Multiqubit Matter-Wave Interferometry under Decoherence and the Heisenberg Scaling Recovery

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Most of the matter-wave interferometry (MWI) schemes for quantum sensing are so far evaluated in ideal situations without noises. In this work, we provide assessments of generic multiqubit MWI schemes under realistic dephasing noises. We find that for certain classes of the MWI schemes, the use of entangled probes could be detrimental to the optimal precision. This result challenges the conventional wisdom that entanglement can enhance the precision in quantum metrology. We initiate the analyses by investigating the optimal precision of multiqubit Sagnac atom interferometry for rotation sensing. And we show that due to the competition between the unconventional interrogation-time quadratic phase accumulation and the exponential dephasing processes, the Greenberger-Horne-Zeilinger (GHZ) state, which is the optimal input state in noiseless scenarios, gives even worse precision than its uncorrelated counterpart. Then our assessments are further extended to generic MWI schemes for quantum sensing with entangled states and under decoherence. Finally, a quantum error-correction logical GHZ state is tentatively analyzed, which could have the potential to recover the Heisenberg scaling and improve the sensitivity.

Introduction.— Matter-wave interferometry (MWI) is sensitive to inertial effects and has been widely used in quantum sensing of physical quantities, including gravitational force, acceleration, and rotation of reference frames [1]. With quantum entanglement as resources, quantum sensing is expected to achieve higher precision and sensitivity, e.g., the Heisenberg limit [2, 3]. Sagnac atom-interferometry gyroscopes (SAIGs) are quantum sensors for rotation frequency based on the Sagnac interferometry [4] of matter waves, where atoms are coherently split and controlled with wave guides (e.g., see Ref. [5]) to enclose a finite area in space and encode the rotation frequency into the Sagnac phase, which can be finally read out from the interference fringes [4–11].

Most of schemes for SAIG utilize both wave nature and spin degrees of freedom (hyperfine states) of atoms. For example, a scheme with uncorrelated and trap-guided atomic clocks was proposed in Ref. [12], and was later generalized to one with multiparticle Greenberger-Horne-Zeilinger (GHZ) state to beat the standard quantum limit (SQL) in Ref. [13]. So far, these proposed schemes were considered in ideal situations where the sensing protocols consisted of perfect unitary quantum channels. However, in realistic experiments, inevitable noises may cause errors which will change the unitary quantum channels and prevent from the expected precision. For example, phase-flip errors may be caused by dephasing noises, which are the dictating noises in cold-atom and trapped-ion sensors, and can be induced, for instance, by laser-field fluctuations and instabilities [14].

In standard Ramsey interferometers for atomic clocks, where the transition frequency $\omega$ between two energy levels of atoms is measured, the phase accumulation is linear in the interrogation time while the dephasing caused by noises is exponential. And the use of entangled states has been proved to only give a constant improvement for the ultimate precision, but still follows the SQL [14–16]. Several strategies and techniques have been proposed and used to protect the precision of atomic clocks from noises [17–21]. Whereas, the evaluation and optimization of generic multiqubit MWI schemes for quantum sensing under decoherence still remain challenging.

In this Letter, we present the first assessment of generic MWI schemes with maximally entangled states and under independent dephasing noises. Start with the SAIG as a prototype, we analyze the competition between the unconventional phase accumulation and the exponential dephasing processes. And we find that for certain classes of the MWI schemes, the use of entangled probes could be detrimental to the optimal precision, if no further steps are taken to suppress noise effects. These classes include most of the current mainstream MWI schemes with atomic clock states and certain time-dependently controlled Hamiltonian systems. Our findings challenge the conventional wisdom that entanglement can enhance or at least does not reduce the precision in quantum metrology [14–16]. Finally, we tentatively analyze a logical GHZ state, which could potentially recover the Heisenberg scaling and improve the sensitivity.

Matter-wave Sagnac interferometry with entangled states.— In order to sense the rotation frequency of a reference frame $R$ rotating in an angular velocity $\Omega$ with respect to an inertial frame $K$, an ensemble of $N$ two-state cold atoms can be initially prepared at the GHZ state, which is the optimal multiparticle input state for unitary quantum channels [22], i.e., $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, where $|0\rangle = |0\rangle^\otimes N$ and $|1\rangle = |1\rangle^\otimes N$, with $|0\rangle = |\uparrow\rangle (|1\rangle = |\downarrow\rangle)$ being the single-atom (pseudo)spin state and eigenstate of Pauli matrix $\sigma_z$ with eigenvalue $+1$ ($-1$). Subsequently, the $|0\rangle$ and $|1\rangle$ components are coherently split by a beam splitter which establishes a state-path entanglement, and then are guided to transport along two distinct paths in real space [23], within an
interrogation time \( \tau \), and are finally recombined at \( t = \tau \).

The state-path entanglement associates the phase shift between two interferometer paths (Sagnac phase) with the relative phase of two atomic internal states (quubit phase), which can be read out from the atomic spectroscopy, e.g., parity measurement [13, 24, 25], after applying a \( \pi/2 \) pulse.

Model.—We assume that the \( N \) two-state bosonic atoms are in the Bose-Einstein condensed (BEC) state, which is described by the mean-field wave function (order parameter) \( \Psi \xi (r, t) \) for the two split components \( | \xi \rangle = |0 \rangle \) and \( | \xi \rangle = |1 \rangle \), respectively. And the wave guide is provided by a ring trap of toroidal geometry [7, 9, 10, 26], with a trapping potential \( \omega_r \) and \( \omega_\theta \), which plays the role of a beam splitter, with \( \theta \) has a kicking operator \( \hat{\Omega}_\theta \) for the single-qubit time evolution operator reads

\[
\hat{U}(\theta) = \exp (\hat{\Omega}_\theta) = \exp \left( \frac{i}{\hbar} \hat{\Omega}_\theta \right),
\]

where \( \hat{\Omega}_\theta \) is the angular momentum operator, \( F \) is the contact interaction strength, with \( F \) have a kicking the two components with \( \pm \nu \) group speed, respectively, as in Refs. [7, 10]. Formally, this can be realized by applying the following kicking operator \( \hat{K}(\nu) = \exp \left( \frac{i}{\hbar} \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \right) \), which plays the role of a beam splitter, with \( L_k = m_F v \) being the kicking angular momentum and \( \sigma_j \) being the Pauli \( Z \) matrix of the \( j \)th particle. Finally, at time \( t = \tau \) when the two components are recombined for the first time [28], the full quantum state is given by

\[
\hat{U}_i (\tau) \hat{K}(\nu) \hat{U}_j (\tau) = \hat{K}(\nu) \hat{U}_j (\tau) \hat{U}_i (\tau),
\]

where \( \hat{U}_i (\tau) \) is the time evolution operator for the \( i \)th qubit. Here for well-defining a Sagnac phase, we assume that the interaction \( \hat{U} = 0 \) during the interrogation. Consequently, the quantum state in the spin subspace after tracing out the orbital degrees of freedom related to \( \Psi \xi (\theta, \tau) \) is given by

\[
|\psi(\phi)\rangle = \left( e^{i\phi/2} |0\rangle + |1\rangle \right)/\sqrt{2} \quad \text{(up to a global phase factor)},
\]

where [29]

\[
\phi_S = \beta N \Omega \tau^2
\]

is the multiparticle Sagnac phase, with \( \beta = 2m v^2 / (\pi \hbar) \). This expression for \( \phi_S \) is equivalent to \( N \) times the well-known single-particle Sagnac phase \( 2m v A / \hbar \), where \( A = \pi R^2 \) is the area of the Sagnac interferometer, and for constant \( v \) we have \( A = v^2 R^2 / \pi \).

As a result, a Sagnac pure phase gate as an unitary operation mapping the initial state of the qubits to the readout state is constructed, which in the GHZ subspace spanned by \(|0\rangle, |1\rangle\) reads

\[
U(\phi_S) = \text{diag} \left[ e^{i\phi_S}, 1 \right],
\]

and the rotation frequency can be extracted from the interference signal of the final state (after applying a \( \pi/2 \) pulse). Following standard quantum metrological protocols, the above Sagnac phase encoding and rotation frequency readout processes are repeated for \( \nu = T/\tau \) times to reach a high precision, where \( T \) is the total resource time (See Fig. 1). And the standard deviation \( \delta \Omega \) for any unbiased estimator \( \Omega \) of the rotation frequency is bounded from below by the quantum Cramér-Rao bound (QCRB) [30, 31], \( \delta \Omega \geq 1/\sqrt{F} \), where \( F \) is the quantum Fisher information (QFI) with respect to \( \Omega \) for the readout state, which is an effective theoretical tool for assessing the performance of various interferometry schemes for quantum sensing [10]. Equivalently, we have \( \delta \Omega \sqrt{T} \geq 1/\sqrt{F/\tau} \). For more introduction to quantum sensing and QFI, see Ref. [29].

From Eq. (2), we see that the accumulated Sagnac phase is linear in the number of qubits \( N \) while it is quadratic in the interrogation time \( \tau \), and the noiseless optimal QFI for the readout state is \( F_0 = (\delta \Omega \phi_S)^2 = (2mN \Omega / \hbar)^2 \) [13], which achieves the Heisenberg scaling and increases monotonically.
with the size of the interferometer. The $A^2$ scaling of $F_0$ results in a $\tau^4$ scaling and may give a higher accuracy [32, 33]. However, this situation will be changed in a noisy environment.

**Optimal sensitivity under decoherence.—**Now we turn to the situation where the atoms are under decoherence during the interrogation which may cause imperfections of the Sagnac phase gate. In SAIG experiments, the dominate errors come from dephasing noises, which may caused by interparticle collisions or laser-field instabilities, so here we mostly focus on the purely dephasing channel, for which the phase accumulation and dephasing operators are commutative. The time evolution of the state $\rho(t)$ in the phase-covariant frame is given by [34]

$$\frac{d\rho(t)}{dt} = \gamma \sum_{i=1}^{N} [\sigma_{iz}\rho(t)\sigma_{iz} - \rho(t)],$$

where $\gamma > 0$ is the dephasing strength. The completely positive and trace preserving (CPTP) map $E$, which is a solution of Eq. (4) and maps $\rho_0$ to $\rho(\tau)$, is $\rho(\tau) = E(\rho_0) = \bigotimes_{i=1}^{N} E_i(\rho_0)$, where $\rho_0 = \rho_0 = \ket{\psi_0}\bra{\psi_0}$ is the initial state and $E_1(\rho_0) = E_{00}\rho_0 E_{00}^{\dagger} + E_{11}\rho_0 E_{11}^{\dagger}$ is the local phase-flip error operator for the $i$th qubit, with $E_{00} = \sqrt{1-p(\tau)}I_2$ and $E_{11} = \sqrt{p(\tau)}\sigma_{iz}$ being the Kraus operators, while $p(\tau) = (1-e^{-\gamma\tau})/2$ is the single-qubit error probability. Then it is straightforward to reach the readout state [35], $\rho(\tau) = E[U(\phi_S)\rho_0 U^{\dagger}(\phi_S)] = \frac{1}{2} \left( |0\rangle\langle 0| + |1\rangle\langle 1| + (e^{-N\gamma\tau}e^{i\delta_S}|0\rangle\langle 1| + h.c.) \right)$, where $\phi_S$ is given by Eq. (2). The QFI with respect to $\Omega$ for this state is [29]

$$F = \beta^2 N^2 \tau^3 e^{-2N\gamma\tau}.$$  

(F/\tau)_{opt} = \beta^2 \left( \frac{3}{2\gamma^3} \right) \frac{1}{N},$$

with $\tau_{opt} = 3/(2N\gamma)$. Therefore the optimal lower precision bound (precision uncertainty) is proportional to $\sqrt{N}$, which is totally different from the noiseless scenarios, where the Heisenberg scaling can be achieved. On the other hand, uncorrelated qubits are used, the QFI is given by $F_{SQL} = N\beta^2 e^{-2N\gamma\tau}$, which is proportional to $N$ for any value of $\tau$. Shown in Fig. 2 is the interrogation-time normalized $F$ for increasing the qubit number $N$ with (a) GHZ probe and (b) uncorrelated qubits, respectively. For a given value of $N$ and increasing $\tau$, the power law $\tau^3$ behavior dominates at the beginning while the exponential prevails after reaching a maximum. While the SQL is achieved in Fig. 2(b) with independent qubits, in contrast, the QFI curves are shrinking with increasing $N$ in Fig. 2(a) for the GHZ probe. So the QFI in the presence of dephasing will be reduced by increasing the number of (maximally) entangled probe qubits and even be lost in the large $N$ limit. The physical reason is that $\phi_S(\tau = \tau_{opt}) \propto N^{-1}$, such that the accumulated phase signal is weakened with increasing $N$.

**Assessments of generic MWI schemes.—**Eq. (6) is one of the main results of this Letter and it may be applied to analyze more generic MWI schemes for quantum sensing with GHZ states and under independent dephasing, which were previously only considered in single-qubit or noiseless scenarios, as long as the accumulated phase in the readout state is $\phi_\chi(\tau) \propto N\chi\tau^2$, where $\chi$ is the physical quantity to be sensed. For example, one may assume that GHZ states are used in the following schemes, e.g., the atom gravimetry considered in Refs. [5, 36–39], with the phase shift $\phi_g(\tau) = k_0 \cdot g(\tau/2)^2$ and the atom free-propagation Sagnac interferometers in Refs. [4, 6, 40, 41], with the encoded single-qubit phase $\phi_{\Omega}(\tau) = k_0 \cdot (v_0 \times \Omega) \tau^2/2$, where $g$ is gravitational acceleration, and $v_0$ and $k_0$ are the semiclassical velocity of
atoms and effective Raman propagation vectors, respectively. In general, for GHZ probes in the presence of independent dephasing, if the accumulated phase is \( \phi_\lambda (\tau) \propto N \gamma \tau^\lambda \) with \( \lambda > 0 \), then the optimal QFI with respect to \( \chi \) is \( (F_\chi / \tau)_{\text{opt}} = O(N^{3-2\lambda}) \), with \( \tau_{\text{opt}} = (2\lambda - 1)/(2\gamma) \) [29]. Therefore, we may conclude that the Heisenberg scaling is actually inaccessible because the condition \( \tau_{\text{opt}} > 0 \) requires \( \lambda > 1/2 \). And for \( \lambda \geq 1 \), the best QFI can be achieved is \( (F_\chi / \tau)_{\text{opt}} = O(N) \) (SQL) with the \( \lambda = 1 \) class [14–16]. For classes with \( \lambda > 2 \), the use of entangled probes could be detrimental to the precision.

It should be noted that most of the current mainstream MWI schemes with atomic clock states [4–6, 36–41] belong to the \( \lambda = 2 \) class (without entanglement), and so do the proposed schemes in Refs. [32, 33] with time-dependently controlled Hamiltonians. In Table I we give a comparison between standard Ramsey interferometers for atomic clocks (\( \lambda = 1 \)) and Sagnac interferometers for rotation sensing (\( \lambda = 2 \)) [42], where the input states for both cases are GHZ states of \( N \) particles and \( H_{\text{single}} \) denotes the single-particle sensing Hamiltonian. As above, \( \tau \) is the interrogation time at which the probe state is measured and the signal is read out.

Table I. Comparison between Ramsey and Sagnac interferometers with GHZ probe states

| Interferometers | Quantity | \( H_{\text{single}} \) Phase | Noiseless \( F/\tau \) | Noiseless \( \tau_{\text{opt}} \) | Dephasing | Noisy \( (F/\tau)_{\text{opt}} \) | Noisy \( \tau_{\text{opt}} \) |
|-----------------|----------|-------------------------------|----------------------|----------------|----------------|-------------------------------|----------------|
| Ramsey | \( \omega \) | \( \hbar \omega \sigma_z/2 \) N\( \omega \tau \) | \( O(N^2) \) | \( T \) | \( e^{-N\gamma \tau} \) | \( O(N) \) [14–16] | \( 1/(2N\gamma) \) |
| Sagnac | \( \Omega \) | \( -\Omega L_z \) \( \beta N \Omega \tau^2 \) | \( O(N^2) \) [13] | \( T \) | \( e^{-N\gamma \tau} \) | \( O(N^{-1}) \) | \( 3/(2N\gamma) \) |

In summary, we have presented an assessment of the optimal precision given by the QCRB for matter-wave interferometers, with multiqubit GHZ input and in the presence of decoherence. Our results show that due to the competition between the unconventional phase accumulation (i.e., \( \lambda \geq 2 \)) and the exponential dephasing, the use of entangled probes could be detrimental to the precision, which challenges the conventional wisdom. Finally, we tentatively analyzed a QEC scheme with logical GHZ states, which could have the potential to protect the Heisenberg scaling. Our work should be instrumental and should find applications in designing and optimizing real-world MWI schemes for high-precision quantum sensing.

**Conclusion.**—In summary, we have presented an assessment of the optimal precision given by the QCRB for matter-wave interferometers, with multiqubit GHZ input and in the presence of decoherence. Our results show that due to the competition between the unconventional phase accumulation (i.e., \( \lambda \geq 2 \)) and the exponential dephasing, the use of entangled probes could be detrimental to the precision, which challenges the conventional wisdom. Finally, we tentatively analyzed a QEC scheme with logical GHZ states, which could have the potential to protect the Heisenberg scaling. Our work should be instrumental and should find applications in designing and optimizing real-world MWI schemes for high-precision quantum sensing.

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I. DERIVATION OF THE MULTIQUBIT SAGNAC PHASE IN EQ. (2) OF THE MAIN TEXT

We assume that the $N$ two-state bosonic atoms are in the Bose-Einstein condensed (BEC) state during the interrogation, which is described by the mean-field wave function (order parameter) $\Psi(\theta,t)$ for the two split components $|\xi\rangle = |0\rangle$ and $|\xi\rangle = |1\rangle$, respectively. The trapping potential in cylindrical coordinates $\{r,\theta,z\}$ reads $V_{\text{trap}}(r,t) = \frac{1}{2}m \left[ \omega^2_r (r-R)^2 + \omega^2_\theta R^2 \theta^2 \Theta(-t) + \omega^2_z z^2 \right]$ [1, 2], where $m$ is the particle mass and $(\omega_r,\omega_\theta,\omega_z)$ are the respective (radial, angular, axial) trapping frequencies, and $R$ is the radius of the circular interferometer. $\Theta(t)$ is the standard step function with the definition $\Theta(t)=1$ for $t \geq 0$ and $\Theta(t)=0$ for $t < 0$. When the radial and axial trapping confinements are sufficiently tight, the dynamics along these directions is frozen and then the time evolution ($t \geq 0$) of the order parameter in the rotating frame $\hat{\mathcal{R}}$ is given by the one-dimensional Gross-Pitaevskii (GP) equation $i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\theta,t) = H_\xi \hat{\Psi}(\theta,t)$, with the mean-field Hamiltonian

$$H_\xi = \frac{\hat{L}_z^2}{2mR^2} + \mathcal{U} |\Psi(\theta,t)|^2 - \Omega \hat{L}_z,$$

where $\hat{L}_z = -i\hbar \frac{\partial}{\partial \theta}$ is the axial angular momentum operator and $\mathcal{U}$ is the contact interaction strength.

For the $\mathcal{U} = 0$ case, the time evolution operator for the $i$th particle reads

$$\hat{U}_i(t) = \exp \left( \Omega t \frac{\partial}{\partial \theta_i} \right) \exp \left[ \frac{i\hbar}{2mR^2} \frac{\partial^2}{\partial \theta_i^2} \right] \otimes \mathcal{I}_2,$$

where $\mathcal{I}_2$ is the two-dimensional identity matrix.

The trapping potential along the angular direction is $V_{\text{trap}}(\theta,t) = \frac{1}{2}m \omega^2_\theta R^2 \theta^2 \Theta(-t)$, and we assume that for the both components, the initial mean-field wave function at $t = 0$ is a Gaussian wave packet, i.e., ground state of the harmonic trap, $\Psi(\theta,0) = \left( \frac{1}{\sqrt{\pi\sigma}} \right)^{\frac{\xi}{2}} \exp \left\{ -\frac{\theta(\theta-\theta(0))^2}{2\sigma^2} \right\}$ for $\theta \in [\theta(0) - \pi, \theta(0) + \pi]$, where $\theta(0) = 0$ and $\sigma = \sqrt{\hbar/(m\omega_\theta)}$, $R \ll \pi$ are the initial center and the width of the wave packet, respectively. Due to the periodicity of the $\theta$ coordinate, the wave function outside this interval can be defined via $\Psi(\theta + 2J\pi,0) = \Psi(\theta,0)$, with $J$ being an integer. The multiqubit initial GHZ state, is given by

$$|\tilde{\psi}(\theta_1,\theta_2,\ldots,\theta_N;0)\rangle = \frac{1}{\sqrt{2}} \prod_{i=1}^{N} \Psi(\theta_i,0) (|0\rangle + |1\rangle),$$

for which the normalization condition is given by $1 = \int d\theta_1 d\theta_2 \cdots d\theta_N \langle \tilde{\psi}(\theta_1,\theta_2,\ldots,\theta_N;0) | \tilde{\psi}(\theta_1,\theta_2,\ldots,\theta_N;0) \rangle$. The interferometer is then launched at $t = 0$ via kicking the two components with $\pm v$ group velocity, respectively, as in Refs. [1, 2]. Formally, this can be realized by applying the following kicking operator $\hat{K}(v) = \exp \left( \frac{i}{\hbar} L_k \sum_{j=1}^{N} \theta_j \sigma_j^z \right)$, which plays the role of a beam splitter, with $L_k = mRv$ being the kicking angular momentum and $\sigma_j^z$ being the Pauli Z matrix of the $j$th particle. We assume that the interaction $\mathcal{U} = 0$ during the interrogation for well-defining a Sagnac phase. Consequently, the full quantum state at time $t = \tau$ when the two components are recombined for the first time, is given by

$$|\tilde{\psi}(\theta_1,\theta_2,\ldots,\theta_N;\tau)\rangle = \hat{K}^\dagger(v) \bigotimes_{i=1}^{N} \hat{U}_i(\tau) \hat{K}(v) |\tilde{\psi}(\theta_1,\theta_2,\ldots,\theta_N;0)\rangle,$$

where $\hat{U}_i(\tau)$ is the time evolution operator for the $i$th qubit, which is given by Eq. (S2).
After applying the kicking operator, the mean-field wave function for the jth particle of $|\xi_j\rangle$ spin state ($\xi = 0, 1$) is given by $\Psi_\xi (\theta_j, 0) = \Psi (\theta_j, 0) \exp \left[ -(1)^2 i L_k \theta_j / \hbar \right]$, which can be directly obtained with $\sigma_{jz} |\xi_j\rangle = (-1)^\xi |\xi_j\rangle$, and $\Psi (\theta_j, 0)$ is the initial Gaussian wave packet. Therefore, the wave function at time $t$ reads

$$\Psi_\xi (\theta_j, t) \otimes |\xi_j\rangle = \hat{U}_j (t) \Psi_\xi (\theta_j, 0) \otimes |\xi_j\rangle . \quad (S5)$$

In addition, the Fourier transform of the initial Gaussian wave packet is given by $\Psi (\theta, 0) = [1/(2\pi)]^{1/2} \sum_{l=-\infty}^{l=+\infty} \hat{\Psi} (l) \exp (i l \theta)$, where

$$\check{\Psi} (l) = [1/(2\pi)]^{1/2} \int_{-\pi}^\pi \Psi (\theta, 0) \exp (-i l \theta) \, d\theta \approx \sigma / \sqrt{\pi} \exp (-\sigma^2 l^2 / 2) \exp \left( \frac{\pi + i \sigma^2 l}{\sqrt{2} \sigma} \right), \quad (S6)$$

where $\text{erf} (z) = 2 / \sqrt{\pi} \int_0^z \exp (-t^2) \, dt$ is the Gaussian error function, for which $\text{erf} (z) \approx 1$ when $\text{Re} z \gg 1$, which is the situation with $\sigma \ll \pi$ here. And by applying the time evolution operator $\hat{U}_j (t)$, one can obtain

$$\Psi_\xi (\theta_j, t) \approx \left( \frac{1}{\sqrt{\pi} \hat{\sigma} (t)} \right)^{1/2} \exp \left\{ -\frac{|\theta_j - \theta^{(\xi)} (t)|^2}{2 \sigma (t)} \right\} \times \exp \left[ -(1)^\xi \theta / \hbar (\Omega t + \theta) \right] \exp \left[ -i L_k^2 / 2 m \hbar R \right] \times \sum_{n=-\infty}^{+\infty} \exp \left\{ 2 \pi i n \kappa - 2 \pi^2 n^2 / |\sigma \hat{\sigma} (t)| \right\} , \quad (S7)$$

where $\hat{\sigma} (t) = \sigma + i L_k / (m R^2 \sigma)$ and $\theta^{(\xi)} (t) = \left[ -(1)^\xi \theta / \hbar - \Omega t \right]$, and $\kappa = -i \left[ \theta_j - \theta^{(\xi)} (t) / |\sigma \hat{\sigma} (t)| \right]$. Furthermore, under the condition $|\hat{\sigma} (t)| \ll \pi$ for $t \in [0, \tau]$, we have $\sum_{n=-\infty}^{+\infty} \exp \left\{ 2 \pi i n \kappa - 2 \pi^2 n^2 / |\sigma \hat{\sigma} (t)| \right\} \approx 1$, and then we obtain

$$|\Psi_\xi (\theta_j, t)|^2 \approx \frac{1}{\sqrt{\pi} |\hat{\sigma} (t)|} \exp \left\{ -\frac{|\theta_j - \theta^{(\xi)} (t)|^2}{|\hat{\sigma} (t)|^2} \right\} . \quad (S8)$$

Therefore, at time $t$ and under the condition $|\hat{\sigma} (t)| \ll \pi$, the wave function in Eq. (S7) describes Gaussian wave packets centered at $\theta^{(\xi)} (t)$, i.e., propagating in group linear velocity $- \xi \theta / \hbar - \Omega R$, for $\xi = 0$ and 1, respectively, and with the same width $|\hat{\sigma} (t)|$. The interrogation time (or collision time) $\tau$, at which the two centers of the counter-propagating Gaussian wave packets are completely overlapped, is given by $\theta^{(0)} (\tau) - \theta^{(1)} (\tau) = 2 \pi$, or equivalently, $\tau = \pi R / \nu$.

With above results, one can obtain the multiparticle readout state $|\tilde{\psi} (\theta_1, \theta_2, ..., \theta_N; \tau)\rangle$ in Eq. (S4) and the corresponding density matrix reads $\tilde{\rho} (\theta_1, \theta_2, ..., \theta_N; \tau) = |\tilde{\psi} (\theta_1, \theta_2, ..., \theta_N; \tau)\rangle \langle \tilde{\psi} (\theta_1, \theta_2, ..., \theta_N; \tau)|$. The reduced density matrix in the spin subspace after tracing out the orbital degrees of freedom related to $\Psi_\xi (\theta, \tau)$ is given by

$$\rho (\tau) = \int d\theta_1 d\theta_2 \cdots d\theta_N \tilde{\rho} (\theta_1, \theta_2, ..., \theta_N; \tau) \approx \frac{1}{2} \left[ |0\rangle \langle 0| + |1\rangle \langle 1| + (e^{i \phi_S} |0\rangle \langle 1| + \text{h.c.}) \right] , \quad (S9)$$

where

$$\phi_S = \beta N \Omega \tau^2 \quad \text{(S10)}$$

is the multiparticle Sagnac phase, with $\beta = 2 m v^2 / (\pi \hbar)$. This expression for $\phi_S$ is equivalent to $N$ times the well-known single-particle Sagnac phase $2 m \Omega A / \hbar$, where $A = \pi R^2$ is the area of the Sagnac interferometer, and for constant $v$ we have $A = v^2 \tau^2 / \pi$. The corresponding spin-subspace quantum state can be written as $|\psi (\tau)\rangle = (e^{i \phi_S} |0\rangle + |1\rangle) / \sqrt{2}$ (up to a global phase factor), with which $\rho (\tau)$ can be given by $\rho (\tau) = |\psi (\tau)\rangle \langle \psi (\tau)|$. 
II. QUANTUM SENSING AND QUANTUM FISHER INFORMATION

The QFI plays a crucial role in quantum metrology and quantum sensing. Our basic quantum resources for a SAIG include \( N \) cold probe (two-level) atoms (qubits), total sensing time \( T \), single-round interrogation time \( \tau \), and the controlling and measurement devices. In a standard metrological scheme, the initial state of the probe is prepared at \( \rho_0 \) and followed by a dynamical evolution \( \rho_0 \xrightarrow{\phi_S(t)} \rho_\chi \), which encodes the quantity \( \chi \) to be sensed into the relative phase \( \phi_\chi(t) \) of qubits, and can be read out by quantum measurements after a single-round time \( t = \tau \). Within the total time \( T \), the number of repetitive rounds of sensing and measurement is \( \nu = T/\tau \). The standard deviation for any unbiased estimator \( \hat{\chi} \) is bounded from below by the quantum Cramér-Rao bound [4, 5],

\[
\delta \hat{\chi} \geq 1/\sqrt{\nu F},
\]

(S11)

where \( F \) is the QFI at \( t = \tau \), or equivalently,

\[
\delta \hat{\chi} \sqrt{T} \geq 1/\sqrt{F/\tau}.
\]

(S12)

Thus, finding the optimal input state and quantum measurement to maximize the QFI is a central problem in high precision quantum sensing. In general, the QFI of \( \chi \) associated with \( \rho_\chi \) is defined by \( F = \text{Tr}(\rho_\chi L^2) \) [4, 5], where \( \text{Tr} \) is the trace operation and \( L \) is the symmetric logarithmic derivative (SLD) operator, which is given by

\[
\partial_\chi \rho_\chi = (\rho_\chi L + L\rho_\chi)/2.
\]

(S13)

Usually, a signal accumulation process is a unitary quantum channel, which gives \( \rho_\chi = U_\chi \rho_0 U_\chi^\dagger \), where \( U_\chi \) is a time and \( \chi \) dependent unitary operator. It has been shown that for a pure state in unitary quantum channels, the QFI can be obtained from the variance of a Hermitian operator \( \mathcal{H} = i (\partial_\chi U_\chi^\dagger) U_\chi \) in \( \rho_0 \), with [6–9]

\[
F = 4(\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2),
\]

(S14)

where \( \langle O \rangle := \text{Tr}(\rho_0 O) \) for any operator \( O \). For an ensemble of \( N \) qubits as the input state in a standard Ramsey experiment, the maximal QFI (\( \propto N^2 \)) is obtained when \( \rho_0 \) is the GHZ state [8], and when the inputs are uncorrelated qubits, \( F \propto N \). So the GHZ state gives the Heisenberg scaling for the sensing precision while the uncorrelated inputs leads to the SQL, according to Eq. (S12). However, in the presence of noises, the unitary quantum channel will be modified by errors, and the corresponding QFI will be reduced or even be lost. As a result, the expected sensing precision may not be achieved. See the main text.

III. CALCULATIONS OF THE NOISY QFI FOR THE READOUT STATES

The spectral decomposition of the density matrix \( \rho \) is given by

\[
\rho = \sum_{i=1}^{d} p_i |\psi_i\rangle \langle \psi_i|,
\]

(S15)

where \( d \) is the dimension of the support set of \( \rho \), and \( p_i \) is the \( i \)th eigenvalue of \( \rho \), with \( |\psi_i\rangle \) being the corresponding \( i \)th eigenvector. With this representation, the QFI with respect to the quantity \( \chi \) can be expressed as [10, 11]

\[
F = \sum_{i=1}^{d} \frac{(\partial_\chi p_i)^2}{p_i} + \sum_{i=1}^{d} 4 p_i \langle \partial_\chi |\psi_i\rangle |\partial_\chi |\psi_i\rangle - \sum_{i,j=1; p_i+p_j \neq 0}^{d} \frac{8 p_i p_j}{p_i + p_j} |\langle \psi_i|\partial_\chi |\psi_j\rangle|^2.
\]

(S16)

For the readout GHZ state

\[
\rho(\tau) = |0\rangle\langle 0| + |1\rangle\langle 1| + (e^{-N\gamma \tau} e^{i\phi_S}|0\rangle \langle 1| + \text{h.c.}) /2
\]

(S17)

at \( t = \tau \), where \( \phi_S = \beta N \Omega \tau^2 \) is the Sagnac phase, the dimension of the support set is \( d = 2 \), with the two eigenvalues

\[
p_\pm = (1 \pm e^{-N\gamma \tau}) /2
\]

and the corresponding eigenvectors \( |\psi_\pm\rangle = (e^{i\phi_S}|0\rangle \pm |1\rangle) / \sqrt{2} \), respectively. The QFI with respect to the rotation frequency \( \Omega \) can be readily calculated from Eq. (S16) and is given by

\[
F = \left( \frac{\partial \phi_S}{\partial \Omega} \right)^2 e^{-2N\gamma \tau} = \beta^2 N^2 \tau^4 e^{-2N\gamma \tau},
\]

(S18)
Figure S1. (Color online) Effects of increasing the dephasing strength $\gamma$ on $F/\tau$ vs $\tau$. We set $\beta^2 = 1$ and the representative value for the qubit number is $N = 5$.

which is Eq. (5) in the main text. In noiseless scenarios with $\gamma = 0$, $F/\tau$ increases monotonically with $\tau$ while it has an optimum for finite dephasing strength. In Fig. S1 we show the effects of increasing the dephasing strength on the interrogation time normalized QFI of the readout state with a fixed qubit number $N$. One sees that both of the optimal $F/\tau$ and optimal interrogation time are decreasing with increasing $\gamma$.

QFI of generic MWI schemes.—For generic MWI schemes with GHZ states and under independent dephasing noises, if the accumulated phase is $\phi_\chi(\tau) \propto N\chi\tau^\lambda (\lambda > 0)$, then following the same procedure one can easily obtain the QFI with respect to the quantity $\chi$, which is given by

$$F_\chi = \left(\frac{\partial \phi_\chi}{\partial \chi}\right)^2 e^{-2N\gamma \tau} \propto N^2\tau^2 e^{-2N\gamma \tau}. \quad \text{(S19)}$$

And the the interrogation-time optimized value of $F_\chi/\tau$ is $(F_\chi/\tau)_{\text{opt}} = \mathcal{O}(N^{3-2\lambda})$, with $\tau_{\text{opt}} = (2\lambda - 1)/(2N\gamma)$. So for $\lambda \geq 1$, the best QFI can be achieved is $(F_\chi/\tau)_{\text{opt}} = \mathcal{O}(N)$ (SQL) with the $\lambda = 1$ class. See the main text.

IV. QUANTUM ERROR CORRECTION WITH THE LOGICAL GHZ STATE

The error probability $p(\tau)$ is exponentially suppressed by replacing the raw GHZ state with a logical one $[12, 13]$, where the coding space $C(S)$ is stabilized by the stabilizer group $S$. The $n$-qubit ($n$ is odd) phase-flip code is defined as $C_n = \{ |0\rangle_L, |1\rangle_L \}$, where $|0\rangle_L = (|+\rangle^\otimes n + |-\rangle^\otimes n)/\sqrt{2}$ and $|1\rangle_L = (|+\rangle^\otimes n - |-\rangle^\otimes n)/\sqrt{2}$ are the bases for each logical qubit block, with $|+\rangle$ $(-\rangle)$ being the eigenstate of the Pauli matrix $\sigma_x$ with eigenvalue $+1$ $(-1)$. The above code is stabilized by the operator $X_\alpha = \prod_{i \in \alpha} \sigma_x$, with $\alpha \subset \{1, 2, 3, ..., n\}$ and $|\alpha|$ = even, and is capable of correcting $(n-1)/2$ phase-flip errors $\{\sigma_x\}$ $[12, 13]$. With $N$ total physical qubits as resources, the number of logical qubits is $N/n$. Furthermore, the error probability is renormalized to the logical level as

$$p_L(\tau) = \sum_{k=0}^{(n-1)/2} \binom{n}{k} p^{n-k}(\tau) [1 - p(\tau)]^k. \quad \text{(S20)}$$

The QFI in the main text can be rewritten in terms of $p(\tau)$ using $e^{-\gamma\tau} = 1 - 2p(\tau)$, and the logical QFI in terms of $p_L(\tau)$ is given by

$$F_L = (n\beta)^2 (N/n)^2 \tau^4 [1 - 2p_L(\tau)]^{2N/n}$$
$$= \beta^2 N^2 \tau^4 [1 - 2p_L(\tau)]^{2N/n}. \quad \text{(S21)}$$

Now the quantum Crâmer-Rao bound for the rotation frequency sensing is $\delta \Omega/\sqrt{T} = 1/\sqrt{F_L/\tau}$. See the main text for plotted information.
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