Monetary Measures of Risk∗

Andreas H. Hamel†

A monetary risk measure is a mathematical tool for quantifying the risk of a random future gain (or loss) which is denoted in (discounted) units of a reference instrument (a currency, for example). As such, it is a real-valued function, and it is convenient to allow for the value $+\infty$. The greater the value of the risk measure, the higher the risk.

Two elementary mathematical properties turned out to be crucial. Both have a straightforward and convincing economic interpretation.

The first one is a monotonicity property: if gain $X$ is not less than gain $Y$ no matter what happens in the world, then the risk of $X$ should not be greater than the risk of $Y$.

The second one is additivity with respect to a riskless reference instrument: if one adds $s$ units of the reference instrument (e.g., cash) to the (discounted) random gain (e.g., as a deposit), then the risk (i.e., the value of the risk measure) decreases by $s$. Because of the immediate interpretation of the value of such risk measures as capital requirements, they are also called monetary measures of risk [27].

This second property, called cash-additivity, has remarkable mathematical consequences. Its economic interpretation, ‘linearity in payments’ has been pointed out already very clearly in [54, p. 101] by Yaari, and it became popular through the work [4] by Artzner et al. Cash-additivity goes by many names, for example “translation invariance,” (already in [53], also [4]), “translation equivariance” ([51]), or just “additivity” ([5, p. 1455]).

Following the famous [38], the variance of a random variable was used as a risk measure in portfolio selection problems (see also 3.2.2.2 and 3.2.2.4). However, it is neither monotone, nor cash-additive. Moreover, it weighs (random) gains and losses in a symmetrical way which is not a desirable property of a (financial) risk measure and, on an even deeper level, the variance is not consistent with important stochastic dominance orders as already discussed, e.g., in [41]. On the other hand, in the financial practice the (cash-additive) so-called Value-at-Risk was (and still is) widely used as a risk measure. Its drawback turned out to be the missing convexity: diversification is not generally supported by the Value-at-Risk. Therefore, a new class of (monotone and cash-additive) risk measures, called coherent, was introduced in [4] (see also 4.6.3.3).

∗This paper was written in 2015 as a contribution to Wiley Encyclopedia of Operations Research and Management Science. It is accepted and will appear in the second edition of EORMS.
†Free University of Bozen-Bolzano, Faculty of Economics and Management, andreas.hamel@unibz.it
1Such labels refer to other articles in EORMS.
1 Risk measures and acceptance sets

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $p \in [0, \infty]$ and $L^p := L^p(\Omega, \mathcal{F}, P)$ be the linear space of all (equivalence classes of) univariate, to the $p$th power (absolutely) integrable random variables where $X_1: \Omega \to \mathbb{R}$ and $X_2: \Omega \to \mathbb{R}$ generate the same element of $L^p$ whenever $P \left( \{ \omega \in \Omega \mid X_1(\omega) = X_2(\omega) \} \right) = 1$. If $p = 0$, $L^0$ is the space of all random variables. If $p = \infty$, $L^\infty := L^\infty(\Omega, \mathcal{F}, P)$ is the space of essentially bounded random variables. An inequality like $X_1 \leq X_2$ for two elements of $X_1, X_2 \in L^p$ is understood $P$-almost surely, i.e. $P \left( \{ \omega \in \Omega \mid X_1(\omega) \leq X_2(\omega) \} \right) = 1$. The element $\mathbb{I} \in L^p$ denotes the function whose value is 1 $P$-almost surely, and $L^p_+ = \{ X \in L^p \mid 0 \leq X \}$.

A function $\varphi: L^p \to \mathbb{R} \cup \{ +\infty \}$ is called monotone if $X, Y \in L^p$, $X \leq Y$ imply $\varphi(Y) \leq \varphi(X)$, and it is called cash-additive if

$$\forall X \in L^p, \forall r \in \mathbb{R}: \varphi(X + r \mathbb{I}) = \varphi(X) - r. \quad (1)$$

Definition 1 A risk measure is a function $\varphi: L^p \to \mathbb{R} \cup \{ +\infty \}$ which is monotone, cash-additive and satisfies $\varphi(0) \in \mathbb{R}$.

Risk measures can also be defined on other linear spaces of random variables, see, for example, [12]. It is remarkable, though mathematically not difficult, that risk measures are basically in one-to-one correspondence with their acceptance sets (see also 3.2.4.1). This fact depends almost entirely on property (1). Here are the necessary concepts. A set $A \subseteq L^p$ is called monotone if $A + L^p_+ \subseteq A$, and it is called directionally closed if $X \in A$, $\{r_n\}_{n=0,1,\ldots} \subseteq \mathbb{R}_+$, $\lim_{n \to \infty} r_n = 0$ and $X + r_n \mathbb{I} \in A$ for all $n = 0, 1, \ldots$ imply $X \in A$.

Definition 2 An acceptance set is a set $A \subseteq L^p$ which is monotone, directionally closed and satisfies $A \cap \mathbb{R} \mathbb{I} \neq \emptyset$ as well as $(L^p \setminus A) - \mathbb{R} \mathbb{I} = L^p$.

The correspondence between acceptance sets and risk measures is established in the following result.

Proposition 3 If $A \subseteq L^p$ is an acceptance set, then the function $\varphi_A$ on $L^p$ defined by

$$\varphi_A(X) = \inf \{ s \in \mathbb{R} \mid X + s \mathbb{I} \in A \} \quad (2)$$

is a risk measure. If $\varphi: L^p \to \mathbb{R} \cup \{ +\infty \}$ is a risk measure, then the set

$$A_\varphi = \{ X \in L^p \mid \varphi(X) \leq 0 \} \quad (3)$$

is an acceptance set. Moreover, it holds $A = A_{\varphi_A}$ and $\varphi = \varphi_{A_\varphi}$.

The condition $A \cap \mathbb{R} \mathbb{I} \neq \emptyset$ means that there is an amount of cash which the financial agent accepts, and $(L^p \setminus A) - \mathbb{R} \mathbb{I} = L^p$ says that there is a limit to cash withdrawals starting from whatever position $X$. Both conditions together make sure that $\varphi_A$ never attains $-\infty$ as a value and that $\varphi_A(0) \in \mathbb{R}$. Everything else being straightforward, one
can verify $A_{\varrho_{A}} \subseteq A$ as follows: If $X \in A_{\varrho_{A}}$, then $\varrho_{A}(X) = \inf \{ s \in \mathbb{R} \mid X + s \mathbb{I} \in A \} \leq 0$, and the very definition of the infimum implies the following: For each $\varepsilon > 0$ there is $s_{\varepsilon} \leq \varepsilon$ such that $X + s_{\varepsilon} \mathbb{I} \in A$. Using the fact that $A$ is monotone one obtains

$$X + \varepsilon \mathbb{I} = X + s_{\varepsilon} \mathbb{I} + (\varepsilon - s_{\varepsilon}) \mathbb{I} \in A + \mathbb{R}_{+} \mathbb{I} \subseteq A.$$  

Since $A$ is directionally closed, $X \in A$, hence $A_{\varrho_{A}} \subseteq A$.

Directional closedness of $A$ implies the closedness of the set $\{ s \in \mathbb{R} \mid X + s \mathbb{I} \in A \}$ which means that the infimum in the definition of $\varrho_{A}(X)$ is attained if it is not $+\infty$: there is $s_{0} \in \mathbb{R}$ such that $\varrho_{A}(X) = s_{0}$ and $X + s_{0} \mathbb{I} \in A$. This implies $X + \varrho_{A}(X) \mathbb{I} \in A$, i.e., $X$ can be made acceptable by depositing $\varrho_{A}(X)$ units of cash (or more).

In most cases, risk measures and acceptance sets have to satisfy further requirements. Again, property (1) provokes one-to-one correspondences between properties of risk measures and acceptance sets:

(a) A risk measure $\varrho$ is convex if, and only if, the “induced” acceptance set $A_{\varrho}$ is convex. This implies that a risk measure is convex if, and only if, it is quasiconvex.

(b) $\varrho$ is positively homogeneous (sublinear) if, and only if, $A_{\varrho}$ is a cone (a convex cone).

(c) $\varrho$ has (only) real values if, and only if, $A_{\varrho} - \mathbb{R}_{+} \mathbb{I} = L^{p}$.

Corresponding statements are obtained for acceptance sets $A$ and the “induced” risk measures $\varrho_{A}$.

Following [4], it became custom in the math finance community to call sublinear risk measures coherent. However, the authors of [4] probably intended to use the word “coherent” in a more literal sense: for example in [33] convex, but not necessarily sublinear risk measures are also called (weakly) coherent. Moreover, “coherent” is also used in the sense which is usually associated with “arbitrage-free”–as in [52].

A few elementary examples for risk measures are the following. The function $X \mapsto E[-X]$ is a linear risk measure on $L^{1}$, the function $X \mapsto -\operatorname{essinf} X$ is a sublinear risk measure on $L^{\infty}$. A large class of risk measures is based on quantiles: For $\alpha \in (0, 1]$, the number

$$q_{\alpha}^{+}(X) = \inf \{ t \in \mathbb{R} \mid P[X \leq t] > \alpha \} = \sup \{ t \in \mathbb{R} \mid P[X < t] \leq \alpha \}$$

is called the upper $\alpha$-quantile of $X$; the function

$$X \mapsto V@R_{\alpha}(X) = -q_{\alpha}^{+}(X) = \inf \{ t \in \mathbb{R} \mid P[X + t \mathbb{I} < 0] \leq \alpha \}$$

is called the Value-at-Risk of $X$ at level $\alpha$; the function

$$X \mapsto AV@R_{\alpha}(X) = \frac{1}{\alpha} \int_{0}^{\alpha} V@R_{\beta}(X)d\beta$$

is called the Average-Value-at-Risk of $X$ at level $\alpha$. Whereas $V@R$ is a positively homogeneous, but in general non-convex risk measure on $L^{0}$, the $AV@R$ is a sublinear risk measure on $L^{1}$, i.e., it is coherent.
The function \( \tau : L^p \to \mathbb{R} \cup \{+\infty\} \) defined by
\[
\tau(X) = \begin{cases} 
-r & : X = r\mathbb{I}, \ r \in \mathbb{R} \ \\
+\infty & : X \text{ is non-constant}
\end{cases}
\]
is sublinear and satisfies the requirements of Definition 1 except for monotonicity. It could be seen as an extreme way to evaluate risk: non-constant payoffs are considered as intolerable risks, and acceptable are only the non-negative constant ones. It turns out that every risk measure has a representation in terms of \( \tau \). Indeed, defining the indicator function (in the sense of convex analysis) \( I_A : L^p \to \mathbb{R} \cup \{+\infty\} \) of an acceptance set \( A \subseteq L^p \) by \( I_A(X) = 0 \) whenever \( X \in A \) and \( I_A(X) = +\infty \) otherwise, formula (2) can be written as
\[
(I_A \square \tau)(X) = \inf \{ I_A(X_1) + \tau(X_2) \mid X_1 + X_2 = X \}
= \inf \{ I_A(X_1) - r \mid X_1 + r\mathbb{I} = X, r \in \mathbb{R} \}
= \inf \{-r \mid X - r\mathbb{I} \in A, r \in \mathbb{R} \} = \varrho_A(X).
\]
Thus, the position \( X \) is split into a constant and a remaining position \( X_1 \) which should be acceptable. If this is possible, one looks for the minimal risk of the constant evaluated by \( \tau \). If such a split is not possible, \( I_A(X_1) = +\infty \) always holds and \( (I_A \square \tau)(X) = +\infty \). Mathematically, \( \varrho_A \) is the infimal convolution of \( I_A \) and \( \tau \). Every risk measure that satisfies the assumptions in Proposition 3 has such a representation: \( \varrho(X) = (I_{A_{\varrho}} \square \tau)(X) \) for all \( X \in L^p \). This is very convenient, in particular for duality purposes, since the two functions \( I_A \) and \( \tau \) are easy to handle.

2 Closedness and dual representation

The space \( L^0 \) is a complete metric, linear space for any Lévy-metric. If \( p \geq 1 \), \( L^p \) is a Banach space with the norm \( \|X\|_p = \left(\int_{\Omega} |X|^p \, dP\right)^{1/p} \) for \( 1 \leq p < \infty \) and \( \|X\|_\infty = \text{esssup} |X| \) for \( p = \infty \). In the following, \( p \in \{0\} \cup [1, \infty] \) is assumed.

Proposition 4 The following statements are equivalent for a risk measure \( \varrho : L^p \to \mathbb{R} \cup \{+\infty\} \):
(a) At each \( X \in L^p \), \( \varrho \) is lower semicontinuous, i.e., \( \varrho(X) \leq \lim \inf_{n \to \infty} \varrho(X_n) \) whenever \( \lim_{n \to \infty} X_n = X \) in \( L^p \).
(b) \( \{X \in L^p \mid \varrho(X) \leq r\} \) is closed for each \( r \in \mathbb{R} \);
(c) \( A_\varrho = \{X \in L^p \mid \varrho(X) \leq 0\} \subseteq L^p \) is closed.
A parallel statement holds for \( \varrho_A \) replaced by a risk measure \( \varrho \) and \( A \) by \( A_\varrho \).

The equivalence of (a) and (b) is standard in variational analysis, while (c) enters the picture because of the cash-additivity (1). A risk measure that satisfies one (and hence) all of the conditions in Proposition 4 is called closed.

For \( p \in [1, \infty) \), the topological dual of \( L^p \) is the Banach space \( L^q \) for \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( q = \infty \) whenever \( p = 1 \). If \( p = \infty \) then \( L^\infty \) is supplied with the weak topology generated
by the dual pair \((L^1, L^\infty)\) (of locally convex spaces, see [3, Section 5.14]), and this ensures that \(L^\infty\) and \(L^1\) become topological duals of each other. Note that the topology on \(L^\infty\) influences the closedness of \(\varrho\): There are functions on \(L^\infty\) which are closed with respect to the norm topology, but not closed with respect to the (weak) topology generated by \(L^1\). A condition that ensures the latter turns out to be equivalent to the so-called Fatou property, see [27, Section 4.3].

Let \(\varrho: L^p \to \mathbb{R} \cup \{+\infty\}\) be a closed, convex risk measure. According to Definition 1, it never attains the value \(-\infty\), and it has at least one real value \(\varrho(0)\). This means that \(\varrho\) is proper in the sense of convex analysis (see [55, p. 39]), and it satisfies all the assumptions of the Fenchel-Moreau theorem [55, Theorem 2.3.3]: it coincides with its Legendre-Fenchel biconjugate \(\varrho^{**}\) which is given by the two formulas

\[
\varrho^*(Y) := \sup_{X \in L^p} \{E[XY] - \varrho(X)\} \quad \text{and} \quad \varrho^{**}(X) := \sup_{Y \in L^q} \{E[XY] - \varrho^*(Y)\}
\]

where the Legendre-Fenchel conjugate \(\varrho^*: L^\infty \to \mathbb{R} \cup \{+\infty\}\) of \(\varrho\) is defined on the topological dual space \(L^q\) of \(L^p\).

The representation \(\varrho = \varrho^{**}\) is useful only if one can determine \(\varrho^*\). It turns out that

\[
\varrho^*(Y) = \begin{cases} 
\sup_{X \in A_\varrho} E[-XY] : E[Y] = 1, Y \in L_+^\infty & \text{for } Y \in L_+^q \\
+\infty & \text{otherwise}
\end{cases}
\]

(4)

This follows from the representation \(\varrho = I_{A_\varrho} \square \tau\) and the fact that the conjugate of an infimal convolution is the sum of the conjugates ([55, Theorem 2.3.1 (ix)]: one has to compute \((I_{A_\varrho})^*\) and \(\tau^*\). While the former is known to be the support function of \(A_\varrho\) (an easy consequence of the definition of the conjugate), the latter is \(I_{\{1_{Y \in L^q}|E[Y] = -1\}}\). Observing that the support function of \(A_\varrho\) attains the value \(+\infty\) whenever \(Y \notin -L_+^q\) (this follows from monotonicity of \(A_\varrho\)) and then replacing \(Y\) by \(-Y\) one obtains (4).

The two conditions for \(Y\) in (4) admit a striking interpretation of the dual representation formula \(\varrho = \varrho^{**}\). To \(Y \in L_+^q\) satisfying \(E[Y] = 1\) one can assign a probability measure \(Q\) by

\[
Q(A) = \int_A Y(\omega) dP \quad \text{for } A \in \mathcal{F}
\]

which is absolutely continuous with respect to \(P\), i.e., \(\frac{dQ}{dP} = Y\). Moreover, the relationship between \(Q\) and \(Y\) is one-to-one. If one denotes the set of such probability measures by \(\mathcal{M}_1(P)\), then the dual representation result for risk measures on \(L^p\) reads as follows.

**Theorem 5** The function \(\varrho: L^p \to \mathbb{R} \cup \{+\infty\}\) is a closed, convex risk measure if, and only if, there exists a non-empty set \(\mathcal{Q}_\varrho \subseteq \mathcal{M}_1(P)\) and a function \(\gamma: \mathcal{Q}_\varrho \to \mathbb{R}\) such that

\[
\forall X \in L^p: \varrho(X) = \sup_{Q \in \mathcal{Q}_\varrho} \{E^Q[-X] - \gamma(Q)\}
\]

Moreover, \(\gamma(Q) = \sup_{X \in A_\varrho} E[-XY] = \sup_{X \in A_\varrho} E^Q[-X]\) whenever \(Q\) is the probability measure generated by \(Y \in L_+^q\) which satisfies \(E[Y] = 1\) and \(\varrho^*(Y) < +\infty\).

If \(\varrho\) is additionally positive homogeneous (hence sublinear), then \(\gamma(Q) = 0\) for \(Q \in \mathcal{Q}_\varrho\).
The worst case risk measure $\varrho_{\text{max}}: L^\infty \to \mathbb{R}$ defined by $\varrho_{\text{max}}(X) = -\text{essinf} X = \inf \{ t \in \mathbb{R} \mid X + t1_{\mathbb{R}} \geq 0 \}$ has the dual representation

$$\varrho_{\text{max}}(X) = \sup_{Q \in \mathcal{M}_1(P)} E^Q[-X],$$

whereas the Average-Value-at-Risk on $L^1$ can be represented as

$$\text{AV@R}_\alpha(X) = \sup \left\{ E^Q[-X] \mid Q \in \mathcal{M}_1(P), \frac{dQ}{dP} \leq \frac{1}{\alpha} \right\}.$$

Both are sublinear (coherent) risk measures. Note that $\frac{dQ}{dP} \in L^\infty$ for $Q \in \mathcal{Q}_{\text{AV@R}}$. A verification of this formula can be given via the representation $\text{AV@R}_\alpha(X) = \frac{1}{\alpha} \log E[\exp(-\beta X)]$ for $\beta > 0$, a risk measure $\varrho_{\text{ent}}: L^\infty \to \mathbb{R}$ is defined; it is convex, but not positively homogeneous. Its dual representation is

$$\varrho_{\text{ent}}(X) = \sup_{Q \in \mathcal{M}_1(P)} \left\{ E^Q[-X] - \frac{1}{\beta} H(Q \mid P) \right\},$$

where $H(Q \mid P) = E^Q[\log \frac{dQ}{dP}]$ is the relative entropy of $Q$ with respect to $P$.

### 3 Law invariance and Kusuoka representation

A risk measure $\varrho: L^p \to \mathbb{R} \cup \{+\infty\}$ is called law invariant if $\varrho(X) = \varrho(Y)$ whenever $X$ and $Y$ have the same distribution under $P$. Standard examples of law invariant risk measures are the quantile based $V@R$ and $AV@R$. For risk measures on $L^\infty$, law invariance has strong implications. A typical result reads as follows.

**Theorem 6** Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space such that $L^2$ is separable. Then, $\varrho: L^\infty \to \mathbb{R}$ is a law invariant convex risk measure if, and only if, there exists a convex function $\pi: \mathcal{M}_1((0, 1]) \to [0, \infty]$ such that

$$\forall X \in L^\infty: \varrho(X) = \sup_{m \in \mathcal{M}_1((0, 1])} \left\{ \int_0^1 \text{AV@R}_\alpha(X) dm(\alpha) - \pi(m) \right\},$$

where $\mathcal{M}_1((0, 1])$ is the set of (Borel) probability measures on $(0, 1]$.

The characterization in Theorem 6 is due to Kusuoka [37] for the sublinear case (in terms of integrated quantile functions) and due to Jouini, Schachermayer and Touzi [36] in the general convex case. It shows the importance of the Average-Value-at-Risk.
4 Constructing risk measures

Translative envelopes. Let \( \phi : L^1 \to \mathbb{R} \cup \{ +\infty \} \) be a monotone function. Then, the function \( \varrho_\phi : L^1 \to \mathbb{R} \cup \{ +\infty \} \) defined by

\[
\varrho_\phi(X) = \inf \{ \phi(X_1) + \tau(X_2) \mid X_1 + X_2 = X \} = \inf \{ \phi(X - rI) - r \mid r \in \mathbb{R} \}
\]

is a risk measure whenever \( \varrho_\phi(0) \in \mathbb{R} \). Note that \( \varrho_\phi \) is nothing else than the infimal convolution of the two functions \( \phi \) and \( \tau \) ([55, Theorem 2.1.3 (ix)]) Moreover, it can be shown that \( \varrho_\phi \) is the pointwise greatest cash-additive function which is pointwise not greater than \( \phi \), thus it may be called the (lower) cash-additive envelope of \( \phi \). This construction has been introduced in [17] in a different context, and for risk measures in [23]. Moreover, the so-called “optimized certainty equivalent” introduced in [5, 6] has the same form in which \( \phi(X) = E[\ell(X)] \) for a monotone (non-increasing) function \( \ell : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \). As shown above, every risk measure is the cash-additive envelope of the indicator function of its “induced” acceptance set: \( I_{A_\phi} \) is monotone since \( A_\phi \) is.

Risk measures associated with loss/utility functions. Let \( \ell : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) be a proper, increasing and not identically constant function and \( r_0 \in \text{int} \ell(\mathbb{R}) \). Define the set \( A_\ell = \{ X \in L^1 \mid E[\ell(-X)] \leq r_0 \} \). The risk measure \( \varrho_\ell \) defined by

\[
\varrho_\ell(X) = \varrho_{A_\ell}(X) = \inf \{ s \in \mathbb{R} \mid E[\ell(-X - sI)] \leq r_0 \}
\]

is called loss-based shortfall risk measure. It is convex if \( \ell \) is convex. If \( \ell \) is real-valued and \( \varrho_\ell \) is considered as a function on \( L^\infty \), then it is weakly closed with dual representation

\[
\forall X \in L^\infty : \varrho_\ell(X) = \max_{Q \in \mathcal{M}_1(P)} \left[ E^Q[-X] - \frac{1}{\lambda > 0} \left( r_0 + E \left[ \ell^* \left( \frac{dQ}{dP} \right) \right] \right) \right]
\]

where \( \ell^* \) is the Fenchel conjugate of \( \ell : \mathbb{R} \to \mathbb{R} \). Shortfall risk measures are law invariant and in some sense dual to divergence risk measures (discussed in [27, Section 4.9], the latter have a primal representation depending on \( \ell^* \)) which in turn also coincide with the “optimized certainty equivalent” introduced by Ben-Tal and Teboulle [5, 6].

Spectral risk measures. The crucial observation is that a convex combination of two risk measures again is a risk measure, and this can even be generalized to mixtures via probability measures on \([0, 1]\), see [1, Proposition 2.2]. Acerbi [1] introduced the following concept. Let \( \phi : [0, 1] \to \mathbb{R} \) be a function satisfying (a) \( \phi(\alpha) \geq 0 \) for all \( \alpha \in [0, 1] \), (b) \( \int_0^1 \phi(\alpha)d\alpha = 1 \), (c) \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \) implies \( \phi(\alpha_1) \geq \phi(\alpha_2) \). Then, the function \( \varrho_\phi : L^\infty \to \mathbb{R} \cup \{ +\infty \} \) defined by

\[
\varrho_\phi(X) = -\int_0^1 \phi(s)q_X(s)ds
\]

is a coherent, law invariant risk measure, and the function \( \phi \) is called a risk spectrum which can chosen by the decision maker. Here, \( q_X(\alpha) = \inf \{ t \in \mathbb{R} \mid F_X(t) \geq \alpha \} \) is the lower
$\alpha$-quantile of $X$. $V@R$ and $AV@R$ turn out to be special spectral risk measures. Compare [12] for further properties, dual representation results and relationships to stochastic dominance orders. Note that already the results of Kusuoka [37, Theorem 7] imply that the class of spectral risk measures on $L^\infty$ over an atomless probability space coincides with the class of all weakly closed, coherent, law invariant and comonotonic risk measures (compare Remark 4.4 in [1]).

5 Relationships to other concepts in risk evaluation

Stochastic dominance orders. Stochastic dominance orders for probability distributions are important tools for risk evaluation. Therefore, a crucial property of a risk measure is monotonicity with respect to these orders. The Average Value at Risk does even characterize the second order stochastic dominance $\preceq_{SSD}$: If $X, Y \in L^1$, then

$$X \preceq_{SSD} Y \iff \forall \alpha \in (0, 1]: AV@R_\alpha(X) \geq AV@R_\alpha(Y).$$

This observation goes back to [42], see also [27, Remark 4.49]. In a similar way, the Value-at-Risk characterize first order stochastic dominance.

Other translative functions. Remarkably, many other functions share property (1). In particular, the sub- and superhedging price of a financial position in an incomplete market are versions of a cash-additive function [27, Section 1.3] and also the so-called good deal bounds [34]. Outside finance, Dempster’s belief functions [14, formula (3.9), p. 363], Choquet integrals [15], imprecise lower/upper expectations [52], insurance premiums as discussed, e.g., in [53], exact functionals and games [39], [40] as well as maxmin expected utility functions [29], among many others, share property (1).

Extensions. (a) The famous Markowitz model for portfolio selection [38] involves the variance as a risk evaluating tool - which is neither monotone, nor cash-additive. On the contrary, it is constant on the linear subspace of $L^2$ formed by the constant functions. This property is shared by deviation measures introduced by Rockafellar, Uryasev and Zabarankin [47], [48] which are basically the difference of a risk measure and the expected value. They may replace the variance in procedures like regression analysis [49] or portfolio selection [48]. See [46] for an overview. (b) Since a cash-additive risk measure is quasiconvex if, and only if, it is convex, weaker versions of (1) were introduced, see [19] and [9]. In [8], [18], a concise motivation, further results on quasiconvex risk measures (called performance or assessment indices) and many examples can be found. (c) Under market conditions, one may want to make available a dynamic risk assessment procedure. The main issue is time-consistency, i.e., a position which is acceptable at some point in time should already be acceptable at earlier times. Extensions of the above concepts to the dynamic case were initiated in [16], [10], [11], [43]. More recently, the $L^0$-module framework was developed mainly motivated by time-dependent, conditional risk measures, see [24] for an overview and references. (d) In markets with transaction costs and illiquidity, the risk of multi-variate positions needs to be evaluated (see also 3.1.5.7). Several
approaches have been pursued: scalar risk measures for multivariate payoffs [7], [20], for example, and vector- and set-valued risk measures [35], [30], [31], [32]. (e) Condition (1) requires the existence of a “non-defaultable” (discountable) numéraire which serves as reference instrument. In the light of recent financial and economic crises, this assumption is questionable. Even more reasons for leaving the framework of “constant numéraires” and alternatives can be found in [21], [22].

References

[1] Acerbi, C. (2002). Spectral measures of risk: a coherent representation of subjective risk aversion. Journal of Banking & Finance 26(7): 1505-1518.

[2] Acerbi, C., & Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. Quantitative Finance, 8(7): 681-692.

[3] Aliprantis, C., & Border, K. (2006). Infinite Dimensional Analysis. Springer Publishers, 3rd edition.

[4] Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. Mathematical Finance, 9(3), 203-228.

[5] Ben-Tal, A., & Teboulle, M. (1986). Expected utility, penalty functions, and duality in stochastic nonlinear programming. Management Science, 32(11), 1445-1466.

[6] Ben-Tal, A., & Teboulle, M. (2007). An old-new concept of convex risk measures: the optimized certainty equivalent. Mathematical Finance, 17(3), 449-476.

[7] Burgert, C., & Rüschendorf, L. (2006). Consistent risk measures for portfolio vectors. Insurance: Mathematics and Economics, 38(2), 289-297.

[8] Cherny, A., & Madan, D. (2009). New measures for performance evaluation. Review of Financial Studies, 22(7), 2571-2606.

[9] Cerreia–Vioglio, S., Maccheroni, F., Marinacci, M., & Montrucchio, L. (2011). Risk measures: rationality and diversification. Mathematical Finance, 21(4), 743-774.

[10] Cheridito, P., Delbaen, F., & Kupper, M. (2004). Coherent and convex monetary risk measures for bounded cadlag processes. Stochastic Processes and their Applications, 112(1), 1-22.

[11] Cheridito, P., Delbaen, F., & Kupper, M. (2005). Coherent and convex monetary risk measures for unbounded cadlag processes. Finance and Stochastics, 9(3), 369-387.

[12] Cheridito, P., & Li, T. (2008). Dual characterization of properties of risk measures on Orlicz hearts. Mathematics and Financial Economics, 2(1), 29-55.
[13] Delbaen, F. (2002). Coherent risk measures on general probability spaces. In: Advances in Finance and Stochastics (pp. 1-37). Springer Publishers.

[14] Dempster, A. P. (1966). New methods for reasoning towards posterior distributions based on sample data. The Annals of Mathematical Statistics, 355-374.

[15] Denneberg, D. (1994). Non-additive measure and integral. Kluwer Academic Publishers Dordrecht.

[16] Detlefsen, K., & Scandolo, G. (2005). Conditional and dynamic convex risk measures. Finance and Stochastics, 9(4), 539-561.

[17] Dolecki, S., & Greco, G. H. (1995). Niveloids. Topological Methods in Nonlinear Analysis, 5(1), 1-22.

[18] Drapeau, S., & Kupper, M. (2013). Risk preferences and their robust representation. Mathematics of Operations Research, 38(1), 28-62.

[19] El Karoui, N., & Ravanelli, C. (2009). Cash subadditive risk measures and interest rate ambiguity. Mathematical Finance, 19(4), 561-590.

[20] Ekeland, I., & Schachermayer, W. (2011). Law invariant risk measures on $L^\infty(\mathbb{R}^d)$. Statistics & Risk Modeling with Applications in Finance and Insurance, 28(3), 195-225.

[21] Farkas, W., Koch-Medina, P. & Munari, C. (2014). Capital requirements with defaultable securities. Insurance: Mathematics and Economics 55: 58-67.

[22] Farkas, W., Koch-Medina, P. & Munari, C. (2014). Beyond cash-additive risk measures: when changing the numraire fails. Finance & Stochastics 18(1): 145-173.

[23] Filipović, D., & Kupper, M. (2007). Monotone and cash-invariant convex functions and hulls. Insurance: Mathematics and Economics, 41(1), 1-16.

[24] Filipović, D., Kupper, M., & Vogelpoth, N. (2012). Approaches to conditional risk. SIAM Journal on Financial Mathematics, 3(1), 402-432.

[25] Föllmer, H., & Schied, A. (2002). Convex measures of risk and trading constraints. Finance and Stochastics, 6(4), 429-447.

[26] Föllmer, H., & Schied, A. (2002). Robust preferences and convex measures of risk. In Advances in Finance and Stochastics (pp. 39-56). Springer Publishers.

[27] Föllmer, H., & Schied, A. (2011). Stochastic Finance: An Introduction in Discrete Time. Walter de Gruyter, 3rd edition.
[28] Frittelli, M., & Rosazza Gianin, E. (2002). Putting order in risk measures. Journal of Banking & Finance, 26(7), 1473-1486.

[29] Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. J. Mathematical Economics, 18(2), 141-153.

[30] Hamel, A. H., & Heyde, F. (2010). Duality for set-valued measures of risk. SIAM J. Financial Mathematics, 1(1):66-95.

[31] Hamel, A. H., Heyde, F., & Rudloff, B. (2011). Set-valued risk measures for conical market models. Mathematics and Financial Economics, 5(1):1-28.

[32] Hamel, A. H., & Kostner, D. (2018). Cone distribution functions and quantiles for multivariate random variables, J. Multivariate Analysis 167, 97-113.

[33] Heath, D. & Ku, H. (2004). Pareto equilibria with coherent measures of risk. Math. Finance, 14(2):163-172.

[34] Jaschke, S., & Küchler, U. (2001). Coherent risk measures and good-deal bounds. Finance and Stochastics, 5(2), 181-200.

[35] Jouini, E., Meddeb, M., & Touzi, N. (2004). Vector-valued coherent risk measures. Finance and Stochastics, 8(4), 531-552.

[36] Jouini, E., Schachermayer, W., & Touzi, N. (2006). Law invariant risk measures have the Fatou property. In Advances in Mathematical Economics (pp. 49-71). Springer Publishers.

[37] Kusuoka, S. (2001). On law invariant coherent risk measures. In: Advances in Mathematical Economics (pp. 83-95). Springer Publishers.

[38] Markowitz, H. (1952). Portfolio selection. The Journal of Finance, 7(1), 77-91.

[39] Maaß, S. (2001). Coherent Lower Previsions as Exact Functionals and their (Sigma-)Core. In ISIPTA, Vol. 1, pp. 230-236.

[40] Maaß, S. (2002). Exact functionals and their core. Statistical Papers, 43(1), 75-93.

[41] Ogryczak, W., & Ruszczynski, A. (1999). From stochastic dominance to mean-risk models: Semideviations as risk measures. European J. Operational Research, 116(1), 33-50.

[42] Ogryczak, W., & Ruszczynski, A. (2002). Dual stochastic dominance and quantile risk measures. International Transactions in Operational Research, 9(5), 661-680.

[43] Riedel, F. (2004). Dynamic coherent risk measures. Stochastic processes and their applications, 112(2), 185-200.
[44] Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. Journal of Risk, 2, 21-42.

[45] Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. Journal of Banking & Finance, 26(7), 1443-1471.

[46] Rockafellar, R. T., & Uryasev, S. (2013). The fundamental risk quadrangle in risk management, optimization and statistical estimation. Surveys in Operations Research and Management Science, 18(1), 33-53.

[47] Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2006). Generalized deviations in risk analysis. Finance and Stochastics, 10(1), 51-74.

[48] Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2006). Optimality conditions in portfolio analysis with general deviation measures. Mathematical Programming, 108(2-3), 515-540.

[49] Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2008). Risk tuning with generalized linear regression. Mathematics of Operations Research, 33(3), 712-729.

[50] Roorda, B., Schumacher, J. M., & Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. Mathematical Finance, 15(4), 589-612.

[51] Ruszczynski, A., & Shapiro, A. (2006). Optimization of convex risk functions. Mathematics of Operations Research, 31(3), 433-452.

[52] Walley, P. (1991). Statistical Reasoning with Imprecise Probabilities. London: Chapman and Hall.

[53] Wang, S. S., Young, V. R., & Panjer, H. H. (1997). Axiomatic characterization of insurance prices. Insurance: Mathematics and Economics, 21(2), 173-183.

[54] Yaari, M. E. (1987). The dual theory of choice under risk. Econometrica, 95-115.

[55] Zălinescu, C. (2002). Convex Analysis in General Vector Spaces. World Scientific Singapore.