Random projections of linear and semidefinite problems with linear inequalities

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Abstract

The Johnson-Lindenstrauss Lemma states that there exist linear maps that project a set of points of a vector space into a space of much lower dimension such that the Euclidean distance between these points is approximately preserved. This lemma has been previously used to prove that we can randomly aggregate, using a random matrix whose entries are drawn from a zero-mean sub-Gaussian distribution, the equality constraints of a Linear Program (LP) while preserving approximately the value of the problem. In this paper we extend these results to the inequality case by introducing a random matrix with non-negative entries that allows to randomly aggregate inequality constraints of an LP while preserving approximately the value of the problem. By duality, the approach we propose allows to reduce both the number of constraints and the dimension of the problem while obtaining some theoretical guarantees on the optimal value. We will also show an extension of our results to certain semidefinite programming instances.

Keywords: random projection, linear programming, semi-definite programming

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1. Introduction

Random matrices are matrices $T \in \mathbb{R}^{k \times m}$ whose entries are drawn from a probability distribution. When the underlying distribution is properly chosen, these matrices can have some very interesting properties: the Johnson-Lindenstrauss Lemma (JLL), $[1, 2]$, states that, if the entries of $T$ are drawn independently from the standard normal distribution $\mathcal{N}(0, \frac{1}{2})$, it is possible to project a set of $n$ points of $\mathbb{R}^m$ into a space of dimension $k = O\left(\frac{\log(n)}{\epsilon^2}\right)$ while preserving approximately (with $\epsilon$ precision) the Euclidean distance between these points with arbitrarily high probability (w.a.h.p.).

Recently, this result has been exploited in $[3]$ to prove that equality constraints of an LP written in standard form with inputs $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, could be randomly aggregated using a random matrix $T$ with $k < m$, into a new LP:

$$
\begin{align*}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{align*}
$$

while preserving, approximately, w.a.h.p. the optimal value of the problem. Although this result makes it possible to aggregate equality constraints of the form “$Ax = b$”, aggregating an inequality constraint of the form “$Ax \leq b$" is a far subtler issue.

For example, one might be tempted to take a look at the duals of the problems above, which leads to problems of the format \[
\max \left\{ b^\top y \mid c - A^\top y \geq 0 \right\} \quad \text{and} \quad \max \{ b^\top (T^\top y_T) \mid c - A^\top T^\top y_T \geq 0 \}.\]

Performing the substitution $z = T^\top y_T$ leads to a dual problem with less variables, but the number of inequality constraints remains the same. So this approach cannot be used to aggregate inequality constraints. Alternatively, one may consider adding slack variables $s \in \mathbb{R}_+^m$ to transform inequality constraints into equality constraints. However, this is unlikely to be efficient in the random projection framework because, although the number of constraints is reduced, the number of variables increases by $m$ in the projected problem. Hence using slack variables is a non-starter and we can not expect to reduce the solving time of the problem.

Indeed, to randomly aggregate a set of inequality constraints “$Ax \leq b$”, we need a random matrix $S$ whose entries $S_{ij}$ are non-negative. In this paper, we propose the first method that allows to randomly aggregate a set of inequality constraints in an LP. More precisely, let us consider the pair:

$\mathcal{P}\left\{ \begin{array}{c}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax \geq b \\
& x \in \mathbb{R}^n
\end{array} \right\}$ \hspace{1cm} $\mathcal{P}_S\left\{ \begin{array}{c}
\min \quad & c^\top x \\
\text{s.t.} \quad & SAx \geq Sb \\
& x \in \mathbb{R}^n
\end{array} \right\}$

with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Here, $S \in \mathbb{R}^{k \times m}$ is a random iid matrix such that $S_{ij} = \frac{1}{\sqrt{k}}T_{ij}^2$ where $T_{ij}$ is drawn from the standard normal distribution $\mathcal{N}(0, 1)$. Although this looks very similar to the equality case it is actually
quite different: the random matrix $S$ does not satisfy the JLL property, and hence, a different analysis should be applied. Notice that since each entry of $S$ is non-negative, $\mathcal{P}_S$ is a relaxation of $\mathcal{P}$; $v(\mathcal{P}_S) \leq v(\mathcal{P})$ holds (where $v(\cdot)$ denotes the optimal value of an optimization problem). The difficult part is to prove that there exists $\delta(k) > 0$ such that, w.a.h.p.,

$$v(\mathcal{P}) - \delta(k) \leq v(\mathcal{P}_S) \leq v(\mathcal{P}),$$

where $\delta(k)$ is a decreasing function of $\varepsilon$ (recall that typically $k = O\left(\frac{\log(m)}{\varepsilon^2}\right)$), which represents the distortion in distance after projection.

More generally, we will consider a pair of linear optimization problems over a cone $K$ which is a product of the non-negative orthant and semidefinite cones, i.e., $K = \mathbb{R}_+^m \times S_+^{p_1} \times \cdots \times S_+^{p_l}$:

$$\begin{align*}
\mathcal{P}_K \{ & \min_x c^T x \\
& Ax - b \in \mathcal{K} \\
& x \in \mathbb{R}^n \} \\
\mathcal{P}_Q \{ & \min_x c^T x \\
& Q(Ax - b) \in \mathcal{Q}(\mathcal{K}) \\
& x \in \mathbb{R}^n \}
\end{align*}$$

where $Q$ is a linear map such that $\mathcal{K} = \mathbb{R}_+^m \times S_+^{p_1} \times \cdots \times S_+^{p_l}$ is mapped to $\mathbb{R}_+^k \times S_+^{q_1} \times \cdots \times S_+^{q_l}$, where $k < m$ and $q_i < p_i$ for all $i$. We will prove that one can build a random map $Q$ such that, with arbitrarily high probability, $v(\mathcal{P}_Q)$ approximates $v(\mathcal{P}_K)$.

We now review some related works. Notice that in all the works using a random projection matrix $T$ to reduce the dimension of a problem, $T$ preserves approximately the distances, i.e., $\|Tx\|_2 \approx \|x\|_2$ with high probability, and its entries $T_{ij}$ have all zero expectation. In numerical linear algebra, random projections are used to compress a matrix into a smaller one where computations can be performed quickly thereby accelerating the solution of the original problem, e.g., [4, 5, 6]. In optimization, random projections are also referred to as sketching, especially if random matrices are used not to reduce the dimension of the problem but to reduce the sample size of some data matrix of the problem. For example in a least-square problem setting, i.e., $\min_{x \in C} \|y - Xx\|_2^2$ where $C \subseteq \mathbb{R}^n$ is a convex set, $X \in \mathbb{R}^{m' \times n}$ is the design matrix and $y$ is the response vector, the sample size $m'$ is reduced using random projections. In [7, 8, 9, 10], random projections are used to reduce the size of some large dimensional least squares problem: the data $(X, Y)$ of the problem is replaced by a lower dimensional sketched-data $(SX, SY)$ where $S$ is a random matrix. In [11] and [12], iterative methods using random projections are proposed. In [13], an application of random projections to the pure semidefinite programming (SDP) case is considered. The random projection map, $M \mapsto TMT^T$, considered in [13], preserves the structure of the semidefinite cone, and hence, in opposition to the LP case, the inequality constraints are not a problem. Later in this paper, we will further discuss the relation between the approach in [13] and ours.

To the best of our knowledge, this is the first work to randomly aggregate linear inequalities. In addition, one of the main contribution of our paper is
| Notation | Convention |
|----------|------------|
| $\|X\|_{\psi_2}$ | the sub-Gaussian norm of a random vector |
| $\|X\|_{\psi_1}$ | the sub-exponential norm of a random vector |
| $C_1$ | absolute constant |
| $S_p$ | the $p \times p$ real symmetric matrices |
| $S_p^+$ | the cone of $p \times p$ real positive semidefinite matrices |
| $x^\top y$ | the Euclidean scalar product between $x$ and $y$ |
| $\langle M_1, M_2 \rangle_F$ | the Frobenius scalar product of matrices $M_1$ and $M_2$ |
| $\|M\|_F, \|M\|_*$ | respectively the Frobenius norm and the nuclear norm of matrix $M$ |
| $\|M\|_i$ | the induced $\| \cdot \|_i$ for matrix $M$ for $i = 1, 2$: $\|M\|_i = \max_{\|x\|_2 = 1} \|Mx\|_i$ |
| $|x|, x \in \mathbb{R}^n$ | vector whose components are the absolute value of those of $x$ |
| $v(P)$ | optimal value of the optimization problem $(P)$ |
| $A \circ B$ | Hadamard product of matrices $A$ and $B$ |
| $D(A)$ | matrix whose diagonal is the vector $a$ |
| $D^{-1}(a)$ | diagonal of matrix $A$ |
| $1$ | the all one vector |

Table 1: Notational conventions

to use a non-negative random matrix for sketching. Indeed, in all papers we have seen using random random projections to reduce the size of a problem, the entries of the random matrices used for the aggregation have zero expectation. We remark that zero expectation matrices cannot be used to deal with inequalities constraints because such matrix would typically have negative entries (approximately half of the entries would be negative) and it would imply that the positive orthant is not usually mapped into another positive orthant.

This work is divided as follows. In Section 2 we will recall some basic facts about concentration inequalities. Then, in Section 3 we will recall some known results and derive some new ones for random matrices. In Section 4 we will give the main result of this paper and in Section 5, we will restrict the LP case and obtain a simpler bound. Finally in Section 6 we will present some numerical results.

We resume all the notations used in the paper in Table 1.

2. Concentration inequalities

In this section we recall some basic facts about concentration inequalities.

**Definition 1** (Sub-Gaussian random variables). Let $X$ be a zero mean random variable such that there exists $K > 0$ such that for all $t > 0$,

$$P(|X| > t) \leq 2 \exp \left( -\frac{t^2}{K^2} \right).$$

(5)
Then $X$ is said to be sub-Gaussian. The sub-Gaussian norm of $X$ is defined to be the smallest $K$ satisfying (5) and is denoted by $\|X\|_{\psi_2}$.

**Remark 2.** A classical example ([14, Examples 2.5.8]) of sub-Gaussian random variable is a Gaussian random variable $X \sim N(0, \sigma^2)$ with $\|X\|_{\psi_2} \leq 2\sigma$.

**Definition 3** (Sub-exponential random variables). Let $X$ be a zero mean random variable and let $K > 0$ such that for all $t > 0$,

$$P(|X| > t) \leq 2 \exp\left(-\frac{t}{K}\right). \quad (6)$$

Then $X$ is said to be sub-exponential. The sub-exponential norm of $X$ is defined to be the smallest $K$ satisfying (6) and is denoted by $\|X\|_{\psi_1}$.

Sub-Gaussian and sub-exponential random variables are closely related, as we can see that any sub-Gaussian random variable is also sub-exponential (only large values of $t$ are relevant). Furthermore, it turns out that the product of two sub-Gaussian random variables is sub-exponential.

**Lemma 4** ([14, Lemma 2.7.7]). Let $X$ and $Y$ be sub-Gaussian random variables, then $XY$ is sub-exponential, furthermore

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_2} \|Y\|_{\psi_2}.$$

Also if $Z$ is sub-exponential, then $\|Z - E(Z)\|_{\psi_1} \leq 2 \|Z\|_{\psi_1}$.

Next we recall the Bernstein inequality whose proof can be found in [14, Theorem 2.8.2].

**Proposition 5** (Bernstein inequality). Let $Y_1, \ldots, Y_N$ be independent, mean zero, sub-exponential random variables. Then, for every $t \geq 0$, we have

$$P\left(|\sum_{i=1}^N Y_i| \geq t \right) \leq 2 \exp\left(-C_1 \min\left(\frac{t^2}{\sum \|Y_i\|_{\psi_1}^2}, \frac{t}{\max \|Y_i\|_{\psi_1}}\right)\right), \quad (7)$$

where $C_1 > 0$ is an absolute constant.

We now recall the notion of $\hat{\varepsilon}$-net:

**Definition 6.** Given a subset $K \subset \mathbb{R}^m$ and $\hat{\varepsilon} > 0$, we say that a subset $\mathcal{N}$ is an $\hat{\varepsilon}$-net of $K$ if every point of $K$ is within $\hat{\varepsilon}$ of a point in $\mathcal{N}$, i.e.,

$$\forall x \in K, \exists y \in \mathcal{N} \text{ s.t. } \|x - y\|_2 \leq \hat{\varepsilon}.$$ 

**Remark 7.** We can find a $\hat{\varepsilon}$-net of the $m$-Euclidean ball of size $(\frac{2}{\hat{\varepsilon}} + 1)^m$ (c.f. [14, Corollary 4.2.13]).

In practice $\hat{\varepsilon}$-nets can be used to bound the operator norm, $\|M\|_2$, of a matrix $M$: 

Lemma 8 ([14, Lemma 4.4.1, Exercise 4.4.3]). Let $M \in \mathbb{R}^{p \times q}$, then for any $\hat{\varepsilon}$-net $N$ ($\hat{\varepsilon} < 1$) of the unit sphere $S^{q-1}$ we have

$$
\sup_{x \in N} \|Mx\|_2 \leq \|M\|_2 \leq \frac{1}{1 - \hat{\varepsilon}} \sup_{x \in N} \|Mx\|_2.
$$

Furthermore, for any $\hat{\varepsilon}$-net $N'$ of $S^{p-1}$ (with $\hat{\varepsilon} < 1/2$), we have

$$
\sup_{x \in N', y \in N'} \langle Mx, y \rangle \leq \|M\|_2 \leq \frac{1}{1 - 2\hat{\varepsilon}} \sup_{x \in N', y \in N'} \langle Mx, y \rangle.
$$

Moreover if $p = q$ and $M$ is symmetric, we have

$$
\sup_{x \in N} |\langle Mx, x \rangle| \leq \|M\|_2 \leq \frac{1}{1 - 2\hat{\varepsilon}} \sup_{x \in N} |\langle Mx, x \rangle|.
$$

3. Properties of random projection matrices

In this section we recall the famous Johnson-Lindenstrauss lemma, which is generalized to sub-Gaussian distribution, and derive some new concentration properties for random matrices.

Lemma 9 (Johnson-Lindenstrauss Lemma (JLL) [15]). Let $Z$ be a set of $h$ points in $\mathbb{R}^l$ and let $G$ be a $k \times l$ random matrix whose entries are independent $N(0, 1)$ random variables, let $0 < \varepsilon < 1$. Then with probability $1 - 2h \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$, we have that for all $z_i, z_j \in Z$

$$
(1 - \varepsilon)\|z_i - z_j\|_2^2 \leq \frac{1}{\sqrt{k}}\|Gz_i - Gz_j\|_2^2 \leq (1 + \varepsilon)\|z_i - z_j\|_2^2,
$$

(8)

In the Johnson-Lindenstrauss Lemma, the term $1 - 2h \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$ means that we can choose $k = O\left(\frac{\log(h)}{\varepsilon^2/2 - \varepsilon^3/3}\right)$ so that Equation (8) is satisfied with probability as small as we want.

The following Lemma, proved in [16], enumerates some consequences of the JLL.

Lemma 10 (c.f. [16, Lemmas 3.1, 3.2, 3.3]). For $G$ defined as in Lemma 9, let $T = \frac{1}{\sqrt{k}}G$. Let $0 < \varepsilon < 1$, then we have

(i) For any $x, y \in \mathbb{R}^l$,

$$
x^\top y - \varepsilon\|x\|\|y\| \leq (Tx)^\top (Ty) \leq x^\top y + \varepsilon\|x\|\|y\|
$$

with probability at least $1 - 4p \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$.

(ii) For any $x \in \mathbb{R}^l$ and $A \in \mathbb{R}^{p \times l}$ whose $i$th row is denoted by $A_i$,

$$
Ax - \varepsilon\|x\| \begin{bmatrix} \|A_1\|_2 \\
\vdots \\
\|A_p\|_2 
\end{bmatrix} \leq AT^\top Tx \leq Ax + \varepsilon\|x\| \begin{bmatrix} \|A_1\|_2 \\
\vdots \\
\|A_p\|_2 
\end{bmatrix}
$$

with probability at least $1 - 4p \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$. 6
(iii) For any two vectors $x, y \in \mathbb{R}^l$ and a square matrix $Q \in \mathbb{R}^{l \times l}$, then with probability at least $1 - 8r \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$, we have

$$x^\top Q y - 3\varepsilon \|x\| \|y\| \|Q\|_F \leq x^\top T^\top TQT^\top T y \leq x^\top Q y + 3\varepsilon \|x\| \|y\| \|Q\|_F,$$

where $r$ is the rank of $Q$.

A consequence of Lemma 10 is that the random mapping $\mathbb{R}^{m \times m} \mapsto \mathbb{R}^{k \times k} : M \mapsto TMT^\top$ “almost” preserves the Frobenius norm, $\|M\|_F$, of $M$.

**Lemma 11.** Let $G$ be defined as in Lemma 9 and let $T = \frac{1}{\sqrt{k}}G$. Let $0 < \varepsilon < 1$. Then for any $A, B \in \mathbb{R}^{l \times l}$, we have that with probability at least $1 - 8r_1r_2 \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$,

$$\|\langle A, B \rangle_F - \langle TAT^\top, TBT^\top \rangle_F\| \leq 3\varepsilon \|A\|_F \|B\|_* ,$$

where $r_1, r_2$ are the ranks of $A$ and $B$, respectively.

**Proof.** Assume first that $B$ has rank one, then there exists unit vectors $x, y \in \mathbb{R}^l$ and $\sigma > 0$ such that $B = \sigma xy^\top$. Then by Lemma 10(iii), we have with probability at least $1 - 8r_1 \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$,

$$\sigma x^\top Ay - 3\varepsilon \sigma \|x\| \|y\| \|A\|_F \leq \sigma x^\top T^\top TAT^\top T y \leq \sigma x^\top Ay + 3\varepsilon \sigma \|x\| \|y\| \|A\|_F.$$

Since

$$\sigma x^\top Ay = \langle \sigma xy^\top, A \rangle_F = \langle B, A \rangle_F$$

and $\varepsilon = \|B\|_*$, this proves the Lemma in the rank one case.

For the general case, we write, using the singular value decomposition of $B$,

$$B = \sum_{i=1}^{r_2} \sigma_i x_i y_i^\top,$$

where $x_i, y_i \in \mathbb{R}^l$ are unit vectors and $\sigma_i > 0$. We conclude by an union bound on $i \in \{1, \cdots, r_2\}$ (we use Lemma 10(iii) for all $x_i, y_i$), using the linearity of the scalar product and the fact that $\|B\|_* = \sum_{i=1}^{r_2} \sigma_i$. \hfill \Box

Notice that in the above lemma, the nuclear norm of $A$ or $B$ should be taken into account in the approximation error. In [13], it has been proven that such random mapping cannot preserve the Frobenius norm of a matrix in a similar fashion as the JLL (hence the error cannot be written as $O(\varepsilon \|A\|_F \|B\|_F)$). The approximation we obtain is tighter than the one obtain in [13] as our error is $O(\varepsilon \|A\|_F \|B\|_*)$ instead of $O(\varepsilon \|A\|_* \|B\|_*)$.

In the next Lemma, we prove a concentration result for random Gaussian matrices.
Lemma 12. Let $a \in \mathbb{R}_{++}^m$ and let $U$ be the random $k \times m$ matrix such that its $j$-th column is a random vector drawn, independently from the other columns, from the $\mathcal{N}(0, a_j I_k)$ distribution. Then for any $0 < \varepsilon \leq 1$, $0 < \delta < \frac{1}{2}$ if
\[
m \geq \frac{9^8}{C_1 \varepsilon^2} (3k - \ln(\delta)),
\]then with probability at least $1 - 2\delta$, we have
\[
\left\| \frac{1}{\| a \|_1} U U^\top - I_k \right\|_2 \leq \frac{\max a_i}{\min a_i} \varepsilon.
\]

Proof. Let us denote, for all $j \leq m$, by $A_j$ the $j$th column of $U$ multiplied by $\frac{1}{\sqrt{\| a \|_1}}$. Let us consider a $\frac{1}{4}$-net, $\mathcal{N}$, of the unit sphere $S^{k-1}$ such that $|\mathcal{N}| \leq 9^k$ (c.f. Remark 7).

Using Lemma 8 with $\hat{\varepsilon} = \frac{1}{4}$ and the fact that the matrix $\frac{1}{\| a \|_1} U U^\top - I_k$ is symmetric, we deduce that
\[
\left\| \frac{1}{\| a \|_1} U U^\top - I_k \right\|_2 \leq 2 \sup_{x \in \mathcal{N}} \left| \left\langle \frac{1}{\| a \|_1} U U^\top x - x, x \right\rangle \right| = 2 \sup_{x \in \mathcal{N}} \left\| \frac{1}{\sqrt{\| a \|_1}} U^\top x \right\|_2^2 - 1.
\]

Let $x \in S^{k-1}$, we can express $\left\| \frac{1}{\sqrt{\| a \|_1}} U^\top x \right\|_2^2$ as a sum of independent random variables:
\[
\left\| \frac{1}{\sqrt{\| a \|_1}} U^\top x \right\|_2^2 = \sum_{i=1}^m (A_i, x)^2,
\]
where the $A_i$ are independent sub-Gaussian vectors distributed under the $\mathcal{N}(0, \frac{a_i}{\| a \|_1} I_k)$ distribution for every $i$. Thus, by Remark 2, $X_i = (A_i, x)$ are independent sub-Gaussian random variables with
\[
E(X_i^2) = \frac{a_i}{\| a \|_1} \quad \text{and} \quad \| X_i \|_{\psi_2} \leq 2 \sqrt{\frac{a_i}{\| a \|_1}}.
\]

Therefore, $Y_i = X_i^2 - \frac{a_i}{\| a \|_1}$ are independent zero means, sub-exponential random variables and by Lemma 4 we have
\[
\| X_i \|_{\psi_1} \leq \| X_i \|_{\psi_2} \| X_i \|_{\psi_2} \leq 2^2 \frac{a_i}{\| a \|_1},
\]

hence by Lemma 4 again,
\[
\left\| X_i^2 - \frac{a_i}{\| a \|_1} \right\|_{\psi_1} \leq 2 \left\| X_i^2 \right\|_{\psi_1} \leq 2^3 \frac{a_i}{\| a \|_1}.
\]
Notice that \( \sum_{i=1}^{m} \frac{a_i}{\|a\|_1} = 1 \). Using Bernstein inequality (7) and (11), we obtain

\[
P \left( \left\| \frac{1}{\sqrt{\|a\|_1}} U^T x \right\|_2 - 1 \geq \frac{1}{2} \max_{i} a_i \varepsilon \right) = P \left( \left\| \sum_{i=1}^{m} (x_i^2 - \frac{a_i}{\|a\|_1}) \right\|_2 \geq \frac{1}{2} \min_{i} a_i \varepsilon \right)
\]

\[
\leq 2 \exp \left( -C_1 \min \left( \frac{\max_{i} a_i \varepsilon}{m \min_{i} a_i}, \frac{\max_{i} a_i \varepsilon}{2 \sqrt{m} \min_{i} a_i} \right) \right).
\]

(12)

In order to bound (12), we use the following inequality with \( a \in \mathbb{R}^m_{++} \):

\[
\sum_{i=1}^{m} \left( 2^{3} a_i \|a\|_1 \right)^2 \geq 2^6 \frac{\sum_{i} a_i^2}{m} + 2 \sum_{i<j} a_i a_j \geq m (\max_{i} a_i)^2
\]

\[
\leq 2^6 \frac{(\max_{i} a_i)^2}{m (\min_{i} a_i)^2}.
\]

(13)

Then, using (13), the fact that \( \max_{i} \left( 2^{3} a_i \|a\|_1 \right) \leq 2^3 \frac{\max_{i} a_i}{\min_{i} a_i} \), we plug all those bounds in (12) to obtain

\[
P \left( \left\| \frac{1}{\sqrt{\|a\|_1}} U^T x \right\|_2 - 1 \geq \frac{1}{2} \max_{i} a_i \varepsilon \right) \leq 2 \exp \left( -C_1 \min \left( \frac{1}{2^{2} m \cdot \varepsilon^2 m}, \frac{1}{2^{4} m \cdot \varepsilon^2 m} \right) \right)
\]

\[
\leq 2 \exp \left( -\frac{C_1}{2^8} \varepsilon^2 m \right).
\]

Now using an union bound on the set \( \mathcal{N} \), we have that

\[
P \left( \left\| \frac{1}{\sqrt{\|a\|_1}} U^T x \right\|_2 - 1 \geq \frac{1}{2} \min_{i} a_i \varepsilon \right) \leq 2^{m} \exp \left( \frac{C_1}{2^{8} \varepsilon^2 m} \right)
\]

\[
\leq 2 \exp \left( 3k - \frac{C_1}{2^{8} \varepsilon^2 m} \right).
\]

From (10) we have

\[
P \left( \left\| \frac{1}{\|a\|_1} U U^T - I_k \right\|_2 \geq \max_{i} a_i \varepsilon \right) \leq P \left( \left\| \sup_{x \in \mathcal{N}} \frac{1}{\sqrt{\|a\|_1}} U^T x \right\|_2 - 1 \geq \frac{1}{2} \min_{i} a_i \varepsilon \right).
\]
Hence,
\[
P\left( \frac{1}{\|a\|_1} UU^\top - I_k \right)_2 \geq \frac{\max a_i}{\min a_i} \varepsilon \leq 2 \exp \left( 3k - \frac{C_1}{2} \varepsilon^2 m \right) \leq 2 \exp(\ln(\delta)) = 2\delta.
\]

In the above Lemma, we proved that \( \frac{1}{\|a\|_1} UU^\top \) concentrates around its expectation, \( \mathbb{E}\left( \frac{1}{\|a\|_1} UU^\top \right) = I_k \). This is a generalization of a result proved in [17] about concentration of Gaussian random matrices.

**Corollary 13.** Let \( a \in \mathbb{R}_+^m \) and let \( U \) be a random Gaussian \( k \times m \) matrix such that its \( j \)-th column is drawn, independently from the \( \mathcal{N}(0, I_k) \) distribution, then for any \( 0 < \varepsilon \leq 1 \), \( 0 < \delta < \frac{1}{2} \) if
\[
m \geq \frac{2^8}{C_1 \varepsilon^2} (3k - \ln(\delta)),
\]
then with probability at least \( 1 - 2\delta \), we have
\[
\left\| UD(a) U^\top - \|a\|_1 I_k \right\|_2 \leq \frac{\max a_i}{\min a_i} \|a\|_1 \varepsilon,
\]
where \( D(a) \) denotes the \( m \times m \) diagonal matrix built from the vector \( a \).

**Proof.** Let \( U' \) be the matrix \( U' = \sqrt{D(a)} U \). The \( j \)-th column of \( U' \) follows the \( \mathcal{N}(0, a_j I_k) \) distribution, hence, by Lemma 12, with probability at least \( 1 - 2\delta \), we have
\[
\left\| \frac{1}{\|a\|_1} U' U'^\top - I_k \right\|_2 \leq \frac{\max a_i}{\min a_i} \varepsilon,
\]
which ends the proofs after multiplying both sides by \( \|a\|_1 \).

4. The projected problem

In this section, we will analyze randomly projected versions of the conic optimization problem discussed in Section 1. However, there are a number of technical assumptions we need to impose and their degree of restrictiveness vary. Strictly speaking, the problems for which our results are valid must have the following shape

\[
P^K \left\{ \begin{array}{l} \min_{x} \ c^\top x \\ Ax - b \in \mathcal{K} \\ Bx - d \in \mathcal{K}' \\ x \in \mathbb{R}^n, \end{array} \right. \tag{14}
\]

where \( \mathcal{K} = \mathbb{R}_+^m \times S_+^{p_1} \times \cdots \times S_+^{p_l} \) and \( \mathcal{K}' \) is some arbitrary self-dual cone\(^1\). In the remaining of the paper, an element \( y \in \mathcal{K} \) will be denoted by \( y = (y_0, M_1, \cdots, M_l) \).

\(^1\mathcal{K}' \) is self-dual if and only if \( \mathcal{K}' = \{ u | u^\top v \geq 0, \forall v \in \mathcal{K}' \} \).
**Hypothesis 14.** We make the following assumptions on \((14)\).

(i) \(\mathcal{K}'\) is a self-dual cone and the set \(\{ x \mid Bx - d \in \mathcal{K}' \}\) is non-empty such that \(\min_{Bx - d \in \mathcal{K}'} c^\top x\) has finite value (for example, if \(c \geq 0\) we can consider the set \(\{ x \in \mathbb{R}^n \mid x \geq 0 \}\)).

(ii) For all \(i \in \{1, \cdots, n\}, c_i \neq 0\).

(iii) All the \(l, m, p, p_1, \cdots, p_l\) are all big-O of \(n\).

(iv) There exists an optimal solution \((y^*, \lambda^*)\) to the dual problem of \((14)\), where \(y_0^*, M_1^*, \cdots, M_l^*\), such that \(y_{0,j}^* \neq 0\), for all \(j \in \{1, \cdots, m\}\).

(vi) The optimal value, \(v(\mathcal{P}_K)\), of \(\mathcal{P}_K\) is non-zero.

**Remark 15.** Notice that (ii), (iv) hold generically (c.f. [18, 19]) and (vi) also holds generically. Furthermore, regarding item (v), in practice we can always consider a point \(\tilde{y}^*\) in a neighborhood of \(y^*\), instead of \(y^*\), such that \(|b^\top y^* - b^\top \tilde{y}^*| \leq \varepsilon\). As for (vi) it is not needed if we consider error bounds in absolute value instead of relative value with respect to \(v(\mathcal{P}_K)\). In any case, we can perturb the vector \(c\) by a random quantity \(\delta c\) of small variance to ensure that the hypothesis holds.

As for (i) it is indeed necessary to prove that the projected problem is bounded.

Let us consider the following random map:

\[
Q : \mathbb{R}^m \times \mathbb{S}^{p_1} \times \cdots \times \mathbb{S}^{p_l} \mapsto \mathbb{R}^k \times \mathbb{S}^{q_1} \times \cdots \times \mathbb{S}^{q_l}.
\]

We have

\[
Q((y_0, M_1, \cdots, M_l)) = (Sy_0, Q_1(M_1), \cdots, Q_l(M_l)),
\]

where \(S \in \mathbb{R}^{k \times m}\) is a random iid matrix such that \(S = T \circ T\) where \(\circ\) denotes the Hadamard product and where \(T_{ij}\) is drawn from the normal distribution \(\mathcal{N}(0, \frac{1}{k})\). Furthermore, we have

\[
Q_i(M_i) = T^{(i)} M_i T^{(i)}\top,
\]

where for all \(i \in \{1, \cdots, l\}, T^{(i)} \in \mathbb{R}^{q_i \times p_i}\) are random iid matrices such that each entry is drawn independently from \(\mathcal{N}(0, \frac{1}{q_i})\).

Notice that \(Q\) and \(\mathcal{K} = \mathbb{R}^n_+ \times \mathbb{S}^{p_1}_+ \times \cdots \times \mathbb{S}^{p_l}_+\) satisfy

\[
Q(\mathcal{K}) = \mathbb{R}^k_+ \times \mathbb{S}^{q_1}_+ \times \cdots \times \mathbb{S}^{q_l}_+,
\]

which is also the product of a non-negative orthant and positive semidefinite cones.
Remark 16. While it is true that LP (1) can be written as an SDP, the method proposed in [13] would not be efficient if applied to (1). This is because the resulting projected problem is an SDP which would not be reducible to an LP again. Hence, we would need to solve the projected problem as an SDP.

We consider the following projected problem:

\[
\mathcal{P}_Q^K \left\{ \begin{array}{l}
\min \quad c^T x \\
Q(A)x - Q(b) \in Q(K) \\
Bx - d \in K' \\
x \in \mathbb{R}^n,
\end{array} \right. \tag{17}
\]

where \(Q(A)\) is the matrix whose columns are \(Q(A_i)\) where \(A_i\) is the \(i\)th column of \(A\). Notice that \(\mathcal{P}_Q^K\) is a relaxation of \(\mathcal{P}_K\), hence \(v(\mathcal{P}_Q^K) \leq v(\mathcal{P}_K)\).

We now derive a lower bound for the value of \(\mathcal{P}_Q^K\).

Let us consider the duals, \(\mathcal{D}_Q^K\) of (14) and \(\mathcal{D}_Q^K\) of (17):

\[
\begin{align*}
\max_{y,\lambda} & \quad b^T y + d^T \lambda \\
& \quad A^T y + B^T \lambda = c \\
y \in K, \lambda \in K' & \quad \mathcal{D}_Q^K \tag{18}
\end{align*}
\]

\[
\begin{align*}
\max_{z,\lambda} & \quad (Q(b))^T z + d^T \lambda \\
& \quad A^T Q^T(z) + B^T \lambda = c \\
z \in Q(K), \lambda \in K' & \quad \mathcal{D}_Q^K \tag{19}
\end{align*}
\]

where \(Q^T\) denotes the dual of the map \(Q\). Let \((y^*, \lambda^*) \in K \times K'\) be an optimal solution of (18). We consider the following “approximated” projected solution, \((z_Q, \lambda^*)\), where

\[
z_Q := Q(y^*) \in Q(K). \tag{20}
\]

We will now prove that \((z_Q, \lambda^*)\) is “almost” feasible for (19). Let us consider the modified dual problem:

\[
\begin{align*}
\max_{z,\lambda} & \quad (Q(b))^T z + d^T \lambda \\
& \quad A^T Q^T(z) + B^T \lambda = c + A^T (Q^T(z_Q) - y^*) \\
z \in Q(K), \lambda \in K' & \quad \mathcal{D}_Q^K \tag{21}
\end{align*}
\]

Notice that by definition of \(\mathcal{D}_Q^K\), \((z_Q, \lambda)\) is a feasible solution for (21). We will now prove that \(Q^T(z_Q) = Q^T(Q(y^*))\) is “close” to \(y^*\), which will be enough to obtain a lower bound on \(v(\mathcal{P}_Q^K)\) in Theorem 22. Let \(E \in \mathbb{R}^n\) denote the “error”:

\[
E := A^T (Q^T(z_Q) - y^*) = A^T (Q^T(Q(y^*)) - y^*). \tag{22}
\]
4.1. Bounding the error \( E \)

Let us write \( y^* = (y_0^*, M_1^*, \ldots, M_l^*) \in \mathbb{R}^m \times S^{p_1} \times \cdots \times S^{p_l} \). We have that

\[
Q^T(Q(y^*)) = \left( S^T S y_0^*, T^{(1)}T^{(1)*}, \ldots, T^{(l)}T^{(l)*} \right),
\]

hence

\[
Q^T(z_Q) - y^* = \begin{pmatrix}
T^{(1)}T^{(1)*} - M_1^* \\
\vdots \\
T^{(l)}T^{(l)*} - M_l^*
\end{pmatrix}.
\]

(23)

Let

\[
A^\top = (A^{(0)} \ A^{(1)} \ \cdots \ A^{(l)})^\top
\]

be the column decomposition of \( A^\top \) such that for all \( y = (y_0, M_1, \ldots, M_l) \), and hence,

\[
A^\top y = A^{(0)}^\top y_0 + \sum_{i=1}^l A^{(i)}^\top M_i,
\]

(25)

where \( A^{(0)} \in \mathbb{R}^{m \times n} \) and \( A^{(i)} \in \mathbb{R}^{p_i \times n} \) for \( 1 \leq i \leq l \). Here we use the notation \( A^{(i)}^\top M_i \) to denote the vector in \( \mathbb{R}^n \) whose \( j \)th component is given by \( (A^{(i)}_j, M_i)_j \), where the \( p_i \times p_i \) matrix \( A^{(i)}_j \) is seen as the \( j \)th column of \( A^{(i)} \).

Using (25), \( E \) in (22) can be written as

\[
E = A^{(0)}^\top (S^T S y_0^* - y_0^*) + \sum_{i=1}^l A^{(i)}^\top \left( T^{(i)}T^{(i)*} - M_i^* \right).
\]

(26)

The goal of this subsection is to prove the following proposition.

**Proposition 17.** Let \( \varepsilon, \delta, m \) be such that \( 0 < \delta < \frac{1}{4}, 0 < \varepsilon < 1 \) and \( m \geq \frac{2^8}{\delta (\varepsilon^2/2 - \varepsilon^3/3)} \). Then, with probability at least \( 1 - \delta - \frac{8m^2 + 4m(n + 1) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))}{\sum_{i=1}^l 8p_i^2(n + 1) \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3))} \),

\[
|E| \leq \varepsilon \alpha(y_0^*, A^{(0)}) \left( \|A^{(0)}\|_2 \|y_0^*\|_2 \right) + 3\varepsilon \left( \max_{i=1, \ldots, l} \|M_i^*\|_2 \sum_{i=1}^l \|M_i\|_F \right) \left( \frac{\|A^{(i)}_j\|_F}{\|A^{(i)}_j\|_2} \right),
\]

(27)

\[
|(Q(b))^\top z_Q - b^\top y^*| \leq \varepsilon \alpha(y_0^*, b_0)\|b_0\|_2 \|y_0^*\|_2 + 3\varepsilon \left( \max_{i=1, \ldots, l} \|M_i^*\|_2 \sum_{i=1}^l \|b_i\|_F \|M_i\|_F \right),
\]

(28)
where $A^{(i)}_j$ denotes the $j$th column of $A^{(i)}$, where $|E|$ is the vector whose components are the absolute value of the components of $E$ and where
\[
\alpha(y^*_0, A^{(0)}) = 16 \left( \max_j \frac{\|A^{(0)}_j\|_1}{k\|A^{(0)}_j\|_2} \left( \frac{\|y^*_0\|_1 \max |y^*_0|}{\|y^*_0\|_2 \min |y^*_0|} (1 + \varepsilon) + k \right) \right),
\]
(29)
\[
\alpha(y^*_0, b_0) = 16 \left( \frac{\|b_0\|_1}{k\|b_0\|_2} \left( \frac{\|y^*_0\|_1 \max |y^*_0|}{\|y^*_0\|_2 \min |y^*_0|} (1 + \varepsilon) + k \right) \right)
\]
(30)
and $b = (b_0, b_1, \ldots, b_l)$.

We will show the proof of Proposition 17 later after presenting a few preliminary results. In particular, to obtain a bound on $E$, we will bound each term in the summation in (26).

The first step is to use item (iii) of Lemma 10 for all $1 \leq i \leq l$ in order to bound the terms
\[
A^{(i)\top} \left( T^{(i)\top} T^{(i)} M^*_T T^{(i)\top} T^{(i)} - M^*_T \right).
\]
Indeed, let us denote by $A^{(i)}_j$ the $j$th column of $A^{(i)}$. $A^{(i)}_j$ is a $p_i \times p_i$ matrix, hence by Lemma 11, for every $\varepsilon \in (0, 1)$ we have that with probability at least $1 - 8p_i^2 \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3))$,
\[
-3\varepsilon \|M^*_T\|_F \|A^{(i)}_j\|_F \leq \left( A^{(i)}_j, \left( T^{(i)\top} T^{(i)} M^*_T T^{(i)\top} T^{(i)} - M^*_T \right) \right)_F \leq 3\varepsilon \|M^*_T\|_F \|A^{(i)}_j\|_F.
\]
(31)
By an union bound, considering all the $j \in \{1, \ldots, n\}$, we have that with probability at least $1 - 8p_i^2 n \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3))$,
\[
|A^{(i)\top} (T^{(i)\top} T^{(i)} M^*_T T^{(i)\top} T^{(i)} - M^*_T)| \leq 3\varepsilon \|M^*_T\|_F \left( \begin{array}{c} \|A^{(i)}_1\|_F \\ \vdots \\ \|A^{(i)}_n\|_F \end{array} \right) \left( \begin{array}{c} \|M^*_T\|_F \\ \vdots \\ \|M^*_T\|_F \end{array} \right).
\]
(32)
Notice that Lemma 9 and Lemma 11 imply
\[
q_i = O \left( \frac{\log(p_i)}{\varepsilon^2} \right).
\]
(33)
Since $p_i = O(n)$ by Hypothesis 14(iii), this ensures that Equation (32) holds w.a.h.p.

In order to complete the task of bounding $E$, next we bound the term $A^{(0)\top} (S^T y^*_0 - y^*_0)$. In what follows, $D(\cdot) : \mathbb{R}^m \to S^m$ denotes the function that maps a vector $y_0 \in \mathbb{R}^m$ into a $m \times m$ diagonal matrix with $y_0$ in its entries. Then, $D^{-1}(\cdot)$ is the function that maps a matrix to its diagonal vector. We have the following lemma.

**Lemma 18.** For any $y_0 \in \mathbb{R}_+^m$, we have that
\[
S y_0 = D^{-1}(T D(y_0) T^\top).
\]
Proof. Let \( U = \sqrt{D(y_0)} \). We have \( TD(y_0)T^\top = (TU)(TU)^\top \), hence the \( i \)th term on the diagonal of \( TD(y_0)T^\top \) is equal to \( \sum_{j=1}^m (TU)_{ij}^2 \). Since \( U \) is a diagonal matrix:

\[
U = \begin{pmatrix} \sqrt{y_0_1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & \sqrt{y_0_m} \end{pmatrix},
\]

we deduce that \( (TU)_{ij} = T_{ij}\sqrt{y_0_j} \). Hence the \( i \)th term on the diagonal of \( TD(y_0)T^\top \) is equal to \( \sum_{j=1}^m T_{ij}^2y_0_j = (S y_0)_i \). \( \square \)

We deduce from Lemma 18 the following corollary.

**Corollary 19.** We have that for any \( y_0 \in \mathbb{R}^m_+ \)

\[
S^\top S y_0 = D^{-1}(T^\top D(D^{-1}(TD(y_0)T^\top))T).
\]

**Proposition 20.** Let \( y_0 \in \mathbb{R}^m \) such that \( y_{0i} \neq 0 \) for all \( i \), let \( 0 < \delta < \frac{1}{4} \).

Assuming that \( 0 < \varepsilon \leq C^3 \), and that \( m \geq \frac{\varepsilon^8}{\sqrt{\varepsilon^2} \delta^6} (3k - \ln(\delta)) \), we have that

\[
\|D(S y_0) - TD(y_0)T^\top\|_2 \leq 16\varepsilon\frac{\max |y_{0i}|}{k \min |y_{0i}|} \|y_0\|_1
\]

holds with probability at least \( 1 - 4\delta \).

**Proof.** We first assume that \( y_0 \in \mathbb{R}^m_+ \). Let \( U \in \mathbb{R}^{k \times m} \) be defined by \( U = \sqrt{k}T \). By Corollary 13, we conclude that with at least probability \( 1 - 2\delta \),

\[
\|UD(y_0)U^\top - \|y_0\|_1IK\|_2 \leq \varepsilon\frac{\max |y_{0i}|}{\min |y_{0i}|} \|y_0\|_1.
\]

Hence we deduce that

\[
\left\|TD(y_0)T^\top - \frac{y_0}{k}IK\right\|_2 \leq \varepsilon\frac{\max |y_{0i}|}{\min |y_{0i}|} \frac{\|y_0\|_1}{k}, \quad (34)
\]

Furthermore, the \( ii \)-th element of \( D(D^{-1}(TD(y_0)T^\top)) \) is given by

\[
e_i^\top TD(y_0)T^\top e_i.
\]

We deduce from (34) that for all \( i \leq k \)

\[
\left|e_i^\top TD(y_0)T^\top e_i - \frac{y_{0i}}{k}\right| \leq \varepsilon\frac{\max |y_{0i}|}{\min |y_{0i}|} \|y_0\|_1
\]

and, since \( D(D^{-1}(TD(y_0)T^\top)) - \frac{y_0}{k}IK \) is diagonal, we have that

\[
\left\|D(D^{-1}(TD(y_0)T^\top)) - \frac{y_0}{k}IK\right\|_2 \leq \varepsilon\frac{\max |y_{0i}|}{\min |y_{0i}|} \|y_0\|_1. \quad (35)
\]
By combining (34), (35), the triangle inequality and Lemma 18 we conclude that for \(y_0 \in \mathbb{R}_+^m\) we have with at least probability \(1 - 2\delta\),

\[
\|D(Sy_0) - TD(y_0)T^\top\|_2 \leq 2\varepsilon \frac{\max \|y_{0i}\|}{\min \|y_{0i}\|}\|y_0\|_1. \tag{36}
\]

For the general case, write \(y_0 = y_0^+ - y_0^-\) where \(y_0^+, y_0^- \in \mathbb{R}_+^m\) are chosen in the following way

- \(y_0^+\) is the sum of the positive part of \(y\) and \(\min(|y_{0i}|)1\),
- \(y_0^-\) is the sum of the negative part of \(y\) and \(\min(|y_{0i}|)1\),

where \(1 \in \mathbb{R}^m\) is the vector having all entries equal to 1. Since \(y_0\) has no zero components we have that \(\|\min_i(|y_{0i}|)1\|_1 \leq \|y_0\|_1\), hence, \(\|y_0^+\|_1 \leq 2\|y_0\|_1\) and \(\|y_0^-\|_1 \leq 2\|y_0\|_1\). Furthermore, \(\max_i y_{0i}^+ \leq 2 \max_j |y_{0j}|\), \(\max_i y_{0i}^- \leq 2 \max_j |y_{0j}|\), \(\min_i y_{0i}^+ \geq \min_j |y_{0j}|\) and \(\min_i y_{0i}^- \geq \min_j |y_{0j}|\) hold.

We have

\[
\|D(S(y_0)) - TD(y_0)T^\top\|_2 = \|D(S(y_0^+)) - D(S(y_0^-))\|_2
\]

+ \(\|D(S(y_0^+)) - D(S(y_0^-))\|_2\).

Since \(y_0^+\) and \(y_0^-\) are positive, we conclude from (36) that with at least probability \(1 - 4\delta^2\):

\[
\|D(S(y_0)) - TD(y_0)T^\top\|_2 \leq 2\varepsilon \left(\frac{\max y_{0i}^+}{\min y_{0i}}\|y_0^+\|_1 + \frac{\max y_{0i}^-}{\min y_{0i}}\|y_0^-\|_1\right) \leq 16\varepsilon \frac{\max \|y_{0i}\|}{\min \|y_{0i}\|}\|y_0\|_1,
\]

where we also used the relations between \(\|y_0^-\|_1, \|y_0^+\|_1\) and \(\|y_0\|_1\) as well the relations between \(\min(|y_{0i}|)\), \(\max(|y_{0i}|)\), \(\min(|y_{0i}^+|)\), \(\max(|y_{0i}^-|)\), \(\max(|y_{0i}^-|)\).

**Proposition 21.** Let \(S\) be as in (15) and let \(y_0^1, y_0^2 \in \mathbb{R}^m\) be such that for all \(i \in \{1, \ldots, m\}, y_{0i}^2 \neq 0\). Assume that \(m \geq \frac{2}{\varepsilon^2(3k - \ln(\delta))}\). Then, with probability at least \(1 - 4\delta - (8m^2 + 4m)\exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))\), we have

\[
|(S(y_0^1))^{\top}(S(y_0^2)) - y_0^{1\top} y_0^2| \leq \varepsilon \alpha(y_0^1, y_0^2)\|y_0\|_2\|y_0^1\|_2\|y_0^2\|_2,
\]

where \(\alpha(y_0^1, y_0^2) = 16 \left(\frac{\|y_0^1\|_2}{\|y_0\|_2} \frac{\max |y_0^1|}{\min |y_0^2|} (1 + \varepsilon) + k\right)\).

\(^2\)This probability is obtained by an union bound.
Proof. We have that \((S_{y_0}^1)^\top(S_{y_0}^2) = y_0^\top(S^\top S_y)^2\), hence by Corollary 19, we have that
\[
(S_{y_0}^1)^\top(S_{y_0}^2) = y_0^\top D^{-1}(T^\top D(D^{-1}(TD(y_0^2)T^\top))T).
\]
Since \(D^{-1}(T^\top D(D^{-1}(TD(y_0^2)T^\top))T)\) is the vector whose \(i\)th component is \((Te_i)^\top D(D^{-1}(TD(y_0^2)T^\top))(Te_i)\), we have that
\[
(S_{y_0}^1)^\top(S_{y_0}^2) = \sum_{i=1}^m y_0^1 (e_i^\top T^\top TD(y_0^2)T^\top Te_i + (Te_i)^\top (D(D^{-1}(TD(y_0^2)T^\top)) - TD(y_0^2)T^\top) (Te_i)).
\]
Hence we have that
\[
|(S_{y_0}^1)^\top(S_{y_0}^2) - y_0^\top y_0^2| \leq \sum_{i=1}^m y_0^1 e_i^\top T^\top TD(y_0^2)T^\top Te_i - y_0^\top y_0^2 + \sum_{i=1}^m y_0^1 ((Te_i)^\top (D(D^{-1}(TD(y_0^2)T^\top)) - TD(y_0^2)T^\top) (Te_i)).
\]
First let us bound the term \(\sum_{i=1}^m y_0^1 e_i^\top T^\top TD(y_0^2)T^\top Te_i\) by using
\[
\sum_{i=1}^m y_0^1 e_i^\top T^\top TD(y_0^2)T^\top Te_i = \langle D(y_0^1), T^\top TD(y_0^2)T^\top \rangle_F = \langle TD(y_0^2)T^\top, TD(y_0^2)T^\top \rangle_F.
\]
Recalling that \((D(y_0^1), D(y_0^2))_F = (y_0^1)^\top y_0^2\), we use Lemma 11 to conclude that with probability at least \(1 - 8m^2 \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))\) we have
\[
|\langle D(y_0^1), D(y_0^2) \rangle_F - TD(y_0^1)T^\top, TD(y_0^2)T^\top \rangle_F| \leq 3\varepsilon \|D(y_0^1)\|_2 \|D(y_0^2)\|_F,
\]
and hence
\[
\sum_{i=1}^m y_0^1 e_i^\top T^\top TD(y_0^2)T^\top Te_i = y_0^\top y_0^2 - \langle TD(y_0^2)T^\top, TD(y_0^2)T^\top \rangle_F.
\]
Now let us bound the second term, \(\sum_{i=1}^m y_0^1 ((Te_i)^\top (D(D^{-1}(TD(y_0^2)T^\top)) - TD(y_0^2)T^\top) (Te_i))\), of the sum in (38). According to Proposition 20, we have that
\[
\|D(D^{-1}(TD(y_0^2)T^\top)) - TD(y_0^2)T^\top\|_2 \leq 16\varepsilon \frac{\max\{y_0^2\}}{k \min\{y_0^1\}} \|y_0^2\|_1.
\]
holds with probability at least $1 - 4\delta$. Hence for all $i \leq m$, we have that

$$|(T_{e_i})^T (D(D^{-1}(TD(y_0^2)T^T)) - TD(y_0^2)T^T) (T_{e_i})| \leq 16\varepsilon \max_{k \min |y_0^2|} \frac{y_0^2}{||y_0^2||_1} ||T_{e_i}||^2.$$ 

Furthermore, by the JLL (Lemma 9), we have that with probability at least $1 - 4m \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$, $||T_{e_i}||^2_2 \leq 1 + \varepsilon$ holds for all $i \leq m$. Hence with probability at least $1 - 4\delta - 4m \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$ \footnote{The probability $1 - 4\delta - 4m \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$ is obtained by an union bound between $1 - 4\delta$ and $1 - 4m \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$, using the fact that $P(E_1 \cap E_2) \geq 1 - (2 - P(E_1) - P(E_2))$ for any events $E_1$ and $E_2$.} (we remind that $T = \frac{1}{\sqrt{k}} G$), we have that

$$|(T_{e_i})^T (D(D^{-1}(TD(y_0^2)T^T)) - TD(y_0^2)T^T) (T_{e_i})| \leq 16\varepsilon \max_{k \min |y_0^2|} \frac{y_0^2}{||y_0^2||_1} ||y_0^2||_1(1 + \varepsilon)$$

and hence that

$$\sum_{i=1}^{m} |y_{0i}(T_{e_i})^T (D(D^{-1}(TD(y_0^2)T^T)) - TD(y_0^2)T^T) (T_{e_i})| \leq 16\varepsilon \left( \frac{||y_0^1||_1 \frac{y_0^2}{||y_0^2||_1} \max_{k \min |y_0^2|} \frac{y_0^2}{||y_0^2||_2} \min_{|y_0^2|} \frac{y_0^2}{|y_0^2|}}{1 + \varepsilon} \right) \frac{y_0^1}{2} \frac{y_0^2}{2}.$$

By combining (38), (39), (40) we have that

$$|(Sy_1^0)(Sy_0^2) - y_0^1 y_0^2| \leq 16\varepsilon \left( \frac{||y_0^1||_1 \frac{y_0^2}{||y_0^2||_2} \max_{k \min |y_0^2|} \frac{y_0^2}{||y_0^2||_2} \min_{|y_0^2|} \frac{y_0^2}{|y_0^2|}}{1 + \varepsilon} + k \right) \frac{y_0^1}{2} \frac{y_0^2}{2}$$

holds with probability $1 - 4\delta - (8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)).$ \hfill \Box

Notice that the term $(8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$ in the probability appearing in Proposition 21 can be made arbitrarily small choosing $k = k_0 \frac{\log(m)}{\varepsilon^2}$, for some constant $k_0$. The proof of Proposition 21 requires indeed $k$ at least equal to $k_0 \frac{\log(m)}{32\varepsilon^4}$, as we have used Lemma 9 for $h = m$ in the proof. We now explain how to choose the constant $k_0$ in the $O(\frac{\log(m)}{\varepsilon^2})$ such that $(8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3))$ is small enough.

Indeed for any $\delta' \in (0,1)$, since $k = k_0 \frac{\log(m)}{\varepsilon^2}$, we have that $(8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) \leq (8m^2 + 4m) \exp(-k/12\varepsilon^2) \leq \delta'$ is equivalent to

$$\frac{(8m^2 + 4m)}{\exp(\varepsilon^2/12k)} = \frac{8m^2 + 4m}{m^{k_0/12}} \leq \delta',$$

which is achieved by taking, for example,

$$k_0 \geq \frac{3 + \ln(12) - \ln(\delta')}{1/12} \geq \frac{3 \ln(m) + \ln(12) - \ln(\delta')}{\ln(m)} \geq \frac{12 \ln(8m^2 + 4m) - \ln(\delta')}{\ln(m)},$$
as \( \ln(m) \geq 1 \). Hence the condition required in Proposition 20 and 21 is equivalent to

\[
m \geq \frac{2^8}{C_1 \varepsilon^2} \left( 3k_0 \frac{\ln(m)}{\varepsilon^2} - \ln(\delta) \right),
\]

which holds for sufficiently large \( m \).

**Proof of Proposition 17.** Let us write \( b = (b_0, b_1, \cdots, b_l) \), such that the scalar product \( b^T y^* \) can be decomposed into

\[
b^T y^* = b_0^T y_0^* + \sum_{i=1}^l \langle b_i, M_i^* \rangle_F.
\]

To bound \( |b^T y^* - Q(b)^T z_Q| = |b^T y^* - b^T Q^T(Q(y^*))| \), we first write

\[
|b^T y^* - b^T Q^T(Q(y^*))| = \left| b_0^T y_0^* - (Sb_0)^T S y_0^* + \sum_{i=1}^l \left( \langle b_i, M_i^* \rangle_F - \left\langle T^{(i)} b_i T^{(i)} , T^{(i)} M_i^* T^{(i)} \right\rangle_F \right) \right|.
\]

Using Lemma 11, for all \( i \leq l \), we can bound the terms

\[
\left\langle T^{(i)} b_i T^{(i)} , T^{(i)} M_i^* T^{(i)} \right\rangle_F - \langle b_i, M_i^* \rangle_F.
\]

In fact, by an union bound, we obtain that with probability at least \( 1 - \sum_{i=1}^l \frac{8p_i^2 \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3))}{F} \),

\[
\left| \sum_{i=1}^l \left( \left\langle T^{(i)} b_i T^{(i)} , T^{(i)} M_i^* T^{(i)} \right\rangle_F - \langle b_i, M_i^* \rangle_F \right) \right| \leq 3\varepsilon \left( \max \left\{ \frac{\|M_i^*\|_F}{\|M_i^*\|_F} \right\} \sum_{i=1}^l \|M_i^*\|_F \|b_i\|_F.\right.
\]

Furthermore, using Proposition 21 with \( y_0^* = b_0 \) and \( y_0^2 = y_0^* \), we have that with probability at least \( 1 - 4\delta - (8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) \)

\[
|b_0^T S^T S y_0^* - b_0^T y_0^*| \leq \varepsilon \alpha(b_0, y_0^*)\|y_0^*\|_2\|b_0\|_2.
\]

Using (44) and (43) with (42) we obtain, by an union bound, that Equation (28) of Proposition 17 holds with probability at least

\[
1 - 4\delta - (8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) - \sum_{i=1}^l \frac{8p_i^2 \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3))}{F}.
\]

Now we will bound the term \( A^{(0)}(S^T S y_0^* - y_0^* \rangle \) from (26). For all \( i \in \{1, \cdots, n\} \), using Proposition 21 with \( y_0^* = A_i^{(0)} \), \( y_0^2 = y_0^* \), and taking the \( \delta \) of Proposition 21 equal to \( \frac{4}{n} \), if

\[
m \geq \frac{2^8}{C_1 \varepsilon^2} \left( 3k - \ln\left( \frac{\delta}{n} \right) \right),
\]

19
we have, by an union bound, that with probability at least

\[ 1 - n \left( 4\frac{\delta}{n} - (8m^2 + 4m) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) \right), \]

\[ |A^{(0)}^\top (S^\top S y_0^* - y_0^*)| \leq \varepsilon \alpha(y_0^*, A^{(0)}) \|y_0^*\|_2 \left( \frac{\|A_1^{(0)}\|_2}{\|A_0^{(0)}\|_2} \right) \]

holds.

We are now ready to bound the error \( E = A^\top (Q^\top (z_Q - y^*)). \) Using (45), (24), (26), (32) we prove by an union bound that with probability at least

\[ 1 - 4\delta - (8m^2 + 4m)n \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) - \sum_{i=1}^{l} 8p_i^2 n \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3)), \]

(27) of Proposition 17 holds. Hence, by an union bound, we prove that the claim of the proposition holds with probability at least

\[ 1 - 8\delta - (8m^2 + 4m)(n+1) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) - \sum_{i=1}^{l} 8p_i^2 (n+1) \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3)). \]

\[ \square \]

4.2. Bounding the projected optimization problem

Now we are ready to show the main theorem.

**Theorem 22.** Let \( \varepsilon, \delta, m \) be such that \( 0 < \delta < \frac{1}{8}, 0 < \varepsilon \leq 1 \) and \( m \geq \frac{2^n}{\varepsilon^2(3k + \ln(n) - \ln(\delta))}. \) With probability at least 1 - \( 8\delta - (8m^2 + 4m)(n + 1) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) - \sum_{i=1}^{l} 8p_i^2 (n+1) \exp(-q_i/2(\varepsilon^2/2 - \varepsilon^3/3)), \) we have

\[ v(\mathcal{P}^N) \left( 1 - \varepsilon \max \left( \alpha(y_0^*, A^{(0)}, b_0), \max_{i=1,\ldots,l} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \left( \max_{j=1,\ldots,n} \left( \frac{1}{\cos(\gamma_j)} \right) \frac{4\|x_Q^*\|_2}{\cos(\beta)} + \frac{3}{\cos(\beta)} \right) \right) \leq v(\mathcal{P}_0^N) \leq v(\mathcal{P}), \]

where

- \( x^*, (y^*, \lambda^*) \) are optimal solutions of \( \mathcal{P}^N \) and \( \mathcal{D}^N, \) respectively,
- \( \alpha(y_0^*, A^{(0)}, b_0) = \max(\alpha(y_0^*, A^{(0)}), \alpha(y_0^*, b_0)), \)
- \( \beta \) is the angle between \( (b, d) \) and \( (y^*, \lambda^*), \) \( \gamma_j \) is the angle between \( (y^*, \lambda^*) \) and the \( j \)th column of the matrix \( \begin{pmatrix} A \\ B \end{pmatrix}, \)
• \( \theta \) is the angle between \( c \) and \( x^* \).

• \( x^*_Q \) is a feasible solution of \( \mathcal{P}^K_Q \) such that \( e^T x^*_Q - v(\mathcal{P}^K_Q) \leq \varepsilon' \) for some \( \varepsilon' \) satisfying

\[
\varepsilon' \leq \varepsilon \max \left( \alpha(y_0^*, A^{(0)}, b_0), \max_{i=1,\ldots,l} \left\| M_i^* \right\|_F \right) \max_{j=1,\ldots,n} \left( \frac{1}{\cos(\gamma_j)} \right) \left\| x^*_Q \right\|_2 \cos(\theta) \left\| x^* \right\|_2 v(\mathcal{P}^K).
\]

Notice that in the case where \( v(\mathcal{P}^K) < 0 \) we have that \( \cos(\beta) \) and \( \cos(\theta) \) are negative, implying that Theorem 22 also holds for the case. Notice that the probability in the above theorem can be made arbitrarily small by considering \( k = O \left( \frac{\log(m)}{\varepsilon^2} \right) \) and \( q_i = O \left( \frac{\log(p_i)}{\varepsilon} \right) \).

**Proof.** Let \( \varepsilon, \delta, m \) be as in the assumptions of theorem and let \( z_Q \) be as in (20). Since \( z_Q \) is a feasible solution of \( \mathcal{D}_Q^* \), we have by Proposition 17, that

\[
v(\mathcal{D}_Q^*) \geq (Q(b))^T z_Q + d^T \lambda^* \geq b^T y^* + d^T \lambda^* - \varepsilon \left( \alpha(y_0^*, b_0)||b_0||_2 ||y_0^*||_2 + 3 \left( \max_{i=1,\ldots,l} \frac{||M_i^*||_F}{M_i^*} \right) \sum_{i=1}^l ||b_i||_F ||M_i^*||_F \right).
\]

Hence, by Hypothesis 14(iv),

\[
v(\mathcal{D}_Q^*) \geq v(\mathcal{P}^K) - \varepsilon \left( \alpha(y_0^*, b_0)||b_0||_2 ||y_0^*||_2 + 3 \left( \max_{i=1,\ldots,l} \frac{||M_i^*||_F}{M_i^*} \right) \sum_{i=1}^l ||b_i||_F ||M_i^*||_F \right).
\]

We recall that \( y^* = (y_0^*, M_1^*, \ldots, M_l^*) \) and \( b = (b_0, b_1, \ldots, b_l) \). Hence

\[
||y^*||_2 = ||y_0^*||_2 + \sum_{i=1}^l ||M_i^*||_F
\]

\[
||b||_2 = ||b_0||_2 + \sum_{i=1}^l ||b_i||_F^2.
\]

Since

\[
||y_0^*||_2 ||b_0||_2 + \sum_{i=1}^l ||M_i^*||_F ||b_i||_F \leq ||y^*||_2 ||b||_2,
\]

we deduce by (46),

\[
v(\mathcal{D}_Q^*) \geq v(\mathcal{P}^K) - 3\varepsilon \max \left( \alpha(y_0^*, b_0), \max_{i=1,\ldots,l} \frac{||M_i^*||_F}{||M_i^*||_F} \right) ||y^*||_2 ||b||_2.
\]

Since \( ||y^*||_2 \leq ||(y^*, \lambda^*)||_2 \) and \( ||b||_2 \leq ||(b, d)||_2 \), we have that

\[
v(\mathcal{D}_Q^*) \geq v(\mathcal{P}^K) - 3\varepsilon \max \left( \alpha(y_0^*, b_0), \max_{i=1,\ldots,l} \frac{||M_i^*||_F}{||M_i^*||_F} \right) ||(y^*, \lambda^*)||_2 ||(b, d)||_2.
\]
Let us denote by $\beta \in [-\pi, \pi]$ the angle between $(b, d)$ and $(y^*, \lambda^*)$. That is, $\beta$ satisfies
\[
v(D^K) = b^\top y^* + d^\top \lambda^* = \cos(\beta) \|(y^*, \lambda^*)\|_2 \|(b, d)\|_2.
\]
By Hypothesis 14(vi) we have $\cos(\beta) \neq 0$. In addition, by Hypothesis 14(iv), we have $v(D^K) = v(P^K)$, therefore
\[
v(D^*_Q) \geq v(P^K) - 3\varepsilon \max \left(\alpha(y^*_0, b_0), \max_{i=1, \ldots, t} \frac{\|M^*_i\|_2}{\|M^*_i\|_F}\right) \frac{1}{\cos(\beta)} (b^\top y^* + d^\top \lambda^*)
= v(P^K) \left(1 - 3\varepsilon \frac{1}{\cos(\beta)} \max \left(\alpha(y^*_0, b_0), \max_{i=1, \ldots, t} \frac{\|M^*_i\|_2}{\|M^*_i\|_F}\right)\right).
\]
By weak duality we deduce that
\[
v(P^K) \left(1 - 3\varepsilon \frac{\varepsilon}{\cos(\beta)} \max \left(\alpha(y^*_0, b_0), \max_{i=1, \ldots, t} \frac{\|M^*_i\|_2}{\|M^*_i\|_F}\right)\right) \leq v(D^*_Q) \leq v(P^*_Q),
\]
where $P^*_Q$ denotes the dual of $D^*_Q$:
\[
P^*_Q \left\{ \min_{x} (c + E)^\top x \right. \\
Q(Ax - b) \in Q(K) \\
Bx - d \in K' \\
x \in \mathbb{R}^n, \right.
\]
where $E$ is defined as in (22).
By definition of $Q$, $P^*_Q$ is a relaxation of $P^K$, hence it is feasible. Let $x^*_Q$ be a feasible solution of $P^*_Q$ such that $c^\top x^*_Q - v(P^*_Q) \leq \varepsilon'$, where
\[
\varepsilon' \leq \varepsilon \max \left(\alpha(y^*_0, A^{(0)}), \max_{i=1, \ldots, t} \frac{\|M^*_i\|_2}{\|M^*_i\|_F}\right) \max \left(\frac{1}{\cos(\gamma_j)}\right) \frac{\|x^*_Q\|_2}{\cos(\theta)\|x^*_Q\|_2} v(P^K).
\]
Such a $x^*_Q$ exists since $P^*_Q$ is feasible and its minimum is bounded by Hypothesis 14(i). Putting such a solution in (48), we have that
\[
v(P^*_Q) \leq c^\top x^*_Q + E^\top x^*_Q.
\]
From $c^\top x^*_Q - v(P^*_Q) \leq \varepsilon'$ we deduce that
\[
v(P^*_Q) \leq v(P^*_Q) + E^\top x^*_Q + \varepsilon'.
\]
By Proposition 17, we have that
\[
|E| \leq \varepsilon \alpha(y^*_0, A^{(0)}) \left(\|A^{(0)}\|_2\|y^*_0\|_2\right) + 3\varepsilon \left(\max_{i=1, \ldots, l} \frac{\|M^*_i\|_2}{\|M^*_i\|_F}\right) \sum_{i=1}^l \|M^*_i\|_F \left(\frac{\|A^{(i)}\|_2}{\|A^{(i)}\|_F}\right).
\]
Hence

\[ |E| \leq 3\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, l} \frac{\|M_i^*\|_F}{\|M_i^*\|} \right) \left( \left\| \frac{A_1^{(0)} y_0^*}{\|A_1^{(0)} y_0^*\|_F} \right\|_2 \right) + \sum_{i=1}^l \|M_i^*\|_F \left( \frac{A_i^{(0)} y_0^*}{\|A_i^{(0)} y_0^*\|_F} \right), \]

Since for all \( j \in \{1, \ldots, n\} \), we have that

\[ \|A_j^{(0)}\|_2 \|y_0^*\|_2 + \sum_{i=1}^l \|M_i^*\|_F \|A_j^{(0)}\|_F \leq \left\| \left( \begin{array}{c} (A_j^{(0)}) \\ \vdots \\ (A_j^{(0)}) \end{array} \right) \right\|_2 \left\| \left( \begin{array}{c} y_0^* \\ \vdots \\ y_0^* \end{array} \right) \right\|_2, \]

we deduce that

\[ |E| \leq 3\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, l} \frac{\|M_i^*\|_F}{\|M_i^*\|} \right) \left( \left\| \frac{A_1^{(0)} y_0^*}{\|A_1^{(0)} y_0^*\|_F} \right\|_2 \right), \]

where \( A_j = \left( \begin{array}{c} (A_j^{(0)}) \\ \vdots \\ (A_j^{(0)}) \end{array} \right) \) is the \( j \)-th column of \( A \).

Consider the columns \( C_1, \ldots, C_n \) of the matrix \( \left( \begin{array}{c} A \\ B \end{array} \right) \) for (14). Since for all \( i \), \( \|A_i\|_2 \leq \|C_i\|_2 \), we have

\[ |E| \leq 3\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, l} \frac{\|M_i^*\|_F}{\|M_i^*\|} \right) \left( \left\| \frac{C_1^{(0)} (y^*, \lambda^*)}{\|C_1^{(0)} (y^*, \lambda^*)\|_F} \right\|_2 \right). \]

Let us consider for all \( j \in \{1, \ldots, n\} \) the angles \( \gamma_j \) between \((y^*, \lambda^*)\) and \( C_j \). Since \( C_j^T (y^*, \lambda^*) = A_j^T y^* + B_j^T \lambda^* = c_j \neq 0 \), we have that \( \cos(\gamma_j) \neq 0 \) for every \( j \). Hence we have

\[ |E| \leq 3\varepsilon \max_{j=1, \ldots, n} \left( \frac{1}{\cos(\gamma_j)} \right) \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, l} \frac{\|M_i^*\|_F}{\|M_i^*\|} \right) \left( \left\| \frac{C_1^{(0)} (y^*, \lambda^*)}{\|C_1^{(0)} (y^*, \lambda^*)\|_F} \right\|_2 \right), \]

\[ = 3\varepsilon \max_{j=1, \ldots, n} \left( \frac{1}{\cos(\gamma_j)} \right) \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, l} \frac{\|M_i^*\|_F}{\|M_i^*\|} \right) \left( \left\| \frac{c_1}{\|c_1\|} \right\|_2 \right). \]

(50)

(51)
Hence, we have
\[
|E^T x_Q^*| \leq \|x_Q^*\|_2 \|E\|_2 \leq 3\varepsilon \max \left( \frac{1}{\cos(\gamma_j)} \right) \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \|x_Q^*\|_2 \|c\|_2 \\
= 3\varepsilon \max \left( \frac{1}{\cos(\gamma_j)} \right) \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \|x_Q^*\|_2 \frac{c^T x^*}{\|x^*\|_2 \cos(\theta)}.
\]

where \(x^*\) is an optimal solution of \(v(P^K)\) and where \(\theta\) is the angle between \(c\) and \(x^*\). By Hypothesis 14(vi) we have \(\cos(\theta) \neq 0\).

Hence,
\[
v(P^L_Q) \leq v(P^L_S) + |E^T x_Q^*| + \varepsilon' \leq v(P^L_S) + 3\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{\|x_Q^*\|_2}{\cos(\theta)} v(P^K) + \varepsilon'.
\]

Now by (49) we have that
\[
v(P^L_S) \leq v(P^L_S) + 4\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{\|x_Q^*\|_2}{\cos(\theta)} v(P^K).
\]

Combining the inequality above with (47), we obtain
\[
v(P^K) \left( 1 - 3 \frac{\varepsilon}{\cos(\beta)} \max \left( \alpha(y_0^*, b_0), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \right) \leq v(P^L_Q) \leq v(P^L_S) \leq v(P^K)
\]
\[
\leq v(P^L_S) + 4\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{\|x_Q^*\|_2}{\cos(\theta)} v(P^K).
\]

Hence
\[
v(P^L_Q) \geq v(P^K) - 3 \frac{\varepsilon}{\cos(\beta)} \max \left( \alpha(y_0^*, b_0), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) v(P^K)

\]
\[
- 4\varepsilon \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{\|x_Q^*\|_2}{\cos(\theta)} v(P^K).
\]

Let \(\alpha(y_0^*, A^{(0)}, b_0) = \max(\alpha(y_0^*, A^{(0)}), \alpha(y_0^*, b_0))\). We have
\[
\max \left( \alpha(y_0^*, A^{(0)}, b_0), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) = \max \left( \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right), \max \left( \alpha(y_0^*, A^{(0)}), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \right),
\]

hence from (52), we obtain that
\[
v(P^L_Q) \geq v(P^K) \left( 1 - \varepsilon \max \left( \alpha(y_0^*, A^{(0)}, b_0), \max_{i=1, \ldots, d} \frac{\|M_i^*\|_*}{\|M_i^*\|_F} \right) \left( \frac{3}{\cos(\beta)} + 4 \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{\|x_Q^*\|_2}{\cos(\theta)} \right) \right),
\]

which finishes the proof.
5. The LP case

In this section we consider the case where we have a pure LP:

\[
\begin{align*}
P & \begin{cases}
   \min_z & c^\top x \\
   & Ax \geq b \\
   & Bx \geq d \\
   & x \in \mathbb{R}^n
\end{cases} \\
\end{align*}
\]

(53)

Under Hypothesis 14, we will show a version of Theorem 22 with a simplified bound.

5.1. A simplified error bound

The idea is to apply some transformations that preserve the optimal value of \( P \) and, then, use Theorem 22 on the transformed problem. First, let \( N \) be an invertible \( n \times n \) matrix and let \( P_N \) be the problem obtained by replacing \( A, B, c \) in \( P \) by \( AN, BN, N^\top c \).

We observe that the optimal value of \( P \) and \( P_N \) are the same. This is because the map \( x \mapsto N^{-1}x \) is a bijection between the sets of feasible solutions of \( P \) and \( P_N \). Furthermore, this map preserves the objective function value since \( c^\top x = (N^\top c)^\top N^{-1}x \).

With that mind we will now construct a specific matrix \( N \). We assume that \( A \) has full row rank \( n \) and without lost of generality, we may assume that the first \( n \) rows of \( A \) are linearly independent. Therefore, \( A \) can be divided in blocks as follows

\[
A = \begin{pmatrix} \hat{A} \\ \tilde{A} \end{pmatrix},
\]

where \( \hat{A} \) is an \( n \times n \) invertible matrix and \( \hat{A} \) is an \((m-n)\times n\) matrix. Let

\[
N = \hat{A}^{-1}.
\]

Hence

\[
AN^\eta = \begin{pmatrix} I_n \\ \hat{A}N \end{pmatrix}.
\]

(54)

This shows that every column of \( AN \) can have at most \( m-n+1 \) nonzero elements. Now, we recall that if a vector \( u \in \mathbb{R}^m \) has at most \( k \) elements, then \( \|u\|_1 \leq \sqrt{k}\|u\|_2 \), which is a consequence of the Cauchy-Schwarz inequality\(^4\).

Let \((AN)_j\) denote the \( j \)th column of \( AN^\eta \). By the preceding discussion we have

\[
\frac{||(AN)_j||_1}{||(AN)_j||_2} \leq \sqrt{m-n+1}, \quad \forall j \in \{1,\ldots,n\}.
\]

(55)

\(^4\)Let \( v \) be a vector such that \( v_i = 1, -1 \) or 0 if \( u_i \) is positive, negative, or null respectively. Then, \( \|u\|_1 = u^\top v \leq \|v\|_2 \|u\|_2 \leq \sqrt{k}\|u\|_2 \).
Next, we will consider the effect of shifting the constants \( b, d \) in \( \mathcal{P}^N \) using a vector \( v \). Let \( \mathcal{P}^{N,v} \) be the problem obtained by replacing \( b, d \) by \( b - ANv, d - ANv \). Then, assuming that \( (N^T c)^\top v = 0 \), we have the optimal values of \( \mathcal{P}, \mathcal{P}^N \) and \( \mathcal{P}^{N,v} \) all coincide: the map \( x \mapsto N^{-1}x - v \) is a bijection between the sets of feasible solutions of \( \mathcal{P} \) and \( \mathcal{P}^{N,v} \). Furthermore, this map preserves the objective function value since \( c^\top x = (N^T c)^\top (N^{-1}x - v) \).

We now select a specific vector \( v \). By adding a small random perturbation to \( c \) we can assume w.l.o.g. that all the components of \( N^T c \) are non zeros. In particular, since \( (N^T c)_1 \neq 0 \), the matrix \( I_c \) obtained by replacing the first row of \( I_n \) by \( (N^T c)^\top \) is still invertible. Let \( v \) be the (unique) solution satisfying

\[
I_c v = (0, b_2, \ldots, b_n). \tag{56}
\]

In view of (54), we have that for all \( j \leq n \), \((ANv)_j = v_j \). Furthermore, by (56), we have that for all \( 2 \leq j \leq n \), \( v_j = b_j \). Hence for all \( 2 \leq j \leq n \), \((ANv)_j = b_j \).

Hence, \( v \) has the property that \((N^T c)^\top v = 0 \) and \((b - ANv)_j = 0 \) for \( j = 2, \ldots, n \). Therefore, \( b - ANv \) has at most \( m - n + 1 \) nonzero elements and we have the bound

\[
\frac{\|b - ANv\|_1}{\|b - ANv\|_2} \leq \sqrt{m - n + 1}. \tag{57}
\]

We recall that we also have that

\[
\frac{\|(AN)\|_1}{\|(AN)\|_2} \leq \sqrt{m - n + 1}, \quad \forall j \in \{1, \ldots, n\}. \tag{58}
\]

We now have all the pieces to prove the following result.

**Proposition 23.** Consider problem the \( \mathcal{P} \) in (53), where it is assumed that \( A \) has full rank. Let \( \varepsilon, \delta, m \) be such that \( 0 < \delta < \frac{1}{8}, \ 0 \leq \varepsilon \leq 1 \) and \( m \geq 2 \left( \frac{\varepsilon^3}{\gamma_{17}} \right) (3k + \ln(n) - \ln(\delta)) \). With probability at least \( 1 - 8\delta - (8m^2 + 4m)(n + 1)\exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) \), we have

\[
v(\mathcal{P}) \left( 1 - 16\varepsilon \sqrt{m - n + 1} \left( \frac{\|y\|_1}{\|y\|_2} \max \frac{|y_i^*|}{\sqrt{\min\{|y_i^*|^\top\}}}(1 + \varepsilon) + 1 \right) \left( \max \left( \frac{1}{\cos(\gamma_j)} \right) \frac{4\|x^*_2\|_2}{\cos(\theta)\|x^*\|_2} + \frac{3}{\cos(\beta)} \right) \right) \leq v(\mathcal{P}^S) \leq v(\mathcal{P}),
\]

where

- \( N \) and \( v \) are such that (57) and (58) hold.
- \( x^*, (y^*, \lambda^*) \) are optimal solutions of \( \mathcal{P}^{N,v} \) and \( \mathcal{D}^{N,v} \) (the dual of \( \mathcal{P}^{N,v} \)), respectively.
- \( \beta \) is the angle between \( (b - ANv, d - ANv) \) and \( (y^*, \lambda^*) \), \( \gamma_j \) is the angle between \( (y^*, \lambda^*) \) and the \( j \)th column of the matrix \( \begin{pmatrix} A^\top \\ BN \end{pmatrix} \).
- \( \theta \) is the angle between \( N^T c \) and \( x^* \),
Proof. By the preceding discussion, the optimal values of $\mathcal{P}$ and $\mathcal{P}_S$ are equal to the optimal values of the transformed problems $\mathcal{P}^{N,v}$, $\mathcal{P}_S^{N,v}$, respectively. With that in mind, we apply Theorem 22 to $\mathcal{P}^{N,v}$.

Notice that in the LP case an optimal solution exists (since the optimal value is finite), hence we can take an optimal solution of the projected problem for $x_Q^*$. To prove the proposition, all we need to do is to bound all the terms $\|(A_N)^j\|_1/\|(A_N)^j\|_2$ for $j \in \{1, \cdots, n\}$, and $\|b\|_2$, that appear in $\alpha(y^*, AN)$ and $\alpha(y^*, b - ANv)$ (see (29), (30)) in Theorem 22 by $\sqrt{m - n + 1}$. These bounds follow from (55) and (57).

Next we consider a special case of $\mathcal{P}$ where $d = 0$ and $B = I_n$, and $c > 0$.

$$\mathcal{P}_{\geq} \begin{cases} \min_x c^T x \\ Ax \geq b \\ x \geq 0 \end{cases} \quad \mathcal{P}_S^{\geq} \begin{cases} \min_x c^T x \\ SAx \geq Sb \\ x \geq 0 \end{cases}$$

We will prove, by slightly modifying the proof of Theorem 22, that we can obtain a bound in this case where the term, $\frac{4\|x_Q\|_2}{\cos(\theta)\|x^*\|_2}$, does not appear in the approximation ratio. We have the following theorem:

**Theorem 24.** Let $\varepsilon, \delta, m$ be such that $0 < \delta < \frac{1}{4}$, $0 < \varepsilon \leq 1$ and $m \geq \frac{2^8}{\varepsilon^2}(3k + \ln(n) - \ln(\delta))$. With probability at least $1 - 8\delta - (8m^2 + 4m)(n + 1)\exp(-k/2(\varepsilon^2/2 - \varepsilon^4/3))$, we have:

$$v(\mathcal{P}_{\geq}) \left(1 - 48 \max \left(\frac{\max_{1 \leq j \leq n} \|A_j\|_1}{\|A\|_2}, \frac{\|b\|_1}{\|b\|_2}\right) \frac{\max \|y^*\|_1}{\|y^*\|_2} k \min \|y^*_j\|(1 + \varepsilon) + 1 \right) \left(\max \left(\frac{1}{\cos(\gamma_j)}\right) + \frac{1}{\cos(\beta)}\right) \leq v(\mathcal{P}_S^{\geq}) \leq v(\mathcal{P}_{\geq}),$$

where

- $x^*, (y^*, \lambda^*)$ are optimal solutions of $\mathcal{P}_{\geq}$ and $\mathcal{D}_{\geq}$, respectively,
- $\beta$ is the angle between $(b, 0)$ and $(y^*, \lambda^*)$, $\gamma_j$ is the angle between $(y^*, \lambda^*)$ and the $j$th column of the matrix $\left(\frac{A}{I_n}\right)$

**Proof.** The proof is basically the same as in Theorem 22:

We define as in Theorem 22

$$\mathcal{P}_S^{\geq} \begin{cases} \min_x (c + E)^T x \\ SAx \geq Sb \\ x \geq 0 \end{cases}$$

(59)

where $E = A^T(S^T Sy^* - y^*)$, since we do not have SDP terms anymore. As in (47), we have that

$$v(\mathcal{P}_{\geq}) \left(1 - \frac{\varepsilon}{\cos(\beta)}\right) \alpha(y^*, b) \leq v(\mathcal{P}_S^{\geq}).$$
where
\[ \alpha(y^*, b) = 16 \frac{\|b\|_1}{\|b\|_2} \left( \frac{\|y^*\|_1 \max |y_i^*|}{\|y^*\|_2 \min |y_i^*|} (1 + \varepsilon) + 1 \right). \]

Hence
\[ v(P_{\ge}) \left( 1 - 10 \frac{\|b\|_1}{\|b\|_2 \cos(\beta)} \left( \frac{\|y^*\|_1 \max |y_i^*|}{\|y^*\|_2 \min |y_i^*|} (1 + \varepsilon) + 1 \right) \right) \leq v(P_S^{\ge}) \tag{60} \]

Now, using the fact that \( c > 0 \), we have, as in (50) that
\[ |E| \leq \varepsilon \max \left( \frac{1}{\cos(\gamma_j)} \right) (3\alpha(y^*, A)) (c_1 \cdots c_n), \]

hence since \( x_Q^* \geq 0 \), we obtain that
\[ |E^T x^*| \leq \varepsilon \max \left( \frac{1}{\cos(\gamma_j)} \right) (3\alpha(y^*, A)) c^T x_Q^*, \]

where \( x_Q^* \) is an optimal solution of \( P_S^{\ge} \). Hence
\[ |E^T x^*| \leq 16\varepsilon \max_{1 \leq j \leq n} \frac{\|A_j\|_1}{\|A_j\|_2} \left( \frac{\|y^*\|_1 \max |y_i^*|}{\|y^*\|_2 \min |y_i^*|} (1 + \varepsilon) + 1 \right) 3 \max \left( \frac{1}{\cos(\gamma_j)} \right) v(P_S^{\ge}). \tag{61} \]

Since
\[ v(P_S^\ge) \leq v(P_S^\ge) + |E^T x_Q^*|, \]

and that \( v(P_S^\ge) \leq v(P_{\ge}) \), we obtain the theorem by combining (60) and (61).

5.2. Interpretation of the error bound

Theorem 24 shows that the approximation ratio \( R = \frac{v(P_{\ge}) - v(P_S^\ge)}{v(P_{\ge})} \) is given by
\[ R = O \left( \varepsilon \sigma \left( \frac{\sqrt{n}}{k} + 1 \right) \max \left( \frac{1}{\cos(\gamma_j)} \right) \right), \]

where \( \sigma \geq \max \left( \frac{\max_{1 \leq j \leq n} \|A_j\|_1}{\|A_j\|_2}, \frac{\|b\|_1}{\|b\|_2} \right) \), as \( \frac{\|y^*\|_1}{\|y^*\|_2} \leq \sqrt{n} \) since \( y^* \) can be chosen to have at most \( n \) non-zero components, and as the other terms in the error bound can be interpreted as constants that do not depend on the dimension \( m, n \) of the problem. Notice that \( \sigma \) is a bound on the sparsity of the columns of \( A \) and the vector \( b \). Furthermore, the probability bound \( 1 - 8\delta - (8m^2 + 4m)(n + 1) \exp(-k/2(\varepsilon^2/2 - \varepsilon^3/3)) \) can be made as close to 1 as we want by choosing \( k = O(\log(m)) \). Let \( \gamma^* \in \arg \min |\cos(\gamma_j)| \), we obtain that
\[ R = \frac{v(P_{\ge}) - v(P_S^\ge)}{v(P_{\ge})} = O \left( \varepsilon \frac{\sigma}{\cos(\gamma^*)} \left( \varepsilon^2 \frac{\sqrt{n}}{\log(m)} + 1 \right) \right). \]
Let us take
$$
\varepsilon = O \left(\sqrt{\frac{\log(m)}{n^{1/2-\alpha}}}\right),
$$
for $0 \leq \alpha \leq \frac{1}{2}$. Then $\varepsilon^2 \frac{\sqrt{\sigma^2 \log(m)}}{n^{1/2-\alpha/2} \cos(\gamma^*)} = O(n^\alpha)$, which implies that
$$
R = O \left(\frac{n^\alpha \sqrt{\sigma^2 \log(m)}}{n^{1/4-\alpha/2} \cos(\gamma^*)}\right) = O \left(\frac{n^{3\alpha/2-1/4} \sqrt{\sigma^2 \log(m)}}{\cos(\gamma^*)}\right).
$$

This suggests that taking $\varepsilon = O \left(\sqrt{\frac{\log(m)}{n^{1/2-\alpha}}}\right)$ allows us to obtain a ratio $R$ if
$$
R \cos(\gamma^*) \geq C n^{3\alpha/2-1/4} \sqrt{\log(m) \sigma^2},
$$
for some constant $C$.

For such $\varepsilon$, we have $k = O \left(\frac{\log(m)}{\varepsilon^2}\right) = O(n^{1/2-\alpha})$. Furthermore, the condition
$$
m \geq \frac{2^8}{C_1 \varepsilon^2} (3k + \log(n) - \ln(\delta))
$$
implies that
$$
m \geq C' \left(\frac{n^{1-2\alpha}}{\log(m)} + \log(n) - \log(\delta)\right),
$$
for some constant $C'$, which holds, for all $0 \leq \alpha \leq 1/2$ as long as $m, n$ are large enough.

The above discussion also suggests that the error bound decreases when the columns of $A$ and the vector $b$ are sparse.

### 6. Numerical results

In this section, we present some preliminary numerical experiments where we generate random instances and we solve both the original formulation and the smaller reduced version. A difficulty in performing these experiments is that, although for $0 < \varepsilon < 1$ fixed, the bound $m \geq \frac{2^8}{C_1 \varepsilon^2} (3k + \ln(m) - \ln(\delta))$ is always satisfied for sufficiently large $m$, such $m$ is too large for the computer we are using to solve the original LP. Nevertheless we still perform some experiments on the pure LP (as in Section 5) with $m$ up to 20000. All results have been obtained using Gurobi called through Julia [20] and the JuMP [21] interface. The specs of the machine are as follows: Intel Core i5 at 3.8GHz with 8 GB of DDR4 RAM.

#### 6.1. Random instances

The random instances considered here are all feasible: $c$ is the all ones vector, $A$ is a random matrix build either from the uniform or the normal distribution, $b = Ax_0 - \eta$ where both $x_0$ and $\eta$ are random positive vectors and
$\{ x \mid Bx - d \in \mathcal{K} \}$ is just the non-negative orthant. The results are summarized in the tables below, “$m$” denotes the number of constraints, “$n$” the number of variables, “$k$” the number of constraints of the projected problem (computed for $\varepsilon = 0.2$), “$d$” the density of matrix $A$, “law” is the probability law used to generate the coefficient of $A$. Here $U(a, b)$ denotes the uniform law in the interval $[a, b]$ and $N(a, b)$ denotes the normal law of mean $a$ and standard deviation $b$. Each line of the table is the average over 10 instances generated with the same $m$, $n$, $d$, law. Furthermore, “meantorg” is the average time to solve the original LP, “stdtorg” is the corresponding standard deviation, “meantproj” and “stdtproj” are respectively the average time and the standard deviation to solve the projected problem, “meanratio” is the average error ratio $\frac{v(P) - v(P_S)}{v(P)}$ and “stdratio” is the corresponding standard deviation.
## Table 2: Numerical results for \(d = 0.1\)

The values where meanratio < 0.2 are written in boldface.
Table 3: Numerical results when the standard deviation of the law changes ($\sigma = 1$)

| m        | n        | k  | law (d) | meanratio | stdratio | meanproj | stdproj | meanratio | stdratio |
|----------|----------|----|---------|-----------|----------|----------|---------|-----------|----------|
| 5000     | 344      | 5000| U(0,1)  | 5.11E-01  | 1.53E+04 | 3.50E-02 | 1.87E-02 | 4.19E-07  |
| 5000     | 344      | 5000| U(0,10) | 1.21E+02  | 1.53E+04 | 1.02E-01 | 3.53E-03 | 1.76E-02  | 3.63E-07 |
| 5000     | 344      | 5000| U(0,50) | 1.11E+02  | 1.53E+04 | 1.02E-01 | 2.22E-04 | 1.80E-02  | 5.42E-07 |
| 5000     | 344      | 5000| U(0,100)| 1.21E+02  | 1.53E+04 | 1.02E-01 | 1.07E-04 | 1.79E-02  | 1.93E-07 |
| 5000     | 344      | 5000| U(0,500)| 8.15E+01  | 1.53E+04 | 1.02E-01 | 9.34E-03 | 1.82E-02  | 3.30E-07 |
| 5000     | 344      | 5000| U(0,1000)| 8.97E+01  | 1.53E+04 | 1.02E-01 | 1.00E-02 | 1.77E-02  | 4.90E-07 |
| 5000     | 344      | 5000| N(0,1)  | 8.59E+01  | 1.19E+04 | 1.02E-01 | 1.24E-03 | 9.94E-01  | 4.88E-05 |
| 5000     | 344      | 5000| N(0,10) | 5.38E+01  | 5.42E+00 | 1.02E-01 | 7.57E-05 | 9.71E-01  | 5.10E-05 |
| 5000     | 344      | 5000| N(0,50) | 5.38E+01  | 1.65E-00 | 1.02E-01 | 1.02E-02 | 9.76E-01  | 2.94E-05 |
| 5000     | 344      | 5000| N(0,100)| 5.38E+01  | 1.65E-00 | 1.02E-01 | 1.85E-04 | 9.73E-01  | 2.03E-05 |
| 5000     | 344      | 5000| N(0,500)| 1.04E+02  | 1.91E+04 | 1.02E-01 | 8.15E-04 | 9.69E-01  | 3.49E-05 |
| 5000     | 344      | 5000| N(0,1000)| 8.58E+02  | 8.21E+03 | 9.30E+01 | 1.85E-01 | 1.42E-02  | 2.21E-07 |
| 10000    | 7000     | 400 | U(0,10)| 8.62E+02  | 7.36E+02 | 9.29E+01 | 5.14E-02 | 1.42E-02  | 2.95E-07 |
| 10000    | 7000     | 400 | U(0,100)| 1.06E+03  | 2.60E+03 | 9.31E+01 | 1.02E-01 | 1.42E-02  | 2.43E-07 |
| 10000    | 7000     | 400 | U(0,1000)| 1.23E+03  | 4.87E+03 | 9.33E+01 | 6.28E-02 | 1.43E-02  | 3.63E-07 |
| 10000    | 7000     | 400 | N(0,10)| 7.15E+02  | 2.94E+02 | 9.33E+01 | 9.72E-02 | 9.85E-01  | 7.17E-06 |
| 10000    | 7000     | 400 | N(0,100)| 7.97E+02  | 5.36E+02 | 9.33E+01 | 1.30E-01 | 9.83E-01  | 2.23E-05 |
| 10000    | 7000     | 400 | N(0,1000)| 8.52E+02  | 1.49E+03 | 9.32E+01 | 6.74E-02 | 9.79E-01  | 1.82E-05 |

First we notice, from Table 2, that the time to solve the projected LP is always shorter than the time to solve the original LP, and furthermore as the number of constraints, $m$, of the original LP increases the gap between the two times increases drastically. Concerning the average error ratio between the values of the two problems, the only striking observation we can make from Table 2 is that the projected problem approximates the original well when the probability law, used to generate the coefficients of the constraint matrix $A$, does not have 0 in expectation. Such ratio are written in boldface in Table 2. They correspond to the case when $\text{meanratio} < 0.2$.

In Table 3 we see how the standard deviation of the probability law affects the ratio. We see that in both cases (the expectation of the probability law is equal to 0 or different from 0) an increase of the standard deviation seems to decrease a little the error ratio.

Next we show, for $m$ fixed, the influence of $n$ on the value of the approximation ratio. Figure 1 shows how the value of the approximation ratio changes, in the case of an uniform law, in function of $n$ for $m$ fixed. Although the approximation ratio is catastrophic, in the case of a normal law, Figure 2 shows how
the value of the approximation ratio changes in function of $n$ for $m$ fixed. As suggested by Theorem 23, we notice that the error bound gets better when $n$ increases for $m$ fixed. Finally we plot the ratio $\frac{\text{meantorg}}{\text{meantproj}}$ in function of $n$ for $m$ fixed in Figure 3. We see that, both in the normal case and in the uniform case the ratio tends to get smaller as $n$ and $m$ increase.

6.2. Real instances

We now discuss numerical results on 3 real instances taken from the Hans Mittelmann collection (http://plato.asu.edu/ftp/lptestset/). All these instances are LP relaxations from Integer Linear Problem (ILP) taken from [22].

The first instance we consider, a2864-99blp, is the LP-relaxation of a clique problem. The problem has $n = 200787$ variables all belonging to $[0, 1]$ and $m = 22117$ constraints. It has a nonzero density equal to $4.52 \times 10^{-3}$ and is originally solved in 1056.23 seconds. Using $\varepsilon = 0.2$ the projected problem has $k = 452$ constraints and was solved in 103.82 seconds for an approximation ratio equal to $8.87 \times 10^{-2}$.

The second instance we consider, rmine15 comes from open pit mining over a cube. The problem has $n = 42438$ variables all belonging to $[0, 1]$ and $m = 358395$ constraints. It has a nonzero density equal to $5.78407 \times 10^{-5}$ and is originally solved in 623.20 seconds. Using $\varepsilon = 0.2$ the projected problem has $k = 577$ constraints and was solved in 18.20 seconds for an approximation ratio equal to $7.72 \times 10^{-1}$.

The third instance scpml1 has $n = 500000$ variables all belonging to $[0, 1]$ and $m = 5000$ constraints. It has a nonzero density equal to $2.50 \times 10^{-3}$ and is originally solved in 25.65 seconds. Using $\varepsilon = 0.2$ the projected problem has
$k = 385$ constraints and has been solved in 537.10 seconds for an approximation ratio equal to $3.79 \times 10^{-1}$. For this instance we notice that it takes much more time to solve the projected problem rather than the original one. One possible explanation is that since the projection matrix is not sparse, the projected problem lose all its sparsity pattern after random projections, which may lead to a greater solving time despite having fewer constraints. Furthermore, the ratio between the number of constraints of the original problem and the projected problem is quite small, compared to the other cases. This might explain why, for this case, reducing the number of constraints did not reduce the solving time.

7. Conclusion and future works

In this paper we applied random projection to reduce the number of constraints of linear optimization problems over a cone $K$ which is a product of the non-negative orthant and semidefinite cones, i.e., $K = \mathbb{R}_+^n \times S_p^1 \times \cdots \times S_p^l$. We considered a random projection matrix $Q$ such that $Q(K)$ is also a product of the non-negative orthant and semidefinite cones, each having smaller dimension. Under some conditions on the original problem, we could obtain some theoretical guarantees on the value of the projected problem.

One possible future work would be to consider sparse projection matrices to further reduce the time to solve the projected problem. One such approach would be to consider Johnson-Lindenstrauss transform:

**Definition 25.** A random $k \times l$ matrix $T$ is a Johnson-Lindenstrauss transform (JLT) with parameters $(\varepsilon, \delta, h)$ if with probability at least $1-\delta$, for any $h$-element subset $Z \subset \mathbb{R}^l$, for all $z_1, z_2 \in Z$ we have

$$
(1-\varepsilon)\|z_1 - z_2\|_2^2 \leq \|Tz_1 - Tz_2\|_2^2 \leq (1+\varepsilon)\|z_1 - z_2\|_2^2.
$$

By Lemma 9, Gaussian random matrices are JLT. More generally, in [23], it is proven that random matrices with independent sub-Gaussian entries are also JLT. In [24, Section 5.1], it is proven that sparse Gaussian matrices, where each entry is non-zero with probability $1-\gamma$, are JLT. However to generalize the approach presented in this paper to JLTs we would need a more general version of Lemma 12.

Another topic for further research is the case of linear conic problems where the underlying cone $K$ is only assumed to be symmetric. This would, for example, allow us to consider second-order cone constraints or to deal more effectively with the case where there is a large number of small-dimensional SDP constraints. However, such generalizations would require more sophisticated tools dealing with random matrices over more general algebras.

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