“K-THEORETIC” ANALOG OF POSTNIKOV-SHAPIRO
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Abstract. In this paper we study a filtered “K-theoretical” analog of a
graded algebra associated to any loopless graph $G$ which was introduced in
[4]. We show that two such filtered algebras are isomorphic if and only if their
graphs are isomorphic. We also study a large family of filtered generalizations
of the latter graded algebra which includes the above “K-theoretical” analog.

1. Introduction

The following square-free algebra $C_G$ associated to an arbitrary vertex labeled
graph $G$ was defined in [4], see also [1]. Let $G$ be a graph without loops on the vertex
set $\{0, \ldots, n\}$. (Below we always assume that all graphs might have multiple edges,
but no loops). Throughout the whole paper, we fix a field $K$ of zero characteristic.
Let $\Phi_G$ be the graded commutative algebra over $K$ generated by the variables
$\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G.$$ 

Let $C_G$ be the subalgebra of $\Phi_G$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1, \ldots, n$, where

$$c_{i,e} = \begin{cases} 
1 & \text{if } e = (i, j), i < j; \\
-1 & \text{if } e = (i, j), i > j; \\
0 & \text{otherwise.} \quad (1)
\end{cases}$$

For the reasons which will be clear soon, we call $C_G$ the spanning forests counting
algebra of $G$. Its Hilbert series and the set of defining relations were calculated in
[5] following the initial paper [6]. Namely, let $J_G$ be the ideal in $K[x_1, \ldots, x_n]$ generated by the polynomials

$$p_I = \left( \sum_{i \in I} x_i \right)^{D_I+1}, \quad (2)$$

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and $D_I = \sum_{i \in I} d_I(i)$, where $d_I(i)$ is the total number of edges connecting a given vertex $i \in I$ with all vertices
outside $I$. Thus, $D_I$ is the total number of edges between $I$ and the complementary
set of vertices $\bar{I}$. Set $B_G := K[x_1, \ldots, x_n]/J_G$.

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Remark 1. Observe that since \( \sum_{i=0}^{n} X_i = 0 \), we can define \( C_G \) as the subalgebra of \( \Phi_G \) generated by \( X_0, X_1, \ldots, X_n \).

We can also define \( B_G \) as the quotient algebra of \( \mathbb{K}[x_0, \ldots, x_n] \) by the ideal generated by \( p_I \), where \( I \) runs over all subsets of \( \{x_0, x_1, \ldots, x_n\} \). This follows from the relation

\[
p_I = \left( \sum_{i \in I} x_i \right)^{D_I+1} = \left( p_{\{0,1,\ldots,n\}} - \sum_{i \in I} x_i \right)^{D_I+1}.
\]

To describe the Hilbert polynomial of \( C_G \), we need the following classical notion going back to W. T. Tutte. Given a simple graph \( G \), fix an arbitrary linear order of its edges. Now, given a spanning forest \( F \) in \( G \) (i.e., a subgraph without cycles which includes all vertices of \( G \)) and an edge \( e \in G \setminus F \) in its complement, we say that \( e \) is externally active for \( F \), if there exists a cycle \( C \) in \( G \) such that all edges in \( C \setminus \{e\} \) belong to \( F \) and \( e \) is minimal in \( C \) with respect to the chosen linear order. The total number of external edges is called the external activity of \( F \). Although the external activity of a given forest/tree in \( G \) depends on the choice of a linear ordering of edges, the total number of forests/trees with a given external activity is independent of this ordering. Now we are ready to formulate the main result of [5].

**Theorem 1.** [Theorems 3 and 4 of [5]] For any simple graph \( G \), algebras \( B_G \) and \( C_G \) are isomorphic. The total dimension of these algebras (as vector spaces over \( \mathbb{K} \)) is equal to the number of spanning subforests in \( G \). The dimension of the \( k \)-th graded component of these algebras equals the number of subforests \( F \) in \( G \) with external activity \( |G| - |F| - k \). Here \( |G| \) (resp. \( |F| \)) stands for the number of edges in \( G \) (resp. \( F \)).

In the above notation, our main object will be the filtered subalgebra \( K_G \subset \Phi_G \) defined by the generators:

\[
Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.
\]

(Notice that we have one more generator here than in the previous case.)

**Remark 2.** Since \( Y_i \) is obtained by exponentiation of \( X_i \), we call \( K_G \) the “K-theoretic” analog of \( C_G \).

Our first result is as follows. Define the ideal \( I_G \) in \( \mathbb{K}[y_0, y_1, \ldots, y_n] \) as generated by the polynomials

\[
q_I = \left( \prod_{i \in I} y_i - 1 \right)^{D_I+1}, \quad (3)
\]

where \( I \) ranges over all nonempty subsets in \( \{0, 1, \ldots, n\} \) and the number \( D_I \) is the same as in [2]. Set \( D_G := \mathbb{K}[y_0, \ldots, y_n]/I_G \).

**Theorem 2.** For any graph \( G \), algebras \( B_G, C_G, D_G \) and \( K_G \) are isomorphic as (non-filtered) algebras.

Moreover, the following stronger statement holds.

**Theorem 3.** For any graph \( G \), algebras \( D_G \) and \( K_G \) are isomorphic as filtered algebras.

Recall that in a recent paper [3] the first author has shown that \( C_G \) contains all information about the matroid of \( G \) and only it. Namely,
Theorem 4. [Theorem 5 of [3]] Given two graphs $G_1$ and $G_2$, algebras $C_{G_1}$ and $C_{G_2}$ are isomorphic if and only if the matroids of $G_1$ and $G_2$ coincide. (The latter isomorphisms can be thought of either as graded or as non-graded, the statement holds in both cases.)

On the other hand, filtered algebras $D_G$ and $K_G$ contain complete information about $G$.

Theorem 5. Given two graphs $G_1$ and $G_2$ without isolated vertices, $K_{G_1}$ and $K_{G_2}$ are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic.

The structure of this paper is as follows. In §2 we prove new results formulated above. In §3 we discuss Hilbert series of similar algebras defined by other sets of generators. In §4 we discuss “K-theoretic” analogs of algebras counting spanning trees. Finally, in §5 we present a number of open problems.

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2. Proofs

To prove Theorem 2, we need some preliminary results.

Lemma 1. For any simple graph $G$, algebras $C_G$ and $K_G$ coincide as subalgebras of $\Phi_G$.

Proof. Since $(X_i)^{d_i+1} = 0$, where $d_i$ is the degree of vertex $i$, then

$$Y_i = \exp(X_i) = 1 + \sum_{j=1}^{d_i} \frac{(X_i)^j}{j!}.$$ 

Hence $Y_i \in C_G$ which means that $K_G \subseteq C_G \subseteq \Phi_G$.

To prove the opposite inclusion, consider $\tilde{Y}_i = Y_i - 1 = \exp(X_i) - 1$. Since $X_i|\tilde{Y}_i$, we get

$$X_i^{d_i+1} = 0.$$ 

Using the relation $X_i = \ln(1 + \tilde{Y}_i) = \sum_{j=1}^{d_i} \frac{(-1)^{j-1}(\tilde{Y}_i)^j}{j!}$, we conclude $X_i \in K_G$. Thus $C_G \subseteq K_G$, implying that $C_G$ and $K_G$ coincide. $\square$

Lemma 2. For any simple graph $G$, algebras $B_G$ and $D_G$ are isomorphic as (non-filtered) algebras.

Proof. First we change the variables in $D_G$ by using $\tilde{y}_i = y_i - 1$, $i = 0, 1, \ldots, n$. The generators of ideal $I_G$ transform as

$$\tilde{q}_I = \left( \prod_{i \in I} (\tilde{y}_i + 1) - 1 \right)^{d_I+1},$$

for any subset $I \subseteq \{0, 1, \ldots, n\}$.

Since for every vertex $i = 0, 1, \ldots, n$,

$$(\tilde{y}_i + 1)^{d_i+1} = \tilde{y}_i^{d_i+1},$$

we can consider $D_G$ as the quotient $\mathbb{K}[[\tilde{y}_0, \ldots, \tilde{y}_n]]/\tilde{I}_G$ of the ring of formal power series factored by the ideal $\tilde{I}_G$ generated by all $\tilde{q}_I$.

Similarly we can consider $B_G$ as the quotient $\mathbb{K}[[x_0, \ldots, x_n]]/\tilde{J}_G$ of the ring of formal power series by the ideal $\tilde{J}_G$ generated by all $p_I$. 

Introduce the homomorphism $\psi : [y_0, \ldots, y_n] \mapsto [x_0, \ldots, x_n]$ defined by:

$$\psi : y_i \mapsto e^{x_i} - 1.$$  

In fact, $\psi$ is an isomorphism, because $\psi^{-1}$ is defined by $x_i \mapsto \ln(1 + y_i)$.

Let us look at what happens with the ideal $\tilde{I}_G$ under the action of $\psi$. For a given $I \subset \{0, 1, \ldots, n\}$, consider the generator $\tilde{q}_I$. Then,

$$\psi(\tilde{q}_I) = \left( \prod_{i \in I} (\psi(y_i) + 1) - 1 \right)^{D_1 + 1} = \left( \prod_{i \in I} e^{x_i} - 1 \right)^{D_1 + 1} =$$

$$= \left( \exp \left( \sum_{i \in I} x_i \right) - 1 \right)^{D_1 + 1} = \left( \sum_{i \in I} x_i \right)^{D_1 + 1} \left( \frac{\exp \left( \sum_{i \in I} x_i \right) - 1}{\sum_{i \in I} x_i} \right)^{D_1 + 1} .$$

The factor $\exp \left( \sum_{i \in I} x_i \right) - 1$ is a formal power series starting with the constant term 1. Hence the last factor in the right-hand side of the latter expression is an invertible power series. Thus, the generator $\tilde{q}_I$ is mapped by $\psi$ to the product $\tilde{p}_I \ast \ast$, where $\ast$ is an invertible series. This implies $\psi(\tilde{I}_G) = \tilde{J}_G$. Hence algebras $D_G$ and $B_G$ are isomorphic.

Proof of Theorem 2. By Lemmas 1 and Theorem 1 we get that all four algebras are isomorphic to each other. Furthermore, by Theorem 1 we know that their total dimension over $K$ is the number of subforests in $G$.

Theorem 3 now follows from Theorem 2.

Proof of Theorem 3. Consider the surjective homomorphism $h : D_G \to K_G$, defined by:

$$h(y_i) = Y_i, \ i = 0, 1, \ldots, n.$$  

(It is indeed a homomorphism because every relation $q_I$ holds for $Y_0, \ldots, Y_n$.) By Theorem 2 we know that these algebras have the same dimension, implying that $h$ is an isomorphism. It is clear that $h$ preserves the filtration.

2.1. Proving Theorem 5. We start with a few definitions.

Given a commutative algebra $A$, its element $t \in A$ is called reducible nilpotent if and only if there exists a presentation $t = \sum u_i v_i$, where all $u_i, v_i$ are nilpotents.

For a nilpotent element $t \in A$, define its degree $d(t)$ as the minimal non-negative integer for which there exists a reducible nilpotent element $h \in A$ such that

$$(t - h)^{d+1} = 0 .$$

Given an element $R \in \Phi_G$, we say that an edge-element $\phi_e$ belongs to $R$, if monomial $\phi_e$ has a non-zero coefficient in the expansion of $R$ as the sum of square-free monomials in $\Phi_G$.

Lemma 3. For any nilpotent element $R \in K_G \subset \Phi_G$, the degree $d(R)$ of $R$ equals the number of edges of $G$ belonging to $R$.  

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Proof. We can write $R$ in terms of $\{X_0, \ldots, X_n\}$. (Observe that $K_G$ and $C_G$ coincide as subsets of $\Phi_G$, but have different graded/filtered structures). Now we can concentrate on the graded structure of $C_G$. Select the part of $R$ which lies in the first graded component of $C_G$. Thus

$$R = R_1 + R' = \sum_{i=0}^{n} a_i X_i + R', $$

where $R'$ is reducible nilpotent because it belongs to the linear span of other graded components. Thus $d(R) = d(R_1)$. The statement of Lemma 5 is obvious for $R_1$. Additionally by construction, an edge-element $\phi_e$ belongs to $R$ if and only if it belongs to $R_1$.

Lemma 4. Given a graph $G$, let $\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ be the set of generators of $K_G$ corresponding to the vertices (i.e., $\tilde{Y}_i = \exp(X_i) - 1$). Then

1. $\{\tilde{Y}_0, \ldots, \tilde{Y}_n\}$ are nilpotents;
2. $\sum_{i=0}^{n} \ln(1 + \tilde{Y}_i) = 0$;
3. for any subset $I \subseteq [0, n]$ and any set of pairwise distinct non-zero numbers $a_i \in \mathbb{R}$ ($i \in I$), the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is equal to the number of edges incident to the vertices belonging to $I$;
4. the number of edges between vertices $i$ and $j$ equals to $\frac{d(\tilde{Y}_i) + d(\tilde{Y}_j) - d(\tilde{Y}_i + \tilde{Y}_j)}{2}$.

Proof. Item (1) is obvious.

To settle (2), observe that $\ln(1 + \tilde{Y}_i) = X_i$ which implies

$$\sum_{i=0}^{n} \ln(1 + \tilde{Y}_i) = \sum_{i=0}^{n} X_i = 0.$$

To prove (3), notice that, by Lemma 5 the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is equal to the number of edges belonging to the sum $\sum_{i \in I} a_i \tilde{Y}_i$. Each edge belongs either to zero, to one or to two generators $\tilde{Y}_i$ from the latter sum. Moreover, if an edge belongs to two generators, then it has coefficients of opposite signs. Since all $a_i$ are different, an edge-element $\phi_e$ belongs to $\sum_{i \in I} a_i \tilde{Y}_i$ if and only if it belongs to at least one $\tilde{Y}_i$, for $i \in I$. Thus the degree $d(\sum_{i \in I} a_i \tilde{Y}_i)$ is the number of edges incident to all vertices from $I$.

To settle (4), notice that if $e$ is an edge between vertices $i$ and $j$, then $\phi_e$ belongs to $\tilde{Y}_i$ and to $\tilde{Y}_j$ with the opposite coefficients. Therefore $\phi_e$ does not belong to $(\tilde{Y}_i + \tilde{Y}_j)$. Using Lemma 5, we get that $d(\tilde{Y}_i) + d(\tilde{Y}_j) - d(\tilde{Y}_i + \tilde{Y}_j)$ equals twice the number of edges between $i$ and $j$. \qed

Our proof of Theorem 5 uses the following technical lemma which should be obvious to the specialists.

Lemma 5 (Folklore). Let $E$ be the set of edges of some graph $G$ without isolated vertices. If we know the following information:

1. which pairs $e_i, e_j \in E$ of edges are multiple, i.e., connect the same pair of vertices;
2. which pairs $e_i, e_j \in E$ of edges have exactly one common vertex;
3. which triples $e_i, e_j, e_k \in E$ of edges form a triangle,

then we can reconstruct $G$ up to an isomorphism.
Proof. Assume the contrary, i.e., that there exist two non-isomorphic graphs $G$ and $G'$ such that there exists a bijection $\psi$ of their edge sets $E$ and $E'$ preserving (1) - (3). Assume that under this bijection an edge $e \in E$ corresponds to the edge $e' \in E'$. Additionally assume that $|V(G')| \geq |V(G)|$.

Now we construct an isomorphism between $G$ and $G'$. Let us split the vertices of $G$ into two subsets: $V(G) = \tilde{V}(G) \cup \hat{V}(G)$, where $\tilde{V}(G)$ are all vertices which are incident to some pair of non-multiple edges.

Let us construct a bijection $\psi$ between the vertices of $G$ and $G'$, which extends the given bijection $\psi$ of edges, i.e., for any $e = uv \in E$, $e' = \psi(e) = \psi(u)\psi(v)$.

At first we define it on $\tilde{V}(G)$. Namely, given a vertex $v \in \tilde{V}(G)$, choose two non-multiple edges $e_i$ and $e_j$ incident to it, and define $\psi(v)$ as a common vertex of $e_i'$ and $e_j'$. We need to show that $\psi(v)$ does not depend on the choice of $e_i$ and $e_j$. It is enough to check it for a pair $e_i$ and $e_k$ where $e_k$ is another edge incident to $v$. Indeed, if $e_k'$ has no common vertex with both $e_i'$ and $e_j'$, then $e_i'$, $e_j'$ and $e_k'$ form a triangle in $G'$ (because $e_k'$ has a common vertex with $e_j'$ and with $e_i'$). Hence, $e_i$, $e_j$ and $e_k$ form a triangle in $G$, but they have a common vertex $v$. Contradiction.

Now we need to extend $\psi$ to vertices belonging to $\hat{V}(G_1)$. Note that each vertex $v \in \hat{V}(G_1)$ has exactly one adjacent vertex. There are two possibilities.

1° Adjacent vertex $u$ of $v$ belongs to $\tilde{V}(G)$. Consider the edge $e_{uv} \in E$. (There might be several such edges, but this is not important, because in $G'$ they are also multiple.) Knowing the image $\psi(e_{uv})$ and the vertex $\psi(u)$, we define $\psi(v)$ as the vertex of $\psi(e_{uv})$ different from $\psi(u)$.

2° Adjacent vertex $u$ of $v$ belongs to $\hat{V}(G)$. Consider edge $e_{uv} \in E$ Knowing $\psi(e_{uv})$, we define $\psi(u)$ and $\psi(v)$ as the vertices of edge $\psi(e_{uv})$ (not important which is mapped to which).

Since $G'$ has no isolated vertices and each edge $e'$ has exactly two incident vertices from $\psi(V)$, we get that $\psi : G \rightarrow G'$ is surjective. Hence, $\psi : G \rightarrow G'$ is an isomorphism (otherwise it must be non-injective on vertices and, hence, $|V(G)| > |V(G')|$). Therefore $G$ and $G'$ are isomorphic.

Proof of Theorem Let $G$ and $G'$ be a pair of graphs such that their filtered algebras $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are isomorphic. Without loss of generality, we can assume that $|E(G)| \leq |E(G')|$. Denote the numbers of vertices in $G$ and $G'$ by $n + 1$ and $n' + 1$ resp.

We consider $\mathcal{K}_G$ as a subalgebra in $\Phi_G$. The elements $\tilde{Y}_i = \exp(X_i) - 1$, $i \in [0, n]$ form a set of generators of $\mathcal{K}_G$. Denote by $\hat{Z}_i \in \mathcal{K}_{G'}$, $i \in [0, n']$ the elements corresponding to the vertices of $G'$ under the isomorphism of filtered algebras. The set $\{\hat{Z}_i, i \in [0, n']\}$ is also a generating set for $\mathcal{K}_{G'}$ which gives the same filtered structure and satisfies the assumptions of Lemma. In order to avoid confusion, we call $\hat{Y}_i$ the $i$-th vertex of graph $G$, and we call $\hat{Z}_j$ the $j$-th vertex of graph $G'$.

Since $\tilde{Y}_i$, $i \in [0, n]$ and $\hat{Z}_i$, $i \in [0, n']$ determine the same graded structure, then, in particular,

$$\text{span}\{1, \tilde{Y}_0, \ldots, \tilde{Y}_n\} = \text{span}\{1, \hat{Z}_0, \ldots, \hat{Z}_{n'}\}.$$  

Additionally, by Lemma $\hat{Y}_i$, $i \in [0, n]$ and $\hat{Z}_i$, $i \in [0, n']$ are nilpotents, implying that

$$\text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\} = \text{span}\{\hat{Z}_0, \ldots, \hat{Z}_{n'}\}.$$  

Firstly, we need to show that each edge-element $\phi_e$ belongs to at most two different $\hat{Z}_i$'s. Assume the contrary, i.e., that $\phi_e$ belongs to $\hat{Z}_i, \hat{Z}_j$ and $\hat{Z}_k$. Then there exist three distinct non-zero coefficients $r_1, r_2, r_3 \in \mathbb{K}$ such that $\phi_e$ does not belong to $r_1\hat{Z}_i + r_2\hat{Z}_j + r_3\hat{Z}_k$. Moreover, for generic distinct non-zero coefficients $r'_1, r'_2, r'_3 \in \mathbb{K}$, element $\phi_{e'}$ ($e' \in E(G)$) belongs to $r'_1\hat{Z}_i + r'_2\hat{Z}_j + r'_3\hat{Z}_k$ if and only if...
\( \phi' \) belongs to at least one of \( Z_i, Z_j \) and \( Z_k \). Hence by Lemma 3,
\[
d(r_1 Z_i + r_2 Z_j + r_3 Z_k) < d(r_1' Z_i + r_2' Z_j + r_3' Z_k).
\]

But at the same time, by Lemma 4 (3), they should coincide, contradiction.

By Lemma 4 for any \( i \in [0, n'] \), the degree \( d(Z_i) \) equals to the valency of \( Z_i \).

Therefore,
\[
2|E(G')| = \sum_{i=0}^{n'} d(Z_i) \leq 2|E(G)|.
\]

because each edge-element is included in at most two \( Z_i \). Since \( |E(G)| \leq |E(G')| \), we conclude that \( |E(G)| = |E(G')| \). Furthermore, by Lemma 4 (2), each element \( \phi_e, e \in E(G) \) belongs exactly to two vertices from \( Z_i, i \in [0, n'] \) with the opposite coefficients. Since \( |E(G)| = |E(G')| \), we can additionally assume that the number of pairs of non-multiple edges which have a common vertex in \( G' \) is bigger than that in \( G \).

So far we have constructed a bijection between the edges of \( G \) and the edges of \( G' \). We want to prove that this bijection provides a graph isomorphism. We will achieve this as a result of the 5 claims collected in the following proposition which is closely related to Lemma 5.

**Proposition 1.** The following facts hold.

1. If \( \phi_{e_1} \) and \( \phi_{e_2} \) have no common vertex in \( G \), then they have no common vertex in \( G' \) as well.
2. If \( \phi_{e_1} \) and \( \phi_{e_2} \) are multiple edges in \( G \), then they are multiple edges in \( G' \) as well.
3. If \( \phi_{e_1} \) and \( \phi_{e_2} \) have exactly one common vertex in \( G \), then they have exactly one common vertex in \( G' \) as well.
4. If \( \phi_{e_1}, \phi_{e_2}, \) and \( \phi_{e_3} \) form a star in \( G \), then they form a star in \( G' \) as well.
   (Three edges form a star if they have one common vertex and their three other ends are distinct.)
5. If \( \phi_{e_1}, \phi_{e_2} \) and \( \phi_{e_3} \) form a triangle in \( G \), then they form a triangle in \( G' \) as well.

**Proof.** To prove (1), assume the contrary, i.e., assume that \( \phi_{e_1} \) and \( \phi_{e_2} \) belong to \( Z_j \) (and denote the corresponding coefficients by \( a \) and \( b \) resp.). Since elements \( \tilde{Y}_0, \ldots, \tilde{Y}_n \) have no monomial \( \phi_{e_1} \phi_{e_2} \), then \( \tilde{Z}_0, \ldots, \tilde{Z}_{n'} \) have no monomial \( \phi_{e_1} \phi_{e_2} \) as well (since their spans coincide). Then \( \ln(1 + \tilde{Z}_j) \) contains the monomial \( \phi_{e_1} \phi_{e_2} \) with the coefficient \( -ab \).

By Lemma 4 (2), we have \( \sum_{i=0}^{n'} \ln(1 + \tilde{Z}_i) \), then there exists \( k \in [0, n'], k \neq i \) such that \( \ln(1 + \tilde{Z}_k) \) contains the monomial \( \phi_{e_1} \phi_{e_2} \) with a non-zero coefficient. Then \( \tilde{Z}_k \) must contain \( \phi_{e_1} \) and \( \phi_{e_2} \) (since \( \tilde{Z}_k \) does not contain \( \phi_{e_1} \phi_{e_2} \)). Hence, \( \tilde{Z}_k \) has \( \phi_{e_1} \) and \( \phi_{e_2} \) with coefficients \( -a \) and \( -b \) resp. Therefore \( \ln(1 + \tilde{Z}_k) \) contains monomial \( \phi_{e_1} \phi_{e_2} \) with the coefficient \( -(a)(-b) = ab \). Thus the sum \( \sum_{j=0}^{n'} \ln(1 + \tilde{Z}_j) \) contains \( \phi_{e_1} \phi_{e_2} \) with coefficient \( -2ab \), contradiction.

To prove (2), consider the map from \( \text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\} \) to \( \mathbb{K}^2 \), sending an element from the span to the pair of coefficients of \( \phi_{e_1} \) and \( \phi_{e_2} \) resp. Since edges \( e_1 \) and \( e_2 \) are multiple in \( G \), the image of this map has dimension 1. If \( \phi_{e_1} \) and \( \phi_{e_2} \) are not multiple in \( G' \), then the image of the map from \( \text{span}\{\tilde{Z}_0, \ldots, \tilde{Z}_{n'}\} = \text{span}\{\tilde{Y}_0, \ldots, \tilde{Y}_n\} \) has dimension 2.

To prove (3), observe that we have already settled Claims 1 and 2, and also we additionally assumed that the number of pairs of edges which have a common
vertex in $G'$ is bigger than that in $G$. Then each such pair of edges from $G$ is mapped to the pair of edges from $G'$ with the same property.

To prove (4), consider the map from span$\{Y_0, \ldots, Y_n\}$ to $\mathbb{K}^3$, sending an element in the span to the triple of coefficients of $\phi_{e_1}$, $\phi_{e_2}$ and $\phi_{e_3}$ resp. The image of this map has dimension 3. However if $\phi_{e_1}$, $\phi_{e_2}$ and $\phi_{e_3}$ form a triangle in $G'$, then the image of the map from span$\{Z_0, \ldots, Z'_n\}$ has dimension 2.

Proof of (5) is similar to that of (4).

Now applying Lemma 5 we finish our proof of Theorem 5.

3. FURTHER GENERALIZATIONS

In this section we will consider the Hilbert series of other filtered algebras similar to $K_G$. (Recall that the Hilbert series of a filtered algebra is, by definition, the Hilbert series of its associated graded algebra.)

Let $f$ be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $F[f]_G \subset \Phi_G$ as generated by 1 together with

$$f(X_i) = f\left(\sum c_{i,e} \phi_e\right), \quad i = 0, \ldots, n.$$ 

**Example 1.** For $f(x) = x$, $F[f]_G$ coincides with $C_G$. For $f(x) = \exp(x)$, $F[f]_G$ coincides with $K_G$.

Obviously, the filtered algebra $F[f]_G$ does not depend on the constant term of $f$. From now on, we assume that $f(x)$ has no constant term, since for any $g$ such that $f - g$ is constant, the filtered algebras $F[f]_G$ and $F[g]_G$ are the same.

**Proposition 2.** Let $f$ be any polynomial with a non-vanishing linear term. Then algebras $C_G$ and $F[f]_G$ coincide as subalgebras of $\Phi_G$.

**Proof.** The argument is the same as in the proof of Lemma 1. We only need to change exp$(x) - 1$ to $f(x)$ and ln$(1 + y)$ to $f^{-1}(y)$.

**Theorem 6.** Let $f$ be any polynomial with non-vanishing linear and quadratic terms. Then given two simple graphs $G_1$ and $G_2$, $F[f]_{G_1}$ and $F[f]_{G_2}$ are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic graphs.

**Proof.** Repeat the proof of Theorem 5.

3.1. **Generic functions $f$ and their Hilbert series.** Since $X_i^{d_i+1} = 0$ for any $i$, we can always truncate any polynomial (or a formal power series) $f$ at degree $|G|+1$ without changing $F[f]_G$. Therefore, for a given graph $G$, it suffices to consider $f$ as a polynomial of degrees less than or equal to $|G|$. To simplify our notation, let us write $HS_{f,G}$ instead of $HS_{F[f]_G}$.

Given a graph $G$, consider the space of polynomials of degree less than or equal to $|G|$ and the corresponding Hilbert series.

**Proposition 3.** In the above notation, for generic polynomials $f$ of degree at most $|G|$, the Hilbert series $HS_{f,G}$ is the same. This generic Hilbert series (denoted by $HS_G$ below) is maximal in the majorization partial order among all $HS_{g,G}$, where $g$ runs over the set of all formal power series with non-vanishing linear term.

Recall that, by definition, a sequence $(a_0, a_1, \ldots)$ is bigger than $(b_0, b_1, \ldots)$ in the majorization partial order if and only if, for any $k \geq 0$,

$$\sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i.$$ 

More information about the majorization partial order can be found in e.g. [2].
Proof. Note that, for a function \( f \), the sum of the first \( k+1 \) entries of its Hilbert series \( HS_{f,G} \) equals the dimension of \( \text{span}\{ f^{\alpha_0}(X_0)f^{\alpha_1}(X_1)\cdots f^{\alpha_n}(X_n) : \sum_{i=0}^n \alpha_i \leq k \} \). It is obvious that, for a generic \( f \), this dimension is maximal. Since all Hilbert series \( HS_{f,G} \) are polynomials of degree at most \( |G|+1 \), then the required property has to be checked only for \( k \leq |G| \). Therefore it is obvious that, for generic \( f \), their Hilbert series is maximal in the majorization order. □

Remark 3. We know that the Hilbert series of the graded algebra \( C_G \) is a specialization of the Tutte polynomial of \( G \). However we can not calculate the Hilbert series of \( K_G \) from the Tutte polynomial of \( G \), because there exists a pair of graphs \( (G,G') \) with the same Tutte polynomial and different \( HS_{K_G} \) and \( HS_{K_G'} \), see Example 2.

Additionally, notice that, in general, \( HS_{exp,G} := HS_{K_G} \neq HS_G \). Analogously we can not calculate generic Hilbert series \( HS_G \) from the Tutte polynomial of \( G \), see Example 2.

![Figure 1](image_url)  
**Figure 1.** Graphs with the same matroid and different “K-theoretic” and generic Hilbert series.

Example 2. Consider two graphs \( G_1 \) and \( G_2 \) presented in Fig. 1. It is well-known that \( G_1 \) and \( G_2 \) have isomorphic matroids and hence, the same Tutte polynomial. Therefore, the Hilbert series of \( C_{G_1} \) and \( C_{G_2} \) coincide. Namely,

\[
HS_{C_{G_1}}(t) = HS_{C_{G_2}}(t) = 1 + 3t + 6t^2 + 9t^3 + 8t^4 + 4t^5 + t^6.
\]

However, the Hilbert series of “K-theoretic” algebras are distinct. Namely

\[
HS_{K_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,
\]

\[
HS_{K_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.
\]

Moreover their generic Hilbert series are also distinct and different from their “K-theoretic” Hilbert series. Namely,

\[
HS_{G_1}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4,
\]

\[
HS_{G_2}(t) = 1 + 4t + 10t^2 + 16t^3 + t^4.
\]

Putting our information together we get,

\[
HS_{C_{G_1}} = HS_{C_{G_2}} \prec HS_{K_{G_1}} \prec HS_{K_{G_2}} = HS_{G_1} \prec HS_{G_2},
\]

where \( \prec \) denotes the majorization partial order.
4. “K-theoretical” analog for spanning trees

For an arbitrary loopless graph $G$ on the vertex set $\{0, \ldots, n\}$, let $\Phi^G_T$ be the graded commutative algebra over a given field $K$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G;$$

$$\prod_{e \in H} \phi_e = 0, \quad \text{for any non-slim subgraph } H \subset G,$$

where a subgraph $H$ is called slim if its complement $G \setminus H$ is connected.

Let $C^G_T$ be the subalgebra of $\Phi^G_T$ generated by the elements

$$X^T_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1, \ldots, n$, where $c_{i,e}$ is given by (1). (Notice that $X^T_i$ and $X_i$ are defined by exactly the same formula but in different ambient algebras.)

Algebra $C^G_T$ will be called the spanning trees counting algebra of $G$ and is, obviously, the quotient of $C^G$ modulo the set of relations $\prod_{e \in H} \phi_e = 0$ over all non-slim subgraphs $H$. Its defining set of relations is very natural and resembles that of (2).

Namely, define the ideal $J^G_T$ in $K[x_1, \ldots, x_n]$ as generated by the polynomials:

$$p^T_I = \left( \sum_{i \in I} x_i \right)^{D_I},$$

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and the number $D_I$ is the same as in (2). Set $B^G_T := K[x_1, \ldots, x_n]/J^G_T$. One of the results of [4] claims the following.

**Theorem 7.** [Theorems 9.1 and Corollary 10.5 of [4]] For any simple graph $G$ on the set of vertices $\{0, 1, \ldots, n\}$, algebras $B^G_T$ and $C^G_T$ are isomorphic. Their total dimension is equal to the number of spanning trees in $G$. The dimension $\dim B^G_T(k)$ of the $k$-th graded component of $B^G_T$ equals the number of spanning trees $T$ in $G$ with external activity $|G| - n - k$.

Similarly to the above, we can define the filtered algebra $K^G_T \subset \Phi^G_T$ which is isomorphic to $C^G_T$ as a non-filtered algebra. Namely, $K^G_T$ is defined by the generators:

$$Y^T_i = \exp(X^T_i) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.$$

The first result of this section is as follows. Define the ideal $I^G_T \subseteq K[y_0, y_1, \ldots, y_n]$ as generated by the polynomials:

$$q^T_I = \left( \prod_{i \in I} y_i - 1 \right)^{D_I},$$

where $I$ ranges over all nonempty proper subsets in $\{0, 1, \ldots, n\}$ and the number $D_I$ is the same as in (2), together with the generator

$$q^T_{\{0,1,\ldots,n\}} = \prod_{i=0}^{n} y_i - 1.$$

Set $D^G_T := K[y_0, \ldots, y_n]/I^G_T$.

We present two results similar to the case of spanning forests.
Theorem 8. For any simple graph $G$, algebras $B^T_G$, $C^T_G$, $D^T_G$ and $K^T_G$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning trees in $G$.

Proof. The proof is similar to that of Theorem 2. Algebras $C^T_G$ and $K^T_G$ coincide as subalgebras of $\Phi^T_G$ (but they have different filtrations); algebras $C^T_G$ and $B^T_G$ are isomorphic by Theorem 7. The proof of the isomorphism between $D^T_G$ and $B^T_G$ is the same as above; we only need to add the variable $x_0 = -(\sum_{i=1}^n x_i)$ to $B^T_G$. \[\Box\]

Theorem 9. For any simple graph $G$, algebras $D^T_G$ and $K^T_G$ are isomorphic as filtered algebras.

Proof. Similar to the above proof of Theorem 8. \[\Box\]

To move further, we need to give a definition.

Definition 1. Let $G$ be a connected graph. We define its $\Delta$-subgraph $\hat{G} \subseteq G$ as the subgraph with following edges and vertices:
- $e \in E(\hat{G})$, if $e$ is not a bridge (i.e., $G\setminus e$ is still connected),
- $v \in V(\hat{G})$, if there is an edge $e \in E(\hat{G})$ incident to $v$.

In general, $\hat{G}$ contains more information about $G$ than its matroid, because there exist graphs with isomorphic matroids and non-isomorphic $\Delta$-subgraphs, see Figure 2. Recall that in a recent paper [3], the first author has shown that $C^T_G$ depends only on the bridge-free matroid of $G$. Namely,

Proposition 4. [Proposition 12 of [3]] For any two connected graphs $G_1$ and $G_2$ with isomorphic bridge-free matroids (matroids of their $\Delta$-subgraphs), algebras $C^T_{G_1}$ and $C^T_{G_2}$ are isomorphic.

Unfortunately, we can not prove the converse implication at present although we conjecture that is should hold as well, see Conjecture 4 in §3. In case of filtered algebra $K^T_{G_1}$ and $K^T_{G_2}$ we can also prove an appropriate result only in one direction, see Proposition 5.

Similarly to §3 we can to define $F[f]^{T}_{G} \subset \Phi_G$. Let $f$ be a univariate polynomial or a formal power series over $K$. We define the subalgebra $F[f]^{T}_{G} \subset \Phi_G$ as generated by 1 and by

$$f\left(x_i^T\right) = f\left(\sum c_{i,e} \phi_{e}\right), \quad i = 0, \ldots, n.$$  

Proposition 5. For univariate polynomial $f$ and any two connected graphs $G_1$ and $G_2$ with isomorphic $\Delta$-subgraphs $\hat{G}_1$ and $\hat{G}_2$, algebras $F[f]^{T}_{G_1}$ and $F[f]^{T}_{G_2}$ are isomorphic as filtered algebras. Additionally, $K^T_{G_1}$ and $K^T_{G_2}$ are isomorphic as filtered algebras.

Proof. Note that if $G$ has a bridge $e$, then filtered algebra $F[f]^{T}_{G}$ is the Cartesian product of filtered algebras $F[f]^{T}_{G'}$ and $F[f]^{T}_{G''}$, where $G'$ and $G''$ are connected component of $G\setminus e$.

Thus filtered algebra $F[f]^{T}_{G}$ is the Cartesian product of such filtered algebras corresponding to the connected components of the $\Delta$-subgraph of $G$.

Therefore if connected graphs $G_1$ and $G_2$ have isomorphic $\Delta$-subgraphs, then their filtered algebras $F[f]^{T}_{G_1}$ and $F[f]^{T}_{G_2}$ are isomorphic. \[\Box\]

Remark 4. In general case we can not prove that these algebras distinguish graphs with different $\Delta$-subgraphs. The proof of Theorem 5 does not work for two reasons. Firstly, $d(Y_i)$ is not the degree of the $i$-th vertex in $G$. Secondly, even if we can
construct a similar bijection between edges, we do not have an analog of Proposition 1. Since in the proof we consider coefficients of monomial $\phi_{e_1}\phi_{e_2}$, in case when $e_1$ and $e_2$ are not bridges and when $\{e_1, e_2\}$ is a cut, this monomial can still lie in the ideal.

It is possible to construct such a bijection in a smaller set of graphs, namely for graphs such that, for any edge $e$ in the graph, there is another edge $e'$ which is multiple to $e$. For such graphs we do not have the second problem, because if $\{e_1, e_2\}$ is a cut, then $e_1$ and $e_2$ are multiple edges. So, instead of the actual converse of Proposition 5, we can prove the converse in the latter situation, but we do not present this result here.

**Proposition 6.** In the above notation, for generic polynomials $f$ of degree at most $|G|$, the Hilbert series $HS_{F[f]}[G]$ is the same. This generic Hilbert series (denoted by $HS_{G^T}$ below) is maximal in the majorization partial order among $HS_{F[g]}[G]$ for $g$ running over the set of power series with non-vanishing linear term.

Proof. See the proof of Proposition 3. $\square$

**Example 3.** Consider two graphs $G_1$ and $G_2$, see Fig. 2. It is easy to check that subgraphs $\tilde{G}_1$ and $\tilde{G}_2$ have isomorphic matroids, implying that algebras $C_{G_1^T}$ and $C_{G_2^T}$ are isomorphic.

![Graphs and their ∆-subgraphs.](image)

**Figure 2.** Graphs and their ∆-subgraphs.

$$HS_{C_{\tilde{G}_1}}(t) = HS_{C_{\tilde{G}_2}}(t) = 1 + 4t + 4t^2.$$ The Hilbert series of “K-theoretic” algebras are distinct, namely

$$HS_{K_{\tilde{G}_1}}(t) = 1 + 5t + 3t^2,$$

$$HS_{K_{\tilde{G}_2}}(t) = 1 + 6t + 2t^2.$$ These graphs are “small”, so their generic Hilbert series coincides with the “K-theoretic” one. Putting our information together, we get

$$HS_{C_{\tilde{G}_1}} = HS_{C_{\tilde{G}_2}} \prec HS_{K_{\tilde{G}_1}} = HS_{G_1^T} \prec HS_{K_{\tilde{G}_2}} = HS_{G_2^T}.$$  

5. Related problems.

At first, we formulate several problems in case of spanning forests; their analogs for spanning trees are straight-forward.

**Problem 1.** For which functions $f$ besides $a + bx$ and $a + be^x$, one can present relations in $F[f][G]$ for any graph $G$ in a simple way? In other words, for which $f$, one can define an algebra similar to $B_G$ and $D_G$?
Since the Hilbert series $HS_{K_G}$ and $HS_G$ are not expressible in terms of the Tutte polynomial of $G$, they contain some other information about $G$.

**Problem 2.** Find combinatorial description of $HS_{K_G}$ and $HS_G$?

**Problem 3.** For which graphs $G$, Hilbert series $HS_{K_G}$ and $HS_G$ coincide? In other words, for which $G$, $\exp$ is a generic function?

**Problem 4.** Describe combinatorial properties of $HS_{f,G}$ when $f$ is a function starting with a monomial of degree bigger than 1, i.e. $f(x) = x^k + \cdots$, $k > 1$? In particular, calculate the total dimension of $F[f]_G$.

The most delicate and intriguing question is as follows.

**Problem 5.** Do there exist non-isomorphic graphs $G_1$ and $G_2$ such that, for any polynomial $f(x)$, the Hilbert series $HS_{f,G_1}$ and $HS_{f,G_2}$ coincide? In other words, does the collection of Hilbert series $HS_{f,G}$ taken over all formal series $f$ determine $G$ up to isomorphism?

The following problems deal with the case of spanning trees only.

**Conjecture 6.** [comp. [3]] Algebras $\mathcal{C}^T_{G_1}$ and $\mathcal{C}^T_{G_2}$ for graphs $G_1$ and $G_2$ are isomorphic if and only if their bridge-free matroids are isomorphic, where the bridge-free matroid is the graphical matroid of $\Delta$-subgraph.

**Problem 7.** Which class of graphs satisfies the property that if two graphs $G_1$ and $G_2$ from this class have isomorphic $K^T_{G_1}$ and $K^T_{G_2}$, then their $\Delta$-subgraphs are isomorphic. In other words, can one classify all pairs $(G_1, G_2)$ of connected graphs, which has isomorphic filtered algebras $K^T_{G_1}$ and $K^T_{G_2}$? (The same problem for $F[f]_{G_1}$ and $F[f]_{G_2}$, where $f(x) = x + ax^2 + \cdots$)

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