THE AMBIGUITY INDEX OF AN EQUIPPED FINITE GROUP

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Abstract. In [10], the ambiguity index $a_{(G,O)}$ was introduced for each equipped
finite group $(G,O)$. It is equal to the number of connected components of a Hurwitz
space parametrizing coverings of a projective line with Galois group $G$ assuming that
all local monodromies belong to conjugacy classes $O$ in $G$ and the number of branch
points is greater than some constant. We prove in this article that the ambiguity
index can be identified with the size of a generalization of so called Bogomolov
multiplier ([8], see also [1]) and hence can be easily computed for many pairs $(G,O)$.

Introduction

Let $G$ be a finite group and $O$ be a subset of $G$ consisting of conjugacy classes $C_i$ of
$G$, $O = C_1 \cup \cdots \cup C_m$, which together generate $G$. The pair $(G,O)$ is called an equipped
group and $O$ is called an equipment of $G$. We fix the numbering of conjugacy classes
contained in $O$. One can associate a $C$-group $(\tilde{G}, \tilde{O})$ to each equipped group $(G,O)$. The $C$-group $\tilde{G}$ is generated by the letters of the alphabet $Y = Y_O = \{y_g \mid g \in O\}$ subject to relations:

$$y_{g_1}y_{g_2} = y_{g_2}y_{g_1}^{-1}g_2y_{g_2}^{-1}y_{g_1}.$$  \(1\)

We assume $\tilde{O} = Y_O$ in the definition of $\tilde{G}$.

There is an obvious natural homomorphism $\beta : \tilde{G} \to G$ given by $\beta(y_g) = g$. It
was shown in [10], that the commutator subgroup $[\tilde{G}, \tilde{G}]$ is finite. The order $a_{(G,O)}$ of
the group $\ker \beta \cap [\tilde{G}, \tilde{G}]$ was called the ambiguity index of the equipped finite group
$(G,O)$.

The notion of equipped groups is related to the description of Hurwitz spaces
parametrizing maps between projective curves with $G$ as the monodromy group and the ambiguity index $a_{G,O}$ is equal to the properly defined “asymptotic” number of
connected components of Hurwitz space parametrizing covering of curves with fixed ramification data. More precisely, let $f : \tilde{E} \to F$ be a morphism of a non-singular complex irreducible projective curve $E$ onto a non-singular projective curve $F$. Let us choose a point $z_0 \in F$ such that $z_0$ is not a branch point of $f$ hence the points $f^{-1}(z_0) = \{w_1, \ldots, w_d\}$, where $d = \deg f$, are simple. If we fix the numbering of points in $f^{-1}(z_0)$ then we call $f$ a marked covering.
Let \( B = \{z_1, \ldots, z_n\} \subset F \) be the set of branch points of \( f \). The numbering of the points of \( f^{-1}(z_0) \) defines a homomorphism \( f_* : \pi_1(F \setminus B, z_0) \to \Sigma_d \) of the fundamental group \( \pi_1 = \pi_1(F \setminus B, z_0) \) to the symmetric group \( \Sigma_d \). Define \( G \subset \Sigma_d \) as \( \text{im} f_* = G \). It acts transitively on \( f^{-1}(z_0) \). Let \( \gamma_1, \ldots, \gamma_n \) be simple loops around, respectively, the points \( z_1, \ldots, z_n \) starting at \( z_0 \). The image \( g_j = f_*(\gamma_j) \in G \) is called a local monodromy of \( f \) at the point \( z_j \). Each local monodromy \( g_j \) depends on the choice of \( \gamma_j \), therefore it is defined uniquely up to conjugation in \( G \).

Denote by \( O = C_1 \cup \cdots \cup C_m \subset G \) the union of conjugacy classes of all local monodromies and by \( \tau_i \) the number of local monodromies of \( f \) belonging to the conjugacy class \( C_i \). The collection \( \tau = (\tau_1 C_1, \ldots, \tau_m C_m) \) is called the monodromy type of \( f \). Assume that the elements of \( O \) generate the group \( G \). Then the pair \((G, O)\) is an equipped group.

Let \( \text{HUR}^m_{d, G, O, \tau}(F, z_0) \) be the Hurwitz space (see the definition of Hurwitz spaces in [4] or in [9]) of marked degree \( d \) coverings of \( F \) with Galois group \( G \subset \Sigma_d \), local monodromies in \( O \), and monodromy type \( \tau \). Hurwitz space \( \text{HUR}^m_{d, G, O, \tau}(F, z_0) \) may consists of a different number of connected components. However it was proved in [9] that for each equipped finite group \((G, O)\), \( O = C_1 \cup \cdots \cup C_m \), there is a number \( T \) such that the number of irreducible components of each non-empty Hurwitz space \( \text{HUR}^m_{d, G, O, \tau}(F, z_0) \) is equal to \( a_{(G,O)} \) if \( \tau_i \geq T \) for all \( i = 1, \ldots, m \). The number \( T \) does not depend on the base curve \( F \) and degree \( d \) of the coverings.

The subgroup \( B_0(G) \subset H^2(G, Q/Z) \) was defined and studied in [1]. It consists of elements of \( H^2(G, Q/Z) \) which restrict trivially onto abelian subgroups of \( G \). It was conjectured in [2] that \( B_0(G) \) is trivial for simple groups. This conjecture was partially solved already in [2] and it was completely solved by Kunyavski in [8], and by Kunyavski-Kang in [7] for a wider class of almost simple groups. The latter consists of groups \( G \) which contain some simple group \( L \) and in turn are contained in the automorphism group \( \text{Aut}L \). Kunyavski in [8] called \( B_0(G) \) as Bogomolov multiplier and we are going to use his terminology here. Denote by \( b_0(G) \) the order of the group \( B_0(G) \) and denote by \( h_2(G) \) the Schur multiplier of the group \( G \), that is, the order of the group \( H_2(G, Z) \).

The aim of this article is to prove

**Theorem 1.** For an equipped finite group \((G, O)\) we have the following inequalities

\[
\text{b}_0(G) \leq a_{(G,O)} \leq h_2(G).
\]

In particular, \( a_{(G,G\setminus\{1\})} = b_0(G) \).

Since, by [8], \( b_0(G) = 1 \) for a finite almost simple group \( G \), we conclude:

**Corollary 1.** Let \( G \) be a finite almost simple group. Then there is a constant \( T \) such that for any projective irreducible non-singular curve \( F \) each non-empty Hurwitz space \( \text{HUR}^m_{d, G, G\setminus\{1\}, \tau}(F, z_0) \) is irreducible if all \( \tau_i \geq T \).
It was shown in [10] that if \( O_1 \subset O_2 \) are two equipments of a finite group \( G \), then \( a_{(G,O_2)} \leq a_{(G,O_1)} \).

For a symmetric group \( \Sigma_d \), the famous Clebsch–Hurwitz Theorem ([3], [6]) states that the ambiguity index \( a_{(\Sigma_d,T)} = 1 \), where \( T \) is the set of transpositions in \( \Sigma_d \), and it was shown in \([11]\) that the ambiguity index \( a_{(\Sigma_d,O)} = 1 \) if the equipment \( O \) contains an odd permutation \( \sigma \in \Sigma_d \) such that \( \sigma \) leaves fixed at least two elements. Theorem 8 (see subsection 3.4) gives the complete answer on the value of \( a \) of \( G,O \) an equipped finite group \((G,O)\) and in Section 4, we give a cohomological description of the ambiguity indices.

In Section 3, we investigate the properties of ambiguity indices of a quasi-cover of \((G,O)\) and in Section 4, we give a cohomological description of the ambiguity indices.

In Section 5, we give examples of finite groups \( G \) which Bogomolov multiplier \( b_0(G) > 1 \). Therefore for such groups \( G \) each non-empty space \( \text{HUR}_{d,G,O,\tau}(\mathbb{F},z_0) \) consists of at least \( b_0(G) > 1 \) irreducible components for any \( \tau = (\tau_1,\ldots,\tau_m) \) with big enough \( \tau_i \).

In this article, if \( \mathbb{F} \) is a free group freely generated by an alphabet \( X \), \( N \) is a normal subgroup of \( \mathbb{F} \), and a group \( G = \mathbb{F}/N \), then a word \( w = w(x_{i_1},\ldots,x_{i_n}) \) in letters \( x_{i_j} \in X \) and their inverses will be considered as an element of \( G \) in case if it does not lead to misunderstanding.

1. **C-GROUPS AND THEIR PROPERTIES**

Let us remind the definition of a \( C \)-group (see, for example, [12]).

**Definition 2.** A group \( G \) is a \( C \)-group if there is a set of generators \( x \in X \) in \( G \) such that the basis of relations between \( x \in X \) consists of the following relations:

\[
x_i^{-1}x_jx_i = x_k, \quad (x_i,x_j,x_k) \in M,
\]

where \( M \) is a subset of \( X^3 \).

Thus the \( C \)-structure of \( G \) is defined by \( X \subset G \) and \( M \subset X^3 \).

Let \( \mathbb{F} \) be a free group freely generated by an alphabet \( X \). Denote by \( N \) the subgroup of \( \mathbb{F} \) normally generated by the elements \( x_i^{-1}x_jx_i^{-1}, (x_i,x_j,x_k) \in M \). The group \( N \) is a normal subgroup of \( \mathbb{F} \). Let \( f: \mathbb{F} \to G = \mathbb{F}/N \) be the natural epimorphism given by presentation 2. In the sequel, we consider each \( C \)-group \( G \) as an equipped group \((G,O)\) with the equipment \( O = f(\mathbb{F}) \) (where \( \mathbb{F} \) is the orbit of \( X \) under the action of the group of inner automorphisms of \( \mathbb{F} \)). The elements of \( O \) are \( C \)-generators of the \( C \)-group \( G \). In particular, the equipped group \((\mathbb{F},\mathbb{F})\) is a \( C \)-group.

A homomorphism \( f: G_1 \to G_2 \) of a \( C \)-group \((G_1,O_1)\) to a \( C \)-group \((G_2,O_2)\) is called a \( C \)-homomorphism if it is a homomorphism of equipped groups, that is, \( f(O_1) \subset O_2 \). In particular, two \( C \)-groups \((G_1,O_1)\) and \((G_2,O_2)\) are \( C \)-isomorphic if they are isomorphic as equipped groups.
Claim 1. (Lemma 3.6 in [12]) Let $N$ be a normal subgroup of $\mathbb{F}$ normally generated by elements of the form $w_i^{-1}x_jw_iw_kx_j^{-1}w_i^{-1}$, where $w_i$ and $w_k$ are some elements of $\mathbb{F}$ and $x_j, x_k \in X$. Let $f : \mathbb{F} \to G \simeq \mathbb{F}/N$ be the natural epimorphism. Then $(G, f(X))$ is a $C$-group and $f$ is a $C$-homomorphism.

To each $C$-group $(G, O)$, one can associate a $C$-graph. By definition, the $C$-graph $\Gamma = \Gamma_{(G, O)}$ of a $C$-group $(G, O)$ is a directed labeled graph whose set of vertices $V = \{v_g \mid g \in O\}$ is in one to one correspondence with the set $O$. Two vertices $v_{g_1}$ and $v_{g_2}$, $g_1, g_2 \in O$, are connected by a labeled edge $e_{v_{g_1}, v_{g_2}, v_g}$ (here $v_{g_1}$ is the tail of $e_{v_{g_1}, v_{g_2}, v_g}$, $v_{g_2}$ is the head of $e_{v_{g_1}, v_{g_2}, v_g}$, and $v_g$ is the label of $e_{v_{g_1}, v_{g_2}, v_g}$) if and only if in $G$ we have the relation $g^{-1}g_1g = g_2$ with some $g \in O$.

A $C$-homomorphism $f : (G_1, O_1) \to (G_2, O_2)$ of $C$-groups induces a map $f_* : \Gamma_{(G_1, O_1)} \to \Gamma_{(G_2, O_2)}$ from the $C$-graph $\Gamma_{(G_1, O_1)}$ in the $C$-graph $\Gamma_{(G_2, O_2)}$, where by definition, $f_*(v_g) = v_{f(g)}$ for each vertex $v_g$ of $\Gamma_{(G_1, O_1)}$ and $f_*(e_{v_{g_1}, v_{g_2}, v_g}) = e_{v_{f(g_1)}, v_{f(g_2)}, v_{f(g)}}$ for each edge $e_{v_{g_1}, v_{g_2}, v_g}$ of $\Gamma_{(G_1, O_1)}$.

The following Claim is obvious.

Claim 2. A $C$-homomorphism $f : (G_1, O_1) \to (G_2, O_2)$ is a $C$-isomorphism if $f_*$ is one-to-one between the sets of vertices of $\Gamma_{(G_1, O_1)}$ and $\Gamma_{(G_2, O_2)}$.

In the sequel, we will consider only finitely presented $C$-groups (as groups without equipment) and $C$-graphs consisting of finitely many connected components. Denote by $m$ the number of connected components of a $C$-graph $\Gamma_{(G, O)}$.

Then it is easy to see that $G/[G, G] \simeq \mathbb{Z}^m$ and any two $C$-generators $g_1$ and $g_2$ are conjugated in the $C$-group $G$ if and only if $v_{g_1}$ and $v_{g_2}$ belong to the same connected component of $\Gamma_{(G, O)}$.

Thus the set $O$ of $C$-generators of the $C$-group $(G, O)$ is the union of $m$ conjugacy classes of $G$ and there is a one-to-one correspondence between the conjugacy classes of $G$ contained in $O$ and the set of connected components of $\Gamma_{(G, O)}$.

Denote by $\tau : G \to H_1(G, \mathbb{Z}) = G/[G, G]$ the natural epimorphism. In the sequel, we fix some numbering of the connected components of $\Gamma_{(G, O)}$. Then the group $H_1(G, \mathbb{Z}) \simeq \mathbb{Z}^m$ obtains a natural base consisting of vectors $\tau(g) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 stands on the $i$-th place if $g$ is a $C$-generator of $G$ and $v_g$ belongs to the $i$-th connected component of $\Gamma_{(G, O)}$. For $g \in G$ the image $\tau(g)$ is called the type of $g$.

Lemma 1. Let $g_1, g_2$ be two $C$-generators of a $C$-group $(G, O)$, $N$ the normal closure of $g_1g_2^{-1}$ in $G$, and $f : G \to G_1 = G/N$ the natural epimorphism. Then

(i) $(G_1, O_1)$ is a $C$-group, where $O_1 = f(O)$, and $f$ is a $C$-homomorphism;
(ii) the map $f_* : \Gamma_{(G, O)} \to \Gamma_{(G_1, O_1)}$ is a surjection.
(iii) if $g_1g_2^{-1}$ belong to the center $Z(G)$ of the group $G$ and let $v_{g_1}$ and $v_{g_2}$ belong to different components of $\Gamma_{(G, O)}$, then

(iii$_1$) the number of connected components of the $C$-graph $\Gamma_{(G_1, O_1)}$ is less than the number of connected components of the $C$-graph $\Gamma_{(G, O)}$;
(iii$_2$) $f : [G, G] \to [G_1, G_1]$ is an isomorphism.
Proof. Claims (i), (ii), and (iii) are obvious.

To prove (iii2), note that $N$ is a cyclic group generated by $g_1g_2^{-1}$, since $g_1g_2^{-1}$ belongs to the center $Z(G)$. The type $\tau((g_1g_2^{-1})^n)$ is non-zero for $n \neq 0$, since $v_{g_1}$ and $v_{g_2}$ belong to different connected components of $\Gamma_{(G,O)}$. Therefore to complete the proof, it suffices to note that the groups $N$ and $[G,G]$ have trivial intersection, since $\tau(g) = 0$ for all $g \in [G,G]$. \hfill \Box

A $C$-group $(G,O)$ is called a $C$-finite group if the set of vertices of $C$-graph $\Gamma_{(G,O)}$ is finite or, the same, if the equipment $O$ of $G$ is a finite set.

**Proposition 1.** ([10]) Let $(G,O)$ be a $C$-finite group. Then the commutator $[G,G]$ is a finite group.

As it was mentioned in Introduction, for each finite equipped group $(G,O)$, one can associate a $C$-group $(\tilde{G}, \tilde{O})$ defined as follows. The group $\tilde{G}$ is generated by the letters of the alphabet $Y = Y_O = \{y_g | g \in O\}$ subject to relations

$$y_{g_1}y_{g_2} = y_{g_2}y_{g_1}^{-1}y_{g_2}^{-1}y_{g_1}. \quad (3)$$

Here $\tilde{O} = Y_O$ and there is a natural epimorphism $\beta_O : \tilde{G} \to G$ given by $\beta_O(y_g) = g$.

Note also that a homomorphism of equipped groups $f : (G_1, O_1) \to (G, O)$ induces a $C$-homomorphism $\tilde{f} : (\tilde{G}_1, \tilde{O}_1) \to (\tilde{G}, \tilde{O})$ such that $f \circ \beta_{O_1} = \tilde{f} \circ \beta_O$.

Let the elements of a subset $S$ of an equipment $O$ of a group $G$ generate the group $G$ and $O = S^G$. Denote by $F_S$ a free group freely generated by the alphabet $Y_S = \{y_g | g \in S\}$ and $R_S$ is the normal subgroup of $F_S$ such that the natural epimorphism $h_S : F_S \to F_S/R_S \simeq G$ gives a presentation of the group $G$.

**Claim 3.** Let $\tilde{R}_S \subset R_S$ be the normal subgroup normally generated by the elements of $R_S$ of the form $w_{i,j}^{-1} y_{g_i} w_{i,j} y_{g_j}^{-1}$, where $w_{i,j} \in F_S$ and $y_{g_i}, y_{g_j} \in Y_S$. Then the $C$-group $(\tilde{G}, \tilde{O})$ has the following presentation: $\tilde{G} \simeq F_S/\tilde{R}_S$ such that the images of the elements of $Y_S$ are $C$-generators of $\tilde{G}$.

**Proof.** Denote by $G_1 = F_S/\tilde{R}_S$. By Claim 1, $G_1$ is a $C$-group with $C$-equipment $O_1 = Y_{\tilde{G}_1}$ and there is a natural epimorphism $\beta_S : ((G_1, O_1) \to (G, O)$ given by $\beta_S(y_g) = g$ for $g \in S$.

Assume that $S$ consists of elements $g_1, \ldots, g_n \in O$. If $S \neq O$ then choose an element $g_{n+1} \in O \setminus S$. It is conjugated to some $g_i \in S$. Denote by $R_{g_{n+1}}$ the set of all presentations of $g_{n+1}$ in the form

$$g_{n+1} = w(g_1, \ldots, g_n)^{-1}gw(g_1, \ldots, g_n), \quad g \in S. \quad (4)$$

Note that if

$$g_{n+1} = w_i(g_1, \ldots, g_n)^{-1}g_iw_i(g_1, \ldots, g_n)$$

and $g_{n+1} = w_j(g_1, \ldots, g_n)^{-1}g_jw_j(g_1, \ldots, g_n)$, then

$$w_jw_i^{-1}g_iw_iw_j^{-1} = g_j,$$
that is,
\[ w_j(y_{g_1}, \ldots, y_{g_n}) w_i^{-1}(y_{g_1}, \ldots, y_{g_n}) y_g w_i(y_{g_1}, \ldots, y_{g_n}) w_j^{-1}(y_{g_1}, \ldots, y_{g_n}) y_g^{-1} \in R_S. \] (5)

Similarly, if \( g_{n+1} = w_i(g_1, \ldots, g_n) \) and \( g_{n+1}^{-1} g_i g_{n+1} = g_j \) for some \( g_i, g_j \in S \), then
\[ w(y_{g_1}, \ldots, y_{g_n})^{-1} y_g w(y_{g_1}, \ldots, y_{g_n}) y_g^{-1} \in R_S. \] (6)

Therefore, if \( S_1 = S \cup \{ g_{n+1} \} \), \( F_{S_1} \) is a free group freely generated by the alphabet \( Y_{S_1} = \{ y_g \mid g \in S_1 \} \), \( R_{g_{n+1}} \) is the set of words of the form
\[ w(y_{g_1}, \ldots, y_{g_n})^{-1} y_g w(y_{g_1}, \ldots, y_{g_n}) y_g^{-1} \]
defined by all relations (4), and \( \tilde{R}_{S_1} \) is the normal closure in \( F_{S_1} \) of the set \( \tilde{R}_S \cup R_{g_{n+1}} \), then \( G_1 \simeq F_{S_1}/\tilde{R}_{S_1} \) in view of relations (4) and (6).

Note that if we have a relation \( g_i^{-1} g_j g_i = g_k \) for some \( g_i, g_j, g_k \in S_1 \) then
\[ y_{g_i}^{-1} y_{g_j} y_{g_k}^{-1} \in \tilde{R}_{S_1}. \] (7)

If \( S_1 \neq O \), then we can repeat the construction described above and obtain a presentation \( G_1 \simeq F_{S_2}/\tilde{R}_{S_2} \), and so on. After several steps we obtain a presentation \( G_1 \simeq F_O/\tilde{R}_O \). Note that, by induction, we deduce that for any relation in \( G \) of the form \( g_i^{-1} g_j g_i = g_k \) for some \( g_i, g_j, g_k \in O \) we have \( y_{g_i}^{-1} y_{g_j} y_{g_k}^{-1} \in \tilde{R}_O \). Therefore there is a natural \( C \)-homomorphism \( f : (\tilde{G}, \tilde{O}) \to (G_1, O_1) \). By Claim 2, \( f \) is a \( C \)-isomorphism.

For an equipped finite group \((C, O)\), consider a presentation of \( G \) of the following form. Let us take a free group \( F = F_O \) freely generated by the alphabet \( X_O = \{ x_g \mid g \in O \} \). Consider a normal subgroup \( R_O \subset F \) such that \( F/R_O \simeq G \). Let \( h_O : F \to F/R_O \simeq G \) be the natural epimorphism.

We can associate to \((G, O)\) a group \( \overline{G} = F/[F, R_O] \).

Denote by \( \alpha_O : \overline{G} \to G \) the natural epimorphism. By Claim 1, \((\overline{G}, \overline{O})\) is a \( C \)-group, where \( \overline{O} = h_O(X_O) \). It is evident that there is the natural epimorphism of \( C \)-groups \( \kappa_O : (\overline{G}, \overline{O}) \to (G, \tilde{O}) \) sending \( \kappa_O(x_g) = y_g \) for all \( g \in O \) and such that \( \alpha_O = \beta_O \circ \kappa_O \).

The \( C \)-group \((\overline{G}, \overline{O})\) is called the universal central \( C \)-extension of the equipped finite group \((G, O)\).

It is easy to see that \( \alpha_O : \overline{G} \to G \) is a central extension of groups, that is, \( \ker \alpha_O \) is a subgroup of the center \( Z(\overline{G}) \).

We have
\[ \ker \alpha_O \cap (\overline{G}, \overline{O}) = (R_O \cap [F, F])/[F, R_O]. \]

By Hopf’s integral homology formula, we have \( H_2(G, \mathbb{Z}) \simeq (R_O \cap [F, F])/[F, R_O] \). Denote by \( h_2(G) \) the order of the group \( H_2(G, \mathbb{Z}) \) and denote by \( K_{(G, O)} \) the subgroup of \( (R_O \cap [F, F])/[F, R_O] \) generated by the elements of \( R_O \) of the form \([w, x_g]\), where \( g \in O \) and \( w \in F \), and let \( k_{(G, O)} \) be its order.
Theorem 3. For an equipped finite group \((G, O)\) we have

\[ h_2(G) = k(G, O) a(G, O). \]

Proof. We have \(\ker \kappa_O \subset \ker \alpha_O\). Therefore \(\ker \kappa_O \subset Z(\tilde{G})\).

Let us show that for some \(n \geq 0\) there is a sequence of \(C\)-groups \(\overline{G}_0 = \mathbb{F}/R_0, \ldots, \overline{G}_n = \mathbb{F}/R_n\), a sequence of \(C\)-homomorphisms

\[ \varphi_i : (\overline{G}_i, \overline{O}_i) \rightarrow (\overline{G}_{i+1}, \overline{O}_{i+1}), \quad 0 \leq i \leq n - 1, \]

where \((\overline{G}_0, \overline{O}_0) = (\overline{G}, \overline{O})\), and a \(C\)-homomorphism \(\tilde{\kappa} : (\overline{G}_n, \overline{O}_n) \rightarrow (\tilde{G}, \tilde{O})\) such that

(i) \(\kappa = \tilde{\kappa} \circ \varphi\), where \(\varphi = \varphi_n \circ \cdots \circ \varphi_0\);

(ii) for each \(i\) the homomorphism \(\varphi_i : [\overline{G}_i, \overline{O}_i] \rightarrow [\overline{G}_{i+1}, \overline{O}_{i+1}]\) is an isomorphism;

(iii) \(\tilde{\kappa}_*\) induces a one-to-one correspondence between the connected components of the \(C\)-graphs \(\Gamma(\overline{G}_n, \overline{O}_n)\) and \(\Gamma(\tilde{G}, \tilde{O})\).

Indeed, let us put \(R_0 = R_O\) and consider the map \(\kappa_*\). If it is induces a one-to-one correspondence between the connected components of the \(C\)-graphs \(\Gamma(\overline{G}, \overline{O})\) and \(\Gamma(\tilde{G}, \tilde{O})\), then \(n = 0\) and it is nothing to prove.

Otherwise, for some \(g \in O\) there is a vertex \(v_{y_g}\) of \(\Gamma(\overline{G}, \overline{O})\) which preimage \(\kappa^{-1}_*(v_{y_g})\) contains at least two vertices, say \(v_{x_g}\) and \(v_{\overline{y}_g}\) (here \(\overline{y}\) is an element of \(X^{\overline{G}}\)), of \(\Gamma(\overline{G}, \overline{O})\) belonging to different connected components of \(\Gamma(\overline{G}, \overline{O})\).

Denote by \(R_1\) the normal closure of \(R_O \cup \{x_g \overline{y}_g^{-1}\}\) in \(G\) and consider the natural homomorphism \(\varphi_0 : \overline{G} \rightarrow \overline{G}_1 = \mathbb{F}/R_1\). The element \(x_g \overline{y}_g^{-1}\), considered as an element of \(\overline{G}\), belongs to \(\ker \kappa\). Therefore, \(x_g \overline{y}_g^{-1} \in Z(\overline{G})\).

Denote by \(\kappa_1 : \overline{G}_1 \rightarrow \tilde{G}\) the homomorphism induced by \(\kappa\). By Lemma 1, the homomorphism \(\varphi_1\) is a \(C\)-homomorphism of \(C\)-groups. It is easy to see that \(\varphi_0 : [\overline{G}_0, \overline{O}_0] \rightarrow [\overline{G}_1, \overline{G}_1]\) is an isomorphism and the number of connected components of the \(C\)-graph \(\Gamma(\overline{G}_1, \overline{O}_1)\) is less than the number of connected components of the \(C\)-graph \(\Gamma(\overline{G}, \overline{O})\).

Assume now that \(\kappa_*\) is not a one-to-one correspondence between the connected components of the \(C\)-graphs \(\Gamma(\overline{G}_1, \overline{O}_1)\) and \(\Gamma(\tilde{G}, \tilde{O})\). Then for some \(g_1 \in O\) there is a vertex \(v_{y_{g_1}}\) of \(\Gamma(\tilde{G}, \tilde{O})\) which preimage \(\kappa_*^{-1}(v_{y_{g_1}})\) contains at least two vertices \(v_{x_{g_1}}\) and \(v_{\overline{y}_{g_1}}\) of \(\Gamma(\overline{G}_1, \overline{O}_1)\) belonging to different connected components of \(\Gamma(\overline{G}_1, \overline{O}_1)\).

Hence we can repeat the construction described above and obtain a \(C\)-group \((\overline{G}_2, \overline{O}_2)\) and \(C\)-homomorphisms \(\varphi_1 : \overline{G}_1 \rightarrow \overline{G}_2 = \mathbb{F}/R_2\) and \(\kappa_2 : \overline{G}_2 \rightarrow \tilde{G}\) such that \(\varphi_1 : [\overline{G}_1, \overline{G}_1] \rightarrow [\overline{G}_2, \overline{G}_2]\) is an isomorphism and the number of connected components of the \(C\)-graph \(\Gamma(\overline{G}_2, \overline{O}_2)\) is less than the number of connected components of the \(C\)-graph \(\Gamma(\overline{G}_1, \overline{O}_1)\). Since the number of connected components of the \(C\)-graph \(\Gamma(\overline{G}, \overline{O})\) is finite, after several \((n)\) steps of our construction we obtain the desired sequences of \(C\)-groups and \(C\)-homomorphisms.

Now, consider the \(C\)-homomorphism \(\tilde{\kappa} : \overline{G}_n \rightarrow \tilde{G}\). The \(C\)-graph \(\Gamma(\tilde{G}, \tilde{O})\) consists of connected components \(\Gamma_1, \ldots, \Gamma_m\). Let \(\{v_{g_{i,1}}, \ldots, v_{g_{i,1}}\}\) be the set of the vertices of
where the group inverses such that

since all $\cup \{v_{g_{i,j}}, v_{\eta_{i,j},1}, \ldots, v_{\eta_{i,j},r_{i,j}}\}, \mathcal{O}_{i,j,k} \in \mathcal{O}_n$ for $1 \leq k \leq r_{i,j}$.

Since the graph $\mathbf{G}_i$ is connected, there are words $w_{i,j,k}$ in letters of $X_O$ and their inverses such that

$$\mathcal{O}_{i,j,k} = w_{i,j,k}x_{g_{i,j}}^{-1}w_{i,j,k}^{-1}, \quad 1 \leq k \leq r_{i,j}.$$  

Obviously, the elements $u_{i,j,k} = [w_{i,j,k}, x_{g_{i,j}}] = \mathcal{O}_{i,j,k}x_{g_{i,j}}^{-1} \in [\mathcal{O}_n, \mathcal{O}_n] \cap \ker \pi$.

Therefore $u_{i,j,k}$, as elements of $\mathbb{F}$ belong to $R_O \cap [\mathbb{F}, \mathbb{F}]$.

Consider the group $\mathcal{O}_{n+1} = \mathbb{F}/R_{n+1}$, where the group $R_{n+1}$ is the normal closure of $R_n \cup \{u_{i,j,k}\}_{1 \leq i \leq m, 1 \leq j \leq l_i, 1 \leq k \leq r_{i,j}}$ in $\mathbb{F}$. Then, by Claim 1 $\mathcal{O}_{n+1} = \mathbb{F}/R_{n+1}$ is a $C$-group and the natural map $\pi_1 : \mathcal{O}_{n+1} \rightarrow \mathcal{G}$, induced by $\pi$, is a $C$-homomorphism. Moreover, $\ker \varphi_n$ of the natural epimorphism $\varphi_n : \mathcal{O}_n \rightarrow \mathcal{O}_{n+1}$ is a subgroup of $[\mathcal{G}, \mathcal{G}] \simeq [\mathbb{F}, \mathbb{F}] / [\mathbb{F}, R_O]$ generated by the elements $u_{i,j,k} = [w_{i,j,k}, x_{g_{i,j}}]$, where $1 \leq i \leq m, 1 \leq j \leq l_i$, and $1 \leq k \leq r_{i,j}$.

To complete the proof of Theorem 3 it suffices to note that $\pi_1$ induces a one-to-one correspondence between the sets of vertices of the $C$-graphs $\Gamma_{(\mathcal{O}_{n+1}, \mathcal{O}_{n+1})}$ and $\Gamma_{(\mathcal{G}, \mathcal{O})}$, since all $u_{i,j,k} = \mathcal{O}_{i,j,k}x_{g_{i,j}}^{-1}$ belong to ker $\varphi_n$. Therefore $\pi_1$ is an isomorphism. \hfill \Box

**Lemma 2.** Let the order of $g \in O$ be $n$ and let $[x_g, w] \in ([\mathbb{F}, \mathbb{F}] \cap R_O)/[\mathbb{F}, R_O] \subset \mathbb{F}/[\mathbb{F}, R_O]$. Then the order of the element $[x_g, w]$ is a divisor of $n$.

**Proof.** The elements $x_g^n$ and $[x_g, w]$ belong to the center of the group $\mathbb{F}/[\mathbb{F}, R_O]$. Therefore

$$[x_g^n, w] = x_g^{n-1}[x_g, w]x_g^{1-n}[x_g^{n-1}, w] = [x_g, w][x_g^{n-1}, w] = \cdots = [x_g, w]^n$$

is the unity of $\mathbb{F}/[\mathbb{F}, R_O]$. \hfill \Box

**Proposition 2.** Let the equipment $O$ of an equipped finite group $(G, O)$ consists of conjugacy classes of elements of orders coprime with $h_2(G)$. Then $a_{(G, O)} = h_2(G)$.

**Proof.** It follows from Lemma 2 and Theorem 3. \hfill \Box

**2. Proof of Theorem 1**

By definition, the **Bogomolov multiplier** $b_0(G)$ of a finite group $G$ is the order of the group

$$B_0(G) = \ker[H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \bigotimes_{A \subset G} H^2(A, \mathbb{Q}/\mathbb{Z})]$$

where $A$ runs over all abelian subgroups of $G$.

**Remark 1.** Note that it suffices to consider only restrictions to abelian groups with two generators in order to define that the element $w \in H^2(G, \mathbb{Q}/\mathbb{Z})$ is contained in $B_0(G)$.
There is a natural duality between $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $H_2(G, \mathbb{Z})$. Both groups are finite for finite groups $G$ and duality implies an isomorphism of $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ as abstract groups.

By Theorem 3 we have the inequality $h_2(G) \geq a_{(G,O)}$ for any equipped finite group $(G, O)$. By Corollary 2 in [10], we have inequality $a_{(G,O)} \geq (a_{(G,G\setminus \{1\})})$ for each equipment $O$ of $G$. Therefore to prove Theorem 1 it suffice to show that for the equipped finite group $(G, G \setminus \{1\})$ its ambiguity index $a_{(G,G\setminus \{1\})}$ is equal to $b_0(G)$.

In notation used in Section 1 and by Theorem 3, we have

$$a_{(G,G\setminus \{1\})} = \frac{h_2(G)}{k_{(G,G\setminus \{1\})}},$$

where $k_{(G,G\setminus \{1\})}$ is the order of the subgroup $K_{G\setminus \{1\}}$ of the group

$$(R_{G\setminus \{1\}} \cap [F_{G\setminus \{1\}}, F_{G\setminus \{1\}}])/[F_{G\setminus \{1\}}, R_{G\setminus \{1\}}] \cong H_2(G, \mathbb{Z})$$

generated by the elements of $R_{G\setminus \{1\}}$ of the form $[w, x_g]$, where $g \in G \setminus \{1\}$ and $w \in F_{G\setminus \{1\}}$.

**Lemma 3.** Let for some $w_1, w_2 \in F_{G\setminus \{1\}}$ the commutator $[w_1, w_2]$ belong to $R_{G\setminus \{1\}}$. Then $[w_1, w_2]$, considered as an element of $F_{G\setminus \{1\}}/[F_{G\setminus \{1\}}, R_{G\setminus \{1\}}]$, belongs to $K_{G\setminus \{1\}}$.

**Proof.** First of all, note that if $[x_g, w] \in K_{G\setminus \{1\}}$, then $[x_g, w] = [w, x_g^{-1}] = [x_g^{-1}, w^{-1}] = [x_g^{-1}, w]$ in $K_{G\setminus \{1\}}$, since $K_{G\setminus \{1\}}$ is a subgroup of the center of the $C$-group $G_{G\setminus \{1\}} = F_{G\setminus \{1\}}/[F_{G\setminus \{1\}}, R_{G\setminus \{1\}}]$ and these four commutators are conjugated to each other in $F_{G\setminus \{1\}}$. Similarly, $[w, x_g] = [x_g, w^{-1}] = [w^{-1}, x_g^{-1}] = [x_g^{-1}, w^{-1}] \in K_{G\setminus \{1\}}$, since $[w, x_g]$ is the inverse element to the element $[x_g, w]$. Note also that for any $w_1$ the element $w_1 [w, x_g] w_1^{-1}$ belongs to $K_{G\setminus \{1\}}$ if $[w, x_g] \in K_{G\setminus \{1\}}$.

Next, the elements $w_1^{-1}$ and $w_2^{-1}$, considered as elements of $G$, are equal to some elements $g_1$ and $g_2$ of $G$. Therefore if $[w_1, w_2] \in R_{G\setminus \{1\}}$ then

$$w_1 x_{g_1}, w_2 x_{g_2}, [x_{g_1}, x_{g_2}], [w_1, x_{g_1}], [w_1, x_{g_2}] \in R_{G\setminus \{1\}}.$$

In addition, we have $[w_1, w_2 x_{g_2}] \in [F_{G\setminus \{1\}}, R_{G\setminus \{1\}}]$ and

$$[w_1, w_2 x_{g_2}] = [w_1, w_2] (w_2 [w_1, x_{g_2}] w_2^{-1}).$$

Therefore $[w_1, w_2] \in R_{G\setminus \{1\}} \cap [F_{G\setminus \{1\}}, F_{G\setminus \{1\}}]$ (as an element of $K_{G\setminus \{1\}}$) is the inverse element to the element $[w_1, x_{g_2}] \in K_{G\setminus \{1\}}$ and hence $[w_1, w_2] \in K_{G\setminus \{1\}}$.

To complete the proof of Theorem 1 note that, by Lemma 3, for each imbedding $i : H \to G$ of an abelian group $H$ generated by two elements the image of $i_* : H_2(H, \mathbb{Z}) \to H_2(G, \mathbb{Z})$ is a subgroup of $K_{G\setminus \{1\}}$ and the group $K_{G\setminus \{1\}}$ is generated by the images of such elements. Therefore the group

$$K_{G\setminus \{1\}} = \{ \varphi \in \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \mid \varphi(w) = 0 \text{ for all } w \in K_{G\setminus \{1\}} \}$$

coincides with the group $B_0(G)$ and its order is equal to $a_{(G,G\setminus \{1\})} = \frac{h_2(G)}{k_{(G,G\setminus \{1\})}}$. 

\qed
3. Quasi-covers of equipped finite groups

In this section we use notations introduced in Sections 1.

3.1. Definitions. Let \( f : (G_1, O_1) \to (G, O) \) be a homomorphism of equipped groups. We say that \( f \) is a cover of equipped groups (or, equivalently, \((G_1, O_1)\) is a cover of \((G, O)\)) if

\[
\begin{align*}
(i) & \text{ \( f \) is an epimorphism such that } f(O_1) = O; \\
(ii) & \ker f \text{ is a subgroup of the center } ZG_1 \text{ of } G_1; \\
(iii) & f_* : H_1(G_1, Z) \to H_1(G, Z) \text{ is an isomorphism.}
\end{align*}
\]

Let \( f : (G_1, O_1) \to (G, O) \) be a homomorphism of equipped finite groups. We say that \( S \subset O_1 \) is a section of \( f \) if \( f|_S : S \to O \) is a one-to-one correspondence. Denote by \( O_S \subset O_1 \) the orbit of \( S \) under the action of the group of the inner automorphisms of \( G_1 \).

Let \( f : (G_1, O_1) \to (G, O) \) be an epimorphism of equipped groups such that \( \ker f \subset ZG_1 \). We say that \( f \) is a quasi-cover of equipped groups (or, equivalently, \((G_1, O_1)\) is a quasi-cover of \((G, O)\)) if there is a section \( S \) of \( f \) such that \( O_S = O_1 \).

Below, we will assume that for a quasi-cover \( f \) of equipped groups a section \( S \) is chosen and fixed.

3.2. Properties of quasi-covers.

Lemma 4. Let \( f : (G_1, O_1) \to (G, O) \) be a cover of equipped finite groups and \( S \subset O_1 \) a section. Then \( G_1 \) is generated by the elements of \( S \).

Proof. Denote by \( G_S \) the subgroup of \( G_1 \) generated by the elements of \( S \). Obviously, \( \varphi = f|_{G_S} : G_S \to G \) is an epimorphism and \( \ker \varphi \subset \ker f \). Therefore, to prove Lemma it suffices to show that \( \ker f \subset G_S \). To show this, let us consider the natural epimorphism \( f_1 : G_1 \to G_2 = G_1/\ker \varphi \) and the natural epimorphism \( \psi : G_2 \to G \) induced by \( f \). Obviously, \( \psi : (G_2, f_1(O_1)) \to (G, O) \) is a cover of equipped finite groups and \( \psi_H : H \to G \) is an isomorphism, where \( H = f_1(G_S) \). Therefore \( G_2 \simeq \ker \psi \times G \). Consequently, \( \ker \psi = 0 \), since \( \psi_* : H_1(G_2, Z) \to H_1(G, Z) \) is an isomorphism. \( \square \)

If \( S \) is a section of a cover \( f : (G_1, O_1) \to (G, O) \), then Lemma 4 implies that \( O_S = S^{G_1} \) is an equipment of \( G_1 \) and \( f : (G_1, O_S) \to (G, O) \) is also a cover of equipped groups.

Below, we fix a section \( S \) of a cover \( f : (G_1, O_1) \to (G, O) \). Then the cover \( f \) can be considered as a quasi-cover.

In notations used in Section 1 consider the universal central C-extension \( \alpha_O : (\overline{G}, O) \to (G, O) \) of an equipped finite group \((G, O)\). We have two natural epimorphisms \( h_O : F_O \to G = F_O/R_O \) and \( \beta_O : F_O \to \overline{G} = F_O/[F_O, R_O] \) such that \( h_O = \alpha_O \circ \beta_O \).
Lemma 5. Let $f : (G_1, O_1) \to (G, O)$ be a quasi-cover of equipped finite groups. Then there is an epimorphism $\alpha_S : (\overline{G}, \overline{O}) \to (G_1, O_S)$ of equipped groups such that $\alpha_O = f \circ \alpha_S$.

Proof. By Lemma 4, there is an epimorphism $h_S : F_O \to G_1$ defined by $h_S(x_g) = \hat{g} \in S$ for all $g \in G$, where $\hat{g} = f_{|S}^{-1}(g)$. Denote by $R_S = \ker h_S$. Obviously, we have $f \circ h_S = h_O$. Therefore $R_S \subset R_O$.

Let us show that the group $[F_O, R_O]$ is a subgroup of $R_S$. Indeed, consider any $w \in R_O$. Then, as an element of $G_1$, the element $w \in \ker f$ and, consequently, $w$ belongs to the center of $G_1$. In particular, it commutes with any generator $\hat{g} \in S$ of $G_1$ and hence $[w, x_g] = \hat{g} \in R_S$, that is, $[F_O, R_O] \subset R_S$.

The inclusion $[F_O, R_O] \subset R_S$ implies the desired epimorphism $\alpha_S$. \hfill \Box

We say that a cover (resp., a quasi-cover) of equipped finite groups $f : (G_1, O_1) \to (G, O)$ is maximal if for any cover of equipped finite groups $f_1 : (G_2, O_2) \to (G_1, O_1)$ such that $f_2 = f \circ f_1$ is also a cover (resp., quasi-cover) of equipped finite groups, the epimorphism $f_1$ is an isomorphism.

Theorem 4. For any cover (resp., quasi-cover) of equipped finite groups $f : (G_1, O_1) \to (G, O)$, there is a maximal cover (resp., quasi-cover) $f_2 : (G_2, O_2) \to (G, O)$ for which there is a cover $f_1 : (G_2, O_2) \to (G_1, O_S)$ such that

(i) $f_2 = f \circ f_1$;
(ii) $\ker f_2 \simeq H_2(G, \mathbb{Z})$ (resp., $[\overline{G}, \overline{G}] \cap \ker f_2 \simeq H_2(G, \mathbb{Z})$).

Proof. Consider the epimorphism $\alpha_S : (\overline{G}, \overline{O}) \to (G_1, O_S)$ defined in the proof of Lemma 5. The group $\ker \alpha_S$ is a subgroup of the center of $\overline{G}$.

Since $(\overline{G}, \overline{O})$ is a C-group and $\overline{O}$ consists of $M$ conjugacy classes, where $M = |O| = \rk F_O$, then $H_1(\overline{G}, \mathbb{Z}) = \overline{G}/[\overline{G}, \overline{G}] = \mathbb{Z}^M$. Let $\tau : \overline{G} \to \mathbb{Z}^M$ be the natural homomorphism (that is, $\tau$ is the type homomorphism $\overline{G} \to H_1(\overline{G}, \mathbb{Z})$, see Introduction). The image $\tau(\ker \alpha_S)$ is a sublattice of maximal rank in $\mathbb{Z}^M$. Let us choose a $\mathbb{Z}$-free basis $a_1, \ldots, a_M$ in $\tau(\ker \alpha_S)$ and choose elements $\overline{y}_i \in \ker \alpha_S$, $1 \leq i \leq M$, such that $\tau(\overline{y}_i) = a_i$.

Denote by $H_S$ a group generated by the elements $\overline{y}_i$, $1 \leq i \leq M$, and denote by $K_S = [\overline{G}, \overline{G}] \cap \ker \alpha_S$. Then it is easy to see that $H_S \simeq \mathbb{Z}^M$ is a subgroup of the center of $\overline{G}$, the intersection $H_S \cap [\overline{G}, \overline{G}]$ is trivial, and $\ker \alpha_S \simeq K_S \times H_S$.

Denote by $G_2 = \overline{G}/H_S$ the quotient group and by $\alpha_{H_S} : \overline{G} \to G_2$, $f_1 : G_2 \to G_1$ the natural epimorphisms. We have $\alpha_S = f_1 \circ \alpha_{H_S}$. Denote also by $O_2 = \alpha_{H_S}(\overline{O})$. Then it is easy to see that $\alpha_{H_S} : (\overline{G}, \overline{O}) \to (G_2, O_2)$ and $f_1 : (G_2, O_2) \to (G_1, O_S)$ are central extensions of equipped groups.

By construction, it is easy to see that $[\overline{G}, \overline{G}] \cap \ker \alpha_{H_S}$ is trivial and $\ker f_1 \subset [G_1, G_1]$ is a subgroup of the center of $G_1$. Therefore the epimorphism $f_1$ is a cover of equipped
groups. In addition, it is easy to see that \( \alpha_O = f_1 \circ \alpha_{H_S} \) and \( f_2 = f \circ f_1 : (G_2, O_2) \to (G, O) \) is a cover (resp., quasi-cover) of equipped groups. We have

\[
K_S \simeq \ker f_1 \subset \alpha_{H_O}([G, G] \cap \ker \alpha_0) = \alpha_{H_O}(H_2(G, Z)) \subset [G_2, G_2].
\]

Therefore, if \( k_{f_i} = |\ker f_i|, i = 1, 2 \), is the order of the group \( \ker f_i \) and \( k_f \) is the order of \( \ker f \), then

\[
h_2(G) = k_{f_2} = k_{f_1}k_f.
\]  

Since we can repeat the construction described above to the cover (resp., quasi-cover) \( f_2 \) and applying again equality (8), where new \( f \) is our \( f_2 \) and new \( f_1 \) is a cover existence of which follows from assumption that old \( f_2 \) is not maximal, we obtain that new \( f_1 \) is an isomorphism, that is, the covering \( f_2 \) is maximal. \( \square \)

In the case then \( f_1 : (G, G \setminus \{1\}) \to (G, G \setminus \{1\}) \) is an isomorphism of equipped finite groups, a maximal cover \( f_2 : (G_2, O_2) \to (G, G \setminus \{1\}) \), constructed in the proof of Theorem 4 will be called a universal maximal cover.

**Corollary 2.** For any equipped finite group \( (G, O) \) there is a maximal cover of equipped groups.

For any cover (resp., quasi-cover) \( f : (G_1, O_1) \to (G, O) \) of equipped finite groups, \( k_f = |\ker f| \leq h_2(G) \) (resp., \( k_f = |\ker f \cap [G_1, G_1]| \leq h_2(G) \)) and \( f \) is maximal if and only if \( k_f = h_2(G) \).

3.3. The ambiguity index of a quasi-cover of equipped group. Let \((\tilde{G}, \tilde{O})\) be the \( C \)-group associated with an equipped group \((G, O)\) and \( \beta_O : (\tilde{G}, \tilde{O}) \to (G, O) \) the natural epimorphism of equipped groups (see definitions in Section 1).

**Theorem 5.** Let \( f : (G_1, O_1) \to (G, O) \) be a quasi-cover of equipped finite groups. Then there is a natural \( C \)-epimorphism \( \kappa_S : (\overline{G}, \overline{O}) \to (\tilde{G}_1, \tilde{O}_S) \) such that \( \kappa_O = \tilde{f} \circ \kappa_S \) and \( \alpha_O = \beta_O \circ \tilde{f} \circ \kappa_S = f \circ \beta_{O_S} \circ \kappa_S \), where the \( C \)-epimorphism \( \kappa_O : (\overline{G}, \overline{O}) \to (\tilde{G}, \tilde{O}) \) is defined in Section 1 and the \( C \)-epimorphism \( \tilde{f} : (\tilde{G}_1, \tilde{O}_S) \to (\tilde{G}, \tilde{O}) \) is associated with \( f \).

**Proof.** In notations used in the proof of Lemma 5, we have an inclusion \( R_S \subset R_O \) of normal subgroups of \( \mathbb{F}_O \) which induces \( f : G_1 = \mathbb{F}_O/R_S \to G = \mathbb{F}_O/R_O \).

Let \( \tilde{R}_S \subset R_S \) be the normal subgroup normally generated by the elements of \( R_S \) of the form \( w_{i,j}^{-1}x_{y_i}w_{i,j}x_{y_j}^{-1} \), where \( w_{i,j} \in \mathbb{F}_O \) and \( x_{y_i}, x_{y_j} \in X_O \). For any \( w \in R_O \) and any generator \( x, g \in O \), the commutator \( [x_g, w] \in R_S \), since \( f \) is a central extension of groups. Therefore

\[
[\mathbb{F}_O, R_O] \subset \tilde{R}_S
\]  

By Claim \( 3 \) \( \tilde{G}_1 \simeq \mathbb{F}_S/\tilde{R}_S \). Therefore inclusion (9) induces an epimorphism \( \kappa_S : \overline{G} = \mathbb{F}_O/[[\mathbb{F}_O, R_O]] \to \mathbb{F}/\tilde{R}_S \simeq \tilde{G}_1 \). Obviously, the \( C \)-epimorphism \( \kappa_S : (\overline{G}, \overline{O}) \to (\tilde{G}_1, \tilde{O}_S) \) satisfies all properties claimed in Theorem 4. \( \square \)
Proof. For quasi-covers \( zx \) that \( \ker \) in the case of maximal covers, and \( f \) splits completely in \( f \). Let \( k_f \) the order of the group \( \ker f \cap [G_1, G_1] \) and by \( k_{fs} \) the order of the group \( \ker \tilde{f}_S \cap [\tilde{G}_1, \tilde{G}_1] \).

Corollary 3. Let \( f : (G_1, O_1) \to (G, O) \) be a quasi-cover of equipped finite groups, \( S \) a section of \( f \). Then

\[ h_2(G) = a_{(G,O)}k_{fs}k_S = k_f a_{(G_1,O_1)}k_S, \]

where \( k_S \) is the order of the group \( \ker k_S \cap [G, G] \).

Corollary 4. Let \( f : (G_1, O_1) \to (G, O) \) be a cover (resp., quasi-cover) of equipped finite groups, \( S \) a section of \( f \). Then for any equipment \( \tilde{O} \) of \( G_1 \) (resp., such that \( O_1 \subset \tilde{O} \)) we have an inequality \( a_{(G_1, \tilde{O})} \leq h_2(G) \).

If \( f \) is maximal, then \( a_{(G_1, \tilde{O})} = 1 \).

Proof. If \( f \) is a cover, then \( f : (G_1, \tilde{O}) \to (G, f(\tilde{O})) \) is also a cover of equipped groups and \( a_{(G_1, \tilde{O})} \leq h_2(G) \) by Corollary 3.

As it was mention in the Introduction, we have \( a_{(G_1, \tilde{O})} \leq a_{(G_1, O_1)} \) if \( O_1 \subset \tilde{O} \) and if \( f \) is a quasi-cover, then \( a_{(G_1, O_1)} \leq h_2(G) \) by Corollary 3.

If \( f \) is maximal, then \( k_f = h_2(G) \) by Corollary 2 and therefore if \( f \) is a cover then \( f : (G_1, \tilde{O}) \to (G, f(\tilde{O})) \) is also maximal. It follows from Corollary 3 that \( a_{(G_1, \tilde{O})} = 1 \) in the case of maximal covers, and \( a_{(G_1, \tilde{O})} \leq a_{(G_1, O_1)} = 1 \) in the case of maximal quasi-covers \( f \).

Let \( f : (G_1, O_1) \to (G, O) \) be a cover of equipped finite groups such that \( f^{-1}(O) = O_1 \). We say that \( f \) splits over a conjugacy class \( C \subset O \) if \( f^{-1}(C) \) consists of at least two conjugacy classes of \( G_1 \). The number \( s_f(C) \) of the conjugacy classes containing in \( f^{-1}(C) \) is called the splitting number of the conjugacy class \( C \) for \( f \). We say that \( f \) splits completely over \( C \) if \( s_f(C) = k_f \), where \( k_f = |\ker f| \).

Let \( C \) be a conjugacy class in \( G \). Consider the subgroups \( K_C \subset K_{G \setminus \{1\}} \) of the group

\[ (R_{G \setminus \{1\}} \cap [\mathbb{Z}_{G \setminus \{1\}}, \mathbb{Z}_{G \setminus \{1\}}]) / \mathbb{Z}_{G \setminus \{1\}}, R_{G \setminus \{1\}} \cong H_2(G, \mathbb{Z}), \]

where \( K_C \) is generated by the elements of \( R_{G \setminus \{1\}} \) of the form \([x_h, x_g], h \in G \setminus \{1\}\). Let \( k_C \) be the order of the group \( K_C \).

Proposition 3. Let \( f : (G_1, O_1) \to (G, G \setminus \{1\}) \) be a universal maximal cover of equipped finite groups and let \( C \) be a conjugacy class in \( G \). Then \( h_2(G) = s_f(C)k_C \).

Proof. For \( g \in C \) the preimage \( f^{-1}(C) \) consists of the conjugacy classes of the elements \( z^x_g \), where \( z \in \ker f = (R_{G \setminus \{1\}} \cap [\mathbb{Z}_{G \setminus \{1\}}, \mathbb{Z}_{G \setminus \{1\}}]) / \mathbb{Z}_{G \setminus \{1\}}, R_{G \setminus \{1\}} \cong H_2(G, \mathbb{Z}) \). Note that \( \ker f \subset ZG_1 \) and \( \ker f \) acts transitively on the set of the conjugacy classes \( C_1, \ldots, C_{k_f(C)} \) involving in \( f^{-1}(C) \), \( z(C_i) = C_j \) if \( z \mathcal{G} \in C_j \) for \( \mathcal{G} \in C_i \).
Let \( x_g \in C_1 \), where \( g \in C \). Then \( z(C_1) = C_1 \) if and only if for some \( w \in G_1 \) we have \( wx_gw^{-1} = zx_g \), that is, \( z = [w,x_g] \).

If \( f(w) = h \) then \( w = z_1x_h \) for some \( z_1 \in \ker f \) and therefore \( z = [x_h,x_g] \), that is, \( z \in K_C \). The inverse statement that each element \( z \in K_C \) leaves fixed the conjugacy class \( C_1 \) is obvious. \qed

**Proposition 4.** Let \( f : (G_1,O_1) \to (G,G \setminus \{1\}) \) be a universal maximal cover of equipped finite groups. Then \( a_{(G,O)} = h_2(G) \) if and only if \( f \) splits completely over each conjugacy class \( C \subset O \).

If \( s_f(C) = 1 \) for some conjugacy class \( C \subset O \) then \( a_{(G,O)} = 1 \).

**Proof.** We have \( k_f = h_2(G) \).

The map \( g \mapsto x_g \) is a section in \( O_1 \). Denote by \( \overline{O} \) the equipment of \( G_1 \) consisting of the elements conjugated to \( x_g, g \in O \). Therefore \( f : (G_1,\overline{O}) \to (G,O) \) is a maximal cover of equipped groups and Proposition 4 follows from Corollary 3. \qed

**Proposition 5.** Let \( f : (G_1,O_1) \to (G,G \setminus \{1\}) \) be a universal maximal cover of equipped finite groups and let \( C_1 \subset O \) and \( C_2 \subset O \) be two conjugacy classes containing in an equipment of \( G \). Then \( a_{(G,O)} = 1 \) if \( s_f(C_1) \) and \( s_f(C_2) \) are coprime.

**Proof.** Follows from Corollary 3 since the group \( \ker f_s \cap [\tilde{G}_1,\tilde{G}_1] \subset H_2(G,\mathbb{Z}) \) contains two subgroups \( K_{C_1} \) and \( K_{C_2} \) whose indices in \( H_2(G,\mathbb{Z}) \) are coprime. \qed

**Proposition 6.** Let \( f : (G_1,O_1) \to (G,G \setminus \{1\}) \) be a universal maximal cover of equipped finite groups and let \( h_2(G) = pq \), where \( p \) and \( q \) are coprime integers. Let \( C_1 \subset O \) be a conjugacy class such that \( s_f(C_1) = q \) and let \( s_f(C) \) is coprime with \( p \) for each conjugacy class \( C \subset O \). Then the ambiguity index \( a_{(G,O)} = p \).

**Proof.** Follows from Corollary 3 since the group \( \ker f_s \cap [\tilde{G}_1,\tilde{G}_1] \subset H_2(G,\mathbb{Z}) \) generated by the subgroups \( K_{C_1} \) of index \( p \) in \( \ker f \) and subgroups of indices also coprime with \( p \). \qed

### 3.4. The ambiguity indices of symmetric groups and alternating groups.

In [5], it was proved the following theorems

**Theorem 6.** (Theorem 3.8 in [5]) Let \( \Sigma_d \) be a maximal cover of the symmetric group \( \Sigma_d \). The conjugacy classes of \( \Sigma_d \) which split in \( \tilde{\Sigma}_d \) are: (a) the classes of even permutations which can be written as a product of disjoint cycles with no cycles of even length; and (b) the classes of odd permutations which can be written as a product of disjoint cycles with no two cycles of the same length (including 1).

**Theorem 7.** (Theorem 3.9 in [5]) Let \( \tilde{A}_d \) be the maximal cover of the alternating group \( \tilde{A}_d \). The conjugacy classes of \( \tilde{A}_d \) which split in \( \tilde{A}_d \) are: (a) the classes of permutations whose decompositions into disjoint cycles have no cycles of even length; and (b) the classes of permutations which can be expressed as a product of disjoint cycles with at least one cycle of even length and with no two cycles of the same length (including 1).
Remind that, by definition, an equipment $O$ of $\Sigma_d$ must contain a conjugacy class of odd permutation since the elements of the equipment must generate the group.

It is well known that for the symmetric group $\Sigma_d$, $d \geq 4$, and for the alternating group $A_d$, $d \neq 6, 7$, $d \geq 4$, the Schur multiplier $h_2(\Sigma_d) = h_2(A_d) = 2$. The following theorems are straightforward consequences of Proposition 3 and Theorems 4 – 7.

**Theorem 8.** Let $O$ be an equipment of a symmetric group $\Sigma_d$. Then $a(\Sigma_d, O) = 2$ if and only if $O$ consists of conjugacy classes of odd permutations such that they can be written as a product of disjoint cycles with no two cycles of the same length (including 1) and conjugacy classes of even permutations such that they can be written as a product of disjoint cycles with no cycles of even length. Otherwise, $a(\Sigma_d, O) = 1$.

**Theorem 9.** Let $O$ be an equipment of an alternating group $A_d$, $d \neq 6, 7$. Then $a(A_d, O) = 2$ if and only if $O$ consists of conjugacy classes of permutations whose decompositions into disjoint cycles have no cycles of even length and the classes of permutations which can be expressed as a product of disjoint cycles with at least one cycle of even length and with no two cycles of the same length (including 1). Otherwise, $a(A_d, O) = 1$.

It is well known that in the case when $d = 6, 7$, the Schur multiplier $h_2(A_d) = 6$.

For $\sigma \in A_d$ denote by $c(\sigma) = (l_1, \ldots, l_m)$ the cycle type of permutation $\sigma$, that is, the collection of lengths $l_i$ of non-trivial (that is $l_i \geq 2$) cycles entering into the factorization of $\sigma$ as a product of disjoint cycles. For a conjugacy class $C$ in $A_d$ the collection $c(C) = c(\sigma)$ is called the cycle type of $C$ if $\sigma \in C$. It is well known that the cycle type $c(C)$ does not depend on the choice of $\sigma \in C$ and there are at most two conjugacy classes in $A_d$ of a given cycle type $c$.

The group $A_d$, $d = 6, 7$, has the following non-trivial conjugacy classes:

(I) two conjugacy classes of each cycle type $(5)$, $(2, 4)$, and (if $d = 7$) $(7)$;
(II) two conjugacy classes of cycle type $(3)$ and one conjugacy class of cycle type $(3, 3)$;
(III) one conjugacy class of cycle type $(2, 2)$ and one conjugacy class of cycle type $(2, 2, 3)$ if $d = 7$.

**Proposition 7.** The ambiguity index $a(A_d, O)$, $d = 6, 7$, takes the following values:

(1) $a(A_d, O) = 6$ if $O$ contains only the elements of conjugacy classes of type (I);

(II) $a(A_d, O) = 2$ if $O$ contains only the elements of conjugacy classes of type (I) and the elements of at least one conjugacy class of type (II);

(III) $a(A_d, O) = 3$ if $O$ contains only the elements of conjugacy classes of type (I) and the elements of at least one conjugacy class of type (III);

(II+III) $a(A_d, O) = 1$ if $O$ contains the elements of at least one conjugacy class of type (II) and the elements of at least one conjugacy class of type (III).
Proof. Let \( f : (G_1, O_1) \rightarrow (A_d, A_d \setminus \{1\}) \) be the universal maximal cover.

Note that, by \( S \), \( a_{(A_d, A_d \setminus \{1\})} = 1 \). Therefore there exist elements \( \sigma_1, \ldots, \sigma_4 \) in \( A_d \) such that \([x_{\sigma_1}, x_{\sigma_2}]\) and \([x_{\sigma_3}, x_{\sigma_4}]\) in \((\mathbb{F}_{A_d \setminus \{1\}}, \mathbb{F}_{A_d \setminus \{1\}}) \cap R_{A_d} / [\mathbb{F}_{A_d \setminus \{1\}}, R_{A_d}]\) have, respectively, order two and three.

It is easy to see that for an element \( \sigma \) belonging to a conjugacy class \( C \) of type (I) the centralizer \( Z(\sigma) \subset A_d \) of the element \( \sigma \) is a cyclic group generated by \( \sigma \). Therefore \( K_C \) is the trivial group and hence \( s_f(C) = h_2(A_d) \). Therefore, by Proposition 4 \( a_{(A_d, O)} = 6 \) if \( O \) contains only the elements of conjugacy classes of type (I).

Let \( \sigma \) is of cycle type \((3, 3)\). Without loss of generality, we can assume that \( \sigma = \sigma_1 \sigma_2 \), where \( \sigma_1 = (1, 2, 3) \) and \( \sigma_2 = (4, 5, 6) \). Then the centralizer \( Z(\sigma) \subset A_d \) of \( \sigma \) is \( K_{1,2} \times \langle \sigma_2 \rangle \), where \( K_{1,2} = \langle \sigma_1 \rangle \times \langle \sigma_3 \rangle \) and \( \sigma_3 = (1, 2)(3, 4) \). We have \([x_\sigma, x_{\sigma_2, \sigma_1}^{-1}] = [x_{\sigma_1}, x_{\sigma_2}] \) in the group \( \mathbb{F}_{A_d \setminus \{1\}} / [\mathbb{F}_{A_d \setminus \{1\}}, R_{A_d}] \). Therefore \( K_C \), where \( C \) has type \((2, 2, 3)\), is a group of order at most two since the order of \( \sigma_1 \) is two (see Lemma 2) and it is of order two if and only if \([x_{\sigma_1}, x_{\sigma_2}]\) is not the unity in \( \mathbb{F}_{A_d \setminus \{1\}} / [\mathbb{F}_{A_d \setminus \{1\}}, R_{A_d}] \).

But, the embeddings \( \langle \sigma_1, \sigma_2 \rangle \subset A_d \subset \Sigma_d \) define a sequence of homomorphisms
\[
H_2(\langle \sigma_1, \sigma_2 \rangle, \mathbb{Z}) \rightarrow H_2(A_d, \mathbb{Z}) \rightarrow H_2(\Sigma_d, \mathbb{Z})
\]
such that the image of the non-trivial element \([x_{\sigma_1}, x_{\sigma_2}]\) in \( H_2(\langle \sigma_1, \sigma_2 \rangle, \mathbb{Z}) \) is a non-trivial in \( H_2(\Sigma_d, \mathbb{Z}) \). Therefore \( s_f(C) = 3 \) for the conjugacy class \( C \) of cyclic type \((2, 2, 3)\) and, similarly, \( s_f(C) = 3 \) for the conjugacy class \( C \) of cyclic type \((2, 2)\), since \( K_C \) is a subgroup of \( H_2(A_d, \mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z} \) generated by the elements of the second order (see Proposition 3) and only the elements of \( K_{C_1} \) and \( K_{C_2} \) can generate the subgroup of order two in \( H_2(A_d, \mathbb{Z}) \).

4. Cohomological description of the ambiguity indices

In notations used in Section 1 for an equipped finite group \((G, O)\) a subgroup \( K_{(G, O)} \) of \( H_2(G, \mathbb{Z}) \) was defined as follows: \( K_{(G, O)} \) is the subgroup of \((R_O \cap [\mathbb{F}_O, \mathbb{F}_O]) / [\mathbb{F}_O, R_O]\) generated by the elements of \( R_O \) of the form \([w, x]\), where \( g \in O \) and \( w \in \mathbb{F}_O \), and \( k_{(G, O)} \) is its order.
Denote

\[ B_{(G,O)} = K_{(G,O)}^1 = \{ \varphi \in \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \mid \varphi(w) = 0 \text{ for all } w \in K_{(G,O)} \} \]

a subgroup of \( H^2(G, \mathbb{Q}/\mathbb{Z}) \) dual to \( K_{(G,O)} \). As in the proof of Theorem \( \text{[1]} \) it is easy to show that

\[ B_{(G,O)} = \ker[H^2(G, \mathbb{Q}/\mathbb{Z}) \to \bigotimes_{A \subset G} H^2(A, \mathbb{Q}/\mathbb{Z})], \]

where \( A \) runs over all abelian subgroups of \( G \) generated by two elements \( g \in O \) and \( h \in G \). Let \( b_{(G,O)} \) be the order of the group \( B_{(G,O)} \). In particular, \( b_{(G,G\setminus\{1\})} = b_0(G) \).

The next theorem immediately follows from Theorem \( \text{[3]} \)

**Theorem 10.** For an equipped finite group \((G,O)\) we have \( a_{(G,O)} = b_{(G,O)} \).

The group \( H^2(G, \mathbb{Q}/\mathbb{Z}) \) is a direct sum of primary components \( H^2(G, \mathbb{Q}/\mathbb{Z}) = \Sigma_p H^2(G, \mathbb{Q}/\mathbb{Z})_p \) where primes \( p \) run through a subset of primes dividing the order of \( H^2(G, \mathbb{Q}/\mathbb{Z}) \) and hence \( G \). Therefore we have the following:

**Proposition 8.** If the set of conjugacy classes \( O \) consists of all classes of power of prime order then \( a_{(G,O)} = b_0(G) \). Moreover it is sufficient to consider such classes only for primes dividing \( h_2(G) \).

Note that \( H^2(G, \mathbb{Q}/\mathbb{Z})_p \) embeds into \( H^2(\text{Syl}_p(G), \mathbb{Q}/\mathbb{Z})_p \) where \( \text{Syl}_p(G) \) is a Sylow \( p \)-subgroup of \( G \). Similarly the \( p \)-primary component \( B_0(G)_p \) is a subgroup of \( B_0(\text{Syl}_p(G)) \).

More explicit versions of Proposition \( \text{[3]} \) for different groups provide with simple methods to compute \( B_O(G) \)

5. **An example of a finite group \( G \) with \( b_0(G) > 1 \)**

The following groups where constructed in the article of Saltman \( \text{[13]} \).

Consider a finite \( p \)-group \( G_p \) which is a central extension of \( \mathbb{Z}_p^4 = A_p \) with generators \( x_i \). The center of \( G_p \) is generated by pairwise commutators \( x_i x_j x_i^{-1} x_j^{-1} = [x_i, x_j] \) with one relation between \([x_1, x_2][x_3, x_4] = 1\). Thus there is natural exact sequence:

\[ 1 \to \mathbb{Z}_p^5 \to G_p \to A_p \to 1 \]

**Lemma 6.** (\( \text{[1]}, \text{[3]} \)) \( B_0(G_p) = \mathbb{Z}/p \).

**Proof.** It is shown in \( \text{[1]} \) using standard spectral sequence that for a central extension \( G \) of an abelian group \( A \) the group \( B_0(G) \) is contained in the image of \( H^2(A, \mathbb{Q}/\mathbb{Z}) \) in \( H^2(G, \mathbb{Q}/\mathbb{Z}) \). The group \( H^2(A_p, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_p^6 \) which is generated by elements \([x_i, x_j]^*\).

The kernel of the map \( H^2(A_p, \mathbb{Q}/\mathbb{Z}) \to H^2(G_p, \mathbb{Q}/\mathbb{Z}) \) is naturally dual to the center \( \mathbb{Z}_p^5 \) of \( G_p \). Thus the image of \( H^2(A_p, \mathbb{Q}/\mathbb{Z}) \) in \( H^2(G_p, \mathbb{Q}/\mathbb{Z}) \) is a cyclic \( p \)-group generated by one element \( w \). Let us show that the latter is in \( B_0(G_p) \). It is enough to check that it is trivial on any abelian subgroup in \( G_p \) which surjects onto rank 2 subgroup
$\mathbb{Z}_p^2 \subset \mathbb{Z}_p^4 = A_p$. However $G_p$ does not contain such subgroups. Indeed assume that the restriction $w$ on a subgroup with generators $x_1, y_1 \in G_p$ is trivial. It means that the commutator $[x, y] = 1$ in $G_p$ where $x, y$ are projections of $x_1, y_1$ into $A_p$. On the other hand the only nontrivial relation between commutators of elements in $A_p$ is $[x_1, x_2][x_3, x_4] = 1$ which is not equal to $[x, y]$ for any pair $x, y \in A_p$. Hence $w$ restricts trivially onto any subgroup with two generators in $G_p$ and generates $B_0(G_p)$.

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