Formality and strongly unique enhancements

Antonio Lorenzin

Universität Innsbruck, Institut für Mathematik, Technikerstraße 25, 6020 Innsbruck, Austria
E-mail address: antonio.lorenzin@uibk.ac.at

Contents

Introduction ......................................................... 1
1. Pretriangulated DG-categories. ................................. 4
2. DG-enhancements ............................................. 11
3. Triangulated formality ....................................... 15
4. Formal standardness ......................................... 21
5. Main result ................................................... 22
6. Free generators ................................................. 25
7. D-standardness and K-standardness .......................... 27
A. On bounded derived categories of exact categories .......... 30
References ......................................................... 32

Abstract  Inspired by the intrinsic formality of graded algebras, we prove a necessary and sufficient condition for strongly uniqueness of DG-enhancements. This approach offers a generalization to linearity over any commutative ring. In particular, we obtain several new examples of triangulated categories with a strongly unique DG-enhancement. Moreover, we also show that the bounded derived category of an exact category has a unique enhancement.

Keywords  triangulated categories; algebraic triangulated categories; strongly unique enhancements; DG-categories; standard categories

Acknowledgements  I am very thankful to my PhD supervisor Alberto Canonaco for his suggestions and corrections. I also wish to acknowledge the help of Francesco Genovese and Alice Rizzardo, whose suggestions clarified some arguments presented here. I would further like to thank Amnon Neeman for his insight into the proof of Proposition 3.10 and Corollary 3.11. Finally, I am grateful to Gustavo Jasso and Noah Olander for our brief discussions on their research.

Introduction

It is well-known that cones of triangulated categories are not functorial. In order to fix the issues coming from such ill behaviour, the theory has developed the idea of enhancing triangulated categories to higher categories.
A triangulated category having a DG-enhancement, i.e. obtained by a DG-category with cones, is called algebraic. As one may hope, many important examples of triangulated categories are algebraic: for instance, all derived categories and homotopy categories of complexes satisfy the definition.

However, being algebraic does not prevent a triangulated category from having a bizarre behaviour. Indeed, a priori its structure may come from different DG-categories, which are not quasi-equivalent. This gives rise to the notion of having a unique DG-enhancement, satisfied by triangulated categories with one and only one DG-shape. Of course, not all algebraic triangulated categories have a unique enhancement. An example is given by $K$-mod, the category of finite dimensional vector spaces over the field $K$. This category becomes triangulated with shift the identity and distinguished triangles generated by short exact sequences. In [Sch02], Schlichting proved that $K$-mod does not have a unique enhancement when $K = \mathbb{F}_p$ (with $p$ prime), giving two explicit examples that are not quasi-equivalent (for the transposition of the result in the DG-world, one may refer to [CS17 Corollary 3.10]). It is important to notice that one of the DG-enhancements is not $K$-linear.

The motivating question that led to the birth of this article is the following: does $K$-mod have a unique $K$-linear enhancement? The answer is yes (Example 6.7). In order to prove such result, we started by noticing that any DG-enhancement of $K$-mod is associated to the graded algebra $K[t, t^{-1}]$, where $t$ has degree 1. After some reading, we found the notion of intrinsic formality of a graded algebra (Definition 3.2), which immediately implies uniqueness of the DG-enhancement for the associated triangulated category, as proved in Proposition 3.3. The conclusion that $K[t, t^{-1}]$ is intrinsically formal follows from [Sai21 Corollary 4.2].

After this discovery, we wanted to understand how intrinsic formality relates to stricter requirements of the uniqueness of enhancements, namely strong uniqueness. This property tells us that the triangulated autoequivalences of the triangulated category come from the DG-world (cf. Proposition 2.5). Very few examples of triangulated categories with a strong unique enhancement are known, the most important one being the bounded derived categories of projective varieties, investigated by Lunts and Orlov in the celebrated article [LO10]. The procedure used to obtain such a result has been generalized for other cases: Canonaco and Stellari worked on coherent sheaves of a quasi-projective scheme supported in a projective subscheme [CS17]; Olander studied the case of smooth proper varieties over a field [Ola20] (and later expanded his result in [Ola22]); Li, Pertusi, and Zhao considered the case of Kuznetsov components [LPZ22].

Currently, there is only one explicit example of a triangulated category with a unique but not strongly unique enhancement (see [JM22 Corollary 5.4.12]). However, we do not know whether the uniqueness of enhancements implies the strong uniqueness of enhancements for derived categories or homotopy categories of complexes. It is worth noting that the examples of triangulated categories with a unique enhancement are by far more general: the most recent paper in this direction is [CNS22], where it is proved that all the derived categories and all the

---

*As it will be made explicit in Convention 0.1 all categories are assumed to be small in an appropriate Grothendieck universe.
homotopy categories of complexes over an abelian category have a unique enhancement.

For this reason, we are motivated to characterize strong uniqueness. With this aim, we define the original notions of triangulated formal DG-category, which is a relaxed version of intrinsic formality, and formally standard DG-category, inspired by D-standard and K-standard categories introduced by Chen and Ye in [CY18]. When we restrict to graded categories, we have the following.

**5.6 Theorem.** Let $\mathcal{B}$ be a graded category. The following are equivalent:

1. $\mathcal{B}$ is triangulated formal and formally standard;
2. $\text{tr}(\mathcal{B})$ has a strongly unique enhancement;
3. $\mathcal{D}(\mathcal{B})^c$ has a strongly unique enhancement.

In the statement, $\text{tr}(\mathcal{B})$ indicates the triangulated category generated by $\mathcal{B}$ and $\mathcal{D}(\mathcal{B})^c$ is the idempotent completion of $\text{tr}(\mathcal{B})$. In fact, the implication $1 \Rightarrow 2, 3$ holds for more general DG-categories (see Proposition 5.1 and Remark 5.2). It is very important to highlight that these results do not require linearity over a field (see Convention 0.1).

First simple applications of the result are discussed in Section 6. In Corollary 6.6 and Example 6.7 we deal with particular cases of periodic triangulated categories, i.e. triangulated categories such that $[n] \cong \text{id}$ for some integer $n$ (see [Sai21]). In Section 7 we show that K-standardness and D-standardness (Definition 7.1) are linked to formal standardness, and prove the following statements.

**7.3 Proposition.** An additive category $\mathcal{A}$ is K-standard if and only if $\mathcal{K}^b(\mathcal{A})$ has a strongly unique enhancement.

**7.7 Theorem.** An exact category $\mathcal{E}$ is D-standard if and only if $\mathcal{D}^b(\mathcal{E})$ has a strongly unique enhancement.

These results follow from the fact that $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{E})$ have a (semi-strongly) unique enhancement for every choice of $\mathcal{A}$ additive and $\mathcal{E}$ exact (see Proposition 3.9 and Proposition 3.12). We emphasize that the theorem above extends [CY18, Theorem 5.10], which is valid for the category of finitely generated modules over a finite-dimensional $k$-algebra, with $k$ a field.

As an application, all bounded derived categories of hereditary abelian categories have a strongly unique enhancement (Corollary 7.9). For instance, $\mathcal{D}^b(\text{Mod}(\mathbb{Z}))$, the bounded derived category of abelian groups, has a strongly unique enhancement. Moreover, all geometric examples provided by the theory show D-standardness in order to conclude that the derived category has a strongly unique enhancement (see [LO10, CS17, Ola20, Ola22, LPZ22]). This follows by proving that the abelian category has an almost ample set, which is a generalization of the ample sequences introduced by Orlov in [Orl97].

**7.13 Proposition.** Let $\mathcal{A}$ be an abelian category with an almost ample set. Then $\mathcal{D}^b(\mathcal{A})$ has a strongly unique enhancement.

**Overview** In Section 1 and Section 2 we recall some basic notions and results of pretriangulated DG-categories and DG-enhancements. Section 3 is devoted to the notion of triangulated
formality of DG-categories, which generalizes intrinsic formality of graded algebras, while Section 4 describes formal standardness. The main result, linking triangulated formality and formal standardness to strong uniqueness of DG-enhancements, is proved in Section 5; applications are discussed in Section 6 and Section 7. Appendix A deals with bounded derived categories of exact categories.

0.1. Convention. All categories are assumed to be $\mathcal{U}$-small for an appropriate Grothendieck universe $\mathcal{U}$.

Moreover, we consider everything to be $\mathbb{k}$-linear, where $\mathbb{k}$ is a commutative ring. We reserve the capital letter $K$ for fields. We adopt the term ring when linearity is considered over a commutative ring, and algebra for linearity over a field. In particular, we will talk about graded rings and graded algebras in the two contexts.

1. Pretriangulated DG-categories

We start by recalling some basics on pretriangulated DG-categories. The non-expert reader may refer to [CC21, Kel06, Toë11, Yek20] for detailed treatments on DG-categories.

1.1. Definition. A DG-category is a $\mathbb{k}$-linear category $\mathcal{C}$ such that $\text{Hom}(X,Y)$ is a $\mathbb{Z}$-graded $\mathbb{k}$-module together with a differential $d : \text{Hom}(X,Y) \to \text{Hom}(X,Y)$ of degree 1 for any $X,Y \in \mathcal{C}$. The compositions $\text{Hom}(X,Y) \otimes \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$ are morphisms of complexes $(X,Y,Z \in \mathcal{C})$.

A morphism $f : X \to Y$ in a DG-category is closed if $d(f) = 0$, whereas it is homogeneous if it belongs to $\text{Hom}(X,Y)^i \subset \text{Hom}(X,Y)$, the set of morphisms of degree $i$, for some $i \in \mathbb{Z}$. We will use $|\cdot|$ to denote the degree of a homogeneous morphism.

A DG-isomorphism is a closed morphism of degree 0 with an inverse.

1.2. Definition. A functor $f : \mathcal{C} \to \mathcal{D}$ between DG-categories is a DG-functor if, for every $X,Y \in \mathcal{C}$,

$$f_{X,Y} : \text{Hom}(X,Y) \to \text{Hom}(FX,FY)$$

is a morphism of degree 0 commuting with the differentials. We say that a DG-functor $f$ is a DG-equivalence if it is fully faithful and every object of $\mathcal{D}$ is DG-isomorphic to $f(X)$ for some $X \in \mathcal{C}$.

1.3. Definition. Let $\mathcal{C}, \mathcal{D}$ be two DG-categories. We define the functor DG-category, denoted by $\mathcal{H}om(\mathcal{C}, \mathcal{D})$, as follows:

- The objects are the DG-functors $f : \mathcal{C} \to \mathcal{D}$;
- A morphism $\eta : f \to g$ of degree $i$ is a family of morphisms $(\eta_X : fX \to gX)_{X \in \mathcal{C}}$ of degree $i$ in $\mathcal{D}$ such that

$$g(h)\eta_X = (-1)^{|\eta||h|}\eta_f(h)$$
for any homogeneous $h : X \to Y$. With this definition, we can create $\text{Hom}(f, g)$ as a graded module whose $i$-th component is the set of all morphisms of degree $i$. The differential $(d\eta)_X := d(\eta_X)$ gives $\mathcal{H}om(C, D)$ the structure of a DG-category.

Morphisms of $\mathcal{H}om(C, D)$ are also called DG-natural transformations. Similarly, its DG-isomorphisms are also called DG-natural isomorphisms.

1.5. Remark. One may show that a DG-functor $f : C \to D$ is a DG-equivalence if and only if there exists an inverse up to DG-natural isomorphisms. This result is analogous to the classical one for the equivalence of categories.

1.6. Definition. Let $C$ be a DG-category. The opposite DG-category $C^o$ has the same objects of $C$, morphisms have opposite direction $\text{Hom}_{C^o}(X, Y) := \text{Hom}_C(Y, X)$, and composition is reversed up to a change of sign: $f \circ_C g := (-1)^{|f||g|} g \circ_C f$, where $f, g$ are homogeneous.

1.7. Definition. The homotopy category $H^0(C)$ of a DG-category $C$ has the same objects of $C$ and the morphisms are defined by $\text{Hom}_{H^0(C)}(X, Y) := H^0(\text{Hom}_C(X, Y))$. Two objects $X, Y$ in $C$ are homotopy equivalent if they are isomorphic in $H^0(C)$.

The graded homotopy category $H^*(C)$ has the same objects of $C$ and $\text{Hom}_{H^*(C)}(X, Y) := H^*(\text{Hom}_C(X, Y))$.

Any DG-functor $f : C \to D$ gives rise to functors $H^0(f) : H^0(C) \to H^0(D)$ and $H^*(f) : H^*(C) \to H^*(D)$.

1.8. Definition. Let $C$ be a DG-category. Its truncation is the couple $(\tau_{\leq 0}C, p_{\leq 0})$, where $\tau_{\leq 0}C$ is the DG-category whose objects are the same of $C$ and

$$\text{Hom}_{\tau_{\leq 0}C}(X, Y)^n := \begin{cases} 
\text{Hom}(X, Y)^n & \text{if } n < 0 \\
\text{ker}^0_{\text{Hom}(X, Y)} & \text{if } n = 0 \\
0 & \text{if } n > 0
\end{cases}$$

for every $X, Y \in C$, while $p_{\leq 0} : \tau_{\leq 0}C \to C$ is the natural DG-functor which is the identity on objects and the inclusion on morphisms.

1.9. Remark. Dually, one may expect to define the truncation $(\tau_{\geq 0}C, p_{\geq 0}) : C \to \tau_{\leq 0}C)$. However, $p_{\geq 0}$ is not always well-defined: pick two composable homogeneous morphisms $f$ and $g$, respectively of degree $-i$ and $n + i$ with $n, i > 0$, such that $gf \neq 0$. Then $gf = p_{\geq 0}(gf) \neq p_{\geq 0}(g)p_{\geq 0}(f) = p_{\geq 0}(g)0 = 0$.

Despite such situation, notice that in the case of $\tau_{< 0}C$, the left truncation exists because the DG-functor $\tau_{< 0}C \to H^0(C)$ is actually well-defined, as the obstruction above cannot happen.

1.10. Remark. Given a DG-functor $f : C \to D$, we have a natural DG-functor $\tau_{\leq 0}f : \tau_{\leq 0}C \to \tau_{\leq 0}D$ satisfying $f_{p_{\leq 0}} = p_{\leq 0}\tau_{\leq 0}f$.

\[\text{The author would like to thank Amnon Neeman for pointing out the impossibility of this dual truncation.}\]
1.11. Definition. Let \( \mathcal{A} \) be a \((k\text{-linear})\) category. The \textit{DG-category of complexes (of } \mathcal{A} \text{)}, denoted with \( \mathcal{C}_{DG}(\mathcal{A}) \), is described as follows:

- Its objects are complexes \( M = (M_i, d_i)_{i \in \mathbb{Z}} \), i.e. sequences

\[
\ldots \rightarrow M^i \rightarrow M^{i-1} \rightarrow \ldots
\]

with \( \{ M_i \}_{i \in \mathbb{Z}} \subset \mathcal{A} \) and \( d_i M_i \rightarrow M_{i-1} = 0 \) for all \( i \).

- Given \( M, N \) complexes, the \( \ell \)-th homogeneous morphisms are

\[
\text{Hom}_{\mathcal{C}_{DG}(\mathcal{A})}(M, N) = \prod_{i \in \mathbb{Z}} \text{Hom}_\mathcal{A}(M_i, N_{i+\ell}).
\]

The differential is defined by

\[
d(f) := d_N \circ f - (-1)^{|f|} f d_M
\]

for any homogeneous element \( f \).

Analogously, we can define the \textit{DG-category of bounded complexes} \( \mathcal{C}^b_{DG}(\mathcal{A}) \) as the full DG-subcategory of \( \mathcal{C}_{DG}(\mathcal{A}) \) whose objects are bounded complexes, i.e. \( M = (M_i, d_i) \) with \( M_i = 0 \) for \( |i| \gg 0 \).

An important example is given by \( \text{Mod}(k) \), the category of all \( \mathbb{k} \)-modules; \( \mathcal{C}_{DG}(\text{Mod}(k)) \) is called the \textit{category of DG-modules}.

1.12. Definition. Let \( \mathcal{C} \) be a DG-category. The \textit{DG-category of (right) DG \( \mathcal{C} \)-modules} is defined by

\[
\text{DGMod}(\mathcal{C}) := \text{Hom}(\mathcal{C}^\text{op}, \mathcal{C}_{DG}(\text{Mod}(\mathbb{k}))).
\]

1.13. Lemma – Yoneda DG-embedding. The \textit{Yoneda embedding}

\[
y : \mathcal{C} \to \text{DGMod}(\mathcal{C}) : X \mapsto \text{Hom}_\mathcal{C}(-, X)
\]

is a fully faithful DG-functor.

1.14. Definition. Let \( \mathcal{C} \) be a DG-category. Given \( X \in \mathcal{C} \), its \textit{suspension} is an object \( Y \in \mathcal{C} \) equipped with two closed morphisms \( \sigma_X : Y \to X \) of degree 1 and \( \tau_X : X \to Y \) of degree \(-1\) such that \( \tau_X \sigma_X = \text{id}_X \) and \( \sigma_X \tau_X = \text{id}_X \). In this case, we will write \( \Sigma(X) := Y \), since \( Y \) is determined up to unique DG-isomorphism.

1.15. Definition. Let \( f : X \to Y \) be a closed morphism of degree 0 in a DG-category \( \mathcal{C} \). The \textit{cone of } \( f \) \text{ is an object } \text{Cone}_{DG}(f) \text{ together with 4 morphisms of degree 0}

\[
\begin{array}{ccc}
Y & \xrightarrow{j} & \text{Cone}_{DG}(f) \\
\mathbb{R} & \xrightarrow{q} & \Sigma(X)
\end{array}
\]

satisfying

\[
p_i = \text{id}, \quad q_j = \text{id}, \quad q_i = 0, \quad p_j = 0, \quad ip + jq = \text{id}
\]
and
\[ d(j) = d(p) = 0, \quad d(i) = jf\sigma_X, \quad d(q) = -f\sigma_Xp. \]

We emphasize that \( j \) and \( p \) are required to be closed, while \( q \) and \( i \) are generally not.

1.16. Definition. A DG-category \( C \) is strongly pretriangulated if it is closed under cones and suspensions, meaning that:

- Any object \( X \in C \) has a suspension \( Y \in C \), and there exists \( Z \in C \) such that \( X \) is a suspension for \( Z \);
- Any closed morphism of degree 0 has a cone.

The previous definitions are inspired by the structure of DG-modules. In fact, one can prove the following well-known result.

1.17. Proposition. For any DG-category \( C \), \( \text{DGMod}(C) \) is strongly pretriangulated.

1.18. Definition. For any DG-category \( C \), its pretriangulated closure is the smallest full DG-subcategory \( C_{\text{pretr}} \) of \( \text{DGMod}(C) \) that contains \( y(C) \) and is strongly pretriangulated. In particular, notice that the Yoneda embedding factors through \( C_{\text{pretr}} \). By an abuse of notation, we call \( y : C \hookrightarrow C_{\text{pretr}} \) the natural embedding so obtained. One may prove that a DG-category \( C \) is strongly pretriangulated if and only if \( y : C \rightarrow C_{\text{pretr}} \) is a DG-equivalence (see [CC21] Lemma 6.2.2).

1.19. Remark. Given a DG-functor \( f : C \rightarrow D \), we have an induced DG-functor \( f_{\text{pretr}} : C_{\text{pretr}} \rightarrow D_{\text{pretr}} \) given by restricting \( \text{Ind}(f) : \text{DGMod}(C) \rightarrow \text{DGMod}(D) \), defined in [Dri04, C.9].

1.20. Example. A direct check shows that \( C_{\text{DG}}(A) \) is a strongly pretriangulated DG-category for any additive category \( A \). The same holds for \( C_{b,\text{DG}}(A) \).

Here we strengthen [BLL04, Proposition 4.11].

1.21. Proposition. Let \( C, D \) be DG-categories, with \( D \) strongly pretriangulated. Then the embedding \( y : C \hookrightarrow C_{\text{pretr}} \) induces a DG-equivalence \( \mathcal{H}om(C_{\text{pretr}}, D) \rightarrow \mathcal{H}om(C, D) \).

More explicitly,

(Essential surjectivity). Any DG-functor \( f : C \rightarrow D \) admits an extension \( f' : C_{\text{pretr}} \rightarrow D \).

(Fully faithfulness). Let us consider another DG-functor \( g : C \rightarrow D \) and an extension \( g' : C_{\text{pretr}} \rightarrow D \). Then any DG-natural transformation \( \eta : f \rightarrow g \) extends to a unique DG-natural transformation \( \eta' : f' \rightarrow g' \). In particular, if \( \eta \) is a DG-natural isomorphism, then \( \eta' \) is a DG-natural isomorphism.

PROOF. An extension \( C_{\text{pretr}} \rightarrow D \) is given by \( y^{-1}f_{\text{pretr}} \), since \( y : D \rightarrow D_{\text{pretr}} \) is a DG-equivalence by definition.

For the sake of simplicity, assume \( D = D_{\text{pretr}} \). We aim to prove that \( \eta \) can be extended to \( \eta' \). For suspensions, we consider \( f'(\Sigma M) \) for \( M \in C \). Then \( f'(\sigma_M) \) and \( f'(\tau_M) \) show that \( f'(\Sigma M) \cong \Sigma(fM) \) (via unique DG-isomorphism). As the same can be said about \( g' \), we define \( \eta_{\text{pretr}} : \Sigma(fM) \rightarrow \Sigma(gM) \) and check that it is a DG-natural isomorphism by extension of the DG-natural isomorphism \( \eta \).

\[ \eta_{\text{pretr}} := \]
The morphisms $\sigma_M$ and $\tau_M$ prove that $\eta$ cannot have another extension, so this is well-defined even if $\Sigma M \in C$. Therefore, from now on we may assume that $C$ is closed under suspensions.

Let $\phi_h : M_h \to N_h$ be a closed morphism of $C$, with $h = 1, 2$. We consider $\text{Cone}_{\text{DG}}(\phi_h)$ with the following notation

$$N_h \xrightarrow{j_h} \text{Cone}_{\text{DG}}(\phi_h) \xrightarrow{p_h} \Sigma M_h$$

and define $\eta'_{\text{Cone}_{\text{DG}}(\phi_h)} := g'(j_h) \eta_N f'(q_h) + g(j_h) \eta_{\Sigma M} f'(p_h)$. In order to conclude the proof, we aim to prove that for any homogeneous morphism $\nu : \text{Cone}_{\text{DG}}(\phi_1) \to \text{Cone}_{\text{DG}}(\phi_2)$, the requirement (1.4) of DG-natural transformation holds. This will be done in two steps.

First of all, we restrict the claim to $\mu : P \to \text{Cone}_{\text{DG}}(\phi_2)$ with $P \in C$. For the sake of simplicity, in the following computations we avoid the subscript $2$.

$$\eta'_{\text{Cone}_{\text{DG}}(\phi_2)} f' = g'(j) \eta_N f'(q) f'(\mu) + g'(i) \eta_{\Sigma M} f'(p) f'(\mu)$$

$$= g'(j) \eta_N f'(q\mu) + g'(i) \eta_{\Sigma M} f'(p\mu)$$

$$= (\nu \cdot \eta|_{\mu|\nu} - \nu \cdot \eta|_{\mu|\nu}) f'(\mu)$$

We are now able to address the general situation of $\nu : \text{Cone}_{\text{DG}}(\phi_1) \to \text{Cone}_{\text{DG}}(\phi_2)$:

$$g'(v) \eta'_{\text{Cone}_{\text{DG}}(\phi_2)} = g'(v) \eta'_{\text{Cone}_{\text{DG}}(\phi_2)} f'(q) + g'(v) \eta_{\Sigma M} f'(p)$$

$$= (\nu \cdot \eta|_{\mu|\nu} - \nu \cdot \eta|_{\mu|\nu}) \eta'_{\text{Cone}_{\text{DG}}(\phi_2)} f'(q) + (\nu \cdot \eta|_{\mu|\nu} - \nu \cdot \eta|_{\mu|\nu}) \eta'_{\text{Cone}_{\text{DG}}(\phi_2)} f'(p)$$

It is important to notice that the same passages also entail $\eta'_{\text{Cone}_{\text{DG}}(\phi_2)} = \eta'_{\text{Cone}_{\text{DG}}(\phi_2)}$ whenever $\text{Cone}_{\text{DG}}(\phi_h) \in C$. The last sentence of the statement follows by the definition of $\eta'$.

1.22. Definition. A DG-functor $f : C \to D$ is

- quasi-fully faithful if the induced graded functor $H^* (f) : H^* (C) \to H^* (D)$ is fully faithful;
- quasi-essentially surjective if $H^0 (f) : H^0 (C) \to H^0 (D)$ is essentially surjective;
- a quasi-equivalence if it is quasi-fully faithful and quasi-essentially surjective.

From now on, $\simeq$ is used to indicate that a DG-functor is a quasi-equivalence.

1.23. Remark. If a DG-functor $f : C \to D$ is a quasi-equivalence, then also $f^{\text{pretr}}$ is a quasi-equivalence (see [Dri04 Proposition 2.5]). Similarly, if $f$ is fully faithful, then $f^{\text{pretr}}$ is fully faithful as well.

Moreover, using suspensions, one can show that a DG-functor $f$ between strongly pretriangulated DG-categories is a quasi-equivalence if and only if $H^0 (f)$ is an equivalence.
1.24. Definition. A quasi-functor \( f : C \to D \) is a (suitable equivalence class of) zig-zag of DG-functors

\[
\begin{array}{ccc}
C & \rightsquigarrow & C_1 \leftarrow \cdots \rightarrow C_2 \leftarrow \cdots \rightarrow D.
\end{array}
\]

A quasi-equivalence quasi-functor is a quasi-functor given by a zig-zag of quasi-equivalences. Two DG-categories are quasi-equivalent if there exists a quasi-equivalence quasi-functor. In particular, a quasi-functor is a morphism of the homotopy category with respect to the model structure on the category of DG-categories with weak equivalences the quasi-equivalences (cf. \[Hov99\] Definition 1.2.1 and \[Tab05\]).

1.25. Remark. According to Remark 1.19 and Remark 1.23, for every quasi-functor \( f : C \to D \) we have an induced quasi-functor \( f^\pretr : C^\pretr \to D^\pretr \).

1.26. Remark. The only quasi-functor \( f : A \to B \) such that \( H^0(f) = 0 \) is the trivial one.

Indeed, if \( H^0(f) = 0 \), then the image of \( f \) is quasi-equivalent to 0. We conclude that \( f = 0 \) by the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \text{im}(f) \simeq 0 \\
\downarrow & & \downarrow \\
B & & B.
\end{array}
\]

1.27. Definition. A DG-category \( C \) is pretriangulated if \( H^0(y) : H^0(C) \to H^0(C^\pretr) \) is an equivalence.

1.28. Proposition. Let \( C \) be a pretriangulated DG-category. Then \( H^0(C) \) is a triangulated category (cf. \[Yek20\] Corollary 5.4.14).

1.29. Remark. The second part of Remark 1.23 can be generalized to quasi-functors between pretriangulated DG-categories using \( H^0(y) \).

1.30. Definition. Let \( C \) be a DG-category. The DG-category of semi-free DG-modules \( \text{SF}(C) \) is the full DG-subcategory of \( \text{D} \text{GMod}(C) \) whose objects have a filtration of free DG \( C \)-modules, i.e.

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots = M
\]

such that \( M_i/M_{i-1} \) is isomorphic to a direct sum of \( \Sigma^n y(X) \) for some \( n \in \mathbb{Z} \) and \( X \in C \).

\[\text{In the literature, quasi-functors are generally a zig-zag of DG-functors as above, and our definition of quasi-functor is often called an isomorphism class of quasi-functors.}\]
1.31. Definition. Let $C$ be a DG-category. The *DG-category of perfect complexes* $\text{Perf}(C)$ is the full DG-subcategory of $\text{SF}(C)$ whose objects are homotopy equivalent to a direct summand of $C^\text{pretr}$.

1.32. Remark. In the same fashion of Remark [1.19] any DG-functor $f : C \to D$ admits extensions $\text{Perf}(f) : \text{Perf}(C) \to \text{Perf}(D)$ and $\text{SF}(f) : \text{SF}(C) \to \text{SF}(D)$, given by restricting the DG-functor $\text{Ind}(f) : \text{DGMod}(C) \to \text{DGMod}(D)$, defined in [Dri04, C.9].

1.33. Remark. The following chain of inclusions of strongly pretriangulated DG-categories holds:

$$C^\text{pretr} \subset \text{Perf}(C) \subset \text{SF}(C) \subset \text{DGMod}(C).$$

1.34. Notation. Let $C$ be a DG-category. We will use the following notation:

$$\text{tr}(C) := H^0(C^\text{pretr}), \quad D(C)^c := H^0(\text{Perf}(C)), \quad D(C) := H^0(\text{SF}(C)).$$

1.35. Remark. The notation $D(C)^c$ comes from the fact that $H^0(\text{Perf}(C))$ is given by all the compact objects of $D(C)$ ([Kel94]). We recall that an object $X$ in a triangulated category with all coproducts is compact if $\text{Hom}(X, -)$ commutes with coproducts.

Another important fact to keep in mind is that $D(C)^c$ is the idempotent closure of $\text{tr}(C)$.

1.36. Definition/Proposition. [Dri04 Theorem 1.6.2]. Let $C$ be a DG-category and $B \subset C$ a full DG-subcategory. A *DG-quotient*, often denoted by $C/B$, is a DG-category $D$ together with a quasi-functor $q : C \to D$ satisfying the following equivalent properties:

1. The functor $H^0(q)$ is essentially surjective and $H^0(q^\text{pretr})$ induces a triangulated equivalence $\text{tr}(C)/\text{tr}(B) \to \text{tr}(D)$.

2. For every DG-category $K$, the category of quasi-functors $D \to K$ is equivalent by composition to the category of quasi-functors $C \to K$ such that $B \to C \to K$ is zero (see [Dri04 Appendix E] for a discussion on these categories).

The DG-quotient is determined up to quasi-equivalence, i.e. given another DG-quotient $D'$ with $q' : C \to D'$, we can find a quasi-equivalence quasi-functor $f : D \to D'$ such that $q' \cong fq$.

1.37. Remark. With the same notation above, we can choose $D$ so that $q$ becomes a DG-functor. Additionally, if $C$ is pretriangulated, then so is $D$. The reader may refer to [LO10 Remark 1.4 and Lemma 1.5].

1.38. Example – Main examples.

- Let $\mathcal{A}$ be any additive category. Let us consider $C_{\text{DG}}(\mathcal{A})$, whose homotopy category is the homotopy category of complexes $K(\mathcal{A})$. By Proposition [1.21] and Remark [1.23] the inclusion $\mathcal{A} \to C_{\text{DG}}(\mathcal{A})$ extends to a fully faithful DG-functor $\mathcal{A}^\text{pretr} \to C_{\text{DG}}(\mathcal{A})$ because $C_{\text{DG}}(\mathcal{A})$ is strongly pretriangulated (as noted in Example [1.20]). Therefore, $\text{tr}(\mathcal{A})$ is equivalent to a triangulated subcategory of $K(\mathcal{A})$. Moreover, it is the triangulated
This suffices to conclude that \( \text{tr}(\mathcal{A}) \cong \mathcal{K}^b(\mathcal{A}) \), since \( \mathcal{K}^b(\mathcal{A}) \) is the triangulated envelope of \( \mathcal{A} \) in \( \mathcal{K}(\mathcal{A}) \). In addition, \( \mathcal{C}^b_{\text{DG}}(\mathcal{A}) \) is quasi-equivalent to \( \mathcal{A}^{\text{pretr}} \).

For a general (\( \mathbb{A} \)-linear) category \( \mathcal{A} \), the same argument proves that \( \text{tr}(\mathcal{A}) \) is equivalent to the bounded homotopy category of complexes in its additive closure.

- Let \( \mathcal{E} \) be an exact category. Let \( \text{Ac}_{\text{DG}}^b(\mathcal{E}) \) be the full DG-subcategory of \( \mathcal{C}^b_{\text{DG}}(\mathcal{E}) \) whose objects are acyclic complexes. Consider the DG-quotient \( \mathcal{D}_{\text{DG}}^b(\mathcal{E}) := \mathcal{C}^b_{\text{DG}}(\mathcal{E}) / \text{Ac}_{\text{DG}}^b(\mathcal{E}) \). Its homotopy category is equivalent to \( \mathcal{D}^b(\mathcal{E}) \) since

  \[
  H^0(\mathcal{D}_{\text{DG}}^b(\mathcal{E})) \cong H^0(\mathcal{C}^b_{\text{DG}}(\mathcal{E}))/H^0(\text{Ac}_{\text{DG}}^b(\mathcal{E})) = \mathcal{K}^b(\mathcal{E}) / \text{Ac}^b(\mathcal{E}),
  \]

and \( \mathcal{D}^b(\mathcal{E}) \) is by definition the Verdier quotient \( \mathcal{K}^b(\mathcal{E}) / \text{Ac}^b(\mathcal{E}) \).

## 2. DG-enhancements

In this section, we state some properties and results about DG-enhancements. In particular, we introduce the notion of lift to discuss the relation between strong uniqueness of enhancements and autoequivalences.

### 2.1. Definition

Let \( \mathcal{T} \) be a triangulated category. A (DG-)enhancement is a couple \((C,E)\) where \( C \) is a pretriangulated DG-category and \( E : H^0(C) \to \mathcal{T} \) is a triangulated equivalence. If a triangulated category admits a (DG-)enhancement, it is called algebraic.

### 2.2. Definition

An algebraic triangulated category \( \mathcal{T} \) has a unique (DG-)enhancement if, given two enhancements \((C,E)\) and \((C',E')\), there exists a quasi-equivalence quasi-functor \( f : C \to C' \).

In other words, there exists a zig-zag of quasi-equivalences

\[
\begin{array}{ccc}
C & \xleftarrow{\sim} & D_1 \\
\downarrow & & \downarrow \\
D_2 & \xleftarrow{\sim} & \ldots \\
\downarrow & & \downarrow \\
& \xleftarrow{\sim} & C'
\end{array}
\]

### 2.3. Definition

An algebraic triangulated category \( \mathcal{T} \) has a strongly unique enhancement (respectively, semi-strongly unique) if, given two enhancements \((C,E)\) and \((C',E')\), there exists a quasi-equivalence quasi-functor \( f : C \to C' \) such that \( E \cong E'H^0(f) \) (respectively, \( E(X) \cong E'H^0(f)(X) \) for all \( X \in C \)).

We want to interpret strong and semi-strong uniqueness of enhancements via autoequivalences of the triangulated category. Let us start with an unconventional definition.

### 2.4. Definition

Let \( F : \mathcal{T} \to \mathcal{T}' \) be a triangulated functor between algebraic triangulated categories. Given enhancements \((C,E)\) and \((C',E')\) of \( \mathcal{T} \) and \( \mathcal{T}' \) respectively, a \((C,E) - (C',E')\)-lift of \( F \) is a quasi-functor \( f : C \to C' \) such that \( F \cong E'H^0(f)E^{-1} \). If \( \mathcal{T} = \mathcal{T}' \) and \((C',E') = (C,E)\),

\[\text{[The notion of triangulated envelope is defined in [HILLM Section 2].}]

we will say that $f$ is a $(C, E)$-lift of $F$. If for every enhancement $(C, E)$ there exists a $(C, E)$-lift for $F$, we say that $F$ has a good DG-lift.

Similarly, a $(C, E) - (C', E')$-semilift of $F$ is a quasi-functor $f : C \to C'$ such that $F(X) \cong E'H^0(f)E^{-1}(X)$ for any $X \in \mathcal{T}$. In the same fashion as above, we define $(C, E)$-semilift and good DG-semilift.

2.5. Proposition. Let $\mathcal{T}$ be an algebraic triangulated category with a unique enhancement. The following are equivalent.

1. $\mathcal{T}$ has a strongly unique enhancement.
2. Every triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a good DG-lift.
3. There exists an enhancement $(C, E)$ such that any triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a $(C, E)$-lift.
4. There exist two enhancements $(C, E)$ and $(C', E')$ such that any triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a $(C, E) - (C', E')$-lift.

Proof. 1 $\Rightarrow$ 4. It suffices to take some arbitrary enhancements $(C, FE)$ and $(C', E')$. Indeed, by hypothesis, we can find $f$ such that $FE \cong E'H^0(f)$, so $F \cong E'H^0(f)E^{-1}$.

4 $\Rightarrow$ 3. Let us consider a quasi-equivalence quasi-functor $g : C' \to C$. This gives an equivalence $G = E'H^0(g)(E')^{-1} : \mathcal{T} \to \mathcal{T}$. We now consider a $(C, E) - (C', E')$-lift for $G^{-1}F$. The quasi-functor $gh : C \to C$ is then a $(C, E)$-lift for $F$:

$$E'H^0(gh)E^{-1} \cong E'H^0(g)H^0(h)E^{-1} \cong E'H^0(g)(E')^{-1}E'H^0(h)E^{-1} \cong GG^{-1}F \cong F$$

3 $\Rightarrow$ 2. Let $(C', E')$ be another enhancement and let $g : C' \to C$ be a quasi-equivalence quasi-functor. We denote with $L$ the autoequivalence $\mathcal{T} \to \mathcal{T}$ given by $E'H^0(g)(E')^{-1}$. From assumption, $LFL^{-1}$ has a $(C, E)$-lift $h$. The quasi-functor $g^{-1}hg : C' \to C'$ is the wanted $(C', E')$-lift:

$$E'H^0(g^{-1}hg)(E')^{-1} \cong E'H^0(g^{-1})H^0(h)H^0(g)(E')^{-1} \cong L^{-1}E'H^0(h)E^{-1}L \cong L^{-1}LFL^{-1}L \cong F.$$ 

2 $\Rightarrow$ 1. Let $(C, E)$ and $(C', E')$ be two enhancements. By assumption, we have a quasi-equivalence quasi-functor $g : C \to C'$. Now we consider the equivalence $E'H^0(g)^{-1}(E')^{-1} : \mathcal{T} \to \mathcal{T}$. Since it has a good DG-lift, we can consider its $(C, E)$-lift $f$, satisfying $E'H^0(f)E^{-1} \cong E'H^0(g)^{-1}(E')^{-1}$. From this isomorphism of functors, we obtain $E'H^0(gf) \cong E$. 

Analogously to the previous proposition, one can show the following.

2.6. Proposition. Let $\mathcal{T}$ be an algebraic triangulated category with a unique enhancement. The following are equivalent.
1. $\mathcal{T}$ has a semi-strongly unique enhancement.
2. Every triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a good DG-semilift.
3. There exists an enhancement $(C, E)$ such that any triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a $(C, E)$-semilift.
4. There exist two enhancements $(C, E)$ and $(C', E')$ such that any triangulated autoequivalence $F : \mathcal{T} \to \mathcal{T}$ has a $(C, E) - (C', E')$-semilift.

2.7. Notation. Let $(C, E)$ be an enhancement of a triangulated category $\mathcal{T}$, and let $\mathcal{S} \subset \mathcal{T}$ be a full subcategory (not necessarily triangulated). With the notation $C_{\mathcal{S}}$ we mean the full DG-subcategory of $C$ whose objects $X$ are such that $E(X) \cong Y \in \mathcal{S}$. In particular, $EH^0(C_{\mathcal{S}})$ is equivalent to $\mathcal{T}$. When there is no misunderstanding, we simply write $C_{\mathcal{S}}$ instead of $C_{\mathcal{S}}$.

2.8. Lemma. Let $(C, E)$ be an enhancement of a triangulated category $\mathcal{T}$ and consider $\mathcal{S} \subset \mathcal{T}$ a full subcategory. Then $C_{\mathcal{S}}$ is closed under homotopy equivalence in $C$.

PROOF. Let $Y \in C$ be homotopy equivalent to $X \in C_{\mathcal{S}}$. In other words, $X$ and $Y$ are isomorphic in $H^0(C)$. Since an equivalence sends isomorphisms to isomorphisms, $E(Y) \cong E(X)$. By definition, there exists $Z \in \mathcal{T}$ such that $E(X) \cong Z$. We obtain $E(Y) \cong Z$, which implies that $Y \in C_{\mathcal{S}}$.

We now investigate some relations between uniqueness of enhancements of the triangulated categories associated to a DG-category. We start with a technical lemma.

2.9. Lemma. Let $\mathcal{T}$ be a triangulated category and $A$ be a DG-category. If

$$F : \mathcal{T} \to H^0(DGMod(A))$$

is a full triangulated functor and $H^0(A) \subset \text{EssIm}(F)$, then $\text{tr}(A) \subset \text{EssIm}(F)$.

Moreover, if $\mathcal{T}$ is idempotent complete and $G : \mathcal{T} \to H^0(DGMod(A))$ is a fully faithful triangulated functor such that $H^0(A) \subset \text{EssIm}(G)$, then $D(A) \subset \text{EssIm}(G)$.

PROOF. Let $X \in \text{tr}(A)$. We aim to show that $X \in \text{EssIm}(F)$. If $X \in H^0(A)$, this is true by hypothesis. Therefore, it suffices to show that $\text{EssIm}(F)$ is closed under shifts and cones (cf. Definition 1.18). A trivial reasoning shows that $X \in \text{EssIm}(F)$ if it is the shift of an object in $\text{EssIm}(F)$. It remains to study the case $X = \text{Cone}(f)$ for a morphism $f : Y_1 \to Y_2$ where $Y_i \in \text{EssIm}(F)$ for $i = 1, 2$. Notice there exist an object $Z_i \in \mathcal{T}$ and an isomorphism $\varphi_i : FZ_i \to Y_i$. Since $F$ is full, we can find $g : Z_1 \to Z_2$ such that $Fg = \varphi_2^{-1}f\varphi_1$. Then $X \cong \text{Cone}(Fg) \cong F(\text{Cone}(g)) \in \text{EssIm}(F)$.

When $\mathcal{T}$ is idempotent complete and $G : \mathcal{T} \to H^0(DGMod(A))$ is a fully faithful triangulated functor, $\text{tr}(A) \subset \text{EssIm}(G)$ directly implies that $D(A) \subset \text{EssIm}(G)$.

2.10. Remark. It directly follows from Lemma 2.9 that every fully faithful triangulated functor $F : \text{tr}(A) \to \text{tr}(A)$ with $H^0(A) \subset \text{EssIm}(F)$ is an equivalence. Analogously, any fully faithful triangulated functor $G : D(A) \to D(A)$ with $H^0(A) \subset \text{EssIm}(G)$ is an equivalence. In particular,
in this situation $G_{||A||} : \text{tr}(A) \to G(\text{tr}(A))$ is an equivalence and $G$ is its idempotent extension up to natural isomorphism by [BS01, Theorem 1.5].

If $H^0(A)$ is a full and essentially wide subcategory of $\text{EssIm}(G_{||A||})$, the inclusion $\text{tr}(A) \subset \text{EssIm}(G_{||A||})$ obtained by Lemma 2.9 is in fact an equivalence. This follows by considering the fully faithful triangulated functor

$$L : \text{tr}(A) \xrightarrow{\text{incl}} \text{EssIm}(G_{||A||}) \xrightarrow{(G_{||A||})^{-1}} \text{tr}(A),$$

which is an equivalence since $H^0(A) \subset \text{EssIm}(L)$ by assumption. Therefore, since the second functor is already an equivalence, incl is an equivalence as well. In particular, we obtain an induced functor $G' : \text{tr}(A) \to \text{tr}(A)$ whose idempotent extension is $G$, since $G'$ is defined as $L^{-1} = (\text{incl})^{-1}G_{||A||}$.

2.11. Proposition. Let $A$ be a DG-category. If $D(A)^c$ has a strongly unique enhancement, then $\text{tr}(A)$ has a strongly unique enhancement.

Proof. Take a triangulated equivalence $F : \text{tr}(A) \to \text{tr}(A)$. Now consider its idempotent extension $F' : D(A)^c \to D(A)^c$, which is unique (up to natural isomorphism) by [BS01, Theorem 1.5]. Take $(C, E)$ an enhancement of $D(A)^c$. Then we have a quasi-functor $f' : C \to C$ which is a $(C, E)$-lift of $F'$. We restrict $f'$ to $C_{||A||}$ (see Notation 2.7). By the fact that $F'_{||A||}$ is $F$, the restiction of $f'$ gives a quasi-functor $f : C_{||A||} \to C_{||A||}$ (cf. Lemma 2.8). We conclude that $f$ is a $(C_{||A||}, E_{||A||})$-lift of $F$.

It remains to show that $\text{tr}(A)$ has a unique enhancement, so that we can apply Proposition 2.5 to conclude. Let $(D, E)$ and $(D', E')$ be two enhancements of $\text{tr}(A)$. Then $\text{Perf}(D)$ and $\text{Perf}(D')$ are enhancements of $D(A)^c$. Indeed, $\text{tr}(D) \cong H^0(D) \cong \text{tr}(A)$, which implies $D(D)^c \cong D(A)^c$ by [BS01, Theorem 1.5]. By hypothesis, we have a quasi-equivalence quasi-functor $g : \text{Perf}(D) \to \text{Perf}(D')$ lifting the identity of $D(A)^c$. We now consider the restrictions $\text{Perf}(D)_{||A||}$ and $\text{Perf}(D')_{||A||}$. Since $g$ is a lift of the identity, it induces a quasi-equivalence quasi-functor $g' : \text{Perf}(D)_{||A||} \to \text{Perf}(D')_{||A||}$. We conclude by the following diagram:

$$\xymatrix{ D \ar[r]^-{=} & \text{Perf}(D)_{||A||} \ar[r]^-g & \text{Perf}(D')_{||A||} \ar[r]^-{=} & D'.}$$

2.12. Remark. The previous result holds also if we replace strongly unique enhancement with semi-strongly unique enhancement. The only difference in the proof is that we need to consider semilifts.

2.13. Proposition. Let $A$ be a DG-category.

1. If $\text{tr}(A)$ has a unique enhancement, then $D(A)^c$ has a unique enhancement.

2. If $D(A)^c$ has a unique enhancement, then $\text{tr}(A)$ has a unique enhancement.

\footnote{i.e. it contains at least one object for each isomorphism class.}
Proof.

1. Let $C$ and $C'$ be two enhancements of $D(A)^c$. Then $C_{|tr(A)}$ and $C'_{|tr(A)}$ are enhancements of $tr(A)$. In particular, there exists a quasi-equivalence quasi-functor $f : C_{|tr(A)} \to C'_{|tr(A)}$.

Consider $g$ as the composition

$$C \xrightarrow{\gamma} \text{Perf}(C) \xrightarrow{\text{Perf(incl)}} \text{Perf}(C_{|tr(A)}) \xrightarrow{\text{Perf}(f)} \text{Perf}(C') \xrightarrow{\phi} C'.$$

Lemma 2.9 shows that $\gamma$ and Perf(incl) are quasi-equivalences for both $C$ and $C'$. Since Perf$(f)$ is a quasi-equivalence as well, $g$ becomes a quasi-equivalence.

2. The proof is very similar to item 1: consider $C$ and $C'$ two enhancements of $D(A)$. By hypothesis, there exists a quasi-equivalence quasi-functor $f : C_{|D(A)^c} \to C'_{|D(A)^c}$. Let $h$ be the composition

$$C \xrightarrow{\phi} \text{SF}(C_{|D(A)^c}) \xrightarrow{\text{SF}(f)} \text{SF}(C'_{|D(A)^c}) \xrightarrow{\phi'} C',$$

where $\phi, \phi'$ are defined as in [LO10, Section 1]. By [LO10, Proposition 1.17], they both are quasi-equivalences. Since $f$ is a quasi-equivalence, the same holds for SF$(f)$ (see [BL94, Theorem 10.12.5.1], or [Kel94, Example 7.2]). Finally, $h$ is a quasi-equivalence.

3. Triangulated formality

In this section we define the original notion of triangulated formal DG-categories and discuss their relation with uniqueness of enhancements (see Figure 3.19 for an overview). This concept is inspired by the property of intrinsically formal graded rings proved in Proposition 3.3 below.

3.1. Notation. Any DG-ring $A$, i.e. a DG-algebra over a commutative ring $\mathbb{k}$, can be thought of as a DG-category with one object. By a slight abuse of notation, $A$ will indicate both the DG-ring and the DG-category. Its unique object will be denoted with $O_A$.

3.2. Definition. We recall that a DG-morphism $f : A \to A'$ (i.e. a DG-functor between DG-rings) is a quasi-isomorphism if $H^*(f)$ is an isomorphism.

Let $B$ be a graded ring. We say that $B$ is intrinsically formal if, for every DG-ring $A$ such that $H^*(A) \cong B$, we are able to find a zig-zag of quasi-isomorphisms

$$A \xrightarrow{\sim} A_1 \xrightarrow{\sim} A_2 \xrightarrow{\sim} \ldots \xrightarrow{\sim} B.$$

3.3. Proposition. Let $B$ be an intrinsically formal graded ring. For any enhancement $(C, E)$ of $tr(B)$, there exists a quasi-equivalence quasi-functor $f : B^{\text{pre}} \to C$ such that $EH^0(f)(O_B) \cong O_B$. 

PROOF. Without loss of generality, we assume that \( C \) is strongly pretriangulated. Consider \( C \in C \) such that \( E(C) \cong O_B \). Since 

\[
H^n(\text{Hom}_C(C,C)) \cong H^0(\text{Hom}_C(C,C)[n]) \cong H^0(\text{Hom}_C(C[C][n]))
\]

\[
\cong \text{Hom}_{H^0(C)}(C,C[n]) \cong \text{Hom}_{H^0}(O_B,O_B[n]) \cong H^n(B) = B^n,
\]

\( \text{Hom}_C(C,C) \) has cohomology \( B \) (the product of \( B \) is recovered from the composition on \( \text{tr}(B) \)). By hypothesis, we obtain a zig-zag of quasi-isomorphisms from \( B \) to \( \text{Hom}_C(C,C) \) extending to a quasi-fully faithful quasi-functor \( f : A^{\text{pretr}} \to C \) by Remark \[1.23\] and Proposition \[1.21\]. Since \( EH^0(f)(O_B) \cong O_B \), Remark \[2.10\] shows that \( H^0(f) \) is an equivalence, so \( f \) is a quasi-equivalence.

The previous proposition motivates the following.

3.4. Definition. A DG-category \( A \) is triangulated formal if

TF For any enhancement \((C,E)\) of \( \text{tr}(A) \), we have a quasi-equivalence quasi-functor \( f : A^{\text{pretr}} \to C \) such that

\[
EH^0(f)(X) \cong X \quad \text{for all } X \in H^0(A).
\]

A DG-category \( A \) is unbounded triangulated formal if

uTF Given any enhancement \((C,E)\) of \( D(A) \), there exists a quasi-equivalence quasi-functor \( f : SF(A) \to C \) such that \((3.5)\) holds.

3.6. Remark. Let \( A \) a DG-category.

1. Triangulated formality is stable under quasi-equivalence.
2. If \( \text{tr}(A) \cong \text{tr}(A') \) via the inclusion \( A \subset A' \), then \( A \) is (unbounded) triangulated formal whenever \( A' \) is (unbounded) triangulated formal.
3. If \( A \) is triangulated formal, then \( \text{tr}(A) \) has a unique enhancement. Analogously, if \( A \) is unbounded triangulated formal, then \( D(A) \) has a unique enhancement.
4. If \( \text{tr}(A) \) has a semi-strongly unique enhancement, then \( A \) is triangulated formal by picking a \((A^{\text{pretr}},\text{id})\)-semilift of the identity \( \text{id} : \text{tr}(A) \to \text{tr}(A) \) for any enhancement \((C,E)\) (we use Proposition \[2.6\]). Analogously, if \( D(A) \) has a semi-strongly unique enhancement, then \( A \) is unbounded triangulated formal.
5. As a matter of fact, \( \text{tr}(A) \) has a semi-strongly unique enhancement if and only if \( A^{\text{pretr}} \) is triangulated formal.

3.7. Example. Intrinsically formal graded rings are examples of triangulated formal graded categories with one object by Proposition \[3.3\]. In particular, all rings are triangulated formal (see, for instance, \[DS04, Lemma 6.6\]).

We now provide a wide range of meaningful examples of triangulated formal DG-categories.

3.8. Proposition. Let \( \mathcal{A} \) be a \( (\mathbb{k} \text{-linear}) \) category. Then it is triangulated formal (cf. \[LO10\] Proposition \[2.6\]).
3 Triangulated formality

PROOF. The idea is to proceed analogously to the proof of a ring being intrinsically formal (cf. [DS04 Lemma 6.6]). Let $(C, E)$ be an enhancement of $\text{tr}(\mathcal{A})$. For the sake of simplicity, assume $C$ to be strongly pretriangulated and consider $C_{\text{id}}$ (recall Notation 2.4). Take $\tau_{\leq 0}(C_{\text{id}})$ as described in Definition 1.8. We have two natural DG-functors given by truncation: $\tau_{\leq 0}(C_{\text{id}}) \to C_{\text{id}}$ and, by Remark 1.9, $\tau_{\leq 0}(C_{\text{id}}) \to H^0(C_{\text{id}}) \cong \mathcal{A}$ (the equivalence $H^0(C_{\text{id}}) \cong \mathcal{A}$ is given by the restriction of $E$). It is easy to prove that these DG-functors are in fact quasi-equivalences, because $H^i(C_{\text{id}}) = 0$ for $i \neq 0$. By Remark 1.23 and Proposition 1.21, we can extend the zig-zag $\mathcal{A} \leftarrow \tau_{\leq 0}(C_{\text{id}}) \to C_{\text{id}}$ to obtain a quasi-equivalence quasi-functor $f : \mathcal{A}_{\text{pretr}} \to C$. The fact that $EH^0(f)(X) \cong X$ for all $X \in \mathcal{A}$ follows from the definition of $f$; indeed, $H^0(f)$ restricted to $\mathcal{A}$ is the inverse of $E$ on objects.

3.9. Proposition. Let $\mathcal{A}$ be an additive category. Then $K^b(\mathcal{A})$ has a semi-strongly unique enhancement.

PROOF. We recall that $K^b(\mathcal{A}) \cong \text{tr}(\mathcal{A})$ by Example 1.38, so $K^b(\mathcal{A})$ has a unique enhancement from Proposition 3.8 and Remark 3.6. To simplify the notation, assume $K^b(\mathcal{A}) = \text{tr}(\mathcal{A})$. We want to show item 3 of Proposition 2.6 for the enhancement $(\mathcal{A}_{\text{pretr}}, \text{id})$. Let $F$ be an autoequivalence of $K^b(\mathcal{A})$. Since $\mathcal{A}$ is triangulated formal by Proposition 3.8, considering the enhancement $(\mathcal{A}_{\text{pretr}}, F)$ of $K^b(\mathcal{A})$, we get a quasi-equivalence quasi-functor $f : \mathcal{A}_{\text{pretr}} \to \mathcal{A}_{\text{pretr}}$ such that $FH^0(f)(X) \cong X$ for all $X \in \mathcal{A}$. Let $G := FH^0(f)$. Then $G_{\text{id}}$ gives an equivalence $\mathcal{A} \to \mathcal{A}$, so we can consider the DG-functor $g := (G_{\text{id}})^{\text{pretr}} : \mathcal{A}_{\text{pretr}} \to \mathcal{A}_{\text{pretr}}$. Of course, $GH^0(g^{-1})$ is the identity when restricted to $\mathcal{A}$. By [CY18 Proposition 3.2], $GH^0(g^{-1})(X) \cong X$ for all $X \in K^b(\mathcal{A})$. By recalling the definition of $G$ and moving the equivalences around, we get $F(X) \cong H^0(g^{-1})(X)$ for all $X \in K^b(\mathcal{A})$. This is exactly item 3 of Proposition 2.6, as wanted.

Proposition 3.10 below is inspired by [KV87 Theorem 3.2]. In that article, the authors used the definition of algebraic triangulated categories via Frobenius categories (see [CS17, Proposition 3.1]). Using DG-categories, we are able to say something more, and provide a proof of uniqueness of enhancements for bounded derived categories of exact categories. Furthermore, the DG-category $\mathcal{E}_{\text{DG}}$ associated is immediately triangulated formal.

3.10. Proposition. Let $\mathcal{T}$ be an algebraic triangulated category and let $\mathcal{E}$ be an admissible exact subcategory, i.e. an extension closed subcategory $\mathcal{E}$ of $\mathcal{T}$ such that $\text{Hom}(X, Y[n]) = 0$ for $n < 0$ (this has an induced exact structure by [Dye03]).

Then, for any enhancement $(C, E)$ of $\mathcal{T}$, there exists a realization functor $\text{real} : D^b(\mathcal{E}) \to \mathcal{T}$ admitting a $(\mathcal{P}^b_{\text{DG}}(\mathcal{E}), \text{id})$-lift\footnote{For the sake of simplicity, we assume $D^b(\mathcal{E}) = H^0(\mathcal{P}^b_{\text{DG}}(\mathcal{E}))$.} where $\mathcal{P}^b_{\text{DG}}(\mathcal{E})$ was defined in Example 1.38

PROOF. As our reasoning will not be affected by the quasi-equivalence inclusion $y : C \to C_{\text{pretr}}$, for the sake of simplicity we assume $C$ to be strongly pretriangulated, and consider the DG-functor $(C_{\text{id}})^{\text{pretr}} \to C$ obtained by Proposition 1.21.
From the natural truncation $\tau_{\leq 0}C_{|\mathcal{E}|} \to C_{|\mathcal{E}|}$, and the quasi-equivalence $\tau_{\leq 0}C_{|\mathcal{E}|} \to H^0(C_{|\mathcal{E}|}) \cong \mathcal{E}$ (this is a quasi-equivalence because $\mathcal{E} \subset \mathcal{F}$ is admissible by assumption), Proposition 1.21 gives rise to a quasi-functor $f : \mathcal{E}^{\text{pretr}} \to (C_{|\mathcal{E}|})^{\text{pretr}} \to C$. At the homotopy level, this defines a triangulated functor $K^b(\mathcal{E}) \to \mathcal{T}$ (recall Example 1.38).

We now want to prove that $A\mathcal{E}^b(\mathcal{E}) \to K^b(\mathcal{E}) \to \mathcal{T}$ is the zero functor. Indeed, given any conflation $0 \to A \to B \to C \to 0$ in $\mathcal{E}$, we obtain a commutative diagram

$$
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{\text{Cone}(f)} & A[1] \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{=} & C & \xrightarrow{=} & A[1]
\end{array}
$$

of distinguished triangles in $\mathcal{T}$. Since $\text{Hom}(A[1], C) = \text{Hom}(A, C[-1]) = 0$, the morphism $\text{Cone}(f) \to C$ is determined by $B \to \text{Cone}(f) \to C$, which is exactly the map appearing in the conflation. Looking at $\text{Cone}(f) \to C$ in $K^b(\mathcal{E})$, its cone is the conflation $0 \to A \to B \to C \to 0$, and its image is zero since $\text{Cone}(f) \to C$ is an isomorphism in $\mathcal{T}$. This shows that conflations are sent to zero via $K^b(\mathcal{E}) \to \mathcal{T}$. By Lemma A.5, we conclude that $A\mathcal{E}^b(\mathcal{E}) \to K^b(\mathcal{E}) \to \mathcal{T}$ is the zero functor.

By Remark 1.26 at the DG-level we have that $A\mathcal{E}_{DG}^b(\mathcal{E}) \to C_{DG}^b(\mathcal{E}) \cong \mathcal{E}^{\text{pretr}} \to C$ is the trivial quasi-functor. Therefore, we obtain an induced quasi-functor $r : D^b_{DG}(\mathcal{E}) \to C$ satisfying the statement.

3.11. Corollary. For any exact category $\mathcal{E}$, $D^b(\mathcal{E})$ has a unique enhancement. More precisely, the full DG-subcategory $\mathcal{E}_{DG} := D^b_{DG}(\mathcal{E})|_{DG}$ is triangulated formal.

In addition, given any enhancement $(C, E)$ of $D^b(\mathcal{E})$, the pretriangulated closure of the truncation $p_{\leq 0} : \tau_{\leq 0}C_{|\mathcal{E}|} \to C_{|\mathcal{E}|}$ gives rise to a DG-quotient.

Proof. Notice that $\text{tr}(\mathcal{E}_{DG}) \cong D^b(\mathcal{E})$ since $\text{tr}(\mathcal{E}_{DG}) \subset H^0(D^b_{DG}(\mathcal{E})) \cong D^b(\mathcal{E})$ and $\text{tr}(\mathcal{E}_{DG})$ is the triangulated envelope of $\mathcal{E}$. By applying Proposition 3.10 to $\mathcal{T} = D^b(\mathcal{E})$, we can construct a quasi-equivalence quasi-functor between any enhancement of $D^b(\mathcal{E})$ and $D^b_{DG}(\mathcal{E})$ (the quasi-functor is a quasi-equivalence by [Pos11] Corollary A.7.1 and Proposition A.7] and [CHZ19, Corollary 2.8], which holds also for exact categories; more details are given in [Lor22b]). Moreover, this quasi-equivalence fixes $\mathcal{E}$, so (3.5) is satisfied. The last part of the statement follows from the construction of the realization functor in the proof of Proposition 3.10.

3.12. Proposition. Let $\mathcal{E}$ be an exact category. Then $D^b(\mathcal{E})$ has a semi-strongly unique enhancement.

Proof. Mimicking the proof of Proposition 3.9 with the natural enhancement obtained by $D^b_{DG}(\mathcal{E})$, the statement follows since the reasoning of [CY18] Proposition 3.2 and Proposition 3.7] can be adapted to this setting. The only detail to be precise about is the fact that $G_{|\mathcal{E}|}$ gives an exact equivalence $\mathcal{E} \to \mathcal{E}$, which gives a DG-functor $C_{DG}^b(\mathcal{E}) \to C_{DG}^b(\mathcal{E})$ inducing a quasi-functor $g : D^b_{DG}(\mathcal{E}) \to D^b_{DG}(\mathcal{E})$ via the property of DG-quotients.
3.13. Remark. Since additive categories and abelian categories are examples of exact categories, Proposition 3.12 in fact generalizes both Proposition 3.9 and the bounded case of [CNS22, Remark 5.4], which shows semi-strong uniqueness of enhancements for the derived categories of abelian categories under any boundedness requirement.

Let us state some results relating triangulated formality with the uniqueness of enhancements. First of all, we motivate why we avoided the notion of triangulated formality for perfect complexes.

3.14. Proposition. A DG-category $A$ is triangulated formal if and only if the following holds $cTF$ For any enhancement $(C, E)$ of $D(A)^c$, we can choose a quasi-equivalence quasi-functor $f : \text{Perf}(A) \to C$ satisfying (3.5).

**Proof.** $TF \Rightarrow cTF.$ Let $(C, E)$ be an enhancement of $D(A)^c$ and consider the restriction $C_{\text{tr}(A)}$ (recall Notation 2.7). By triangulated formality, we get a quasi-equivalence quasi-functor $f' : A_{\text{pretr}} \to C_{\text{tr}(A)}$ satisfying (3.5). Denote by $f$ the following composition:

$$\text{Perf}(A) \xrightarrow{\phi_{\text{Perf}(A)}} \text{Perf}(A_{\text{pretr}}) \xrightarrow{\phi_{f'}} \text{Perf}(C_{\text{tr}(A)}) \xrightarrow{\phi_{\text{incl}}} \text{Perf}(C) \xrightarrow{\gamma} C$$

Notice that $EH^0(f)(X) \cong X$ for all $X \in H^0(A)$, and all DG-functors considered in the composition are quasi-fully faithful. Lemma 2.9 shows that $f$ is a quasi-equivalence.

$cTF \Rightarrow TF.$ Given any enhancement $(D, F)$ of $\text{tr}(A)$, then $\text{Perf}(D)$ is an enhancement of $D(A)^c$ with the unique extension of $F$ (we recall Remark 1.35). Moreover, $D \cong \text{Perf}(D)_{\text{tr}(A)}$ via inclusion. By $cTF$, we have a quasi-equivalence quasi-functor $g : \text{Perf}(A) \to \text{Perf}(D)$ satisfying (3.5). Restricting $g$ to $A_{\text{pretr}}$, by Remark 2.10 we get a quasi-equivalence $A_{\text{pretr}} \to \text{Perf}(D)_{\text{tr}(A)}$ satisfying (3.5).

3.15. Remark. From Proposition 3.14 $D(A)^c$ has a unique enhancement for any triangulated formal DG-category $A$.

3.16. Proposition. A triangulated formal DG-category $A$ is also unbounded triangulated formal.

**Proof.** Let $(C, E)$ be an enhancement of $D(A)$ and consider $C' := C_{D(A)^c}$. By Proposition 3.14, we obtain a quasi-equivalence quasi-functor $f : \text{Perf}(A) \to C'$ satisfying (3.5). Let us define $h$ as the composition

$$\text{SF}(A) \xrightarrow{\phi_{\text{Perf}(A)}} \text{SF}(\text{Perf}(A)) \xrightarrow{\phi_{f}} \text{SF}(C') \xrightarrow{\phi_{C'}} C,$$

where $\phi_{\text{Perf}(A)}$ and $\phi_{C'}$ are quasi-functors described in [LO10, Section 1]. Then $h$ is a quasi-equivalence as explained in the proof of item 2 of Proposition 2.13.

We are reduced to check that (3.5) is satisfied. In [LO10], $\phi_{\text{Perf}(A)}$ and $\phi_{C'}$ are obtained by a Yoneda embedding and a restriction functor, both of which do not affect the subcategory associated (Perf(A) and $C'$ respectively). Therefore, since $\text{SF}(f)$ is an extension of $f$, $h$ fulfills (3.5).
We can prove the converse of Proposition 3.16 in a special case.

**3.17. Proposition.** An unbounded triangulated formal DG-category $A$ is also triangulated formal if the following holds:

$\text{EE}$ For any enhancement $(C, E)$ of $\text{tr}(A)$, $E$ extends to a triangulated equivalence $E': D(C) \to D(A)$ up to natural isomorphism, i.e. $E'_{|H^0(C)} \cong E$.

(EE stands for Extending Enhancements).

**Proof.** Let $(C, E)$ be an enhancement of $\text{tr}(A)$. By assumption, we can consider the enhancement $(\text{SF}(C), E')$ of $D(A)$ such that $E'_{|H^0(C)} \cong E$. Since $A$ is unbounded triangulated formal, there exists $f: \text{SF}(A) \to \text{SF}(C)$ such that $E'H^0(f)(X) \cong X$ for all $X \in H^0(A)$.

We now want to show that $\text{EssIm}(f_{\text{Apretr}})$ is contained in $\overline{C}$, the homotopy closure of $C$ in $\text{SF}(C)$. Let $Y \in \text{EssIm}(f_{\text{A}})$. Then there exists $X \in H^0(A)$ such that $Y \cong H^0(f)(X)$. By applying $E'$, we have $E'Y \cong E'H^0(f)(X) \cong X$. Since $E'_{|H^0(C)} \cong E$, we can choose $Y' \in H^0(C)$ such that $E'Y' \cong X \cong E'Y$. In particular, $Y$ is homotopy equivalent to $Y'$. From this, we have $f_{\text{A}}: A \to \overline{C}$. Being $H^0(\overline{C})$ triangulated, $\text{EssIm}(f_{\text{Apretr}}) \subseteq \overline{C}$, as wanted.

Consider the quasi-functor $g: A_{\text{pretr}} \to \overline{C} \leftarrow C$, where the first map is the quasi-functor $f_{\text{Apretr}}$ and the second is a Yoneda embedding. By Remark 2.10, we conclude that $g$ is a quasi-equivalence, so $A$ is triangulated formal.

**3.18. Remark.** Assume $A$ is a DG-category for which EE holds and $D(A)$ has a semi-strongly unique enhancement. Then $\text{Perf}(A)$ is unbounded triangulated formal because $\text{SF}(\text{Perf}(A)) \cong \text{SF}(A)$ by [LO10, Proposition 1.17]. Since $A$ satisfies EE, from [BS01, Theorem 1.5] one can prove that $\text{Perf}(A)$ also satisfies EE. By Proposition 3.17, $\text{Perf}(A)$ is triangulated formal, meaning that $D(A)^c$ has a semi-strongly unique enhancement by Remark 3.6.

**3.19. Figure.** Relation between triangulated formality and uniqueness of enhancements for a DG-category $A$.

- $D(A)^c$ has a semi-strongly unique enhancement
- $\text{tr}(A)$ has a semi-strongly unique enhancement
- $D(A)$ has a semi-strongly unique enhancement
- $A$ is triangulated formal
- $A$ is unbounded triangulated formal
- $\text{tr}(A)$ has a unique enhancement
- $D(A)$ has a unique enhancement
- $D(A)^c$ has a unique enhancement

**Remark 3.12.**

**Remark 3.6.**

**Proposition 1.10.**

**Remark 3.17.**

**EE.**

**Remark 3.18.**

**Proposition 2.13.**

**Proposition 3.17.**

**Remark 3.6.**
4. Formal standardness

4.1. Proposition. Let \( \mathcal{F} \) be a triangulated category and consider \( \mathcal{I} \subset \mathcal{F} \) a full subcategory. Then any triangulated equivalence \( F : \mathcal{I} \to \mathcal{F} \) such that \( FX = X \) for \( X \in \mathcal{I} \) is naturally isomorphic to a triangulated equivalence \( G \) such that \( GX = X \) for \( X \in \mathcal{I} \).

PROOF. Let us consider the family of isomorphisms \( \eta := (\eta_X) \), where \( \eta_X : X \to FX \) is an isomorphism if \( X \in \mathcal{I} \), while \( \eta_X = \text{id} \) if \( X \notin \mathcal{I} \). Then \( G_{XY} := \eta_Y^{-1}F_{XY}\eta_X \) describes the wanted triangulated equivalence \( G \), and \( \eta \) becomes a natural isomorphism \( G \to F \).

4.2. Definition. Let \( A \) be a DG-category and consider a triangulated autoequivalence \( (F, \eta) \) on \( \text{tr}(A) \) such that
\[
FX = X \quad \text{for any } X \in H^0(A) .
\]

Its graded restriction is a graded functor \( F^g_{H^*(A)} : H^*(A) \to H^*(A) \) defined by \( F^g_{H^*(A)}(X) = X \) and
\[
\Hom_{H^*(A)}(X,Y) \xrightarrow{\eta_X} \text{Hom}_{H^*(A)}(X,Y)
\]
\[
\bigoplus_i \Hom_{\text{tr}(A)}(X,Y[i]) \xrightarrow{\oplus F_{X,Y[i]}} \bigoplus_i \Hom_{\text{tr}(A)}(FX,F(Y[i])) \xrightarrow{\oplus \eta_X^i} \bigoplus_i \Hom_{\text{tr}(A)}(X,Y[i])
\]

for any \( X, Y \in A \), where the vertical arrows are obtained via the following isomorphisms:
\[
\Hom_{H^*(A)}(X,Y) \cong \bigoplus_i H^i(\Hom_A(X,Y)) \cong \bigoplus_i H^0(\Hom_A(X,Y[i])) \cong \bigoplus_i \Hom_{H^0(A)}(X,Y[i]).
\]

4.4. Definition. A DG-category \( A \) is
- **formally standard** if, given two triangulated equivalences \( F, G : \text{tr}(A) \to \text{tr}(A) \) satisfying (4.3) and \( F^g_{H^*(A)} \cong G^g_{H^*(A)} \), there is a natural isomorphism \( F \cong G \).
- **lifted** if for every triangulated equivalence \( F : \text{tr}(A) \to \text{tr}(A) \) for which (4.3) holds, we have a quasi-functor \( f : A \to A \) such that \( H^*(f) \cong F^g_{H^*(A)} \).

The notion of lifted is introduced to treat at the same time the two following examples, which are crucial for applications.

4.5. Example.
- Every graded category \( B \) is lifted: indeed, given a triangulated equivalence \( F : \text{tr}(B) \to \text{tr}(B) \), the quasi-functor \( f \) required is simply given by the graded restriction \( F^g_{B} \).

\[\text{This definition explicitly requires the axiom of choice, since we choose an isomorphism for every object in } \mathcal{I} .\]
\[\text{The first and the last isomorphisms are given by definition, while the second one is obtained by iterated composition of the closed morphisms associated to the suspensions.}\]
• Let $\mathcal{E}$ be an exact category and consider $\mathcal{E}_{DG} := D^b_{DG}(\mathcal{E})$ as in Corollary 3.11. We claim that $\mathcal{E}_{DG}$ is lifted: indeed, for any triangulated equivalence $F : \text{tr}(\mathcal{E}_{DG}) \rightarrow \text{tr}(\mathcal{E}_{DG})$, we obtain an exact equivalence $\mathcal{E} \rightarrow \mathcal{E}$. This induces an equivalence $C^b_{DG}(\mathcal{E}) \rightarrow C^b_{DG}(\mathcal{E})$; via quotient we get a quasi-functor $D^b_{DG}(\mathcal{E}) \rightarrow D^b_{DG}(\mathcal{E})$, and finally a quasi-functor $f : \mathcal{E}_{DG} \rightarrow \mathcal{E}_{DG}$. Moreover, the exact equivalence $\mathcal{E} \rightarrow \mathcal{E}$ uniquely determines what happens on $H^\ast(\mathcal{E}_{DG})$; this is Proposition 4.6. We conclude that $H^\ast(f) \cong F_{\text{gr}} H^\ast(\mathcal{E}_{DG})$.

4.6. Remark. A DG-category $A$ is formally standard if and only if any triangulated equivalence $F : \text{tr}(A) \rightarrow \text{tr}(A)$, such that (4.3) holds and $F_{\text{gr}} H^\ast(\text{tr}(A)) \cong \text{id}$, is naturally isomorphic to the identity.

4.7. Remark. Replacing $\text{tr}(A)$ with $D(A)^c$ in Definition 4.4, one could be tempted to define perfect formally standard and perfect lifted DG-categories. However, this is not useful, since the equivalences $F : D(A)^c \rightarrow D(A)^c$ satisfying (4.3) are determined by $F_{\text{tr}(A)}$ up to natural isomorphism by [BS01] Theorem 1.5, and Remark 2.10 shows that $\text{tr}(A) \subset \text{EssIm}(F_{\text{tr}(A)})$, so that $F_{\text{tr}(A)}$ can be thought of as an equivalence $\text{tr}(A) \rightarrow \text{tr}(A)$. In other words, the only equivalences admitting a graded restriction on $D(A)^c$ are equivalences restricting to $\text{tr}(A)$.

5. Main result

In this section, we show how the new notions introduced so far (triangulated formality, formal standardness and liftedness) are connected to strong uniqueness of enhancements. In the case of graded categories $B$, we are able to provide a characterization of strong uniqueness of enhancements for $\text{tr}(B)$ and $D(B)^c$ (see Theorem 5.6).

Let us start with a sufficient condition for strong uniqueness of enhancements.

5.1. Proposition. Let $A$ be a lifted, triangulated formal and formally standard DG-category. Then $\text{tr}(A)$ has a strongly unique enhancement.

Proof. Let $F : \text{tr}(A) \rightarrow \text{tr}(A)$ be any triangulated equivalence. Let us consider the enhancement $(A_{\text{pretr}}, F)$ of $\text{tr}(A)$. By triangulated formality, there exists $f : A_{\text{pretr}} \rightarrow A_{\text{pretr}}$ such that $FF^0(f)(X) \cong X$ for all $X \in H^0(A)$. Up to natural isomorphism, we can assume that $G := FH^0(f)$ satisfies (4.3) by Proposition 4.1. We aim to prove that $G$ has an $(A_{\text{pretr}}, \text{id})$-lift. This suffices to conclude that $F$ has an $(A_{\text{pretr}}, \text{id})$-lift as well. Since $A$ is lifted, we have a quasi-functor $g : A \rightarrow A$ such that $H^\ast(g) \cong G_{|H^\ast(A)_{\text{gr}}}$, up to natural isomorphism, notice that $H^0(g_{\text{pretr}})$ satisfies (4.3). It remains to prove that $G_{|H^0(A)_{\text{gr}} \rightarrow H^0(A)} \cong H^0(g_{\text{pretr}})_{|H^\ast(A)_{\text{gr}}}$, which follows from direct computations. Formal standardness implies that $G \cong H^0(g_{\text{pretr}})$.

Since any triangulated equivalence $F : \text{tr}(A) \rightarrow \text{tr}(A)$ has an $(A_{\text{pretr}}, \text{id})$-lift, Remark 3.6 and Proposition 2.5 show that $\text{tr}(A)$ has a strongly unique enhancement. 

5.2. Remark. By Proposition 3.14 and Remark 4.7, we can modify the proof of Proposition 5.1 to prove that $D(A)^c$ has a strongly unique enhancement under the same requirements.
We now aim to prove the converse implication of Proposition 5.1. In order to do so, we need to restrict to graded categories and state a technical result (Lemma 5.4). First, we recall the following.

5.3. Fact. [Hov99, Theorem 1.2.10] and [Tab05, Théorème 2.1, Remarque 1]. Every DG-category is quasi-equivalent to a cofibrant DG-category. Moreover, let $C$ be a cofibrant DG-category. Every quasi-functor $f : C \to D$ can be represented by a DG-functor $f' : C \to D$. In particular, this means that $H^0(f) \cong H^0(f')$.

5.4. Lemma – Presentation via a cofibrant DG-category. Let $A$ be a DG-category and let $C$ be a cofibrant DG-category with a quasi-equivalence quasi-functor $H_0 \colon C \to A$. Then we can construct a quasi-equivalence $G : A \to C$ such that $(G|_A)^\sim = H^0|_A$. Indeed, if this is true, given any autoequivalence $A \to A$, then $Y \in A$ is homotopy equivalent to an object of $A$, then $Y \in A$. Notice the functor $i : H^0(A) \to H^0(A|_A) = \text{tr}(A)$, obtained from the inclusion $A \to A|_A$, is an equivalence by Lemma 5.2.9 so $(A', i)$ is an enhancement of $\text{tr}(A)$.

Let us now consider the quasi-equivalence quasi-functor $f' : C \to A|_A \leftarrow A$. By Fact 5.3, we can assume $f'$ to be a DG-functor. By definition, we have that $f'(D) \subseteq A$ by the choice of $A'$. Finally, the restriction $f'|_D : D \to A$ is the wanted quasi-equivalence $h$.

5.5. Proposition. Let $B$ be a graded category. If $\text{tr}(B)$ has a strongly unique enhancement, then $B$ is triangulated formal and formally standard.

Proof. By Remark 5.6, we are reduced to check that $B$ is formally standard. For this purpose, we will show that any autoequivalence $F$ on $\text{tr}(B)$ satisfying (4.3) is naturally isomorphic to $H^0(F|_B)^\pretr$. Indeed, if this is true, given any autoequivalence $G$ such that (4.3) holds, $F|_B \cong G|_B$ implies that $(F|_B)^\pretr \cong (G|_B)^\pretr$ by Proposition 1.2.11 from which $F \cong H^0((F|_B)^\pretr) \cong H^0((G|_B)^\pretr) \cong G$, as wanted. We divide our reasoning in two steps:
1. We choose a presentation $A$ according to Lemma 5.4 and a "well-behaved" associated enhancement $(C, H^0(e))$. The meaning of its well-behaviour will become clear in the second part of the proof.
2. We describe a DG-functor $f'$, which is a $(B|_A, \text{id})$-lift of $F$. We conclude that $F \cong H^0(f') \cong H^0((F|_B)^\pretr)$.

We prove item 1. Let $(C, E)$ be a cofibrant enhancement of $\text{tr}(B)$ and define $A := C_{(H^0(B))}$. We consider the DG-functors $j : A|_E \to C|_E$, induced by the inclusion, and $h : A \to B$, associated to a $(C, E) \sim (B|_A, \text{id})$-lift of the identity (which exists by Proposition 2.5) as expressed in Lemma 5.4. We define $e$ to be the (quasi-equivalence) quasi-functor given by the following
composition
\[ C \xymatrix{ \ar[r]^y & C_{\text{pretr}} \ar[l]_j & A_{\text{pretr}} \ar[r]^{h_{\text{pretr}}} & B_{\text{pretr}} }, \]
where \( y \) is the Yoneda embedding. We want to consider the enhancement \((C, H^0(\varepsilon))\). The "well-behaviour" discussed above is motivated by the fact that \( A = C_{H^0(\varepsilon)} \). Let us prove it. By the definition of the functors, for every \( X \in H^0(\varepsilon) \) we have
\[
H^0(\varepsilon)(X) \cong H^0(h_{\text{pretr}}^*)H^0(j^{-1})H^0(\varepsilon)(X) \cong H^0(h_{\text{pretr}}^*)H^0(j^{-1})(X) \\
\cong H^0(h_{\text{pretr}}^*)(X) \cong Y \in \operatorname{H}^1(B). 
\]
This implies that \( A \subset C_{H^0(\varepsilon)} \). Conversely, let \( X \in C_{H^0(\varepsilon)} \). Then we have \( H^0(\varepsilon)(X) \cong Y \in \operatorname{H}^0(B) \). Since \( h : A \to B \) is a quasi-equivalence, there exists \( Z \in A \) such that \( H^0(h)(Z) \cong Y \).

From the definitions of \( j \) and \( y \), we have \( H^0(\varepsilon)(Z) \cong Y \). In particular, \( X \) and \( Z \) are homotopy equivalent. Since \( A \) is closed under homotopy equivalence by Lemma 2.8, \( X \in A \) as wanted.

We prove item 2. By Proposition 2.5, \( F \) has a \((C, H^0(\varepsilon))\)-lift, and by Fact 5.3, this lift can be chosen to be a DG-functor \( f \). Since \( A = C_{H^0(\varepsilon)} \), we have that \( f(A) \subset A \) because \( H^0(\varepsilon)H^0(f)(X) \cong FH^0(\varepsilon)(X) \cong H^0(\varepsilon)(X) \) for every \( X \in H^0(A) \). We define \( f_A := f|_A : A \to A \). Notice that \( f_{\text{pretr}}^* \cong j_{\text{pretr}}^* \) by Proposition 1.21.

Moreover, since \( B \) is graded and \( h : A \to B \) is a quasi-equivalence, we have that \( H^*(h) : H^*(A) \to H^*(B) = B \) is a graded equivalence. Therefore, for the sake of simplicity, we can replace \( B \) with \( H^*(A) \) and \( h \) with \( H^*(h)^{-1}h \), so that \( H^*(h) = \text{id} \). From the definition of \( H^* \), we obtain the commutative diagram of DG-categories
\[
\begin{array}{ccc}
A & \xrightarrow{f_A} & A \\
\downarrow h & & \downarrow h \\
B & \xrightarrow{H^*(f_A)} & B 
\end{array}
\]
which can be extended to the pretriangulated closures by Proposition 1.21. Let \( f' = (H^*(f_A)_{\text{pretr}} \). We have the following situation
\[
\begin{array}{ccc}
B_{\text{pretr}} & \xrightarrow{h_{\text{pretr}}} & A_{\text{pretr}} & \xrightarrow{j} & C_{\text{pretr}} & \xrightarrow{y} & C \\
\downarrow f' & & \downarrow e_{\text{pretr}} & & \downarrow e_{\text{pretr}} & & \downarrow e_{\text{pretr}} \\
B_{\text{pretr}} & \xrightarrow{h_{\text{pretr}}} & A_{\text{pretr}} & \xrightarrow{j} & C_{\text{pretr}} & \xrightarrow{y} & C \\
\downarrow f & & \downarrow f' & & \downarrow f' & & \downarrow f' \\
\end{array}
\]
In particular, this diagram shows that \( f' \) is a \((B_{\text{pretr}}, \text{id})\)-lift of \( F \). We have a natural isomorphism \( \eta : F \to H^0(f') \), which gives a graded isomorphism \( \mu : F_{B_{\text{pretr}}} \to H^*(f_A) \). Being \( B \) a DG-category with trivial differential, \( \mu \) is a DG-natural isomorphism, so we can extend it to a unique
DG-natural isomorphism on $B^\text{pretr}$ by Proposition 1.21. In particular, $(F^\text{gr})^\text{pretr} \simeq f'$ up to DG-isomorphism, so $F \simeq H^0((F^\text{gr})^\text{pretr})$.

5.6. Theorem. Let $B$ be a graded category. The following are equivalent:

1. $B$ is triangulated formal and formally standard;
2. $\text{tr}(B)$ has a strongly unique enhancement;
3. $D(B)^c$ has a strongly unique enhancement.

Proof. Proposition 5.1 (remembering Example 4.5) and Proposition 5.5 prove 1 $\iff$ 2. Remark 5.2 deals with 1 $\implies$ 3, while Proposition 2.11 shows 3 $\implies$ 2.

6. Free generators

Our aim is to apply Theorem 5.6. We start with some simple examples, described by the following non-canonical definition.

6.1. Definition. A DG-category $A$ is a free generator if every object $X \in D(A)^c$ is isomorphic to a direct summand of

$$\bigoplus_{i \in \mathbb{Z}} \left( \bigoplus_{j=1}^{n_i} C_{i,j}[i] \right)$$

for some $C_{i,j} \in H^0(A)$ and $n_i \neq 0$ for finitely many $i$’s.

6.3. Example. In a trivial way, any pretriangulated DG-category is a free generator, but this example is far from our idea of application, since we want to study it for graded categories (i.e. DG-categories with trivial differential). Let us give some meaningful examples.

- The DG-category $R$ given by a semisimple ring is a free generator. Indeed, a finitely generated $R$-module is a direct summand of a free $R$-module. As a consequence, $D(R)^c$ is obtained by cones of closed morphisms

$$\bigoplus_{i \in \mathbb{Z}} R^n[i] \to \bigoplus_{j \in \mathbb{Z}} R^m[j],$$

which are simply given by kernels and cokernels, again expressed via finitely generated $R$-modules.

- Consider an algebraic finite triangulated category $\mathcal{T}$ as defined in [Mur20] and let $\Lambda$ be the algebra of endomorphisms associated to a basic additive generator $X$. Given an enhancement $(C,E)$ of $\mathcal{T}$, any object $Y \in E^{-1}(X)$, together with its endomorphisms, defines a free generator DG-category with one object (this follows from the fact that $\mathcal{T}$ is equivalent to the category of finitely generated projective (right) $\Lambda$-modules).

Roughly speaking, the following lemma tells us that whenever a ring is a free generator, then it is a free generator also for its associated periodic triangulated categories (the notion of periodic triangulated category is given, for instance, in the introduction of [Sat21]).
6.4. **Lemma.** Let $R$ be a ring and consider $R[t, t^{-1}]$ with $t$ homogeneous of positive degree. If $R$ is a free generator, then so is $R[t, t^{-1}]$.

**Proof.** Let us consider the inclusion $R \rightarrow R[t, t^{-1}]$, which is a (differential) graded morphism. Then we can extend it to a DG-functor $\pi : \text{Perf}(R) \rightarrow \text{Perf}(R[t, t^{-1}])$. Set $n := \deg(t)$, and for any $k \in \mathbb{Z}$ use $\tilde{k} \in \{0, \ldots, n - 1\}$ for the representative of $k$ in $\mathbb{Z}/n\mathbb{Z}$.

The objects in $\text{DGMOD}(R)$ are functors $M : R^e \rightarrow \text{CDG}(\text{Mod}(k))$, so they are DG-modules $M(R^n) := (M^k, d^k)_{k \in \mathbb{Z}}$. Similarly, an object $M_n \in \text{DGMOD}(R[t, t^{-1}])$ can be thought of as the DG-module $M_n(R[t, t^{-1}]^n) := (M_n^k, d_n^k)_{k \in \mathbb{Z}}$. The morphism $M_n(t) : M_n \rightarrow M_n$ has degree $n$ and an inverse, which shows that $M_n^j \cong M_n^{j + n}$ for every $j \in \mathbb{Z}$. In particular, $M_n$ is identified with $(M_n^k, d_n^k)_{k = 0}^{n - 1}$, with $d_n^{i - 1} : M_n^{i - 1} \rightarrow M_n^i$. Then $\pi$ can be explicitly expressed by (cf. [Sai21, Definition 3.7])

$$\pi(M)^k := \bigoplus_{\ell = k} M^\ell, \quad d^k(M) := \bigoplus_{\ell = k} d^\ell_M$$

(the behaviour of $\pi$ on morphisms is expressed accordingly). From this description, it is easy to notice that $\pi$ is essentially surjective. Since $\pi$ is also additive and preserves suspensions, the statement follows. 

6.5. **Proposition.** Let $A$ be a free generator DG-category. Then $A$ is formally standard.

**Proof.** Define $A_{\text{add}}$ the full DG-subcategory of $A^{\text{pret}}$ whose objects are of the form (6.2). Notice that $H^0(A_{\text{add}}) \subset \text{tr}(A)$. Let $F : \text{tr}(A) \rightarrow \text{tr}(A)$ a triangulated equivalence such that its graded restriction is naturally isomorphic to the identity. Therefore, $F_{H^0(A_{\text{add}})}$ is naturally isomorphic to the identity. Let $G$ be the composition

$$H^0(A_{\text{add}}) \xrightarrow{F_{H^0(A_{\text{add}})}} \text{tr}(A) \xrightarrow{} D(A)^c$$

Since $A$ is a free generator, $D(A)^c$ is the idempotent completion of $H^0(A_{\text{add}})$; [BSO1] Proposition 1.3] gives a unique extension $H : D(A)^c \rightarrow D(A)^c$ of $G$, which is therefore an extension of $F$. Further, from the same proposition the natural isomorphism $F_{H^0(A_{\text{add}})} \rightarrow \text{id}$ extends to a natural isomorphism $H \rightarrow \text{id}$. By restricting this natural isomorphism, $F$ is naturally isomorphic to the identity. Remark 1.8 concludes the proof.

6.6. **Corollary.** Let $\mathbb{K}$ be a field. Given a free generator $\mathbb{K}$-algebra $\Lambda$ with finite projective dimension $d$ as an $\Lambda$-bimodule, then $D(\Lambda[t, t^{-1}])^c$ has a strongly unique $\mathbb{K}$-linear enhancement for any $t$ homogeneous of degree greater or equal than $d$.

**Proof.** Under these assumptions, [Sai21, Corollary 4.2] shows that $\Lambda[t, t^{-1}]$ is intrinsically formal. By Example 5.7 Lemma 6.4 Proposition 6.5 and Theorem 5.6 we conclude.

6.7. **Example.** Here we list some examples following from the previous corollary.
• The triangulated category $\mathbb{K}$-mod with suspension the identity is $\text{tr}(\mathbb{K}[t,t^{-1}])$ with $t$ of degree 1, so it has a strongly unique $\mathbb{K}$-linear enhancement by Corollary 6.6. Notice this is not the case when $\mathbb{K} = \mathbb{F}_p$ (with $p$ a prime) and we consider linearity over $\mathbb{Z}$ (see [Sch02] and [CS17 Corollary 3.10]). In particular, $\mathbb{F}_p[t,t^{-1}]$ is not intrinsically formal as a $\mathbb{Z}$-linear graded ring.

• Let $\mathbb{K}$ be a perfect field and let $\Lambda$ be a semisimple finite-dimensional $\mathbb{K}$-algebra. In this case, $\Lambda$ is a projective $\Lambda$-bimodule by [Pie82, Corollary b, p. 192]. Then, by Example 6.3 and Corollary 6.6, $\text{D}(\Lambda[t,t^{-1}]^c)$ has a strongly unique $\mathbb{K}$-linear enhancement for any homogeneous $t$ of positive degree.

6.8. Remark. Let us briefly discuss the example of a non-unique $\mathbb{K}$-linear enhancement provided by Rizzardo and Van den Bergh in [RvdB19].

Let $\mathbb{K}$ be a field and $\mathbb{F} := \mathbb{K}(x_1, \ldots, x_n)$ with $n > 0$ even. Then $\text{D}(\mathbb{F}[t,t^{-1}])^c$ with $\text{deg}(t) = n$ has a non-unique enhancement, as shown in [RvdB19]. We notice that this example carefully avoids any situation described above. As discussed in Example 6.3 and Corollary 6.6, $\text{D}(\Lambda[t,t^{-1}])^c$ has a strongly unique $\mathbb{K}$-linear enhancement for any homogeneous $t$ of positive degree.

As one may expect from the viewpoint depicted in this article, the proof in [RvdB19] shows explicitly that $\mathbb{F}[t,t^{-1}]$ is not intrinsically formal by deforming the graded algebra into a different minimal $A_\infty$-algebra.

7. D-standardness and K-standardness

In this section we consider the notions of D-standard and K-standard categories introduced in [CY18], and show that for a very large class of examples they are, in fact, equivalent to strongly unique enhancement. We emphasize that these results hold for $\mathbb{K}$-linearity, where $\mathbb{K}$ is any commutative ring.

7.1. Definition. Let $\mathcal{E}$ be an exact category and consider $F : \text{D}^b(\mathcal{E}) \to \text{D}^b(\mathcal{E})$ a triangulated equivalence. Then $\mathcal{E}$ is D-standard if the following implication holds:

(♠) Whenever $F(\mathcal{E}) \subseteq \mathcal{E}$ and $\eta_0 : F\mid_{\mathcal{E}} \to \text{id}_{\mathcal{E}}$ is a natural isomorphism, there exists a natural isomorphism $\eta : F \to \text{id}$ extending $\eta_0$.

An additive category $\mathcal{A}$ is K-standard if it is D-standard as an exact category (see Example A.4).

7.2. Lemma. Let $F : \text{D}^b(\mathcal{E}) \to \text{D}^b(\mathcal{E})$, with $\mathcal{E}$ an exact category, be a triangulated equivalence. Then (♠) holds if and only if the following is satisfied.

(♣) Whenever $F(\mathcal{E}) \subset \mathcal{E}$ and $F\mid_{\mathcal{E}} \cong \text{id}_{\mathcal{E}}$, then $F \cong \text{id}$.

PROOF. This result is analogous to [CY18 Lemma 3.5]. The fact that (♣) implies (♠) is obvious. Conversely, let $\eta_0 : F\mid_{\mathcal{E}} \to \text{id}_{\mathcal{E}}$. By (♠), there exists $\mu : F \to \text{id}$. In particular, $\mu : F\mid_{\mathcal{E}} \to \text{id}_{\mathcal{E}}$, so we can consider $\text{D}^b(\text{id}_{\mathcal{E}} \mu \mid_{\mathcal{E}}^{-1}) : F \to \text{id}$: by definition, such natural isomorphism restricted to $\mathcal{E}$ is $\eta_0$. This concludes the proof. □
7.3. Proposition. An additive category $\mathcal{A}$ is $K$-standard if and only if $K^{b}(\mathcal{A})$ has a strongly unique enhancement.

Proof. We recall that $\mathcal{A}$ is triangulated formal by Proposition 3.8. Then Remark 4.6 and Lemma 7.2 shows that $\mathcal{A}$ is $K$-standard if and only if it is formally standard. Theorem 5.6 concludes the proof.

7.4. Example. A Krull-Schmidt additive category $\mathcal{A}$ is an Orlov category if
1. The endomorphism ring of each indecomposable is a division ring;
2. There is a degree function $\deg: \text{ind}(\mathcal{A}) \to \mathbb{Z}$, where $\text{ind}(\mathcal{A})$ is the set of all indecomposables, such that $\deg X \leq \deg Y$ implies $\text{Hom}(X, Y) = 0$ whenever $X \not\cong Y$.

As proved in [CY18, Proposition 4.6], an Orlov category $\mathcal{A}$ is $K$-standard, so $K^{b}(\mathcal{A})$ has a strongly unique enhancement.

7.5. Lemma. Let $E$ be an exact category, and let $E^{\text{DG}} := D^{b}(E)$ be the DG-category as in Example 4.5. Then $E$ is D-standard if and only if $E^{\text{DG}}$ is formally standard.

Proof. By Lemma 7.2 and Proposition 4.1, $E$ is D-standard if and only if any triangulated equivalence $F$ such that $F(X) = X$ for $X \in E$ and $F|_{E} \cong \text{id}_{E}$ is naturally isomorphic to the identity. By Remark 4.6, we are reduced to prove that $F_{|_{E}} \cong \text{id}_{E}$ if and only if $F_{|_{E}}^{gr} \cong \text{id}_{E}^{gr}$. This follows from Proposition A.6.

We now aim to show the analogous of Proposition 7.3 for derived categories.

7.6. Proposition. Let $\mathcal{E}$ be an exact category, and consider $F$ a triangulated autoequivalence of $D^{b}(\mathcal{E})$ such that $F(\mathcal{E}) \subset \mathcal{E}$ and $F|_{\mathcal{E}} \cong \text{id}_{\mathcal{E}}$. Let $(C, E)$ be any enhancement of $D^{b}(\mathcal{E})$. If $F$ has a $(C, E)$-lift, then $F$ is naturally isomorphic to the identity.

Moreover, the identity of $C$ is the only quasi-functor lifting the identity of $D^{b}(\mathcal{E})$. Consequently, any autoequivalence of $D^{b}(\mathcal{E})$ has at most one $(C, E)$-lift.

Proof. Without loss of generality, assume $C$ is a cofibrant DG-category. By Fact 5.3 there exists a DG-functor $f: C \rightarrow C$ which is a $(C, E)$-lift of $F$. We now consider $f|_{E}: C|_{E} \rightarrow C|_{E}$. Defined the quasi-equivalence quasi-functor
\[(C|_{E})^{\text{pretr}} \xrightarrow{j} C^{\text{pretr}} \xleftarrow{y} C,
\]
where $j$ is induced by inclusion, notice that $yf \cong f^{\text{pretr}}y$ and $j(f|_{E})^{\text{pretr}} \cong f^{\text{pretr}}j$ by Proposition 1.21.

Moreover, we are able to construct the following commutative diagram
\[
\begin{array}{ccc}
H^{0}(C|_{E}) & \xrightarrow{\tau_{\leq 0}C|_{E}} & C|_{E}^{e} \\
\downarrow\text{id} & & \downarrow\tau_{\leq 0}f|_{E} \\
H^{0}(C|_{E}) & \xrightarrow{\tau_{\leq 0}C|_{E}} & C|_{E}
\end{array}
\]
by assumption (indeed $H^0(f|_E) = F|_E \cong \text{id}$). The commutative diagram obtained by taking the pretriangulated closures shows that $(f|_E)^\text{pretr}$ is the identity quasi-functor by the universal property of the DG-quotient (see item 1 of Definition/Proposition 1.36 and Corollary 3.11). This suffices to show that $f$ is the identity as well because $f_y^{-1}j \cong y^{-1}(f|_E)^\text{pretr} \cong y^{-1}j$, where $j$ and $y$ are quasi-equivalences. In particular, $F \cong E H^0(f|_E)^{-1} \cong EE^{-1} \cong \text{id}$, as wanted.

7.7. Theorem. Let $\mathcal{E}$ be an exact category. Then $\mathcal{E}$ is $D$-standard if and only if $D^b(\mathcal{E})$ has a strongly unique enhancement.

PROOF. Recall that $\mathcal{E}_{DG}$ is triangulated formal by Corollary 3.11. If $\mathcal{E}$ is $D$-standard, by Lemma 7.5 $\mathcal{E}_{DG}$ is also formally standard. Proposition 5.1 shows that $D^b(\mathcal{E})$ has a strongly unique enhancement, since $\mathcal{E}_{DG}$ is always lifted (see Example 4.5). The converse implication follows from Proposition 7.6 and Proposition 2.5.

7.8. Proposition. Let $\mathcal{A}$ be an abelian category with enough projective objects. We denote with $\text{Proj}(\mathcal{A})$ its subcategory of projective objects. If $K^b(\text{Proj}(\mathcal{A}))$ has a strongly unique enhancement, then $D^b(\mathcal{A})$ has a strongly unique enhancement.

PROOF. It immediately follows from Proposition 7.3 [CY18, Theorem 6.1] and Theorem 7.7.

We now state some examples obtained by applying Theorem 7.7.

7.9. Corollary. Let $\mathcal{A}$ be a hereditary abelian category (i.e. $\text{Ext}^i = 0$ for $i > 1$). Then its bounded derived category $D^b(\mathcal{A})$ has a strongly unique enhancement. (cf. [CY18, Corollary 5.6]).

7.10. Example. Let $R$ be any ring (recall Convention 0.1). If $R$ is (right) hereditary, the bounded derived category of (all right) $R$-modules has a strongly unique enhancement. If $R$ is (right) semihereditary and Noetherian, the bounded derived category of finitely generated $R$-modules has a strongly unique enhancement. Dually, the result holds for left modules.

To prove that bounded derived categories of smooth projective varieties have a strongly unique enhancement, Lunts and Orlov in fact showed $D$-standardness using the notion of ample sequence [LO10]. Here we consider a generalization due to Canonaco and Stellari [CS14, Definition 2.9].

7.11. Definition. Given an abelian category $\mathcal{A}$ and a set $I$, we say that $\{P_i\}_{i \in I} \subset \mathcal{A}$ is an almost ample set if, for any $A \in \mathcal{A}$, there exists $i \in I$ such that

1. There is a natural number $k$ such that $P_i^{\otimes k} \to A$ is an epimorphism;
2. $\text{Hom}(A, P_i) = 0$.

7.12. Example. Given an algebraic space $X$ proper over an Artinian ring with depth $\geq 1$ at every closed point, the category of coherent sheaves $\text{Coh}(X)$ has an almost ample set (see [Ola22, Lemma 3.3.2]). Another class of examples is given by [CS14, Proposition 2.12].
Finally, we have the following result for derived categories with a geometric flavour.

7.13. Proposition. Let $\mathcal{A}$ be an abelian category with an almost ample set. Then $\mathcal{D}^b(\mathcal{A})$ has a strongly unique enhancement.

Sketch of proof. We want to show that $\mathcal{A}$ is $\mathcal{D}$-standard and apply Theorem 7.7. This is proved analogously to [CST14 Proposition 3.7], which in turn mimics [Orl03 Proposition 3.4.6] with some crucial adjustments.

A. On bounded derived categories of exact categories

A.1. Definition. Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair (in $\mathcal{A}$) is a pair $(i, p)$ of composable morphisms such that $i$ is a kernel of $p$ and $p$ is a cokernel of $i$.

For a class of kernel-cokernel pairs $E$, a morphism $i$ is an admissible monic if there exists $p$ such that $(i, p) \in E$. Dually, we define admissible epic. We say that a class of kernel-cokernel pairs $E$ is an exact structure if it is closed under isomorphisms and the following axioms are satisfied:

1. All identities are admissible monics and admissible epics.
2. The composition of two admissible monics (resp. epics) is admissible monic (resp. epic).
3. The push-out of an admissible monic and an arbitrary morph ism exists and gives an ad-

An exact category $\mathcal{E}$ is given by a couple $(\mathcal{A}, E)$, where $\mathcal{A}$ is an additive category and $E$ is an exact structure on $\mathcal{A}$. A kernel-cokernel pair in $\mathcal{E}$ is called conflation and it is represented as a short exact sequence in the context of abelian categories, i.e. $0 \to A \to B \to C \to 0$. For the sake of simplicity, $\mathcal{E}$ will also denote the underlying additive category $\mathcal{A}$.

A.2. Theorem – Gabriel-Quillen embedding. ([TT90, Theorem A.7.1]). A (small) category $\mathcal{E}$ is exact if and only if it is a full extension closed additive subcategory of an abelian category $\mathcal{A}$. In particular, conflations are short exact sequences in $\mathcal{A}$.

A.3. Definition. A complex

$$\ldots \to X_{i-1} \overset{d^{i-1}}{\to} X_i \overset{d^i}{\to} X_{i+1} \to \ldots$$

is acyclic if, for all $i \in \mathbb{Z}$, $d^i$ factor through an object $C_i \in \mathcal{E}$ and $0 \to C_{i-1} \to X_i \to C_i \to 0$ is a conflation.

For an exact category $\mathcal{E}$, the bounded derived category is defined by the Verdier quotient $\mathcal{D}^b(\mathcal{E}) = \mathcal{K}^b(\mathcal{E}) / \mathcal{A}^b(\mathcal{E})$, where $\mathcal{K}^b(\mathcal{E})$ is the bounded homotopy category of complexes and $\mathcal{A}^b(\mathcal{E})$ is the full subcategory of bounded acyclic complexes.

A.4. Example.
1. An additive category with split exact sequences is exact, and the derived category associated to it is simply the homotopy category of complexes. This follows from the fact that split exact sequences are always homotopy equivalent to 0.

2. An abelian category with all its short exact sequences is exact, and the derived category associated is the expected one.

The following generalizes the last part of the statement of [Kra15, Lemma 3.1].

**A.5. Lemma.** Let $E$ be an exact category. Then $\text{Ac}^b(E) \subset K^b(E)$, the category of acyclic complexes, is the triangulated envelope of the full subcategory given by conflations (intended as complexes).

**PROOF.** Let $X = (X^i, d^i)$ be a bounded acyclic complex. Up to shift, we can assume $X^i = 0$ for $i < 1$ and $i > n$ for some $n > 3$. Let us write

$$X := \cdots \to 0 \to X^1 \to X^2 \to \cdots \to X^n \to 0 \to \cdots$$

$$X^{\leq n-2} := \cdots \to 0 \to X^1 \to X^2 \to \cdots \to X^{n-2} \xrightarrow{\rho} \text{coker}(d^{n-3}) \to 0 \to \cdots$$

$$X^{\geq n-1}[−1] := \cdots \to 0 \to \text{coker}(d^{n-3}) \xrightarrow{j} X^{n-1} \xrightarrow{d^{n-1}} X^n \to 0 \to \cdots$$

where the composition of $\rho : X^{n-2} \to \text{coker}(d^{n-3})$ and $j : \text{coker}(d^{n-3}) \to X^{n-1}$ gives $-d^{n-2}$.

From direct computations, we have that $X \cong \text{Cone}(f)$, where $f : X^{\geq n-1}[−1] \to X^{\leq n-2}$ is defined to be the identity on $\text{coker}(d^{n-3})$ and 0 elsewhere. The conclusion now follows from an induction. □

We recall that the Ext-groups of an exact category $E$ are given exactly as in the case of abelian categories. The $n$-extensions of $A$ and $B$ are, up to the same equivalence relation, acyclic complexes of length $n + 2$ of the form:

$$0 \to B \xrightarrow{\xi_0} X_1 \xrightarrow{\xi_1} X_2 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_{n-1}} X_n \xrightarrow{\xi_n} A \to 0$$

For a more precise discussion, the reader may consult [Lor22a, Definition A.4].

**A.6. Proposition.** Let $E$ be an exact category, and consider $F : \mathcal{D}^b(E) \to \mathcal{D}^b(E)$ a triangulated equivalence such that $F(X) = X$ for all $X \in E$. Then $F_{|E}$ determines uniquely $F_{|H^*_{DG}(E)}$.

**PROOF.** By [Pos11, Corollary A.7.1 and Proposition A.7], $H^*(E_{DG})$ is simply the category of the Ext-groups, i.e.

$$\text{Hom}_{H^*(E_{DG})}(X, Y) = \bigoplus_i \text{Ext}^i(X, Y)[-i]$$

for every $X, Y \in E$. Since every morphism in $\text{Hom}(X, Y[1])$ is associated to a conflation (as expressed in [Dye05]), given another triangulated autoequivalence $(G, \mu)$ such that $G_{|E} = F_{|E}$ (so $G(X) = X$ for all $X \in E$ as well), we have the following isomorphism of distinguished
where the first two vertical arrows are the identity because $G_{\xi,\varepsilon} = F_{\xi,\varepsilon}$. From the universal property of the cokernel $X$ in $\mathcal{E}$, the dashed morphism is also the identity, so that $\eta_Y Fh = \mu_Y Gh$. As every extension is obtained by Yoneda products of elements in the first Ext-group (cf. the beginning of the proof of [Lor22a Proposition A.7]), a simple induction concludes the proof.

References

[BL94] J. Bernstein and V. Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[BLL04] A. I. Bondal, M. Larsen, and V. A. Lunts. Grothendieck ring of pretriangulated categories. *Int. Math. Res. Not.*, 2004(29):1461–1495, 2004.

[BS01] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236(2):819–834, 2001.

[CC21] X. Chen and X.-W. Chen. An informal introduction to dg categories, 2021. arXiv 1908.04599.

[CHZ19] X. Chen, Z. Han, and Y. Zhou. Derived equivalences via HRS-tilting. *Adv. Math.*, 354:106749, 26, 2019.

[CNS22] A. Canonaco, A. Neeman, and P. Stellari. Uniqueness of enhancements for derived and geometric categories. *Forum Math. Sigma*, 10:65, 2022. Id/No e92.

[CS14] A. Canonaco and P. Stellari. Fourier-Mukai functors in the supported case. *Compos. Math.*, 150(8):1349–1383, 2014.

[CS17] A. Canonaco and P. Stellari. A tour about existence and uniqueness of dg enhancements and lifts. *J. Geom. Phys.*, 122:28–52, 2017.

[CY18] X.-W. Chen and Y. Ye. The $D$-standard and $K$-standard categories. *Adv. Math.*, 333:159–193, 2018.

[Dri04] V. Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.

[DS04] D. Dugger and B. Shipley. $K$-theory and derived equivalences. *Duke Math. J.*, 124(3):587–617, 2004.

[Dye05] M. J. Dyer. Exact subcategories of triangulated categories, 2005.

[Hov99] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[JM22] G. Jasso and F. Muro. The Triangulated Auslander–Iyama Correspondence, 2022. With an appendix by B. Keller. arXiv 2208.14413.

[Kel94] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[Kel06] B. Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[Kra15] H. Krause. Deriving Auslander’s formula. *Doc. Math.*, 20:669–688, 2015.

[KV87] B. Keller and D. Vossieck. Sous les catégories dérivées. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(6):225–228, 1987.
References

[LO10] V. A. Lunts and D. O. Orlov. Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.*, 23(3):853–908, 2010.

[Lor22a] A. Lorenzin. Compatibility of t-structures in a semiorthogonal decomposition. *Appl. Categ. Structures*, 30(4):755–778, 2022.

[Lor22b] A. Lorenzin. Some developments on existence and uniqueness of DG-enhancements. PhD thesis, University of Pavia and Milano-Bicocca, 2022.

[LPZ22] C. Li, L. Pertusi, and X. Zhao. Derived categories of hearts on Kuznetsov components, 2022. arXiv 2203.13864.

[Mur20] F. Muro. Enhanced finite triangulated categories. *J. Inst. Math. Jussieu*, pages 1–43, June 2020.

[Ola20] N. Olander. Orlov’s Theorem in the Smooth Proper Case, 2020.

[Ola22] N. Olander. Resolutions, Bounds, and Dimensions for Derived Categories of Varieties. ProQuest LLC, Ann Arbor, MI, 2022. PhD thesis, Columbia University.

[Orl97] D. O. Orlov. Equivalences of derived categories and K3 surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Algebraic geometry, 7.

[Orl03] D. O. Orlov. Derived categories of coherent sheaves and equivalences between them. *Uspekhi Mat. Nauk*, 58(3(351)):89–172, 2003.

[Pue82] R. S. Pierce. Associative algebras, volume 9 of *Studies in the History of Modern Science*. Springer-Verlag, New York-Berlin, 1982.

[Pos11] L. Positselski. Mixed Artin-Tate motives with finite coefficients. *Mosc. Math. J.*, 11(2):317–402, 407–408, 2011.

[RvdB19] A. Rizzardo and M. Van den Bergh. A note on non-unique enhancements. *Proc. Amer. Math. Soc.*, 147(2):451–453, 2019.

[Sai21] S. Saito. Tilting objects in periodic triangulated categories, 2021. arXiv 2011.14096.

[Sch02] M. Schlichting. A note on K-theory and triangulated categories. *Invent. Math.*, 150(1):111–116, 2002.

[Tab05] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.

[Toë07] B. Toën. The homotopy theory of dg-categories and derived Morita theory. *Invent. Math.*, 167(3):615–667, 2007.

[Toë11] B. Toën. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011.

[TT90] R. W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Yek20] A. Yekutieli. Derived categories, volume 183 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.