On Duality Rotations in Light-Like Noncommutative Electromagnetism

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ABSTRACT: We study electric-magnetic duality rotations for noncommutative electromagnetism (NCEM). We express NCEM as a nonlinear commutative $U(1)$ gauge theory and show that it is self-dual when the noncommutativity parameter $\theta$ is light-like (e.g. $\theta^{0i} = \theta^{1i}$). This implies, in the slowly varying field approximation, self-duality of NCEM to all orders in $\theta$.

Keywords: Duality; Born-Infeld; Noncommutative Gauge Theory

Introduction

Field theories on noncommutative spaces have received renewed interest since their relevance in describing Dp-branes effective actions (see [1] and references therein). Noncommutativity in this context is due to a nonvanishing NS background two-form on the Dp-brane. Initially space-like (magnetic) backgrounds ($B^{ij} \neq 0$) were considered, then NCYM theories also with time noncommutativity ($B^{0i} \neq 0$) have been studied [2]. It turns out that unitarity of NCYM holds only if $B$ is space-like or light-like (e.g. $B_{0i} = -B_{1i}$) and that these are precisely the NCYM theories that can be obtained from open strings in the decoupling limit $\alpha' \to 0$ [4]. Following [1], gauge theory on a Dp-brane with constant two-form $B$ can be described via a commutative Lagrangian and field strength $\mathcal{L}(F + B)$ or via a noncommutative one $\hat{\mathcal{L}}(\hat{F})$, where $\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu]$. (Here $\star$ is the star product, on coordinates $[x^\mu \star x^\nu] = x^\mu x^\nu - x^\nu x^\mu = i\theta^{\mu\nu}$, where $\theta$ depends on $B$ and the metric on the Dp-brane). These two descriptions are complementary and are

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related by Seiberg-Witten map (SW map). In the $\alpha' \to 0$ limit \[1\] the exact effective electromagnetic theory on a D$p$-brane is NCEM, this is equivalent, via SW map, to a nonlinear commutative $U(1)$ gauge theory. For a D3-brane, in the slowly varying field approximation, we give an explicit expression of this nonlinear $U(1)$ theory and we show that it is self-dual when $B$ (or $\theta$) is light-like. Via SW map solutions of $U(1)$ nonlinear electromagnetism are mapped into solutions of NCEM, so that duality rotations are also a symmetry of NCEM, i.e. NCEM is self-dual. When $\theta$ is space-like we do not have self-duality and the S-dual of space-like NCYM is a noncommutative open string theory decoupled from closed strings \[5\]. Related work appeared in \[6\]. Self-duality of NCEM was initially studied in \[3\] to first order in $\theta$.

We show self-duality of NCEM using Gaillard-Zumino approach \[7\] to study duality rotations in nonlinear electromagnetism, we thus provide [see (29)] a new example of Lagrangian satisfying Gaillard-Zumino self-duality condition. We present here the case where the axion and the dilaton are zero. It is also possible to include arbitrary constant axion and dilaton as well as a kinetic and interaction term for Higgs fields. Higgs fields in the noncommutative description result minimaly coupled to the gauge field. Formally (covariant derivatives, minimal couplings) NCEM resembles commutative $U(N)$ YM, and on tori with rational $\theta$ the two theories are $T$-dual \[8\]. Self-duality of NCEM then hints to a possible duality symmetry of the equations of motion of $U(N)$ YM.

This paper is organized as follows. We first review Gaillard-Zumino self-duality condition and see that the D3-brane Lagrangian is self-dual \[3\]. We then present a simple argument showing why we need $B$ and $\theta$ light-like in the zero slope limit. Finally we discuss self-duality of noncommutative Born-Infeld theory and NCEM.

**Duality Rotations**

Consider in four dimension and with metric $g_E$ the Lagrangian density (Lagrangian for short)

$$\mathcal{L}(F_{\mu\nu}, g_E\mu\nu, \chi^i) = \sqrt{-g_E} L(F_{\mu\nu}, g_E\mu\nu, \chi^i)$$

where $\chi^i$ are some constant parameters that can possibly have space-time indices. If we define\[4\]

\[\Omega_{\mu\nu} = \frac{1}{2} \sqrt{-g_E} \epsilon_{\mu\nu\rho\sigma} \Omega^{\rho\sigma}, \quad \epsilon^{0123} = -\epsilon_{0123} = 1; \quad g_E \text{ has signature } (-, +, +, +).\]

We have $\Omega^{*\mu\nu} = -\Omega_{\mu\nu}$, $\Omega^{*\mu\nu} = \frac{1}{2} \sqrt{-g_E} \epsilon^{\mu\nu\rho\sigma} \Omega_{\rho\sigma}$.
\[ K^{*\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}} \quad \left( \frac{\partial F^\rho_{\sigma}}{\partial F_{\mu\nu}} = \delta^\rho_\mu \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma \right) \]

then the Bianchi identity and the equations of motion (EOM) for \( \mathcal{L} \) read

\[ \partial_\mu (\sqrt{-g} F^{*\mu\nu}) = \partial_\mu \tilde{F}^{\mu\nu} = 0 , \]

\[ \partial_\mu (\sqrt{-g} K^{*\mu\nu}) = \partial_\mu \tilde{K}^{\mu\nu} = 0 , \]

where \( \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) and \( \tilde{K}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} K_{\rho\sigma} \). Under the infinitesimal transformations

\[ \delta \begin{pmatrix} K \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} K \\ F \end{pmatrix} \]

the system of EOM (3) is mapped into itself. We can also allow the parameters \( \chi \) to vary

\[ \delta \chi^i = \xi^i(\chi) , \]

then under (4),(5) the \( \mathcal{L}_\chi \) EOM (3) are mapped into the \( \mathcal{L}_{\chi+\delta\chi} \) EOM. Consistency of (4),(3) with the definition of \( K \), i.e. \( K + \delta K = \frac{\partial}{\partial (F+\delta F)} \mathcal{L}(F+\delta F, g, \chi+\delta \chi) \) holds in particular if \( \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \) belongs to the Lie algebra of \( SL(2,R) \), and the variation of the Lagrangian under (4),(5) can be written as

\[ \delta \mathcal{L} \equiv (\delta F + \delta \chi) \mathcal{L} = \frac{1}{4} (BF\tilde{F} + CK\tilde{K}) . \]

When (8) holds, under a finite \( SL(2,R) \) rotation a solution \( F \) of \( \mathcal{L}_\chi \) is mapped into a solution \( F' \) of \( \mathcal{L}_{\chi'} \) and we say that \( \mathcal{L}_\chi \) is self-dual. A self-duality condition equivalent to (8) is obtained using (4) to evaluate the \( \delta_F \mathcal{L} \) term in (3)

\[ \delta_\chi \mathcal{L} = \frac{1}{4} BF\tilde{F} - \frac{1}{4} CK\tilde{K} - \frac{1}{2} DF\tilde{K} . \]

If the \( \chi \) parameters are held fixed (strict self-duality) the maximal duality group is \( U(1) \) [7], see also the nice review [11]. Viceversa we can always extend a \( U(1) \) self-dual Lagrangian to a \( SL(2,R) \) self-dual one introducing two real valued scalars \( S = S_1 + iS_2 \) (axion and dilaton). If the Lagrangian \( \mathcal{L}(F) \) is self dual under \( U(1) \), then the new Lagrangian

\[ \tilde{\mathcal{L}}(F,S) \equiv \mathcal{L}(S_2^2 F) + \frac{1}{4} S_1 F\tilde{F} , \]
where $\tilde{F} \equiv F_{\mu\nu} \tilde{F}^{\mu\nu}$, is self-dual under $SL(2, R)$ provided that $S = S_1 + iS_2$ transforms as

$$S' = \frac{aS + b}{cS + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R) . \quad (9)$$

The term in (8) proportional to $S_1$ is a total derivative if $S_1$ is constant; this term does not affect the EOM but enters the definition of $K$. To recover the original Lagrangian just set $S_1 = 0$ and $S_2 = 1$; the duality group is then $U(1)$, the $SL(2, R)$ subgroup that leaves $S_1 = 0$ and $S_2 = 1$ invariant.

We now discuss self-duality of the D3-brane effective action in a IIB supergravity background with constant axion, dilaton NS and RR two-forms. The background two-forms can be gauged away in the bulk and we are left with the field strength $\mathcal{F} = F + B$ on the D3-brane. Here $B$ is defined as the constant part of $\mathcal{F}$, or $B = \mathcal{F}|_{\text{spatial } \infty}$ since $F$ vanish at spatial infinity. For slowly varying fields the Lagrangian, in string and in Einstein frames, respectively reads

$$\mathcal{L} = -\frac{1}{\alpha' g_s} \sqrt{-\det(g + \alpha' \mathcal{F})} + \frac{1}{4} C \mathcal{F} \tilde{\mathcal{F}}$$

$$= -\frac{1}{\alpha'} \sqrt{-\det(g_E + \alpha' S_2^{1/2} \mathcal{F})} + \frac{1}{4} S_1 \mathcal{F} \tilde{\mathcal{F}}$$

$$= -\frac{1}{\alpha'} g_E \sqrt{1 + \frac{\alpha'^2}{2} S_2 \mathcal{F}^2 - \frac{\alpha'^4}{16} S_2^2 (\mathcal{F} \mathcal{F}^*)^2} + \frac{1}{4} S_1 \mathcal{F} \tilde{\mathcal{F}} \quad (10)$$

in the second line $S = S_1 + iS_2 = C + i g_s$ while, in the last line, we have simply expanded the 4x4 determinant and $\mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \mathcal{F} \mathcal{F}^* \equiv \mathcal{F}_{\mu\nu} \mathcal{F}^{*\mu\nu}$.

Under the $SL(2, R)$ rotation

$$\begin{pmatrix} \mathcal{K}' \\ \mathcal{F}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{K} \\ \mathcal{F} \end{pmatrix}, \quad \begin{pmatrix} aS + b \\ cS + d \end{pmatrix} \quad (\alpha')' = \alpha' \quad (11)$$

where $\tilde{\mathcal{K}} = \frac{\partial}{\partial \mathcal{F}} \mathcal{L}$, it is not difficult to directly check that the Lagrangian $\mathcal{L}$ satisfies the self-duality condition (9) (with $F, K$ replaced by $\mathcal{F}, \mathcal{K}$). For simplicity, we will later set $S_1 = 0$ and $S_2 = g_s = 1$. Then the duality group reduces to $U(1)$, moreover

\footnote{we omit the RR four-form $C_4$ because it is invariant under $SL(2, R)$ duality rotations. If $\mathcal{L}$ is self-dual then also $\mathcal{L}_{D3} = \mathcal{L} + \tilde{\mathcal{C}}_4$ is self-dual (here $\tilde{\mathcal{C}}_4 = \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$).}
string and Einstein frames coincide. The Lagrangian reduces to the Born-Infeld Lagrangian

$$\mathcal{L}_{BI} = -\frac{1}{\alpha'^2} \sqrt{- \det (g + \alpha' \mathcal{F})}.$$ (12)

Using (8) we can always recover the more general situation (10). The explicit expression of $\mathcal{K}$ is

$$\mathcal{K}_{\mu\nu} = \frac{\mathcal{F}^*_{\mu\nu} + \frac{\alpha'^2}{4} \mathcal{F} \mathcal{F}^* \mathcal{F}_{\mu\nu}}{\sqrt{1 + \frac{\alpha'^2}{2} \mathcal{F}^2 - \frac{\alpha'^4}{16} (\mathcal{F} \mathcal{F}^*)^2}}.$$ (13)

From (11) and (13), one can extract how $B$ (the constant part of $\mathcal{F}$) transforms

$$B'_{\mu\nu} = \cos \gamma B_{\mu\nu} - \sin \gamma \frac{B^*_{\mu\nu} + \frac{\alpha'^2}{4} BB^* B_{\mu\nu}}{\sqrt{1 + \frac{\alpha'^2}{2} B^2 - \frac{\alpha'^4}{16} (BB^*)^2}};$$ (14)

this transformation is independent from the slowly varying fields approximation.

**Open/closed strings and light-like noncommutativity**

The open and closed string parameters are related by (see [1], the expressions for $G$ and $\theta$ first appeared in [10])

$$\frac{1}{g + \alpha' B} = G^{-1} + \frac{\theta}{\alpha'}$$

$$g^{-1} = (G^{-1} - \theta/\alpha') G (G^{-1} + \theta/\alpha') = G^{-1} - \alpha'^{-2} \theta G \theta$$

$$\alpha' B = -(G^{-1} - \theta/\alpha') \theta/\alpha' (G^{-1} + \theta/\alpha')$$

$$G_s = g_s \sqrt{\frac{\det G}{\det (g + \alpha' B)}} = g_s \sqrt{\det G \det (G^{-1} + \theta/\alpha')} = g_s \sqrt{\det g^{-1} \det (g + \alpha' B)}$$ (15)

The decoupling limit $\alpha' \to 0$ with $G_s, G, \theta$ nonzero and finite [1] leads to a well defined field theory only if $B$ is space-like or light-like [3]. Looking at the closed and open string coupling constants it is easy to see why one needs this space-like or light-like condition on $B$. Consider the coupling constants ratio $G_s/g_s$, that expanding the 4x4 determinant reads (here $B^2 = B_{\mu\nu} B_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}, \theta^2 = \theta^{\mu\nu} \theta^{\rho\sigma} G_{\mu\rho} G_{\nu\sigma}$ and so on)

$$\frac{G_s}{g_s} = \sqrt{1 + \frac{\alpha'^{-2}}{2} \theta^2 - \frac{\alpha'^{-4}}{16} (\theta \theta^*)^2} = \sqrt{1 + \frac{\alpha'^2}{2} B^2 - \frac{\alpha'^4}{16} (BB^*)^2}.$$ (16)
Both $G_s$ and $g_s$ must be positive; since $G$ and $\theta$ are by definition finite for $\alpha' \to 0$ this implies $\theta \theta^* = 0$ and $\theta^2 \geq 0$. Now $\theta \theta^* = 0 \Leftrightarrow \det \theta = 0 \Leftrightarrow \det B = 0 \Leftrightarrow BB^* = 0$. In this case from (16) we also have $\theta^2 = \alpha'^4 B^2$. In conclusion the $\alpha' \to 0$ limit defined by keeping $G_s, G, \theta$ nonzero and finite [1], is well defined iff

$$B^2 \geq 0, \quad BB^* = 0 \quad \text{i.e.} \quad \theta^2 \geq 0, \quad \theta \theta^* = 0 \quad (17)$$

This is the condition for $B$ (and $\theta$) to be space-like or light-like. Indeed with Minkowski metric (17) reads $\vec{B}^2 - \vec{E}^2 \geq 0$ and $\vec{E} \perp \vec{B}$.

If we now require the $\alpha' \to 0$ limit to be compatible with duality rotations, we immediately see that we have to consider only the light-like case $B^2 = BB^* = 0$. Indeed under $U(1)$ rotations the electric and magnetic fields mix up, in particular under a $\pi/2$ rotation (14) a space-like $B$ becomes time-like.

In the light-like case, relations (15) simplify considerably. The open and closed string coupling constants coincide: $G_s = g_s = S_2^{-1} = 1$. Use of the relations

$$\Omega_{\mu\rho}^* \Omega^{*\rho\nu} - \Omega_{\mu\rho} \Omega^{\rho\nu} = \frac{1}{2} \Omega^2 \delta^\nu_\mu, \quad \Omega_{\mu\rho} \Omega^{*\rho\nu} = \Omega^*_{\mu\rho} \Omega^{\rho\nu} = -\frac{1}{4} \Omega \Omega^* \delta^\nu_\mu \quad (18)$$

valid for any antisymmetric tensor $\Omega$, shows that any two-tensor at least cubic in $\theta$ (or $B$) vanishes. It follows that $g^{-1} G \theta = \theta$ and that the raising or lowering of the $\theta$ and $B$ indices is independent from the metric used. We also have

$$B_{\mu\nu} = -\alpha'^{-2} \theta_{\mu\nu} \quad . \quad (19)$$

**Self-duality of NCBI and NCEM**

We now study duality rotations for noncommutative Born-Infeld (NCBI) theory and its zero slope limit that is NCEM. The relation between the NCBI and the BI Lagrangians is [1]

$$\hat{L}_{\text{BI}}(\hat{F}, G, \theta, G_s) = L_{\text{BI}}(F + B, g) + O(\partial F) + \text{tot. der.} \quad (20)$$

where $O(\partial F)$ stands for higher order derivative corrections, $\hat{F}$ is the noncommutative $U(1)$ field strength and we have set $g_s = 1$. The NCBI Lagrangian is

$$\hat{L}_{\text{BI}}(\hat{F}, G, \theta, G_s) = \frac{-1}{\alpha'^2 G_s} \sqrt{-\det(G + \alpha' \hat{F})} + O(\partial \hat{F}) \quad . \quad (21)$$

In the slowly varying field approximation the action of duality rotations on $\hat{L}_{\text{BI}}$ is derived from self-duality of $L_{\text{BI}}$. If $\hat{F}$ is a solution of the $\hat{L}_{\text{BI}}^{G_s, G, \theta}$ EOM then $\hat{F}^\nu$
obtained via $\hat{F} \overset{\text{SW map}}{\leftrightarrow} F \overset{\text{duality rot.}}{\leftrightarrow} F' \overset{\text{SW map}}{\leftrightarrow} \hat{F}'$ is a solution of the $\hat{\mathcal{L}}_{\text{BI}}^{G',G',\theta'}$ EOM where $G', G', \theta'$ are obtained using (15) from $g', B'$ and $g' = g_s = 1$.

In the light-like case we have $G_s = g_s = 1$, the $B$ rotation (14) simplifies to

$$B'_{\mu\nu} = \cos\gamma B_{\mu\nu} - \sin\gamma B^*_{\mu\nu} ,$$

using (11) and (22) the $U(1)$ duality action on the open string variables is

$$G' = G , \quad \theta'^{\mu\nu} = \cos\gamma \theta^{\mu\nu} - \sin\gamma \theta^*^{\mu\nu} .$$

For $\theta$ light-like, solutions $\hat{F}$ of $\hat{\mathcal{L}}^{G,\theta}$ are mapped into solutions $\hat{F}'$ of $\hat{\mathcal{L}}^{G',\theta'}$ and therefore $\hat{\mathcal{L}}^{G,\theta}$ is self-dual. Moreover, by a rotation in three dimensional space we can map $\theta'$ into $\theta$. In order to show self-duality of NCEM we consider the zero slope limit of (20) and verify that the resulting lagrangian on the r.h.s. of (20) is self-dual. We rewrite $\mathcal{L}_{\text{BI}}$ in terms of the open string parameters $G, \theta$

$$\mathcal{L}_{\text{BI}} = -\frac{1}{\alpha'^2} \sqrt{-\det(g + \alpha' F)} = -\frac{1}{\alpha'^2} \sqrt{\frac{\det(g + \alpha' B + \alpha' F)}{\det(g + \alpha' B)}}$$

$$= -\frac{1}{\alpha'^2} \sqrt{-\det(G + \alpha' F + G\theta F)} .$$

(24)

The determinant in the last line can be evaluated as sum of products of traces (Newton-Leverrier formula). Each trace can then be rewritten in terms of the six basic Lorentz invariants $F^2, FF^*, F\theta, F\theta^*, \theta^2 = \theta\theta^* = 0$, explicitly

$$\det G^{-1} \det (G + \alpha' F + G\theta F) = (1 - \frac{1}{2} \theta F)^2 + \alpha'^2 \left[ \frac{1}{2} F^2 + \frac{1}{4} \theta F^* FF^* \right] - \alpha'^4 \left( \frac{1}{4} FF^* \right)^2$$

Finally we take the $\alpha' \to 0$ limit of (24), drop the infinite constant and total derivatives and denote by $\mathcal{L}_\theta^{\alpha' \to 0}$ the resulting Lagrangian

$$\mathcal{L}_\theta^{\alpha' \to 0} = -\frac{1}{4} F^2 - \frac{1}{2} \theta F^* FF^* .$$

(25)

We thus have an expression for NCEM in terms of $G, \theta$ and $F$

$$\hat{\mathcal{L}}_{\text{EM}} \equiv -\frac{1}{4} \hat{F}^2 = \mathcal{L}_\theta^{\alpha' \to 0} + O(\partial F) + \text{tot. der.}$$

(26)

The Lagrangian (25) satisfies the self-duality condition (7) with $\chi = \theta$ and $A=D=0, \ C= -B$ and therefore NCEM is self-dual under the $U(1)$ duality rotations (23) and $\hat{F}' = \cos\gamma F - \sin\gamma K$.
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