Several Extended Mean-variance Fuzzy Portfolio Selection Models Based on Possibility Theory

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Abstract. Classical Mean-Variance model pioneered by Markowitz 1952 has proposed the model to be one of the most important researches in modern finance. However the classical Markowitz’s Mean-Variance method does not match to represent the practicality of the uncertainties in stock market problem in portfolios. Seven types of extended mean-variance method of portfolio selection model based on the possibilitic return and possibilistic risk is presented in this paper. The returns of assets obey trapezoidal and pentagonal fuzzy numbers. The objective of this paper is to compare the optimal portfolio composition and performance using trapezoidal, pentagonal fuzzy numbers by considering whether the data is assumed normal, using real distribution and follow the trading day situations. A numerical example is presented to illustrate the usability of the model. The result will provide on how the investors could improve on their investment strategy.

1. Introduction
The classical mean-variance portfolio optimization model proposed by H. Markowitz in 1952 has been one of the principal methods of financial theory and portfolio selection to model behavior of investment under uncertainty with combination of probability and optimization techniques. Since then, with the development of financial fields, many alternative models have been studies which are based on probability theory to characterize risk and return. With the introduction of fuzzy set theory by [1] and development in [2], many researchers have tried to employ fuzzy variables to manage selection problem. Many theories of uncertainty have been proposed such as, possibility theory, credibility theory, random fuzzy theory, fuzzy random theory and others.

Possibility theory based on possibilistic distribution is proposed by Zadeh [3]and then advanced by [4]. The possibility theory is introduced to allow reasoning to be carried out on imprecise or vague knowledge. Possibility is normally associated with fuzziness, either in the background knowledge on which possibility is based, or in the set for which possibility is stated[5]. There are many elements that can be fuzzified such as risk and return [6–10]. Possibilistic portfolio selection model was firstly proposed by [11] where fuzzy variables are associated with exponential possibility distribution. Then, [12] propose upper and lower possibility distribution which can be used in portfolio selection problem to reflect experts’ knowledge. [13] proposed two kind of portfolio selection models that based on possibility probabilities and fuzzy distribution. Alternatively, [14] introduced the notion of lower and upper possibilistic mean and variance of fuzzy variables. The crisp possibilistic mean and variance was also been introduced. Based on that, [15] find an optimal mean-variance portfolio selection model with...
highest utility score under assumption that the return of securities are trapezoidal fuzzy variables. Later, [16] presented the notion of lower and upper possibilistic variance and covariance of fuzzy variables. A lower and upper possibilistic mean-variance model in portfolio selection was proposed by [17] when the investment proportion has lower bound constrained. Afterward, [18] discussed the portfolio selection model problem for a bounded security on the base of the general possibilistic mean-variance utility function with a parameter quadrating programming problem. Then, [19] studied the possibilistic mean-variance portfolio selection problem considering the securities obeying LR-type possibility distributions. Then [20] introduced a possibilistic mean-variance portfolio selection model where it is formulate with background risk by assuming that the possibility distribution of security returns is LR-type. The models and approaches introduced by researchers have it own strength and weakness.

The objective of this paper is to compare the performance of seven methods of extended fuzzy mean-variance model. Numerical examples are given to illustrate the application of all the six models. This paper is organized as follows: in section 2 the review of mathematical models is given. In section 3 the data and methodology used are explained. In section 4, numerical examples are given to demonstrate the application of the models. Finally, section 6 concludes the paper.

2. Methodology

This paper focuses on the portfolio selections based on possibilistic mean, possibilistic variance, and the possibilistic covariance under assumption that the returns of assets are trapezoidal and pentagonal fuzzy numbers. The returns of an asset are considered as trapezoidal and pentagonal fuzzy numbers. Each model with trapezoidal and pentagonal fuzzy numbers will be divided into model with assumption of normal distribution, model with the actual distribution and model with trading day.

2.1. Fuzzy mean-variance model with trapezoidal fuzzy numbers

The mean, variance and covariance for mean-variance with trapezoidal fuzzy numbers are calculated using possibility approach. The model is divided into three types of distribution representation of model with actual distribution, by assuming the data is normal and with trading day. The basic model for Fuzzy mean-variance with trapezoidal fuzzy number is shown in (1) below.

$$\text{Min } \sigma_p^2 = \sum w_i \left( \frac{(r_i - r_k)^2}{4} + \frac{(r_i - r_k + r_l - r_p)^2}{8} + \frac{(r_i - r_k)(r_i - r_k + r_l - r_p)}{3} \right) +$$

$$\sum_{i=1}^{n} w_i^2 \left( \frac{T_{i1}^2}{4} + \frac{T_{i2}^2}{8} + 2\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \frac{T_{i1} T_{j1}}{4} + \frac{T_{i2} T_{j2}}{6} + \frac{T_{i3} T_{j3}}{8} \right)$$

respectively

Subject to:  
$$R_p \geq \sum w_i \left( \frac{1}{6} (r_l + r_k) + \frac{1}{3} (r_2 + r_3) \right)$$

$$\sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0$$

(1)

with $T_{ik} = r_{ik} - r_{ik} \cdot T_{jk} = r_{jk} - r_{jk} + r_{jk} - r_{ik}, \quad k = i, j$

where:

$R_p$ is a portfolio return, $\sigma_p^2$ is a portfolio risk, $r_1$, $r_2$, $r_3$ and $r_4$ is asset $i^{th}$ rate return.
2.1.1. Method I. In method I, the main model in (1) is used and the return representation will follow Vercher’s [21] approach. The percentile is used to determine the return of asset in data. In this method, the data is assumed normal. where \( r_i \) is asset return at 40\(^{th}\) percentile, \( r_2 \) is asset return spread between 40\(^{th}\) percentile and 5\(^{th}\) percentile, \( r_3 \) is asset return at 60\(^{th}\) percentile and \( r_4 \) is asset return between 95\(^{th}\) percentile and 60\(^{th}\) percentile.

Method II. This method will consider the actual situation of the data. In real situation, the skewness of the distribution will affect the percentile selected to represent the fuzzy number. The appropriate central tendency when the existence of skewness is mode because it is situated on the peak of the distribution. Hence the trapezoidal can be formed based on the mode as the reference. The percentile of the mode is determined using formula in (2) and the entire percentile to form a trapezoidal is then composed. Each distribution curve is separated into two parts. First part is the left distribution from 5\(^{th}\) percentile to mode – 10\(^{th}\) percentile and second part is the right distribution, mode + 10\(^{th}\) percentile to 95\(^{th}\) percentile. Figure 1 shows the assets return distribution on percentile of mode. The fuzzy return is obtained will be used in the formula for the fuzzy mean-variance model as shown in (1).

\[
\text{Percentile} = \frac{(N \leq + 0.5N)}{N_T} \times 100
\]  

(2)

where : \( N_i \) is the number of score below the mode, \( N \) is the number of scores equal to mode
\( N_T \) is the total number of scores.

2.1.2. Method III. Instead of using percentile in method I and II, trading day price is used to represent the return of trapezoidal fuzzy number. The possible maximum return, the minimum return and the average return of the security is defined in (3). The four parameters need to be observed are the closing price \( p_{it}^{close} \), the high price \( p_{it}^{high} \), the low price \( p_{it}^{low} \) and the opening price \( p_{it}^{open} \), where \( i = 1,2,3,...,N \) the number of the type of securities is and \( t = 1,2,...,T \) is the number of trading day.

\[
R_{it}^{max} = \frac{p_{it}^{high} - p_{it}^{low}}{p_{it}^{low}}, \quad R_{min} = \frac{p_{low} - p_{high}}{p_{high}}, \quad R_{av1} = \frac{p_{low} - p_{open}}{p_{open}}, \quad R_{av2} = \frac{p_{close} - p_{low}}{p_{low}}
\]  

(3)

2.2. Fuzzy mean-variance model with pentagonal fuzzy numbers

The model proposed in [22] will be the main model in order to applied method IV, V, VI and VII. The main model is as in (4).

2.2.1. Method IV. In method IV, the main model in (4) is used and the return representation will use percentile approach. The percentile is used to determine the return of asset in data. The return represent the pentagonal fuzzy numbers is divided by using percentile of 5\(^{th}\), 25\(^{th}\), 40\(^{th}\), 50\(^{th}\), 75\(^{th}\) and 95\(^{th}\). The percentile is based on normal distribution and the pentagonal fuzzy number is constructed based on the percentile yield by the value of mean. In this method, the data is assumed normal.

2.2.2. Method V. This method is considering the actual data distribution. The method is similar to method II in trapezoidal fuzzy numbers in determining the percentile of the mode. The difference is in this method it will have five percentile representations. The percentile of the mode is determined using formula in (2) and the entire percentile to form a pentagonal is then composed. Each distribution curve is separated into two parts and one middle point. First part is the left distribution from 5\(^{th}\) percentile to mode minus the 10\(^{th}\) percentile and second part is the right distribution, mode plus to the 10\(^{th}\) percentile to 95\(^{th}\) percentile.
Min $\sigma^2_p = \sum_{i=1}^{n} w_i^2 \left( \frac{r_i^2}{12} - \frac{r_{i2}^2}{3} + \frac{r_{i3}^2}{6} - \frac{r_{i4}^2}{3} + \frac{2r_{i5}^2}{3} - r_{i2}r_{i3} + \frac{r_{i2}r_{i4}}{3} - r_{i2}r_{i4} + \frac{r_{i3}r_{i4}}{3} + \frac{r_{i3}r_{i4}}{6} + \frac{r_{i4}r_{i5}}{12} \right) + \sum_{i=1}^{n} \left( \frac{r_{i2}}{6} - \frac{r_{i3}}{6} + \frac{5r_{i4}}{6} - \frac{r_{i5}}{6} \right) \sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$

subject to: $R_p \geq \sum w_i \left( \frac{r_i^2}{6} - \frac{r_{i3}}{6} + \frac{5r_{i4}}{6} - \frac{r_{i5}}{6} \right)$

with $T_{ik} = r_{ik} - r_{ik}$, $T_{2k} = r_{4k} - r_{3k} + r_{2k} - r_{1k}$, $k = i, j$

2.2.3. Method VI. The calculation of return is defined by using the price of opening price, closing price, the high price and the low price. Then the possible maximum return, the minimum return and the average return of the security is defined. The asset return was derived as a pentagonal of fuzzy number based on the observed price of the trading day. In trapezoidal fuzzy numbers only four returns is defined. While in pentagonal fuzzy numbers, the five trading day return to represent pentagonal fuzzy numbers is defined in (5)

$$R_{\text{max}} = \frac{P_{\text{high}}}{P_{\text{low}}} - P_{\text{low}}, R_{\text{min}} = \frac{P_{\text{low}} - P_{\text{high}}}{P_{\text{high}}}, R_{\text{av1}} = \frac{P_{\text{low}} - P_{\text{open}}}{P_{\text{high}}}, R_{\text{av2}} = \frac{P_{\text{close}} - P_{\text{low}}}{P_{\text{low}}}, R_{\text{av3}} = \frac{R_{\text{av1}} - R_{\text{av2}}}{2}$$

2.2.4. Method VII. The difference in this method is the average part of pentagonal representation of fuzzy numbers. Instead of using the average of $r_2$ and $r_3$ in order to define $r_3$ in model VI, in this model $r_3$ is define by using the average of $r_{\text{min}}$ and $r_{\text{max}}$ (6).

$$R_{\text{max}} = \frac{P_{\text{high}} - P_{\text{low}}}{P_{\text{mean}}}, R_{\text{min}} = \frac{P_{\text{low}} - P_{\text{high}}}{P_{\text{mean}}}, R_{\text{av1}} = \frac{P_{\text{low}} - P_{\text{open}}}{P_{\text{mean}}}, R_{\text{av2}} = \frac{P_{\text{close}} - P_{\text{low}}}{P_{\text{low}}}, R_{\text{av3}} = \frac{R_{\text{av1}} - R_{\text{av2}}}{2}$$

3. Empirical results

3.1. Portfolio Composition

A numerical example is given to illustrate the application of the models and the efficiency of the models is tested using daily data prices of 17 stocks included in the Kuala Lumpur Composite Index (KLCI) starting from October 2012 to March 2017. Table 1 indicates the daily portfolio composition of the models. Weight of the stock with non-zero value indicates that the shares will be selected from the model while stocks with zero weight will not being selected by the model to form optimum portfolio. The optimum portfolio composition is the value of weight which gives optimum value for certain model. Method I for trapezoidal fuzzy numbers consists of GENTING BHD only. Portfolios generated using method II, III, for trapezoidal fuzzy numbers and method I for pentagonal fuzzy numbers are highly invested in KLCI. If method II in trapezoidal fuzzy numbers is chosen to be employed, 48.2725% of fund will be invested. While Method III for trapezoidal fuzzy numbers, method I for pentagonal fuzzy
numbers is utilized then 38.9467% and 12.766% will be invested respectively. Alternatively if method I is chosen to be employ, 66.11% will be invested in MISC. KLCC is the only selected stock should be invested in the optimal portfolio if one chooses to employ method III and IV in pentagonal fuzzy numbers.

**Table 1. Portfolio composition**

| Stock          | Fuzzy mean-variance with trapezoidal fuzzy numbers | Fuzzy mean-variance with pentagonal fuzzy numbers |
|----------------|--------------------------------------------------|--------------------------------------------------|
| MISC           | Method I: -                                      | Method II: -                                     |
| Digi           | Method III: 4.2259                               | Method IV: 0.7553                                |
| KLCC           | Method I: 48.2725                                 | Method II: 12.7666                               |
| HL             | Method III: -                                    | Method IV: -                                    |
| FINANCIAL      | Method I: -                                      | Method III: -                                   |
| PPB            | Method I: 5.1509                                 | Method III: -                                   |
| GENTING BHD    | Method I: 1                                     | Method III: -                                   |
| CIMB           | Method I: -                                      | Method III: -                                   |
| HL BANK        | Method I: 2.4896                                 | Method III: -                                   |
| SIME DARBY     | Method I: 1.0424                                 | Method III: -                                   |
| AMMB           | Method I: 6.5743                                 | Method III: 11.8523                              |
| YTL            | Method I: -                                      | Method III: 1.9087                               |
| GENTING        | Method I: -                                      | Method III: -                                   |
| PETRONAS       | Method I: 6.4078                                 | Method III: -                                   |
| GAS            | Method I: -                                      | Method III: -                                   |
| IJM            | Method I: -                                      | Method III: -                                   |
| PET. DAGANG    | Method I: 1.0788                                 | Method III: -                                   |
| PUBLIC BANK    | Method I: 39.4913                                 | Method III: 26.8004                              |
| TELEKOM        | Method I: 7.1307                                 | Method III: 4.6254                              |
| AXIATA         | Method I: -                                      | Method III: 5.2881                               |
| MAYBANK        | Method I: 5.6672                                 | Method III: 1.9618                              |
| KL. KEPONG     | Method I: -                                      | Method III: 1.6369                              |
| BERHAD         | Method I: -                                      | Method III: 9.0313                              |
| TENAGA         | Method I: -                                      | Method III: 1.0337                              |
| HAP SENG       | Method I: -                                      | Method III: 9.0313                              |
| RHB CAPITAL    | Method I: -                                      | Method III: 9.0313                              |
| BAT            | Method I: -                                      | Method III: 9.0313                              |

**3.2. Portfolio performance**

Table 2 provides the summary statistics of the seven different methods of optimal portfolios in trapezoidal and pentagonal fuzzy numbers. As described in table 2, it is found that the highest mean return is given by method I by trapezoidal fuzzy numbers with an average return of 0.9269%, followed by method II and IV in pentagonal fuzzy numbers and method II in trapezoidal fuzzy numbers with 0.6895%, 0.6825% and 0.182% respectively. Method III in trapezoidal fuzzy numbers, method I and II in pentagonal fuzzy numbers gives negative return with -0.0012%, -0.2097% and -0.6211%. Method II pentagonal fuzzy numbers show less risky portfolio (0.0002%) followed by method I pentagonal fuzzy numbers (0.0016%), method I trapezoidal fuzzy numbers (0.1752%), method II in pentagonal fuzzy numbers (0.2178%), method IV pentagonal fuzzy numbers (0.2221%) and method III for trapezoidal(0.6704%). The most risky asset is method II trapezoidal fuzzy numbers with 0.8125%.
Overall, method I trapezoidal fuzzy numbers is able to produce the highest performance among the seven methods. Optimum portfolio for method I and method II for trapezoidal fuzzy numbers, method I and IV for pentagonal fuzzy numbers indicate positive ratio of the mean return to standard deviation of the portfolio. Method III for trapezoidal fuzzy numbers and method I and II pentagonal fuzzy numbers indicate negative ratio of the mean return to standard deviation of the portfolio.

Table 2. Summary Statistics

| Method  | Fuzzy mean-variance with trapezoidal fuzzy numbers | Fuzzy mean-variance with pentagonal fuzzy numbers |
|---------|--------------------------------------------------|-------------------------------------------------|
| I       | Mean return: 0.9269                             | Mean return: -0.2097                            |
| II      | Mean risk: 0.1752                                | Mean risk: 0.0016                               |
| III     | Mean Portfolio Performance: 5.2922               | Mean Portfolio Performance: -0.5304             |

4. Conclusion

This paper discussed the portfolio optimization models of possibilistic means, variances and covariance in Markowitz’s mean-variance model with trapezoidal and pentagonal fuzzy numbers respectively. Fuzzy mean-variance with trapezoidal fuzzy numbers and pentagonal fuzzy numbers consist of three and four method respectively. According to table 1, investors will make different decision on which stocks they prefer to invest. The portfolio’s performances and compositions were compared for seven methods in fuzzy mean-variance with trapezoidal and pentagonal fuzzy numbers. Among the seven methods, method I gives the best performance. However, method I is not considered the actual distribution or actual market situation and method I only considered one stock to invest. Even though method VI and VII are based on the actual situation by using the trading day price, the result shows that there is only one stock should be invested. Hence, it can be concluded that method II gives the most appropriate portfolio optimization model among the seven methods with consideration of the actual data.

5. References

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