Perturbative matching of staggered four-fermion operators with hypercubic fat links

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We calculate the one-loop matching coefficients between continuum and lattice four-fermion operators for lattice operators constructed using staggered fermions and improved by the use of fattened links. In particular, we consider hypercubic fat links and SU(3) projected Fat-7 links, and their mean-field improved versions. We calculate only current-current diagrams, so that our results apply for operators whose flavor structure does not allow “eye-diagrams”. We present general formulae, based on two independent approaches, and give numerical results for the cases in which the operators have the taste (staggered flavor) of the pseudo-Goldstone pion. We find that the one-loop corrections are reduced down to the 10-20% level, resolving the problem of large perturbative corrections for staggered fermion calculations of matrix elements.

I. INTRODUCTION

It has recently become clear that improving staggered fermions through the use of fattened links greatly reduces both the breaking of taste symmetry and the size of one-loop matching coefficients. For example, two of us (WL and SS) calculated the one-loop matching corrections for all bilinear operators constructed on a hypercube, and found them to be reduced by fattening from as large as ~ 50% to the ~ 10% level for various choices of fattening. The greatest reduction is for “Fat-7” and hypercubic fat (HYP) links, the latter after mean-field improvement. Based on this, and on the result that HYP links lead to a greater reduction in taste symmetry breaking in the pion multiplet than other local fattening choices, we are undertaking numerical calculations of weak matrix elements using staggered fermions with HYP fattened links. The weak matrix elements include those relevant for predicting CP violation in the kaon system. These matrix elements involve four-fermion operators, and an essential adjunct to the numerical calculations is a determination of the one-loop matching coefficients for such operators. In this paper we present the results for these matching coefficients for operators, constructed using HYP links, in which only “current-current” diagrams contribute. This completes one-loop calculation for the operators relevant for $B_K$ and the $\Delta I = 3/2$ component of the $K \to \pi\pi$ amplitudes. For the most general four-fermion operator, one also needs to calculate the contribution of “penguin” diagrams, results of which we hope to present soon.

Calculations with unimproved staggered fermions suggest that one-loop corrections for four-fermion operators will be of similar size to those for the bilinears from which they are constructed. For example, for gauge-invariant unimproved four-fermion operators, the corrections range up to 100%, roughly twice those for the corresponding bilinears. Our results confirm this expectation for improved operators. One-loop contributions to the matching coefficients are at the 10% level for nearly all operators, small enough that the systematic error in results for matrix elements due to the missing higher loop contributions will likely be smaller than those from other sources, at least in the next few years.

This paper is organized as follows. In Sec. II we present the action and composite operators made of staggered fermions along with fat links, and discuss the Feynman rules that follow. In Sec. III we discuss the calculation of the renormalization of the lattice operators, using two independent methods. General results are given in four appendices, and the specific numerical values of renormalization constants for operators of particular interest are given in Tables I–XVIII. Matching with continuum operators is discussed in Sec. IV and we close with some conclusions in Sec. V. Some preliminary results from this paper were presented in Ref. [7]. This work is done as a part of the staggered $\epsilon'/\epsilon$ project [8].

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1 We adopt the recently proposed name for what was previously known as “staggered flavor” symmetry. We also refer to “fattened” rather than “smeared” links, in order to distinguish our operators from the smeared operators introduced in Ref. [9], which involved smearing of the fermion fields.

2 See Ref. [5] for an extensive study of the properties of fattened links for other quantities.
II. ACTION, OPERATORS AND FEYNMAN RULES

The calculations required when using fattened links are a straightforward extension of those needed for unimproved (“naive”) staggered fermions. In particular, Feynman rules for the latter, in the notation we use here, can be found in Refs. [1, 8, 9, 10], while matching factors for four-fermion operators with naive staggered fermions are calculated in Refs. [2, 5]. Furthermore, many of the additional features introduced by fattening the links have been presented in the calculation of matching factors for bilinears [2, 3, 5]. In view of this, we give only a bare-bones summary of the action, operators and Feynman rules, emphasizing those features special to the present calculation.

The fermion action has the standard staggered form

\[ S = a^4 \sum_n \left[ \frac{1}{2 a} \sum_\mu \eta_\mu(n) \left( \tilde{\chi}(n)V_\mu(n)\chi(n) + \tilde{\chi}(n+\mu)V_\mu^\dagger(n)\chi(n) \right) + m\tilde{\chi}(n)\chi(n) \right], \quad (1) \]

where \( n = (n_1, n_2, n_3, n_4) \) is the lattice coordinate and \( \eta_\mu(n) = (-1)^{n_1+\cdots+n_\mu-1} \), except that original “thin” links \( U_\mu \) are replaced with fattened links \( V_\mu \). Note, however, that we continue to use the Wilson plaquette gauge action constructed out of the thin links.

We use HYP links for the \( V_\mu \). HYP links are defined by three stages of fattening, and at each stage one must choose a fattening parameter. We consider two choices of fattening parameters: those determined by Hasenfratz and Knechtli using a non-perturbative optimization [4]; and the values which arise when implementing the Symanzik improvement program [4, 5]. It turns out that the latter choice is equivalent, in one-loop calculations, to using Fat-7 links if, in addition, one projects the links back into \( SU(3) \) (which we call \( \text{Fat7 links} \)) [11].

The key features of HYP links are (i) that they have reduced coupling to the high-momentum gluons which lead to taste-breaking transitions; (ii) that they are local in the sense of involving only gauge links contained in hypercubes connected to the original link; and (iii) that they lead to (or, in the case of Fat7 links, are conjectured to lead to) a large reduction in taste symmetry breaking in the spectrum. Several useful properties of HYP and Fat7 links have been discussed recently by one of us [15].

The detailed definitions of \( V_\mu \) are given in the original references and need not be repeated here. What is important for us are the following properties shared by both choices of fattened links. The usual definition of gauge fields

\[ U_\mu(x) = \exp \left(i a A_\mu(x+\frac{\mu}{2})\right) \quad (2) \]

can be extended to the fattened fields, since the latter live in \( SU(3) \):

\[ V_\mu(x) = \exp \left(i a B_\mu(x+\frac{\mu}{2})\right). \quad (3) \]

The “blocked gauge fields” \( B_\mu \) can be expressed in terms of the usual gauge fields as

\[ B_\mu = \sum_{n=1}^\infty B_\mu^{(n)}, \quad (4) \]

where \( B^{(n)} \) contains all terms with \( n \) factors of \( A \). In our one-loop calculation we need only \( B_\mu^{(1)} \). Although \( B_\mu^{(2)} \) does enter in one-loop “tadpole” diagrams, these contributions vanish, since it follows from the \( SU(3) \) projection that \( B_\mu^{(2)} \) has the form of a commutator [1, 15, 16]. Thus all we need to know is the coefficient in the linear relation (written, for later convenience, in momentum space)

\[ \tilde{B}_\mu^{(1)}(k) = \sum_\nu h_{\mu\nu}(k) \tilde{A}_\nu(k). \quad (5) \]

The end result [15] is that, to generalize the one-loop calculations from those for naive staggered fermions (where \( V_\mu \to U_\mu \)), we need only replace the gauge-field propagator with

\[ \langle \tilde{B}_\mu^{(1)}(k) \tilde{B}_\nu^{(1)}(c)(-k) \rangle = \sum_{\rho,\sigma} \langle h_{\mu\rho}(k) h_{\nu\sigma}(-k) \rangle \langle \tilde{A}_\rho^{(1)}(k) \tilde{A}_\sigma^{(1)}(c)(-k) \rangle = \delta^{bc} \sum_\rho \langle h_{\mu\rho}(k) h_{\nu\rho}(-k) \rangle \left[ \sum_\beta \frac{4}{a^2} \sin^2 \left(\frac{1}{2} \beta k_\beta\right) \right]^{-1}, \quad (6) \]
where \( b, c \) are color indices, and the last line holds only in Feynman gauge, which we use in our calculations. The simplicity of perturbation theory with HYP or SU(3) projected links has also been emphasized in Ref. 3.

A convenient general form for \( h_{\mu\nu}(k) \) is 2

\[
\begin{align*}
\delta_{\mu\nu} D_{\mu}(k) &+ (1 - \delta_{\mu\nu}) G_{\mu\nu}(k) \\
D_{\mu}(k) &= 1 - d_1 \sum_{\nu \neq \mu} \bar{s}_\nu^2 + d_2 \sum_{\nu, \rho \neq \mu} \bar{s}_\nu^2 \bar{s}_\rho^2 - d_3 \bar{s}_\nu^2 \bar{s}_\rho^2 \bar{s}_\sigma^2 - d_4 \sum_{\nu \neq \mu} \bar{s}_\nu^4 \\
G_{\mu\nu}(k) &= \bar{s}_\mu \bar{s}_\nu \tilde{G}_{\mu\nu}(k) \\
\tilde{G}_{\mu\nu}(k) &= d_1 - d_2 \left( \frac{\bar{s}_\mu^2 + \bar{s}_\nu^2}{2} \right) + d_3 \frac{\bar{s}_\mu^2 \bar{s}_\nu^2}{3} + d_4 \bar{s}_\mu^2.
\end{align*}
\]  

Here, the coefficients \( d_i \) distinguish different choices of fat links.\(^3\) Note that \( h_{\mu\nu}(-k) = h_{\mu\nu}(k) \).

We consider here the following choices of coefficients:

(i) Unimproved links (naive staggered fermions):

\[
d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0.
\]  

(ii) HYP links:

\[
d_1 = (2/3)\alpha_1 (1 + \alpha_2 (1 + \alpha_3)), \quad d_2 = (4/3)\alpha_1 \alpha_2 (1 + 2\alpha_3), \quad d_3 = 8\alpha_1 \alpha_2 \alpha_3, \quad d_4 = 0,
\]  

where \( \alpha_1-3 \) are fattening parameters. One choice was determined in Ref. 4 using a non-perturbative optimization procedure: \( \alpha_1 = 0.75, \quad \alpha_2 = 0.6 \quad \alpha_3 = 0.3 \). This gives

\[
d_1 = 0.89, \quad d_2 = 0.96, \quad d_3 = 1.08, \quad d_4 = 0,
\]  

and we call these links “HYP(I)”.

(iii) HYP links with one-loop Symanzik-improved coefficients, i.e. coefficients chosen to remove \( O(a^2) \) taste symmetry breaking couplings at tree level. This choice gives

\[
d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 0.
\]  

These turn out to be identical to the coefficients for Fat-7 links, and thus we call these links “HYP(II)/Fat7”.

We now turn to the four-fermion operators. We construct these from standard hypercube bilinears 17, in which the spin and tastes of the four continuum fermions are spread over a hypercube. It is useful to recall the form of the gauge-invariant bilinears:

\[
[S \times F](y) = \frac{1}{N_f} \sum_{A,B} [\bar{\chi}_b(y + A)(\gamma_S \otimes \xi_F)_{AB} \chi_c(y + B)] \gamma_{bc}(y + A, y + B). \]  

where \( y \) denotes the particular \( 2^4 \) hypercube, and \( A, B \) are “hypercube vectors” denoting the positions within the hypercube. The normalization factor is chosen so that this operator goes over, in the continuum limit, to a continuum bilinear with standard normalization. The matrices \((\gamma_S \otimes \xi_F)_{AB}\) are standard; see, e.g., Refs. 2 4. The spin \((S)\) and taste \((F)\) of the bilinear can each be scalar, \( S \), vector, \( V_{\mu} \), tensor, \( T_{\mu\nu} \), axial vector, \( A_{\mu} \), or pseudoscalar, \( P \).

The only new feature of these operators compared to those used with unimproved staggered fermions lies in the links used to make them gauge invariant. The factor \( \gamma_{bc}(y + A, y + B) \) is constructed by averaging over all of the shortest paths between \( y + A \) and \( y + B \), and for each path forming the product of the fattened gauge links \( V_{\mu} \). When constructing the operators, we use the same fattened links as in the action, ensuring the conservation of the current \([V \times S]\). These are the operators whose one-loop matching factors were found to be small in Ref. 2.

\(^3\) We include \( d_4 \) for completeness although it is not needed in our present calculations. It is non-zero for fully \( O(a^2) \) one-loop improved staggered fermions 14.
For four-fermion operators we need to distinguish the two ways of joining color indices to make gauge invariant operators:

\[ [S \times F][S' \times F']_I(y) \equiv \frac{1}{N_f^2} \sum_{A,B,C,D} \chi^{(1)}_y (y + A) (\gamma_s \otimes \xi_F)_{AB} \chi^{(2)}_y (y + B) [\chi^{(3)}_y (y + C) (\gamma_s' \otimes \xi_F')_{CD} \chi^{(4)}_y (y + D)] \]

\[ [S \times F][S' \times F']_{II}(y) \equiv \frac{1}{N_f^2} \sum_{A,B,C,D} \chi^{(1)}_y (y + A) (\gamma_s \otimes \xi_F)_{AB} \chi^{(2)}_y (y + B) [\chi^{(3)}_y (y + C) (\gamma_s' \otimes \xi_F')_{CD} \chi^{(4)}_y (y + D)] \]

The subscripts \( I \) and \( II \) indicate the number of color traces resulting if each bilinear is contracted with a different external operator. We refer to these two types of operator as one-color-trace and two-color-trace, respectively. The superscripts \((1-4)\) label different flavors (not tastes) of staggered fermions—choosing them all different, as we do here, forbids “penguin” diagrams.

Because staggered fermions represent four tastes, there is considerable redundancy in the transcription of continuum operators onto the lattice. The choice we have made—the so-called “two-spin-trace” operators, to be distinguished from the one-spin-trace operators of Ref. [10]—is that made in most previous work on staggered weak matrix element calculations [9, 18], and in our present numerical studies [7].

For bilinear operators with HYP links, a further reduction in the size of one-loop matching contributions was achieved using mean-field (or “tadpole”) improvement [2]. Mean-field improvement for HYP links is completely analogous to that for the usual links [19], which was implemented for staggered fermions in Refs. [1, 6, 12]. Links are rescaled and the fields are renormalized as follows:

\[ \chi \rightarrow \psi = \sqrt{u_0} \chi \quad \bar{\chi} \rightarrow \bar{\psi} = \sqrt{u_0} \bar{\chi} \quad V_{\mu} \rightarrow \hat{V}_\mu = \frac{V_{\mu}}{u_0} . \]

The mean-field scaling factor \( u_0 \) is determined from the plaquette composed of fattened links

\[ u_0 = \left[ \frac{1}{3} \text{Re} \langle \text{Tr} V_{\text{Plaq}} \rangle \right]^{1/4} = 1 - \frac{g^2}{(4\pi)^2} C_F I_{MF} + O(g^4) \]

The integral \( I_{MF} \) was called \( T^c_{\Delta=2} \) in Ref. [2], and is given by

\[ I_{MF} = \int_k B \bar{s}_2 \sum_\alpha h_{1\alpha}(k)h_{1\alpha}(k) - \bar{s}_1 \sum_\alpha h_{1\alpha}(k)h_{2\alpha}(k) . \]

The numerical values are \( I_{MF} = 0.57826(1) \) for HYP(I) links and \( I_{MF} = 1.05382(3) \) for HYP(II)/Fat7 links. These are substantially smaller than the corresponding factor for the thin link, \( \pi^2 \), but are nevertheless significant.

### III. Renormalization of the Lattice Operators

The different types of one-loop diagrams leading to the renormalization of lattice four-fermion operators with distinct flavors are shown in Figs. [1] and [2]. These are generically referred to as “current-current” diagrams. Note that the non-locality of the lattice operators leads to many diagrams in which one or two gluons emanate from the operators themselves. One of the aims of fattening is to reduce their contribution, thus making the lattice operators more “continuum-like”.

The one-loop matrix elements are in general infrared divergent. Following common practice, we regularize this divergence by introducing a gluon mass \( \lambda \), i.e. we add \( \lambda^2 \) to the denominator of the gluon propagator in Eq. (6). This allows us to set both the quark masses and the external momenta to zero.

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4 We do not consider operators made gauge invariant by fixing to Landau gauge and leaving out links, since, when one considers penguin diagrams, these mix with lower dimension operators, requiring additional non-perturbative subtractions [13]. This likely makes them impractical for general studies of matrix elements.
FIG. 1: Examples of the different types of one-loop diagrams contributing to matching factors for “two-color trace” operators. The boxes indicate how taste indices are contracted, and the dashed lines indicate how color indices are contracted.

It is useful to label operators with a vector notation \( \vec{\mathcal{O}} \):
\[
\vec{\mathcal{O}}_{Latt}^i = \begin{pmatrix} \mathcal{O}_{i,Latt}^i \\ \mathcal{O}_{i,Latt}^{I,I} \end{pmatrix},
\]
where \( i \) runs over the 16\(^4\) different choices for \( S, S', F \) and \( F' \). This notation reflects the factorization of color factors from the spin-taste part of the diagram, and is convenient for expressing the results in a relatively compact way.

The general form of the one-loop matrix element of the four-fermion operators \( \vec{\mathcal{O}}_{Latt}^i \) can be written
\[
\langle \vec{\mathcal{O}}_{Latt}^i \rangle^{(1)} = \left\{ \delta_{ij} + \frac{g^2}{(4\pi)^2} \left[ \tilde{\gamma}_{ij} \log(a\lambda) + \tilde{c}_{ij} \right] \right\} \langle \vec{\mathcal{O}}_{Latt}^j \rangle^{(0)} + O(a)
\]
(22)
where $\langle \tilde{\mathcal{O}}_{\text{latt}}^{\text{L}(0)} \rangle$ is the tree level matrix element of the $j$'th operator, $\hat{\gamma}_{ij}$ is the anomalous dimension matrix\(^5\) and $\hat{C}_{ij}$ is the finite part of the correction. Both $\hat{\gamma}$ and $\hat{C}$ are matrices in the two-dimensional “color-index” space, as indicated by their “hats”, as well as in the explicit indices $i$ and $j$.

Note that, although $\hat{C}$ is a very large matrix $[(16^4) \times (16^4)$ in its $i$, $j$ indices], we are actually interested only in small blocks within it. This will be discussed explicitly in the following section. Nevertheless, we quote the general results since it takes no more work to do so.

The implementation of mean-field improvement for fattened link operators follows exactly the same steps as for those with unfattened links. The details are given in Ref. \cite{Ref}, and we do not repeat them here. The one-loop finite

\footnote{The anomalous dimension matrix ($\hat{\gamma}$) is related to the matrix ($\hat{\Gamma}$) of Ref. \cite{Ref} by $\hat{\Gamma} = -\hat{\gamma}$.}

FIG. 2: Examples of the different types of one-loop diagrams contributing to matching factors for “one-color trace” operators. The boxes indicate how taste indices are contracted, and the dashed lines indicate how color indices are contracted.
parts are shifted according to

\[ \left( \hat{C}_{ij}^{\text{Latt}} \right)^{MF} = \hat{C}_{ij}^{\text{Latt}} + C_F I_{MF} \tilde{T}_{ij}, \]  

(23)

where \( \tilde{T}_{ij} \) are numerical factors which are given below for the cases of interest, and \( I_{MF} \) is given in Eq. (20).

In the Appendices we give the expressions for the contributions to \( \hat{C}_{ij}^{\text{Latt}} \) for general \( i \) and \( j \). We have calculated these using two independent methods. The first, results of which are presented in Appendices [A, [C] is a generalization of that used in Ref. [6, 10]. The second makes maximal use of previous calculations with bilinears, reducing the number of additional integrals required. Results using this method are presented in Appendix D for mixing in which all bilinears (before and after mixing) have the same taste \( F \), which are the cases of interest for our numerical calculations. It is a non-trivial check of our results that these two methods give results consistent within the errors of the numerical evaluation of integrals.

We have done two further checks of our results for \( \hat{C}_{ij}^{\text{Latt}} \). First, we have checked that they are consistent with the \( U(1)_A \) symmetry, which implies that the renormalization of the following operators should be identical:

\[ [S \times F][S' \times F'], \quad [S \times F][S'5 \times F'5], \quad [S5 \times F5][S' \times F'], \quad [S5 \times F5][S'5 \times F'5]. \]  

(24)

Second, we have checked the Fierz identities explained in Appendices A and B of Ref. [6]. Using these identities, any one-color-trace operator can be represented as a linear combination of two-color-trace operators and vice versa, leading to many non-trivial relations between renormalization constants.

In Tables [IX] we give the numerical values for \( \hat{C}_{ij}^{\text{Latt}} \), and the mean-field factors \( \tilde{T}_{ij} \) for the following five operators:

\[
\begin{align*}
\bar{O}_1^{\text{Latt}} &= [V_\mu \times P][V_\mu \times P] + [A_\mu \times P][A_\mu \times P] \\
\bar{O}_2^{\text{Latt}} &= [V_\mu \times P][V_\mu \times P] - [A_\mu \times P][A_\mu \times P] \\
\bar{O}_3^{\text{Latt}} &= -2\left([S \times P][S \times P] - [P \times P][P \times P]\right) \\
\bar{O}_4^{\text{Latt}} &= [S \times P][S \times P] + [P \times P][P \times P] \\
\bar{O}_5^{\text{Latt}} &= -\frac{1}{2}\left([S \times P][S \times P] + [P \times P][P \times P] - \sum_{\mu < \nu}[T_{\mu \nu} \times P][T_{\mu \nu} \times P]\right). 
\end{align*}
\]

(25)

Here the color-trace indices are omitted on the right-hand-side. The first three operators \( (\bar{O}_i^{\text{Latt}}, i = 1, 2, 3) \) contribute to CP-violating transitions of kaons in the standard model; the remaining two contribute in models of physics beyond the standard model.

The impact of fattening the links can be seen clearly from the table. Consider first matching between operators with different tastes (i.e. coefficients in the tables with tastes other than \( P \)). Since taste is conserved in the continuum, these coefficients are the full matching factors. They indicate the size of taste-breaking due to the lattice action. As the tables show, these coefficients are typically reduced by an order of magnitude when the links are fattened. Taste-conserving coefficients are also reduced substantially, but, since here there are additional contributions to the matching factors, we postpone further discussion until the following section.

IV. MATCHING WITH CONTINUUM OPERATORS

The continuum operators we wish to match to are:

\[
\begin{align*}
\mathcal{O}_1 &= (\bar{\psi}_1 \gamma_\mu \psi_2)(\bar{\psi}_3 \gamma_\mu \gamma_5 \psi_4) \\
\mathcal{O}_2 &= (\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2)(\bar{\psi}_3 \gamma_\mu \psi_4) \\
\mathcal{O}_3 &= -2(\bar{\psi}_1 \psi_2)(\bar{\psi}_3 \gamma_5 \psi_4) \\
\mathcal{O}_4 &= (\bar{\psi}_1 \gamma_5 \psi_2)(\bar{\psi}_3 \psi_4) \\
\mathcal{O}_5 &= -\frac{1}{8}\left(1 + \frac{3}{4} \epsilon\right) \sum_{\mu, \nu}(\bar{\psi}_1 \gamma_\mu \gamma_\nu \gamma_5 \psi_2)(\bar{\psi}_3 \gamma_\nu \gamma_5 \gamma_\mu \psi_4). 
\end{align*}
\]

(26)

Here \( L, R = 1 \pm \gamma_5 \), and color indices are not shown. As on the lattice, the operators come in color types I and II. Matrix elements of the operators are regularized using the \( \overline{\text{MS}} \) scheme, with \( \epsilon = (4 - d)/2 \). The particular basis
we have chosen is the “practical basis” of Ref. [20], in which the peculiar factors multiplying \( O_5 \) are required to maintain four-dimensional Fierz relations among renormalized operators. For more discussion on this point, and for the conditions which fully define the \( \overline{\text{MS}} \) scheme, see Ref. [20].

Our aim is to find the lattice operators which, at one-loop level, match onto the positive parity parts of these operators. To do this we must face the fact that our lattice theory has the additional taste degree of freedom. We do so in the manner laid out in Refs. [18, 21, 22, 23]. First, we divide each quark loop by a factor of four, corresponding to taking the fourth-root of the determinant in simulations with dynamical quarks. This step is not needed here, since we do not consider penguin contractions or two-loop diagrams. Second, we match lattice matrix elements with particular external tastes to continuum matrix elements. To do this we must divide out by appropriate taste factors in the lattice matrix elements. To do this we must divide out by appropriate taste factors in the lattice matrix elements. To do this we must divide out by appropriate taste factors in the lattice matrix elements. To do this we must divide out by appropriate taste factors in the lattice matrix elements. To do this we must divide out by appropriate taste factors in the lattice matrix elements.

In this second stage, there is considerable redundancy corresponding to the freedom to pick different tastes. We have resolved this redundancy by choosing operators with the taste of the lattice pseudo-Goldstone bosons.

A useful way of thinking about this matching is the following three stage process. First, we match lattice operators with typical lattice spacings \((a \sim 0.1 \text{ fm})\) onto lattice operators with infinitessimal lattice spacing \((a \to 0)\). This can be done at the level of operators, and is straightforward in principle, aside from potential theoretical issues arising from taking the fourth-root of the determinant. Second, we match onto continuum operators defined in our \( \overline{\text{MS}} \) scheme at an extremely high scale \((\mu \to \infty)\). This matching must be done between matrix elements (or contractions in general), but it is straightforward to remove the taste factors since one can work at tree level. Third, we match these very high scale continuum operators onto those at a typical scale, \( \mu \sim 2 \text{ GeV} \). This is again straightforward. This three stage process is an adaptation of that laid out by Ji for normal matching factors [24]. We implement it here at the one-loop level.

To carry out the third stage of this process we need the one-loop renormalization of the continuum operators defined above. This has been worked out in Ref. [6, 20], and we recall the essential results here. Using the same vector notation for the different choices of color indices defined in Eq. (21), the renormalized one-loop matrix elements are

\[
\langle \bar{O}_i \rangle_{\text{Cont}(1)} = \langle \bar{O}_i \rangle_{\text{Cont}(0)} + \frac{g^2}{(4\pi)^2} \left[ \gamma_{ij} \log \left( \frac{\lambda}{\mu} \right) + \tilde{C}^\text{Cont}_{ij} \right] \langle \bar{O}_j \rangle_{\text{Cont}(0)}.
\]  

The finite constants can be written as follows:

\[
\tilde{C}^\text{Cont}_{ij} = \left( \begin{array}{cc} -1/6 & \frac{1}{3} \\ 0 & 4/3 \end{array} \right) \otimes \mathcal{M}_a^i + \left( \begin{array}{cc} 4/3 & 0 \\ 1/2 & -1/6 \end{array} \right) \otimes \mathcal{M}_b^i + \left( \begin{array}{cc} -1/6 & 1/2 \\ 1/2 & -1/6 \end{array} \right) \otimes \mathcal{M}_c^i,
\]

where the first matrix in each tensor product acts on the color-trace indices. The mixing matrices which act on the operator indices are

\[
\mathcal{M}_a = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -2 \end{array} \right),
\]

\[
\mathcal{M}_b = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array} \right),
\]

\[
\mathcal{M}_c = \left( \begin{array}{cccc} -11 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -5 \end{array} \right).
\]

We see that the continuum finite constants are typically of magnitude \( \sim 5 \).

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6 We could equally well consider the negative parity parts. Matching to these is accomplished by considering lattice operators obtained by multiplying by appropriate factors of \( \gamma_5 \times \xi_5 \).
7 When we consider operators in which two or more of the four flavors are the same, then, in general, we have to match each contraction of the continuum operator onto a particular matrix element of a different lattice operator.
The matching coefficients are obtained by equating the one-loop matrix elements calculated using the continuum and lattice regularizations, after appropriate taste factors have been divided out of the latter. Although we actually match matrix elements, we present the results in the form of an operator matching, since this is more familiar. The result is:

\[
\hat{\mathcal{C}}^\text{Cont}_i = \sum_j \left[ \delta_{ij} - \frac{g^2}{(4\pi)^2} \hat{c}_{ij} \log(\mu a) + \frac{g^2}{(4\pi)^2} \hat{c}_{ij} \right] \hat{\mathcal{C}}^\text{Latt}_j
\]

(32)

\[
\hat{c}_{ij} = \left( \hat{C}^\text{Cont}_{ij} - \hat{C}^\text{Latt}_{ij} \right)
\]

(33)

The numerical values of matching coefficients can be obtained, for taste \(\xi_5\), by combining the result of Eq. \(28\) with those in the Tables. We close this section by giving some numerical examples of the matching corrections for the operators of phenomenological importance. For definiteness, we consider lattice operators with mean-field improved HYP(II)/Fat7 links. In order to facilitate comparison with previous work, we quote the same blocks of the matching matrices as given in Ref. \[22\] for specific staggered operators.

For the first example we quote the square sub-matrix with indices running over the operators

\[
k, l = \{ (O_1)_I, (O_1)_{II}, (O_2)_I, (O_2)_{II} \}.
\]

(34)

Note that the matching of continuum operators of this form actually involves many other lattice operators than these four, but the others have different tastes, and thus do not contribute to matrix elements with external lattice pseudo-Goldstone bosons until \(O(g^4)\). We find (with \(L = \log(\mu a)\)):

\[
c_{kl} - \gamma_{kl} L = \begin{pmatrix}
4.5785 + 2L & -5.8506 - 6L & -0.2559 & 0.7676 \\
-5.8161 - 6L & 4.4209 + 2L & 0 & 2.4824 \\
-0.2559 & 0.7676 & 11.1493 + 16L & -3.0345 \\
0 & 2.4824 & -0.1839 + 6L & 2.5435 - 2L
\end{pmatrix},
\]

(35)

where the error in the last digit is approximately \(\pm 2\).

The second example is a rectangular submatrix, having indices

\[
k = \left\{ \frac{1}{2}(O_3)_I, \frac{1}{2}(O_3)_{II} \right\},
\]

(36)

and

\[
l = \left\{ \frac{1}{2}(O_3)_I, \frac{1}{2}(O_3)_{II}, (O_4)_I, (O_4)_{II} \right\}.
\]

(37)

Again, these are the only contributions to mixing with operators of the same taste. We find:

\[
c_{kl} - \gamma_{kl} L = \begin{pmatrix}
2.9622 - 2L & -1.0019 + 6L & 0.9403 & -2.8208 \\
-3 & 10.9530 + 16L & 0 & -7.4011
\end{pmatrix},
\]

(38)

where the error in the last digit is approximately \(\pm 2\).

Note that these two matching matrices are sufficient for the calculation of \(B_K, B_7^{3/2}\) and \(B_8^{3/2}\), for which the relevant operators are, respectively, \((O_1)_I + (O_1)_{II}, (O_2)_I + (O_3)_I\), and \((O_2)_I + (O_3)_{II}\).

The numerical values of the matching coefficients should be multiplied by the factor \(c_{B_K}(1/a)/4\pi\). Taking, as an example, \(1/a = 2\) GeV, for which this factor is \(\approx 1/66\) we find that a typical coefficient of \(\sim 5\) gives rise to a \(\sim 10\%\) correction, while the largest coefficients are roughly twice this size. These corrections are small enough that one-loop perturbation theory is reasonably convergent, which is not the case for unfattened links.

In making these estimates we are assuming that the appropriate scale (often called \(q^*\)) is \(\mu \approx 1/a\), so that \(L = 0\). This is reasonable for fattened (and mean-field improved) links which have reduced couplings to high momentum gluons. A more detailed test of this point can, however, be made by calculating \(q^*\), an issue which is subtle for operators with anomalous dimensions \[23\ \[24\ \[25\].

V. CONCLUSION

The main result of this paper is that, by using fattened links, a long-standing obstacle to studying many weak matrix elements of interest using staggered fermions has been removed. In particular, one can use the standard
methodology of Refs. 21, 23, which employs external lattice pseudo-Goldstone bosons, and operators spread out over the unit hypercube, as long as the links in the operators and the action are appropriately fattened. In this case, the one-loop corrections in matching factors are typically ∼ 10%, and range up to ∼ 20%, for all operators. This is the same size as the corrections for other choices of fermion discretization, in which one can use ultra-local operators which do not contain links. We have shown this for HYP links (or Fat-7 links including SU(3) projection), but we expect a similar result to hold for other fattening choices which reduce taste-symmetry breaking.

Strictly speaking, our calculation is only complete for operators which do not have penguin contractions. It thus applies for the phenomenologically interesting quantities $B_K$, $B_7^{3/2}$ and $B_8^{3/2}$, but only a part for $\Delta I = 0$ kaon decays. However, the remaining contribution from penguin diagrams is small for unfattened links 13, and we do not see any reason to expect this to be changed by fattening. We are presently checking this expectation.

A particularly encouraging feature of our results is the across-the-board reduction in mixing of operators with different tastes. This provides a thorough test of the idea that, with fattened links, the staggered fermion operators are much closer to their continuum counterparts.

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APPENDIX A: TWO-COLOR-TRACE OPERATORS: FACTORIZABLE CONTRIBUTIONS

Here we give the expressions for the contributions to the renormalization of two-color-trace operators from factorizable diagrams. Examples of each type of such diagrams are given in Fig. II(a)-(c). These contributions are the same as those for bilinears, and thus can be extracted from the results given in Ref. 2. We present them again here, but using a different notation, that which we use in subsequent appendices to express the contributions from non-bilinear-like diagrams.

We use the following abbreviations for lattice integrals:

\[
\int_k \equiv (4\pi)^2 \prod_{\mu=1}^4 \int_{-\pi}^{+\pi} \frac{dk_\mu}{2\pi},
\]

\[
s_\mu \equiv \sin(k_\mu), \quad s_\mu \equiv \sin(k_\mu/2)
\]

\[
c_\mu \equiv \cos(k_\mu), \quad c_\mu \equiv \cos(k_\mu/2)
\]

\[
B \equiv \frac{1}{4\sum_\mu s_\mu^2 + (a\lambda)^2}, \quad F \equiv \frac{1}{\sum_\mu s_\mu^2}
\]

\[
x \equiv -2\ln(a\lambda) + F_{0000} - \gamma_E + 1
\]

(1)

where $F_{0000} = 4.36923(1)$ and $\gamma_E = 0.577216\ldots$. Here $\lambda$ is the “gluon mass” that regulates the infra-red divergences. We set $a\lambda \to 0$ except for the logarithms contained in $x$.

We consider the renormalization of the two-color-trace four-fermion operator with general spins and tastes: $[S \times F][S' \times F']_{11}$. Factorizable diagrams lead only to two-color-trace operators. The contribution from corrections to one of the bilinears, say $[S \times F]$, can be written

\[
G_{1(a)} = \frac{g^2}{(4\pi)^2} C_F \delta_{ab} \delta_{a' b'} \left[ \sigma_S \cdot (\gamma_S \otimes \bar{\gamma}_F)_{CD} + \sum_{M,N} \sum_{\mu,\nu,\beta,\rho} X^{\nu,\beta,\rho}_{MN} (\gamma_\mu MSN_{\nu,\rho} \otimes \xi_{MFN})_{CD} (\gamma_{S'} \otimes \bar{\gamma}_{F'})_{C'D'} \right]
\]

(2)

\[
G_{1(b)} = \frac{g^2}{(4\pi)^2} C_F \delta_{ab} \delta_{a' b'} \sum_{M,N} \sum_{\mu,\nu,\rho,\alpha} Y^{\mu,\nu,\alpha,\rho}_{MN} (S_\nu + F_\nu)_{\text{mod } 2}
\]

\[
\left[ (\gamma_\nu MSN_{\alpha,\rho} \otimes \xi_{\nu,5MFN})_{CD} + (\gamma_\mu MSN_{\nu,\rho} \otimes \xi_{\nu,5MFN})_{CD} \right] (\gamma_{S'} \otimes \bar{\gamma}_{F'})_{C'D'}
\]

(3)

\[
G_{1(c)} = \frac{g^2}{(4\pi)^2} C_F \delta_{ab} \delta_{a' b'} \frac{1}{2} \left[ -\Delta_{SF} \cdot \gamma_\nu + T_{\Delta_{SF}} \gamma_\nu \right] (\gamma_{S'} \otimes \bar{\gamma}_{F'})_{CD}(\gamma_{S'} \otimes \bar{\gamma}_{F'})_{C'D'}
\]

(4)
Here we have given the contribution from all diagrams of the given type. We use the definitions $\sigma_S = (4, 1, 0)$ for spins $S = (1 \text{ or } \gamma_5, \gamma_\mu \text{ or } \gamma_\mu \gamma_5, \sigma_{\mu\nu})$, $\Delta_{SF} = \sum_\mu (S_\mu - F_\mu)_{\text{mod } 2}$ is the distance between the quark and anti-quark fields in the $[S \times F]$ bilinear and $C, D, (C', D')$ correspond to the bilinear $[S 	imes F]$ ($[S' \times F']$). The matrices $(\gamma_S \otimes \xi_F)_{CD}$ are standard; see, for example, Ref. [3].

There are analogous contributions from diagrams which correct the $[S' \times F']$ bilinear, and which can be obtained from those above by the interchanges $abCDFS \leftrightarrow a'b'C'D'S'F'$.

The finite loop integrals appearing in these expressions are

\begin{align}
X^{\mu\nu, \beta \rho}_{MN} &= \int_k \left[ BF^2 E_M(k) E_N(-k) + s_\mu s_\nu s_\rho s_\sigma \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} - \frac{1}{4} \delta_{M,0} \delta_{N,0} \delta_{\mu\nu} \delta_{\beta \rho} B^2 \right] \\
Y^{\mu\nu, \alpha}_{MN} &= \int_k \left[ BF(i s_\alpha) \bar{c}_\rho \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \frac{1}{12} \sum_{j, j' = 1}^{4} E_M(\theta^{(j)}_{\nu}) E_N(-\theta^{(j')}_{\mu}) \right] \\
T_\mu &= \int_k \left[ B \exp(ik \cdot n) \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \right] = \int_k \left[ B \sum_{\alpha} h_{\mu\lambda} \prod_{\alpha} \cos(k_n n_\alpha) \right] \\
T_\Delta &= \int_k \left[ B \bar{s}_\mu \bar{s}_\nu \bar{g}_\Delta(k) \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \right] \\
\bar{g}_\Delta(k) &= (0, 0, 1, 2 + c_\rho, 3 + 2c_\rho + c_\rho c_\sigma) \quad \text{for} \quad \rho \neq \sigma \neq \nu \neq \mu
\end{align}

where $h_{\mu\nu}(k)$ is given in Eq. (7) of the text, and

\begin{align}
E_M(k) &= \prod_{\mu = 1}^{4} \frac{1}{2} \left( \exp(-ik_\mu/2) + (-1)^{\bar{M}_\mu} \exp(+ik_\mu/2) \right) \\
\bar{M}_\mu &= \sum_{\alpha \neq \mu} M_{\alpha} \\
\phi &= \sum_\rho \phi_\rho \bar{\rho} \\
\theta^{(1)}_{\mu\nu}(\phi) &= \frac{1}{2} \phi_\mu \bar{\mu}, \\
\theta^{(2)}_{\mu\nu}(\phi) &= \frac{1}{2} \phi_\nu \bar{\nu}, \\
\theta^{(3)}_{\mu\nu}(\phi) &= \phi - \frac{1}{2} \phi_\nu \bar{\nu}, \\
\theta^{(4)}_{\mu\nu}(\phi) &= \phi - \frac{1}{2} \phi_\mu \bar{\mu} - \phi_\nu \bar{\nu}.
\end{align}

Finally, the finite loop integrals needed for the self-energy diagrams are

\begin{align}
I^{(1)}_\mu &= \int_k \left[ BF \bar{s}_\mu \sum_\rho \bar{c}_\rho s_\rho \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \right] \\
I^{(2)}_\mu &= \int_k \left[ BF c_\mu \bar{c}_\mu \sum_\lambda h_{\mu\lambda}^2 - B^2 \right] \\
K^{(2)}_\beta &= \int_k \left[ BF c_\beta \sum_\mu \bar{c}_\rho c_\rho \sum_\lambda h_{\mu\lambda}^2 - 4B^2 \right] \\
I^{(4)}_\mu &= \int_k \left[ BF \bar{c}_\mu \bar{s}_\mu \sin(2k_\mu) \sum_\lambda h_{\mu\lambda}^2 - \frac{1}{2} B^2 \right]
\end{align}
\[ K_\beta^{(4)} = \int k \left[ BFFs_\beta \sin(2k_\beta) \sum_\mu \tilde{c}_\mu^2 \sum_\lambda h_{\mu\lambda}^2 - 2B^2 \right] \]
\[ L_{\mu\nu}^{(4)} = \int k \left[ BFF\bar{c}_\mu c_\nu s_\mu \sin(2k_\mu) \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \right] \quad \text{where} \quad \mu \neq \nu. \]  

**APPENDIX B: TWO-COLOR-TRACE OPERATORS: NON-FACTORIZABLE CONTRIBUTIONS**

Here we give expressions for the non-factorizable corrections to two-color-trace operators, i.e. those from Figs. 1(f-h). These diagrams lead to mixing with one-color-trace operators, and introduce new integrals particular to four-fermion operators.

The results are

\[ G_{1(f)} = \frac{g^2}{(4\pi)^2} \left( -\frac{1}{2N_c} \delta_{ab}\delta_{a'b'} + \frac{1}{2} \delta_{ab'}\delta_{a'b} \right) \cdot \sum_{\mu,\nu,\alpha,\beta} \sum_{M,N,L} U_{NML}^{\mu\nu\beta} \cdot \left[ (S_\nu + F_\nu)_{mod 2} \left( \gamma_{[\mu5\bar{M}S]} \otimes \xi_{\nu5[M]\bar{F}} \right)_{CD} \right] \left( \frac{\gamma_{[\mu5\bar{M}S]} \otimes \xi_{\nu5[M]\bar{F}}}{C'} \right) \]

\[ G_{1(g)} = -\frac{g^2}{(4\pi)^2} \left( -\frac{1}{2N_c} \delta_{ab}\delta_{a'b'} + \frac{1}{2} \delta_{ab'}\delta_{a'b} \right) \cdot \sum_{\mu,\nu,\alpha,\beta} \sum_{M,N} (X_{\mu\nu,\alpha,\beta}^{MN} + \frac{1}{4} \delta_{\mu\nu}\delta_{\alpha\beta} \delta_{M,0}\delta_{N,0}) \cdot \left[ (\gamma_{[\nu\alpha\bar{M}S]} \otimes \xi_{\mu\nu5[M]\bar{F}})_{CD} \right] \left( \gamma_{[\nu\alpha\bar{M}S]} \otimes \xi_{\mu\nu5[M]\bar{F}} \right)_{C'D'} \]

\[ G_{1(h)} = -\frac{g^2}{(4\pi)^2} \left( -\frac{1}{2N_c} \delta_{ab}\delta_{a'b'} + \frac{1}{2} \delta_{ab'}\delta_{a'b} \right) \sum_{\mu,\nu} \sum_{M,N,K,L} V_{MNL}^{\mu\nu} \cdot \left[ (S_\mu + F_\mu)_{mod 2} \cdot (S_\nu + F_\nu)_{mod 2} \cdot (\gamma_{[\nu\alpha5\bar{M}S]} \otimes \xi_{\mu\nu5[M]\bar{F}}) \right] \left( \gamma_{[\nu\alpha5\bar{M}S]} \otimes \xi_{\mu\nu5[M]\bar{F}} \right)_{C'D'} \]

where the finite loop integrals \( U_{NML}^{\mu\nu\beta} \) and \( V_{MNL}^{\mu\nu} \) are defined as

\[ U_{NML}^{\mu\nu\beta} = \int k \left[ B F \cdot i\bar{c}_\mu s_\beta \cdot \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \cdot \frac{1}{12} \sum_{\alpha \neq \nu, j=1}^4 E_N(\theta_{\nu\alpha}^{(j)}) E_M(k - \theta_{\nu\alpha}^{(j)}) E_L(-k) \right] \]

\[ V_{MNL}^{\mu\nu} = \int k \left[ B \cdot \sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \cdot \frac{1}{12} \sum_{\rho \neq \mu}^4 E_M(k - \theta_{\mu\rho}^{(j)}) E_N(\theta_{\rho\nu}^{(j)}) \cdot \frac{1}{12} \sum_{\sigma \neq \nu}^4 E_K(-k + \theta_{\rho\nu}^{(j)}) E_L(-\theta_{\rho\nu}^{(j)}) \right]. \]

**APPENDIX C: ONE-COLOR-TRACE OPERATORS**

Finally, we give the results for one-color-trace operators. The contributions from diagrams of the type of Fig. 2(a) are the same as the corresponding two-color-trace contributions aside from the color factor:

\[ G_{2(a)} = G_{1(a)} \left( C_F \delta_{ab}\delta_{a'b'} \rightarrow \left\{ -\frac{1}{2N_c} \delta_{ab}\delta_{a'b'} + \frac{1}{2} \delta_{ab'}\delta_{a'b} \right\} \right). \]

The self-energy diagrams, Figs. 2(d-e) also differ only by color factors, e.g.

\[ G_{2(e)} = G_{1(e)}(\delta_{ab}\delta_{a'b'} \rightarrow \delta_{ab'}\delta_{a'b}) \]  

The remaining diagrams give

\[ G_{2(b)} = -\frac{g^2}{(4\pi)^2} C_F \delta_{ab'} \delta_{a'b} \sum_{\mu,\nu} \sum_{M,N} \frac{1}{2} \left[ Y_{\mu\nu,\beta}^{MN} + Y_{\mu\nu,\beta}^{MN} \right]. \]
\[
G_{2(c)} = \frac{1}{2(4\pi)^2} \cdot C_F \delta_{ab} \delta_{a'b'} \cdot T_{0000}^{1} \cdot \\
\left[ 4(\gamma S \otimes \xi F_{P})C_{D}(\gamma S^{*} \otimes \xi F_{P})C_{D'} + (\gamma S^{*} \otimes \xi F_{P})C_{D}(\gamma S \otimes \xi F_{P})C_{D'} + (\gamma S^{*} \otimes \xi F_{P})C_{D}(\gamma S^{*} \otimes \xi F_{P})C_{D'} \right] (C3)
\]
\[
\tilde{T}_{M}^{\mu\nu} = \int \sum_{\lambda} h_{\mu}\lambda h_{\nu}\lambda \cdot \left\{ \frac{1}{2} \delta_{M,0} + \frac{1}{6} \sum_{j=1}^{3} E_{M}(\psi_{\mu\nu}^{(j)}) \psi_{\mu\nu}^{(j)} \right\} (C5)
\]
\[
G_{2(f)} = \frac{g^2}{(4\pi)^2} \left( -\frac{1}{2N_{c}} \delta_{ab} \delta_{a'b'} + \frac{1}{2} \delta_{ab} \delta_{a'b'} \right) \cdot \sum_{\mu,\nu,\beta} \sum_{L,M,N} \frac{1}{2} \left\{ U_{[54],MN}^{\mu\nu,\beta} - U_{[54],LM}^{\mu\nu,\beta} \right\} \cdot \\
\left[ (\gamma_{\mu S L} \otimes \xi_{N F L})C_{D}(\gamma S^{*} \otimes \xi_{M F})C_{D'} + (\gamma_{S M} \otimes \xi_{F M})C_{D}(\gamma_{S^{*} N^{*} \beta^{*}} \otimes \xi_{L F N})C_{D'} + (\gamma_{S^{*} M} \otimes \xi_{F M})C_{D}(\gamma_{S N^{*} \beta^{*}} \otimes \xi_{L F N})C_{D'} \right] (C7)
\]
\[
G_{2(g)} = \frac{g^2}{(4\pi)^2} \sum_{\mu,\nu} \sum_{\rho,\sigma} \sum_{M,N} \left\{ X_{\mu\nu,\rho\sigma}^{MN} + \frac{1}{4} \delta_{\rho\sigma} \delta_{\mu\nu} \delta_{M,0} \delta_{N,0} \right\} \cdot \\
\left[ C_{F} \delta_{ab} \delta_{a'b'} \cdot \left\{ (\gamma_{S M \rho \mu} \otimes \xi_{F M})C_{D}(\gamma_{S^{*} N S^{*}} \otimes \xi F_{P})C_{D'} + (\gamma_{M S \rho \mu} \otimes \xi_{F M})C_{D}(\gamma_{S^{*} N S^{*}} \otimes \xi F_{P})C_{D'} \right\} \right. \\
\left. - \left( -\frac{1}{2N_{c}} \delta_{ab} \delta_{a'b'} + \frac{1}{2} \delta_{a'b'} \delta_{a'b'} \right) \cdot \\
\left\{ (\gamma_{S M \rho \mu} \otimes \xi_{F M})C_{D}(\gamma_{S^{*} N S^{*}} \otimes \xi F_{P})C_{D'} + (\gamma_{M S \rho \mu} \otimes \xi_{F M})C_{D}(\gamma_{S^{*} N S^{*}} \otimes \xi F_{P})C_{D'} \right\} \right] (C8)
\]
\[
G_{2(h)} = -\frac{g^2}{(4\pi)^2} \left( -\frac{1}{2N_{c}} \delta_{ab} \delta_{a'b'} + \frac{1}{2} \delta_{a'b'} \delta_{a'b'} \right) \cdot \sum_{\mu,\nu} \sum_{M,N} \sum_{K,L} \frac{1}{4} (\gamma_{LS M} \otimes \xi_{L F M})C_{D}(\gamma_{S N S^{*} K} \otimes \xi F_{P} K)C_{D'} \cdot 
\]
APPENDIX D: AN ALTERNATIVE METHOD FOR CALCULATING RENORMALIZATION FACTORS

We have done a second, independent, calculation in order to check our results. The second method originates from an idea of Martinelli [28], and has been generalized to Landau-gauge staggered four-fermion operators in Ref. [13], and general local operators in Ref. [20]. Here we generalize it further to staggered gauge invariant operators. The idea is to use the information available from the renormalization of bilinear operators as fully as possible. It turns out that only two types of diagram are specific to four-fermion operators, and these one cannot avoid calculating.

Although this method allows a full calculation of mixing, we have only done the calculation for specific cases. We consider initial operators in which the tastes of the two component bilinears are the same \( F_i' = F_i \), and calculate their mixing only into operators having the same tastes \( F_j' = F_j = F_i \). We call this “taste-diagonal” mixing. This is the subset of mixing coefficients that are needed in our numerical calculations, in which we use external states of definite taste. It is legitimate to exclude mixing with “taste off-diagonal” operators because this leads to an error of \( O(g^4) \), the same size as the error we are making anyway by using 1-loop matching factors.

The restriction to taste-diagonal mixing allows a number of integrals to be dropped, and allows us to obtain relatively compact expressions for the final result.

The basic observation is that almost all diagrams contributing to renormalization of lattice four-fermion operators have already been calculated as part of the matching factors for bilinears. The qualifier “as part of” is important: one needs to know the results for individual subdiagrams. To explain this in more detail we use the notation of Ref. [1, 13] for the different classes of diagrams: X diagrams involve gluon exchange between external quark lines \{e.g. Figs. 1(a,g) and 2(a,g)\}; Y diagrams involve gluon exchange between external quark lines and the links in the operator \{e.g. Figs. 1(b,f) and 2(b,f)\}; Z diagrams are self-energy corrections excluding tadpoles \{e.g. Figs. 1(c,h) and 2(c,h)\}; ZT diagrams are tadpole self-energy corrections \{e.g. Figs. 1(d) and 2(d)\}; and T diagrams are tadpoles in which the gluon begins and ends on (in general different) links within in the operator \{e.g. Figs. 1(e) and 2(e)\}. This classification applies to corrections for both bilinear and four-fermion operators.

The self-energy diagrams, both types Z and ZT, are independent of the operator under consideration, and thus can be taken over from the bilinear calculation without change.

Contributions from X diagrams can also be expressed in terms of bilinear X diagram corrections, using Fierz transformations and charge conjugation. This has been explained in Ref. [13], and we do not repeat this discussion here. There is it also shown how the X, Z and ZT diagrams come with color factors such that they combine as for the bilinear calculations.

The Y and T diagrams divide into three classes. The first, exemplified by Figs. 1(b,c), contribute to the renormalization of the individual bilinears in the color two-trace form. The second, exemplified by Figs. 1(b,c), contribute to the renormalization of bilinears after Fierzing the one color trace four fermion operator into two color trace form. These two sets of diagrams can be combined with the X, Z and ZT diagrams in the way described in the following subsection. They do not require calculations beyond those need for bilinears.

The third set of Y and T diagrams are intrinsic to four-fermion operators. For two-color-trace operators these are exemplified by Figs. 1(h); these, however, can be shown to contribute only to taste off-diagonal mixing. Thus for two-color-trace operators the bilinear calculations are sufficient. For one-color-trace operators the diagrams exemplified by Figs. 2(f,h); these, however, do contribute to taste-diagonal mixing and have to be calculated explicitly. We discuss these calculations in subsections [D2 [Fig. 2(h)]] and [D3 [Fig. 2(f)]], respectively. Note that these two contributions are not affected by tadpole improvement.

1. Bilinear-like diagrams

Here we discuss how the “bilinear-like” diagrams can be calculated using input only from calculations of matching factors for bilinears.

First note that we only consider initial four-fermion operators, \( \bar{O}^{\text{Latt}} \), in which both bilinears have the same spin and taste. Our restriction to mixing which is taste-diagonal turns out to imply mixing only with operators of the same type, i.e. the spin may change, but it does so for both bilinears in the same way. The label \( i \) of such operators obviously also labels the spin-taste of the individual bilinears. The taste-diagonal part of the bilinear corrections, which is all that we need, are denoted by \( X_i, Y_i, Z, ZT \) and \( T_i \) for the various diagrams. The convention here is that the bilinear color factor \( (C_F = 4/3) \) and a factor of \( g^2/(4\pi)^2 \) are not included. The self-energy contributions, \( Z \) and \( ZT \), do not have a label since they are the same for all operators. We also need the Fierz matrix for such four-fermion operators. The basic observation is that almost all diagrams contributing to renormalization of lattice four-fermion operators have already been calculated as part of the matching factors for bilinears. The qualifier “as part of” is important: one needs to know the results for individual subdiagrams. To explain this in more detail we use the notation of Ref. [1, 13] for the different classes of diagrams: X diagrams involve gluon exchange between external quark lines \{e.g. Figs. 1(a,g) and 2(a,g)\}; Y diagrams involve gluon exchange between external quark lines and the links in the operator \{e.g. Figs. 1(b,f) and 2(b,f)\}; Z diagrams are self-energy corrections excluding tadpoles \{e.g. Figs. 1(c,h) and 2(c,h)\}; ZT diagrams are tadpole self-energy corrections \{e.g. Figs. 1(d) and 2(d)\}; and T diagrams are tadpoles in which the gluon begins and ends on (in general different) links within in the operator \{e.g. Figs. 1(e) and 2(e)\}. This classification applies to corrections for both bilinear and four-fermion operators.

The self-energy diagrams, both types Z and ZT, are independent of the operator under consideration, and thus can be taken over from the bilinear calculation without change.

Contributions from X diagrams can also be expressed in terms of bilinear X diagram corrections, using Fierz transformations and charge conjugation. This has been explained in Ref. [13], and we do not repeat this discussion here. There is it also shown how the X, Z and ZT diagrams come with color factors such that they combine as for the bilinear calculations.

The Y and T diagrams divide into three classes. The first, exemplified by Figs. 1(b,c), contribute to the renormalization of the individual bilinears in the color two-trace form. The second, exemplified by Figs. 1(b,c), contribute to the renormalization of bilinears after Fierzing the one color trace four fermion operator into two color trace form. These two sets of diagrams can be combined with the X, Z and ZT diagrams in the way described in the following subsection. They do not require calculations beyond those need for bilinears.

The third set of Y and T diagrams are intrinsic to four-fermion operators. For two-color-trace operators these are exemplified by Figs. 1(h); these, however, can be shown to contribute only to taste off-diagonal mixing. Thus for two-color-trace operators the bilinear calculations are sufficient. For one-color-trace operators the diagrams exemplified by Figs. 2(f,h); these, however, do contribute to taste-diagonal mixing and have to be calculated explicitly. We discuss these calculations in subsections [D2 [Fig. 2(h)]] and [D3 [Fig. 2(f)]], respectively. Note that these two contributions are not affected by tadpole improvement.
operators, $F_{ij}$, and the charge conjugation matrix, $C_{ij}$. The calculation of both is straightforward and is discussed in Ref. [13].

To express our results we first define two useful matrices

$$B_{ij} = \delta_{ij} (X_i + Y_i + Z + ZT + T_i) ;$$  \hspace{1cm} (D1)

$$B'_{ij} = \delta_{ij} (X_i + Z + ZT) .$$ \hspace{1cm} (D2)

The former has the full bilinear corrections along the diagonal; in the latter the contributions involving the gauge links in the operator are removed. The result is then

$$\tilde{C}_{ij}^{Latt} \text{ (bilin. part)} = 2 \times \begin{pmatrix} (-1/6) & (1/2) \\ 0 & (4/3) \end{pmatrix} \begin{pmatrix} B'_{ij} & 0 \\ 0 & B_{ij} \end{pmatrix}$$

$$+ 2 \times \begin{pmatrix} (4/3) & 0 \\ (1/2) & (-1/6) \end{pmatrix} \begin{pmatrix} F_{ik}B_{kl}F_{lj} & 0 \\ 0 & F_{ik}B'_{kl}F_{lj} \end{pmatrix}$$

$$- 2 \times \begin{pmatrix} (-1/6) & (1/2) \\ (1/2) & (-1/6) \end{pmatrix} \times C_{ik}F_{kl}B'_{lm}F_{mn}C_{nj} .$$ \hspace{1cm} (D3)

The factors of two are from the presence of two diagrams of each type. The color matrices are taken from Ref. [13].

Expressions for the bilinear contributions can be determined from Ref. [2] as follows. The $X_i$ are related to the mixing matrix $X_{jk}$ of Ref. [2] by

$$X_i = X_{ii} + x \sigma S .$$ \hspace{1cm} (D4)

where $x$ is defined in Eq. A11 and $\sigma S = (4, 1, 0)$ for spins $S = (1 \ or \ \gamma_5, \gamma_\mu \ or \ \gamma_\mu \gamma_5, \sigma_{\mu\nu})$. The $Y_i$ are exactly as given in Ref. [2]; the result depends only on the distance $\Delta_i$ of the bilinear and is labelled $Y_{\Delta_i}$. The $T_i$ and $ZT$ are combined in Ref. [2], and given in terms of two quantities $T^a_{\Delta_i}$ and $T^b_{\Delta_i}$, which also depend only on the distance. The combinations we want are

$$T_i = T^a_{\Delta_i+1} + T^b_{\Delta_i} = \Delta_i T^a_{\Delta=2} + T^b_{\Delta_i} , \hspace{1cm} (D5)$$

$$ZT = T^a_{\Delta=0} .$$ \hspace{1cm} (D6)

Finally, the non-tadpole self-energy is given by

$$Z = -X_{(\gamma_\mu \otimes 1)} - Y_{\Delta=1} .$$ \hspace{1cm} (D7)

Mean-field improvement is simple to implement in this method: one simply applies it to the bilinear corrections and then uses the same formulae. This has the following effects:

$$T_i \rightarrow T_i + T^\Delta_{\Delta_i+1} = T_i + \Delta_i T^a_{\Delta=2} = T_i + \Delta_i I_{MF} , \hspace{1cm} (D8)$$

$$ZT \rightarrow ZT + T^a_{\Delta=0} = ZT - I_{MF} , \hspace{1cm} (D9)$$

where $T^a_{\Delta}$ is given in Ref. [2].

### 2. Crossed tadpole diagrams

Here we report the contribution of Fig. 2(h) to taste-diagonal mixing. In fact, one can see that the taste-diagonal contributions are also diagonal in the spins of the bilinears, and thus are completely diagonal. Following Ref. [2], we first define

$$P\mu(k) \equiv D\mu(k)^2 + \sum_{\nu\mu} G_{\nu,\mu}(k)^2 ,$$ \hspace{1cm} (D10)

$$4\tilde{\delta}_\mu \tilde{\delta}_\nu O_{\mu\nu} = D\mu G_{\nu,\mu} + D\nu G_{\mu,\nu} + \sum_{\nu \neq (\mu, \rho)} G_{\nu,\mu} G_{\nu,\rho} ,$$ \hspace{1cm} (D11)

which are, respectively, the diagonal and off-diagonal parts of the propagator from unfattened link to unfattened link. The integrals that arise are then

$$T_{CP}^{\mu}(\delta) = \frac{1}{16} \int k B(k) P\mu(k) V\mu(H, \delta, k) V_T(H, \delta, k) ,$$ \hspace{1cm} (D12)

$$T_{CO}^{\mu}(\delta) = \frac{1}{16} \int k B(k) O_{\mu\nu}(\tilde{\delta}_\mu \tilde{\delta}_\nu \tilde{\delta}_\rho \tilde{\delta}_\sigma) V_T(H, \delta, k) ,$$ \hspace{1cm} (D13)
from these diagrams is the crossed tadpoles. It can be written in terms of the two integrals

tadpoles. In the end, however, if one considers only taste-diagonal contributions, the final form is similar to that for

\( V \)

where the notation is as follows:

- The hypercube vector \( \delta \) is \( \delta = 2 \ S - F \), where \( S \) and \( F \) are the hypercube vectors representing the spin and taste of each bilinear in the four-fermion operator.
- The indices \( \mu, \nu, \rho \) and \( \sigma \) are unequal—a point reinforced by the prime on the sum in \( T_C^L \).
- \( H \) is a hypercube vector, representing the displacement (mod-2) between the quark in one bilinear and the antiquark in the other. For one-color trace operators, the gauge links span this distance. When one considers the four-fermion operators, \( H \) is averaged over, as the expression indicates.
- To write the result in a way which applies to all \( H \), we have used the notation

\[
\tilde{c}_\mu = \cos(H_\mu k_\mu/2), \quad \tilde{s}_\mu = \sin(H_\mu k_\mu/2) = H_\mu \bar{s}_\mu.
\]  

(D14)

Note in particular that \( s'_\mu = 0 \) if \( H_\mu = 0 \).
- The “vertex factor” \( V_\mu \) arises from the product of links of total displacement \( H \) from which the gluon emanates—one on each end of the gluon propagator:

\[
V_\mu(H, \delta, k) = (\tilde{c}_\mu \tilde{c}_\sigma \tilde{s}_\sigma)^2 + \frac{1}{9} [(\tilde{c}_\mu \tilde{s}_\rho \tilde{s}_\sigma)^2 (-\delta_\mu + \delta_\sigma + \text{perms}].
\]  

(D15)
- Finally, the “transverse” form factor \( T \) contains dependence on momenta perpendicular to \( H \):

\[
V_T(H, \delta, k) = \prod_{c=1,4} \cos([1 - H_c] \delta_c k_c).
\]  

(D16)
- Due to the permutation symmetry between indices, these integrals in fact depend only on the “distance” \( |\delta| \) of each bilinear, so that there are five independent integrals.

Adding in the color factor resulting from one gluon exchange, we find that the contribution to the mixing coefficients from these diagrams is

\[
\tilde{C}^L_{i\mu}(\text{Crossed tadpoles}) = \left( \begin{array}{ccc} -1/6 & 1/2 & 0 \\ 0 & 0 & 0 \end{array} \right) \left[ T_{C}^P(|\delta_i|) + T_{C}^O(|\delta_i|) \right].
\]  

(D17)

This contribution is not affected by mean-field improvement.

3. Crossed Y diagrams

The calculation for the crossed Y diagrams, exemplified by Fig. 2(f), is more involved than that for the crossed tadpoles. In the end, however, if one considers only taste-diagonal contributions, the final form is similar to that for the crossed tadpoles. It can be written in terms of the two integrals

\[
Y_{C}^P(|\delta|) = \frac{1}{16} \sum_{H} \sum_{\mu} (-\delta_\mu) H_\mu \int k B(k) F(k) P_\mu(k) s^2 \tilde{V}_\mu(H, \delta, k) V_T(H, \delta, k),
\]  

(D18)

\[
Y_{C}^O(|\delta|) = \frac{1}{16} \sum_{H} \sum_{\mu \nu} (-\delta_\mu) H_\mu \int k B(k) F(k) 4 O_{\mu\nu}(\bar{s}_\nu \bar{s}_\sigma)^2 s^2 \tilde{V}_\mu(H, \delta, k) V_T(H, \delta, k).
\]  

(D19)

where the notation is as for the crossed tadpoles, except that the vertex factor changes to

\[
\tilde{V}_\mu(H, \delta, k) = (\tilde{c}_\mu \tilde{c}_\nu \tilde{c}_\sigma)^2 + \frac{1}{3} [(\tilde{c}_\mu \tilde{s}_\rho \tilde{s}_\sigma)^2 (-\delta_\mu + \delta_\sigma + \text{perms}].
\]  

(D20)
in which the second factor is three times larger than in \( V_\mu \), Eq. (D14). Note that the sum over \( \nu \) in the expression for \( Y_{C}^O \) is constrained to avoid \( \mu \), but is otherwise free. In particular, the absence of a prime on \( s^2 \nu \) means that there is a contribution even if \( H_\nu \) vanishes.
Adding in the color factor resulting from one gluon exchange, the overall factor of 2 because the gluon can originate from either link factor, and including the overall sign, we find that the contribution to the mixing coefficients from these diagrams is

\[
\hat{C}_{ii}^{\text{Latt}}(\text{Crossed } Y^\dagger) = \left( \begin{array}{cc} -1/6 & 1/2 \\ 0 & 0 \end{array} \right) \times (-2) \times [Y_{C_1}^P(\delta_i) + Y_{C_1}^Q(\delta_i)] .
\]

(D21)

Again the result depends only of the distance $|\delta|$.

This contribution is not affected by mean-field improvement.
TABLE I: Renormalization constants for $O_j = (O_j^{\text{latt}})_I = [V_\mu \times P][V_\mu \times P]_I + [A_\mu \times P][A_\mu \times P]_I$. The anomalous dimension matrix $\tilde{\gamma}_{ij}$ and the finite constants $\hat{C}_{ij}^{\text{latt}}$ are defined in Eq. (22). The coefficients $\hat{T}_{ij}$ are needed for mean-field improvement, as defined in Eq. (24). All Greek indices are implicitly summed, with the condition that they are unequal. Results are accurate to ±2 in the last digit quoted.

| $O_j^{\text{latt}}$ | color | trace | $\tilde{\gamma}_{ij}$ $H^{\text{NAIVE}}$ | $\tilde{\gamma}_{ij}$ $H^{\text{HYP(I)}}$ | $\tilde{\gamma}_{ij}$ $H^{\text{HYP(II)}}$ $\hat{T}_{ij}$ |
|---------------------|-------|-------|-------------------------------|-----------------------------|-----------------------------|
| $[S \times V_\mu][S \times V_\mu]$ | I | 0 | -14.395 | -1.756 | -2.391 | +1 |
| $[S \times V_\mu][S \times V_\nu]$ | II | 0 | -3.992 | -0.712 | -0.615 | 0 |
| $[S \times V_\mu][S \times A_\mu]$ | I | 0 | -1.110 | -0.238 | -0.194 | 0 |
| $[S \times V_\mu][S \times A_\nu]$ | I | 0 | -0.313 | +0.011 | +0.071 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II | 0 | -1.219 | -0.034 | -0.061 | 0 |
| $[S \times A_\mu][S \times A_\nu]$ | I | 0 | -0.119 | -0.003 | -0.008 | 0 |
| $[V_\mu \times S][V_\mu \times S]$ | I | 0 | +0.342 | +0.008 | +0.022 | 0 |
| $[V_\mu \times S][V_\mu \times V_\mu]$ | II | 0 | -0.539 | -0.010 | -0.030 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | +1.397 | -0.008 | -0.071 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\rho}]$ | II | 0 | -2.170 | -0.122 | -0.131 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\rho}]$ | I | 0 | +0.565 | +0.029 | +0.023 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\nu}]$ | II | 0 | -0.499 | -0.026 | -0.021 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\nu}]$ | I | 0 | -0.051 | -0.004 | -0.003 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\nu}]$ | II | 0 | +0.154 | +0.011 | +0.009 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | -0.051 | -0.004 | -0.003 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\rho}]$ | II | 0 | +0.154 | +0.011 | +0.009 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\rho}]$ | I | 0 | +1.205 | +0.091 | +0.108 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\rho}]$ | II | 0 | -1.972 | -0.114 | -0.113 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\nu}]$ | I | 0 | +0.502 | +0.027 | +0.022 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\nu}]$ | II | 0 | -0.307 | -0.022 | -0.018 | 0 |
| $[V_\mu \times P][V_\mu \times P]$ | I | 0 | -24.1705 | -3.6940 | -5.2994 | +2 |
| $[V_\mu \times P][V_\mu \times P]$ | II | 0 | +6.70295 | +0.8481 | +0.4170 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | +18.270 | +1.864 | +2.448 | -1 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | -4.544 | -0.733 | -0.662 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\nu]$ | I | 0 | -1.952 | -0.400 | -0.337 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\nu]$ | II | 0 | +2.527 | +0.488 | +0.427 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | I | 0 | -0.233 | -0.072 | -0.064 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | II | 0 | +0.669 | +0.215 | +0.191 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | I | 0 | -0.233 | -0.072 | -0.064 | 0 |
| $[T_{\mu\nu} \times Y_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | +0.669 | +0.215 | +0.191 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | I | 0 | -15.294 | -1.375 | -2.010 | +1 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | II | 0 | -4.387 | -0.730 | -0.652 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$ | I | 0 | +1.110 | +0.237 | +0.194 | 0 |
TABLE II: Renormalization constants for \( \mathcal{O}_i^{\text{Latt}} \) (continued from Table I).

| \( \mathcal{O}_i^{\text{Latt}} \) | color trace | \( \hat{\gamma}_{ij} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{T}_{ij} \) |
|------------------|-------------|------------------|------------------|------------------|------------------|------------------|
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | +0.018 | −0.005 | −0.007 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II 0 | −1.061 | −0.031 | −0.051 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | −0.055 | −0.001 | −0.007 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II 0 | −0.191 | −0.005 | −0.004 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | +0.032 | +0.001 | +0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II 0 | −0.096 | −0.002 | −0.002 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | +0.032 | +0.001 | +0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II 0 | −0.096 | −0.002 | −0.002 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | +0.742 | +0.027 | +0.045 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II 0 | −1.219 | −0.034 | −0.061 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I 0 | +0.119 | +0.003 | +0.008 | 0 |

| \( [A_{\mu} \times S] [A_{\mu} \times S] \) | I 0 | +0.342 | +0.008 | +0.022 | 0 |
| \( [A_{\mu} \times S] [A_{\mu} \times S] \) | II 0 | −0.539 | −0.010 | −0.030 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | +1.397 | −0.007 | −0.071 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II 0 | −2.170 | −0.122 | −0.131 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | +0.399 | +0.020 | +0.016 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | −0.083 | −0.004 | −0.004 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II 0 | +0.249 | +0.013 | +0.011 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | −0.083 | −0.004 | −0.004 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II 0 | +0.249 | +0.013 | +0.011 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | +1.336 | +0.097 | +0.120 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II 0 | −2.367 | −0.131 | −0.150 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I 0 | +0.399 | +0.020 | +0.016 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II 0 | −24.775 | −4.2156 | −5.81 | +2 |
| \( [A_{\mu} \times P] [A_{\mu} \times P] \) | I 0 | +5.2234 | +0.7166 | +1.1181 | 0 |

| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | I 0 | +17.108 | +2.232 | +2.804 | −1 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | II 0 | −4.149 | −0.716 | −0.625 | 0 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | I 0 | +0.805 | +0.107 | +0.078 | 0 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | II 0 | +0.915 | +0.391 | +0.350 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | I 0 | +1.073 | +0.010 | −0.034 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | II 0 | −1.061 | −0.031 | −0.051 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | I 0 | +0.183 | +0.005 | +0.009 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | II 0 | −0.191 | −0.005 | −0.004 | 0 |
TABLE III: Renormalization constants for $O_i = (O_i^\text{Latt})_{II} = [V_\mu \times P][V_\mu \times P]_{II} + [A_\mu \times P][A_\mu \times P]_{II}$.

| $O_i^\text{Latt}$                                      | color trace $\tilde{\gamma}_{ij}$ | $\tilde{C}_{ij}^\text{Latt}$ | $\tilde{C}_{ij}^\text{Latt}$ | $\tilde{C}_{ij}^\text{Latt}$ | $\tilde{T}_{ij}$ |
|------------------------------------------------------|-----------------------------------|-------------------------------|-------------------------------|-------------------------------|-----------------|
| $[S \times V_\mu][S \times V_\mu]$                   | I                                 | 0                            | -5.587                        | -1.140                        | -1.058          |
| $[S \times V_\nu][S \times V_\nu]$                   | II                                | 0                            | +1.862                        | +0.380                        | +0.353          |
| $[S \times V_\mu][S \times V_\nu]$                   | I                                 | 0                            | +1.112                        | +0.346                        | +0.312          |
| $[S \times V_\nu][S \times V_\mu]$                   | II                                | 0                            | -0.371                        | -0.115                        | -0.104          |
| $[S \times A_\mu][S \times A_\mu]$                   | I                                 | 0                            | -1.311                        | -0.028                        | -0.028          |
| $[S \times A_\mu][S \times A_\mu]$                   | II                                | 0                            | +0.437                        | +0.009                        | +0.009          |
| $[S \times A_\mu][S \times A_\nu]$                   | I                                 | 0                            | -0.236                        | -0.006                        | -0.007          |
| $[S \times A_\mu][S \times A_\nu]$                   | II                                | 0                            | +0.079                        | +0.002                        | +0.002          |
| $[V_\mu \times S][V_\mu \times S]$                   | I                                 | 0                            | -1.107                        | -0.022                        | -0.066          |
| $[V_\mu \times S][V_\mu \times S]$                   | II                                | 0                            | +0.369                        | +0.007                        | +0.022          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | I                                 | 0                            | -1.106                        | -0.058                        | -0.053          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | II                                | 0                            | +0.369                        | +0.019                        | +0.018          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | I                                 | 0                            | +0.150                        | +0.008                        | +0.006          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | II                                | 0                            | -0.050                        | -0.003                        | -0.002          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | I                                 | 0                            | +0.403                        | +0.024                        | +0.019          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | II                                | 0                            | -0.134                        | -0.008                        | -0.006          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | I                                 | 0                            | -2.777                        | -0.185                        | -0.226          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | II                                | 0                            | +0.926                        | +0.062                        | +0.075          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | I                                 | 0                            | +0.150                        | +0.008                        | +0.006          |
| $[V_\mu \times T_{\mu \nu}][V_\mu \times T_{\mu \nu}]$ | II                                | 0                            | -0.050                        | -0.003                        | -0.002          |
| $[V_\mu \times P][V_\mu \times P]$                   | I                                 | +6                            | -4.4997                       | -0.0548                       | +0.3161         |
| $[V_\mu \times P][V_\mu \times P]$                   | II                                | -2                            | -58.5269                      | -7.9545                       | -10.6903        |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | -2.766                        | -0.454                        | -0.376          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | +0.922                        | +0.151                        | +0.125          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | -0.416                        | -0.089                        | -0.073          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | +0.139                        | +0.030                        | +0.024          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | +0.764                        | +0.218                        | +0.192          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | -0.255                        | -0.072                        | -0.064          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | +0.764                        | +0.218                        | +0.192          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | -0.255                        | -0.072                        | -0.064          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | -5.972                        | -1.000                        | -0.920          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | +1.991                        | +0.333                        | +0.307          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | I                                 | 0                            | +1.945                        | +0.524                        | +0.458          |
| $[T_{\mu \nu} \times V_\mu][T_{\mu \nu} \times V_\mu]$ | II                                | 0                            | -0.648                        | -0.175                        | -0.153          |
TABLE IV: Renormalization constants for \((O_{j}^{\text{Latt}})_{ii}\) (continued from Table III).

| \(O_{j}^{\text{Latt}}\) | color trace | \(\hat{\gamma}_{ij}\) | \(\hat{c}_{ij}^{\text{Latt}}\) | \(\hat{c}_{ij}^{\text{Latt}}\) | \(\hat{c}_{ij}^{\text{Latt}}\) | \(\hat{T}_{ij}\) |
|-----------------|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | I           | 0               | -0.772          | -0.017          | -0.019          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | II          | 0               | +0.257          | +0.006          | +0.006          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | I           | 0               | -0.045          | -0.001          | -0.003          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | II          | 0               | +0.015          | +0.000          | +0.001          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | I           | 0               | -1.705          | -0.057          | -0.112          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | II          | 0               | +0.568          | +0.019          | +0.037          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | I           | 0               | -0.529          | -0.013          | -0.008          | 0               |
| \([T_{\mu} \times A_{\mu}] [T_{\mu} \times A_{\mu}]\) | II          | 0               | +0.176          | +0.004          | +0.003          | 0               |

\(|A_{\mu} \times S\) | [A_{\mu} \times S] | I           | 0               | -0.479          | -0.008          | -0.026          | 0               |

\(|A_{\mu} \times S\) | [A_{\mu} \times S] | II          | 0               | +0.160          | +0.003          | +0.009          | 0               |

\(|A_{\mu} \times T_{\mu}\) | [A_{\mu} \times T_{\mu}] | I           | 0               | -1.106          | -0.058          | -0.053          | 0               |

\(|A_{\mu} \times T_{\mu}\) | [A_{\mu} \times T_{\mu}] | II          | 0               | +0.369          | +0.019          | +0.018          | 0               |

\(|A_{\mu} \times T_{\mu}\) | [A_{\mu} \times T_{\mu}] | I           | 0               | +0.150          | +0.008          | +0.006          | 0               |

\(|A_{\mu} \times T_{\mu}\) | [A_{\mu} \times T_{\mu}] | II          | 0               | -0.050          | -0.003          | -0.002          | 0               |

\(|A_{\mu} \times T_{\rho}\) | [A_{\mu} \times T_{\rho}] | I           | 0               | -2.776          | -0.185          | -0.226          | 0               |

\(|A_{\mu} \times T_{\rho}\) | [A_{\mu} \times T_{\rho}] | II          | 0               | +0.925          | +0.062          | +0.075          | 0               |

\(|A_{\mu} \times T_{\rho}\) | [A_{\mu} \times T_{\rho}] | I           | 0               | -0.656          | -0.041          | -0.033          | 0               |

\(|A_{\mu} \times T_{\rho}\) | [A_{\mu} \times T_{\rho}] | II          | 0               | +0.219          | +0.014          | +0.011          | 0               |

\(|A_{\mu} \times P\) | [A_{\mu} \times P] | I           | +6              | -4.4997         | -0.0548         | +0.3161         | 0               |

\(|A_{\mu} \times P\) | [A_{\mu} \times P] | II          | -2              | +1.4999         | +0.0183         | -0.1052         | 0               |

\(|P \times V_{\nu}\) | [P \times V_{\nu}] | I           | 0               | -3.153          | -0.314          | -0.237          | 0               |

\(|P \times V_{\nu}\) | [P \times V_{\nu}] | II          | 0               | +1.051          | +0.105          | +0.079          | 0               |

\(|P \times V_{\nu}\) | [P \times V_{\nu}] | I           | 0               | +0.416          | +0.089          | +0.073          | 0               |

\(|P \times V_{\nu}\) | [P \times V_{\nu}] | II          | 0               | -0.139          | -0.030          | -0.024          | 0               |

\(|P \times A_{\mu}\) | [P \times A_{\mu}] | I           | 0               | -0.377          | -0.011          | -0.029          | 0               |

\(|P \times A_{\mu}\) | [P \times A_{\mu}] | II          | 0               | +0.126          | +0.004          | +0.010          | 0               |

\(|P \times A_{\mu}\) | [P \times A_{\mu}] | I           | 0               | +0.045          | +0.001          | +0.003          | 0               |

\(|P \times A_{\mu}\) | [P \times A_{\mu}] | II          | 0               | -0.015          | +0.000          | -0.001          | 0               |
TABLE V: Renormalization constants for $\mathcal{O}_i = (\mathcal{O}_2^{\text{att}})_{ij} = [V_\mu \times P][V_\mu \times P]_i - [A_\mu \times P][A_\mu \times P]_i$.

| $\mathcal{O}_j^{\text{att}}$ | color trace $\tilde{\gamma}_{ij}$ | $\mathcal{C}_ij^{\text{Latt}}$ | $\tilde{\mathcal{C}}_ij^{\text{Latt}}$ | $\tilde{\mathcal{T}}_ij$ |
|-----------------------------|-------------------------------|--------------------------------|-----------------|-----------------|
| $[S \times V_\mu][S \times V_\mu]$ | I 0 | +23.854 | +2.698 | +3.103 | -1 |
| $[S \times V_\mu][S \times V_\mu]$ | II 0 | +0.842 | +0.398 | +0.377 | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | I 0 | -0.139 | -0.030 | -0.024 | 0 |
| $A_\mu \times P]$ | II 0 | +0.416 | +0.089 | +0.073 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I 0 | -1.443 | -0.022 | -0.015 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II 0 | -0.842 | -0.023 | -0.033 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I 0 | +0.015 | +0.000 | +0.001 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II 0 | -0.045 | -0.001 | -0.003 | 0 |

| $[V_\mu \times S][V_\mu \times S]$ | I 0 | +0.515 | +0.004 | +0.021 | 0 |
| $[V_\mu \times S][V_\mu \times S]$ | II 0 | +0.254 | +0.006 | +0.016 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | -3.978 | -0.128 | -0.052 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | +1.064 | +0.064 | +0.078 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | -0.216 | -0.011 | -0.009 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | +0.648 | +0.034 | +0.027 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | -0.051 | -0.004 | -0.003 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | +0.154 | +0.011 | +0.009 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | -0.051 | -0.004 | -0.003 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | +0.154 | +0.011 | +0.009 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | -3.486 | -0.088 | -0.001 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | -0.409 | -0.054 | -0.076 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | I 0 | +0.053 | +0.005 | +0.004 | 0 |
| $[V_\mu \times T_\mu][V_\mu \times T_\mu]$ | II 0 | -0.158 | -0.014 | -0.012 | 0 |

| $[V_\mu \times P][V_\mu \times P]$ | I 0 | -16.0 -13.6711 | -3.5661 | -6.0369 | +2 |
| $[V_\mu \times P][V_\mu \times P]$ | II 0 | -2.5928 | -0.7933 | -0.7331 | 0 |

| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -24.728 | -2.923 | -3.324 | +1 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +1.781 | +0.279 | +0.286 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | +0.980 | +0.192 | +0.167 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | -2.941 | -0.577 | -0.500 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -0.223 | -0.072 | -0.064 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +0.669 | +0.215 | +0.191 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -0.223 | -0.072 | -0.064 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +0.669 | +0.215 | +0.191 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -23.590 | -2.740 | -3.136 | +1 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | -1.623 | -0.275 | -0.276 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -0.139 | -0.030 | -0.024 | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +0.416 | +0.089 | +0.073 | 0 |
TABLE VI: Renormalization constants for \( \mathcal{O}_i^{\text{Latt}} \) (continued from Table V).

| \( \mathcal{O}_i^{\text{Latt}} \) | color trace | \( \tilde{\gamma}_{ij} \) | \( \tilde{c}_{ij}^{\text{Latt}} \) | \( \tilde{h}_{ij}^{\text{Latt}} \) | \( \tilde{h}_{i} \) | \( \tilde{h}_{ii} \) | \( \tilde{h}_{i} \) |
|-----------------|-------------|----------------|----------------|----------------|-------------|-------------|-------------|
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +1.820 | +0.035 | +0.037 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | −0.289 | −0.014 | −0.032 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +0.049 | +0.001 | +0.000 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | −0.146 | −0.003 | −0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +0.032 | +0.001 | +0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | −0.096 | −0.002 | −0.002 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +0.032 | +0.001 | +0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | −0.096 | −0.002 | −0.002 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +1.575 | +0.024 | +0.012 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | +0.446 | +0.017 | +0.043 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | I | 0 | +0.015 | +0.000 | +0.001 | 0 |
| \( [T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}] \) | II | 0 | −0.045 | −0.001 | −0.003 | 0 |
| \( [A_{\mu} \times S] [A_{\mu} \times S] \) | I | 0 | −0.515 | −0.004 | −0.021 | 0 |
| \( [A_{\mu} \times S] [A_{\mu} \times S] \) | II | 0 | −0.254 | −0.006 | −0.016 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I | 0 | +3.978 | +0.128 | +0.052 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II | 0 | −1.064 | −0.064 | −0.078 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I | 0 | +0.050 | +0.003 | +0.002 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II | 0 | −0.150 | −0.008 | −0.006 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I | 0 | −0.083 | −0.004 | −0.004 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | II | 0 | +0.249 | +0.013 | +0.011 | 0 |
| \( [A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}] \) | I | 0 | −0.083 | −0.004 | −0.004 | 0 |
| \( [A_{\mu} \times T_{\nu\rho}] [A_{\mu} \times T_{\nu\rho}] \) | I | 0 | +0.249 | +0.013 | +0.011 | 0 |
| \( [A_{\mu} \times T_{\nu\rho}] [A_{\mu} \times T_{\nu\rho}] \) | II | 0 | +3.355 | +0.083 | +0.012 | 0 |
| \( [A_{\mu} \times T_{\nu\rho}] [A_{\mu} \times T_{\nu\rho}] \) | I | 0 | +0.804 | +0.071 | +0.114 | 0 |
| \( [A_{\mu} \times T_{\nu\rho}] [A_{\mu} \times T_{\nu\rho}] \) | II | 0 | +0.050 | +0.003 | +0.002 | 0 |
| \( [A_{\mu} \times T_{\nu\rho}] [A_{\mu} \times T_{\nu\rho}] \) | I | 0 | −0.150 | −0.008 | −0.006 | 0 |
| \( [A_{\mu} \times P] [A_{\mu} \times P] \) | I | +16 | +14.2732 | +4.0877 | +6.5486 | −2 |
| \( [A_{\mu} \times P] [A_{\mu} \times P] \) | II | 0 | +0.7236 | −0.7715 | −0.8020 | 0 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | I | 0 | +24.468 | +2.964 | +3.358 | −1 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | II | 0 | −0.999 | −0.402 | −0.388 | 0 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | I | 0 | −0.166 | −0.101 | −0.092 | 0 |
| \( [P \times V_{\mu}] [P \times V_{\mu}] \) | II | 0 | +0.497 | +0.302 | +0.277 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | I | 0 | −1.952 | −0.037 | −0.034 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | II | 0 | +0.685 | +0.020 | +0.022 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | I | 0 | −0.079 | −0.002 | −0.002 | 0 |
| \( [P \times A_{\mu}] [P \times A_{\mu}] \) | II | 0 | +0.236 | +0.006 | +0.007 | 0 |
TABLE VII: Renormalization constants for $O_i = (O_2^{\text{Latt}})_{II} = [V_\mu \times P][V_\mu \times P]_{II} - [A_\mu \times P][A_\mu \times P]_{II}$.

| $O_j^{\text{Latt}}$ | color trace $\tilde{\gamma}_{ij}$ | $\hat{\mathcal{C}}_{ij}^{L_{\text{Latt}}}$ | $\hat{\mathcal{C}}_{ij}^{H_{\text{Y}}(I)}$ | $\hat{\mathcal{C}}_{ij}^{H_{\text{Y}}(II)/\text{Fat}}$ | $\hat{T}_{ij}$ |
|---------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|----------------|
| $[S \times V_\mu][S \times V_\mu]$ | I 0 | +5.587 | +1.140 | +1.058 | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | II 0 | -1.862 | -0.380 | -0.353 | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | I 0 | -1.112 | -0.346 | -0.312 | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | II 0 | +0.371 | +0.115 | +0.104 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I 0 | -1.311 | -0.028 | -0.028 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II 0 | +0.437 | +0.009 | +0.009 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I 0 | -0.236 | -0.006 | -0.007 | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II 0 | +0.079 | +0.002 | +0.002 | 0 |

| $[V_\mu \times S][V_\mu \times S]$ | I 0 | +1.107 | +0.022 | +0.066 | 0 |
| $[V_\mu \times S][V_\mu \times S]$ | II 0 | -0.369 | -0.007 | -0.022 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I 0 | -1.106 | -0.058 | -0.053 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II 0 | +0.369 | +0.019 | +0.018 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I 0 | +0.150 | +0.008 | +0.006 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II 0 | -0.050 | -0.003 | -0.002 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I 0 | +0.403 | +0.024 | +0.019 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II 0 | -0.134 | -0.008 | -0.006 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I 0 | +0.150 | +0.008 | +0.006 | 0 |
| $[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II 0 | -0.050 | -0.003 | -0.002 | 0 |

| $[V_\mu \times P][V_\mu \times P]$ | I -6 | +4.4997 | +0.0548 | -0.3161 | 0 |
| $[V_\mu \times P][V_\mu \times P]$ | II +2 | -61.5267 | -7.9910 | -10.4796 | +4 |

| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -2.766 | -0.454 | -0.376 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +0.922 | +0.151 | +0.125 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -0.416 | -0.089 | -0.073 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +0.139 | +0.030 | +0.024 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | +0.764 | +0.218 | +0.192 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | -0.255 | -0.073 | -0.064 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | +0.764 | +0.218 | +0.192 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | -0.255 | -0.073 | -0.064 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | -5.972 | -1.000 | -0.920 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | +1.991 | +0.333 | +0.307 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I 0 | +1.945 | +0.524 | +0.458 | 0 |
| $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II 0 | -0.648 | -0.175 | -0.153 | 0 |
**TABLE VIII:** Renormalization constants for \((\mathcal{O}_2^{\text{Latt}})_{II}\) (continued from Table VII).

| \(\mathcal{O}_2^{\text{Latt}}\) | color trace | \(\hat{\zeta}_{ij}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{T}_{ij}\) |
|-------------------------------|-------------|-----------------|----------------|----------------|----------------|----------------|
| \([T_{\mu \nu} \times A_\mu][T_{\mu \nu} \times A_\mu]\) | I           | 0               | +0.772         | +0.017         | +0.019         | 0              |
| \([T_{\mu \nu} \times A_\mu][T_{\mu \nu} \times A_\mu]\) | II          | 0               | −0.257         | −0.006         | −0.006         | 0              |
| \([T_{\mu \nu} \times A_\mu][T_{\mu \nu} \times A_\nu]\) | I           | 0               | +0.045         | +0.001         | +0.003         | 0              |
| \([T_{\mu \nu} \times A_\mu][T_{\mu \nu} \times A_\nu]\) | II          | 0               | −0.015         | +0.000         | −0.001         | 0              |
| \([T_{\mu \nu} \times A_\rho][T_{\mu \nu} \times A_\sigma]\) | I           | 0               | +1.705         | +0.057         | +0.112         | 0              |
| \([T_{\mu \nu} \times A_\rho][T_{\mu \nu} \times A_\sigma]\) | II          | 0               | −0.568         | −0.019         | −0.037         | 0              |
| \([T_{\mu \nu} \times A_\rho][T_{\mu \nu} \times A_\sigma]\) | I           | 0               | +0.529         | +0.013         | +0.008         | 0              |
| \([T_{\mu \nu} \times A_\rho][T_{\mu \nu} \times A_\sigma]\) | II          | 0               | −0.176         | −0.004         | −0.003         | 0              |

\[\color{red}A_\mu \times S][A_\mu \times S]\]
I: 0; +0.479; −0.008; −0.026; 0
II: 0; +0.160; +0.003; +0.009; 0

\[\color{red}A_\mu \times S][A_\mu \times S]\]
I: 0; +1.106; +0.058; +0.053; 0
II: 0; −0.369; −0.019; −0.018; 0

\[\color{red}A_\mu \times T_{\mu \nu}][A_\mu \times T_{\mu \nu}\]
I: 0; −0.150; −0.008; −0.006; 0
II: 0; +0.050; +0.003; +0.002; 0

\[\color{red}A_\mu \times T_{\mu \nu}][A_\mu \times T_{\mu \nu}\]
I: 0; +2.777; +0.185; +0.226; 0
II: 0; −0.926; −0.062; −0.075; 0

\[\color{red}A_\mu \times T_{\mu \nu}][A_\mu \times T_{\mu \nu}\]
I: 0; +0.656; +0.041; +0.033; 0
II: 0; −0.219; −0.014; −0.011; 0

\[\color{red}A_\mu \times T_{\mu \nu}][A_\mu \times T_{\mu \nu}\]
I: 0; +6.4997; −0.0548; +0.3161; 0
II: 0; −2; +1.4999; +0.0183; −0.1055; 0

\[\color{red}P \times V_\mu][P \times V_\nu]\]
I: 0; +3.154; +0.314; +0.237; 0
II: 0; −1.051; −0.105; −0.079; 0

\[\color{red}P \times V_\mu][P \times V_\nu]\]
I: 0; −0.416; −0.089; −0.073; 0
II: 0; +0.139; +0.030; +0.024; 0

\[\color{red}P \times A_\mu][P \times A_\mu]\]
I: 0; −0.377; −0.011; −0.029; 0
II: 0; +0.126; +0.004; +0.010; 0

\[\color{red}P \times A_\mu][P \times A_\mu]\]
I: 0; +0.045; +0.001; +0.003; 0
II: 0; −0.015; +0.000; −0.001; 0
TABLE IX: Renormalization constants for $O_i = (O_i^{\text{Latt}})_I = 2[P \times P][P \times P]_I - 2[S \times P][S \times P]_I$.

| $O_i^{\text{Latt}}$ | color trace | $\hat{c}_{ij}$ | $\hat{c}_{ij}^{\text{Latt}}$ | $\hat{c}_{ij}^{\text{Latt}}$ | $\hat{c}_{ij}^{\text{Latt}}$ | $\hat{c}_{ij}^{\text{Latt}}$ | $T_{ij}$ |
|---------------------|-------------|----------------|------------------|------------------|------------------|------------------|--------|
| $[S \times S][S \times S]$ | I | 0 | -0.154 | -0.005 | -0.014 | 0 |
| $[S \times S][S \times S]$ | II | 0 | -0.507 | -0.011 | -0.031 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I | 0 | -1.068 | -0.004 | +0.039 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II | 0 | -0.458 | -0.002 | +0.017 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I | 0 | -0.349 | -0.018 | -0.014 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II | 0 | -0.150 | -0.008 | -0.006 | 0 |
| $[S \times P][S \times P]$ | I | $2 \times (-2)$ | $2 \times (+25.1879)$ | $2 \times (+3.2311)$ | $2 \times (+4.6654)$ | $2 \times (-2)$ |
| $[S \times P][S \times P]$ | II | $2 \times (+6)$ | $2 \times (+3.9775)$ | $2 \times (+2.2370)$ | $2 \times (+2.3189)$ | 0 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I | 0 | +33.749 | +3.621 | +4.821 | -2 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II | 0 | -2.263 | -0.586 | -0.547 | 0 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I | 0 | -0.580 | +0.210 | +0.208 | 0 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II | 0 | +0.193 | -0.070 | -0.069 | 0 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I | 0 | +1.249 | +0.267 | +0.219 | 0 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II | 0 | -0.416 | -0.089 | -0.073 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I | 0 | -0.403 | -0.033 | -0.082 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II | 0 | +0.776 | +0.031 | +0.060 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I | 0 | +0.462 | +0.007 | +0.012 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II | 0 | +0.198 | +0.003 | +0.005 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I | 0 | -0.064 | -0.002 | -0.001 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II | 0 | +0.191 | +0.005 | +0.004 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I | 0 | -0.064 | -0.002 | -0.001 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II | 0 | +0.191 | +0.005 | +0.004 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I | 0 | +0.104 | +0.003 | +0.007 | 0 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II | 0 | +0.045 | +0.001 | +0.003 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | +0.815 | -0.053 | -0.115 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | -2.065 | -0.144 | -0.211 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | +0.449 | +0.023 | +0.018 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | -0.150 | -0.008 | -0.006 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | -0.449 | -0.023 | -0.018 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | +0.150 | +0.008 | +0.006 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | +0.102 | +0.007 | +0.006 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | -0.307 | -0.022 | -0.018 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | +0.102 | +0.007 | +0.006 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | -0.307 | -0.022 | -0.018 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | I | 0 | -0.551 | +0.065 | +0.140 | 0 |
| $[T_{\mu \nu} \times T_{\mu \nu}][T_{\mu \nu} \times T_{\mu \nu}]$ | II | 0 | +1.275 | +0.110 | +0.136 | 0 |
TABLE X: Renormalization constants for ($\mathcal{O}_5^{\text{Latt}})_I$ (continued from Table X).

| $\mathcal{O}_5^{\text{Latt}}$ | color trace | $\tilde{c}_{ij}$ | $\tilde{c}_{ij}^{\text{Latt}}$ | $\tilde{c}_{ij}^{\text{Latt}}$ | $\tilde{c}_{ij}^{\text{Latt}}$ | $\tilde{T}_{ij}$ |
|-------------------------------|-------------|------------------|-----------------|-----------------|-----------------|-----------------|
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +32.136 | +3.228 | +4.449 | −2 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | +2.578 | +0.593 | +0.568 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | −0.451 | +0.163 | +0.161 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | −0.193 | +0.070 | +0.069 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +0.304 | +0.130 | +0.117 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | −0.913 | −0.391 | −0.350 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +0.304 | +0.130 | +0.117 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | −0.913 | −0.391 | −0.350 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +0.971 | +0.208 | +0.170 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | +0.416 | +0.089 | +0.073 | 0 |

| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | I 0 | +0.220 | −0.010 | −0.035 | 0 |
| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | II 0 | −1.090 | −0.038 | −0.080 | 0 |
| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | I 0 | +0.594 | +0.009 | −0.015 | 0 |
| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | II 0 | −0.198 | −0.003 | +0.005 | 0 |
| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | I 0 | +0.134 | +0.003 | +0.009 | 0 |
| $[A_\mu \times A_\nu][A_\mu \times A_\nu]$ | II 0 | −0.045 | −0.001 | −0.003 | 0 |

| $[P \times S][P \times S]$ | I 0 | +0.154 | +0.005 | +0.014 | 0 |
| $[P \times S][P \times S]$ | II 0 | +0.507 | +0.011 | +0.031 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I 0 | +1.068 | +0.004 | −0.039 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II 0 | +0.458 | +0.002 | −0.017 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I 0 | +0.349 | +0.018 | +0.014 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II 0 | +0.150 | +0.008 | +0.006 | 0 |

| $[P \times P][P \times P]$ | I 2×(+2) 2×(−39.8785) 2×(−5.2542) 2×(−6.5460) 2×(+2) |
| $[P \times P][P \times P]$ | II 2×(−6) 2×(+40.0945) 2×(+3.8324) 2×(+3.3226) 0 |
TABLE XI: Renormalization constants for $O_i = (O_i^{\text{Latt}})_{11} = 2[P \times P][P \times P]_{11} - 2[S \times P][S \times P]_{11}$.

| $O_i^{\text{Latt}}$ | color trace | $\tilde{\gamma}_{ij}$ | $\tilde{\gamma}_{ij}^{\text{Latt}}$ | $\tilde{\gamma}_{ij}^{\text{Latt}}$ | $\tilde{\gamma}_{ij}^{\text{Latt}}$ | $\tilde{\gamma}_{ij}^{\text{Latt}}$ | $\tilde{\gamma}_{ij}^{\text{Latt}}$ |
|---------------------|-------------|------------------------|--------------------------|------------------|---------------------|--------------------------|------------------|
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | I | 0 | -0.916 | -0.004 | +0.033 | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | II | 0 | +0.305 | +0.001 | -0.011 | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | I | 0 | -0.299 | -0.015 | -0.012 | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | II | 0 | +0.100 | +0.005 | +0.004 | 0 |
| $[S \times P] [S \times P]$ | II | 2 × (+16) | 2 × (+95.6216) | 2 × (+14.6233) | 2 × (+19.1180) | 2 × (-6) |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | I | 0 | +0.682 | +0.017 | +0.028 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | II | 0 | -0.227 | -0.006 | -0.009 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | I | 0 | +0.396 | +0.006 | -0.010 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | II | 0 | -0.132 | -0.002 | +0.003 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | I | 0 | +0.191 | +0.005 | +0.004 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | II | 0 | -0.064 | -0.002 | -0.001 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | I | 0 | +0.191 | +0.005 | +0.004 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | II | 0 | -0.064 | -0.002 | -0.001 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | I | 0 | +0.089 | +0.002 | +0.006 | 0 |
| $[V_\mu \times A_\mu] [V_\mu \times A_\mu]$ | II | 0 | -0.030 | -0.001 | -0.002 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | I | 0 | +0.625 | +1.231 | +1.227 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | II | 0 | -2.008 | -0.410 | -0.409 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | I | 0 | -0.387 | +0.140 | +0.138 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | II | 0 | +0.129 | -0.047 | -0.046 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | I | 0 | -1.528 | -0.435 | -0.385 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | II | 0 | +0.509 | +0.145 | +0.128 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | I | 0 | -1.528 | -0.435 | -0.385 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | II | 0 | +0.509 | +0.145 | +0.128 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | I | 0 | +0.833 | +0.179 | +0.146 | 0 |
| $[A_\mu \times V_\mu] [A_\mu \times V_\mu]$ | II | 0 | -0.278 | -0.060 | -0.049 | 0 |
| $[P \times T_{\mu\nu}] [P \times T_{\mu\nu}]$ | I | 0 | +0.916 | +0.004 | -0.033 | 0 |
| $[P \times T_{\mu\nu}] [P \times T_{\mu\nu}]$ | II | 0 | -0.305 | -0.001 | +0.011 | 0 |
| $[P \times T_{\mu\nu}] [P \times T_{\mu\nu}]$ | I | 0 | +0.299 | +0.015 | +0.012 | 0 |
| $[P \times T_{\mu\nu}] [P \times T_{\mu\nu}]$ | II | 0 | -0.100 | -0.005 | -0.004 | 0 |
| $[P \times P] [P \times P]$ | II | 2 × (-16) | 2 × (+111.266) | 2 × (+8.250) | 2 × (+6.925) | 2 × (-2) |
TABLE XII: Renormalization constants for $\mathcal{O}_i = (\mathcal{O}_i^{\text{latt}})_f = [P \times P][P \times P]_f + [S \times P][S \times P]_f$.

| $\mathcal{O}_i^{\text{latt}}$ | color | trace | $\hat{\gamma}_{ij}$ | $\hat{\mathcal{C}}_{ij}^{\text{latt}}$ | $\hat{\mathcal{C}}_{ij}^{\text{latt}}$ | $\hat{\mathcal{C}}_{ij}^{\text{latt}}$ | $\hat{T}_{ij}$ |
|------------------------------|-------|-------|----------------------|-------------------|-------------------|-------------------|------------------|
| $[S \times S][S \times S]$ | I     | 0     | -0.077               | -0.003            | -0.007            | 0                 |
| $[S \times S][S \times S]$ | II    | 0     | -0.254               | -0.006            | -0.016            | 0                 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I     | 0     | -0.977               | -0.043            | -0.036            | 0                 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II    | 0     | -0.419               | -0.019            | -0.015            | 0                 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I     | 0     | -0.099               | -0.003            | +0.001            | 0                 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II    | 0     | -0.042               | -0.001            | +0.000            | 0                 |
| $[S \times P][S \times P]$ | I     | 0     | -25.1880             | -3.2311           | -4.6654           | +2                |
| $[S \times P][S \times P]$ | II    | 0     | -3.9774              | -2.2370           | -2.3189           | 0                 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I     | 0     | +16.875              | +1.811            | +2.410            | -1                |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II    | 0     | -1.132               | -0.293            | -0.274            | 0                 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I     | 0     | +2.865               | +0.419            | +0.341            | 0                 |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II    | 0     | -0.955               | -0.140            | -0.114            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I     | 0     | +0.201               | +0.017            | +0.041            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II    | 0     | -0.388               | -0.015            | -0.030            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I     | 0     | +0.032               | +0.001            | +0.001            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II    | 0     | -0.096               | -0.002            | -0.002            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I     | 0     | +0.032               | +0.001            | +0.001            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II    | 0     | -0.096               | -0.002            | -0.002            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | I     | 0     | -0.524               | -0.012            | -0.017            | 0                 |
| $[V_{\mu} \times A_{\mu}][V_{\mu} \times A_{\mu}]$ | II    | 0     | -0.225               | -0.005            | -0.007            | 0                 |
| $[T_{\mu \nu \times S}[T_{\mu \nu \times S}$ | I     | 0     | -0.222               | -0.003            | -0.011            | 0                 |
| $[T_{\mu \nu \times S}[T_{\mu \nu \times S}$ | II    | 0     | -0.095               | -0.001            | -0.005            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | +0.407               | -0.027            | -0.058            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | -1.033               | -0.072            | -0.105            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | +1.374               | +0.060            | +0.044            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | -0.458               | -0.020            | -0.015            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | -0.127               | -0.004            | +0.001            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | +0.042               | +0.001            | +0.000            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | -0.127               | -0.004            | +0.001            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | +0.042               | +0.001            | +0.000            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | -0.051               | -0.004            | -0.003            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | +0.154               | +0.011            | +0.009            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | -0.051               | -0.004            | -0.003            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | +0.154               | +0.011            | +0.009            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | I     | 0     | +0.276               | -0.032            | -0.070            | 0                 |
| $[T_{\mu \nu \times T_{\mu \nu}][T_{\mu \nu \times T_{\mu \nu}}$ | II    | 0     | -0.638               | -0.055            | -0.068            | 0                 |
| $[T_{\mu \nu \times P][T_{\mu \nu \times P}$ | I     | 0     | +14/3                | -3.4998           | -0.0426           | +0.2458           | 0                 |
| $[T_{\mu \nu \times P][T_{\mu \nu \times P}$ | II    | 0     | +2                  | -1.4999           | -0.0183           | +0.1054           | 0                 |
TABLE XIII: Renormalization constants for $\hat{C}_{ij}^{\text{Latt}}$ (continued from Table XII).

| $O_{ij}^{\text{Latt}}$ | color trace | $\hat{\gamma}_{ij}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{T}_{ij}$ |
|------------------------|-------------|----------------------|-----------------------------|-----------------------------|-----------------------------|----------------|
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -16.068 | -1.614 | -2.224 | +1 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | II | 0 | -1.289 | -0.297 | -0.284 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -2.229 | -0.326 | -0.265 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | II | 0 | -0.955 | -0.140 | -0.114 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -0.152 | -0.065 | -0.058 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | II | 0 | +0.457 | +0.196 | +0.175 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -0.152 | -0.065 | -0.058 | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | II | 0 | +0.457 | +0.196 | +0.175 | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | I | 0 | +0.110 | -0.005 | -0.018 | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | II | 0 | -0.545 | -0.019 | -0.040 | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | I | 0 | +0.673 | +0.016 | +0.021 | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | II | 0 | -0.224 | -0.005 | -0.007 | 0 |
| $[P \times S][P \times S]$ | I | 0 | -0.077 | -0.003 | -0.007 | 0 |
| $[P \times S][P \times S]$ | II | 0 | -0.254 | -0.006 | -0.016 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I | 0 | -0.977 | -0.043 | -0.036 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II | 0 | -0.419 | -0.019 | -0.015 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I | 0 | -0.099 | -0.003 | +0.001 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II | 0 | -0.042 | -0.001 | +0.000 | 0 |
| $[P \times P][P \times P]$ | I | 0 | +2 | -39.8787 | -5.2542 | -6.5459 | +2 |
| $[P \times P][P \times P]$ | II | 0 | -6 | +40.0946 | +3.8324 | +3.3226 | 0 |
| $\mathcal{O}_0^{\text{Latt}}$ | color trace | $\tilde{\gamma}_{ij}$ | $\tilde{\gamma}_{ij}^{\text{NAIVE}}$ | $\tilde{\gamma}_{ij}^{\text{HYP(I)}}$ | $\tilde{\gamma}_{ij}^{\text{HYP(II)/Fat}}$ | $\tilde{T}_{ij}$ |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ | I 0 | -0.837 | -0.037 | -0.031 | 0 |
| $[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ | II 0 | +0.279 | +0.012 | +0.010 | 0 |
| $[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ | I 0 | -0.085 | -0.003 | +0.001 | 0 |
| $[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ | II 0 | +0.028 | +0.001 | +0.000 | 0 |
| $[S \times P][S \times P]$ | II | -16 | -95.6216 | -14.6233 | -19.1180 | +6 |
| $[V_\mu \times A_\nu][V_\mu \times A_\mu]$ | I 0 | -0.341 | -0.009 | -0.014 | 0 |
| $[V_\mu \times A_\nu][V_\mu \times A_\mu]$ | II 0 | +0.114 | +0.003 | +0.005 | 0 |
| $[V_\mu \times A_\nu][V_\mu \times A_\nu]$ | I 0 | -0.449 | -0.010 | -0.014 | 0 |
| $[V_\mu \times A_\nu][V_\mu \times A_\nu]$ | II 0 | +0.150 | +0.003 | +0.005 | 0 |
| $[V_\mu \times A_\nu][V_\mu \times A_\nu]$ | I 0 | -0.096 | -0.002 | -0.002 | 0 |
| $[V_\mu \times A_\nu][V_\mu \times A_\nu]$ | II 0 | +0.032 | +0.001 | +0.001 | 0 |
| $[T_{\mu\nu} \times S][T_{\mu\nu} \times S]$ | I 0 | -0.190 | -0.003 | -0.010 | 0 |
| $[T_{\mu\nu} \times S][T_{\mu\nu} \times S]$ | II 0 | +0.063 | +0.001 | +0.003 | 0 |
| $[T_{\mu\nu} \times P][T_{\mu\nu} \times P]$ | I | -4 | -2.9999 | -0.0366 | +0.2107 | 0 |
| $[T_{\mu\nu} \times P][T_{\mu\nu} \times P]$ | II | +4/3 | +1.0000 | +0.0122 | -0.0702 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | -3.013 | -0.615 | -0.613 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | +1.004 | +0.205 | +0.204 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | -1.910 | -0.279 | -0.227 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | +0.637 | +0.093 | +0.076 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +0.764 | +0.218 | +0.192 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | -0.255 | -0.073 | -0.064 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | I 0 | +0.764 | +0.218 | +0.192 | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II 0 | -0.255 | -0.073 | -0.064 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I 0 | -0.837 | -0.037 | -0.031 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II 0 | +0.279 | +0.012 | +0.010 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | I 0 | -0.085 | -0.003 | +0.001 | 0 |
| $[P \times T_{\mu\nu}][P \times T_{\mu\nu}]$ | II 0 | +0.028 | +0.001 | +0.000 | 0 |
| $[P \times P][P \times P]$ | II | -16 | +111.2658 | +8.2503 | +6.9246 | -2 |
TABLE XV: Renormalization constants for $O_i = (O^L_{ij})_I = -\frac{1}{2} [P \times P][P \times P]_I - \frac{1}{2} [S \times P][S \times P]_I + \frac{1}{2} \sum_{\mu < \nu} [T_{\mu \nu} \times P][T_{\mu \nu} \times P]_I$.

| $O^L_{ij}$ | color trace $\tilde{\gamma}_{ij}$ | $\hat{\hat{C}}^L_{ij}$ (NAIVE) | $\hat{\hat{C}}^L_{ij}$ (HYP(I)) | $\hat{\hat{C}}^L_{ij}$ (HYP(II)) | $\hat{T}_{ij}$ |
| --- | --- | --- | --- | --- | --- |
| $[S \times S][S \times S]$ | I | 0 | -0.294 | -0.003 | -0.013 | 0 |
| $[S \times S][S \times S]$ | II | 0 | -0.016 | +0.001 | +0.001 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I | 0 | +2.001 | +0.066 | +0.027 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II | 0 | -0.568 | -0.038 | -0.041 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I | 0 | +0.037 | +0.003 | +0.003 | 0 |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II | 0 | -0.111 | -0.010 | -0.009 | 0 |
| $[S \times P][S \times P]$ | I | +6 | +6.3443 | +1.5516 | +2.7015 | -1 |
| $[S \times P][S \times P]$ | II | +6 | -0.2612 | +1.0911 | +1.3175 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | I | 0 | -11.779 | -1.394 | -1.603 | +1/2 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | II | 0 | -0.866 | -0.063 | -0.034 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | I | 0 | -11.630 | -1.340 | -1.542 | +1/2 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | II | 0 | -1.320 | -0.227 | -0.217 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | I | 0 | -0.210 | -0.041 | -0.036 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | II | 0 | +0.631 | +0.122 | +0.107 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | I | 0 | -0.210 | -0.041 | -0.036 | 0 |
| $[V_\mu \times V_\nu][V_\mu \times V_\nu]$ | II | 0 | +0.631 | +0.122 | +0.107 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | I | 0 | +0.910 | +0.015 | +0.012 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | II | 0 | -0.143 | +0.000 | +0.004 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | I | 0 | +0.990 | +0.019 | +0.021 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | II | 0 | -0.385 | -0.012 | -0.024 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | I | 0 | -0.016 | +0.000 | +0.000 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | II | 0 | +0.048 | +0.001 | +0.001 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | I | 0 | -0.016 | +0.000 | +0.000 | 0 |
| $[V_\mu \times A_\mu][V_\mu \times A_\mu]$ | II | 0 | +0.048 | +0.001 | +0.001 | 0 |
TABLE XVI: Renormalization constants for $(\mathcal{O}_j^{\text{Latt}})_I$ (continued from Table XV).

| $\mathcal{O}_j^{\text{Latt}}$ | color trace | $\hat{c}_{ij}$ | $\hat{c}_{ij}$ | $\hat{c}_{ij}$ | $\hat{c}_{ij}$ | $\bar{T}_{ij}$ |
|--------------------------|-------------|----------------|----------------|----------------|----------------|--------------|
| $[T_{\mu\nu} \times S][T_{\mu\nu} \times S]$ | I | 0 | +0.358 | +0.004 | +0.017 | 0 |
| $[T_{\mu\nu} \times S][T_{\mu\nu} \times S]$ | II | 0 | −0.174 | −0.003 | −0.010 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | −1.761 | −0.055 | −0.023 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.151 | +0.007 | +0.030 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | −1.602 | −0.039 | +0.004 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.627 | −0.041 | −0.051 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.042 | +0.002 | +0.002 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.125 | −0.007 | −0.005 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.042 | +0.002 | +0.002 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.125 | −0.007 | −0.005 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.014 | +0.000 | +0.000 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.042 | −0.001 | +0.000 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.097 | +0.005 | +0.003 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.292 | −0.014 | −0.010 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.026 | +0.002 | +0.001 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.077 | −0.005 | −0.004 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | +0.026 | +0.002 | +0.001 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.077 | −0.005 | −0.004 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | I | 0 | −1.695 | −0.052 | −0.017 | 0 |
| $[T_{\mu\nu} \times T_{\rho\delta}][T_{\mu\nu} \times T_{\rho\delta}]$ | II | 0 | −0.349 | −0.002 | +0.012 | 0 |
| $[T_{\mu\nu} \times P][T_{\mu\nu} \times P]$ | I | −26/3 | −6.6186 | −1.8938 | −3.1543 | +1 |
| $[T_{\mu\nu} \times P][T_{\mu\nu} \times P]$ | II | +2 | −1.9161 | −0.0643 | +0.0410 | 0 |
TABLE XVII: Renormalization constants for $\langle O_\mu^{\text{latt}} \rangle_I$ (continued from Table XVI).

| $O_\mu^{\text{latt}}$ | color trace $\hat{\gamma}_{ij}$ | $\hat{C}_{ij}^{\text{latt}}$ | $\hat{C}_{ij}^{\text{latt}}$ | $\hat{C}_{ij}^{\text{latt}}$ | $\hat{T}_{ij}$ |
|---------------------|--------------------------------|------------------|------------------|------------------|----------------|
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | I 0 | +12.330 | +1.436 | +1.624 | −1/2 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | II 0 | −0.788 | −0.061 | −0.029 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | I 0 | +12.483 | +1.490 | +1.685 | −1/2 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | II 0 | −1.241 | −0.225 | −0.212 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | I 0 | +0.076 | +0.033 | +0.029 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | II 0 | −0.228 | −0.098 | −0.087 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | I 0 | +0.076 | +0.033 | +0.029 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | II 0 | −0.228 | −0.098 | −0.087 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | I 0 | −0.223 | −0.072 | −0.064 | 0 |
| \[ A_\mu \times V_\mu \] $\times \[ A_\mu \times V_\mu \]$ | II 0 | +0.669 | +0.215 | +0.191 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | I 0 | −0.841 | −0.016 | −0.016 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | II 0 | −0.064 | +0.002 | +0.009 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | I 0 | −0.760 | −0.012 | −0.007 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | II 0 | −0.306 | −0.010 | −0.019 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | I 0 | +0.016 | +0.000 | +0.000 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | II 0 | −0.048 | +0.001 | −0.001 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | I 0 | +0.016 | +0.000 | +0.000 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | II 0 | −0.048 | +0.001 | −0.001 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | I 0 | +0.032 | +0.001 | +0.001 | 0 |
| \[ A_\mu \times A_\mu \] $\times \[ A_\mu \times A_\mu \]$ | II 0 | −0.096 | −0.002 | −0.002 | 0 |
| \[ P \times S \] $\times \[ P \times S \]$ | I 0 | −0.294 | −0.003 | −0.013 | 0 |
| \[ P \times S \] $\times \[ P \times S \]$ | II 0 | −0.016 | +0.001 | +0.001 | 0 |
| \[ P \times T_{\mu \nu} \] $\times \[ P \times T_{\mu \nu} \]$ | I 0 | +2.067 | +0.069 | +0.033 | 0 |
| \[ P \times T_{\mu \nu} \] $\times \[ P \times T_{\mu \nu} \]$ | II 0 | −0.765 | −0.047 | −0.060 | 0 |
| \[ P \times T_{\mu \nu} \] $\times \[ P \times T_{\mu \nu} \]$ | I 0 | −0.014 | +0.000 | +0.000 | 0 |
| \[ P \times T_{\mu \nu} \] $\times \[ P \times T_{\mu \nu} \]$ | II 0 | +0.042 | +0.001 | +0.000 | 0 |
| \[ P \times P \] $\times \[ P \times P \]$ | I 6 | +14.6896 | +2.5631 | +3.6417 | −1 |
| \[ P \times P \] $\times \[ P \times P \]$ | II 6 | −22.2972 | −1.9436 | −1.5033 | 0 |
TABLE XVIII: Renormalization constants for $\mathcal{O}_i = (\mathcal{O}_{\mu}^{\text{Latt}})_{ij} = -\frac{1}{2}[P \times P][P \times P]_{ij} - \frac{1}{2}[S \times P][S \times P]_{ij} + \frac{1}{2} \sum_{\mu \nu} [T_{\mu \nu} \times P][T_{\mu \nu} \times P]_{ij}$.  

| $\mathcal{O}_{\mu}^{\text{Latt}}$ | color trace $\hat{\gamma}_{ij}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $\hat{T}_{ij}$ |
|----------------------------------|--------------------------------|-----------------|-----------------|-----------------|---------|
| $[S \times S][S \times S]$      | I                             | 0               | -0.286          | -0.004          | -0.015   | 0       |
| $[S \times S][S \times S]$      | II                            | 0 +0.095        | +0.001          | +0.005          | 0       |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I                             | 0 +0.419        | +0.019          | +0.015          | 0       |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II                            | 0 -0.140        | -0.006          | -0.005          | 0       |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | I                             | 0 +0.042        | +0.001          | +0.000          | 0       |
| $[S \times T_{\mu \nu}][S \times T_{\mu \nu}]$ | II                            | 0 -0.014        | +0.000          | +0.000          | 0       |
| $[S \times P][S \times P]$      | I                             | 0 +6 -4.998     | -0.0548         | +0.3161         | 0       |
| $[S \times P][S \times P]$      | II                            | 0 +6 +49.3107   | +7.3299         | +9.4537         | -3      |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I                             | 0 -2.864        | -0.419          | -0.341          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II                            | 0 +0.955        | +0.140          | +0.114          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I                             | 0 -3.417        | -0.587          | -0.534          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II                            | 0 +1.139        | +0.196          | +0.178          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I                             | 0 +0.382        | +0.109          | +0.096          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II                            | 0 -0.127        | -0.036          | -0.032          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I                             | 0 +0.382        | +0.109          | +0.096          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II                            | 0 -0.127        | -0.036          | -0.032          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | I                             | 0 +0.764        | +0.218          | +0.192          | 0       |
| $[V_{\mu} \times V_{\mu}][V_{\mu} \times V_{\mu}]$ | II                            | 0 -0.255        | -0.073          | -0.064          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | I                             | 0 +0.170        | +0.004          | +0.007          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | II                            | 0 -0.057        | -0.001          | -0.002          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | I                             | 0 +0.225        | +0.005          | +0.007          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | II                            | 0 -0.075        | -0.002          | -0.002          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | I                             | 0 +0.048        | +0.001          | +0.001          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | II                            | 0 -0.016        | +0.000          | +0.000          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | I                             | 0 +0.048        | +0.001          | +0.001          | 0       |
| $[V_{\mu} \times A_{\nu}][V_{\mu} \times A_{\nu}]$ | II                            | 0 -0.016        | +0.000          | +0.000          | 0       |
TABLE XIX: Renormalization constants for \( (\mathcal{O}^{\text{Latt}}_{\mu\nu})_{II} \) (continued from Table XVIII).

| \( \mathcal{O}^{\text{Latt}}_{\mu\nu} \) | color trace | \( \hat{\gamma}_{ij} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{C}_{ij}^{\text{Latt}} \) | \( \hat{T}_{ij} \) |
|---------------------------------|--------------|----------------|----------------|----------------|----------------|----------------|
| \( [T_{\mu\nu} \times S][T_{\mu\nu} \times S] \) | I | 0 | 0.0095 | 0.0001 | 0.0005 | 0 |
| \( [T_{\mu\nu} \times S][T_{\mu\nu} \times S] \) | II | 0 | -0.032 | 0.0000 | -0.002 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -1.335 | -0.059 | -0.045 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.445 | 0.020 | 0.015 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -1.712 | -0.102 | -0.117 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.571 | 0.034 | 0.039 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -0.201 | -0.012 | -0.010 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.067 | 0.004 | 0.003 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -0.201 | -0.012 | -0.010 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.067 | 0.004 | 0.003 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -0.445 | -0.025 | -0.019 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.148 | 0.008 | 0.006 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -0.042 | -0.001 | 0.000 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.014 | 0.000 | 0.000 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | I | 0 | -1.335 | -0.059 | -0.045 | 0 |
| \( [T_{\mu\nu} \times T_{\mu\nu}][T_{\mu\nu} \times T_{\mu\nu}] \) | II | 0 | 0.445 | 0.020 | 0.015 | 0 |
| \( [T_{\mu\nu} \times P][T_{\mu\nu} \times P] \) | I | -2 | 1.4999 | 0.0183 | -0.1054 | 0 |
| \( [T_{\mu\nu} \times P][T_{\mu\nu} \times P] \) | II | +10/3 | -15.1221 | -1.9317 | -2.4186 | +1 |
TABLE XX: Renormalization constants for \((O_4^{\text{Latt}})_{IJ}\) (continued from Table XXX).

| \(O_4^{\text{Latt}}\) | color trace | \(\hat{\gamma}_{ij}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{C}_{ij}^{\text{Latt}}\) | \(\hat{T}_{ij}\) |
|------------------------|-------------|-------------------|-----------------|-----------------|-----------------|-----------------|
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | I           | 0                 | 1.506           | 0.308           | 0.307           | 0               |
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | II          | 0                 | -0.502          | -0.103          | -0.102          | 0               |
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | I           | 0                 | 0.955           | 0.140           | 0.114           | 0               |
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | II          | 0                 | -0.318          | -0.047          | -0.038          | 0               |
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | I           | 0                 | -0.382          | -0.109          | -0.096          | 0               |
| \([A_\mu \times V_\mu][A_\mu \times V_\mu]\) | II          | 0                 | 0.127           | 0.036           | 0.032           | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | I           | 0                 | -0.673          | -0.016          | -0.021          | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | II          | 0                 | 0.224           | 0.005           | 0.007           | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | I           | 0                 | -1.014          | -0.032          | -0.059          | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | II          | 0                 | 0.338           | 0.011           | 0.020           | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | I           | 0                 | -0.143          | -0.003          | -0.003          | 0               |
| \([A_\mu \times A_\mu][A_\mu \times A_\mu]\) | II          | 0                 | 0.048           | 0.001           | 0.001           | 0               |
| \([P \times S][P \times S]\) | I           | 0                 | -0.286          | -0.004          | -0.015          | 0               |
| \([P \times S][P \times S]\) | II          | 0                 | 0.095           | 0.001           | 0.005           | 0               |
| \([P \times T_{\mu \nu}][P \times T_{\mu \nu}]\) | I           | 0                 | 0.419           | 0.019           | 0.015           | 0               |
| \([P \times T_{\mu \nu}][P \times T_{\mu \nu}]\) | II          | 0                 | -1.40           | -0.006          | -0.005          | 0               |
| \([P \times T_{\mu \nu}][P \times T_{\mu \rho}]\) | I           | 0                 | 0.042           | 0.001           | 0.000           | 0               |
| \([P \times T_{\mu \nu}][P \times T_{\mu \rho}]\) | II          | 0                 | -0.014          | 0.000           | 0.000           | 0               |
| \([P \times P][P \times P]\) | I           | 6                 | -4.4998         | -0.0548         | 0.3161          | 0               |
| \([P \times P][P \times P]\) | II          | 6                 | -54.1330        | -4.1069         | -3.5677         | +1              |