HEEGNER POINTS AT EISENSTEIN PRIMES AND TWISTS OF ELLIPTIC CURVES

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Abstract. For any elliptic curve $E$ with a rational 3-isogeny, we show that a positive proportion of quadratic twists of $E$ have analytic rank 0 (resp. 1). In other words, the weak Goldfeld conjecture holds for $E$. We also show that the weak Goldfeld conjecture holds for the sextic twists family of $j$-invariant 0 curves. To prove these results, we establish a general criterion for the non-triviality of the $p$-adic logarithm of Heegner points at an Eisenstein prime $p$, in terms of the relative $p$-class numbers of certain number fields and then apply this criterion to the special case $p = 3$. As a by-product, we also prove the 3-part of the Birch and Swinnerton-Dyer conjecture for many elliptic curves of $j$-invariant 0.

1. Introduction

1.1. Goldfeld’s conjecture. Let $E$ be an elliptic curve over $\mathbb{Q}$. We denote by $r_{an}(E)$ its analytic rank. By the theorem of Gross–Zagier and Kolyvagin, the rank part of the Birch and Swinnerton-Dyer conjecture holds whenever $r_{an}(E) \in \{0, 1\}$. One can ask the following natural question: how is $r_{an}(E)$ distributed when $E$ varies in families? The simplest (1-parameter) family is given by the quadratic twists family of a given curve $E$. For a fundamental discriminant $d$, we denote by $E^{(d)}$ the quadratic twist of $E$ by $\mathbb{Q}(\sqrt{d})$. The celebrated conjecture of Goldfeld [Gold ] asserts that $r_{an}(E^{(d)})$ tends to be as low as possible (compatible with the sign of the function equation). Namely in the quadratic twists family $\{E^{(d)}\}$, $r_{an}$ should be 0 (resp. 1) for 50% of $d$’s. Although $r_{an} \geq 2$ occurs infinitely often, its occurrence should be sparse and accounts for only 0% of $d$’s. More precisely,

Conjecture 1.1 (Goldfeld). Let

$$N_r(E, X) = \{ |d| < X : r_{an}(E^{(d)}) = r \}.$$ 

Then for $r \in \{0, 1\}$,

$$N_r(E, X) \sim \frac{1}{2} \sum_{|d| < X} 1, \quad X \to \infty.$$ 

Here $d$ runs over all fundamental discriminants.

Goldfeld’s conjecture is widely open: we do not yet know a single example $E$ for which Conjecture 1.1 is valid. One can instead consider the following weak version (replacing 50% by any positive proportion):

Conjecture 1.2 (Weak Goldfeld). For $r \in \{0, 1\}$, $N_r(E, X) \gg X$.

Remark 1.3. Heath-Brown ([Hea04, Thm. 4]) proved Conjecture 1.2 conditional on GRH.
Remark 1.4. Katz–Sarnak [KS99] conjectured the analogue of Conjecture 1.1 for the 2-parameter family \( \{E_{A,B} : y^2 = x^3 + Ax + B\} \) of all elliptic curves over \( \mathbb{Q} \). The weak version in this case is now known unconditionally due to the recent work of Bhargava–Skinner–W. Zhang [BSZ14]. However, their method does not directly apply to quadratic twists families.

The curve \( E = X_0(19) \) is the first known example for which Conjecture 1.2 is valid (see James [Jam98] for \( r = 0 \) and Vatsal [Vat98] for \( r = 1 \)). Later many authors have verified Conjecture 1.2 for infinitely many curves (see [Vat99], [BJK09] and [Kri16]). However, all these examples are a bit special, as they are all covered by our first main result:

Theorem 1.5. The weak Goldfeld Conjecture is true for any \( E \) with a rational 3-isogeny.

Remark 1.6. Theorem 1.5 gives so far the most general results for Conjecture 1.2. There is only one known example for which Conjecture 1.2 is valid and is not covered by Theorem 1.5: the congruent number curve \( E : y^2 = x^3 - x \) (due to the recent work of Smith [Smi16] and Tian–Yuan–S. Zhang [TYZ14]).

For an elliptic curve \( E \) of \( j \)-invariant 0 (resp. 1728), one can also consider its cubic or sextic (resp. quartic) twists family. The weak Goldfeld conjecture in these cases asserts that for \( r \in \{0, 1\} \), a positive proportion of (higher) twists should have analytic rank \( r \). Our second main result verifies the weak Goldfeld conjecture for the sextic twists family. More precisely, consider the elliptic curve

\[
E = X_0(27) : y^2 = x^3 - 432
\]

of \( j \)-invariant 0 (isomorphic to the Fermat cubic \( X^3 + Y^3 = 1 \)). For a 6th-power-free integer \( d \), we denote by

\[
E_d : y^2 = x^3 - 432d
\]

the \( d \)-th sextic twist of \( E \).

Theorem 1.7 (Theorem 5.9). The weak Goldfeld conjecture is true for the sextic twists family \( \{E_d\} \): for \( r \in \{0, 1\} \), there exists a positive proportion of 6th-power-free integers \( d \) such that \( r_{an}(E_d) = r \).

Remark 1.8. For explicit lower bounds for the proportion in Theorems 1.5 and 1.7, see the more precise statements in Theorems 4.3, 4.4 and 5.9 and Example 4.7.

Remark 1.9. For a wide class of elliptic curves of \( j \)-invariant 0, we can also construct many (in fact \( \gg X/\log^{7/8} X \)) cubic twists of analytic rank 0 (resp. 1). However, these cubic twists do not have positive density. See Theorem 6.1 for a precise statement and Example 6.3.

1.2. Heegner points at Eisenstein primes. The above results on weak Goldfeld conjecture are applications of a more general \( p \)-adic criterion for non-triviality of Heegner points on \( E \) (applied to \( p = 3 \)). To be more precise, let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \). Let \( K = \mathbb{Q}(\sqrt{d_K}) \) denote an imaginary quadratic field of fundamental discriminant \( d_K \). We assume that \( K \) satisfies the Heegner hypothesis for \( N \):

each prime factor \( \ell \) of \( N \) is split in \( K \).

For simplicity, we also assume that \( d_K \neq -3, -4 \) so that \( \mathcal{O}_K = \{-1, 1\} \), and that \( d_K \) is odd (i.e. \( d_K \equiv 1 \) (mod 4)). We denote by \( P \in E(K) \) the corresponding Heegner point, defined up to sign and torsion with respect to a fixed modular parametrization \( \pi_E : X_0(N) \to E \) (see [Gro84]). Let

\[
f(q) = \sum_{n=1}^{\infty} a_n(E)q^n \in S^\text{new}_2(\Gamma_0(N))
\]
be the normalized newform associated to $E$. Let $\omega_E \in \Omega^1_{E/Q} := H^0(E/Q, \Omega^1)$ such that
\[ \pi^*_E(\omega_E) = f(q) \cdot dq/q. \]
We denote by $\log_{\omega_E}$ the formal logarithm associated to $\omega_E$. Notice $\omega_E$ may differ from the Néron differential by a scalar when $E$ is not the optimal curve in its isogeny class.

For a finite order Galois character $\psi : G_Q := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$, we abuse notation and denote by $\psi : (\mathbb{Z}/f(\psi)\mathbb{Z})^\times \to \mathbb{C}^\times$ the corresponding Dirichlet character, where $f(\psi)$ is its conductor. The generalized (first) Bernoulli number is defined to be
\[ B_{1,\psi} := \frac{1}{f(\psi)} \sum_{m=1}^{f(\psi)} \psi(m)m. \]

Let $\varepsilon_K$ be the quadratic character associated to $K$. We consider the even Dirichlet character
\[ \psi_0 := \begin{cases} \psi, & \text{if } \psi \text{ is even}, \\ \psi \varepsilon_K, & \text{if } \psi \text{ is odd}. \end{cases} \]

Now suppose $p$ is an Eisenstein prime for $E$ (i.e., $E[p]$ is a reducible $G_Q$-representation, or equivalently, $E$ admits a rational $p$-isogeny), we prove the following criterion for the non-triviality of the $p$-adic logarithm of Heegner points, in terms of the $p$-indivisibility of Bernoulli numbers.

**Theorem 1.10** (Theorem [2.1]). Suppose $p$ is an odd prime such that $E[p]$ is a reducible $G_Q$-representation. Write $E[p]^{\text{ss}} \cong \mathbb{F}_p(\psi) \oplus \mathbb{F}_p(\psi^{-1}\omega)$, for some character $\psi : G_Q \to \text{Aut}(\mathbb{F}_p) \cong \mu_{p-1}$ and the mod $p$ cyclotomic character $\omega$. Assume that
1. $\psi(p) \neq 1$ and $(\psi^{-1}\omega)(p) \neq 1$.
2. $E$ has no primes of split multiplicative reduction.
3. If $\ell \neq p$ is an additive prime for $E$, then $\psi(\ell) \neq 1$ and $(\psi^{-1}\omega)(\ell) \neq 1$.

Let $K$ be an imaginary quadratic field satisfying the Heegner hypothesis for $N$. Assume $p$ splits in $K$. Assume
\[ B_{1,\psi_0^{-1}\varepsilon_K} \cdot B_{1,\psi_0\omega^{-1}} \neq 0 \pmod{p}. \]

Then
\[ \frac{|\tilde{E}^{\text{ns}}(\mathbb{F}_p)|}{p} \cdot \log_{\omega_E} P \neq 0 \pmod{p}. \]

In particular, $P \in E(K)$ is of infinite order and $E/K$ has analytic and algebraic rank 1.

Notice that the two odd Dirichlet characters $\psi_0^{-1}\varepsilon_K$ and $\psi_0\omega^{-1}$ cut out two abelian CM fields (of degree dividing $p-1$). When the relative $p$-class numbers of these two CM fields are trivial, it follows from the relative class number formula that the two Bernoulli numbers in Theorem 1.10 are nonzero mod $p$ (see [3]), hence we conclude $r_{\text{an}}(E/K) = 1$. When $p = 3$, the relative $p$-class numbers becomes the 3-class numbers of two quadratic fields. Our final ingredient to finish the proof of Theorems 1.3 and 1.10 is Davenport–Heilbronn’s theorem ([DH71]) (enhanced by Nakagawa–Horie [NH88] with congruence conditions), which allows one to find a positive proportion of twists such that both 3-class numbers in question are trivial.

**Remark 1.11.** Although the methods are completely different, the final appearance of Davenport–Heilbronn type theorem is a common feature in all previous works ([Jam98], [Vat98], [Vat99], [BJK09]), and also ours.
1.3. A by-product: the 3-part of the BSD conjecture. The Birch and Swinnerton-Dyer conjecture predicts the precise formula

\[ \frac{L^{(r)}(E/Q, 1)}{r! \Omega(E/Q) R(E/Q)} = \prod_p c_p(E/Q) \cdot |\text{III}(E/Q)| \frac{1}{|E(Q)_{\text{tor}}|^2} \]

for the leading coefficient of the Taylor expansion of \( L(E/Q, s) \) at \( s = 1 \) (here \( r = r_{an}(E) \)) in terms of various important arithmetic invariants of \( E \) (see [Gro11] for detailed definitions). When \( r \leq 1 \), both sides of the BSD formula (2) are known to be positive rational numbers. To prove that (2) is indeed an equality, it suffices to prove that it is an equality up to a \( p \)-adic unit, for each prime \( p \). This is known as the \( p \)-part of the BSD formula (BSD\((p)\) for short). Much progress has been made recently, but only in the case \( p \) is semi-stable and non-Eisenstein (for \( r = 0 \), [Kat04], [SU14], [Wan14]; for \( r = 1 \): [Zha14], [SZ14], [BBV15], [JSW15]). We establish the following new results on BSD(3) for many sextic twists \( E_d : y^2 = x^3 - 432d \), in the case \( p = 3 \) is additive and Eisenstein.

**Theorem 1.12** (Theorem 5.10). Suppose \( K \) is an imaginary quadratic field satisfies the Heegner hypothesis for \( 3d \). Assume that

1. \( d \) is a fundamental discriminant.
2. \( d \equiv 2, 3, 5, 8 \pmod{9} \).
3. If \( d > 0 \), \( h_3(-3d) = h_3(d_Kd) = 1 \). If \( d < 0 \), \( h_3(d) = h_3(-3d_Kd) = 1 \).
4. The Manin constant of \( E_d \) is coprime to 3.

Then \( r_{an}(E_d/K) = 1 \) and BSD(3) holds for \( E_d/K \). (\( h_3(D) \) denotes the 3-class number of \( \mathbb{Q}(\sqrt{D}) \)).

**Remark 1.13.** Since the curve \( E_d \) has complex multiplication by \( \mathbb{Q}(\sqrt{-3}) \), we already know that BSD\((p)\) holds for \( E_d/Q \) if \( p \neq 2, 3 \) (when \( r = 0 \)) and if \( p \neq 2, 3 \) is a prime of good reduction (when \( r = 1 \)) thanks to the works [Rub91], [PRS87], [Kob13], [PR04].

1.4. Structure of the paper. In §2 we establish the non-triviality criterion for Heegner points at Eisenstein primes, in terms of \( p \)-indivisibility of Bernoulli numbers (Theorem 1.10). In §3 we explain the relation between the Bernoulli numbers and relative class numbers. In §4 we combine our criterion and the Nakagawa–Horie theorem to prove the weak Goldfeld conjecture for curves with a 3-isogeny (Theorem 1.5). In §5 we give applications to the sextic twists family (Theorems 1.7 and 1.12). Finally, in §6 we give an application to cubic twists families (Theorem 6.1).

2. Heegner points at Eisenstein primes

Throughout, let \( E/Q \) be an elliptic curve of conductor \( N = N_{\text{split}}N_{\text{nonsplit}}N_{\text{add}} \), where \( N_{\text{split}} \) is only divisible by primes of split multiplicative reduction, \( N_{\text{nonsplit}} \) is only divisible by primes of nonsplit multiplicative reduction, and \( N_{\text{add}} \) is only divisible by primes of additive reduction.

2.1. Notations and conventions. Fix an algebraic closure \( \overline{Q} \) of \( Q \), and view all number fields \( L \) as embedded \( L \subset \overline{Q} \). Fix an algebraic closure \( \overline{Q}_p \) of \( Q_p \) (which amounts to fixing a prime of \( \overline{Q} \) above \( p \)). Let \( \mathbb{C}_p \), be the \( p \)-adic completion of \( \overline{Q}_p \), and let \( L_p \) denote the \( p \)-adic completion of \( L \subset \mathbb{C}_p \). All Dirichlet (i.e. finite order Hecke) characters \( \psi : \mathbb{A}_Q^\times \) will be primitive, and we denote the conductor (an ideal in \( \mathbb{Z} \)) by \( f(\psi) \), and let \( |f(\psi)| \) denote the smallest positive integer in \( f(\psi) \).

We may equivalently view \( \psi \) as a character \( \psi : (\mathbb{Z}/f(\psi))^\times \rightarrow \overline{Q}^\times \) via

\[ \psi(x \text{ mod } f(\psi)) = \prod_{\ell | f(\psi)} \psi_\ell(x) = \prod_{\ell | f(\psi)} \psi_\ell^{-1}(x) \]
where \(\psi_\ell : \mathbb{Q}_\ell \to \mathbb{C}^\times\) is the local character at \(\ell\). Following convention, we extend \(\psi\) to \(\mathbb{Z}/f(\psi) \to \mathbb{Q}\), defining \(\psi(a) = 0\) if \((a, f(\psi)) \neq 1\). Given Dirichlet character \(\psi_1\) and \(\psi_2\), we let \(\psi_1 \psi_2\) denote the unique primitive Dirichlet character such that \(\psi_1 \psi_2(a) = \psi_1(a) \psi_2(a)\) for all \(a \in \mathbb{Z}\) with \((a, f(\psi)) = 1\). We define the Gauss sum of \(\psi\) by

\[
g(\psi) := \sum_{a \in (\mathbb{Z}/f(\psi)\mathbb{Z})^\times} \psi(a) e^{2\pi i (a/f(\psi))}.
\]

For a finite prime \(\ell\), we define the Gauss sum of the local character \(\psi_\ell : \mathbb{Q}_\ell^\times \to \mathbb{C}^\times\) similarly: letting \(|f(\psi)| = \prod \ell^\ell_{\ell}\), we put

\[
g_\ell(\psi) := \psi_\ell(\ell^{\ell_{\ell}}) \sum_{a \in (\mathbb{Z}_\ell/\ell^{\ell_{\ell}}\mathbb{Z}_\ell)^\times} \psi_\ell^{-1}(a) e^{2\pi i (a/f(\psi))},
\]

where \(\{x\}_\ell\) denotes the \(\ell\)-fractional part of \(x \in \mathbb{Z}_\ell\). Throughout, for a given \(p\), let \(\omega : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mu_{p-1}\) denote the mod \(p\) cyclotomic character. Let \(N_{\mathbb{Q}} : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times\) denote the norm character, normalized to have infinity type \(-1\). For a number field \(K\), let \(N_{\mathbb{K}/\mathbb{Q}} : \mathbb{A}_\mathbb{K}^\times \to \mathbb{A}_{\mathbb{Q}}^\times\) denote the idelic norm, and let \(N_{\mathbb{K}} := N_{\mathbb{Q}} \circ N_{\mathbb{K}/\mathbb{Q}} : \mathbb{A}_\mathbb{K}^\times \to \mathbb{C}^\times\). Suppose we are given an imaginary quadratic field \(K\). For any Dirichlet character \(\psi\) over \(\mathbb{Q}\), let

\[
\psi_0 := \begin{cases} 
\psi & \text{if } \psi \text{ even}, \\
\psi_{\mathbb{K}} & \text{if } \psi \text{ odd}.
\end{cases}
\]

Finally, for any number field \(L\), let \(h_L\) denote its class number.

### 2.2. Main theorem

We will show, by direct \(p\)-adic integration, the following generalization of [Kri16, Theorem 13]. Our generalization, in particular, does not require \(p \nmid N\).

**Theorem 2.1.** Suppose \(p\) is a prime \(E[p]\) is a reducible \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-representation, or equivalently, \(E[p]^{ns} \cong \mathbb{F}_p(\psi) \oplus \mathbb{F}_p(\psi^{-1}\omega)\), and \(\psi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mu_{p-1}\) is some Dirichlet character. Let \(K\) be an imaginary quadratic field such that \(p\) splits in \(K\), \(K\) satisfies the Heegner hypothesis with respect to \(E\). Suppose further that either the following four conditions hold

1. \(\psi(p) \neq 1\) and \((\psi^{-1}\omega)(p) \neq 1\),
2. \(N_{\text{split}} = 1\),
3. \(\ell \neq p, \ell | N_{\text{add}}\) implies either \(\psi(\ell) \neq 1\) and \(\ell \notin \psi(\ell) \pmod{p}\), or \(\psi(\ell) = 0\),
4. \(p \nmid B_{1,\psi_0^{-1}\epsilon_{\mathbb{K}}}: B_{1,\psi_0\omega^{-1}}\),

or the following three conditions hold

1. \(\psi = 1\),
2. \(N\) is a power of \(p\),
3. \(p \nmid h_K\).

Then

\[
\frac{|\bar{E}_{ns}(\mathbb{F}_p)|}{p} \cdot \log_{\omega_{E}} P \neq 0 \pmod{p}.
\]

In particular, \(P \in E(K)\) is of infinite order and \(E/K\) has algebraic and analytic rank 1.

**Remark 2.2.** When \(p = 2\), we must have \(\psi = 1\).

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1. Here our generalization also corrects a self-contained typo in the statement of Theorem 13 in loc. cit., where part of condition (3) was mistranscribed from Theorem 4 in loc. cit.: “\(\ell \neq -1 \pmod{p}\)” should be “\(\ell \neq \psi(\ell) \pmod{p}\)”. 
Remark 2.3. Note that when \( p = 3 \) or \( \psi \) is quadratic, condition (3) in the statement of Theorem 2.1 is equivalent to \( \ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 1 \pmod{3} \) implies that \( \psi(\ell) = -1 \), and \( \ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 2 \pmod{3} \) implies that \( \psi(\ell) = 0 \).

2.3. Stabilization operators. Here, we recall the definition of “stabilization operators”, as in [KL16 §3.2]. We will use Katz’s notion of \( p \)-adic modular forms as rules on the space of isomorphism classes of ordinary test triples (see [KL16 Definition 3.1 and 3.2]). Let \( \hat{M}_k^{p-\text{adic}}(\Gamma_0(N)) \) denote the space of weak \( p \)-adic modular forms of level \( N \) and \( M_k^{p-\text{adic}}(\Gamma_0(N)) \) and the space of \( p \)-adic modular forms of level \( N \), respectively. (See the paragraph after Definition 3.2 in loc. cit.) Note that \( M_k^{p-\text{adic}}(\Gamma_0(N)) \subset \hat{M}_k^{p-\text{adic}}(\Gamma_0(N)) \).

From now on, let \( N \) denote the minimal level of \( F \) (i.e. the smallest \( N \) such that \( F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N)) \)). When a larger level \( N' \) is clear from context, we will often view \( F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N')) \) by identifying \( F(A,C,\omega) = F(A,C[N],\omega) \).

Fix \( N^\# \in \mathbb{Z}_{>0} \) such that \( N | N^\# \), so that we can view \( F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N^\#)) \), and further suppose \( \ell^2 | N^\# \) where \( \ell \) is a prime (not necessarily different from \( p \)). Take the base ring \( S = \mathcal{O}_{C_p} \). Then the operator on \( \hat{M}_k^{p-\text{adic}}(\Gamma_0(N^\#)) \) given on \( q \)-expansions by

\[
F(q) \mapsto F(q^\ell)
\]

has a moduli-theoretic interpretation given by “dividing by \( \ell \)-level structure”. That is, we have an operation on test triples \((A,C,\omega)\) defined over \( p \)-adic \( \mathcal{O}_{C_p} \)-algebras \( R \) given by

\[
V_\ell(A,C,\omega) = (A/C[\ell], \pi(C), \tilde{\pi}^*\omega)
\]

where \( \pi : A \to A/C[\ell] \) is the canonical projection and \( \tilde{\pi} : A/C[\ell] \to A \) is its dual isogeny.

Thus \( V_\ell \) induces a form \( V_\ell^*F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N^\#)) \) defined by

\[
V_\ell^*F(A,C,\omega) := F(V_\ell(A,C,\omega)).
\]

For the Tate curve test triple \((\text{Tate}(q), \mu_{N^\#}, du/u)\), one sees that \((\mu_{N^\#})[\ell] = \mu_\ell \) and \( \pi : \text{Tate}(q) \to \text{Tate}(q^\ell) \). Since \( \pi : \hat{G}_m = \text{Tate}(q) \to \text{Tate}(q^\ell) = \hat{G}_m \) is multiplication by \( \ell \), we have \( \pi^*du/u = \ell \cdot du/u \), and so \( \tilde{\pi}^*du/u = du/u \). Thus one sees that \( V_\ell \) acts on \( q \)-expansions by

\[
V_\ell^*F(q) = V_\ell^*F(\text{Tate}(q), \mu_{N^\#}, du/u) = F(\text{Tate}(q^\ell), \mu_{N^\#/\ell}, du/u) = F(q^\ell).
\]

If \( F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N^\#)) \), then \( V_\ell^*F \in M_k^{p-\text{adic}}(\Gamma_0(N^\#)) \), and the \( q \)-expansion principle then implies that \( V_\ell^*F \) is the unique \( p \)-adic modular form of level \( N^\# \) with \( q \)-expansion \( F(q^\ell) \).

Now we define the stabilization operators as operations on rules on the moduli space of isomorphism classes of test triples. Let \( F \in \hat{M}_k^{p-\text{adic}}(\Gamma_0(N^\#)) \), and let \( a_\ell(F) \) denote the coefficient of the \( q^\ell \) term in the \( q \)-expansion \( F(q) \). Then up to permutation there is a unique pair of numbers \((\alpha_\ell(F), \beta_\ell(F)) \in \mathbb{C}_p^2 \) such that \( a_\ell(F) + \beta_\ell(F) = a_\ell(F), a_\ell(F)\beta_\ell(F) = \ell, \) and \( \text{ord}_p(\alpha_\ell(F)) \leq \text{ord}_p(\beta_\ell(F)) \). We henceforth fix such a choice of \((\alpha_\ell, \beta_\ell)\) (which is uniquely determined when \( \text{ord}_p(\alpha_\ell(F)) < \text{ord}_p(\beta_\ell(F)) \), as it will be in our application to Eisenstein series). When \( \ell \nmid N \), we define the \((\ell)^+\)-stabilization of \( F \) as

\[
F^{(\ell)^+} = F - \beta_\ell(F)V_\ell^*F,
\]

the \((\ell)^-\)-stabilization of \( F \) as

\[
F^{(\ell)^-} = F - \alpha_\ell(F)V_\ell^*F,
\]
and the \((\ell^0)\)-stabilization for \(F\) as
\[
F^{(\ell^0)} = F - a_\ell(F)V^*_\ell F + \ell^{-1}V^*_\ell V^*_\ell F.
\]
When \(\ell|N\), we define the \((\ell^0)\)-stabilization of \(F\) as
\[
F^{(\ell^0)} = F - a_\ell(F)V^*_\ell F.
\]
We have \(F^{(\ell^+,-,0)} \in M_k^{p\text{adic}}(\Gamma_0(N^\#))\). If \(\ell \nmid N\), then \(F^{(\ell^+,-,0)}\) have \(q\)-expansions
\[
F^{(\ell^+)}(q) := F(q) - \beta_\ell(F)F(q^\ell),
\]
\[
F^{(\ell^-)}(q) := F(q) - \alpha_\ell(F)F(q^\ell),
\]
\[
F^{(\ell^0)}(q) := F(q) - a_\ell(F)F(q^\ell) + \ell^{-1}F(q^{\ell^2}).
\]
In fact, if \(F\) is a \(T_n\)-eigenform where \(\ell \nmid n\), then \(F^{(\ell^+,-,0)}\) is still an eigenform for \(T_n\). If \(F\) is a \(T_\ell\) eigenform, one verifies by direct computation that \(T_\ell F^{(\ell^+)} = \alpha_\ell F\), \(T_\ell F^{(\ell^-)} = \beta_\ell F\), and \(T_\ell F = 0\). Similarly, if \(\ell|N\), then
\[
F^{(\ell^0)}(q) := F(q) - a_\ell(F)F(q^\ell).
\]
If \(F\) is a \(U_n\)-eigenform where \(\ell \nmid n\), then \(F^{(\ell^0)}\) is still an eigenform for \(U_n\). If \(F\) is a \(U_\ell\)-eigenform, one verifies by direct computation that \(U_\ell F = 0\). Note that for \(\ell_1 \neq \ell_2\), the stabilization operators \(F \mapsto F^{(\ell_1^+,-,0)}\) and \(F \mapsto F^{(\ell_2^+,-,0)}\) commute. Then we define, for integers \(N_+ = \prod \ell_+, N_- = \prod \ell_-\), \(N_0 = \prod \ell_0\), we define the \((N^+)(N^-)(N_0)\)-stabilization of \(F\) as
\[
F^{(N^+)(N^-)(N_0)^0} := F^{(\ell^+)^0}\prod (\ell^0)^0.
\]

### 2.4. Stabilization operators at CM points

Let \(K\) be an imaginary quadratic field satisfying the Heegner hypothesis with respect to \(N^\#\). Assume that \(p\) splits in \(K\), and let \(\mathfrak{p}\) be prime above \(p\) determined by the embedding \(K \subset \mathbb{C}_p\). Let \(\mathfrak{m}^\# \subset \mathcal{O}_K\) be a fixed ideal such that \(\mathcal{O}/\mathfrak{m}^\# = \mathbb{Z}/N^\#\), and if \(p|\mathfrak{m}^\#\), we specify that \(p\mathfrak{m}^\#\). Let \(A/\mathcal{O}_\mathbb{C}_p\) be an elliptic curve with CM by \(\mathcal{O}_K\). By the theory of complex multiplication and Deuring’s theorem, \((A, A[\mathfrak{m}^\#], \omega)\) is an ordinary test triple over \(\mathcal{O}_\mathbb{C}_p\).

Given an ideal \(\mathfrak{a} \subset \mathcal{O}_K\), we define \(A_{\mathfrak{a}} = A/A[\mathfrak{a}]\), which is also an elliptic curve over \(\mathcal{O}_\mathbb{C}_p\) which has CM by \(\mathcal{O}_K\). Note that if \(\mathfrak{a}\) is principal, then \(A_{\mathfrak{a}} \cong A\). If \(\mathfrak{a}\) is prime to \(\mathfrak{m}^\#\), we can define
\[
\mathfrak{a} \ast (A, A[\mathfrak{m}^\#], \omega) := (A_{\mathfrak{a}}, A_{\mathfrak{a}}[\mathfrak{m}^\#], \omega_{\mathfrak{a}})
\]
where \(\phi_{\mathfrak{a}} : A \to A_{\mathfrak{a}}\) is the canonical projection, and \(\omega_{\mathfrak{a}} \in \Omega_{A_{\mathfrak{a}}/\mathbb{C}_p}^1\) is such that \(\phi_{\mathfrak{a}}^*\omega_{\mathfrak{a}} = \omega\). Note that \(\omega_{\mathfrak{a}}\) might not be defined over \(\mathcal{O}_\mathbb{C}_p\), and in fact \(\omega_{\mathfrak{a}} \in \Omega_{A_{\mathfrak{a}}/\mathcal{O}_\mathbb{C}_p}^1\) if and only if \(p \nmid \mathfrak{a}\). Note that this definition of the Shimura action on CM triples is slightly different from the action of \(\mathfrak{a}\) on CM triples given in Section 3.4 of loc. cit.

One verifies that gives ideals \(\mathfrak{a}, \mathfrak{a}' \subset \mathcal{O}_K\) which are prime to \(\mathfrak{m}^\#\), the canonical isomorphism \(\mathfrak{a} \ast (A' \ast A) \cong \mathfrak{a}' \ast A\) induces an isomorphism \(\mathfrak{a} \ast (A' \ast (A, A[\mathfrak{m}^\#], \omega)) \cong \mathfrak{a}' \ast (A, A[\mathfrak{m}^\#], \omega)\). Hence there is an induced action of \(\mathcal{C}(\mathcal{O}_K)\) on the set of isomorphism classes \([A, A[\mathfrak{m}^\#], \omega])\), given by \([\mathfrak{a}] \ast ((A, A[\mathfrak{m}^\#], \omega]) = [\mathfrak{a} \ast (A, A[\mathfrak{m}^\#], \omega])\). Finally, note that for any level dividing \(N^\#\), the Shimura reciprocity law also induces an action of \(\mathcal{C}(\mathcal{O}_K)\) on isomorphism classes of ordinary CM test triples of that level in an analogous way.

**Lemma 2.4.** For a prime \(\ell\), let \(v|\mathfrak{m}^\#\) be the corresponding prime ideal of \(\mathcal{O}_K\) above it, and let \(\mathfrak{a} \subset \mathcal{O}_K\) be an ideal prime to \(\mathfrak{m}^\#\). Then for any \(\omega \in \Omega_{A/\mathcal{O}_\mathbb{C}_p}^1\), we have
\[
[V_{\ell}(\mathfrak{a}\mathfrak{m}^\# \ast (A, A[\mathfrak{m}^\#], \omega))] = [\sigma_{\ell}^{-1} \mathfrak{a}\mathfrak{m}^\# \ast (A, A[\mathfrak{m}^\# v^{-1}], \omega)]
\]
and

\[(8) \quad [V_\ell(V_\ell(\alpha A# (A, A[A#], \omega)))] = \{A A# (A, A[A#] v^{-2}, \omega)\}.
\]

As a consequence, if \( F \in \tilde{\mathcal{M}}_{\ell}^p(\Gamma_0(\mathcal{N})) \), when \( \ell \nmid N \) we have

\[(9) \quad F^{(\ell^2)}(\alpha A# (A, A[A#], \omega))
= F(\alpha A# (A, A[A#], \omega)) - \beta_\ell(F)F(\tau^{-1} \alpha A# (A, A[A#], \omega)),
\]

\[(10) \quad F^{(\ell^2)}(\alpha A# (A, A[A#], \omega))
= F(\alpha A# (A, A[A#], \omega)) - \alpha_\ell(F)F(\tau^{-1} \alpha A# (A, A[A#], \omega)),
\]

where the last equality, and hence (7) follows, once we prove the claim that under the canonical dual isogeny, we have

\[(12) \quad F^{(\ell^2)}(\alpha A# (A, A[A#], \omega)) = F(\alpha A# (A, A[A#], \omega)) - \alpha_\ell(F)F(\tau^{-1} \alpha A# (A, A[A#], \omega)).
\]

Proof. Note that \((A, A[A#])[\ell] = A_{a[A#]}[v] \). Hence

\[
[V_\ell(V_\ell(\alpha A# (A, A[A#], \omega)))] = [\alpha A# (A, A[A#], \omega)]
= [\alpha A# (A, A[A#], \omega)]
= \{A A# (A, A[A#] v^{-1}, \omega)\}
= \{A A# (A, A[A#] v^{-1}, \omega)\}
= \{A A# (A, A[A#] v^{-1}, \omega)\}
= \{A A# (A, A[A#] v^{-1}, \omega)\}
\]

where the last equality, and hence (7) follows, once we prove the claim that under the canonical isomorphism \( i : A_{(\ell)^0} \sim A \), we have

\[(\hat{\phi}_v^* \omega)_\tau = i^* \omega.
\]

To show this, note that the above identity is equivalent to

\[(\hat{\phi}_v^* \omega)_\tau = i^* \omega.
\]

which follows if we can show

\[(\hat{\phi}_v^* \omega)_\tau = i^* \omega.
\]

By the definitions, \( \ker(i \circ \phi \circ \phi_v) = A[\ell] \) and \( \ker(\phi_v \circ i \circ \phi) = A[\ell] \). Hence by uniqueness of the dual isogeny, we have \( \hat{\phi}_v = i \circ \phi_v \), from which our claim follows.

The identity (8) follows by the same argument as above, replacing \( \mathcal{N} \) with \( \mathcal{N} v^{-1} \). In particular, viewing \( F \) as a form of level \( N \) and using \( (7) \) and \( (8) \), \( (9) \), \( (10) \) and \( (12) \) follow from \( (3) \), \( (4) \), \( (5) \) and \( (6) \), respectively.

By “\( p \)-adic Hecke character \( \chi : \mathbb{A}_K^* \to \mathbb{C}_p^* \)”, we mean a \( p \)-adic character arising from an algebraic Hecke character. For a place \( v \) of \( K \), let \( \chi_v : K_v^* \to \mathbb{C}_p^* \) denote the local character at \( v \).
Lemma 2.5. Suppose \( F \in \tilde{M}_k^{p\text{-adic}}(\Gamma_0(N^\#)) \), and let \( \chi : \mathbb{A}_K^\times \to \mathbb{C}_p^\times \) be a \( p \)-adic Hecke character such that \( \chi|_{\mathcal{O}_K^\times} = 1 \) (which in particular implies that \( \chi \) is unramified), and \( \chi_\alpha(a) = \alpha^k \) for any \( \alpha \in K^\times \). Let \( \{a\} \) be a full set of integral representatives of \( \mathcal{O}(\mathcal{O}_K) \) where each \( a \) is prime to \( \mathfrak{n}^\# \). If \( \ell \nmid N \), we have

\[
\sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F^{(\ell)^+}(a \ast (A, A[\mathfrak{n}]), \omega)) = (1 - \beta_\ell(F)\chi^{-1}(\overline{\theta})) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)),
\]

\[
\sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F^{(\ell)^-}(a \ast (A, A[\mathfrak{n}]), \omega)) = (1 - \alpha_\ell(F)\chi^{-1}(\overline{\theta})) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)),
\]

\[
\sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F^{(\ell)^0}(a \ast (A, A[\mathfrak{n}]), \omega)) = \left(1 - \alpha_\ell(F)\chi^{-1}(\overline{\theta}) + \frac{\chi^{-2}(\overline{\theta})}{\ell}\right) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)).
\]

and if \( \ell|N \), we have

\[
\sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F^{(\ell)^0}(a \ast (A, A[\mathfrak{n}]), \omega)) = (1 - \alpha_\ell(F)\chi^{-1}(\overline{\theta})) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)).
\]

Proof. First note that by our assumptions on \( \chi \), for any \( G \in \tilde{M}_k^{p\text{-adic}}(\Gamma_0(N^\#)) \), the quantity

\[
\chi^{-1}(a)G(a \ast (A, A[\mathfrak{n}]), \omega))
\]

depends only on the ideal class \([a] \) of \( a \). Since \( \{a\} \) of integral representatives of \( \mathcal{O}(\mathcal{O}_K) \), \( \{a\mathfrak{n}^\#\} \) is also a full set of integral representatives of \( \mathcal{O}(\mathcal{O}_K) \). By summing over \( \mathcal{O}(\mathcal{O}_K) \) and applying Lemma 2.4, we obtain

\[
\sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F^{(\ell)^0}(a \ast (A, A[\mathfrak{n}]), \omega)) = \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)) - \alpha_\ell(F) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a\mathfrak{n}^\#)F(\overline{\theta}^{-1}a\mathfrak{n}^\# \ast (A, A[\mathfrak{n}]), \omega)) - \frac{1}{\ell} \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a\mathfrak{n}^\#)F(\overline{\theta}^{-2}a\mathfrak{n}^\# \ast (A, A[\mathfrak{n}]), \omega))
\]

\[
= \left(1 - \alpha_\ell(F)\chi^{-1}(\overline{\theta}) + \frac{\chi^{-2}(\overline{\theta})}{\ell}\right) \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \chi^{-1}(a)F(a \ast (A, A[\mathfrak{n}]), \omega)).
\]

when \( \ell \nmid N \). Similarly, we obtain the other identities for \((\ell)^+\) and \((\ell)^-\)-stabilization when \( \ell \nmid N \), as well as the identity for \((\ell)^0\)-stabilization when \( \ell|N \). \( \square \)
2.5. Proof of Main Theorem 2.1 We may assume without loss of generality that \( \psi \neq \omega \). As in the proof of Theorem 13 in [Kri16], the argument relies on establishing an Eisenstein congruence. More precisely, let \( f \) be the normalized weight 2 \( \Gamma_0(N) \)-level newform over \( \mathbb{Z} \) associated with \( E \). Recall the weight 2 Eisenstein series \( E_{2,\psi} \) defined by the \( q \)-expansion (at \( \infty \))

\[
E_{2,\psi}(q) := \delta(\psi) \frac{L(0, \psi)}{2} + \sum_{n=1}^{\infty} \sigma_{\psi,\psi^{-1}}(n) q^n
\]

where \( \delta(\psi) = 1 \) if \( \psi = 1 \) and \( \delta(\psi) = 0 \) otherwise, and

\[
\sigma_{\psi,\psi^{-1}}(n) = \sum_{0<d|n} \psi(n/d) \psi^{-1}(d)d.
\]

This determines a \( \Gamma_0(f(\psi)^2) \)-level algebraic modular form of weight 2 over \( \mathbb{C} \), in Katz’s sense (see [Kat76, Chapter II]). The assumption that \( E[p] \) is reducible implies, by [Kri16, Theorem 34 (2)], that we may write \( N = N_+N_0N_0 \) so that \( N_+N_0 \) is the square-free part of \( N \), \( N_0 \) is the square-full part of \( N \), and

\[
\ell|N_+ \implies a_\ell(f) \equiv \psi(\ell) \pmod{p},
\]

\[
\ell|N_- \implies a_\ell(f) \equiv \psi^{-1}(\ell)\ell \pmod{p},
\]

\[
\ell|N_0 \implies a_\ell(f) = 0.
\]

Thus, viewing \( f \) and the stabilization \( E_{2,\psi}^{(N_+)+N_-}\psi^{-1}(N_0)^0 \) as a \( p \)-adic \( \Gamma_0(N) \)-level modular forms over \( \mathcal{O}_{\mathbb{C}_p} \), the \( q \)-expansion principle implies that

\[
\theta^j f \equiv \theta^j E_{2,\psi}^{(N_+)+N_-}\psi^{-1}(N_0)^0 \pmod{p\mathcal{O}_{\mathbb{C}_p}}
\]

for all \( j \geq 1 \). Let \( A_0 \) is the elliptic curve defined over the Hilbert class field of \( K \) and complex points \( A_0(\mathbb{C}) = \mathbb{C}/\mathcal{O}_K \), and fix an ideal \( \mathfrak{N} \subset \mathcal{O}_K \) such that \( \mathcal{O}_K/\mathfrak{N} = \mathbb{Z}/N \) and \( p|\mathfrak{N} \) if \( p|N \). Again, since \( p \) is split in \( K \), this translates to congruences on \( CM \) points corresponding to \( \mathcal{O}_K \), which implies (as is explained in Section 3 of [KL16]), by a generalization of Coleman’s theorem ([LZZ15, Proposition A.1, see also [KL16, Theorem 3.2]]) that (for any generator \( \omega \in \Omega_1^{A_0/\mathcal{O}_{\mathbb{C}_p}} \))

\[
\frac{|\tilde{E}_{ns}(\mathbb{F}_p)|}{p} \cdot \log_{\omega_E} P_E = \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \theta^{-1} f(p)(a \ast (A_0, A_0[\mathfrak{N}], \omega))
\]

\[
\equiv \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \theta^{-1} E_{2,\psi}^{(N_+)+N_-}\psi^{-1}(p^2N_0)^0 (a \ast (A_0, A_0[\mathfrak{N}], \omega))
\]

\[
= \prod_{\ell|N_+} (1 - \psi^{-1}(\ell)) \prod_{\ell|N_-, \ell \neq p} \left(1 - \frac{\psi(\ell)}{\ell}\right) \prod_{\ell|N_0, \ell \neq p} \left(1 - \psi^{-1}(\ell)\right) \left(1 - \frac{\psi(\ell)}{\ell}\right)
\]

\[
\cdot \sum_{[a] \in \mathcal{O}(\mathcal{O}_K)} \theta^{-1} E_{2,\psi}^{(p^2)^0} (a \ast (A_0, A_0[\mathfrak{N}], \omega)) \pmod{p\mathcal{O}_{\mathbb{C}_p}}
\]

where the final equality follows from Lemma 2.5, applied to successive stabilizations of \( E_{2,\psi} \). We claim that the final sum is interpolated by the Katz \( p \)-adic \( L \)-function. Indeed, let \( \chi_j \) be the unramified Hecke character of infinity type \((h_K j, -h_K j)\) defined on ideals by

\[
\chi_j(a) = (a/P)^j
\]

where \((a) = a^h_K\), and \( h_K \) is the class number of \( K \). Choose a good integral model \( A_0 \) of \( A_0 \) at \( p \), choose an identification \( \iota: \tilde{A}_0 \to \tilde{G}_m \) (unique up to \( \mathbb{Z}_p^\times \)), and let \( \omega_{can} := \iota^* du / u \) where \( u \) is the
coordinate on \( \hat{G}_m \). This choice of \( \omega_{\text{can}} \) determines \( p \)-adic and complex periods \( \Omega_p \) and \( \Omega_\infty \) as in Section 3 of [Kr10]. Then by looking at \( q \)-expansions and invoking the \( q \)-expansion principle, it is apparent that the above sum is given by

\[
\sum_{[a] \in C\ell(\mathcal{O}_K)} \theta^{-1} E_{2,\psi}^{(p^2 \rho \psi)} (a * (A_0, A_0[\mathfrak{N}], \omega_{\text{can}}))
= \lim_{j \to -1} \left( \frac{\Omega_p}{\Omega_\infty} \right)^{2h_{K,j}} \sum_{[a] \in C\ell(\mathcal{O}_K)} (\chi_j^{-1} N_{K}^{h_{K,j}})(a) \theta^{-1 + h_{K,j}} E_{2,\psi}^{(p^2 \rho \psi)} (a * (A_0, A_0[\mathfrak{N}], \omega_{\text{can}}))
= \lim_{j \to -1} \left( \frac{\Omega_p}{\Omega_\infty} \right)^{2h_{K,j}} \sum_{[a] \in C\ell(\mathcal{O}_K)} (\chi_j^{-1} N_{K}^{h_{K,j}})(a) \theta^{-1 + h_{K,j}} E_{2,\psi} (a * (A_0, A_0[\mathfrak{N}], \omega_{\text{can}})) \cdot (1 - \psi(p)(\chi_j^{-1} N_{K})(\mathfrak{P}))(1 - \psi(p)\chi_j^{-1}(\mathfrak{P}))
\]

since \( \chi_j^{-1} N_{K}^{h_{K,j}} \to 1 \) as \( j \to 0 = (0,0) \in \mathbb{Z}/(p - 1) \times \mathbb{Z}_p \); here the last equality again follows from Lemma 2.5 applied to \( F = E_{2,\psi} \). Let \( f \mathfrak{N} \) such that \( \mathcal{O}/f = Z/f(\psi) \). Then by Theorem 21 and Proposition 36 of loc. cit. (with \( \psi_1 = \bar{\psi}_2^{-1} = \psi \) and \( u = t = 1, \mathfrak{N} = f \)), since we assume \( d_K \) is odd, viewing \( E_{2,\psi} \) as an algebraic modular form over \( \mathbb{C} \), and \( (A_0, A_0[\mathfrak{N}], \omega_{\text{can}}) \) as a triple over \( \mathbb{C} \) (by identifying \( A_0[\mathfrak{N}] \) with \( \mathbb{C} \mathbb{Z} \) under the complex uniformization \( \mathbb{C}/\mathcal{O}_K \cong A_0 \), we have the following identity of values in \( \mathbb{C} \) for \( j \geq 1 \): 

\[
\sum_{[a] \in C\ell(\mathcal{O}_K)} (\chi_j^{-1} N_{K}^{h_{K,j}})(a) \theta^{-1 + h_{K,j}} E_{2,\psi} (a * (A_0, A_0[\mathfrak{N}], 2\pi idz)) \cdot (1 - \psi(p)(\chi_j^{-1} N_{K})(\mathfrak{P}))(1 - \psi(p)\chi_j^{-1}(\mathfrak{P})) = \frac{f(\psi)^2 \Gamma(1 + h_{K,j})\psi^{-1}(-\sqrt{d_K})(\chi_j^{-1} N_{K})(\mathfrak{P})}{(2\pi i)^{1 + h_{K,j}} g(\psi^{-1})(\sqrt{d_K})^{-1 + h_{K,j}}} L((\psi \circ N_{K/Q}) \chi_j^{-1} N_{K}, 0)
\]

where \( \psi^{-1}(-\sqrt{d_K}) \) denotes the Dirichlet character \( \psi^{-1} \) evaluated at the unique class \( b \in (\mathbb{Z}/f(\psi))^{\times} \) such that \( b + \sqrt{d_K} \equiv 0 \pmod{f} \). (In particular, note that the above complex-analytic calculation does not use the assumptions \( p > 2 \) or \( p \nmid f(\psi) \).) From the interpolation property of \( L_p^{\text{Katz}}(\chi, s) \) (see [HT93] Theorem II], where \( L_p^{\text{Katz}}(\chi, s) \) is normalized as in [Gro80], we have, under a fixed isomorphism \( i : \mathbb{C} \cong \mathbb{C}_p \) (which identifies a complex algebraic Hecke character \( \chi \) with its \( p \)-adic avatar),

\[
L_p^{\text{Katz}}((\psi \circ N_{K/Q}) \chi_j^{-1} N_{K}, 0) = 4 \cdot \text{Local}_p((\psi \circ N_{K/Q}) \chi_j^{-1} N_{K}) \left( \frac{\Omega_p}{\Omega_\infty} \right)^{2h_{K,j}} \cdot \left( \frac{2\pi i}{\sqrt{d_K}} \right)^{-1 + h_{K,j}} \Gamma(1 + h_{K,j})(1 - \psi(p)(\chi_j^{-1} N_{K})(\mathfrak{P}))(1 - \psi(p)\chi_j^{-1}(\mathfrak{P})) L((\psi \circ N_{K/Q}) \chi_j^{-1} N_{K}, 0)
\]

for all \( j \geq 1 \), where \( \text{Local}_p(\chi) = \text{Local}_p(\chi, \Sigma, \delta) \) is defined as in [Kat78, 5.2.26] with \( \Sigma = \{ p \} \) and \( \delta = \sqrt{d_K}/2 \) (or as denoted \( W_p(\lambda) \) in [HT93, 0.10]). In this situation, by directly plugging in \( \chi = (\psi \circ N_{K/Q}) \chi_j^{-1} N_{K} \) to the definitions, we have

\[
\text{Local}_p((\psi \circ N_{K/Q}) \chi_j^{-1} N_{K}) = \psi_p(\sqrt{d_K}) \frac{|f(\psi)_p|}{g_p(\psi)}.
\]
Letting $f(\psi)_p$ denote the $p$-primary part of the conductor $f(\psi)$ of $\psi$, $f$ denote the prime-to-$p$ part of $f(\psi)$, we have from the above series of identities that for each $j \geq 1$,

$$
\sum_{[a] \in C(\mathcal{O}_K)} (x_j^{-1}N_K^{h_j}) \theta^{-1+h_j} \left( \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \right) \frac{|f|}{4(\prod_{\ell \mid \ell \neq p} \psi^{-1}(\sqrt{d_K})g_{\ell}(\psi))} L_p(Katz((\psi \circ Nm_{K/Q})N^{-1}K,0)).
$$

Taking the limit $j \to -1 = (-1,0) \in \mathbb{Z}/(p-1) \times \mathbb{Z}_p$, $\chi_j^{-1}N_K \to N_K$ and $N_K(\overline{\psi}) = |f(\psi)|^{-1}$. Thus, we have

$$
\sum_{[a] \in C(\mathcal{O}_K)} \theta^{-1} \left( \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \right) \frac{|f|}{4(\prod_{\ell \mid \ell \neq p} \psi^{-1}(\sqrt{d_K})g_{\ell}(\psi))} L_p(Katz((\psi \circ Nm_{K/Q})N^{-1}K,0)).
$$

Applying Gross’s factorization theorem (see [Kri16] Theorem 28) for the general auxiliary conductor case and standard special value formulas for the Kubota-Leopoldt $p$-adic $L$-function (see Section 4.3 of loc. cit.), we finally obtain, if $\psi \neq 1$,

\begin{equation}
\frac{\sum_{[a] \in C(\mathcal{O}_K)} \theta^{-1} \left( \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \right) \frac{|f|}{4(\prod_{\ell \mid \ell \neq p} \psi^{-1}(\sqrt{d_K})g_{\ell}(\psi))} L_p(Katz((\psi \circ Nm_{K/Q})N^{-1}K,0))}{P_{E}} \cdot \log_{\omega_E} P_{E} \equiv \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \left( 1 - \psi(\ell) \right) \left( 1 - \psi(\ell) \right) \frac{1}{4} \left( (1 - \psi^{-1}(\ell))(1 - (\psi \omega^{-1}(p))(\ell))B_{1,\psi^{-1}(\ell)}B_{1,\psi \omega^{-1}(\ell)} \right) \mod p\mathcal{O}_p,
\end{equation}

and if $\psi = 1$,

\begin{equation}
\frac{\sum_{[a] \in C(\mathcal{O}_K)} \theta^{-1} \left( \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \prod_{\ell \mid \ell \neq p} \left( 1 - \psi^{-1}(\ell) \right) \right) \frac{|f|}{4(\prod_{\ell \mid \ell \neq p} \psi^{-1}(\sqrt{d_K})g_{\ell}(\psi))} L_p(Katz((\psi \circ Nm_{K/Q})N^{-1}K,0))}{P_{E}} \cdot \log_{\omega_E} P_{E} \equiv \prod_{\ell \mid \ell \neq p} 0 \prod_{\ell \mid \ell \neq p} \left( 1 - \frac{1}{\ell} \right) \prod_{\ell \mid \ell \neq p} 0 \cdot \frac{p - 1}{2p} \log_{\overline{\alpha}} \overline{\alpha} \mod p\mathcal{O}_p,
\end{equation}

where $(\overline{\alpha}) = \mathfrak{p}^{h_K}$ and $\log_p$ is the Iwasawa $p$-adic logarithm (i.e. the branch of the $p$-adic logarithm with $\log_p p = 0$). For any $x \in \mathfrak{p}$, by writing $x = a + 2\pi$ where $\pi \in \mathfrak{p}\mathcal{O}_K_p$ (so in particular $a$ is a $p$-unit), and taking the binomial expansion to get $x^{h_K} = a^{h_K} + 2h_Ka^{h_K-1}\pi + \text{terms with } p\text{-order} \geq \text{ord}_p(2h_K\pi)$, one sees that $\text{ord}_p(\log_p \overline{\alpha}) = \text{ord}_p(2h_K) + 1$. Hence

\begin{equation}
\text{ord}_p \left( \frac{p - 1}{2p} \log_{\overline{\alpha}} \overline{\alpha} \right) = \text{ord}_p(h_K).
\end{equation}

When $\psi = 1$ and $p \geq 3$, [Yoo15] Theorem 2.2 implies that $N_+N_0 \neq 1$, so note that the right hand side of (14) in this case is 0 unless $N = N_+N_0$ is a power of $p$. When $\psi = 1$ and $p = 2$, each factor in the three products on the right hand side of the congruence (14) are 0 (mod 2), so (15) shows that the right hand side of (14) is always 0 (mod 2) in this case. Hence, in any case when $\psi = 1$, the right-hand side of (14) vanishes mod $p$ if and only if $N = N_+N_0$ is a power of $p$ and $p \not\mid h_K$.

The rest of the theorem now follows from studying when the right hand sides of the (13) does not vanish mod $p$.

**Remark 2.6.** Note that our proof uses a direct method of $p$-adic integration, and in particular does not go through the construction of the Bertolini-Darmon-Prasanna (BDP) $p$-adic $L$-function as in the proof of the main theorem of loc. cit. In particular, it does not recover the more general congruence of the BDP and Katz $p$-adic $L$-functions established when $p$ is of good reduction in [Kri16] (also for higher weight newforms). We expect that our method should extend to higher weight newforms, in
particular establishing congruences between images of generalized Heegner cycles under appropriate $p$-adic Abel-Jacobi images and quantities involving higher Bernoulli numbers and Euler factors.

3. Bernoulli numbers and relative class numbers

When $p = 3$, all Dirichlet characters in Theorem 2.1 are quadratic. Note that for a quadratic character $\psi$ over $\mathbb{Q}$, by the analytic class number formula we have

$$B_{1,\psi} = -2 \frac{h_{K_\psi}}{|O_{K_\psi}^\times|}$$

where $K_\psi$ is the imaginary quadratic field associated with $\psi$. So the 3-indivisibility criteria of the theorem becomes a question of 3-indivisibility of quadratic class numbers. This fact will be employed in our applications to Goldfeld’s conjecture.

More generally, for $p \geq 3$, we can find a sufficient condition for non-vanishing mod $p$ of the Bernoulli numbers $B_{1,\psi} - 10 \epsilon_K B_{1,\psi} 0 \omega^{-1}$ in terms of non-vanishing mod $p$ of the relative class numbers of the CM fields cut out by $\psi^{-1} \in R$ and $\psi \omega^{-1}$. Let us first observe the following simple lemma. Let $\Phi : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ denote the Euler totient function, and for a Dirichlet character $\psi$, let $\mathbb{Z}[\psi]$ denote the finite extension of $\mathbb{Z}$ generated by the values of $\psi$. Let $\chi_f : \text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q}) \sim \mu_{\Phi(f)}$ denote the mod $f$ cyclotomic character, i.e. the character which is defined via the relation

$$\sigma \cdot \zeta = \zeta^{\chi_f(\sigma)}$$

for $\sigma \in \text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$ and $\zeta \in \mu_{\Phi(f)}$.

**Lemma 3.1.** Suppose $\psi : (\mathbb{Z}/f)^{\times} \rightarrow \mu_{\Phi(f)}$ is a Dirichlet character, and assume $\psi^{-1} \neq \chi_f$, or equivalently, assume there exists some $a \in (\mathbb{Z}/f)^{\times}$ such that $\psi(a)a \not\equiv 1 \pmod{f \mathbb{Z}[\psi]}$. Then

$$B_{1,\psi} \in \mathbb{Z}[\psi].$$

**Proof.** By our assumption, there exists some $a \in (\mathbb{Z}/f)^{\times}$ such that $\psi(a)a \not\equiv 1 \pmod{f}$. Then we have

$$\sum_{m=1}^{f} \psi(m)m \equiv \sum_{m=1}^{f} \psi(am)am = \psi(a)a \sum_{m=1}^{f} \psi(m)m \pmod{f \mathbb{Z}[\psi]}$$

$$\Rightarrow (1 - \psi(a)a) \sum_{m=1}^{f} \psi(m)m \equiv 0 \pmod{f \mathbb{Z}[\psi]} \Rightarrow \sum_{m=1}^{f} \psi(m)m \equiv 0 \pmod{f \mathbb{Z}[\psi]}.$$  

Now our conclusion follows from the formula for the Bernoulli numbers [1]. \hfill \square

For an odd Dirichlet character $\psi$, let $K_\psi$ denote the abelian CM field cut out by $\psi$. Consider the relative class number $h_{K_\psi}^{-} = h_{K_\psi}/h_{K_\psi}^{+}$, where $K_\psi^{+}$ is the maximal totally real subfield of $K_\psi$. The relative class number formula ([Was97 4.17]) gives

$$h_{K_\psi}^{-} = Q \cdot w \cdot \prod_{\chi \text{ odd}} \left( -\frac{1}{2} B_{1,\chi} \right)$$

(17)
where \( \chi \) runs over all odd characters of \( \text{Gal}(K_{\psi}/Q) \), \( w \) is the number of roots of unity in \( K_{\psi} \) and \( Q = 1 \) or 2 (see \cite{Was97} 4.12). By Lemma 3.1, assuming that \( \psi^{-1} \neq \chi f \), we see that

\[
p \nmid h_{K_{\psi}}^+ \implies p \nmid B_{1,\psi}.
\]

In particular, for \( \psi \) such that \( \psi^{-1} \varepsilon_K \neq \chi f \) and \( \psi_0 \omega^{-1} \neq \chi f \), we have

\[
p \nmid h_{K_{\psi_0}\varepsilon_K}^+ \cdot h_{K_{\psi_0}^{-1}\omega}^+ \implies p \nmid B_{1,\psi_0}\varepsilon_K \cdot B_{1,\psi_0}^{-1}\omega.
\]

4. Goldfeld’s conjecture for elliptic curves with a 3-isogeny

The goal in this section is to prove Theorem 1.5. We will need some Davenport-Heilbronn type class number divisibility results due to Horie-Nakagawa and Taya. For any \( x \geq 0 \), let \( K^+(x) \) denote the set of real quadratic fields \( k \) with fundamental discriminant \( d_k < x \) and \( K^-(x) \) the set of imaginary quadratic fields \( k \) with fundamental discriminant \( |d_k| < x \). Put

\[
K^+(x, m, M) := \{ k \in K^+(x) : d_k \equiv m \pmod{M} \}
\]

\[
K^-(x, m, M) := \{ k \in K^-(x) : d_k \equiv m \pmod{M} \}.
\]

Horie and Nakagawa prove the following.

**Theorem 4.1** (\cite{NH88}). Suppose that \( m \) and \( M \) are positive integers such that if \( \ell \) is an odd prime number dividing \((m, M)\), then \( \ell^2 \) divides \( M \) but not \( m \). Further, if \( M \) is even, suppose that

1. \( 4 | M \) and \( m \equiv 1 \pmod{4} \), or
2. \( 16 | M \) and \( m \equiv 8 \) or 12 (mod 16).

Then

\[
\sum_{k \in K^+(x, m, M)} h_3(d_k) \sim \frac{4}{3} |K^+(x, m, M)| \quad (x \to \infty)
\]

\[
\sum_{k \in K^-(x, m, M)} h_3(d_k) \sim 2 |K^-(x, m, M)| \quad (x \to \infty).
\]

Furthermore,

\[
|K^+(x, m, M)| \sim |K^-(x, m, M)| \sim \frac{3x}{x^{\pi \Phi(M)}} \prod_{\ell | M} \frac{q}{\ell + 1} \quad (x \to \infty).
\]

Here \( f(x) \sim g(x) \quad (x \to \infty) \) means that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \), \( \ell \) ranges over primes dividing \( M \), \( q = 4 \) if \( \ell = 2 \), and \( q = \ell \) otherwise.

Now put

\[
K^+_e(x, m, M) := \{ k \in K^+(x, m, M) : h_3(d_k) = 1 \}
\]

\[
K^-_e(x, m, M) := \{ k \in K^-(x, m, M) : h_3(d_k) = 1 \}.
\]

Taya \cite{Tay00} proves the following bound using Theorem 4.1.
Proposition 4.2. Suppose \( m \) and \( M \) satisfy the conditions of Theorem 4.1. Then

\[
\lim_{x \to \infty} \frac{|K^+(x, m, M)|}{|K^+(x, 1, 1)|} \geq \frac{5}{6\Phi(M)} \prod_{\ell \mid M} \frac{q}{\ell + 1}
\]

\[
\lim_{x \to \infty} \frac{|K^-(x, m, M)|}{|K^-(x, 1, 1)|} \geq \frac{1}{2\Phi(M)} \prod_{\ell \mid M} \frac{q}{\ell + 1}
\]

where given a prime \( \ell \), \( q = 4 \) if \( \ell = 2 \) and \( q = \ell \) for \( \ell \) odd. In particular, the of real (resp. imaginary) quadratic fields \( k \) such that \( d_k \equiv m \pmod{M} \) and \( h_3(d_k) = 1 \) has positive density in the set of all real (resp. imaginary) quadratic fields.

Proof. This follows from the above asymptotic estimates and the fact that \( |K^\pm(x, 1, 1)| \sim \frac{3\pi}{\ell^2} \) by Theorem 4.1. □

We have the following slight strengthening of Theorem 54 of loc. cit.

Theorem 4.3. Suppose \( E/\mathbb{Q} \) is any elliptic curve whose mod 3 Galois representation \( E[3] \) is reducible and \( E[3]^\text{ss} \cong \mathbb{F}_3 \oplus \mathbb{F}_3(\omega) \). Let \( \psi = \psi_d \) be the quadratic character of conductor \( d \) (a fundamental discriminant) such that

1. \( \psi(3) \neq 1 \) and \( (\psi^{-1}\omega)(3) \neq 1 \);
2. \( h_3(-3d) = 1 \) if \( \psi(-1) = 1 \), and \( h_3(d) = 1 \) if \( \psi(-1) = -1 \);
3. \( \ell \not\equiv 3, \ell \mid N_{\text{split}} \) implies \( \psi(\ell) = -1 \);
4. \( \ell \not\equiv 3, \ell \mid N_{\text{nonsplit}} \) implies \( \psi(\ell) = 1 \);
5. \( \ell \not\equiv 3, \ell \mid N_{\text{add}}, \ell \equiv 1 \pmod{3} \) implies \( \psi(\ell) = -1 \);
6. \( \ell \not\equiv 3, \ell \mid N_{\text{add}}, \ell \equiv 2 \pmod{3} \) implies \( \psi(\ell) = 0 \);
7. \( 4 \mid N \) implies \( d \equiv 8 \) or \( 12 \pmod{16} \).

If \( \ell \nmid \text{lcm}(N, d^2) \) let \( r = 1 \), if \( \ell \mid \text{lcm}(N, d^2) \) let \( r = 3 \), and if \( 4 \mid \text{lcm}(N, d^2) \) let \( r = \nu_2(\text{lcm}(N, d^2, 16)) \).

Given a prime \( \ell \), let \( q = 4 \) for \( \ell = 2 \), and \( q = \ell \) for \( \ell \) odd. Then for a positive proportion of at least

\[
\frac{1}{2^{r+1}} \prod_{\ell \mid N_{\text{split}}, \ell \mid d, \ell \text{ odd}, \ell \not\equiv 3} \frac{1}{2} \prod_{\ell \mid N_{\text{add}}, \ell \mid d, \ell \text{ odd}, \ell \not\equiv 3} \frac{1}{2} \prod_{\ell \mid N, \ell \not\equiv 3} \frac{1}{2\ell} \prod \frac{q}{\ell + 1}
\]

of imaginary quadratic fields \( K \), \( K \) satisfies the Heegner hypothesis with respect to \( E[d] \) and the Heegner point \( P_{E[d]}(K) \) is non-torsion.

Proof. Again we will apply Proposition 4.2 as well as Theorem 4.1. Using the Chinese remainder theorem, choose a positive integer \( m \) such that

1. \( m \equiv 2 \pmod{3} \) or \( m \equiv 3 \pmod{9} \),
2. \( \ell \) odd prime, \( \ell \mid N_{\text{split}} \implies m \equiv [\text{quadratic non-residue unit}] \pmod{\ell} \),
3. \( 2 \mid N_{\text{split}} \implies m \equiv 5 \pmod{8} \),
4. \( \ell \) prime, \( \ell \mid N_{\text{nonsplit}} \implies m \equiv [\text{quadratic residue unit}] \pmod{\ell} \),
5. \( 2 \mid N_{\text{nonsplit}} \implies m \equiv 1 \pmod{8} \),
6. \( \ell \) odd prime, \( \ell \mid N_{\text{add}}, \ell \mid d \implies m \equiv [\text{quadratic non-residue unit}] \pmod{\ell} \),
7. \( \ell \) odd prime, \( \ell \mid N_{\text{add}}, \ell \mid d \implies m \equiv 0 \pmod{\ell} \) where \( \frac{M}{d} \equiv [\text{quadratic residue unit}] \pmod{\ell} \),
8. \( 4 \mid N \implies m \equiv d \pmod{16} \).

Here (and throughout the rest of the proof) conditions (1)-(7), in tandem with the assumed conditions (1)-(7) on \( \psi \) in the statement of the theorem, give the desired conditions on \( K \) to apply
Theorem 2.1 and (8) is an extra condition which we will need to apply Horie-Nakagawa’s theorem. Let $N'$ denote the prime-to-3 part of $N$. Given such an $m$, let a positive integer $M$ be defined as follows:

1) if $m \equiv 2 \pmod{3}$, let $M = 3\lcm(N', d^2)$ if $2 \nmid \lcm(N', d^2)$ is odd, $M = 3\lcm(N', d^2, 8)$ if $2\mid \lcm(N', d^2)$, and $M = 3\lcm(N', d^2, 16)$ if $4\mid \lcm(N', d^2, 8)$;

2) if $m \equiv 3 \pmod{9}$, let $M = 9\lcm(N', d^2)$ if $2 \nmid \lcm(N', d^2)$ is odd, $M = 9\lcm(N', d^2, 8)$ if $2\mid \lcm(N', d^2)$, and $M = 9\lcm(N', d^2, 16)$ if $4\mid \lcm(N', d^2, 8)$.

Suppose $d > 0$ (resp. $d < 0$). Suppose $K$ is any imaginary quadratic field such that $dd_K \equiv m \pmod{M}$ where $m \equiv 2 \pmod{3}$ if $d \neq 0 \pmod{3}$ and $m \equiv 3 \pmod{9}$ if $d \equiv 0 \pmod{3}$ (resp. $-3dd_K \equiv m \pmod{M}$ where $m \equiv 3 \pmod{9}$ if $d \neq 0 \pmod{3}$, and $-dd_K/3 \equiv m \pmod{M}$ where $m \equiv 2 \pmod{3}$ if $d \equiv 0 \pmod{3}$). Since $d_K$ is odd, we must have $d_K \equiv 1 \pmod{4}$, and this is compatible with condition (6) which forces $d_K \equiv 1 \pmod{8}$, which in turn forces 2 to split in $K$.

The above congruence conditions in particular imply that $d_K$ is odd and that $(d, d_K) = 1$. Then the above congruence conditions on $m$, along with the congruence conditions of our assumptions, imply

1) $\psi(3) = -1$ or $\psi(3) = 0$, $\varepsilon_K(3) = 1$, and $(\psi\varepsilon_K)(3) = -1$ or 0;

2) $\ell$ prime, $\ell \nmid N_{\text{split}}$, $\ell \mid d \implies \psi(\ell) = -1, \varepsilon_K(\ell) = 1, (\psi\varepsilon_K)(\ell) = -1$;

3) $\ell$ prime, $\ell \nmid N_{\text{non-split}}$, $\ell \mid d \implies \psi(\ell) = 1, \varepsilon_K(\ell) = 1, (\psi\varepsilon_K)(\ell) = 1$;

4) $\ell$ odd prime, $\ell \nmid N_{\text{odd}}$, $\ell \mid d \implies \psi(\ell) = -1, \varepsilon_K(\ell) = 1, (\psi\varepsilon_K)(\ell) = -1$;

5) $\ell$ odd prime, $\ell \mid d \implies \psi(\ell) = 0, \varepsilon_K(\ell) = 1, (\psi\varepsilon_K)(\ell) = 0$;

6) $4\mid N_{\text{odd}} \implies \psi(2) = 0, \varepsilon_K(2) = 1, (\psi\varepsilon_K)(2) = 0$;

7) if $2\mid N$, then $4\mid M$ and $m \equiv 1 \pmod{4}$;

8) if $4\mid N$, then $16\mid M$ and $m \equiv 8$ or 12 $\pmod{16}$.

Thus for imaginary quadratic $K$ such that $dd_K \equiv m \pmod{M}$ where $m \equiv 2 \pmod{3}$ if $d \neq 0 \pmod{3}$ and $m \equiv 3 \pmod{9}$ if $d \equiv 0 \pmod{3}$ (resp. $-3dd_K \equiv m \pmod{M}$ where $m \equiv 3 \pmod{9}$ if $d \neq 0 \pmod{3}$, and $-dd_K/3 \equiv m \pmod{M}$ where $m \equiv 2 \pmod{3}$ if $d \equiv 0 \pmod{3}$), $(E, 3, \psi, K)$ satisfies all the congruence conditions of Theorem 2.1 except for possibly $h_3(dd_K) = 1$ (resp. $h_3(-3dd_K) = 1$). Moreover, the congruence conditions above imply that $m$ and $M$ are valid positive integers for Theorem 4.1. Thus, by Proposition 4.2

$$\lim_{x \to \infty} \frac{|K_+(x, m, M)|}{|K_-(x, 1, 1)|} \geq \frac{1}{2\Phi(M)} \prod_{\ell \mid M} \frac{q}{\ell} + 1$$

and so a positive proportion of imaginary quadratic $K$ satisfy $d_{Q(\sqrt{dd_K})} \equiv m \pmod{M}$ and $3 \nmid h_{Q(\sqrt{dd_K})}$. Thus, for these $K$, $(E, 3, \psi, K)$ satisfies all the congruence conditions of Theorem 2.1 and so $P_{E(\psi)}(K)$ is non-torsion. Moreover, noticing that the congruence conditions (1)-(6) on $m$ above are independent (again by the Chinese remainder theorem), we have

$$\prod_{\ell \mid N_{\text{split}}, N_{\text{non-split}}, \ell \mid d, \ell \text{ odd}} \frac{\ell - 1}{2} \prod_{\ell \mid N_{\text{odd}}, d, \ell \text{ odd}} \frac{\ell(\ell - 1)}{2} \prod_{\ell \mid d, \ell \text{ odd}} \frac{\ell - 1}{2}$$

choices for residue classes of $m \pmod{M}$. Combining all the above and summing over each valid residue class $m \pmod{M}$, we immediately obtain our lower bound for the proportion of valid $K$. □

Similarly, we have the slight strengthening of Theorem 53 of loc. cit.
Theorem 4.4. Suppose \((N_{\text{split}}, N_{\text{nonsplit}}, N_{\text{add}})\) is a triple of pairwise coprime integers such that 
\(N_{\text{split}}, N_{\text{nonsplit}}\) is square-free, \(N_{\text{add}}\) is square-full and \(N_{\text{split}}N_{\text{nonsplit}}N_{\text{add}} = N\). If \(2 \nmid N\) let \(r = 1\), if \(2 | N\) let \(r = 3\), and if \(4 | N\) let \(r = v_2(\text{lcm}(N,16))\). Finally, let \(c(r) = 1\) if \(r \leq 3\) and \(c(r) = 0\) if \(r \geq 4\). Given a prime \(\ell\), let \(q = 4\) for \(\ell = 2\), and \(q = \ell\) for \(\ell\) odd. Then a proportion of at least

\[
\frac{1}{2^{r+c(r)}} \prod_{\ell | N_{\text{split}}, \ell \text{ odd}, \ell \neq 3} \frac{1}{2} \prod_{\ell | N_{\text{add}}, \ell \equiv 1 \pmod{3}} \frac{\ell + 2}{2\ell} \prod_{\ell | N_{\text{add}}, \ell \equiv 2 \pmod{3}} \frac{1}{2\ell} \prod_{\ell | N, \ell \neq 3} \frac{q}{\ell + 1}
\]

of even (resp. odd) quadratic characters \(\psi\) of conductor \(d\) (a fundamental discriminant) satisfy

1. \(\psi(3) \neq 1\) and \((\psi^{-1}\omega)(3) \neq 1\);
2. \(h_3(-3d) = 1\) (resp. \(h_3(d) = 1\));
3. \(\ell \neq 3, \ell | N_{\text{split}}\) implies \(\psi(\ell) = -1\);
4. \(\ell \neq 3, \ell | N_{\text{nonsplit}}\) implies \(\psi(\ell) = 1\);
5. \(\ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 1 \pmod{3}\) implies \(\psi(\ell) = -1\);
6. \(\ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 2 \pmod{3}\) implies \(\psi(\ell) = 0\);
7. \(4 | N\) implies \(d \equiv 8\) or \(12 \pmod{16}\).

Moreover, we have the following:

1. \(3/4\) (resp. \(1/4\)) of the above fundamental discriminants \(d > 0\) (resp. \(d < 0\)) satisfy \(d \equiv 3 \pmod{9}\),
2. \(1/12\) (resp. \(1/4\)) satisfy \(d \equiv 2 \pmod{9}\),
3. \(1/6\) (resp. \(1/2\)) satisfy \(d \equiv 5, 8 \pmod{9}\).

Proof. The proof is essentially the same as the proof of Theorem 53 in loc. cit. We will apply Proposition 4.2. Using the Chinese remainder theorem, choose a positive integer \(m\) which satisfies the following congruence conditions:

1. \(m \equiv 3 \pmod{9}\) or \(m \equiv 2 \pmod{3}\),
2. \(\ell\) odd prime, \(\ell \neq 3, \ell | N_{\text{split}}\) \(\implies m \equiv -3[\text{quadratic non-residue unit}] \pmod{\ell}\),
3. \(2 | N_{\text{split}}\) \(\implies m \equiv 1 \pmod{8}\),
4. \(\ell\) odd prime, \(\ell \neq 3, \ell | N_{\text{nonsplit}}\) \(\implies m \equiv -3[\text{quadratic residue unit}] \pmod{\ell}\),
5. \(2 | N_{\text{nonsplit}}\) \(\implies m \equiv 5 \pmod{8}\),
6. \(\ell\) odd prime, \(\ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 1 \pmod{3}\) \(\implies m \equiv -3[\text{quadratic non-residue unit}] \pmod{\ell}\)
   or \(m \equiv 0 \pmod{\ell}\) and \(m \neq 0 \pmod{\ell^2}\),
7. \(\ell\) odd prime, \(\ell \neq 3, \ell | N_{\text{add}}, \ell \equiv 2 \pmod{3}\) \(\implies m \equiv 0 \pmod{\ell}\) and \(m \neq 0 \pmod{\ell^2}\),
8. \(4 | N_{\text{add}}\) \(\implies m \equiv 8\) or \(12 \pmod{16}\).

Here (and throughout the proof) again, conditions (1)-(7) correspond to the desired conditions (1)-(7) on \(\psi\) in the statement of the theorem, and (8) is an extra condition required to apply the theorem of Horie-Nakagawa. Let \(N'\) denote the prime-to-3 part of \(N\). Given such an \(m\), let a positive integer \(M\) be defined as follows:

1. if \(m \equiv 3 \pmod{9}\), let \(M = 9N'\) if \(2 \nmid N\), \(M = 36N'\) if \(2 | N\), and \(M = 9\text{lcm}(N',16)\) if \(4 | N\);
2. if \(m \equiv 2 \pmod{3}\), let \(M = 3N'\) if \(2 \nmid N\), \(M = 12N'\) if \(2 | N\), and \(M = 3\text{lcm}(N',16)\) if \(4 | N\).

Suppose \(d > 0\) (resp. \(d < 0\)) is a fundamental discriminant with \(d \neq 0 \pmod{3}\) (resp. \(d \equiv 0 \pmod{3}\)), and \(-3d \equiv m \pmod{M}\) (resp. \(d \equiv m \pmod{M}\)) where \(m \equiv 3 \pmod{9}\) (resp. \(m \equiv 2 \pmod{3}\)), or \(d \equiv 0 \pmod{3}\) (resp. \(d \equiv 0 \pmod{3}\)) and \(-d/3 \equiv m \pmod{M}\) (resp \(d \equiv m \pmod{M}\)) if \(m \equiv 2 \pmod{3}\) (resp. \(m \equiv 3 \pmod{9}\)). Let \(\psi\) be the quadratic character associated with \(d\). Then the above congruence conditions on \(m\) along with our assumptions imply
(1) \( \psi(3) = -1 \) or 0;
(2) \( \ell \neq 3 \) prime, \( \ell | N_{\text{split}} \implies \psi(\ell) = -1; \)
(3) \( \ell \neq 3 \) prime, \( \ell | N_{\text{nonsplit}} \implies \psi(\ell) = 1; \)
(4) \( \ell \) odd prime, \( \ell | N_{\text{add}}, \ell \equiv 1 \pmod{3} \implies \psi(\ell) = -1 \) or 0;
(5) \( \ell \) odd prime, \( \ell | N_{\text{add}}, \ell \equiv 2 \pmod{3} \implies \psi(\ell) = 1; \)
(6) \( 4 | N_{\text{add}} \implies \psi(2) = 0; \)
(7) if \( 2 | N \), then \( 4 | M \) and \( m \equiv 1 \pmod{4} \);
(8) if \( 4 | N \), then \( 16 | M \) and \( m \equiv 8 \) or 12 \pmod{16}.

Thus for \( d > 0 \) (resp. \( d < 0 \)) such that \( d_{\mathbb{Q}(\sqrt{-d})} \equiv m \pmod{M} \) (resp. \( d \equiv m \pmod{M} \)), \( \psi \) satisfies all the desired congruence conditions except for possibly \( h_3(-3d) = 1 \) (resp. \( h_3(d) = 1 \)). Moreover, the congruence conditions above imply that \( m \) and \( M \) are valid positive integers for Theorem 4.1 (in particular implying that \( 4 | d \) if \( 4 | N \)). (Note that in congruence conditions (2) and (3) above, we do not allow \( m \equiv 0 \pmod{\ell} \), i.e. \( \psi \) is ramified at \( \ell \), because the resulting pair \( m \) and \( M \) would violate the auxiliary hypothesis of Theorem 4.1.) Thus, by Proposition 4.2,

\[
\lim_{x \to \infty} \frac{|K_\ell(x, m, M)|}{|K^-(x, 1, 1)|} \geq \frac{1}{2 \Phi(M)} \prod_{\ell | M} \frac{q}{\ell + 1}
\]

and so at least this positive proportion of \( d > 0 \) (resp. \( d < 0 \)) satisfy \( d_{\mathbb{Q}(\sqrt{-d})} \equiv m \pmod{M} \) (resp. \( d \equiv m \pmod{M} \)) and \( h_3(-3d) = 1 \) (resp. \( h_3(d) = 1 \)). Moreover, noticing that the congruence conditions on \( m \) above are independent (again by the Chinese remainder theorem), we have

\[
\prod_{\ell | N_{\text{split}}, \ell \text{ odd}, \ell \neq 3} \frac{\ell - 1}{2} \prod_{\ell | N_{\text{nonsplit}}, \ell \text{ odd}, \ell \neq 3} \frac{\ell - 1}{2} \cdot \prod_{\ell | N_{\text{add}}, \ell \text{ odd}, \ell \equiv 1 \pmod{3}} \frac{(\ell + 2)(\ell - 1)}{2} \prod_{\ell | N_{\text{add}}, \ell \text{ odd}, \ell \equiv 2 \pmod{3}} (\ell - 1) \prod_{4 | N_{\text{add}}} 2
\]

valid choices of residue classes for \( m \pmod{M} \). Combining all the above and summing over each valid residue class \( m \pmod{M} \), we immediately obtain our lower bounds for the proportions of valid \( d \).

The final results follow from considering residue classes \( m \pmod{M = 9N'} \) and using the calculations above. \( \square \)

**Remark 4.5.** Note that for each \( d \) produced by Theorem 4.4, Theorem 4.3 shows that there is a positive proportion of imaginary quadratic \( K \) such that \( P_{E^{(d)}}(K) \) is non-torsion. In particular, for each such \( d \) there is at least one \( K \), such that \( P_{E^{(d)}}(K) \) is non-torsion. In particular, \( r_{\text{an}}(E^{(d)}/K) = 1 \) by the Gross–Zagier formula.

**Proof of Theorem 4.4.** Suppose \( E[3] \) is reducible, i.e. \( E[3]^{ss} \cong \mathbb{F}_3(\psi) \oplus \mathbb{F}_3(\psi^{-1} \omega) \) for some Dirichlet character \( \psi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mu_2 \). Twisting by the quadratic character \( \psi^{-1} \), we may assume without loss of generality that \( E[3]^{ss} \cong \mathbb{F}_3 \oplus \mathbb{F}_3(\omega) \). Now we apply Theorems 4.3 and 4.3 to produce positive proportions of real and imaginary quadratic twists \( d \) with \( r_{\text{an}}(E^{(d)}/K) = 1 \). It follows that one of \( r_{\text{an}}(E^{(d)}) \) and \( r_{\text{an}}(E^{(dd)K}) = 0 \) and the other is 1, according to their root numbers.

The root number \( w(E^{(d)}) \) is computed via changes in the local root numbers as follows (see Bal14 Table 1):

1. if \( \ell \nmid Nd \), then \( w_\ell(E^{(d)}) = w_\ell(E) \);
2. if \( \ell | N, \ell \nmid d \), then \( w_\ell(E^{(d)}) = \psi_\ell(\ell)w_\ell(E) \);
3. if \( \ell \nmid N, \ell | d \), then \( w_\ell(E^{(d)}) = \psi_\ell(-1)w_\ell(E) \);
(4) if $\ell|(N,d)$, then $w_\ell(E^{(d)}) = -\psi_\ell(-1)w_\ell(E)$.

So

$$w(E^{(d)}) = -\prod_{\ell<\infty} w_\ell(E^{(d)}) = -\left(\prod_{\ell|d} \psi_\ell(-1)\right) \left(\prod_{\ell|N,\ell|d} \psi_\ell(\ell)\right) (-1)^{\#\{\ell|(N,d)\}} w(E)$$

$$= -\psi(-1) \left(\prod_{\ell|N,\ell|d} \psi_\ell(\ell)\right) (-1)^{\#\{\ell|(N,d)\}} w(E).$$

The $d$ produced by Theorem 4.4 are chosen so that for each $\ell | d \implies \ell||N \implies w_\ell(E)\psi_\ell(\ell) = a_\ell(E)\psi_\ell(\ell) = 1$ (since for $\ell||N$, $w_\ell(E) = -a_\ell(E)$). So the above is equal to

$$-\psi(-1) (-1)^{\#\{\ell|(N,d)\}} w(E),$$

where $\psi$ is the quadratic character associated with $d$. The $d$ produced by Theorem 4.4 also satisfy

$$\#\{\ell|(N,d)\} = \#\{\ell \text{ prime} : \ell|\text{add}, \ell \equiv 2 \pmod{3} \cup \{\ell \text{ prime} : \ell|(3,N,d)\} : = S(N) + c_3(d)$$

where $c_3(d) = 0$ if $d \equiv 2 \pmod{3}$ or $3 | N$, and $c_3(d) = 1$ if $d \equiv 0 \pmod{3}$ and $3|N$. So we finally have

$$w(E^{(d)}) := -\psi(-1)(-1)^{S(N)+c_3(d)} w(E).$$

It follows that $w(E^{(d)})$ is +1 (resp. -1) for a positive proportion of $d$’s under consideration. Hence a positive proportion of real and imaginary quadratic twists of $E$ have algebraic and analytic rank 0 and a positive proportion of real and imaginary quadratic twists of $E$ have algebraic and analytic rank 1, with explicit lower bounds on these proportions given in the statements of Theorems 4.4 and 4.3.

**Remark 4.6.** It is possible to refine the casework in the proofs of Theorems 4.4 and 4.3 in order to achieve better lower bounds of twists with ranks 0 or 1.

**Example 4.7.** Consider the elliptic curve

$$E = 19a1 : y^2 + y = x^3 + x^2 - 9x - 15$$

in Cremona’s labeling. Then $E(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$, so we take $p = 3$ and obtain $E[3]^{ss} = \mathbb{F}_3 \oplus \mathbb{F}_3(\omega)$. Notice that $N = N_{\text{split}} = 19$ and the root number $w(E) = +1$. Consider the set of fundamental discriminant $d > 0$ (resp. $d < 0$) such that

1. $\psi_d(3) \neq 1$ and $\psi_d(\omega)(3) \neq 1$.
2. $\psi_d(19) = -1$.
3. $\omega_3(-3d) = 1$ (resp. $h_3(d) = 1$).

The first few such $d > 0$ are

$$d = 8, 12, 21, 41, 53, 56, 65, 84, 89, 129, 164, 165, 185, 189, \cdots$$

and the first few such $d < 0$ are

$$d = -4, -7, -24, -28, -43, -55, -63, -115, -123, -159, -163, -168, -172, -175, -187, -195, \cdots$$

Then by Theorem 4.3 and notice that the root number $w(E^{(d)}) = \psi_d(-19) = -1$ (resp. +1), we know that

$$r_{an}(E^{(d)}) = \begin{cases} 0, & d < 0, \\ 1, & d > 0. \end{cases}$$
The explicit lower bounds in Theorem 4.4 shows that at least \( \frac{19}{160} = 11.875\% \) of real quadratic twists of \( E \) have rank 1, and at least \( \frac{19}{240} = 7.917\% \) in \[Jam98, p. 640\]

5. The sextic twists family

5.1. The curves \( E_d \). In this section we consider the elliptic curve of \( j \)-invariant 0,

\[
E = 27a1 = X_0(27) : y^2 = x^3 - 432.
\]

We remind the reader that \( E \) has CM by the ring of integers \( \mathbb{Z}[\zeta_3] \) of \( \mathbb{Q}(\sqrt{-3}) \) and is isomorphic to the Fermat cubic curve \( X^3 + Y^3 = 1 \) via the transformation

\[
X = \frac{36 - y}{6x}, \quad Y = \frac{36 + y}{6x}.
\]

Definition 5.1. For \( d \in \mathbb{Z} \), we denote \( E_d \) the \( d \)-th sextic twist of \( E \),

\[
E_d : y^2 = x^3 - 432d.
\]

Notice that the \( d \)-th quadratic twist \( E^{(d)} \) of \( E \) is given by

\[
E_d^{(3)} = E^{(d)} : y^2 = x^3 - 432d^3,
\]

and the \( d \)-th cubic twist of \( E \) is given by

\[
E_d^{(2)} : y^2 = x^3 - 432d^2.
\]

Lemma 5.2. We have an isomorphism of \( G_{\mathbb{Q}} \)-representations

\[
E_d[3]^{ss} \cong \mathbb{F}_3(\psi_d) \oplus \mathbb{F}_3(\psi_d\omega).
\]

Here \( \psi_d : G_{\mathbb{Q}} \to \text{Aut}(\mathbb{F}_3) = \{\pm 1\} \) is the quadratic character associated to the extension \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \) and \( \omega = \psi_{-3} : G_{\mathbb{Q}} \to \text{Aut}(\mathbb{F}_3) = \{\pm 1\} \).

Proof. Notice that \( E_d \cong E_{d^2} \) is the \( d^4 \)-th sextic twist of the curve \( E_{d^3} \), which is the same as the \( d^2 \)-cubic twist of the quadratic twist \( E^{(d)} \). Since \( E(\mathbb{Q})[3] \cong \mathbb{Z}/3\mathbb{Z} \), we know that

\[
E[3]^{ss} \cong \mathbb{F}_3 \oplus \mathbb{F}_3(\omega).
\]

Hence

\[
E^{(d)}[3]^{ss} \cong \mathbb{F}_3(\psi_d) \oplus \mathbb{F}_3(\psi_d\omega).
\]

Since cubic twisting does not change the semi-simplification of the mod 3 \( G_{\mathbb{Q}} \)-representation (equivalently, the associated modular forms are congruent mod 3 under cubic twisting), the result then follows.

Lemma 5.3. Assume that:

1. \( d \) is a fundamental discriminant.
2. \( d \equiv 2 \pmod{3} \).

Then the root number of \( E_d \) is given by

\[
w(E_d) = \begin{cases} +1, & d \equiv 2 \pmod{9}, \\ -1, & d \equiv 5, 8 \pmod{9}. \end{cases}
\]

Proof. We use the closed formula for the local root numbers \( w_\ell(E_d) \) in \[Liv95, \S 9\].
(1) Since \( d \) is a fundamental discriminant, we have either \( d \equiv 1 \pmod{4} \) or \( d = 4d' \) for some \( d' \equiv 2, 3 \pmod{4} \). In the first case, we have \( -432d = 2^4 \cdot (-27d) \), with \( 2 \nmid (-27d) \) and in the second case we have \( -432d = 2^6 \cdot (-27d') \), with \( 2 \nmid (-27d') \). In both cases the odd part of \(-432d\) is not \( \equiv 1 \pmod{4} \), and so the local root number

\[
w_2(E_d) = +1.
\]

(2) Notice that \(-432d = 3^3 \cdot (-16d)\). Its prime-to-3 part \(-16d\) satisfies \(-16d \pmod{9} \in \{2, 7, 1\}\) if and only if \( d \pmod{9} \in \{8, 1, 5\}\). It follows that the local root number

\[
w_3(E_d) = \begin{cases} +1, & d \equiv 2 \pmod{9}, \\ -1, & d \equiv 5, 8 \pmod{9}. \end{cases}
\]

(3) Since \( d \equiv 2 \pmod{3} \), we know that the number of prime factors \( \ell | d \) such that \( \ell \geq 5 \) and \( \ell \equiv 2 \pmod{3} \) is odd. Hence

\[
\prod_{\ell \geq 5} w_\ell(E_d) = -1.
\]

Now the result again follows from the product formula.

\[\square\]

**Lemma 5.4.** Assume that:

(1) \( d \) is a fundamental discriminant.

(2) \( d \equiv 0 \pmod{3} \).

Then the root number of \( E_d \) is given by

\[
w(E_d) = \begin{cases} -1, & d \equiv 3 \pmod{9}, \\ +1, & d \equiv 6 \pmod{9}. \end{cases}
\]

**Proof.** The proof is similar to Lemma 5.3 using [Liv95, §9].

(1) Since \( d \) is a fundamental discriminant, we again have \( w_2(E_d) = +1 \).

(2) Let \( d = 3d' \). Then \(-432d = 3^4 \cdot (-16d')\), with \( 3 \nmid -16d' \). Since the exponent of 3 is 4, which is \( \equiv 1 \pmod{3} \), we know that \( w_3(E_d) = +1 \).

(3) The number of prime factors \( \ell | d \) such that \( \ell \geq 5 \) and \( \ell \equiv 2 \pmod{3} \) is odd if and only if \( d' \equiv 2 \pmod{3} \). Hence

\[
\prod_{\ell \geq 5} w_\ell(E_d) = \begin{cases} +1, & d \equiv 3 \pmod{9}, \\ -1, & d \equiv 6 \pmod{9}. \end{cases}
\]

Now the result again follows from the product formula.

\[\square\]

5.2. **Heegner points on** \( E_d \). Since \( E_d \) is CM, we know that its conductor \( N(E_d) = N_{\text{add}}(E_d) \).

When \( d \) is a fundamental discriminant, the curve \( E_d \) has additive reduction exactly at the prime factors of \( 3d \).

**Definition 5.5.** For any non-square integer \( D \), we denote by \( h_3(D) := |\text{Cl}(\mathbb{Q}(\sqrt{D}))|\) the 3-class number of the quadratic field \( \mathbb{Q}(\sqrt{D}) \).

**Theorem 5.6.** Let \( K = \mathbb{Q}(\sqrt{d_K}) \) be an imaginary quadratic field satisfying the Heegner hypothesis with respect to \( 3d \). Let \( P_d \in E_d(K) \) be the associated Heegner point. Assume that:

(1) \( d \) is a fundamental discriminant.

(2) \( d \equiv 2 \pmod{3} \) or \( d \equiv 3 \pmod{9} \).
If $d > 0$, then $h_3(-3d) = h_3(d_Kd) = 1$. If $d < 0$, then $h_3(d) = h_3(-3d_Kd) = 1$. 

Then

$$\log_{\omega_{E_d}} P_d \not\equiv 0 \pmod{3}. \tag{18}$$

In particular, $P_d$ is of infinite order and $E_d/K$ has both analytic and algebraic rank one.

**Remark 5.7.** The conclusion (18) may fail when removing any of the three assumptions.

**Proof.** It follows by applying Theorem 2.1 for $p = 3$ and noticing that $|\tilde{E}_d^s(F_3)| = 3$ since $E_d$ has additive reduction at 3. It remains to check that all the assumptions of Theorem 2.1 are satisfied. By Lemma 5.2 we have $E[3]$ is reducible with $\psi = \psi_d$. Since $d$ is a fundamental discriminant, we know that $f(\psi) = d$, which divides $N_{\text{add}}(E_d)$. The condition that $\psi(3) \neq 1$ and $(\psi^{-1}\omega)(3) \neq 1$ is equivalent to that $d \equiv 2 \pmod{3}$ or $d \equiv 3 \pmod{9}$. For $\ell \neq 3$ and $\ell | N_{\text{add}}(E_d)$, we have $\ell | d$, so $\psi_d(\ell) = 0$. Finally, the requirement on the trivial 3-class numbers is exactly the assumption that $3 \not| B_1,\psi_0 \cdot \epsilon_{K} B_1,\psi_0 \omega^{-1}$ by noticing that

$$(\psi_d)_0 = \begin{cases} \psi_d, & d > 0 \\ \psi_{d_Kd}, & d < 0, \end{cases}$$

and using the formula for the Bernoulli numbers (16). \hfill \square

**Corollary 5.8.** Assume we are in the situation of Theorem 5.6.

(1) If $d \equiv 2 \pmod{9}$, then $E_d/Q$ has analytic rank zero and $E_{d_K}/Q$ has analytic rank one.

(2) If $d \equiv 3, 5, 8 \pmod{9}$, then $E_d/Q$ has analytic rank one and $E_{d_K}/Q$ has analytic rank zero.

**Proof.** It follows immediately from Lemmas 5.3, 5.4 and Theorem 5.6. \hfill \square

### 5.3. Goldfeld’s conjecture.

**Theorem 5.9.** The weak Goldfeld’s conjecture holds for the sextic twists family $\{E_d\}$. In fact,

(1) $E_d$ has analytic rank 0 for at least $1/48$ of positive fundamental discriminants $d$ and at least $1/16$ of negative fundamental discriminants $d$.

(2) $E_d$ has analytic rank 1 for at least $11/48$ of positive fundamental discriminants $d$ and at least $3/16$ of negative fundamental discriminants $d$.

**Proof.** Part (1) follows from part (2) of the final part of Theorem 4.4, Theorem 2.1, and Remark 4.5. Part (2) follows from adding parts (1) and (3) of the final part of Theorem 4.4, Theorem 2.1, and Theorem 5.8. \hfill \square

### 5.4. The 3-part of the BSD conjecture over $K$.

**Theorem 5.10.** Assume we are in the situation of Theorem 5.6. Assume the Manin constant of $E_d$ is coprime to 3. Then BSD(3) is true for $E_d/K$.

By the Gross–Zagier formula, the BSD conjecture for $E_d/K$ is equivalent to the equality ([GZ86, V.2.2])

$$u_K \cdot c_{E_d} \cdot \prod_{\ell | N(E_d)} c_{\ell}(E_d) \cdot |\Pi(E_d/K)|^{1/2} = [E_d(K) : \mathbb{Z}P_d], \tag{19}$$
where \( u_K = \{ \mathcal{O}_K^*/\{\pm 1\} \} \), \( c_{E_d} \) is the Manin constant of \( E_d/\mathbb{Q} \), \( c_\ell(E_d) = \left[ E_d(\mathbb{Q}_\ell) : E^0_d(\mathbb{Q}_\ell) \right] \) is the local Tamagawa number of \( E_d \) and \( [E_d(K) : \mathbb{Z}P_d] \) is the index of the Heegner point \( P_d \in E_d(K) \). Since 3 splits in \( K \), we know \( K \neq \mathbb{Q}(\sqrt{-1}) \) or \( \mathbb{Q}(\sqrt{-3}) \), so \( u_K = 1 \). Therefore the BSD conjecture for \( E_d/K \) is equivalent to the equality
\[
\prod_{\ell \mid N(E_d)} c_\ell(E_d) \cdot [\text{III}(E_d/K)]^{1/2} = \frac{[E_d(K) : \mathbb{Z}P_d]}{c_{E_d}}.
\]
We will prove BSD(3) by computing the 3-part of both sides of (20) explicitly.

**Lemma 5.11.** We have \( E_d(K)[3] = 0 \).

**Proof.** By Lemma 5.2 we have \( E_d[3]_{\text{ss}} \cong F_3(\psi_d) \oplus F_3(\psi_d\omega) \). Since neither \( \psi_d \) nor \( \psi_d\omega \) becomes trivial when restricted to \( G_K \), we know that \( E_d(K)[3] = 0 \).

**Lemma 5.12.** If \( \ell \mid N(E_d) \) and \( \ell \neq 3 \) (equivalently, \( \ell \mid d \)), then \( 3 \nmid c_\ell(E_d) \).

**Proof.** Since \( E_d \) has additive reduction at \( \ell \) and \( \ell \neq 3 \), we know that the 3-part of \( c_\ell(E_d) \) is equal to the size of \( E_d[3](\mathbb{Q}_\ell) \). By Lemma 5.2 we have \( E_d[3]_{\text{ss}} \cong F_3(\psi_d) \oplus F_3(\psi_d\omega) \). Since \( \psi_d \) and \( \psi_d\omega \) are both ramified at \( \ell \), we know that \( E_d[3](\mathbb{Q}_\ell) = 0 \), and thus the 3-part of \( c_\ell(E_d) \) is trivial.

**Definition 5.13.** Let \( F \) be any number field. Let \( \mathcal{L} = \{ \mathcal{L}_v \} \) be a collection of subspaces \( L_v \subseteq H^1(F_v, E_d[3]) \), where \( v \) runs over all places of \( L \). We say \( \mathcal{L} \) is a collection of local conditions if for almost all \( v \), we have \( \mathcal{L}_v = H^1_{\text{ur}}(F_v, E_d[3]) \) is the unramified subspace. We define the Selmer group cut out by the local conditions \( \mathcal{L} \) to be
\[
H^1_{\mathcal{L}}(F, E_d[3]) := \{ x \in H^1(F, E_d[3]) : \text{res}_v(x) \in \mathcal{L}_v, \text{for all } v \}.
\]
We will consider the following four types of local conditions:

1. The Kummer conditions \( \mathcal{L} \) given by \( \mathcal{L}_v = \text{im} \left( E(F_v)/3E(F_v) \rightarrow H^1(F_v, E_d[3]) \right) \). The 3-Selmer group \( \text{Sel}_3(E_d/F) = H^1_{\mathcal{L}}(F, E_d[3]) \) is cut out by the Kummer conditions.

2. The unramified conditions \( \mathcal{U} \) given by \( \mathcal{U}_v = H^1_{\text{ur}}(F_v, E_d[3]) \).

3. The strict conditions \( \mathcal{S} \) given by \( \mathcal{S}_v = \mathcal{U}_v \) for \( v \nmid 3 \) and \( \mathcal{S}_v = 0 \) for \( v | 3 \).

4. The relaxed conditions \( \mathcal{R} \) given by \( \mathcal{R}_v = \mathcal{U}_v \) for \( v \nmid 3 \) and \( \mathcal{R}_v = H^1(F_v, E_d[3]) \) for \( v | 3 \).

**Lemma 5.14.** \( H^1_{\mathcal{L}}(K, E_d[3]) = H^1_{\mathcal{S}}(K, E_d[3]) = 0 \).

**Proof.** By inflation-restriction, we have
\[
H^1_{\mathcal{L}}(K, E_d[3]) \cong H^1_{\mathcal{L}}(\mathbb{Q}, E_d[3]) \oplus H^1_{\mathcal{L}}(\mathbb{Q}, E_d^{(d_K)}[3]).
\]
By Lemma 5.2 that \( E_d[3]_{\text{ss}} \cong F_3(\psi_d) \oplus F_3(\psi_d\omega) \). Hence by class field theory, we know that
\[
|H^1_{\mathcal{L}}(\mathbb{Q}, E_d[3])| = h_3(d) \cdot h_3(-3d), \quad |H^1_{\mathcal{L}}(\mathbb{Q}, E_d^{(d_K)}[3])| = h_3(d_Kd) \cdot h_3(-3d_Kd).
\]
By the assumptions on the 3-class numbers in Theorem 5.6 and Scholz’s reflection theorem ([Sch32], see also [Was97, 10.2]), we know that the four 3-class numbers above are all trivial. Hence \( H^1_{\mathcal{L}}(K, E_d[3]) = 0 \). Since by definition we have
\[
H^1_{\mathcal{S}}(K, E_d[3]) \subseteq H^1_{\mathcal{L}}(K, E_d[3]),
\]
we also know that \( H^1_{\mathcal{S}}(K, E_d[3]) = 0 \).

**Lemma 5.15.** \( \dim H^1_{R}(K, E_d[3]) = 2 \).
Proof. It follows from [DDT97] Theorem 2.18 that
\[ \dim H^1_K(K, E_d[3]) - \dim H^1_K(K, E_d[3]) = \frac{1}{2} \sum_{\nu | 3} \dim \mathcal{R}_\nu. \]

Consider \(\nu|3\). Since 3 is split in \(K\), we know that \(H^1(K_v, E_d[3]) \cong H^1(\mathbb{Q}_3, E_d[3])\). By Lemma 5.2 that \(E_d[3]^\text{ns} \cong \mathbb{F}_3(\psi_d) \oplus \mathbb{F}_3(\psi_d \omega)\). Since \(\psi_d(3) \neq 1\) and \(\psi_d \omega(3) \neq 1\), we know that
\[ H^0(\mathbb{Q}_3, E_d[3]) = H^2(\mathbb{Q}_3, E_d[3]) = 0. \]

It follows from the Euler characteristic formula that
\[ \dim (H^1(\mathbb{Q}_3, E_d[3])) = 2. \]

Namely, \(\dim \mathcal{R}_\nu = 2\). The result then follows from Lemma 5.14.

Lemma 5.16. \(\text{Sel}_3(E_d/K) \cong \mathbb{Z}/3\mathbb{Z}\). In particular, \(\text{III}(E_d/K)[3] = 0\).

Proof. We claim that \(\mathcal{L}_\nu = \mathcal{U}_\nu\) for any \(\nu \nmid 3\). In fact:
1. If \(\nu \nmid 3\), then \(E_d\) has good reduction at \(\nu\) and so \(\mathcal{L}_\nu = H^1_{\text{ur}}(K_v, E_d[3])\) by [GP12] Lemma 6.
2. If \(\nu|\infty\), then \(\nu\) is complex and \(H^1(K_v, E_d[3]) = 0\). So \(\mathcal{L}_\nu = H^1_{\text{ur}}(K_v, E_d[3]) = 0\).
3. If \(\nu|d\), then \(\nu\) is split in \(K\) and thus \(K_\nu \cong \mathbb{Q}_\ell\). By Lemma 5.12 \(c_\ell(E)\) is coprime to 3. It follows that \(\mathcal{L}_\nu = H^1_{\text{ur}}(K_v, E_d[3])\) by [GP12] Lemma 6.

It follows from the claim that
\[ \text{Sel}_3(E_d/K) \subseteq H^1_K(K, E_d[3]). \]

So \(\dim \text{Sel}_3(E_d/K) \leq 2\) by Lemma 5.13.

By the Heegner hypothesis, the root number of \(E_d/K\) is \(-1\). Since the 3-parity conjecture is known for elliptic curves with a 3-isogeny ([DD11] Theorem 1.8), we know that \(\dim \text{Sel}_3(E_d/K)\) is odd and thus must be 1. Hence \(\text{Sel}_3(E_d/K) \cong \mathbb{Z}/3\mathbb{Z}\) as desired.

Lemma 5.17. We have
\[ \text{ord}_3(c_3(E_d)) = \begin{cases} 1, & d \equiv 2 \pmod{9}, \\ 0, & d \equiv 3, 5, 8 \pmod{9}. \end{cases} \]

In either case we have \(\text{ord}_3(c_3(E_d)) = \text{ord}_3\left(\frac{(E_d(K) : \mathbb{Z}P_d)}{c_{E,d}}\right)\).

Proof. The first part follows directly from Tate’s algorithm [Sil94 IV.9] (see also the formula in [Sat86 0.5]).

Suppose \(\text{ord}_3(c_3(E_d)) = 0\). We need to show that \(\text{ord}_3(E_d(K) : \mathbb{Z}P_d) = 0\). If not, then since \(E_d(K)[3] = 0\) (Lemma 5.11), we know that there exists some \(Q \in E_d(K)\) such that \(3Q = nP_d\) for some \(n\) coprime to 3. Let \(\omega_{E_d}\) be the Néron differential of \(E_d\) and let \(\log_{E_d} := \log_{E_d}\). By the very definition of the Manin constant we have \(c_{E_d} \cdot \omega_{E_d} = \omega_{E_d}\) and \(c_{E_d} \cdot \log_{E_d} = \log_{E_d}\). Since \(c_{E_d}\) is assumed to be coprime to 3, we have up to a 3-adic unit,
\[ \frac{|E_d^{\text{ns}}(\mathbb{F}_3)| \cdot \log_{E_d} P_d}{3} = \frac{|E_d^{\text{ns}}(\mathbb{F}_3)| \cdot \log_{E_d} P_d}{3} = |E_d^{\text{ns}}(\mathbb{F}_3)| \cdot \log_{E_d}(Q). \]

On the other hand, \(c_3(E_d) \cdot |E_d^{\text{ns}}(\mathbb{F}_3)| \cdot Q\) lies in the formal group \(E_d(3\mathcal{O}_K)\) and \(\text{ord}_3(c_3(E_d)) = 0\), we know that
\[ |E_d^{\text{ns}}(\mathbb{F}_3)| \cdot \log_{E_d}(Q) \in 3\mathcal{O}_K. \]
which contradicts the formula (18).

Now suppose \( \text{ord}_3(c_3(E_d)) = 1 \). The same argument shows as the previous case shows that \( \text{ord}_3([E_d(K) : \mathbb{Z}P_d]) \leq 1 \). We need to show that
\[
\text{ord}_3([E_d(K) : \mathbb{Z}P_d]) \neq 0.
\]
Assume otherwise, then the image of \( P_d \) in \( E_d(K)/3E_d(K) \) is nontrivial, and hence its image in \( \text{Sel}_3(E_d/K) \cong \mathbb{Z}/3\mathbb{Z} \) is a generator. We now analyze its local Kummer image at 3 and derive a contradiction.

Since \( c_3(E_d) = 3 \) and \( \hat{E}_d^{ns}(\mathbb{F}_3) = \mathbb{Z}/3\mathbb{Z} \), we know that \( E_d(\mathbb{Q}_3)/\hat{E}_d(3\mathbb{Z}_3) \) is a group of order 9, so
\[
E_d(\mathbb{Q}_3)/\hat{E}_d(3\mathbb{Z}_3) \cong \mathbb{Z}/9\mathbb{Z} or \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.
\]
Since \( \text{dim} H^1(\mathbb{Q}_3, E_d[3]) = 2 \) and the local Kummer condition is a maximal isotropic subspace of \( H^1(\mathbb{Q}_3, E_d[3]) \), we know that \( E_d(\mathbb{Q}_3)/3E_d(\mathbb{Q}_3) = \mathbb{Z}/3\mathbb{Z} \). So the only possibility is that
\[
E_d(\mathbb{Q}_3)/\hat{E}_d(3\mathbb{Z}_3) \cong \mathbb{Z}/9\mathbb{Z}.
\]
Now by the formula (18), we know that \( \hat{P}_d \in \hat{E}_d(3\mathbb{O}_K) \) but \( 3P_d \notin \hat{E}_d(3\mathbb{O}_K) \). It follows that \( P_d \in 3E_d(K_\ell) \). The local image of \( P_d \) in \( E_d(K_\ell)/3E_d(K_\ell) \) is trivial.

It follows that \( \text{Sel}_3(E_d/K) \) is equal to the strict Selmer group \( H^1_S(K, E_d[3]) \), a contradiction to Lemmas 5.14 and 5.16 \( \square \)

**Proof of Theorem 5.10.** Theorem 5.10 follows immediately from the equivalent formula (20) and Lemmas 5.12 5.16 and 5.17 \( \square \)

### 6. Cubic twists families

In this section we consider the elliptic curve \( E_d/\mathbb{Q} : y^2 = x^3 - 432d \) of \( j \)-invariant 0, where \( d \) is any 6th-power-free integer. Recall that for a cube-free integer \( D \), the \( D \)-th cubic twist \( E_d \) is the curve \( E_dD^2 \) (cf. Definition 5.1). For \( r \geq 0 \), we define
\[
C_r(E_d, X) = \{ |D| < X : D \text{ cube-free, } r_{an}(E_dD^2) = r \}
\]
to be the counting function for the number of cubic twists of \( E_d \) of analytic rank \( r \). Recall that by Lemma 5.2 \( E_d[3]^{ns} \cong \mathbb{F}_3(\psi_d) \oplus \mathbb{F}_3(\psi_d\omega) \).

**Theorem 6.1.** Assume for any prime \( \ell | N(E_d) \), we have \( \psi_d(\ell) \neq 1 \) and \( \psi_d\omega(\ell) \neq 1 \). Assume there exists an imaginary quadratic field \( K \) satisfying the Heegner hypothesis for \( N(E_d) \) such that
(1) \( 3 \) is split in \( K \).
(2) If \( d > 0 \), then \( h_3(-3d) = h_3(dKd) = 1 \). If \( d < 0 \), then \( h_3(d) = h_3(-3dKd) = 1 \).
Then for \( r \in \{0, 1\} \), we have
\[
C_r(E_d, X) \gg \frac{X}{\log^{7/8}(X)}.
\]

**Remark 6.2.** Notice that when \( 3 \nmid d \) is a fundamental discriminant, the condition \( \psi_d(\ell) \neq 1 \) and \( \psi_d\omega(\ell) \neq 1 \) for \( \ell | N(E_d) \) are automatically satisfied.

**Proof.** We consider the following set \( S \) consisting of primes \( \ell \nmid 6N(E_d) \) such that
(1) \( \ell \) is split in \( K \).
(2) \( \psi_d(\ell) = -1 \) (\( \ell \) is inert in \( \mathbb{Q}(\sqrt{d}) \)).
(3) \( \omega(\ell) = 1 \) (\( \ell \) is split in \( \mathbb{Q}(\sqrt{-3}) \)).
Since our assumption implies that the three quadratic fields \( K, \mathbb{Q}(\sqrt{d}) \) and \( \mathbb{Q}(\sqrt{-3}) \) are linearly disjoint, we know that the set of primes \( S \) has density \( \alpha = (\frac{1}{3})^3 = \frac{1}{27} \) by Chebotarev’s density theorem.

Let \( \mathcal{N} \) be the set of integer consisting of square-free products of primes in \( S \). Then for any \( D \in \mathcal{N} \). We have \( E_{dD}[3]^{ss} \cong \mathbb{F}_3(\psi_d) \oplus \mathbb{F}_3(\psi_3 \omega) \). For any \( \ell | N(E_{dD^2}) \), we have \( \psi_d(\ell) \neq 1 \) and \( \psi_3 \omega(\ell) \neq 1 \) by construction. The imaginary quadratic field \( K \) also satisfies the Heegner hypothesis for \( N(E_{dD^2}) \). Since the relevant 3-class numbers are trivial, we can apply Theorem 2.1 \((p = 3)\) to \( E_{dD^2} \) and conclude that \( r_{an}(E_{dD^2}/K) = 1 \). The root number \( w(E_{dD^2}) \) is \(+1\) (resp. \(-1\)) for a positive proportion of \( D \in \mathcal{N} \), so we have for \( r \in \{0,1\} \),

\[
C_r(E_d, X) \gg \# \{ D \in \mathcal{N} : |D| < X \}.
\]

By a standard application of Ikehara’s tauberian theorem (see [KL16, 4.2]), we know that

\[
\# \{ D \in N : |D| < X \} \sim c \cdot \frac{X}{\log^{1-\alpha} X},
\]

for some \( c > 0 \). Here \( \alpha = \frac{1}{3} \) is the density of the set of primes \( S \). The results then follow. \( \square \)

**Example 6.3.** Consider \( d = 2^2 \cdot 3^3 = 108 \). Then \( E_d = 144a1 : y^2 = x^3 - 1 \). The field \( K = \mathbb{Q}(\sqrt{-23}) \) satisfies the Heegner hypothesis for \( N = 144 \) and 3 is split in \( K \). We compute the 3-class numbers \( h_3(-3d) = h_3(-1) = 1 \) and \( h_3(dkd) = h_3(-69) = 1 \). So the assumptions of Theorem 6.1 are satisfied. The set \( \mathcal{N} \) in the proof of Theorem 6.1 consists of square-free products of the primes

\[
31, 127, 139, 151, 163, 211, 223, 271, 307, 331, 439, 463, 487, 499, \ldots
\]

Notice that \( D \in \mathcal{N} \) implies that \( D \equiv 1 \pmod{3} \). One can compute the root number of the cubic twist

\[
E_{dD^2} : y^2 = x^3 - D^2
\]

to be

\[
w(E_{dD^2}) = \begin{cases} +1, & D \equiv 1, 4 \pmod{9} \\ -1, & D \equiv 7 \pmod{9} \end{cases}
\]

We conclude that for \( D \in \mathcal{N} \),

\[
r_{an}(E_{dD^2}) = \begin{cases} 0, & D \equiv 1, 4 \pmod{9} \\ 1, & D \equiv 7 \pmod{9} \end{cases}
\]

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