Network topology: detecting topological phase transitions in the Kitaev chain and the rotor plane

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We propose a novel network measure of topological invariants, called small-worldness, for identifying topological phase transitions of quantum and classical spin models. Small-worldness is usually defined in the study of social networks based on the best known discovery that one can find a short chain of acquaintances connecting almost any two people on the planet. Here we demonstrate that the small-world effect provides a useful description to distinguish topologically trivial and non-trivial phases in the Kitaev chain and accurately capture the Kosterlitz-Thouless transition in the rotor plane. Our results further suggest that the small-worldness containing both locality and non-locality of the network topology can be a practical approach to extract characteristic quantities of topological states of matter.

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I. INTRODUCTION

Determining the topological order of an interacting quantum system from its microscopic many-body entanglement is one of the recent goals of condensed matter theory. Traditional phases of matter and phase transitions are usually distinguished by local order parameters. Consider, for instance, a second-order phase transition, the critical point is accompanied by a diverging correlation length in Landau’s symmetry breaking framework \cite{1}. However, it becomes clear that some topologically ordered phases do not fall into this framework, such as fractional quantum Hall states \cite{2}, quantum spin liquids \cite{3} and recently discovered topological insulators \cite{4,5}. Moreover, characterizing topological phase transitions between them is a difficult task due to the absence of local order parameters \cite{6,7}. There are recent advances in diagnosing the presence of topological order from knowledge of many-body ground states, according to entanglement entropy \cite{8,9}, entanglement spectrum \cite{10,11}, modular transformation \cite{12}, \textit{Z}_2 topological invariant \cite{13,14,15} and momentum polarization \cite{16}. They all have brought us closer to being able to tackle such important questions.

Another interesting question is the nature of phase transitions between topologically ordered states. Some of those methods mentioned above to identify a topological phase are usually not unique when several topological phases can have the same topological invariants \cite{17}. Furthermore, these methods cannot provide more complete information about all phase transitions incorporating traditional Landau’s symmetry breaking picture. To address this issue, we attempt to map a physical model to a network possibly offers an alternative understanding of the essence of phase transitions in condensed matter physics.

The complex network theory originating from the graph theory in mathematics has become one standard tool to analyze the structure and dynamics of real-world systems, which consist of overwhelming information \cite{18,22}. The building blocks of the complex network include nodes (system elements) and links (the relation between two elements). Via the unique patterns of connections, the essential features of a collection of interacting elements can be unveiled by the direct visualization and the topological analysis. Its application is prevailing in many fields, such as sociology, biology, informatics and many other interdisciplinary studies \cite{23}. Thus, this generic description is reasonably applicable to condensed matter systems.

A naive question in the condensed matter theory and the application of network analysis is whether or not the topological phase transitions can be detected by complex network topology. Out of many properties in real-world networks, the small-world effect is a common phenomenon, which is characterized by a small average length of the shortest paths between two nodes. The characteristics of strong clustering and shortest path length, proposed by Watts and Strogatz decades ago \cite{24}, establishes enough long-range connections in network space. In light of extensive studies in various real-world networks \cite{25,28}, it will be interesting to explore the point of view of the small-world effect in many-body systems.

The focus of the present paper is to construct the weighted networks for the Kitaev chain and the two-dimensional (2D) classical rotor model. The weights of network links we define carry quantum and classical correlations between lattice sites in the Kitaev chain and the rotor plane, respectively. An interesting observation is that the two topologically distinct phases in the Kitaev chain can be distinguished by the novel network topology instead of typical topological invariants. The topology arising from the weighted networks is illustrated by the small-world phenomenon. Using the network property in the rotor plane, we find that it can also extract the Kosterlitz-Thouless (KT) transition point obtained by conventional quantities, \textit{e}., spin stiffness. In order to further quantify the small-world network, we propose a network quantity, small-worldness, as an order parameter in these two models. The small-worldness exhibits an obvious change while these systems undergo a topological phase transition. These phases of matter are encoded in the network representation, from which we can extract the universal properties of the topolog-

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II. NETWORK MEASURES

In the language of complex network, each network of \( N \) nodes is described by its \( N \times N \) adjacency matrix representation \( \hat{A} \) [3]. A number of real systems, e.g. transportation networks, neural networks and so on, are better captured by the weighted network in which the links use weights to quantify their strengths. In these two models, we consider lattice sites as nodes of the weighted network, with each weighted link between nodes \( i \) and \( j \) expressed by the element of the adjacency matrix \( \hat{A}_{ij} \). The link carries the weight containing detailed information about the relationship between particles at different sites. Thus the weights of network links are assigned by quantum and classical correlations between particles in the Kitaev chain and the rotor plane, respectively. Note that the symmetric adjacency matrix \( \hat{A}_{ij} \) in our examples only has real entries. Following the convention in weighted complex networks [29], here we take the absolute value of \( \hat{A}_{ij} \) normalized by the maximum weight in the network, \( \frac{\hat{A}_{ij}}{\max |\hat{A}_{ij}|} \).

Two concepts, the clustering coefficient and the path length, play a key role in the development of network science in the last decades. They can be easily evaluated from the adjacency matrix \( \hat{A}_{ij} \). The former refers to the local property of a network. More precisely, if the neighbors of a given node connect to each other, a local cluster will be formed in an unweighted network. As for a weighted network, the degree of clustering of the network can be captured by the weighted clustering coefficient \( C \) [30], given by

\[
C = \frac{1}{N} \sum_{i} \left[ \left( \frac{\hat{A}^{(1/3)}}{k_i(k_i - 1)} \right)^3 \right]_{ii}.
\]

Here \( \hat{A}^{(n)} \) is a matrix obtained from \( \hat{A} \) by taking the n-th root of its individual elements. \( k_i(= \sum_{j=1}^{N} \hat{A}_{ij}) \) represents the node strength for each node \( i \). Notice that according to the definition of the adjacency matrix, the unweighted and weighted clustering coefficients are equivalent as the weights become binary. Thus one can regard Eq. (1) as the probability with two neighbors of a randomly selected node linking to each other.

We now turn to another fundamental concept in graph theory which is the shortest path of nodes. Physical distance in a network is usually irrelevant, and should be replaced by the path length. A path is a route that runs along the links of a network. The path length is thus defined by the inverse of the link weights the path contains. This definition seizes the intuitive idea that strongly coupled nodes are close to each other. It is noteworthy, then, that the path length can represent the non-local property of a network. Hence the path length \( D \) is the average of the shortest path lengths in a network [31], defined as

\[
D = \frac{2}{N(N-1)} \sum_{i<j} \hat{d}_{ij},
\]

where \( \hat{d}_{ij} \) is the sum of \( \hat{A}_{\mu
u}^{-1} \) along the shortest weighted path including nodes \( \mu \) and \( \nu \) between nodes \( i \) and \( j \). Here we compute the matrix \( \hat{d}_{ij} \) by using the well-known graph search algorithm which is called Dijkstra’s algorithm [32].

In most real networks, a majority of nodes may not be neighbors but can reach each other by a small number of steps which means relatively small path length. This is called small-world phenomenon [33]. However, although the random network also shows the small-world effect, it still fails to reproduce some important features of real networks, such as clustering. A small-world network including not only high clustering but also short path length has thus been introduced to describe many real networks by Watts and Strogatz [24]. A network measure of the small-world property called "small-worldness" has been proposed as well [34]. The definition is based on the maximal tradeoff between high clustering (large \( C \)) and short path length (small \( D \)). We can further define the small-worldness as

\[
S = \frac{C}{D}.
\]

A network with larger \( S \) has a higher small-world level [35]. If a network is complete, i.e. all nodes are connected with equal link weights, both \( C \) and \( D \) will approach 1, and then \( S \rightarrow 1 \) (which means an extremely small world). Therefore, the small-worldness can simultaneously contain the local and non-local properties of a given network topology. Later we will show that the small-worldness behaves as an order parameter in the topological phase transitions.

III. KITAEV CHAIN

We start off with one-dimensional (1D) quantum Ising model in a chain of length \( L \) with periodic boundaries [1]. The Ising Hamiltonian with transverse magnetic field is written as

\[
H_{\text{Ising}} = -g \sum_{i} \hat{\sigma}_{i}^{x} - \sum_{i} \hat{\sigma}_{i}^{x} \cdot \hat{\sigma}_{i+1}^{x},
\]

where \( \hat{\sigma}_{i}^{x} \) and \( \hat{\sigma}_{i}^{z} \) are the usual Pauli matrices and \( g \) represents a dimensionless magnetic field. In the Ising model, the \( Z_{2} \) spin reflection symmetry is spontaneously broken while the quantum phase transition takes place at the critical field \( (g_{c} = 1) \). For much larger magnetic field \( g \), the ground state is a quantum paramagnet with all spins polarized along the field, whereas for small \( g \), there are two degenerate ferromagnetic ground states with all spins pointing either "up" or "down" perpendicular to the magnetic field.
The 1D quantum Ising model can be re-written in terms of spinless fermion by using the Jordan-Wigner transformation \[ \hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_i^\dagger \hat{c}_{i+1} + H.c. + \mu \hat{c}_i^\dagger \hat{c}_i, \] where $\mu (= 2g)$ is chemical potential. Hereafter the particle-hole symmetry allows us to consider only the case $\mu \geq 0$. The simplest superconducting (SC) model system shows the two-fold ground-state degeneracy stemming from an unpaired Majorana fermion at the end of the chain with open boundary conditions (OBC). A characteristic feature of the topological order is thus encoded in the Majorana zero mode \[ G_{ij}. \] Kitaev showed that this model has two phases sharing the same physical symmetries: a topologically trivial phase for $\mu > 2$ and a topologically non-trivial phase for $\mu < 2$ \[ 14]. The transition between them is the topological phase transition identified by the presence or absence of unpaired Majorana fermions localized at each end.

The Hamiltonian in momentum space is quadratic of fermionic operators $\hat{c}_k$, which has the form:

\[
\sum_k \left( \hat{c}_k^\dagger \hat{c}_{-k} \right) \left( \begin{array}{cc}
-\frac{\mu}{2} - \cos k & -i \sin k \\
i \sin k & \frac{\mu}{2} + \cos k
\end{array} \right) \left( \begin{array}{c}
\hat{c}_k \\
\hat{c}_{-k}^\dagger
\end{array} \right).
\] (6)

Note that the periodic boundaries of the spin chain become anti-periodic boundary condition for the spinless fermion. By using the standard Bogoliubov transformation, $\gamma_k = \cos (\theta_k/2) \hat{c}_k - i \sin (\theta_k/2) \hat{c}_{-k}^\dagger$ where $\tan \theta_k = - \sin k/(\frac{\mu}{2} + \cos k)$, Eq. (6) can be diagonalized. The excitation spectrum of the form, $E_k = \sqrt{2(2\cos k + \mu)^2 + \sin^2 k}$, remains fully gapped except for the critical point $\mu_c (\approx 2)$. The SC ground state is the state annihilated by all $\gamma_k$ \[ 38 \], given by

\[
|\Psi_{GS}\rangle = e^{\frac{i}{2} \sum_{i,j} G_{ij} \hat{c}_i^\dagger \hat{c}_j} |0\rangle.
\] (7)

$G_{ij}$ represents the pairing amplitude defined by the Fourier transform of $\tan(\theta_k/2)$. It has been proven that for the Kitaev chain the reduced density matrices can be determined from the properties of the pairing amplitude \[ 39 \]. Thus the link weights of the Kitaev-chain network are assigned by the pairing amplitude in which the non-local property between spinless fermions is concealed.

The Kitaev chain admits gapless excitations only when the Fermi level coincides with the top of the conduction band ($\mu = \mu_c$). The two gapped phases are intuitively different in the regimes with $\mu < \mu_c$ and $\mu > \mu_c$ owing to closing the excitation gap at the critical point. Figure 1(a)-(c) show network topologies at different chemical potential $\mu$ in the Kitaev chain. Below the critical point $\mu_c$, the SC ground state corresponds to a weak pairing regime in which the size of the Cooper pair is infinite. For $\mu = 0$, which gives rise to the unpaired Majorana fermions at each end of the chain with OBC, the recombination of the unpaired Majorana fermions generates an ordinary fermion with a highly non-local property. We here observe a trivial complete network at $\mu = 0$, where each node is connected to all other nodes with equal link weights. As increasing $\mu$, the Majorana end states decay exponentially into the bulk of the chain and the Cooper pair size seems to be slowly reduced. The topologically non-trivial phase thus displays irregular patterns of the network shown in Fig 1(a).

At the critical point, the critical phase has power-law correlations at large distances. Interestingly, we find that the nodes with the largest link weight begin to form a "chain-like" structure in Fig 1(b). The obvious change of the topology of the network is intimately related to the critical behavior observed in real space. Above the critical point, i.e., $\mu > \mu_c$, the ground state is instead in a strong pairing regime where the pairing amplitude is exponentially decaying with distances. The Cooper pairs form molecules from two fermions bound in real space over a length scale. Exponentially decaying pairing amplitude in real space results in the strongest links between neighboring nodes in network space. In Fig 1(c), the topologically trivial phase demonstrates that a clear ring structure comprised of the nodes with the largest link weight emerges in the network pond. Similar physics would appear in the well-known "BEC-BCS crossover" in $s$-wave superconductors without any sharp transition \[ 41, 42 \].

As mentioned above, the weak and strong pairing phases are obviously distinct and separated by a topological phase transition at which the bulk gap closes. One can express the topological invariant distinguishing them, such as the Majorana number \[ 14 \]. However, we here propose an entirely different point of view from complex network analysis to detect the topological phase transition. We calculate the small-worldness $S$ in the Kitaev chain. Surprisingly, in Fig 1(d) the small-worldness drops to zero when the SC system comes from the topologically non-trivial phase to the topologically
trivial phase across the critical point \( \mu_c \). In particular, it shows less finite size dependence than other network measures, e.g. the weighted clustering coefficient \( C \) (not shown). As a result, we illustrate that the network topology enables the small-worldness, akin to an order parameter in the theory of conventional phase transitions, to expose the change of nontrivial topology inherent in the weak pairing regime.

In order to further explore the implications of the network topology, we investigate the probability distribution \( P(w) \) of the weights of network links in the Kitaev chain (see Fig.2(a)). In the case of \( \mu < \mu_c \), the link weights \( w \) distribute like a delta function due to the long-range Cooper pairs in real space. Namely, the link weights homogeneously distribute in the network space. As further increasing \( \mu \), the position of the peak of the distribution are moved left but still remains nearly homogeneous. It is noteworthy that the distribution at the critical point possesses a decaying function with a heavy tail and much broader width than others at different \( \mu \). Hence the weight distribution of network links becomes more heterogeneous. On the other hand, in the strong pairing regime \( (\mu > \mu_c) \) the weight distribution of network links moves to the weight \( w \sim 0 \) and recovers the sharp peak. The tail of the distribution now looks much more heterogeneous as a result of the formation of “molecule-like” Cooper pairs. The sudden change of the weight distribution at the critical point makes it easier to classify the network links in both topologically trivial and non-trivial phases so that we can have a clear order parameter to identify the phase transition.

To examine the novel idea at the critical point, we compare the small-worldness with the common order parameter in the 1D quantum Ising model, spontaneous magnetization \( M \), defined as

\[
M = \left\langle \left| \sum_{i} \hat{\sigma}_i^z \right|^2 \right\rangle. \tag{8}
\]

It is well-known that the quantum phase transition in the chain belongs to the universality class of the 2D classical Ising model, which has been analytically solved by Onsager \[43\]. In the thermodynamic limit, the singular behaviors of the spontaneous magnetization near the critical external magnetic field \( g_c \) can be described by the scaling form: \( (g_c - g)^\beta \) with the critical exponent \( \beta = \frac{1}{4} \). In Fig.3(b), we extrapolate the critical exponent \( \gamma \approx 1.83 \) of the small-worldness \( S \propto (\mu_c - \mu)^\gamma \) from the finite size analysis, which is unexpectedly bigger than \( \beta \). Notably, \( \gamma \) is very close to the critical exponent describing the divergence of magnetic susceptibility, whose value is \( \frac{7}{4} \). This result strongly suggests that near the critical point the small-worldness behaves as the second derivative of the free energy with respect to some twist.

This finding inspires us to carefully check whether or not the new order parameter, small-worldness, is much easier to capture the critical point of the classical Ising model on a square lattice. Notice that here the definition of the adjacency matrix is replaced by the spin-spin correlation function, \( \hat{A}_{ij} = \frac{1}{\text{max}(|\hat{\sigma}_i^z|, |\hat{\sigma}_j^z|)} \). In Fig.3(a), one can see that given a lattice size the small-worldness indeed needs much less effort to extract the critical point than the spontaneous magnetization (compare to Fig.3(b)). This result can be understood as a consequence for the bigger critical exponent of the small-worldness. In practice, we thus present the efficiency of the small-worldness for the numerical simulation of the order parameter in the 2D classical Ising model.
The other example is the classical rotor plane, sometimes called 2D classical XY model, in a square lattice of size \( N \) described by [45, 46].

\[
H_{XY} = -\sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -\sum_{\langle i,j \rangle} \cos (\theta_i - \theta_j),
\]

where \( \theta_i \) is the angle of the 2D spin vector \( \vec{S}_i \) at site \( i \). Conventional long-range order, like ferromagnetism or a crystal, is common in three dimensional systems. However, in 2D systems with continuous symmetry as the model introduced above, the true long-range order is completely washed out by thermal or quantum fluctuations and only its topology remains.

In fact, the low-temperature phase forms a quasi-long-range order originating from the power-law correlation decay. There is a phase transition from this phase to the high-temperature disordered phase whose correlations decay exponentially with distances. Such a transition is known as the KT transition as illustrated in Fig.4(a). When temperature approaches the KT transition point, some nodes begin to cluster and a "crystallike" structure is visualized as shown in Fig.4(b). The reason behind the structure is that near \( T_c \), the short-range correlations become more dominant than the quasi-long-range order at low temperatures. At much higher temperature \( (T = 2.0) \), a square-lattice structure composed of the strongest links appears in Fig.4(c). Extremely short-range correlations of the high-temperature disordered phase indicate much larger link weights between the nearest-neighboring nodes so that the network structure resembles a square lattice.

Spin stiffness has usually been used to determine the KT transition temperature [48]. Here we demonstrate that the small-worldness \( S \) defined in Eq.(3) also have an ability to detect the KT transition. In Fig.4(d), we show the critical behavior of the small-worldness vs temperature. One can see that the temperature dependence of the small-worldness has a linear decrease at low temperatures and an apparent deviation from linearity around the KT transition. We thus define the critical temperature \( T_c \) where the small-worldness begins to deviate from its linear behavior at low temperatures. In Fig.5(a), we estimate \( T_c \) (\( \approx 0.8940 \)) by finite size scaling, which is very close to the value \( (T_c \approx 0.8935) \) adapted from the universal jump of the spin stiffness [44]. This agreement convinces us that a similar "jump" can be defined by using the small-worldness as well. Moreover, similar to the Kitaev chain, this result also implies that the small-worldness should be related to the physical quantity corresponding to the sec-

**IV. ROTOR PLANE**

FIG. 4: Network representations of the classical rotor plane at different temperature: (a) \( T = 0.1 \), (b) \( T = 0.9 \) and (c) \( T = 2.0 \). The lattice size \( N = 400 \). The thickness of links represents the amplitude of \( C_{ij} \). The size of nodes stands for the node strength. Here only 100 nodes are plotted because of periodic boundary condition. (d) Small-worldness \( S \) of the classical rotor plane as a function of temperature \( T \) for different lattice size \( N \). The pink vertical line indicates the critical temperature \( T_c \) estimated by spin stiffness [44].

FIG. 5: (a) The finite size scaling of the critical temperature \( T_c \) determined from the deviation of the linear \( T \) dependence of \( S \). (b) The probability distribution \( P(w) \) of the link weight \( w \) for different \( T \).
ond derivative of the free energy with respect to a twist. As a result, we suggest that the small-worldness can be also considered as a useful quantity to characterize the phase transition, instead of spin stiffness.

Let us now discuss the weight distribution of network links in the classical rotor plane. Unlike the Kitaev chain, Fig.5[b] shows that at low temperature ($T = 0.1$) the weight distribution is no longer similar to delta function but more like the log-normal distribution. Most links center their weights around $w \sim 1$, and thus display the homogeneous distribution. As further increasing temperature, the mean and variance of the distributions in Fig.5(b) becomes smaller and larger, respectively. It is noteworthy that the heterogeneity of the weight distributions appears before the KT transition. At high temperature, the peak of the distribution is moved to $w \sim 0$ with a long tail so that the weight distributions become much more heterogeneous. As compared to the topological phase transition in the Kitaev chain, the rotor plane shows much broader weight distribution of network links for all temperatures in the network space. It turns out that it is rather difficult to come up with an order parameter in the classical rotor plane. The same reasoning from the network topology can be applied to other many-body systems without local order parameters.

V. CONCLUSION AND OUTLOOK

In the Kitaev chain, we have illustrated that the small-worldness clearly distinguish the topological non-trivial phase ($\mu < \mu_c$) and the topological trivial phase ($\mu > \mu_c$) in the absence of symmetry distinction. In addition to the Chern number [13] and the Majorana number [14], we have formulated another relevant topological quantity, small-worldness, extracted from the network space allowing one to characterize the topological phases. As compared to the critical phenomena of the 2D classical Ising model, the critical exponent of the small-worldness near $\mu_c$ is very close to the one of the inverse of magnetic susceptibility, implying that the new order parameter is within the universality Ising class and much easier to capture the critical point than the spontaneous magnetization. In addition, the transition of the weight distribution of network links across the critical point can be also applied to other phase transitions with the existence of conventional long-range orders.

In the classical rotor plane, we have found that the small-worldness provides a shortcut to estimate the KT transition temperature. According to our definition for the "universal jump" of the small-worldness near the critical point, we have shown that the transition temperature $T_c$ we obtained is almost the same as the one estimated from the spin stiffness by previous large-scale Monte Carlo simulations [44]. Again, this result supports the finding in the Kitaev chain that the small-worldness should be closely related to susceptibility or response to the external field. We have also confirmed that the topological phase transition in the 2D classical rotor model can be characterized by using the weight distribution of network links in the network representation.

In summary, we have proposed a novel complex network analysis for computing the new topological invariant in 1D Kitaev model and identifying the KT transition in 2D classical rotor model. By defining the small-world network which have strongly coupled small-world properties in these models, we have found that the critical behavior of the many-body systems can be described by the change of the weighted network topology. A network measure including both local and non-local features, called small-worldness, has been proven to be easier to investigate the quantum and classical phase transitions. In particular, the picture behind the weight distribution of network links provides significant information to comprehend the generic phase transitions with/without local order parameters in condensed matter systems. Based on the success of the complex network analysis in this work, a very interesting direction that we leave for the future is further detecting other topologically ordered phases with the same topological entanglement entropy in microscopic Hamiltonians, such as the toric code [50] and the double semion [51] model.

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