THE NORM ESTIMATE OF THE DIFFERENCE BETWEEN
THE KAC OPERATOR AND SCHRÖDINGER SEMIGROUP II:
THE GENERAL CASE INCLUDING THE RELATIVISTIC CASE

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Abstract More thorough results than in our previous paper in Nagoya Math. J. are given
on the $L_p$-operator norm estimates for the Kac operator $e^{-tV/2}e^{-tH_0}e^{-tV/2}$ compared with the
Schrödinger semigroup $e^{-t(H_0+V)}$. The Schrödinger operators $H_0 + V$ to be treated in this paper
are more general ones associated with the Lévy process, including the relativistic Schrödinger
operator. The method of proof is probabilistic based on the Feynman-Kac formula. It differs
from our previous work in the point of using the Feynman-Kac formula not directly for these
operators, but instead through subordination from the Brownian motion, which enables us to
deal with all these operators in a unified way. As an application of such estimates the Trotter
product formula in the $L_p$-operator norm, with error bounds, for these Schrödinger semigroups
is also derived.

Keywords Schrödinger operator, Schrödinger semigroup, relativistic Schrödinger operator,
Trotter product formula, Lie-Trotter-Kato product formula, Feynman-Kac formula, subordi-
nation of Brownian motion, Kato’s inequality

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1. Introduction

By the Kac operator we mean an operator of the kind $K(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2}$, where $H = H_0 + V \equiv -\Delta/2 + V(x)$ is the nonrelativistic Schrödinger operator in $L_2(\mathbb{R}^d)$ with mass 1 with scalar potential $V(x)$ bounded from below. This $K(t)$ may correspond to the transfer operator for a lattice model in statistical mechanics studied by M. Kac [Ka]. There it is one of the important problems to know asymptotic spectral properties of $K(t)$ for $t \downarrow 0$. To this end, in [H1, H2] Helffer estimated the $L_2$-operator norm of the difference between $K(t)$ and the Schrödinger semigroup $e^{-tH}$ to be of order $O(t^2)$ for small $t > 0$, if $V(x)$ satisfies $|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|^2)^{(2-|\alpha|)+}/2$ for every multi-index $\alpha$ with a constant $C_\alpha$. Then such norm estimates may be applied to get spectral properties of $K(t)$ in comparison with those of $H$.

In [I-Tak1] and [I-Tak2] we have extended his result to the case of more general scalar potentials $V(x)$ even in the $L_p$-operator norm, $1 \leq p \leq \infty$, making a probabilistic approach based on the Feynman-Kac formula. In [I-Tak2] we have also considered this problem for both the nonrelativistic Schrödinger operator $H = H_0 + V$ and the relativistic Schrödinger operator $H_r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V(x)$ with light velocity 1. The $L_p$-operator norm of this difference is estimated to be of order $O(t^a)$ of small $t > 0$ with $a \geq 1$, though the relativistic case shows for small $t > 0$ a slightly different behavior from the nonrelativistic case. As another application of these results the Trotter product formula for the nonrelativistic and relativistic Schrödinger operators in the $L_p$-operator norm with error bounds is obtained. There are also related $L_2$ results with operator-theoretic methods, for which we refer to [D-I-Tam].

The aim of this paper is to generalize and refine the result of [I-Tak2] in the relativistic case, admitting of more general operators than the free relativistic Schrödinger operator $H_0^r = \sqrt{-\Delta + 1} - 1$ as well as relaxing the conditions for the potentials $V(x)$. We use the probabilistic method with Feynman-Kac formula, though observing everything in a unified way through subordination from the Brownian motion. In this respect the present method differs from that in [I-Tak2] used for the relativistic Schrödinger operator $H^r$, which made the best of the explicit expression of the integral kernel of $e^{-tH_0^r}$.

The more general operator we have in mind is the following operator

$$H_0^\psi = \psi\left(\frac{1}{2}(-\Delta + 1)\right) - \psi\left(\frac{1}{2}\right),$$

which will play the same role as the relativistic Schrödinger operator

$$H_0^r = \sqrt{-\Delta + 1} - 1$$

in [I-Tak2]. Obviously, $H_0^\psi$ is a selfadjoint operator in $L_2(\mathbb{R}^d)$. Here $\psi(\lambda)$ is a continuous increasing function on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(\infty) = \infty$ expressed as

$$\psi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda t})n(dl), \quad \lambda \geq 0,$$

where $n(dl)$ is a Lévy measure on $(0, \infty)$ (i.e. a measure on $(0, \infty)$ such that $\int_{(0,\infty)} l \wedge 1 n(dl) < \infty$) with $n((0,\infty)) = \infty$. It is clear that

$$\psi\left(\lambda + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) = \int_{(0,\infty)} (1 - e^{-\lambda t})e^{-t/2}n(dl).$$

2
As a special case of $H_0^\psi$ we have for $\psi(\lambda) = (2\lambda)^{\alpha}$, $0 < \alpha < 1$, the operator

$$H_0^{(\alpha)} = (-\Delta + 1)^{\alpha} - 1,$$  \hspace{1cm} (1.5)

which reduces to the relativistic Schrödinger operator when $\alpha = 1/2$: $H_0^{(1/2)} = H_0^\psi$. In this case the Lévy measure is $n(dl) = \{2^{\alpha\gamma}/\Gamma(1 - \alpha)\}l^{-1-\alpha} dl$.

To formulate our results we are going to describe what kind of function $V(x)$ is. Let $0 < \gamma, \delta \leq 1$, $0 \leq \kappa \leq 1$, $0 \leq \mu, \nu, \rho < \infty$, $0 \leq C_1, C_2, c_1, c_2 < \infty$ and $0 < c < \infty$. Let $V : \mathbb{R}^d \rightarrow [0, \infty)$ be a continuous function satisfying one of the following five conditions:

\begin{itemize}
  \item[(A)_0] $|V(x) - V(y)| \leq C_1|x - y|\gamma$;
  \item[(A)_1] $V$ is a $C^1$-function such that
    \begin{itemize}
      \item[(i)] $|\nabla V(z)| \leq C_1(1 + V(z)^{1-\delta})$, \hspace{0.5cm} (ii) $|\nabla V(x) - \nabla V(y)| \leq C_2|x - y|\kappa$;
    \end{itemize}
  \item[(A)_2] $V$ is a $C^1$-function such that
    \begin{itemize}
      \item[(i)] $|\nabla V(z)| \leq C_1(1 + V(z)^{1-\delta})$,
      \item[(ii)] $|\nabla V(x) - \nabla V(y)| \leq C_2\{V(x)(1-2\delta)^+ (1 + |x - y|^\rho) + 1 + |x - y|^\mu\}|x - y|$;
    \end{itemize}
  \item[(V)_1] $V$ is a $C^1$-function such that
    \begin{itemize}
      \item[(i)] $V(z) \geq c(z)^\rho$, \hspace{0.5cm} (ii) $|\nabla V(z)| \leq c_1(z)^{(\rho - 1)+}$;
    \end{itemize}
  \item[(V)_2] $V$ is a $C^2$-function such that
    \begin{itemize}
      \item[(i)] $V(z) \geq c(z)^\rho$, \hspace{0.5cm} (ii) $|\nabla V(z)| \leq c_1(z)^{(\rho - 1)+}$,
      \item[(iii)] $|\nabla^2 V(z)| \leq c_2(z)^{(\rho - 2)+}$.
    \end{itemize}
\end{itemize}

Here $\langle z \rangle := \sqrt{1 + |z|^2}$.

Conditions (A)_0, (A)_1 and (A)_2 on $V(x)$ are used in [Tak] and are more general than in [I-Tak1,2], while conditions (V)_1 and (V)_2 are used in [D-I-Tam]. But these conditions may not be best possible. A simple example of a function which has property (A)_0, (A)_1 or (A)_2 is, needless to say, $V(x) = |x|^r$ ($0 < r < \infty$), and a slightly complicated one $V(x) = |x|^r(2 + \sin \log |x|)$, according as $0 < r \leq 1$, $1 < r < 2$ or $r \geq 2$. Also $V(x) = 1 + |x_1 - x_2|^r$ ($x = (x_1, x_2, \ldots, x_d)$) satisfies (A)_0, (A)_1 or (A)_2 with the same $r$ as above, but neither (V)_1 nor (V)_2. To the contrary $V(x) = 1 + |x|^r \int_0^1 (1 + \sin(\theta^2)) d\theta$ satisfies (V)_1, but neither (V)_2, (A)_0, (A)_1 nor (A)_2.

The operator $H_0^\psi + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, and so its unique selfadjoint extension is also denoted by the same $H_0^\psi + V$. The semigroup $e^{-t(H_0^\psi + V)}$ on $L_2(\mathbb{R}^d)$ is extended to a strongly continuous semigroup on $L_p(\mathbb{R}^d)$ ($1 \leq p < \infty$) and $C_\infty(\mathbb{R}^d)$, to be denoted by the same $e^{-t(H_0^\psi + V)}$. Here $C_\infty(\mathbb{R}^d)$ is the Banach space of the continuous functions on $\mathbb{R}^d$ vanishing at infinity. To be complete, these and further facts are proved in Appendix.
As for the Lévy measure $n(dl)$ introduced in (1.3) and (1.4), we make the following assumption:

(L) For some $\alpha \in [0, 1]$, $n((\cdot, \infty))$ is regularly varying at zero with exponent $-\alpha$, i.e., there exists a slowly varying function $L(\lambda)$ at infinity such that

$$n((t, \infty)) \sim t^{-\alpha} L(\frac{1}{t}) \quad \text{as } t \downarrow 0.$$ 

(1.6)

Here a positive function $L(\cdot)$ is called slowly varying at infinity if for any $c > 0$,

$$\lim_{\lambda \to \infty} \frac{L(c\lambda)}{L(\lambda)} = 1.$$

Let $\phi^{-1}(\cdot)$ be the inverse function of $\phi(\lambda) := \psi(\lambda + 1/2) - \psi(1/2)$. (Note that $\phi$ is strictly increasing.) Under the above assumption, set

$$L_1(\lambda) := \begin{cases} \Gamma(1 - \alpha) L(\lambda) & \text{if } 0 \leq \alpha < 1 \\ \int_0^{1/\lambda} n((s, \infty))ds & \text{if } \alpha = 1, \end{cases}$$

$$L_2(x) := L_1(\phi^{-1}(x))^{-1/\alpha} \quad \text{if } 0 < \alpha \leq 1.$$

These two functions are slowly varying at infinity, and we have $\phi(\lambda) \sim \lambda^\alpha L_1(\lambda)$ as $\lambda \to \infty$ and $\phi^{-1}(x) \sim x^{1/\alpha} L_2(x)$ as $x \to \infty$, as will be seen from Fact in Section 6, so that $\int_0^1 (\phi^{-1}(\theta))^{-\alpha} d\theta$ ($0 < \alpha < 1$) is also slowly varying at infinity.

Now we state the main results of this paper, which generalize the results in [I-Tak2]. In the following $\| \cdot \|_{p \to p}$ stands for the $L_p$-operator norm for $1 \leq p < \infty$ and the supremum norm on $C_\infty(\mathbb{R}^d)$ for $p = \infty$.

**Theorem 1.** Suppose assumption (L) and let $1 \leq p \leq \infty$. Then the following estimates (i), (ii) and (iii) hold for small $t > 0$.

(i) Under $\text{(A)}_0$,

$$\| e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)} \|_{p \to p},$$

$$\| e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)} \|_{p \to p},$$

$$\| e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)} \|_{p \to p}$$

$$= \begin{cases} O(t^2) & \text{if } \alpha < \gamma/2 \\ O(t^2 \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = \gamma/2 \\ O(t^{1+\gamma/2\alpha} L_2(\frac{1}{t})^{-\gamma/2}) & \text{if } \gamma/2 < \alpha. \end{cases}$$

(ii) Under $\text{(A)}_1$,

$$\| e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)} \|_{p \to p}$$

4
Under a consequence of Theorem 1 is the following Trotter product formula in the $L_p$-operator norm with error bounds.\[\|e^{-tV} e^{-tH_0^{\phi}} - e^{-t(H_0^{\phi} + V)}\|_{p \to p},\]
\[\|e^{-tH_0^{\phi}/2} e^{-tV} e^{-tH_0^{\phi}/2} - e^{-t(H_0^{\phi} + V)}\|_{p \to p} = O(t^{1+1/2\delta}) \]
\[\|e^{-tV/2} e^{-tH_0^{\phi}} e^{-tV/2} - e^{-t(H_0^{\phi} + V)}\|_{p \to p} = O(t^{1+1/2\delta}).\]

(iii) Under (A)$_2$,
\[\|e^{-tV} e^{-tH_0^{\phi}} - e^{-t(H_0^{\phi} + V)}\|_{p \to p},\]
\[\|e^{-tV} e^{-tH_0^{\phi}} - e^{-t(H_0^{\phi} + V)}\|_{p \to p},\]
\[\|e^{-tH_0^{\phi}/2} e^{-tV} e^{-tH_0^{\phi}/2} - e^{-t(H_0^{\phi} + V)}\|_{p \to p}
= \left\{ \begin{array}{ll}
O(t^{1+\delta}) & \text{if } \alpha < 1/2 \\
O(t^{1+\delta}) & \text{if } \alpha = 1/2 \\
O(t^{1+1/2\delta} L_2(1/2)^{-1/2}) & \text{if } 1/2 < \alpha.
\end{array} \right.
\]

In fact, the first estimate in (iii) holds independent of (L).

A consequence of Theorem 1 is the following Trotter product formula in the $L_p$-operator norm with error bounds.

**Theorem 2.** Suppose assumption (L) and let $1 \leq p \leq \infty$. Then the following estimates (i), (ii), (iii) and (iv) hold uniformly on each finite $t$-interval on $[0, \infty)$.

(i) Under (A)$_0$,
\[\|e^{-tV/2n} e^{-tH_0^{\phi}/n} e^{-tV/2n} - e^{-t(H_0^{\phi} + V)}\|_{p \to p} = O(n^{-1}),\]
\[\|e^{-tV/n} e^{-tH_0^{\phi}/n} - e^{-t(H_0^{\phi} + V)}\|_{p \to p},\]
\[\|e^{-tH_0^{\phi}/2n} e^{-tV/n} e^{-tH_0^{\phi}/2n} - e^{-t(H_0^{\phi} + V)}\|_{p \to p}
= \left\{ \begin{array}{ll}
O(n^{-1}) & \text{if } \alpha < \gamma/2 \\
O(n^{-1} \int_0^n (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = \gamma/2 \\
O(n^{-\gamma/2\alpha} L_2(n)^{-\gamma/2}) & \text{if } \gamma/2 < \alpha.
\end{array} \right.
\]
(ii) Under (A)₁,
\[
\begin{align*}
&\|(e^{-tV/2}n e^{-tH_0^\psi / n} e^{-tV/2})^n - e^{-t(H_0^\psi + V)}\|_{p-p}, \\
&\|(e^{-tV/n} e^{-tH_0^\psi / n} )^n - e^{-t(H_0^\psi + V)}\|_{p-p},
\end{align*}
\]
\[
\begin{align*}
&\|(e^{-tH_0^\psi / 2n} e^{-tV/n} e^{-tH_0^\psi / 2n})^n - e^{-t(H_0^\psi + V)}\|_{p-p} \\
&= \begin{cases} 
O(n^{-1/2\delta}) & \text{if } \alpha < (1 + \kappa)/2 \text{ or } \kappa = 1 \\
O(n^{-2\delta}) + O(n^{-1} \int_0^n (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = (1 + \kappa)/2 < 1 \\
O(n^{-2\delta}) + O(n^{-(1+\kappa)/2} L_2(n)^{(1+\kappa)/2}) & \text{if } (1 + \kappa)/2 < \alpha.
\end{cases}
\end{align*}
\]

(iii) Under (A)₂,
\[
\begin{align*}
&\|(e^{-tV/2}e^{-tH_0^\psi / n} e^{-tV/2})^n - e^{-t(H_0^\psi + V)}\|_{p-p}, \\
&\|(e^{-tV/n} e^{-tH_0^\psi / n} )^n - e^{-t(H_0^\psi + V)}\|_{p-p},
\end{align*}
\]
\[
\begin{align*}
&\|(e^{-tH_0^\psi / 2n} e^{-tV/n} e^{-tH_0^\psi / 2n})^n - e^{-t(H_0^\psi + V)}\|_{p-p} \\
&= O(n^{-1/2\delta}),
\end{align*}
\]

(iv) Under (V)ₐ (i = 1, 2),
\[
\begin{align*}
&\|(e^{-tV/2}e^{-tH_0^\psi / n} e^{-tV/2})^n - e^{-t(H_0^\psi + V)}\|_{p-p}, \\
&\|(e^{-tV/n} e^{-tH_0^\psi / n} )^n - e^{-t(H_0^\psi + V)}\|_{p-p},
\end{align*}
\]
\[
\begin{align*}
&\|(e^{-tH_0^\psi / 2n} e^{-tV/n} e^{-tH_0^\psi / 2n})^n - e^{-t(H_0^\psi + V)}\|_{p-p} \\
&= O(n^{-1/2\delta}).
\end{align*}
\]

In fact, the asymptotic estimates (iii) and (iv) hold independent of (L).

Notice here that though the estimates with small \(t\), in Theorem 1, for \(e^{-tV} e^{-tH_0^\psi}\) and \(e^{-tH_0^\psi / 2} e^{-tV} e^{-tH_0^\psi / 2}\) are of worse order than that for \(e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2}\), one has, in Theorem 2, the same error bounds with large \(n\) for these three products.

Finally we give a comment on what kind of operators are to be covered by our \(H_0^\psi + V\). To this end we briefly illustrate how our result reads on the Trotter product formula in the case \(H_0^\psi + V\) with \(H_0^\psi = (-\Delta + 1)^\alpha - 1, 0 < \alpha < 1\), in (1.5). In this case, we have \(n(t, \infty) = (2^\alpha / \Gamma(1-\alpha)) t^{-\alpha}\), or \(L_2(\cdot) \equiv 2^{-1}\), so that
\[
\int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta \sim 2^\alpha \log x \quad \text{as } x \to \infty.
\]

Therefore Theorem 2 says that for \(1 \leq p \leq \infty\) and uniformly on each finite \(t\)-interval in \([0, \infty)\),
\[
\begin{align*}
&\|(e^{-tV/2}e^{-tH_0^\psi / n} e^{-tV/2})^n - e^{-t(H_0^\psi + V)}\|_{p-p}, \\
&\|(e^{-tV/n} e^{-tH_0^\psi / n} )^n - e^{-t(H_0^\psi + V)}\|_{p-p},
\end{align*}
\]
\[
\begin{align*}
&\|(e^{-tH_0^\psi / 2n} e^{-tV/n} e^{-tH_0^\psi / 2n})^n - e^{-t(H_0^\psi + V)}\|_{p-p}
\end{align*}
\]
\[
\begin{aligned}
&\begin{cases}
O(n^{-1}) & \text{if } \alpha < \gamma/2 \\
O(n^{-1} \log n) & \text{if } \alpha = \gamma/2 \quad \text{under (A)0,}
\end{cases} \\
&\begin{cases}
O(n^{-\gamma/2\alpha}) & \text{if } \gamma/2 < \alpha \\
O(n^{-1^\delta}) & \text{if } \alpha < (1 + \kappa)/2 \\
O(n^{-1} \log n) & \text{if } \alpha = (1 + \kappa)/2 \text{ and } 1/2 \leq \delta \leq 1 \\
O(n^{-2\delta}) & \text{if } \alpha = (1 + \kappa)/2 \text{ and } 0 < \delta < 1/2 \\
O(n^{-2^{\delta}} (1 + \kappa)/2\alpha) & \text{if } (1 + \kappa)/2 < \alpha
\end{cases}
\end{aligned}
\]

An important remark is the following. In the above example, the case \(\alpha = 1\) is missing. This is equivalent to the nonrelativistic case \(H_0 + V = -\Delta/2 + V(x)\), treated in [Tak] (cf. [I-Tak1,2]). However we may think that this case is also implicitly contained in our results, Theorems 1 and 2, for \(\alpha = 1/2\). Indeed, by using \(H_0^c(c) = \sqrt{-c^2 \Delta + c^4 - c^2}\) with light velocity \(c\) restored in place of \(H_0^c\) in (1.2), we can obtain the case \(\alpha = 1/2\) so as to involve the parameter \(c\) (light velocity).

Since, in the nonrelativistic limit \(c \to \infty\), the relativistic Schrödinger semigroup \(e^{-t(H_0^c + V)}\) is strongly convergent to the nonrelativistic Schrödinger semigroup \(e^{-t(H_0 + V)}\) uniformly on each finite \(t\)-interval in \([0, \infty)\) (e.g. [I2]), we can reproduce the nonrelativistic result in [Tak] (cf. Remark following Theorem 2.3).

In Section 2, we state our results in more general form: we generalize Theorems 1 and 2 to Theorems 2.1 and 2.2 /2.3 by introducing the subordinator \(\sigma_t\), namely, a time-homogeneous Lévy process associated with the Lévy measure \(e^{-l/2}n(dl)\). Moreover we state Theorem 2.4 on asymptotics of the moments of the process \(\sigma_t\). Once we know these asymptotics, we can obtain Theorems 1 and 2 from Theorems 2.1 and 2.2 /2.3. These four theorems are proved in Sections 3–6.

In Appendix, we give a full study of the semigroups \(e^{-t(H_0^c + V)}\), \(t \geq 0\), on \(L^p(\mathbb{R}^d), 1 \leq p < \infty\) and \(C_\infty(\mathbb{R}^d)\) defined through the Feynman-Kac formula. We show they constitute a strongly continuous contraction semigroup there. It is also shown that its infinitesimal generator \(\Phi_0^c, V\) has \(C_\infty(\mathbb{R}^d)\) as a core, by establishing Kato’s inequality for the operator \(H_0^c\). Some of these results seem to be new.

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2. General results

In this section we shall prove the theorems in a little more general setting based on probability theory. To describe it we introduce some notations and notions. For a continuous function \(V : \mathbb{R}^d \to [0, \infty)\), set
\[
K(t) := e^{-tV/2} e^{-tH_0^c} e^{-tV/2},
\]
\[
G(t) := e^{-tV} e^{-tH_0^c},
\]

7
\[ R(t) := e^{-tH_0^V/2}e^{-tV}e^{-tH_0^V/2} \]

and

\[
Q_K(t) := K(t) - e^{-t(H_0^V + V)}, \\
Q_G(t) := G(t) - e^{-t(H_0^V + V)}, \\
Q_R(t) := R(t) - e^{-t(H_0^V + V)}.
\]

Suppose we are given the independent random objects \( N(\cdot) \) and \( B(\cdot) \) on some probability space \((\Omega, \mathcal{F}, P)\):

(i) \( N(dsdl) \) is a Poisson random measure on \([0, \infty) \times (0, \infty)\) such that \( \mathbb{E}[N(dsdl)] = dse^{-l/2}n(dl)\);

(ii) \( (B(t))_{t \geq 0} \) is a \( d \)-dimensional Brownian motion starting at 0.

Set

\[
\sigma_t := \int_0^{t+} \int_{(0, \infty)} l N(dsdl).
\]

Then \( (\sigma_t)_{t \geq 0} \) is a time-homogeneous Lévy process with increasing paths such that

\[
\mathbb{E}[e^{-\lambda \sigma_t}] = e^{-t(\psi(\lambda+1/2)-\psi(1/2))}
\]

(e.g. Note 1.7.1 in [It-MK]). Note that \( \sigma_t \) has moments of all order (cf. (6.1)), which is to be seen at the beginning of Section 6. We use a subordination of \( B(\cdot) \) by a subordinator \( \sigma_\cdot \), i.e., a process \( (B(\sigma_t))_{t \geq 0} \) on \( \mathbb{R}^d \). This is a Lévy process such that

\[
\mathbb{E}[e^{\sqrt{t}p, B(\sigma_t)}] = e^{-t(\psi(|p|^2+1/2)-\psi(1/2))},
\]

which corresponds to the semigroup \( \{e^{-tH_0^V}\}_{t \geq 0} \) with generator \( H_0^V \) in (1.1).

We prove the following generalization of Theorems 1 and 2.

**Theorem 2.1.** Let \( 1 \leq p \leq \infty \) and \( t \geq 0 \).

(i) Under \( (A)_0 \),

\[
\|Q_K(t)\|_{p\to p}, \|Q_G(t)\|_{p\to p}, \|Q_R(t)\|_{p\to p} \leq \text{const}(\gamma, d) C_1 t \mathbb{E}[\sigma_t^{\gamma/2}].
\]

(ii) Under \( (A)_1 \),

\[
\|Q_K(t)\|_{p\to p} \leq \text{const}(\delta, \kappa, d) \left[ C_1^2 (t^2 + t^{2\delta}) \mathbb{E}[\sigma_t] + \sum_{j=1} C_2 t^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}] \right],
\]

\[
\|Q_G(t)\|_{p\to p}, \|Q_R(t)\|_{p\to p} \leq \text{const}(\delta, \kappa, d) \sum_{j=1} \left\{ C_1^j (t^j + t^{j\delta}) \mathbb{E}[\sigma_t^{j/2}] + C_2 t^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}] \right\}.
\]
(iii) Under (A)\(_2\),

\[
\|Q_K(t)\|_{p-p} \leq \text{const}(\delta, \mu, \nu, d) \left[ C_1^2 \left( t^2 + t^{2\delta} \right) \mathbb{E}[\sigma_t] + \sum_{j=1}^{2} \left\{ (C_2 t)^j \mathbb{E}[\sigma_t^j] \right. \right.
\]
\[
+ \left. (C_2 t)\mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right] + (C_2 t)\mathbb{E}[\sigma_t^{j(1+\nu/2)}] \right) \right] + (C_2 t)\mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right) \right] \right],
\]

\[
\|Q_C(t)\|_{p-p}, \|Q_R(t)\|_{p-p} \leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^{2} \left\{ C_1^j \left( t^j + t^{j\delta} \right) \mathbb{E}[\sigma_t^j] + (C_2 t)^j \mathbb{E}[\sigma_t^j] \right. \right.
\]
\[
+ \left. (C_2 t)\mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right] + (C_2 t)\mathbb{E}[\sigma_t^{j(1+\nu/2)}] \right) \right] + (C_2 t)\mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right) \right].
\]

Theorem 2.2. Let \(1 \leq p \leq \infty\), \(t \geq 0\) and \(n \in \mathbb{N}\).

(i) Under (A)\(_0\),

\[
\|e^{-tv/2n} e^{-tH_0^\gamma/n} e^{-tv/2n} \|_{p-p},
\]
\[
\|e^{-tv/n} e^{-tH_0^\gamma/n} \|_{p-p},
\]
\[
\|e^{-tH_0^\gamma/2n} e^{-tv/n} e^{-tH_0^\gamma/2n} \|_{p-p}
\]
\[
\leq \text{const}(\gamma, d) C_1 t \mathbb{E}[\sigma_t^{j/2}].
\]

(ii) Under (A)\(_1\),

\[
\|e^{-tv/2n} e^{-tH_0^\gamma/n} e^{-tv/2n} \|_{p-p}
\]
\[
\leq \text{const}(\delta, \kappa, d) \left[ C_1^2 \left( \left( \frac{\delta}{n} \right)^2 + \left( \frac{\delta}{n} \right)^{2\delta} \right) \mathbb{E}[\sigma_t/n] + \sum_{j=1}^{2} (C_2 \frac{\delta}{n})^j n \mathbb{E}[\sigma_t/n] \right]
\]
\[
\left. + \left. (C_2 \frac{\delta}{n})^j n \mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right) \right] + C_1 \left( \frac{\kappa}{n} + \left( \frac{\kappa}{n} \right)^\delta \right) \mathbb{E}[\sigma_t^{j/2}] + C_1^2 \left( \left( \frac{\delta}{n} \right)^2 + \left( \frac{\delta}{n} \right)^{2\delta} \right) \mathbb{E}[\sigma_t/n] + \sum_{j=1}^{2} \left( C_2 \frac{\delta}{n} \right)^j n \mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right].
\]

(iii) Under (A)\(_2\),

\[
\|e^{-tv/2n} e^{-tH_0^\gamma/n} e^{-tv/2n} \|_{p-p}
\]
\[
\leq \text{const}(\delta, \mu, \nu, d) \left[ C_1^2 \left( \left( \frac{\mu}{n} \right)^2 + \left( \frac{\mu}{n} \right)^{2\delta} \right) \mathbb{E}[\sigma_t/n] + \sum_{j=1}^{2} \left( C_2 \frac{\mu}{n} \right)^j n \mathbb{E}[\sigma_t/n] \right]
\]
\[
+ \left. (C_2 \frac{\mu}{n})^j n \mathbb{E}[\sigma_t^{(1+\nu/2)}] \right] + (C_2 \frac{\mu}{n})^{1+\delta} \mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right] \right] + (C_2 \frac{\mu}{n})^{1+\delta} \mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right].
\]
\begin{align*}
&\| (e^{-tV/n} e^{-tH_0^\delta/n}) e^{-tV/n} r - e^{-tV/n} r \|_{p,p}, \\
&\| (e^{-tH_0^\delta/2n} e^{-tV/2n} e^{-tH_0^\delta/2n}) e^{-tV/n} r - e^{-tH_0^\delta/2n} e^{-tV/n} r \|_{p,p} \\
&\leq \text{const}(\delta, \mu, \nu, d) \left[ \frac{1}{n} \big( C_1 (t + t^\delta) \mathbb{E}[\sigma_t^{1/2}] + C_2 t^{1\wedge \delta} (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right] \\
&\quad + C_2 t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) + C_1 \left( \frac{1}{n} + (\frac{1}{n})^j \right) \mathbb{E}[\sigma_t^{1/2}] + C_2^2 \left( \frac{1}{n}^2 + (\frac{1}{n})^{2\delta} \right) n \mathbb{E}[\sigma_t/n] \\
&\quad + 2 \sum_{j=1}^2 \left\{ (C_2 \frac{1}{n})^j n \mathbb{E}[\sigma_t^j] + (C_2 \frac{1}{n})^j n \mathbb{E}[\sigma_t^{j(1+\mu/2)}] + (C_2 \frac{1}{n})^{1\wedge 2\delta} n \mathbb{E}[\sigma_t^j] \\
&\quad + (C_2 \frac{1}{n})^{1\wedge 2\delta} n \mathbb{E}[\sigma_t^{j(1+\mu/2)}]) \right\}.
\end{align*}

**Theorem 2.3.** Let $1 \leq p \leq \infty$ and $t \geq 0$.

(i) Under $(V)_1$ for $n \geq 2^{2(2\nu/\rho)}$,

\begin{align*}
&\| (e^{-tV/2n} e^{-tH_0^\delta/2n}) e^{-tV/2n}) e^{-tV/2n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\leq \text{const}(\rho, c, c_1, d) n^{-1/2\nu} \left[ t^{2/(\rho \wedge 2) \wedge 1} - 1 + (t^2 + t^{2(1\wedge (\rho \wedge 2))}) n \mathbb{E}[\sigma_t/n] \\
&\quad + 2 \sum_{j=1}^2 \left( (t^j + t^{2j/2\nu}) n \mathbb{E}[\sigma_t^j] + t^j n \mathbb{E}[\sigma_t^{j(2\nu/\rho)/2}] \right) \right],
\end{align*}

\begin{align*}
&\| (e^{-tV/n} e^{-tH_0^\delta/n}) e^{-tV/n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\| (e^{-tH_0^\delta/2n} e^{-tV/2n} e^{-tH_0^\delta/2n}) e^{-tV/n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\leq \text{const}(\rho, c, c_1, d) n^{-1/2\nu} \left[ t^{2/(\rho \wedge 2) \wedge 1} - 1 + (t + t^{1 \wedge (\rho \wedge 2) \wedge 1/2\nu}) n \mathbb{E}[\sigma_t^{1/2}] \\
&\quad + (t^{2/\nu} n \mathbb{E}[\sigma_t] + (t^2 + t^{2(1\wedge (\rho \wedge 2))}) n \mathbb{E}[\sigma_t/n] \\
&\quad + 2 \sum_{j=1}^2 \left( (t^j + t^{2j/2\nu}) n \mathbb{E}[\sigma_t^j] + t^j n \mathbb{E}[\sigma_t^{j(2\nu/\rho)/2}] \right) \right].
\end{align*}

(ii) Under $(V)_2$ for $n \geq 1$,

\begin{align*}
&\| e^{-tV/2n} e^{-tH_0^\delta/2n} e^{-tV/2n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\leq \text{const}(\rho, c, c_1, c_2, d) n^{-2/2\nu} \left[ (t^2 + t^{2/1\nu}) n \mathbb{E}[\sigma_t/n] \\
&\quad + 2 \sum_{j=1}^2 \left( (t^j + t^{2j/2\nu}) n \mathbb{E}[\sigma_t^j] + t^j n \mathbb{E}[\sigma_t^{j(2\nu/\rho)/2}] \right) \right],
\end{align*}

\begin{align*}
&\| (e^{-tV/n} e^{-tH_0^\delta/n}) e^{-tV/n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\| (e^{-tH_0^\delta/2n} e^{-tV/2n} e^{-tH_0^\delta/2n}) e^{-tV/n} r - e^{-t(H_0^\delta+V)} r \|_{p,p} \\
&\leq \text{const}(\rho, c, c_1, c_2, d) \left[ n^{-2/2\nu} \left( (t + t^{1/1\nu}) n \mathbb{E}[\sigma_t^{1/2}] + (t + t^{2/2\nu}) n \mathbb{E}[\sigma_t] + t \mathbb{E}[\sigma_t^{2\nu/\rho}/2] \\
&\quad + (t^2 + t^{2/1\nu}) n \mathbb{E}[\sigma_t/n] + 2 \sum_{j=1}^2 \left( (t^j + t^{2j/2\nu}) n \mathbb{E}[\sigma_t^j] + t^j n \mathbb{E}[\sigma_t^{j(2\nu/\rho)/2}] \right) \right) \right].
\end{align*}
+ n^{-1/1\rho}E^\sigma_{1/2}(t + t^{1/1\rho}).

**Remark.** As noted at the end of Section 1, the nonrelativistic case for \( H_0 + V = \Delta/2 + V \), being equivalent to the case \( \alpha = 1 \) which Theorems 1 and 2 fail to cover, can be thought to be implicitly contained in the relativistic case, of the above three theorems, for the relativistic Schrödinger operator \( H^0(c) = \sqrt{-c^2\Delta + c^4 - c^2} \) with the light velocity \( c \geq 1 \) restored. We have \( H^0(c) = H^0(0) \), where this \( \psi(\lambda) \) is a \( c \)-dependent function (1.3) given by:

\[
\psi(\lambda; c) = \sqrt{2c^2 \lambda + c^4 - c^2 - \sqrt{c^4 - c^2}} \text{ associated with the } \lambda \text{-dependent Lévy measure } e^{-1/2} n(dl; c) = (2\pi)^{-1/2} ce^{-c^2 l^2 / 2l^3/2} dl.
\]

In this case, Theorem 2.1 and Theorems 2.2 / 2.3 hold with the corresponding \( c \)-dependent subordinator \( \sigma_t(c) \), just as they stand, namely, only with \( E[s^a] \) replaced by \( E[s^a(c)^a] \) for each respective \( s > 0 \) and \( a > 0 \). Then the nonrelativistic case in question is obtained as the nonrelativistic limit \( c \to \infty \) of this \( c \)-dependent relativistic case, turning out to be just Theorems 2.1 and 2.2 / 2.3 with \( E[s^a] \) replaced by \( s^a \). This is because one can show that, as \( c \to \infty \), the relativistic Schrödinger semigroup \( e^{-t(H^0(c) + V)} \) on the LHS converges strongly to the nonrelativistic Schrödinger semigroup \( e^{-t(H_0 + V)} \) uniformly on each finite \( t \)-interval in \( [0, \infty) \) (cf. [I2]), and \( E[\sigma_t(c)^a] \) on the RHS tends to \( t^a \). Then taking the most dominant contribution on the RHS for small \( t \) or large \( n \) reproduces the same nonrelativistic result as in [Tak].

Theorems 1 and 2 follow immediately from Theorems 2.1 and 2.2 / 2.3, if one knows the asymptotics for \( t \downarrow 0 \) of the moments of \( \sigma_t \) to investigate which of the terms on the RHS makes a dominant contribution for small \( t \) or large \( n \). These asymptotics are given by the following theorem.

**Theorem 2.4.** Suppose assumption (L). Let \( a > 0 \).

(i) If \( \alpha < a \) or \( a \geq 1 \), then \( \int_{(0, \infty)} t^a e^{-1/2} n(dl) < \infty \) and

\[
E[\sigma_t^a] \sim t \int_{(0, \infty)} t^a e^{-1/2} n(dl) \quad \text{as } t \downarrow 0.
\]

In fact, for \( a \geq 1 \) this always holds independent of (L).

(ii) If \( \alpha = a \) and \( a < 1 \), then

\[
E[\sigma_t^a] \sim \frac{1}{\Gamma(1 - \alpha)} t \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } t \downarrow 0.
\]

(iii) If \( 0 < a < \alpha \), then

\[
E[\sigma_t^a] \sim \frac{\Gamma(1 - \frac{\alpha}{a})}{\Gamma(1 - a)} t^{\alpha/a} L_2(\frac{1}{a})^{-a} \quad \text{as } t \downarrow 0.
\]

The proofs of Theorems 2.1, 2.2, 2.3 and 2.4 are given in Sections 3, 4, 5 and 6, respectively. To show Theorem 2.1, in fact, we prove estimates of the integral kernels of \( Q_K(t) \), \( Q_G(t) \) and \( Q_B(t) \) by a finite positive linear combination of \( t^n \mathbb{E}[|x - y|^n \sigma_t^n p(\sigma_t, x - y)] \), where \( p(t, x - y) \) is the heat kernel (see (A.2)). Such estimates of the integral kernels of the three operators of difference in Theorems 2.2 / 2.3 also can be obtained (cf. [Tak]), but are omitted.
3. Proof of Theorem 2.1

It is easily seen (see (A.6)) that for \( f \in C_0^\infty(\mathbb{R}^d) \)
\[
K(t)f(x) = \mathbb{E}\left[ \exp\left( -\frac{t}{2}(V(x) + V(x + X_t)) \right) f(x + X_t) \right],
\]
(3.1)
\[
G(t)f(x) = \mathbb{E}\left[ \exp\left( -tV(x) \right) f(x + X_t) \right],
\]
(3.2)
\[
R(t)f(x) = \mathbb{E}\left[ \exp\left( -tV(x + X_{t/2}) \right) f(x + X_t) \right],
\]
(3.3)
and generally
\[
K(\frac{t}{n})^n f(x) = \mathbb{E}\left[ \exp\left( -\frac{t}{n} \sum_{k=1}^{n} (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n})) \right) f(x + X_t) \right],
\]
(3.4)
\[
G(\frac{t}{n})^n f(x) = \mathbb{E}\left[ \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(k-1)t/n}) \right) f(x + X_t) \right],
\]
(3.5)
\[
R(\frac{t}{n})^n f(x) = \mathbb{E}\left[ \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(2k-1)t/2n}) \right) f(x + X_t) \right].
\]
(3.6)

Further, for \( f \in C_0^\infty(\mathbb{R}^d) \) we have (see (A.13))
\[
Q_K(t)f(x) = \int_{\mathbb{R}^d} df(y) \mathbb{E}_\sigma \left[ \mathbb{E}_B[v_K(t, x, y; \sigma)] p(\sigma_t, x - y) \right],
\]
(3.7)
\[
Q_G(t)f(x) = \int_{\mathbb{R}^d} df(y) \mathbb{E}_\sigma \left[ \mathbb{E}_B[v_G(t, x, y; \sigma)] p(\sigma_t, x - y) \right],
\]
(3.8)
\[
Q_R(t)f(x) = \int_{\mathbb{R}^d} f(y) dy \mathbb{E}_\sigma \left[ \mathbb{E}_B[v_R(t, x, y; \sigma)] p(\sigma_t, x - y) \right],
\]
(3.9)
where \( \mathbb{E}_\sigma \) and \( \mathbb{E}_B \) are the expectations with respect to \( \sigma \) and \( B \), respectively,
\[
v_K(t, x, y; \sigma) := \exp\left( -\frac{t}{2}(V(x) + V(y)) \right) - \exp\left( -\int_0^t V(B_{0,x}^{\sigma,y}(\sigma_s)) ds \right),
\]
(3.10)
\[
v_G(t, x, y; \sigma) := \exp\left( -tV(x) \right) - \exp\left( -\int_0^t V(B_{0,x}^{\sigma,y}(\sigma_s)) ds \right),
\]
(3.11)
\[
v_R(t, x, y; \sigma) := \exp\left( -tV(B_{0,x}^{\sigma,y}(\sigma_{t/2})) \right) - \exp\left( -\int_0^t V(B_{0,x}^{\sigma,y}(\sigma_s)) ds \right),
\]
(3.12)
and, for \( \tau > 0, x, y \in \mathbb{R}^d \) and \( 0 \leq \theta \leq \tau \)
\[
B_{0,x}^{\tau,y}(\theta) := x + \frac{\theta}{\tau}(y - x) + B_0^\tau(\theta)
\]
(3.13)
\[
B_0^\tau(\theta) := B(\theta) - \frac{\theta}{\tau} B(\tau).
\]

Since
\[
e^a - e^b = (a - b)e^b + (a - b)^2 \int_0^1 (1 - \theta)e^{\theta a} e^{(1-\theta)b} d\theta, \quad a, b \in \mathbb{R},
\]
we have

\[ v_K(t, x, y; \sigma) = w_K(t, x, y; \sigma) \exp\left(-\frac{1}{2}(V(x) + V(y))\right) \]

\[ - w_K(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta\right) \times \exp\left(- (1 - \theta) \frac{1}{2}(V(x) + V(y))\right) d\theta \]

\[ =: v_{K1}(t, x, y; \sigma) + v_{K2}(t, x, y; \sigma), \quad (3.14) \]

\[ v_G(t, x, y; \sigma) = w_G(t, x, y; \sigma) \exp\left(-tV(x)\right) \]

\[ - w_G(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta\right) \times \exp\left(- (1 - \theta)tV(x)\right) d\theta \]

\[ =: v_{G1}(t, x, y; \sigma) + v_{G2}(t, x, y; \sigma), \quad (3.15) \]

\[ v_R(t, x, y; \sigma) = w_R(t, x, y; \sigma) \exp\left(-tV(B_{0,x}^{\sigma, y}(\sigma_{t/2}))\right) \]

\[ - w_R(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta\right) \times \exp\left(- (1 - \theta)tV(B_{0,x}^{\sigma, y}(\sigma_{t/2}))\right) d\theta \]

\[ =: v_{R1}(t, x, y; \sigma) + v_{R2}(t, x, y; \sigma), \quad (3.16) \]

where

\[ w_K(t, x, y; \sigma) := \frac{1}{2}(V(x) + V(y)) + \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta, \quad (3.17) \]

\[ w_G(t, x, y; \sigma) := -tV(x) + \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta, \quad (3.18) \]

\[ w_R(t, x, y; \sigma) := -tV(B_{0,x}^{\sigma, y}(\sigma_{t/2})) + \int_0^t V(B_{0,x}^{\sigma, y}(\sigma_s))d\theta. \quad (3.19) \]

When \( V \) is further a \( C^1 \)-function, since

\[ V(z) - V(w) = \langle \nabla V(w), z - w \rangle + \int_0^1 \langle \nabla V(w + \theta(z - w)) - \nabla V(w), z - w \rangle d\theta, \]

we have

\[ w_K(t, x, y; \sigma) = \frac{1}{2} \langle \nabla V(x) - \nabla V(y), y - x \rangle \int_0^t \frac{\sigma_s}{\sigma_t} ds \]

\[ + \frac{1}{2} \langle \nabla V(y), y - x \rangle \left( \int_0^t \frac{\sigma_s}{\sigma_t} ds - \int_0^t \frac{\sigma_s - \sigma_x}{\sigma_t} ds \right) \]

\[ + \frac{1}{2} \langle \nabla V(x) + \nabla V(y), \int_0^t B_{0,x}^{\sigma_t}(\sigma_s) ds \rangle \]
In this subsection, we suppose condition (A)\textsuperscript{2}.

### Case (A)\textsuperscript{1}

In the following we shall prove Theorem 2.1 only in Cases (A)\textsuperscript{1} (A)\textsubscript{0}. The proof of Case (A)\textsubscript{1} is omitted; it is similar to that of (A)\textsubscript{2}.

#### 3.1. Case (A)\textsubscript{2}

In this subsection, we suppose condition (A)\textsubscript{2} on \( V(x) \).

**Claim 3.1.**

\[
\left| E_\sigma \left[ E_B [v_{K1}(t, x, y; \sigma)] p(\sigma_t, x - y) \right] \right| \\
\leq \text{const}(\delta, \mu, \nu, d) \left[ t^{1+2\delta} \left( E_\sigma [ |x - y|^2 p(\sigma_t, x - y)] + E_\sigma [\sigma_t p(\sigma_t, x - y)] \right) \\
+ E_\sigma [ |x - y|^{2+\mu} p(\sigma_t, x - y)] + E_\sigma [\sigma_t^{1+\mu/2} p(\sigma_t, x - y)] \right) \\
+ t \left( E_\sigma [ |x - y|^2 p(\sigma_t, x - y)] + E_\sigma [\sigma_t p(\sigma_t, x - y)] + E_\sigma [ |x - y|^{2+\nu} p(\sigma_t, x - y)] \right) \\
+ E_\sigma [\sigma_t^{1+\nu/2} p(\sigma_t, x - y)] \right].
\]
Proof. In view of (3.14) and (3.20), we set

\[ v_{K1}(t, x, y; \sigma) = \sum_{j=1}^{5} w_{Kj}(t, x, y; \sigma) e^{-t(V(x)+V(y))/2} \]

\[ =: \sum_{j=1}^{5} v_{K1j}(t, x, y; \sigma). \quad (3.23) \]

Clearly

\[ \mathbb{E}_B[w_{K3}(t, x, y; \sigma)] = \frac{1}{2} \langle \nabla V(y), \nabla V(y), \mathbb{E}_B[B_{0}^{\alpha_t}(\sigma_s)] ds \rangle = 0, \]

and hence \( \mathbb{E}_B[v_{K13}(t, x, y; \sigma)] = 0 \). By the fact \( (\sigma_t - \sigma_{t-s})_{0 \leq s \leq t} \approx (\sigma_t)_{0 \leq s \leq t} \),

\[ \mathbb{E}_\sigma[w_{K2}(t, x, y; \sigma)p(\sigma_t, x - y)] \]

\[ = \frac{1}{2} \langle \nabla V(y), y - x \rangle \left( \mathbb{E}_\sigma \left[ \int_0^t \frac{\partial \sigma}{\partial t} ds p(\sigma_t, x - y) \right] - \mathbb{E}_\sigma \left[ \int_0^t \frac{\partial \sigma - \sigma_{t-s}}{\partial \sigma - \sigma_{t-s}} ds p(\sigma_t - \sigma_{t-s}, x - y) \right] \right) \]

\[ = 0, \]

and hence \( \mathbb{E}_\sigma[\mathbb{E}_B[v_{K12}(t, x, y; \sigma)]p(\sigma_t, x - y)] = \mathbb{E}_\sigma[v_{K12}(t, x, y; \sigma)p(\sigma_t, x - y)] = 0 \). By (A)2(ii)

\[ |v_{K11}(t, x, y; \sigma)| = |w_{K1}(t, x, y; \sigma)| e^{-t(V(x)+V(y))/2} \]

\[ \leq \frac{1}{2} |\nabla V(x) - \nabla V(y)| |x - y| t e^{-t(V(x)+V(y))/2} \]

\[ \leq \frac{C_2}{2} \{ V(x)^{(1-2\delta)} + (1 + |x - y|^\mu) + 1 + |x - y|^{2+\mu} \} |x - y|^2 t e^{-tV(x)/2} \]

\[ \leq \frac{C_2}{2} \{ V(x)^{(1-2\delta)} e^{-tV(x)/2} t(2|x - y|^2 + |x - y|^{2+\mu}) + t(|x - y|^2 + |x - y|^{2+\nu}) \}

\[ \leq \frac{C_2}{2} \left\{ \left( \frac{2(1-2\delta)}{e} \right) \{ (1-2\delta) + 1 \} \right\} (|x - y|^2 + |x - y|^{2+\nu}) + t(|x - y|^2 + |x - y|^{2+\nu}) \}. \quad (3.24) \]

Here (and hereafter) the following inequality has been (will be) used:

\[ e^b e^{-t} \leq \left( \frac{b}{t} \right)^b, \quad t \geq 0, \quad b \geq 0, \quad (3.25) \]

where for \( b = 0 \) we understand \((0/e)^0 := 1\). By (A)2(ii) and (3.25) again

\[ |v_{K14}(t, x, y; \sigma)| = |w_{K4}(t, x, y; \sigma)| e^{-t(V(x)+V(y))/2} \]

\[ \leq \frac{1}{2} \int_0^t ds \int_0^1 |\nabla V(x + \theta(y - x) + B_{0}^{\alpha_t}(\sigma_s)) - \nabla V(x)| \]

\[ \times |\frac{\sigma}{\sigma_t}(y - x) + B_{0}^{\alpha_t}(\sigma_s)| d\theta e^{-tV(x)/2} \]

\[ \leq \frac{C_2}{2} \int_0^t \left\{ V(x)^{(1-2\delta)} e^{-tV(x)/2} \left( |\frac{\sigma}{\sigma_t}(y - x) + B_{0}^{\alpha_t}(\sigma_s)|^2 + |\frac{\sigma}{\sigma_t}(y - x) + B_{0}^{\alpha_t}(\sigma_s)|^{2+\mu} \right) \right\} \]
Claim 3.2.

Similarly to (3.26), we have

\[
\left| v_{K15}(t, x, y; \sigma) \right| \leq C_2 \left\{ \left( \frac{2(1-2\delta)}{e} \right)^{(1-2\delta)+} t^{1+2\delta} \right. \\
\times \left( 3|x-y|^2 + 6C(2, d)\sigma_t + 3^{1+\mu}|x-y|^{2+\mu} + 3^{1+\mu}2C(2+\mu, d)\sigma_t^{1+\mu/2} \right) \\
+ t \left( 3|x-y|^2 + 6C(2, d)\sigma_t + 3^{1+\nu}|x-y|^{2+\nu} + 3^{1+\nu}2C(2+\nu, \sigma_t)^{1+\nu/2} \right) \left\}. \right.
\]

Collecting all the above into (3.23) yields the estimate in Claim 3.1 and the proof is complete. \( \square \)
\[ + C_2^2 l^2 \left( \mathbb{E}_\sigma [ |x - y|^4 p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t^2 p(\sigma_t, x - y)] \right) + \mathbb{E}_\sigma [|x - y|^{4+2\nu} p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t^{2+\nu} p(\sigma_t, x - y)] \right) \].

**Proof.** By (A)\(2(\text{i})\)

\[
\left| \sum_{j=1}^3 w_{K_j}(t, x, y; \sigma) \right| = \left| \frac{1}{2} \left\langle \nabla V(x), \int_0^t \left( \frac{\sigma_x}{\sigma_t} (y - x) + B_0^{\sigma_x}(\sigma_s) \right) ds \right\rangle \right.
\]
\[
+ \left. \frac{1}{2} \left\langle \nabla V(y), \int_0^t \left( \frac{\sigma_x}{\sigma_t} (x - y) + B_0^{\sigma_x}(\sigma_s) \right) ds \right\rangle \right|
\]
\[
\leq \frac{C_1}{2} \left\{ (1 + V(x)^{1 - \delta}) \int_0^t \left| \frac{\sigma_x}{\sigma_t} (y - x) + B_0^{\sigma_x}(\sigma_s) \right| ds \right.
\]
\[
+ \left. (1 + V(y)^{1 - \delta}) \int_0^t \left| \frac{\sigma_x}{\sigma_t} (x - y) + B_0^{\sigma_x}(\sigma_s) \right| ds \right\}. \quad (3.29)
\]

This estimate together with (3.26) and (3.27) gives us that

\[
|w_K(t, x, y; \sigma)| e^{-\theta t (V(x) + V(y))/4}
\]
\[
\leq \left| \sum_{j=1}^3 w_{K_j}(t, x, y; \sigma) \right| e^{-\theta t (V(x) + V(y))/4} + \sum_{j=4}^5 \left| w_{K_j}(t, x, y; \sigma) \right| e^{-\theta t (V(x) + V(y))/4}
\]
\[
\leq \frac{C_2}{2} \left\{ 1 + \left( \frac{4(1 - \delta)}{e} \right)^{1 - \delta} \theta^{-1 + \delta} t^{1 + \delta} \right\}
\]
\[
\times \int_0^t \left( \left| \frac{\sigma_x}{\sigma_t} (y - x) + B_0^{\sigma_x}(\sigma_s) \right|^2 + \left| \frac{\sigma_x}{\sigma_t} (x - y) + B_0^{\sigma_x}(\sigma_s) \right|^{2+\mu} \right)
\]
\[
\left. + \left| \sigma_x - \sigma_s \right| (x - y) + B_0^{\sigma_x}(\sigma_s) \right|^2 + \left| \sigma_x - \sigma_s \right| (y - x) + B_0^{\sigma_x}(\sigma_s) \right|^{2+\mu} \right) \right.
\]
\[
+ \left. \left| \sigma_x - \sigma_s \right| (x - y) + B_0^{\sigma_x}(\sigma_s) \right|^2 + \left| \sigma_x - \sigma_s \right| (y - x) + B_0^{\sigma_x}(\sigma_s) \right|^{2+\nu} \right). \quad (3.29)
\]

By the Schwarz inequality, it follows that

\[
\left( \left| w_K(t, x, y; \sigma) \right| e^{-\theta t (V(x) + V(y))/4} \right)^2
\]
\[
\leq 12 \left[ \left( \frac{C_2}{2} \right)^2 \left( t + \left( \frac{4(1 - \delta)}{e} \right)^{2(1 - \delta)} \theta^{-2 + 2\delta} t^{2 - 2\delta} \right) \right.
\]
\[
\left. \times \left( \int_0^t \left| \frac{\sigma_x}{\sigma_t} (y - x) + B_0^{\sigma_x}(\sigma_s) \right|^2 ds + \int_0^t \left| \sigma_x - \sigma_s \right| (x - y) + B_0^{\sigma_x}(\sigma_s) \right|^2 ds \right) \right)
\]
\[
+ \left( \frac{C_2}{2} \right)^2 \left( \left( \frac{4(1 - 2\delta)}{e} \right)^{2(1 - 2\delta)} \theta^{-2(1 - 2\delta)} t^{2(1 - 2\delta)} - 1 \right) \right]
\]

17
Claim 3.3.

Similarly to what is done in (3.29), (3.26) and (3.27), we have

\[ \times \left( \int_0^t \frac{\sigma_t}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t \frac{\sigma_t}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds \right) \]

\[ + \int_0^t \frac{\sigma_t-x}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t \frac{\sigma_t-x}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds \]

\[ + t \left( \int_0^t \frac{\sigma_t}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t \frac{\sigma_t}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds \right) \]

\[ + \int_0^t \frac{\sigma_t}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^4 ds + \int_0^t \frac{\sigma_t}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s)|^{4+2\mu} ds \right] \].

Take expectation \( \mathbb{E}_B \) above, and integrate in \( \theta \). Then

\[ \mathbb{E}_B \left[ |v_{K2}(t, x, y; \sigma)| \right] \]

\[ \leq \mathbb{E}_B \left[ |w_{K}(t, x, y; \sigma)|^2 \int_0^1 \theta e^{-\theta t(V(x)+V(y))/2} d\theta \right] \]

\[ = \int_0^1 \theta \mathbb{E}_B \left[ \left( |w_{K}(t, x, y; \sigma)|e^{-\theta t(V(x)+V(y))/4} \right)^2 \right] d\theta \]

\[ \leq 12 \left( \frac{C_1}{2} \right)^2 \left( 3^2 + \frac{(4(1-\delta))}{e} \right) \left( \frac{4(1-2\delta)}{e} \right)^2 \left( \frac{1}{e} \right)^2 \left( \frac{2(1-\delta)}{e} \right) \]

\[ \times [3^2(|x-y|^4 + 2C(4, d)\sigma_t^2) + 3^2 + 2\mu (|x-y|^{4+2\mu} + 2C(4 + 2\mu, d)\sigma_t^{2+\mu})] \]

\[ + t^2 \left[ 3^2(|x-y|^4 + 2C(4, d)\sigma_t^2) + 3^2 + 2\mu (|x-y|^{4+2\mu} + 2C(4 + 2\nu, d)\sigma_t^{2+\nu}) \right] \],

whence follows immediately the estimate in Claim 3.2. \( \square \)

Claim 3.3.

\[ \mathbb{E}_\sigma \left[ \mathbb{E}_B \left[ |v_G(t, x, y; \sigma)|p(\sigma_t, x - y) \right] \right], \quad \mathbb{E}_\sigma \left[ \mathbb{E}_B \left[ |v_R(t, x, y; \sigma)|p(\sigma_t, x - y) \right] \right] \]

\[ \leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left[ C_1^j (t^j + x^j) \left( \mathbb{E}_\sigma[|x-y|^j p(\sigma_t, x - y)] + \mathbb{E}_\sigma[\sigma_t^{j/2} p(\sigma_t, x - y)] \right) \right. \]

\[ + C_2^j t^{j(1/2+\delta)} \left( \mathbb{E}_\sigma[|x-y|^{2j} p(\sigma_t, x - y)] + \mathbb{E}_\sigma[\sigma_t^{j(1+\mu/2)} p(\sigma_t, x - y)] \right) \]

\[ + \mathbb{E}_\sigma[|x-y|^{j(2+\mu)} p(\sigma_t, x - y)] + \mathbb{E}_\sigma[\sigma_t^{j(1+\nu/2)} p(\sigma_t, x - y)] \]

\[ + \mathbb{E}_\sigma[|x-y|^{j(2+\nu)} p(\sigma_t, x - y)] + \mathbb{E}_\sigma[\sigma_t^{j(1+\nu/2)} p(\sigma_t, x - y)] \left] \right. \right]. \]

Proof. Similarly to what is done in (3.29), (3.26) and (3.27), we have

\[ |w_{G1}(t, x, y; \sigma)|e^{-rtV(x)} \]

\[ \leq C_1 (1 + \frac{(1-\delta)}{e} \frac{1-\delta}{e} + (1+\delta)) \int_0^t \frac{\sigma_t}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)|ds, \quad (3.30) \]
Lemma 3.1. We are now in a position to prove Theorem 2.1(iii). To do so, we need the following lemma.

By (3.15), (3.16), (3.21) and (3.22), note that

$$|w_{G2}(t,x,y;\sigma)|e^{-rtV(x)}$$

$$\leq C_2 \left[ (\frac{1-2\delta_1}{e})^{1-2\delta} (rt)^{-(1-2\delta)} \right.$$ 

$$\times \int_0^t \left( |\frac{\alpha_s}{\sigma_t}(y-x)+B_0^\sigma(\sigma_s)|^2 + |\frac{\alpha_s}{\sigma_t}(y-x)+B_0^\sigma(\sigma_s)|^{2+\mu} \right) ds$$

$$\left. + \int_0^t \left( |\frac{\alpha_s}{\sigma_t}(y-x)+B_0^\sigma(\sigma_s)|^2 + |\frac{\alpha_s}{\sigma_t}(y-x)+B_0^\sigma(\sigma_s)|^{2+\nu} \right) ds \right],$$

$$31$$

$$|w_{R1}(t,x,y;\sigma)|e^{-rtV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))}$$

$$\leq C_1 (1 + (\frac{1-\delta}{e})^{1-\delta} (rt)^{1+\delta}) \int_0^t |B_0^{\sigma_t,y}(\sigma_s) - B_0^{\sigma_t,y}(\sigma_{t/2})| ds,$$

$$33$$

$$|w_{R2}(t,x,y;\sigma)|e^{-rtV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))}$$

$$\leq C_2 \left[ (\frac{1-2\delta_1}{e})^{1-2\delta} (rt)^{-(1-2\delta)} \right.$$ 

$$\times \int_0^t \left( |B_0^{\sigma_t,y}(\sigma_s) - B_0^{\sigma_t,y}(\sigma_{t/2})|^2 + |B_0^{\sigma_t,y}(\sigma_s) - B_0^{\sigma_t,y}(\sigma_{t/2})|^{2+\mu} \right) ds$$

$$\left. + \int_0^t \left( |B_0^{\sigma_t,y}(\sigma_s) - B_0^{\sigma_t,y}(\sigma_{t/2})|^2 + |B_0^{\sigma_t,y}(\sigma_s) - B_0^{\sigma_t,y}(\sigma_{t/2})|^{2+\nu} \right) ds \right].$$

By (3.15), (3.16), (3.21) and (3.22), note that

$$|v_{G}(t,x,y;\sigma)|$$

$$\leq |w_{G1}(t,x,y;\sigma)|e^{-tV(x)} + |w_{G2}(t,x,y;\sigma)|e^{-tV(x)}$$

$$+ \int_0^1 \theta \left( |w_{G1}(t,x,y;\sigma)|e^{-\theta tV(x)/2} + |w_{G2}(t,x,y;\sigma)|e^{-\theta tV(x)/2} \right) d\theta,$$

$$34$$

$$|v_{R}(t,x,y;\sigma)|$$

$$\leq |w_{R1}(t,x,y;\sigma)|e^{-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))} + |w_{R2}(t,x,y;\sigma)|e^{-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))}$$

$$+ \int_0^1 \theta \left( |w_{R1}(t,x,y;\sigma)|e^{-\theta tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))/2} \right. 

$$\left. + |w_{R2}(t,x,y;\sigma)|e^{-\theta tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))/2} \right) d\theta.$$

$$35$$

Also note that for $a > 0$ and $0 \leq \theta_1, \theta_2 \leq \tau$ ($\tau > 0$) (cf. (3.28))

$$\mathbb{E}_B \left[ |B_{0,x}^{\sigma_t,y}(\theta_1) - B_{0,x}^{\sigma_t,y}(\theta_2)|^a \right] \leq 3^{(a-1)+} (|x-y|^a + 2C(a,d)\tau^{a/2}).$$

$$36$$

Collecting all the above yields the estimate in Claim 3.3 immediately.}

We are now in a position to prove Theorem 2.1(iii). To do so, we need the following lemma.

**Lemma 3.1.** Let $1 \leq p \leq \infty$. Then, for $a, b \geq 0$ with $C(a,d) = \int_{\mathbb{R}^d} |y|^a p(1,y) dy$,

$$f_{a,b}(t) := \left\lVert \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma [\cdot - y]^a \sigma_t^b p(\sigma_t, \cdot - y) dy \right\rVert_p$$
\[ \leq C(a, d)\mathbb{E}_\sigma[\sigma_t^{n/2+b}] \|f\|_p, \quad f \in L_p(\mathbb{R}^d). \]

**Proof.** For \( p = \infty \), the described estimate is obvious. So let \( 1 \leq p < \infty \). First we note the Minkowski inequality for integrals: If \( h(x, y) \) is a measurable function on a \( \sigma \)-finite product measure space \((X \times Y, \alpha(dx) \times \beta(dy))\), then
\[
\left( \int_Y \left( \int_X |h(x, y)|\alpha(dx) \right)^p \beta(dy) \right)^{1/p} \leq \int_X \left( \int_Y |h(x, y)|^p \beta(dy) \right)^{1/p} \alpha(dx).
\]

Note also that for \( c \geq 0 \)
\[
\left\| \int_{\mathbb{R}^d} |f(y)| \cdot -y|^c p(\tau, -y) dy \right\|_p \leq C(c, d)\tau^{c/2} \|f\|_p.
\]

By these inequalities, the estimate is obtained as follows:
\[
\left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma[|\cdot -y|^c \sigma_t^b] p(\sigma_t, -y) dy \right\|_p \leq \mathbb{E}_\sigma \left[ \left\| \int_{\mathbb{R}^d} |f(y)| \cdot -y|^c \sigma_t^b p(\sigma_t, -y) dy \right\|_p \right]
\leq C(a, d)\mathbb{E}_\sigma[\sigma_t^{n/2+b}] \|f\|_p. \quad \Box
\]

**Proof of Theorem 2.1(iii).** By Claims 3.1, 3.2 with (3.7)
\[
\|Q_K(t)f\|_p \leq \left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma[\mathbb{E}_B[v_{K1}(t, \cdot, y; \sigma)] p(\sigma_t, -y)] dy \right\|_p
\leq \text{const}(\delta, \mu, \nu, d) \left[ C_1^2(t^2 + t^{2\delta})(f_{2,0}(t) + f_{0,1}(t)) \right.
\left. + \sum_{j=1}^2 \left\{ C_2^j t^{j(1+2\delta)} (f_{2j,0}(t) + f_{0,j}(t)) + f_{j(2+\mu),0}(t) + f_{0,j(1+\mu/2)}(t) \right\} \right].
\]

By Claim 3.3 with (3.8), (3.9)
\[
\|Q_R(t)f\|_p \leq \left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma[\mathbb{E}_B[v_{C_R}(t, \cdot, y; \sigma)] p(\sigma_t, -y)] dy \right\|_p
\leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left[ C_1^j (t^j + t^{j\delta})(f_{j,0}(t) + f_{0,j/2}(t)) \right.
\left. + C_2^j t^{j(1+2\delta)} (f_{2j,0}(t) + f_{0,j}(t)) + f_{j(2+\mu),0}(t) + f_{0,j(1+\mu/2)}(t) \right].
\]

Combining these with Lemma 3.1 we have the assertion of Theorem 2.1(iii). \quad \Box

20
3.2. Case (A)₀

In this subsection, we suppose condition (A)₀ on \( V(x) \). In this case
\[
|v_K(t, x, y; \sigma)| \leq |w_K(t, x, y; \sigma)| \\
\leq \frac{C_0}{2} \int_0^t |\frac{\sigma_t}{\sigma}(y - x) + B^t_0(\sigma_s)|^\gamma ds + \frac{C_0}{2} \int_0^t |\frac{\sigma_t - \sigma_s}{\sigma}(x - y) + B^t_s(\sigma_s)|^\gamma ds,
\]
\[
|v_G(t, x, y; \sigma)| \leq |w_G(t, x, y; \sigma)| \\
\leq C_1 \int_0^t |\frac{\sigma_t}{\sigma}(y - x) + B^t_0(\sigma_s)|^\gamma ds,
\]
\[
|v_R(t, x, y; \sigma)| \leq |w_R(t, x, y; \sigma)| \\
\leq C_1 \int_0^t |B^t_{0,x}(\sigma_s) - B^t_{0,x}(\sigma_{t/2})|^\gamma ds.
\]
Here taking expectation \( \mathbb{E}_B \), we have by (3.28) or (3.36),
\[
\mathbb{E}_B[|v_K(t, x, y; \sigma)|], \mathbb{E}_B[|v_G(t, x, y; \sigma)|], \mathbb{E}_B[|v_R(t, x, y; \sigma)|] \\
\leq C_1 t (|x - y|^\gamma + 2C(\gamma, d)\sigma_t^{\gamma/2})
\]
and hence, by (3.7), (3.8) and (3.9)
\[
|Q_K(t)f(x)|, |Q_G(t)f(x)|, |Q_R(t)f(x)| \\
\leq C_1 t \left\{ \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma \left[ |x - y|^\gamma p(\sigma_t, x - y) \right] dy \\
+ 2C(\gamma, d) \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma \left[ \sigma_t^{\gamma/2} p(\sigma_t, x - y) \right] dy \right\}.
\]
From this and Lemma 3.1 the assertion of Theorem 2.1(i) follows immediately.

4. Proof of Theorem 2.2

For notational simplicity we set \( H_0 := H^0_0 \) and \( H := H_0 + V \), in the following, so that \( K(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2}, G(t) = e^{-tV}e^{-tH_0} \) and \( R(t) = e^{-tH_0/2}e^{-tV}e^{-tH_0/2} \).

4.1. Proof of Theorem 2.2 for \( K(t) \)

Since \( K(t) \) and \( e^{-sH} \) are contractions, we have
\[
\|K(t/n)^n - e^{-tH}\|_{\rho-p} = \| \sum_{k=0}^{n-1} K(t/n)^{n-1-k}(K(t/n) - e^{-tH/n})e^{-ktH/n} \|_{\rho-p} \\
\leq \sum_{k=0}^{n-1} \|K(t/n) - e^{-tH/n}\|_{\rho-p} \\
= n\|QK(t/n)\|_{\rho-p}.
\]
Combined with the estimates for \( QK(t) \) in Theorem 2.1, the desired bound for \( K(t/n)^n - e^{-tH} \) in Case (A)₀, (A)₁ or (A)₂ is obtained immediately.
4.2. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case (A)$_0$

In the same way as above,

\[
\|G(\frac{t}{n}) - e^{-tH}\|_{p-p} \leq n \|QG(\frac{t}{n})\|_{p-p}, \\
\|R(\frac{t}{n}) - e^{-tH}\|_{p-p} \leq n \|QR(\frac{t}{n})\|_{p-p},
\]

from which together with Theorem 2.1(i), the desired bounds follow immediately.

4.3. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case (A)$_1$ or (A)$_2$

In this subsection we suppose that $V(x)$ satisfies (A)$_1$ or (A)$_2$.

We first observe that for $t \geq 0$ and $n \in \mathbb{N}$

\[
G(\frac{t}{n}) - e^{-tH} = e^{-tV/2n}(K(\frac{n-1}{n}t \frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n})e^{-tV/2n}e^{-tH_0/n} \\
+ e^{-tH_0/2n}e^{-tV/2n}(K(\frac{n-1}{n}t \frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n})e^{-tV/2n}e^{-tH_0/2n} \\
+ e^{-tH_0/2n}e^{-(n-1)tH/n}e^{-tV/2n}e^{-tH_0/2n}e^{-tV/n}e^{-tH_0/2n} + e^{-(n-1)tH/n}QR(\frac{t}{n}),
\]

where $[A, B] = AB - BA$. Hence

\[
\|G(\frac{t}{n}) - e^{-tH}\|_{p-p} \leq \|K(\frac{n-1}{n}t \frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p-p} \\
+ \|e^{-tV/2n}, e^{-(n-1)tH/n}\|_{p-p} + \|QG(\frac{t}{n})\|_{p-p}, \tag{4.1}
\]

\[
\|R(\frac{t}{n}) - e^{-tH}\|_{p-p} \leq \|K(\frac{n-1}{n}t \frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p-p} \\
+ \|e^{-tV/2n}, e^{-(n-1)tH/n}\|_{p-p} + \|e^{-tH_0/2n}, e^{-(n-1)tH/n}\|_{p-p} \\
+ \|QR(\frac{t}{n})\|_{p-p}. \tag{4.2}
\]

As for the first term on the RHS of (4.1) and (4.2), we see by Theorem 2.2 which was proved in Section 4.1

\[
\|K(\frac{n-1}{n}t \frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p-p}
\]

\[
\leq \begin{cases} 
\text{const}(\delta, \kappa, d) \left[ C_1^2 \left( \frac{t}{n} \right)^2 + \left( \frac{t}{n} \right)^2 \delta \right] (n - 1) E[\sigma_{t/n}] + \sum_{j=1}^{2} (C_2 \frac{t}{n}^j) (n - 1) E[\sigma_{t/n}^{j+\frac{1}{2}}] 
\end{cases}
\]

\[
\leq \begin{cases} 
\text{const}(\delta, \mu, \nu, d) \left[ C_1^2 \left( \frac{t}{n} \right)^2 + \left( \frac{t}{n} \right)^2 \delta \right] (n - 1) E[\sigma_{t/n}] + \sum_{j=1}^{2} \left\{ (C_2 \frac{t}{n}^j) (n - 1) E[\sigma_{t/n}^{j+\frac{1}{2}}] + \left( C_2 \frac{t}{n}^{1+\mu/2} \right) \right\}, 
\end{cases}
\]

in Case (A)$_1$, and

in Case (A)$_2$.
As for the third term on the RHS of (4.1) and the fourth term of (4.2), we see by Theorem 2.1

\[ \|Q_C(\frac{k}{n})\|_{p \rightarrow p}, \|Q_R(\frac{k}{n})\|_{p \rightarrow p} \]

\[ \leq \begin{cases} \text{const} (\delta, \kappa, d) \sum_{j=1}^{2} \left\{ C_1^j \left( \frac{L}{n} \right)^j + \left( \frac{L}{n} \right)^j \mathbb{E}[\sigma_t^{j/2}] \right\}, & \text{in Case (A)_1,} \\ \text{const} (\delta, \mu, \nu, d) \sum_{j=1}^{2} \left\{ C_1^j \left( \frac{L}{n} \right)^j + \left( \frac{L}{n} \right)^j \mathbb{E}[\sigma_t^{j/2}] \right\} + C_2 \left( \frac{L}{n} \right)^j \mathbb{E}[\sigma_t^{j/(1+\nu/2)}], & \text{in Case (A)_2.} \end{cases} \]

Therefore we need to estimate the middle terms of (4.1) and (4.2).

**Claim 4.1.** Let \( s \geq 0 \) and \( t > 0 \). Then

\[ \|e^{-sV}, e^{-tH}\|_{p \rightarrow p}, \|e^{-sH_0}, e^{-tH}\|_{p \rightarrow p} \]

\[ \leq \begin{cases} \text{const} (\delta, \kappa, d) \left[ C_1^j \left( 1 + t^{-1+\delta/2} \right) \mathbb{E}[\sigma_t^{j/2}] + C_2 \mathbb{E}[\sigma_t^{j/(1+\kappa/2)}] \right], & \text{in Case (A)_1,} \\ \text{const} (\delta, \mu, \nu, d) \left[ C_1^j \left( 1 + t^{-1+\delta/2} \right) \mathbb{E}[\sigma_t^{j/2}] + C_2 t^{-1-2\delta} \mathbb{E}[\sigma_t^{j/(1+\nu/2)}] \right], & \text{in Case (A)_2.} \end{cases} \]

**Proof.** First we estimate the \( L_p \)-operator norm of \([e^{-sV}, e^{-tH}]\). We have (by (A.13)) that for \( f \in C_0(\mathbb{R}^d) \)

\[ [e^{-sV}, e^{-tH}] f(x) = \int_{\mathbb{R}^d} f(y) (e^{-sV(x)} - e^{-sV(y)}) \mathbb{E} \left[ \exp \left( - \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) \, dr \right) p(\sigma_t, x-y) \right] dy. \]

Hence we have

\[ \| [e^{-sV}, e^{-tH}] f(x) \| \leq s \int_{\mathbb{R}^d} |f(y)| \mathbb{E} \left[ |V(y) - V(x)| \exp \left( - \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) \, dr \right) p(\sigma_t, x-y) \right] dy. \]  \[ \text{To estimate the integrand in (4.3), note by Taylor's theorem that} \]

\[ V(y) - V(x) = \int_0^t \langle \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_r)), y-x \rangle \frac{dr}{t} \]

\[ + \int_0^1 d\theta \int_0^t \langle \nabla V(x + \theta(y-x)) - \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_r)), y-x \rangle \frac{d\theta}{t}. \]
In Case (A), it follows that

\[ |V(y) - V(x)| \leq \int_0^t C_1(1 + V(B_{0,x}(\sigma_r)^{1-\delta}) \frac{dr}{t})|x-y| \]

\[ + \int_0^1 d\theta \int_0^t C_2(B_{0,x}(\sigma_r)^{\sup_{0\leq\sigma\leq\tau}}(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma_r)^{\mu}) \]

\[ \leq C_1 \left( 1 + t^{-1+\delta} \left( \int_0^t V(B_{0,x}(\sigma_r)^{1-\delta})dr \right)^{1-\delta} \right)|x-y| \]

\[ + C_2 t^{-1+\delta} \left( \int_0^t V(B_{0,x}(\sigma_r)^{1-\delta})dr \right)^{1-\delta} \int_0^1 \left( \sup_{0\leq\sigma\leq\tau} |(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma)| \right) d\theta|x-y| \]

where the last inequality is due to Jensen’s inequality. In Case (A)_2

\[ |V(y) - V(x)| \]

\[ \leq \int_0^t C_1(1 + V(B_{0,x}(\sigma_r)^{1-\delta}) \frac{dr}{t})|x-y| \]

\[ + \int_0^1 d\theta \int_0^t C_2 \left\{ V(B_{0,x}(\sigma_r)^{(1-2\delta)} + (1 + |(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma_r)^{\mu}) \right\} |(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma_r)|^{1+\mu} \frac{dr}{t}|x-y| \]

\[ \leq C_1 \left( 1 + t^{-1+\delta} \left( \int_0^t V(B_{0,x}(\sigma_r)^{1-\delta})dr \right)^{1-\delta} \right)|x-y| \]

\[ + C_2 t^{-1+\delta} \left( \int_0^t V(B_{0,x}(\sigma_r)^{1-\delta})dr \right)^{1-\delta} \int_0^1 \left( \sup_{0\leq\sigma\leq\tau} |(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma)| \right) d\theta|x-y| \]

\[ + C_2 t^{-1+\delta} \left( \int_0^t \left( |(\frac{\partial}{\partial \sigma} - \theta)(y-x) + B_{0}^{\sigma_t}(\sigma_r)^{\mu}) \right) dr|x-y| \].

By (3.25), (4.4) and (4.5) imply the desired estimate:

\[ |V(y) - V(x)| \exp \left( - \int_0^t V(B_{0,x}(\sigma_r)^{1-\delta})dr \right) \]
Next we estimate the $L^1$-operator norm of $[e^{-sH_0}, e^{-tH}]$ by Lemma 3.1 with (4.3).

Hence follows the desired bound for $[e^{-sV}, e^{-tH}]$ by Lemma 3.1 with (4.3).

Next we estimate the $L_p$-operator norm of $[e^{-sH_0}, e^{-tH}]$.

First we suppose that $V : \mathbb{R}^d \to [0, \infty)$ is in $C^\infty$ and all its derivatives have polynomial growth. Then it is easily verified that (cf. Claim A.2 and its Remark)

(i) $e^{-tH}(S(\mathbb{R}^d)) \subset S(\mathbb{R}^d)$, in particular, $e^{-tH_0}(S(\mathbb{R}^d)) \subset S(\mathbb{R}^d)$, and

(ii) $S(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} \mathcal{D}(\mathfrak{g}_p^{\psi,V}) \cap \bigcap_{1 \leq p \leq \infty} \mathcal{D}(\mathfrak{g}_p^{\psi,0})$ and $\mathfrak{g}_p^{\psi,V} = \mathfrak{g}_p^{\psi,0} - V$ on $S(\mathbb{R}^d)$.

Here $\mathfrak{g}_p^{\psi,V}$ $(1 \leq p < \infty)$ is the infinitesimal generator of $\{e^{-t(H_0+V)}\}$ on $L_p(\mathbb{R}^d)$ and $\mathfrak{g}_\infty^{\psi,V}$ the one on $C_\infty(\mathbb{R}^d)$. By these facts the following formula holds in $L_p(\mathbb{R}^d)$ ($1 \leq p < \infty$) and $C_\infty(\mathbb{R}^d)$:
For each \( f \in \mathcal{S}(\mathbb{R}^d) \)
\[
[e^{-sH_0}, e^{-tH}]f = \int_0^s e^{-uH_0}[V, e^{-tH}]e^{-(s-u)H_0}f du.
\]

Hence, taking \( L_\rho\)-norm in the above yields that for each \( f \in \mathcal{S}(\mathbb{R}^d) \)
\[
\| [e^{-sH_0}, e^{-tH}]f \|_\rho \leq \int_0^s \| [V, e^{-tH}]e^{-(s-u)H_0}f \|_\rho du.
\] (4.8)

Now let \( V \) satisfy (A)1 or (A)2. In this case \( V \) is not necessarily smooth. So, take a nonnegative \( h \in C_0^\infty \) with support in \( \{ x \in \mathbb{R}^d : |x| \leq 1 \} \) and \( \int_{\mathbb{R}^d} h(x)dx = 1 \). Set \( V^\varepsilon = V * h_\varepsilon \) with \( h_\varepsilon(x) = (1/\varepsilon)^d h(x/\varepsilon) \). Then \( V^\varepsilon \) is in \( C^\infty(\mathbb{R}^d \to [0, \infty)) \), and satisfies condition (A)1 or (A)2 with the same \( \text{const} \)’s as \( V \) does. Further, by (A)1(i) or (A)2(ii) all the derivatives of \( V^\varepsilon \) have polynomial growth. Hence, by (4.7) and Lemma 3.1 it holds that for \( g \in \mathcal{S}(\mathbb{R}^d) \)
\[
\| [V^\varepsilon, e^{-t(H_0 + V^\varepsilon)}]g \|_\rho
\]
\[
\begin{cases}
\text{const}(\delta, \kappa, d) [C_1(1 + t^{-1+\delta})E[\sigma_t^{1/2}] + C_2E[\sigma_t^{(1+\kappa)/2}]])\|g\|_\rho, & \text{in Case (A)1,} \\
\text{const}(\delta, \mu, d) [C_1(1 + t^{-1+\delta})E[\sigma_t^{1/2}] + C_2t^{-(1-2\delta)} + (E[\sigma_t] + E[\sigma_t^{1+\mu/2}])]
\quad \\
+ C_2(E[\sigma_t] + E[\sigma_t^{1+\mu/2}]) \|g\|_\rho, & \text{in Case (A)2.}
\end{cases}
\]

Since (4.8) holds with \( V = V^\varepsilon \), by combining this with the above we have
\[
\| [e^{-sH_0}, e^{-t(H_0 + V^\varepsilon)}]f \|_\rho
\]
\[
\begin{cases}
\text{const}(\delta, \kappa, d) [C_1(1 + t^{-1+\delta})E[\sigma_t^{1/2}] + C_2E[\sigma_t^{(1+\kappa)/2}]])\|f\|_\rho, & \text{in Case (A)1,} \\
\text{const}(\delta, \mu, d) [C_1(1 + t^{-1+\delta})E[\sigma_t^{1/2}] + C_2t^{-(1-2\delta)} + (E[\sigma_t] + E[\sigma_t^{1+\mu/2}])]
\quad \\
+ C_2(E[\sigma_t] + E[\sigma_t^{1+\mu/2}]) \|f\|_\rho, & \text{in Case (A)2.}
\end{cases}
\]

Finally let \( \varepsilon \downarrow 0 \). Since \( V^\varepsilon \to V \) compact uniformly, we see by the Feynman-Kac formula (A.6) that \( e^{-t(H_0 + V^\varepsilon)}f \to e^{-t(H_0 + V)}f \) boundedly pointwise, so that \( [e^{-sH_0}, e^{-t(H_0 + V^\varepsilon)}]f \to [e^{-sH_0}, e^{-t(H_0 + V)}]f \) pointwise. Hence the desired bound for \( [e^{-sH_0}, e^{-t(H_0 + V)}] \) follows immediately by the Fatou inequality.

We return to estimate \( G(t/n)^n - e^{-tH} \) and \( R(t/n)^n - e^{-tH} \). By Claim 4.1
\[
\| [e^{-tV^{2n}}, e^{-(n-1)tH/n}]_p \|_{p \to p}, \| [e^{-tH_0^{2n}}, e^{-(n-1)tH/n}]_p \|_{p \to p}
\]
\[
\begin{cases}
\text{const}(\delta, \kappa, d) \frac{1}{n} [C_1(t + t^\delta)E[\sigma_t^{1/2}] + C_2tE[\sigma_t^{(1+\kappa)/2}]], & \text{in Case (A)1,} \\
\text{const}(\delta, \mu, d) \frac{1}{n} [C_1(t + t^\delta)E[\sigma_t^{1/2}] + C_2t^{1+2\delta}(E[\sigma_t] + E[\sigma_t^{1+\mu/2}])]
\quad \\
+ C_2t(E[\sigma_t] + E[\sigma_t^{1+\mu/2}]), & \text{in Case (A)2.}
\end{cases}
\]

Therefore, collecting all the estimates above yields the desired bounds for \( G(t/n)^n - e^{-tH} \) and \( R(t/n)^n - e^{-tH} \).
5. Proof of Theorem 2.3

As in the previous section, we are setting $H_0 = H_0^{\delta}$ and $H = H_0 + V$.

5.1. Case (V)$_2$

Condition (V)$_2$ implies (A)$_2$ with $\delta = 1 \land 1/\rho$, $C_1 = c_1 e^{-\langle 1-1/\lambda/\rho \rangle}$, $C_2 = c_2 2^{(\rho-3)+}$, $\langle 1/2 \rangle c^{-2(L(1/\lambda/\rho))_{+} \lor 1}$, $\mu = 0$ and $\nu = \langle \rho-2 \rangle_{+}$. So this case follows immediately from Theorem 2.2(iii).

5.2. Case (V)$_1$

In this subsection we suppose condition (V)$_1$ on $V(x)$.

Let us adopt an idea in [D-I-Tam]. Take again a nonnegative $h \in C^\infty_0$ with support in $\{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} h(x)dx = 1$. For $0 < \varepsilon \leq 1/4$, set

$$V_\varepsilon(x) := \left( \frac{1}{\varepsilon} \right)^d \int_{\mathbb{R}^d} h \left( \frac{x-y}{\varepsilon} \right) V(y)dy,$$

where $\eta := ((\rho-1) \lor 0) \land 1$. Then $V_\varepsilon$ is a smooth function and it satisfies the following:

**Lemma 5.1.** (i) $V_\varepsilon(x) \geq c'(x)\rho$ where $c' = c/4^\rho$.

(ii) $|V_\varepsilon(x) - V(x)| \leq C'\varepsilon(x)^{\langle \rho-1 \rangle_{+} + 1}$ where $C' = c_1(5/4)^{\langle \rho-1 \rangle_{+}}$.

(iii) $|\nabla V_\varepsilon(x)| \leq c_1(x)^{\langle \rho-1 \rangle_{+}}$ where $c_1 = c_1(5/4)^{\alpha 1}$.

(iv) $|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \leq (1/\varepsilon)c_2' \left\{ (x)^{\langle \rho-2 \lambda \rangle_{+}} + |x-y|^\langle \rho-2 \lambda \rangle_{+} \right\} |x-y|$ where $\lambda := (1+\eta)/2$ and $c_2' = c_1(5/4)^{\langle \rho-1 \rangle_{+}}2^{\langle \rho-3 \rangle_{+}}(5d/16 + 2)$.

The proof is not difficult, so is omitted (cf. [Tak]).

As a consequence of Lemma 5.1, it is easily seen that $V_\varepsilon$ satisfies condition (A)$_2$, i.e.

(A)$_{2,\varepsilon}$ $|\nabla V_\varepsilon(x)| \leq C'_1 V_\varepsilon(x)^{1-\langle 1-1/\lambda/\rho \rangle}$,

$$|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \leq \frac{1}{\varepsilon} C'_2 \left\{ V_\varepsilon(x)^{1-2\langle 1-1/\lambda/\rho \rangle_{+}} + |x-y|^\langle \rho-2 \rangle_{+} \right\} |x-y|$$

where $C'_1 = c'_1 d^{\langle 1-1/\lambda/\rho \rangle}$ and $C'_2 = c'_2 d^{\langle 1-2\langle 1-1/\lambda/\rho \rangle_{+} \lor 1 \rangle}$. Indeed, by the definition of $\lambda$, we have $\rho - \rho \land \lambda \geq (\rho-1)_{+}$, $\rho - 2(\rho \land \lambda)_{+} = (\rho - 2\lambda)_{+} = (\rho - 2)_{+}$. Hence (A)$_{2,\varepsilon}$ follows, because, by (i) with $\langle x \rangle \geq 1$,

$$V_\varepsilon(x)^{1-\langle 1-1/\lambda/\rho \rangle} \geq (c')^{1-\langle 1-1/\lambda/\rho \rangle} \geq (c')^{1-\langle 1-1/\lambda/\rho \rangle} \langle x \rangle^{(\rho-1)_{+}},$$

$$V_\varepsilon(x)^{1-2\langle 1-1/\lambda/\rho \rangle_{+}} \geq (c')^\langle (1-2\langle 1-1/\lambda/\rho \rangle)_{+} \rangle \langle x \rangle^{\langle \rho-2(\rho \land \lambda) \rangle_{+}} = (c')^\langle (1-2\langle 1-1/\lambda/\rho \rangle)_{+} \rangle \langle x \rangle^{\langle \rho-2 \lambda \rangle_{+}}.$$

In what follows we write $c, C, c_1, c_2, C_1$ and $C_2$ simply for $c', C', c_1', c_2', C_1'$ and $C_2'$.

Now let $K_\varepsilon(t):= e^{-tV_\varepsilon/2}e^{-tH_0}e^{-tV_\varepsilon/2}$, $G_\varepsilon(t):= e^{-tV_\varepsilon}e^{-tH_0}$ and $R_\varepsilon(t):= e^{-tH_0/2}e^{-tV_\varepsilon}e^{-tH_0/2}$. 

27
Claim 5.1. Let \( t \geq 0 \) and \( n \in \mathbb{N} \). Then with \( H^\varepsilon = H_0 + V_\varepsilon \)

\[
\| K_\varepsilon \left( \frac{t}{n} \right) \|^n - e^{-tH^\varepsilon} \|_{p-p} \\
\leq \text{const}(\rho, d) \left[ C_1^2 \left( \frac{t^2}{n^2} \right) + \left( \frac{t}{n} \right)^{2(1+c\rho)} n^2 \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^n \left\{ (C_2 \frac{t}{n})^j n^2 \mathbb{E}[\sigma_{t/n}]^j \right\} + (C_2 \frac{t}{n})^j n^2 \mathbb{E}[\sigma_{t/n}]^j \right],
\]

\[
\| G_\varepsilon \left( \frac{t}{n} \right) \|^n - e^{-tH^\varepsilon} \|_{p-p}, \| R_\varepsilon \left( \frac{t}{n} \right) \|^n - e^{-tH^\varepsilon} \|_{p-p}
\leq \text{const}(\rho, d) \left[ \frac{1}{t} \left( 1 + \frac{1}{n^2} \right) \mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^2] + \mathbb{E}[\sigma_t^3] + \mathbb{E}[\sigma_t^4] \right],
\]

This is obvious from (A)_{2,\varepsilon} and Theorem 2.2(iii).

Claim 5.2. Let \( t \geq 0 \) and \( n \in \mathbb{N} \). Then

\[
\| e^{-tH} - e^{-tH^\varepsilon} \|_{p-p}, \| K_\varepsilon \left( \frac{t}{n} \right) \|^n - K_\varepsilon \left( \frac{t}{n} \right)^n \|_{p-p}, \| G_\varepsilon \left( \frac{t}{n} \right) \|^n - G_\varepsilon \left( \frac{t}{n} \right)^n \|_{p-p}, \| R_\varepsilon \left( \frac{t}{n} \right) \|^n - R_\varepsilon \left( \frac{t}{n} \right)^n \|_{p-p}
\leq \text{const}(C, c, \rho) \varepsilon t^{2/((\rho^\varepsilon)^2+1)-1}.
\]

Proof. Let \( f \in C_0^\infty(\mathbb{R}^d) \). By (3.4), (3.5) and (3.6) with (A.6),

\[
| (e^{-tH} - e^{-tH^\varepsilon}) f(x) |
\leq \mathbb{E} \left[ \left| \exp \left( - \int_0^t V(x + X_s) ds \right) - \exp \left( \int_0^t V_\varepsilon(x + X_s) ds \right) \right| f(x + X_t) \right], \quad (5.1)
\]

\[
| (K_\varepsilon \left( \frac{t}{n} \right) \|^n - K_\varepsilon \left( \frac{t}{n} \right)^n ) f(x) |
\leq \mathbb{E} \left[ \left| \exp \left( - \frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n})) \right) - \exp \left( - \frac{t}{2n} \sum_{k=1}^n (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n})) \right) \right| f(x + X_t) \right], \quad (5.2)
\]

\[
| (G_\varepsilon \left( \frac{t}{n} \right) \|^n - G_\varepsilon \left( \frac{t}{n} \right)^n ) f(x) |
\leq \mathbb{E} \left[ \left| \exp \left( - \frac{t}{n} \sum_{k=1}^n V(x + X_{(k-1)t/n}) \right) - \exp \left( - \frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(k-1)t/n}) \right) \right| f(x + X_t) \right], \quad (5.3)
\]

28
\[ |(R(\frac{t}{n})^n - R_\varepsilon(\frac{t}{n})^n)f(x)| \]
\[ \leq \mathbb{E} \left[ \left| \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(2k-1)t/2n}) \right) \right| \left| f(x + X_t) \right| \right]. \quad (5.4) \]

By a formula
\[ e^{-\alpha} - e^{-b} = \int_{0}^{1} (b - a)e^{-\theta a}e^{-(1-\theta)b}d\theta, \quad a, b \in \mathbb{R} \]
and Lemma 5.1, we have
\[ \left| \exp\left( -\int_{0}^{t} V(x + X_s)ds \right) - \exp\left( -\int_{0}^{t} \varepsilon V(x + X_s)ds \right) \right| \]
\[ \leq \int_{0}^{1} d\theta \int_{0}^{t} |V(\varepsilon (x + X_s)) - V(x + X_s)|ds \]
\[ \times \exp\left( -\theta \int_{0}^{t} V(x + X_s)ds \right) \exp\left( -(1-\theta) \int_{0}^{t} \varepsilon V(x + X_s)ds \right) \]
\[ \leq C\varepsilon \int_{0}^{t} (x + X_s)^{(\rho-1)+\eta}ds \exp\left( -c \int_{0}^{t} (x + X_s)^\rho ds \right), \]
\[ \left| \exp\left( -\frac{t}{2n} \sum_{k=1}^{n} (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n})) \right) \right| \]
\[ - \exp\left( -\frac{t}{2n} \sum_{k=1}^{n} (\varepsilon V(x + X_{(k-1)t/n}) + \varepsilon V(x + X_{kt/n})) \right) \]
\[ \leq \int_{0}^{1} d\theta \left( \frac{t}{2n} \sum_{k=1}^{n} \left| V(\varepsilon (x + X_{(k-1)t/n})) - V(x + X_{(k-1)t/n}) \right| \right. \]
\[ \left. + \left| V(\varepsilon (x + X_{kt/n}) - V(x + X_{kt/n}) \right| \right) \]
\[ \times \exp\left( -\theta \frac{t}{2n} \sum_{k=1}^{n} (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n})) \right) \]
\[ \times \exp\left( -(1-\theta) \frac{t}{2n} \sum_{k=1}^{n} (\varepsilon V(x + X_{(k-1)t/n}) + \varepsilon V(x + X_{kt/n})) \right) \]
\[ \leq C\varepsilon \left( \frac{t}{2n} \sum_{k=1}^{n} (x + X_{(k-1)t/n})^{(\rho-1)+\eta} + \frac{t}{2n} \sum_{k=1}^{n} (x + X_{kt/n})^{(\rho-1)+\eta} \right) \]
\[ \times \exp\left( -c \left( \frac{t}{2n} \sum_{k=1}^{n} (x + X_{(k-1)t/n})^{\rho} \right) \exp\left( -c \left( \frac{t}{2n} \sum_{k=1}^{n} (x + X_{kt/n})^{\rho} \right) \right). \]
Similarly
\[
\left| \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(k-1)t/n}) \right) - \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V_\varepsilon(x + X_{(k-1)t/n}) \right) \right| \\
\leq C\varepsilon^{\frac{t}{n}} \sum_{k=1}^{n} \langle x + X_{(k-1)t/n} \rangle^{(\rho-1)+\eta} \exp\left( -c_4 \frac{t}{n} \sum_{k=1}^{n} \langle x + X_{(k-1)t/n} \rangle^{\rho} \right),
\]
\[
\left| \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(2k-1)t/2n}) \right) - \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V_\varepsilon(x + X_{(2k-1)t/2n}) \right) \right| \\
\leq C\varepsilon^{\frac{t}{n}} \sum_{k=1}^{n} \langle x + X_{(2k-1)t/2n} \rangle^{(\rho-1)+\eta} \exp\left( -c_4 \frac{t}{n} \sum_{k=1}^{n} \langle x + X_{(2k-1)t/2n} \rangle^{\rho} \right).
\]

By Jensen’s inequality and (3.25),
\[
\left| \exp\left( -\int_{0}^{t} V(x + X_s) ds \right) - \exp\left( -\int_{0}^{t} V_\varepsilon(x + X_s) ds \right) \right|,
\]
\[
\left| \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(k-1)t/n}) \right) - \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V_\varepsilon(x + X_{(k-1)t/n}) \right) \right|,
\]
\[
\left| \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V(x + X_{(2k-1)t/2n}) \right) - \exp\left( -\frac{t}{n} \sum_{k=1}^{n} V_\varepsilon(x + X_{(2k-1)t/2n}) \right) \right|,
\]
\[
\leq C\varepsilon t^{1-(\rho-1)+\eta}/\rho \left( \frac{[\rho-1]+\eta}{\rho+1} \right)^{(\rho-1)+\eta}/\rho,
\]
\[
\left| \exp\left( -\frac{t}{2n} \sum_{k=1}^{n} (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n})) \right) - \exp\left( -\frac{t}{2n} \sum_{k=1}^{n} (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n})) \right) \right|,
\]
\[
\leq C\varepsilon^{\frac{1}{2}}(\frac{t}{\sqrt{n}})^{1-(\rho-1)+\eta}/\rho \left( \frac{[\rho-1]+\eta}{\rho+1} \right)^{(\rho-1)+\eta}/\rho,
\]
where for \(\rho = 0\) we understand \(([\rho-1]+\eta)/\rho = 0\). Substituting these into (5.1), (5.2), (5.3) and (5.4), respectively, we have
\[
\left| (e^{-tH} - e^{-tH^\varepsilon}) f(x), \right|,
\]
\[
\left| (K(\frac{t}{n})^n - K_\varepsilon(\frac{t}{n})^n) f(x)\right|, \left| (G(\frac{t}{n})^n - G_\varepsilon(\frac{t}{n})^n) f(x)\right|, \left| (R(\frac{t}{n})^n - R_\varepsilon(\frac{t}{n})^n) f(x)\right|,
\]
\[
\leq \text{const}(C, c, \rho) t^{2/((\rho^2+1)^{\eta})} - 1 \mathbb{E}[\|f(x + X_t)\|],
\]
which imply the estimates in Claim 5.2 and the proof is complete. \(\square\)

**Proof of Theorem 2.3(i).** By Claims 5.1 and 5.2
\[
\|K(\frac{t}{n})^n - e^{-tH}\|_{p-p}
\]
\[
\leq \|K_{1/n}^n - K_{\varepsilon}(1/n)^n\|_{p-p} + \|K_{\varepsilon}(1/n)^n - e^{-tH\varepsilon}\|_{p-p} + \|e^{-tH\varepsilon} - e^{-tH}\|_{p-p}
\]
\[
\leq \text{const}(\rho, C, c, d) \left[ \varepsilon t^{2/((\rho^2)\nu(1)) - 1} + C_1^2 ((\frac{1}{n})^2 + (\frac{1}{n})^2(1+\lambda/\rho))n\mathbb{E}[\sigma_{t/n}] 
\right.
\]
\[
+ \sum_{j=1}^2 \left\{ (C_2 \frac{1}{n})^{1/2} j n\mathbb{E}[\sigma_{t/n}^j] + (C_2 \frac{1}{n})^{1/2} j n\mathbb{E}[\sigma_{t/n}^{(2\nu)j/2}] 
\right.
\]
\[
+ (C_2 \frac{1}{n})^{1/2} j n\mathbb{E}[\sigma_{t/n}^{(2\nu)j/2}] \right\}, 
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\leq \text{const}(\rho, C, c, d) \left[ \varepsilon t^{2/((\rho^2)\nu(1)) - 1} + (t^2 + t^2((\rho^2)\nu(1)/2))n\mathbb{E}[\sigma_{t/n}] 
\right.
\]
\[
+ \sum_{j=1}^2 \left\{ (t^j + t^{j/2}\nu j n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{(2\nu)j/2}] \right\}, 
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\leq \text{const}(\rho, C, c, d) \left[ \varepsilon t^{2/((\rho^2)\nu(1)) - 1} + (t^2 + t^2((\rho^2)\nu(1)/2))n\mathbb{E}[\sigma_{t/n}] 
\right.
\]
\[
+ \sum_{j=1}^2 \left\{ (t^j + t^{j/2}\nu j n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{(2\nu)j/2}] \right\}, 
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\leq \text{const}(\rho, C, c, d) \left[ \varepsilon t^{2/((\rho^2)\nu(1)) - 1} + (t^2 + t^2((\rho^2)\nu(1)/2))n\mathbb{E}[\sigma_{t/n}] 
\right.
\]
\[
+ \sum_{j=1}^2 \left\{ (t^j + t^{j/2}\nu j n\mathbb{E}[\sigma_{t/n}^j] + t^j n\mathbb{E}[\sigma_{t/n}^{(2\nu)j/2}] \right\}, 
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]

Now let \( n \geq 2^{2(2\nu)} \) and \( \varepsilon := n^{-(1/2)\nu(\lambda/\rho)} = n^{-1/2} \nu \). Then \( \varepsilon \leq 1/4, \) \( \varepsilon^{-1} n^{-1/2} \nu = n^{-1/2} \nu \) and \( \varepsilon^{-1} n^{-1} \leq n^{-1/2} \nu \). Therefore we have
\[
\|K_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]
\[
\|G_{1/n} - e^{-tH}\|_{p-p} \quad \|R_{1/n} - e^{-tH}\|_{p-p}
\]

6. Proof of Theorem 2.4

For \( a > 0 \), the proof will be given, divided into the three cases \( a = 1, a > 1 \) and \( 0 < a < 1 \).

First we note that for every \( a > 0 \)
\[
\mathbb{E}[\sigma_{t/n}^{a}] < \infty. \tag{6.1}
\]
In fact, it is enough to show when \( a = \nu \) is a positive integer. To do so, let \( \varphi_t \) be the characteristic function of \( \sigma_t \), i.e., \( \varphi_t(\xi) = \mathbb{E}[e^{\sqrt{T_\xi} \sigma_t}] \). We have \( \varphi_t(\xi) = e^{-t f(\xi)} \), where

\[
f(\xi) = \int_{(0, \infty)} (1 - e^{\sqrt{T_\xi} l}) e^{-l/2} n(dl).
\]

Since smoothness of \( \varphi_t(\xi) \) near \( \xi = 0 \) implies existence of moments of \( \sigma_t \) (cf. Exercise 2.6(viii) in [It]), we have only to show that \( \varphi_t \) or \( f \) is in \( C^1 \) near \( \xi = 0 \). But this is obvious, because, by a property of the Lévy measure \( n \), the integral \( \int_{(0, \infty)} l^a e^{-l/2} n(dl) \) is convergent, so that by the Lebesgue convergence theorem

\[
\left( \frac{4}{l^2} \right)^{\nu} f(\xi) = -\int_{(0, \infty)} (\sqrt{-1} l)^{\nu} e^{\sqrt{-1} T_\xi l} e^{-l/2} n(dl).
\]

By Itô’s formula (e.g. [Ik-Wa]),

\[
\sigma_t^a = \int_0^{t+} \int_{(0, \infty)} \left\{ (\sigma_s + l)^a - \sigma_s^a \right\} N(dsdl)
\]

\[
= \int_0^{t+} \int_{(0, \infty)} a \int_0^1 (\sigma_s + \theta l)^{a-1} d\theta \, l \, N(dsdl),
\]

and hence, by taking expectation \( \mathbb{E} \)

\[
\mathbb{E}[\sigma_t^a] = \int_0^t ds \int_{(0, \infty)} l e^{-l/2} n(dl) a \int_0^1 \mathbb{E}[(\sigma_s + \theta l)^{a-1}] d\theta. \tag{6.2}
\]

This is further, by the change of variable \( r = \frac{s}{t} \), rewritten as

\[
\frac{1}{t} \mathbb{E}[\sigma_t^a] = \int_0^1 dr \int_{(0, \infty)} l e^{-l/2} n(dl) a \int_0^1 \mathbb{E}[(\sigma_{tr} + \theta l)^{a-1}] d\theta. \tag{6.3}
\]

6.1. The case \( a = 1 \)

By (6.3), it is clear that

\[
\frac{1}{t} \mathbb{E}[\sigma_t] = \int_{(0, \infty)} l e^{-l/2} n(dl) \in (0, \infty). \tag{6.4}
\]

6.2. The case \( a > 1 \)

By (6.1) and (6.3), \( \mathbb{E}[(\sigma_{tr} + \theta l)^{a-1}] \) is of course integrable on \( (0, \infty) \times [0, 1] \times [0, 1] \) w.r.t. \( l e^{-l/2} n(dl) dr d\theta \). Since \( \sigma_t \) is increasing in \( t \) with \( \sigma_{0+} = \sigma_0 = 0 \) and \( a - 1 > 0 \), we have \( (\sigma_{tr} + \theta l)^{a-1} \downarrow \theta^{a-1} \) as \( t \downarrow 0 \). It follows by the Lebesgue convergence theorem that

\[
\frac{1}{t} \mathbb{E}[\sigma_t^a] \downarrow \int_{(0, \infty)} l^a e^{-l/2} n(dl) \in (0, \infty). \tag{6.5}
\]
6.3. The case $0 < a < 1$

By the same reason as above (but in this case, $a - 1 < 0$), we have $(\sigma_{t^r} + \theta t)^{a-1} \uparrow \theta^{a-1} t^{a-1}$ as $t \downarrow 0$, and hence, by the monotone convergence theorem

$$\frac{1}{n} \mathbb{E}[\sigma_t^2] \uparrow \int_{(0,\infty)} l^a e^{-l/2} n(dl) \in (0,\infty).$$  \hspace{1cm} (6.6)

This time the integral on the RHS is not always convergent. To find the exact asymptotics we suppose assumption (L).

We start with a remark on (L) and $\psi(\lambda)$ defined by (1.3):

Fact. (i) If $0 \leq \alpha < 1$, then

$$\psi(\lambda) \sim \Gamma(1 - \alpha) \lambda^\alpha L(\lambda) \text{ as } \lambda \uparrow \infty.$$  

(ii) If $\alpha = 1$, then $\int_0^t n((s, \infty))ds$ is slowly varying at zero, $L(1/t) = o(\int_0^t n((s, \infty))ds)$ as $t \downarrow 0$ and

$$\psi(\lambda) \sim \lambda \int_0^{1/\lambda} n((s, \infty))ds \text{ as } \lambda \uparrow \infty.$$  

Proof. First of all note that

$$\int_{(0,\infty)} l \wedge 1 n(dl) = \int_0^1 n((t, \infty))dt,$$  \hspace{1cm} (6.7)

$$\psi(\lambda) = \lambda \int_0^\infty e^{-\lambda t} d\left(\int_0^t n((s, \infty))ds\right).$$  \hspace{1cm} (6.8)

By (1.6), $n((1/y, \infty)) \sim y^\alpha L(y)$ as $y \uparrow \infty$, and by (6.7),

$$\int_x^\infty \frac{1}{y^2} n((1/y, \infty))dy = \int_0^{1/x} n((s, \infty))ds < \infty \text{ for any } x > 0.$$  

Let us apply Lemma and Theorem 1 of §VIII.9 in [Fe]. These say that $\int_x^\infty 1/y^2 n((1/y, \infty))dy$ is regularly varying with exponent $-1 + \alpha$ and

$$\frac{(1/x)n((1/x, \infty))}{\int_x^\infty 1/y^2 n((1/y, \infty))dy} \rightarrow 1 - \alpha \text{ as } x \uparrow \infty.$$  

Combining these with (1.6), we see that when $0 \leq \alpha < 1$

$$\int_0^t n((s, \infty))ds \sim \frac{1}{1 - \alpha} t n((t, \infty)) \sim \frac{1}{1 - \alpha} t^{1-\alpha} L\left(\frac{1}{t}\right) \text{ as } t \downarrow 0,$$

and that when $\alpha = 1$, $\int_0^t n((s, \infty))ds$ is slowly varying at zero and $L(1/t) = o(\int_0^t n((s, \infty))ds)$ as $t \downarrow 0$.

By virtue of (6.8), if we apply the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), the asymptotics of $\psi$ follow from those of $\int_0^t n((s, \infty))ds$.  \hspace{1cm} \Box
Remark. Conversely, when $0 \leq \alpha < 1$, we have (1.6) by Fact (i) by the Tauberian theorem.

Recall functions $\phi$, $L_1$ and $L_2$ around assumption (L) in Section 1. By Fact, $L_1$ is slowly varying at infinity and

$$\phi(\lambda) \sim \lambda^\alpha L_1(\lambda) \quad \text{as } \lambda \uparrow \infty.$$ \hfill (6.9)

As $\psi$ is strictly increasing with $\psi(0) = 0$ and $\psi(\infty) = \infty$, so is $\phi$, so that the inverse $\phi^{-1}$ exists. By (6.9), if $0 < \alpha \leq 1$,

$$\phi^{-1}(x) \sim x^{1/\alpha} L_2(x) \quad \text{as } x \uparrow \infty.$$ \hfill (6.10)

Since, by (6.9) again, $\phi$ is regularly varying with exponent $\alpha$, so is $\phi^{-1}$ with exponent $1/\alpha$, and hence $L_2$ and $\int_0^\infty (\phi^{-1}(\theta))^{-\alpha} d\theta$ ($0 < \alpha < 1$) are also slowly varying at infinity.

Now we are in a position to show the asymptotics of $\mathbb{E}[\sigma_t^a]$ for $0 < a < 1$.

Claim 6.1. (i) If $0 < a < \alpha$,

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1 - a)} t^{\alpha/a} L_2(1/t)^{-a} \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1 - a)} \phi^{-1}(1/t)^{-a} \quad \text{as } t \downarrow 0.$$  

(ii) If $a = \alpha$,

$$\mathbb{E}[\sigma_t^a] \sim \frac{1}{\Gamma(1 - a)} t \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } t \downarrow 0.$$  

(iii) If $\alpha < a < 1$, then $\int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \in (0, \infty)$ and

$$\mathbb{E}[\sigma_t^a] \sim t \frac{a}{\Gamma(1 - a)} \int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \quad \text{as } t \downarrow 0.$$  

Proof. To rewrite (6.2), we see first with (2.2)

$$\mathbb{E}[(\sigma_s + \theta l)^{a-1}] = \frac{1}{\Gamma(1 - a)} \int_0^\infty \lambda^{-a} e^{-\lambda \theta l} \mathbb{E}[e^{-\lambda \sigma_s}] d\lambda$$

$$= \frac{1}{\Gamma(1 - a)} \int_0^\infty \lambda^{-a} e^{-\lambda \theta l} e^{-s \phi(\lambda)} d\lambda,$$

and then we have

$$\mathbb{E}[\sigma_t^a] = \frac{a}{\Gamma(1 - a)} \int_0^t ds \int_0^\infty \lambda^{-a} e^{-s \phi(\lambda)} d\lambda \int_{(0, \infty)} (1 - e^{-\lambda}) e^{-l/2} n(dl)$$

$$= \frac{a}{\Gamma(1 - a)} \int_0^t ds \int_0^\infty \lambda^{-a} \phi(\lambda) e^{-s \phi(\lambda)} d\lambda.$$  

The $\lambda$-integral in the last line is further computed by the change of variable $\lambda = \phi^{-1}(x)$ as follows:

$$\int_0^\infty \lambda^{-1-a} \phi(\lambda) e^{-s\phi(\lambda)} d\lambda$$

$$= \int_0^\infty (\phi^{-1}(x))^{-1-a} x e^{-sx} (\phi^{-1})'(x) dx$$

$$= \left[-\frac{1}{a} (\phi^{-1}(x))^{-a} x e^{-sx}\right]_0^\infty + \frac{1}{a} \int_0^\infty (\phi^{-1}(x))^{-a} (e^{-sx} - sxe^{-sx}) dx$$

$$= \frac{1}{a} \left\{ \int_0^\infty (\phi^{-1}(x))^{-a} e^{-sx} dx - s \int_0^\infty (\phi^{-1}(x))^{-a} x e^{-sx} dx \right\}$$

$$= \frac{1}{a} \left\{ \int_0^\infty e^{-sx} \left[ f_0^x (\phi^{-1}(\theta))^{-a} d\theta \right] - s \int_0^\infty e^{-sx} \left[ f_0^x (\phi^{-1}(\theta))^{-a} d\theta d\theta \right] \right\}$$

$$= \frac{1}{a} \left\{ L\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) - sL\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta d\theta \right) \right\}.$$  

Here $L(\cdot, G)$ denotes the Laplace transform of a right-continuous increasing function $G : [0, \infty) \to [0, \infty)$: $L(s, G) := \int_0^\infty e^{-sx} dG(x)$. The last fourth and third equalities are respectively because $0 \leq (\phi^{-1}(x))^{-a} x e^{-sx} \leq (\psi' (1/2)) a x^{1-a} e^{-sx} \to 0$ as $x \to 0$, and because for $b > a - 1,$

$$\int_0^\infty (\phi^{-1}(x))^{-a} x^b e^{-sx} dx \leq (\psi' (1/2)) a \int_0^\infty x^{b-a} e^{-sx} dx = (\psi' (1/2)) a s^{a-b-1} \Gamma(b-a+1) < \infty.$$  

Hence (6.2) is rewritten as follows:

$$\mathbb{E}[\sigma_t^a] = \frac{1}{\Gamma(1-a)} \int_0^t \left[ L\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) - sL\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta d\theta \right) \right] ds.$$  

\[\text{1°}\] The case $0 < a < \alpha$. Then $0 < \alpha \leq 1$. By (6.10), $(\phi^{-1}(\cdot))^{-a}$ is regularly varying with exponent $-a/\alpha \in (-1, 0)$. By Theorem 1 of §VIII.9 in [Fe],

$$\frac{x^2 (\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a} d\theta} \to 2 - \frac{a}{\alpha} > 0, \quad \frac{x (\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a} d\theta} \to \frac{a - a}{\alpha} > 0$$

as $x \to \infty$. Hence, by combining this with (6.10),

$$\int_0^x (\phi^{-1}(\theta))^{-a} d\theta \sim \frac{1}{2 - a/\alpha} x^2 (\phi^{-1}(x))^{-a} \sim \frac{1}{2 - a/\alpha} x^{2-a/\alpha} L(x)^{-a}$$

and

$$\int_0^x (\phi^{-1}(\theta))^{-a} d\theta \sim \frac{\alpha}{\alpha - a} x (\phi^{-1}(x))^{-a} \sim \frac{\alpha}{\alpha - a} x^{1-a/\alpha} L(x)^{-a}$$

as $x \to \infty$. By applying the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), this implies that

$$L\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) \sim \frac{\Gamma(2 - a/\alpha + 1)}{2 - a/\alpha} s^{-2+a/\alpha} L(\frac{1}{s})^{-a} = \Gamma(2 - \frac{a}{\alpha}) s^{-2+a/\alpha} L(\frac{1}{s})^{-a},$$

$$L\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) \sim \frac{\alpha}{\alpha - a} \Gamma(2 - \frac{a}{\alpha}) s^{-1+a/\alpha} L(\frac{1}{s})^{-a}$$

as $s \to 0$, and hence

$$L\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) - sL\left(s, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta d\theta \right) \sim \left(\frac{\alpha}{\alpha - a} - 1\right) \Gamma(2 - \frac{a}{\alpha}) s^{-1+a/\alpha} L(\frac{1}{s})^{-a}$$

$$= \frac{\alpha}{\alpha} \Gamma(\frac{a-a}{\alpha}) s^{-1+a/\alpha} L(\frac{1}{s})^{-a}$$

as $s \to 0$.  

35
Now if, for simplicity, we set
\[ Z(x) := L(\frac{1}{2}, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta) - \frac{1}{2} L(\frac{1}{2}, \int_0^x (\phi^{-1}(\theta))^{-a} d\theta), \]
then, by (6.11)
\[ \mathbb{E}[\sigma_t^a] = \frac{1}{\Gamma(1-a)} \int_0^t Z(\frac{s}{t}) ds = \frac{1}{\Gamma(1-a)} \int_{1/t}^\infty x^{-2} Z(x) dx \]
and also,
\[ Z(x) \sim \frac{2a}{\alpha} \Gamma(\frac{\alpha-a}{\alpha}) x^{1-a/\alpha} L_2(x)^{-a} \quad \text{as } x \uparrow \infty. \]
Therefore, applying Theorem 1 of §VIII.9 in [Fe] again, we have
\[ \frac{(1/t)^{-2+1} Z(1/t)}{\mathbb{E}[\sigma_t^a]} \rightarrow \Gamma(1-a) \frac{\alpha}{\alpha} \quad \text{as } t \downarrow 0, \]
and consequently
\[ \mathbb{E}[\sigma_t^a] \sim \frac{\Gamma((\alpha-a)/\alpha)}{\Gamma(1-a)} t^{\alpha/\alpha} L_2(\frac{1}{t})^{-a}, \]
which is just the assertion (i).

2° The case \( a = \alpha \). Then \( 0 < \alpha < 1 \) and hence, by (6.10), \( (\phi^{-1}(\cdot))^{-\alpha} \) is regularly varying with exponent \( -1 \). Once again, by Theorem 1 of §VIII.9 in [Fe],
\[ \int_0^x (\phi^{-1}(x))^{-\alpha} \theta d\theta \rightarrow 1, \quad \int_0^x (\phi^{-1}(x))^{-\alpha} d\theta \rightarrow 0 \]
as \( x \uparrow \infty \), and \( \int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta \) is slowly varying at infinity. By combining this with (6.10)
\[ \int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta \sim x^2 (\phi^{-1}(x))^{-\alpha} \sim x L_2(x)^{-\alpha}, \]
\[ L_2(x)^{-\alpha} \sim x (\phi^{-1}(x))^{-\alpha} = o(\int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta) \]
as \( x \uparrow \infty \), and hence, by the Abelian theorem
\[ L\left(s, \int_0^s (\phi^{-1}(\theta))^{-\alpha} d\theta \right) \sim s^{-1} L_2(\frac{1}{s})^{-\alpha}, \]
\[ L\left(s, \int_0^s (\phi^{-1}(\theta))^{-a} d\theta \right) \sim \int_0^{1/s} (\phi^{-1}(\theta))^{-a} d\theta \]
as \( s \downarrow 0 \). Therefore
\[ Z(\frac{1}{s}) = L\left(s, \int_0^s (\phi^{-1}(\theta))^{-\alpha} d\theta \right) - sL\left(s, \int_0^s (\phi^{-1}(\theta))^{-\alpha} d\theta \right) \]
\[ \sim \int_0^{1/s} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } s \downarrow 0. \]
In exactly the same way as in 1° we eventually have
\[
\frac{(1/t)^{-2+1} Z(1/t)}{E[\sigma^2_t]} \to \Gamma(1 - \alpha) \quad \text{as } t \downarrow 0,
\]
from which the assertion (ii) is easily seen.

3° The case \( \alpha < a < 1 \). Then \( 0 \leq \alpha < 1 \). By (6.6), it is enough to show that
\[
\int_{(0,\infty)} l^a e^{-l/2} n(dl) = \frac{a}{\Gamma(1-a)} \int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda < \infty.
\]
First this identity is seen from the following computation:
\[
\int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda = \int_0^\infty \lambda^{-1-a} \left( \psi(\lambda + \frac{1}{2}) - \psi(\frac{1}{2}) \right) d\lambda = \int_{(0,\infty)} e^{-l/2} n(dl) \int_0^\infty \lambda^{-1-a} (1 - e^{-\lambda}) d\lambda
\]
\[
= \frac{\Gamma(1-a)}{a} \int_{(0,\infty)} l^a e^{-l/2} n(dl).
\]
Next this integral is convergent. Indeed, since \( \phi(\lambda) \leq \psi'(1/2) \lambda (\lambda \geq 0) \),
\[
\int_0^R \lambda^{-1-a} \phi(\lambda) d\lambda \leq \psi'(\frac{1}{2}) \int_0^R \lambda^{-a} d\lambda = \psi'(\frac{1}{2}) \frac{R^{1-a}}{1-a} < \infty
\]
for any \( R > 0 \). On the other hand, since \( \phi(\lambda) \sim \lambda^a L_1(\lambda) \) as \( \lambda \uparrow \infty \), and \( L_1(\cdot) \) is slowly varying at infinity, there exists an \( R_\varepsilon > 0 \) such that \( \phi(\lambda) \leq 2 \lambda^a L_1(\lambda) \) for \( \lambda \geq R_\varepsilon \). Hence
\[
\int_{R_\varepsilon}^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \leq \int_{R_\varepsilon}^\infty \lambda^{-1-a} 2 \lambda^{a+\varepsilon} d\lambda = \frac{2}{a - \alpha - \varepsilon} \left( \frac{1}{R_\varepsilon} \right)^{a-\alpha-\varepsilon} < \infty.
\]

\[\square\]

Appendix: Semigroups \( e^{-t(H_0^*+V)} \) and their generators in \( L_p(\mathbb{R}^d) \) and \( C_\infty(\mathbb{R}^d) \)

In this appendix we suppose only that \( V : \mathbb{R}^d \to [0, \infty) \) is a continuous function. The main result is Theorem A.1, which follows from Lemma A.2 (Kato’s inequality).

Let \( M(dsdx) \) be a Poisson random measure on \([0, \infty) \times (\mathbb{R}^d \setminus \{0\})\) with intensity measure \( dsJ(dx)\), where
\[
J(dx) := \int_{(0,\infty)} e^{-l/2} p(l, x) n(dl) dx,
\]
\[
p(l, x) := \left( \frac{1}{2\pi l} \right)^{d/2} \exp \left( -\frac{|x|^2}{2l} \right).
\]
This \( M(\cdot) \) may be defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) as in Section 2. Note that for \( p \in [1, \infty) \) the \( 2p \)-th order absolute moment of \( J \) is finite, i.e.,
\[
\int_{\mathbb{R}^d \setminus \{0\}} |x|^{2p} J(dx) < \infty.
\]
Following the notation in [Ik-Wa], we set
\[ \tilde{M}(dsdx) := dsJ(dx), \quad \tilde{M}(dsdx) := M(dsdx) - \tilde{M}(dsdx) \]
and define an \( \mathbb{R}^d \)-valued right-continuous process \( (X_t)_{t \geq 0} \) by
\[
X_t := \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} x1_{|x| \geq 1} M(dsdx) + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} x1_{|x| < 1} \tilde{M}(dsdx),
\]
(A.4)
where the second term on the RHS is a stochastic integral w.r.t. \( \tilde{M} \). This is a \( d \)-dimensional time-homogeneous Lévy process starting at the origin such that
\[
\mathbb{E}[e^{-T(p, X_t)}] = e^{-t((\psi((|p|)^2+1)/2) - \psi(1/2))},
\]
which is easily seen by Itô’s formula (cf. [Ik-Wa]), so that
\[
(X_t)_{t \geq 0} \overset{\mathcal{L}}{\sim} (B(\sigma_t))_{t \geq 0}.
\]
We now define a system of operators \( P^\psi_t V f, t \geq 0 \), by the Feynman-Kac formula:
\[
P^\psi_t V f(x) := \mathbb{E}\left[\exp\left(-\int_0^t V(x + X_s)ds\right) f(x + X_t)\right].
\]
(A.6)
From this definition the following is easily seen:
(i) If \( f \) is a nonnegative Borel measurable function, so is \( P^\psi_t V f \), and it satisfies
\[
P^\psi_t V P^\psi_s V f = P^\psi_{t+s} V f,
\]
(A.7)
\[
\int_{\mathbb{R}^d} |P^\psi_{t+s} V f(x)|^p dx \leq \int_{\mathbb{R}^d} |f(x)|^p dx, \quad 1 \leq p < \infty.
\]
(A.8)
(ii) If \( f \in C_\infty(\mathbb{R}^d) \), then \( P^\psi_t V f \in C_\infty(\mathbb{R}^d) \) and
\[
\|P^\psi_t V f\|_\infty \leq \|f\|_\infty,
\]
(A.9)
\[
\|P^\psi_t V f - f\|_\infty \to 0 \quad \text{as} \ t \downarrow 0.
\]
(A.10)
(iii) For two nonnegative Borel measurable functions \( f, g \)
\[
\int_{\mathbb{R}^d} P^\psi_{t+s} V f(x) g(x) dx = \int_{\mathbb{R}^d} f(x) P^\psi_{t+s} V g(x) dx.
\]
(A.11)
By (i) and (ii), \( \{P^\psi_t V\}_{t \geq 0} \) is a strongly continuous contraction semigroup on \( C_\infty(\mathbb{R}^d) \). By the Riesz-Banach theorem there exists a finite measure \( P^\psi_t V(t, x, dy) \) on \( \mathbb{R}^d \) such that
\[
P^\psi_t V f(x) = \int_{\mathbb{R}^d} f(y) P^\psi_t V(t, x, dy), \quad f \in C_\infty(\mathbb{R}^d).
\]
(A.12)
Indeed, by noting (A.5), \( P^\psi,V(t, x, dy) \) is absolutely continuous w.r.t. the Lebesgue measure \( dy \) on \( \mathbb{R}^d \) and expressed as

\[
P^\psi,V(t, x, dy) = \mathbb{E}\left[ \exp\left( - \int_0^t V(B^\tau_{0,x}^\psi(s)) ds \right) p(\sigma_t, x - y) \right] dy,
\]

where \( B^\tau_{0,x}^\psi(\theta) \) is defined in (3.13).

By (i) and (ii) again \( P^\psi,V_t \) is uniquely extended to a bounded operator on \( L_p(\mathbb{R}^d) \), which is denoted by the same \( P^\psi,V_t \), and thus \( \{ P^\psi,V_t \}_{t \geq 0} \) is a strongly continuous contraction semigroup on \( L_p(\mathbb{R}^d) \). Clearly, for \( f \in L_p(\mathbb{R}^d) \)

\[
P^\psi,V_t f(x) = \mathbb{E}\left[ \exp\left( - \int_0^t V(x + X_s) ds \right) f(x + X_t) \right] \quad \text{a.e. } x
\]

and, when \( p = 2 \), \( P^\psi,V_t \) is symmetric.

Let \( \mathfrak{g}_p^\psi,V \) be the infinitesimal generator of \( \{ P^\psi,V_t \}_{t \geq 0} \) on \( L_p(\mathbb{R}^d) \) for \( 1 \leq p < \infty \), and on \( C_0(\mathbb{R}^d) \) for \( p = \infty \). Their domains are denoted by \( \mathfrak{D}(\mathfrak{g}_p^\psi,V) \).

Put

\[
H_0^\psi f(x) := - \int_{\mathbb{R}^d \setminus \{0\}} \left\{ f(x + y) - f(x) - \langle y, \nabla f(x) \rangle 1_{|y| < 1} \right\} J(dy),
\]

\[
H^\psi f(x) := H_0^\psi f(x) + V(x) f(x).
\]

**Claim A.1.** (i) For \( f \in \mathcal{S}(\mathbb{R}^d) \), \( H_0^\psi f \) is in \( \mathcal{S}(\mathbb{R}^d) \), and hence, for \( f \in C_0^\infty(\mathbb{R}^d) \), \( H^\psi f \in C_0^\infty(\mathbb{R}^d) \cap \bigcap_{1 \leq p < \infty} L_p(\mathbb{R}^d) \).

(ii) For \( f \in C_0^\infty(\mathbb{R}^d) \cap L_p(\mathbb{R}^d) \) (where \( 1 \leq p < \infty \)), \( H_0^\psi f \) is well-defined, i.e., the integral in (A.14) is convergent for a.e. \( x \), and \( H^\psi f \in L_p^{\text{loc}}(\mathbb{R}^d) \). Also, for \( f \in C_0^\infty(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \), the integral in (A.14) is convergent for every \( x \) and \( H_0^\psi f \in C(\mathbb{R}^d) \).

For the proof, cf. [11].

**Claim A.2.** \( C_0^\infty(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} \mathfrak{D}(\mathfrak{g}_p^\psi,V) \), and for \( f \in C_0^\infty(\mathbb{R}^d) \), \( \mathfrak{g}_p^\psi,V f = -H^\psi f \).

**Proof.** Let \( f \in C_0^\infty(\mathbb{R}^d) \).

We start with the proof that

\[
\frac{1}{t} (P^\psi,V_t f - f) \xrightarrow{t \downarrow 0} -H^\psi f \quad \text{in } C_0^\infty(\mathbb{R}^d).
\]

Since \( H^\psi f \in C_0^\infty(\mathbb{R}^d) \) by Claim A.1, it is enough to check pointwise convergence (cf. Lemma 31.7 in [Sa]). To do so we apply Itô’s formula for (A.4) to obtain

\[
\exp\left( - \int_0^t V(x + X_s) ds \right) f(x + X_t)
\]
\[
= f(x) - \int_0^t \exp \left( - \int_0^s V(x + X_r) \, dr \right) V(x + X_s) f(x + X_s) \, ds
\]
\[
+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} \exp \left( - \int_0^s V(x + X_r) \, dr \right) \left( f(x + X_{s-} + y) - f(x + X_{s-}) \right) 1_{|y| \geq 1} M(dy) ds dy
\]
\[
+ \int_0^{t+} \exp \left( - \int_0^s V(x + X_r) \, dr \right) \left( f(x + X_{s-} + y) - f(x + X_{s-}) \right) 1_{|y| < 1} \tilde{M}(dy)
\]
\[
+ \int_0^t \exp \left( - \int_0^s V(x + X_r) \, dr \right) \left( f(x + X_s + y) - f(x + X_s) \right)
\]
\[
- \langle y, \nabla f(x + X_s) \rangle 1_{|y| < 1} \tilde{M}(dy).
\]

Note that the third term on the RHS is a martingale, so that the expectation is zero. Taking expectation and changing the variable \( s = t\sigma \) we have
\[
\frac{1}{t} (P_t^{\psi, V} f(x) - f(x)) + \int_0^1 \mathbb{E} \left[ \exp \left( - \int_0^{t\sigma} V(x + X_r) \, dr \right) (Vf)(x + X_{t\sigma}) \right] d\sigma
\]
\[
= \int_0^1 d\sigma \int_{|y| \geq 1} \mathbb{E} \left[ \exp \left( - \int_0^{t\sigma} V(x + X_r) \, dr \right) \left( f(x + X_{t\sigma} + y) - f(x + X_{t\sigma}) \right) \right] J(dy)
\]
\[
+ \int_0^1 d\sigma \int_{0 < |y| < 1} \mathbb{E} \left[ \exp \left( - \int_0^{t\sigma} V(x + X_r) \, dr \right) \left( f(x + X_{t\sigma} + y) - f(x + X_{t\sigma}) \right)
\]
\[
- \langle y, \nabla f(x + X_{t\sigma}) \rangle \right] J(dy)
\]
\[
= \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{E} \left[ \exp \left( - \int_0^{t\sigma} V(x + X_r) \, dr \right)
\]
\[
\times \int_0^1 (1 - \theta) \langle y, \nabla^2 f(x + X_{t\sigma} + \theta y) y \rangle d\theta \right] J(dy),
\]
where the second equality is due to Taylor’s theorem with the aid of symmetry of \( J(dy) \). On letting \( t \downarrow 0 \) in the first equality of (A.17) we have (A.16) pointwise.

Next we prove for \( 1 \leq p < \infty \) that
\[
\frac{1}{t} (P_t^{\psi, V} f - f) \xrightarrow{t \downarrow 0} -H^\psi f \quad \text{in } L_p(\mathbb{R}^d).
\]
\[
(A.18)
\]

Since \( H^\psi f \in L_p(\mathbb{R}^d) \) by Claim A.1, it is enough to check weak convergence (cf. Lemma 32.3 in [Sa]).

First of all, we note by (A.17) that
\[
\sup_{t > 0} \|\frac{1}{t} (P_t^{\psi, V} f - f)\|_p < \infty \quad \text{for } 1 \leq p < \infty
\]
\[
(A.19)
\]

and that
\[
\lim_{R \to \infty} \limsup_{t \downarrow 0} \int_{|x| > R} \left| \frac{1}{t} (P_t^{\psi, V} f(x) - f(x)) \right| dx = 0.
\]
\[
(A.20)
\]
Indeed, by the second equality of (A.17)
\[ |\frac{1}{T}(P^\psi_t f(x) - f(x))| \leq \int_0^1 \mathbb{E}[(V f)(x + X_{t\sigma})] d\sigma \]
\[ + \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \int_0^1 (1 - \theta)\mathbb{E}[\nabla^2 f(x + X_{t\sigma} + \theta y)] d\theta. \]

(A.21)

Hence, by Minkowski’s inequality, Jensen’s inequality and Fubini’s theorem
\[ \left( \int_{\mathbb{R}^d} |\frac{1}{T}(P^\psi_t f(x) - f(x))|^p dx \right)^{1/p} \leq \|V f\|_p + \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \|\nabla^2 f\|_p, \]
which shows (A.19). To show (A.20), take \( R_0 > 0 \) such that \( \text{supp} \; f \subset \{ x \in \mathbb{R}^d ; |x| < R_0 \} \), and let \( R > R_0 \). Note that \( 1_{|x|>R}h(x+y) = 1_{|x|>R}h(x+y)1_{|y|<R-R_0} \) for \( h = f, \nabla f \) or \( \nabla^2 f \). Hence, by (A.21),
\[ \int_{|x|>R} |\frac{1}{T}(P^\psi_t f(x) - f(x))| dx \]
\[ \leq \int_0^1 \mathbb{E} \left[ \int_{|x|>R} |(V f)(x + X_{t\sigma})| dx ; |X_{t\sigma}| \geq R - R_0 \right] d\sigma \]
\[ + \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \int_0^1 (1 - \theta)d\theta \]
\[ \times \mathbb{E} \left[ \int_{|x|>R} |\nabla^2 f(x + X_{t\sigma} + \theta y)| dx ; |X_{t\sigma} + \theta y| \geq R - R_0 \right] \]
\[ \leq \|V f\|_1 \int_0^1 \mathbb{P}(|X_{t\sigma}| \geq R - R_0) d\sigma \]
\[ + \frac{1}{2} \|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \mathbb{P}(|X_{t\sigma}| + |y| \geq R - R_0). \]

Since \( \lim_{t \downarrow 0} X_{t\sigma} = 0 \) a.s., by the Lebesgue-Fatou inequality
\[ \limsup_{t \downarrow 0} \int_{|x|>R} |\frac{1}{T}(P^\psi_t f(x) - f(x))| dx \]
\[ \leq \frac{1}{2} \|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \limsup_{t \downarrow 0} \mathbb{P}(|X_{t\sigma}| + |y| \geq R - R_0) \]
\[ \leq \frac{1}{2} \|\nabla^2 f\|_1 \int_{|y| \geq R - R_0} |y|^2 J(dy), \]
and thus (A.20) follows.

Now we show weak convergence in \( L_p(\mathbb{R}^d) \) of (A.18). When \( 1 < p < \infty \), let \( q \) be the conjugate exponent of \( p \). For each \( g \in L_q(\mathbb{R}^d) \) and \( R > 0 \)
\[ |\frac{1}{T}(P^\psi_t f - f) + H^\psi f, g| \leq \|\frac{1}{T}(P^\psi_t f - f) + H^\psi f\|_\infty \int_{|x| \leq R} |g(x)| dx \]

41
Lemma A.1. Let \( H \) be a selfadjoint operator on \( L^2(\mathbb{R}^d) \) and \( \psi \) be a function in \( C_c^\infty(\mathbb{R}^d) \) such that \( H^\psi \) is essentially selfadjoint on \( C_0^\infty(\mathbb{R}^d) \). Then, \( H^\psi \) is closable as an operator in \( L^p(\mathbb{R}^d) \), \( 1 \leq p < \infty \), and \( C_\infty(\mathbb{R}^d) \). It is natural to ask whether or not its smallest closed extension agrees with \(-H^\psi\). The following theorem is an affirmative answer.

Theorem A.1. The smallest closed extension of \( H^\psi = -\mathfrak{S}_0^{\psi, V} \) on \( C_0^\infty(\mathbb{R}^d) \) in \( L_p(\mathbb{R}^d) \) \((\text{resp. } H^\psi = -\mathfrak{S}_\infty^{\psi, V} \text{ in } C_\infty(\mathbb{R}^d))\) agrees with \(-\mathfrak{S}_0^{\psi, V}\) \((\text{resp. } -\mathfrak{S}_\infty^{\psi, V})\). In other words, \( C_0^\infty(\mathbb{R}^d) \) is a core of \( \mathfrak{S}_0^{\psi, V} \) \((1 \leq p \leq \infty)\).

By Claim A.2, \( H^\psi \) on \( C_0^\infty(\mathbb{R}^d) \) is closable as an operator in \( L_p(\mathbb{R}^d) \), \( 1 \leq p < \infty \), or \( C_\infty(\mathbb{R}^d) \). It is natural to ask whether or not its smallest closed extension agrees with \(-\mathfrak{S}_0^{\psi, V}\). The following theorem is an affirmative answer.

Lemma A.1. Let \( 1 \leq q \leq \infty \). Suppose \( u \in L_q(\mathbb{R}^d) \) is such that \( H_0^\psi u \in L_1^\text{loc}(\mathbb{R}^d) \), i.e., for some \( f \in L_1^\text{loc}(\mathbb{R}^d) \) it holds that for any \( \varphi \in C_0^\infty(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} u(x)H_0^\psi \varphi(x)dx.
\]

Then \( H_0^\psi u_\delta \to H_0^\psi u \) in \( L_1^\text{loc}(\mathbb{R}^d) \) as \( \delta \downarrow 0 \).
Proof. Since \( u \in L_q(\mathbb{R}^d) \), \( u^\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d) \). By Claim A.1, \( H_0^\psi u^\delta \in L_q^{\text{loc}}(\mathbb{R}^d) \) or \( C(\mathbb{R}^d) \) according as \( 1 \leq q < \infty \) or \( q = \infty \), and hence \( H_0^\psi u^\delta \in L_1^{\text{loc}}(\mathbb{R}^d) \). For the proof, it is enough to check that \( H_0^\psi u^\delta = (H_0^\psi u)^\delta \).

By (A.22)

\[
(H_0^\psi u)^\delta(x) = \int_{\mathbb{R}^d} (H_0^\psi u)(y)\rho_\delta(x - y)dy
\]

\[
= \int_{\mathbb{R}^d} u(y)H_0^\psi \rho_\delta(x - \cdot)(y)dy
\]

\[
= \int_{\mathbb{R}^d} u(y)dy - \int_{\mathbb{R}^d \setminus \{0\}} \left\{ \rho_\delta(x - y - z) - \rho_\delta(x - y) - \langle z, \nabla \rho_\delta(x - \cdot)(y) \rangle_{1 < |z| < 1} \right\} J(dz).
\]

(A.23)

The integral on the RHS is convergent, because with \( \rho_\delta(x - \cdot) =: g \in C^\infty_0(\mathbb{R}^d) \), it is bounded by \( \int_{|z| \geq 1} J(dz) \|u\|_q^2 \|g\|_{q/(q - 1)}^2 + (1/2) \int_{|z| < 1} |z|^2 J(dz) \|u\|_q \|\nabla^2 g\|_{q/(q - 1)} \). Here when \( q = 1 \) or \( \infty \) we understand \( \|\cdot\|_{q/(q - 1)} = \|\cdot\|_\infty \) or \( \|\cdot\|_1 \). Hence by noting that \( \nabla \rho_\delta(x - \cdot)(y) = -\langle \nabla \rho_\delta(x), \cdot \rangle \), Fubini’s theorem gives us that

\[
(H_0^\psi u)^\delta(x) = -\int_{\mathbb{R}^d \setminus \{0\}} J(dz) \left( \int_{\mathbb{R}^d} u(y)\rho_\delta(x - z - y)dy - \int_{\mathbb{R}^d} u(y)\rho_\delta(x - y)dy \right.
\]

\[
- \left. 1_{|z| < 1} \langle -z, \int_{\mathbb{R}^d} u(y)(\nabla \rho_\delta)(x - y)dy \rangle \right)
\]

\[
= -\int_{\mathbb{R}^d \setminus \{0\}} \left( u^\delta(x + z) - u^\delta(x) - 1_{|z| < 1}(z, \nabla u^\delta(x)) \right) J(dz)
\]

\[
= H_0^\psi u^\delta(x),
\]

where the symmetry of \( J(dz) \) has been used. The proof is complete. \( \square \)

Lemma A.2. (Kato’s inequality). Let \( 1 \leq q \leq \infty \). Suppose \( u \in L_q(\mathbb{R}^d) \) is such that \( H_0^\psi u \in L_1^{\text{loc}}(\mathbb{R}^d) \). Then the following distributional inequality holds:

\[
\text{sgn } u H_0^\psi u \geq H_0^\psi |u|,
\]

i.e. for any \( \varphi \in C^\infty_0(\mathbb{R}^d) \) with \( \varphi \geq 0 \),

\[
\int_{\mathbb{R}^d} (\text{sgn } u)(x)H_0^\psi u(x)\varphi(x)dx \geq \int_{\mathbb{R}^d} |u(x)|H_0^\psi \varphi(x)dx.
\]

Here \( \text{sgn } u \) is a bounded function on \( \mathbb{R}^d \) defined by

\[
(\text{sgn } u)(x) := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}
\]
Proof. First let $u \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$. By Claim A.1 $H_0^\psi u \in L^{\text{loc}}_q(\mathbb{R}^d)$ or $C(\mathbb{R}^d)$ according as $1 \leq q < \infty$ or $q = \infty$. For $\varepsilon > 0$, set $u_\varepsilon(x) := \sqrt{|u(x)|^2 + \varepsilon^2}$. Clearly $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $u_\varepsilon \geq \varepsilon$.

Since $|u(x)| |u(x+y)| \leq u_\varepsilon(x) u_\varepsilon(x+y) - \varepsilon^2$, we have

$$-|u(x)| |u(x+y)| + u(x)^2 \geq -u_\varepsilon(x) u_\varepsilon(x+y) + u_\varepsilon(x)^2.$$ 

By noting that $2u(x)\nabla u(x) = \nabla|u(x)|^2 = \nabla u_\varepsilon(x)^2 = 2u_\varepsilon(x)\nabla u_\varepsilon(x)$, this inequality gives us that

$$-u(x\{u(x+y) - u(x) - \langle y, \nabla u(x) \rangle 1_{|y|<1}$$

$$= -u_\varepsilon(x) u_\varepsilon(x+y) + u_\varepsilon(x)^2 + \langle y, u_\varepsilon(x) \nabla u_\varepsilon(x) \rangle 1_{|y|<1}$$

$$\geq -u_\varepsilon(x) u_\varepsilon(x+y) + u_\varepsilon(x)^2 + \langle y, u_\varepsilon(x) \nabla u_\varepsilon(x) \rangle 1_{|y|<1}$$

$$= -u_\varepsilon(x) \{u_\varepsilon(x+y) - u_\varepsilon(x) - \langle y, \nabla u_\varepsilon(x) \rangle 1_{|y|<1}\}.$$ 

Integrating both sides by $J(dy)$, we have $u(x)H_0^\psi u(x) \geq u_\varepsilon(x)H_0^\psi u_\varepsilon(x)$, or

$$\frac{u(x)}{u_\varepsilon(x)} H_0^\psi u(x) \geq H_0^\psi u_\varepsilon(x). \quad \text{(A.24)}$$

Second let $u \in L_q(\mathbb{R}^d)$ be such that $H_0^\psi u \in L^{\text{loc}}_1(\mathbb{R}^d)$. Since $u^\delta = u * \rho_\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$, it holds by (A.24) that

$$\int_{\mathbb{R}^d} \frac{u^\delta(x)}{(u^\delta)_\varepsilon(x)} H_0^\psi u^\delta(x) \varphi(x) \, dx \geq \int_{\mathbb{R}^d} H_0^\psi (u^\delta)_\varepsilon(x) \varphi(x) \, dx$$

$$= \int_{\mathbb{R}^d} (u^\delta)_\varepsilon(x) H_0^\psi \varphi(x) \, dx \quad \text{(A.25)}$$

for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$. In (A.25) let $\delta \downarrow 0$ first and $\varepsilon \downarrow 0$ second. As $\delta \downarrow 0$, $H_0^\psi u^\delta \rightarrow H_0^\psi u$ in $L^{\text{loc}}_1(\mathbb{R}^d)$ by Lemma A.1, and $u^\delta \rightarrow u$ in $L^{\text{loc}}_1(\mathbb{R}^d)$. By taking a subsequence if necessary we may suppose that $u^\delta \rightarrow u$ a.e. Since $|(u^\delta)_\varepsilon - u_\varepsilon| \leq |u^\delta - u|$ and $|u^\delta/(u^\delta)_\varepsilon| \leq 1$, $u^\delta/(u^\delta)_\varepsilon \rightarrow u/u_\varepsilon$ boundedly a.e. Hence, letting $\delta \downarrow 0$ in (A.25), we have

$$\int_{\mathbb{R}^d} \frac{u(x)}{u_\varepsilon(x)} H_0^\psi u(x) \varphi(x) \, dx \geq \int_{\mathbb{R}^d} u_\varepsilon(x) H_0^\psi \varphi(x) \, dx. \quad \text{(A.26)}$$

Finally, by $|u_\varepsilon - u| \leq \varepsilon$ and $|u/u_\varepsilon| \leq 1$, we obtain that $u/u_\varepsilon \rightarrow \text{sgn} u$ boundedly as $\varepsilon \downarrow 0$. Consequently, letting $\varepsilon \downarrow 0$ in (A.26) yields that

$$\int_{\mathbb{R}^d} (\text{sgn} u)(x) H_0^\psi u(x) \varphi(x) \, dx \geq \int_{\mathbb{R}^d} |u(x)| H_0^\psi \varphi(x) \, dx$$

and the proof is complete. \(\square\)

**Proof of Theorem A.1.** First consider the $L_p$-case, $1 \leq p < \infty$. It suffices to show that $\text{Im} (H_0^\psi + V + 1) = (H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d))$ is dense in $L_p(\mathbb{R}^d)$. By the Hahn-Banach theorem, this is further reduced to show the following: Let $q$ be the conjugate exponent of $p$. If $v \in L_q(\mathbb{R}^d)$ satisfies that $\langle v, (H_0^\psi + V + 1)\varphi \rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $v = 0$ in $L_q(\mathbb{R}^d)$. 

44
Let \( v \in L_q(\mathbb{R}^d) \) be as above. Then \( H_0^\psi v = -(V + 1)v \) and hence \( H_0^\psi v \in L_1^{loc}(\mathbb{R}^d) \). By Lemma A.2, it is seen that for any nonnegative \( \varphi \in C_0^\infty(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} |v(x)|H_0^\psi \varphi(x)dx \leq \int_{\mathbb{R}^d} (\text{sgn} \, v)(x)H_0^\psi v(x)\varphi(x)dx
\]

\[
= -\int_{\mathbb{R}^d} (V(x) + 1)|v(x)|\varphi(x)dx,
\]

and hence

\[
\int_{\mathbb{R}^d} |v(x)|(H_0^\psi + 1)\varphi(x)dx \leq 0.
\] (A.27)

Each \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) can be approximated by a sequence \( \{\varphi_n\}_{n=1}^\infty \) of \( C_0^\infty(\mathbb{R}^d) \) in the sense that \( \varphi_n \to \varphi \) and \( (H_0^\psi + 1)\varphi_n \to (H_0^\psi + 1)\varphi \) in \( L_p(\mathbb{R}^d) \). If \( \varphi \) is moreover nonnegative, so are \( \{\varphi_n\}_{n=1}^\infty \). Therefore (A.27) is valid for nonnegative \( \varphi \in \mathcal{S}(\mathbb{R}^d) \).

Now note that the resolvent \( (1 - \mathcal{G}_p^{\psi, 0})^{-1} \) is expressed as

\[
(1 - \mathcal{G}_p^{\psi, 0})^{-1}f(x) = \int_0^\infty e^{-t}\mathbb{E}[f(x + X_t)]dt.
\]

Then it is not difficult to check that if \( f \in \mathcal{S}(\mathbb{R}^d) \), then \( (1 - \mathcal{G}_p^{\psi, 0})^{-1}f \in \mathcal{S}(\mathbb{R}^d) \) and further, if \( f \) is nonnegative, so is \( (1 - \mathcal{G}_p^{\psi, 0})^{-1}f \). Also, by Remark following Claim A.2 (with \( V(x) \equiv 0 \)) \( f = (1 - \mathcal{G}_p^{\psi, 0})(1 - \mathcal{G}_p^{\psi, 0})^{-1}f = (H_0^\psi + 1)(1 - \mathcal{G}_p^{\psi, 0})^{-1}f \). Hence, by (A.27)

\[
\int_{\mathbb{R}^d} |v(x)|f(x)dx \leq 0,
\]

whence it immediately follows that \( v = 0 \) and the proof in the \( L_p \)-case is complete.

Next let us consider the \( C_\infty \)-case. In the same reason as above we have only to show that \( (H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d)) \) is dense in \( C_\infty(\mathbb{R}^d) \). For this let \( \nu \in C_\infty(\mathbb{R}^d)^* \), the dual of \( C_\infty(\mathbb{R}^d) \), be such that \( \langle \nu, (H_0^\psi + V + 1)\varphi \rangle = 0 \) for any \( \varphi \in C_0^\infty(\mathbb{R}^d) \). By the Riesz-Banach theorem, \( \nu \) is regarded as a finite signed Borel measure on \( \mathbb{R}^d \), and thus

\[
\int_{\mathbb{R}^d} (H_0^\psi + V + 1)\varphi(x)\nu(dx) = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^d).
\] (A.28)

Let \( \nu^\delta = \nu * \rho_\delta \), i.e., \( \nu^\delta(x) := \int_{\mathbb{R}^d} \rho_\delta(x - y)\nu(dy), x \in \mathbb{R}^d \). Then \( \nu^\delta \in C_0^\infty(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \). It follows by Claim A.1 and by (A.28) that \( H_0^\psi \nu^\delta \in C(\mathbb{R}^d) \) and

\[
H_0^\psi \nu^\delta = -\nu^\delta(1 + V) - \int_{\mathbb{R}^d} (V(y) - V(x))\rho_\delta(x - y)\nu(dy).
\]

By Lemma A.2, this implies that for nonnegative \( \varphi \in C_0^\infty(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} |\nu^\delta(x)|H_0^\psi \varphi(x)dx
\]

\[
\leq \int_{\mathbb{R}^d} (\text{sgn} \, \nu^\delta)(x)H_0^\psi \varphi(x)dx
\]

\[
\leq -\int_{\mathbb{R}^d} |\nu^\delta(x)|\varphi(x)dx + \int_{\mathbb{R}^d} \varphi(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x - y)|\nu|(dy),
\]

45
where $|\nu|$ is the total variation of $\nu$, and hence we have that for nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$
\[
\int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi(x)dx \leq \int_{\mathbb{R}^d} \varphi(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy).
\] (A.29)

Let $f \in \mathcal{S}(\mathbb{R}^d)$ be nonnegative. Then, $\varphi := (1 - \mathcal{G}_\infty^\psi) - f$ is in $\mathcal{S}(\mathbb{R}^d)$ and nonnegative, and $(H_0^\psi + V)\varphi = f$. Set $\varphi_n(x) := \varphi(x)\chi(|x|^2/n^2)$ with a $\chi \in C_0^\infty([0, \infty) \to \mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(t) = 1$ ($0 \leq t \leq 1$) and $\chi(t) = 0$ ($t \geq 2$). Clearly $\varphi_n \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \varphi_n \leq \varphi$ and supp $\varphi_n \subset \{x; |x| \leq \sqrt{2n}\}$. Moreover $\|\varphi_n - \varphi\|_\infty$ and $\|H_0^\psi \varphi_n - H_0^\psi \varphi\|_\infty \to 0$ as $n \to \infty$. From (A.29) and this observation it follows that
\[
\int_{\mathbb{R}^d} |\nu^\delta(x)|f(x)dx = \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi_n(x)dx
\]
\[
= \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi_n(x)dx + \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)(\varphi - \varphi_n)(x)dx
\]
\[
\leq \int_{\mathbb{R}^d} \varphi_n(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy)
\]
\[
+ \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty|\nu|(\mathbb{R}^d)
\]
\[
\leq \|\varphi\|_\infty \int_{|x| \leq \sqrt{2n}} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy)
\]
\[
+ \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty|\nu|(\mathbb{R}^d).
\]

Here, recalling that $\rho_\delta(z)$ has support in $\{z; |z| \leq \rho\}$, we see that for each $n \in \mathbb{N}$
\[
\int_{|x| \leq \sqrt{2n}} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy)
\]
\[
\leq \int_{|y| \leq \sqrt{2n} + \delta} |\nu|(dy) \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)dx
\]
\[
= \int_{|y| \leq \sqrt{2n} + \delta} |\nu|(dy) \int_{|x| \leq 1} |V(y) - V(y + \delta x)|\rho(x)dx
\]
\[
\to 0 \quad \text{as } \delta \downarrow 0.
\]

On the other hand, noting that $\nu^\delta(x)dx \to \nu(dx)$ weakly, we see that
\[
\int_{\mathbb{R}^d} |\nu^\delta(x)|f(x)dx \geq \int_{\mathbb{R}^d} f(x)|\nu^\delta(x)dx| \to \int_{\mathbb{R}^d} f(x)|\nu(dx)| \quad \text{as } \delta \downarrow 0.
\]

Therefore it follows that $\int_{\mathbb{R}^d} f(x)|\nu(dx)| = 0$ for $f \in \mathcal{S}(\mathbb{R}^d)$, $f \geq 0$, which implies that $\nu = 0$, and the proof in the $C_\infty$-case is complete. \hfill \Box

In this paper we have denoted the semigroups $P_t^\psi:0$ and $P_t^\psi:V$ by $e^{-tH_0^\psi}$ and $e^{-t(H_0^\psi + V)}$, respectively, taking Theorem A.1 into account. With the general theory ([Trot], [Ch]) we have taken for granted that the Trotter product formula holds in the strong topology of $L_p(\mathbb{R}^d)$ or $C_\infty(\mathbb{R}^d)$:
\[
s^{-}\lim_{n \to \infty} \left( e^{-tH_0^\psi/n}e^{-tV/n} \right)^n = s^{-}\lim_{n \to \infty} \left( e^{-tV/2n}e^{-tH_0^\psi/n}e^{-tV/2n} \right)^n
\]
\[
= s^{-}\lim_{n \to \infty} \left( e^{-tH_0^\psi/2n}e^{-tV/n}e^{-tH_0^\psi/2n} \right)^n
\]
\[
= e^{-t(H_0^\psi + V)}.
\]
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