NARROW ESCAPE PROBLEM IN THE PRESENCE OF THE FORCE FIELD

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Abstract. This paper considers the narrow escape problem of a Brownian particle within a three-dimensional Riemannian manifold under the influence of the force field. We compute an asymptotic expansion of mean sojourn time for Brownian particles. As a auxiliary result, we obtain the singular structure for the restricted Neumann Green’s function which may be of independent interest.

1. Introduction

Let us consider a Brownian particle confined to a bounded domain by a reflecting boundary, except for a small absorbing part which is thought of as a target. The narrow escape problem deals with computing the mean sojourn time of the aforementioned Brownian particle. Mathematically, this can be formulated as follows. Let \((M, g, \partial M)\) be a compact, connected, orientable Riemannian manifold with non-empty smooth boundary \(\partial M\). Additionally, let \((X_t, \mathbb{P}_x)\) be the Brownian motion on \(M\) generated by differential operator

\[
\Delta_g + g(F, \nabla_g \cdot)
\]

where \(\Delta_g = -d^*d\) is the (negative) Laplace-Beltrami operator, \(\nabla_g\) is the gradient, \(F\) is a force field given by the potential \(\phi\), that is \(F = \nabla_g \phi\). We use \(\Gamma_{\varepsilon} \subset \partial M\) to denote the absorbing window through which the \((X_t, \mathbb{P}_t)\) can escape, we use \(\varepsilon\) to denote the size of the window and we denote by \(\tau_{\Gamma_{\varepsilon}}\) the first time the Brownian motion \(X_t\) hits \(\Gamma_{\varepsilon}\), that is

\[
\tau_{\Gamma_{\varepsilon}} := \inf\{t \geq 0 : X_t \in \Gamma_{\varepsilon}\}.
\]

As said earlier, we wish to derive asymptotics as \(\varepsilon \to 0\) for the mean sojourn time which is denoted by \(u_{\varepsilon}\) and is given by \(\mathbb{E}[\tau_{\Gamma_{\varepsilon}} | X_0 = x]\). Another quantity of interest is the spatial average of the mean sojourn time:

\[
|\text{M}|^{-1} \int_M \mathbb{E}[\tau_{\Gamma_{\varepsilon}} | X_0 = x] d_g(x).
\]

Here \(|\text{M}|\) denotes the Riemannian volume of \(M\) with respect to the metric \(g\).

Initially, this problem was mentioned in the context of acoustics in [18] (1945). Much later (2004), the interest to this problem was renewed due to relation to molecular biology and biophysics; see [7]. Many problems in cellular biology may be formulated as mean sojourn time problems; a collection of analysis methods, results, applications, and references may be found in [8] and [2]. For example, cells have been
modelled as simply connected two-dimensional domains with small absorbing windows on the boundary representing ion channels or target binding sites; the quantity sought is then the mean time for a diffusing ion or receptor to exit through an ion channel or reach a binding site [22, 7, 17]. All of this lead to the narrow escape theory in applied mathematics and computational biology; see [21, 26, 24].

There has been much progress for this problem in the setting of planar domains, and we refer the readers to [7, 17, 26, 1] and references therein for a complete bibliography. An important contribution was made in the planar case by [1] to introduce rigor into the computation of [17]. The use of layered potential in [1] also cast this problem in the mainstream language of elliptic PDE and facilitates some of the approach we use in this article.

Few results exists for three dimensional domains in \( \mathbb{R}^n \) or Riemannian manifolds; see [4, 21, 25, 5, 16] and references therein. The additional difficulties introduced by higher dimension are highlighted in the introduction of [1] and the challenges in geometry are outlined in [25]. In the case when \( M \) is a domain in \( \mathbb{R}^3 \) with Euclidean metric and \( \Gamma_{\varepsilon,a} \) is a single small disk absorbing window, [21, 25] gave an expansion for the average of the expected first arrival time, averaged over \( M \), up to an unspecified \( O(1) \) term. These results were improved upon in [16], by using geometric microlocal analysis. Namely, the authors derived the bounded term and estimated the remaining term, moreover, they obtain these results for general Riemannian manifolds. The case when \( \Gamma_{\varepsilon,a} \) is a small elliptic window was also addressed in [21, 25, 16]. We also mention [20], where the author gave a short review of related works (up to 2012).

When \( M \) is a three dimensional ball with multiple circular absorbing windows on the boundary, an expansion capturing the explicit form of the \( O(1) \) correction in terms of the Neumann Green’s function and its regular part was done in [4]. The method of matched asymptotic used there required the explicit computation of the Neumann Green’s function, which is only possible in special geometries with high degrees of symmetry/homogeneity. In these results one does not see the full effects of local geometry. This result was also rigorously proved in [3] but with a better estimate for the error term.

Much less has been done for the case of non-zero force field. For instance, all works we mentioned above, except [1], deal with the diffusion without a force field, that is \( F = 0 \). We could find two works concerning this case: [23] and [1]. Both these works consider \( M \) being a domain in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). In [23], the authors generalize the method of [21, 26, 24] to obtain the leading-order term of the average of the expected first arrival time for two and three dimensional cases. For planar domains, the authors in [1], by using layer potential techniques, derive asymptotic expansion up to \( O(\varepsilon) \) term.

In this paper, we derive all the main terms of the expected value of the first arrival time for Riemannian manifolds of dimension three in the presence of a force field. The window or target, \( \Gamma_{\varepsilon} \), is considered to be a small geodesic ellipse of eccentricity \( \sqrt{1-a^2} \) and size \( \varepsilon \to 0^+ \) (to be made precise later). To investigate \( \mathbb{E}[\tau_{\Gamma_{\varepsilon}}|X_0=x] \), we needed to know a singularity structure of the Neumann Green’s function, which is given by the equation

\[
\begin{align*}
\left\{ \begin{array}{ll}
\Delta_{g,z} G(x,z) - \text{div}_{g,z}(F(z)G(x,z)) = -\delta_z(z); \\
\partial_{\nu_z} G(x,z) - F(z) \cdot \nu_z G(x,z) \Big|_{\partial M} = -\frac{1}{|\partial M|}; \\
\int_{\partial M} e^{-\phi(z)}G(x,z)\,d_\mathbb{M}(z) = 0.
\end{array} \right.
\end{align*}
\]
allows us to define the following metric $h$ is classical in the Euclidean case (see [19]) but we could not find a reference for the $E$ tools we have developed. The appendix characterizes the expected first arrival time function. Finally, in Section 5 we carry out the asymptotic calculation using the per. Section 4 deals with computing the singular structure of the Neumann Green's function. In Section 3, we formulate the problem, state and discuss the main results of this paper. Multiple windows but we present the single window case to simplify notations. Our method extends to the mean first arrival time of a Brownian particle on a Riemannian manifold with a single absorbing window which is a small geodesic ellipse. Our method extends to the general case of a Riemannian manifold with boundary.

The paper is organized as follows. In Section 2, we introduce the notations. In Section 3, we formulate the problem, state and discuss the main results of this paper. Section 4 deals with computing the singular structure of the Neumann Green's function. Finally, in Section 5 we carry out the asymptotic calculation using the tools we have developed. The appendix characterizes the expected first arrival time $E[T_{x,a}|X_0 = x]$ as the solution of an elliptic mixed boundary value problem. This is classical in the Euclidean case (see [19]) but we could not find a reference for the general case of a Riemannian manifold with boundary.

2. Preliminaries

2.1. Notation. Throughout this paper, $(M, g, \partial M)$ be a compact connected orientable Riemannian manifold with non-empty smooth boundary. The corresponding volume and geodesic distance are denoted by $d_g(\cdot)$ and $d_g(\cdot, \cdot)$, respectively. By $|M|$ we denote the volume of $M$.

Let $\iota_{\partial M} : \partial M \to M$ is the trivial embedding of the boundary $\partial M$ into $M$. This allows us to define the following metric $h := \iota_{\partial M}^* g$ be the metric on the boundary $\partial M$. Set $d_h(\cdot)$ and $d_h(\cdot, \cdot)$ be the volume and the geodesic distance on the boundary given by metric $h$. By $|\partial M|$ we denote the volume of $\partial M$ with respect to $d_h$.

For $x \in \partial M$, let $E_1(x), E_2(x) \in T_x \partial M$ be the unit eigenvectors of the shape operator at $x$ corresponding respectively to the principal curvatures $\lambda_1(x), \lambda_2(x)$. We will drop the dependence in $x$ from our notation when there is no ambiguity. We choose $E_1$ and $E_2$ such that $E_1^\flat \wedge E_2^\flat \wedge \nu^\flat$ is a positive multiple of the volume form $d_g$ (see p.26 of [12] for the “musical isomorphism” notation of $^\flat$ and $^\sharp$). Here we use $\nu$ to denote the outward pointing normal vector field. By $H(x)$, we denote the mean curvature of $\partial M$ at $x$. We also set

$$\Pi_x(V) := \Pi_x(V, V), \quad V \in T_x \partial M$$

to be the scalar second fundamental quadratic form (see pages 235 and 381 of [12] for definitions). Note that, in defining $\Pi$ and the shape operator, we will follow geometry literature (e.g. [12]) and use the inward pointing normal so that the sphere embedded in $\mathbb{R}^3$ would have positive mean curvature in our convention.

2.2. Boundary normal coordinates. In this work, we will often use the boundary normal coordinates. Therefore, we briefly recall its construction. For a fixed $x_0 \in \partial M$, we will denote by $B_h(\rho; x_0) \subset \partial M$ the geodesic disk of radius $\rho > 0$ (with respect to the metric $h$) centered at $x_0$ and $D_\rho$ to be the Euclidean disk in $\mathbb{R}^2$ of radius $\rho$ centered at the origin. In what follows $\rho$ will always be smaller than the injectivity radius of $(\partial M, h)$. Letting $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, we will construct a coordinate system $x(t; x_0)$ by the following procedure:
Write $t \in \mathbb{R}^3$ near the origin as $t = (t', t_3)$ for $t' = (t_1, t_2) \in \mathbb{D}_\rho$. Define first

$$x((t', 0); x_0) := \exp_{x_0, h}(t_1 E_1 + t_2 E_2),$$

where $\exp_{x_0, h}(V)$ denotes the time 1 map of $h$-geodesics with initial point $x_0$ and initial velocity $V \in T_{x_0} \partial M$. The coordinate $t' \in \mathbb{D}_\rho \mapsto x((t', 0); x_0)$ is then an $h$-geodesic coordinate system for a neighborhood of $x_0$ on the boundary surface $\partial M$. We can extend this to become a coordinate system for points in $M$ near $x_0$ so that $t \mapsto x(t; x_0)$ is a boundary normal coordinate system with $t_3 > 0$ in $M$ as the boundary defining function. Readers wishing to know more about boundary normal coordinates can refer to [11] for a brief recollection of the basic properties we use here and Prop 5.26 of [12] for a detailed construction.

For convenience we will write $x(t'; x_0)$ in place of $x((t', 0); x_0)$. The boundary coordinate system $t \mapsto x(t; x_0)$ has the advantage that the metric tensor $g$ can be expressed as

$$g_{j,k}(t)dt_jdt_k = \sum_{\alpha,\beta=1}^2 h_{\alpha,\beta}(t', t_3)dt_\alpha dt_\beta + dt_3^2,$$

where $h_{\alpha,\beta}(t', 0) = h_{\alpha,\beta}(t')$ is the expression of the boundary metric $h$ in the $h$-geodesic coordinate system $x(t'; x_0)$.

We will also use the rescaled version of this coordinate system. For $\varepsilon > 0$ sufficiently small we define the (rescaled) $h$-geodesic coordinate by the following map

$$x^\varepsilon(\cdot; x_0) : t' = (t_1, t_2) \in \mathbb{D} \mapsto x(\varepsilon t'; x_0) \in B_h(\varepsilon; x_0),$$

where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$.

3. THE MAIN RESULTS

Here we state and discuss the main results of this paper. We begin with formulating the problem. Let $(X_t, \mathbb{P}_x)$ be the Brownian motion on $M$ starting at $x$, generated by the differential operator

$$u \mapsto \Delta_g u + g(F, \nabla_g u),$$

where $F$ is a force field given by potential $\phi$, that is $F = \nabla_g \phi$. For $x^* \in \partial M$ and $\varepsilon > 0$, let $\Gamma_{\varepsilon,a} \subset \partial M$ be a small geodesic ellipse define as

$$\Gamma_{\varepsilon,a} := \{\exp_{x^*, h}(\varepsilon t_1 E_1(x^*) + \varepsilon t_2 E_2(x^*)) \mid t_1^2 + a^{-2} t_2^2 \leq 1\}.$$

Denote by $\tau_{\Gamma_{\varepsilon,a}}$ the first time the Brownian motion $X_t$ hits $\Gamma_{\varepsilon,a}$, that is

$$\tau_{\Gamma_{\varepsilon,a}} := \inf\{t \geq 0 : X_t \in \Gamma_{\varepsilon,a}\}.$$

We aim to investigate the mean sojourn time, that is the expected value

$$u_{\varepsilon,a}(x) := \mathbb{E}[\tau_{\Gamma_{\varepsilon,a}} | X_0 = x],$$

and its average expected value over $M$

$$|M|^{-1} \int_M \mathbb{E}[\tau_{\Gamma_{\varepsilon,a}} | X_0 = x]d_g(x).$$

Namely, we want to derive asymptotic expansion for these quantities as $\varepsilon \to 0$. 
In the Appendix, we show that the mean sojourn time, \( u_{\varepsilon,a} \), satisfies the following elliptic mixed boundary value problem

\[
\begin{aligned}
\Delta_g u_{\varepsilon,a} + g(F, \nabla_g u_{\varepsilon,a}) &= -1; \\
\left. u_{\varepsilon,a} \right|_{\Gamma_{\varepsilon,a}} &= 0; \\
\left. \partial_\nu u_{\varepsilon,a} \right|_{\partial M \setminus \Gamma_{\varepsilon,a}} &= 0.
\end{aligned}
\]

(3.2)

Let \( G(\cdot, \cdot) \in \mathcal{D}'(M \times M) \) solve (1.1). For \( x \in M^o \), Greens formula in conjunction with (1.1) and (3.2) yields the following integral representation for the mean sojourn time \( u_{\varepsilon,a} \)

\[
u_{\varepsilon,a}(x) = G(x) + \int_{\partial M} G(x, z) \partial_\nu u_{\varepsilon,a}(z) d_h(z) + C_{\varepsilon,a}
\]

(3.3)

where

\[
C_{\varepsilon,a} := \frac{1}{|\partial M|} \int_{\partial M} u_{\varepsilon,a}(z) d_h(z), \quad G(x) := \int_M G(x, z) d_h(z).
\]

Where \( G \) satisfies the following boundary value problem

\[
\begin{aligned}
\Delta_g G(x) + g(F, \nabla_g G(x)) &= -1; \\
\left. \partial_\nu G(x) \right|_{x \in \partial M} &= -\frac{\Phi(x)}{|\partial M|}; \\
\int_{\partial M} G(x) d_h(x) &= 0.
\end{aligned}
\]

(3.4)

Where \( \Phi \) is a weighted volume defined in Theorem 3.2.

In order to derive an asymptotic expansion for \( u_{\varepsilon,a} \), we need to derive asymptotics for \( C_{\varepsilon,a} \). The first step within said program is to exploit the vanishing Dirichlet boundary condition of \( \Gamma_{\varepsilon,a} \). In doing so, we restrict \( x \in M^o \) to \( x \in \Gamma_{\varepsilon,a} \) which yields

\[
0 = G(x) \bigg|_{\Gamma_{\varepsilon,a}} + \left( \int_{\partial M} G(x, z) \partial_\nu u_{\varepsilon,a}(z) d_h(z) \right) \bigg|_{\Gamma_{\varepsilon,a}} + C_{\varepsilon,a}.
\]

What follows, is the definition of the restricted Neumann Greens function, defined as the Schwartz kernel to the operator

\[
G_{\partial M} : f \mapsto \left( \int_{\partial M} G(x, y) f(y) d_h(y) \right) \bigg|_{\partial M}.
\]

Here \( G_{\partial M} : C^\infty(\partial M) \to C^\infty(\partial M) \) can be extended to \( H^k(\partial M) \). Using a parametrix construction, in conjunction with Fourier techniques and homogeneous distributions, we can show that the kernel \( G_{\partial M} \) attains the following form for \( x, y \in \partial M \) near the diagonal.

**Proposition 3.1.** There exists an open neighbourhood of the diagonal

\[
\text{Diag} := \{(x, y) \in \partial M \times \partial M \mid x = y\}
\]

such that in this neighbourhood, the singularity structure of \( G_{\partial M}(x, y) \) is given by:

\[
G_{\partial M}(x, y) = \frac{1}{2\pi} d_g(x, y)^{-1} - \frac{1}{4\pi} (H(x) + \partial_\nu \phi(x)) \log d_h(y, x) + \frac{1}{16\pi} \left( \Pi_x \left( \frac{1}{\exp_{x,h}(y)} \right) - \Pi_x \left( \frac{1}{\exp_{x,h}(y)} \right) \right)
\]

(3.5)

\[
- \frac{1}{4\pi} h_x \left( F^\parallel(x), \frac{\exp_{x,h}(y)}{\exp_{x,h}(y)} \right) + R(x, y),
\]
where \( F \) is the tangential part of the force field \( F \) and \( R(\cdot, \cdot) \in C^{0, \mu}(\partial M \times \partial M) \), for all \( \mu < 1 \), is called the regular part of the Green’s function and \( * \) is the Hodge-star operator (i.e. rotation by \( \pi/2 \) on the surface \( \partial M \)).

We will use the formula in Proposition 3.1 to derive the mean first arrival time of a Brownian particle on a Riemannian manifold with a single absorbing window which is a small geodesic ellipse. As mentioned earlier, our method extends to multiple windows but we present the single window case to simplify notations. We first state the result when the window is a geodesic disk of the boundary \( \partial M \) around a fixed point since the statement is cleaner:

**Theorem 3.2.** Let \( (M, g, \partial M) \) be a smooth compact Riemannian manifold of dimension three with boundary. Fix \( x^* \in \partial M \) and let \( \Gamma_\varepsilon \) be a boundary geodesic ball centered at \( x^* \) of geodesic radius \( \varepsilon > 0 \).

i) For each \( x \notin \Gamma_\varepsilon \),
\[
\mathbb{E}[\tau_{\Gamma_\varepsilon}|X_0 = x] = C_\varepsilon + G(x) - \Phi(x^*)G(x^*, x) + r_\varepsilon(x),
\]
with \( r_\varepsilon \in C^k(K) \) for some integer \( k \) and compact set \( K \subset \overline{M} \) which does not contain \( x^* \). The function \( G \) is the solution of (3.1). The constant \( C_\varepsilon \) is given by
\[
C_\varepsilon = \frac{\Phi(x^*)}{4\varepsilon} - \frac{(H(x^*) + \partial_v \phi(x^*))\Phi(x^*)}{4\pi} \log \varepsilon + R(x^*, x^*)\Phi(x^*) - G(x^*)
\]
\[
- \frac{(H(x^*) + \partial_v \phi(x^*))\Phi(x^*)}{4\pi} \left( 2 \log 2 - \frac{3}{2} \right) + O(\varepsilon \log \varepsilon),
\]
where \( R(x^*, x^*) \) is the evaluation at \( (x, y) = (x^*, x^*) \) of the kernel \( R(x, y) \) in Proposition 3.1, and
\[
\Phi(x) := \int_M e^{\phi(z) - \phi(x)} d_g(z).
\]

ii) One has that the integral of \( \mathbb{E}[\tau_{\Gamma_\varepsilon,a}|X_0 = x] \) over \( M \) satisfies
\[
\int_M \mathbb{E}[\tau_{\Gamma_\varepsilon,a}|X_0 = x] d_g(x) = C_\varepsilon|M| + \int_M G(x) d_g(x) - \Phi(x^*) \int_M G(x, x^*) d_g(x).
\]

Theorem 3.2 does not realize the full power of Proposition 3.1 as it does not see the non-homogeneity of the local geometry at \( x^* \) (only the mean curvature \( H(x^*) \) shows up). This is due to the fact that we are looking at windows which are geodesic balls. If we replace geodesic balls with geodesic ellipses, we see that the second fundamental form term in (3.5) contributes to a term in \( \mathbb{E}[\tau_{\Gamma_\varepsilon,a}|X_0 = x] \) which is the difference of principal curvatures.

**Theorem 3.3.** Let \( (M, g, \partial M) \) be a smooth Riemannian manifold of dimension three with boundary. Fix \( x^* \in \partial M \) and let \( \Gamma_\varepsilon \) be a boundary geodesic ellipse given by (3.1) with \( \varepsilon > 0 \).

i) For each \( x \in M \setminus \Gamma_\varepsilon \),
\[
\mathbb{E}[\tau_{\Gamma_\varepsilon}|X_0 = x] = C_\varepsilon + G(x) - \Phi(x^*)G(x^*, x) + r_\varepsilon(x),
\]
Therefore, by the last condition in (4.1), we obtain
\[ C_{e,a} = \frac{K_e}{4\pi^2a} - \frac{1}{4\pi} - \frac{4}{4\pi} \int_D \frac{1}{(1 - |s'|^2)^{1/2}} \int_D \frac{1}{(t_1 - s_1)^2 + a^2(t_2 - s_2)^2} \frac{1}{(1 - |t'|^2)^{1/2}} dt' ds' \]
\[ + O(\varepsilon \log \varepsilon), \]
where \( R(x^*, x^*) \) is the evaluation at \((x, y) = (x^*, x^*)\) of the kernel \( R(x, y) \) in Proposition 3.7. \( \mathbb{D} \) is the two dimensional unit disk centered at the origin, and
\[ \Phi(x) := \int_M e^{\phi(z) - \phi(x)} d_g(z), \quad K_e := \frac{\pi}{2} \int_0^{2\pi} \left( \cos^2 \theta + \frac{\sin^2 \theta}{a^2} \right)^{-1/2} d\theta \]
\[ \text{ii) One has that the integral of } E[\tau_{T_{e,a}} | X_0 = x] \text{ over } M \text{ satisfies} \]
\[ \int_M E[\tau_{T_{e,a}} | X_0 = x] d_g(z) = C_e |M| + \int_M G(x) d_g(x) - \Phi(x) \int_M G(x, x^*) d_g(x). \]

4. The Neumann Green’s Function

Here, we investigate the Neumann Green’s function. Namely, we derive its singular structure on the boundary near the diagonal. By Neumann Green’s function, \( G(\cdot, \cdot) \in \mathcal{D}'(M \times M) \), we mean the solution to the following equation
\[ \begin{cases} \Delta_{g,z} G(x, z) - \text{div}_{g,z} (F(z)G(x, z)) = -\delta_z(z); \\ \partial_{\nu_z} G(x, z) - F(z) \cdot \nu_z G(x, z) \big|_{\partial M} = -\frac{1}{|\partial M|}; \\ \int_{\partial M} e^{-\phi(z)} G(x, z) d_h(z) = 0. \end{cases} \]

By using Green’s identity to
\[ e^{-\phi(y)} G(z, y) = -\int_M e^{-\phi(x)} G(z, x) (\Delta_{g,z} G(y, x) - \text{div}_{g,z} (F(x)G(y, x))) d_g(x), \]
we obtain
\[ e^{-\phi(y)} G(z, y) - e^{-\phi(z)} G(y, z) = \frac{1}{|\partial M|} \left( \int_{\partial M} e^{-\phi(x)} G(y, x) d_h(x) - \int_{\partial M} e^{-\phi(x)} G(z, x) d_h(x) \right). \]

Therefore, by the last condition in (4.1), we obtain
\[ G(z, y) = e^{\phi(y) - \phi(z)} G(y, z). \]

Therefore, we can check
\[ \int_{\partial M} G(z, y) d_h(z) = \int_{\partial M} e^{\phi(y) - \phi(z)} G(y, z) d_h(z) = 0, \]
\[ \partial_{\nu_z} G(z, y) = e^{\phi(y) - \phi(z)} (\partial_{\nu_z} G(y, z) - F(z) \cdot \nu_z G(y, z) \big|_{\partial M}) = -\frac{e^{\phi(y) - \phi(z)}}{|\partial M|} \]
for $z \in \partial M$, and finally
\[
\Delta_{g,z}G(z,y) + g(F(z), \nabla_{g,z}G(z,y)) = e^{\phi(y) - \phi(x)} \left( g(\nabla_{g,z}\phi(z), \nabla_{g,z}\phi(z))G(y, z) - \Delta_{g,z}\phi(z)G(y, z) - 2g(\nabla_{g,z}\phi(z), \nabla_{g,z}G(y, z)) + \Delta_{g,z}G(y, z) - g(F(z), \nabla_{g,z}\phi(z))G(y, z) + g(F(z), \nabla_{g,z}G(y, z)) \right).
\]

Therefore, we conclude that $G(z,y)$ satisfies the differential equation (with respect to the first variable)
\[
\begin{cases}
\Delta_{g,z}G(z,y) + g(F(z), \nabla_{g,z}G(z,y)) = -\delta_g(z); \\
\partial_{\nu_z}G(z,y) \mid_{z \in \partial M} = -\frac{e^{\phi(y) - \phi(x)}}{|\partial M|}; \\
\int_{\partial M} G(z,x) d\nu(z) = 0.
\end{cases}
\]

This indicates that $G$ solves the problem \((3.4)\). Now, we let $f \in C^\infty(\partial M)$. Using this $f$, we introduce $u_f$, the solution to the following auxiliary problem
\[
\Delta_g u_f(z) + g_z(F(z), \nabla_g u_f(z)) = 0, \quad u_f \mid_{z \in \partial M} = f \in C^\infty(\partial M).
\]

By using Green’s identity and the Divergence form theorem to
\[
u_f(x) = -\int_M u_f(z) (\Delta_{g,z}G(x,z) - \text{div}_{g,z}(F(z)G(x,z))) d\nu_g(z),
\]
we compute
\[
u_f(x) = -\int_M \Delta_g u_f(z)G(x,z)d\nu_g(z) - \int_{\partial M} (u_f(z)\partial_{\nu_z}G(x,z) - \partial_{\nu_z}u_f(z)G(x,z)) d\nu_h(z)
- \int_M G(x,z)g_z(F(z), \nabla_g u_f(z))d\nu_g(z) + \int_{\partial M} u_f(z)G(x,z)F(z) \cdot \nu_z d\nu_h(z).
\]

Since functions $u_f, G(x,z)$ satisfy \((4.4), (4.4)\) respectively, we conclude that
\[
u_f(x) = \int_{\partial M} G(x,z)\partial_{\nu_z}u_f(z)d\nu_h(z) + \frac{1}{|\partial M|} \int_{\partial M} f(z) d\nu_h(z).
\]

Restricting $x$ to $\partial M$, we have that
\[
\begin{equation}
\label{4.6}
 f(x) \mid_{\partial M} = \left( \int_{\partial M} G(x,z)\partial_{\nu_z}u_f(z)d\nu_h(z) \right) \mid_{\partial M} + \frac{1}{|\partial M|} \int_{\partial M} f(z) d\nu_h(z)
\end{equation}
\]

Let $\Lambda_F \in \Psi^1_{\partial M}(\partial M)$ be the Dirichlet-to-Neumann map associated to the boundary value problem \((4.4)\) and $G_{\partial M}(x,y)$ be the Schwartz kernel of the operator
\[
f \rightarrow \left( \int_{\partial M} f(y)G(x,y)d\nu_h(y) \right) \mid_{\partial M}
\]
which takes $C^\infty(\partial M) \rightarrow C^\infty(\partial M)$. Then we can rewrite \((4.6)\) in the following way
\[
f(x) = G_{\partial M}\Lambda_F f + Pf
\]
where $P$ is a smoothing operator, that is $P \in \Psi^{-\infty}(\partial M)$. In operator form this is
\[
I = G_{\partial M}\Lambda_F + P.
\]

Since $\Lambda_F$ is an elliptic pseudo-differential operator, we can construct $G_{\partial M}$ via a standard left parametrix construction.
4.1. Symbolic Expansion for the symbol of the Dirichlet-to-Neumann map.
We compute here the first two terms of the asymptotic expansion for the symbol of
the Dirichlet-to-Neumann map. We will use this to obtain the corresponding terms
for the symbol of \(G_{\partial M}\). We will follow \cite{13} and adapt some of their results for the
drift case.

In boundary normal coordinates, we decompose our differential operator in the
following way

\[
-\Delta_g - g_x(F, \nabla_g) = D_{x^3}^2 + i\tilde{E}(x)D_{x^3} + \tilde{Q}(x, D_{x'})
\]

where

\[
\tilde{E}(x) := -\frac{1}{2} \sum_{\alpha,\beta} h^{\alpha\beta}(x) \partial_{x^\alpha} h_{\alpha\beta}(x) - F^\alpha(x)
\]

\[
\tilde{Q}(x, D_{x'}) := \sum_{\alpha,\beta} h^{\alpha\beta}(x) D_{x^\alpha} D_{x^\beta}
\]

\[
- i \sum_{\alpha,\beta} \left( \frac{1}{2} h^{\alpha\beta}(x) \partial_{x^\alpha} \log \delta(x) + \partial_{x^\alpha} h^{\alpha\beta}(x) \right) D_{x^\beta} + iF^\beta(x)h_{\beta}^\alpha(x)D_{x^\alpha}.
\]

We will need the following modification of Proposition 1.1 in \cite{13}

**Proposition 4.1.** There exists a pseudo-differential operator \(A_F(x, D_{x'}) \in \Psi_{\mathcal{C}}^1(\partial M)\)
which depends smoothly on \(x^3\) such that

\[
-\Delta_g - g_x(F, \nabla_g) = \left( D_{x^3}^2 + i\tilde{E}(x) - iA_F(x, D_{x'}) \right) \left( D_{x^3} + iA_F(x, D_{x'}) \right),
\]

modulo a smoothing operator.

**Proof.** We construct an asymptotic series for the symbol of \(A_F(x, D_{x'})\) using a ho-
mogeneity argument. The proposition can be re-stated as the construction of some
pseudo-differential operator \(A_F(x, D_{x'})\) modulo \(\Psi^{-\infty}\) which satisfies the following statement

\[
0 = \Delta_g + g_x(F, \nabla_g) + \left( D_{x^3} + i\tilde{E}(x) - iA_F(x, D_{x'}) \right) \left( D_{x^3} + iA_F(x, D_{x'}) \right)
\]

Due to decomposition \cite{13}, the problem becomes the construction of a classical, first
order pseudo-differential operator \(A_F(x, D_{x'})\) which satisfies the following operator
equation

\[
A_F^2 - \tilde{Q} + i[D_{x^3}, A_F] - \tilde{E} A_F = 0
\]

modulo a smoothing operator.

Reduction of the above operator equation to the pseudo-differential symbol calculus
yields the following equation (modulo \(S^{-\infty}\)).

\[
\sum_{|\mu| \geq 0} \frac{1}{\mu!} \partial_{\xi}^\mu a D_x^\mu a - \tilde{q} + \partial_{x^3} a - \tilde{E} a = 0
\]

where \(a\) is the full symbol of \(A_X\) and \(\tilde{q}\) is the full symbol of \(\tilde{Q}\) given by

\[
\tilde{q}(x, \xi') = \sum_{\alpha,\beta} h^{\alpha\beta}(x) \xi_{\alpha} \xi_{\beta} - i \sum_{\alpha,\beta} \left( \frac{1}{2} h^{\alpha\beta}(x) \partial_{x^\alpha} \log \delta(x) + \partial_{x^\alpha} h^{\alpha\beta}(x) - F^\alpha(x)h_{\alpha}^\beta(x) \right) \xi_{\beta}
\]

\[
= : \tilde{q}_2(x, \xi') + \tilde{q}_1(x, \xi')
\]
Let us write
\[ a(x, \xi') \sim \sum_{j \leq 1} a_j(x, \xi') \]
where \( a_j \in S^j_{1,0}(T^*\partial M) \), and is homogeneous of degree \( j \) in \( \xi' \). Collecting terms which are homogeneous of degree 2 in (4.9) yields the following
\[ a_1^2 - \tilde{q}_2 = 0 \implies a_1 = \pm \sqrt{\tilde{q}_2} \]
For consistency, we will choose \( a_1 := -\sqrt{\tilde{q}_2} \). Next, we will collect the terms which are homogeneous of degree 1 in (4.9) as follows
\[ 2a_0a_1 + \sum_{\alpha} \partial_{\xi^\alpha} a_1 D_{x^{\alpha}} a_1 - \tilde{q}_1 + \partial_{x^3} a_1 - \tilde{E} a_1 = 0 \]
Solving for \( a_0 \), we have that
\[ a_0 = -\frac{1}{2a_1} \left( \sum_{\alpha} \partial_{\xi^\alpha} a_1 D_{x^{\alpha}} a_1 - \tilde{q}_1 + \partial_{x^3} a_1 - \tilde{E} a_1 \right) \]
\[ = \frac{1}{2\sqrt{\tilde{q}_2}} \left( \sum_{\alpha} \partial_{\xi^\alpha} \sqrt{\tilde{q}_2} D_{x^{\alpha}} \sqrt{\tilde{q}_2} - \tilde{q}_1 - \partial_{x^3} \sqrt{\tilde{q}_2} + \tilde{E} \sqrt{\tilde{q}_2} \right) \]
We can apply the same recursive argument indefinitely for all degrees of homogeneity \( 1 - j \) in order to obtain \( a_{1-j} \) for every \( j \geq 1 \). For our purposes, the construction of \( a_0 \) is sufficient. This completes the proof. \( \square \)

Proposition 1.2 of [13] yields the required calculation for the Dirichlet to Neumann operator, given by
\[ \Lambda_F f = \delta^{1/2} A_F f dx^1 \wedge dx^2 \mod \Omega^2(\partial M) \]
Via a one-to-one correspondence, we can associate to this \( n - 1 \) differential form a symbol, denoted by \( b(x, \xi') \in S^1_{d}(T^*\partial M) \) given by
(4.10)
\[ b(x, \xi') = \delta^{1/2} a(x, \xi'). \]

4.2. Explicit Calculation for the Neumann Green’s function. In this subsection, we will extract the singular part of \( G_{\partial M} \) on the diagonal.
Since \( \Lambda_F \) is elliptic and (4.7), we construct \( G_{\partial M} \) as a standard left parametrix of order \(-1\). Let \( p(x, \xi') \in S^{-1}_d(\partial M) \) be its symbol with the following asymptotic expansion
\[ p(x, \xi') \sim \sum_{j \geq 1} p_{-j}(x, \xi'), \]
where \( p_{-j} \in S^{-j}(T^*\partial M) \) for each \( j \geq 1 \). From (4.7), we deduce that
\[ 1 = (p#b)(x, \xi') + S^{-\infty}(T^*\partial M) \]
\[ = \sum_{|\mu| \geq 0} \frac{1}{\mu!} \partial_{\xi}^\mu p D_x^\mu \delta^{1/2} a + S^{-\infty}(T^*\partial M), \]
where \( a \) is the symbol constructed in Section 4.1. By matching the terms with the same orders, we obtain
\[ p_{-1}(x, \xi') = \frac{\chi(\xi')}{\delta^{1/2} \sqrt{\tilde{q}_2}(x, \xi')} \]
Here, $\chi$ is a smooth cutoff function, non-zero outside of some sufficiently large neighbourhood of the origin. The choice of terms $p_{-j}$ for $j \geq 3$ can be done via standard, iterative parametrix arguments which were used to obtain $p_{-1}$ and $p_{-2}$. For the sake of brevity, such computations shall be omitted and are unnecessary for our purposes. It should be noticed that for $x = x^*$, the origin of our geodesic disk, as a result of boundary normal co-ordinates, the symbol terms are reduce to

$$p_{-1}(x^*, \xi') = \frac{\chi(\xi')}{|\xi'|}.$$

In addition, we have that

$$p_{-2}(x^*, \xi') = \frac{\chi(\xi')}{|\xi'|} \left( \frac{\chi(\xi') a_0(x^*, \xi')}{|\xi'|} - \sum_{|\mu|=1} \partial^\mu_{\xi'} \frac{\chi(\xi')}{\sqrt{q_2(x, \xi')}} D^\mu_x \sqrt{q_2(x, \xi')} \right).$$

Notice that in $p_{-2}$, the $\delta^{1/2}$ term is peeled off by the product rule and vanishes as a result of $\partial_x h_{ij} = 0$ at the origin.

The calculation of the principle symbol for $G_{\partial M}$ as well as the the next highest order term yield the following asymptotic for the kernel centred at $x^*$, evaluated at $x \in B_h(\rho; x^*) \subset \partial M$

$$G_{\partial M}(x^*, x) = \Phi^* G_{\partial M}(0, t')$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} p_{-1}(x^*, \xi') d\xi' + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} p_{-2}(x^*, \xi') d\xi' + \Psi^{-3}(\partial M).$$

The first term evaluates to

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} p_{-1}(x^*, \xi') d\xi' = \frac{1}{4\pi^2 \delta^{1/2}} \int_{\mathbb{R}^2} e^{-i \xi' \cdot \chi(\xi')} d\xi'.$$

Furthermore, we can split the above integral into a singular and regular part as follows

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} \frac{1}{|\xi'|} d\xi' + I_{\text{reg}}.$$

The regular part is of no interest to us and will be lost in the error as this computation is done in order to isolate the most prevalent singularities occurring in $G_{\partial M}$. This idea can be carried forth in the computation for the second integral. We have that

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} \frac{1}{|\xi'|} d\xi' = \mathcal{F}(|\xi'|^{-1})(t') = \frac{1}{2\pi} |t'|^{-1}.$$

Next, we use the previous arguments and proceed to split the second term up as follows

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} a_0(0, \xi') \frac{1}{|\xi'|^2} d\xi'$$

$$- \sum_{|\mu|=1} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i \xi' \cdot t'} \frac{1}{|\xi'|} \partial^\mu_{\xi'} (q_2(x, \xi'))^{-1/2} \big| \big|_{x=x^*} D^\mu_x \sqrt{q_2(x, \xi')} \big| \big|_{x=x^*} d\xi'.$$
We recall that we have that $\tilde{q}_2(x, \xi') = h^{\alpha\beta}(x)\xi_\alpha\xi_\beta$. Thus, we have the following identity

$$D_x^\mu \sqrt{h^{\alpha\beta}(x)\xi_\alpha\xi_\beta} = \frac{D_\xi h^{\alpha\beta}(x)}{2q_1(x, \xi')}$$

In boundary normal co-ordinates, centred at $x = x^*$, this identity evaluates to 0. Thus, we have that the second integral is vanishing. We are now left to compute the following integral

$$I(t') := \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{1}{|\xi'|^3} \left( \sum_\alpha \partial_{\xi_\alpha} \sqrt{q_2} D_{\xi^\alpha} \sqrt{q_2} - \tilde{q}_1 - \partial_x \sqrt{E}\sqrt{q_2} + \tilde{E}\sqrt{q_2} \right) \bigg|_{x=x^*} d\xi'.$$

Furthermore, since $D_{\xi^\alpha} \sqrt{q_2} = 0$ and $\tilde{q}_1(x, \xi') = iX^\alpha(x)\xi_\alpha$ in boundary normal co-ordinates, we have that $I$ is given by

$$I(t') = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{1}{|\xi'|^3} \left( \tilde{E}|\xi'| - \partial_x \sqrt{h^{\alpha\beta}(x)\xi_\alpha\xi_\beta} - iF^\alpha(x)\xi_\alpha \right) \bigg|_{x=x^*} d\xi'$$

$$= \frac{\tilde{E}(x^*)}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{1}{|\xi'|^3} d\xi' - \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{\partial_x \sqrt{h^{\alpha\beta}(x)\xi_\alpha\xi_\beta}}{|\xi'|^3} d\xi'$$

$$- \frac{i}{8\pi^2} F^\alpha(x^*) \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{\xi_\alpha}{|\xi'|^3} d\xi'.$$

We calculate each term in the integral in order of increasing difficulty. Using proposition 8.17 [Lee] we can ascertain the following

$$\tilde{E}(x^*) = -\frac{1}{2} \sum_{\alpha, \beta} h^{\alpha\beta}(x)\partial_x h_{\alpha\beta}(x) \bigg|_{x=x^*} - F^3(x^*) = 2H(x^*) - F^3(x^*).$$

Where $H(x^*)$ denotes the mean curvature of $(M, g, \partial M)$ at $x^*$. Thus, we have that

$$\frac{\tilde{E}(x^*)}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{1}{|\xi'|^2} d\xi' = \frac{2H(x^*) - F^3(x^*)}{2} \mathcal{F}(|\xi'^{-1}| - 2) = -\frac{2H(x^*) - F^3(x^*)}{4\pi} \log |t'|.$$

Next, we calculate the third term by making the following observation

$$\partial_{\xi_\alpha} |\xi'|^{-1} = -\frac{\xi_\alpha}{|\xi'|^3}.$$

We can re-write the third term as follows

$$-\frac{i}{8\pi^2} \sum_\alpha F^\alpha(x^*) \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \frac{\xi_\alpha}{|\xi'|^3} d\xi' = \frac{i}{8\pi^2} \sum_\alpha F^\alpha(x^*) \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} \partial_{\xi_\alpha} |\xi'|^{-1} d\xi'$$

$$= \frac{1}{8\pi^2} \sum_\alpha t_\alpha F^\alpha(x^*) \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi} |\xi'|^{-1} d\xi'$$

$$= \frac{1}{2} \sum_\alpha t_\alpha F^\alpha(x^*) \mathcal{F}(|\xi'^{-1}|)(t')$$

$$= \sum_\alpha F^\alpha(x^*) \frac{t_\alpha}{4\pi |t'|}.$$
Lastly, we compute the second term. It should be noted that via rotation with basis $E_1 \wedge E_2 \wedge \nu'$, we find that the scalar second fundamental form is diagonalised in our co-ordinate system. Thus, we have the following

\[-\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \nu} \frac{\partial_{x^2}^2 |\xi'|^2 |x|^2}{|\xi'|^2} d\xi' \]

\[= -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \nu} (\partial_x h^{11}(x) + 2 \partial_x h^{12}(x) + \partial_x h^{22}(x)) |x=x' d\xi' \]

\[= -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot \nu} \lambda_1(x') \xi_1^2 + \lambda_2(x') \xi_2^2 d\xi' \]

\[= \frac{1}{4\pi^2} \left( \frac{\lambda_1(x^*) + \lambda_2(x^*)}{2} \pi \log |t'| - \frac{\pi}{4} \left( \frac{\lambda_2(x^*) t_1^2 + \lambda_1(x^*) t_2^2}{|t'|^2} - \frac{\lambda_1(x^*) t_1^2 + \lambda_2(x^*) t_2^2}{|t'|^2} \right) \right) \]

Here $\lambda_1, \lambda_2$ denote the associated principle curvatures. Therefore, we have that $I(t')$ is given by

\[I(t') = -\frac{2H(x^*) - F^3(x^*)}{4\pi} \log |t'| \]

\[+ \frac{1}{4\pi^2} \left( H(x^*) \pi \log |t'| - \frac{\pi}{4} \left( \frac{\lambda_2(x^*) t_1^2 + \lambda_1(x^*) t_2^2}{|t'|^2} - \frac{\lambda_1(x^*) t_1^2 + \lambda_2(x^*) t_2^2}{|t'|^2} \right) \right) \]

\[+ \frac{1}{4\pi} \sum_{\alpha} F^a(x^*) t_\alpha \frac{1}{|t'|} \]

Therefore, we have that

\[I(\exp_{x^*}^{-1}(x)) = -\frac{H(x^*)}{4\pi} \log d_h(x^*, x) + \frac{F^3(x^*)}{4\pi} \log d_h(x^*, x) \]

\[+ \frac{1}{16\pi} \left( \Pi_{x^*} \left( \frac{\exp_{x^*}^{-1}(x)}{|\exp_{x^*}^{-1}(x)|_h} \right) - \Pi_{x^*} \left( \frac{^* \exp_{x^*}^{-1}(x)}{|^* \exp_{x^*}^{-1}(x)|_h} \right) \right) \]

\[+ \frac{1}{4\pi} h_{x^*} \left( F^\parallel(x^*), \frac{\exp_{x^*}^{-1}(x)}{|\exp_{x^*}^{-1}(x)|_h} \right) \]

This yields the following asymptotic for $G_{\partial M}$

\[G_{\partial M}(x^*, x) = \frac{1}{2\pi} d_h(x^*, x)^{-1} - \frac{H(x^*)}{4\pi} \log d_h(x^*, x) + \frac{X^3(x^*)}{4\pi} \log d_h(x^*, x) \]

\[+ \frac{1}{16\pi} \left( \Pi_{x^*} \left( \frac{\exp_{x^*}^{-1}(x)}{|\exp_{x^*}^{-1}(x)|_h} \right) - \Pi_{x^*} \left( \frac{^* \exp_{x^*}^{-1}(x)}{|^* \exp_{x^*}^{-1}(x)|_h} \right) \right) \]

\[+ \frac{1}{4\pi} h_{x^*} \left( F^\parallel(x^*), \frac{\exp_{x^*}^{-1}(x)}{|\exp_{x^*}^{-1}(x)|_h} \right) + \Psi^{-3}(\partial M). \]
We can make a further refinement on the above series by invoking Corollary 2.5 from [16] in order to write the following

$$G_{\partial M}(x^*, x) = \frac{1}{2\pi} d_g(x^*, x)^{-1} - \frac{H(x^*)}{4\pi} \log d_h(x^*, x) + \frac{F^3(x^*)}{4\pi} \log d_h(x^*, x)$$

$$+ \frac{1}{16\pi} \left( \Gamma(x^*) \left( \frac{\exp_{x^*}^{-1}(x)}{\exp_{x^*}^{-1}(x)_h} \right) - \Pi_x \left( \frac{\exp_{x}^{-1}(x)}{\exp_{x}^{-1}(x)_h} \right) \right)$$

$$+ \frac{1}{4\pi} h_{x^*} \left( \frac{F^3(x^*)}{\exp_{x^*}^{-1}(x)_h} \right) + \Psi^{-3}(\partial M).$$

The above work yields an expression for $G_{\partial M}$ centred at the origin of the window $x^* \in \Gamma_{\epsilon,a}$. We however need an expansion for $G_{\partial M}(x, y)$ for $x, y$ close in $\Gamma_{\epsilon,a}$. In particular, we need an expression given by (2.2). In order to do so, we simply invoke the following proposition, which was proven in [16].

**Proposition 4.2.** Let $x_0 \in \partial M$ and $\lambda_1(x_0)$ and $\lambda_2(x_0)$ be the eigenvalues of the shape operator at $x_0$. Assume that $x = x^*(s'; x_0)$, $y = x^*(t'; x_0)$, $r = |s' - t'|$ and $t' = s + r\omega$. Then, for $\epsilon > 0$ sufficiently small, we have that

$$d_g(x, y)^{-1} = \epsilon^{-1}r^{-1} + \epsilon r^{-1} A^1(\epsilon, s', r, \omega),$$

$$d_h(x, y)^{-1} = \epsilon^{-1}r^{-1} + \epsilon r^{-1} A^2(\epsilon, s', r, \omega),$$

$$\Pi_x \left( \frac{\exp_{x^*}^{-1}(y)}{\exp_{x^*}^{-1}(y)_h} \right) = \left( \lambda_1(x_0) \left( \frac{s_1 - t_1}{r^2} \right)^2 + \lambda_2(x_0) \left( \frac{s_2 - t_2}{r^2} \right)^2 \right) + \epsilon R^1_\epsilon(t', \omega, r),$$

$$\Pi_x \left( \frac{\exp_{x^*}^{-1}(y) \ast \exp_{x^*}^{-1}(y)}{\exp_{x^*}^{-1}(y)_h} \right) = \left( \lambda_2(x_0) \left( \frac{s_1 - t_1}{r^2} \right)^2 + \lambda_1(x_0) \left( \frac{s_2 - t_2}{r^2} \right)^2 \right) + \epsilon R^2_\epsilon(t', \omega, r),$$

$$h_x \left( \frac{F^3(x^*)}{\exp_{x}^{-1}(y)_h} \right) = \frac{F_1(t')(s_1 - t_1) + F_2(t')(s_2 - t_2)}{r} + \epsilon R^3_\epsilon(t', \omega, r),$$

where $A^1$, $A^2$, and $A^3$ are smooth in $[0, \epsilon] \times D \times \mathbb{R} \times S^1$ and $R^1_\epsilon$, $R^2_\epsilon$, $R^3_\epsilon$ are smooth in $D \times S^1 \times [0, \rho]$ with derivatives of all orders uniformly bounded in $\epsilon$.

**Proof.** The first two statements were proved in Corollaries 2.6 and 2.3 of [16], respectively. The next two results were proved in Corollary 2.9 of [16]. The last one follows from Lemma 2.8. \qed

5. PROOF OF THEOREMS 3.2 AND 3.3

In this section we give a proof for Theorems 3.2 and 3.3. We recall that the mean sojourn time, $u_{\epsilon,a}$, satisfies the elliptic mixed boundary value problem (3.2), this is proved in the Appendix. Therefore, by using Green’s identity, we show that $u_{\epsilon,a}$ time satisfies the integral equation

$$u_{\epsilon,a}(x) = G(x) + \int_{\partial M} G(x, z) \partial_x u_{\epsilon,a}(z) d_h(z) + C_{\epsilon,a},$$

where $x \in M^0$, $G$ is the Neumann Green function we discussed in Section 4 and

$$C_{\epsilon,a} := \frac{1}{|\partial M|} \int_{\partial M} u_{\epsilon,a}(z) d_h(z), \quad G(x) := \int_M G(x, z) d_h(z).$$
From (4.3), it follows that \( \mathcal{G} \) satisfies
\[
(5.2) \begin{cases} 
\Delta_g \mathcal{G}(x) + g(F(x), \nabla_g \mathcal{G}(x)) = -1; \\
\partial_{\nu} \mathcal{G}(x) \big|_{x \in \partial M} = -\frac{\phi(x)}{|\partial M|}; \\
\int_{\partial M} \mathcal{G}(x) \, dh(x) = 0.
\end{cases}
\]

By taking the trace of the integral equation (5.1) to \( x \in \Gamma_{\varepsilon,a} \), we obtain
\[0 = \mathcal{G}(x) + \int_{\partial M} G_{\partial M}(x, z) \partial_{\nu, \varepsilon,a}(z) \, dh(z) + C_{\varepsilon,a}.
\]

Therefore, Proposition 3.1 gives
\[-\mathcal{G}(x) - C_{\varepsilon,a} = \frac{1}{2\pi} \int_{\Gamma_{\varepsilon,a}} d_g(x, y)^{-1} \partial_{\nu, \varepsilon,a}(y) \, dh(y)
- \frac{H(x) - \partial_{\nu, \phi}(x)}{4\pi} \int_{\Gamma_{\varepsilon,a}} \log d_h(x, y) \partial_{\nu, \varepsilon,a}(y) \, dh(y)
+ \frac{1}{16\pi} \int_{\Gamma_{\varepsilon,a}} \left( \Pi_x \left( \frac{\exp^{-1}(y)}{|\exp^{-1}(y)|_h} \right) - \Pi_x \left( \frac{\exp^{-1}(y)}{|\exp^{-1}(y)|_h} \right) \right) \partial_{\nu, \varepsilon,a}(y) \, dh(y)
+ \frac{1}{4\pi} h_x \left( F^g(x), \frac{\exp_{\varepsilon,a}(y)}{|\exp_{\varepsilon,a}(y)|_h} \right) \partial_{\nu, \varepsilon,a}(y) \, dh(y)
+ \int_{\Gamma_{\varepsilon,a}} R(x, y) \partial_{\nu, \varepsilon,a}(y) \, dh(y).
\]

Since \( F = \nabla_g \phi \), the fact that \( u_{\varepsilon,a} \) satisfies (5.2) implies
\[\text{div}_g(e^\phi \nabla_g u_{\varepsilon,a}) = e^\phi(\Delta_g u_{\varepsilon,a} + F \cdot \nabla_g u_{\varepsilon,a}) = -e^\phi.\]

By the divergence form theorem, we know that
\[\int_M \text{div}_g(e^\phi \nabla_g u_{\varepsilon,a})(z) \, d_g(z) = \int_{\partial M} e^\phi \partial_{\nu, \varepsilon,a}(z) \, dh(z).
\]

Therefore, by integrating the penultimate equation, we derive the following compatibility condition
\[
(5.4) \int_{\partial M} e^\phi \partial_{\nu, \varepsilon,a}(z) \, dh(z) = -\int_M e^\phi(z) \, d_g(z).
\]

We will use the coordinate system given by
\[
\mathbb{D} \ni (s_1, s_2) \mapsto x^\varepsilon(s_1, a; s_2; x^*) \in \Gamma_{\varepsilon,a},
\]
where \( x^\varepsilon(\cdot; x^*) : \mathcal{E}_a \to \Gamma_{\varepsilon,a} \) is the coordinate defined in Section (2). To simplify notation we will drop the \( x^* \) in the notation and denote \( x^\varepsilon(\cdot; x^*) \) by simply \( x^\varepsilon(\cdot) \).

Note that in these coordinates the volume form for \( \partial M \) is given by
\[
d_h = a\varepsilon^2(1 + \varepsilon^2 Q_{\varepsilon}(s^*)) \, ds_1 \wedge ds_2, \ s^* \in \mathbb{D}
\]
for some smooth function \( Q_{\varepsilon}(s^*) \) whose derivatives of all orders are bounded uniformly in \( \varepsilon \). We denote
\[
(5.7) \psi_{\varepsilon}(s^*) := \partial_{\nu, \varepsilon,a}(x^\varepsilon(s_1, a; s_2)).
\]

Then, in this coordinate system, the compatibility condition become
\[
(5.8) \int_{\mathbb{D}} e^{\phi(x^\varepsilon(s^*))} \psi_{\varepsilon}(s^*)(1 + \varepsilon^2 Q_{\varepsilon}(s^*)) \, ds^* = -\frac{1}{a\varepsilon^2} \int_M e^\phi(z) \, d_g(z).
\]
We can re-write this as follow
\[ e^{\phi(x^*)} \int_{\mathbb{D}} \psi(x) dx + \int_{\mathbb{D}} \left[ e^{\phi(x^*)} - e^{\phi(y^*)} \right] \psi_{\epsilon}(s') ds' = -\frac{1}{\alpha^2} \int_{M} e^{\phi(z)} d_\nu(z). \]

Since \( \phi \) is smooth, we conclude that
\[ \int_{\mathbb{D}} \psi(x) dx = -\frac{\Phi(x^*)}{\alpha^2} + \epsilon \int_{\mathbb{D}} \tilde{Q}_\epsilon(s') \psi_{\epsilon}(s') ds', \]
where \( \Phi \) is the function defined in Theorem (3.2) and \( \tilde{Q}_\epsilon(s') \) is some smooth function whose derivatives of all orders are bounded uniformly in \( \epsilon \).

Next, we re-write (5.3) in this coordinate system given by (5.5). To do this, let us first introduce the following operators. Consider
\[ L_a f = a \int_{\mathbb{D}} \frac{f(s')}{((t_1 - s_1)^2 + a^2(t_2 - s_2)^2)^{1/2}} ds', \]
acting on functions of the disk \( \mathbb{D} \). By [21] we have that
\[ L_a (K_a^{-1}(1 - |t'|^2)^{-1/2}) = 1, \]
on \( \mathbb{D} \) where
\[ K_a = \frac{\pi}{2} \int_{0}^{2\pi} \frac{1}{(\cos^2 \theta + \sin^2 \theta)^{1/2}} d\theta. \]

By (4.4) in [16], this is the unique solution in \( H^{1/2}(\mathbb{D})^* \) to \( L_a u = 1 \).

Next we denote
\[ R_{\log_a} f(t') := a \int_{\mathbb{D}} \log((t_1 - s_1)^2 + a^2(t_2 - s_2)^2)^{1/2} f(s') ds', \]
\[ R_{\infty,a} f(t') := a \int_{\mathbb{D}} \frac{(t_1 - s_1)^2 - a^2(t_2 - s_2)^2}{(t_1 - s_1)^2 + a^2(t_2 - s_2)^2} f(s') ds', \]
\[ R_{F,a} f(t') := a \int_{\mathbb{D}} \frac{F^1(0)(t_1 - s_1) + aF^2(0)(t_2 - s_2)}{(t_1 - s_1)^2 + a^2(t_2 - s_2)^2} f(s') ds'. \]

Remark 5.1. In [16], it was showed that the operators \( R_{\log,a} \) and \( R_{\log,a} \) are bounded maps from \( H^{1/2}(\mathbb{D})^* \) to \( H^{3/2}(\mathbb{D}) \). By repeating the arguments, one can show that this is also true for \( R_{F,a} \).

We unwrap the right hand side of (5.3) term by term in the following five lemmas. Note that the first four of them are proved in [16]. We repeat them here for the convenience of the readers.

Lemma 5.2. We have the following identity
\[ \int_{\Gamma_{\epsilon,a}} d_\nu(x, y)^{-1} \partial_x u_{\epsilon,a}(y) ds_h(y) = \epsilon L_a \psi_{\epsilon}(t') + \epsilon^3 A_{\epsilon} \psi_{\epsilon}(t') \]
with \( x = x^\epsilon(t') \) for some \( A_{\epsilon} : H^{1/2}(\mathbb{D}; ds_\epsilon) \to H^{1/2}(\mathbb{D}; ds') \) with operator norm bounded uniformly in \( \epsilon \).
Proof. By using Proposition 4.2 and (5.6), we see that in the coordinate system (5.5) it follows
\[
\int_{\Gamma_{L,a}} d_g(x, y)^{-1} \partial_{\nu} u_{\varepsilon, a}(y) d_h(y) = a \varepsilon \int_D \frac{1}{((s_1 - t_1)^2 + a(s_2 - t_2)^2)^{1/2}} \psi_{\varepsilon}(s') ds' + a \varepsilon^3 \int_D A_1'(s', t')(1 + \varepsilon^2 Q_{\varepsilon}(s')) + Q_{\varepsilon}(s') \psi_{\varepsilon}(s') ds'.
\]

Therefore, by Lemma 2.11 of [16], the second term of the right-hand side can be written as \( \varepsilon^3 A_{\varepsilon} \psi \), for operator \( A_{\varepsilon} \) which satisfies the requirement of the statement. □

From now on, we will denote by \( A_{\varepsilon} \) any operator which takes \( H^{1/2}(\mathbb{D}; ds')^* \rightarrow H^{1/2}(\mathbb{D}; ds') \) whose operator norm is bounded uniformly in \( \varepsilon \).

For the second term of the right-hand side of (5.3), the following lemma holds.

**Lemma 5.3.** We have the following identity
\[
(H(x) - \partial_{\nu} \phi(x)) \int_{\Gamma_{L,a}} \log d_h(x, y) \partial_{\nu} u_{\varepsilon, a}(y) d_h(y) = -(H(x^*) - \partial_{\nu} \phi(x^*)) \Phi(x^*) \log \varepsilon
+ \varepsilon^2 (H(x^*) - \partial_{\nu} \phi(x^*)) R_{\log, a} \psi_{\varepsilon}(t') + O_{H^{1/2}(\mathbb{D})}(\varepsilon \log \varepsilon) + \varepsilon^3 \log \varepsilon A_{\varepsilon} \psi_{\varepsilon},
\]
where \( x = x^*(t') \) and \( \Phi \) is the function defined in Theorem 3.2.

**Proof.** By using Proposition 4.2 and (5.6), we obtain
\[
(H(x) - \partial_{\nu} \phi(x)) \int_{\Gamma_{L,a}} \log d_h(x, y) \partial_{\nu} u_{\varepsilon, a}(y) d_h(y)
= a \varepsilon^2 \log \varepsilon (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \psi_{\varepsilon}(s') ds'
+ a \varepsilon^2 (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \log [(t_1 - s_1)^2 + a(t_2 - s_2)^2)^{1/2}] \psi_{\varepsilon}(s') ds'
- a \varepsilon^2 (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \log (1 + \varepsilon^2 A(s', t')) \psi_{\varepsilon}(s') ds'
- a \varepsilon^4 (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \log (1 + \varepsilon^2 A(s', t')) Q_{\varepsilon}(s') \psi_{\varepsilon}(s') ds'
- a \varepsilon^4 (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \log (1 + \varepsilon^2 A(s', t')) Q_{\varepsilon}(s') \psi_{\varepsilon}(s') ds'.
\]

Apart the first and second terms, all terms of the right-hand side can be written as \( \varepsilon^3 A_{\varepsilon} \psi_{\varepsilon} \). The first term, by (5.9), is equals to
\[
-(H(x^*) - \partial_{\nu} \phi(x^*)) \Phi(x^*) \log \varepsilon + \varepsilon^3 \log \varepsilon (H(x^*(t')) - \partial_{\nu} \phi(x^*(t')))) \int_D \tilde{Q}_{\varepsilon}(s') \psi_{\varepsilon}(s') ds'
+ (H(x^*) - \partial_{\nu} \phi(x^*) - H(x^*(t')) + \partial_{\nu} \phi(x^*(t'))) \log \varepsilon \Phi(x^*),
\]
consequently, by Lemma 2.11 of [16], it is equal to
\[
-(H(x^*) - \partial_{\nu} \phi(x^*)) \Phi(x^*) \log \varepsilon + O_{H^{1/2}(\mathbb{D})}(\varepsilon \log \varepsilon) + \varepsilon^3 \log \varepsilon A_{\varepsilon} \psi_{\varepsilon}.
\]
While the second term is
\[ \varepsilon^2 (H(x^*) - \partial_x \phi(x^*)) R_{\log, a} \psi_\varepsilon(t') \]
\[ + \varepsilon^2 (H(x^c(t')) - \partial_x \phi(x^c(t')) - H(x^*) + \partial_x \phi(x^*)) R_{\log, a} \psi_\varepsilon(t'). \]
Since \( H \) and \( \phi \) are smooth functions, we derive that the last term of the above expression is \( \varepsilon^3 A_\varepsilon \psi_\varepsilon \).

For the forth term of (5.3), we have the following lemma.

**Lemma 5.4.** We have the following identity
\[ \int_{\Gamma_{x,a}} \left( I_\varepsilon \left( \frac{\exp_x^{-1}(y)}{\exp_x^{-1}(y)|y|} \right) \right) - I_\varepsilon \left( * \frac{\exp_x^{-1}(y)}{\exp_x^{-1}(y)|y|} \right) \partial_x u_{x,a}(y) dy = \varepsilon^2 (\lambda_1(x^*) - \lambda_2(x^*)) R_{\infty,a} \psi_\varepsilon(t') + \varepsilon^3 A_\varepsilon \psi_\varepsilon. \]

**Proof.** By Proposition 4.2 and (5.6), we see that
\[ \int_{\Gamma_{x,a}} \left( I_\varepsilon \left( \frac{\exp_x^{-1}(y)}{\exp_x^{-1}(y)|y|} \right) \right) - I_\varepsilon \left( * \frac{\exp_x^{-1}(y)}{\exp_x^{-1}(y)|y|} \right) \partial_x u_{x,a}(y) dy \]
\[ = a(\lambda_1(x^*) - \lambda_2(x^*)) \varepsilon^2 \int_{\mathbb{D}} \frac{(s_1 - t_1)^2 - a^2(s_2 - t_2)^2}{(s_1 - t_1)^2 + a^2(s_2 - t_2)^2} \psi_\varepsilon(s') ds' + a \varepsilon^3 \int_{\mathbb{D}} (R^1(t', s') - R^2(t', s')) (1 + \varepsilon^2 Q(s')) \psi_\varepsilon(s') ds' \]
\[ + a(\lambda_1(x^*) - \lambda_2(x^*)) \varepsilon^4 \int_{\mathbb{D}} \frac{(s_1 - t_1)^2 - a^2(s_2 - t_2)^2}{(s_1 - t_1)^2 + a^2(s_2 - t_2)^2} Q(s') \psi_\varepsilon(s') ds'. \]
We use Lemma 2.11 of [16] to complete the proof. □

Next, we study the fourth term of (5.3).

**Lemma 5.5.** The following is true
\[ \int_{\Gamma_{x,a}} h_x \left( F^{\parallel}(x), \frac{\exp_{x,h}(y)}{\exp_{x,h}(y)|y|} \right) \partial_x u_{x,a}(y) dy = \varepsilon^2 R_{F,a} \psi_\varepsilon + \varepsilon^3 A_\varepsilon \psi_\varepsilon. \]

**Proof.** By Proposition 4.2 and (5.6), we get
\[ \int_{\Gamma_{x,a}} h_x \left( F^{\parallel}(x), \frac{\exp_{x,h}(y)}{\exp_{x,h}(y)|y|} \right) \partial_x u_{x,a}(y) dy \]
\[ = a \varepsilon^2 \int_{\mathbb{D}} \frac{F_1(t')(s_1 - t_1) + F_2(t')(s_2 - t_2)}{((s_1 - t_1)^2 + a^2(s_2 - t_2)^2)^{1/2}} \psi_\varepsilon(s') ds' + a \varepsilon^3 \int_{\mathbb{D}} R^3(t', s')(1 + \varepsilon^2 Q(s')) \psi_\varepsilon(s') ds' \]
\[ + a \varepsilon^4 \int_{\mathbb{D}} \frac{F_1(t')(s_1 - t_1) + F_2(t')(s_2 - t_2)}{((s_1 - t_1)^2 + a^2(s_2 - t_2)^2)^{1/2}} Q(s') \psi_\varepsilon(s') ds'. \]
Finally, Lemma 2.11 of [16] implies that the last two terms are \( \varepsilon^3 A_\varepsilon \psi_\varepsilon \). □

Finally, let us look to the last term of (5.3). By Lemma 5.1 in [16], we know that for operator \( T_\varepsilon : C^\infty_c(\mathbb{D}) \to \mathcal{D}'(\mathbb{D}) \) defined by the integral kernel
\[ R(x^c(t'; x^*), x^c(s'; x^*)) - R(x^*, x^*), \]
we have \( ||T_\varepsilon||_{H^{1/2}(\mathbb{D}^*) \to H^{1/2}(\mathbb{D})} = O(\varepsilon \log \varepsilon) \). Therefore, by using Lemmas 5.2,5.5 we re-write (5.13) in the following way

\[
-G(x^*) - C_{\varepsilon,a} = \frac{\varepsilon}{2\pi} L_a \psi_\varepsilon + \frac{1}{4\pi} (H(x^*) - \partial_\nu \phi(x^*)) \Phi(x^*) \log \varepsilon \\
- \frac{\varepsilon^2}{4\pi} \left( \frac{H(x^*) - \partial_\nu \phi(x^*)}{R_{\log,a} - \frac{\lambda_1 - \lambda_2}{8} R_{\infty,a} + \frac{1}{2} R_{F,a}} \right) \psi_\varepsilon \\
- R(x^*, x^*) \Phi(x^*) + \varepsilon^2 a T_\varepsilon \psi_\varepsilon + O_{H^{1/2}(\mathbb{D})}(\varepsilon \log \varepsilon) + \varepsilon^3 \log \varepsilon A_\varepsilon \psi_\varepsilon.
\]

This is equivalent to

\[
\frac{2\pi}{\varepsilon} \left( R(x^*, x^*) \Phi(x^*) - G(x^*) - C_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu \phi(x^*)) \Phi(x^*) \log \varepsilon}{4\pi} \right) = \left( L_a - \varepsilon R \right) \psi_\varepsilon \\
+ \varepsilon a T_\varepsilon \psi_\varepsilon + O_{H^{1/2}(\mathbb{D})}(\log \varepsilon) + \varepsilon^2 \log \varepsilon A_\varepsilon \psi_\varepsilon,
\]

where

\[
R := \frac{H(x^*) - \partial_\nu \phi(x^*)}{2} R_{\log,a} - \frac{\lambda_1 - \lambda_2}{8} R_{\infty,a} + \frac{1}{2} R_{F,a}.
\]

Applying \( L_a^{-1} \) to both sides, we obtain

\[
\frac{2\pi}{\varepsilon} \left( R(x^*, x^*) \Phi(x^*) - G(x^*) - C_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu \phi(x^*)) \Phi(x^*) \log \varepsilon}{4\pi} \right) L_a^{-1} = \left( I - \varepsilon L_a^{-1} R + \varepsilon \log \varepsilon \right) \psi_\varepsilon + O_{H^{1/2}(\mathbb{D})}(\log \varepsilon),
\]

for some \( T'_\varepsilon : H^{1/2}(\mathbb{D})^* \to H^{1/2}(\mathbb{D})^* \) with operator norm \( O(\varepsilon \log \varepsilon) \). As we mentioned in Remark 5.1, \( R_{\log,a}, R_{\infty,a}, \) and \( R_{F,a} \) are bounded maps from \( H^{1/2}(\mathbb{D})^* \) to \( H^{3/2}(\mathbb{D}) \). Therefore, the right side can be inverted by Neumann series to deduce

\[
\psi_\varepsilon = -\frac{2\pi C_{\varepsilon,a}}{\varepsilon} L_a^{-1} 1 + C_{\varepsilon,a} O_{H^{1/2}(\mathbb{D})^*}(1) + O_{H^{1/2}(\mathbb{D})^*}(\varepsilon^{-1} \log \varepsilon).
\]

Let us integrate this over \( \mathbb{D} \) and use (5.9), then we derive

\[
-\frac{2\pi C_{\varepsilon,a}}{\varepsilon} \int_{\mathbb{D}} L_a^{-1} 1(s') ds' + C_{\varepsilon,a} O(1) + O(\varepsilon^{-1} \log \varepsilon) = -\frac{\Phi(x^*)}{a \varepsilon^2}.
\]

Note that

\[
\int_{\mathbb{D}} L_a^{-1} 1(s') ds' = \frac{1}{K_a} \int_{\mathbb{D}} \frac{1}{(1 - |s'|^2)^{1/2}} ds' = \frac{2\pi}{K_a},
\]

and hence, the previous equation gives

\[
C_{\varepsilon,a} = \frac{K_a \Phi(x^*)}{4\pi^2 a \varepsilon} + C_{\varepsilon,a}'
\]

with \( C_{\varepsilon,a}' = O(|\log \varepsilon|) \). We put this into (5.13), to obtain

\[
\psi_\varepsilon = -\frac{K_a \Phi(x^*)}{2\pi a \varepsilon^2} L_a^{-1} 1 + \psi_\varepsilon',
\]

where \( \|\psi_\varepsilon'\|_{H^{1/2}(\mathbb{D}, ds')} = O(\varepsilon^{-1} \log \varepsilon) \). Let us insert this into (5.9), then we obtain

\[
\int_{\mathbb{D}} \psi_\varepsilon'(s') ds' = -\frac{K_a \Phi(x^*)}{2\pi a \varepsilon^2} \int_{\mathbb{D}} \hat{Q}(s') L_a^{-1} 1(s') ds' + \varepsilon \int_{\mathbb{D}} \hat{Q}(s') \psi_\varepsilon'(s') ds'
\]

where

\[
\hat{Q}(s') = \frac{1}{ \varepsilon \varepsilon} \left( (e^{\phi(x^*))} - e^{\phi(x^*+\varepsilon)} + \varepsilon^2 e^{\phi(x^*+\varepsilon)} Q_\varepsilon(s') \right)
\]
and $Q$ is the function involved into volume form \eqref{5.6}. If we use Taylor expansion to $e^\phi$ at $x = x^*$, we obtain
\[
e^\phi = e^{\phi(x^*)} + \varepsilon g_{x^*} (e^{\phi(x^*)} \nabla_x \phi(x) |_{x=x^*}, t_1 E_1 + t_2 E_2) + O(\varepsilon^2).
\]
Therefore
\[
\int_D e^{\phi(x^*)} [L_a^{-1}] (t) dt = e^{\phi(x^*)} \int_D [L_a^{-1}] (t) dt
+ \varepsilon g_{x^*} (e^{\phi(x^*)} \nabla_x \phi(x) |_{x=x^*}, t_1 E_1) \int_D t_1 [L_a^{-1}] (t) dt
+ \varepsilon g_{x^*} (e^{\phi(x^*)} \nabla_x \phi(x) |_{x=x^*}, t_2 E_2) \int_D t_2 [L_a^{-1}] (t) dt
+ O(\varepsilon^2).
\]
Noting that
\[
\int_D t_1 [L_a^{-1}] (t) dt = \int_D t_2 [L_a^{-1}] (t) dt = 0,
\]
we obtain
\[
\int_D e^{\phi(x^*)} [L_a^{-1}] (t) dt = e^{\phi(x^*)} \int_D [L_a^{-1}] (t) dt + O(\varepsilon^2).
\]
Therefore, from \eqref{5.17} it follows that
\[
\int_D \psi^*(s') ds' = O(1) + O(\log \varepsilon) = O(\log \varepsilon).
\]
Next, we put \eqref{5.15} and \eqref{5.16} into \eqref{5.12} to obtain
\[
\frac{2\pi}{\varepsilon} \left( R(x^*, x^*) \Phi(x^*) - G(x^*) - C_{x,a}^* - \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*) \log \varepsilon}{4\pi} \right) [L_a^{-1}]
= \frac{K_a \Phi(x^*)}{2\pi a \varepsilon} L_a^{-1} RL_a^{-1} - \frac{K_a \Phi(x^*)}{2\pi a \varepsilon} L_a^{-1} T_\varepsilon L_a^{-1}
+ \psi^* - \varepsilon (L_a^{-1} R - L_a^{-1} T_\varepsilon) \psi' + O_{H^{1/2}(D)^*} (\log \varepsilon).
\]
Therefore, recalling that
\[
\|T_\varepsilon^*\|_{H^{1/2}(D)^* \rightarrow H^{1/2}(D)^*} = O(\varepsilon \log \varepsilon), \quad \|R\|_{H^{1/2}(D)^* \rightarrow H^{3/2}(D)^*} = O(1),
\]
we derive
\[
\frac{2\pi}{\varepsilon} \left( R(x^*, x^*) \Phi(x^*) - G(x^*) - C_{x,a}^* - \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*) \log \varepsilon}{4\pi} \right) [L_a^{-1}]
= \frac{K_a \Phi(x^*)}{2\pi a \varepsilon} L_a^{-1} RL_a^{-1} + \psi^* + O_{H^{1/2}(D)^*} (\log \varepsilon).
\]
Let us integrate this over $D$ and take into account \eqref{5.19}, \eqref{5.14}, then
\[
C_{x,a}^* = - \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*)}{4\pi} \log \varepsilon
+ R(x^*, x^*) \Phi(x^*) - G(x^*) - \frac{K_a^2 \Phi(x^*)}{8\pi^3 a} \int_D [L_a^{-1} RL_a^{-1} (t) dt + O(\varepsilon \log \varepsilon).
Since \( L_a^{-1} \) is self-adjoint, we can express the last integral more explicitly:

\[
\int_D L_a^{-1} \mathcal{R} L_a^{-1} 1(s') ds' = K_a^{-2} (1 - |s'|^2)^{-1/2}, \mathcal{R} (1 - |s'|^2)^{-1/2}.
\]

Moreover,

\[
\int_D \left( \frac{s_1}{1 - |s'|^2} \right)^{1/2} \int_D \left( \frac{1}{(s_1 - t_1)^2 + a^2(s_2 - t_2)^2} \right)^{1/2} \frac{1}{(1 - |t'|^2)^{1/2}} dt' ds' = 0.
\]

Indeed, consider the following two changes of variables for the left-hand side

\[
(s_1, s_2, t_1, t_2) = (r \cos \alpha, r \sin \alpha, \rho \cos \beta, \rho \sin \beta),
\]

\[
(s_1, s_2, t_1, t_2) = (-r \cos \alpha, r \sin \alpha, -\rho \cos \beta, \rho \sin \beta).
\]

The results differ by multiplying by \(-1\), which means that the left-hand side is 0. Therefore, we know that

\[
\int_D L_a^{-1} R_{F,a} L_a^{-1} 1(s') ds' = 0.
\]

Finally, recalling the definition of \( \mathcal{R} \) and the relation between \( C_{\varepsilon,a} \) and \( C_{\varepsilon,a} \), we obtain

\[
C_{\varepsilon,a} = \frac{K_a \Phi(x^*)}{4\pi^2 a \varepsilon} - \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*)}{4\pi} \log \varepsilon + R(x^*, x^*) \Phi(x^*) - \mathcal{G}(x^*)
\]

\[
- \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*)}{16\pi^3} \int_D \left( \frac{1}{1 - |s'|^2} \right)^{1/2} \int_D \left( \frac{1}{(1 - |t'|^2)^{1/2}} \right)^{1/2} dt' ds'
\]

\[
+ \frac{\lambda_1(x^*) - \lambda_2(x^*)}{64\pi^3} \int_D \left( \frac{1}{1 - |s'|^2} \right)^{1/2} \int_D \left( \frac{1}{(1 - |t'|^2)^{1/2}} \right)^{1/2} \frac{1}{(1 - |t'|^2)^{1/2}} dt' ds'
\]

\[
+ O(\varepsilon \log \varepsilon).
\]

In case of the disc, that is \( a = 1 \), the last two terms explicitly calculated in Lemmas 4.5 and 4.6 in [16]. Therefore, we have

\[
(5.20)
\]

\[
C_{\varepsilon} := C_{\varepsilon,1} \Phi(x^*) = \frac{4\pi}{4\varepsilon} - \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*)}{4\pi} \log \varepsilon + R(x^*, x^*) \Phi(x^*) - \mathcal{G}(x^*)
\]

\[
- \frac{(H(x^*) - \partial_x \phi(x^*)) \Phi(x^*)}{4\pi} \left( 2 \log 2 - 3 \right) + O(\varepsilon \log \varepsilon).
\]

Next, let us recall that

\[
u_{\varepsilon,a}(x) = \mathbb{E}[\tau_{x,a} | X_0 = x] = \mathcal{G}(x) + C_{\varepsilon,a} - \Phi(x^*) G(x^*, x) + r_{\varepsilon}(x)
\]

for each \( x \in M \setminus \Gamma_{\varepsilon,a} \). Here the remainder \( r_{\varepsilon} \) is given by

\[
(5.22)
\]

\[
r_{\varepsilon}(x) = \int_{\partial M} (G(x, y) - G(x, x^*)) \partial_y u_{\varepsilon}(y) d\nu(y).
\]

Let \( K \subseteq M \) be a compact subset of \( M \) which has positive distance from \( x^* \) and consider \( x \in K \). Writing out this integral in the \( x^* \)-coordinate system and using (5.7), (5.10), and the expression of \( \psi_{\varepsilon} \) derived in (5.10), we get

\[
(5.23)
\]

\[
r_{\varepsilon}(x) = \epsilon \int_D \frac{-|M|}{2\pi (1 - |s'|^2)^{1/2}} L(x, \varepsilon) (1 + \varepsilon^2 Q_{\varepsilon}(s')) ds' \]

\[
+ a \varepsilon^3 \int_D (\psi_{\varepsilon}'(s')) L(x, \varepsilon)(1 + \varepsilon^2 Q_{\varepsilon}(s')) ds'
\]
for some function $L(x, s')$ jointly smooth in $(x, s') \in K \times \mathbb{D}$. The second integral formally denotes the duality between $H^{1/2} (\mathbb{D})^*$ and $H^{1/2} (\mathbb{D})$. The estimate for $\psi'$ derived in [5,10] now gives for any integer $k$ and any compact set $K$ not containing $x^*$, $\| r_e \|_{C^k(K)} \leq C_{k,k}$. This gives us the first parts of Theorems 3.2 and 3.3.

Finally, we compute the average expected value over $M$. Let us write

$$v(x) = \int_{\Gamma_{\nu,a}} G(x, y) \partial_{\nu} u_{\epsilon,a}(y) d_h(y).$$

Then

$$v(x) = u_{\epsilon,a}(x) - \mathcal{G}(x) - C_{\epsilon,a},$$

so that

$$\begin{cases}
\Delta_g v(x) + g(F(x), \nabla_g v(x)) = 0; \\
v(x) \mid_{\partial M} = \int_{\Gamma_{\nu,a}} G_{\partial M}(x, y) \partial_{\nu} u_{\epsilon,a}(y) d_h(y).
\end{cases}$$

Since $G_{\partial M} \in \Psi^{-1}_d(\partial M)$ we know that $v(x) \mid_{\partial M} \in H^{1/2}(\partial M)$.

Let $\{ f_j \}_{j=1}^{\infty}$ be a sequence of smooth functions such that $f_j \rightarrow \partial_{\nu} u_{\epsilon,a}$ in $H^{-1/2}(\partial M)$.

Let $\{ v_j \}_{j=1}^{\infty}$ be functions which satisfy

$$\begin{cases}
\Delta_g v_j(x) + g(F(x), \nabla_g v_j(x)) = 0; \\
v_j(x) \mid_{\partial M} = \int_{\Gamma_{\nu,a}} G_{\partial M}(x, y) f_j(y) d_h(y).
\end{cases}$$

Then $v_j \rightarrow v$ in $H^1(M)$. Therefore, we compute

$$\int_M \int_{\Gamma_{\nu,a}} G(x, y) \partial_{\nu} u_{\epsilon,a}(y) d_h(y) d_g(x) = \lim_{j \rightarrow \infty} \int_M \int_{\partial M} G(x, y) f_j(y) d_h(y) d_g(x)$$

$$= \lim_{j \rightarrow \infty} \int_{\partial M} f_j(y) \int_M G(x, y) d_g(x) d_h(y)$$

$$= \int_{\Gamma_{\nu,a}} \partial_{\nu} u_{\epsilon,a}(y) \int_M G(x, y) d_g(x) d_h(y).$$

Recalling (4.2), we derive

$$\int_{\Gamma_{\nu,a}} \partial_{\nu} u_{\epsilon,a}(y) \int_M G(x, y) d_g(x) d_h(y) = \int_{\Gamma_{\nu,a}} \partial_{\nu} u_{\epsilon,a}(y) \int_M e^{\phi(y) - \phi(x)} G(y, x) d_g(x) d_h(y)$$

$$= - \int_M e^{\phi(z)} d_g(z) \int_M e^{-\phi(x)} G(x^*, x) d_g(x) + O(\epsilon)$$

$$= - \Phi(x^*) \int_M e^{\phi(x^*) - \phi(x)} G(x^*, x) d_g(x) + O(\epsilon)$$

$$= - \Phi(x^*) \int_M G(x, x^*) d_g(x) + O(\epsilon)$$

Therefore, (5.1) implies

$$\int_M u_{\epsilon,a}(x) = \int_M \mathcal{G}(x) - \Phi(x^*) \int_M G(x, x^*) d_g(x) + M |C_{\epsilon,a}|.$$
In this appendix we show that \( u(x) := \mathbb{E}[\tau_\Gamma | X_0 = x] \) satisfies the boundary value problem (3.2). This is standard material but we could not find a suitable reference which precisely addresses our setting. As such we are including this appendix for the convenience of the reader.

Let \((M, g, \partial M)\) be an orientable compact connected Riemannian manifold with non-empty smooth boundary oriented by \(d_g\). Let us consider the operator

\[
u \rightarrow \Delta_g \nu + g(F, \nabla_g \nu),
\]

where \(F\) is a force field, which is given by \(F = \nabla_g \phi\) for a smooth up to the boundary potential \(\phi\). We can re-write this operator in the following way

\[
\Delta_g^F := \Delta_g \cdot + g(F, \nabla_g \cdot) = \frac{1}{e^\phi} \text{div}_g(e^\phi \nabla_g \cdot).
\]

Note that \(e^\phi\) is a smooth positive function on \(M\). Moreover, there exist constants such that

\[
(A.1) \quad 0 < c_0 < e^{\phi(x)} < c_1 < \infty, \quad x \in M.
\]

According to [6], the operator \(\Delta_g^F\) is called by weighted Laplace operator and the pair \((M, \mu)\), where \(\mu(x) := e^{\phi(x)}d_g(x)\), is called a weighted manifold. Note that the operator \(\Delta_g^F\) with initial domain \(C_0^\infty(M)\) is essentially self-adjoint in \(L^2(M, \mu)\) and non-positive definite.

Let \((X_t, \mathbb{P}_x)\) be the Brownian motion on \(M\) starting at \(x\), generated by the weighted Laplace operator \(\Delta_g^F\). Let \(\Gamma\) be a geodesic ball on \(\partial M\) with radius \(\varepsilon > 0\). We denote by \(\tau_\Gamma\) the first time the Brownian motion \(X_t\) hits \(\Gamma\), that is

\[
\tau_\Gamma := \inf\{t \geq 0 : X_t \in \Gamma\}.
\]

We set

\[
\mathcal{P}_\Gamma(t, x) := \mathbb{P}[\tau_\Gamma \leq t | X_0 = x].
\]

Let us note that \(\mathcal{P}_\Gamma(t, x)\) is the probability that the Brownian motion hits \(\Gamma\) before or at time \(t\), and therefore, satisfies

\[
(A.2) \quad \mathcal{P}_\Gamma(0, x) = 0, \quad x \in M \setminus \Gamma,
\]

\[
(A.3) \quad \mathcal{P}_\Gamma(t, x) = 1, \quad (t, x) \in [0, \infty) \times \Gamma.
\]

Note that, for any compact subset \(\Gamma \subset M\), it follows

\[
\text{Cap}(\Gamma, M) := \inf_{u \in C_0^\infty(M), u|_{\partial M} = 0} \int_M |\nabla_g u(x)|^2 e^{\phi(x)}d_g(x) = 0.
\]

Note that in [6] and [9], the authors consider the manifold together with its boundary, and \(C_0^\infty(M)\), \(C_0^\infty(M)\) denote the set of smooth (up to the boundary) functions with compact support. In case of compact manifold, these sets coincide with \(C^\infty(M)\). This implies that that \((M, \mu)\) is parabolic, that is, the probability that the Brownian motion ever hits any compact set \(K\) with non-empty interior is 1. Since \(\Gamma \subset \partial M\) is connected with non-empty interior on \(\partial M\), we can extend \(M\) to a compact connected Riemannian manifold \(\tilde{M}\) such that \(M \setminus \tilde{M}\) is compact with non-empty interior and...
$M \setminus M \cap M = \Gamma$. Note that, the Brownian motion, starting at any point $M \setminus \Gamma$, hits $M \setminus M$ if and only if it hits $\Gamma$. Therefore, the parabolicity condition of $(M, \mu)$ gives

$$
\lim_{t \to \infty} \mathcal{P}_t(t, x) = 1, \quad x \in M.
$$

Further, let us define the mean first arrival time $u$, as

$$
u(x) := \mathbb{E}[\tau_\Gamma | X_0 = x] := \int_0^\infty t d\mathcal{P}_t(t, x),
$$

where the integral is a Riemann-Stieltjes integral. To investigate $u$, let us recall some properties of $\mathcal{P}_t$. By Remark 2.1 in [6], it follows that

$$
1 - \mathcal{P}_t(t, x) = \left(e^{t\Delta_{mix}^F} \mathbf{1}\right)(x),
$$

where $e^{t\Delta_{mix}^F}$ is the semigroup with infinitesimal generator $\Delta_{mix}^F$, and $\Delta_{mix}^F$ is the weighted Laplace operator $\Delta_g^F$ corresponding to the Dirichlet boundary condition on $\Gamma$ and Neumann boundary condition on $\partial M \setminus \Gamma$, which is defined as follows

$$
\Delta_{mix}^F := \{ u \in H^1(M) : \Delta_g^F u \in L^2(M, \mu) \ u|_\Gamma = 0, \ \partial_\nu u|_{\Gamma^c} = 0 \}
$$

Moreover, for $u \in H^1(M)$, the conditions $\Delta_g^F u \in L^2(M, \mu)$ and $\partial_\nu u|_{\Gamma^c}$ are equivalent. Therefore, $D(\Delta_{mix}^F) = D(\Delta_{mix})$, where $\Delta_{mix}$ is the classical Laplace operator $\Delta_g$ corresponding to the Dirichlet boundary condition on $\Gamma$ and Neumann boundary condition on $\partial M \setminus \Gamma$:

$$
\Delta_{mix} := \{ u \in H^1(M) : \Delta_g u \in L^2(M) \ u|_\Gamma = 0, \ \partial_\nu u|_{\Gamma^c} = 0 \}
$$

In (A.6) and (A.8), we define $\partial_\nu u \in H^{-1/2}(\partial M)$ using the same method for defining the Dirichlet to Neumann map. That is, for $u \in H^1(M)$ such that $\Delta_g u \in L^2(M)$, the distribution $\partial_\nu u|_{\partial M} \in H^{-1/2}(\partial M)$ acts on $f \in H^{1/2}(\partial M)$ via

$$
\langle \partial_\nu u|_{\partial M}, f \rangle := \int_M \Delta_g u(z) v_f(z) d_g(z) + \int_M g(\nabla_g u(z), \nabla_g v_f(z)) d_g(z),
$$

where $v_f \in H^1(M)$ is the harmonic extension of $f$. We say that $\partial_\nu u|_{\partial M} = 0$, for non-empty open set $\omega \subset \partial M$, if $\langle \partial_\nu u|_{\partial M}, f|_{\partial M} \rangle = 0$ for all $f \in H^{1/2}(\partial M)$ such that $f|_{\partial M \setminus \omega} = 0$. Note that if $u$ sufficiently regular, for instance $u \in H^2(M)$, then $\langle \partial_\nu u|_{\partial M}, f \rangle$ is equal to the boundary integral of $\partial_\nu u f|_{\partial M}$ and $f$.

Next, we note that

$$
(\Delta_{mix}^F u, u)_{L^2(M, \mu)} = - \int_M e^{g(z)} g(\nabla_g u(z), \nabla_g u(z)) d_g(z),
$$

and therefore, (A.1) implies that

$$
c_1(\Delta_{mix}^F u, u)_{L^2(M)} < (\Delta_{mix}^F u, u)_{L^2(M, \mu)} < c_0(\Delta_{mix}^F u, u)_{L^2(M)}
$$

for $u \in D(\Delta_{mix})$, recall that $D(\Delta_{mix}^F) = D(\Delta_{mix})$. Note that $\Delta_{mix}$ is a self-adjoint operator with discrete spectrum, consisting of negative eigenvalues accumulating at $-\infty$; see for instance Proposition 7.1 in [16]. Therefore, the above inequality implies
that the spectrum of $\Delta_{mix}^F$ consists eigenvalues with finite multiplicity accumulating at $-\infty$. Hence, $\Delta_{mix}^F$ satisfies the quadratic estimate
\[
\int_0^\infty \|t \Delta_{mix}^F (1 + t^2 (\Delta_{mix}^F)^2)^{-1} u\|_{L^2(M,\mu)}^2 \frac{dt}{t} \leq C \|u\|_{L^2(M,\mu)}^2,
\]
for some $C > 0$ and all $u \in L^2(M,\mu)$; see for instance [14] p. 221. Therefore, $\Delta_{mix}^F$ admits the functional calculus defined in [15].

**Remark 6.1.** The functional calculus in [15] is defined for a concrete operator, which is denoted by $T$ in the notation used in that article. However, $\Delta_{mix}^F$ satisfy all necessary conditions to admit this functional calculus.

Therefore, the semigroup $e^{t\Delta_{mix}^F}$, which is contracting by Hille-Yosida theorem [10] Theorem 8.2.3, can be defined as follows
\[
e^{t\Delta_{mix}^F}u = \frac{1}{2\pi i} \int_{\gamma_{a,\alpha}} e^{\xi} (\zeta - \Delta_{mix}^F)^{-1} u d\zeta, \quad u \in L^2(M),
\]
where $a \in (\tau, 0)$, $\alpha \in (0, \frac{\pi}{2})$, and $\gamma_{a,\alpha}$ is the anti-clockwise oriented curve:
\[
\gamma_{a,\alpha} := \{ \zeta \in \mathbb{C} : \text{Re} \zeta \leq a, \quad |\text{Im} \zeta| = |\text{Re} \zeta - a| \tan \alpha \}.
\]
Let $\varepsilon > 0$ such that $a + \varepsilon < 0$. Then $\Delta_{mix}^F + \varepsilon$ is also a negative self-adjoint operator, and hence generates contracting semigroup, $e^{(\Delta_{mix}^F + \varepsilon)}$, as above.

By definition, we obtain, for $u \in L^2(M,\mu)$,
\[
e^{t\Delta_{mix}^F}u = \frac{1}{2\pi i} \int_{\gamma_{a,\alpha}} e^{\xi} (\zeta - \Delta_{mix}^F)^{-1} u d\zeta
\]
\[= e^{-t\varepsilon} \frac{1}{2\pi i} \int_{\gamma_{a,\alpha}} e^{t(\zeta + \varepsilon)} (\zeta + \varepsilon - (\Delta_{mix}^F + \varepsilon))^{-1} u d\zeta
\]
\[= e^{-t\varepsilon} \frac{1}{2\pi i} \int_{\gamma_{a+\varepsilon,\alpha}} e^{t(\xi - (\Delta_{mix}^F + \varepsilon))^{-1} u d\xi} = e^{-t\varepsilon} e^{t(\Delta_{mix}^F + \varepsilon)} u,
\]
where $\gamma_{a+\varepsilon,\alpha} = \gamma_{a,\alpha} + \varepsilon \subset \{ \text{Re} \xi < 0 \}$. Let $f_1$ the constant function on $M$ equals 1. By Theorem 8.2.2 in [10], we know, for $\lambda > 0$,
\[
(\lambda - (\Delta_{mix}^F + \varepsilon))^{-1} f_1 = \int_0^\infty e^{-\lambda t} e^{t(\Delta_{mix}^F + \varepsilon)} f_1 dt.
\]
Let us choose $\lambda = \varepsilon$, then, by using (A.11), we obtain
\[
-((\Delta_{mix}^F))^{-1} f_1 = \int_0^\infty e^{-\varepsilon t} e^{t(\Delta_{mix}^F + \varepsilon)} f_1 dt = \int_0^\infty e^{t\Delta_{mix}^F} f_1 dt
\]
and hence,
\[
\int_0^\infty 1 - \mathcal{P}_\Gamma(t, x) dt = -((\Delta_{mix}^F))^{-1} f_1(x) < \infty.
\]
Therefore, the dominated convergence theorem implies
\[
\lim_{b \to \infty} \int_0^b (\mathcal{P}_\Gamma(b, x) - \mathcal{P}_\Gamma(t, x)) dt = \int_0^\infty 1 - \mathcal{P}_\Gamma(t, x) dt < \infty.
\]
Hence, by using (A.5) and integration by parts, we obtain
\[
\begin{align*}
  u(x) &= \lim_{b \to \infty} \left( \mathcal{P}_\Gamma(b, x) b - \int_0^b \mathcal{P}_\Gamma(t, x) dt \right) = \lim_{b \to \infty} \int_0^b \left( \mathcal{P}_\Gamma(b, x) - \mathcal{P}_\Gamma(t, x) \right) dt < \infty \\
  &= \int_0^\infty 1 - \mathcal{P}_\Gamma(t, x) dt.
\end{align*}
\]
Therefore, by (A.12), we obtain
\[
\Delta F_{\text{mix}} u = -f_1 = -1.
\]
In particular, \( u \in D(\Delta_{\text{mix}}) \), and hence,
\[
u \partial u |_{\partial M \setminus \Gamma} = 0.
\]
We see that (3.2) is satisfied.

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