The embedding structure of unitary $N = 2$ minimal models

Matthias Dörrozapf *

Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138, USA

ABSTRACT

We derive the embedding structure of unitary $N = 2$ minimal models and show as a result that these representations have a degeneration of uncharged singular states. This corrects some earlier mistakes made in the literature. We discuss the connexion to the $N = 2$ character formulae and finally give a proof for the embedding diagrams.
1 Introduction

Conformal and superconformal symmetry is the underlying structure of many models in very different areas of physics. This includes statistical systems, random walk models, percolation models and also string theory. In particular, the $N = 2$ superconformal algebras play a crucial rôle for string theory, since they supply the underlying symmetry for the $N = 2$ string. The representation theory of the Virasoro algebra, the algebra of generators of local two-dimensional conformal symmetry, is well-understood. Based on it, the generalisation to the $N = 1$ superconformal algebras is straightforward. At first, it seemed as if the representation theory of $N = 2$ superconformal algebras followed similar patterns and character formulae as well as embedding diagrams were therefore derived applying methods used for the Virasoro algebra assuming they would work for the representations of the $N = 2$ superconformal algebras as well. However, we have shown in Ref. 8 and Ref. 9 that the $N = 2$ algebras are more complicated than originally believed. We have proven that the existing embedding diagrams are wrong for some crucial models and we could show that in some $N = 2$ cases one obtains degenerated singular vectors. The discrete series of unitary cases or unitary minimal models is one class of representations for which the previous embedding patterns fail.

Starting from the $N = 2$ determinant formula and taking the degenerated singular vector spaces into account, we obtained a full set of embedding diagrams containing all the singular vectors guaranteed by the determinant formula. Our classification follows the Feigin and Fuchs “I-II-III pattern” and can be found in Ref. 9. But the $N = 2$ superconformal algebra turned out to be all the more interesting as subsingular vectors were recently discovered by Gato-Rivera and Rosado. Therefore it is not sufficient to follow only the vanishing curves of the determinant formula in order to derive the embedding structure. The unitary embedding diagrams, however, as we shall see at the end of this paper, do not contain any subsingular vectors and our results of Ref. 9 hold for at least these cases. There are intriguing similarities between these $N = 2$ embedding diagrams and embedding diagrams of class IV of the affine Lie superalgebra $\hat{sl}(2,1;\mathbb{C})$, as shown in Ref. 5. Just recently, the embedding diagrams of the $N = 2$ superconformal algebra were derived in an alternative way in Ref. 21. The authors of Ref. 21 show that the embedding structure of $\hat{sl}(2)$ entails the embedding structure of the $N = 2$ superconformal algebra and ultimately rederive many of our results of Ref. 9.

In this paper we present the embedding structure of the most important highest weight representations: the unitary representations. There are two classes of unitary representations, the continuous class and the discrete series. The embedding diagrams for the continuous class given earlier by Dobrev are correct. The problems arise for the discrete series. These are the cases we want to concentrate on. After a short introduction to the notation and some necessary conventions in Sec. 2, we review in Sec. 3 some results of our Ref. 8 which are necessary for the reader to understand the difference in structure between the Virasoro algebra and the $N = 2$ algebra. In Sec. 4 we present the embedding structure of the unitary minimal models which is generalised to a larger class of representations in Sec. 5. In Sec. 6 we follow the idea of Eholzer and Gaberdiel to show that the embedding diagrams imply the correct character formulae that finally proves the embedding diagrams of Sec. 4 and Sec. 5.
2 \( N = 2 \) superconformal algebra and its unitary representations

In a quantum field theory, unitarity is fundamental in that it is the condition of conservation of probability. Hence, from the viewpoint of a quantum field theory the most interesting representations of the \( N = 2 \) superconformal algebras are the unitary representations. The unitary representations for the \( N = 2 \) superconformal algebras have been identified by Boucher, Friedan and Kent for all different sectors of the \( N = 2 \) superconformal algebra. Similarly to the Virasoro case, one finds a discrete series, called “the unitary minimal models”, and a continuous class of unitary representations. The continuous class of unitary highest weight representations has been analysed correctly by Dobrev. Therefore we shall concentrate on the discrete series of unitary representations. Related to this, Eholzer and Gaberdiel demonstrated recently another important feature of the \( N = 2 \) algebra. They showed in Ref. 11 that all rational \( N = 2 \) superconformal theories may be unitary. Where a theory is called rational if it has only finitely many irreducible highest weight representations, and if the highest weight space of each of them is finite dimensional. In Sec. 6 we shall use their method of deriving character formulae out of the embedding diagrams for the unitary minimal models.

We denote the \( N = 2 \) superconformal algebra in the Neveu-Schwarz (or antiperiodic) modeling by \( \mathfrak{sc}(2) \). It is given by the Virasoro algebra, the Heisenberg algebra plus two anticommuting subalgebras with the (anti-)commutation relations \(^a\):

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12} (m^3 - m) \delta_{m+n,0} , \\
[L_m, G_r^\pm] &= \left( \frac{1}{2} m - r \right) G_{m+r}^\pm , \\
[L_m, T_n] &= -n T_{m+n} , \\
[T_m, T_n] &= \frac{1}{3} C m \delta_{m+n,0} , \\
[T_m, G_r^\pm] &= \pm G_{m+r}^\pm , \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r - s)T_{r+s} + \frac{C}{3} (r^2 - \frac{1}{4}) \delta_{r+s,0} , \\
[L_m, C] &= [T_m, C] = [G_r^\pm, C] = 0 , \\
\{G_r^+, G_s^+\} &= \{G_r^+, G_s^-\} = \{G_r^-, G_s^-\} = 0 , \\
&\quad \text{for } m, n \in \mathbb{Z}, \ r, s \in \mathbb{Z}_{\frac{1}{2}} .
\end{align*}
\]

We can write \( \mathfrak{sc}(2) \) in its triangular decomposition: \( \mathfrak{sc}(2) = \mathfrak{sc}(2)_- \oplus \mathcal{H}_2 \oplus \mathfrak{sc}(2)_+ \), where \( \mathfrak{sc}(2)_\pm = \text{span}\{L_{\pm n}, T_{\pm n}, G_{\pm r}^\pm : n \in \mathbb{N}, r \in \mathbb{N}_{\frac{1}{2}}\} \), and \( \mathcal{H}_2 = \text{span}\{L_0, T_0, C\} \) is the grading preserving Cartan subalgebra\(^b\). A simultaneous eigenvector \( |h, q, c\rangle \) of \( \mathcal{H}_2 \) with \( L_0, T_0 \) and \( C \) eigenvalues \( h, q \) and \( c \) respectively and vanishing \( \mathfrak{sc}(2)_+ \) action \( \mathfrak{sc}(2)_+ |h, q, c\rangle = 0 \), is called a highest weight vector. The Verma module \( \mathcal{V}_{h,q,c} \) is defined as the \( \mathfrak{sc}(2) \) left module \( U(\mathfrak{sc}(2)) \otimes \mathcal{H}_2 \otimes \mathfrak{sc}(2)_+ |h, q, c\rangle \), where \( U(\mathfrak{sc}(2)) \) denotes the universal enveloping algebra of \( \mathfrak{sc}(2) \). Finally, we call a vector singular in \( \mathcal{V}_{h,q,c} \) if it is not proportional to the highest weight vector but still satisfies the highest weight vector conditions: \( \Psi_{n,p} \in \mathcal{V}_{h,q,c} \) is called singular if

\(^a\) We write \( \mathbb{N} \) for \( \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 \) for \( \{0, 1, 2, \ldots\} \), \( \mathbb{N}_{\frac{1}{2}} \) for \( \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\} \) and \( \mathbb{Z}_{\frac{1}{2}} \) for \( \{\ldots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\} \).

\(^b\) There are four-dimensional abelian subalgebras of \( \mathfrak{sc}(2) \). However, these triangular decompositions of the algebra are not consistent with our \( \mathbb{Z}_2 \)-grading.
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$L_0 \Psi_{n,p} = (h + n)\Psi_{n,p}, \ T_0 \Psi_{n,p} = (q + p)\Psi_{n,p}$ and $\mathfrak{sc}(2)_+ \Psi_{n,p} = 0$ for some $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. If a vector is an eigenvector of $L_0$ we call its eigenvalue $h$ its conformal weight and similarly its eigenvalue of $T_0$ is called its $U(1)$-charge.$^c$

The determinant formula given by Boucher, Friedan and Kent$^3$ makes it apparent that the Verma module $\mathcal{V}_{h_r,s(t,q),q,c(t)}$ has for positive, integral $r$ and positive, even $s$ an uncharged singular vector at level $\frac{s+2}{2}$ which we shall call $\Psi_{r,s} = \Theta_{r,s} | h_{r,s}, q, c \rangle$, with $\Theta_{r,s} \in \mathfrak{sc}(2)_-$. We use the parametrisation

$$c(t) = 3 - 3t \quad , \quad h_{r,s}(t, q) = \frac{(s - rt)^2}{8t} - \frac{q^2}{2t} - \frac{t}{8}. \quad (2)$$

We can find $\mp 1$ charged singular vectors $\Psi_k^\pm = \Theta_k^\pm | h_k^\pm, q, c \rangle$ (with $\Theta_k^\pm \in \mathfrak{sc}(2)_-$) in the Verma module $\mathcal{V}_{h_k^\pm(t,q),q,c(t)}$ at level $k$ for $k \in \mathbb{N}_{\frac{1}{2}}$. The conformal weight $h_k^\pm$ is

$$h_k^\pm(t, q) = \pm kq + \frac{1}{2}t(k^2 - \frac{1}{4}). \quad (3)$$

For convenience we shall frequently use $\tilde{t} = \frac{t}{2}$ and $\tilde{s} = \frac{s}{2}$. If we use the parametrisation

$$a^2 = 4\tilde{t}h + \tilde{t}^2 + q^2 \quad , \quad k^a = -\frac{q}{2\tilde{t}} + \frac{a}{2\tilde{t}}, \quad k^b = -\frac{q}{2\tilde{t}} - \frac{a}{2\tilde{t}}, \quad (4)$$

then the determinant expression factorises (for $t \neq 0$) proportional to the following product ($n$ refers to the level and $m$ to the charge):

$$\det M_{n,m} \propto \prod_{1 \leq r \leq s \leq n, r, s \in \mathbb{Z}} (\tilde{s} - r\tilde{t} + a)^{P(n-r\tilde{s},m)} \prod_{k \in \mathbb{Z}_{\frac{1}{2}}} [ (k - k^a)(k - k^b) ]^{\tilde{P}(n-|k|,m-\text{sgn}(k),k)}, \quad (5)$$

where $P$ and $\tilde{P}$ are the corresponding partition functions$^3$. In order to analyse the singular vectors guaranteed by the determinant, we simply need to find integer solutions $(r, \tilde{s})$ on the straight line $\tilde{s} = r\tilde{t} - a$ for the uncharged sector leading us to $\Psi_{r,s}$ and we need to verify if $k^a$ or $k^b$ is half-integral for the charged sector. If $k^a$ or $k^b$ is $\in \mathbb{Z}_{\frac{1}{2}}$, then there is a $\text{sgn}\{k^a\}$ or $\text{sgn}\{k^b\}$ charged singular vector at level $|k^a|$ or $|k^b|$ respectively. Once we have found singular vectors in a Verma module, we can then analyse the submodules built on top of these singular vectors by computing the new parameters $a, k^a$ and $k^b$ for the embedded modules. However, singular vectors of embedded modules may be trivial in the original Verma module. Using Eqs. (11-13) of the following section, allows us to verify whenever this happens.

As shown by Gato-Rivera and Rosado$^{13}$, representations of the $N = 2$ superconformal algebra contain in some cases vectors which become singular in the quotient module of the Verma module with a submodule but are not singular in the original Verma module. This kind of vectors is called subsingular vectors. Each submodule of a Verma module is generated by singular and subsingular vectors.

In the Neveu-Schwarz sector of the algebra, for an element of the discrete series of unitary representations there exists a number $m \in \mathbb{N}, m \geq 2$ such that $t = \frac{2}{m}$. Besides, there exist two

$^c$For a singular vector $\Psi_{n,p} \in \mathcal{V}_{h,q,c}$ we may simply say its charge $p$ and its level $n$ rather than $U(1)$-charge $q + p$ and conformal weight $h + n$. 


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half-integral numbers $j, k \in \mathbb{N}_\frac{1}{2}$ and $0 < j, k, j + k \leq m - 1$ such that the conformal weight is given by $h = \frac{jk - \frac{1}{2}}{m}$ and the $U(1)$-charge by $q = \frac{j - k}{m}$.

Schwimmer and Seiberg showed\textsuperscript{20} that the $N = 2$ Ramond algebra is simply a rewriting of the $N = 2$ Neveu-Schwarz algebra using the algebra isomorphism $\mathcal{U}_\theta$ called the spectral flow.

\begin{align*}
\mathcal{U}_\theta L_m \mathcal{U}_-\theta &= L_m + \theta T_m + \frac{c}{6} \theta^2 \delta_{m,0} \\
\mathcal{U}_\theta T_m \mathcal{U}_-\theta &= T_m + \frac{c}{3} \theta \delta_{m,0} \\
\mathcal{U}_\theta G^\pm_m \mathcal{U}_-\theta &= G^\pm_{m,\theta}
\end{align*}

(6)

The Neveu-Schwarz and the Ramond algebras are connected via the spectral flow using $\theta = \frac{1}{2}$ or $\theta = -\frac{1}{2}$.

Ramond highest weight representations consist of two independent sectors (the $+$ and the $-$ sector). Each of them built on highest weight vectors satisfying an additional condition that $G^\pm_0$ or $G^\pm_0$ annihilates the highest weight vector. Only for $h = \frac{2}{m}$ we find highest weight vectors satisfying both constraints but also highest weight vectors satisfying none of the additional constraints. The latter ones do not need to be considered for unitary representations since they contain level 0 singular states. Dividing these out leads us again to the $+$ and $-$ sectors.

The discrete series of unitary representations in the Ramond $\pm$ sectors are given\textsuperscript{3} by integers $m \geq 2$, $J$ and $K$ such that $t = \frac{2}{m}$, $h = \frac{4K}{m} + \frac{c}{24}$ and $q = \pm \frac{2K}{m}$ with $0 \leq J - 1, K, J + K \leq m - 1$. It can easily be verified that the spectral flow maps the unitary representations of the Ramond and of the Neveu-Schwarz algebra onto each other such that $J = j + \frac{1}{2}$ and $K = k - \frac{1}{2}$. The Ramond embedding diagrams coincide with the ones for the Neveu-Schwarz algebra, taking into account that the level of the charged singular vectors are shifted by $\frac{1}{2}$ and therefore some of the uncharged singular vectors may be at the same level as the charged singular vectors although in the case of the Neveu-Schwarz algebra they appear at different levels. How to use the topological twists in order to transform singular vector embedding structures from the Neveu-Schwarz sector to the topological algebra is described in Ref. 13. We shall therefore focus on the Neveu-Schwarz sector.

3 Fermionic uncharged singular vectors

In Ref. 8 we showed that the Verma modules $\mathcal{V}_{h,q,t}$ are embedded in the analytically continued space $\tilde{\mathcal{V}}_{h,q,t}$, in which the operators $L_{-1}$ are continued to arbitrary complex powers $L^a_{-1}$, for $a \in \mathbb{C}$. $L^a_{-1}$ together with $\mathfrak{sc}(2)_-$ generates $\mathfrak{sc}(2)_-$. We therefore obtain the generalised algebra $\mathfrak{sc}(2) = \mathfrak{sc}(2)_- \oplus \mathcal{H}_{2} \oplus \mathfrak{sc}(2)_+$. The generalised (anti-)commutation relations\textsuperscript{8} of $\mathfrak{sc}(2)$ contain infinite but countable sums in powers of $L_{-1}$. The extended Verma modules $\tilde{\mathcal{V}}_{h,q,c} = U(\mathfrak{sc}(2)) \otimes \mathcal{H}_{2} \otimes \mathfrak{sc}(2)_+$ $|h, q, c\rangle$ decomposes just like the original Verma module $\mathcal{V}_{h,q,c}$ in infinitely many $(L_0, T_0)$-grade spaces, however, this time the grade spaces themselves are infinite dimensional and furthermore the level of a grade space is given by a complex number and there is a non-trivial infinite dimensional grade space for each complex number.

A generalised singular vector\textsuperscript{8} in $\tilde{\mathcal{V}}_{h,q,c}$ is a vector such that its cut off vectors of order $M$ satisfy the highest weight conditions up to order $M$, for all $M \in \mathbb{N}$. A singular vector $\Psi \in \mathcal{V}_{h,q,t}$...
is therefore also singular in this generalised space and conversely the finite generalised singular vectors \( \tilde{\Psi}^f \in \hat{V}_{h,q,c} \) are exactly the ones satisfying \( sc(2)_{+} \tilde{\Psi}^f = 0 \). However, this does not necessarily mean that \( \tilde{\Psi}^f \in V_{h,q,c} \), but if it was the case that \( \tilde{\Psi}^f \in V_{h,q,c} \) then \( \tilde{\Psi}^f \) would be singular in \( V_{h,q,c} \).

For \( t \neq 0 \) the singular vectors of \( \hat{V}_{h,q,c} \) are all known. They can be constructed by using products of analytic continuations \(^8\) of the operators \( \Theta^\pm_k \) for values of \( k \in \mathbb{C} \). We find that the space of uncharged singular vectors in \( V_{h,q,c} \) is exactly two-dimensional for each level \( a \in \mathbb{C} \). This two-dimensional space is spanned by

\[
\Delta_{r,s}(0,1) = \frac{1}{2^{\frac{d}{2}-1}} \prod_{m=0}^{\frac{d}{2}-1} \Theta^\pm_{-\frac{r-t+q+2m}{2}} (t, q) |h_{r,s}, q, c\rangle ,
\]

\[
\Delta_{r,s}(1,0) = \frac{1}{2^{\frac{d}{2}-1}} \prod_{m=0}^{\frac{d}{2}-1} \Theta^\pm_{-\frac{s-rt+q}{t}} (t, q) |h_{r,s}, q, c\rangle ,
\]

for any complex numbers \( r, s \) such that \( \frac{r}{2} = a \). For the charged singular vectors we find at each level in \( \mathbb{C} \) that there is an exactly one-dimensional vector space of +1 charged singular vectors and a one-dimensional vector space of -1 charged singular vectors. These vectors can easily be constructed by using products of the generalised operators \( \Theta^\pm_k \) together with the vectors \( \Delta_{r,s}(0,1) \) and \( \Delta_{r,s}(1,0) \). This implies that in \( V_{h,q,c} \) there are at most two linearly independent uncharged singular vectors at the same level and also \( \pm 1 \) charged singular vectors are unique modulo scalar multiples. There cannot be any singular vectors in \( V_{h,q,c} \) with charges different to 0 and \( \pm 1 \).

Using the leading terms of the singular vectors \( \Psi_{r,s} \in V_{h,r,s,q,c} \) we can identify these vectors among the generalised singular vectors \( (r \in \mathbb{N}, s \in 2\mathbb{N}) \):

\[
\Psi_{r,s} = \epsilon_{r,s}^+(t, q) \theta_{r,s}(1,0) |h_{r,s}, q, c\rangle + \epsilon_{r,s}^-(t, q) \theta_{r,s}(0,1) |h_{r,s}, q, c\rangle ,
\]

with

\[
\epsilon_{r,s}^\pm(t, q) = \prod_{n=1}^{r} (\pm \frac{s-rt}{2t} + \frac{q}{t} + \frac{1}{2} \pm n) ,
\]

and \( \theta_{r,s}(1,0), \theta_{r,s}(0,1) \in \widetilde{sc(2)}_+ \) defined in such a way that \( \theta_{r,s}(1,0) |h_{r,s}, q, c\rangle = \Delta_{r,s}(1,0) \) and \( \theta_{r,s}(0,1) |h_{r,s}, q, c\rangle = \Delta_{r,s}(0,1) \).

\( \Delta_{r,s}(1,0) \) and \( \Delta_{r,s}(0,1) \) are in general not finite and therefore they are certainly not singular vectors in \( V_{h,q,c} \). However, their sum in Eq. (9) has to be finite and in \( V_{h,q,c} \). As a consequence, whenever \( \epsilon_{r,s}^+(t, q) \) is trivial we find that \( \Delta_{r,s}(0,1) \in V_{h,q,c} \) and thus \( \Delta_{r,s}(0,1) \) is singular in \( V_{h,q,c} \). Similarly, whenever \( \epsilon_{r,s}^-(t, q) \) is trivial we have \( \Delta_{r,s}(1,0) \) singular in \( V_{h,q,c} \). At the intersection points of \( \epsilon_{r,s}^+(t, q) = 0 \) and \( \epsilon_{r,s}^-(t, q) = 0 \), however, both \( \Delta_{r,s}(1,0) \) and \( \Delta_{r,s}(0,1) \) are singular in \( V_{h,q,c} \) and we obtain two singular vectors at the same level and charge \(^8\). In the following section, we will show that most singular vectors of the discrete series of unitary representations satisfy

\(^d\)Since the order in the product is significant, we define \( \prod_{m=a}^{b} f(m) = f(b)f(b-1)\ldots f(a+1)f(a) \).

\(^e\)We could also imagine these conditions for different pairs \((r, s)\) and \((\tilde{r}, \tilde{s})\) if by coincidence \( rs = \tilde{r}\tilde{s} \).
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this condition. Therefore the embedding structure of the unitary minimal models always shows this degeneracy of uncharged singular vectors. The converse, that the unitary minimal models are the only ones showing this degeneracy is not true.

One of the major differences of the superconformal algebras to the Virasoro algebra is that the product of two operators may be trivial due to the anticommuting property of the fermionic operators. For the understanding of the embedding diagrams it is of particular interest to know whether or not products of singular vector operators are trivial. In the Virasoro case products of singular vector operators are never trivial and one obtains the embedding diagrams simply by descending down from low level singular vectors that are guaranteed by the determinant formula.

In Ref. 8 we obtained multiplication rules for singular vector operators. Using these rules and taking into account that $\theta(0,0) \equiv 0$ we find:

$$\Theta_{k,+}^+, \theta_{r_+,s_+}(1,0) \equiv 0 \quad , \quad \Theta_{k,-}^-, \theta_{r_-,s_-}(0,1) \equiv 0 \quad , \quad (11)$$

$$\theta_{r_+,s_+}(0,1) \Theta_{j_+}^+ \equiv 0 \quad , \quad \theta_{r_-,s_-}(1,0) \Theta_{j_-}^- \equiv 0 \quad , \quad (12)$$

$$\theta_{r_2,s_2}(1,0) \theta_{r_1,s_1}(0,1) = 0 \quad , \quad \theta_{r_2,s_2}(0,1) \theta_{r_1,s_1}(1,0) = 0 \quad , \quad (13)$$

assuming that the product of singular vector operators is defined: $h_{k,\pm}^\pm(t,q) = h_{r_\pm,s_\pm}(t,q) + \frac{\Theta^\pm}{2}$, $h_{r_\pm,s_\pm}(t,q \pm 1) = h_{j_\pm}(t,q) + j^\pm$ or $h_{r_2,s_2}(t,q) = h_{r_1,s_1}(t,q) + \frac{r_1s_1}{2}$ respectively.

Uncharged singular vectors of the type $\Delta_{r,s}(1,0)$ are thus annihilated by singular vector operators $\Theta^+_k$ provided the corresponding weight relation for $h^\pm_k$ is uncharged and of type $\Delta(1,0)$. We shall call a vector of type $\Delta(1,0)$ left fermionic uncharged singular vector. Similarly, the vectors of type $\Delta(0,1)$ are called right fermionic uncharged singular vector. The embedded Verma module built on a fermionic uncharged singular vector is not complete. Therefore, it is of particular interest to distinguish fermionic uncharged singular vectors from “normal” uncharged singular vectors in the embedding diagrams. We denote left or right fermionic uncharged singular vectors in embedding diagrams and formulae by $\langle$ or $\rangle$ respectively.

Charged singular vectors are trivially fermionic in the sense that they always satisfy condition (12). In addition, we know that there are no charge $\pm 2$ singular vectors. Therefore, for cases where the determinant formula would naively imply a $\pm 2$ charged singular vector descending from $\Psi^\pm_k$ by the operator $\Theta^\pm_j$, then we know that $\Psi^\pm_j \Psi^\pm_k \equiv 0$. It is a simple exercise to verify that if $\Psi^\pm_k$ is guaranteed by the determinant formula (either directly or indirectly as descendant) with parameter $k^a = \pm l$ or $k^b = \pm l$, then there exists an operator $\Theta^\pm_l$ which annihilates $\Psi^\pm_k$. This is due to the fact that the module built on top of $\Psi^\pm_k$ has again $k^a = \pm l$ or $k^b = \pm l$. The set of operators annihilating $\Psi^\pm_k$ defines a submodule in $V_{h^\pm_{k+q_1}+k,q_{1,t}}$, which we shall call the kernel of $\Psi^\pm_k$. As a submodule of $V_{h^\pm_{k+q_1}+k,q_{1,t}}$, the kernel of $\Psi^\pm_k$ is generated by singular and subsingular vectors of $V_{h^\pm_{k+q_1}+k,q_{1,t}}$. It may even be generated by a singular vector with relative level smaller than $l$. In the unitary cases this does not happen. It will be crucial for our later considerations to show that there are neither any subsingular vectors in $V_{h^\pm_{k+q_\pm}+k,q_{1,t}}$ for the unitary minimal models.

\[\text{Note that } \mathfrak{sc}(2) \Psi^\pm_k = \frac{\mathfrak{sc}(2)(t)}{\text{kernel } \Psi^\pm_k} \text{ is embedded in } V_{h^\pm_{k+q_{1,t}}}\]
A module generated by a charged singular vector is smaller than a module generated by an uncharged singular vector without any constraints. The partition functions for such modules can be found in Ref. 3. We shall need them in Sec. 6.

4 Embedding diagrams

Let us consider an element of the discrete series given by the parameters \( j, k, \) and \( m \), with \( m \in \mathbb{N}, m \geq 2 \), such that \( j, k \in \mathbb{N}_\frac{1}{2}, t = \frac{1}{m} \) and \( 0 < j, k, j + k \leq m - 1 \), then the conformal weight is given by \( h = \frac{j k - \frac{1}{4}}{m} \) and the \( U(1) \)-charge by \( q = \frac{j - k}{m} \). We easily find the parameter \( a = \frac{j + k}{m} \) and \( k^a = k \in \mathbb{N}_\frac{1}{2}, k^b = -j \in -\mathbb{N}_\frac{1}{2} \). In the notation of Ref. 9 these cases are always of type \( \Pi I^0 A^+ B^- \).

**Theorem 4.A** The unitary minimal models of the \( \mathfrak{sc}(2) \) algebra have the following embedding structure:

![Diagram](image)

**Fig. 1** Discrete series of unitary representations: singular vectors and their levels.

We split the proof of theorem (4.A) into two parts. We will spend most of this section to show that the embedding diagram necessarily contains the structure of Fig. 1. That this structure is also sufficient will be the result of Sec. 6. In order to show necessity, we first analyse the singular vector structure that is guaranteed by the determinant formula Eq. (5). Factorising the determinant expression leads to uncharged singular vectors for integer pairs \((r, s)\) on the line \( \tilde{s} = r\tilde{t} - a \) with \( \tilde{t} = \frac{1}{m} \) and \( a = \frac{j + k}{m} \). The relevant integer points are thus \( r_n = j + k + nm, s_n = n \) for \( n \in \mathbb{Z} \setminus \{0\} \). Therefore, we find uncharged singular vectors \( \Psi_{r_n, s_n} \) at the levels \( r_n s_n \). As given straight line intersects the \( r \)-axis at an integer point, we find that descendant singular vectors imply secondary singular vectors at exactly the same levels. We shall demonstrate this for descendants of the singular vector \( \Psi_{r-1, s-1} \) at level \( r-1 \tilde{s} - 1 = m - j - k \). Factorising the determinant for \( \Psi_{r-1, s-1} \) as highest weight state gives uncharged singular vectors for integer points on the straight line \( \tilde{s}' = r'\tilde{t} - a' \) with \( a' = 2 - \frac{j + k}{m} \). We then obtain descendant singular vectors of \( \Psi_{r-1, s-1} \) at the levels \( m - j - k + r' s'_n \) with \( r'_n = 2m - j - k + nm \) and \( s'_n = n \).
where \( n \in \mathbb{Z} \setminus \{-2, -1, 0\} \). Obviously, these levels agree with \( r_n \tilde{s}_n \). This should remind us of the Virasoro case\(^\text{12}\), although here for the \( N = 2 \) case the fact that the levels are the same does not imply that these singular vectors are linearly dependent, since the space of uncharged singular vectors can be two-dimensional. We therefore need to analyse the dimensions of the singular vector spaces. This is done by verifying Eqs. (10). If both \( \epsilon^{+}_{r_n, z_n}(t, q) \) and \( \epsilon^{-}_{r_n, z_n}(t, q) \) vanish, then the space of the singular vectors guaranteed by the determinant expression is two-dimensional. In addition, it would be spanned by a left fermionic and a right fermionic uncharged singular vector. Substituting the solutions \( r_n, \tilde{s}_n \) into Eqs. (10) shows that both expressions vanish for \( n \in \mathbb{N} \), but that they are non-trivial for \( n \in -\mathbb{N} \). We thus obtain in the uncharged sector alternately a one-dimensional and a two-dimensional singular vector space at levels \( n(nm - j - k) \) and \( n(nm + j + k) \) respectively, derived from the determinant formula for \( n \in \mathbb{N} \). However, if we now perform the same calculation for the descendant singular vectors of \( \Psi_{r_{-1}, s_{-1}} \), we find that the singular vector spaces which were before one-dimensional become two-dimensional and vice versa. We note that \( \Psi_{r_{-1}, s_{-1}} \) is neither left fermionic nor right fermionic. As a consequence, all the singular vector spaces at the levels \( r_n, s_n \) are two-dimensional in \( V_{h, q, c} \) and contain one left fermionic and one right fermionic uncharged singular vector with the single exception of \( \Psi_{r_{-1}, s_{-1}} \), which stays one-dimensional and is not uncharged fermionic.

Let us now consider the charged singular vectors. \( k^a = k \) and \( k^b = -j \) show that there is a \(-1\) charged singular vector at level \( j \) and a \(+1\) charged singular vector at level \( k \). Descendant singular vectors of \( \Psi_k^+ \) in the \(+1\) charged sector correspond to integer points on the straight line \( \tilde{s}^+ = r^+ \tilde{t} - a^+ \), where \( a^+ = a + 1 \). The solutions to this problem satisfy again similar fermionic conditions \( \epsilon_{r^+, s^+} \). However, as descendants of \( \Psi^+_k \), only the left fermionic components are non-trivial in the original Verma module according to Eq. (13), and there exists a left fermionic component for all solutions. Therefore, we obtain \(+1\) charged singular vectors at the levels \( k + r_n^+ \tilde{s}_n^+ \) with \( r_n^+ = j + k + (n + 1)m \) and \( \tilde{s}_n^+ = n \) for \( n \in \mathbb{Z} \setminus \{-1, 0\} \). Once again we do not obtain more singular vectors in the charged sector by analysing more descendant vectors of \( \Psi^+_r, s_n^+ \), since the given straight line intersects the \( r \)-axis at an integer point. Already at this stage, we ought to mention, that by analysing charged descendant singular operators of \( \Psi^+_r, s_n^+ \), we find constraints on the embedded Verma modules built on top of \( \Psi^+_r, s_n^+ \), since there are no charge 2 singular vectors in the original Verma module. For \( \Psi^+_r, s_n^+ \) we easily find these constraints at descendant relative levels \( k + nm \) for \( n \in \mathbb{N} \) and \( -j - (n + 1)m \) for \( n \in -\mathbb{N} \setminus \{-1\} \), simply by computing the new factors \( k^a \) and \( k^b \). It is important to realise that these are the descendant charged singular operators at lowest levels\(^\text{3}\), exactly as they appear in the fermionic partition function of the determinant. All other singular vectors in the kernel of \( \Psi_k^+ \) descend from these operators, so that for our later considerations in addition to \( \Psi_k^+ \) we only need to be concerned about possible subsingular vectors in the kernel in order to generate the whole kernel. Similar results are valid for \( \Psi^-_k \).

\(^3\)Is is also important to note that the first uncharged singular vector is neither left nor right fermionic. Therefore, the charged singular vector given by the new factor \( k^a \) or \( k^b \) defines the lowest level operator annihilating \( \Psi^+_r, s_n^+ \).
We have now shown that the singular vectors given in Fig. 1 are necessarily contained in the embedding structure at the levels given in Tab. 1. In order to verify the embedding homomorphisms drawn in Fig. 1, we simply need to descend down from the vectors given in Tab. 1. For the uncharged sector we explained this briefly above. Therefore, the only cases left are the lines connecting charged and uncharged sectors. If we find a match in level for such a connexion and if the vectors are not annihilated by fermionic conditions, then due to Eqs. (11-12) we know that the \(-1\) charged singular vectors can only connect to left fermionic uncharged singular vectors, whilst the \(+1\) charged singular vectors connect to right fermionic uncharged singular vectors and vice versa. We are now left with proving that the levels match in the way indicated in Fig. 1. We shall show this for lines from right fermionic uncharged singular vectors to \(+1\) charged singular vectors only. The determinant parameters for modules embedded on top of \(\Psi_{r_n,s_n}^\dagger\) are \(k^{a}_{r_n,s_n} = k + nm\) which for \(n \in \mathbb{N}\) implies a charge \(+1\) singular vector. As \(\Psi_{r_n,s_n}^\dagger\) is right fermionic, the descendant singular vector is non-trivial in the original Verma module at level \(n(j + k + nm) + k + nm\) which matches with the level of \(\Psi_{r_n,s_n}^{-}\). In the case of negative \(n\), \(k^{b}_{r_n,s_n} = -j - nm\) leads to the vector \(\Psi_{r_{n+1},s_{n+1}}^{+}\). This completes the proof that the embedding diagram in Fig. 1 is at least a subset of the embedding diagram for the discrete series of unitary representations. That it is also sufficient and that there are no subsingular vectors shall be shown in the following sections.

### 5 The representations \(Q^{m_{ij}+l}\)

In order to prove the embedding diagrams of the previous section we first need to generalise theorem 4.A to a larger class of modules for which the same embedding diagram is valid. We consider a particular class of representations, again given by \(\tilde{t} = \frac{1}{m}\) with \(m \in \mathbb{N}\backslash\{1\}\), \(h = \frac{jk - \frac{1}{2}}{m}\) and \(q = \frac{j - k}{m}\) with \(k, j \in \mathbb{Z}\), and \(m\) not dividing \(j + k\), in symbols: \(m \nmid j + k\). Obviously, we also find \(a = \frac{j + k}{m}, k^{a} = k > 0\) and \(k^{b} = -j < 0\). The discrete series of unitary representations are certainly of this type but the restriction \(j, k, j + k \leq m - 1\) does no longer apply. We shall call these highest weight representations \(Q^{m_{ij}+l}_{m,1}\) with \(m \in \mathbb{N}\backslash\{1\}\), following conventions\(^5\) of Ref. 11. It is rather simple to show that the same analysis as performed in the previous section holds. We therefore ultimately come back to the same embedding patterns. The only difference is that the lowest level uncharged singular vector is not at level \(m - j - k\) but at level \((\beta + 1)(\beta + 1)m - j - k\) where \(\beta\) repre-

\(^{5}\)(m, 1) refers to \(\tilde{t} = \frac{1}{m}\). Similarly, \((m, m')\) denotes \(\tilde{t} = \frac{m'}{m}\).
resents the integer value of \( \frac{\beta + k}{m} \). For uncharged singular vectors with negative index \( n \) we have to replace \( r_n \) and \( \tilde{s}_n \) by \( r_n = (n - \beta)m + j + k \) and \( \tilde{s}_n = n - \beta \), and similarly for charged singular vectors.

\[
q : \quad -1 \quad 0 \quad +1
\]

Fig. ii Embedding diagrams for representations of type \( \mathcal{Q}_{m,1}^{m'j+k} \).

In fact, the embedding diagrams shown in Fig. ii are valid for a much larger class of representations\(^{11,4} \): for all the representations \( \mathcal{Q}_{m,m'}^{m'j+k} \) with coprime \( m, m' \), \( k^a = k > 0 \), \( k^b = -j < 0 \) and \( m, m' \in \mathbb{N}, k, j \in \mathbb{N}_{\frac{1}{2}} \), and \( m \) not dividing \( j + k \), as described in Ref. 11. This is difficult to understand by just descending down along vanishing curves of the determinant. \( \text{A priori} \) one would believe that the charged singular vectors do not join the uncharged singular vectors any more, what clearly violates a generalised asymptotic Ore-condition\(^{17} \) for \( N = 2 \). This belief is based on the conditions of Eqs. (11-13). However, in this case you have to apply two of such conditions successively. The first condition annihilates a submodule assumed to contain the operator connecting to the uncharged sector. But even this connexion is subject to one of the vanishing conditions. Therefore, the removed kernel does not contain the connexion back to the uncharged sector. For the purpose of this paper, we do not need to be concerned by these generalisations though, but our proof would equally well hold for these cases.

6 Character formulae

Eholzer and Gaberdiel showed in Ref. 11 that the embedding diagram of Fig. ii reveals the correct vacuum character formula. In the same manner, we shall briefly explain how to deduce generally characters out of the embedding diagrams until we reach the familiar character formulae. Finally, this will also complete the proof for the embedding diagram of Fig. ii. We define the partition function over a \( (L_0, T_0) \)-graded module \( \mathcal{V}_{h,q} \) as the formal power series \( P_{\mathcal{V}} = x^{-h} y^{-q} \text{tr}_{\mathcal{V}} (x^{L_0} y^{T_0}) \). We first write down all the partition functions for the modules built on top of the singular vectors\(^{i} \) given in Tab. a. For convenience, we label modules \( \mathcal{V} \) of type \( \mathcal{Q}_{m,1}^{m'j+k} \) simply by their parameters \( j, k \) and \( m \) as \( \mathcal{V}_{j,k,m} \) and partition functions of a module

\(^{i}\text{We will not need partition functions for } \Psi_{r_n,s_n}^{\leq} \text{ and } \Psi_{r_n,s_n}^{\geq} \text{ but for unconstrained singular vectors at the same levels which we denote by } \Psi_{r_n,s_n}^{\leq} \text{.} \)
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built on top of a singular vector $\Psi$ simply by $P_\Psi$. These partition functions were first given by Boucher, Friedan and Kent\(^3\).

\[
P_{\nu_{j,k,m}} = \prod_{l=1}^{\infty} \frac{(1 + x^{l-\frac{1}{2}} y) (1 + x^{l-\frac{1}{2}} y^{-1})}{(1 - x^l)^2}, \quad (14)
\]

\[
P_{\Psi_{r-n,s-n}} = x^{n(nm-j-k)} P_{\nu_{j,k,m}}, \quad n \in \mathbb{N}, \quad (15)
\]

\[
P_{\Psi_{r-n}} = x^n(nm+j+k) P_{\nu_{j,k,m}}, \quad n \in \mathbb{N}, \quad (16)
\]

\[
P_{\Psi_{r-n,s-n}}^+ = \frac{x^{k+n((n+1)m+j+k)} y P_{\nu_{j,k,m}}}{1 + x^{k+nm} y}, \quad n \in \mathbb{N}_0, \quad (17)
\]

\[
P_{\Psi_{r-n-1,s-n-1}}^+ = \frac{x^{k+(n+1)(nm-j-k)} y P_{\nu_{j,k,m}}}{1 + x^{-n} y}, \quad n \in \mathbb{N}, \quad (18)
\]

\[
P_{\Psi_{r-n,s-n}}^- = \frac{x^{j+n((n+1)m+j+k)} y^{-1} P_{\nu_{j,k,m}}}{1 + x^{j+nm} y^{-1}}, \quad n \in \mathbb{N}_0, \quad (19)
\]

\[
P_{\Psi_{r-n-1,s-n-1}}^- = \frac{x^{j+(n+1)(nm-j-k)} y^{-1} P_{\nu_{j,k,m}}}{1 + x^{-k+nm} y^{-1}}, \quad n \in \mathbb{N}. \quad (20)
\]

Let us start with $\nu_{j,k,m}$ being a module of the discrete unitary series. Our aim is to compute the partition function for

\[
Q_{j,k,m} = \frac{\nu_{j,k,m}}{U(\mathfrak{sc}(2))\Psi_{r-1,s-1} + U(\mathfrak{sc}(2))\Psi_{r}^+ + U(\mathfrak{sc}(2))\Psi_{r}^-} \quad (21)
\]

and show that this reveals the correct character formula. Therefore, we first construct the partition function for $Q_{j,k,m}^+ = \frac{\nu_{j,k,m}}{U(\mathfrak{sc}(2))\Psi_{r}^+ + U(\mathfrak{sc}(2))\Psi_{r}^-}$ which is simply $P_{Q_{j,k,m}} = P_{\nu_{j,k,m}} - P_{\Psi_{r}^-}$ since the submodules generated by $\Psi_{r}^+$ and $\Psi_{r}^-$ do not intersect\(^1\). $Q_{j,k,m}$ contains $\nu_{j,k,m}$ and they are related by $Q_{j,k,m} = \frac{\nu_{j,k,m}}{U(\mathfrak{sc}(2))\Psi_{r-1,s-1}}$. Complications arise due to the fact that $U(\mathfrak{sc}(2))\Psi_{r-1,s-1}$ intersects with the submodules generated by $\Psi_{r}^+$ and $\Psi_{r}^-$. If we simply subtract $P_{\Psi_{r-1,s-1}}$ from $P_{Q_{j,k,m}}^+$, then the intersection module generated by $\Psi_{r+1,s_1}$, $\Psi_{r+2,s_2}^+$ and $\Psi_{r-2,s_2}^-$ is subtracted already twice, and thus needs to be added back in. It is important to note that $\Psi_{r+1,s_1}$ is the lowest level uncharged descendant vector of $\Psi_{r-1,s-1}$ and is therefore a complete module (neither left fermionic nor right fermionic). By adding the partition functions of these modules back in, we again obtain an intersection module generated by $\Psi_{r+2,s_2}$, $\Psi_{r+2,s_2}^+$ and $\Psi_{r-2,s_2}^-$ which this time has not been subtracted overall. Proceeding in this manner leads to an alternating pattern of subtracting and adding partition functions of the type of Eqs. (14-20). This pattern coincides with the pattern given by Dobrev\(^6\), even though he started from the wrong embedding diagrams, neglecting the degeneration of uncharged singular vectors in left and right fermionic uncharged singular vectors\(^k\).

\(^1\)Note that at an intersection point we would need to find a common singular vector.

\(^k\)Note that the signs in the uncharged sector of Fig. iii refer to complete uncharged Verma modules.
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We finally obtain the following partition function for $Q_{j,k,m}$:

$$P_{Q_{j,k,m}} = P_{V_{j,k,m}} \{1 - \sum_{n \in \mathbb{N}_0} P_{\Phi_n^+} - \sum_{n \in \mathbb{N}_0} P_{\Psi_n^-} - \sum_{n \in \mathbb{N}} P_{\Psi_n^+} + \sum_{n \in \mathbb{N}} P_{\Phi_n^-} \} , \quad (22)$$

$$= P_{V_{j,k,m}} \left\{ 1 + \sum_{n=0}^{\infty} x^{n^2 m} (x^{n(j+k)} - x^{-n(j+k)}) + \sum_{n=0}^{\infty} x^{(n+1)nm-j} y \left( \frac{x^{2(n+1)m-(n+1)j-(n+1)k}}{1 + x^{(n+1)m-j} y} - \frac{x^{(n+1)j+(n+1)k}}{1 + x^{k+nm} y} \right) + \sum_{n=0}^{\infty} x^{(n+1)nm-k} y^{-1} \left( \frac{x^{2(n+1)m-(n+1)j-(n+1)k}}{1 + x^{(n+1)m-k} y^{-1}} - \frac{x^{(n+1)j+(n+1)k}}{1 + x^{j+nm} y^{-1}} \right) \right\} . \quad (23)$$

Eq. (23) coincides with known formulae for the $N = 2$ characters $^{11,1,15,19,6}$. Deriving similar expressions for the representations of type $Q_{m,1}$ is straightforward by adding appropriate factors of $\beta$, and is left to the reader. Before we finish the proof for the embedding diagrams, we make some important remarks:

**Theorem 6.A** $\Upsilon$ is a subsingular vector in the Verma module $V$. If $\Upsilon$ lies in a submodule $W$ of $V$, then $\Upsilon$ is also subsingular in $W$.

Proof: We have nothing to show if $\Upsilon$ is subsingular with respect to a singular vector $\Psi \in W$. Thus, assume that $\Psi$ lies outside $W$. The module built on top of $\Psi$ cannot be disjoint from

$^{1}$Strictly speaking, we have to assume that for the coset construction the $N = 2$ algebra is indeed the coset and not only a subalgebra of the coset. No independent proof of this is known to us. We are grateful to Matthias Gaberdiel and Wolfgang Eholzer for pointing this out.

$^{m}$Instead of having only one vector $\Psi$, $\Upsilon$ can also be subsingular with respect to many singular and subsingular vectors. The proof still holds.
We summarise our results in the following theorem: 

Theorem 6.B For a representation of type $Q_{m,1}^{nlj+k}$ the uncharged singular vector at lowest level $\Psi_{r-1,s-1}$ and the charged vectors $\Psi^+_k$ and $\Psi^-_j$ define as unconstrained highest weight modules $V_{hr-1,s-1+r-1\bar{s}-1,q,t}$, $V_{h^+_k+r+k+1,t}$ and $V_{h^-_j+j,q-1,t}$ again representations of type $Q_{m,1}^{nlj+k}$.

Proof: If the given model has the parameters $k^a = k$ and $k^b = -j$, $j \in \mathbb{N}_{\frac{1}{2}}$, then the factors $k^a_{-1}$ and $k^b_{-1}$ for the module $V_{hr-1,s-1+r-1\bar{s}-1,q,t}$ with $r-1 = j+k-(\beta+1)m$, $s = -(\beta+1)$ are:

$k^a_{-1} = k-(\beta+1)m$, $k^b_{-1} = -j+(\beta+1)m$. Which is again a representation of type $Q_{m,1}^{nlj+k}$.

Similarly, $k^a_+ = k$, $k^b_+ = -j-m$, $k^a_- = k+m$ and $k^b_- = -j$.

We summarise our results in the following theorem:

Theorem 6.C The embedding diagrams of the unitary $N = 2$ minimal models and modules of type $Q_{m,1}^{nlj+k}$ are given by Fig. i and Fig. ii. There are no subsingular vectors in these

*Note that $(\beta+1)m \geq j,k.$
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representations. Furthermore, the singular vectors $\Psi_{r,s}^{\pm} $ have constraints generated by the singular operators given in Tab. and their kernels do not contain any subsingular vectors.

Just recently subsingular vectors were discovered for some highest weight representations of the $\mathfrak{sc}(2)$ algebra by Gato-Rivera and Rosado\textsuperscript{13,14}. As we can easily verify, in the case of the unitary minimal models these subsingular vectors turn into ordinary singular vectors already contained in Fig. i. It is also easy to verify that the subsingular vectors given in Ref. 10 are not representations of type $Q_{m,1}^{m|j+k}$.

7 Conclusion

In this paper we have presented the embedding structure for singular vectors of the unitary minimal models and for representations of type $Q_{m,1}^{m|j+k}$. These embedding patterns show a degeneration of uncharged singular vectors. We have therefore corrected previous publications in this area by different authors\textsuperscript{6,16,18}. The character expressions derived from our embedding patterns coincide with the correct character formul\ae for these representations. Surprisingly enough, the character formul\ae also agrees with the ones derived earlier by the above authors using wrong embedding structures. We finally gave a proof for the embedding diagram using the character formula. This shows, that the $N = 2$ unitary minimal models do not contain any subsingular vectors. Nevertheless, there are other representations of the $N = 2$ superconformal algebra which do contain subsingular vectors as recently shown by Gato-Rivera and Rosado\textsuperscript{13}. The fact that just the unitary representations are not concerned by subsingular vectors raises the question whether this holds for a much larger class of algebras.

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