}} \theta(r_0 - r) \quad (3.134) \]

where \( \Phi_n \) are independent of \( r \) with prime denoting derivatives wrt \( \frac{1}{r} \) henceforth; and the radial coordinate is measured at the horizon \( r_h \) and the cutoff \( r_c(1 - \zeta) \) as before. The explicit expressions for the other coefficients listed above, namely, \( \tilde{G}^{mn}, \tilde{M}^{mn}, \tilde{a}_{(a)}, \tilde{J}_m, \tilde{b}_{m(j)} \) and \( \tilde{b}_{m(j)} \) can be worked out easily from our earlier analysis (see Appendix C for one specific example). Note that \( X[r_{(a)}] \) is a function independent of \( \Phi_m \) and appears for generic renormalised action.

Now the generic form for the energy momentum tensor is evident from looking at the linear terms in the above action \((3.134)\). This is then given by:

\[
T^{mm}_0 = \int \frac{d^4q}{(2\pi)^4} \left[ H_{(a)}^{mn} + H_{(a)}^{nm} s^{(4)(\beta)}_{mn} - 4(K_{(a)}^{mn} + K_{(a)}^{nm}) s^{(4)(\beta)}_{mn} + (K_{(a)}^{mn} + K_{(a)}^{nm}) s^{(5)(\beta)}_{mn} \right. \\
+ \left. \left( \sum_{j=0}^{\infty} \frac{\tilde{b}_{m(j)}}{r_{(a)}} \right) \tilde{J}_m \delta_{nm} \right] \left( 1 - \frac{r_h^4}{r_c^4(1 - \zeta)^4} \right)^{-\frac{1}{2}} \theta(r_0 - r_c) \quad (3.135) \]

at \( r = r_c \) (we ignore the result at the horizon) and sum over \((\alpha, \beta)\) is implied. This result should be compared to the ones derived in [45] [13] [14] [15] [16] which doesn’t have any \( r_c \) dependence. This is of course the first line of the above result. For the case studied in [43] [13] [14] [15] [16] the boundary theory is defined with infinite degrees of freedom at UV (which we have been calling as the parent cascading theory). How do we then reproduce the results of those papers? Before we go about elucidating this, notice that the second line also has a \( r_c \) independent additive constant. This additive term is irrelevant for our purpose because the energy-momentum can always be shifted by a constant to absorb this factor. Thus once we specify the cutoff \( r_c \) and the UV degrees of freedom, then our result shows that the energy-momentum tensor not only inherits the universal behavior of the parent cascading theory but there are additional corrections coming precisely from the added UV degrees of freedom at \( r = r_c \). These corrections go as \( r_c^{-1} \) or as \( e^{-\mathcal{N}_{\text{eff}}} \) with \( \mathcal{N}_{\text{eff}} \) being the UV degrees of freedom at the cut-off (see \((3.23)\)). As long as these UV degrees of freedom are not infinite, they contribute to small corrections to the energy-momentum tensors of the various gauge theories.

The above description is one of the key points of our paper, and tells us how we can distinguish our results from the standard AdS/QCD answers. Therefore let us elaborate this in little more details. This will justify what we have been saying so far about UV caps, and put everything in a rigorous mathematical framework at least from the action point-of-view. In the process we will also be able to reconcile with the results of [13] [14] [15] [16]. The first important issue here is that we can study
infinite number of UV completed theories in our full F-theory set-up. All of these
theories have good boundary descriptions and have same degrees of freedom as the
parent cascading theory at certain specified scales. The simplest UV complete theory
(however with infinite degrees of freedom at UV) is of course the parent cascading
theory that we will discuss in a moment. The question now is to construct other
possible theories by defining the degrees of freedom at the boundary. To do this
observe that we defined the boundary theory using the identification:

\[ S_{\text{ren}}^{(1)} \big| r_h = S_{\text{ren}}^{(1)} \big| r_c + S_{\text{ren}}^{(1)} \big| r_c \]

where the boundary is at \( r \to \infty \). For the boundary cascading theory the above
expression simply means that

\[ S_{\text{ren}}^{(1)} \big| r_h = \int \frac{d^4q}{(2\pi)^4} \left( \sum_{j=0}^{\infty} \tilde{b}_{n(j)}^{(a)} \tilde{J}^n \theta(r_0 - r_c) \right) - \int \frac{d^4q}{(2\pi)^4} \left( \sum_{j=0}^{\infty} \tilde{B}_{n(j)}^{(a)} \frac{r^{4j}_h}{r_c^{4j}(\alpha)} \right) \Phi_n \]

\[ + \frac{\tilde{b}_{n(j)}^{(a)}, \mathcal{O}(r^{4j}_h)}{\infty} \text{ factors} \]

where the sign is crucial and sum over \( \alpha \) is again implied (note (a) the cut-off de-
pendence, and (b) \( r_c^{(\beta)} \) is some function of \( r_c^{(\alpha)} \) that one can determine easily). We
now see that the contributions from the UV cap give the following values for \( B_j \):

\[ B_{n(0)}^{(a)} = B_{n(1)}^{(a)} = B_{n(2)}^{(a)} = B_{n(3)}^{(a)} = 0 \]

\[ B_{n(4)}^{(a)} = \frac{1}{2} \left[ \tilde{b}_{n(0)}^{(a)} \tilde{J}^n \theta(r_0 - r_c) + \mathcal{O}(H_m^{nm} K_{nm}^{(a)}, s^{(a)}_{mn}) \right], \ldots. \]

this means that in the limit of small \( r_h \) we can ignore the contributions coming from
\( r_h^{4j} \). For this section such an assumption will not change any of the results, so we
will stick with this. From the next section onwards, we will restore back the \( \mathcal{O}(r_h) \)
dependences.

Taking all of the above considerations, this implies that the contribution from
\( r > r_c \) exactly cancels the \( \mathcal{O}(1/r_c) \) contributions coming in from the action measured
from \( r_h \leq r \leq r_c \), giving us the boundary energy-momentum tensor

\[ \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha, \beta} \left[ (H_m^{nm} + H_m^{nm}) s_n^{(4)[\beta]} - 4(K_m^{nm} + K_m^{nm}) s_n^{(4)[\beta]} + (K_m^{nm} + K_m^{nm}) s_n^{(5)[\beta]} \right] \]

which is the result derived in [15], [13], [14], [15], [16]. The above way of reinterpreting
the boundary contribution should tell us precisely how we could modify the boundary
degrees of freedom to construct distinct UV completed theories. There are two
possible ways we can achieve this:
• From the geometrical perspective we can cutoff the deformed conifold background at $r = r_c$ and attach an appropriate UV “cap” from $r = r_c$ to $r \to \infty$ by carefully modifying the geometry at the neighborhood of the junction point. As an example, this UV cap could as well be another AdS background from $r_c$ to $r \to \infty$. There are of course numerous other choices available from the F-theory limit. Each of these caps would give rise to distinct UV completed gauge theories.

• From the action perspective we could specify precisely the value of the action measured from $r_c$ to $r \to \infty$, i.e $[S^{(1)}_{\text{ren}}]_{r_c}$. The simplest case where this zero gives rise to a boundary theory whose degrees of freedom at the UV is the one given in (3.21). We will give an example of this towards the end of this section. To study more generic cases, we need to see how much constraints we can put on our integral.

One immediate constraint is the holographic renormalisability of our theory. This will not have finite actions. This in turn implies

$$[S^{(1)}_{\text{ren}}]_{r_c} = \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha} \left\{ \left( \sum_{j=0}^{\infty} \frac{\tilde{A}_{mn(j)}}{r_{c(\alpha)}} \right) \tilde{G}^{mn} \Phi \Phi_n + \left( \sum_{j=0}^{\infty} \frac{\tilde{E}_{mn(j)}}{r_{c(\alpha)}} \right) \tilde{M}^{mn} \right\} (3.140)$$

where by specifying the coefficients $\tilde{A}_{mn(j)}^{(\alpha)}$, $\tilde{E}_{mn(j)}^{(\alpha)}$ and $\tilde{B}_{m(j)}^{(\alpha)}$ we can specify the precise UV degrees of freedom! The finite terms are $r_c$ independent and therefore would only provide finite shifts to our observables. They could therefore be scaled to zero. Notice also that the contributions from (3.140) only renormalises the coefficients $\tilde{a}_{mn(j)}^{(\alpha)}$, $\tilde{e}_{mn(j)}^{(\alpha)}$ and $\tilde{b}_{m(j)}^{(\alpha)}$ in (3.134), and therefore the final expressions for all the physical variables for various UV completed theories could be written directly from (3.134) simply by replacing the $1/r_c$ dependent coefficients by their renormalised values. This is thus our precise description of how to specify the UV degrees of freedom for various gauge theories in our setup (see figure 11 below).

Once the UV descriptions are properly laid out, we can determine the form for $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$ by writing Callan-Symanzik type equations for them. These are classical equations and do not capture any quantum behavior. Nevertheless they tell us how $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$ would behave with the scale $r_c$ or equivalently $\mu_c$. For $\tilde{A}$ the equation is

$$\frac{\partial \tilde{A}_{mn(j)}^{(\alpha)}}{\partial \mu_c} = j \left[ 1 - \epsilon_{(\alpha)} \right] \left[ \tilde{A}_{mn(j)}^{(\alpha)} + \tilde{a}_{mn(j)}^{(\alpha)} \right] (3.141)$$

52Remember that there are additional $O(r_h)$ contributions to the coefficients. For small $r_h$ they will only make the UV contributions more involved without changing any of the underlying physics as we saw above. We will therefore ignore them.

53The following equation is derived from the scale-invariance of $[S^{(1)}_{\text{ren}}]_{r_h}$.  

\[ \text{[natural_text]} \]
Figure 11: The equivalence between two different ways of viewing the boundary theory at zero temperature. To the left we add finite UV degrees of freedom at \( r = r_c \) of the deformed conifold geometry. Such a process is equivalent to the figure on the right where we cut-off the deformed conifold geometry at \( r = r_c \) and add a UV cap from \( r = r_c \) to \( r = \infty \). The boundary theory on the right has \( \mathcal{N}_{\text{uv}} \) degrees of freedom at \( r = \infty \) and all physical quantities computed in either of these two pictures would only depend on \( \mathcal{N}_{\text{uv}} \) but not on \( r = r_c \). At non-zero temperature the UV descriptions remain unchanged. Of course, for both cases one need to put in the corrections to IR from UV deformations.

with similar equations for \( \tilde{B}^{(\alpha)} \) and \( \tilde{E}^{(\alpha)} \). These equations tell us that physical quantities are independent of scales. The parent cascading theory is defined as the scale-invariant limits of (3.141), i.e:

\[
\tilde{A}^{(\alpha)}_{mn(j)} = -\hat{a}^{(\alpha)}_{mn(j)}; \quad \tilde{E}^{(\alpha)}_{mn(j)} = -\hat{e}^{(\alpha)}_{mn(j)}; \quad \tilde{B}^{(\alpha)}_{m(j)} = -\hat{b}^{(\alpha)}_{m(j)} \tag{3.142}
\]

The above relation gives us a hint how to express \( \tilde{A}^{(\alpha)}, \tilde{B}^{(\alpha)} \) and \( \tilde{E}^{(\alpha)} \) in terms of \( \mathcal{N}_{\text{eff}} \), the effective degrees of freedom at \( r = r_c \) and \( \mathcal{N}_{\text{uv}} \), the effective degrees of freedom at \( r = \infty \) i.e the boundary:

\[
\tilde{A}^{(\alpha)}_{mn(j)} = -\hat{a}^{(\alpha)}_{mn(j)} + \hat{a}^{(\alpha)}_{mn(j)} e^{-j[\mathcal{N}_{\text{uv}}-(1-\epsilon(\alpha))\mathcal{N}_{\text{eff}}]}
\]

\[
\tilde{E}^{(\alpha)}_{mn(j)} = -\hat{e}^{(\alpha)}_{mn(j)} + \hat{e}^{(\alpha)}_{mn(j)} e^{-j[\mathcal{N}_{\text{uv}}-(1-\epsilon(\alpha))\mathcal{N}_{\text{eff}}]}
\]

\[
\tilde{B}^{(\alpha)}_{m(j)} = -\hat{b}^{(\alpha)}_{m(j)} + \hat{b}^{(\alpha)}_{m(j)} e^{-j[\mathcal{N}_{\text{uv}}-(1-\epsilon(\alpha))\mathcal{N}_{\text{eff}}]} \tag{3.143}
\]

where the actual boundary degrees of freedom are specified by knowing \( \hat{a}^{(\alpha)}_{mn(j)} \), \( \hat{e}^{(\alpha)}_{mn(j)} \) and \( \hat{b}^{(\alpha)}_{m(j)} \) as well as \( \mathcal{N}_{\text{uv}} \). Since \( j \) goes from 0 to \( \infty \), there are infinite possible UV complete boundary theories possible\(^{54}\). For very large \( \mathcal{N}_{\text{uv}} \) (i.e \( \mathcal{N}_{\text{uv}} \to e^{-n}, n >> 1 \))

\(^{54}\)The connection of \( j \) with UV completions come from the coefficients \( \hat{a}^{(\alpha)}_{mn(j)} \), \( \hat{e}^{(\alpha)}_{mn(j)} \) and \( \hat{b}^{(\alpha)}_{mn(j)} \) etc. that depend on \( j \). For different choices of these coefficients we can have different UV completions. In this sense \( j \) and UV completions are related. See also the F-theory discussion presented towards the end of section 3.1.
the boundary theories are similar to the original cascading theory. The various choices of $(\hat{a}^{(\alpha)}_{mn(j)}(\vec{q}), \hat{e}^{(\alpha)}_{mn(j)}(\vec{q}), \hat{b}^{(\alpha)}_{m(j)}(\vec{q}))$ tell us how the degrees of freedom change from $N_{uv}$ to $N_{\text{eff}}$ under RG flow. The $\vec{q}$ dependence of all the quantities will tell us how the UV degrees of freedom affect IR physics. This is to be expected: addition of irrelevant operators do change IR physics, but not the far IR\textsuperscript{55}.

Therefore with this understanding of the boundary theories we can express the energy-momentum tensor at the boundary with $N_{uv}$ degrees of freedom at the boundary purely in terms of gauge theory variables, as:

$$
T_{0}^{nm} \equiv \int \frac{d^4q}{(2\pi)^4} \left[ (H_{[\alpha]}^{mn} + H_{[\alpha]}^{nm}) s_{mn}^{(4)[\beta]} - 4(K_{[\alpha]}^{mn} + K_{[\alpha]}^{nm}) s_{mn}^{(4)[\beta]} + (K_{[\alpha]}^{mn} + K_{[\alpha]}^{nm}) s_{mn}^{(5)[\beta]} + \sum_{j=0}^{\infty} \hat{b}^{(\alpha)}_{m(j)}(\vec{q}) \hat{J}^n e^{-jN_{uv}\delta_{nm}} \right] \quad (3.144)
$$

where sum over $\alpha$ is again implied, and the first line is the universal property of the parent cascading theory inherited by our gauge theory. This part, common to all the theories, will be numerically different. The second line specifies the precise degrees of freedom that we add at $r = r_c$ to describe the UV behavior of our theory at the boundary $r \to \infty$. Using this procedure, the final results of any physical quantities should be expressed only in terms of $N_{uv}$ i.e the UV degrees of freedom\textsuperscript{56}.

The above analysis of holographic renormalisation tells us something very interesting. The final result of the renormalisation procedure is almost identical to the renormalisation procedure done by taking the highest positive integer power of $r$ and ignoring the log $r$ part! The only difference is the presence of coefficients like $H_{[\alpha]}^{mn}$ that depend explicitly on flavor degrees of freedom. This will be useful for us when we study shear viscosity in the next section.

Another question to ask regarding holographic renormalisation is the issue of corrections for finite values of $g_s N_f, g_s M^2/N$. This is probably more difficult to tackle. However an easier question would be to ask what happens if the UV behavior is governed by $r^{-p}$ instead of $r_{-p}^{(\alpha)}$ for a certain choice of an UV cap?

To evaluate this observe that we can take the following ansatze for the background

\textsuperscript{55}The $\vec{q}$ dependences of the UV caps are also one-to-one correspondence to the changes in the local geometries near the cut-off radius $r_c$, as we discussed before. All in all this fits nicely with what one would have expected from UV degrees of freedom. Of course it still remains to verify the story from an actual supergravity calculation. We need to analyse the metric near the junction by studying the continuity and differentiability of the metric and see how far below $r = r_c$ we expect deformations from the UV caps. Various types of deformations will signal various set of irrelevant operators. Needless to say, the far IR physics remain completely unaltered.

\textsuperscript{56}Restoring back the $O(r_h)$ contributions would mean that there should be an additional contribution to (3.144) of the form $\sum_{j=0}^{\infty} G(\hat{b}^{(\alpha)}_{m}, H_{[\alpha]}^{mn}, K_{[\alpha]}^{mn}, s_{mn}) T^4 e^{-jN_{uv}}$ where $G$ is a function whose functional form could be inferred from the UV integral (3.140).
ground warp factor for large $r$ on the slice (3.14)\textsuperscript{57}:

$$h(r) = \frac{L^4}{r^4} \left[ 1 + \sum_{m,n,p} \left( \frac{g_s M^2}{N} \right)^n \log^{p} r \right] \to \frac{L^4}{r^4} \left[ 1 + \sum_{p=1}^{p_0} \frac{c_p}{r^{p-4}} \right]$$

(3.145)

where $p_0$ is some given integer, $c_p$ are functions of $g_s N_f, g_s M^2/N$ and the arrow is motivated by the fact that, in the presence of a suitable UV cap the background is given by F-theory construction \[30\] that guarantees that the warp factor goes from $\log r$ at small $r$ to inverse powers of $r$ at large $r$ to avoid the Landau poles.

In this background we expect the fluctuation to go like:

$$\Phi^{[1]}_m(r,q) = \int \frac{d^4x}{(2\pi)^4 \sqrt{g(r_c)}} e^{i(q_0 t - q_1 x - q_2 y - q_3 z)} L^{mm}(t,r,x,y,z)$$

(3.146)

which is similar to the definition of $\Phi^{[1]}_m(r,q)$ given earlier in (3.117) with $q_0 = \omega \sqrt{g(r_c)}$ as before. Our next step would be to write the above action as an EOM part and a boundary part like as in (3.120) with appropriate changes. Once we do this we would require the Gibbons-Hawking boundary terms to cancel any unwanted $\delta \Phi^{[1]}_m$ parts from the action.

Taking all these into considerations the boundary action takes the usual form (3.123). Now the coefficients $C_{1m}^{mn}, A_{1m}^{mn}, B_{1m}^{mn}, E_1^m$ and $F_1^m$ are again divergent, and from our previous consideration we know that the divergences will be controlled by the highest integer power of $r$. Therefore we need to regularise and renormalise the action again. The divergences are:

1. $C_{1m}^{mn}(\zeta,q) - A_{1m}^{mn}(\zeta,q) = \sum_{p=1}^{p_0} \tilde{H}_p^{mn}(q) r_c^p (1 - \zeta)^p + \text{finite terms}$

2. $B_{1m}^{mn}(\zeta,q) - A_{1m}^{mn}(\zeta,q) = \sum_{p=1}^{p_0} \tilde{K}_p^{mn}(q) r_c^{p+1} (1 - \zeta)^{p+1} + \text{finite terms}$

3. $E_1^m(\zeta,q) - F_1^m(\zeta,q) = \sum_{p=1}^{p_0} \tilde{I}_p^m(q) r_c^p (1 - \zeta)^p + \text{finite terms}$

(3.147)

To start off then, we write the metric perturbation $\Phi^{[1]}_m$ as a power series in $\zeta$ where $\zeta$ is defined with the equation $r = r_c (1 - \zeta)$:

$$\Phi^{[1]}_m(\zeta) = \theta(\zeta - \zeta_0) \sum_{k=0}^{\infty} \frac{s_{mn}^{(k)}}{r_c^k (1 - \zeta)^k}$$

(3.148)

\textsuperscript{57}The equation (3.145) could be motivated by adding a suitable UV cap that allows an inverse $r$ dependences in (3.35) which in turn stems from the finiteness of F-theory. At large $r$ the axio-dilaton contributions will come from (3.40) that will back react on the geometry to give a finite asymptotic geometry. There are infinite such geometries possible. For a given geometry the coefficients of $r^{-p}$ terms would be fixed accordingly (by solving the sugra equations with axio-dilaton and three and five-fluxes as sources) provided these fluxes are also asymptotically finite. For more details see the F-theory arguments presented earlier.
which can again be easily derived by plugging (3.108) in (3.146).

Once we have the mode expansions of the field, we can use this expansion directly in the action (3.116) to determine the possible counter-terms. The infinities come from (3.147), much like the situation with (3.124). Therefore its no surprise that we get the following counter-terms:

\[
S^{(2)}_{\text{counter}} = \int \frac{d^4q}{(2\pi)^4} \sum_p \left\{ \tilde{H}^{\text{ren}}(\zeta - \zeta_0) \left[ s_{mm}^{(0)} s_{nn}^{(0)} r_c^p (1 - \zeta)^p \right. \\
+ \left. \left( s_{mm}^{(1)} + s_{mm}^{(1)} \right) r_c^{p-1} (1 - \zeta)^{p-1} + \left( s_{mm}^{(2)} + s_{mm}^{(2)} \right) r_c^{p-2} (1 - \zeta)^{p-2} + \left( s_{mm}^{(3)} s_{nn}^{(3)} + s_{mm}^{(3)} s_{mm}^{(3)} + s_{mm}^{(3)} s_{mm}^{(3)} \right) r_c^{p-3} (1 - \zeta)^{p-3} \right. \\
- \frac{2s_{mm}^{(1)} s_{mm}^{(1)}}{r_c^2} (1 - \zeta)^{p-2} - \frac{3s_{mm}^{(2)} s_{mm}^{(2)} + s_{mm}^{(1)} s_{mm}^{(1)}}{r_c^3} (1 - \zeta)^{p-3} \right\} \\
\right.
\]

\[
\left. \left. \left. + \frac{2s_{mm}^{(1)} s_{mm}^{(1)}}{r_c^2} (1 - \zeta)^{p-2} + s_{mm}^{(3)} s_{mm}^{(3)} r_c^{p-3} (1 - \zeta)^{p-3} \right] \right\} \right\} (3.149)
\]

with of course another set of counter-terms defined at the horizon \( r_h \) that we do not write here. Observe that we have written the counterterms wrt the highest power of \( p \). Finally the renormalised action after some manipulations can be written, similar to (3.134) but with different coefficients (whose explicit values will not be relevant for the present case), as:

\[
S^{(2)}_{\text{ren}} = \int \frac{d^4q}{(2\pi)^4} \sum_p \left\{ \left[ \sum_{j=0}^{\infty} \dfrac{\tilde{C}^{(j)}}{r_c^j (1 - \zeta)^j} \right] \tilde{L}^{mn} \Phi_m \Phi_n + \left[ \sum_{j=0}^{\infty} \dfrac{\tilde{f}^{(j)}}{r_c^j} \right] \tilde{M}^{mn} (\Phi_m \Phi_n^{(p+1)}) \\
+ \Phi_m^{(p+1)} \Phi_n + \tilde{H}^{\text{ren}} \left[ s_{mm}^{(p)} \Phi_m + s_{mm}^{(p)} \Phi_n \right] + \tilde{K}^{mn} \left[ -4s_{mm}^{(p)} \Phi_m - 4s_{mm}^{(p)} \Phi_n + s_{mm}^{(p+1)} \Phi_m^{(p+1)} \right] \\
+ \left( \sum_{j=0}^{\infty} \dfrac{\tilde{D}^{(j)}}{r_c^j (1 - \zeta)^j} \right) \tilde{Q}^{mn} \Phi_m + Y (r) \right\} \theta(\zeta - \zeta_0) \right) \right) (3.150)
\]

from here we can extract the energy-momentum tensor \( T^{mn} \) once we add the right boundary degrees of freedom. This is exactly as we did before. The contribution from UV cap now will be:

\[
[S^{(2)}_{\text{ren}}]_\zeta = \int \frac{d^4q}{(2\pi)^4} \left\{ \left[ \sum_{j=0}^{\infty} \dfrac{\tilde{C}^{(j)}}{r_c^j (1 - \zeta)^j} \right] \tilde{L}^{mn} \Phi_m \Phi_n + \left[ \sum_{j=0}^{\infty} \dfrac{\tilde{f}^{(j)}}{r_c^j} \right] \tilde{M}^{mn} (\Phi_m \Phi_n^{(p+1)}) \\
+ \Phi_m^{(p+1)} \Phi_n + \left( \sum_{j=0}^{\infty} \dfrac{\tilde{D}^{(j)}}{r_c^j (1 - \zeta)^j} \right) \tilde{Q}^{mn} \Phi_m \right\} \theta(\zeta - \zeta_0) + \text{finite terms (3.151)}
\]
where by the range $\zeta$ to $\infty$ we mean the coordinate range $r_c(1 - \zeta)$ to $r \to \infty$. As before, specifying the quantities $\tilde{C}^{(\alpha)}$, $\tilde{F}^{(\alpha)}$ and $\tilde{D}^{(\alpha)}$ will in turn specify the boundary degrees of freedom. We also expect similar Callan-Symanzik type equations for them. Indeed:

$$
\mu_c \frac{\partial \tilde{D}_{m(j)}}{\partial \mu_c} = j \left[ \tilde{D}_{m(j)} + \tilde{d}_{m(j)} \right]
$$

(3.152)

with similar relations for the others. This way we can compute the total energy-momentum tensor of the system once we know the precise value of $p_0$, $c_p$ for the exact background. On the other hand, for the case that we know very well (3.11), the energy-momentum tensor that we should substitute in (3.64) to compute the wake will be:

$$
T_{\text{medium+quark}}^{mn} = \int \frac{d^4 q}{(2\pi)^4} \sum_{\alpha,\beta} \left\{ (H_{\alpha}^{mn} + H_{\alpha}^m) s_{nn}^{(4)[\beta]} - 4(K_{\alpha}^{mn} + K_{\alpha}^m) s_{nn}^{(4)[\beta]} + (K_{\alpha}^{mn} + K_{\alpha}^m) s_{nn}^{(5)[\beta]} + \sum_{j=0}^{\infty} \hat{b}_{n(j)}^{(\alpha)} \tilde{f}_n \delta_{nm} e^{-jN_{uv}} + \mathcal{O}(T e^{-N_{uv}}) \right\}
$$

(3.153)

where $(\hat{b}_{n(j)}^{(\alpha)}, \hat{d}_{n(j)}^{(\alpha)}, N_{uv})$ together will specify the full boundary theory for a specific UV complete theory. Observe that the result is completely independent of the cut-off that we imposed to do our analysis. In the limit $N_{uv} \to \epsilon^{-n}$ with $n >> 1$ i.e when the boundary degrees of freedom go to infinity as this way, we reproduce precisely the result of the parent cascading theory.

Before we end this section let us make the following observation. This is a slightly more non-trivial example where $N_{uv} \approx N_{\text{eff}}$ with $N_{\text{eff}}$ being the degrees of freedom at $r = r_c$ given earlier in (3.21). This means that the degrees of freedom don’t change significantly as we go from far UV to the scale $\Lambda_c$. From the gravity dual perspective this is like adding a UV cap given by an AdS geometry. For such a theory (3.153) can be exactly determined (both $\hat{b}_{n(j)}^{(\alpha)}, \hat{d}_{n(j)}^{(\alpha)}$ can be scaled to identity), and so precise prediction can be made.\(^{58}\)

In the next two sections where we deal with shear viscosity and viscosity-to-entropy ratio, we will find that all the techniques that we discussed in this section will become very useful.

### 3.4 Shear Viscosity

In this section we will compute the shear viscosity of the four dimensional theory following some of the recent works \([4, 59]\). Our basic idea would be to use the Kubo

\(^{58}\)Interestingly for small enough $\Lambda_c$ we are almost in the IR where we expect the gauge theory to confine (at least at zero temperature). The UV of the gauge theory has very slow running, which can be constructed from our F-theory model by suitably choosing the coefficients $a_i, b_j$ in (3.41) such that axio-dilaton vanish. With vanishing axio-dilaton the geometry at UV approaches AdS. Therefore this set-up is almost like large $N$ flavored QCD with vanishing Beta function at UV and confinement at IR! Our method could then be used to evaluate the thermal quantities of this theory.
formula (60):

$$
\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d^3 x \ e^{i\omega t} \langle [T_{23}(x), T_{23}(0)] \rangle = -\lim_{\omega \to 0} \frac{\text{Im} \ G^R(\omega, 0)}{\omega}
$$

(3.154)

where $G^R(\omega, \vec{q})$ is the momentum space retarded propagator for the operator $T_{23}$ at finite temperature, defined by

$$
G^R(\omega, \vec{q}) = -i \int d^3 x e^{i(\omega t - \vec{x} \cdot \vec{q})} \theta(t) \langle [T_{23}(x), T_{23}(0)] \rangle
$$

(3.155)

In the following, we will compute the Minkowski propagator $\langle T_{23}(t, \vec{x}) T_{23}(0, \vec{x}') \rangle$ using gauge/gravity duality\textsuperscript{59}. However before we compute this explicitly, let us take a small detour to evaluate the higher order corrections to the effective action from the wrapped D7 brane in our theory. We have already given a brief discussion of this in (3.92). It is now time to deal with this in some more details.

In the case of a D7 brane, the disc level action contains the term (61):

$$
S_{\text{disc}}^{D7} = \frac{1}{192 \pi g_s} \cdot \frac{1}{(4\pi \alpha')^2} \int_{M^8} [C_4 \wedge \text{tr } (R \wedge R) - e^{-\phi} \text{tr } (R \wedge *R)]
$$

(3.156)

where $C_4$ is the four-form, $R$ is the curvature two-form, $\phi$ is the dilaton and $M^8$ is a non-trivial eight manifold forming the world-volume of the D7 brane. The above action is $SL(2, \mathbb{Z})$ invariant which one can show by doing an explicit analysis [61]. Since for our case the D7 wrap a non-trivial four-cycle, we can dimensionally reduce it over the four-cycle to get the following action:

$$
S_{\text{disc}}^{D7} = \frac{1}{16 \pi^2} \int_{M^4} \text{Re} [\log \eta(\tau) \ \text{tr } (R \wedge *R - iR \wedge R)]
$$

(3.157)

where $\eta(\tau)$ is the Dedekind function, and $\tau$ is the modular parameter defined by:

$$
\tau = \frac{1}{g_s(4\pi \alpha')^2} \left( \int_{S^4} C_4 + i \mathcal{V}_4 \right) \equiv \frac{1}{g_s}(\tau_1 + i\tau_2)
$$

(3.158)

with $\mathcal{V}_4$ being the volume of the four-cycle on which we have the wrapped D7 brane.

An interesting point here is that the above action can be derived from the following action that has two parts, one CP-even and the other CP-odd [62]:

$$
\frac{1}{32 \pi^2} \int_{M^4} \log |\eta(\tau)|^2 \text{tr } (R \wedge *R) - \frac{i}{32 \pi^2} \int_{M^4} \log \frac{\eta(\tau)}{\eta(\bar{\tau})} \ \text{tr } (R \wedge R)
$$

(3.159)

\textsuperscript{59}Note that our prescription (3.65) computes a path integral $\langle O_\phi \rangle_T \sim \int [\mathcal{D}O]_T e^{\int_T \mathcal{L}_{\text{SUGRA}}}$, (where $T$ is the time ordering) with a classical action $S_{\text{SUGRA}}$, unaware of the time ordering. Therefore computing any commutator is rather subtle here. For this reason, we compute only the correlator $\langle T_{23}(\tau, \vec{x}) T_{23}(0, \vec{x}) \rangle$ and not its commutator. Using this correlator, we will eventually obtain information about commutators and finally the viscosity $\eta$. 


where the first part is CP-even and the second part is CP-odd. To compare (3.159) with (3.157) note that the Dedekind \( \eta \) function has the following expansion in terms of \( q \equiv e^{2\pi i \tau} \):

\[
\log |\eta(\tau)|^2 = -\frac{\pi}{6} \tau_2 - \left[ q + \frac{3q^2}{2} + \frac{4q^3}{3} + \ldots + \text{c.c} \right]
\]

\[
\log \frac{\eta(\tau)}{\eta(\bar{\tau})} = +i\frac{\pi}{6} \tau_1 - \left[ q + \frac{3q^2}{2} + \frac{4q^3}{3} + \ldots - \text{c.c} \right]
\]  

(3.160)

Combining everything together we see that, up to powers of \( q \) (3.159) and (3.157) are equivalent. However writing the action in terms of (3.159) instead of (3.157) has a distinct advantage: from D7 point of view (3.159) captures the D3 instanton corrections in the system [62, 63]. But there is a even deeper reason for writing the action as (3.159). The CP-even and CP-odd terms can be

\[
S_{\text{CP-even}} = -\alpha_1 \int_{M^8} e^{-\phi} \mathcal{L}_{\text{GB}} - T_7 \int_{M^8} e^{-\phi} \left[ \sqrt{G} - \frac{(4\pi^2 \alpha')^2}{24} \mathcal{L}_R + \mathcal{O}(\alpha'^4) \right]
\]  

(3.161)

where \( \alpha_1 \) is a constant, and \( \mathcal{L}_{\text{GB}} \) and \( \mathcal{L}_R \) are respectively the Gauss-Bonnet and the curvature terms defined in the following way:

\[
\mathcal{L}_{\text{GB}} = \frac{\sqrt{G}}{32\pi^2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2 \right)
\]

\[
\mathcal{L}_R = \frac{\sqrt{G}}{32\pi^2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta} R^{\alpha\beta} - R_{ab\gamma\delta} R^{ab\gamma\delta} + 2R_{ab} R^{ab} \right)
\]  

(3.162)

In the above note that the three curvature terms \( R_{\alpha\beta} R^{\alpha\beta}, R_{ab} R^{ab} \) and \( R^2 \) are not the pull-backs of the bulk Ricci tensor. Furthermore we have used the notations \((\alpha, \beta)\) to denote the world-volume coordinates, and \((a, b)\) to denote the normal bundle.

Thus from the CP-even terms, the coefficient of \( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \) is given by:

\[
c_3 \equiv \frac{e^{-\phi} \sqrt{G}}{32\pi^2} \left( \frac{4\pi^4 \alpha'^2}{3} - \alpha_1 \right)
\]  

(3.163)

which has an overall plus sign because \( \alpha_1 \) in many interesting cases tend to be zero (see [62] for details on this). However in general for certain exotic compactifications we expect \( \alpha_1 \ll \frac{4\pi^4 \alpha'^2}{3} \). If we now compare this to [64] we see that \( c_3 \), which is the

\[60\text{Note that } q \text{ can be made small because both } C_4 \text{ and } V_4 \text{ in (3.158) can be made small.}
\[61\text{The Gauss-Bonnet term is in general a topological invariant in four dimensions, but it is a total derivative at quadratic order in all dimensions. Therefore it wouldn’t contribute to the equation of motion.} \]
coefficient of $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ in [7], is positive. This would clearly mean that adding fundamental flavors lowers the viscosity to entropy bound!\footnote{We were motivated to carry out the above analysis from a comment by Aninda Sinha. His paper [9] dealing with the violation of viscosity to entropy bound has appeared recently and has some overlap with this section.}

The CP-odd term on the other hand has a standard expansion of the following form for the D7 brane [14, 51, 52]:

$$S_{\text{CP-odd}} = T_7 \int_{M^8} \left( C_8 + \frac{\pi^2 \alpha' \alpha'^2}{24} C_4 \wedge \text{tr} \, R \wedge R \right)$$  \hfill (3.164)

where the first term gives the standard dual axionic charge of the D7 brane. Combining (3.161) and (3.164) we get the full back reactions of the D7 brane up to $O(\alpha'^2)$.

Having computed the back reactions of the embedded D7 brane, we can use this result to compute the shear viscosity. There are three scenarios from which we can compute the viscosity using the gravity dual now:

- Allow a gravity dual that has no running (i.e no RG running in the gauge theory side), but incorporates the back reaction of the D7 brane (3.161).
- Allow a gravity dual that shows the RG running of the gauge theory, but does not incorporate the back reaction of the D7 brane (although may allow the CP-odd part (3.164)).
- Allow a gravity dual that not only shows the RG running of the gauge theory, but also incorporates both the CP-even as well as CP-odd parts (3.161) and (3.164).

The first part has recently been done in [4, 8, 9]. The background remains AdS, but now there would be terms from the D7 brane (3.161) that would lower the viscosity to entropy bound. However the second and the third part is not yet been addressed in the literature. As we show below, the second part is rather easy to do following the calculations of [7] because we have already constructed the background in the previous section.

The third part on the other hand is rather subtle because incorporating the higher order (curvature)$^2$ corrections would change all the results that we derived earlier. In particular the mass and drag of the quark would need to be modified along with the wake of the quark. Plus the holographic renormalisability would also get modified because of the extra derivative terms from these corrections. In this paper we will deal mostly with the second part of the above list and some calculations of the third part, but leave a more detailed analysis of the third part for the sequel [53].

To start off the analysis, we need the correlation function of $T_{23}(x)$ and $T_{23}(0)$ to use it in the Kubo formula (3.154). From gauge/gravity duality we know that switching on $T_{23}$ in the gauge theory is equivalent to switching on graviton mode along $x^2 = x$ and $x^3 = y$ directions. Thus in the given OKS-BH background (3.91),
we switch on the following off-diagonal part:

\[
\begin{pmatrix}
ge_{00} & e_{0x} & e_{0y} & e_{0z} & e_{0r} \\
e_{x0} & e_{xx} & e_{xy} & e_{xz} & e_{xr} \\
e_{y0} & e_{yx} & e_{yy} & e_{yz} & e_{yr} \\
e_{z0} & e_{zx} & e_{zy} & e_{zz} & e_{zr} \\
e_{r0} & e_{rx} & e_{ry} & e_{rz} & e_{rr}
\end{pmatrix} \cdot 
\begin{pmatrix}
-g(r) & 0 & 0 & 0 & 0 \\
0 & 1 & \phi(r, t) & 0 & 0 \\
0 & \phi(r, t) & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{r^2 h}{g(r)}
\end{pmatrix}
\]

(3.165)

where \( h = h(r, \pi, \pi) \) and the internal space metric is independent of the radial coordinate \( r \) as shown in (3.91)\(^{63}\). Note that the off-diagonal gravitons are propagating in OKS-BH geometry with energy \( \omega \).

Since our goal is to compute the Fourier transform of \( \langle T_{23}(t, \vec{x})T_{23}(0, \vec{x}) \rangle \), we can do this by first writing the supergravity action in momentum space, treating it as a functional of Fourier modes for \( \phi(r, t) \) where:

\[
\phi(r, t) = \tilde{\phi}(r, t) \tilde{\phi}(t) \equiv \int d\omega \ e^{-i\omega t} \phi(r, \omega) = \int d\omega \ e^{-i\omega \sqrt{g} \phi(r, \omega)}
\]

(3.166)

\[
\phi(r, \omega) = \tilde{\phi}(r, |\omega|) \tilde{\phi}(\omega)
\]

where as before, we defined the Fourier transform using the curved space time \( \tau \equiv \sqrt{g(r_c) t} \) and not simply \( t \). Although this definition is precise for the theory at the cut-off \( r = r_c \) only, we will use it also for \( r < r_c \) because in the end we will only provide description at the boundary (i.e. \( r \to \infty \)) where the results would be independent of the choice of the cut-off.

The way we proceed now is the following\(^{64}\). We consider the metric fluctuation as in (3.163) and plug this in the effective action (3.92) but with \( c_n = b_n = 0 \) (we will modify this soon). We will also take a convention where \( g(r_c) = 1 \) in the subsequent analysis to avoid clutter. In the final result we will substitute the exact value of \( g(r_c) \). Finally, we will call this resulting action as \( S_{SG}^{(2)} \) where the subscript \( (2) \) involves writing the action in terms of quadratic \( \phi(r, \omega) \). The reason for doing

\(^{63}\)Note that the above choice of the warp factor means that we have instinctively chosen the minima where \( \theta_1 = \theta_2 = \pi \). Needless to say, since we are analysing the effect of the flavor on the viscosity, we took the point close to where the quark string originally was. Again a more detailed analysis could be performed directly from the ten-dimensional point of view, but such a analysis do not reveal any new physics.

\(^{64}\)This is almost similar to the procedure of [7] whom we refer the readers for more details. Notice however that the theory considered by [8] has no running but contains higher curvature-squared corrections, as mentioned above.
this is because there exists a very useful relation for computing the shear viscosity (see for example \[65\]):

\[
\lim_{\omega \to 0} \text{Im} \ G_{11}^{SK}(\omega, \vec{q}) = \lim_{\omega \to 0} \frac{2T}{\omega} \text{Im} \ G_{11}^{R}(\omega, \vec{0})
\]

(3.167)

where \(G_{ij}^{SK}\) is the Schwinger-Keldysh propagator \[65, 66\]. Comparing this with our earlier Kubo formula (3.154) we see that the shear viscosity is nothing but:

\[
\eta = -\frac{1}{2T} \lim_{\omega \to 0} \text{Im} \ G_{11}^{SK}(\omega, \vec{0})
\]

(3.168)

Thus if we can write our effective supergravity action in the following way:

\[
S_{SG}^{(2)}[\phi(r_0, \omega)] = \frac{1}{2} \int \frac{d\omega d^3q}{(2\pi)^4} \phi_i(r_0, \omega) G_{ij}^{SK}(|\omega|, \vec{q}) \phi_j(r_0, -\omega)
\]

(3.169)

where \(r_0\) is a specified point, then taking the \(G_{11}^{SK}(\omega, \vec{0})\) part and using (3.168) we can easily get our shear viscosity\(^{65}\). Saying it a little differently, we will be taking two functional derivatives of \(S_{SG}^{(2)}[\phi(r_0, \omega)]\) with respect to \(\phi(r_0, \omega)\) and thus are only interested in terms quadratic in \(\phi(r_0, \omega)\) in the action. Of course as mentioned above, in real time formalism, we are concerned with the Schwinger-Keldysh propagator \(G_{ij}^{SK}\) of the doublet fields \(\phi_i(r, t), \phi_j(r, t)\). In the context of gauge/gravity duality, we follow the procedure outlined by \[66\] for AdS/CFT correspondence and treat \(\phi_1(r, t), \phi_2(r, t)\) as the perturbation \(\phi(r, t)\) and it’s doublet in the four dimensional Minkowski space\(^{66}\). In ten dimensional gravity theory, \(\phi_1(r) = \phi(r)\) is the field in the R quadrant of the Penrose diagram while \(\phi_2(r)\) is the field in the L quadrant.

For more details see \[66\] \[70\] \[71\] \[72\] \[73\] and \[68\].

Let us make this a bit more precise. Our aim is to get the effective action in the form (3.169). To this effect we take our metric (3.165) and plug it in the five dimensional effective action (3.92). The net result is the following action:

\[
S_{SG}^{(2)} = \frac{1}{8\pi G_N \sqrt{g(r_c)}} \int d\omega d^3q (2\pi)^4 \int_{r_h}^{r_c} dr \left[ A(r)\phi(r, -\omega)\phi''(r, \omega) + B(r)\phi'(r, -\omega)\phi'(r, \omega) + C(r)\phi'(r, -\omega)\phi'(r, \omega) + D(r)\phi(r, -\omega)\phi(r, \omega) \right]
\]

(3.170)

\(^{65}\)Notice that there would be an overall volume factor of \(T^{1.1}\) that would appear with the effective action. This factor is harmless and just modifies the Newton’s constant in five dimensions.

\(^{66}\)We are a little sloppy here. Our background is a deformation of the AdS space and therefore one might not expect the arguments of \[66\] to carry over exactly as above. However although the gauge/gravity duality argument for our case is more involved, we can still consider the \(\phi_1\) perturbations to compute the Schwinger-Keldysh propagator because by definition a propagator always appears sandwiched between the fields. See also \[68\] for more generic approach.
where prime denotes derivative with respect to \( r \) and the explicit expressions for \( A, B, C, D \) are given in Appendix E. The five dimensional Newton’s constant is given by:

\[
G_N \equiv \frac{\kappa_{10}^2 L_5^5}{4\pi V_T^{1.1}} \tag{3.171}
\]

where volume of \( T^{1.1} \) i.e \( V_T^{1.1} \) is dimensionful and \( \kappa_{10} \) is proportional to ten dimensional Newton’s constant.

The fluctuation \( \phi(r, \omega) \) is not anything arbitrary of course. It satisfies the following Euler-Lagrange equation of motion:

\[
\phi''(r, \omega) + \frac{A'(r) - B'(r)}{A(r) - B(r)} \phi'(r, \omega) + \frac{2D(r) - C'(r) + A''(r)}{2 [A(r) - B(r)]} \phi(r, \omega) = 0 \tag{3.172}
\]

which we can derive from (3.170) above. Once we plug in the values of \( A, B \) etc., the above Euler-Lagrange equation takes the following form:

\[
\phi''(r, \omega) + \left[ \frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right] \phi'(r, \omega) + \left[ \frac{\omega^2 g(r_c) \bar{h}(r)}{g(r)^2} + \mathcal{J}(r) \right] \phi(r, \omega) = 0 \tag{3.173}
\]

where \( \mathcal{J}(r) \) and \( \mathcal{M} \) are added to allow for most generic conditions on the scalar fields\(^{67}\). As before, primes in (3.172) and (3.173) denote derivatives wrt the five dimensional radial coordinate.

Now as we mentioned above in (3.166), \( \phi(r, \omega) \) can be decomposed in terms of \( \tilde{\phi}(r, |\omega|) \) and \( \bar{\phi}(\omega) \). Then as a trial solution, just like in \([7]\), we first try \( \tilde{\phi}(r, |\omega|) = g(r)\gamma \) and look at (3.173) for \( r \) near \( r_h \) where \( g(r) \to 0 \). Plugging this in (3.173) with \( g(r) = 0 \) we obtain the following expression for \( \gamma \):

\[
\gamma = \pm i |\omega| \sqrt{\frac{h(r_h) g(r_c)}{16} r_h} = \pm i \frac{|\omega|}{4 \pi T_c} \tag{3.174}
\]

where in the last step we have used the definition of temperature \( T_c \) as in (3.18).

To get the solution with \( g(r) \neq 0 \) we propose the following ansatze for the solution to (3.173)\(^{68}\):

\[
\phi(r, \omega) = g(r)^{\pm i |\omega|/4 \pi T_c} F(r, |\omega|) \bar{\phi}(\omega) \tag{3.175}
\]

\(^{67}\)In special cases we expect \( \mathcal{J}(r) \) and \( \mathcal{M} \) to vanish (see for example \([58]\)). However when this is not the case (as may arise for non-trivial UV completions of our model) we could expect a non-minimally coupled scalar field.

\(^{68}\)Incidentally, this form for \( \phi(r, \omega) \) can be shown to be exactly like (3.125) discussed earlier. This will become clearer as we go along.
Plugging this in (3.173) we see that the equation satisfied by $F(r, |\omega|)$ can be expressed in terms of $\gamma$ and $\gamma^2$ in the following way:

\[
F''(r, |\omega|) + \left( \frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M} \right) F'(r, |\omega|) + \left( \frac{\omega^2 g(r_c) \bar{h}}{g^2(r)} + \mathcal{J}(r) \right) F(r, |\omega|) + \gamma \left\{ \frac{2g'(r)}{g(r)} F'(r, |\omega|) + \left[ \frac{g''(r)}{g(r)} + \left( \frac{5}{r} + \mathcal{M} \right) \frac{g'(r)}{g(r)} \right] F(r, |\omega|) \right\} + \gamma^2 \frac{g^2(r)}{g^2(r)} F(r, |\omega|) = 0
\]

(3.176)

where the $\gamma^2$ terms come from both the last term in the above equation as well as the $|\omega|^2$ term above. Furthermore, note that the source $\mathcal{J}(r) \sim \mathcal{O}(g_s) + \mathcal{O}(g_s^3)$, so in the limit $g_s \to 0$ we find that (3.176) has a solution of the form $F(r, |\omega|) = c_1 + c_2 g(r)^{-2\gamma}$ with $c_1, c_2$ constants. Then we expect the complete solution for $g_s \neq 0$ to be $F(r, |\omega|) = c_1 + c_2 g(r)^{-2\gamma} + f(r, |\omega|)$. Demanding that $F(r, |\omega|)$ be regular at the horizon $r = r_h$ forces $c_2 = 0$ as $g(r_h) = 0$. We choose $c_1 = 1$ and $f = \mathcal{G} + \gamma \mathcal{H} + \gamma^2 \mathcal{K} + ...$ as a series solution in $\gamma$. Then our ansatz for the solution to (3.176) becomes

\[
F(r, |\omega|) = 1 + \mathcal{G}(r) + \gamma \mathcal{H}(r) + \gamma^2 \mathcal{K}(r) + ...
\]

(3.177)

Once we plug in the ansatz (3.177) in (3.176) we see that the resulting equation can be expressed as a series in $\gamma$:

\[
\mathcal{G}'' + \left( \frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{G}' + \mathcal{J}(1 + \mathcal{G})
+ \gamma \left\{ \mathcal{H}'' + \left( \frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{H}' + \mathcal{J} \mathcal{H} + \frac{2g'}{g} \mathcal{G}' + \left[ \frac{g''}{g} + \left( \frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] (1 + \mathcal{G}) \right\}
+ \gamma^2 \left\{ \mathcal{K}'' + \left( \frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{K}' + \mathcal{J} \mathcal{K} + \frac{2g'}{g} \mathcal{H}' + \left[ \frac{g''}{g} + \left( \frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] \mathcal{H}
+ \left( \kappa_0 + \frac{g^2}{g^2} \right) (1 + \mathcal{G}) \right\}
+ \gamma^3 \left\{ \frac{2g'}{g} \mathcal{K}' + \left[ \frac{g''}{g} + \left( \frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] \mathcal{K} + \left( \kappa_0 + \frac{g^2}{g^2} \right) \mathcal{H} + ....... \right\} + \mathcal{O}(\gamma^4) = 0
\]

(3.178)

where we have avoided showing the explicit $r$ dependences of the various parameters to avoid clutter. We have also defined $\kappa_0$ in terms of the variables of (3.174) in the following way:

\[
\kappa_0 \equiv -\frac{16}{T^2 g^2}
\]

(3.179)

Although the above equation (3.178) may look formidable there is one immediate simplification that could be imposed, namely, putting the coefficients of $\gamma_0, \gamma, \gamma^2, ...$
individually to zero. This is possible because one can view $\gamma$ to be an arbitrary parameter that can be tuned by choosing the graviton energy $\omega$ or the temperature $T_c$. This means that the zeroth order in $\gamma$ we will have the following equation:

$$G''(r) + \left[ \frac{g'(r)}{g(r)} + \frac{5}{r} + M(r) \right] G'(r) + J(r) [1 + G(r)] = 0 \quad (3.180)$$

In the above equation observe that the source $J(r)$, for cases where it is non-zero, has a complicated structure with logarithms and powers of $r$. To simplify the subsequent expressions, let us choose to work near the cut-off $r = r_c$. This is similar to the spirit of the previous section where we eventually analysed the system from the boundary point of view. Then to solve (3.180) near $r \sim r_c$ we can switch to following coordinate system

$$r = r_c (1 - \zeta) \quad (3.181)$$

Taylor expanding all the terms $J(r), g(r), \frac{1}{r_c^2 (1 - \zeta)^n}$ in (3.180) about $\zeta = 0$ and using similar arguments as the previous section, we obtain a power series solution for $G$ as:

$$G(r) = \sum_\alpha \sum_{i=0}^{\infty} \frac{a_i^{(\alpha)}}{r_c^{4i} (1 - \zeta)^{4i}} \equiv \sum_{i=0}^{\infty} a_i \zeta^i \quad (3.182)$$

Since (3.180) is a second order differential equation, we can fix two coefficients and we choose $a_0 = a_1 = 0$. Then the rest of $a_i$’s are determined by equating coefficients of $\zeta^i$ on both sides of equation (3.180). The exact solutions are listed in Appendix E. Note that all $a_i$ are proportional to $g_s$ and in the limit $g_s \to 0$, $G \to 0$.

To next order in $\gamma$ we have an equation for $H$ that also depends on the solution that we got for $G$. The equation for $H(r)$ can be taken from (3.178) as:

$$H''(r) + \left[ \frac{g'(r)}{g(r)} + \frac{5}{r} + M(r) \right] H'(r) + J(r) H(r) = -\frac{2g'(r)}{g(r)} G'(r)$$

$$- \left\{ \frac{g''(r)}{g(r)} + \left[ \frac{5}{r} + M(r) \right] \frac{g'(r)}{g(r)} \right\} [1 + G(r)] \quad (3.183)$$

To solve this we make the coordinate transformation (3.181) and plug in the series solution for $G(r)$ given above. The final result for $H$ can again be expressed as a series solution in $\zeta$ in the following way:

$$H(r) = \sum_\alpha \sum_{i=0}^{\infty} \frac{\tilde{b}_i^{(\alpha)}}{r^{2i} c(\alpha) (1 - \zeta)^{2i}} \equiv \sum_{i=0}^{\infty} b_i \zeta^i \quad (3.184)$$

We again set $b_0 = b_1 = 0$ and following similar ideas used to solve for $G$, we determine all $b_i$’s by equating coefficients in (3.183). The exact solution is given in Appendix E. Again note that all $b_i$ are of at least $O(g_s)$ and thus with $g_s \to 0$, $H \to 0$. 


Finally the second order in $\gamma$ is a much more involved equation that uses results of the previous two equations to determine $\mathcal{K}$. This is given by:

$$
\mathcal{K}''(r) + \left( \frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right) \mathcal{K}'(r) + \mathcal{J}(r) \mathcal{K}(r) = \frac{-2g'(r)}{g(r)} \mathcal{H}'(r) \tag{3.185}
$$

$$
- \left[ \frac{g''(r)}{g(r)} + \left( \frac{5}{r} + \mathcal{M}(r) \right) \frac{g'(r)}{g(r)} \right] \mathcal{H}(r) - \left( \kappa_0 + \frac{g^2(r)}{g'(r)} \right) [1 + \mathcal{G}(r)]
$$

which could also be solved using another series expansion in $\zeta^i$ (we haven’t attempted it here). Therefore combining (3.182) and (3.184) we finally have the solution for the metric perturbation:

$$
\tilde{\phi}(r, |\omega|) \pm = g(r)^{\pm |\omega|} \left[ 1 + \mathcal{G}(r) \pm i \frac{|\omega|}{4\pi T_c} \mathcal{H}(r) - \frac{|\omega|^2}{16\pi^2 T_c^2} \mathcal{K}(r) + ... \right] \tag{3.186}
$$

We can analyse this in the regime where the gravitons have very small energy, i.e $\omega \to 0$ or equivalently $\gamma \to 0$. In this limit we can Taylor expand $\tilde{\phi}(r, |\omega|)$ about $\gamma = 0$ to give us the two possible solutions:

$$
\tilde{\phi}(r, |\omega|) \pm = 1 + \mathcal{G}(r) \pm i \frac{|\omega|}{4\pi T_c} \left\{ \mathcal{H}(r) + [1 + \mathcal{G}(r)] \log g(r) \right\} \tag{3.187}
$$

$$
- \frac{|\omega|^2}{16\pi^2 T_c^2} \left\{ \mathcal{K}(r) + \mathcal{H}(r) \log g(r) + [1 + \mathcal{G}(r)] \log^2 g(r) \right\} + \mathcal{O}(|\omega|^3)
$$

which consequently means that to the first order in $\omega$ the off diagonal gravitational perturbation at low energy is given by two possible solutions corresponding to positive and negative frequencies as:

$$
\phi(r, \omega) \pm = [1 + \mathcal{G}(r)] \tilde{\phi}(\omega) \pm i \frac{|\omega|}{4\pi T_c} \left\{ \mathcal{H}(r) + [1 + \mathcal{G}(r)] \log g(r) \right\} \tilde{\phi}(\omega) \tag{3.188}
$$

As is well known following, say, [74] [60], we can define field on the right $\mathbf{R}$ and left $\mathbf{L}$ quadrant of the Kruskal plane in terms of $\phi_+(r, \omega)$ and $\phi_-(r, \omega)$ in the following way:

$$
\phi_{\mathbf{R}, \pm}(\omega, r) = \phi_\pm(\omega, r) \quad \text{in} \quad \mathbf{R}
$$

$$
\phi_{\mathbf{L}, \pm}(\omega, r) = \phi_\pm(\omega, r) \quad \text{in} \quad \mathbf{L}
$$

$$
\phi_{\mathbf{R}, \pm}(\omega, r) = 0 \quad \text{in} \quad \mathbf{L}
$$

$$
\phi_{\mathbf{L}, \pm}(\omega, r) = 0 \quad \text{in} \quad \mathbf{R}
$$

(3.189)

Now $\phi_{\mathbf{R}, \pm}, \phi_{\mathbf{L}, \pm}$ contain positive and negative frequency modes but a certain linear combination of $\phi_{\mathbf{R}, \pm}, \phi_{\mathbf{L}, \pm}$ gives purely positive or purely negative frequency modes in the entire Kruskal plane [74] [60]. Furthermore imposing that positive frequency modes are infalling at the horizon in $\mathbf{R}$ quadrant and negative frequency modes are outgoing at the horizon in $\mathbf{R}$ fixes two combinations:

$$
\phi_{\text{pos}} = e^{\omega/2T_c} \phi_{\mathbf{R}, -}(\omega, r) + e^{\omega/2T_c} \phi_{\mathbf{L}, +}(\omega, r)
$$

$$
\phi_{\text{neg}} = \phi_{\mathbf{R}, +}(\omega, r) + e^{\omega/2T_c} \phi_{\mathbf{L}, -}(\omega, r)
$$

(3.190)
With (3.190) we see that we can define fields in \( R(L) \) quadrant as linear combination of positive and negative frequency modes

\[
\phi_R(\omega, r) \equiv \tilde{a}_0 [\phi_{R,+}(\omega, r) - e^{\omega/T_c} \phi_{R,-}(\omega, r)] \equiv \phi_1 \\
\phi_L(\omega, r) \equiv \tilde{a}_0 e^{\omega/2T_c} [\phi_{L,+}(\omega, r) - \phi_{L,-}(\omega, r)] \equiv \phi_2
\]

(3.191)

where we have identified \( \phi_R(\phi_L) \) with the thermal field \( \phi_1(\phi_2) \) defined on the complex time contour which familiarly appears in the Schwinger-Keldysh propagators of real time thermal field theory. Here \( \tilde{a}_0 \) is a constant. The final physical quantity that we will extract from here will only depend on \( T \), as we will show soon.

Having got the graviton fluctuations \( \phi(r, \omega) \equiv \phi_R(\omega, r) \), we are almost there to compute the viscosity \( \eta \) using (3.168). Our next step would be to compute the Schwinger-Keldysh propagator \( G_{11}^{SK}(0, \vec{q}) \). All we now need is to write the action (3.170) as (3.169) and from there extract the Schwinger-Keldysh propagator. This analysis is almost similar to the one that we did in the previous section, so we could be brief (see also [29]). The action (3.170) can be used to get the boundary action once we shift \( \phi(r, \omega) \) to \( \phi(r, \omega) + \delta \phi(r, \omega) \) in the following way:

\[
S_{SG}^{(2)}(\phi + \delta \phi) = \frac{g(r_c)^{-1/2}}{8\pi G_N} \int \frac{d\omega d^3q}{(2\pi)^4} \int_{r_h}^{r_c} dr \left\{ A(r) \phi(r, -\omega) \phi''(r, \omega) + B(r) \phi'(r, -\omega) \phi'(r, \omega) + \right.
\]

\[
\left. C(r) \phi(r, -\omega) \phi'(r, \omega) + D(r) \phi(r, -\omega) \phi(r, \omega) + \left[ 2A(r) \phi''(r, \omega) - 2B(r) \phi''(r, \omega) - 2B'(r) \phi'(r, \omega) - C'(r) \phi(r, \omega) + 2D(r) \phi(r, \omega) + A''(r) \phi(r, \omega) + 2A'(r) \phi'(r, \omega) \right] \delta \phi(r, -\omega) + \partial_r \left[ 2B(r) \phi'(r, \omega) \phi(r, -\omega) - C(r) \phi(r, \omega) \phi'(r, -\omega) + A(r) \phi(r, \omega) \phi'(r, -\omega) - \partial_r (A(r) \phi(r, \omega)) \delta \phi(r, -\omega) \right] \right\}
\]

(3.192)

Plugging in the background value of \( \phi(r, \omega) \) will tell us that only the boundary term survives. And as before, to cancel the \( A(r) \phi(r, \omega) \delta \phi'(r, -\omega) \) we will have to add the Gibbons-Hawking term to the action [18]. The net result is the following boundary action:

\[
S_{SG}^{(2)} = \frac{g(r_c)^{-1/2}}{8\pi G_N} \int \frac{d\omega d^3q}{(2\pi)^4} \phi(r, -\omega) \left\{ \frac{1}{2} \left[ C(r) - A'(r) \right] + \left[ B(r) - A(r) \right] \frac{\phi'(r, -\omega)}{\phi(r, -\omega)} \right\} \phi(r, \omega) \right|_{r_h}^{r_c}
\]

\[
\equiv \frac{1}{8\pi G_N \sqrt{g(r_c)}} \int \frac{d\omega d^3q}{(2\pi)^4} F(\omega, r) \left|_{r_h}^{r_c} \right.
\]

(3.193)

Now comparing (3.169) with (3.193) we see that the terms between the braces combine to give us the required Schwinger-Keldysh propagator:

\[
G_{11}^{SK}(0, \vec{q}) = \lim_{\omega \to 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} \frac{F(\omega, r)}{\phi_1(r, \omega) \phi_1(r, -\omega)} \left|_{r_h}^{r_c} \right.
\]

\[
= \lim_{\omega \to 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} \left\{ \frac{1}{2} \left[ C(r) - A'(r) \right] + \left[ B(r) - A(r) \right] \frac{\phi_1'(r, -\omega)}{\phi_1(r, -\omega)} \right\} \left|_{r_h}^{r_c} \right.
\]

(3.194)
where we assume\footnote{At this point one might worry that the solution for \( \phi_1 \) is only known around \( r_c \). That this is not the case can be seen in the following way: Integration by parts gives (3.194) which says one only needs to know the value of the field \( \phi_1 \) at \( r_c \) and \( r_h \). The solution for \( \phi_1 = \frac{\partial \phi_2}{\partial \lambda} \) is given in (3.186) from which it is clear that \( \phi_1(r_h) = 0 \) as \( g(r_h) = 0 \). Furthermore to know \( \eta \) we only need to know the imaginary part of (3.194), which is evaluated using (3.196) in (3.194) and using boundary values of \( \phi_1(r_c) \) and \( \phi_1(r_h) \).} that \( \phi_1(r_h, \omega) = \tilde{a}_0 \left[ \phi_{R,+}(r_h, \omega) - e^{\omega/T_c} \phi_{R,-}(r_h, \omega) \right] \). Now to evaluate the shear viscosity from the above result we need to perform two more steps:

- Evaluate the contributions from the UV cap that we attach from \( r = r_c \) to \( r = \infty \).
- Take the imaginary part of the resulting total Schwinger-Keldysh propagator. This should give us result independent of the cut-off.

To evaluate the first step i.e contributions from the UV cap, we need to see precisely the singularity structure of \( S_{SG}^{(2)} \). The second step would then be to extract the imaginary part of SK propagator from there. Since the imaginary part can only come from the second term of (3.193), we only need to evaluate:

\[
\lim_{\omega \to 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} \left[ B(r) - A(r) \right] \frac{\phi_1'(r, -\omega)}{\phi_1(r, -\omega)} \bigg|_{r_c}^\infty \tag{3.195}
\]

with \( \phi(r, -\omega) \) being the graviton fluctuation in the regime \( r > r_c \). To analyse this let us first consider a case where \( g_s \to 0 \) and \( (G(r), H(r), S_K(r), \ldots) \to 0 \). In this limit we expect for \( r_h \leq r \leq r_c \):

\[
B(r) - A(r) = -\frac{1}{2g_s^2} g(r) r^5 + \mathcal{O}(g_s N_f) \tag{3.196}
\]

\[
\phi_1(r, \omega) = \tilde{a}_0 \left[ -\frac{\omega}{T_c} \left( 1 + \mathcal{G} - i \frac{|\omega|}{4\pi T_c} \mathcal{H} \right) + i \frac{|\omega|}{2\pi T_c} \mathcal{H} + i \frac{|\omega|}{2\pi T_c} \log g(1 + \mathcal{G}) \right]
\]

\[
\phi_1'(r, -\omega) = \frac{g'(r)}{g(r)} \left( \frac{2\pi}{4\pi^2 + \log^2 g(r)} \right)
\]

The above considerations would mean that the contribution to the viscosity, \( \eta_1 \), for this simple case without incorporating the UV cap will be:

\[
\eta_1 = \frac{r_h^4}{2\pi T_c g_s^2 G_N \sqrt{g(r_c)}} \left( \frac{1}{4\pi + \frac{1}{\pi} \log^2 g(r_c)} \right) = \frac{T_c^3 L^2}{2g_s^2 G_N} \left( \frac{1}{4\pi + \frac{1}{\pi} \log^2 g(r_c)} \right) \tag{3.197}
\]

where we have used the relations \( \pi T_c \sqrt{g(r_c)} = \left[ r_h \sqrt{h(r_h)} \right]^{-1} \) and \( h(r_h) \approx \frac{L^4}{r_h^4} \) in this limit. This helps us to write everything in terms of \( T \) and not the scale dependent temperature \( T_c \). In fact as we show below, once we incorporate the contributions from the UV cap, the \( r_c \) dependence of the above formula will also go away and
the final result will be completely independent of the cut-off. Note that in the limit $r_c \to \infty$ we recover the result for the cascading theory.

Combining all the ingredients together, the contribution to the viscosity in the limit where $(G(r), H(r), K(r), \ldots)$ etc are non-zero can now be presented succinctly as (although $\eta_1$ below doesn’t have any real meaning on the gauge theory side as this is an intermediate quantity):

$$
\eta_1 = \frac{r_c^5 \sqrt{h(r_c)}}{2g_s^2 G_N} \left\{ \frac{1 + \frac{r_c^5 g(r_c)}{4r_c^4} \left[ \frac{h'}{1+g} - \frac{hG'}{(1+g)^2} \right]}{4\pi + \frac{1}{\pi} \left[ \log g(r_c) + \frac{h}{1+g} \right]^2} \right\}
$$

(3.198)

Note that the above expression is exact for our background at least in the limit where we take the leading order $r_c^5$ singularity of the background. This is motivated from our detailed discussion that we gave in the previous section. Note that the second term in the action (3.193) is exactly the second equation of the set (3.124) whose singularity structure has been shown to be renormalisable. Thus taking the leading order singularity $r_c^5$ instead of the actual $r_c^5(\alpha)$ will not change anything if we carefully compensate the coefficients with appropriate $g_s N_f, g_s M^2 / N_f$ factors!

But this is still not the complete expression as we haven’t added the contributions from the UV cap. Before we do that, we want to re-address the singularity structure of the above expression. The worrisome aspect is the existence of $r_c^5$ factor in (3.198).

Does that create a problem for our case?

The answer turns out to be miraculously no, because of the form of $H$ and $G$ given in (3.184) and (3.182). This, taking only the leading powers of $r_c$, yields:

$$
H' = -\frac{4\hat{b}_1}{r_c^5} - \frac{8\hat{b}_2}{r_c^9} + \ldots, \quad G' = -\frac{4\hat{a}_1}{r_c^5} - \frac{8\hat{a}_2}{r_c^9} + \ldots
$$

(3.199)

killing the $r_c^5$ dependence in (3.198)\footnote{It is now easy to see why $\hat{b}_1 = \hat{a}_1 = 0$ is consistent. For non-zero $\hat{b}_1, \hat{a}_1$ there would have been additional log $r$ terms from $r_c^5(\alpha)$. These would have made the theory non-renormalisable. Thus holographic renormalisability would demand $\hat{b}_1 = \hat{a}_1 = 0$ from the very beginning – consistent with what we choose earlier.}. This would make $\eta_1$ completely finite and all the $r_c$ dependences would go as $O(1/r_c)$. Therefore we expect the contribution to the viscosity from the UV cap to go like:

$$
\eta_2 \equiv \eta_{\mid r_c^\infty} = \sum_{i=0}^{\infty} \frac{G_i}{r_c^{4i}}
$$

(3.200)

where the total viscosity will be defined as $\eta \equiv \eta_1 + \eta_2$. As this is a physical quantity we expect it to be independent of the scale. Therefore

$$
\frac{\partial \eta}{\partial r_c} = 0
$$

(3.201)
which will give us similar Callan-Symanzik type equations, as discussed in the previous section, from where we could derive the precise forms for $G_i$ in (3.200). Finally when the dust settles, the result for shear viscosity can be expressed as:

$$
\eta = \frac{T^5 \sqrt{\bar{h}(T)}}{2 g_s^2 G_N} \left[ \frac{1}{4 \pi} + \frac{1}{4} \log \left( 1 - T^4 e^{-4 \bar{N}_{uv}} \right) \right] 
$$

(3.202)

where $\alpha_k$ are functions of $T$ that can be easily determined from the coefficients $(\bar{a}_i, \bar{b}_i)$ in (3.182) and (3.184) or $(a_i, b_i)$ worked out in Appendix E; and $\bar{h}(T) \equiv \frac{T^4}{\bar{T}} + \mathcal{O}(g_s, N_f, M)$. Observe that the final result for shear viscosity is completely independent of $r_c$ and $T_c$; and only depend on $T$ and the degrees of freedom at the UV i.e through $e^{-\bar{N}_{uv}}$. Needless to say, for large enough $N_{uv}$ (which is always the case for our case because $N_{uv} \to \epsilon^{-n}, n \geq 1$), the shear viscosity is only sensitive to the characteristic temperature $T$ of the cascading theory. The interesting thing however is that the shear viscosity with finite but large enough $N_{uv}$ can be smaller than or equal to the shear viscosity with $N_{uv} \to \epsilon^{-n}, n >> 1$ i.e for the parent cascading theory provided:

$$
\alpha_k \leq \frac{1}{4 \pi^2} \sum_{n \in \mathbb{Z}} \frac{T^{4k}}{n(k - n)}, \quad n \leq k, \quad k \in \mathbb{Z}
$$

(3.203)

in the limit of small characteristic temperature $T$. This will have effect on the viscosity to entropy ratio, to which we turn next.

### 3.5 Viscosity to entropy ratio

Our final set of analysis will be to calculate the viscosity to the entropy ratio for the above two cases i.e one with only RG flow, and the other with both RG flow and curvature squared corrections. As usual the former is easier to handle so we discuss this first.

Starting with the type IIB supergravity action in ten dimension i.e the $S_{OKS-BH}$ in (3.67) the entropy is given by the Wald’s formula \cite{76, 77, 78}:

$$
S = -2\pi \int dxdydz d^5 \mathcal{M} \sqrt{\mathcal{P}} \frac{\partial \mathcal{L}_{10}}{\partial R_{abcd}} \epsilon^{ab} \epsilon^{cd}
$$

(3.204)

where the integral is over the eight dimensional surface of the horizon at $r = r_h$, $\mathcal{L}_{10}$ is the lagrangian density of the action in (3.67), $\mathcal{P}_{ab}, a, b = 1, 8$ is the induced $8 \times 8$ metric at horizon, $\epsilon_{ab}$ is the binormal normalized to $\epsilon_{ab} \epsilon^{ab} = -2$. Finally using explicit expression for the metric (3.4) and (3.5), we have

$$
s = \frac{S}{V_3} = -\frac{\pi r_h^5}{108 V_3 h_{10}^2} \int dxdydz \, d^5 \mathcal{M} \sin \theta_1 \sin \theta_2 \sqrt{h(r_h, \theta_1, \theta_2)} \frac{\partial \mathcal{L}_{10}}{\partial R_{abcd}} \epsilon^{ab} \epsilon^{cd}
$$

$$
= \frac{r_h^3 L^2}{2 g_s^2 G_N} \left\{ 1 + 3 g_s M^2 \left[ 1 + \frac{3 g_s N_f}{2 \pi} \left( \log r_h + \frac{1}{2} \right) - \frac{g_s N_f}{4 \pi} \right] \log r_h \right\}^{1/2}
$$

(3.205)
where $V_3$ is the infinite three dimensional volume and we have used the definition of five dimensional Newton’s constant $G_N$ introduced in (3.171) as well as $\bar{h}(r_h)$ introduced in (3.173). The relation above is consistent with the notion that the effective five dimensional warp factor is well approximated with taking a slice $\theta_i = \pi, \phi_i = \psi = 0$. The results would only differ by a $O(g_s N_f)$ term which, in our approximation, is very small. Note that the definition of temperature depends on the effective five dimensional warp factor and as the approximation in (3.205) is consistent with it, our computation of entropy and temperature are consistent.

Once we replace $r_h$ by the characteristic temperature $T$, we see that the entropy is only sensitive to the temperature and is independent of any other scale of the theory. Since the above result is also independent of $N_{uv}$ it would seem that the Wald formula only gives the entropy for the theory with $N_{uv} = \infty$ i.e for the parent cascading theory\textsuperscript{71}. The interesting question now would be to ask what is the entropy for the theory whose UV description is different from the parent cascading theory? In other words, what is the effect of the UV cap attached at $r = r_c$ on the entropy?

To evaluate this, observe first that in finite temperature gauge theory, entropy density of a thermalized medium having stress tensor $\langle T^{\mu \nu} \rangle = \text{diagonal}(\epsilon, P, P, P)$ is given by

$$s = \frac{\epsilon + P}{T} \quad (3.206)$$

where $\epsilon$ is the energy density, $P \equiv P_x = P_y = P_z$ is the pressure of the medium and $T$ being the temperature. With our gravity dual we can compute the stress tensor $\langle T_{pq}^{\text{med}} \rangle$ (and thus the energy $\epsilon = \langle T_{00}^{\text{med}} \rangle$ and the pressure $P = \langle T_{11}^{\text{med}} \rangle$) of the medium through equation of the form (3.66), i.e

$$\langle T_{pq}^{\text{med}} \rangle = \frac{\delta_b S_{\text{total}}}{\delta_b g_{pq}} \quad (3.207)$$

where again $p, q = 0, 1, 2, 3$ and $g_{pq}$ is the four dimensional metric obtained from the ten dimensional OKS-BH metric $g_{ij}, i, j = 0, 1, \ldots, 9$; and $\delta_b$ operation has been defined earlier. There are two ways by which we could get a four-dimensional metric from the corresponding ten-dimensional one. The first way is to integrate out the $\theta_i, \phi_i$ directions to get the four-dimensional effective theory. This is because the warp factor for our case is dependent on the $\theta_i$ directions. The second way is to work on a slice in the internal space. The slice is coordinated by choosing some specific values for the internal angular coordinates. Such a choice is of course ambiguous, and we can only rely on it if the physical quantities that we want to extract from our theory is not very sensitive to the choice of the slice. Clearly the first way is much more robust but unfortunately not very easy to implement. We will therefore follow the second

\textsuperscript{71}This can be argued by observing that fact that in a renormalisable theory, like ours, the dependences on degrees of freedom go like $O\left(e^{-N_{uv}}\right)$ corrections as we saw in the previous sections.
way by choosing the the five dimensional slice as $\theta_1 = \theta_2 = \pi, \psi = \phi_1 = \phi_2 = 0$ and thus obtaining
\[
\mathbf{g}_{\mu\nu} \equiv g_{\mu\nu}(\theta_i = \pi, \psi = \phi_i = 0) \tag{3.208}
\]
with $\mu, \nu = 0, 1, 2, 3, 4$. The next step would be to evaluate all the fluxes and the axio-dilaton on the slice. To do this we define:
\[
|H_3|^2 = |H_3|^2(\theta_i = \pi, \psi = \phi_i = 0); \quad |F_3|^2 = |\tilde{F}_3|^2(\theta_i = \pi, \psi = \phi_i = 0)
\]
\[
|F_5|^2 = |\tilde{F}_5|^2(\theta_i = \pi, \psi = \phi_i = 0); \quad |F_1|^2 = |F_1|^2(\theta_i = \pi, \psi = \phi_i = 0)
\]
\[
\Phi = \Phi(\theta_i = \pi, \psi = \phi_i = 0) \tag{3.209}
\]
Once the fluxes have been defined, we need the description for $S_{\text{total}}$ in (3.208). This is easily obtained from (3.67) as:
\[
S_{\text{total}} = \frac{1}{2\kappa_5^2} \int d^5x \ e^{-2\Phi} \sqrt{-g} \left( R - 4 \partial_i \Phi \partial^i \Phi - \frac{1}{2} |H_3|^2 \right)
\]
\[
- \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left( |F_1|^2 + |F_3|^2 + \frac{1}{2} |F_5|^2 \right) \tag{3.210}
\]
with $R$ being the Ricci-scalar for $g_{\mu\nu}$ and $g = \det \mathbf{g}_{\mu\nu}$. Note that in the definition for the slice sources $H_3, F_1, F_3, F_5$ and $R$, we still have $g_{ij}, i, j \geq 5$ which we evaluate at $\theta_i = \pi, \psi = \phi_i = 0$, treating them simply as functions and not metric degrees of freedom.

To complete the background we need the line element. Here we will encounter some subtleties regarding the choice of the black-hole factors and the corresponding $g_sN_f$ type corrections to them. With the definition of $g_{\mu\nu}$ the line element is:
\[
ds^2 = -\frac{\bar{g}_1(r)}{\sqrt{h(r, \pi, \pi)}} dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{\bar{g}_2(r)} dr^2 + \frac{1}{\sqrt{h(r, \pi, \pi)}} d\bar{x}^2 \tag{3.211}
\]
\[
\bar{g}_1(r) = g_1(r, \theta_1 = \pi, \theta_2 = \pi) = 1 - \frac{r_h^4}{r^4} + \sum_{i,j=0}^{\infty} \alpha_{ij} \frac{\log^i(r)}{r^j} = 1 + \sum_{j,\alpha} \sigma_{j}^{(\alpha)} r_{j}^{(\alpha)}
\]
\[
\bar{g}_2(r) = g_2(r, \theta_1 = \pi, \theta_2 = \pi) = 1 - \frac{r_h^4}{r^4} + \sum_{i,j=0}^{\infty} \beta_{ij} \frac{\log^i(r)}{r^j} = 1 + \sum_{j,\alpha} \kappa_{j}^{(\alpha)} r_{j}^{(\alpha)}
\]
where $\alpha_{ij}, \beta_{ij}$ are all of $\mathcal{O}(g_sN_f, g_sM)$ and only involve the parameters of the theory namely, $r_h, L$ and $\mu$ from the embedding equation (3.49) and guarantees that $\alpha_{ij}, \beta_{ij}$ are dimensionless. On the other hand $\sigma_{j}^{(\alpha)}, \kappa_{j}^{(\alpha)}$ can incorporate zeroth orders in $g_sN_f$. However note that so far we have been assuming $g_1(r) \approx g_2(r) = g(r)$, ignoring their inherent $\theta_i$ dependences, and also the inequality stemming from the choices of $\alpha_{ij}$ and $\beta_{ij}$. This will be crucial in what follows, so we will try to keep the black hole
factors unequal. These considerations do not change any of our previous results of course.

Now looking at the form of the metric, knowing the warp factor \( h(r, \pi, \pi) \) and \( \bar{g}_i(r) \), just like before we can expand the line element as AdS line element plus \( \mathcal{O}(g_s N_f, g_s M) \) corrections. We can then rewrite the line element as:

\[
ds^2 = -\frac{r^2}{L^2}[g(r) + l_1] dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{\bar{g}_2(r)} dr^2 + \frac{r^2}{L^2} (1 + l_2) d\vec{x}^2
\]

where again \( \gamma_{ij}, \zeta_{ij} \) are of \( \mathcal{O}(g_s N_f, g_s M) \) and we are taking \( h(r, \pi, \pi) = \frac{L^4}{r^4} + \mathcal{O}(g_s N_f, g_s M) \).

Such a way of writing the local line element tells us that there are two induced four-dimensional metrics at any point \( r \) along the radial direction:

\[
\begin{align*}
g^{(0)}_{pq} &\equiv \text{diagonal } (-g(r), 1, 1, 1), \\
g^{(1)}_{pq} &\equiv \text{diagonal } (-l_1, l_2, l_2, l_2)
\end{align*}
\]

where we haven’t shown the \( r^2/L^2 \) dependences. The reason for specifically isolating the four-dimensional part is to show that we can study the system from boundary point of view where the dynamics will be governed by our choice of the boundary degrees of freedom. It should also be clear, from four-dimensional point of view, the metric choice \( g^{(0)}_{pq} \) is directly related to the AdS geometry whereas the other choice \( g^{(1)}_{pq} \) is the deformation due to extra fluxes and seven branes. This decomposition is similar to the decomposition that we studied earlier.

The above decomposition also has the effect of simplifying our calculations of the energy momentum tensor \( \langle T_{\text{med}}^{pq} \rangle \). We can rewrite the total energy momentum tensor as the sum of two parts, one coming from the AdS space and the other coming from the deformations, in the following way:

\[
\langle T_{\text{med}}^{pq} \rangle = \frac{\delta_b S^{[0]}_{\text{total}}}{\delta_b g^{0}_{pq}} + \frac{\delta_b S^{[1]}_{\text{total}}}{\delta_b g^{1}_{pq}}
\]

where \( S^{[0]}_{\text{total}} \) is zeroth order in \( g_s N_f, g_s M \) and \( S^{[1]}_{\text{total}} \) is higher order in \( g_s N_f, g_s M \).

Note that \( \langle T_{\text{med}}^{pq} \rangle_{\text{AdS}} = \frac{\delta_b S^{[0]}_{\text{total}}}{g_0 g^{0}_{pq}} \) is the well known AdS/CFT result obtained from the analysis of [13] [14] [15] [16] in the limit \( r_c \to \infty \). With the \( \mathcal{O}(1/r) \) series expansion of our metric \( g^{0}_{00} = 1 - r_h^4/r^4, g^{0}_{11} = g^{0}_{22} = g^{0}_{33} = 1 \), the result at the boundary is:

\[
\begin{align*}
\langle T_{\text{med}}^{00} \rangle_{\text{AdS}} &= \frac{r_h^4}{2g_4^2 G_N} = \frac{T^4}{2g_4^2 G_N} \\
\langle T_{\text{med}}^{mn} \rangle_{\text{AdS}} &= 0 \quad m, n = 1, 2, 3
\end{align*}
\]
This only gives the CFT stress tensor as we evaluate the tensor on the AdS boundary at infinity, reproducing the expected first term of (3.205). How do we then evaluate the $O(g_s N_f, g_s M)$ contributions from the deformed AdS part i.e the energy momentum tensor $\langle T^p_q \rangle_{\text{def}}$ at any $r = r_c$ cut-off in the geometry?

In fact the procedure to evaluate exactly such a result has already been discussed in the last two sections: namely the wake analysis in section 3.3 and shear viscosity analysis in section 3.4. Therefore without going into any details, the final answer after integrating by parts, adding appropriate Gibbons-Hawkings terms and then using the equation of motion for $g_{pq}$, we have

$$S_{\text{total}}^{[1]} = \frac{1}{8\pi G_N} \int \frac{d^4q}{(2\pi)^4} \sqrt{|g(r_{\text{max}})|} \left\{ \tilde{C}_{mn}^{\alpha\beta}(r, q) - \tilde{A}_{mn}^{\alpha\beta}(r, q) \right\} \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \left[ \tilde{B}_{1}^{mn}(r, q) - \tilde{A}_{1}^{mn}(r, q) \right] \left[ \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) \right] + \left( \tilde{E}_{1}^{m} - \tilde{E}_{1}^{n} \right) \Phi_m^{[1]}(r, q) \right\}
$$

where $r_{\text{max}} \equiv r_c(1 - \zeta)$. The values of the coefficients are given in Appendix F. The above form is exactly as we had before, and so all we now need is to get the mode expansion for $\Phi_m^{[1]}$. Note however that the subscript $m$ can take only two values, namely $m = 0, 1$ as there are only two distinct fields $g_{00}^{[1]}$ and $g_{11}^{[1]} = g_{22}^{[1]} = g_{33}^{[1]}$. Therefore our proposed mode expansion is:

$$\Phi_m^{[1]} = g_{mn}^{[1]} = \sum_{\alpha} \sum_{i=0}^{\infty} \frac{s_{m\alpha}^{(i)[\alpha]}}{r_c^{(i)}(1 - \zeta)^i} (3.217)$$

Just like the wake and shear viscosity analysis, the action in (3.216) is divergent due to terms of $O(r_c^2), O(r_c^3)$ and hence we need to renormalise the action. The equations for renormalisation are identical to the set of equations (3.126) -- (3.131), and therefore we analogously subtract the counter terms to obtain the following renormalised action:

$$S_{\text{ren}}^{[1]} = \frac{1}{8\pi G_N} \int \frac{d^4q}{(2\pi)^4} \left[ 1 - \frac{r_h^4}{r_c^4(1 - \zeta)^4} \right] \sum_{\alpha,\beta} \left\{ \left( \sum_{i=0}^{\infty} \tilde{A}_{mn}^{(i)[\alpha]}(r_{c\alpha}) \right) \tilde{g}_{mn}^{[1]} \Phi_m \Phi_n + X[r_{c\alpha}] + \left( \sum_{i=0}^{\infty} \tilde{g}_{mn}^{[1]}(r_{c\alpha}) \right) \tilde{h}_{mn}^{[1]} \Phi_m \Phi_n + H_{[\alpha]}^{[1]} \left[ s_{mn}^{(4)[\beta]} \Phi_m + s_{mn}^{(5)[\beta]} \Phi_n + s_{mm}^{(5)[\beta]} \Phi_m \Phi_n \right] + K_{[\alpha]}^{[1]} \left[ -4s_{mn}^{(4)[\beta]} \Phi_m - 4s_{mn}^{(5)[\beta]} \Phi_n + s_{mn}^{(5)[\beta]} \Phi_m + s_{mm}^{(5)[\beta]} \Phi_n \right] \right\}
$$

where the radial coordinate is measured at the two boundaries $r_h$ and $r_c(1 - \zeta)$ and $\Phi_m$ are independent of $r$ as before. Note that $X[r_{c\alpha}]$ is a function independent of $\Phi_m$ and appears for generic renormalised action.
Now the generic form for the energy momentum tensor is evident from looking at the linear terms in the above action (3.218). This is again the same as before. However now we also need the entropy from the energy-momentum tensor as in (3.200). The result for the energy-momentum tensor at \( r = r_c \) is given by:

\[
\langle T_{mn}^{\text{med}} \rangle \equiv \frac{1}{8\pi G_N} \int \frac{d^4q}{(2\pi)^4} \frac{1}{\sqrt{g(r_c)}} \sum_{\alpha,\beta} \left[ (H_{[\alpha]}^{mn[1]} + H_{[\alpha]}^{nm[1]}) s_{nm}^{(4)[\beta]} - 4(K_{[\alpha]}^{nm[1]}) s_{nm}^{(4)[\beta]} \right] + K_{[\alpha]}^{nm[1]} s_{nm}^{(4)[\beta]} + (K_{[\alpha]}^{mn[1]} + K_{[\alpha]}^{nm[1]}) s_{nm}^{(5)[\beta]} + \left( \sum_{i=0}^{\infty} \frac{T_{n(i)[1]}}{r_{c}(\alpha)} \right) \delta_{nm} \right] (3.219)
\]

The explicit expressions for the coefficients listed above, namely, \( H_{[\alpha]}^{mn[1]} \), \( K_{[\alpha]}^{mn[1]} \), \( \tilde{b}_{n(i)[1]}^{(\alpha)} \) and \( s_{nm}^{(i)[1]} \) are given in Appendix F.

To complete the story we need the contribution from the UV cap. This is similar to our earlier results. The final expression for the ratio of the energy-momentum tensor to the temperature takes the simple form:

\[
\frac{\langle T_{mn}^{\text{med}} \rangle}{T_b} \equiv \frac{\pi T\sqrt{h(T)}}{8\pi G_N} \int \frac{d^4q}{(2\pi)^4} \sum_{\alpha,\beta} \left[ (H_{[\alpha]}^{mn[1]} + H_{[\alpha]}^{nm[1]}) s_{nm}^{(4)[\beta]} - 4(K_{[\alpha]}^{nm[1]}) s_{nm}^{(4)[\beta]} \right] + K_{[\alpha]}^{nm[1]} s_{nm}^{(4)[\beta]} + (K_{[\alpha]}^{mn[1]} + K_{[\alpha]}^{nm[1]}) s_{nm}^{(5)[\beta]} + \sum_{j=0}^{\infty} \frac{\tilde{b}_{n(j)[1]}^{(\alpha)}}{r_{c}(\alpha)} \delta_{nm} e^{-jN_{uv}} \right] (3.220)
\]

We would like to make a few comments here: First, observe that the final result is independent of our choice of cut-off. Secondly, in the string frame there should be a \( 1/g_s^2 \) dependence. Finally, we can pull out a \( T^4 \) term because the coefficients have an explicit \( r_h^4 \) dependence (see Appendix F). This means that both from the AdS and the deformed calculations performed above we can show that the entropy is of the form:

\[
s = \frac{T^5 \sqrt{h(T)}}{2g_s^2 G_N} \left[ 1 + \mathcal{O}(g_s N_f, g_s M, e^{-N_{uv}}) \right] \quad (3.221)
\]

where the first part is from (3.218) and the second part is from (3.220). The result for the parent cascading theory is (3.205), and so we should regard (3.221) as the entropy for the theory with \( N_{uv} \) degrees of freedom at the boundary. Of course in the limit \( N_{uv} \rightarrow \epsilon^{-n}, n >> 1 \) we should recover the entropy formula (3.205) for the parent theory. All in all we see that the correction due to \( N_{uv} \) degrees of freedom only goes as \( e^{-N_{uv}} \), so in practice this is always small for the type of \( N_{uv} \) that we consider here. This means that we can use the entropy for the parent cascading theory to estimate the viscosity by entropy ratio for a system with \( N_{uv} \) degrees of freedom at the UV as:

\[
\frac{\eta}{s} = \left[ \frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4kN_{uv}}}{4\pi + \frac{1}{\pi} \log^2 (1 - T^4 e^{-4N_{uv}})} \right] \quad (3.222)
\]
where we see that the boundary entropy term (3.205) neatly cancels the \( T^3 \) coefficient in the viscosity (3.202) to give us the precise bound of \( \frac{1}{4\pi} \) when \( N_{uv} \to \infty \). Of course from our other analysis (3.221) we might expect a \( O(g_s M, g_s N_f, e^{-N_{uv}}) \) contribution that would make (3.222) saturate the celebrated bound \( \frac{1}{4\pi} \) if the total entropy density factors compensate the factors coming from the viscosity. This would seem consistent with, for example, [79]. In fact our conjecture would be for non-zero \( M, N_f \) and \( N_{uv} \to \epsilon, n >> 1 \), the bound is exactly saturated\(^{73}\) i.e \( \eta_s = \frac{1}{4\pi} \).

Our second and final step would be to incorporate both the RG flow as well as curvature square corrections. As we discussed before the curvature squared corrections are typically of the form \( c_3 R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \) with \( c_3 \) being the coefficient (3.163) that we computed before.

The crucial point here is that (see [4] where this has also been emphasised) in the presence of curvature squared corrections the five dimensional metric itself changes to:

\[
 ds^2 = \frac{-g_1(r)}{\sqrt{h(r, \pi, \pi)}} dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{g_2(r)} dr^2 + \frac{d \vec{x}^2}{\sqrt{h(r, \pi, \pi)}} \tag{3.223}
\]

where the black hole factors \( g_i \) are no longer given by (3.13) or its simplified version (3.86). They take the following forms:

\[
 g_1(r) = 1 - \frac{r^4}{r^4} + \alpha + \gamma \frac{r^8}{r^4} + \tilde{\alpha}_{mn} \frac{\log^m r}{r^n} \\
 g_2(r) = 1 - \frac{r^4}{r^4} + \alpha + \gamma \frac{r^8}{r^4} + \tilde{\beta}_{mn} \frac{\log^m r}{r^n} \tag{3.224}
\]

where \( \tilde{\alpha}_{mn}, \tilde{\beta}_{mn} \) are all of \( O(g_s N_f, g_s M) \) and can be worked out with some effort (we will not derive their explicit forms here). Similarly we could also express (3.224) in terms of inverse powers of \( r \) to have good asymptotic behavior. Observe that we can still impose \( g_1 \approx g_2 \) because the corrections are to \( O(g_s N_f, g_s M) \), although all our previous analysis have to be changed in the presence of curvature corrections because the explicit values of \( g_i(r) \) have changed. We will address these issues in the sequel [55]. Finally (\( \alpha, \gamma \)) are given by

\[
 \alpha = \frac{4c_3 \kappa}{3L^2}, \quad \gamma = \frac{4c_3 \kappa}{T^2} \tag{3.225}
\]

At this point one might get worried that the metric perturbation on this background would become very complicated. On the contrary our analysis becomes rather simple

\(^{72}\)Provided of course if we assume that \( \alpha_k \)'s are more general now, being functions of \( T, g_s M, g_s N_f \). This way even for non-zero \( M, N_f \), whenever we have \( N_{uv} \to \infty \) the bound is exactly \( \frac{1}{4\pi} \).

\(^{73}\)Note that any possible deviations from \( \frac{1}{T} \) due to (3.199) in (3.198) cannot happen because the underlying holographic renormalisability will make \( \alpha_1 = \beta_1 = 0 \), as discussed earlier. Thus the bound in itself is a rather strong result.

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once we ignore terms of $O(c_3 g_s M, c_3 g_s N_f)$ (which is a valid approximation with $c_3 << 1$). In this limit the metric perturbation can be written simply as a linear combination of the terms proportional to $c_3$ in $\Phi$ which appears in \[7\] and our solution (3.186, 3.187) derived for RG flow. The final result is:

$$\tilde{\phi}(r, |\omega|)_{\pm, R^2} = 1 \pm i \frac{|\omega|}{4\pi T_c} \left\{ H(r) + \frac{\alpha r^8 + \gamma r^8_h}{r^8 g(r)} - \alpha + 4\gamma r^4_h \right\}$$

$$+ G(r) - \frac{|\omega|^2}{16\pi^2 T_c^2} \left\{ \mathcal{K}(r) + H(r) \log g(r) + \frac{[1 + G(r)]\log^2 g(r)}{} \right\}$$

$$+ O(|\omega|^3) + O(c_3 g_s N_f) + O(c_3 g_s M)$$

(3.226)

where we have written (3.13) as:

$$g_1(r) = g(r) + O(c_3 g_s N_f) + O(c_3 g_s M)$$

$$g_2(r) = g(r) + O(c_3 g_s N_f) + O(c_3 g_s M)$$

(3.227)

with $g(r)$ being the usual black hole factor defined in (3.86). Of course as emphasised above, this is valid only in the limit $c_3 << 1$, which at least for our background seems to be the case (see (3.163)).

The above corrections are not the only changes. The entropy computed earlier also gets corrected and therefore the horizon can no longer be at $r = r_h$. To evaluate the correction to entropy we again ignore the terms of $O(c_3 g_s M, c_3 g_s N_f)$. In this limit the correction terms are precisely given by the analysis of [7] and are proportional to the $c_3$ factor (3.163) as expected. This means that the final result for $\eta/s$ including all the ingredients i.e RG flows, Riemann square corrections as well as the contributions from the UV caps; is given by:

$$\frac{\eta}{s} = \left[ \frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4k N_{uv}}}{4\pi + \frac{1}{4\pi} \log^2 (1 - T^4 e^{-4N_{uv}})} \right]$$

$$- \frac{c_3 \kappa}{3L^2 (1 - T^4 e^{-4N_{uv}})^{3/2}} \left[ B_o (4\pi^2 - \log^2 C_o) + 4\pi A_o \log C_o \right]$$

$$+ 16\pi^2 \log^2 C_o$$

(3.228)

where we see two things: one, the bound is completely independent of the cut-off $r = r_c$ in the geometry, and two, the bound decreases in the presence of curvature square corrections even when $N_{uv} \to \epsilon^{-n}$ with $n = O(1)$.\textsuperscript{74} The constants appearing

\textsuperscript{74}In [24], non-relativistic systems that appear to have no lower bound were constructed. However, these are systems which necessarily require large chemical potentials and low temperature. In highly relativistic system created at high energy colliders such as the RHIC or the LHC, chemical potentials are small and temperature is high. Our discussion here assumes that the system under discussion has such properties so that the use of thermodynamic identity $\epsilon + P = Ts$ is valid. Hence, our discussion here is in no direct conflict with the models constructed in [24].
in (3.228) are defined as:

\begin{align*}
C_0 &= 1 - T^4 e^{-4N_{uv}} \\
A_0 &= -18 T^8 e^{-8N_{uv}} + (3T^8 e^{-8N_{uv}} - 47T^4 e^{-4N_{uv}}) \log C_0 + 26T^4 e^{-4N_{uv}} \\
B_0 &= -88\pi T^8 e^{-8N_{uv}} + 48\pi T^4 e^{-4N_{uv}} + 48
\end{align*}

(3.229)

This is consistent with [7], and the only violation of \( \eta/s \) may be entirely from the \( c_3 \) factor provided the increase in bound from the first term of (3.228) is negligible, as we discussed earlier for (3.222). This means in particular:

\[ \frac{\eta}{s} = \frac{1}{4\pi} - n_b c_3 + O(\mathcal{E} e^{-N_{uv}}) \]

(3.230)

where \( n_b \) can be extracted from (3.228). In this paper however we will not study the subsequent implication of this result, for example whether there exists a causality violation in our theory due to the curvature corrections as in [8]. We hope to address this in the sequel [55].

To see the explicit behavior of \( \eta/s \) we can plot the function (3.228) (see figure 12). Of course our result is valid for infinitely large UV degrees of freedom i.e \( N_{uv} \to \epsilon^{-n}, n \geq 1 \), but we can extrapolate the result to see what properties we get for small enough UV degrees of freedom. Incidentally with the geometry cut-off at small \( r_c \) and an AdS cap attached from \( r = r_c \) to \( r = \infty \) would start resembling standard QCD as we briefly mentioned earlier. For such a scenario we expect the UV degrees of freedom to be smaller than the UV degrees of freedom for the parent cascading theory i.e \( N_{uv} \) would approach infinity at a smaller rate.

4. Conclusions

In this work we have studied the dynamic response of a strongly-coupled, strongly-interacting medium to a fast quark. The drag coefficient and the wake resulting from the parton-medium interaction was evaluated. The calculations were performed by constructing a gravity dual inspired by the Klebanov-Strassler model with D7 branes to take into account fundamental quarks. The gauge dual then has a running coupling constant, unlike models stemming from a pure AdS geometry, with features close to that of QCD. This procedure in fact allows the consideration of a family of gauge theories which have a well-defined completion in the UV, beyond a cut-off scale. We could also show that physical results were independent of the choice of this cut-off. We have applied this model to the calculation of the drag force and the wake left by a moving quark in the strongly interacting medium were computed. In addition, we have evaluated the ratio of shear viscosity to entropy density - \( \eta/s \) - and shown the violation of the bound conjectured in Ref. [4], for a range of parameter values. It
is important to test and verify the robustness of this limit, and we view the current work as contributing to this effort. More work is needed in order to identify the size and extent of this violation, at the moment this violation is only parametric and our parameters need to have better defined physical origins. In the end, one may need to rely on the empirical identification of key quantities, like transport coefficients for example \[80\]. In this regard, the role of heavy ion experiments at RHIC and at the LHC can’t be overestimated.

$$\eta/s$$ with and without Riemann Square term. The x-axis is defined as \(k = \frac{1}{N_{uv}}\) where \(N_{uv}\) is the UV degrees of freedom and \(T\) is the characteristic temperature of the cascading theories. For the parent cascading theory \(k \to \infty\) and we see a violation of the bound (the solid blue line). As \(k\) decreases (assuming this is possible) the red dashed line dips slightly below the \(1/4\pi\) axis, but the solid blue line remains considerable below the \(1/4\pi\) axis. For \(k\) sufficiently small the bound is not violated. However all the models that we studied in this paper can only realise the large \(k\) limit.

**Figure 12:** Plot for \(\eta/s\) with and without Riemann Square term. The x-axis is defined as \(k = \frac{1}{N_{uv}}\) where \(N_{uv}\) is the UV degrees of freedom and \(T\) is the characteristic temperature of the cascading theories. For the parent cascading theory \(k \to \infty\) and we see a violation of the bound (the solid blue line). As \(k\) decreases (assuming this is possible) the red dashed line dips slightly below the \(1/4\pi\) axis, but the solid blue line remains considerable below the \(1/4\pi\) axis. For \(k\) sufficiently small the bound is not violated. However all the models that we studied in this paper can only realise the large \(k\) limit.

**Note added:** In our recent paper \[81\], we have given a concrete example of a UV complete theory that consistently proves all the statements that we made here. Our new geometry has a UV cap given by an asymptotic AdS-Schwarzchild geometry. The IR dynamics of our theory is captured by the OKS-BH geometry. The deformation of the OKS-BH geometry at the boundary is indeed captured by an *interpolating* geometry that ties smoothly, in the presence of sources and fluxes, to the asymptotic AdS-Schwarzchild geometry. All the relevant details like changes in stress-tensor,
entropy etc due to the UV cap can now be calculated. In addition to that we have been able to study zero temperature linear confinement as well as high temperature quarkonium suppression and melting from our UV complete theory. Our results are also consistent with many of the earlier predictions made in the literature using completely different techniques than ours [82, 83, 84, 85].

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A. Back reaction effects in the AdS Black-Hole background: A toy example

The following analysis, although independent of the main calculations in the paper, serves as an interesting warm-up example where we can study metric perturbations due to fluxes and D7 brane in a controlled AdS background. This will prepare us for the analysis of the next section where we study metric perturbations in the non-trivial OKS-BH geometry due to fluxes, D7 brane and strings.

To start-off consider an $AdS_5$ Black Hole (AdS-BH) metric given in the following way:

$$ds^2 = \frac{1}{u^2} \left[ -f(u) dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{du^2}{f(u)} \right]$$

$$= g^{f\mu\nu}(u) dx^\mu dx^\nu$$  \hspace{1cm} (A.1)

with $f(u) = 1 - \frac{u^4}{u_h^4}$, $u_h$ is black hole horizon, $u = 0$ is the boundary, and $\mu$ or $\nu = 0, 1, 2, 3, 4$. The above metric is asymptotically AdS [16] as

$$u^2 g^{f\mu\nu}|_{u=0} = \eta_{ij}$$  \hspace{1cm} (A.2)

where $\eta_{ij}$ is the Minkowski metric, with $i$ or $j = 0, 1, 2, 3$. Note that we have expressed the metric in terms of $u \equiv \frac{1}{r}$ coordinate. We could as well express everything in terms of $r$ coordinates (as we did in the main text).

To this background, first let us now add a D7 brane by switching off the black-hole factor. The D brane Born Infeld (DBI) action for D7 brane in 5 dimensional AdS-BH background is given by [17]

$$S_{\text{DBI}} = \int d^5x \mathcal{L}(x) = g_s \int d^5x \sqrt{g(x)} \cos^3 \Phi(x) \sqrt{1 + g^{\mu\nu}(x) \partial_\mu \Phi(x) \partial_\nu \Phi(x)}$$

$$= \sqrt{\frac{g^{\mu\nu}}{g^{\alpha\beta}}} \partial_\mu \Phi \partial_\nu \Phi$$  \hspace{1cm} (A.3)

where the metric $g_{\mu\nu}$ is the metric of the resulting background, i.e $g_{\mu\nu}(x) = g^{f=1}_{\mu\nu}(x)$. In the above, $g_s$ is coupling constant, and $\Phi(x)$ is a scalar field describing the D7 brane. The Euler-Lagrangian equation for the $\Phi(x)$ field leads to:

$$\Box \Phi + 3 \tan \Phi - \frac{1}{2} \frac{g^{\mu\nu} \partial_\mu \partial_\nu \left( g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right)}{1 + g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi} = 0$$

$$= 0$$  \hspace{1cm} (A.4)

where $\Box = \frac{\partial_\mu \left( \sqrt{g^{\mu\nu}} \partial_\nu \right)}{\sqrt{g}}$. As (A.4) is a second order non linear differential equation, the solution can be written in the form [17]:

$$\Phi(u, \vec{x}, t) = u \sum_{i=0}^{\infty} \left[ \phi_i(\vec{x}, t) u^i + \psi_i(\vec{x}, t) u^i \log u \right]$$

$$= \phi(\vec{x}, t) u + \psi(\vec{x}, t) u \log u$$  \hspace{1cm} (A.5)
Now considering first order $O(g_s)$ in the perturbation and terms only upto $O(u^3)$, we find (see [17] for details),

$$\Phi(u, \vec{x}, t) = cu + \frac{c^3 u^3}{6}$$  \hspace{1cm} (A.6)

where $c$ is a constant. Using the lagrangian (A.3) and the solution (A.6), we can find the stress energy tensor for the D7 brane:

$$T^\mu_\nu(x) = \frac{\partial L}{\partial (\partial_\mu \Phi(x))} \partial_\nu \Phi(x) - L \delta^\mu_\nu$$

$$= \sum_{i=1}^{\infty} g_s^i T^{(i)}_\mu(x)$$  \hspace{1cm} (A.7)

where in the last line we have expressed this in powers of $g_s$ to make it more generic.

In fact, as we will see below, the energy momentum tensor can be assumed to come from various sources that include the D7 brane as well as background fluxes.

Once we introduce D7 brane as well as the black-hole factor $f$ we need to modify the scenario. In particular there will be non-trivial axio-dilaton background that we have been ignoring. This will generally back-react on the geometry and make the system non-AdS. To preserve the AdS like configuration we will need more D7 branes (and also orientifold planes so that we could lift the system to a F-theory configuration [30]) as well as other background fluxes. Let us denote the action of all the fluxes etc. in five dimensional space as $S_{\text{fluxes}}$, the action of the O-planes as $S_{\text{planes}}$ and the action of the string from the D7 branes to the black-hole as $S_{\text{strings}}$.

This means that our total action is:

$$S_{\text{total}} = \int d^5x \sqrt{G} R + S_{\text{DBI}} + S_{\text{fluxes}} + S_{\text{planes}} + S_{\text{strings}}$$  \hspace{1cm} (A.8)

where $R$ is the Ricci scalar for the metric $G_{\mu\nu} = g^f_{\mu\nu} + g_s h_{\mu\nu}$, and $S_{\text{DBI}}$ now include the total action of all the D7 branes. We then expect the Euler-Lagrange equation for $G_{\mu\nu}$ using the action (A.8) determines the Einstein tensor:

$$G_{\mu\nu}(x) = \sum_{i=1}^{\infty} g_s^i G^{(i)}_{\mu\nu}(x) = \sum_{i=1}^{\infty} g_s^i T^{(i)}_{\mu\nu}(x)$$  \hspace{1cm} (A.9)

where the energy-momentum tensor is determined by varying the actions $S_{\text{DBI}} + S_{\text{fluxes}} + S_{\text{planes}}$ wrt the background metric $G_{\mu\nu}$. Now equating coefficients of $g_s$ in the above expansion, we have the following equation for $h_{\mu\nu}$:

$$G^{(1)}_{\mu\nu}[h_{\mu\nu}(x)] = T^{(1)}_{\mu\nu}(x)$$  \hspace{1cm} (A.10)

Notice that as $h_{\mu\nu}$ is symmetric tensor, it has fifteen degrees of freedom. We use coordinate transformations to fix ten degrees of freedom and treat $h_{\mu\nu}$ as a diagonal
matrix. Consider the coordinate transformation $x^a \rightarrow x'^a = x^a + g_a e^a$. Under this transformation

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} - D_\mu e_\nu - D_\nu e_\mu$$  \hspace{1cm} (A.11)

which is a gauge transformation. Thus we can fix five components (say $h_{01} = h_{02} = h_{03} = h_{04} = h_{12} = 0$) with the above transformations. Just like in electromagnetism, this does not completely fix the gauge. We can add to $e^a$ another set of functions $e'^a$ which obey $\Box e'^a = 0$ and leaves Einstein tensor unchanged. Demanding that $e'^a$ also leaves the fixed components invariant, we can further fix another five components (say $h_{13} = h_{14} = h_{23} = h_{24} = h_{34} = 0$). Thus we have diagonal $h_{\mu\nu}$. Observe that this is not the most generic choice, but will suffice for this toy example provided we can have the energy-momentum tensors to be diagonal. We will also assume that the fluxes and branes/planes are independent of $x', t$ and thus the metric perturbations $h_{\mu\nu}$ they induce should only depend on $u$ and we should have $h_{11} = h_{22} = h_{33}$. With all these, we have

$$h_{\mu\nu} = \begin{pmatrix} k(u) & 0 & 0 & 0 & 0 \\ 0 & h(u) & 0 & 0 & 0 \\ 0 & 0 & h(u) & 0 & 0 \\ 0 & 0 & 0 & h(u) & 0 \\ 0 & 0 & 0 & 0 & l(u) \end{pmatrix}$$  \hspace{1cm} (A.12)

We observe that (A.10) is a second order non-linear differential equation for $h_{\mu\nu}$. This will have a general solution:

$$m(u) = \frac{1}{u} \sum_{j}^{\infty} \left[ m_j u^j + \tilde{m}_j u^j \log u \right]$$  \hspace{1cm} (A.13)

where $m(u) = k(u), h(u), l(u)$ and $m_j = k_j, h_j, l_j$, $\tilde{m}_j = \tilde{k}_j, \tilde{h}_j, \tilde{l}_j$ are constants.

As we mentioned before the above form of the metric (A.12) will make sense if the sources can also be made diagonal. In the presence of fluxes, branes, planes and non-trivial profile of the strings this doesn’t look very difficult to achieve. We will therefore assume that the energy-momentum tensors are also diagonal, and the right hand side of (A.10) can be written as a Lorentz series as:

$$\mathcal{T}^{(1)}_{\mu\nu}(u) = \frac{t^{(1)}_{\mu\nu(-7)}}{u^7} + \frac{t^{(1)}_{\mu\nu(-6)}}{u^6} + \cdots + t^{(1)}_{\mu\nu(3)} u^3 + \cdots$$  \hspace{1cm} (A.14)
Once (A.14) is made diagonal, we can satisfy (A.10) easily. The explicit expressions for the Einstein tensor are:

\[
G^{(1)}_{00}(u) = \frac{3}{4a^2u_h^2} \left[ \left( -4u^8 u_h^8 + 2u^{12} u_h^4 + 2u^4 u_h^{12} \right) \frac{d^2h(u)}{du^2} + \left( 6u^{11} u_h^4 - 8u^7 u_h^8 + 2u^3 u_h^{12} \right) \frac{dh(u)}{du} \\
- 8 \left( u_h^4 - u^4 \right) u^2 u_h^8 h(u) + \left( 2u^3 u_h^{12} - 6u^7 u_h^8 + 6u^{11} u_h^4 - 2u^{15} \right) \frac{dl(u)}{du} + \left( 12u^2 u_h^{12} \\
- 36u^6 u_h^8 + 36u^{10} u_h^4 - 12u^{14} - 8(u_h^4 - u^4) (2u^2 u_h^8 - 2u^6) \right) l(u) - 8u^2 u_h^{12} k(u) \right] \tag{A.15}
\]

\[
G^{(1)}_{ii}(u) = \frac{1}{B(u)} \left[ \left( -4u^8 u_h^{16} + 4u^{16} u_h^4 - 12u^{12} u_h^8 + 12u^8 u_h^{12} \right) \frac{d^2h(u)}{du^2} \\
+ \left( -4u^3 u_h^{16} + 28u^7 u_h^{12} - 4u^{11} u_h^8 + 20u^{15} u_h^4 \right) \frac{dh(u)}{du} \\
+ \left( 80u^6 u_h^{12} - 32u^2 u_h^{16} - 64u^{10} u_h^8 + 16u^{14} u_h^4 - (96u^4 u_h^{16} (u_h^4 - u^4)^4) B^{-1}(u) \right) h(u) \\
+ \left( 20u^7 u_h^{12} - 6u^3 u_h^{16} - 24u^{11} u_h^8 + 12u^{15} u_h^4 - 2u^{19} \right) \frac{dl(u)}{du} + \left( -36u^6 u_h^{16} - 176u^{10} u_h^8 \\
+ 88u^4 u_h^{14} + 136u^6 u_h^{12} - 12u^{12} u_h^{18} - (192u^4 u_h^{12} (u_h^4 - u^4)^5) B^{-1}(u) \right) l(u) \\
+ \left( 2u^4 u_h^{16} + 2u^{12} u_h^8 - 4u^8 u_h^{12} \right) \frac{d^2k(u)}{du^2} + \left( -2u^{11} u_h^8 + 2u^3 u_h^{16} \right) \frac{dk(u)}{du} \\
+ \left( 40u^2 u_h^{16} - 24u^6 u_h^{12} - (96u^4 u_h^{20} (u_h^4 - u^4)^3) / B(u) \right) k(u) \right] \tag{A.16}
\]

\[
G^{(1)}_{44}(u) = \frac{3}{C'(u)} \left[ \left( -2u^7 + 6u^3 u_h^4 \right) \frac{dh(u)}{du} + \left( 32u^4 u_h^8 C^{-1}(u) \right) k(u) \\
+ \left( -4u^6 - 4u^2 u_h^4 + 64u^4 u_h^4 (-u_h^4 - u^4) C^{-1}(u) \right) h(u) \right] \tag{A.17}
\]

where we have defined \( B(u) \) and \( C(u) \) in the following way:

\[
B(u) = 4u^2 u_h^8 (u_h^4 - u^4)^2, \quad C(u) = -4u^2 (u_h^4 - u^4) \tag{A.18}
\]

with \( ii = 11, 22, 33 \) from here on due to our assumed diagonal nature of the perturbation. Once the perturbations become non-diagonal the analysis will become more involved as we saw in section 3.3.

By matching coefficients of various powers of \( u \) in (A.10) we can solve for \( h_{\mu\nu} \) order by order. We obtain equations for the constants \( m_j, \tilde{m}_j \) by plugging in (A.13) in (A.10) and using the explicit expressions (A.13) onwards and the expansion (A.14). Doing so, we observe that it is sufficient to have \( m_j, \tilde{m}_j = 0 \) for \( j < -6 \).

The equations obtained by matching coefficients of \( \frac{1}{d} \) for \( -6 \leq j \leq -3 \) are:

\[
\frac{4_{\mu=0\nu=j-1}}{3} = 2(-3 - 2j + j^2) h_j + (-6 + 2j)(l_j - 8k_j + 2(j + 2) \tilde{h}_j + 2 \tilde{l}_j
\]
\[ 4 t^{(1)}_{ii(j-1)} = -4(3 - 2j + j^2)h_j - 6(-3 + j)l_j + 2(-23 - 2j + j^2)k_j \]
\[ -4(-2 + j + j^2)\tilde{h}_j - 6\tilde{l}_j + 2(-2 + j + j^2)\tilde{k}_j \]

\[ \frac{4 t^{(1)}_{44(j-1)}}{3} = -(2 + 6j)h_j + (6 + 2j)k_j - 6\tilde{h}_j + 2\tilde{k}_j \]  

(A.19)

where because of the form of \( G_{\mu\nu} \) all the above equations are independent of \( u_h \). This will not be the case for \( j > -3 \). Finally, matching coefficients of \( \log u \) again for \(-6 \leq j \leq -3 \) gives us:

\[ 2(-3 - j)\tilde{h}_j + 2(-3 + j)\tilde{l}_j - 8\tilde{k}_j = 0 \]
\[ -4(2 - j)\tilde{h}_j - 6(-3 + j)\tilde{l}_j + 2(-23 - j)\tilde{k}_j = 0 \]
\[ (2 + 6j)\tilde{h}_j + (6 + 2j)\tilde{k} = 0 \]  

(A.20)

For a given \(-6 \leq j \leq -3 \), we have six equations and six unknowns in (A.19) and (A.20) and thus we can solve exactly. For \( j > -3 \), as we mentioned earlier, equations governing \( m_j, \tilde{m}_j \) will in general depend\(^75\) on \( m_{j-p}, \tilde{m}_{j-p} \) for \( 0 < p \leq j + 6 \). Once we solve for \( m_{j-p}, \tilde{m}_{j-p} \), we can obtain exact solutions for all \( m_j, \tilde{m}_j \) with \( j > -3 \). This way we have exact solutions for all \( j \). For \(-6 \leq j \leq -3 \) the solutions are:

\[ h_j = \frac{(3 + j)}{(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{00(j-1)} + \frac{j^3 - 9j^2 - 61j + 5}{(2j^3 - 3j^2 - 62j - 5)(-3 + j)}t^{(1)}_{ii(j-1)} + \frac{(-1 + 3j)}{(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{44(j-1)} \]  

(A.21)

\[ l_j = \frac{(3 + j)}{3(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{00(j-1)} - \frac{(-5 - 21j + j^2 + j^3)}{((2j^3 - 3j^2 - 62j - 5)(-3 + j))}t^{(1)}_{ii(j-1)} + \frac{(-1 + 3j)}{3(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{44(j-1)} \]  

(A.22)

\[ k_j = \frac{(-2j - 2j - 35)}{3(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{00(j-1)} + \frac{(45 + 68j - 30j^2 - 4j^3 + j^4)}{((-3 + j)(2j^3 - 3j^2 - 62j - 5))}t^{(1)}_{ii(j-1)} + \frac{(-2j - 15 + j^2)}{3(2j^3 - 3j^2 - 62j - 5)}t^{(1)}_{44(j-1)} \]  

(A.23)

\[ \tilde{h}_j = \frac{((2j^3 + 9j^3 + 11j^2 - 21j - 181)}{2((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))}t^{(1)}_{00(j-1)} - \frac{(107j^5 + 324j^4 - 1932j^3 - 3614j^2 + 697j - 1790)}{2((j^2 + 30j + 1)(2j^3 - 3j^2 - 62j - 5))}t^{(1)}_{ii(j-1)} + \frac{(6j^4 + 7j^3 - 17j^2 + 7j + 77)}{2((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))}t^{(1)}_{44(j-1)} \]  

(A.24)

\(^75\) Also on \( u_h \). This can be easily checked by plugging in the mode expansion for \( m_j, \tilde{m}_j \) in \( G_{\mu\nu} \) given above.
\[ \tilde{l}_j = \frac{(2j^4 + 9j^3 + 11j^2 - 21j - 181)}{6((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))} t_{100(j-1)}^{(1)} \]
\[ - \frac{(33j^5 + 32j^4 - 444j^3 + 1286j^2 + 515j - 590)}{2((j^2 + 30j + 1)(-3 + j)^2(2j^3 - 3j^2 - 62j - 5))} t_{111(j-1)}^{(1)} \]
\[ + \frac{(6j^4 + 7j^3 - 17j^2 + 7j + 77)}{6((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))} t_{144(j-1)}^{(1)} \] (A.25)

\[ \tilde{k}_j = \frac{(-13j^3 - 99j^2 + 177j + 2131)}{6((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))} t_{100(j-1)}^{(1)} \]
\[ - \frac{(2j^7 + 50j^6 - 343j^5 - 1342j^4 + 5682j^3 - 2384j^2 - 4869j + 6980)}{2((j^2 + 30j + 1)(-3 + j)^2(2j^3 - 3j^2 - 62j - 5))} t_{111(j-1)}^{(1)} \]
\[ - \frac{(31j^3 - 33j^2 + 71j + 907)}{6((2j^3 - 3j^2 - 62j - 5)(j^2 + 30j + 1))} t_{144(j-1)}^{(1)} \] (A.26)

This way we get the background in the limit where the back reactions from fluxes, branes, planes and strings are small. In the presence of off-diagonal energy-momentum tensor, the analysis will have to change but the underlying physics will remain unchanged.
B. Operator equations for metric fluctuations

As we discussed in section 3.3, in the limit where the background fluxes, including the effects of the D7 brane, are very small the equation (3.96) can be presented as an operator equation of the following form:

$$\kappa \triangle^{\alpha \beta} l_{\alpha \beta}(x) \approx T_{\mu \nu}^{\text{string}}(x)$$

(B.1)

where \(x\) is a generic five-dimensional coordinate and \(T_{\mu \nu}^{\text{string}}(x)\) is the energy momentum of the string. In this final form, the operator \(\triangle^{\alpha \beta}\) is a second order differential operator derived from (3.95), with \(\mu, \nu, \alpha, \beta = 0, 1, 2, 3, 4\).

Recall now that where we have used coordinate transformations to fix the five components of \(l_{\mu \nu}\), namely \(l_{4 \mu} = 0, \mu = 0, 1, 2, 3, 4\). An additional residual gauge transformation allow us to eliminate another five degrees of freedom and we end up with five physical degrees of freedom i.e. five independent metric perturbation using which all other components could be expressed. Alternatively one can use certain combinations of the fifteen components to write five independent degrees of freedom for the metric fluctuations.

The above result is easy to demonstrate for the AdS space, as has already been discussed in [45]. For non-AdS spaces this is not so easy to construct. Therefore in the following we will take the complete set of ten components and using them we will determine the triangle operator \(\triangle^{\alpha \beta}\) in (3.96). If the ten components that can be labelled as a set:

\[ l_n = \{l_{00}, \ l_{01}, \ l_{02}, \ l_{03}, \ l_{11}, \ l_{12}, \ l_{13}, \ l_{22}, \ l_{23}, \ l_{33} \} \]  

(B.2)

then the operator \(\triangle^{\alpha \beta}\) in (3.96) gives rise to 77 equations that we present below. The warp factor \(h\) appearing in these equations can be taken to be \(h = h(r, \pi, \pi)\) because we will analyse fluctuations close to the string and therefore our choice of background will be:

\[ ds^2 = L_{\mu \nu} dx^\mu dx^\nu \]

\[
L_{00}(t, r, x, y, z) = \frac{-g(r) + \kappa l_{00}(t, r, x, y, z)}{h(r)^{1/2}}
\]

\[
L_{01}(t, r, x, y, z) = \frac{\kappa l_{01}(t, r, x, y, z)}{h(r)^{1/2}}
\]

\[
L_{02}(t, r, x, y, z) = \frac{\kappa l_{02}(t, r, x, y, z)}{h(r)^{1/2}}
\]

\[
L_{03}(t, r, x, y, z) = \frac{\kappa l_{03}(t, r, x, y, z)}{h(r)^{1/2}}
\]

\[
L_{11}(t, r, x, y, z) = \frac{h(r)^{1/2} + \kappa l_{11}(t, r, x, y, z)}{g(r)}
\]

\[
L_{12}(t, r, x, y, z) = \frac{\kappa l_{12}(t, r, x, y, z)}{h(r)^{1/2}}
\]
\[ L_{13}(t, r, x, y, z) = \frac{\kappa l_{13}(t, r, x, y, z)}{h(r)^{1/2}} \]
\[ L_{22}(t, r, x, y, z) = \frac{1 + \kappa l_{22}(t, r, x, y, z)}{h(r)^{1/2}} \]
\[ L_{23}(t, r, x, y, z) = \frac{\kappa l_{23}(t, r, x, y, z)}{h(r)^{1/2}} \]
\[ L_{33}(t, r, x, y, z) = \frac{1 + \kappa l_{33}(t, r, x, y, z)}{h(r)^{1/2}} \]  
(B.3)

Therefore using all the considerations, the explicit forms for \( \Delta_{\mu\nu}^{\alpha\beta} \) can be presented as:

\[ \Delta_{00}^{00} = -\frac{1}{16h^2} \left( 12gh^{\frac{3}{2}} \frac{\partial^2 h}{\partial r^2} - 21 \left( \frac{\partial h}{\partial r} \right)^2 g\sqrt{h} + 6 \frac{\partial h}{\partial r} \frac{\partial g}{\partial r} \right) \]

\[ \Delta_{11}^{11} = -\frac{1}{16h^2} \left( 12g^2 \frac{\partial^2 h}{\partial r^2} + 8gh^{\frac{3}{2}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 24g^2 \frac{\partial h}{\partial r} + 6g^2 \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} + 6gh \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{22}^{22} = -\frac{1}{16h^2} \left( 8gh^{\frac{3}{2}} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 8g^2h^{\frac{3}{2}} \frac{\partial^2}{\partial r^2} - 10g^2h^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} + 4gh^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{33}^{33} = -\frac{1}{16h^2} \left( 8gh^{\frac{3}{2}} \frac{\partial^2}{\partial x^2} + 8gh^{\frac{3}{2}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 10g^2h^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} + 4gh^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{01}^{12} = -\frac{1}{16h^2} \left( 16g^2h^{\frac{3}{2}} \frac{\partial^2}{\partial r \partial x} + 20gh^2h^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial x} - 8gh^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial x} \right) \]

\[ \Delta_{13}^{13} = -\frac{1}{16h^2} \left( 16g^2h^{\frac{3}{2}} \frac{\partial^2}{\partial r \partial y} + 20gh^2h^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial y} - 8gh^{\frac{3}{2}} \frac{\partial h}{\partial r} \frac{\partial r}{\partial y} \right) \]

\[ \Delta_{02}^{02} = \frac{1}{8h^2} \left( 8h^2 \frac{\partial^2}{\partial t \partial x} \right) \]

\[ \Delta_{03}^{03} = \frac{1}{8h^2} \left( 8h^2 \frac{\partial^2}{\partial t \partial y} \right) \]

\[ \Delta_{10}^{10} = \frac{1}{8h^2} \left( -4h^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) - 3 \frac{\partial h}{\partial r} \frac{\partial g}{\partial r} + 3g \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{11}^{11} = \frac{1}{8h^2} \left( -4h^2 \frac{\partial^2}{\partial x^2} + 4h^2 g \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) - 3g^2 \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} + 2gh \frac{\partial g}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{12}^{22} = \frac{1}{8h^2} \left( -4h^2 \frac{\partial^2}{\partial x^2} + 4h^2 g \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - 3g^2 \frac{\partial h}{\partial r} \frac{\partial r}{\partial r} + 2gh \frac{\partial g}{\partial r} \frac{\partial r}{\partial r} \right) \]

\[ \Delta_{13}^{23} = \frac{1}{8h^2} \left( -8h^2 g \frac{\partial^2}{\partial x \partial y} \right) \]

\[ \Delta_{13}^{13} = \frac{1}{8h^2} \left( 6g^2 \frac{\partial h}{\partial r} \frac{\partial r}{\partial y} - 4gh \frac{\partial g}{\partial r} \frac{\partial r}{\partial y} \right) \]
\[\Delta_{11}^{12} = -\frac{1}{8hg^2} \left( 6g^2 \frac{\partial h}{\partial r} \frac{\partial}{\partial x} - 4gh \frac{\partial g}{\partial r} \frac{\partial}{\partial x} \right)\]

\[\Delta_{11}^{01} = -\frac{1}{8hg^2} \left( 6g \frac{\partial h}{\partial r} \frac{\partial}{\partial t} \right)\]

\[\Delta_{11}^{11} = \frac{1}{16h^2 g^2} \left( -4g^2 h^4 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + 6g^3 h \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - 24g^3 \left( \frac{\partial g}{\partial r} \right)^2 + 18g^2 h \frac{\partial g}{\partial r} \frac{\partial}{\partial r} \right.\]

\[+ 8g^2 h^3 \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2} \right) + 12g^3 h \frac{\partial^2 h}{\partial r^2} - 8g^2 h^2 \frac{\partial^2 g}{\partial r^2} \right)\]

\[\Delta_{22}^{22} = \frac{1}{16h^2 g^2} \left( 21g^3 \sqrt{h} \left( \frac{\partial h}{\partial y} \frac{\partial}{\partial y} \right)^2 - 16h^2 g^2 \frac{\partial h}{\partial y} \frac{\partial}{\partial t} - 12g^3 h \frac{\partial^2 h}{\partial r^2} + 8g^2 h^2 \frac{\partial^2 g}{\partial r^2} \right)\]

\[\Delta_{22}^{13} = \frac{1}{16h^2 g^2} \left( 20g^3 h^3 \frac{\partial h}{\partial r} - 16g^2 h^3 \frac{\partial g}{\partial y} \frac{\partial}{\partial y} - 16g^3 h \frac{\partial^2}{\partial y \partial y} \right)\]

\[\Delta_{22}^{01} = \frac{1}{16h^2 g^2} \left( -20h^2 g^2 \frac{\partial}{\partial t} + 8g h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial t} + 16h^2 g^2 \frac{\partial^2}{\partial t \partial t} \right)\]

\[\Delta_{22}^{00} = \frac{1}{16h^2 g^2} \left( 10h^2 g^2 \frac{\partial h}{\partial r} \frac{\partial}{\partial r} + 4g h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} - 4h^2 \left( \frac{\partial g}{\partial r} \right)^2 - 10g h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} \right.\]

\[\left. - 8h^2 g^2 \frac{\partial^2}{\partial r^2} - 8h^2 \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) + 8g h^2 \frac{\partial^2 g}{\partial r^2} \right)\]

\[\Delta_{22}^{33} = \frac{1}{16h^2 g^2} \left( 8g^2 h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + 8g^3 h^2 \frac{\partial^2}{\partial r^2} - 8h^2 g^2 \frac{\partial^2}{\partial t^2} + 8g^2 h^2 \frac{\partial^2}{\partial z^2} - 10g^3 h \frac{\partial h}{\partial r} \frac{\partial}{\partial r} \right)\]

\[\Delta_{22}^{12} = \frac{1}{16h^2 g^2} \left( -16h^2 g^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} - 16g^3 h \frac{\partial^2}{\partial r \partial y} \right)\]

\[\Delta_{22}^{03} = \frac{1}{16h^2 g^2} \left( 16h^2 g \frac{\partial^2}{\partial t \partial y} \right)\]

\[\Delta_{33}^{12} = \frac{1}{16h^2 g^2} \left( 20g^3 h^3 \frac{\partial h}{\partial r} - 16g^2 h^3 \frac{\partial g}{\partial r} \frac{\partial}{\partial x} - 16g^3 h \frac{\partial^2}{\partial x \partial x} \right)\]

\[\Delta_{33}^{22} = \frac{1}{16h^2 g^2} \left( -10g^3 h^2 \frac{\partial h}{\partial r} \frac{\partial}{\partial r} + 8g^2 h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + 8g^3 h^2 \frac{\partial^2}{\partial r^2} + 8h^2 g^2 \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) \right)\]

\[\Delta_{33}^{33} = \frac{1}{16h^2 g^2} \left( 21g^3 \sqrt{g} \left( \frac{\partial h}{\partial r} \right)^2 - 16g^2 h^3 \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - 12g^3 h \frac{\partial^2 h}{\partial r^2} + 8h^2 g^2 \frac{\partial^2 g}{\partial r^2} \right)\]

\[\Delta_{33}^{11} = \frac{1}{16h^2 g^2} \left( -4g^2 h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + 6g^3 h \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - 24gh^3 \left( \frac{\partial h}{\partial r} \right)^2 + 18g^2 h \frac{\partial h}{\partial r} \frac{\partial}{\partial r} \right.\]

\[\left. - 8g^2 h^2 \frac{\partial^2 g}{\partial r^2} + 12g^2 \frac{\partial^2 h}{\partial r^2} + 8h^2 g^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) \right)\]

\[\Delta_{33}^{01} = \frac{1}{16h^2 g^2} \left( -20h^2 g^2 \frac{\partial h}{\partial r} \frac{\partial}{\partial t} + 8g h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial t} + 16h^2 g^2 \frac{\partial^2}{\partial r \partial t} \right)\]
\[ \Delta_{00}^{00} = \frac{1}{16h^2g^2} \left( 10h^2g^2 \frac{\partial h}{\partial r} \frac{\partial h}{\partial r} + 4gh^2 \frac{\partial g}{\partial r} \frac{\partial g}{\partial r} - 4h^2 \left( \frac{\partial g}{\partial r} \right)^2 - 16h^2g \frac{\partial g}{\partial r} \frac{\partial h}{\partial r} ight) \\
- 8h^2g \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) - 8h^2g^2 \frac{\partial^2}{\partial r^2} + 8h^2 \frac{\partial^2 g}{\partial r^2} \right) \]

\[ \Delta_{02}^{02} = \frac{1}{16h^2g^2} \left( 16h^2g \frac{\partial^2}{\partial t \partial x} \right) \]

\[ \Delta_{33}^{12} = \frac{1}{16h^2g^2} \left( -16h^2g^3 \frac{\partial^2}{\partial r \partial x} \right) \]

\[ \Delta_{01}^{01} = \frac{1}{16h^3g} \left( 8h^3g \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - 21g^2 \left( \frac{\partial h}{\partial r} \right)^2 + 12g^2h \frac{\partial^2 h}{\partial r^2} + 6gh \frac{\partial h \partial g}{\partial r \partial r} \right) \]

\[ \Delta_{02}^{12} = \frac{1}{16h^3g} \left( -8h^3g \frac{\partial^2}{\partial t \partial x} \right) \]

\[ \Delta_{01}^{13} = \frac{1}{16h^3g} \left( -8h^3g \frac{\partial^2}{\partial t \partial y} \right) \]

\[ \Delta_{01}^{02} = \frac{1}{16h^3g} \left( 8h^3 \frac{\partial g}{\partial r} \frac{\partial h}{\partial r} - 8h^3g \frac{\partial^2}{\partial r \partial x} \right) \]

\[ \Delta_{01}^{13} = \frac{1}{16h^3g} \left( 8h^3 \frac{\partial g}{\partial r} \frac{\partial h}{\partial r} - 8h^3g \frac{\partial^2}{\partial r \partial y} \right) \]

\[ \Delta_{01}^{22} = \frac{1}{16h^3g} \left( 8h^3g \frac{\partial^2}{\partial t \partial r} - 4h^3 \frac{\partial g}{\partial r} \frac{\partial}{\partial t} \right) \]

\[ \Delta_{01}^{33} = \frac{1}{16h^3g} \left( -8h^3g \frac{\partial g}{\partial r} + 8h^3g \frac{\partial^2}{\partial r \partial t} \right) \]

\[ \Delta_{11}^{11} = \frac{1}{16h^3g} \left( 6h^2g \frac{\partial h}{\partial r} \frac{\partial h}{\partial r} \right) \]

\[ \Delta_{22}^{02} = \frac{1}{16h^3g} \left( 8h^3 \frac{\partial^2}{\partial t \partial y} \right) \]

\[ \Delta_{33}^{02} = \frac{1}{16h^3g} \left( -8h^3 \frac{\partial^2}{\partial t \partial y} \right) \]

\[ \Delta_{02}^{02} = \frac{1}{16h^3g} \left( -8h^3 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 21g^2 \left( \frac{\partial h}{\partial r} \right)^2 - 8gh^2 \frac{\partial^2}{\partial r^2} \\
- 12g \frac{\partial^2 h}{\partial r^2} - 16h \frac{\partial g \partial h}{\partial r \partial r} + 10gh \frac{\partial h \partial g}{\partial r \partial r} + 8h^2 \frac{\partial^2 g}{\partial r^2} \right) \]

\[ \Delta_{02}^{03} = \frac{1}{16h^3g} \left( 8h^3 \frac{\partial^2}{\partial x \partial y} \right) \]

\[ \Delta_{11}^{02} = \frac{1}{16h^3g} \left( -8h^2 \frac{\partial^2}{\partial t \partial x} \right) \]

\[ \Delta_{01}^{02} = \frac{1}{16h^3g} \left( 8gh^2 \frac{\partial^2}{\partial r \partial x} - 10gh \frac{\partial h \partial g}{\partial r \partial x} + 8h^2 \frac{\partial g}{\partial r} \frac{\partial x}{} \right) \]
\[ \Delta_{12}^{12} = \frac{1}{16h^3} \left( 8gh^2 \frac{\partial^2}{\partial r \partial t} - 10gh \frac{\partial h}{\partial r} \frac{\partial}{\partial t} \right) \]

\[ \Delta_{03}^{12} = \frac{1}{16h^3} \left( 8h^3 \frac{\partial^2}{\partial x \partial y} \right) \]

\[ \Delta_{23}^{22} = \frac{1}{16h^3} \left( -8h^3 \frac{\partial^2}{\partial t \partial y} \right) \]

\[ \Delta_{03}^{03} = \frac{1}{16h^3} \left( -8h^3 \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) + 21g \left( \frac{\partial h}{\partial r} \right)^2 - 8gh^2 \frac{\partial^2}{\partial r^2} - 12gh \frac{\partial^2 h}{\partial r} \right. \]

\[ \left. + 10gh \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - 16h \frac{\partial h}{\partial r} \frac{\partial h}{\partial r} + 8h^2 \frac{\partial^2 g}{\partial r^2} \right) \]

\[ \Delta_{03}^{23} = \frac{1}{16h^3} \left( -8h^3 \frac{\partial^2}{\partial h \partial x} \right) \]

\[ \Delta_{11}^{11} = \frac{1}{16h^3} \left( -8h^3 \frac{\partial^2}{\partial h \partial y} \right) \]

\[ \Delta_{03}^{01} = \frac{1}{16h^3} \left( 8h^2 g \frac{\partial^2}{\partial r \partial y} - 10gh \frac{\partial h}{\partial r} \frac{\partial}{\partial y} + 8h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial y} \right) \]

\[ \Delta_{03}^{13} = \frac{1}{16h^3} \left( 8h^2 g \frac{\partial^2}{\partial r \partial t} - 10gh \frac{\partial h}{\partial r} \frac{\partial}{\partial t} \right) \]

\[ \Delta_{12}^{12} = \frac{1}{16h^3 g^2} \left( -8h^3 g^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + 21g^3 \left( \frac{\partial h}{\partial r} \right)^2 + 8h^3 g^2 \frac{\partial^2 g}{\partial r^2} \]

\[ - 12g^3 h \frac{\partial^2 h}{\partial r^2} + 8h^2 g^2 \frac{\partial^2 g}{\partial r^2} - 16h g^2 \frac{\partial h}{\partial r} \frac{\partial g}{\partial r} \]

\[ \Delta_{12}^{13} = \frac{1}{16h^3 g^2} \left( 8h^3 g^2 \frac{\partial^2}{\partial x \partial y} \right) \]

\[ \Delta_{12}^{33} = \frac{1}{16h^3 g^2} \left( -8h^3 g^2 \frac{\partial^2}{\partial r \partial x} \right) \]

\[ \Delta_{12}^{23} = \frac{1}{16h^3 g^2} \left( 8h^3 g^2 \frac{\partial^2}{\partial r \partial y} \right) \]

\[ \Delta_{12}^{02} = \frac{1}{16h^3 g^2} \left( -8h^3 g \frac{\partial}{\partial r \partial t} \right) \]

\[ \Delta_{12}^{00} = \frac{1}{16h^3 g^2} \left( -4h^3 g \frac{\partial}{\partial r \partial x} + 8h^3 g \frac{\partial^2}{\partial r \partial x} \right) \]

\[ \Delta_{12}^{01} = \frac{1}{16h^3 g^2} \left( -8h^3 g \frac{\partial}{\partial t \partial x} \right) \]

\[ \Delta_{12}^{11} = \frac{1}{16h^3 g^2} \left( -6h^3 \frac{\partial h}{\partial r} \frac{\partial}{\partial r \partial x} + 4h^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r \partial x} \right) \]

\[ \Delta_{12}^{13} = \frac{1}{16h^3 g^2} \left( -8h^3 g \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) - 21g^3 \left( \frac{\partial h}{\partial r} \right)^2 - 8h^3 g \frac{\partial^2 g}{\partial r^2} \right. \]

\[ + 12g^3 h \frac{\partial^2 h}{\partial r^2} - 8h^2 g^2 \frac{\partial^2 g}{\partial r^2} + 16h g^2 \frac{\partial h}{\partial r} \frac{\partial g}{\partial r} \]
\[ \Delta_{12}^{13} = -\frac{1}{16h^3g^2} \left( -8h^3g^2 \frac{\partial^2}{\partial x \partial y} \right) \]
\[ \Delta_{13}^{22} = -\frac{1}{16h^3g^2} \left( 8h^3g^2 \frac{\partial^2}{\partial r \partial y} \right) \]
\[ \Delta_{13}^{03} = -\frac{1}{16h^3g^2} \left( 8h^3g \frac{\partial^2}{\partial r \partial t} \right) \]
\[ \Delta_{13}^{00} = -\frac{1}{16h^3g^2} \left( 4h^3g \frac{\partial}{\partial r} - 8h^3g \frac{\partial^2}{\partial r \partial y} \right) \]
\[ \Delta_{13}^{23} = -\frac{1}{16h^3g^2} \left( -8h^3g^2 \frac{\partial^2}{\partial r \partial x} \right) \]
\[ \Delta_{13}^{13} = -\frac{1}{16h^3g^2} \left( 8h^3g \frac{\partial^2}{\partial t \partial y} \right) \]
\[ \Delta_{13}^{13} = -\frac{1}{16h^3g^2} \left( 6h^2g \frac{\partial h}{\partial r} \frac{\partial}{\partial y} - 4h^2g \frac{\partial g}{\partial r} \frac{\partial}{\partial y} \right) \]
\[ \Delta_{23}^{23} = -\frac{1}{16h^3g^2} \left( -8h^3 \frac{\partial^2}{\partial t^2} + 8h^3g \frac{\partial^2}{\partial z^2} - 21g^28g^3 \left( \frac{\partial h}{\partial r} \right)^2 + 8g^2h^2 \frac{\partial^2}{\partial r^2} + 12g^2h^2 \frac{\partial^2 h}{\partial r^2} \right) \]
\[ - 10g^2h \frac{\partial h}{\partial r} \frac{\partial}{\partial r} + 8gh^2 \frac{\partial g}{\partial r} \frac{\partial}{\partial r} + 16gh \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - 8h^2g \frac{\partial^2 g}{\partial r^2} \]
C. An example with diagonal perturbations in OKS-BH background

As we discussed in the previous section, the full analysis with all ten components of the metric fluctuations is rather difficult. In section 3.3 we did present a partial analysis taking all the components into account. However in that section we couldn’t provide precise numerical answers to the metric fluctuations because of the underlying complexity of the problem. In this section we will take a middle path where we will only consider diagonal perturbations for the metric fluctuations much like what we did in Appendix A for the AdS case. This means that with the choice of coordinates and the string profile given by (3.57) and (3.58) we can now formally obtain the source in (B.1) provided we take into account not only the energy momentum tensor of the string but also other sources (see below). However since the string moves in the $x$ direction, the perturbation created in $y$ and $z$ directions are equal and therefore we can demand:

$$l_{22} = l_{33} \quad (C.1)$$

This would mean that we have only four independent components of $l_{\mu \nu}$ and it is enough to consider four independent equations (B.1) sourced by the energy momentum tensors that can come from various sources like the string, fluxes, D7 branes as well as O7 planes:

$$T_{\mu \nu}^{\text{total}} = T_{\mu \nu}^{\text{string}} + T_{\mu \nu}^{\text{fluxes}} + T_{\mu \nu}^{\text{planes}} + T_{\mu \nu}^{\text{branes}} \quad (C.2)$$

At this stage we will assume that the total energy momentum tensor from all the above sources guarantee a diagonal perturbation in the system. We will also assume that the sources are all expressed in terms of the variable $u$ where in terms of $u$, the UV is at $u = 0$ whereas IR is at $u = \infty$. Therefore in the regime close to $u = \infty$ we expect certain aspects of QCD to be revealed from the gravity dual (3.101).

We will now solve (B.1) order by order in $g_s N_f, g_s M^2 / N$ and $g_s^2 N_f M$. At zeroth order in $g_s N_f, g_s^2 N_f M, g_s M^2 / N$, the warp factor becomes $h(u) = L^4 u^4$ and the metric (3.101) reduces to that of AdS$_5$. Hence at zeroth order in $g_s N_f, g_s M^2 / N, g_s^2 MN_f$ our analysis will be similar to the AdS/CFT calculations \cite{45}. As (B.1) is a second order non linear partial differential equation, we can solve it by Fourier decomposing $x, y, z, t$ dependence of $l_{\mu \nu}$ and writing it as a Taylor series in $u$ in the following way:

$$l_{\mu \nu} = l_{\mu \nu}^{[0]} + g_s N_f (a + b g_s M) l_{\mu \nu}^{[1]} \quad (C.3)$$

where the subscript $[0], [1]$ refer to the zeroth and the first order in $(g_s N_f)$.\textsuperscript{76} The zeroth order can then be succinctly presented as a Fourier series in the following

\textsuperscript{76}Henceforth it also means we are keeping terms upto $O(g_s M^2 / N, g_s^2 N_f M)$. 

way\textsuperscript{77}:
\[
l_{\mu\nu}^{[0]}(t, u, x, y, z) = \sum_{k=0}^{\infty} \int \frac{dq_1 dq_2 dq_3 d\omega}{(2\pi)^4} \left[ e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} s_{\mu\nu}^{(k)[0]}(\omega, q_1, q_2, q_3) u^k \right]
\] (C.4)

where \( s_{\mu\nu}^{(k)[0]} \) are expansion coefficients of the solution \( l_{\mu\nu}^{[0]} \). The constant coefficients \((a, b)\) in \( l_{\mu\nu}^{[1]} \) can be worked out easily.

Similarly, we can also write the source in Fourier space as:
\[
T_{\mu\nu}^{\text{total}} = T_{\mu\nu}^{[0]\text{total}} + T_{\mu\nu}^{[1]\text{total}}
\] (C.5)

where as before, \([0, 1]\) refer to the zeroth and first orders in \((g_s N_f)\) respectively. The zeroth order can then be written as:
\[
T_{\mu\nu}^{[0]\text{total}}(t, u, x, y, z) = \int \frac{dq_1 dq_2 dq_3 d\omega}{(2\pi)^4} e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} t_{\mu\nu}^{[0]}(\omega, u, q_1, q_2, q_3)
\] (C.6)

where \( t_{\mu\nu}^{[0]} \) are expansion coefficients of source \( T_{\mu\nu}^{[0]\text{total}} \) at zeroth order in \( g_s N_f \). These coefficients are obtained by using explicit expressions for \( T_{\mu\nu}^{\text{total}}(x^\mu) \) given above in (C.2). In terms of matrices we then expect the metric fluctuations \( l_{ii}, i = 0, 1, 2, 3 \) to take the following form:
\[
\begin{pmatrix}
0 & \Delta_{01}^{11} & \Delta_{01}^{22} & \Delta_{01}^{33} \\
0 & \Delta_{02}^{11} & 0 & \Delta_{02}^{33} \\
0 & \Delta_{03}^{11} & \Delta_{03}^{22} & \Delta_{03}^{33} \\
\Delta_{12}^{00} & \Delta_{12}^{11} & 0 & \Delta_{12}^{33}
\end{pmatrix}
\begin{pmatrix}
l_{00} \\
l_{11} \\
l_{22} \\
l_{33}
\end{pmatrix}
= \frac{1}{\kappa}
\begin{pmatrix}
T_{01} \\
T_{02} \\
0 \\
T_{12}
\end{pmatrix}
\] (C.7)

where we have only switched on the 01, 02 and 12 components of the sources; and \( \Delta_{\mu\nu}^{0} \) are now defined wrt the variable \( u \) instead of \( r \), the radial coordinate.

Manipulating the above matrix equation and using the explicit expressions for \( t_{\mu\nu}^{[0]} \), we can extract a relation for all \( s_{22}^{(k)[0]} \) at zeroth order in \( g_s N_f, g_s M \) (B.1) as:
\[
\sum_k \left[ \frac{3k u}{2} + \left( -\frac{u^2}{4g} \frac{dg}{du} - \frac{3u}{g} + \frac{u^2}{4} \frac{dg}{du} + \frac{g}{2} \right) \right] u^k s_{22}^{(k)[0]} = - \frac{i A^0(\omega, u, \vec{q})}{\omega u^4 L^4}
\] (C.8)

\textsuperscript{77}Note that in all the subsequent mode expansions we will be ignoring the \( \sqrt{g(u_c)} \) dependences. Therefore for us \( u_c \) is close to the actual boundary so that \( \tau \approx t \). A more careful analysis has been presented in section 3.3 wherein we took all the subtleties into account.
where $A^0$ can be given in terms of a series in $u^j$ in the following way:

$$A^0 = \delta(\omega - vq_1)\theta(u - u_0)\sum_{j=4}^{\infty} \tilde{\zeta}_j u^j$$

(C.9)

where $\tilde{\zeta}_j$ are $u$-independent constants that could be determined from (C.2) once we have the explicit expressions for all the terms in (C.2). We can now use the above equations (C.8) and (C.7) to write a relation between $s_{11}^{(k)[0]}$ and $s_{22}^{(k)[0]}$ in the following way:

$$\sum_k \left[ s_{11}^{(k)[0]} + \frac{s_{22}^{(k)[0]}}{g} \right] u^k = B^0(\omega, u, \vec{q})$$

(C.10)

where again $B^0$ can also be expressed in series like (C.9). The above two set of equations have infinite number of variables. They can be solved if we know a generating function. Such a function could be determined for $s_{33}^{(k)[0]}$ in terms of $s_{11}^{(k)[0]}$ and $s_{22}^{(k)[0]}$ as:

$$s_{33}^{(k)[0]} = C^0(\omega, u, \vec{q}) - g s_{11}^{(k)[0]} - s_{22}^{(k)[0]}$$

(C.11)

Thus if we know $s_{11}^{(k)[0]}$ and $s_{22}^{(k)[0]}$ we can determine $s_{33}^{(k)[0]}$. In the following we will determine these coefficients using Green’s function. However before we go into it, let us write the last equation relating $s_{33}^{(k)[0]}$ to the other coefficients:

$$\sum_k \left[ \left( -\frac{2q_1 u}{g} \frac{dg}{du} + 4q_1 k \right) s_{33}^{(k)[0]} + \left( 2gq_1 \frac{dg}{du} - 12g^2 q_1 \right) s_{11}^{(k)[0]} u^k - 8gq_1 k s_{33}^{(k)[0]} \right] u^{k-1} = \frac{D^0(\omega, u, \vec{q})}{L^4 u^6}$$

(C.12)

where $C^0, D^0$ are determined from the source (C.2) like (C.9) above. Note that we could also write (C.12) in terms of $s_{11}^{(k)[0]}$ and $s_{22}^{(k)[0]}$ using the generating function (C.11).

To solve the set of equations (C.8) to (C.12) we will be using Green’s functions. Since all the equations are given in terms of series in $u^k$, we first write the delta function as

$$\delta(u) = \sum_i b_i u^i$$

(C.13)

so that we can equate coefficients on both sides of the equation. Using this it is straightforward to show that the the Green’s function for (C.8) is given by:

$$G_{22}^0(u, \omega, \vec{q}) = \sum_{i=4}^{\infty} c_i^0(\omega, \vec{q}) u^i$$

(C.14)

$^{78}$Such a way of expressing the delta function can be motivated from the standard completeness relation in quantum mechanics. Here of course the coefficient $b_i$ are some specified integers.
where \( c_i^0 \) are given, in terms of the \( b_i \) coefficients appearing in the delta function (C.13), in the following way:

\[
\begin{align*}
   c_{-4}^0 &= -\frac{2ib_0}{\omega L^4}, \quad c_{-3}^0 = -\frac{2ib_1 + 36ib_0}{\omega L^4}; \quad c_{-2}^0 = -\frac{2ib_2 + 30ib_1 + 540ib_0}{\omega L^4} \\
   c_{-1}^0 &= -\frac{2ib_3 + 36ib_2 + 360ib_1 + 648ib_0}{\omega L^4} \\
   c_0^0 &= \frac{2ib_4 + 18ib_3 + 216ib_2 + 3240ib_1 + 58320ib_0}{\omega L^4}
\end{align*}
\]

Observe that the lower limit of the sum is from \( i = -4 \) because of the \( u^{-4} \) suppression in the LHS of (C.8).

It is now easy to write down the solution for the metric perturbation \( l_{22}^{[0]} \) using the Green’s function as:

\[
l_{22}^{[0]}(t, u, x, y, z) = \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} \int_0^u du' A^0(u') \mathcal{G}_{22}^{[0]}(u', \omega, \vec{q}) \\
\equiv \sum_{k=0}^{\infty} \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} s_{22}^{(k)[0]}(\omega, \vec{q}) \theta(u - u_0)u^k
\]

where explicit expressions for \( s_{22}^{(k)[0]} \) are given below. Observe that with \( s_{22}^{(k)[0]} \) known, we obtain \( s_{11}^{(k)[0]} \) using (C.10). Then knowing \( s_{22}^{(k)[0]}, s_{11}^{(k)[0]} \) we obtain \( s_{33}^{(k)[0]} \) using (C.11) and finally \( s_{00}^{(k)[0]} \) using (C.12). The explicit expressions for \( s_{22}^{(k)[0]} \) are:

\[
\begin{align*}
   s_{22}^{(0)[0]} &= \delta(\omega - q_1) \left[ -\zeta_4 c_{-4}^0 u_0 + (\bar{\zeta}_4 c_{-3}^0 + \bar{\zeta}_5 c_{-4}^0) \frac{u_0^2}{2} - (\bar{\zeta}_4 c_{-2}^0 + \bar{\zeta}_5 c_{-3}^0) \frac{u_0^3}{3} \right. \\
   &\quad \left. + (\bar{\zeta}_4 c_{-1}^0 + \bar{\zeta}_5 c_{-2}^0) \frac{u_0^4}{4} - (\bar{\zeta}_4 c_{0}^0 + \bar{\zeta}_5 c_{-1}^0 + \bar{\zeta}_8 c_{-4}^0) \frac{u_0^5}{5} \right] \\
   s_{22}^{(1)[0]} &= \delta(\omega - q_1) \bar{\zeta}_4 c_{-4}^0 \\
   s_{22}^{(2)[0]} &= \frac{1}{2} \delta(\omega - q_1)(\bar{\zeta}_4 c_{-3}^0 + \bar{\zeta}_5 c_{-4}^0) \\
   s_{22}^{(3)[0]} &= \frac{1}{3} \delta(\omega - q_1)(\bar{\zeta}_4 c_{-2}^0 + \bar{\zeta}_5 c_{-3}^0) \\
   s_{22}^{(4)[0]} &= \frac{1}{4} \delta(\omega - q_1)(\bar{\zeta}_4 c_{-1}^0 + \bar{\zeta}_5 c_{-2}^0) \\
   s_{22}^{(5)[0]} &= \frac{1}{5} \delta(\omega - q_1)(\bar{\zeta}_4 c_{0}^0 + \bar{\zeta}_5 c_{-1}^0 + \bar{\zeta}_8 c_{-4}^0)
\end{align*}
\]

Observe also that all \( s_{22}^{(k)[0]} \) are proportional to \( \delta(\omega - q_1 u) \) which will eventually produce the Mach cone. Finally, using (C.16) in (C.10), (C.11), and (C.12), we can obtain rest of the metric perturbations at zeroth order in \( g_s \) (although we do not present the explicit expressions for the rest of the zeroth order perturbations here).

Now we solve for the metric perturbation at linear order in \( g_s N_f, g_s M \) where it is easiest to switch to coordinate \( u = \frac{1}{r_c(1-\zeta)} \), so that the entire manifold is now
described by \(0 \leq \zeta \leq 1\) with \(r_c\) arbitrarily large and we can get a meaningful Taylor series expansion of the logarithms and other functions appearing in the equation. As we did for the zeroth order cases, we can decompose the first order in \(g_s N_f, g_s M\) fluctuations via the following Fourier series:

\[
l^{[1]}_{\mu\nu}(t, \zeta, x, y, z) = \sum_{k=0}^{\infty} \int dq_1 dq_2 dq_3 d\omega (2\pi)^4 \left[ e^{-i(\omega t-q_1 x-q_2 y-q_3 z)} s^{(k)[1]}_{\mu\nu}(\omega, q_1, q_2, q_3) \zeta^k \right]
\]

(C.18)

where \(s^{(k)[1]}_{\mu\nu}\) are the corresponding Fourier modes. These Fourier modes will eventually appear in the final equations for \(l^{[1]}_{\mu\nu}\). In fact we would also need the zeroth order perturbations \(l^{[0]}_{\mu\nu}\) in the equations. Therefore we decompose \(l^{[0]}_{\mu\nu}\) as:

\[
l^{[0]}_{\mu\nu}(t, \zeta, x, y, z) = \int d\omega dq (2\pi)^4 e^{-i(\omega t-q x-q y-q z)} l^{(k)[0]}_{\mu\nu}(\omega, \zeta, q) \zeta^k
\]

(C.19)

where \(l^{(k)[0]}_{\mu\nu}\) are now the Fourier modes for \(l^{[0]}_{\mu\nu}\). These modes have one-to-one correspondence with \(s^{(k)[0]}_{\mu\nu}\) given earlier and can be related by coordinate transformations. Using these equation (C.7) reads:

\[
\sum_k \left[ \frac{3i\omega \tilde{h}}{2r_c \zeta} l^{(k)[0]}_{22} + \frac{3i\omega k L^4}{2r_c^5 \zeta(1-\zeta)^4} s^{(k)[1]}_{22} \right] + \frac{i\omega l^{(k)[0]}_{22}}{r_c} \left( -\frac{\tilde{h}}{4 g d \zeta} - \frac{3}{4 g d \zeta} + \frac{\tilde{h} d g}{2} + \frac{gh r_c}{2} \right) + \frac{i\omega s^{(k)[1]}_{22} L^4}{r_c(1-\zeta)^4} \left( -\frac{1}{4 r_c g d \zeta} - \frac{3}{r_c(1-\zeta) g} + \frac{1}{2 r_c g d \zeta} + \frac{g}{2} \right) \zeta^k = A^1(\omega, \tilde{q})
\]

(C.20)

where as before as in (C.8), we could separate the equations relating \(s^{(k)[1]}_{22}\) and \(l^{(k)[0]}_{22}\) from the rest of the other Fourier modes. We have also defined \(\tilde{h}\) as:

\[
\tilde{h} = \frac{L^4}{r_c^4(1-\zeta)^4} \left[ A \log r_c(1-\zeta) - B \log^2 r_c(1-\zeta) \right]
\]

(C.21)

with \(A\) and \(B\) are defined in (3.101). The other variables appearing above have already been defined earlier. Using these, and using the appropriate Green’s function we can easily determine these Fourier modes. For example the explicit expressions for \(s^{(k)[1]}_{22}\) can be determined by first writing the delta function as:

\[
\delta(1/r) = \sum_{i=0}^{\infty} \frac{\tilde{b}_i}{r^i} \equiv \sum_{j=0}^{\infty} \tilde{b}_j \zeta^j
\]

(C.22)

with \(\tilde{b}_j\) defined in the following way:

\[
\tilde{b}_j = \sum_{i=0}^{\infty} \frac{i(i+1)\ldots(i+1-j)\tilde{b}_i}{r^i}
\]

(C.23)
Then solve for the Greens function for (C.20) to obtain

\[ G_{22}^{(1)}(\zeta, \omega, \vec{q}) = \sum_{i=1}^{\infty} c_i^{(1)}(\omega, \vec{q}) \zeta^i \]  

(C.24)

where the coefficients appearing above are defined as:

\[ c_1 = -l_{22}^{(1)[0]} [A \log(r_c) - B \log^2(r_c)] - \frac{2i\bar{b}_1 r_c^5}{3\omega L^4} \]

\[ c_2 = -4c_1 - l_{22}^{(1)[0]} [4A \log(r_c) - 4B \log^2(r_c) - A + 2B \log(r_c)] - 2l_{22}^{(2)[0]} \frac{A \log(r_c) - B \log^2(r_c)}{r_c^4} \]

\[ + \frac{2c_1 r_c}{3} \left[ \frac{r_h^4}{r_c^5 g(r_c)} - \frac{3}{4r_c g(r_c)} - \frac{4r_c^4}{r_c^5} \frac{g(r_c)}{2} \right] + \frac{2l_{22}^{(1)[0]} r_c}{3} \left[ \frac{A \log(r_c) - B \log^2(r_c)}{r_c^4} \right] \]

\[ - \frac{12}{3} \frac{g(r_c) [A \log(r_c) - B \log^2(r_c)] - 3A + 6B \log(r_c)}{4r_c g(r_c)} + \frac{4}{2} \frac{g(r_c) [A \log(r_c) - B \log^2(r_c)]}{r_c^4} \]

\[ + \frac{4}{2} \frac{g(r_c) [A \log(r_c) - B \log^2(r_c)]}{r_c^4} \]

(C.25)

Once everything is laid up, we can write the source in (C.20) as a power series in \( \zeta \) i.e. \( A^4 = \sum_j \tilde{a}_j \zeta^j \) with \( \tilde{a}_j \) derivable from (C.2). This would finally give us the required Fourier coefficients to first order in \( g_s N_f \) as\(^79\):

\[ s_{22}^{(0)[1]} = \delta(\omega - \nu_1) \left[ -\frac{c_1^0 \tilde{a}_0 \zeta_0^2}{2} - \frac{(c_2^0 \tilde{a}_0 + c_1^0 \tilde{a}_1) \zeta_0^3}{3} \right] \]

\[ s_{22}^{(2)[1]} = \delta(\omega - \nu_1) \left[ \frac{c_1^0 \tilde{a}_0}{2} \right] \]

\[ s_{22}^{(3)[1]} = \delta(\omega - \nu_1) \left[ \frac{(c_2^0 \tilde{a}_0 + c_1^0 \tilde{a}_1)}{3} \right] \]

(C.26)

with the following Fourier decomposition:

\[ l_{22}^{[1]}(t, \zeta, x, y, z) = \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} \int_0^\zeta d\zeta' A^4(\zeta') G_{22}^{(1)}(\zeta', \omega, \vec{q}) \]

\[ \equiv \sum_{k=0}^{\infty} \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i(\omega t - q_1 x - q_2 y - q_3 z)} s_{22}^{(k)[1]}(\omega, \vec{q}) \theta(\zeta - \zeta_0) \zeta^k \]  

(C.27)

Once we know these modes, we can use them to write the relation for \( s_{11}^{(k)[1]} \) in the following way:

\[ \sum_k \left[ s_{11}^{(k)[1]} + \frac{s_{22}^{(k)[1]}}{g} \right] \zeta^k = B^4(\omega, \vec{q}) \]  

(C.28)

\footnotesize \(^79\)We are only solving upto \( O(\zeta^3) \).
with $B^1$ given in the appendix. Observe that the above equation has exactly the same form as (C.10) except that these modes are written for linear order perturbations. It is then no surprise that the generating function for $s_{33}^{(k)[1]}$ takes exactly the same form as in (C.11):

$$s_{33}^{(k)[1]} = C^1(\omega, \vec{q}) - g s_{11}^{(k)[1]} - s^{(k)[1]}$$  \hspace{1cm} (C.29)

Finally once we know the Fourier modes $s_{ii}^{(k)[1]}$ with $i = 1, 2, 3$ we can use them to write the equation for $s_{00}^{(k)[1]}$. The equation turns out to be rather involved with all zeroth and first order coefficients appearing together. Nevertheless one can present the following form for the equation:

$$\sum_k \left[ -2 \frac{d g}{d \zeta} \tilde{h} q_1^{(k)[0]} \zeta^k - 2 \frac{d g}{d \zeta} \frac{L^4 q_1}{r_c^5 (1 - \zeta)^4} s_{00}^{(k)[1]} \zeta^k \right. + 4 q_1 \tilde{h} l_{00}^{(k)[0]} k \zeta^{k-1} + 4 q_1 L^4 \frac{1}{r_c^5 (1 - \zeta)^4} s_{00}^{(k)[1]} k \zeta^{k-1} + 2 g \frac{d g}{d \zeta} \tilde{h} q_1 l_{11}^{(k)[0]} \zeta^k \\
+ 2 g \frac{d g}{d \zeta} \frac{L^4}{r_c^5 (1 - \zeta)^4} q_1 s_{11}^{(k)[1]} \zeta^k - 3 g^2 \frac{d h}{d \zeta} q_1 l_{11}^{(k)[0]} \zeta^k \\
- 12 g^2 \frac{L^4}{r_c^5 (1 - \zeta)^5} q_1 s_{11}^{(k)[1]} \zeta^k - 8 g \tilde{h} q_1 l_{11}^{(k)[0]} \zeta^k \\
- 8 \frac{g L^4 q_1}{r_c^5 (1 - \zeta)^4} s_{33}^{(k)[1]} \zeta^{k-1} \right] = D^1(\omega, \zeta, \vec{q})  \hspace{1cm} (C.30)$$

where $D^1, C^1$ is determined from the source (C.2). Plugging in the other Fourier modes, we have been able to solve for all the $s_{00}^{(k)[1]}$ modes using the corresponding Green’s function (we don’t present the results here).

To conclude therefore, using the equations (C.28), (C.29) and (C.30) we find $s_{00}^{(k)[1]}, s_{11}^{(k)[1]}$ and $s_{33}^{(k)[1]}$ etc. which in turn give us $l_{00}^{[1]}(t, \zeta, x, y, z), l_{11}^{[1]}(t, \zeta, x, y, z)$ and $l_{33}^{[1]}(t, \zeta, x, y, z)$ etc. using equations like (C.27).
D. Coefficients in (3.116) and (3.120)

A sample of the coefficients appearing in (3.120) are given below for the simplified OKS-BH geometry with approximate diagonal perturbations:

\[
A_{1j} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (16h^2g^2)
\]

\[
A_{1j} = e^{-2\phi} \frac{16h^2g^4}{16h^2g^4}
\]

\[
A_{1j} = A_{2j} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (16h^2g^3)
\]

\[
A_{1j} = A_{1j} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (32h^2g^3)
\]

\[
A_{1j} = A_{2j} = e^{-2\phi} \frac{16g^3h^2}{16g^3h^2}
\]

\[
A_{1j} = A_{1j} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (16g^3h^2)
\]

\[
A_{1j} = A_{1j} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (16g^3h^2)
\]

\[
C_{10} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-40h^2g \frac{dg}{dr} - 24h \frac{dh}{dr} g^2)
\]

\[
C_{11} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (8h^2g^4 \frac{dg}{dr} + 16h \frac{dh}{dr} g^4)
\]

\[
C_{12} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-8h^2g^3 \frac{dg}{dr} + 24h \frac{dh}{dr} g^4)
\]

\[
C_{13} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-24h \frac{dh}{dr} g^3)
\]

\[
C_{14} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-24h \frac{dh}{dr} g^3 - 8h^2g^2 \frac{dg}{dr})
\]

\[
C_{15} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-48h \frac{dh}{dr} g^3 - 16h^2g^2 \frac{dg}{dr})
\]

\[
C_{16} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-48h \frac{dh}{dr} g^3)
\]

\[
C_{17} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (48h \frac{dh}{dr} g^4 - 24g^3h^2 \frac{dg}{dr})
\]

\[
C_{18} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (-16h \frac{dh}{dr} g^3 + 8h^2g^2 \frac{dg}{dr})
\]

\[
C_{19} = e^{-2\phi} \frac{32g^3h^{13/4}}{32g^3h^{13/4}} (32h^2g \frac{dg}{dr} - 24h \frac{dh}{dr} g^3)
\]
where $h$ is the warp factor measured on the slice (3.14), and $\phi$ is the dilaton. To get the explicit expressions for $A^{ij}_0, C^{ij}_0$ in (3.116) we need to replace $h$ with $L^4/r^4$ and $e^{-2\phi}$ with $1/g_s^2$ in the above expressions. We have written the nonzero $A^{ij}_1, C^{ij}_1$ and therefore the terms not appearing above are all zeroes. Note also that $A^{ij}_k = B^{ij}_k, k = 0,1$ and since we don’t explicitly know the sources and the full background, we don’t know the explicit expressions for the terms $D^{ij}_k, E^i_k, F^i_k$ etc.
E. Detailed Viscosity Analysis

As in the previous section, here we work out the coefficients in the quadratic action in \( (3.170) \) up to \( \mathcal{O}(g_s N_f, g_s M^2/N) \):

\[
A = \frac{1}{g_s^2} 2g(r) r^5 \left( 1 - \frac{3 N_f g_s \log r}{2 \pi} + \frac{N_f g_s}{\pi} \right) \\
B = \frac{1}{g_s^2} 3 \frac{g(r) r^5}{2} \left( 1 - \frac{3 N_f g_s \log r}{2 \pi} + \frac{N_f g_s}{\pi} \right) \\
C = \frac{1}{2g_s^2} \left[ r^4 \left( \frac{3 g_s M^2}{2 \pi N} \right) + 24 r^4 - 8 r_h^4 + r_h^4 \left( \frac{3 g_s M^2}{2 \pi N} \right) + \frac{N_f g_s}{4 \pi} \left\{ 150 r^4 \left( \frac{3 g_s M^2}{2 \pi N} \right) \log r - 96 r^4 + 6 r^4 g \left( \frac{3 g_s M^2}{2 \pi N} \right) + 32 r_h^4 \log r - 6 r_h^4 \left( \frac{3 g_s M^2}{2 \pi N} \right) \log r \right\} \right] \\
D = \frac{\tilde{V}}{g_s^2} \left[ -g(r) r^3 \left( \frac{3 g_s M^2}{2 \pi N} \right) \left\{ 1 - \left( \frac{3 g_s M^2}{2 \pi N} \right) \log r \right\} + \frac{N_f g(r) r^3}{16 \pi} \left( \frac{3 g_s M^2}{2 \pi N} \right) (-28 \log r - 5) + \frac{81 g_s^2 M^2 g(r) r^3 \alpha^2}{8 L^4} + \frac{1}{4 r^3} \left\{ 16 + 48 \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 (\log r)^2 - 2 \left( \frac{3 g_s M^2}{2 \pi N} \right) + 8 \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 \log r + \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 \right\} \right] \\
\mathcal{M}(r) = -\frac{3 g_s N_f}{2 \pi r} - g_s^2 N_f^2 \left( \frac{3 \log (r) - 2}{2 \pi} \right) \left[ \frac{3 \log (r) - 2}{2 \pi} \left( g'(r) + \frac{5 g(r)}{r} \right) + \frac{3}{2 \pi r} \right] \\
\mathcal{J}(r) = \frac{1}{r^2} \left[ d_0 + d_1 \log (r) + d_2 (\log (r))^2 \right] + \frac{r h^4}{r^6} \left[ c_0 + c_1 \log (r) + d_2 (\log (r))^2 \right] + \frac{r h^4 f_0}{r^10}
\]

where \( \mathcal{J}(r) \) and \( \mathcal{M}(r) \) appear in \( (3.173) \). The coefficients appearing in the equations above are defined as:

\[
\begin{align*}
d_0 &= -4 g_s N_f + \frac{729 g_s^2 M^4 \alpha^2}{16 L^4 \pi^2 N^2} + \frac{75 g_s^2 M^2 N_f}{16 \pi N} - \frac{122 g_s N_f}{4 \pi} \\
d_1 &= 4 \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 - \frac{28 g_s^2 M^2 N_f}{16 \pi N} - \frac{159 g_s^2 M^2 N_f}{4 \pi^2 N} - \frac{12 g_s N_f}{4 \pi} \\
d_2 &= -120 \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 + 30 \left( \frac{3 g_s^2 M^2 N_f}{8 \pi^2 N} \right) \\
c_0 &= -\frac{6 g_s M^2}{2 \pi N} + 15 \frac{g_s^2 N_f M^2}{8 \pi^2 N} - \frac{122 g_s N_f}{4 \pi} + \left( \frac{3 g_s M^2}{2 \pi N} \right)^2 \left[ \frac{1}{2} - \frac{24 N_f g_s + 9 g_s^2 M^2 N_f}{2 \pi N} \right]
\end{align*}
\]

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\[
e_1 = 2 \left( \frac{3g_s M^2}{2\pi N} \right)^2 - \frac{159g_s^2 M^2 N_f}{4\pi^2 N} - \frac{12g_s N_f}{4\pi}
\]
\[
f_0 = \left( \frac{3g_s M^2}{2\pi N} \right)^2 \frac{1}{2} - \frac{24N fg_s + \frac{9g_s^2 M^2 N_f}{2\pi N}}{4\pi}
\]

Finally the perturbation coefficients \(a_i, b_j\) appearing in the mode expansions for \(G\) and \(H\) in (3.182) and (3.184) respectively are given to \(\mathcal{O}(g_s N_f, g_s M^2 / N)\) as:

\[
a_0 = a_1 = 0
\]
\[
a_2 = \frac{1}{2} \left[ \frac{6g_s M^2}{\pi N} + \frac{122N fg_s + 12N fg_s \log r_c}{4\pi} + \frac{r_h^4}{r_c^4} \left( \frac{3g_s M^2}{\pi N} + \frac{122N fg_s + 12N fg_s \log r_c}{4\pi} \right) \right]
\]
\[
b_0 = b_1 = b_2 = 0
\]
\[
b_3 = \frac{8r_h^4}{3r_c^4} \left( 1 + \frac{r_h^4}{r_c^4} \right) a_2
\]
F. Detailed Entropy Analysis

The coefficients in (3.216), imposing an UV cut-off at \( r = r_c \), are:

\[
C_{11}^{11} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} \left( 80 r h g^2 + 8 r^2 h g^2 g_2^{-1} \frac{dg_2}{dr} - 4 r^2 g g_2^{-1} \frac{dh}{dr} - 48 r^2 h \frac{dg}{dr} \right)
\]
\[
C_{11}^{12} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} \left( -240 g^3 h r + 12 \frac{dh}{dr} g^3 r^2 - 24 r^2 h g^3 g_2^{-1} \frac{dg_2}{dr} + 24 r^2 h g^3 \frac{dg}{dr} \right)
\]
\[
\bar{C}_1^{21} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} \left( -240 g^3 h r + 12 \frac{dh}{dr} g^3 r^2 - 24 r^2 h g^3 g_2^{-1} \frac{dg_2}{dr} + 48 r^2 h g^3 \frac{dg}{dr} \right)
\]
\[
C_{12}^{22} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} \left( -240 g^4 h r + 12 \frac{dh}{dr} g^4 r^2 - 24 r^2 h g^4 \frac{dg}{dr} - 24 r^2 h g^4 g_2^{-1} \frac{dg_2}{dr} \right)
\]
\[
A_{11}^{11} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} (16 g^2 r^2 h)
\]
\[
A_{11}^{11} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} (-48 g^4 r^2 h)
\]
\[
\bar{A}_1^{21} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} (-48 g^3 r^2 h)
\]
\[
\bar{A}_1^{12} = \bar{A}_1^{21}
\]
\[
\bar{B}_1^{11} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} (16 g^2 r^2 h)
\]
\[
\bar{B}_1^{12} = \frac{r^6 e^{-2\phi}}{32 \sqrt{gg_2^{-1} h g^3 L^{3/4} r^{1/4}}} (-48 g^3 r^2 h)
\]
\[
E_1^{1} = \frac{r^6 e^{-2\phi}}{8 h g^2 r^4 L^{3/4} r^{1/4}} \left( -40 g^2 h r + 8 r^2 h g \frac{dg}{dr} + 2 g^2 r^2 \frac{dh}{dr} - 4 r^2 g g_2^{-1} \frac{dg_2}{dr} \right)
\]
\[
\bar{E}_1^{1} = \frac{r^6 e^{-2\phi}}{8 h g^2 r^4 L^{3/4} r^{1/4}} \left( 6 g^3 r^2 \frac{dh}{dr} - 120 g^3 h r - 12 g^3 g_2^{-1} h r^2 \frac{dg_2}{dr} - 12 r^2 g^2 h \frac{dg}{dr} \right)
\]
\[
\bar{F}_1^{1} = \frac{r^6 e^{-2\phi}}{8 h g^2 r^4 L^{3/4} r^{1/4}} (-8 g^2 r^2 h)
\]
\[
\bar{F}_1^{2} = \frac{r^6 e^{-2\phi}}{8 h g^2 r^4 L^{3/4} r^{1/4}} (-24 g^3 r^2 h)
\]

The nonzero coefficients in (3.219) are (taking the approximation \( g_1 = g_2 = 1 - \frac{r^4}{\alpha} + \alpha + \gamma \frac{r^4}{\tau} \))

\[
H^{11[1]} = \frac{1}{32 g_s^3 L^5 (1 + A + B \log^2 r)^{1/4}} \left[ 80 \alpha^2 - \frac{4 \alpha^2}{1 + A + B \log^2 r} \right] - \frac{2}{L^5 g_s^2 \alpha (1 + A + B \log^2 r)^{1/4}} \left( -4 (1 + A + B \log^2 r) \right)
\]
The other coefficients \( \tilde{b}_{n(i)[1]} \) in (3.219) are defined as:

\[
\begin{align*}
\tilde{b}_{0(0)[1]} &= \frac{4gr_4}{2L^5g_2^2(1 + A \log r + B \log^2 r)^{1/4}} \\
\tilde{b}_{0(4)[1]} &= -\frac{32g\kappa c_3/L^2r_h^8}{2L^5g_2^2(1 + A \log r + B \log^2 r)^{1/4}} \\
\tilde{b}_{1(0)[1]} &= \frac{1}{8L^5g_2^2(1 + A \log r + B \log^2 r)^{5/4}} \left[ -6r_h^4 \{ -4(1 + A \log r + B \log^2 r) + A + 2B \log r \} \
+120r_h^4(1 + A \log r + B \log^2 r) - 96r_h^4(1 + A \log r + B \log^2 r) + 6r_h^4 \{ -4(1 + A \log r + B \log^2 r) \
+A + 2B \log r \} + \frac{72g\kappa c_3}{2\pi} r_h^4(1 + A \log r + B \log^2 r) \right] \\
\tilde{b}_{1(4)[1]} &= \frac{1}{8L^5g_2^2(1 + A \log r + B \log^2 r)^{5/4}} \left[ 24\kappa c_3 L^{-2}r_h^8 \{ -4(1 + A \log r + B \log^2 r) + A + 2B \log r \} \
-480r_h^8\kappa c_3 L^{-2}(1 + A \log r + B \log^2 r) + 384r_h^8\kappa c_3 L^{-2}(1 + A \log r + B \log^2 r) \
-24\kappa c_3 L^{-2}r_h^8 \{ -4(1 + A \log r + B \log^2 r) + A + 2B \log r \} \
-144(1 + A \log r + B \log^2 r) g_\kappa N_f L^{-2}r_h^8\kappa c_3/\pi \right]
\end{align*}
\]

where \( A = \frac{3g_r M^2}{N} \left(1 + \frac{3g_n N_L}{4\pi} \right) \), \( B = \frac{9g_r^2 M^2 N_f}{8\pi^2 N} \), \( \alpha = 1 + \alpha = 1 + \frac{4g_{\kappa c}}{3L^2} \). We also have

\[
\begin{align*}
s_{00}^{(0)[1]} &= -\frac{1}{2} (A \log r + B \log^2 r) \\
\tilde{s}_{00}^{(4)[1]} &= \frac{r_h^4}{2} (A \log r + B \log^2 r) \\
\tilde{s}_{11}^{(0)[1]} &= -A \log r - B \log^2 r
\end{align*}
\]

with every other \( s_{mn}^{(i)[1]} = 0 \). Note that (a) all coefficients are suppressed as \( O(g_\kappa N_f, g_r M^2/N) \) as expected, and (b) an appropriate UV cap will make all \( s_{mn}^{(k)[1]} \) independent of \( r \), as we discussed in the main text.
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