The Solutions of the equations of pulse propagation in optical fibers

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We have found a solution of the optical pulses propagation equation in optical fibers in quadratures. We have derived an expanded equation of optical pulses propagation in silica optical fibers and found its localized solution. The derived solution of the extended equation of optical pulses propagation in optical fibers for arbitrary function of the nonlinear medium response to an external harmonic disturbance has been found in quadratures.

1. Introduction.

The optical pulse field which is distributed in a single-mode optical fiber supporting the linear polarization condition has the form [1]:

\[ E(r, t) = e_z F(x, y) A(z, t) \exp\left\{i \left(\beta_z z - \omega_z t\right)\right\}. \]  

For this function we have derived a propagation equation [1], [2]:

\[ \frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = i(\Delta\beta)A, \]  

where \( \beta_1 = 1/v_g \) - is the value of the reverse group velocity. \( n \) the field of waveguide transparency \( \Delta\beta \) is the real function of \( |A|^2 \).

The equation (2) holds true for pulses with duration \( > 0.1nC \) which corresponds to quasi-monochromatic approximation approximation. This equation belongs to Ginsburg – Landau class of equations. In this abridged form (without the second spatial derivative) it can be reduced to a nonlinear Schrödinger equation, in which the coordinate and time switch places. Most of the known publications about optics use a standard approach to the solution of the nonlinear Schrödinger equation. The aim of this work is to apply a non-standard approach to the equation solution (2).

2. Method for solving equation of optical fibers pulses propagation in quadratures.

The solution will be conducted in the laboratory frame of reference without specifying the nonlinear response functions \( \Delta\beta \). Substitution into (2)

\[ A(z, t) = \sqrt{T} \exp\left\{i \frac{qz}{2}\right\}, \]  

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 were $q$ - is a small correction to the central wave number $\beta_o$ in (2) after separating the real and imaginary parts will result in the following two equations for envelope optical intensity $I(z, t) = |A(z, t)|^2$:

\[
\frac{\partial I}{\partial z} + \beta_1 \frac{\partial I}{\partial t} = 0
\]  

(4)

\[
\beta_2 \left[ 2I \frac{\partial^2 I}{\partial t^2} - \left( \frac{\partial I}{\partial t} \right)^2 \right] = -8qI^2 + 8I^2 \Delta \beta(I).
\]  

(5)

The general solution of the linear homogeneous equation (4) is an arbitrary differentiable function $I = I(s)$, where $s(z, t) = z - z_o - v_g t$. Going over in (5) to a form of the differential equation, we get:

\[
2I \frac{d^2 I}{ds^2} - \left( \frac{dI}{ds} \right)^2 = \frac{8I^2}{\beta_2 v_g^2} [\Delta \beta(I) - q].
\]  

(6)

A quadratic equation (6) is an equation of Euler - Lagrange for mechanical analogue-particle. We have found the solution in quadratures:

\[
\int \frac{dl}{I \sqrt{(G(I)/I-q)/\beta_2}} = \frac{2\sqrt{2}}{v_g} s.
\]  

(7)

Thus, the solution of the problem of optical pulses propagation in single-mode optical fibers described by the differential equation in partial derivatives (2) is reduced to the calculation of the primitive in the left-hand side of equation (7) and to the treatment of the obtained expression that is certainly an easier task.

3. The solution of the extended equation of optical fibers pulses propagation

Let $\gamma$ - be the nonlinearity coefficient, then, from (2):

\[
\frac{\partial A}{\partial z} + \frac{1}{v_g} \frac{\partial A}{\partial t} + i\beta_2 \frac{\partial^2 A}{\partial t^2} = i\gamma |A|^2 A.
\]  

(8)

All the above parameters are known for single mode quartz fibers and their numerical values are given in [1], [2], [6]. But the solution (7) is flawed because the distribution of the pulse velocity has the form of a Delta-function, and because this rate fixed for all impulses does not depend on the pulse peak intensity.

We have derived an extended equation of pulses propagation and we propose its solution to solve this problem. We have considered an equation for the field strength in a spectral representation [1].

\[
\nabla^2 \tilde{E}(\tilde{r}, \omega - \omega_o) + \varepsilon(\omega)k_o^2 \tilde{E}(\tilde{r}, \omega - \omega_o) = 0.
\]  

(9)

where \(\tilde{E}(\tilde{r}, \omega - \omega_o) = \int E(\tilde{r}, t) \exp\{i(\omega - \omega_o)\} dt\), \(\varepsilon(\omega) = (n + \Delta n)^2 \equiv n^2 + 2n\Delta n\) - is dielectric permeability.

The equation (4) is solved by the standard method of separation of variables, by substituting equation (9) into the solution. Let $\overline{\beta^2}$ - be a separation constant, then we obtain two equations:

\[
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \left[ \varepsilon(\omega)k_o^2 - \overline{\beta^2} \right] F = 0,
\]  

(10)

\[
\frac{\partial^2 \tilde{A}}{\partial z^2} + 2i\beta_o \frac{\partial \tilde{A}}{\partial z} + \left( \overline{\beta^2} - \beta_o^2 \right) \tilde{A} = 0.
\]  

(11)
(10) determines the distribution of the mode field $F(x, y)$ and the correction $\Delta \beta$ to the propagation constant in linear approximation $\beta(\omega) = \beta(\omega) + \Delta \beta$.

While considering (11) we will not follow instructions [1], [2], [6] to neglect the second derivative in the equation (11) because of assumptions of the slow function changeability $\delta \omega \delta A$, so we get the complete solution from the equation (11)

$$i \left( \frac{\partial A}{\partial z} + \beta \frac{\partial A}{\partial t} \right) + \frac{1}{2 \beta_o} \frac{\partial^2 A}{\partial z^2} - \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \gamma |A|^2 A = 0.$$  \hspace{1cm} \text{(12)}

It is interesting to find solutions to this equation and study various limiting cases. We have derived the solution of this equation:

$$A(z, t) = \frac{E_o \exp \left\{ i y E_o^2 / 2 \right\}}{\cosh \left( z - z_o - vt \right) \left[ \frac{\gamma E_o^2 \beta_o}{1 + |\beta_2| \beta_o v^2} \right]}.$$  \hspace{1cm} \text{(13)}

This solution represents a wave packet moving with a constant speed which depends on the tension peak values and it has a central wave number. This radically distinguishes it from formula (3). Within the limit formula (13) goes over into (3), i.e. it is more general.

4. The solution of the extended equation of optical fibers pulses propagation at arbitrary nonlinearity in quadratures

Now let us consider the solution of equation (12) in quadratures for an arbitrary function of the nonlinear response. The function $\Delta \beta(|A|^2)$ describing the nonlinear response of the medium to external harmonic perturbation, belongs to one of three classes: competing, saturating and transitional nonlinearity [7]. Let in (12):

$$A(z, t) = R(z, t) \exp \left\{ \pm iqz \right\}$$  \hspace{1cm} \text{(14)}

were $R$ - the real function, $q$ - arbitrary nonnegative parameter, which is associated with field peak strength.

Thus, the envelope is a running wave of unchanged profile that moves with constant speed $\nu = \nu_g \left( 1 \pm q / \beta_o \right)$. Profile of this wave is determined by the equation

$$\left( \frac{1}{2 \beta_o} - \frac{\beta_2 v_g^2}{2} \right) \frac{d^2 R}{ds^2} = \left[ \pm q + \frac{q^2}{2 \beta_o} - \Delta \beta(R^2) \right] R,$$  \hspace{1cm} \text{(15)}

which is soluble in quadratures:

$$\int \frac{dR}{\sqrt{\left( \pm q + q^2 / 2 \beta_o \right) R^2 - B(R^2) + C_1}} = z - z_o - vt,$$  \hspace{1cm} \text{(16)}

where $(-z_o)$ is an arbitrary constant of integration. Constant $C_1$ is determined by the behavior of the envelope at infinity. We should notice that parameter $q$ is a correction to the wave number $\beta_o$, which should not exceed half of the value, so the sign of the expression inside the parentheses under the radical is determined by the sign before $q$.

5. Conclusions

The solution of the problem of optical pulses propagation in single-mode optical fibers that support the status of linear polarization is reduced to the calculation of the primitive in the left-hand side of (16). It is certainly technically easier than the solution of equations in partial derivatives. As graphical
analysis does not require treatment of expression which is obtained from (16), the envelope graph can be constructed in the case when the left part of the formula (16) is not integrable in elementary functions.

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