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A study of elliptic gamma function and allies

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Abstract

We study analytic and arithmetic properties of the elliptic gamma function

\[ \prod_{m,n=0}^{\infty} \frac{1 - x^{-1} q^{m+1} p^{n+1}}{1 - x q^m p^n}, \quad |q|, |p| < 1, \]

in the regime \( p = q \), in particular, its connection with the elliptic dilogarithm and a formula of S. Bloch. We further extend the results to more general products by linking them to non-holomorphic Eisenstein series and, via some formulae of D. Zagier, to elliptic polylogarithms.

Keywords: Theta function, Elliptic gamma function, Elliptic dilogarithm, Elliptic polylogarithm

1 Introduction

For complex \( z \) and \( \tau \) with \( \text{Im} \tau > 0 \), set \( x = e^{2\pi iz} \) and \( q = e^{2\pi i \tau} \). Transformation properties of the so-called short theta function

\[ \theta_0(z; \tau) := \prod_{m=0}^{\infty} (1 - x^{-1} q^{m+1})(1 - x q^m) \]

under the action of the modular group are well understood. In view of its transparent invariance under translation \( \tau \mapsto \tau + 1 \), the main source of the modular action originates from the \( \tau \)-involution

\[ z \mapsto \hat{z} = \frac{z}{\tau}, \quad \tau \mapsto \hat{\tau} = -\frac{1}{\tau}. \]  

(1)

The related classical transformation of \( \theta_0(z; \tau) \) can be recorded as

\[ q^{1/12} x^{-1/2} \theta_0(z; \tau) = i e^{-\pi i \hat{z}} \hat{q}^{1/12} \hat{x}^{-1/2} \theta_0(\hat{z}; \hat{\tau}) \]  

(2)

(see, for example, [3, Section 2]), where we define \( \hat{x} = e^{2\pi i \hat{z}} \) and \( \hat{q} = e^{2\pi i \hat{\tau}} \). Less is known about modular properties of the related product

\[ \theta_1(z; \tau) := \prod_{m=0}^{\infty} \frac{(1 - x^{-1} q^{m+1} p^{n+1})}{(1 - x q^m p^n)}, \]

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which naturally comes as the $\sigma = \tau$ specialisation of the elliptic gamma function
\[
\Gamma(z; \tau, \sigma) := \prod_{m,n=0}^{\infty} \frac{1 - x^{-1}q^{m+1}p^{n+1}}{1 - xq^{m}p^{n}}, \quad \text{where} \quad p = e^{2\pi i\sigma},
\]
introduced by Ruijsenaars [5] (see also [3,4]). Namely, we have
\[
\theta_1(z; \tau) = \theta_0(z; \tau) \Gamma(z; \tau, \tau) = \Gamma(z + \tau; \tau, \tau).
\]

A known functional equation of the elliptic gamma function [3, Theorem 4.1] represents an $\text{SL}_3(\mathbb{Z})$ symmetry of $\Gamma(z; \tau, \sigma)$. The problem of determining its behaviour in the regime $\sigma = \tau$ under $\text{SL}_2(\mathbb{Z})$ transformations is specifically addressed in [2], where the (logarithm of the) infinite product is related to the elliptic dilogarithm via a formula of S. Bloch [1].

Our principal aim in this note is recasting analytic and arithmetic (modular) properties of the function $\theta_1(z; \tau)$ and its relatives, in particular, linking them to non-holomorphic Eisenstein series and the elliptic dilogarithm. This programme is carried out in Sects. 2–4; it gives a new proof of Bloch’s formula and related results from [2]. In Sect. 5 we go further to discuss similar features of products that generalise ones for $\theta_0$ and $\theta_1$; their relationship with non-holomorphic Eisenstein series and formulae from [7] allow us to link them to elliptic polylogarithms.

For future record, notice that iterating the transformation $(z, \tau) \mapsto (\hat{\tau}, \hat{\tau})$ twice maps $(z, \tau)$ to $(-z, \tau)$ and that
\[
\theta_1(-z; \tau) = \frac{1}{\theta_1(z; \tau)} \quad \text{and} \quad \theta_0(-z; \tau) = -x^{-1} \theta_0(z; \tau).
\] (3)

2 Period functions
A natural way of measuring failure of weight $k$ modular behaviour under the transformation $(z, \tau) \mapsto (\hat{z}, \hat{\tau})$ for a function $f(z, \tau)$ is through the period function
\[
g(z, \tau) = g_k(z, \tau) := f(\hat{z}, \hat{\tau}) - \tau^k f(z, \tau).
\]

Lemma 1 We have
\[
t^k g(\hat{z}, \hat{\tau}) + (-1)^k g(z, \tau) = t^k \left( f(-z, \tau) - (-1)^k f(z, \tau) \right).
\]

Observe that the expression in the parentheses on the right-hand side measures the failure of $k$-parity of $f(z, \tau)$.

Proof We only use $(\hat{z}, \hat{\tau}) = (-z, \tau)$ and $\tau \hat{\tau} = -1$:
\[
t^k g(\hat{z}, \hat{\tau}) - g(z, \tau) = t^k \left( f(-z, \tau) - t^k f(\hat{z}, \hat{\tau}) \right) + (-1)^k \left( f(\hat{z}, \hat{\tau}) - t^k f(z, \tau) \right)
\]
\[
= t^k \left( f(-z, \tau) - (-1)^k f(z, \tau) \right). \quad \square
\]

The lemma and the parity relation for $\ln \theta_1(z; \tau)$ in (3) imply the following.

Lemma 2 The function
\[
T(z; \tau) = \tau \ln \theta_1(z; \tau) - \ln \theta_1(z; \hat{\tau})
\] (4)
satisfies the functional equation
\[
T(\hat{z}; \hat{\tau}) = \tau^{-1} T(z; \tau).
\]
Furthermore, we can relate the function $T(z; \tau)$ to the dilogarithm function

$$\text{Li}_2(x) = -\int_0^x \ln(1 - t) \frac{dt}{t}. $$

**Lemma 3** The function (4) admits the following representation:

$$T(z; \tau) = \frac{\pi i(\tau - 2z)(1 + 2\tau z - 2z^2)}{12\tau} + z \ln \theta_0(z; \tau) - \frac{1}{2\pi i} \sum_{m=0}^{\infty} \left( \text{Li}_2(x^{-1} q^{m+1}) - \text{Li}_2(x q^m) \right).$$

**Proof** As shown in the proof of Theorem 5.2 in [3],

$$\ln \theta_1(z; \tau) = \ln \theta_0(z; \tau) + \ln \Gamma(z; \tau, \tau)$$

$$= -\pi i \lambda(z; \tau) + \ln \theta_0(z; \tau)$$

$$+ (t - \bar{z}) \sum_{k=1}^{\infty} \frac{(\hat{\kappa} - 1) q^k}{k(1 - \hat{q}^k)} - \frac{1}{6} \sum_{k=1}^{\infty} \frac{\hat{\kappa}^k}{k(1 - \hat{q}^k)^2}$$

where

$$\lambda(z; \tau) = \frac{z^3}{3\tau^2} - \frac{2\tau - 1}{2\tau^2} z^2 + \frac{(\tau - 1)(5\tau - 1)}{6\tau^2} z - \frac{\tau - 2}{12\tau}$$

and the assumptions $|\hat{\kappa}|, |\hat{\kappa}^{-1} q| < 1$ are made to ensure convergence. (The latter can be dropped in the final result by appealing to the analytic continuation in $z$.) Recalling the transformation (2), using

$$\frac{1}{1 - \hat{q}^k} = \sum_{m=0}^{\infty} \hat{q}^{mk} \quad \text{and} \quad \frac{\hat{q}^k}{(1 - \hat{q}^k)^2} = \sum_{m=0}^{\infty} m \hat{q}^{mk},$$

interchanging summation and summing over $k$, we obtain

$$\ln \theta_1(z; \tau) = -\pi i \left( \lambda(z; \tau) - \frac{1}{2} \frac{z^2}{\tau} + \frac{\tau}{6} - z + \frac{1}{6\tau} \right)$$

$$+ \bar{z} \sum_{m=0}^{\infty} \left( \ln (1 - \hat{\kappa}^{-1} q^{m+1}) + \ln (1 - \hat{\kappa} q^m) \right)$$

$$- \bar{\tau} \sum_{m=0}^{\infty} (m + 1) \ln (1 - \hat{\kappa}^{-1} q^{m+1}) - m \ln (1 - \hat{\kappa} q^m)$$

$$- \frac{1}{2\pi i} \sum_{m=0}^{\infty} \left( \text{Li}_2 \left( \hat{\kappa}^{-1} q^{m+1} \right) - \text{Li}_2 \left( \hat{\kappa} q^m \right) \right)$$

$$= \frac{\pi i}{12} \left( 1 + 2z \right) - \frac{2z(1 + z)(1 + 2z)}{\tau^2} + \bar{z} \ln \theta_0(\bar{z}; \bar{\tau}) - \frac{1}{\tau} \ln \theta_1(\bar{z}; \bar{\tau})$$

$$- \frac{1}{2\pi i} \sum_{m=0}^{\infty} \left( \text{Li}_2 \left( \hat{\kappa}^{-1} q^{m+1} \right) - \text{Li}_2 \left( \hat{\kappa} q^m \right) \right).$$
(This formula can be alternatively derived from logarithmically differentiating identity (2) with respect to $\tau$ and further integrating the result with respect to $z$.) Substituting $(z/\tau, -1/\tau)$ for $(z, \tau)$ translates the result into

$$\tau \ln \theta_1(z; \tau) - \ln \theta_1(\hat{z}; \hat{\tau}) = \frac{\pi i (\tau - 2z)(1 + 2\tau z - 2z^2)}{12\tau} + z \ln \theta_0(z; \tau) - \frac{1}{2\pi i} \sum_{m=0}^{\infty} (\operatorname{Li}_2(x^{-1}q^{m+1}) - \operatorname{Li}_2(q^m)),$$

the desired relation. □

3 Non-holomorphic modularity

Denote

$$A = A(z, \tau) := \frac{z - \bar{z}}{\tau - \bar{\tau}} \in \mathbb{R},$$

so that

$$\hat{A} = A(\hat{z}, \hat{\tau}) := \frac{\bar{z} - \bar{\tau} \bar{z}}{\tau - \bar{\tau}} \in \mathbb{R}$$

and $z = A\tau - \hat{A}$. Define

$$Q(z; \tau) := q^{B_3(A)/3} \prod_{m=0}^{\infty} \frac{(1 - xq^m)^{m+A}}{(1 - xq^{m+1})^{m+1-A}} = \frac{q^{B_3(A)/3} \theta_0(z; \tau)^A}{\theta_1(z; \tau)},$$

where $B_3(t) := t^3 - \frac{3}{2} t^2 + \frac{1}{2} t$ is the third Bernoulli polynomial, $B_3(1 - t) = -B_3(t)$, and

$$F_+(z; \tau) := \ln Q(\hat{\tau}; \hat{\tau}) - \tau \ln Q(z; \tau), \quad F_-(z; \tau) := \ln Q(z; \tau) - \tau \ln Q(z; \tau).$$

It follows then from Lemma 1 and the parity relations (3) that

$$\tau F_+(\hat{z}; \hat{\tau}) - F_+(z; \tau) = \tau (\ln Q(-z; \tau) + \ln Q(z; \tau))$$

$$= \frac{2\pi i}{3} (B_3(-A) + B_3(A)) \tau^2 + 2\pi i A \tau - \pi i A \tau$$

$$= -\pi i A (2A - A(z - 1)) \tau = -\pi i A (2\hat{A} + 1) \tau$$

and

$$\tau F_-(\hat{z}; \hat{\tau}) - F_-(z; \tau) = \tau (\ln Q(-z; \tau) + \ln Q(z; \tau))$$

$$= -\frac{2\pi i}{3} (B_3(-A) + B_3(A)) \tau \bar{\tau} - 2\pi i A \bar{\tau} \tau + \pi i A \tau$$

$$= \pi i A (2 (A \tau - \bar{\tau}) + 1) \tau = \pi i A (2\hat{A} + 1) \tau.$$

We summarise our finding in the following claim.

Lemma 4 We have

$$\tau F_+(\hat{z}; \hat{\tau}) - F_+(z; \tau) = -\pi i A (2\hat{A} + 1) \tau,$$

$$\tau F_-(\hat{z}; \hat{\tau}) - F_-(z; \tau) = \pi i A (2\hat{A} + 1) \tau.$$

Lemma 3 leads to the following expansions of the functions $F_+$ and $F_-$. 
Theorem 1 We have

\[ F_+(z; \tau) = S(z, \tau) - \frac{1}{2\pi i} L(z, \tau), \]
\[ F_-(z; \tau) = -\frac{2\pi i (\tau - \bar{\tau})}{3} B_3(A) + S(z, \tau) + \frac{1}{\pi} U(z, \tau) + \frac{1}{2\pi i} L(z, \tau), \]

where

\[ L(z, \tau) := \sum_{m=0}^{\infty} (L_{i2} (x^{-1} q^{m+1}) - L_{i2} (x q^m)), \]
\[ U(z, \tau) := \sum_{m=0}^{\infty} (\ln |x^{-1} q^{m+1}| \ Li_1 (x^{-1} q^{m+1}) - \ln |x q^m| \ Li_1 (x q^m)), \]
\[ S(z, \tau) := -\frac{\pi i}{12} (2A - 1) (6z^2 - 12Az + 6z + 8A^2 \tau^2 - 2A \tau^2 - 6A \tau + 1). \]

Proof For \( F_+ \) substitute the expression of \( T(z; \tau) \) from Lemma 3 into the computation

\[ F_+(z; \tau) = \ln Q(\tau; \ell) - \tau \ln Q(z; \tau) \]
\[ = \frac{2\pi i}{3} (B_3(\mathcal{A}) \ell - B_3(A) \tau^2) + \mathcal{A} \ln \theta_0 (\tau; \ell) - (\mathcal{A} + z) \ln \theta_0 (z; \tau) \]
\[ + \tau \ln \theta_1 (z; \tau) - \ln \theta_1 (\ell; \ell). \]

This leads to the formula

\[ F_+(z; \tau) = S(z, \tau) - \frac{1}{2\pi i} L(z, \tau) \]

with

\[ S(z, \tau) = \frac{2\pi i}{3} (B_3(\mathcal{A}) \ell - B_3(A) \tau^2) + \mathcal{A} \pi i \left( \frac{\tau}{6} - \frac{\ell}{6} + z \ell - \frac{1}{2} - z + \ell \right) \]
\[ + \frac{\pi i}{12 \tau} (\tau - 2z)(1 + 2z - 2z^2), \]

and the latter simplifies to the expression given in the statement of Theorem 1 by elementary manipulation.

For \( F_- \) we proceed as follows. We have

\[ \ln Q(z; \tau) = \frac{2\pi i \ell B_3(A)}{3} - \sum_{m=0}^{\infty} ((m + 1 - A) Li_1 (x^{-1} q^{m+1}) - (m + A) Li_1 (x q^m)). \]

Multiply this expression by \( \tau - \bar{\tau} = 2i \Im \tau \) and use \( A(\tau - \bar{\tau}) = 2i \Im z \) to get

\[ (\tau - \bar{\tau}) \ln Q(z; \tau) = \frac{2\pi i \ell (\tau - \bar{\tau}) B_3(A)}{3} - \frac{1}{\pi} U(z, \tau). \]

Now, notice

\[ (\tau - \bar{\tau}) \ln Q(z; \tau) = F_-(z; \tau) - F_+(z; \tau) \]

to deduce the expression for \( F_- \) as in the theorem.

A consequence of this expansion is the invariance of

\[ F(z; \tau) := \frac{F_+(z; \tau) + F_-(z; \tau)}{2} = \ln |Q(\ell; \ell)| - \tau \ln |Q(\tau; \tau)| \]

under translation \( \tau \mapsto \tau + 1. \)
Lemma 5 We have

\[ F_+(z; \tau + 1) - F_+(z; \tau) = -(F_-(z; \tau + 1) - F_-(z; \tau)). \]

Proof The functions \( L(z, \tau) \) and \( U(z, \tau) \) (hence their complex conjugates) are clearly invariant under translation \( \tau \mapsto \tau + 1 \). The result follows from noticing that

\[ 2 \Re S(z, \tau) + \frac{2\pi i (\tau - \bar{\tau})}{3} B_3(A) = -\frac{\pi i (\tau - \bar{\tau})^2 A (1 - A) (1 - 2A)}{6} \]

is also invariant under the transformation.

We summarise the results in this section as follows.

Theorem 2 The weight 1 period function

\[ F(z; \tau) = \ln |Q(\hat{z}; \hat{\tau})| - \tau \ln |Q(z; \tau)| \]

\[ = \frac{1}{2\pi} \sum_{m=0}^{\infty} \left( \ln |x^{-1} q^m| \ln |x q^m| - \ln |x q^m| \ln |x q^m| \right) - \frac{\pi i (\tau - \bar{\tau})^2}{6} B_3(A) - \frac{1}{2\pi i} \Im \sum_{m=0}^{\infty} \left( \ln |x q^m| - \ln |x q^m| \right) \]

of \( \ln |Q(z; \tau)| \) satisfies

\[ \tau F(\hat{z}; \hat{\tau}) = F(z; \tau) \quad \text{and} \quad F(z; \tau) = F(z; \tau + 1). \]

In other words, it behaves like a Jacobi form of weight 1 on \( SL_2(\mathbb{Z}) \).

4 Elliptic dilogarithm

Theorem 2 provides a natural link between the period function \( F(z; \tau) \) and the elliptic dilogarithm [7]

\[ D(q; x) := \sum_{m \in \mathbb{Z}} D(x q^m) = \sum_{m=0}^{\infty} (D(x q^m) - D(x^{-1} q^{m+1})) \]

together with its companion

\[ J(q; x) := \sum_{m=0}^{\infty} (J(x q^m) - J(x^{-1} q^{m+1})) + \frac{\log^2 |q|}{3} B_3 \left( \frac{\log |x|}{\log |q|} \right), \]

where

\[ D(x) := \ln |x| \arg(1 - x) + \Im Li_2(x) = - \ln |x| \Im Li_1(x) + \Im Li_2(x) \]
denotes the Bloch–Wigner dilogarithm and

\[ J(x) := \ln |x| \ln |1 - x| = - \ln |x| \Re Li_1(x) \]

its companion. Namely, the expansion in the theorem can be stated as

\[ F(z; \tau) = \frac{1}{2\pi i} \left( D(q; x) + iJ(q; x) \right). \]
This is essentially the result discussed in [2, Section 1].

Viewing now \( z \) as an element of the lattice \( \mathbb{R} + \mathbb{R} \tau \), so that \( A \) and \( \hat{A} \) in the representation \( z = -\hat{A} + A \tau \) are fixed, we find out that the \( \tau \)-derivative

\[
\frac{1}{2\pi i} \frac{d}{d\tau} \ln Q(z; \tau) = q \frac{d}{dq} \ln Q(z; \tau)
\]

is the Eisenstein series

\[
\frac{i}{4\pi^3} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} (m\tau + n)^3
\]

of weight 3, where the notation \( \sum' \) indicates omitting the term \( m = n = 0 \) from the summation. Integrating we obtain

\[
\ln Q(z; \tau) = \frac{1}{4\pi^2} \sum'_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\hat{A} + nA)}}{m(m\tau + n)^2}
\]

implying

\[
\ln |Q(z; \tau)| = \frac{1}{2} \left( \ln Q(z; \tau) + \ln Q(\hat{z}; \hat{\tau}) \right)
\]

\[
= \frac{1}{8\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} \left( \frac{1}{m(m\tau + n)^2} - \frac{1}{m(m\tau + n)^2} \right)
\]

\[
= \frac{1}{2\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} \left( \frac{1}{m(m\tau + n)^2} - \frac{1}{m(m\tau + n)^2} \right)
\]

\[
= i \frac{\ln \tau}{\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} (m \Re \tau + n) \frac{1}{m(m\tau + n)^2}.
\]

This is equation (7) in [2]. Since \( \hat{z} = z/\tau = A - A/\tau = A + \hat{A} \), it follows that

\[
\ln |Q(\hat{z}; \hat{\tau})| = i \frac{\ln \hat{\tau}}{2\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + n\hat{A})} (m \Re \hat{\tau} + n) \frac{1}{m(m\hat{\tau} + n)^2}
\]

\[
= i \frac{\ln \hat{\tau}}{2\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} - n\hat{A})} (m \Re \hat{\tau} + n) \frac{1}{m(m\hat{\tau} + n)^2}
\]

\[
= i \frac{\ln \hat{\tau}}{2\pi^2} \sum'_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} - n\hat{A})} (m \Re \hat{\tau} + n) \frac{1}{m(m\hat{\tau} + n)^2}
\]

implying

\[
\ln |Q(\hat{z}; \hat{\tau})| - \tau \ln |Q(z; \tau)| = \ln \frac{e^{2\pi i (m\hat{A} + nA)}}{e^{2\pi i (m\hat{A} + n\hat{A})}} \frac{1}{m(m\tau + n)^4}.
\]

The latter is a (non-holomorphic) modular form of weight 1, and combined with equation (6) is the formula of Bloch mentioned previously.
**Theorem 3** (Bloch’s formula [1,2,7]) For \( z = A\tau - \hat{A}, \) we have

\[
F(z; \tau) = \frac{1}{2\pi i} \left( D(q; x) + i f(q; x) \right) = \frac{(\text{Im } \tau)^2}{2\pi^2} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)(m\tau + n)}.
\]

**5 General weight**

A natural generalisation of the product in (5) is

\[
Q_k(z; \tau) := q^B_{k+2(A)/(k+2)} \prod_{m=0}^{\infty} (1 - xq^m)^{A}(1 - x^{-1} q^{m+1})^{-1}(1 - A)^k(1 - A)^k,
\]

where \( k = 0, 1, 2, \ldots \) and \( B_k(t) \) stands for the \( k \)th Bernoulli polynomial. Then \( Q_0(z; \tau) \) is an arithmetic normalisation of the short theta function \( \theta_0(z; \tau) \) (a Siegel modular unit) and \( Q_1(z; \tau) \) coincides with (5). Following the earlier recipe, define

\[
\begin{align*}
F_+(z; \tau) &= F_{k+}(z; \tau) := \ln Q_k(z; \tau) - \tau^{k+2} \ln Q_k(z; \tau), \\
F_-(z; \tau) &= F_{k-}(z; \tau) := \ln \frac{Q_k(z; \tau)}{Q_k(z; \tau)}
\end{align*}
\]

and \( F_k(z; \tau) := \frac{1}{2} (F_{k+}(z; \tau) + F_{k-}(z; \tau)) \). Then from Lemma 1 we deduce the following generalisation of Lemma 4.

**Lemma 6** We have, for \( k \geq 1, \)

\[
\begin{align*}
\tau^k F_+(z; \tau) + (-1)^k F_-(z; \tau) &= (-1)^k \pi i A^k (2\hat{A} + 1) \tau^k, \\
\tau^k F_-(z; \tau) + (-1)^k F_+(z; \tau) &= (-1)^k \pi i A^k (2\hat{A} + 1) \tau^k.
\end{align*}
\]

**Proof** Apply Lemma 1 and the relation

\[
B_{k+2}(-t) - (-1)^k B_{k+2}(t) = (-1)^k(k + 2) t^{k+1}.
\]

We further use that the \( \tau \)-derivative of \( \ln Q_k(z; \tau) \) is an Eisenstein series.

**Lemma 7** For \( k \geq 1, \)

\[
\ln Q_k(z; \tau) = \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} m(m\tau + n)^{k+1}.
\]

where \( z = -\hat{A} + A\tau. \)

**Proof** Consider \( \hat{Q}_k(A, \hat{A}; \tau) := Q_k(A\tau - \hat{A}; \tau) \) as a function of real variables \( A, \hat{A} \) and complex variable \( \tau. \) The \( \tau \)-derivative

\[
G_{k+2}(A, \hat{A}; \tau) := \frac{1}{2\pi i} \frac{d}{d\tau} \ln Q_k(A, \hat{A}; \tau) = \frac{d}{dq} \ln Q_k(A, \hat{A}; \tau)
\]

is seen to be the Eisenstein series

\[
E_{k+2}(A, \hat{A}; \tau) := \frac{(-1)^k(k + 1)!}{(2\pi i)^{k+2}} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\hat{A} + nA)} (m\tau + n)^{k+2}.
\]
of weight $k + 2$. This is true for $k = 1$ (see Sect. 4), while for $k \geq 1$ we observe the functional equation
\[
\frac{\partial}{\partial \tilde{A}} E_{k+3}(A, \tilde{A}; \tau) = \frac{\partial}{\partial \tau} E_{k+2}(A, \tilde{A}; \tau).
\]

The equality $G_{k+2}(A, \tilde{A}; \tau) = E_{k+2}(A, \tilde{A}; \tau)$ then follows by induction on $k$ using the fact that the constant terms of both functions at $\tau = \infty$ (or $q = 0$) agree.

Integrating we obtain
\[
\ln Q_k(A, \tilde{A}; \tau) = \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{m(m\tau + n)^{k+1}}.
\]

Since both sides continuously depend on $A$ and $\tilde{A}$, the formula remains valid also for $\ln Q_k(z; \tau)$.

As in our computation in Sect. 4 we obtain
\[
\ln |Q_k(z; \tau)| = \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{m(m\tau + n)^{k+1}} \left( \frac{1}{m(m\tau + n)^{k+1}} - \frac{1}{m(m\tau + n)^{k+1}} \right)
\]
\[
= \frac{(-1)^k k!}{(2\pi i)^{k+1}} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^{k+1}} \sum_{j=0}^{k} (m\tau + n)^j (m\tau + n)^{k-j}
\]
\[
= -\frac{i^k k! \text{ Im } \tau}{(2\pi i)^{k+1}} \sum_{j=0}^{k} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^{k-j+1}} (m\tau + n)^{j+1}
\]

and
\[
\ln |Q_k(z; \tau)| = -\frac{i^k k! \text{ Im } \tau}{(2\pi i)^{k+1} |\tau|^2} \sum_{j=0}^{k} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^{j+1}} (n - m\tau)^{j+1} (n - m\tau)^{k-j+1}
\]
\[
= -\frac{i^k k! \text{ Im } \tau}{(2\pi i)^{k+1}} \sum_{j=0}^{k} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^{k-j+1}} (m\tau + n)^{j+1}.
\]

Thus,
\[
F_k(z; \tau) = \ln |Q_k(z; \tau)| - r^k \ln |Q_k(z; \tau)|
\]
\[
= \frac{i^k k! \text{ Im } \tau}{(2\pi i)^{k+1}} \sum_{j=0}^{k} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^{j+1}} (m\tau + n)^{k-j+1}
\]
\[
= \frac{i^k k!}{(2\pi i)^{k+1} |\tau|^2} \sum_{j=1}^{k} \tau^{j-1} (\tau^j - \tau^j) \sum_{m,n \in \mathbb{Z}} (m\tau + n)^{j+1} (m\tau + n)^{k-j+1}
\]
\[
= \frac{i^k k!}{(4\pi \text{ Im } \tau)^k} \sum_{j=1}^{k} \tau^{j-1} \text{ Im } (\tau^j) \sum_{m,n \in \mathbb{Z}} (m\tau + n)^{j+1} (m\tau + n)^{k-j+1}
\]
\[
= \frac{i^k k!}{(4\pi \text{ Im } \tau)^k} \sum_{j=1}^{k} \tau^{j-1} \text{ Im } (\tau^j) \sum_{m,n \in \mathbb{Z}} (m\tau + n)^{j+1} (m\tau + n)^{k-j+1}.
\]

where
\[
D_{a,b}(q; x) := \frac{(\tau - \tau)^a (\tau + x)^b}{2\pi i} \sum_{m,n \in \mathbb{Z}} \frac{e^{2\pi i (m\lambda + nA)}}{(m\tau + n)^a (m\tau + n)^b}
\]
for positive integers $a$ and $b$.

Finally, observe that the non-holomorphic Eisenstein series (8) can be identified with the elliptic polylogarithms using a formula of Zagier [7, Proposition 2]. This leads to the following general result.

**Theorem 4** For $k \geq 1$ and $z = A\tau - \hat{A}$, we have

$$\ln |Q_k(z; \tau)| - \tau^k \ln |Q_k(z; \tau)| = \frac{i k!}{(4\pi \Im \tau)^k} \sum_{j=1}^{k} \tau^{k-j} \Im(\tau') D_{j+1,k-j+1}(q;x),$$

where

$$D_{a,b}(q;x) = \sum_{m=0}^{\infty} (D_{a,b}(xq^m) + (-1)^{a+b} D_{a,b}(x^{-1}q^{m+1})) + \frac{(4\pi \Im \tau)^{a+b-1}}{(a+b)!} B_{a+b}(A)$$

and

$$D_{a,b}(x) = (-1)^{a-1} \sum_{\ell=a}^{a+b-1} 2^{a+b-\ell-1} \binom{\ell-1}{a-1} \frac{(-\ln |x|)^{a+b-\ell-1}}{(a+b-\ell-1)!} \Li_{\ell}(x)$$

$$+ (-1)^{b-1} \sum_{\ell=b}^{a+b-1} 2^{a+b-\ell-1} \binom{\ell-1}{b-1} \frac{(-\ln |x|)^{a+b-\ell-1}}{(a+b-\ell-1)!} \Li_{\ell}(x).$$

**6 Conclusion**

This final (and very short!) part is devoted to highlighting some directions for further research.

In spite of generalisability of the story in Sects. 2–4 to the function

$$F_k(z; \tau) = \ln |Q_k(z; \tau)| - \tau^k \ln |Q_k(z; \tau)|,$$

where $k \geq 1$ and the product $Q_k(z; \tau)$ is defined in (7), the case $k = 1$ remains the only one, which is invariant under translation $\tau \mapsto \tau + 1$. At the same time, Lemma 6 implies the transformation

$$\tau^k F_k(z, \tau) = (-1)^{k-1} F_k(z, \tau) \quad \text{for } k = 1, 2, \ldots$$

This consideration does not exclude, however, a possibility for modified products (7) and related functions $F_k$ to exist such that the latter ones have true modular behaviour for each $k \geq 1$. It sounds to us a nice problem to determine such modular objects.

Several arithmetic problems related to the case $k = 1$ (originating from the elliptic gamma function) are still open. Our personal favourites include connection of (5) with the Mahler measure and mirror symmetry; see, for example, observation in [6].

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On behalf of all authors, the corresponding author Wadim Zudilin states that there is no conflict of interest.

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