General Relativistic Relation Between Density Contrast and Peculiar Velocity

Reza Mansouri*
Sohrab Rahvar†
Department of Physics, Sharif University of Technology, P.O.Box 11365–9161, Tehran, Iran
Institute for Studies in Theoretical Physics and Mathematics, P.O.Box 19395–5531, Tehran, Iran
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Abstract

Concepts like peculiar velocity, gravitational force, and power spectrum and their interrelationships are of utmost importance in the theories of structure formation. The observational implementation of these concepts is usually based on the Newtonian hydrodynamic equations, but used up to scales where general relativistic effects come in. Using a perturbation of FRW metric in harmonic gauge, we show that the relativistic effects reduce to light cone effects including the expansion of the universe.

Within the Newtonian gravitation, the linear perturbation theory of large scale structure formation predicts the peculiar velocity field to be directly proportional to gravitational force due to the matter distribution. The corresponding relation between peculiar velocity field and density contrast has been given by Peebles. Using the general relativistic perturbation we have developed, this familiar relation is modified by doing the calculation on the light cone in contrast to the usual procedure of taking a space-like slice defined at a definite time. The velocity and density–spectrum are compared to the familiar Newtonian expressions. In particular, the relativistic $\beta$–value obtained is reduced and leads to an increased bias factor or a decreased expected amount of the dark matter in a cluster.

1 Introduction

Although cosmology has been one of the first areas of application of general relativity, in physical cosmology we encounter rarely general relativistic considerations. Relativistic effects are commonly restricted to either areas of very high density, like black holes or neutron stars, or global cosmological effects, like global dynamics of FRW models. In

*E-mail: mansouri@theory.ipm.ac.ir
†E-mail: rahvar@theory.ipm.ac.ir
theoretical implementations of observational cosmology, such as theoretical interpretation of the Two-degree Field (2dF) – and the Sloan Digital Sky Survey (SDSS) – projects, the Stromlo-APM red-shift survey [1] and the Las Campanas red-shift survey [2] one usually rely on basic Newtonian dynamics [3], [4], [5] and [6]. The correct theoretical interpretation of these surveys needs a thorough understanding of structure formation processes in the universe, mainly objects of scales greater than galaxies and at cosmological distances. It is obvious that at such scales one can not use the full formalism of general relativity to study the dynamics of structures and one has to rely on some approximation methods. In relativistic jargon, the Newtonian limit means gravity in the vicinity of objects having 'small' mass, or expressing it more exactly, where the ratio of the Schwarzschild radius to object radius is a very small number [7]. But one never discuss the relativistic effects on cosmological parameters in the intermediate scales where the relativistic effects due to the mass concentration is negligible. This is exactly the case in the theory of structure formation. For small density contrasts it is used to apply the Newtonian dynamics to the linear theory of structure formation. But even if one tries to apply the linear theory of structure formation to evaluate the observational data, general relativistic effects due to the large extension of the objects or their distances come in. This is exactly the point which has been neglected in theoretical studies up to now and we are going to consider it.

The problem is to formulate an approximation method within general relativity for the cases where we have small density perturbation in a FRW model for intermediate regions smaller than the Hubble radius but large enough to be forced to consider light cone effects. Specifically, we will calculate the relation between the density contrast and the peculiar velocity in clusters within the general relativity and compare it to the well known Newtonian relations. The formalism developed here shed some light on the methodology of the calculation of different so-called light cone effects which has been published [8], [9], [10], [11], [12]. The relativistic calculation helps us to understand more systematically the complex role of different cosmological parameters which come in once. In Section 2 we review the basic hydrodynamical equations based on the Newtonian gravity. In Section 3 a perturbation formalism for FRW universes in harmonic gauge is given. Using this approximation formalism, we obtain a relativistic relation between the density contrast and the peculiar velocity. In Section 4 we obtain relativistic derivation from density contrast. Section 5 is devoted to the power spectrum. The general relativistic $\beta$-value is then calculated in Section 6. And finally in Section 7 we summarize the main points of this study.

Throughout the paper we choose the signature $(-, +, +, +)$, and assume $c = 1$, except where indicated.

2 Peculiar Velocity Field in the Newtonian Hydrodynamics

Basic hydrodynamical equations used to study the peculiar velocity of structures in the linear regime are the continuity and the Poisson equation:

$$\dot{\delta} + \nabla \cdot \mathbf{v} = 0,$$

(1)
\[ \nabla \cdot g = 4\pi G \rho \delta, \]  

(2)

where \( \delta = \frac{\rho - \rho_0}{\rho_0} \) is the density contrast in the homogeneous and isotropic universe with the background density \( \rho_0 \). The peculiar velocity is defined by \( v(r) = u(r) - H_0 r \), where \( u \) is the velocity of cosmic fluid, \( H_0 \) is the Hubble constant, and \( r \) is the physical distance. Within the Newtonian gravitational theory, the following relation between the density contrast and the peculiar velocity is then derived by Peebles [3]:

\[
\vec{v}(\vec{r}) = \frac{H_0 f(\Omega)}{4\pi b} \int d^3 r' \delta(r') \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3},
\]

(3)

where \( b \) is the biasing factor defined by

\[
\delta_{\text{galaxy}} = b\delta,
\]

and \( f(\Omega) = \frac{d\ln D}{d\ln a} \approx \Omega^{0.6} \). Eq.(3) can also be written as follows:

\[
\nabla \cdot \vec{v} = -\frac{H_0 f(\Omega)}{b} \delta.
\]

(4)

Note that in general \( f(\Omega) \) depends also on \( \Lambda \). However it has been shown by Lahav [13] that the effect of \( \Lambda \) is negligible.

### 3 Peculiar Velocity in the Relativistic Hydrodynamics

Our aim is to study the peculiar velocity of a linear structure in FRW space–time. The structure is assumed to be large enough to take part in the cosmological expansion, yet smaller than the present Hubble radius, with a density contrast small enough to justify the use of the linear theory.

This is in contrast to the familiar textbook approach which uses Newtonian gravity for the linear regime in the scales less than the Hubble radius. As a result of our general relativistic calculations we will distinguish between intermediate scales where relativistic effects in some reasonable approximations have to be taken into account and small scales where the Newtonian approximation is valid within the observational accuracy. Therefore, we perturb linearly a FRW metric, or what is the same, we expand a general metric linearly around a FRW space–time.

Over the past decades, perturbation of FRW space–times have been studied using different gauges like synchronous, Poisson and restricted–gauge (for review see [14], [15] and [16]). These gauges are, however, not suitable for our purposes. As we know from the comparison of the weak limit of general relativity to Newtonian gravity, the harmonic gauge is the most suitable one playing the role of the Lorentz gauge in electromagnetism as contrasted to the Coulomb gauge. Therefore, we try to formulate the corresponding gauge in the perturbed FRW space–time.

Consider a small perturbation of FRW metric in the form

\[
g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},
\]

\[
T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu},
\]

(5)
where \( g^{(0)}_{\mu\nu} \) is the background FRW metric, \( T^{(0)}_{\mu\nu} \) the stress tensor of the cosmic fluid, \( h_{\mu\nu} \ll g^{(0)}_{\mu\nu} \) the metric perturbation, and \( \delta T_{\mu\nu} \ll T^{(0)}_{\mu\nu} \) energy momentum tensor of the structures. The structures we are considering are in a matter dominated universe. Therefore, the only non-zero component of the energy momentum tensor in the comoving reference frame is \( T^{(0)}_{00} = \rho_0 \). Taking into account the general form of the energy–momentum tensor of a homogeneous perfect fluid:

\[
T_{\mu\nu} = (p + \rho)u^\mu u^\nu + pg_{\mu\nu},
\]

where \( u^\mu = \frac{dx^\mu}{d\tau} \) is the four-velocity of the perturbed cosmic fluid. we may write its perturbed component as follows:

\[
T_{00} = (\rho_0 + \delta \rho) = \rho_0 (1 + \delta)
\]

\[
T_{0i} = \rho_0 u^i
\]

\[
T_{ij} = 0,
\]

where we ignored higher order of perturbation. We introduce Harmonic gauge in FRW metric as follows:

\[
\frac{1}{2} h_{\mu\nu} \cdot \nu = h_{\mu\nu},
\]

For the \( \nu = 0 \) and \( \nu = i \) in the expression (10), harmonic gauge can be written in the FRW as follows:

\[
\frac{1}{2} h_{0,0,0}^0 + h_{0,i}^i - \frac{1}{2} h_{i,0}^i + 3 \frac{\dot{a}}{a} h_{0,0}^0 - \frac{\dot{a}}{a} h_{i}^i = 0 \quad \nu = 0
\]

\[
3 \frac{\ddot{a}}{a} h_{0} + \frac{1}{2} h_{j,i}^j - \frac{1}{2} h_{0,i}^0 + h_{i,j}^i - h_{i,0}^0 = 0 \quad \nu = i
\]

we obtain the (00)–component of the perturbed Einstein equation \( \delta G^{0}_{0} = 8\pi G \delta T^{0}_{0} \), after some lengthy calculations, in the following form:

\[
h_{0,0,0}^i + 3 \frac{\dot{a}}{a} h_{0,i}^i - \frac{1}{2} h_{0,i}^i - 2 \frac{\dot{a}}{a} h_{i}^i - 2 \left( \frac{\dot{a}}{a} \right)^2 h_{i}^i - \frac{1}{2} h_{i,0}^0 \]

\[+ 3 \frac{\dot{a}}{a} h_{i,0}^0 - \frac{3}{2} \frac{\dot{a}}{a} h_{0,0}^0 \]

\[+ \frac{3}{2} \frac{\dot{a}^2}{a^2} h_{0,0}^0 = 4\pi G \rho \delta
\]

Substituting \( \nu = 0 \) component of harmonic gauge expression (11) into (13), Einstein equation obtain as follows :

\[
- \frac{1}{2} h_{00,\mu}^\mu + 3 \frac{\dot{a}}{a} h_{00,0} - \frac{1}{2} \frac{\dot{a}}{a} h_{i,0}^i + \frac{9}{2} \left( \frac{\dot{a}}{a} \right)^2 h_{00} + \frac{\ddot{a}}{a} (3h_{00} - h_{i}^i) = 4\pi G \rho \delta
\]

Note that for \( a = constant \), i.e. the Minkowski space time, the familiar Poison equation is recovered. Now, for scales smaller than the Hubble radius, i.e. \( \lambda < H^{-1} = \frac{a}{\tau} \), it is easily seen that all the terms in the left hand side of (14) can be ignored relative to the first term. In fact, we have to compare the Hubble time \( t_H = \frac{a}{\tau} \) with the characteristic time of changes in the cluster, \( \tau \), which is less that \( \lambda \). It is then obvious that the second,third and fourth term which are of the order of \( (t_H \tau)^{-1} \) and fifth term is in the order of \( t_H^2 \) are ignorable relative to the first term which is of the order of \( \tau^{-2} \). Therefore, taking the definition of the gravitational potential of the perturbing field, \( \phi \), as \( h_{00} = -2\phi \), we obtain finally for the perturbed field equation:

\[
\Box \phi = 4\pi G \rho \delta,
\]
where spatial derivatives of $\phi$ are in the terms of the physical coordinates $\vec{r}$. We see that the dynamics of perturbing potential obeys the same equation as in the case of relativistic radiation, taking into account the difference between comoving and physical–length. We may write therefore the potential in the retarded form:

$$\phi(\vec{r}) = - \int \frac{G \rho(t - |\vec{r} - \vec{r}'|) \delta(t - |\vec{r} - \vec{r}'|)}{|\vec{r}' - \vec{r}|} d^3r'. \quad (16)$$

Now, the gravitational acceleration $\vec{g} = - \vec{\nabla} \phi$ due to the above gravitational potential is given by:

$$g(\vec{r}) = - \int \frac{G(\vec{r} - \vec{r})}{|\vec{r}' - \vec{r}|^2} \rho(t - |\vec{r} - \vec{r}'|) \delta(t - |\vec{r} - \vec{r}'|) d^3r'$$

$$- \int \frac{G(\vec{r} - \vec{r})}{c |\vec{r}' - \vec{r}|^2} \delta(t - |\vec{r} - \vec{r}'|) \rho(t - |\vec{r} - \vec{r}'|) + \delta(t - |\vec{r} - \vec{r}'|) \dot{\rho}(t - |\vec{r} - \vec{r}'|) d^3r'. \quad (17)$$

Note that the background density in the FRW universe is $\rho \propto a^{-3}$, and in the linear regime for the structure formation, we have $\delta \propto a$. It is then easily seen that the ratio of second term in (17) with respect to the first one is of the order of $\frac{\lambda}{d_H}$. Therefore, the second term may be ignored and we obtain finally

$$g(\vec{r}) = - \int \frac{G(\vec{r} - \vec{r})}{|\vec{r}' - \vec{r}|^3} \rho \delta(t - |\vec{r} - \vec{r}'|) d^3r'. \quad (18)$$

One may therefore interpret the gravitational effect of such linear perturbations at intermediate scales as the Newtonian one with two modifications:

i. All coordinate lengths should be understood as physical and therefore have to be multiplied by the cosmological scale factor $a(t)$.

ii. The time coordinate should be replaced by the retarded time which means taking into account the finite velocity of light or doing the calculations on the light cone.

This justifies the recent modifications to the Newtonian calculations of different cosmological parameters related to the structure formation which are coined with the term 'light cone effects', see [8], [9], [10].

The expression (17) may also be written in the differential form. By taking the divergence of (17) and ignoring terms of the order of $\frac{\lambda}{d_H}$, we obtain:

$$\nabla \cdot g = 4\pi G \rho \delta, \quad (19)$$

which is of the same form as the Newtonian Poisson equation except for the modifications discussed above.

The corresponding continuity equation can be obtained through the Bianchi identities. We therefore consider the conservation of energy-momentum tensor in the harmonic gauge. Now, using (13, 14), the zero component of the Bianchi identities as the conservation of the energy-momentum,

$$T^{\mu\nu}_{\;\;\;\;\;\;\;\;\mu} = T^{\mu\nu}_{\;\;\;\;\;\;\;\;\mu} + \Gamma^{\nu}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\beta} T^{\nu\beta} = 0, \quad (20)$$
leads to
\[ \dot{\delta} + \nabla \cdot u - \frac{3}{2} h_{00,0} + h_{0,i} = 0. \] (21)
The third and fourth terms are small relative to the first and second term. This can best be seen by rewriting expression (21) in the Fourier space:
\[ \dot{\delta}_k + \frac{ik}{a} \cdot u_k - \frac{3}{2} h_{00,0} + \frac{ik}{a} h^{(k)}_0 = 0. \] (22)
In the linear regime, the first term is proportional to \(\frac{\dot{a}}{a} \delta_k\). From expression (21) it is easily seen that the second term is of the same order of magnitude as the first one. To estimate the third term we note first that \(h^{(k)}_0\) is essentially the potential energy obeying the Newtonian relation
\[ \nabla^2 \phi = 4\pi G \rho \delta. \]
Taking the time derivative of the corresponding equation in Fourier space we easily obtain:
\[ h^{(k)}_{00,0} = 3 \left( \frac{\dot{a}}{a} \right)^2 \lambda^2 \delta_k - 3 \left( \frac{\dot{a}}{a} \right)^3 \lambda^2 \delta_k \] (23)
The two terms on the right hand side of this equation are of the order of \((\lambda/d_H)^2 H \delta_k\).
Therefore, the third term in Eq.(21) is of the order of \((\lambda/d_H)^2\) times the first or the second term and is therefore ignorable.
To estimate the forth term in (21) we consider the \(i0\) component of the Einstein equation by taking into account that \(\lambda < d_H\):
\[ - \frac{1}{2} h_{i0},\mu = 8\pi G \rho \left( \frac{1}{2} h_{i0} - u_i \right) \] (24)
Taking the Fourier transformation of the above equation and using the expression (4), we obtain:
\[ \frac{kh^{(k)}_{i0}}{a} \simeq \left( \frac{\lambda}{d_H} \right)^2 H \delta_k, \] (25)
which is small relative to the first and second term and can be ignored. Therefore (21) may be written as:
\[ \dot{\delta} + \nabla \cdot u = 0. \] (26)
We have therefore seen that by perturbing the matter dominated FRW universe in harmonic gauge the same Newtonian hydrodynamic Eqs. (11) and (12) are obtained with the two modifications discussed above. Combining the (18) and (26) and using the fact that in the linear regime \(\delta \propto a\) we obtain as follows:
\[ \nabla \cdot v(r) = \frac{H(t) f(\Omega)}{4\pi G \rho(t)} \nabla \cdot g(r). \] (27)
Omitting \(\nabla\) from both sides and substituting \(g\) from (18), we obtain for the peculiar velocity
\[ \vec{v}(r, t) = \frac{H(t) f(\Omega)}{4\pi G b \rho(t)} \int G \rho(t - \frac{|r - r'|}{c}) \delta(r', t - \frac{|r - r'|}{c}) \frac{(\vec{r'} - \vec{r})}{|r' - r|} d^3r'. \] (28)
In the following sections we will rewrite the above relativistic expression in terms of observational parameters and will compare it to the corresponding Newtonian results.
4 Relativistic Derivation of the Peculiar Velocity From Density Contrast

Assume a cluster of galaxies, $G_1, G_2, ... G_n$, at a large distance from us. Take two typical galaxies $G_1$ and $G_2$ (Fig.1). For the purpose of calculating the relation between density contrast and peculiar velocity we need the gravitational action of $G_2$ on $G_1$. The situation is best visualized in a space–time diagram. The world lines of $G_1$ and $G_2$ cross our past light-cone at the observation event $O$ at $P$ and $Q$ respectively. These events correspond to our observation of the galaxies. But the action of $G_2$ on $G_1$ is defined through another event: crossing of the past light-cone of $P$ and the world-line of $G_2$. Call this event $R$ (Fig.1). Times corresponding to each of the events $P, Q,$ and $R$ are characterized by the corresponding subscripts. Our observation time at the event $O$ is given by $t_0$.

Now, we have to calculate the relation between $v_{pec}$ and $\delta(x)$ using observed data on $P$ and $Q$. Note first the following relations between time and space coordinate $s$ of the events $P$ and $Q$, taking again $c = 1$:

$$t_P = t_0 - |\vec r_P|, \qquad (29)$$
$$t_Q = t_0 - |\vec r_Q|. \qquad (30)$$

Let $G_1$ be the reference galaxy. The action of any other galaxy, like $G_2$, on $G_1$ is along the past light-cone of events on the world line of $G_1$. It is easily seen that the time $G_2$ acts on $G_1$ is given by

$$t_R = t_0 - |\vec r_P| - |\vec r_R - \vec r_P|, \qquad (31)$$

where $r_R$ is defined as the space coordinate of the event $R$. Within the approximation we are interested in, one can replace $r_R$ by $r_Q \simeq r_R$. Therefore the above relation can be written as

$$t_R = t_0 - |\vec r_P| - |\vec r_Q - \vec r_P|. \qquad (32)$$

Now, Eq.(28) can be written in terms of space time coordinates at $P$ and $R$:

$$\vec v(r_P, t_P) = \frac{H(t_P)f(\Omega)}{4\pi Gb\rho(t_P)} \int d^3r_R G\rho(t_R)\delta(r_R, t_R)\frac{(\vec r_R - \vec r_P)}{|\vec r_R - \vec r_P|^3}. \quad (33)$$

Quantities depending on $R$ have to be reformulated as functions of space time coordinates of $Q$. This is done in two steps. First we expand $H(t_P)$ in terms of $H_0$ and $\rho(t_R)$ in the terms of $\rho(t_P)$:

$$H(t_P) = H_0(1 + \frac{|\vec r_P|}{H_0^{-1}}) \qquad (34)$$
$$\rho(t_R) = \rho(t_P)(1 + 2\frac{|\vec r_R - \vec r_P|}{H_0^{-1}}), \qquad (35)$$

where we have used the FRW equations

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho \qquad (36)$$
$$\dot \rho + 3H(\rho + P) = 0 \qquad (37)$$
Substituting now the Eqs. (34) and (35) into (33), we obtain:

$$\bar{\mathbf{v}}(t_P, t_P) = \frac{H_0 f(\Omega)}{4\pi b} \int \delta(r_R, t_R) \frac{(\mathbf{r}_R - \mathbf{r}_P)}{|\mathbf{r}_R - \mathbf{r}_P|^3} d^3 r_R$$

$$+ \frac{H_0 |\mathbf{r}_P| H_0 f(\Omega)}{c 4\pi b} \int \delta(r_R, t_R) \frac{(\mathbf{r}_R - \mathbf{r}_P)}{|\mathbf{r}_R - \mathbf{r}_P|^3} d^3 r_R$$

$$+ \frac{3H_0 H_0 f(\Omega)}{c 4\pi b} \int \delta(r_R, t_R) \frac{(\mathbf{r}_R - \mathbf{r}_P)}{|\mathbf{r}_R - \mathbf{r}_P|^3} d^3 r_R.$$  \hspace{1cm} (38)

Note that the range of integration is taken as large as desired. Now, terms depending on the event $R$ have to be replaced by those depending on $Q$. Denoting the relative velocity of $G_2$ with respect to us as $\vec{V}$, we obtain

$$\mathbf{r}_R = \mathbf{r}_Q - \vec{V}(t_Q - t_R), \hspace{1cm} (39)$$

where

$$\vec{V} = H_0 \mathbf{r}_Q + \mathbf{v}_{pec} - \mathbf{v}_0.$$  \hspace{1cm} (40)

Here $v_0$ is the peculiar velocity of us with respect to CMBR and $v_{pec}$ is the peculiar velocity of $G_2$. Using the relations (31) and (32) and (30), the expression (38) may be written in the form:

$$\mathbf{r}_R = \mathbf{r}_Q + \left[ (v_0 - v_{pec}) - H_0 r_Q \right] \left[ |r_P| + |r_P - r_Q| - |r_Q| \right]\hspace{1cm} (41)$$

For the large scales we are considering, the term proportional to $v_0 - v_{pec}$ can be neglected relative to $H_0 r_Q$. We may also use the following relation which is easily understood:

$$\frac{1}{|\mathbf{r}_R - \mathbf{r}_P|^3} = \frac{1}{|\mathbf{r}_Q - \mathbf{r}_P|^3} \left[ 1 + 3H_0 |\mathbf{r}_Q| (1 - \frac{\mathbf{r}_Q \cdot \mathbf{r}_P}{|\mathbf{r}_Q - \mathbf{r}_P|^2}) \right].$$  \hspace{1cm} (42)

Now, let us expand the density contrast $\delta(t_R, \mathbf{r}_R)$ around $(t_Q, \mathbf{r}_Q)$:

$$\delta(t_R, \mathbf{r}_R) = \delta(t_Q, \mathbf{r}_Q) - \frac{\partial \delta(t_Q, \mathbf{r}_Q)}{\partial t} (t_Q - t_R) + \nabla \delta(t_Q, \mathbf{r}_Q) \cdot (\mathbf{r}_R - \mathbf{r}_Q) + ...$$  \hspace{1cm} (43)

Assuming the density contrast to be proportional to the scale factor $a(t)$, it is seen that the time derivation of the density contrast is given by:

$$\frac{\partial \delta(t_Q, \mathbf{r}_Q)}{\partial t_Q} = \frac{2 \delta(t_Q, \mathbf{r}_Q)}{3 t_Q}.$$  \hspace{1cm} (44)

Substituting (44) into (13), we obtain:

$$\delta(t_R, \mathbf{r}_R) = \delta(t_Q, \mathbf{r}_Q) - \frac{2 \delta(t_Q, \mathbf{r}_Q)}{3 t_Q} (t_Q - t_R) + \nabla \delta(t_Q, \mathbf{r}_Q) \cdot \mathbf{v} + ...$$  \hspace{1cm} (45)

Substituting for $t_R$ and $t_Q$ from Eqs. (34) and (32), we obtain finally:

$$\delta(t_R, \mathbf{r}_R) = \delta(t_Q, \mathbf{r}_Q) - \frac{2 \delta(t_Q, \mathbf{r}_Q)}{3H_0^{-1}} \left[ |r_P| + |r_P - r_Q| - |r_Q| \right] - \nabla \delta(t_Q, \mathbf{r}_Q) \cdot \mathbf{v}$$

$$\times \left[ (|r_P| + |r_P - r_Q| - |r_Q|) - \frac{\mathbf{r}_Q \cdot \nabla \delta(t_Q, r_Q)}{H^{-1}} \right] (|r_P| + |r_P - r_Q| - |r_Q|).$$  \hspace{1cm} (46)
We have still to write the measure of the integral in (33), \(d^3r_R\), in terms of the observed volume \(d^3r_Q\):

\[
d^3r_R = \left| \frac{\partial r_R}{\partial r_Q} \right| d^3r_Q. \tag{47}
\]

From Eq.(44), ignoring higher order terms, the Jacobian is obtained to be:

\[
\left| \frac{\partial r_R}{\partial r_Q} \right| = 1 - \nabla \cdot v(|r_P| + |r_P - r_Q| - |r_Q|) + \vec{v} \cdot \vec{k} - 3H_0(|r_P| + |r_P - r_Q| - |r_Q|) + H_0\vec{r}_Q \cdot \vec{k}, \tag{48}
\]

where

\[
k^i = \frac{r_Q^i}{|r_Q|} - \frac{x_P^i}{|r_Q - x_P|}. \tag{49}
\]

It can easily be seen that, within the approximation we are considering, it is allowed to replace in the above relation the divergence of the velocity by the density contrast from expression (4) to obtain the final result:

\[
d^3r_R = \left(1 + \frac{f(\Omega)}{6H_0} \right) \delta(r_Q, t_Q)(|r_P| + |r_P - r_Q| - |r_Q|) - \vec{v} \cdot \vec{k} - 3H_0(|r_P| + |r_P - r_Q| - |r_Q|) - H_0\vec{r}_Q \cdot \vec{k})d^3r_Q. \tag{50}
\]

Replacing now all terms corresponding to the point \(R\) with those defined at the point \(Q\), we obtain finally for the peculiar velocity

\[
\vec{v}(r_P) = \frac{H_0f(\Omega)}{4\pi b} \int \frac{\delta(t_Q, r_Q)}{|r_P - r_Q|} (\vec{r}_Q - \vec{r}_P) d^3r_Q + \tilde{G}(r_P), \tag{51}
\]

where

\[
\tilde{G}(r_P) = \bar{F}_1(r_P) + \bar{F}_2(r_P) + \bar{F}_3(r_P) + \bar{F}_4(r_P) + \bar{F}_5(r_P) + \bar{F}_6(r_P) + \bar{F}_7(r_P) + \bar{F}_8(r_P) + \bar{F}_9(r_P), \tag{52}
\]

and

\[
\bar{F}_1(r_P) = \frac{H_0^2 f(\Omega)}{4\pi b} \int \delta(r_Q, t_Q) \frac{(\vec{r}_Q - \vec{r}_P)}{|\vec{r}_Q - \vec{r}_P|^2} \left( |r_P| + |r_Q - r_P| - |r_Q| \right) \left( \frac{|r_Q|^2}{|r_Q - r_P|^2} \right) d^3r_Q, \tag{53}
\]

\[
\bar{F}_2(r_P) = \frac{|r_P| H_0^2 f(\Omega)}{4\pi b} \int \delta(r_Q, t_Q) \frac{(\vec{r}_Q - \vec{r}_P)}{|\vec{r}_Q - \vec{r}_P|^2} d^3r_Q, \tag{54}
\]

\[
\bar{F}_3(r_P) = \frac{2H_0^2 f(\Omega)}{4\pi b} \int \delta(r_Q, t_Q) \frac{(\vec{r}_Q - \vec{r}_P)}{|\vec{r}_Q - \vec{r}_P|^2} d^3r_Q, \tag{55}
\]

\[
\bar{F}_4(r_P) = -\frac{11H_0^2 f(\Omega)}{3\pi b} \int \frac{\delta(t_Q, r_Q)}{|r_P - r_Q|^3} (\vec{r}_Q - \vec{x}_P)(|r_P| + |r_Q - r_P| - |r_Q|) d^3r_Q, \tag{56}
\]

\[
\bar{F}_5(r_P) = -\frac{H_0^2 f(\Omega)}{4\pi b} \int \frac{\vec{r}_Q \cdot \nabla \delta(t_Q, r_Q)}{|r_P - r_Q|^3} (\vec{r}_Q - \vec{r}_P)(|r_P| + |r_Q - r_P| - |r_Q|) d^3r_Q. \tag{57}
\]
\begin{align*}
\bar{F}_0(r_P) & = -\frac{H_0^2 f(\Omega)}{4\pi b} \int \frac{\delta(t_Q, r_Q)}{|r_P - r_Q|^3} (|r_P| + |r_Q| - |r_P - r_Q|) r_Q^3 d^3 r_Q, \\
\bar{F}_7(r_P) & = -\frac{H_0 f(\Omega)}{4\pi b} \int \frac{\delta(t_Q, r_Q)}{|r_P - r_Q|^3} (|r_P| + |r_Q - r_P| - |r_Q|) \delta(t_Q, r_Q) d^3 r_Q, \\
\bar{F}_8(r_P) & = +\frac{H_0 f(\Omega)}{4\pi b} \int \frac{v(r_Q) \cdot \hat{k}}{|r_P - r_Q|^3} (r_Q - r_P) \delta(t_Q, r_Q) d^3 r_Q, \\
\bar{F}_9(r_P) & = -\frac{H_0 f(\Omega)}{4\pi b} \int \frac{v(r_Q) \cdot \nabla \delta(t_Q, r_Q)}{|r_P - r_Q|^3} (r_Q - r_P) (|r_P| + |r_Q - r_P| - |r_Q|) d^3 r_Q. \\
\end{align*}

To simplify the results and to have an estimation of the corrections to the Newtonian expressions, we take a closer look at different terms in (62). Let us first distinguish between \(F_1 - F_4\), and \(F_6\) on one side and \(F_7 - F_9\) on the other side. The latter terms are typically of the order of formers times the fraction of the peculiar velocity divided by the Hubble velocity, being of the order of \(10^{-2}\). Therefore, the latter terms may be neglected. The term \(F_5\) contains the factor \(\vec{r}_Q \cdot \nabla \delta\) which is vanishing in general. To see this, consider a cluster around \(r_Q\). Because of the symmetric distribution of matter in the cluster the gradient term accept positive and negative values while \(r_Q\) is almost constant. Therefore, one may assume that the integral in \(F_5\) is vanishing. The same is true about the contribution of the term \(\frac{\vec{r}_Q \cdot r_P}{|r_P - r_Q|^3}\) in \(F_1\). This is due to the large extent of our integration domain represented by \(r_Q\). Now, for Stromlo–APM– and Las Campanas–red-shift survey we may assume the distance to the cluster to be of the same order as the extent of the cluster, \(L\), so that we have \(|\vec{r}_P - \vec{r}_Q| \simeq |\vec{r}_P| \simeq |\vec{r}_Q| \simeq L \approx 100 Mpc\). A closer look at the remaining terms shows that \(F_{1,3,4,6}\) are of the order of or bigger than \(F_2\). We may therefore write

\begin{equation}
\bar{G} \simeq \frac{4LH_0}{3} \bar{W}, \tag{62}\end{equation}

where

\begin{equation}
\bar{W} = \frac{H_0 f(\Omega)}{4\pi b} \int \frac{\delta(t_Q, r_Q)}{|r_P - r_Q|^3} (r_Q - r_P) d^3 r_Q. \tag{63}\end{equation}

Substituting now \(\bar{G}\) in expression (51) we obtain finally:

\begin{equation}
\bar{v} \simeq \bar{W}(1 + \frac{4L}{3cH_0}), \tag{64}\end{equation}

where \(L\) is the size of structure and \(c\) is explicitly inserted again. In the limit \(c \to \infty\) or \(L \to 0\) we obtain the Newtonian value for the peculiar velocity, \(v_N\), which is just the first term on the right hand side of (51). Therefore, the relative relativistic correction to the peculiar velocity up to the first order of \(\frac{1}{c}\) is given by

\begin{equation}
\frac{v_{rel} - v_N}{v_N} = \frac{4LH_0}{3c}, \tag{65}\end{equation}

where we have taken the absolute values for the velocities and added the subscripts \(rel\) and \(N\) to emphasize the relativistic corrections. Taking the divergence of the (51), we obtain a relation between the density contrast and the divergence of the peculiar velocity:

\begin{equation}
\frac{b}{H_0 f(\Omega)} \nabla \cdot \bar{v}(r_P) = \delta(r_P) - \frac{b}{cH_0 f(\Omega)} \nabla \cdot \bar{G}(r_P). \tag{66}\end{equation}
Here again, in the limit $c \to \infty$ the second term on the right hand side vanishes and we get the familiar Newtonian expression for the density contrast, which will be denoted by $\delta_N(r_P)$. Now, to distinguish the density contrast appearing on the right hand side of (66) from the Newtonian value, we call it $\delta_{rel}$. The expression (66) is now written as:

$$\delta_{rel}(r_P) = \delta_N(r_P) + \frac{b}{H_0 f(\Omega)} \nabla \cdot \vec{G}(r_P). \quad (67)$$

In the following sections we will calculate the density and velocity power spectra for both Newtonian and relativistic cases.

## 5 Power Spectrum: Newtonian Versus Relativistic Case

Take the correlation function of density contrast as

$$\xi(r) = \int \delta(r + r')\delta(r') \frac{d^3 r'}{V}. \quad (68)$$

defined in a cosmological volume $V$. For simplicity we will omit hereafter the subscript $P$ in $r_P$. The power spectrum is given by the Fourier transform of the correlation function:

$$\xi(r) = \int P(k)e^{ikr}d^3k. \quad (69)$$

Using relation (67) between the Newtonian and relativistic density contrast, substituting it in Eq.(68) and ignoring higher order terms in $\delta$, we obtain the following relation between the Newtonian– and relativistic–correlation functions:

$$\xi_N(r) = \xi_{rel}(r) - \frac{b}{c H_0 f(\Omega)} \int \delta(r' + r)\nabla \cdot \vec{G}(r')d^3 r' - \frac{b}{c H_0 f(\Omega)} \int \delta(r')\nabla \cdot \vec{G}(r' + r)d^3 r'. \quad (70)$$

The divergence term on the right hand side is calculated in the Appendix A and is given by

$$\nabla \cdot \vec{G}(r) = -\frac{4H_0^2 f(\Omega)}{3b}\delta(r,t) |r| \quad (71)$$

Using this relation and taking the Fourier transform of expression (70), the following relation between the Newtonian– and relativistic–power spectrum is easily obtained:

$$P_N(k) = P(k)_{rel}(1 + \frac{16\pi H_0}{3ck}). \quad (72)$$

In the limit $c \to \infty$, the Newtonian value is regained.

## 6 $\beta$–Value: Newtonian Versus Relativistic

Let us now calculate the relativistic velocity power spectrum. The peculiar velocity is now expressed in terms of its Fourier components:

$$v(r) = \frac{V}{(2\pi)^{3/2}} \int v_k e^{ikr}d^3k, \quad (73)$$
where $V$ is the space volume. Its power spectrum is defined as $P_V(k) = \langle |v_{kx}|^2 + |v_{ky}|^2 + |v_{kz}|^2 \rangle$. If $v(r)$ is an isotropic Gaussian field, then the different Fourier components are uncorrelated and the power spectrum provides a complete statistical description of the field. The Velocity spectrum is given by

$$V^2(k) = \frac{1}{2\pi^2}k^3P_V(k). \quad (74)$$

As we have shown in the appendix B, the Fourier transform of (65) leads then to the following relation between the Newtonian– and relativistic– velocity spectrum:

$$V^2_{rel}(k) = V^2_N(k)(1 + \frac{16\pi H_0^2}{3ck}). \quad (75)$$

The Newtonian spectrum is independently obtained by taking the Fourier transform of (4):

$$V^2_N(k) = \frac{1}{2\pi^2}\beta^2H_0^2kP(k). \quad (76)$$

where $\beta = \frac{\Omega_b}{\Omega}$. These relations are obtained in terms of real space expressions in contrast to power spectrum data which are obtained in the red-shift space [17]. It is used to change the factor $\beta^2$ in front of the term on the right hand side of (76) to $F(\beta)$ to account for the transformation between red-shift– and real-space. Therefore we may write the final equation in terms of red-shift data in the form

$$V^2_N(k) = \frac{1}{2\pi^2}F^2_N(\beta)H_0^2kP(k), \quad (77)$$

where

$$F^2_N(\beta) = \frac{\beta^2}{1 + 2\beta/3 + \beta^2/5}. \quad (78)$$

Now, using expression (78) the relativistic velocity spectrum will be given in the form

$$V^2_{rel}(k) = \frac{1}{2\pi^2}F^2_{rel}(\beta)H_0^2kP(k), \quad (79)$$

where

$$F^2_{rel}(\beta) = F^2_N(\beta)(1 + \frac{16\pi H_0^2}{3ck}). \quad (80)$$

7 Discussion

Any observation in cosmology is carried along the light cone and not on a space-like slice defined by a definite observer time. However the difference between the two procedures are usually ignorable, which leads to the usual ignorance of the light cone effects and the finite signal velocity. At cosmological distances this difference can leads however to observable effects.

Here we have calculated the peculiar velocity–density contrast relation in a cluster of galaxy taking into account the finite signal velocity. The result differs from the familiar Newtonian relation through a lengthly term which can however be simplified for a typical cluster. The estimation based on this simplification depends on the extend of the cluster; the larger the extent the bigger the difference.
Appendix A

Let us calculate the term $\nabla \cdot \vec{G}(r)$ appearing in (67) and (70). Using eqs.(53-61) we may write:

$$\nabla \cdot \vec{G}(r_P) = \frac{3H_0^2 f(\Omega)}{4\pi b} \int \frac{\vec{r}_P \cdot \vec{r}_Q}{|r_P||r_P - r_Q|^3} \delta(r_Q)d^3r_Q - \frac{3H_0^2 f(\Omega)}{4\pi b} \int \frac{|r_P|}{|r_P - r_Q|^3} \delta(r_Q)d^3r_Q - \frac{4H_0^2 f(\Omega)}{3b} |r_P| \delta(r_P), \quad (81)$$

The symmetric distribution of matter in large scales leads to positive and negative values for $\vec{r}_P \cdot \vec{r}_Q$ in the first integral which makes it vanishing. The second integral on the right hand side may be written as \(\frac{3H_0^2 f(\Omega)}{4\pi b} |r_P| \int \frac{\delta(r_Q)}{|r_P - r_Q|^3}d^3r_Q \simeq \frac{3H_0^2 f(\Omega)}{4\pi b} < \delta > L\). The average value of density contrast, \(< \delta >\), taking the domain of integration sufficiently large, does vanish too. We then obtain finally:

$$\nabla \cdot \vec{G}(r_P) = -\frac{4H_0^2 f(\Omega)}{3b} |r_P| \delta(r_P). \quad (82)$$

Appendix B

In order to derive the Newtonian power spectrum we substitute expression (71) into (70):

$$\xi_N(r) = \xi_{Rel}(r) + \frac{8\pi H_0}{3c} \int \delta(r' + r)\delta(r') |r'| d^3r'. \quad (83)$$

Taking the Fourier transformation of it we obtain

$$\int P_N(k)e^{ik \cdot r}d^3k = \int P_{Rel}(k)e^{ik \cdot r}d^3k + \frac{1}{(2\pi)^3} \frac{8H_0}{3c} \int \delta_k \delta_k' e^{ik \cdot (r + r')r'}d^3r'd^3k'd^3k'. \quad (84)$$

Putting for simplicity \(r' = \frac{2\pi}{k}\) we may write \(\int e^{ik \cdot (r + k')}d^3r' = (2\pi)^3\delta^3(k + k')\), which leads to

$$P_N(k) = P_{Rel}(k)(1 + \frac{16\pi H_0}{3kc}). \quad (85)$$

Calculation of the velocity power spectrum is similar to above procedure. One start with (73), by substituting it into (74) and taking Fourier transformation of it and using Eq. (74), it is easily seen that:

$$V_{rel}^2(k) = V_N^2(k)(1 + \frac{16\pi H_0}{3kc}). \quad (86)$$
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8 Figure Captions

Fig. 1. Our past light cone crosses the world-lines of galaxies $G_1$ and $G_2$ at $P$ and $Q$ respectively. The event $R$, on the intersection of the world line of $G_2$ and the past light cone of $P$, gives the moment at which $G_2$ acts on $G_1$ being effective at $P$. 
