Solution of diffusively monitored linear quantum systems

P. Warszawski,1 H. M. Wiseman,2 and A. C. Doherty1

1Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, Sydney, NSW 2006, Australia
2Centre for Quantum Computation and Communication Technology, Centre for Quantum Dynamics, Griffith University, Brisbane, Queensland 4111, Australia

(Dated: April 7, 2020)

We study solutions to the evolution of diffusively monitored N-mode linear quantum systems possessing a time-independent Hamiltonian. Here linearity means the Hamiltonian is quadratic and the Lindblad operators (including those corresponding to the diffusive monitoring) linear in the mode annihilation and creation operators. In cases where we can solve the associated 2N-degree polynomials, we provide an analytical solution for initial states that are arbitrary (i.e. not necessarily Gaussian). The solution takes the form of an evolution operator, with the measurement-result dependence captured in 2N stochastic integrals over these classical random signals. The solutions also allow the POVM characterizing the measurement to be determined. To illustrate our results, we solve some single-mode example systems, with the obtained POVMs being of practical relevance to quantum state tomography. Our key tool is the thermo-entangled state representation of mixed states of quantum mechanical oscillators as state vectors rather than state matrices (albeit in a larger Hilbert space). Together with methods from Lie algebra, this allows a more straightforward manipulation of the exponential operators comprising the system evolution than is possible in the original Hilbert space.

I. INTRODUCTION

The evolution of an open quantum system undergoing measurement upon the environments with which it interacts is governed by a stochastic master equation (SME), with the system state following a quantum trajectory. Due to their ubiquitous nature and importance to emerging quantum technologies, such systems have been studied extensively, both from a theoretical [1–13] and, increasingly, experimental perspective [14–30]. In this paper we focus on a particular class of open quantum systems, being linear quantum systems subject to diffusive monitoring (terms that will be defined shortly), which are of wide practical importance, as well as being amenable to analytic techniques [31–43]. Physical systems that can be modeled in such a way include multimodal light fields, optical and optomechanical systems (including squeezing), microwave resonators and Bose-Einstein condensates. Further motivation for their study arises due to recent interest in the control of such bosonic systems, potentially using feedback [4, 38, 44–46]. Our specific research goal is to find an evolution operator, that can be applied to arbitrary (not necessarily Gaussian) initial states; in other words, to solve the SMEs belonging to the class under consideration.

Let us now define, incrementally, what we mean by a ‘linear quantum system subject to diffusive monitoring’. A ‘linear quantum system’ refers to the dynamics of the system configuration being linear. An equivalent description is that a complete set of observables (that is, the system configuration) is governed by linear quantum Langevin equations [4]. A necessary requirement for this to be true is that the observables have unbounded spectra and, consequently, describe bosonic modes. A complete set of observables is provided by a canonically conjugate pair of position and momentum observables, one pair for each bosonic mode. Equivalently, an annihilation and creation operator for each mode can be used instead.

In the absence of monitoring (or by ignoring the measurement results) the dynamics of the system configuration will be linear given two restrictions. Firstly, the Hamiltonian evolution must be at most quadratic in the bosonic annihilation and creation operators. Although not necessary for the preservation of linearity, in our work we will take the quadratic terms as being time-independent (or made to be so by transformation to a new frame), so that analytic results are possible. Secondly, the Lindblad operators describing the irreversible evolution must be linear in the annihilation and creation operators.

When measurement of the environment is included, further restrictions must be placed to retain the linearity of the evolution when conditioning upon the measurement results. Specifically, the monitoring must be ‘diffusive’, in contrast to jump-like trajectories. The jump class of trajectories arise when the measurement record is a point processes in which a detector ‘click’ is accompanied by a finite change in the conditioned state matrix. The diffusive class, by contrast, is one in which the stochasticity of the measurement results is described by a Wiener increment and the conditioned state evolves continuously (though non-differentially) in time. Prominent examples of the diffusive class are homodyne and heterodyne detection, which can be realized by coherently combining the light leaking out of an optical cavity with a very strong local oscillator before detection. As almost all detection events are due to the local oscillator, the effect of each one on the system state becomes infinitesimal and a continuous description arises. In this paper we will model the most general form of such dyne (diffusive) unravel-
We treat completely arbitrary initial states. For initial conditions, it is common to assume that the initial system state is Gaussian. This is not assumed in our work. We treat completely arbitrary initial states. For initial Gaussian states, the system solution is well known, being governed by a Kalman filter. Therefore, the extension of our work provides is that of a more general solution to linear quantum systems undergoing diffusive measurement-induced evolution, being applicable to such initial non-Gaussian states as ‘cat’ or Fock states.

The solution to a SME naturally involves classical random variables, as it represents the description of a particular quantum trajectory. This is distinct from master equation solutions which are deterministic and provide a description that is inherently averaged over all possible trajectories. By ‘analytical solution’ of a SME, we therefore aim to find an expression for the system state at time $t$ in terms of a stochastic evolution operator that contains a finite number of stochastic integrals; this evolution operator will be independent of the initial state. That is, rather than defining the evolved system state in terms of the infinity of numbers constituting the entire continuous measurement record, we will show that the final state is only dependent upon $2N$ complex-valued stochastic integrals.

The solution of the SME, given as a function of a finite number of stochastic integrals, has a number of uses, as we now discuss. A SME solution allows calculation of expectation values conditional upon the measurement results which are, in general, distinct from the values obtained from the average system behavior (described by the master equation). Thus, a SME solution will be essential in state-based feedback control, by which knowledge of the system state is used for its accurate future control.

Another benefit of the SME solution is that it allows a characterization of the measurement, by defining the relevant POVM. The POVM and related theoretical constructs, such as Bayesian inference, are of use in many contexts. For example, they allow the optimal inference of the input system state via state tomography. The motivation for solving the SME in was to know the POVMs relating to optomechanical position measurement with parametric amplification. The method used there can be turned into a general method of solving SMEs, which is detailed in this paper. To make the link more explicit to the previous work, and to provide more detail regarding those calculations, we here consider the relevant optomechanical system as a specific single-mode example.

A related use of the POVM is that it allows a calculation of the probability density of obtaining a measurement sequence. In combination with the system solution, we therefore have knowledge of the type of states obtainable under measurement and the probability distribution of such states. This is extremely powerful: to simulate the system state at some specific future time one needs only to sample the state distribution, rather than integrating the SME. We stress that this applies to non-Gaussian initial states that cannot be fully tracked by their first and second-order moments. Potential specific applications include facilitating the investigation of the rate of decoherence of quantum superpositions or entanglement dynamics.

Before closing this Introduction, we briefly discuss the methods that we use to obtain SME solutions. In order to make a solution tractable we use a linear SME, in which some of the information concerning the probability of a measurement sequence occurring is contained in the norm (trace) of the density matrix. It is important to note that the SME for the normalized quantum state is nonlinear, even when the system belongs to the class of ‘diffusively monitored linear quantum systems’ which, by definition, possess linear quantum Langevin equations for the system configuration. The use of a linear SME removes the measurement-induced nonlinearity and provides us with a pathway to calculate the POVM. Our work in many regards generalizes that of Wiseman, and of Jacobs and co-workers, which provided a general method of calculating the evolution operator for the stochastic Schrödinger equation (SSE). We extend the class of solutions to include arbitrary dyne measurements in systems requiring a mixed state description (that is, a SME rather than a SSE). Also influential is the application of group theory methods developed by such practitioners as Gilmore and Yuan. Wilson and co-workers have obtained analytic solutions to master equations using Lie methods. Much of the problem of obtaining a practicable SME solution is contained in operator disentangling and re-ordering tasks. There is a considerable literature devoted to these topics, for example. The final method that will be mentioned here is that of the thermo-entangled state representation (sometimes called non-equilibrium thermo field dynamics). This key technique transforms the superoperators of the standard formulation of the SME into operators acting in a larger Hilbert space. We can then utilize powerful group theoretic tools to re-organize the infinite string of time slice evolutions.

It is well known that in the absence of measurement the solution of linear quantum systems is possible via phase space methods, but there has been considerable interest in providing new methods of solution to the deterministic Gaussian master equation, so we note that our method of solution of the SME naturally subsumes non-stochastic systems and does so at the very general level described above.
Our paper is organized as follows. In Sec. II, we first sketch the steps that will be followed in order to solve the linear SME. After providing the system specification and some necessary background material, these steps are then carried out. In Sec. III, the POVM pertaining to the compiled measurement of finite duration is obtained. The adjoint equation approach to finding the POVM is also discussed. Mathematical details that are omitted earlier, for readability, are supplied in Sec. IV. Reduction to a single mode, so that more details may be provided in this highly relevant case, is also performed in this section. Some single mode examples are provided in Sec. V. The paper concludes with a discussion in Sec. VI.

II. SOLVING THE STOCHASTIC MASTER EQUATION

A. Solution Sketch

To be conceptually clear, a sketch of the solution method is first provided. We start with the most general form of linear density matrix evolution, that of a completely positive (but not necessarily trace preserving) quantum channel, $\mathcal{O}$, with an operator-sum representation $[83]$. The linearity exploited here (and wherever else in this paper that a linear SME is discussed) is a calculational tool, and does not relate at all to linear system dynamics (a term that we reserve as referring to linear quantum Langevin equations). We wish to treat monitored open quantum systems, so conditioning upon the measurement result, $y$, is indicated by subscript. Given a measurement result $y$, the density matrix is updated as

$$\rho_y(t + \Delta t) = \mathcal{O}_y \rho(t) = \sum_j \tilde{\Omega}_{y,j} \rho(t) \tilde{\Omega}_{y,j}^\dagger, \quad (1)$$

under the superoperator action of the operation $\mathcal{O}_y$, for some set of Kraus operators $\{\tilde{\Omega}_{y,j} : j\}$. Note that $\rho_y(t + \Delta t)$ is an unnormalized density matrix. When we directly consider the linear SME we will be interested in continuous measurements, but it is easiest to elucidate our solution method by first envisioning discrete measurements and a correspondingly finite time, $\Delta t$. If $\rho(t)$ were a normalized density matrix, then the norm of $\rho_y(t + \Delta t)$ would give the probability of the result $y$ being obtained:

$$\varphi_y = \text{Tr}[\rho_y(t + \Delta t)] = \text{Tr}[\mathcal{O}_y \rho(t)], \quad (2)$$

which could be used to give a normalized state, $\mathcal{O}_y \rho(t)/\text{Tr}[\mathcal{O}_y \rho(t)]$, if desired.

The POVM, which characterizes a quantum measurement, is defined as the set of Hermitian operators, $\{\hat{W}_y : y\}$, such that for all (normalized) $\rho$

$$\varphi_y = \text{Tr}[\rho_y(t + \Delta t)] = \text{Tr}[\hat{W}_y \rho(t)]. \quad (3)$$

$\hat{W}_y$ is known as the ‘effect’ for the measurement result $y$. Note the two different ‘pictures’ (the Schrödinger and Heisenberg pictures) represented in Eq. (3). The first places the time evolution (including measurement update) into the quantum state, whilst the second shifts it to an effect operator. Further discussion of this is provided in the next subsection, as the Heisenberg picture provides an alternative pathway to solving for the time evolution. In terms of Kraus operators, the effect operator is given by

$$\hat{W}_y = \sum_j \hat{\Omega}_{y,j}^\dagger \hat{\Omega}_{y,j}. \quad (4)$$

Importantly, it is possible to formulate quantum measurement theory slightly differently from that indicated by Eq. (1), by introducing an ‘ostensible’ probability, $\varphi_{\text{ost}}(y)$, for the result $y$. This necessitates the use of a rescaled operator $\mathcal{O}_y = \mathcal{O}_y/\varphi_{\text{ost}}(y)$, and then the actual probability of getting the result $y$ becomes

$$\varphi_y = \text{Tr}[\rho_y(t + \Delta t)]/\varphi_{\text{ost}}(y), \quad (5)$$

where $\rho_y(t + \Delta t) = \mathcal{O}_y \rho(t)$. The bar over $\rho_y$ in Eq. (4) is meant to alert the reader to the fact that the norm of $\rho_y$ is not to be interpreted in the sense of Eq. (2), due to the introduction of ostensible statistics for $y$. The utility of such a formulation is that there is freedom of choice regarding the ostensible distribution; this will be exploited so as to make the linear equation for $\rho_y(t + \Delta t)$ as simple as possible. In the next section, we will take advantage of this when solving the linear SME.

We can represent the linear operation of $\mathcal{O}_y$ (or $\mathcal{O}_y$) as left-only matrix multiplication in a larger space. For a finite dimensional basis, this could be achieved simply by column stacking the elements of $\rho$ to form a vector. That it can be achieved for infinite dimensional bosonic modes, via the thermo-entangled state representation $[81]$, will be discussed when the detailed solution method is provided in the next section (also see Sec. II.C). For the moment, we note that it is possible to ‘vectorize’ $\rho$ in a larger space and it is evolved as per

$$|\rho(t + \Delta t)\rangle_y = V_y |\rho(t)\rangle, \quad (6)$$

with the evolution enacted by the operation $\mathcal{O}_y$ now represented by the nonunitary evolution operator $V_y$.

Now consider a string of measurement results

$$y = (y_1, y_2, \ldots, y_J), \quad (7)$$

with $J \Delta t = t$, that represents the discrete sequence of measurement results obtained when measurement is repeated at intervals of $\Delta t$. It is then clear that the unnormalized density matrix at time $t$ can be written in terms of that at the initial time as

$$|\rho(t)\rangle_y = V_{y_J} V_{y_{J-1}} \cdots V_{y_1} |\rho(0)\rangle \equiv V_y |\rho(0)\rangle. \quad (8)$$

By finding the evolution operator, $V_y$, the evolution is solved. If desired, the vectorization to form $|\rho(t)\rangle$ could
Although complications exist in infinite dimensional settings where Eq. (11) is well defined, we note that in all practical systems of interest there will some finite dimensional representation of the system state or, alternatively, an unnormalized system density matrix or, alternatively, an approximation of the system state is conditioned upon the measurement record. Note that the measurement results at the discretized times could actually comprise (column) vectors of results, as would be appropriate when there exists more than one measurement channel. In that case, we generalize Eq. (7) to

\[ Y \equiv (y_1, y_2, \ldots, y_J), \]  

so that \( Y \) becomes a matrix.

The above solution sketch describes three major steps. Firstly, the evolution must be specified via a description of \( O \) or, equivalently, the linear SME. Next, evolution is represented in a larger space where the density matrix, \( \rho \), is treated as a vector |\( \rho \rangle \). Thirdly, the evolution operator representing infinitesimal conditioned evolution in time \( dt \) is found. By multiplying a long string of these together, the operator describing finite evolution can be approximated. The final task is therefore to organize this string into a useful and concise operator \( \hat{V}_Y \). These tasks are performed, beginning in Sec. II B.

1. The adjoint channel

Before specifying the system of interest, and solving the SME, we briefly discuss the adjoint equation to Eq. (1), with the adjoint equation being analogous to the Heisenberg picture of evolution. Below Eq. (3), it was described how the probability of measurement result outcomes could be viewed in terms of an updated (and unnormalized) system density matrix or, alternatively, an updated effect matrix. This is an example of an ‘adjoint’ channel, \( O^\dagger \), which is defined for two operators \{\( A, B \)\} on the Hilbert space, as

\[ \text{Tr} [\hat{B} O [\hat{A}]] = \text{Tr} [O^\dagger [\hat{B}] \hat{A}]. \]  

Although complications exist in infinite dimensional Hilbert space, regarding the domain of \{\( A, B \)\}, for which Eq. (11) is well defined, we note that in all physical systems of interest there will some finite dimensional restriction, that accurately models the system, for which these difficulties may be resolved. To make explicit contact with the previous section, \( O \) can be taken to be the quantum channel from Eq. (1), with the operator \( \hat{A} \) representing \( \rho \), while \( \hat{B} \) is the effect operator. A feature of the adjoint channel is that, given forwards-in-time evolution of the operator \( A \) (enacted by \( O \)), \( O^\dagger \) will lead to backwards-in-time evolution of the operator \( \hat{B} \) — this can be seen by considering two successive updates and then using the cyclic properties of the trace operation to obtain the adjoint channel. The two different pictures represent two different ways to proceed; either we can solve for the time evolution of the system state or of the effect operator. Despite the duality provided by the adjoint channel, it is clear from Eq. (11) that there is, in general, a loss of information regarding the Kraus decomposition if only the effect operator is solved for. This arises due to the final condition of \( I \) being implicit in Eq. (1) (to clarify, in our work we take the ‘final condition’ for a backwards-in-time differential equation as referencing the starting point for the integration and, consequently, the latest absolute time). In other words, the final state matrix cannot be uniquely inferred from the POVM. The approach taken is to formulate the update for an arbitrary effect matrix (thus preserving generality, by not imposing a final condition). Whilst preserving generality, the authors do not provide an analytic solution to the adjoint equation.

Given the two equivalent paths of solving for the evolution induced by the quantum channel \( O \), we will choose to solve directly for the system state (making no assumption about its initial gaussianity), which is of interest in and of itself, and then infer the POVM.

B. System specification

In this subsection, the procedure outlined in Sec. II A is initiated, with the provision of the mathematical description of the system. This is in the form of the incremental evolution, \( d\hat{\rho}(t) \). More specifically, the system is described by a linear SME for an \( N \)-mode bosonic system subject to an arbitrary number, \( L \), of completely general ‘dyne’ measurements. Homodyne and heterodyne type measurements are two, experimentally prevalent, examples included within this framework. For the system to have linear dynamics (as opposed to having a linear SME, which is an always available calculational tool), we also require that the Hamiltonian be at most a quadratic function of the bosonic annihilation and creation operators. The \( L \) Lindblad operators, which we write in column vector form as \( \hat{e} \equiv (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_L)^T \), are likewise limited to being arbitrary linear combinations of those operators. Consequently, in units where \( \hbar = 1 \), the system state is conditioned upon the measurement results \( y(t) \), measured at time \( t \), as}

\[ d\hat{\rho}(t) = -i \left[ \hat{H}, \hat{\rho}(t) \right] dt + \mathcal{D} [\hat{e}] \hat{\rho}(t) dt + y^T(t) dt \hat{\rho}(t), \]  

where the superoperators are defined by

\[ \mathcal{D} [\hat{e}] \equiv \sum_{k=1}^{L} \mathcal{D} [\hat{c}_k], \quad \mathcal{D} [\hat{e}] \rho \equiv \hat{e} \rho \hat{e}^\dagger - \frac{1}{2} \hat{e}^\dagger \hat{e} \rho - \frac{1}{2} \rho \hat{e}^\dagger \hat{e}, \]  

and

\[ \mathcal{H} [\hat{e}] \rho \equiv \hat{e} \rho + \rho \hat{e}^\dagger, \]  

with \( \hat{e}^\dagger \equiv (\hat{e}^T)^\dagger \) and \( \hat{e}^\dagger \equiv (\hat{e}_1^\dagger, \hat{e}_2^\dagger, \ldots, \hat{e}_L^\dagger) \). For compactness of notation, a ‘bar’ over \( \rho(t) \) is used here to indicate...
two important aspects. Firstly, it implies that the norm of \( \tilde{\rho}(t) \) is to be interpreted in the sense of Eq. (13); that is, \( y \) is assigned ostensible statistics. Secondly, the bar is taken to indicate conditioning on the set of all prior measurement results, \( Y(t) \). This avoids the cumbersome notation whereby conditioning is indicated by subscript and the density matrix on the LHS would have to be written as \( d\rho_{Y(t)}(t) \) and that on the RHS as \( \tilde{\rho}_{Y(t)}(t) \), with \( t^{\ast} \) meaning all times strictly prior to \( t \). Note that the bar appearing in \( H \) is used to indicate a linear form of the commonly utilized superoperator \( H \) [3]: its previous appearance in the literature [86] was also in the context of an ostensibly chosen linear SME.

There are several other features of Eq. (12) that need explanation. The size of the column vector \( y(t) \) is \( 2L \times 1 \). Its length is twice that of \( L \) as we allow for heterodyne-style measurement currents that can be decomposed into two real-valued components (we will often refer to \( y(t) \) as a measurement ‘current’). The complex matrix \( M \), of size \( L \times 2L \), parameterizes the unraveling and defines the measurements being conducted. The set of allowed \( M \) is identified by the constraint \( MM^\dagger \in \delta \) where \( \delta \) is

\[
\delta = \{ H = \text{diag}(\eta) \} \forall k, \eta_k \in [0, 1]\}.
\]  

(15)

Note that \( H \) (capital \( \eta \)) is a diagonal matrix of detector efficiencies (not the system Hamiltonian operator, \( \hat{H} \)). The reader will observe that the matrix \( M \) generalizes the scalar detector efficiency factor \( \sqrt{\eta} \) that would appear in a standard single channel homodyne linear SME. Eq. (12) is known as the \( M \)-representation of the linear SME [59].

As mentioned earlier, the notation \( \tilde{\rho}(t) \) not only implies that this represents an unnormalized density matrix, but also that, in writing Eq. (12), an ostensible distribution has been chosen for \( y \equiv y(t) \). That is, a Girsanov transformation [57] on \( y \) is made in order to assign it a white-noise ostensible distribution

\[
\varphi_{\text{ost}}(y) = \left( \frac{dt}{2\pi} \right)^L \exp \left[ -\frac{1}{2} y^T y dt \right].
\]  

(16)

The ostensible first and second order moments of \( y \) (as implied by Eq. (15)) are:

\[
\tilde{E} \{ y \} = 0_{2L},
\]

\[
dt E \{ y y^\dagger \} = 1_{2L}.
\]  

(17)

(18)

Here, to match the bar on \( \tilde{\rho}(t) \), \( \tilde{E} \) indicates an expectation that is taken on the ostensible distribution, while \( 0_{2L} \) is a vector of \( 2L \) zeros and \( 1_{2L} \) is the \( 2L \times 2L \) identity matrix. In fact, as Itô’s rule tells us, the explicit ensemble average in Eq. (18) can be removed. This leads to

\[
dt y y^\dagger = 1_{2L}.
\]  

(19)

a fact that is central to our derivation.

The actual distribution for \( y \) is also Gaussian, but does not have zero-mean:

\[
E \{ y \} = (M^\dagger \hat{c} + M^T \hat{c}^\dagger),
\]

\[
dt E \{ y y^\dagger \} = dt y y^\dagger = 1_{2L}.
\]  

(20)

(21)

The absence of a ‘bar’ indicates that the expectation is being taken over the actual distribution. The quantum expectation value, on the RHS of Eq. (20), requires the use of a normalized density matrix. Note that Itô’s rule allows the removal of the averaging in Eq. (21), just as it did for the ostensible distribution.

Having detailed both the ostensible and actual distribution for \( y \), we can now make a very important observation about the utility of the solution to Eq. (12) that will be derived. Despite Eq. (12) being a linear SME, and involving the ostensibly Gaussian distributed random variable \( y(t) \), it still provides the experimentalist, possessing actual values of \( y(t) \) (which will not in general have zero mean), with the valid conditional density matrix (after normalization). That is, if experimentally obtained measurement results are used in Eq. (12), then the state could simply be normalized when required. The reader might question whether this correspondence (dual purpose) will hold after the linear SME is integrated. We answer in the affirmative, as in the derivation of the integrated linear SME that follows, we only use statistical properties that are true for both the ostensibly and actually distributed measurement current — that Eq. (15) holds for \( y(t) \) regardless of whether it obeys ostensible or actual statistics.

As previously stated, the Hamiltonian, \( \hat{H} \), is quadratic at most (in the bosonic annihilation and creation operators), whilst the Lindblad operators, \( \hat{c} \), are linear. It is standard procedure to write these operators in terms of pairs of canonically conjugate quadrature operators, with a single pair for each mode, \( \hat{q}_n, \hat{p}_n \), having commutation relation \( [\hat{q}_n, \hat{p}_n] = i \). Thus, \( \hat{q}_n, \hat{p}_n \) are related to the annihilation and creation operator, \( \hat{a}_n, \hat{a}^\dagger_n \), of each mode via

\[
\begin{pmatrix}
\hat{q}_n \\
\hat{p}_n
\end{pmatrix}
= \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 \\
i & -i
\end{pmatrix}
\begin{pmatrix}
\hat{a}^\dagger_n \\
\hat{a}_n
\end{pmatrix},
\]  

(22)

with \( [\hat{a}_m, \hat{a}^\dagger_n] = \delta_{mn} \). A vector of operators

\[
\hat{x} = (\hat{q}_1, \hat{p}_1, ..., \hat{q}_N, \hat{p}_N)^T
\]

(23)

is defined, so that \( [\hat{x}_m, \hat{x}_n] = i \Sigma_{mn}, \) where the \( 2N \times 2N \) symplectic matrix is given by

\[
\Sigma = \bigoplus_{n=1}^{N}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]  

(24)

For later use, we also define the column vector of annihilation operators,

\[
\hat{a} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_N)^T,
\]  

(25)
from which follows the definition of $\hat{a}^\dagger$. Having laid the notational groundwork, we can then state the quadratic Hamiltonian as

$$\hat{H} = \frac{1}{2} \hat{x}^T G \hat{x} - \hat{x}^T \Sigma B u,$$  

with the $2N \times 2N$ matrix $G$ real and symmetric, $u$ a classical drive, and a matrix, $B$, that is also real. To allow a formal analytic solution of the linear SME, we have assumed a time-independent Hamiltonian. By making a canonical transformation, and then considering a shifted vacuum state, it is possible to remove the linear Hamiltonian term and also any constants in the Lindblad operators $^74$. Consequently, without further loss of generality, the Hamiltonian is taken to be

$$\hat{H} = \frac{1}{2} \hat{x}^T G \hat{x}$$

and the vector of Lindblad operators is

$$\dot{c} = C \hat{x},$$

for the $L \times 2N$ matrix $C$.

The evolution described by Eq. (12), with the specification of a quadratic Hamiltonian and linear Lindblad operators, is special in that it admits a Gaussian state as its solution. That is, given an initial state possessing a Gaussian Wigner function, the system Wigner function will remain Gaussian at all future times. The evolution of the Gaussian state can be tracked just with the first and second order moments of the quadrature operators. The equations governing these moments are jointly known as the generalized Kalman filter; the equation for the coherence evolution operator as acting via left-only multiplication is introduced in order to convert ensemble average calculations into equivalent pure state expressions. It is important to realize that in our work we go beyond this and treat arbitrary (that is, possibly non-Gaussian) initial states.

C. Brief Review of the Thermo-Entangled State Representation

To solve the linear SME provided in Eq. (12), the density matrix, $\hat{\rho}(t)$ is mapped to a vector, $|\hat{\rho}(t)\rangle$. This requires working in a larger Hilbert space, but leads to the evolution operator as acting via left-only multiplication upon $|\hat{\rho}(t)\rangle$. To achieve the mapping from $\hat{\rho}(t)$ to $|\hat{\rho}(t)\rangle$, we utilize the thermo-entangled state representation $^72$ $^74$ $^76$ $^51$ $^82$. This representation is based on prior work by Takahashi and Umezawa, relating to thermo field dynamics $^76$ $^77$, in which a fictitious field is introduced in order to convert ensemble average calculations into equivalent pure state expressions. Similarly, we introduce a fictitious mode, via the thermo-entangled state representation, as it allows a vectorization of the SME leading to expressions that are more easily manipulated. As the thermo-entangled state representation may be unfamiliar to some readers, a brief review is provided for completeness.

Given a physical system described by a Hilbert space, $\mathcal{H}$, an ancillary, unphysical, Hilbert space is introduced $\mathcal{H}$, which houses the unphysical modes. For an arbitrary operator $\hat{A}$ acting on vectors of the Hilbert space $\mathcal{H}$, there is a ‘tilde conjugate’ operator $\tilde{\hat{A}}$ that acts identically on vectors of the Hilbert space $\mathcal{H}$. Without loss of generality, we can define the relationship between tilde and non-tilde operators as

$$\tilde{\hat{A}} = A(\hat{a}, \hat{a}^\dagger), \quad \tilde{A} = A^*(\hat{a}, \hat{a}^\dagger).$$  

Note that $\hat{a} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_N)^T$ has been introduced and that taking the tilde conjugate of a matrix does not alter its dimensions. This generalizes to the case where the object to be tilde-conjugated is itself a matrix of operators, $\tilde{\hat{A}} = A(\hat{a}, \hat{a}^\dagger)$, and the the matrix dimensions are left unaltered:

$$\tilde{\hat{A}} = A^*(\hat{a}, \hat{a}^\dagger).$$  

From Eq. (29), and the requirement that $(\tilde{\hat{A}})^\dagger = \tilde{\hat{A}}$, the following ‘tilde-conjugation’ rules may be inferred

$$\left(\tilde{\hat{A}}_1 \hat{A}_2 + z_1 \hat{A}_2 \hat{A}_1^\dagger + z_2 \hat{A}_2^\dagger \hat{A}_1 \right) = \tilde{\hat{A}}_1 \hat{A}_2 + z_1 \hat{A}_2 \hat{A}_1^\dagger + z_2 \hat{A}_2^\dagger \hat{A}_1,$$

for complex numbers $z_1, z_2$.

Similarly to multimode coherent states, the ‘mixed’-mode operator

$$\hat{\beta} = \hat{a} - \hat{a}^\dagger$$

(recall our previous definition, applying equally to tilde operators, that $\hat{a}^\dagger = (\hat{a}^\dagger)^\dagger$) may be defined, which has eigenstates:

$$\hat{\beta} |\beta\rangle = \beta |\beta\rangle, \quad \hat{\beta}^\dagger |\beta\rangle = \beta^\dagger |\beta\rangle, \quad \beta \in \mathbb{C}^N.$$  

Note the use of the non-operator expression $\beta^\dagger = (\beta^\ast)^T$, where $\beta$ is a vector of complex numbers. Like position or momentum eigenstates, the states $|\beta\rangle$ are not normalizable. From Eq. (33) we deduce that

$$\hat{a} |\beta = 0\rangle = \hat{a}^\dagger |\beta = 0\rangle, \quad \hat{a}^\dagger |\beta = 0\rangle = \hat{a} |\beta = 0\rangle.$$  

Going forward, the cumbersome notation of $|\beta = 0\rangle$, indicating the zero eigenvector of $\hat{\beta}$, will be dropped and merely written as $|0\rangle \equiv |\beta = 0\rangle$. To distinguish this zero-valued eigenvector of $\hat{\beta}$ — which we will often refer to as the thermo-entangled state vacuum — from the $2N$-mode vacuum state of the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, we will denote the latter by $|\text{VAC}\rangle$.

It is shown in $^73$ that the thermo-entangled states are given by:

$$|\beta\rangle = \exp \left(-\frac{|\beta|^2}{2} + \hat{a}^\dagger \beta - \beta^\dagger \hat{a}^\ast + \hat{a}^\dagger \hat{a}^\ast \right) |\text{VAC}\rangle.$$
As an aside, the thermo-entangled states are precisely the common eigenstates of the relative co-ordinate and total momentum of two particles, which are central to the original Einstein, Podolsky and Rosen (EPR) scheme [58, 59]. These states allow us to represent a mixed $N$-mode system state $\rho$ by a vector in the larger, $2N$-mode, Hilbert-space, via
\[ |\rho\rangle \equiv \rho \otimes \mathbf{1} |0\rangle , \]
where the $N$-mode identity is acting on the unphysical modes.

### D. Solving the linear SME

As previously indicated, we will work in the thermo-entangled state representation, whereby the state matrix is vectorized according to $|\tilde{\rho}\rangle \equiv \tilde{\rho} \otimes \mathbf{1} |0\rangle$. That is, Eq. (12) is simply right multiplied by $|0\rangle$ and then the relations of Eqs. (29) + (30) and Eq. (31) are used. It is found that
\[ d|\tilde{\rho}\rangle = \left[ \hat{S} \left[ -i\hat{H} + \hat{D} \hat{c} \right] dt + y^T(t) dt \hat{S} \left[ M^I \hat{c} \right] \right] |\tilde{\rho}\rangle . \]  
(37)

The operator $\hat{D} \hat{c}$ is defined to represent the unconditional decoherence terms
\[ \hat{D} \hat{c} = e^T \hat{c} - \frac{1}{2} \hat{c} \hat{c}^\dagger - \frac{1}{2} \left( \hat{c} \hat{c}^\dagger \right)^\sim , \]
and we have also introduced notation for the sum of an operator and its tilde conjugate:
\[ \hat{S} \left[ M^I \hat{c} \right] = M^I \hat{c} + \left( M^I \hat{c} \right)^\sim . \]  
(39)

The RHS of Eq. (37)’s being invariant under tilde conjugation ensures that the hermiticity of the density matrix is preserved.

The next step is to factorize the evolution — this will allow the calculation of an evolution operator. It also opens up the possibility of using techniques from Lie Algebra. An expression accurate to $O(dt)$ is formally achieved with
\[ |\tilde{\rho}(t + dt)\rangle = \exp \left[ \hat{S} \left[ -i\hat{H} + \hat{D} \hat{c} \right] dt + y^T(t) dt \hat{S} \left[ M^I \hat{c} \right] \right] |\tilde{\rho}(t)\rangle , \]  
(40)
\[ \frac{1}{2} \left( y^T(t) dt \hat{S} \left[ M^I \hat{c} \right] \right)^2 |\tilde{\rho}(t)\rangle , \]  
(41)
where factors linear in the annihilation and creation operators are contained in $dS_L$, and quadratic terms are contained in $\mathcal{L}_Q$. To verify Eq. (40), note that $y^T(t) dt$ is of order $\sqrt{dt}$ so that its second order contribution is required for accuracy to $O(dt)$. The form of Eq. (41) is now equivalent to that of Eq. (39).

The finite evolution operator is formed by
\[ |\tilde{\rho}(t)\rangle = \prod_{j=1}^J \exp \left[ \mathcal{L}_Q(jdt) dt + dS_L(jdt) \right] |\tilde{\rho}(0)\rangle , \]  
(42)
where $Jdt = t$ and the product is enumerated with $j$ increasing from right to left. This expression for the state matrix (in vectorized form) is analogous to that for the state vector obtained in [53], thus allowing a similar fundamental approach (but requiring different techniques). The form of Eq. (38) has also been achieved.

To derive a practicable solution for $|\tilde{\rho}(t)\rangle$ the infinitesimal evolutions of Eq. (42) must be re-ordered and summed so as to provide a finite number of factorized terms, as indicated by Eq. (39). It is also required that each of these terms can be disentangled into exponentials of single operator terms rather than linear combinations (for example, via Baker-Campbell-Hausdorff relations in the simplest non-trivial case). This is a task in bosonic algebra and methods of Lie groups are used. In order to make analytic progress we assume that the quadratic parameters (possibly in a transformed frame) and the measurement strengths are time-independent.

To make further contact with Lie algebra, note that the multimode bosonic operators form a closed algebra, specifically a subalgebra of the symplectic algebra $sp(2m + 2; \mathbb{R})$ where $m = 2N$ is the total number of both physical and unphysical modes [57, 59]. The size of the algebra has been restricted by demanding that the Hamiltonian be at most quadratic and the Lindblad operators be linear. It is important to emphasize that unphysical tilde modes (represented by operators $\tilde{a}, \tilde{a}^\dagger$) are on the same footing as physical modes (represented by operators $\hat{a}, \hat{a}^\dagger$) in terms of bosonic algebra calculations. To reinforce this, and for notational convenience, the $\tilde{a}$ and $\tilde{a}$ are placed into a single column vector of length $2N$:
\[ \hat{b} = (\hat{a} ; \tilde{a}) = (\hat{a}_1, ..., \hat{a}_N, \tilde{a}_1, ..., \tilde{a}_N)^T . \]  
(43)

Eq. (42) can be simplified using the following properties of the multimode bosonic algebra: the quadratic terms form a subalgebra and the linear terms form an ideal. Thus, the commutator of a quadratic and linear term gives a non-zero linear term only. Stated another way, exponentials of quadratic terms are left unchanged when they move through exponentials of linear terms. After noting that $\exp \left[ \mathcal{L}_Q(jdt) dt + dS_L(jdt) \right] = \exp \left[ \mathcal{L}_Q(jdt) dt \right] \exp \left[ dS_L(jdt) dt \right]$ to $O(dt)$, we can use these properties of the multimode bosonic algebra to perform a prototypical re-ordering of terms appearing in Eq. (42)
\[ e^{dS_L(jdt)} = \prod_{k=1}^\infty \exp \left[ \mathcal{L}_Q(jdt) dt \right] e^{dS_L(jdt)} e^{\mathcal{L}_Q(jdt)} , \]  
(44)
with the interim time $jdt$ labeled as $\tau$. Note that the linear term has been modified, as indicated by the addition of the prime, and that the time independence of $\mathcal{L}_Q$ has been utilized in combining all the infinitesimal quadratic exponentials. This allows Eq. (42) to be written as
\[ |\tilde{\rho}(t)\rangle = e^{\mathcal{L}_Q(t)} e^{dS_L(t)} |\tilde{\rho}(0)\rangle , \]  
(45)
In the limit \( dt \to 0 \), \( \mathcal{S}_L(t) \) is a stochastic Itô integral. Arbitrarily it has been chosen to move the quadratic terms to the left (the direction of increasing time). Linear and quadratic terms have thus been separated and summed. This is in the required form of Eq. (39). In the next two subsections we will provide more detail concerning \( \mathcal{S}_L(t) \) and obtain a more convenient expression for \( e^{L_Q t} \).

1. Investigating \( e^{\mathcal{S}_L(t)} \)

To progress further, \( L_Q \) and \( d\mathcal{S}_L(t) \) are written in terms of the bosonic vector \( \hat{b} \) (see Eq. (44)) that contains both physical and unphysical modes

\[
\begin{align*}
L_Q t &= \hat{b}^\dagger \hat{R} \hat{b}^\dagger + \hat{b}^\dagger \hat{D} \hat{b} + \hat{b}^\dagger \hat{L} \hat{b} \\
\mathcal{S}_L(t) &= \hat{b}^\dagger \hat{d} \hat{r} + \hat{d} \hat{b},
\end{align*}
\]

(46)

(47)

with the \( 2N \times 2N \) matrices \( \mathbf{R}, \mathbf{D}, \mathbf{L} \), the \( 2N \times 1 \) column vector, \( \hat{d} \hat{r} \), and the \( 1 \times 2N \) row vector, \( \hat{d} \hat{b} \), all constrained by the Hermiticity preservation of the evolved state matrix. Rather than use \( \{ \hat{r}, \hat{l} \} \) we have introduced \( \{ \hat{d} \hat{r}, \hat{d} \hat{l} \} \), in the linear evolution containing the measurement noise, to emphasize that they are infinitesimal and are of \( \mathcal{O}(\sqrt{\hat{d} \hat{b}}) \). The relationship between the system description in Eq. (12), that is given in terms of \( \{ \mathbf{G}, \mathbf{C}, \mathbf{M} \} \), and the parameterization of \( \{ \mathbf{R}, \mathbf{D}, \mathbf{L}, \hat{d} \hat{r}, \hat{d} \hat{l} \} \) of Eqs. (46)–(47) is provided in App. A.

The explicit expression for \( \mathcal{S}_L(t) \) is not difficult to derive using a finite dimensional representation of the algebra \( \text{sp}(2m + 2; \mathbb{R}) \) (see Sec. XV for details of this calculation), but in this section it is given heuristically as

\[
e^{\mathcal{S}_L(t)} = \prod_{j=1}^J e^{d\mathcal{S}_L(jdt)} = \prod_{j=1}^J e^{\hat{b}^\dagger \hat{d} \hat{r}^{j} + \hat{d} \hat{b}^{j}},
\]

(48)

with the primes indicating that a linear transformation of the stochastic vectors \( \hat{d} \hat{r} \) and \( \hat{d} \hat{b} \) has taken place when quadratic terms have been shifted through. Note that \( \hat{d} \hat{r}^{j} \) is itself a vector, with the index \( j \) indicating which time step it is evaluated at. The form of the transformation is given in Sec. XV and specific single mode examples will be provided in Sec. XV.A.

In Eq. (48), the operators present consist of \( m \) copies of the Heisenberg-Weyl algebra, and the rearrangements can be treated simply with a multimode Baker-Campbell-Hausdorff formula (91). Depending on the value of the index \( j \), we will have a different number of linear terms to commute through when grouping annihilation and creation operators; this is the origin of the double summation in the following

\[
\prod_{j=1}^J e^{d\mathcal{S}_L(jdt)} = \exp \left[ \frac{1}{2} \sum_{j=1}^J \sum_{k=1}^J \hat{d} \hat{r}^{j} \hat{d} \hat{r}^{k} + \sum_{j=1}^J \sum_{k=1}^J \hat{d} \hat{r}^{j} \hat{d} \hat{r}^{k} \right] \times \exp \left[ \hat{b}^\dagger \hat{d} \hat{r}^j \right] \exp \left[ \sum_{j=1}^J \hat{d} \hat{r}^j \hat{b}^\dagger \right].
\]

(49)

\( \hat{d} \hat{r}^{j} \) and \( \hat{d} \hat{r}^{j} \) contain the stochastic Wiener increments of \( \mathcal{O}(\sqrt{\hat{d} \hat{b}}) \). The single index summation that is quadratic in \( \{ \hat{d} \hat{r}^{j}, \hat{d} \hat{r}^{j} \} \) (thus quadratic in noise) will have contributing summations of \( \mathcal{O}(dt) \) and be of a deterministic nature (see Eq. (13)). The double index summation will contain both stochastic (\( k \neq j \)) and deterministic sums (\( k = j \)), as will the sums attached to the operators. The summations should be identified with integrals, but in converting them we need to take care where the summand will lead to an integrand that is itself a function of the noise. This is not the case for the single summations. For the double summation, there are terms containing the product of correlated stochastic increments. To convert to Itô integrals we must adapt the summand to separate these terms (92). That is,

\[
\begin{align*}
\sum_{j=1}^J \sum_{k=1}^J \hat{d} \hat{r}^{j} \hat{d} \hat{r}^{k} &= \sum_{j=1}^J \sum_{k=1}^J \left( \hat{d} \hat{r}^{j} + \sum_{k=1}^{j-1} \hat{d} \hat{r}^{k} \right) \\
&= \int_0^t \hat{d} \hat{r}(\tau) \hat{d} \hat{r}(\tau) + \int_0^t \hat{d} \hat{r}(\tau) \hat{d} \hat{r}(\tau)
\end{align*}
\]

with

\[
r'(\tau) = \int_0^\tau \hat{d} \hat{r}(s)
\]

(51)

for notational convenience. By Itô’s rule \( d\hat{r}(\tau) \hat{d} \hat{r}(\tau) \) is \( \mathcal{O}(\tau) \) so the first integral is deterministic, whilst the second term retains its stochasticity. For clarity, the complete linear component of Eq. (45) is

\[
e^{\mathcal{S}_L(t)} = \exp \left[ \frac{3}{2} \int_0^t \hat{d} \hat{r}(\tau) \hat{d} \hat{r}(\tau) + \int_0^t \hat{d} \hat{r}(\tau) \hat{d} \hat{r}(\tau) \right]
\]

\[
\times \exp \left[ \hat{b}^\dagger \hat{d} \hat{r}(t) \right] \exp \left[ \hat{d} \hat{r}(t) \hat{b} \right]
\]

(52)

\[
= \hat{e}^h \hat{b}^\dagger \hat{d} \hat{r}(t) \hat{b} \hat{e}^h,
\]

(53)

with \( h \) being a scalar non-Gaussian complex-valued stochastic integral. The explicit time dependence of \( \{ \hat{r}', \hat{r}' \} \) has been suppressed for display purposes. The relevance (or lack thereof) of the presence of the non-Gaussian random variable, \( h \), will be discussed in detail in relation to the POVM, in Sec. XIII. For the moment we note that \( h \) has no effect upon the system state, as the non-operator multiplicative factor, \( e^h \), will be removed when the state is normalized. The, perhaps unexpected, factor of \( \frac{3}{2} \) appearing in Eq. (52) is, in part, due to our choice of a normal ordering for \( e^{\mathcal{S}_L(t)} \).

2. Disentangling \( e^{L_Q t} \)

The final result required to complete the SME solution is that, in the mathematical physics literature, is termed
a ‘disentanglement’ of the exponentiated quadratic piece of Eq. (15), \( e^{LQ_t} \). Disentanglement involves splitting the exponential into pieces, with a convenient ordering chosen to allow the calculation of, for example, expectation values. The choice we will make is that which pushes towards normal order. Once again only the heuristic form is provided in this section as the explicit result is obtained from the finite dimensional representation of \( \text{sp}(2m+2, \mathbb{R}) \) (see Sec. [IV] and Sec. [IVA] for details and examples). Using the form of \( L_Q \) given in Eq. (16) we state

\[
e^{LQ_t} = \exp \left[ b^\dagger R b + b^\dagger Db + b^\dagger Lb \right] = \exp \left[ b^\dagger R' b^\dagger \right] \exp \left[ b^\dagger D'b + \delta^\dagger \right] \exp \left[ b^\dagger L'b \right],
\]

with the primes indicating a different function of system parameters. Note the appearance of the scalar \( \delta^\dagger \), due to the disentanglement involving the commutation of quadratic terms.

An obvious objection to our solution (formed by Eq. (53), Eq. (55) and Eq. (45), together with forthcoming explicit expressions for primed variables in Sec. [IV]) is that it still contains the unphysical mode operators (\( \hat{a} \) and \( \hat{a}^\dagger \), within the mixed mode operators \( \hat{b} \) and \( \hat{b}^\dagger \)). These can be removed in the following way. Recall that \( |\tilde{\rho}(0)\rangle = \hat{\tilde{\rho}}(0) \otimes 1 |0\rangle \), where \( \hat{\tilde{\rho}}(0) \) is the physical mode density matrix. Thus all tilde mode operators commute through it to act on \( |0\rangle \). Then Eq. (34) is used to convert all tilde operators to physical mode operators. As the system state is mixed in general (that is, impure) we cannot separately factorize the exponentials of physical and unphysical mode operators, so to perform the conversion we would have to power series expand the exponentials. Alternatively we could express our solution for \( \hat{\tilde{\rho}}(t) \) using physical mode superoperators. Example \( \hat{a}^\dagger \hat{a} \tilde{\rho} = \hat{a}^\dagger \hat{a} \hat{\rho} = \hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} \). We prefer to leave the solution in the thermo-entangled state representation, which can easily be worked with — for example, to find expectation values of a system operator \( \hat{A}(\hat{a}, \hat{a}^\dagger) \) we use

\[
\text{Tr}[\hat{A}(\hat{a}, \hat{a}^\dagger)\hat{\tilde{\rho}}(t)] = \langle 0 \left| \hat{A}(\hat{a}, \hat{a}^\dagger) \right| \hat{\tilde{\rho}}(t) \rangle.
\]

It is also important to comment on the feasibility of our method of solution. In the above derivation, the question of analytic tractability has been focused into the issue of whether the primed functions \( \{ R', L', dr', dl' \} \) can be found. Their form follows from the finite dimensional representation calculations, that are detailed in Sec. [IV] but ultimately 2N x 2N matrices must be characterized and manipulated, with N being the number of physical modes. This involves solving polynomials of degree 2N which becomes insurmountable beyond \( N = 2 \), in general. Thus, we do not claim to provide a practical analytic solution for all Gaussian bosonic SMEs, but rather only those for which the necessary degree 2N polynomials can be solved.

III. PROBABILITY DENSITY OF |\tilde{\rho}(t)⟩ — FINDING THE POVM

Given an initial state, \( |\tilde{\rho}(0)\rangle \) (that can be non-Gaussian), we have shown how to find \( |\tilde{\rho}(t)\rangle \). It has been seen that there is a deterministic factor, \( e^{LQ_t} \), and, in addition, terms involving the stochastic integrals \( \{ h, L', r' \} \). Given recorded measurement currents, the experimentalist can therefore follow the system state, which, when normalized, will be independent of \( h \). However, it is of interest to perform calculations as to the expected characteristics of the system evolution; for this, one requires the probability density of the state at time \( t \), given by \( \varphi(h, L', r') \). For an arbitrary normalized initial state, a POVM, \( \{ \hat{W}_{h, L', r'} : h, L', r' \} \), achieves this via

\[
\varphi(h, L', r'|\tilde{\rho}(0)) = \text{Tr} \left[ \hat{W}_{h, L', r'} \hat{\tilde{\rho}}(0) \right] = \langle 0 \left| e^{LQ_t} e^{S_h(t)} \right| \tilde{\rho}(0) \rangle \times \varphi_{\text{out}}(h, L', r'),
\]

where Eq. (56) has been used with \( A(\hat{a}, \hat{a}^\dagger) = \hat{1} \), and also Eq. (5) and Eq. (53). For clarity, we remind the reader that the thermo-entangled state vacuum, \( |0\rangle \), is the zero-valued eigenvector of \( \hat{g} \) (see Eq. (22)), which differs from the multimode ground state, \( |\text{VAC}\rangle \).

To obtain the POVM from Eqs. (57)–(59), the operator \( e^{LQ_t} e^{S_h(t)} \) needs to be converted into one that only contains physical mode operators. This can be achieved by acting the tilde mode operators backwards onto \( |0\rangle \), via Eq. (34). The POVM, \( \{ \hat{W}_{h, L', r'} : h, L', r' \} \), represents the compiled measurement up until a time \( t \). That is, the information obtained relating to the system at the initial time, \( t = 0 \), can be fully summarized in terms of a finite number of integrals over the continuous measurement record. Later it will be shown that the POVM can be made independent of \( h \), without loss of predictive (or retrodictive) power with regards to measurement outcomes, but for now we proceed with its direct calculation in the thermo-entangled state representation (which has been used elsewhere with regards to retrodiction of the quantum state, see [22]). Comments relating to an alternative method of calculating the POVM, via the adjoint equation (57), will also be given later.

As indicated, we wish to act all unphysical mode operators onto the thermo-entangled state vacuum, so as to obtain physical mode operators: \( |0\rangle \hat{a} = |0\rangle \hat{a}^\dagger \) and \( |0\rangle \hat{a}^\dagger = |0\rangle \hat{a} \). This is non-trivial to achieve, due to the non-commutativity of the terms containing tilde mode operators. Additionally, once all the tilde mode operators are converted we will re-order the POVM towards normal order. These two processes can lead to apparently very complex expressions for the disentangling parameters that can be difficult to simplify.

To proceed with greater simplicity and elegance to the
POVM, we use the following calculational trick. Instead of disentangling \( e^{L \rho(t)} \) to obtain the solution for \( \rho(t) \), as per Eq. (60), we note that \( \langle 0 | e^{L \rho(t)} = 0 \rangle \) is required for the POVM. The thermo-vacuum can be expanded in terms of the standard coherent vacuum (for which \( \hat{a} | \text{VAC} \rangle = \hat{a}^T | \text{VAC} \rangle = 0 \) giving:

\[
\langle 0 | e^{L \rho(t)} = \langle \text{VAC} | e^{\hat{a}^T \hat{a} e^{L \rho(t)}} \text{VAC} \rangle.
\]

It is then clear that the disentanglement required is of \( e^{\hat{a}^T \hat{a} e^{L \rho(t)}} \), which we choose to order as:

\[
e^{\hat{a}^T \hat{a} e^{L \rho(t)}} = \exp \left( \hat{b}^T R^t \hat{b} \right) \exp \left[ \hat{b}^T D^t \hat{b} + \delta'' \right] \times \exp \left[ \hat{a}^T \hat{a} \right] \exp \left( \hat{b}^T L'' \hat{b} \right)
\]

Double primes indicate that a different form (from the single prime matrices) is expected due to the inclusion and embedding of \( e^{\hat{a}^T \hat{a}} \) (appearing as the second last exponential). As a reminder, the vector \( \hat{b} \) contains both tilde and non-tilde operators, whilst \( \hat{a} \) and \( \hat{a} \) are individually ‘unmixed’. The expression, on the RHS of Eq. (61), achieves a simultaneous disentangling and re-ordering of the product of the two exponentials on the LHS of Eq. (61). The explicit form of it is once again found by using the finite dimensional representation of \( \text{sp}(2m + 2; \mathbb{R}) \) (see Sec. [IV] and App. [D2a]). The advantage of this particular disentanglement is that most of the terms annihilate against the multimode coherent vacuum that appears in Eq. (64). This simplifies the disentanglement procedure as we only need to solve for one of the parameters (\( L'' \)). It also removes the need for the tedious and complexifying re-ordering of exponentials when finding the POVM. Up to a constant factor we are thus left with:

\[
\langle \text{VAC} | \exp \left[ \hat{a}^T \hat{a} \right] \exp \left( \hat{b}^T L'' \hat{b} \right) = \langle 0 | \exp \left( \hat{b}^T L'' \hat{b} \right)
\]

where the thermo-entangled vacuum has been reconstituted on the RHS. As the POVM element is an operator, not a superoperator, we aim to express it in terms of physical mode operators only. To facilitate this, \( L'' \) is now written in block form, separating the physical and unphysical modes:

\[
L'' = \begin{bmatrix} L & \hat{L} \\ \hat{L} & L^* \end{bmatrix}.
\]

The requirement that \( L'' \) preserve state matrix hermiticity has been used, which also implies that \( \hat{L}^T = \hat{L} \). Note that Roman font has been used to distinguish the \( N \times N \) block matrices of \( L'' \) from the full matrix (which is \( 2N \times 2N \)), with the double primes dropped for notational simplicity. Additionally, we take \( L''^{\text{physical}} = L'' \), without loss of generality. This allows us to write:

\[
\hat{b}^T L'' \hat{b} = \hat{a}^T \hat{L} \hat{a} + 2 \hat{a}^T \hat{L} \hat{a} + \hat{a}^T \hat{L}^* \hat{a},
\]

so that we can act the tilde exponentials onto \( \langle 0 | \) and convert them to physical mode operators. The ordering of the (commuting) exponential terms is chosen so that the disentanglement parameter matrices \( \{ L, \hat{L} \} \) are altered as little as possible when then conversion takes place:

\[
\langle 0 | e^{2 \hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \hat{L}^* \hat{a}} = \langle 0 | e^{\hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \hat{L}^* \hat{a}} = \langle 0 | e^{\hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \ln(1 + N) \hat{L} \hat{a}} e^{\hat{a}^T \hat{L} \hat{a}}
\]

where the operator identity:

\[
\hat{b}^T r'' = \hat{a}^T r' + \hat{a}^T r''
\]

\[
l'' \hat{b} = l' \hat{a} + l'' \hat{a}.
\]

Then the tilde terms must be brought into contact with \( \langle 0 | \) (or \( | 0 \rangle \), as they could be moved through the physical mode density operator) and converted. In Eqs. (63)–(69), the Hermiticity preservation constraint has been used, and Roman font indicates the block vectors of size \( N \). Then we perform a normal re-ordering of the linear pieces using the finite dimensional representation. Thus, we arrive at the Hermitian POVM element:

\[
\hat{W}_{h, l', r'} = e^{\Delta e^{\hat{a}^T \hat{L} \hat{a}} e^{\hat{a}^T \ln(1 + N) \hat{L} \hat{a}} e^{\hat{a}^T \hat{L} \hat{a}} + d^T L \hat{a}} e^{h} \times \varphi_{\text{ost}}(h, l', r'),
\]

where \( e^{\Delta} \) collects all the constant terms (non-stochastic, non-operator) that have been picked up along the path of our derivation. We have also introduced the stochastic vector, \( d \), defined by:

\[
d = l'^T + 2 L \hat{r}' + \left( 1 + 2 L^T \right) r'.
\]

An explanation of the way in which \( d \) is found is given in Sec. [IV]. Examples of explicit POVMs will be given in Sec. [IV A].

There are two significant issues with the expression for the POVM in Eq. (70). Firstly, the joint ostensible distribution for \( \varphi_{\text{ost}}(h, l', r') \) will be difficult to determine analytically due to the non-Gaussian nature of \( h \). Secondly, all calculations should actually be independent of \( h \) as it does not affect the system state. As an example, the system state is retrodicted via the application of Bayes’ rule,

\[
\varphi(\rho(0)| h, l', r') = \frac{\varphi(h, l', r'| \rho(0)) \varphi(\rho(0))}{\varphi(h, l', r')},
\]

by which the scalar factor \( e^h \) cancels from numerator and denominator, so that the RHS in independent of \( h \). In fact, we see that the dependence of the RHS upon \( \{ l', r' \} \)
is contained in the single stochastic integral, $d$. In [55], the authors suggest resorting to numeric calculation of the ostensible statistics of $h$. Rather than take that approach, we now try to analytically determine $\hat{W}_d$, which represents a minimalistic POVM that, nonetheless, contains all relevant statistics.

A. A simplified POVM, $\hat{W}_d$

The POVM with $h$ absent is straightforward to formally write:

$$\hat{W}_d = \int \hat{W}_{h,d} d^2 h$$

$$= e^{\Delta e^a d + a L^a + e^a \ln(1_N + 2L)a_e a^T L a + d^a}$$

$$\times \int e^h \psi_{\text{opt}}(h,d) d^2 h,$$

(74)

however, once again, the non-Gaussianity of $h$ makes this difficult to evaluate. (Note that in writing Eq. (74) in terms of $d$ rather than $\{h, r\}$ we have just replaced the factor $\psi_{\text{opt}}(h, r')$ by $\psi_{\text{opt}}(h, d)$, as indicated by the form of Eq. (5).) Furthermore, it is far from obvious that the above POVM, when viewed only in a mathematical sense, is capable of giving Gaussian statistics for $d$. Despite this, given an initial Gaussian state, it would be highly surprising if $\varphi(d|\tilde{\rho}(0))$ was not Gaussian, as the SME maps Gaussian states to Gaussian states. To explain further, $d$ is a linear integral of the measurement currents, as is clear from Eq. (71). In turn, the statistics of the measurement currents are given by Eqs. (23)–(24), which have white noise added to linear functions of the first order moments. It is a well-known property of the Kalman filter that these currents will have Gaussian statistics for initial Gaussian states. Given conviction from physical arguments, in App. [3] we show mathematically how $\int e^h \psi_{\text{opt}}(h,d) d^2 h$ does, in fact, provide Gaussian statistics. The essence of the argument is that when the integrals that define $h$ are discretized, it can be seen that $h$ is a linear combination of chi-squared random variables. The integral in Eq. (74) then reduces to Gaussian integrals, which of course provide a Gaussian outcome. In what follows, an explicit expression for the POVM will be found, without having to calculate the integral over $h$ directly.

In order to determine $\hat{W}_d$, we will find its $Q$-function, $\langle \alpha | \hat{W}_d | \alpha \rangle = \varphi(d|\alpha)$. Here, $|\alpha\rangle$ is the N-mode coherent state of amplitude $\alpha$ (rather than a 2-N-mode thermodynamical state). As the $Q$-function of an Hermitian operator is unique, finding it will be sufficient to specify $\hat{W}_d$. From Eq. (74), the $\alpha$-dependent factors of $\langle \alpha | \hat{W}_d | \alpha \rangle$ are simple to find. However, there is $d$-dependence arising from the integral over $h$, and this prevents us immediately inferring the precise Gaussian form of $\langle \alpha | \hat{W}_d | \alpha \rangle$. Instead, we will use the fact that the $\alpha$-dependent factors are known in order to first find $\varphi(d|\alpha)$. In turn, this will fix $\varphi(d|\alpha)$ via an application of Bayes’ theorem:

$$\varphi(d|\alpha) = \frac{\varphi(\alpha|d)\varphi(d)}{\varphi(\alpha)}, \quad (75)$$

To be clear, in this section we are not ultimately interested in performing retrodiction. The utilization of Bayes’ theorem is as a mathematical tool to step from a quantity that we can more easily to determine, to the quantity that is desired.

To infer $\varphi(\alpha|d)$, a normalized Gaussian distribution for $\alpha$, all that is needed is the mean, $(\alpha)$, covariance, $\Gamma$, and pseudo-covariance, $\Upsilon$. These can be determined by equating the $\alpha$-dependent pieces of $\langle \alpha | \hat{W}_d | \alpha \rangle$ with the general form of a multidimensional complex normal distribution $95$

$$\varphi(\alpha|d) = N_1 \exp \left[-(\alpha - (\alpha)_d)^T (P^{-1})^* (\alpha - (\alpha)_d) \right.$$

$$\left. + \Re (\Gamma - (\alpha)_d)^T Q^T (P^{-1})^* (\alpha - (\alpha)_d)) \right],$$

(76)

with $Q = \Upsilon \Gamma^{-1}$, $P = \Gamma^* - Q \Upsilon$ and $(\alpha)_d$ being a function of $d$. $N_1$ is a normalization that depends only on $Q, \Upsilon$. Comparing Eq. (76) with Eq. (74) we can read off the relations for the distribution parameters

$$\frac{1}{2}(P^{-1})^* = L$$

$$\frac{1}{2}Q^T (P^{-1})^* = L$$

$$2L(\alpha)_d - (L^T + L) (\alpha)_d = d$$

(77)

(78)

(79)

$$N_1 = (2/\pi)^N \sqrt{\det(L - L^T L)} \det(L^*)$$

(80)

where the last two equations have been simplified with the use Eqs. (77)–(78). As per the comments below Eq. (56), the feasibility of obtaining explicit analytic expressions depends on the ability of characterizing matrices; in this case it is necessary to invert matrices to find $(\alpha)_d$ from Eq. (79). If $(\alpha)_d$ can be found, an analytic expression for $\varphi(\alpha|d)$ results. For a single mode, the matrices $(L, L^*)$ are, of course, just scalars.

Having found $\varphi(\alpha|d)$, we now consider the two remaining factors in Eq. (75) that are required to fix $\varphi(d|\alpha)$: $\varphi(\alpha)$ and $\varphi(d)$.

For simplicity, we assume no knowledge of $\alpha$ exists before measurement begins; the prior distribution for $\alpha$ is flat and can be represented by a (multi-dimensional) Gaussian of infinite variance. In Eq. (75), $\varphi(\alpha)$ can consequently be treated as a constant factor independent of $\alpha$.

The expression for $\varphi(d)$ (the final factor on the RHS of Eq. (75) yet to be determined) is given by

$$\varphi(d) = \int \varphi(d|\alpha)\varphi(d) d^2 \alpha,$$

(81)

where both $\varphi(d|\alpha)$ and $\varphi(\alpha)$ are Gaussian distributions, as we have argued. As such, the integral over the $N$-dimensional complex plane can be carried out analytically. The only aspect of that we need is that the
resulting Gaussian, for \( \wp(d) \), will have an infinite variance. This follows because if the mean of \( d \) is linearly dependent upon \( \alpha \), and \( \alpha \) has a flat distribution, then \( d \) itself will also have a flat distribution. This linear relationship is inferred from Eq. (79). Consequently, \( \wp(d) \) can be treated as a constant factor independent of \( d \).

Using Eq. (75), we can draw together our knowledge of \( \wp(\alpha|d) \), \( \wp(\alpha) \) and \( \wp(d) \) to write

\[
\wp(d|\alpha) = N_2 \wp(\alpha|d),
\]

with \( \wp(\alpha|d) \) given by Eq. (76) and \( \langle \alpha \rangle_d \) expressed as a function of \( d \) through Eq. (79). \( N_2 \) is a new normalization constant independent of \( \{\alpha, d\} \).

Having determined the \( Q \)-function, \( \langle \alpha \rangle \dot{W}_d | \alpha \rangle \), the operator form can be inferred by inspection of Eq. (73). To explain further, the operator dependence is determined from the \( \alpha \) terms, while the terms without powers of \( \alpha \) provide the scalar factors. We deduce that

\[
\dot{W}_d = N \exp \left[ -2\langle \alpha \rangle_d L(\alpha|d) + \langle \alpha \rangle_d^T L(\alpha|d) \
+ \langle \alpha \rangle_d^T L(\alpha|d) \right] \exp \left[ \dot{a}^\dagger d + \dot{a} L \dot{a}^\dagger \right] \times : \exp \left[ 2\dot{a}^\dagger L \dot{a} + d \dot{a} \right],
\]

with \( N \) a function of \( \{L, \dot{L} \} \) that is fixed by normalization of \( \wp(d|\alpha) \) with respect to \( d \). The normal ordering could be removed via Eq. (67). Note the similarity of Eq. (83) to Eq. (79); the operator factors have remained the same. However, more than just the normalization has been found (from the perspective of \( d \)) as there is quadratic dependence upon \( d \) contained in the exponent of the first exponential. This piece originates from the integral over the non-Gaussian variable, \( \int e^{ib \wp_{\text{ost}}(h, \{l, \dot{l}\})} d^2h \), that we have avoided evaluating directly. If only the operator dependence of \( \dot{W}_d \) had been found, then the correct statistics of \( d \) would remain unknown.

An interesting aspect of Eq. (83) is that, for its specification, only one stochastic integral, \( d \), needs to be tracked. In contrast, the values of \( \{l, \dot{l}\} \) must be known in order to determine the operator \( \wp_{\text{eff}} \). In other words, the probability of obtaining a particular measurement record depends only on \( d \), whereas the system state conditional on a recorded measurement current is dependent upon \( \{l, \dot{l}\} \) (which together are sufficient to fix \( d \)). When examples are provided, we will see that there do exist cases for which one of the two stochastic integrals, \( \{l, \dot{l}\} \), is strictly zero. Thus, there is a natural classification of systems subjected to dyne measurement: whether or not \( \dot{W}_d \) is sufficient to specify \( \wp_{\text{eff}} \).

We conclude this subsection by noting that the POVM is of a Gaussian form, yielding Gaussian statistics for \( d \). Given that a Gaussian operator can be characterized by a vector of means and a covariance matrix, it is natural to wonder whether the result could have been obtained by a more direct approach. In the next subsection, we describe an existing method to obtain the POVM that proceeds in such a manner.

### B. Finding the POVM via the adjoint equation

The linear SME, provided in Eq. (12), describes how the unnormalized system state is updated in time, given the measurement record \( y(t) \). It is possible to use the linear SME to derive a backwards-in-time equation of motion for the effect operator, \( \dot{W} \). Using the backwards increment notation \( d\dot{W}(t) = \dot{W}(t - dt) - \dot{W}(t) \) (with \( dt \) positive), it is straightforward to derive that

\[
d\dot{W}(t) = i \left[ \dot{H}, \dot{W}(t) \right] dt + D^\dagger [\dot{\epsilon}] \dot{W}(t) dt
+ y^T(t - dt) dt \dot{H} \left[ M^T \dot{\epsilon}^\dagger \right] \dot{W}(t).
\]

This generalizes the adjoint equation contained in Eq. (82) by allowing for completely general measurement parameterizations, \( M \). To obtain the POVM element applicable to the entire measurement record, the effect operator needs to be integrated from the final (absolute) measurement time backwards to the time at which measurement is physically turned on, \( t_0 \). Working backwards in time, the appropriate starting value of \( \dot{W} \) is the identity operator, 1, as no measurement data has yet been used in a mathematical sense. The superoperator \( D^\dagger \) is defined by

\[
D^\dagger [\dot{\epsilon}] = \sum_{k=1}^L D^\dagger [\epsilon_k], \quad D^\dagger [\dot{\epsilon}] \rho = \dot{\epsilon}^\dagger \rho \dot{\epsilon} - \frac{1}{2} \dot{\epsilon}^\dagger \dot{\epsilon} \rho - \frac{1}{2} \rho \dot{\epsilon}^\dagger \dot{\epsilon}.
\]

Note that the adjoint equation for \( \dot{W} \), Eq. (84), is a linear equation that preserves Hermiticity. That is, it is the same general form as the linear SME, Eq. (12). Consequently, the methods we have described in our paper for solving the linear SME could be applied to solve the adjoint equation. Specifically, by using a thermo-entangled state representation and then applying techniques from Lie algebra. However, other authors have taken a different approach, designed to avoid much of the complication that we have considered. We now describe this approach, and highlight both its achievements and limitations.

The equation for the effect operator is not trace-preserving, which does not pose a problem for retrodiction of the prior quantum state, as the scalar norm of the effect operator is removed by renormalization (see the discussion below Eq. (72)). However, the approach taken in [31, 32, 41, 96] is to apply a phase space representation of the effect operator, and its first and second order moments can only be found from the normalized phase space distribution. As such, Eq. (84) can be put in an explicitly trace-preserving form:

\[
d\dot{W}(t) = i \left[ \dot{H}, \dot{W}(t) \right] dt
+ \left( D^\dagger [\dot{\epsilon}] \dot{W}(t) + \text{Tr} \left[ (\dot{\epsilon}^\dagger \dot{\epsilon} - \dot{\epsilon}^\dagger \dot{\epsilon}^\dagger) \dot{W} \right] \dot{W} \right) dt
+ s^T (t - dt) dt \dot{H} \left[ M^T \dot{\epsilon}^\dagger \right] \dot{W}(t).
\]
the stochastic term now has the factor \( s \), which is a vector of Wiener increments, related to \( y \) via

\[
s dt = y dt - \text{Tr} \left[ (M^T \dot{c} + M \dot{c}^T) W \right] dt, \quad (87)
\]
as can be inferred from Eqs. (20)–(21). The reader will note the clear analogy of Eq. (86) with the nonlinear SME (see 85), which is explored further in 85. Equation (86) disregards the scalar factor that may depend on the measurement record, which we know exists from Eq. (83). In order to retrieve it, the procedure described in Sec. IIIA could be undertaken.

The duality between Eq. 86 and the nonlinear SME, means that it maps Gaussian effect operators to Gaussian effect operators 85. Given the identity operator \( \hat{W} \), it is straightforward to find dynamical equations for \( x \) and \( V \):

\[
dx dt = x(t) - x(t)
\]

\[
dV dt = \left[ A x dt + (2VB^T + S^T) s dt \right]
\]

\[
(88)
\]

\[
(89)
\]

for matrices \( A = \Sigma (G + \text{Im} \{C^T C\}) \), \( B = \text{Re} \{M^T C\} \), \( S = \text{Im} \{M^T C\} \Sigma^T \) and \( E = \Sigma \text{Re} \{C^T C\} \Sigma^T \).

Eqs. (88) (89) once again represent a generalization of 85 to account for arbitrary ‘dyne’ unravelings. Specifically, a diagonal matrix of detector efficiencies has been replaced by the matrix \( M \), the allowable form of which is dictated by Eq. (15). This generalization also appears in 11, albeit using the U-representation 84 of diffusive monitorings. The U-representation is alternative, but equivalent, to the M-representation that is used in this paper (see discussion around Eq. (15)).

Given that we wish to assign a value of infinity to the variance as a starting point for the backwards-in-time integration, it is worth noting that this will lead to infinite ‘kicks’ to the mean. In order to perform the backwards integration accurately 31, 41 it is, therefore, necessary to introduce a new variable, \( \mathbf{w} = \Lambda x \) for \( \Lambda = V^{-1} \). Both \( \mathbf{w} \) and \( \Lambda \) have a starting value of zero 97. The backwards-in-time equation for \( \mathbf{w} \) is

\[
dw dt = \left[ A + 2S^T B - (E - S^T S) \Lambda \right] \Lambda^T \mathbf{w} dt + \left( 2B^T + \Lambda S^T \right) \mathbf{w} dt.
\]

\[
(90)
\]

For simplicity, it has been written in terms of \( y dt \) rather than \( s dt \). In Sec. IV Eq. (90) will be used to solve the adjoint equation when investigating a single mode example.

Eqs. (88) (89) (or more conveniently Eq. (90)) are Kalman filter equations and are amenable to analytic solution provided sufficiently small system dimension or other simplifying features. It is clear that the derivation here is much simpler than that which we used to obtain Eq. (83). Consequently, we now wish to discuss whether we have arrived at exactly the same object (that is, the POVM for the compiled measurement) and, if not, whether it is of the same utility.

Firstly, as noted above, it lacks the scalar factor that depends on the measurement record in Eq. (83). This stochastic factor is not relevant to retrodiction of the past quantum state, as has been mentioned, because it cancels in the numerator and denominator of Eq. (72). Additionally, the operator part of the POVM can be used to infer the stochastic scalar piece, using the same procedure as Sec. IIIA so that the complete POVM could be retrieved. What has been irretrievably lost in Eqs. (88) (89), as compared to the thermofield method, is the final system state upon completion of the measurement. By using a final condition, namely the identity, for the adjoint equation, some of its generality has been lost; there is not a one-to-one correspondence between the POVM and the final system state.

In summary, finding the POVM via a phase space representation of the adjoint equation provides a simple way to obtain a great deal of information but it does not complete the state-inclusive analysis of the compiled measurement.

IV. METHODS FROM LIE ALGEBRA

In this section, we describe in more detail how to use finite dimensional representations to obtain the primed expressions \( \{R', D', L', dr', dl'\} \) that arise in Sec. IIID. The primes indicate that the process of disentangling and re-ordering operator exponentials has modified these parameters from their form in Eqs. (69)–(71). To be clear, disentanglement generically refers to the breaking up of an exponential of a sum of operator terms into the product of exponentials of simpler terms.

Finite dimensional representations have long been understood as calculational tools in quantum optics 57, 59, 98, and are appropriate to our problem as the re-ordering and disentanglement of exponentiated operators is only a function of the operator commutator relations. A finite dimensional matrix representation that respects the commutators of the algebra can transform an infinite dimensional problem (in the energy eigenbasis of an oscillator) into something much more tractable. Results obtained by explicit calculation in the representation can then be lifted to the abstract level.

The starting point is to identify the relevant algebra (or group) to be represented. We have constructed our target problem with this in mind: \( L_Q \) and \( dS_L(t) \) are at most quadratic functions of the annihilation and creation operators in the physical and unphysical modes. The
operators

\[
\left\{ 1, \mathbf{b}_{\mu}, \mathbf{b}_{\mu}^\dagger, \mathbf{b}_{\mu} \mathbf{b}_{\nu}, \mathbf{b}_{\mu}^\dagger \mathbf{b}_{\nu}, \mathbf{b}_{\mu} \mathbf{b}_{\nu} + \frac{2}{\delta_{\mu,\nu}} \right\}, \quad (91)
\]

for \( \mu, \nu \in 1, \ldots, 2N \), close under commutation to form a subalgebra of the symplectic algebra \( \text{sp}(2m+2; \mathbb{R}) \), where \( m = 2N \) is the total number of modes \( \left[57, 90\right] \) and \( \delta_{\mu,\nu} \) is the Kronecker delta. As a reminder, the use of the vector \( \mathbf{b} \) indicates that the unphysical (tilde) modes introduced via the thermo-entangled state representation are enumerated alongside physical modes, as per Eq. (43). The algebra of interest contains further structure: \( \left\{ 1, \mathbf{b}_{\mu}, \mathbf{b}_{\mu}^\dagger \right\} \) form an ideal and \( \left\{ \mathbf{b}_{\mu} \mathbf{b}_{\nu}, \mathbf{b}_{\mu}^\dagger \mathbf{b}_{\nu}, \mathbf{b}_{\mu} \mathbf{b}_{\nu} + \frac{2}{\delta_{\mu,\nu}} \right\} \) a subalgebra. Also, note that \( \left\{ \mathbf{b}_{\mu}^\dagger \mathbf{b}_{\nu} + \frac{2}{\delta_{\mu,\nu}} \right\} \) alone forms a closed algebra, \( \mathfrak{u}(m) \left[57\right] \).

The smallest faithful finite dimensional representation of \( \text{sp}(4N + 2; \mathbb{R}) \) are square matrices of size \( (4N + 2) \). A smaller representation would mean that we lose distinction between some algebra elements and would not be ‘faithful’. In this paper, the real valued, size \( (4N + 2) \), representation from \( \left[57\right] \) is used (see App. \( \left[C\right] \) where it is specified for the reader’s convenience). This representation of the algebra has the property that each element is either nilpotent (with index 2) or idempotent (when columns and rows entirely of zeroes are stripped away),

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
N_{10} & N_{11} & N_{1-1} & 0 \\
N_{-10} & N_{-11} & N_{-1-1} & 0 \\
N_{00} & N_{01} & N_{0-1} & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
r' - 2R'e^{-D^T}l^T & e^D - 4R'e^{-D^T}l' & 2R'e^{-D^T}J & 0 \\
-2e^{-D^T}l^T & -2J e^{-D^T}l' & J e^{-D^T}J & 0 \\
-2\Delta' + r'Te^{-D^T}l^T & -l' + 2r'Te^{-D^T}l' & -r'Te^{-D^T}J & 1
\end{pmatrix},
\quad (94)
\]

where on the LHS the row and column subscript symbols \( \pm 0 \) are unidimensional labels and the \( \pm 1 \) are \( 2N \) dimensional \( \left[57\right] \). That is, the \( N \)’s represent matrices, vectors and a scalar depending on the subscript labels (and should not be confused with \( N \), the number of physical modes). On the RHS, the \( 2N \times 2N \) matrix \( J \) has been used, and is defined as

\[
J_{\mu\nu} = \begin{cases} 
1, & \text{if } \mu + \nu = n + 1 \\
0, & \text{otherwise}
\end{cases}
\quad (95)
\]

which is a matrix of ones on the anti-diagonal (and zeroes elsewhere). Note that our definitions of the matrices \( \{ \mathbf{R}', \mathbf{L}', r', l' \} \) differ slightly from those of \( \left[57, 90\right] \). Specifically, our matrices have elements distributed in the standard fashion, for example \( \mathbf{R}'_i \) is located in the \( i \)th row and \( j \)th column. From Eq. (43), the disentanglement parameters \( \{ \mathbf{R}', \mathbf{L}', r', l', \Delta' \} \) can easily be found in terms of the \( N \)’s (by block matrix manipulation) and are given in App. \( \left[D\right] \). The RHS of Eq. (43) follows from the Taylor expansion of each of the 3 exponentials of Eq. (93), thus making the exponentiation simple, via a Taylor expansion. Our clear target is to be able to work with \( e^{L_{0t}} \) and \( e^{dS_{L(t)}} \), which form elements of the Lie group \( \text{Sp}(4N + 2; \mathbb{R}) \). To this end, we first consider a general element, \( G \in \text{Sp}(4N + 2; \mathbb{R}) \), and its towards normal order disentanglement:

\[
G = \exp \left[ \mathbf{b}^\dagger \mathbf{r} + \mathbf{b}^\dagger \mathbf{R} b + \mathbf{b}^\dagger \mathbf{D} b + \mathbf{b}^\dagger \mathbf{L} b + \mathbf{b}^\dagger \mathbf{b} \right] \quad (92)
\]

\[
= \exp \left[ \mathbf{b}^\dagger (\mathbf{r}' + \mathbf{R}' b) \right] \exp \left[ \mathbf{b}^\dagger \mathbf{D} b + \mathbf{D}^\dagger \mathbf{b} \right] \times \exp \left[ \mathbf{b}^\dagger \mathbf{L} b + \mathbf{L}^\dagger \mathbf{b} \right],
\quad (93)
\]

where primes indicate that different functional forms are anticipated. The second (towards normal ordered) expression for \( G \) contained in Eq. (43) is what we desire and \( \{ \mathbf{R}', \mathbf{L}', r', l', \Delta' \} \) are fixed by demanding equality with Eq. (92). Without loss of generality, \( \{ \mathbf{R}, \mathbf{L}, \mathbf{r}, \mathbf{l}, \Delta \} \) are written in a symmetric form.

Given a particular choice of finite dimensional representation, \( G \) can now be calculated. We will make progress by doing this somewhat heuristically in the multimode case and then being more definite in the case of a single mode (see Sec. \( \left[IVA\right] \)). This is necessary as the multimode calculation involves solution of degree 2 \( N \) polynomials and becomes intractable for large \( N \). The structure of \( G \) when expressed as Eq. (92) (LHS) and Eq. (93) (RHS), in the chosen representation of \( \left[57, 90\right] \), is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\mathbf{d}r & 1 & 0 & 0 \\
-\mathbf{J} d\mathbf{l}^T & 0 & 1 & 0 \\
\mathbf{d}l \mathbf{d}r - \mathbf{d}l - \mathbf{dr}^T & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\quad (96)
\]

which either terminate at first order or are diagonal, in the chosen representation.

As \( e^{L_{0t}} \) and \( e^{dS_{L(t)}} \) are elements of \( \text{Sp}(4N + 2; \mathbb{R}) \), they have the same form as \( G \) but with, respectively, either the linear or quadratic pieces missing. This can be directly seen from Eqs. (10)–(17), and thus immediately gives a disentanglement of \( e^{L_{0t}} \) as specified in App. \( \left[1\right] \) in terms of the \( N \) block matrices. To find \( dS_{L(t)} \), Eq. (14) will be turned into a matrix equation by substituting the finite dimensional representations of \( e^{L_{0t}} \) and \( e^{dS_{L(t)}} \). Using Eq. (17), and by inspection of Eq. (94), it should be clear that this can be written as
where we have chosen to represent $e^{LQt}$ in terms of the blocks of its exponentiated form rather than by its disentanglement parameters (which can be calculated in terms of the former anyway, see App. D1). The ideal nature of $dS'_L(\tau)$ has been used, as evidenced by the quadratic term being unmodified when shifting through the linear term. Note that the $N$’s are also a function of $\tau$. The primed variables can now be solved, giving

$$
\begin{align*}
\dot{d}' = &dN_{11} + dr^TN_{-11} \\
\dot{d}'' = &N'_{41}d' + N'_{-11}dr.
\end{align*}
$$

These form the increment $dS'_L(jdt)$, as per Eq. (17). Note that there are redundant equations in Eq. (96) from which the primed variables can be solved. This embodies the fact that the $N$’s are interrelated.

In Sec. III, the finite dimensional representation was also used to calculate a disentanglement of $e^{\hat{a}_1^\dagger \hat{a}_1} e^{LQt}$ (see Eq. (61)). This is simply achieved by multiplying the LHS of Eq. (21) by the representation of $e^{\hat{a}_1^\dagger \hat{a}_1}$ and then solving for primed parameters, as detailed further in App. D2.2

The calculation of $d$, which appears in Eqs. (20–21), is similarly done by solving a matrix equation. However, there exists the simplification that the reordering is amongst physical mode operators only, so that $\mathrm{Sp}(2N+2; \mathbb{R})$ elements need be considered, rather than $\mathrm{Sp}(4N+2; \mathbb{R})$.

### A. Further details for a single mode

Reduction to a single physical mode, $N = 1$, will allow further clarification of our methods and also prepare us to directly calculate some specific examples. It is important to note that we have shunted the difficulty of the multimode calculation into the determination and manipulation of the matrices, $N$, contained on the LHS of Eq. (94). As the number of modes grows, this difficulty will become insurmountable, in general, as it will involve the solution of higher order polynomials (beyond quartic). Thus, the single mode, which can be solved with quadratics, represents the most basic application and deserves attention. Firstly, the general form of the single mode linear SME solution and POVM is found and then, in Sec. IV, two examples are considered.

For $N = 1$, the relevant algebra is the two-mode (one of which is unphysical) double (and single)-photon algebra, which is a semisimple subalgebra of $\mathrm{sp}(6; \mathbb{R})$. It is 15 dimensional, consisting of the elements:

$$\{ \hat{1}, \hat{a}, \hat{a}^\dagger, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger, \hat{a}_3, \hat{a}_3^\dagger, \hat{a}_4, \hat{a}_4^\dagger, \hat{a}_5, \hat{a}_5^\dagger, \hat{a}_6, \hat{a}_6^\dagger \}. \tag{98}$$

The single mode versions of equations Eq. (40) and Eq. (17) (tilde operators will be kept explicit rather than combined into a vector with non-tildes) are

$$\begin{align*}
\mathcal{L}_Q = &L\hat{a}^2 + L^*\hat{a}_1^2 + 2\hat{L}\hat{a}\hat{a}_1 + R\hat{a}_1^2 + R^*\hat{a}_1^2 + 2\hat{R}\hat{a}_1^\dagger \hat{a}_1^\dagger \\
&+ D\hat{a}\hat{a}_1^\dagger + D^*\hat{a}_1^\dagger \hat{a}_1 + D\hat{a}_1^\dagger + D^*\hat{a}_1^\dagger. \tag{99}
\end{align*}$$

$$
\mathcal{L}_S(t) = d\hat{a}\hat{a} + d\hat{a}^\dagger \hat{a} + r\hat{a}_1^\dagger + r^* \hat{a}_1^\dagger, \tag{100}
$$

with the relationship of $\{R, L, dr, dl\}$ to the system parameters of the SME given in App. A. Note that the Hermiticity preservation conditions of the SME have been utilized; there is only one independent parameter in each of $\{dl, dr\}$, which we have written as $\{dl, dr\}$. That is, $\hat{d}_l = dl_1$ and $\hat{d}^*= dl_2$. Similarly, we have set $L = L_{11}, \hat{L} = L_{12}$, together with analogous assignments for $\{R, D\}$. Note that here we use $\hat{L}$ as compared with $\hat{L}'$, which is used for $L''$.

When $e^{LQt}$ is calculated in the real, finite $(4N+2) \times (4N+2)$ dimensional representation of $\hat{L}$ (with $N = 1$) it has the following symmetry (due to Hermiticity preservation of the evolution):

$$e^{LQt} = \exp \left[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & D & \hat{D} & 2\hat{R} & 2R & 0 \\
0 & \hat{D}^* & D^* & 2\hat{R}^* & 2R^* & 0 \\
0 & -2\hat{L} & -2L^* & -D^* & -D & 0 \\
0 & -2\hat{L} & -2L & -\hat{R}^* & -\hat{R} & 0 \\
-2C & 0 & 0 & 0 & 0 & 0
\end{array} \right]$$

$$= \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 & 0 \\
0 & s & t & u & v & 0 \\
0 & t^* & s^* & v^* & u^* & 0 \\
0 & w & x & y & z & 0 \\
0 & x^* & w^* & z^* & y^* & 0 \\
c & 0 & 0 & 0 & 0 & 1
\end{array} \right]. \tag{101}$$

The $\tau$ dependent matrix elements $s, t, u, v, w, x, y, z$ (which form the $N$ matrices of Eq. (94)) are inter-related and, in general, have a complicated dependence upon $\{L, R, d\}$. There is no difficulty in their accurate calculation with a symbolic manipulator although their length prohibits their display here. From Eq. (101), it can be seen that the calculation reduces to the exponentiation of the inner $4 \times 4$ block. Further simplifying matters is that the characteristic polynomial is a depressed quartic (no cubic term). This is important as the Cayley-Hamilton theorem states that a matrix satisfies its own eigenvalue equation. The power series expansion for the exponential has no higher than cubic powers of the matrix. The eigenvalues of the matrix define the time scales of the deterministic evolution. As it will often prove useful to consider the simplification that the representation of $e^{LQt}$ is real valued (the system quadratic parameters being real), we provide the eigenvalues for this case. The 4 eigenvalues $(\pm \lambda_{\pm})$ of the inner block of $\mathcal{L}_Q$ are then defined by

$$\lambda_{\pm} = \sqrt{(D + \hat{D})^2 - 4(L \pm \hat{L})(R \pm \hat{R})}. \tag{102}$$

The division of the 4 eigenvalues into 2 pairs illustrates the Hermiticity symmetry which leads to the collapsing
of the $4 \times 4$ matrix analysis into multiple $2 \times 2$ operations. This can be made explicit, when $L_Q \tau$ is real valued, by a unitary transform of the representation of $L_Q \tau$ to a $2 \times 2$ block diagonal form.

The other pieces of Eq. (44), $e^{dS_L(\tau)}$ and $e^{dS_L(\gamma)}$, can be calculated similarly in the finite dimensional representation, which leads us to the analogous expressions to Eq. (97)

\[
\begin{align*}
\tilde{d}l' &= s dl + t^* dl^* + w dr + x^* dr^*, \\
\tilde{d}r' &= u dl + v^* dl^* + y dr + z^* dr^*,
\end{align*}
\]

together with $dl'^* = dl'_2$ and $dr'^* = dr'_2$.

As it represents an important result of this paper, we give the single mode linear SME solution:

\[
\begin{align*}
|\tilde{\rho}(t)\rangle &= e^{L_Q t} e^{S_L(t)} |\tilde{\rho}(0)\rangle \\
e^{L_Q t} &\cong \exp \left[ 2 \tilde{R} \tilde{a} \tilde{a}^\dagger + R \tilde{a}^2 + R^* \tilde{a}^2 \right. \\
&\left. \times \exp \left[ 2 |D' \tilde{a} \tilde{a}^\dagger + D \tilde{a} \tilde{a}^\dagger + D^* \tilde{a} \tilde{a}^\dagger \right. \\
&\left. \times \exp \left[ 2 \tilde{L} \tilde{a} \tilde{a}^\dagger + L \tilde{a} \tilde{a}^\dagger + L^* \tilde{a} \tilde{a}^\dagger \right] \\
e^{S_L(t)} &\cong \exp \left[ r' \tilde{a}^\dagger + r' \tilde{a} \right] \exp \left[ l' \tilde{a} + l' \tilde{a}^\dagger \right],
\end{align*}
\]

where, as described below Eq. (53), the multiplicative non-operator factors contained in $e^{L_Q t}$ and $e^{S_L(t)}$ have been dropped (leading to the use of the '$\cong$' symbol) as they do not affect the system state. The quadratic disentanglement, in terms of $s, t, u, v, w, x, y, z$, is given in App. D.11. Finally, we give the single mode POVM element

\[
\tilde{W}_d = \mathcal{N} \exp \left[ -2 \langle \alpha_d \rangle^2 \tilde{L}^\prime + \langle \alpha_d \rangle^2 \tilde{L}'' + \langle \alpha_d \rangle^2 \tilde{L}''^* \right] \\
\times \exp \left[ d \tilde{a} \tilde{a}^\dagger + L'' \tilde{a}^\dagger \right] : \exp \left[ 2 \tilde{L} \tilde{a} \tilde{a}^\dagger \right] : \\
\times \exp \left[ L''^* \tilde{a}^\dagger + d^* \tilde{a} \right],
\]

with the real and imaginary parts of $\langle \alpha_d \rangle$ related to those of $d$ by

\[
\langle \alpha_d \rangle_{r,i} = \frac{d_{r,i}}{2 \left( L'' - L''_{01} \right)}
\]

\[
|\tilde{\rho}(t + dt)\rangle = \exp \left[ -\frac{\gamma}{2} (K + 1) \left( 1 + \tilde{a} \tilde{a}^\dagger + \tilde{a}^\dagger \tilde{a} + \frac{\frac{\eta}{2(K + 1)} + \frac{(K + 1)(\tilde{a} + \tilde{a}^\dagger) - K(\tilde{a}^\dagger + \tilde{a})}{2K + 1}}{2(K + 1)} \right) dt \\
+ \frac{\gamma}{2} (K + 1) \tilde{a} \tilde{a}^\dagger \right) dt + y(t)dt \right] \left( \sqrt{\frac{\gamma}{2K + 1}} \left( (K + 1)(\tilde{a} + \tilde{a}^\dagger) - K(\tilde{a}^\dagger + \tilde{a}) \right) \right) |\tilde{\rho}(t)\rangle.
\]

To obtain the quadratic, $L_Q$, and linear, $dS_L(\tau)$, evolution components, terms are grouped by operator and comparison with Eqs. (43)–(100) is made. Next, the calculations in the real finite dimensional representation are performed. We note that all the parameters of Eq. (113)

\[
\langle \alpha_d \rangle_{d,\beta} = \frac{d_{\beta}}{2 \left( L'' - L''_{01} \right)}
\]

are real-valued so that the simplifications of App. D.11 can be used. It is interesting that in the case of a zero-temperature ($K = 0$) thermal bath $r' = 0$ and knowledge of the stochastic integral $l'$, that determines the POVM effect, through $d$, is sufficient to specify the system state.
at the end of the measurement period (see Eq. (106)). The disentanglement expressions are readily calculable, although we do not display them here. Instead we study the POVM of the compiled measurement (see Sec. III).

To calculate the POVM parameters \( \{ L'', \tilde{L}'', d \} \), we use the matrix elements of \( e^{L'\omega t} \) and the expressions in App. [D]. This leads to

\[
\tilde{L}'' = L'' = -\frac{(1 - e^{-\gamma t}) \eta}{2 + 4K(1 - \eta(1 - e^{-\gamma t}))} \int_0^t e^{-\frac{1}{2}\gamma y(\tau)} y(\tau) d\tau,
\]

(114)

which are to be substituted into Eq. (107) to obtain the POVM element. We note that [99] provides a general ‘dyne’ POVM for the instantaneous measurement result, but here we form the composite measurement up to a finite time \( t \).

The POVM defined by Eq. (114) can be compared with the literature [3] in the limit that an initial pure state is kept pure; that is, perfect efficiency detection and a zero temperature bath. In this scenario, a stochastic Schrödinger equation (SSE) rather than SME is sufficient. For \( \eta = 1 \) and \( K = 0 \), the POVM simplifies to

\[
\tilde{W}_{\text{pure}} = \exp \left[ \sqrt{\gamma} \int_0^t e^{-\frac{1}{2}\gamma y(\tau)} d\tau \hat{a} - \frac{1}{2}(1 - e^{-\gamma t}) \hat{a}^2 \right] \times \exp \left[ -\gamma \hat{a}^\dagger \hat{a} \right] \times \exp \left[ -\frac{1}{2}(1 - e^{-\gamma t}) \hat{a}^2 + \sqrt{\gamma} \int_0^t e^{-\frac{1}{2}\gamma y(\tau)} d\tau \right],
\]

(115)

which agrees with the result contained in [3].

We began this section by assuming that the \( x \)-quadrature was being measured, but it is easy to retrieve any quadrature can be analyzed. Initially, a canonical transformation such as

\[
\hat{a} \to \hat{a} e^{i\Phi} \quad \text{and} \quad \hat{a}^\dagger \to \hat{a}^\dagger e^{-i\Phi},
\]

(116)

together with the implied tilde conjugate transformation (based on Eq. (31)), could have been made to absorb the quadrature phase. This preserves the commutation relations of \( \hat{a}, \hat{a}^\dagger, \hat{a}^\dagger e^{-i\Phi} \). As the disentanglement and reordering of the group elements is a function only of the commutation relations, this is permissible and is undone at the end of the calculation. Thus, with the replacement \( \hat{a} \to \hat{a} e^{i\Phi} \) and \( \hat{a}^\dagger \to \hat{a}^\dagger e^{-i\Phi} \) in our POVM results (or SME solutions), measurement of any quadrature can be analysed.

1. Adjoint equation

The POVM can also be investigated via the adjoint equation, as per Sec. III[B]. We now take this approach for the case of homodyne detection in the presence of a thermal input bath (that is, the linear SME specified in Eq. (111)). The matrix \( C \) is implicitly defined by Eq. (112), and this allows the calculation of the matrices \( A, B, S \) and \( E \). Using these matrices in Eqs. (88–89) provides the backwards-in-time equations of motion for the phase space representation of the effect operator:

\[
\begin{bmatrix}
\frac{dx}{dt} = \gamma \cdot \begin{bmatrix}
x \\
p 
\end{bmatrix} & \cdot \begin{bmatrix}
2V_{xx} + 1 + 2K \\
2V_{px}
\end{bmatrix} s(t) dt
\end{bmatrix}
\]

(117)

\[
\begin{bmatrix}
\dot{V}_{xx} \\
\dot{V}_{pp}
\end{bmatrix} = \gamma \begin{bmatrix}
1 + 2K & 1 - \eta & 0 \\
1 - \eta & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
V_{xx} & V_{xp} - \eta & V_{px} \\
V_{px} & V_{xx} & \eta V_{px} \\
\eta V_{px} & \eta V_{xx} & \eta V_{px}
\end{bmatrix} - \gamma \begin{bmatrix}
\eta V_{xx} & \eta V_{px} & \eta V_{xx} \\
\eta V_{px} & \eta V_{xx} & \eta V_{px}
\end{bmatrix}.
\]

(118)

For clarity, we will use a backwards-in-time integration variable \( \tau \), and a measurement completion time \( t_m \) (so that \( \tau \leq t_m \)). After the backwards-in-time integration has been performed we can set \( \tau = 0 \) and \( t_m = t \) to obtain the effect operator for a measurement compiled over the same time frame \([0, t]\) as Eq. (114).

From Eq. (118), it can be seen that the equation for \( \dot{V}_{xx} \) is decoupled. It can be solved using an infinite variance Gaussian, having no \( x-p \) correlation, as the final condition (which is a suitable approximation to the identity matrix for the effect operator). We obtain

\[
V_{xx} = \gamma \left( 1 + 2K \right) \left( \frac{1}{\eta (1 - e^{-\gamma (t_m - t)})} - 1 \right).
\]

(119)

In contrast, \( V_{pp} \) does not become finite when it is integrated backwards-in-time, which is consistent with homodyne detection of the \( x \)-quadrature providing no information about the \( p \)-quadrature. By inspection, we also observe that the matrix Riccati equation will not evolve the values of \( V_{xp}, V_{px} \) away from zero.

The solution of Eq. (117) for \( x \) is made problematic due to the divergence of \( V_{xx} \) at \( \tau = t_m \). Consequently, we use Eq. (90) and instead work with \( w_x = x V^{-\dagger}_{xx} \). We do not need to consider \( w_p \) as it is decoupled from \( w_x \), given our final condition together with Eq. (118). Using Eq. (90),

\[
dw_x = -\gamma \left[ \frac{1}{2} (1 - 2\eta) + \Lambda_{xx} (1 - \eta) (1 + 2K) \right] w_x dt \\
+ \sqrt{\frac{2\gamma \eta}{1 + 2K} \left( 1 + \frac{\Lambda_{xx}}{2} (1 + 2K) \right) \eta y} dt,
\]

(120)

where \( \Lambda_{xx} = V^{-\dagger}_{xx} \) and is determined by Eq. (119). After
solving Eq. (120), using \( w_x(t_m) = 0 \) [97], we obtain
\[
w_x = \sqrt{\frac{2\gamma}{\eta}} \frac{e^{\gamma \tau/2}}{1 + 2K \frac{1 - \eta}{1 - e^{-\gamma(t_m - \tau)}}} \times \int_0^{t_m} e^{-\gamma \tau'/2} y(\tau') d\tau',
\]
which can be verified by inspection. Together with Eq. (119), this determines the \( x \)-quadrature mean of the POVM via \( x = v_{xx}^{-1} w_x \). Before providing the expression for \( x \), we extend the integration back to \( \tau = 0 \) and set the measurement turn-off time to \( t_m = t \), as appropriate for considering the compiled measurement from \([0, t] \). This leads to
\[
x = \sqrt{\frac{(1 + 2K)}{2\eta}} \frac{1}{1 - e^{-\gamma t}} \int_0^t e^{-\gamma \tau'/2} y(\tau') d\tau'.
\] (122)

The reader is invited to observe how Eq. (120) dramatically simplifies for \( \eta \to 1 \). It is therefore perhaps surprising that the only manifestation of non-perfect efficiency detection in the expression for \( x \) is in the scalar coefficient.

We can verify the accuracy of Eq. (122) by comparison with the POVM obtained via the thermo-entangled state representation. That is, Eq. (83) is used together with the POVM parameters of Eq. (114). To perform the comparison we note that \( \hat{W}_d \) of Eq. (83) is most easily converted to a Q-phase space function while the adjoint equation method here gives the first and second moments of the Wigner function (resulting from a symmetric ordering of operators). The variances of these distributions are related by the Q-function variance being half a unit of vacuum noise larger than that of the Wigner function. After this adjustment, the variances using the two methods agree. We also find the mean values agree, with \( 2\sqrt{2} x = -d/L' \).

This approach, of using the adjoint equation, reproduces the mean and variance, but does not directly reproduce the non-operator stochastic dependence of the POVM. This is because a normalized version of the adjoint equation, with a specific final condition, is being treated. However, the non-operator dependence could be obtained via the methods of Sec. 11A if desired.

**B. Optomechanical position measurement with squeezing**

In recent work [51], two of the current authors analyzed optomechanical position measurement, with a primary focus on quantum state tomography of the initial state of the mechanical oscillator. The authors worked, for example, in the bad-cavity regime at both zero and blue-detuning [101], in which it is possible to obtain a linear SME for the mechanics alone. By incorporating squeezing (parametric amplification) alongside the measurement, it was found that effectively a homodyne limited measurement can be performed on the mechanical oscillator, despite operating in the weak measurement regime. This is in contrast to the heterodyne limited measurement performed in the absence of squeezing, as the weak measurement does not allow localization of the mechanical position on the timescale of its period of motion. The purpose of the current subsection is to use the theory developed in this paper to obtain expressions describing the quality of the optomechanical tomographic measurement in the zero-detuned limit. To do so, we frame the problem in terms of the generic solutions we have provided in this paper.

‘Zero-detuning’ refers to the local oscillator being on resonance with the cavity that it illuminates. The cavity is then coupled via radiation pressure with the mechanical oscillator (see figure 1(a) in [51]). The linear SME for the mechanical oscillator, after the optical cavity has been adiabatically eliminated, with measurement strength \( \mu \) and in the presence of a thermal bath (coupling \( \gamma \) and thermal phonon occupation \( K_{th} \)) and parametric amplification of strength \( \chi \) inducing squeezing in the \( x \)-quadrature, is [101, 102]

\[
d\hat{\rho}(t) = \frac{\chi}{4} \left[ e^{-i\theta} \hat{a}^2 \hat{a}^\dagger - e^{i\theta} \hat{a}^\dagger \hat{a}^2, \hat{\rho}(t) \right] dt + \left[ (\gamma K + 1) + \mu' \right] \hat{D}[\hat{a}] \hat{\rho}(t) dt + (\gamma K + \mu') \hat{D}[\hat{a}^\dagger] \hat{\rho}(t) dt + \sqrt{\mu'} (\hat{x} \hat{\rho}(t) + \hat{\rho}(t) \hat{x}) y_x(t) dt + \sqrt{\mu'} (\hat{\gamma} \hat{\rho}(t) + \hat{\rho}(t) \hat{\gamma}) y_\rho(t) dt.
\] (123)

Here \( \mu' = \mu \eta \) represents an effective measurement strength, \( K = K_{th} + \mu (1 - \eta)/\gamma \) is an effective bath temperature, \( \hat{x} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}, \hat{\rho} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2} \) are quadrature operators and \( \{y_x, y_\rho\} \) are real-valued stochastic quadrature measurement results. The parameters \( \{\mu', K\} \) are introduced in order to simplify resultant expressions. To place Eq. (123) in the \( M \)-representation, we specify a 3-component column vector of Lindblad operators

\[
\hat{c} = C \hat{x}
\]
\[
= \begin{bmatrix}
\sqrt{\gamma K + \mu'} & 0 \\
0 & \sqrt{\gamma K + \mu'} \\
\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}
\end{bmatrix}
\hat{x}
\]
\[
= \sqrt{\gamma K + \mu'} \begin{bmatrix}
\hat{x} \\
\hat{\rho} \\
\sqrt{\gamma K + \mu'} \hat{a}
\end{bmatrix}
\] (124)

and an \( M \)-matrix that is of size \( 3 \times 6 \) having all zeroes except for \( M_{1,1} = M_{2,2} = \sqrt{\mu'/\gamma K + \mu'} \).

Vectorizing and factorizing, after canonical transformation with \( \Phi = \theta/2 \), allows the coefficients of \( \mathcal{L}_Q \) and
\[ dS_L(t) \]

\[
L = \chi/4, \\
\dot{L} = \gamma(K + 1)/2, \\
R = -\chi/4, \\
\dot{R} = \gamma K/2, \\
D = -\gamma(2K + 1)/2 - 2\mu', \\
\dot{D} = 0, \\
dl = \sqrt{\mu'} \delta x e^{i\theta/2}, \\
dr = \sqrt{\mu'} \delta y e^{-i\theta/2},
\]

(125)

where we have written the complex Wiener increment \( dy = (dy_x + idy_y)/\sqrt{2} \). These expressions are then used in the matrix that is exponentiated in Eq. (101); this results in the matrix elements on the RHS of that equation. From App. D.11 we can then state the solution to the linear SME as per Eqs. (103)–(106). As the POVM is of direct utility in [51], we provide the POVM parameters

\[
L'' = 2\mu' \left( \frac{1}{\gamma + 4\mu' - \chi + \Gamma_- \coth(\Gamma_+ t/2)} - \frac{1}{\gamma + 4\mu' + \chi + \Gamma_+ \coth(\Gamma_+ t/2)} \right),
\]

(126)

\[
\dot{L}'' = 2\mu' \left( \frac{1}{\gamma + 4\mu' - \chi + \Gamma_- \coth(\Gamma_+ t/2)} + \frac{1}{\gamma + 4\mu' + \chi + \Gamma_+ \coth(\Gamma_+ t/2)} \right),
\]

(127)

\[
d = \frac{\sqrt{\mu'}(\gamma - \Gamma_+ + 4\mu' - \chi + 4\gamma K)}{2\gamma K - \chi} \int_0^t e^{-\Gamma_+ \tau/2} y_x(\tau) d\tau
\]

\[-\frac{\sqrt{\mu'}(\gamma - \Gamma_- + 4\mu' + \chi + 4\gamma K)}{2\gamma K + \chi} \int_0^t e^{-\Gamma_- \tau/2} y_p(\tau) d\tau,
\]

(128)

with rates \( \Gamma_{\pm} = \sqrt{(\gamma \pm \chi)^2 + 8\mu'(1 + 2K) + 16\mu'^2} \). For simplicity, \( d \) is provided in the large \( t \) limit, which represents a measurement carried out until the final state is uncorrelated with the initial state. The canonical transformation has been unwound in these expressions.

The approach taken in [51] to assess the tomographic effectiveness of various measurement schemes, was to consider the variance in the retrodictive estimates of the \( x \) and \( p \)-quadratures for an initial coherent state, which we denote as \( \sigma^2_x, p \). Bearing in mind that \( \{L'', \dot{L}'\} \in \mathbb{R} \), it follows from Eq. (126) that

\[
\sigma^2_x = \frac{1}{2} \left( \dot{L}'' - L'' \right) = \frac{1}{2} + \frac{\gamma + \chi + \Gamma_+ \coth(\Gamma_+ T/2)}{8\mu'},
\]

\[
\sigma^2_p = \frac{1}{2} \left( \dot{L}'' + L'' \right) = \frac{1}{2} + \frac{\gamma - \chi + \Gamma_- \coth(\Gamma_- T/2)}{8\mu'},
\]

(129)

which are the results used extensively in [51].

VI. DISCUSSION AND CONCLUSION

In this paper we have shown how the evolution of linear quantum systems subjected to diffusive monitoring can be solved (provided they possess a time-independent Hamiltonian). The treatment of mixed quantum states represents a non-trivial generalization of the previously existing literature [3, 53, 54], which was limited to solving the stochastic Schrödinger equation. As a corollary to obtaining the stochastic master equation (SME) solution, the deterministic master equation (for linear quantum systems) is also solved, using a non-phase space method. Our method of solution was to use the thermo-entangled state representation together with techniques from Lie algebra. The obtained SME solution is in the form of an evolution superoperator that is dependent upon \( 2N \) complex-valued measurement-dependent stochastic integrals (\( N \) being the number of physical modes).

There are a number of uses for such an analytic solution, some of which we now detail. Firstly, one can calculate the possible states, and their probability distribution, resulting from combined Hamiltonian and measurement dynamics. Likewise, a simple method to numerically simulate Lindbladian evolution is obtained by sampling the distribution, in contrast to integrating the SME in the infinite dimensional Hilbert space (where some form of truncation is required). This should also facilitate applications in quantum control, in particular state-based feedback control. Finally, the statistics of the compiled measurement record are captured in a POVM, which is directly utilized in quantum state tomography, by which the initial state is determined via the measurement record. Given that there has been much experimental progress towards manipulating quantum systems, verification of their states is of paramount importance. One of the major techniques to achieve this is homodyne tomography, so our POVM would have direct application. It is worth noting that a SME POVM has been investigated previously in the literature, using the adjoint equation approach; we have shown that the two approaches agree for a simple case, as expected.

The reader may be curious as to why the multimode SME was chosen as the starting point given that its analytic tractability is limited in concert with the solubility of the relevant higher order polynomials (which are found to be of degree \( 2N \) for \( N \) physical modes). Apart from the symmetries of particular higher degree systems leaving the possibility of solution, a reason is that Eq. (124) may provide a very efficient launching pad for numeric investigation of multimode SME bosonic systems. In [57] the infinite dimensional representation of multimode bosonic operators is given and the benefits of numerical integration within the finite representation of the group are detailed. The context in that work is unitary evolution but there is no barrier preventing the extension to stochastic non-unitary dynamics. The alternative of using a Fock space representation is not feasible since the size of the space grows exponentially with the number of modes, and
even then an approximation must be made limiting the
evolution to a finite number of basis states (in say, the
energy eigenbasis). By contrast, the proposed algebraic
approach avoids any energy cut-off and utilizes matrices
that grow in size only quadratically in \( N \), specifically as
\((4N + 2) \times (4N + 2)\). In this context it is worth noting
the work by Galitski \cite{103} which explores Lie methods
in multi mode systems and emphasizes that a trajectory
within a Lie algebra is in some ways more natural than
a Hilbert space representation.

An obvious consideration for future work is whether
the same methods could be applied to other classes of
single mode SMEs, such as for a spin \( \frac{1}{2} \) particle. For ex-
ample, the Pauli operators provide an obvious representa-
tion to work in. However, the time-dependent evolutions (e.g.
measurement terms) are now present in a more com-
plex group, \( SU(2) \), than the Heisenberg-Weyl algebra. It
is worth noting that the thermo-entangled state repre-
sentation for fermions has been developed and is only a
trivial extension to the bosonic case \cite{104}. Another gen-
eralization that could be considered is photo-detection
instead of ‘dyne’ detection. With photo-detection, the
evolution between detection events is deterministic so
can be solved similarly to a standard master equation.
For example, in \cite{74} the waiting time distribution is cal-
culated utilizing a thermo-entangled state approach for a
particular system. In \cite{20} general deterministic evolution
of a Lindblad form is treated, but the evolution between
detections is not of this type.

It has been commented that the adjoint equation to the
SME provides an alternative path for finding the POVM.
It would be of interest to perform a detailed comparison,
for higher dimensional systems, of the difficulty of solv-
ing the Kalman filter as compared with using the finite
dimensional representation to find the POVM param-
ters. This could be both from an analytic and numeric
perspective. Numerically, the Kalman filter has efficient
solution algorithms \cite{105}, which is perhaps not surprising
given that it also reduces the system to a finite number
of variables.

When using the adjoint equation, evolution is attrib-
uted to the system operators rather than the sys-
tem state, in complete analogy to the Heisenberg and
Schrödinger pictures. As a final consideration for future
work, we query whether the class of soluble SMEs could be
expanded (beyond quadratic Hamiltonian with linear
measurements) by moving to an interaction picture, for
which the evolution is split between operators and the
state. The purpose would be to isolate more simple Lie
groups that could be independently solved, before recom-
bining the evolution to obtain the system solution.

**ACKNOWLEDGMENTS**

We wish to thank Kiarn Laverick for very helpful
discussions regarding the adjoint equation. This work
was supported by the Australian Research Council via
discovery project number DP130103715, via the Centre
of Excellence in Engineered Quantum Systems (EQuS),
project number CE110001013 and CE170100009, and via
the Centre for Quantum Computation and Communication
Technology (CQC2T), project number CE110001027
and CE170100012, and by the University of Sydney Fac-
ulty of Science via a Postgraduate Scholarship. We ac-
knowledge the traditional owners of the land on which
this work was undertaken at Griffith University, the Yugg-
gera people, and the traditional owners of the land on
which this work was undertaken at the University of Syd-
ney, the Gadigal people of the Eora Nation.

**Appendix A: Switching between \{\( \hat{H}, \hat{c}, M \)\} and
\{R, D, L, dl, dr\} system descriptions**

In this appendix we give the relationships between
\{G, C, M\} and \{R, D, L, dl, dr\}, with the former being
the initial parameterization of the linear SME given in
Eq. (12) and the latter set being a convenient parameter-
ization used for its solution.

To begin, we move to thermo-entangled state represen-
tation of Eq. (12), given in Eq. (40). Equating the RHS of
Eq. (12) to the initial parameterization of the linear SME given in
Eq. (40) will allow the relationships
\begin{equation}
\{\hat{H}, \hat{c}, M\} \rightarrow \{R, D, L, dl, dr\},
\end{equation}
to be inferred.

Assuming that we are given \( \hat{c} \) in the form
\( \hat{c} = C \hat{x} \), we need to express \( \hat{x} \) in terms of \( \{\hat{b}, \hat{b}^\dagger\} \).
To this end, we write
\begin{equation}
\hat{x} = X\hat{b} + X^*\hat{b}^\dagger,
\end{equation}
with \( X \) a \( 2N \times 2N \) matrix whose elements are defined by
\begin{equation}
\sqrt{2}X_{mn} = \begin{cases} 1, & \text{if } m = 2n - 1 \\ -i, & \text{if } m = 2n \\ 0, & \text{otherwise} \end{cases},
\end{equation}
which, of course, allows us to write
\begin{equation}
\hat{c} = C \left(X\hat{b} + X^*\hat{b}^\dagger\right).
\end{equation}
It is convenient to define the matrix
\begin{equation}
\hat{I} = \begin{bmatrix} 0 & 1_N \\ 1_N & 0 \end{bmatrix},
\end{equation}
such that
\begin{equation}
\hat{b} = \hat{I}\hat{b}.
\end{equation}
For compactness, we also define
\begin{equation}
B = C^TC \quad \text{(A6)}
\end{equation}
\begin{equation}
F = C^TM^*M^TC \quad \text{(A7)}
\end{equation}
\begin{equation}
K = C^TM^*M^TC^* \quad \text{(A8)}
\end{equation}
and, for any matrix $A$ (excluding $A = I$, for which the barring expression is defined in Eq. (A14)),

$$A_T = X^T A X$$  \hspace{1cm} (A9)
$$A_1 = X^T A X^*$$  \hspace{1cm} (A10)
$$A_0 = X^A X^*$$  \hspace{1cm} (A11)
$$A_d = I A I$$  \hspace{1cm} (A12)

Note that the order of operation is defined as superscript first, then subscript followed lastly by 'baring', so that taking the transpose, Hermitian conjugate or complex conjugate will only act on the $A$ matrix and leave the $X$ matrices unaffected. Conjugation by $I$ is performed as a final step. For example, $\bar{Q}$ shows that tribute to the normalization only. Thus, it is sufficient to collect terms in order to compare with Eqs. (10)–(17). The results are

$$L = -\frac{i}{2} (\mathcal{G}_T - \mathcal{G}_d) + \bar{I}B_D - \frac{1}{2}B_T$$
$$-\frac{1}{2} (\bar{B}_d^* + F_T + K_I^* + I K^*_d + \bar{F}^T_1)$$  \hspace{1cm} (A14)
$$R = -\frac{i}{2} (\mathcal{G}_d - \mathcal{G}_T) + \bar{I}B_1 - \frac{1}{2}B_d$$
$$-\frac{1}{2}B_T^* + F_d + K_D^* + \bar{I} K^*_T + \bar{F}^T_1$$  \hspace{1cm} (A15)
$$D = -i (\mathcal{G}_0 - \mathcal{G}_d) + \bar{I}B_T + B^*_1 I$$
$$-\frac{1}{2} (\bar{B}_D + B^*_D + \bar{B}_1 + B^*_1)$$
$$-F_0 - K_D^* - \bar{I} K^*_T - \bar{F}^T_1$$  \hspace{1cm} (A16)
$$d_l = y^T dt \left( M' C X + M^T C^* X^* I \right)$$  \hspace{1cm} (A17)
$$d_r = y^T dt \left( M' C X^* + M^T C^* X I \right).$$  \hspace{1cm} (A18)

**Appendix B: Integrating out $b$ leaves Gaussian statistics for $d$**

We wish to show that for an initial multi-mode coherent state, $d$ has Gaussian statistics. The reason for this choice of initial state is that it evaluates the $Q$-function of the POVM. If the $Q$-function is Gaussian, then the POVM is Gaussian, by definition. From Eq. (14), assuming $\rho_0 = |\alpha_0 \rangle \langle \alpha_0|$, 

$$\varphi(d|\alpha_0) = e^{\Delta + \alpha_0^* d + \alpha_0^* L^* \alpha_0^* + 2 \alpha_0^* L \alpha_0 + \alpha_0^* L^* \alpha_0 + d^* \alpha_0}$$
$$\times \int e^b \varphi_{\text{ost}}(h,d) d^2 h.$$  \hspace{1cm} (B1)

The first exponential contains terms that are linear in $d$, which will shift the Gaussian mean of the distribution (provided that the remaining factors are Gaussian, of course), and also terms independent of $d$, which contribute to the normalization only. Thus, it is sufficient to show that 

$$\varphi'(d|\alpha_0) = \int e^b \varphi_{\text{ost}}(h,d) d^2 h.$$  \hspace{1cm} (B2)

is Gaussian in $d$, in order to prove that $\varphi(d|\alpha_0)$ is itself Gaussian in $d$.

Let us begin by writing the ostensible distribution for the random variables $\{h,d\}$ as an integral over all possible measurement records:

$$\varphi_{\text{ost}}(h,d) = \int \varphi_{\text{ost}}(Y) \delta^2 (h - h_s) \delta^2 (d - d_s) Y dt,$$  \hspace{1cm} (B3)

where $\{h_s, d_s\}$ are complex valued stochastic integrals, detailed in Eq. (B2) (which defines $h_s$) and Eq. (71). The measurement record over all time is represented by $Y$, see Eq. (10), although there has been a minor abuse of notation. In Eq. (B3) $Y dt$ represents a product of the infinitesimal quantities that compose the matrix $Y dt$, rather than the matrix itself (it is anticipated that the reader can quickly observe from context the correct meaning). The bold font Dirac delta function represents a product of delta functions, one for each of the vector argument’s components; for an arbitrary vector $v$, of length $N$, it is given by 

$$\delta(v) = \delta(v_1) \delta(v_2), ..., \delta(v_N).$$  \hspace{1cm} (B4)

The expression for $\varphi_{\text{ost}}(Y)$ is given by Eq. (10), but with a further generalization to a finite duration of time (this will be made explicit shortly). Substituting Eq. (B3) into Eq. (12), we see that the integrals over $h$ are collapsed by the Dirac delta functions. This leaves 

$$\varphi'(d|\alpha_0) = \int e^{h_s \varphi_{\text{ost}}(Y)} \delta^2 (d - d_s) Y dt.$$  \hspace{1cm} (B5)

To progress, the integral is discretized into a very large number, $J$, of time slices, such that $t = J dt$. The ostensible distribution for the entire measurement current is a product of Gaussians

$$\varphi_{\text{ost}}(Y) = \left( \frac{dt}{2\pi} \right)^{J L} \exp \left[ -\frac{1}{2} \sum_{j=1}^{J} y_j^T y_j dt \right],$$  \hspace{1cm} (B6)

where $y_j = \{(y_{j,1}, y_{j,2}...y_{j,2L})^T\}$ is a column vector of measurement results at the time $j dt$ corresponding to the monitoring of the $L$ Lindblad channels.

The stochastic integrals $\{h_s, d_s\}$ can also be discretized. We drop any deterministic contributions to $\{h_s, d_s\}$ that affect the normalization of $\varphi(d|\alpha_0)$ only (as opposed to its moments) and express them as 

$$h_s = \sum_{j,k=1}^{J} y_j^T dt H_{j,k} y_k dt$$  \hspace{1cm} (B7)
$$d_s = \sum_{j=1}^{J} D_j y_j dt.$$  \hspace{1cm} (B8)

From Eq. (17), we see that $h_s$ is the sum of chi-squared random variables. As we are only trying to prove that $d$ has Gaussian statistics, and not find what the mean and
variance actually are, we do not try to specify \( \{ H_{j,k}, D_j \} \), apart from noting that there are deterministic. The dimensions of \( H_{j,k} \), for given \( j,k \), are \( 2L \times 2L \), while \( D_j \) is an \( N \times 2L \) matrix.

The remaining \( 2N \) Dirac delta functions in Eq. (B5) can be used to collapse a further \( 2N \) of the measurement record integrals. For simplicity, we choose to collapse the first time slice, corresponding to the integrals over \( y_1 \), and also only consider \( N = L \), so that all the \( y_1 \) integrals are collapsed (and no others). The case where \( N \neq L \) presents only further notational difficulties. For simplicity, we assume that \( D_1 \) has a left inverse, so that the Dirac delta functions collapse the integrals onto the following value

\[
y_1 dt = D_1^{-1} \left( \mathbf{d} - \sum_{j=2}^{J} D_j y_j dt \right).
\]  

(B9)

The important point is that now \( y_1 \) is set as a linear function of both \( \mathbf{d} \) and the remaining measurement records. In Eqs. (B6)–(B7), there are terms that are respectively linear and quadratic (and independent) in \( y_1 \), which will lead to terms that are linear and quadratic in \( \mathbf{d} \). Substituting for \( y_1 \) in Eqs. (B6)–(B7), and then placing these expressions back into Eq. (B5), we obtain the following form

\[
\varphi' ( \mathbf{d} | \mathbf{a}_0 ) = \int \exp \left[ -\frac{1}{2} \sum_{j=2}^{J} y_j^T \mathbf{U}_j y_j dt + \sum_{j=2}^{J} \mathbf{v}_j^T ( \mathbf{d} ) y_j dt + w ( \mathbf{d} ) \right] Y' dt.
\]  

(B10)

with \( Y' \equiv y_2, \ldots, y_J \). The (here unspecified) \( \{ \mathbf{U}_{j,k}, \mathbf{v}_j ( \mathbf{d} ), w ( \mathbf{d} ) \} \) are independent of the measurement record, but \( \{ \mathbf{v}_j ( \mathbf{d} ), w ( \mathbf{d} ) \} \) do have dependence upon \( \mathbf{d} \). \( \mathbf{v}_j ( \mathbf{d} ) \) will be at most linear in \( \mathbf{d} \), while \( w ( \mathbf{d} ) \) will be at most quadratic. To make contact with standard multidimensional Gaussian integrals, we re-express Eq. (B10) as

\[
\varphi' ( \mathbf{d} | \mathbf{a}_0 ) = \int \exp \left[ -\frac{1}{2} \mathbf{y}'^T \mathbf{U}' \mathbf{y}' dt + \mathbf{v}'^T ( \mathbf{d} ) \mathbf{y}' dt + w ( \mathbf{d} ) \right] Y' dt,
\]  

(B11)

where the dimensions of \( \mathbf{y}', \mathbf{U}', \mathbf{v}' \) are respectively \( 2L(J-1) \times 1, 2L(J-1) \times 2L(J-1), 1 \times 2L(J-1) \). The multidimensional Gaussian integral in Eq. (B11) can be evaluated, giving

\[
\varphi' ( \mathbf{d} | \mathbf{a}_0 ) = \sqrt{\frac{(2\pi)^{2L(J-1)}}{\det \mathbf{U}'}} \times \exp \left[ w ( \mathbf{d} ) + \frac{1}{2} \mathbf{v}'^T ( \mathbf{d} ) \mathbf{U}'^{-1} \mathbf{v}' ( \mathbf{d} ) \right].
\]  

(B12)

Due to \( \mathbf{v}'^T ( \mathbf{d} ) \) being linear in \( \mathbf{d} \) and \( w ( \mathbf{d} ) \) being quadratic in \( \mathbf{d} \), \( \varphi' ( \mathbf{d} | \mathbf{a}_0 ) \) is a Gaussian distribution, which in turn, implies that \( \varphi ( \mathbf{d} | \mathbf{a}_0 ) \) is also Gaussian in \( \mathbf{d} \).

| TABLE I. Matrix representation of bosonic operators |
|-----------------------------------------------|
| Operator | Matrix representation |
|---------|----------------------|
| \( b^\dagger_\mu b_\nu + \frac{1}{2} \delta_{\mu,\nu} \) | \( M_{\mu\nu} - M_{-\nu-\mu} \) |
| \( b^\dagger_\mu b_\nu \) | \( M_{\mu\nu} + M_{-\nu-\mu} \) |
| \( b^\dagger_\mu b_\mu \) | \( 2M_{\mu\mu} \) |
| \( b_\nu b_\nu \) | \( -M_{\mu\nu} - M_{-\nu-\mu} \) |
| \( b^\dagger_\mu b_\mu \) | \( -2M_{\mu\mu} \) |
| \( b_\nu b_\nu \) | \( -M_{\mu0} - M_{-\nu0} \) |
| \( \mathbb{1} \) | \( -2M_{00} \) |

(a) In this appendix, the matrices \( M \) are of dimension \( (4N + 2) \times (4N + 2) \) and have their rows and columns labelled from 0 to \( 2N \) and then from \(-2N \) to \(-0\). \( M_{\mu\nu} \) denotes a matrix that has only a single non-zero component, being equal to one, at the \( \mu \)th row and \( \nu \)th column. Note that a sign correction, to the table contained in [57], has been made in the second row.

Appendix C: Finite dimensional representation

In this appendix, a \( (4N + 2) \times (4N + 2) \) matrix representation of the operators

\[
\left\{ 1, b^\dagger_\mu b_\mu, b^\dagger_\mu b_\nu, b^\dagger_\mu b^\dagger_\nu, b^\dagger_\mu b_\nu + \frac{1}{2} \delta_{\mu,\nu} \right\},
\]  

(C1)

for \( \mu, \nu = 1, \ldots, 2N \), is given. These operators represent a subalgebra of the symplectic algebra \( \text{sp}(4N + 2; \mathbb{R}) \). The representation is faithful in the sense that it preserves the usual bosonic commutation relations of the annihilation and creation operators of the multiple modes. We use the same representation as Ref. [57], but detail it here, in Table I for the reader’s convenience.

To help reduce confusion, we given an example operator matrix for \( N = 1 \). In this case there are two creation operators, one each for the physical and unphysical modes. Arbitrarily, we choose to display the representation of \( \hat{a}^\dagger \hat{a}^\dagger \), which is given by

\[
\hat{a}^\dagger \hat{a}^\dagger = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]  

(C2)

as can be determined from Table II

Appendix D: \( \text{SP}(4N+2;\mathbb{R}) \) disentanglement

1. \( e^{\xi Q^*} \)

   a. Multimode

In this appendix we solve Eq. (D4) for the disentanglement parameters \( \{ R', D', L', d^* \}, dr' \) in terms of the \( N \) matrices. This effectively allows us to factorize the
quadratic evolution $e^{L_Q t}$ (indeed a general quadratic bosonic group element) towards normal order. It is found that

\[ D^{T} = -J \ln \mathbf{N}_{-1-1} J, \]
\[ 2R' = N_{-1}(N_{-1})^{-1} J, \]
\[ 2L' = -J(N_{-1})^{-1} N_{-11}, \]
\[ r^{T} = -N_{-0-1}(N_{-1})^{-1} J, \]

with the matrix $J$ defined in Eq. (95).

b. Single mode

Given the disentangling ansatz of Eq. (105), the expressions in terms of the matrix elements of $e^{L_Q t}$ (see Eq. (101)) are

\[ D' = -\frac{1}{2} \log \left( y^2 - z^2 \right), \]
\[ \dot{D}' = \frac{1}{2} \log \left( \frac{y - z}{y + z} \right), \]
\[ R' = \frac{u y - u z}{2(y^2 - z^2)}, \]
\[ \dot{R}' = \frac{w y - w z}{2(y^2 - z^2)}, \]
\[ L' = \frac{w z - x y}{2(y^2 - z^2)}, \]
\[ \dot{L}' = \frac{x z - w y}{2(y^2 - z^2)}, \]

where we have assumed that any complex parameters in $L_Q$ can be made real by transformation (see the end of Sec. V A). This assumption is performed for simplicity, not necessity.

2. $e^{\tilde{a}^T \tilde{a}} e^{L_Q t}$

a. Multimode

The disentanglement of $e^{\tilde{a}^T \hat{a}} e^{L_Q t}$ that we use in calculating the POVM (see Eq. (61)) can be found from that of $e^{L_Q t}$ with minimal work. If we write $\tilde{a}^T \hat{a} = \frac{1}{2} \hat{b}^T \bar{I} \hat{b}$ (see Eq. (A4) for $\bar{I}$) then the finite dimensional representation of $e^{\tilde{a}^T \hat{a}} e^{L_Q t}$ is

\[ e^{\tilde{a}^T \hat{a}} e^{L_Q t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \]

The RHS of Eq. (61) actually has the same functional form as that of Eq. (55) as the $e^{\hat{b}^T \bar{I} \hat{b}/2}$ term can be absorbed to give $e^{\hat{b}^T (L' + 1/2) \hat{b}}$. Thus the disentanglement is the same as in App. D 1 but with the following replacements:

\[ N_{-11} \rightarrow N_{-11} - \bar{I} N_{11}, \]
\[ N_{-11} \rightarrow N_{-11} - \bar{I} N_{11}, \]
\[ L' \rightarrow L'' + \bar{I}/2. \]

b. Single mode

The representation of $e^{\tilde{a}^T \hat{a}}$ term multiplies the representation of $e^{L_Q t}$ to give

\[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & t & u & v & 0 \\ 0 & t^* & s^* & v^* & u^* & 0 \\ 0 & w - s & x - t & y - u & z - v & 0 \\ 0 & x^* - t^* & w^* - s^* & z^* - v^* & y^* - u^* & 0 \\ c & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \]

The disentanglement order chosen allows the $e^{\tilde{a}^T \hat{a}}$ term to be absorbed into $e^{2L' \tilde{a}^T \tilde{a}}$ to give $e^{(2L' + 1) \tilde{a}^T \tilde{a}}$, so that the disentanglement is the same as in App. D 1 but with the following replacements:

\[ w \rightarrow w - s, \quad x \rightarrow x - t, \quad y \rightarrow y - u, \quad z \rightarrow z - v, \quad L' \rightarrow L'' + \frac{1}{2}. \]

[1] V. Belavkin, Rep. Math. Phys. 43, A405 (1999).
[2] V. Belavkin and P. Staszewski, Phys. Lett. A 140, 359 (1989).
[3] H. Carmichael, An Open Systems Approach to Quantum Optics: Lectures Presented at the Universit Libre de Bruxelles, October 28 to November 4, 1991 (Lecture Notes in Physics Monographs) 1993 edition by Carmichael, Howard (2014) Paperback, 1993rd ed. (Springer, 1600).
[4] H. M. Wiseman and G. J. Milburn, Quantum Measurement and Control 1st ed. (Cambridge University Press, 2014).
[5] H. M. Wiseman, Quantum and Semiclass. Opt. 8, 205 (1996)
[63] H.-Y. Fan and L.-Y. Hu, Commun. Theor. Phys. 51, 321 (2009).
[64] H.-Y. Fan and L.-Y. Hu, Commun. Theor. Phys.(Beijing, China) 51, 506 (2009).
[65] J. Vargas-Martinez, H. Moya-Cessa, and M. Fernandez Guasti, Revista mexicana de fisica E 52, 13 (2006).
[66] H.-Y. Fan and L.-Y. Hu, Commun. Theor. Phys. (Beijing, China) 51, 506 (2009).
[67] J. Vargas-Martinez, H. Moya-Cessa, and M. Fernandez Guasti, Revista mexicana de fisica E 52, 13 (2006).
[68] F. M. Fernandez, Phys. Rev. A 40, 41 (1989).
[69] A. Wünsche, J. Opt. B: Quantum and Semiclass. Opt. 4, 1 (2001).
[70] R. Wilcox, J. Math. Phys. 8, 962 (1967).
[71] A. DasGupta, Am. J. Phys. 64, 1422 (1996).
[72] J. Twamley, Phys. Rev. A 48, 2627 (1993).
[73] J. Zhou, H.-Y. Fan, and J. Song, Int. J. Theor. Phys. 50, 3149 (2011).
[74] L.-Y. Hu and H.-Y. Fan, Int. J. Theor. Phys. 48, 3396 (2009).
[75] H.-Y. Fan and H.-L. Lu, Mod. Phys. Lett. B 21, 183 (2007).
[76] D. S. Kosov, arXiv:1605.02170 (2016).
[77] H.-Y. Fan and Y. Fan, Phys. Lett. A 246, 242 (1998).
[78] T. Arimitsu and H. Umezawa, Prog. Theo. Phys. 74, 429 (1985).
[79] H. Umezawa, H. Matsumoto, and M. Tachiki, (1982).
[80] J. F. Corney and P. D. Drummond, Phys. Rev. A 68, 063822 (2003).
[81] H.-Y. Fan and T. H. Seligman, J. Phys. A: Math. Theo. 43, 392004 (2010).
[82] M. R. Bzerraian, S. M. Ashrafi, and F. Naghdi, Chin. Phys. Lett. 31, 070303 (2014).
[83] H.-Y. Fan and L.-Y. Hu, Opt. Comm. 281, 5571 (2008).
[84] M. Ban, J. Mod. Opt. 56, 577 (2009).
[85] M. A. Nielsen and I. Chuang, Quantum computation and quantum information (Cambridge University Press, 2002).
[86] S. Attal, Institut Camille Jordan, University of Lyon (2014), online lecture notes: Quantum channels.
[87] G. Welch and G. Bishop, An introduction to the Kalman filter (University of North Carolina, Department of Computer Science, 1995).