A LEFT-RIGHT SYMMETRIC MODEL
A LA CONNES-LOTT

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Abstract

We present a left-right symmetric model with gauge group $U(2)_L \times U(2)_R$ in the Connes-Lott non-commutative frame work. Its gauge symmetry is broken spontaneously, parity remains unbroken.

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A Yang-Mills-Higgs model is specified by the choice of a Lie group $G$, representations $\phi$, $\psi_L$, $\psi_R$ for scalars, left and right handed fermions, and a Higgs potential $V(\phi)$. From these data, one then computes the mass matrices (i.e. masses and mixing angles) of the gauge bosons, fermions and scalars. In the Connes-Lott scheme [1], the input consists of an involution algebra $\mathcal{A}$ (which contains the group $G$), of (algebra) representations $\psi_L$, $\psi_R$ and of the fermion mass matrix. The rest, that is the boson mass matrix and the complete Higgs sector, is output. So far only few models have been computed in detail with the Connes-Lott algorithm, the standard model $G = SU(3) \times SU(2) \times U(1)$ [1,2,3], a “chiral version” of electromagnetism $G = U(1)$ [1], a left-right symmetric model with $G = U(1)_L \times U(1)_R$ [1] and a model with $G = U(2) \times U(1)$ [4]. The purpose of the present work is twofold. To get acquainted with the fine points of the Connes-Lott algorithm it seems indispensable to study further examples. Secondly, the most striking success of the Connes-Lott scheme is certainly the geometric explanation of the spontaneous break down of gauge symmetry. It natural to ask whether parity violation can be explained at the same time. To this end, we study the left-right symmetric $U(2)_L \times U(2)_R$ model. As a warm up, we also consider a $U(2)$ model. To get started let us briefly summarize our notations [4].

## 1 Notations

Let $\mathcal{A}$ be a finite dimensional involution algebra (associative, with unit 1 and involution $^*$) and $\rho$ a faithful representation of $\mathcal{A}$ on a finite dimensional Hilbert space $\mathcal{H}$. Let $\chi$, “the chirality”, be a self adjoint operator on $\mathcal{H}$, with $\chi^2 = 1$ and let $\mathcal{D}$, “the (internal) Dirac operator”, be another self adjoint operator on $\mathcal{H}$. Furthermore we suppose that $\rho(a)$ is even:

$$\rho(a)\chi = \chi\rho(a)$$

for all $a \in \mathcal{A}$ and that $\mathcal{D}$ is odd:

$$\mathcal{D}\chi = -\chi\mathcal{D}.$$ 

In other words the representation $\rho$ is reducible and decomposes into a left handed and a right handed part $\rho_L$ and $\rho_R$ living on the left handed and right handed Hilbert spaces

$$\mathcal{H}_L := \frac{1 - \chi}{2} \mathcal{H},$$

$$\mathcal{H}_R := \frac{1 + \chi}{2} \mathcal{H}.$$ 

We can always pick a basis such that

$$\chi = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix}. $$
Then
\[ \rho = \begin{pmatrix} \rho_L & 0 \\ 0 & \rho_R \end{pmatrix}, \]
\[ \mathcal{D} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \]
with \( M \) a matrix of size \( \dim \mathcal{H}_L \times \dim \mathcal{H}_R \), “the mass matrix”. The triple \((\mathcal{H}, \chi, \mathcal{D})\) plays an important role in non-commutative geometry where it is called K-cycle.

The representation \( \rho \) is extended from the algebra \( \mathcal{A} \) to its universal differential envelop \( \hat{\Omega} \mathcal{A} \) by
\[ \pi(a_0 \delta a_1 \ldots \delta a_p) := (-i)^p \rho(a_0)[\mathcal{D}, \rho(a_1)] \ldots [\mathcal{D}, \rho(a_p)]. \]
Although \( \rho \) is faithful, \( \pi \) is not. The central piece of Connes’ theory is the differential algebra
\[ \Omega \mathcal{A} = \bigoplus_{p \in \mathbb{N}} \Omega^p \mathcal{A} \]
defined by
\[ \Omega^0 \mathcal{A} := \rho(\mathcal{A}), \]
\[ \Omega^1 \mathcal{A} := \pi(\hat{\Omega}^1 \mathcal{A}), \]
\[ \Omega^p \mathcal{A} := \frac{\pi(\hat{\Omega}^p \mathcal{A})}{\pi(\delta(\ker \pi)^{p-1})}, \quad p \geq 2. \]

The involution \( * \) is extended to the universal differential envelop \( \hat{\Omega} \mathcal{A} \) by
\[ (\delta a)^* := \delta(a^*) =: \delta a^*, \quad a \in \mathcal{A} \]
and passes to the quotient \( \Omega \mathcal{A} \). We warn the reader that the same symbols, \( \delta \) for the differential and \( * \) for the involution, are used in both differential algebras \( \hat{\Omega} \mathcal{A} \) and \( \Omega \mathcal{A} \).

Since the elements of \( \pi(\hat{\Omega} \mathcal{A}) \) are operators on the Hilbert space \( \mathcal{H} \), i.e. concrete matrices, they have a natural scalar product defined by
\[ < \dot{\phi}, \dot{\psi} > := \text{tr}(\dot{\phi}^* \dot{\psi}), \quad \dot{\phi}, \dot{\psi} \in \pi(\hat{\Omega}^p \mathcal{A}) \]
for forms of equal degree and by zero for the scalar product of two forms of different degree. With this scalar product, the quotient \( \Omega \mathcal{A} \) is a subspace of \( \pi(\hat{\Omega} \mathcal{A}) \), the one orthogonal to the “junk” \( J := \delta \ker \pi \). As a subspace \( \Omega \mathcal{A} \) inherits a scalar product which deserves a special name \( (\ , \ ) \). It is given by
\[ (\phi, \psi) = \text{tr}(\phi^* P \psi), \quad \phi, \psi \in \Omega^p \mathcal{A} \]
where $P$ is the orthogonal projector in $\pi(\hat{\Omega}A)$ onto the ortho-complement of $J$ and $\phi$ and $\psi$ are any representatives in their classes. Again the scalar product vanishes for forms with different degree.

A Higgs (multiplet) or gauge potential $H$ is by definition an antihermitian element of $\Omega^1A$. The Higgses carry an affine representation of the group of unitaries

$$G = \{g \in A, \quad gg^* = g^*g = 1\}$$

defined by

$$H^g := \rho(g)H\rho(g^{-1}) + \rho(g)\delta\rho(g^{-1})$$

$$= \rho(g)H\rho(g^{-1}) + (-i)\rho(g)[D, \rho(g^{-1})]$$

$$= \rho(g)(H - iD)\rho(g^{-1}) + iD.$$  \hspace{1cm} (1)

$H^g$ is the “gauge transformed of $H$”. To motivate the term gauge potential, we note that every $H$ defines a covariant derivative $\delta + H$. This covariant derivative operates on $\Omega A$:

$$^g\psi := \rho(g)\psi, \quad \psi \in \Omega A$$

and is covariant under the left action of $G$:

$$(\delta + H^g) ^g\psi = ^g[(\delta + H)\psi].$$

As usual we define the curvature $C$ of $H$ by

$$C := \delta H + H^2 \in \Omega^2A.$$  

Note that here and later $H^2$ is considered as element of $\Omega^2A$ which means it is the projection $P$ applied to $H^2 \in \pi(\hat{\Omega}^2A)$. The curvature $C$ is a hermitian 2-form with homogeneous gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g)C\rho(g^{-1}).$$

We define the “preliminary Higgs potential” $V_0(H)$, a functional on the space of Higgses, by

$$V_0(H) := (C, C) = \text{tr}[(\delta H + H^2)P(\delta H + H^2)].$$

It is a polynomial of degree 4 in $H$ with real, non-negative values. Furthermore it is gauge invariant, $V_0(H^g) = V_0(H)$ because of the homogeneous transformation property of the curvature $C$ and because the orthogonal projector $P$ commutes with all gauge transformations $\rho(g)P = P\rho(g)$. The transformation law for $H$, equation (1), motivates the following change of variables

$$\Phi := H - iD.$$  \hspace{1cm} (2)
The new variable $\Phi$ transforms homogeneously

$$\Phi^g = \rho(g)\Phi \rho(g^{-1})$$

where the differential is, of course, considered gauge invariant $D^g = D$.

The central result of Connes’ scheme is the following: Let us repeat the procedure outlined above where we tensorize the (finite dimensional) internal algebra $\mathcal{A}$ with the (infinite dimensional) algebra of functions on spacetime and where we tensorize the internal Dirac operator with the genuine Dirac operator. Then the Higgs (in the now infinite dimensional space of 1-forms) consists of a multiplet of scalar fields $H(x)$ and a genuine $G$-gauge potential, i.e. a differential 1-form valued in the Lie algebra of the group of unitaries $G$ of the internal algebra. Furthermore after a suitable regularisation of the trace, (at this point spacetime has to be supposed compact and with Euclidean signature), the preliminary Higgs potential in the infinite dimensional space can be computed and it is the complete bosonic action of a Yang-Mills-Higgs model with, in general, spontaneously broken gauge symmetry $G$:

$$\int \text{tr} F^{\mu \nu} F_{\mu \nu} \sqrt{g} \, d^4 x + \int \text{tr} D^\mu \Phi^* D_\mu \Phi \sqrt{g} \, d^4 x + \int V(H) \sqrt{g} \, d^4 x$$

where (full) Higgs potential is given by

$$V(H) = V_0(H) - \langle \alpha C, \alpha C \rangle = \text{tr} \left[ (C - \alpha C)^2 \right].$$

Here $\alpha$ is the linear map

$$\alpha : \Omega^2 \mathcal{A} \longrightarrow \left[ \rho(\mathcal{A}) + \pi(\delta(\ker \pi)^1) \right]^\mathbb{C}$$

determined by the two equations

$$\langle R, C - \alpha C \rangle = 0 \quad \text{for all } R \in \rho(\mathcal{A})^\mathbb{C}, \quad (3)$$

$$\langle K, \alpha C \rangle = 0 \quad \text{for all } K \in \pi(\delta(\ker \pi)^1)^\mathbb{C}. \quad (4)$$

The scalar product is the finite dimensional one in $\pi(\hat{\Omega}^2 \mathcal{A})$, the $x$ dependence of $C$ can be ignored. Consequently the (full) Higgs potential is still a non-negative, invariant polynomial of degree 4.

2 A $U(2)$ model

We choose as internal algebra $\mathcal{A} = M_2(\mathbb{C})$, the algebra of complex $2 \times 2$ matrices. Both left and right handed fermions come in $N$ generations of doublets, i.e. the fermions are elements of the Hilbert space

$$\mathcal{H} := \mathcal{H}_L \oplus \mathcal{H}_R = \mathbb{C}^2 \otimes \mathbb{C}^N \oplus \mathbb{C}^2 \otimes \mathbb{C}^N.$$
This Hilbert space carries the representation
\[ \rho(a) = \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \rho_R(a) \end{pmatrix} = \begin{pmatrix} a \otimes 1_N & 0 \\ 0 & a \otimes 1_N \end{pmatrix}, \ a \in A \]

The internal Dirac operator is
\[ D := \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \]
where we choose the fermion mass matrix of block diagonal form
\[ M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = e \otimes m_1 + (1 - e) \otimes m_2 = e \otimes \mu + 1 \otimes m_2, \]

\( m_1 \) and \( m_2 \) are complex \( N \times N \) matrices which should be thought of as mass matrices of the quarks of electric charge \( 2/3 \) and \(-1/3 \) and we suppose them different, \( m_1 \neq m_2 \). The total mass matrix \( M \) is chosen block diagonal to ensure conservation of electric charge. We introduced the shorthands
\[ e := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]
\[ \mu := m_1 - m_2 \neq 0. \]

We have to compute \( \Omega^1 A \) and \( \Omega^2 A \). In degree 1, we have
\[
\pi(a_0 \delta a_1) = (-i) \left( \rho_R(a_0)[M^* \rho_L(a_1) - \rho_R(a_1)M^*] \rho_L(a_0)[M \rho_R(a_1) - \rho_L(a_1)M] \right)
\]
\[
= i \left( a_0[a_1e - ea_1] \otimes \mu^* \right) \left( a_0[a_1e - ea_1] \otimes \mu^* \right)
\]
and in degree 2
\[
\pi(a_0 \delta a_1 \delta a_2) = i^2 \left( a_0[a_1e - ea_1][a_2e - ea_2] \otimes \mu \mu^* \right)
\]
\[
= a_0[a_1e - ea_1][a_2e - ea_2] \otimes \mu \mu^* \left( a_0[a_1e - ea_1][a_2e - ea_2] \otimes \mu^* \mu \right).
\]

Let us now show that
\[ \pi(\delta(\ker \pi)^1) = \{0\}. \]

In fact, a general element in \( \pi(\delta(\ker \pi)^1) \) consists of a sum of matrices of the form
\[ i^2 \left( a_0[a_1e - ea_0][a_1e - ea_1] \otimes \mu \mu^* \right)
\]
with the constraint
\[ a_0[a_1e - ea_1] \otimes \mu = a_0[a_1e - ea_1] \otimes \mu^* = 0. \]

Multiplication of the first term in equation (5) by \( 1 \otimes \mu^* \) on the right and by \( e \otimes 1 \) on the left shows that the upper left coefficient in (3) vanishes and by symmetry the entire matrix (3) is
zero. In a similar fashion, the images of all higher kernels are shown to vanish. Therefore, we have in this example

$$\Omega A = \pi(\hat{\Omega} A)$$

to be contrasted to the vector-like model $m_1 = m_2 = 1$ where $\Omega A$ vanishes in all positive degrees.

To compute the Higgs potential, we need an explicit expression of the differential $\delta$ in degree zero and one:

$$\delta \left( \begin{array}{cc} a \otimes 1 & 0 \\ 0 & a \otimes 1 \end{array} \right) = i \left( \begin{array}{cc} [ae - ea] \otimes \mu^* & 0 \\ 0 & 0 \end{array} \right),$$

$$\delta \left( \begin{array}{cc} 0 & ih \otimes \mu \\ ih \otimes \mu^* & 0 \end{array} \right) = \left( \begin{array}{cc} [ehe - (1 - e)h(1 - e)] \otimes \mu \mu^* & 0 \\ 0 & [ehe - (1 - e)h(1 - e)] \otimes \mu^* \mu \end{array} \right).$$

A Higgs is given by

$$H = i \left( \begin{array}{cc} 0 & h \otimes \mu \\ h \otimes \mu^* & 0 \end{array} \right) \in \Omega^1 A, \quad h = h^*.$$  

Its curvature is the hermitian 2-form

$$C := \delta H + H^2 =: \left( \begin{array}{cc} c \otimes \mu \mu^* & 0 \\ 0 & c \otimes \mu^* \mu \end{array} \right) \in \Omega^2 A$$

with

$$c = ehe - (1 - e)h(1 - e) - h^2.$$  

In our example the gauge group is

$$G = \{ g \in A, \; gg^* = g^* g = 1 \} = U(2).$$

Under a gauge transformation $g$, the Higgses transform inhomogeneously

$$H^g := \rho(g) H \rho(g^{-1}) + \rho(g) \delta \rho(g^{-1})$$

$$= \rho(g) H \rho(g^{-1}) + (-i) \rho(g) [D, \rho(g^{-1})]$$

$$= i \left( \begin{array}{cc} 0 & h^g \otimes \mu \\ h^g \otimes \mu^* & 0 \end{array} \right)$$

with

$$h^g = g(h - e)g^{-1} + e$$

while the curvature transforms homogeneously

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g) C \rho(g^{-1}) := \left( \begin{array}{cc} c^g \otimes \mu \mu^* & 0 \\ 0 & c^g \otimes \mu^* \mu \end{array} \right).$$
with
\[ e^g = gcg^{-1}. \]

With the change of variables, equation (2)
\[ \Phi := i \begin{pmatrix} 0 & \phi \otimes \mu^* \\ \phi \otimes \mu & 0 \end{pmatrix} := H - i \begin{pmatrix} 0 & e \otimes \mu \\ e \otimes \mu^* & 0 \end{pmatrix}, \]
we get
\[ \phi = h - e, \quad \text{and} \quad \phi^g = g\phi g^{-1}. \]

In this example the (translated) Higgses sit in the adjoint representation of the gauge group. In terms of \( \phi \), the curvature becomes
\[ c = ehe - (1 - e)h(1 - e) - h^2 = -\phi(1 + \phi) \]
and the preliminary Higgs potential reads
\[ V_0(H) = 2\text{tr} \left( (\mu\mu^*)^2 \right) \text{tr} \left[ \phi^2(1 + \phi)^2 \right]. \]

Here, with \( \pi(\delta(\ker \pi)^1) = \{0\} \), \( \alpha \) is the linear map
\[ \alpha : \Omega^2 \mathcal{A} \to \rho(\mathcal{A}) \]
determined by the equation
\[ \langle R, C - \alpha C \rangle = 0 \quad \text{for all} \ R \in \rho(\mathcal{A}). \]

Therefore
\[ \alpha C = \frac{\text{tr} (\mu \mu^*)}{N} \begin{pmatrix} c \otimes 1 & 0 \\ 0 & c \otimes 1 \end{pmatrix} \]
and the Higgs potential is
\[ V(H) = 2 \left( \text{tr} \left( (\mu \mu^*)^2 \right) - \frac{\left(\text{tr} \mu \mu^*\right)^2}{N} \right) \text{tr} \left[ \phi^2(1 + \phi)^2 \right]. \]

It has two minima, \( \phi = 0 \) and \( \phi = -1 \). Both are gauge invariant and the gauge group is not broken spontaneously. Note that for one generation, \( N = 1 \), the Higgs potential vanishes identically.

3 \quad G = U(2)_L \times U(2)_R

Now our internal algebra is
\[ \mathcal{A} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \ni (a, b). \]
The fermions live in $N$ generations of doublets

$$\mathcal{H} := \mathcal{H}_L \oplus \mathcal{H}_R = \mathbb{C}^2 \otimes \mathbb{C}^N \oplus \mathbb{C}^2 \otimes \mathbb{C}^N,$$

with representation

$$\rho(a, b) = \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \rho_R(b) \end{pmatrix} = \begin{pmatrix} a \otimes 1_N & 0 \\ 0 & b \otimes 1_N \end{pmatrix}, \quad (a, b) \in \mathcal{A}.$$ 

Internal Dirac operator and mass matrix are as in section 1:

$$\mathcal{D} := \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = e \otimes m_1 + (1 - e) \otimes m_2.$$ 

We assume that $m_2$ is not a multiple of $m_1$. In order to compute $\pi(\hat{\Omega}^p \mathcal{A})$, we need the commutator

$$[\mathcal{D}, \rho(a, b)] = \begin{pmatrix} 0 & M \rho_R(b) - \rho_L(a) M \\ M^* \rho_L(a) - \rho_R(b) M^* & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & [e a - b e] \otimes m_1^* + [(b e - a e) - (b - a)] \otimes m_2^* \\ [e b - a e] \otimes m_1 + [(a e - b e) - (a - b)] \otimes m_2 & 0 \end{pmatrix}.$$ 

We find in degree 1

$$\pi(\delta(a, b)) =$$

$$i \begin{pmatrix} 0 & (a e - b e) \otimes m_1 + (a(1 - e) - (1 - e)b) \otimes m_2 \\ (b e - a e) \otimes m_1^* + (b(1 - e) - (1 - e)a) \otimes m_2^* & 0 \end{pmatrix},$$

$$\pi((a_0, b_0) \delta(a_1, b_1)) = \rho(a_0, b_0) \pi(\delta(a_1, b_1))$$

$$= i \begin{pmatrix} 0 & h_1 \otimes m_1 + h_2 \otimes m_2 \\ h_1^* \otimes m_1^* + h_2^* \otimes m_2^* & 0 \end{pmatrix} \quad (7)$$

with

$$h_1 = a_0(a_1 e - e b_1)$$

$$h_2 = a_0[(a_1 - b_1) - (a_1 e - e b_1)].$$

The $\tilde{h}_j$ are obtained from the $h_j$ by interchanging $a$'s and $b$'s. In degree 2, we have

$$\pi(\delta(a_0, b_0) \delta(a_1, b_1)) = \pi(\delta(a_0, b_0)) \pi(\delta(a_1, b_1))$$

$$= \begin{pmatrix} \sum_{j,k=1}^2 x_{jk} \otimes m_j m_k^* & 0 \\ 0 & \sum_{j,k=1}^2 \tilde{x}_{jk} \otimes m_j^* m_k \end{pmatrix}$$

with
\[ x_{11} = (a_0e - eb_0)(ea_1 - b_1e), \quad (8) \]
\[ x_{12} = (a_0e - eb_0) [(a_1 - b_1) - (ea_1 - b_1e)], \quad (9) \]
\[ x_{21} = [(a_0 - b_0) + (a_0e - eb_0)] (ea_1 - b_1e), \quad (10) \]
\[ x_{22} = [(a_0e - eb_0) - (a_0 - b_0)] (ea_1 - b_1e) \\
+ [(a_0 - b_0) - (a_0e - eb_0)] (a_1 - b_1). \quad (11) \]

The \( \tilde{x}_{jk} \) are obtained from the \( x_{jk} \) by interchanging \( a \)'s and \( b \)'s. Trying to rewrite these formulas in terms of the \( h \)'s, we get

\[ x_{11} + x_{22} = h_1e + h_2(1 - e) + e\tilde{h}_1 + (1 - e)\tilde{h}_2, \]
\[ x_{12} = h_1(1 - e) + e\tilde{h}_2, \]
\[ x_{21} = h_2e + (1 - e)\tilde{h}_1, \]

but

\[ x_{11} = a_0ea_1 - a_0a_1e + (h_1 + h_2)e + e\tilde{h}_1 \]

cannot be expressed in terms of the \( h \)'s entirely because of the term \( a_0ea_1 - a_0a_1e \) and similarly for \( x_{22} \).

To write

\[ MM^* = 1 \otimes \Sigma + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \Delta \]

and

\[ M^*M = 1 \otimes \tilde{\Sigma} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \tilde{\Delta}, \]

we introduce the following shorthands

\[ \Sigma := \frac{1}{2}(m_1m_1^* + m_2m_2^*), \]
\[ \Delta := \frac{1}{2}(m_1m_1^* - m_2m_2^*), \]
\[ \tilde{\Sigma} := \frac{1}{2}(m_1^*m_1 + m_2^*m_2), \]
\[ \tilde{\Delta} := \frac{1}{2}(m_1^*m_1 - m_2^*m_2). \]

We are now ready to compute the junk \( J^2 := \pi(\delta(\ker \pi)^1) \). The 1-form in equation (7) vanishes if and only if \( h_1 = h_2 = \tilde{h}_1 = \tilde{h}_2 = 0 \), because by assumption \( m_2 \) is not a multiple of \( m_1 \). In this case we have by equations (8-11)

\[ x_{11} = -x_{22} = a_0ea_1 - a_0a_1e \quad \text{and} \quad x_{11} = x_{22} = 0. \]

Thus,

\[ J^2 = \left\{ \left( \sum_i x_{jk}^i \otimes m_j^* m_k \right), \sum_i \tilde{x}_{jk}^i \otimes m_j^* m_k \right\}, \quad \sum_i h_i^1 = \sum_i h_i^2 = \sum_i \tilde{h}_i^1 = \sum_i \tilde{h}_i^2 = 0 \} \]
\[
\sum \left\{ \left( \sum_i a_i^0 e a_i^1 \otimes \Delta, \sum_i b_i^0 e b_i^1 \otimes \Delta \right), \sum_i a_i^0 a_i^1 = \sum_i b_i^0 b_i^1 = 0 \right\} \\
= \left\{ \left( x \otimes \Delta, 0 \right), x, \bar{x} \in M_2(\mathbb{C}) \right\}.
\]

Equation (12)

To prove the last equality in equation (12), we note that the subspace is a two-sided ideal in the rhs. Furthermore the subspace contains non-zero elements, for instance if:

\[ a_0 := \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad a_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \]

then \( a_0 a_1 = 0 \) and \( a_0 e a_1 \neq 0 \). The algebra \( M_2(\mathbb{C}) \) being simple, the subspace coincides with the whole algebra. Consequently

\[ J^2 = \left\{ \left( x \otimes \Delta, 0 \right), x, \bar{x} \in M_2(\mathbb{C}) \right\}. \]

We need an explicit expression for the orthogonal projector on the ortho-complement of \( J^2 \) in \( \pi(\Omega^2 A) \):

\[
P(x_j \otimes m_j m_k^* \otimes \bar{x}_j \otimes \bar{m}_j m_k) \\
= \left( (x_{11} + x_{22}) \otimes \Sigma' + x_{12} \otimes m_{12}' + x_{21} \otimes m_{21}' \right) \otimes 0 \\
\left( (\bar{x}_{11} + \bar{x}_{22}) \otimes \bar{\Sigma}' + \bar{x}_{12} \otimes \bar{m}_{12}' + \bar{x}_{21} \otimes \bar{m}_{21}' \right)
\]

with

\[ m_{12} := m_1 m_2^*, \quad m_{21} := m_2 m_1^*. \]

The corresponding expressions with tildes are obtained from the ones without by tilding all \( h \)'s or by interchanging all \( m_j \) and \( m_j^* \). The prime denotes projecting out \( \Delta \) and \( \bar{\Delta} \):

\[ \bullet' := \bullet - \frac{\text{tr}(\bullet \Delta)}{\text{tr}(\Delta^2)} \Delta, \]

\[ \bar{\bullet}' := \bar{\bullet} - \frac{\text{tr}(\bar{\bullet} \bar{\Delta})}{\text{tr}(\bar{\Delta}^2)} \bar{\Delta}. \]

Now as operator in \( \Omega^2 A = P\pi(\Omega^2 A) \), the derivation \( \delta : \Omega^1 A \to \Omega^2 A \) takes the following form:

\[
\delta \pi ((a_0, b_0)\delta(a_1, b_1)) = iP\delta \left( h_{j} \otimes m_{j}^* \otimes 0 \right) \\
= \left( x \otimes \Sigma' + x_{12} \otimes m_{12}' + x_{21} \otimes m_{21}' \right) \otimes 0 \\
\left( \bar{x} \otimes \bar{\Sigma}' + \bar{x}_{12} \otimes \bar{m}_{12}' + \bar{x}_{21} \otimes \bar{m}_{21}' \right)
\]

where

\[ x := h_1 e + \bar{h}_1 + h_2 (1 - e) + (1 - e) h_2, \]

\[ x_{12} := h_1 (1 - e) + \bar{h}_2, \]
\[ x_{21} := h_2e + (1 - e)\tilde{h}_1. \]

The Higgs is in \( \Omega^1 A \)
\[ H = i \begin{pmatrix} 0 & h_1 \otimes m_1 + h_2 \otimes m_2 \\ h_1^* \otimes m_1^* + h_2^* \otimes m_2^* & 0 \end{pmatrix}. \]

Its curvature is the 2-form in \( \Omega^2 A \)
\[ C := \delta H + H^2 \]
\[ = \begin{pmatrix} c \otimes \Sigma' + c_{12} \otimes m'_{12} + c_{21} \otimes m'_{21} & 0 \\ 0 & \bar{c} \otimes \bar{\Sigma}' + \bar{c}_{12} \otimes \bar{m}'_{12} + \bar{c}_{21} \otimes \bar{m}'_{21} \end{pmatrix} \]
with
\[ c := h_1e + eh_1^* + h_2(1 - e) + (1 - e)h_2^*, \]
\[ c_{12} := h_1(1 - e) + eh_2^*, \]
\[ c_{21} := h_2e + (1 - e)h_1^*. \]

Now the expressions with tildes are obtained from the ones without upon replacing all \( h_j \) by \( h_j^* \).

Under a gauge transformation
\[ g = (g_L, g_R) \in G = \{ g \in A, \, gg^* = g^*g = 1 \} = U(2)_L \times U(2)_R \]
the Higgs transforms as
\[ H^g = i \begin{pmatrix} 0 & h_1^g \otimes m_1 + h_2^g \otimes m_2 \\ h_1^{g*} \otimes m_1^{g*} + h_2^{g*} \otimes m_2^{g*} & 0 \end{pmatrix} \]
with
\[ h_1^g = g_L h_1 g_R^{-1} - g_L e g_R^{-1} + e, \]
\[ h_2^g = g_L h_2 g_R^{-1} - g_L (1 - e) g_R^{-1} + (1 - e). \]

Again, we pass to the homogeneous Higgs variables
\[ \phi_1 := h_1 - e, \]
\[ \phi_2 := h_2 - (1 - e), \]
\[ \phi_j^g = g_L \phi_j g_R^{-1}, \quad j = 1, 2 \]
and
\[ \Phi = i \begin{pmatrix} 0 & \phi_1 \otimes m_1 + \phi_2 \otimes m_2 \\ \phi_1^* \otimes m_1^* + \phi_2^* \otimes m_2^* & 0 \end{pmatrix} = H - iD. \]

In these variables, the curvature becomes
\[ c = 1 - \phi_1 \phi_1^* - \phi_2 \phi_2^*. \]
\[ c_{12} = -\phi_1 \phi_2^*, \]
\[ c_{21} = -\phi_2 \phi_1^*. \]

There is one and only one gauge invariant point in the space of Higgses namely \( \Phi = 0 \). The curvature of this point is different from zero because \( c = 1 \). The preliminary Higgs potential is

\[
V_0(H) = (C, C) = 2[\text{tr}(c^2) \text{tr}(\Sigma''^2) + \text{tr}(c_{12}^2) \text{tr}(m_{12}^2) + \text{tr}(c_{21}^2) \text{tr}(m_{21}^2)]
\]

and

\[
2\text{tr}(cc_{12}) \text{tr}(\Sigma' m_{12}') + 2\text{tr}(cc_{21}) \text{tr}(\Sigma' m_{21}') + 2\text{tr}(c_{12}c_{21}) \text{tr}(m_{12}' m_{21}')]
\]

and breaks the gauge symmetry spontaneously. The Higgs potential

\[
V(H) = V_0(H) - <\alpha C, \alpha C> = \text{tr}[(C - \alpha C)^2]
\]

is computed with the linear map

\[
\alpha : \Omega^2 \mathcal{A} \rightarrow \rho(A) + \pi(\delta(\ker \pi)^1)
\]
determined by the two equations (3,4). A straightforward calculation yields

\[
\alpha C = \left( [\text{tr}\Sigma' + c_{12}\text{tr}m_{12}' + c_{21}\text{tr}m_{21}'] \otimes Y \begin{pmatrix} 0 \\ \text{tr}\Sigma' + \tilde{c}_{12}\text{tr}m_{12}' + \tilde{c}_{21}\text{tr}m_{21}' \end{pmatrix} \otimes \tilde{Y} \right)
\]

with

\[
Y := \frac{1}{N - (\text{tr}\Delta)^2 / \text{tr}(\Delta^2)} \left( 1 - \frac{\text{tr}\Delta}{\text{tr}(\Delta^2)\Delta} \right).
\]

Therefore also the Higgs potential breaks the gauge symmetry spontaneously (unless there is a numerical accident in the fermionic mass matrix). The vacuum expectation value of \( \Phi \) is any point in the orbit of \( \Phi - H = -i\mathcal{D} \), for instance

\[
\phi_1 = e, \quad \phi_2 = 1 - e,
\]

and the left handed gauge bosons acquire the same masses as the right handed ones. Indeed, parity remains unbroken because the Higgs representation consists of two complex \((2_L, 2_R)\), \( \phi_1 \) and \( \phi_2 \) [5].

4 Conclusion

The main motivation of this work was to find a Connes-Lott model, that enjoys spontaneous breaking of gauge symmetry and parity simultaneously. This hope was spoiled. In a general left-right symmetric model, e.g. \( \mathcal{A} = M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \), we are unable to compute explicitly the junk and the differential \( \delta \) from \( \Omega^1 \mathcal{A} \) to \( \Omega^2 \mathcal{A} \) and we are therefore unable to decide whether the
gauge symmetry is broken or not. Nevertheless, it is pretty clear that any vacuum expectation, that might come out, will be an element of $\rho_L \otimes \rho_R$ and parity preserving.

Chamseddine & Fröhlich [6] have considered the left-right symmetric, grand unified $SO(10)$ model in the Connes-Lott setting. Without computing the junk $J^2$, they also conclude that parity breaking does not occur.

There are two alternative algorithms applying non-commutative geometry to particle physics. One is due Dubois-Violette, Madore & Kerner [7]. In their scheme the differential algebra $\Omega A$ is defined in terms of derivations and does not depend on fermion representations. The other algorithm, due to Coquereaux [8], takes the differential algebra as starting point and is thereby more flexible. Both algorithms also yield spontaneous breakdown of gauge symmetry and it would be interesting to know if they can accommodate spontaneous parity violation.

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