TOPOLOGICAL PRESSURE FOR THE COMPLETELY IRREGULAR SET OF BIRKHOFF AVERAGES

Xueting Tian
School of Mathematical Sciences, Fudan University
Shanghai 200433, China
(Communicated by Sébastien Gouëzel)

Abstract. It is well-known that for certain dynamical systems (satisfying specification or its variants), the set of irregular points w.r.t. a continuous function \( \phi \) (i.e. points with divergent Birkhoff ergodic averages observed by \( \phi \)) either is empty or carries full topological entropy (or pressure, see [6, 17, 36, 37] etc. for example). In this paper we study the set of irregular points w.r.t. a collection \( D \) of finite or infinite continuous functions (that is, points with divergent Birkhoff ergodic averages simultaneously observed by all \( \phi \in D \)) and obtain some generalized results. As consequences, these results are suitable for systems such as mixing shifts of finite type, uniformly hyperbolic diffeomorphisms, repellers and \( \beta \)-shifts.

1. Introduction. In this paper, a dynamical system \((X, f)\) means always that \(X\) is a compact metric space and \(f : X \to X\) is a continuous map. Let \(\phi : X \to \mathbb{R}\) be a continuous observable function. A point \(x \in X\) is called to be \(\phi\)-regular, if the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))
\]

exists. Otherwise, \(x\) is called \(\phi\)-irregular (the orbit of \(x\) is called to have historic behavior in [30]). Define the \(\phi\)-regular set to be the set of all \(\phi\)-regular points, that is,

\[
R(\phi,f) := \{ x \in X | x \text{ is } \phi\text{-regular} \};
\]

and define the \(\phi\)-irregular set

\[
I(\phi,f) := X \setminus R(\phi,f).
\]

Let \(I(f) := \bigcup_{\phi \in C^0(X)} I(\phi,f)\), called irregular set, where \(C^0(X)\) denotes the space of all continuous functions on \(X\). Its complementary set \(X \setminus I(f)\) is called regular set, denoted by \(R(f)\). Note that \(R(f) = \bigcap_{\phi \in C^0(X)} R(\phi,f)\). By Birkhoff’s ergodic theorem, the set \(I(f)\) is always of zero measure for any invariant measure.

Pesin and Pitskel [26] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols from

2010 Mathematics Subject Classification. 37C50, 37C45, 37B10, 37D20.

Key words and phrases. Topological entropy and topological pressure, ergodic average, specification property, subshifts of finite type and \(\beta\)-shifts, hyperbolic systems.

The author is supported by National Natural Science Foundation of China (grant no. 11671093, 11301088) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130071120026).
the dimensional perspective. Barreira, Schmeling, etc. studied the irregular set in the setting of shifts of finite type and beyond, see [6, 12, 37, 12, 35, 36, 29, 38, 24, 13] etc. Ruelle uses the terminology in [30] ‘historic behavior’ to describe irregular point and in contrast to dimensional perspective, Takens asks in [34] for which smooth dynamical systems the points with historic behavior has positive Lebesgue measure. Moreover, many researchers studied irregular set from topological or geometric viewpoint that irregular set forms dense $G_δ$ set, for example, see [21, 20, 16, 2, 18, 23, 3].

It was usual to study $φ$–irregular set (in most known results) under one observable function $φ$ but few results discussed $φ$–irregular set under a collection of observable functions as far as the author know. There was a such result in [6] to study finite observable functions.

**Theorem 1.1.** [6, Theorem 2.1] Suppose $(X, f)$ is a topological mixing subshift of finite type (or repeller or horseshoe). Then the set $\bigcap_{i=1}^{k} I(ϕ_i, f)$ under finite observable Hölder functions $ϕ_1, ϕ_2, \cdots, ϕ_k (k \geq 1)$ either is empty or carries full topological entropy.

In the present paper, we are mainly aiming to generalize Theorem 1.1 to a more refined version, dealing with the intersection of $ϕ_i$–irregular sets under infinite (and even uncountable) observable functions $\{ϕ_i\}$. Now we start to state our main result.

1.1. **Main result.** Let $\hat{C}_f^0(X) = \{ϕ ∈ C^0(X) | I(ϕ, f) \neq ∅\}$. Remark that

$\hat{C}_f^0(X) = ∅ \iff R(f) = X.$

Obviously any uniquely ergodic system satisfies this relation but this relation also applies for some non-uniquely ergodic systems, for example, the south-north map. We will recall some basic equivalent conditions for $I(ϕ, f) \neq ∅$ in Section 3.1.

**Definition 1.2.** (Completely-irregular set) define completely-irregular set as

$CI(f) := \bigcap_{ϕ ∈ \hat{C}_f^0(X)} I(ϕ, f).$

The point of $CI(f)$ is called to be completely-irregular point.

Remark that for any $ϕ ∈ \hat{C}_f^0(X)$, $CI(f) \subseteq I(ϕ, f) \subseteq I(f)$. Let $M(X)$ denote the space of all probability measures. Let $M_f(X)$ and $M_γ(X)$ denote the space of all $f$–invariant probability measures and the space of all $f$–ergodic measures respectively. Given a set $Γ ⊆ X$, we say $Γ$ to have full topological pressure, if for any $ϕ ∈ C^0(X)$,

$P(Γ, ϕ, f) = P(X, ϕ, f) = \sup\{h_μ(f) + ∫ ϕdμ | μ ∈ M_f(X)\},$

where $P$ denotes the topological pressure.

**Theorem 1.3.** Suppose that $(X, f)$ is not uniquely ergodic. If $f$ satisfies $γ$–almost product property and uniform separation property, then the completely-irregular set $CI(f)$ is not empty and carries full topological pressure.

**Remark 1.** By the definitions of topological pressure and BS-dimension (see Section 2.2 below for the definitions), it is not difficult to see that for any set $Z ⊆ X$, the BS-dimension of $Z$ is a unique foot of Bowen’s equation

$P(Z, −sϕ, f) = 0$, i.e., $s = BS(Z, ϕ, f).$ (1)
Let the conclusion of Theorem 1.3 can be stated

\[ BS(CI(f), \varphi, f) = BS(X, \varphi, f) = \sup \left\{ \frac{b_\mu(f)}{\int \varphi d\mu} \mid \mu \in M_f(X) \right\} \]

for any strictly positive function \( \varphi \in C^0(X) \).

This paper is organized as follows. In Section 2 we recall some definitions and in Section 3 we give some basic lemmas. We will prove Theorem 1.3 in Section 4. In the final Section we will give some remarks, for example, adding refined information about how points with irregular and regular behavior can be large in the sense of topological pressure.

2. Preliminary.

2.1. Specification-like properties. Firstly we recall the definition of (almost) specification, see [15, 32, 9, 10, 28, 36, 37].

**Definition 2.1.** We say that \((X, f)\) satisfies specification property, if the following holds: for any \( \epsilon > 0 \) there exists an integer \( M(\epsilon) \) such that for any \( k \geq 2 \), any \( k \) points \( x_1, \ldots, x_k \), any integers \( a_1 \leq b_1 < a_2 \leq b_2 \leq \cdots \leq a_k \leq b_k \) with \( a_{i+1} - b_i \geq M(\epsilon) \) \((1 \leq i \leq k-1)\), there exists a point \( x \in X \) such that

\[ d(f^j(x), f^j(x_i)) < \epsilon, \quad \text{for } a_i \leq j \leq b_i, \quad 1 \leq i \leq k. \]

(2)

The original definition of specification, due to Bowen, was stronger.

**Definition 2.2.** We say that \((X, f)\) satisfies Bowen’s specification property, if under the assumptions of Definition 2.1 and for any integer \( p \geq M(\epsilon) + b_k - a_1 \), there exists a point \( x \in X \) with \( T^p(x) = x \) satisfying (2).

Recall that almost specification introduced in [37] is slightly different from \( g \)-almost product property in [28] (Almost specification is slightly weaker). And their main ideas are same: one requires only partial shadowing of the specified orbit segments, contrary to specification property. Therefore, in present paper we treat almost specification same as \( g \)-almost product property and we only introduce the definition of \( g \)-almost product property as follows. People who want to know the detailed difference, see [37, 28]. A striking and typical example of \( g \)-almost product property (and almost specification) is that it applies to every \( \beta \)-shift [37, 28]. In sharp contrast, the set of \( \beta \) for which the \( \beta \)-shift has specification property has zero Lebesgue measure \([11, 31]\).

Let \( \Lambda_n = \{0, 1, 2, \ldots, n-1\} \). The cardinality of a finite set \( \Lambda \) is denoted by \(|\Lambda|\). Let \( x \in X \). The dynamical ball \( B_n(x, \varepsilon) \) is the set

\[ B_n(x, \varepsilon) := \{ y \in X \mid \max \{d(f^j(x), f^j(y)) \mid j \in \Lambda_n \} \leq \varepsilon \} \]

**Definition 2.3.** Let \( g : \mathbb{N} \rightarrow \mathbb{N} \) be a given nondecreasing unbounded map with the properties \( g(n) < n \) and \( \lim_{n \rightarrow \infty} \frac{g(n)}{n} = 0 \). The function \( g \) is called blowup function. Let \( x \in X \) and \( \varepsilon > 0 \). The \( g \)-blowup of \( B_n(x, \varepsilon) \) is the closed set \( B_n(g; x, \varepsilon) := \{ y \in X \mid \exists \Lambda \subseteq \Lambda_n, |\Lambda_n \setminus \Lambda| \leq g(n) \text{ and } \max \{d(f^j(x), f^j(y)) \mid j \in \Lambda \} \leq \varepsilon \} \).

**Definition 2.4.** We say that \((X, f)\) satisfies \( g \)-almost product property with blowup function \( g \), if there is a nonincreasing function \( m : \mathbb{R}^+ \rightarrow \mathbb{N} \), such that for any
where $M_0 := 0, M_i := n_1 + \cdots + n_i, i = 1, 2, \cdots, k - 1$.

2.2. Entropy and pressure. Now we recall the definition of uniform separation property. Set

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta f_i(x).$$

Let $\mathcal{M}_x(f)$ be the set of all limits of $\{\mathcal{E}_n(x)\}_{n \in \mathbb{N}}$ in weak$^*$ topology. For $\delta > 0$ and $\varepsilon > 0$, two points $x$ and $y$ are $(\delta, n, \varepsilon)$-separated if

$$|\{j : d(f^j x, f^j y) > \varepsilon, j \in \Lambda_n\}| \geq \delta n.$$

A subset $E \subseteq X$ is $(\delta, n, \varepsilon)$-separated if any pair of different points of $E$ are $(\delta, n, \varepsilon)$-separated. Let $F \subseteq \mathcal{M}(X)$ be a neighborhood of $\nu \in \mathcal{M}_f(X)$. Define $X_{n,F} := \{x \in X|\mathcal{E}_n(x) \in F\}$, and define

$$N(F; \delta, n, \varepsilon) := \maximal cardinality of a (\delta, n, \varepsilon) - separated subset of X_{n,F}.$$

Let $\xi = \{V_i\}_{i=1, 2, \cdots, k}$, be a finite partition of measurable sets of $X$. The entropy of $\nu \in \mathcal{M}(X)$ with respect to $\xi$ is

$$H(\nu, \xi) := - \sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i).$$

We write $f^{\nu_n} \xi := \bigvee_{k \in \Lambda} f^{-k} \xi$. The entropy of $\nu \in \mathcal{M}_f(X)$ with respect to $\xi$ is

$$h(f, \nu, \xi) := \lim_{n \to \infty} \frac{1}{n} H(\nu, f^{\nu_n} \xi),$$

and the metric entropy of $\nu$ is

$$h_{\nu}(f) := \sup_{\xi} h(f, \nu, \xi).$$

Definition 2.5. We say that $(X, f)$ satisfies uniform separation property, if following holds. For any $\eta > 0$, there exist $\delta^* > 0, \varepsilon^* > 0$ such that for $\mu$ ergodic and any neighborhood $F \subseteq \mathcal{M}(X)$ of $\mu$, there exists $n_{F, \mu, \eta}^*$ such that for $n \geq n_{F, \mu, \eta}^*$,

$$N(F; \delta^*, n, \varepsilon^*) \geq 2^{n(h_{\nu}(f) - n)}.$$

Now we recall the definition of topological pressure and entropy. Let $E \subseteq X$, $\varphi \in C^0(X)$ and $\mathcal{F}_n(E, \varepsilon)$ be the collection of all finite or countable covers of $E$ by sets of the form $B_m(x, \varepsilon)$ with $m \geq n$. We set

$$C(E; t, \varphi, n, \varepsilon, f) := \inf \left\{ \sum_{B_m(x, \varepsilon) \in \mathcal{C}} 2^{-tm + \sup_{y \in B_m(x, \varepsilon)} \sum_{i=0}^{m-1} \varphi(f^i(x))) \mid \mathcal{C} \in \mathcal{F}_n(E, \varepsilon) \right\},$$

and

$$C(E; t, \varphi, n, \varepsilon, f) := \lim_{n \to \infty} C(E; t, \varphi, n, \varepsilon, f).$$

Then

$$P(E, \varphi, \varepsilon, f) := \inf \{t : C(E; t, \varphi, \varepsilon, f) = 0\} = \sup \{t : C(E; t, \varphi, \varepsilon, f) = \infty\}$$

and the topological pressure of $E$ is defined as

$$P(E, \varphi, f) := \lim_{\varepsilon \to 0} P(E, \varphi, \varepsilon, f).$$
In particular, if $\phi = 0$, then the topological entropy of $E$ is defined as

$$h_{top}(f, E) = P(E, 0, f).$$

Now we recall BS-dimension which was introduced by Barreira and Schmeling in \[6\]. If $\phi$ is a strictly positive continuous function, then for each $E \subseteq X$ and each number $\epsilon > 0$, define

$$N(E; t, \phi, \epsilon, f) := \lim_{n \to \infty} N(E; t, \phi, n, f),$$

where

$$N(E; t, \phi, n, \epsilon, f) := \inf \{ \sum_{B_m(x, \epsilon) \in C} 2^{-t} \sup_{y \in B_m(x, \epsilon)} \sum_{i=0}^{n-1} \phi(f^i(x)) : C \in \mathcal{F}_n(E, \epsilon) \}.$$ 

Set

$$BS(E, \phi, \epsilon, f) := \inf\{ t : N(E; t, \phi, \epsilon, f) = 0 \} = \sup\{ t : N(E; t, \phi, \epsilon, f) = \infty \}$$

and the BS-dimension of $E$ is defined as

$$BS(E, \phi, f) := \lim_{\epsilon \to 0} BS(E, \phi, \epsilon, f).$$

Remark that if $\phi = 1$, then $BS(E, 1, f) = h_{top}(f, E)$.  

2.3. Saturated set. Firstly we recall a result of \[28, 25\] on saturated sets (see \[28\] for topological entropy and see \[25\] for topological pressure). We say that $f : X \to X$ is saturated, if for any $\phi \in C^0(X)$ and any compact connected nonempty set $K \subseteq \mathcal{M}_f(X)$,

$$P(G_K, \phi, f) = \inf \{ h_\mu(f) + \int \phi \, d\mu \mid \mu \in K \},$$

where $G_K = \{ x \in X \mid \mathcal{M}_x(f) = K \}$ (called saturated set of $K$).

Lemma 2.6. (Saturated property) Suppose that $(X, f)$ satisfies $g$-almost product property and uniform separation property. Then $f$ is saturated.

3. Basic Facts.

3.1. Irregular points. It was worth recalling a result of \[36\] Lemma 1.6 that an equivalent condition for $I(\phi, f) \neq \emptyset$ is that the function is not in the closure (in the sup norm) of the coboundaries when the system has specification property (see \[37\] for the case of almost specification). Let $\phi_1, \phi_2 \in C^0(X)$. We say $\phi_1$ is cohomologous to $\phi_2$ if they differ by a coboundary, i.e. there exists $h \in C^0(X)$ such that $\phi_1 = h - h \circ f + \phi_2$. For a constant $c$, let $Cob(X, f, c)$ denote the space of functions cohomologous to $c$ and $\overline{Cob}(X, f, c)$ be the closure of $Cob(X, f, c)$ in the sup norm.

Lemma 3.1. Let $(X, f)$ be a dynamical system.

(1) The following are equivalent for $\phi \in C^0(X)$:

1. $\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))$ does not converge pointwise to a constant;
2. $\inf_{\mu \in \mathcal{M}_f(X)} \int \phi(x) \, d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \phi(x) \, d\mu$;
3. $\inf_{\mu \in \mathcal{M}_f^s(X)} \int \phi(x) \, d\mu < \sup_{\mu \in \mathcal{M}_f^s(X)} \int \phi(x) \, d\mu$;
4. $\phi \notin \bigcup_{c \in \mathbb{R}} \overline{Cob}(X, f, c)$;
5. $\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))$ does not converge uniformly to a constant;

(2) For any $\phi \in \hat{C}_f^0(X)$, all items (1.1)-(1.5) hold.

(3) Suppose that $(X, f)$ has almost specification. Then $\phi \in \hat{C}_f^0(X) \Leftrightarrow$ all items (1.1)-(1.5).
Proof. (1) and (2) were from Lemma 1.6 of [36].

(3) For the part of ‘⇐’, see the paragraph behind of Lemma 2.1 in [37], as a corollary of Lemma 2.1 and Theorem 4.1 there, P. 5397 (see Lemma 1.6 of [36] for the case of specification). For the case of ‘⇒’, it is item (2).

**Lemma 3.2.** Let \((X, f)\) be a dynamical system. Let \(\phi \in C^0(X)\) and \(x \in X\). Then
\[
\phi \in \hat{C}^0_f(X), \quad x \in I(\phi, f) \iff \inf_{\mu \in M_\mu(f)} \int \phi(x) d\mu < \sup_{\mu \in M_\mu(f)} \int \phi(x) d\mu.
\]

**Proof.** On one hand, fix \(\phi \in \hat{C}^0_f(X)\) and \(x \in I(\phi, f)\). By definition there are two sequences of \(n_j, m_j \uparrow +\infty\) such that the following limits exist and
\[
\lim_{j \to \infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \phi(f^i(x)) = \lim_{j \to \infty} \frac{1}{m_j} \sum_{i=0}^{m_j-1} \phi(f^i(x)).
\]

By weak* topology one can take two convergence subsequences (if necessary) of \(\{\frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i(x)}\}_{n_j \in \mathbb{N}}, \{\frac{1}{m_j} \sum_{i=0}^{m_j-1} \delta_{f^i(x)}\}_{m_j \in \mathbb{N}}\) and then the two limits of \(\mu_1\) and \(\mu_2\) are in \(M_{x}(f)\) and satisfy that
\[
\int \phi d\mu_1 = \lim_{j \to \infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \phi(f^i(x)) \neq \lim_{j \to \infty} \frac{1}{m_j} \sum_{i=0}^{m_j-1} \phi(f^i(x)) = \int \phi d\mu_2.
\]

On the other hand, Let \(\phi \in C^0(X)\) and \(x \in X\) satisfy
\[
\inf_{\mu \in M_\mu(f)} \int \phi(x) d\mu < \sup_{\mu \in M_\mu(f)} \int \phi(x) d\mu.
\]

Take two measures \(\mu_1, \mu_2 \in M_\mu(f)\) such that \(\int \phi(x) d\mu_1 < \int \phi(x) d\mu_2\). Then we can take two convergence subsequences of
\[
\{\frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i(x)}\}_{n_j \in \mathbb{N}}, \{\frac{1}{m_j} \sum_{i=0}^{m_j-1} \delta_{f^i(x)}\}_{m_j \in \mathbb{N}}
\]
such the limits are \(\mu_1\) and \(\mu_2\). So
\[
\lim_{j \to \infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \phi(f^i(x)) = \int \phi d\mu_1 \neq \int \phi d\mu_2 = \lim_{j \to \infty} \frac{1}{m_j} \sum_{i=0}^{m_j-1} \phi(f^i(x)).
\]

Hence, \(x \in I(\phi, f)\) and thus \(\phi \in \hat{C}^0_f(X)\). \(\square\)

### 3.2. Cardinality of \(\hat{C}^0_f(X)\).

We show that the set \(\hat{C}^0_f(X)\) is uncountable for any dynamical system with almost specification.

**Proposition 1.** Suppose that \((X, f)\) satisfies almost specification. Then the following conditions are equivalent.

1. \(f\) is not uniquely ergodic;
2. \(\hat{C}^0_f(X) \neq \emptyset\);
3. \(\hat{C}^0_f(X)\) is open and dense in \(C^0(X)\).

Firstly we prove a general lemma.

**Lemma 3.3.** Let \((X, f)\) be a dynamical system. If \(\hat{C}^0_f(X) \neq \emptyset\), then \(f\) is not uniquely ergodic and \(\hat{C}^0_f(X)\) is open and dense in \(C^0(X)\).
Proof. Take \( \phi_0 \in \hat{C}_f^0(X) \) and \( x \in I(\phi_0, f) \). By Lemma 3.2, \( f \) is not uniquely ergodic. Now we start to prove that \( \hat{C}_f^0(X) \) is open and dense in \( C^0(X) \).

On one hand, we show \( C^0(X) \) is dense in \( C^0(X) \setminus \hat{C}_f^0(X) \). Fix \( \phi \in C^0(X) \setminus \hat{C}_f^0(X) \). Then \( I(\phi, f) = \emptyset \) so that \( R(\phi, f) = X \). Take \( \phi_n = \frac{1}{n} \phi_0 + \phi \), \( n \geq 1 \). Then \( \phi_n \) converges to \( \phi \) in sup norm. By construction, it is easy to check that \( x \in I(\phi_n, f) \), \( n \geq 1 \). That is, \( \phi_n \in \hat{C}_f^0(X) \).

On the other hand, we prove that \( \hat{C}_f^0(X) \) is open. Fix \( \phi \in \hat{C}_f^0(X) \) and \( y \in I(\phi, f) \). Then by Lemma 3.2, there are two different invariant measures \( \mu_1, \mu_2 \in M_y(f) \) such that \( \int \phi \, d\mu_1 < \int \phi \, d\mu_2 \). By continuity of sup norm, we can take an open neighborhood of \( \phi \), denoted by \( U(\phi) \), such that for any \( \varphi \in U(\phi) \), \( \int \varphi \, d\mu_1 < \int \varphi \, d\mu_2 \). Notice that \( \mu_1, \mu_2 \in M_y(f) \). By Lemma 3.2, \( y \in I(\varphi, f) \), \( \forall \varphi \in U(\phi) \). This implies \( \varphi \in \hat{C}_f^0(X) \), \( \forall \varphi \in U(\phi) \). \( \square \)

Now we start to prove Proposition 1. Item (3) implying item (2) is obvious. By Lemma 3.3, we only need to prove that item (1) implies (2). By assumption, there are two different invariant measures \( \mu_1, \mu_2 \). By weak* topology, there is a continuous function \( \phi \) such that \( \int \phi \, d\mu_1 \neq \int \phi \, d\mu_2 \). By Lemma 3.1, item (3), \( I(\phi, f) \neq \emptyset \). \( \square \)

4. Proof of Theorem 1.3

4.1. Topological properties of completely-irregular set. In a Baire space, a set is residual if it contains a countable intersection of dense open sets. Some results showed that certain irregular sets can also be large from the topological point of view. For example, Albeverio, Pratsiovytyi and Torbin \[2\], Hyde et al \[18\] and Olsen \[23\] proved that some kinds of irregular sets associated with integer expansion are residual. Baek and Olsen \[3\] discussed the set of extremely non-normal points of self-similar set from the topological point of view. Li and Wu \[20\] proved that the set of divergence points of self-similar measure with the open set condition is either residual or empty, and they also proved in \[21\] that

**Theorem 4.1.** Suppose that \((X, f)\) satisfies specification. Then for any continuous function \( \phi : X \to \mathbb{R} \), the \( \phi \)-irregular set \( I(\phi, f) \) either is empty or residual in \( X \).

This result has been generalized into the case of asymptotic average shadowing \[16\], which is weaker than specification and almost specification. Its proof is mainly based on the existence of saturated sets. The existence of saturated set was first proved by Sigmund \[32\] and has been generalized into non-uniformly hyperbolic diffeomorphisms \[22\], \( C^1 \) generic diffeomorphisms \[33\] and topological dynamical systems with asymptotic average shadowing \[16\]. Since in the present paper we deal with almost specification which is stronger than asymptotic average shadowing, we just state \[16\] Theorem 1.4 and Corollary 1.6 in the case of almost specification.

**Theorem 4.2.** Suppose that \((X, f)\) satisfies almost specification. Then

1. for every non-empty connected compact set \( K \subseteq M_f(X) \), there exists a point \( y \in X \) such that \( K = M_y(f) \) (that is \( G_K \neq \emptyset \)). Moreover, the set \( G_K \) is dense in \( \Delta_K = \bigcup_{v \in K} \text{supp}(v) \).
2. If \( \bigcup_{v \in M_f(X)} \text{Support}(v) = X \), then \( G_{\text{max}} \) is residual in \( X \) where \( G_{\text{max}} := \{ x \in X | M_x(f) = M_f(X) \} \).
For convenience of readers, we explain some proof ideas of this theorem in the case of specification [15, Proposition 21.14]. Given a non-empty connected compact set $K \subseteq \mathcal{M}_f(X)$ and for any invariant measure in $K$, by Ergodic Decomposition Theorem, its space average can be approximated by the time average along finitely many orbit segments (in the case of ergodic measure, allowed to be approximated by the time average along a true orbit). Then one can use specification property to construct one true orbit to shadow all above orbit segments. In this process one can use specification property infinitely many times by induction so that the scale used to shadow is close to zero as the time goes to infinity. Then by weak* topology the time average of this true orbit can reflect the set $K$. This is just a rough idea since the precise proof is complicated and needs careful construction, see [32, 15, 16, 22] for details on different dynamical systems.

Now we use Theorem 4.2 to show that the completely-irregular set is still very “large” from geometric or topological perspective.

**Theorem 4.3.** Suppose that $(X, f)$ satisfies almost specification and is not uniquely ergodic. Then

1. the set $CI(f)$ is not empty and dense in $\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu)$.
2. if there is an invariant measure with full support, then $CI(f)$ is residual in $X$.

Before proof we give a basic relation on $CI(f)$ and saturated sets. Define $G_{\text{max}} := \{x \in X | \mathcal{M}_x(f) = \mathcal{M}_f(X)\}$.

**Lemma 4.4.** Suppose that $(X, f)$ is a topological dynamical system. Then for any $\phi \in \hat{C}_0^0 f(X)$, $G_{\text{max}} \subseteq \bigcap_{\phi \in \hat{C}_0^0 f(X)} I(\phi, f)$.

**Proof.** Fix $\phi \in \hat{C}_0^0 f(X)$. By Lemma 3.2 there are two invariant measures $\mu_1, \mu_2$ such that $\int \phi d\mu_1 \neq \int \phi d\mu_2$. For any $z \in G_{\text{max}}$, by definition there are two subsequences of $\{\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(z)}\}_{n \in \mathbb{N}}$ converging to $\mu_1, \mu_2$. Then by weak* topology $\int \phi d\mu_1 \neq \int \phi d\mu_2$ implies that the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(z))$$

does not exist. That is, $z \in I(\phi, f)$.

**Proof of Theorem 4.3.** Recall that $\hat{C}_0^0 f(X) \neq \emptyset$ is from Proposition 1. By Lemma 4.4 $G_{\text{max}} \subseteq \bigcap_{\phi \in \hat{C}_0^0 f(X)} I(\phi, f) = CI(f)$. By item (1) of Theorem 4.2 we know $G_{\text{max}}$ is not empty and dense in $\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu)$ and so do $CI(f)$. This finishes the proof of item (1).

If further there is an invariant measure with full support, then

$$\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu) = X.$$

By item (2) of Theorem 4.2 we know $G_{\text{max}}$ is dense in $\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu)$ and so do $CI(f)$. Now we finish the proof of item (2).

Now we study a basic property of any completely-irregular point. A point $x \in X$ is called transitive point, if $\omega_f(x) = X$. Let $\text{Tran}_f$ denote the space of all transitive points.
Theorem 4.5. Suppose that \((X, f)\) satisfies almost specification. Then
(1) for any \(x \in \text{CI}(f)\),
\[
\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu) \subseteq \omega_f(x),
\]
where \(\omega_f(x)\) denotes the \(\omega\)-limit set of \(x\). In other words, \(\mu(\omega_f(x)) = 1\) holds for
any \(x \in \text{CI}(f)\) and any invariant measure \(\mu \in \mathcal{M}_f(X)\).
(2) If there is an invariant measure with full support, then \(\text{CI}(f) \subseteq \text{Tran}_f\).

Proof. It is easy to check that item (1) implies item (2). So we only need to prove item (1).
Assume by contradiction that there is a point \(x \in \text{CI}(f)\),
\[
\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu) \setminus \omega_f(x) \neq \emptyset.
\]
This implies that there exists an invariant measure \(\mu\) and a point \(y \in \text{Support}(\nu) \setminus \omega_f(x)\). Since \(\omega_f(x)\) is a closed invariant set, we can choose \(\epsilon > 0\) such that the open
ball centered on \(y\) with radius \(2\epsilon\) \(B_{2\epsilon}(y)\) satisfies
\[
B_{2\epsilon}(y) \cap \omega_f(x) = \emptyset.
\]
Then we can define a continuous function \(\phi : X \to [0, 1]\) such that the values of \(\phi\)
restricted on \(\omega_f(x)\) are zero and the values of \(\phi\) restricted on the closed ball \(\overline{B}_\epsilon(y) := \text{Closure}(B_\epsilon(y))\) are 1. Note that \(y \in \text{Support}(\nu)\) implies \(\mu(\overline{B}_\epsilon(y)) \geq \mu(B_\epsilon(y)) > 0\).
Then
\[
\int \phi d\mu \geq \int_{\overline{B}_\epsilon(y)} \phi d\mu = \mu(\overline{B}_\epsilon(y)) > 0.
\]
If we take an invariant measure \(\nu\) supported on \(\omega_f(x)\), then \(\int \phi d\nu = 0\). So \(\int \phi d\nu < \int \phi d\mu\). By Lemma 3.1, \(I(\phi, f) \neq \emptyset\). So \(\phi \in \mathcal{C}_f^0(X)\) and thus \(x \in I(\phi, f)\). However,
\(\phi|_{\omega_f(x)} \equiv 0\) implies the limit
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))
\]
exists and equals to 0. It contradicts to \(x \in I(\phi, f)\). \(\square\)

For the case of specification, from [14] we know minimal points are dense in \(X\) and thus in this case \(\bigcup_{\nu \in \mathcal{M}_f(X)} \text{Support}(\nu) = X\). So by Theorem 4.3 and 4.5 we have

Corollary 1. Suppose that \((X, f)\) satisfies specification and is not uniquely ergodic.
Then \(\text{CI}(f)\) is residual in \(X\) and for any \(x \in \text{CI}(f)\), \(\omega(x) = X\) (that is, \(\text{CI}(f) \subseteq \text{Tran}_f\)).

4.2. The topological pressure of \(\bigcap_{\phi \in D} I(\phi, f)\). We state a more general version of our result, which implies Theorem 1.3 and may possibly apply to a larger class of dynamical systems.

Theorem 4.6. Suppose that \((X, f)\) is saturated and \(\mathcal{C}_f^0(X) \neq \emptyset\). Then
(1) for any subsets \(D \subseteq \mathcal{C}_f^0(X)\), the set \(\bigcap_{\phi \in D} I(\phi, f)\) carries full topological pressure. In particular, \(\text{CI}(f)\) carries full topological pressure;
(2) for any subsets \(D \subseteq \mathcal{C}^0(X)\), the set \(\bigcap_{\phi \in D} I(\phi, f)\) either is empty or carries full topological pressure.
Proof. It is easy to check item (1) implies item (2) so that we only need to prove item (1).

Fix $\epsilon > 0$ and $\varphi \in C^0(X)$. By classical Variational Principle, we can take an ergodic measure $\mu$ such that $h_\mu(f) + \int \varphi d\mu > P(X, \varphi, f) - \epsilon$. Take $\theta \in (0,1)$ close to 1 such that

$$\theta(h_\mu(f)) + \int \varphi d\mu > P(X, \varphi, f) - \epsilon$$

and $(1 - \theta)\|\varphi\| < \epsilon$, where $\|\varphi\| = \max_{x \in X} |\varphi(x)|$. Since $D \subseteq C^0_f(X)$, then for any $\phi \in D$, there is an invariant measure $\mu_\phi$ such that $\int \phi d\mu_\phi \neq \int \phi d\mu$. If take $\nu_\phi := \theta\mu + (1 - \theta)\mu_\phi$, then

$$h_{\nu_\phi}(f) + \int \varphi d\nu_\phi \geq \theta(h_\mu(f)) + \int \varphi d\mu - (1 - \theta)\|\varphi\| > P(X, \varphi, f) - 2\epsilon.$$  

Remark that for any $\phi \in D$, $\int \phi d\nu_\phi \neq \int \phi d\mu$. By continuity of functions, for $\phi \in D$, there is an open neighborhood of $\phi$, $U_\phi \subseteq C^0(X)$ such that for any $\psi \in U_\phi$, $\int \psi d\nu_\phi \neq \int \psi d\mu$. So $\{U_\phi | \phi \in D\}$ forms an open cover of $D$. Since $C^0(X)$ has countable topological basis and $D$ is a subset of $C^0(X)$, we can take a (at most) countable subcover of $D$, $\{U_{\phi_i}\}_{i=1}^\infty$, with respect to countable functions $\phi_i \in D$. So for each $\psi \in D$, there is $i \geq 1$ such that $\psi \in U_{\phi_i}$ and thus

$$\int \psi d\nu_{\phi_i} \neq \int \psi d\mu. \tag{3}$$

Let $\nu_1 = \nu_{\phi_1}$. Take a strictly increasing sequence of real numbers $\theta_1 \in (\theta, 1)$ converging to 1 and define $\omega_i := \theta_i\mu + (1 - \theta_i)\nu_1$. Then for each $\psi \in D$, by (3) there is $i \geq 1$ such that $\int \psi d\omega_i \neq \int \psi d\chi_i$. Remark that

$$h_{\omega_i}(f) + \int \varphi d\omega_i \geq \theta_i(h_\mu + \int \varphi d\mu) + (1 - \theta_i)h_{\nu_1}(f) - (1 - \theta_i)\|\varphi\| \geq \min\{\theta(h_\mu + \int \varphi d\mu), h_\mu + \int \varphi d\mu\} - (1 - \theta)\|\varphi\| > P(X, \varphi, f) - 2\epsilon.$$  

Let $K := \{\mu\} \cup \bigcup_{i=1}^\infty \{t\omega_i + (1 - t)\omega_{i+1} | t \in [0,1]\}$. Since $\omega_i$ converge to $\mu$, it is easy to check that $K$ is connected and compact and every measure $\nu \in K$ satisfies that $h_{\nu}(f) + \int \varphi d\nu > P(X, \varphi, f) - 2\epsilon$.

Since $f$ is saturated, then for above $K$ one has

$$P(G_K, \varphi, f) = \inf\{h_{\nu}(f) + \int \varphi d\nu | \nu \in K\} \geq P(X, \varphi, f) - 2\epsilon.$$  

So the left to end the proof is only to show $G_K \subseteq \bigcap_{\psi \in D} I(\psi, f)$. Fix $x \in G_K$ and $\psi \in D$. Then there is $i \geq 1$ such that $\int \psi d\omega_i \neq \int \psi d\mu$. By definition of $G_K$, $M_\varphi(f) = K \supseteq \{\mu, \omega_i\}$. Then there are two sequences of $n_j, m_j \uparrow +\infty$ such that in weak* topology $\lim_{j \to \infty} \frac{1}{n_j} \sum_{l=0}^{n_j-1} \delta_{f^l(x)} = \mu$, $\lim_{j \to \infty} \frac{1}{m_j} \sum_{l=0}^{m_j-1} \delta_{f^l(x)} = \omega_i$. Then

$$\lim_{j \to \infty} \frac{1}{n_j} \sum_{l=0}^{n_j-1} \psi(f^l(x)) = \lim_{j \to \infty} \psi d\mu \neq \lim_{j \to \infty} \psi d\omega_i = \lim_{j \to \infty} \frac{1}{m_j} \sum_{l=0}^{m_j-1} \psi(f^l(x)).$$

This implies $x \in I(\psi, f)$. Now we finish the proof. \hfill \Box

Corollary 2. Suppose that $(X, f)$ is saturated, not uniquely ergodic and has positive topological entropy. Then $CI(f)$ is not empty and carries full topological entropy.
Proof. We first show that $\hat{C}_0(X) \neq \emptyset$. By variational principle, there is an ergodic measure $\mu$ with positive metric entropy. Since the system is not uniquely ergodic, then there is another invariant measure $\mu' \neq \mu$. Take $\nu = \frac{1}{2}\mu + \frac{1}{2}\mu'$, then $\nu \neq \mu$. Take $K = \{\tau \mu + (1 - \tau)\nu | \tau \in [0,1]\}$, then $K$ is compact and connected and each measure in $K$ has positive metric entropy larger or equal to $\frac{1}{2}h_\mu(f)$. By saturated property, $h_{top}(G_K, f) = \inf_{\omega \in K} h_\omega(f) \geq \frac{1}{2}h_\mu(f) > 0$. This implies $G_K \neq \emptyset$. Take one point $x_0 \in G_K$. Then $\nu, \mu \in K = M_{x_0}(f)$. Since $\nu \neq \mu$, by weak* topology, there is $\phi \in C^0(X)$ such that $\int \phi d\mu \neq \int \phi d\nu$. By Lemma 3.2, $x_0 \in I(\phi, f)$ and $\phi \in \hat{C}_0(X)$.

By item (1) of Theorem 4.6, $CI(f)$ carries full topological pressure for all potentials so that it also carries full topological entropy (taking zero potential function). Since the system has positive entropy, then $CI(f)$ has positive topological entropy and thus is not empty. \hfill $\square$

4.3. Proof of Theorem 1.3. By assumption and Lemma 2.6, $f$ is saturated. By Proposition 1 $\hat{C}_0(X) \neq \emptyset$. If $D = \hat{C}_0(X)$, then $\bigcap_{\phi \in D} I(\phi, f) = CI(f)$. By Theorem 4.6, $CI(f)$ carries full topological pressure. Note that $CI(f) \neq \emptyset$ by Theorem 4.3. Now we complete the proof of Theorem 1.3. \hfill $\square$

4.4. Examples. The above consequence can be applicable to a large class of classical dynamical systems.

Theorem 4.7. For any one of following systems, the completely-irregular set $CI(f)$ is residual in $X$ and carries full topological pressure (in particular, topological entropy) and BS-dimension:

(A). $f: X \to X$ is a mixing subshift of finite type.

(B). $f: X \to X$ is a subsystem of a $C^1$ diffeomorphism $f: M \to M$ over a compact Riemannian manifold $M$ where $X$ is a topological mixing locally maximal hyperbolic set.

(C). $f: X \to X$ is a subsystem of a $C^1$ map $f: M \to M$ over a compact Riemannian manifold $M$ where $X$ is a topological mixing locally maximal expanding invariant set (called repeller).

Proof. It is known that any system $f$ in Theorem 4.7 is not uniquely ergodic, has positive entropy and satisfies shadowing and specification (topological mixing + shadowing property $\Rightarrow$ specification). From [28], we know that specification implies $g$–almost product property [28, Proposition 2.1]. Note that the systems of (A)- (C) are all expansive. Thus, Theorem 4.7 can be deduced from Corollary 4 for residual property, Theorem 1.3 for full pressure and Remark 1 for full BS dimension. \hfill $\square$

Recall a result that for $C^{1+\delta}$ conformal repellers, it was proved in [17] that an intersection of finite $\phi$-irregular sets is either empty or carries full Hausdorff dimension. In this case if $\varphi = \log \|df\|$, then $BS(Z, \varphi, f) = dim_H(Z)$ for every $Z \subseteq X$. So Theorem 4.7 (C) implies that

Corollary 3. Let $f: X \to X$ be a subsystem of a conformal $C^{1+\delta}$ map $f: M \to M$ over a compact Riemannian manifold $M$ where $X$ is a topological mixing and expanding invariant set (called conformal repeller). Then for any subset $D \subseteq C^0(X)$, the set $\bigcap_{\phi \in D} I(\phi, f)$ either is empty or carries full Hausdorff dimension.
This corollary generalizes the result of [17] from observation of finite functions to any number of functions.

Let \((\Sigma_\beta, \sigma_\beta)\) denote \(\beta\)-shift \((\beta > 1)\) (cf. [40] Chapter 7.3 for the definition). It is known that the topological entropy of \(\beta\)-shift \((\beta > 1)\) is \(\log \beta\). By Variational Principle, there exists an ergodic measure with positive entropy. Note that the Dirac measure supported on the fixed point \(x = \{0\}_{\infty}^1 \in \Sigma_\beta\) has zero entropy. So every \(\beta\)-shift is not uniquely ergodic.

It is known that every \(\beta\)-shift \((\beta > 1)\) \((\Sigma_\beta, \sigma_\beta)\) is expansive (as a subshift) and satisfies \(g\)-almost product property from [28] (see the Example on P.934). So the completely-irregular set of every \(\beta\)-shift has full topological entropy \((= \log \beta)\), full topological pressure and BS-dimension by Theorem 1.3 and Remark 1. It is known that the unique maximal entropy measure of \(\beta\)-shifts always carries full support \([39, Theorem 13 (ii)]\) so that by item (2) of Theorem 4.3 the completely-irregular set of every \(\beta\)-shift is residual in \(\Sigma_\beta\).

Moreover, from [37, Lemma 5.3] every subset \(Z \subseteq \Sigma_\beta\) satisfies that
\[
\dim_H(Z) = \frac{1}{\log \beta} h_{\text{top}}(\sigma_\beta, Z)
\]
so that \(\dim_H(\bigcap_{\phi \in \hat{C}^0(\Sigma_\beta)} I(\phi, f)) = 1\). That is,

**Theorem 4.8.** \((\beta\)-shift\) For \(\beta > 1\), let \((\Sigma_\beta, \sigma_\beta)\) denote the \(\beta\)-shift. Then the completely-irregular set \(\mathcal{IC}(\sigma_\beta) = \bigcap_{\phi \in \hat{C}^0(\Sigma_\beta)} I(\phi, \sigma_\beta)\) is residual in \(\Sigma_\beta\) and carries full topological entropy, topological pressure, BS-dimension and full Hausdorff dimension.

5. Remarks.

5.1. The case without uniform separation. It is known from [7, 11] that any topologically mixing interval map satisfies Bowen’s specification. For example, Jakobson [19] showed that there exists a set of parameter values \(\Lambda \subseteq [0, 4]\) of positive Lebesgue measure such that if \(\lambda \in \Lambda\), then the logistic map \(f_\lambda(x) = \lambda x(1-x)\) is topologically mixing. Remark that it is unknown whether any topologically mixing interval map has uniform separation. In this case there is still a result for finitely many observable functions, which extends Theorem 1.1 and has been obtained in [37] for the case of topological entropy.

**Theorem 5.1.** Suppose that \((X, f)\) satisfies almost specification. Then
\(\begin{align*}
(1) & \text{ For any finite functions } \phi_1, \cdots, \phi_k \in C^0(X)(k \geq 1), \text{ the set } \bigcap_{j=1}^k I(\phi_j, f) \text{ either is empty or carries full topological pressure.} \\
(2) & \text{ Suppose that } \hat{C}^0(X) \neq \emptyset. \text{ If } \phi_i \subseteq \hat{C}^0(X), \text{ the set } \bigcap_{j=1}^k I(\phi_j, f) \text{ is not empty and carries full topological pressure.}
\end{align*}\)

**Proof.** Here we need to consider multiple observable functions different from [36, 37] which only considers one function. But the idea is just to adapt the proof of [36, 37] (Remark that the result of [37] is for the case of topological entropy but by slight modification its idea is still valid for topological pressure). So we only give a sketch of the proof.

(1) Assume \(\bigcap_{j=1}^k I(\phi_j, f) \neq \emptyset\). Take a point \(x\) in this set. Then by definition of \(I(\phi_i, f)\) and weak* topology (see Lemma 3.2), there are invariant measures \(\mu_{1,j}, \mu_{2,j} \in \mathcal{M}_x(f)(1 \leq j \leq k)\) such that \(\int \phi_j d\mu_{1,j} \neq \int \phi_j d\mu_{2,j}, 1 \leq j \leq k\).
For a fixed $1 \leq j \leq k$, obviously the solutions $(\lambda_1, \cdots, \lambda_k)$ of linear equation
\[
\sum_{i=1}^k (\int \phi_j d\mu_{1,i} - \int \phi_j d\mu_{2,i})\lambda_i = 0
\]
form a $k - 1$ dimensional closed linear subspace of $\mathbb{R}^k$, denoted by $P_j$. Notice that $\bigcup_{1 \leq j \leq k} P_j$ is the union of finite $k - 1$ dimensional closed linear subspaces so that $\mathbb{R}^k \setminus (\bigcup_{1 \leq j \leq k} P_j)$ is open and dense in $\mathbb{R}^k$. One can take $k$ positive numbers of $\lambda_1, \cdots, \lambda_k$ such that for any $1 \leq j \leq k$,
\[
\sum_{i=1}^k (\int \phi_j d\mu_{1,i} - \int \phi_j d\mu_{2,i})\lambda_i \neq 0.
\]
For example, if $k = 3$, $(\lambda_1, \lambda_2, \lambda_3)$ is chosen in the first octant of $\mathbb{R}^3$.

Let $\theta_j = \frac{\lambda_j}{\sum_{i=1}^k \lambda_i}$. Then all $\theta_j$ are positive and $\sum_{j=1}^k \theta_j = 1$. If we define $\mu_l = \sum_{i=1}^k \theta_i \mu_{l,i}, \ l = 1, 2$, then for any $1 \leq j \leq k$, $\int \phi_j d\mu_1 \neq \int \phi_j d\mu_2$.

Fix $\epsilon > 0$ and $\varphi \in C^0(X)$. By classical Variational Principle, we can take an ergodic measure $\mu$ such that $h_\mu(f) + \int \varphi d\mu > P(X, \varphi, f) - \epsilon$. Take $\theta \in (0, 1)$ close to 1 such that
\[
\theta(h_\mu(f) + \int \varphi d\mu) > P(X, \varphi, f) - \epsilon
\]
and $(1 - \theta)\|\varphi\| < \epsilon$, where $\|\varphi\| = \max_{x \in X} |\varphi(x)|$. Then the two invariant measures $\nu_l := \theta \mu + (1 - \theta)\mu_l (l = 1, 2)$ satisfy
\[
h_{\nu_l}(f) + \int \varphi d\nu_l \geq \theta(h_\mu(f) + \int \varphi d\mu) - (1 - \theta)\|\varphi\| > P(X, \varphi, f) - 2\epsilon (l = 1, 2).
\]
Remark that for any $1 \leq j \leq k$, $\int \phi_j d\nu_1 \neq \int \phi_j d\nu_2$.

By [27], when $f$ has the almost specification, $f$ satisfies entropy-dense property. So we can find two sequence of ergodic measures $\nu_{l,i} \in \mathcal{M}_f(X)$ such that
\[
h_{\nu_{l,i}}(f) \to h_{\nu_l}, \text{ and } \nu_{l,i} \to \nu_l (l = 1, 2)
\]
in weak* topology. Therefore, we can take two measures belonging to these two sequence which we called $\rho_1$ and $\rho_2$ respectively such that
\[
h_\mu(f) + \int \varphi d\rho_l > P(X, \varphi, f) - 2\epsilon (l = 1, 2)
\]
and for any $1 \leq j \leq k$, $\int \phi_j d\rho_1 \neq \int \phi_j d\rho_2$. This is the first step of [36, 37] to choose two good ergodic measures but crucial because the ergodicity is important in the proof of [36, 37] to avoid the use of uniform separation.

Then one can follow the proof of [36, 37] to complete the proof. Roughly speaking, using the above two ergodic measures to construct a set $F \subseteq \bigcap_{j=1}^k I(\phi_j, f)$ such that the topological pressure of $F$ is larger than $P(X, \varphi, f) - 2\epsilon$. In this process, Entropy Distribution Principle plays an important role. Here we omit the details.

(2) If $\phi_1 \subseteq \tilde{C}_f^0(X)$, then $\bigcap_{i=1}^k I(\phi_i, f) \supseteq CI(f)$. In this case $CI(f) \neq \emptyset$ by Theorem 4.3 Thus $\bigcap_{i=1}^k I(\phi_i, f) \neq \emptyset$ and thus by item (1) we complete the proof.  \[\square\]
5.2. The topological pressure of $(\bigcap_{\phi \in D_1} I(\phi, f)) \cap (\bigcap_{\psi \in D_2} R(\psi, f))$.

**Theorem 5.2.** Suppose that $(X, f)$ is saturated.

1. If the entropy function $h_\nu(f) : \mathcal{M}_f(X) \to \mathbb{R}$, $\nu \mapsto h_\nu(f)$ is upper semi-continuous, then for any subsets $D_1, D_2 \subseteq C^0(X)$, the set
   
   $$(\bigcap_{\phi \in D_1} I(\phi, f)) \cap (\bigcap_{\psi \in D_2} R(\psi, f))$$

   either is empty or carries full topological pressure.

2. For any subsets $D_1, D_2 \subseteq C^0(X)$, if the set $D_1$ is at most countable, then the set
   
   $$(\bigcap_{\phi \in D_1} I(\phi, f)) \cap (\bigcap_{\psi \in D_2} R(\psi, f))$$

   either is empty or carries full topological pressure.

**Remark 2.** The item (1) of Theorem 5.2 with Lemma 2.6, Proposition 3.3, Theorem 4.3 and Proposition 1 can also deduce Theorem 1.3. Let us explain more precisely. By assumption and Lemma 2.6, the entropy function is upper semi-continuous. By Proposition 1, $C_i^0(X) \neq \emptyset$ and by Theorem 4.3, $CI(f) \neq \emptyset$. If $D_1 = C_1^0(X)$ and $D_2$ is composed of a constant function, then $(\bigcap_{\phi \in D_1} I(\phi, f)) \cap (\bigcap_{\psi \in D_2} R(\psi, f)) = CI(f)$. By item (1) of Theorem 5.2, $CI(f)$ carries full topological pressure so that we complete the proof of Theorem 1.3.

**Proof.** Assume $D_1, D_2 \subseteq C^0(X)$, and $(\bigcap_{\phi \in D_1} I(\phi, f)) \cap (\bigcap_{\psi \in D_2} R(\psi, f)) \neq \emptyset$. Take a point $x_0$ in this set. Then by definition of $R(\psi, f)$, for any $\nu_1, \nu_2 \in \mathcal{M}_{x_0}(f)$ and $\psi \in D_2$,

$$\int \psi d\nu_1 = \int \psi d\nu_2 (= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x_0))).$$

By definition of $I(\phi, f)$ and weak* topology (see Lemma 3.2), for any $\phi \in D_1$, there are invariant measures $\mu_{1, \phi}, \mu_{2, \phi} \in \mathcal{M}_{x_0}(f)$ such that $\int \phi d\mu_{1, \phi} \neq \int \phi d\mu_{2, \phi}$. Note that $\bigcup_{\phi \in D_1} \{\mu_{1, \phi}, \mu_{2, \phi}\} \subseteq \mathcal{M}_{x_0}(f)$. Thus for any $\psi \in D_2$,

$$\inf_{\phi \in D_1} \int \psi d\mu_{1, \phi} = \inf_{\phi \in D_1} \int \psi d\mu_{2, \phi} \leq \sup_{\phi \in D_1} \int \psi d\mu_{1, \phi} \geq \sup_{\phi \in D_1} \int \psi d\mu_{2, \phi}. \quad (4)$$

Fix $\epsilon > 0$ and $\varphi \in C^0(X)$. By classical Variational Principle, we can take an ergodic measure $\mu$ such that $h_\mu(f) + \int \varphi d\mu > P(X, \varphi, f) - \epsilon$. Take $\theta \in (0, 1)$ close to 1 such that

$$\theta h_\mu(f) + \int \varphi d\mu > P(X, \varphi, f) - \epsilon$$

and $(1 - \theta)\|\varphi\| < \epsilon$, where $\|\varphi\| = \max_{x \in X} |\varphi(x)|$. For any $\phi \in D_1$, if take $\nu_{l, \phi} := \theta \mu + (1 - \theta)\mu_{l, \phi}$ ($l = 1, 2$), then

$$h_{\nu_{l, \phi}}(f) + \int \varphi d\nu_{l, \phi} \geq \theta h_\mu(f) + \int \varphi d\mu - (1 - \theta)\|\varphi\| > P(X, \varphi, f) - 2\epsilon \quad (l = 1, 2).$$

Remark that for any $\phi \in D_1$, $\int \phi d\nu_{1, \phi} \neq \int \phi d\nu_{2, \phi}$ and for any $\psi \in D_2$, by $4$,

$$\inf_{\phi \in D_1} \int \psi d\nu_{1, \phi} = \inf_{\phi \in D_1} \int \psi d\nu_{2, \phi} = \sup_{\phi \in D_1} \int \psi d\nu_{1, \phi} = \sup_{\phi \in D_1} \int \psi d\nu_{2, \phi}. \quad (5)$$

Now we give the proof one by one.
(1). Let $K' := \bigcup_{\phi \in D_1} \{\nu_1, \nu_2\}$ and let $K$ be the convex hull of $K'$. That is,

$$K = \{ \sum_{i=1}^{k} \lambda_i \nu_i \mid \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \in (0, 1), \nu_i \in K' \}. $$

Since any $\nu \in K'$ satisfies $h_{\nu}(f) + \int \varphi \, d\nu > P(X, \varphi, f) - 2\epsilon$, we have $h_{\nu}(f) + \int \varphi \, d\tau > P(X, \varphi, f) - 2\epsilon$ for any $\tau = \sum_{i=1}^{k} \lambda_i \nu_i$ where $\lambda_i \in (0, 1), \nu_i \in K', \sum_{i=1}^{k} \lambda_i = 1$.

Since the entropy function is upper semi-continuous and $\rho \mapsto \int \varphi \, d\rho$ is continuous, then $h_{\nu}(f) + \int \varphi \, d\nu \geq P(X, \varphi, f) - 2\epsilon$ holds for any $\nu \in K$. Similarly, by (5) and the continuity of functions $\psi \in D_2$, we have

$$\inf_{\nu \in K} \int \psi \, d\nu = \sup_{\nu \in K} \int \psi \, d\nu. \quad (6)$$

Since $f$ is saturated, then for the above $K$

$$P(G_K, \varphi, f) = \inf \{ h_{\nu}(f) + \int \varphi \, d\nu \mid \nu \in K \} \geq P(X, \varphi, f) - 2\epsilon.$$

So the left to end the proof is only to show $G_K \subseteq \bigcap_{\phi \in D_1} I(\phi, f) \cap \bigcap_{\psi \in D_2} R(\psi, f)$.

By weak* topology, it is obvious from (6) that $G_K \subseteq \bigcap_{\psi \in D_2} R(\psi, f)$.

Fix $x \in G_K$ and $\phi \in D_1$. By definition of $G_K$, $M_x(f) = K \supseteq \{\nu_1, \nu_2\}$. Then there are two sequences of $n_j, m_j \uparrow +\infty$ such that in weak* topology

$$\lim_{j \to \infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i(x)} = \nu_1, \lim_{j \to \infty} \frac{1}{m_j} \sum_{i=0}^{m_j-1} \delta_{f^i(x)} = \nu_2.$$

Then

$$\lim_{j \to \infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \phi(f^i(x)) = \int \phi \, d\nu_1 \neq \int \phi \, d\nu_2, \lim_{j \to \infty} \frac{1}{m_j} \sum_{i=0}^{m_j-1} \phi(f^i(x)) = \int \phi \, d\nu_2 \neq \int \phi \, d\nu_1.$$

This implies $x \in I(\phi, f)$.

(2). Suppose $D_1 = \{\phi_i\}_{i=1}^{\infty}$. Let $\nu_{1,i} = \nu_{1,\phi_i}$ and $\nu_{2,i} = \nu_{2,\phi_i}$. Then for $i = 1, 2, 3, \cdots$,

$$\int \phi_i \, d\nu_{1,i} \neq \int \phi_i \, d\nu_{2,i} \quad (7)$$

and for any $\psi \in D_2$, by (5)

$$\inf_i \int \psi \, d\nu_{1,i} = \inf_i \int \psi \, d\nu_{2,i} = \sup_i \int \psi \, d\nu_{1,i} = \sup_i \int \psi \, d\nu_{2,i}. \quad (8)$$

Let $\omega = \nu_{1,1}, \omega' = \nu_{2,1}$. Then by (7), $\int \phi_i \, d\omega \neq \int \phi_i \, d\omega'$. For $i \geq 2$, by (7) we can take $\omega' \in \{\nu_{1,i}, \nu_{2,i}\}$ such that $\int \phi_i \, d\omega' \neq \int \phi_i \, d\omega$. By (8) for any $\psi \in D_2$,

$$\inf_i \int \psi \, d\omega' = \sup_i \int \psi \, d\omega' = \int \psi \, d\omega. \quad (9)$$

Remark that $h_{\omega}(f) + \int \varphi \, d\omega > P(X, \varphi, f) - 2\epsilon, h_{\omega'}(f) + \int \varphi \, d\omega' > P(X, \varphi, f) - 2\epsilon$. Take a strictly increasing sequence of real numbers $\theta_i \in (0, 1)$ converging to 1 and define $\omega_i := \theta_i \omega + (1 - \theta_i) \omega', i = 1, 2, \cdots$. Then

$$h_{\omega_i}(f) + \int \varphi \, d\omega_i > P(X, \varphi, f) - 2\epsilon.$$

Remark that every $\omega_i$ satisfies $\int \phi_i \, d\omega \neq \int \phi_i \, d\omega_i$ and by (9)

$$\inf_i \int \psi \, d\omega_i = \sup_i \int \psi \, d\omega_i = \int \psi \, d\omega. \quad (10)$$
Let \( K := \{ \omega \} \cup \bigcup_{i=1}^{\infty} \{ t \omega_i + (1-t) \omega_{i+1} \mid t \in [0,1] \} \). Since \( \omega_i \) converges to \( \omega \), it is easy to check that \( K \) is connected and compact and every measure \( \nu \in K \) satisfies that \( h_{\nu}(f) + \int \varphi d\nu > P(X, \varphi, f) - 2\epsilon \).

Since \( f \) is saturated, then for above \( K \) one has

\[
P(G_K, \varphi, f) = \inf \{ h_{\nu}(f) + \int \varphi d\nu \mid \nu \in K \} \geq P(X, \varphi, f) - 2\epsilon.
\]

So the left to end the proof is only to show \( G_K \subseteq \bigcap_{i=1}^{\infty} I(\phi_i, f) \cap (\bigcap_{\psi \in D_2} R(\psi, f)) \). Fix \( x \in G_K \) and \( i \geq 1 \). By definition of \( G_K \), \( \mathcal{M}_x(f) = K \supseteq \{ \omega, \omega_i \} \). Then there are two sequences of \( n_j, m_j \uparrow +\infty \) such that in weak* topology \( \lim_{j \to \infty} \frac{1}{m_j} \sum_{l=0}^{m_j-1} \delta_{f^l(x)} = \omega \), \( \lim_{j \to \infty} \frac{1}{n_j} \sum_{l=0}^{n_j-1} \delta_{f^l(x)} = \omega_i \). Then

\[
\lim_{j \to \infty} \frac{1}{n_j} \sum_{l=0}^{n_j-1} \phi_i(f^l(x)) = \int \phi_i d\omega \neq \int \phi_i d\omega_i = \lim_{j \to \infty} \frac{1}{m_j} \sum_{l=0}^{m_j-1} \phi_i(f^l(x)).
\]

This implies \( x \in I(\phi_i, f) \). By \( \Box \)

\[
\inf_{\nu \in K} \int \psi d\nu = \sup_{\nu \in K} \int \psi d\nu = \int \psi d\omega.
\]

By weak* topology, it implies that \( G_K \subseteq \bigcap_{\psi \in D_2} R(\psi, f) \). Now we finish the proof of item (2).

**Corollary 4.** Suppose that \((X, f)\) is saturated.

1. If the entropy function \( h_{\nu}(f) : \mathcal{M}_f(X) \to \mathbb{R}, \mu \mapsto h_{\mu}(f) \) is upper semi-continuous, then for any subset \( D \subseteq C^0(X) \), the set \( \bigcap_{\psi \in D} I(\phi, f) \) either is empty or carries full topological pressure.

2. For any subset \( D \subseteq C^0(X) \), if the set \( D_1 \) is at most countable, then the set \( \bigcap_{\phi \in D} I(\phi, f) \) either is empty or carries full topological pressure.

3. Suppose \( f \) satisfies \( g \)-almost product property and \( \hat{C}^0_f(X) \neq \emptyset \). If \( D \subseteq \hat{C}^0_f(X) \) is finite, then the set \( \bigcap_{\phi \in D} I(\phi, f) \setminus C(f) \) is not empty and carries full topological pressure.

**Proof.** Note that

\[
I(\phi, f) \setminus C(f) = I(\phi, f) \setminus \left( \bigcap_{\psi \in \hat{C}^0_f(X)} I(\psi, f) \right)
= I(\phi, f) \cap \left( \bigcup_{\psi \in \hat{C}^0_f(X)} R(\psi, f) \right) = \bigcup_{\psi \in \hat{C}^0_f(X)} (I(\phi, f) \cap R(\psi, f)).
\]

For any subset \( D \subseteq C^0(X) \), if the set \( \bigcap_{\phi \in D} I(\phi, f) \setminus C(f) \) is not empty, then by \( \Box \) there is \( \psi \in \hat{C}^0_f(X) \) such that \( \bigcap_{\phi \in D} I(\phi, f) \cap R(\psi, f) \neq \emptyset \). By Theorem 5.2 we complete the proof of item (1) and (2).

Now we start to prove item (3). By item (2) of Corollary 4 we only need to prove \( \bigcap_{\phi \in D} I(\phi, f) \setminus C(f) \neq \emptyset \). We state it as a proposition as follows.

**Proposition 2.** Suppose that \((X, f)\) satisfies \( g \)-almost product property and \( \hat{C}^0_f(X) \neq \emptyset \). Then for any finite functions \( \phi_1, \ldots, \phi_k \in \hat{C}^0_f(X)(k \geq 1) \), there is some \( \psi \in \hat{C}^0_f(X) \) such that \( \bigcap_{j=1}^{k} I(\phi_j, f) \cap R(\psi, f) \neq \emptyset \). In particular, \( \bigcap_{j=1}^{k} I(\phi_j, f) \setminus C(f) \neq \emptyset \).
Remark 3. This proposition implies that $CI(f)$ can not be written an intersection of finite $\phi$-irregular sets.

Proof. By the observation of (11) we only need to find some $\psi \in \hat{C}_f^0(X)$ such that
\[
\bigcap_{j=1}^k I(\phi_j, f) \cap R(\psi, f) \neq \emptyset. \quad \text{Before that we need the first part of [27 Proposition 2.3] and [27 Theorem 2.1].}
\]

Lemma 5.3. Suppose that $(X, f)$ satisfies $g-$almost product property.

(1) [27 Proposition 2.3] Let $\mu$ be an ergodic measure. Then for any neighborhood $G \subseteq M(X)$ of $\mu$, there exists a closed $f$-invariant subset $Y \subseteq X$ and an integer $N_G > 0$ such that for any $n \geq N_G$ and $y \in Y$, $\mathcal{E}_n(y) \in G$.

(2) [27 Theorem 2.1] $f$ has entropy-dense property, that is, for any $\nu \in \mathcal{M}_f(X)$, any neighborhood $G \subseteq M(X)$ of $\mu$ and any $h_* < h_*(T)$, there exists an ergodic measure $\mu \in G \cap \mathcal{M}_f(X)$ such that $h_*(T) > h_*$. 

Recall $\mu_1, \mu_2$ to be the two ergodic measures in the proof of Theorem 5.1 and for any $1 \leq j \leq k$, $\int \phi_j \, d\mu_1 \neq \int \phi_j \, d\mu_2$. By item (2) of Lemma 5.3 $f$ has entropy-dense property, thus one can choose another ergodic measure $\mu_3$ close to $\frac{1}{2}(\mu_1 + \mu_2)$ enough in weak* topology such that for any $1 \leq j \leq k$,
\[
\min_{l=1,2} \int \phi_l \, d\mu_l < \int \phi_j \, d\mu_3 < \max_{l=1,2} \int \phi_l \, d\mu_l.
\]

Take three closed neighborhoods $G_1, G_2, G_3$ of $\mu_1, \mu_2, \mu_3$ respectively such that $G_1, G_2, G_3$ are pairwise disjoint and for any measures $\omega_i \in G_i(i = 1, 2, 3)$, any $1 \leq j \leq k$,
\[
\min_{l=1,2} \int \phi_j \, d\omega_l < \int \phi_j \, d\omega_3 < \max_{l=1,2} \int \phi_j \, d\omega_l.
\]

Then by item (1) of Lemma 5.3 one can take three closed $f$-invariant subsets $Y_1, Y_2, Y_3 \subseteq X$ and a common integer $N$ such that for any $n \geq N$ and $y \in Y_i$, $\mathcal{E}_n(y) \in G_i$. Remark that $Y_1, Y_2, Y_3$ are pairwise disjoint so that there exists a continuous function $\psi : X \to \mathbb{R}$ such that
\[
\psi|_{Y_1 \cup Y_2} = 0, \quad \psi|_{Y_3} = 1.
\]

Take $\nu_i(i = 1, 2, 3)$ to be three ergodic measures supported on $Y_i$. Then by Birkhoff ergodic theorem, each $\nu_i$ is a limit point of $\mathcal{E}_n(y_i)$ for some point $y_i \in Y_i$ so that we have $\nu_i \in G_i(i = 1, 2, 3)$. Remark that $\nu_1, \nu_2, \nu_3$ satisfy that for any $1 \leq j \leq k$,
\[
\min_{l=1,2} \int \phi_j \, d\nu_l < \int \phi_j \, d\nu_3 < \max_{l=1,2} \int \phi_j \, d\nu_l.
\]

Note that $\int \psi \, d\nu_1 = 0 < 1 = \int \psi \, d\nu_3$ so that by Lemma 3.1 $I(\psi, f) \neq \emptyset$. That is, $\psi \in \hat{C}_f^0(X)$.

Let $K = \{\tau \nu_1 + (1-\tau) \nu_2\}$. Then it is compact and connected so that by Theorem 4.2 there is some $x \in X$ such that $\mathcal{M}_x(f) = K$. That $\int \phi_j \, d\nu_1 \neq \int \phi_j \, d\nu_2$ implies for any $1 \leq j \leq k$,
\[
\inf_{\mu \in \mathcal{M}_x(f)} \int \phi_j(x) \, d\mu = \inf_{\mu \in K} \int \phi_j(x) \, d\mu < \sup_{\mu \in K} \int \phi_j(x) \, d\mu = \sup_{\mu \in \mathcal{M}_x(f)} \int \phi_j(x) \, d\mu.
\]

So by Lemma 3.2 $x \in \bigcap_{j=1}^k I(\phi_j, f)$. Notice that $\int \psi(x) \, d\nu_1 = \int_{K_1} \psi(x) \, d\nu_1 = 0 = \int_{K_2} \psi(x) \, d\nu_1 = \int \psi(x) \, d\nu_2$ so that
\[
\inf_{\mu \in \mathcal{M}_x(f)} \int \psi(x) \, d\mu = \sup_{\mu \in \mathcal{M}_x(f)} \int \psi(x) \, d\mu = 0.
\]
Thus by Lemma 3.2, $x \in R(\psi, f)$. We complete the proof.

Acknowledgments. The author would like to thank anonymous referee for careful reading of the paper and valuable suggestions and comments.

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Received May 2016; revised January 2017.

E-mail address: xuetingtian@fudan.edu.cn