Diffusion of Inhomogeneous Vortex Tangle and Decay of Superfluid Turbulence

Sergey K. Nemirovskii
Institute of Thermophysics, Lavrentyev ave., 1, 630090, Novosibirsk, Russia
Novosibirsk State University, Pirogova 2, 630090, Novosibirsk, Russia

The theory describing the evolution of inhomogeneous vortex tangle at zero temperature is developed on the bases of kinetics of merging and splitting vortex loops. Vortex loops composing the vortex tangle can move as a whole with some drift velocity depending on their structure and their length. The flux of length, energy, momentum etc. executed by the moving vortex loops takes a place. Situation here is exactly the same as in usual classical kinetic theory with the difference that the "carriers" of various physical quantities are not the point particles, but extended objects (vortex loops), which possess an infinite number of degrees of freedom with very involved dynamics. We offer to fulfill investigation basing on supposition that vortex loops have a Brownian structure with the only degree of freedom, namely, lengths of loops \( l \). This conception allows us to study dynamics of the vortex tangle on the basis of the kinetic equation for the distribution function \( n(l,t) \) of the density of a loop in the space of their lengths. Imposing the coordinate dependence on the distribution function \( n(l,r,t) \) and modifying the "kinetic" equation with regard to inhomogeneous situation, we are able to investigate various problem on the transport processes in superfluid turbulence. In this paper we derive relation for the flux of the vortex line density \( \mathcal{L}(x,t) \). The corresponding evolution of quantity \( \mathcal{L}(x,t) \) obeys the diffusion type equation as it can be expected from dimensional analysis. The according diffusion coefficient is evaluated from calculation of the (size dependent) free path of the vortex loops. We use this equation to describe the decay of the vortex tangle at very low temperature. We compare that solution with recent experiments on decay of the superfluid turbulence.

INTRODUCTION AND SCIENTIFIC

BACKGROUND

Idea that inhomogeneous vortex tangle evolves in a diffusive-like manner appeared quite long ago. Thus, the authors of paper \[1\], who observed the regions of high vortex line densities \( \mathcal{L}(r,t) \) — “plugs” in the channel with the counterflowing He II proposed that this phenomenon appeared due to diffusion of quantity \( \mathcal{L}(r,t) \). An attempt to describe theoretically these processes was made by \[1\], and \[2\], who proposed to introduce the term proportional \( \nabla^2 \mathcal{L}(r,t) \) into the Vinen equation. Authors were not able to restore the value of the diffusion coefficient from experiment (these results are reviewed and discussed in \[3\]).

In series of works (see \[4\] and references therein) both the diffusion process and the diffusion coefficient we obtained from the so called Non Equilibrium Thermodynamics principles developed by the authors earlier. In paper \[5\] the spatial diffusion of an inhomogeneous vortex tangle had been studied numerically (see Fig. 1). Analyzing their results the authors determined the diffusion constant to be of order of \( 0.1\kappa \) (\( \kappa \) is the quantum of circulation). Dynamics of the inhomogeneous vortex tangle had been studied numerically also in paper \[6\], however, since the authors studied the dilute tangle they observed the "ballistic" regime rather then pure diffusion.

Especially interest to the diffusion processes arises in context of the problem on decay of the vortex tangle at zero (extremely low) temperature. Indeed, the most apparent mechanism of dissipation in quantum fluids - the mutual friction between the vortices and the normal component -disappears. At the same time in a series of experimental works \[7\],\[8\],\[9\] the decay of quantum turbulence is observed, and the question arises what is the mechanism for dissipation at zero temperature. Various approaches and ideas such as a cascade like break down of the loops, Kelvin waves cascade, acoustic radiation, reconnection loss etc. have been discussed in details in recent review \[10\]. The mechanism of the vortex tangle spreading with the subsequent degeneration is usually ignored. In fact, contribution of diffusion had been discussed in the cited experimental work, but grounding on the value \( 0.1\kappa \) for the diffusion constant, the authors concluded that this small diffusion coefficient did not lead to correct time of decay. In the present paper we develop the theory describing the evolution of an inhomogeneous vortex tangle on the bases of kinetics of the merging and breaking down vortex loops. We showed that evolution of a weakly inhomogeneous vortex tangle obeys the diffusion equation with the coefficient equal to about \( 2.2\kappa \), which exceeds approximately twentyfold as large the value obtained in \[5\]. We present arguments that the diffusion constant would be significantly underestimated in \[5\] due to especial procedure used by the authors. We used the diffusion equation to describe the decay of the vortex tangle at very low temperature. Comparison with the recent experiments on decay of the superfluid turbulence \[7\],\[8\],\[9\] is made.
Vortex loops composing the vortex tangle can move as a whole with some drift velocity $V_l$ depending on their structure and their length. The flux of the line length, energy, momentum etc., executed by the moving vortex loops takes place. In the case of inhomogeneous vortex tangle the net flux due to gradient of concentration of the vortex line density appears. Situation here is exactly the same as in classical kinetic theory with the difference that the "carriers" are not the point particles but the extended objects (vortex loops), which possess an infinite number of degrees of freedom with very involved dynamics. In addition, while collision (or self-intersection) of elements of filaments the so called reconnection of the lines occurs, and loops either merge or split, losing their individuality and turning into other loops. The full statement of this problem requires some analog of the secondary quantization method for extended objects, or the string field theory, the problem of incredible complexity. Clearly, this problem can be hardly resolved in the nearest future. Some approach crucially reducing the number of degrees of freedom is required.

We offer to fulfill investigation basing on supposition that vortex loops have the Brownian structure (see [14]). In accordance with this approach the average loop can be imagined as consisting of many arches with the mean radius of curvature equal $\xi_0$ randomly (but smoothly) connected to each other. Quantity $\xi_0$ is important parameter of the approach. It plays a role of the "elementary step" in the theory of polymer. It is low cut-off of the approach developed, the theory does not describe scales smaller then $\xi_0$. Statistics of Brownian loops is described by the generalized Wiener distribution with the only parameter $\xi_0$ for all loops. Thus we consider the vortex tangle as a collection of vortex loops of various lengths $l$, which is only degree of freedom. Let us introduce distribution function $n(l,t)$ of the density of a loop in the "space" of their lengths. It is defined as the number of loops (per unit volume) with lengths lying between $l$ and $l+dl$. Knowing quantity $n(l,t)$ and statistics of each personal loop we are able to evaluate various properties of real vortex tangle. The distribution function $n(l,t)$ obeys the Boltzmann type "kinetic" equation. Study of exact solution to this "kinetic" equation allowed us to develop a theory of superfluid turbulence, which quantitatively describes the main features of this phenomenon [11],[12].

This approach turns out to be useful for the study of inhomogeneous vortex tangle. In this case we have to impose the coordinate dependence on the distribution function and on parameter $\xi_0$, that is to put $n(l, r, t), \xi_0(r, t)$, and to modify the "kinetic" equation with regard to inhomogeneous situation. In fact, in this work we restrict ourselves to a bit more modest problem, namely we study the question of the spatial and temporal evolution of the vortex line density $L(r, t)$. The corresponding theory can be developed in spirit of the classical kinetic theory with the difference that the transport processes are executed with the extended objects - vortex loops. Accordingly the key questions is to evaluate the drift velocity $V_l$ and the free path for the loop of size $l$. The drift velocity $V_l$ is defined via an averaged quadratic velocity of the line elements (simple average velocity vanishes due to symmetry)

$$V_l = \left\langle \frac{1}{2} \int \langle \xi(t)^2 \rangle d\xi \right\rangle. \quad (1)$$

The averaging $\langle \cdot \rangle$ should be done with use of the Gaussian model briefly described above [14]. The result of according calculations is that $V_l = \beta / \sqrt{\xi_0}$. Quantity $\beta$ is $C_v(\kappa/4\pi) \ln(1/a_0)$, where $a_0$ is the core radius and $C_v$ is a constant about unity. Velocity $V_l$ can be also estimated from the following qualitative consideration. Consequently considering the average loop as consisting of $n = l/\xi_0$ arches with the mean radius of curvature equal to $\xi_0$ randomly (but smoothly) connected to each other, we take its velocity as the resulting velocity of all arches composing the loop. Since the arches randomly connected to each other, and have the velocity as for rings, $V_{arch} = \beta / \xi_0$ (directed along the normal), the resulting averaged velocity is the "random walking" average, $V_l = \frac{1}{\sqrt{n}} V_{arch} = \beta / \sqrt{\xi_0}$.

The drift motion is realized until the loops collide with other loops with subsequent formation of larger loops. Number of collisions $P_{coll}(dt)$ per small interval $dt$ can be estimated from the "kinetic equation" for the distribution function $n(l)$ of density of loops in space of their lengths $l$ (see [11],[12]). The rate of change of density $n(l)$ due

FIG. 1: Diffusion of a vortex tangle at t=0 sec(a), t=10.0 sec(b), t=20.0 sec(c) and t=30.0 sec(d),(Tsubota et, al. [5].

**THE EVOLUTION OF INHOMOGENEOUS VORTEX TANGLE**

**Free path**
to collisions is
\[
\frac{\partial n(l,t)}{\partial t} = -2 \int \int A(l_1,l,l_2)\delta(l_2-l_1-l)n(l_1)dl_1dl_2. 
\]  
(2)

We omit the processes of the loops breakdown due to the self-intersection. The reason is that the migration of loops is performed mainly by small loops, the large ones undergo reconnections without any essential drift. The scattering cross-section \(A(l_1,l,l_2)\) describes the rate (number of events per unit volume and unit time) of collision of two loops with lengths \(l_1\) and \(l_2\) and forming the loop of length \(l_1 + l = l_2\). It is evaluated in papers [13] \(A(l_1,l,l_2) = b_m V[l]l\). Here \(b_m\) is the numerical factor approximately equal to \(b_m \approx 0.2\) and \(V_l = \beta/\xi_0\) is the velocity of approaching of vortex line elements (similar result for quantity \(b_m \sim 0.1 \div 0.2\) was obtained qualitatively in the context of cosmic strings [16]). Then probability \(P_{Col}(dt)\) for loop to collide with other loops (and reconnect) in small interval \(dt\) is:
\[
P_{Col}(dt) = \Lambda dt, 
\]  
(3)

where quantity \(\Lambda\) is evaluated with the use of relation (2)
\[
\Lambda = 2 \int \int A(l_1,l,l_2)\delta(l_2-l_1-l)n(l_1)dl_1dl_2. 
\]  
(4)

We calculate the collision probability \(\Lambda\) with the use of the distribution function \(n(l)\), obtained early \(n(l) = C l^{-5/2}\) (see [11],[12]). Here coefficient \(C\) can be ascertained from the normalization condition that the vortex line density \(\mathcal{L}\) is just \(\mathcal{L} = \int n(l)dl\) (we used while integration the fact that quantity \(\xi_0\) serves as the low cut-off ). Simple calculation leads to relation:
\[
\Lambda(l) = 2\beta b_m L \sqrt{\frac{l}{\xi_0}}. 
\]  
(5)

In the usual way we conclude that probability \(P(x)\) for the loop of length \(l\) to fly distance \(x\) without collision is
\[
P(x) = 2\mathcal{L} \exp(-2\beta b_m \mathcal{L}x). 
\]  
(6)

It can be seen from relation (6) that the free path for loop of length \(l\) is \(1/2\beta b_m \mathcal{L}\).

The flux of length and the diffusion equation

Knowing the \(l\) and \(\xi_0\)-dependent averaged velocity of loops and the probability \(P(x)\) we can evaluate the spatial flux of the vortex line density \(\mathcal{L}\) executed by the loops. The procedure is very close to the one made in the classical kinetic theory with the difference that the carriers have different sizes, this requires additional integration over the loop lengths.

Let us consider the small area element placed at some point \(x\) and orientated perpendicularly to axis \(x\) (see Fig.2). The \(x\)-component of flux of line length executed by loops of sizes \(l\) placed in \(\theta, \varphi\) direction (from the left and right sides, correspondingly) and remote from the area element at distance \(R\) can be written as
\[
J_x = \int (d\mathcal{J}_+(\theta,\varphi,l,R) - d\mathcal{J}_-(\theta,\varphi,l,R)) \frac{\sin \theta d\theta d\varphi}{4\pi} dl dR. 
\]  
(7)

Taking that spacial density of the loops is the function of coordinate \(x\) (i.e. \(n(l,x)\)) we obtain after simple integration
\[
J_x = -\frac{1}{24}\beta L b_m \left[ \frac{1}{\xi_0^2} \frac{\partial}{\partial x} \xi_0 + \frac{1}{\xi_0} \frac{\partial}{\partial x} L \right]. 
\]  
(8)

There are possible the following three variants: i). The mean radius of curvature \(\xi_0\) is the fixed quantity (this case can be realized while injecting the loops (rings) of some fixed size into volume). ii).Quantity \(\xi_0(x)\) is an independent variable, which can appear, when the own vortex line (kelvin wave propagation) dynamics prevails collisions of loop, determining thereby the structure of the vortex tangle. In this case the mean radius of curvature \(\xi_0\) should be connected to spectrum of Kelvin waves[17] iii). Mean curvature \(\xi_0(x)\) and vortex line density \(\mathcal{L}(x)\) are not independent, the relation between these two quantities appears due to the balancing of fusion and splitting of the loops and can be determined from the ”kinetic” equation [11], it is \(\xi_0 \approx 0.27\mathcal{L}^{-1/2}\). The fact that the interline space is of the order of the mean radius of curvature had been firstly discovered numerically in [18]. Using this connection we come to conclusion that \(J_x = -D_n \partial \mathcal{L}/\partial x\) and correspondingly, the spatial-temporal evolution of quantity \(\mathcal{L}\) obeys the diffusion type equation
\[
\frac{\partial \mathcal{L}}{\partial t} = D_v \nabla^2 \mathcal{L}, 
\]  
(9)

where the diffusion coefficient \(D_v\) is equal \(C_{dm}\). Our approach is fairly crude to claim for good quantitative
description. However, if to adopt the data grounded on exact solution to the Boltzmann type "kinetic" equation ([12]) we conclude that $C_d \approx 2.2$. As it was discussed above, the result that evolution of VLD obeys the diffusion type equation with the diffusion constant of order of $\kappa$ is expected. The actual interest is connected to prefactor $C_d$. Early, the only quantitative result was obtained numerically by Tsubota et al., and it was $C_d \approx 0.1$, in 22 times smaller. We will discuss the probable reason for this large discrepancy later.

**BOUNDARY CONDITIONS**

To construct the boundary conditions we have to state the problem specifically.

1. Smearing of the tangle. Let us consider, first, the case when the vortex tangle is placed in some restricted domain of superfluid helium. Let us consider also that vortex line density $\mathcal{L}$ is not too high, this allows the relatively large loop to be radiated. They move slowly, the smaller loops run down larger loop, then collide and reconnect with them. So, outside of initial domain the well developed tangle is formed. This, secondary, vortex tangle smoothly joins the initial tangle inside the domain. This implies that in this case no boundary conditions are required at all, and evolution of the vortex line density obeys equation (9) in infinite space with initial distribution $\mathcal{L}(r, 0)$ inside this domain.

2. Radiation of loops. The second situation can be realized when the radiated vortex loops run away does not influence the initial vortex tangle. It can happen, for instance if the initial tangle is very dense, so it can radiate only very small loops, which propagate rapidly. These loops run away without interaction with each other and with initial tangle where they are radiated from. Other hypothetic variant appears when there is some trap on the boundary absorbing vortex loops. Thus the vortex loops escaped from the initial domain do not influence on the original vortex tangle. In both case the boundary conditions can be found assuming that diffusive like flux of length near boundary $J = -D_v \nabla \mathcal{L}(x_b, t)$ coincides with the flux executed by vortex loops radiated through the (right) boundary $J_{\text{rad}}(x_b, t)$. The latter is evaluated by small modernization of (7)

$$J_r = \int \ln(l, x_b + R \cos \theta)(v_l \cos \theta)P(R) \frac{\sin \theta d\theta d\varphi}{4\pi} dl dR.$$  

(10)

Contribution due to remote loops (term $R \cos \theta$ in argument for $n(l, x_b + R \cos \theta)$) is evaluated as early giving obvious result $J = -(1/2)D_v \nabla \mathcal{L}(x_b, t)$. To evaluate contribution from the first one $J_{\text{rad}}$ we use again solution $n(l) = C l^{-5/2}$ and relation (6). Simple calculation lead to result that flux $J_{\text{evap}}$ of length due to evaporation (radiation) is

$$J_{\text{evap}} = C_{\text{rad}}/\mathcal{L}^2,$$

(11)

with $C_{\text{rad}} \approx 0.47$. The sum of $J_x$ and $J_{\text{evap}}$ gives the flux through the boundary, which has to be equal to $-D_v \nabla \mathcal{L}(x_b, t)$ at the boundary. Then we finally have the following boundary condition

$$C_{\text{rad}} \mathcal{L}^2 + \frac{1}{2}D_v \nabla \mathcal{L}(x_b, t) = 0.$$  

(12)

3. Solid walls. In case of solid wall which corresponds to some experiments the situation is more involved. Vortices can annihilate on the solid wall, they can undergo pinning and depinning radiating vortices back to the bulk of helium. Surely this requires a special treatment which goes beyond the scope of the work. One possible ways is to consider the solid wall as a "partial" trap, which catches the loops and re-emits a part of them back into the volume. Formally it can be written as condition (12) with additional term $J_{\text{back}}$ describing the back flux. Without detailed analysis it can be supposed that the back flux is proportional to the vortex line density on boundary $J_{\text{back}} = -C_{\text{back}} \mathcal{L}(x_b, t)$ with coefficient $C_{\text{back}}$ depending on dynamics of line on the wall (jumps between pinning sites, Kelvin waves dynamics near the wall etc.). Thus the boundary condition can be written in form.

$$C_{\text{rad}} \mathcal{L}^2 + \frac{1}{2}D_v \nabla \mathcal{L}(x_b, t) - C_{\text{back}} \mathcal{L} = 0.$$  

(13)

**DECAY OF THE VORTEX TANGLE**

Let us discuss the question how the approach developed can be applied to the problem of decay of the vortex tangle at zero temperature. As it was discussed in the Introduction, the contribution of diffusion (or radiation of loops) is usually ignored, mainly due to smallness of the diffusion constant offered in paper by Tsubota et al [5]. Let us examine this work more thoroughly. In paper [5] there was studied one-dimensional evolution (spacial spreading) of the vortex tangle concentrated initially in some domain of space and having nonuniform distribution there as it is depicted in the first picture in Fig. 3.

In the rest pictures distribution of the VLD at different moments of time is shown. To describe this evolution of the VLD it had been supposed that quantity $\mathcal{L}(x, t)$ obeys equation (9) with additional term $-\chi_2 (\kappa/2\pi)^2 \mathcal{L}$ in the right hand side. In turn this term was introduced to describe the decay of the vortex tangle in previous numerical simulation made by Tsubota et al. [15]. Because of this additional term contribution of diffusion to the whole decay would be significantly underestimated. We
FIG. 3: Development of the vortex line density distribution. The dotted line shows the results of the numerical simulation of the diffusion equation with the diffusion constant is equal to $C_d \approx 0.1 \times 10^{-3} \text{cm}^2/\text{s}$ and with an auxiliary term $-\chi^2 (\kappa/2\pi) L^2$, (Tsubota et al. [5]).

FIG. 4: Evolution of the vortex line density calculated with use of equation (9) without an auxiliary term, the diffusion constant is equal to $C_d \approx 2.2 \times 10^{-3} \text{cm}^2/\text{s}$.

would like to note that there is possible to choose another reasoning, namely, to consider that decay of the vortex tangle in both cited works occurs mainly by the diffusion process (with the diffusion coefficient calculated in the present work). We calculated spatial-temporal evolution of vortex tangle (under condition of numerical experiment by Tsubota et al.) with the use of equation (9). Results depicted in Fig. 4 enables us to conclude that the approach developed describes satisfactorily the evolution of vortex tangle without any additional supposition.

Thus, we demonstrated that evolution of the vortex line density in numerical experiment [15] can be described in terms of pure diffusion of the vortex tangle with the diffusion coefficient $C_d \approx 2.2 \times 10^{-3} \text{cm}^2/\text{s}$.

Let us now discuss two recent experiments on decay of the vortex tangle at very small temperature [8] and [9]. The authors reported attenuation of vortex line density in superfluid turbulent helium, $^3$He-B in paper [8] and $^4$He in paper [9]. They attribute the decay of the vortex tangle to the classical turbulence mechanisms. Without discussion of this variant we would like just to estimate the contribution into attenuation of the vortex line density due to the pure diffusion mechanism.

In the upper picture of Fig. 5 we displayed the Fig 2. of work [8] showing results of measurements on the temporal behavior of the average vortex line density $L(t)$ (solid curves, see for details [8]). In the lower picture of Fig. 5 we depicted the evolution the same quantity (for initial condition $L = 10^8 \text{ 1/cm}^2$) due to diffusion process described in the present paper. The initial domain of high vortex line density was created in the volume $^3$He-B, so its diffusion like behavior should satisfy the first type boundary condition. The straight line in the lower figure exactly corresponds to line A in the upper one (which was named by authors of [8] as “limiting behavior”).

In contrast to work [8] in the paper [9] the decay of vortex tangle in He-II was observed in the closed cube with solid walls. In the upper picture of Fig. 6 there is depicted the temporal behavior of the average vortex line
density $\mathcal{L}_{\text{ave}}(t)$.

We calculated the same dependence on the base of diffusion equation (9), result is shown in the lower picture of Fig. 6. As discussed above we can (in the frame of the present paper) determine the boundary condition only up to coefficient of re-emission $C_{\text{back}}$. Considering it as a fitting parameter we have chosen the value of $C_{\text{back}} \approx 0.9$. It can be seen that the decay of the vortex tangle due to diffusion reproduces some feature observed in experiment.

In particular, there is some plateau anticipatory decay of the tangle. Full decay of the tangle occurs in time of the order of few thousands seconds as it was observed in experiments. The slope of curve in interval of most intensive decrease shows the dependence close to $\sim t^{-3/2}$ (The straight line, its position coincides with position of the straight line in upper picture). Thus, there is again very good agreement between experimental data and theoretical predictions. There is one very interesting by-product of the consideration exposed above. It is easy to notice that for small values of vortex line density the bend appears on curve $\mathcal{L}(t)$. It corresponds to the fact that the diffusive-like flux of the length vanishes (because of vanishing the gradient $\nabla \mathcal{L}(x_b, t)$) and the only first and third terms in the boundary condition (13) survive. Let us recall that these terms correspond to radiation of loops from the bulk to the boundary and to the re-emission of loops from the wall into bulk of helium. Comparing this terms (for the chosen value $C_{\text{back}} \approx 0.9$) we obtain that equilibrium is reached for value of the vortex line density $\mathcal{L}$ of order $50 \div 100$ $1/cm^2$. This value can be considered as a "background" value of pre-existing vortices in helium.

**CONCLUSION**

In summary, the theory describing evolution of inhomogeneous vortex tangle at zero temperature was developed on the bases of kinetics of merging and splitting vortex loops. Using the Gaussian model for vortex loops we calculated the (size dependent) free path and mean quadratic velocity of vortex loops. With the use of these quantities we calculated the flux of the vortex line density $\mathcal{L}(x, t)$ in inhomogeneous vortex tangle and demonstrated that under certain circumstances it satisfies to the diffusion like equation with the coefficient equal approximately to $2.2\kappa$. We used this equation to describe the decay of the vortex tangle at very low temperature. We compare solution with the recent experiments on decay of the superfluid turbulence. The good agreement with the experimental data allowed us to conclude that the diffusion processes give the significant contribution in the free decay of the vortex tangle at absence of normal component.

**ACKNOWLEDGMENTS**

This work was partially supported by grant 07-02-01124 from the RFBR and grant of President Federation on the state support of leading scientific schools RF NSH-6749.2006.8. I am grateful to participants of the LT 25 (Amsterdam, 2008), especially Prof. M. Tsubota for useful discussion of this work.

---

[1] van Beelen, H., W. van Joolingen, and K. Yamada, Physica B, 153, 248, (1988).
[2] Geurst, J.A., Physica B 1541, 327, (1989).
[3] S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. 67, 37 (1995).
[4] M.S. Mongiov’i and D. Jou and M., Phys. Rev. B 75, 024507 (2007).
[5] M. Tsubota, T. Araki and W.F. Vinen, Physica B 329-333, 224 (2003).
[6] Carlo F. Barenghi and David C. Samuels, PRL, 89, 155302,(2002).
[7] S.I. Davis, P.C. Hendry and P.V.E. McClintock, Physica B 280 B (2000) 43-44.
[8] D.I. Bradley, D.O. Chubb, S.N. Fisher, A.M. Guenault, R.P. Haley, C.J. Matthews, G.R. Pickett, V. Tsepelin and K. Zaki, Phys. Rev. Lett. 96, 035301,(2006).
[9] P. M. Walsmsley, A. I. Golov, H. E. Hall, A. A. Levchenko, and W. F. Vinen, Phys. Rev. Lett. 99, 265302 (2007)
[10] C. F. Barenghi, Physica D, 237 2195 (2008).
[11] Sergey K. Nemirovskii, Phys. Rev. Lett., 96, 015301, (2006).
[12] Sergey K. Nemirovskii Phys. Rev B 77, 214509, (2008).
[13] Sergey K. Nemirovskii, J. Low Temperature Physics, Vol. 142, Nos.5/6, (2006).
[14] S.K. Nemirovskii, Phys. Rev B 57, 5792, (1997).
[15] M. Tsubota, T. Araki and S. K. Nemirovskii, Phys. Rev. B 62, 11751 (2000).
[16] E.J.Copeland, T.W.B.Kibble and D.A.Steer, Phys. Rev. D, 58, 043508, (1998).
[17] B. V. Svistunov, Phys. Rev. B 52, 3646 (1995).
[18] K.W.Schwarz, Phys. Rev. B 38, 2398 (1988).