AC Response of Thin Film Superconductors at Various Temperatures and Magnetic Fields

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Abstract: A common method for measuring the electromagnetic properties of superconductors is to measure their complex magnetic ac susceptibility \( \chi \) as a function of frequency \( \omega \) and amplitude \( H_0 \), and of temperature \( T \) and applied dc magnetic field \( H_{dc} \) as parameters. The basic theory of the linear \( \chi(\omega) \) and nonlinear \( \chi(H_0, \omega) \) is outlined for various geometries, e.g. disks, rings, and strips of thin films or thicker platelets in a perpendicular magnetic field. It is shown how \( \chi(\omega) \) explicitly depends on the linear resistivity \( \rho_{ac}(\omega) = E/J \) or penetration depth \( \lambda_{ac}(\omega) \), and \( \chi(H_0, \omega) \) on the nonlinear current–voltage law \( E(J, B) \) where \( E(r), J(r), B(r) \) are the local electric field, current density, and induction. The dependence of \( E(J, B) \) on \( T \) and on various material properties like pinning forces or pinning energies, structural defects and granularity, leads to an implicit dependence of \( \chi \) on these parameters.

1. Introduction

The magnetic response of type-II superconductors may be considered under various aspects. First, one may be interested in the microscopic motion of the Abrikosov vortices, i.e. in their pinning, thermally activated depinning, crossing of surface barriers, and mutual annihilation. Second, within continuum theory these microscopic processes are described by a current-voltage characteristic \( E = E(J, B) \) (the local electric field caused by the current density \( J \) and depending also on the local induction \( B \)) and by a reversible magnetization curve \( H = H(B) \), which in sufficiently large \( B \) may be approximated by \( H = B/\mu_0 \), but at smaller \( B \) may lead to a geometric barrier for flux penetration (Zeldov et al. 1994, Brandt 1999). In general, these material laws are anisotropic when the material is anisotropic, when anisotropic defects are introduced, or when the Hall effect is taken into account. In the isotropic case one has \( H = H(B)H/H \) and (in simple geometries) \( E = \rho(J, B)J \) with \( \rho = E/J \). Third, the geometry of the experiment, i.e. the specimen shape and orientation of the applied magnetic field and/or transport current have to be accounted for. The geometry determines the local profiles \( B(r) \) and \( J(r) \) and the global response, i.e. the magnetic moment and voltage drop. In some cases the geometry modifies the appropriate law \( E(J, B) \). For example, in isotropic films the critical current density \( J_c \) becomes anisotropic when a strong magnetic field is applied parallel to
the film (Schuster et al. 1997). Fourth, one may consider the vortex state as a “phase” which undergoes transitions from a strong pinning state (“vortex glass”) to a thermally depinned state when in the BT plane an “irreversibility line” $T_{rev}(B)$ or “glass line” $T_g(B)$ is crossed (Blatter et al. 1993, Brandt 1995), or which changes from solid to “liquid” at a “melting line” $T_m(B)$ where the flux-line lattice melts (Zeldov et al. 1995, Majer et al. 1995, Schilling et al. 1996, Welp et al. 1996, Roulin et al. 1998, Sasagawa et al. 1998, Dodgson et al. 1998). In such a “universal” description the detailed microscopic behavior of the vortices remains less clear; in particular, the microscopic interpretation of the length and time scales which diverge at the glass temperature $T_g(B)$ is still open.

At low temperatures $T$, the current–voltage curve $E(J, B)$ is highly nonlinear, typically $E = E_c \cdot (J/J_c)^n$ with large exponent $n(B, T) \gg 1$, and one arrives at the usual Bean model (Bean 1964, Campbell and Evetts 1972) with critical current density $J_c(B, T)$. With increasing $T$, the exponent $n$ decreases and one observes flux creep, i.e. nonlinear relaxation of currents and magnetic moment, see Sec. 2. At high $T$ above the irreversibility line, thermal depinning leads to a linear $E(J)$ which in general is frequency dependent and complex. An interesting task is the detailed measurement of $E(J, B)$ of thin superconductor films at various temperatures. A nontrivial reversible $H(B)$ with a jump at the lower critical field $H_{c1}$ (calculated, e.g. from the Ginzburg-Landau equations) leads to surface barriers, which may be microscopic as predicted by Bean and Livingston (1964), or geometric, caused by the non-ellipsoidal shape of the superconductor.

Some geometry problems have been solved exactly, e.g. the Bean model for long superconductors in parallel field (Bean 1964) and for thin disks (Mikheenko and Kuzovlev 1993) and long strips (Brandt et al. 1993) in perpendicular field, and in strips with transport current without (Norris 1970) and with (Brandt and Indenbom 1993, McElfresh et al. 1994) applied magnetic field. Known are further the linear (Clem et al. 1976, Kes et al. 1989) and nonlinear (Rhyner 1993, Gilchrist and Dombre 1994) relaxation (or flux diffusion) in longitudinal geometry where demagnetization effects are absent, and the linear and nonlinear reversible response of homogeneous ellipsoids, which is obtained from the longitudinal results by introducing a demagnetization factor. The magnetic response of thin strips and disks (Brandt 1994a,b) in a perpendicular field is easily computed by solving a one-dimensional (1D) integral equation for the current density, while thin rectangular (or arbitrarily shaped) films and thick strips and disks are two-dimensional (2D) problems, which still can be computed on a PC (Brandt 1995a, 1996b, 1998).

Recently two different algorithms were presented (Labusch and Doyle 1997, Doyle et al. 1997, Brandt 1999) which account for both constitutive laws $E = E(J, B, r)$ and $H = H(B, r)$ and which apply to all geometries, i.e. to superconductors of any shape. These universal methods allow to compute the geometric “edge barrier”, (Zeldov et al. 1994), e.g. in strips and circular disks (or cylinders) of finite thickness in perpendicular field. For the reversible magnetization curve $H(B)$ any model may be used which approximates the exact Ginzburg-Landau result (Brandt 1997a). If one is interested in the behavior not too close to the upper critical field $H_{c2}$, one may use a simple hyperbola:
\( H(B) \approx [H_{c1}^2 + (B/\mu_0)^2]^{1/2}\), for \( B > \mu_0 H_{c1} \) and with undefined reversible field \(-H_{c1} \leq H \leq H_{c1} \) at \( B = 0 \).

As mentioned above, really 3D problems (e.g. the superconducting cube) have not been computed so far. The difficulty is not only of numerical nature, but it is even not clear which material laws \( \mathbf{E}(\mathbf{J}, B) \) are reasonable when \( \mathbf{J} \) is not parallel to \( \mathbf{B} \). One may introduce two critical current densities \( J_{c\perp} \) and \( J_{c\parallel} \) for currents \( \perp \mathbf{B} \) and \( \parallel \mathbf{B} \). In general, at low \( B \ll B_{c2} \) (the upper critical field) one should have \( E \propto B \) since each vortex contributes independently to the resistivity.

Strictly spoken, even the flux penetration into thin films is a 3D problem if the specimen is not of exactly circular shape or is not an infinitely long strip (Mikitik and Brandt 1999). This is so since during the penetration of flux, the orientation of the sheet current \( \mathbf{J}_s(x, y) = \int \mathbf{J}(x, y, z) \, dz \) changes with time. Therefore, the orientation of the vortices which penetrate from the flat surfaces also changes, since these vortices are perpendicular to \( \mathbf{J}_s \). This means that at any point \( x, y \) away from the symmetry axes and from the specimen edge or center, in principle the vortex orientation may be a function of the depth \( z \), i.e. the vortex arrangement may be twisted. This rotation is most pronounced at moderate magnetic fields; it is absent when the applied field is above the field of full penetration, or when the film thickness \( d \) is smaller than the magnetic penetration depth \( \lambda \), since in these cases the vortices are nearly along \( z \).

2. Definition of AC Susceptibilities

The global magnetic response of a superconductor of arbitrary shape in a time dependent homogeneous magnetic field \( H_a(t) \) may be characterized by its (dipolar) magnetic moment

\[
\mathbf{m}(t) = \frac{1}{2} \int \mathbf{r} \times \mathbf{J}(\mathbf{r}, t) \, d^3r , \tag{1}
\]

were \( \mathbf{J}(\mathbf{r}, t) \) is the current density inside the specimen of volume \( V \). Extensions to inhomogeneous \( H_a(\mathbf{r}, t) \) and to higher (e.g. quadrupolar) moments usually are not required when an experiment is properly designed to obtain the material laws of the specimen. These material laws may be linear or nonlinear. In the linear case, the conductor is completely described by its linear resistivity \( \rho_{ac}(\omega) = i\omega\mu_0\lambda_{ac}^2 \), which in general is complex and frequency dependent, or equivalently, by a conductivity \( \sigma_{ac}(\omega) = 1/\rho_{ac}(\omega) \) or a complex penetration depth \( \lambda_{ac}(\omega) = (\rho_{ac}/i\omega\mu_0)^{1/2} \). For example, an applied field \( H_a(t) = H_0e^{i\omega t} \) generates a local current density \( \mathbf{J}(\mathbf{r}, t) = \mathbf{J}_0(\mathbf{r})e^{i\omega t} \) and an electric field \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})e^{i\omega t} \) where \( \mathbf{J}_0(\mathbf{r}) \) and \( \mathbf{E}_0(\mathbf{r}) = \rho_{ac}\mathbf{J}_0(\mathbf{r}) \) are complex amplitudes. The magnetic field then penetrates into a conducting half space \( x > 0 \) as \( H(x, t) = H_a(t) \exp[-x/\lambda_{ac}(\omega)] \). For magnetic materials with linear permeability \( \mu(\omega) = B/(\mu_0H) \), in these formulae \( \mu_0 \) has to be replaced by \( \mu(\omega)\mu_0 \). A superconductor may exhibit such a linear response above a temperature where thermal depinning of vortices, or thermally assisted flux flow (TAFF) occurs (Kes et al. 1989).

If pinning is strong, or if the reversible magnetization curve \( H = H(B) \) with a finite lower critical field \( H_{c1} \) is accounted for, the magnetic response is nonlinear. At sufficiently large
induction $B \gg \mu_0 H_{c1}$ one may put $B = \mu_0 H$. In this case pinning and flux creep may be modeled by a nonlinear resistivity law $E = \rho(J)J$, e.g. by the model $\rho(J) = \rho_0 (J/J_c)^{n-1}$ with critical current density $J_c(B)$ and creep exponent $n(B) > 1$, as mentioned above. This model results when in the electric field $E(J) = E_0 \exp[-U(J)/kT]$ an activation energy of the form $U(J) = U_0 \ln(J_c/J)$ is inserted, yielding $E = E_c(J/J_c)^n$ with $n = U_0/kT$.

Interestingly, when $E \propto J^n$ then during creep in the fully penetrated state, after some transient time $t_1$ the electric field exhibits universal behavior, i.e. the $r$ and $t$ dependences separate: $E(r, t) \propto f(r)/(t_1 + t)^{n/(n-1)} \approx f(r)/t$, where the profile $f(r)$ depends only on the specimen shape (Gurevich and Brandt 1994, Brandt 1996a). As a consequence, $J(r, t) \propto f(r)^{1/n}/(t_1 + t)^{1/(n-1)} \propto \text{const} - \ln t$ if $n \gg 1$, and thus the magnetic moment relaxes as $m \propto \text{const} - \ln t$. During flux creep the magnetic response to a small ac magnetic field is linear, though the underlying constitutive law $E \propto J^n$ is highly nonlinear (Gurevich and Brandt 1997, Brandt and Gurevich 1996).

From the calculated current density $J(r, H_a)$ the magnetic moment $m(H_a)$ (1) is obtained. If $H_a(t)$ is cycled sinusoidally, $H_a(t) = H_0 \sin(\omega t)$, then $m(t)$ (the component of $m$ along $H_a$) performs a hysteresis loop from which nonlinear complex ac susceptibilities $\chi_\mu(H_0, \omega) = \chi'_\mu - i\chi''_\mu \quad (\mu = 1, 2, 3, \ldots)$ may be defined as

$$
\chi_\mu(H_0, \omega) = \frac{i}{\pi H_0} \int_0^{2\pi} m(t) e^{-i\mu \omega t} d(\omega t).
$$

For small amplitudes $H_0 \to 0$ this yields $\chi_1(H_0, \omega) = m'(0) = \lim_{H_a \to 0} \partial m(H_a)/\partial H_a$. A convenient normalization of the $\chi_\mu(H_0, \omega)$ is thus $\chi_\mu \to \chi_\mu/|m'(0)|$, which at $H_0 \to 0$ yields the ideal diamagnetic susceptibility $\chi_1(0, \omega) = -1$ if $m'(0) < 0$. In the critical state model, with general $J_c(B)$ or with Bean’s $J_c =$ const, the susceptibilities $\chi_\mu(H_0, \omega)$ do not depend on $\omega$, i.e. all cycling rates give the same loop $m(H_a)$ and same $\chi_\mu(H_0)$.

In the opposite limit of linear response, no higher harmonics are generated, i.e. $\chi_\mu = 0$ for $\mu \neq 1$, and $\chi_1 = \chi(\omega)$ does not depend on the amplitude. In complex notation with $H_a(t) = H_0 e^{i\omega t}$ (only the real parts of $H_a, H, B, E, J,$ and $m$ have physical meaning) the linear ac susceptibility $\chi(\omega) = \chi' - i\chi''$ is defined as

$$
\chi(\omega) = \frac{1}{\pi H_0} \int_0^{2\pi} m(t) e^{-i\omega t} d(\omega t).
$$

In the general nonlinear case the $\chi_\mu(H_0, \omega)$ depend on both $H_0$ and $\omega$. In the useful model $E(J) = E_c(J/J_c)^n$, the $\chi_\mu(H_0, \omega)$ depend only on combinations of the form $H_0/\omega^{1/(n-1)}$ or $\omega/H_0^{n-1}$ or any function of these ratios. Thus, $\chi(H_0, \omega)$ for different frequencies $\omega$ is obtained by rescaling the amplitude axis, $\chi(H_0, c\omega) = \chi(H_0/c^{1/(n-1)}, \omega)$ for any constant $c$. This scaling to a good approximation applies also to other $E(J)$ laws if these are sufficiently nonlinear and if the effective exponent $n$ is defined as $n = \partial(\ln E)/\partial(\ln J)$ taken at $J = J_c$ where $J_c$ is the typical current density of the experiment. The nonlinear susceptibility thus depends only on one variable combining amplitude and
frequency, and further on an effective exponent \( n \) and on the geometry.

3. Some Linear AC Susceptibilities

The linear susceptibilities \( \chi(\omega) = \chi' - i\chi'' \) or permeabilities \( \mu(\omega) = \chi(\omega) + 1 \) of conductors with arbitrary shape and arbitrary complex ac resistivity \( \rho_{ac} \) or penetration depth \( \lambda_{ac} = [\rho_{ac}(\omega)/i\omega\mu_0]^{1/2} \) in a homogeneous magnetic field \( H_a(t) = H_0 e^{i\omega t} \) may be obtained by solving a linear diffusion equation for \( J(r,t) \) or \( H(r,t) \) with diffusivity \( D = \rho_{ac}/\mu_0 \). In this way one finds for infinite slabs (width \( 2a \)) in a parallel field (Kes et al. 1989) and long cylinders (Clem et al. 1976) or spheres of radius \( a \) (London 1961) in an axial field:

\[
\chi_{\text{slab}}(\omega) = \frac{\tanh u}{u} - 1, \\
\chi_{\text{cyl}}(\omega) = \frac{2I_1(u)}{u I_0(u)} - 1, \\
\chi_{\text{sphere}}(\omega) = \frac{3 \coth u}{u} - \frac{3}{u^2} - 1,
\]

where \( u = a/\lambda_{ac} = [i\omega\mu_0 a^2/\rho(\omega)]^{1/2} \) is complex and the definition \( \chi(\omega) = m(\omega)/|m(\omega \to \infty)| \) was used. \( I_0 \) and \( I_1 \) are modified Bessel functions. At high frequencies \( (|u| \gg 1) \) one has for \( \mu = \chi + 1 \): \( \mu_{\text{slab}} \approx 1/u, \mu_{\text{cyl}} \approx 2/u, \) and \( \mu_{\text{sphere}} \approx 3/u \).

Interestingly, long cylinders in perpendicular \( H_a(t) \) yield the same normalized susceptibility \( \chi_{\text{cyl}}(\omega) \) (5) as cylinders in parallel field, but the magnetic moment is twice as large: \( m_{\text{cyl}}^\perp(t) = 2m_{\text{cyl}}^\parallel = 2\pi a^2 H_a(t) \chi_{\text{cyl}}(\omega) \). Since this response is linear, one may superimpose both solutions and thus finds the magnetic moment for long cylinders in an applied field \( H_a(t) \) inclined at an arbitrary angle \( \theta \) away from the cylinder axis, \( m_{\text{cyl}}(t;\theta) = \pi a^2 H_a(t) \chi_{\text{cyl}}(\omega)(\cos \theta, 2 \sin \theta) \).

The current density in the cylinder is \( \mathbf{J} = (J_\varphi, J_\parallel) \),

\[
\mathbf{J}(r, \varphi, t;\theta) = \frac{H_a(t)}{\lambda_{ac}} \frac{I_1(r/\lambda_{ac})}{I_0(a/\lambda_{ac})} (\cos \theta, 2 \sin \theta \cos \varphi).
\]

In general, the linear susceptibility for any geometry may be written as an infinite sum, or approximated by a finite sum, of the form

\[
\chi(\omega) = -w \sum_{\nu} \frac{\Lambda_\nu b_\nu^2}{w + \Lambda_\nu} / \sum_{\nu} \Lambda_\nu b_\nu^2.
\]

Here \( \Lambda_\nu \) \( (\nu = 1, 2, \ldots) \) are the eigenvalues of an eigenvalue problem, the \( b_\nu \) (“dipole moments”) are integrals over the eigenfunctions \( f_\nu(r) \), and the complex variable \( w \) is proportional to \( i\omega/\rho_{ac}(\omega) \). The sum in the denominator of Eq. (9) provides the normalization \( \chi(\omega \to \infty) = -1 \) (ideal diamagnetic screening).
For thin film disks and strips with diameter or width $2a$ and thickness $2b \ll 2a$ in perpendicular field the $\Lambda_\nu$ and $b_\nu$ are tabulated by Kötzler et al. 1994 and Brandt 1994c, and the main variable is

$$w = \frac{i\omega\mu_0 ab}{\pi \rho_{ac}(\omega)} = \frac{ab}{\pi \lambda_{ac}^2} = i\omega \tau(\omega).$$  \hspace{1cm} (10)

Tables of the $\Lambda_\nu$ and $b_\nu$ for thick disks or cylinders of arbitrary length in axial field are given by Brandt 1998. For infinite bars with rectangular cross-section $|x| \leq a, |y| \leq b$, in perpendicular $H_a(t)$ along $y$, the variable $w$ (10) applies also and the eigenvalue problem reads (Brandt 1996b)

$$f_\nu(r) = -\frac{\Lambda_\nu}{ab} \int_{-a}^{a} dx' \int_{-b}^{b} dy' \ln |r - r'| f_\nu(r'),$$  \hspace{1cm} (11)

where $r = (x, y), r' = (x', y')$. With the normalization $\int f_\mu f_\nu d^2r = \delta_{\mu\nu}$ the “oscillator strengths” are in this case

$$b_\nu = \frac{\pi}{ab} \int x f_\nu(x, y) d^2r.$$  \hspace{1cm} (12)

For practical purposes, a finite number of terms $\nu = 1 \ldots N$ in the sum (9) is sufficient. When the (real and positive) numbers $\Lambda_\nu$ and $b_\nu$ are known for a given geometry, then $\chi(\omega)$ (9) may be calculated for any complex resistivity $\rho_{ac}(\omega)$. By inverting this relationship between the two complex functions $\chi(\omega)$ and $\rho_{ac}(\omega)$ numerically, the complex resistivity $\rho_{ac}(\omega)$ may be obtained from measured ac susceptibilities as done by Kötzler et al. 1994.

The linear $\chi'(\omega)$ and $\chi''(\omega)$ of disks or cylinders with Ohmic resistivity $\rho_{ac}(\omega) = \rho =$ const in axial $H_a(t)$ are shown in Fig. 1 for various aspect ratios $b/a = 0 \ldots 10 (a =$ radius, $2b =$ length) versus the reduced frequency $\omega \tau$ with $\tau = \mu_0 ab/(\pi \rho)$. Note that in this double logarithmic plot for sufficiently thin disks ($b \ll a$) the $\mu(\omega) = \chi(\omega) + 1$ with increasing frequency crosses over from the behavior in perpendicular geometry (nonlocal flux diffusion) (Brandt 1994c),

$$\mu' = \chi' + 1 \propto 1/\omega, \quad \mu'' = \chi'' \propto \ln(\text{const} \cdot \omega \tau)/\omega,$$  \hspace{1cm} (13)

to the parallel geometry (local flux diffusion),

$$\mu' \approx \mu'' \propto 1/\sqrt{\omega}.$$  \hspace{1cm} (14)

At all aspect ratios $b/a > 0$ the real and imaginary parts at large frequencies coincide, $\mu'(\omega) = \mu''(\omega)$. The physical reason for this finding is that above the frequency where the skin depth $\delta = \sqrt{2} \lambda_{ac} = (2\rho/i\omega\mu_0)^{1/2}$ coincides with the half thickness $b$, the Ohmic conductor nearly behaves like an ideal diamagnet (or superconductor in the Meissner state), screening almost all magnetic flux from the interior of the conductor. The magnetic field lines thus have to flow around the bar or cylinder such that $\mathbf{B}$ is parallel to the specimen surface everywhere. Therefore, at high frequencies, thin (and thick) Ohmic
(and non-Ohmic) conductors in a perpendicular magnetic ac field behave as if the field were applied parallel.

4. Nonlinear Magnetic Response of Thin Films

The nonlinear susceptibility of thin superconductor films with given material law in a perpendicular magnetic field, in general may be obtained only numerically, e.g. by time-integrating an integral equation for the current density $J(r, t)$, inserting this into Eq. (1) for the magnetic moment $m(t)$, and then evaluating the Fourier integral (2) for $\chi(H_0, \omega)$. Such results are presented for thin film disks and rings by Brandt 1997b and for thick disks (or short cylinders) by Brandt 1998.

For narrow rings or thin-walled hollow cylinders, analytical expressions for $\chi(H_0)$ are available in the Bean limit (Brandt 1997b). In this case (and also when a ring contains one or several weak links) the virgin magnetization curve is a straight line, $m(H_a) = -H_a m_{sat}/H_p$ (ideal screening or Meissner state) which saturates to a horizontal line, $|m| = m_{sat}$ at $|H_a| \geq H_p$, the field of first penetration of flux into the hole. This is so since in the current in the ring is limited to a critical value $I_c$. For a narrow ring with width $w = a - a_1 \ll R = (a + a_1)/2$ ($a, a_1$ are the outer and inner radius) and thickness $d \ll R$ one has $I_c = J_{c wd}, m_{sat} = \pi R^2 I_c, H_p \approx I_c \ln(5R/x)/\pi R$ where the inductivity $L \approx \mu_0 R \ln(5R/w)$ was used (Brandt 1997b). For such rings or tubes, in cycled $H_a(t) = H_0 \sin \omega t$ the magnetization loop at small amplitudes $H_0 \leq H_p$ degenerates to a straight line, yielding a constant and real $\chi(H_0) = -m_{sat}/H_p$, while at large $H_0 \geq H_p$ the loop $m(H_a)$ is a parallelogramme with height $2m_{sat}$. This yields the normalized ac susceptibility $\chi(h) = \chi(H) H_p/m_{sat} = \chi' - i\chi''$, with $h = H_0/H_p$ and $s = 2/h - 1$,

$$\chi'(h) = \begin{cases} 1, & h \leq 1 \\ 0, & h > 1 \end{cases}$$

$$\chi''(h) = \begin{cases} -\frac{1}{2} - \frac{1}{\pi} \arcsin s - \frac{1}{\pi} s \sqrt{1 - s^2}, & h \leq 1, \\ \frac{4h - 1}{\pi h^2} = \frac{1 - s^2}{\pi}, & h > 1 \end{cases}$$

(15)

see the curve $a_1/a = 1$ in Fig. 2. The polar plot ($\chi''$ versus $\chi'$ with $h$ as parameter) of the ring susceptibility (15) is symmetric, i.e. $\chi''(\chi')$ yields the same curve as $\chi''(-1 - \chi')$.

The maximum of $\chi''_{max} = 1/\pi = 0.318$ occurs at $h = 2$ (at $s = 0$). For large amplitudes $h = H_0/H_p \gg 1$ one has $\chi'(h) \approx -1.69/h^{3/2}$ and $\chi''(h) \approx 4/(\pi h)$.

Very good approximate expressions are available within the Bean critical state model for thin strips (width $2a$, thickness $d \ll a$, length $L \gg a$), disks (radius $a$), and ellipses (semi-axes $a$ and $b$, eccentricity $e = b/a \leq 1$) (Mikitik and Brandt 1999). In these three cases and also for square and rectangular films, the virgin magnetization curve with less than 1% error is

$$m(H_a) = -m_{sat} \tanh(H_a/H_1),$$

(16)
with \( H_1 = m_{\text{sat}}/|m'(0)| \), and explicitly,

\[
m_{\text{sat}} = \begin{cases} J_c d L a^2, & H_1 = J_c d / \pi \quad \text{(strip)}, \\ \pi J_c d a^3, & H_1 = \pi J_c d / 8 \quad \text{(disk)}, \\ 4 J_c d a b^2 \cos(e \pi/2) / \left(1 - e^2\right), & H_1 = J_c d E(k) \cos(e \pi/2) / \left(1 - e^2\right) \quad \text{(ellipse)}. 
\]

The formulae (19) for ellipses with \( e = b/a \leq 1 \) reproduce the results for disks \((e \to 1, \text{radius } a = b)\) and strip \((e \to 0, \text{length } 2a, \text{width } \sqrt{8/3} b)\). From the virgin curve \( m(H_a) \) (16), the Bean magnetization loops with amplitude \( H_0 \) follow as

\[
m_{\downarrow}(H_a, H_0) = m(H_0) + 2m\left(H_a - H_0 \right),
\]

(decreasing \( H_a \)), and \( m_{\uparrow}(H_a, H_0) = -m_{\downarrow}(-H_a, H_0) \) (increasing \( H_a \)). Inserting this into definition (2) one obtains the universal nonlinear ac susceptibility \( \tilde{\chi}(h) \) of films normalized to \( \tilde{\chi}(0) = -1 \) and depending on \( h = H_0 / H_1 \):

\[
\tilde{\chi}(h) = \frac{2i}{\pi h} \int_{-\pi/2}^{\pi/2} \left[ 2 \tanh \left( \frac{1 - \sin \varphi}{2} \right) - \tanh h \right] e^{-i \varphi} \, d\varphi.
\]

From this universal function the not normalized \( \chi(H_0) \) is obtained as

\[
\chi(H_0) = \frac{m_{\text{sat}}}{H_1} \tilde{\chi}\left(\frac{H_0}{H_1}\right).
\]

Figure 2 shows some normalized ac susceptibilities \( \tilde{\chi}(h) \) for thin film rings and for the disk and strip in the Bean limit, i.e. for large creep exponent \( n \to \infty \). Note that the curves for disks and strips practically coincide.

The nonlinear ac susceptibility of disks with various creep exponents \( n \geq 3 \) is depicted in Fig. 3. Here \( E(J) = E_c(J/J_c)^n \) was assumed, the circular frequency was \( \omega = \omega_1 \equiv 2E_c/(\mu_0 J_c d a) \), and the amplitude \( H_0 \) is in units \( J_c d \) \((d = \text{tickness}, a = \text{radius of the disk})\). The plots in Fig. 3 apply to any frequency \( \omega \) if the unit of \( H_0 \) is changed to \((\omega/\omega_1)^{1/(n-1)} J_c d\).

5. Final Remarks

The measured ac susceptibilities depend on the temperature \( T \) indirectly via the \( T \) dependence of the material properties. The nonlinear \( \chi(H_0, \omega) \) depends on \( J_c(B, T) \) and on the creep exponent \( n(B, T) = U_0 / kT \), see above. The linear \( \chi(\omega) \) depends on the variable \( w \), Eq. (10), which contains the linear complex ac resistivity \( \rho_{ac}(\omega, B, T) \), see e.g. Coffey and Clem 1990 and Brandt 1990 for theories of \( \rho_{ac}(\omega) \) of type-II superconductors. The linear \( \chi(\omega) \) usually is measured in a bias dc magnetic field \( H_a^{dc} \gg H_0 \), but the nonlinear \( \chi(H_0, \omega) \) discussed above assumed the absence of a bias field and has to be recomputed if \( H_a^{dc} > 0 \).
Furthermore, if $H_a(t)$ is too small, the effects of a finite $H_{c1}$ should be considered, which were disregarded here but may lead to a geometric barrier, in particular in thin films, as discussed in the introduction. The edge barrier may be suppressed by using small measuring coils close to the film, see e.g. Gilchrist and Brandt 1996 for a detailed theory and analytic formulae. The numerical program described by Brandt 1999 in principle accounts for arbitrary reversible magnetization $H(B)$ and thus for $H_{c1}$ and for the edge barrier in strips or disks of constant thickness. It may also be generalized to allow for dc and ac transport currents in strips.

The problem of ac losses in thin strip superconductors with transport current is related to the problem of ac susceptibilities. It is conceptually difficult since with increasing ac amplitude the losses may be caused first by flux jumping over the edge barrier, then by bulk pinning (Bean critical state), then (after full penetration) by the rapidly increasing electric field $E \propto J^n$. The hysteresis losses caused by the edge barrier in superconductor strips without bulk pinning were recently calculated by Clem and Benkraouda 1998.

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Figure 1: The real (top) and imaginary (bottom) parts of the linear ac permeability $\mu = \chi + 1 = \mu' - i\mu''$ of Ohmic cylinders with rectangular cross-section $2a \times 2b$ in an axial magnetic ac field for aspect ratios $b/a = 0$ (thin disk), $b/a = 0.01$, $0.03$, $0.1$, $0.3$, $1$, $2$, and $3$ (short cylinders) computed from the sum (9) (solid lines). The dashed curves give the analytic expression (5) for long cylinders with $b/a = 3$, $5$, and $10$, with the variable $u = a/\lambda_{ac} = (i\omega\tau\pi a/b)^{1/2}$ inserted, where $\tau = \mu_0 ab/(\pi \rho)$ is the time scale. For $b/a = 3$ the numerical and analytical curves are very close. To facilitate comparison, some $\mu'$ curves are repeated as dotted lines in the lower plot.
Top: Real and imaginary parts of the nonlinear complex susceptibility $\chi(H_0) = \chi' - i\chi''$ of thin strips, disks, and rings with various ratios $a_1/a$ of the inner and outer radius, plotted versus the amplitude $H_0$ of the ac field. Bean model with constant $J_c$. The unit of $H_0$ is $H_1 = J_c d/\pi$ for the strip and $H_1 = \pi J_c d/8$ for the disk. For rings $H_0$ is in units of a penetration field $H'_p \approx H_p$ which gives best fit to the sharp rise in $\chi''$ of the ideal ring: $H'_p/J_c d = 0.126 (0.210)$ for $a_1/a = 0.9 (0.8)$, while from Table 1 in Brandt 1997b one has $H_p/J_c d = 0.136 (0.239)$. For the narrow ring with width $w = a - a_1 \ll a$ one has $H'_p = H_p$, $H_p/J_c d \approx (w/\pi a) \ln(5a/w)$. Bottom: Polar plots of the same $\chi' - i\chi''$. For the narrow ring $\chi'$ and $\chi''$ [Eq. (15)] are discontinuous at $H_0 = H_p$ and the polar plot is symmetric.
Figure 3: Top: Real and imaginary parts of the nonlinear complex susceptibility $\chi(H_0) = \chi' - i\chi''$ of a thin disk plotted versus the ac amplitude $H_0$ for various creep exponents $n = 3, 5, 11, 51,$ and $201$ in the law $E = E_c(J/J_c)^n$ with constant $J_c$. Here $H_0$ is in units $J_c d$. Bottom: Polar plots of the same $\chi = \chi' - i\chi''$. The dashed line shows the Bean model ($n \to \infty$) for the disk.