Proxy Convexity: A Unified Framework for the Analysis of Neural Networks Trained by Gradient Descent

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Abstract
Although the optimization objectives for learning neural networks are highly non-convex, gradient-based methods have been wildly successful at learning neural networks in practice. This juxtaposition has led to a number of recent studies on provable guarantees for neural networks trained by gradient descent. Unfortunately, the techniques in these works are often highly specific to the particular setup in each problem, making it difficult to generalize across different settings. To address this drawback in the literature, we propose a unified non-convex optimization framework for the analysis of neural network training. We introduce the notions of proxy convexity and proxy Polyak-Łojasiewicz (PL) inequalities, which are satisfied if the original objective function induces a proxy objective function that is implicitly minimized when using gradient methods. We show that gradient descent on objectives satisfying proxy convexity or the proxy PL inequality leads to efficient guarantees for proxy objective functions. We further show that many existing guarantees for neural networks trained by gradient descent can be unified through proxy convexity and proxy PL inequalities.

1 Introduction
Understanding the ability of gradient-based optimization algorithms to find good minima of non-convex objective functions has become an especially important problem due to the success of gradient descent (GD) in learning deep neural networks. Although there exist non-convex objective functions and domains for which GD will necessarily lead to sub-optimal local minima, it appears that for many problems of interest in deep learning, across domains as varied as natural language and images, these worst-case situations do not arise. Indeed, a number of recent works have developed provable guarantees for GD when used for objective functions defined in terms of neural networks, despite the non-convexity of the underlying optimization problem [Bru+18; ALL19; CG20; JT20; FCG20; FCG21]. To date, however, there has not been a framework which could unify the variegated approaches for guarantees in these settings.

In this work, we introduce the notion of proxy convexity and demonstrate that many existing guarantees for minimizing neural network objective functions with gradient-based optimization fall into a problem satisfying proxy convexity. Consider the following optimization problem,

$$\min_{w \in W} f(w),$$

(1.1)
where \( \mathcal{W} \subset \mathbb{R}^p \) is a parameter domain and \( f: \mathbb{R}^p \to \mathbb{R} \) is a loss function. We are interested in guarantees using the standard gradient descent algorithm,

\[
w^{(t+1)} = w^{(t)} - \eta \nabla f(w^{(t)}),
\]

where \( \eta > 0 \) is a fixed learning rate. We now introduce the first notion of proxy convexity we will consider in the paper.

**Definition 1.1 (Proxy convexity).** We say that a function \( f: \mathbb{R}^p \to \mathbb{R} \) satisfies \((g, h)\)-proxy convexity if there exist functions \( g, h: \mathbb{R}^p \to \mathbb{R} \) such that for all \( w, v \in \mathbb{R}^p \),

\[
\langle \nabla f(w), w - v \rangle \geq g(w) - h(v).
\]

Clearly, every convex function \( f \) satisfies \((f, f)\)-proxy convexity. We next introduce the analogy of proxy convexity for the Polyak–Łojasiewicz (PL) inequality [KNS16].

**Definition 1.2 \((g, \xi)-\text{proxy} , \xi\)-optimal PL inequality).** We say that a function \( f: \mathbb{R}^p \to \mathbb{R} \) satisfies a \((g, \xi, \alpha, \mu)\)-proxy PL inequality with parameters \( \alpha > 0 \) and \( \mu > 0 \) (in short, \( f \) satisfies the \((g, \xi, \alpha, \mu)\)-PL inequality) if there exists a function \( g: \mathbb{R}^p \to \mathbb{R} \) and scalars \( \xi \in \mathbb{R}, \mu > 0 \) such that for all \( w \in \mathbb{R}^p \),

\[
\|\nabla f(w)\|^\alpha \geq \frac{1}{2} \mu (g(w) - \xi).
\]

As we shall see below, the proxy PL inequality is a natural extension of the standard PL inequality. Our main contributions are as follows.

1. When \( f \) satisfies \((g, h)\)-proxy convexity, and \( f \) is either Lipschitz or satisfies a particular proxy-smoothness assumption, then for any norm bound \( R > 0 \), GD run for polynomial (in \( 1/\varepsilon \) and \( R \)) number of iterations satisfies the following,

\[
\min_{t<T} g(w^{(t)}) \leq \min_{\|w\| \leq R} h(w) + \varepsilon.
\]

2. When \( f \) satisfies a \((g, \xi, \alpha, \mu)\)-proxy PL inequality and has Lipschitz gradients, GD run for a polynomial (in \( 1/\varepsilon \)) number of iterations satisfies the following,

\[
\min_{t<T} g(w^{(t)}) \leq \xi + \varepsilon.
\]

3. We demonstrate that many previous guarantees for neural networks trained by gradient descent can be unified in the framework of proxy convexity.

As we will describe in more detail below, if a loss function \( \ell \) is \((g, h)\)-proxy convex or satisfies a \(g\)-proxy PL inequality, then the optimization problem is straightforward and the crux of the problem then becomes connecting guarantees for the proxy \( g \) with approximate guarantees for \( f \).

**Notation.** We use uppercase letters to refer to matrices, and lowercase letters will either refer to vectors or scalars depending on the context. For vectors \( w \), we use \( \|w\| \) to refer to the Euclidean norm, and for matrices \( W \) we use \( \|W\| \) to refer to the Frobenius norm. We use the standard \( O(\cdot) \), \( \Omega(\cdot) \) notations to hide universal constants, with \( \tilde{O}(\cdot) \) and \( \tilde{\Omega}(\cdot) \) additionally hiding logarithmic factors.
2 Proxy Convexity in Comparison to Other Non-convex Optimization Frameworks

In this section, we describe how proxy convexity and proxy PL-inequalities relate to other notions in non-convex optimization. In Section 6, we will discuss additional related work. First, recall that a function $f$ is $(g, h)$-proxy convex if there exist functions $g$ and $h$ such that for all $w, v$,

$$\langle \nabla f(w), w - v \rangle \geq g(w) - h(v).$$

One notion from the non-convex optimization literature that is related to our notion of proxy convexity is that of *invexity* [Han81]. A function $f$ is invex if it is differentiable and there exists a vector-valued function $k(w, v)$ such that for any $w, v$,

$$\langle \nabla f(w), k(w, v) \rangle \geq f(w) - f(v).$$

It has been shown that a smooth function $f$ is invex if and only if every stationary point of $f$ is a global minimum [CG85]. However, for many problems of interest involving neural networks, it is not the case that every stationary point will be a global optimum, which makes invexity a less appealing framework for understanding neural networks. Indeed, we shall see in Section 4.2 below that if one considers the problem of learning a single ReLU neuron $x \mapsto \sigma(\langle w, x \rangle) = \max(0, \langle w, x \rangle)$ under the squared loss, it is not hard to see that there exist stationary points which are not global minima (e.g., $w = 0$ assuming the convention $\sigma'(0) = 0$). By contrast, we shall see that the single ReLU neuron does satisfy a form of proxy convexity that enables GD to find approximately (but not globally) optimal minima. Thus even the simplest neural networks induce objective functions which are proxy convex and non-invex. We shall see in Section 3.3 that proxy convexity appears in the objective functions induced by wide and deep neural networks as well.

To understand how the proxy PL inequality compares to other notions in the optimization literature, recall that an objective function $f$ satisfies the standard PL inequality [Pol63; Loj63] if there exists $\mu > 0$ such that

$$\|\nabla f(w)\|_2^2 \geq \mu^2 [f(w) - f^*] \quad \forall w,$$

where $f^* = \min_w f(w)$. Clearly, any stationary point of an objective satisfying the standard PL inequality is globally optimal. Thus, the presence of local minima among stationary points in neural network objectives makes the standard PL inequality suffer from the same drawbacks that invexity does for understanding neural networks trained by gradient descent. This further applies to any of the conditions which are known to imply the PL inequality, like weak strong convexity, the restricted secant inequality, and the error bound condition [KNS16].\footnote{Karimi, Nutini, and Schmidt [KNS16] shows that these conditions imply the PL inequality under the assumption that the objective function has Lipschitz-continuous gradients.}

In comparison, the $(g, \xi, \alpha, \mu)$-proxy PL inequality is satisfied if there exists a function $g$ and constants $\xi > 0$, $\alpha > 0$ and $\mu > 0$ such that

$$\|\nabla f(w)\|_2^\alpha \geq \frac{\mu}{2} [g(w) - \xi] \quad \forall w.$$

It is clear that if a function $f$ satisfies the standard PL inequality, then it satisfies the $(f, f^*, 2, \mu)$ proxy PL inequality. Stationary points $w^*$ of objective functions satisfying the proxy PL inequality have $\|\nabla f(w^*)\| = 0$ which imply $g(w^*) \leq \xi$. In the case that $g = f$, the slack error term $\xi$ allows for the proxy PL inequality framework to accommodate the possibility that stationary points may not be globally optimal (i.e. have objective value $f^* = \min_w f(w)$), but could be approximately optimal by, for example, having objective
value at most $\xi = C \cdot f^*$ or $\xi = C \cdot \sqrt{f^*}$ for some constant $C \geq 1$. When $g \neq f$, the proxy PL inequality allows for the possibility of analyzing a proxy loss function $g$ which is implicitly minimized when using gradient-based optimization of the objective $f$.

At a high level, proxy convexity and the proxy PL inequality are well-suited to situations where stationary points may not be globally optimal, but may be approximately optimal with respect to a related optimization objective. The proxy convexity framework allows for one to realize this through developing problem-specific analyses that connect the proxy objective $g$ to the original objective $f$. As we shall see below, rich function classes like neural networks are often more easily analyzed by considering a proxy objective function that naturally appears when one analyzes the gradient of the loss.

Finally, we note that Liu, Zhu, and Belkin [LZB21] introduced a different generalization of the PL inequality, namely the PL* and PL* ε inequalities, which relaxes the standard PL inequality definition so that the PL condition only needs to hold on a subset of the domain. In particular, a function $f : \mathbb{R}^p \to \mathbb{R}$ satisfies the PL* inequality on a set $S \subset \mathbb{R}^p$ if there exists $\mu > 0$ such that

$$\|\nabla f(w)\|^2 \geq \mu f(w) \quad \forall w \in S.$$  

Likewise, $f$ satisfies the PL* ε inequality on $S$ if there exists a set $S$ and $\varepsilon \geq 0$ such that the PL* inequality holds on the set $S_\varepsilon = \{w \in S : f(w) \geq \varepsilon\}$. One can see that if $f$ satisfies the PL* ε inequality on $S$, then the function $g(w) := f(w) + \varepsilon$ satisfies the $g$-proxy, $\varepsilon$-optimal PL inequality on $S$.

We wish to emphasize the differences in the framing and motivation of the PL* ε inequality by Liu, Zhu, and Belkin [LZB21] and that of proxy convexity and the proxy PL inequality in this paper. Liu, Zhu, and Belkin [LZB21] focus on the geometry of optimization in the overparameterized setting where one has a fixed set of samples $\{(x_i, y_i)\}_{i=1}^n$ and a parametric model class $g(x; w)$ (for $w \in \mathbb{R}^p$, $p > n$) and the goal is to solve $g(x_i; w) = y_i$ for all $i \in [n]$. In this setting one can view the optimization problem as a nonlinear least squares system with $p$ unknowns and $n$ equations, and Liu, Zhu, and Belkin [LZB21] use geometric arguments to show that when $p > n$ the PL* condition is satisfied throughout most of the domain. They extend the PL* condition to the PL* ε condition with the motivation that in underparameterized settings, or when performing early stopping, there may not exist interpolating solutions. By contrast, in our work we consider general optimization problems (rather than objective functions defined in terms of an empirical average over samples) which hold regardless of ‘overparameterization’. Furthermore, our aim in this work is to develop a framework that allows for formal characterizations of optimization problems where stationary points are not globally optimal with respect to the original objective but are approximately optimal with respect to proxy objective functions. We will demonstrate below that such a framework can help unify a number of works on learning with neural networks trained by gradient descent.

## 3 Proxy PL Inequality Implies Proxy Objective Guarantees

In this section, we show that for loss functions satisfying a proxy PL inequality, gradient descent efficiently minimizes the proxy. We then go through different examples of neural network optimization problems where the proxy PL inequality is satisfied.

### 3.1 Main Result

We present our main theorem for the proxy PL inequality below. We leave the proofs for Section 5.

**Theorem 3.1.** Suppose $f(w)$ satisfies the $(g(\cdot), \xi, \alpha, \mu)$-proxy PL inequality for some function $g(\cdot) : \mathbb{R}^p \to \mathbb{R}$. Assume that $f$ is non-negative and has $L_2$-Lipschitz gradients. Then for any $\varepsilon > 0$, provided $\eta < 1/L_2$,
GD with fixed step size $\eta$ and run for $T = 2\eta^{-1}(\mu\varepsilon/2)^{-2/\alpha}f(w^{(0)})$ iterations results in the following guarantee,

$$\min_{t < T} g(w^{(t)}) \leq \xi + \varepsilon. \tag{3.1}$$

To get a feel for how a proxy PL inequality might be useful for learning neural networks, consider a classification problem with labels $y \in \{\pm 1\}$, and suppose that $N(w; x)$ is a neural network function parameterized by some vector of weights $w$ (we concatenate all weights into one weight vector). A standard approach for learning neural networks is to minimize the cross-entropy loss $\ell(yN(w; x)) = \log(1 + \exp(-yN(w; x)))$ using gradient descent on the empirical risk induced by a set of $n$ i.i.d. samples $\{(x_i, y_i)\}_{i=1}^n$. Using the variational form of the norm, we have

$$\left\| \nabla \left( \frac{1}{n} \sum_{i=1}^n \ell(y_iN(w; x_i)) \right) \right\| = \sup_{\|u\| = 1} \left\langle \nabla \left( \frac{1}{n} \sum_{i=1}^n \ell(y_iN(w; x_i)) \right), u \right\rangle$$

\[ \geq \frac{1}{n} \sum_{i=1}^n -\ell'(y_iN(w; x_i)) \cdot y_i \langle \nabla N(w; x_i), v \rangle, \tag{3.2} \]

where $v$ is any vector satisfying $\|v\| = 1$. Now, although the function $-\ell'$ is not an upper bound for $\ell$ (indeed, $-\ell' < \ell$), it is an upper bound for a constant multiple of the zero-one loss, and can thus serve as a proxy for the classification error. This is because for convex and decreasing losses $\ell$, the function $-\ell'$ is non-negative and decreasing, and so we can bound $\mathbb{1}(z \leq 0) \leq \ell'(z)/\ell'(0)$.

Thus, if one can bound the risk under $-\ell'$ (and $\ell'(0) \neq 0$), one has a bound for the classification error. Indeed, this property has been used in a number of recent works on neural networks [CG20; FCG19; JT20; FCG21]. This lets the $\ell'$ term in (3.2) represent the desired proxy $g$ in the definition of the $(g, \xi, \alpha, \mu)$-proxy PL inequality. For neural network classification problems, this reduces the problem of showing the neural network has small classification error to that of constructing a vector $v$ that allows for the quantity $y_i \langle \nabla N(w; x_i), v \rangle$ to be large and non-negative for each sample $(x_i, y_i)$. The quantity $y_i \langle \nabla N(w; x_i), v \rangle$ can be thought of as a margin function that is large when the gradient of the neural network loss points in a good direction. Although we shall see below that in some instances one can derive a lower bound for $y \langle \nabla N(w; x), v \rangle$ that holds for all $w$, $x$, and $y$, a more general approach would be to show that along the gradient descent trajectory $\{w^{(t)}\}$, a lower bound for $y_i \langle \nabla N(w^{(t)}; x_i), v \rangle$ holds for each $i$.\footnote{Although our results as stated would not immediately apply in this setting, the proof would be the same up to trivial modifications.}

In the remaining subsections, we will show that a number of recent works on learning neural networks with gradient descent utilized proxy PL inequalities. In our first example, we consider recent work by Charles and Papailiopoulos [CP18] that directly used a (standard) PL inequality.

### 3.2 Standard PL Inequality for Single Leaky ReLU Neurons and Deep Linear Networks

Charles and Papailiopoulos [CP18] showed that the standard PL inequality holds in two distinct settings. The first is that of a single leaky ReLU neuron $x \mapsto \sigma(\langle w, x \rangle)$, where $\sigma(z) = \max(c_\sigma z, 0)$ for $c_\sigma \neq 0$. They showed that if $s_{\min}(X)$ is the smallest singular value of the matrix $X \in \mathbb{R}^{n \times d}$ of $n$ samples, then for a $\lambda$-strongly convex loss $\ell$, the loss $f(w) = \ell(\sigma(Xw))$ satisfies the standard $\mu$-PL inequality, i.e., the $(f, f^*, 2, \mu)$-proxy PL inequality for $\mu = \lambda s_{\min}(X)^2 c_\sigma^2$ [CP18, Theorem 4.1].

The same authors also showed that under certain conditions the standard PL inequality holds when the neural network takes the form $N(w; x) = W_L \cdots W_1 x$ and the loss is the squared loss,
$f(w) = \frac{1}{2}\|Y - N(w; X)\|_F^2$, where $X \in \mathbb{R}^{n \times d}$ is the feature matrix and $Y \in \mathbb{R}^n$ are the labels. In particular, they showed that if $s_{\min}(W_l) \geq \tau > 0$ throughout the gradient descent trajectory, then $f$ satisfies the standard $\mu$-PL inequality for $\mu = L^2 - 2L^2/(XX^\top)^{-1} \|X\|_F^2$ [CP18, Theorem 4.5].

The standard PL inequality has been used by a number of other authors in the deep learning theory literature, see e.g. Xie, Liang, and Song [XLS17, Theorem 1], Hardt and Ma [HM17, Eq. 2.3], Zhou and Liang [ZL17, Theorem 1], Shamir [Sha19, Theorem 3].

In our next example, we show that a proxy PL inequality holds for deep neural networks in the neural tangent kernel (NTK) regime.

### 3.3 Proxy PL Inequality for Deep Neural Networks in NTK Regime

Consider the class of deep, $L$-hidden-layer ReLU networks, either with or without residual connections:

$$N_1(w; x) = \sigma(W_1x), \quad N_l(w; x) = s_lN_{l-1}(w; x) + \sigma(W_lN_{l-1}(w; x)), \quad l = 2, \ldots, L,$$

$$N(w; x) = \sum_{j=1}^m a_j[N_L(w; x)]_j,$$

where $s_l = 0$ for fully-connected networks and $s_l = 1$ for residual networks, and we collect the parameters $W_1, \ldots, W_L$ into the vector $w$. Cao and Gu [CG20, Theorem 4.2], Frei, Cao, and Gu [FCG19, Lemma 4.3], and Zou et al. [Zou+19, Lemma B.5] have shown that under certain distributional assumptions and provided the iterates of gradient descent stay close to their initializations, for samples $\{(x_i, y_i)\}_{i=1}^n$ and objective function

$$f(w) := \frac{1}{n} \sum_{i=1}^n \ell(y_i, N(w; x_i)), \quad \ell(z) = \log(1 + \exp(-z)), \quad (3.3)$$

one can guarantee that the following proxy PL inequality holds:

$$\|\nabla f(w)\| \geq C_1 \cdot \frac{1}{n} \sum_{i=1}^n -\ell'(y_i, N(w; x_i)) =: C_1g(w). \quad (3.4)$$

One can see that the loss $f$ satisfies the $(g, 0, 1, 2C_1)$-proxy PL inequality, which shows that approximate stationary points of the original objective have small $g$ loss. Since $\mathbb{1}(z < 0) \leq 2 \cdot -\ell'(z)$, small $g$ loss implies small classification error. Note that since the ReLU is not smooth, the loss $f$ will not have Lipschitz gradients, and thus a direct application of Theorem 3.1 is not possible. Instead, the authors show that in the NTK regime, the loss obeys a type of semi-smoothness that still allows for an analysis similar to that of Theorem 3.1.

### 3.4 Proxy PL Inequality for One-Hidden-Layer Networks Outside NTK Regime

Consider a one-hidden-layer network with activation function $\sigma$, parameterized by $w = \text{vec}(W)$, where $W \in \mathbb{R}^{m \times d}$ has rows $w_j$,

$$N(w; (x, y)) = \sum_{j=1}^m a_j\sigma(w_j, x), \quad (3.5)$$

Above, the second layer weights $\{a_j\}_{j=1}^m$ are randomly initialized and fixed at initialization, but $w = \text{vec}((w_1, \ldots, w_m))$ are trained. Assume $\sigma$ satisfies $\sigma'(z) \geq c_\sigma > 0$ for all $z$ (e.g., the leaky ReLU activation). Consider training with gradient descent on the empirical average of the logistic loss defined in
terms of i.i.d. samples \( \{(x_i, y_i)\}_{i=1}^n \) as in (3.3). Frei, Cao, and Gu have shown [FCG21, Lemma 3.1] that there exists a vector \( v \in \mathbb{R}^{md} \) with \( \|v\| = 1 \) such that for ‘well-behaved’ distributions, there are absolute constants \( C_1, C_2 > 1 \) such that with high probability it holds that for any \( w \) and \( i \),
\[
y_i \langle \nabla N(w; x_i), v \rangle \geq C_1 |c_\sigma - C_2 \sqrt{\text{OPT}}|
\]
where \( \text{OPT} \) is the best classification error achieved by a halfspace over \( D \). Since \( |\ell'| \leq 1 \), this establishes the following proxy-PL inequality,
\[
\|\nabla f(w)\| = \sup_{\|z\|=1} \langle \nabla f(w), z \rangle \\
\geq \frac{1}{n} \sum_{i=1}^n \ell'(y_i N(w; x_i)) : y_i \langle \nabla N(w; x_i), v \rangle \\
\geq C_1 c_\sigma \cdot \left[ \frac{1}{n} \sum_{i=1}^n \ell'(y_i N(w; x_i)) - c_\sigma^{-1} C_2 \sqrt{\text{OPT}} \right].
\]

As in Section 3.3, by defining \( g(w) = \frac{1}{n} \sum_{i=1}^n \ell'(y_i N(w; x_i)) \), the above inequality shows that \( f \) satisfies the \((g, c_\sigma^{-1} C_2 \sqrt{\text{OPT}}, 1, 2C_1 c_\sigma)\)-proxy PL inequality. Thus, provided we can show that \( f \) has \( L_2 \)-Lipschitz gradients for some constant \( L_2 > 0 \), Theorem 3.1 shows that for \( T \) large enough,
\[
\min_{t<T} \frac{1}{n} \sum_{i=1}^n \mathbb{1}(y_i \neq \text{sgn}(N(w^{(t)}; x_i))) \\
\leq \min_{t<T} \frac{1}{\ell'(0)} \cdot \frac{1}{n} \sum_{i=1}^n \ell'(y_i N(w^{(t)}; x_i)) \\
\leq \frac{c_\sigma^{-1} C_2 \sqrt{\text{OPT}}}{-\ell'(0)} + \varepsilon.
\]

Provided \( \sigma \) is such that \( \sigma' \) is continuous and differentiable, then \( f \) has \( L_2 \)-Lipschitz gradients and thus the guarantees will follow. In particular, this analysis follows if \( \sigma \) is any smoothed approximation to the leaky ReLU which satisfies \( \sigma'(z) \geq c_\sigma > 0 \).

Note that the above optimization analysis is an original contribution of this work as we utilize a completely different proof technique than that of Frei, Cao, and Gu [FCG21]. In that paper, the authors utilize a Perceptron-style proof technique that analyzes the correlation \( \langle w^{(t)}, v \rangle \) of the weights found by gradient descent and a reference vector \( v \). Their proof relies crucially on the homogeneity of the (non-smooth) leaky ReLU activation, namely that \( z \sigma'(z) = \sigma(z) \) for \( z \in \mathbb{R} \), and cannot accommodate more general smooth activations. By contrast, the proxy PL inequality proof technique in this example relies upon the smoothness of the activation function and is more similar to smoothness-based analyses of gradient descent.

## 4 Proxy Convexity Implies Proxy Objective Guarantees

In this section, we show that if \( f \) satisfies \((g, h)\)-proxy convexity, we can guarantee that by minimizing \( f \) with gradient descent, we find weights \( w \) for which \( g(w) \) is at least as small as the smallest-possible loss under \( h \). We then go through examples of neural network optimization problems that satisfy proxy convexity.
4.1 Main Result

We present two versions of our result: one that relies upon fewer assumptions on the loss $f$ but needs a small step size, and another that requires a proxy smoothness assumption on $f$ but allows for a constant step size. The proofs for the theorem are given in Section 5.

**Theorem 4.1.** Suppose that $f : \mathbb{R}^p \to \mathbb{R}$ is $(g, h)$-proxy convex.

(a) Assume there exists $L_1 > 0$ such that for all $w$, $\|\nabla f(w)\|_2^2 \leq L_1^2$. Then for any $v \in \mathbb{R}^p$ and any $\varepsilon > 0$, performing GD on $f(w)$ from an arbitrary initialization $w^{(0)}$ with fixed step size $\eta \leq \varepsilon L_1^{-2}$ for $T = \eta^{-1} \varepsilon^{-1/2} \|w^{(0)} - v\|^2$ iterations implies that,

$$\min_{t < T} g(w^{(t)}) \leq h(v) + \varepsilon.$$  

(b) Assume there exists $L_2 > 0$ such that for all $w$, $\|\nabla f(w; z)\|_2^2 \leq 2 L_2 g(w)$. Then for any $v \in \mathbb{R}^p$ and any $\varepsilon > 0$, performing GD on $f(w)$ from an arbitrary initialization with fixed step size $\eta \leq L_2^{-1}/2$ for $T = \eta^{-1} \varepsilon^{-1/2} \|w^{(0)} - v\|^2$ implies that,

$$\min_{t < T} g(w^{(t)}) \leq (1 + 2 \eta L_2) h(v) + \varepsilon.$$  

In order for $(g, h)$-proxy convexity to be useful, there must be a way to relate guarantees for $g$ into guarantees for the desired objective function $f$. In the remainder of this section, we will discuss two neural network learning problems which satisfy proxy convexity and for which the proxy objectives are closely related to the original optimization problem. Our first example is the problem of learning a neural network with a single nonlinear unit.

4.2 Single ReLU Neuron Satisfies Proxy Convexity

Consider the problem of learning a single neuron $x \mapsto \sigma(\langle w, x \rangle)$ under the squared loss, where $\sigma$ is the ReLU activation $\sigma(z) = \max(0, z)$. The objective function of interest is

$$F(w) = \mathbb{E}_{(x, y) \sim D} \frac{1}{2} (\sigma(\langle w, x \rangle) - y)^2,$$

where $D$ is a distribution over $(x, y) \in \mathbb{R}^p \times \mathbb{R}$. Denote

$$F^* := \min_{\|w\| \leq 1} F(w).$$

It is known that $F$ is non-convex [YS20]. Under the assumption that learning sparse parities with noise is computationally hard, it is known that no polynomial time algorithm can achieve risk $F^*$ exactly when $D$ is the standard Gaussian; moreover, it is known that (unconditionally) the standard gradient descent algorithm cannot achieve risk $F^*$ [GKK19],\(^3\) However, Frei, Cao, and Gu [FCG20] showed that although $F$ is non-convex and no algorithm can achieve risk $F^*$, $F$ does satisfy a form of proxy convexity that allows for gradient descent to achieve risk $O(\sqrt{F^*})$. They showed that for samples $(x_i, y_i) \sim D$, the loss function

$$f_i(w) = \frac{1}{2} (\sigma(\langle w, x_i \rangle) - y_i)^2$$

\(^3\)This stands in contrast to learning a single leaky ReLU neuron $x \mapsto \max(\alpha x, x)$ for $\alpha \neq 0$, which as we showed in Section 3.2 can be solved using much simpler techniques.
satisfies \((g_i, h_i)\)-proxy convexity along the trajectory of gradient descent, where
\[
\begin{align*}
g_i(w) &= 2 |\sigma(\langle w, x_i \rangle) - \sigma(\langle v^*, x_i \rangle)|^2 \sigma'(\langle w, x_i \rangle), \\
h_i(v) &= |\sigma(\langle v^*, x_i \rangle) - y_i| = \sqrt{2f_i(v^*)},
\end{align*}
\]
where \(v^*\) is the population risk minimizer of \(F(w)\) (see their Eq. (3.13)). Moreover, they showed that for some \(L_1 > 0\), (see their Eq. (3.9)),
\[
\|\nabla f_i(w)\|^2 \leq L_1^2.
\]
Thus, given \(n\) i.i.d. samples \(\{(x_i, y_i)\}_{i=1}^n\) i.i.d. \(\mathcal{D}\), if one considers the empirical risk minimization problem and defines
\[
\begin{align*}
f(w) &:= \frac{1}{n} \sum_{i=1}^n f_i(w), \quad g(w) := \frac{1}{n} \sum_{i=1}^n g_i(w), \quad h(w) := \frac{1}{n} \sum_{i=1}^n h_i(w),
\end{align*}
\]
then \(f\) satisfies \((g, h)\)-proxy convexity. Thus, Theorem 4.1 implies that GD with step size \(\eta \leq \varepsilon L_1^{-2}\) and \(T = \eta^{-1} \varepsilon^{-1} \|w^{(0)} - v^*\|^2\) iterations will find a point \(w^{(t)}\) satisfying
\[
g(w^{(t)}) \leq h(v^*) + \varepsilon.
\]
By using uniform convergence-based arguments, it is not hard to show that with enough samples, the empirical quantities \(h(w)\) and \(g(w)\) are close to their expected values,
\[
\begin{align*}
\mathbb{E}[g(w^{(t)})] &\approx G(w^{(t)}) := 2\mathbb{E}\left[ |\sigma(\langle w^{(t)}, x \rangle) - \sigma(\langle v^*, x \rangle)|^2 \sigma'(\langle w^{(t)}, x \rangle) \right], \\
\mathbb{E}[h(v)] &\approx H(v) := \mathbb{E}[|\sigma(\langle v^*, x \rangle) - y|] \leq \sqrt{\mathbb{E}[(\sigma(\langle v^*, x \rangle) - y)^2]} = \sqrt{2F^*}.
\end{align*}
\]
In particular, proxy convexity allows for \(G(w^{(t)}) \leq O(\sqrt{F^*})\). The authors then show that under some distributional assumptions on \(\mathcal{D}\), \(G(w^{(t)}) = O(\sqrt{F^*})\) implies \(F(w^{(t)}) = O(\sqrt{F^*})\) [FCG20, Lemma 3.5]. Thus, the optimization problem for \(F\) induces a proxy convex optimization problem defined in terms of \(G\) which yields guarantees for \(G\) in terms of \(H\), and this in turn leads to approximate optimality guarantees for the original objective \(F\).

In our next example, we show that a number of works on learning one-hidden-layer ReLU networks in the neural tangent kernel regime [JGH18] can be cast as problems satisfying proxy convexity.

### 4.3 Proxy Convexity for One-Hidden-Layer ReLU Neural Networks in the NTK Regime

Consider the class of one-hidden-layer ReLU networks consisting of \(m\) neurons,
\[
N(w; (x, y)) = \sum_{j=1}^m a_j \sigma(\langle w_j, x \rangle),
\]
where the \(\{a_j\}_{j=1}^m\) are randomly initialized and fixed at initialization, but \(w = \text{vec}((w_1, \ldots, w_m))\) are trained. Suppose we consider a binary classification problem, where \(y_i \in \{\pm 1\}\) and we minimize the cross-entropy loss \(\ell(z) = \log(1 + \exp(-z))\) for samples \(\{(x_i, y_i)\}_{i=1}^n\), so that the objective function is,
\[
f_i(w) = \ell(y_i N(w; (x_i, y_i))), \quad f(w) = \frac{1}{n} \sum_{i=1}^n f_i(w).
\]
Ji and Telgarsky [JT20, Proof of Lemma 2.6] showed that there exists a function \( \tilde{h}(w, v) \) such that the iterates of gradient descent satisfy

\[
\langle \nabla f(w), w - v \rangle \geq f(w) - \tilde{h}(w, v).
\]

Under the assumption that the iterates of gradient descent stay close to the initialization (i.e., the neural tangent kernel regime), they show that \( \tilde{h}(w, v) \leq \varepsilon \) under distributional assumptions, and thus \( f(w) \) will satisfy \((f, \tilde{h} \equiv \varepsilon)\)-proxy convexity. Moreover, we have the following smoothness property,

\[
\|\nabla f(w)\|^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(y_i, N(w; (x_i, y_i))) \right\|^2
\leq \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla \ell(y_i, N(w; (x_i, y_i))) \right\|^2
\leq 1 \cdot \frac{1}{n} \sum_{i=1}^{n} \left[ \ell'(y_i, N(w; (x_i, y_i))) \right]^2
\leq L_2 \cdot \frac{1}{n} \sum_{i=1}^{n} \left[ \ell'(y_i, N(w; (x_i, y_i))) \right]^2
\]

Above, inequality \((i)\) uses Jensen’s inequality while \((ii)\) uses that the ReLU activation is Lipschitz, and \( L_2 \) is a quantity that depends on the (fixed) values of \( \{a_j\} \). Since the logistic loss satisfies \([\ell'(z)]^2 \leq \ell(z)\), the above shows that \(\|\nabla f(w)\|^2 \leq L_2 f(w)\), and thus we can apply Theorem 4.1(b) to show that GD with large step sizes can achieve \(\min_{t < T} f(w^{(t)}) \leq \varepsilon\) for the cross-entropy loss.

In another problem of learning one-hidden-layer networks, Allen-Zhu, Li, and Liang [ALL19, Proof of Lemma B.4] show that there exists a proxy loss function \(g(w)\) such that provided the neural network weights stay close to their initialized values, \( f(w) \) satisfies \((g, g + \varepsilon)\) proxy convexity. Using a similar argument as above, since the cross-entropy loss satisfies \([\ell'(z)]^2 \leq \ell(z)\), part (b) of Theorem 4.1(b) is applicable so that GD-trained neural networks in the NTK regime satisfy \(\min_{t < T} f(w^{(t)}) \leq \min_v g(v) + \varepsilon\). They further show that the proxy loss \(g\) is close to the cross entropy loss, implying a bound of the form \(\min_{t < T} f(w^{(t)}) \leq \min_v f(v) + \varepsilon\).

## 5 Proof of the Main Results

In this section we provide the proofs of the theorems given in Sections 3 and 4.

We first give the proof of Theorem 3.1 which provides guarantees for learning with objectives satisfying proxy PL inequalities.

**Proof of Theorem 3.1.** Since \( f \) has \( L_2 \)-Lipschitz gradients, we have for any \( w, w' \),

\[
f(w) \leq f(w') + \langle \nabla f(w'), w - w' \rangle + \frac{L_2}{2} \|w - w'\|^2.
\]

Taking \( w = w^{(t+1)} \), \( w' = w^{(t)} \),

\[
f(w^{(t+1)}) \leq f(w^{(t)}) - \eta \left\| \nabla f(w^{(t)}) \right\|^2 + \frac{\eta^2 L_2}{2} \left\| \nabla f(w^{(t)}) \right\|^2
\leq f(w^{(t)}) - \eta \left[ 1 - \eta L_2 / 2 \right] \left\| \nabla f(w^{(t)}) \right\|^2.
\]

(5.1)
Since \( \eta < 1/L_2 \), we have \((1 - \eta L_2/2)^{-1} \leq 2\), and thus we can rearrange the above to get
\[
\left\| \nabla f(w^{(t)}) \right\|^2 \leq \frac{1}{\eta(1 - \eta L_2/2)} [f(w^{(t)}) - f(w^{(t+1)})] \\
\leq \frac{2}{\eta} [f(w^{(t)}) - f(w^{(t+1)})].
\] (5.2)

Summing the above from \( t = 0 \) to \( t = T - 1 \) and using that \( f \) is non-negative, we get
\[
\frac{1}{T} \sum_{t=0}^{T-1} \left\| \nabla f(w^{(t)}) \right\|^2 \leq 2\frac{f(w^{(0)}) - f(w^{(T)})}{\eta T} = \frac{2f(w^{(0)})}{\eta T}.
\]

Using the definition of proxy PL inequality, this implies
\[
\frac{1}{T} \sum_{t=0}^{T-1} (\mu/2)^{2/\alpha} (g(w^{(t)}) - \xi)^{2/\alpha} \leq \frac{1}{T} \sum_{t=0}^{T-1} \left\| \nabla f(w^{(t)}) \right\|^2 \leq \frac{2f(w^{(0)})}{\eta T}.
\]

Taking the minimum over \( t < T \) and re-arranging terms, this means
\[
\min_{t<T} (g(w^{(t)}) - \xi)^{2/\alpha} \leq \frac{2f(w^{(0)})}{\eta T (\mu/2)^{2/\alpha}}.
\]

Therefore, we have
\[
\min_{t<T} g(w^{(t)}) \leq \xi + \frac{2}{\mu} \left( \frac{2f(w^{(0)})}{\eta T} \right)^{\alpha/2}.
\]

Taking \( T = 2\eta^{-1} f(w^{(0)}) (\mu \varepsilon / 2)^{-2/\alpha} \), we get (3.1).

We next prove guarantees for GD when the objective satisfies proxy convexity.

**Proof of Theorem 4.1.** By the definition of proxy convexity,
\[
\left\| w^{(t)} - v \right\|^2 - \left\| w^{(t+1)} - v \right\|^2 = 2\eta \langle \nabla f(w^{(t)}), w^{(t)} - v \rangle - \eta^2 \left\| \nabla f(w^{(t)}) \right\|^2 \\
\geq 2\eta [g(w^{(t)}) - h(v)] - \eta^2 \left\| \nabla f(w^{(t)}) \right\|^2 \\
= 2\eta [g(w^{(t)}) - h(v) - (\eta/2)\left\| \nabla f(w^{(t)}) \right\|^2].
\] (5.3)

For case (a), we have \( \left\| \nabla f(w^{(t)}) \right\| \leq L_1 \), so that the above becomes
\[
\left\| w^{(t)} - v \right\|^2 - \left\| w^{(t+1)} - v \right\|^2 \geq 2\eta [g(w^{(t)}) - h(v) - \eta L_1^2/2].
\]

Dividing both sides by \( 2\eta T \) and summing from \( t = 0, \ldots, T - 1 \), we get
\[
\frac{1}{T} \sum_{t=0}^{T-1} g(w^{(t)}) \leq \frac{1}{T} \sum_{t=0}^{T-1} h(v) + \frac{\eta L_1^2}{2} + \frac{\left\| w^{(0)} - v \right\|^2 - \left\| w^{(t)} - v \right\|^2}{2\eta T}.
\]
Dropping the $-\|w^{(t)} - v\|^2$ term,
\[
\min_{t < T} g(w^{(t)}) \leq \frac{1}{T} \sum_{t=0}^{T-1} g(w^{(t)}) \leq h(v) + \frac{\eta L_2}{2} + \frac{\|w^{(0)} - v\|^2}{2\eta T}.
\]
In particular, for $\eta \leq \varepsilon L_1^{-2}$ and $T = \eta^{-1} \varepsilon^{-1} \|w^{(0)} - v\|^2$, we get
\[
\min_{t < T} g(w^{(t)}) \leq h(v) + \varepsilon.
\]
For case (b), $\|\nabla f(w^{(t)})\|^2 \leq 2L_2 g(w^{(t)})$ so that (5.3) becomes
\[
\|w^{(t)} - v\|^2 - \|w^{(t+1)} - v\|^2 \geq 2\eta [g(w^{(t)}) - h(v) - \eta L_2 g(w^{(t)})] = 2\eta \left[ (1 - \eta L_2) g(w^{(t)}) - h(v) \right].
\]
Dividing both sides by $2\eta T(1 - \eta L_2)$ and summing from $t = 0, \ldots, T - 1$,
\[
\frac{1}{T} \sum_{t=0}^{T-1} g(w^{(t)}) \leq \frac{1}{1 - \eta L_2} h(v) + \frac{\|w^{(0)} - v\|^2 - \|w^{(t)} - v\|^2}{2\eta T(1 - \eta L_2)}
\leq (1 + 2\eta L_2) h(v) + \frac{(1 + 2\eta L_2)\|w^{(0)} - v\|^2}{2\eta T},
\]
where in the last line we have used that $\eta \leq L_2^{-1}/2$ and that $1/(1 - x) \leq 1 + 2x$ on $[0, 1/2]$. In particular, for $T = \eta^{-1} \varepsilon^{-1} \|w^{(0)} - v\|^2$, we have
\[
\min_{t < T} g(w^{(t)}) \leq \frac{1}{T} \sum_{t=0}^{T-1} g(w^{(t)}) \leq (1 + 2\eta L_2) h(v) + \frac{1}{2}(1 + 2\eta L_2) \varepsilon \leq (1 + 2\eta L_2) h(v) + \varepsilon.
\]

6 Additional Related Work

The Polyak–Lojasiewicz inequality can be dated back to the original works of Polyak [Pol63] and Lojasiewicz [Loj63]. Recent work by Karimi, Nutini, and Schmidt [KNS16] proved linear convergence under the PL condition and showed that the PL condition is one of the weakest assumptions under which linear convergence is possible. In particular, they showed that the error bound inequality [LT93], essential strong convexity [Liu+15], weak strong convexity [NNG19], and the restricted secant inequality [ZY13] are all assumptions under which linear convergence is possible and that each of these assumptions implies the PL inequality.

As we described in Section 2, the standard PL condition was shown to hold under certain assumptions for neural network objective functions [HM17; XLS17; ZL17; CP18]. In addition to those covered in this paper, there are a number of other provable guarantees for generalization of SGD-trained networks which rely on a variety of different techniques, such as tensor methods [LMZ20] and utilizing connections with partial differential equations by way of mean field approximations [MMN18; CB18; MMM19; Che+20].
In the optimization literature, recent work has shown that SGD can efficiently find stationary points and can escape saddle points [Ge+15; FLZ19]. As the proxy PL inequality implies guarantees for the proxy objective function at stationary points of the original optimization objective, our framework can naturally be used for other optimization algorithms that are known to efficiently find stationary points, such as SVRG [AH16; Red+16], Natasha2 [All18], SARAH/SPIDER [Ngu+17; Fan+18], and SNVRG [ZXG18].

7 Conclusion

In this paper we have introduced the notion of proxy convexity and proxy PL inequality and developed guarantees for learning with stochastic gradient descent under these conditions. We demonstrated that many recent works in the learning of neural networks with gradient descent can be framed in terms of optimization problems that satisfy either proxy convexity or a proxy PL inequality. We believe the proxy convexity and proxy PL inequality approaches can be used to derive new optimization guarantees for structured non-convex optimization problems.

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