THE $T$ AND $T^*$ COMPONENTS OF $\Lambda$ - MODULES AND LEOPOLDT’S CONJECTURE

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Abstract. The conjecture of Leopoldt states that the $p$ - adic regulator of a number field does not vanish. It was proved for the abelian case in 1967 by Brumer, using Baker theory. A conjecture, due to Gross and Kuz’min will be shown here to be in a deeper sense a dual of Leopoldt’s conjecture with respect to the Iwasawa involution. We prove both conjectures for arbitrary number fields $K$. The main ingredients of the proof are the Leopoldt reflection, the structure of quasi - cyclic $Z_p[\text{Gal}(K/Q)]$ - modules of some of the most important $\Lambda[\text{Gal}(K/Q)]$ - modules occurring ($T$ acts on them like a constant in $Z_p$), and the Iwasawa skew symmetric pairing.

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1. **Introduction**

Let $\mathbb{K}/\mathbb{Q}$ be a finite galois extension with group $G$. Dirichlet’s unit theorem states that, up to torsion made up by the roots of unity $W(\mathbb{K}) \subset \mathbb{K}^\times$, the units $E = \mathcal{O}(\mathbb{K})^\times$ are a free $\mathbb{Z}$ - module of $\mathbb{Z}$ - rank $r_1 + r_2 - 1$. As usual, $r_1$ and $r_2$ are the numbers of real, resp. pairs of complex conjugate embeddings $\mathbb{K} \hookrightarrow \mathbb{C}$. Let $p$ be a rational prime. We consider the set $P = \{ \wp \subset \mathcal{O}(\mathbb{K}) \colon (p) \subset \wp \}$ of distinct prime ideals above $p$ and let

$$
\mathfrak{K}_p = \mathfrak{K}_p(\mathbb{K}) = \prod_{\wp \in P} \mathbb{K}_\wp = \mathbb{K} \otimes \mathbb{Q} \mathbb{Q}_p
$$

be the product of all completions of $\mathbb{K}$ at primes above $p$. Let $\iota : \mathbb{K} \hookrightarrow \mathfrak{K}_p$ be the diagonal embedding. We write $\iota_\wp(x)$ for the projection of $\iota(x)$ in the completion at $\wp \in P$. If $y \in \mathfrak{K}_p$, then $\iota_\wp(y)$ is simply the component of $y$ in $\mathbb{K}_\wp$. If $U \subset \mathfrak{K}_p^\times$ is the group of units, thus the product of local units at the same completions, then $E$ embeds diagonally via $\iota : E \hookrightarrow U$. Furthermore one can use $\iota$ for inducing a galois structure on $\mathfrak{K}_p$ (see §2.1).

Let $E = \iota(E) \subset U$ be the closure of $\iota(E)$; this is a $\mathbb{Z}_p$ - module with $\mathbb{Z}_p$-rk$(E) \leq \mathbb{Z}$-rk$(E) = r_1 + r_2 - 1$. The difference

$$
D(\mathbb{K}) = (\mathbb{Z}$-rk$(E)) - (\mathbb{Z}_p$-rk$(E))
$$

is called the **Leopoldt defect**. The defect is positive if relations between the units arise in the local closure, which are not present in the global case. Equivalently, if the $p$ - adic regulator of $\mathbb{K}$ vanishes.

Leopoldt suggested in [21] that $D(\mathbb{K}) = 0$ for all number fields $\mathbb{K}$. This conjecture of Leopoldt was proved for abelian extensions by Brumer [7] in 1967, using a result of Ax [5] and a local version of Baker’s linear forms in
logarithms \[6\]. It is still open for arbitrary non abelian extensions. Since 1967 various attempts have been made for extending the results of \[7\] to non abelian extensions, using class field theory, Diophantine approximation or both. The following very succinct list is intended to give an overview of various approaches rather than a extensive list of results on Leopoldt’s conjecture. In \[12\], Greenberg notes for the first time a relation between the Leopoldt Conjecture and a special case of the Greenberg Conjecture: he shows that Leopoldt’s Conjecture implies that \( B \) (see \( \S 1.1 \) for the definitions) is finite for totally real fields, i.e. the Greenberg Conjecture for holds for the \( T \) - part. The Conjectures of Leopoldt and Gross (see below) are equivalent to Greenberg’s Conjecture for the \( T \) and \( T^* \) - parts of \( A \); see also \( \S 6 \) for more details on the connections between the Greenberg Conjecture and those treated in this paper.

The works of Federer \[9\] and and Federer and Gross \[10\] use Iwasawa theory and introduce the related Conjecture known as Gross’s Conjecture, which will be proved in Theorem \( A \) and again, together with the Proposition \( 5 \). In an earlier paper, Kuz’mín \[18\] had stated the ’Hypothesis 3’ asserting that \( A'/(A')^T \) is finite. It is this version of the Gross Conjecture that we shall prove; since it is identical to Hypothesis 3, we shall speak of the Gross-Kuz’mín Conjecture. Jaulent relates the Leopoldt and Gross Conjectures in the wider context of a Conjecture which carries his name \[16\]; the Conjecture of Jaulent is beyond the scope of this paper. Emsalem, Kissilewski and Wales \[8\] use group representations and Baker theory for proving the Conjecture for some small non abelian groups; this direction of research has been continued by Emsalem or Emsalem and coauthors, in some further papers. Jaulent proves in \[15\] the Conjecture for some fields of small discriminants, using the phantom field \( \Phi \) which we shall define below. The strongest result based on Diophantine approximation was achieved by Waldschmidt \[24\], who proved that if \( r \) is the \( \mathbb{Z} \) - rank of the units in the field \( K \), then the Leopoldt defect verifies \( D(K) \leq r/2 \).

It is easy to show that if \( K'/Q \) is a field such that Leopoldt’s Conjecture holds for some galois extension \( K/Q \) which contains \( K' \), then it holds for \( K' \). See for instance \[19\], the final remark on p. 108. Likewise, if the Gross - Kuz’mín Conjecture holds for \( K \), then it holds for \( K' \). We may thus concentrate on galois extensions of \( Q \) and shall assume in the rest of this paper that \( K/Q \) is galois and contains the \( p \)-th roots of unity; in particular \( K \) is complex. The Dirichlet number is \( r = r_2 - 1 \) and the \( p \) - adic rank of \( E \) is \( r_p = r - D(K) \). Furthermore, we assume that \( K \) is such that all the primes above \( p \) are completely ramified in the \( \mathbb{Z}_p \) - cyclotomic extension \( K_\infty/K \) and the Leopoldt defect is constant for all intermediate fields of this extension. This can be assumed by choosing \( K \) sufficiently large, according to \[25\], Lemma 13.30; although the field \( F \) is assumed in the whole section

\footnote{See also, the ’generalized Gross Conjecture’, (iii) in \[20\], p. 854}
of Washington’s book to be totally real, the proof of 13.30 does not depend on this assumption, but see also Appendix A, proof of Lemma 13.30.

The main purpose of the paper is to prove:

**Theorem 1.** The Conjectures of Leopoldt and Gross - Kuz’min hold for all number fields $\mathbb{K}/\mathbb{Q}$.

The philosophy of this paper is reflected in the following targets:

A. Avoid $\mu$-parts and pseudo-isomorphisms as much as possible, by considering adequate fields and their galois groups.

B. If $X$ is a $\Lambda$-module, find explicite embeddings of abstract modules like the dual $X^\bullet$ and the adjoint $\alpha(X)$ and the double adjoint $\alpha(\alpha(X))$, whenever possible in $X$ itself. For this, we construct explicite radicals for finite and infinite Kummer extensions and develop infinite Kummer extensions together with their radicals as projective modules, from scratch. This is done in §2.4 and yields also a solution to the embedding problem.

C. Give a precise description of intermediate extensions of $L/K_\infty$ for arbitrary extensions $L$ such that $\text{Gal}(L/K_\infty)$ is a $\Lambda$-module. This is done in §2.3

D. Assume that the Leopoldt Conjecture is false; there is then a sequence of fields, which emerge naturally from conditions imposed by reflection and class field theory, and which all have $\mathbb{Z}_p$-free groups of rank $D(K_n)$. These shall be called phantom fields. The necessary properties resulting from Iwasawa and class field theory shall be followed from one consequence to another, eventually leading to a contradiction. This contradiction does indeed arise and it is caught in Proposition 5.

In view of this program, some of the theoretical constructs go beyond the immediate requirements for the proof of the Conjectures under investigation. Rather than introducing artificial reduction of generality, we chose to defer the proof of technical and elementary facts to the Appendices which grow accordingly.

1.1. **General notations.** Let $\mathbb{K}_\infty/\mathbb{K}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{K}$ and $\mathbb{K}_n$ the intermediate fields of level $n$. The ground field is $\mathbb{K}$, a complex galois extension which contains the $p^{k+1}$-th but not the $p^{k+2}$-th roots of unity. The constant $k$ will be fixed in Definition 3 below, such that the $\Lambda$-modules related to $\mathbb{K}$ have some useful additional properties; we shall denote then by $\kappa$ the particular value chosen. We write $\mathbb{K} = \mathbb{K}_0 = \mathbb{K}_1 = \mathbb{K}_2 = \ldots = \mathbb{K}_k$. As usual, we let $\tau$ be a topological generator of $\Gamma = \text{Gal}(\mathbb{K}_\infty/\mathbb{K})$ and $T = \tau - 1$, $\Lambda = \mathbb{Z}_p[[T]]$. We shall assume that the Leopoldt defect $D(\mathbb{K}_n)$ is constant for all $n \geq k$.

For all $n \geq 0$ we let $A_n$ be the $p$-Sylow subgroups of the class group $\mathcal{C}(\mathbb{K}_n)$ and $A$ the projective limit, a $\Lambda$-module. The groups $A_n', A'$ are defined as $A_n, A$, with respect to the class groups of $p$-integers: see also
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the numeration respects the same rule as for the intermediate fields. The norms $N_{m,n} = N_{K_m/K_n}$ for $m > n \geq k$ are supposed to be surjective as maps $A_m \to A_n$ and $A'_m \to A'_n$. The product of all $\mathbb{Z}_p$ - extensions of $K$ is $\mathbb{M}$ and $\Delta = \text{Gal}(\mathbb{M}/K)$; thus $K_{\infty} \subset \mathbb{M}$. We let further $\mathbb{H}, \Omega$ be the maximal $p$ - abelian extensions of $K_{\infty}$, which are unramified, respectively $p$ - ramified. Note that we do not use the index $\infty$ for $H$ and will write instead $H(K) = H_0$ for the Hilbert class field of $K$. For some field $K$ we write $E(K), E'(K)$ for the units respectively the $p$ - units of $K$. We shall consider the following additional subfields of $\Omega$:

$$\Omega_E = \bigcup_{n \geq 0} K_n \left[ E(K_n)^{1/p^{n+1}} \right], \quad \Omega_{E'} = \bigcup_{n \geq 0} K_n \left[ E'(K_n)^{1/p^{n+1}} \right],$$

so $K_{\infty} \subset \Omega_E \subset \Omega_{E'} \subset \Omega$.

If $X$ is some infinite abelian group, we write $X^\circ$ for its $\mathbb{Z}_p$ - torsion (which may include some torsion $\Lambda$ - module of infinite $p$ - rank) and for $\Omega \supseteq F \supseteq K_{\infty}$, some infinite extension, we shall write

$$F = X^\circ \text{Gal}(F/K_{\infty})^\circ$$

for the fixed field of the $\mathbb{Z}_p$ - torsion of galois group of this field. Thus $\text{Gal}(F/K_{\infty})$ is a free $\mathbb{Z}_p$ - module, possibly of infinite rank. This construction also cancels eventual $\mu$ - parts in our modules, so we need no assumptions on the vanishing of the $\mu$ - part of $A$. In general $\mathbb{H} \neq \mathbb{H}$ and $\Omega = \Omega$ iff $\mu(K) = 0$, while $\Omega_E = \Omega_E$ and $\Omega_{E'} = \Omega_{E'}$. We may thus use both simple and barred notations for these fields.

The general notations from Iwasawa theory which we use here are:

- $p$ A rational prime,
- $X^\circ$ The $\mathbb{Z}_p$ - torsion of the abelian group $X$,
- $\zeta_{p^n}$ Primitive $p^n$-th roots of unity with $\zeta_{p^n}^k = \zeta_{p^n-1}$ for all $n > 0$,
- $\mu_{p^n}$ \{ $\zeta_{p^n}^k, \, k \in \mathbb{N}$ \},
- $K$ A galois extension of $\mathbb{Q}$ containing the $p$-th roots of unity
- $K_{\infty}, K_n \in \text{The cyclotomic } \mathbb{Z}_p \text{- extension of } K$, resp. its $n$-th intermediate field,
- $\text{Gal} \left( K/K_{\infty} \right)$,
- $s$ The number of primes above $p$ in $K$,
- $\Gamma$ $\text{Gal}(K_{\infty}/K) = \mathbb{Z}_p \tau, \quad \tau$ a topological generator of $\Gamma$
- $T$ $\tau - 1$,
- $\ast$ Iwasawa’s involution on $\Lambda$ induced by $T^* = (p^{k+1} - T)/(T + 1)$,
- $\Lambda$ $\mathbb{Z}_p[[T]], \quad \Lambda_n = \Lambda/(\omega_n \Lambda),$
- $\omega_n$ $(T + 1)^{p^{n-(k+1)} - 1}, \quad (K_{\infty})^{\omega_n} = \{ 1 \}$,
- $N_{m,n}$ $N_{K_m/K_n} = N_{K_n/K_m}; \quad N_p = N_{K_{\infty}/K},$
- $K_{\infty}$ $\cup_{n \geq 0} K_n$ : The cyclotomic $\mathbb{Z}_p$ - extension of $K$. 
\[ A_n = A(\mathbb{K}_n) \quad \text{The } p \text{- part of the ideal class group of } \mathbb{K}_n, \]
\[ A = \lim \frac{A_n}{A_n}, \]
\[ A'_n = A'(\mathbb{K}_n) \quad \text{The } p \text{- part of the ideal class group of the } p \text{- integers of } \mathbb{K}_n, \]
\[ A' = \lim \frac{A'_n}{A'_n}, \]
\[ \mathcal{D}(\mathbb{K}) \quad \text{The Leopoldt defect of the field } \mathbb{K}, \]
\[ \mathcal{B} \quad \{b = (b_n)_{n \in \mathbb{N}} \in A : \text{The classes } b_n \text{ contain products of ramified primes}\}, \]
\[ \Omega \quad \text{The maximal } p \text{- abelian } p \text{- ramified extension of } \mathbb{K}_\infty, \]
\[ \Omega_E \quad \cup_{n=0}^\infty \mathbb{K}_n[E(\mathbb{K}_n)^1/p^{n+1}] = \mathbb{K}_\infty[E^1/p^\infty], \]
\[ \Omega_{E'} \quad \cup_{n=0}^\infty \mathbb{K}_n[E'(\mathbb{K}_n)^1/p^{n+1}] = \mathbb{K}_\infty[E'^1/p^\infty], \]
\[ \mathbb{M} \quad \text{The product of all } \mathbb{Z}_p \text{ extensions of } \mathbb{K}, \]
\[ \mathbb{H} \quad \text{The maximal } p \text{- abelian unramified extension of } \mathbb{K}_\infty. \]

Some galois groups repeatedly used, are
\[
\begin{align*}
\mathfrak{X} &= \text{Gal}(\mathbb{H}/\mathbb{K}_\infty), & \overline{\mathfrak{X}} &= \text{Gal}(\mathbb{\overline{H}}/\mathbb{K}_\infty) \\
\mathfrak{Y} &= \text{Gal}(\Omega/\mathbb{K}_\infty), & \overline{\mathfrak{Y}} &= \text{Gal}(\overline{\Omega}/\mathbb{\overline{K}}_\infty) \\
\mathfrak{H} &= \text{Gal}(\mathbb{O}/\mathbb{\overline{M}}), & \overline{\mathfrak{H}} &= \text{Gal}(\overline{\Omega}/\overline{\mathbb{M}}) \\
\mathfrak{Z} &= \text{Gal}(\Omega/\overline{\mathbb{M}}_{E'}), & \overline{\mathfrak{Z}} &= \text{Gal}(\overline{\Omega}/\overline{\mathbb{M}}_{E'}) \\
\mathfrak{Y}_E &= \text{Gal}(\overline{\Omega}_E/\mathbb{K}_\infty), & \mathfrak{Y}_{E'} &= \text{Gal}(\overline{\Omega}_{E'}/\mathbb{K}_\infty)
\end{align*}
\]

At finite levels, we mostly write \( \mathbb{F}_n/\mathbb{K}_n \) for the maximal Kummer subextension of exponent \( p^{n+1} \) included in \( \mathbb{F} \). One exception may occur because of the ambiguous notation \( \Omega_n \): it may be the maximal Kummer subextension of \( \Omega \) over \( \mathbb{K}_n \) but it may also be the maximal \( p \) - abelian \( p \) - ramified extension of \( \mathbb{K}_n \). We choose the first and denote the second, consistently, by \( \mathbb{M}_n \). The previous groups will then be indexed like in
\[
\overline{\mathfrak{X}}_n = \text{Gal}(\overline{\mathbb{M}}_n/\mathbb{K}_n), \quad \mathfrak{Y}_{E,n} = \text{Gal}(\overline{\Omega}_{E,n}/\mathbb{K}_\infty), \quad \text{etc.}
\]

If \( F(T) \) is the characteristic polynomial of \( A \) and \( f|F \) is a distinguished polynomial, we let \( \Omega_f \subset \Omega \) be the maximal subfield with galois group \( \mathfrak{Y}_f = \text{Gal}(\Omega_f/\mathbb{K}_\infty) \) annihilated by \( f \) and free as a \( \mathbb{Z}_p \) - module. Thus \( \Omega_f, \overline{\Omega}_f \) are subfields of \( \overline{\mathbb{H}} \) corresponding to some factors of the group \( \text{Gal}(\overline{\mathbb{H}}/\mathbb{K}_\infty) \) of essential \( \Lambda \) - rank \( r_2 \), much like finite subextensions of a \( \mathbb{Z}_p \) - extension correspond to factors by the compact subgroups \( p^n \cdot \mathbb{Z}_p \subset \mathbb{Z}_p \). The field \( \overline{\Omega}_f \subset \Omega \) is defined like \( \Omega_f \), with respect to \( \overline{\mathbb{H}} \): the maximal subfield of \( \Omega \) with group \( \mathfrak{H}_f = \text{Gal}(\overline{\Omega}_f/\overline{\mathbb{H}}) \) annihilated by \( f \). Note that \( \text{Gal}(\overline{\Omega}/\overline{\mathbb{H}}) \) is \( \mathbb{Z}_p \) - torsion of infinite rank, so it is not annihilated by \( f \), thus \( \Omega_f \subset \overline{\Omega} \). The fields \( \mathbb{H}_f = \Omega_f \cap \overline{\mathbb{H}} \), by definition. These definitions are given with proofs of existence in §3.

We assume that the primes above \( p \) are completely ramified in \( \mathbb{K}_\infty/\mathbb{K} \) and \( \varphi \subset \mathbb{K} \) is one such prime. We let \( \pi \in \mathbb{K} \) be fixed by the decomposition group \( D_\varphi \subset G \) of \( \varphi \) and such that \( (\pi) = \varphi^{\text{ord}(\varphi)} \). With this, we fix \( \Pi = \{\pi^{s} : \sigma \in G/D_\varphi\} \subset \mathbb{K} \), thus \( s = |\Pi| \) is the number of primes above \( p \) in \( \mathbb{K} \), like above; the choice of \( \Pi \) is not canonical, but \( \overline{\Omega}_E[\Pi^{1/p^\infty}] = \overline{\Omega}_{E'}, \) so \( \Pi \) is
a notational simplification. Here is a complementary list of notations which we introduce:

- $\mathcal{P}$: \{\(\sigma \varphi : \sigma \in G, \text{and } \varphi \text{ a prime of } \mathbb{K} \text{ above } p\)\},
- $D_\varphi \subset G$: The decomposition group of $\varphi$,
- $C$: Coset representatives for $G/D_\varphi$,
- $\Pi$: \{\(\sigma \pi : \sigma \in C, \pi \in \mathbb{K}, (\pi) = \varphi^{\text{ord}(\varphi)}\)\},
- $\mathfrak{R}(\mathbb{K})$: $\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, for global fields $\mathbb{K}$
- $\iota_\varphi$: The projection from $\mathfrak{R}(\mathbb{K})$ in the completion $\mathbb{K}_p$,
- $U(\mathbb{K})$: The units of $\mathfrak{R}(\mathbb{K})$,
- $E(\mathbb{K}), E'(\mathbb{K})$: The units and $p$-units of some global field $\mathbb{K}$,
- $U^{(1)}(\mathbb{K})$: The one - units in $U$, $U_n = U^{(1)}(\mathbb{K}_n)$,
- $U'(\mathbb{K})$: The one units of absolute norm 1 in $\mathfrak{R}(\mathbb{K})$, up to torsion,
- $E(\mathbb{K})$: The completion of $E(\mathbb{K}) \hookrightarrow U(\mathbb{K})$,
- $\overline{\mathbb{H}}$: $\overline{\mathbb{H}}^{\varphi(A)}$,
- $\overline{\mathbb{H}}_E$: $\overline{\mathbb{H}} \cap \Omega_E$,
- $\Omega_{E_f}$: $\bigcup_{n=0}^{\infty} \mathbb{K}_n^{[E(\mathbb{K})^{1/p^{n+1}}]} = \mathbb{K}_\infty^{[E(\mathbb{K})^{1/p^\infty}]} \subset \Omega_E$,
- $\Omega_f$: $\bigcup_{L \in \mathcal{P}, (Gal(\mathbb{K}/\mathbb{K})/\{1\}) \subset \mathbb{L}_n}$,
- $\Omega^\prime_f$: $\bigcup_{L \in \mathcal{P}, (Gal(\mathbb{K}/\mathbb{K})/\{1\}) \subset \mathbb{L}_n}$,
- $\mathcal{Y}_f$: $Gal(\Omega_f/\mathbb{K}_\infty)$, $\mathcal{Y}_f^\prime = Gal(\Omega_f^\prime/\overline{\mathbb{H}})$,
- $\Phi$: $\Omega^\prime_{E_1} \cap \overline{\mathbb{H}}_\infty$,
- $\mathbb{H}_f$: The maximal subfield of $\overline{\mathbb{H}}$ with group fixed by $f$,
- $\mathbb{M}_n$: The product of all $\mathbb{Z}_p$ extensions of $\mathbb{K}_n$, $\mathbb{M} = \mathbb{M}(\mathbb{K})$
- $\Delta_n$: $Gal(\mathbb{M}_n/\mathbb{K}_n)$, $\Delta_n = Gal(\mathbb{M}_n/\mathbb{K}_\infty)$
- $\mathbb{M}_E$: $\Omega_E \cap \mathbb{M}$.

The symbols $\mathbb{K}_\infty^{[E^{1/p^\infty}]}$, $\mathbb{K}_\infty^{[E^{1/p^\infty}]}$ are well defined, however symbols like $\Omega_E[A^{1/p^\infty}]$ and even more so, $\mathbb{K}_\infty^{[A^{1/p^\infty}]}$ have lead to some ambiguity. The use of radicals of classes will be developed in much detail in the third chapter, where all such symbols are either rigorously defined or discarded; the last holds for $\mathbb{K}_\infty^{[A^{1/p^\infty}]}$ which has a possible non canonic definition, but is not of great help and can thus be avoided. We then may write things like $\Omega_E[A^{1/p^\infty}], \overline{\mathbb{H}}_E[A^{1/p^\infty}], a \in A$ or $\Omega_E[B^{1/p^\infty}], B \subset A'$, etc.

1.2. **Connection to Iwasawa theory.** We shall take here an approach using class field and Iwasawa theory. Let $T \rightarrow T^*$ be the Iwasawa involution (see [10] for a definition); the main fields of our interest will be $\Omega_T$ and $\Omega_{T^*}, \Omega_{T^*}$. Note that in this notation, $\Omega_T = \mathbb{M}$, since by definition $Gal(\Omega_T/\mathbb{K}_\infty)$ is also $\mathbb{Z}_p$ torsion - free.

In general, for arbitrary polynomials $f$, one may prove by using the Iwasawa skew symmetric pairing that $Gal(\Omega_f/\mathbb{M}_{f^*}) \sim Gal((\overline{\mathbb{H}} \cap \Omega_{f^*})/\mathbb{K}_\infty)^*$; this and more consequences of Iwasawa’s work are proved in Theorem 3 and lay the ground for the proof of the final result. Although the general structure of $\Omega_f$ and $\Omega_f$ is beyond the scope of this paper, this and similar facts will be exposed for the sake of completeness.
Assuming that Leopoldt’s Conjecture is false for $K$, we shall show that $\Omega_{T^*} \cap H_\infty = \Phi \subset \Omega_E$ is a non trivial extension with group of $\mathbb{Z}_p$- rank $D(K)$: its existence is thus equivalent to the failing of Leopoldt’s Conjecture\(^2\). This and two other extensions with the same property are denoted by phantom - fields, for obvious reason: they encrypt a constant which should be zero.

The $\mathbb{Z}_p$- ranks of the groups $\mathcal{Y}_f$ can be calculated exactly by means of class field theory and it turns out that $\mathbb{Z}_p$-rk($\mathcal{Y}_f$) = $r_2 + s - 1$, where $s$ denotes like usual the number of primes above $p$ in $K$. We then use the tower (25) as a filter for the extension $\Omega_{T^*}/H_\infty$: the sum of the ranks of the different intersections must add up to the total of $r_2 + s - 1$. Some direct investigations of ranks show the following equalities:

\[
\begin{align*}
\mathbb{Z}_p\text{-rk}(\text{Gal}(\Omega_{T^*} \cap \Omega_{E'})) &= r_2 + s - 1, \\
\mathbb{Z}_p\text{-rk}(\text{Gal}(\Omega_{T^*} \cap \Omega_{E'})) &= r_2 + s - 1 - D(K), \\
\mathbb{Z}_p\text{-rk}(\text{Gal}(\Omega_{T^*} \cdot \Omega_{E'})) &= D(K).
\end{align*}
\]

The last equation in (2) is a consequence of the first two, based on the Lemma\(^1\). The proof of the second identity in (2) is given in Section 5 and is the crucial step in proving the main result of this paper. The central observation of the proof is based on the remark that the field $\Phi$ is one with Kummer radical annihilated by $T$ and group annihilated by $T^*$. This has as consequence that a tower of extensions of the type $K_n \subset L_n \subset L_n'$, in which $L_n \subset H_f$ and $L_n' \subset \Omega_f$, while $L_n'/K_n$ is abelian, cannot exist. These towers are possible for $f \neq T^*$, but the radicals of $L_n' \cdot K_{2n}$ come from $K_{2n}$ in that case, as shown in Example 2 in Appendix C. For $f = T^*$, they must come from $K$, which leads to a contradiction and proves the Theorem\(^1\).

1.3. Structure. The development of the material in the paper is rather smooth and we introduce the basic concepts in more detail then directly necessary; in part, proofs which are not of direct importance, are given in the Appendices.

Since the Conjecture is related to $T$ and $T^*$ - parts, the idea is to follow all possible extensions over $K_\infty$ or $\mathbb{H}$ with galois groups or radicals annihilated by $T$ and bring their relations and ranks in evidence. It becomes a pattern, that in all cases some phantom field with galois group of rank $D(K)$ will emerge. In Chapter 2 and the related parts of Appendix A we treat the structure of $\mathbb{Z}_p[G]$ - modules and their isomorphy classes in the case when $G$ is non commutative, and develop the theory of local and global Minkowski units. In §2.3 we consider in detail the growth of $\Lambda$ - modules, obtaining some general results on Weierstrass modules – i.e. $\Lambda$ - torsion modules of finite rank and $\mathbb{Z}_p$ - torsion - free. These are then used as a base for presenting a projective structure on radicals and defining the Leopoldt reflection for $\Lambda [G]$ - modules. The action of $G$ on our modules is always taken into consideration in addition to the one of $\Lambda$.

\(^2\)The field $\Phi$ was often noticed in the literature: for example in Jaulent’s recent paper [15], treating a special case of Leopoldt’s Conjecture.
The Chapter 3 treats the extension $\Omega/\Omega_E$ and we define in this context radicals as classes; this allows embedding adjoints and duals of submodules of $A$ in $A$ itself and opens the way for the deep results on duality based on the Iwasawa skew symmetric pairing, which are presented in §3.3. The section 3.4 gives the rank estimates for (2) and some new explicit radical constructions; these bring a new perspective on duality in the non CM case, when it appears as a useful counterpart for the lacking complex conjugation.

In Chapter 4 we prove the Gross - Kuz’min Conjecture, using an approach which relays strongly on duality; the CM case is however quite simple. In Chapter 5 we then prove the Proposition 5, from which the Main Theorem follows. The final Chapter treats some consequences, like the fact that the proven conjectures are the special cases of the Greenberg Conjecture, for the $T$ and $T^*$ parts of $A'$.

2. Groups, modules and fields

Most of the modules that we shall encounter are $\mathbb{Z}[/math][G]$-modules, with $G = \text{Gal}(K/\mathbb{Q})$ or $G = G_n = \text{Gal}(K_n/\mathbb{Q})$. The first section of this Chapter will treat the structure of such modules. Next we shall recall some properties of the completions of a field at primes above $p$, and consider local and global Minkowski units in some detail.

Since most groups we encounter are finite or infinite $p$-groups, we define here some notions that we repeatedly use:

**Definition 1 (Group-related constants).** Let $X$ be an abelian $p$-group written additively. If $X$ is a $\mathbb{Z}[/math][G]$-torsion module, the exponent $\exp(X) = \min\{p^i : p^i X = 0\}$, and $p^i$ is an exponent for $X$ iff $i \geq \exp(X)$. Suppose that $X$ has exponent $p^n$ and $p$-rank $r$. We say that $X$ has sub-exponent $p^m \leq p^n$ for some $m > 0$, if there is a subgroup $Y \subset X$ with $p$-rk($Y$) = $p$-rk($X$) = $r$ and for all $y \in Y \setminus pY$ we have $p^{m-1}y \neq 0$.

The quotient $X/pX$ is an $\mathbb{F}_p$-vector space with respect to which the $p$-rank is defined by $p$-rk($X$) = dim($X/pX$). If $X$ is finite, then $p$-rk($X$) = dim($X/pX$) = dim($S(X)$), where $S(X)$ is the first socle of $X$. In general, the $m$-th socle is $S_m(X) = \text{Ker}(p^m : X \rightarrow X)$. If $X$ is infinite, the essential $p$-rank is ess. $p$-rk($X$) = lim$_{m \rightarrow \infty}$ $p$-rk($p^m X$).

Let $X$ be a $\Lambda$-module. By Nakayama’s Lemma, the number of elements in a minimal set of generators $x_i \in X \setminus X^M$ of $X$ is the $p$-rank of $X^{\omega_1}$ and does not depend on the choice of the $x_i$. We denote this number by $\Lambda$-rank of $X$, so

$\Lambda - \text{rank}(X) = p$-rk($X^{\omega_1}$).

2.1. Idempotents, supports, annihilators and components in $\mathbb{Z}[/math][G]$.

In the context of Leopoldt’s Conjecture we are interested in ranks and not in torsion of modules over rings. It is thus a useful simplification to tensor these modules with fields, so we introduce the following
Definition 2. Let $G$ be a finite group and $A, B$ a $\mathbb{Z}$-, respectively a $\mathbb{Z}_p$-module, which are torsion free. Let $a \in A, b \in B$. We denote
\[ \hat{A} = A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \hat{a} = a \otimes 1, \quad \hat{B} = B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \hat{b} = a \otimes 1. \]
We note that $\mathbb{Z}$-rk$(A) = \mathbb{Q}$-rk$(\hat{A})$ and $\mathbb{Z}_p$-rk$(B) = \mathbb{Q}_p$-rk$(\hat{B})$.

The major advantage of this notation is that it will allow to consider, in a first step, annihilator modules as images of some idempotents in $\mathbb{Q}_p[G]$. If $X$ is some $\mathbb{Z}_p[G]$-module, we write
\[ \hat{X}^\top = \{ y \in \mathbb{Q}_p[G] : \hat{x}y = 1, \forall \hat{x} \in \hat{X} \}, \]
the annihilator module of $\hat{X}$. The annihilator module of $X$ will then be
\[ X^\top = \hat{X}^\top \cap \mathbb{Z}_p[G]. \]

From class field theory, one has ([22], Chapter 5, Theorem 5.1):
\[ \text{Gal}(\mathbb{M}/\mathbb{H}(\mathbb{K})) \cong p \text{- part of } \mathbb{U}^{(1)}(\mathbb{K})/E(\mathbb{K}). \]
and the global Artin symbol extends to a covariant $\mathbb{Q}_p[G]$-isomorphism
\[ \varphi : \mathbb{U}^{(1)}(\mathbb{K})/\mathbb{E} \to \tilde{\Delta}. \]
Alternatively, we may consider $\varphi$ as a surjective $\mathbb{Q}_p[G]$-homomorphism $\varphi : \mathbb{U}^{(1)}(\mathbb{K}) \to \tilde{\Delta}$ with kernel $\mathbb{E}$. The Artin map is an isomorphism $\varphi : \mathbb{A}_n \to \text{Gal}(\mathbb{H}_n/\mathbb{K}_n)$ which extends in the projective limit to an isomorphism of $\Lambda$-modules, $\varphi : \mathbb{A} \to \text{Gal}(\mathbb{H}/\mathbb{K}_\infty)$.

It is known that there is a Minkowski unit $\delta \in E$ ([23], lemma 5.27), i.e. a unit such that
\[ \mathbb{Z}\text{-rk} \left( \delta^G \right) = r. \]

We develop in the first subsection of the Appendix A the notions of support and annihilator for non commutative group rings. This is important for the understanding of extensions which are not CM, and will be developed also in subsequent papers for a more in depth study of duality in number fields.
2.2. Completions and local units. If $K$ is a finite or infinite number field, we let $\mathfrak{A} = K \otimes \mathbb{Q}$, and may also write $\mathfrak{A}(K)$ when the global fields need precisation. The following is proved here for completeness. See also Jaulent [16] §1.a (ii) for an alternative treatment of this construction.

**Fact 1.** Let $\mathbb{K} = \mathbb{Q}[x]$, $p, P$ and $\mathfrak{R}_p$ be like in the introduction, suppose that $f \in \mathbb{Z}[X]$ is a minimal polynomial of $x$ and $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is the natural embedding. Then

$$\mathfrak{R}_p = \mathbb{Q}_p[X]/(\iota(f))$$

is a galois algebra with group $G = \text{Gal}(\mathbb{K}/\mathbb{Q})$ and the embedding $\iota$ extends to an embedding $\mathbb{K} \hookrightarrow \mathfrak{R}_p$ which commutes with the galois action. The image $\iota(\mathbb{K}) \subset \mathfrak{R}_p$ is dense in the product topology.

The proof is direct and is given in Appendix A. The group under consideration is thus the multiplicative subgroup of idèles which are trivial at all places except for the primes above $p$. By the Chinese Remainder Theorem we identify $u \in U$ with $(\iota_p(u))_{p \in P}$.

For arbitrary fields $K$, the units $U^{(1)}(K)$ are the products of $U^{(1)}(K_p) = \{u \in U : u \equiv 1 \mod \pi\}$ for some uniformizer $\pi$ of the completion $K_p$. For $K = \mathbb{K}_n$ we simply write $U_n = U^{(1)}(\mathbb{K}_n)$. If $K/\mathbb{Q}_p$ is a finite local extension, we also write $K_n = K[\mu_{p^n + 1}]$ and

$$U'_n = \{u \in U^{(1)}(K_n) : N_{K_n/K}(u) = 1\}.$$

Then

**Lemma 1.** Let $K/\mathbb{Q}_p$ be a local finite galois extension with $K \cap \mathbb{Q}_p[\mu_\infty] = \mathbb{Q}_p[\mu_{p^{k+1}}]$. Then the system $(U'_n)_{n \in \mathbb{N}}$ defined above is norm coherent and the norm is surjective at all levels, that is

$$N_{K_n/K_m}(U'_m) = U'_n, \quad \forall m > n > 0.$$

**Proof.** This follows from class field theory: $\text{Gal}(K/\mathbb{Q}_p)$ acts by conjugation on the groups $\Gamma_{n,m} = \text{Gal}(K_n/K_m)$, which are fixed under this action. By class field theory,

$$\Gamma_{n,m} \cong K_m^\times/N_{K_n/K_m}(K_n^\times).$$

But $K_n/K_m$ is totally ramified, so we have

$$K_m^\times/N_{K_n/K_m}(K_n^\times) = U_m^{(1)}/N_{K_n/K_m}(U_n^{(1)}),$$

and [5] implies that the norm residue group is $\text{Gal}(K/\mathbb{Q}_p)$ - invariant, and it thus is a quotient of $\mathbb{Z}_p$. Since $N(U'_n) = \{1\}$ by definition, it follows that the restriction of the norm to $U'_n$ is indeed surjective.

We now show that there are local Minkowski units and describe their relation to the global ones. Serre proves in [23], §1.4, Proposition 3, in the case when $K/\mathbb{Q}_p$ is a local field, that the group $U^{(1)}(K)$ contains a cyclic $\mathbb{Z}_p[G]$ module of finite index, which is isomorphic to $\mathbb{Z}_p[G]$: thus $U^{(1)}(K)$ is quasi - cyclic.
Using this result one obtains units generating \( \mathbb{Z}_p[G] \) modules of finite index in \( U \), which we call local Minkowski units for \( K \). Let \( \wp \in P \) be fixed and \( \upsilon \in \mathbb{K}_p \) be a local Minkowski unit, according to Serre. Then we define \( \xi = \xi(\upsilon) \in U \) and \( \tilde{\wp} \in U \) by:

\[
\tau_\wp(\xi) = \begin{cases} 
\upsilon & \text{for } \tau = 1, \\
1 & \text{for } \tau \in G, \tau \neq 1.
\end{cases}
\]

(6)

\[
\tau_\wp(\tilde{\wp}) = \begin{cases} 
1 & \text{for } \tau = 1, \\
0 & \text{for } \tau \in G, \tau \neq 1.
\end{cases}
\]

(7)

Let \( D_\wp \) be the decomposition group of \( \wp \) and \( C = D_\wp \setminus G \) be coset representatives. Then \( C \) acts on \( \xi \) and for \( \sigma \in C \), the unit \( \xi_\sigma \) satisfies:

\[
\tau_\wp(\xi_\sigma) = \begin{cases} 
\upsilon & \text{for } \tau = \sigma, \\
1 & \text{for } \tau \in G, \tau \neq \sigma.
\end{cases}
\]

We denote units the generators \( u \in U \) for \( U \) by local Minkowski units. The previous construction shows that such units exist. Since \( N_{K^n/\mathbb{Q}}(E_n) = \{1\} \), local units of norm one are interesting for the embedding \( E_n \hookrightarrow U_n \); we define:

\[
U' = \{ u \in U^{(1)} : N_{\mathbb{Q}_p/\mathbb{Q}}(u) = 1 \}
\]

which is a quasi-cyclic \( \mathbb{Z}_p[G] \) submodule of \( U \) with \( U^{(1)}/U' = U^{(1)}(\mathbb{Z}_p) \cong \mathbb{Z}_p \). Therefore \( \tilde{U}' \cong (1 - N_{\mathbb{Q}/\mathbb{Q}}(G))\mathbb{Q}_p[G] \), the last being a two sided module in \( \mathbb{Q}_p[G] \). For any \( K \) we have \( \bar{E}(K) \subset U' \) and therefore \( U^{(1)}(\mathbb{Z}_p) \) is mapped injectively in \( \Delta \) by the Artin map. Let \( \xi_0 \in U'_0 \setminus U''_0 \) be a local Minkowski unit for \( U'_0 \). From Lemma \( \square \) we conclude that there is a norm coherent system \( (\xi_n)_{n \in \mathbb{N}} \), \( \xi_n \in U'_n \) in which \( \xi_n \) are local Minkowski units for \( U'_n \). The question about norm coherent systems of global Minkowski units is related to the capitulation of the primes above \( p \), which makes its investigation more involved. We will not address it here.

By choosing a global Minkowski unit \( \delta \in E \), one can find a local one \( \xi \in U' \) such that

\[
\dot{\xi}^\alpha = \delta, \text{ with } \alpha^2 = \alpha \in \mathbb{Q}_p[G].
\]

(9)

This corresponds to a map \( \phi_E : U' \to \bar{\iota}(E) \), according to the general theory on isomorphy classes of idempotents in Appendix A.

### 2.3. The growth of \( \Lambda \)-modules

Let \( C_n = \mathbb{Q}[[\zeta_{p^n+1}]] \), \( n \geq 0 \) be the \( p^{n+1} \)-th cyclotomic extension, \( C_n = \text{Gal}(C_{p^n}/\mathbb{Q}) \). We identify \( C_0 \) with its lift to \( \text{Gal}(C_n/\mathbb{Q}) \) which generates a group of order \( p - 1 \) and let \( \mathbb{B}_n = C_{p^n}^{C_0} \), the intermediate fields of the \( \mathbb{Z}_p \)-cyclotomic extension \( \mathbb{B}/\mathbb{Q} \). Let \( \hat{\Gamma} = \text{Gal}(\mathbb{B}/\mathbb{Q}) \cong \text{Gal}(C_{\infty}/C_0) \) and \( \hat{\tau} \in \hat{\Gamma} \) be a topological generator, acting by restriction on \( C_n \) and \( \mathbb{B}_n \).
We assumed that $\mathbb{K}$ is a galois extension of $\mathbb{Q}$ with group $G$, which contains the $C_k$ but not $C_{k+1}$. Let

$$K_{-1} = \bigcap_{L \subset K : K = C_k \cdot L} L \subset K$$

be the smallest galois extension which generates $\mathbb{K}$ by compositum with $C_k$ and $G_{-1} = \text{Gal}(K_{-1}/\mathbb{Q})$. Then $\text{Gal}(K_n/\mathbb{Q}) = C_n \times G_{-1}$ and in particular $G = C_k \times G_{-1}$. If $K_{-1}$ and $C_1$ are linearly disjoint, the product is a direct one. We write $K_n = K_n \cdot \mathbb{Q}_n$ for $n \geq k$ and $K_\infty = \mathbb{Q}_\infty \cdot K$, while $K_j = K$ for $0 \leq j \leq k$.

Since $C_k \subset K$, the group $\text{Gal}(K_\infty/K) \cong \hat{\Gamma}^k$. Our ground field will be $K$, so we let $\Gamma = \hat{\Gamma}^k$ and $\tau = \hat{\tau}^k$ be a topological generator; $\Gamma_n = \text{Gal}(\mathbb{Q}_n/\mathbb{Q}_k) \cong \hat{\Gamma}/\hat{\tau}^k$. We further assume that all the primes above $p$ ramify completely in $K_\infty/K$ and the Leopoldt defect is constant in $K_n$, $n \geq k$; since the Leopoldt defect is bounded (see [25], Lemma 13.30), this can be achieved by choosing $k$ sufficiently large.

Like usual we let $T = \tau - 1$ and $\Lambda = \mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[\Gamma]$. For $n = k + l > k$ we let $\nu_n = (T + 1)^{p^k} - 1 = \gamma^{p^{n+1}} - 1$ and $\Lambda_n = \mathbb{Z}_p[\Gamma_n] \cong \Lambda/(\Lambda^{\nu_n})$. The norms $N_{m,n} = N_{K_m/K_n} = \omega_m/\omega_n$ for $m > n \geq 0$ are sometimes denoted by $\nu_{n,m}$, but we use $\nu$ in connection with the group $\Delta$ here. The notations $T, \hat{T}$ are self-explained and may not be necessary. Let the cyclotomic character act on $\Lambda$ by $k(\tau) = (q + 1)\tau$, where $q = p^{k+1}$. The Iwasawa involution is defined by

$$T = \tau - 1 \mapsto T^* = \frac{q - T}{T + 1}$$

Let $(M_n)_{n \in \mathbb{N}}$ be a family of abelian $p$-groups, finite or infinite, which are closed under the action of $\Gamma_n$ and which form a projective system under a family of maps $f_{m,n} : M_m \mapsto M_n, m > n \geq k$, which shall mostly be the norms $N_{m,n} = N_{K_m/K_n}$. All the $\Lambda$-modules we encounter are projective limits $M = \lim M_n$ of such systems. The structure of $\Lambda$-modules is well known (e.g. [14], §1, [25], Chapter 7). Some of the most important groups we shall encounter are $(E_n)_{n \in \mathbb{N}}, (E'_n)_{n \in \mathbb{N}}$, units and $p$-units of $\mathbb{K}$; possible projective systems on these units will be considered below. The groups $A = (A_n)_{n \in \mathbb{N}}, A' = (A'_n)_{n \in \mathbb{N}}$ with $A_n, A'_n$ the $p$-parts of the class groups of the integers, resp. $p$-integers of $\mathbb{K}$; these form classical projective systems with limits $\Lambda, A'$.

If $X$ is a $\Lambda$-module, it is customary to write $X^\ast$ for the module endowed with action of $\Lambda$ twisted by the Iwasawa involution, [14], §8.1. The notation $X^\ast$ will receive below a more general definition, which takes the action of $G$ and embedding in $X$ into account. We may use the short-lived notation $X^I$ for the Iwasawa dual.

Taking the action of $G_n = \text{Gal}(K_n/\mathbb{Q})$ under consideration, there is an involution of $\mathbb{Z}_p[G_n]$-modules, the Leopoldt reflection involution, defined as follows: let $\alpha = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{Z}_p[G_n]$, with $a_\sigma \in \Lambda$. Then the Leopoldt
reflection involution is an automorphism of \( \mathbb{Z}_p[\Gamma_n] \) defined by

\[
\alpha \mapsto \alpha' = \sum_{\sigma \in \Gamma_n} a_\sigma \cdot \chi(\sigma) \cdot \sigma^{-1},
\]

(11)

with \( \chi \) the Teichmüller (cyclotomic) character: \( \zeta_{\chi(\sigma)}^p = \sigma(\zeta_{p^n+1}). \)

The following lemma generalizes in its first two points an observation of Fukuda [11]; it gives an overview of the asymptotic growth of torsion \( \Lambda \)-modules:

**Proposition 1.** Let the notations be like above, and \( k = 0 \) for ease of notation\(^3\). Let \( M = \limleftarrow M_n \) be a torsion - \( \Lambda \)-module such that the maps \( f_{m,n} \) are all surjective for \( m > n \geq 0 \). Then there is an \( n_0 > 0 \) such that for all \( n \geq n_0 \) the following hold:

1. If \( |M_n| = |M_{n+1}| \), then \( M_m = M_n \) for all \( m > n \geq 0 \).
2. If \( p \text{-rk}(M_n) = p \text{-rk}(M_{n+1}) \) then \( p \text{-rk}(M_m) = p \text{-rk}(M_n) = R \) for all \( m > n \geq 0 \); furthermore there is a constant \( \lambda(M) \leq R \) such that \( |M_{n+1}| - |M_n| = p\lambda \) for all \( n \) sufficiently large.
3. Suppose that the maps \( f_{m,n} \) are the relative norms \( N_{n+1, n} \), which are all surjective and let \( \iota : M_n \to M_{n+1} \) be the natural embedding. Then

\[
x^p = \iota(N_{n+1, n}(x)), \quad \iota(M_n) = pM_{n+1}, \quad \text{for all} \quad x \in M_{n+1}.
\]

Furthermore, if \( M \) is like in point 2., and \( R = \lambda(M) \) then the map \( \iota \) is injective.

The proof of this proposition is given in Appendix A. A typical application is the case when \( M \subset A \) is some \( \Lambda \)-submodule. The first two points are unsurprising, but the third gives useful additional information about the stationary growth of \( \Lambda \)-modules of finite rank. For such modules \( M \), the \( \mathbb{Z}_p \)-torsion is a canonical finite submodule, but the infinite part is primarily just a factor. We shall denote \( \Lambda \)-modules of finite \( p \)-rank and \( \mathbb{Z}_p \)-torsion free by **Weierstrass modules**. Their structure is given by

**Lemma 2.** Let \( X \) be a finitely generated Weierstrass module written additively, with characteristic polynomial \( F \) and \( F = \prod_{j=1}^d f_j \) be a decomposition in pairwise coprime distinguished polynomial. Then \( X_j = X^{F/f_j} \) are canonic submodules with

\[
X = \bigoplus_{j=1}^d X_j.
\]

For every distinguished polynomial \( f|F \) there is a canonic \( f \)-component \( X_f = X^{F/f} \subset X \) and if \( f \) and \( F/f \) are coprime, then \( X_f \) is a canonic direct term in \( X \).

For Weierstrass modules we have the following consequence of Proposition 1.
Corollary 1. Let \( M \subset A \) be a Weierstrass \( \Lambda \)-submodule. Then for \( n > n_0 \) the capitulation kernel \( \text{Ker}(\iota : M_n \to M_{n+1}) \) is trivial and

\[
x^p = \iota(N_{n+1,n}(x)), \quad \forall \ x \in M_{n+1}.
\]

Furthermore, there is an integer \( z(x) \) which is independent on \( n \) and such that \( v_p(\text{ord}(x_n)) = n + 1 + z(x) \) for all \( n \) sufficiently large. As a map \( z : A \to \mathbb{Z} \), we have \( z(xy) \leq \max(z(x), z(y)) \) and if \( c = \max\{v_p(\text{ord}(x_0)) - (k' + 1) : x_0 \in A(\mathbb{K})\} \), with \( k' \) such that \( \mathbb{K} \cap \mathbb{C}_{k'+1} = \mathbb{C}_{k'} \), then \( z(A) \in \mathbb{Z}_{\leq c} \). Finally, the map \( z : A \to A \) can be defined by

\[
z(x) = \begin{cases} -\infty & \text{if } x \in A^\circ \\ \lim_{n \to \infty} v_p(\text{ord}(x_n)) - (n + 1) & \text{otherwise.} \end{cases}
\]

The next lemma shows that pure \( \mu \)-modules have no capitulation. The proofs of the Lemma 2, Corollary 1 and Lemma 3 are in the Appendix A.

Lemma 3. Let \( a \subset A^\circ \) be such that \( \Delta a \subset A \) is a \( \Lambda \)-module of unbounded \( p \)-rank. Then for sufficiently large \( n \) the capitulation kernel \( \text{Ker}(\iota_{n,n+1}) = \{1\} \) and

\[
v_p(|\Delta a_n|) = \mu p^n - \nu, \quad \nu \geq 0.
\]

We are now in position to choose our base field \( \mathbb{K} \) such as to avoid some unpleasant phenomena at finite levels:

**Fact 2.** Let \( \mathbb{K} \) be a galois extension of \( \mathbb{Q} \) containing the \( p \)-th roots of unity and \( \mathbb{K}_n = \mathbb{K} \cdot \mathbb{B}_n \). Then there is an index \( n_0 \) such that for all \( n \geq n_0 \) the following facts are true:

1. The Leopoldt defect \( D(\mathbb{K}_n) \) is constant, all the primes above \( p \) ramify completely and the norm \( A_n \to A_{n_0} \) is surjective.
2. The \( p \)-ranks of all submodules \( B \subset A \) of finite rank are stable: \( p\text{-rk}(B_n) = p\text{-rk}(B_{n_0}) \). Furthermore, there is a constant \( \mu \) such that

\[
|A_{n_0}^\circ| = |A_n^\circ| \cdot (p^{\mu(p^n-p^{n_0})}).
\]

and the constant \( p^{n_0} \) annihilates \( A^\circ \).
3. The capitulation kernels \( \text{ker}(B_n \to B_{n+1}) \) are trivial for all Weierstrass submodules \( B \subset A \) and for all cyclic \( \Lambda \)-modules of unbounded \( p \)-rank.
4. The function \( z(x) \) in Corollary 1 is bounded by \( n_0 \) and

\[
x^p = \iota_{n,n+1}(N_{n+1,n}(x)) \quad \text{for all} \quad x \in A_{n+1}.
\]

Furthermore, for \( x = (x_n)_{n \in \mathbb{N}} \in A \) and \( n \) such that \( x_n \neq 1 \), the value \( z(x_n) = v_p(\text{ord}(x_n)) - (n + 1) = z(x) \) is constant.

This follows from the Corollary and Proposition above. Since \( A \) is a finitely generated \( \Lambda \)-module, one may choose a minimal set of generators \( a_i \); for each module \( \Lambda a_i \) there is a value \( m_i \) for which the conditions above are fulfilled; set \( n_0 \) to be larger or equal to the maximal value for all \( i \).
We define additionally \( \hat{A}_n = \{ x \in A_n : z(a_n) \leq 0 \} \) and \( \hat{A} = \{ x \in A : z(a) \leq 0 \} \); since \( z(a_n) = z(a) \) for all \( n \) in the present definition of \( \mathbb{K} \), it follows that \( (\hat{A})_n = \hat{A}_n \) for all \( n \).

The following lemma describes two important shift maps derived from the bound \( z(A) \leq \kappa \).

**Lemma 4.** Let \( \varsigma : A \to A : a \mapsto a^{p^\kappa} \) and the restrictions \( A_n \to A_n : a_n \mapsto a_n^{p^\kappa} \) be also denoted by \( \varsigma \). Then \( \varsigma \) annihilates \( A^\circ \) and \( \varsigma(\kappa) \subset \hat{A}_n \).

For \( d > 0 \) we let \( A(d) \) be the set of sequences \( a = (a_n)_{n \in \mathbb{N}} \) with \( a_n \in A_n \) and such that \( a(d) := (a_m)_{m \geq d} \) is norm coherent and there is a \( b \in A \) such that \( a(d) = b(d) \); the elements \( (a_n)_{n < d} = 1 \). Then \( A(d) \) is a \( \Lambda \) - module and for \( d := 2\kappa \), there is a map \( \iota_{\mathbb{K}} : A \to A(d) \) such that

\[
A \ni a = (a_n)_{n \in \mathbb{N}} \mapsto b = (b_m)_{m \in \mathbb{N}}, \quad b_m = \begin{cases} 1 & \text{if } m < 2\kappa, \\ \iota_{m-\kappa,m}(a_{m-\kappa}) & \text{otherwise}. \end{cases}
\]

The kernels are \( \text{Ker}(\iota_{\mathbb{K}}) = A \) and \( \text{Ker}(\varsigma) = A^\circ \). Furthermore, \( \varsigma(A) \) is a Weierstrass module and \( \iota_{\mathbb{K}}(A) \) has no finite \( \mathbb{Z}_p \) - torsion.

**Proof.** The map \( \varsigma \) annihilates \( A^\circ \) since \( p^\kappa \) is by definition an exponent thereof. It is a straight forward verification that \( A(d) \) is a \( \Lambda \) - module and \( \iota_{\mathbb{K}} \) is an homomorphism of \( \Lambda \) - modules. By choice of \( \kappa \), all ideals in classes of \( \hat{A}_n \) capitulate in \( A_{n+\kappa} \), so \( A \subset \text{Ker}(\iota_{\mathbb{K}}) \). On the other hand if \( a \in A^\circ \setminus A \), then \( \Lambda a \) has unbounded \( p \) - rank and by Lemma \( \mathbb{K} \) and the choice of \( \kappa \), the map \( \iota_{m,n+\kappa} \) is injective on \( \Lambda a \), so \( a \notin \text{Ker}(\iota_{\mathbb{K}}) \). Since \( p^\kappa \) annihilates the \( \mathbb{Z}_p \) - torsion, \( \text{Ker}(\varsigma) = A^\circ \) and \( \varsigma(A) \) is \( \mathbb{Z}_p \) - torsion free, so it is a Weierstrass module. The fact that \( \iota_{\mathbb{K}}(A) \) has no finite \( \mathbb{Z}_p \) - torsion follows from Lemma \( \mathbb{K} \) and point 1. of Proposition \( \mathbb{K} \). \( \square \)

It will make sense to extend the map \( \varsigma \) also on local and global units, so for arbitrary abelian \( p \) - groups \( X \) we let

\[
\varsigma : x \mapsto x^{p^\kappa}, \quad \forall x \in X.
\]

### 2.4. Radicals and pairings.

Let \( \mathbb{L} \supset \mathbb{K}_\infty \) be some infinite abelian extension and \( \mathbb{L}_n \subset \mathbb{K}_n \) be the maximal Kummer subextension of degree \( p^{n+1} \) contained in \( \mathbb{L} \). The radical \( \text{rad}(\mathbb{L}_n/\mathbb{K}_n) \) is often considered as a group \( \mathbb{K}_n^{\times} \subset \text{rad}(\mathbb{L}_n/\mathbb{K}_n) \subset (\mathbb{K}_n^{\times})^{p^{n+1}} \), such that \( \mathbb{L}_n = \mathbb{K}_n[\text{rad}(\mathbb{L}_n/\mathbb{K}_n)^{1/p^{n+1}}] \). The injectivity condition \( \mathbb{L}_n \subset \mathbb{L}_{n+1} \) translates into

\[
\text{rad}(\mathbb{L}_n/\mathbb{K}_n)^{p} \subset \text{rad}(\mathbb{L}_{n+1}/\mathbb{K}_{n+1}).
\]

The relevance of the notion of subexponent for such extensions is given by:
Lemma 5. Let $X = \text{Gal}(\mathbb{L}/\mathbb{K}_\infty)$ be a torsion $\Lambda$-module of $p$-rank $r$ and free as a $\mathbb{Z}_p$-module. Suppose that $B_n \subset \mathbb{K}_n^\times$ are groups such that $\mathbb{L}_n' := \mathbb{K}_n[B_n^{1/p^{n+1}}] \subset \mathbb{L}_n$ and $\mathbb{L}_{n+1}' \supset \mathbb{L}_n$ for all sufficiently large $n$. If for all such $n$, the $p$-rk$(B_n) = r$ and the subexponents of $B_n$ diverge, then $\bigcup_n \mathbb{L}_n' \sim \mathbb{L}$.

Proof. The injective limit $\mathbb{L}' = \bigcup_n \mathbb{L}_n'$ exists, from the hypothesis. Since the subexponents diverge, we have $\mathbb{Z}_p$-rk$(\text{Gal}(\mathbb{L}'/\mathbb{K}_\infty)) = \lim p$-rk$(\text{Gal}(\mathbb{L}_n'/\mathbb{K}_n)) = r$. Therefore $\text{Gal}(\mathbb{L}'/\mathbb{K}_\infty) \subset \text{Gal}(\mathbb{L}/\mathbb{K}_\infty)$ is a $\mathbb{Z}_p$-free submodule of the same rank. It follows that $\mathbb{L} = \mathbb{L}'$.

The radicals $\text{rad}(\mathbb{L}_n/\mathbb{K}_n)$ are endowed with an obvious structure of injective systems. Can a projective system be loaded upon these radicals, maybe using some modified definition? The answer is affirmative for a wide range of cases, but the notion of radical introduced above is not adapted for such a structure. We may use the alternative definition

\begin{equation}
\text{RAD}(\mathbb{L}_n/\mathbb{K}_n) = \mathbb{K}_n^\times \langle \text{rad}(\mathbb{L}_n/\mathbb{K}_n)^{1/p^{n+1}} \rangle / \mathbb{K}_n^\times,
\end{equation}

where $\mathbb{K}_n^\times \langle \text{rad}(\mathbb{L}_n/\mathbb{K}_n)^{1/p^{n+1}} \rangle \subset \mathbb{L}_n$ is the multiplicative group spanned by $\langle \text{rad}(\mathbb{L}_n/\mathbb{K}_n)^{1/p^{n+1}} \rangle$ over $\mathbb{K}_n^\times$; this definition is frequently used, for instance in cogalois theory, e.g. Albu’s monograph [1] and the specialized paper [2]. The injectivity condition is now $\text{RAD}(\mathbb{L}_n/\mathbb{K}_n) \subset \text{RAD}(\mathbb{L}_{n+1}/\mathbb{K}_{n+1})$ and if $\text{Gal}(\mathbb{L}_n/\mathbb{K}_n)$ is a $\Lambda$-module, so are both radicals defined above. A third version of radicals, which we shall not explicitly use is $\text{rad}((\mathbb{L}_n/\mathbb{K}_n)/(\mathbb{K}_n^\times)^{p^{n+1}})$. All three definitions describe the same extension; the first $\text{rad}(\mathbb{L}_n/\mathbb{K}_n)$ is useful as a subgroup of $\mathbb{K}_n^\times$. But it is large; the last one is the ‘active part’ of it, it has the advantage of being isomorphic to $\text{Gal}(\mathbb{L}_n/\mathbb{K}_n)$ as a group (but only via twisted action as a $\Lambda$-module), but it is a factor. The radical $\text{RAD}(\mathbb{L}_n/\mathbb{K}_n)$ is isomorphic to $\text{Gal}(\mathbb{L}_n/\mathbb{K}_n)$ and is not a factor; furthermore it has the advantage of yielding projective systems.

We let now $\mathbb{L}'/\mathbb{L}$ be an extension in the tower $\mathbb{K}_\infty \subset \mathbb{L} \subset \mathbb{L}' \subset \Omega$, such that both $\mathbb{L}$ and $\mathbb{L}'$ are galois over $\mathbb{K}$ and $\text{Gal}(\mathbb{L}'/\mathbb{L})$ is torsion $\Lambda$-module of finite $p$-rank and $\mathbb{Z}_p$-torsion-free. The radicals $\text{RAD}(\mathbb{L}_n'/\mathbb{L}_n)$ have the nice property of being endowed with a structure of projective systems, as stated in the next proposition. The proof will be given in the Appendix A, since the result is not used in the proof of Theorem [1].

Proposition 2. Let $\mathbb{L}'/\mathbb{L}$ be an extension in the tower $\mathbb{K}_\infty \subset \mathbb{L} \subset \mathbb{L}' \subset \Omega$, such that both $\mathbb{L}$ and $\mathbb{L}'$ are galois over $\mathbb{K}$ and $\text{Gal}(\mathbb{L}'/\mathbb{L})$ is a torsion $\Lambda$-module. Then the radicals $\text{RAD}(\mathbb{L}_n'/\mathbb{L}_n)$ form a projective system with respect to the relative norms. The projective limit is $\text{RAD}(\mathbb{L}'/\mathbb{L}) = \varprojlim \text{RAD}(\mathbb{L}_n'/\mathbb{L}_n)$. 


For \( L, L' \) like in the proposition and \( m > n > n_0 \) we have the compatible systems of Kummer pairings:

\[
\begin{align*}
\text{Gal}(L_m/L_m) & \times \text{RAD}(L'_m/L_m) \rightarrow \mu_{p^{n+1}} \\
\downarrow & \downarrow \\
\text{Gal}(L'_n/L_n) & \times \text{RAD}(L'_n/L_n) \rightarrow \mu_{p^{n+1}},
\end{align*}
\]

in which the arrows are induced by the norms \( N \). Let \( T(\mu) = \lim \mu_{p^{n+1}} \) be the Tate module for the group \( \mu \) of all the \( p^n \)-th roots of unity (\cite{Iwasawa}, \S10). The projective limits in (16) are then \( \text{Gal}(L'/L), \text{RAD}(L'/L), T(\mu) \), respectively. In view of \( \text{RAD}(L_n/K_n) = K_n^\times (\text{rad}(L_n/K_n)^{1/p^{n+1}}) / \mathbb{K}_n^\times \), for Kummer extensions \( L_n/K_n \) of exponent \( p^{n+1} \), we may consider the limit \( \text{RAD}(L/K) \) as a Tate module for the quotients \( \text{rad}(L_n/K_n)/(\mathbb{K}_n^\times)^{p^{n+1}} \). This is done by Iwasawa in \cite{Iwasawa} \S9.3, for the case when \( K_n \supset \Omega_{E,n} \), a case we shall discuss below.

Independently of the understanding of \( \text{RAD}(L'/L) \) as a Tate module, there is thus a projective - projective Kummer pairing at infinity, given by the projective limit of this diagram. We shall denote this pairing, in analogy with the Iwasawa skew symmetric pairing also by \([,] : \text{Gal}(L'/L) \times \text{RAD}(L'/L) \rightarrow T(\mu) \). The pairing \([,] \) enjoys all the properties of the Kummer pairings at finite levels:

**Theorem 2.** The pairing \([,] : \text{Gal}(L'/L) \times \text{RAD}(L'/L) \rightarrow T(\mu) \) defined above is bilinear, non degenerate and for all \((\sigma, b) \in \text{Gal}(L'/L) \times \text{RAD}(L'/L)\) and \( g \in \text{Gal}(\mathbb{K}_\infty/\mathbb{Q}) \) we have

\[
[\sigma, b]^g = [\sigma^g, b^g].
\]

**Proof.** The proof follows from Proposition 2 and the compatibility in (16). The fact that the properties of Kummer pairing are conserved in the projective limit can be verified for the three types of galois groups \( \text{Gal}(L'/L) \) considered in the proof of Proposition 2 below. \( \square \)

### 2.5. Reflection and duality.

We may define \( \Lambda[G] \) as the \( \mathbb{Z}_p \) - group ring of \( \Gamma \ltimes G_0 \) and

\[
\Lambda[G] = \left\{ x = \sum_{g \in G} a_g \times g, a_g \in \Lambda \right\}.
\]

Although \( \Lambda \) and \( G \) interact in general, the development of \( x \) above is unique: if \( W = \prod_{i=1}^{w} \tau^{e_i} g_i \) with \( e_i > 0 \) and \( g_i \in \text{Gal}(\mathbb{K}/C_\kappa) \), then the word has a unique reduction to a product \( W_1 = \tau^e g \); indeed, there is an \( f_w \) such that

---

4The existence of \( n_0 \) is given in the proof of the Proposition. It is likely that in the range of fields considered, the value of \( n_0 \) is uniformly bounded by \( \kappa \) or some larger value, but we do not investigate this fact here. The obstruction is the free \( \Lambda \) - module in \( \text{Gal}(\Omega_\kappa/\mathbb{K}_\infty) \); but one may always restrict to considering at most a finite family of \( \Lambda \)-torsion modules at a time, for which \( n_0 \) exists

5The pairings are related but not identical and the relation will be investigated in depth further on
\[\tau^f g \tau^{-1} = g \tau^f \tau^{-1}, \text{ since } G_\infty = \Gamma \times G_0, \text{ with } G_0 = \text{Gal}(\mathbb{K} / \mathbb{C}_p).\]
Replacing \(g \tau^{-1}\) by \(g \tau^{-1} \cdot g\) and \(e \tau^{-1}\) by \(e \tau^{-1} + f\), we obtain \(W = \prod_{i=1}^{w-1} \tau^i g_i\) a word of smaller length. The claim follows by induction on \(w\). For \(\alpha \in \Lambda\) and \(g \in G_0\), we have
\[(\alpha g)^\prime = \alpha^* \cdot g'.\]
Furthermore, the word reduction above is compatible with the Leopoldt involution, which makes it into an automorphism of \(\Lambda[G]\). However, \(\Lambda[G]\) is in general not a group ring, since \(\alpha g \neq g\alpha\) for all pairs \((\alpha, g)\). It is one, if \(\mathbb{K}_\infty\) and \(\mathbb{B}\) are linearly disjoint. The reflection automorphism of \(\Lambda[G]\) is defined by:
\[\sum_{g \in G} a_g \times g \mapsto \sum_{g \in G} a_g \chi(g) \times g^{-1}, \quad a_g \in \Lambda.\]
We extend the definition of \(X^\ast\) to \(\Lambda[G]\) - modules as follows

**Definition 4 (Duals).** Let \(X\) be an abelian \(p\) - group which is a compact \(\Gamma\) - module on which \(G\) acts, making it into a \(\Lambda[G]\) - module. We define the dual module \(\hat{X}\) by
\[
\Gamma \times X \rightarrow \hat{X} : (\gamma, x) \mapsto \gamma \circ x = \chi(\gamma) \gamma^{-1} x,
\]
\[
G_0 \times X \rightarrow \hat{X} : (g, x) \mapsto g \circ x = \chi(g) g^{-1} x.
\]
The module \(\Lambda[G]\) acts on \(\hat{X}\) via Leopoldt’s involution on \(X\). Let \(Y \subset X\) be a submodule and suppose that there is a pseudo-isomorphic embedding \(Y^\ast \subset X\). If \(\hat{Y}\) such that \(\hat{Y} \sim Y^\ast = Y^\ast \cap X\), then \(Y^\ast\) is the embedded dual of \(Y\) in \(X\). Not all submodules have embedded duals.

We say that \(X\) is self - dual, if there is and embedded dual \(X^\ast \subset X\) such that:
\[X^\ast \subset X \quad \text{and} \quad X^\ast \cap X \sim X \sim X^\ast \sim \hat{X}.
\]
The product of all submodules \(Y \subset X\) which have an embedded dual \(Y^\ast \subset X\) is the self - dual part of \(X\), a canonic submodule, up to pseudo-isomorphisms.

The following example explains the difference between duals, Iwasawa duals and embedded duals.

**Example 1.** Let \(\zeta(A)\) have characteristic polynomial \(F\) and \(f|F\) be a distinguished polynomial. Since \(\zeta(A)\) is a Weierstrass module, there is a canonic maximal submodule \(A_f \subset \zeta(A)\) annihilated by \(f\); however, already its Iwasawa dual denoted above by \(A^\ast_f\) is only embedded in \(\zeta(A)\) iff \(f^\ast|F\). The fact that the same holds for the Leopoldt dual \(\hat{A}_f\) which will be stated explicitly below.

The \(p\)-th cyclotomic field \(\mathbb{K} = \mathbb{C}_0\) yields a concrete case. Let \(\varepsilon_k, k = 0, 1, \ldots, p - 2\) denote like usual the orthogonal idempotents of \(\mathbb{Z}_p[\text{Gal}(\mathbb{C}_0/\mathbb{Q})]\) \((25, \S 6.3)\) and suppose that \(a \in \varepsilon_k A\) with \(k\) odd is a class with \(f \in \mathbb{Z}_p[G]\) the characteristic polynomial of \(A_a\). Then the Iwasawa dual \((\Lambda a)^\ast\) is annihilated by \((\varepsilon_k, f^*\), and it is embedded iff \(f^*\) divides the characteristic polynomial of
The Leopoldt dual is more natural in the sense that \( \Lambda \) is annihilated by \((1 - \varepsilon_p - k, f^*)\), thus duality acts also on the group of the base field. Such modules only arise, if the Greenberg Conjecture is false. The Conjecture for this simple field is in fact equivalent to the fact that the self-dual part of \( \Lambda \) is trivial.

Finally, if \( \Lambda \) is a \( \mathbb{Z}_p \)-torsion module of infinite \( p \)-rank, then it is pseudo-isomorphic to its Iwasawa dual, but this does not imply the existence of an embedded Leopoldt dual.

The main property of reflection is the following: let \( \mathbb{L}'/\mathbb{L}/\mathbb{K} \) be a Kummer extension of \( \mathbb{K} \in \{ \mathbb{K}_n : n \in \mathbb{N} \cup \{ \infty \} \} \), such that \( \mathbb{L}, \mathbb{L}' \) are galois over \( \mathbb{Q} \), so \( \text{Gal}(\mathbb{L}/\mathbb{L}) \) is a \( \Lambda[\mathbb{G}] \)-module which has exponent \( p^{n+1} \), if \( \mathbb{K} = \mathbb{K}_n \). If \( \mathbb{K} = \mathbb{K}_\infty \), then we assume that \( \text{Gal}(\mathbb{L}/\mathbb{L}) \) is a \( \Lambda - \)module. For arbitrary \( b \in \text{RAD}(\mathbb{L}'/\mathbb{L}), \sigma \in \text{Gal}(\mathbb{L}'/\mathbb{L}) \) and \( \alpha \in \Lambda[\mathbb{G}] \) we have:

\[
(19) \langle \sigma, b^\alpha \rangle = \langle \sigma^\alpha, b \rangle, \quad \text{if} \quad \mathbb{K} = \mathbb{K}_n \quad \text{and} \quad \langle \sigma, b^\alpha \rangle = \left[ \begin{array}{c} \sigma^\alpha \end{array}, b \right], \quad \text{otherwise}.
\]

This is easily verified at finite levels, from the properties of the Kummer pairing. For \( g \in \mathbb{G} \) we have \( \langle a^g, \nu \rangle = \langle (a^g, \nu) \rangle^g \chi(g) \). By definition of the cyclotomic character, \( g^{-1} \chi(g) \) acts like identity on the roots of unity and the galois equivariance of the Kummer pairing yields \( \langle (a^g, \nu) \rangle^g \chi(g) = \left( (a^{g^{-1}}, \nu^{g^{-1}}) \right)^{\chi(g)} \) while the bilinearity of the Kummer pairing implies \( \langle a^g, \nu \rangle = \langle a, \nu^{g'} \rangle \), with \( g' = g^{-1} \chi(g) \). It is an important property of extensions \( \mathbb{L}/\mathbb{K}_\infty \) which are galois over \( \mathbb{K} \), that their multiplicative groups are large self-dual \( \Lambda[\mathbb{G}] \)-modules. In particular, the Leopoldt duals of radicals are always embedded, so at infinity we have

\[
(20) \quad \text{RAD}(\mathbb{L}'/\mathbb{L})^* = \text{Gal}(\mathbb{L}'/\mathbb{L}).
\]

This follows from Theorem \( [2] \) by taking projective limits in \( (19) \). Nevertheless, by acting with the Leopoldt involution on this identity, we obtain in general only \( \text{RAD}(\mathbb{L}'/\mathbb{L}) = \text{Gal}(\mathbb{L}'/\mathbb{L}) \), since the galois group may not have an embedded dual, even relative to \( \text{Gal}(\Omega/\mathbb{L}) \).

The properties of the Kummer pairing at infinity and reflection will become fruitful in the next section, where we prove isomorphisms between subgroups of \( \mathbb{A} \) and the Kummer radicals of extensions \( \mathbb{L}/\mathbb{M}_E \).

A module \( X \) is a \( \Lambda[\mathbb{G}] \)-module iff it is a \( \mathbb{Z}_p \)-module closed under the action of \( \Gamma \times G_0 \). We are mainly interested here in \( T \) and \( T^* \); the associated \( \Lambda[\mathbb{G}] \)-modules will be particularly simple. We investigate at the end some properties of these particular modules.

If \( Y \) is a \( \Lambda[\mathbb{G}] \)-module fixed by \( T \), then it is isomorphic to a \( \mathbb{Z}_p[\mathbb{G}] \) module on which \( \tau \) acts trivially; if \( Y \) is fixed by \( T^* \), it is still isomorphic to a \( \mathbb{Z}_p[\mathbb{G}] \)-module, but \( \tau \) acts via the cyclotomic character, according to \( (11) \), while \( G \) acts via \( G' \), namely \( (g, y) \in G \times Y \mapsto \chi(g)g^{-1}y \). In particular, the dual of \( \Lambda[\mathbb{G}]^T \), the submodule fixed by \( T \), is always embedded and

\[
\Lambda[\mathbb{G}]^T \cong \mathbb{Z}_p[\mathbb{G}], \quad (\Lambda[\mathbb{G}]^T)^* \cong \mathbb{Z}_p[\mathbb{G}'].
\]
If $Y$ is cyclic as a $\mathbb{Z}_p[G]$ module and $Y$ is embedded in a larger cyclic $\Lambda[G]$ module $X$, then there is an embedding $Y \cdot Y \hookrightarrow X$ with image $X_0$, a cyclic $\mathbb{Z}_p[G]$ module. The following Lemma gives details about generators in the case of such cyclic $\Lambda[G]$ modules.

**Lemma 6.** Let $\mathbb{K}_\infty \subset \mathbb{L} \subset \mathbb{L}' \subset \Omega$ be a tower with $\mathbb{L}, \mathbb{L}'$ galois over $\mathbb{K}$ and $X = \text{Gal}(\mathbb{L}/\mathbb{L})$ be a cyclic $\Lambda[G]$ module with $X^T = \{1\}$. Then $\text{RAD}(\mathbb{L}/\mathbb{L})^+ = \{1\}$ and $\text{RAD}(\mathbb{L}/\mathbb{L})^\bullet \cong \text{Gal}(\mathbb{L}/\mathbb{L})$ is a $\mathbb{Z}_p[G]$ module on which $\tau$ acts by the cyclotomic character.

If $X$ is any cyclic $\Lambda[G]$ module annihilated by $T$ and $x \in X$ a generator, then there is a subset $G_x \subset G$ such that $\{x^g : g \in G_x\}$ is a base for $X$ as a $\mathbb{Z}_p[G]$ module. Moreover, support and annihilator of $X$ and $\hat{X}$ are dual to each other:

$$(X^\perp)^\bullet = (\hat{X}^\perp)^\perp, \quad (X^\perp)^\bullet = (\hat{X}^\perp)^\perp.$$

The proof is given in Appendix A. The following application is important for fields which are not CM. We shall show in (4) that

$$Z_p\text{-rk}(\text{Gal}(\Omega_E \cap M/\mathbb{K}_\infty)) = r_2 \quad \text{and} \quad \text{rad}(\Omega_E \cap M/\mathbb{K}_\infty) = \cup_{n=0}^{\infty} E_n^{N_n} \cdot E_n^p^{\kappa + 1}.$$ 

Consider now the group $V^- \subset \text{Gal}(\Omega_E \cap M/\mathbb{K}_\infty)$ as a $\mathbb{Z}_p[G]$ module. Then we may define its canonic supports and annihilators by Definition 5. They induce a decomposition $\mathbb{Q}_p[G] = \mathbb{Q}_p[G]^{e+} \oplus \mathbb{Q}_p[G]^{e-}$ with $\mathbb{Q}_p[G]^{e+} = (V^-)^\perp$ and $\mathbb{Q}_p[G]^{e-} = (V^-)^\perp$. We let $\alpha^{e+}, \alpha^{e-} \in \mathbb{Q}_p[G]$ be the respective canonic idempotents. From the above, we see that

$$(Q_p[G]^{e+})^\perp = (Q_p[G]^{e+})^\bullet = Q_p[G]^{e-}, \quad \text{and} \quad (Q_p[G]^{e-})^\perp = (Q_p[G]^{e-})^\bullet = (Q_p[G]^{e-})^\perp.$$

Furthermore, we have the fundamental property that

$$(23) \quad |E_n^{e-}| < \infty, \quad \forall n \geq 0.$$

Moreover, is $X$ is an arbitrary quasi-cyclic $\mathbb{Z}_p[G]$ module, then there is a decomposition in components induced by $\alpha^{e+}$ and $\alpha^{e-}$:

$$(24) \quad \tilde{X} = \tilde{X}^{e+} \oplus \tilde{X}^{e-}.$$

We prove the above statements in:

**Corollary 2.** There is a canonic decomposition of $\mathbb{Q}_p[G]$ in isomorphy classes $R^{e+}, R^{e-}$ of submodules of $\mathbb{Q}_p$ - rank $r_2$, such that $R^{e-} = (R^{e+})^\bullet$ and $\mathbb{Q}_p[G] = R^{e+} \oplus R^{e-}$. This induces a canonic decomposition of arbitrary quasi-cyclic $\mathbb{Z}_p[G]$ modules $X$ according to (24). The module $X = E_n$, is essentially equal to its $e+$-component, according to (23).
3. Classes as radicals and first applications

In this section we consider the Kummer radicals of extensions \( \mathbb{L}'/\mathbb{L} \) in which \( \mathbb{L} \in \{ \Omega_E, \Omega_{E'} \} \) and show their relations to the class group \( A \). We investigate Kummer pairings and radicals of various relative extensions between \( \bar{\mathbb{K}}_{\infty} \) and \( \Omega \). For this we shall keep the fundamental tower

\[
\mathbb{K}_{\infty} \subset \Omega_E \subset \Omega_{E'} \subset \Omega \subset \Omega(25)
\]

in sight, together with the intersections of \( \mathbb{H} \) and \( \bar{\mathbb{H}} \) with the fields of tower. This gives also valuable insights on the structure of \( \Lambda \)-submodules of \( A \) and \( A' \). If \( \mathbb{F} \subset \Omega \) is any extension, it decomposes with respect to the tower into

\[
\mathbb{K}_{\infty} \subset \mathbb{F} \cap \Omega_E \subset \mathbb{F} \cap \Omega_{E'} \subset \mathbb{F} \cap \Omega \subset \mathbb{F}.
\]

We may also write \( \mathbb{F}_E = \mathbb{F} \cap \Omega_E, \mathbb{F}_{E'} = \mathbb{F} \cap \Omega_{E'}, \mathbb{F} = \mathbb{F} \cap \Omega, \bar{\mathbb{F}} = \mathbb{F} \). The extension \( \mathbb{F}/\mathbb{F}_E \) will be, with a slight abuse of language, equal to \( \mathbb{F}:\Omega_E/\Omega_E \); the groups are most of the times isomorphic; likewise for \( \mathbb{F}/\mathbb{F}_{E'} \) and \( \mathbb{F}/\bar{\mathbb{F}} \). We shall discuss whether this rule applies or there are deviations, in particular for every new type of fields \( \mathbb{F} \) which we encounter.

Let \( n > \kappa \); we define the intermediate levels of \((25)\) as follows: \( \Omega_{E',n} \) is the largest subextension of exponent \( p^{n+1} \) with

\[
\Omega_{E',n} \subset \Omega_{E'}^{\text{Gal}(\mathbb{F}_E/\mathbb{K}_{\infty})^n}.
\]

Thus \( \Omega_{E',n}/\mathbb{K}_n \) is a Kummer extension of maximal exponent, while the extension \( \Omega_{E'}^{\text{Gal}(\mathbb{F}_E/\mathbb{K}_{\infty})^n} \) is infinite. The fields \( \Omega_n, \bar{\Omega}_n \) are defined similarly, with respect to \( \Omega \) and \( \bar{\Omega} \): if \( \mu = 0 \), then \( \bar{\Omega}_n = \Omega_n \).

Although we restrict our attention in this paper mainly to the barred fields, in this Chapter we derive some facts on the structure of \( \Lambda \) which deserve a treatment in full generality. In particular, most of the general results are not directly necessary for the proof of Leopoldt’s Conjecture. Since at this point the distinction between \( T, T^* \)-parts and other \( \Lambda \)-modules is somewhat artificial, we choose to treat the general case in the text rather than in an Appendix: most phenomena are identical with the case of the \( T \) and \( T^* \)-parts. For \( \mathbb{F} \) like above, we let also \( \mathbb{F}_n \subset \mathbb{F}^{\text{Gal}(\mathbb{F}/\mathbb{K}_{\infty})^n} \) be the largest subextension with group of exponent \( p^{n+1} \). At finite levels, the tower \((25)\) and the related filtration of \( \mathbb{F}_n \) are thus

\[
\mathbb{K}_n \subset \Omega_{E,n} \subset \Omega_{E',n} \subset \Omega_n \subset \bar{\Omega}_n,
\]

\[
\mathbb{K}_n \subset \mathbb{F}_{E,n} \subset \mathbb{F}_{E',n} \subset \mathbb{F} \subset \bar{\mathbb{F}} = \bar{\mathbb{F}},
\]

together with the respective intersections with \( \mathbb{H}_n \) and \( \bar{\mathbb{H}}_n \), which are also defined as the maximal Kummer extensions contained in the Hilbert class field of \( \mathbb{K}_n \), respectively its intersection with \( \bar{\mathbb{H}} \).
3.1. The symbols $\Omega_E, n[a^{1/p^{n+1}}], \overline{\Omega}_E[A^{1/p^{\infty}}]$ and their radicals. If $\mathbb{L} = \overline{\Omega}_E$, then the radicals $\text{rad}(\mathbb{L}_n'/\mathbb{L}_n)$ are defined up to units and it will be possible to relate them to subgroups of $A_n$. Consider a norm coherent sequence $a = (a_n)_{n \in \mathbb{N}} \in A$. The notations $\overline{\Omega}_E[a^{1/p^{n+1}}]$ and $\overline{\Omega}_E[A^{1/p^{\infty}}]$ are meaningful, but require certain precision. We define them in full detail, in the way that we shall use them, below. In conformity with Definition 8 we let

\begin{equation}
\hat{A} := \{ a \in A : \varphi(a) \leq 0 \} \supset A^{\nu} = \zeta(A),
\end{equation}

Then $\hat{A}$ is a subgroup and even a $\Lambda$-module. Assume first that $\text{ord}(a_n) \leq p^{n+1}$ for all $n$ and let $\mathfrak{B} \in \mathfrak{A}_n$ and $\beta \in \mathbb{K}_n$ with $(\beta) = \mathfrak{B}^{p^{n+1}}$, a principal ideal. The Kummer extension $\mathbb{L} = \Omega_{E,n}[B^{1/p^{n+1}}]$ is uniquely determined by $a_n$ and we shall denote it by $\Omega_{E,n}[a^{1/p^{n+1}}]$. Indeed, let $\mathfrak{B}' = (x)\mathfrak{B} \in a_n$ be an ideal from the same class (which needs not differ from $\mathfrak{B}$) and $(\beta') = (\mathfrak{B}')^{p^{n+1}} = x^{p^{n+1}} \cdot (\beta)$; then there is a unit $e$ with $\beta' = e\beta x^{p^{n+1}}$ and

$$\Omega_{E',n}[\beta'^{1/p^{n+1}}] = \Omega_{E',n}[e x^{p^{n+1}} \beta^{1/p^{n+1}}] = \Omega_{E',n}[\beta^{1/p^{n+1}}] = \mathbb{L}.$$  

The field $\mathbb{L} = \Omega_{E,n}[a_n^{1/p^{n+1}}]$ is thus well defined for $\varphi(a) \leq 0$. In this case, we note that for any $a_n \in A_n$ which generates the same cyclic subgroup in $A_n$ we also have $\mathbb{L} = \Omega_{E,n}[a_n^{1/p^{n+1}}]$. The converse holds too, by Kummer theory, so it might be more accurate then to write $\Omega_{E,n}[(\mathbb{Z}_p a_n)^{1/p^{n+1}}]$. We keep though notation simple along the usual lines and observe that $\Omega_{E,n}[a_n^{1/p^{n+1}}]$ is well defined for all $a_n \in \hat{A}_n$. Since $\hat{A}$ is a $\Lambda$-module, the notation $\Omega_{E,n}[B_n^{1/p^{n+1}}]$ is also defined for all subgroups or submodules $B \subset \hat{A}$. Let now $a \in \hat{A} \setminus A^\nu$; we show that $\mathbb{L}_n := \Omega_{E,n}[a_n^{1/p^{n+1}}]$ form an injective system. This follows from (12) and the injectivity of $\iota$ in Proposition 12.

$$\mathbb{L}_{n+1} \supset \overline{\Omega}_{E,n+1}[(a_{n+1}^p)^{1/p^{n+2}}] = \overline{\Omega}_{E,n+1}[\iota(a_n)^{1/p^{n+1}}] = \overline{\Omega}_{E,n+1} \cdot \mathbb{L}_n.$$  

Consequently there is a field $\mathbb{L} = \cup_{n=0}^{\infty} \mathbb{L}_n \subset \Omega$, which we shall denote by $\overline{\Omega}_E[a^{1/p^{\infty}}]$. The next lemma shows that $\mathbb{A}$ is inoffensive in our context:

**Lemma 7.**

$$\cup_{n=0}^{\infty} \Omega_{E,n}[A_n^{1/p^{n+1}}] = \overline{\Omega}_E.$$  

**Proof.** Let $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{A}$ and $\mathbb{L}_n = \Omega_{E,n}[a_n^{1/p^{n+1}}]$. For any $\mathfrak{B}_n \in a_n \in \mathbb{A}_n$ and $(\beta_n) = \mathfrak{B}_n^{p^{n+1}}$, by point 3. of Proposition 12 and the choice of $\kappa$, the ideal $\mathfrak{B}_n \cdot \mathbb{O}(\mathbb{K}_{n+\kappa}) = (a_{n+\kappa})$ is principal. Therefore $\beta_n = a_{n+\kappa}^{p^{n+1}} \cdot e$ for some $q \geq p$ and a unit $e \in E_{n+M}$, so

$$\overline{\Omega}_{E,n+\kappa}[a_n^{1/p^{n+1}}] = \overline{\Omega}_{E,n+\kappa} \cdot \mathbb{L}_n = \Omega_{E,n} \subset \overline{\Omega}_E.$$  

This holds for all $a_n$, so the claim follows.  

---

\[6\]This should not be mistaken for Iwasawa's notation $\hat{X} = X^\ast$
Remark 1. Since \( \Omega_E[A_n^{1/p^n+1}] = \Omega_E[\hat{A} = \Omega_E, \text{ the extensions } \Omega_{E,n}[A_n^{1/p^{n+1}}] \) are volatile in a particular way: they arise at level \( n \) and vanish before level \( n + \kappa \); nevertheless, at each finite level there is one such extension of constant size. Therefore the \( p \) - rank of the galois groups \( \text{Gal}(\Omega_n/\Omega_{E,n}) \) at finite levels are larger than the \( \mathbb{Z}_p \) - rank of \( \text{Gal}(\Omega/\Omega_E)\).

We also observe that the extensions \( \Omega_E[a^{1/p^n}] \) and \( \Omega_{E,n}[a_n^{1/p^{n+1}}] \) are well defined for all \( a = (a_n)_{n \in \mathbb{N}} \in A^{\circ} \), also when \( \Lambda \) has infinite \( p \) - rank.

The \( p \) - ramified extensions which are at the center of our attention, are characterized by the following common fact from Kummer theory:

Fact 3. Let \( \mathbb{L} = \bigcup_{n=1}^{\infty} \mathbb{L}_n \) with \( \mathbb{L}_n/\mathbb{K}_n \) Kummer extensions of exponent dividing \( p^{n+1} \) and \( \mathbb{L}_{n+1} \supset \mathbb{L}_n \). If \( \mathbb{L}/\mathbb{K}_\infty \) is \( p \) - ramified, then there are Kummer radicals \( B_n \in \mathbb{K}_n^* \) with \( (\mathbb{K}_n^*)^{p^{n+1}} \subset B_n \), such that

1. \( \mathbb{L}_n = \mathbb{K}_n[B_n^{1/p^n}] \).
2. For each \( a_n \in B_n \) there is an ideal \( \mathfrak{B} \subset \mathcal{O}(\mathbb{K}_n) \) and an ideal \( \mathfrak{p} \) which is divisible only by primes above \( p \), such that \( (a) = \mathfrak{p} \cdot \mathfrak{B}^{p^n} \). In particular, \( a_n \) may be a unit.
3. If \( \mathbb{L} \subset \mathbb{M} \), then \( a_n^{T^*} \in (\mathbb{K}_n^*)^{p^n} \).

Proof. Point 1 is a consequence of \( \mathbb{L}_n \) being Kummer extensions. Since \( \mathbb{L}_n \) is \( p \) - ramified, we deduce point 2. Finally, if \( \mathbb{L} \subset \mathbb{M} \), it is by definition abelian over \( \mathbb{K} \). Therefore, if \( \alpha \in \text{Gal}(\mathbb{L}_m/\mathbb{K}_m) \) is a generator, then \( \alpha^T = 1 \) and Kummer pairing yields

\[
\langle a, \alpha^T \rangle = \langle a^{T^*}, \alpha \rangle = 1,
\]

which confirms point 3, the Kummer pairing being non - degenerate. \( \square \)

In view of Fact 3 almost every cyclic extension \( \mathbb{L}/\Omega_{E,n}, \mathbb{L} \subset \hat{\Omega}_n \) arises in \( \Omega_{E,n}[A_n^{1/p^{n+1}}] \). The almost is related to the fact that in general \( A_n \not\subseteq A_n \); the difference arises from the sequences \( a \in A \) with \( 0 < z(a) < \kappa \), but the good news is that \( z(a) \) is uniformly bounded. We needed \( A_n \) in the definition of \( A_n^{1/p^{n+1}} \) in order to have compatible group structures. However, if \( a = (a_n)_{n \in \mathbb{N}} \in A \) has \( z(a) > 0 \), it also gives raise to extensions in \( \Omega_n \) defined in a similar way to the ones above. Let \( \mathfrak{B} \in a_n \); then \( \mathfrak{B}^{p^{n+1+z(a)}} = (\beta) \) is a principal ideal and \( z(a) \) is the least positive integer for which this holds. Arguing like before, we see that \( \mathbb{L} = \Omega_{E,n}[\beta^{1/p^{n+1}}] \) is an extension that does not depend on \( \beta \) or \( \mathfrak{B} \) but only on the class \( a_n \). One might write it as \( \Omega_{E,n}[a_n^{p^{n+1+z(a)}}] \); it is a radical extension related to the group \( A_n \): the canonic definition above corresponds to \( \Omega_{E,n}[a_n^{p^{n+1+z(a)}}] \) and only applies when \( z(a) \leq 0 \). It is useful to keep this ugly notation for a short time, in order to clarify the point where we need – and develop – a canonical work around for the case when \( z(a) \neq -\infty \).
The next lemma shows that all cyclic extensions of $\Omega_{E,n}$ arise from roots of $\hat{A}_n$:

**Lemma 8.** Let $n \geq \kappa$ and $\mathbb{L} \subset \hat{\mathcal{O}}_n$ be a cyclic extension of $\Omega_{E,n}$. Then there is a class $a_n \in A_n$ such that $\mathbb{L} = \Omega_{E,n}[a_n^{p^{n+1}}]$.

**Proof.** Let $L = \Omega_{E,n}[\beta^{1/p^{n+1}}]$ for some $\beta \in \Omega_{E,n}$. Then it follows from Fact $\mathfrak{B}$ that $(\beta) = \left(\mathfrak{B}^{rad(\mathfrak{B})}\right)^{p^n} \cdot \pi$ with $\pi \in E_n'$ and $0 \leq m \leq n + 1$; if $a_n = [\mathfrak{B}]$ is the class of $\mathfrak{B}$, it follows that $L = \Omega_{E,n}[a_n^{p^{n+1}}]$. \hfill $\blacksquare$

We have thus a one-to-one map between the $RAD(\Omega_n/\Omega_{E,n})$ and fractional powers of $A_n$. As a map $v : A_n \to RAD(\Omega_n/\Omega_{E,n})$ it is however not a group homomorphism, since, for $m = 0$ in the above proof, we may have $v(a_n) = a_n^{p^{n+1}}$ and $v(b_n) = b_n^{p^{n+1}}$, but $v(a_nb_n) = (a_nb_n)^{p^{n+1}}$, and since $z$ is not a homomorphism, neither can $v$ be. We have thus a canonical homomorphism from a subgroup of $A_n$ to a subgroup of the radical of $\Omega_n$ and a map to the full radical, which is not a homomorphism.

The work around is based on the uniform bound on $z(a)$ and uses the fact that $\varsigma(A_n) \subset \hat{A}_n$, and thus $a_n^{p^{n+1}} = (a^{p^{n+1}})^{1/p^{n+1}}$ in 'ugly' notation, corresponds to the canonical root $\varsigma(a_n)^{1/p^{n+1}}$. From Lemma $\mathfrak{B}$ we have

$$\Omega_{n,S}(\Gamma(\Omega_n/\Omega_{E,n})) = \Omega_{E,n}[\varsigma(A_n)^{1/p^{n+1}}] \subset \Omega_{E,n}[\hat{A}_{n}^{1/p^{n+1}}],$$

where $S_{\kappa}$ is the $\kappa$-th socle in Definition $\mathfrak{D}$ we have an exact sequence

$$1 \to \varsigma(A_n) \to RAD(\Omega_n/\Omega_{E,n}) \to C_n \to 1,$$

where $C_n$ is a group of exponent bounded by $p^\kappa$. At infinity, we obtain

$$\Omega = \cup_n \Omega_n = \cup_n \mathbb{K}_n \cdot \Omega_n = \mathbb{K}_\infty \cdot \Omega_n = \cup_n \mathbb{K}_n[\hat{A}_{n}^{1/p^{n+1}}] = \cup_n \mathbb{K}_\infty[\hat{A}_{n}^{1/p^{n+1}}].$$

We have the following back doors to the canonical radicals: $\varsigma$ annihilates the full $\mathbb{Z}_p$ - torsion, which may be more than we wish, but $\iota_{\mathcal{K}}$ only annihilates $A_n$, thus the volatile part of $\Omega_n/\Omega_{E,n}$. We have

$$\text{Gal}(\hat{\Omega}_n/\Omega_n) = \text{Gal}(\Omega_n[A_n^\varnothing]/\Omega_n) \approx \text{Gal}(\Omega_{E,n+\kappa}[\iota_{\mathcal{K}}(A_n)^{1/p^{n+1}}]/\Omega_{E,n+\kappa}),$$

$$\text{Gal}(\Omega_n/\Omega_{E,n}) \approx \text{Gal}(\Omega_{E,n+\kappa}[\iota_{\mathcal{K}}(A_n)^{1/p^{n+1}}]/\Omega_{E,n+\kappa}).$$

Finally, since $\varsigma(A_n) \subset \hat{A}_n$ while $\iota_{\mathcal{K}}(A_n) \subset \hat{A}_{n+\kappa}$ and the radical $\hat{A}_{n+\kappa}^{1/p^{n+1}}$ is canonical for all $n$, we obtain the following consistent definition of radicals of class groups:
Definition 5 (Class-group Radicals). Notations being like above, the symbols $\Omega_{E,n}[s/(a_n)^{1/p^{n+1}}]$ and $\Omega_{E,n+\kappa}[t_{K}(A_n)^{1/p^{n+\kappa+1}}]$ are well defined for all $a_n \in A_n$ and

$$RAD(\Omega_{n}/\Omega_{E,n})^{p^n} \subset \varsigma(\Omega_{n})^{1/p^{n+1}} \subset RAD(\Omega_{n}/\Omega_{E,n}),$$

$$RAD((\Omega_{n} \cdot \mathbb{K}_{n+\kappa})/\Omega_{E,n+\kappa}) = t_{\mathbb{K}}(A_n)^{1/p^{n+\kappa+1}} \subset \hat{A}^{1/p^{n+\kappa+1}}.$$

Furthermore, $\overline{\Omega}_{E}[s/(A)^{1/p^{\infty}}] = \cup \Omega_{E,n}[s/(A_n)^{1/p^{n+1}}]$ and

$$\overline{\Omega}_{E}[t_{\mathbb{K}}(A)^{1/p^{\infty}}] = \cup \Omega_{E,n}[t_{\mathbb{K}}(A_{n-\kappa})^{1/p^{n+1}}]$$

are well defined notations too, and

$$(30) \quad \overline{\Omega} = \overline{\Omega}_{E}[s/(A)^{1/p^{\infty}}], \quad \Omega = \overline{\Omega}_{E}[t_{\mathbb{K}}(A)^{1/p^{\infty}}].$$

Herewith, we define the Class-group Radicals by

$$C-RAD(\overline{\Omega}_{E}) = t_{\mathbb{K}}(A)^{1/p^{\infty}}, \quad C-RAD(\overline{\Omega}/\overline{\Omega}_{E}) = \varsigma(A)^{1/p^{\infty}},$$

$$C-RAD(\Omega_{n}/\Omega_{E,n}) = t_{\mathbb{K}}(A_{n-\kappa})^{1/p^{n+1}}, \quad C-RAD(\Omega_{n-\kappa}/\overline{\Omega}_{E,n-\kappa}) = \varsigma(A)^{1/p^{n+1}}.$$

We also have $\hat{\Omega}_{n} = \overline{\Omega}_{n}[t_{\mathbb{K}}(A_{n-\kappa})^{1/p^{\kappa}}]$ for all $n \geq 2\kappa$ and $\Omega = \overline{\Omega}_{[\mathbb{K}]}(A)^{1/p^{\kappa}}$.

By the choice of index, we see that all the class field radicals $C-RAD(\overline{\Omega}/\overline{\Omega}_{E})$ are canonically isomorphic, at finite and infinite levels with the plain radicals $RAD(\overline{\Omega}/\overline{\Omega}_{E})$ in all the situations covered by the definition. This yields a solid interpretation to the notation $\mathbb{F}[A^{1/p^{\infty}}]$ used by Lang in [22, §6.2], at least for the case then $\overline{\Omega}_{E} \supset \Omega_{E,n}$ or $\overline{\Omega}_{E} \supset \Omega_{E}$. The notation $\mathbb{K}_{\infty}[A^{1/p^{\infty}}]$ must be understood as an arbitrary choice of $\beta_n \in \mathbb{K}_n$ such that $(\beta_n) = \mathfrak{B}_n^{p^{n+1}}$, with $\mathfrak{B}$, $\beta_n$ like above: this is in general not a canonical choice, but there are exceptions. For instance, if $\mathbb{K}$ is CM and we restrict to the minus part $\Omega^{-}$, i.e. the fixed field of $\mathfrak{F}^+$, then, for odd $p$, the radicals $(A^{-})^{1/p^{\infty}}$ are well defined over $\mathbb{K}_{\infty}$.

The radicals $\varsigma(A)$ are $\mathbb{Z}_p$-torsion destructive, while $t_{\mathbb{K}}(A)$ conserves the $\mu$-part. According to the interest or disinterest for the $\mu$-part, one may choose one or the other, both being canonical. By the above, they are not distinct kinds of radicals, they are distinguished by the choice between $\Omega_n$ and $\hat{\Omega}_n$, resp. $\Omega$ or $\hat{\Omega}$ as upper field.

3.2 Embedding adjoints. Let $B \subset A$ be the projective limits of ideal classes $a_n \in A_n$ containing products of ramified primes above $p$, following Greenberg’s notation [12] and $D = A/(A^T \cdot B)$. Then the simplest result in our direction is

$$\overline{\Omega}_{E} = \cup_{n=1}^{\infty} \Omega_{E,n}[A^{1/p^n}] = \Omega_{E,n}[B^{1/p^n}].$$

The inclusion $\overline{\Omega}_{E} \supset \Omega_{E,n}[B^{1/p^n}]$ is clear. The other direction follows from the definition of the $p$-units $E'$. We may thus proceed with the investigation of extensions over $\Omega_{E',n}$ – but will encounter $\overline{\Omega}_{E}/\overline{\Omega}_{E}$ once again, when discussing unramified extensions.
Since the \( p \) - units are powers in \( \Omega_{E',n} \), it follows that
\[
\Omega_n = \Omega_{E',n}[\varsigma(\bar{A}'_n)^{1/p^{n+1}}], \quad \Omega = \Omega_{E'}[\varsigma(\bar{A}')^{1/p^\infty}].
\]

We shall fix a \( \Lambda \) - module \( A^1 \subset A' \) which is a complement of the \( \mathbb{Z}_p \) - torsion: \( A' = A^1 \oplus (A')^0 \). Let \( F(T) \in \mathbb{Z}_p[T] \) be the characteristic polynomial of \( A' \) – which is also the one of \( A^1 \) and \( f|F \) be a distinguished polynomial. Since \( \varsigma(A') \) is a Weierstrass module, by Lemma \( \mathbb{2} \) there is a canonical submodule \( \varsigma(A')_f \subset \varsigma(A') \) with \( \varsigma(A')_f \cong (A'/(A')^f)^{p^n} \). This shows a relation to the double adjoint of \( A'/A')^f \); adjoints were introduced by Iwasawa in \([13]\) in relation to Kummer pairing. Later, in \([14]\) \( \S \) \( \alpha \), it follows Tate and the definition is free of concern for an explicite embedding of the adjoint \( \alpha(X), X \subset A \) in \( A \). We do not need adjoints here, but are interested in the observation that the main property of \( \alpha(X) \sim X^* \), namely that of being a canonical module free of finite \( \mathbb{Z}_p \) - torsion, is close to our \( \varsigma(A') \). Indeed, if \( X \subset A' \) has finite \( p \) - rank, we see that
\[
\varsigma(X) \cong (\alpha \circ \alpha(X))^{p^n}(X) \cong (\alpha(\alpha(X))^{p^n} \cong (\alpha(\alpha(X)).
\]

For arbitrary \( X \), we have \( i_X(X) \cong (\alpha(X)). \)

Restricted to \( \text{RAD}(\Omega/\Omega_{E'}) \), it follows that \( \varsigma(A')_f \cong (\alpha(A'/A')^f) \), so we may regard \( \varsigma(A')_f \) as an 
\textit{explicit embedding} of \( \alpha(A'/A')^f \) in \( A' \). Furthermore, for any choice of \( A^1 \), there is a unique Weierstrass submodule \( A^1_f \subset A^1 \) such that \( \varsigma(A^1_f) = \varsigma(A')_f \), that is, a \( p^n \)-th root of \( \varsigma(A')_f \) in \( A^1 \); there is no such canonical root, but the possible choices are all reduced to the one choice of \( A^1 \). We may write with a slight abuse of language
\[
\Omega_{E'}[\varsigma(A')_f^{1/p^\infty}] = \Omega_{E'}[(A')_f^{1/p^\infty}].
\]

We now construct subfields of \( \Omega \) and \( \Omega \) with group fixed by \( f \). Since
\[
\cup_n (f^p \mathbb{Z}_p) = f^p \mathbb{Z}_p \quad \text{and} \quad \cup_n (f^p \mathbb{Z}_p) = f^p \mathbb{Z}_p,
\]

the subgroups \( f^p \mathbb{Z}_p \subset \mathbb{Z}_p \) and \( f^p \mathbb{Z}_p \subset \mathbb{Z}_p \) are compact and there are fixed fields \( \Omega_f := \Omega_{E'}^f \) and \( \Omega_f := \Omega_{E'}^f \), which are the maximal subfields of \( \Omega \) with Galois groups over \( \mathbb{K}_\infty \), resp. \( \mathbb{H}_n \), annihilated by \( f \).

\textbf{Definition 6} (The fields \( \Omega_f, \Omega_f \) and the modules \( A_f, \varsigma(A_f) \)). Let \( F \in \mathbb{Z}_p[T] \) be the characteristic polynomial of \( A \) and \( f|F \). The fields \( \Omega_f, \Omega_f \) are defined by
\[
\Omega_f^\mathbb{N}, \quad \text{and} \quad \Omega_f := \Omega_f^\mathbb{N}.
\]

The intermediate fields \( \Omega_{n,f}, \Omega_{n,f} \) are the maximal Kummer extensions of exponent \( p^{n+1} \) over \( \mathbb{K}_n \), respectively \( \mathbb{H}_n \), which are included in \( \Omega_n \), resp. \( \Omega_f \).

For \( f(T) \neq T \), we define \( \varsigma(A)_f = \{ x \in \varsigma(A) : x^f = 1 \} \) and we have in fact \( \varsigma(A)_f \subset \varsigma(A) \); furthermore \( A_f \subset A^1 \) is the unique submodule in the fixed Weierstrass module \( A^1 \), such that \( \varsigma(A)_f = \varsigma(A) \). In this case we have \( \varsigma(A)_f = \varsigma(A')_f = A_f \).
For $f(T) = T$ we let $\zeta(A')_T = \{ x \in \zeta(A') : x^T = 1 \}$ and $\zeta(A)_T = \zeta(B) \cdot \zeta(A')_T$. The modules $A_T, A'_T$ are defined by intersection with $A^\dagger \cap A'$ and $A^\dagger$, like before.

The galois groups are

$$\text{Gal}(\Omega_f/\mathbb{K}_{\infty}) = \mathcal{Y}_f \quad \text{Gal}(\tilde{\Omega}_f/\mathbb{K}_{\infty}) = \breve{\mathcal{Y}}_f,$$

$$\text{Gal}(\Omega_{n.f}/\mathbb{K}_n) = \mathcal{Y}_{f,n} \quad \text{Gal}(\tilde{\Omega}_{f,n}/\mathbb{H}_n) = \breve{\mathcal{Y}}_{f,n}$$

We have $\Omega_T = \mathbb{M}$; the further fields of interest in this paper are $\Omega_{T^*}$ and $\tilde{\Omega}_{T^*}$ together with the modules $A_T, A_{T^*}$, resp. $\zeta(A')_T, \zeta(A)_{T^*}$.

3.3. The principal self-dual module of Iwasawa theory. We come to consider the intersections of $\mathbb{H}$ with $\Omega_f$. By definition, the group $\text{Gal}(\mathbb{H}/\mathbb{K}_{\infty}) \cong A/A^0 \cong \zeta(A)$ is a Weierstrass module, and the Artin map induces an isomorphism $\tilde{\varphi} : \zeta(A) \to \text{Gal}(\mathbb{H}/\mathbb{K}_{\infty})$. For $f|F$ as above we let $\mathbb{H}_f = \mathbb{H} \cap \Omega_f$ and $E = \mathbb{H} \cap \tilde{\Omega}_f = \mathbb{H} \cap \tilde{\Omega}_E$; the last equality is a consequence of the fact that $\tilde{\Omega}_E/\tilde{\Omega}_f$ is totally ramified. Let $A_E \subset \zeta(A)$ be the group such that $\tilde{\varphi}(A_E)$ fixes $\mathbb{H}_E$ in $\mathbb{H}$. Note that the Artin map $\varphi$ only determines a non-canoncic counterpart $A^\dagger_E \subset A^\dagger$, which explains the use of $\tilde{\varphi}$.

Let $a, b \in \zeta(A)$; ramified and unramified extension have the following product rule: if $\Omega_E[a^{1/p^\infty}]$ and $\mathbb{H}[b^{1/p^\infty}]$ are unramified, then so is $\Omega_E[(ab)^{1/p^\infty}]$. If only one is ramified and one is unramified, then $\Omega_E[(ab)^{1/p^\infty}]$ is ramified. If both are ramified, nothing can yet be said about $\Omega_E[(ab)^{1/p^\infty}]$. There is a maximal unramified subextension $\Omega_f/\Omega_E$, and we denote by $A_{nr} \subset \zeta(A)$ its class - radical: the maximal submodule such that $\Omega_E[A_{nr}^{1/p^\infty}]$ is totally unramified. Since $\text{RAD}(\Omega_E/\mathbb{H}_E/\mathbb{K}_{\infty})$ is a $\Lambda$ - cyclic Weierstrass module by definition of $\Omega_E$ and $\mathbb{H}$, it follows that

$$\zeta(A)/\zeta(A_E) \cong \text{Gal}(\Omega_E/\mathbb{H}_E/\mathbb{K}_{\infty})$$

is also a $\Lambda$ - cyclic Weierstrass module. We let $A_r \subset \zeta(A)$ be a radical for $\Omega_f/\Omega_E \cdot \mathbb{H}$. Then $A_r \cong \zeta(A)/A_{nr}$, yet the choice of $A_r$ is in general not canoncic; but see also the proof of Theorem 3. It is not difficult to verify that the notation $\mathbb{H}_E[A_{nr}^{1/p^\infty}]$ is well defined and the field diagram

$$\Omega_E \longrightarrow \Omega_E[A_{nr}^{1/p^\infty}]$$

commutes. Indeed, let $a = (a_n)_{n \in \mathbb{N}} \in A_{nr}$. At finite levels, if $\Omega_{E,n}[a_n^{1/p^{n+1}}] = \Omega_{E,n}[\beta_n^{1/p^{n+1}}]$ with $(\alpha_n) = (\beta_n) = (\beta_n) = 2^{p^{n+1}}$ and $[\beta] \in a_n$, then the extension is unramified and we find that $\mathbb{H}_n[A_{nr}^{1/p^\infty}] \subset \mathbb{H}_{E,n}$, which shows that $\mathbb{H}_E[A_{nr}^{1/p^\infty}]$ is well defined. By definition of $A_E$ we have

$$\tilde{\varphi}(A_E) \cong \text{Gal}(\Omega_{nr}/\Omega_E) = \text{Gal}(\Omega_E[A_{nr}^{1/p^\infty}]/\Omega_E) \cong \hat{A}_{nr},$$
the last isomorphism being by (20). Since the Artin map is covariant, it follows by comparing the first and last terms above that $A_{nr}$ and $E$ have embedded duals in $A$, which are dual to each other: $A_{nr} = A_E$ and $A_E^* = A_{nr}$. Using Iwasawa’s skew symmetric pairing we have the stronger result:

**Theorem 3.** Notation being like above,

1. The groups $A_E = A_{nr}$, being thus self - dual; $\text{Gal}(\Omega_{nr}/\Omega_E) \cong A_E$ is a self - dual group, isomorphic to $A_{nr}$.

2. For $f|F$ and $\varsigma(A')_f$ defined in Definition 6 we have

$$\Omega_{E'} \cdot \Omega_f = \Omega_{E'}[\varsigma(A')_f \cap A_{nr}]^{1/p^\infty},$$

$$\Omega_{E'} \cdot \Omega_f = (\Omega_{E'}[\varsigma(A')_f \cap A_{nr}]^{1/p^\infty}.$$

3. The ramified extension $\Omega/(\Omega_{E'} \cdot \Omega_f)$ is related to $\Omega_E$ by:

$$\text{Gal}(\Omega/(\Omega_{E'} \cdot \Omega_f)) \cong \text{Gal}(\Omega_E/\Omega_{\infty}) \cong \varsigma(A)/A_E \cong A_f.$$

4. Let $F_E = \chi_{1}/A_{E}$ be the characteristic polynomial; if $f$ is a Weierstrass polynomial such that $f|F_E$ and $(f,F/f) = 1$, then $\Omega_{E'}[\varsigma(A')_f^{1/p^\infty}] / \Omega_{E'}$ is totally ramified.

5. In particular, $\Omega_{E'}[\varsigma(A')_f^{1/p^\infty}] / \Omega_{E'}$ and $\Omega_{E'}[\varsigma(A')_f^{1/p^\infty}] / \Omega_{E'}$ are totally ramified.

We review in Appendix B the Iwasawa skew symmetric pairing and give a proof of the theorem by using this pairing.

3.4. **Complex conjugation, explicite radicals and estimation of ranks.**

If $K$ is a CM field and $f \subset \text{Gal}(K_{\infty}/Q)$ is the complex conjugation, then for any $\Lambda$ - module $X$ there is a canonic splitting

$$X = X^+ \oplus X^- = X^{(1+i)/2} \oplus X^{(1-i)/2},$$

and the denominator can be omitted for $p > 2$. Accordingly, if $L/K_\infty$ is a CM extension, then $L^+ = L^{\text{Gal}(L/K_\infty)^-}, L^- = L^{\text{Gal}(L/K_\infty)^+}$. An important property of complex conjugation is

**Fact 4.** If $K$ is CM, then the capitulation kernels $\text{Ker}(A_{nr} \to A^-) = \{1\}$ for all $n$.

We gather some technical results on explicite radicals. The core question is the following: let $f|F$ and $\Omega_n,f \subset \Omega_n$ be the maximal Kummer extensions of $K_n$ of exponent $p^{n+1}$ included in $\Omega_f$. How can the radicals $\text{rad}(\Omega_n,f/K_n)$ be expressed explicitly, in the filtration (20)? The answer is in general indicated by the following observation: let $p^{(n)}$ be the least power of $p$ in the ideal $(f^*(T), \omega_n(T))$ and suppose explicitly that

$$f^*(T) \cdot u_n(T) + \omega_n(T)v(T) = p^{(n)}w_n(T), \quad u_n, v_n, w_n \in \Lambda.$$

Then $u_{n+1}(T) \equiv u_n(T) \mod \omega_n(T)$. If $R_n = \text{rad}(\Omega/K_\infty)$ then, essentially, $\text{rad}(\Omega_n,f/K_n) = R_n^{(w_n,u_n)}$, where some additional work has to be done about the powers of $p$. For estimating ranks of totally ramified extensions, we are
also interested in $U_f$ and its intermediate groups $U_{n,f}$, where $U := \cup_n U_n$. The idea is common for both questions and the technical details are related to the issue of eliminating eventual $p$-powers. In our case, the interesting value is $f(T) = T$. The results are used, among other, for determining $Z_p$-ranks of the interesting extensions in $[2]$. Some of the results of this section will be proved in Appendix B. The Lemma 14 in Appendix A is a central instrument of proof. It has the following important consequence, proved in Appendix B:

**Proposition 3.** For each $n > \kappa$ we have

$$p\text{-}rk(\tilde{H}_{T^n,n}) = r_2 + s - 1,$$

and in the limit,

$$Z_p\text{-}rk(\tilde{H}_T) = r_2 + s - 1.$$

Recall that $K$ is a complex galois extension, so $r_1 = 0$. The second application of Lemma 14 is a constructive proof of the known result (35); the constructive approach yields additional information about the radicals, which brings insights on duality in $Z_p[G]$.

**Proposition 4.** Let $E_n = E_n^{N_n,n}E_n^{p^{n+1}}$ and $E = \cup_{n=\kappa+1}^\infty E_n^{1/p^{n+1}}/E_n$. Then $p\text{-}rk(E) = r_2$ and $M \cap \Omega_E = K_\infty[E] = \cup_{n=\kappa+1}^\infty E_n^{1/p^{n+1}}$; moreover

$$Z_p\text{-}rk(\text{Gal}(M \cap \Omega_E/K_\infty)) = r_2.$$

When $K$ is a CM field, by class field theory ([22], Chapter 5, Theorem 5.1) and since $E^-$ is finite, being equal to the group of roots of unity, (35) specializes to

$$M^- \subset \Omega_E.$$

In both cases, Leopoldt’s Conjecture is equivalent to

$$M \subset \Omega_E.$$

We see from (35) and (36) that, independently on the truth of Leopoldt’s Conjecture, the intersection $M \cap \Omega_E$ is a canonical subfield of $M$. In the CM case, it coincides with $M^-$, while the global Artin symbol $\varphi$ is bijective as a map $U(K)^-/(U(K)^-) \to \text{Gal}(M^-/K_\infty)$.

In the case when Leopoldt’s Conjecture is false, $Z_p\text{-}rk(\text{Gal}(M/K_\infty)) = r_2 + \mathcal{D}(K)$ and it follows from (35) that $Z_p\text{-}rk(\text{Gal}(M/\Omega_E)) = \mathcal{D}(K)$. There is a submodule $A_\ast \subset A_T^*$ of rank $\mathcal{D}(K)$ such that $M \cdot \Omega_E = \Omega_E[\varsigma(A_\ast)^1/p^\infty]$; indeed, the radical of $\text{RAD}(M/\Omega_E)$ is a class field radical by definition, and duality implies that it is annihilated by $T^*$. We shall write $\Phi_r = \Omega_E \cdot M$, a *phantom* extension with group over $\Omega_E$ having the rank equal to the Leopoldt defect. Let $\Phi = \tilde{H}_T^*$ and $D = \text{RAD}(\Phi/K_\infty)$; then $D^T = \{1\}$. Now $D$ is a Weierstrass module, and from Fact $[3]$ and the finiteness of $A_0$ we deduce that $\varsigma(D) \subset E(K)$. It is not difficult to see that $\lim_n p\text{-}rk(i(E) \cap U^{p^n}) = \mathcal{D}(K)$, but the following lemma gives a construction of the radicals $\text{rad}(\Phi_n/K_n)$:
Lemma 9. Assume that the Leopoldt defect \( r = D(\mathcal{K}) > 0 \). For every \( n > 0 \) there is a \( \mathbb{Z} \)-submodule \( D_n \subset E \) such that \( (D_n \cdot E^p)/E^p \) has \( p \)-rank \( r \) and \( \iota(D_n) \subset U^{p^n+1} \). Furthermore, \( D_{n+k} \subset D_n \cdot E^{p^{n+k+1}} \) and \( (D_n \cdot E^{p^n+1})/E^{p^n+1} \) is a group with exponent and sub-exponent \( p^{n+1} \). In particular, the extensions \( \mathbb{K}_n[D_n^{1/p^{n+1}}]/\mathbb{K}_n \) are unramified and form an injective sequence with limit \( \cup_n \mathbb{K}_n[D_n^{1/p^{n+1}}] = \Phi \).

It follows that the field \( \Phi = \mathbb{H}_T^* \) has group of rank \( D(\mathcal{K}) \): it is the first phantom field mentioned in the introduction. Furthermore, since \( \text{Gal}(\mathbb{H}/\mathbb{K}_\infty) \) is a Weierstrass - module, we have \( A_T^* \cong \text{Gal}(\Phi/\mathbb{K}_\infty) \) so \( A_* = A_{T^*} \), having the same rank.

The last radicals which we need to consider for the sequel are \( \text{rad}(\Omega_{T^*,n}/\mathbb{K}_n) \). The following result concerns these radicals.

Lemma 10. Let \( \mathbb{K} \) be above, let \( n \geq 3\kappa \) and \( \mathbb{L}_n \subset \Omega_{E,n} \) be a cyclic subextension with group annihilated by \( T^* \) and \( [\mathbb{L}_n : \mathbb{K}_n] = p^\beta \leq p^n+1 \), \( m \geq \kappa \). If \( \mathbb{L}_n = \mathbb{K}_n[e_n^{1/p^m}] \), then the unit \( e_n \in E(\mathbb{K}_n) \) verifies
\[
(38) \quad e_n = c_n \cdot w_n^{p^{m-\kappa+1}}, c_n \in E'(\mathbb{K}) \setminus (E'(\mathbb{K}))^p, \quad w_n \in E_{2\kappa} \cdot (E'_n)^T.
\]

Proof. It follows by duality from the definition of \( \mathbb{L}_n, e_n \), that \( e_n^T = d_n \) and thus \( N_n(d)^{p^m} = 1 \), so \( N_n(d) = \xi \in \mu_{p^\kappa+1} \subset \mathbb{K} \). By Hilbert 90, it follows that \( d_n^{p^\kappa+1} = w_n^T, w_n \in \mathbb{K}_n \) and thus \( (e_n/w_n^{p^{m-\kappa+1}})^T = 1 \), so \( e_n = c_n \cdot w_n^{p^{m-\kappa+1}} \) with \( c_n \in \mathbb{K} \). Furthermore, \( w_n^T \in E_n \) and by Lemma 16 it follows that \( w_n = d_1 d_2^2 \) with \( d_1 \in E_{2\kappa}, d_2 \in E_n'(\mathbb{K}) \). In particular, \( c_n \in E'(\mathbb{K}) \) and since \( e_n \notin E'_n \), we must have \( c_n \notin (E'(\mathbb{K}))^p \), which completes the proof.

As a consequence concerning the filtration of \( \Omega_{T^*} \), it follows that \( \Omega_{E^*} \cap \Omega_{T^*} = \mathbb{K}_\infty[(E')^{1/p^\infty}] \). We shall write from now on \( \Omega_{E_1} = \mathbb{K}_\infty[E(\mathbb{K})^{1/p^\infty}] \) and \( \Omega_{E'_1} = \mathbb{K}_\infty[E_1^{1/p^\infty}] = \Omega_{E_1}[\Pi^{1/p^\infty}] \). It thus follows from Lemma 10 that:
\[
(39) \quad \Omega_{T^*} \cap \Omega_{E^*} = \Omega_{E_1}[\Pi^{1/p^\infty}] = \Omega_{E'_1}.
\]

4. The filtration of \( \mathcal{M}/\mathbb{K}_\infty \) and the Conjecture of Gross-Kuz’min

We recall from Theorem 3 the main result of interest for the sequel:

Lemma 11. The fields \( \Omega_{E^*}[\zeta(A)^{1/p^\infty}] \), \( \Omega_{E^*}[\zeta(A')^{1/p^\infty}] \) are totally ramified over \( \Omega_{E^*} \) and \( \mathbb{H}_T^* \subset \mathcal{M}_E^* \).

Let \( R_n = E(\mathbb{K})^{n-\kappa} \cdot N_n(E_n) \) for \( n \geq \kappa \); then \( R_n \supset R_{n+1} \) and the quotients \( \mathcal{E}_n := E(\mathbb{K})/R_n \) form a projective system with limit \( \mathcal{E} := \varprojlim E(\mathbb{K})/R_n \).

Then

Lemma 12. Let \( \mathcal{E} := \varprojlim E(\mathbb{K})/R_n \), like above. There is an injective map of \( \mathbb{Z}_p[G] \) -modules \( \zeta(A')_T \to \zeta(\mathcal{E}) \) with \( \zeta(\mathcal{E}) \cong \zeta(A')_T \).
Proof. We give an explicite construction of the map \( v \). Let \( a'_1, a'_2, \ldots, a'_k \in A'_T \) be a minimal set of generators. Then \( z(a'_i) \geq 1 - \kappa \); let \( a_i = (a'_i)^{p^{z(a'_i)}} \), so \( z(a_i) = -(1 + \kappa) \) for all \( i \). We shall prove that for any \( a = (a_n)_{n \in \mathbb{N}} \in \{ a_i : i = 1, 2, \ldots, k \} \) and each \( n > 0 \), there is a unique \( e_n \in E(\mathbb{K}) \) depending on \( a_n \) and such that \( e_n \not\in N_n(E_n) \); furthermore \( e_{n+1} \equiv e_n \mod N_n(E_n) \).

Let \( n > 2\kappa, A_n \in a_n \) and \((A_n) = \mathfrak{A}_n^{p^{-\kappa}} \); since \( a_n = 1 \), there is a \( \nu_n \in \mathbb{K}_\times^\times \) with \( \mathfrak{A}^T_n = (\nu_n) \) and \( \alpha_n^T = d_n \nu_n^{p^{-\kappa}}, d_n \in E_n \). Let \( c := N_n(\nu_n) \in E(\mathbb{K}) \); then \( N_n(c) = \nu_n^{p\kappa - x} = N_n(\nu_n)^{p\kappa - x} = N_n(e_n)^{-1} \); thus by Hilbert 90, \( e_n = c \cdot u^T \) and \( \alpha_n^T \equiv e_n \mod ((\mathbb{K}^\times)^T \cdot (\mathbb{K}^\times)^{p\kappa^{-\kappa}}) \). In this form, one can verify that \( e_n \) does not change upon choosing different representants for the principal ideal \( \mathfrak{A}_n^{p^{-\kappa}} \) or other ideals from the class \( a_n \). Assuming that \( e_n \in N_n(E_n) \), then \( N_n(\delta \nu) = e_n \cdot e_n^{-1} = 1 \) and there is a unit \( \delta \in E_n \) such that \( \nu = \delta \cdot x^T \). It follows that \((\mathfrak{A}_n/\langle x \rangle) = (1) \) and \( \mathfrak{A}_n \) is an ambig ideal, thus a product of ramified primes and ideals from \( \mathbb{K} \) which capitated in \( \mathbb{K}_\times^\times \). Since \( n > 2\kappa \) all ideals from \( \mathbb{K} \) belonging to classes in coherent sequences of bounded order are principal in \( \mathbb{K}_\times^\times \), but not \( a_n \): it remains that \( a_n \in \mathfrak{B}_n \) and \( \alpha_n \) is a \( p \)-unit, which contradicts the assumption \( a \in A' \). Therefore \( e_n \in N_n(E_n) \) iff \( a_n \) is the principal class and there is thus an injective map \( v_n : A'_T, n \rightarrow \mathfrak{C}_n \) which extends \( a_n \mapsto e_n \) by \( \mathbb{Z}/(p^{-\kappa} \cdot \mathbb{Z}) \)-linearity.

In order to show that \( e_n \in \mathfrak{C}_n \) form a projective system, let \( m > n > \kappa \) and \( \Omega_m \in a_m \) be a prime that splits completely over \( \mathbb{Q} \) and \( \alpha_m, \nu_m, e_m \) be defined with respect to \( \Omega_m \). Then we have seen that \( e_m = N_m(\nu_m)^{-1} \) independently of the choice of the generator \( \nu_m \) of \( \mathfrak{A}_m^T \). Since \( (N_{m,n}(\nu_m)) = N_{m,n}(\Omega_m)^T = \Omega_n^T = (\nu_n) \), it follows from the uniqueness of \( e_m, e_n \), that \( e_m \equiv e_n \mod N_n(E_n) \). Note also that \( \nu_n(a_n^m) = N_n(\nu_n)^{p\kappa} = \nu_n(a_n)^2 \), both for \( x \in \mathbb{N} \) and for \( x \in G \). Thus the maps \( v_n \) can extend in the projective limit to an injective morphism of \( \mathbb{Z}_p[\mathcal{G}] \) - modules \( v : A'_T \hookrightarrow \mathfrak{C} \). The initial choice of \( a \) deserves some remark: in order to achieve \( \text{ord}(a_n) = p^{\kappa^{-\kappa}} \) uniformly on all an – which simplifies the notation – we assumed that \( a = a^q \) for some \( p \)-power \( p \) and \( a \in A'_T \subset (A'_T)^p \). We may define \( \nu_n, e_n \) with respect \( a_n \), and the map \( v \) extends naturally to \( a \); it follows that \( v(a) = v(a)^q \). Thus \( v \) is defined on a minimal set of generators for \( A'_T \) and extends to an injective homomorphism of \( \mathbb{Z}_p[\mathcal{G}] \) - modules, as claimed.

Conversely, let \( e = (e_n)_{n \in \mathbb{N}} \in \mathfrak{C} \) and \( n > \kappa \). By the Hasse Norm Theorem for cyclic extensions, there is an \( x_n \in \mathbb{K}_n \) with \( N_n(x_n) = e_n \) and, considering the prime ideal decomposition of \( x_n \), we find that there is some ideal \( \mathfrak{X}_n \) such that \( (x_n) = \mathfrak{X}_n^{T} \). The definitions above show that \( e_n = v_n([\mathfrak{X}_n]) \); let \( x_n = [\mathfrak{X}_n] \) and \( x = (x_n)_{n \in \mathbb{N}} \in A \). Obviously \( x^T = 1 \); furthermore, \( \text{ord}(x) = \infty \) since otherwise \( \text{ord}(\mathfrak{X}_n) \leq \kappa \) and one deduces that \( e_n^p \in N_n(E_n) \) for all \( n \) and thus \( \varsigma(e) = 1 \). The restriction of \( v \) to \( \varsigma(A') \) is thus bijective, which completes the proof. \( \square \)

We are prepared for the
Theorem 4. The Conjecture of Gross - Kuz'min holds for $\mathbb{K}$.

Proof of the Conjecture of Gross-Kuz'min. For completing the argument, we consider first the simpler CM case. Since $\mathbb{H}_T \subset \Omega_E$, by Lemma 11 it follows by reflection, in the CM case, that $A'_T \subset A^-$. However, $E(\mathbb{K})^- = \mu_p$ is a torsion group, and the previous lemma implies that $(A'_T)^{-} = A'_T = \{1\}$ in this case, which completes the proof. Using the Lemma 18 this argument extends to the non-CM case.

We give an alternative proof for the non CM case, using Lemma 12. Formally, $E_n$ are submodules of factors of $E(\mathbb{K})$; the extensions $\Omega'_n = \mathbb{K}_n[\zeta(\mathcal{E})_n^{1/p^{n+1}}]$ are well defined, since $E(\mathbb{K})^{1/p^{n+1}} \subset R_n$. Let $\Omega' = \cup_n \Omega'_n = \mathbb{K}_\infty[\zeta(\mathcal{E})^{1/p\infty}]$ and $\mathcal{Y}' = \text{Gal}(\Omega'/\mathbb{K}_\infty)$. By reflection we have $(\mathcal{Y}')^{T\ast} = \{1\}$, and thus $\mathcal{Y}' \subset \Omega_{T\ast} \cap \Omega_E$ while $\Omega'' := \Omega_E[\zeta(A'_1)^{1/p\infty}] \subset \Omega_{T\ast}$ too. The radicals of these extensions are isomorphic and $\nu$ induces by duality an isomorphism $\text{Gal}(\Omega'/\mathbb{K}_\infty) \cong \text{Gal}(\Omega_E[\zeta(A'_1)^{1/p\infty}]/\Omega_E)$.

We know little about $\Omega_{T\ast}$ but $\Omega_{T\ast} = \text{Gal}(\tilde{\Omega}_T/\mathbb{H})$ is a quasi-cyclic $\mathbb{Z}_p[G]$ module of essential $\mathbb{Z}_p$ - rank $r_2 + s - 1$, by Proposition 3 also $\Omega_E[\zeta(A'_1)^{1/p\infty}]/\Omega_E$ is totally unramified by Lemma 11. The quasi-cyclicity of $\Omega_{T\ast}$ implies that $\Omega' \cap \tilde{\Omega}_{T\ast} = \mathbb{K}_\infty$. It follows that $\Omega' \subset \Omega_{T\ast} \cap \mathbb{H} = \Phi$. We have the following diagram of fields:

\[
\begin{array}{c}
\mathbb{K}_\infty \\
\Omega_E \\
\Omega_E[\zeta(A'_1)^{1/p\infty}] \\
\tilde{\Omega}_{T\ast} \cdot \Omega_E
\end{array}
\]

in which the field extensions on the right hand side have isomorphic groups and radicals, while both lower and upper lines are $\mathbb{Z}_p[G]$ - quasi cyclic extensions. Since $\text{Gal}(\Phi/\mathbb{K}_\infty) \cong A_{T'\ast}$, it follows that $\zeta(A'_p) \leftarrow \zeta(A_{T'\ast})$. But this contradicts the Lemma 11, thus completing the proof of the Gross-Kuz'min Conjecture for non CM fields. \qed

We might try dualization for proving the Conjecture of Leopoldt. The natural dual of $E(\mathbb{K})$ appears to be $\cup_n E_n^{N^n}$. The condition $N_n(\nu_\ast) \notin N_n(E_n)$ in the proof of Lemma 12 becomes $\nu_\ast^{N_n} \notin E_n^{N^n}$. However, in turns out in this case that $x_n := \nu_\ast^{N^n}$ is nothing else but $\zeta(\alpha_n)$, up to units. Using thus $x_n$ to define a new map $\nu'$ is thus not fruitful, since the image of this map is essentially $\zeta(A_{T'\ast})$ itself. The approach for the Leopoldt Conjecture will therefore take another path.

5. The filtration of $\Omega_{T'\ast}, \tilde{\Omega}_{T\ast}$ and the proof of the Main Theorem

We have shown by class field theory in Lemma 3 that $\mathbb{Z}_p$-rk$(\text{Gal}(\tilde{\Omega}_{T\ast}/\mathbb{H}_{T\ast})) = r_2 + s - 1$. We now traverse the filtration (25) from upside down: $\Omega_E' \cdot \Omega_{T\ast} = \cdots \cdots$
$\Omega_{E'}[s(A)^{1/p^{\infty}}] = \Omega_{E'}$ in view of the proof of the Conjecture of Gross-Kuz'min. Note that we only need the first step of the proof for this, namely the argument based on self-duality of $A_{nr}$, Lemma \[12\]. We thus have $\Omega_{T^*} \subset \Omega_{E'}$. From \[9\] we gather that $\Omega_{T^*} \subset \Omega_{E'_1}$. Conversely, $\text{Gal}(\Omega_{E'_1}/\mathbb{K}_{\infty})^{T^*} = \{1\}$, so we have

$$\Omega_{E'_1} = \Omega_{T^*}.$$  

It is a straightforward verification that

$$Z_p\text{-}rk(\text{Gal}(\Omega_{E'_1}/\mathbb{K}_{\infty})) = p\text{-}rk(\Pi) + p\text{-}rk(E(\mathbb{K})) = s + r_2 - 1 = Z_p\text{-}rk(\text{Gal}(\tilde{\Omega}_{T^*}/\Phi)).$$

Since $\Omega_{T^*} \cap \mathbb{H} = \Phi$ and the groups $\mathcal{V}_{T^*}$ and $\mathcal{V}_{T^*}$ are torsion-free by definition, a rank computation yields for the intersection $\Omega_s = \Omega_{T^*} \cap \tilde{\Omega}_{T^*}$ that

$$Z_p\text{-}rk(\text{Gal}(\Omega_s/\mathbb{K}_{\infty})) = r_2 + s - 1 - D(\mathbb{K})$$

while $\Omega_s \subset \Omega_{E'_1}$ since both fields are included in $\Omega_{E'_1}$. The radical $\text{rad}(\Omega_s/\mathbb{K}_{\infty}) \cong E'_1/\text{rad}(\Phi)$. However, since $Z_p\text{-}rk(\tilde{\mathcal{V}}_{T^*}) = r_2 + s - 1$ too, it remains that there is a totally ramified extension $\Phi_s/\Phi$ with $\mathbb{L} \subset \Omega_{E'}$ and $\text{Gal}(\mathbb{L}/\Phi)^{T^*} = \{1\}$, while $Z_p\text{-}rk(\text{Gal}(\Phi_s/\Phi)) = D(\mathbb{K})$. This is the last phantom extension we have to consider; we shall show by a detailed investigation at finite levels, that it cannot exist, and this implies the Main Theorem.

Like usual, we denote by $\tilde{\Omega}_{T^*,n}$ the maximal Kummer extension of $\mathbb{K}_n \cdot (\mathbb{H} \cap \tilde{\Omega}_{T^*})$ of exponent $p^{n+1}$ included in $\tilde{\Omega}_{T^*}$. Since it is a subfield of $\Omega_{E'_1}$, it is important to observe that $\text{Gal}\left(\tilde{\Omega}_{T^*,n}/\mathbb{K}_n\right)$ is abelian and it may have larger exponent than $p^{n+1}$. We shall give a proof of the Theorem \[10\] by showing that there are no abelian extensions of $\mathbb{K}_n$ which are ramified and contain $\Phi_n$. This implies that $\Phi_s$ cannot exist and $D(\mathbb{K}) = 0$, thus the Leopoldt Conjecture.

The extension $(\tilde{\Omega}_{T^*} \cap \Omega_{E})/\mathbb{K}_{\infty}$ is defined for every distinguished polynomial; if $f$ does not divide the characteristic polynomial $F_E(T) = \text{char}(\text{Gal}(\mathbb{H} \cap \Omega_{E}))$, then $\tilde{\Omega}_{f^*} = \Omega_{f^*}$. The polynomial of interest in relation with Leopoldt’s Conjecture is $f(T) = T$ and we know that $T|F_E(T)$ iff $D(\mathbb{K}) > 0$. We shall derive a contradiction in Proposition \[5\] below, which implies $D(\mathbb{K}) = 0$: we show that $\text{Gal}(\tilde{\Omega}_{T^*/\mathbb{H}_{T^*}})$ cannot have the rank proved by class field theory in Lemma \[3\] a fact based on Lemma \[12\] and the resulting particularity of $f(T) = T$, which makes that

$$\text{rad}(\tilde{\Omega}_{T^*, n} \cdot \mathbb{K}_{2n})/\mathbb{K}_{2n}) \subset E'_1 \cdot (\mathbb{K}_{2n}^\times)^{p^{2n+1}}.$$  

Since this is the central point of the proof of Leopoldt’s Conjecture, it may be useful, before proceeding to the technical proof of the Proposition \[5\] below, to compare to cases $f|F_E$, with $f(T) \neq T$, which are cases which do occur. We illustrate in Appendix C on the example of $\mathbb{K} = \mathbb{Q}[\zeta_p]$, the fact that for some polynomials $f(T) \neq T$, one has in general extensions $\mathbb{K}_n \subset \mathbb{F}_n \subset \mathbb{L}_n$ such that $\mathbb{F}_n/\mathbb{K}_n$ is unramified, $\mathbb{L}_n/\mathbb{F}_n$ is $p$-ramified and $\mathbb{L}_n/\mathbb{K}_n$
abelian. Furthermore both groups $\text{Gal}(\mathbb{F}_n/\mathbb{K}_n)$, $\text{Gal}(\mathbb{L}_n/\mathbb{F}_n)$ are annihilated by $f(T^*)$. As predicted, the difference consists in the fact that in the general case, the radicals

$$\text{rad}((\check{\Omega}^*_{f^*, n} \cdot \mathbb{K}_2 n)/\mathbb{K}_2 n) \subset E'_2 n \cdot (\mathbb{K}^\times_n)^{p^{2n+1}}.$$ 

and the effective part $\text{rad}((\check{\Omega}^*_{f^*, n} \cdot \mathbb{K}_2 n)/\mathbb{K}_2 n)/(\mathbb{K}^\times_n)^{p^{2n+1}} \not\subset \mathbb{K}^\times_n$, modulo $p^{2n+1}$ powers. It turns out that $\check{\Omega}^*_{f^*, n} \not\subset \Omega^*_{f^*, n}$ but $\check{\Omega}^*_{f^*, n} \subset \Omega^*((f^*)^2 n)$; however, only the $\mathbb{F}_n$ form an injective sequence! This general phenomenon is analyzed in detail in Example 2.

Turning back to $f(T) = T$ we prove:

**Proposition 5.** Notations being like above,

$$(41) \quad \text{Z}_p \text{-rk} \left( \text{Gal}(\check{\Omega} \cap \Omega_{E'})/\mathbb{H} \right) = r_2 - 1 + s - \mathcal{D}(\mathbb{K}).$$

**Proof.** We have seen in Lemma 10 that $\text{rad}(\Omega^*_{T^*, n}/\mathbb{K}_n)$ stems from $E'(\mathbb{K})$. On the other hand, Lemma 3 implies that the maximal $p$-abelian extension $\Omega^*_{T^*, n}/\mathbb{K}_n$, which is totally ramified at $p$ above $\mathbb{H}_n$, has group $\check{\Omega}^*_{T^*, n} = \text{Gal}(\check{\Omega}^*_{T^*, n}/\mathbb{H}_n)$ with $p\text{-rk}(\check{\Omega}^*_{T^*, n}) = r_2 + s - 1$ and sub-exponent at least $p^{(n-k)/2}$. If $\Omega^*_{T^*, n} \subset \Omega^*_{T^*, n}$, since there are $\mathcal{D}(\mathbb{K})$ independent unramified extensions of $\mathbb{K}_n$ in $\Omega^*_{T^*, n}$, namely $\Phi_n \subset \Omega^*_{T^*, n}$, the claim already follows. If the claim is false, we must thus have $\check{\Omega}^*_{T^*, n} \not\subset \Omega^*_{T^*, n}$. 

**Fig. 1:** Overview of the main extensions of $\mathbb{K}_\infty$.

The values across lines are $\mathbb{Z}_p$ - ranks of galois groups.
The previous example shows that the fields in $\hat{\Omega}_{T^*,n}$ may be extensions of $\Omega_{T^*,n}$, with group over $\mathbb{K}_n$ which is annihilated by $(T^*)^2$ rather than $T^*$. It suffices to consider subfields of $\Omega_{E^*}$; we shall show that

$$\hat{\Omega}_{T^*,n} \cap \Omega_{E,n} = \hat{\Omega}_{T^*,n} \cap \Omega_{E_1,n},$$

and the Galois group has $p$-rank $r_2 - 1 - D(\mathbb{K})$ over $\Phi_n$; it follows then that $\hat{\Omega}'_{n} = \hat{\Omega}_{T^*,n} \cap \Omega_{E_1,n}[\Pi^{1/p^{\infty}}]$ has a group of $p$-rank $r_2 - 1 + s - D(\mathbb{K})$ over $\mathbb{K}_n$ and

$$\mathbb{Z}_p^{\text{rk}(\text{Gal}(\hat{\Omega}'/\mathbb{K}_n))} = r_2 - 1 + s - D(\mathbb{K}).$$

Let $\delta \in E(\mathbb{K})$ be a Minkowski unit with annihilator $\theta \in \mathbb{Q}_p[G]$, let $q'$ be the $p$-power dividing $|G|$ and $n$ be sufficiently large. For $M > 4(n + 1)$, let $\alpha_M, \theta_M \in \mathbb{Z}[G]$ be approximants to the $p^M$-th order of $q'^{\alpha}, q'^{\theta} \in \mathbb{Z}_p[G]$ as in the Lemma [9] so $\alpha$ generates the annihilator ideal of $\text{Gal}(\hat{\Phi}/\mathbb{K}_\infty)$. We suppose that $M$ is such that $E(\mathbb{K})^{\otimes M} \subset U(\mathbb{K})^{\omega}$. Using the approximants above, we have $\Phi_n \subset \Omega^{\theta_M}_{E_1,n} = \mathbb{K}_n[\delta^{\theta_M/p^{\infty} + 1}]$, an abelian unramified extension with group of $p$-rank $D(\mathbb{K})$. Therefore $\Omega^{\theta_M}_{E_1,n}/\Phi_n$ must contain $D(\mathbb{K})$ independent cyclic extensions of sub-exponent $p^{(n-k)/2}$ and with group annihilated by $T^*$; this follows from Lemma [14].

We restrict ourselves for simplicity to one maximal cyclic extension $L_n/\mathbb{K}_n$ with $L_n \subset \Omega^{\theta_M}_{n,T^*} \cap \Omega_{E}$: as observed above, if the claim of the proposition was false, such an extension must exist. Let $L_n \cap \Phi_n = F_n = \mathbb{K}_n[d^{1/p^{\infty} + 1}]$ for some $d \in D_M$, the radical of $\Phi_n$. Then $p^m : = [L_n/F_n] \geq p^{n/2-\kappa}, m \leq n + 1$ and $\text{Gal}(L_n/F_n)^{T^*} = \{1\}$. Furthermore, $L_n$ is cyclic over $\mathbb{K}_n$. We shall show that the two conditions are incompatible. Assuming that such an extension exists, then $L'_n = \mathbb{K}_{2n} \cdot L_n$ is an abelian extension of $\mathbb{K}_n$ and Kummer over $\mathbb{K}_{2n}$.

Let $e \in E_{2n}$ generate its radical, so $L'_n = \mathbb{K}_{2n}[e^{1/p^{\infty} + m}]$. Since $\mathbb{K}_{2n} \cdot F_n = \mathbb{K}_{2n}[d^{1/p^{\infty} + 1}] \subset L'_n$, we may assume by Kummer theory that $e = d \cdot u^{p^{m+1}}$ for some $u \in E_{2n}$. Since $(\text{Gal}(L'_n/\mathbb{K}_{2n}[d^{1/p^{\infty} + 1}]))^{T^*} = \{1\}$, we must have by duality also that $(du^{p^{m+1}})^T = (u^{p^{m+1}})^T \equiv (u^{p^{m+1}})^{p^m} \mod E_{2n}^{p^{n+1}}$ so $u^{p^{n+1}}T \in E_{2n}^m \cdot E_{2n}^{p^{n+m+1}}$. Moreover, $L'_n/\mathbb{K}_n$ is abelian, so $(du^{p^{m+1}})^{\omega_n} \in E_{2n}^{p^{n+m+1}}$. We now apply (50) to $u$ and $d$ independently:

$$u^{p^{n+1}}w_n \equiv u^{\omega_n p^{n+1}} \equiv u^{p^{m+n-k}} \mod E_{2n}^{p^{n+m+1}}, \quad t \in \Lambda^\kappa,$$

while $d^{\omega_n} = d^{p^{n+1}}, c \in \mathbb{Z}_p^\kappa$. Combining the two, we obtain

$$(du^{p^{n+1}})^{\omega_n} \equiv d^{(c+tN_p p^{m+n-k}} \mod E_{2n}^{p^{n+m+1}}.$$}

Therefore $L'_n/\mathbb{K}_n$ can not be abelian for $m \geq (n-k)/2 > (k + 1)$, i.e. $n \geq 3(k + 1)$.

For $n$ sufficiently large, there is thus no $p$-ramified extension $L_n/F_n$ of degree $p^m, m > (n-k)/2$, with group annihilated by $T^*$ and with $L_n/\mathbb{K}_n$ abelian. Since this holds for all extensions above $\Phi_n$, it follows that $D(\mathbb{K})$
independent cyclic unramified extensions in $\Phi_n$ have no cyclic continuations over $\mathbb{K}_n$ that are $p$-ramified over $\Phi_n$.

It follows that $p$-rk($\text{Gal}(\tilde{\Omega}_n,T^*/\mathbb{H}_n)$) = $r_2 - 1 - D(\mathbb{K})$, which completes the proof of $\Box$.

The main Theorem 1 follows from the Proposition, as explained above: since $\tilde{\Omega}_T \cap \Omega_{E'}$ has group of rank $r_2 + s - 1 - D(\mathbb{K})$ while $\tilde{\Omega}_T \subset \Omega_{E'}$ and by Lemma 3 $\mathbb{Z}_p$-rk($\text{Gal}(\tilde{\Omega}_T^*/\mathbb{K}_\infty)$) = $r_2 + s - 1$, it follows that $D(\mathbb{K}) = 0$.

The Example 2 is continued by an illustration of the main steps of the proof for the case $\mathbb{K} = \mathbb{Q}[\zeta_p]$, which was known from the Theorem of Baker and Brumer.

![](image)

**Fig. 2:** The unamified (marked: - ) and ramified (marked =) extensions at finite levels.

### 6. Consequences

The results in the previous section give a complete picture of the $T$ and $T^*$ parts of the class groups and $p$-abelian extensions in the cyclotomic $\mathbb{Z}_p$-extension of arbitrary galois fields.

The following Conjecture is a natural generalization of the Greenberg Conjecture to arbitrary fields:

\textbf{7} Note also that we might consider $\mathbb{F}, \mathbb{L}$ as extensions of $\Omega_{c,n} = \mathbb{K}_n[[\Pi^{1/p^{n+1}}]]$, since this extension is independent of the Leopoldt Conjecture for all choices of $\Pi$. In this case, we may even assume $d_1 \in E(\mathbb{K})^{\theta \mu}$, so $\mathbb{L}_n$ would also be unramified.
**Conjecture 1.** Let $\mathbb{K}$ be a number field, $\mathbb{K}_\infty$ its cyclotomic $\mathbb{Z}_p$-extension and $\mathbb{H} = \mathbb{H}_\infty^{\varphi}(A^\varphi)$. Then

\[(42) \quad \mathbb{H} \subset \Omega_E.\]

Note that $\text{Gal}(\mathbb{H}/\mathbb{K}_\infty)$ is a $\Lambda$-torsion module by definition, so $(42)$ has no information on $\mu(\mathbb{K})$. The Conjecture $\mu(\mathbb{K}) = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of arbitrary number field is thus independent of the $(42)$.

For the case when $\mathbb{K}$ is totally real, we may adjoin roots of unity to $\mathbb{K}$ and find that $\mathbb{H} \cap \Omega_E = \mathbb{K}_\infty$, since $\text{Gal}(\Omega_E/\mathbb{K}_\infty)^{1+i} = \{1\}$: thus the Conjecture $\mu = 0$ is equivalent to Greenberg for CM fields $\mathbb{K}$. If $\mathbb{K}$ is abelian, the Greenberg Conjecture is equivalent to $(42)$, since $\mu = 0$ was proved by Ferrero and Washington [25], §7.5.

If $f(T)$ divides the characteristic polynomial of $\text{Gal}(\mathbb{H}/\mathbb{K}_\infty)$, we say that Greenberg’s Conjecture holds for the $f(T)$ - part of $A$, if $\mathbb{H}_f \subset \Omega_E$, with $\mathbb{H}_f$ in Definition 6. With this we have proved:

**Theorem 5.** Let $\mathbb{K}$ be a complex galois extension. Then Greenberg’s Conjecture holds for the $T$ and $T^\ast$ parts of $A$ and $A/A^T^\ast$ is finite.

**Proof.** The Gross - Kuz'min Conjecture says that $A_T = \{1\}$ and $A_T^\ast = \{1\}$ is Leopoldt's Conjecture. The last also implies that $\mathbb{M} \subset \Omega_E$ and therefore $A_T \sim B \sim \text{Gal}(\mathbb{H}_T \cap \Omega_E)/\mathbb{K}_\infty)$; but $\mathbb{H}_T \subset \mathbb{M} \subset \Omega_E$, which confirms also the Greenberg Conjecture for the $T$ - part. \[\square\]

Note that $B$ is not necessarily finite for complex fields $\mathbb{K}$. In the case when $\mathbb{K}$ is CM we give below a precise description of $A_T$, since in this case $A_T \cong B^\ast$, canonically.

**Proposition 6.** Let $\mathbb{K}/\mathbb{Q}$ be a CM galois extension and $\mathbb{K}_n, \mathbb{K}_\infty, \mathbb{A}_n, \mathbb{A}$ be defined as previously. Let $\varphi \subset \mathcal{O}(\mathbb{K}^\ast)$ be any prime above $p$ and let

\[g' = \begin{cases} 
0 & \text{if } \varphi \text{ is unsplit in } \mathbb{K}/\mathbb{K}^\ast, \\
g(\varphi) = \frac{[\mathbb{K}^\ast : \mathbb{Q}]}{|D_\varphi|} & \text{otherwise}, \end{cases}\]

here $D_\varphi \subset \text{Gal}(\mathbb{K}^\ast/\mathbb{Q})$ is the decomposition group of $\varphi$. Then the module $B^\ast$ is a free $\mathbb{Z}_p$-module of rank $g'$.

**Proof.** Since ess. $p$-rk$(A/A^T) = \mathbb{Z}_p$-rk$(B)$, it suffices to consider primes $\varphi \subset \mathbb{K}$ which ramify in ideals $\varphi_n \subset \mathbb{K}_n$ with diverging orders in the ideal class group. We know that $A/(A^T^\ast)$ is finite and since $\mathbb{K}$ is a CM field, this implies that $B^\ast$ is finite and ess. $p$-rk$(B) = \mathbb{Z}_p$-rk$(B^\ast)$, where $B^\ast$ is a torsion-free module. It follows in particular that $B$ is finite if the primes above $p$ are unsplit in $\mathbb{K}/\mathbb{K}^\ast$, which explains that $g' = 0$ in this case.

Now if $\varphi$ splits in $\mathbb{K}/\mathbb{K}^\ast$, we have $\mathbb{Z}_p$-rk$(B^\ast) = \mathbb{Z}_p$-rk$((\text{Gal}(\mathbb{M}^\ast \cap \mathbb{H}_T)/\mathbb{K}_\infty)$, so it suffices to evaluate the second rank. Let $L \subset \mathbb{M}^\ast \cap \mathbb{H}_T$ be a $\mathbb{Z}_p$-extension of $\mathbb{K}_\infty$. Then $L_{\varphi}/\mathbb{K}_{\varphi}|\mu_\infty$ is either a trivial extension or the unique unramified $\mathbb{Z}_p$-extension of $\mathbb{K}_{\varphi}$: its group is in particular $G$ - invariant. For each $\tau \in G$ there is exactly one such extension, and since for each
ν ∈ Gal(\(\mathbb{M}^-/\mathbb{K}_\infty\)) we have ν^{1+j} = 1, we see from the definition of \(g'\) that there are at most \(g'\) independent extensions \(L\) with this property. In order to verify that there are exactly \(g'\) extensions \(L\), we observe that \(\mathbb{M}^- = \mathbb{M}\) as a consequence of Leopoldt’s Conjecture and thus \(Z_p\text{-rk}(Gal(\mathbb{M}^-/\mathbb{K}_\infty)) = r_2\) and \(Z_p\text{-rk}(Gal(\mathbb{M}_{\tau_\mathbb{P}}/\mathbb{K}_{\tau_\mathbb{P}}[\mu_\infty]) = [\mathbb{K}_{\tau_\mathbb{P}} : \mathbb{Q}_p].\) In particular, there is a \(G\) - invariant local subextension for each \(\tau\); the count now follows from \(\nu^{1+j}\) for \(\nu_\tau \in \Delta\) an automorphism which generates \(Gal(\mathbb{M}_{\tau_\mathbb{P}}/\mathbb{K}_{\tau_\mathbb{P}}).\) Therefore \(Z_p\text{-rk}(B) = g'\), which completes the proof.

In [16], Jaulent generalizes this expression to fields \(\mathbb{K}\) which do not contain the \(p\)-th root of unity.

In Fig. 3: The a posteriori field diagram, for \(D(\mathbb{K}) = 0\)

7. Appendix A: Basics

The first section of this Appendix is a technical extension of §2.1 and deals with the difficult problem of defining annihilators for \(Z_p[G]\) - modules in the non commutative case.

7.1. Annihilators, supports and components in the non-commutative case. If \(G\) is a finite group and \(X\) a \(Z_p[G]\) - module, then \(X\) is quasi - cyclic if there is some \(x \in X \setminus X^p\) such that \([X : x^{Z_p[G]}] < \infty\), or, equivalently, \(X = \tilde{x}^{Z_p[G]}\). Then \(x\) is called a generator for \(X\); if \(x \in X^p\), we may denote it by weak generator for \(X\) and such generators may occur when considering norm coherent sequences of modules. The module is cyclic, if the index is 1. Let \(q = p^{v_p([G])}\). For any idempotent \(\alpha \in \mathbb{Q}_p[G]\), we have \(q\alpha \in \mathbb{Z}_p[G]\). The Theorem of Maschke [2], p. 116 asserts that each submodule \(A \subset \mathbb{Q}_p[G]\) is a direct sum of irreducible modules. Furthermore, one sees from the proof of the theorem, that there is an idempotent \(\alpha \in \mathbb{Q}_p[G]\) such
that $A = (\alpha) = \alpha \mathbb{Q}_p[G]$. Modules and their idempotents are too numerous, so we will identify isomorphy classes of modules. It is then known that $A$ is isomorphic to an algebra of column - matrices over $\mathbb{Q}_p$; more precisely, we have the following standard fact ([3], Chapters 5 and 6).

**Fact 5.** Let $G$ be a finite group. Then $\mathbb{Q}_p[G]$ has a canonic decomposition as a sum of bilateral submodules $\mathbb{Q}_p[G] = \sum_\psi 1_\psi \mathbb{Q}_p[G]$, where $\psi$ are the central irreducible idempotent of $\mathbb{Q}_p[G]$, associated to the irreducible characters $\psi$. If $\psi$ is any irreducible character and $f_\psi$ is the dimension of the associated representation, then there is canonic decomposition

$$
\begin{align*}
1_\psi \mathbb{Q}_p[G] &= \oplus_{i=1}^{f_\psi} A_{\psi,i}, \\
\mathbb{Q}_p[G] &= \bigoplus_{\psi} 1_\psi \oplus_{i=1}^{f_\psi} A_{\psi,i}
\end{align*}
$$

with $A_{\psi,i}$ being the modules of column $f_\psi \times f_\psi$ matrices having 0 in all but the $i$-th column. Furthermore, every submodule $A \subset \mathbb{Q}_p[G]$ is isomorphic to the direct sum of a collection of $A_{\psi,i}$’s. In particular, there are exactly $2^f$ isomorphy classes of modules in $\mathbb{Q}_p[G]$, where $f = \sum_\psi f_\psi$. Since $\mathbb{Q}_p$ is not algebraically close, the Parseval formula $|G| = \sum_\psi f_\psi^2$ may not hold over $\mathbb{Q}_p$.

Using this fact, we shall develop the notions of isomorphy classes of $\mathbb{Q}_p[G]$-modules, their supports, annihilators, generating idempotents and components. This is, among other, of importance for dealing with fields which are not CM and defining a canonic orthogonal pairing on $U(K)$, with respect to which the Leopoldt involution can be used in the cases of interest, in order to define 'plus' and 'minus' parts of cyclic $\mathbb{Z}_p[G]$ - modules, which are reminiscent of complex conjugation. The proof of the Gross - Kuz’mín Conjecture for non CM fields is the only place where we explicitly need this construction. The reader who wishes to proceed quickly to the proof of the main Theorem may jump this and related parts and follow the line of the proof for the simpler CM case, in which we may distinguish between plus and minus parts of modules by means of the complex conjugation.

The relation between modules and idempotents is explained in detail in the following

**Proposition 7.** Let $R \subset \mathbb{Q}_p[G]$ be some non trivial submodule. If $\alpha, \beta \in \mathbb{Q}_p[G]$ are two idempotents, they both generate the same module $R = \alpha \mathbb{Q}_p[G] = \beta \mathbb{Q}_p[G]$ iff there is a nilpotent $\nu \in \mathbb{Q}_p[G]$ with

$$
\nu^2 = 0, \quad \nu \alpha = 0, \quad \text{and} \quad \alpha \nu = \alpha.
$$

Furthermore, if $R' \subset \mathbb{Q}_p[G]$ is an other submodule, then $R' \cong R$ iff there is a unit $u \in \mathbb{Q}_p[G]^{\times}$ and an idempotent $\alpha \in \mathbb{Q}_p[G]$ such that

$$
R = \alpha \mathbb{Q}_p[G], \quad R' = \alpha' = \mathbb{Q}_p[G] \quad \text{with} \quad \alpha' = u \cdot \alpha \cdot u^{-1}.
$$

**Proof.** Let $\alpha, \beta$ be like in the hypothesis and $\nu = \alpha - \beta$. Since $\beta \in \alpha \mathbb{Q}_p[G]$, there is a $u \in \mathbb{Q}_p[G]$ with $\beta = \alpha u$ and since $\alpha \in \beta \mathbb{Q}_p[G]$, it follows that $u$ is
a unit. Using the idempotent property, we have

\[ \beta \alpha = \beta \beta u^{-1} = \beta u^{-1} = \alpha, \quad \alpha \beta = \alpha \alpha u = \alpha u = \beta, \quad \text{thus} \]
\[ \nu^2 = (\alpha - \beta)^2 = \alpha - \alpha \beta - \beta \alpha + \beta = 0, \]
\[ \nu \alpha = (\alpha - \beta) \alpha = \alpha - \beta \alpha = 0, \quad \alpha \nu = \alpha - \alpha \beta = \alpha - \beta = \nu. \]

Conversely, if \( R = (\alpha)Q_p[G] \) and \( \nu \) has the properties in (44), then \( \beta = \alpha - \nu = \alpha - \alpha \nu = \alpha(1 - \nu) \) and \( u = (1 - \nu) = (1 + \nu)^{-1} \in Q_p[G]^\times \), so \( \beta Q_p[G] = R \). We also have

\[ \beta^2 = \alpha(1 - \nu) \cdot \alpha(1 - \nu) = \alpha \cdot (1 - \nu) - \nu \alpha(1 - \nu) = \alpha(1 - \nu) - 0 = \beta, \]

which confirms the first claim.

Let \( \alpha \in Q_p[G] \) be an idempotent and \( R = \alpha Q_p[G] \); for any \( u \in Q_p[G]^\times \), \( \beta := u \alpha u^{-1} \) is an idempotent and we claim that \( R' = \beta Q_p[G] \cong R \). Indeed the map \( \psi_0 : Q_p[G] \to Q_p[G], x \mapsto ux \) is an automorphism of \( Q_p[G] \) and induces an isomorphism \( R \to R' : \alpha y \mapsto \beta \psi_0(y) \). Conversely, if \( R = \alpha Q_p[G] \cong R' = \beta Q_p[G] \) and \( \psi : R \to R' \), then \( \psi(\alpha) = \beta \cdot \nu \); furthermore, \( \beta \nu \in (R')^\times \), since \( \psi \) is surjective. Let \( u \in Q_p[G]^\times \) be such that \( \beta u = \beta v \) and define \( \alpha' = u \alpha u^{-1} \). Let \( \alpha' = u \alpha u^{-1} \); by definition of \( \psi \), it follows that \( \psi(1 - \alpha) = \beta u(1 - \alpha) = 0 = \beta(1 - \alpha') \), so \( \beta \in \alpha' \mathbb{Z}_p[G] \), with \( \alpha' = u \alpha u^{-1} \) and by comparing ranks, the two modules must be isomorphic. The proposition implies that \( \beta - \alpha' \) verifies (44) and there is a further unit \( u' = u(1 - \nu) \) (with \( \nu = 1 \) also possible), such that \( \beta = u' \alpha(u')^{-1} \), which completes the proof.

As a consequence,

**Corollary 3.** Let \( A = (\alpha)Q_p[G] \subset Q_p[G] \) be an irreducible module with \( \alpha \in I_0 \). Then \( A \cong A_{\psi,i} \) for some central irreducible idempotent \( \psi \) and \( 1 \leq i \leq f_\psi \), and there is a canonical representant \( \alpha_{(c)} \) of the image of \( \alpha \) in \( I/\cong \), such that \( \overline{\alpha} Q_p[G] = A_{\psi,i} \). In general, every isomorphism class \( \overline{\alpha} \in I/\cong \) has exactly one canonic representant \( \alpha_{(c)} \) with

\[
\alpha_{(c)}(\overline{\alpha})Q_p[G] = \bigoplus_{(\psi,i) \in I/I} A_{\psi,i}, \quad \text{for some} \quad \{(\psi,i) : \psi \text{ irredu. centr. idpt. and } 1 \leq i \leq f_\psi\}.
\]

**Proof.** See [3], Chapter 5, Theorem 6 for the isomorphism of \( A \) with an elementary module of column matrices. The existence and unicity of the canonic representant follows from the map between isomorphism classes of idempotents and classes of isomorphic modules. Finally, the statement for general modules follows from Maschke’s theorem and the decomposition in irreducible modules.

In view of the above proposition and its corollary, we define the following relations among idempotents:
Definition 7 (Classes of idempotents). Let \( \alpha, \beta \in \mathbb{Z}_p[G] \) be two idempotents. Then
\[
\alpha \equiv \beta \quad \text{iff} \quad \nu := \alpha - \beta \text{ verifies } (44),
\]
\[
\alpha \cong \beta \quad \text{iff there is a } u \in \mathbb{Z}_p[G]^\times \text{ with } \beta = u\alpha u^{-1}.
\]
We say that \( \alpha \) is congruent to \( \beta \) if \( \alpha \equiv \beta \) and they are isomorphic, if \( \alpha \cong \beta \).

Let \( \mathcal{I}_0 \subset \mathbb{Z}_p[G] \) be the set of all idempotents. Then \( \mathcal{I} := \mathcal{I}_0/ \mathcal{I} \equiv \) is in one - to - one correspondence with the submodules of \( \mathbb{Z}_p[G] \), while \( \mathcal{I}/ \mathcal{I} \equiv \) parameterizes the isomorphism classes of submodules of \( \mathbb{Z}_p[G] \). By abuse of language, we shall identify \( \alpha \in \mathcal{I}_0 \) with its class in \( \mathcal{I} \). The canonic representant of \( \alpha \in \mathcal{I} \) is \( \alpha_{(c)} \) and it verifies (47).

For \( \alpha' = u\alpha u^{-1} \) we have \( 1 - \alpha' = u(1 - \alpha)u^{-1} \); therefore complementarity can be defined also for isomorphy classes of idempotents. Let \( \mathcal{I} \subset \mathbb{Q}_p[G] \) be the set of all idempotents and \( \mathcal{I}/ \mathcal{I} \equiv \) the set of isomorphy classes of idempotents and denote the class of \( \alpha \in \mathcal{I} \) by \( \hat{\alpha} \). Then we may build complements on isomorphy classes by \( \hat{\alpha}^\top = (1 - \hat{\alpha}) \) and the notation \( 1 - \hat{\alpha} \) lays at hand for this complement; in general we may write \( 1 - \hat{\alpha} \) for \( \alpha \in \mathcal{I} \) an isomorphisms class. For intersections and direct sums, the definition needs additional clauses, in order to assure compatibility under 'conjugation' with units.

Let \( X \) be a \( \mathbb{Q}_p[G] \)-module; for each \( x \in X \) we write \( x^\top = \{ a \in \mathbb{Q}_p[G] : x^a = 1 \} \) for its annihilator module. Defining the annihilator of \( X \) in general is more delicate, but we may use the canonic idempotents; we restrict to the case when \( X \) is a cyclic \( \mathbb{Q}_p[G] \)-module of rank \( r \leq |G| \) as a \( \mathbb{Q}_p \)-module. We write \( X_0 \subset X \) for the set of all generators of \( X \) as a \( \mathbb{Q}_p[G] \)-module and \( X = \{ x^p : x \in X_0 \} \). By abuse of notation, we identify elements \( x \in X \) with their classes \( x^p \in X \). If \( X \) is a quasi - cyclic \( \mathbb{Z}_p[G] \)-module, then \( X = \{ x \in X \setminus X^p : x^p \in X \} \); therefore \( X \) contains uniquely determined elements of \( X \) in this case. We would expect from a proper definition of the annihilator module \( X^\top \subset \mathbb{Q}_p[G] \), that it is a space of dimension \( |G| - r \) over \( \mathbb{Q}_p \). This is in general not the case, if we use the naive definition \( X^\top = \{ a \in \mathbb{Q}_p[G] : x^a = 1, \text{ for all } x \in X \} \). If \( \gamma \in X \) is any generator, then \( \gamma^\top \) verifies naturally the rank condition, but it depends on the choice of \( \gamma \). If \( G \) is commutative, then the annihilator \( X^\top \) is well defined and the support is
\[
X^\perp = \oplus_{\chi \in 1 \chi, \chi \neq (1)} 1X.
\]
where \( 1 \chi \) are central idempotents belonging to irreducible characters \( \chi \); this yields an embedding \( X \hookrightarrow \mathbb{Z}_p[G] \).

In the non commutative case, for any generator \( \gamma' \) there is a unit \( u \in \mathbb{Z}_p[G] \) such that \( \gamma' = \gamma^u \); therefore the annihilator modules \( \gamma^\top = (\gamma')^\top \) as submodules of \( \mathbb{Q}_p[G] \). There is an isomorphy class \( I_X \in \mathcal{I}/ \mathcal{I} \equiv \) and a map \( \phi_X : I_X \rightarrow X \) such that \( \phi_X(\alpha)^\top = \alpha \mathbb{Q}_p[G] \); in view of (44), \( \phi_X(\alpha) = \phi_X(\beta) \) iff \( \alpha \cong \beta \). Furthermore, for each \( \alpha \in I_\alpha \), we have \( (\phi_X(\alpha)^\top)^\top = (1 - \phi_X(\alpha)^\top)^\top \)
\( \alpha \) \( \mathbb{Q}_p[G] \) and thus \( X \cong \alpha \mathbb{Z}_p[G] \) for all \( \alpha \in \mathcal{I}_X \). Moreover, there is a canonic annihilation idempotent \( \alpha_{(c)} \in \mathcal{I} \), which corresponds to a uniquely determined generator \( x \in X \) such that \( x^\perp = \alpha_{(c)} \mathbb{Q}_p[G] \). We shall say that \( X^\perp = \alpha_{(c)} \mathbb{Q}_p[G] \) is the canonic annihilator of \( X \), and write also \( \alpha_{(c)} = \alpha^\perp \). We note that it is defined via a well determined choice of a generator \( x \in X \). More generally, one may view annihilators also as isomorphism classes of idempotents together with a map to the set of generators of \( X \). The support is then \( X^\perp = (X^\perp)^\perp \) and we write \( \alpha^\perp \) for its canonic generator. One verifies that for each elementary module \( A = A_{\psi,i} \subset \mathbb{Q}_p[G] \) we have \( AX \neq 0 \iff AX^\perp \neq 0 \). By definition, we have \( X^\perp \oplus X^\perp \). Like the annihilators, supports can also be viewed as isomorphism classes of modules together with a map to the generators of \( X \).

Finally, for \( \alpha_{(c)} \) the canonic idempotent of some module \( A \subset \mathbb{Q}_p[G] \), we define the \( \alpha_{(c)} \) component of \( X \) by letting \( \beta \) be a canonic idempotent generating \( \alpha_{(c)} \mathbb{Q}_p[G] \cap X^{\perp} \) and

\[
X_\alpha = \beta \cdot X, \quad X = X_\alpha \oplus X_{1-\alpha}.
\]

The following computation explains the decomposition of \( X \) in complementary components: let \( x \in X \) be the generator such that \( x^{\perp} = \alpha^{\perp} \) and suppose that \( A \subset X^{\perp} \). Since \( A \) is a sum of elementary modules, it has a complement \( A' \) which is also sum of elementary modules and we let \( B = A' \cap X^{\perp} \) and \( \beta \) be a canonic idempotent generating \( B \). Suppose that \( X_\alpha \cap X_{1-\alpha} \neq 0 \) and let \( t \in \mathbb{Q}_p[G] \) be such that \( t = x^\alpha u = x^\beta v \in X_\alpha \cap X_{1-\alpha} \). Then \( \alpha u + \beta v \in X^\perp \) and by choice of \( \alpha, \beta \), it follows that \( u, v = 0 \) and thus \( t = 0 \).

The next definition synthesizes the above constructions:

**Definition 8 (Annihilators and Supports).** Let \( X \) be a cyclic (right) \( \mathbb{Q}_p[G] \) module. Then there are two canonic idempotents \( \alpha^\perp \) and \( \alpha^{\perp} \) and a uniquely determined \( \alpha \in X \) such that \( x^{\perp} = \alpha^\perp \mathbb{Q}_p[G] \) and \( x^{\perp} = \alpha^{\perp} \mathbb{Q}_p[G] \). Furthermore, if \( \mathcal{I}_X \subset \mathcal{I} \) is the isomorphy class of \( \alpha^\perp \), there is a map \( \phi_X : \mathcal{I}_X X \) and the two are related by

\[
\phi_X(\beta)^\perp = \beta \mathbb{Q}_p[G], \quad \forall \beta \in \mathcal{I}_X.
\]

If \( X \) is a cyclic \( \mathbb{Z}_p[G] \), then the pair \((\mathcal{I}_X, \phi_X)\) is defined with respect to \( X \) and the relation \((\mathcal{I}_X, \phi_X)\) becomes

\[
\phi_X(\alpha)^\perp = \alpha \mathbb{Q}_p[G] \cap \mathbb{Z}_p[G], \quad \forall \alpha \in \mathcal{I}_X.
\]

Finally, if \( A \subset X^{\perp} \) is an elementary module with generator \( \alpha \), then there is a well defined \( \alpha \) component \( X_\alpha \) with support \( X_\alpha^{\perp} = A \) and its complement in \( X \) is \( X_{1-\alpha} \). Components may be defined as isomorphy classes of submodules, using the map \( \phi_X \) above.

Since supports and annihilators come with canonic choices of generators of \( X \), one may induce maps between two cyclic \( \mathbb{Q}_p[G] \) modules having the same support and annihilators. Furthermore, a decomposition of \( \mathbb{Q}_p[G] \) in components induces decompositions of all cyclic \( \mathbb{Q}_p[G] \) modules.
the case of $\mathbb{Z}_p[G]$ - modules $X$, one defines the annihilators, supports and components with respect to $\hat{X}$ and then takes intersections with $\mathbb{Z}_p[G]$.

In the case of quasi - cyclic $\mathbb{Z}_p[G]$ - modules $X$, there are always $p$ - powers involved in the passage from classes $\hat{x}^p \in \hat{X}$ to $x \in X$; of course, these powers depend strongly upon $X$, its deviation from a cyclic $\mathbb{Z}_p[G]$ and $v_p(|G|)$. The next lemma describes the deviation of quasi - cyclic modules from cyclicity.

**Lemma 13.** Let $X$ be a quasi - cyclic module and $x'$ a generator. Then there is a further generator $x$ depending on $x'$ and such that $X^q \subset x^{z_p(G)}$. In particular, if $v_p(|G|) = 1$, then the all quasi - cyclic modules are cyclic.

**Proof.** Let $\alpha \in \mathcal{T}$ generate the annihilator $\hat{x}^T$ and $(1 - \alpha)q_p[G] = \oplus_{i} \mathcal{K}_p[G]$ be a decomposition of $\hat{x}^{\perp}$ in a direct sum of irreducible modules, generated by the idempotents $\beta_i$. Let $x'_i = (x')^{q_p} \in X$ and let $x_i \in X \setminus X^p$ such that $x'_i = x_i^{p^{e_i}}$ for some $e_i \geq 0$. We obviously have $\hat{X} = \oplus_{i} \hat{x}_i^{q_p[G]}$ and $X = \hat{X} \cap X$. Furthermore, in every component, $\hat{x}_i^{q_p[G]} \subset X$. Letting $x = \prod_i x_i$, we see that $\hat{x}^{q_p} \in X$ and $\hat{x}^{q_p} \cap \hat{x}^{q_p} \subset \hat{x}^{q_p \mathbb{Z}_p[G]}$, since $\beta_i \mathbb{Z}_p[G]$ are irreducible. If $q = 1$, it follows that $X = x^{z_p(G)}$, so $X$ is cyclic. □

As an important consequence, when $p$ is not coprime to $|G|$, the decomposition of $X$ in disjoint components works only up to a power of $p$.

### 7.2. Arithmetic in $\Lambda$. We gather in this section several elementary computations related to the arithmetic in $\Lambda$. Since the numbering of the intermediate fields starts with $\mathbb{K}_0 = \mathbb{K}_n$ (see §2.3 Definition 3 for the precise definition of $\kappa$), we have $\omega_n = (T + 1)^{p^{n-\kappa}} - 1$. Then

$$(49) \quad N_n := N_{\mathbb{K}_n/\mathbb{K}_0} = \omega_n/T \quad \text{and} \quad N_{m,n} = N_{\mathbb{K}_m/\mathbb{K}_n} = \omega_m/\omega_n,$$

for $m > n \geq \kappa$. The action of the Iwasawa involution on $\omega_n$ is seen from

$$(T + 1)^{p^{n-\kappa}} \omega_n + \omega_n = (p^n - 1)^{p^{n-\kappa}} - 1; \quad \text{it follows that:}$$

$$(50) \quad \omega_n + t \omega_n^p = p^{n-\kappa}c, \quad t \in \Lambda_n^\times, c \in \mathbb{Z}_p^\times.$$  

We shall use the following observation of B. Anglès [4], Lemma 2.1, (2): let $m = \kappa + l$ and $l' = [l/2]$. Then

$$\omega_m(T) = TN_m \in \left(p^{l'}, T^{p^{l'+1}}\right).$$

We may thus choose $a, b \in \Lambda_m$ with $a \in \Lambda_m^\times$ such that

$$(51) \quad N_m^a = ap^{l'} + bN_l^{l'+1}.$$  

For the $\mathbb{F}_p$ - modules $R_n = \Lambda_n/p\Lambda_n$ we have the isomorphism $R_n \cong \mathbb{F}_p[T]/(T)^{p^{n-\kappa}}$, so the image of $T$ is a nilpotent in these modules. Furthermore, $T \cong cT^n \mod p\Lambda, c \in \Lambda^\times$. These facts will be used below for the study of modules of infinite $p$ - rang. A further set of identities which we use repeatedly is the following: let $\vartheta$ generate $\text{Gal}(\mathbb{K}_{n+1}/\mathbb{K}_n)$ and $\theta = \vartheta - 1$. Then
the following useful expansions:

$$N_{n+1,n} = p + \theta \cdot \left( \sum_{k=2}^{p} \left( \frac{p}{k} \right) \theta^{k-1} \right)$$

(52) $$= p + \theta h(\theta) = p + \theta \cdot \left( \frac{p}{2} + h_1(\theta) \right)$$

$$= p \cdot h_3(\theta) + \theta^{p-1}, \quad h_1(\theta), h_2(\theta) \in \mathbb{F}_p[\theta], h_3(\theta) \in (\mathbb{F}_p[\theta])^\times.$$

The following Lemma is a general result on duality:

**Lemma 14.** Let X be a Λ-mod module, $X_n = X^{T_n}$ be the submodule fixed by $\omega_n$ and $X_{n,T^*} = \{ x \in X_n : x^{T^*} \equiv 1 \mod X_n^{p^{n-\kappa}} \}$; we assume that $N_{m,n} : X_m \to X_n$ is surjective. If $\mu_n \subset X^\circ$ is the maximal submodule of exponent $p^{n-\kappa}$ which is annihilated by $T^*$ modulo $X_n^{p^{n-\kappa}}$, then

$$X_{n,T^*}/X_n^{p^{n-\kappa}} = (X_{n}^\times : \mu_n \cdot X_n^{p^{n-\kappa}})/(X_n^{p^{n-\kappa}}).$$

Moreover, if $X$ is quasi-cyclic of $\Lambda^\circ$ rank r as a Λ-mod module, then for $n > \kappa$, $X_{n,T^*}/\mu_n$ are quasi-cyclic finite $\mathbb{Z}_p[G]$-modules of $p$-rank r and subexponent $p^{(n-\kappa)/2}$. They form an injective system with limit $X_{T^*} = \lim N_{m,n} X_{n,T^*}/\mu_n$ which is a quasi-cyclic $\mathbb{Z}_p[G]$-module of rank r. Furthermore, if $\iota_{n,m} : X_n \to X_m$ for $m > n$, then $\iota_{n,m}(X_{n,T^*}) \subset X_{m,T^*}^\times$.

**Proof.** Apply (54): if $x \in X_n^\times$, then $x^{T^*} = y^{\iota^\times} = y^{c^\iota^{p^{n-\kappa}}}$. Since $\mu_n \subset X_n^{T^*}$ by definition, this shows the inclusion ‘⊃’. For the other direction, consider $x \in X_n$ with $x^{T^*} = y^{p^{n-\kappa}}$; then $x^{T^*} N_n^\times = x^{c^\iota^{p^{n-\kappa}}} = y^{N_n^\times y^{p^{n-\kappa}}}$ and thus $\left( x/\left( y^{c^\iota^{p^{n-\kappa}}} \right) \right)^{p^{n-\kappa}} = 1$, so $x = y^{c^\iota^{p^{n-\kappa}}} \in \mu_n$, with $\xi^{p^{n-\kappa}} = 1$; from $x^{T^*} \in X_n^{p^{n-\kappa}}$ it follows that $\xi \in \mu_n$. This implies also ‘⊂’ and completes the proof of (53).

We now prove the properties of $X_{T^*}$. Let $G_n = \text{Gal}(\mathbb{K}_n/\mathbb{Q})$, $X_n = X_{n,T^*}/\mu_n$ and $g = |G|$. An element $\alpha \in \mathbb{Z}[G_n]$ acting on $X_n$ has the following dual development in the group ring:

$$\alpha = \sum_{i=0}^{g-1} A_i(T^*) \cdot \tau_i, \quad \tau_i \in \text{Gal}(\mathbb{K}/\mathbb{Q}),$$

where the $A_i \in \mathbb{Z}[x]$ are polynomials of degree $\deg(A_i) < p^p$. Let $\alpha_0 = \sum_{i=0}^{g-1} A_i(0) \tau_i$. We show that $p\text{-rk}(X_{n,T^*}/\mu_n) = r$.

We shall construct a subset $D' \subset X_m$ such that $(X_m^\times D')/(X_m^\times)$ is an $\mathbb{F}_p$-space of maximal rank r. Let $\delta_n \in X_n$ be a generator for the Λ-cyclic modules $X_n$ and $\delta_0 = N_n(\delta_n)$, a power of a generator for $X_0$; consider $H \subset G_n \cup \{1\}$, a maximal subset such that $\delta_0^H \subset X_0$ is a free $\mathbb{Z}_p$-module of rank $r-1$. Let $D_n = \{ \delta_n^\sigma : \sigma \in H \cup \{1\} \}$ be a system of relative generators for $X_n/X_0$ and $D_n \subset X_n$ be their $\mathbb{Z}_p$-span; the identity automorphism accounts
for the pre-image $N^{-1}_{X_n/K}(1) \subset X_n$ of 1 in $X_n$. The system $D_n$ has $p$-rank $r$; consider $D_n^{N_n} \subset X_n^{N_n}$. From \[\text{(51)}\] we deduce that $p\text{-rk}(D_n^{N_n}/(D_n^{N_n})^{p^{n-k}}) = r$; a fortiori, $p\text{-rk}(x_n) \leq r$: it may happen that $D_n^{N_n} \subset X_n^{p^{n-k}}$. We show that this is not the case for sufficiently large $n$, and the two ranks are then equal.

Let thus $T_n = X_n^{N_n}/D_n^{N_n}$ be the torsion and $t_1, t_2, \ldots, t_y \in T_n$ be a minimal system of generators with $y \leq r$ and decreasing orders in the torsion group $T_n$, so $\text{ord}(t_1) \geq \text{ord}(t_2) \geq \ldots \geq t_y$. We shall identify the $t_i$ with a set of representatives in $x_n$ and let $d_i' = t_i^{\text{ord}(t_i)} \in D_n^{N_n}$, $i = 1, 2, \ldots, y$. Then $d_i'$ are $\mathbb{Z}_p[G]$- independent; we may choose $d_j' \in D_n^{N_n}$, $y < j \leq r$ such that $X_n^{N_n} = \text{Span}(t_i, d_j') \mathbb{R}$, $1 \leq y < j \leq r$.

The set $F = \{t_i : 1 \leq y \leq y'\} \cup \{d_j' : y' < j \leq x_2\}$ is then a set of $\mathbb{Z}_p[G]$- generators for $X_n^{N_n}$ and this shows that $X_n^{N_n}$ has the $p$-rank $r$. By construction, $(\text{Span}(F) \cdot X_n^p)/X_n^p$ has also the rank $r$ as an $\mathbb{F}_p$- vector space and thus,

$$p\text{-rk}\left((X_n^{N_n} \cdot X_n^{p^{n-k}})/X_n^{p^{n-k}}\right) = p\text{-rk}(X_n^{N_n}) = r.$$  

We finally show that the exponents of $x_n := X_n^{N_n}/X_n^{p^{n-k}N_n}$ are diverging. For this we use \[\text{(51)}\]. Let $x \in X_n \setminus \text{Span}(F)^p$, so $x = z^{N_n}$, $z \in X_n \setminus X_n^p$, and set $l' = [\frac{n-k}{2}]$. The formula \[\text{(51)}\], in which we choose $a$ to be a unit, implies that $x \notin X_n^{p^{l'+1}}$ and therefore $x$ generates a cyclic group of order at least $p^{l'}$. Since $(\text{Span}(F) \cdot X_n^p)/X_n^p$ has rank $r$, it follows that there is a subgroup $W_n \subset X_n$ with $W_n \cong (C_{p^{l'}})^r$, thus $W_n$ has $p$-rank $r$ and subexponent $l' = [\frac{n-k}{2}]$. Finally we have to show that $x_n$ form a projective system. Let $x_{n+1} \in X_{n+1}$ and $x_n = N_{n+1,n}(x_{n+1})$. In view of \[\text{(50)}\], we have $N_{n+1,n} \equiv c_1N_{n+1,n} \mod p^{n-k}$ with $c_1 \in \Lambda^x$, and thus

$$N_{n+1}^{N_n} = N_{n+1,n} \cdot N_n \equiv c_1N_{n+1,n} \cdot N_n \mod p^{n-k}.$$  

If follows that $x_{n+1}^{N_{n+1,n}} \equiv x_n^{c_1N_n} \mod X_n^{p^{n-k}}$ and thus $X_{n,T^*}$ form an injective system. One verifies with the same methods that $\iota_{n+1,n}(X_{n,T^*}) = N_{n+1,n}(X_{n+1,T^*})^p$, where $N_{n+1,n}$ is regarded as an endomorphism of $X_{n+1}$; this induces also a projective structure which is best understood in terms of “radicals” $X_{n,T^*}^{1/p^{n-k}}$.

\[\square\]

7.3. Units.

Proof of Fact\[\square\] Let $e, f, g$ denote as usual, the ramification index, the degree of the residual fields and the splitting index of the primes above $p$. The polynomial $\iota(f(X))$ is separable over $\mathbb{Q}_p$ and splits in $g$ polynomials of degree $e$. Thus $\mathbb{R}_p = \mathbb{Q}_p[X]/(\iota(f))$ is the product of $g$ isomorphic local, unramified extensions of degree $e$. Each completion $\mathbb{K}_p \cong \mathbb{K}$ is a ramified extension of degree $e$ of the unramified extension $\mathbb{K}/\mathbb{Q}_p$ of degree $f$.

It follows from the Chinese Remainder Theorem that $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ extends to an embedding $\iota : \mathbb{K} \hookrightarrow \mathbb{Q}_p[X]/(\iota(f))$ and that the image of $\mathbb{K}$ is dense in
is a matrix $Q$. This can be verified either by tensoring with $B$ and any pair $r$ minimal set of generators of the map $A$, a unimodular change of base in the vector space $\mathbb{Q}$ such that $t = \text{rank}(A)$. Indeed, we have the equality of $B/\mathbb{Q}$ which approximate $h$ to the power $p^n$, so $\lim_{n \to \infty} t = \text{rank}(A)$. We see $t_n = h_n(\alpha) \in \mathbb{K}$, we also have $\iota(t_n) = \iota(h_n(\alpha)) \to \iota(\alpha)$; hence $\mathbb{K}$ is dense in $A$. For any $\sigma \in H$, we define $\sigma(\alpha) = \iota(\iota(\alpha))$. This action is well defined and commutes with the embedding, since for $t \in \mathbb{K}$ we have $\iota(\sigma(t)) = \iota(h(\alpha)) = \iota(\iota(\alpha)) = \sigma(\iota(\alpha))$.

7.4. $\Lambda$ - modules, radicals and duality. The following elementary lemma will be used for the proof of Proposition [11]

**Lemma 15.** Let $A$ and $B$ be finitely generated abelian $p$-groups denoted additively, an let $N : B \to A$, $\iota : A \to B$ two $\mathbb{Z}_p$ - linear maps such that:

1. $N$ is surjective;
2. The $p$-ranks of $A$ and $B$ are both equal to $r$ and $|B|/|A| = p^r$.
3. $N(\iota(a)) = pa$, $\forall a \in A$ and $\iota$ is rank preserving;

Then $\iota$ is injective, $\iota(A) = pB$ and $\text{ord}(x) = p \cdot \text{ord}(\chi x)$ for all $x \in B$.

**Proof.** We start by noting that for any finite abelian $p$-group $A$ of $p$-rank $r$ and any pair $\alpha_i, \beta_i$; $i = 1, 2, \ldots, r$ of minimal systems of generators there is a matrix $E \in \text{Mat}(r, \mathbb{Z}_p)$ which is invertible over $\mathbb{Z}_p$, such that

$$ (55) \quad \beta = E\alpha. $$

This can be verified either by tensoring with $\mathbb{Q}_p$, or directly by extending the map $\alpha_i \mapsto \beta_i$ linearly to $A$ and, since $(\beta_i)^r_{i=1}$ is also a minimal system of generators, deducing that the map is invertible, thus regular. It represents a unimodular change of base in the vector space $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The maps $\iota$ and $N$ induce maps

$$ \tau : A/pA \to B/pB, \quad \overline{N} : B/pB \to A/pA. $$

From 1, we see $\overline{N}$ is surjective and since, by 2., it is a map between finite sets of the same cardinality, it is actually an isomorphism. But 3. implies that $\overline{N} \circ \tau : A/pA \to A/pA$ is the trivial map and since $\overline{N}$ is an isomorphism, $\tau$ must be the trivial map, hence $\iota(A) \subseteq pB$.

Assume now that $\iota$ is rank preserving and let $b_i$, $i = 1, 2, \ldots, r$ be a minimal set of generators of $B$: thus the images $b_i$ of $b_i$ in $B/pB$ form an $\mathbb{F}_p$ - base of this algebra. Let $a_i = N(b_i)$; since $p \cdot \text{rk}(B/pB) = p \cdot \text{rk}(A/pA)$, the set $(a_i)$ also forms a minimal set of generators for $A$. We claim that $|B/\iota(A)| = p^r$.

Pending the proof of this equality, we show that $\iota(A) = pB$ and $\iota$ is injective. Indeed, we have the equality of $p$-ranks:

$$ |B/pB| = |A/pA| = |B/\iota(A)| = p^r, $$
implying that \(|pB| = |\iota(A)|\); since \(\iota(A) \subset pB\) and the \(p\)-ranks are equal, the two groups are equal, which is the first claim. The second claim will be proved after showing that \(|B/\iota(A)| = p^r\). Since \(|B/|A| = p^r\), it follows that \(|A| = |\iota(A)|\), so \(\iota\) is injective.

Let \(S(X)\) denote the socle of the finite abelian \(p\)-group \(X\). There is the obvious inclusion \(S(\iota(A)) \subset S(B) \subset B\) and since \(\iota\) is rank preserving, \(p\)-rk \((A) = p\)-rk \((S(A)) = p\)-rk \((B) = p\)-rk \((S(B)) = p\)-rk \((S(\iota(A)))\), thus \(S(B) = S(\iota(A))\). Let \((a_i)_{i=1}^r\) be a minimal set of generators for \(A\) and \(a'_i = \iota(a_i) \in B, i = 1, 2, \ldots, r\); the \((a'_i)_{i=1}^r\) form a minimal set of generators for \(\iota(A) \subset B\). We choose in \(B\) two systems of generators in relation to \(a'_i\) and the matrix \(E\) will map these systems according to (55).

First, let \(b_i \in B\) be such that \(p^{e_i}b_i = a'_i\) and \(e_i > 0\) is maximal among all possible choices of \(b_i\). From the equality of socles and \(p\)-ranks, one verifies that the set \((b_i)_{i=1}^r\) spans \(B\) as a \(\mathbb{Z}_p\)-module; moreover, \(\iota(A) \subset B^p\) implies \(e_i \geq 1\). On the other hand, the norm being surjective, there is a minimal set of generators \(b'_i \in B, i = 1, 2, \ldots, r\) such that \(N(b'_i) = a_i\). Since \(b_i, b'_i\) span the same finite \(\mathbb{Z}_p\)-module \(B\), (55) in which \(\vec{a} = \vec{b}\) and \(\vec{\beta} = \vec{\beta}'\) defines a matrix with \(\vec{b} = E \cdot \vec{\beta}\). On the other hand,

\[
\iota(\vec{a}) = \vec{a}' = \text{Diag}(p^{e_i})\vec{b} = \text{Diag}(p^{e_i})E \cdot \vec{\beta}.
\]

The linear map \(N : B \to A\) acts component-wise on vectors \(\vec{x} \in B^r\). Therefore,

\[
N\vec{b} = N\vec{b}_i = N(E\vec{\beta}) = N \left( \prod_j b'_j^{\sum_{i,j} e_{i,j}} \right)_{i=1}^r = \left( \prod_j (N_{b'_j})^{e_{i,j}} \right)^r_{i=1} = \left( \prod_j (a_{i,j})^{e_{i,j}} \right)^r_{i=1} = E(\vec{a}).
\]

We obtain thus two expressions for \(N\vec{a}'\) as follows

\[
N\vec{a}' = p\vec{a} = pI \cdot \vec{a} = N \left( \text{Diag}(p^{e_i})\vec{b} \right) = \text{Diag}(p^{e_i}) \cdot N\vec{b} = \text{Diag}(p^{e_i}) \cdot E\vec{a},
\]

and thus

\[
\vec{a} = \text{Diag}(p^{e_i - 1}) \cdot E\vec{a}.
\]

The \(a_j\) form a minimal system of generators and \(E\) is regular over \(\mathbb{Z}_p\); therefore \((\alpha) := (a_{j})_{j=1}^r = E\vec{a}\) is also minimal system of generators of \(A\) and the last identity above becomes

\[
\vec{a} = \text{Diag}(p^{e_i - 1}) \cdot \vec{\alpha}.
\]

If \(e_i > 1\) for some \(i\), then the right hand side is not a generating system of \(A\) while the left side is; it follows that \(e_i = 1\) for all \(i\). Therefore \(|B/\iota(A)| = p^r\) and we have shown above that this implies the injectivity of \(\iota\).
Finally, let \( x \in B \) and \( q = \text{ord}(Nx) \geq p \). Then \( qN(x) = 1 = N(qx) \), and since \( qx \in \iota(A) \), it follows that \( N(qx) = pqx = 1 \) and thus \( pq \) annihilates \( x \). Conversely, if \( \text{ord}(x) = pq \), then \( pqx = 1 = N(qx) = qN(x) \), and \( \text{ord}(Nx) = q \). Thus \( \text{ord}(x) = p \cdot \text{ord}(Nx) \) for all \( x \in B \) with \( \text{ord}(x) > p \). If \( \text{ord}(x) = p \), then \( x \in S(B) = S(\iota(A) \subset \iota(A)) \) and \( Nx = px = 1 \), so the last claim holds in general. \( \square \)

This small exercise in linear algebra avoids a deeper investigation of \( M \) as a sum of irreducible \( \Lambda \) - modules, and the afferent pseudo-isomorphisms which may arise. We can non give the

**Proof of Proposition 5.** We assume without restriction of generality that \( 0 \) is the least integer \( n \) for which \( |M_n| = |M_{n+1}| \), for point 1., respectively \( p \cdot \text{rk}(M_n) = p \cdot \text{rk}(M_{n+1}) \), for point 2. Let

\[
Y = \{ x = (x_n)_{n \in \mathbb{N}} \in M : f_{n,0}(x_n) = 1 \text{ for all } n \geq 0 \} \subset M.
\]

One verifies from the definition that \( Y \) is a \( \Lambda \) - submodule. For \( n > 0 \) the map \( f_{n,0} \) is surjective with \( \text{Ker} (f_{n,0}) = \{ x \in M_n : f_{n,0}(x) = 1 \} = \nu_{0,n}Y \) and we thus have a commutative diagram in which \( M_n \to M_0 \) is induced by the map \( f_{n,0} \) while the horizontal isomorphism are deduced from the previous remark:

\[
\begin{array}{ccc}
M_n & \cong & M/\nu_{0,n}Y \\
\downarrow & & \downarrow \\
M_0 & \cong & M/Y.
\end{array}
\]

(56)

For the first point we assume \( |M_1| = |M_0| \). Then \( M_1 \to M_0 \) is an isomorphism; therefore \( \nu_{0,1}Y = Y \). Since \( \nu_{0,1} \in (p,T) \subset \Lambda \), the unique maximal ideal, and since \( Y \) is finitely generated over \( \Lambda \), it follows from Nakayama’s lemma that \( Y = 0 \). Consequently \( M \cong M_0 \) is finite and \( M_n \cong B_0 \cong M \) for all \( n \geq 0 \). This proves the assertion 1.

Suppose now that \( p \cdot \text{rk}M_1 = p \cdot \text{rk}M_0 \). Then \( M_1/pM_1 \cong M_0/pM_0 \) and thus \( M/(\nu_{0,1}Y + pM) \cong M/(Y + pM) \) and \( \nu_{0,1}Y + pM = Y + pM \). Letting \( Z = (Y + pM)/pM \), we have

\[
\nu_{0,1}Z = (\nu_{0,1}Y + pM)/pM = (Y + pM)/pM = Z.
\]

By Nakayama’s lemma we conclude that \( Z = 0 \) and \( Y \subset pM \). Therefore

\[
p \cdot \text{rk}(M_0) = p \cdot \text{rk}(M/\nu_{0,n}Y) = p \cdot \text{rk}(M/(\nu_{0,n}Y + pM)) = p \cdot \text{rk}(M/pM) = Z_{p \cdot \text{rk}}(M)
\]

for all \( n \geq 0 \). By Iwasawa’s formula, for \( n \) sufficiently large we have

\[
|M_n| = p^{\mu p^n + \lambda n + \nu},
\]

and since the rank stabilizes, we see that \( \mu(M) = 0 \) and \( |M_{n+1}| - |M_n| = p^\lambda \). This proves assertion 2.

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8 We follow here Fukuda [11] and use Iwasawa’s notation \( \nu_{n,m} = N_{m,n} = \omega_m/\omega_n \) for \( m > n \geq 0 \).
For point 3. we need Lemma 15. We have seen in the proof of the Lemma, that \( \iota(M_n) \subset pM_{n+1} \) in general, while in the case when \( \iota \) is rank preserving we have equality. The kernel \( \text{Ker} (\iota) \) is the capitulation kernel and has thus exponent \( p \). Therefore, \( \iota \) is rank preserving iff the subexponent of \( M_n \) (see Definition 1) is larger than \( p \). Suppose thus that \( \iota \) is rank preserving and \( \iota(M_n) = pM_{n+1} \) and let \( N' = \iota \circ N \).

For proving \( x^p = \iota(N(x)) = N'(x) \), we make explicite use of the structure of \( \Gamma \). Let \( t = \omega_n = (T + 1)^{p^n} - 1 \) and

\[
N' = N_{\mathbb{K}_{n+1}/\mathbb{K}_n} = p + t \cdot v = p + t \left( \frac{p}{2} + tw \right), \quad v, w \in \mathbb{Z}[t],
\]

as follows from the Newton development of \( N = \frac{(t+1)^{p^n} - 1}{t} \) (see also [52]). By definition, \( t \) annihilates \( \iota(M_n) \) and \( x_0 := x^p \in \iota(M_n) \) from our assumption. Thus \( (x^t)^p = x_0^p = 1 \) and \( x^t \in S(M_{n+1}) = S(\iota(M_n)) \subset \iota(M_n) \), as shown in the proof of the Lemma 15. In view of the above development then

\[
N't = x^p \cdot (x^t)^{(p^2) + tw} = x^p,
\]

which completes the proof.

Suppose now that \( \iota \) is not rank preserving, so the subexponent is \( p \) for all \( n \). Then \( M = M' \oplus M'' \), with \( M'' \) a module of exponent \( p \) and \( M' \) of subexponent larger than \( p \). The result holds for \( M' \) by the above, while for \( M'' \) we have \( x^p = 1 \) for all \( x \in M_{n+1} \). Consequently, \( M_{n+1} \) is an \( \mathbb{F}_p[T] \) -module of finite rank annihilated by the image of \( \omega_{n+1} \) in \( \mathbb{F}_p[T] \), which is \( \omega_{n+1} = (T + 1)^{p^{n+1}} - 1 \mod p\mathbb{Z}[T] = T^{p^n+1} \). The rank being finite, we see that \( x^t = 1 \) too, for sufficiently large \( n \). Consequently \( N'(x) = x^p \) for all \( M \) of finite rank and \( n \) sufficiently large.

The first point is concerned with pure capitulation. Indeed, we deduce from \( N'(x) = x^p \) and \( |M_n| = |M_{n+1}| \) that \( p \cdot \text{rk}(\text{Ker}(\iota)) = p \cdot \text{rk}(M_n) = R \), so the whole socle of \( M_n \) must capitulate. In the second case, assuming in addition that \( R = \lambda(M) \) we have \( v_p(|M_n|) = \mu p^n + \lambda n + \nu \) and since the ranks are bounded, \( \mu = 0 \). From \( v_p(|M_{n+1}|) - v_p(|M_n|) = \lambda \) and \( |M_{n+1}| = p|\iota(M_n)| \), so \( |M_{n+1}| - |\iota(M_n)| = p^R \). In the case when \( R = \lambda \), it follows that \( |\iota(M_n)| = |M_n| \) so the two groups are isomorphic, \( \iota \) is injective and \( M \) is a Weierstrass module. This completes the proof. \( \square \)

We see that the growth of infinite \( \Lambda \) - modules of finite rank is very simple, at levels beyond the rank stabilization. The whole taxonomy of these modules is established by the growth prior to this stabilization and this may be very complex. Even the question whether capitulation in Weierstrass modules can be excluded also before rank stabilization is not easy to answer. We therefore chose to avoid this question by adequate choice of the base field. It is however remarkable that the weaker condition that \( \iota \) be rank preserving already determines regular conditions of growth: the relation \( \text{ord}(x)/\text{ord}(\iota(N(x))) = p \) depends only on this condition and we may only have \( \text{ord}(x) > p \cdot \text{ord}(\iota(N(x))) \) if \( \iota(N(x)) = 1 \), for \( x \in M_{n+1} \setminus M_{n+1}^p \) with
\(\iota(N(x)) = 1\). Some additional work using the objects in the proof of Lemma 15 shows that in this case \(\text{ord}(x) \leq p^2\).

**Remark 2.** Like any phenomenon that is not impossible, this does in fact happen, as shown by the following example due to Kraft and Schoof [17]. In the cyclotomic \(\mathbb{Z}_2\) - extension of \(\mathbb{K}_0 = \mathbb{Q}(\sqrt[n]{5329})\), for the transition \(A_1 \to A_2\) there is an \(a \in A_2\) of order \(p^2\) with \(N'(a) = 1\). The groups \(A_1, A_2\) have \(p\)-ranks 2 and are of types \((9,3)\) and \((9,9)\). There is total capitulation, and thus \(\iota^*\) is not rank preserving; indeed, \(p\)-rk \((\iota^*_1(A_1)) = 1 < p\)-rk \((A_1)\).

This phenomenon is not in contradiction with (12), since there the bound \(n\) to be sufficiently large – in particular, beyond total capitulation of the finite part of \(M_n\). There, the regularity is recovered. We now consider the results on Weierstrass modules:

**Proof of Lemma** [2]. Let \(Y \subset X\) be a \(\Lambda\) - submodule. Since \(X\) is free as a \(\mathbb{Z}_p\) - module and \(\mathbb{Z}_p\) has no finite subgroups, \(Y\) is also Weierstrass. Therefore finite intersections of Weierstrass modules are trivial. This implies the direct sum decomposition \(X = \oplus_j X_j\) in the claim. \(\square\)

**Proof of Corollary** [1]. Since \(M\) is a Weierstrass module, we can apply point 3. in Proposition 1, which implies the claim 13. Let \(n > n_0\) be the least integer such that \(x_n \neq 1\) and define \(z(x) = v_p(\text{ord}(x_n)) - (n + 1)\). Then 13 and the injectivity of \(\iota\) implies \(v_p(\text{ord}(x_m)) - (m + 1) = z(x)\) for all \(m > n\). It makes therefore sense to use the notation \(z(a_n)\) for \(a_n \neq 1\). The ultrametric inequality \(z(xy) \leq \max(z(x), z(y))\) follows from \(\text{ord}(x_ny_n) \leq \max(\text{ord}(x_n), \text{ord}(y_n))\), and from this, the surjectivity of the norm implies the bound \(c\). Indeed, if \(\{a_i = (a_{i,n})_{n \in \mathbb{N}} \in M : i = 1, 2, \ldots r\}\) is a minimal set of generators of \(M\), then by Nakayama’s lemma we have \(a_{i,0} \neq 1\) and thus \(z(a_i) \leq c\) by the definition of \(c\). Since the \(a_i\) generate \(M\), the ultrametric inequality implies \(z(x) \leq c\) for all \(x \in M\). Note that \(c\) is constant also for all intermediate fields \(\mathbb{K}_n\): replacing \(\mathbb{K}\) by a larger intermediate extension does not increase the value of the constant.

Finally we show that \(z\) extends to a map defined on \(A\) by (14). If \(x \in A\) has infinite order, then the module \(\Lambda x^{p^d}\) is \(\mathbb{Z}_p\) - torsion free, for sufficiently large \(d\), so it is Weierstrass. Thus \(z(x^{p^d})\) is well defined and the limit \(\lim_{n \to \infty} v_p(\text{ord}(x_n)) - (n + 1)\) exists, which completes the proof. As a map on \(A\), one may regard \(p^{-z}\) as a degenerated ultrametric on \(A\), with kernel \(A^0\). For any \(a_n \in A_n\) with \(a_n \neq 1\) and any lifts \(a \in A\) which project on \(a_n\) at the \(n\)-th level, the value \(z(a_n) = v_p(\text{ord}(a_n)) - (n + 1) = z(a)\) is constant. \(\square\)

Finally we consider the \(\mu\) - part, giving the

**Proof of Lemma** [3]. Let \(a = (a_n)_{n \in \mathbb{N}} \in A^0\) be such that \(\Lambda b\) has unbounded \(p\)-rank and let \(\lambda, \mu, \nu\) be the Iwasawa constants of this module; then \(\lambda = 0\) and, for \(n > n_0\),

\[
(57) \quad v_p(|\Lambda a_n|) = \mu p^n + \nu,
\]
Let $b = a^{p^\nu-1}$. Since $\Lambda a \sim \Lambda/(p^\mu)$, it follows that the module $B := \Lambda b \sim \Lambda/(p)$ has also unbounded $p$-rank. We claim that $\ker (B_n \to B_{n+1}) = \{1\}$ for all $n > n_0$. Since $\ord(b) = p$, it follows that $B$ is an $\mathbb{F}_p[[T]]$-module and from

$$\omega_n \equiv T^{p^n} \mod p\mathbb{F}_p[[T]],$$

it follows that $p\rk(B_n) \leq p^n$. For $n > n_0$ we have $|B_n| = p^{p\rk(B_n)}$, so $\nu \leq 0$. Comparing with (57), in which $\mu = 1$, we see that $a_n T^j, j \leq p^n + \nu$ form a base for the $\mathbb{F}_p$-vector space $B_n$, for all $n > n_0$. We assume that there is capitalization for $n > n_0$, so $\iota_{n,n+1}(b_{T_k}) = 1$ for some $k < p^n + \nu$; let

$$N = N_{n+1,n} \equiv T^{p^{n+1}(p-1)} \mod p\mathbb{F}_p[T],$$

where the congruence follows from (58). Since the norm is surjective in $A$, we have $a_n = N(a_{n+1})$; let $\mathfrak{A}_{n+1} \in a_{n+1}$ be a prime that splits completely above $\mathbb{Q}$ and $\mathfrak{A}_n = N(\mathfrak{A}_{n+1}) = \mathfrak{A}_n^{p^{n+1}(p-1)} \in a_n$. Consequently, $\mathfrak{A}_{n+1}$ is a principal ideal of $K_{n+1}$ and thus

$$p^{n+1} + \nu = p\rk(B_{n+1}) \leq (p-1)p^n + k,$$

and $k \geq p^n + \nu$. But $p\rk(B_n) = p^n + \nu$, so in this case $a_{n+1}^{T_k} = 1$ anyhow: there is no capitalization for $n > n_0$.

We now proceed to radicals and prove first an auxiliary lemma:

**Lemma 16.** Let $n > 3\kappa$ and $y \in K_\kappa^\times$ be such that $y^T \in E_n'$, and if $c \in K$ divides $y$, then $c \in E'(K)$. Moreover, $y = d \cdot e^T$ with $d \in E_{2\kappa}, e \in E_n'$.

**Proof.** Since $y^T$ is a $p$-unit, it follows that $(y)$ is an ambig ideal fixed by $\tau$. Let $\mathfrak{A} \subset K_n$ with $[\mathfrak{A}] = a_n \in A_n$ and $a_n$ lifting to a sequence $a_\alpha \in A$ be any ambig ideal fixed by $\Gamma$. If $a$ has finite order, $\mathfrak{A}$ capitulated in $K_{2\kappa}$, since $p^\kappa$ is an exponent for $A^\kappa$, and thus $\mathfrak{A} = (\alpha), \alpha \in K_{2\kappa}$. In particular, if $(\mathfrak{A},p) = (1)$, then this must be the case. Let $(y) = \mathfrak{B} \cdot \mathfrak{P}$ be a splitting such that $\mathfrak{B}(p) = (1)$ and $\mathfrak{P}$ is a product of primes above $p$. By the above, $\mathfrak{B} = (\beta), \beta \in K_{2\kappa}$ and since $\mathfrak{B}$ is principal, so must be $\mathfrak{P}$. Thus $\mathfrak{P}$ is a $p$-unit; let $\varphi_m \subset K_m$ be the primes above $K \supset \varphi \supset (p)$ and $a_m = [\varphi_m]$. Recall that $C = G/D_\varphi$ is a set of $s$ coset representatives of the quotient of $G$ by the decomposition group of some prime $\varphi$ like above. Then $\mathfrak{P} = \varphi_m^\theta$ with $\theta = \sum_{\sigma \in C} a_{\sigma} \sigma \in \mathbb{Z}[C]$; if $a = (a_m)$ has infinite order, then $\ord(a_n) > p^{n-\kappa}$ and $\varphi_{m,\ord(a_n)} = \varphi_{\ord(a_n)} = (\pi_0)$. We may split in $\theta = \theta_s + \theta_l$ with $\theta_s = \sum_{\sigma \in C: v_p(a_{\sigma}) < v_p(\ord(a_n))} a_{\sigma} \sigma$ and herewith, $\mathfrak{P} = \mathfrak{P}_s \cdot \mathfrak{P}_l$, where $\mathfrak{P}_s := \varphi_m^{\theta_s}$ and $\mathfrak{P}_l = \varphi_m^{\theta_l}$. Now by definition of $\mathfrak{P}_l$, it is a principal ideal $(\pi_l) \subset K_l$, and it remains that so must be $\mathfrak{P}_s$. The classes $[\varphi_m^{\theta_s}]$ have bounded order for $n \to \infty$ and thus $\mathfrak{P}_s = (\pi_s)$ with $\pi_s \in K_{2\kappa}$. Altogether, $(y) = (\beta)(\pi_s)(\pi_l)$ and $y = e \cdot \beta \cdot \pi_s \cdot \pi_l$, with $e \in E_n$. But $(\beta \cdot \pi_s)^T \in E_{2\kappa}$ and $\pi_l^T = 1$, so it remains that $y^T = d \cdot e^T, d \in E_{2\kappa}$, which completes the proof.

\[ \square \]
We proceed to the

Proof of Proposition 2. Let $L', L$ be like in the hypothesis. We shall give the proof for elementary $\Lambda$ - modules of one of the three types: finite $\mathbb{Z}_p$ - torsion, infinite $\mathbb{Z}_p$ - torsion and Weierstrass modules. Since every $\Lambda$ - torsion group $\text{Gal}(L'/L)$ in the given tower is the product of modules of these types, the general case follows.

Assume that $\text{Gal}(L'/L)$ is a Weierstrass module and $B_n := \text{RAD}(L'_{n}/L_n)$. We have shown that $B_n$ are $\Lambda$ - modules of bounded rank, at all finite levels and $B_n \subset B_{n+1} \subset \cap B_{n} \subset L'$. The condition $B_n \subset B_{n+1}$ is equivalent to $L'_{n}L_{n+1} \subset L_{n+1}$.

Our task is to prove that the norm $N_{n+1,n} : B_{n+1} \rightarrow B_n$ is surjective, so we cannot apply the Proposition 1 directly, but we may apply it to the dual galois groups. Indeed, since $L, L'$ are galois over $K$, it follows for instance from Iwasawa, [14] §3.1, that $\text{Gal}(L/K_\infty), \text{Gal}(L'/K_\infty)$ are $\Lambda$ - modules. Then so is $L = \text{Gal}(L'/L)$ as a factor of $\Lambda$ - modules, and by assumption it has finite $p$ - rank. The Galois groups $L_n = \text{Gal}(L'_{n}/L_n)$ form an infinite projective system with limit $L = \text{Gal}(L'/L)$, under the norm maps. Let $\iota_{n,n+1}$ be any lift $L_n \rightarrow L_{n+1}$. One can choose compatible chains of lifts such that for $n < m < m'$, we have $\iota_{n,m} \circ \iota_{m,m'} = \iota_{n,m'}$. The group $\text{Gal}(K_{m}/K_n)$ acts on $L_m$ by conjugation and it fixes $\iota_{n,m}(L_n)$, independently of the lift chosen. Therefore the condition $N_{n+1,n} \circ \iota_{n,n+1} = p$ as endomorphisms of $L_n$ is fulfilled and we may apply Proposition 1 to $L$. It follows that for $n > n_0$, $\iota_{n,n+1}(L_n) = L_{n+1}^p$ and thus $\text{Gal}(L_{n+1}((L_{n+1} \cdot L_n'))$ is an $\mathbb{F}_p[T]$ - module of rank $r = p\text{-rk}(L)$ for $n > n_0$. Furthermore, $\iota_{n,n+1}$ is injective, so

\begin{equation}
|\text{Gal}(L_{n+1}L_n)/L_{n+1}| = |L_n|.
\end{equation}

Let $o \in \text{Gal}(K_{n+1}/K_n)$ be a generator; setting $t = o - 1$, we have $N = p + tv$, as in [52]. By choosing $n_0$ large enough, we may assume that for $n > n_0$, the element $t \equiv T^p \mod p\Lambda$ annihilates $\text{Gal}(L_{n+1}((L_{n+1} \cdot L_n'))$. By duality, $\overline{B}_{n+1} := \text{RAD}(L'_{n+1}/(L_{n+1} \cdot L_n')) is also a group of exponent $p$ annihilated by $t'$; however $T' \equiv T \mod p$, so $t$ is also an annihilator of $\overline{B}_{n+1}$ and therefore, taking norms, it follows that $N_{n+1,n}(B_{n+1}) \subset \text{RAD}(L'_{n+1} \cdot L_n')/L_{n+1}$; the inclusion '⊂' follows from the fact that $L_{n+1}[N_{n+1,n}(B_{n+1})]$ is the maximal subfield of $L'_{n+1}$ which is fixed by $o$. It remains to show that $\text{RAD}(L'_{n+1}/L_{n+1}) \equiv \text{RAD}(L'/L_n)$; this follows from (59). This completes the proof for this case. If $L$ is a finite $\mathbb{Z}_p$ - torsion module, then by the first point of Proposition 1 for $n > n_0$ we have $\iota_{n,n+1}(L_n) = L_{n+1}^p$ and we deduce like before, that $N_{n+1,n}B_{n+1} = B_n$.

Finally, suppose that $L$ is an infinite $\mathbb{Z}_p$ - torsion module. We may restrict ourselves, without loss of generality, to the case when $L = \Lambda \ell$ is cyclic of exponent $p^\mu$ and $\iota_{n,n+1}$ is injective for $n > n_0$; this discards finite $\mathbb{Z}_p$ - torsion, which can be achieved by taking complements. Moreover, we assume that $\mu = 1$, in order to avoid modules over $\mathbb{Z}/(p^\mu \cdot \mathbb{Z})$. The general case follows by induction on $\mu$ and the number of generators of $L$. 

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Applying Lemma 3 and its proof, we see that $L_n$ are $\mathbb{F}_p[T]$ - modules of finite rank $p^n - \nu$ and $N_{n+1,n}L_{n+1} = L_{n+1}^{p-1}$. Note that in this case the exponent $p$ is replaced by $o \equiv o_1 \mod p$, as observed in the previous section, it follows by duality that $N_{n+1,n}B_{n+1} = B_{n+1}^{o_1-1}$. Since $\iota_{n,n}B_{n+1} = \iota_{n,n+1}B_{n+1}$. This completes the proof. □

We now pass to reflection, giving the

Proof of Lemma 6. We have seen that the dual of $X$ is a $\mathbb{Z}[G]$ - module on which the action of $\tau$ is determined by $\tau^* = 1$. The claim follows after proving the existence of $\alpha$. Let $x \in X$ be a generator. The existence of the set $G_x$ is a consequence of the Steinitz lemma in linear algebra. Let $\theta = \alpha^\top \in \mathbb{Q}_p[G]$ and $\alpha = 1 - \theta$; we claim $\alpha$ satisfies the condition. Indeed $\tilde{X} = \alpha\mathbb{Q}_p[G]$ is the minimal submodule of $\mathbb{Q}_p[G]$ annihilated by $\theta\mathbb{Q}_p[G]$, so $\tilde{x}^\top \sim \tilde{x}^\top$, which confirms the claim. □

We indicate here the general proof for the stabilization of the Leopoldt defect.

Proof of Lemma 13.30. From Proposition 4, $\mathbb{Z}_p$-rk$((\text{Gal}(\Omega/\Omega_E) \cap \mathbb{M}_n)/\mathbb{K}_n) = r_2$ for arbitrary base fields $\mathbb{K}$ containing $p$-th roots of unity. The proof is constructive, since it amounts to $\mathbb{M}_n \cap \Omega_E = \bigcup_{m>n}\mathbb{K}_\infty[E_n^{*m,n}/p^m]$. Therefore $(\mathbb{M}_n \cdot \Omega_E)/\Omega_E$ is an extension of $\mathbb{Z}_p$ - rank $D(\mathbb{K}_n)$. At the same time, $\mathbb{M}_n \subset \Omega$ and the last is a field attached to a fixed base field $\mathbb{K}$. Since $\text{Gal}(\mathbb{M}_n \cdot \Omega_E)/\Omega_E$ is a free $\mathbb{Z}_p$ - group, it follows that this rank is upperbounded by $\lambda(\mathbb{K})$. Plainly $D(\mathbb{K}_n) \leq \lambda(\mathbb{K})$ for all $n$, which completes the proof. The same idea is used in Washington’s proof. □

8. APPENDIX B: FACTS AND PROOFS FOR SECTION 3

We start with the exposition of the Iwasawa skew symmetric pairing which allows the use of duality in the extension $\overline{\Omega}/\overline{\Omega}_E'$.

8.1. The Iwasawa skew symmetric pairing. In this section we recall Iwasawa’s skew symmetric pairing [14], §§9-11 and prove the Theorem 3. For the ease of the reader we give a small table translating from our notation to the one used by Iwasawa in his seminal paper.

| $\Omega$ | $M$ | $\mathbb{H}$ | $L$ | $\Omega_E$ | $N$ | $\overline{\Omega}_E'$ | $N'$ | $\text{Gal}(\overline{\Omega}/\overline{\Omega}_E')$ | $X$ | $\text{Gal}(\overline{\Omega}/(\overline{\Omega}_E' \cdot \mathbb{H}')$ | $Y$ |
|---------|-----|-------------|-----|------------|-----|----------------------|------|----------------------|-----|----------------------|------|
| $\overline{\Omega}_E$ | $\mathbb{F}_p$ | $\mathbb{H}$ | $L$ | $\overline{\Omega}_E'$ | $N'$ | $\mathbb{H}'$ | $L'$ | This translation is pseudo-isomorphic; more precisely, Iwasawa also allows for some finite $\mathbb{Z}_p$ - torsion parts, which we do not consider in our notation. He carefully detaches only the $\mu$ part of the galois group $\mathcal{Y}$. This detail is however not important for the arguments below. We define here $\mathbb{H}' \subset \mathbb{H}$ to be the maximal subfield of $\mathbb{H}$ which splits all primes above $p$. In Iwasawa’s notation, $\mathbb{H}' = L'$. |
We recall the fundamental properties of Iwasawa's skew symmetric pairing. It is a map \([a, b]: \mathbb{X} \times \mathbb{X} \to \mathbb{Z}_p\), written additively, with kernel \(\omega^{-1}(Y) = \cup_n \omega^{-1}(Y)\), i.e. \([X, x] = 0 \Leftrightarrow x \in \omega^{-1}(Y)\): this important property is based on Lemma 14, §9. The pre-image \(\omega^{-1}(Y)\) defined in [14], §9.3 corresponds to the totally ramified subextensions of \(\Omega/\Omega_E\). The pairing is related to the Kummer pairing as follows: at finite levels \(n > 0, n \in \mathbb{N}\), Iwasawa associates to \(x \in X\) a class \(c(x, n) \in \mathbb{A}',\) such that \(\Omega_E[c(x, n)^{1/p^{n+1}}] \) is unramified. Then the finite level pairing is induced from Kummer pairing by:

\[
[x, x']_n = \langle x, c(x', n) \rangle
\]

The skew symmetric pairing is the projective limit of these pairings, switching to additive notation. The kernel of the map \(c: \mathbb{X} \to \mathbb{X}'\) is exactly \(\omega^{-1}(Y)\), and it induces the kernel of the pairing. The magnificent work of Iwasawa consists in proving that the pairing is skew symmetric. In particular, \([x, x'] \neq 0\) iff \([x', x] \neq 0\). As a consequence, we can show

**Lemma 17.** Notations being like above, let \(f \in \mathbb{Z}_p[T]\) be a distinguished polynomial which divides the characteristic polynomial of \(A'\) and \(a \in A_f\), the adjoint embedding in Definition [2]. Suppose that \(\Omega_E[\zeta(a)^{1/p^\infty}]/\Omega_E\) is an unramified \(\mathbb{Z}_p\) - extension which splits the primes above \(p\). Then there is a \(b \in A_f\) such that \(\Omega_E[\zeta(b)^{1/p^\infty}]/\Omega_E\) is unramified, \([a, b] = [\overline{b}, a]\) and \(a, b\) generate reciprocally the galois groups of the above fields, via Artin symbol.

**Proof.** Let \(a = (a_n)_{n \in \mathbb{N}} \in A_f\) and \(L = \Omega_E[a^{1/p^\infty}]\). Since \(L/\Omega_E\) splits the primes above \(p\), we may apply the Iwasawa skew symmetric pairing. It follows that there is a \(b = (b_n)_{n \in \mathbb{N}} \in A_f\), such that

\[
[a_n, b_n]_n = [\overline{b_n}, a_n]_n = \zeta_{p^{m(n)}}, \quad p^{m(n)} = \text{ord}(b_n) \to \infty.
\]

By definition of the skew symmetric pairing, there are \(\alpha_n, \beta_n \in \Omega_E, \) and ideals \(\mathfrak{A} \in \mathbb{A}, \mathfrak{B} \in \mathbb{B} \) such that \((\alpha_n) = \mathfrak{A}^{\text{ord}(a_n)}\), \((\beta_n) = \mathfrak{B}^{\text{ord}(b_n)}\) such that \(\Omega_E[\alpha_n^{1/p^{n+1}}] = \Omega_E[\mathfrak{A}^{1/p^{n+1}}],\) \(\Omega_E[\beta_n^{1/p^{n+1}}] = \Omega_E[\mathfrak{B}^{1/p^{n+1}}],\) and

\[
[a_n, b_n]_n = \langle \varphi(a_n), \beta_n \rangle; \quad [\overline{b_n}, a_n]_n = \langle \varphi(b_n), \alpha_n \rangle.
\]

Since \(\langle \varphi(a_n), \beta_n \rangle = \zeta_{p^{m(n)}},\) the above relation implies \(\langle \varphi(b_n), \alpha_n \rangle = \overline{\zeta}_{p^{m(n)}}\).

In particular, \(b \notin \text{Ker}(c)\) and the extension \(\Omega_E[b^{1/p^\infty}]\) must be unramified. Furthermore, it follows from [30] that \(\varphi(b) \cdot \mathbb{Z}_p = \text{Gal}(L/\Omega_E)\) and \(\varphi(a) \cdot \mathbb{Z}_p = \text{Gal}(\Omega_E[b^{1/p^\infty}]/\Omega_E).\) This completes the proof. \(\square\)

With this, we can give the

**Proof of Theorem** [3] It follows from the Lemma [17] that the radical \(A_{nr}\) is selfdual, and since \(A_{E} = \mathbb{A}'_{nr}\), it follows that \(A_{E} = A_{nr}\) and \(\Omega_E[\zeta(A_E)]\) is the maximal unramified extension of \(\Omega_E\) in \(\overline{\Omega}\). This proves point 1. Note that we cannot make any statement about \(\Omega/\overline{\Omega}:\) there may be ramified \(\mu\) - extensions, unramified ones or both; however, these extensions have finite exponent.
For \( f \mid F \) like in the hypothesis of the Proposition, the radical \( \zeta(A^2) \mathcal{f} \) splits into a ramified and an unramified part: the first generates a subfield of \( \overline{\mathbb{H}}_f \), the second of \( \overline{\Omega}_f \). The non canonical choice of \( A_r \) has here the same role as \( A^1 \): the intersection \( \zeta(A^1) \mathcal{f} \cap A_r \) is a well defined radical which generates the extension \( \overline{\mathbb{E}}_E \mathcal{f} \cdot \overline{\Omega}_f \) over \( \overline{\mathbb{E}}_E \mathcal{f}[\Omega^1_{n_f}/p^\infty] = \overline{\mathbb{H}}_f \cdot \overline{\mathbb{E}}_E \). This is point 2.

Point 3 is directly verified from the points above: by considering radicals, we have \( A_E = A_{nr} \) and thus. The definition of \( A_E \) yields \( \zeta(A)/A_E^* \cong \text{Gal}(\overline{\mathbb{H}}_E/k_{\infty})^* \) while \( A_E^* = A_E \) and \( A_r^* \cong \text{Gal}(\overline{\Omega}/(\overline{\mathbb{E}}_E \cdot \overline{\mathbb{H}})) \) explains the last two isomorphisms. It follows that \( A_r \cong \zeta(A/A_E^*) \).

For point 4, let \( f \mid F \) be such that \( \overline{\mathbb{H}}_f \subset \Omega_E \), then we use the fact that \( \text{rad}(\Omega_E/k_{\infty}) \) is \( \Lambda \) quasi - cyclic and thus \( \text{Gal}(\overline{\mathbb{H}}_f \cap \Omega_E/k_{\infty}) \) is a cyclic \( \Lambda \)-torsion Weierstrass module. If the \( f \)-part of \( \zeta(A) \) is cyclic or \( f \) is coprime to \( F/F_E \), then \( \zeta(A)_{f} \cong \text{Gal}(\overline{\mathbb{H}}_f \cap \Omega_E/k_{\infty}) \) is well defined. Applying point 3, it follows that \( \overline{\mathbb{E}}_E [\zeta(A)_{f}^{1/p^\infty}] / \overline{\mathbb{E}}_E \) is totally ramified. Note that in this case \( \zeta(A)_{f} \) is a canonic component of \( A_r \). The obstruction to a canonic definition of \( A_r \) arises thus only from \( f \)-components which are not cyclic \( \Lambda \)-modules.

In particular, for \( f = T^* \) and \( f = T \), which are both prime polynomials such that the \( f \)-primary part \( \Omega_f \) is cyclic, the point 4. applies and it follows that \( \overline{\mathbb{E}}_E [\zeta(A)_{T^*}^{1/p^\infty}] / \overline{\mathbb{E}}_E \) and \( \overline{\mathbb{E}}_E [\zeta(A_{T}^{1/p^\infty}] / \overline{\mathbb{E}}_E \) are both totally ramified, which is point 5. This completes the proof of Theorem 3.

The skew symmetric property of the Iwasawa pairing is proved by classical reciprocity. It is intimately related to the Leopoldt reflection, as the following simple example illustrates. Suppose that \( f \) is a prime polynomial with \( f(T) \not\in \{T, T^*\} \) and \( \zeta(A) \cong A/A^f \oplus A/A^f \). We may also assume that \( k \) is CM and thus \( f^* \) annihilates \( A^+ \). The \( f \)-part of \( A \) is \( A_f \) and has rank 2, but there is a canonic (up to finite torsion) cyclic submodule \( A_0 = A_0 \subset A_f \) such that \( \overline{\mathbb{E}}_E \cdot \overline{\mathbb{H}} = \overline{\mathbb{E}}_E [\zeta(A)^{1/p^\infty}] \), while reflection requires that \( \overline{\mathbb{E}}_E [\zeta(A)^{1/p^\infty}] / \overline{\mathbb{E}}_E \) be unramified. Therefore \( A_{nr} = A_0 \cdot A_f^* \). We certainly have \( A_f^* \subset A_E \), since \( (A/A_E)^{1+j} = \{1\} \); but is \( A_0^* \cong A_f^* \) under Kummer pairing? This is a direct consequence of the Iwasawa skew symmetric pairing. Showing that \( A_0 \subset A_E \) by means of the Leopoldt reflection is the difficult part, since it arises canonically as a radical, but non-canonical as a factor. However, if \( A_0 \not\subset A_E \), since \( A_E \) is a Weierstrass module and \( f \) is assumed to be prime, it follows that \( A_0 \cap A_E = \{1\} \); comparing ranks, we find that \( \text{Gal}(\overline{\mathbb{H}}_f/k_{\infty}) \cong \varphi(A_0)|_{\overline{\mathbb{H}}_f} \). We now invoke the Artin symbol, by which

\[
\text{RAD}(\overline{\mathbb{H}}_E/k_{\infty}) \cong \text{RAD}(\Omega_E/k_{\infty})/\text{RAD}(\Omega_E/\overline{\mathbb{H}}_E) \cong U_\infty^{+}/U_{\infty} \cong \text{Gal}((\overline{\mathbb{H}} \cdot \overline{\mathbb{H}}_E)) \cong A_r^*.
\]

It follows that \( A_r^* \cong \text{Gal}(\overline{\mathbb{H}}_E/k_{\infty})^* \) and \( A_r \cong A/A_E \cong A_0 \). But then \( (\overline{\mathbb{E}}_E \cdot \overline{\mathbb{H}})^{1/p^\infty} \) is unramified, in contradiction with the choice of \( A_0 \). This
proof scheme is easily generalized in the CM case; for the non CM case we need the Corollary below. One can prove thus by mere Leopoldt reflection, that \( A_E = A^\bullet_n \), which then implies all the other statements in the Theorem. In particular, one may view the Kummer pairing as a pairing \( A_E \times A_E \to \mathbb{Z}_p \); it induces a symmetric quadratic form \( Q : a \to [a, a^\bullet] \). Deducing a map \( c : A_E \to A_E \) such that

\[
[a, c(a)^\bullet] = \overline{c(a)^\bullet, a} = [c(a)^{-1}, a^\bullet],
\]

which induces Iwasawa’s skew symmetric pairing, requires some more work which we skip here, since we may use directly the results of \([14]\).

8.2. Complex multiplication and duality.

Proof of Fact. Suppose that for an \( n \geq 0 \), the map \( \iota_n : \mathbb{Z}_n \to \mathbb{A}_n \) is not injective and let \( \Omega \subset \mathbb{A}_n \) be a prime of order \( p \) which capitulates in \( \mathbb{K}_n \). Let \( (\beta) = \mathbb{O}_n^p, \beta \in \mathbb{K}_n \) and \( \mathcal{O}(\mathbb{K}_n)\Omega = (b) \). Then there is a unit \( e \in E(\mathbb{K}_n) \) with \( \beta = eb^p \) and \( \beta^{1-j} = (e/\gamma) \cdot b^{p(1-j)} \); but then \( e/\gamma = \xi^c \) is a root of unity and after eventually multiplying \( \beta \) by \( \xi^{-c/2} \) we find \( (\beta/b^{p^{1-j}} = 1 \). Since \( \mathbb{K}_n+1 = \mathbb{K}_n[\xi^{1/p}] \), it follows by Kummer theory that \( \beta^{1-j} = \beta^{1-j/p} \in \mathbb{K}_n+1 \) and \( \beta^{1-j} = \xi^{c'/p} \cdot \beta_p \). This implies that \( \Omega \) is principal and \( a_n = 1 \), so there is no capitulation.

We use now the Lemma in the previous Appendix for proving the Proposition and Proposition. We use class field theory; let \( \Omega', \Omega_n' \) denote here the maximal \( p \)-ramified \( p \)-abelian extension of \( \mathbb{K}_n \); thus \( \Omega_n \subset \Omega_n' \) is the maximal Kummer extension contained in \( \Omega_n' \). Then \( \tilde{\Omega}_{T^{*},n} \subset \Omega_n' \) is the maximal subextension with group over \( \mathbb{H}_n \) annihilated by \( T^* \) by class field theory, writing \( V_n = U_n^{(1)} / \mathbb{E}_n \), we have

\[
\tilde{\Omega}_{T^{*},n} = Gal(\tilde{\Omega}_{T^{*},n}/\mathbb{H}_n) \cong V_n/V_n^{T^*}.
\]

We apply Lemma to \( V_n \). Now \( U_\infty = \lim U_n^{(1)} \) is a \( \Lambda \)-module with a cyclic part of \( \Lambda \) rank \( r = 2r_2 \) while \( E_\infty = \lim E_n^{(1)} \) has the essential \( \Lambda \)-rank \( r_2 \). Since the Leopoldt defect is bounded – and in our hypothesis stable from \( \mathbb{K} \) upwards – the limit \( U_\infty / E_\infty \) has the essential \( \Lambda \) rank \( r_2 \) and \( p\)-rank \( V_n \geq r_2 p^{n-\kappa} \).

We consider the torsion part of \( V_n/T^* \); this is \( \mu_{p^{n+1}} \) for \( \mathbb{E}_n \), while \( U_n \) has a \( p \)-torsion part \( T_n \cong (\mathbb{C}_{p^{n+1}})^s \), where \( s \) is the number of ramified primes above \( p \); this is thus a group of exponent and sub-exponent \( p^{n+1} \). The torsion is generated by the roots of unity in the various completions at primes above \( p \). Indeed, let \( p = \tilde{p}(\zeta_{p^{n+1}} - 1) + 1 \in U_n \), with \( \tilde{p} \) defined in \([17]\). Then \( \iota_p(\rho) = \zeta_{p^{n+1}} \) while \( \iota_{p^j}(\rho) = 1 \) for \( j > 1 \); since \( U_p[\zeta_{p^{n+1}}] = U_n[\rho] \), we see that \( \Gamma \) fixes \( U_n[\rho] \) and \( \rho^{T^*} = 1 \). The torsion is thus \( T_n = (\rho^{p^{n+1}}[C] \cong (\mathbb{C}_{p^{n+1}})^s \) and annihilated by \( T^* \); it follows that, in the notation of Lemma, we have

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We would then have \( \tilde{\Omega} \) converging subexponents. Therefore \( E \) is the structure by this field, which naturally belongs to the module of unity of \( K \). Groups \( Y \) is annihilated by \( K \), according to (23): the annihilator module \( \tilde{\Omega} \) is quasi - cyclic of \( p \) - rank \( r_2 \) and subexponent \( p^m \) with \( m \geq \left[ \frac{n}{2} \right] \).

We have shown in Lemma 14 that \( \tilde{\Omega}_{n,T^*,E} \cdot \kappa_\ell = \left( \tilde{\Omega}_{m,T^*,E} \right) \varphi(Y_{m}^{\tilde{\Omega}_{n,T^*,E}}). \) In particular, the fields \( \tilde{\Omega}_{n,T^*,E} \) form an injective sequence of fields with groups of constant \( p \) - rank and in the limit we have

\[
\tilde{\Omega}_{T^*} = \tilde{\Omega}_{T^*,E}^\varphi(\mu_\infty(V)).
\]

Putting the two pieces together, \( \tilde{\Omega}_{T^*} \subset \tilde{\Omega}_{T^*} \subset \tilde{\Omega}_{T^*} \) and \( \text{Gal}(\tilde{\Omega}_{T^*}/\tilde{\Omega}_{T^*}) \) is \( \mathbb{Z}_p[G] \) quasi - cyclic of \( p \) - rank \( r_2 \) while \( \text{Gal}(\tilde{\Omega}_{T^*}/\tilde{\Omega}_{T^*}) = \varphi(\mu_\infty(V)) \) is a group on which the transitivity and which is annihilated by the norm of \( \kappa_\ell \), having \( p \) - rank \( s - 1 \). It is interesting to observe that \( \kappa_\ell[p^{1/p^\infty}] \subset \tilde{\Omega}_{T^*} \) is the subfield with group fixed by \( G \); one may ‘shift’ the structure by this field, which naturally belongs to the \( p \) - unit field \( \kappa_\ell[p^{1/p^\infty}] \) – this would correspond to the denominator \( \mu_n(E_\ell) \) in \( \mu_n(V_{\ell}) \).

We would then have \( \tilde{\Omega}_{T^*} = \tilde{\Omega}_{T^*,E}[p^{1/p^\infty}] \), for a new \( \tilde{\Omega}_{T^*,E} \) with group of rank \( r_2 - 1 \). The rank computation is the same and the difference consists in the side to which \( \kappa_\ell[p^{1/p^\infty}] \) is counted. \( \square \)

We proceed with the simpler

**Proof of Proposition 4.** The radical \( \text{RAD}((\Omega_{E} \cap M)/\kappa_\infty) \subset E^{1/p^\infty} \) is annihilated by \( T^* \). At finite levels we may apply the Lemma 14 to the finite \( \Lambda \) - modules \( V_\ell := E_\ell/(E_\ell^{p^{n+1}}) \). The torsion part is \( \mu_n(V_{\ell}) = \mu_\ell^{p^{n+1}} \), the \( p \) - roots of unity of \( K \). They correspond to the extension \( \kappa_\infty \); hence, the contribution of the torsion vanishes at infinity and we have \( \Omega_{E} \cap M = \cup_n \kappa_\ell[E_\ell/(E_\ell^{p^{n+1}})]. \)

By Lemma 14 the fields \( \kappa_\ell[(E_\ell^{p^{n+1}})^{1/p^\infty}] \) build an injective sequence with groups \( \mathcal{Y}_{T,n} \) which are \( \mathbb{Z}_p[G] \) - quasi cyclic, of constant \( p \) - rank \( r_2 \) and diverging subexponents. Therefore \( \mathcal{Y}_{T} = \text{Gal}(\Omega_{E} \cap M) \) is a quasi - cyclic \( \mathbb{Z}_p[G] \) - module of rank \( r_2 \), free as a \( \mathbb{Z}_p \) - module. \( \square \)

As a useful consequence, we have at least the following generalization of Fact 4.

**Lemma 18.** Suppose that the field \( K \) is not CM. Let \( B \subset A \) be a \( \Lambda[G] \) - submodule such that \( \tilde{B}^{e^+} = \{1\} \). Then the capitulation kernel \( \text{Ker} (\varsigma(B_{\ell}) \rightarrow \varsigma(B)) = \{1\} \).

**Proof.** Like in the proof of Fact 4 we choose \( a_n \in B_{\ell} \) and \( \Omega \in a_n \), \( (\beta) = \Omega^a = (b) \) with \( b \in \mathbb{K}_n \), so \( \beta = eb^p \). Now we use annihilation of \( E \) by \( R^{e^-} \), according to (23): the annihilator module \( \varsigma(\ell)^p \subset R^{e^-} \) is disjoint from \( R^{e^+} \), which annihilates \( a_n \). There is then an \( \alpha \in \mathbb{F}_p[G] \) such that \( a_n^\alpha \not\in \varsigma(B)^p \)
but $e^{p^r\alpha} \in E^{p^{r+1}}_n$ and thus $\beta^{p^r\alpha} \in \mathbb{K}^{p^{r+1}}_{n+1}$. The same Kummer argument as before implies that $\beta^{p^r\alpha} = x^{p^{r+1}} \in \mathbb{K}^{p^{r+1}}_n$ and thus $((\Omega/(x))^{\alpha})^{p^{r+1}} = (1)$, so $\Omega^{p^r\alpha}$ is principal, in contradiction with our assumption. □

We proceed now to the proof of the simple Lemma 9.

Proof of Lemma 9. Let $\delta \in E$ be a Minkowski unit and $\theta \in \mathbb{Z}_p[G]$ such that $\theta/|G| \in \mathbb{Q}_p[G]$ is an idempotent which generates the annihilator ideal $\delta^+ = \mathbb{Q}_p[G]$. Let $\theta = \theta_m + p^{m+1}r_m$, with $\theta \equiv \theta_m \mod p^{m+1}\mathbb{Z}_p[G]$, so $\theta_m$ are the rational approximants of $\theta$ to the $p^m$-th order. Let $H \subset G$ be a minimal subset such that $\theta\mathbb{Z}_p[G] = \theta\mathbb{Z}_p[H]$. We first define $D_n' = \text{Span}(\delta^{\theta_m+1}\sigma)_{\sigma \in H}$, where Span denotes here the $\mathbb{Z}$-span. Then $D_n' \subset U^{p^{n+1}}$ by construction. However the condition that $(D_n\cdot E^p)/E^p$ has $p$-rank $r$ may not be fulfilled, so we shall need to perform some change of generators. This will be done by combining $D_n'$ with radicals from $D_n'_{j>0}$; the approach is similar to the one in the proof of Lemma.

The set $S_1 = ((\theta_i\mathbb{Z}_p[H]) \cdot (p\mathbb{Z}_p[G]))/(p\mathbb{Z}_p[G])$ is finite and $D_1' \equiv \delta S_1$ mod $(D_1')^p$. Let $i(x) : E \to N$ be the $p$-index, so $i(x) = k \iff x \in E^p \setminus E^{p+1}$; there is then a finite $k = \max(i(\delta^s) : s \in S_1)$. If $k = 0$, then we may define $D_n = D_n'$. Otherwise, let $r' < r$ be the $p$-rank of $(D_1'E^k)/E^k$ and $r_1 = r - r'$. Let

$$d_j' \in D_1', e_j \in E : d_j' = e_j^k, j = 1, 2, \ldots, r_1$$

be a system of $\mathbb{Z}$-independent units and let $t_j \in \mathbb{Z}[G]$ be such that $d_j' = \delta t_j$. Then we define

$$d_{j,n} = \delta^{\theta_{t_j}}/p.$$ 

By construction, we see that $d_{j,n} \in E \setminus E^p$ and $d_{j,n} \in U^{p^{n+1}}$. Let $D_{1,n} = \text{Span}(d_{j,n})_{j=1}^{r_1}$. We proceed by induction as follows: let $H_1 \subset H$ be a maximal subset such that $\delta^H \mathbb{Z}[H]$ and $D_{1,1}$ are $\mathbb{Z}$-independent, thus $|H_1| = r-r_1$. Let $S_2$ be defined with respect to $H_1$ by $S_2 = ((\theta_i\mathbb{Z}_p[H_1]) \cdot (p\mathbb{Z}_p[G]))/(p\mathbb{Z}_p[G])$ and $k_1 = \max(i(\delta^s) : s \in S_2)$. If $k_1 = 0$, then we let $D_n = D_{1,n} \cdot \delta^{\theta_{t_j}}$. The systems $D_n \subset E$ fulfill the required properties by construction. If $k_1 \neq 0$, we proceed like in the previous step and since $k_1 < k$, the procedure will eventually end for a value $k_h = 0$. Thus, we obtain systems of units $D_n \subset E$ with $p\text{-rk}(D_nE^p/E^p) = r$, $D_n \subset U^{p^{n+1}}$ and $D_{n+1} \subset D_n \cdot E^{p^{n+1}}$. The sub-exponent $p^{n+1}$ for $D_n/D_n^{p^{n+1}}$ follows from the fact that

$$p\text{-rk}(D_n/D_n^{p^{n+1}}) = p\text{-rk}(D_n \cdot E^{p^{n+1}}/E^{p^{n+1}}),$$

which holds by construction. □

9. Appendix C: Complements to Sections 4 and 5

We give an example of an extension $\tilde{\Omega}_{f^*}/\mathbb{H}_{f^*}$ with a polynomial $f(T) \neq T$:
Example 2. Let $\mathbb{K} = \mathbb{Q}[\zeta]$ be the $p$-th cyclotomic extension. Then $s = 1$ and $v = (p - 1)/2$. Thus $\mathbb{Z}_p$-rk($\text{Gal}(\Omega \mathbb{K}/\mathbb{K})$) = $v$ and

$$\Omega_{T^*} = \Omega_E[\Pi^{1/p^\infty}].$$

Suppose now that $p$ is such that Vandiver’s Conjecture holds and the irregularity index is 1. Let then $A = A^{-} = \Lambda a$ and assume additionally, that the minimal polynomial of $a$ and $r$ is a situation which occurs often. The cyclotomic units $C_n = E_n = O(\mathbb{K})$ and the local units $U'_n$ are norm coherent and the norm is surjective on both systems of units; let $\xi_k$ be the orthogonal idempotent with $c_{p-k}A \neq \{1\}$ and $\chi \in \mathbb{Z}[G]$ approximate $\varepsilon_k$, the reflected idempotent, to order $p^M$, for some large $M$; in particular, $k$ is even. There is for $n \leq M$ a system of local and global Minkowski units $\xi_n \in U_n, \eta_n \in \mathbb{Q} \cap \mathbb{K}$ such that

$$\xi_n f(T) = \eta_n^x \cdot p^M, \quad x_n \in U_n.$$  

In particular $\xi_n^x, \eta_n^x$ generate one dimensional $\Lambda_n/p^M \Lambda_n$ - modules. Let $n$ be fixed with $2n < M$; by choice of $f$, the classes in $A_n$ have order $p^{n+1}$, so there is a cyclic unramified extension $F_n/\mathbb{K}$ of degree $p^{n+1}$. By the proof of Lemma 14, class field theory requires that there also be a $p$ - ramified extension $L_n/\mathbb{K}_n$ of degree $p^m$ with $p^{n+1} \geq p^m \geq p^{[n/2]}$ and galois group in the $\varepsilon_{p-k}$ component of $\text{Gal}(\Omega_n/\mathbb{K}_n)$, annihilated by $f(T^*)$.

The Lemma concerns in fact only the polynomial $f(T) = T$, but the case when $f$ is an arbitrary polynomial is proved similarly. In general, if $f(T)$ is a polynomial of degree $d$, there exist for $n$ sufficiently large $g_n, h_n \in \Lambda$ such that

$$g_n \cdot f + h_n \cdot \omega_n = p^{n+1}.$$ 

It follows that $(U_n^{p^n} \cdot U_n^{p^{n+1}})/U_n^{p^{n+2}}$ has $p$ - rank $d \cdot (2v)$ and is annihilated by $f$. Defining $f$ like above and $\Omega_f$ by Definition 10, it follows that $p$-rk$(\varepsilon_{p-k} \text{Gal}(\Omega_n, f^2/\mathbb{K}_n)) = 2 = \text{deg}(f^2)$. In our example, the ramified extension must be a cyclic extension of $F_n$ and $\mathbb{L}'_n = \mathbb{K}_{n+m} \mathbb{L}_n$ is a Kummer cyclic extension which is abelian over $\mathbb{K}_n$ and $F_n = F_n \cdot \mathbb{K}_{n+m} \subset \mathbb{L}'_n$.

Let $\mathbb{L}' = \mathbb{K}_{n+m}[e^{1/p^{n+1}}]$ and $\nu \in \text{Gal}(\mathbb{L}'_n/\mathbb{K}_{n+m})$ be a generator. Then $\nu^{p^{n+1}}$ is a generator for the ramified extension $\mathbb{L}'_n/\mathbb{F}'_n$; by hypothesis we must have $\nu^{p^{n+1}} \cdot f(T^*) = 1$. Furthermore, $\nu$ generates by restriction $\text{Gal}(\mathbb{F}'_n/\mathbb{K}_{n+m})$ and the hypothesis implies that $\nu^f(T^*)$ fixes $\mathbb{F}'_n$, thus $\nu^f(T^*) \in \nu^{p^{n+1}}$. Assembling the two conditions, we deduce that $\nu^f(T^*)^2 = 1$. It follows that $\mathbb{L}_n \subset \varepsilon_{p-k} \mathbb{K}_{n,f^2}$, a $p$ - abelian, $p$ - ramified extension of $p$ - rank 2, where idempotents act on fields by acting on galois groups fixing these fields:

$$\varepsilon_{p-k} \mathbb{K}_{n,f^2} = \mathbb{K}_{n,(1-\varepsilon_{p-k}) \text{Gal}(\Omega_n/\mathbb{K}_n)}.$$

We now consider Kummer radicals. Reflection implies for $e$ that

$$e^f(T^*)^2 \in \mathbb{E}_{n+m}^{p^{n+m+1}}.$$
Furthermore, since \( L_n/K_n \) is abelian, we have the condition
\[
e^{\mu_n} \in E^{p^m+1}_{n+m}.
\]

Additionally, \( F_n \) is Kummer over \( K_n \), so there are \( e_0 \in E_n \) and \( u \in E_{n+m} \) with
\[
e = e_0 \cdot u^{p^{n+1}}, \quad e_0 \in U^{p^{n+1}}, \quad f(T) \in E_n^{p^{n+1}}.
\]

The three conditions must have a solution, as required by class field theory. Let \( e = \eta_{n+m}^\lambda \), with \( \lambda \in \Lambda \). Then \((64)\) yields \( \lambda = N_{n+m,n}a(T)+p^{n+1}b(T) \) for some \( a(T), b(T) \in \Lambda \setminus p\Lambda \) such that
\[
\begin{align*}
a(T) \cdot f(T) &\in (\omega_n, p^{n+1})\Lambda \\
N_{n+m,n} \cdot a(T) \cdot f^2(T) + p^{n+1}b(T) \cdot f^2(T) &\in (\omega_{n+m}, p^{n+m+1})\Lambda \\
N_{n+m,n} \cdot a(T) \cdot \omega_n^* + p^{n+1}b(T) \cdot \omega_n^* &\in (\omega_{n+m}, p^{n+m+1})\Lambda \\
N_{n+m,n} \cdot a(T) \cdot f(T) + p^{n+1}b(T) \cdot f(T) &\not\in (\omega_{n+m}, p^{n+2})\Lambda.
\end{align*}
\]

The last condition stems from \( \eta_{n+m} = \xi_{n+m}^{f(T)} \), which is \((61)\), and implies that \( \mathbb{L}'/F_n' \) is ramified. A solution arises by using \((60)\) and the general fact that, for coprime polynomials \( f, g \in \mathbb{Z}_p[T] \), the ideal \((f, g)\) is of finite index in \( \Lambda \) and there is a linear combination \( uf + vg = p^s \) with \( p^s \geq \max(f(0), g(0)) \). Let \( g_n f + x_n \omega_n = p^{n+1} \). The first condition implies that \( a(T) \) is a multiple of \( g_n \), say \( a(T) = g_n(T)a'(T) \). The second and the last conditions become then
\[
\begin{align*}
a'(T) + b(T) f(T) &\in \Lambda \setminus (p, \omega_{n+m})\Lambda, \\
a'(T) f(T) + b(T) f^2(T) &\in (\omega_{n+m}, p^m)\Lambda,
\end{align*}
\]
while the third becomes, via \((67)\),
\[
g_n(T)a'(T) + b(T) \cdot u_n \omega_n \in (\omega_{n+m}, p^m)\Lambda.
\]

Finally the resulting system can be solved as follows: first find a couple \( a_1'(T), b_1(T) \in \Lambda \setminus p\Lambda \) with minimal valuations and such that the condition \((66)\) is fulfilled. Set \( a'(T) = a_2(T) \cdot a_1(T) \) and \( b(T) = b_1(T) \cdot p^s \cdot a_2(T) \) and solve \((65)\) with respect to \( a_2(T) \) and \( s \). A possible solution arises by setting \( s = 0 \) and \( g'(T) \in (p, \omega_{n+m})\Lambda \), such that \( g'(T)f(T) + y(T)\omega_{n+m} \in p^m\Lambda \). Then let \( \lambda' = g_n(T)a_1(T) + b_1(T)f(T) \), which is the right hand side in the first condition of \((69)\). We may assume that \( \lambda' \not\in p\Lambda \), since both terms are not \( p \)-multiples and if the sum is, one may always add a multiple of \( p^m \) to \( b_1(T) \), achieving the required result. Thus we solve
\[
a_2(T)\lambda' \in (g'(T), p^m)\Lambda.
\]

Then neither \( e_0 \) nor \( u \) are \( p \)-powers and the resulting \( e \) verifies all the required conditions, including the fact that \( \mathbb{L}'/F_n' \) is ramified.

After having shown the existence of the extension towers \( K_n \subset F_n \subset L_n \), it is interesting to consider the picture at infinity. We have shown that the galois groups \( \text{Gal}(\Omega_n/F_n) \) are norm coherent. The extensions \( F_n \) form an injective system, so let \( F = \bigcap_n F_n \). Since \( \varepsilon_{p-k}\Omega_{(f^*)^2} \) has group of \( \mathbb{Z}_p \)-
rank 2, there is a \( \mathbb{Z}_p \) - extension \( \mathbb{K}_\infty \subset \mathbb{F} \subset \varepsilon_{p-k} \Omega((s')\!^2) \) which is linearly disjoint from \( \mathbb{F} \) and with galois group annihilated by \( \mathfrak{f}^2 \) but not by \( \mathfrak{f} \). Since \( \text{Gal}(\mathbb{L}/\mathbb{F}_n) \) form a projective system, it follows that \( \mathbb{F} \cdot \mathbb{L}_n \) are injective and \( \mathbb{L} = \bigcup_n \mathbb{L}_n \cdot \mathbb{F} \) is a \( \mathbb{Z}_p \) - extension of \( \mathbb{F} \) with \( \text{Gal}(\mathbb{L}/\mathbb{F})f(T^*) = \{1\} \), as required. Furthermore, \( \mathbb{L}_n \subset \Omega_n(f^2) \) for all \( n \), so it follows that \( \mathbb{L} \subset \Omega_{(f')^2} \). Although \( \mathbb{L}_n/\mathbb{K}_n \) are cyclic for all \( n \), the system \( (\mathbb{K}_\infty \cdot \mathbb{L}_n)_{n \in \mathbb{N}} \) is not injective. This can also be verified from the explicite construction above. For \( f(T) \neq T \) one thus observes that in the case when \( \mathbb{H}_f \neq \mathbb{K}_\infty \) there is a \( p \) - abelian and totally \( p \) - ramified extension \( \mathbb{L}/\mathbb{H}_f \) with galois group annihilated by \( f \). In this case \( \mathbb{L} \subset \Omega_{(f')^2} \setminus \Omega_f \).

The same arguments require that \( \Omega'_{(f')^2}/\mathbb{H} \) with \( \text{Gal}\left(\Omega'_{(f')^2}/\mathbb{H}\right)^{\varepsilon_{p-k}(f')^2} = \{1\} \) has \( \mathbb{Z}_p \) - rank 2. The rank loss propagates and there must be a \( \mathbb{Z}_p \) - subextension \( \mathbb{L}' \subset \Omega'_{(f')^2} \) with \( \mathbb{L}' \subset \Omega_{(f')^2} \setminus \Omega_{(f')^2} \). Since \( \Omega \) contains a free \( \Lambda \) - submodule, the rank loss is absorbed at infinity.

Next we illustrate the arguments of the proof of Proposition 5 on the example of the \( p \) - cyclotomic tower, used in the previous example. Thus, \( \mathbb{K} = \mathbb{Q}[\zeta] \). In this example, we may assume that Vandiver’s Conjecture holds for \( p \), so the units \( E(\mathbb{K}_n) \) are cyclotomic and \( \mathbb{N}_{m,n}(E_m) = E_n \). Let \( \varepsilon_k = \frac{1}{p-1} \sum_{\sigma \in G} \omega^k(\sigma)^{-1} \) be the orthogonal idempotents of \( \mathbb{Z}_p[G] \) and assume that Leopoldt’s Conjecture is false. Then there is an even number \( p - k \) such that \( \varepsilon_{p-k} E = \{1\} \); the construction of \( \Phi \) shows that \( \varepsilon_k A/(A^{T^*}) \) is infinite. Let \( \chi \in \mathbb{Z}[G] \) approximate \( \varepsilon_{p-k} \) to the \( p^M \)-th power for a large \( M \), so \( \eta^\chi \in U(\mathbb{K})^{p^M} \), with \( \eta \) a real cyclotomic unit generating \( E(\mathbb{K}) \) as a \( \mathbb{Z}_p[G] \) - module. Let \( M/4 > n > 0 \) and \( \Phi_n = \mathbb{K}_n[\eta^{\chi/p^{n+1}}] \), an unramified extension. Proposition 5 shows that there is no totally ramified extension \( \mathbb{L}_n/\Phi_n \) of degree \( p^{n/2} \leq \mathfrak{p}^n = [\mathbb{L}_n : \Phi_n] \leq p^{n+1} \), such that \( \mathbb{L}_n/\mathbb{K}_n \) is abelian – here the Leopoldt case differs from Example 2. It follows from Lemma 3 that \( \mathbb{L}_n/\Phi_n \) must be totally ramified; however the Iwasawa skew symmetric pairing shows that this extension is unramified. Therefore it must be trivial and the Leopoldt defect vanishes.

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**References**

[1] T. Albu. *Cogalois theory*. Number 252 in Monographs and textbooks in pure and applied mathematics. Marcel Dekker Inc., 2003.
[2] T. Albu. Field theoretic cogalois theory via abstract cogalois theory. J. Pure Appl. Algebra, 208(1):101–106, 2007.

[3] J. Alperin and R. Bell. Groups and Representations, volume 162 of Graduate Texts in Mathematics. Springer, 1995.

[4] B. Anglès. On the $p$-adic Leopoldt transformation of a power series. Acta Arithmetica, 134(4):349–368, 2008.

[5] J. Ax. On the units of an algebraic number field. Illinois Journal of Mathematics, 9:584–589, 1965.

[6] A. Baker. Linear forms in the logarithms of algebraic numbers I, II, III. Mathematika, 13:14204–216; 102–107, 220–228, 1966, 67.

[7] A. Brumer. On the units of algebraic number fields. Mathematika, 14:121–124, 1967.

[8] M. Emsalem, H. Kisilevsky, and D. Wales. Indépendance linéaire sur $\mathbb{Q}$ de logarithmes $p$-adiques de nombres algébriques et rang $p$-adique du groupe des unités d’un corps de nombres. Journal of Number Theory, 19:384–391, 1984.

[9] L. Federer. The nonvanishing of Gross’ $p$-adic regulator Galois cohomologically. In Journées Arithmétiques de Besançon (1985), volume 147-48 of Astérisque, pages 71–77, 1987.

[10] L. Federer and B. Gross. Regulators and Iwasawa modules. Invent. Math., 62(3):443–457, 1981.

[11] T. Fukuda. Remarks on $\mathbb{Z}_p$-extensions of number fields. Proc. Japan Acad. Ser. A Math. Sci., 70(8):264–266, 1994.

[12] R. Greenberg. On the Iwasawa invariants of totally real fields. American Journal of Mathematics, 98:263–284, 1976.

[13] K. Iwasawa. On $\gamma$-finite modules. Ann. Math. Second Series, 70:290 – 312, 1959.

[14] K. Iwasawa. On $\mathbb{Z}_\ell$-extensions of number fields. Ann. Math. Second Series, 98:247 – 326, 1973.

[15] J. Jaulent. Note sur la conjecture de Leopoldt. http://front.math.ucdavis.edu/0712.2995, 2007.

[16] J. Jaulent. Sur les conjectures de Leopoldt et Gross. In Journées Arithmétiques de Besançon (1985), volume 147-48 of Astérisque, pages 107–120, 1987.

[17] J. Kraft and R. Schoof. Computing Iwasawa modules of real quadratic number fields. Compositio Mathematica, 97:135–155, 1995.

[18] L. Kuz’m’in. The Tate module for algebraic number fields. Math. USSR Izvestija, 6(2):263–321, 1972.

[19] M. Laurent. Rang $p$-adique d’unités et action de groupes. J. reine angew. Math., 399:81–108, 1989.

[20] M. Le Floc’h, A. Movahhedi, and T. Nguyen Quang Do. On Capitulation Cokernels in Iwasawa Theory. American Journal of Mathematics, 127(5):851–877, 2005.

[21] H. Leopoldt. Zur Arithmetik in Abelschen Zahlkörpern. J. Reine Angew. Math, 209:54–71, 1962.

[22] S. Lang. Cyclotomic fields I and II, volume 121 of Graduate Texts in Mathematics. Springer, combined Second Edition edition, 1990.

[23] J. Serre. Local Class Field Theory. In Cassels and Fröhlich, editors, Algebraic Number Theory, pages 129–161. Academic Press, 1967.

[24] M. Waldschmidt. Transcendance et éxponentielles en plusieurs variables. Inventiones Mathematicae, 63, 1981.

[25] L. Washington. Introduction to Cyclotomic Fields, volume 83 of Graduate Texts in Mathematics. Springer, 1996.