Gravitating self-dual fields in New Massive Gravity

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A class of three dimensional contact manifolds is shown to describe certain types of superconductors. specializing to a particular para-Sasakian metric on the three-sphere shows it to be a vacuum solution of three-dimensional massive gravity. The geometry is Lorentzian and the manifold enjoys local isometry given by the Heisenberg algebra. The background is shown to support self-gravitating force-free electromagnetic fields of Maxwell-Chern-Simons theory or Abelian Higgs theory.

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INTRODUCTION

There is a long tradition on applications of geometrical methods in physics; over the years branches as Hamiltonian dynamics, geometric optics, fluid dynamics and General Relativity have benefited from the techniques developed with a geometric perspective [1–6]. In the case of pseudo-Riemannian manifolds, this becomes even more true with the use of para-contact geometry [7–12]. More specifically, para-Sasakian geometry has paved its way into General Relativity as a tool to analyze Ricci solitons, lightlike hypersurfaces, Killing vectors and associated horizons [13–18]. In this Letter, we discuss further implications of para-Sasakian structures within the realm of New Massive Gravity (NMG) [19]. In particular, we show that the structure gives rise to a distinguished Trkalian flow; a special type of Beltrami flow.

Beltrami fields where originally introduced in the realm of hydrodynamics to describe flows whose stream lines are parallel to their vorticity. In the case of electromagnetism, these have received the name of force-free magnetic fields. That is, magnetic fields such that the induced currents in a conducting medium experience a vanishing Lorentz force. This feature is, indeed, a property of the medium and it is described by London’s constitutive relations. In this letter we show that such relations are completely captured by the geometry of a class of three dimensional metric contact manifolds whose structural elements give rise to propagating force-free fields. In particular, we study the case of \(S^3\) endowed with a contact structure together with an associated metric such that the generators of the contact distribution together with the Reeb vector field generate a null triad for a \((2+1)\) spacetime. We explicitly obtain the conditions for the metric to be a solution to New Massive Gravity for the vacuum, Maxwell-Chern-Simons and Abelian-Higgs cases.

BELTRAMI FIELDS AND SUPERCONDUCTORS

Let us begin by considering a 3-dimensional manifold endowed with a \(\varepsilon\)-contact metric structure [20]. That is, a contact metric manifold such that the contact 1-form \(\eta\) and the metric \(g\) satisfy the relation

\[ \ast \eta = -\ell d\eta, \]  

(1)

where \(\ast\) is the Hodge star associated with a metric \(\tilde{g}\) in the conformal class defined by \(\tilde{g} = \ell^2 g\) with \(\ell \neq 0\). Here, without loss of generality, we have written (1) for Lorentzian para-Sasakian geometries such as the one in equation (13).

To each contact 1-form \(\eta\) there is a distinguish vector field \(\xi\) defined by the conditions \(\eta(\xi) = 0\) and \(i_\xi d\eta = 0\). If the metric \(g\) satisfies the condition \(g(\xi) = \eta\), then the field \(\xi\) is a Beltrami field, namely \(\xi = \ell \text{curl}(\xi)\).

It is straightforward to verify that

\[ (\delta d - 1/\ell^2)\eta = 0, \quad \text{and} \quad \delta \eta = 0. \]  

(2)

The first equation is reminiscent of the sourceless Proca field equation, in which case \(A = \eta\). Then, the second of the two equations above is the Lorenz gauge. This interpretation means that equation (1) is a sort of “square root” of the Proca field equation. The first time this concept was explored was in reference [21]. Therein, the concept of self-duality for gauge fields was extended to odd dimensions. In three dimensions self-duality is defined by

\[ \ast F = \frac{1}{\ell} A, \]  

(3)

but this is merely equation (1) under our gauge field interpretation. Moreover, equations (2) allow us to write

\[ (\Box - 1/\ell^2) \ast F = 0, \]  

(4)

which is exactly the force-free field equation associated with superconducting media [22, 23]. In our case, an
effective material medium is described by (13). In particular, this provides us with a direct relation between the conformal parameter \( \ell \) and the penetration depth in the medium. Additionally, the inhomogeneous Maxwell equation together with (1) yields the relation

\[
d \star F = J = \frac{1}{\ell} F, \tag{5}\]

where \( J \) is the induced current 2-form in the medium. In this sense, equation (1) corresponds to a constitutive relation for a conductive medium, that is, a relation between the induced current and the field strength characterizing the response of the medium to electromagnetic stimuli. Indeed, (1) is a metric relation between the contact form and its exterior derivative for the para-Sasakian class of contact metric manifolds. Moreover, at this point, we are ready to rewrite equation (1) as

\[
d \star J = \frac{1}{\ell^2} F, \tag{6}\]

which is no other than London’s constitutive relation for superconducting media. This is a remarkable result, since we have not yet used any particular form of the metric. That is, an \( \varepsilon \)-contact 3-manifold represents a superconducting medium.

**CONTACT METRIC STRUCTURE**

Let us consider a topological three-dimensional sphere endowed with its standard contact structure, parametrized by the one-form

\[
\eta = \frac{1}{2} (d\psi + \cos \theta d\phi), \tag{7}\]

where we have used Euler angles \( 0 \leq \psi \leq 4\pi, 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \) to coordinate the manifold. Notice that for the contact sphere \( w = \psi/2, q = \phi/2 \) and \( \sigma = -\cos \theta \) are the set of local coordinates in the Darboux theorem [24]. This is not surprising when one keeps in mind the Hopf fibration of the hypersphere. A contact form \( \eta \) defines a unique vector field \( \xi \) satisfying the conditions \( i_\xi d\eta = 0 \) and \( i_\xi \eta = 1 \) known as the Reeb vector field. Indeed, in the case of \( S^3 \) equipped with the contact form (7) the Reeb field is tangent to the Hopf circle fiber.

A three-dimensional contact structure is a completely non-integrable distribution of two-dimensional planes in the tangent bundle. The generators of the contact distribution are vector fields annihilated by the contact 1-form. Thus, in the present case, these are given by

\[
Q = 2 \left( \frac{\partial}{\partial \phi} - \cos \theta \frac{\partial}{\partial \psi} \right), \quad \text{and} \quad P = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}, \tag{8}\]

which, in particular, show that the contact distribution given by (7) is bracket-generating [25] as we have

\[
[P, Q] = \xi, \quad [\xi, Q] = 0, \quad \text{and} \quad [\xi, P] = 0. \tag{9}\]

This is to say, \( P \) and \( Q \) together with their iterated Lie brackets generate a basis for the tangent bundle. Moreover, equation (9) exhibits the fact that the non-coordinate basis \( \{P, Q, \xi\} \) satisfies the Heisenberg algebra.

There is a certain freedom in choosing a metric associated with a contact structure [26]. In the present work, we consider a metric satisfying

\[
g(\xi, \xi) = 1 \quad \text{and} \quad g(\xi, P) = g(\xi, Q) = 0, \tag{10}\]

together with

\[
g(Q, Q) = g(P, P) = 0. \tag{11}\]

Conditions (10) merely state that the metric is compatible with the contact 1-form, that is, the Reeb vector field is normalized and is orthogonal to the generators of the contact distribution; conditions (11) imply that the metric is Lorentzian. These are the defining properties of an associated metric to the almost para-contact structure

\[
\varphi(\xi) = 0, \quad \varphi(Q) = Q, \quad \text{and} \quad \varphi(P) = -P, \tag{12}\]

representing a reflection in the \( \theta \) direction of the contact distribution. Therefore, \( (S^3, \eta, \xi, \varphi, g) \), is a para-contact manifold. Here, the metric \( g = \eta \otimes \eta - d\eta \odot (\varphi \otimes 1) \) on the contact sphere, whose line element in local coordinates is given by

\[
ds^2 = \frac{1}{4} \left( d\psi^2 + 2 \cos \theta d\psi d\phi - 4 \sin \theta d\theta d\phi + \cos^2 \theta d\phi^2 \right), \tag{13}\]

defines a (2+1) spacetime where \( (P, Q, \xi) \) is its Newmann-Penrose null triad. Moreover, the congruences associated with the null vector fields \( P \) and \( Q \) are geodesics and have no expansion, shear nor twist, therefore, \( g \) defines a Kundt spacetime. Notice that \( \xi \) corresponds to the spacelike vector field \( m \) whilst \( P \) and \( Q \) to the null vector fields \( l \) and \( n \), respectively. From this, it is straightforward to construct the orthonormal triad

\[
\sqrt{2} e_0 = P + Q = -2 \cos \theta \frac{\partial}{\partial \psi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} + 2 \frac{\partial}{\partial \phi}, \tag{14}\]

\[
\sqrt{2} e_1 = P - Q = 2 \cos \theta \frac{\partial}{\partial \psi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} - 2 \frac{\partial}{\partial \phi}, \tag{15}\]

\[
e_2 = \xi = 2 \frac{\partial}{\partial \psi}, \tag{16}\]

so that, using the coframe \( \{e^0, e^1, e^2\} \), the line element of the geometry is expressed as

\[
ds^2 = -e^0 e^0 + e^1 e^1 + e^2 e^2 \]

where the Lorentz signature is manifest. Hence, the geometry is not given by the standard round Riemannian metric. Nor is equation (13) the canonical Lorentz metric on the sphere. Even though we have chosen the standard contact structure on the hypersphere.

The present contact metric sphere is not Einstein, however, it is \( \eta \)-Einstein, that is, the Ricci tensor satisfies

\[
\text{Ric} = \frac{1}{2} g - \eta \otimes \eta. \tag{17}\]
It has been established that three-dimensional $\eta$-Einstein Sasakian manifolds must have constant sectional curvature when restricted to planes in the contact distribution [27]. Furthermore, if the value of this constant is -3 the metric is nil (a.k.a. Heisenberg) [27, 28]. For metric (13) this constant is equal to 3 (when scaled for compatibility with [27]). This suggests to us that the metric is nil (a.k.a. Heisenberg) [27, 28]. For metric (13), the Cotton tensor, see for example [35]. We find the answer to be positive. Moreover, we mention that the Jordan normal form of the metric’s Cotton tensor reveals the spacetime to be of “Petrov” type D [35, 36].

The $\eta$-Einstein nature of the metric greatly simplifies the equations of motion, as second order quantities, e.g.,

$$S[g] = \int d^3x \sqrt{-g} \left( R - 2\Lambda + \beta_1 R^2 + \beta_2 R_{\mu\nu} R^{\mu\nu} \right).$$

We find that the metric is a solution of the theory whenever

$$\Lambda = \frac{1}{8\ell^2} \text{ and } \beta_1 = -\ell^2 - 3\beta_2.$$  

However, Gauss-Bonnet is a ghost-free theory, thus a closer analogy between theories is provided by New Massive Gravity, which is also ghost-free [19]. The action is given by

$$S[g] = \int d^3x \sqrt{-g} \mathcal{L}_{NMG},$$

with

$$\mathcal{L}_{NMG} = \frac{1}{2\kappa^2} \left[ R - 2\Lambda - \frac{1}{m^2} \left( |\text{Ric}|^2 - \frac{3}{8} R^2 \right) \right].$$

Here $\Lambda$ is the cosmological constant and $m$ is the mass of the propagating degrees of freedom. It is a famous result that this theory is equivalent at the linearized level to the (unitary) Fierz-Pauli action for a massive field with spin two. The equations of motion are

$$\text{Ric} - \frac{1}{2} R g + \Lambda g - \frac{1}{2m^2} K = 0$$

where $K$ is a tensor with components

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - R g_{\mu\nu} + 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta} - \frac{3}{2} RR_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}.$$  

It can be directly verified that the metric (13) of our para-Sasakian 3-sphere is a solution to these equations provided that

$$\Lambda = \frac{1}{8\ell^2}, \text{ and } m^2 = 21\Lambda.$$  

One might also wonder if the metric is a solution of the theory when the action is additionally coupled to the Cotton tensor, see for example [35]. We find the answer to be positive. Moreover, we mention that the Jordan normal form of the metric’s Cotton tensor reveals the spacetime to be of “Petrov” type D [35, 36].

Higher dimensional para-Sasakian geometries analogous to (13) have been found to be Einstein-Gauss-Bonnet vacua [34]. However, in three dimensions the quadratic-curvature Gauss-Bonnet term vanishes. For this reason we consider instead the most general quadratic-curvature theory in three dimensions

$$S[g] = \int d^3x \sqrt{-g} \left( R - 2\Lambda + \beta_1 R^2 + \beta_2 R_{\mu\nu} R^{\mu\nu} \right).$$  

We find that the metric is a solution of the theory whenever

$$\Lambda = \frac{1}{8\ell^2} \text{ and } \beta_1 = -\ell^2 - 3\beta_2.$$  

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The $\eta$-Einstein nature of the metric greatly simplifies the equations of motion, as second order quantities, e.g.,
the Ricci tensor, become zero-order cf. (17). This also occurs for fourth order quantities such as
\[ \square \text{Ric} = \frac{1}{2\ell^4} q - \frac{3}{2\ell^2} q \otimes \eta. \] (31)

Now, equations (30) tell us that the Heisenberg group with one of its left-invariant metrics is a solution of NMG. Moreover, we have checked that the Euclidean metric is also a solution to the equations of motion (28). This is one of the eight Thurston geometries. These geometries have been studied in mathematicians and physics. In string theory, these geometries have been studied in the framework of string dualities [37]. Therein, the geometries have been found to be dual amongst themselves with one exception, the sol geometry. They have also been studied in string-inspired three-dimensional gravity [38]. In this light, it is a natural question if all Thurston geometries are solutions to NMG. Furthermore, when considering Lorentzian signature there are, instead of eight, four relevant geometries [40], two of which have constant sectional curvature: Minkowski and AdS. The remaining two are the Lorentzian versions of the nil and sol geometries. We intend to report the full details concerning these solutions in a forthcoming article.

**Coupling to the Maxwell-Chern-Simons and Abelian Higgs fields**

Drawing from our previous examination of equation (1), we define a gauge potential given by \( A = -2q \eta \) so that the field strength is given by
\[ F = dA = q \sin \theta d\theta \wedge d\phi, \] (32)
the standard homogeneous field strength on a two-sphere.

To understand how a field like (32) is supported by the para-Sasakian 3-sphere and what is the nature of the corresponding induced current, we consider two obvious choices, namely, NMG coupled to Maxwell-Chern-Simons theory and alternatively coupled to the Abelian Higgs model.

Consider the NMG action functional (26) coupled to Maxwell-Chern-Simons theory (MCS)
\[ S[g, A, \Phi] = \int d^3x \sqrt{-g} \mathcal{L}_{\text{NMG}} + S_{\text{MCS}}, \] (33)
where
\[ S_{\text{MCS}} = \frac{1}{2} \int -(F \wedge \ast F + \mu A \wedge F). \] (34)

Thus, the Maxwell equation (5) is satisfied provided \( \ell = 1/\mu \), yielding
\[ (\square - \mu^2) \ast F = 0, \] (35)
for equation (4). As usual, the Chern-Simons coupling constant \( \mu \) determines the mass of the gauge field. In the present case it also fixes the characteristic length scale \( \ell \) of spacetime. Since the field is Trkalian, cf. (35), then the topologically massive gauge theory (34) is gauge invariant.

For this NMG-MCS theory the gauge field is supported by the para-Sasakian provided
\[ \Lambda = \frac{\mu^2}{8} (1 - 4\kappa^2 \mu^2 q^2), \quad \text{and} \quad m^2 = \frac{21\mu^2}{8(1 + 4\kappa^2 \mu^2 q^2)}. \] (36)

We now move on to the Abelian Higgs theory, which generalizes the Ginzburg-Landau theory where superconductors were originally described by Abrikosov [41]. We also refer the reader to [42] for a closer analogue of the following configuration and to [43] for recent work on gravitating superconducting configurations.

We now couple the Abelian Higgs theory to New Massive Gravity
\[ S[g, A, \Phi] = \int d^3x \sqrt{-g} \mathcal{L}_{\text{NMG}} + S_{\text{AH}}, \] (37)
with
\[ S_{\text{AH}} = \int -\frac{1}{2} F \wedge \ast F + \frac{1}{2} D\Phi \wedge \ast D\Phi^\dagger - \ast V(|\Phi|^2). \] (38)

The Higgs field \( \Phi \) is in general complex-valued and its covariant derivative is given by \( D \Phi = d\Phi - iA\Phi \). The Higgs field satisfies the equation of motion
\[ \ast D \ast D\Phi = V'. \] (39)

Here, we consider a contribution from the scalar field to the energy momentum tensor, so that
\[ T^\mu_\nu = -D_\mu \Phi D_\nu \Phi + \frac{1}{2} \eta_{\mu\nu} (D_\alpha \Phi D^\alpha \Phi - V). \] (40)

Additionally, the Higgs field’s electric current is given by
\[ \ast J = -iD\Phi D^\Phi. \] (41)

By considering a constant real-valued Higgs field \( \Phi = h \) the previous equation becomes a London equation \( \ast J = h^2 A \) which when compared to (6) shows that the value of the Higgs field plays the role of \( \mu \) in the MCS case. Indeed, considering \( \ell = 1/h \) is sufficient for the Maxwell equations to hold. Moreover, the Higgs equation (39) fixes the self-interaction potential to
\[ V(|\Phi|) = \frac{\lambda}{4} |\Phi|^4. \] (42)

The scalar field’s equation of motion also fixes the charge of the Maxwell field (32) through \( 4q^2 = \lambda \). When compared to the MCS configuration above, it possesses a rigid
Maxwell field which compensates the extra degree of freedom coming from the Higgs field.

The Maxwell and Higgs fields self-gravitate on the background with the fine-tuning

$$\Lambda = \frac{h^2}{8} (1 - \kappa^2 \lambda h^2), \quad m^2 = \frac{21h^2}{8(1 + 2\kappa^2 \lambda h^2)}.$$  \hspace{1cm} (43)

Notice that there is a critical value of the Higgs field $h$ for which the cosmological constant changes sign [similar to $q$ in (36)]. In the critical case we have $\Lambda = 0$, of course, and $h$ fixed by the coupling constants of the theory, in particular $h^2 = 8m^2/7$. Another special value of the Higgs field is given by $h^2 = 13/16\kappa^2 \lambda$ for which $\Lambda > 0$ and the mass of the propagating degrees of freedom in NMG and of the Abelian Higgs theory coincide, i.e., $h^2 = m^2$. This is to say we have

$$(\Box - m^2)A = 0$$  \hspace{1cm} (44)

for $m$ given in (27).

**CLOSING REMARKS**

In this manuscript, we showed that the metric relation (1) – which holds for every 3-dimensional manifold equipped with an $\varepsilon$-contact structure – serves as a constitutive relation for electromagnetic fields such that their potential 1-form is proportional to the contact form of the manifold. This result does not rely on the particular case explored in this manuscript and constitutes a general result for 3-dimensional electromagnetic fields. That is, a material medium described by a metric constitutive relation is a superconductor whenever the manifold is a 3-dimensional $\varepsilon$-contact metric structure. Motivated by this result, we studied the self-gravitating Maxwell-Chern-Simons and the Abelian-Higgs model coupled to New Massive Gravity in the case of $(S^3, g)$ and found the conditions this particular spacetime must satisfy in order to be a solution.

We studied a non-standard metric structure for $S^3$. In particular, by considering the 3-sphere as a contact manifold, we obtained a locally homogeneous, left invariant, Lorentzian Heisenberg metric $g$, cf. equation (13). Considering the pair $(S^3, g)$ as a 3-dimensional spacetime, we verified that it is a Petrov D, Kundt spacetime. Furthermore, to the best of our knowledge, this is a new vacuum solution for New Massive Gravity. In addition, when coupled to the Maxwell-Chern-Simons action, written solely in terms of geometric objects directly linked to the contact structure of $S^3$, we obtained a new charged solution to NMG-MCS.

This exercise has shown us that metric contact manifolds might play an important role in the exploration of 3-dimensional field theories. Moreover, it has open new ways to understand 3-dimensional superconductors in terms of Beltrami fields on a $\varepsilon$-contact manifold. This constitutes a completely geometric picture of the macroscopic phenomenon of superconductivity which may shed some light on its higher dimensional counterpart.

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[1] A. Trautman. Gauge and optical aspects of gravitation. *Class. Quant. Grav.*, 16:A157–A175, 1999.

[2] Igor Robinson and Andrzej Trautman. **CAUCHY-RIEMANN STRUCTURES IN OPTICAL GEOMETRY.** In *4th Marcel Grossmann Meeting on the Recent Developments of General Relativity*, pages 317–324. North-Holland, 1986.

[3] Igor Robinson and Andrzej Trautman. OPTICAL GEOMETRY. In *XI Warsaw Symposium on Elementary Particle Physics*, pages 457–497. World Scientific, 1998.

[4] Arnol’d V.I. *Mathematical Methods of Classical Mechanics*. Springer Verlag, 1989.

[5] Igor Robinson and Andrzej Trautman. The generalized Taub-NUT congruence in Minkowski spaces. *Gen. Relat. Gravit.*, 25:225–244, 1993.

[6] Ghrist, R. and Komendarczyk, R. Topological Features of Inviscid Flows: An Introduction to the Geometry and Topology of Fluid Flows. Cambridge University Press, 2000.

[7] I. Sato. On a structure similar to the almost contact structure. *Tensor (N.S.),* 30:219–224, 1976.

[8] T. Adati and K. Matsumoto. On conformally recurrent and conformally symmetric P-Sasakian manifolds. *TRU Mathematics*, 13:25–32, 1977.

[9] Soji Kaneyuki and Floyd L. Williams. Almost paraccontact and parahodge structures on manifolds. *Nagoya Math. J.*, 99:173–187, 1985.

[10] T. Adati and K. Matsumoto. On $(\phi, \xi, \eta)$-connections in P-Sasakian manifolds. *TRU Mathematics*, 21:195–199, 1985.

[11] S. Zamkovoy. Canonical connections on paracontact manifolds. *Ann. Glob. Anal. Geom.*, 36:37–60, 2009.

[12] C.L. Bejanc and M. Crasmaru. Second order parallel tensors and Ricci solitons in 3-dimensional normal paraccontact geometry. *Ann. Glob. Anal. Geom.*, 46:117–127, 2014.

[13] K. Matsumoto. Conformal Killing vector fields in a P-Sasakian manifold. *J. Korean Math. Soc.*, 14:135–142, 1977.

[14] M.S. Rahman. A study of para-sasakian manifolds. *International Report IC/95/212, International Centre for Theoretical Physics (1995).*

[15] K.L. Duggal and A. Bejancu. **Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.** Kluwer Academic, 1996.

[16] K.L. Duggal and B. Sahin. **Differential Geometry of Lightlike Submanifolds.** Frontiers in Mathematics, 2010.

[17] Giovanni Calvaruso and Antonella Perrone. Ricci solitons in three-dimensional paraccontact geometry. *Journal of
[18] Blaga A. Solitons and geometrical structures in a perfect fluid spacetime. *Rocky Mountain J. Math.*, 50:41–53, 2020.

[19] Eric A. Bergshoeff, Olaf Hohm, and Paul K. Townsend. Massive Gravity in Three Dimensions. *Phys. Rev. Lett.*, 102:201301, 2009.

[20] Angel Murcia and C.S. Shahbazi. Contact metric three manifolds and lorentzian geometry with torsion in six-dimensional supergravity. *Journal of Geometry and Physics*, page 103868, 2020.

[21] P.K. Townsend, K. Pilch, and P. van Nieuwenhuizen. Selfduality in Odd Dimensions. *Phys. Lett. B*, 136:38, 1984. [Addendum: Phys.Lett.B 137, 443 (1984)].

[22] S Chandrasekhar and P C Kendall. On Force-Free Magnetic Fields. *Astrophys. J*, 126:457, September 1957.

[23] Arkady L Kholodenko. *Applications of Contact Geometry and Topology in Physics*. World Scientific, 2013.

[24] David E. Blair. *Riemannian geometry of contact and symplectic manifolds*. Progress in Mathematics 203. Birkhäuser Basel, 2 edition, 2010.

[25] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry, Volume 1*. Wiley Classics Library. Wiley-Interscience, 1996.

[26] C S López-Monsalvo, F Nettel, Pineda-Reyes V, and Escamilla-Herrera L F. Symplectic Polarizations and Legendre Transformations in Contact Geometric Thermodynamics. arXiv: 2006.12754.

[27] Charles P. Boyer, Krzysztof Galicki, and Paola Matzeu. On eta-Einstein Sasakian geometry. *Commun. Math. Phys.*, 262:177–208, 2006.

[28] Florin Alexandru Belgun. On the metric structure of non-Kähler complex surfaces. *Math. Ann.*, 317:1–40, 2000.

[29] S. Rahmani. Métriques de Lorentz sur les groupes de Lie unimodulaires, de dimension trois. *J. Geom. Phys.*, 9:295–302, 1992.

[30] Shûkichi Tanno. Sasakian manifolds with constant φ-holomorphic sectional curvature. *Tohoku Math. J. (2)*, 21(3):501–507, 1969.

[31] G. D’Ambra. Isometry groups of Lorentz manifolds. *Invent. math.*, 92:555–565, 1988.

[32] Eloy Ayón-Beato, Cristian Martínez, and Jorge Zanelli. Birkhoff’s theorem for three-dimensional AdS gravity. *Phys. Rev. D*, 70:044027, 2004.

[33] Maximo Bañados, Claudio Teitelboim, and Jorge Zanelli. The Black hole in three-dimensional space-time. *Phys. Rev. Lett.*, 69:1849–1851, 1992.

[34] A Bravetti and C S López-Monsalvo. Para-sasakian geometry in thermodynamic fluctuation theory. *Journal of Physics A: Mathematical and Theoretical*, 48(12):125206, mar 2015.

[35] Alberto A. García-Díaz. *Exact Solutions in Three-Dimensional Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2017.

[36] Alberto García, Friedrich W. Hehl, Christian Heinicke, and Alfredo Macías. The Cotton tensor in Riemannian space-times. *Class. Quant. Grav.*, 21:1099–1118, 2004.

[37] J. Gegenberg, S. Vaidya, and Justin F. Vazquez-Poritz. Thurston geometries from eleven-dimensions. *Class. Quant. Grav.*, 19:L199–L204, 2002.

[38] J. Gegenberg and G. Kunstatter. Using 3-D stringy gravity to understand the Thurston conjecture. *Class. Quant. Grav.*, 21:1197–1208, 2004.

[39] We thank Eloy Ayón-Beato for suggesting this question to us.

[40] Sorin Dumitrescu and Abdelghani Zeghib. Géométries Lorentziennes de dimension 3 : classification et complétude. *Geometriae Dedicata*, 149:243–273, 2010.

[41] A.A. Abrikosov. The magnetic properties of superconducting alloys. *Journal of Physics and Chemistry of Solids*, 2(3):199 – 208, 1957.

[42] Holger Bech Nielsen and P. Olesen. Vortex Line Models for Dual Strings. *Nucl. Phys. B*, 61:45–61, 1973.

[43] Fabrizio Canfora, Alex Giacomini, Marcela Lagos, Seung Hun Oh, and Aldo Vera. Gravitating cosmic superconducting tubes in the Einstein gauged non-linear σ-model in (3+1)-dimensions. arXiv: 2001.11910.