NON-LOCALITY OF EQUIVARIANT STAR PRODUCTS ON $T^*\mathbb{RP}^n$

RANEE BRYLINSKI

Abstract. Lecomte and Ovsienko constructed $SL_{n+1}(\mathbb{R})$-equivariant quantization maps $Q_\lambda$ for symbols of differential operators on $\lambda$-densities on $\mathbb{RP}^n$.

We derive some formulas for the associated graded equivariant star products $\star_\lambda$ on the symbol algebra $\text{Pol}(T^*\mathbb{RP}^n)$. These give some measure of the failure of locality.

Our main result expresses (for $n$ odd) the coefficients $C_p(\cdot, \cdot)$ of $\star_\lambda$ when $\lambda = \frac{1}{2}$ in terms of some new $SL_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operators $Z_p(\cdot, \cdot)$ on $T^*\mathbb{CP}^n$ and the operators $(E + \frac{2}{s} \pm s)^{-1}$ where $E$ is the fiberwise Euler vector field and $s \in \{1, 2, \cdots, \lceil \frac{p}{2} \rceil \}$.

1. Introduction

Lecomte and Ovsienko ([L-O]) constructed $SL_{n+1}(\mathbb{R})$-equivariant quantization maps $Q_\lambda$ for symbols of differential operators on $\lambda$-densities on $\mathbb{RP}^n$.

We derive some formulas for the associated graded equivariant star products $\phi \star_\lambda \psi = \phi \psi + \sum_{p=1}^\infty C_p^\lambda(\phi, \psi)t^p$ on the symbol algebra $\text{Pol}_\infty(T^*\mathbb{RP}^n)$. The star products $\star_\lambda$ is “algebraic” in that (Proposition 3.1) it restricts to the subalgebra $R$ generated by the momentum functions $\mu^x$, $x \in \mathfrak{sl}_{n+1}(\mathbb{R})$.

We compute some special values of $\phi \star_\lambda \psi$ in Proposition 4.1. We conclude in Corollary 4.2 that $C_p^\lambda(\cdot, \cdot)$ fails to be bidifferential, except if $\lambda = \frac{1}{2}$ and $p = 1$. The reason is that $C_p^\lambda(\cdot, \cdot)$ involves operators of the form $(E + r)^{-1}$ where $E$ is the fiberwise Euler vector field on $T^*\mathbb{RP}^n$ and $r$ is a positive number.

In our main result (Theorem 5.1), we write, for $n$ odd, the coefficients $C_p^\lambda(\cdot, \cdot)$ when $\lambda = \frac{1}{2}$ in terms of some new $SL_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operators $Z_p(\cdot, \cdot)$ on $\mathbb{CP}^n$ and the operators $(E + \frac{2}{s} \pm s)^{-1}$ where $s \in \{1, 2, \cdots, \lceil \frac{p}{2} \rceil \}$. Our proofs in §4 and §5 are applications of the formulas in [L-O], §5.5] for $Q_\lambda$.

The operator $Z_p(\cdot, \cdot)$ ($p \geq 2$) is quite subtle as it has total homogeneous degree $-p$. It is not the $p$th power of the Poisson tensor (with respect to some coordinates) because we can show that the total order of $Z_p(\cdot, \cdot)$ is too large. It would be very interesting to find a way to construct $Z_p$ using the method of Levasseur and Stafford ([L-S]).

I thank Christian Duval and Valentin Ovsienko for several interesting discussions.

2. The Lecomte-Ovsienko quantization maps

In [L-O], Lecomte and Ovsienko constructed, for each $\lambda \in \mathbb{C}$, an $SL_{n+1}(\mathbb{R})$-equivariant (complex linear) quantization map $Q_\lambda$ from $A = \text{Pol}_\infty(T^*\mathbb{RP}^n)$ to $B^\lambda = \mathcal{D}_\infty(\mathbb{RP}^n)$. Here $A = \bigoplus_{d=0}^\infty A^d$ is the graded Poisson algebra of smooth complex-valued functions on $T^*\mathbb{RP}^n$ which are polynomial along the cotangent fibers, and $B^\lambda = \bigcup_{d=0}^\infty B^\lambda_d$ is the filtered algebra of smooth (linear) differential operators on $\lambda$-densities on $\mathbb{RP}^n$. Then
$Q_\lambda$ is a quantization map in the sense that $Q_\lambda$ is a vector space isomorphism and $\phi$ is the principal symbol of $Q_\lambda(\phi)$ if $\phi \in \mathcal{A}^d$.

The natural action of $SL_{n+1}(\mathbb{R})$ on $\mathbb{R}^n$ lifts canonically to a Hamiltonian action on $T^*\mathbb{R}^n$ with moment map $\mu: T^*\mathbb{R}^n \rightarrow \mathfrak{sl}_{n+1}(\mathbb{R})^*$. The density line bundle on $\mathbb{R}^n$ is homogeneous for $SL_{n+1}(\mathbb{R})$. This geometry produces natural (complex linear) representations of $SL_{n+1}(\mathbb{R})$ on $\mathcal{A}$ and $B^\lambda$; $Q_\lambda$ is equivariant for these representations.

The procedure of Lecomte and Ovsienko was to construct ([L-O, Thm. 4.1]) an $\mathfrak{sl}_{n+1}(\mathbb{R})$-equivariant quantization map $Q_\lambda$ from $\text{Pol}_\infty(T^*\mathbb{R}^n)$ to $\mathcal{D}_\infty^\lambda(\mathbb{C}^n)$, where $\mathbb{R}^n$ is the big cell in $\mathbb{R}^n$. They show their map is unique. Then $Q_\lambda$ restricts to a quantization map from $\mathcal{A}$ to $B^\lambda$ ([L-O, Cor. 8.1]).

We can represent points in $\mathbb{R}^n$ in homogeneous coordinates $[u_0, \ldots, u_n]$. Then $u_1, \ldots, u_n$ are linear coordinates on the big cell $\mathbb{R}^n$ defined by $u_0 = 1$. These, together with the conjugate momenta $\xi_1, \ldots, \xi_n$, give Darboux coordinates on $T^*\mathbb{R}^n$.

For any vector field $\eta$ on $\mathbb{R}^n$, let $\mu_\eta \in A^1$ be its principal symbol and let $\eta_\lambda$ be its Lie derivative acting on $\lambda$-densities so that $\eta_\lambda \in B^1_\lambda$. Then $Q_\lambda(\mu_\eta) = \eta_\lambda$; this follows by [L-O, §4.3].

The quantization map $Q_\lambda$ defines a star product; see [L-O, §8.2]. For $\phi, \psi \in \mathcal{A}$, we put $\phi \ast_\lambda \psi = Q_\lambda^{-1}(Q_{\lambda t}(\phi)Q_{\lambda t}(\psi))$ where $Q_{\lambda t}$ is the linear map $\mathcal{A} \rightarrow B^\lambda[t]$ such that $Q_{\lambda t}(\phi) = t^d Q_\lambda(\phi)$ if $\phi \in \mathcal{A}^d$. Then $\ast_\lambda$ makes $\mathcal{A}[t]$ into an associative algebra over $\mathbb{C}[t]$. This satisfies

$$\phi \ast_\lambda \psi = \sum_{p=0}^{\infty} C^\lambda_p(\phi, \psi)t^p$$

where $C^\lambda_0(\phi, \psi) = \phi \psi$ and $C^\lambda_1(\phi, \psi) = C^\lambda_1(\psi, \phi) = \{\phi, \psi\}$. Also $C^\lambda_p(\phi, \psi) \in \mathcal{A}^{j+k-p}$ if $\phi \in \mathcal{A}^j$ and $\psi \in \mathcal{A}^k$. So $\ast_\lambda$ is a graded star product on $\mathcal{A}$.

We say that $\ast_\lambda$ has parity iff $C^\lambda_p(\phi, \psi) = (-1)^p C^\lambda_p(\psi, \phi)$; then $C^\lambda_1(\phi, \psi) = \frac{1}{2}\{\phi, \psi\}$.

**Lemma 2.1.** $\ast_\lambda$ has parity iff $\lambda = \frac{1}{2}$.

**Proof.** Let $\beta: B^\lambda \rightarrow B^{1-\lambda}$ be the canonical algebra anti-isomorphism and let $\alpha: \mathcal{A} \rightarrow \mathcal{A}$ be the Poisson algebra anti-involution defined by $\phi^\alpha = (-1)^d \phi$ if $\phi \in \mathcal{A}^d$. Then $Q_\lambda(\phi^\alpha) = Q_{1-\lambda}(\phi)$ by [L-O, Lem. 6.5]. This implies $C^\lambda_p(\phi, \psi) = (-1)^p C^\lambda_{1-\lambda}(\psi, \phi)$. So we have parity if $\lambda = \frac{1}{2}$. Otherwise parity is violated, already for $C^\lambda_1$. Indeed, if $\phi \in \mathcal{A}^0$ and $\mu \in \mathcal{A}^1$, then $\phi \ast_\lambda \mu = \phi \mu + \lambda\{\phi, \mu\}t$, and so $C^\lambda_1(\phi, \mu) = \lambda\{\phi, \mu\}$ while $C^\lambda_1(\mu, \phi) = -C^\lambda_1(\phi, \mu) = (\lambda - 1)\{\phi, \mu\}$. □

### 3. Algebraicity of $\ast_\lambda$

Each $x \in \mathfrak{sl}_{n+1}(\mathbb{R})$ defines a vector field $\eta^x$ on $T^*\mathbb{R}^n$. The principal symbols $\mu^x = \mu_{\eta^x}$ are the momentum functions for $SL_{n+1}(\mathbb{R})$. The $SL_{n+1}(\mathbb{R})$-equivariance of $Q_\lambda$ is equivalent to $\mathfrak{sl}_{n+1}(\mathbb{R})$-equivariance, i.e., $Q_\lambda(\{\mu^x, \phi\}) = [\eta^x, Q_\lambda(\phi)]$. Then $Q_\lambda$ is $\mathfrak{sl}_{n+1}(\mathbb{C})$-equivariant, where we define $\mu^x$ and $\eta^x$ for $x \in \mathfrak{sl}_{n+1}(\mathbb{C})$ by $\mu^{x+iy} = \mu^x + i\mu^y$ and so on.

The algebra $R(T^*\mathbb{C}P^n)$ of regular functions (in the sense of algebraic geometry) on (the quasi-projective complex algebraic variety) $T^*\mathbb{C}P^n$ identifies, by restriction, with a subalgebra $\mathcal{R}$ of $\mathcal{A}$. Similarly the algebra of $\mathcal{D}^\lambda(\mathbb{C}P^n)$ of twisted algebraic (linear)
differential operators for the formal λth power of the canonical bundle \( K \) identifies with a subalgebra \( D^\lambda \) of \( B^\lambda \).

Then \( \mathcal{R} \) is generated by the momentum functions \( \mu^x \), \( D^\lambda \) is generated by the operators \( \eta^x_\lambda \), and \( \text{gr} \, D^\lambda = \mathcal{R} \). These statements follow, for instance, by [Bo-Bi, Lem. 1.4 and Thm. 5.6], since the proofs of the relevant results there generalize immediately to the twisted case. We get natural identifications \( \mathcal{R} = \mathcal{S}/I \) and \( D^\lambda = \mathcal{U}(\mathfrak{g})/J \) where \( I \) is graded Poisson ideal in the symmetric algebra \( \mathcal{S} = \mathcal{S}(\mathfrak{sl}_{n+1}(\mathbb{C})) \), \( J \) is a two-sided ideal in the enveloping algebra \( \mathcal{U} = \mathcal{U}(\mathfrak{sl}_{n+1}(\mathbb{C})) \), and \( \text{gr} \, J = I \).

Notice \( \mathcal{R} \) carries a natural representation of \( SL_n(\mathbb{C}) \), which then extends the \( SL_{n+1}(\mathbb{R}) \)-symmetry it inherits from \( \mathcal{A} \).

**Proposition 3.1.** For every \( \lambda \), \( *_\lambda \) restricts to a graded \( G \)-equivariant star product on the momentum algebra \( \mathcal{R} \).

**Proof.** It suffices to check that \( Q_\lambda \) maps \( \mathcal{R} \) onto \( D^\lambda \) (which is stated for \( \lambda = 0 \) in [L-O, §1.5, Remark (c)]). This follows easily in any number of ways. For instance, the formula for \( Q_\lambda \) in [L-O, (4.15)] implies \( Q_\lambda(\xi_1 \cdots \xi_n) = \frac{\partial^\lambda}{\partial u_1^\lambda} \cdots \frac{\partial^\lambda}{\partial u_n^\lambda} \). But \( \{\xi^d\}_{d=0}^\infty \) is a complete set of lowest weight vectors in \( \mathcal{R} \) and \( D^\lambda \).

**Remark 3.2.** The restriction of \( *_\lambda \) to \( \mathcal{R} \) has parity iff (i) \( \lambda = \frac{1}{2} \) or (ii) \( n = 1 \); see [A-Bi, §3]. Notice that (ii) does not contradict the proof of Lemma 2.1, as \( \mathcal{R}^0 = \mathbb{C} \).

4. SOME SPECIAL VALUES OF \( \phi *_\lambda \psi \)

Pol\(_\infty(T^*\mathbb{R}^n)\) is the tensor product of two maximal Poisson commutative subalgebras, namely the algebra \( \mathbb{C}_\infty[u] = \mathbb{C}_\infty[u_1, \ldots, u_n] \) of smooth functions on the big cell \( \mathbb{R}^n \) and the polynomial algebra \( \mathbb{C}[\xi] = \mathbb{C}[\xi_1, \ldots, \xi_n] \). Let \( E \) be the fiberwise Euler vector field \( \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i} \). Set \( D = \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \).

**Proposition 4.1.** If \( \phi \in \mathbb{C}_\infty[u] \) and \( \psi \in \mathbb{C}[\xi] \) then \( \phi *_\lambda \psi = g_\lambda(\phi \psi) \) where

\[
g_\lambda = 1 + \sum_{d=1}^\infty g_{\lambda,d} D^d t^d \quad \text{and} \quad g_{\lambda,d} = \frac{1}{d!} \prod_{j=0}^{d-1} \frac{-E - j - \lambda(n + 1)}{2E + j + n + 1} \quad (4.1)
\]

**Proof.** Let \( Q_{\text{norm}} : \text{Pol}_\infty(T^*\mathbb{R}^n) \to \mathcal{D}_\infty(\mathbb{R}^n) \) be the normal ordering quantization map. The construction of \( Q_\lambda \) in [L-O] gives \( Q_\lambda = Q_{\text{norm}} h_\lambda \) where \( h_\lambda = 1 + \sum_{d=1}^\infty h_{\lambda,d} D^d t^d \) and \( h_{\lambda,d} \) are certain operators. Here \( \mathcal{D}_\infty(\mathbb{R}^n) \) identifies with \( \mathcal{D}_\infty^\lambda(\mathbb{R}^n) \) in the usual way.

In [L-O, Th. 4.1] they give a very nice formula for the \( h_{\lambda,d} \) when \( \lambda = \frac{1}{2} \). Going back to [L-O, (4.15)], we get a similar formula for all \( \lambda \). We find

\[
h_{\lambda,d} = \frac{1}{d!} \prod_{j=0}^{d-1} \frac{E + j + \lambda(n + 1)}{2E + j + n + d} \quad (4.2)
\]

Thus for \( \phi, \psi \in \text{Pol}_\infty(T^*\mathbb{R}^n) \) we have

\[
\phi *_\lambda \psi = g_\lambda(h_\lambda(\phi) \# h_\lambda(\psi)) \quad (4.3)
\]

where \( \# \) denotes the graded star product defined by \( Q_{\text{norm}} \) and \( g_\lambda = h_\lambda^{-1} \). We find, directly from [L2] or using [L-O (4.10)], that \( g_\lambda \) is given by (1.1).
More succinctly, $C_g p_r T_i \phi \psi = \sum_{p=0}^{\infty} N_p(\phi, \psi) t^p$ where $N_k(\phi, \psi) = \frac{1}{k!} \sum_{\alpha \in \{1, \ldots, n\}^k} \partial^k \phi \partial^k \psi \frac{\partial^k \phi \partial^k \psi}{\partial x_\alpha \partial u_{\alpha}}$. Now, for $\phi \in \mathbb{C}_\infty[u]$ and $\psi \in \mathbb{C}[\xi]$, \((1.3)\) gives $\phi \ast \lambda \psi = g_{\lambda}(\phi \psi)$.

\[ \Box \]

**Corollary 4.2.** None of the operators $C^\lambda_p (p \geq 1, \lambda \in \mathbb{C})$ is bidifferential on $T^* \mathbb{R}^n$, with one exception: $2C_1^\frac{1}{2}$ is the Poisson bracket.

**Proof.** We just showed that $C^\lambda_p(\phi, \psi) = g_{\lambda} p D_p(\phi \psi)$ if $\phi \in \mathbb{C}_\infty[u]$ and $\psi \in \mathbb{C}[\xi]$. This implies, if $C^\lambda_p$ is bidifferential, that $g_{\lambda} p$ is a differential operator on $T^* \mathbb{R}^n$. Looking at our expression for $g_{\lambda} p$, we deduce $E + j + \lambda(n + 1) = E + \frac{1}{2} + \frac{1}{2}(n + 1)$ for $j = 0, \ldots, p - 1$. But this forces $p = 1$ and $\lambda = \frac{1}{2}$. By parity, $C^1_1 \frac{1}{2} \{ \cdot, \cdot \}$.

The corollary contradicts the claim in \cite{LO}, \[8.2\]. They no doubt meant that for each pair $j, k$, the restricted map $C^\lambda_p : \mathcal{A}^j \times \mathcal{A}^k \to \mathcal{A}^{j+k-p}$ is given by some bidifferential operator.

\[ 5. \text{ Coefficients } C^\lambda_p \text{ for } \lambda = \frac{1}{2} \]

In this section, we set $\lambda = \frac{1}{2}$ and suppress the corresponding super(sub)scripts. We put $E' = E + n \frac{1}{2}$ where $E$ is the fiberwise Euler vector field on $T^* \mathbb{R}^n$. See \cite{AB} for an interpretation of the shift $\frac{1}{2}$.

We put $T_p = \prod_{i=1}^{\frac{n}{2}} (E' + i)$ and $S_p = \prod_{i=1}^{\frac{n}{2}} (E' - i)$. These are both invertible on $\mathcal{A}$ if $n$ is odd. Our main result is

**Theorem 5.1.** Assume $n$ is odd and let $p \geq 1$. Then $C_p$ has the form

\begin{equation}
C_p(\phi, \psi) = \frac{1}{T_p} Z_p \left( \frac{1}{S_p} \phi \frac{1}{S_p} \psi \right), \quad \phi, \psi \in \mathcal{A}
\end{equation}

where $Z_p$ is an $\text{SL}_{n+1}(\mathbb{R})$-invariant bidifferential operator on $T^* \mathbb{R}^n$.

$Z_p$ is uniquely determined by \((5.1)\), even if we just take $\phi, \psi \in \mathcal{R}$. Thus $\ast$ is uniquely determined by its restriction to $\mathcal{R}$, once we know that $(\phi, \psi) \mapsto T_p C_p(S_p \phi, S_p \psi)$ is bidifferential.

Finally, $Z_p$, like $E'$, extends uniquely to an $\text{SL}_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operator on $T^* \mathbb{C}^n$.

**Proof.** We return to the proof of Proposition \[(1.1)\]. Let $g_d = g_d d^d$ and $h_d = h_d d^d$, with $g_0 = h_0 = 1$. Writing out \[(1.3)\] termwise, we get, for $p \geq 1$,

\[ C_p(\phi, \psi) = \sum_{i+j+k+m=p} g_m N_k(h_i \otimes h_j) \]

More succinctly, $C_p = \sum_{i+j+k+m=p} g_m N_k(h_i \otimes h_j)$.

For $\lambda = \frac{1}{2}$, the formula \[(1.2)\] simplifies in that $\left[ \frac{d+1}{2} \right]$ factors cancel out. Then $h_d = U_d V^{-1}_d$ where $U_d = \frac{1}{2^n d!} \prod_{i=1}^{\frac{d}{2}} (E' + i - \frac{1}{2})$ and $V_d = \prod_{i=[\frac{d}{2}+1]}^{\frac{d}{2}+1} (E' + i)$. Then $h_d = U_d V^{-1}_d d^{d} = U_d d^{d} S^{-1}_d$. This is a formal relation, valid for $n$ odd since then $S_d$ is invertible. Similarly, \[(1.1)\] gives $g_d = T^{-1}_d F_d d^d$ where $F_d = \frac{1}{2^n d!} \prod_{i=[\frac{d}{2}+1]}^{\frac{d}{2}+1} (-E' - i - \frac{1}{2})$. We put $U_0 = F_0 = 1$.

4
We put $Z_p(\phi, \psi) = T_p C_p(S_p \phi, S_p \psi)$. Let $T_{p;j} = T_p T_j^{-1}$ and $S_{p;j} = S_j^{-1} S_p$. Now (5.2) gives $Z_p = \sum_{i+j+k+m=p} Z^{mkij}$ where
\[
Z^{mkij} = T_{p;m} F_m D_m N_k \left( U_i D_i S_{p;i} \otimes U_j D_j S_{p;j} \right)
\]
Each $Z^{mkij}$, and so also their sum $Z_p$, is a bidifferential operator on $T^* \mathbb{R}^n$ with polynomial coefficients. I.e., $Z_p$ lies in $\mathcal{E} \otimes_p \mathcal{E}$ where $\mathcal{E} = \mathbb{C}[u_i, \xi_j, \frac{\partial}{\partial u_i}, \frac{\partial}{\partial \xi_j}]$ and $\mathcal{P} = \mathbb{C}[u_i, \xi_j]$.

Now $Z_p$ is invariant under $\mathfrak{sl}_{n+1}(\mathbb{R})$; this is clear since $T_p, C_p$ and $S_p$ are all invariant. It follows by projective geometry (as in [L-O, §8.1]) that $Z_p$ extends uniquely to a global $SL_{n+1}(\mathbb{R})$-invariant bidifferential operator on $T^* \mathbb{R}^n$.

We have $\{C_p(\phi, \psi) \mid \phi, \psi \in \mathcal{R}\} \to \{Z_p(\phi, \psi) \mid \phi, \psi \in \mathcal{R}\} \to \{Z_p(\phi, \psi) \mid \phi, \psi \in \mathcal{A}\}$ where the arrows indicate that one set of values completely determines the next set. The middle arrow follows because any bidifferential operator on $T^* \mathbb{R}^n$ is completely determined by its values on $\mathcal{R}$ ([E, Lemma 5.1]).

Clearly $Z_p$ extends naturally (and uniquely) to an algebraic differential operator $\tilde{Z}_p$ on $T^* \mathbb{C}^n$; this amounts to replacing our Darboux coordinates $u_i, \xi_j$ by their holomorphic counterparts $z_i, \zeta_j$. Then $\tilde{Z}_p$ is $\mathfrak{sl}_{n+1}(\mathbb{C})$-invariant and (by projective geometry again) extends to $T^* \mathbb{C}^n$.

Notice that this proof gives an explicit formula (in the coordinates $u_i, \xi_j$) for $Z_p$.

**Remarks 5.2.** (i) Suppose $n$ is even. Then this proof still shows that the formula $Z_p(\phi, \psi) = T_p C_p(S_p \phi, S_p \psi)$ defines an operator $Z_p$ in $\mathcal{E} \otimes_p \mathcal{E}$. Then (5.1) is valid as long as $\phi$ and $\psi$ lie in $\mathcal{A}^* = \oplus_{d=0}^{\infty} [\xi]_{d+n+1} \mathcal{A}^d$. We can show that all the other results in Theorem [5.1] are still true, so that (5.1) determines $Z_p$ uniquely even for $\phi, \psi \in \mathcal{R} \cap \mathcal{A}^*$.

(ii) The maps $Q_{\text{norm}}$ and $h_{\lambda}$ are equivariant with respect to only a parabolic subgroup $P$ of $SL_{n+1}(\mathbb{R})$, even though their product $Q_{\lambda} = Q_{\text{norm}} h_{\lambda}$ is equivariant for $SL_{n+1}(\mathbb{R})$. Here $P$ is the subgroup of the affine transformations of $\mathbb{R}^n$ (i.e., the one which fixes the subspace $(u_0 = 0)$ in $\mathbb{R}^n$). Our formula (5.1) is manifestly equivariant for $SL_{n+1}(\mathbb{R})$.

6. **Operators $C_p(\phi, \cdot)$ for $\lambda = \frac{1}{2}$**

Next we recover part of the results found for $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ in [A-B1], Prop. 4.2.3 and [A-B2], Thm. 6.3 and Cor. 8.2.

**Corollary 6.1.** Let $n \geq 1$. For any momentum function $\mu^x$, $x \in \mathfrak{sl}_{n+1}(\mathbb{C})$, we have
\[
C_2(\mu^x, \psi) = \frac{1}{E'(E'+1)} L^x(\psi), \quad \psi \in \mathcal{A}
\] (6.1)

where $L^x$ is an order 4 differential operator on $T^* \mathbb{R}^n$.

Neither $E'$ nor $E'+1$ left divides $L^x (x \neq 0)$ over $T^*U$ for any open set $U$ in $\mathbb{R}^n$.

Hence $C_2(\mu^x, \cdot)$ is not a differential operator on $T^*U$.

Finally, $L^x$ extends uniquely to an algebraic differential operator on $T^* \mathbb{C}^n$.

**Proof.** Suppose $n$ is odd. For $\psi \in \mathcal{A}$, (5.1) gives
\[
C_2(\mu^x, \psi) = \frac{1}{E'+1} Z_2 \left( \frac{1}{E'-1} \mu^x, \frac{1}{E'-1} \psi \right) = \frac{2}{nE'(E'+1)} Z_2(\mu^x, \psi)
\] (6.2)

The last equality follows because the operator $Z_2(\mu^x, \cdot)$ is graded of degree $-1$. 


For \( n \) even, (5.2) is still true on account of Remark 5.2(i), except in the case where \( n = 2 \) and \( \psi \notin \bigoplus_{d=1}^{\infty} \mathcal{A}^d \). But if \( \psi \in \mathcal{A}^0 \) then both \( C_2(\mu^x, \psi) \) and \( Z_2(\mu^x, \psi) \) vanish for degree reasons and so the first and third expressions in (5.2) are still equal.

This proves (6.1), for all \( n \), where \( L^x = \frac{2}{n} Z_2(\mu^x, \cdot) \). Then \( L^x \) extends to an algebraic differential operator on \( T^* \mathbb{C}P^n \); this follows since both \( Z_2 \) and \( \mu^x \) so extend.

The \( L^x \), for \( x \neq 0 \), all have the same order. This follows because the \( L^x \), like the \( \mu^x \), transform in the adjoint representation of \( SL_{n+1}(\mathbb{C}) \). We can choose \( \mu^x = \xi_m \) (the choice of \( m \in \{1, \ldots, n\} \) is arbitrary). Let \( L^{(m)} \) be the corresponding operator \( L^x \).

Using (5.2) we find after some calculation

\[
C_2(\cdot, \xi_m) = -\frac{1}{16 E'} \frac{1}{(E' + 1)} \xi_m D^2 + \frac{1}{8 E'} \frac{1}{1 \partial u_m} D
\]

So \( L^{(m)} = -\frac{1}{16} (\xi_m D - 2 E' \frac{\partial}{\partial u_m}) D \). Clearly \( L^{(m)} \) has order 4. Using principal symbols, we see that \( L^{(m)} \) has no left factors of the form \( E' + c \) if \( n \geq 2 \). For \( n = 1 \), (5.3) gives \( L^{(m)} = \frac{1}{16} (E' + \frac{1}{2}) \frac{\partial^3}{\partial \xi_1^3} \), and so the only such factor is \( E' + \frac{1}{2} \).

**Corollary 6.2.** Assume \( n \) is odd and let \( p \geq 1 \). If \( \phi \in \mathcal{A}^d \) then

\[
C_p(\phi, \cdot) = \frac{1}{\prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (E' + i)(E' - i + p - d)} L_p^\phi
\]

where \( L_p^\phi \) is a differential operator on \( T^* \mathbb{R}P^n \). If \( \phi \in \mathcal{R} \), then \( L_p^\phi \) is an algebraic differential operator on \( T^* \mathbb{C}P^n \).

**Proof.** This follows because \( L_p^\phi = Z_p(S_p^{-1} \phi, \cdot) \).

**References**

[A-B1] A. Astashkevich, R. Brylinski, *Exotic Differential Operators on Complex Minimal Nilpotent Orbits*, Advances in Geometry, Progress in Mathematics, Vol. 172, Birkhauser, 1998, 19–51.

[A-B2] A. Astashkevich and R. Brylinski, *Non-Local equivariant star product on the minimal nilpotent orbit*, posted at http://front.math.ucdavis.edu on QA, SG, RT.

[A-B3] A. Astashkevich, R. Brylinski, *Geometric quantization of classical complex minimal nilpotent orbits*, in preparation.

[B] R. Brylinski, *Equivariant Deformation Quantization for the Cotangent Bundle of a Flag Manifold*, posted at http://front.math.ucdavis.edu on QA, SG, RT.

[Bo-Br] W. Borho and J-L. Brylinski, *Differential operators on homogeneous spaces I. Irreducibility of the associated variety for annihilators of induced modules*, Invent. Math. 69 (1982), 437–476.

[D-L-O] C. Duval, P. Lecomte and V. Ovsienko, *Methods of equivariant quantization*, in Noncommutative Differential Geometry and its Applications to Physics, Shonan-Kokusaimura, Japan, Kluwer, 1999

[L-O] P. B. A. Lecomte and V. Yu. Ovsienko, *Projectively equivariant symbol calculus*, Letters in Math. Phys. 49 (1999), 173–196

[L-S] T. Levasseur and J.T. Stafford, *Differential operators on some nilpotent orbits*, Rep. Theory 3 (1999), 457–473

Department of Mathematics, Penn State University, University Park 16802

E-mail address: rkb@math.psu.edu

URL: www.math.psu.edu/rkb

6