Classification of isomonodromy problems on elliptic curves

A. M. Levin, M. A. Olshanetsky, and A. V. Zotov

Abstract. This paper describes isomonodromy problems in terms of flat $G$-bundles over punctured elliptic curves $\Sigma_\tau$ and connections with regular singularities at marked points. The bundles are classified by their characteristic classes, which are elements of the second cohomology group $H^2(\Sigma_\tau, \mathcal{Z}(G))$, where $\mathcal{Z}(G)$ is the centre of $G$. For any complex simple Lie group $G$ and any characteristic class the moduli space of flat connections is defined, and for them the monodromy-preserving deformation equations are given in Hamiltonian form together with the corresponding Lax representation. In particular, they include the Painlevé VI equation, its multicomponent generalizations, and the elliptic Schlesinger equations. The general construction is described for punctured complex curves of arbitrary genus. The Drinfeld–Simpson (double coset) description of the moduli space of Higgs bundles is generalized to the case of the space of flat connections. This local description makes it possible to establish the Symplectic Hecke Correspondence for a wide class of monodromy-preserving problems classified by the characteristic classes of the underlying bundles. In particular, the Painlevé VI equation can be described in terms of $\text{SL}(2, \mathbb{C})$-bundles. Since $\mathcal{Z}(\text{SL}(2, \mathbb{C})) = \mathbb{Z}_2$, the Painlevé VI equation has two representations related by the Hecke transformation: 1) as the well-known elliptic form of the Painlevé VI equation (for trivial bundles); 2) as the non-autonomous Zhukovsky–Volterra gyrostat (for non-trivial bundles).

Bibliography: 123 titles.

Keywords: monodromy-preserving deformations, Painlevé equations, flat connections, Schlesinger systems, Higgs bundles.

Contents

1. Introduction 37

Chapter I. General approach to the isomonodromy problem 41

This work was supported by the Russian Foundation for Basic Research (grant no. 12-02-00594 and grant no. 12-01-33071-мол-а-вед for young researchers) and by the Programme "Leading Scientific Schools" (grant no. НШ-4724.2014.2). The first author was also supported by the Laboratory of Algebraic Geometry and its Applications at the National Research University “Higher School of Economics” (Agreement 11.G34.31.0023 of the Government of the Russian Federation). The third author was also supported by Dmitrii Zimin’s “Dynasty” Foundation.

AMS 2010 Mathematics Subject Classification. Primary 34M56, 14H60; Secondary 14H70, 17B80.

© 2014 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
2. Flat bundles. General case
   2.1. The moduli space of flat bundles over smooth curves
   2.2. The moduli space of flat bundles with quasi-parabolic structure
3. Flat bundles and characteristic classes
   3.1. The moduli space of holomorphic bundles via the double coset construction
   3.2. Characteristic classes
   3.3. Flat bundles
   3.4. Hecke transformations
4. Hamiltonian approach to the isomonodromy problem
   4.1. Deformation of complex structures on curves
   4.2. The equations of motion and the isomonodromy problem
   4.3. Contribution of marked points
   4.4. Symplectic reduction
   4.5. Isomonodromic deformations and integrable systems
Chapter II. Isomonodromy problems on elliptic curves
5. The moduli space of flat bundles over elliptic curves
   5.1. Holomorphic bundles over elliptic curves
   5.2. Decomposition of Lie algebras
   5.3. The moduli space of holomorphic bundles
6. Monodromy-preserving equations on elliptic curves
   6.1. Deformations of elliptic curves
   6.2. Symplectic reduction
   6.3. Lax matrix
   6.4. Classical $r$-matrix
   6.5. Symmetries of the phase space
   6.6. Hamiltonians
   6.7. $M$-operators
   6.8. Painlevé–Schlesinger equations
   6.9. KZB equations
   6.10. Painlevé field theory
7. Symplectic Hecke Correspondence
8. The Painlevé VI equation
   8.1. Three forms of the Painlevé VI equation
   8.2. Linear problems
   8.3. Elliptic form of rational connection
   8.4. Symplectic Hecke Correspondence
Appendix A. Simple Lie groups
Appendix B. Generalized sine basis in simple Lie algebras
Appendix C. Elliptic functions
Appendix D. Characteristic classes and conformal groups
Bibliography
1. Introduction

In this paper we propose a classification of isomonodromy problems for flat $G$-bundles over elliptic curves $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ with logarithmic singularities of connections at marked points $z_a$, $a = 1, \ldots, n$. The Lie group $G$ is simple and complex. As a preliminary, let us briefly recall that the isomonodromy problem on a genus-$g$ complex curve $\Sigma_g$ with $n$ marked points leads to the following (monodromy-preserving or isomonodromy) equations:

$$\partial_t_j L(z) - \partial_z M_j(z) = [L(z), M_j(z)],$$  \hspace{1cm} (1.1)

where $z$ is a local coordinate on $\Sigma_g$, the $t_j$ are the moduli of $\Sigma_{g,n} = \Sigma_g \setminus \{z_1, \ldots, z_n\}$ ($j = 1, \ldots, 3g - 3 + n$ for $g > 1$), and $L(z)$ and $M_j(z)$ are $g = \text{Lie}(G)$-valued functions. Equations (1.1) are compatibility conditions for the set of linear problems

$$\begin{cases}
(\partial_z + L(z))\Psi = 0,

(\partial_{t_j} + M_j(z))\Psi = 0,
\end{cases} \hspace{1cm} \Psi = \|\psi_1 \cdots \psi_{\dim V}\|;$$  \hspace{1cm} (1.2)

where $V$ is a finite-dimensional $G$-module and the $\psi_k \in \Gamma(EG)$, $k = 1, \ldots, \dim V$, are sections of the vector bundle $EG = \mathcal{P} \times_G V$ associated with a principal $G$-bundle $\mathcal{P}$. The lower equations arise from the requirement that the monodromies of the solutions of the upper equation for circuits around marked points and fundamental cycles of $\Sigma_g$ are independent of the moduli of the curve $\Sigma_{g,n}$.

Equations (1.1) hold identically with respect to $z$ and describe finite-dimensional non-autonomous Hamiltonian systems. The Painlevé equations, the Schlesinger systems, and their generalizations can be represented in this form (the Lax form or zero curvature form). Historically, the study of isomonodromy problems was motivated by the discovery of the list of the Painlevé equations [1]–[3]. It turned out that the second-order non-linear ordinary differential equations in that list can be written as compatibility conditions for a pair of linear equations on the punctured Riemann sphere [4]–[6]. The latter result was rediscovered around 1980 [7]–[11]. There is now an extensive literature on the isomonodromy problem, the Painlevé equations, and their numerous applications in mathematical physics [12]–[24].

The aim of this paper is to classify the equations (1.1), (1.2) for the case $g = 1$, that is, for elliptic curves $\Sigma_\tau$. The isomonodromy problems on general compact Riemann surfaces and, in particular, on the torus were studied in [25]–[27] by analytic methods. Here we follow the approach developed in [28]–[30]. It is based on the non-autonomous version of Hitchin systems [31] on punctured elliptic curves [32]–[35] (see also [36]–[38]). A construction of the Painlevé equations close in spirit was proposed in [39]. The idea was to reduce the space of flat connections to their moduli space by the action of the gauge group $G$ of symmetries which is generated by the automorphisms of $EG$. The corresponding moment map produced an equation for $L(z)$ of the form

$$\bar{\partial}L(z) = \sum_{a=1}^{n} S^a \delta^{(2)}(z - z_a),$$  \hspace{1cm} (1.3)

where the $S^a$ are the residues of $L(z)$ at the marked points. Equation (1.3) together with the boundary conditions

$$\Psi(z + 1) = \mathcal{D}(z)\Psi(z), \hspace{1cm} \Psi(z + \tau) = \Lambda(z)\Psi(z)$$  \hspace{1cm} (1.4)
for the solutions of (1.1) fixes $L(z)$. The matrices $\mathcal{L}(z)$ and $\Lambda(z)$ are the transition functions of the bundle $E_G$. They satisfy the cocycle condition

$$\mathcal{L}(z)\Lambda^{-1}(z)\mathcal{L}^{-1}(z+\tau)\Lambda(z+1) = 1.$$ (1.5)

We mention again that the same equations (1.3)–(1.5) describe integrable (Hitchin) systems on an elliptic curve. Their solutions (in some particular cases) are known as elliptic Gaudin models [40], [32]. The corresponding matrices $L(z)$ were constructed via a natural generalization of Krichever’s Ansatz for Lax pairs of elliptic integrable systems [41]. The phase spaces of the integrable systems consist of two parts: the set of $S^\alpha$ (the ‘spin’ part) and the moduli of solutions of (1.5), the configuration space connected with the ‘many-body’ degrees of freedom. The isomonodromy versions of integrable systems include non-autonomous generalizations of elliptic Calogero systems, the elliptic top, and the elliptic Schlesinger systems [30], [42], [43]. The phenomenon in which the same pair of matrices satisfies both the Lax equation of an integrable system and the monodromy-preserving deformation equation (1.1) was observed in [44] and is known as the Painlevé–Calogero Correspondence.

In the general case solutions of (1.5) are classified by the characteristic classes of the bundles, that is, elements of the second cohomology group $H^2(\Sigma_\tau, \mathcal{L}(G))$, where $\mathcal{L}(G)$ is the centre of $G$. A classification of these solutions, of the underlying Higgs bundles, and of the corresponding integrable systems (in the case of a single marked point) is given in our recent papers [45] and [46] (see also [47]–[49]). The classification arises from the fact that the transition matrices $\mathcal{L}(z)$ and $\Lambda(z)$ can be chosen to be $z$-independent up to a scalar multiple (of $\mathcal{L}$ and $\Lambda$). Then (1.5) is replaced (with regard to the scalar multiple) by

$$\mathcal{L}\Lambda^{-1}\mathcal{L}^{-1}\Lambda = \zeta, \quad \zeta \in \mathcal{L}(G).$$ (1.6)

Most representative for $SL(N, \mathbb{C})$-bundles$^1$ are $\zeta = 1$ and $\zeta = \exp(2\pi \sqrt{-1}/N)$. Corresponding integrable systems are the Calogero–Moser elliptic model and the elliptic top, respectively. Non-trivial intermediate cases arise when $N = pl$. The characteristic classes $\zeta = \exp((2\pi \sqrt{-1}/N)p)$ or $\zeta = \exp((2\pi \sqrt{-1}/N)l)$ correspond to the so-called models of interacting elliptic tops [51], [52]. The phase space in these cases has the same dimension as the phase space of the Calogero model with spin variables, but with fewer particles ($n < N$) and a greater number of spin degrees of freedom.

In this paper we extend the classification of Higgs bundles and integrable systems by characteristic classes [45], [46] to the case of flat connections and an arbitrary number of punctures on elliptic curves.$^2$

We use two descriptions of the moduli space of flat connections. The first [30] is a natural generalization of the group-theoretic description of the moduli space of Higgs bundles in the framework of the Hitchin approach to integrable

$^1$The classification of bundles over elliptic curves in this case was proposed by Atiyah [50]. In our terms his result is formulated as follows: $H^2(\Sigma_\tau, \mathcal{L}(SL(N, \mathbb{C}))) \sim \mathcal{L}(SL(N, \mathbb{C})) = \mathbb{Z}/N\mathbb{Z}$, that is, bundles are classified by $N$th roots of unity.

$^2$An alternative classification of Higgs bundles was proposed recently in [53]. The comparison of these results with [45], [46] needs further elucidation. The non-compact case of real Lie groups was discussed recently in [54], [55].
systems [32], [35]. It is given as the quotient space

\[ \text{FBun}(\Sigma_g, G) = \text{Conn}(\Sigma_g, G)/\mathcal{G} = \text{FConn}(\Sigma_g, G)/\mathcal{G} \]  

(1.7)
of the space of flat connections

\[ \text{FConn}(\Sigma_g, G) = \left\{ d + \mathcal{A} \mid d\mathcal{A} + \frac{1}{2} \mathcal{A} \wedge \mathcal{A} = 0 \right\} \]  

(1.8)

modulo the gauge group \( \mathcal{G} \). The space \( \text{Conn}(\Sigma_g, G) = \{ d + \mathcal{A} \} \) of smooth connections on \( E_G \) is equipped with the well-known (Atiyah–Bott) symplectic structure \( \omega = \frac{1}{2} \int_{\Sigma_g} \langle \delta \mathcal{A} \wedge \delta \mathcal{A} \rangle \) [56]. After the reduction it becomes the symplectic (and the Poisson) structure of a finite-dimensional Hamiltonian system. Its phase space \( \text{FBun}(\Sigma_g, G) \) is the principal homogeneous space \( PT^* \text{Bun}(\Sigma_g, G) \) over \( T^* \text{Bun}(\Sigma_g, G) \).

The second description of \( \text{FBun}(\Sigma_g, G) \) is from the Drinfeld–Simpson description [57], [58] for the moduli of Higgs bundles. It is formulated in terms of local data (see §3). The moduli space of flat bundles with quasi-parabolic structures at the marked points is given as the double coset space

\[ \text{Bun}(\Sigma_{g,n}, G) = G(\Sigma_{g,n}) \setminus G(D^\times)/G(D), \]  

(1.9)

where \( G(X) \) consists of holomorphic maps from \( X \subset \Sigma_g \) to \( G \), \( D^\times \) is the disjoint union of small punctured disks around the marked points (where the bundle is trivialized), and \( G(D) \) consists of special maps preserving the flags corresponding to the quasi-parabolic structures at the marked points. At the level of \( \text{FBun}(\Sigma_g, G) \) this construction means that we are working with trivial bundles over \( \Sigma_{g,n} \) with regular singularities at the marked points. It is known that a regular singularity can be transformed into a Fuchsian singularity by a meromorphic gauge transformation on a disc. Therefore, it is natural to equip a connection with a meromorphic gauge transformation that is determined up to right multiplication by a holomorphic transformation on the disc. Thus, the action of \( G(D^\times) \) can transform a regular singularity into a Fuchsian one. We can treat such a meromorphic gauge transformation as a transition function for a non-trivial bundle with a Fuchsian singularity. We thus obtain a description of \( \text{FBun}(\Sigma_g, G) \) similar to (1.9).

The latter description is very natural for the definition of Hecke operators (or modifications of bundles) [59], [60] which connect the linear problems (1.2) and the equations (1.1) for different characteristic classes. On the level of the connection component \( L(z) \) the modification acts by a gauge transformation

\[ L \overset{\Xi}{\rightarrow} L^\text{mod}, \quad L^\text{mod}(z)\Xi(z) = \Xi(z)L(z) - \partial_z \Xi(z), \]  

(1.10)

which is degenerate at some point \( z_0 \), that is, \( \det \Xi(z, z_0) \sim z - z_0 \) near \( z_0 \). In this way we extend the *Symplectic Hecke Correspondence* introduced in [61] to the case of monodromy-preserving deformation equations. As an example we consider the Painlevé VI equation. It can be described in terms of \( \text{SL}(2, \mathbb{C}) \)-bundles. Since \( \mathcal{Z}(\text{SL}(2, \mathbb{C})) = \mathbb{Z}_2 \), the equation has two representations related by the Hecke transformation:
1) the elliptic form of the Painlevé VI equation [62], [63],

\[ \frac{d^2 u}{d\tau^2} = \sum_{a=0}^{3} \nu_a^2 \phi'(u + \omega_a); \]  

(1.11)

2) the non-autonomous analogue of the Zhukovsky–Volterra gyrostat [64],

\[ \partial_\tau S = [S, J(S)] + [S, \nu'], \]  

(1.12)

where \( S \) is an \( \mathfrak{sl}^*(2, \mathbb{C}) \)-valued dynamical variable, and \( J \) and \( \nu' \) are non-dynamical, but \( \tau \)-dependent. The \( 2 \times 2 \) linear problems for (1.11) and (1.12) were described in [65] and [64], respectively (details in §8). In the general case the Hecke transformation (1.10) can be regarded as a Bäcklund transformation for monodromy-preserving equations. In other words, it is an analogue of a discrete-time shift, or of the Schlesinger transformation [10].

The monodromy-preserving equations can be considered as deformations of integrable systems. Integrable hierarchies corresponding to the systems considered here were constructed in [45], [46]. The correspondence between integrable systems and isomonodromic deformations, called in [44] the Painlevé–Calogero correspondence, was used already by Boutroux to investigate solutions of the Painlevé equations [66]. Similarly, Garnier constructed an autonomous analogue of the Schlesinger equations [67], and thereby arrived at the isospectral problem. Flaschka and Newell in [7], [9] and recently Krichever in [68] developed the Boutroux–Garnier programme and found that the WKB approximation with respect to the ‘Planck constant’ \( \kappa \) converts the isomonodromy problem into an isospectral problem.

Our paper consists of two chapters and four appendices. The first chapter is devoted to the general approach to the isomonodromy problem on \( \Sigma_{g,n} \). In §§2 and 3 we describe the moduli space of flat connections in two ways. First, in terms of the connections (1.8) and the gauge group. Second, in terms of local data, as the double coset space (1.9). The latter description is used for defining the characteristic classes and the Hecke transformations at the end of §3. In §4 the isomonodromy problem is described and the Hamiltonian approach is given. In the second chapter we consider the case of an elliptic curve in detail. We extend the previously obtained classification of elliptic integrable systems to monodromy-preserving equations. We also briefly review the related Knizhnik–Zamolodchikov–Bernard equations and the field-theoretic generalizations of the equations obtained. The classification is demonstrated with the example of the Painlevé VI equation (§8). In the appendices we give necessary short summaries about Lie groups and elliptic functions. We also describe the generalized sine algebra basis in Lie algebras, which is convenient for classification by characteristic classes. Finally, we describe conformal versions of Lie groups in order to connect characteristic classes with degrees of bundles.

We are grateful to V. Pobereznyi for useful discussions.
Chapter I

General approach to the isomonodromy problem

Let $G$ be a complex simple Lie group and let $E_G$ be a flat $G$-bundle over a curve $\Sigma_{g,n}$ of genus $g$ with $n$ marked points. A complex structure on $\Sigma_{g,n}$ defines a polarization of connections acting on sections in $\Gamma(E_G)$. Locally, in the complex coordinates $(z, \bar{z})$ a connection $d + \mathcal{A}$ has the form $((\partial + A) \otimes dz, (\overline{\partial} + \overline{A}) \otimes d\bar{z})$. By flatness, the following system is consistent:

$$
\begin{align*}
(\partial + A)\psi &= 0, \\
(\overline{\partial} + \overline{A})\psi &= 0,
\end{align*}
$$

$\psi \in \Gamma(E_G)$.

We assume that the monodromies of solutions $\psi$ are independent of the complex structure on $\Sigma_{g,n}$. The independence conditions are differential equations that in some cases can be written explicitly. Our main objects are flat $G$-bundles over elliptic curves $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. But first we consider the case of curves of arbitrary genus.

2. Flat bundles. General case

Let $\mathcal{P}$ be a principal $G$-bundle over $\Sigma_{g,n}$, let $V$ be a finite-dimensional $G$-module, and let $E_G = \mathcal{P} \times_G V$.

We will consider two cases:
1) smooth proper (compact) curves ($n = 0$);
2) smooth proper curves with punctures (marked points) ($n \neq 0$).

2.1. The moduli space of flat bundles over smooth curves. For smooth curves we define the space $\text{Conn}(\Sigma_g, G) = \{d + \mathcal{A}\}$ of smooth connections on $E_G$. The automorphism group $\mathcal{G}$ of $E_G$ (the gauge group) acts on connections by affine transformations:

$$
\mathcal{G}: \mathcal{A} \rightarrow f^{-1} df + f^{-1} \mathcal{A} f.
$$

Let $\text{FConn}(\Sigma_g, G)$ be the space of flat connections:

$$
\text{FConn}(\Sigma_g, G) = \left\{ d + \mathcal{A} \mid \text{d}\mathcal{A} + \frac{1}{2} \mathcal{A} \wedge \mathcal{A} = 0 \right\}.
$$

The group $\mathcal{G}$ preserves flatness. Then the moduli space of flat connections is the quotient

$$
\text{FBun}(\Sigma_g, G) = \text{FConn}(\Sigma_g, G)/\mathcal{G}.
$$

On the other hand, $\text{FBun}(\Sigma_g, G)$ can be described as the result of Hamiltonian reduction of the symplectic space $\text{Conn}(\Sigma_g, G)$ of all smooth connections by the action of the gauge group $\mathcal{G}$. The symplectic form on $\text{Conn}(\Sigma_g, G)$ is

$$
\omega = \frac{1}{2} \int_{\Sigma_g} \langle \delta \mathcal{A} \wedge \delta \mathcal{A} \rangle,
$$

where $\langle \cdot, \cdot \rangle$ denotes the Killing form and $\delta \mathcal{A}$ is a Lie($G$)-valued 1-form on $\Sigma_g$. That is, $\langle \delta \mathcal{A}, \delta \mathcal{A} \rangle$ is a 2-form, and the integral is well defined. This form is
gauge-invariant. Hamiltonian reduction with respect to this group action leads to the moment map

$$\text{Conn}(\Sigma_g, G) \to \mathcal{G}^* \sim \Omega^2_{C^\infty}(\Sigma_g, \text{Lie}^*(G)),$$

$$\mathcal{A} \mapsto F_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

Hence, the pre-image of 0 under the moment map is the space \(\text{FConn}(\Sigma_g, G)\) of flat connections (2.2). The choice of a complex structure on \(\Sigma_g\) defines a polarization on \(\text{Conn}(\Sigma_g, G)\). Then a connection is decomposed into the \((1, 0)\)- and \((0, 1)\)-parts:

$$A = (A, A).$$

We can write the \((1, 0)\)- and \((0, 1)\)-components of the connection in the local coordinates \((z, \bar{z})\):

$$\text{Conn}(\Sigma_g, G) = \{d' = (\partial + A) \otimes dz, \ d'' = (\bar{\partial} + \bar{A}) \otimes d\bar{z} \} \quad (\partial = \partial_z, \ \bar{\partial} = \partial_{\bar{z}}).$$

In this description the \(\omega\) in (2.4) assumes the form

$$\omega = \int_{\Sigma_g} \langle \delta A \wedge \delta \bar{A} \rangle. \quad (2.6)$$

We define the moduli space of holomorphic bundles. A section \(s \in \Gamma(E_G)\) is holomorphic if it is annihilated by the operator \(\partial + A\). The moduli space of holomorphic bundles is the quotient

$$\text{Bun}(\Sigma_g, G) = \{\partial + A\}/\mathcal{G}.$$

Let \(b\) be a point in \(\text{Bun}(\Sigma_g, G)\) and let \(\mathcal{V}_b\) be the corresponding bundle \(E_G\) on \(\Sigma_g\). Denote by \(\text{ad}_b\) the bundle of endomorphisms of \(\mathcal{V}_b\) with fibres isomorphic to a Lie algebra. Let \(\text{Flat}_b\) be the space of flat holomorphic connections on the holomorphic bundle \(\mathcal{V}_b\). In other words, in local coordinates

$$\text{Flat}_b = \{\partial + A \mid F(A, \bar{A}) = 0\}. \quad (2.8)$$

The tangent space to \(\text{Bun}(\Sigma_g, G)\) at the point \(b = E_G\) is canonically isomorphic to the first cohomology group \(H^1(\Sigma_g, \text{ad}_b)\). The space \(\text{Flat}_b\) of holomorphic connections on \(\mathcal{V}_b\) is an affine space over the vector space \(H^0(\Sigma_g, \text{ad}_b \otimes \Omega^1)\) of holomorphic \(\text{ad}_b\)-valued 1-forms, since the difference between any two connections is an \(\text{ad}_b\)-valued differential form. The vector spaces \(H^1(\Sigma_g, \text{ad}_b)\) and \(H^0(\Sigma_g, \text{ad}_b \otimes \Omega^1)\) are dual.

We consider the map from \(\text{FBun}(\Sigma_g, G)\) onto the moduli space \(\text{Bun}(\Sigma_g, G)\). The fibre of the projection \(\text{FBun}(\Sigma_g, G) \to \text{Bun}(\Sigma_g, G)\) over a point \(b\) is naturally isomorphic to \(\text{Flat}_b\):

$$p: \text{FBun}(\Sigma_g, G) \text{Flat}_b \to \text{Bun}(\Sigma_g, G).$$

These fibres are Lagrangian with respect to \(\omega\) in (2.6). From the Riemann–Roch theorem we have

$$\dim(\text{Bun}(\Sigma_g, G)) = \dim(H^1(\Sigma_g, \text{ad}_b)) = \dim(G)(g - 1),$$

$$\dim(\text{FBun}(\Sigma_g, G)) = 2 \dim(G)(g - 1). \quad (2.10)$$

Let \(K\) be the canonical class of \(\Sigma_g\). The Higgs bundle is defined to be the pair

$$(E_G, \Omega^0(\Sigma_g, \text{End}(E_G) \otimes K)).$$
In local coordinates it is represented by the pair \((\nabla + A, \Phi)\), where
\[
\Phi \in \Omega^0(\Sigma, \text{End}(E_G) \otimes K)
\]
is called the \emph{Higgs field}. The Higgs bundle is a symplectic manifold equipped with the symplectic form
\[
\omega^{\text{higgs}} = \int_{\Sigma} \langle \delta \Phi \wedge \delta A \rangle
\]  
(2.11)
(cf. (2.4)). The form is invariant under the action of the gauge group \(\Phi \rightarrow \text{Ad}_f \Phi, \ A \rightarrow f^{-1} \delta f + f^{-1} \delta A f\). After the symplectic reduction we arrive at the cotangent bundle \(T^* \text{Bun}(\Sigma, G)\) to the moduli space. It is the phase space of the Hitchin integrable systems [31].

2.2. The moduli space of flat bundles with quasi-parabolic structure.

We now consider curves \(\Sigma_{g,n}\) with marked points \(x = (x_1, \ldots, x_n)\). Let \(P_a (a = 1, \ldots, n)\) be parabolic subgroups, \(P_a \subset G\). We associate the \(G\)-flag varieties \(\text{Flag}_a \sim G/P_a\) with the marked points, and then say that \(E_G\) has a \emph{quasi-parabolic structure} (qp-structure) [69]. The group \(\mathcal{G}_P\) of automorphisms of this bundle preserves the flags at the marked points. This means that in a neighbourhood \(\mathbb{U}_a\) of \(x_a\)
\[
\mathcal{G}_P = \left\{ f \in \mathcal{G} \mid f|_{\mathbb{U}_a} = P_a + O(z - x_a) \right\}.
\]  
(2.12)
We replace the moduli space \(\text{Bun}(\Sigma, G)\) in (2.7) by the moduli space \(\text{Bun}(\Sigma_{g,n}, x, G)\) of \(G\)-bundles \(E_G\) with qp-structure
\[
\text{Bun}(\Sigma_{g,n}, x, G) = \{ \nabla + A \}/\mathcal{G}_P.
\]  
(2.13)
We have the natural ‘forgetful’ projection
\[
\pi: \text{Bun}(\Sigma, G) \xrightarrow{\Pi_a \text{Flag}_a} \text{Bun}(\Sigma, G).
\]  
(2.14)
The bundle of epimorphisms with these data is the bundle \(\text{Lie}(\mathcal{G}_P) = \text{ad}_{b,\text{Flag}}(-x)\) of endomorphisms whose sections behave near the marked points as follows:
\[
\varphi(z - x_a) = \varphi_a^0 + (z - x_a)\varphi_a^1 + \cdots, \quad \varphi_a^0 \in \text{Lie}(P_a).
\]
The tangent space to \(\text{Bun}(\Sigma, x, G)\) is isomorphic to \(H^1(\Sigma, \text{ad}_{b,\text{Flag}}(-x))\). The dual space to this space is \(H^0(\Sigma, \text{ad}_{b,\text{Flag}}^*(x) \otimes \Omega^1)\).

We consider the coadjoint \(G\)-orbits in the Lie coalgebra \(g^*\) located at the marked points:
\[
\mathcal{O}_a = \{ S_a = \text{Ad}_g S_a^0, \ g \in G, \ S_a^0 \in g^* \} \quad (a = 1, \ldots, n).
\]  
(2.15)
The variables \(S_a\) are called the \emph{spin variables}. The coadjoint orbit \(\mathcal{O}\) is fibered over the flag variety \(\text{Flag}\), and the fibers are the principal homogeneous spaces \(PT^*\text{Flag}\) over the cotangent bundles \(T^*\text{Flag}\). It is a symplectic space with the Kirillov–Kostant symplectic form
\[
\omega^{\text{KK}} = \delta(S^0, \delta g g^{-1}) = \langle S, g^{-1} \delta g \wedge g^{-1} \delta g \rangle.
\]  
(2.16)
Consider the space \(\text{Conn}(\Sigma_{g,n} G)\) of smooth connections with singularities at the marked points. In small neighbourhoods \(\mathbb{U}_a\) of the marked points we can gauge \(A\).
away \((\overline{A} = 0)\). Assume that \(A\) has holomorphic first-order poles at the \(x_a\) with residues taking values in the orbits \(\mathcal{O}_a\). Then the space of connections is defined as

\[
\text{Conn}(\Sigma_{g,n}, G) = \left\{ (A, \overline{A}) \middle| \begin{align*}
\overline{A}|_{\mathcal{U}_a} &= 0, \\
A &= S_a (z_a - x_a)^{-1} + O(1), \quad S_a \in \mathcal{O}_a
\end{align*} \right\}. \tag{2.17}
\]

The flatness condition \((2.2)\) takes into account the poles:

\[
\text{FConn}(\Sigma_{g,n}, G) = \left\{ \mathcal{A} \in \text{Conn}(\Sigma_{g,n}, G) \mid F_{\mathcal{A}} = \sum_{a=1}^{n} S_a \delta^{(2)}_{x_a} \right\}. \tag{2.18}
\]

Consequently, in order to get a symplectic variety we must replace the affine space \(\text{Flat}_b\) of flat holomorphic connections by the affine space \(\text{Flat}_b(\log)\) of flat holomorphic connections with logarithmic singularities. As a result, we get the moduli space (see \((2.3)\)) of pairs (holomorphic bundle, holomorphic connection with logarithmic singularities with residues at marked points in the form of the orbits \(\mathcal{O}_a\)):

\[
\text{FBun}(\Sigma_g, x, G) = \text{FConn}(\Sigma_{g,n}, G) / \mathcal{G}_P = \text{Conn}(\Sigma_{g,n}, G) / / \mathcal{G}_P. \tag{2.19}
\]

The symplectic form \((2.6)\) is now

\[
\omega = \int_{\Sigma_g} \langle \delta A \wedge \delta \overline{A} \rangle + \sum_{a=1}^{n} \omega^{KK}_a. \tag{2.20}
\]

As in the case of \((2.14)\) we have the projection

\[
\pi: \text{FBun}(\Sigma_g, x, G) \xrightarrow{\Pi_a \theta_a} \text{FBun}(\Sigma_g, G) \tag{2.21}
\]

and the Lagrangian projection (cf. \((2.9)\))

\[
p: \text{FBun}(\Sigma_g, x, G) \to \text{Bun}(\Sigma_g, x, G). \tag{2.22}
\]

The fiber of this projection is an affine space over \(H^0(\Sigma, \text{ad}^*_b \text{Flag}(x) \otimes \Omega^1)\). From \((2.10)\) we get that

\[
\dim(\text{Bun}(\Sigma_g, x, G)) = \dim(G)(g - 1) + \sum_{a=1}^{n} \dim(\text{Flag}_a),
\]

\[
\dim(\text{FBun}(\Sigma_g, x, G)) = 2 \dim(G)(g - 1) + \sum_{a=1}^{n} \dim(\mathcal{O}_a). \tag{2.23}
\]

Therefore, to obtain a non-trivial moduli space of bundles over an elliptic curve we should have at least one marked point.

### 3. Flat bundles and characteristic classes

#### 3.1. The moduli space of holomorphic bundles via the double coset construction.

We also need another description of the moduli spaces \(\text{Bun}(\Sigma_{g,n}, x, G)\) and \(\text{FBun}(\Sigma_{g,n}, x, G)\). They can be defined in the following way. A \(G\)-bundle can
be trivialized over a disjoint union $D = \bigcup_{a=1}^{n} D_a$ of small discs around the marked points and over $\Sigma_{g,n} \setminus \mathbf{x}$. Thus, $E_G$ is defined by holomorphic transition functions on $D^\times = \bigcup_{a=1}^{n} D_a^\times$, where $D_a^\times = D_a \setminus x_a$. If $G(X)$ consists of holomorphic maps from $X \subset \Sigma_g$ to $G$, then the isomorphism classes of holomorphic bundles are defined as the double coset spaces

$$\text{Bun}(\Sigma,g,G) = \mathcal{G}_{\text{out}} \setminus G(D^\times)/\mathcal{G}_{\text{int}}, \quad \mathcal{G}_{\text{out}} = G(\Sigma_g \setminus \mathbf{x}), \quad \mathcal{G}_{\text{int}} = G(D).$$

Here $G(D^\times)$ and $G(D)$ are defined as products of loop groups. Let $t_a$ be local coordinates in the disks $D_a$. We replace $G(D)$ and $G(D^\times)$ by the formal series

$$G(D) = \prod_{a=1}^{n} G(D_a) \to \prod_{a=1}^{n} G \otimes \mathbb{C}[[t_a]], \quad (3.1)$$

$$G(D^\times) \to \prod_{a=1}^{n} L_a(G), \quad L_a(G) = G \otimes \mathbb{C}[t_a^{-1}, t_a] \quad (A.60). \quad (3.2)$$

The quotient is again isomorphic to $\text{Bun}(\Sigma,g,G)$:

$$\text{Bun}(\Sigma,g,G) = \mathcal{G}_{\text{out}} \setminus \prod_{a=1}^{n} L_a(G)/\mathcal{G}_{\text{int}}, \quad \mathcal{G}_{\text{out}} = G(\Sigma_g \setminus \mathbf{x}), \quad \mathcal{G}_{\text{int}} = \prod_{a=1}^{n} G \otimes \mathbb{C}[[t_a]]. \quad (3.3)$$

The connection $d''$ in (2.5) can be reconstructed from the holomorphic transition functions $g_a \in L_a(G)$. Then the double coset space (3.3) is equivalent to the definition (2.7).

Let us fix $G$-flags at the fibers over the marked points. Recall that the qp-structure of the $G$-bundle means that $G(D)$ preserves these $G$-flags. In other words, $\mathcal{G}_{\text{int},P} = \prod_{a=1}^{n} L_a^+(G)$, where

$$L_a^+(G) = \{ g_0 + g_1 t + \cdots, g_0 \in P \} \quad (A.61). \quad (3.4)$$

The moduli space of holomorphic bundles with qp-structures at the marked points is the double coset space

$$\text{Bun}(\Sigma_{g,n}, \mathbf{x}, G) = G(\Sigma_g \setminus \mathbf{x}) \setminus G(D^\times)/\mathcal{G}_{\text{int},P}. \quad (3.5)$$

We prove below that this definition is equivalent to the previous one (2.13).

### 3.2. Characteristic classes.

For simplicity we first consider in (3.2) the one-point case $\mathbf{x} = x_0$. Let $t$ be a local coordinate in the disc $D_{x_0}$ ($t = 0 \sim x_0$). In the representation (3.5) replace $g(t) \in G(D^\times)$ by $g(t)h(t)$, where $h(t) \in D_{x_0}$. By (3.5), $h(t)$ is defined up to multiplication on the right by an $f(t) \in G(\mathbb{C}[[t]])$. On the other hand, since the original transition function $g(t)$ is defined up to multiplication on the right by an element of $G(\mathbb{C}[[t]])$, $h(t)$ is an element of the double coset space

$$L^+(G)/L(G)/L^+(G). \quad (3.6)$$

For $g(t) = \tilde{w} = wt^\gamma$ this double coset space is the affine Schubert cell $C_{\tilde{w}}$ (A.71) in the affine flag variety $\text{Flag}^{\text{aff}} = L(G^{\text{ad}})/L^+(G^{\text{ad}})$ in (3.6). The dimension of $C_{\tilde{w}}$ is $l(\tilde{w})$ (see (A.72)).
By (A.66)–(A.68) the moduli space (3.5) is the union of sectors defined by the affine Weyl groups. In particular, for the $G^{\text{ad}}$-bundles

$$\text{Bun}(\Sigma_{g,n}, x, G^{\text{ad}}) = \bigcup_{\hat{\omega} \in \hat{W}_P} \text{Bun}_{\hat{\omega}}(\Sigma_{g,n}, x, G^{\text{ad}}),$$

(3.7)

$$\text{Bun}_{\hat{\omega}}(\Sigma_{g,n}, x, G^{\text{ad}}) = G^{\text{ad}}(\Sigma_g \setminus x) \backslash G^{\text{ad}}(D_x^\times) \hat{\omega}/G^{\text{ad}}(D).$$

(3.8)

We can write this representation for the groups $\overline{G}$ and $G_l$ in (3.18).

**Proposition 3.1.** The double coset construction of the moduli space (3.7) is equivalent to its Dolbeault construction (2.13).

**Proof.** Let $g(t) \in G \otimes \mathbb{C}[t^{-1}, t]]$ be a transition function defining the bundle $E_G$, and consider the decomposition (A.68) of $g(t)$ on a small disc $\bar{D}_{x_0} \supset D_{x_0}$:

$$g(t) = g_- \hat{w} g_+(t), \quad g_- \in N^-(G) \quad (A.62),$$

$$g_+(t) \in G(D_{x_0}) = L^+(G) \quad (3.1).$$

(3.9)

For $\mathbb{CP}^1$ it coincides with the Birkhoff decomposition (A.66)–(A.68) [70]. This means that any vector bundle $E_G$ over $\mathbb{CP}^1$ is isomorphic to the direct sum $\bigoplus_{i=1}^l \mathcal{L}_{\gamma_i}$ of line bundles, where $\mathcal{L}_{\gamma_i}$ is defined by the transition function $t^{\gamma_i}, \gamma = (\gamma_1, \ldots, \gamma_l)^3$.

To describe the moduli space in a general case we modify the construction in [35], where it was applied to the Čech description of $\text{Bun}(\Sigma_g)$. Let $g(t)$ in (3.9) be represented as

$$g(t) = h_{\text{out}}^{-1} \hat{w} h_{\text{int}}.$$  

(3.10)

Any $g(t) \in L(G)$ (A.60) can be represented in this form. As an example, consider the Bruhat representation (A.68) of $g(t) = g_- \hat{w} g_+(t)$. Since $N^-(G)$ is a unipotent group, its logarithm $\log(g_-)$ is well defined. Note that $g_-$ as well as $\log(g_-)$ are true holomorphic functions on the punctured disk $\bar{D}_{x_0}^\times$. With the help of a smooth function

$$\chi(t, \bar{t}) = \begin{cases} 0, & t \notin \bar{D}_{x_0}, \\ 1, & t \in D_{x_0}, \end{cases}$$

(3.11)

we define a (non-holomorphic) extension of the function (3.9) from $\bar{D}_{x_0}^\times$ to $\Sigma_g$:

$$g(t) = g_- \hat{w} g_+(t) \rightarrow g(t) = h_{\text{out}}^{-1} \hat{w} h_{\text{int}}, \quad h_{\text{int}} = g_+, \quad h_{\text{out}} = \exp(\chi(t, \bar{t}) \log(g_-)).$$

(3.12)

This representation has the following interpretation. Let $(e^{\text{hol}})$ be a basis in the space of sections of a trivial holomorphic $G$-bundle over $D_{x_0}$, and let $(e^{C^\infty})$ be a basis in the space of sections of a trivial $C^\infty$ $G$-bundle over $D_{x_0}$. The transformation $h_{\text{int}}$ can be considered as a transformation from $(e^{\text{hol}})$ to $(e^{C^\infty})$. Therefore, $h_{\text{int}}$ is defined in (3.12) up to multiplication by $g_+$ on the right and by $f_{\text{int}}$ on the left. The transformation $h_{\text{out}}$ plays a similar role. We call this representation the non-holomorphic Birkhoff decomposition. Multiplication on the right by holomorphic transformations

$$h_{\text{int}} \rightarrow h_{\text{int}} g_+ , \quad g_+ \in G(D_{x_0}), \quad h_{\text{out}} \rightarrow h_{\text{out}} g_-, \quad g_- \in G(\Sigma_g \setminus x_0),$$

(3.13)

In fact, bundles with $\gamma \neq 0$ are unstable.
corresponds to the actions of $G(D_{x_0})$ and $G(\Sigma_g \setminus x_0)$ in the double coset representation (3.8).

We define trivial connections on $D_{x_0}$ and $\Sigma_g \setminus x_0$:

$$\overline{A}_{\text{int}} = \overline{\partial} h_{\text{int}} h_{\text{int}}^{-1}, \quad \overline{A}_{\text{out}} = \overline{\partial} h_{\text{out}} h_{\text{out}}^{-1}.$$ (3.14)

It follows from the holomorphicity of $g(t)$ that

$$\overline{A}_{\text{int}} = \hat{w} \overline{A}_{\text{out}} \hat{w}^{-1}.$$ This says that by means of $(h_{\text{int}}, h_{\text{out}})$ we define the connection $d''$ on sections of the non-trivial $G$-bundle. Multiplications on the right by holomorphic transformations (3.13) do not change $A_{\text{int}}$ and $A_{\text{out}}$. On the other hand, multiplications on the left by smooth transformations $h_{\text{int}} \to f_{\text{int}} h_{\text{int}}$, $f_{\text{int}} \in G(D_{x_0})$, $h_{\text{out}} \to f_{\text{out}} h_{\text{out}}$, $f_{\text{out}} \in G(\Sigma_g \setminus x_0)$, (3.15) such that

$$f_{\text{out}} = \hat{w} f_{\text{int}} \hat{w}^{-1},$$ (3.16)

do not change $g(t)$, but act as gauge transformations on connections:

$$\overline{A}_{\text{int}} \to \overline{\partial} f_{\text{int}} f_{\text{int}}^{-1} + f_{\text{int}} \overline{A}_{\text{int}} f_{\text{int}}^{-1}, \quad \overline{A}_{\text{out}} \to \overline{\partial} f_{\text{out}} f_{\text{out}}^{-1} + f_{\text{out}} \overline{A}_{\text{out}} f_{\text{out}}^{-1}.$$ Summarizing, we have introduced the space of pairs $(h_{\text{out}}, h_{\text{int}})$ in the decomposition (3.12):

$$\mathcal{T} (\Sigma_g, G) = \{(h_{\text{out}}, h_{\text{int}}) \in \mathcal{G} (D_{x_0}) \times \mathcal{G} (\Sigma_g \setminus x_0)\}.$$ There are groups

$$\mathcal{G}^{\text{hol}} \sim G(D_{x_0}) \times G(\Sigma_g \setminus x_0)$$

and

$$\mathcal{G}^{C^{\infty}} = \{(f_{\text{int}}, f_{\text{out}}): f_{\text{int}} \in \mathcal{G} (D_{x_0}), f_{\text{out}} \in \mathcal{G} (\Sigma_g \setminus x_0), f_{\text{out}} = \hat{w} f_{\text{int}} \hat{w}^{-1}\},$$

of holomorphic and smooth automorphisms acting on $\mathcal{T} (\Sigma_g, G)$ according to the formulae (3.13) and (3.15), respectively. Thus, starting with the space $\mathcal{T} (\Sigma_g, G)$ of pairs, we obtain the moduli space in the Dolbeault description (2.7) or the double coset description (3.8):

Let $P$ be a parabolic subgroup and let $f_{\text{int}} = P + O(t, \overline{t})$. Then we obtain the Dolbeault description of the moduli space of bundles with qp-structures (2.13). This construction can be generalized to the case of several marked points, that is, to Bun$(\Sigma_g, x, G)$. \(\square\)
We now pass to a coarser scale in our study of the moduli space \( \text{Bun}(\Sigma_{g,n}, x_0, G) \). Consider the monodromies of the transition functions \( g(t) \in G(D^x) \) around \( x_0 \): 
\( g(t \exp(2\pi i)) = \zeta g(t) \). For the component labelled by \( \hat{w} = wt^\gamma \) the monodromy is \( \zeta = \exp(2\pi i \gamma) \). If \( \gamma \in Q^\vee \), then the monodromy is trivial: \( \zeta = 1 \). For general \( \gamma \in P^\vee \), \( \zeta = \exp(2\pi i \gamma) \) is an element of the centre \( \mathcal{Z}(\mathcal{G}) \) (A.34). Then by passing to the quotient \( P^\vee/Q^\vee \sim \mathcal{Z}(\mathcal{G}) \) in (3.7) we obtain the decomposition

\[
\text{Bun}(\Sigma_{g,n}, x_0, G) = \bigcup_{\zeta \in \mathcal{Z}(\mathcal{G})} \text{Bun}_\zeta(\Sigma_{g,n}, x_0, G). \tag{3.17}
\]

If \( \zeta = 1 \), then \( g(t) \) can serve as a transition function for the \( \mathcal{G} \)-bundle \( E_{\mathcal{G}} \). Thus, non-trivial monodromies are obstructions to a lifting of \( G \)-bundles to \( \mathcal{G} \)-bundles.

We can write a similar decomposition for other quotient groups (A.22). In the cases of \( A_{n-1} \) (with \( n = pl \) non-prime) and \( D_n \), the centre \( \mathcal{Z}(\mathcal{G}) \) has non-trivial subgroups \( \mathcal{Z}_l \sim \mu_l = \mathbb{Z}/l\mathbb{Z} \). Then the quotient groups

\[
G_l = \mathcal{G}/\mathcal{Z}_l, \quad G_p = G_l/\mathcal{Z}_p, \quad G^{\text{ad}} = G_l/\mathcal{Z}(G_l) \tag{3.18}
\]

exist, where \( \mathcal{Z}(G_l) \sim \mu_p = \mathcal{Z}(\mathcal{G})/\mathcal{Z}_l \). The group \( \mathcal{G} = \text{Spin}_{4n}(\mathbb{C}) \) has a non-trivial centre

\[
\mathcal{Z}(\text{Spin}_{4n}) = (\mu_2^L \times \mu_2^R), \quad \mu_2 = \mathbb{Z}/2\mathbb{Z},
\]

and the subgroups of \( \text{Spin}_{4n}(\mathbb{C}) \) are described by the diagram (A.23). Therefore, in the general case the following monodromies are obstructions to a lifting of bundles:

\[
\zeta \in \mathcal{Z}(\mathcal{G}) \text{is an obstruction to a lifting of } E_{G^{\text{ad}}} \text{ to } E_{\mathcal{G}},
\]

\[
\zeta \in \mathcal{Z}_l \text{is an obstruction to a lifting of } E_{G_l} \text{ to } E_{\mathcal{G}},
\]

\[
\zeta \in \mathcal{Z}(G_l) \text{is an obstruction to a lifting of } E_{G^{\text{ad}}} \text{ to } E_{G_l}.
\]

A cohomological interpretation of the obstructions emerges from the three short exact sequences

\[
1 \to \mathcal{Z}(\mathcal{G}) \to \mathcal{G}(\mathcal{O}_\Sigma) \to G^{\text{ad}}(\mathcal{O}_\Sigma) \to 1,
\]

\[
1 \to \mathcal{Z}_l \to \mathcal{G}(\mathcal{O}_\Sigma) \to G_l(\mathcal{O}_\Sigma) \to 1,
\]

\[
1 \to \mathcal{Z}(G_l) \to G_l(\mathcal{O}_\Sigma) \to G^{\text{ad}}(\mathcal{O}_\Sigma) \to 1.
\]

They imply the following long exact cohomology sequences with coefficients in analytic sheaves:

\[
\to H^1(\Sigma_g, \mathcal{O}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G^{\text{ad}}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}(\mathcal{G})) \sim \mathcal{Z}(\mathcal{G}) \to 0, \tag{3.19}
\]

\[
\to H^1(\Sigma_g, \mathcal{O}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}_l) \sim \mu_l \to 0, \tag{3.20}
\]

\[
\to H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G^{\text{ad}}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}(G_l)) \sim \mu_p \to 0. \tag{3.21}
\]

The first cohomology group \( H^1(\Sigma_g, G(\mathcal{O}_\Sigma)) \) defines the tangent space to \( \text{Bun}(\Sigma_g, G) \). The elements of \( H^2 \) are obstructions to liftings of bundles, namely,

\[
H^2(\Sigma_g, \mathcal{Z}(\mathcal{G})) \text{ contains obstructions to a lifting of } E_{G^{\text{ad}}} \text{ to } E_{\mathcal{G}},
\]

\[
H^2(\Sigma_g, \mathcal{Z}_l) \text{ contains obstructions to a lifting of } E_{G_l} \text{ to } E_{\mathcal{G}},
\]

\[
H^2(\Sigma_g, \mathcal{Z}(G_l)) \text{ contains obstructions to a lifting of } E_{G^{\text{ad}}} \text{ to } E_{G_l}.
\]
Definition 3.1. The images $\zeta(E_G)$ (as elements of $H^1(\Sigma_g, G(\mathcal{O}_\Sigma))$ in $H^2(\Sigma_g, \mathcal{Z})$ are called the characteristic classes of the bundles $E_G$.

Now we consider the case of several marked points. We associate transformations of the affine Weyl group $W_P$ in (A.14) with the marked points:

$$\vec{w} = (w_1t^{\gamma_1}, \ldots, w_nt^{\gamma_n}), \quad \gamma_a \in \mathbb{P}^\vee.$$

Using the decomposition (3.7), we define the sector

$$\text{Bun}_\zeta(\Sigma_{g,n}, x, G^{ad}) = G^{ad}(\Sigma_g \setminus x)G^{ad}(D^x)\vec{w}/G^{ad}(D)$$

in the moduli space (3.5) and the decomposition of the moduli space

$$\text{Bun}(\Sigma_{g,n}, x, G^{ad}) = \bigcup_{\vec{\gamma} \in \mathbb{P}^\vee} \text{Bun}_\zeta(\Sigma_{g,n}, x, G^{ad}). \quad (3.22)$$

In the case of several marked points the sector $\text{Bun}_\zeta(\Sigma_{g,n}, x, G^{ad})$ is defined as in (3.17) by the local transition functions $g(t_a)$ with monodromies $\zeta_a$ around the points $x_a$ such that $\zeta = \prod_{a=1}^n \zeta_a \in \mathcal{Z}(\mathcal{G})$. This means that in (3.22) we identify components $\text{Bun}_\zeta$ and $\text{Bun}_{\vec{\gamma}}$ if $\sum_{a=1}^n (\gamma_a - \gamma'_a) \in Q^\vee$. Then we obtain the decomposition of the moduli space (3.17) into topological sectors:

$$\text{Bun}(\Sigma_{g,n}, x, G^{ad}) = \bigcup_{\zeta \in \mathcal{Z}(\mathcal{G})} \text{Bun}_\zeta(\Sigma_{g,n}, x, G^{ad}). \quad (3.23)$$

3.3. Flat bundles. We consider first the double coset construction for the moduli space of Higgs bundles $T^* \text{Bun}(\Sigma_{g,n}, x, G^{ad})$. The Higgs bundles are defined as the set of pairs $(\Phi_a, g_a), (a = 1, \ldots, n)$, where $g_a \in L_a(G)$ and $\Phi_a \in \text{Lie}^*(L_a(G)) \otimes dt_a$ are the Higgs fields. It is a symplectic manifold with the symplectic form

$$\omega = \sum_a \oint_{\Gamma_a} \langle \delta(\Phi_a g_a^{-1}) \wedge \delta g_a \rangle, \quad (3.24)$$

where the contour $\Gamma_a$ is a subset of $D^x_a$. The action of $\mathcal{G}_{in} = G(D)$ and $\mathcal{G}_{out} = G(\Sigma_g \setminus x)$ on these pairs lifts to $T^* \text{Bun}(\Sigma_{g,n}, x, G^{ad})$:

$$\mathcal{G}_{a,int} : \Phi_a \rightarrow f_{a,int}^{-1}\Phi_a f_{a,int}, \quad g \rightarrow g f_{a,int}, \quad (3.25)$$

$$\mathcal{G}_{out} : \Phi_a \rightarrow \Phi_a, \quad g \rightarrow f_{out}g. \quad (3.26)$$

By the symplectic reduction we define the moduli space

$$\mathcal{G}_{out} \left\| \bigcup_{a=1}^n (\Phi_a, g_a) \right/ \mathcal{G}_{a,int}.$$  

This is a double symplectic quotient which can be shown to be isomorphic to $T^* \text{Bun}(\Sigma_{g,n}, x, G)$. Since the moduli space of Higgs bundles $\text{Bun}(\Sigma_{g,n}, x, G)$ is the union of sectors (3.23), the cotangent bundle is also the union

$$T^* \text{Bun}(\Sigma_{g,n}, x, G) = \bigcup_{\vec{\gamma} \in \mathbb{P}^\vee} T^* \text{Bun}_\zeta(\Sigma_{g,n}, x, G^{ad}) = \bigcup_{\zeta \in \mathcal{Z}(\mathcal{G})} T^* \text{Bun}_\zeta(\Sigma_{g,n}, x, G).$$
Our main interest is the moduli space of flat bundles. We will prove that it is the unions
\[ \text{FBun}(\Sigma_{g,n}, x, G) = \bigcup_{\xi} \text{FBun}_{\xi}(\Sigma_{g,n}, x, G^\text{ad}) = \bigcup_{\zeta} \text{FBun}_{\zeta}(\Sigma_{g,n}, x, G). \]

As above, we replace the space of connections \( \text{Conn}(\Sigma, G) \) by the set \( \mathcal{R} \) of pairs on \( G(D^x) \),
\[ \mathcal{R} = \{ ((\partial_{t_a} + X_a) \otimes dt_a, g_a), \ a = 1, \ldots, n \}, \]
where \( g_a \) is in \( G(D^x) \) and can have a non-trivial monodromy, and \( X_a \in g(D^x) = g \otimes \mathbb{C}[t_a^{-1}, t_a] \). The component \( (\partial_{t_a} + X_a) \otimes dt_a \) belongs to the principal homogeneous space \( PH/T^*G(D^x) \) over the cotangent bundle \( T^*G(D^x) \).

The form \( \omega \) in (2.6) can be rewritten in this parametrization as (cf. (3.24))
\[ \omega = \sum_a \oint_{\Gamma_a} \langle \delta(X_a g_a^{-1}) \wedge \delta g_a \rangle + \frac{1}{2} \oint_{\Gamma_a} \langle g_a^{-1} \delta g_a \wedge \partial_t(g^{-1} \delta g_a) \rangle. \]

Remark 3.1. In fact, the pairs \( ((\partial_{t_a} + X_a) \otimes dt_a, g_a) \) should be replaced by the pairs \( ((\kappa_a \partial_{t_a} + X_a) \otimes dt_a, (g_a, \lambda_a)) \), where \( \lambda_a \) is a central extension of the loop group \( G(D^x) \), and the \( \kappa_a \in \mathbb{C} \) are dual co-central extensions [71]. In this case \( \omega \) acquires the additional term \( \sum_a \delta \kappa_a \wedge \delta \log \lambda_a \). It can be proved that these scalar fields are non-dynamical and do not contribute to the monodromy-preserving equations. For this reason we will not consider the co-central extensions \( \lambda_a \). Nevertheless, we keep the parameter \( \kappa \), because in the limit \( \kappa \to 0 \) in (4.5) we obtain integrable systems.

The symplectic form (3.29) is invariant under gauge transformations in \( \mathcal{G}_{a, \text{int}} = L^+_a(G) \) and \( \mathcal{G}_{\text{out}} = G(\Sigma_g \setminus x) \):
\[ \mathcal{G}_{a, \text{int}} : X_a \to \text{Ad}_{f_{a, \text{int}}}^* \partial_{t_a} f_{a, \text{int}} \]
\[ + f_{a, \text{int}}^{-1} X_a f_{a, \text{int}}, \quad g_a \to g_a f_{a, \text{int}} \]
\[ \mathcal{G}_{\text{out}} : X_a \to X_a, \quad g_a \to f_{\text{out}} g_a \]
They are generated by the Hamiltonians
\[ F_{a, \text{int}} = \oint_{\Gamma_a} \langle \varepsilon_{a, \text{int}}, X_a \rangle, \quad F_{\text{out}} = \sum_a \oint_{\Gamma_a} \langle \varepsilon_{\text{out}}, (g X_a g^{-1} - \partial_{t_a} g g^{-1}) \rangle. \]

Here \( \varepsilon_{\text{out}} \in \text{Lie}(G(\Sigma_g \setminus x)) \) and
\[ \varepsilon_{a, \text{int}} \in \text{Lie}(L^+_a(G)), \quad \varepsilon_{a, \text{int}} = x_{a,0} + t_a x_{a,1} + \cdots, \quad x_{a,0} \in p_a = \text{Lie}(P_a). \]

3.3.1. \( \mathcal{G}_{a, \text{int}} \)-action. We shall follow the finite-dimensional construction for the orbits of the coadjoint action (A.47)–(A.55). The moment map corresponding to the \( \mathcal{G}_{a, \text{int}} \)-action is equal to \( \mu_{\text{int}} = \sum_a \mu_a, \quad \mu_a = \text{Pr}|_{\text{Lie}^*(L^+_a(G))}(X_a) \), where the dual space is defined as
\[ \text{Lie}^*(L^+_a G) = \{ y_{a,0} t_a^{-1} + y_{a,1} t_a^{-2} + \cdots, \ y_{a,0} \in p_a^* \}. \]
From (A.36) and (A.63) we find that \( p_a^* = g_a^* \oplus b_a^* \oplus n_a^- \). We take the moment value as
\[
\mu_a = \text{Pr}|_{\text{Lie}^*(L_a^+)} (X_a) = \nu_a t_a^{-1}, \quad \nu_a \in \mathfrak{g}_a^* \quad (\mathfrak{g}_a^* \text{ is the Levi subalgebra (A.36)}).
\]

In other words, the field \( A \) has simple poles and
\[
X_a(t_a) = (\nu_a + \xi_{a,-1}) t_a^{-1} + \xi_{a,0} + O(t_a) = \nu_a t_a^{-1} + \xi(t_a), \quad \xi_{a,-1} \in \mathfrak{n}^+.
\]

This means that \( \xi(t_a) \) is an arbitrary element of the space \( b(t_a) = \mathfrak{n}^+ t_a^{-1} + \mathfrak{g} \otimes \mathbb{C}[t_a] \).

Note that \( \mathcal{G}_{a,\text{int}} \) preserves the term \( \nu_a t_a^{-1} \) and acts freely on \( b(t_a) \) for \( \nu_a \neq 0 \):
\[
\text{Ad}_{L_a^+}^* X_a(t_a) = \nu_a t_a^{-1} + \xi'(t_a), \quad \xi'(t_a) = \xi_{a,-1} t_a^{-1} + \xi_{a,0} + O(t_a).
\]

From (3.30) and (3.34) we find that the symplectic quotient is given by the set of pairs
\[
(g_a(t_a), \nu_a t_a^{-1} + \xi(t_a)) \quad (a = 1, \ldots, n)
\]
with the equivalence relation
\[
(g_a(t_a), \nu_a t_a^{-1} + \xi(t_a)) \sim (g_a(t_a) f(t_a), \text{Ad}_{f(t_a)}^* (\nu_a t_a^{-1} + \xi(t_a))), \quad f(t_a) \in G(D_a).
\]

Let us fix the gauge of the \( G(D_a) \)-action in (3.35) by putting \( \xi(t_a) = 0 \). The Levi subgroup \( L_a \subset \mathcal{G}_{a,\text{int}} \) (A.38) preserves the gauge:
\[
L_a = \{ f(t_a) \in G(D_a) \mid (\text{Ad}_{f(t_a)}^*)^{-1} \nu_a = \nu_a \}.
\]

It follows from (3.35) that \( g_a(t_a) \) is defined up to multiplication by \( L_a \) on the right. Therefore, the reduced space in this case is the coadjoint orbit
\[
\mathcal{O}_a^\text{aff} = G(D_a^\times)/L_a = \{ g(t_a) \in G(D_a^\times) \mid X_a(t_a) = (\text{Ad}_{g(t_a)}^*)^{-1} \nu_a t_a^{-1} \}
\]
\[
= -\partial_{t_a} g(t_a) g^{-1}(t_a) + g(t_a) \nu_a t_a^{-1} g^{-1}(t_a) \}
\]

This space is the principal homogeneous space over the cotangent bundles on the affine flag varieties \( PH/T^* \text{Flag}_a^\text{aff} \) (Flag\(_a^\text{aff} \sim G(D_a^\times)/G(D_a) \)). The form (3.29) on \( \mathcal{O}_{P_a} \) becomes
\[
\omega_{\mathcal{O}_{P_a}^\text{aff}} = \oint_{\Gamma_a} \langle \delta(\nu_a t_a^{-1} g_a^{-1}) \wedge \delta g_a \rangle + \frac{1}{2} \oint_{\Gamma_a} \langle (g_a^{-1} \delta g_a) \wedge \partial_{t_a} (g^{-1} \delta g_a) \rangle.
\]

Thus, upon Hamiltonian reduction by \( \mathcal{G}_{\text{int}} = \prod_a \mathcal{G}_{a,\text{int}} \) we go from \( \mathcal{R} \) in (3.28) to the space
\[
\bigcup_{a=1}^n \mathcal{O}_a^\text{aff} = \mathcal{R} \parallel \prod_a \mathcal{G}_{a,\text{int}}.
\]

### 3.3.2. \( \mathcal{O}_{out} \)-action.

The \( \mathcal{O}_{out} \)-action on the symplectic space (3.36), (3.37) defines the moment
\[
\mu_{out} = \text{Pr}|_{\text{Lie}^*(G(\Sigma_g \setminus x_0))} X(t) = 0, \quad X(t) = g^\nu t g^{-1} - \partial_t gg^{-1}.
\]

Let \( G(t, z) \) be the Cauchy kernel on the punctured curve \( \Sigma_g \setminus x_0 \). It is a function in the first variable and a 1-form in the second variable. It is regular off the diagonal, on which it has a first-order pole. It exists because the punctured curve is affine.
Proposition 3.2. Let $\Gamma_0$ be a small contour around $x_0$. Then the form
\[
\tilde{A}(z) = \text{Res}_{t=0}(X(t)G(t,z)),
\]
where $X(t)$ satisfies (3.38), is holomorphic on $\Sigma_g \setminus \Gamma_0$. Its asymptotic expression as $z \to \Gamma_0$ coincides with $X(t)$.

Proof. Let us calculate the $k$th coefficient of the Laurent expansion of the form $\tilde{A}$. It is equal to
\[
\frac{1}{2\pi i} \oint_{\Gamma_0} s^{-k-1} \tilde{A}(s) = \frac{1}{2\pi i} \oint_{\Gamma_0} s^{-k-1} \left( \text{Res}_{t=0}(X(t)G(t,s)) \right) = \text{Res}_{t=0} \left( X(t) \frac{1}{2\pi i} \oint_{\Gamma_0} s^{-k-1} G(t,s) \right).
\]
Here interchanging the order of integration and taking the residue is correct. We take the residue at zero with respect to $t$, hence we calculate the integral under the assumption that $|t| < |s|$. The integrand has singularities at $s = t$ and $s = 0$. The contribution from the first singularity equals $t^{-k-1}$ and the contribution from the second is equal to $(1/k!)(\partial_s^k G(t,s)) |_{s=0}$. The residue of the product of $\mu_{\text{out}}$ with the first term gives the $k$th coefficient of $\mu_{\text{out}}$, and the residue of the product with the second term vanishes, since $(1/k!)(\partial_s^k G(p,s) |_{s=0})$ is regular away from the punctures. \(\Box\)

We define on $\Sigma_g \setminus x_0$ the holomorphic connection
\[
A_{\text{out}} = h_{\text{out}}(\partial + \tilde{A})h_{\text{out}}^{-1} = h_{\text{out}}\tilde{A}h_{\text{out}}^{-1} - \partial_z h_{\text{out}}h_{\text{out}}^{-1},
\]
where $h_{\text{out}}$ is defined in the decomposition (3.12). On $D_x^\times$
\[
A_{\text{out}} = h_{\text{out}}X(t)h_{\text{out}}^{-1} - \partial_t h_{\text{out}}h_{\text{out}}^{-1},
\]
where $X(t)$ is from (3.38). It follows from (3.14) and Proposition 3.2 that in $\Sigma_g \setminus x_0$ the curvature vanishes:
\[
F(A_{\text{out}}, \overline{A}_{\text{out}}) = [\partial + A_{\text{out}}, \overline{\partial} + \overline{A}_{\text{out}}] = h_{\text{out}}[\partial + \tilde{A}, \overline{\partial}]h_{\text{out}}^{-1} = 0.
\]
We define $A_{\text{int}}$ as (see (3.14) and (3.38))
\[
A_{\text{int}} = h_{\text{int}}\nu t^{-1}h_{\text{int}}^{-1} - \partial_t h_{\text{int}}h_{\text{int}}^{-1}.
\]
It follows from (3.40) that
\[
A_{\text{out}} = \hat{w}A_{\text{int}}\hat{w}^{-1}.
\]

The curvature on $D_x^0$ takes the form
\[
F(A_{\text{int}}, \overline{A}_{\text{int}}) = (h_{\text{int}}\nu h_{\text{int}}^{-1})\overline{\partial}t^{-1}.
\]
Since $h_{\text{int}}(t,\overline{t}) = h_0 + O(t,\overline{t})$, we get that
\[
F(A_{\text{int}}, \overline{A}_{\text{int}}) = S\delta(t), \quad S = h_0\nu h_0^{-1}.
\]

Thus, starting with $	ext{Conn}(\Sigma_g, G) = R$ in (3.28), we obtain flat bundles with a Fuchsian singularity.
3.4. Hecke transformations. The Hecke transformation is a singular gauge transformation that intertwines sections of bundles corresponding to different components in the sense of the Weyl elements in (3.22) or the elements of the centre in (3.23).

To avoid additional complications we take a curve with one marked point \( \Sigma_g \setminus x_0 \). A holomorphic bundle \( E_G \) is given by a transition function \( g(t) \) in the double coset construction (3.22) or (3.23). We replace \( g(t) \) by \( g(t)h(t) \), where \( h(t) \in L(G) \) can have a non-trivial monodromy around \( t = 0 \). By (3.5), \( h(t) \) is defined up to multiplication on the right by an \( f(t) \in L^+(G) \). As explained in §3.2, this transformation can change the characteristic class of \( E_G \). It follows from (A.68) that as a representative of this double coset we can take \( h(t) = \hat{w} \in W_{t(G)} \), that is,

\[
g(t) \to g(t)\hat{w}, \quad \hat{w} = wt^\gamma, \quad t^\gamma = e(\log(t\gamma)), \quad \gamma \in t(G), \quad (3.42)
\]

where \( t(G) \) is the lattice of coweights (A.31). The monodromy of \( t^\gamma \) is \( \exp(-2\pi i\gamma) \). Since \( \langle \alpha, \gamma \rangle \in \mathbb{Z} \), for any \( x \in \mathfrak{g} \) we have \( \text{Ad}_{\exp(-2\pi i\gamma)} x = x \). Then \( \exp(-2\pi i\gamma) \) is an element of \( \mathscr{Z}(\hat{G}) \) in (A.34). If the transition matrix \( g(t) \) defining \( E_G \) has a trivial monodromy, then the new transition matrix (3.42) acquires a non-trivial monodromy. In this way we obtain a new bundle \( \hat{E}_G \) with a non-trivial characteristic class \( \zeta(\hat{E}_G) \). The bundle \( \hat{E}_G \) is said to be modified. Another name of the modification is the Hecke transformation. If \( \gamma \in Q^\vee \), then \( \zeta = 1 \) and \( \hat{E}_G \) has the same characteristic class as \( E_G \). It follows from (A.68) and (3.6) that the space of Hecke transformations of type \( \gamma \) coincides with the Schubert cell \( C_{\hat{w}} \) in (A.71). Its dimension is equal to \( l(\hat{w}) \) (A.72).

For transformations of sections we use the notation

\[
\Xi(\zeta) : \Gamma(E_G) = \Gamma(E(\zeta = 1)) \to \Gamma(\hat{E}_G) = \Gamma(E(\zeta)), \quad (3.43)
\]

If \( d + \mathscr{A} \) is a connection on \( E_G \) and \( d + \hat{\mathscr{A}} \) is a connection on \( \hat{E}_G \), then

\[
(d + \hat{\mathscr{A}})\Xi(\zeta) = \Xi(\zeta)(d + \mathscr{A}). \quad (3.44)
\]

We give some restrictions on the choice of \( \gamma \in P^\vee \) for bundles with qp-structure at \( x_0 \). Assume that the modification preserves the qp-structure at the marked point \( x_0 \). In this case \( \hat{w} \) should normalize \( G(D) = L^+(G) \) in the decomposition (3.7):

\[
\hat{w}^{-1}L^+(G)\hat{w} = L^+(G). \quad (3.45)
\]

We recall that \( \gamma \) gives a parabolic subalgebra \( \mathfrak{p}_\gamma \subset \mathfrak{g} \) (see (A.43)), and thus a qp-structure.

**Proposition 3.3.** Let \( \gamma \in \hat{\mathfrak{t}}^\vee \) be an admissible fundamental coweight (A.45), and let \( \mathfrak{p}_\gamma = \mathfrak{s} + \mathfrak{n}^+ \) be the admissible parabolic subalgebra with respect to \( \gamma \) (A.44) in the decomposition \( \text{Lie}(L^+(G)) = \mathfrak{p} + t\mathfrak{g} + o(t) \). If

\[
\text{Ad}_w \mathfrak{n}^+ = \mathfrak{n}^-, \quad (3.46)
\]

then \( \hat{w} \) normalizes \( L^+(G) \).
Proof. Let \( x(t) \in \text{Lie}(L^+(G)) \), \( x(t) = a + b + t(a_1 + b_1 + c_1) + o(t) \), where \( a, a_1 \in s \), \( b, b_1 \in n^+ \), and \( c_1 \in n^- \). Then by (A.44)

\[
\text{Ad}_{\tilde{\theta}} = \text{Ad}_w \left( (a + c_1) + ta_1 + t^2b_1 + \cdots \right).
\]

If \( \text{Ad}_w n^+ = n^- \), then \( \text{Ad}_w (a + c_1) \in p = s + n^+ \) and \( \text{Ad}_w b_1 \in n^- \), that is, the \( qp \)-structure is recovered. \( \square \)

For generic maximal parabolic subgroups such transformations do not exist. For the classical groups it exists for \( \text{SL}(2n, \mathbb{C}) \) and the Levi subgroup \( S(\text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})) \), and also in the \( B \) and \( C \) cases and in the \( D \) case for \( q = \varpi \). (see Table 4 in Appendix B). However, if \( P \) is not maximal \( (P \subset P_\gamma) \), then the Hecke transformation can preserve the \( qp \)-structure in more cases. For example, if \( P = B \) is a Borel subgroup, then by taking \( w : \alpha \rightarrow -\alpha \) in (3.42) we get (3.45).

The Hecke transformations of holomorphic bundles can be lifted to Hecke transformations of flat bundles. In this case instead of flag varieties associated with the marked points we have the coadjoint orbits defined by the Levi subalgebras \( s(\Pi') \) in (A.36). For this reason, instead of (3.46) we have

\[
\text{Ad}_w s = \tilde{s}.
\]

Here \( \tilde{w} = wt_\gamma \) and \( \gamma \in \hat{\Pi'} \). Then \( \tilde{s} = \text{Ad}_w s \). Although this transformation changes the Levi subgroup \( L \rightarrow \tilde{L} \), it preserves the orbit \( \theta' = G/L \). Therefore, for flat bundles we only need \( \gamma \in \hat{\Pi} \).

4. Hamiltonian approach to the isomonodromy problem

4.1. Deformation of complex structures on curves. The complex structure on \( \Sigma_g \) is defined by the operator \( \partial \). We consider a deformation of it determined by the change of variables

\[
w = z - \varepsilon(z, \bar{z}), \quad \bar{w} = \bar{z} - \varepsilon(z, \bar{z}),
\]

(4.1)

where \( \varepsilon(z, \bar{z}) \) is small. Up to a common multiplier the partial derivatives assume the form

\[
\begin{align*}
\partial_w &= \partial_z + \mu \partial_{\bar{z}}, \\
\partial_{\bar{w}} &= \partial_{\bar{z}} + \mu \partial_z,
\end{align*}
\]

where

\[
\mu = \frac{\overline{\partial} \varepsilon}{1 - \partial \varepsilon} \sim \partial \varepsilon
\]

(4.2)

is the Beltrami differential \( \mu \in \Omega^{(-1,1)}(\Sigma_g) \). We pass from the local coordinates \((w, \bar{w})\) in (4.1) to the chiral coordinates \((w, \tilde{w})\),

\[
w = z - \varepsilon(z, \bar{z}), \quad \tilde{w} = \bar{z},
\]

(4.3)

because the \( \overline{\partial} \)-dependence is inessential in our construction. The pair \((w, \tilde{w})\) is also a pair of local coordinates on \( \Sigma_g \). In these coordinates the partial derivatives take the form

\[
\begin{align*}
\partial_w &= \partial_z, \\
\partial_{\tilde{w}} &= \partial_{\bar{z}} + \mu \partial_z.
\end{align*}
\]

(4.4)
It is important to stress that \( \partial \bar{w} \) annihilates holomorphic functions: \( \partial \bar{w} f(w) = 0 \). A smooth function \( w(z, \bar{z}) \in C^\infty(\Sigma_g) \) defines a global diffeomorphism of \( \Sigma_g \). We say that the Beltrami differential \( \mu(z, \bar{z}) \) is equivalent to \( \mu'(z, \bar{z}) = \mu(w(z, \bar{z}), \bar{w}(z, \bar{z})) \). The equivalence relation in \( \Omega^{(-1,1)}(\Sigma_g) \) under the action of \( \text{Diff}_{C^\infty}(\Sigma_g) \) is the moduli space \( \mathcal{M}(\Sigma_{g,n}) \) of complex structures on \( \Sigma_g \). The tangent space to the moduli space is the Teichmüller space \( \mathcal{T}_g \cong H^1(\Sigma_g, \Gamma) \), where \( \Gamma \in T\Sigma_g \). From the Riemann–Roch theorem we have
\[
\dim(\mathcal{T}_g) = 3(g-1). \tag{4.5}
\]

Let \( (\mu_1^0, \ldots, \mu_l^0) \) be a basis in the vector space \( H^1(\Sigma_g, \Gamma) \). Then
\[
\mu = \sum_{l=1}^{3g-3} \tau_l \mu_l^0, \tag{4.6}
\]
where the local coordinates \( \tau_l \) will play the role of times in the isomonodromic deformation problem.

**Moving marked points.** Consider the moduli space \( \mathcal{M}(\Sigma_{g,n}) \) of complex structures on curves \( \Sigma_{g,n} \) with marked points. This space is fibred over the moduli space \( \mathcal{M}(\Sigma_g) \) of complex structures on compact curves, with fibers \( \mathcal{U} \subset \mathbb{C}^n \) corresponding to the moving marked points. The moduli space \( \mathcal{M}(\Sigma_{g,n}) \) consists of the classes of the equivalence relation in the space \( \Omega^{(-1,1)}(\Sigma_{g,n}) \) of differentials under the action of the group \( \text{Diff}_{C^\infty}(\Sigma_{g,n}) \) of diffeomorphisms vanishing at the marked points.

We consider local coordinates on a fibre. Let \((z, \bar{z})\) be local coordinates in a neighbourhood of the marked points \( x_0^a \) \((a = 1, \ldots, n)\) and let \( \mathcal{U}_a \) be neighbourhoods of \( x_0^a \) such that \( \mathcal{U}_b \cap \mathcal{U}_a = \emptyset \). We define a \( C^\infty \)-function
\[
\chi_a(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_a' \subset \mathcal{U}_a, \\ 0, & z \notin \mathcal{U}_a. \end{cases} \tag{4.7}
\]
The moving points \((x_0^a \rightarrow x_a)\) correspond to the following local deformation of the local coordinates \((4.1)\):
\[
w = z - \sum_{a=1}^n \varepsilon_a(z, \bar{z}), \quad \varepsilon_a(z, \bar{z}) = -t_a \chi_a(z, \bar{z}) + \sum_{j>0} t_a^{(j)}(z - x_0^a)^j \chi_a(z, \bar{z}),
\]
\[
t_a = x_a - x_0^a. \tag{4.8}
\]
The action of \( \text{Diff}_{C^\infty}(\Sigma_{g,n}) \) lets us put \( t_a^{(j)} = 0 \) for \( j > 0 \). Thus, in general we have only \( n \) times \( t_a \). The part of the Beltrami differential connected with the marked points takes the form
\[
\mu = \sum_{a=1}^n \tau_1 \mu_{a}^{(0)}, \quad \mu_a^{(0)} = \partial \chi_a(z, \bar{z}). \tag{4.9}
\]
Thus, the local coordinates on \( \mathcal{F}(\Sigma_{g,n}) \) are
\[
\{(\tau_1, \ldots, \tau_{3(g-1)}, t_1, \ldots, t_n)\}, \tag{4.10}
\]
and
\[
\dim(\mathcal{F}(\Sigma_{g,n})) = 3(g-1) + n. \tag{4.11}
\]
4.2. The equations of motion and the isomonodromy problem. We fix a complex structure on $\Sigma_{g,n}$ and replace the connections (2.5) by the pairs
\[(\kappa \partial_z + A) \otimes dz, (\overline{\partial} + \overline{A}) \otimes d\overline{z} \].

Following Deligne, we introduce here the notion of $\kappa$-connections, where $\kappa$ is a small parameter (in a certain sense it resembles the Planck constant). The moduli space of $\kappa$-connections was investigated in [72]. These $\kappa$-connections let us pass in the quasi-classical limit to the Higgs fields $\Phi = \lim_{\kappa \to 0} (\kappa \partial_z + A) \otimes dz \in \Omega^0(\Sigma_g, \text{Lie} G \otimes K)$, where $K$ is a canonical class on $\Sigma_g$. In this limit the monodromy-preserving equations become equations of motion for integrable systems of Hitchin type. The corresponding linear problem for the latter systems is an isospectral problem instead of the isomonodromy problem before the limit is taken. We discuss this procedure below in detail.

Consider the polarization (4.3) in the deformed coordinates:
\[(\kappa \partial_w + A) \otimes dw, (\partial_{\overline{w}} + \overline{A}) \otimes d\overline{w} \].

The component $\overline{A}'$ of the connection in the deformed coordinates is defined as
\[
\overline{A}' = \overline{A} - \frac{1}{\kappa} \mu A, \quad (\overline{\partial} + \mu \partial + \overline{A}) \otimes d\overline{w} = (\partial_{\overline{w}} + \overline{A}) \otimes d\overline{w}.
\]

(4.12)

The form $\omega$ in (2.6) can be rewritten as
\[
\omega = \int_{\Sigma_g} (\delta A \wedge \delta \overline{A}) - \frac{1}{\kappa} \int_{\Sigma_g} (A, \delta A) \delta \mu.
\]

This form $\omega$ can be regarded as the differential of the Poincaré–Cartan 1-form ($\omega = \omega_{\text{PC}} = \delta \vartheta_{\text{PC}}$) [73] on the extended space $(A, \overline{A}, \mu)$. In the canonical coordinates on the phase space $(p_j, q_j)$ and for Hamiltonians $H_l(\vec{p}, \vec{q}; t_1, \ldots, t_n)$ it takes the form
\[
\vartheta_{\text{PC}} = \sum_j p_j \delta q_j - \sum_l H_l(\vec{p}, \vec{q}; t_1, \ldots, t_n) \delta t_l.
\]

The connections $(A, \overline{A})$ play the role of canonical coordinates on $\text{Conn}(\Sigma_g, G)$, while the second term can be regarded as differentials of quadratic Hamiltonians pairing with the corresponding times. More precisely, taking (4.6) into account, we rewrite $\omega$ as
\[
\omega_{\text{PC}} = \omega_0 - \frac{1}{\kappa} \sum_{l=1}^{3g-3} \delta H_l \delta \tau_l, \quad H_l = \frac{1}{2} \int_{\Sigma_g} (A, A) \mu^0_l, \quad \omega_0 = \int_{\Sigma_g} (\delta A \wedge \delta \overline{A}).
\]

(4.13)

The Poincaré–Cartan form gives rise to the action functional
\[
S = \sum_{l=1}^{3g-3} \int_0^\infty \left( \int_{\Sigma_g} (A, \partial \overline{A}) - \frac{1}{2\kappa} (A, A) \mu^0_l \right) \, d\tau_l \quad (\partial_l = \partial_{\tau_l}).
\]

The equations of motion following from this action (or from the Hamiltonians) are
\[
\partial_t \overline{A} = \frac{1}{\kappa} A \mu^0_l, \quad \partial_t A = 0.
\]

(4.14)
These equations are compatibility conditions for the linear system

\[ (\kappa \partial_w + A)\psi = 0, \]  
\[(\partial_{\overline{w}} + \overline{A})\psi = 0, \]  
\[\kappa \partial_\tau \psi = 0, \]  

where \( \partial_{\overline{w}} = \overline{\partial} + \mu \partial \) and \( \psi \in \Omega^{(0)}(\Sigma_g, \text{Aut} E_G) \). The equations of motion (4.14) for \( A \) and \( \overline{A} \) are consistency conditions for (4.15a)&(4.15c) and (4.15b)&(4.15c). The monodromy of \( \psi \) is the transformation

\[ \psi \rightarrow \psi Y, \quad Y \in \text{Rep}(\pi_1(\Sigma_g) \rightarrow G). \]

Equation (4.15c) means that the monodromy is independent of the times. The consistency condition for (4.15a) and (4.15b) is the flatness constraint (2.2). The form \( \omega_{\text{PC}} \) is defined on the bundle \( \mathcal{P}(G) \) over the Teichmüller space \( \mathcal{T}_g \). Its fibers are the space \( \text{Conn}(\Sigma_g, G) \) of smooth connections:

\[ \mathcal{P}(G) \]
\[ \downarrow \text{Conn}(\Sigma_g, G) \]
\[ \mathcal{T}_g \]  

The linear equations (4.14) describe a free motion on \( \text{Conn}(\Sigma_g, G) \). They become non-trivial on \( \text{FBun}(\Sigma_g, G) \).

4.3. Contribution of marked points. Assume that the \((1, 0)\)-component of the connection (as in (2.17)) has simple poles at the marked points:

\[ \text{Res}_{z = x_a} A = S^a, \quad \overline{S} = (S^1, \ldots, S^n). \]

The bundle \( \mathcal{P}(G) \) is now defined over the Teichmüller space \( \mathcal{T}_{g,n} \). Its fibers are the space \( \text{Conn}(\Sigma_{g,n}, G) \) of smooth connections with singularities at the marked points. Therefore, the local coordinates on \( \mathcal{P}(G) \) are

\[ (A, \overline{A}, \overline{S}, t), \quad t = (\tau_1, \ldots, \tau_{3g-3}; t_1, \ldots, t_n). \]

The bundle \( \mathcal{P}(G) \) plays the role of the extended phase space, while the space \( \text{Conn}(\Sigma_{g,n}, G) \) is the usual phase space and \( t \) is the set of times. The form (4.13) acquires the additional terms

\[ \omega_{\text{PC}} = \omega_{\text{Conn}} - \frac{1}{\kappa} \left( \sum_{l=1}^{3g-3} \delta H_l \delta \tau_l + \sum_{a=1}^n \delta H_a \delta t_a \right), \]
\[ \omega_{\text{Conn}} = \omega_0 + \sum_{a=1}^n \omega^a_{\text{KK}}, \]

\[ H_a = \frac{1}{2} \int_{\gamma_a} (A, A) \overline{\partial} \chi_a(z, \overline{z}). \]
This form $\omega^{PC}$ generates $(3g-3+n)$ vector fields $D_s$ such that $\omega(D_s, \cdot) = 0$, where

$$D_l = \partial_{\tau_l} + \frac{1}{\kappa} \{H_l, \cdot\}_\omega \quad (l = 1, \ldots, 3g-3),$$
$$D_a = \partial_{t_a} + \frac{1}{\kappa} \{H_a, \cdot\}_\omega \quad (a = 1, \ldots, n).$$

The Poisson brackets corresponding to $\omega_0$ are Darboux brackets, and those corresponding to $\omega_a^{KK}$ are Poisson–Lie brackets. The latter are non-degenerate on the fibers. The vector fields $D_s$ define the equations of motion for any function $f$ on $\mathcal{P}(G)$:

$$\frac{df}{dr_s} = \partial_{r_s} f + \frac{1}{\kappa} \{H_s, f\}, \quad r_s = \tau_s \text{ or } r_s = t_s.$$  

Equations (4.14) are particular examples of them. The compatibility conditions are the so-called Whitham equations [74]:

$$\kappa \partial_{r_s} H_r - \kappa \partial_r H_s + \{H_r, H_s\} = 0. \quad (4.18)$$

In the rational case the $\tau$-functions were investigated in [9]. For the Painlevé I–VI equations they were studied in [75].

The Hamiltonians are quadratic Hitchin Hamiltonians and commute with respect to the Poisson brackets. This means that the following generating function (the tau-function) exists:

$$H_s = \frac{\partial}{\partial t_s} \log \tau, \quad \tau = \exp \left( \frac{1}{2} \sum_{s=1}^j \int_{\Sigma_g, n} \langle A^2 \rangle \mu_s \right). \quad (4.19)$$

4.4. Symplectic reduction. Up to now the equations of motion, the linear problem, and the tau-function have been trivial. Substantive equations arise after imposing the corresponding constraints (2.18) and fixing a gauge. The form $\omega^{PC}$ in (4.17) is invariant under the action of the gauge group $\mathcal{G}_P$ (2.12):

$$A \rightarrow f^{-1} \kappa \partial f + f^{-1} A f, \quad \bar{A} \rightarrow f^{-1} \bar{\partial} f + f^{-1} \bar{A} f. \quad (4.20)$$

Let us fix $\bar{A}$ in such a way that $\bar{L}$ parametrizes generic orbits of the $\mathcal{G}_B$-action:

$$\bar{A} = f(\bar{\partial} + \mu \partial) f^{-1} + f \bar{L} f^{-1}. \quad (4.21)$$

Then the field $L$ dual to $\bar{L}$ is obtained from $A$ by the action of the same element $f$, which is defined up to left multiplication by elements preserving $\bar{L}$:

$$L = f^{-1} \kappa \partial f + f^{-1} A f. \quad (4.22)$$

Thus, in local coordinates the moment equation takes the form (see (2.18))

$$(\bar{\partial} + \partial \mu) L - \kappa \partial \bar{L} + [\bar{L}, L] = 2\pi i \sum_{a=1}^n S^a \delta(x_a). \quad (4.23)$$
The gauge fixation (4.21) and the condition for the moment map give the reduced space \( \text{FBun}(\Sigma_{g,n}, G) = \{ L, \overline{L}, S \} \). It becomes finite-dimensional (see (2.23)), as does the bundle \( \mathcal{P}(G) \):

\[
\dim(\mathcal{P}(G)) = \dim(\text{FBun}(\Sigma_{g,n}, x, G)) + \dim(\mathcal{P}(G)) = 2 \dim(G)(g - 1) + \sum_{a=1}^{n} \dim(\mathcal{O}_a) + 3g - 3 + n.
\]

By the invariance of \( \omega \) in (4.17) the form on \( \text{FBun}(\Sigma_{g,n}, x, G) \) is

\[
\omega = \omega_{\text{FBun}} - \frac{1}{\kappa} \sum_{s=1}^{3g-3+n} \delta H_s \delta \tau_s, \quad H_s(L) = \frac{1}{2} \int_{\Sigma_{g,n}} \langle L^2 \rangle \mu^0_s,
\]

\[
\omega_{\text{FBun}} = \omega_0 + \sum_{a=1}^{n} \omega^r_{a}, \quad \omega_0 = \int_{\Sigma_{g,n}} \langle dL, d\overline{L} \rangle.
\]

In view of (4.23) the system is no longer free, because \( L \) depends on the moduli space of flat bundles and \( S \). Moreover, since \( H_s \) depends explicitly on the times, the reduced system is non-autonomous.

Let \( M_s = \partial_s f f^{-1} \). It follows from (4.21) and (4.22) that the equations of motion (4.14) on the space \( \text{FBun}(\Sigma_{g,n}, x, G) \) take the form

\[
\kappa \partial_\tau L - \kappa \partial_\mu M_s + [M_s, L] = 0 \quad (s = 1, \ldots , l), \quad (4.26a)
\]

\[
(\overline{\partial} + \partial \mu) M_s = -L \mu^0_s. \quad (4.26b)
\]

The equations (4.26a) are the Lax equations. The essential difference from integrable systems is the additional term \( \kappa \partial_\mu M_s \). On \( \text{FBun}(\Sigma_{g,n}, G) \) the linear system (4.15) assumes the form

\[
(\kappa \partial + L) \psi = 0, \quad (4.27a)
\]

\[
(\partial_\pi + \overline{L}) \psi = 0, \quad (4.27b)
\]

\[
(\kappa \partial_\tau + M_s) \psi = 0 \quad (s = 1, \ldots , l_2). \quad (4.27c)
\]

Equations (4.26a) and (4.26b) are consistency conditions for the linear problems (4.27a) & (4.27c) and (4.27b) & (4.27c), while (4.23) is a consistency condition for (4.27a) & (4.27b). Equations (4.27c) provide the isomonodromy property of the system (4.27a), (4.27b) with respect to variations of the times \( t_s \). We refer to the non-linear equations (4.26) as the hierarchy of isomonodromic deformations.

The Hecke transformations act on the sections \( \psi (3.43) \): \( \tilde{\psi} = \Xi(\zeta) \psi \). As a result we go from the system (4.27) to the modified system

\[
(\kappa \partial + L) \tilde{\psi} = 0, \quad (4.28a)
\]

\[
(\partial_\pi + \overline{L}) \tilde{\psi} = 0, \quad (4.28b)
\]

\[
(\kappa \partial_\tau + M_s) \tilde{\psi} = 0 \quad (s = 1, \ldots , l_2). \quad (4.28c)
\]

The transformation

\[
\Xi(\zeta): (L, M_s) \rightarrow (\overline{L}, \overline{M}_s)
\]

is a generator of discrete-time transformations. In this sense the Hecke transformations give Bäcklund transformations for the monodromy-preserving equations.
4.5. Isomonodromic deformations and integrable systems. As noted above, the isomonodromy equations can be interpreted as a deformation (Whitham quantization) of integrable equations. The deformation parameter is \( \kappa \). This approach was developed in [76] for Schlesinger systems.

We introduce the independent times \( t_s^0 = (\tau_l^0, x_a^0) \) as \( \tau_l = \tau_l^0 + \kappa t_l, \ t_a = \kappa t_a \) for \( \kappa \to 0 \). This means that \( t_s = (t_l, t_a) \) play the role of local coordinates in a neighbourhood of the point \( (\tau_l^0, x_a^0) \in \Sigma_{g,n} \). In this limit the equations of motion (4.26a) are the standard Lax equations

\[
\partial_s L^{(0)} + [M_s^{(0)}, L^{(0)}] = 0 \quad (s = 1, \ldots, l),
\]

where \( L^{(0)} = L(t_s^0) \ (M_s^{(0)} = M_s(t_s^0)) \). The linear problem for these systems is obtained from the linear problem for the isomonodromy problem (4.27) by analogy with the quasi-classical limit in quantum mechanics. Let the Baker–Akhiezer function be represented in the form of the WKB approximation

\[
\psi = \Phi \exp \left( \frac{\mathcal{J}^{(0)}}{\kappa} + \mathcal{J}^{(1)} \right)
\]

and then substitute (4.30) in the linear system (4.27). If \( \partial_\bar{z} \mathcal{J}^{(0)} = 0 \) and \( \partial_{\tau_l^0} \mathcal{J}^{(0)} = \partial_{x_a^0} \mathcal{J}^{(0)} = 0 \), then the terms of order \( \kappa^{-1} \) vanish. In the quasi-classical limit we put \( \partial \mathcal{J}^{(0)} = \lambda \). In the zero-order approximation we obtain the linear system

\[
(\lambda + L^{(0)}(z, \tau_0))Y = 0, \quad (4.31a)
\]
\[
\partial_\bar{z}Y = 0, \quad (4.31b)
\]
\[
(\partial_{t_s} + M_s^{(0)}(z, \tau_0))Y = 0. \quad (4.31c)
\]

The Baker–Akhiezer function \( Y \) takes the form

\[
Y = \Phi \exp \left( \sum_s t_s \frac{\partial}{\partial t_s^0} \mathcal{J}^{(0)} \right).
\]

The consistency condition for (4.31a) and (4.31b) is the Lax equation (4.29).

Chapter II

Isomonodromy problems on elliptic curves

5. The moduli space of flat bundles over elliptic curves

The moduli space of flat \( G \)-bundles plays the role of the phase space in an isomonodromy problem. For trivial bundles this space was described in [77]–[79], and for non-trivial bundles in [80], [81].

5.1. Holomorphic bundles over elliptic curves. Here and below we consider the moduli spaces \( \text{Bun}(\Sigma, x, G) \) of holomorphic bundles and the moduli spaces \( \text{FBun}(\Sigma, x, G) \) of flat bundles over elliptic curves realized in the form \( \Sigma \sim \mathbb{C}/(\tau \mathbb{C} \oplus \mathbb{C}) \) (\( \text{Im} \tau > 0 \)). There are two generators \( \rho_1 \) and \( \rho_\tau \) of the fundamental group \( \pi_1(\Sigma) \) corresponding to the shifts \( z \to z + 1 \) and \( z \to z + \tau \) and satisfying the relation

\[
\rho_1 \rho_\tau \rho_1^{-1} \rho_\tau^{-1} = 1. \quad (5.1)
\]
The sections of the $G$-bundle $E_G(V)$ over $\Sigma_\tau$ satisfy the quasi-periodicity conditions
\[ \psi(z + 1) = \mathcal{D}\psi(z), \quad \psi(z + \tau) = \Lambda\psi(z), \]  
(5.2)
where the transition operators $(\mathcal{D}, \Lambda)$ in turn satisfy (5.1):
\[ \mathcal{D}(z)\Lambda^{-1}(z)\mathcal{D}^{-1}(z + \tau)\Lambda(z + 1) = 1. \]  
(5.3)
By definition, these transition operators define a topologically trivial bundle. Let $\zeta$ be an element of $\mathcal{D}(G)$. To obtain a non-trivial bundle, we replace (5.3) by the equation
\[ \mathcal{D}(z)\Lambda^{-1}(z)\mathcal{D}^{-1}(z + \tau)\Lambda(z + 1) = \zeta. \]  
(5.4)
It follows from (A.33) that the right-hand side can be represented in the form $\zeta = e(\gamma)$ ($e(x) = \exp(2\pi ix)$) with $\gamma \in P^\vee$ (A.10). Then (5.4) takes the form
\[ \mathcal{D}(z)\Lambda^{-1}(z)\mathcal{D}^{-1}(z + \tau)\Lambda(z + 1) = e(\gamma). \]  
(5.5)
A bundle $\tilde{E}$ is equivalent to $E$ if its sections $\tilde{s}\psi$ are related to $\psi$ by $\tilde{\psi}(z) = f(z)\psi(z)$, where $f(z)$ is an invertible operator acting on $V$. It follows from (5.5) that the transformed transition operators have the form
\[ \mathcal{D}^f = f(z + 1)\mathcal{D}f^{-1}(z), \quad \Lambda^f = f(z + \tau)\Lambda f^{-1}(z). \]  
(5.6)
These transformations form the automorphism group $\mathcal{G}$ (the gauge group) of the bundle $E_G$ over $\Sigma_\tau$. It follows from [82] that the transition operators can be chosen as constants. Therefore, we have
\[ [\mathcal{D}, \Lambda^{-1}] = e(\gamma) \quad ([\mathcal{D}, \Lambda^{-1}] = \mathcal{D}\Lambda^{-1}\mathcal{D}^{-1}\Lambda). \]  
(5.7)
Thus, $(\mathcal{D}, \Lambda)$ form a projective representation of $\pi_1(\Sigma_\tau)$. The moduli space of stable holomorphic bundles over $\Sigma_\tau$ can be defined as
\[ \text{Bun}_\zeta(\Sigma_\tau, G) = \{[\mathcal{D}, \Lambda^{-1}] = \zeta\}/\mathcal{G}, \quad \zeta \in \mathcal{D}(G). \]  
(5.8)
It was proved in [77]–[79] that $\text{Bun}_\zeta(\Sigma_\tau, G)$ is isomorphic to a weighted projective space for $\zeta = 1$. Let $G = \overline{G}$ be a simply connected group. We fix a Cartan subgroup $\mathcal{H}_G \subset \overline{G}$. Assume that $\mathcal{D}$ is semisimple and therefore conjugate to a generic element in $\mathcal{H}_G$. By neglecting non-semisimple transition functions we define a big cell $\text{Bun}_0^0(\Sigma_\tau, G)$ in the moduli space, that is, we replace (5.8) by
\[ \text{Bun}_0^0(\Sigma_\tau, G) = \{[\mathcal{D}, \Lambda^{-1}] = \zeta\}/\mathcal{G}, \quad \mathcal{D} \in \mathcal{H}_G. \]  
(5.9)
**Proposition 5.1.** Solutions of (5.9) have the following description.$^4$

- The element $\Lambda$ has the form $\Lambda = \Lambda^0e(\mathbf{u})$, where $\Lambda^0$ is uniquely determined by the coweight $\gamma$ ($\Lambda^0 = \Lambda^0_\gamma$). It is a special element in the Weyl group $W$, and it preserves the extended coroot system $\Pi^\vee_{\text{ext}} = \Pi^\vee \cup \alpha^\vee_0$ and thus is a symmetry of the extended Dynkin diagram.

\[ ^4 \text{See [45] for the proof.} \]
Let $\mathfrak{h}_0 \subseteq \mathfrak{h}$ be a Cartan subalgebra preserved by $\Lambda^0$ ($\lambda(\mathfrak{h}_0) = \mathfrak{h}_0$, $\lambda = \text{Ad}_{\Lambda^0}$). Then $\mathfrak{u} \in \mathfrak{h}_0$.

The element $Q$ has the form $Q = Q_0$, where

$$Q_0 = \exp(2\pi i \varrho U), \quad \varrho = \rho^\vee h \in \mathfrak{h}, \quad (5.10)$$

with $h$ the Coxeter number and $\rho^\vee = \frac{1}{2} \sum_{\alpha^\vee \in (R^\vee)^+} \alpha^\vee$, and with $U$ and $\Lambda^0$ commuting with each other.\(^5\)

Remark 5.1. For Spin$(4n)$ the centre $Z(\text{Spin}(4n)) \sim \mu_2 \times \mu_2$ has two generators $\zeta_1$ and $\zeta_2$ corresponding to the fundamental weights $\varpi_a$ and $\varpi_b$ of the left and the right spinor representations. Arguing as above, we find two solutions $\Lambda_a$ and $\Lambda_b$ of (5.9), with $Q$ the same in both cases.

Remark 5.2. If $\xi \in Q^\vee$, then $\zeta = \text{Id}$. Thus, the transformation $\Lambda_0$ is trivial: $\Lambda_0 = \text{Id}$ and $\mathfrak{h}_0 = \mathfrak{h}$. In this case the bundle has a trivial characteristic class, but there are holomorphic moduli that combine to form the vector $\mathfrak{u} \in \mathfrak{h}$.

5.2. Decomposition of Lie algebras. Let us take an element $\zeta \in \mathfrak{Z}(\mathfrak{G})$ of order $l$ and the corresponding $\Lambda^0 \in W$ from Proposition 5.1. Then $\Lambda^0$ generates a cyclic group $\mu_l = (\Lambda^0, (\Lambda^0)^2, \ldots, (\Lambda^0)^l = 1)$ isomorphic to a subgroup $\mathfrak{Z}_l \subseteq \mathfrak{Z}(\mathfrak{G})$. Note that $l$ is a divisor of $\text{ord}(\mathfrak{Z}(\mathfrak{G}))$. We consider the action of $\Lambda^0$ on $\mathfrak{g}$. Since $(\Lambda^0)^l = \text{Id}$, we have an $l$-periodic grading

$$\mathfrak{g} = \bigoplus_{j=0}^{l-1} \mathfrak{g}_j, \quad \lambda(\mathfrak{g}_j) = \omega^j \mathfrak{g}_j, \quad \omega = \exp\left(\frac{2\pi i}{l}\right), \quad \lambda = \text{Ad}_{\Lambda^0}, \quad (5.11)$$

$$[\mathfrak{g}_j, \mathfrak{g}_k] = \mathfrak{g}_{j+k} \pmod{l}, \quad (5.12)$$

where $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and the subspaces $\mathfrak{g}_a$ are representations of it. Moreover, in these terms the Killing form on $\mathfrak{g}$ is

$$(\mathfrak{g}, \mathfrak{g}) = \sum_{j=0}^{l-1} \langle \mathfrak{g}_j, \mathfrak{g}_{l-j} \rangle. \quad (5.13)$$

The $\lambda$-invariant subalgebra $\mathfrak{g}_0$ contains a simple subalgebra $\tilde{\mathfrak{g}}_0$:

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus V. \quad (5.14)$$

The components of this decomposition are orthogonal with respect to the Killing form (B.14):

$$\langle \mathfrak{g}_0, \mathfrak{g}_0 \rangle = \langle \tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_0 \rangle + \langle V, V \rangle, \quad (5.15)$$

and $V$ is a representation space of $\tilde{\mathfrak{g}}_0$. Below we find $\mathfrak{g}_0$ explicitly for all simple Lie algebras in our list. The coroot system $\tilde{\Pi}$ of $\tilde{\mathfrak{g}}_0$ is constructed by averaging over the orbits of the $\lambda$-action on $\Pi^\vee \text{ext} [45]$. We summarize the information about invariant subalgebras in Table 4 of Appendix B.

\(^5\)The first statement can be found in [83] (Proposition 5 in VI.3.2).
5.3. The moduli space of holomorphic bundles. We describe a $G$-bundle $E_G(V)$ by the transition operators $\Lambda = \Lambda^0 e(p)$ and $\mathcal{D} = e(\rho^\vee/h + q)$, where $\Lambda^0 = \Lambda^0(\gamma)$ corresponds to the coweight $\gamma \in P^\vee$. The topological type of $E$ is defined by an element $\zeta = e(\varpi^\vee) \in P^\vee/t(G)$.

Let us transform $(\Lambda, \mathcal{D})$ by taking $f = e(-qz)$ in (5.6). Since $f$ commutes with $\Lambda^0$, we obtain new transition operators $\mathcal{D} = e(q + q) \rightarrow \mathcal{D} = e(q)$ and $\Lambda \rightarrow \Lambda^0 e(p - q\tau)$. Let $p - q\tau = u$. The sections of $E_G(V)$ have the quasi-periodicity properties

$$
\psi(z + 1) = \pi(e(q))\psi(z), \quad \psi(z + \tau) = \pi(e(u)\Lambda^0)\psi(z). \quad (5.16)
$$

Thus, we obtain the transition operators

$$
\mathcal{D} = e(q), \quad \Lambda = e(u)\Lambda^0. \quad (5.17)
$$

Here $u \in \tilde{h}_0$ plays the role of a parameter in the moduli space $\tilde{h}_0$ (Proposition 5.1). In this subsection we describe it in detail. In fact, $\tilde{h}_0$ covers the genuine moduli space which we are going to describe.

Fixing $\mathcal{D}$ and $\Lambda$, we still have a residual gauge symmetry preserving $\tilde{h}_0$. Let $\tilde{W}_0$ be the Weyl group of the Lie algebra $\tilde{g}_0$ and let $t(\tilde{G}_0)$ be the group (A.31) of cocharacters of the invariant subgroup $\tilde{G}_0$ (A.29): $t(\tilde{G}_0) = \{ \mathbb{C}^* \rightarrow \tilde{H}_0 \} = \{ \chi_\gamma(z) = e(\gamma z) \}$. The cocharacters have the quasi-periodicity properties

$$
\chi_\gamma(z + 1) = \chi_\gamma(z), \quad \chi_\gamma(z + \tau) = \chi_\gamma(\tau)\chi_\gamma(z), \quad \gamma \in t(\tilde{G}_0). \quad (5.18)
$$

Let $\mathcal{G}_0 = \{ f(z) \}$ be the group of maps $\Sigma_\tau \rightarrow \tilde{H}_0$, and let

$$
\mathcal{G}_0 = \left\{ f(z) = \sum_{\gamma \in P^\vee} c_\gamma \chi_\gamma(z) \right\}, \quad (5.19)
$$

where $c_\gamma = 0$ for almost all $\gamma$ and

$$
\tilde{\mathcal{R}}^\vee = \begin{cases} 
\tilde{Q}^\vee, & \tilde{G}_0 = \tilde{G}_0, \\
\tilde{P}^\vee, & \tilde{G}_0 = \tilde{G}_0^{\text{ad}}, \\
\tilde{t}(G_1), & \tilde{G}_0 = \tilde{G}_0, \quad (A.31).
\end{cases}
$$

It follows from (5.18) that $f(z + 1) = f(z)$. The group of residual gauge transformations is the semidirect product

$$
\mathcal{G}_{\tilde{W}_0} = \tilde{W}_0 \ltimes \tilde{G}_0. \quad (5.20)
$$

The transformations of the sections by $f(z) = \chi_\gamma(z)$ can be written in the form

$$
\psi(z) \rightarrow e(\gamma z)\psi(z), \quad \gamma \in \tilde{\mathcal{R}}^\vee. \quad (5.21)
$$

For the transition operators we find that (see (5.6))

$$
\mathcal{D}^f = e(\gamma z) \mathcal{D} e(-\gamma z) = \mathcal{D}, \quad \Lambda^f_0 = e(\gamma \tau) e(\gamma z) \Lambda_0 e(-\gamma z) = e(\gamma \tau) \Lambda_0. \quad (5.22)
$$
Since $\gamma \tau \in \tilde{\mathfrak{h}}_0$, the action of $\mathcal{G}_{\tilde{\mathcal{H}}_0}$ on $u$ takes the form
\[
u \to u' = \begin{cases} su, & s \in \tilde{W}_0, \\ u + \tau \gamma_1 + \gamma_2, & \gamma = (\gamma_1, \gamma_2) \in \tilde{\mathcal{H}}^\vee. \end{cases} \tag{5.23}
\]
Thus, the transition operators defined by the parameters $u$ and $u'$ describe equivalent bundles. The semidirect product of the Weyl group $\tilde{W}_0$ and the lattice $\tau \tilde{\mathcal{H}}^\vee \oplus \tilde{\mathcal{H}}^\vee$ is called the Bernstein–Schwarzman group \cite{77}–\cite{79}. It is an elliptic analogue of the affine Weyl group $W_{BS}(\mathfrak{g}) = fW_0 \ltimes (\tau eR^\vee \oplus eR^\vee)$.

Thus, $u$ can be taken in the fundamental domain $C(\mathfrak{g})$ of $W_{BS}(\mathfrak{g})$: $\text{Bun}_0^0(\Sigma, G) = \tilde{\mathfrak{h}}_0/W_{BS} = C(\mathfrak{g})$, the moduli space of holomorphic $G$-bundles. \tag{5.24}

Let $\mathcal{Z}_l(\mathcal{G})$ be a cyclic group of order $N$ and let $\mathcal{Z}_l \subseteq \mathcal{Z}_l(\mathcal{G})$ be a subgroup of it of order $l$. This is the case for the groups $\mathcal{G} = \text{SL}(N, \mathbb{C})$ or $\text{Spin}_{2n+2}$. Thus, for $\mathcal{G}$, $G_l$, $G_p$, $G_{ad}$-bundles we have the following relations between their moduli spaces:

\[
\begin{array}{ccc}
\text{Bun}_1^0(\Sigma, G) & \xrightarrow{} & \text{Bun}_l^0(\Sigma, G_l) \\
\text{Bun}_{\zeta}(\Sigma, G) & \xrightarrow{} & \text{Bun}_p^0(\Sigma, G_p) \\
\text{Bun}_{\zeta}(\Sigma, G^{ad}) & \xleftarrow{} & \text{Bun}_{\zeta}(\Sigma, G)
\end{array}
\tag{5.25}
\]

Here the arrows denote coverings. Note that $\text{Bun}_1^0(\Sigma, G)$, $\text{Bun}_{\zeta}(\Sigma, G^{ad})$ and also $\text{Bun}_{\zeta_l}(\Sigma, G_l)$, $\text{Bun}_{\zeta_p}(\Sigma, G_p)$ are dual to each other in the sense that the lattices defining them are dual. We have a similar picture for $\text{Spin}_{4n}$, where $\mathcal{Z}(\text{Spin}_{4n}) \sim (\mu_2 \oplus \mu_2)$.

\section*{5.3.1. Holomorphic bundles with quasi-parabolic structures.}

As in the general case, we associate with the marked points $x = \{x_a\}$ the $G$-flags $\text{Flag}_a = G/P_a$ ($P_a$ is a parabolic subgroup of $G$). In this way we extend the space $\tilde{\mathfrak{h}}_0$ as
\[
\tilde{\mathfrak{h}}_0 \times \bigcup_{a=1}^n \text{Flag}_a.
\]

Taking into account the action of $\mathcal{G}_{\tilde{\mathcal{H}}_0}$, we find that a big cell $\text{Bun}_\zeta^0(\Sigma, x, G)$ in the moduli space $\text{Bun}_{\zeta}(\Sigma, x, G)$ of holomorphic bundles with quasi-parabolic structures is the quotient
\[
\text{Bun}_\zeta^0(\Sigma, x, G) = \left(\tilde{\mathfrak{h}}_0 \times \bigcup_{a=1}^n \text{Flag}_a\right)/\mathcal{G}_{\tilde{\mathcal{H}}_0}, \tag{5.26}
\]
where the action of $\mathcal{G}_{\mathcal{H}_0}$ on $Y_a \in \text{Flag}_a$ takes the form

$$Y_a \rightarrow \begin{cases} s(Y_a), & s \in \mathcal{W}_0, \\ \text{Ad}_{\chi(x_a)}(Y_a), & \gamma \in \mathcal{R}. \end{cases}$$

We thus parametrize $\text{Bun}_0^0(\Sigma, x, G)$ as

$$\text{Bun}_0^0(\Sigma, x, G) = \{u \in C(\mathcal{R}^\vee); Y_a \in \text{Flag}_a / \mathcal{G}_{\mathcal{H}_0}, a = 1, \ldots, n\}. \quad (5.28)$$

### 5.3.2. Flat bundles with quasi-parabolic structures.

As in the general case (2.9), we describe the moduli space of flat bundles $\text{FBun}(\Sigma, x, G)$ over elliptic curves $\Sigma$ as the principal homogeneous spaces $P T^* \text{Bun}_0^0(\Sigma, x, G)$ over the cotangent bundles $T^* \text{Bun}_0^0(\Sigma, x, G)$. Let $v$ be a vector in the fibers of the projection

$$\pi: \text{FBun}_0^0(\Sigma, G) \xrightarrow{\text{Flat}} \text{Bun}_0^0(\Sigma, G). \quad (5.29)$$

The vector $v$ dual to $u$ is defined by the transition operator $\Lambda(u)$ (5.17). In these terms the form $\omega_0$ in (4.25) becomes

$$\omega = (\delta v \wedge \delta u) + \sum_{a=1}^n \omega_a^{KK}. \quad (5.31)$$

It turns out that this form is annihilated by the vector fields generated by $\mathcal{G}_{\mathcal{H}_0}$ (5.19). Therefore, the structure of the moduli space of flat bundles is described. The moduli space of flat bundles over elliptic curves with quasi-parabolic structure is the symplectic quotient space

$$\text{FBun}_0^0(\Sigma, x, G) = \text{FBun}_0^0(\Sigma, x, G) / / \mathcal{G}_{\mathcal{H}_0} = \{(v, u); S^a \in \mathcal{O}_a / / \mathcal{G}_{\mathcal{H}_0}\}. \quad (5.32)$$

In these terms the $\omega$ in (5.31) is

$$\omega_{\text{FBun}} = (\delta v \wedge \delta u) + \sum_{a=1}^n \omega_a^{KK}, \quad (5.33)$$

where $\omega_a^{KK}$ is the form on $\mathcal{O}_a / / \mathcal{G}_{\mathcal{H}_0}$.

### 6. Monodromy-preserving equations on elliptic curves

In this section we generalize the results of [45] and [46] in two directions: first, to the case of several marked points. The corresponding integrable systems are
generalizations of the elliptic models of Gaudin, which were described by Nekrasov in [32], where the simplest case was considered: an \( SL(N, \mathbb{C}) \)-bundle with trivial characteristic class. Second, we describe isomonodromy problems instead of integrable systems. The corresponding models are non-autonomous Hamiltonian systems generalizing the autonomous Gaudin models. Here we apply the approach in [28]–[30] to the isomonodromy equations, where these equations are treated as non-autonomous generalizations of Hitchin systems.

### 6.1. Deformations of elliptic curves.

Consider the torus, that is, \( \mathbb{T}^2 = \{(x, y) \in \mathbb{R} \mid x, y \in \mathbb{R}/\mathbb{Z}\} \). The complex structure on \( \mathbb{T}^2 \) is defined by the complex coordinate \( z = x + \tau_0 y \), \( \text{Im} \tau_0 > 0 \). In this way we realize the elliptic curve \( \Sigma_{\tau_0} \sim \mathbb{C}/(\mathbb{Z} + \tau_0 \mathbb{Z}) \).

It follows from (4.11) that \( \mathbb{T}^2 \) must have at least one moving point. Since there is a \( \mathbb{C} \)-action on \( \Sigma_{\tau_0} \) by the shifts \( x \rightarrow x + c \), we can take \( x_0 = 0 \) as a marked point, and there remains a single modulus of the deformation \( \Sigma_{\tau_0} \rightarrow \Sigma_{\tau} \sim \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \). We assume that the difference \( \tau - \tau_0 \) is small and the marked points do not coincide.

A big cell \( \mathfrak{T}_{1,n}^0 \) in the space of times is the Teichmüller space \( \mathfrak{T}_{1,n} \), defined as

\[
\mathfrak{T}_{1,n}^0 = (\mathcal{H}^+ = \{\text{Im} \tau > 0\}) \times \{(x_0 = 0, \ldots, x_n), \ x_k \neq x_j \ (\text{mod } \langle \tau, 1 \rangle)\}.
\]

According to the general prescription, we introduce the new coordinate

\[
w = z - \frac{\tau - \tau_0}{\rho_0} (\overline{z} - z), \quad \text{where } \rho_0 = \tau_0 - \overline{\tau}_0.
\]

Under the shifts \( z \rightarrow z + 1 \) and \( z \rightarrow z + \tau_0 \) the coordinate \( w \) is transformed into \( w + 1 \) and \( w + \tau \). Thus, \( w \) is a well-defined holomorphic coordinate on \( \Sigma_{\tau} \). But this deformation moves the points \( x_a \). Instead, we should consider

\[
w = z - \frac{\tau - \tau_0}{\rho_0} (\overline{z} - z) \left(1 - \sum_{a=1}^{n} \chi_a(z, \overline{z})\right),
\]

where \( \chi(z, \overline{z}) \) is the characteristic function (4.7) and \( \chi_a(z, \overline{z}) = \chi(z - x^0_a, \overline{z} - \overline{x}^0_a) \).

As in the general case, we use the coordinates

\[
\begin{cases}
w = z - \frac{\tau - \tau_0}{\rho_0} (\overline{z} - z) \left(1 - \sum_{a=1}^{n} \chi_a(z, \overline{z})\right), \\
\tilde{w} = \overline{z}.
\end{cases}
\]

Note that

\[
z + \tau_0 \rightarrow (w + \tau, \tilde{w} + \overline{\tau}_0).
\]

Taking (4.9) into account, we define

\[
\varepsilon(z, \overline{z}) = \frac{t_\tau}{\rho_0} (\overline{z} - z) \left(1 - \sum_{a=1}^{n} \chi_a(z, \overline{z})\right) + \sum_{a=1}^{n} t_a \chi_a(z, \overline{z}).
\]

The deformed operator has the form \( \partial_{\tilde{w}} = \partial_{\tilde{z}} + \mu \partial_z \), where \( \mu \) (see (4.9)) is represented as the sum

\[
\mu = t_\tau \mu^{(0)} + \sum_{a=1}^{n} t_a \mu^{(0)}.
\]
From (4.2) and (6.2) we find $\mu^{(0)}_{\tau}$:

$$
\mu^{(0)}_{\tau} = \frac{1}{\rho_0} \overline{\partial}(z - \tau) \left( 1 - \sum_{a=1}^{n} \chi_a(z, \bar{z}) \right), \quad t_{\tau} = \tau - \tau_0,
$$

(6.4)

and from (6.52) we find $\mu^{(0)}_a$:

$$
\mu^{(0)}_a = \overline{\partial} \chi_a(z, \bar{z}), \quad t_a = x_a - x^0_a.
$$

(6.5)

The basis dual to the Beltrami-differentials basis $\mu^{(0)}_{\tau}, \{ \mu^{(0)}_a, a = 1, \ldots, n \}$ with respect to integration over $\Sigma_{\tau}$ consists of the first Eisenstein functions $E_1(z - x_a)$ in (C.2) together with 1. There is only one time $t_{\tau}$ in the case of a single marked point (see (4.11)).

The moduli space $\mathcal{M}_1$ of elliptic curves is the result of factorization of the upper half-plane $\mathcal{H}^+$ by the action of $\text{SL}(2, \mathbb{Z})$ as Möbius transformations:

$$
\mathcal{M}_1 = \mathcal{H}^+ / \text{SL}(2, \mathbb{Z}) \quad \left( \tau \to \frac{a\tau + b}{c\tau + d} \right).
$$

A big cell $\mathcal{M}_{1,n}^0$ in the moduli space is defined as the quotient

$$
\mathcal{M}_{1,n}^0 = \mathcal{T}_{1,n}^0 / \text{SL}(2, \mathbb{Z}) \quad (x_a \to x_a(c\tau + d)^{-1})
$$

(6.6)

with $\mathcal{T}_{1,n}^0$ in (6.1).

**6.2. Symplectic reduction.** For generic configurations of connections on the bundle over elliptic curves the $(0, 1)$-part of a connection can be gauged away (see (4.21)):

$$
\mathcal{L} := f^{-1} \partial \bar{w} f + f^{-1} \mathcal{A} f = 0.
$$

(6.7)

Then $f$ acts on $A$ similarly: $f^{-1} \partial \bar{w} f + f^{-1} A f = \mathcal{L}$. Therefore, the moment equation takes the form

$$
\partial \bar{w} L_{\zeta} = \sum_{a=1}^{n} \mathcal{S}^a \delta(w - x_a).
$$

(6.8)

This means that $L$ is meromorphic and has simple poles at the marked points, with residues

$$
\text{Res}_{w=x_a} L = \mathcal{S}^a \in \mathcal{O}_a \subset \mathfrak{g}^*, \quad \mathcal{S} = (\mathcal{S}^1, \ldots, \mathcal{S}^n).
$$

(6.9)

To define the space $\widehat{\text{FBun}}^0_\zeta(\Sigma_{\tau}, x, G)$ (5.30) we must take into account the quasi-periodicity conditions

$$
L_{\zeta}(w + 1) = \text{Ad}_{\mathcal{D}} L(w), \quad L_{\zeta}(w + \tau) = \text{Ad}_{\Lambda(u)} L(w) \quad ([\mathcal{D}, \Lambda^{-1}(u)] = \zeta).
$$

(6.10)

Then

$$
\widehat{\text{FBun}}^0_\zeta(\Sigma_{\tau}, x, G) = \{ \text{solutions of (6.8) with the conditions (6.10)} \},
$$

(6.11)

$$
\text{FBun}_\zeta^0(\Sigma_{\tau}, x, G) = \{ \text{solutions of (6.8) with the conditions (6.10)} \} / \mathcal{A}_{\mathcal{H}_0^+}.
$$
The linear problem (4.27) on $\text{FBun}_0^\zeta$ can be written as
\begin{align}
(\kappa \partial + L)\psi &= 0, \\
\partial_\tilde{w}\psi &= 0, \\
(\kappa \partial_s + M_s)\psi &= 0 \quad (s = 1, \ldots, n + 1),
\end{align}
where
\[ \partial_\tilde{w} = \partial_z + \left( t_\tau \mu_\tau^{(0)} + \sum_{a=1}^n t_a \mu_a^{(0)} \right) \partial_z. \]

Finally, the Lax equation has the form
\[ \kappa \partial_s L - \kappa \partial M_s + [M_s, L] = 0. \]

6.3. Lax matrix. To calculate $L$ we use the GS-basis (see Appendix B). Let $(t_k^k, h_k^\alpha)$ be the basis (B.5), (B.12) in the $k$-component in (5.11), and let $h_0^\alpha$ be the Cartan generators of the invariant subalgebra $h_0'$. We consider the residues (6.9) of $L$:
\[ S^a = (S^a)^{l,-k}_\beta t^k_\beta + (S^a)^{h,-k}_\alpha h^k_\alpha + (S^a)^{h,0}_\alpha h^0_\alpha. \]
Let the $L$-operator be represented as the sum of the Cartan and the root parts:
\[ L(w) = + L_h(w) + L_0^h(w) + L_R(w), \]
Then an $L$ satisfying (6.8) and (6.10) has the form
\begin{align*}
L_R(w) &= \frac{1}{2} \sum_{a=1}^n \sum_{l=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi^k_{\beta}(u, w - x_a) (S^a)^{l,-k}_\beta t^k_\beta, \\
L_h(w) &= \sum_{\alpha \in \Pi} \sum_{k=1}^{l-1} \sum_{a=1}^n \phi\left( \frac{k}{l}, w - x_a \right) (S^a)^{h,-k}_\alpha h^k_\alpha, \\
L_0^h(w) &= \sum_{\alpha \in \Pi} \left( v^\alpha_\alpha + \sum_{a=1}^n E_1(w - x_a) (S^a)^{h,0}_\alpha \right) h^0_\alpha.
\end{align*}
Here the functions $\varphi^k_{\beta}(u, z)$ are defined in (C.23), and $E_1(z)$ is the Eisenstein function (C.2). The needed properties of $L$ follow from (C.19)–(C.21).

The Poisson structure. For the spaces $\text{FBun}_0^\zeta(\Sigma_\tau, x, G)$ and $\text{FBun}^0_\zeta(\Sigma_\tau, x, G)$ in (5.32) we describe their Poisson structure. The manifold indicated is the Poisson manifold $\mathcal{P}$ with the canonical brackets for $v, u$ and Poisson–Lie brackets for $S$:
\[ \mathcal{P} = T^*C \times \bigcup_a \theta_a = \{v, u, S^a \in \theta_a\}. \]
It has dimension $\sum_a \text{dim}(\theta_a) + 2 \text{dim}(\tilde{h}_0)$. We describe the Poisson structure on $\text{FBun}_0^\zeta$ as the result of Poisson reduction with respect to the action of $\tilde{G}_0$ in (5.19).
We consider the Poisson algebra $\mathcal{A} = C^\infty(\mathbb{P})$ of smooth functions on $\mathbb{P}$. Let $\varepsilon \in \mathfrak{h}_0$, let $\gamma$ be a small contour around $z = 0$, and consider the function

$$\mu_\varepsilon = \oint_{\gamma} (\varepsilon, L(v, u, S)) = (\varepsilon, S^b_0), \quad S^b_0 = \sum_{j=1}^{p} S^b_j e_j.$$ 

It generates on $\mathbb{P}$ the vector field

$$V_\varepsilon : L(v, u, S) \to \{\mu_\varepsilon, L(v, u, S)\} = [\varepsilon, L(v, u, S)].$$

Let $\mathcal{A}^{\text{inv}}$ be an invariant subalgebra of $\mathcal{A}$ under the $V_\varepsilon$ action. Then $I = \{\mu_\varepsilon F(v, u, S) \mid F(v, u, S) \in \mathcal{A}\}$ is the Poisson ideal in $\mathcal{A}^{\text{inv}}$. The reduced Poisson algebra is the quotient algebra

$$\mathcal{A}^{\text{red}} = \mathcal{A}^{\text{inv}} / I = \mathcal{A} / \mathfrak{h}_0 (\mathfrak{h}_0 = \exp \mathfrak{h}_0).$$

The reduced Poisson manifold $\mathbb{P}^{\text{red}}$ is determined by the moment condition

$$\sum_{a=1}^{n} (S^a)_\alpha^b = 0 \quad (6.17)$$

and $\dim \mathfrak{h}$ is determined by the conditions fixing the gauge on the spin variables (we do not specify these conditions):

$$\mathbb{P}^{\text{red}} = \mathbb{P} / \mathfrak{h}_0 = \mathbb{P}(S^b = 0) / \mathfrak{h}_0, \quad \dim(\mathbb{P}^{\text{red}}) = \dim(\mathbb{P}) - 2 \dim(\mathfrak{h}_0) = \dim(\mathcal{O}). \quad (6.18)$$

Because of the moment condition we pass from (6.15) to the Lax operator that has the correct (quasi-)periodicity properties. It depends on the variables $\{v, u, S^a\}$ in $\mathbb{P}^{\text{red}}$. Here the $S^a$ are not in general independent, because of the gauge fixing. Thus, after the reduction we obtain the Poisson manifold $\mathbb{P}^{\text{red}}$, which has the dimension of the coadjoint orbit $\mathcal{O}$. The Poisson brackets on $\mathbb{P}^{\text{red}}$ are the Dirac brackets [84], [85].

On $\mathbb{P}^{\text{red}}$ the equations of motion are written in the Lax form (6.13). In the limit as $\kappa \to 0$ the isospectral flows that arise become completely integrable, because the number of commuting involutive integrals of motion is equal to $(1/2) \dim(\mathbb{P}^{\text{red}})$ [45].

### 6.4. Classical $r$-matrix.

Let us describe the Poisson structure on the unreduced space $\mathbb{P}$ in terms of the classical dynamical $r$-matrix. The Poisson brackets on $\mathbb{P}$ (see (6.16) are presented in the form of the direct sum of the Poisson–Lie brackets
at each marked point and the canonical brackets for \(v\) and \(u\):

\[
\{ (S^a)^{l,k}_\alpha, (S^b)^{l,m}_\beta \} = \delta^{ab} \left\{ \begin{array}{l}
\frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{ms} C_{\alpha,\lambda^s \beta} (S^a)^{l+k+m}_{\alpha+\lambda^s \beta}, \quad \alpha \neq -\lambda^s \beta, \\
\frac{P_{\alpha}}{\sqrt{l}} \omega^{sm} (S^a)^{h,k+m}_\alpha, \quad \alpha = -\lambda^s \beta,
\end{array} \right.
\]

(6.19)

\[
\{ (S^a)^{h,k}_\alpha, (S^b)^{l,m}_\beta \} = \delta^{ab} \left\{ \begin{array}{l}
1 \sqrt{l - 1} X_{s=0} \omega^{sm} \left( C_{\alpha,\lambda^s \beta} (S^a)^{l+k+m}_{\alpha+\lambda^s \beta} + P_{\alpha} \omega^{sm} (S^a)^{h,k+m}_\alpha \right), \quad \alpha \neq -\lambda^s \beta, \\
\frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} (\tilde{\alpha}, \lambda^s \beta),
\end{array} \right.
\]

\[
\{ v_\alpha, (S^a)^{l,k}_\alpha \} = \{ v_\alpha, (S^a)^{h,k}_\alpha \} = \{ u_\alpha, (S^a)^{h,k}_\alpha \} = 0,
\]

\[
\{ u_\alpha, (S^a)^{l,k}_\alpha \} = \{ u_\alpha, (S^a)^{h,k}_\alpha \} = \{ u_\alpha, (S^a)^{h,k}_\alpha \} = 0.
\]

We note that the Poisson–Lie brackets at each marked point are dual to the commutation relations (B.39), (B.40) in the Lie algebra. The Poisson brackets (6.19) are generated by the dynamical \(r\)-matrix structure with the \(r\)-matrix of the form

\[
r(u, z, w) = r(u, z - w) = r_\beta(u, z - w) + r_R(u, z - w),
\]

(6.20)

where

\[
r_R(u, z) = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi^k_{\alpha}(u, z) t^k_\alpha \otimes t^{-k}_\alpha,
\]

(6.21)

\[
r_\beta(u, z) = \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \varphi^k_{\alpha}(u, z) h^k_\alpha \otimes h^{-k}_\alpha.
\]

(6.21)

More precisely, the following two statements were proved in [45].

**Proposition 6.1.** The \(r\)-matrix (6.20)–(6.21) and the Lax operator (6.14)–(6.15) described above define the Poisson brackets (6.19) via the RLL-equation:

\[
\{ L(w) \otimes 1, 1 \otimes L(w) \} = [L(w) \otimes 1 + 1 \otimes L(w), r(z, w)]
\]

\[
- \frac{\sqrt{l}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_u \varphi^k_{\alpha}(u, z - w) S^h_{\alpha} t^k_\alpha \otimes t^{-k}_\alpha.
\]

(6.22)

The Jacobi identity for the brackets (6.22) follows from the next result.
Proposition 6.2. The $r$-matrix $(6.20)$–$(6.21)$ satisfies the classical Yang–Baxter dynamical equation:

\[
[r_{12}(z, w), r_{13}(z, x)] + [r_{12}(z, w), r_{23}(w, x)] + [r_{13}(z, x), r_{23}(w, x)] - \sqrt{l} \sum_{k=0}^{l-1} \sum_{\alpha \in \mathbb{R}} \left( \frac{|\alpha|^2}{2} t_{\alpha}^k \otimes t_{-\alpha}^{-k} \otimes \tilde{h}_{\alpha}^0 \partial_{u} \varphi_{\alpha}^k(u, z - w) \right.
\]

\[
- \frac{|\alpha|^2}{2} t_{\alpha}^k \otimes \tilde{h}_{\alpha}^0 \otimes t_{-\alpha}^{-k} \partial_{u} \varphi_{\alpha}^k(u, z - x) + \frac{|\alpha|^2}{2} \tilde{h}_{\alpha}^0 \otimes t_{\alpha}^k \otimes t_{-\alpha}^{-k} \partial_{u} \varphi_{\alpha}^k(u, w - x) \right) = 0. \quad (6.23)
\]

The last term on the right-hand side of $(6.22)$ prevents the system from being integrable on $P$. However, the action by the Cartan subgroup $H'_0$ of the invariant subgroup $G'_0 \subset G$ generates the moment map condition

\[
\sum_{a=1}^{n} (S^a)_{\alpha}^\beta_{\alpha} = 0 \quad \forall \alpha \in \tilde{\Pi}. \quad (6.24)
\]

As already explained, after the reduction with respect to $\mathcal{H}_0$ in (6.18) we get $P^{\text{red}}$, on which the term hindering integrability vanishes. Then

\[
\{L^{\text{red}}(w) \otimes 1, 1 \otimes L^{\text{red}}(w)\} = [L^{\text{red}}(w) \otimes 1 + 1 \otimes L^{\text{red}}(w), \tilde{r}(z, w)]. \quad (6.25)
\]

Here the $r$-matrix is replaced by $\tilde{r}$ because the Poisson structure on $P^{\text{red}}$ differs from the Poisson structure on $P$. The difference is because of the Dirac terms coming from the reduction $P^{\text{red}} = P // \mathcal{H}_0$. We do not need its explicit form.

The classical dynamical $r$-matrices corresponding to trivial bundles were constructed in [86]. In this case the dynamical parameter $u$ belongs to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The problem of classifying $r$-matrices when $u \in \mathfrak{h} \subset \mathfrak{h}$ was posed in [86]. A solution of it was presented in [87] for the trigonometric case without the spectral parameter, in terms of symmetries of extended Dynkin diagrams. The corresponding elliptic version was considered in [88], [89], and [45].

6.5. Symmetries of the phase space. The explicit form of the Lax operator (6.15) makes it possible to describe the action of the Bernstein–Schwarzman group on the dynamical variables. As explained above, this action is generated by the gauge subgroup $G_{\tilde{H}_0}$ in (5.20). In addition we consider the action of the modular group $\text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$. In order to calculate the modular transformations of the spin variables, we use the moment equation $\sum_{a} (S^a)_{\alpha}^\beta_{\alpha} = 0$ and the relations (C.42)–(C.44). Note that the Lax operators are not invariant with respect
to the modular group. Indeed, from (C.42)–(C.44) we have

\[
\varphi^k_a \left( \frac{u}{ct + d}, \frac{z}{ct + d}, \frac{a\tau + b}{ct + d} \right) = \exp(2\pi i z(\varrho, \alpha)) \phi \left( \langle u, \alpha \rangle + \tau \langle \varrho, \alpha \rangle + \frac{k}{l}, z \right) \\
= \exp \left( 2\pi i z \left( \frac{c(u, \alpha)}{ct + d} + \langle \varrho, \alpha \rangle a + \frac{k}{l} c \right) \right) \phi \left( \langle u, \alpha \rangle + \left( \langle \varrho, \alpha \rangle, \frac{k}{l} \right) \left( \frac{a}{c} \frac{b}{d} \right) \left( \frac{\tau}{1}, z \right) \right) \\
= \exp \left( 2\pi i z \frac{c(u, \alpha)}{ct + d} \right) \varphi^{k'}_{\alpha'}(u, z | \tau),
\]

where the last equality should be considered as the definition of \( k' \) and \( \alpha' \), that is,

\[
\langle u, \alpha' \rangle + \tau \langle \varrho, \alpha' \rangle + \frac{k'}{l} = \langle u, \alpha \rangle + \left( \langle \varrho, \alpha \rangle, \frac{k}{l} \right) \left( \frac{a}{c} \frac{b}{d} \right) \left( \frac{\tau}{1} \right).
\]

However, the Lax operators become invariant with respect to the modular group if the latter acts together with a certain gauge transformation. The results presented in Table 1 are given with such a transformation taken into account. First, we can gauge away the factor \( \exp \left( 2\pi i z \frac{c(u, \alpha)}{ct + d} \right) \) by some transformation \( h = h(u, z) \) in the Cartan subgroup. Second, we can act by some \( g \) such that \( \text{Ad}_g t^k_{\alpha} = t^{k'}_{\alpha'} \). Finally, we have

\[
\kappa\partial_z + L \left( S, v, \frac{u}{ct + d}, \frac{z}{ct + d}, \frac{a\tau + b}{ct + d} \right) = \text{Ad}_{gh} \left( \kappa\partial_z + L(S', v', u', z | \tau) \right),
\]

where the primed variables are given in the last (right-hand) column of Table 1.

| Table 1. Symmetry properties of the phase space variables |
|-----------------------------------------------|
| \( \dot{W}_0 = \{ s \} \) | \( \dot{R}^\gamma \oplus \tau R^\gamma \) | \( \text{SL}(2, \mathbb{Z}) \) |
|-----------------|-----------------|-----------------|
| \( v \) | \( sv \) | \( v + \kappa \gamma \) |
| \( u \) | \( su \) | \( u + \gamma_1 + \gamma_2 \tau \) |
| \( (S^a)^{b,-k}_{-\alpha} \) | \( (S^a)^{b,-k}_{-\alpha} \) | \( (ct + d)(S^a)^{b,-k'}_{-\alpha} \) |
| \( (S^a)^{b,0}_{\alpha} \) | \( (S^a)^{b,0}_{\alpha} \) | \( (ct + d)(S^a)^{b,0}_{\alpha} \) |
| \( (S^a)^{l,k}_{-\alpha} \) | \( (S^a)^{l,k}_{-\alpha} \) | \( \chi(\gamma, \alpha)(x_a)(S^a)^{l,k}_{-\alpha} \) |
| \( \tau \) | \( \tau \) | \( \tau \) |
| \( x_a \) | \( x_a \) | \( x_a(\tau | ct + d)^{-1} \) |

It follows from the last column that we can consider the independent variables taking values in the moduli space

\[
\mathcal{M}_{1,n} = \mathcal{F}_{1,n} / \text{SL}(2, \mathbb{Z})
\]
of elliptic curves with marked points. The Poincaré–Cartan bundle (4.16) becomes non-trivial over \( \mathcal{M}_{1,n} \):

\[
\begin{array}{c}
P(G) \\
\downarrow \mathcal{B}u_{\zeta}(\Sigma, x, G) \\
\mathcal{M}_{1,n}
\end{array}
\] (6.11)

6.6. Hamiltonians. Let us consider the structure of the extended phase space \( \mathcal{P} \) in (4.16) for bundles over elliptic curves. We replace the Teichmüller space \( H^+ = \{ \tau \mid \text{Im} \tau > 0 \} \) by its quotient space \( H^+/\text{SL}(2, \mathbb{Z}) \). Let

\[ \mathcal{M}_n = \{ (\tau, x = (x_1, \ldots, x_n)) \mid \tau \in H^+/\text{SL}(2, \mathbb{Z}), x_a \neq x_b \}. \]

Then the space \( \mathcal{P} \) has the structure

\[
\begin{array}{c}
P(G) \\
\downarrow \mathcal{R}_{\zeta}(\Sigma, x, G) \\
\mathcal{M}_n
\end{array}
\] (6.28)

The local coordinate on the fiber \( \mathcal{R}_{\zeta}(\Sigma, x, G) \) is \( (v, u, \vec{S}) \) with the symplectic form \( \omega^\mathcal{P} \) in (5.31).

We calculate the Hamiltonians (4.24) using the form of the Beltrami differential (6.4), (6.5). In the coordinates \( (w, e^w) \) the density of the measure on \( \Sigma_{\tau} \) is

\[ d\sigma = \frac{dz \, d\bar{z}}{\rho_0} = \frac{dw \, d\bar{w}}{\rho}, \quad \rho_0 = \tau_0 - e^{-\tau_0} = \tau - \tau_0. \]

It follows from the structure of \( L \) that under the condition (6.24) \( \text{tr}(L^2(w)) \) is a periodic function on \( \Sigma_{\tau} \) with second-order poles at the marked points (6.9). Therefore, the Hamiltonians can be computed from the decomposition

\[
\frac{1}{2} \text{tr}(L^2(w)) = H_\tau + \sum_{a=1}^{n}(H_a E_1(w - x_a) + C_2^a E_2(w - x_a))
\] (6.29)

or in terms of the integrals

\[ H_\tau = \frac{1}{2} \int_{\Sigma_{\tau}} \text{tr}(L^2) \mu_0^0, \quad H_a = \frac{1}{2} \int_{\Sigma_{\tau}} \text{tr}(L^2) \mu_a^0. \]

Then the quadratic Casimir functions (corresponding to the orbits at the points \( x_a \)) are

\[
C_2^a = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 (S^a)^{\alpha,k}_{\alpha} (S^a)^{l-k}_{-\alpha} + \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} (S^a)^{\alpha,h,k}_{\alpha} (S^a)^{h,k}_{-\alpha}, \quad a = 1, \ldots, n.
\] (6.30)

The ‘Gaudin’ (or ‘Schlesinger’) Hamiltonians have the form

\[ H_a = l \sum_{\alpha \in \Pi} v_\alpha^h (S^a)^{h,0}_{\alpha} + \sum_{c \neq a}^{n} \left( \frac{1}{2} \sum_{k=0}^{l-1} |\alpha|^2 \varphi_\alpha^k (u, z_a - z_c) (S^a)^{l,k}_{\alpha} (S^c)^{l,k}_{-\alpha} \right) \]

The classification of isomonodromy problems on elliptic curves.
\[ + \sum_{\alpha \in \Pi} \varphi^k_0(u, z_a - z_c)(S^a)^{h, k}_\alpha (S^c)^{h, -k}_\alpha. \tag{6.31} \]

Finally, the zero Hamiltonian corresponding to particle interaction is

\[ H_\tau = \frac{l}{2} \sum_{\alpha \in \Pi} \sum_{s=0}^{l-1} h^h_{\alpha v_s} + n \sum_{b \neq d} \sum_{k=0}^{l-1} \left( \frac{1}{2} \sum_{\alpha \in R} |\alpha|^2 f^{k}_{\alpha}(u, z_b - z_d)(S^b)^{l, k}_\alpha (S^d)^{l, -k}_\alpha \right. \]
\[ + \sum_{\alpha \in \Pi} f^{k}_{0}(u, z_b - z_d)(S^b)^{h, k}_\alpha (S^d)^{h, -k}_\alpha \tag{6.32} \]

The functions \( f^{k}_{\alpha}(u, z) \) are defined in (C.26)–(C.28).

6.7. M-operators. The \( M_s \)-operators can be calculated from the Lax equations (6.13) since the \( L \)-operator is already known. However, following [90], we construct the \( M \)-operators from the \( r \)-matrix structure (6.22) with the \( r \)-matrix (6.20)–(6.21).

First we take the autonomous case (4.29). We define the Lax pairs for the integrable flows described by the Hamiltonians (6.31), (6.32). Let us compute the (off-shell) brackets

\[ \frac{1}{2} \text{tr}_2 \{ 1 \otimes L^2(z), L(z) \otimes 1 \} = [L(z), M_w(z)] + \Delta(z, w), \tag{6.33} \]

where \( \text{tr}_2 \) denotes the trace in the second component of the tensor product

\[ M_w(z) = - \text{tr}_2 (r(z, w)L_2(w)), \tag{6.34} \]

and \( \Delta(z, w) \) comes from the last term in (6.22):

\[ \Delta(z, w) = \text{tr}_2 \left( \frac{\sqrt{l}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_u \varphi^k_\alpha(u, z - w) \overline{S^h_{\alpha t_k}} \otimes t^-_{-\alpha} L_2(z) \right). \tag{6.35} \]

This term vanishes on the shell of the first class constraint (6.24). Therefore, on (6.24) we have

\[ \frac{1}{2} \text{tr}_2 \{ 1 \otimes L^2(z), L(z) \otimes 1 \} \big|_{\text{on the shell \,(6.24)}} = [L(z), M_w(z)]. \tag{6.36} \]

Taking (6.29) into account, we conclude that the \( M \)-operators of the Hamiltonians (6.31) and (6.32) can be computed by expansion of \( M_w(w) \) in (6.34) as in the case of (6.29). In this way we obtain a set of Lax equations

\[ \partial_{t_a} L(z) = [L(z), M_a(z)], \quad a = 1, \ldots, n, \tag{6.37} \]

and

\[ \partial_{t_\tau} L(z) = [L(z), M_\tau(z)] \tag{6.38} \]

on the constraint surface (6.24). These \( M \)-operators have the form

\[ M_a(z) = (M^a)_R(z) + (M^a)_b(z) + (M^a)_b^0(z) \tag{6.39} \]
with
\[
(M^a)_R(z) = -\frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \phi_\beta^k(u, z - x_a)(S^a)_\beta^{-k} t_\beta^k, \\
(M^a)_b(z) = -\sum_{k=1}^{l-1} \sum_{\alpha \in \Pi} \phi_\alpha \left( \frac{k}{2}, z - x_a \right) (S^a)_\alpha^{-k} h_\alpha^k, \\
(M^a)_b^0(z) = -\sum_{\alpha \in \Pi} E_1(z - x_a)(S^a)_\alpha^0 h_\alpha^0,
\]
and
\[
M^\tau(z) = (M^\tau)_R(z) + (M^\tau)_b(z) + (M^\tau)_b^0(z)
\]
with
\[
(M^\tau)_R(z) = \frac{1}{2} \sum_{a=1}^{n} \sum_{k=0}^{l-1} \sum_{\beta \in R} |\beta|^2 f_\beta^k(u, z - x_a)(S^a)_\beta^{-k} t_\beta^k, \\
(M^\tau)_b(z) = \sum_{a=1}^{n} \sum_{k=1}^{l-1} \sum_{\alpha \in \Pi} f_\alpha \left( \frac{k}{2}, z - x_a \right) (S^a)_\alpha^{-k} h_\alpha^k, \\
(M^\tau)_b^0(z) = \frac{1}{2} \sum_{a=1}^{n} \sum_{\alpha \in \Pi} (E_1^2(z - x_a) - \varphi(z - x_a))(S^a)_\alpha^0 h_\alpha^0.
\]

As noted, the constraints (6.24) must be supplemented by conditions for fixation of a gauge (of the action of the Cartan subgroup \( H_0 \) of the invariant subgroup). The full reduction \( \mathbf{P}^{\text{red}} = \mathbf{P} / \tilde{H}_0 \) (generally speaking) changes the \( M \)-operators and \( r \)-matrices (see the example of the spinless Calogero model in [91]). We do not consider full reductions here because there is not any ‘good’ choice of the gauge fixation.

6.8. Painlevé–Schlesinger equations. At the beginning of this section we mentioned that the monodromy-preserving equations arise in our approach as non-autonomous generalizations of Hitchin systems [30]. On the level of the Lax equation or zero curvature equation this means that the Lax pairs of matrices satisfying (6.37) or (6.38) satisfy also the monodromy-preserving equations
\[
\partial_{x_a} L(z) - \partial_z M_a(z) = [L(z), M_a(z)], \quad a = 1, \ldots, n, \quad (6.43)
\]
or
\[
\partial_\tau L(z) - \partial_z M_\tau = [L(z), M_\tau(z)], \quad (6.44)
\]
respectively. This phenomenon is known as the Painlevé–Calogero Correspondence [44] (see also [92] for the developed version of this idea, and [93]–[96] for the quantum version).

Note that the time variables \( t_a \) in (6.37) are replaced by the coordinates of the marked points \( x_a \) in (6.43), while the matrices \( M_a(z) \) are the same in both sets of equations. Similarly, the time variable \( t_\tau \) in (6.38) is replaced by the modulus \( \tau \) in (6.44), while \( M_\tau \) is the same for both cases. Technically, the Painlevé–Calogero Correspondence follows from
\[
d_{x_a} L(z) = \partial_z M_a(z), \quad (6.45)
\]
where $d_{x_a}$ is the derivative only with respect to the explicit dependence on $x_a$, and
\[ d_\tau L(z) = \partial_z M_\tau, \tag{6.46} \]
where $d_\tau$ is the derivative only with respect to the explicit dependence on $\tau$. Equations (6.45) follow trivially from the type of dependence $L(z, x_a) = L(z - x_a)$, $M_\tau(z, x_a) = M_\tau(z - x_a)$ while (6.46) follows from the heat equation (C.29).

We thus get the Painlevé–Schlesinger equations (6.43) and (6.44). We also remark that the $M$-operator can be derived in terms of the modification $\Xi(z)$ from the assumption that the Painlevé–Calogero Correspondence holds [97].

6.9. KZB equations.

6.9.1. KZB equations on elliptic curves. The Knizhnik–Zamolodchikov–Bernard (KZB) equations arise as a condition for horizontality of certain sections (conformal blocks) for the so-called KZB connection. The KZB equations (see (6.52) below) can be regarded as the quantization of the monodromy-preserving equations. The KZB connection can be constructed on an elliptic curve by following [98]. For an arbitrary characteristic class it is given by the following differential operators [48]:

\[ \nabla_a = \partial_{z_a} + \hat{\partial}^a + \sum_{c \neq a} r^{ac}, \tag{6.47} \]
\[ \nabla_\tau = 2\pi i \partial_\tau + \Delta + \frac{1}{2} \sum_{a, c} f^{ac}, \tag{6.48} \]

with

\[ r^{ac} = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi^{k}_{\alpha}(u, z_a - z_c) t^{k,a}_{\alpha} t^{-k,c}_{\alpha} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \varphi^{k}_{0}(u, z_a - z_c) \delta^{k,a}_{\alpha} \delta^{k,c}_{\alpha}, \tag{6.49} \]
\[ f^{ac} = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 f^{k}_{\alpha}(u, z_a - z_c) t^{k,a}_{\alpha} t^{-k,c}_{\alpha} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} f_{0}(u, z_a - z_c) \delta^{k,a}_{\alpha} \delta^{k,c}_{\alpha}, \tag{6.50} \]

where $t^{k,a}_{\alpha} = 1 \otimes \cdots 1 \otimes t^k_{\alpha} \otimes 1 \cdots 1$ (with $t^k_{\alpha}$ at the $a$th position), and similarly for the generators $\delta^{k,a}_{\alpha}$ and $\delta^{k,c}_{\alpha}$. Here we use the notation

\[ \hat{\partial}^a = l \sum_{\alpha \in \Pi} \delta^{0,a}_{\alpha} \partial_{\alpha}; \quad \Delta = \frac{l}{2} \sum_{\alpha \in \Pi} \sum_{s=0}^{l-1} \partial_{u_{\alpha}} \partial_{u_{\lambda s\alpha}}. \]

From the definition it follows that $r^{ac} = -r^{ca}$ and $f^{ac} = f^{ca}$. We note that

\[ f^{cc} = -\sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi^{k}_{\alpha} t^{k,c}_{\alpha} t^{-k,c}_{\alpha} - \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \delta^{k,c}_{\alpha} \delta^{-k,c}_{\alpha} - 2l\eta_1 C_2, \tag{6.51} \]

\[ \text{For brevity we write } t^{k,a}_{\alpha} \text{ and } \delta^{k,a}_{\alpha} \text{ instead of representations of these generators in the spaces } V_{\mu_{\alpha}}. \]
where \( C^c \) is the Casimir operator acting on the \( c \)-th component. The KZB equations have the form

\[
\begin{cases}
\nabla_a F = 0, & a = 1, \ldots, n, \\
\nabla_{\tau} F = 0.
\end{cases}
\] (6.52)

There are two types of the compatibility conditions for these equations:

\[
\begin{cases}
[\nabla_a, \nabla_b] F = 0, & a, b = 1, \ldots, n, \\
[\nabla_a, \nabla_{\tau}] F = 0, & a = 1, \ldots, n.
\end{cases}
\] (6.53)

It is important to point out that the solutions \( F \) of (6.52) are assumed to satisfy the condition (cf. (6.24))

\[
\sum_{c=1}^{n} h^0_{\alpha,c} F = 0 \quad \text{for each } \alpha \in \tilde{\Pi}.
\] (6.54)

**Proposition 6.3.** The upper equations \([\nabla_a, \nabla_b] = 0\) in (6.53) are valid for the \( r \)-matrix (6.20) on the space of solutions of (6.52) satisfying (6.54). They follow from the classical Yang–Baxter dynamical equations:

\[
[r^{ab}, r^{ac}] + [r^{ab}, r^{bc}] + [\hat{\nabla}^{a}, r^{bc}] + [\hat{\nabla}^{c}, r^{ab}] + [\hat{\nabla}^{b}, r^{ca}] = 0.
\] (6.55)

**Proposition 6.4.** The lower equations \([\nabla_a, \nabla_{\tau}] = 0\) in (6.53) are also valid for the \( r \)-matrix (C.23) on the space of solutions of (6.52) satisfying (6.54).

The proofs of these statements are given in [48].

### 6.9.2. The KZB equations and the isomonodromy problem

The KZB equations (6.47), (6.48) given here for elliptic curves can also be written for curves of arbitrary genus [99]–[101]. Their connections with the isomonodromy problem on a sphere were investigated in [102] and [103]. In the general case, as in the genus-1 case, the KZB equations have the form of the non-stationary Schrödinger equation

\[
(\kappa \partial_s + \hat{H}_s) \Psi = 0
\] (6.56)

with quantum quadratic Hitchin Hamiltonians \( \hat{H}_s \). To pass to the classical limit we replace the wave function (the conformal block) by its quasi-classical expression

\[
\Psi = \exp \left( \frac{iS}{\kappa} \right),
\]

where \( S \) is the classical action (\( S = \log \tau \) (4.19)). The classical limit \( \kappa \to 0 \) leads to the Hamilton–Jacobi equations for \( S \) [104]. They are equivalent to the equations of motion (4.26a). On the other hand, putting \( \kappa = 0 \) in (6.56) and staying with the quantum Hamiltonians, we get the Schrödinger equation \( \hat{H}_s \Psi = 0 \). On the classical level the limit \( \kappa \to 0 \) recalls the passage from isomonodromy flows to isospectral
flows, as described in §4.5:

\[
\begin{align*}
\text{KZB eqs., } (\kappa, \Sigma_{g,n}, G) & \quad \xrightarrow{\kappa \to 0} \quad \text{KZB eqs. on the critical level, } (\Sigma_{g,n}, G) \\
(\kappa \partial_s + \hat{H}_s)\Psi = 0 & \quad \text{on } (s = 1, \ldots, \dim(\Sigma_{g,n})) \\
(\kappa \partial_s + \hat{H}_s)\Psi = 0 & \quad \text{on } (s = 1, \ldots, \dim(\Sigma_{g,n}))
\end{align*}
\]

\[
\begin{align*}
\text{Isomonodromy deformations on } \Sigma_{g,n} & \quad \xrightarrow{\kappa \to 0} \quad \text{Hitchin systems on } \Sigma_{g,n} \\
\end{align*}
\]

6.10. Painlevé field theory. We recall once again that in previous sections we obtained the monodromy-preserving equations as non-autonomous versions of integrable systems in mechanics. More precisely, in the elliptic case we replaced the linear problem and its compatibility condition (known as the Lax equations)

\[
\begin{align*}
L(z)\Psi &= \Lambda\Psi, \\
(\partial_t + M(z))\Psi &= 0,
\end{align*}
\]

by another linear problem and the corresponding compatibility condition (that is, the monodromy-preserving equations)

\[
\begin{align*}
(\partial_z + L(z))\Psi &= 0, \\
(\partial_t + M(z))\Psi &= 0,
\end{align*}
\]

\[
\partial_t L(z) - \partial_z M(z) = [L(z), M(z)].
\] (6.58)

At the same time it is well known that integrable systems in mechanics can be ‘generalized’ to (1+1) integrable field theories described by the following linear problem and compatibility condition (Zakharov–Shabat equations)

\[
\begin{align*}
(\partial_z + \bar{L}(z))\Psi &= 0, \\
(\partial_t + \bar{M}(z))\Psi &= 0,
\end{align*}
\]

\[
\partial_t \bar{L}(z) - \partial_z \bar{M}(z) = [\bar{L}(z), \bar{M}(z)].
\] (6.59)

where the phase space consists of $\mathbb{S}^1$-valued fields, and $x$ is the coordinate on the unit circle $\mathbb{S}^1$. For Hitchin systems the transition $(6.57) \rightarrow (6.59)$ was described in [61] (see also [105], [106]).

In [107] we proposed a field-theoretic generalization of the monodromy-preserving equations. It arises from the linear problem

\[
\begin{align*}
(\partial_z + \partial_x + L'(z))\Psi &= 0, \\
(\partial_t + M'(z))\Psi &= 0
\end{align*}
\]

with the compatibility condition

\[
\partial_t L'(z) - \partial_x M'(z) - \partial_z M'(z) = [L'(z), M'(z)].
\] (6.61)

For example, the field generalization of the Painlevé VI equation has the following form. Consider the four three-dimensional vector fields $S_\alpha^b(x)$, $\alpha \in (\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus (0,0)$,
b ∈ ℤ_2 × ℤ_2, and let \( J^I(β, ∂x, τ) \) and \( J^{II}(β, b, c, ∂x, τ) \) be the pseudodifferential operators

\[
J^I(α, b, c, ∂x, τ) = E_2\left(ω_α - \frac{κ}{2πi}∂x, τ\right), \quad J^{II}(α, b, c, ∂x, τ) = f_α\left(\frac{κ}{2πi}∂x, ω_c - ω_b\right).
\]

Then the field equation has the form of four interacting non-autonomous Euler–Arnold tops \cite{108}, \cite{109} corresponding to the loop group \( L(\text{SL}(2, \mathbb{C})) \),

\[
\frac{∂}{∂τ} S^α_b(x) = \sum_{β \neq α} \left( S^{α−β}_b(x) J^I(α, β, ∂x, τ) S^β_b(x) \right)
+ \sum_{c \neq b} \left( S^{α−β}_b(x) J^{II}(α, β, b, c, ∂x, τ) S^β_c(x) \right),
\]

and subject to the constraints \( S^α_b(x) = (-1)^{b×α} S^α_b(−x) \). The equation (6.63) itself can be regarded as a field generalization of the elliptic Schlesinger system \cite{42}, and in the particular case of an sgl-connection and four marked points at half-periods of the elliptic curve and zero, and also on the constraint equations, it generalizes the Painlevé VI equation. Namely, it can be shown that the dynamics of the zero modes of (6.63) on the constraints is described by the equation for the non-autonomous Zhukovsky–Volterra gyrostat (8.6) for the three variables \( S^α \): the zero modes of \( S^α \). The latter is one of the possible forms of the Painlevé VI equation, as we will see below (in §8).

### 7. Symplectic Hecke Correspondence

We consider the Hecke transformation in the case \( g = 1, n = 1 \). The phase space has the dimension \( 2 \sum_{j=1}^{\text{rank } G} (d_j - 1) \) of a coadjoint orbit. In this case \( L \) satisfies the conditions

\[
L(z + 1) = \mathcal{D}L(z) \mathcal{D}^{-1}, \quad L(z + τ) = ΛL(z) Λ^{-1},
\]

where \( \mathcal{D} \) and \( Λ \) are solution of (5.4) and

\[
\overline{∂}L(z) = S∂(z, τ).
\]

In other words, \( \text{Res}_{z=0} L(z) = S \). These conditions fix \( L \). To underscore the dependence on the characteristic class \( ζ(E_G) \), we will write \( L^\varphi \) for the Lax matrix constructed from the quasi-periodicity conditions with \( Λ = Λ^\varphi \) and \( \mathcal{D}^\varphi \) which are solutions of (5.4) with \( ζ = e(−\varphi) \), \( \varphi ∈ P^\varphi \).

The modification \( Ξ(γ) \) of \( E_G \) changes the characteristic class. It acts on \( L^\varphi \) as follows:

\[
L^\varphi Ξ(γ) = Ξ(γ)L^\varphi + γ.
\]

It is the singular gauge transformation mentioned in the Introduction. The action (7.3) lets us write a condition on \( Ξ(γ) \). Since \( L^\varphi \) has a simple pole at \( z = 0 \), the modified Lax matrix \( L^\varphi + γ \) also has a simple pole at \( z = 0 \). Let us decompose \( L^\varphi \) and \( L^\varphi + γ \) in the Chevalley basis (A.15), (B.2):

\[
L^\varphi = L_β(z) + \sum_{α∈R} L_α(z) E_α, \quad L^\varphi + γ = \tilde{L}_β(z) + \sum_{α∈R} \tilde{L}_α(z) E_α.
\]
We expand $\alpha$ in the basis of simple roots (A.2): $\alpha = \sum_{j=1}^{l} f_{j}^{\alpha} \alpha_{j}$ and $\gamma$ in the basis of fundamental coweights: $\gamma = \sum_{j=1}^{l} m_{j} \varpi_{j}^{\gamma}$. We assume that $\langle \gamma, \alpha_{j} \rangle \geq 0$ for simple $\alpha_{j}$. In other words, $\gamma$ is a dominant coweight. Then $\langle \gamma, \alpha \rangle = \sum_{j=1}^{l} m_{j} n_{j}^{\alpha}$ is an integer, positive for $\alpha \in R^{+}$ and negative for $\alpha \in R^{-}$. From (7.3) we find that

$$L_{b}^{\varpi_{j}^{\gamma}+\gamma}(z) = L_{b}^{\varpi_{j}^{\gamma}}(z), \quad L_{\alpha}^{\varpi_{j}^{\gamma}+\gamma}(z) = z^{\langle \gamma, \alpha \rangle} L_{\alpha}^{\varpi_{j}^{\gamma}}(z). \quad (7.4)$$

In a neighbourhood of $z = 0$ the operator $L_{\alpha}(z)$ has the form

$$L_{\alpha}^{\varpi_{j}^{\gamma}}(z) = a_{\langle \gamma, \alpha \rangle} z^{-\langle \gamma, \alpha \rangle} + a_{\langle \gamma, \alpha \rangle+1} z^{-\langle \gamma, \alpha \rangle+1} + \cdots \quad (\alpha \in R^{-}), \quad (7.5)$$

otherwise the transformed Lax operator becomes singular. This means that the type of the modification $\gamma$ is not arbitrary but depends on the local behavior of the Lax operator, which enables us (in principle) to find the dimension of the space of Hecke transformations.

Now consider the global behaviour of $L(z)$ (7.1). Then we find that $\Xi(\gamma)$ intertwines the quasi-periodicity conditions:

$$\Xi(\gamma, z + 1) Q_{\varpi_{j}^{\gamma}} = Q_{\varpi_{j}^{\gamma}+\gamma} \Xi(\gamma, z), \quad \Xi(\gamma, z + \tau)L_{\varpi_{j}^{\gamma}} = L_{\varpi_{j}^{\gamma}+\gamma} \Xi(\gamma, z).$$

For $G = \text{SL}(N, \mathbb{C})$, $\gamma = \varpi_{1}^{\gamma}$, and special residues of $L(z)$ the solutions of these equations were found in [61]. On the reduced space $\mathbf{P}^{\text{red}}$ the equations of motion corresponding to the Hamiltonians $H_s$ have the Lax form

$$\kappa \partial_{t_s} L - \kappa \partial_{z} M_{t_s} + [M_{t_s}, L] = 0.$$

The operators $M_{t_s}$ are constructed from the $L$- and $r$-matrices.

**Example.** Let us demonstrate the modification procedure with the example of the group $\text{SL}(N, \mathbb{C})$. The characteristic classes of the underlying bundles are elements of $\mathbb{Z}_{N}$. The values $+1$ and $\exp \frac{2\pi i}{N}$ correspond to the most different bundles and dynamical systems. These are the elliptic Calogero model and the elliptic top model [61]. In the isomonodromic case both models become non-autonomous [42], [43], [110]. However, the Lax pairs are the same as in autonomous integrable mechanics because of the Painlevé–Calogero Correspondence. The Lax matrices of the models are fixed by their residues at the single marked point $z = 0$ on the elliptic curve $\Sigma_{\tau}$:

$$\text{Res}_{z=0} L^\text{Cal}(z) = \nu \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix},$$

$$\text{Res}_{z=0} L^\text{top}(z) = S = \sum_{m,n} T_{mn} S_{mn},$$

$$\text{Spec}(S) = \text{diag}(\nu, \ldots, \nu, -(N-1)\nu),$$

and by the boundary conditions

$$L^\text{Cal}(z + 1) = L^\text{Cal}(z), \quad L^\text{top}(z + 1) = Q L^\text{top}(z) Q^{-1},$$

$$L^\text{Cal}(z + \tau) = e(-u) L^\text{Cal}(z) e(u), \quad L^\text{top}(z + \tau) = \Lambda L^\text{top}(z) \Lambda^{-1}, \quad (7.7)$$
where

\[ u_{ij} = \delta_{ij} u_i, \quad \Lambda_{ij} = \delta^{(\text{mod } N(i+1), j)}, \quad Q_{ij} = \delta_{ij} e_{\frac{i}{N}}, \]  

and

\[
\left\{ T_{mn} = e\left(\frac{mn}{2N}\right) Q^m A^n \right\}, \quad m, n \in \mathbb{Z}_N,
\]

is the sine algebra basis. The Lax matrices are

\[
L^{\text{Cal}}(z) = \sum_{i,j=1}^{N} E_{ij} (\delta_{ij} v_i + \nu(1 - \delta_{ij}) \Phi(z, u_i - u_j)), \quad (E_{ij})_{ab} = \delta_{ia} \delta_{jb},
\]

\[
L^{\text{top}}(z) = \sum_{n^2 + m^2 \neq 0} S_{mn} T_{mn} \varphi_{mn}(z),
\]

and they generate the Hamiltonian functions

\[
H^{\text{Cal}} = \frac{1}{2} \sum_{k=1}^{N} v_k^2 - \sum_{i<j} \nu^2 \varphi(u_i - u_j),
\]

\[
H^{\text{top}} = \frac{1}{2} \sum_{n^2 + m^2 \neq 0} S_{mn} S_{-m, -n} \varphi\left(\frac{m + n\tau}{N}\right).
\]

The modification is given by the matrix [61]

\[
\Xi(z, U, \tau) = \theta \left[ \frac{i}{N} - \frac{1}{2} \right] (z - Nu_i, N\tau) D_i, \quad D_i = (-1)^l \prod_{j<k; j,k \neq l} \varphi^{-1}(u_k - u_j, \tau).
\]

It acts as a gauge transformation and therefore establishes the following relation between the models:

\[
L^{\text{top}}(z) = \Xi(z) L^{\text{Cal}}(z) \Xi^{-1}(z) - \kappa \partial_z \Xi(z) \Xi^{-1}(z) + \frac{\kappa}{N} E_1(z).
\]

The last term can be incorporated in the definition of \(L^{\text{top}}\) or \(L^{\text{Cal}}\), since it is proportional to the identity matrix.

8. The Painlevé VI equation

8.1. Three forms of the Painlevé VI equation. In this section we describe the relations between three forms of the Painlevé VI equation and the corresponding linear problems. These three forms are as follows.

1. The rational form is the original one, found by Gambier [3], [1], [14], [15]. It proved to be the last and the most general in the list of second-order ordinary differential equations having the Painlevé property:

\[
\frac{d^2 X}{dT^2} = \frac{1}{2} \left( \frac{1}{X^2} + \frac{1}{X - 1} + \frac{1}{X - T} \right) \left( \frac{dX}{dT} \right)^2 - \left( \frac{1}{T} + \frac{1}{T - 1} + \frac{1}{X - T} \right) \frac{dX}{dT}
\]
It contains four free complex constants $\alpha$, $\beta$, $\gamma$, $\delta$.

2. The elliptic form was found by Painlevé [62], [63]:

$$\frac{d^2u}{d\tau^2} = \sum_{a=0}^{3} \nu_a^2 \phi'(u + \omega_a), \quad \{\omega_a, k = 0, \ldots, 3\} = \left\{0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\right\}. \quad (8.2)$$

The equation is defined on an elliptic curve $\Sigma_\tau$ with half-periods $\omega_a$ and modulus $\tau$, which is the time variable here. The relation between (8.1) and (8.2) is given by the change of variables $(u, \tau) \rightarrow (X(u, \tau) = \frac{\phi(u) - e_1}{e_2 - e_1}, T(\tau) = \frac{e_3 - e_1}{e_2 - e_1}), \quad e_\alpha \equiv \phi(\omega_\alpha), (8.3)$

In such a form the equation is obviously Hamiltonian. The mechanics model is described by the Hamiltonian function

$$H^{\text{PVI}} = \frac{1}{2} v^2 - \sum_{a=0}^{3} \nu_a^2 \phi(u + \omega_a) \quad (8.4)$$

and the canonical Poisson brackets

$$\{v, u\} = 1. \quad (8.5)$$

The mechanics is non-autonomous, since two of the half-periods and the Weierstrass $\phi$-function depend on $\tau$.

3. The non-autonomous version of the Zhukovsky–Volterra gyrostat was introduced and proved to be equivalent to the Painlevé VI equation in [64]:

$$\partial_\tau S = [S, J(S)] + [S, \nu'], \quad (8.6)$$

where $S$ is an $\mathfrak{sl}_2^n$-valued dynamical variable. In the basis of Pauli matrices

$$S = \sum_{\alpha=1}^{3} S_\alpha \sigma_\alpha, \quad J(S) = \sum_{\alpha=1}^{3} J_\alpha(\tau) S_\alpha \sigma_\alpha, \quad J_\alpha(\tau) = \phi(\omega_\alpha) = e_\alpha, \quad \alpha = 1, 2, 3, (8.7)$$

$$\nu' = \sum_{a=1}^{3} \nu'_a \sigma_\alpha, \quad \nu'_a = -\tilde{\nu}_a \exp(-2\pi i \omega_a \partial_\tau \omega_a) \left(\frac{\phi'(0)}{\phi(\omega_a)}\right)^2, \quad a = 0, 1, 2, 3, \quad \text{where the } \tilde{\nu}_a \text{ are four free complex constants.}$$

The autonomous version of (8.6) (when the time variable is not related to the modulus $\tau$) is known as the Zhukovsky–Volterra gyrostat [111], [112]. The vector $(\nu'_1, \nu'_2, \nu'_3)$ plays the role of the gyrostatic
momentum, while the \( J_\alpha(\tau) \) are the inverse components of the inertia tensor in the principal axes of inertia.

The equation (8.6) is a Hamiltonian equation with Hamiltonian function

\[
H^{ZVG} = \frac{1}{2} \sum_{\alpha=1}^{3} (J_\alpha S^2_\alpha + S_\alpha \nu'_\alpha) \tag{8.8}
\]

and the Poisson–Lie brackets on \( \mathfrak{sl}_2^* \) are

\[
\{S_\alpha, S_\beta\} = \varepsilon_{\alpha\beta\gamma} S_\gamma. \tag{8.9}
\]

The relation between (8.6) and (8.2) is given by the change of variables

\[
S_1 = -v \frac{\theta_2(0) \theta_2(2u)}{\theta'(0) \theta(2u)} - \frac{\kappa}{2} \frac{\theta_2(0) \theta_2'(2u)}{\theta'(0) \theta(2u)} + \tilde{\nu}_0 \frac{\theta_2^2(0)}{\theta'(0) \theta_4(0)} \frac{\theta_3(2u) \theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1 \frac{\theta_2^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2 \frac{\theta_2(0) \theta_2(2u) \theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3 \frac{\theta_2(0) \theta_2(2u) \theta_3(2u)}{\vartheta^2(2u)}, \tag{8.10}
\]

\[
iS_2 = \frac{\theta_3(0) \theta_3(2u)}{\theta'(0) \theta(2u)} + \frac{\kappa}{2} \frac{\theta_3(0) \theta_3'(2u)}{\theta'(0) \theta(2u)} - \tilde{\nu}_0 \frac{\theta_3^2(0)}{\theta_2(0) \theta_4(0)} \frac{\theta_2(2u) \theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_1 \frac{\theta_3(0) \theta_3(2u) \theta_2(2u)}{\vartheta^2(2u)} - \tilde{\nu}_2 \frac{\theta_3(0) \theta_3(2u) \theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_3 \frac{\theta_2^2(2u)}{\vartheta^2(2u)}\]

\[
S_3 = -v \frac{\theta_4(0) \theta_4(2u)}{\theta'(0) \theta(2u)} - \frac{\kappa}{2} \frac{\theta_4(0) \theta_4'(2u)}{\theta'(0) \theta(2u)} + \tilde{\nu}_0 \frac{\theta_4^2(0)}{\theta_3(0) \theta_3(0)} \frac{\theta_2(2u) \theta_3(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1 \frac{\theta_4(0) \theta_2(2u) \theta_3(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2 \frac{\theta_3^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3 \frac{\theta_4(0) \theta_4(2u) \theta_3(2u)}{\vartheta^2(2u)}
\]

and the following relations for the constants:

\[
\tilde{\nu}_0 = \frac{1}{2}(\nu_0 + \nu_1 + \nu_2 + \nu_3),
\]

\[
\tilde{\nu}_1 = \frac{1}{2}(\nu_0 + \nu_1 - \nu_2 - \nu_3),
\]

\[
\tilde{\nu}_2 = \frac{1}{2}(\nu_0 - \nu_1 + \nu_2 - \nu_3),
\]

\[
\tilde{\nu}_3 = \frac{1}{2}(\nu_0 - \nu_1 - \nu_2 + \nu_3). \tag{8.11}
\]

Note that the three constants \( (\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3) \) appear in (8.6) explicitly, while the last constant \( \nu'_0 \) is connected with the value of the Casimir function of the brackets (8.9):

\[
\frac{1}{2} \sum_{\alpha=1}^{3} S^2_\alpha = \tilde{\nu}_0^2 = \nu'_0^2, \tag{8.12}
\]

that is, three linear combinations of the four Painlevé VI constants in (8.2) are combined to form the gyrostatic momentum vector and the last one is the length of the angular momentum vector.
8.2. Linear problems. We describe the linear problems for the above three forms of the Painlevé VI equation.

1. Rational form. The linear problem for the Painlevé equation in the rational form arises naturally from the $\mathfrak{sl}_2$-Schlesinger system [6], [9]–[11] on $\mathbb{CP}^1 \setminus \{0, 1, T, \infty\}$. It describes the behaviour of the connection

$$ \left( \partial_\zeta + A(\zeta) \right) d\zeta = \left( \partial_\zeta + \frac{A^0}{\zeta} + \frac{A^1}{\zeta - 1} + \frac{A^T}{\zeta - T} \right) d\zeta \quad (8.13) $$

with logarithmic singularities at $\{0, 1, T, \infty\}$ and $\mathfrak{sl}_2^*$-valued residues. The isomonodromy equation is the compatibility condition for the linear problem

$$ \begin{cases} \left( \partial_\zeta + A(\zeta) \right) \Psi = 0, \\ \left( \partial_T + M(\zeta) \right) \Psi = 0, \end{cases} \quad M(\zeta) = -\frac{A^T}{\zeta - T}, \quad (8.14) $$

and it has the form

$$ \partial_T A(\zeta) - \partial_\zeta M(\zeta) = [A(\zeta), M(\zeta)]. \quad (8.15) $$

To get the Painlevé VI equation one must perform the reduction

$$ \mathcal{O}_0 \times \mathcal{O}_1 \times \mathcal{O}_T \times \mathcal{O}_\infty \longrightarrow \mathcal{O}_0 \times \mathcal{O}_1 \times \mathcal{O}_T \times \mathcal{O}_\infty / / \text{SL}(2, \mathbb{C}), \quad (8.16) $$

where $\mathcal{O}^a$ is the coadjoint orbit, that is, the space of the coalgebra $A^a$ with some fixed eigenvalues which are free constants. The reduction (8.16) is generated by the global ($\zeta$-independent) coadjoint action

$$ A^c \rightarrow \text{Ad}_{\text{SL}(2, \mathbb{C})}^* A^c, \quad c = 0, 1, T, \infty. $$

This gives the moment map equation

$$ A^0 + A^1 + A^T + A^\infty = 0. \quad (8.17) $$

We must also choose some coordinates on the reduced space. For the case of $2 \times 2$ Lax pairs the recipe for the canonical variables is very well-known: let $A_{12}(X) = 0$ and $P = A_{11}(X)$, and then $\{P, X\} = 1$ (see, for instance, [113]). After fairly long calculations we get (8.1).

2. Elliptic form. The $2 \times 2$ linear problem was posed in [65]. It is formulated in terms of the connection $\left( \partial_z + L^{\text{PVI}}(z) \right) dz$ in the holomorphic bundle over the elliptic curve with transition functions $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g_\tau = \begin{pmatrix} e(u) & 0 \\ 0 & e(-u) \end{pmatrix}$:

$$ L^{\text{PVI}}(z) = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{c=0}^3 L^c_{\text{PVI}}, \quad (8.18) $$

$$ L^c_{\text{PVI}} = \tilde{\nu}_c \begin{pmatrix} 0 & \varphi_c(2u, z + \omega_c) \\ \varphi_c(-2u, z + \omega_c) & 0 \end{pmatrix}, $$

with the $\tilde{\nu}_c$ defined in (8.11). Together with the $M$-operator

$$ M^{\text{PVI}}(z) = \sum_{c=0}^3 M^c_{\text{PVI}}, \quad M^c_{\text{PVI}} = \tilde{\nu}_c \begin{pmatrix} 0 & f_c(2u, z + \omega_c) \\ f_c(-2u, z + \omega_c) & 0 \end{pmatrix}, \quad (8.19) $$
the Lax matrix (8.18) yields the Painlevé VI equation (8.2):

$$\partial_\tau L^{PVI}(z) - \partial_z M^{PVI}(z) = \frac{1}{\kappa}[L^{PVI}(z), M^{PVI}(z)].$$

(8.20)

Note that the Lax matrix (8.18) is a section of the bundle over $\Sigma_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Alternatively, one can obtain the Painlevé VI equation from the Lax pair

$$L^{PVI}(z) = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{c=0}^3 L_c^{PVI},$$

(8.21)

$$M^{PVI}(z) = \sum_{c=0}^3 M_c^{PVI},$$

(8.22)

with the $\nu_a$ in (8.2). Here the Lax matrix (8.21) is a section of a holomorphic bundle over the doubled elliptic curve $\Sigma_{2,2\tau} = \mathbb{C}/2\mathbb{Z} + 2\mathbb{Z}\tau$.

3. Non-autonomous Zhukovsky–Volterra gyrostat. The linear problem with a spectral parameter on an elliptic curve was posed in [64]. As in the previous case it is formulated in terms of the connection $(\partial_z + L^{ZVG}(z))dz$ in the holomorphic bundle over the elliptic curve $\Sigma_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ with transition functions $g_1 = -Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $g_\tau = -e^{-\frac{\tau}{4} - \frac{z}{2}}\Lambda = -e^{-\frac{\tau}{4} - \frac{z}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$L^{ZVG}(z) = -\frac{\kappa}{2} \partial_z \log(\vartheta(z; \tau))\sigma_0 + \sum_{\alpha=1}^3 (S_\alpha \varphi_\alpha(z) + \nu_\alpha \varphi_\alpha(z - \omega_\alpha))\sigma_\alpha,$$

(8.23)

$$M^{ZVG}(z) = -\frac{\kappa}{2} \partial_\tau \log(\vartheta(z; \tau))\sigma_0 - \sum_{\alpha=1}^3 S_\alpha \frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{\varphi_\alpha(z)}\sigma_\alpha + E_1(z)L^{ZVG}(\kappa = 0).$$

(8.24)

Then the monodromy-preserving condition

$$\partial_\tau L^{ZVG}(z) - \partial_z M^{ZVG}(z) = \frac{1}{\kappa}[L^{ZVG}(z), M^{ZVG}(z)]$$

(8.25)

is equivalent to equation (8.6).

Trigonometric and rational degenerations of Lax pairs for Calogero–Gaudin–Schlesinger type systems were investigated in [114]–[117]. One more $2 \times 2$ elliptic Lax pair for the Painlevé VI equation arises in the quantum version of the Painlevé–Calogero Correspondence [95], [96]. In this case the matrix element $L_{12}(z)$ has two simple zeros at $\pm u$ and does not have poles.

7The scalar multipliers are not very important here, since $\partial_z + L(z)$ is a connection in the adjoint bundle.
8.3. Elliptic form of rational connection. To relate the rational and elliptic connections we must lift the bundle over \(\mathbb{C}P^1\) to some bundle over an elliptic curve. A way to do this was proposed in [118]. Let us perform the analogue of the substitution (8.3) for (8.13), that is, let us make the following change of variables for the spectral parameter in the rational connection

\[
A(\zeta) = \left( \frac{A^0}{\zeta} + \frac{A^1}{\zeta - 1} + \frac{A^T}{\zeta - T} \right) d\zeta:
\]

\[
\zeta = \zeta(z, \tau) = \frac{\wp(z/2) - e_1}{e_2 - e_1}, \quad T = T(\tau) = \frac{e_3 - e_1}{e_2 - e_1}.
\]  

(8.26)

This gives

\[
A(\zeta) d\zeta = \sum_{\gamma=0,1,T} \frac{A_\gamma \wp'(z/2) + \wp(\omega_\gamma)}{2} dz,
\]

(8.27)

where the half-periods \(\omega^0, \omega^1, \omega^T\) of \(\Sigma_\tau\) are connected with the \(\omega_a\) in (8.2) as follows:

\[
\omega^0 = \omega_1 = \frac{1}{2}, \quad \omega^1 = \omega_2 = \frac{1 + \tau}{2}, \quad \omega^T = \omega_3 = \frac{\tau}{2}.
\]

We note that

\[
E_1 \left( \frac{z}{2} + \omega_\gamma \right) - E_1 \left( \frac{z}{2} \right) - E_1 (\omega_\gamma) = -\wp_\alpha(z) - \wp_\beta(z)
\]  

(8.28)

for any different indices \(\alpha, \beta, \gamma\) running over \(\{1, 2, 3\}\). Indeed, both sides of (8.28) are doubly periodic functions on the doubled elliptic curve \(\Sigma_{2,2\tau}\) with simple poles at its half-periods \((0, 1, \tau+1, \tau)\). It remains only to compare the residues. Therefore, we can rewrite (8.27) as

\[
A(\zeta) d\zeta = \sum_{\alpha=1}^{3} B^\alpha \wp_\alpha(z) dz,
\]  

(8.29)

where

\[
\begin{align*}
B^1 &= -A^1 - A^T, \\
B^2 &= -A^0 - A^T, \\
B^3 &= -A^0 - A^1
\end{align*}
\]

or

\[
\begin{align*}
A^\infty &= \frac{1}{2} (+B^1 + B^2 + B^3), \\
A^0 &= \frac{1}{2} (+B^1 - B^2 - B^3), \\
A^1 &= \frac{1}{2} (-B^1 + B^2 - B^3), \\
A^T &= \frac{1}{2} (-B^1 - B^2 + B^3).
\end{align*}
\]

(8.30)

The expression

\[
L^{\text{ell}}(z) = \sum_{\alpha=1}^{3} B^\alpha \wp_\alpha(z)
\]  

(8.31)

is a doubly periodic function on the doubled elliptic curve \(\Sigma_{2,2\tau}\) with simple poles at the half-periods \((0, 1, \tau+1, \tau)\). The residues are equal to \(2A^\infty, 2A^0, 2A^1, 2A^T\), respectively.
8.4. Symplectic Hecke Correspondence. The modification of bundles can be regarded as a procedure connecting the bundles with different characteristic classes and describing the Symplectic Hecke Correspondence [61, 45]. In the case of \( \text{SL}(2, \mathbb{C}) \) the characteristic classes are elements of \( \mathbb{Z}_2 \), that is, \( \pm 1 \). The trivial class \( +1 \) corresponds to the system (8.18)–(8.20), while \( -1 \) corresponds to (8.23)–(8.25). The Symplectic Hecke Correspondence connects the elliptic Lax pairs (8.18)–(8.20) and (8.23)–(8.25) generated by the modification (7.14). In the case of \( \text{SL}(2, \mathbb{C}) \) it has the form

\[
\Xi(z) = \begin{pmatrix} \theta_3(z - 2u, 2\tau) & -\theta_3(z + 2u, 2\tau) \\ -\theta_2(z - 2u, 2\tau) & \theta_2(z + 2u, 2\tau) \end{pmatrix}.
\]

(8.32)

It maps the holomorphic bundle for \( P \) to the one for \( Z \). Therefore, the corresponding connections are related as follows:

\[
L^Z(z) = \Xi(z)L^P(z)\Xi^{-1}(z) - \kappa \partial_z \Xi(z)\Xi^{-1}(z).
\]

(8.33)

This relation gives the change of variables (8.10) (see [64]).

Similarly, the modification (8.32) relates the connections (8.21) and (8.31) on the curve \( \Sigma_{2, 2\tau} \). The connection (8.21) has a non-trivial automorphism: conjugation by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) when \( z \to -z \) (the latter corresponds to changing the branch of the elliptic curve realized as the two-fold covering of \( \mathbb{C}^1 \) ramified at four points). It can be verified that the modification (8.32) trivializes this automorphism. Finally, we have

\[
L^\gamma(z) - \frac{\kappa}{2} E_1(z) \sigma_0 = \Xi(z)L^P(z)\Xi^{-1}(z) - \kappa \partial_z \Xi(z)\Xi^{-1}(z).
\]

(8.34)

This relation provides a non-trivial parametrization of the space \( \mathcal{C}^0 \times \mathcal{C} \times \mathcal{C}^T \times \mathcal{C}^\infty / \text{SL}(2, \mathbb{C}) \) in terms of the canonical variables \( v \) and \( u \), which means that the \( B^\alpha \) in (8.31) and therefore also \( A^0, A^1, A^T, \) and \( A^\infty \) are obtained as functions of \( v \) and \( u \). Explicit expressions were found in [118]. In particular,

\[
\left. B^1 \right|_{\kappa = 0} = \begin{pmatrix} -\nu_1 \frac{\theta_2(0)\theta_2(u)}{\nu_3(0)} - \nu_3 \frac{\theta_2(0)\theta_2(u)}{\nu_3(0)} & \nu_3 \frac{\theta_2(0)\theta_2(u)}{\nu_3(0)} - \nu_4 \frac{\theta_2(0)\theta_2(u)}{\nu_4(0)} \\ \nu_3 \frac{\theta_2(0)\theta_2(u)}{\nu_3(0)} + \nu_4 \frac{\theta_2(0)\theta_2(u)}{\nu_4(0)} & \nu_1 \frac{\theta_2(0)\theta_2(u)}{\nu_1(0)} + \nu_1 \frac{\theta_2(0)\theta_2(u)}{\nu_1(0)} \end{pmatrix}
\]

Appendix A. Simple Lie groups

Most of the facts and notation here are taken from [83] and [119].

Roots and weights. \( V \) is a vector space over \( \mathbb{R} \), \( \dim V = n; \) \( V^* \) is its dual space, and \( \langle \cdot, \cdot \rangle \) is the pairing between \( V \) and \( V^* \).

\( R = \{ \alpha \} \) is a root system in \( V^* \). The dual system \( R^\vee = \{ \alpha^\vee \} \) is a root system in \( V \). If \( V \) and \( V^* \) are identified by a scalar product \( \langle \cdot, \cdot \rangle \), then \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \).
The group of automorphisms of $V^*$ generated by the reflections

$$s_\alpha : x \mapsto x - \langle x, \alpha \rangle \alpha^\vee$$

is the Weyl group $W(R)$.

Simple roots $\Pi = (\alpha_1, \ldots, \alpha_l)$ form a basis in $R$:

$$\alpha = \sum_{j=1}^n f_j^\alpha \alpha_j, \quad f_j^\alpha \in \mathbb{Z},$$

and all the $f_j^\alpha$ are positive (in this case $\alpha \in R^+$ is a positive root), or all are negative ($\alpha \in R^-$ is a negative root); $R = R^+ \cup R^-$. The level of $\alpha$ is the sum

$$f_\alpha = \sum_{\alpha_j \in \Pi} f_j.$$  

The Cartan matrix is

$$a_{jk} = \langle \alpha_j, \alpha_k^\vee \rangle, \quad \alpha_j \in \Pi, \quad \alpha_k^\vee \in \Pi^\vee.$$  

The simple roots generate the root lattice $Q = \sum_{j=1}^n n_j \alpha_j$ ($n_j \in \mathbb{Z}, \alpha_j \in \Pi$) in $V^*$. There exists a unique maximal root $-\alpha_0 \in R^+$, given by

$$-\alpha_0 = \sum_{\alpha_j \in \Pi} n_j \alpha_j.$$  

Its level is equal to $h - 1$, where

$$h = 1 + \sum_{\alpha_j \in \Pi} n_j$$

is the Coxeter number. The positive Weyl chamber is

$$C^+ = \{x \in V \mid \langle x, \alpha \rangle > 0, \ \alpha \in R^+\}.$$  

The Weyl group acts transitively on the set of Weyl chambers. The set $\Pi^\vee = (\alpha_1^\vee, \ldots, \alpha_l^\vee)$ of simple coroots forms a basis in $V$ and generates the coroot lattice

$$Q^\vee = \sum_{j=1}^n n_j \alpha_j^\vee \subset V, \quad n_j \in \mathbb{Z}.$$  

The weight lattice

$$P = \sum_{j=1}^n m_j \omega_j \subset V^*, \quad m_j \in \mathbb{Z},$$

is dual to the coroot lattice (A.8). The fundamental coweights

$$\Upsilon^\vee = \{\omega_j^\vee \in \mathfrak{h}, \ j = 1, \ldots, n \mid \langle \alpha_k, \omega_j^\vee \rangle = \delta_{kj}, \ \alpha_j \in \Pi\}$$  

are dual to the simple roots. They generate the \textit{coweight lattice}

\[ P^\vee = \sum_{j=1}^{l} m_j \varpi_j^\vee, \quad m_j \in \mathbb{Z}, \tag{A.10} \]

dual to the root lattice \( Q \). The half-sum of the positive roots is

\[ \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \frac{1}{2} \sum_{j=1}^{n} \varpi_j. \]

For the dual vector in \( V \) we have

\[ \rho^\vee = \frac{1}{2} \sum_{\alpha \in R^\vee+} \alpha^\vee = \sum_{j=1}^{n} \varpi_j^\vee. \tag{A.11} \]

\textbf{Affine Weyl group.} The affine Weyl group \( W_a \) is \( Q^\vee \circ W_a \):

\[ W_Q = Q^\vee \rtimes W, \quad x \rightarrow x - \langle \alpha, x \rangle \alpha^\vee + k\beta^\vee, \quad \alpha^\vee, \beta^\vee \in R^\vee, \quad k \in \mathbb{Z}. \tag{A.12} \]

The \textit{Weyl alcoves} are the connected components of the set \( V \setminus \{ \langle \alpha, x \rangle \in \mathbb{Z} \} \). Their closures are fundamental domains of the \( W_a \)-action. An alcove belongs to \( C^+ \) \( (A.7): \)

\[ C_{\text{alc}} = \{ x \in V \mid \langle \alpha, x \rangle > 0, \alpha \in \Pi, \langle \alpha_0, x \rangle > -1 \}. \tag{A.13} \]

The shift operator \( x \rightarrow x + \gamma, \gamma \in P^\vee \), generates a semidirect product

\[ W_P = P^\vee \rtimes W. \tag{A.14} \]

The factor group is isomorphic to the centre: \( W_P/W_Q \sim P^\vee/Q^\vee \sim \mathcal{Z}(G) \).

\textbf{Chevalley basis in} \( \mathfrak{g} \). Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) of rank \( n \) and let \( \mathfrak{h} \) be a Cartan subalgebra. Let \( \mathfrak{h} = V + iV \), where \( V \) is the vector space defined above with the root system \( R \). The algebra \( \mathfrak{g} \) has the root decomposition

\[ \mathfrak{g} = \mathfrak{h} + \mathfrak{l}, \quad \mathfrak{l} = \sum_{\beta \in R} \mathfrak{R}_\beta, \quad \dim_{\mathbb{C}} \mathfrak{R}_\beta = 1. \tag{A.15} \]

The Chevalley basis in \( \mathfrak{g} \) is formed by the generators

\[ \{ E_{\beta_j} \in \mathfrak{R}_{\beta_j}, \beta_j \in R; H_{\alpha_k} \in \mathfrak{h}, \alpha_k \in \Pi \}, \tag{A.16} \]

where the \( H_{\alpha_k} \) are defined by the commutation relations

\begin{align*}
[H_{\alpha_k}, E_{-\alpha_k}] &= H_{\alpha_k}, \quad [H_{\alpha_k}, E_{\pm\alpha_j}] = a_{kj} E_{\pm\alpha_k}, \quad \alpha_k, \alpha_j \in \Pi, \\
[H_{\alpha_j}, E_{\alpha_k}] &= a_{kj} E_{\alpha_k}, \quad [E_{\alpha}, E_{\beta}] = C_{\alpha, \beta} E_{\alpha + \beta}, \tag{A.17} \end{align*}

where the \( C_{\alpha, \beta} \) are the structure constants of \( \mathfrak{g} \). They possess the properties

\begin{align*}
C_{\alpha, \beta} &= -C_{\beta, \alpha}, \\
C_{\lambda \alpha, \beta} &= C_{\alpha, \lambda^{-1} \beta}, \quad \lambda \in W, \\
C_{\alpha + \beta, -\alpha} &= \frac{|\beta|^2}{|\alpha + \beta|^2} C_{-\alpha, -\beta}. \tag{A.18} \end{align*}
Table 2. The centres of universal covering groups ($\mu_N = \mathbb{Z}/N\mathbb{Z}$)

| $\mathcal{G}$         | $\text{Lie}(\mathcal{G})$ | $\mathcal{Z}(\mathcal{G})$ |
|------------------------|----------------------------|-----------------------------|
| $\text{SL}(n, \mathbb{C})$ | $A_{n-1}$                  | $\mu_n$                     |
| $\text{Spin}_{2n+1}(\mathbb{C})$ | $B_n$                      | $\mu_2$                     |
| $\text{Sp}(n, \mathbb{C})$   | $C_n$                      | $\mu_2$                     |
| $\text{Spin}_{4n}(\mathbb{C})$ | $D_{2n}$                   | $\mu_2 \oplus \mu_2$       |
| $\text{Spin}_{4n+2}(\mathbb{C})$ | $D_{2n+1}$                | $\mu_4$                     |
| $E_6(\mathbb{C})$            | $E_6$                      | $\mu_3$                     |
| $E_7(\mathbb{C})$            | $E_7$                      | $\mu_2$                     |

If $(\cdot, \cdot)$ is a scalar product in $\mathfrak{h}$, then the $H_\alpha$ can be identified with coroots as $H_\alpha = \alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$, and

$$ (H_\alpha, H_\beta) = \frac{4(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} = 2 \frac{a_{\alpha, \beta}}{(\alpha, \alpha)}. \quad (A.19) $$

The Killing form on the subspace $\mathfrak{l}$ is expressed in terms of $(\alpha, \alpha)$:

$$ (E_\alpha, E_\beta) = \delta_{\alpha, -\beta} \frac{2}{(\alpha, \alpha)}. \quad (A.20) $$

**Centres of simple groups.** In all the cases other than $G_2$, $F_4$, and $E_8$, a simply connected group $\mathcal{G}$ has a non-trivial centre $\mathcal{Z}(\mathcal{G}) \sim P^\vee / Q^\vee$.

The group $\mathcal{Z}(\mathcal{G})$ is a cyclic group except for $\mathfrak{g} = D_{4l}$, and $\text{ord}(\mathcal{Z}(\mathcal{G})) = \text{det}(a_{kj})$, where $(a_{kj})$ is the Cartan matrix. Moreover,

$$ G^{\text{ad}} = \mathcal{G} / \mathcal{Z}(\mathcal{G}). \quad (A.21) $$

In the cases $A_{n-1}$ ($n$ non-prime) and $D_n$ the centre $\mathcal{Z}(\mathcal{G})$ has non-trivial subgroups $\mathfrak{z}_l \sim \mu_l = \mathbb{Z}/l\mathbb{Z}$. Then the factor groups

$$ G_l = \mathcal{G} / \mathfrak{z}_l, \quad G_p = G_l / \mathfrak{z}_p, \quad G^{\text{ad}} = G_l / \mathcal{Z}(G_l) \quad (A.22) $$

exist, where $\mathcal{Z}(G_l)$ is the centre of $G_l$ and $\mathcal{Z}(G_l) \sim \mu_p = \mathcal{Z}(\mathcal{G}) / \mathfrak{z}_l$.

The group $\mathcal{G} = \text{Spin}_{4n}(\mathbb{C})$ has a non-trivial centre

$$ \mathcal{Z}(\text{Spin}_{4n}) = (\mu_2^L \times \mu_2^R), \quad \mu_2 = \mathbb{Z}/2\mathbb{Z}, $$

where the three subgroups can be described in terms of their generators as

$$ \mu_2^L = \{(1, 1), (-1, 1)\}, \quad \mu_2^R = \{(1, 1), (1, -1)\}, \quad \mu_2^{\text{diag}} = \{(1, 1), (-1, -1)\}. $$
Therefore, there are three intermediate subgroups between $G = \text{Spin}_{4n} (\mathbb{C})$ and $G^{\text{ad}}$:

\[
\begin{array}{ccc}
\text{Spin}_{4n} & \xrightarrow{\Gamma^L} & \text{SO}(4n) = \text{Spin}_{4n} / \Gamma^{\text{diag}} \\
\text{Spin}^R_{4n} = \text{Spin}_{4n} / \Gamma^L & \text{SO}(4n) = \text{Spin}_{4n} / \Gamma^{\text{diag}} & \text{Spin}^L_{4n} = \text{Spin}_{4n} / \Gamma^R \\
G^{\text{ad}} = \text{Spin}_{4n} / (\mu_2^L \times \mu_2^R) & \\
\end{array}
\]

(A.23)

**Characters and cocharacters.** Let $\mathcal{H}$ be a Cartan subgroup, $\mathcal{H} \subset G$. We define the group of characters

\[
\Gamma(G) = \{ \chi : \mathcal{H} \to \mathbb{C}^* \}.
\]

(A.24)

This group can be identified with a lattice group in $\mathfrak{h}^*$ as follows. Let $x = (x_1, x_2, \ldots, x_n)$ be an element of $\mathfrak{h}$ and let $\exp(2\pi i x) \in \mathcal{H}$. We define $\gamma \in V^*$ such that $

\chi_\gamma = \exp(2\pi i \langle \gamma, x \rangle) \in \Gamma(G).$

Then

\[
\Gamma(G) = P, \quad \Gamma(G^{\text{ad}}) = Q
\]

(A.25)

and $\Gamma(G^{\text{ad}}) \subseteq \Gamma(G_l) \subseteq \Gamma(\mathcal{G})$. The fundamental weights $\varpi_k$ ($k = 1, \ldots, n$) and the simple roots $\alpha_k$ form bases in $\Gamma(\mathcal{G})$ and $\Gamma(G^{\text{ad}})$, respectively. Let $\mathcal{W}(\mathcal{G})$ be a cyclic group and let $p$ be a divisor of $\text{ord}(\mathcal{W}(\mathcal{G}))$ such that $l = \text{ord}(\mathcal{W}(\mathcal{G}))/p$. Then the lattice $\Gamma(G_l)$ is

\[
\Gamma(G) = Q + \varpi \mathbb{Z}, \quad p\varpi \in Q.
\]

(A.26)

We define the dual groups $t(G) = \Gamma^*(G)$ of cocharacters as holomorphic maps:

\[
t(G) = \{ \mathbb{C}^* \to \mathcal{H} \}.
\]

(A.27)

Equivalently,

\[
t(G) = \{ x \in \mathfrak{h} \ | \ \chi(e^{2\pi ix}) = 1 \}.
\]

(A.28)

A generic element of $t(G)$ has the form

\[
z^\gamma = \exp(2\pi i \gamma \log z) \in \mathcal{H}_G, \quad \gamma \in \Gamma^*(G), \quad z \in \mathbb{C}^*.
\]

(A.29)

In particular, the groups $t(\mathcal{G})$ and $t(G^{\text{ad}})$ are identified with the coroot and the coweight lattices:

\[
t(\mathcal{G}) = Q^\vee, \quad t(G^{\text{ad}}) = P^\vee,
\]

(A.30)

and $t(\mathcal{G}) \subseteq t(G_l) \subseteq t(G^{\text{ad}})$. It follows from (A.26) that

\[
t(G) = Q^\vee + \varpi^\vee \mathbb{Z}, \quad l\varpi^\vee \in Q^\vee.
\]

(A.31)

The sublattice $t(G_l) \subset P^\vee$ gives the affine Weyl group

\[
W_{t(G)} = t(G) \rtimes W
\]

(A.32)

---

\textsuperscript{8}Holomorphic maps of $\mathcal{H}$ to $\mathbb{C}^*$ such that $\chi(xy) = \chi(x)\chi(y)$ for $x, y \in \mathcal{H}$.
(see (A.12), (A.14)). The centre $Z(G)$ of $G$ is isomorphic to the quotient:

$$Z(G) \sim P^\vee/t(G),$$

(A.33)

and $\pi_1(G) \sim t(G)/Q^\vee$. In particular,

$$Z(\overline{G}) = P^\vee/t(\overline{G}) \sim P^\vee/Q^\vee.$$

(A.34)

Similarly, the fundamental group of $G^{\text{ad}}$ is $\pi_1(G^{\text{ad}}) \sim t(G^{\text{ad}})/Q^\vee \sim P^\vee/Q^\vee$. The triple $(R, t(G), \Gamma(G))$ is called the root data.

**Parabolic subgroups and flag varieties.** Let $\Pi'$ be a subset of the simple roots $\Pi(\mathfrak{g})$ and let $\mathfrak{g}'$ be the semisimple subalgebra of $\mathfrak{g}$ corresponding to $\Pi'$. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$. Then the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ has a decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \tilde{\mathfrak{h}}$. Similarly, for the coalgebras we have

$$\mathfrak{h}^* = \mathfrak{h}'^* \oplus \tilde{\mathfrak{h}}^*.$$  

(A.35)

These data define a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Let $R'$ be the set of roots corresponding to $\Pi'$ and let $\tilde{R} = R \setminus R'$. The parabolic subalgebra is the semidirect sum of a reductive subalgebra (the Levi subalgebra) and a nilpotent ideal:

$$\mathfrak{p} = s(\Pi') \oplus \mathfrak{n}^+(\tilde{R}^+), \quad s = \mathfrak{g}'(\Pi') \oplus \tilde{\mathfrak{h}}, \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in \tilde{R}^+} c_{\alpha}E_{\alpha}.$$  

(A.36)

We consider the decomposition of $\mathfrak{g}$ into a parabolic and a nilpotent subalgebra:

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}^-, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \tilde{R}^-} c_{\alpha}E_{\alpha}.$$  

(A.37)

Let $P \subset G$ be the parabolic subgroup determined by $\mathfrak{p}$, let $N^+$ be a unipotent subgroup, let $\mathfrak{n}^+ = \text{Lie}(N^+)$, and let $L$ be the Levi subgroup with Lie algebra $\text{Lie}(L) = s = \mathfrak{g}' \oplus \tilde{\mathfrak{h}}$  

(A.36).

The parabolic subgroup $P$ is the semidirect product

$$P = L \ltimes N^+, \quad \text{Lie}(N^+) = \mathfrak{n}^+.$$  

(A.39)

If $\Pi' = \emptyset$, then $P = B$ is a Borel subgroup. For $G$ we have the Bruhat decompositions

$$G = \bigcup_{w \in W} N^- wP = \bigcup_{w \in W} N^+ wP, \quad \text{Lie}(N^-) = \mathfrak{n}^-.$$  

(A.40)

The quotient $G/P = \text{Flag}$ is called a $G$-flag variety. The $N^+$-orbits of $w$ in Flag are Schubert cells. If $P = B$ is a Borel subgroup, then the flag $\text{Flag} = G/B$ is a full $G$-flag. Let $G^{\text{comp}}$ be the compact form of the complex group $G$. There exists the Iwasawa decomposition

$$G = G^{\text{comp}} P, \quad G^{\text{comp}} \cap P = L^{\text{comp}}, \quad L^{\text{comp}} = L \cap G^{\text{comp}}.$$  

(A.41)
Thus, the flag varieties are orbits of the compact groups:

\[ \text{Flag} = G/P \sim G^{\text{comp}}/L^{\text{comp}}. \] (A.42)

The parabolic subalgebras can be defined by means of fundamental coweights. Let \( \gamma \in \Upsilon^\vee \) (A.9). Its Lie algebra has the form (see (A.36))

\[ \text{Lie}(P_\gamma) = p_\gamma = s_\gamma(\Pi') \oplus n_\gamma^+(\tilde{R}^+), \quad \Pi' = \{ \alpha \in \Pi \mid \langle \alpha, \gamma \rangle = 0 \}. \] (A.43)

The corresponding parabolic subgroup \( P_\gamma \) is maximal, and the semisimple subalgebra \( g' \) (A.38) has rank \( n - 1 \). We need a special class of parabolic subalgebras. We say that parabolic subalgebras and the corresponding Levi algebras are \textit{admissible} if

\[ \langle \gamma, \alpha \rangle = 1 \quad \forall \alpha \in \tilde{R}^+. \] (A.44)

Admissible fundamental coweights are defined by

\[ \tilde{\Upsilon}^\vee = \{ \gamma \in \Upsilon^\vee \mid \gamma \notin \Pi^\vee, \langle \gamma, \alpha \rangle = 1 \ \forall \alpha \in \tilde{R}^+ \}. \] (A.45)

Note that for classical algebras any non-trivial fundamental coweight \( \gamma \in \Upsilon^\vee \) with \( \gamma \notin \Pi^\vee \) determines an admissible parabolic subalgebra and a Levi subalgebra. This is not so for the exceptional algebras \( E_6 \) and \( E_7 \).

Using the notation in [83] for roots and weights, we list the admissible fundamental coweights and semisimple components of admissible Levi algebras of simple Lie groups with non-trivial centres (Table 3).

| \( G \) | \( \tilde{\Upsilon}^\vee \) | \( g' \) | \( G/P \) |
|-------|---------|-------|---------|
| \( \text{SL}(n, \mathbb{C}) \) | \( \Upsilon^\vee \) | \( \text{sl}(n - p) \oplus \text{sl}(p) \) | \( \text{Gr}(n, p) = \text{S}(U(n - p) \times U(p)) \) |
| \( \text{Spin}_{2n+1}(\mathbb{C}) \) | \( \omega_1^\vee \) | \( \text{so}(2n + 1) \) | \( \text{SO}(2n + 1, \mathbb{R})/\text{SO}(2n - 1, \mathbb{R}) \times \text{SO}(2) \) |
| \( \text{Sp}_n(\mathbb{C}) \) | \( \omega_1^\vee \) | \( \text{sp}(n) \) | \( \text{Sp}(n, \mathbb{R})/\text{Sp}(n - 1, \mathbb{R}) \times \text{SO}(2) \) |
| \( \text{Spin}_{2n}(\mathbb{C}) \) | \( \omega_2^\vee \) | \( \text{so}(2n - 2) \) | \( \text{SO}(2n, \mathbb{R})/\text{SO}(2n - 2, \mathbb{R}) \times \text{SO}(2) \) |
| \( \text{Spin}_{2n}(\mathbb{C}) \) | \( \omega_{n-1,n}^\vee \) | \( \text{sl}(n) \) | \( \text{SO}(2n, \mathbb{R})/\text{SU}(n - 1) \times \text{SO}(2) \) |
| \( E_6(\mathbb{C}) \) | \( \omega_1^\vee \) | \( \text{so}(10) \) | \( E_6^{\text{comp}}/\text{SO}(10) \times \text{SO}(2) \) |
| \( E_7(\mathbb{C}) \) | \( \omega_7^\vee \) | \( E_6 \) | \( E_7^{\text{comp}}/E_6^{\text{comp}} \times \text{SO}(2) \) |

If we weaken the condition (A.44) to

\[ \langle \gamma, \alpha \rangle = 1 \quad \text{or} \quad 0 \quad \forall \alpha \in \tilde{R}^+, \] (A.46)

then any parabolic subalgebra of an admissible parabolic algebra is admissible. In particular, Borel subalgebras are admissible.

Coadjoint orbits. The cotangent bundle \( T^*G \) to \( G \) is equipped with the symplectic form

\[ \omega = \delta(a, g^{-1}\delta g), \quad a \in \mathfrak{g}^*, \quad g \in G. \] (A.47)
This form is invariant under the parabolic subgroup actions $f_{\text{out}}, f_{\text{int}} \in P$:

$$
\text{Ad}_{f_{\text{int}}}^*(a) = f_{\text{int}}^{-1}af_{\text{int}}, \quad g \rightarrow gf_{\text{int}},
$$

(A.48)

$$
a \rightarrow a, \quad g \rightarrow f_{\text{out}}g.
$$

(A.49)

The elements $\varepsilon_{\text{int}}, \varepsilon_{\text{out}} \in \mathfrak{p}$ generate the vector fields determined by the transformations (A.48), (A.49). Their Hamiltonians $F_{\text{int}}, F_{\text{out}}$ have the form

$$
F_{\text{int}} = \langle \varepsilon_{\text{int}}, a \rangle, \quad F_{\text{out}} = \langle \varepsilon_{\text{out}}, gag^{-1} \rangle.
$$

(A.50)

The moments of these actions take values in $\mathfrak{p}^* = s \oplus \mathfrak{n}^+$ (see (A.36)). The moment corresponding to the int-action is equal to $\mu_{\text{int}} = \text{Pr}_{\mathfrak{p}^*}(a)$. We fix its value as $\mu_{\text{int}} = \nu \in s$. This means that the solution of the moment equation is

$$
a = \nu + \xi, \quad \xi \in \mathfrak{n}^+,
$$

(A.51)

where $\nu$ is fixed and $\xi$ is an arbitrary element of $\mathfrak{n}^+$.

The coadjoint action of $P$ preserves $\nu \in s$ and the reduced symplectic manifold

$$
T^*G/P = \mu_{\text{int}}^{-1}(\nu)/P.
$$

(A.52)

It follows from (A.48) and (A.51) that the symplectic quotient $T^*G/P$ is given by the pairs $(g, \nu + \xi)$ with $g \in G$ and $\xi \in \mathfrak{n}^+$, with the equivalence relation

$$
(gb, \text{Ad}_{b}^* (\nu + \xi)) \sim (g, \nu + \xi), \quad b \in P.
$$

(A.53)

Note that the coadjoint action of $P$ on $\xi \in \mathfrak{n}^+$ is an affine action due to the $\nu$ term. The group $P$ acts freely on $\mathfrak{n}^+$. This means that $T^*G/P$ is the principal homogeneous space $PH/T^*(G/P)$ over the cotangent bundle to the flag variety $G/P$. The cotangent bundle corresponds to the choice $\nu = 0$.

Let us fix a gauge of the $P$-action by the choice $\xi = 0$. It follows from (A.36) and (A.39) that the Levi subgroup $L$ preserves $\xi = 0$. Then from (A.53) we find that

$$
\mathcal{O}_{\nu} = (\text{Ad}^*)^{-1}_G \nu = G/L.
$$

(A.54)

The form $\omega$ on $T^*G$ in (A.47) becomes the Kirillov–Kostant form

$$
\omega_{\text{KK}} = \langle \nu, g^{-1} \delta gg^{-1} \delta g \rangle.
$$

(A.55)

The dimension of the orbit is

$$
\text{dim}(\mathcal{O}_{\nu}) = \#(\tilde{R}).
$$

(A.56)

**Affine Lie algebras** [71]. The affine root system is defined as

$$
R^\text{aff} = \left\{ \tilde{\alpha} = \alpha + n \left| \begin{array}{c}
\alpha \in R \cup \{0\}, \ n \in \mathbb{Z} \setminus \{0\}, \\
\alpha \in R, \quad n = 0
\end{array} \right. \right\},
$$

$$
R^\text{aff}_+ = R^\text{aff} \text{ for } n > 0 \text{ or for } \alpha \in R^+, \ n = 0, \quad R^\text{aff}_- = R^\text{aff} \setminus R^\text{aff}_+.
$$

(A.57)
The affine Lie algebra can be represented as $L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where
\[
L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] = \left\{ \sum_k x_k t^k, \ x_k \in \mathfrak{g} \right\}, \quad (A.58)
\]
c is the generator of the centre, and d is a derivation of $L(\mathfrak{g})$. Affine roots correspond to affine root subspaces of $L(\mathfrak{g})$:
\[
E_{\tilde{\alpha}} = E_\alpha t^n, \quad H_{\tilde{\alpha}} = H_\alpha t^n \quad (n \neq 0).
\]
Let $\mathcal{H}$ be a Cartan subgroup of $G$. We realize the Weyl group $W$ of $\mathfrak{g}$ as the quotient $\mathcal{N}(\mathcal{H})/\mathcal{C}(\mathcal{H})$, where $\mathcal{N}(\mathcal{H})$ ($\mathcal{C}(\mathcal{H})$) is the normalizer (centralizer) of $\mathcal{H}$. We define two types of affine Weyl groups:
\[
\begin{align*}
W_P &= \{ \tilde{w} = wt^j, \ w \in W, \gamma \in P^\vee \}, \\
W_Q &= \{ \tilde{w} = wt^j, \ w \in W, \gamma \in Q^\vee \}. \quad (A.59)
\end{align*}
\]
They act on root vectors as $E_{\tilde{\alpha}} = E_\alpha t^n \rightarrow E_{\tilde{w}(\tilde{\alpha})} = E_{w(\alpha)} t^{n+\langle \gamma, \alpha \rangle}$.

**Loop groups** [70]. Let $L(G)$ be the loop group corresponding to the loop Lie algebra (A.58):
\[
L(G) = G \otimes \mathbb{C}[t^{-1}, t] = \left\{ \sum_k g_k t^k, \ g_k \in G \right\}. \quad (A.60)
\]
We define the loop subgroups
\[
\begin{align*}
L^+(G) &= \{ g_0 + g_1 t + \cdots = g_0 + tL[t] \}, \\
N^-(G) &= \{ n_- + g_1 t^{-1} + \cdots = n_- + t^{-1}L[t^{-1}] \}, \\
N^+(G) &= \{ n_+ + g_1 t + \cdots = n_+ + tL[t] \},
\end{align*}
\]
where $\text{Lie}(N^+) = n^+ = \sum_{\alpha \in R^+} c_\alpha E_\alpha$ (A.36). It follows from (A.63) that
\[
L^+(G) = L \ltimes N^+(G), \quad (A.64)
\]
where $L$ is a Levi subgroup.

There are loop analogues of the Bruhat decomposition (A.40). The decomposition (A.35) of the Cartan subalgebra gives the coweight sublattice
\[
\tilde{P}^\vee = \{ \gamma \in P^\vee \ | \ \langle \gamma, \alpha \rangle = 0, \ \alpha \in \mathfrak{h}' \}. \quad (A.65)
\]
The subgroup $\tilde{W}_P$ of the affine Weyl group $W_P$ is generated by the Weyl group $W' = W'(\mathfrak{g}')$, with $\mathfrak{g}'$ in (A.35) and the sublattice $\tilde{P}^\vee \subset P^\vee$. Let us consider the quotient $\tilde{W}_P = W_P/W'_P$. Similarly, we define $\tilde{W}_Q = W_Q/W'_Q$ and $\tilde{W}_{t(G)} = W_{t(G)}/W'_{t(G)}$. In analogy with (A.40) the affine Bruhat decomposition assumes the form
\[
L(G^{\text{rad}}) = \bigcup_{\tilde{w} \in \tilde{W}_P} N^-(G^{\text{rad}}) \tilde{w} L^+(G^{\text{rad}}) = \bigcup_{\tilde{w} \in \tilde{W}_P} N^+(G^{\text{rad}}) \tilde{w} L^+(G^{\text{rad}}), \quad (A.66)
\]
\[
L(\mathcal{G}) = \bigcup_{\hat{w} \in \mathcal{W}_Q} N^- (\mathcal{G}) \hat{w} L^+ (\mathcal{G}) = \bigcup_{\hat{w} \in \mathcal{W}_Q} N^+ (\mathcal{G}) \hat{w} L^+ (\mathcal{G}), 
\]
(A.67)

\[
L(G) = \bigcup_{\hat{w} \in \mathcal{W}_{i(G)}} N^- (G) \hat{w} L^+ (G) = \bigcup_{\hat{w} \in \mathcal{W}_{i(G)}} N^+ (G) \hat{w} L^+ (G). 
\]
(A.68)

An element \( g(t) \in L(G) \) can have the monodromy \( g(te^{2\pi i}) = g(t)\zeta, \zeta = e(\xi) \), where \( \xi \) is a representative in the quotient \( P^\vee / t(G) \sim \mathcal{Z}(G) \) (A.33). We define the subset \( L_\zeta(G) \) of loops homotopic to \( e(\xi) \). Then we obtain the decomposition

\[
L(G) = \bigcup_{\zeta \in \mathcal{Z}(G)} L_\zeta(G). 
\]
(A.69)

The affine flag variety is defined to be the quotient

\[
\text{Flag}^{\text{aff}} = L(G)/L^+(G). 
\]
(A.70)

The \( N^+(G) \)-orbit of an element \( \hat{w} \) in \( \text{Flag}^{\text{aff}} \),

\[
C_{\hat{w}} = \{ n(t)\hat{w}L^+(G) \mid n(t) \in N^+(G) \},
\]
(A.71)

is called an affine Schubert cell. From (A.66)–(A.68) we find that

\[
\text{Flag}^{\text{aff}} = \bigcup_{\hat{w} \in \mathcal{W}_{P,Q}} C_{\hat{w}}.
\]

The dimension of the affine Schubert cell is

\[
\dim C_{\hat{w}} = l(\hat{w}),
\]
(A.72)

where \( l(\hat{w}) \) is the length of \( \hat{w} \). It is equal to the number of negative affine roots (A.57), which \( \hat{w} \) transforms into positive roots.

**Appendix B. Generalized sine basis in simple Lie algebras**

In this appendix we briefly describe the construction in [45].

Let \( \mathfrak{g} \) be a complex simple Lie algebra, let \( \mathfrak{h} \) be a Cartan subalgebra, and let \( R \) the corresponding root system. Then we have the decomposition

\[
\mathfrak{g} = \mathfrak{h} + \mathfrak{l}, \quad \mathfrak{l} = \sum_{\beta \in R} \mathfrak{R}_\beta, \quad \dim \mathfrak{C} \mathfrak{R}_\beta = 1.
\]
(B.1)

The Chevalley basis in \( \mathfrak{g} \) is generated by the collection

\[
\{ E_{\beta_j} \in \mathfrak{R}_{\beta_j}, \beta_j \in R; H_{\alpha_k} \in \mathfrak{h}, \alpha_k \in \Pi \},
\]
(B.2)

where the \( H_{\alpha_k} \) are determined by the commutation relations

\[
[H_{\alpha_j}, E_{\alpha_k}] = a_{kj} E_{\alpha_k}, \quad \[H_{\alpha_k}, E_{\pm \alpha_j}\] = a_{kj} E_{\pm \alpha_k}, \quad \alpha_k, \alpha_j \in \Pi,
\]
(B.3)
Let us pass from the Chevalley basis (B.2) to a new basis that is more convenient for describing bundles with non-trivial characteristic classes. We call it the generalized sine basis (GS-basis), because for the case of $A_n$ and degree-1 bundles it coincides with the sine algebra basis (see, for example, [120]).

We take an element $\zeta \in \mathcal{Z}(\mathcal{G})$ of order $l$ and the corresponding $\Lambda^0 \in W$ in (5.4). Then, as in §5.2, $\Lambda^0$ generates a cyclic group $\mu_l = \langle \Lambda^0, (\Lambda^0)^2, \ldots, (\Lambda^0)^l = 1 \rangle$ isomorphic to a subgroup of $\mathcal{Z}(\mathcal{G})$. Since $\Lambda^0 \in W$, it preserves the root system $R$.

We define the quotient set $\mathcal{T}_l = \mathcal{R}/\mu_l$. Then $\mathcal{R}$ is represented as a union of $\mu_l$-orbits: $\mathcal{R} = \bigcup_{\beta} \mathcal{O}$. We denote by $\mathcal{O}(\beta)$ the orbit starting from the root $\beta$:

$$\mathcal{O}(\beta) = \{\beta, \lambda(\beta), \ldots, \lambda^{l-1}(\beta)\}, \quad \beta \in \mathcal{T}_l.$$

The number of elements in an orbit $\mathcal{O}$ (the length of $\mathcal{O}$) is $l/l_\alpha = l_\alpha$, where $p_\alpha$ is a divisor of $l$. Let $\nu_\alpha$ be the number of orbits $\mathcal{O}(\alpha)$ of length $l_\alpha$. Then $\sharp(\mathcal{R}) = \sum \nu_\alpha l_\alpha$. Note that if $\mathcal{O}(\beta)$ has length $l_\beta (l_\beta \neq 1)$, then the elements $\lambda^k \beta$ and $\lambda^{k+l_\beta} \beta$ coincide.

**Basis in l (B.1).** We first transform the root basis $\mathcal{E} = \{E_\beta, \beta \in \mathcal{R}\}$ in $l$. We define the orbit $\mathcal{E}(\beta) = \{E_\beta, E_{\lambda(\beta)}, \ldots, E_{\lambda^{l-1}(\beta)}\}$ in $\mathcal{E}$ corresponding to $\mathcal{O}(\beta)$. Again, $\mathcal{E} = \bigcup_{\beta} \mathcal{E}(\beta)$.

We define the set of integers

$$J_{p_\alpha} = \{a = mp_\alpha \mid m \in \mathbb{Z}, \ a \ is \ defined \ modulo \ l\} \quad (p_\alpha = \frac{l}{l_\alpha}). \quad \text{(B.4)}$$

The ‘Fourier transform’ of the root basis on the orbit $\mathcal{O}(\beta)$ is defined as

$$t_k^\beta = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^ma E_{\lambda^m(\beta)}, \quad \omega = \exp\left(\frac{2\pi i}{l}\right), \quad a \in J_\beta. \quad \text{(B.5)}$$

This transformation is invertible:

$$E_{\lambda^k(\beta)} = \frac{1}{\sqrt{l}} \sum_{a \in J_\beta} \omega^{-ka} t_k^\beta,$$

and therefore there is a one-to-one map $\mathcal{E}(\beta) \leftrightarrow \{t_k^\beta, \ a \in J_\beta\}$. Thus, we have defined the new basis

$$\{t_k^\beta (a \in J_\beta, \ \beta \in \mathcal{T}_l)\}. \quad \text{(B.6)}$$

Since $\lambda(E_\alpha) = E_{\lambda(\alpha)}$, for $\Lambda \mathbf{e}(\mathbf{u})$ we have ($\mathbf{u} \in \tilde{\mathfrak{h}}_0$)

$$\text{Ad}_\Lambda(t_k^\beta) = e\left(\langle \mathbf{u}, \beta \rangle - \frac{a}{l}\right) t_k^\beta, \quad e(x) = \exp(2\pi ix). \quad \text{(B.7)}$$

This means that $t_k^\beta (\beta \in \mathcal{T}_l)$ is a part of a basis in $\mathfrak{g}_{l-a}$ (5.11). Moreover,

$$\text{Ad}_\varnothing(t_k^\beta) = e(\langle \kappa, \beta \rangle) t_k^\beta. \quad \text{(B.8)}$$
These relations follow from (5.3). We also take into account that $\mathcal{Q}$ and $\Lambda$ commute in the adjoint representation and $e(x)E_\alpha e(-x) = e(\langle x, \alpha \rangle)E_\alpha$ for $x \in \tilde{h}_0$.

Picking another element $\Lambda'$ generating a subgroup $\mathcal{Z}'_l$ ($l' \neq l$), we get another set of orbits and another basis. We have as many types of bases as there are non-isomorphic subgroups in $\mathcal{Z}(\mathcal{G})$.

The Killing form. We consider two orbits $\mathcal{O}(\overline{\alpha})$ and $\mathcal{O}(\overline{\beta})$ passing through $E_\alpha$ and $E_\beta$. Assume that there exists an integer $r$ such that $\alpha = -\lambda^r(\beta)$. This implies that the elements of the two orbits are connected by the relation $\lambda^n(\alpha) = -\lambda^m(\beta)$ for $m - n = r$. In other words, $-\beta \in \mathcal{O}(\overline{\alpha})$. In particular, the orbits therefore have the same length. It follows from (B.5) and (A.20) that

$$\langle t_{\alpha_1}^{l_1}, t_{\alpha_2}^{l_2} \rangle = \delta_{\alpha_1, -\lambda^r(\beta)} \delta^{(c_1 + c_2, 0 \pmod{l})} \omega^{-rc_1} \frac{2p_\alpha}{(\alpha, \alpha)},$$

where $p_\alpha = l/l_\alpha$ and $l_\alpha$ is the length of $\mathcal{O}(\overline{\alpha})$. In particular, $(t_{\alpha_1}^l, t_{-\alpha_1}^{-l}) = 2p_\alpha/(\alpha, \alpha)$.

In what follows we need the dual basis $\overline{\mathcal{X}}^b_\alpha$:

$$(\overline{\mathcal{X}}^{b_1}_{\alpha_1}, \overline{\mathcal{X}}^{b_2}_{\alpha_2}) = \delta^{(b_1 + b_2, 0 \pmod{l})} \delta_{\alpha_1, -\alpha_2}, \quad \overline{\mathcal{X}}^b_\alpha = \frac{t_{-\alpha}}{2p_\alpha}.$$  \hspace{1cm} (B.10)

The Killing form in this basis is inverse to (B.9):

$$(\overline{\mathcal{X}}^{a_1}_{\alpha_1}, \overline{\mathcal{X}}^{a_2}_{\alpha_2}) = \delta_{\alpha_1, -\lambda^r(\alpha_2)} \delta^{(a_1 + a_2, 0 \pmod{l})} \omega^{ra_1} \frac{(\alpha_1, \alpha_1)}{2p_\alpha_1}.$$  \hspace{1cm} (B.11)

A basis in the Cartan subalgebra. Almost the same construction exists in $\mathfrak{h}$. Again let $\Lambda^0$ generate the group $\mu_l$. Since $\Lambda^0$ preserves the extended Dynkin diagram, its action preserves the extended coroot system $\Pi^{\text{ext}} = \Pi^{\vee} \cup \alpha^\vee_0$ in $\mathfrak{h}$. Consider the quotient $\mathcal{H}_l = \Pi^{\text{ext}}/\mu_l$. We define an orbit $\mathcal{H}(\overline{\alpha})$ of length $l_\alpha = l/p_\alpha$ in $\Pi^{\text{ext}}$ passing through $H_\alpha \in \Pi^{\text{ext}}$:

$$\mathcal{H}(\overline{\alpha}) = \{ H_\alpha, H_{\lambda(\alpha)}, \ldots, H_{\lambda^{l-1}(\alpha)} \}, \quad \overline{\alpha} \in \mathcal{H}_l = \Pi^{\text{ext}}/\mu_l.$$  \hspace{1cm}

The set $\Pi^{\text{ext}}$ is a union of orbits $\mathcal{H}(\overline{\alpha})$:

$$\Pi^{\text{ext}} = \bigcup_{\overline{\alpha} \in \mathcal{H}_l} \mathcal{H}(\overline{\alpha}).$$

We define the ‘Fourier transform’

$$\mathfrak{h}_\alpha^k = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mk} H_{\lambda^m(\alpha)}, \quad \omega = \exp\left(\frac{2\pi i}{l}\right), \quad k \in J_\alpha.$$  \hspace{1cm} (B.4)

The basis $\mathfrak{h}_\overline{\alpha}^k$ ($c \in J_\alpha$, $\overline{\alpha} \in \mathcal{H}_l$) is over-complete in $\mathfrak{h}$. Namely, let $\mathcal{H}(\overline{\alpha}_0)$ be the orbit $\{ H_\alpha, H_{\lambda(\alpha_0)}, \ldots, H_{\lambda^{l-1}(\alpha_0)} \}$ passing through the minimal coroot. Then the
element $h^0_{\alpha_0}$ is a linear combination of elements $h^{0\alpha}_\alpha (\alpha \in \Pi)$ and we should exclude it from the basis. We replace the basis $\Pi^\vee$ in $\mathfrak{h}$ by

$$h^c_{\alpha} \quad (c \in J_\alpha), \quad \begin{cases} \alpha \in \widetilde{\mathcal{H}} = \mathcal{H} \setminus \mathcal{H}(\alpha_0), & c = 0, \\ \alpha \in \mathcal{H}_l, & c \neq 0. \end{cases} \quad \text{(B.13)}$$

As before, there is a one-to-one map $\Pi^\vee \leftrightarrow \{h^c_{\alpha}\}$. The elements $(h^k_{\alpha}, t^l_{\alpha})$ form the GS-basis in $\mathfrak{g}_{\l-\alpha}$ (5.11).

The Killing form. The Killing form in the basis (B.13) can be found from (A.19):

$$(h^k_{\alpha}, h^l_{\beta}) = \delta((k+1, 0 \text{~mod~} l)) \omega^{k}_{\alpha, \beta} = \frac{2}{(\beta, \beta)} \sum_{s=0}^{l-1} \omega^{-sk} a_{\beta, \lambda_s(\alpha)}, \quad \text{(B.14)}$$

where $a_{\alpha, \beta}$ is the Cartan matrix (A.4). The dual basis is generated by the elements $h^l_{\alpha}$:

$$(h^k_{\alpha}, h^l_{\beta}) = \delta((k+1, 0 \text{~mod~} l)) \delta_{\alpha, \beta},$$

$$h^k_{\alpha} = \sum_{\beta \in \Pi} (\omega^{k}_{\alpha, \beta})^{-1} h^{-k}_{\beta}, \quad h^l_{\beta} = \sum_{\alpha \in \Pi} \omega^{-k} h^{-k}_{\alpha}. \quad \text{(B.15)}$$

The Killing form in the dual basis has the form

$$(h^{k_1}_{\alpha_1}, h^{k_2}_{\alpha_2}) = \delta((k_1+k_2, 0 \text{~mod~} l)) (\omega^{k}_{\alpha_1, \alpha_2})^{-1}. \quad \text{(B.16)}$$

In summary, we have defined the GS-basis in $\mathfrak{g}$:

$$\{t^k_{\alpha}, h^l_{\beta}; \quad (k, \beta, j, \alpha) \text{~are defined in (B.6), (B.13)}\} \quad \text{(B.17)}$$

and the dual basis

$$\{h^k_{\alpha}, h^l_{\beta}; \quad (k, \beta, j, \alpha) \text{~are defined in (B.10), (B.15)}\} \quad \text{(B.18)}$$

along with the Killing forms.

Commutation relations. The commutation relations in the GS-basis can be found from the commutation relations in the Chevalley basis (A.17). If we take into account the invariance of the structure constants with respect to the Weyl group action, $C_{\lambda, \lambda\beta} = C_{\alpha, \beta}$, then it is not difficult to derive the commutation relations in the GS-basis using the definition in the Chevalley basis (B.5), (B.12). In the case of root-root commutators we get the relations

$$[t^a_{\alpha}, t^b_{\beta}] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{hs} C_{\alpha, \lambda^s\beta} t^{a+b}_{\alpha+\lambda^s\beta}, \quad \alpha \neq -\lambda^s\beta, \quad \text{(B.19)}$$

$$[h^a_{\alpha}, t^b_{\beta}] = \frac{p_{\alpha}}{\sqrt{l}} \omega^{sb} h^{a+b}_{\alpha}, \quad \alpha = -\lambda^s\beta.$$
Here we denote by $\tilde{\alpha}$ the duals to the simple roots in the Cartan subalgebra:

$$(\tilde{\alpha}_i, \beta_j) = \delta_{ij}. \tag{B.21}$$

It is more convenient to use the following normalized basis for the Cartan subalgebra:

$$\tilde{\mathcal{h}}_\alpha^k = \frac{(\alpha, \alpha)}{2} \mathcal{h}_\alpha^k, \quad \tilde{\mathcal{H}}_\alpha^k = \frac{2}{(\alpha, \alpha)} \mathcal{h}_\alpha^k. \tag{B.22}$$

This reparametrization leads to the commutation relations

$$[[\tilde{\mathcal{h}}_\alpha^k, t^m_\beta], t^m_\beta] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} (\tilde{\alpha}, \lambda^s \beta) t^{k+m}_\beta, \tag{B.23}$$

$$[[\tilde{\mathcal{H}}_\alpha^k, t^m_\beta], t^m_\beta] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} (\tilde{\alpha}, \lambda^s \beta) t^{k+m}_\beta. \tag{B.24}$$

The following simple formula expresses the decomposition of a Cartan element in the basis of simple roots:

$$\tilde{\mathcal{h}}_\beta^k = \sum_{\alpha \in \Pi} (\tilde{\alpha}, \beta) \tilde{\mathcal{h}}_\alpha^k, \quad \beta \in R. \tag{B.25}$$

The connection between the dual bases is clear from the expression

$$\sum_{\beta \in \Pi} (\tilde{\alpha}, \beta) \tilde{\mathcal{h}}_\beta^k = \sum_{\beta \in \Pi} (\alpha, \beta) \tilde{\mathcal{H}}_\beta^k. \tag{B.26}$$

The Cartan elements have the symmetry properties

$$\tilde{\mathcal{h}}_{-\alpha}^k = -\tilde{\mathcal{h}}_\alpha^k, \quad \tilde{\mathcal{H}}_{-\alpha}^k = -\tilde{\mathcal{H}}_\alpha^k. \tag{B.27}$$

**Invariant subalgebra.** We consider the invariant subalgebra $\mathfrak{g}_0$. It is generated by the basis $\{t_\beta^0, \tilde{h}_\alpha^0\}$ (B.17). In particular, $\{\mathcal{h}_\alpha^0\}$ (B.12), (B.13) forms a basis in the Cartan subalgebra $\tilde{\mathfrak{h}}_0 \subset \mathfrak{h}$ (dim $\tilde{\mathfrak{h}}_0 = p < n$).

We pass from $\{\mathcal{h}_\alpha^0\}$ to a special basis in $\tilde{\mathfrak{h}}_0$:

$$\tilde{\Pi}^\vee = \{\tilde{\alpha}_k^\vee \mid k = 1, \ldots, p\}. \tag{B.28}$$

It is constructed in the following way. Consider the subsystem of simple coroots

$$\Pi_\vee^\gamma = \Pi^\text{ext} \setminus \mathcal{O}(\alpha_0^\vee) \tag{B.29}$$

(see (B.13)). In other words, $\Pi_\vee^\gamma$ is a subset of simple coroots that does not contain simple coroots in the orbit passing through $\alpha_0$.

For $A_{N-1}$, $B_n$, $E_6$, and $E_7$ the coroot basis $\tilde{\Pi}^\vee$ (B.27) is the result of averaging over $\lambda$-orbits in $\Pi_\vee^\gamma$:

$$\tilde{\alpha}^\vee = \sum_{m=1}^{l-1} H_{\lambda^m(\alpha)}, \quad H_{\alpha} \in \Pi_\vee^\gamma. \tag{B.29}$$

In the cases of $C_n$ and $D_n$ this construction is valid for almost all coroots except the last one in the Dynkin diagram (see Remark 10.1 in [46]).

We consider the set $\tilde{\Pi} = \{\tilde{\alpha}_k \mid k = 1, \ldots, p, \langle \tilde{\alpha}_k, \tilde{\alpha}_k^\vee \rangle = 2\}$ of dual vectors in $\tilde{\mathfrak{h}}_0^*$. 

\[ \]
Proposition B.1. The set
\[ \tilde{\Pi} = \{ \tilde{\alpha}_k | k = 1, \ldots, p \} \]  
(B.30)
of vectors in \( \tilde{h}_0^* \) is a system of simple roots of the simple Lie subalgebra \( \tilde{g}_0 \subset g_0 \) given by the root system \( \tilde{R} = \tilde{R}(\tilde{\Pi}) \) and the Cartan matrix \( \langle \tilde{\alpha}_k, \tilde{\alpha}_j^\vee \rangle \).

The proof of this statement is carried out in [46] case by case.

Let \( R_1 = R_1(\Pi_1) \) be the subset of roots generated by the simple roots \( \Pi_1 = \Pi^\text{ext} \setminus \mathcal{O}(\alpha_0) \). It is invariant under the \( \lambda \) action. The root system \( \tilde{R} \) in \( \tilde{g}_0 \) corresponds to the \( \lambda \)-invariant set \( R_1 \). Let \( \mathcal{R}_1 = R_1(\Pi_1) \) be the subset of roots generated by the simple roots \( \Pi_1 = \Pi_\text{ext} \setminus \mathcal{O}(\alpha_0) \). It is invariant under the \( \lambda \) action. The root system \( \tilde{R} \) in \( \tilde{g}_0 \) corresponds to the \( \lambda \)-invariant set \( R_1 \). Let \( \mathcal{R}_1 = R_1(\Pi_1) \) be the subset of roots generated by the simple roots \( \Pi_1 = \Pi_\text{ext} \setminus \mathcal{O}(\alpha_0) \). It is invariant under the \( \lambda \) action. The root system \( \tilde{R} \) in \( \tilde{g}_0 \) corresponds to the \( \lambda \)-invariant set \( R_1 \).

Let \( \mathcal{R}_1 = R_1(\Pi_1) \) be the subset of roots generated by the simple roots \( \Pi_1 = \Pi_\text{ext} \setminus \mathcal{O}(\alpha_0) \). It is invariant under the \( \lambda \) action. The root system \( \tilde{R} \) in \( \tilde{g}_0 \) corresponds to the \( \lambda \)-invariant set \( R_1 \).

This is a subset among all the orbits \( \mathcal{R}_1 = R_1(\Pi_1) \). Therefore, \( \mathcal{R}_1 = \tilde{R} \cup \mathcal{R}_1' \). The \( \lambda \)-invariant subalgebra \( g_0 \) contains the subspace \( V = \left\{ \sum_{\beta \in \mathcal{R}_1'} a_\beta t_\beta^0, \ a_\beta \in \mathbb{C} \right\} \).

Then \( g_0 \) is the sum of \( \tilde{g}_0 \) and \( V \):
\[ g_0 = g_0' \oplus V. \]  
(B.33)
The components of this decomposition are orthogonal with respect to the Killing form (B.14) and \( V \) is a representation space of \( g_0' \). We give below in Table 4 the explicit forms of \( g_0' \) for all simple algebras from our list.

We summarize the information about invariant subalgebras in Table 4.

Remark B.1. For any \( \xi \in Q^\vee \) a solution of (5.7) is \( \Lambda = \text{Id} \). In this case \( g_0' = g \) and the GS-basis is the Chevalley basis.

The GS-basis from a canonical basis in \( \mathfrak{h} \). Let \( (e_1, e_2, \ldots, e_n) \) be a canonical basis in \( \mathfrak{h} \) \( ((e_j, e_k) = \delta_{jk}) \). Since \( \Lambda \) preserves \( \mathfrak{h} \), we can consider the action of \( \mu_l \) on the canonical basis. We define an orbit of length \( l_s = l/p_s \) passing through \( e_s \): \( \mathcal{O}(s) = \{ e_s, \lambda(e_s), \ldots, \lambda^{l-1}(e_s) \} \).

The Fourier transform with respect to \( \mathcal{O}(s) \) has the form
\[ \mathfrak{h}_s^c = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mc} \chi^m(e_s), \quad c \in J_{p_s}, \quad \omega = \exp \left( \frac{2\pi i}{l} \right), \]  
(B.36)
\[ ^{9}\text{For root systems corresponding to } A_n \text{ and } E_6 \text{ it is convenient to choose canonical bases in } \mathfrak{h} \oplus \mathbb{C}. \]
Table 4. Invariant subalgebras $\mathfrak{g}_0 = \mathfrak{g}_{\Pi}$ and $\tilde{\mathfrak{g}}_0$ of simple Lie algebras. The coweights generating central elements are displayed in column 3.

| $\Pi$ | $\mathcal{Z}(G)$ | $\varpi^\vee_j$ | $\Pi_1$ | $l = \text{ord}(\Lambda)$ | $\tilde{\mathfrak{g}}_0$ | $\mathfrak{g}_0$ |
|-------|----------------------------------|----------------|--------|----------------|----------------|----------------|
| $A_{N-1}$ $(N = pl)$ | $\mu_N$ | $\varpi^\vee_{N-1}$ | $\bigcup_{p=1}^l A_{p-1}$ | $N/p$ | $\mathfrak{sl}_p$ | $\mathfrak{sl}_p \bigoplus \mathfrak{gl}_p$ |
| $B_n$ | $\mu_2$ | $\varpi^\vee_m$ | $\text{so}_{2n-1}$ | $2$ | $\text{so}(2n-1)$ | $\text{so}(2n)$ |
| $C_{2l}$ $(l > 1)$ | $\mu_2$ | $\varpi^\vee_{2l}$ | $A_{2l-1}$ | $2$ | $\mathfrak{so}(2l)$ | $\mathfrak{gl}_{2l}$ |
| $C_{2l+1}$ | $\mu_2$ | $\varpi^\vee_{2l+1}$ | $A_{2l}$ | $2$ | $\mathfrak{so}(2l+1)$ | $\mathfrak{gl}_{2l+1}$ |
| $D_{2l+1}$ $(l > 1)$ | $\mu_4$ | $\varpi^\vee_{2l+1}$ | $A_{2l-2}$ | $4$ | $\mathfrak{so}(2l-1)$ | $\mathfrak{so}(2l) \oplus \mathfrak{so}(2l) \oplus 1$ |
| $D_{2l}$ $(l > 2)$ | $\mu_2 \oplus \mu_2$ | $\varpi^\vee_{2l}$ | $A_{2l-1}$ | $2$ | $\mathfrak{so}(2l)$ | $\mathfrak{so}(2l) \oplus 1$ |
| $E_6$ | $\mu_3$ | $\varpi^\vee$ | $D_4$ | $3$ | $\mathfrak{g}_2$ | $\mathfrak{so}(8) \oplus 2 \cdot 1$ |
| $E_7$ | $\mu_2$ | $\varpi^\vee$ | $e_6$ | $2$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6 \oplus 1$ |

where $J_p = \{c = mp \text{ (mod } l) \mid m \in \mathbb{Z}\}$. We consider $\mathcal{C}_l = (e_1, e_2, \ldots, e_n)/\mu_l$. Then we can pass from the canonical basis to the GS-basis:

$$(e_1, e_2, \ldots, e_n) \longleftrightarrow \{h^s, s \in \mathcal{C}_l\}.$$

The Killing form is obtained from (B.36):

$$(h^i_s, h^j_s) = \delta(s_1, s_2) \delta^{ij}.$$

Then the dual generators are

$$\mathfrak{g}_s^k = h^{k-s}.$$

The commutation relations in $\mathfrak{g}$ are

$$[h^k_s, t^k_\beta] = \frac{1}{\sqrt{l}} \sum_{r=0}^{l-1} \omega^{-rk} \langle \lambda^r(\beta), e_s \rangle t^{k_1+k_2},$$

$$[t^k_\alpha, t^k_\beta] = \frac{1}{p_\alpha \sqrt{l}} \sum_s (\alpha^\vee, e_s) h_s^{k_1+k_2} \quad \text{if } \alpha = -\lambda^r(\beta) \text{ for some } r.$$
We obtain the last relation from (B.5) and the expansion $h^k_\alpha = \sum_s (\alpha^s, e_s) h^k_s$. Alternatively, the same relations can be written as given in (B.19)–(B.20):

\[
[t^k_\alpha, t^j_\beta] = \begin{cases} 
\frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{js} C_{\alpha, \lambda} \beta (\alpha, \beta) t^{k+j}_\alpha, & \alpha \neq -\lambda \beta, \\
\frac{p_\alpha}{\sqrt{l}} \omega^{s_2} h^k_\alpha, & \alpha = -\lambda \beta, 
\end{cases} 
\]

(B.40)

Appendix C. Elliptic functions

Most of the facts and notation are taken from [121], [122]. The identities can be proved directly by comparing residues and quasi-periodicity conditions on the lattice $\mathbb{Z} + \tau \mathbb{Z}$.

Notation:

\[
e(x) = \exp(2\pi ix), \quad q = e\left(\frac{1}{2}\tau\right),
\]

$\omega_1, \omega_2$ are the fundamental half-periods, $\tau = \omega_2/\omega_1$.

The theta function:

\[
\vartheta(z \mid \tau) = \sum_{n \in \mathbb{Z}} (-1)^{n-1/2} e\left(\frac{1}{2} \left( \left( n + \frac{1}{2} \right) \tau + (2n+1)z \right) \right)
\]

\[
= 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin \pi z.
\]

The Eisenstein functions:

\[
E_1(z \mid \tau) = \partial_z \log \vartheta(z \mid \tau), \quad E_1(z \mid \tau) \big|_{z \to 0} \sim \frac{1}{z} - 2\eta_1 z; 
\]

\[
E_2(z \mid \tau) = -\partial_z E_1(z \mid \tau) = \partial_z^2 \log \vartheta(z \mid \tau), \quad E_2(z \mid \tau) \big|_{z \to 0} \sim \frac{1}{z^2} + 2\eta_1.
\]

Here

\[
\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m,n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} = \frac{24}{2\pi^2} \frac{\eta'(\tau)}{\eta(\tau)},
\]

(C.4)

where $\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n)$ is the Dedekind function.

Relation to the Weierstrass functions:

\[
\zeta(z, \tau) = E_1(z, \tau) + 2\eta_1(\tau) z,
\]

(C.5)
The function $\phi$:

$$\phi(u, z) = \frac{\partial(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \quad (C.7)$$

which has a pole at $z = 0$:

$$\phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \varphi(u)) + \ldots. \quad (C.8)$$

Let

$$f(u, z) = \partial_u \phi(u, z) = \phi(u, z)(E_1(u + z) - E_1(u)). \quad (C.9)$$

The addition formulae:

$$\phi(u, z)f(v, z) - \phi(v, z)f(u, z) = (E_2(v) - E_2(z))\phi(u + v, z), \quad (C.10)$$

$$\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u), \quad (C.11)$$

$$\phi(u, z)\phi(-u, w) = \phi(u, z - w)[E_1(u + z - w) - E_1(u) + E_1(w) - E_1(z)]. \quad (C.12)$$

Let $\sum_{a=1}^{n} c_a = 0$. Then

$$\left(\sum_{a=1}^{n} c_a E_1(w-x_a)\right)^2 = \sum_{a=1}^{n} \left( c_a^2\varphi(w-x_a) + \sum_{b \neq a} c_ac_b E_1(x_a-x_b)E_1(w-x_a) \right). \quad (C.13)$$

The heat equation:

$$\partial_\tau \phi(u, w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u, w) = 0. \quad (C.14)$$

Parity:

$$\phi(u, z) = \phi(z, u), \quad \phi(-u, -z) = -\phi(u, z); \quad (C.15)$$

$$E_1(-z) = -E_1(z), \quad E_2(-z) = E_2(z); \quad (C.16)$$

$$f(-u, -z) = f(u, z). \quad (C.17)$$

Quasi-periodicity:

$$\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-1/2}e^{-2\pi iz}\vartheta(z); \quad (C.18)$$

$$E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i; \quad (C.19)$$

$$E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z); \quad (C.20)$$

$$\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi iz}\phi(u, z); \quad (C.21)$$

$$f(u, z + 1) = f(u, z), \quad f(u, z + \tau) = e^{-2\pi iz}f(u, z) - 2\pi i\phi(u, z). \quad (C.22)$$

The most important object for the construction of Lax operators and $r$-matrices is the function

$$\varphi^k_{\alpha}(u, z) = e^{2\pi i(q, \alpha)z}\phi\left(\langle u + q\tau, \alpha \rangle + \frac{k}{l}, z\right), \quad \left( q = \frac{\rho}{h} \right), \quad (C.23)$$
Substituting here we find that
\[ \varphi_0^k(u, z) = \phi \left( \frac{k}{L}, z \right), \quad (C.24) \]
\[ \varphi_0^0(u, z) = E_1(z), \quad (C.25) \]
and
\[ f_\alpha^k(u, z) = e^{2\pi i(\varphi, \alpha)z} f \left( \langle u + \rho \tau, \alpha \rangle + \frac{k}{l}, z \right), \quad (C.26) \]
\[ f_\alpha^0(u, 0) = -E_2 \left( \langle u + \rho \tau, \alpha \rangle + \frac{k}{l} \right) = -\phi \left( \langle u + \rho \tau, \alpha \rangle + \frac{k}{l} \right) - 2\eta_1, \quad (C.27) \]
\[ f_\alpha^0(u, z) = f_0^0(z) = \frac{1}{2} \left( E_1^2(z) - \varphi(z) \right). \quad (C.28) \]

The heat equation takes the form
\[ 2\pi i \partial_{\tau} \varphi_\alpha^k(u, z) = \partial_z f_\alpha^k(u, z). \quad (C.29) \]

For brevity we sometimes omit the \( u \)-dependence of functions in the formulae below.

**Fay identity:**
\[ \phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0. \quad (C.30) \]

Differentiating with respect to \( u_2 \), we find that
\[ \phi(u_1, z_1)f(u_2, z_2) - \phi(u_1 + u_2, z_1)f(u_2, z_2 - z_1) = \phi(u_2, z_2 - z_1)f(u_1 + u_2, z_1) + \phi(u_1, z_1 - z_2)f(u_1 + u_2, z_2). \quad (C.31) \]

Substituting here
\[ u_1 = \langle u + \rho \tau, \alpha + \beta \rangle + \frac{k + m}{l}, \quad u_2 = -\langle u + \rho \tau, \beta \rangle - \frac{m}{l}, \quad (C.32) \]
\[ z_1 = z_a - z_c = z_{ac}, \quad z_2 = z_b - z_c = z_{bc} \]
and multiplying by a suitable exponential factor, we obtain
\[ \varphi_\alpha^k(z_{ac})f_\beta^m(z_{ab}) - \varphi_\beta^m(z_{ab})f_\alpha^k(z_{ac}) + \varphi_\alpha^{k+m}(z_{ab})f_\alpha^k(z_{cb}) - \varphi_\alpha^{k+m}(z_{ac})f_{-\beta}^m(z_{bc}) = 0. \quad (C.33) \]

Setting \( m = 0 \) and \( \beta = 0 \) and using the expansion
\[ \phi(u, z) \sim \frac{1}{u} + E_1(z) + uf_0^0(z) + \cdots, \quad (C.34) \]
we find that
\[ \varphi_\alpha^k(z_{ac})f_0^0(z_{ab}) - E_1(z_{ab})f_\alpha^k(z_{ac}) + \varphi_\alpha^k(z_{ab})f_\alpha^k(z_{bc}) - \varphi_\alpha^k(z_{ac})f_0^0(z_{cb}) = \frac{1}{2} \partial_u f_\alpha^k(z_{ac}). \quad (C.35) \]

More Fay identities:
\[ \varphi_\alpha^k(z_{ac})f_\beta^m(z_{ac}) - \varphi_\beta^m(z_{ac})f_\alpha^k(z_{ac}) = \varphi_\alpha^{k+m}(z_{ac})(\varphi_\alpha^k - \varphi_\beta^m), \quad (C.36) \]
\[ \varphi^m_{\beta}(z_{ac}) f^{-m}_{-\beta}(z_{ac}) - \varphi^{-m}_{-\beta}(z_{ac}) f^m_{\beta}(z_{ac}) = E^m_{2\beta}, \tag{C.37} \]
\[ \varphi^k_{\beta}(z_{ac}) \varphi^{k}_{\beta} - \varphi^{k}_{\beta}(z_{ac}) f^0_{\beta}(z_{ac}) + E_1(z_{ac}) f^k_{\beta}(z_{ac}) = \frac{1}{2} \partial_u f^k_{\beta}(z_{ac}). \tag{C.38} \]

The last one follows from
\[ \partial_u \phi(u, z) = \phi(u, z)(E_1(z + u) - E_1(u)) \tag{C.39} \]
and
\[ (E_1(z + u) - E_1(u) - E_1(z))^2 = \varphi(z) + \varphi(u) + \varphi(z + u). \tag{C.40} \]

**Modular properties.** Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), that is, \( a, b, c, d \in \mathbb{Z} \) and \( \text{ad} - bc = 1 \). Then
\[ \theta \left( \frac{z}{c\tau + d} \bigg| \frac{a\tau + b}{c\tau + d} \right) = \varepsilon e^{\pi i/4(c\tau + d)^{1/2}} \exp \left( \frac{i\pi cz^2}{c\tau + d} \right) \theta(z \mid \tau) \quad (\varepsilon^8 = 1), \tag{C.41} \]
\[ E_1 \left( \frac{z}{c\tau + d} \bigg| \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d) E_1(z \mid \tau) + 2\pi icz, \tag{C.42} \]
\[ E_2 \left( \frac{z}{c\tau + d} \bigg| \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(z \mid \tau) - 2\pi ic(c\tau + d), \tag{C.43} \]
\[ \phi \left( \frac{u}{c\tau + d}, \frac{z}{c\tau + d} \bigg| \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d) \exp \left( 2\pi i \frac{czu}{c\tau + d} \right) \phi(u, z \mid \tau). \tag{C.44} \]

**Appendix D. Characteristic classes and conformal groups**

In this appendix we briefly describe the construction in [45] of conformal groups.

**Conformal groups.** Here we introduce an analogue of the group \( \text{GL}(N, \mathbb{C}) \) for simple groups other than \( \text{SL}(N, \mathbb{C}) \). Let
\[ \phi: \mathcal{Z}(\overline{G}) \hookrightarrow (\mathbb{C}^*)^r \tag{D.1} \]
be the embedding of the centre \( \mathcal{Z}(\overline{G}) \) in the algebraic torus \( (\mathbb{C}^*)^r \) of minimum dimension \( (r = 1) \) for a cyclic centre and \( (r = 2) \) for \( \mu_2 \times \mu_2 \). We note that any two embeddings of \( \mathcal{Z}(\overline{G}) \) \( (\overline{G} \neq \text{SL}(N, \mathbb{C})) \) are left-conjugate: \( \phi_1 = A \phi_2 \) for some automorphism \( A \) of the torus \( (\mathbb{C}^*)^r \). For these groups we deal with \( \mu_2, \mu_3, \mu_4, \) or \( \mu_2 \times \mu_2 \). In these cases the non-trivial roots of unity either coincide or are inverse to each other. In the latter case \( A: x \rightarrow x^{-1} \).

Consider the ‘anti-diagonal’ embedding \( \mathcal{Z}(\overline{G}) \rightarrow \overline{G} \times (\mathbb{C}^*)^r, \zeta \mapsto (\zeta, \phi(\zeta)^{-1}), \zeta \in \mathcal{Z}(\overline{G}) \). The image of this map is a normal subgroup, since \( \mathcal{Z} \) is the centre of \( \overline{G} \).

**Definition D.1.** The quotient
\[ \overline{C} \overline{G} = (\overline{G} \times (\mathbb{C}^*)^r) / \mathcal{Z}(\overline{G}) \]
is called the conformal version of \( \overline{G} \).
In a similar way the conformal version can be defined for any \( G \) with a non-trivial centre. If the centre of \( G \) is trivial, as for \( G^{\text{ad}} \), then \( CG = G \times \mathbb{C}^* \).

The group \( CG \) does not depend on the way the centre is embedded in \( \mathbb{C}^r \), because of the above remark about the conjugacy of \( \phi \) maps. We have a natural embedding \( CG \subseteq C\mathcal{G} \). Consider the quotient torus \( Z^\vee = (\mathbb{C}^*)^r / \mathcal{Z}(\mathcal{G}) \sim (\mathbb{C}^*)^r \). The last isomorphism is given by \( \lambda \rightarrow \lambda^N \) for a cyclic centre and \( (\lambda_1, \lambda_2) \rightarrow (\lambda_2^1, \lambda_2^2) \) for \( D_{\text{even}} \). The sequence

\[
1 \rightarrow G \rightarrow CG \rightarrow Z^\vee \rightarrow 1 \tag{D.2}
\]

is an analogue of the sequence

\[
1 \rightarrow \text{SL}(N, \mathbb{C}) \rightarrow \text{GL}(N, \mathbb{C}) \rightarrow \mathbb{C}^* \rightarrow 1.
\]

On the other hand, we have the embedding \( (\mathbb{C}^*)^r \rightarrow C\mathcal{G} \) with the quotient \( C\mathcal{G}/(\mathbb{C}^*)^r = G^{\text{ad}} \). Then the sequence

\[
1 \rightarrow (\mathbb{C}^*)^r \rightarrow C\mathcal{G} \rightarrow G^{\text{ad}} \rightarrow 1 \tag{D.3}
\]

is similar to the sequence

\[
1 \rightarrow \mathbb{C}^* \rightarrow \text{GL}(N, \mathbb{C}) \rightarrow \text{PGL}(N, \mathbb{C}) \rightarrow 1.
\]

Let \( \pi \) be an irreducible representation of \( \mathcal{G} \) and let \( \chi \) be a character of the torus \( (\mathbb{C}^*)^r \). It follows from (D.2) that an irreducible representation \( \widetilde{\pi} \) of \( C\mathcal{G} \) is defined by

\[
\widetilde{\pi} = \pi \boxtimes \chi((\mathbb{C}^*)^r) \quad \text{such that} \quad \pi_{|_{\mathcal{Z}(\mathcal{G})}} = \chi \phi \quad (\phi \text{ was defined in (D.1)}). \tag{D.4}
\]

Assume for simplicity that \( \pi \) is a fundamental representation. This means that the highest weight \( \nu \) of \( \pi \) is a fundamental weight. Let \( \varpi^\vee \) be a fundamental coweight generating \( \mathcal{Z}(\mathcal{G}) \) for \( r = 1 \). In other words, \( \zeta = e(\varpi^\vee) \) is a generator of \( \mathcal{Z}(\mathcal{G}) \) \( (\zeta^N = 1, \ N = \text{ord}(\mathcal{Z}(\mathcal{G}))) \). Then \( \pi_{|_{\mathcal{Z}(\mathcal{G})}} \) acts as a scalar \( e(\langle \varpi^\vee, \nu \rangle) \). The highest weight can be expanded in the basis of simple roots: \( \nu = \sum_{\alpha \in \Pi} c_\alpha^\nu \alpha \). Then the coefficients \( c_\alpha^\nu \) are rows of the inverse Cartan matrix. They have the form \( k/N \), where \( k \) is an integer. Therefore, the scalar

\[
e(\langle \varpi^\vee, \nu \rangle) = e\left( \sum_{\alpha \in \Pi} c_\alpha^\nu \delta(\varpi^\vee, \alpha) \right) \tag{D.5}
\]

is a root of unity. On the other hand, let \( \chi_m(\mathbb{C}^*) = w^m \ (w \in \mathbb{C}^*) \) be a character of \( \mathbb{C}^* \) and let \( \phi(\zeta) = e(l/N) \). In terms of weights the definition of \( \widetilde{\pi} \) (D.4) takes the form \( e(\langle \varpi^\vee, \nu \rangle) = e(ml/N) \). It follows from this construction that the characters of the group \( C\mathcal{G} \) are defined by the weight lattice \( P \) and the integer lattice \( \mathbb{Z} \), with the additional condition

\[
\chi_{(\gamma, m)}(x, w) = \exp(2\pi i \langle \gamma, x \rangle)w^m, \quad \langle \gamma, \varpi^\vee \rangle = \frac{ml}{N} + j, \quad \gamma \in P, \ m, j \in \mathbb{Z}, \ x \in \mathfrak{h}.
\]

The case of \( D_{\text{even}} \) (for example, \( r = 2 \)) can be treated in a similar way.
Remark D.1. Simple groups can be defined as subgroups of \( \text{GL}(V) \) preserving certain multilinear forms on \( V \). For example, in the defining representations these forms are bilinear symmetric forms for SO, bilinear antisymmetric forms for Sp, trilinear forms for \( E_6 \), and forms of fourth order for \( E_7 \). In a generic situation \( G \) is defined as a subgroup of \( \text{GL}(V) \) preserving a three-tensor in \( V^* \otimes V^* \otimes V \) [123]. The conformal versions of these groups can be alternatively defined as transformations preserving the forms up to dilations. We prefer to use here the algebraic construction, which justifies the term ‘conformal version’.

Conformal versions can also be defined in terms of faithful representations of \( \overline{G} \). Let \( V \) be such a representation space and assume that \( \mathcal{X}(\overline{G}) \) is a cyclic group. Then \( \mathcal{CG} \) is the subgroup of \( \text{GL}(V) \) generated by \( G \) and the dilations of \( \mathbb{C}^* \). The character \( \det V \) is equal to \( \lambda \dim V \), where \( \lambda \) is equal to (D.5) for fundamental representations.

For \( D_{\text{even}} \) we use two representations, for example, the left and right spinors \( \text{Spin}^{L,R} \). The conformal group \( C\text{Spin}_{4k} \) is the subgroup of \( \text{GL}(\text{Spin}^{L} \oplus \text{Spin}^{R}) \) generated by \( \text{Spin}_{4k} \) and \( \mathbb{C}^* \times \mathbb{C}^* \), where the first factor \( \mathbb{C}^* \) acts by dilations on \( \text{Spin}^{L} \), and the second factor acts on \( \text{Spin}^{R} \). The characters \( \det \text{Spin}^{L} \) and \( \det \text{Spin}^{R} \) are equal to \( \lambda_1 \dim(\text{Spin}_{4k}^{L}) \) and \( \lambda_2 \dim(\text{Spin}_{4k}^{R}) \), respectively (\( \dim(\text{Spin}_{4k}^{L,R}) = 2^{2k-1} \)).

Characteristic classes and degrees of vector bundles. From the exact sequence (D.3) and the vanishing \( H^2(\Sigma, \theta^*) = 0 \) of the second cohomology of \( \Sigma \) with coefficients in an analytic sheaf we get that any \( G^{ad}(\theta) \)-bundle (even a topologically non-trivial one, with \( \zeta(G^{ad}(\theta)) \neq 0 \)) can be lifted to a \( \mathcal{CG}(\theta) \)-bundle.

Let \( V \) be a faithful representation space (irreducible or the sum \( \text{Spin}^{L} \oplus \text{Spin}^{R} \) for \( D_{2k} \)). Then from (D.1) we have an embedding of \( \mathcal{X}(\overline{G}) \) in the automorphisms of \( V \):

\[
\phi_V : \mathcal{X}(\overline{G}) \hookrightarrow (\mathbb{C}^*)^r = \text{Aut}_{\overline{G}}(V). \tag{D.6}
\]

In the particular case when \( V \) is a fundamental representation space the centre acts by multiplication by (D.5). Let \( \mathcal{P}_{\overline{G}} \) be a principal \( \mathcal{CG}(\theta) \)-bundle. Denote by \( E(V) = \mathcal{P}_{\overline{G}} \otimes_{\overline{G}} V \) (or \( E(\text{Spin}^{L,R}) \)) the vector bundle induced by the representation space \( V \) (\( \text{Spin}^{L,R} \) for \( D_{\text{even}} \)).

Theorem D.1. Let \( E_{\text{ad}} = E(\text{Ad}) \) be the adjoint bundle with characteristic class \( \zeta(E_{\text{ad}}) \). The image of \( \zeta(E_{\text{ad}}) \) under the action of \( \phi_V \) (D.6) is

\[
\phi_V(\zeta(E_{\text{ad}})) = \begin{cases} 
\exp \left( -\frac{2\pi i \deg(E_{\overline{G}}(V))}{\dim V} \right), & \mathcal{X}(\overline{G}) \text{ is cyclic}, \\
\exp \left( -\frac{2\pi i \deg(E_{\text{Spin}^{L,R}})}{2^{2k-1}} \right), & \mathcal{X}(\overline{G}) \text{ is cyclic}.
\end{cases}
\]

For \( G = \text{GL}(N, \mathbb{C}) \) this theorem was proved in [82].
Proof. Consider the commutative diagram

\[
\begin{array}{cccc}
1 & 1 & & 1 \\
& & & & \\
1 & \rightarrow & Z^\vee(\partial_\Sigma) & \rightarrow & Z^\vee(\partial_\Sigma) & \rightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & \left[\mu \right] & & \left[\mu \right] & & \left[\mu \right] \\
1 & \rightarrow & \left(\partial_\Sigma^*\right)^r & \rightarrow & C\tilde{G}(\partial_\Sigma) & \rightarrow & G^{\text{ad}}(\partial_\Sigma) & \rightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & \left[\mu \right] & & \left[\mu \right] & & \left[\mu \right] & & \left[\mu \right] \\
1 & \rightarrow & \mathcal{X}(G) & \rightarrow & G(\partial_\Sigma) & \rightarrow & G^{\text{ad}}(\partial_\Sigma) & \rightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & \left[\mu \right] & & \left[\mu \right] & & \left[\mu \right] & & \left[\mu \right] \\
& & 1 & & 1 & & 1 & & 1
\end{array}
\]

and the corresponding diagram of Čech cochains. Let \(\psi\) be a 1-cocycle with values in \(G^{\text{ad}}(\partial_\Sigma)\). We consider its pre-image as a cocycle with values in \(G^{\text{ad}}(\partial_\Sigma)\). By the definition of \(C\tilde{G}\) this cocycle is a pair of cochains \((\Psi, \nu)\) with values in \(G(\partial_\Sigma)\) and \(\left(\partial_\Sigma^*\right)^r\) such that \(\phi_V(d\Psi) \cdot d\nu = 1 \in \left(\partial_\Sigma^*\right)^r\), where \(d\) is the Čech coboundary operator. The cohomology class of \(d\Psi\) by definition is the characteristic class \(c\), that is, \(\phi_V\) of it is the inverse of the class of \(d\nu\): \(\phi_V(\zeta(E_{\text{ad}})) = (d\nu)^{-1}\). Since \(\nu\) acts in \(V\) as the scalar \(\nu^{\text{dim}(V)}\), it is a 1-cocycle as a determinant of this action. It represents the determinant of the line bundle \(E(V)\). Hence, \(\nu\) is a pre-image of the cocycle \(\nu^{\text{dim}(V)}\) under the map taking the \(N = \text{dim}(V)\)th power \(\partial_\Sigma^* \rightarrow \partial_\Sigma^*, \nu \rightarrow \nu^N\), \(N = \text{dim}(V)\).

We consider the long exact sequence

\[1 \rightarrow \mu_N \rightarrow \partial_\Sigma^* \leftarrow \left[\mu \right] \rightarrow \partial_\Sigma^* \rightarrow 1 \quad (\mu_N = \mathbb{Z}/N\mathbb{Z}).\]

It induces a map \(H^1(\Sigma, \partial_\Sigma^*) \rightarrow H^2(\Sigma, \mu_N)\). The cocycle \(d\nu\) lies in the cohomology class which is the image of the class of \(\text{det} E(V) = \nu^N\) under the coboundary map \(H^1(\Sigma, \partial_\Sigma^*) \rightarrow H^2(\Sigma, \mu_N)\). We denote it by \(\text{Inv}_N = \text{Image}(\text{det} E(V))\). Thus, by definition, the class of \(d\nu\) is equal to \(\text{Inv}_N(\text{det} E(V)) = \text{Inv}_N(\zeta_1(E(V)))\).

The assertion of the theorem is a consequence of the following proposition.

**Proposition D.1.** Let \(\gamma\) be a 1-cocycle with values in \(\partial_\Sigma^*\). Then \(\text{Inv}_N(\gamma) = \exp((1/N) \cdot 2\pi i \text{deg}(\gamma))\).

**Proof.** Consider the diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \mu_N & \rightarrow & \partial_\Sigma^* \leftarrow \left[\mu \right] \rightarrow \partial_\Sigma^* & \rightarrow & 0 \\
& & \uparrow{\exp} & & \uparrow{\exp} & & \uparrow & & \uparrow \\
0 & \rightarrow & \partial_\Sigma & \times N & \rightarrow & \partial_\Sigma & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & 2\pi i\mathbb{Z} & \times N & \rightarrow & 2\pi i\mathbb{Z}
\end{array}
\]
Let $\gamma$ be a 1-cocycle of $\mathcal{O}_\Sigma^\ast$. By definition, its image in $H^2(X, \mu_N)$ is equal to the coboundary of the 1-cochain $\gamma^{1/N}$ of $\mathcal{O}_\Sigma^\ast$, $(\gamma^{1/N})^N = \gamma$. Let $\log(\gamma)$ be the pre-image of the cycle $\gamma$ under the exponential map; $\log(\gamma)$ is a 1-cochain of $\mathcal{O}_\Sigma^\ast$, and its coboundary equals the degree of $\gamma$ times $2\pi i$. Since multiplication by $N$ is invertible on $\mathcal{O}_\Sigma^\ast$, the cochain $(1/N)\log(\gamma)$ is well-defined. By the commutativity of the diagram we can choose $\exp((1/N)\log(\gamma))$ as $\gamma^{1/N}$. Hence, the image of $\gamma$ in $H^2(X, \mu_N)$ equals the coboundary of $\exp((1/N)\log(\gamma))$.

The case $r = 2$, can be treated similarly. □

As above, let $\varpi^\lor$ be a fundamental coweight generating the centre $\mathcal{Z}(G)$ and let $\nu$ be the weight of the representation of $G$ on $V$. Then it follows from Theorem D.1 and (D.5) that

$$\deg E(V) = \dim(V)(\langle \varpi^\lor, \nu \rangle + k), \quad k \in \mathbb{Z}. \quad (D.7)$$

It follows from our considerations that the transition matrix change

$$\Lambda \to \tilde{\Lambda}(z) = e^{\left(\langle \varpi^\lor, \nu \rangle \left(z + \frac{\tau}{2}\right)\right)} \Lambda$$

defines the bundle of the conformal group $CG$ of degree (D.7). For a fundamental representation of $G$ the formula (D.7) is illustrated in Table 5.

Table 5. The degrees of bundles for conformal groups. $\text{Mp}_n(\mathbb{C})$ is the universal covering of $\text{Sp}_n(\mathbb{C})$

| $G$      | $\nu$  | $V$   | $\deg(E(V))$         |
|----------|--------|-------|-----------------------|
| $\text{SL}(n, \mathbb{C})$ | $\varpi^\lor$ | $n$    | $-1 + kn$            |
| $\text{Spin}_{2n+1}(\mathbb{C})$ | $\varpi^\lor$ | $2^n$  | $2^{n-1}(1 + 2k)$    |
| $\text{Mp}_n(\mathbb{C})$     | $\varpi^\lor$ | $2n$   | $n(1 + 2k)$          |
| $\text{Spin}^L_{4n}(\mathbb{C})$ | $\varpi^\lor_{n,n-1}$ | $2^{2n-1}$ | $2^{2n-2}(1 + 2k)$ |
| $\text{Spin}_{4n+2}(\mathbb{C})$ | $\varpi^\lor$ | $2^n$  | $2^{n-2}(1 + 4k)$   |
| $E_6(\mathbb{C})$             | $\varpi^\lor$ | $27$   | $9(1 + 3k)$          |
| $E_7(\mathbb{C})$             | $\varpi^\lor$ | $56$   | $28(1 + 2k)$         |

$(k \in \mathbb{Z})$

Bibliography

[1] P. Painlevé, “Mémoire sur les équations différentielles dont l’intégrale générale est uniforme”, Bull. Soc. Math. France 28 (1900), 201–261.
[2] P. Painlevé, “Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme”, Acta Math. 25:1 (1902), 1–85.
[3] B. Gambier, “Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critique fixes”, Acta Math. 33:1 (1910), 1–55.
Classification of isomonodromy problems on elliptic curves

[4] R. Fuchs, “Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen”, Math. Ann. 63:3 (1907), 301–321.

[5] R. Garnier, “Sur des équations différentielles du troisième ordre dont l’intégrale générale est uniforme et sur une classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses points critique fixés”, Ann. Sci. École Norm. Sup. (3) 29 (1912), 1–126.

[6] L. Schlesinger, “Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten”, J. Reine Angew. Math. 141 (1912), 96–145.

[7] H. Flaschka and A. C. Newell, “Monodromy — and spectrum-preserving deformations. I”, Comm. Math. Phys. 76:1 (1980), 65–116.

[8] H. Flaschka and A. C. Newell, “Multiphase similarity solutions of integrable evolution equations”, Phys. D 3:1-2 (1981), 203–221.

[9] M. Jimbo, T. Miwa, and K. Ueno, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function”, Phys. D 2:2 (1981), 306–352.

[10] M. Jimbo and T. Miwa, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II”, Phys. D 2:3 (1981), 407–448.

[11] M. Jimbo and T. Miwa, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III”, Phys. D 4:1 (1981), 26–46.

[12] E. L. Ince, Ordinary differential equations, Dover Publications, New York 1956, viii+558 pp.

[13] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida, From Gauss to Painlevé. A modern theory of special functions, Aspects Math., vol. E16, Friedr. Vieweg & Sohn, Braunschweig 1991, xii+347 pp.

[14] R. Conte (ed.), The Painlevé property. One century later, CRM Ser. Math. Phys., Springer-Verlag, New York 1999, xxvi+810 pp.

[15] A.R. Its and V.Yu. Novokshenov, The isomonodromic deformation method in the theory of Painlevé equations, Lecture Notes in Math., vol. 1191, Springer-Verlag, Berlin 1986, iv+313 pp.

[16] A.S. Fokas, A.R. Its, A.A. Kapaev, and V. Yu. Novokshenov, Painlevé transcendents. The Riemann–Hilbert approach, Math. Surveys Monogr., vol. 128, Amer. Math. Soc., Providence, RI 2006, xii+553 pp.

[17] D. Levi and P. Winternitz (eds.), Painlevé transcendents. Their asymptotics and physical applications, Proceedings of the NATO Advanced Research Workshop (Sainte-Adèle, Canada 1990), NATO Adv. Sci. Inst. Ser. B: Phys., vol. 278, Springer-Verlag, New York, NY 1992, xxvi+446 pp.

[18] B. Dubrovin, “Geometry of 2D topological field theories”, Integrable systems and quantum groups (Montecatini Terme 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin 1996, pp. 120–348.

[19] А.А. Болибрух, Фуксовы дифференциальны уравнения и голоморфны расслоения, МЦНМО, М. 2000, 127 с. [A.A. Bolibruch, Fuchsian differential equations and holomorphic bundles, Moscow Centre for Continuous Mathematical Education, Moscow 2000, 127 pp.]

[20] A.A. Bolibruch, “On isomonodromic deformations of Fuchsian systems”, J. Dynam. Control Syst. 3:4 (1997), 589–604.

[21] S.Yu. Slavyanov and W. Lay, Special functions. A unified theory based on singularities, with a foreword by A. Seeger, Oxford Math. Monogr., Oxford Univ. Press, Oxford 2000, xvi+293 pp.
112

A. M. Levin, M. A. Olshanetsky, and A. V. Zotov

[22] I. Krichever, “Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations”, *Mosc. Math. J.* 2:4 (2002), 717–752.

[23] P. Boalch, “From Klein to Painlevé via Fourier, Laplace and Jimbo”, *Proc. London Math. Soc.* (3) 90:1 (2005), 167–208.

[24] Р. Р. Гонцов, В. А. Побережный, Г. Ф. Хельминк, “Деформации систем линейных дифференциальных уравнений”, *УМН* 66:1(397) (2011), 65–110; English transl., R. R. Gontsov, V. A. Poberezhnyi, and G. F. Helminck, “On deformations of linear differential systems”, *Russian Math. Surveys* 66:1 (2011), 63–105.

[25] K. Okamoto, “On Fuchs’ problem on a torus”, *Japan–United States seminar on ordinary differential and functional equations* (Kyoto 1971), Lecture Notes in Math., vol. 243, Springer, Berlin 1971, pp. 277–280.

[26] K. Iwasaki, “Moduli and deformation for Fuchsian projective connections on a Riemann surface”, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 38:3 (1991), 431–531.

[27] S. Kawai, *Deformation of complex structures on a torus and monodromy preserving deformation*, preprint, University of Tokyo 1995.

[28] A. M. Levin and M. A. Olshanetsky, *Classical limit of the Knizhnik–Zamolodchikov–Bernard equations as hierarchy of isomonodromic deformations. Free fields approach*, 1997 (v3 – 1998), 40 pp., arXiv:hep-th/9709207.

[29] M. A. Olshanetsky, “Generalized Hitchin systems and the Knizhnik–Zamolodchikov–Bernard equation on elliptic curves”, *Lett. Math. Phys.* 42:1 (1997), 59–71; 1995, 11 pp., arXiv:hep-th/9510143.

[30] A. Levin and M. Olshanetsky, “Hierarchies of isomonodromic deformations and Hitchin systems”, *Moscow seminar in mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 191, Amer. Math. Soc., Providence, RI 1999, pp. 223–262.

[31] N. Hitchin, “Stable bundles and integrable systems”, *Duke Math. J.* 54:1 (1987), 91–114.

[32] N. Nekrasov, “Holomorphic bundles and many-body systems”, *Comm. Math. Phys.* 180:3 (1996), 587–603; 1995, 21 pp., arXiv:hep-th/9503157.

[33] A. Gorsky and N. Nekrasov, *Elliptic Calogero–Moser system from two dimensional current algebra*, 1994, 8 pp., arXiv:hep-th/9401021.

[34] B. Enriquez and V. Rubtsov, “Hitchin systems, higher Gaudin operators and r-matrices”, *Math. Res. Lett.* 3:3 (1996), 343–357.

[35] A. Levin and M. Olshanetsky, “Double coset construction of moduli space of holomorphic bundles and Hitchin systems”, *Commun. Math. Phys.* 188:2 (1997), 449–466; 1996, 19 pp., arXiv:alg-geom/9605005.

[36] D. Korotkin and H. Samtleben, “On the quantization of isomonodromic deformations on the torus”, *Internat. J. Modern Phys. A* 12:11 (1997), 2013–2029.

[37] G. Kuroki and T. Takebe, “Twisted Wess–Zumino–Witten models on elliptic curves”, *Comm. Math. Phys.* 190:1 (1997), 1–56; 1996 (v4 – 1997), 55 pp., arXiv:q-alg/9602033.

[38] K. Takasaki, “Gaudin model, KZ equation, and isomonodromic problem on torus”, *Lett. Math. Phys.* 44:2 (1998), 143–156; 1997, 15 pp., arXiv:hep-th/9711058.

[39] J. Harnad and M. A. Wisse, “Loop algebra moment maps and Hamiltonian models for the Painlevé transcendent”, *Mechanics day* (Waterloo, ON 1992), Fields Inst. Commun., vol. 7, Amer. Math. Soc., Providence, RI 1996, pp. 155–169; 1993, 14 pp., arXiv:hep-th/9305027.

[40] А. Г. Рейман, М. А. Семенов-Тян-Шанский, “Алгебры Ли и лаксовы уравнения со спектральным параметром на эллиптической кривой”, *Вопросы квантовой теории поля и статистической физики*. 6, Зап. науч. сем. ЛОМИ, 150,
Classification of isomonodromy problems on elliptic curves

113

Изд-во “Наука”, Ленинград. отд., Л. 1986, с. 104–118; English transl., A.G. Reiman and M.A. Semenov-Tian-Schanskii, “Lie algebras and Lax equations with spectral parameter on elliptic curve”, J. Soviet Math. 46:1 (1989), 1631–1640.

[41] И.М. Кричевер, “Эллиптические решения уравнения Кадомцева–Петвиашвили и интегрируемые системы частиц”, Функц. анализ и его прил. 14:4 (1980), 45–54; English transl., I.M. Krichever, “Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles”, Funct. Anal. Appl. 14:4 (1980), 282–290.

[42] Yu. Chernyakov, A.M. Levin, M. Olshanetsky, and A. Zotov, “Elliptic Schlesinger system and Painlevé VI”, J. Phys. A 39:39 (2006), 12083–12101; 2006, 16 pp., arXiv:nlin/0602043.

[43] M.A. Olshanetsky and A.V. Zotov, “Isomonodromic problems on elliptic curve, rigid tops and reflection equations”, Elliptic integrable systems, Rokko Lectures in Math., vol. 18, Kobe Univ., Kobe, Japan 2005, pp. 149–171.

[44] A.M. Levin and M.A. Olshanetsky, “Painlevé–Calogero correspondence”, Calogero–Moser–Sutherland models (Montréal, QC 1997), CRM Ser. Math. Phys., Springer, New York 2000, pp. 313–332; 1997, 17 pp., arXiv:alg-geom/9706010.

[45] A. Levin, M. Olshanetsky, A. Smirnov, and A. Zotov, “Characteristic classes and Hitchin systems. General construction”, Comm. Math. Phys. 316:1 (2012), 1–44; Characteristic classes and integrable systems. General construction, 2010, 52 pp., arXiv:1006.0702.

[46] A. Levin, M. Olshanetsky, A. Smirnov, and A. Zotov, “Calogero–Moser systems for simple Lie groups and characteristic classes of bundles”, J. Geom. Phys. 62:8 (2012), 1810–1850; Characteristic classes and integrable systems for simple Lie groups, 2010, 51 pp., arXiv:1007.4127.

[47] A.V. Zotov, A.V. Smirnov, “Модификации расслоений, эллиптические интегрируемые системы и связанные задачи”, TMФ 177:1 (2013), 3–67; English transl., A.V. Zotov and A.V. Smirnov, “Modifications of bundles, elliptic integrable systems, and related problems”, Theoret. and Math. Phys. 177:1 (2013), 1281–1338.

[48] A.M. Levin, M.A. Olshanetsky, A.V. Smirnov, and A.V. Zotov, “Hecke transformations of conformal blocks in WZW theory. I. KZB equations for non-trivial bundles”, SIGMA 8 (2012), 095, 37 pp.; 2012, 37 pp., arXiv:1207.4386.

[49] A.M. Levin, M.A. Olshanetsky, and A.V. Zotov, “Monopoles and modifications of bundles over elliptic curves”, SIGMA 5 (2009), 022, 12 pp.; 2008 (v2 – 2009), 22 pp., arXiv:0811.3056.

[50] M.F. Atiyah, “Vector bundles over an elliptic curve”, Proc. London Math. Soc. (3) 7 (1957), 414–452.

[51] A.V. Zotov, A.M. Levin, “Интегрируемая система взаимодействующих эллиптических волчков”, TMФ 146:1 (2006), 55–64; English transl., A.V. Levin and A.M. Zotov, “Integrable model of interacting elliptic tops”, Theoret. and Math. Phys. 146:1 (2006), 45–52.

[52] A.M. Levin, M.A. Olshanetsky, A.V. Smirnov, and A.V. Zotov, “Characteristic classes of SL(N, C)-bundles and quantum dynamical elliptic R-matrices”, J. Phys. A 46:3 (2013), 035201, 25 pp.; 2012, 27 pp., arXiv:1208.5750.

[53] E. Franco, O. Garcia-Prada, and P.E. Newstead, Higgs bundles over elliptic curves, 2013, 41 pp., arXiv:1302.2881.

[54] N. Hitchin, Higgs bundles and characteristic classes, 2013, 18 pp., arXiv:1308.4603.
[55] D. Baraglia and L. P. Schaposnik, *Real structures on moduli spaces of Higgs bundles*, 2013, 19 pp., arXiv:1309.1195.

[56] M. F. Atiyah and R. Bott, “The Yang–Mills equations over Riemann surfaces”, *Philos. Trans. Roy. Soc. London Ser. A* 308:1505 (1983), 523–615.

[57] V. G. Drinfeld and C. Simpson, “B-structures on G-bundles and local triviality”, *Math. Res. Lett.* 2:6 (1995), 823–829.

[58] A. Beauville and Y. Laszlo, “Un lemme de descente”, *C. R. Acad. Sci. Paris Sér. I Math.* 320:3 (1995), 335–340.

[59] D. Arinkin and S. Lysenko, “Isomorphisms between moduli spaces of SL(2)-bundles with connections on \( \mathbb{P}^1 \setminus \{x_1, \ldots, x_4\} \)”, *Math. Res. Lett.* 4:2 (1997), 181–190.

[60] D. Arinkin and S. Lysenko, “On the moduli spaces of SL(2)-bundles with connections on \( \mathbb{P}^1 \setminus \{x_1, \ldots, x_4\} \)”, *Int. Math. Res. Not.* 1997:19 (1997), 983–999.

[61] A. M. Levin, M. A. Olshanetsky, and A. Zotov, “Hitchin systems–symplectic Hecke correspondence and two-dimensional version”, *Comm. Math. Phys.* 236:1 (2003), 93–133; 2001 (v3 – 2002), 39 pp., arXiv:math/0110045.

[62] P. Painlevé, “Sur les équations différentielles du second ordre à points critiques fixes”, *C. R. Acad. Sci. (Paris)* 143 (1906), 1111–1117.

[63] Yu. I. Manin, “Sixth Painlevé equation, universal elliptic curve, and mirror of \( \mathbb{P}^2 \)”, *Geometry of differential equations*, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI 1998, pp. 131–151.

[64] A. M. Levin, M. A. Olshanetsky, and A. Zotov, “Hitchin systems–symplectic Hecke correspondence and two-dimensional version”, *Comm. Math. Phys.* 268:1 (2006), 67–103; 2005 (v2 – 2006), 32 pp., arXiv:math/0508058.

[65] A. Zotov, “Elliptic linear problem for Calogero–Inozemtsev model and Painlevé VI equation”, *Lett. Math. Phys.* 67:2 (2004), 153–165; 2003, 13 pp., arXiv:hep-th/0310260.

[66] P. Boutroux, “Recherches sur les transcendantes de M. Painlevé et l’étude asymptotique des équations différentielles du second ordre”, *Ann. Sci. École Norm. Super.* (3) 30 (1913), 255–375; suite 31 (1914), 99–159.

[67] R. Garnier, “Étude de l’intégrale générale de l’équation VI de M. Painlevé dans le voisinage de ses singularités transcendantes”, *Ann. Sci. École Norm. Sup.* (3) 34 (1917), 239–353.

[68] I. M. Krichever, “Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations”, *Mosc. Math. J.* 2:4 (2002), 717–752; 2001, 38 pp., arXiv:hep-th/0112096.

[69] C. T. Simpson, “Harmonic bundles on noncompact curves”, *J. Amer. Math. Soc.* 3:3 (1990), 713–770.

[70] A. Pressley and G. Segal, *Loop groups*, Oxford Math. Monogr., Oxford Univ. Press, New York 1986, viii+318 pp.

[71] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, Cambridge 1990, 400 pp.

[72] D. Arinkin, *Moduli of connections with a small parameter on a curve*, 2004, 19 pp., arXiv:math/0409373.

[73] В. И. Арнольд, *Математические методы классической механики*, Наука, М. 1974, 431 с.; English transl., V. Arnold, *Mathematical methods of classical mechanics*, Grad. Texts in Math., vol. 60, Springer-Verlag, New York–Heidelberg 1978, x+462 pp.

[74] И. М. Кричевер, “Метод усреднения для двумерных «интегрируемых» уравнений”, *Функц. анализ и его прил.* 22:3 (1988), 37–52; English transl.,
I. M. Krichever, “Method of averaging for two-dimensional “integrable” equations”, *Funct. Anal. Appl.* 22:3 (1988), 200–213.

[75] K. Okamoto, “On the τ-function of the Painlevé equations”, *Phys. D* 2:3 (1981), 525–535.

[76] K. Takasaki, “Spectral curves and Whitham equations in isomonodromic problems of Schlesinger type”, *Asian J. Math.* 2:4 (1998), 1049–1078; 1997, 41 pp., arXiv: solv-int/9704004.

[77] E. Looijenga, “Root systems and elliptic curves”, *Invent. Math.* 38:1 (1976/77), 17–32.

[78] И.Н. Бернштейн, О.В. Шварцман, “Теорема Шевалле для комплексных кристаллографических кокстеровских групп”, Функц. анализ и его прил. 12:4 (1978), 79–80; English transl., I. N. Bernstein and O. V. Schwarzman, “Chevalley’s theorem for complex crystallographic Coxeter groups”, *Funct. Anal. Appl.* 12:4 (1978), 308–310.

[79] J. Bernstein and O. Schwarzman, “Complex crystallographic Coxeter groups and affine root systems”, *J. Nonlinear Math. Phys.* 13:2 (2006), 163–182.

[80] R. Friedman and J. W. Morgan, *Holomorphic principal bundles over elliptic curves*, 1998, 68 pp., arXiv: math/9811130.

[81] C. Schweigert, “On moduli spaces of flat connections with non-simply connected structure group”, *Nuclear Phys. B* 492:3 (1997), 743–755.

[82] M. S. Narasimhan and C. S. Seshadri, “Stable and unitary vector bundles on a compact Riemann surface”, *Ann. of Math.* (2) 82:3 (1965), 540–567.

[83] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 4–6, transl. from the 1968 French original by A. Pressley, *Elem. Math.* (Berlin), Springer-Verlag, Berlin 2002, xi+300 pp.

[84] P. A. M. Dirac, *Lectures on quantum mechanics*, Belfer Grad. Sch. Sci. Monogr. Ser., vol. 2, Academic Press, Inc., New York 1967, v+87 pp.

[85] H. W. Braden, V. A. Dolgushev, M. A. Olshanetsky, and A. V. Zotov, “Classical r-matrices and the Feigin–Odesskii algebra via Hamiltonian and Poisson reductions”, *J. Phys. A* 36:5 (2003), 6979–7000; 2003, 27 pp., arXiv: hep-th/0301121.

[86] P. Etingof and A. Varchenko, “Geometry and classification of solutions of the classical dynamical Yang–Baxter equation”, *Comm. Math. Phys.* 192:1 (1998), 77–120.

[87] O. Schiffmann, “On classification of dynamical r-matrices”, *Math. Res. Lett.* 5:1-2 (1998), 13–30.

[88] P. Etingof and O. Schiffmann, “Twisted traces of intertwiners for Kac–Moody algebras and classical dynamical r-matrices corresponding to generalized Belavin–Drinfeld triples”, *Math. Res. Lett.* 6:5-6 (1999), 593–612; 1999, 18 pp., arXiv: math/9908115.

[89] L. Fehér and B. G. Pusztai, “Generalizations of Felder’s elliptic dynamical r-matrices associated with twisted loop algebras of self-dual Lie algebras”, *Nuclear Phys. B* 621:3 (2002), 622–642; 2001, 22 pp., arXiv: math/0109132.

[90] O. Babelon and C.-M. Viallet, “Hamiltonian structures and Lax equations”, *Phys. Lett. B* 237:3-4 (1990), 411–416.

[91] E. Billey, J. Avan, and O. Babelon, “The r-matrix structure of the Euler–Calogero–Moser model”, *Phys. Lett. A* 186:1-2 (1994), 114–118.

[92] K. Takasaki, “Painlevé–Calogero correspondence revisited”, *J. Math. Phys.* 42:3 (2001), 1443–1473.
[93] Б. И. Сулейманов, “Гамильтоновость уравнений Пенлеве и метод изомонодромных деформаций”, Дифференц. уравнения 30:5 (1994), 791–796; English transl., B.I. Suleimanov, “The Hamilton property of Painlevé equations and the method of isomonodromic deformations”, Differential Equations 30:5 (1994), 726–732.

[94] A. Zabrodin and A. Zotov, “Quantum Painlevé–Calogero correspondence”, J. Math. Phys. 53:7 (2012), 073507, 19 pp.; 2011, 55 pp., arXiv:1107.5672.

[95] А. Т. Забродин, А. В. Зотов, “Классическое-квантумное соответствие и функциональные отношения для уравнений Пенлеве”, Phys. Particles Nuclei 37:3 (2006), 400–443.

[96] G. Aminov, S. Arthamonov, A. Smirnov, and A. Zotov, “Classical-quantum correspondence and functional relations for Painlevé equations”, 2012, 38 pp., arXiv:1212.5813.

[97] N. Reshetikhin, “The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem”, Lett. Math. Phys. 26:3 (1992), 167–177.

[98] V.I. Arnold and B.A. Khesin, Topological methods in hydrodynamics, Appl. Math. Sci., vol. 125, Springer-Verlag, New York 1998, xvi+374 pp.

[99] B. Khesin, A. Levin, and M. Olshanetsky, “Bihamiltonian structures and their field-theoretical generalizations”, Phys. Particles Nuclei 37:3 (2006), 400–443.

[100] В.И. Арнольд, “Гамильтоновость уравнений Эйлера динамики твердого тела и идеальной жидкости”, УМН 24:3 (147) (1969), 225–226. [V.I. Arnold, “The Hamiltonian property of the Euler equations in the dynamics of a rigid body and of an ideal fluid”, Uspekhi Mat. Nauk 24:3(147) (1969), 225–226.]
[111] Н. Е. Жуковский, “О движении твердого тела, имеющего полости, наполненные однородной капельной жидкостью. I, II, III”, Журнал Русского физико-химического общества 17 (1885), 81–113, 145–199, 231–280.
[N. E. Zhukovskii (Zhukovsky), “The motion of a rigid body with cavities filled by a homogeneous droplet liquid. I, II, III”, Zh. Russk. Fiz.-Khim. Obshch. 17 (1885), 81–113, 145–199, 231–280.]

[112] V. Volterra, “Sur la théorie des variations des latitudes”, Acta Math. 22:1 (1899), 201–357.

[113] Е. К. Скилянин, “Separation of variables. New trends”, Quantum field theory, integrable models and beyond (Kyoto 1994), Progr. Theoret. Phys. Suppl., 1995, no. 118, 35–60; 1995, 33 pp., arXiv:solv-int/9504001.

[114] А. В. Зотов, Ю. Б. Черняков, “Интегрируемые многочастичные системы, полученные с использованием предела Иноземцева”, ТМФ 129:2 (2001), 258–277; English transl., A. V. Zотов and Yu. B. Chernyakov, “Integrable many-body systems via the Inosemtsev limit”, Theoret. and Math. Phys. 129:2 (2001), 1526–1542; 2001, 16 pp., arXiv:hep-th/0102069.

[115] А. В. Смирнов, “Интегрируемые $sl(N,\mathbb{C})$-волчки как системы Калоджеро–Мозера”, ТМФ 158:3 (2009), 355–369; English transl., A. V. Smirnov, “Integrable $sl(N,\mathbb{C})$ tops as Calogero–Moser systems”, Theoret. and Math. Phys. 158:3 (2009), 300–312; Correspondence between Calogero–Moser systems and integrable $SL(N,\mathbb{C})$ Euler–Arnold tops, 2008, 14 pp., arXiv:0809.2187.

[116] Г. Аминов и С. Артамонов, New $2 \times 2$-matrix linear problems for the Painlevé equations, 2011 (v2 – 2012), 18 pp., arXiv:1112.4688.

[117] Г. А. Аминов, С. Б. Артамонов, “Вырождение эллиптической системы Шлезингера”, ТМФ 174:1 (2013), 3–24; English transl., G. Aminov and S. Arthamonov, “Degenerating the elliptic Schlesinger system”, Theoret. and Math. Phys. 174:1 (2013), 1–20.

[118] A. Levin and A. Zotov, “On rational and elliptic forms of Painlevé VI equation”, Moscow Seminar on mathematical physics. II, Amer. Math. Soc. Transl. Ser. 2, vol. 221, Amer. Math. Soc., Providence, RI 2007, pp. 173–183.

[119] Э. Б. Винберг, А. Л. Онищен, Семинар по группам Ли и алгебраическим группам, Наука, М. 1988, 344 с.; English transl., A. L. Onishchik and È. B. Vinberg, Lie groups and algebraic groups, Springer Ser. Soviet Math., Springer-Verlag, Berlin 1990, xx+328 pp.

[120] D. B. Fairlie, P. Fletcher, and C. K. Zachos, “Infinite dimensional algebras and a trigonometric basis for the classical Lie algebras”, J. Math. Phys. 31:5 (1990), 1088–1094.

[121] D. Mumford, Tata lectures on theta, vol. I, Progr. Math., vol. 28, Birkhäuser Boston, Inc., Boston, MA 1983, xiii+235 pp.; vol. II, Progr. Math., vol. 43, 1984, xiv+272 pp.

[122] A. Weil, Elliptic functions according to Eisenstein and Kronecker, Ergebn. Math. Grenzgeb., vol. 88, Springer-Verlag, Berlin–New York 1976, ii+93 pp.
[123] N. L. Gordeev and V. L. Popov, “Automorphism groups of finite dimensional simple algebras”, *Ann. of Math.* (2) **158**:3 (2003), 1041–1065.

Andrei M. Levin
Laboratory of Algebraic Geometry,
National Research University
“Higher School of Economics”;
Institute of Theoretical and Experimental Physics
E-mail: alevin57@gmail.com

Mikhail A. Olshanetsky
Institute of Theoretical and Experimental Physics;
Moscow Institute of Physics and Technology
E-mail: olshanet@itep.ru

Andrei V. Zotov
Steklov Mathematical Institute
of Russian Academy of Sciences;
Institute of Theoretical and Experimental Physics;
Moscow Institute of Physics and Technology
E-mail: zotov@mi.ras.ru

Received 15/NOV/13