The generating functional for the electromagnetic interaction in the strong gravitational field

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Abstract

This article is devoted problems of electromagnetic interaction in curved spacetime. Such problems exist, in particular, when we investigate electromagnetic quantum processes near black holes. The generalization of reduction formalism permits to find formulas for scattering matrices. For the free Dirac and electromagnetic fields corresponding generating functionals are calculated. Next we have found the generating functional for the interaction of these fields. This result holds the central position in our investigation. By means of generating functionals and reduction formulas we have obtained the scattering amplitudes for elementary electrodynamic processes: Compton scattering and the annihilation of electron-positron pair (for the tree-level approximation). On the base of these results we have formulated the generalized Feynman rules for the electromagnetic interaction in curved spacetime. Another electrodynamic processes can be studied by means of these Feynman rules and crossing-symmetry. The generating functional can be used to study problems of quantum statistics in curved spacetime.

1. Introduction

The tendency to unification of quantum field theory with gravity theory obtained real incarnation as yet 87 years ago. The generalization of the Dirac equation for curved spacetime was considered in the pioneer works Fock V.A. & Ivanenko D.D. (1929), Schrödinger E. (1932). Investigations of Schrödinger E.(1939), DeWitt B.S.(1953), Takahashi Y. & Umezawa H.(1957), Imamura T.(1960), Parker L.(1966) et al. were devoted to particle creation due to curvature of spacetime. DeWitt B.S.(1965), Bunch T.S. & Parker L.(1979) had found the expansion for the propagators of the scalar and Dirac...
fields. The thermal emission for black holes was discovered by Hawking S.W. in 1974 and investigated by Hawking S.W.(1975), DeWitt B.S.(1975), Hartle J.B.(in common with Hawking S.W. in 1976), Christensen S.M.& Fulling S.A (1977) et al. 

A lot of articles were devoted to problems of interaction for quantum fields in curved spacetime. Questions of forming S-matrices, the interaction of scalar fields (including renormalization) as well as particle production due to this interaction are considered in [1] ch.9. The massless theory of scalar field in asymptotically flat spacetime is studied in [13]. Renormalization in the massless quantum electrodynamics (in de Sitter space) is analysed in [3, 5, 12]. The first part of [4] contains quantum electrodynamics in curved spacetime taking into account particle production in the strong gravity field. The second part describes quantum processes in conformally flat spacetime.

The given paper is devoted to problems of quantum electrodynamics in curved spacetime: to the interaction of photons with fermions. We shall consider Compton scattering and the annihilation of electron-positron pair. Such problems exist, in particular, when we investigate electromagnetic quantum processes near black holes [6].

First of all we have found the corresponding generating functionals for the free Dirac and electromagnetic fields as well as for the interaction of these fields. Using these generating functionals and reduction formalism, we have calculated the scattering matrices for elementary processes: Compton scattering and the annihilation of electron-positron pair (for the tree-level approximation). On the base of these results we have formulated the generalized Feynman rules for electrodynamic processes in curved spacetime. The generating functional can be applied for problems of quantum statistics in curved spacetime.

2. Fermions and photons in curved spacetime

Preparatory to analysing the interaction of photons with fermions, we consider basic concepts of quantum electrodynamics in curved spacetime [1, 8]. By \( g_{ij}(x) \) denote the metric tensor of curved spacetime, and \( g(x) = \text{det}(g_{ij}), dV_x = d^4x\sqrt{-g} \). A covariant derivative is designated by \( \nabla_\mu \).

Let \( \mu(x) \) be the unit measure concentrated at the point \( x = 0 \), that is,

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1There is the comprehensive bibliography concerning problems of quantum field theory in curved spacetime (see [1, 8]).
Then this Dirac unit measure corresponds to the delta-function \(\delta(x)\) determined by the linear functional (see [11] ch.2):

\[
\int \phi(x) \delta(x) dV_x = \phi(0).
\]  

(2)

We use as well notations of quantum field theory corresponding [9].

2.1. The Dirac equation

In theory of the Dirac field the tetrad formalism is usually used to go to a system of locally inertial Cartesian coordinates (see [14] ch.12).

Let \(X\) be a point in curved spacetime. In the neighborhood of this point we determine local coordinates \(y^{(a)}\), \(a = 0, 1, 2, 3\). Then the metric tensor for local coordinates is \(\eta_{(a)(b)}\). Quantities

\[
e^\mu_{(a)}(X) = \left( \frac{\partial x^\mu}{\partial y^{(a)}} \right)_{x^\mu = X^\mu}, \quad e^\mu_{(a)}(X) = \left( \frac{\partial y^{(a)}}{\partial x^\mu} \right)_{x^\mu = X^\mu}
\]  

(3)

form a tetrad. The tetrad satisfies the relations:

\[
e^\mu_{(a)}(X)e^\nu_{(b)}(X) = \delta^\mu_{(a)}e^\nu_{(b)}(X) = \delta^a_b,
\]  

(4)

and

\[
e^\mu_{(a)}(X)e^\nu_{(b)}(X)g_{\mu\nu}(X) = \eta_{(a)(b)}, \quad e^\mu_{(a)}(X)e^\nu_{(b)}(X)\eta_{(a)(b)} = g_{\mu\nu}(X).
\]  

(5)

As indicated in [14] ch.12, the covariant derivative of the spinor field \(\psi(x)\) is

\[\tilde{\nabla}_\nu \psi(x) = [\partial_\nu + \Gamma_\nu(x)]\psi(x),\]

(6)

where

\[\Gamma_\nu(x) = \frac{1}{8} \omega^{(a)(b)}(x)[\gamma^{(a)}, \gamma^{(b)}].\]

(7)

\footnote{We use Latin indices \(a, b, c\) for local coordinates.}
The coefficients $\omega_{(a)(b)\nu}(x)$ are the components for the spin connection. It can be indicated that

$$\omega_{(a)(b)\nu}(x) = e^\mu_{(a)}(x)e_{(b)\mu\nu}(x).$$

The covariant derivative $\tilde{\nabla}_{(a)}$ can be calculated by means of the expression

$$\tilde{\nabla}_{(a)} = e^\mu_{(a)}(x)\tilde{\nabla}_\mu.$$  \hspace{1cm} (9)

The action $S$ for the Dirac field is described as

$$S = \int \mathcal{L}(x)dV_x,$$  \hspace{1cm} (10)

where

$$\mathcal{L}(x) = i\bar{\psi}\gamma^{(a)}e^\mu_{(a)}\tilde{\nabla}_\mu\psi - m\bar{\psi}\psi = i\bar{\psi}\tilde{\gamma}^\mu(x)\tilde{\nabla}_\mu\psi - m\bar{\psi}\psi.$$  \hspace{1cm} (11)

In the expression (11) we utilize a set of matrices

$$\tilde{\gamma}^\mu(x) = e^\mu_{(a)}(x)\gamma^{(a)}.$$  \hspace{1cm} (12)

Using the variation of the action $S$, we derive the covariant Dirac equation

$$i\tilde{\gamma}^\mu(x)\tilde{\nabla}_\mu\psi - m\psi = 0.$$  \hspace{1cm} (13)

There exists a set of solutions $U_{ps}(x), V_{ps}(x)$ of the equation (13) for a fermion with momentum $p$ and spin $s$. Then we can use the next expansions for $\psi(x)$ and $\tilde{\psi}(x)$:

$$\psi(x) = \int d^3p \sum_s [b(p, s)U_{ps}(x) + d^\dagger(p, s)V_{ps}(x)],$$  \hspace{1cm} (14)

$$\tilde{\psi}(x) = \int d^3p \sum_s [b^\dagger(p, s)\tilde{U}_{ps}(x) + d(p, s)\tilde{V}_{ps}(x)].$$  \hspace{1cm} (15)

The operators $b^\dagger(p, s), d^\dagger(p, s)$ are creation operators for electrons and positrons respectively, and the operators $b(p, s), d(p, s)$ represent corresponding annihilation operators.
The Green function \( S_F(x - x') \) of the Dirac field in curved spacetime is defined as

\[
i S_F(x - x') = <0|T\psi(x)\bar{\psi}(x')|0>
\]
and satisfies the following equation

\[
[i\gamma^\mu(x)\vec{\nabla}_\mu - m]\delta S_F(x - x') = \delta(x - x').
\]

2.2. The electromagnetic field

By \( A_\nu \) denote the vector potential of the electromagnetic field. The action \( S \) for the electromagnetic field is described using the Maxwell field strength tensor \( F_{\mu\nu} \):

\[
S = \int \mathcal{L} dV_x,
\]

where

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu}.
\]

Variation of the action \( S \) yields the condition\(^3\)

\[
F^{\mu\nu}_{;\nu} = 0.
\]

Consequently, it can write

\[
F^{\mu\nu}_{;\nu} = \nabla_\nu \nabla^\mu A^\nu - \nabla^\mu A\nabla_\nu A^\nu = \nabla^\mu \nabla_\nu A^\nu + R^\mu_{\nu\gamma\delta} A^\nu - \nabla^\nu \nabla^\mu A^\nu = 0,
\]

where \( R^\mu_{\nu\gamma\delta} \) is the Ricci tensor\(^4\). In the Lorentz gauge \( \nabla_\nu A^\nu = 0 \), and we obtain the equation

\[
\nabla_\nu \nabla^\mu A^\nu - R^\mu_{\nu\gamma\delta} A^\nu = 0.
\]

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\(^3\)See [7] sec.22.4.
\(^4\)Ibid.
By $\mathcal{R}^\nu$ denote the operator De Rham that satisfies the relation

$$
\mathcal{R}^\nu A = -\nabla_\mu \nabla^\mu A^\nu + R^\nu_\mu A^\mu.
$$

(24)

Let $A_{k\lambda}^\alpha(x)$ be a set of solutions of the equation (23) for a photon with momentum $k$ and polarization $\lambda$.

Then the expansion for $A^\alpha(x)$ has the form

$$
A^\alpha(x) = \int d^3k \sum_\lambda \left[ a_k(\lambda)A_{k\lambda}^\alpha(x) + a_k^\dagger(\lambda)A_{k\lambda}^{\alpha*}(x) \right].
$$

(25)

We can interpret $a_k(\lambda)$ and $a_k^\dagger(\lambda)$ as annihilation and create operators of photons.

The Green function $D_{F}^{\mu\nu}(x - x')$ of the electromagnetic field in curved spacetime is determined as

$$
i D_{F}^{\mu\nu}(x - x') = <0|TA^\mu(x)A^\nu(x')|0>
$$

(26)

and satisfies the following equation (in the Lorentz gauge):

$$
-\nabla_\rho \nabla^\rho D_{F}^{\mu\nu}(x - x') + R^\rho_\mu D_{F}^{\mu\nu}(x - x') = \delta(x - x').
$$

(27)

3. Reduction formalism

The Lagrangian density for interacting fields takes the form

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu(x) \tilde{\nabla}_\mu \psi - m \bar{\psi} \psi - e \bar{\psi} \gamma^\mu(x) \gamma^5 A_\mu.
$$

(28)

The latter term in (28) describes the interaction of the electromagnetic field with fermions. In- and out-states of fermions are determined as

$$
\lim_{x^0 \to -\infty} \psi(x) = \psi_{in}(x),
$$

(29)

$$
\lim_{x^0 \to \infty} \psi(x) = \psi_{out}(x).
$$

(30)

Using (14) and (15), we obtain the expansions for $\psi_{in}(x)$ and $\tilde{\psi}_{in}(x)$:
\[ \psi_{\text{in}}(x) = \int d^3 p \sum_s [b_{\text{in}}(p, s)U_{ps}(x) + d_{\text{in}}^\dagger(p, s)V_{ps}(x)], \quad (31) \]

\[ \tilde{\psi}_{\text{in}}(x) = \int d^3 p \sum_s [b_{\text{in}}^\dagger(p, s)\bar{U}_{ps}(x) + d_{\text{in}}(p, s)\bar{V}_{ps}(x)]. \quad (32) \]

By means of the production operators \( b_{\text{in}}^\dagger(p, s) \) and \( d_{\text{in}}^\dagger(p, s) \) we can form an arbitrary in-state. Similar expansions can be produced for out-states.\footnote{In general, the solutions \( U_{ps}(x), V_{ps}(x) \) for out-states will not be equal to the corresponding solutions for in-states.}

Replacing in [2] sec.16.9

\[ \partial_\mu \rightarrow \bar{\nabla}_\mu, \gamma^\mu \rightarrow \bar{\gamma}^\mu, \quad (33) \]

we find that

\[ < \beta \text{ out}(ps), \alpha \text{ in } > - \beta - (ps)\text{out}|\alpha \text{ in } > - \frac{i}{\sqrt{Z_2}} \int dV_x < \beta \text{ out} | \bar{\psi}(x)|\alpha \text{ in } > (-i\bar{\gamma} \cdot \bar{\nabla} - m) U_{ps}. \quad (34) \]

The first member describes the contribution only for the elastic scattering. The second member defines the amplitude of inelastic scattering. If the in-state contains an antiparticle, then we state for the amplitude of inelastic scattering

\[ \frac{i}{\sqrt{Z_2}} \int dV_x \bar{V}_{ps} (i\bar{\gamma} \cdot \bar{\nabla} - m) < \beta \text{ out} | \psi(x)|\alpha \text{ in } > . \quad (35) \]

For out-states of particles and antiparticles we have respectively:

\[ -\frac{i}{\sqrt{Z_2}} \int dV_x \bar{T}_{p's'} (i\bar{\gamma} \cdot \bar{\nabla} - m) < \beta \text{ out} | \psi(x)|\alpha \text{ in } >, \quad (36) \]

\[ \frac{i}{\sqrt{Z_2}} \int dV_x < \beta \text{ out} | \bar{\psi}(x)|\alpha \text{ in } > (-i\bar{\gamma} \cdot \bar{\nabla} - m) V_{p's'}. \quad (37) \]

Consider now the electromagnetic field. The expansion for an in-state takes the form
\[ A^\alpha_{in}(x) = \int d^3k \sum_\lambda [a_{in}(k, \lambda) A^{\alpha}_{k\lambda}(x) + \dagger a_{in}(k, \lambda) A^{\alpha*}_{k\lambda}(x)] \] (38)

By means of the operators \( a_{in}^\dagger(k, \lambda) \) we can form an arbitrary in-state with \( n \)-photons.

The reduction formula for removing a photon from an out-state is

\[ \langle \gamma(k') \text{out} | \phi(x) | \alpha \text{in} > = \langle \gamma \text{out} | T(A_\mu(y) \phi(x)) | \alpha \text{in} > - i \frac{1}{\sqrt{Z_3}} \int dV_y < \gamma \text{out} | T(A_\mu(y) \phi(x)) | \alpha \text{in} > \Re_y A^\mu_{k'\lambda'}(y). \] (39)

After we shall remove all particles from in- and out-states, we shall arrive a vacuum state. Consider, in particularly, the scattering of photons by electrons. The S-matrix takes the form

\[ \langle p's'; k'\lambda' \text{out} | ps; k\lambda \text{in} > = \delta_{ij} - \frac{1}{Z_2Z_3} \int dV_x \int dV_{x'} \int dV_z \int dV_{z'} \times \]

\[ \times A^\nu_{k\lambda}(x) \Re_x \bar{U}_{p's'}(z')(i\gamma(z') \cdot \nabla_{z'} - m) \times \]

\[ < 0 | T(\psi(z')A_\mu(x')\bar{\psi}(z)A_\nu(x)) | 0 > \] \( (40) \)

Since we are restricted to the tree-level approximation in this article and therefore do not consider the procedure of renormalization, we shall take \( Z_2 = Z_3 = 1 \). The first term in (40) corresponds to the forward scattering.

4. Generating functionals

To build Green functions for quantum fields, we shall utilize generating functionals. The functional derivative is defined as

\[ \frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F[f(x) + \varepsilon \delta(x - y)] - F[f(x)]}{\varepsilon}. \] (41)

Functional differentiation in this section is based on the relation

\(^6\) We have substituted \( \Re\nu A \) for \( \Box A^\nu \) in [2] sec.16.10.
\[
\frac{\delta f(x)}{\delta f(y)} = \delta(x - y)
\] 

(42)
as the consequence from (41).

4.1. Free fields

Consider the free Dirac field. The generating functional for the Dirac field can be represented as

\[
Z_D^0[\eta, \bar{\eta}] = \frac{1}{\mathcal{N}} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\{i \int dV_x \times \bar{\psi}(x) (i\tilde{\gamma} \cdot \tilde{\nabla} - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\}\text{.}
\]

(43)

where \(\bar{\eta}(x)\) is a source for the field \(\psi(x)\), \(\eta(x)\) is a source for the field \(\bar{\psi}(x)\), \(\mathcal{N}\) is a number.

Let \(S\) be the operator determined as

\[
S^{-1} = i\tilde{\gamma} \cdot \tilde{\nabla} - m.
\]

(44)

Then

\[
Z_D^0[\eta, \bar{\eta}] = \frac{1}{\mathcal{N}} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\{i \int dV_x (\bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta)\}.
\]

(45)

Using reasoning in [10] sec. 6.7, we derive \(^7\)

\[
Z_D^0[\eta, \bar{\eta}] = \exp\{-i \int dV_x dV_y \bar{\eta}(x)S(x - y)\eta(y)\}.
\]

(46)

To find the Dirac propagator, we carry out functional differentiation:

\[
\tau(x, y) = -\left[\frac{\delta^2 Z_D^0[\eta, \bar{\eta}]}{\delta \bar{\eta}(x) \delta \eta(y)}\right]_{\eta = \bar{\eta} = 0} = -\frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)}
\]

\[
\exp\left\{ -i \int dV_x dV_y \bar{\eta}(x') S(x' - y') \eta(y') \right\}_{\eta = \bar{\eta} = 0} = iS(x - y). \tag{47}
\]

\(^7\)Taking into account that the normalization condition is \(Z_D^0[0, 0] = 1\).
Since

\[ -\left[ \frac{\delta^2 Z_0^D[\eta, \bar{\eta}]}{\delta \bar{\eta}(x) \delta \eta(y)} \right]_{\eta = \bar{\eta} = 0} \leq 0|T\psi(x)\bar{\psi}(y)|0 \geq iS_F(x - y), \tag{48} \]

then \( S_F(x - y) = S(x - y) \), and

\[ Z_0^D[\eta, \bar{\eta}] = \exp \left[ -i \int dV_x dV_y \bar{\eta}(x) S_F(x - y) \eta(y) \right]. \tag{49} \]

Now we can move on to the electromagnetic field. The generating functional for such field can be represented as

\[ Z_0^E[J] = \frac{1}{N} \int DA \exp \left\{ i \int dV_x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\nu A^\nu \right] \right\}, \tag{50} \]

where \( J \) is a source for the field \( A(x) \), \( N \) is a number.

At first we shall transform the integral of the action:

\[ S = -\frac{1}{4} \int dV_x F_{\mu\nu} F^{\mu\nu} = \]

\[ = -\frac{1}{4} \int dV_x [\nabla_\mu A_\nu - \nabla_\nu A_\mu] [\nabla^\mu A^\nu - \nabla^\nu A^\mu]. \tag{51} \]

After integration by parts \(^8\):

\[ S = \frac{1}{2} \int dV_x [A_\nu (\nabla_\mu \nabla^\mu A^\nu - \nabla_\mu \nabla^\nu A^\mu) = \]

\[ = \frac{1}{2} \int dV_x A_\nu (\nabla_\mu \nabla^\mu A^\nu - \nabla^\mu \nabla_\mu A^\nu - R^\nu_{\mu\alpha} A^\alpha). \tag{52} \]

Using the Lorentz gauge, we state the next relation

\[ S = -\frac{1}{4} \int dV_x F_{\mu\nu} F^{\mu\nu} = \]

\[ = \frac{1}{2} \int dV_x A_\nu (\nabla_\mu \nabla^\mu A^\nu - R^\nu_{\mu\alpha} A^\alpha). \tag{53} \]

\(^8\)The covariant form of the Gauss theorem is described in \[14\]: the formula (4.7.8). Besides we use the relation (16.6) from \[7\].
After substitution $S$ in (50) the generation functional for the electromagnetic field gets the form

$$Z_0^E[J] = \frac{1}{N} \int DA \exp\left\{ i \int dV_x \left[ \frac{1}{2} A_\nu (\nabla_\mu A_\nu - R_\mu^\nu A_\mu) + J_\nu A_\nu \right] \right\}.$$  

(54)

Taking into account (24), we obtain

$$\frac{1}{2} A_\nu (\nabla_\mu A_\nu - R_\mu^\nu A_\mu) + J_\nu A_\nu = -\frac{1}{2} A^T \Re A + J^T A. \quad (55)$$

Let D be the operator determined as

$$D = -\Re^{-1}. \quad (56)$$

We can transform (54) using the functional extension for the relation (6.26) from [10] sec. 6.2

$$Z_0^E[J] = \exp\left\{ -\frac{i}{2} \int J^T(x) D(x-y) J(y) dV_x dV_y \right\}. \quad (57)$$

Functional differentiation yields

$$\tau(x, y) = -\left[ \frac{\delta^2 Z_0^E[J]}{\delta J(x) \delta J(y)} \right]_{J=0} =$$

$$= -\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \exp\left\{ -\frac{i}{2} \int dV_{x_1} dV_{x_2} J^T(x_1) D(x_1 - x_2) J(x_2) \right\}_{J=0} =$$

$$= -\frac{\delta}{\delta J(x)} \left[ -\frac{i}{2} \int dV_{x_1} D(y - x_2) J(x_2) - \frac{i}{2} \int dV_{x_1} J^T(x_1) D(x_1 - y) \right] \times$$

$$\times \exp\left\{ -\frac{i}{2} \int dV_{x_1} dV_{x_2} J^T(x_1) D(x_1 - x_2) J(x_2) \right\}_{J=0} = i D(x - y) \quad (58)$$

Since

$$-\left[ \frac{\delta^2 Z_0^E[J]}{\delta J(x) \delta J(y)} \right]_{J=0} = < 0 | TA(x) A(y) | 0 > = i D_F(x - y), \quad (59)$$

then $D_F(x - y) = D(x - y)$, and

$$Z_0^E[J] = \exp\left\{ -\frac{i}{2} \int J^T(x) D_F(x - y) J(y) dV_x dV_y \right\}. \quad (60)$$

\(^9\text{We use as well the normalization condition } Z_0^E[0] = 1.\)
4.2. Interacting fields

The generating functional for the interaction of photons with fermions takes the form

\[ Z[\eta, \bar{\eta}, J] = \frac{1}{N} \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi \mathcal{D}A \exp\{iS_0 + i \int dV_x [J_\mu A^\mu + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] + i \int \mathcal{L}_{int} dV_x \}, \quad (61) \]

where

\[ S_0 = \int \{ i\bar{\psi}\gamma^\mu(x)\tilde{\nabla}_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_\mu \nu F^{\mu \nu} \} dV_x, \quad (62) \]

\[ \mathcal{L}_{int} = -e\bar{\psi}\gamma_\mu \psi A^\mu. \quad (63) \]

The numerator for the generating functional of the free fields \( Z_0[\eta, \bar{\eta}, J] \) is determined by the expression

\[ Z_0[\eta, \bar{\eta}, J] = \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi \mathcal{D}A \exp\{iS_0 + i \int dV_x [J_\mu A^\mu + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \}. \quad (64) \]

Using the first three members in the expansion

\[ \exp(i \int \mathcal{L}_{int} dV_x) = 1 - ie \int \bar{\psi}\gamma_\mu \psi A^\mu dV_x + \frac{1}{2!}(ie \int \bar{\psi}\gamma_\mu \psi A^\mu dV_x)^2 + \ldots \quad (65) \]

and substituting in (61), we obtain \cite{10}

\[ Z[\eta, \bar{\eta}, J] = \frac{1}{N} \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi \mathcal{D}A [1 - (ie \int \bar{\psi}\gamma_\mu \psi A^\mu dV_x) + \frac{1}{2!}(ie \int \bar{\psi}\gamma_\mu \psi A^\mu dV_x)^2] \times \]

\[ \times \exp\{iS_0 + i \int dV_x [J_\mu A^\mu + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \} = \]

\cite{10}Thereafter we can omit indices for \( J, \tilde{\gamma}, \) and \( D_F. \)
\[ Z[\eta, \bar{\eta}, J] = \frac{1}{N} \left[ 1 - \left( ie \int \left( \frac{1}{i \delta \eta} \right) \tilde{\gamma}(z) \left( \frac{1}{i \delta \bar{\eta}} \right) \left( \frac{1}{i \delta J(z)} \right) dV_z \right) + \frac{1}{2!} \left( ie \int \left( \frac{1}{i \delta \eta} \right) \tilde{\gamma}(z) \left( \frac{1}{i \delta \bar{\eta}} \right) \left( \frac{1}{i \delta J(z)} \right) dV_z \right)^2 \right] Z_0[\eta, \bar{\eta}, J] (66) \]

Let \( z_1 \) and \( z_2 \) be the points of collision and decay. Then

\[ Z_0[\eta, \bar{\eta}, J] = \exp \left\{ -i \int dV_x dV_y [\bar{\eta}(x) S_F(x - y) \eta(y)] + \frac{1}{2} J(x) D_F(x - y) J(y) \right\} \tag{68} \]

After functional differentiation we have\(^{11}\)

\[ \left( \frac{1}{i \delta J(z)} \right) Z_0[\eta, \bar{\eta}, J] = - \int D_F(z - x) J(x) dV_x \exp \{ ... \} \]

\[ \left( \frac{1}{i \delta \eta(z)} \right) \left( \frac{1}{i \delta J(z)} \right) Z_0[\eta, \bar{\eta}, J] = \int D_F(z - x) J(x) dV_x \int S_F(z - y) \eta(y) dV_y \exp \{ ... \} \]

\[ \left( \frac{1}{i \delta \bar{\eta}(z)} \right) \tilde{\gamma}(z) \left( \frac{1}{i \delta J(z)} \right) Z_0[\eta, \bar{\eta}, J] = \tilde{\gamma}(z) \left[ \frac{1}{i} S_F(0) \right] \int D_F(z - x) J(x) dV_x - \int D_F(z - x) J(x) dV_x \int S_F(z - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z) dV_z \exp \{ ... \} \tag{69} \]

Substituting (69) in (67), we state for the members of the first order:

\(^{11}\)We shall write further \( \exp \{ ... \} \) instead of \( \exp \{ -i \int dV_x dV_y [\bar{\eta}(x) S_F(x - y) \eta(y)] + \frac{1}{2} J(x) D_F(x - y) J(y) \} \).
the first differentiation, we derive as well the members of the second order according to (67). Fulfilling do not yield the contribution to the scattering matrix. Therefore, we shall

\[ Z^{(1)}[\eta, \bar{\eta}, J] = ie \int dV_x \bar{\gamma}(z) \{-iS_F(0) \int D_F(z - x)J(x)dV_x - \]

\[ - \int D_F(z - x)J(x)dV_x \int S_F(z - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z)dV_x \} \exp\{\ldots\}\] (70)

We shall establish further in sec.5.1 that the members of the first order do not yield the contribution to the scattering matrix. Therefore, we shall consider as well the members of the second order according to (67). Fulfilling the first differentiation, we derive

\[ \left( \frac{1}{i \delta J(z_2)} \right) Z^{(1)}[\eta(z_1), \bar{\eta}(z_1), J(z_1)] = ei \int dV_z \bar{\gamma}(z_1) \times \]

\[ \times [-S_F(0)D_F(z_1 - z_2) + i\bar{\gamma}_\mu S_F(0) \int D_F(z_1 - x)J(x)dV_x \int D_F(z_2 - x)J(x)dV_x + \]

\[ + iD_F(z_1 - z_2) \int S_F(z_1 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_1)dV_x + \]

\[ + \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_1)dV_x \times \]

\[ \times \int D_F(z_2 - x)J(x)dV_x] \exp\{\ldots\}\] (71)

Using the second differentiation, we have

\[ \left( \frac{1}{i \delta \bar{\eta}(z_2)} \right) \left( \frac{1}{i \delta J(z_2)} \right) Z^{(1)}[\eta(z_1), \bar{\eta}(z_1), J(z_1)] = ei \int dV_z \bar{\gamma}(z_1) \times \]

\[ \times [-iS_F(0) \int D_F(z_1 - x)J(x)dV_x \int D_F(z_2 - x)J(x)dV_x \int S_F(z_2 - y)\eta(y)dV_y + \]

\[ + S_F(0)D_F(z_1 - z_2) \int S_F(z_2 - y)\eta(y)dV_y + \]

\[ + D_F(z_1 - z_2)S_F(z_2 - z_1) \int S_F(z_1 - y)\eta(y)dV_y - \]

\[ -iD_F(z_1 - z_2) \int S_F(z_1 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_1)dV_x \int S_F(z_2 - y)\eta(y)dV_y - \]

\[ -iS_F(z_2 - z_1) \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int D_F(z_2 - x)J(x)dV_x - \]

\[ -iS_F(z_2 - z_1) \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int D_F(z_2 - x)J(x)dV_x - \]

\[ -iS_F(z_2 - z_1) \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int D_F(z_2 - x)J(x)dV_x - \]
After the third differentiation we obtain

\[- \int D_F(z_1 - x) J(x) dV_x \int S_F(z_1 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_1) dV_x \times \]

\[\times \int D_F(z_2 - x) J(x) dV_x \int S_F(z_2 - y) \eta(y) dV_y \exp\{\ldots\}\]

\[(72)\]

After the third differentiation we obtain

\[
\left\{ \frac{1}{i} \frac{\delta}{\delta \eta(z_2)} \right\} \tilde{\eta}(z_2) \left\{ \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(z_2)} \right\} \left\{ \frac{1}{i} \frac{\delta}{\delta J(z_2)} \right\} Z^{(1)}[\eta(z_1), \bar{\eta}(z_1), J(z_1)] =
\]

\[= e i \int dV_z \tilde{\eta}(z_2) \tilde{\eta}(z_1) \left[ -i S_F(0) D_F(z_1 - z_2) S_F(0) - \right.
\]

\[- S_F(0) D_F(z_1 - z_2) \int S_F(z_2 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_2) dV_x +
\]

\[\left. - S_F(0) S_F(0) \int D_F(z_1 - x) J(x) dV_x \int D_F(z_2 - x) J(x) dV_x + \right.
\]

\[+ i S_F(0) \int D_F(z_1 - x) J(x) dV_x \int D_F(z_2 - x) J(x) dV_x \int S_F(z_2 - y) \eta(y) dV_y \times \]

\[\times \int \bar{\eta}(x) S_F(x - z_2) dV_x -
\]

\[- i D_F(z_1 - z_2) S_F(z_2 - z_1) S_F(z_1 - z_2) - \]

\[- D_F(z_1 - z_2) S_F(z_2 - z_1) \int S_F(z_1 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_2) dV_x - \]

\[- D_F(z_1 - z_2) S_F(z_1 - z_2) \int \bar{\eta}(x) S_F(x - z_1) dV_x \int S_F(z_2 - y) \eta(y) dV_y - \]

\[- D_F(z_1 - z_2) S_F(0) \int S_F(z_1 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_1) dV_x + \]

\[+ i D_F(z_1 - z_2) \int S_F(z_1 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_1) dV_x \times \]

\[\times \int S_F(z_2 - y) \eta(y) dV_y \int \bar{\eta}(x) S_F(x - z_2) dV_x -
\]

\[- S_F(z_2 - z_1) S_F(z_1 - z_2) \int D_F(z_1 - x) J(x) dV_x \int D_F(z_2 - x) J(x) dV_x + \]

\[+ i S_F(z_2 - z_1) \int D_F(z_1 - x) J(x) dV_x \int S_F(z_1 - y) \eta(y) dV_y \times \]

\[\times \int D_F(z_2 - x) J(x) dV_x \int \bar{\eta}(x) S_F(x - z_2) dV_x + \]
\[ iS_F(z_1 - z_2) \int D_F(z_1 - x)J(x)dV_x \int \bar{\eta}(x)S_F(x - z_1)dV_x \times \]

\[ \times \int D_F(z_2 - x)J(x)dV_x \int S_F(z_2 - y)\eta(y)dV_y + \]

\[ + iS_F(0) \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_1)dV_x \times \]

\[ \times \int D_F(z_2 - x)J(x)dV_x + \]

\[ + \int D_F(z_1 - x)J(x)dV_x \int S_F(z_1 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_1)dV_x \times \]

\[ \times \int D_F(z_2 - x)J(x)dV_x \int S_F(z_2 - y)\eta(y)dV_y \int \bar{\eta}(x)S_F(x - z_2)dV_x \exp\{...\} \]

(73)

Since according to (73) \( N \neq 1 \), we present \( N \) as \( N = 1 - e^2B \), where \( B \) consists of the corresponding members of the numerator in the expression (67) subtracted from \( Z_0 \). These members have not integrals containing functions \( J, \eta, \bar{\eta} \) before \( \exp\{...\} \) and therefore are not equal to 0 for \( J = \eta = \bar{\eta} = 0 \).

Let \( A_0 \) and \( A_1 \) be the members of the numerator in (67) containing integrals before \( \exp\{...\} \). These members vanish for \( J = \eta = \bar{\eta} = 0 \). Then in the second approximation

\[ Z[\eta, \bar{\eta}, J] = \frac{[1 - ieA_0 - e^2(A_1 + B)]Z_0[\eta, \bar{\eta}, J]}{1 - e^2B} \approx [1 - ieA_0 - e^2(A_1 + B)](1 + e^2B)Z_0[\eta, \bar{\eta}, J] \approx [1 - ieA_0 - e^2A_1)]Z_0[\eta, \bar{\eta}, J] \]

(74)

Consequently, the term \( B \) vanishes from \( Z[\eta, \bar{\eta}, J] \).

Using (67), (69), and (73) in common with (74), we can in the next section to build the corresponding Green functions for elementary processes.

5. Elementary processes of quantum electrodynamics

5.1. The Green function for the interaction of fermions with photons

To find the matrix scattering by means of expression (40), we need the Green function \( \tau(x_1, x_2, x_3, x_4) \). Consider the function

\[ \tau(x_1, x_2, x_3, x_4) = \left[ \frac{\delta}{\delta \bar{\eta}(x_4)} \frac{\delta}{\delta J(x_3)} \frac{\delta}{\delta \eta(x_2)} \frac{\delta}{\delta J(x_1)} Z[\eta, \bar{\eta}, J] \right]_{\eta=\bar{\eta}=J=0} \]
that corresponds to Compton scattering:

\[ e^{-\gamma} \rightarrow e^{-\gamma} \]  

(76)

where \( J(x_1) \) and \( J(x_3) \) are sources for the input and output photons respectively, and \( \eta(x_2) \) and \( \bar{\eta}(x_4) \) are sources for the input and output electrons.

Consider operations of differentiation according to (75). The result depends on the presence of integrals containing functions \( J, \eta, \bar{\eta} \) before \( \exp\{...\} \) in the members of sums (70) and (73) (after differentiation)\(^{12}\):

1. If integrals with \( J \) are absent, then after two differentiations of \( \exp\{...\} \) wrt \( J \) the propagator of the free electromagnetic field appears (see sec.4.1)\(^{13}\).

2. If there is one integral with \( J \), then after the first differentiation wrt \( J \) it disappears, and after the second differentiation of \( \exp\{...\} \) wrt \( J \) the integral with \( J \) appears.

3. If there are two integrals with \( J \), then these integrals disappear after two differentiations wrt \( J \).

4. If there are more two integrals with \( J \), then all integrals can not disappear.

5. If integrals containing functions \( \eta, \bar{\eta} \) are absent, then after the first and second differentiation of \( \exp\{...\} \) wrt \( \eta, \bar{\eta} \) the propagator of the free Dirac field appears (see sec.4.1).

6. If there is one pair of integrals containing functions \( \eta, \bar{\eta} \), then these integrals disappear after the first and second differentiation wrt \( \eta, \bar{\eta} \).

7. If there are two pairs of integrals, then after the first and second differentiation wrt \( \eta, \bar{\eta} \) only one pair disappears.

According to the items 1-7 (taking account of (67), (69), and (73) in common with (74)), we can do such conclusions:

- the member \( Z_0[\eta, \bar{\eta}, J] \) leads to the propagators of free fields;
- the members from (70) do not be included to the Green function (75);

\(^{12}\) Integrals containing functions \( \eta, \bar{\eta} \) may be only paired: the integral with \( \eta \) and the integral with \( \bar{\eta} \).

\(^{13}\) Hereafter we shall use the abbreviation ”wrt” instead of ”with respect to”.

17
• the members from (73) do not be taken into account if they contain $S_F(0)$ and therefore form loop diagrams;

• the members including $D_F(z_1 - z_2)S_F(z_1 - z_2)$ do not be taken into account since they describe in common a virtual photon and fermion;

• the member

$$e^2 \frac{i}{2} \int dV_z dV_{z_2} \tilde{\gamma}(z_1) \tilde{\gamma}(z_2) S_F(z_2 - z_1) \int D_F(z_1 - x) J(x) dV_x \times$$

$$\times \int S_F(z_1 - x) \eta(x) dV_x \int D_F(z_2 - x) J(x) dV_x \int S_F(x - z_2) \bar{\eta}(x) dV_x \exp\{\ldots\}$$ (77)

yields two contributions to the Green function (75):

$$\tau(x_1, x_2, x_3, x_4) = i e^2 \int dV_{z_1} dV_{z_2} \tilde{\gamma}(z_1) \tilde{\gamma}(z_2) S_F(z_2 - z_1) \times$$

$$\times [D_F(z_1 - x_1) D_F(z_2 - x_3) + D_F(z_1 - x_3) D_F(z_2 - x_1)] S_F(z_1 - x_2) S_F(x_4 - z_2)$$ (78)

The factor $\frac{1}{2}$ vanishes since there is another member in (73) that is equal to (77) after the substitution $z_1 \leftrightarrow z_2$.

Thus, we receive the result corresponding to classical quantum electrodynamics. However, instead of usual Green functions of QED it should be used Green functions for curved spacetime.

Now we can move on to calculating the scattering matrix for elementary quantum electrodynamic processes.

5.2. The interaction of fermions with photons

To find the scattering matrix for the reaction (76), we substitute the relation (78) in the expression (40) taking into account that $x_1 = x, x_2 = z, x_3 = x', x_4 = z'$. At first consider the first member in (78):

$$- \int dV_{z_1} dV_{x'} dV_{z'} A_{k\lambda}^\mu(x) \overrightarrow{R}_{x'} \overleftarrow{U}_{p's}(z') (i \tilde{\gamma}(z') \cdot \nabla_{z'} - m) \times$$

$$\times i e^2 \int dV_{z_1} dV_{z_2} \tilde{\gamma}_\mu(z_1) \tilde{\gamma}_\nu(z_2) S_F(z_2 - z_1) D_F(z_2 - x') S_F(z_1 - z) S_F(z' - z_2) \times$$

$$\times (-i \tilde{\gamma}(z) \cdot \nabla_{z} - m) U_{ps}(z) \overleftarrow{R}_{x'} A_{k'\lambda'}^{\mu'}(x')$$ (79)
We have the relations:

\[ D_F(z_2 - x') \overset{\sim}{=} \delta(z_2 - x'), \tag{80} \]

\[ \Re_x D_F(z_1 - x) = \delta(z_1 - x), \tag{81} \]

\[ (i\tilde{\gamma}(z') \cdot \tilde{\nabla}_{x'} - m) S_F(z' - z_2) = \delta(z' - z_2), \tag{82} \]

\[ S_F(z_1 - z) (-i\tilde{\gamma}(z) \cdot \nabla_z - m) = \delta(z_1 - z). \tag{83} \]

Taking into account (80) - (83), the first member has the form

\[ -ie^2 \int dV_{z_1} dV_{z_2} \bar{U}_{p's'}(z_2) \tilde{\gamma}_\nu(z_2) A_{k'\lambda'}(z_2) S_F(z_2 - z_1) \tilde{\gamma}_\mu(z_1) A_{k\lambda}(z_1) U_{p\nu}(z_1). \tag{84} \]

For the second member in (78) \( z_2 = x, z_1 = x'. \) Therefore, this member is equal to

\[ -ie^2 \int dV_{z_1} dV_{z_2} \bar{U}_{p's'}(z_2) \tilde{\gamma}_\nu(z_2) A_{k\lambda}(z_2) S_F(z_2 - z_1) \tilde{\gamma}_\mu(z_1) A_{k'\lambda'}(z_1) U_{p\nu}(z_1). \tag{85} \]

Thus,

\[ S_{fi} = -ie^2 \int dV_{z_1} dV_{z_2} [\bar{U}_{p's'}(z_2) \tilde{\gamma}_\nu(z_2) A_{k'\lambda'}(z_2) S_F(z_2 - z_1) \tilde{\gamma}_\mu(z_1) A_{k\lambda}(z_1) U_{p\nu}(z_1) + \]

\[ + \bar{U}_{p's'}(z_2) \tilde{\gamma}_\nu(z_2) A_{k\lambda}(z_2) S_F(z_2 - z_1) \tilde{\gamma}_\mu(z_1) A_{k'\lambda'}(z_1) U_{p\nu}(z_1)]. \tag{86} \]

For Minkowski space we can present \( U_{p\nu}, \bar{U}_{p's'}, A_{k\lambda}, A_{k'\lambda'}^\nu \) in terms of plane waves. Substituting in (86)

\[ S_F(z_2 - z_1) = \frac{1}{(2\pi)^4} \int dq \exp(-iq(z_2 - z_1)) \tilde{S}_F(q), \tag{87} \]

where \( \tilde{S}_F(q) \) is the Fourier transform for \( S_F(z) \), we shall find that all signs of integrals in (86) vanish, and we shall obtain the usual expression for Compton scattering (see [9] sec.5.5).
5.3. The annihilation of electron-positron pair

The annihilation of electron-positron pair corresponds to the reaction

\[ e^- e^+ \rightarrow \mu^- \mu^+ . \]  

(88)

Using reduction formalism, we shall establish that the matrix scattering gets the form

\[ < p' s'; q' r' \text{out} | p s; q r \text{in} > = \delta_{ij} - \int dV_x \int dV_{x'} \int dV_z \int dV_{z'} \times \]
\[ \times U_{p' s'}(x')(i \gamma(x') \cdot \nabla_{x'} - m_\mu) V_{q r}(z)(i \gamma(z) \cdot \nabla_z - m_e) \times \]
\[ \times < 0 | T(\bar{\psi}_\mu^-(z') \psi_\mu^-(x') \psi_{e+}(z) \bar{\psi}_{e-}(x)) | 0 > \times \]
\[ \times (-i \gamma(z') \cdot \nabla_{z'} - m_\mu) V_{q' r'}(z') (-i \gamma(x) \cdot \nabla_x - m_e) U_{p s}(x). \]  

(89)

To find the matrix scattering by means of the expression (89), we need the Green function \( \tau(x_1, x_2, x_3, x_4) \). Consider the function

\[ \tau(x_1, x_2, x_3, x_4) = \left[ \frac{\delta}{\delta \bar{\eta}(x_4)} \frac{\delta}{\delta \eta(x_3)} \frac{\delta}{\delta \bar{\eta}(x_2)} \frac{\delta}{\delta \eta(x_1)} Z[\eta, \bar{\eta}, J] \right]_{\eta \bar{\eta} J = 0}, \]  

(90)

where \( \bar{\eta}(x_1) \) and \( \eta(x_2) \) are sources for an electron and positron respectively, and \( \eta(x_3) \) and \( \bar{\eta}(x_4) \) are sources for muons.

The result in (90) depends on the presence of integrals containing functions \( J, \eta, \bar{\eta} \) before \( exp\{\ldots\} \) in the members of sums (70) and (73) (after differentiation) \[14\].

1. If integrals containing \( \eta, \bar{\eta} \) are absent, then after the first and second differentiation of \( exp\{\ldots\} \) wrt \( \eta, \bar{\eta} \) the propagator of the free Dirac field appears (see sec.4.1). After the third and fourth differentiation of \( exp\{\ldots\} \) another propagator of the free Dirac field appears.

2. If there is one pair of integrals containing \( \eta, \bar{\eta} \), then these integrals disappear after the first and second differentiation wrt \( \eta, \bar{\eta} \). After the third and fourth differentiation of \( exp\{\ldots\} \) the propagator of the free Dirac field appears.

\[14\] We recall that integrals containing \( \eta, \bar{\eta} \) may be only paired.
3. If there are two pairs of integrals, then all integrals disappear after all differentiations wrt $\eta, \bar{\eta}$.

According to the items 1-3 (using (67), (69), and (73) in common with (74)) we can do such conclusions: only the member
\[
e^{2i} \int dV_1 dV_2 \bar{\gamma}(z_2) \gamma(z_1) D_F(z_1 - z_2) \int S_F(z_1 - y) \eta(y) dV_2 \int \bar{\eta}(x) S_F(x - z_1) dV_x \times
\]
\[
\times \int S_F(z_2 - y) \eta(y) dV_2 \int \bar{\eta}(x) S_F(x - z_2) dV_x(91)
\]
gives the contribution to the Green function:
\[
\tau(x_1, x_2, x_3, x_4) = \frac{ie^2}{2} \int dV_1 dV_2 \bar{\gamma}(z_2) \gamma(z_1) D_F(z_2 - z_1) [S_F(z_1 - x_2) S_F(x_1 - z_1) \times
\]
\[
\times S_F(z_2 - x_3) S_F(x_4 - z_2) + S_F(z_2 - x_2) S_F(x_1 - z_2) S_F(z_1 - x_3) S_F(x_4 - z_1)](92)
\]

Bath members in (92) are equal after the substitution $z_1 \leftrightarrow z_2$. Therefore,
\[
\tau(x_1, x_2, x_3, x_4) = ie^2 \int dV_1 dV_2 \bar{\gamma}(z_2) \gamma(z_1) D_F(z_2 - z_1) S_F(z_1 - x_2) S_F(x_1 - z_1) \times
\]
\[
\times S_F(z_2 - x_3) S_F(x_4 - z_2)(93)
\]

Thus, we state the result corresponding to classical quantum electrodynamics. However, instead of usual Green functions of QED it should be applied Green functions for curved spacetime.

To find the scattering matrix, we substitute the relation (93) in the expression (89) taking into account that $x_1 = x, x_2 = z, x_3 = x', x_4 = z'$:
\[
< p's'; q'r' out|ps; qr, in > = \delta_{ij} - ie^2 \int dV_1 dV_2 \bar{\gamma}(z_2) \gamma(z_1) \int dV_x \int dV_y \int dV_z \times
\]
\[
	imes \bar{U}_{p's'}(x') (i \bar{\gamma}(x') \cdot \nabla_{x'} - m_\mu) \bar{V}_{q'r}(z) (i \gamma(z) \cdot \nabla_z - m_e) \times
\]
\[
	imes D_F(z_2 - z_1) S_F(z_1 - z) S_F(x - z_1) S_F(z_2 - x') S_F(z' - z_2) \times
\]
\[
\times \left[ (-i \bar{\gamma}(z') \cdot \nabla_{z'} - m_\mu) V_{q'r'}(z') \right] (-i \gamma(x) \cdot \nabla_x - m_e) U_{ps}(x)(94)
\]

We have the relations:
\( (i\tilde{\gamma}(z) \cdot \hat{\nabla}_z - m_e)S_F(z_1 - z) = \delta(z_1 - z), \) \hspace{1cm} (95) \\
\( (i\tilde{\gamma}(x') \cdot \hat{\nabla}_{x'} - m_\mu)S_F(z_2 - x') = \delta(z_2 - x'), \) \hspace{1cm} (96) \\
\( S_F(z' - z_2) (-i\tilde{\gamma}(z') \cdot \hat{\nabla}_{z'} - m_\mu) = \delta(z' - z_2), \) \hspace{1cm} (97) \\
\( S_F(x - z_1) (-i\tilde{\gamma}(x) \cdot \hat{\nabla}_x - m_e) = \delta(x - z_1). \) \hspace{1cm} (98) \\

Taking into account (95) - (98), we find that \\

\[
S_{fi} = -ie^2 \int dV_{z_1} dV_{z_2} V_{qr}(z_1) \tilde{\gamma}_\mu(z_1) U_{ps}(z_1) D_{F}^{\mu\nu}(z_2 - z_1) U_{p's'}(z_2) \tilde{\gamma}_\nu(z_2) V_{q'r'}(z_2). \tag{99}
\]

For Minkowski space we can present \( U_{ps}, \tilde{U}_{p's'}, V_{qr}, V_{q'r'} \) in terms of plane waves. Substituting in (99)

\[
D_{F}^{\mu\nu}(z_1 - z_2) = \frac{1}{(2\pi)^4} \int dq \exp(-iq(z_1 - z_2)) \tilde{D}_{F}^{\mu\nu}(q), \tag{100}
\]

where \( \tilde{D}_{F}^{\mu\nu}(q) \) is the Fourier transform for \( D_{F}^{\mu\nu}(z) \), we shall find that all signs of integrals in (99) vanish, and we shall obtain the usual expression for the annihilation of electron-positron pair (see [9] sec.5.1).

5.4. The generalized Feynman rules

On the base of results of sec. 5.2 and 5.3 we can formulate the generalized Feynman rules for quantum electrodynamics in curved spacetime \(^{15}\):

1. For each internal fermion line the factor \( iS_F(x - y) \).
2. For each internal photon line the factor \( iD_F(x - y) \).
3. For each vertex the factor

\[
-ie \int dV_z \tilde{\gamma}(z). \tag{101}
\]

\(^{15}\)We recall that, in the general case, the solutions \( U_{ps}(x), V_{ps}(x) \) for out-states will not be equal to the corresponding solutions for in-states.
4. For each external photon line with momentum $k$ and polarization $\lambda$ the factor $A_{k\lambda}(x)$ in the case of in-line and the factor $A^*_{k\lambda}(x)$ for out-line.

5. For each external fermion line with momentum $p$ and spin $s$ the factor $U_{ps}(x)$ in the case of in-line and the factor $\bar{U}_{ps}(x)$ for out-line.

6. For each external antifermion line with momentum $p$ and spin $s$ the factor $\bar{V}_{ps}(x)$ in the case of in-line and the factor $V_{ps}(x)$ for out-line.

For Minkowski space these rules are the Feynman rules in [9], Appendix A1.

6. Conclusion

1. We have found the generating functional for the interaction of Dirac and electromagnetic fields.
2. Using reduction formalism, we have calculated the scattering matrices for elementary processes of quantum electrodynamics in curved spacetime: Compton scattering and the annihilation of electron-positron pair (for the tree-level approximation).
3. On the base of these results we have formulated the generalized Feynman rules for the electromagnetic interaction in curved spacetime.
4. Another electrodynamic processes can be studied by means of these Feynman rules and crossing-symmetry.

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