A Frequency Domain Bootstrap for General Multivariate Stationary Processes

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Abstract: For many relevant statistics of multivariate time series, no valid frequency domain bootstrap procedures exist. This is mainly due to the fact that the distribution of such statistics depends on the fourth-order moment structure of the underlying process in nearly every scenario, except for some special cases like Gaussian time series. In contrast to the univariate case, even additional structural assumptions such as linearity of the multivariate process or a standardization of the statistic of interest do not solve the problem. This paper focuses on integrated periodogram statistics as well as functions thereof and presents a new frequency domain bootstrap procedure for multivariate time series, the multivariate frequency domain hybrid bootstrap (MFHB), to fill this gap. Asymptotic validity of the MFHB procedure is established for general classes of periodogram-based statistics and for stationary multivariate processes satisfying rather weak dependence conditions. A simulation study is carried out which compares the finite sample performance of the MFHB with that of the moving block bootstrap.

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1. Introduction

Developing valid bootstrap methods for time series has been a particularly challenging problem since the mid-1980s. Soon after Efron’s seminal paper (cf. Efron (1979)) on the bootstrap for i.i.d. observations the first attempts towards an extension to dependent data were made. While for univariate time series a variety of proposals exists that are asymptotically valid under certain assumptions on

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the dependence structure of the underlying process and for certain types of statistics, very little progress has been made for multivariate time series. This is due to the fact that in the multivariate context the distribution of most relevant statistics depends on the fourth-order moment structure of the underlying process, which many bootstrap methods developed for univariate time series are not able to imitate. Even under the assumption of a linear time series, that is, an $\mathbb{R}^m$-valued, strictly stationary stochastic process $(X(t))_{t \in \mathbb{Z}}$ which fulfills

$$X(t) = \sum_{j=-\infty}^{\infty} B_j e(t - j), \quad t \in \mathbb{Z}, \quad (1.1)$$

for certain $m \times m$ real matrices $B_j$ and an i.i.d. white noise process $(e(t))_{t \in \mathbb{Z}}$, the distribution of most statistics of interest depends on the fourth-order moment structure and many bootstrap methods fail. In contrast to this, for univariate linear time series – that is for processes $(1.1)$ with dimension $m = 1$ – there are a number of scenarios in which the distribution of some statistic of interest depends only on first and second-order moments and established univariate bootstrap methods are successful. It should be emphasized at this point that whenever we use the term linear time series (or linear process) in this work, we refer to a process as given by $(1.1)$ including the i.i.d. assumption on the innovations $e(t)$, as it is done in many standard references for time series, cf. Brockwell and Davis (1991), among others. A subclass of $(1.1)$ is that of the causal and invertible linear processes. In this case a sequence $\{A_j, j \in \mathbb{N}\}$ of $m \times m$ real matrices exists with $\sum_{j=1}^{\infty} \|A_j\|_F < \infty$, such that $(1.1)$ also can be expressed as $X(t) = \sum_{j=1}^{\infty} A_j X(t-j) + e(t). \{X(t), t \in \mathbb{Z}\}$ is then called a linear VAR($\infty$) process. Here $\| \cdot \|_F$ refers to the Frobenius norm of a matrix. Notice that any process following expression $(1.1)$ but with non-i.i.d. white noise innovations, i.e. where the $e_t$ are uncorrelated with zero mean but not independent, is nonlinear.

Bootstrap methods for time series are usually formulated either in the time domain or in the frequency domain, with a few hybrid methods that combine both approaches. In the following paragraphs we will give a short overview of existing methods for the univariate and the multivariate setup, and discuss their respective limitations. Prominent examples in the time domain are the block bootstrap and variants thereof, the AR sieve bootstrap and the linear process bootstrap, among others. Block bootstrap methods are rough tools which are valid for a wide class of processes but typically their performance heavily depends on the block size. For univariate time series, the AR sieve bootstrap is known to be valid exclusively in those situations where the distribution of interest depends on the first and second-order characteristics of the process only, cf. Kreiss, Paparoditis and Politis (2011). There are some relevant statistics for which this is the case, like the sample mean for general stationary processes or sample autocorrelations for linear processes while sample autocovariances (even for linear processes) are already outside the range of its validity. The same remarks also can be made for the linear process bootstrap. To overcome some of the aforementioned limitations for univariate time series, Frangeskou and Pa-
Meyer and Paparoditis (2019) proposed a procedure that involves a wild bootstrap scheme to generate pseudo innovations which asymptotically correctly imitate the first, the second and the fourth-order moment structure of the true innovations. This extends the validity of the AR sieve bootstrap beyond the class of linear, causal and invertible processes.

However, if one switches to multivariate processes the fourth-order moment structure shows up in the asymptotic distribution of almost any relevant statistic. This makes the bootstrap estimation problem much more involved. In particular, the AR sieve and the linear process bootstrap fail for multivariate non-invertible linear processes (1.1) and for such basic statistics as sample cross-correlations; see Jentsch and Kreiss (2010) and Meyer and Kreiss (2015) for more details. Thus, beyond causal and invertible linear processes, and apart from special cases like Gaussian time series, it seems that validity of the vector AR sieve bootstrap is essentially restricted to very elementary statistics like the sample mean. Even the aforementioned extension of the AR sieve via wild bootstrap-generated pseudo innovations is not available in the multivariate context. Consequently, for a wide class of stationary multivariate processes, including linear processes (1.1), and for many interesting statistics, the only available bootstrap method is essentially the time domain block bootstrap and variants thereof.

Concerning frequency domain bootstrap methods for univariate time series, the situation is very similar. Interest is here focused on so-called integrated periodogram statistics. These are statistics which are obtained by integrating over all frequencies the periodogram multiplied with some function of interest. Many time domain statistics like autocorrelations or autocovariances have a frequency domain analogue, that is they can also be expressed as (functions of) integrated periodograms. Hurvich and Zeger (1987), Franke and Härdle (1992) and Dahlhaus and Janas (1996) developed for univariate time series a multiplicative bootstrap procedure for the periodogram. By construction, this scheme is capable of imitating the variance of periodogram ordinates at different frequencies but not their dependence structure across frequencies. Since for periodogram-based statistics, the fourth-order structure of the process is transmitted to their distribution through the weak dependence of the periodogram ordinates across different frequencies, the multiplicative periodogram bootstrap suffers from the same limitations as some of the time domain procedures discussed so far. More specifically, it is valid exclusively in those situations where the distribution of interest only depends on the spectral density, that is on the second-order structure of the process. Dahlhaus and Janas (1996) showed that a subclass of standardized integrated periodograms, the so-called ratio statistics, falls into this category, but only under the assumption that the underlying time series is linear. Kreiss and Paparoditis (2012) proposed a modification of the multiplicative periodogram bootstrap which extends its validity for integrated periodograms to the entire class of univariate linear time series. Recently, Meyer et al. (2020) proposed a hybrid method which is valid for a very general class of weakly dependent univariate processes and which goes far beyond the linear process class.
However, for multivariate time series the related inference problems are much more involved and no valid frequency domain bootstrap method exists so far. In principle, an analogue of the univariate multiplicative approach (Franke and Härdle (1992) and Dahlhaus and Janas (1996)), can be formulated in the multivariate setup by using the fact that – for a wide class of stationary processes – the periodogram matrices at any fixed set of frequencies are asymptotically independent and have a Wishart distribution; see for instance Brillinger (1981). However, applying such a bootstrap approach alone will fail in imitating the dependence across periodogram ordinates since it only can capture the second-order properties of the underlying multivariate process. Therefore, there seem to be no relevant statistics for multivariate time series for which such a procedure will be valid. This is true even for the linear process class (1.1), for which the distribution of statistics like auto or cross-correlations depends on the fourth-order moment structure of the process. In other words, for multivariate time series, any frequency domain bootstrap scheme that ignores the weak dependence structure of the periodogram matrix across frequencies, would be systematically failing to properly capture the distribution of a large class of relevant statistics.

To summarize, a frequency domain bootstrap method that is capable of handling large classes of periodogram based statistics for general multivariate processes is not available so far. It is the purpose of the present paper to close this gap. Towards this end we introduce the concept of the multivariate frequency domain hybrid bootstrap (MFHB). This procedure is composed of two ingredients. The first is the multivariate analogue of the previously mentioned multiplicative bootstrap approach proposed for univariate processes. This ingredient is used to imitate features of the distribution of interest that depend on the second-order structure of the process. The second ingredient of our MFHB uses an adaptation of the convolved subsampling idea (cf. Tewes et al. (2019)) to the multivariate frequency domain context. It is used to capture those features of the same distribution that the first ingredient is systematically missing out on, and which are crucially determined by the fourth-order structure of the process. Notice that the MFHB procedure we propose is structurally related to the hybrid procedure that was proposed in Meyer et al. (2020) for univariate time series. However, the MFHB for multivariate time series has to take a number of obstacles into account that are not present in the univariate situation and that have to be solved in a novel way. In particular, the limiting distribution of multivariate integrated periodogram statistics – as well as of functions thereof – is a multivariate complex normal distribution where both covariance and relation matrix depend on the fourth-order moment structure of the underlying process. A valid frequency domain bootstrap procedure therefore has to imitate correctly not only the covariance but also the relation matrix of the corresponding distribution. Moreover, additional difficulties arise in the multivariate context from applying a frequency domain bootstrap to the class of smooth functions of integrated periodograms. Such an extension of the MFHB turns out to be important because it allows for applications to interesting classes of statistics, like for instance, to cross-correlations. In order to make such an extension possi-
ble, the second and the fourth-order characteristics of the covariance and of the relation matrix of the related smooth functions have to be separated and appropriately imitated by the two ingredients of the MFHB procedure. As we will see, in contrast to integrated periodograms, this separation can not be explicitly calculated and therefore the bootstrap scheme has to be appropriately modified. We show that the proposed MFHB procedure simultaneously establishes correct covariance and relation matrix estimation for both aforementioned classes of statistics. Moreover, the MFHB will be proven to be valid for integrated periodogram statistics as well as for smooth functions thereof and for a wide class of stochastic processes which includes many known linear and nonlinear multivariate time series models.

The remainder of this paper is organized as follows. Section 2 states the assumptions we impose on the multivariate process and recalls some definitions and limiting results for multivariate integrated periodogram statistics. Section 3 presents the MFHB procedure for integrated periodograms and for functions thereof, discusses some of its features and establishes its asymptotic validity. Section 4 presents simulations that investigate the finite sample performance of the MFHB and compares it with that of the moving block bootstrap. The proofs of our main results are presented in the Appendix of this paper while the proofs of some technical results are deferred to the Supplementary Material.

2. Preliminaries and basic results

Consider an \( \mathbb{R}^m \)-valued, weakly stationary stochastic process \( (X(t))_{t \in \mathbb{Z}} \) with mean zero and component processes \( (X_r(t))_{t \in \mathbb{Z}}, \; r = 1, \ldots, m \). We denote the autocovariance matrix of the process by \( \Gamma(h) = E(X(t+h)X(t)^\top) \in \mathbb{R}^{m \times m} \), for \( h \in \mathbb{Z} \), with entries \( \gamma_{rs}(h) = \text{Cov}(X_r(t+h), X_s(t)) \), \( r, s \in \{1, \ldots, m\} \), which fulfill \( \gamma_{rs}(h) = \gamma_{sr}(-h) \). Furthermore, the \( k \)-th order cumulant of \( (X(t))_{t \in \mathbb{Z}} \) is denoted, for any \( r_1, \ldots, r_k \in \{1, \ldots, m\} \) and \( t, h_1, \ldots, h_{k-1} \in \mathbb{Z} \), by

\[
\text{cum}(X_{r_1}(t+h_1), X_{r_2}(t+h_2), \ldots, X_{r_{k-1}}(t+h_{k-1}), X_{r_k}(t)).
\]

(2.1)

The following stationarity and weak dependence assumptions are imposed on the process under consideration.

**Assumption 1.** \( (X(t))_{t \in \mathbb{Z}} \) is eighth-order stationary, i.e. for all \( r_j \in \{1, 2, \ldots, m\} \) all joint cumulants of the process up to order eight do not depend on the time point \( t \). We therefore write \( c_{r_1r_2...r_8}(h_1, h_2, \ldots, h_{8-k}) \) for (2.1). Furthermore, it holds for all \( r_1, \ldots, r_8 \in \{1, \ldots, m\} \)

\[
(i) \quad \sum_{h \in \mathbb{Z}} (1 + |h|)|\gamma_{r_1r_2}(h)| < \infty,
\]

\[
(ii) \quad \sum_{h_1, h_2, h_3 \in \mathbb{Z}} (1 + |h_1| + |h_2| + |h_3|)|c_{r_1r_2r_3r_4}(h_1, h_2, h_3)| < \infty,
\]

\[
(iii) \quad \sum_{h_1, \ldots, h_7 \in \mathbb{Z}} |c_{r_1...r_7}(h_1, h_2, \ldots, h_7)| < \infty.
\]

The above assumption is satisfied for a large class of stochastic processes which includes, for instance, the multivariate linear processes class (1.1) under
certain moment assumptions on the innovations $\mathbf{e}_t$ and summability conditions on the matrices $\mathbf{B}_j$; see Brillinger (1981).

Notice that since $\sum_{h \in \mathbb{Z}} |\gamma_{r_1, r_2}(h)| < \infty$, for all $r_1, r_2 \in \{1, 2, \ldots, m\}$, the process $(\mathbf{X}(t))_{t \in \mathbb{Z}}$ possesses a spectral density matrix $\mathbf{f} : [-\pi, \pi] \to \mathbb{C}^{m \times m}$ with entries

$$f_{rs}(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma_{rs}(h) e^{-ih\lambda}, \quad \lambda \in [-\pi, \pi],$$

for $r, s \in \{1, 2, \ldots, m\}$, which is Hermitian, i.e., $f_{rs}(\lambda) = f_{sr}(-\lambda)$ and $\overline{f}(\lambda) = f(\lambda)$. Here, and throughout this work, $\overline{\mathbf{A}}$ denotes the conjugate transpose of a complex-valued matrix $\mathbf{A}$. Furthermore, $\mathbf{f}$ is bounded from above and is a continuous function of the frequency $\lambda$. In the following we additionally assume that the eigenvalues of the spectral density matrix $\mathbf{f}$ are all bounded away from zero over all frequencies.

**Assumption 2.** A constant $\delta > 0$ exists such that the eigenvalues of the spectral density matrix, i.e., $\sigma(\mathbf{f}(\lambda))$ satisfy

$$\min \left(\sigma(\mathbf{f}(\lambda))\right) > \delta,$$

for all frequencies $\lambda \in (-\pi, \pi]$.

Given an $m$-dimensional vector time series $\mathbf{X}(1), \mathbf{X}(2), \ldots, \mathbf{X}(n)$ stemming from $(\mathbf{X}(t))_{t \in \mathbb{Z}}$, a common moment estimator of the cross covariances $\gamma_{rs}(h)$, for $-n < h < n$, is given by

$$\hat{\gamma}_{rs}(h) = \begin{cases} \frac{1}{n} \sum_{t=0}^{n-h} \mathbf{X}_r(t+h)\mathbf{X}_s(t), & \text{for } 0 \leq h \leq n-1 \\ \frac{1}{n} \sum_{t=|h|}^{n-|h|} \mathbf{X}_r(t)\mathbf{X}_s(t+|h|), & \text{for } -(n-1) \leq h \leq -1 \end{cases}.$$

In the following, the periodogram matrix $\mathbf{I} : [-\pi, \pi] \to \mathbb{C}^{m \times m}$ with $\mathbf{I}(\lambda) = \mathbf{d}(\lambda) \overline{\mathbf{d}(\lambda)}$ is used as a basic statistic, where

$$\mathbf{d}(\lambda) = (d_1(\lambda), \ldots, d_m(\lambda))^\top,$$

and the $m$-dimensional vector of finite Fourier transforms $d_r$ is given by

$$d_r(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \mathbf{X}_r(t) e^{-it\lambda}, \quad r = 1, 2, \ldots, m.$$

Notice that $I_{rs}(\lambda) = d_r(\lambda) d_s(-\lambda)$ while $I_{rr}(\lambda)I_{ss}(\lambda) = I_{rs}(\lambda)I_{sr}(\lambda)$. Furthermore, it holds

$$I_{rs}(\lambda) = \frac{1}{2\pi} \sum_{h=-\lfloor(n-1)/2\rfloor}^{\lfloor(n-1)/2\rfloor} \hat{\gamma}_{rs}(h) e^{-ih\lambda}. $$

Let $\lambda_{j,n} = 2\pi j/n$, $j \in \mathcal{G}(n)$, be the Fourier frequencies based on a sample size $n$, where

$$\mathcal{G}(n) := \{j \in \mathbb{Z} : 1 \leq |j| \leq \lfloor n/2 \rfloor\}. \quad (2.2)$$
Denote further for \( p, q, r, s \in \{1, 2, \ldots, m\} \), by
\[
f_{pqrs}(\lambda, \mu, \eta) = \frac{1}{(2\pi)^3} \sum_{h_1, h_2, h_3 \in \mathbb{Z}} c_{pqrs}(h_1, h_2, h_3) e^{-i(h_1 \lambda + h_2 \mu + h_3 \eta)}
\]
the fourth-order cumulant spectral density of \((X(t))_{t \in \mathbb{Z}}\). The following lemma, which describes the covariance structure of the elements of the periodogram matrix at the Fourier frequencies, is very useful for our subsequent analysis.

**Lemma 2.1.** Let \((X(t))_{t \in \mathbb{Z}}\) have finite fourth moments and be fourth-order stationary satisfying the summability conditions (i) and (ii) of Assumption 1. Then it holds for all Fourier frequencies \( \lambda_{j,n}, \lambda_{k,n} \in [0, \pi] \), and for all \( r, s, v, w \in \{1, \ldots, m\} \)
\[
\text{Cov}(I_{rs}(\lambda_{j,n}), I_{vw}(\lambda_{k,n})) = S_1 + S_2 + S_3,
\]
where
\[
S_1 = \frac{2\pi}{n} f_{rsvw}(\lambda_{j,n}, -\lambda_{j,n}, -\lambda_{k,n}) + O\left(\frac{1}{n^2}\right),
\]
\[
S_2 = \begin{cases} f_{rv}(\lambda_{j,n}) \cdot f_{sw}(\lambda_{j,n}) + O\left(\frac{1}{n}\right), & \lambda_{j,n} = \lambda_{k,n}, \\ O\left(\frac{1}{n^2}\right), & \lambda_{j,n} \neq \lambda_{k,n}, \end{cases}
\]
\[
S_3 = \begin{cases} f_{rw}(\lambda_{j,n}) \cdot f_{sv}(\lambda_{j,n}) + O\left(\frac{1}{n}\right), & \lambda_{j,n} = \lambda_{k,n} \in \{0, \pi\}, \\ O\left(\frac{1}{n^2}\right), & \text{else} \end{cases}
\]
and where all \(O(\cdot)\) bounds are uniform over all Fourier frequencies.

As it is seen from the above result, the covariance between the elements of the periodogram matrix at different frequencies vanishes by the order \(1/n\) and depends on the fourth-order cumulant spectral density \(f_{rsvw}\). We will see that this rate is not fast enough so that the covariance between periodogram ordinates – and consequently the fourth-order structure of the process – show up in the limiting distribution of integrated periodogram statistics. This class of statistics, which we will consider in the following, is defined as functions of the periodogram matrix. In particular, we require:

**Assumption 3.** For some \( J \in \mathbb{N} \), \( \varphi_j : [-\pi, \pi] \to \mathbb{C}, j = 1, 2, \ldots, J \) are square-integrable functions which are bounded in absolute value.

Now, the vector of integrated periodogram statistics we consider is defined as
\[
M_n = \left(M(\varphi_j, I_{rs}) = \int_{-\pi}^{\pi} \varphi_j(\lambda) I_{rs}(\lambda) d\lambda, j = 1, 2, \ldots, J\right)^T.
\]

Note that \( M_n \) is an estimator of the following vector of **spectral means**
\[
M = \left(M(\varphi_j, f_{rs}) = \int_{-\pi}^{\pi} \varphi_j(\lambda) f_{rs}(\lambda) d\lambda, j = 1, 2, \ldots, J\right)^T.
\]
Before proceeding with some limiting results regarding the behaviour of the estimators \( M_n \), let us look at some examples.

**Example 2.2.** The sample cross-covariance \( \hat{\gamma}_{rs}(h) \) at lag \( 0 \leq h < n \) is an integrated periodogram statistic. This is due to the fact that choosing \( \varphi(\lambda) = e^{ih\lambda} \) it follows from straightforward calculations that \( \hat{\gamma}_{rs}(h) = M(\varphi, I_{rs}) \) as well as \( \gamma_{rs}(h) = M(\varphi, f_{rs}) \). Notice that for \( -n < h < 0 \), \( \hat{\gamma}_{rs}(h) = \hat{\gamma}_{rs}(-h) \) is an estimator of \( \gamma_{rs}(h) \).

**Example 2.3.** The sample cross-correlation \( \hat{\rho}_{rs}(h) \) at lag \( 0 \leq h < n \) is a function of integrated periodograms. To elaborate, let \( \varphi_1(\lambda) = e^{ih\lambda} \), \( \varphi_2(\lambda) = \varphi_3(\lambda) = 1 \) and consider the corresponding three-dimensional vector of spectral means. Then,

\[
\hat{\rho}_{rs}(h) = M(\varphi_1, I_{rs})/\sqrt{M(\varphi_2, I_{rs})M(\varphi_3, I_{ss})}
\]

is an estimator of \( \rho_{rs}(h) = M(\varphi_1, f_{rs})/\sqrt{M(\varphi_2, f_{rr})M(\varphi_3, f_{ss})} \), the lag \( h \) cross-correlation. Notice that \( \rho_{rs}(h) \) and \( \hat{\rho}_{rs}(h) \) are functions of the elements of the vectors \( M = (M(\varphi_1, f_{rs}), M(\varphi_2, f_{rr}), M(\varphi_3, f_{ss}))^T \) and \( M_n = (M(\varphi_1, I_{rs}), M(\varphi_2, I_{rr}), M(\varphi_3, I_{ss}))^T \), respectively. The limiting distribution of \( \hat{\rho}_{rs}(h) \) will be derived in Example 2.4.

For practical calculations the integral in the expression for \( M(\varphi, I_{rs}) \) is commonly replaced by a Riemann sum using the Fourier frequencies. The corresponding approximation of \( M(\varphi, I_{rs}) \) is given by

\[
M_{\mathcal{G}_n}(\varphi, I_{rs}) = \frac{2\pi}{n} \sum_{l \in \mathcal{G}_n} \varphi(\lambda_l, n) I_{rs}(\lambda_l, n).
\]

Before discussing the asymptotic properties of \( M_n \), we evaluate on some properties of the complex normal distribution which are important for our subsequent discussion. An \( m \)-dimensional complex random vector \( X = (X_1, X_2, \ldots, X_m)^T \) is called complex normal (or complex Gaussian) if and only if the \( 2m \)-dimensional vector of real and imaginary parts

\[
(\text{Re}(X)^T, \text{Im}(X)^T)^T := (\text{Re}(X_1), \ldots, \text{Re}(X_m), \text{Im}(X_1), \ldots, \text{Im}(X_m))^T
\]

has a \( 2m \)-dimensional (real) normal distribution. The complex normal distribution is determined by three parameters: expectation \( \mu_X = E(X) \), covariance matrix \( \Sigma_X \), and relation matrix \( \Gamma_X \), where

\[
\Sigma_X = E([X - \mu_X] [\overline{X} - \overline{\mu_X}]) , \quad \Gamma_X = E([X - \mu_X] [X - \mu_X]^T),
\]

recalling that \( \overline{A} \) denotes the conjugate transpose of any matrix or vector \( A \). We therefore write

\[
X \sim \mathcal{N}_m^c(\mu_X, \Sigma_X, \Gamma_X).
\]

We are particularly interested in the case of centered vectors, i.e. \( \mu_X = 0 \). It is then easy to see that the covariance matrix of the real normal vector
\((\text{Re}(X))^{\top}, \text{Im}(X)^{\top}\) can be directly deduced from the parameters of the complex normal vector \(X\) via

\[
X \sim \mathcal{N}_m^c(0, \Sigma_X, \Gamma_X) \iff \begin{pmatrix} \text{Re}(X) \\ \text{Im}(X) \end{pmatrix} \sim \mathcal{N}_{2m}(0, G_X),
\]

where

\[
G_X = E \begin{bmatrix} \text{Re}(X) \\ \text{Im}(X) \end{bmatrix}^{\top} = \begin{pmatrix} \frac{1}{2}(\text{Re}(\Sigma_X) + \text{Re}(\Gamma_X)) & \frac{1}{2}(-\text{Im}(\Sigma_X) + \text{Im}(\Gamma_X)) \\ \frac{1}{2}(\text{Im}(\Sigma_X) + \text{Im}(\Gamma_X)) & \frac{1}{2}(\text{Re}(\Sigma_X) - \text{Re}(\Gamma_X)) \end{pmatrix}.
\]

The above expression shows that \(G_X\) is fully determined by \(\Sigma_X\) and \(\Gamma_X\) and vice versa. In particular, denote by \(G_{X_{ij}}, i, j \in \{1, 2\}\), the four matrices appearing in the \(i\)-th row and \(j\)-th column of the above block matrix \(G_X\). Then

\[
\Sigma_X = (G_{X_{11}} + G_{X_{22}}) + i(G_{X_{21}} - G_{X_{12}}) \quad \text{and} \quad \Gamma_X = (G_{X_{11}} - G_{X_{22}}) + i(G_{X_{21}} + G_{X_{12}}).
\]

The equivalence from (2.5) also carries over to weak convergence: For a sequence of complex random vectors \((X_n)_{n \in \mathbb{N}}\) we have \(X_n \xrightarrow{d} \mathcal{N}_m^c(0, \Sigma_X, \Gamma_X)\) if and only if \((\text{Re}(X_n))^{\top}, \text{Im}(X_n)^{\top}) \xrightarrow{d} \mathcal{N}_{2m}(0, G_X)\).

For \(\mu_X = 0\) there are two particularly important special cases where the distribution is completely determined by the matrix \(\Sigma_X\), the real normal case and the circularly symmetric case. The centered random vector \(X\) is almost surely real-valued if and only if \(\Sigma_X = \Gamma_X\), as a simple calculation shows. In this case one may switch to the usual notation for real-valued random vectors:

\[
X \sim \mathcal{N}_m^c(0, \Sigma_X, \Sigma_X) \iff X \sim \mathcal{N}_m(0, \Sigma_X).
\]

Another important special case is the circularly symmetric case. \(X\) is called circularly symmetric if for all \(\varphi \in (-\pi, \pi]\) the distribution of \(e^{i\varphi}X\) equals the distribution of \(X\). This is the case if and only if \(\mu_X = 0\) and \(\Gamma_X = 0\). In this case the joint distribution of real and imaginary parts takes a very specific form, as can be seen from (2.5):

\[
X \sim \mathcal{N}_m^c(0, \Sigma_X, 0) \iff \begin{pmatrix} \text{Re}(X) \\ \text{Im}(X) \end{pmatrix} \sim \mathcal{N}_{2m} \begin{pmatrix} 0 \\ 0 \end{pmatrix} 

\frac{1}{2} \begin{pmatrix} \text{Re}(\Sigma_X) & -\text{Im}(\Sigma_X) \\ \text{Im}(\Sigma_X) & \text{Re}(\Sigma_X) \end{pmatrix}.
\]

In particular, \(\text{Re}(X)\) and \(\text{Im}(X)\) then are identically distributed.

Under Assumption 1 and some additional weak dependence conditions on the process \((X(t))_{t \in \mathbb{Z}}\) it is known that, for any \(r, s \in \{1, \ldots, m\}\), \(M(\varphi, I_{rs})\) is a consistent estimator for \(M(\varphi, I_{rs})\) and that the following central limit theorem holds true for \(V_n := \sqrt{n}(\mathbf{M}_n - \mathbf{M})\):

\[
V_n = \sqrt{n} \begin{pmatrix} M(\varphi_j, I_{rs,j}) - M(\varphi_j, I_{rs,j}) \end{pmatrix}^{\top} \xrightarrow{d} V, \tag{2.6}
\]
where \( \mathbf{V} \) is a \( J \)-dimensional complex normal random vector with mean zero, covariance matrix \( \Sigma \), and relation matrix \( \Gamma \). That is,
\[
\mathbf{V} = (V_{r_j s_j}(\varphi_j), j = 1, \ldots, J) \top \sim \mathcal{N}_J^{c}(\mathbf{0}, \Sigma, \Gamma).
\]
\( \mathbf{V} \) fulfills
\[
\overline{V_{r_j s_j}(\varphi_j(\cdot))} = V_{r_j s_j}(\overline{\varphi_j(\cdot)}), \tag{2.7}
\]
and covariance and relation parameters are given as follows. The covariance matrix decomposes into \( \Sigma = \Sigma_1 + \Sigma_2 \) where the \((j, k)\)-th element of the matrix \( \Sigma \) is given by
\[
\Sigma_{jk} = \text{Cov}(V_{r_j s_j}(\varphi_j), V_{r_k s_k}(\varphi_k)) = \Sigma_{1;jk} + \Sigma_{2;jk} \tag{2.8}
\]
with
\[
\Sigma_{1;jk} = 2\pi \int_{-\pi}^{\pi} \varphi_j(\lambda) \overline{\varphi_k(\lambda)} f_{r_j r_k}(\lambda) f_{s_j s_k}(-\lambda) d\lambda \tag{2.9}
\]
\[
+ 2\pi \int_{-\pi}^{\pi} \varphi_j(\lambda) \overline{\varphi_k(-\lambda)} f_{r_j r_k}(\lambda) f_{s_j s_k}(-\lambda) d\lambda,
\]
and
\[
\Sigma_{2;jk} = 2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi_j(\lambda_1) \overline{\varphi_k(-\lambda_2)} f_{r_j s_j r_k s_k}(\lambda_1, -\lambda_1, -\lambda_2) d\lambda_1 d\lambda_2. \tag{2.10}
\]
The relation matrix decomposes into \( \Gamma = \Gamma_1 + \Gamma_2 \) where the \((j, k)\)-th element is given by
\[
\Gamma_{jk} = \text{Cov}(V_{r_j s_j}(\varphi_j), V_{r_k s_k}(\varphi_k)) = \Gamma_{1;jk} + \Gamma_{2;jk} \tag{2.11}
\]
with
\[
\Gamma_{1;jk} = 2\pi \int_{-\pi}^{\pi} \varphi_j(\lambda) \overline{\varphi_k(-\lambda)} f_{r_j r_k}(\lambda) f_{s_j s_k}(-\lambda) d\lambda \tag{2.12}
\]
\[
+ 2\pi \int_{-\pi}^{\pi} \varphi_j(\lambda) \overline{\varphi_k(\lambda)} f_{r_j s_k}(\lambda) f_{s_j r_k}(-\lambda) d\lambda,
\]
and
\[
\Gamma_{2;jk} = 2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi_j(\lambda_1, -\lambda_1) \varphi_k(\lambda_2) f_{r_j s_j r_k s_k}(\lambda_1, -\lambda_1, -\lambda_2) d\lambda_1 d\lambda_2. \tag{2.13}
\]
Observe that due to (2.7) it holds \( \Gamma_{jk} = \text{Cov}(V_{r_j s_j}(\varphi_j(\cdot)), V_{r_k s_k}(\overline{\varphi_k(-\cdot)})) \).
Hence, \( \Gamma \) can be obtained from \( \Sigma \) by replacing \( \varphi_k(\cdot) \) with \( \overline{\varphi_k(-\cdot)} \), and we stated the explicit form merely for convenience reasons. We refer to Rosenblatt (1963), Brillinger (1981), Dahlhaus (1985) and Taniguchi and Kakizawa (2000). As it is seen from the above expressions the terms \( \Sigma_2 \) and \( \Gamma_2 \) appearing in the
covariance and relation matrices of the limiting complex Gaussian distribution depend on the entire fourth-order moment structure of the process \((X(t))_{t \in \mathbb{Z}}\), and this dependence is due to the covariance of periodogram ordinates across frequencies; see Lemma 2.1.

As argued in (2.5), the weak convergence \(V_n \xrightarrow{d} V\) stated in (2.6) is equivalent to the statement that the \(2J\)-dimensional real random vector \((\text{Re}(V_n)^\top, (\text{Im}(V_n)^\top)^\top)\) converges weakly to a \(2J\)-dimensional real normal distribution with mean zero and covariance matrix

\[
G = E\left[ \begin{pmatrix} \text{Re}(V) \\ \text{Im}(V) \end{pmatrix} \left( \begin{array}{c} \text{Re}(V)^\top \\ \text{Im}(V)^\top \end{array} \right) \right] \\
= \begin{pmatrix} \frac{1}{2}(\text{Re}(\Sigma) + \text{Re}(\Gamma)) & \frac{1}{2}(\text{Im}(\Sigma) + \text{Im}(\Gamma)) \\ \frac{1}{2}(\text{Im}(\Sigma) + \text{Im}(\Gamma)) & \frac{1}{2}(\text{Re}(\Sigma) - \text{Re}(\Gamma)) \end{pmatrix}.
\]

(2.14)

Notice that the limiting distribution \(V\) is not necessarily real-valued even if the functions \(\varphi_j, j = 1, 2, \ldots, J\) are real-valued. However, if the functions \(\varphi_j\) satisfy \(\varphi_j(-\lambda) = \varphi_j(\lambda)\) for \(j = 1, 2, \ldots, J\), then \(\Sigma = \Gamma\) and \(V\) is real valued, that is \(V \sim \mathcal{N}_J(0, \Sigma)\).

**Example 2.4** (Ex. 2.3 continued). With the central limit theorem given for \(V_n\) above, we can now state the limiting distribution of the sample cross-correlation. For the particular vectors \(M_n\) and \(M\) from Example 2.3 it can easily be seen that \(V_n\) is real-valued and converges to a real Gaussian random vector \(V\). Applying the delta method to this CLT it follows after straightforward but tedious computations that

\[
\sqrt{n} (\hat{\rho}_{rs}(h) - \rho_{rs}(h)) \xrightarrow{d} \mathcal{N}(0, \tau^2),
\]

where

\[
\tau^2 = \sum_{j \in \mathbb{Z}} \left\{ \rho_{rr}(j)\rho_{ss}(j) + \rho_{rs}(j + h)\rho_{sr}(j - h) + \frac{c_{rrrr}(j, j - h, 0)}{\gamma_{rr}(0)\gamma_{ss}(0)} \right. \\
+ \frac{1}{2} \rho_{rs}(h)^2 \left( \rho_{rr}(j)^2 + \rho_{ss}(j)^2 + 2\rho_{rs}(j)^2 \right) \\
- 2\rho_{rs}(h) \left[ \rho_{rr}(j)\rho_{sr}(j - h) + \rho_{rs}(j)\rho_{ss}(j - h) \right] \\
+ \frac{1}{4} \rho_{rs}(h)^2 \left[ \frac{c_{rrrr}(j, j, 0)}{\gamma_{rr}(0)^2} + \frac{c_{ssss}(j, j, 0)}{\gamma_{ss}(0)^2} + \frac{2c_{rrss}(j, j, 0)}{\gamma_{rr}(0)\gamma_{ss}(0)} \right] \\
- \frac{\rho_{rs}(h)}{\sqrt{\gamma_{rr}(0)\gamma_{ss}(0)}} \left[ \frac{c_{rrss}(j, j - h, 0)}{\gamma_{rr}(0)} + \frac{c_{ssss}(j, j - h, 0)}{\gamma_{ss}(0)} \right] \right\}.
\]

The above dependence of the variance of the limiting Gaussian distribution on the fourth-order structure of the process does not simplify even in the case of linear processes as this is the case for univariate linear processes. The following simple example illustrates this fact.
Example 2.5 (Example 2.4 continued). Consider the following simple bivariate linear process, which is a vector moving average process of order 1:

\[
X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_1(t-1) \\ \varepsilon_2(t-1) \end{pmatrix},
\]

(2.15)

where \((\varepsilon_1(t))_{t \in \mathbb{Z}}\) and \((\varepsilon_2(t))_{t \in \mathbb{Z}}\) are independent univariate i.i.d. white noise processes, with \(E(\varepsilon_j(t)) = 0\), \(E(\varepsilon_j(t)^2) = 1\), and kurtosis \(E(\varepsilon_j(t)^4) =: \eta_j\), for \(j = 1, 2\). We take a look at the cross-correlation at lag \(h = 0\), and derive the limiting variance of \(\sqrt{n}(\hat{\rho}_{12}(0) - \rho_{12}(0))\). For the process (2.15) it holds \(\rho_{12}(0) = 0\), and the expression \(\tau^2\) from Example 2.4 simplifies considerably to

\[
\tau^2 = \rho_{11}(0)\rho_{22}(0) + 2\rho_{11}(1)\rho_{22}(1) + 2\rho_{12}(1)\rho_{21}(1) + \sum_{j \in \mathbb{Z}} \frac{c_{1212}(j,j,0)}{\gamma_{11}(0)\gamma_{22}(0)}
\]

\[
= 1 + \frac{\eta_1 - 3}{9} + \frac{\eta_2 - 3}{9}.
\]

As it is seen – and in contrast to what happens for univariate linear processes – in the multivariate context the limiting variance depends on the fourth-order structure of the underlying white noise which can in general not be expressed in terms of second-order quantities of the process \((X(t))\).

The considerations in this section made it clear that for a frequency domain bootstrap to be successful in the multivariate context, it has to appropriately imitate the second and the fourth order structure of the underlying stochastic process. This is what the MFHB procedure achieves.

3. The multivariate frequency domain hybrid bootstrap (MFHB)

We discuss the frequency domain procedure proposed in this paper in two parts. First we motivate and describe the MFHB procedure for integrated periodograms. Then we present a modification of the MFHB so that in can be successfully applied to functions of integrated periodograms.

3.1. The MFHB for integrated periodograms

The MFHB procedure for integrated periodograms generates two sets of bootstrap pseudo random variables which will be denoted by the superscripts * and +, respectively. The procedure is divided into three main steps which are denoted by Step I, Step II and Step III. In Step I independent pseudo periodogram matrices are generated which are denoted by \(\mathbf{I}^*\). This is done using the asymptotic complex Gaussian distribution of the vector of finite Fourier transforms and their asymptotic independence across frequencies. In Step II the idea of convolved bootstrap of subsamples, cf. Tewes et al. (2019), is adopted to develop an algorithm which generates a second set of pseudo periodogram matrices, denoted by \(\mathbf{I}^+\), which are independent of \(\mathbf{I}^*\). The pseudo periodogram
matri
ts $I_1$ correctly imitate the weak dependence of the ordinary periodogram matrices $I$ across frequencies within subsamples. Step III merges the integrated periodogram statistics based on the two bootstrapped periodograms $I^+$ and $I^+$ in an appropriate way. The merging ensures that replicates of the integrated periodograms based on $I^+$ imitate all features of the corresponding distribution up to those depending on the fourth-order structure of the process. The fourth-order features of this distribution are contributed by the corresponding statistics based on the pseudo periodograms $I^+$. Notice that the MFHB bootstrap approximations are designed in such a way that the covariance and the relation matrix of the limiting complex Gaussian distribution of the integrated periodograms is consistently estimated, see Remark 3.2. Furthermore, the bootstrap random matrices $I^+$ and $I^+$ are defined on the same probability space, with probability measure $P^*$, and are independent from each other. Consequently, all bootstrap expectations, variances and covariances are denoted in the following by $E^*$, $\text{Var}^*$, and $\text{Cov}^*$, respectively.

The following algorithm implements the previously discussed ideas.

**Step I.1** Let $\hat{f}$ be an estimator of the spectral density matrix $f$ and denote by $f_{rs}(\lambda)$ the $(r,s)$-th element of $f(\lambda)$.

**Step I.2** Generate, independently for $j = 1, 2, \ldots, N := [n/2]$, pseudo Fourier transforms

$$d^*(\lambda_{j,n}) = (d^*_1(\lambda_{j,n}), d^*_2(\lambda_{j,n}), \ldots, d^*_m(\lambda_{j,n}))^\top \sim \mathcal{N}_m^c(0, \hat{f}(\lambda_{j,n}), 0)$$

and calculate the pseudo periodogram matrices

$$I^*(\lambda_{j,n}) = \begin{cases} d^*(\lambda_{j,n})d^*(\lambda_{j,n})^\top & \text{for } j = 1, 2, \ldots, N, \\ I^*(\lambda_{j,n})^\top & \text{for } j = -1, -2, \ldots, -N \end{cases}$$

with entries denoted by $I^*_{rs}(\lambda_{j,n})$, $r, s \in \{1, 2, \ldots, m\}$.

**Step I.3** Let

$$V_n^* = \sqrt{n} \left( M_{G(n)}(\varphi_j, I^*_{rs,j}) - M_{G(n)}(\varphi_j, \hat{f}_{rs,j}) \right), j = 1, 2, \ldots, J \right)^\top,$$ be the analogue of $V_n$ based on $I^*$.

**Step II.1** Select a positive integer $b < n$ and consider the set of all periodogram matrices based on subsamples of length $b$. Denote by $I_t(\lambda_{j,b})$ the periodogram matrix of the subsample $X(t), X(t+1), \ldots, X(t+b-1)$ at frequency $\lambda_{j,b} = 2\pi j/b$, $\lambda_{j,b} \in G(b)$. Let $\tilde{f}(\lambda_{j,b}) = (n-b+1)^{-1} \sum_{t=1}^{n-b+1} I_t(\lambda_{j,b})$ be the average of the periodogram matrices of the subsamples at frequency $\lambda_{j,b}$.

**Step II.2** Define the set of frequency domain residual matrices

$$U_t(\lambda_{j,b}) = \tilde{f}^{-1/2}(\lambda_{j,b})I_t(\lambda_{j,b})\tilde{f}^{-1/2}(\lambda_{j,b}),$$

where $j = 1, 2, \ldots, [b/2]$, $t = 1, 2, \ldots, n - b + 1$ and $A^{-1/2}$ denotes the square root of the inverse matrix $A^{-1}$, i.e., $A^{-1/2}A^{-1/2} = A^{-1}$. 

Step II.3 Let \( k = [n/b] \) and generate i.i.d. bootstrap random variables \( i_1, i_2, \ldots, i_k \) with a discrete uniform distribution on the set \( \{1, 2, \ldots, n - b + 1\} \). Let

\[
I^+(\lambda_{j,b}) = \frac{1}{k} \sum_{\ell=1}^{k} I^+_{i_\ell}(\lambda_{j,b}),
\]

where \( I^+_{i_\ell}(\lambda_{j,b}) = \hat{F}^{1/2}(\lambda_{j,b}) U_{i_\ell}(\lambda_{j,b}) \hat{F}^{1/2}(\lambda_{j,b}) \). We write \( I^+_{i_\ell}(\lambda_{j,b}) \) for \( I^+_{i_\ell}(\lambda_{j,b}) \).

The entries of the pseudo periodogram matrices \( I^+(\lambda_{j,b}) \) are denoted \( I^+_{rs}(\lambda_{j,b}) \).

Step II.4 Let

\[
V^+_n = \sqrt{kb} \left( M_G(b) (\varphi_j, I^+_{r_j,s_j}) - M_G(b) (\hat{\varphi}_j, \hat{f}_{r_j,s_j}) \right), j = 1, 2, \ldots, J
\]

be the analogue of \( V_n \) based on \( I^+ \).

Step III.1 Let

\[
G^*_n = E^* \left[ \begin{pmatrix} \text{Re}(V^+_n) \\ \text{Im}(V^+_n) \end{pmatrix} \begin{pmatrix} \text{Re}(V^+_n)^\top, \text{Im}(V^+_n)^\top \end{pmatrix} \right],
\]

\[
G^+_n = E^* \left[ \begin{pmatrix} \text{Re}(V^+_n) \\ \text{Im}(V^+_n) \end{pmatrix} \begin{pmatrix} \text{Re}(V^+_n)^\top, \text{Im}(V^+_n)^\top \end{pmatrix} \right].
\]

Furthermore, denote by \( \Sigma^+_1, n \) the \( J \times J \) matrix with \( (j,k) \)-th element given by

\[
\sigma^+_{jk} = 4\pi^2 \frac{[b/2]}{b} \sum_{l=-[b/2]}^{[b/2]} \varphi_j(\lambda_{l,b}) \varphi_k(\lambda_{l,b}) S_{r_j,s_j,r_k s_k}(\lambda_{l,b})
\]

\[
+ 4\pi^2 \frac{[b/2]}{b} \sum_{l=-[b/2]}^{[b/2]} \varphi_j(\lambda_{l,b}) \varphi_k(-\lambda_{l,b}) S_{r_j,s_j,r_k s_k}(\lambda_{l,b}),
\]

and \( \Gamma^+_1, n \) the \( J \times J \) matrix with \( (j,k) \)-th element given by

\[
\gamma^+_{jk} = 4\pi^2 \frac{[b/2]}{b} \sum_{l=-[b/2]}^{[b/2]} \varphi_j(\lambda_{l,b}) \varphi_k(-\lambda_{l,b}) S_{r_j,s_j,r_k s_k}(\lambda_{l,b})
\]

\[
+ 4\pi^2 \frac{[b/2]}{b} \sum_{l=-[b/2]}^{[b/2]} \varphi_j(\lambda_{l,b}) \varphi_k(\lambda_{l,b}) S_{r_j,s_j,r_k s_k}(\lambda_{l,b}),
\]

where

\[
S_{rsuw}(\lambda) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} (\hat{I}_{t,r}(\lambda) - \hat{f}_{r}(\lambda)) (\hat{I}_{t,u}(\lambda) - \hat{f}_{u}(\lambda)),
\]
and $\tilde{I}_{t,rs}(\lambda)$ is the $(r,s)$-th element of the matrix 

$$\tilde{I}_t(\lambda) = \hat{f}^{1/2}(\lambda)U_t(\lambda)\hat{f}^{1/2}(\lambda).$$

Finally, define the matrix 

$$C^+_n = \begin{pmatrix}
\frac{1}{2}(\text{Re}(\Sigma^+_{1,n}) + \text{Re}(\Gamma^+_{1,n})) & \frac{1}{2}(\text{Im}(\Sigma^+_{1,n}) + \text{Im}(\Gamma^+_{1,n})) \\
\frac{1}{2}(\text{Im}(\Sigma^+_{1,n}) + \text{Im}(\Gamma^+_{1,n})) & \frac{1}{2}(\text{Re}(\Sigma^+_{1,n}) - \text{Re}(\Gamma^+_{1,n}))
\end{pmatrix}.
$$

**Step III.2** Calculate 

$$G^\circ_n = G^+_n + (G^+_n - C^+_n)$$

and 

$$\begin{pmatrix}
\text{Re}(V^\circ_n) \\
\text{Im}(V^\circ_n)
\end{pmatrix} = (G^+_n)^{1/2}(G^+_n)^{-1/2} \begin{pmatrix}
\text{Re}(V^\circ_n) \\
\text{Im}(V^\circ_n)
\end{pmatrix}.$$  \tag{3.1}

**Step III.3** Approximate the distribution of $V_n$ by that of 

$$V^\circ_n = \text{Re}(V^\circ_n) + i\text{Im}(V^\circ_n).$$

The following series of remarks clarifies several aspects of the above bootstrap procedure.

**Remark 3.1.** In Step I.2, the problem of generating a complex normal random vector $d^*(\lambda_{j,n})$ can in practice be reduced to that of generating a real normal random vector (which is usually a pre-implemented routine). Since $\mathcal{N}_m(0,\hat{f}(\lambda_{j,n})^{-1}0)$ is a circularly symmetric complex normal distribution, one may generate 

$$\begin{pmatrix}
\text{Re}(d^*(\lambda_{j,n})) \\
\text{Im}(d^*(\lambda_{j,n}))
\end{pmatrix} \sim \mathcal{N}_{2m} \begin{pmatrix}
0 \\
0
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
\text{Re}(\hat{f}(\lambda_{j,n})) \\
\text{Im}(\hat{f}(\lambda_{j,n}))
\end{pmatrix},$$

and then set $d^*(\lambda_{j,n}) = \text{Re}(d^*(\lambda_{j,n})) + i \cdot \text{Im}(d^*(\lambda_{j,n})).$

**Remark 3.2.**

(i) As already mentioned, the pseudo periodogram matrices $\Gamma(\lambda_{j,n})$ in Step I.2 are generated using the fact that the $m$-dimensional vector of finite Fourier transforms converges towards a circular symmetric complex normal distribution. Notice that the $\Gamma(\lambda_{j,n})$ generated in this step are independent across frequencies. Therefore, the integrated periodogram statistic $V^\circ_n$ obtained in Step I.3 and based on these pseudo periodogram matrices, is only able to imitate the parts $\Sigma_1$ and $\Gamma_1$ of the limiting variance and relation matrices $\Sigma$ and $\Gamma$, respectively. This will be proven in Lemma 3.9 (i). Recall that $\Sigma_1$ and $\Gamma_1$ only depend on the spectral density matrix $f$ of the underlying process.
(ii) In Step II.1 the periodogram matrices \( \mathbf{I}_t(\lambda_{j,b}) \) of the subsamples of length \( b \) are used. In Step II.2 frequency domain residual matrices \( \mathbf{U}_t(\lambda_{j,b}) \) are defined which are i.i.d. resampled in Step II.3 to obtain the pseudo periodogram matrix \( \mathbf{I}^+(\lambda_{j,b}) \). Notice that the latter matrix is an average of \( k \) independent matrices \( \mathbf{I}^+_t(\lambda_{j,b}) \) calculated using \( k \) randomly selected residual matrices \( \mathbf{U}_t(\lambda_{j,b}) \) and after pre- and post-multiplying them with the square root of the estimated spectral density matrix \( \hat{\mathbf{f}}(\lambda_{j,b}) \). Observe that all quantities in Step II are calculated at the Fourier frequencies \( \lambda_{j,b} = 2 \pi j/b, j \in \mathcal{G}(b) \) corresponding to the length \( b \) of the subsamples.

(iii) The integrated periodogram statistic \( \mathbf{V}^+_n \) based on \( \mathbf{I}^+(\lambda_{j,b}) \) is used to imitate the missing terms \( \mathbf{\Sigma}_2 \) and \( \mathbf{\Gamma}_2 \) which depend on the fourth order moment structure of the process. To do this appropriately, notice first that the pseudo statistic \( \mathbf{V}^+_n \) imitates asymptotically correct the entire covariance and relation matrices \( \mathbf{\Sigma} \) and \( \mathbf{\Gamma} \) as will be proven in Lemma 3.9 (ii). Since we use \( \mathbf{V}^+_n \) generated in Step I to imitate the distribution of \( \mathbf{V}_n \) and the matrices \( \mathbf{\Sigma}_1 \) and \( \mathbf{\Gamma}_1 \), we have to subtract from the covariance and relation matrix of \( \mathbf{V}^+_n \) the corresponding parts \( \mathbf{\Sigma}^+_{1,n} \) and \( \mathbf{\Gamma}^+_{1,n} \) so that only the desired estimators of the components \( \mathbf{\Sigma}_2 \) and \( \mathbf{\Gamma}_2 \) are left. This is done in Step III.1 and Step III.2. In particular, in Step III.1 the elements \( \sigma^+_{jk} \) and \( c^+_{jk} \) of the matrices \( \mathbf{\Sigma}^+_{1,n} \) and \( \mathbf{\Gamma}^+_{1,n} \) are explicitly calculated and the corresponding matrix \( \mathbf{C}^+_{1,n} \) is obtained. The latter matrix is then subtracted from the matrix \( \mathbf{G}^+_{1,n} \), so that the obtained matrix \( \mathbf{G}^+_{1,n} - \mathbf{C}^+_{1,n} \) contains the elements of the covariance and relation matrix of \( \mathbf{V}^+_n \) which only depend on the fourth-order structure of the process, that is the parts \( \mathbf{\Sigma}^+_{2,n} \) and \( \mathbf{\Gamma}^+_{2,n} \). These are used to estimate \( \mathbf{\Sigma}_2 \) and \( \mathbf{\Gamma}_2 \). Now, adding to this difference the matrix \( \mathbf{G}^+_{2,n} \) which correctly imitates both parts of \( \mathbf{\Sigma} \) and \( \mathbf{\Gamma} \), compare also Lemma 3.9 (ii). We stress here the fact that in \( \mathbf{G}^+_{2,n} \), the parts corresponding to \( \mathbf{\Sigma}_1 \) and \( \mathbf{\Gamma}_1 \) are contributed by the bootstrap procedure based on the asymptotic Gaussianity of the finite Fourier transforms, that is by \( \mathbf{\Sigma}^+_{1,n} \) and \( \mathbf{\Gamma}^+_{1,n} \), while the parts \( \mathbf{\Sigma}_2 \) and \( \mathbf{\Gamma}_2 \) by the convolved bootstrap procedure, that is by \( \mathbf{\Sigma}^+_{2,n} \) and \( \mathbf{\Gamma}^+_{2,n} \).

(iv) In Step III.2 the matrix \( \mathbf{G}^+_{2,n} \) is used in equation (3.1) to appropriately rescale the bootstrap vector \( \mathbf{V}^+_n \). The resulting bootstrap complex vector \( \mathbf{V}^\circ_n \) is finally used in Step III.3 to approximate the distribution of the complex vector \( \mathbf{V}^+_n \).

**Remark 3.3.** In the definition of the frequency domain residual matrices \( \mathbf{U}_t(\lambda_{j,b}) \) in Step II.2, the pre- and post-multiplication with the matrix \( \hat{\mathbf{f}}^{-1/2}(\lambda_{j,b}) \) ensures that the resampled residual matrices \( \mathbf{U}_t(\lambda_{j,b}) \) satisfy

\[
E^\circ(\mathbf{U}_t(\lambda_{j,b})) = \hat{\mathbf{f}}^{-1/2}(\lambda_{j,b}) \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{I}_t(\lambda_{j,b}) \hat{\mathbf{f}}^{-1/2}(\lambda_{j,b}) = \mathbf{I}_d_m,
\]

where \( \mathbf{I}_d_m \) denotes the \( m \times m \) unit matrix. Therefore,

\[
E^\circ(\mathbf{I}^+(\lambda_{j,b})) = E^\circ(\mathbf{I}^+_t(\lambda_{j,b})) = \hat{\mathbf{f}}^{1/2}(\lambda_{j,b}) \cdot \mathbf{I}_d_m \cdot \hat{\mathbf{f}}^{1/2}(\lambda_{j,b}) = \hat{\mathbf{f}}(\lambda_{j,b}).
\]
Furthermore, by the consistency of \( \hat{f}(\lambda) \) as an estimator of \( f(\lambda) \), see the proof of Lemma 3.9, it holds true that for any fixed frequency \( \lambda \in (0, \pi) \), \( U_t(\lambda) = \hat{f}^{-1/2}(\lambda) I_t(\lambda) f^{-1/2}(\lambda) \stackrel{D}{\rightarrow} U(\lambda) \), as \( b \to \infty \), where \( U(\lambda) \) has the complex Wishart distribution of dimension \( m \) and one degree of freedom, i.e., \( U(\lambda) \sim W_m^C(1, \text{Id}_m) \); see Brillinger (1981), Section 4.2. Recall that if \( U(\lambda) \sim W_m^C(1, \text{Id}_m) \) then \( \text{E}(U(\lambda)) = \text{Id}_m \) and \( \text{Cov}(U_{jk}(\lambda), U_{lm}(\lambda)) = \delta_{j,l} \delta_{k,m} \) for any two elements \( U_{jk}(\lambda) \) and \( U_{lm}(\lambda) \) of the matrix \( U(\lambda) \).

**Remark 3.4.** The terms \( S_{rsuw}(\lambda) \) which appear in the expressions of \( \sigma_{jk}^+ \) and \( c_{jk}^+ \) in Step III.1 are obtained by evaluating the covariance expressions \( \text{Cov}^+(I_{t,rs}(\lambda_{t,\ell}), I_{t,rs}(\lambda_{t,\ell})) \) where \( I_{t,rs}(\lambda_{t,\ell}) \) denotes the \((r,s)\)-th element of \( I_{t}(\lambda_{t,\ell}) \). To elaborate on the parameters they are estimating, consider the simplified form,

\[
\tilde{S}_{rsuw}(\lambda) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \left( I_{t,rs}(\lambda) - f_{rs}(\lambda) \right) \left( I_{t,ws}(\lambda) - f_{ws}(\lambda) \right),
\]

obtained after replacing estimated by true quantities and where \( I_{t,rs}(\lambda) \) denotes the \((r,s)\)-th of the matrix \( I_{t}(\lambda) \) defined in Step II.1. See also the proof of Lemma 3.9 (ii). It can then be shown that

\[
\tilde{S}_{rsuw}(\lambda,j,b) = f_{rw}(\lambda,j,b)f_{su}(\lambda,j,b) + o_P(1) = f_{rw}(\lambda,j,b)f_{ws}(\lambda,j,b) + o_P(1). \tag{3.2}
\]

To see this observe that

\[
\text{E}(\tilde{S}_{rsuw}(\lambda,j,b)) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \text{E} \left[ \left( I_{t,rs}(\lambda,j,b) - f_{rs}(\lambda,j,b) \right) \left( I_{t,ws}(\lambda,j,b) - f_{ws}(\lambda,j,b) \right) \right]
\]

\[
= \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \text{Cov} \left( I_{t,rs}(\lambda,j,b), I_{t,ws}(\lambda,j,b) \right) + O(b^{-1})
\]

\[
= f_{rw}(\lambda,j,b)f_{su}(\lambda,j,b) + o(1),
\]

where the second equality follows because \( \text{E}(I_{t,rs}(\lambda,j,b)) = f_{rs}(\lambda,j,b) + O(b^{-1}) \) with the \( O(b^{-1}) \) term independent of \( t \), and the last equality follows using Lemma 2.1. By Lemma 3.8 below, it can further be shown that \( \text{Var}(\tilde{S}_{rsuw}(\lambda,j,b)) \to 0 \) (see also the proof of Lemma 3.9), which justifies expression (3.2).

**Remark 3.5.** Notice that if the limiting distribution of \( V_n \) is real-valued, i.e. if \( V \sim \mathcal{N}(0, \Sigma, \Sigma) \), then Step III.1 and Step III.2 simplify. In particular, in this case one can set the matrices \( \Sigma^+ \) and \( \Sigma^+ \) as real-valued and \( J \times J \) dimensional according to \( \Sigma^+_n = E^+(V_n^+(V_n^+)^\top) \) and \( \Sigma^+ = E^+(V_n^+(V_n^+)^\top) \). Consequently, the matrix \( \Sigma^+ \) can then also be set as real-valued and \( J \times J \) dimensional according to \( \Sigma^+ = \Sigma^+_1 \).

### 3.2 Smooth functions of integrated periodograms

The MFHB bootstrap procedure proposed – appropriately modified – also can be applied to estimate the distribution of statistics which are functions of inte-
grated periodograms such as, for instance, sample cross-correlations. To elaborate, suppose that
\[ g = (g_1, g_2, \ldots, g_L): C^J \rightarrow C^L \]  
(3.3)
is some (smooth) function and that the statistic of interest is given by \( \sqrt{n}(R_n - R) \) where \( R_n = g(M_n) \) and \( R = g(M) \) for the vector of integrated periodograms \( M_n \) and spectral means \( M \) given in (2.3) and (2.4). Sample cross-correlations can be expressed in this way as can be seen from Example 2.3.

Our bootstrap procedure from the previous section can be adapted to approximate the distribution of \( \sqrt{n}(R_n - R) \). We first impose some smoothness assumption on the function \( g \). To do so, we can of course interpret \( g \) as a function defined on \( R^2 \) via the identification \( g(z) = g(\text{Re}(z), \text{Im}(z)) \) \( \forall z \in C^J \).

Splitting up real and imaginary parts also for the values of \( g \) leads to the accompanying function \( \tilde{g}: R^2 \rightarrow R^2_L \) given by
\[ \tilde{g}(x) := \left( \frac{\text{Re}(g(x))}{\text{Im}(g(x))} \right) \quad \forall x \in R^2 \].

**Assumption 4.** The function \( \tilde{g}: R^2 \rightarrow R^2_L \) is continuously differentiable (in the real sense) in some neighbourhood around \( (\text{Re}(M)^T, \text{Im}(M)^T)^T \) with Jacobi matrix \( J_{\tilde{g}}((\text{Re}(M)^T, \text{Im}(M)^T)^T) \).

Note that – although considering complex-valued random variables and complex functions \( g \) – we require only real differentiability of the accompanying function \( \tilde{g} \). This is a much less restrictive condition than assuming complex differentiability of \( g \). We define the random vectors
\[ \tilde{R}_n := \left( \frac{\text{Re}(R_n)}{\text{Im}(R_n)} \right), \quad \tilde{R} := \left( \frac{\text{Re}(R)}{\text{Im}(R)} \right). \]

Now, applying the delta method to (2.6), respectively (2.14), we get the limiting result
\[ \sqrt{n}(\tilde{R}_n - \tilde{R}) \xrightarrow{d} J_{\tilde{g}}((\text{Re}(M)^T, \text{Im}(M)^T)^T) N_{2J}(0, \Sigma_{\tilde{R}}, \Gamma_{\tilde{R}}). \]  
(3.4)

From the continuous mapping theorem, applied for \( h(x_1, x_2) := x_1 + i \cdot x_2, \) \( x_1, x_2 \in R^L \), it follows
\[ \sqrt{n}(R_n - R) \xrightarrow{d} W \sim N_L^C(0, \Sigma_R, \Gamma_R), \]  
(3.5)
for suitable matrices \( \Sigma_R \) and \( \Gamma_R \) which can be obtained from (3.4) (the precise form of \( \Sigma_R \) and \( \Gamma_R \) is not needed for the ensuing bootstrap algorithm and its validity results).

We propose the following MFHB approximation of the distribution of \( \sqrt{n}(R_n - R) \).
Step 1  Apply Steps I, II, III.1, and III.2 of the algorithm from Section 3.1 to obtain the matrix $G_n^*$ and the pseudo periodogram matrices $I^*(\lambda_{j,n})$, $j \in \mathcal{G}(n)$.

Step II  Let

$$M_n^* := \left( M_{\mathcal{G}(n)}(\varphi_j, I_{r_j,s_j}^*), j = 1, 2, \ldots, J \right)^\top,$$

$$\tilde{M}_n := \left( M_{\mathcal{G}(n)}(\varphi_j, \tilde{f}_{r_j,s_j}), j = 1, 2, \ldots, J \right)^\top,$$

and

$$\left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) := \sqrt{n} \left[ \hat{g} \left( \text{Re}(M_n^*) \right) - \tilde{g} \left( \text{Im}(\tilde{M}_n) \right) \right].$$

Step III  Let

$$\tilde{G}_n^* := E^* \left[ \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right] - E^* \left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) \cdot E^* \left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right)^\top,$$

and

$$\tilde{G}_n^\circ := J \tilde{g} \left( \text{Re}(\tilde{M}_n) \right) \times G_n^\circ \times J \tilde{g} \left( \text{Im}(\tilde{M}_n) \right)^\top.$$

Step IV  Calculate

$$\left( \begin{array}{c}
\text{Re}(W_n^\circ) \\
\text{Im}(W_n^\circ)
\end{array} \right) := \left( \tilde{G}_n^\circ \right)^{1/2} \left( \tilde{G}_n^* \right)^{-1/2} \left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right).$$

Step V  Approximate the distribution of $\sqrt{n}(R_n - \bar{R})$ by that of

$$W_n^\circ := \text{Re}(W_n^\circ) + i \text{Im}(W_n^\circ).$$

Remark 3.6.

(i) The algorithm above works on real and imaginary parts separately. This has the advantage that we do not have to impose complex differentiability assumptions on the function $g$; it suffices to assume real differentiability of $\tilde{g}$.

(ii) In Step II the bootstrap vector $(\text{Re}(W_n^*), \text{Im}(W_n^*)^\top)^\top$ imitates the structure of

$$\sqrt{n}(\bar{R}_n - \bar{R}) = \sqrt{n} \left[ \hat{g} \left( \text{Re}(M_n) \right) - \tilde{g} \left( \text{Re}(M) \right) \right].$$

But as a comparison of (5.12) in the proof of Theorem 3.11 with (3.4) shows, the limiting distribution of $\sqrt{n}(\bar{R}_n - \bar{R})$ is only partially captured.
by \((\text{Re}(W_n^*)^T, \text{Im}(W_n^*)^T)^T\): While the factor determined by the Jacobi matrix is established properly, the matrix \(G\) entering the covariance structure of the normal distribution is not captured. This is corrected in Steps III and IV where the bootstrap random vectors are first standardized via multiplication with \((\tilde{G}_n^*)^{-1/2}\). Then the proper variance is established by \((\tilde{G}_n^*)^{1/2}\) which simultaneously approximates both \(G\) and the Jacobi matrix \(J_{\tilde{g}}((\text{Re}(M))^T, \text{Im}(M))^T)^T\).

**Example 3.7.** One very important statistic of interest that can be written as a smooth function of integrated periodograms is the sample cross-correlation. Examples 2.3 and 2.4 show how the function \(g\) can be defined such that \(\sqrt{n}(\hat{\rho}_{rs}(h) - \rho_{rs}(h))\) takes the form \(\sqrt{n}(R_n - R)\). The limiting variance of this expression in general takes the form indicated in (3.5), and for this particular example it is given by the expression \(\tau^2\) from Example 2.4. Since the function \(g\) fulfills Assumption 4, our bootstrap algorithm \(W_n^*\) is asymptotically valid for this statistic (as is proven in Theorem 3.11), and successfully imitates the rather complicated expression \(\tau^2\).

### 3.3. Bootstrap validity

We need the following consistency assumption for the spectral density estimator \(\hat{f}\) used in Step I of the bootstrap procedure in Section 3.1.

**Assumption 5.** The spectral density estimator \(\hat{f}\) is Hermitian, positive definite and satisfies

\[
\sup_{\lambda \in [-\pi, \pi]} \|\hat{f}(\lambda) - f(\lambda)\|_F \overset{P}{\to} 0.
\]

We refer to Wu and Zaffaroni (2018) for estimators of the spectral density matrix and for general classes of multivariate processes for which the above assumption is satisfied. We next establish the following two lemmas, the proofs of which are given in the Supplementary Material.

**Lemma 3.8.** Let Assumption 1 be fulfilled. Denote by \(I_{r,s}(\lambda)\) the \((r,s)\)-th element of the periodogram matrix \(I_1(\lambda)\) based on the subsample \(X(t), X(t + 1), \ldots, X(t + b - 1)\). Then,

(i) \[
\sum_{\ell \in G(b)} \left| f_{rs}(\lambda_{\ell,b}) - E(I_{1;r,s}(\lambda_{\ell,b})) \right| = O_P(\sqrt{b^5/(n - b + 1)}) ,
\]

(ii) \[
\sum_{\ell_1, \ell_2 \in G(b)} \left| \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} I_{t;r,s}(\lambda_{\ell_1,b})I_{t,r_k}\lambda_{\ell_2,b}) - E(I_{1;r,s}(\lambda_{\ell_1,b})I_{1;r_k}\lambda_{\ell_2,b}) \right| = O_P(\sqrt{b^5/(n - b + 1)}) ,
\]
and
\[
\sum_{\ell \in \mathcal{G}(b)} \left| \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} I_{1: r_j s_j} (\lambda_{t,b}) I_{1: r_k s_k} (\lambda_{t,b}) \right| = \mathcal{O}_p \left( \sqrt{b^3/(n-b+1)} \right).
\]
where the \( \mathcal{O}_p \) terms are uniformly in \( r, s \), respectively \( r_j, s_j, r_k, s_k \).

In order to formulate our theoretical results dealing with the asymptotic properties of the bootstrap approximations proposed, we state the following assumption which summarizes our requirements regarding the behavior of the subsampling parameter \( b \).

**Assumption 6.** \( b = b(n) \to \infty \) as \( n \to \infty \) such that \( b^3/(n-b+1) \to 0 \).

The following lemma investigates the asymptotic properties of the covariance and relation matrices of the random vectors \( V_n^* \) and \( V_n^+ \) generated in the MFHB procedure and derives the limiting distribution of \( V_n^* \).

**Lemma 3.9.** If Assumptions 1, 2, 3, 5 and 6 are satisfied, then,

(i) \( \text{Cov}^* (V_n^*) \xrightarrow{p} \Sigma_1 \) and \( E^* (V_n^* (V_n^*)^\top) \xrightarrow{p} \Gamma_1 \).

(ii) \( G_n^+ - C_n^+ \xrightarrow{p} \left( \begin{array}{cc} \frac{1}{2} (\text{Re}(\Sigma_2) + \text{Re}(\Gamma_2)) & \frac{1}{2} (\text{Im}(\Sigma_2) + \text{Im}(\Gamma_2)) \\ \frac{1}{2} (\text{Im}(\Sigma_2) + \text{Im}(\Gamma_2)) & \frac{1}{2} (\text{Re}(\Sigma_2) - \text{Re}(\Gamma_2)) \end{array} \right) \).

(iii) \( V_n^* \xrightarrow{d} N(J, \Sigma_1, \Gamma_1) \) in \( P \)-probability.

Lemma 3.9 leads to the following result which establishes consistency of the MFHB procedure in estimating the distribution and the second-order moments of the random vector \( V_n \).

**Theorem 3.10.** If Assumptions 1, 2, 3, 5 and 6 are satisfied, then,

(i) \( \text{Cov}^* (V_n^0) \xrightarrow{p} \Sigma_0 \) and \( E^* (V_n^0 (V_n^0)^\top) \xrightarrow{p} \Gamma_0 \).

(ii) \( V_n^0 \xrightarrow{d} V_0 \) in \( P \)-probability.

The next result establishes validity of the MFHB procedure for smooth functions of integrated periodograms.

**Theorem 3.11.** Let Assumptions 1 to 6 be satisfied. Then, as \( n \to \infty \),

\( W_n^0 \xrightarrow{d} W \) in \( P \)-probability, where \( W \) is the limiting distribution of \( \sqrt{n} (R_n - R) \) given in (3.5).
4. Simulations

4.1. Choice of the MFHB parameters

The practical implementation of the MFHB procedure requires the choice of two parameters. The first is the spectral density estimator \( \hat{f} \) and the second the subsampling parameter \( b \). Assumption 5 and Assumption 6 state our general requirements on these parameters focusing on the asymptotic properties they have to satisfy in order for the proposed bootstrap procedure to be consistent. Certainly, the choice of these parameters for a given sample size \( n \) is an important venue of feature research. In the following we discuss some rather practical rules on how to choose these parameters.

Regarding the spectral density estimator \( \hat{f} \), a variety of estimators exists which can be used in our procedure; see Brillinger (1981). As a simple approach, we use in the following kernel estimators obtained by locally averaging the periodogram matrix over frequencies close to the frequency of interest, i.e., \( \hat{f}(\lambda) = (nh)^{-1} \sum_j K((\lambda - \lambda_{j,n})/h)I_n(\lambda_{j,n}) \). \( K \) denotes the kernel function which determines the weights assigned to the ordinates of the periodogram matrix, while \( h \) is the bandwidth that controls the number of periodogram ordinates effectively taken into account in order to obtain the kernel estimator \( \hat{f} \). To select the parameter \( h \) in practice, cross-validation type approaches have been proposed and investigated in the literature which also can be applied in our setting; see Robinson (1991) for details.

For the subsampling parameter \( b \), Assumption 6 solely states the required conditions on the rate at which this parameter has to increase to infinity with respect to the sample size \( n \) in order to ensure consistency of the MFHB procedure. Our simulation experience with the choice of this parameter shows that the results obtained are not very sensitive with respect to the choice of \( b \), provided that this parameter is not chosen too small. This motivates the suggestion of the following practical rule for selecting \( b \). Select this parameter as the smallest integer which is larger or equal to \( 3 \cdot n^{0.30} \). This rule satisfies the requirements of Assumption 6 and at the same time delivers a value of \( b \) which is large enough for the MFHB procedure to perform well in practice.

Notice that the numerical results presented in the next section are reported for different combinations of bandwidth and block size parameters, \( h \) and \( b \). On the one hand this avoids a further increase of the computational burden caused by a cross-validation type choice of the bandwidth \( h \). On the other hand it allows us to investigate the sensitivity of the bootstrap estimates with respect to different choices of the parameters involved.

4.2. Numerical results

In this section we investigate the finite sample performance of the MFHB procedure and compare it with that of the time domain moving block bootstrap (MBB). Note that, in view of our discussion in the Introduction, popular time
domain bootstrap methods other than the MBB (and its variations) can not be considered as competitors for our MFHB since these procedures are asymptotically non-valid for a large class of statistics in the context of multivariate linear and non-linear time series. We consider time series of length \( n = 100 \) stemming from the following two bivariate processes considered in Tsay (2014). The first is a VAR(1) process driven by i.i.d. innovations, i.e.,

**Model I:** \( X(t) = \Phi X(t-1) + e(t), \quad \Phi = \begin{pmatrix} 0.8 & 0.4 \\ -0.3 & 0.6 \end{pmatrix}, \)

where \( e_t \sim \mathcal{N}_2(0, S_e) \) and \( S_e = (\sigma_{i,j})_{i,j=1,2} \), with \( \sigma_{1,1} = 2.0, \sigma_{1,2} = 0.5 \) and \( \sigma_{2,2} = 1 \). The second model is a bivariate VARMA(2,1) process the innovations of which follow a bivariate GARCH-type process, that is,

**Model II:** \( X(t) = \Phi_1 X(t-1) + \Phi_2 X(t-2) + \Theta u(t-1) + u(t), \)

with parameter matrices

\[ \Phi_1 = \begin{pmatrix} 0.816 & -0.623 \\ -1.116 & 1.074 \end{pmatrix}, \]

\[ \Phi_2 = \begin{pmatrix} -0.643 & 0.592 \\ 0.615 & -0.133 \end{pmatrix} \]

and \( \Theta = \begin{pmatrix} 0 & -1.248 \\ -0.801 & 0 \end{pmatrix}. \)

Furthermore, the innovations \( u(t) \) are generated as \( u(t) = S_t^{1/2} e(t) \), where the \( e(t) \)'s are i.i.d. \( \mathcal{N}_2(0, \text{Id}_2) \) distributed, and the volatility matrix \( S_t \) evolves according to a BEKK(1,1) model, i.e. \( S_t = A_0 A_0^\top + A_1 u(t-1) u(t-1)^\top A_1^\top + B_1 S_{t-1} B_1^\top \), where

\[ A_0 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.15 & 0.20 \\ 0.06 & 0.40 \end{pmatrix} \] and \( B_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}. \)

Observe that for the VAR(1) model with Gaussian innovations the fourth-order cumulant spectral densities equal zero, so the expressions of the covariance and relation matrices \( \Sigma \) and \( \Gamma \) simplify. This is not the case for the VARMA(2,1) process which is nonlinear due to the BEKK(1,1) generated innovations. The parameter matrices of this BEKK(1,1) model are very close to those of the same model fitted to the IBM stock and S&P composite index in Tsay (2014), p. 418, Table 7.3. See also Francq and Zakoian (2016), Section 6, for a similar parametrization.

We consider the problem of estimating the standard deviation of the cross-correlation estimates \( \hat{\rho}(h) \) for the values \( h = -1,0, +1 \). Recall that the estimators considered also can be written as functions of integrated periodograms; see Example 2.3. We have generated 10,000 replications of both models in order to estimate the exact standard deviation of the sample cross-correlations considered. As already mentioned, the spectral density estimator \( \hat{f} \) used in the MFHB procedure is a kernel estimator obtained via smoothing the periodogram matrix with bandwidth \( h \) and using the Bartlett-Priestley kernel; see Priestley (1981). Furthermore, and in order to see the sensitivity of the bootstrap methods
compared with respect to the choice of the bootstrap parameters, the MFHB procedure has been applied using different choices of the bandwidth $h$ and of the subsampling parameter $b$. The same values of $b$ have also been used as block sizes in the MBB procedure. We also report results for $b = 12$ which is the value of the subsampling parameter selected according to the rule proposed in Section 4.1.

Table 1 presents the results for both models considered. The results reported in this table are based on $R = 500$ repetitions where $B = 300$ bootstrap replications have been applied for each repetition. As this table shows, the MFHB procedure performs quite well for both models considered and outperforms the MBB. In particular, comparing the performance of both bootstrap procedures for the same subsampling parameter, respectively block size $b$, the MFHB mean square errors are in almost all cases considered, and independent of the choice of $h$, lower than those of the MBB procedure. Furthermore, the MFHB estimates seem to be less sensitive with respect to the choice of the subsampling parameter $b$ than the MBB procedure is with respect to the choice of the block size $b$.

5. Appendix: Proofs

Proof of Lemma 3.9 (i): The $(j,k)$-th entry of the covariance matrix $\text{Cov}^*(\mathbf{V}_n^*)$ is given by

$$\text{Cov}^*(V_{n,j}^*, V_{n,k}^*) = \text{Cov}^* \left( \frac{2\pi}{n} \sum_{l_1 \in G(n)} \varphi_j(\lambda_{l_1, n}) (I_{r_j s_j}^* (\lambda_{l_1, n}) - \hat{f}_{r_j s_j}(\lambda_{l_1, n})), \frac{2\pi}{n} \sum_{l_2 \in G(n)} \varphi_k(\lambda_{l_2, n}) (I_{r_k s_k}^* (\lambda_{l_2, n}) - \hat{f}_{r_k s_k}(\lambda_{l_2, n})) \right)$$

which equals

$$\frac{4\pi^2}{n} \sum_{l_1, l_2 \in G(n)} \varphi_j(\lambda_{l_1, n}) \varphi_k(\lambda_{l_2, n}) \text{Cov}^* (I_{r_j s_j}^* (\lambda_{l_1, n}), I_{r_k s_k}^* (\lambda_{l_2, n})). \quad (5.1)$$

The last expression can be decomposed, using $\lambda_{-l,n} = -\lambda_{l,n}$ and $I_{r_k}^*(-\lambda_{l,n}) = I_{r_k}^*(\lambda_{l,n})$, into

$$\frac{4\pi^2}{n} \sum_{l_1, l_2 = 1}^{N} \left\{ \varphi_j(\lambda_{l_1, n}) \varphi_k(\lambda_{l_2, n}) \text{Cov}^* (I_{r_j s_j}^* (\lambda_{l_1, n}), I_{r_k s_k}^* (\lambda_{l_2, n})) + \varphi_j(-\lambda_{l_1, n}) \varphi_k(-\lambda_{l_2, n}) \text{Cov}^* (I_{s_j r_j}^* (\lambda_{l_1, n}), I_{s_k r_k}^* (\lambda_{l_2, n})) + \varphi_j(-\lambda_{l_1, n}) \varphi_k(\lambda_{l_2, n}) \text{Cov}^* (I_{s_j r_j}^* (\lambda_{l_1, n}), I_{r_k s_k}^* (\lambda_{l_2, n})) + \varphi_j(\lambda_{l_1, n}) \varphi_k(-\lambda_{l_2, n}) \text{Cov}^* (I_{r_j s_j}^* (\lambda_{l_1, n}), I_{s_k r_k}^* (\lambda_{l_2, n})) \right\}. \quad (5.2)$$
For the first of the four similar summands in this expression we can calculate the following, where we used that $\Gamma^*(\lambda_{1,n})$ and $\Gamma^*(\lambda_{2,n})$ are independent for $l_1 \neq l_2$ and $1 \leq l_1, l_2 \leq N$, and that $\Gamma^*(\lambda_{1,n})$ has a complex Wishart $W^C_m(1, \hat{f}(\lambda_{1,n}))$ distribution (the covariance structure of which can be obtained from, e.g., Brillinger (1981), Section 4.2):

\[
\frac{4\pi^2}{n} \sum_{l_1,l_2=1}^N \varphi_j(\lambda_{l_1,n})\overline{\varphi_k(\lambda_{l_2,n})} \text{Cov}^*(I_{r_j s_j}^*(\lambda_{l_1,n}), I_{r_k s_k}^*(\lambda_{l_2,n}))
\]

\[
= \frac{4\pi^2}{n} \sum_{l_1}^N \varphi_j(\lambda_{l_1,n})\overline{\varphi_k(\lambda_{l_1,n})} \text{Cov}^*(I_{r_j s_j}^*(\lambda_{l_1,n}), I_{r_k s_k}^*(\lambda_{l_1,n}))
\]

\[
= \frac{4\pi^2}{n} \sum_{l_1}^N \varphi_j(\lambda_{l_1,n})\overline{\varphi_k(\lambda_{l_1,n})} \hat{f}_{r_j r_k}(\lambda_{l_1,n}) f_{s_j s_k}(\lambda_{l_1,n})
\]

\[
= \frac{4\pi^2}{n} \sum_{l_1}^N \varphi_j(\lambda_{l_1,n})\overline{\varphi_k(\lambda_{l_1,n})} (\hat{f}_{r_j r_k}(\lambda_{l_1,n}) f_{s_j s_k}(\lambda_{l_1,n}) - f_{r_j r_k}(\lambda_{l_1,n}) f_{s_j s_k}(\lambda_{l_1,n})),
\]

The second summand on the last right-hand side vanishes asymptotically due to Assumption 5 and $N = O(n)$. Since the first summand is a Riemann sum the last right-hand side converges in probability to

\[
2\pi \int_0^{\pi} \varphi_j(\lambda)\overline{\varphi_k(\lambda)} f_{r_j r_k}(\lambda) f_{s_j s_k}(\lambda) d\lambda.
\]

With analogous calculations the other three summands in (5.2) yield

\[
2\pi \int_0^{\pi} \varphi_j(\lambda)\overline{\varphi_k(\lambda)} f_{s_j s_k}(\lambda) f_{r_j r_k}(\lambda) d\lambda
\]

\[
+ 2\pi \int_0^{\pi} \varphi_j(\lambda)\overline{\varphi_k(\lambda)} f_{r_j s_k}(\lambda) f_{s_j r_k}(\lambda) d\lambda
\]

Substituting $\lambda$ with $-\lambda$ in the first and second of these three terms shows that the limit in probability of (5.2) is given by $\Sigma_{1:jk}$.

As for the relation matrix, observe that the $(j,k)$-th entry of $E^*(V_n^* V_n^{*\top})$ is given by

\[
\text{Cov}^*(V_{n,j}^*, V_{n,k}^*) = \frac{4\pi^2}{n} \sum_{l_1,l_2 \in G(n)} \varphi_j(\lambda_{l_1,n})\overline{\varphi_k(\lambda_{l_2,n})} \text{Cov}^*(I_{r_j s_j}^*(\lambda_{l_1,n}), I_{r_k s_k}^*(\lambda_{l_2,n})),
\]

which is (5.1) if one replaces $\varphi_k(\cdot)$ with $\overline{\varphi_k(\cdot)}$. Therefore, one can follow along the lines of the calculation for $\Sigma_{1:jk}$ above to see that the $(j,k)$-th entry of $E^*(V_n^* V_n^{*\top})$ converges in probability to $\Gamma_{1:jk}$. 

\[\square\]
Proof of Lemma 3.9 (ii): We first establish the uniform (over the Fourier frequencies) consistency of \( \tilde{f}^{-1/2} \) and \( \hat{f}^{1/2} \) as estimators of \( f^{-1/2} \) and \( f^{1/2} \), respectively. We first show that

\[
\max_{\ell \in G(b)} \| \tilde{f}^{-1}(\lambda_{\ell,b}) - f^{-1}(\lambda_{\ell,b}) \|_F \overset{P}{\to} 0.
\] (5.3)

For this notice first that

\[
\max_{\ell \in G(b)} \| \tilde{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F \overset{P}{\to} 0.
\] (5.4)

This follows since \( E(I_{1:rs}(\lambda_{\ell,b})) = f_{rs}(\lambda_{\ell,b}) + O(b^{-1}) \) uniformly in \( r, s \) and

\[
\max_{\ell \in G(b)} \| \tilde{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F \leq \max_{\ell \in G(b)} \| \tilde{f}(\lambda_{\ell,b}) - E\tilde{f}(\lambda_{\ell,b}) \|_F + \max_{\ell \in G(b)} \| E\tilde{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F,
\]

\[
= \max_{\ell \in G(b)} \left\{ \sum_{r=1}^{m} \sum_{s=1}^{m} |f_{rs}(\lambda_{\ell,b}) - E\tilde{f}_{rs}(\lambda_{\ell,b})|^2 \right\}^{1/2} + \max_{\ell \in G(b)} \left\{ \sum_{r=1}^{m} \sum_{s=1}^{m} |E(I_{1:rs}(\lambda_{\ell,b})) - f_{rs}(\lambda_{\ell,b})|^2 \right\}^{1/2}
\]

\[
\leq \sum_{r=1}^{m} \sum_{s=1}^{m} \left| f_{rs}(\lambda_{\ell,b}) - E\tilde{f}_{rs}(\lambda_{\ell,b}) \right| + O(b^{-1})
\]

\[
= O_P(\sqrt{b^3/(n - b + 1)}) + O(b^{-1}) \to 0,
\]

where the last convergence holds true because of Lemma 3.8 (i) and Assumption 6. Now to see (5.3) notice that \( \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) \|_F^2 = O(1) \) and that

\[
\max_{\ell \in G(b)} \| \tilde{f}^{-1}(\lambda_{\ell,b}) - f^{-1}(\lambda_{\ell,b}) \|_F = \max_{\ell \in G(b)} \| \tilde{f}^{-1}(\lambda_{\ell,b}) (f(\lambda_{\ell,b}) - \tilde{f}(\lambda_{\ell,b})) f^{-1}(\lambda_{\ell,b}) \|_F
\]

\[
\leq \max_{\ell \in G(b)} \| \tilde{f}^{-1}(\lambda_{\ell,b}) \|_F \max_{\ell \in G(b)} \| f(\lambda_{\ell,b}) - \tilde{f}(\lambda_{\ell,b}) \|_F \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) \|_F
\]

\[
\leq \left( \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) \|_F \right) + \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) - f^{-1}(\lambda_{\ell,b}) \|_F
\]

\[
\times \max_{\ell \in G(b)} \| \tilde{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) \|_F
\] (5.5)

By (5.4) and for \( n \) large enough such that

\[
\max_{\ell \in G(b)} \| \tilde{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F \max_{\ell \in G(b)} \| f^{-1}(\lambda_{\ell,b}) \|_F < 1,
\]

expression (5.5) leads to

\[
\max_{\ell \in G(b)} \| \tilde{f}^{-1}(\lambda_{\ell,b}) - f^{-1}(\lambda_{\ell,b}) \|_F
\]
the bound

\[ \text{Cov}^{+}(V^+_{n,i}, V^+_{n,j}) = \text{Cov}^{+}(\bar{V}_{n,i}, \bar{V}_{n,j}) + o_P(1), \]

where

\[ \bar{V}_{n,j} = \sqrt{kb} \frac{2\pi}{b} \sum_{i \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) (\bar{I}_{r,s,j}(\lambda_{\ell,b}) - \bar{f}_{r,j,s}(\lambda_{\ell,b})) \]

and \( \bar{I}_{r,s}(\lambda_{\ell,b}) \) denotes the \((r,s)\)-th element of the matrix \( k^{-1} \sum_{m=1}^{\ell} I_m(\lambda_{\ell,b}). \) Let \( I_{i_1,r,s}(\lambda_{\ell,b}) \) denote the \((r,s)\)-th element of \( I_{i_1}(\lambda_{\ell,b}). \) We then have

\[ \text{Cov}^{+}(\bar{V}_{n,i}, \bar{V}_{n,j}) = \frac{4\pi^2}{b} \sum_{i_1, i_2 \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) \varphi_k(\lambda_{\ell,b}) \text{Cov}^{+}(I_{i_1,r,s,j}(\lambda_{\ell,b}), I_{i_1,r,s,k}(\lambda_{\ell,b})) \]

Assumption 5 together with (5.3) implies that

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{1/2}(\lambda_{\ell,b}) - f^{1/2}(\lambda_{\ell,b}) \|_F \overset{P}{\to} 0 \]

and

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{-1/2}(\lambda_{\ell,b}) - f^{-1/2}(\lambda_{\ell,b}) \|_F \overset{P}{\to} 0. \]

To see (5.6) notice that since \( \bar{f} \) is positive definite, we get by Assumption 2 and equation (1.3) in Schmitt (1992), that

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{1/2}(\lambda_{\ell,b}) - f^{1/2}(\lambda_{\ell,b}) \|_F \leq \frac{1}{\sqrt{\sigma}} \max_{\ell \in \mathcal{G}(b)} \| \bar{f}(\lambda_{\ell,b}) - f(\lambda_{\ell,b}) \|_F. \]

For (5.7) we get by the same arguments as above and since \( A^{-1/2} = (A^{-1})^{1/2}, \) the bound

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{-1/2}(\lambda_{\ell,b}) - f^{-1/2}(\lambda_{\ell,b}) \|_F \leq \sqrt{\sigma_{\max}} \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{-1}(\lambda_{\ell,b}) - f^{-1}(\lambda_{\ell,b}) \|_F, \]

where \( \sigma_{\max} = \max_{\lambda \in [0,\pi]} \sigma(f(\lambda)). \) Equation (5.7) follows then by (5.3).

Assertions (5.6) and (5.7) imply that

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{1/2}(\lambda_{\ell,b}) - f^{-1/2}(\lambda_{\ell,b}) \|_F - \text{Id}_m \|_F \overset{P}{\to} 0 \]

and

\[ \max_{\ell \in \mathcal{G}(b)} \| \bar{f}^{-1/2}(\lambda_{\ell,b}) - f^{1/2}(\lambda_{\ell,b}) \|_F \overset{P}{\to} 0. \]

Using (5.8) and (5.9) it follows by straightforward calculations that

\[ \text{Cov}^{+}(V^+_{n,j}, V^+_{n,k}) = \text{Cov}^{+}(\bar{V}_{n,j}, \bar{V}_{n,k}) + o_P(1), \]

\[ \varphi_j(\lambda_{\ell,b}) (\bar{I}_{r,s,j}(\lambda_{\ell,b}) - \bar{f}_{r,j,s}(\lambda_{\ell,b})) \]
we then conclude that
\[ G_{\ell_1, r_1, s_1}(\lambda_{\ell_1}, b), I_{\ell_1, r_2, s_2}(\lambda_{\ell_2}, b) \]
and Lemma 3.8. By the same arguments it follows that
\[ E^*(V_{n,j}^+ V_{n,k}^+) = E^*(\tilde{V}_{n,j} \tilde{V}_{n,k}) + O_P(1), \]
and that
\[ E^*(\tilde{V}_{n,j} \tilde{V}_{n,k}) \overset{P}{\rightarrow} \Gamma_{1;jk}. \]

The above results show that \( G_n^+ \overset{P}{\rightarrow} G \). To conclude the proof we have to show that \( \Sigma_{1,n}^+ \overset{P}{\rightarrow} \Sigma_1 \) and that \( \Gamma_{1,n}^+ \overset{P}{\rightarrow} \Gamma_1 \). Using (5.8) and (5.9) again, it follows that
\[
\sigma_{jk}^+ = \frac{4\pi^2}{b} \sum_{\ell \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) \varphi_k(\lambda_{\ell,b}) \tilde{S}_{r_1 s_1 r_2 s_2}(\lambda_{\ell,b}) + \frac{4\pi^2}{b} \sum_{\ell \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) \varphi_k(-\lambda_{\ell,b}) \tilde{S}_{r_1 s_1 r_2 s_2}(\lambda_{\ell,b}) + O_P(1).
\]

and
\[
c_{jk}^+ = \frac{4\pi^2}{b} \sum_{\ell \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) \varphi_k(-\lambda_{\ell,b}) \tilde{S}_{r_1 s_1 r_2 s_2}(\lambda_{\ell,b}) + \frac{4\pi^2}{b} \sum_{\ell \in \mathcal{G}(b)} \varphi_j(\lambda_{\ell,b}) \varphi_k(\lambda_{\ell,b}) \tilde{S}_{r_1 s_1 r_2 s_2}(\lambda_{\ell,b}) + O_P(1),
\]
where
\[
\tilde{S}_{r_1 s_1 r_2 s_2}(\lambda) = \frac{1}{n - b + 1} \sum_{t=1}^{n - b + 1} (I_{t, r, s}(\lambda) - \tilde{f}_{r s}(\lambda))(I_{t, u, w}(\lambda) - \tilde{f}_{u w}(\lambda)).
\]

From Lemma 3.8 we then conclude that \( \sigma_{jk}^+ \overset{P}{\rightarrow} \Sigma_{1;jk} \) and that \( c_{jk}^+ \overset{P}{\rightarrow} \Gamma_{1;jk} \). □
Proof of Lemma 3.9 (iii): We use the notation $D^*_r(\lambda) = I^*_r(\lambda) - \hat{f}_r(\lambda)$. To show that the $J$-dimensional complex vector
\[
V^*_n = \left( \frac{2\pi}{\sqrt{n}} \sum_{l \in \mathcal{G}(n)} \varphi_j(\lambda_{t,n})D^*_{r,s_j}(\lambda_{t,n}), j = 1, 2, \ldots, J \right)^T,
\]
converges weakly to the complex normal variable $N_j^*(0, \Sigma_1, \Gamma_1)$, it suffices to show that the $2J$-dimensional real vector
\[
V^*_{n,T} = \frac{2\pi}{\sqrt{n}} \sum_{l \in \mathcal{G}(n)} \left( \begin{array}{c}
\Re\{\varphi_1(\lambda_{t,n})D^*_{r,s_1}(\lambda_{t,n})\} \\
\vdots \\
\Re\{\varphi_J(\lambda_{t,n})D^*_{r,s_J}(\lambda_{t,n})\} \\
\Im\{\varphi_1(\lambda_{t,n})D^*_{r,s_1}(\lambda_{t,n})\} \\
\vdots \\
\Im\{\varphi_J(\lambda_{t,n})D^*_{r,s_J}(\lambda_{t,n})\}
\end{array} \right)
\]
converges to the $2J$ dimensional, real normal distribution $N_{2J}^*(0, \Sigma_V)$, where $\Sigma_V$ denotes the limit of $E^*(V^*_{n,T}V^*_{n,T})$, which in view of assertion (i) is well defined. Notice that the covariance matrix $\Sigma_1$ and the relation matrix $\Gamma_1$ together specify $\Sigma_V$ and vice versa. To simplify the presentation we give only the proof for the case where the functions $\varphi_j$ are real-valued. The case of complex-valued $\varphi_j$ can be proved along the same lines but with a much more complicated notation. Write $V^*_{n,T} = \sum_{i=1}^N Z^*_i$, where the $j$-th component of $Z^*_i$ is given for $j = 1, 2, \ldots, J$, by
\[
\frac{2\pi}{\sqrt{n}} \left( \Re\{\varphi_j(\lambda_{t,n})D^*_{r,s_j}(\lambda_{t,n})\} + \Re\{\varphi_j(-\lambda_{t,n})D^*_{r,s_j}(-\lambda_{t,n})\} \right)
\]
and for $j = J + 1, \ldots, 2J$, by
\[
\frac{2\pi}{\sqrt{n}} \left( \Im\{\varphi_{j-J}(\lambda_{t,n})D^*_{r,s_j}(\lambda_{t,n})\} + \Im\{\varphi_{j-J}(-\lambda_{t,n})D^*_{r,s_j}(-\lambda_{t,n})\} \right).
\]
Observe that the random vectors $Z^*_i$ are independent. Thus in view of assertion (i), to establish the desired weak convergence, it suffices to show that Lyapunov’s condition is satisfied, that is, that $\sum_{i=1}^N E^*\|Z^*_i\|^{2+\delta} \rightarrow 0$ for some $\delta > 0$. Choose $\delta = 2$. Using $(a + b)^2 \leq 2a^2 + 2b^2$ and $\varphi(\lambda)\Re\{D^*_r(\lambda)\}^2 + \varphi(\lambda)\Im\{D^*_r(\lambda)\}^2 = \varphi(\lambda)|D^*_r(\lambda)|^2$ we have
\[
\sum_{i=1}^N E^*\|Z^*_i\|^4 \leq \frac{(2\pi)^4}{n^2} \sum_{i=1}^N E^* \left( 2 \sum_{j=1}^J \varphi_j^2(\lambda_{t,n})|D^*_{r,s_j}(\lambda_{t,n})|^2 \right) + \frac{2}{n^2} \sum_{j=1}^J \varphi_j^2(-\lambda_{t,n})|D^*_{r,s_j}(-\lambda_{t,n})|^2 \right)^2
\leq \frac{4(2\pi)^4}{n^2} \max_{1 \leq j \leq J} \sup_{\lambda \in [-\pi, \pi]} \varphi_j^2(\lambda) \sum_{i=1}^N E^* \left( \sum_{j=1}^J |D^*_{r,s_j}(\lambda_{t,n})|^2 \right)
\begin{equation*}
+ \sum_{j=1}^{J} |D_{r_j s_j}^* (-\lambda_{l,n})|^2
\leq \frac{8(2\pi)^4}{n^2} \max_{1 \leq j \leq J} \sup_{\lambda \in [-\pi, \pi]} \varphi_j^2(\lambda) \left( \sum_{l=1}^{N} \sum_{j=1}^{J} E^* |D_{r_j s_j}^* (\lambda_{l,n})|^4 \right.
+ \sum_{l=1}^{N} \sum_{j=1}^{J} E^* |D_{r_j s_j}^* (-\lambda_{l,n})|^4 \right).
\end{equation*}

Now, recall the definition of $D_{r_j s_j}^* (\lambda)$ and verify by straightforward calculations that

\begin{equation*}
E^* |I_{r_{s}}^*(\lambda) - \tilde{f}_{r_{s}}(\lambda)|^4 = E^* [I_{r_{s}}^*(\lambda)I_{r_{s}}^*(\lambda)^2] + 2\tilde{f}_{r_{s}}(\lambda)^2 \tilde{f}_{r_{s}}(\lambda)^2
- 2\tilde{f}_{r_{s}}(\lambda)E^* [I_{r_{s}}^*(\lambda)I_{r_{s}}^*(\lambda)^2] - 2\tilde{f}_{r_{s}}(\lambda)E^* [I_{r_{s}}^*(\lambda)^2I_{r_{s}}^*(\lambda)]
+ \tilde{f}_{r_{s}}(\lambda)^2E^* [I_{r_{s}}^*(\lambda)^2] + \tilde{f}_{r_{s}}(\lambda)^2E^* [I_{r_{s}}^*(\lambda)^2]
+ 4\tilde{f}_{r_{s}}(\lambda)\tilde{f}_{r_{s}}(\lambda)E^* [I_{r_{s}}^*(\lambda)I_{r_{s}}^*(\lambda)]
- 2\tilde{f}_{r_{s}}(\lambda)\tilde{f}_{r_{s}}(\lambda)^2E^* [I_{r_{s}}^*(\lambda)] - 2\tilde{f}_{r_{s}}(\lambda)^2\tilde{f}_{r_{s}}(\lambda)E^* [I_{r_{s}}^*(\lambda)].
\end{equation*}

Now since $I^*(\lambda_{l,n})$ has for every $l = 1, 2, \ldots, N$ a complex Wishart distribution with parameters $1$ and $\hat{\lambda}(\lambda_{l,n})$, for short, $I^*(\lambda_{l,n}) \sim \text{Wishart}(1, \hat{\lambda}(\lambda_{l,n}))$, we get using Assumption 5 and expressions for the moments of the complex Wishart distribution, see Withers and Nadarajah (2012), that $E^* |I_{r_{s}}^*(\lambda) - \tilde{f}_{r_{s}}(\lambda)|^4 \overset{P}{\to} E|S_{r_{s}}(\lambda) - \tilde{f}_{r_{s}}(\lambda)|^4 = O(1)$, for all $r, s \in \{1, 2, \ldots, m\}$, where the random variable $S(\lambda) = (S_{r_{s}}(\lambda))_{r,s=1,2,\ldots,m}$ has the Wishart$(1, f(\lambda))$ distribution. Therefore,

\begin{equation*}
\sum_{l=1}^{N} E^* \|Z_l\|^4 \leq \frac{8(2\pi)^4}{n^2} \max_{1 \leq j \leq J} \sup_{\lambda \in [-\pi, \pi]} \varphi_j^2(\lambda) \cdot N \cdot O_P(1) = O_P(n^{-1}) \to 0.
\end{equation*}

\[\Box\]

**Proof of Theorem 3.10:** For assertion (i), note that $E^* (V_n^\circ) = E^* (V_n^\circ) = 0$ which implies $\text{Cov}^* ((\text{Re}(V_n^\circ)^T, \text{Im}(V_n^\circ)^T)^T) = G_n^\circ$. From the definition of $C_n^+$ one can see that $C_n^+$ is symmetric, and thus also $G_n^\circ$, are symmetric. It follows from (3.1) that

\begin{equation*}
\text{Cov}^* \left[ \begin{array}{c}
\text{Re}(V_n^\circ) \\
\text{Im}(V_n^\circ)
\end{array} \right] = (G_n^\circ)^{1/2} (G_n^\circ)^{-1/2} G_n^* (G_n^\circ)^{-1/2} (G_n^\circ)^{1/2} = G_n^\circ \overset{P}{\to} G,
\end{equation*}

due to Lemma 3.9, (2.5) and (2.14). Again using (2.5) yields assertion (i). Since $(\text{Re}(V_n^\circ)^T, \text{Im}(V_n^\circ)^T)^T$ is asymptotically normal due to Lemma 3.9 (iii), $(\text{Re}(V_n^\circ)^T, \text{Im}(V_n^\circ)^T)^T$ is asymptotically normal with covariance matrix $G$ which gives assertion (ii). \[\Box\]
PROOF of Theorem 3.11: First note that \( \sqrt{n}(M_n^* - \tilde{M}_n) \) equals \( V_n^* \) from Step I.3. Hence, Lemma 3.9 (iii) together with (2.5) yields
\[
\sqrt{n} \left[ \begin{array}{c}
\text{Re}(M_n^*) \\
\text{Im}(M_n^*)
\end{array} \right] \xrightarrow{d} \mathcal{N}_{2J}(0, G_1)
\] (5.10)
in \( P \)-probability, where \( G_1 \) is defined analogous to \( G \) in (2.14) but with \( \Sigma \) replaced by \( \Sigma_1 \) and \( \Gamma \) replaced by \( \Gamma_1 \). We next apply the delta method to (5.10). Invoking the mean value theorem for all \( 2L \) component functions of \( \tilde{g} \) leads to
\[
\left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) = \left( \begin{array}{c}
\nabla \tilde{g}_i(\xi_i) \\
\nabla \tilde{g}_i(\xi_i)
\end{array} \right)_{i=1,...,2L} \times \sqrt{n} \left[ \begin{array}{c}
\text{Re}(M_n^*) \\
\text{Im}(M_n^*)
\end{array} \right] = \left( \begin{array}{c}
\text{Re}(\tilde{M}_n^*) \\
\text{Im}(\tilde{M}_n^*)
\end{array} \right),
\]
(5.11)
where \( \nabla \tilde{g}_i(\cdot) \) denotes the gradient (as a row vector) of the \( i \)-th component function of \( \tilde{g} \), and \( \xi_i \in \mathbb{R}^{2J} \) denotes a vector on the line segment between \( (\text{Re}(M_n^*)^\top, \text{Im}(M_n^*)^\top)^\top \) and \( (\text{Re}(\tilde{M}_n^*)^\top, \text{Im}(\tilde{M}_n^*)^\top)^\top \). We now have from (5.10) that \( \|\xi_i - (\text{Re}(\tilde{M}_n^*)^\top, \text{Im}(\tilde{M}_n^*)^\top)^\top\| \to 0 \) in \( P^* \)-probability for all \( i \in \{1, \ldots, 2L\} \), and Assumption 5 together with convergence of Riemann sums yields \( \|\tilde{M}_n - M\| \overset{P}{\to} 0 \). Therefore (5.11) and (5.10) imply
\[
\left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) \xrightarrow{d} J_{\tilde{g}}((\text{Re}(M)^\top, \text{Im}(M)^\top)^\top) \mathcal{N}_{2J}(0, G_1)
\] (5.12)
in \( P \)-probability. We only need the asymptotic normality from this statement. Note that \( G_n^* \) is the covariance matrix of the last left-hand side and therefore
\[
(G_n^*)^{-1/2} \left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) \xrightarrow{d} \mathcal{N}_{2L}(0, \text{Id}_{2J})
\] (5.13)
in probability. From the proof of Theorem 3.10 we have \( G_n^* \overset{P}{\to} G \). Also, the above considerations together with continuity of \( J_{\tilde{g}} \) yield \( J_{\tilde{g}}((\text{Re}(\tilde{M}_n)^\top, \text{Im}(\tilde{M}_n)^\top)^\top) \overset{P}{\to} J_{\tilde{g}}((\text{Re}(M)^\top, \text{Im}(M)^\top)^\top) \). Thus,
\[
G_n^* \overset{P}{\to} J_{\tilde{g}} \left( \begin{array}{c}
\text{Re}(M) \\
\text{Im}(M)
\end{array} \right) \times G \times J_{\tilde{g}} \left( \begin{array}{c}
\text{Re}(M) \\
\text{Im}(M)
\end{array} \right)^\top.
\]
Therefore we have
\[
\left( \begin{array}{c}
\text{Re}(W_n^*) \\
\text{Im}(W_n^*)
\end{array} \right) \xrightarrow{d} J_{\tilde{g}}((\text{Re}(M)^\top, \text{Im}(M)^\top)^\top) \mathcal{N}_{2J}(0, G)
\] (5.14)
in probability. Applying the continuous mapping theorem with \( h(x_1, x_2) := x_1 + i \cdot x_2 \), for \( x_1, x_2 \in \mathbb{R}^L \), if it follows
\[
W_n^o \xrightarrow{d} W \sim \mathcal{N}_{L}^o(0, \Sigma_R, \Gamma_R)
\] (5.15)
in probability, for the matrices \( \Sigma_R, \Gamma_R \) stated in (3.5) (the exact form of which does not need to be specified for the desired assertion to hold). \( \square \)
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| MODEL I | Mean | Std   | MSE×10 | Mean | Std   | MSE×10 | Mean | Std   | MSE×10 |
|--------|------|-------|--------|------|-------|--------|------|-------|--------|
| MFHB   |      |       |        |      |       |        |      |       |        |
| h=0.10 |      |       |        |      |       |        |      |       |        |
| b=6    | 0.794| 0.123 | 0.158  | 1.014| 0.129 | 0.176  | 1.142| 0.167 | 0.283  |
| b=8    | 0.788| 0.130 | 0.172  | 1.024| 0.140 | 0.206  | 1.153| 0.188 | 0.358  |
| b=10   | 0.787| 0.129 | 0.168  | 1.011| 0.146 | 0.214  | 1.143| 0.192 | 0.368  |
| b=12   | 0.788| 0.136 | 0.188  | 1.000| 0.160 | 0.256  | 1.130| 0.192 | 0.367  |
| b=16   | 0.771| 0.142 | 0.203  | 0.958| 0.171 | 0.300  | 1.104| 0.212 | 0.456  |
| h=0.12 |      |       |        |      |       |        |      |       |        |
| b=6    | 0.824| 0.124 | 0.187  | 1.027| 0.140 | 0.207  | 1.159| 0.172 | 0.308  |
| b=8    | 0.828| 0.122 | 0.178  | 1.045| 0.126 | 0.181  | 1.171| 0.170 | 0.300  |
| b=10   | 0.821| 0.117 | 0.162  | 1.030| 0.129 | 0.181  | 1.152| 0.171 | 0.297  |
| b=12   | 0.815| 0.127 | 0.179  | 1.023| 0.147 | 0.226  | 1.156| 0.170 | 0.295  |
| b=16   | 0.799| 0.145 | 0.243  | 1.005| 0.167 | 0.283  | 1.140| 0.191 | 0.365  |
| MBB    |      |       |        |      |       |        |      |       |        |
| b=6    | 0.906| 0.112 | 0.323  | 1.016| 0.162 | 0.273  | 1.092| 0.152 | 0.248  |
| b=8    | 0.849| 0.122 | 0.201  | 0.977| 0.181 | 0.332  | 1.055| 0.185 | 0.406  |
| b=10   | 0.832| 0.143 | 0.237  | 0.955| 0.198 | 0.414  | 1.048| 0.199 | 0.475  |
| b=12   | 0.795| 0.145 | 0.217  | 0.933| 0.208 | 0.477  | 1.020| 0.205 | 0.548  |
| b=16   | 0.750| 0.164 | 0.271  | 0.875| 0.208 | 0.559  | 0.966| 0.217 | 0.743  |
| MODEL II |      |       |        |      |       |        |      |       |        |
| MFHB   |      |       |        |      |       |        |      |       |        |
| h=0.10 |      |       |        |      |       |        |      |       |        |
| b=6    | 1.164| 0.232 | 0.570  | 1.160| 0.166 | 0.300  | 1.195| 0.196 | 0.424  |
| b=8    | 1.153| 0.263 | 0.620  | 1.168| 0.168 | 0.301  | 1.196| 0.195 | 0.409  |
| b=10   | 1.140| 0.235 | 0.633  | 1.148| 0.181 | 0.331  | 1.193| 0.183 | 0.353  |
| b=12   | 1.113| 0.240 | 0.785  | 1.115| 0.205 | 0.428  | 1.193| 0.216 | 0.491  |
| b=16   | 1.113| 0.248 | 0.742  | 1.084| 0.229 | 0.536  | 1.144| 0.245 | 0.601  |
| h=0.12 |      |       |        |      |       |        |      |       |        |
| b=6    | 1.167| 0.224 | 0.544  | 1.195| 0.173 | 0.337  | 1.222| 0.195 | 0.431  |
| b=8    | 1.158| 0.222 | 0.547  | 1.193| 0.160 | 0.311  | 1.216| 0.185 | 0.399  |
| b=10   | 1.184| 0.230 | 0.538  | 1.200| 0.176 | 0.357  | 1.227| 0.186 | 0.387  |
| b=12   | 1.149| 0.227 | 0.578  | 1.159| 0.194 | 0.394  | 1.221| 0.203 | 0.498  |
| b=16   | 1.134| 0.235 | 0.661  | 1.111| 0.208 | 0.437  | 1.168| 0.232 | 0.538  |
| MBB    |      |       |        |      |       |        |      |       |        |
| b=6    | 1.176| 0.208 | 0.454  | 1.229| 0.255 | 0.784  | 1.191| 0.221 | 0.524  |
| b=8    | 1.157| 0.245 | 0.655  | 1.189| 0.273 | 0.782  | 1.187| 0.246 | 0.623  |
| b=10   | 1.096| 0.235 | 0.730  | 1.131| 0.268 | 0.718  | 1.162| 0.263 | 0.695  |
| b=12   | 1.048| 0.238 | 0.999  | 1.090| 0.271 | 0.766  | 1.116| 0.269 | 0.735  |
| b=16   | 1.024| 0.245 | 1.002  | 1.057| 0.289 | 0.870  | 1.080| 0.290 | 0.888  |

**TABLE 1**: Bootstrap estimates of the standard deviation of the sample cross-correlations \( \hat{\rho}(h) \) for lags \( h \in \{-1, 0, +1\} \) for time series of length \( n = 100 \) stemming from Model I and Model II. MFHB refers to the estimates of the multivariate frequency domain hybrid bootstrap and MBB to those of the moving block bootstrap.
A FREQUENCY DOMAIN BOOTSTRAP FOR GENERAL MULTIVARIATE STATIONARY PROCESSES

– SUPPLEMENTARY MATERIAL –

Proof of Lemma 2.1: The proof generalizes calculations from Rosenblatt (1985) and Krogstad (1982) regarding the covariance structure of univariate periodogram ordinates. The assertion of Lemma 2.1 remains true if one switches from Fourier frequencies \( \lambda_{j,n}, \lambda_{k,n} \) to two fixed frequencies \( \lambda_1, \lambda_2 \in [0, \pi] \); we will state at the end of this proof which arguments have to be adapted in this situation.

A direct calculation yields the decomposition of the covariance into three major components:

\[
\text{Cov}(I_{rs}(\lambda_{j,n}), I_{vw}(\lambda_{k,n})) = \frac{1}{4\pi^2 n^2} \sum_{t_1, t_2, t_3, t_4=1}^{n} \text{cum}(X_r(t_1), X_w(t_2), X_v(t_3), X_w(t_4)) e^{-i(t_1-t_2)\lambda_{j,n}+i(t_3-t_4)\lambda_{k,n}}
\]

\[
+ \frac{1}{4\pi^2 n^2} \sum_{t_1, t_2, t_3, t_4=1}^{n} \gamma_{rv}(t_1-t_3)\gamma_{sw}(t_2-t_4) e^{-i(t_1-t_2)\lambda_{j,n}+i(t_3-t_4)\lambda_{k,n}}
\]

\[
+ \frac{1}{4\pi^2 n^2} \sum_{t_1, t_2, t_3, t_4=1}^{n} \gamma_{rw}(t_1-t_4)\gamma_{sv}(t_2-t_3) e^{-i(t_1-t_2)\lambda_{j,n}+i(t_3-t_4)\lambda_{k,n}}
\]

= : \( S_1 + S_2 + S_3 \).

For \( S_1 \) we obtain with an index shift for three summands

\[
\frac{n}{2\pi} S_1 = \frac{1}{(2\pi)^3 n} \sum_{t_1, t_2, t_3, t_4=1}^{n} c_{rsuw}(t_1-t_4, t_2-t_4, t_3-t_4) e^{-i(t_1-t_2)\lambda_{j,n}+i(t_3-t_4)\lambda_{k,n}}
\]

\[
= \frac{1}{(2\pi)^3 n} \sum_{t_4=1}^{n} \sum_{t_1, t_2, t_3=1-t_4}^{n-t_4} c_{rsuw}(t_1, t_2, t_3) e^{-i(t_1-t_2)\lambda_{j,n}+i(t_3-t_4)\lambda_{k,n}}.
\]

By merging like summands the last expression can be seen to be equal to

\[
\frac{1}{(2\pi)^3 n} \sum_{h_1, h_2, h_3=-(n-1)}^{n-1} q_n(h_1, h_2, h_3) c_{rsuw}(h_1, h_2, h_3) e^{-i(h_1-h_2)\lambda_{j,n}+i(h_3-\lambda_{k,n})},
\]

(5.16)

where \( q_n(h_1, h_2, h_3) \) counts the number of times the respective summand appears in \((n/2\pi)S_1\). It holds

\[
q_n(h_1, h_2, h_3) = \left( n - \max\{|h_1|, |h_2|, |h_3|, |h_1 - h_2|, |h_1 - h_3|, |h_2 - h_3|\} \right)_+.
\]
We can replace the \( g_n(\cdot) \) term in (5.16) by \( n \) since the resulting remainder term vanishes with rate \( O(n^{-1}) \) as the following bound shows:

\[
\left| \frac{1}{(2\pi)^3} \sum_{h_1,h_2,h_3=-\infty}^{n-1} (g_n(h_1,h_2,h_3) - n) c_{rsvw}(h_1,h_2,h_3) e^{-i\cdot} \right| \leq \frac{1}{(2\pi)^3} \sum_{h_1,h_2,h_3=-\infty}^{\infty} 2 \max\{|h_1|,|h_2|,|h_3|\} |c_{rsvw}(h_1,h_2,h_3)| \leq \frac{1}{(2\pi)^3} \sum_{h_1,h_2,h_3=-\infty}^{\infty} (1 + |h_1| + |h_2| + |h_3|) |c_{rsvw}(h_1,h_2,h_3)|,
\]

which is of order \( O(n^{-1}) \). Now, after replacing \( g_n(\cdot) \) with \( n \) in (5.16), the difference of the remaining term and \( f_{rsvw}(\lambda_j,n_1-\lambda_j,n,-\lambda_k,n) \) can be bounded by

\[
\left| \frac{1}{(2\pi)^3} \sum_{h_1,h_2,h_3=-\infty}^{\infty} \mathbf{1}_{\{\max\{|h_1|,|h_2|,|h_3|\} \geq n\}} |c_{rsvw}(h_1,h_2,h_3)| \right|,
\]

which also vanishes with rate \( O(n^{-1}) \) under the imposed summability assumption on \( |c_{rsvw}(h_1,h_2,h_3)| \). This yields the desired assertion for \( S_1 \).

To handle \( S_2 \) and \( S_3 \) we have to introduce some notation and preliminaries. We define the functions \( g_n(\lambda) := \sum_{k=1}^{n} e^{i k \lambda} \). For all \( \lambda \not\in 2\pi \mathbb{Z} \), we can use the geometric sum formula to express \( g_n \) as

\[
g_n(\lambda) = e^{i \lambda} \frac{e^{i n \lambda} - 1}{e^{i \lambda} - 1} = e^{i(n+1)\lambda/2} \frac{\sin(n\lambda/2)}{\sin(\lambda/2)}.
\]

Note that for all \( \lambda \in \mathbb{R} \) we have \( g_n(\lambda) g_n(-\lambda) = 2\pi n F_n(\lambda) \), where \( F_n \) is the (continuously extended) Fejér kernel:

\[
F_n(\lambda) := \begin{cases} 
\frac{1}{2\pi n} \frac{\sin^2(n\lambda/2)}{\sin^2(\lambda/2)}, & \lambda \not\in 2\pi \mathbb{Z} \\
\frac{n}{2\pi}, & \lambda \in 2\pi \mathbb{Z}
\end{cases}
\]

Integrals over these functions can be handled in an elegant way by determining their Fourier coefficients. For arbitrary \( \lambda_1, \lambda_2 \) define

\[
z_{n,\lambda_1,\lambda_2}(\alpha) := g_n(\alpha - \lambda_1) g_n(-\alpha - \lambda_2).
\]

The \( \ell \)-th Fourier coefficient of this function is given by

\[
z_{n,\lambda_1,\lambda_2}[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\alpha - \lambda_1) g_n(-\alpha + \lambda_2) e^{-i\ell \alpha} d\alpha = \frac{1}{2\pi} \sum_{p,q=1}^{n} e^{-ip\lambda_1} e^{iq\lambda_2} \int_{-\pi}^{\pi} e^{i(p-q-\ell) \alpha} d\alpha.
\]
\[ e^{-ip\lambda_1}e^{iq\lambda_2} = \sum_{p,q=1}^{n} e^{-ip\lambda_1}e^{iq\lambda_2} \mathbb{1}_{\{p-q=\ell\}}. \] (5.17)

In particular it holds \( \tilde{z}_{n,\lambda_1,\lambda_2}[\ell] = 0 \) for all \( |\ell| \geq n \).

With this notation, and using the fact that \( \gamma_{rs}(h) = \int_{-\pi}^{\pi} e^{ih\alpha} f_{rs}(\alpha) d\alpha \), we can write

\[ S_2 = \frac{1}{4\pi^2 n^2} \int_{-\pi}^{\pi} z_{n,\lambda_1,\lambda_2,\lambda_k,\lambda_k}(\alpha) f_{rv}(\alpha) d\alpha \cdot \int_{-\pi}^{\pi} z_{n,-\lambda_1,-\lambda_2,\lambda_k,\lambda_k}(\alpha) f_{sw}(\alpha) d\alpha. \] (5.18)

Consider first the case with \( j = k \), that is, with \( \lambda_j,n = \lambda_k,n \). Then the last expression simplifies to

\[ S_2 = \int_{-\pi}^{\pi} F_n(\alpha - \lambda_j,n) f_{rv}(\alpha) d\alpha \cdot \int_{-\pi}^{\pi} F_n(\alpha + \lambda_j,n) f_{sw}(\alpha) d\alpha. \]

Since \( F_n(\cdot - \lambda_j,n) = z_{n,\lambda_1,\lambda_2,\lambda_k,\lambda_k}(\cdot)/(2\pi n) \), its \( \ell \)-th Fourier coefficient can be obtained from \( (5.17) \), it is

\[ \hat{F}_n(\cdot - \lambda_j,n)[\ell] = \frac{(n - |\ell|) e^{-i\ell\lambda_j,n}}{2\pi n} \mathbb{1}_{\{|\ell| \leq n\}}. \]

The Fourier coefficients of \( f_{rv} \) are given by

\[ \hat{f}_{rv}[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{rv}(\alpha) e^{-i(\ell)\alpha} d\alpha = \frac{\gamma_{rv}(\ell)}{2\pi}. \]

Then we can apply Parseval’s Theorem to calculate the first integral in \( S_2 \) as

\[ \int_{-\pi}^{\pi} F_n(\alpha - \lambda_j,n) f_{rv}(\alpha) d\alpha = 2\pi \sum_{\ell=-\infty}^{\infty} \hat{f}_{rv}[\ell] \hat{F}_n(\cdot - \lambda_j,n)[\ell] = \frac{1}{2\pi} \sum_{\ell=-(n-1)}^{n-1} \gamma_{rv}(\ell) \left(1 - \frac{|\ell|}{n}\right) e^{-i\ell\lambda_j,n}. \]

Of course, the last result could have been obtained by direct calculations for the expression \( S_2 \) as well. The calculation via Fourier coefficients presented here serves as the basis for the upcoming other cases to consider. In those cases the proposed way via Fourier coefficients seems much more elegant and shorter than a direct calculation.

The difference between the right-hand side of the last equation and \( f_{rv}(\lambda_j,n) \) can be shown to be of the order \( O(n^{-1}) \) with standard arguments, using \( \sum_{h \in \mathbb{Z}} (1 + |h|) |\gamma_{rv}(h)| < \infty \). The second integral in \( S_2 \) behaves as \( f_{sw}(-\lambda_j,n) \) by analogous arguments. Together we have

\[ |S_2 - f_{rv}(\lambda_j,n) \cdot f_{sw}(-\lambda_j,n)| = O(n^{-1}) \]
for the case $\lambda_{j,n} = \lambda_{k,n}$.

Next we consider the case with $j \neq k$, that is, with $\lambda_{j,n} \neq \lambda_{k,n}$ different Fourier frequencies within $[0, \pi]$. In this case $\lambda_{k,n} - \lambda_{j,n} = 2\pi(k - j)/n$ is a Fourier frequency with $\lambda_{k,n} - \lambda_{j,n} \notin 2\pi\mathbb{Z}$, and thus $\sum_{q=1}^{n} e^{iq(\lambda_{k,n} - \lambda_{j,n})} = 0$. We can use this fact together with (5.17) to obtain a bound for the Fourier coefficients of $z_{n,\lambda_{j,n},\lambda_{k,n}}$ for all $|\ell| < n$ (all other coefficients are zero anyway):

$$\left| \hat{z}_{n,\lambda_{j,n},\lambda_{k,n}}[\ell] \right| = \left| \sum_{q=1}^{n-|\ell|} e^{iq(\lambda_{k,n} - \lambda_{j,n})} \right| = \sum_{q=n-|\ell|+1}^{n} e^{iq(\lambda_{k,n} - \lambda_{j,n})} \leq |\ell|. \quad (5.19)$$

With this bound the first integral in (5.18) can be bounded via Parseval’s Theorem by

$$\left| \int_{-\pi}^{\pi} z_{n,\lambda_{j,n},\lambda_{k,n}}(\alpha) \overline{f_{\text{vr}}(\alpha)} \, d\alpha \right| = 2\pi \sum_{\ell=-\infty}^{\infty} \left| \hat{z}_{n,\lambda_{j,n},\lambda_{k,n}}[\ell] \right| \left| \overline{f_{\text{vr}}[\ell]} \right| \leq \sum_{\ell=-\infty}^{\infty} (1 + |\ell|) |\gamma_{\text{vr}}(\ell)| < \infty .$$

The second integral in (5.18) can be bounded analogously. Therefore, $S_2$ vanishes with rate $O(n^{-2})$ in this case.

The $S_3$ term can be treated similar – but not completely analogous – to the $S_2$ term. It holds

$$S_3 = \frac{1}{4\pi^2 n^2} \int_{-\pi}^{\pi} z_{n,\lambda_{j,n},-\lambda_{k,n}}(\alpha) f_{\text{rw}}(\alpha) \, d\alpha \cdot \int_{-\pi}^{\pi} z_{n,-\lambda_{j,n},\lambda_{k,n}}(\alpha) f_{\text{sv}}(\alpha) \, d\alpha . \quad (5.20)$$

In the case $\lambda_{j,n} = \lambda_{k,n} = 0$, one can proceed analogously to the $S_2$ case and see that $S_3$ simplifies to

$$S_3 = \int_{-\pi}^{\pi} F_n(\alpha) f_{\text{rw}}(\alpha) \, d\alpha \cdot \int_{-\pi}^{\pi} F_n(\alpha) f_{\text{sv}}(\alpha) \, d\alpha ,$$

for which we have

$$\left| S_3 - f_{\text{rw}}(0) \cdot f_{\text{sv}}(0) \right| = O(n^{-1}) .$$

Now let $\lambda_{j,n} = \lambda_{k,n} = \pi$. This case is slightly more subtle because both integrals in (5.20) have to be treated together to see that certain terms cancel out. We then get

$$S_3 = \int_{-\pi}^{\pi} \frac{1}{2\pi n} \frac{\sin \left( -\frac{n\pi}{2} + \frac{n\alpha}{2} \right) \sin \left( -\frac{n\pi}{2} - \frac{n\alpha}{2} \right)}{\sin \left( -\frac{n\pi}{2} + \frac{n\pi}{2} \right) \sin \left( -\frac{n\pi}{2} - \frac{n\pi}{2} \right)} f_{\text{rw}}(\alpha) \, d\alpha \quad (5.21)$$

$$\times \int_{-\pi}^{\pi} \frac{1}{2\pi n} \frac{\sin \left( -\frac{n\pi}{2} + \frac{n\alpha}{2} \right) \sin \left( -\frac{n\pi}{2} - \frac{n\alpha}{2} \right)}{\sin \left( -\frac{n\pi}{2} + \frac{n\pi}{2} \right) \sin \left( -\frac{n\pi}{2} - \frac{n\pi}{2} \right)} f_{\text{sv}}(\alpha) \, d\alpha .$$
Observe that the function \( \sin(-n\pi/2 + \cdot) \) is even if \( n \) is odd and odd if \( n \) is even. Therefore it holds
\[
\sin\left(-\frac{n\pi}{2} + \frac{n\alpha}{2}\right) \sin\left(-\frac{n\pi}{2} + \frac{n\alpha}{2}\right) = \sin^2\left(\frac{n(\alpha - \pi)}{2}\right) s(n),
\]
where \( s(n) = -1 \) if \( n \) is even and \( s(n) = 1 \) if \( n \) is odd. Thus (5.21) simplifies to
\[
S_3 = \int_{-\pi}^{\pi} \frac{s(n)}{2\pi n} \sin^2\left(\frac{n(\alpha - \pi)}{2}\right) f_{rw}(\alpha) d\alpha \cdot \int_{-\pi}^{\pi} \frac{s(n)}{2\pi n} \sin^2\left(\frac{n(\alpha - \pi)}{2}\right) f_{sv}(\alpha) d\alpha
\]
\[
= \int_{-\pi}^{\pi} F_n(\alpha - \pi) f_{rw}(\alpha) d\alpha \cdot \int_{-\pi}^{\pi} F_n(\alpha - \pi) f_{sv}(\alpha) d\alpha,
\]
which implies
\[
|S_3 - f_{rw}(\pi) \cdot f_{sv}(\pi)| = O(n^{-1}),
\]
for the case \( \lambda_{j,n} = \lambda_{k,n} = \pi \).

For all other cases, that is, for \( \lambda_{j,n} = \lambda_{k,n} \notin \{0, \pi\} \) or \( \lambda_{j,n} \neq \lambda_{k,n} \) we have \( \pm(\lambda_{j,n} + \lambda_{k,n}) \notin 2\pi\mathbb{Z} \) and we get
\[
|\tilde{\gamma}_{n,\lambda_{j,n},-\lambda_{k,n}}[\ell]| \leq |\ell| \text{ and } |\tilde{\gamma}_{n,-\lambda_{j,n},\lambda_{k,n}}[\ell]| \leq |\ell|,
\]
which implies \( |S_3| = O(n^{-2}) \), analogous to the calculations for \( S_2 \) before. Moreover, an inspection of this proof shows that in all cases the \( O(\cdot) \) bounds are uniform over all frequencies. Also, if one is interested in \( \text{Cov}(I_{rs}(\lambda_1), I_{rw}(\lambda_2)) \) for two fixed frequencies \( \lambda_1, \lambda_2 \in [0, \pi] \) instead of Fourier frequencies \( \lambda_{j,n}, \lambda_{k,n}, \) one can in large parts use the same arguments presented here. The only argument that changes is (5.19). For fixed frequencies \( \lambda_1 \neq \lambda_2 \) one gets instead of (5.19)
\[
|\tilde{\gamma}_{n,\lambda_1,\lambda_2}[\ell]| = \left| \sum_{q=1}^{n-|\ell|} e^{i(q(\lambda_2 - \lambda_1))} \right| \leq C_{\lambda_1, \lambda_2},
\]
where the finite constant \( C_{\lambda_1, \lambda_2} \) depends only on \( \lambda_1, \lambda_2 \) but not on \( n \). With this bound one can proceed as in the case of Fourier frequencies. \( \square \)

**Proof of Lemma 3.8 (i):** For the variance of \( \tilde{f}_{rs}(\lambda_{t,b}) \) we have
\[
\text{Var}(\tilde{f}_{rs}(\lambda_{t,b})) = \frac{1}{(n-b+1)^2} \sum_{t_1, t_2=1}^{n-b+1} \text{Cov}(I_{t_1;r}(\lambda_{t,b}), I_{t_2;r}(\lambda_{t,b}))
\]
\[
\leq \frac{1}{(n-b+1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{b_1, b_2=0}^{b} \left| \begin{array}{l}
\gamma_{rr}((t_1 - t_2) + (g_1 - g_3)) \\
\gamma_{ss}((t_1 - t_2) + (g_2 - g_4))
\end{array} \right| \left| \begin{array}{l}
\gamma_{ss}((t_1 - t_2) + (g_1 - g_3)) \\
\gamma_{rr}((t_1 - t_2) + (g_2 - g_4))
\end{array} \right|
\]

\[\frac{1}{(n-b+1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{b_1, b_2=0}^{b} \left| \gamma_{rr}((t_1 - t_2) + (g_1 - g_3)) \right| \left| \gamma_{ss}((t_1 - t_2) + (g_2 - g_4)) \right| \]
For the first term on the right hand side of the last expression, we get by first summing over \( g_4 \) and then over \( t_2 \) that this term is \( \mathcal{O}(b/(n - b + 1)) \). The \( \mathcal{O}(b/(n - b + 1)) \) bound is obtained for the second term by the same arguments while for the last term we get by summing first over \( g_1 \), then over \( g_3 \) and then over \( t_1 \) that this term is \( \mathcal{O}(1/(n - b + 1)) \). Now, since \( E(\bar{f}_{rs}(\lambda_{\ell,b})) = E(I_{1;rs}(\lambda_{\ell,b})) \), we get that

\[
E\left( \sum_{t \in \mathcal{G}(b)} |\bar{f}_{rs}(\lambda_{\ell,b}) - E(I_{1;rs}(\lambda_{\ell,b}))| \right) \leq \sum_{t \in \mathcal{G}(b)} \sqrt{\text{Var}(\bar{f}_{rs}(\lambda_{\ell,b}))} = \mathcal{O}\left( \sqrt{b^2/(n - b + 1)} \right)
\]

which completes the proof. \( \square \)

**Proof of Lemma 3.8 (ii):** Notice that

\[
\frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} \left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) \right) I_{t;r,k,s_k} (\lambda_{\ell_2,b}) - E\left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) I_{t;r,k,s_k} (\lambda_{\ell_2,b}) \right)
\]

\[
= E\left( I_{1;r,k,s_k} (\lambda_{\ell_2,b}) \right) \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} \left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) - E\left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) \right) \right)
\]

\[
+ E\left( I_{1;r,j,s_j} (\lambda_{\ell_1,b}) \right) \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} \left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) - E\left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) \right) \right)
\]

\[
+ \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} \left\{ \left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) - E\left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) \right) \right) \left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) - E\left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) \right) \right)
\]

\[
- E\left( \left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) - E\left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) \right) \right) \left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) - E\left( I_{t;r,k,s_k} (\lambda_{\ell_2,b}) \right) \right) \right) \right\}
\]

\[
=: T_{1,n}(\ell_1,\ell_2) + T_{2,n}(\ell_1,\ell_2) + T_{3,n}(\ell_1,\ell_2)
\]

with an obvious notation for \( T_{j,n}(\ell_1,\ell_2) \), \( j = 1, 2, 3 \). Using \( E\left( I_{1;r,s}(\lambda_{\ell,b}) \right) = f_{rs}(\lambda_{\ell,b}) + \mathcal{O}(1/b) \) and assertion (i) of the lemma, we get

\[
\sum_{\ell_1,\ell_2 \in \mathcal{G}(b)} |T_{1,n}(\ell_1,\ell_2)| \leq \sum_{m_2 \in \mathcal{G}(b)} |E\left( I_{1;r,k,s_k} (\lambda_{\ell_2,b}) \right)|
\]

\[
\times \sum_{\ell_1 \in \mathcal{G}(b)} \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} \left| I_{t;r,j,s_j} (\lambda_{\ell_1,b}) - E\left( I_{t;r,j,s_j} (\lambda_{\ell_1,b}) \right) \right|
\]

\[
= \mathcal{O}_P\left( \sqrt{b^2/(n - b + 1)} \right).
\]
By analogous arguments, the same bound is obtained for the term $T_{2,n}(\ell_1,\ell_2)$. For the term $T_{3,n}(\ell_1,\ell_2)$ observe that $E(T_{3,n}(\ell_1,\ell_2)) = 0$ and that $\sum_{\ell_1,\ell_2\in G(b)} E|T_{3,n}(\ell_1,\ell_2)| \leq \sum_{\ell_1,\ell_2\in G(b)} \sqrt{\text{Var}(T_{3,n}(\ell_1,\ell_2))}$. Using the notation $I_{t;rs}(\lambda) = I_{t;rs}(\lambda) - E(I_{t;rs}(\lambda))$ we get for the variance

$$\text{Var}(T_{3,n}(\ell_1,\ell_2)) = \frac{1}{(n-b+1)^2} \sum_{\ell_1,\ell_2=1}^{n-b+1} \left\{ E(I_{t_1;r_1}s_1)(\lambda t_1, b) T_{t_2;r_2}s_2(\lambda t_2, b) E(I_{t_1;r_1}s_1)(\lambda t_3, b) T_{t_2;r_2}s_2(\lambda t_3, b) \right\} + E(I_{t_1;r_1}s_1)(\lambda t_1, b) T_{t_2;r_2}s_2(\lambda t_2, b) E(I_{t_1;r_1}s_1)(\lambda t_3, b) T_{t_2;r_2}s_2(\lambda t_3, b) + \text{cum}(I_{t_1;r_1}s_1)(\lambda t_1, b) T_{t_2;r_2}s_2(\lambda t_2, b) T_{t_3;r_3}s_3(\lambda t_3, b)$$

$$=: V_{1,n}(\ell_1,\ell_2) + V_{2,n}(\ell_1,\ell_2) + V_{3,n}(\ell_1,\ell_2),$$

with an obvious notation for $V_{1,n}(\ell_1,\ell_2)$, $j = 1, 2, 3$. We show that each one of the terms $V_{j,n}(\ell_1,\ell_2)$ is of the order $O(b/(n-b+1))$. This implies that

$$\sum_{\ell_1,\ell_2\in G(b)} \sqrt{V_{1,n}(\ell_1,\ell_2) + V_{2,n}(\ell_1,\ell_2) + V_{3,n}(\ell_1,\ell_2)} = O(\sqrt{b/(n-b+1)}).$$

The terms $V_{1,n}(\ell_1,\ell_2)$ and $V_{2,n}(\ell_1,\ell_2)$ can be treated similarly, so we only consider $V_{1,n}(\ell_1,\ell_2)$. For this term we have using

$$\left| E(I_{t_1;r_1}s_1)(\lambda t_1, b) T_{t_2;r_2}s_2(\lambda t_1, b) \right| \leq \frac{1}{4\pi^2 b^2} \sum_{g_1, g_2, g_3, g_4=1}^{b} \left\{ \right.$$

$$|\gamma_{r_1,s_1}((t_1 - t_2) + (g_1 - g_3)) \gamma_{s_2,s_1}((t_1 - t_2) + (g_2 - g_4))|$$

$$+ |\gamma_{r_1,s_1}((t_1 - t_2) + (g_1 - g_4)) \gamma_{s_2,s_1}((t_1 - t_2) + (g_2 - g_3))|$$

$$+ |c_{r_1,s_1,r_2,s_1}(t_1 - t_2 + g_1 - g_4, t_1 - t_2 + g_2 - g_4, g_3 - g_4)|\left. \right\}$$

that

$$\left| V_{1,n}(\ell_1,\ell_2) \right| \leq \frac{1}{16\pi^2 b^4} \sum_{t_1, t_2=1}^{n-b+1} C_j(t_1, t_2) C_k(t_1, t_2),$$

where

$$C_j(t_1, t_2) := \sum_{g_1, g_2, g_3, g_4=1}^{b} \left\{ \right.$$}

$$|\gamma_{r_1,s_1}((t_1 - t_2) + (g_1 - g_3)) \gamma_{s_2,s_1}((t_1 - t_2) + (g_2 - g_4))|$$

$$+ |\gamma_{r_1,s_1}((t_1 - t_2) + (g_1 - g_4)) \gamma_{s_2,s_1}((t_1 - t_2) + (g_2 - g_3))|$$

$$+ |c_{r_1,s_1,r_2,s_1}(t_1 - t_2 + g_1 - g_4, t_1 - t_2 + g_2 - g_4, g_3 - g_4)|\left. \right\}.$$
Evaluating the \( C_j(t_1, t_2) C_k(t_1, t_2) \) term in (5.23) leads to the consideration of terms which are similar to the following three:

\[
I_{1,n} = \frac{1}{16\pi^4 b^4 (n - b + 1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{g_1, g_2, g_3, g_4=1}^{b} \sum_{v_1, v_2, v_3, v_4=1}^{b} \left| \gamma_{r_1, r_2}((t_1 - t_2) + (g_1 - g_2)) \right| \left| \gamma_{s_1, s_2}((t_1 - t_2) + (g_2 - g_4)) \right| \times \left| \gamma_{s_3, s_4}((t_1 - t_2) + (v_1 - v_4)) \right|,
\]

\[
I_{2,n} = \frac{1}{16\pi^4 b^4 (n - b + 1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{g_1, g_2, g_3, g_4=1}^{b} \sum_{v_1, v_2, v_3, v_4=1}^{b} \left| \gamma_{r_1, r_2}((t_1 - t_2) + (g_1 - g_3)) \right| \left| \gamma_{s_1, s_2}((t_1 - t_2) + (g_2 - g_4)) \right| \times \left| c_{r_3, s_1} c_{r_4, s_2} (t_1 - t_2 + v_1 - v_4, t_1 - t_2 + v_2 - v_3 - v_4) \right|,
\]

and

\[
I_{3,n} = \frac{1}{16\pi^4 b^4 (n - b + 1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{g_1, g_2, g_3, g_4=1}^{b} \sum_{v_1, v_2, v_3, v_4=1}^{b} \left| c_{r_1, s_1} c_{r_2, s_2} (t_1 - t_2 + g_1 - g_4, t_1 - t_2 + g_2 - g_4, g_3 - g_4) \right| \times \left| c_{r_3, s_1} c_{r_4, s_2} (t_1 - t_2 + v_1 - v_4, t_1 - t_2 + v_2 - v_3 - v_4) \right|.
\]

For \( I_{1,n} \) we get summing first over \( v_2 \), then over \( v_1 \), then over \( g_2 \) and finally over \( t_1 \), that this term is \( O(b/(n - b + 1)) \). For the term \( I_{2,n} \) we get by summing first over \( v_3 \), then over \( v_2 \), then over \( v_1 \), then over \( g_2 \) and finally over \( t_1 \) that this term is \( O(1/(n - b + 1)) \). For \( I_{3,n} \) we get summing first over \( v_3 \), then over \( v_2 \), then over \( v_1 \), then over \( g_3 \), then over \( g_2 \) and finally over \( t_1 \), that \( I_{3,n} = O(1/(b(n - b + 1))) \). From this we conclude that \( V_{1,n}(\ell_1, \ell_2) = O(b/(n - b + 1)) \).

It remains to show that \( V_{3,n}(\ell_1, \ell_2) = O(b/(n - b + 1)) \). For this we get

\[
|V_{3,n}(\ell_1, \ell_2)| \leq \frac{1}{16\pi^4 b^4 (n - b + 1)^2} \sum_{t_1, t_2=1}^{n-b+1} \sum_{g_1, g_2, \ldots, g_8=1}^{b} \left| \sum (X_{r_1}(t_1 + g_1 - 1) \cdot X_{s_1}(t_1 + g_2 - 1), X_{r_2}(t_1 + g_3 - 1) \cdot X_{s_1}(t_1 + g_4 - 1), X_{r_3}(t_2 + g_5 - 1) \cdot X_{s_2}(t_2 + g_6 - 1), X_{r_4}(t_2 + g_7 - 1) \cdot X_{s_2}(t_2 + g_8 - 1)) \right|.
\]

The above cumulant term can be expressed as the sum of products of cumulants over all indecomposable partitions of the two dimensional table:

\[
\begin{align*}
t_1 + g_1 - 1 & \quad t_1 + g_2 - 1 \\
t_1 + g_3 - 1 & \quad t_1 + g_4 - 1 \\
t_2 + g_5 - 1 & \quad t_2 + g_6 - 1 \\
t_2 + g_7 - 1 & \quad t_2 + g_8 - 1;
\end{align*}
\]
see for instance Brillinger (1981), Theorem 2.3.2. Investigation of this sum shows that it is dominated by those indecomposable partitions which contain only pairs with a typical term of such a partitions given by

$$\frac{1}{16\pi^4 b^4} \frac{1}{(n - b + 1)^2} \sum_{t_1, t_2 = 1}^{n-b+1} \sum_{g_1, g_2, \ldots, g_8 = 1}^{b}$$

$$\left| \text{cum}(X_{r_j}(t_1 + g_1 - 1), X_{s_j}(t_2 + g_6 - 1)) \right| \times \left| \text{cum}(X_{r_k}(t_1 + g_3 - 1), X_{s_k}(t_2 + g_8 - 1)) \right|$$

$$\times \left| \text{cum}(X_{s_k}(t_1 + g_4 - 1), X_{r_k}(t_2 + g_7 - 1)) \right|$$

$$= \frac{1}{16\pi^4 b^4} \frac{1}{(n - b + 1)^2} \sum_{t_1, t_2 = 1}^{n-b+1} \sum_{g_1, g_2, \ldots, g_8 = 1}^{b}$$

$$\left| \gamma_{r_j,s_j}((t_1 - t_2) + (g_4 - g_6)) \right| \left| \gamma_{r_k,s_k}((t_1 - t_2) + (g_3 - g_8)) \right|$$

$$\times \left| \gamma_{s_k,r_k}((t_1 - t_2) + (g_4 - g_7)) \right| \left| \gamma_{s_j,r_j}((t_1 - t_2) + (g_2 - g_5)) \right|. $$

Summing first over $g_4$, then over $g_2$, then over $g_3$ and then over $t_1$ we get that this term is $O(b/(n - b + 1)).$ \qed