Abstract

Adams and Conway have stated without proof a result which says, roughly speaking, that the representation ring $R(G)$ of a compact, connected Lie group $G$ is generated as a $\lambda$-ring by elements in 1-to-1 correspondence with the branches of the Dynkin diagram. In this note we present an elementary proof of this.

§1. Introduction

Let $G$ be a 1-connected, compact Lie group. A celebrated theorem of Weyl’s describes the representation ring $R(G)$ as a polynomial ring

$$R(G) = [V_1, \ldots, V_r],$$

where the "fundamental representations" $V_i$ are in 1-to-1 correspondence with the nodes of the Dynkin diagram. Therefore, in order to understand the representations of $G$, one basically needs to understand $r$ of them.

In his 1985 article [2], J. F. Adams presents a method, which he ascribes to J. H. Conway, to cut down on this number. The idea, simply enough, is to exploit the structure of $\lambda$-ring which $R(G)$ possesses since it is possible to take exterior powers of representations (see [4] for this – we shall not need anything from the general theory of $\lambda$-rings, though).

The key result announced by Adams is the following. Suppose that the Dynkin diagram of $G$ has an arm of length $k$; by this we mean that there are nodes $v_1, v_2, \ldots, v_k$ such that $v_i$ is connected to $v_{i+1}$ by a single bond for $1 \leq i < k$, and such that no other edge runs to any of $v_1, v_2, \ldots, v_{k-1}$. Then one has:

**Theorem 1.1** – Suppose that the Dynkin diagram of the 1-connected, compact Lie group $G$ has an arm of length $k$. Let $V_i$ be the representation corresponding to $v_i$. Then for $1 \leq i \leq k$, one has

$$i^i V_1 = V_i \oplus \text{(lower terms)}$$

where the lower terms are irreducible representations $W < V_i$.

In this statement we order the irreducible representations of $G$ according to their respective maximal weights.
For any $G$-module $M$, we let $\mathcal{Top}(M)$ denote the smallest submodule of $M$ having the same maximal weights, with the same multiplicities. In other words, $\mathcal{Top}(M)$ is the smallest $G$-submodule of $M$ satisfying $M = \mathcal{Top}(M) \oplus$ (lower terms) (here we mean that any irreducible factor in the "lower terms" is dominated by at least one irreducible factor in $\mathcal{Top}(M)$). With this terminology, theorem 1.1 can be reformulated thus:

$$V_i = \mathcal{Top}(l^iV_1).$$

As another illustration of the $\mathcal{Top}$ notation, which is due to Adams, we point out that the traditional proof of Weyl’s theorem (eg as in [1]) shows that any irreducible representation of $G$ is of the form $\mathcal{Top}(V_1 \otimes V_2 \otimes \cdots \otimes V_r).$

With this result in hand, one can show easily that $l^iV_1$ may replace $V_i$ in the statement of Weyl’s theorem (this is more or less trivial, but requires a certain amount of notation – we shall come back to this in due time). Therefore, if we see $R(G)$ as a $\lambda$-ring, the whole arm of the Dynkin diagram is accounted for by the single generator $V_1.$

Adams (unlike Conway) had a proof of theorem 1.1 ready, but wanted to check if anyone else had previously obtained the result before publishing it. His intention seemed to include the proof in a book he had in preparation on exceptional Lie groups. This was sadly prevented by Adams’s untimely death. The book has appeared [3], based on his lecture notes, but does not include this proof.

In this quick note we present an elementary proof of the main theorem, and give a few applications.

§2. Notations

$G$ is a 1-connected, compact Lie group, with maximal torus $T$ and Weyl group $W.$ We fix a $W$-invariant inner product on $L(T),$ the real Lie algebra tangent to $T,$ denoted by $\langle -, - \rangle.$

The roots of $G$ are non-zero weights of the adjoint representation. They come in $m$ pairs $\pm \theta_1, \pm \theta_2, \ldots, \pm \theta_m$, and we assume that the signs are arranged so that the "fundamental Weyl chamber"

$$FWC = \{ v \in L(T) : \theta_i(v) > 0 \}$$

is nonempty. We call $\theta_1, \ldots, \theta_m$ the positive roots; we assume that the indices are so chosen that the simple roots are $\theta_1, \theta_2, \ldots, \theta_r$ (where $r = \dim T$).

Dually, the fundamental dual Weyl chamber is

$$FDWC = \{ \lambda \in L(T)^* : \langle \lambda, \theta_i \rangle > 0 \}.$$

To each simple root $\theta_i$ corresponds the fundamental weight $\phi_i$ characterized by

$$\frac{2\langle \phi_i, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \delta_{ij}$$

for any simple root $\theta_j.$

We partially order weights by requiring $\lambda_1 < \lambda_2$ if and only if $\lambda_1(v) < \lambda_2(v)$ for all $v \in FWC.$
For any root $\theta_i$, there is an element $\varphi_i \in W$ which induces on $L(T)$ the orthogonal reflection in $\ker \theta_i$. For convenience, $\varphi_0$ will denote the identity. Very explicitly, for $\lambda \in L(T)$ or $\lambda \in L(T)^{\ast}$:

$$\varphi_i(\lambda) = \lambda - \frac{2\langle \lambda, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} \theta_i.$$  

Finally, we assume, of course, that the Dynkin diagram of $G$ has an arm of length $k$, as explained in the introduction. For simplicity, we arrange the notations so that $\theta_1, \theta_2, \ldots, \theta_k$ are the simple roots corresponding to the nodes $v_1, v_2, \ldots, v_k$. We write $V_i$ for the representation associated with $\phi_i$ (or with $v_i$, if you want). We will also write $V$ for $V_1$ and $\phi$ for $\phi_1$.

It may be useful to recall that two simple roots $\theta_i$ and $\theta_j$ correspond to nodes with a simple bond between them precisely when $\langle \theta_i, \theta_j \rangle = \langle \theta_j, \theta_j \rangle$ and

$$\frac{-2\langle \theta_i, \theta_j \rangle}{\langle \theta_i, \theta_i \rangle} = 1.$$  

For example, we shall use repeatedly $\varphi_i \theta_{i+1} = \varphi_{i+1} \theta_i = \theta_i + \theta_{i+1}$ for $i < k$.

§3. Main proof

Our first objective is to show that the weight $\phi_i$ occurs in $l^i V$.

**Lemma 3.1** — For $1 \leq i < k$, we have

$$\phi_{i+1} = \phi + \varphi_1 \phi + \varphi_2 \phi + \cdots + \varphi_i \varphi_1 \cdots \varphi_1 \phi.$$  

**Proof.** Let $\lambda$ denote the right hand side. We shall compute $\langle \lambda, \theta_j \rangle$.

If $j > i + 1$ then we have

$$\langle \varphi_i \cdots \varphi_1 \phi, \theta_j \rangle = \langle \phi, \varphi_i \cdots \varphi_1 \phi \rangle = \langle \phi, \theta_j \rangle = 0$$  

for $s \leq i$ and it follows that $\langle \lambda, \theta_j \rangle = 0$.

If now $j < i + 1$, the equations above still hold when $s < j - 1$. The two terms in $\langle \lambda, \theta_j \rangle$ obtained for $s = j - 1$ and $s = j$ cancel each other, for $\varphi_i \theta_j = -\theta_j$. Finally, for $s \geq j + 1$ we obtain

$$\langle \phi, \varphi_1 \cdots \varphi_j \theta_j \rangle = \langle \phi, \varphi_1 \cdots \varphi_j \theta_{j+1} \rangle = \langle \phi, \theta_{j+1} \rangle = 0$$  

where we use $\varphi_j \varphi_{j+1} \theta_j = \varphi_j (\theta_j + \theta_{j+1}) = -\theta_j + \theta_j + \theta_{j+1} = \theta_{j+1}$. So $\langle \lambda, \theta_j \rangle = 0$ in this case also.

It remains to compute $\langle \lambda, \theta_{i+1} \rangle$. Arguing as above, we see that all terms vanish except

$$\langle \phi, \varphi_1 \cdots \varphi_i \theta_{i+1} \rangle = \langle \phi, \theta_1 + \theta_2 + \cdots + \theta_{i+1} \rangle = \langle \phi, \theta_1 \rangle$$  

and we obtain $\langle \lambda, \theta_{i+1} \rangle = \langle \phi, \theta_1 \rangle = \frac{1}{2} \langle \theta_1, \theta_1 \rangle = \frac{1}{2} \langle \theta_{i+1}, \theta_{i+1} \rangle$. It follows that $\lambda = \phi_{i+1}$.

As a corollary, the weights $\varphi_i \varphi_{i-1} \cdots \varphi_1 \phi = \phi_{i+1} - \phi_i$, for various values of $i$, are pairwise distinct. It follows that, if $v \in V$ is a non-zero eigenvector under the action of $T$ with weight $\phi$, then $v \land \varphi_1 v \land \cdots \land \varphi_i v \land \varphi_1 v \in l^{i+1} V$ is a non-zero eigenvector with weight $\phi_{i+1}$.

Let us prove another technical lemma before proceeding.
Lemma 3.2 — Let $i_1, i_2, \ldots, i_t$ be integers, where $1 \leq t < k$. Then

$$
\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_t}\emptyset = \varphi_s\varphi_{s-1}\cdots\varphi_1\emptyset
$$

for some $s \leq t$.

More precisely, for $s < k - 1$, the weight $\varphi_s\varphi_{s-1}\cdots\varphi_1\emptyset$ is fixed by any $\varphi_u$ unless $u = s$ or $u = s + 1$.

Proof. Clearly the first equality follows by repeated use of the more precise statement. Let $\lambda = \varphi_s\varphi_{s-1}\cdots\varphi_1\emptyset = \emptyset_{s+1} - \emptyset_s$. Then

$$
\langle \varphi_u\lambda, \theta_j \rangle = \langle \lambda, \varphi_u\theta_j \rangle = \langle \lambda, \theta_j \rangle + c_j\langle \lambda, \theta_u \rangle
$$

for some constant $c_j$. If $u$ is neither equal to $s$ nor $s + 1$, then $\langle \emptyset_{s+1} - \emptyset_s, \theta_u \rangle = 0$ and we conclude that $\langle \varphi_u\lambda, \theta_j \rangle = \langle \lambda, \theta_j \rangle$ for all $j$. \hfill $\Box$

It follows in particular that $\varphi_i\emptyset = \emptyset$ if $i \geq 2$. An immediate induction then gives

$$
\varphi_i\varphi_{i-1}\cdots\varphi_1\emptyset = \emptyset - (\theta_1 + \theta_2 + \ldots + \theta_i)
$$

which will be handy in the sequel.

We are now ready to prove that $\emptyset_t$ is maximal in $l'V$. Let us start by ordering the weights in the $W$-orbit of $\emptyset$.

Lemma 3.3 — Let $v \in FWC$. Then

$$
\emptyset(v) > \varphi_1\emptyset(v) > \varphi_2\emptyset(v) > \cdots > \varphi_{k-1}\varphi_{k-2}\cdots\varphi_1\emptyset(v)
$$

and $\varphi_{k-1}\varphi_{k-2}\cdots\varphi_1\emptyset(v) > \varphi_0(v)$ for any other value of $\varphi_0(v)$ with $\varphi \in W$.

Proof. The first inequalities are trivial, and follow from the last expression given for $\varphi_i\varphi_{i-1}\cdots\varphi_1\emptyset$.

It is well-known ([1], 6.26) that $\varphi_0(v) < \emptyset(v)$ as soon as $\varphi_0 \neq \emptyset$. Now, let $\varphi \in W$ be such that $\varphi_0(v)$ reaches its second highest value. Then $\varphi_0 \neq \emptyset$ so $\varphi_0$ is not in $CIFDWC$, and therefore there is an $u$ such that $\langle \varphi_0, \theta_u \rangle < 0$. Compute then

$$
\varphi_u\varphi_0(v) = \varphi_0(v) - \frac{2\langle \varphi_0, \theta_u \rangle}{\langle \theta_u, \theta_u \rangle}\theta_u(v) > \varphi_0(v).
$$

By definition of $\varphi$, this implies $\varphi_u\varphi_0(v) = \emptyset(v)$, so that $\varphi_u\varphi_0 = \emptyset$ and $\varphi_0 = \varphi_u\emptyset$. By the last lemma, $\varphi_u\emptyset$ can only be $\varphi_1\emptyset$, so we conclude that $\varphi_1\emptyset$ is the second highest value of $\varphi_0(v)$ for $\varphi \in W$, and that it is only reached when $\varphi_1\varphi$ fixes $\emptyset$.

We would prove similarly that $\varphi_2\varphi_1\emptyset(v)$ is the third highest value of $\varphi_0(v)$, which is only reached when $\varphi_2\varphi_1\varphi$ fixes $\emptyset$. The result follows by induction. \hfill $\Box$

In order to deal with the weights in $V$ which are not in the $W$-orbit of $\emptyset$, we shall need\(^1\) the following.

Lemma 3.4 — Let $M$ be an irreducible $G$-module with maximal weight $\lambda$.

\(^1\)such weights do exist, cf the 26-dimensional representation of $F_4$ which has two zero weights.
1. Let \( m \in M \) be an eigenvector for the weight \( \alpha \). For each positive root \( \theta_i \), let \( X_{\theta_i} \in L(G) \) be an eigenvector for \( \theta_i \). Then \( \alpha = \lambda \) if and only if \( X_{\theta_i} m = 0 \) for each \( i \).

2. All the weights occurring in \( M \) are of the form

\[
\lambda - \sum n_s \theta_s,
\]

where \( n_s \) is a nonnegative integer.

Proof. Half of (1) is trivial: if \( m \) is an eigenvector for \( \lambda \), then \( X_{\theta_i} m \) is an eigenvector for \( \lambda + \theta_i > \lambda \), so it must be zero. We prove the converse and (2) at the same time.

Suppose then that \( X_{\theta_i} m = 0 \) for each positive root \( \theta_i \). Since \( M \) is irreducible, it is generated as an \( L(G) \)-module by \( m \) alone. Therefore \( M \) is spanned by elements of the form

\[
X_s X_{s-1} \cdots X_1 m
\]

with \( X_j \in L(G) \). Splitting \( L(G) \) according to the adjoint action of the maximal torus \( T \), we may assume that each \( X_j \) is an eigenvector associated to the root \( \lambda_j \), or to the zero weight. A vector as above is then an eigenvector in \( M \) with weight \( \alpha + \lambda_1 + \cdots + \lambda_s \).

We claim that \( M \) is in fact generated by elements as above for which each \( \lambda_i \) is a negative root (or zero). Granting this, \( \alpha \) is clearly maximal, so \( \alpha = \lambda \), and the other weights in \( M \) are as announced in (2).

To prove the claim, we show by induction on \( s \) how to replace the generating vectors. The case \( s = 1 \) is our assumption, while the induction proceeds easily using \( X_i X_j = X_j X_i + [X_i, X_j] \).

Theorem I.1 now follows from this:

**Proposition 3.5** - \( \omega_{i+1} \) is the highest weight is \( l^{i+1} V \) (equivalently, it is the only maximal weight).

Proof. By induction on \( i \) (the case \( i = 0 \) being given). So let

\[
\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_{i+1}
\]

be a maximal weight occurring in \( l^{i+1} V \), where each \( \lambda_j \) is a weight in \( V \).

**First case.** Suppose to start with that for each \( \lambda_j \) and each \( v \in FWC \) we have \( \lambda_j(v) \geq \varphi_i \varphi_{i-1} \cdots \varphi_1 \varphi(v) \). By the preceding lemma, we have

\[
\lambda_j = \varphi - \sum n_s \theta_s
\]

while \( \varphi_i \varphi_{i-1} \cdots \varphi_1 \varphi = \varphi - \theta_1 - \theta_2 - \cdots - \theta_i \). It follows that

\[
\sum n_s \theta_s(v) \leq \theta_1(v) + \cdots + \theta_i(v)
\]

for all \( v \in FWC \), or even for \( v \in ClFWC \). Applying this with \( v = \omega_s \) (and identifying \( L(G) \) with its dual) we conclude that each \( n_s \) is 0 or 1 when \( 1 \leq s \leq i \) (and zero otherwise).
So if \( \lambda_j \neq \emptyset \), let \( t \) denote the largest value of \( s \) such that \( n_s = 1 \), and write

\[
\lambda_j = \varphi_{i_1}\varphi_{l-1} \cdots \varphi_{1}\emptyset + \sum_{s<t} m_s\theta_s,
\]

with \( m_s = 1 - n_s \). If there exists an eigenvector in \( V \) for this weight, then there is also one for the weight

\[
\alpha = \emptyset + \varphi_1\varphi_2 \cdots \varphi_t \left( \sum_{s<t} m_s\theta_s \right).
\]

Here \( \varphi_1\varphi_2 \cdots \varphi_t\theta_s = \varphi_1 \cdots \varphi_{s+1}\theta_s = \theta_{s+1} \) (as already observed during the proof of \( \Theta \)). It follows that \( \alpha > \emptyset \) if any coefficient \( m_s \) is nonzero. As this is impossible, we conclude that \( \lambda_j = \varphi_{i_1}\varphi_{l-1} \cdots \varphi_{1}\emptyset \).

It follows that \( t \) must be different for each \( j \), and that \( \lambda = \theta_{i+1} \).

Second case. Suppose now that there is a \( v \in FWC \) and a \( j \) such that \( \lambda_j(v) < \varphi_{i}\varphi_{l-1} \cdots \varphi_{1}\emptyset(v) \), say \( j = i + 1 \). We shall derive a contradiction (recall that \( \lambda \) is assumed to be maximal).

Now if \( \lambda_{i+1} \) is in the \( W \)-orbit of \( \emptyset \), we draw from \( \Theta \) that \( \lambda_j(v) < \varphi_{i}\varphi_{l-1} \cdots \varphi_{1}\emptyset(v) \) actually holds for all \( v \in FWC \). Then by the induction hypothesis, we have

\[
\lambda_1(v) + \lambda_2(v) + \ldots + \lambda_i(v) \leq \emptyset(v) + \varphi_i\emptyset(v) + \ldots + \varphi_{i-1}\varphi_{i-2} \cdots \varphi_{1}\emptyset(v),
\]

so \( \lambda(v) < \theta_{i+1}(v) \) and we are done.

If on the other hand \( \lambda_{i+1} \) is not in the \( W \)-orbit of \( \emptyset \), we shall proceed quite differently. Let \( x \in V \) be an eigenvector associated to \( \emptyset \), and let \( y \in V \) be an eigenvector for \( \lambda_{i+1} \). Put

\[
m = x \wedge \varphi_1 x \wedge \ldots \wedge \varphi_{i-1}\varphi_{i-2} \cdots \varphi_1 x \wedge y \in l^{i+1}V.
\]

Then \( m \) is non-zero, and is an eigenvector for a weight which is \( \geq \lambda \) (using the induction hypothesis), so by maximality of \( \lambda \), we see that \( m \) is an eigenvector for \( \lambda \). Using lemma \( \Theta \) we conclude that there exists a positive root \( \theta_j \) and an eigenvector \( X = X_{\theta_j} \in L(G) \) such that \( Xy \neq 0 \). However by the same lemma, \( Xx = 0 \) and \( Xm = 0 \) (\( m \) being an eigenvector for the maximal weight of some irreducible submodule of \( l^{i+1} \)). Write

\[
0 = Xm = X(x \wedge \varphi_1 x \wedge \ldots \wedge \varphi_{i-1}\varphi_{i-2} \cdots \varphi_1 x) \wedge y \\
\pm x \wedge \varphi_1 x \wedge \ldots \wedge \varphi_{i-1}\varphi_{i-2} \cdots \varphi_1 x \wedge Xy
\]

The first term here is 0 by induction, so we conclude that \( Xy \) is in the linear span of \( x, \varphi_1 x, \ldots, \varphi_{i-1}\varphi_{i-2} \cdots \varphi_1 x \). Since \( Xy \) is a (nonzero) eigenvector for the action of \( T \), it follows that \( Xy \) is in fact (a scalar multiple of) one of these. In conclusion, \( \lambda_{i+1} + \theta_j = \varphi_s \varphi_{s-1} \cdots \varphi_1 \emptyset \) for some \( s \) with \( 1 \leq s \leq i - 1 \), or equivalently \( \lambda_{i+1} = (\theta_{s+1} - \emptyset) - \theta_j \).

Now, the contradiction will come from the fact that, by maximality of \( \lambda \), we must have \( \lambda_{i+1} \in ClFDWC \), so that \( \varphi_j \lambda_{i+1} \leq \lambda_{i+1} \). This amounts to

\[
\frac{\langle \theta_{s+1} - \emptyset, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} \geq 1.
\]

For this to happen, we would need a positive root \( \theta_j \) which, when written as a sum of positive simple roots \( \theta_j = \sum n_i\theta_i \), has \( n_{s+1} - n_s \geq 2 \). This never
happens. You may either inspect the existing root systems, or fill in the details of the following sketch: let \( S \) be the set of positive roots such that \( |n_{s+1} - n_s| \leq 1 \) for all \( 1 \leq s + 1 \leq k - 1 \). Then \( S \) contains the positive simple roots, and is stable under each reflection \( \varphi_u \) for \( u \leq r \). Hence \( S \) is stable under \( W \) and consequently, it is equal to the set of all positive roots.

**Corollary 3.6** — We may replace \( V_i \) by \( l^i V \) in the statement of Weyl’s theorem, for \( 1 \leq i \leq k \). More generally, if \( \text{Top}(V'_i) = V_i \) for each \( i \), then

\[
R(G) = \mathbb{Z}[V'_1, V'_2, \ldots, V'_r]
\]

and moreover any irreducible representation of \( G \) is of the form

\[
\text{Top}(V'_1 \otimes V'_2 \otimes \cdots \otimes V'_r).
\]

**Proof.** Run through a proof of Weyl’s theorem and replace \( V_i \) by \( V'_i \).

As this is short enough to write out, we give the details. For each weight \( \lambda \in \text{ClFDWC} \), let \( S(\lambda) \) denote the elementary symmetric sum

\[
S(\lambda) = \sum_{\alpha \in W\lambda} e^{2i\pi \alpha}.
\]

These form a \( \mathbb{Z} \)-basis for \( R(G) = R(T)^W \) (viewed as the character ring rather than the representation ring).

Now, if \( \lambda = n_1 \omega_1 + \ldots + n_r \omega_r \), we clearly have

\[
\chi(V'_1 \otimes V'_2 \otimes \cdots \otimes V'_r) = S(\lambda) + \sum_{\lambda_i < \lambda} m_i S(\lambda_i)
\]

where \( \chi \) means the character (use [1], 6.36). If we suppose by induction that each \( S(\lambda_i) \) can be written in the form \( \chi(P(V'_1, \ldots, V'_r)) \) (with \( P \) a polynomial), then the same can be said of \( S(\lambda) \).

As a result, \( V'_1, V'_2, \ldots, V'_r \) generate \( R(G) \) as a ring. They are clearly algebraically independent (for example, consider the transcendence degree of \( R(G) \)).

The rest is easy.

§4. Applications

All the information on exceptional Lie groups used below can be found in Adams’s book [3].

**The representation ring of \( F_4 \).**

This group has the following Dynkin diagram:

```
1 ---- 2 ---- 3 ---- 4
short short long long
```

The group \( F_4 \) may be defined as the group of automorphisms of the (Jordan) algebra \( J \) of \( 3 \times 3 \) hermitian matrices over the octonions, the multiplication being \( A, B \mapsto \frac{1}{4}(AB + BA) \). The real dimension of \( J \) is 27, and \( F_4 \) acts on the subspace of matrices with trace 0. Complexifying, this affords a 26-dimensional
representation $U$ of $F_4$. This $U$ turns out to be irreducible and its highest weight corresponds to the node 1.

The (complexified) adjoint representation $Ad$, on the other hand, corresponds to the node 4.

It follows from the main theorem in this paper that

$$R(F_4) = \mathbb{Z}[Ad, l^2 Ad, U, l^2 U].$$

The $E$ family.

Adams originally considered the case of $E_8$ in [2]. This group has the following Dynkin diagram:

![Dynkin diagram for $E_8$]

Adams concludes that

$$R(E_8) = \mathbb{Z}[\alpha, l^2 \alpha, l^3 \alpha, l^4 \alpha, \beta, l^2 \beta, \gamma, \delta]$$

where $\delta$ can be taken to be either $l^5 \alpha$, or $l^3 \beta$, or $l^2 \gamma$. It is easy to see that the representation $\alpha$ is the adjoint representation, and Adams in loc cit gives some concrete information on $\beta$ and $\gamma$ (eg, the way they sit in $\alpha \otimes \alpha$). I do not know a direct, concrete definition of $\beta$ and $\gamma$.

Similarly, for $E_7$ we have the following diagram:

![Dynkin diagram for $E_7$]

So that

$$R(E_7) = \mathbb{Z}[a, l^2 a, l^3 a, b, l^2 b, c, d]$$

where $d$ can be taken to be either $l^4 a$, or $l^3 b$, or $l^2 c$. Here however, the adjoint representation does not correspond to any of $a$, $b$, $c$. We can identify nevertheless the representation $a$ as the complexification of the 56-dimensional real representation $W$ of $E_7$ which can be used to define $E_7$ as the group of linear automorphisms of $W$ preserving both a quadratic form and a quartic form.

The situation for $E_6$ is analogous. The Dynkin diagram has three branches, so $R(E_6)$ is generated by 3 elements as a $\lambda$-ring. I am not aware of any explicit description of the three fundamental representations involved. They are neither given by the adjoint representation, nor by the representation $W$ used to define $E_6$ as a group of maps.
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