A CERTAIN CLASS OF LAPLACE TRANSFORMS WITH APPLICATIONS TO REACTION AND REACTION-DIFFUSION EQUATIONS

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Abstract. A class of Laplace transforms is examined to show that particular cases of this class are associated with production-destruction and reaction-diffusion problems in physics, study of differences of independently distributed random variables and the concept of Laplacianess in statistics, $\alpha$-Laplace and Mittag-Leffler stochastic processes, the concepts of infinite divisibility and geometric infinite divisibility problems in probability theory and certain fractional integrals and fractional derivatives. A number of applications are pointed out with special reference to solutions of fractional reaction and reaction-diffusion equations and their generalizations.

1 Introduction

We start with simple examples and then will proceed into generalizations. Consider a gamma density

$$f_1(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad x > 0, \beta > 0, \text{Re}(\alpha) > 0,$$

where $\text{Re}(\cdot)$ denotes the real part of (\cdot). Its Laplace transform, with parameter $p$, is given by the following:
\[
L_{f_1}(p) = \int_0^\infty e^{-px} f_1(x) \, dx = (1 + \beta p)^{-\alpha}, \quad 1 + \beta p > 0. \tag{1}
\]

For \(\alpha = 1\) we have the Laplace transform of the exponential density. Consider the Laplace transform of the Laplace density

\[
f_2(x) = \frac{1}{2\beta} e^{-|x|/\beta}, \quad -\infty < x < \infty, \quad \beta > 0.
\]

Its Laplace transform is given by the following

\[
L_{f_2}(p) = (1 - \beta^2 p^2)^{-1}. \tag{2}
\]

This Laplace transform \(L_{f_2}(p)\) can also arise from a production-destruction or input-output model of the following type: Consider the residual effect of an input-output type of situation \(u = x_1 - x_2\) where \(x_1\) and \(x_2\) are independently and identically distributed gamma type input and gamma type output variables respectively. Then the Laplace transform of the density of \(u\), denoted by \(L_u(p)\), is given by

\[
L_u(p) = L_{x_1}(p) L_{x_2}(-p) = (1 + \beta p)^{-\alpha} (1 - \beta p)^{-\alpha} = (1 - \beta^2 p^2)^{-\alpha}. \tag{3}
\]

This residual variable \(u\) has applications in many different areas. A few of the applications are mentioned in Mathai (1993a) and in Mathai, Provost, and Hayakawa (1995). The Laplace transform in (3) is also associated with the concept of Laplacianness in statistics, see Mathai (1993). This concept is connected to bilinear forms, quadratic form and the concept of chi-squaredness of quadratic forms, which is the basis for making inference in analysis of variance, analysis of covariance, regression and general model building areas.

If there are several independent input variables \(x_1, \ldots, x_n\) such as the situation in reaction or production problems, and if there are several independent output variables \(x_{m+1}, \ldots, x_{m+n}\) and if they are all gamma type variables with different parameters then the residual \(u = x_1 + \cdots + x_m - x_{m+1} - \cdots - x_{m+n}\) has the Laplace transform

\[
L_u(p) = (1 + \beta_1 p)^{-\alpha_1} \cdots (1 + \beta_m p)^{-\alpha_m} \times (1 - \beta_{m+1} p)^{-\alpha_{m+1}} \cdots (1 - \beta_{m+n} p)^{-\alpha_{m+n}}. \tag{4}
\]
The density of \( u \), corresponding to the Laplace transform in (4), is also available in the literature, see for example Mathai and Provost (1992). Now consider the Mittag-Leffler function

\[
E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + k\alpha)}, \quad x > 0, \Re(\alpha) > 0,
\]

and let

\[
f_3(x) = E_\alpha(-x^\alpha),
\]

then the Laplace transform

\[
L_{f_3}(p) = \int_0^\infty e^{-px}E_\alpha(-x^\alpha)dx = p^{\alpha-1}(1 + p^\alpha)^{-1}. \quad (5)
\]

Haubold and Mathai (1995, 2000) considered a reaction problem where the number density of the reacting particles is a function of time \( t \). For the \( i^{th} \) particle let the number density be \( N_i = N_i(t) \) with \( N_i(t=0) = N_0 \). If the production rate is proportional to the number density then we have the simple differential equation, dropping \( i \),

\[
\frac{d}{dt}N(t) = a \cdot N(t),
\]

where \( a \) is a constant. If some particles produced are also destroyed or consumed and if the destruction rate is given by

\[
\frac{d}{dt}N(t) = -b \cdot N(t),
\]

then the residual effect is given by

\[
\frac{d}{dt}N(t) = -c \cdot N(t), \quad c = b - a,
\]

and then the solution is

\[
N(t) = N_0 e^{-at}.
\]

But if a fractional integral, instead of the full integral, is used then the reaction equation is given by

\[
N(t) - N_0 = -c'_{0}D_{t}^{-\nu}N(t), \quad \nu > 0, \quad (6)
\]

where \( c \) is replaced by \( c' \) and its solution has the form
\[ N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{k\nu}}{\Gamma(1 + k\nu)} = N_0 E_{\nu}(-c^\nu t^\nu), \]

where the fractional integral is defined as follows

\[ aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t - u)^{\nu-1} f(u) du, \quad \nu > 0, \quad (7) \]

with \( aD_t^0 f(t) = f(t) \). The Laplace transform of \( N(t) \), coming from (6), is then,

\[ L_{N(t)}(p) = \frac{N_0}{p[1 + (\frac{c}{p})^\nu]}, \quad (8) \]

which is a constant multiple of (5) for \( c = 1, \nu = \alpha \).

2 Generalizations

If the basic fractional production-destruction equation in (6) is modified to the form

\[ N(t) - N_0 t^{\mu-1} = -c^\nu aD_t^{-\nu} N(t), \mu > 0, \nu > 0, \quad (9) \]

then the Laplace transform is given by

\[ L_{N(t)}(p) = \frac{N_0 \Gamma(\mu)}{p^\nu[1 + (\frac{c}{p})^\nu]} = \frac{N_0 \Gamma(\mu)}{p^{\mu-\nu}[p^\nu + c^\nu]} \quad (10) \]

\[ = \frac{N_0 \Gamma(\nu)}{[p^\nu + c^\nu]} \quad \text{for} \quad \mu = \nu. \quad (11) \]

This Laplace transform in (11) for \( c = 1 \) is associated with infinitely divisible and geometrically infinitely divisible distributions, \( \alpha \)-Laplace and Linnik distributions in probability theory and \( \alpha \)-Laplace and Mittag-Leffler stochastic processes, see the works of R.N. Pillai and his associates, a summary of which is available from Jose and Seetha Lekshmi (2004). A comprehensive presentation of this field of processes is given in Uchaikin and Zolotarev (1999). The class of Laplace transforms relevant in geometrically infinitely divisible and \( \alpha \)-Laplace distributions is of the form
\[ L(p) = \frac{1}{1 + \eta(p)}, \quad (12) \]

where \( \eta(p) \) satisfies the condition \( \eta(bp) = b^{\alpha} \eta(p) \) where \( \eta(p) \) is a periodic function for fixed \( \alpha \). Observe that (11) satisfies the above condition with \( \eta(p) = p^{\nu} \) and \( \alpha = \nu \). Thus the basic reaction equation in (9) is also associated with certain stochastic processes and infinitely divisible distributions.

Another observation that can be made is that the inverse of (10) or the number density \( N(t) \), which gives rise to (10), is of the following general form

\[ N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\nu}), \quad (13) \]

where \( E_{\nu,\mu}(\cdot) \) is a generalized Mittag-Leffler function,

\[ E_{\nu,\mu}(-c^{\nu} t^{\nu}) = \sum_{k=0}^{\infty} \frac{(-1)^k (c^{\nu} t^{\nu})^k}{\Gamma(\mu + k \nu)}, \quad \mu > 0, \quad \nu > 0. \quad (14) \]

Note that (14) is also connected to the F-function introduced by Hartley and Lorenzo (1998) and Lorenzo and Hartley (1999). Their F-function is the following,

\[ F_q(-a, t) = t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma(q + nq)} = t^{q-1} E_{q,\mu}(-at^q), \quad Re(q) > 0. \quad (15) \]

When the variable is restricted to the interval \( t \geq \delta \) for some \( \delta \), then (15) reduces to the R-function of Lorenzo and Hartley (1999), which is defined as

\[ R_{\nu,\mu}(a, \delta, t) = \sum_{n=0}^{\infty} \frac{a^n (t - \delta)^{(n+1)\nu - \mu - 1}}{\Gamma[(n + 1) \nu - \mu]}, \quad t > \delta > 0 \]

\[ = (t - \delta)^{\nu - \mu - 1} E_{\nu,\mu - \mu}[a(t - \delta)^{\nu}], \quad t > \delta > 0. \quad (16) \]

If we consider another modification of the basic reaction equation in (9) to the form

\[ N(t) - N_0 \ t^{\mu - 1} E_{\nu,\mu}(-c^{\nu} t^{\nu}) = -c^{\nu} D_t^{-\nu} N(t), \quad (17) \]

where \( E_{\nu,\mu}(\cdot) \) is a further generalized form of the Mittag-Leffler function given by
\[ E_{\nu,\mu}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \frac{x^k}{\Gamma(\mu + k\nu)} \quad Re(\mu) > 0, \ Re(\nu) > 0, \]  
\[ (\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1), \ \gamma \neq 0, (\gamma)_0 = 1, \]  
then the Laplace transform is the following

\[ L_{N(t)}(p) = \frac{N_0}{p^{\mu-\nu(\gamma+1)}[\varphi^\nu + p^\nu]^{\gamma+1}}. \]  

The inverse of (19) or the number density in this case is given by

\[ N(t) = N_0 t^{\mu-1} E_{\nu,\mu}(t^\alpha). \]  

The most general form of Laplace transform associated with Mittag-Leffler functions is that in (19), which is a special case of the general class of Laplace transforms associated with \( \alpha \)-Laplace stochastic processes and geometrically infinitely divisible statistical distributions.

If the Mittag-Leffler function is further generalized then the Laplace transforms will enter the family of Fox’s H-function. For example, let us consider a simple generalization of the Mittag-Leffler function to the following form

\[ g_1(t) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k}{(\beta_1)_k k!} \frac{t^k}{\Gamma(\beta + k\alpha)}. \]  

Then the Laplace transform of \( t^{\beta-1} g_1(t^\alpha) \), denoted by \( L_1(p) \), is given by the following

\[ L_1(p) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k}{(\beta_1)_k k!} p^{-(\alpha k + \beta)} = p^{-\beta} \pFq{1}{1}{(\gamma_1; \beta_1; p^{-\alpha})}, \]  

where \( \pFq{1}{1} \) is a confluent hypergeometric function. If the factor \( t^{\beta-1} \) is not incorporated then the Laplace transform will be a special case of an H-function. For a description and properties of Fox’s H-function see Mathai and Saxena (1978), Mathai (1993a), and Kilbas and Saigo (2004).

A direct generalization of a Mittag-Leffler function and the most generalized form in this category is Wright’s function given by

\[ p^\Psi_q \left[ \left( a_1, A_1, \cdots, a_p, A_p \right); z \right] = \sum_{k=0}^{\infty} \frac{\left\{ \prod_{j=1}^{p} \Gamma(a_j + A_j k) \right\} z^k}{\left\{ \prod_{j=1}^{q} \Gamma(b_j + B_j k) \right\} k!}, \]  

6
with \(1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0\).

Wright’s function is again a particular case of Fox’s H-function. Note that the Mittag-Leffler function

\[
E_{\alpha,\beta}(z) = 1_{\psi_{1}}^{(1,1)}[z] = H_{1,2}^{1,1}
\left[
\begin{array}{c}
(0,1) \\
(\beta,\alpha) \\
\end{array}
\right]
\left[
\begin{array}{c}
-\left(\begin{array}{c}
B_j \\
A_j \\
\end{array}\right) \\
\left(\begin{array}{c}
a_j \\
1-a_j \\
\end{array}\right) \\
\end{array}\right],
\]

(23)

where \(H(\cdot)\) denotes an H-function, and the H-function is defined in terms of the following Mellin-Barnes type representation

\[
H_{|z|}^{m,n} = \frac{1}{2\pi i} \int_L g(s) z^{-s} ds, i = \sqrt{-1},
\]

(24)

where

\[
g(s) = \left\{ \prod_{j=1}^{n} \Gamma(b_j + B_j s) \right\} \left\{ \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s) \right\}
\left\{ \prod_{j=m+1}^{p} \Gamma(1 - b_j - B_j s) \right\} \left\{ \prod_{j=n+1}^{p} \Gamma(a_j + A_j s) \right\},
\]

(25)

with \(A_j, j = 1, \cdots, p\) and \(B_j, j = 1, \cdots, q\) being positive real numbers and \(a_j\)'s and \(b_j\)'s are complex quantities. Existence conditions, the types of contour \(L\) and properties are available from Mathai and Saxena (1978) and Kilbas and Saigo (2004).

Observe that for \(\gamma = 1\) in (18) we have \(E_{1,\mu}^{\nu,\mu} = E_{\nu,\mu}^{\nu,\mu}\). Hence for \(\gamma = 1, 2, 3 \cdots\), for positive integer values, one can write \(E_{\nu,\mu}^{\nu,\mu}\) as a linear function of \(E_{.,.}(\cdot)\). Thus, certain linear functions of Laplace transforms of the type in (10) can be summed up to a Laplace transform of the type in (19). Some examples of this type may be seen from a series of papers by Haubold, Mathai, and Saxena, for example, see Saxena, Mathai, and Haubold (2004). One such example can be stated as follows: If \(c > 0, \mu > 0, \nu > 0\) then the solution of the fractional integral equation for the number density \(N(t)\),

\[
N(t) - N_0 \ t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\nu}) = -c^{\nu} 0D_t^{-\nu} N(t),
\]

(26)

is given by

\[
N(t) = \frac{N_0 \ t^{\mu-1}}{\nu} \left[ E_{\nu,\mu-1}(-c^{\nu} t^{\nu}) + (1 - \mu + \nu) E_{\nu,\mu}(-c^{\nu} t^{\nu}) \right].
\]

(27)
3 Reaction Equation

If the Laplace transform has the structure of a product of various factors then such a case can also be handled without much difficulty. As an example, consider the production-destruction fractional integral model for the number density $N(t)$ of the following type

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}(-d^\nu t^\nu) = -c^\nu_0 D_t^{-\nu} N(t). \quad (28)$$

Then it is not difficult to see that the Laplace transform of $N(t)$, with parameter $p$, denoted by $L_2(p)$ is the following

$$L_2(p) = \frac{N_0}{p^\mu [1 + (\frac{c}{p})^\nu][1 + (\frac{d}{p})^\nu]}.$$

(29)

When $c = d$ then this case reduces to the Laplace transform of a generalized Mittag-Leffler function, which can also be written as a linear combination of simple Mittag-Leffler functions as explained in (26) and (27). When $c \neq d$ then we may consider the following identity

$$\frac{1}{(p^\nu + c^\nu)(p^\nu + d^\nu)} = \frac{1}{(c^\nu - d^\nu) \left[ \frac{1}{p^\nu + d^\nu} - \frac{1}{p^\nu + c^\nu} \right]}, c \neq d. \quad (30)$$

Hence,

$$L_2(p) = \frac{1}{(c^\nu - d^\nu)} \left\{ \frac{1}{p^\mu - \nu} \left[ \frac{1}{1 + (\frac{d}{p})^\nu} - \frac{1}{1 + (\frac{c}{p})^\nu} \right] \right\}. \quad (31)$$

Then taking the inverse Laplace transform, we have,

$$N(t) = N_0 \frac{t^{\mu-\nu-1}}{(c^\nu - d^\nu)} \left[ E_{\nu,\mu-\nu}(-d^\nu t^\nu) - E_{\nu,\mu-\nu}(-c^\nu t^\nu) \right]. \quad (32)$$

Now, let us look into another situation where the Laplace transforms are of the following types

$$L_3(p) = \frac{p^{\alpha-1}}{p^\alpha + ap^\beta + b} \text{ and } L_4(p) = \frac{p^{\beta-1}}{p^\alpha + ap^\beta + b}, \quad (33)$$

where $a, b, \alpha, \beta$ are constants. Such Laplace transforms can be handled by using the following procedure. For example, consider
\begin{align*}
L_3(p) &= \frac{p^{\alpha-1}}{p^\alpha + ap^\beta + b} = \frac{p^{\alpha-\beta-1}}{(p^{\alpha-\beta} + bp^{-\beta})(1 + \frac{a}{p^{\alpha-\beta} + bp^{-\beta}})} \\
&= \sum_{r=0}^{\infty} (-a)^r \frac{p^{-\alpha} r^{-1}}{(1 + \frac{b}{p^\alpha})^{r+1}}.
\end{align*}

Comparing with the Laplace transform of a generalized Mittag-Leffler function we have the inverse given by the following

\begin{equation}
L^{-1}\left[\frac{p^{\alpha-1}}{p^\alpha + ap^\beta + b}\right] = \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} E_{\alpha,\alpha-\beta}(r+1)(-bt^\alpha).
\end{equation}

\section{Reaction-Diffusion Equation}

A problem recently considered by Saxena, Mathai, and Haubold (2005) is a reaction-diffusion system occurring in various areas of pattern formation in biology, chemistry, physics, see for example Henry and Wearne (2000, 2002), Henry, Langlands, and Wearne (2005), and Manne, Hurd, and Kenkre (2000). The basic reaction-diffusion equation has the form

\begin{equation}
\frac{\partial N}{\partial t} = \mu \frac{\partial^2 N}{\partial x^2} + \lambda f(N), N = N(x,t),
\end{equation}

where $\mu$ is the diffusion constant, $\lambda$ is a constant and $f(N)$ is a nonlinear function of $N$. When $f(N) = \delta N(1-N)$, where $\delta$ is a constant, then (36) reduces to the real Ginsburg-Landau equation. An equation in this category examined by Manne, Hurd, and Kenkre (2000) is the following

\begin{equation}
\frac{\partial^2 N}{\partial t^2} + a \frac{\partial N}{\partial t} = \nu^2 \frac{\partial^2 N}{\partial x^2} \xi^2 N(x,t),
\end{equation}

where $\xi$ indicates the strength of nonlinearity in the system. For solving (37) one can adopt the procedure of taking the Laplace transform with respect to $t$ and Fourier transform with respect to $x$, simplifying and then inverting the Laplace-Fourier transform to obtain $N(x,t)$. Details of the procedure may be seen from Saxena, Mathai, and Haubold (2004, 2005). In this procedure, the crucial step is the inversion of the Laplace transform of the type $L_4(p)$.
in (33). From (35) one can see that the inverse can be written as a series of Mittag-Leffler functions

\[
L^{-1}\left[p^{\beta - 1} \frac{p^{\beta - 1}}{p^\alpha + ap^\beta + b}\right] = \sum_{r=0}^{\infty} (-a)^r t^{(\alpha - \beta)(r+1)} E_{\alpha, (\alpha - \beta)(r+1)+1}(-bt^\alpha). \tag{38}
\]

5 Conclusions

Linear and nonlinear reaction, diffusion, and reaction-diffusion equations, respectively, are used to model spatio-temporal processes in physical systems, for example astrophysical fusion plasmas (Wilhelmsson and Lazzaro, 2001; Kulsrud, 2005). Such equations give rise to dissipative structures and self-organization phenomena (Nicolis and Prigogine, 1977; Haken 2004). Attempts have been made to employ them to discover a time-dependent mechanism in solar nucleosynthesis (Haubold and Mathai, 1995, 2000).

The linear reaction (relaxation) equation can be used to explore fundamental principles of Boltzmann-Gibbs statistical mechanics and its nonlinear generalization leads to new insights into nonextensive statistical mechanics (Tsallis, 2004). This approach has also been extended to discover the connection between Tsallis’ nonextensive maximum entropy formalism and the nonlinear reaction-diffusion equation (Tsallis and Bukman, 1996).

In this paper we used Laplace transform techniques to derive closed-form solutions of reaction equations starting from its simplest form and its fractional version in eq. (6) through generalizations of the equation given in (9), (17), (26), and (28). Along this way of generalizing the linear reaction equation we provide their respective closed-form representations in terms of Mittag-Leffler functions. Gradually the link between the Mittag-Leffler function and Wright’s function and, subsequently, Fox’s function has been shown. This transition can be characterized as a transition from the standard exponential behavior (”normal” relaxation) to a power-law behavior (”anomalous” relaxation) of the solutions of the respective reaction equations. For the reaction-diffusion equations (36) and (37) we indicate that the same Laplace transform techniques, as successfully employed for deriving closed-form solutions of reaction equations, can also be used for finding solutions of reaction-diffusion equations.
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