The failure of the uncountable non-commutative Specker Phenomenon

By
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Abstract

Higman proved in 1952 that every free group is non-commutatively slender, this is to say that if $G$ is a free group and $h$ is a homomorphism from the countable complete free product $\times_{i\in\omega}\mathbb{Z}$ to $G$, then there exists a finite subset $F \subseteq \omega$ and a homomorphism $\tilde{h} : \ast_{i\in F}\mathbb{Z} \rightarrow G$ such that $h = \tilde{h}\rho_{F}$, where $\rho_{F}$ is the natural map from $\times_{i\in\omega}\mathbb{Z}$ to $\ast_{i\in F}\mathbb{Z}$. Corresponding to the abelian case this phenomenon was called the non-commutative Specker Phenomenon. In this paper we show that Higman’s result fails if one passes from countable to uncountable. In particular, we show that for non-trivial groups $G_{\alpha}$ ($\alpha \in \lambda$) and uncountable cardinal $\lambda$ there are $2^{2^{\lambda}}$ homomorphisms from the complete free product of the $G_{\alpha}$’s to the ring of integers.

0 Introduction

Higman proved in 1952 [H] that every free group $F$ is non-commutatively slender, where slenderness means that any homomorphism $h$ from the countable complete free product $\times_{i\in\omega}\mathbb{Z}$ of the integers to $F$ depends only on finitely many coordinates. A similar result was proved by Specker in 1950 [S] for abelian groups. Specker showed that any homomorphism from the countable product $\Pi_{\omega}\mathbb{Z}$ to the integers depends only on finitely many entries. These two phenomena were called the commutative and the non-commutative Specker Phenomenon. Eda extended Higman’s result in 1992 in [E1] by showing that for any non-commutatively slender group $S$, non-trivial groups $G_{\alpha}$ ($\alpha \in I$) and any

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homomorphism $h$ from the free $\sigma$-product of the $G_\alpha$’s to $\mathbb{Z}$ there exist a finite subset $F$ of $I$ and a homomorphism $h : *_{i \in F} G_i \to S$ such that $h = h_F$, where $\rho_F$ is the natural map from $\mathcal{X}_{i \in I} G_i$ to $*_{i \in F} G_i$. Motivated by this result Eda asked the question [E1][Question 3.8] whether the non-commutative Specker Phenomenon still holds if one passes from countable to uncountable. In this paper we will answer this question to the negative by constructing for a given uncountable cardinal $\lambda$ and non-trivial groups $G_\alpha$ ($\alpha \in \lambda$), a homomorphisms $h$ from the complete free product of the $G_\alpha$’s to $\mathbb{Z}$ for which the non-commutative Specker Phenomenon fails. In particular, we will show that there are $2^\lambda$ of these homomorphisms, hence the size of the set of all homomorphisms from $\mathcal{X}_{\alpha \in \lambda} G_\alpha$ to the integers is as large as possible.

1 Basics and notations

Let $I$ be an arbitrary index set. For groups $G_i$ ($i \in I$), the free product is denoted by $*_{i \in I} G_i$ (see [M] for details on free products). If $J$ is a finite subset of $I$ then we write $J \subseteq I$. For $X \subset Y \subset I$ let $\rho_{XY} : *_{i \in Y} G_i \to *_{i \in X} G_i$ be the canonical homomorphism. Then, the set $\{*_{i \in X} G_i : X \in I\}$ together with the homomorphisms $\rho_{XY}$ ($X \subset Y \in I$) form an inverse system and its inverse limit $\lim(*_{i \in X} G_i, \rho_{XY} : X \subset Y \in I)$ is called the unrestricted free product (see [E1]). Eda [E1] introduced an infinite version of free products and defined the complete free product $\mathcal{X}_{i \in I} G_i$ of the groups $G_i$ which is isomorphic to the subgroup $\bigcap_{F \subseteq I} \{*_{i \in F} G_i \cdot \lim(*_{i \in X} G_i, \rho_{XY} : X \subset Y \in I)\}$ of the unrestricted free product. To get familiar with the complete free product we recall the definition of words of infinite length and some basic facts about $\mathcal{X}_{i \in I} G_i$ from [E1].

**Definition 1.1** Let $G_i$ ($i \in I$) be non-trivial groups such that $G_i \cap G_j = \{e\}$ for $i \neq j \in I$. Elements of $\bigcup_{i \in I} G_i$ are called letters. A word $W$ is a function $W : \bar{W} \to \bigcup_{i \in I} G_i$ such that $\bar{W}$ is a linearly ordered set and $W^{-1}(G_i)$ is finite for any $i \in I$. In case the cardinality of $\bar{W}$ is countable, we say that $W$ is a $\sigma$-word. The class of all words is denoted by $W(G_i : i \in I)$ (abbreviated by $W$) and the class of all $\sigma$-words is denoted by $W^\sigma(G_i : i \in I)$ (abbreviated by $W^\sigma$).

Two words $U$ and $V$ are said to be isomorphic ($U \cong V$) if there exists an isomorphism $\varphi : \bar{U} \to \bar{V}$ as linearly ordered sets such that $U(\alpha) = V(\varphi(\alpha))$ for all $\alpha \in U$. It is easily seen that $W$ is a set and that for words of finite length the above definition coincides with the usual definition of words. For a subset $X \subseteq I$ the restricted word (or subword) $W_X$ of $W$ is given by the
function $W_X : \bigcup_{i \in X} G_i$, where $W_X = \{ \alpha \in \hat{W} : W(\alpha) \in \bigcup_{i \in X} G_i \}$ and $W_X(\alpha) = W(\alpha)$ for all $\alpha \in \hat{W}$. Hence $W_X \in \mathcal{W}$. Now an equivalence relation is defined on $\mathcal{W}$ by saying that two words $U$ and $V$ are equivalent $(U \sim V)$ if $U_F = V_F$ for all $F \subseteq I$, where we regard $U_F$ and $V_F$ as elements of the free product $\ast_{i \in F} G_i$. The equivalence class of a word $W$ is denoted by $[W]$ and the composition of two words as well as the inverse of a word are defined natural. Thus $\mathcal{W}/ \sim = \{ [W] : W \in \mathcal{W} \}$ becomes a group.

**Definition 1.2** The complete free product $\ast_{i \in I} G_i$ is the group $\mathcal{W}(G_i : i \in I)/ \sim$. The free $\sigma$-product $\ast_{i \in I}^\sigma G_i$ is the group $\mathcal{W}^\sigma(G_i : i \in I)/ \sim$, which is a subgroup of $\ast_{i \in I} G_i$. In case every $G_i$ is isomorphic to $G$, we abbreviate $\ast_{i \in I} G_i$ by $\ast I G$ and similarly for free $\sigma$-products.

Obviously, $\ast_{i \in I} G_i$ and $\ast_{i \in I}^\sigma G_i$ are isomorphic to $\ast_{i \in I}^\sigma G_i$ if $I$ is finite. By [E], Proposition 1.8 the complete free product $\ast_{i \in I} G_i$ is isomorphic to the subgroup $\bigcap_{F \subseteq I} \{ \ast_{i \in F} G_i \ast \lim_{\tau} \ast_{i \in X} G_i, \rho_XY : X \subset Y \subseteq I \}$ of the unrestricted free product. Moreover, Eda proved in [E] that each equivalence class $[W]$ is determined uniquely by a reduced word. A word $W \in \mathcal{W}(G_i : i \in I)$ is called reduced, if $W \equiv UXV$ implies $[X] \not= e$ for any non-empty word $X$, where $e$ is the identity, and for any neighboring elements $\alpha$ and $\beta$ of $\hat{W}$ it never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same $G_i$.

**Lemma 1.3 (Eda, [E])** For any word $W \in \mathcal{W}(G_i : i \in I)$, there exists a reduced word $V \in \mathcal{W}(G_i : i \in I)$ such that $[W] = [V]$ and $V$ is unique up to isomorphism.

Furthermore, Eda showed in [E] the following lemma where a word $W \in \mathcal{W}(G_i : i \in I)$ is called quasi-reduced if the reduced word of $W$ can be obtained by multiplying neighboring elements without cancelation.

**Lemma 1.4 (Eda, [E])** For any two reduced words $W, V \in \mathcal{W}(G_i : i \in I)$ there exist reduced words $V_1, W_1, M \in \mathcal{W}(G_i : i \in I)$ such that $W \equiv W_1 M$, $V \equiv M^{-1} V_1$ and $W_1 V_1$ is quasi-reduced.

We would like to remark that a free $\sigma$-product $\ast_{i \in I}^\sigma Z_i$ is isomorphic to the fundamental group and the free complete product $\ast_{i \in I} Z_i$ is isomorphic to the big fundamental group of the Hawaiian earring with $I$-many circles (see [C]). Hence free complete products are also of topological interest.

## 2 The uncountable Specker Phenomenon

In 1950, E. Specker [3] proved that for any homomorphism $h$ from the countable direct product $Z^\omega$ to the ring of integers $Z$, there exist a finite subset $F$ of $\omega$
and a homomorphism \( \tilde{h} : \mathbb{Z}^F \to \mathbb{Z} \) satisfying \( h = \tilde{h}\rho_F \), where \( \rho_F : \mathbb{Z}^\omega \to \mathbb{Z}^F \) is the canonical projection. This phenomenon is called the Specker Phenomenon and it can easily be seen that Specker’s result still holds if one replaces \( h : \mathbb{Z}^\omega \to \mathbb{Z} \) by \( g : \mathbb{Z}^\omega \to G \), where \( G \) is any free abelian group. For generalizations to products of larger cardinalities and the resulting definition of slenderness for abelian groups we refer to [EM] or [F1]. In [E2] Eda introduced a non-commutative version of slenderness.

**Definition 2.1** A group \( G \) is **non-commutatively slender** if for any homomorphism \( h : \times_{i \in I} G_i \to G \) there exists a natural number \( n \) such that \( h(\times_{i \in I\setminus \{1, \ldots, n\}} \mathbb{Z}) = \{e\} \).

Eda proved that every non-commutatively slender group is torsion-free and that in the abelian case non-commutative slenderness is equivalent to the commutative slenderness (see [E1, Theorem 3.3. and Corollary 3.4.]). Moreover, he proved that non-commutative slender groups behave nice in the following sense for non-trivial groups \( G_i (i \in I) \).

**Proposition 2.2** (Eda, [E1]) Let \( S \) be a non-commutative slender group and \( h : \times_{i \in I} G_i \to S \) be a homomorphism. Then, there exist a finite subset \( F \) of \( I \) and a homomorphism \( \tilde{h} : \ast_{i \in F} G_i \to S \) such that \( h = \tilde{h}\rho_F \), where \( \rho_F \) is the natural map from \( \times_{i \in I} G_i \) to \( \ast_{i \in F} G_i \).

Moreover, if \( S_j (j \in J) \) are non-commutatively slender groups then also the restricted direct product and the free product of the \( S_j \)’s are non-commutatively slender (see [E1, Theorem 3.6.]). The first fundamental result on the class of non-commutatively slender groups was already obtained by Higman in [H] where he proved the following theorem.

**Theorem 2.3** (Higman, [H]) Every free group is non-commutatively slender.

In contrast to Higman’s result we will show that if one replaces countable by uncountable then the non-commutative Specker Phenomenon fails. In particular we show that there are \( 2^{\lambda^\kappa} \) homomorphisms from the complete free product \( \times_{\alpha \in \lambda} G_\alpha \) to the ring of integers if \( \lambda \) is uncountable and the \( G_\alpha \)’s are non-trivial groups. For the convenience of the reader we first construct one homomorphism for which the Specker Phenomenon fails and then modify the construction to obtain our main result.

**Theorem 2.4** Let \( \lambda \) be any uncountable cardinal and \( G_\alpha (\alpha \in \lambda) \) non-trivial groups. For each \( \kappa \leq \lambda \) regular, uncountable there exists a homomorphism \( \varphi_\kappa : \times_{\alpha \in \lambda \setminus \kappa} G_\alpha \to \mathbb{Z} \) for which the Specker Phenomenon fails.
Proof. Let $G_\alpha$ ($\alpha < \lambda$) be a collection of non-trivial groups and choose $e_\alpha \neq g_\alpha \in G_\alpha$, where $e_\alpha$ is the identity of $G_\alpha$ ($\alpha < \lambda$). We define the following words $M_\kappa \in \times_{\alpha \in \lambda} G_\alpha$ for any regular, uncountable cardinal $\kappa < \lambda$.

$$M_\kappa : (\kappa, <) \to \bigcup_{\alpha<\lambda} G_\alpha \text{ via } \beta \mapsto g_\beta$$

where $<$ is the natural ordering of $\lambda$. Note that $M_\kappa$ is a word of uncountable cofinality since $\kappa$ is regular and uncountable. For $\beta < \kappa$ we let $M_{\kappa, \beta}$ be the subword $M_\kappa \upharpoonright [\beta, \kappa)$ of $M_\kappa$. Now let $X$ be any reduced word in $\times_{\alpha \in \lambda} G_\alpha$ and recall that a subset $J \subseteq (\lambda, <)$ is called convex if $x < y < z$ and $x, z \in J$ implies $y \in J$. We put

$$\text{Occ}_\kappa^+(X) := \{ J \subseteq (\lambda, <) : J \text{ is convex and } X \upharpoonright J \cong M_{\kappa, \beta} \text{ for some } \beta < \kappa \}.$$ 

Thus $\text{Occ}_\kappa^+(X)$ counts the occurrences of end segments of $M_\kappa$ in $X$. Similarly we let

$$\text{Occ}_\kappa^-(X) := \{ J \subseteq (\lambda, <) : J \text{ is convex and } X \upharpoonright J \cong M_{\kappa, \beta}^{-1} \text{ for some } \beta < \kappa \}.$$ 

In order to avoid counting subsets of $(\lambda, <)$ several times we define an equivalence relation on $\text{Occ}_\kappa^+(X)$ and $\text{Occ}_\kappa^-(X)$ in the following way. Two convex subsets $J_1, J_2$ of $(\lambda, <)$ are said to be equivalent if they have a common end segment, i.e. $J_1 \sim_\kappa J_2$ if there exist $j_i \in J_i$ such that $X \upharpoonright S_1 \cong X \upharpoonright S_2$, where $S_i = \{ j \in J_i : j \geq j_i \}$. First we prove that two subsets $J_1, J_2 \in \text{Occ}_\kappa^+(X)$ are either disjoint or equivalent. Therefore assume that $J_1, J_2 \in \text{Occ}_\kappa^+(X)$ are not disjoint, hence there exists $j^* \in J_1 \cap J_2$. We let $h_i : M_{\kappa, \beta_i} \to X \upharpoonright J_i$ be isomorphisms for some $\beta_i < \lambda$ ($i = 1, 2$). Thus we can find $\gamma_i \geq \beta_i$ such that $h_i(\gamma_i) = j^*$ and therefore $X(\gamma_i) = g_\gamma$, for $i = 1, 2$. Hence $\gamma_1 = \gamma_2$ and by transfinite induction we conclude $X \upharpoonright T_1 \cong X \upharpoonright T_2$, where $T_i = \{ j \in J_i : j \geq j^* \}$. Note that $h_i$ is an isomorphism of linearly ordered sets, hence $h_i$ commutes with limits and the successor-function. Similarly two subsets $J_1, J_2 \in \text{Occ}_\kappa^-(X)$ are either disjoint or equivalent. Next we will show that the sets $\text{Occ}_\kappa^+(X) / \sim_\kappa$ and $\text{Occ}_\kappa^-(X) / \sim_\kappa$ are finite. Therefore assume that there exist infinitely many pairwise non-equivalent $J_n \in \text{Occ}_\kappa^+(X)$ ($n \in \omega$). Hence $J_n$ and $J_m$ are disjoint for $n \neq m$. We let $X \upharpoonright j_n \cong M_{\kappa, \beta_n}$ for some $\beta_n < \kappa$ and $n \in \omega$. Then $\beta = \bigcup_{n \in \omega} \beta_n$ is strictly less than $\kappa$ since $\kappa$ is regular and uncountable, hence $\text{cf}(\kappa) > \aleph_0$. Since $\beta \in [\beta_n, \kappa)$ for all $n \in \omega$ we can find $j_n \in J_n$ such that

$$X(j_n) = M_{\kappa, \beta_n}(\beta) = M_{\kappa, \beta}(\beta).$$ 

for $n \in \omega$. But all $J_n$ are pairwise disjoint and therefore $X^{-1}(G_\beta)$ is infinite which is a contradiction. Thus $\text{Occ}_\kappa^+(X) / \sim_\kappa$ and similarly $\text{Occ}_\kappa^-(X) / \sim_\kappa$ are finite sets. We now define $\varphi_\kappa : \times_{\alpha \in \lambda} G_\alpha \to \mathbb{Z}$ as follows:

$$X \mapsto |\text{Occ}_\kappa^+(V) / \sim_\kappa| - |\text{Occ}_\kappa^-(V) / \sim_\kappa|$$

5
where \( V \) is the reduced word corresponding to \( X \).
Note that \( \varphi_\kappa \) is well-defined by Lemma 1.3. Moreover, by definition \( \varphi_\kappa(X^{-1}) = -\varphi_\kappa(X) \) and obviously the Specker Phenomenon fails for \( \varphi_\kappa \). All we have to show is that \( \varphi_\kappa \) is a homomorphism. Therefore let \( X \) and \( Y \) be reduced words.
By Lemma 1.4 there exist reduced words \( X_1, Y_1 \) and \( M \) such that \( X \sim X_1M \) and \( Y \sim M^{-1}Y_1 \) and \( X_1Y_1 \) is quasi-reduced. Now it is easy to check that
\[
\varphi_\kappa(XY) = \varphi_\kappa(X_1Y_1) = \varphi_\kappa(X_1) + \varphi_\kappa(Y_1) = \varphi_\kappa(X) + \varphi_\kappa(Y),
\]
since \( X_1Y_1 \) is quasi-reduced.

\( \square \)

We would like to remark that the uncountable cofinality of \( \lambda \) in Theorem 2.4 is essential and can not be avoided by Higman’s theorem. Modifying the proof of Theorem 2.4 we obtain

**Theorem 2.5** Let \( \lambda \) be an uncountable cardinal and \( G_\alpha (\alpha \in \lambda) \) be non-trivial groups. Then there are \( 2^{2^\lambda} \) homomorphisms from the complete free product of the \( G_\alpha \)’s to the ring of integers.

**Proof.** Let \( M_\alpha \) be a reduced word in \( \times_{\alpha \in \lambda} G_\alpha \) of uncountable cofinality \( \lambda \), i.e. \( M_\alpha = (\lambda, <) \). Recall from the proof of Theorem 2.4 that by \( M_{\alpha, \beta} \) we mean the subword \( M_\alpha \mid_{[\beta, \lambda)} \) for \( \beta \in \lambda \). Assume that we have a family of such words \( M_\alpha (\alpha \in 2^\lambda) \) satisfying the following condition for a convex subset \( J \subseteq \lambda \) and a reduced word \( X \):

\[
X \mid_J \cong M_{\alpha, \beta} \text{ for } \beta \in \lambda \implies X \mid_J \not\cong M_{\alpha, \beta} \text{ for all } \alpha \not= \gamma \in 2^\lambda, \beta \in \lambda \quad (\ast)
\]

Then it is well-known that we can choose \( 2^{2^\lambda} \) almost disjoint families \( F_\alpha = \{ M_\beta : \beta \in I_\alpha \} \) such that \( I_\alpha \) has size \( \lambda \). We now define for \( \alpha \in 2^{2^\lambda} \)

\[
\text{Occ}_\alpha^+(X) := \{ J \subseteq (\lambda, <) : J \text{ convex}, \ X \mid_J \cong M_{\beta, \gamma} \text{ for some } \beta \in I_\alpha, \gamma < \lambda \}
\]

and similarly

\[
\text{Occ}_\alpha^-(X) := \{ J \subseteq (\lambda, <) : J \text{ convex}, \ X \mid_J \cong M_{\beta, \gamma}^{-1} \text{ for some } \beta \in I_\alpha, \gamma < \lambda \}.
\]

As in the proof of Theorem 2.4 we can see that the sets \( \text{Occ}_\alpha^+(X)/\sim_\lambda \) and \( \text{Occ}_\alpha^-(X)/\sim_\lambda \) are finite for any reduced word \( X \) and \( \alpha \in 2^{2^\lambda} \). Moreover, the maps \( \varphi_\alpha : \times_{\beta \in I_\delta} G_\beta \longrightarrow \mathbb{Z} \) defined by

\[
X \mapsto |\text{Occ}_\alpha^+(V)/\sim_\lambda| - |\text{Occ}_\alpha^-(V)/\sim_\lambda|
\]
where $V$ is the reduced word corresponding to $X$, are well-defined homomorphisms and $\varphi_\alpha \neq \varphi_\beta$ for $\alpha, \beta \in 2^\lambda$ since the families $F_\delta$ are almost disjoint and satisfy condition $(\ast)$. Hence the size of all homomorphisms from the complete free product of the $G_\alpha$’s to the integers is $2^{2^\lambda}$ as claimed. It remains to show the existence of the words $M_\alpha$ satisfying $(\ast)$.

We start with any partition of $\lambda$ into two sets, i.e. with a function $g : \lambda \to \{0, 1\}$. Note that there are $2^\lambda$ of those functions. Moreover, we choose elements $e_\alpha \neq h_\alpha \in G_\alpha (\alpha \in \lambda)$ and define the word $M'_g \in \times_{\alpha \in \lambda} G_\alpha$ by

$$M'_g(\beta) = h_{\beta + 2g(\beta)}$$

Then $M'_g$ is a reduced word and we let $M_g$ be the composition of $M'_g$ with itself $\omega_1$ times. Then $M_g$ is still reduced and for different $g, g' : \lambda \to \{0, 1\}$ condition $(\ast)$ is satisfied for $M_g$ and $M_{g'}$. Thus the family

$$F = \{M_g : g : \lambda \to \{0, 1\}\}$$

is a family of reduced words of size $2^\lambda$ satisfying $(\ast)$ as desired. \hfill \Box

Moreover, the authors would like to mention that modifying the proof of Theorem 2.4 Conner and Eda proved a more general result in [CE].

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