The Storage vs Repair-Bandwidth Trade-off for Clustered Storage Systems

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Abstract

We study a generalization of the setting of regenerating codes, motivated by applications to storage systems consisting of clusters of storage nodes. There are $n$ clusters in total, with $m$ nodes per cluster. A data file is coded and stored across the $mn$ nodes, with each node storing $\alpha$ symbols. For availability of data, we demand that the file is retrievable by downloading the entire content from any subset of $k$ clusters. Nodes represent entities that can fail, and here we distinguish between intra-cluster and inter-cluster bandwidth-costs during node repair. Node-repair is accomplished by downloading $\beta$ symbols each from any set of $d$ other clusters. The replacement-node also downloads content from any set of $\ell$ surviving nodes in the same cluster during the repair process. We first identity the optimal trade-off between storage-overhead and inter-cluster (IC) repair-bandwidth under functional repair, and also present optimal exact-repair code constructions for a class of parameters. The new trade-off is strictly better than what is achievable via space-sharing existing coding solutions, whenever $\ell > 0$. We then obtain lower bounds on the necessary intra-cluster repair-bandwidth to achieve optimal trade-off. Our bounds reveal the interesting fact that while it is beneficial to increase the number of local helper nodes $\ell$ in order to improve the storage-vs-IC-repair-bandwidth trade-off, a high value of $\ell$ also demands a higher amount of intra-cluster repair-bandwidth. Under functional repair, random linear network codes (RLNCs) simultaneously optimize usage of both inter and intra-cluster repair bandwidth; simulation results based on RLNCs suggest optimality of the bounds on intra-cluster repair-bandwidth. We also analyze resiliency of the clustered storage system against passive eavesdropping by providing file-size bounds and optimal code constructions.

I. INTRODUCTION

We consider the problem of designing efficient erasure codes for fault-tolerant data storage in a clustered network of storage nodes. We assume a fully connected network where communication links are present between any two nodes in the network. We say that two nodes appearing within the same cluster are connected via an intra-cluster link; where two nodes appearing in different clusters are connected via an inter-cluster link. A clustered network is characterized by the differences in communication costs within, and across clusters; intra-cluster communication cost is typically much less expensive than inter-cluster communication costs. From an availability perspective, it is of interest to encode and store a data file such that access to a subset of clusters allows recovery of the entire uncoded file. We consider nodes as failure domains, and demand efficient repair of failed nodes in any cluster. Repair of a failed node in a cluster is performed by downloading content from other clusters, and also a subset of surviving nodes in the host-cluster. We permit the possibility that a cluster which aids in the repair of a failed node combines helper-data from its various nodes, before sending it to the target-node, in order to decrease inter-cluster repair-bandwidth costs. The model described above is applicable in several practical scenarios. For instance, the clusters could represent geographically separated data centers of a cloud-service provider or a content-delivery network. Another application, motivated by security reasons, is one where a file is never stored entirely in any single cloud-service provider; rather it is coded and distributed across multiple service providers to prevent leakage of information to a passive eavesdropper who taps into a subset of the service providers.

A straightforward solution that guarantees availability of data, and minimizes inter-cluster repair-bandwidth costs, is to use a product code, consisting of two Maximum Distance Separable (MDS) codes - one across the clusters, and another within a cluster. While the solution entirely eliminates any inter-cluster bandwidth cost, it suffers from poor storage-overhead due to need to have redundancy in every cluster. At the other extreme, it is also possible
to achieve highly optimized storage cost, at the expense of inter-cluster repair-bandwidth, by using a maximum storage regenerating (MSR) code across the clusters. For instance, if we assume there are \(n\) clusters in total, then an \((n, k)\) MSR code across the clusters ensures that data is retrievable by accessing (entire) content of any \(k\) clusters. However, the solution suffers from high inter-cluster repair-bandwidth cost. Our goal in this work, at a high level, is to explore alternate solutions which can smoothly trade-off storage-overhead against inter-cluster repair-bandwidth for given availability requirements. We show that it is indeed possible to achieve (see Fig. 6) operating points which are strictly better than those obtained by space-sharing the above two conventional schemes. Further, we also characterize the amount of intra-cluster repair-bandwidth needed for achieving the optimal trade-off between storage-overhead and inter-cluster repair bandwidth.

A. System Model

We propose a natural generalization of the setting of regenerating codes [2] for clustered storage networks. The network consists of \(n\) clusters, with \(m\) nodes in each cluster. A data file of size \(B_m\) symbols is encoded into \(nm\alpha\) symbols, and stored across the \(nm\) nodes such that each node stores \(\alpha\) symbols. The symbols are assumed to come a finite field \(\mathbb{F}_q\) of \(q\) elements. For data-collection (availability constraint), we demand that (entire) content of any \(k\) clusters is sufficient to recover the original data file (Fig. 1). As mentioned before, nodes represent failure domains. We restrict ourselves to the case of efficient recovery from single node failure. Node repair is parametrized by three parameters \(d, \beta\) and \(\ell\). We assume that the replacement of a failed node is placed in the same cluster as the failed node. The replacement node downloads \(\beta\) symbols each from any set of \(d\) other clusters. The \(\beta\) symbols from any of the helper clusters is possibly a function of the \(m\alpha\) symbols present in the cluster - we assume any one of the nodes in the cluster takes responsibility for computing these \(\beta\) symbols before passing the helper data. Further, we also permit the replaced node to download (entire) content from any set of \(\ell\) surviving nodes in its host cluster, during the repair process. The quantity \(d\beta\) represents the inter-cluster (IC) repair-bandwidth. We refer to the overall code as the generalized regenerating code \(C_m\) with parameters \(\{(n, k, d)(\alpha, \beta)(m, \ell)\}\). The generalization reduces to the classical setup of regenerating codes, when \(m = 1\) (in which case, \(\ell = 0\) automatically). The generalization has two additional parameters \(\ell\) and \(m\) when compared with the classical set up of regeneration in [2]. We consider both the notions of functional, and exact repair, like in the classical setup. Under exact repair, the content of the repaired node is identical to that of the failed node; while in functional repair, the repair content permits data collection and repair of additional failed nodes. The first goal of the paper is to obtain a trade-off between storage-overhead and inter-cluster repair-bandwidth for an \(\{(n, k, d)(\alpha, \beta)(m, \ell)\}\) generalized regenerating code. In our model, whenever \(d > 0\), we make the assumption that the content of the \(m\) nodes of any given cluster have no local dependence.
among them, i.e., the entropy (with respect to log \( q \)) of any cluster is \( m \alpha \). As we will see, the case \( d = 0 \) is a special one where local dependence is necessary. Unlike the classical setting, the generalized setting permits \( d < k \).

The model described above does not consider intra-cluster bandwidth incurred during repair. Intra-cluster bandwidth is needed firstly to compute the \( \beta \) symbols in any helper cluster, and secondly to download content from any set of \( \ell \) surviving nodes in the host cluster. In order to characterize the amount of intra-cluster bandwidth that is needed to establish optimal trade-off between storage-overhead and inter-cluster repair-bandwidth, we consider the more involved repair model shown in Fig. 1(b). In this model, the replacement node downloads at most \( \gamma \) symbols from each of the \( \ell \) nodes from the host-cluster. With regard to a helper cluster, we assume that the \( \beta \) symbols contributed by it are only a function of at most \( \ell' \) symbols from each of the \( \ell \) nodes of the helper cluster. We make the assumption that any set of \( \ell' \) nodes can be used to compute the \( \beta \) symbols. Further, we limit the amount of data that each of these \( \ell' \) nodes can contribute to at most \( \gamma' \) symbols. A second goal of this paper is to identify necessary requirements on the parameters \( \gamma, \ell, \gamma' \) that are needed to guarantee optimal trade-off between storage-overhead and inter-cluster repair-bandwidth.

B. Related Work

Regenerating codes were originally introduced in [2] for simultaneously optimizing storage overhead and repair bandwidth for flat storage systems, where all communication links are equally costly. There has since then been significant progress in the area of classical regenerating codes in terms of code-constructions, finding optimal trade-offs under exact repair, and practical implementation. Below, we review variations of regenerating codes that have been proposed for non-flat topologies, and see how our model relates to these existing variations. We will also comment on how the model of locally repairable codes [3] relates to our model.

1) Regenerating Code Variations for Clustered Topologies: Regenerating code variations for data-center like topologies consisting of racks and nodes are considered in [4]–[8]. In [4], [5] and [6], the authors distinguish between inter-rack (inter-cluster) and intra-rack (intra-cluster) communication costs. Further, the works [4] and [5] permit pooling of intra-rack helper data to decrease inter-rack bandwidth. Also, all three works allow taking help from host-rack nodes during repair. However, for data collection all three works simply demand file retrievability from any set of \( k \) nodes irrespective of which racks (clusters) they belong to. In other words, the notion of clustering applies only to repair, and not data-collection, and this is a major difference with respect to our model. The work in [7] is a variation of that in [6] for a two-rack model, where the per-node storage capacity of the two racks differ. In [8], the authors consider a two-layer storage setting like ours consisting of several blocks of storage nodes. A different clustering approach is followed for both data collection and node repair. For data-collection, one accesses \( k_c \) nodes each from any of \( b_c \) blocks. Though the work focuses on node repair, the model assumes possible unavailability of the whole block where the failed node resides, and as such uses only nodes from other blocks for repair. The authors assume no difference between inter-block and intra-block communication, like we do in this work. The framework of twin-codes introduced in [9] is also related to our model and implicitly contains the notion of clustering. In this work, nodes are divided into two sets; for data collection, one connects to \( k \) nodes of any one set. Recovery of a failed node in one set is accomplished by connecting to \( d \) nodes in the other set. However, all links are considered equal in terms of communication cost, and this becomes the main difference with our model.

2) Regenerating Code Variations for Heterogeneous Systems: Several works [10]–[15] study variations of regenerating codes in very general settings, where a combination of node capacities, link costs, and amount of data-download-per-node lack uniformity across the network. Though none of these works explicitly consider clustering of nodes while performing data-collection like we do (which becomes the major difference with respect to our model), we would like to highlight some main features of the models considered in these works. In [10], the authors introduce flexible regenerating codes, where uniformity of download is enforced neither during data-collection nor during node-repair. However, all links have equal cost while computing the overall repair cost. The works of [11], [12] consider systems where the storage and repair-download costs are non-uniform across the various nodes. The authors of [11] like in [10], allow a replacement node to download an arbitrary amount of data from each helper node. In the work of [13], nodes are divided into two sets based on the cost incurred while these nodes aid during repair. As noted in [7], this is different from a clustered network, where the repair cost incurred due to a specific helper node depends on which cluster the replacement node belongs to. The works of [14] and [15] focus
on minimizing regeneration time rather regeneration bandwidth in systems with non-uniform point-to-point link
capacities. Essentially, each helper node is expected to find the optimal path, perhaps via other intermediate nodes,
to the replacement node such that the various link capacities are used in the best possible way to transfer all the
helper data needed for repair. It is interesting to note both of these works permit pooling of data at an intermediate
node which combines any relayed data with its own helper data. Recall that our model (and the one in [4]) also
considers pooling of data within a helper cluster, before passing on to the target cluster.

3) Locally Repairable Codes: Locally repairable codes (LRCs), introduced in [3], inherently has the motivation
of carrying out efficient node repair in hierarchical storage systems, like in data centers. The subject of LRCs, like
regenerating codes, has attained significant attention both in theory and practice, since its introduction. Recovery
of a node failure within a rack is first attempted locally, in order to minimize cross-rack bandwidth; if too many
nodes in the rack are unavailable, global parity nodes spread across various rack aid in recovery. However, LRCs do
not model clustering of nodes while carrying out data collection, and this is once again the major difference with
our model. In our model, during the repair of a node, the helper data from the ℓ nodes from the same cluster can
be considered as local-helper data. However, except for a very special case (d = 0), in this work we will assume
no dependence among the contents of the m nodes in any cluster, i.e, the entropy of contents of any cluster is
maximum. The case d = 0 corresponds to one having no helper clusters, and where repair is carried out entirely locally.
The difference in this case of d = 0 with model of LRCs is notion of clustering of nodes, while performing data
collection. For any d > 0, by avoiding local parities we reduce storage overhead when compared to LRCs; at the
same time, by utilizing the helper data from the remaining local nodes in the host-cluster, we perform better than
simply stacking classical regenerating codes.

C. Our Results

1) Upper Bound on File Size $B_m$: Under the setting of functional repair, the file-size $B_m$ is shown to be upper
bounded by

$$B_m \leq B_m^* = \ell k\alpha + (m - \ell) \sum_{i=0}^{k-1} \min\{\alpha, (d - i)^+ \beta\},$$

(1)

where we use the notation $a^+$ to mean $\max(a, 0)$, for any integer $a$. The bound is shown by considering the
information-flow graph under functional-repair, and calculating the minimum cut. For any finite information-flow
graph, the achievability of (1) follows from results in network coding [16]. This establishes the optimality of (1)
when we know an upper bound on the number of node failures that occur during the life-time of the system. In
practice, random linear network codes (RLNCs) [17] can be used to achieve near-optimal operating points.

For fixed values of $B_m = B_m^*, n, k, d > 0, \ell, m$, (1) gives a trade-off (see Fig. 2) between storage-overhead
$n/\alpha B_m$ and IC-repair-bandwidth-overhead $d\beta/\alpha$ (see Section IV-A on why we refer to $d\beta/\alpha$ as overhead). For
the case when $\ell = 0$, the trade-off is exactly same as that of the classical regenerating codes [2]. When $\ell > 0, m > 0$,
the trade-off is strictly better than that of the classical setup.

2) Optimal Code Constructions: We present optimal codes for the minimum storage-overhead (MSR) and the
minimum inter-cluster (IC) bandwidth-overhead (MBR) operating points, under the setting of exact repair. In both
cases, the code is constructed by suitably combining $\ell [n, k]$ vector MDS codes over $\mathbb{F}_q$ and $(m - \ell)$ classical
exact-repair regenerating codes. We also present an optimal code construction for functional-repair, which tolerates
an arbitrary number of failures\(^1\), for the case $\ell = m - 1, d \geq k$. The code is constructed by combining $m - 1 [n, k]$ vector MDS codes over $\mathbb{F}_q$, and an $\{n, k, d, (\alpha, \beta)\}$ functional-repair code from [18]. The code construction in
[18] does not depend on the number of repairs that can happen over the life-time, and since we directly use these
codes in our construction, the resultant functional-repair code retains this property.

\(^1\)The network-coding based achievability works only if there is a known upper bound on the number of repairs that occur over the lifetime of the system.
3) Lower bound on Intra-Cluster Bandwidth: We calculate lower bounds on the intra-cluster-bandwidth related parameters $\gamma, \ell', \gamma'$, shown in Fig. 1, under assumption that (1) is achieved with equality. While studying the impact of any one of these parameters, we ignore the effects of the other two; for example, the lower bound on $\gamma$ is obtained under the assumption that $\ell' = m$ and $\gamma' = \alpha$, etc.

Under functional repair with $d > 0$, the per-node intra-cluster bandwidth needed from the host-cluster is lower bounded by

$$\gamma \geq \gamma^* = \alpha - (d - k + 1)^+ \beta. \quad (2)$$

When $d < k$, the bound gives $\gamma \geq \alpha$, i.e. the entire content of the local helper nodes must be used. For $d \geq k$, at the MBR point characterized by $\alpha = d\beta$, the bound gives $\gamma \geq (k - 1)\beta$, and at the MSR point characterized by $\alpha = (d - k + 2)\beta$, the bound gives $\gamma \geq 0$. The trivial bound at the MSR point is indeed optimal, since optimal (achieving equality in (1)) codes at the MSR point can be achieved by simply stacking $m$ classical $(n, k, d)$, $(\alpha, \beta)$ MSR codes. In this case, no local help is needed for repair, and hence $\gamma = 0$ is indeed optimal at the MSR point.

In fact, under functional repair, the bound in (2) is optimum not just at the MSR point; we prove the converse statement that as long as $\gamma \geq \gamma^*$, it is indeed possible to achieve the optimal file-size in (1) for any set of parameters, as long as there is a known upper bound on the number of repairs in the system.

We provide bounds for the parameters $\ell'$ and $\gamma'$ which characterize intra-cluster bandwidth from helper cluster under the assumption that $\alpha \geq (d - k + 2)\beta$, and $d \geq k$. Specifically we show the parameter $\ell'$ is no less than $m$, i.e., $\ell' = m$, and

$$\gamma' \geq \frac{\beta}{m - \ell}. \quad (3)$$

Under functional repair, random linear network codes simultaneously optimize usage of both inter-cluster and intra-cluster bandwidths. Our simulations based on RLNCs indicate the tightness (achievability) of the bound in (3), under functional repair (see Fig. 3). We do not have an analytical converse for this bound.

The bounds on $\ell'$ and $\gamma'$ highlight the necessary trade-off between the system capacity $B_m$ and the external helper intra-cluster bandwidth $\ell' \gamma' = m \gamma'$, via parameter $\ell$, the key parameter that distinguishes our model from the classical model. Our bounds reveal the interesting fact that while it is beneficial to increase the number of local helper nodes $\ell$ in order to improve the storage-vs-IC-repair-bandwidth trade-off, a high value of $\ell$ also demands a higher amount of intra-cluster repair-bandwidth. For example, if we consider optimal MBR codes having minimum IC-repair-bandwidth, we see that their storage-overhead approaches that of MSR codes for large $m$ as $\ell$ gets close

The values of $\alpha$ in the range $(d - k + 2)\beta < \alpha \leq (d - k + 1)\beta$ corresponds to the region in the trade-off between the minimum-storage operating point, and the next corner point. The bound in (3) do not apply when $\alpha$ is in this range.
(a) Effect of local cluster repair-bandwidth $\gamma$, for an $(n = 3, k = 2, d = 2), (\ell = 2, m = 3), (\alpha = 8, \beta = 4), \ell' = m, \gamma' = \alpha$ system.

(b) Effect of helper cluster number of nodes $\ell'$, for an $(n = 3, k = 2, d = 2), (\ell = 1, m = 3), (\alpha = 8, \beta = 4)$ system.

(c) Effect of helper cluster repair-bandwidth $\gamma'$, for an $(n = 3, k = 2, d = 2), (\ell = 1, m = 3), (\alpha = 8, \beta = 4)$ system.

Fig. 3. Simulation results showing probability of successful data collection against number of node-repairs performed, for a clustered storage system employing random linear network codes (RLNCs). The three figures respectively indicate the impact of the intra-cluster-bandwidth related parameters $\gamma, \ell'$ and $\gamma'$ on the probability of decoding.

to $m$. However, a high value of $\ell$ also increases the helper cluster bandwidth; clearly, $m\gamma'$ surges enormously as $m - \ell$ approaches 1, see Fig. 4 for an illustration.

4) Security under Passive Eavesdropping - Bounds and Codes: We study resiliency of the clustered storage system against passive eavesdropping. An eavesdropper (say, Eve) gains access to the entire content of any subset of $e$ clusters, where $1 \leq e \leq k$. Eve also gets to observe all the helper data that gets downloaded for repair of nodes in these $e$ clusters. The properties of data collection and disk repair remain same as in the case of no eavesdropper. The setting is along the lines of that considered in [19], where authors study security under the classical regenerating code framework. The maximum file size $B_{m}^{(s)}$ that can be securely stored such that Eve does not gain any information about the file is shown to be upper bounded by

$$B_{m}^{(s)} \leq \ell(k - e)\alpha + (m - \ell) \sum_{i=e}^{k-1} \min\{\alpha, (d - i)\beta\}.$$  \hspace{1cm} (4)

We also present explicit optimal secure codes for the MBR point under the setting of exact repair. Like in the case of no security, an optimal secure code is constructed by suitably precoding a combination of $m$ component codes, but this time the component codes themselves are secure codes. A code at the MBR point is constructed by combining $\ell$ secure MDS codes for the wiretap-II channel [20] [21], and $(m - \ell)$ classical exact-repair secure MBR codes [19] [22]. Our security results are straightforward extensions of the security results for classical regenerating codes, given our own results for the no-security case.

D. An Example

Consider a system consisting of $n = 4$ clusters, with $m = 4$ nodes/cluster, where we have the availability requirement that content from any set of $k = 3$ clusters suffice for data-collection. Let us consider three coding options, which permit single node repair: (i) A product code consisting of two $[4, 3]$ simple parity check codes, one across the clusters, and another within a cluster. In this case, repair happens entirely within a cluster, and there is no inter-cluster bandwidth. Note that this corresponds to the case of $d = 0$ in our framework. (ii) Stacking $m$ classical $(n, k, d = 3)$ regenerating codes - by this we mean that corresponding nodes from the $n$ clusters employ the regenerating code. This corresponds to the case of $\ell = 0$ in our framework. (iii) A $(n, k, d = 3)(m = 4, \ell = 3)$ generalized regenerating code, which is constructed as follows (see Fig. 5): We stack three $[4, 3]$ simple parity check codes (over the vector alphabet $\mathbb{F}_q^3$) to populate coded data in all but the last nodes of all the clusters. In the last column (corresponding to the last node of all clusters), we take the sum of these stacked codes along with a
Fig. 4. Illustrating the impact of number of local helper nodes on the various performance metrics. We operate at the minimum inter-cluster repair-bandwidth (MBR) point, with parameters \( \left\{ (n = 12, k = 8, d = n - 1)(\alpha = d\beta, \beta = 2) \right\} \). Storage-overhead is \( \frac{mn\alpha}{B^*} \), where \( B^* \) is calculated using (1). Inter-cluster BW is \( d\beta \). Local and helper intra-cluster BWs are respectively calculated using (2) and (3). We see that while \( \ell = m - 1 \) is ideal in terms of optimizing storage and inter-cluster BW, it imposes the maximum burden on intra-cluster BW.

![Graph showing storage overhead and inter-cluster bandwidth](image)

Cluster 1
Cluster 2
Cluster n

\( C_1 \)
\( C_{m-1} \)
\( (m - 1) \text{ stacked } [n, k] \text{ MDS codes} \)

Classical MBR code

Fig. 5. Illustration of an \((n = 4, k = 3, d = 3)(m = 4, \ell = 3)\) generalized regenerating code attaining minimum intra-cluster repair-bandwidth. Classical \((4, 3, 3)(\alpha = d\beta, \beta = 1)\) MBR code. Constructions of classical MBR codes appear in [23]. Data-collection property of the code is straightforward. For repair of any node, the last node of each of the helper clusters (with the help of the remaining 3 nodes) first extracts the MBR code, and passes the helper data for the MBR code. The replacement node regenerates the MBR code content first, and uses the helper data from the 3 local nodes to finally recover the original stored content. The storage-overhead vs IC-repair-bandwidth trade-off achieved by these three options is shown in Fig. 6. We see that the framework of generalized regenerating codes introduced here, offers operating points which are strictly better than those that can be achieved by space sharing the first two options.

The rest of the document is organized as follows. To get further insights into the set up, and the results, we first analyze certain special cases of the problem in Section II. In Section III, we present the information-flow graph (IFG) models used in the derivation of the bounds. The file-size bound under functional repair for general set of parameters is discussed in Section IV. The exact-repair and functional-repair code constructions both appear
in Section V. Bounds on parameters relating to intra-cluster bandwidth under functional repair, are discussed in Section VI. Section VII considers security under passive eaves-dropping. Our conclusions and directions for future work appear in Section VIII.

II. FILE SIZE BOUND FOR SPECIAL CASES

In this section, we consider certain special cases of the setting of generalized regenerating codes, and identity the corresponding storage vs IC-repair-bandwidth trade-offs. The following cases are considered 1) \( d = 0 \), which corresponds to the case when repair of a node is carried out entirely with the help of other nodes in the cluster. 2) \( \ell = 0 \), which corresponds to the case when repair of a node is carried out solely with the help of other clusters, without taking any help from surviving nodes in the host-cluster. As mentioned previously, except for the case \( d = 0 \), we assume that content of the nodes of a cluster are independent.

A. Case \( d = 0 : \) No Inter-Cluster Help for Repair

When \( d = 0 \), node repair is accomplished by contacting any set of \( \ell \) other nodes in the host cluster. Since any set of \( k \) clusters should be sufficient to decode the whole file, it follows that the file-size \( B_m \) is upper bounded by

\[
B_m \leq \ell k \alpha. \tag{5}
\]

The achievability of the above bound follows by using an \([n, k] \times [m, \ell] \) product code, with both component codes being MDS codes over \( \mathbb{F}_q^\alpha \). In fact, the parameter \( \alpha \) is redundant for this case; one may choose \( \alpha = 1 \) in practice. It is clear that there is no storage vs IC-repair-bandwidth trade-off offered by this case.

B. Case \( \ell = 0 : \) No local side information

We next consider the special case when repair is performed without any help from nodes in the host-cluster. We present a simple proof of the optimal trade-off in this case. Let \( C_m \) denote a generalized regenerating code of file-size \( B_m \) with parameters \( \{(n, k, d)(\alpha, \beta)(m, \ell = 0)\} \). Let \( B^*_m \) denote that maximum possible file-size under the setting. In this notation, \( C_1 \) denotes a classical regenerating code, and under functional-repair, we know that \[2\]

\[
B^*_1 = \sum_{i=0}^{k-1} \min \{\alpha, (d - i)\beta\}. \tag{6}
\]
Theorem 2.1: The optimal file size under the setting of functional-repair generalized regenerating codes, for the case of no local side information, is given by

\[ B_m^* = m B_1^* = m \sum_{i=0}^{k-1} \min\{\alpha, (d-i)\beta\}. \] (7)

In other words, the storage-overhead vs IC-repair-bandwidth-overhead trade-off for functional-repair generalized regenerating code with any number of nodes per cluster, for the case of no local side information during repair, is identical to that of the functional-repair classical regenerating code.

Proof: The achievability part of the proof is straightforward; the optimal code is constructed by simply stacking \( m \) classical codes \( C_i \) each of which achieves the bound in (6). By stacking, we mean that the code \( C_1 \) is deployed across the corresponding nodes from all \( n \) clusters. In this case, note that during node-repair, there is no pooling of content from various nodes of a helper cluster; repair happens as though there is only one code \( C_1 \) in the system.

For showing the upper bound on the file size, we note that given a code \( C_m \) with file-size \( B_m \) having parameters \( \{(n, k, d)(\alpha, \beta)(m, \ell = 0)\} \), one can construct a functional-repair classical regenerating code \( \hat{C}_1 \), also with file-size \( B_m \), and having parameters \( \{(n, k, d)(\alpha, \beta)\} \), where \( \hat{\beta} \leq m\beta \). For this, we simply assume the contents of all \( m \) nodes of any cluster \( i \) of \( C_m \), to be the contents of node \( i \) of \( \hat{C}_1 \), \( 1 \leq i \leq n \). Clearly \( \hat{C}_1 \) retains the data-collection property. For node repair in \( \hat{C}_1 \), we perform individual repairs of each of the \( m \) nodes, but with the same set of \( d \) helper clusters. In this case, we know that

\[ B_m \leq \sum_{i=0}^{k-1} \min\{m\alpha, (d-i)\hat{\beta}\} \leq \sum_{i=0}^{k-1} \min\{m\alpha, (d-i)m\beta\} = m \sum_{i=0}^{k-1} \min\{\alpha, (d-i)\beta\}. \] (8)

(9)

(10)

III. INFORMATION FLOW GRAPH MODEL

In this section, we describe the information flow graph (IFG) models used to derive the various bounds in this work. The models are generalizations of the one used in [2] for the case of classical regenerating codes. Under functional repair, the problem is one of multicasting the source file to an arbitrary number of data collectors over the IFG. The IFG characterizes the data flows from the source to a data collector, and also reflects the sequence of failures and repairs in the storage system. Two models of IFGs will be used; the first one will be used in two scenarios: 1) to derive the trade-off between storage-overhead and inter-cluster repair-bandwidth overhead. In this, we ignore the effects of intra-cluster bandwidth, 2) to find the optimal local helper node intra-cluster bandwidth \( \gamma \), which is needed to establish the optimal trade-off between storage-overhead and inter-cluster repair-bandwidth overhead. We wish to note that while obtaining the bound on \( \gamma \), we do not impose any limitations on \( \gamma' \), \( \ell' \), i.e., we assume that \( \gamma' = \alpha \) and \( \ell' = m \). A second related model will be used while deriving the lower bounds on the parameters \( \ell', \gamma' \), which relate to the intra-cluster repair bandwidth needed in the helper clusters. In this second model, we will assume that \( \gamma = \alpha \), i.e., we ignore the effects of limited local helper-node intra-cluster bandwidth, while calculating bounds on external helper-node intra-cluster bandwidth. We describe the two models next.

A. IFG Model for Storage vs Inter-Cluster Repair Bandwidth Trade-off

Let \( X_{i,j} \) denote the physical node \( j \) in cluster \( i \), \( 1 \leq i \leq n, 1 \leq j \leq m \). Recall that capacity of any node is \( \alpha \). In the IFG, the physical node is represented by the pair of nodes \( X_{i,j}^{in} \) and \( X_{i,j}^{out} \), with an edge of capacity \( \alpha \) going from \( X_{i,j}^{in} \) to \( X_{i,j}^{out} \). We will write \((X_{i,j}^{in}) \rightarrow (X_{i,j}^{out}) \) to denote that there is an edge from going from \( X_{i,j}^{in} \) to \( X_{i,j}^{out} \). With a slight abuse of notation, we will let \( X_{i,j} \) to also denote the pair \((X_{i,j}^{in}, X_{i,j}^{out})\) of the graph nodes. Cluster
replacement node, say $\hat{\omega}$, subset of $k$ of infinite capacity. The sink node $X$ with a slight abuse of notation, we will use $(\hat{\omega})$ with new identical pair of nodes regenerated by downloading $\beta$ also has an additional external node, denoted as $X_i^{ext}$. Each out-node $X_{i,j}^{out}$, $1 \leq j \leq m$ is connected to $X_i^{ext}$ via an edge of capacity $\alpha$. The external node $X_i^{ext}$ is used to transfer data outside the cluster, and thus serves two purposes: 1) it represents a single point of contact to the cluster, for a data collector which connects to this cluster, and 2) it represents the compute unit which generates the $\beta$ symbols for repair of any node in a different cluster.

The source node $S$ represents the original placement of the encoded source file into the $n m$ storage nodes. $S$ connects to the in-nodes of all physical storage nodes in their original state ($S \rightarrow X_i^{in}$), $\forall i \in [n], \forall j \in [m]$, via links of infinite capacity. The sink node $T$ represents a data collector, it connects to the external nodes of an arbitrary subset of $k$ clusters ($X_i^{ext} \rightarrow T$) also via links of infinite capacity.

Each cluster at any moment has $m$ active nodes. When a physical node $X_{i,j}$ fails, it becomes inactive, and its replacement node, say $\hat{X}_{i,j}$, becomes active instead (see Fig. 7 for an illustration). The replacement node $\hat{X}_{i,j}$ is regenerated by downloading $\beta$ symbols from any $d$ nodes in the set $\{X_i^{ext}, 1 \leq i' \leq n, i' \neq i\}$. The replacement node also connects to any subset of $\ell$ nodes in the set $\{X_{i,j}^{out}, 1 \leq j' \leq m, j' \neq j\}$. The capacity of the links $\{(X_{i,j}^{out} \rightarrow \hat{X}_{i,j}^{in}), 1 \leq j' \leq m, j' \neq j\}$ depend on whether we use the model for finding the IC-bandwidth vs storage trade-off, or we use it for finding bounds on local helper bandwidth $\gamma$. These links have capacity $\alpha$ and $\gamma$ in the former and latter cases, respectively.

In our model, recall that we focus on one repair at a time. In this scenario, along with the replacement of $X_{i,j}$ with $\hat{X}_{i,j}$, we will also copy all the remaining $m - 1$ nodes, as they are, in the cluster $i$, and represent them with new identical pair of nodes $(X_{i,j}', X_{i,j}'')$, $1 \leq j' \leq m, j' \neq j$. We will also a have a new external node for the cluster, which connects to the new $m$ out-nodes. Thus, in the IFG modeling, we say that the entire old cluster (where the failed node resides) becomes inactive, and gets replaced by a new active cluster. For either data collection or repair, we connect to external nodes of only the active clusters. Note that at any point in time, a physical cluster contains only one active cluster in the IFG, and $f_i$ inactive clusters in the IFG, where $f_i \geq 0$ denotes the total number of failures and repairs experienced by the various nodes in the cluster. We will use the notation $\mathcal{X}_i(t), 0 \leq t \leq f_i$ to denote the cluster that appears in IFG after the $t^{th}$ repair associated with cluster $i$. The clusters $\mathcal{X}_i(0), \ldots, \mathcal{X}_i(f_i - 1)$ are inactive, while $\mathcal{X}_i(f_i)$ is active, after $f_i$ repairs. The nodes of $\mathcal{X}_i(t)$ will be denoted by $X_{i,j}^{in}(t), X_{i,j}^{out}(t), X_{i,j}^{ext}(t), 1 \leq j \leq m$. With a slight abuse of notation, we will let $\mathcal{X}_i(t)$ to also denote the collection of all $2m + 1$ nodes in this cluster. We write $X_{i,j}(t)$ to denote the pair $(X_{i,j}^{in}(t), X_{i,j}^{out}(t))$; again, with a slight abuse of notation, we will use $X_{i,j}(t)$ to also denote the node $j$ in cluster $i$ after the $t^{th}$ repair (in cluster $i$). We will further use notation $Fam(i)$ to denote the union of all nodes in all inactive clusters, and the active cluster, corresponding to the physical cluster $i$, i.e., $Fam(i) = \bigcup_{t=0}^{f_i} \mathcal{X}_i(t)$. 

Fig. 7. An example of the information flow graph model representing the notion of generalized regenerating codes, when intra-cluster bandwidth is ignored. In this figure, we assume $(n = 3, k = 2, d = 2)(m = 2, \ell = 1)$. The model is a generalization of one the used in [2] for the setting of classical regenerating codes.
Failed node

Fig. 8. An example of the second information flow graph model, with limited external helper cluster intra-cluster bandwidth $\gamma'$. The model assumes that $\gamma = \alpha$. In this example, we assume $(n = 3, k = 2, d = 2)(m = 2, \ell = 1)(\ell' = 1)$.

### B. IFG Model for Finding Bounds on Intra-Cluster Repair Bandwidth

We now describe the model used to obtain lower bounds on the parameters $\gamma'$, $\ell'$. Unlike in the case of bounds for file-size and $\gamma'$, where we also show converses, for $\gamma'$ and $\ell'$ we do not provide converses to the lower bounds. When only dealing with lower bounds, we can significantly simplify the model described above. Of course, additional structure needs to be added to the model to enable usage of $\gamma'$ and $\ell'$. We describe these changes next.

In the second model, each physical node $X_{i,j}$ is again represented by the pair of nodes $(X_{i,j}^{\text{in}}, X_{i,j}^{\text{out}})$, such that the edge $(X_{i,j}^{\text{in}} \rightarrow X_{i,j}^{\text{out}})$ has capacity $\alpha$. Unlike the first model, we do not add an external node to a cluster, unless the cluster aids in either data-collection or repair of another node. Whenever a physical node $X_{i,j}$ fails, it becomes inactive, and its replacement node, say $\hat{X}_{i,j} = (\hat{X}_{i,j}^{\text{in}}, \hat{X}_{i,j}^{\text{out}})$, becomes active in the same cluster. The remaining $m-1$ nodes are not replicated, as in the previous model. Thus, in the second model, there is only a single graph cluster corresponding to a physical cluster, and no cluster ever becomes inactive. On the graph image, the replacement node is visually linked to the replaced failed node with a red dotted line (see Fig. 8 for an example).

The replacement node $\hat{X}_{i,j}$ connects to any subset of $\ell$ active nodes in the same cluster via links of capacity $\alpha$. It also downloads $\beta$ symbols each from $d$ other clusters via their external nodes. If replacement node $\hat{X}_{i,j}$ downloads helper data from cluster $i'$, an external node $X_{i',i,j}^{\text{ext}}$ is added to the IFG, such that the edge $(X_{i',i,j}^{\text{ext}} \rightarrow \hat{X}_{i,j}^{\text{in}})$ has capacity $\beta$. External node $X_{i',i,j}^{\text{ext}}$ also connects locally to a subset of $\ell'$ active out-nodes of cluster $i'$ via links of capacity $\gamma'$. Note how we index the external node of cluster $i'$ that aids in the repair of $X_{i,j}$. Thus, every time cluster $i'$ acts as an external helper cluster toward the repair of any node, we add a new external node in a manner similar to $X_{i',X_{i,j}}^{\text{ext}}$. Finally, a data collector $T$ connects to cluster $i$ via the external node $X_{i,T}^{\text{ext}}$, which connects to all $m$ active out-nodes in the cluster via links of capacity $\alpha$. We index the external node also with $T$ since the $m$ active nodes that form part of the cluster evolves over time.

In this model, we do not time-index the sequence of failures, like in the previous model. This is primarily because, in our proof of bounds for $\gamma'$ and $\ell'$, we only consider system evolutions in which each node fails at most once. In this case, since we do not replicate non-failed nodes, we find it convenient to simply denote the replacement node of $X_{i,j}$ as $\hat{X}_{i,j}$. Finally, we feel that the usage of the second model for analyzing file-size and $\gamma$ as well, would have complicated their converse proofs, and this is primary reason to have two models.

### IV. File Size Bound for General Parameters

In this section, we derive the file-size bound in (1) under the setting of functional repair, for arbitrary set of code parameters. We further use this bound to characterize the storage-overhead vs IC-repair-bandwidth-overhead trade-off. Intra-cluster bandwidth is ignored in this section. Thus, for the repair of any node, the entire content of $\ell$ local helper nodes can be used; similarly, entire content ($m\alpha$ symbols) of each helper cluster is used to generate its $\beta$ helper symbols.
Theorem 4.1: The file size $B_m$ of a functional repair generalized regenerating code having parameters $\{(n, k, d) \ (\alpha, \beta) \ (m, \ell)\}$ is upper bounded by

$$B_m \leq B_m^* \triangleq \ell k \alpha + (m - \ell) \sum_{i=0}^{k-1} \min\{\alpha, (d - i)^+ \beta\}. \quad (11)$$

Further, if there is a known upper bound on the number of repairs that occur over the lifetime of the system, the above bound is sharp, i.e., $B_m^*$ gives the functional repair storage capacity of the system.

Proof: The proof technique is similar to the proof of the bound under functional repair for the setting of classical regenerating codes [2]. The problem of functional repair is one of multicasting, and thus for finding the desired upper bound on the file-size, it is enough if we exhibit a cut in an IFG, for a specific sequence of failures and repairs, which separates the source from the sink, such that the value of the cut is the desired upper bound. We will then show that for any valid IFG, independent of the specific sequence of failures and repairs, $B_m^*$ is indeed a lower bound on the minimum possible value of any $S - T$ cut. The achievability result, when there is known upper bound on the number of failures and repairs, will then follow from results in network coding [16].

We begin with the proof of the upper bound. We consider a sequence of $k(m - \ell)$ failures and repairs, as follows: Physical nodes $X_{i,\ell+1}, X_{i,\ell+2}, \ldots X_{i,m}$ fail in this order in cluster $i = 1$, then in cluster $i = 2$, and so on, until cluster $i = k$. In the IFG, (see Section III-A), this corresponds to the sequence of failures of nodes $X_{1,\ell+1}(0), X_{1,\ell+2}(1), \ldots, X_{1,m}(m - \ell - 1), X_{2,\ell+1}(0), \ldots, X_{2,m}(m - \ell - 1), \ldots, X_{k,m}(m - \ell - 1)$, in the respective order. The replacement node $X_{i,\ell+t}(t)$ for $X_{i,\ell+t}(t - 1), 1 \leq t \leq m - \ell$ draws local helper data from $X_{i,1}(t - 1), X_{i,2}(t - 1), \ldots, X_{i,\ell}(t - 1)$, and external helper data from the clusters $X_1(m - \ell), \ldots, X_{i-1}(m - \ell)$, and from some set of $d - \min\{i - 1, d\} = (d - i + 1)^+$ other active clusters in the IFG. An example is shown in Fig. 9 for a set of system parameters that is same as those used in Fig. 7.

Let data collector $T$ connect to clusters $X_1(m - \ell), \ldots, X_k(m - \ell)$. Consider the $S - T$ cut consisting of the following edges of the IFG:

- $\{(X_{i,j}^{in}(0) \rightarrow X_{i,j}^{out}(0)), i \in [k], j \in [\ell]\}$. Total capacity of these edges is $k \alpha$.
- For each $i \in [k], t \in [m - \ell]$, either the set of edges $\{(X_{i,t}^{ext}(0) \rightarrow X_{i,\ell+t}(t)), t' \in \{\text{helper cluster indices for the replacement node } X_{i,\ell+t}(t)\}\} \setminus \{\min\{i - 1, d\}\}$, or the edge $\{(X_{i,\ell+t}(t) \rightarrow X_{i,\ell+t}(t))\}$. Among the two possibilities, we pick the one which has smaller capacity. In this case, the total capacity of this part of the cut is given by $\sum_{i=1}^{k} \sum_{t=\ell+1}^{m} \min\{\alpha, (d - \min\{i - 1, d\})^+ \beta\} = (m - \ell) \sum_{i=1}^{k} \min\{\alpha, (d - i + 1)^+ \beta\}$. The value of the cut is given by $k \alpha + (m - \ell) \sum_{i=1}^{k} \min\{\alpha, (d - i + 1)^+ \beta\} = B_m^*$, which proves our upper bound.

We next show that for any valid IFG (independent of the specific sequence of failures and repairs), $B_m^*$ is indeed a lower bound on the minimum possible value of any $S - T$ cut. Let $C$ denote any $S - T$ cut, and let IFG$_S$ and IFG$_T$ denote the two resultant disconnected parts of the IFG corresponding to the nodes $S$ and $T$, respectively.
Next, consider the subsequence of \( E \) without loss of generality, we assume that \( Y \) order. Further, the node \( X_{i,a}(t_{i,a}) \) is a replacement node in the IFG.

Since node \( T \) connects to \( k \) external nodes via links of infinite capacity, we only consider cuts such that IFG\(_T\) has at least \( k \) external nodes corresponding to active clusters. Next, we observe that the IFG is a directed acyclic graph, and hence there exists a topological sorting of nodes of the graph such that an edge exists between two nodes \( A \) and \( B \) of the IFG only if \( A \) appears before \( B \) in the sorting. It is straightforward to see that one can consider a sorting with the following properties, for any \( i, t: 1 \) the \( 2m + 1 \) nodes of the cluster \( X_i(t) \) appear together in the sorted order, and \( 2 \) the nodes of the cluster \( X_i(t) \) appear ahead the nodes of the cluster \( X_i(t + 1) \) in the sorted order.

Now, consider the sequence \( E \) of all the external nodes (which are part of both active and inactive clusters) in IFG\(_T\) in their sorted order. Let \( Y_1 \) denote the first node in this sequence. Without loss of generality let \( Y_1 \in Fam(1) \). Next, consider the subsequence of \( E \) which is obtained after excluding all the external nodes in \( Fam(1) \) from \( E \). Let \( Y_2 \) denote the first external node in this subsequence. We continue in this manner until we find the first \( k \) external nodes \( \{Y_1, Y_2, \ldots, Y_k\} \in E \), such that each of the \( k \) nodes corresponds to a distinct physical cluster. Once again without loss of generality, we assume that \( Y_i \in Fam(i), 2 \leq i \leq k \). Let us assume that \( Y_i = X_i^{\text{ext}}(t_i) \), for some \( t_i \). Now, consider the \( m \) out-nodes \( X_{i,1}^{\text{out}}(t_i), \ldots, X_{i,m}^{\text{out}}(t_i) \) that connect to \( X_i^{\text{ext}}(t_i) \). Among these \( m \) out-nodes, let \( a_i, 0 \leq a_i \leq m \) denote the number of out-nodes that appear in IFG\(_S\). Without loss of generality let these be the nodes \( X_{i,1}^{\text{out}}(t_i), X_{i,2}^{\text{out}}(t_i), \ldots, X_{i,a_i}^{\text{out}}(t_i) \). Next, corresponding to the out-node \( X_{i,j}^{\text{out}}(t_i) \), \( a_i + 1 \leq j \leq m \), consider its past versions \( \{X_{i,j}^{\text{out}}(t), t < t_i\} \) in the IFG, and let \( X_{i,j}^{\text{out}}(t_{i,j}) \), for some \( t_{i,j} \leq t_i \) denote the first sorted node that appears\(^3\) in IFG\(_T\). Without loss of generality let us also assume that the nodes \( \{X_{i,j}^{\text{out}}(t_{i,j}), a_i + 1 \leq j \leq m\} \) are sorted in the order \( X_{i,a_i+1}^{\text{out}}(t_{i,a_i+1}), X_{i,a_i+2}^{\text{out}}(t_{i,a_i+2}), \ldots, X_{i,m}^{\text{out}}(t_{i,m}) \). An illustration is provided in Fig. 10.

![](https://example.com/fig10.png)

**Fig. 10.** An example of how any \( S - T \) cut in the IFG affects nodes in \( Fam(i) \). In the example, we assume \( m = 4 \). With respect to the description in the text, \( a_i = 2 \). Further, the node \( X_{i,a}(t_{i,a}) \) is a replacement node in the IFG.

A lower bound on the value of the \( S - T \) cut can be obtained based on the following observations:

- The \( a_i \) edges \( \{(X_{i,j}^{\text{out}}(t_i) \rightarrow X_{i}^{\text{ext}}(t_i)), 1 \leq j \leq a_i\} \) are part of the cut. These contribute a total value of \( a_i \alpha \).
- For any node \( X_{i,j}^{\text{out}}(t_{i,j}) \), \( a_i + 1 \leq j \leq m \), if the corresponding in-node \( X_{i,j}^{\text{in}}(t_{i,j}) \) belongs to IFG\(_S\), then the edge \( \{X_{i,j}^{\text{in}}(t_{i,j}) \rightarrow X_{i,j}^{\text{out}}(t_{i,j})\} \) appears in the cut, and contributes a value of \( \alpha \). Now, consider the case when

\(^3\)It may be noted that even though \( X_{i,j}^{\text{out}}(t_{i,j}) \) appears in IFG\(_T\), the corresponding external node \( X_{i}^{\text{ext}}(t_i) \) appears in IFG\(_S\). This is due to our assumption that \( Y_i = X_{i}^{\text{ext}}(t_i) \) is the first external node, corresponding to physical cluster \( i \), that appears in IFG\(_T\).
the in-node $X_{i,j}^{in}(t_{i,j})$ belongs to IFG$_T$. In this case, consider the following two sub cases:

- The node $X_{i,j}(t_{i,j})$ is not a replacement node: In this case, either the edge \{(\text{X}^{out}_{i,j}(t_{i,j} - 1) \rightarrow X_{i,j}^{in}(t_{i,j}))\} appears in the cut, if $t_{i,j} > 0$, or the edge \{S \rightarrow X_{i,j}^{in}(t_{i,j})\} appears in the cut, if $t_{i,j} = 0$. In any case, the contribution to the overall value of the cut is at least $\alpha$.

- The node $X_{i,j}(t_{i,j})$ is a replacement node of $X_{i,j}(t_{i,j} - 1)$: We know that $\ell$ local helper nodes and $d$ external nodes are involved in repair. Based on our assumptions so far, it is straightforward to see that out of the $\ell$ local helper nodes, at most $(j - 1)$ belong to IFG$_T$. To see this, note that the potential candidates for the local helper nodes that appear in IFG$_T$ correspond to the physical nodes $X_{i,1}, X_{i,2}, \ldots, X_{i,j-1}$. Node $X_{i,j'}, j' > j$ appear in IFG$_S$, because of our definition of the nodes $X_{i,j'}(t_{i,j'})$. It may also be noted that we count the physical nodes $X_{i,1}, X_{i,2}, \ldots, X_{i,a_i}$ among the possible set of local helpers, although we assume that $X_{i,j}(t_j), 1 \leq j \leq a_i$ appears in IFG$_S$. This is because, we cannot discount the possibility that $X_{i,j}(t_{i,j'} - 1)$ appears in IFG$_T$, for $j \leq a_i, j' > a_i$. Next, note that out of the $d$ external nodes, at most $(i - 1)$ belong to IFG$_T$. In this case, the contribution to the value of the cut, due to the edges that aid in repair, is lower bounded by $(\ell - j + 1)^+ \alpha + (d - (i - 1))^+ \beta$.

Based on the observations above, the value of the cut is lower bounded by

\[
\min\text{cut}(S - T) \geq \sum_{i=1}^{k} \left( a_i \alpha + \sum_{j=a_i+1}^{m} \min(\alpha, (\ell - j + 1)^+ \alpha + (d - (i - 1))^+ \beta) \right) \tag{12}
\]

\[
= a_i k \alpha + \sum_{i=1}^{k} \sum_{j=a_i+1}^{\ell} \alpha + \sum_{i=1}^{k} \sum_{j=\max(\ell,a_i)+1}^{m} \min(\alpha, (d - (i - 1))^+ \beta) \tag{13}
\]

\[
= \max(a_i, \ell) k \alpha + (m - \max(a_i, \ell)) \sum_{i=1}^{k} \min(\alpha, (d - (i - 1))^+ \beta) \tag{14}
\]

\[
\geq \ell k \alpha + (m - \ell) \sum_{i=1}^{k} \min(\alpha, (d - (i - 1))^+ \beta), \tag{15}
\]

for any $a_i, 0 \leq a_i \leq m$. This completes the proof of the converse.

**Definition 1 (Optimal Code):** Code $C_m$ is said to be optimal or capacity achieving, if its file-size $B_m = B^*_m$, where $B^*_m$ is as given in Theorem 4.1.

### V. Code Constructions

In this section, we describe our optimal code constructions. Two constructions will be presented; the first one is an instance of an exact repair code, and results in optimal codes at the MSR and MBR points under the setting of generalized regenerating codes. In our second construction, we present a functional-repair regenerating code that can withstand any number of repairs during the lifetime of operation. Recall that the network-coding based achievability proofs work only if there is a known upper bound on the number of repairs that occur over the lifetime of the system. In both cases, we rely on existing optimal classical regenerating codes that are linear. By a
linear regenerating code\(^4\), we mean that both encoding and repair are performed via linear combinations of either the input or the coded symbols. The first construction generates an optimal \((n, k, d)(\alpha, \beta)(m, \ell)\) code for any \(m, \ell < m - 1\), \(1 \leq d \leq n - 1\), whenever an optimal \((n, k, \min(k, d))(\alpha, \beta)\) classical exact repair linear regenerating code exists. Our functional repair code construction is limited to the case \(\ell = m - 1\), \(d \geq k\), and relies on the construction in [18].

A. Exact Repair Code Construction

We begin with a description of the code, and then show its data collection and repair properties. The construction itself is a generalization of the example presented in Section I-D.

We say that two linear codes are identical if they have same generator matrix.

Construction 5.1: Let \(C_j, 1 \leq j \leq \ell\) denote \([n, k]\) MDS array codes over \(\mathbb{F}_q^\alpha\). We assume that all these codes \(\ell\) codes have the same generator matrix. The amount of data that can be encoded with these \(\ell\) codes is \(\ell k \alpha\). Next, let \(C_j, \ell + 1 \leq j \leq m\) denote \((n, k, d') = \min(d, k))(\alpha, \beta)\) classical exact repair linear regenerating codes, each having a file size \(B' = \sum_{i=0}^{k-1} \min(\alpha, (d' - i)\beta)\). We demand that all the codes \(C_j, \ell + 1 \leq j \leq m\) are identical. For encoding, we first divide the data file of size \(B_m = \ell k \alpha + (m - \ell)B'\) into \(m\) stripes, such that first \(\ell\) have size \(k \alpha\), and the last \(m - \ell\) have size \(B'\). Stripe \(j, 1 \leq j \leq m\) is encoded by \(C_j\) to generate the coded symbols \(c_j = [c_{1,j}, c_{2,j}, \ldots, c_{\alpha,j}]^T\). Next, consider an \(m \times m\) invertible matrix \(A\) over \(\mathbb{F}_q\) such that the first the \(\ell\) rows of \(A\) generate an \([m, \ell]\) MDS code \(\mathbb{F}_q\). Let matrix \(A\) be decomposed as

\[
A_{m \times m} = \begin{bmatrix} E_{\ell \times m} \\ F_{m - \ell \times m} \end{bmatrix}
\]

The coded data stored in the various clusters is generated as follows:

\[
[c_1', c_2', \ldots, c_m'] = [c_1, c_2, \ldots, c_m]A_{m \times m}.
\]

The content of node \(j\) in cluster \(i\) is given by \([c_{i(j-1)\alpha+1, j}, c_{i(j-1)\alpha+2, j}, \ldots, c_{i\alpha, j}]^T, 1 \leq i \leq n, 1 \leq j \leq m\). This completes the description of the construction. A pictorial overview of the description appears in Fig. 11.

The following lemma will be used in the proof of optimality of the above construction. The proof is straightforward, and is omitted.

\(^4\)The usual definition of linear codes does not necessarily imply linearity of repair operation. So we take linearity of repair as an extra assumption.
Lemma 5.2: Let \( \{C_i, 1 \leq i \leq s, s \geq 1\} \) denote identical \((n, k, d)(\alpha, \beta)\) classical linear exact repair regenerating codes. Define a new \( n \)-length array code \( C \) over \( \mathbb{F}_q^\alpha \) as \( C = \sum_{i=1}^{s} a_i C_i, a_i \in \mathbb{F}_q \). By this, it is meant that the \( j^{th} \) vector-symbol of \( C \) is given by \( c_j = \sum_{i=1}^{s} c_{i,j}, 1 \leq j \leq n \), where \( c_{i,j} \) denotes the \( j^{th} \) vector-symbol of \( C_i \). We assume that not every \( a_i, 1 \leq i \leq s \) is 0. Then, the code \( C \) is an \((n, k, d)(\alpha, \beta)\) classical linear exact repair regenerating code over \( \mathbb{F}_q \), and is identical to \( C_i, 1 \leq i \leq a \).

We are now ready to prove optimality property of Construction 5.1.

Theorem 5.3: The code described in Construction 5.1 is an optimal exact repair generalized regenerating code, for any \( m, \ell \leq m \). The optimal code can be constructed whenever an optimal \((n, k, d') = \min(d, k))(\alpha, \beta)\) exact repair linear regenerating code exists, having a file size \( B' = \sum_{i=0}^{k-1} \min(\alpha, (d' - i)\beta) \).

Proof: It is clear that the code in Construction 5.1 has a file size \( B^*_m \), where \( B^*_m \) is as given in Theorem 4.1. Further, the data collection property of the code is also straightforward to check, and this essentially follows from the facts that 1) the matrix \( A \) is invertible, and 2) each of the codes \( C_i, 1 \leq i \leq m \) is uniquely decodable given its coded data belonging to any \( k \) clusters. Towards examining the repair property of the code let us rewrite (17) as follows:

\[
[c'_1 \ c'_2 \ \cdots \ c'_m] = [c_1 \ c_2 \ \cdots \ c_m] A_{m \times m}
\]

(18)

Further, the data collection property of the code is also straightforward to check, and this essentially follows from the facts that 1) the matrix \( A \) is invertible, and 2) each of the codes \( C_i, 1 \leq i \leq m \) is uniquely decodable given its coded data belonging to any \( k \) clusters. Towards examining the repair property of the code let us rewrite (17) as follows:

\[
[c'_1 \ c'_2 \ \cdots \ c'_m] = [c_1 \ c_2 \ \cdots \ c_m] A_{m \times m}
\]

(19)

Further, the data collection property of the code is also straightforward to check, and this essentially follows from the facts that 1) the matrix \( A \) is invertible, and 2) each of the codes \( C_i, 1 \leq i \leq m \) is uniquely decodable given its coded data belonging to any \( k \) clusters. Towards examining the repair property of the code let us rewrite (17) as follows:

\[
[c'_1 \ c'_2 \ \cdots \ c'_m] = [c_1 \ c_2 \ \cdots \ c_m] A_{m \times m}
\]

(20)

where \( C_{MDS} = [c_1 \ \cdots \ c_\ell] \) and \( C_{regen} = [c_{\ell+1} \ \cdots \ c_m] \). The matrices \( c_{MDS}^{(i)} \) and \( c_{regen}^{(i)} \), \( 1 \leq i \leq n \) denote rows \((i - 1)\alpha + 1, \ldots, i\alpha \) of \( C_{MDS} \) and \( C_{regen} \), respectively. Let us also expand the decomposition of matrix \( A \) in (16) further as follows:

\[
A_{m \times m} = E_{\ell \times m}
\]

(1)

\[
= E_{\ell \times m}
\]

(21)

where \( e_j^T \) and \( f_j^T, 1 \leq j \leq m \) denote the \( j^{th} \) column of the matrices \( E \) and \( F \), respectively. Based on (20) and (22), it can be seen that the content of node \( j \) in cluster \( i \) is given by

\[
\begin{bmatrix}
    c_{MDS}^{(i)} \\
    c_{regen}^{(i)}
\end{bmatrix} \begin{bmatrix}
    e_j^T \\
    f_j^T
\end{bmatrix}
\]

(23)

Given the notation above, without loss of generality, consider repairing node \( \ell + 1 \) in cluster 1 with the help of 1) the first \( \ell \) local nodes in cluster 1 and 2) clusters \( 2, \ldots, d' + 1 \). Let us first examine the role of the \( \ell \) local nodes in the repair process. Let \( E' \) and \( F' \) denote the first \( \ell \) columns of \( E \) and \( F \), respectively. By assumption, \( E \) generates an \([m, \ell]\) MDS code, and hence the submatrix \( E' \) is invertible. In this case, the content from the \( \ell \) local nodes can be put together to generate

\[
\begin{bmatrix}
    e_1^{(1)} \\
    c_{regen}^{(1)}
\end{bmatrix} \begin{bmatrix}
    E' \\
    F'
\end{bmatrix} E'^{-1} e_{\ell+1} = \begin{bmatrix}
    e_1^{(1)} \\
    c_{regen}^{(1)}
\end{bmatrix} \begin{bmatrix}
    e_{\ell+1} \\
    \hat{e}_{\ell+1}^T
\end{bmatrix},
\]

(24)

where \( \hat{e}_{\ell+1}' = F'E'^{-1}e_{\ell+1}' \). Thus, the local helper nodes serve to recover the part corresponding to the MDS-codes’ components; however the process generates interference corresponding to the regenerating-codes’ components.
Let us next examine the role of the $d'$ helper clusters. We know that the data stored in cluster $i$ is given by $c_{regen}^i$ since matrix $A$ is invertible, the vector $c_{regen}^i$ can be recovered from this. We also know, from Lemma 5.2, that the code $\hat{C}_i = \sum_{j=1}^{m-\ell} (f_j - \hat{f}_j)C_{\ell+j}$ is an $(n, k, d')(\alpha, \beta)$ linear regenerating code, which is identical to $C_j, \ell + 1 \leq j \leq m$. Thus, cluster $i, 2 \leq i \leq d'+1$ generates and passes the helper data corresponding to the first vector symbol, for the code $\hat{C}_i$. The replacement node regenerates $c_{regen}^{(1)}(f_T^j - \hat{f}_j^T)$ using the helper data from the $d'$ external clusters, and combines it with the local helper data (see (24)) to restore the content of the lost node. A pictorial illustration of the repair process is shown in Fig. 12.

**B. A Functional Repair Code for Arbitrary number of Failures**

In this section, we will show the existence of optimal functional repair codes over a finite field that can tolerate an unbounded number of failures and repairs. We show the existence for any $(n, k, d)(\alpha, \beta)$, $m, \ell = m - 1$ The code construction is identical to the one used in the example in Section I-D, and combines $m-1$ MDS array codes $C_1, \ldots, C_{m-1}$ with a $(n, k, d)(\alpha, \beta)$ functional repair code $C_m$ for the classical setting. The code $C_m$ is one which can tolerate an unbounded number of failures and repairs, whose existence over a finite field is shown in [18]. The following lemma is a direct consequence of Theorem 3 and the description in Section V of [18], and guarantees the existence of the code $C_m$ that we use here.

**Lemma 5.4:** For any $(n, k, d)(\alpha, \beta)$, there exists an optimal deterministic classical functional repair linear regenerating code over $F_q$ that can handle an unbounded number of repairs, whenever $q > q_0$, where $q_0$ is entirely determined by the parameters $(n, k, d)(\alpha, \beta)$, and is independent of the number of repairs performed over the lifetime of the code.

While Theorem 3, [18] shows the existence of codes as in the above lemma via a randomized procedure, in Section V, [18] the authors describe how to choose the coding coefficients in a deterministic way. The fact the code is deterministic ensures that every repair process is also deterministic, i.e., the regenerated data is uniquely determined given the content of the helper nodes. As we will see, this feature of the code $C_m$ is essential to ensure the data-collection property of our functional repair construction.

Below, we first describe the code construction, along with the repair procedure, and then show the optimality property of the code. In the following construction, we assume that the characteristic of the finite field $F_q$ is 2; i.e., $q = 2^w$ for some $w > 0$. The assumption is made only for the ease of description of the construction, the construction can be easily modified to accommodate finite fields of any characteristic.

**Construction 5.5:** Let $C_j, 1 \leq j \leq m-1$ denote $[n, k]$ MDS array codes over $F_q$. The amount of data that can be encoded with these $m$ codes is $(m-1)k\alpha$. Next, let $C_m$ denote an $(n, k, d' = \min(d, k))(\alpha, \beta)$ classical functional
repair linear regenerating code whose existence is guaranteed by Lemma 5.4. The code $C_m$ has a file size $B' = \sum_{i=0}^{k-1} \min(\alpha, (d'-i)\beta)$. For encoding, we first divide the data file of size $B'_m = \ell k\alpha + (m-\ell)B'$ into $m$ stripes, such that first $m-1$ have size $k\alpha$, and the last one has size $B'$. Stripe $j, 1 \leq j \leq m$ is encoded by $C_j$ to generate the coded symbols $\hat{c}_j = [c_{1,j}, c_{2,j}, \ldots, c_{\alpha_0,j}]^T$. The arrangement of coded data in the various nodes is same as in the example in Section I-D. Thus, node $j, 1 \leq j \leq m-1$ in cluster $i$ stores the vector $[c_{(i-1)\alpha+1,j}, c_{(i-1)\alpha+2,j}, \ldots, c_{\alpha_0,j}]^T$. Node $m$ in cluster $i$ stores the sum of all the $m-1$ other nodes with symbols $[c_{(i-1)\alpha+1,m}, c_{(i-1)\alpha+2,m}, \ldots, c_{\alpha_0,m}]^T$ of the regenerating code $C_m$, i.e., content of node $m$ is given by $\sum_{j=1}^{m} c_{(i-1)\alpha+1,j}, \sum_{j=1}^{m} c_{(i-1)\alpha+2,j}, \ldots, \sum_{j=1}^{m} c_{\alpha_0,j}]^T$.

This completes the description of the initial layout of the coded data. Since the code is a functional repair code, the code description is not complete unless we specify the procedure for node repair, as well. We do this next.

**Node Repair:** Let $Y_{i,j}(t) \in \mathbb{F}_q^m$ denote the content of node $j$ in cluster $i$, after the $t$th repair, $t \geq 0$, in the system. The quantities $\{Y_{i,j}(0), 1 \leq i \leq n, 1 \leq j \leq m\}$ denote the initial content present in the system, and is as described above. Since we assume that repairs are performed one at a time, there is at most one pair $(i, j)$ such that $Y_{i,j}(t) \neq Y_{i,j}(t+1), t \geq 0$. The repair procedure is such that the vector $[\sum_{j=1}^{m} Y_{1,j}(t), \sum_{j=1}^{m} Y_{2,j}(t), \ldots, \sum_{j=1}^{m} Y_{n,j}(t)]^T$ remains as a valid codeword of the functional repair regenerating code $C_m$, for every $t \geq 0$ (to be proved in Theorem 5.6). Clearly, the above statement is true for $t = 0$. The repair procedure can be described recursively as follows:

Let the $t$th repair be associated with node $i'$ in cluster $j'$. Each of the $d'$ helper clusters, say $i$, internally computes $\sum_{j=1}^{m} Y_{i,j}(t-1)$, and passes the $\beta$ symbols toward the repair of $\sum_{j=1}^{m} Y_{i,j}(t-1)$. The replacement node first all of regenerates $\hat{Y}_{i'}(t-1)$, as the replacement of $\sum_{j=1}^{m} Y_{i,j}(t-1)$, given the helper data from the $d$ external clusters. Next, since $\ell = m-1$, the replacement node gets access to local helper data $\{Y_{i',j}(t-1), 1 \leq j \leq m, j \neq j'\}$. The content that is eventually stored in the replacement node is computed as follows:

$$Y_{i',j'}(t) = \sum_{j=1, j \neq j'}^{m} Y_{i',j}(t-1) + \hat{Y}_{i'}(t-1).$$

(25)

Also, for any $(i, j) \neq (i', j')$, we assume that

$$Y_{i,j}(t) = Y_{i,j}(t-1).$$

(26)

This completes the description of the repair process, and hence the code construction as well.

In the following theorem, we argue the optimality property of the above construction. Specifically, we show that the code retains the functional repair and data collection properties, after every repair. We assume that the data collector is aware of the entire repair-history of the system. By this we mean that the data collector is aware of

1) the exact sequence of $t$ failures and repairs that has happened in the system, and
2) the indices of the helper clusters that aided in each of the $t$ repairs.

**Theorem 5.6:** The code described in Construction 5.5 is an optimal $(n, k, \ell)(\alpha, \beta)(m, \ell = m-1)$ functional repair generalized regenerating code.

**Proof:** It is clear that the code in Construction 5.5 has a file-size $B'_m$, as given by Theorem 4.1. Toward showing that the code retains functional repair property, it is sufficient if we show that the vector $[\sum_{j=1}^{m} Y_{1,j}(t), \sum_{j=1}^{m} Y_{2,j}(t), \ldots, \sum_{j=1}^{m} Y_{n,j}(t)]^T$ remains as a valid codeword of the functional repair regenerating code $C_m$, for every $t \geq 0$. We do this inductively. Clearly, the statement is true for $t = 0$. Let us next assume that the statement is true for $t-1 \geq 1$, and show its validity for $t$. Assume that the $t$th repair is associated with node $j'$ in cluster $i'$. The content of the various nodes before and after the $t$th repair are related as in (25) and (26). In this case, the quantities $\{\sum_{j=1}^{m} Y_{i,j}(t), 1 \leq i \leq n\}$ are given by

$$\sum_{j=1}^{m} Y_{i',j}(t) \overset{(a)}{=} \hat{Y}_{i'}(t-1),$$

(27)

$$\sum_{j=1}^{m} Y_{i,j}(t) = \sum_{j=1}^{m} Y_{i,j}(t-1), 1 \leq i \leq n, i \neq i',$$

(28)

where $(a)$ follows from our assumption that the finite field $\mathbb{F}_q$ has characteristic 2. Now, recall that $\hat{Y}_{i'}(t-1)$ is the replacement of $\sum_{j=1}^{m} Y_{i',j}(t-1)$, which is regenerated using the helper data generated using $d$ elements of the
set \( \{ \sum_{j=1}^{m} Y_{i,j}(t-1), 1 \leq i \leq n, i \neq i' \} \). Combining with the induction hypothesis for \( t-1 \), it follows that the statement holds good for \( t \) as well. This completes the proof of functional repair property of the code.

Let us next see how data collection is accomplished after \( t, t \geq 0 \) repairs in the system. Without loss of generality assume that a data collector (DC) connects to clusters \( 1, 2, \ldots, k \), and accesses \( \{ Y_{i,j}(t), 1 \leq i \leq k, 1 \leq j \leq m \} \). The DC as a first step computes the vector \( \{ \sum_{j=1}^{m} Y_{1,j}(t), \sum_{j=1}^{m} Y_{2,j}(t), \ldots, \sum_{j=1}^{m} Y_{k,j}(t) \}^{T} \), and uses this to decode the data corresponding to the code \( C_m \). Now, recall our assumptions that 1) the code \( C_m \) is deterministic, and 2) the DC is aware of the entire repair-history of the system. In this case, having decoded \( C_m \), using (25) and (26), the DC can iteratively recover \( \{ Y_{i,j}(t'), 1 \leq k \leq 1 \leq j \leq m \} \), for \( t \geq t' \geq 0 \) by starting at \( t' = t \) and proceeding backwards until the content at \( t' = 0 \) is recovered. Finally, from Construction 5.5, we know that the content \( \{ Y_{i,j}(0), 1 \leq k \leq 1 \leq j \leq m-1 \} \) is precisely the stacked coded data corresponding to the \( m-1 \) \([n,k]\) MDS code \( C_1, \ldots, C_{m-1} \), and thus these codes can also be decoded. This completes the proof of data-collection,

and also the theorem.

 VI. Intra-cluster Bandwidth under Optimal Trade-off

We now turn our attention to calculate the amount of intra-cluster repair bandwidth that is needed for a \((n, k, d)(\alpha, \beta)(m, \ell)\) code \( C_m \) to have optimal file-size \( B_m^* \), given by Theorem 4.1. As discussed in Section I-A, there are two contributors to intra-cluster repair bandwidth: 1) the local helper bandwidth \( \gamma \), which is amount of data that each of the local helper nodes contributes to repair, and 2) the external helper bandwidth \( \gamma' \), which is amount of the data that each of the \( \ell' \) nodes of a helper cluster contribute toward computing the \( \beta \) symbols of the helper cluster. In this section, we will individually study the minimum requirements on the parameters \( \gamma \), \( \gamma' \) and \( \ell' \). For obtaining lower bound on \( \gamma \), we continue to work with the IFG model in Section III-A, except for the fact that links that connect the local helper out-nodes to the in-node of the replacement node, will have a capacity \( \gamma \), instead of \( \alpha \). The IFG model in Section III-B will be used when we compute lower bound on \( \gamma' \) and \( \ell' \). We also will prove the tightness of the bound on \( \gamma \) via a converse; however, no such converse is known to us regarding the bound on \( \gamma' \). We note that while computing the bound on \( \gamma \), we ignore the effects of limited \( \gamma' \) and \( \ell' \) (and vice versa), i.e., we assume that \( \ell' = m \) and \( \gamma' = \alpha \). Further, the bounds on \( \gamma \) and \( \gamma' \) (or \( \ell' \)) are obtained under the assumptions that \( d > 0 \) and \( d \geq k \), respectively.

A. Bound on Local Helper Node Intra-cluster Repair Bandwidth, \( \gamma \)

**Theorem 6.1:** For an optimal functional repair generalized regenerating code with parameters \( \{(n,k,d > 0), (\alpha, \beta), (m, \ell)\} \), \( \gamma' = \alpha, \ell' = m \), local helper node bandwidth \( \gamma \) is lower-bounded by

\[
\gamma \geq \gamma^* \triangleq \alpha - (d-k+1)^+\beta.
\] (29)
Further, if there is a known upper bound on the number of repairs that occur over the life-time of the system, the above bound is sharp; i.e., the functional repair capacity of the system remains as $B_m^*$ as long as $\gamma \geq \gamma^*$.

Proof: For the bound, we consider a system evolution similar to that used in proof of Theorem 4.1, and demonstrate a cut-set whose value depends on $\gamma$. The lower bound on $\gamma$ follows from the observation that the value of this cut is necessarily lower bounded by $B_m^*$ for a capacity-achieving code. We will then prove that as long as $\gamma \geq \gamma^*$, the min-cut of any valid IFG is necessarily lower bounded by $B_m^*$; in this case, like in the proof of Theorem 4.1, we know that the functional repair capacity remains as $B_m^*$, as long as there is a known upper bound on the number of repairs in the system. We start with the proof of bound. Consider the same system evolution as in the proof of the bound in Theorem 4.1, except for the $k$-th cluster accessed by the data collector. Thus, physical nodes $X_{i,t+1},X_{i,t+2},\ldots,X_{i,m}$ fail in this order in cluster $i=1$, then in cluster $i=2$, and so on, until cluster $i=k-1$. Note that each of the first $k-1$ clusters experiences a total of $m-\ell$ node failures. For cluster $k$, we consider failure of $m-\ell+1$ nodes, corresponding to physical nodes $X_{k,t+1},X_{k,t+2},\ldots,X_{k,m},X_{k,1}$ in this respective order. In terms of the notation introduced in III-A, the sequence of failures in the $k$-th cluster correspond to IFG nodes $X_{k,t+1}(0),X_{k,t+2}(1),\ldots,X_{k,m}(m-\ell),X_{k,1}(m-\ell+1)$. For the repair of $X_{k,t+1}(0)$, the local helper nodes used are $X_{k,1}(0),X_{k,2}(0),\ldots,X_{k,t}(0)$. For the repair of any of the remaining nodes $X_{k,(t+\ell) mod (m-1)}(t),1 \leq t \leq m-\ell$, the local helper nodes used are $X_{k,2}(t),X_{k,3}(t),\ldots,X_{k,t}(t)$. Also, clusters $X_{k,m}(m-\ell),X_{k,2}(m-\ell),\ldots,X_{k,\min(d,k-1)}(m-\ell)$ are included in the set of external clusters that aid in the repair of the $m-\ell+1$ nodes in the $k$-th cluster. An illustration of the IFG, for the $k$-th cluster is shown in Fig. 13. Note in this figure that the edges corresponding to local help have capacity $\gamma$.

Let data collector $T$ connect to clusters $X_{k,m}(m-\ell),\ldots,X_{k-1,m}(m-\ell),X_k(m-\ell+1)$. Consider an $S$-$T$ cut in the IFG that partitions the graph nodes in clusters $1,\ldots,k$ in the same way as in 4.1; however differs in way the nodes of cluster $k$ are partitioned. The overall set of edges in the cut-set is as given below:

$\text{Cluster 1, } \ldots, k-1$:
- \{(\text{in}^{i,j}\rightarrow \text{out}^{i,j},i \in [k-1],j \in [\ell])\}. Total capacity of these edges is $(k-1)\alpha$.
- For each $i \in [k-1],\ t \in [m-\ell]$, either the set of edges $\{(\text{in}^{i,t}\rightarrow \text{out}^{i,t'},i' \in \text{helper cluster indices for the replacement node X}_{i,t'}^{\text{in}}(t))\}$ or the edge $\{(\text{in}^{i,t'}\rightarrow \text{out}^{i,t},t' \in \text{helper cluster indices for the replacement node X}_{i,t'}^{\text{in}}(t))\}$. Among the two possibilities, we pick the one which has smaller sum-capacity. In this case, the total capacity of this part of the cut is given by $\sum_{i=1}^{k-1}\sum_{j=\ell+1}^{\ell+1} \min \{\alpha, (d - \min\{i-1, d\}) \} = (m-\ell)\sum_{i=1}^{k-1} \min \{\alpha, (d-i+1)^+\}$.

$\text{Cluster } k$:
- $\{(\text{out}^{i,j}\rightarrow \text{in}^{i,j},j \in [2,\ell])$. Total capacity of these edges is $(\ell-1)\alpha$.
- Either the set of edges $\{(\text{in}^{i,t}\rightarrow \text{out}^{i,t'},i' \in \text{helper cluster indices for the replacement node X}_{i,t'}^{\text{in}}(t)\}$ or the set of edges $\{(\text{in}^{i,t}\rightarrow \text{out}^{i,t'},i' \in \text{helper cluster indices for the replacement node X}_{i,t'}^{\text{in}}(t)\} \circ \{\min\{i-1, d\} \leq t \leq (m-\ell)\}$. Among the two sets, we pick the one which has smaller sum-capacity. In this case, the total capacity of these edges is $(m-\ell+1)\min\{\alpha, (d-k+1)\}$.

The value (say, $C_{\text{cut}}$) of the cut is given by

$$C_{\text{cut}} = (k-1)\ell\alpha + (m-\ell)\sum_{i=0}^{k-2} \min \{\alpha, (d-i)^+\} + \gamma + (\ell-1)\alpha + (m-l+1)\min \{\alpha, (d-k+1)^+\}$$

$$= k\ell\alpha + (m-\ell)\sum_{i=0}^{k-1} \min \{\alpha, (d-i)^+\} - \alpha + \min \{\alpha, (d-k+1)^+\} + \gamma$$

$$= B_m^* - \alpha + \min \{\alpha, (d-k+1)^+\} + \gamma.$$
proof, except with the change that the edges corresponding to local help have capacity $\gamma$. In this case, it can be seen that instead of (12), we get the following lower bound on min-cut:

$$\text{mincut}(S - T) \geq \sum_{i=1}^{k} \left( a_i \alpha + \sum_{j=a_i+1}^{m} \min(\alpha, (\ell - j + 1)^+ \gamma + (d - (i - 1))^+ \beta) \right)$$  \hspace{1cm} (30)

In the above expression, observe that if $\gamma \geq \gamma^*$, we have

$$(\ell - j + 1)^+ \gamma + (d - (i - 1))^+ \beta \geq \alpha,$$  \hspace{1cm} (31)

whenever $j \leq \ell, i \leq k$. In this case, it follows that (30) can be written as (13). It is then clear that mincut$(S - T)$ is indeed lower bounded by $B^*_m$ as long as $\gamma \geq \gamma^*$. This completes the proof of the converse, and also the theorem.

**B. Bounds on $\ell', \gamma'$**

In this section, we provide bounds on the parameters $\gamma'$ and $\ell'$. We use the second IFG model in Section III-B here. Recall that in our setting, for any of the helper clusters, we allow any subset of $\ell'$ nodes in the helper cluster to be used to generate the $\beta$ symbols contributed by the cluster. Also, in this section, we make the assumption that the number of helper clusters $d \geq k$.

**Theorem 6.2:** For an optimal functional repair generalized regenerating code with parameters $\{(n, k, d \geq k) \ (\alpha, \beta), (m, \ell)\}$, $\gamma = \alpha, \ell' = m$, the external helper-node repair bandwidth $\gamma'$ is lower-bounded by

$$\gamma' \geq \beta/(m - \ell),$$  \hspace{1cm} (32)

whenever $\alpha \geq (d - k + 2)\beta$.

**Proof:** We consider data collection from clusters 1 to $k$. Before data collection, the system experiences $k(m - \ell)$ repairs. Nodes $\ell + 1, \ldots, m$ fail and get repaired in cluster 1 in this respective order. This is followed by failure and repair of nodes $\ell + 1, \ldots, m$ in cluster 2. We proceed like this until we consider repairs in cluster $k$. In terms of physical nodes, it may be noted that this is the same sequence of failures that was considered in the proof of Theorem 4.1; however, in here, we will impose additional restrictions on the choice of the external helper clusters.
External help is taken from the set of the first $d + 1$ clusters, excluding the cluster that failed. Thus, for the repair of $X_{i,j}$, the indices of helper clusters are $\{1, \ldots, i - 1, i + 1, \ldots, k, k + 1, \ldots, d + 1\}$. The choice of local helper nodes remain same as in the proof of Theorem 4.1, where we used the first $\ell$ nodes in the cluster. An illustration of the IFG is shown in Figure 14.

It can be seen that the following cut-set separates the source from the data collector:

- $\{(X_{i,j}^{in} \rightarrow X_{i,j}^{out}), i \in [k], j \in [\ell]\}$. Total capacity of these edges is $k\ell\alpha$.
- For each $i, 1 \leq i \leq k$, the edge set with smaller capacity out of $A_1(i) \cup A_2(i)$ and $A_3(i)$ where
  - $A_1(i) = \{(X_{i,j}^{ext} \rightarrow \hat{X}_{i,j}^{in}), i' \in [k + 1, d + 1], j' \in [\ell + 1, m]\}$. Total capacity of edges in $A_1(i)$ is $(d - k + 1)(m - \ell)\beta$
  - $A_2(i) = \{(X_{i,j}^{out} \rightarrow X_{i,j}^{ext}), j \in [\ell + 1, m], i' \in [i + 1, k], j' \in [\ell + 1, m]\}$. Total capacity of edges in $A_2(i)$ is $(m - \ell)(k - i)(m - \ell)\gamma'$
  - $A_3(i) = \{(\hat{X}_{i,j}^{in} \rightarrow \hat{X}_{i,j}^{out}), j \in [\ell + 1, m]\}$. Total capacity of edges in $A_3(i)$ is $(m - \ell)\alpha$.

The capacity of the cut-set is given by

$$C_{\text{cut}} = k\ell\alpha + (m - \ell) \sum_{i=1}^{k} \min\{\alpha, (d - k + 1)\beta + (k - i)(m - \ell)\gamma'\}. \quad (34)$$

Since we consider optimal codes, we have

$$C_{\text{cut}} \geq B_m^* = k\ell\alpha + (m - \ell) \sum_{i=1}^{k} \min\{\alpha, (d - i + 1)\beta\}. \quad (35)$$

In this case, since we assume that $\alpha \geq (d - k + 2)\beta$, we claim that

$$\gamma' \geq \beta/(m - \ell). \quad (36)$$

To see why (36) is true, if we suppose on the contrary that $(m - \ell)\gamma' < \beta$, we have

$$(d - k + 1)\beta + (k - i)(m - \ell)\gamma' < (d - i + 1)\beta, \quad 1 \leq i \leq k - 1. \quad (37)$$

The above equation implies that

$$\min\{\alpha, (d - k + 1)\beta + (k - i)(m - \ell)\gamma'\} \leq \min\{\alpha, (d - i + 1)\beta\}, 1 \leq i \leq k - 2 \quad (38)$$
$$\min\{\alpha, (d - k + 1)\beta + (k - i)(m - \ell)\gamma'\} < \min\{\alpha, (d - i + 1)\beta\}, i = k - 1, \quad (39)$$

where (39) follows from the assumption that $\alpha \geq (d - k + 2)\beta$. Clearly, adding up the two corresponding sides (L.H.S. and R.H.S.) of (38) and (39), and comparing them contradicts the fact that $C_{\text{cut}} \geq B_m^*$. Thus, it must be true that $\gamma' \geq \beta/(m - \ell)$, whenever $\alpha \geq (d - k + 2)\beta$.

The following theorem establishes the necessary condition on $\ell'$ for optimal codes. The proof is along the lines of proof of Theorem 6.2, and is omitted.

**Theorem 6.3:** For an optimal functional repair generalized regenerating code with parameters $\{(n, k, d \geq k) (\alpha, \beta), (m, \ell)\}$, $\gamma = \gamma' = \alpha$, whenever $\alpha \geq (d - k + 2)\beta$, each helper cluster must necessarily access all the $m$ nodes in the cluster while generating the $\beta$ symbols; i.e., the parameter $\ell' = m$.

**VII. Security Under Passive Eavesdropping**

In this section, we analyze resiliency of the clustered storage system against passive eavesdropping. Our model of clustered storage systems is in part motivated by the need to provide security against an eavesdropper who may gain access to a subset of the clusters. In this context, we extend result in Theorem 4.1 to settings that demand security. Below, we first introduce the model for security, then present the revised file bound. We will also present optimal code constructions, which are analogous to those in Construction 5.1.
A. Passive Eavesdropper Model

The security model is along the lines of the passive eavesdropper model considered in [19], where authors study security under the classical regenerating code framework. An eavesdropper (say, Eve) gains access to the entire content of any subset of $e$ clusters, where $1 \leq e \leq k$. Eve also gets to observe all the helper data that gets downloaded for repair of any node in these $e$ clusters. Eve is passive in the sense that Eve does not change any stored or repair data. The properties of data collection and disk repair remain same as in the case of no eavesdropper (see Section I-A). In this model we 1) ignore the effects of intra-cluster bandwidth, and 2) restrict ourselves to the setting of deterministic exact repair codes. By deterministic exact repair code, we mean that the helper data for the repair of any node is uniquely determined given the indices of the failed node, local helper nodes and helper clusters. We avoid the possibility that the same set of helpers can pass two possible sets of helper data for the repair of the same node.

We wish to store a file such that Eve does not gain any information about it, by having eavesdropped into any subset $E$ of $e$ clusters. To be precise, let $\mathcal{F}(s)$ denote the random variable corresponding to the data file that gets stored or repair data. We assume the file $\mathcal{F}(s)$ to be uniformly distributed over $\mathbb{F}_q^{B(s)}$, and thus $B(s)$ denotes the file-size. We wish to ensure that the mutual information $\mathcal{I}(\mathcal{F}(s); \text{data observed by Eve}) = 0$. Note that the data observed by Eve not only includes the content of the $E$ clusters, but also any inter-cluster helper data that is received toward the repair of nodes in these clusters. We will write $C_{m}^{(s)}$ to denote a secure generalized regenerating code, and its parameter set will be identified with $\{(n, k, d, (\alpha, \beta), (m, \ell), (e))\}$.

B. File Size Under Exact Repair

In this section, we will obtain an upper bound on the file-size $B(s)$ of the exact repair secure generalized regenerating code $C_{m}^{(s)}$. To derive the bound, we use information theoretic techniques similar to those used in [24], [19]. We begin with useful notation. Let $Y_{i,j} \in \mathbb{F}_q^{n}$, $1 \leq i \leq n$, $1 \leq j \leq m$ denote the content stored in node $j$ of cluster $i$. We write $Y_i$ to denote $[Y_{i,1} \ldots Y_{i,m}]$, $1 \leq i \leq n$. The property of data collection demands that

$$
H\left(\mathcal{F}(s) | \{Y_i, i \in S\}\right) = 0 \quad \forall S \subset [n], |S| = k,
$$

(40)

where $H(.)$ denote the entropy function computed with respect to $\log q$. Next, consider the repair of node $j$ in cluster $i$. Let $H \subset [n]\{i\}$, $|H| = d$, and $L \subset [m]\{j\}$, $|L| = \ell$ respectively denote the indices of helper clusters and local nodes that aid in the repair process. Let $Z_{i,j}^{H,L}$ denote helper data passed by cluster $i'$. Recall our assumption that the exact repair code is deterministic, and thus $Z_{i,j}^{H,L}$ is uniquely determined as a function of $(i,j)$, $Y_{i,j'}, j' \in L$ and $Y_{i'}, i' \in H$. The property of exact repair is jointly characterized by the following set of inequalities:

$$
H\left(Z_{i,j}^{H,L} | Y_{i'}\right) = 0,
$$

(41)

$$
H\left(Z_{i,j}^{H,L}\right) \leq \beta,
$$

(42)

$$
H\left(Y_{i,j} | \{Z_{i,j'}^{H,L}, Y_{i,j'}, i' \in H, j' \in L\}\right) = 0, \quad \forall H \subset [n]\{i\}, |H| = d, \forall L \subset [m]\{j\}, |H| = \ell.
$$

(43)

Next, define $U_i$ to denote the collection of all the inter-cluster helper data ever received toward the repair of nodes in cluster $i$, i.e.,

$$
U_i = \{Z_{i,j}^{H,L} \text{ for all choices of } i', j, H, L\}.
$$

(44)

The property of being secure against the passive eavesdropper Eve is equivalent to saying that

$$
\mathcal{I}(\mathcal{F}(s), \{Y_i, U_i, i \in E\}) = 0, \quad \forall E \subset [n], |E| = e.
$$

(45)

The following theorem characterizes the file-size bound under passive eavesdropping.

**Theorem 7.1:** The file-size of a secure exact-repair deterministic generalized regenerating code having parameters $\{(n, k, d, (\alpha, \beta), (m, \ell), (e))\}$ is upper bounded by

$$
B_m^{(s)} \leq \ell(k - e)\alpha + (m - \ell) \sum_{i=e}^{k-1} \min\{\alpha, (d - i) + \beta\}.
$$

(46)
\textbf{Proof:} Without loss of generality, let us assume that \( E = \{1, 2, \ldots, e\} \). Using (45), the file-size \( B^{(s)} \) is given by

\[
B^{(s)}_m = H(F^{(s)}) = H(F^{(s)}|\{Y_i, U_i, i \in E\}) \leq H(F^{(s)}|\{Y_i, i \in E\}).
\]

From (40), we further get that

\[
H(F^{(s)}|\{Y_i, i \in E\}) \leq H(\{Y_i, i \in \{e+1, \ldots, k\}\}|\{Y_i, i \in E\}).
\]

Combining (47) and (48), we get

\[
B^{(s)}_m \leq H(\{Y_i, i \in \{e+1, \ldots, k\}\}|\{Y_i, i \in E\})
\]

\[
= \sum_{i=e+1}^{k} H(Y_i|Y_1, \ldots, X_{i-1})
\]

\[
= \sum_{i=e+1}^{k} \sum_{j=1}^{m} H(Y_{i,j}|Y_{i,1}, \ldots, Y_{i,j-1}, Y_1, \ldots, Y_{i-1})
\]

\[
= \sum_{i=e+1}^{k} \sum_{j=1}^{\ell} H(Y_{i,j}|Y_{i,1}, \ldots, Y_{i,j-1}, Y_1, \ldots, Y_{i-1}) + \sum_{i=e+1}^{k} \sum_{j=\ell+1}^{m} H(Y_{i,j}|Y_{i,1}, \ldots, Y_{i,j-1}, Y_1, \ldots, Y_{i-1})
\]

\[
\leq \ell(k-e)\alpha + \sum_{i=e+1}^{k} \sum_{j=1}^{m} H(Y_{i,j}|Y_{i,1}, \ldots, Y_{i,j-1}, Y_1, \ldots, Y_{i-1})
\]

\[
\leq \ell(k-e)\alpha + \sum_{i=e+1}^{k} \sum_{j=\ell+1}^{m} \min(\alpha, (d-(i-1))^{+}\beta)
\]

\[
= \ell(k-e)\alpha + (m-\ell) \sum_{i=e}^{k-1} \min\{\alpha, (d-i)^{+}\beta\},
\]

where (54) follows from (41)-(43). This completes the proof of the upper bound. \( \blacksquare \)

As before, whenever \( d > 0 \), we associate the operating point \( \alpha = d\beta \) with the minimum repair bandwidth secure regenerating codes. Construction 5.1 can be easily adapted to construct optimal secure exact repair codes at the MBR point. In the modified construction, one combines \( \ell \) secure MDS codes for the wiretap-II channel [20] [21], and \( (m-\ell) \) classical exact-repair secure MBR codes [19] [22]. The construction and proof of optimality are very similar to Construction 5.1; we avoid a full description here. A pictorial illustration of the secure code construction appears in Fig. 15.

\section{VIII. Conclusions}

To conclude, we studied the problem of storage-overhead vs repair-bandwidth overhead in clustered storage systems. The notion of clustering is used in both data collection and node repair. For data-collection, we demand retrievability using content from any set of \( k \) clusters. For node repair, we take the help of surviving local nodes in the host cluster, as well as from other external clusters. We first characterized the optimal file-size that is achievable while ignoring intra-cluster bandwidth costs, and then obtained bounds on intra-cluster bandwidth costs that is needed to achieve this file-size. Our results show that while it is beneficial to increase the number of local helper nodes \( (\ell) \) during repair in order to simultaneously improve both storage and inter-cluster bandwidth costs, increasing \( \ell \) has an adverse effect on intra-cluster repair bandwidth. Our bounds on file-size and intra-cluster bandwidth give guidelines for choosing the desired number of local helper nodes in practice, based on the relative costs of the various metrics. We presented constructions of optimal exact and functional repair codes, and they enable operating points for clustered systems which are not achievable via previously known coding solutions. We also analyzed the resiliency of the system against passive eavesdropping.
Fig. 15. Illustration of the exact repair secure code construction. We first stack \( \ell \) secure-MDS codes for the wiretap-II channel, and \((m - \ell)\) classical secure MBR codes, and then transform each row via the invertible matrix \( A \). The first \( \ell \) rows of the matrix \( A \) generates an \([m, \ell]\) MDS code.

Two key questions remain at the end of this work. Firstly, our bounds on intra-cluster bandwidth were derived under the assumption of functional repair. It is unclear if these bounds hold under exact repair; specifically at the minimum-inter-cluster bandwidth (MBR) operating point. The exact repair constructions in this paper, though have optimal file-size (and inter-cluster bandwidth), incur the maximum possible intra-cluster bandwidth. Secondly, the bound on any one of the intra-cluster bandwidth related parameters (say, \( \gamma \)) was derived without limiting the other two other parameters (\( \gamma', \ell' \)). It is of special interest to know if the bounds in (2) and (3) hold simultaneously. We believe that a first step in this direction would be to prove a converse statement (achievability) to (3). Achievability of (3) is indeed suggested by RLNC-based simulations.

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