The spectral excess theorem for distance-regular graphs having distance-$d$ graph with fewer distinct eigenvalues

M.A. Fiol

Universitat Politècnica de Catalunya, BarcelonaTech
Dept. de Matemàtica Aplicada IV, Barcelona, Catalonia
(e-mail: fiol@ma4.upc.edu)

Abstract

Let $\Gamma$ be a distance-regular graph with diameter $d$ and Kneser graph $K = \Gamma_d$, the distance-$d$ graph of $\Gamma$. We say that $\Gamma$ is partially antipodal when $K$ has fewer distinct eigenvalues than $\Gamma$. In particular, this is the case of antipodal distance-regular graphs ($K$ with only two distinct eigenvalues), and the so-called half-antipodal distance-regular graphs ($K$ with only one negative eigenvalue). We provide a characterization of partially antipodal distance-regular graphs (among regular graphs with $d$ distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance $d$ from every vertex. This can be seen as a general version of the so-called spectral excess theorem, which allows us to characterize those distance-regular graphs which are half-antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

Keywords: Distance-regular graph; Kneser graph; Partial antipodality; Spectrum; Predis- tance polynomials.

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1 Preliminaries

Let $\Gamma$ be a distance-regular graph with adjacency matrix $A$ and $d+1$ distinct eigenvalues. In the recent work of Brouwer and the author [2], we studied the situation where the distance-$d$ graph $\Gamma_d$ of $\Gamma$, or Kneser graph $K$, with adjacency matrix $A_d = p_d(A)$, has fewer distinct eigenvalues. In this case we say that $\Gamma$ is partially antipodal. Examples are the so-called half antipodal ($K$ with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs ($K$ being disjoint copies of a complete graph). Here we generalize such a study to the case when $\Gamma$ is a regular graph with $d+1$ distinct eigenvalues. The main result of this paper is a characterization of partially antipodal distance-regular graphs, among regular graphs with $d+1$ distinct eigenvalues, in terms of the spectrum
and the mean number of vertices at maximal distance \(d\) from every vertex. This can be seen as a general version of the so-called spectral excess theorem, and allows us to characterize those distance-regular graphs which are half antipodal, antipodal, bipartite, or with Kneser graph being strongly regular. Other related characterizations of some of these cases were given by the author in [8, 9, 10]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [1], Brouwer and Haemers [3], and Van Damm, Koolen and Tanaka [6].

Let \(\Gamma\) be a regular (connected) graph with degree \(k\), \(n\) vertices, and spectrum \(\text{sp } \Gamma = \{\lambda_0, \lambda_1, \ldots, \lambda_d\}\), where \(\lambda_0(= k) > \lambda_1 > \cdots > \lambda_d\), and \(m_0 = 1\). In this work, we use the following scalar product on the \((d + 1)\)-dimensional vector space of real polynomials modulo \(m(x) = \prod_{i=0}^{d}(x - \lambda_i)\), that is, the minimal polynomial of \(A\).

\[
\langle p, q \rangle_\Gamma = \frac{1}{n} \text{tr}(p(A)q(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i), \quad p, q \in \mathbb{R}[x]/(m(x)).
\]  

(1)

This is a special case of the inner product of symmetric \(n \times n\) real matrices \(M, N\), defined by \(\langle M, N \rangle = \frac{1}{n} \text{tr}(MN)\). The predistance polynomials \(p_0, p_1, \ldots, p_d\), introduced by the author and Garriga [13], are a sequence of orthogonal polynomials with respect to the inner product \([\ ]\), normalized in such a way that \(\|p_i\|_\Gamma^2 = \pi_i(k)\). Then, it is known that \(\Gamma\) is distance-regular if and only if such polynomials satisfy \(p_i(A) = A_i\) (the adjacency matrix of the distance-\(i\) graph \(\Gamma_i\)) for \(i = 0, \ldots, d\), in which case they turn out to be the distance polynomials. In fact, we have the following strongest proposition, which is a combination of results in [13, 7].

**Proposition 1.** A regular graph \(\Gamma\) as above is distance-regular if and only if there exists a polynomial \(p\) of degree \(d\) such that \(p(A) = A_d\), in which case \(p = p_d\). \(\square\)

Moreover, the Hoffman polynomial \(H\), such that \(H(\lambda_i) = n\delta_{n0}\) and \(H(A) = J\), turns out to be \(H = p_0 + p_1 + \cdots + p_d\). Also, as in the case of distance-regular graphs, the multiplicities of \(\Gamma\) can be obtained from the values of \(p_d\) since,

\[
(-1)^i p_d(\lambda_i) \pi_i m_i = p_d(\lambda_0) \pi_0, \quad i = 1, \ldots, d.
\]  

(2)

where \(\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|\). Indeed, let \(L_i(x) = \prod_{j \neq 0,i}(x - \lambda_j)/\prod_{j \neq 0,i}(\lambda_i - \lambda_j)\). Then, since \(\text{dgr } L_i = d - 1\), [2] follows from \(\langle L_i, p_d \rangle_\Gamma = 0\) for \(i = 1, \ldots, d\). Some interesting consequences of the above, together with other properties of the predistance polynomials are the following (for more details, see [4]):

- The values of \(p_d\) at \(\lambda_0, \lambda_1, \ldots, \lambda_d\) alternate in sign.
- Using the values of \(p_d(\lambda_i), i = 0, \ldots, d\), given by [2], in the equality \(\|p_d\|_\Gamma^2 = p_d(\lambda_0)\), and solving for \(p_d(\lambda_0)\) we get the so-called spectral excess

\[
p_d(\lambda_0) = n \left(\frac{\pi_0^2}{m_0 p_1^2}\right)^{-1}.
\]  

(3)
For every $i = 0, \ldots, d$, (any multiple of) the sum polynomial $q_i = p_0 + \cdots + p_i$ maximizes the quotient $r(\lambda_0)/\|r\|_\Gamma$ among the polynomials $r \in \mathbb{R}[x]$ (notice that $q_i(\lambda_0)^2/\|q_i\|_\Gamma^2 = q_i(\lambda_0)$), and (1) $q_0(\lambda_0) < q_1(\lambda_0) < \cdots < q_d(\lambda_0)$ (or $H(\lambda_0) = n$).

Let $\Gamma$ have $n$ vertices, $d+1$ distinct eigenvalues, and diameter $D(\leq d)$. For $i = 0, \ldots, D$, let $k_i(u)$ be the number of vertices at distance $i$ from vertex $u$. Let $s_i(u) = k_0(u) + \cdots + k_i(u)$. Of course, $s_0(u) = 1$ and $s_D(u) = n$. The following result can be seen as a version of the spectral excess theorem, due to Garriga and the author [13] (for short proofs, see Van Dam [5], and Fiol, Gago and Garriga [12]):

**Theorem 2.** Let $\Gamma$ be a regular graph with spectrum $\text{sp} \Gamma = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $\overline{s_i} = \frac{1}{n} \sum_{u \in V} s_i(u)$ be the average number of vertices at distance at most $i$ from every vertex in $\Gamma$. Then, for any polynomial $r \in \mathbb{R}_d[x]$ we have

$$\frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} \leq \overline{s_{d-1}},$$

with equality if and only if $\Gamma$ is distance-regular and $r$ is a nonzero multiple of $q_{d-1}$.

*Proof.* Let $S_{d-1} = I + A + \cdots + A_{d-1}$. As $\text{dgr}(r) \leq d - 1$, $\langle r(A), J \rangle = \langle r(A), S_{d-1} \rangle$. But $\langle r(A), J \rangle = \langle r, H \rangle_\Gamma = r(\lambda_0)$. Thus, Cauchy-Schwarz inequality gives

$$r^2(\lambda_0) \leq \|r(A)\|^2 \|S_{d-1}\|^2 = \|r\|_\Gamma^2 \overline{s_{d-1}},$$

whence (4) follows. Besides, in case of equality we have that $r(A) = \alpha S_{d-1}$ for some nonzero constant $\alpha$. Hence, the polynomial $p = H - (1/\alpha)r$ satisfies $p(A) = J - S_{d-1} = A_d$ and, from Proposition 1, $\Gamma$ is distance-regular, $p = p_d$, and $r = q_d$. The converse in clear from $s_{d-1} = n - k_d = H(\lambda_0) - p_d(\lambda_0) = q_d(\lambda_0)$.

In fact, as it was shown in [11], the above result still holds if we change the arithmetic mean of the numbers $s_{d-1}(u), u \in V$, by its harmonic mean.

## 2 The results

As commented above, in [2] we studied the situation where the distance-$d$ graph $\Gamma_d$, of a distance-regular graph $\Gamma$ with diameter $d$, has fewer distinct eigenvalues. Now, we are interested in the case when $\Gamma$ is regular and with $d + 1$ distinct eigenvalues. In this context, $p_d$ is the highest degree predistance polynomial and, as $p_d(A)$ is not necessarily the distance-$d$ matrix $A_d$ (usually not even a 0-1 matrix), we consider the distinct eigenvalues of $p_d(A)$ vs. those of $A$. More precisely, given a set $H \subset \{0, \ldots, d\}$, we give conditions for all $p_d(\lambda_i)$ with $i \in H$ taking the same value. Notice that, because the values of $p_d$ at the mesh $\lambda_0, \lambda_1, \ldots, \lambda_d$ alternate in sign, the feasible sets $H$ must have either even or odd numbers.
The case $\lambda_0 \notin H$

We first study the more common case when $\lambda_0 \notin H$. For $i = 1, \ldots, d$, let $\phi_i(x) = \prod_{j \neq 0,i}(x-\lambda_i)$, and consider again the Lagrange interpolating polynomial $L_i(x) = \phi_i(x)/\phi_i(\lambda_i)$, satisfying $L_i(\lambda_j) = \delta_{ij}$ for $j \neq 0$, and $L_i(\lambda_0) = (-1)^{i+1}\frac{\pi_0}{\pi_i}$, where $\pi_i = |\phi_i(\lambda_i)|$.

**Theorem 3.** Let $\Gamma$ be a regular graph with degree $k$, $n$ vertices, and spectrum $\text{sp} \Gamma = \{\lambda_0, \lambda_1^m, \ldots, \lambda_d^m\}$, where $\lambda_0(= k) > \lambda_1 > \ldots > \lambda_d$. Let $H \subseteq \{1, \ldots, d\}$. For every $i = 0, \ldots, d$, let $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$. Let $k_d = \frac{1}{n} \sum_{u \in V} k_d(u)$ be the average number of vertices at distance $d$ from every vertex in $\Gamma$. Then,

$$k_d \leq \frac{n \sum_{i \in H} m_i}{\left(\sum_{i \in H} \frac{m_i}{\pi_i}\right)^2 + \sum_{i \notin H} \frac{\pi_0^2}{m_i \pi_i^2}} \sum_{i \in H} m_i,$$

(5)

and equality holds if and only if $\Gamma$ is a distance-regular graph with constant $P_{id} = p_d(\lambda_i)$ for every $i \in H$.

**Proof.** The clue is to apply Theorem 2 with a polynomial $r \in \mathbb{R}_{d-1}[x]$ having the desired properties of $q_{d-1}$. To this end, first notice that, as $q_{d-1} = H - p_d$, we have $q_{d-1}(\lambda_i) = -p_d(\lambda_i)$ for any $i \neq 0$. Thus, we take the polynomial $r$ with values $r(\lambda_i) = -t$ for $i \in H$, and $r(\lambda_i) = -p_d(\lambda_i)$ for $i \notin H, i \neq 0$. Then, using (2),

$$r(x) = -t \sum_{i \in H} L_i(x) - \sum_{i \notin H, i \neq 0} p_d(\lambda_i) L_i(x),$$

$$r(\lambda_0) = -t \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i} - \sum_{i \notin H, i \neq 0} p_d(\lambda_i)(-1)^{i+1} \frac{\pi_0}{\pi_i},$$

$$= -t \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i} - p_d(\lambda_0)^2 \sum_{i \notin H, i \neq 0} \frac{\pi_0^2}{m_i \pi_i^2},$$

$$n \|r\|^2 = r(\lambda_0)^2 + t^2 \sum_{i \in H} m_i + \sum_{i \notin H, i \neq 0} m_i p_d(\lambda_i)^2.$$  

Thus, (4) yields

$$\Phi(t) = \frac{r(\lambda_0)^2}{\|r\|^2} = \frac{n(\alpha + \beta)^2}{(\alpha + \beta)^2 + \sigma t^2 + \gamma} \leq s_{d-1},$$

(6)

where

$$\alpha = \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i}, \quad \beta = -p_d(\lambda_0) \sum_{i \notin H, i \neq 0} \frac{\pi_0^2}{m_i \pi_i^2}, \quad \gamma = \sum_{i \notin H, i \neq 0} m_i p_d(\lambda_i)^2 = \sum_{i \notin H, i \neq 0} \frac{p_d(\lambda_0)^2 \pi_0^2}{m_i \pi_i^2} = -p_d(\lambda_0) \beta, \quad \sigma = \sum_{i \in H} m_i.$$

(7)

(8)
Now, to have the best result in (6) (and since we are mostly interested in the case of equality), we have to find the maximum of the function $\Phi(t)$, which is attained at $t_0 = \alpha \gamma / \beta \sigma$. Then,

$$\Phi_{\text{max}} = \Phi(t_0) = \frac{n(\alpha^2 \gamma + \beta^2 \sigma)}{\alpha^2 \gamma + \beta^2 \sigma + \gamma \sigma} \leq s_{d-1} = n - k_d.$$

Thus, using (7)-(8) and simplifying we get (5). In case of equality, we know, by Theorem 2, that $\Gamma$ is distance-regular with $r(x) = \alpha q_{d-1}(x)$ for some constant $\alpha$. If $i \not\in H, i \neq 0$, $r(\lambda_i) = -p_d(\lambda_i) = \alpha q_{d-1}(\lambda_i) = -\alpha p_d(\lambda_i)$, so that $\alpha = 1$ since $p_d(\lambda_i) \neq 0$. Then, for every $i \in H$, we get

$$P_{id} = p_d(\lambda_i) = H(\lambda_i) - q_{d-1}(\lambda_i) = -r(\lambda_i) = t_0.$$

Conversely, if $\Gamma$ is distance-regular, we have that $k_d = k_d$, and, if $P_{id}$ is a constant, say, $\tau$ for every $i \in H$, we obtain, from [2], that $\sigma = \frac{k_d}{p_d(\lambda_i)} \sum_{i \in H} (-1)^i \frac{\pi_0}{\pi_i} = \frac{k_d}{\tau} \alpha$, whence $\tau = \frac{1}{\sigma} k_d \alpha$. Moreover,

$$nk_d = \|p_d\|_2^2 = \sum_{i \in H} m_i p_d(\lambda_i)^2 + \sum_{i \in H} m_i \tau^2 = k_d^2 \sum_{i \in H} \frac{\pi_0}{\pi_i} + k_d^2 \left( \frac{\sum_{i \in H} \pi_0}{\sum_{i \in H} \pi_i} \right)^2,$$

and equality in (10) holds.

As mentioned above, when $\Gamma$ is already a distance-regular graph, Brouwer and the author [2] gave parameter conditions for partial antipodality, and surveyed known examples. The different examples given here are withdrawn from such a paper.

**Example 4.** The Odd graph $O_5$, on $n = 126$ vertices, has intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$, so that $k_d = 60$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, with $H = \{2, 4\}$, the function $\Phi(t)$ is depicted in Fig. 1. Its maximum is attained for $t_0 = 6$, and its value is $\Phi(6) = 66 = s_{d-1}$. Then, $P_{24} = P_{44}.$

![Figure 1: The function $\Phi(t)$ for $O_5$ with $H = \{2, 4\}$](image)
Notice that if, in the above result, $H$ is a singleton, there is no restriction for the values of $p_d$, and then we get the so-called spectral excess theorem (originally proved by Garriga and the author [13]).

**Corollary 5** (The spectral excess theorem). Let $\Gamma$ be a regular graph with spectrum $\text{sp} \Gamma$ and average number $k_d$ as above. Then $\Gamma$ is distance-regular if and only if
\[
\bar{k}_d = p_d(\lambda_0) = n \left( \sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}.
\]

*Proof.* Take $H = \{i\}$ for some $i \neq 0$ in Theorem 3.

As mentioned before, in [2] a distance-regular graph $\Gamma$ was said to be half antipodal if the distance-$d$ graph has only one negative eigenvalue (i.e., $P_{id}$ is a constant for every $i = 1, 3, \ldots$). Then, a direct consequence of Theorem 3 by taking $H = H_{\text{odd}} = \{1, 3, \ldots\}$ is the following characterization of half antipodality.

**Corollary 6.** Let $\Gamma$ be a regular graph as above. Then,
\[
\bar{k}_d \leq \frac{n \sum_{i \text{ odd}} m_i}{\left( \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ odd}} m_i},
\]
and equality holds if and only if $\Gamma$ is a half antipodal distance-regular graph.

Recall that a regular graph is strongly regular if and only if it has at most three distinct eigenvalues (see e.g. [15]). Then, we can apply Theorem 3 with $H_{\text{even}} = \{2, 4, \ldots\}$ and $H_{\text{odd}} = \{1, 3, \ldots\}$ (and add up the two inequalities obtained) to obtain a characterization of those distance-regular graphs having strongly regular distance-$d$ graph.

**Corollary 7.** Let $\Gamma$ be a regular graph as above. Then,
\[
\bar{k}_d \leq \frac{n^2}{\left( \sum_{i \text{ even}} \frac{\pi_0}{\pi_i} \right)^2 + \left( \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ odd}} m_i + \sum_{i \text{ odd}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ even}} m_i},
\]
and equality holds if and only if $\Gamma$ is a distance-regular graph with strongly regular distance-$d$ graph $\Gamma_d$.

**Example 8.** The Wells graph, on $n = 32$ vertices, has intersection array $\{5, 4, 1, 1; 1, 1, 4, 5\}$ and spectrum $5^1, \sqrt{5}^8, 1^{10}, -\sqrt{5}^8, -3^5$. This graph is 2-antipodal, so that $k_d = 1$. Then, Fig. 2 shows the functions $\Phi_0(t)$ with $H_0 = \{2, 4\}$, and $\Phi_1(t)$ with $H_1 = \{1, 3\}$. Their (common) maximum value is attained for $t_0 = 1$ and $t_1 = -1$, respectively, and it is $\Phi_0(1) = \Phi_1(-1) = 31 = s_{d-1}$. Then, $P_{24} = P_{44}$ and $P_{14} = P_{34}$. 
Figure 2: The functions $\Phi_0(t)$ (in red) with $H_0 = \{2, 4\}$, and $\Phi_1(t)$ (in blue) with $H_1 = \{1, 3\}$ of the Wells graph.

In fact, the above expression can be simplified because $\sum_{i \text{ even}} m_i + \sum_{i \text{ odd}} m_i = n$, $\sum_{i \text{ even}} \frac{\pi_i}{\pi_i} = \sum_{i \text{ odd}} \frac{\pi_i}{\pi_i}$ (see [9]), and, from (3), $\sum_{i \text{ even}} \frac{\pi_i}{m_i \pi_i} + \sum_{i \text{ odd}} \frac{\pi_i}{m_i \pi_i} = n/p_d(\lambda_0)$.

Anyway, we have written (10) as it is to emphasize the 'symmetries' between even and odd terms.

The following result was used in [2, 10] for the case of distance-regular graphs (where $p_d(\lambda_i) = p_d$).

**Corollary 9.** Let $\Gamma$ be a regular graph with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $H \subset \{1, \ldots, d\}$. Then, $p_d(\lambda_i) = p_d(\lambda_j)$ for every $i, j \in H$ if and only if $\sum_{i \neq j} (m_i \pi_i m_j \pi_j)^2 = 0$.

**Proof.** Notice that the right hand side of (5) is just the spectral excess $p_d(\lambda_0)$, which is given by (3). Then, the result follows from from equating both expressions and simplifying.

**The case $\lambda_0 \in H$**

To deal with this case, we could proceed as above by defining conveniently a degree $d - 1$ polynomial $r$. Then the proof is similar to the one for Theorem 3. If $\lambda_0 \in H$ then $p(\lambda_i) = p(\lambda_0)$ for any $i \in H$. Moreover, the odd indexes, cannot belong to $H$. In particular $1 \not\in H$. For instance, a possible choice for $r \in \mathbb{R}_{d-1}[x]$ is:

- $r(\lambda_0) = n - p_d(\lambda_0)$, $r(\lambda_i) = -p_d(\lambda_0)$ for $i \in H, i \neq 0$.
- $r(\lambda_i) = -tp_d(\lambda_i)$ for $i \not\in H, i \neq 1$.

Hovewer, we can follow a more direct approach by using (6). First, the following result was proved in [2]:


Proposition 10 ([2 Prop. 8]). Let $\Gamma$ be a distance regular graph with diameter $d$. If $P_{0d} = P_{id}$ then $i$ is even. Let $i > 0$ be even. Then $P_{0d} = P_{id}$ if and only $\Gamma$ is antipodal, or $i = d$ and $\Gamma$ is bipartite. \hfill $\square$

Theorem 11. Let $\Gamma$ be a regular graph with $n$ vertices, spectrum $\text{sp} \Gamma$ as above, and mean excess $k_d$. Then, for every $i = 1, \ldots, d$,
\[
\bar{k}_d \leq \frac{n \left( m_i + \sum_{j \neq 0, i} \frac{\pi_0^d}{m_j \pi_j} \right)}{\left( \frac{\pi_0}{n} + \sum_{j \neq 0, i} \frac{\pi_0^d}{m_j \pi_j} \right)^2 + m_i + \sum_{j \neq 0, i} \frac{\pi_0^d}{m_j \pi_j}}.
\] (11)

Moreover:

(a) Equality holds for some $i \neq d$ if and only it holds for any $i = 1, \ldots, d$ and $\Gamma$ is an antipodal distance-regular graph.

(b) Equality holds only for $i = d$ if and only if $\Gamma$ is a bipartite, but not antipodal, distance-regular graph.

Proof. The inequality (11) follows from (6) by taking $H = \{i\}$ for some even $i \neq 0$, and choosing $t = p_d(\lambda_0)$. Then, in case of equality, Theorem 3 tells us that $\Gamma$ is distance-regular. Then, $\Gamma_d$ is a regular graph with equal eigenvalues $P_{0d}$ and $P_{id}$. So, the result follow from Proposition 10. \hfill $\square$

Example 12. For the Wells graph the right hand expression of (11) gives $1(= k_4)$ for any $i = 1, \ldots, 4$, in concordance with its antipodal character. In contrast, the folded 10-cube $FQ_{10}$, on $n = 512$ vertices, has intersection array $\{10, 9, 8, 7, 6; 1, 2, 3, 4, 10\}$ and spectrum $10^4, 6^{45}, 2^{210}, -2^{210}, -6^{45}, -10^4$. Then, the right hand expression of (11) gives $234.16, 293.36, 293.36, 234.16$ for $i = 1, 2, 3, 4$, respectively, and $126(= k_5)$ for $i = 5$, showing that $FQ_{10}$ is a bipartite distance-regular graph, but not antipodal.

Another characterization of antipodal distance-regular graphs was given by the author in [8] by assuming that the distance $d$-graph of a regular graph is already antipodal.

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