Quasilinear hyperbolic equations with hysteresis

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Abstract. This paper is devoted to the summary of results about a quasilinear hyperbolic partial differential equation of first order with a hysteresis operator

$$\frac{\partial (u + v)}{\partial t} + \frac{\partial u}{\partial x} = 0.$$ 

Hysteresis is represented by functional describing adsorption and desorption on the particles of the substance and is a possibly discontinuous generalized play operator. The results can be extended to possibly discontinuous generalized Prandtl-Ishlinskii operators of play type.

1. Introduction

We study the partial differential equation

$$\frac{\partial (u + v)}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad v = F[u] \text{ in } (0, L) \times [0, T],$$

as a generic model for the transport and adsorption of a chemical of concentration $u(x, t)$ carried in a solution with constant unit velocity in a tube $x \in (0, L)$ for $t > 0$. Here $F[\cdot]$ is a general functional describing adsorption and desorption of the chemical on the particles of solid filling up the tube. In the classical case $F[\cdot]$ is given by an isotherm, a real-valued function corresponding to an experimentally determined relation between the concentration $v(x, t)$ of the adsorbed species on the surface of the particles and the concentration $u(x, t)$ in solution. In the general situation considered here, the adsorption-desorption functional $F[\cdot]$ exhibits hysteresis, i.e., the relations between $u$ and $v$ for the case when $u$ is increasing (adsorption) and decreasing (desorption) follow different curves. The motivation for our study comes from applications in chemical and geological engineering. Hysteresis coupled to transport phenomena occurs also in the modelling of oil-water interaction and of waste treatment in subsurface reservoirs [6]. The phenomenon of hysteresis in adsorption has been observed and studied for many years (see [4, 7]), frequently in parallel with capillary condensation hysteresis (see [5]).

The equation (1) was studied in [8, 9, 18, 20, 21]. Visintin investigated the Cauchy problem for equation (1) with the hysteresis functional represented by a possibly discontinuous generalized play operator by using the semigroup approach and got the existence and uniqueness of the integral solution in $L^1$. These results can be extended to possibly discontinuous generalized Prandtl-Ishlinskii operators of play type, which includes the case of possibly discontinuous Preisach operators. In his book [20] it was posed as an open problem whether the integral solution of (1) with hysteresis satisfies an entropy condition introduced by Kružkov. Such an
entropy condition was derived by Kopfová [8]. Visintin also considered the Cauchy problem for the completed relay operator and its regularization and proved the existence of a weak solution. Showalter and Peszynska [17] obtained the existence and uniqueness of differentiable solutions by the theory of nonlinear semigroups in a Hilbert space $L^2$. They assumed that hysteresis is represented by a classical play operator and also by a more general case of a convex adsorption-desorption hysteresis functional. Kordulová [10] resumed the investigation of continuous solutions with a slightly modified classical play operator and found that even a small change of the hysteresis operator can cause a discontinuity in the solution. In [11] an asymptotic stability of the solution of the equation (1) in $L^1$ is derived.

The results can be extended to the more general quasilinear partial differential hyperbolic equation

$$\frac{\partial (u + v)}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} (b_j u) + cu = f,$$

where $b_j$ and $c$ are given smooth functions, [9, 20].

The paper is organized as follows. In Section 2 the definitions of hysteresis operators are recalled. In Section 3 Visintin’s results about the existence and uniqueness of an integral solution and the existence of a weak solution are summed up. The entropy condition for equation (1) is established. Section 4 is devoted to the existence and uniqueness of a differentiable solution of (1) with a classical play operator. Finally the asymptotic behaviour of the solutions is characterized in Section 5 and results of [10] are described.

2. Hysteresis operators

In this section the definitions of hysteresis operators are briefly reviewed, for more details see [12, 20].

We consider equation (1) and couple it with the hysteresis relation

$$v(x, t) = [\mathcal{E}(u(x, .), v_0(x)) (t) \quad \text{in} \quad [0, T], \ \text{a.e. in} \ \Omega \ .$$

Here $\mathcal{E}$ is a multivalued functional, $v_0(x)$ a given initial value. Its values depend not only on the current value of $u(., t)$ at $t > 0$, but on the past history $u(., s), 0 < s < t$. We consider at first $\mathcal{E}$ to be a generalized play operator.

We set $\mathbb{R} := [-\infty, +\infty], \mathbb{R}^+ := [0 + \infty], \mathbb{R}^- := [-\infty, 0]$.

Let $\gamma_l, \gamma_r : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be maximal monotone (possibly multivalued) functions with

$$\inf \gamma_r(u) \leq \sup \gamma_l(u), \ \forall u \in \mathbb{R}. \quad (4)$$

Let $\Lambda$ denote the hysteresis region, i.e., the subset of $\mathbb{R}^2$ of admissible pairs $(u, v)$ such that $\inf \gamma_r(u) \leq v \leq \sup \gamma_l(u)$.

Now, given $v_0 \in \mathbb{R}$, we construct the hysteresis operator $\mathcal{E}(\cdot, v_0)$ as follows. Let $u$ be any continuous, piecewise linear function on $\mathbb{R}^+$ such that $u$ is linear on $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots$. We then define $v := \mathcal{E}(u, v_0) : \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$v(t) := \begin{cases} \min\{\gamma_l(u(0)), \max\{\gamma_r(u(0)), v_0\}\} & \text{if} \quad t = 0 \\ \min\{\gamma_l(u(t)), \max\{\gamma_r(u(t)), v(t_{i-1})\}\} & \text{if} \quad t \in (t_{i-1}, t_i], i = 1, 2, \ldots. \end{cases}$$

Note that $v(0) = v_0$ only if $\gamma_r(u(0)) \leq v_0 \leq \gamma_l(u(0))$. 

As proved in Visintin [20], Section III.2, \( \mathcal{E}(\cdot, v_0) \) can be extended to \( C^0([0,T]) \) by continuity. This operator is called a generalized play [12] (see Figure 1).

A special example of a generalized play operator, where \( \gamma_r = u - r, \gamma_l = u + r \) is called the classical play operator (Figure 2).

Let us fix any pair \( \rho := (\rho_1, \rho_2) \in \mathbb{R}^2, \) with \( \rho_1 < \rho_2. \) If the right hysteresis boundary curve is defined as

\[
\gamma_r(u) \in \begin{cases} 
-1 & \text{if } u < \rho_2 \\
[-1, 1] & \text{if } u = \rho_2 \\
1 & \text{if } u > \rho_2,
\end{cases}
\]  
(5)

and the left boundary curve is given by

\[
\gamma_l(u) \in \begin{cases} 
-1 & \text{if } u < \rho_1 \\
[-1, 1] & \text{if } u = \rho_1 \\
1 & \text{if } u > \rho_1,
\end{cases}
\]  
(6)

then the generalized play operator is the completed relay operator [21] (see Figure 3).

To define the generalized Prandtl-Ishlinskii operator of play type, let us assume that we are given a measure space \( (\mathcal{P}, \mathcal{A}, \mu) \), where \( \mu \) is a finite Borel measure. For almost any \( \sigma \in \mathcal{P} \), let \( (\gamma_{sl}, \gamma_{sr}) \) be a pair of functions \( \mathbb{R} \to \mathbb{R}, \) satisfying (4), and for each \( \sigma \in \mathcal{P} \) let \( v_{\sigma 0} \in \mathbb{R}, \) be a given initial value. Let \( \mathcal{E}_\sigma(\cdot, v_{\sigma 0}) \) be the corresponding generalized play operator corresponding
to the couple \((\gamma_{\sigma l}, \gamma_{\sigma r})\). Then the operator defined as
\[
\tilde{\mathcal{E}}_\mu (u, \{v_{\sigma 0}\} \sigma \in \mathcal{P}) = \int_{\mathcal{P}} \mathcal{E}_\sigma (u, v_{\sigma 0}) d\mu(\sigma)
\]
is called a generalized Prandtl-Ishlinskii operator of play type.

If boundary curves \(\gamma_{\sigma r}\) and \(\gamma_{\sigma l}\) are defined as (5) and (6) then the generalized Prandtl-Ishlinskii operator corresponds with the Preisach operator [20].

3. Integral solution
Visintin formulated equation (1) by using a system of differential inclusions containing an accretive operator for a possibly discontinuous generalized play operator. Equation (1) is equivalent to the Cauchy problem:

\[
\begin{cases}
\frac{\partial U}{\partial t} + \mathcal{A}(U) + \mathcal{R}(U) & \ni 0, \quad \text{in } \Omega \times [0, T] \\
U(0) & = U_0,
\end{cases}
\]

where

\[
\begin{align*}
\Omega & := (0, L) \\
\mathcal{D}(\mathcal{A}) & := \{U := (u, v) \in \mathbb{R}^2 : \inf \gamma_r(u) \leq v \leq \sup \gamma_l(u)\} \\
\mathcal{A}(U) & := \{\langle \xi, -\xi \rangle \in \mathbb{R}^2 : \xi \in \phi(U) \cap \mathbb{R}\} \forall U \in \mathcal{D}(\mathcal{A}) \\
\mathcal{B}(u) & := \frac{\partial u}{\partial x} \\
\mathcal{R}(U) & := (\mathcal{B}(u), 0) \\
\mathcal{D}(\mathcal{R}) & := \{U \in L^1(\Omega, \mathbb{R}_1^2) : \mathcal{B}u \in L^1(\Omega)\} \\
\mathcal{Q}(U) & := \mathcal{A}(U) + \mathcal{R}(U) \\
\mathcal{D}(\mathcal{Q}) & := \{U := (u, v) \in L^1(\Omega, \mathbb{R}_1^2) : U \in \mathcal{D}(\mathcal{A}) \text{ a.e. in } \Omega, \quad u \in W^{1,1}(\Omega), u(0) = 0\}.
\end{align*}
\]

By setting 
\[
U := (u, v), \quad U_0 := (u_0, v_0),
\]
\[
\phi(u, v) = \begin{cases}
+\infty & \text{if } v < \inf \gamma_r(u) \\
\mathbb{R}^+ & \text{if } v \in \gamma_r(u) \setminus \gamma_l(u) \\
\{0\} & \text{if } \sup \gamma_r(u) < v < \inf \gamma_l(u) \\
\mathbb{R}^- & \text{if } v \in \gamma_l(u) \setminus \gamma_r(u) \\
-\infty & \text{if } v > \sup \gamma_l(u) \\
\mathbb{R} & \text{if } v \in \gamma_l(u) \cap \gamma_r(u),
\end{cases}
\tag{11}
\]

the following result is proved in [20]:

**Theorem 3.1** Assume that \(\gamma_l(u), \gamma_r(u)\) fulfil (4) and are affinely bounded, that is, there exist constants \(C_1, C_2 > 0\), such that \(\forall w \in \mathbb{R}, \forall z \in \gamma_h(w)\)

\[
|z| \leq C_1|w| + C_2, \quad (h = l, r).
\tag{12}
\]

Then the operator \(A + \mathcal{R}\) is \(m\)- and \(T\)-accretive in \(L^1(\Omega, \mathbb{R}^2)\).

Classical results of the theory of nonlinear semigroups can be applied yielding the following existence and uniqueness result.

**Theorem 3.2** Let \(\Omega\) be an open subset of \(\mathbb{R}^N (N \geq 1)\) of Lipschitz class. Let \(L^1(\Omega, \mathbb{R}^2)\) be endowed with the norm

\[
\|U\|_{L^1(\Omega, \mathbb{R}^2)} := \int_\Omega (\|u(x)\| + \|v(x)\|)dx
\]

\[
\forall U := (u, v) \in L^1(\Omega, \mathbb{R}^2).
\]

Assume that (12) holds. Take any \(U_0 := (u_0, v_0) \in L^1(\Omega, \mathbb{R}^2)\) such that \(U_0 \in \mathcal{D}(\phi)\) a.e. in \(\Omega\).

Then the Cauchy problem (7) has one and only one integral solution \(U : [0, T] \to L^1(\Omega, \mathbb{R}^2)\), which depends continuously on data \(u_0, v_0\). Moreover, if \(\mathcal{R}(u_0) \in L^1(\Omega)\), then \(U\) is Lipschitz continuous.

The theory of nonlinear semigroups is used to prove the existence and uniqueness of the integral solution of (7). The exact result arises from Theorem 3.2. It was expected, see [20], that the integral solution of (1) fulfills a condition of the type introduced by Kružkov. Visintin formulated it as an open problem in his book. Kopfová [8] showed that an integral solution of (1) defined by the semigroup theory (Theorem 3.2) satisfies an entropy condition introduced by Kružkov if \(v = \mathcal{F}[u]\) is represented by a (possibly discontinuous) generalized play or Prandtl-Ishlinskii hysteresis operator of play type. This is proved without the assumption of the symmetry of hysteresis boundary curves, unlike in [8].

**Theorem 3.3** Let the assumptions of Theorem 3.2 hold. Let \(A_0U = A(U) + \mathcal{R}(U)\) on \(\mathcal{D}(A_0)\), and let \(S(t) = (u, v)\) be the corresponding semigroup of contractions.

Let \(w \in \mathcal{D}(A)\) and \(t \geq 0\). Then if \(w = (u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)\),

\[
\int_0^T \int_\Omega |u - k|\psi_l(x, t)dx \, dt + \int_0^T \int_\Omega |v - \tilde{k}|\psi_l(x, t)dx \, dt + \int_0^T \int_\Omega |u - k|\psi_k(x, t)dx \, dt \geq 0
\]

for every \(\psi(x, t) \in C_0^\infty((0, T) \times \Omega)\) such that \(\psi \geq 0\) and every \(k, \tilde{k} \in \Lambda\) and \(T > 0\).
A generalization of this theorem, which also includes the equation (2), is proved in [9].

Visintin dealt with the Cauchy problem for the quasilinear hyperbolic equation of first order that contains an either continuous or discontinuous hysteresis operator $$\mathcal{F}[u]$$:

$$\begin{align*}
&\frac{\partial}{\partial t}(u + \mathcal{F}[u]) + \frac{\partial u}{\partial x} = f \quad \text{in} \quad \mathbb{R} \times (0, T) \\
&(u + v)(x, 0) = u(0) + v(0) \quad \text{in} \quad \mathbb{R},
\end{align*}$$

(13)

where $$\mathcal{F}[u]$$ is represented either by a relay operator or by its regularization, see Figure 4.

![Figure 4. Regularized relay operator](image)

Because of the discontinuity of the relay it is necessary to replace it by its closure with respect to appropriate topologies, i.e., the completed relay. For more details, see [20]. The existence of a weak solution is proved for $$\mathcal{F}[u]$$ equal either to a completed relay operator or to its regularization. Further, under the assumption that $$f \equiv 0$$, continuous monotone dependence of the solution on the initial data, and hence uniqueness, is proved. These results might be extended to the larger class of either continuous or discontinuous Preisach operators.

The precise result is based on the equivalent definition of the relay operator (Figure 3)

$$\begin{align*}
|v| &\leq 1,
\begin{cases}
(v - 1)(u - \rho_2) \geq 0 \\
(v + 1)(u - \rho_1) \geq 0,
\end{cases}
\text{a.e. in } (0, T),
\end{align*}$$

(14)

$$\int_0^t u \, dv \geq \int_0^t \rho_2 \, dv^+ - \int_0^t \rho_1 \, dv^- =: \Psi_\rho^0(v; [0, t]) \quad \forall t \in (0, T)$$

(15)

(these are Stieltjes integrals). Notice that $$\Psi_\rho^0(v; [0, t])$$ is finite whenever $$\frac{\partial v}{\partial t} \in C^0([0, T])$$.

The reformulated problem looks like:

**Problem 3.1** Find $$u \in L^\infty(0, T; L^2(\mathbb{R}))$$ and $$v \in L^\infty(\mathbb{R}_T)$$ such that

$$\begin{align*}
|v| &\leq 1 \text{ a.e. in } \mathbb{R}_T, \quad \frac{\partial v}{\partial t} \in C^0(\mathbb{R}_T),
\int \int_{\mathbb{R}_T} \left( (u + v - u(0) - v(0)) \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + f v \right) \, dx \, dy = 0
\end{align*}$$

$$\forall w \in H^1(\mathbb{R}_T) \cap W^{1,1}(\mathbb{R}_T), \quad w(\cdot, T) = 0,$$

(16)

(17)
\[
\left\{ \begin{array}{l}
(v - 1)(u - \rho_2) \geq 0 \\
(v + 1)(u - \rho_1) \geq 0,
\end{array} \right. \quad \text{a.e. in } (\mathbb{R}_T),
\]
\[
\frac{1}{2} \int_{\mathbb{R}} [u(x,t)^2 - u(x,0)^2] \, dx + \int_{\mathbb{R}} \Psi_0^0(v; [0,t]) \, dx \leq \int_{\mathbb{R}_t} fu \, dx \, d\tau
\]
\[
\quad \text{for a.e. } t \in (0,T).
\]

Visintin proved the existence of a solution of Problem 3.1 via time discretization, and the derivation of a priori estimates.

**Theorem 3.4**

Set \( \mathbb{R}_t := \mathbb{R} \times (0,t) \) for any \( t > 0 \), fix any \( T > 0 \), and assume that
\( u(0), v(0) \in L^2(\mathbb{R}), \quad |v(0)| \leq 1 \) a.e. in \( \mathbb{R} \), \( f \in L^1(\mathbb{R}_T) \cap L^2(\mathbb{R}_T) \).

Then Problem 3.1 has a solution. Moreover, for any \( p \in [1, +\infty] \),
\[
\begin{align*}
u(0), v(0) & \in BV(\mathbb{R}), \quad f \in L^p(0,T; BV(\mathbb{R})) \Rightarrow \\
u, v & \in L^\infty(0,T; BV(\mathbb{R})) \cap W^{1,p}(0,T; C^0(\mathbb{R})).
\end{align*}
\]

**4. Differentiable solution**

Showalter and Peszynska [18] considered an equation in the form
\[
\frac{\partial (u + v)}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad v = F[u], \quad x \in (0,L), \quad t > 0
\]
with the boundary condition
\[
u(0,t) = \varphi(t), \quad t > 0,
\]
and the initial condition
\[
u(x,0) = v(x,0) = 0, \quad x \in (0,L).
\]

They assumed that \( v = F[u] \) is the elementary case of those hysteresis functionals whose sides are bounded by parallel lines, i.e., the classical play. Firstly, they computed an explicit solution. Then they proved the existence and uniqueness of a differentiable solution by using the theory of nonlinear semigroups in \( L^2 \). This is possible because of the special form of the hysteresis operator.

The classical play operator is characterized by the evolution equation
\[
v_t + \text{sgn}^{-1}(v - u) \ni 0, \quad t > 0
\]
\( \text{sgn}^{-1}(\cdot) \) is a maximal monotone graph in \( \mathbb{R} \).

In addition to the usual formulation of the existence-uniqueness results, a second one is suggested by switching the variables \( x \) and \( t \) to represent the solution by a semigroup of breakthrough curves. The spatial variable \( x \) is treated as another time variable, and therefore it is renamed: \( x = \tau \). Then the system is given by
\[
u(0, t) = \varphi(t),
\]
\[
u(\tau, 0) = 0 = v(\tau, 0).
\]
In order to define $\mathcal{A}$, set $\mathcal{D}(\mathcal{A}) = \{u \in H^1(0,T) : u(0) = 0\}$. Note that for each such $u \in \mathcal{D}(\mathcal{A})$, the Cauchy problem

$$
(v - u)_t + \text{sgn}^{-1}(v - u) \geq -u_t,
$$

$$(v - u)(0) = 0
$$

has a unique solution $v \in H^1(0,T)$. Then $\mathcal{A}$ is defined as $\mathcal{A}(u) = u_t + v_t$ with $u \in \mathcal{D}(\mathcal{A})$ and $v$ as given.

It can be proved that $\mathcal{A}$ is $m$–accretive on $L^2(0,T)$ see [18] and they have the following result:

**Theorem 4.1** If $\varphi \in W^{1,2}(0,T) = H^1(0,T)$ there is a unique solution of (23 - 26) with $u_x \in L^\infty(0,L;L^2(0,T))$, and for all $x$, $u(x,.) \in H^1(0,T)$, $u(x,0) = 0$.

5. Discontinuity and stability

Now there is a question whether we can get discontinuous solutions, similar to the quasilinear case without hysteresis, see [19]. We consider the partial differential equation (1) with the given initial conditions

$$
u(x,0) = u_0(x), \quad v(x,0) = v_0(x).
$$

In order to illustrate the typical features of hysteresis, the cases were studied where the functional is first represented by the play hysteresis operator whose sides are bounded by parallel lines [18]. Secondly a slightly different (nonconvex) hysteresis model is considered. Thirdly we presumed a play operator with a skip. The purpose was to study the corresponding classical initial value problem where the functional $\mathcal{F}[u]$ is represented by a convex play operator and additionally by a nonconvex hysteresis operator. The solution of this problem was investigated coupled with each model, i.e., we computed an explicit solution by the method of characteristics.

It follows from direct computations that in the case of a regularized relay (see Figure 4) the equation exhibits similar properties as a quasilinear hyperbolic equation of first order, i.e., the characteristics cross and the solution has to be discontinuous.

The paper [11] is concerned with the asymptotic behaviour of the solution of (1). We used the theory of nonlinear semigroups in $L^1$ and an asymptotic result of Wittbold [22]. The precise result is as follows.

**Theorem 5.1** Suppose all conditions of Theorem 3.1 are satisfied, i.e., the operator $\mathcal{A} + \mathbb{R}$ is $m$- and $T$-accretive in $L^1(\Omega,\mathbb{R}_+^2)$. Suppose also that $u_0 \in \overline{\mathcal{D}(\mathcal{A} + \mathbb{R})}$. Then there exists $v_\infty(x)$ dependent on $x$ only, such that for the solution

$$
U = \begin{pmatrix} u \\ v \end{pmatrix}
$$

of

$$
\frac{\partial U}{\partial t} + (\mathcal{A} + \mathbb{R})(U) = 0, \quad U(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},
$$

the following holds

$$
\|u\|_1 - \lim_{t \to \infty} u(x,t) = 0,
$$

$$
\|v\|_1 - \lim_{t \to \infty} v(x,t) = v_\infty(x).
$$
These results can be extended to possibly discontinuous generalized Prandtl-Ishlinskii operators of play type. This includes the case of possibly discontinuous Preisach operators.

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