A note on the geometric modeling of the full two body problem
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Abstract

The two full body problem concerns the dynamics of two spatially extended rigid bodies (e.g. rocky asteroids) subject to mutual gravitational interaction. In this note we deduce the Euler-Poincaré and Hamiltonian equations of motion using the geometric mechanics formalism.

Keywords: full two body problem, Euler-Poincaré reduction, Hamiltonian, Poisson bracket

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1 Introduction

It is well known that the classical two body problem, in which the bodies are idealized as mass points, can be analysed with almost elementary methods. Once the “mass-point” assumption is dropped, one is faced with a significantly more complex problem: a coupled, nonlinear 12 degrees of freedom system with a configuration space given by the product of two $SO(3)$ Lie groups and two copies of $\mathbb{R}^3$. The main inconvenience in modeling resides in the lack of a global chart for $SO(3)$; for this reason, even for a single rigid body, most classical mechanics textbooks use Euler angles or alike, leading to an intricate presentation; see for example, [Jacob (1980)].

Anticipating future developments in the aerospace industry, the full two body problem was studied extensively in the last decades; see for instance, [Maciejewski (1995)], [Koon et al. (2004)], [Scheeres (2006)], [Bellerose and Scheeres (2008)], [Scheeres (2009)], [Hou and Xin (2018)] and references within. The modeling of the problem within the geometric mechanics framework is developed in [Cendra and Marsden (2004)]. However, this presentation uses extensively the geometric formalism at an abstract level. In this note we provide a description of the full two body problem within the geometric mechanics framework working directly in the full two body problem phase space, and thus avoiding abstract generalizations.

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We start our modeling by assuming that the reduction due to the linear translation symmetry has already been performed and that the centre of mass coincides with the origin of the inertial system of coordinates. We write the Lagrangian, observe the SO(3) symmetry and state and prove the appropriate (Euler-Poincaré) reduction theorem. We continue by computing the Euler equations. Next, we apply the reduced Legendre transform and deduce the Poisson structure of the reduced space, the Hamiltonian, and the equations of motion. Finally, we deduce the Casimir invariant as a consequence of the conservation of the size of the spatial angular momentum. We also include a small appendix with some formulae concerning the potential.

2 Modeling and equations of motion

Consider two rigid bodies moving freely in space, with a coupling (gravitational) potential $V$ depending on the orientations of the bodies and the relative position $r$ of their centres of mass. Choose a spatial coordinate system with origin at the centre of mass of the entire system, which we assume remains fixed. Let $r_i$ be the vector from the centre of mass of the system to the centre of mass of body $i$, for each $i$. Let $r = r_2 - r_1$. Let $B_1$ and $B_2$, both subsets of $\mathbb{R}^3$, be the reference configurations of the two rigid bodies, each equipped with a reference frame defining body coordinates, with origin at the body’s centre of mass. A configuration of the system is determined by $(R_1, R_2, r)$, where $R_i$ specifies a rotation of body $i$ from its reference configuration, around its own centre of mass (see, for instance, Marsden and Ratiu (1999)). The configuration space of the system is $Q := SO(3) \times SO(3) \times \mathbb{R}^3 \setminus \{\text{collisions}\}$, where $SO(3)$ denotes the Lie group of spatial rotations.

Let $\mu_i$ be the mass measure for body $i$, for $i = 1, 2$. Then the total mass of body $i$ is

$$m_i := \int_{B_i} d\mu_i.$$ 

The translational kinetic energy of body $i$ is $\frac{1}{2}m_i \|\dot{r}_i\|^2$. Following the centre of mass reduction, the reduced mass is $m := \frac{m_1 m_2}{m_1 + m_2}$ and the total translational kinetic energy of the system is $\frac{1}{2}m \|\dot{r}\|^2$.

The coefficient of inertia matrix of body $i$, with respect to its own centre of mass, is

$$\mathbb{J}_i := \int_{B_i} XX^t d\mu_i(X),$$

where $(\cdot)^t$ denotes the matrix transpose. The body angular velocities are $\hat{\Omega}_i := R_i^{-1} \dot{R}_i$. The rotational kinetic energy of body $i$ is

$$K_i = \frac{1}{2} \langle \dot{R}_i, \dot{R}_i \rangle_i := \frac{1}{2} \text{tr} \left( \dot{R}_i \mathbb{J}_i \dot{R}_i^t \right) = \frac{1}{2} \text{tr} \left( \left( R_i^{-1} \dot{R}_i \right) \mathbb{J}_i \left( R_i^{-1} \dot{R}_i \right)^t \right) = \frac{1}{2} \text{tr} \left( \dot{\Omega}_i \mathbb{J}_i \dot{\Omega}_i^t \right) = \frac{1}{2} \langle \dot{\Omega}_i, \dot{\Omega}_i \rangle_i.$$ 

The moment of inertia tensors are

$$I_i := \text{tr} (\mathbb{J}_i) \text{Id}_3 - \mathbb{J}_i,$$ 

where $\text{Id}_3$ is the $3 \times 3$ identity matrix. Using the usual identification of the Lie algebra $so(3)$ with $\mathbb{R}^3$ via the hat map $\hat{\cdot} : \mathbb{R}^3 \to so(3)$,

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) \to \hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix},$$

we can also write

$$K_i = \frac{1}{2} \text{tr} (\hat{\Omega}_i \mathbb{J}_i \hat{\Omega}_i^t) = \frac{1}{2} \Omega_i \mathbb{J}_i \Omega_i.$$
For further reference, recall that for any matrices \( \hat{\Omega}, \hat{\Lambda} \in so(3) \) corresponding to the vectors \( \Omega, \Lambda \in \mathbb{R}^3 \), we have

\[
[\hat{\Omega}, \hat{\Lambda}] = \Omega \times \Lambda
\]

where \([\cdot, \cdot]\) denotes the matrix Lie-bracket (i.e. \([A, B] = AB - BA\)).

In coordinates on the tangent bundle \( T(SO(3) \times SO(3) \times \mathbb{R}^3 \setminus \{\text{collisions}\}) \), the dynamics is given by the Lagrangian

\[
L(R_1, R_2, r, \dot{R}_1, \dot{R}_2, \dot{r}) = \frac{1}{2} \left\langle \dot{R}_1, \dot{R}_1 \right\rangle_1 + \frac{1}{2} \left\langle \dot{R}_2, \dot{R}_2 \right\rangle_2 + \frac{1}{2} m \|\dot{r}\|^2 - V(R_1, R_2, r).
\]

(2)

The spatial action of \( SO(3) \) on the configuration space is the diagonal left multiplication action,

\[
A \cdot (R_1, R_2, r) = (AR_1, AR_2, Ar), \quad A \in SO(3).
\]

(3)

Since \( L \) is invariant under this action, the dynamics may be retrieved from a reduced system. Indeed, describing the motion in the coordinates of one of the bodies allows us to render the equations as a reduced system on a smaller dimensional phase space (the reduced space), together with the so-called reconstruction equation that lifts the reduced dynamics back into the unreduced phase space.

For future reference, we note that the infinitesimal action of \( so(3) \) to \( SO(3) \times SO(3) \times \mathbb{R}^3 \) is (see Holm & al. (2009)):

\[
\hat{\Omega}_{SO(3) \times SO(3) \times \mathbb{R}^3} \cdot (R_1, R_2, r) = (\hat{\Omega}R_1, \hat{\Omega}R_2, \hat{\Omega}r)
\]

(4)

Denote the relative orientation matrix of \( B_2 \) with respect to body \( B_1 \), and the relative position of the centre of the mass of the system, respectively, by

\[
R := R_1^{-1}R_2 \quad \text{and} \quad \Gamma := R_1^{-1}r.
\]

(5)

We then calculate the tangent vector (velocity corresponding to the relative orientation) \( \dot{R} \in T_RSO(3) \) and the advected relative velocity (i.e. the velocity corresponding to the relative vector) \( \dot{\Gamma} \)

\[
\dot{R} = R\dot{\Omega}_2 - \dot{\Omega}_1R \quad \text{and} \quad \dot{\Gamma} = R_1^{-1}\dot{r} - \dot{\Omega}_1\Gamma.
\]

(6)

Recalling that \( \dot{R}_i = R_i\hat{\Omega}_i, \ i = 1, 2 \), and using the above we calculate

\[
L(R_1, R_2, r, \dot{R}_1, \dot{R}_2, \dot{r}) = L(R_1^{-1}R_1, R_1^{-1}R_2, R_1^{-1}r, R_1^{-1}\dot{R}_1, R_1^{-1}\dot{R}_2, R_1^{-1}\dot{r})
\]

\[
= L \left( R_1^{-1}R_1, R_1^{-1}R_2, R_1^{-1}(R_1\Gamma), R_1^{-1}(R_1\hat{\Omega}_1), R_1^{-1}(R_2\hat{\Omega}_2), R_1^{-1}R_1(\dot{\Gamma} + \dot{\Omega}_1\Gamma) \right)
\]

\[
= L(\text{Id}_3, R, \Gamma, \hat{\Omega}_1, \hat{\Omega}_2, \dot{\Gamma} + \dot{\Omega}_1\Gamma)
\]

from where we define the reduced lagrangian

\[
l : SO(3) \times so(3) \times so(3) \times T (\mathbb{R}^3 \setminus \{\text{collisions}\}) \to \mathbb{R}
\]

\[
l(R, \hat{\Omega}_1, \hat{\Omega}_2, \Gamma, \dot{\Gamma}) := L(\text{Id}_3, R, \Gamma, \hat{\Omega}_1, \hat{\Omega}_2, \dot{\Gamma} + \dot{\Omega}_1\Gamma)
\]

(7)

that takes the form

\[
l(R, \hat{\Omega}_1, \hat{\Omega}_2, \Gamma, \dot{\Gamma}) = \frac{1}{2} \left\langle \hat{\Omega}_1, \dot{\Omega}_1 \right\rangle_1 + \frac{1}{2} \left\langle \hat{\Omega}_2, \dot{\Omega}_2 \right\rangle_2 + \frac{1}{2} m \|\dot{\Gamma} + \dot{\Omega}_1\Gamma\|^2 - V(R, \Gamma).
\]

(8)

Let \( \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \) be the usual dot product on \( \mathbb{R}^3 \). Thus, for all \( \mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3) \in \mathbb{R}^3 \simeq so(3)^* \) and \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3 \simeq so(3) \) we have

\[
\langle \mathbf{\Pi}, \Omega \rangle_{\mathbb{R}^3} = \mathbf{\Pi} \cdot \Omega = \Pi_1 \Omega_1 + \Pi_2 \Omega_2 + \Pi_3 \Omega_3.
\]
We denote the pairing between \( so(3)^* \) and \( so(3) \) in matrix notation by \( \langle \cdot , \cdot \rangle \) (no subscript!), and define the ‘breve’ map, \( \breve{\cdot} : \mathbb{R}^3 \to so(3)^* \), by \( \langle \breve{\Pi}, \breve{\Omega} \rangle = \langle \Pi, \Omega \rangle_{\mathbb{R}^3} \). It can be shown that

\[
\langle \breve{\Pi}, \breve{\Omega} \rangle = \frac{1}{2} \text{tr}(\breve{\Pi}^t \breve{\Omega}) = \frac{1}{2} \text{tr}(\breve{\Pi} \breve{\Omega}^t)
\]

for all \( \breve{\Pi} \in so(3)^* \) and \( \Omega \in so(3) \).

\[
\langle \dot{M}, R \breve{\Omega} \rangle = \text{tr} \left( \dot{M}^t (R \breve{\Omega}) \right) = \text{tr} \left( (R \dot{M})^t \breve{\Omega} \right) = \left\langle R \hat{M}^t, \breve{\Omega} \right\rangle = \left\langle \left( R \hat{M} - \hat{M} R \right)^t, \breve{\Omega} \right\rangle
\]

for all \( \breve{\Omega} \in so(3) \), where a matrix subscript \( A \) denotes the anti-symmetric part of that matrix. (We use here the fact that the trace pairing of any symmetric matrix with an antisymmetric matrix vanishes.) Similarly,

\[
\langle \dot{M}, \breve{\Omega} R \rangle = \text{tr} \left( \dot{M}^t (\breve{\Omega} R) \right) = \text{tr} \left( (\dot{M} R)^t \breve{\Omega} \right) = \left\langle \dot{M} R^t, \breve{\Omega} \right\rangle = \left\langle (\dot{M} R)^t A, \breve{\Omega} \right\rangle = \left\langle \left( \dot{M} R^t - R \dot{M}^t \right)^t, \breve{\Omega} \right\rangle.
\]

We are ready now to state the main theorem.

**Theorem 2.1** Consider a Lagrangian \( L : T(SO(3) \times SO(3) \times D) \to \mathbb{R}, D \subset \mathbb{R}^3 \) open,

\[
L = L \left( R_1, R_2, r, \dot{R}_1, \dot{R}_2, \dot{r} \right).
\]

For any given curves \( (R_1(t), R_2(t)) \in SO(3) \times SO(3) \) and \( r(t) \in \mathbb{R}^3 \), let \( R(t) = R_1^{-1}(t) R_2(t) \), \( \Gamma(t) = R_1(t) r(t) \) and

\[
\breve{\Omega}_i(t) := R_i(t)^{-1} \dot{R}_i(t) \in so(3).
\]

Consider

\[
l(R, \breve{\Omega}_1, \breve{\Omega}_2, \Gamma, \dot{\Gamma}) := L(Id_3, R, \Gamma, \breve{\Omega}_1, \breve{\Omega}_2, \dot{\Gamma} + \breve{\Omega}_1 \dot{\Gamma})
\]

and let \( R(t) \) be the solution of the non-autonomous differential equation

\[
\ddot{R} = R(t) \breve{\Omega}_2(t) - \breve{\Omega}_1(t) R(t), \quad R(0) = R_0.
\]

where \( R_0 = R_1(0)^{-1} R_2(0) \). The following statements are equivalent:

(i) \( (R_1(t), R_2(t), r(t)) \) satisfies the Euler-Lagrange equations for the Lagrangian \( L \).

(ii) The variational principle

\[
\delta \int_a^b L \left( R_1(t), R_2(t), r(t), \dot{R}_1(t), \dot{R}_2(t), \dot{r}(t) \right) dt = 0
\]

holds for variations with fixed endpoints.

(iii) The reduced variational principle

\[
\delta \int_a^b l \left( R(t), \breve{\Omega}_1(t), \breve{\Omega}_2(t), \Gamma(t), \dot{\Gamma}(t) \right) dt = 0
\]

holds using variations of the form

\[
\delta \breve{\Omega}_i = \hat{\Sigma}_i + [\breve{\Omega}_i, \hat{\Sigma}_i] \quad \text{and} \quad \delta \Gamma = \Lambda - \hat{\Sigma}_1 \Gamma
\]

where the \( \hat{\Sigma}_i(t) \) are arbitrary paths in \( so(3) \) which vanish at the endpoints, i.e. \( \hat{\Sigma}_i(a) = \hat{\Sigma}_i(b) = 0 \), \( i = 1, 2 \), and \( \Lambda(t) \) is an arbitrary path in \( \mathbb{R}^3 \) with \( \Lambda(a) = \Lambda(b) = 0_{\mathbb{R}^3} \).
(iv) The (left invariant) “Euler-Poincaré” equations hold:

\[ \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega} \right) = \left[ \frac{\delta l}{\delta \Omega}, \tilde{\Omega} \right] + \left( R \left( \frac{\delta l}{\delta R} \right)^t - \frac{\delta l}{\delta R} R^t \right), \]

\[ \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) = \left[ \frac{\delta l}{\delta \Omega_2}, \tilde{\Omega}_2 \right] + \left( R^t \frac{\delta l}{\delta R} - \left( \frac{\delta l}{\delta R} \right)^t R \right), \]

\[ \frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) = \frac{\delta l}{\delta \Gamma}. \]

**Proof.** The equivalence of (i) and (ii) is a restatement of Hamilton’s principle. To show that (ii) and (iii) are equivalent, we compute the variations \( \delta \tilde{\Omega}_1, \delta \tilde{\Omega}_2, \) and \( \delta \Gamma \) and induced by the variations \( \delta R_1, \delta R_2, \) and \( \delta \tilde{r}. \)

Given that \( \tilde{\Omega}_i = R_i^{-1} \tilde{r}_i \) and denoting \( \hat{\Sigma}_i := R_i^{-1} \delta R_i \in so(3), \) \( i = 1, 2 \) we calculate:

\[
\delta \tilde{\Omega}_i = (\delta R_i^{-1})\dot{\tilde{r}}_i + R_i^{-1} \delta \dot{\tilde{r}}_i = -(R_i^{-1} \delta R_i R_i^{-1}) \dot{\tilde{r}}_i + R_i^{-1} \delta \dot{\tilde{r}}_i
\]

\[
= -(\delta R_i^{-1} \delta R_i)(R_i^{-1} \dot{\tilde{r}}_i) + R_i^{-1} \frac{d}{dt}(\delta R_i) = -\tilde{\Sigma}_i \dot{\tilde{r}}_i + R_i^{-1} \frac{d}{dt}(\delta R_i)
\]

\[
= -\tilde{\Sigma}_i \dot{\tilde{r}}_i + \frac{d}{dt}(R_i^{-1} \delta R_i) - R_i^{-1} \delta \dot{r}_i = -\hat{\Sigma}_i \dot{\tilde{r}}_i + \frac{d\hat{\Sigma}_i}{dt} + (R_i^{-1} \dot{\tilde{r}}_i R_i^{-1}) \delta R_i
\]

\[
= -\hat{\Sigma}_i \dot{\tilde{r}}_i + \frac{d\hat{\Sigma}_i}{dt} + (R_i^{-1} \dot{\tilde{r}}_i)(R_i^{-1}) \delta R_i = \frac{d\hat{\Sigma}_i}{dt} - \hat{\Sigma}_i \dot{\tilde{r}}_i + \hat{\Sigma}_i \dot{\tilde{r}}_i
\]

\[
= \frac{d\hat{\Sigma}_i}{dt} + [\hat{\Sigma}_i, \dot{\tilde{r}}_i].
\]

Thus we have

\[
\delta \tilde{\Omega}_i = \frac{d\hat{\Sigma}_i}{dt} + [\hat{\Sigma}_i, \dot{\tilde{r}}_i], \quad i = 1, 2.
\]

The variation of \( \Gamma \) is

\[
\delta \Gamma = \delta (R_i^{-1} r) = \delta (R_i^{-1} r) + R_i^{-1} \delta r = -R_i^{-1} (\delta R_i R_i^{-1} r) + R_i^{-1} \delta r
\]

(15)

Denoting \( \Lambda := R_i^{-1} \delta r, \) the above reads:

\[
\delta \Gamma = \Lambda - \hat{\Sigma}_i \Gamma.
\]

(16)

To complete the proof we show the equivalence of (iii) and (iv). First note that since

\[
\delta R = \delta (R_i^{-1} R_2) = \delta (R_i^{-1} R_2) + R_i^{-1} \delta r_2 = -(R_i^{-1} (\delta R_1 R_i^{-1}) R_2 + R_i^{-1} R_2 R_i^{-1} \delta r_2)
\]

\[
= -(R_i^{-1} \delta R_1)(R_i^{-1} R_2) + (R_i^{-1} R_2)(R_i^{-1} \delta R_2) = -\hat{\Sigma}_i R_2 + R \hat{\Sigma}_2
\]

we have

\[
\delta R = R \hat{\Sigma}_2 - \hat{\Sigma}_1 R.
\]

Now we calculate

\[
\delta \int_a^b \left( l(R, \Gamma, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Gamma}) \right) dt = \int_a^b \left( \frac{\delta l}{\delta R}, \delta R \right) + \left( \frac{\delta l}{\delta \Gamma}, \delta \Gamma \right) + \sum_{i=1}^2 \left( \frac{\delta l}{\delta \tilde{\Omega}_i}, \delta \tilde{\Omega}_i \right) + \left( \frac{\delta l}{\delta \tilde{\Gamma}}, \delta \tilde{\Gamma} \right) dt
\]

\[
= \int_a^b \left( \frac{\delta l}{\delta R}, R \hat{\Sigma}_2 - \hat{\Sigma}_1 R \right) dt + \left( \frac{\delta l}{\delta \tilde{\Gamma}}, \tilde{\Gamma} \right) + \sum_{i=1}^2 \int_a^b \left( \frac{\delta l}{\delta \tilde{\Omega}_i}, \dot{\tilde{\Omega}}_i + [\hat{\Sigma}_i, \dot{\tilde{r}}_i] \right) dt + \left( \frac{\delta l}{\delta \tilde{\Gamma}}, \dot{\tilde{\Gamma}} \right) dt
\]

(17)
Using the relations (9) and (10), the first term of (17) becomes
\[
\int_a^b \left\langle \frac{\delta l}{\delta R}, R \dot{\Sigma}_2 - \hat{\Sigma}_1 R \right\rangle dt = \int_a^b \left\langle \frac{\delta l}{\delta R}, R \dot{\Sigma}_2 \right\rangle dt - \int_a^b \left\langle \frac{\delta l}{\delta R}, \hat{\Sigma}_1 R \right\rangle dt
\]
\[
= \int_a^b \left\langle \left( R \frac{\delta l}{\delta R} - \left( \frac{\delta l}{\delta R} \right)^t \right), \dot{\Sigma}_2 \right\rangle dt - \int_a^b \left\langle \left( \frac{\delta l}{\delta R} R^t - R \left( \frac{\delta l}{\delta R} \right)^t \right), \hat{\Sigma}_1 \right\rangle dt
\]

Using that \( \hat{\Pi} \in so^*(3) \) we have \( \left\langle \hat{\Pi}, [\hat{\Sigma}, \hat{\Omega}] \right\rangle = \left\langle [\hat{\Omega}, \hat{\Pi}], \hat{\Sigma} \right\rangle \) for all \( \hat{\Sigma}, \hat{\Omega} \in so(3) \). That \( \delta (d/dt) = (d/dt) \delta \), integrating by parts and taking into account the boundary conditions, the third term of (17) becomes:
\[
\int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_1} \right) + \left[ \frac{\delta l}{\delta \Omega_1}, \hat{\Sigma}_1 \right], \dot{\Sigma}_2 \right\rangle dt + \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) + \left[ \frac{\delta l}{\delta \Omega_2}, \hat{\Sigma}_2 \right], \dot{\Sigma}_2 \right\rangle dt
\]

Finally, define \( \Gamma \circ P \in so^*(3) \) via \( \left\langle \Gamma \circ P, \hat{\Sigma} \right\rangle := \left\langle P, \hat{\Sigma} \Gamma \right\rangle = \left\langle P, \hat{\Sigma} \right\rangle \Gamma = (\Gamma \times P, \Sigma) \Gamma \) for all \( P, \Gamma \in \mathbb{R}^3 \), and \( \hat{\Sigma} \in so(3) \). Substituting (16) the second and the last terms of (17) transform to
\[
\int_a^b \left\langle \frac{\delta l}{\delta \Gamma}, \delta \Gamma \right\rangle + \left\langle \frac{\delta l}{\delta \Gamma}, \delta \Gamma \right\rangle \mathbb{R}^3 = \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) + \frac{\delta l}{\delta \Gamma}, \Lambda - \hat{\Sigma}_1 \Gamma \right\rangle \mathbb{R}^3
\]
\[
= \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) + \frac{\delta l}{\delta \Gamma}, \Lambda \right\rangle - \int_a^b \left\langle \Gamma \circ -\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) + \frac{\delta l}{\delta \Gamma}, \hat{\Sigma}_1 \right\rangle \mathbb{R}^3 dt.
\]

Thus we obtain
\[
\delta \int_a^b l \left( R, \Gamma, \hat{\Omega}_1, \hat{\Omega}_2, \hat{\Gamma} \right) dt
\]
\[
= \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_1} \right) + \left[ \frac{\delta l}{\delta \Omega_1}, \hat{\Sigma}_1 \right] - \left( \frac{\delta l}{\delta R} R^t - R \left( \frac{\delta l}{\delta R} \right)^t \right) - \left( \frac{\delta l}{\delta \Omega_2} \right)^t - \Gamma \circ -\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) + \frac{\delta l}{\delta \Gamma}, \hat{\Sigma}_1 \right\rangle dt
\]
\[
+ \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) + \left[ \frac{\delta l}{\delta \Omega_2}, \hat{\Sigma}_2 \right] + \left( \frac{\delta l}{\delta R} R^t - R \left( \frac{\delta l}{\delta R} \right)^t \right), \hat{\Sigma}_2 \right\rangle dt + \int_a^b \left\langle -\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) + \frac{\delta l}{\delta \Gamma}, \Lambda \right\rangle \mathbb{R}^3 dt.
\]

Since \( \hat{\Sigma}_1, \hat{\Sigma}_2 \) and \( \Lambda \) are arbitrary, the conclusion follows.

Recall that any orthogonal matrix \( R \) can be expressed as \( R := [\alpha_1, \alpha_2, \alpha_3] \) with \( \alpha_i \in \mathbb{R}^3, i = 1, 2, 3 \), such that \( \alpha_i^2 = 1 \) and \( \alpha_i \cdot \alpha_j = 0 \) for \( i \neq j \). Then for any function depending on \( R \in SO(3) \), i.e., \( f = f(R, \cdot) \to \mathbb{R} \) the vector representation of
\[
\hat{T}_1 := R \left( \frac{\delta f}{\delta R} \right)^t - \frac{\delta f}{\delta R} R^t \quad \text{and} \quad \hat{T}_2 := R^t \frac{\delta f}{\delta R} - \left( \frac{\delta f}{\delta R} \right)^t R
\]
is
\[
T_1 = \sum_{i=1,2,3} \alpha_i \times \frac{\delta f}{\delta \alpha_i} \quad \text{and} \quad T_2 = - \sum_{i=1,2,3} \alpha_i \times \frac{\delta f}{\delta \alpha_i},
\]
respectively. Note that in the above, we calculate \( \frac{\delta f}{\delta R} \) as the matrix
\[
\frac{\delta f}{\delta R} = \begin{bmatrix} \frac{\partial f}{\partial \alpha_1} & \frac{\partial f}{\partial \alpha_2} & \frac{\partial f}{\partial \alpha_3} \end{bmatrix}
\]
where for the vector $\mathbf{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})^t$ we have $\frac{\partial f}{\partial \mathbf{\alpha}_i} = \left( \frac{\partial f}{\alpha_{i2}}, \frac{\partial f}{\alpha_{i1}}, \frac{\partial f}{\alpha_{i3}} \right)^t$. This allows to writing the vector form of the reduced equations of motion (14):

$$\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_1} \right) = \frac{\delta l}{\delta \Omega_1} \times \Omega_1 + \sum_{i=1,2,3} \mathbf{\alpha}_i \times \frac{\delta l}{\delta \mathbf{\alpha}_i}$$

(18)

$$\frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) = \frac{\delta l}{\delta \Omega_2} \times \Omega_2 - \sum_{i=1,2,3} \mathbf{\alpha}_i \times \frac{\delta l}{\delta \mathbf{\alpha}_i}$$

(19)

$$\frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) = \frac{\delta l}{\delta \Gamma}$$

(20)

This above system is completed by the relative orientation equation (11).

Specializing the Lagrangian to the full two body problem, the reduced Lagrangian is given by (8).

In order to obtain the reduced Hamiltonian we use the reduced Legendre transform. First we calculate the momenta as usual:

$$\Pi_1 = \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_1} \right) = \mathbb{I}_1 \Omega_1 + m \mathbf{\Gamma} \times \left( \mathbf{\hat{\Gamma}} + \Omega_1 \times \mathbf{\Gamma} \right)$$

(27)

$$\Pi_2 = \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) = \mathbb{I}_2 \Omega_2$$

(28)

$$\mathbf{P} = \frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) = m \left( \mathbf{\hat{\Gamma}} + \Omega_1 \times \mathbf{\Gamma} \right)$$

(29)

3 Hamiltonian formulation

The Hamiltonian of the full two body problem may be obtained by applying the Legendre transform to the Lagrangian (2) and it reads:

$$H : T^*SO(3) \times T^*SO(3) \times T^*\mathbb{R}^3 \setminus \{ \text{collisions} \} \to \mathbb{R}$$

$$H(R_1, \pi_{R_1}, R_2, \pi_{R_2}, \mathbf{r}, \mathbf{p}) = \frac{1}{2} \langle \pi_{R_1}, \pi_{R_1} \rangle^* + \frac{1}{2} \langle \pi_{R_2}, \pi_{R_2} \rangle^* + \frac{1}{2m} \mathbf{p}^2 + V(R, \mathbf{r}),$$

(25)

where the pairings $\langle \cdot, \cdot \rangle^*$ on $T^*_R(SO(3))$ for fixed $R_i$, $i=1,2$ correspond to the kinetic terms in (2), and, as usual:

$$\pi_{R_i} = \frac{\partial L}{\partial \dot{R}_i} \in T^*_R SO(3), \; i = 1, 2 \quad \text{and} \quad \mathbf{p} = \frac{\partial L}{\partial \mathbf{r}} \in T^*_{\mathbf{r}} \mathbb{R}^3 \simeq \mathbb{R}^3.$$

(26)

In order to obtain the reduced Hamiltonian we use the reduced Legendre transform. First we calculate the momenta

$$\Pi_1 = \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_1} \right) = \mathbb{I}_1 \Omega_1 + m \mathbf{\Gamma} \times \left( \mathbf{\hat{\Gamma}} + \Omega_1 \times \mathbf{\Gamma} \right)$$

(27)

$$\Pi_2 = \frac{d}{dt} \left( \frac{\delta l}{\delta \Omega_2} \right) = \mathbb{I}_2 \Omega_2$$

(28)

$$\mathbf{P} = \frac{d}{dt} \left( \frac{\delta l}{\delta \Gamma} \right) = m \left( \mathbf{\hat{\Gamma}} + \Omega_1 \times \mathbf{\Gamma} \right).$$

(29)
Next we calculate the reduced Hamiltonian via

\[ H(R, \Pi_1, \Pi_2, \Gamma, P) = \langle \Pi_1, \Omega_1(\Pi_1, \Pi_1, \Gamma, P) \rangle_{R^3} + \langle \Pi_2, \Omega_2(\Pi_1, \Pi_1, \Gamma, P) \rangle_{R^3} \]

\[ + \langle P, \hat{\Gamma}(\Pi_1, \Pi_1, \Gamma, P) \rangle_{R^3} - l \left( \Omega_1(R, \Pi_1, \Pi_1, \Gamma, P), \Omega_2(\Pi_1, \Pi_1, \Gamma, P), \Gamma, \hat{\Gamma}(\Pi_1, \Pi_1, \Gamma, P) \right) \]

and obtain the reduced Hamiltonian of the full two body problem

\[ H : SO(3) \times so^*(3) \times T^*R^3 \rightarrow \mathbb{R}, \]

\[ H(R, \Pi_1, \Pi_2, \Gamma, P) = \frac{1}{2} \langle \Pi_1 + \Gamma \times P, \Pi^{-1}_1(\Pi_1 + \Gamma \times P) \rangle_{R^3} + \frac{1}{2} \langle \Pi_2, \Pi^{-1}_2 \Pi_2 \rangle_{R^3} \]

\[ + \frac{1}{2} m \langle P, P \rangle_{R^3} + V(R, \Gamma). \]

The dynamics is given by the Poisson bracket

\[ \{ F, H \}(R, \Pi_1, \Pi_2, \Gamma, P) = -\langle \Pi_1, \frac{\delta F}{\delta \Pi_1} \times \frac{\delta H}{\delta \Pi_1} \rangle_{R^3} - \langle \Pi_2, \frac{\delta F}{\delta \Pi_2} \times \frac{\delta H}{\delta \Pi_2} \rangle_{R^3} + \left( \frac{\delta F}{\delta \Pi} \frac{\delta H}{\delta P} - \frac{\delta H}{\delta \Pi} \frac{\delta F}{\delta P} \right) \]

\[ - \left( \frac{\delta F}{\delta R} \frac{\delta H}{\delta \Pi_1} R - R \frac{\delta H}{\delta \Pi_2} \right) + \left( \frac{\delta H}{\delta \Pi_1} R - R \frac{\delta F}{\delta \Pi_2} \right). \]

This is deduced by considering the composition of real valued (smooth) functions \( F : SO(3) \times so^*(3) \times so^*(3) \times T^*R^3 \rightarrow \mathbb{R} \) with the Poisson map

\[ \lambda : T^*SO(3) \times T^*SO^*(3) \times T^*R^3 \rightarrow SO(3) \times so^*(3) \times so^*(3) \times T^*R^3 \]

\[ \lambda(R_1, \pi_{R_1}, R_2, \pi_{R_1}, \Gamma, P) = (R^{-1}_1 R_2, R^{-1}_1 \pi_{R_1}, R^{-1}_2 \pi_{R_2}, \Gamma, P); \]

using the chain rule, the canonical bracket on \( T^*SO(3) \times T^*SO^*(3) \times T^*R^3 \) becomes the Poisson bracket

\[ \text{(31)} \quad \text{for details on this kind of techniques, see [Krishnaprasad and Marsden (1987)]}. \]

The equations of the reduced dynamics are:

\[ \dot{\Pi}_1 = \Pi_1 \times [\Pi^{-1}_1(\Pi_1 + \Gamma \times P)] + \sum_{i=1,2,3} \alpha_i \times \frac{\delta V}{\delta \alpha_i} \]

\[ \dot{\Pi}_2 = \Pi_2 \times \Pi^{-1}_2 \Pi_2 - \alpha_i \times \frac{\delta V}{\delta \alpha_i} \]

\[ \dot{\Gamma} = \frac{1}{m} P + \Gamma \times [\Pi^{-1}_1(\Pi_1 + \Gamma \times P)] \]

\[ \dot{P} = P \times [\Pi^{-1}_1(\Pi_1 + \Gamma \times P)] - \frac{\partial V}{\partial \Gamma} \]

together with the reconstruction (orientation) equation:

\[ \dot{R} = R \hat{\Omega}_2 - \hat{\Omega}_1 R. \]

where \( R = [\alpha_1, \alpha_2, \alpha_3] \) and \( \hat{\Omega}_1 \) and \( \hat{\Omega}_2 \) are calculated via the inverse of \( \text{(27)}-\text{(29)}. \)

**Remark 3.1** Note that with the choice of \( B_1 \) as reference frame, \( \Pi_1 \) is the sum of the angular momentum \( \Pi_1 \Omega_1 \) of the rigid body \( B_1 \) and the angular momentum \( \Gamma \times P \) of the relative vector, both in the body coordinates of \( B_1 \):

\[ \Pi_1 = \Pi_1 \Omega_1 + \Gamma \times P. \]
Remark 3.2  The change of variable

\[(\Pi_1, \Pi_2, P) = (A_1, A_2, P) := (\Pi_1 - \Gamma \times P, \Pi_2, P),\]

is a Poisson map (see [Marsden (1992)], Section 3.7) and it leads to the Hamiltonian of the two full
body problem as used by [Maciejewski (1995)] and [Cendra and Marsden (2004)]:

\[H(R, A_1, A_2, \Gamma, P) = \frac{1}{2} \langle A_1, \Pi_1^{-1} A_1 \rangle_{\mathbb{R}^3} + \frac{1}{2} \langle A_2, \Pi_2^{-1} A_2 \rangle_{\mathbb{R}^3} + \frac{1}{2m^2} P^2 + V(R, \Gamma). \]  \(\text{(42)}\)

The equations of motion are

\[\dot{A}_1 = A_1 \times \Pi_1^{-1} A_1 + \sum_{i=1,2,3} \alpha_i \times \frac{\delta V}{\delta \alpha_i} + \Gamma \times \frac{\partial V}{\partial \Gamma} \]  \(\text{(43)}\)

\[\dot{A}_2 = \Pi_2^{-1} A_2 - \sum_{i=1,2,3} \alpha_i \times \frac{\delta V}{\delta \alpha_i} \]  \(\text{(44)}\)

\[\dot{\Gamma} = \frac{1}{m} P + \Gamma \times \Pi_1^{-1} A_1 \]  \(\text{(45)}\)

\[\dot{P} = P \times \Gamma_1 - \frac{\partial V}{\partial \Gamma}. \]  \(\text{(46)}\)

Note that this equations coincide to those in [Maciejewski (1995)].

The spatial total angular momentum corresponds to the right \(SO(3)\) action on the phase space it is
given by

\[J : T^* (SO(3) \times SO(3) \times \mathbb{R}^3 \setminus \{\text{collisions}\}) \to so^*(3)
J (R_1, R_2, r, \pi_1, \pi_2, p) \mapsto (\pi_1 R_1^t + \pi_2 R_2^t + r \times p). \]  \(\text{(47)}\)

where we deduced the above using the cotangent bundle momentum map formula (see [Holm & al. (2009)]
page 284) and the infinitesimal generator \(\mathfrak{l}\). Since the Hamiltonian \(\mathfrak{H}\) is invariant under the afore-
mentioned action, by Noether’s theorem, the spatial angular momentum is conserved along any tra-
jectory. Denoting \(A_1, A_2\) the body angular momenta of \(B_1\) and \(B_2\), respectively (i.e., \(A_1 = \Pi_1 \Omega_1\) and
\(A_2 = \Pi_2 \Omega_2\)) we have

\[\|\pi_1 R_1^t + \pi_2 R_2^t + r \times p\| = \|\hat{\Omega}_1 \Pi_1 R_1^t + \hat{\Omega}_2 \Pi_2 R_2^t + (R_1 \Gamma \times R_1 P)^\perp\|
= \|(R_1 \Pi_1 \hat{\Omega}_1)^t + (R_2 \Pi_2 \hat{\Omega}_2)^t + (R_1 (\Gamma \times P))^\perp\|
= \|R_1 \Pi_1 \Omega_1 + R_2 \Pi_2 \Omega_2 + R_1 (\Gamma \times P)\|
= \|\Pi_1 \Omega_1 + (R_1^{-1} R_2) \Pi_2 \Omega_2 + \Gamma \times P\|
= \|A_1 + RA_2 + \Gamma \times P\| = \|\Pi_1 + RI_2\| \]  \(\text{(48)}\)

where we used that the relationship between the spatial and body rigid body angular momenta \(\pi = \hat{\Omega} \Pi\)
(see [Holm & al. (2009)] Section 1.5). The composition of the spatial momentum map with the Casimir
\(C : so^*(3) \mapsto \mathbb{R}, C(x) = \|x\|^2\) leads to the Casimir

\[C(R, A_1, A_2, \Gamma, P) = \|A_1 + RA_2 + \Gamma \times P\|^2 = \|\Pi_1 + RI_2\|^2 \]  \(\text{(49)}\)

and further, any function of the form \(\Phi(\|\Pi_1 + RI_2\|^2)\) is a Casimir for the reduced dynamics.

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5 Appendix

We append this note with some formulas on the interacting potential. In concordance with most physical situations, we may assume that the distance between the bodies is much larger than the bodies dimensions. Thus we consider the potential truncated to the third order (Maciejewski (1995)):

\[
V(R, \Gamma) = -\frac{Gm_1 m_2}{|\Gamma|} (m_1 \operatorname{Tr} I_1 + m_1 \operatorname{Tr} I_2) + \frac{3G}{2|\Gamma|^3} (m_1 \langle \Gamma, I_1 \Gamma \rangle_{\mathbb{R}^3} + m_2 \langle R \Gamma, I_2 R \Gamma \rangle_{\mathbb{R}^3})
\] (50)

where the rotation matrix \( R \in SO(3) \) is represented by \( R = [\alpha_1, \alpha_2, \alpha_3] \) with \( \alpha_i \) (column) vectors such that \( \alpha_2^2 = 1 \) and \( \alpha_i \cdot \alpha_j = 0 \) for \( i \neq j \). Next we calculate the terms \( \alpha_i \times \partial V/\partial \alpha_i \) and \( \Lambda \times \partial V/\partial \Lambda \) occurring in the equations of motion. Denoting \( \mathbb{I}_2 := \text{diag}(I_{21}, I_{22}, I_{23}) \), we obtain:

\[
\alpha_1 \times \partial V/\partial \alpha_1 = 2\Gamma^2 \begin{pmatrix}
\alpha_2 \alpha_3 (I_{23} - I_{22}) \\
-\alpha_3 \alpha_1 (I_{23} - I_{21}) \\
\alpha_1 \alpha_2 (I_{21} - I_{22})
\end{pmatrix} + 2\Gamma_1 (\alpha_1 \times \mathbb{I}_2 \alpha_2 + \alpha_1 \times \mathbb{I}_2 \alpha_3)
\] (51)

and circular combinations. Further

\[
\frac{\partial V}{\partial \Lambda} = \left[ \frac{G m_1 m_2}{|\Gamma|^2} - \frac{3G}{2|\Gamma|^4} (m_1 \operatorname{Tr} I_1 + m_1 \operatorname{Tr} I_2) - \frac{15G}{2|\Gamma|^6} (m_1 \langle \Gamma, I_1 \Gamma \rangle_{\mathbb{R}^3} + m_2 \langle R \Gamma, I_2 R \Gamma \rangle_{\mathbb{R}^3}) \right] \frac{\Lambda}{|\Lambda|}
\] (52)

and so

\[
\Lambda \times \frac{\partial V}{\partial \Lambda} = \frac{3G}{|\Gamma|^3} \left[ m_1 \Lambda \times \mathbb{I}_1 \Lambda + m_2 \Lambda \times (R^t \mathbb{I}_2 R \Lambda) \right].
\] (53)

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