Median regression with differential privacy *

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Abstract: Median regression analysis has robustness properties which make it attractive compared with regression based on the mean, while differential privacy can protect individual privacy during statistical analysis of certain datasets. In this paper, three privacy preserving methods are proposed for median regression. The first algorithm is based on a finite smoothing method, the second provides an iterative way and the last one further employs the greedy coordinate descent approach. Privacy preserving properties of these three methods are all proved. Accuracy bound or convergence properties of these algorithms are also provided. Numerical calculation shows that the first method has better accuracy than the others when the sample size is small. When the sample size becomes larger, the first method needs more time while the second method needs less time with well-matched accuracy. For the third method, it costs less time in both cases, while it highly depends on step size.

keywords: median regression, differential privacy, \( l_1 \) sensitivity, Laplace mechanism

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1 Introduction

Personal privacy information may be exposed with the unprecedented availability of datasets, so there is increasing requirement that statistical analysis of such datasets should protect individual privacy. As [6] describes, differential privacy addresses the paradox of learning nothing about an individual while learning useful information about a population. Over the past few years, differential privacy has been investigated in machine learning [1] and has been applied in the real world, see for example [8]. Recently, [3] formulates a general lower bound argument for minimax risks with differential privacy constraints, and applies this argument to high-dimensional mean estimation and linear regression problems.

In this paper, three privacy preserving methods are proposed for median regression, which is a special case of quantile regression. Quantile regression was first introduced in [12], which aims to estimate and conduct inference about conditional quantile functions. In recent years, quantile regression has become a comprehensive method for statistical analysis of response models and it has been widely used in reality, such as survival analysis and economics, see for example, [14], [20] and [15]. The fact that the median regression takes least absolute deviation as its objective function to estimate parameters has been known among statisticians [12].

Denote a dataset of \( n \) i.i.d. samples about independent variables as \( X \) and each observation contains \( d \) variables \( x_1, x_2, \ldots, x_d \). In the regression setting, we assume \( Y_i \) is the response for case \( i \), \( x_{ij} \) is the value of predictor \( j \) for case \( i \), and \( \beta_j \) is the regression coefficient corresponding to predictor \( j \), where \( 1 \leq i \leq n, 1 \leq j \leq d \). In this paper, we consider the linear \( l_1 \) regression problem, i.e., minimizing the following function:

\[
F(\mu, \beta) = \frac{1}{n} \sum_{i=1}^{n} |r_i(\mu, \beta)|, \tag{1}
\]

where \( r_i(\mu, \beta) = \mu + X_i \beta - Y_i (i = 1, 2, \ldots, n) \) and \( X_i \) represents \( i \)-th row of \( X \), and \( \beta = (\beta_1, \ldots, \beta_d)^T \). Without loss of generality, assume that \( ||Y_i||_1 \leq B \) (\( B \) is a positive number) and \( ||X_i||_1 \leq 1 \) for \( i = 1, \ldots, n \). In a vector form, \( r(\mu, \beta) = \mu 1 + X \beta - Y \) represents a set of linear functions in \( \mathbb{R}^n \) with \( Y = (y_1, \ldots, y_n)^T \), where \( 1 \) is an \( n \)-dimensional column vector whose all elements are 1. In addition, the ridge penalized regression is more stable than simple linear regression and its objective function can be viewed as
minimizing the criterion

\[ L(\mu, \beta) = F(\mu, \beta) + \frac{\lambda}{2} \beta^T \beta, \]  

(2)

where \( \lambda \) is a fixed regularization parameter.

2 Backgrounds and definitions

We consider a dataset \( x \) as a collection of observations from a universe \( \mathcal{X} \). It is convenient to represent databases by their histograms: \( x \in \mathbb{N}^{||X||} \), in which each entry \( x_i \) represents the number of elements in the database \( x \) of type \( i \in \mathcal{X} \). For example, the universe \( \mathcal{X} \) contains 5 records and we denote them by \( \{1, 2, 3, 4, 5\} \). If a dataset \( x \) consists of three records 1, 1 and 4, we can denote \( x \) as a 5-dimensional vector \((2, 0, 0, 1, 0)\), where the first element is 2 since record 1 appears twice. A 5-dimensional vector \((2, 0, 1, 1, 0)\) represents another dataset \( y \) with 4 records, respectively.

Differential privacy is based on the neighbourhood of a database, when applying differential privacy into practical use, it is key to define the precise condition under which two databases \( x \) and \( y \) are considered to be neighbouring. There are two possible choices and thus producing two types of differential privacy, one is called unbounded differential privacy \cite{5} and the other is called bounded differential privacy \cite{7}. Bounded differential privacy assumes that both \( x \) and \( y \) have the same size \( n \) and \( y \) can be obtained from \( x \) by replacing exactly one record. While unbounded differential privacy does not require \( x \) and \( y \) have the same fixed size, it holds the view that \( y \) can be obtained from \( x \) by adding or deleting exactly one record. In this paper, we adopt bounded differential privacy as our choice and use the notation \( x \nabla y \) if \( x \) and \( y \) are neighboring.

**Definition 2.1.** A randomized algorithm \( M \) with domain \( \mathbb{N}^{||X||} \) is \((\epsilon, \delta)\)-differentially private if for all \( S \subseteq \text{Range} (M) \) and for all datasets \( x, y \in \mathbb{N}^{||X||} \) and \( x \nabla y \):

\[ Pr(M(x) \in S) \leq exp(\epsilon) Pr(M(y) \in S) + \delta. \]

By intuition, this definition guarantees that a randomized algorithm behaves similarly on slightly different input datasets, which achieves the purpose of protecting individual privacy in some sense. Next, a randomized
algorithm named Laplace mechanism, which is an effective method for privacy preserving, will be introduced. Firstly, we need a concept named $l_1$ sensitivity.

**Definition 2.2.** The $l_1$ sensitivity of a function $f : \mathbb{N}^{|X|} \rightarrow \mathbb{R}^k$ is:

$$\Delta f = \max_{x,y \in \mathbb{N}^{|X|}, x \neq y} \| f(x) - f(y) \|_1.$$ 

The $l_1$ sensitivity of a function $f$ captures the magnitude by which a single individual’s data can change the function $f$ in the worst case. It is noteworthy that $\Delta f$ is an important value in the Laplace mechanism.

**Definition 2.3.** Given any function $f : \mathbb{N}^{|X|} \rightarrow \mathbb{R}^k$, the Laplace mechanism is defined as:

$$M_L(x, f(\cdot), \epsilon) = f(x) + (Y_1, \ldots, Y_k),$$

where $Y_i (i = 1, \ldots, k)$ are i.i.d random variables drawn from the Laplace distribution $Lap(\frac{\Delta f}{\epsilon})$. The density function of the Laplace distribution (centered at 0) $Lap(c)$ is:

$$Lap(x|c) = \frac{1}{2c} \exp(-\frac{|x|}{c}).$$

The following Lemma can be seen in textbooks, see for example Theorem 3.6 of [6].

**Lemma 2.1.** The Laplace mechanism preserves $(\epsilon, 0)$-differential privacy.

## 3 Algorithms

In this section, we put forward three privacy preserving algorithms for $l_1$ regression and calculate their privacy parameters respectively.

### 3.1 Algorithm 1

The finite smoothing method is an important tool to solve nondifferentiable problem, for instance, median regression proposed in [16]. In addition, [16] proves that the solution of smooth function can estimate the solution of original function well. This idea is applied in algorithm 1 by an analogous technique.
Since the absolute value function is not differentiable at the cuspidal point, a smooth method for minimizing function (2) is considered. Let $\gamma$ be a nonnegative parameter which indicates the degree of approximation. Define

$$\rho_\gamma(t) = \begin{cases} \frac{t^2}{2\gamma} & \text{if } |t| \leq \gamma, \\ |t| - \frac{1}{2}\gamma & \text{if } |t| > \gamma. \end{cases}$$

Then the nondifferentiable function $F(\mu, \beta)$ is approximated by the Huber M-estimator (see [2]).

Denote $F_\gamma(\mu, \beta) = \frac{1}{n} \sum_{i=1}^{n} \rho_\gamma(r_i(\mu, \beta))$ and $L_\gamma(\mu, \beta) = F_\gamma(\mu, \beta) + \frac{1}{2} \beta^T \beta$. The sign vector $s_\gamma(\mu, \beta) = (s_1(\mu, \beta), \ldots, s_n(\mu, \beta))^T$ is given by

$$s_i(\mu, \beta) = \begin{cases} -1; & \text{if } r_i(\mu, \beta) < -\gamma, \\ 0; & \text{if } -\gamma \leq r_i(\mu, \beta) \leq \gamma, \\ 1; & \text{if } r_i(\mu, \beta) > \gamma. \end{cases}$$

Let $w_i(\mu, \beta) = 1 - s_i^2(\mu, \beta)$, then

$$\rho_\gamma(r_i(\mu, \beta)) = \frac{1}{2\gamma} w_i(\mu, \beta) r_i^2(\mu, \beta) + s_i(\mu, \beta) \left[ r_i(\mu, \beta) - \frac{1}{2}\gamma s_i(\mu, \beta) \right].$$

Denote $W_\gamma(\mu, \beta)$ as the diagonal $n \times n$ matrix whose diagonal elements are $w_i(\mu, \beta)$. So $W_\gamma(\mu, \beta)$ has value 1 in those diagonal elements related to small residuals and 0 elsewhere. For $\mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$, the derivation of $F_\gamma(\mu, \beta)$ is

$$\frac{\partial F_\gamma(\mu, \beta)}{\partial \beta} = \frac{1}{n} X^T \left[ \frac{1}{\gamma} W_\gamma(\mu, \beta) r(\mu, \beta) + s_\gamma(\mu, \beta) \right],$$

and

$$\frac{\partial F_\gamma(\mu, \beta)}{\partial \mu} = \frac{1}{n} 1^T \left[ \frac{1}{\gamma} W_\gamma(\mu, \beta) r(\mu, \beta) + s_\gamma(\mu, \beta) \right].$$

It can be verified that $L_\gamma(\mu, \beta)$ is convex and a minimizer of $L(\mu, \beta)$ is close to a minimizer of $L_\gamma(\mu, \beta)$ when $\gamma$ is close to zero. Furthermore, according to Theorem 1 in [16], the $l_1$ solution can be detected when $\gamma > 0$ is small enough, i.e., it is not necessary to let $\gamma$ converge to zero in order to find a minimizer of $L_\gamma(\mu, \beta)$. This observation is essential for the efficiency and the numerical stability of the algorithm to be described in this paper. In addition, refer to the algorithm in [4], the first privacy preserving algorithm for median regression is stated as follows.
Algorithm 1:
Inputs: privacy parameter $\epsilon$, design matrix $X$, response vector $Y$, regularization parameter $\lambda$ and approximation parameter $\gamma$.
Generate a random vector $b$ from the density function $h(b) \propto \exp(-\frac{\epsilon}{\lambda}\|b\|_1)$.
To implement this, pick the $l_1$ norm of $b$ from the Gamma distribution $\Gamma(d + 1, \frac{1}{\epsilon})$, and the direction of $b$ uniformly at random.
Compute $(\mu^*, \beta^*) = \arg\min_{\mu, \beta} L_\gamma(\mu, \beta^*) + \frac{\beta^T \omega}{n} + \frac{\beta^2}{\sqrt{n}}$, where $\omega = (\mu, \beta)$ is a $d + 1$ dimensional vector, and $n$ is the number of rows of $X$.
Output $(\mu^*, \beta^*)$.

This algorithm is very similar to the smoothing median regression convex program in [16], and therefore its running time is similar to that of smoothing regression. In fact, $(\mu^*, \beta^*)$ can be obtained by the interior point method. Similar to the proof in [4], we can show that Algorithm 1 is privacy preserving.

Theorem 3.1. Given a set of $n$ samples $X_1, \ldots, X_n$ over $\mathbb{R}^d$, with labels $Y_1, \ldots, Y_n$, where for each $i$, $\|X_i\|_1 \leq 1$ and $\|Y_i\|_1 \leq B$, the output of Algorithm 1 preserves $(\epsilon, 0)$-differential privacy.

Proof. Let $a_1$ and $a_2$ be two row vectors over $\mathbb{R}^d$ with $l_1$ norm at most 1 and $y_1, y_2 \in [-B, B]$. Consider the two inputs $D_1$ and $D_2$ where $D_2$ is obtained from $D_1$ by replacing one record $(a_1, y_1)$ into $(a_2, y_2)$. For convenience, assume the first $n - 1$ records are same. For any output $\omega^* = (\mu^*, \beta^*)$ by Algorithm 1, there is a unique value of $b$ that maps the input to the output. This uniqueness holds, because both the regularization function and the loss functions are differentiable everywhere. Denote $\tilde{a}_1$ as $(1, a_1)$ and $\tilde{a}_2$ as $(1, a_2)$ . Let the values of $d + 1$ dimensional vector $b$ for $D_1$ and $D_2$ respectively, be $b_1$ and $b_2$. Since $\omega^*$ is the value that minimizes both the optimization problems, the derivative of both optimization functions at $\omega^*$ is 0. This implies that for every $b_1$ in the first case, there exists a $b_2$ in the second case such that:

$$b_1 + \tilde{a}_1^T(1/\gamma W_\gamma(\mu^*, \beta^*) (\mu^* + a_1^T \beta^* - y_1) + S_\gamma(\mu^*, \beta^*)) = b_2 + \tilde{a}_2^T(1/\gamma W_\gamma(\mu^*, \beta^*) (\mu^* + a_2^T \beta^* - y_2) + S_\gamma(\mu^*, \beta^*)).$$

According to the definitions of $W_\gamma(\mu^*, \beta^*)$ and $S_\gamma(\mu^*, \beta^*)$, it is clear that

$$-1 \leq 1/\gamma W_\gamma(\mu^*, \beta^*) (\mu^* + a_1^T \beta^* - y_1) + S_\gamma(\mu^*, \beta^*) \leq 1$$

and

$$-1 \leq 1/\gamma W_\gamma(\mu^*, \beta^*) (\mu^* + a_2^T \beta^* - y_2) + S_\gamma(\mu^*, \beta^*) \leq 1.$$
Since \( \| \mathbf{a}_1 \|_1 \leq 2 \) and \( \| \mathbf{a}_2 \|_1 \leq 2 \), we have \( \| \mathbf{b}_1 - \mathbf{b}_2 \|_1 \leq 4 \), which implies that
\[-4 \leq \| \mathbf{b}_1 \|_1 - \| \mathbf{b}_2 \|_1 \leq 4. \]
Therefore, for any \((\mathbf{a}_1, y_1)\) and \((\mathbf{a}_2, y_2)\),
\[
P((\mu^*, \beta^*)|X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, X_n = \mathbf{a}_1, Y_n = y_1) = \frac{h(\mathbf{b}_1)}{h(\mathbf{b}_2)} = e^{-\frac{1}{4}(\| \mathbf{b}_1 \|_1 - \| \mathbf{b}_2 \|_1)},
\]
where \( h(\mathbf{b}_i) \) for \( i = 1, 2 \) is the density of \( \mathbf{b}_i \). Since \(-4 \leq \| \mathbf{b}_1 \|_1 - \| \mathbf{b}_2 \|_1 \leq 4\),
this ratio is at most \( \exp(\epsilon) \).

According to Lemma 1 in [1], theoretical results for accuracy of parameter estimation is given for Algorithm 1.

**Lemma 3.1.** Let \( G(\omega) \) and \( g(\omega) \) be two convex functions, which are continuous and differentiable at all points. If \( \omega_1 = \arg\min_\omega G(\omega) \) and \( \omega_2 = \arg\min_\omega G(\omega) + g(\omega) \), then \( ||\omega_1 - \omega_2||_1 \leq \frac{d n}{G_2} \). Here, \( G_1 = \max_\omega ||\nabla g(\omega)||_1 \) and \( G_2 = \min_\omega \min_\omega \mathbf{v}^T \nabla^2 G(\omega) \mathbf{v} \), for any unit vector \( \mathbf{v} \).

The main idea of the proof is to examine the gradient and the Hessian of the functions \( G \) and \( g \) around \( \omega_1 \) and \( \omega_2 \).

**Lemma 3.2.** If \( ||\mathbf{b}||_1 \) is a random variable drawn from \( \Gamma(d+1, \frac{4}{\epsilon}) \), then with possibility \( 1 - \alpha \), \( ||\mathbf{b}||_1 \leq \frac{4(d+1)\log(\frac{d+1}{\alpha})}{\epsilon} \).

**Proof.** Since a random variable drawn from \( \Gamma(d+1, \frac{4}{\epsilon}) \) can be written as the sum of \( d + 1 \) independent identically distributed random variables, each of which is distributed as an exponential random variable with mean \( \frac{4}{\epsilon} \). Using an union bound, we see that with probability \( 1 - \alpha \), the values of all \( d + 1 \) of these variables are upper bounded by \( \frac{4\log(\frac{d+1}{\alpha})}{\epsilon} \). Therefore, with probability at least \( 1 - \alpha \), \( ||\mathbf{b}||_1 \leq \frac{4(d+1)\log(\frac{d+1}{\alpha})}{\epsilon} \). \( \square \)

**Theorem 3.2.** Given an \( l_1 \) regression problem with regularization parameter \( \lambda \), let \( \omega_1 \) be the classifier that minimizes \( L_\gamma(\mu, \beta) + \frac{\mu^2}{\sqrt{n}} \), and \( \omega_2 \) be the classifier output by Algorithm 1 respectively. Then, with probability \( 1 - \alpha \), \( ||\omega_1 - \omega_2||_1 \leq \frac{4(d+1)\log(\frac{d+1}{\alpha})}{n \min(\lambda, \frac{\epsilon}{\sqrt{n}})} \).

**Proof.** According to Lemma 3.1, we take \( G(\omega) = L_\gamma(\mu, \beta) + \frac{\mu^2}{\sqrt{n}} \) and \( g(\omega) = \frac{\beta^T \beta}{\sqrt{n}} \). Because \( F_\gamma(\mu, \beta) \) is a convex function, if we define the second derivative of \( F_\gamma(\mu, \beta) \) is 0 at nondifferentiable points, then the hessian matrix of
$F_\gamma(\mu, \beta)$ is positive semidefinite. Notice that $\nabla^2 (\frac{\mu^2}{\sqrt{n}}) = \frac{2}{\sqrt{n}}$ and $\nabla^2 (\frac{\lambda}{2} \beta^T \beta) = \lambda I$, where $I$ is an identity matrix with size $d \times d$. Hence, for any unit vector $v$, $G_2 = \min_\omega \min_\omega v^T \nabla^2 G(\omega)v \geq \min(\lambda, \frac{2}{\sqrt{n}})$ and $g_1 = \frac{\|b\|_1}{n}$, $\|\omega_1 - \omega_2\|_1 \leq \frac{\|b\|_1}{n \min(\lambda, \frac{2}{\sqrt{n}})}$. Since $b$ is a random variable drawn from $\Gamma(d + 1, \frac{1}{\epsilon})$, according to Lemma 3.2, with possibility $1 - \alpha$, $\|b\|_1 \leq \frac{4(d+1)\log(d+1)}{\epsilon \alpha}$, then the theorem is obtained.

When $n$ is sufficient large, $\omega_2$ approximates $\omega_1$ well and $\omega_1$ is close to true parameter of $\arg\min_\omega L_\gamma(\omega)$.

### 3.2 Algorithm 2

The second algorithm is based on the iterative algorithm, which was first proposed in [17]. This iterative technique combines absolute deviations regression with least square regression. Hence, at the heart of the technique is any standard least squares curve fitting algorithm.

The basic least squares algorithm minimizes the criterion

$$I = \frac{1}{n} \sum_{i=1}^{n} w_i r_i^2(\mu, \beta) + \frac{\lambda}{2} \beta^T \beta,$$

where the weighting factors $w_i$ are positive real numbers. Based on the Lagrange multiplier approach, for a fixed $\lambda$, there exists a unique value $v$ such that minimizing equation (6) is equivalent to minimizing the following equation.

$$I = \frac{1}{n} \sum_{i=1}^{n} w_i r_i^2(\mu, \beta),$$

$$s.t. \quad \beta^T \beta \leq v.$$

Considering the $(t + 1)$-th iteration, we take $w_i$ as $\frac{1}{|r(t)_i|+e}$, where $r(t)_i$ is the residual of $i$-th sample at the $t$-th iteration. Then the iterative process can be written as

$$I(t + 1) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{|r(t)_i|+e} r_i^2(t + 1) + \frac{\lambda}{2} \beta^T \beta.$$  

(7)
Algorithm 2 preserves $(\epsilon, 0)$-differential privacy.

**Algorithm 2:**

Inputs: privacy parameter $\epsilon$, design matrix $X$, response vector $Y$, regularization parameter $\lambda$, tolerance parameter $\tau$ and the number of iteration $N_0$

Initialize the algorithm with $\hat{\mu}(0)$ and $\hat{\beta}(0)$

$$(\hat{\mu}(1), \hat{\beta}(1)) = \text{argmin}_{\mu, \beta} I(1)$$

for $t = 1, \ldots, N_0 - 1$ do

while $||\hat{\mu}(t) - \hat{\mu}(t-1)||_1 > \tau$ or $||\hat{\beta}(t) - \hat{\beta}(t-1)||_1 > \tau$ do

$$(\hat{\mu}(t+1), \hat{\beta}(t+1)) = \text{argmin}_{\mu, \beta} I(t+1)$$

else do

Output $$(\hat{\mu}(N_0), \hat{\beta}(N_0)) := (\hat{\mu}(t), \hat{\beta}(t))$$

break

end while

end for

Output $$(\hat{\mu}, \hat{\beta}) := (\hat{\mu}(N_0), \hat{\beta}(N_0)) + U,$$

where $U$ is a $d + 1$ dimensional Laplace random variable with parameter

$$c = \frac{8}{n \min(2, \sqrt{d^2 + \beta^2} + \epsilon)} (\sqrt{dv} + B)$$

**Theorem 3.3.** Given a set of $n$ samples $X_1, \ldots, X_n$ over $\mathbb{R}^d$, with labels $Y_1, \ldots, Y_n$, where for each $i$ $(1 \leq i \leq n)$, $||X_i||_1 \leq 1, |Y_i| \leq B$, the output of Algorithm 2 preserves $(\epsilon, 0)$-differential privacy.

**Proof.** Denote $\omega = (\hat{\mu}(N_0), \hat{\beta}(N_0))$ and the $l_1$ sensitivity of $\omega$ as $s(\omega)$. Let $a_1$ and $a_2$ be two vectors over $\mathbb{R}^d$ with $l_1$ norm at most $1$ and $y_1, y_2 \in [-B, B]$. Consider the two inputs $D_1$ and $D_2$ where $D_2$ is obtained from $D_1$ by changing one record $(a_1, y_1)$ into $(a_2, y_2)$. For convenience, assume the first $n - 1$ records are same. According to Lemma 3.1, let $G(\omega) = I(N_0)$ and

$$g(\omega) = \frac{1}{n} w_2(\hat{\mu}(N_0) + a_1^T \hat{\beta}(N_0) - y_2)^2 - \frac{1}{n} w_1(\hat{\mu}(N_0) + a_1^T \hat{\beta}(N_0) - y_1)^2.$$ Similar to the proof in Theorem 3.2, we can achieve that

$$g_1 = \max_\omega ||\nabla g(\omega)||_1 \leq \frac{2}{n} w_1 ||\hat{\mu}(N_0)||_1 + a_1^T \hat{\beta}(N_0) + |y_1| + \frac{2}{n} w_2 ||\hat{\mu}(N_0)||_1 + a_2^T \hat{\beta}(N_0) + |y_2|.$$ Notice that $(\hat{\mu}(N_0), \hat{\beta}(N_0)) = \text{argmin}_{\mu, \beta} I(N_0)$, then $\frac{\partial I(N_0)}{\partial \mu} = 0$ at $\mu = \hat{\mu}(N_0)$,
\[ \sum_{i=1}^{n} w_i (\mu(N_0) + X_i \hat{\beta}(N_0) - Y_i) = 0 \]
\[ \iff \hat{\mu}(N_0) = -\frac{\sum_{i=1}^{n} w_i (X_i \hat{\beta}(N_0) - Y_i)}{\sum_{i=1}^{n} w_i}. \]

Since \(0 < w_i \leq 1/e\), \(||y_i||_1 \leq B(i = 1, \cdots, n)\) and \(||\hat{\beta}(N_0)||_1 \leq \sqrt{d}||\hat{\beta}(N_0)||_2 \leq \sqrt{dv}\), we have \(||\hat{\mu}(N_0)||_1 \leq \sqrt{dv} + B\). Notice that above inequalities are still true in \(t\)-th (\(\geq 2\)) iteration and hence \(\frac{1}{2(\sqrt{dv} + B) + e} \leq w_i \leq \frac{1}{e}\). Then we can achieve that
\[ g_1 = \max_{\omega} ||\nabla g(\omega)||_1 \leq \frac{8(\sqrt{dv} + B)}{ne}. \]

In addition, denote \(F_e(\omega) = \frac{1}{n} \sum_{i=1}^{n} w_i r_i^2(\mu, \beta)\). It can be checked that \(F_e(\omega)\) is convex and \(\frac{\partial^2 F_e(\omega)}{\partial \mu^2} = \frac{2}{n} \sum_{i=1}^{n} w_i \leq \frac{2}{2(\sqrt{dv} + B) + e},\) \[ \nabla^2 \left( \frac{\lambda}{2} \beta^T \beta \right) = \lambda I, \]
where \(I\) is an identity matrix with size \(d \times d\), then \(G_2 \geq \min\left(\frac{2}{2(\sqrt{dv} + B) + e}, \lambda\right)\) and \(s(\omega) \leq \frac{8}{n \min\left(\frac{2}{2(\sqrt{dv} + B) + e}, \lambda\right)}(\sqrt{dv} + B) \cdot \)

According to lemma 2.1, the result is obtained directly from the composition theorem. \(\square\)

For \(\epsilon > 0\), define a perturbation of \(L(\mu, \beta)\) as
\[ L_\epsilon(\mu, \beta) = \sum_{i=1}^{n} |r_i(\mu, \beta)| - \frac{\epsilon}{2} \ln(\epsilon + |r_i(\mu, \beta)|) + \frac{\lambda}{2} \beta^T \beta. \]

[10] proves that iterative least square algorithm without adding noise is a special case of Majorization-Minimization (MM) algorithms (see [11]) for objective function \(L_\epsilon(\mu, \beta)\) and obtained convergence results.

**Proposition 3.1.** For linear median regression with a full-rank covariate matrix \(X\), the iterative least square algorithm without adding noise converges to the unique minimizer of \(L_\epsilon(\mu, \beta)\).

**Proposition 3.2.** If \((\hat{\mu}_\epsilon, \hat{\beta}_\epsilon)\) minimizes \(L_\epsilon(\mu, \beta)\), then any limit point of \((\hat{\mu}_\epsilon, \hat{\beta}_\epsilon)\) as \(\epsilon\) tends to \(0\) minimizes \(L(\mu, \beta)\). If \(L(\mu, \beta)\) has a unique minimizer \((\hat{\mu}, \hat{\beta})\), then \(\lim_{\epsilon \to 0}(\hat{\mu}_\epsilon, \hat{\beta}_\epsilon) = (\hat{\mu}, \hat{\beta}).\)

The proof of above propositions can be seen in [10].
Theorem 3.4. Given a l_1 regression problem with regularization parameter λ, let ω_1 be the classifier that minimizes L_{e}(µ, β), and ω_2 be the classifier output by Algorithm 2 respectively. Then, with probability 1 − α, \[ ||ω_1 - ω_2||_1 \leq \frac{8(\sqrt{dv} + B)(d+1)\log\left(\frac{d+1}{\alpha}\right)}{\epsilon \min(\frac{1}{2(\sqrt{dv}+B)+\epsilon} + \lambda)ne}. \]

Proof. Since \[ ||b||_1 \] is a random variable drawn from \( \Gamma(d+1, \frac{8(\sqrt{dv} + B)(d+1)\log\left(\frac{d+1}{\alpha}\right)}{\epsilon \min(\frac{1}{2(\sqrt{dv}+B)+\epsilon} + \lambda)ne}) \), with possibility 1 − α, \[ ||b||_1 \leq \frac{8(\sqrt{dv} + B)(d+1)\log\left(\frac{d+1}{\alpha}\right)}{\epsilon \min(\frac{1}{2(\sqrt{dv}+B)+\epsilon} + \lambda)ne} \], the theorem is obtained.

Therefore, for fixed small \( \epsilon \), if \( n \) is sufficient large, accuracy can be ensured in practice.

3.3 Algorithm 3

In [1], the authors argue that adding noise to the estimated parameters after optimization would destroy the utility of the learned model. Hence, we prefer a more sophisticated method to control the influence of the training data during the training process, especially in the stochastic gradient decent computation. [19] declares that greedy coordinate descent is an effective method for l_1 regression, where l_1 regression means median regression. So we apply this idea to minimize objective function \( L(µ, β) \) in a similar way. Although \( L(µ, β) \) is nondifferentiable, it does possess directional derivatives along each forward or backward coordinate direction. For example, if \( e_k \) is the coordinate direction along which \( β_k \) varies, then the objective function (2) has directional derivatives

\[ d_{e_k}^+ L(µ, β) = \lim_{\tau \to 0^+} \frac{L(µ, β + \tau e_k) - L(µ, β)}{\tau} = d_{e_k}^+ F(µ, β) + \lambda β_k \]

and

\[ d_{e_k}^- L(µ, β) = \lim_{\tau \to 0^-} \frac{L(µ, β + \tau e_k) - L(µ, β)}{\tau} = d_{e_k}^- F(µ, β) + \lambda β_k. \]

In l_1 regression, the coordinate direction derivatives are

\[ d_{e_k}^\pm F(µ, β) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} -x_{ik}, & r_i(µ, β) < 0, \\ x_{ik}, & r_i(µ, β) > 0, \\ |x_{ik}|, & r_i(µ, β) = 0, \end{cases} \]
and

\[ d_{e_k} F(\mu, \beta) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} x_{ik}, & r_i(\mu, \beta) < 0, \\ -x_{ik}, & r_i(\mu, \beta) > 0, \\ |x_{ik}|, & r_i(\mu, \beta) = 0. \end{cases} \]  

(9)

In greedy coordinate descent progress [9], we update the direction of parameter \( \beta_k \) based on \( \min\{d_{e_k} L(\mu, \beta), d_{e_k} L(\mu, \beta)\} \). If both coordinate directional derivatives are nonnegative, the update of \( \beta_k \) stops. In addition, \( \hat{\mu} = \frac{1}{n_0} \sum_{i=1}^{n_0} (Y_i - X_i \hat{\beta}) \), where \( n_0 = n/N_0 \). And by the method of batch gradient [18], the \( t \)-th iteration only employs records with batch size \( n_0 \), which means \( L(\hat{\mu}(t), \hat{\beta}(t)) \) in the algorithm is calculated by subset \((X(t), Y(t))\).

The algorithm is described as follows.

**Algorithm 3:**

**Inputs:** privacy parameters \( \epsilon \), design matrix \( X \), response vector \( Y \), regularization parameter \( \lambda \), positive number \( \ell \) and the number of iterations \( N_0 \).

- Randomly split \((X, Y)\) into \( N_0 \) disjoint subsets of size \( n_0 \).
- Initialize the algorithm with a vector \((\hat{\mu}(0), \hat{\beta}(0))\) (such as the solution of \( l_2 \) regression).

  for \( t = 0, 1, 2, ..., N_0 - 1 \) do
  
  \( \eta_t = \frac{\ell}{t+1} \)

  for \( k = 1, 2, \ldots, d \) do

  \( \hat{\beta}_k(t+0.5) = \hat{\beta}_k(t) - \eta_t \min \{d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t)), d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t))\} \),

  \( \hat{\beta}_k(t+1) = \hat{\beta}_k(t+0.5) + U_t, \) where \( U_t \sim \text{Lap} \left( \frac{2n_0}{\epsilon \lambda_0} \right), \)

  end for

  \( \hat{\mu}(t+1) = \frac{1}{n_0} \sum_{i=1}^{n_0} (Y_i - X_i \hat{\beta}(t+1)) \).

  end for

**Output** \( \hat{\beta} := \hat{\beta}(N_0), \hat{\mu} = \hat{\mu}(N_0) \).

**Theorem 3.5.** Given a set of \( n \) samples \( X_1, \ldots, X_n \) over \( \mathbb{R}^d \) with labels \( Y_1, \ldots, Y_n \), where for each \( i (1 \leq i \leq n) \), \( ||X_i||_1 \leq 1 \) and \( ||Y_i||_1 \leq B \), the output of Algorithm 3 preserves \((\epsilon, 0)\)-differential privacy.

**Proof.** Because of sample splitting, for \((x, y) \in (X(t), Y(t))\) for some \( 0 \leq t \leq N_0 - 1 \), it suffices to prove the privacy guarantee for the \( t \)-th iteration of the algorithm: any iteration prior to the \( t \)-th one does not depend on \((x, y)\), while any iteration after the \( t \)-th one is differentially private by post-processing [6].
At the $t$-th iteration, the algorithm first updates the non-sparse estimate of $\beta_k$:

$$\hat{\beta}_k(t + 0.5) = \hat{\beta}_k(t) - \eta_t( \min d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t)), d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t))).$$

Let $a_1$ and $a_2$ be two vectors over $\mathbb{R}^d$ with $l_1$ norm at most 1 and $y_1, y_2 \in [-B, B]$. Consider the two inputs $D_1$ and $D_2$ where $D_2$ is obtained from $D_1$ by changing one record $(a_1, y_1)$ into $(a_2, y_2)$. For convenience, assume the first $n_0 - 1$ records are same. Denote $Dir_1(t)$ as the direction derivation (min $\{d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t)), d_{e_k} L(\hat{\mu}(t), \hat{\beta}(t))\}$) for the dataset $D_1$ and $Dir_2(t)$ for the dataset $D_2$. Notice that $\hat{\beta}(t)$ does not depend on $(X(t), Y(t))$, so $\hat{\beta}(t+1)$ would be $(\epsilon, 0)$-differentially private if it can be shown that: for every pair $D$ and $D'$, we have

$$||\eta_t/n_0 [Dir_1(t) - Dir_2(t)]||_1 \leq \frac{2\eta_t}{n_0}.$$ 

This is true, since $||\eta_t/n_0 [Dir_1(t) - Dir_2(t)]||_1 \leq \eta_t/n_0(||a_1||_1 + ||a_2||_1) \leq \frac{2\eta_t}{n_0}$, then the privacy guarantee for $\beta$ is proved by Lemma 2.1. In addition, since $\hat{\mu} = \frac{1}{n_0} \sum_{i=1}^{n_0} (Y_i - X_i\hat{\beta})$, it is differentially private by post-processing [19]. Then the theorem is obtained. \[\square\]

[19] said that coordinate descent may fail for a nondifferentiable function since all directional derivatives must be nonnegative at a minimum point. However, if we can obtain a suitable approximate value quickly, this shortcoming can be accepted in practice. The following theorem shows that estimated parameters would be stable when the number of iteration $N_0$ is large.

**Theorem 3.6.** Given a set of $n$ samples $X_1, \ldots, X_n$ over $\mathbb{R}^d$ with labels $Y_1, \ldots, Y_n$ (for each $i$, $||X_i|| \leq 1$ and $||Y_i||_1 \leq B$), Algorithm 3 is convergent in probability with rate $O(1/t)$.

**Proof.** Consider the $t$-th iteration for $\beta_k$, since $|Dir(t)| \leq \frac{1}{n_0} \sum_{i=1}^{n_0} ||x_{ik}||_1 \leq 1$ and $\hat{\beta}_k(t+1) = \hat{\beta}_k(t) - \eta_t Dir(t) + U_t$, $|\beta_k(t+1) - \beta_k(t)| \leq |\eta_t| + |U_t| = O_p(1/t)$, where $O_p(1/t)$ indicates that it converges in probability with rate $O(1/t)$. Since $\hat{\mu} = \frac{1}{n_0} \sum_{i=1}^{n_0} (Y_i - X_i\hat{\beta})$, it is convergent in probability with rate $O(1/t)$, too. Then the theorem is obtained. \[\square\]
4 Simulated results

Denote \( n \) as the number of samples. Here let \( n \) take two values: 5000 and 5000000. In fact, when \( n \) is small, such as 100, Algorithm 1 can perform well, but Algorithm 2 requires \( n \) bigger (otherwise the noise added would be big which would result in big estimation error). Consider the following example with three independent variables \( x_1, x_2, x_3 \), where \( y_i = 2 + 3x_{i1} - 4x_{i3} + u_i \) and \( u_i \) obeys the Laplace distribution \( \text{Lap}(2) \), for \( i = 1, \ldots, n \). We assume that, for each \( i (1 \leq i \leq n) \), \( l_1 \) norm of \( X_i \) is less than 1 and \( l_1 \) norm of \( Y_i \) is less than 2. In practice, we take \( \lambda \) as 0.002 in the objective function. In Algorithm 1, parameter \( \gamma \) is taken as 0.05. In Algorithm 2, we set parameter \( e = 0.2 \), tolerance parameter \( \tau = 10^{-6} \) and the number of iteration \( N_0 = 200 \). In fact, Algorithm 2 tends to converge with less than 30 iterations. In Algorithm 3, we set \( \ell = 0.1 \), step size \( \eta_t = \frac{\ell}{t+1} \) and the number of iteration \( N_0 = 40 \). In addition, privacy parameters \( (\epsilon, \delta) = (0.1, 0) \) for all the above algorithms. The results are listed in Table 1 and Table 2. It shows that Algorithm 1 performs better than the others when \( n = 5000 \). However, when \( n \) becomes much bigger, Algorithm 1 costs much more time. Notice that when \( n = 5000000 \), the noise added to Algorithm 2 becomes small and it makes the estimated result precise. In addition, Algorithm 3 costs less time in both cases, but it highly depends on initial value and step size \( \eta_t \), which is a common problem for the gradient descent method [18].

|               | Algorithm 1 | Algorithm 2 | Algorithm 3 | True value |
|---------------|-------------|-------------|-------------|------------|
| \( \mu \)     | 2.0684      | 1.9440      | 1.8204      | 2          |
| \( \beta_1 \) | 3.0007      | 0.9227      | 2.6914      | 3          |
| \( \beta_2 \) | -0.0295     | 13.2762     | -0.6099     | 0          |
| \( \beta_3 \) | -4.0835     | -14.2089    | -3.2283     | -4         |
| **time(s)**   | 0.7113      | 0.3143      | 0.1220      |            |
Table 2: Estimated results with sample size 5000000

|           | Algorithm 1 | Algorithm 2 | Algorithm 3 | True value |
|-----------|-------------|-------------|-------------|------------|
| $\mu$     | 1.9538      | 1.9417      | 1.8405      | 2          |
| $\beta_1$ | 3.0152      | 3.0327      | 3.1727      | 3          |
| $\beta_2$ | 0.0073      | 0.0029      | -0.2881     | 0          |
| $\beta_3$ | -3.9460     | -3.9205     | -3.9918     | -4         |
| time(s)   | 80.2314     | 20.0123     | 0.5587      |            |

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