Abstract—We propose a novel system identification technique, based on a least-mean square algorithm, allowing for the estimation of a linear channel by using an unknown-response measurement channel. The key of the technique is a memoryless nonlinear function working as uncoupling block between the estimated and observation channels, conforming a Wiener-Hammerstein function working as uncoupling block between the estimated and observation channels. We prove that this estimation, only differing from the actual channel response by a scaling factor and a temporal shift, does not depend on the observation channel bandwidth. As a consequence, this technique enables the usage of low-cost measurement devices as feedback channel. We present numerical examples of the method, supporting the proposal and displaying excellent results.

I. INTRODUCTION

System identification, also known as channel estimation, refers to the different techniques used to estimate the response of a system starting from measured data of its input and output [1], [2]. In a few words, given certain model for the system, a mathematical function relating the input and output signals, system identification consists in finding the parameters of that function that best fit the actual behavior of the channel.

In many situations, the first approach to the estimation of an unknown-response system is performed by the optimization of a linear model. In these cases, system identification is carried out by means of an algorithm able to find the linear function that best emulates the actual response of the system. Moreover, this function commonly provides a suitable and simple model of the channel.

In this work we focus in the linear estimation case, for which the channel of interest (channel to be estimated) is strictly linear. We consider a particular problem associated to the linear estimation: the measurement of the output through some other unknown-response system, also assumed to be linear. As the measurement system requires to be estimated, this task becomes a mathematical problem: how to perform the system identification of a chain of two unknown linear systems. As we explain in the next section, the decomposition of the different channels of the chain is not possible from a mathematical point of view. However, we show that if a nonlinear function is added between the two channels, it will work as an effective uncoupling block between them, allowing for the estimation of both systems. In general, we show that the channel responses can be estimated unless two coefficients: one related to a scale factor and other associated to a temporal shift. If the nonlinear function, expressed as a polynomial function, has at least one even term and one odd term, the scale factor is always the unity. In addition, if both channels are causal and not delayed, the coefficient of temporal-shift is zero.

The mathematical proofs provided in this work are only valid for scalar-real signals, but can be used as the starting point for a further analysis, including vectorial and complex input-output signals. Our main goal is to illustrate not only that the simultaneous estimation of the channel of interest and the measurement system is possible with a nonlinear function working as uncoupling device, but also an additional advantage of this scheme: the possibility of using ultra-low bandwidth measurement channels. This feature is commonly associated to low-cost measure devices, which are naturally expected to pose a limited frequency response and a low sample rate. A clear example is found in the coherent optical systems, where a high-performance coherent detector is usually required in the feedback channel; results derived in this work show that the coherent detector can be replaced with a low-cost nonlinear device such as a common photodiode. The numerical results presented in the last section are focused in proving this striking feature of the proposed method.

II. SYSTEM IDENTIFICATION: THE OBSERVATION CHANNEL PROBLEM

The digital model of a simple linear channel relates two scalar time-discrete signals, the input sequence \( x[n] \) with the output sequence \( y[n] \), through the sum given by

\[
y[n] = \sum_m L[m] x[n - m] = \sum_m L[n - m] x[m], \tag{1}
\]

where \( L[m] \) are real coefficients defining the response of the channel of interest and, in the general case, \( m \) varies over all the integers. An useful alternative representation of this simple part.

1 The impulse response of the linear system has a nonzero instantaneous part.
By applying this transformation on Eq. 1 we obtain

\[ L[m] = \arg \min \sum_e \hat{m}^2 \],

which is compared with the actual output sequence to calculate the error signal

\[ e_y[n] = \hat{y}[n] - y[n]. \]

The LMS algorithm allows for the minimizing of the mean squared error (MSE)

\[ \hat{\mathcal{E}} = E\{e_y^2[n]\}, \]

where \( E\{\cdot\} \) is the expectation operator. The MSE \( \hat{\mathcal{E}} \) is assumed to be a continuous and differentiable function of the estimated coefficients \( \hat{L}[m] \). Moreover, the algorithm allows for the calculation of the coefficients that minimize Eq. 6

\[ \hat{L}[m] = \arg \min \hat{\mathcal{E}}. \]

It is easy to prove that for the scheme of Fig. 1 the minimum MSE is zero and it is achieved when \( \hat{L}[m] = L[m] \).

A. Linear Observation Channel

A common obstacle to implement a simple LMS algorithm, is the fact that the output sequence is not directly accessible. An observation channel, such as a transducer or any data-acquisition device, is used to obtain information about the system output. In the simplest scheme, the observation channel is a linear system that can be modeled by means of the coefficients \( O[m] \), as shown in Fig. 2 leading to the distorted version or indirect measurement of the output signal, the measurement sequence

\[ z[n] = \sum_m O[m]y[n-m]. \]

The most typical situation is that of the response of this observation channel is not completely known. As a consequence, the system identification must include its estimation, \( \hat{O}[m] \), as displayed in Fig. 2. However, this scheme presents two several limitations. In Appendix A we show that once the algorithm reaches the stationary state, for which the error signal is suppressed, the estimated systems and the actual channels satisfy

\[ \hat{L}(\Omega)\hat{O}(\Omega) = \hat{L}(\Omega)\hat{O}(\Omega), \]

where \( \hat{L}(\Omega) \) and \( \hat{O}(\Omega) \) are the Fourier transforms of \( \hat{L}[m] \) and \( \hat{O}[m] \), respectively. This result suggests that the simultaneous estimation of the channel of interest and the observation channel can not be performed, as Eq. 9 does not imply \( \hat{L}[m] = L[m] \) or \( \hat{O}[m] = O[m] \). Moreover, there exist an infinity number of combinations of \( \hat{L}[m] \) and \( \hat{O}[m] \) satisfying Eq. 9 and being completely uncorrelated with the actual response of the channels. On the other hand, even for the case in which the observation channel is assumed to be known, its bandwidth must be at least equal to that of the \( \hat{L}(\Omega) \), in order to do not miss the information about high-frequency components of the channel response. In many cases, this condition seriously deepens the requirements on the observation channel, augmenting its cost and complexity.

III. The Nonlinear Observation Channel

In order to solve the observation channel problems presented in the previous section, we propose the modified scheme...
shown in Fig. 3. A known memoryless nonlinear function, described by the real coefficients \( a_q \), works as an uncoupling block between the estimated channel and the observation channel. Particularly, the nonlinear function produces the output
\[ s[n] = \sum_{q=1}^{Q} a_q y^q[n], \tag{10} \]
where \( Q \) is the function degree and \( a_q \) is assumed to be nonzero for at least one \( q > 1 \). This structure, conforming by the nonlinear function between two linear channels, is known as a Wiener-Hammerstein filter and is commonly found in the modeling of linear plants with nonlinear-output sensors, such as photodiodes, pressure gauges, thermocouples, between others [4]. The coefficients \( a_q \) are assumed to be known as they commonly describe the transduction principle of the measurement, based in a well-known physical law. For instance, a simple photodiode is assumed to follow a quadratic behavior, based in a well-known physical law. For more detail, see [4].

An useful investigation, similar to that done to obtain Eq. 10, is the frequency-domain expression of the measurement signal. In Appendix C we show that the stationary state \( e[n] = 0 \), in the case of a nonlinear observation channel, involves the frequency-domain relation
\[ \hat{O}(\Omega_i^q) \prod_{i=1}^{Q} \hat{L}(\Omega_i) = \hat{O}(\Omega^q) \prod_{i=1}^{Q} \hat{L}(\Omega_i) \quad \forall \{q|a_q \neq 0\}, \tag{16} \]
where \( \Omega^q = \sum_{i=1}^{Q} \Omega_i \). We observe that, unlike in the case of Eq. 10, a low-bandwidth observation channel does not preclude the information about high-frequency components of the estimated channel. This is because for any \( q > 1 \), there is always a combination of high frequencies \( \Omega_i \) such that \( \Omega^q \) lies within the observation channel bandwidth, leading to accessible information about the components \( \hat{L}(\Omega_i) \).

In order to find a more direct relation between \( \hat{L}[m] \) and \( L[m] \) we rewrite Eq. 16 as
\[ A(\Omega^q) \prod_{i=1}^{Q} B(\Omega_i) = 1, \tag{17} \]
where \( A(\Omega) = \hat{O}(\Omega)/\hat{O}(\Omega) \) and \( B(\Omega) = \hat{L}(\Omega)/\hat{L}(\Omega) \). By derivation of Eq. 17 with respect to \( \Omega_j \), with \( j \in \{1,2,..,q\} \), and dividing the result by the left member of Eq. 17 we obtain
\[ \frac{B'(\Omega_j)}{B(\Omega_j)} = -\frac{A'(\Omega^q_j)}{A(\Omega^q_j)} = c, \tag{18} \]
where \( c \) is an arbitrary complex constant. The solution of this differential equation leads to
\[ \begin{cases} B(\Omega) = e^{\Omega+cB} \\ A(\Omega) = e^{-c\Omega+cA} \end{cases}, \tag{20} \]
where \( c_B \) and \( c_A \) are also arbitrary complex numbers. By replacing the definitions of \( A(\Omega) \) and \( B(\Omega) \) in Eq. 20 we obtain
\[ \begin{cases} \hat{L}(\Omega) = \hat{L}(\Omega)e^{\Omega+cB} \\ \hat{O}(\Omega) = \hat{O}(\Omega)e^{-c\Omega+cA} \end{cases}, \tag{21} \]
Taking into account that \( \hat{L}(\Omega) \) and \( \hat{O}(\Omega) \) are Fourier transforms of real sequences, they must satisfy \( \hat{L}(\Omega) = \hat{L}^*(-\Omega) \) and \( \hat{O}(\Omega) = \hat{O}^*(-\Omega) \). Consequently, \( c \) is shown to be an imaginary number, i.e. \( c = it \), and \( c_A \) and \( c_B \) are shown to be real numbers. Thus we have
\[ \begin{cases} \hat{L}(\Omega) = \alpha_B \hat{L}(\Omega)e^{it\Omega} \\ \hat{O}(\Omega) = \alpha_A \hat{O}(\Omega)e^{-it\Omega}, \end{cases} \tag{22} \]
with $\alpha_B = e^{cB}$ and $\alpha_A = e^{cA}$. On the other hand, by replacing Eq. (22) in Eq. (16) we find that
\[ \alpha_A (\alpha_B)^q = 1 \quad \forall \quad \{q|a_q \neq 0\}. \] (23)
Thus, if the nonlinear function has at least two coefficients satisfying $a_q \neq 0$, one even and other odd, then necessarily $\alpha_A = \alpha_B = 1$. Otherwise, if the nonlinear function poses only one nonzero coefficient, $\alpha_A$ and $\alpha_B$ remain unknown values satisfying Eq. (23) In what follows we assume, for simplicity, that $\alpha_A = \alpha_B = 1$.

By applying the inverse Fourier transform of Eq. (22) we obtain
\[ \hat{L}[m] = L(t + \tau T_s) \big|_{t=mT_s}, \quad \hat{O}[m] = O(t + \tau T_s) \big|_{t=mT_s}, \] (24)
where $1/T_s$ is the sampling frequency, while $L(t)$ and $O(t)$ are continuous-time versions of $L[m]$ and $O[m]$, respectively, defined as
\[ L(t) = \sum_n L[n] \text{sinc} \left( \pi \frac{t - nT_s}{T_s} \right), \quad O(t) = \sum_n O[n] \text{sinc} \left( \pi \frac{t - nT_s}{T_s} \right), \] (25)
with $\text{sinc}(x) = \frac{\sin(x)}{x}$. In other words, the estimation of the channel is a temporal-shifted version of the oversampled response of the actual channel response. It must be noted that this conclusion is not affected by the bandwidth of the observation channel, suggesting that this estimation can be performed even for an arbitrarily ultra-low bandwidth $\hat{O}(\Omega)$. In addition, if both channels are assumed to be causal ($L[m < 0] = 0$ and $O[m < 0] = 0$) and not delayed ($L[0] \neq 0$ and $O[0] \neq 0$), the only possibility to satisfy Eq. (24) is $\tau = 0$, being the estimation completely exact.

IV. NUMERICAL RESULTS

In order to show an example of application of the proposed algorithm, we perform numerical simulations of the system shown in Fig. 3 for a particular case. The estimated channel is given by the arbitrary response
\[ L[m] = \frac{1}{N_L} \left( \frac{1}{4} + \sin(rm) + \sin(3rm/2) \right) e^{-25m/N}, \] (26)with $1 \leq m \leq N$, $r = 0.4$ and $N = 256$. $N_L$ is a normalization factor making $\sum_m L^2[m] = 1$. The observation channel is given by
\[ O[m] = \frac{1}{N_O} \left( \text{sinc}(Bm) + \frac{1}{2} e^{-2Bm} \right), \] (27)
where $B$ is associated to the observation channel bandwidth and takes three different values, $B = \{1/8, 1/32, 1/128\}$, and $N_O$ is the normalization factor ensuring $\sum_m O^2[m] = 1$. The nonlinear function is described by the coefficients $a_2 = 1$ and $a_3 = -2$; the rest of the coefficients are assumed to be zero. The input sequence is obtained as
\[ x[n] = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} x_0[n - m] \text{sinc}(m/3), \] (28)
where $x_0[m]$ is an stochastic variable with normal distribution $\mathcal{N}(0, 1)$. This definition ensures the oversampling of the input sequence (Eq. (45)), i.e. $\hat{x}(\Omega) = 0 \quad \forall \quad |\Omega| > \pi/Q = \pi/3$. The estimation channels are initialized as
\[ \hat{L}[m] = \hat{O}[m] = \delta_{m,0}, \] (29)
where $\delta_{m,n}$ is the Kronecker delta. The adaptation step of the gradient descent algorithm is set to $\beta = 0.1$.

Figure 4 shows the evolution of the mean error square for the three cases simulated, corresponding to different values of channel observation bandwidth. On the other hand, Fig. 5 displays the obtained estimations in comparison with the actual channel frequency-responses. It must be noted that the system identification is accurate even for the extreme case $B = 1/128$, for which the observation channel bandwidth is ultra-low with respect to the estimated channel bandwidth. This is a clear sign that the convergence of the algorithm does not depend on the $O[m]$ bandwidth.

The nonlinear function of the previous example has one odd coefficient and another even, allowing for an exact estimation of the system. However, as the input-sequence power is reduced, the relevance of the third-order term is also diminished, having an effective quadratic feedback with only one coefficient, $a_2$. This situation is illustrated in Fig. 5 where we show a vector sample of the output sequence $\hat{y}[n]$ compared with the nonlinear function. We observe that the signal excursion

![Diagram of the proposed LMS algorithm](image-url)
drawback can be circumvent if an additional constrain is added
do not fit with that of the actual channels. However, this
where the feedback channel is implemented with a common
the frequency responses is correctly estimated, but their scale
is enough to provide a cubic behavior of the relation between
s[n] and y[n]. On the other hand, when the output sequence
is obtained with a lower-power input sequence, the nonlinear
block works as a single-term quadratic function. Consequently,
the estimation of the channels is given by Eq. 22, where
\( \alpha_A \) and \( \alpha_B \) are two unknown scaling factors satisfying \( \alpha_A \alpha_B = 1 \).
We repeat the simulations for the case \( B = 1/8 \) but reducing
the input-sequence power. As shown in Fig. 7 the shape of
the frequency responses is correctly estimated, but their scale
do not fit with that of the actual channels. However, this
drawback can be circumvent if an additional constrain is added
to the algorithm. For instance, in coherent optical devices
where the feedback channel is implemented with a common
photodiode, the nonlinear function is naturally modeled with
a simple quadratic term, leading to this scaling problem. By
the simple measurement of the mean optical power of the
output (\( E\{y^2[n]\} \)), the estimation \( \hat{L}[m] \) can be re-scaled to
obtain \( E\{\hat{y}^2[n]\} = E\{y^2[n]\} \), thus obtaining the exact factor
correction.

V. CONCLUSIONS

We shown that the system identification of a linear channel
with an unknown-response observation channel is possible as
long as a nonlinear memoryless function, uncoupling both
systems, is introduced. In particular, the estimation of the
channel is a scaled and temporal-shifted version of the actual
channel, being the scaling and shifting factors two unknown
coefficients. However, if the nonlinear function, expressed
as a polynomial sum, has at least one term of even order and
other of odd order, the scaling factor is necessarily the
unity. In addition, if the channels are assumed to be causal
and not delayed, the shifting factor is zero, allowing for a
completely faithful estimation of both systems. Moreover,
unlike in the standard scheme of a chain of linear systems,
the estimation does not depend on the measurement system
bandwidth. Actually, by means of a few illustrating numerical
examples, the proposed LMS algorithm was shown to be
effective even for ultra-low bandwidth observation channels,
in comparison with the bandwidth of the channel of interest.
These results suggest that this scheme can be implemented
with low-cost measurement device. A clear example can be
found in the coherent optical devices, where the observation
channel can be reduced to a low-cost standard photodiode.

APPENDIX A

STATIONARY STATE OF THE LMS ALGORITHM, INCLUDING
THE ESTIMATION OF THE OBSERVATION CHANNEL

We first derive a frequency-domain expression for the measure-
ment sequence \( z[n] \). By replacing Eq. 1 into Eq. 8 we obtain
\[
\hat{z}[n] = \sum_{m,r} L[m] \hat{O}[r] x[n - m - r].
\]
Using the inverse Fourier transform we find that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{z}(\Omega) e^{j\Omega n} d\Omega = \frac{1}{(2\pi)^3} \sum_{r} \sum_{m} \int_{-\pi}^{\pi} \hat{L}(\Omega_1) \hat{\hat{O}}(\Omega_2) \hat{x}(\Omega) \times
\]
\[
e^{j\Omega_1 m + j\Omega_2 r + j\Omega (n - m - r)} d\Omega_1 d\Omega_2 d\Omega. \tag{31}
\]
We sum the variables \( m \) and \( r \), taking into account the relation
\[
\sum_{m} e^{j\Omega m} = 2\pi \sum_{u} \delta(\Omega + 2\pi u), \tag{32}
\]
where \( \delta(\cdot) \) is the Dirac delta function, to obtain
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{z}(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\Omega) \hat{\hat{O}}(\Omega) \hat{x}(\Omega) e^{j\Omega n} d\Omega. \tag{33}
\]
Thus we find that
\[
\hat{z}(\Omega) = \hat{L}(\Omega) \hat{\hat{O}}(\Omega) \hat{x}(\Omega). \tag{34}
\]
A similar procedure can be performed to obtain the Fourier transform
of the estimated measure signal \( \hat{z}[n] \), that is
\[
\hat{z}(\Omega) = \hat{L}(\Omega) \hat{\hat{O}}(\Omega) \hat{x}(\Omega). \tag{35}
\]
In the stationary state of the LMS algorithm the error signal is zero,
\( e[n] \equiv \hat{z}[n] - z[n] = 0 \). As a consequence, Eq. 34 must be equal to
Eq. 55 proving the relation
\[
\hat{L}(\Omega) \hat{\hat{O}}(\Omega) = \hat{L}(\Omega) \hat{\hat{O}}(\Omega). \tag{36}
\]

APPENDIX B

DERIVATIVES OF THE GRADIENT DESCENT ALGORITHM

We calculate the derivatives of the mean error square, \( \mathcal{E}_m = \{e^2[n]\} \), where \( e[n] = \hat{z}[n] - z[n] \). On one hand, its derivative with
respect to the estimation of the observation channel coefficients is
given by
\[
\frac{\partial \mathcal{E}_m}{\partial \hat{O}[m]} = 2 \left\{ e[n] \frac{\partial e[n]}{\partial \hat{O}[m]} \right\} = 2 \left\{ e[n] \frac{\partial \hat{z}[n]}{\partial \hat{O}[m]} \right\} =
\]
\[
2 \left\{ e[n] \frac{\partial}{\partial \hat{O}[m]} \sum_r \hat{\hat{O}}[r] s[n - r] \right\} = 2 \left\{ e[n] s[n - m] \right\}. \tag{37}
\]
On the other hand, the derivative with respect to the coefficients of the estimation of the channel is given by

\[
\frac{\partial E_n}{\partial L[m]} = 2 \left\{ e[n] \frac{\partial e[n]}{\partial L[m]} \right\} = 2 \left\{ e[n] \frac{\partial \hat{y}[n]}{\partial L[m]} \right\} = 2 \left\{ e[n] \sum_r \hat{O}[r] \frac{\partial \hat{y}[n-r]}{\partial L[m]} \right\} = 2 \left\{ e[n] \sum_r \hat{O}[r] \sum_{q=1}^Q a_q \hat{y}^{q-1}[n-r] \frac{\partial \hat{y}[n-r]}{\partial L[m]} \right\} = 2 \left\{ e[n] \sum_r \hat{O}[r] \sum_{q=1}^Q a_q \hat{y}^{q-1}[n-r] \frac{\partial \hat{y}[n-r]}{\partial L[m]} \right\}
\]

By using the inverse Fourier transform of \( L[m] \) and \( x[m] \) we find

\[
z[n] = \sum_m O[m] s[n-m] = \sum_m O[m] \sum_{q=1}^Q a_q y^q[n-m] = \sum_m O[m] \sum_{q=1}^Q a_q \left( \sum_r L[r] x[n-m-r] \right)^q.
\]

By summing over \( r \), and taking into account Eq. (38) we obtain

\[
z[n] = \sum_m O[m] \sum_{q=1}^Q a_q \left( \int_{-\pi}^{\pi} \hat{L}^q(\Omega) d\Omega \right)^q
\]

**APPENDIX C**

**STATIONARY STATE OF THE LMS ALGORITHM WITH A NONLINEAR OBSERVATION CHANNEL**

We start from the derivation of a complete expression of the measurement sequence \( z[n] \). By following the scheme of Fig. 5 and the definition of each block we obtain

\[
z[n] = \sum_m O[m] \sum_{q=1}^Q a_q \left( \int_{-\pi}^{\pi} \hat{L}^q(\Omega) d\Omega \right)^q
\]
We assume the input signal to be oversampled in a factor $Q$. By using the inverse Fourier transform of $O[m]$ we find that

$$z[n] = \sum_{q=1}^{Q} a_q \sum_{m} O[m] \frac{1}{(2\pi)^q} \int \int_{-\pi}^{\pi} \int \int_{-\pi}^{\pi} \hat{O}(\Omega) e^{i\Omega m} \times$$

$$\left[ \prod_{i=1}^{Q} \tilde{L}(\Omega_i) \tilde{z}(\Omega_i) e^{i\Omega_i n} \right] d\Omega_1 \ldots d\Omega_q.$$  

(43)

By using the inverse Fourier transform of $O[m]$ we find that

$$z[n] = \sum_{q=1}^{Q} a_q \sum_{m} O[m] \frac{1}{(2\pi)^q} \int \int_{-\pi}^{\pi} \int \int_{-\pi}^{\pi} \hat{O}(\Omega) e^{i\Omega m} \times$$

$$e^{-i\beta_2 m} \left[ \prod_{i=1}^{Q} \tilde{L}(\Omega_i) \tilde{z}(\Omega_i) e^{i\Omega_i n} \right] d\Omega_1 \ldots d\Omega_q.$$  

(44)

We assume the input signal to be oversampled in a factor $Q$, that is

$$\hat{z}(\Omega) = 0 \quad \forall \quad |\Omega| \geq \pi/Q.$$  

(45)

Consequently, by summing Eq.(44) over $m$ we obtain

$$z[n] = \sum_{q=1}^{Q} a_q \sum_{m} O[m] \frac{1}{(2\pi)^q} \int \int_{-\pi}^{\pi} \int \int_{-\pi}^{\pi} \hat{O}(\Omega) e^{i\Omega m} \times$$

$$\left[ \prod_{i=1}^{Q} \tilde{L}(\Omega_i) \tilde{z}(\Omega_i) \right] d\Omega_1 \ldots d\Omega_q.$$  

(46)

Analogously, we can prove that the estimated measure sequence, $\hat{z}[n]$ can be expressed as

$$\hat{z}[n] = \sum_{q=1}^{Q} a_q \sum_{m} O[m] \frac{1}{(2\pi)^q} \int \int_{-\pi}^{\pi} \hat{O}(\Omega) e^{i\Omega m} \times$$

$$\left[ \prod_{i=1}^{Q} \tilde{L}(\Omega_i) \tilde{z}(\Omega_i) \right] d\Omega_1 \ldots d\Omega_q.$$  

(47)

At the stationary state of the LMS algorithm, the measure sequence is assumed to be equal to its estimation, i.e. $z[n] = \hat{z}[n]$. By replacing Eqs. (46) and (47) in this equality we find that

$$\hat{O}(\Omega^*_q) \prod_{i=1}^{Q} \tilde{L}(\Omega_i) = \tilde{O}(\Omega^*_q) \prod_{i=1}^{Q} \tilde{L}(\Omega_i) \quad \forall \quad \{ q | a_q \neq 0 \}.$$  

(48)

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Fig. 7. Estimation of the channel of interest (top) and the observation channel (bottom) with $B = 1/8$ with a lower-power input sequence, leading to a single quadratic term effective nonlinear function. The estimation differs of the actual channel by two unknown scaling factors. In coherent optical devices, this drawback can be compensated by the simple measuring of the mean output power.