Integral points on the rational curve

\[ y = \frac{x^2 + bx + c}{x + a}; \ a, b, c \text{ integers.} \]

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1. Introduction

In this work, we investigate the subject of integral points on rational curves of the form
\[ y = \frac{x^2 + bx + c}{x + a}; \]
where \( a, b, c \) are fixed or given integers. An integral point is a point \((x,y)\) with both coordinates \( x \) and \( y \) being integers. Not that if \((i_1, i_2)\) is an integral point with \( i_1 \neq -a\); on the graph of the above rational function. Then \((i_1, i_2)\) also satisfies the algebraic equation, of degree 2;
\[ x^2 - xy + bx - ay + c = 0. \]
And conversely, if \((i_1, i_2)\) is an integer pair with \( i_1 \neq -a\); satisfying this algebraic equation. It then lies on the graph of the rational function \( y = \frac{x^2 + bx + c}{x + a}. \) \( \tag{1} \)

Since \((-1)^2 - 4(0) = 1 > 0\) (coefficient of the \( x^2 \) term is 1; coefficient of the \( xy \) term is -1; and coefficient of the term \( y^2 \) is 0); the above degree 2 algebraic equation describes a hyperbola on the \( x-y \) plane. In L.E. Diskson’s book *History of the Theory of Numbers, Vol. II* (see reference [1]); there is a wealth of historic information on Diophantine equations of degree 2; in two or three variables. In reference [2], the reader will find an article published in 2009; and dealing with some special cases of integral points on hyperbolas.

In Section 2, we prove a key proposition, Proposition 1. Proposition 1 lays out the precise (i.e. necessary and sufficient conditions, for a quadratic trinomial with integer coefficients to have two integer roots or zeros. This proposition plays a key role in establishing Theorem 1 in Section 5; the main theorem of this paper. According to Theorem 1, if the integer \( a^2 - ab + c \) is not zero (which is to say that the integer \(-a\) is not a zero of the quadratic trinomial \( x^2 + bx + c \)). Then the graph of the rational function in (1) contains exactly \( 4N \) distinct integral points; provided that \( a^2 - ab + c \) is not equal to an integer or perfect square; or minus an integer square. If on the other hand \( a^2 - ab + c = k^2 \); or \( -k^2 \); where \( k \) is a positive integer. Then the graph of \( y=f(x) \) contains exactly \( 4N-2 \) distinct integral points. In both cases above, \( N \) stands for the number of positive integer divisors of \( |a^2 - ab + c| \); divisors not exceeding \( \sqrt{|a^2 - ab + c|} \).

In Section 6, we apply Theorem 1, in the case were \( |a^2 - ab + c| = 1, p, p^2 \), or \( p_1p_2 \). Where \( p \) is a prime; and \( p_1, p_2 \) are distinct primes.
In Section 3, we present an analysis of the graph of \( y=f(x) \) in the special case \( b^2-4c=0 \). In this case the rational function \( f(x) \) reduces to \( f(x) = \frac{(x+d)^2}{x+a} \); where \( d \) is an integer such that \( b=2d \) and \( c=d \).

We analyze (in Section 3) this case from the Calculus point of view. As it becomes evident, there are really only two types of graph of \( y=f(x) \); in the case \( b^2-4c = 0 \).

See Figures 1, 2, 3, 4, and 5

2. A Proposition and its Proof

The following proposition, Proposition 1, is the key result used in establishing Theorem 1 in Section 5. This proposition and its proof can also be found in reference [3] (article authored by this author). But, for reasons of completeness and convenience for the readers of this paper; we include it here. We will make use of a well-known lemma in number theory, (see [4]).

Lemma 1 (Euclid’s Lemma)

Let \( n_1, n_2, n_3 \) be nonzero integers such that \( n_1 \) is a divisor of the product \( n_2, n_3 \). Then if \( n_1 \) is relatively prime to \( n_2 \); \( n_1 \) must divide \( n_3 \).

Proposition 1

Let \( a, b, c \) be integers; with the integer \( a \) being nonzero.

Consider the quadratic trinomial \( g(x) = ax^2 + bx + c \).

Then,

(i) The trinomial \( g(x) \) has either two rational zeros (or roots) or (otherwise) two irrational zeros.

(ii) The trinomial \( g(x) \) has two rational zeros (or roots) if, and only if, the discriminant \( b^2 - 4ac \) is an integer square.

(iii) The trinomial \( g(x) \) has two integer zeros if, and only if, the following conditions are satisfied:

\[
\begin{align*}
b^2 - 4ac &= k^2, \text{ for some integer } k. \\
\text{And, the integer } a \text{ is a divisor of both integers } c \text{ and } b \\
\left(\text{Equivalently, } a \text{ is a divisor of the greatest common divisor of } b \text{ and } c\right)
\end{align*}
\]
(iv) If $a = 1$, then $g(x)$ has two integer zeros if and only if $b^2 - 4c$ is an integer square.

If $a = -1$, then $g(x)$ has two integer zeros if and only if $b^2 + 4c$ is an integer square.

Proof

(i) If $r_1$ and $r_2$ are the zeros of $g(x)$. Then, $r_1 + r_2 = -\frac{b}{a}$; and also $r_1r_2 = \frac{c}{a}$.

Since $-\frac{b}{a}$ is a rational number; it is clear from the last equation that if one of $r_1, r_2$ is rational; so is the other.

(ii) First, suppose that $b^2 - 4ac$ is the square of the square of an integer; $b^2 - 4ac = D^2$, where $D$ is a nonnegative integer.

The two zeros or roots of the trinomial $g(x)$; are the real numbers $-\frac{b + \sqrt{D^2}}{2a}$ and $-\frac{b - \sqrt{D^2}}{2a}$; that is, the numbers $-\frac{b + \sqrt{D}}{2a}$ and $-\frac{b - \sqrt{D}}{2a}$; which are both rational since $b$, $D$, and $a$ are all integers. Now the converse. Let $r_1$ and $r_2$ be the rational zeros of $g(x)$; and $T$ the discriminant.

We have, $r_1 = -\frac{b + \sqrt{T}}{2a}, r_2 = -\frac{b - \sqrt{T}}{2a}$; and $T = b^2 - 4ac$ \(\text{(2)}\)

We write the rational number $r_1$ in lowest terms:

\[
\begin{aligned}
\{ r_1 &= \frac{u}{v}, \text{where } u \text{ and } v \text{ are relatively prime integers and } v \text{ nonzero} \\
\} \text{ (3)}
\end{aligned}
\]

From (2) and (3) we obtain,

\[
(2au - bv)^2 = v^2 \cdot T \text{ (4)}
\]

If $T = 0$, then $T = b^2$ and we are done: $T$ is an integer square.

If $T$ is nonzero, then since $v$ is also nonzero; and so by (4), so is the integer $2au + bv$. According to (4), the nonzero integer square $v^2$ divides the nonzero integer square $(2au + bv)^2$. This implies that positive integer $|v|$ must divide the positive integer $|2au + bv|$. (This last inference can typically be found as an
exercise in elementary number theory books. It can also be found in reference [2]). Therefore the integer is a divisor of the integer $2au + bv$. And so,

$$\begin{align*}
2au + bv &= t \cdot v, \\
&\text{for some integer } t \\
&\text{or equivalently,} \\
2au &= v \cdot (t - b)
\end{align*}$$

(5)

If $t = b$, then by (5) we get $2au = 0$ which implies $u = 0$ in view of $a$ being nonzero. But then $r_1 = 0$ by (3).

Since $g(x)$ has zero as one of its roots. It follows that the constant term $c$ must zero; $c = 0$, which implies $T = b^2$ by (2). Again $T$ is an integer square. If $t \neq b$ in (5) Then since all three integers $a, v$, and $t - b$ are nonzero; so must be the integer $u$ by (5). So all four integers $a, u, v, t - b$ are nonzero. By Lemma 1 (Euclid’s Lemma) since $v$ is a divisor of the product $2au$; and $v$ is a relatively prime to $u$. It follows that $v$ must be a divisor of the integer $2a$. So that,

$$\begin{align*}
2a &= v \cdot w, \text{ for some nonzero integer } w
\end{align*}$$

(6)

Combining (6) and (4) yields,

$$v^2 \cdot (w + b)^2 = v^2 \cdot T;$$

$$T = (w + b)^2,$$

which proves that $T$ is integer square.

The proof is complete. 

(iii) Suppose that the trinomial $g(x)$ has two integer roots and zeros.

Then by part (ii) it follows that,

$$\begin{align*}
b^2 - 4ac &= k^2, \\
&\text{for some nonnegative integer } k
\end{align*}$$

(7)

And the two zeros of $g(x)$ are the numbers
\[
\begin{align*}
&\left\{ \frac{-b+k}{2a} \text{ and } \frac{-b-k}{2a} = r_2 \right\} \\
&\text{where } r_1 \text{ and } r_2 \text{ are integers}
\end{align*}
\]

So, from (8) we have,

\[
\begin{align*}
2ar_i + b &= k; \\
(2ar_i + b)^2 &= k^2
\end{align*}
\]

From (9) and (7) we get,

\[
b^2 + 4abr_i + 4a^2r_i^2 = b^2 - 4ac; \text{ and since } a \neq 0,
\]

\[
br_i + a^2r_i^2 = -c
\]

From (8), we also have,

\[
2a(r_i + r_2) = -2b;
\]

\[-a(r_i + r_2) = b,
\]

Which shows that \(a\) is a divisor of \(b\). Thus,

\[
b = \rho \cdot a, \text{ for some integer } \rho
\]

By (11) and (10) we obtain,

\[
\rho \cdot a \cdot r_i + a \cdot r_i^2 = -c;
\]

\[-a \cdot r_i \cdot (\rho + r_i) = c; \text{ which proves}
\]

That \(a\) is also a divisor of \(c\).

We have shown that the integer \(a\) is a common divisor of \(b\) and \(c\).

Now the converse. Suppose that \(a\) is a common divisor of both \(b\) and \(c\); and that \(b^2 - 4ac = k^2\), for some \(k \in \mathbb{Z}\). Then, \(b = a \cdot b_i\) and \(c = a \cdot c_i\), for some integers \(b_i\) and \(c_i\).
And so,

\[ a^2 b_1^2 - 4a^2 c_i = k^2; \tag{12} \]
\[ a^2 (b_1^2 - 4c_i) = k^2 \]

According to (12); \( a^2 \) is a divisor of \( k^2 \) which implies that \( a \) is a divisor of \( k \). And so \( k = ak_1 \), for some \( k_1 \in \mathbb{Z} \).

Using \( b = ab_1, c = ac_1, k = ak_1 \); and the formulas in (8). We see that the two zeros \( r_1 \) and \( r_2 \) of \( g(x) \) are the numbers

\[ \left( r_1 = \frac{-b_1 + k_1}{2} \text{ and } r_2 = \frac{-b_1 - k_1}{2} \right) \tag{13} \]

Also from, (12), and \( k = ak_1 \); we have

\[ b_1^2 - 4c_i = k_1^2 \tag{14} \]

Equation (14) shows that \( b_1 \) and \( k_1 \) have the same parity; they are either both odd or both even.

Hence by (13), it follows that both rational numbers \( r_1 \) and \( r_2 \); are actually integers.

(iv) This part is an immediate consequence of part (iii). We omit the details.

3. The Case \( b^2 - 4c = 0 \): A Calculus Based Analysis

Then function \( f(x) = \frac{x^2 + bx + c}{x+a} \) has domain, \( D_f = \{ x \in \mathbb{R} \text{ and } x \neq -a \} = (-\infty,-a)U(-a,+-\infty). \)

Now, suppose that the integers \( b \) and \( c \) satisfy,

\[ b^2 - 4c = 0; \]
\[ b^2 = 4c \]

The integer \( b \) must be even; \( b = 2d \), for some integer \( d \).

Which yields \( c = d^2 \). We have,
\[ f(x) = \frac{x^2 + 2dx + d^2}{x + a}; \]
\[ f(x) = \frac{(x + d)^2}{x + a}. \]

**Case 1:** \( a = d = \frac{b}{2} \)

In this case the function \( f \) reduces to,
\[ f(x) = x + a \]

The graph of the \( y = f(x) \) is the graph of the straight line with equation \( y = x + a \) but with the point \((-a, 0)\) removed.

**Figure 1** \( y = f(x) = \frac{(x + a)^2}{x + a}; \) with \( a < 0 \)

**Figure 2** \( y = f(x) = \frac{x^2}{x} \)

\( a = 0 \)

**Figure 3**
\[ y = f(x) = \frac{(x+a)^2}{x+a} \quad \text{with } a < 0 \]

Case 2: \( a \neq d = \frac{b}{2} \)

\[ f(x) = \frac{(x+d)^2}{x+a} \]

(i) **Asymptotes**

The graph of \( y = f(x) \) has no horizontal asymptote. It has one vertical asymptote, the vertical line \( x = -a \). Also, \( \lim_{x \to -a} f(x) = -\infty \), while \( \lim_{x \to -a} f(x) = +\infty \).

The graph of \( y = f(x) \) also has an oblique asymptote; the slant line with equation \( y = x + 2d - a \).

If we perform long division or synthetic division of \((x+d)^2\) with \(x+a\); we obtain

\[
(x+d)^2 = (x+a) \cdot (x+2d-a) + (d-a)^2; \\
\Rightarrow f(x) = \frac{(x+d)^2}{x+a} = x + 2d - a + \frac{(d-a)^2}{x+a}; \quad \text{for } x \neq -a.
\]

\[
\lim_{x \to +\infty} [f(x) - (x+2d-a)] = 0 = \lim_{x \to -\infty} [f(x) - (x+2d-a)].
\]

So that

(ii) **Intercepts**

The point \((-d, 0)\) is the \(x\) intercept on the graph of \(y=f(x)\); while the point \(0, \left(\frac{d^2}{a}\right)\) is the \(y\) intercept.

(iii) **First Derivative, Open Intervals of Increase/Decrease, and Points of Relative Extremum.**

We use the quotient rule to calculate the derivative \(f'(x)\) of \(f(x)\):
\[ f(x) = \frac{(x + d)^2}{x + a}; \]
\[ f'(x) = \frac{2(x + d)\cdot(x + a) - (x + d)^2}{(x + a)^2} = \frac{(x + d)[2(x + a) - (x + d)]}{(x + a)^2}; \quad (15) \]
\[ f''(x) = \frac{(x + d)[x + 2a - d]}{(x + a)^2}. \]

We see from (15) that the function \( f \), has two critical numbers in its domain \((-\infty, -a) \cup (-a, +\infty)\).

These are the numbers \(-d\) and \(-(2a-d)=d-2a\). Note that these two numbers are distinct since \(a\) and \(d\) are distinct. Furthermore, when \(a<d\), then \(-d<-a<d-2a\); and so the function \(f\) is increasing on the open intervals \((-\infty,-d)\) and \((d-2a, +\infty)\); decreasing on the open intervals \((-d,-a)\) and \((-a,d-2a)\).

The point \((-d,0)\) on the graph of \(y=f(x)\); is a point of relative maximum. While the point \((d-2a, 4(d-a))\)

is a point of relative maximum. Note that \(f(d-2a) = \frac{(2d-2a)^2}{d-a} = 4(d-a)\).

On the other hand, if \(d<a\). Then \(d-2a<-a<-d\). The function \(f\) increases on the open intervals \((-\infty,d-2a)\) and \((-d, +\infty)\); and it decreases on the open intervals \((d-2a,-a)\) and \((-a,-d)\).

The point \((d-2a,4(d-a))\) on the graph of \(y = f(x)\); is a point of relative maximum; while the point \((-d,0)\) is a point of relative minimum when \(d<a\) and vice-versa when \(a>d\).

(iv) **Second derivative, open intervals of concavity, and inflection points**

As we will see below, the graph of \(y = f(x)\) has no inflection points. We compute the second derivative. First we write the numerator in (15); in expanded form. \(f'(x) = \frac{x^2 + 2ax + d(2a-d)}{(x + a)^2}\).

Applying the quotient rule gives,

\[ f''(x) = \frac{2(x+a)\cdot(x+a)^2 - 2(x+a)\cdot[x^2 + 2ax + d(2a-d)]}{(x + a)^4}; \]
\[ f''(x) = \frac{2(x+a)^2 - 2[x^2 + 2ax + d(2a-d)]}{(x + a)^3}; \]
\[ f''(x) = \frac{2[x^2 + 2ax + a^2 - x^2 - 2ax - d(2a-d)]}{(x+a)^3}; \]

\[ f''(x) = \frac{2(a-d)^2}{(x+a)^3}; \text{ and since } (a-d)^2 > 0, \]

On account of \( a \neq d \). It clear that \( f''(x) > 0 \) on the open interval \((-a, +\infty)\); while \( f''(x) < 0 \) on the open interval \((-\infty, -a)\). Thus, the graph of \( y = f(x) \) is concave downwards over the open interval \((-\infty, -a)\); and it concave upwards over the open interval \((-a, +\infty)\). There are no inflection points on the graph of \( y = f(x) \).

(v) Two graphs

Figure 4

a < d. And so, \(-2da + a < d < a < a < d < 2a\)

\[ y = f(x) = \frac{(x+d)^2}{x+a} \]
Figure 5  

\[ d < a \]. And so,  
\[ d - 2a < -a < -d < a - 2d \]
4. Integral points in the case $b^2 - 4c = 0$

As we have seen in the previous section; when $b^2 = 4c$, then $b = 2d$ and $c = d^2$; $d$ an integer.

The function $y = f(x)$ reduces to,

$$y = f(x) = \frac{(x+d)^2}{x+a}; \text{ with } x \neq -a \quad (16)$$

Case 1: $a=d$. We have $y = x+a$; and so obviously there are infinitely many integral points on the graph of $y = f(x)$. Set of integral points, $S = \{(x,y)|x = t \text{ and } y = t+a; t \in \mathbb{Z}, \text{ and } t \neq -a\}$

Case 1: The integers $a$ and $d$ are distinct

Equation (16) is equivalent to,

$$\begin{align*}
\left\{ \begin{array}{l}
y \cdot (x+a) = (x+d)^2 \\
\text{with } x \neq -a
\end{array} \right\} (17)
\end{align*}$$

Clearly, the pair $(-d,0)$ is a solution to (17); note that $-d \neq -a$, since $a \neq d$. Now, if $x \neq -d$ in (17). Then $(x+d)^2 > 0$. So either $y$ and $x+a$ are both positive integers; or they are both negative integers.

So we split the process into two cases.

Case 2a:

$$\begin{align*}
\left\{ \begin{array}{l}
y \cdot (x+a) = (x+d)^2; \\
x, y \text{ are integers such that } y \geq 1, x+a \geq 1
\end{array} \right\} (17a)
\end{align*}$$

Let $\rho$ be the greatest common divisor of $y$ and $x+a$.

Then,

$$\begin{align*}
\left\{ \begin{array}{l}
y = \rho \cdot z_1, x+a = \rho \cdot z_2; \\
\text{for some relatively prime positive integers}
\end{array} \right\} (17b)
\end{align*}$$

From (17a) and (17b) we obtain

$$\rho^2 \cdot z_1 \cdot z_2 = (x+d)^2 \quad (17c)$$
According to \((17c)\), the positive integer \(\rho^2\) divides the positive integer \((x + d)^2\); which implies that \(\rho\) must divide \(|x + d|\)
\[
\left\{\begin{array}{l}
|x + d| = \rho \cdot z_3, \\
\text{for some positive integer } z_3
\end{array}\right\} \quad (17d)
\]
Combining \((17c)\) with \((17d)\) yields,
\[
z_1 \cdot z_2 = z_3^2 \quad (17e)
\]
Since the positive integers \(z_1\) and \(z_2\) are relatively prime and their product, according to \((17e)\), is equal to a perfect square or integer square; each of them must be an integer square. Recall that more generally; if the product of two relatively prime positive integers is equal to an nth integer power; each of those positive integers must equal an nth integer power. This result can easily be found in number theory books. For examples, see reference \([4]\).

Thus by \((17e)\) we must have,
\[
\left\{\begin{array}{l}
z_1 = v_1, z_2 = v_2^2, z_3 = v_1 v_2; \\
\text{for relatively prime positive integers } v_1 \text{ and } v_2
\end{array}\right\} \quad (17f)
\]
Going back to \((17d)\) and \((17b)\) to obtain
\[
\left( y = \rho \cdot v_1, x = -a + \rho \cdot v_2^2, |x + d| = \rho v_1 v_2 \right) \quad (17g)
\]
First, suppose that \(a < d\). And so \(-a > -d\).

Since \(x > -a\). We have \(x > -d\), and so by \((17g)\) it follows that \(|x + d| = x + d = \rho v_1 v_2\). Which further implies by \((17g)\) again; that
\[
\begin{align*}
-a + \rho \cdot v_2 v^2 &= \rho v_1 v_2 - d; \\
\rho \cdot v_2 \cdot (v_1 - v_2) &= d - a; \\
v_2 \cdot (v_1 - v_2) &= \frac{d - a}{\rho}; \quad \text{with } v_1 - v_2 \geq 1 \text{ and } d - a \geq 1.
\end{align*}
\]
By setting \(m = v_2\) and \(v_1 - v_2 = n; v_1 = m + n\). We obtain \(y = \rho \cdot (m + n)^2, x = -a + \rho \cdot m^2\); with \(m\) and \(n\) being relatively prime integers. Clearly the conditions \(y \geq 1\) and \(x + a \geq 1\) in \((17a)\) are satisfied. Clearly \(\gcd(m, n) = 1\); since \(\gcd(v_1, v_2) = 1\). Next, suppose that \(d < a\); and so \(-a < -d\). There are two
possibilities to consider in this case. One possibility is \(-a < -d < x\). The other possibility is \(-a < x < -d\); which requires that \(\sqrt{-d - (-a)} = |a - d| = a - d \geq 2\). For if \(a - d = 1\), then there is no integer in the open interval \((-a, -d)\). Also, clearly \(x \neq -d\), as implied by the conditions in (17a).

So, if \(-a < -d < x\). Then \(x + d > 0; |x + d| = x + d\). So (17g) implies \(x = -a + \rho \cdot v_2^2 = -d + \rho v_1 v_2\);
\[
v_2 \cdot (v_2 - v_1) = \frac{a - d}{\rho}; \text{with } v_2 > v_2 - v_1 \geq 1.
\]
By setting \(m = v_2\) and \(v_2 - v_1 = n; v_1 = m - n\). We obtain \(y = \rho \cdot (m - n)^2, x = -a + \rho \cdot m^2\); under the conditions \(m > n \geq 1\); and with \(\gcd(m, n) = 1\). Note that the conditions \(y \geq 1\) and \(x + a \geq 1\) in (17a) are satisfied. Lastly, if \(-a < x < -d; with \ a - d \geq 2\). We have \(x + d < 0; x + d \leq -1\) (since \(x + d\) is an integer). And then (17g) gives,
\[
x = -a + \rho \cdot v_2^2 = -d - \rho v_1 v_2
\]
And so,
\[
v_2 \cdot (v_1 + v_2) = \frac{a - d}{\rho}
\]
We get, \(v_2 = n, v_1 = m - n\); with \(m\) and \(n\) being relatively positive integers with \(m > n \geq 1\). We obtain \(y = \rho \cdot (m - n)^2, x = -a + \rho \cdot n^2\).

We can now state Result 1, which summarizes the results in Case 2a (the case with \(y \geq 1\))

**Result 2**

Consider the rational function \(y = f(x) = \frac{(x + d)^2}{x + a}\), where \(a\) and \(d\) are distinct integers.

The set of all integral points \((x, y)\) on the graph of \(y = f(x)\); with \(y \geq 1\); is a finite set which can be described as follows:

(i) If \(d > a\). Then the set of all integral points on the graph of \(y = f(x)\); with \(y \geq 1\); and thus with \(x + a \geq 1\) as well; can be parametrically described in the following manner: \(y = \rho \cdot (m + n)^2\),
\[ x = -a + \rho \cdot m^2; \text{ where } m \text{ and } n \text{ are relatively prime positive integers such that } m \cdot n = \frac{d-a}{\rho}; \]

and \( \rho \) is a positive integer divisor of \( d-a \).

(ii) If \( d < a \). Then set of all integral points on the graph of \( y = f(x) \); with \( y \geq 1 \) can be parametrically described in the following manner. Those integral points with \( y \geq 1 \) and \( x > -d \) are given by, \( y = \rho \cdot (m-n)^2, x = -a + \rho \cdot m^2; \) where \( m \) and \( n \) are relatively prime integers such that \( m > n \geq 1; \) and with \( m \cdot n = \frac{a-d}{\rho}; \) \( \rho \) a positive divisor of \( a-d \).

And those integral points (such points exist only when \( a-d \geq 2 \)) with \( y \geq 1 \) and \(-a < x < -d \) can be described as follows: \( y = \rho \cdot (m-n)^2, x = -a + \rho \cdot n^2 \) Where \( \rho \) is a positive divisor of \( a-d \); and \( m, n \) are relatively prime positive integers such that, \( m \cdot n = \frac{a-d}{\rho}; \) and with \( m > n \geq 1 \).

**Remark 1**

Note that when \( d > a \), and \( m=n=1, \rho = d-a \). We obtain the integral point 

\((x, y) = (d-2a, 4(d-a)); \) which is one of the two points of extremum that the curve

\[ y = f(x) = \frac{(x+d)^2}{x+a}; \text{ as we saw in Section 3.} \]

Now, we go back to (17) and we consider the second case.

**Case 2b:**

\[
\begin{bmatrix}
y \cdot (x+a) = (x+d)^2; \\
x, y \text{ are integers such that } \\
y \leq -1, \ x+a \leq -1
\end{bmatrix} \tag{17h}
\]

If \( \rho = \gcd(y, x+a) \). Then,

\[
\begin{bmatrix}
y = -\rho \cdot z_1, \ x+a = -\rho \cdot z_2; \\
z_1, z_2 \text{ are relatively prime positive integers.}
\end{bmatrix} \tag{17i}
\]

From 17h and 17i one obtains,
\[
\begin{bmatrix}
z_1z_2 = z_3^2, \quad \rho z_3 = |x + d|; \\
z_3 \text{ a positive integer.}
\end{bmatrix}
\] (17j)

From this point on the rest of the analysis/proof/procedure for this care (Case 2b) is very similar to that of Case 2a. We omit the details and state of Result 2.

**Result 2**

Consider the rational function \( y = f(x) = \frac{(x + d)^2}{x + a} \), where \( a \) and \( d \) are distinct integers.

The set of all integral points \((x, y)\) on the graph of \( y = f(x) \); with \( y \leq -1 \); is a finite set which can be described as follows:

(i) If \( d < a \), then the set of all integral points on the graph of \( y = f(x) \); with \( y \leq -1 \), and thus with \( x + a \leq -1 \) as well; can be parametrically in the following manner:

\[
y = -\rho(m + n)^2, \quad x = -a - \rho \cdot m^2; \quad \text{where } \rho \text{ is a positive divisor of } a - d; \text{ and } m, n \text{ are relatively prime positive integers such that } m \cdot n = \frac{a - d}{\rho}.
\]

(ii) If \( a < d \), then the set of all integral points on the graph of \( y = f(x) \); with \( y \leq -1 \); can be parametrically described in the following manner.

Those integral points with \( y \leq -1 \) and \( x < -d \) are given by \( y = -\rho(m - n)^2, \quad x = -a - \rho m^2; \text{ where } m \text{ and } n \text{ are relatively prime positive integers such that } m > n \geq 1. \)

And those integral points (such points exist only when \( d - a \geq 2 \)) with \( y \leq -1 \) and \( -d < x < -a \) can be described as follows:

\[
y = -\rho(m - n)^2, \quad x = -a - \rho n^2; \quad \text{where } m \text{ and } n \text{ are relatively prime positive integers such that } 1 \leq n < m.
\]

We conclude this section by stating the obvious result below.
Result 3

Let \( f(x) = \frac{(x+d)^2}{x+a} \); where \( a \) and \( d \) are integers.

(i) If \( a \neq d \), the only integral point on the graph of \( y = f(x) \); with \( y \)-coordinate zero; is the point \((-d, 0)\).

(ii) If \( a = d \), the set \( S \) of integral points on the graph of \( y = f(x) \) is the set,

\[
S = \{(x, y) \mid x = t, \ y = a + t; \ t \neq -a, \ t \ an \ integer\}.
\]

5. The General Case and the Main Theorem

We now go back to \( y = \frac{x^2 + bx + c}{x+a} \). If \( -a \) is a zero of the trinomial \( x^2 + bx + c \); i.e. if \( a^2 + ab + c = 0 \).

Then, the other zero is the integer \(-b+a\). So in this case we have

\[
y = \frac{(x-(a-b))(x+a)}{x+a} = x-(a-b) = x+b-a.\ All \ integral \ points \ in \ this \ case \ on \ the \ above \ curve \ are
\]

the points of the form \((x, y) = (t, t+b-a); t \) can be any integer other than \(-a\). Next, assume that \( a^2 + ab + c \)
is not zero. An integral point \((i_1, i_2)\) will be on the curve precisely when,

\[
\begin{align*}
&i_1^2 + i_1 \cdot (b-i_2) + c - a \cdot i_2 = 0, \\
&\text{and } i_1 \neq -a
\end{align*}
\]

According to (18), \( i_1 \) is an integer zero of the quadratic trinomial (with integer coefficients

\( 1, b-i_2, c-a \cdot i_2 \) \( t^2 + (b-i_2) + c - a \cdot i_2 \); and so the other zero must also be integral. Thus by

Proposition 1(iii) we must have (and only then):

\[
\begin{align*}
&\left\{(b-i_2)^2 - 4 \cdot (c - a \cdot i_2) = K^2, \\
&\text{for some nonegative integer } K. \\
&\text{The zeros of (18) are } \frac{-(b-i_2) + K}{2} \text{ and } \frac{-(b-i_2) - K}{2}, \\
&\text{and } i_1 \text{ one of these two zeros.}
\end{align*}
\]

(18a)

From (18a), we have, \( i_1^2 - 2 \cdot (b-2a) \cdot i_2 - (4c + K^2) + b^2 = 0 \) (18b)
Equation (18b) shows that since the integer \( i_2 \) is one of its two zeros; so must be the other zero. Applying Proposition 1(iii) again,

\[
4(b - 2a)^2 - 4\left[ b^2 - (4c + K^2) \right] = 4M^2,
\]

for some nonnegative integer \( M \).

The zeros of (18c) are the integers \( b - 2a + M \) and \( b - 2a - M \);
\( i_2 \) is one of the two integers.

Taking the equation in (18c) further we get,

\[
b^2 - 4ab + 4a^2 - b^2 + 4c + K^2 = M^2; \quad 4(a^2 - ab + c) = M^2 - K^2 = (M - K)(M + K) \tag{18d}
\]

Recall that \( a^2 - ab + c \) is nonzero. It is clear from (18d) that the nonnegative integers \( M \) and \( K \) must have the same parity: either they are both odd; or they are both even. Moreover, since \( M \geq 0 \) and \( K \geq 0 \); we have \( M + K \geq 0 \).

Therefore (18d) implies \( 4 \cdot |a^2 - ab + c| = (M + K) \cdot |M - K| \). \( \tag{18e} \)

Since \( M, K \) have the same parity, \( M+K \) and \( M-K \) are both even integers. Also \( |a^2 - ab + c| \) is a positive integer, since \( a^2 - ab + c \) is nonzero. Thus we must have,

\[
\begin{cases}
M + K = 2d_1 \text{ and } |M - K| = 2d_2 \\
\text{where } d_1 \text{ and } d_2 \text{ are positive divisors of } |a^2 - ab + c| \\
such that } d_1d_2 = |a^2 - ab + c|; \\
\text{and with } 1 \leq d_2 \leq d_1
\end{cases} \tag{18f}
\]

Note that \( d_2 \) cannot exceed \( d_1 \), since \( M + K \geq |M - K| \), on account of the fact that both \( M \) and \( K \) are nonnegative. The equal sign holds only in the case \( K=0 \) and \( M \geq 1 \); or in the case \( M = 0 \) and \( K \geq 1 \) (clearly at most one of \( K, M \) can be zero, by (18e) and the fact that \( |a^2 - ab + c| \geq 1 \)). Also, the cases \( K = 0, M \geq 1 \); or \( K \geq 1, M = 0 \) can only occur in the case in which \( |a^2 - ab + c| \) is perfect or integer
square. When this happens, \(d_2\) can equal \(d_1\): \(1 \leq d_1 = d_2 = \sqrt{a^2 - ab + c}\); and with either \(K = 0\) and \(M \geq 1\); or alternatively \(K \geq 1\) and \(M = 0\).

Otherwise, when \(|a^2 - ab + c|\) is not a perfect square. Both \(M\) and \(K\) are positive integers; and so, \(M + K > |M - K|\); and \(1 \leq d_2 < d_1\). Looking at (18f) we see that in all cases, \(|a^2 - ab + c| = d_1 d_2 \geq d_2 \cdot d_2 = d_2^2 \geq 1\). And so, \(1 \leq d_2 \leq \sqrt{a^2 - ab + c}\).

Thus all the possible such pairs of divisors of \(d_2\) and \(d_1\); are obtained by choosing a positive divisor \(d_2\) of \(|a^2 - ab + c|\); not exceeding \(\sqrt{a^2 - ab + c}\). Then \(d_1 = \frac{a^2 - ab + c}{d_2}\).

Furthermore from (18f) we must have,
\[
\begin{align*}
M = d_1 + d_2 \text{ and } K = d_1 - d_2; \\
\text{when } 0 \leq K < M \\
\text{While, } M = d_1 - d_2 \text{ and } K = d_1 + d_2; \\
\text{when } 0 \leq M < K
\end{align*}
\]

(18g)

Combining (18g), (18c), and (18d) we see that, when actually \(a^2 - ab + c\) is positive; we must have
\[
\begin{align*}
M = d_1 + d_2, \ K = d_1 - d_2; \text{ and with} \\
\text{either } i_2 = b - 2a + M = b - 2a + (d_1 + d_2). \\
\text{or alternatively, } i_2 = b - 2a - M = b - 2a - (d_1 + d_2)
\end{align*}
\]

(18h)

And so from (18h) and (18a), we also get,
\[
\begin{align*}
either i_1 = & \frac{-[b - (b - 2a) - (d_1 + d_2)] + (d_1 - d_2)}{2} \\
or i_1 = & \frac{-[b - (b - 2a) - (d_1 + d_2)] - (d_1 - d_2)}{2} \\
or i_1 = & \frac{-[b - (b - 2a) + (d_1 + d_2)] + (d_1 - d_2)}{2} \\
or i_1 = & \frac{-[b - (b - 2a) + (d_1 + d_2)] - (d_1 - d_2)}{2}
\end{align*}
\]

(18i)
The above four possibilities in (18i) really reduce to the following four possibilities as a straightforward calculation shows:

\[
\begin{align*}
  i_1 &= -a + d_1 \text{ or } i_1 = -a + d_2 \\
  \text{or } i_1 &= -a - d_2 \text{ or } i_1 = -a - d_1
\end{align*}
\]  

(18j)

Combining (18j) with (18h), we see that when \( a^2 - ab + c \) is a positive integer; then for each pair of divisors \( d_1 \) and \( d_2 \) in (18f). Four integral points are generated. These are,

\[
\begin{align*}
  (i_1, i_2) &= (-a + d_1, b - 2a + (d_1 + d_2)), \quad (-a + d_2, b - 2a - (d_1 + d_2)), \\
  (-a - d_2, b - 2a + (d_1 + d_2)), \quad (-a - d_1, b - 2a - (d_1 + d_2)).
\end{align*}
\]  

(18k)

By inspection, these four integral points are distinct; with exception of the case where \( a^2 - ab + c \) is a perfect square; with the choice \( d_1 = d_2 = \sqrt{a^2 - ab + c} \). The four points reduce to two distinct points then.

What happens when \( a^2 - ab + c \) is a negative integer? By (18d) we must have \( 0 \leq M < K \). And so by (18g) we have \( M = d_1 - d_2 \) and \( K = d_1 + d_2 \).

Combining (18g), (18c), and (18d) one gets either \( i_2 = b - 2a + (d_1 - d_2) \) or \( i_2 = b - 2a - (d_1 - d_2) \). After that one combines this with (18a) to obtain the four integral points corresponding to the choice of divisors \( d_1 \) and \( d_2 \) satisfying (18f) (see part (iii) of Theorem 1 stated below). We omit the details.

Also, if we choose another pair of divisors \( \rho_1 \) and \( \rho_2 \) satisfying (18f); with the pair \( (\rho_1, \rho_2) \) being distinct from the pair \( (d_1, d_2) \). Then it is clear that the four points generated by the pair \( (\rho_1, \rho_2) \); will be distinct from the four points generated by the pair \( (d_1, d_2) \). This follows from (18k) (and the corresponding formulas in the case \( a^2 - ab + c < 0 \)) and (18f). We omit the details.
We now state Theorem 1.

**Theorem 1**

Consider the function \( f(x) = \frac{x^2 + bx + c}{x + a} \), with domain all reals except \(-a\); \( a, b, c \) being integers.

Then,

(i) If \( a^2 - ab + c = 0 \). There are infinitely many integral points on the graph of \( y = f(x) \). These are the points of the form, \((x, y) = (t, t + b - a), \) \( t \) an integer, \( t \neq -a \).

(ii) If \( a^2 - ab + c > 0 \). When there are exactly \( 4N \) integral points on the graph of \( y = f(x) \). Except in the case where \( a^2 - ab + c \) is a perfect square, in which case there are exactly \( 4N - 2 \) such points.

Where \( N \) (in either case) is the number of positive divisors of the integer \( a^2 - ab + c \); divisors which do not exceed \( \sqrt{a^2 - ab + c} \). All \( 4N \) integral points can be parametrically described by the formulas,

\[ (x, y) = (-a + d_1, b - 2a + (d_1 + d_2)), \]
\[ (-a + d_2, b - 2a - (d_1 + d_2)), \]
\[ (-a - d_2, b - 2a + (d_1 + d_2)), \]
\[ (-a - d_1, b - 2a - (d_1 + d_2)). \]

Where \( d_1, d_2 \) are positive divisors of \( a^2 - ab + c \); such that \( 1 \leq d_2 \leq d_1 \) and \( d_1d_2 = a^2 - ab + c \)

(iii) If \( a^2 - ab + c < 0 \), there are exactly \( 4N \) integral points on the graph of \( y = f(x) \), unless \( a^2 - ab + c \) is equal to minus a perfect square, in which case there are exactly \( 4N - 2 \) such points. Where \( N \) (in either case) is the number of positive divisors of the integer \( |a^2 - ab + c| \); divisors which do not exceed \( \sqrt{a^2 - ab + c} \). All \( 4N \) integral points can be described by the formulas,

\[ (x, y) = (-a + d_1, b - 2a + (d_1 - d_2)), \]
\[ (-a - d_2, b - 2a + (d_1 - d_2)). \]
\[(a + d_2, b - 2a - (d_1 - d_2)),\]
\[(-a - d_1, b - 2a - (d_1 - d_2))\]

Where \(d_1\) and \(d_2\) are positive integers such that \(1 \leq d_2 \leq d_1\) and \(d_1d_2 = |a^2 - ab + c|\)

6. An application of Theorem 1: Theorem 2

Theorem 2 below is a direct application of Theorem 1. Consider the cases \(|a^2 - ab + c| = 1, p,\) or \(p^2\) where \(p\) is a prime.

Since, in Theorem 1, we have \(|a^2 - ab + c| = d_1d_2,\) with \(1 \leq d_2 < d_1.\) It follows that when \(|a^2 - ab + c| = 1;\) then \(d_1 = d_2 = 1\)

When \(|a^2 - ab + c| = p;\) then \(d_2 = 1\) and \(d_1 = p.\) When \(|a^2 - ab + c| = p^2;\) then either \(d_2 = 1\) and \(d_1 = p;\)
or \(d_1 = d_2 = p.\)

We state Theorem 2 without further elaborating.

**Theorem 2**

Let \(a, b, c\) be integers and consider the function \(f(x) = \frac{x^2 + bx + c}{x + a}\) with domain \((-\infty, -a) \cup (-a, +\infty).\)

(a) If \(a^2 - ab + c = 1,\) then the graph of \(y = f(x)\) contains exactly four integral points. These are \((-a + 1, b - 2a + 2), (-a + 1, b - 2a - 2), (-a - 1, b - 2a + 2),\) and \((-a - 1, b - 2a - 2)\)

(b) If \(a^2 - ab + c = -1,\) then the graph of \(y = f(x)\) contains exactly two integral points. These are: \((-a + 1, b - 2a), (-a - 1, b - 2a)\)

(c) If \(a^2 - ab + c = -p,\) \(p\) a prime. When the graph of \(y = f(x)\) contains exactly four integral points:
\((-a + p, b - 2a + (p + 1)), (-a + 1, b - 2a - (p + 1)), (-a - 1, b - 2a + (p + 1)),
\(-a - p, b - 2a - (p + 1))\)

(d) \(\text{If } a^2 - ab + c = -p, \ \text{a prime. When the graph of } y = f(x) \text{ contains exactly four integral points: }\)
\((-a + p, b - 2a + (p - 1)), (-a - 1, b - 2a + (p - 1)), (-a + 1, b - 2a - (p - 1)),
\(-a - p, b - 2a - (p - 1))\)

(e) \(\text{If } a^2 - ab + c = p^2, \ \text{a prime. When the graph of } y = f(x) \text{ contains exactly eight integral points: }\)
\((-a + p^2, b - 2a + (p^2 + 1)), (-a + 1, b - 2a - (p^2 + 1)),
\(-a - 1, b - 2a + (p^2 + 1)), (-a - p^2, b - 2a - (p^2 + 1)),
\(-a + p, b - 2a + 2p), (-a + p, b - 2a - 2p),
\(-a - p, b - 2a + 2p), (-a - p, b - 2a - 2p)\)

(f) \(\text{If } a^2 - ab + c = -p^2, \ \text{a prime. When the graph of } y = f(x) \text{ contains exactly six integral points. These are: }\)
\((-a + p^2, b - 2a + (p^2 - 1)), (-a - 1, b - 2a + (p^2 - 1)),
\(-a + 1, b - 2a - (p^2 - 1)), (-a - p^2, b - 2a - (p^2 - 1)),
\(-a + p, b - 2a), (-a - p, b - 2a)\)
References

[1]  L.E. Dickson, History of the Theory of Numbers, Vol II, 803 pp., Chelsea Publishing Company 1992. ISBN: 0-8128-1935-6. 
For Pythagorean triangles see pp. 165-190.
For Diophantine equations of degree 2, see pp. 341-428.

[2]  Konstantine Zelator, “Integral points on hyperbolas: a special case”, arXiv:0908.3866, August 2009, 17 pages, no figures.

[3]  Konstantine Zelator, “Integer roots of quadratic and cubic polynomials with integer coefficients”, arXiv: 1110.6110, October 2011, 14 pages.

[4]  W. Sierpinski, Elementary Theory of Numbers, 480 pp. Warsaw, 1964. ISBN: 0-598-52758-3.
For Euclid’s lemma (Lemma 1) see Theorem 5 on page 14.
For Pythagorean triples, see pp. 38-43.
Also, see Theorem 8 on page 17.