The Vlasov-Poisson dynamics as the mean-field limit of rigid charges

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Abstract

The paper treats the validity problem of the non-relativistic Vlasov-Poisson equation in $d \geq 2$ dimensions. It is shown that the Vlasov-Poisson dynamics can be derived as a combined mean-field and point-particle limit of an $N$-particle Coulomb system of rigid charges. This requires a sufficiently fast convergence of the initial empirical distributions. If the electron radius decreases slower than $N^{-\frac{1}{4}}$, the corresponding initial configurations are typical. This result also entails propagation of chaos for the Vlasov-Poisson dynamics.

1 Introduction

We are interested in a microscopic derivation of the Vlasov-Poisson dynamics in $d \geq 2$ spatial dimensions. This is the system of equations

$$\partial_t f + p \cdot \nabla_q f + (k \ast \rho_t) \cdot \nabla_p f = 0,$$

(1)

where $k$ is the Coulomb kernel

$$k(q) := \sigma \frac{q}{|q|^d}, \quad \sigma = \{\pm 1\}$$

(2)

and

$$\rho_t(q) = \rho[f_t](q) = \int d^3p \ f(t,q,p)$$

(3)

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is the charge-density induced by the distribution \( f(t,p,q) \geq 0 \), which describes the microscopic density of particles with position \( q \in \mathbb{R}^d \) and momentum \( p \in \mathbb{R}^d \). Here, units are chosen such that all constants, in particular the mass and charge of the particles, are equal to 1.

The Vlasov-Poisson dynamics provides an effective description of a collisionless plasma with electrostatic (\( \sigma = +1 \)) or gravitational (\( \sigma = -1 \)) interactions. In the gravitational case, the equation is also known as Vlasov-Newton.

Kinetic equations of the Vlasov-type are usually conceived of as mean-field equations, i.e. effective descriptions of a large number of microscopic particles in which the N-particle interactions can be approximated by an “average” effect determining the time-evolution of \( f \).

Classical results, dealing with simplified models with Lipschitz-continuous forces, prove a statement of the following type: If for an initial microscopic configuration \( X = (q_1,p_1; q_2,p_2; \ldots; q_N,p_N) \) the empirical distribution \( \mu^N_0[X] = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i} \delta_{p_i} \) is well approximated by a continuous density \( f_0(q,p) \), then, at time \( t > 0 \), the time-evolved distribution \( \mu^N_t = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i(t)} \delta_{p_i(t)} \) is well approximated by \( f_t(p,q) \), where \( f_t \) is a solution of a Vlasov-equation of the form (Braun and Hepp, 1977 \[4\], Dobrushin, 1979 \[7\], Neunzert, 1984 \[19\].)

However, generalization of the corresponding techniques to singular forces have been unsuccessful, even when suitable (N-dependent) cut-offs are employed. The understanding that has grown in recent years, is that such deterministic statements are in fact too strong for more realistic systems, the reason being that there exist “bad” initial conditions leading to the clustering of particles and hence to significant deviations from the typical mean-field behavior.

The strategy employed by Hauray and Jabin in \[11\] is thus to impose additional constraints on the initial configurations, subsequently showing that the set of “good” initial conditions (for which these constraints are satisfied) approaches measure 1 as \( N \to \infty \). In this way, they are able to treat systems with singular potentials up to - but not including - the Coulomb case and N-dependent cut-off of order \( N^{-1/2d+\epsilon} \).

Recently, Boers and Pickl proposed a novel method for deriving mean-field equations which is designed for stochastic initial conditions, thus aiming directly at a typicality result \[1\]. In \[14\], this method was generalized to include the Coulomb interactions with a microscopic cut-off width of order \( N^{-1/d+\epsilon} \).

In this paper, we present an alternative derivation of the (full)
Vlasov-Poisson dynamics from a regularized Coulomb-system on the microscopic level. While our method requires a relatively large cut-off \((N^{-1/4d+\epsilon} \text{ in the best case}),\) it yields better rates of convergence than \([14]\) and is arguably much simpler than the approach in \([11]\). The probabilistic nature of the result enters through the requirement of a sufficiently fast convergence of the initial distribution \(\mu_0\) to the Vlasov-density \(f_0\). This assumption assures that the initial configurations are “well placed” so to speak, preventing, in particular, local clustering of particles. The proposed method thus offers another perspective on how deterministic mean-field limits can be replaced by typicality results to treat more general interactions.

Moreover, rather than invoking some arbitrary regularization, we consider as a microscopic model N-particle Coulomb systems of rigid charges. The cut-off parameter \(r_N\) thus has a straightforward physical interpretation as a finite electron radius. An analogous model was used by Golse (2012) to derive a regularized version of the Vlasov-Maxwell system \([9]\) (c.f. also Rein, 2004 \([21]\)).

In the light of persistent singularities troubling all standard formulations of electrodynamics, both classical and quantum, there are scientists willing to entertain the idea of rigid charges on a fundamental physical level (see e.g. \([17]\) and references therein), though the main objection arises from special relativity, as the shape of a spatially extended particle wouldn’t be Lorentz invariant. Empirically, experiments put the upper bound on an electron radius to \(10^{-22}\text{m}\) \([5]\).

2 The Wasserstein distances

We recall some basic facts about the Wasserstein distances and optimal transportation. For details and proofs, we refer the reader to the book of Cédric Villani \([24]\) chapter 6.

We denote by \(\mathcal{P}(\mathbb{R}^k)\) the set of probability measures on \(\mathbb{R}^k\) (equipped with its Borel algebra). If \((\mu_n)_{n \in \mathbb{N}}\) is a sequence in \(\mathcal{P}(\mathbb{R}^k)\) and \(\mu\) another element, we denote by \(\mu_n \rightharpoonup \mu\) the weak convergence of \((\mu_n)_{n \in \mathbb{N}}\) to \(\mu\), meaning

\[
\int \phi(x) \, d\mu_n(x) \to \int \phi(x) \, d\mu(x), \quad n \to \infty,
\]

for all bounded and continuous functions \(\phi\).

For given \(\mu, \nu \in \mathcal{P}(\mathbb{R}^k)\) let \(\Pi(\mu, \nu)\) be the set of all probability measures \(\mathbb{R}^k \times \mathbb{R}^k\) with marginal \(\mu\) and \(\nu\) respectively.
Let $p \in [1, +\infty)$. We denote by $\mathcal{P}_p(\mathbb{R}^k) \subset \mathcal{P}(\mathbb{R}^k)$ the set of probability measures with finite $p$-th moment, that is

$$\mathcal{P}_p(\mathbb{R}^k) = \{\mu \in \mathcal{P}(\mathbb{R}^k) : \int |x|^p \, d\mu < \infty\},$$

(4)

$\mathcal{P}_p(\mathbb{R}^k)$ is also called the $p$-th Wasserstein space.

On $\mathcal{P}_p(\mathbb{R}^k)$, we define the Wasserstein distance of order $p$ by

$$W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int |x - y|^p \, d\pi(x, y) \right)^{1/p}.$$

(5)

A direct application of Hölder's inequality yields the relation

$$p < q \Rightarrow W_p(\mu, \nu) \leq W_q(\mu, \nu).$$

A crucial result is the Kantorovich-Rubinstein duality:

$$W_p^p(\mu, \nu) = \sup_{\Phi_1, \Phi_2 \in L^1(\mu) \times L^1(\nu)} \left\{ \int \Phi_1(x) \, d\mu(x) - \int \Phi_2(y) \, d\nu(y) \right\},$$

(6)

For any integrable function $\Phi$, we define its $d^p$-conjugate $\Phi^c$ by

$$\Phi^c(y) := \sup_x \{\Phi(x) - |x - y|^p\}$$

(7)

It’s easy to see that this is the smallest function satisfying $\Phi(x) - \Phi^c(y) \leq |x - y|^p$, $\forall x, y \in \mathbb{R}^3$. Hence, the Kantorovich duality formula is equivalent to

$$W_p^p(\mu, \nu) = \sup_{\Phi \in L^1(\mu)} \left\{ \int \Phi(x) \, d\mu(x) - \int \Phi^c(y) \, d\nu(y) \right\},$$

(8)

A particularly useful case is the first Wasserstein distance, for which the problem reduces further to

$$W_1(\mu, \nu) = \sup_{\Phi \in L^1(\mu)} \left\{ \int \Phi(x) \, d\mu(x) - \int \Phi(x) \, d\nu(x) \right\},$$

(9)

where $\|\Phi\|_{Lip} := \sup_{x \neq y} \frac{\Phi(x) - \Phi(y)}{|x - y|}$, to be compared with the bounded Lipschitz distance

$$d_{BL}(\mu, \nu) = \sup \left\{ \int \Phi(x) \, d\mu(x) - \int \Phi(x) \, d\nu(x) ; \|\Phi\|_{Lip} = \|\Phi\|_{\infty} = 1 \right\}.$$

(10)
Our main interest in the Wasserstein distances is as a way to metrize weak convergence of probability measures. In fact, convergence in Wasserstein distance also implies convergence of the $p$-th moment. Explicitly, we have:

$$W_p(\mu_n, \mu) \to 0 \iff \mu_n \rightharpoonup \mu \text{ and } \lim_{n \to \infty} \int |x|^p \, d\mu_n(x) = \int |x|^p \, d\mu(x).$$

### 3 The charge-cloud limit

In our discussion, $\Omega = \mathbb{R}^d \times \mathbb{R}^d$ and two types of measures are of particular interest. We call $\nu \in \mathcal{P}$ a macroscopic profile if it is absolutely continuous, i.e.

$$\nu(x, \xi) = f(x, \xi) \, dx \, d\xi, \quad (11)$$

with a continuous density $f \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

We call $\mu \in \mathcal{P}$ the $N$-particle empirical distribution for the microscopic configuration $X \in \mathbb{R}^{6N}$, if it’s of the form

$$\mu[X](x, \xi) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - q_i) \delta(\xi - p_i) \quad (12)$$

for $X = ((q_1, p_1), ..., (q_N, p_N)) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ with $q_i \neq q_j$, if $i \neq j$. It describes the distribution of $N$ identical particles with positions and momenta $(q_i, p_i)_{i=1,...,N}$.

In any case, a probability measure $\nu$ on $\mathbb{R}^d \times \mathbb{R}^d$ induces a charge-density on $\mathbb{R}^d$ by

$$\rho[\nu](x) = \int \nu(x, \xi) \, d\xi. \quad (13)$$

The main difficulty in deriving kinetic equations such as the Vlasov-Poisson equation or the Vlasov-Maxwell equation is that electromagnetic interactions are singular for point-particles. In fact, the textbook Maxwell-Lorentz theory of classical electrodynamics is ill-defined on a fundamental level, while in the present electrostatic case, the equations are well-defined only because they omit self-interactions. Still, the Coulomb force and its derivative are unbounded, which precludes a direct application of the “classical” mean-field results.

To have a more well-behaved macroscopic theory, we will thus consider particles of finite extension, i.e. Coulomb systems of rigid charges. Finally, we will let the electron radius $r_N$ go to zero in the limit $N \to \infty$, as the empirical density of the rigid charges approaches
a macroscopic profile. We call the combined limit $N \to \infty$, $r_N \to 0$ the charge-cloud limit. Intuitively, it describes the regime in which a large system of small (but extended) particles can be treated as a continuous charge-cloud.

Size and shape of the rigid charges are described by a form-factor given by smooth, non-negative, compactly supported function $\chi \in C_0^\infty(\mathbb{R}^d)$. For simplicity, we shall assume that $\chi$ is scaled such that

i) $\text{supp}(\chi) \subseteq B(1; 0) = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$.

ii) $\|\chi\|_\infty = \sup_{x \in \mathbb{R}} |\chi(x)| = 1$.

iii) $\|\chi\|_1 = \int \chi(x)dx = 1$.

For the point-particle limit, we define a rescaled form-factor as follows:

**Definition 3.1.**

We call a sequence $(r_N)_{N \in \mathbb{N}}$ of positive real numbers a rescaling sequence if it is monotonously decreasing with $r_1 = 1$ and $\lim_{N \to \infty} r_N = 0$.

Let $(r_N)_{N \in \mathbb{N}}$ be a rescaling sequence as in the previous definition. For any $N \in \mathbb{N}$, we define

$$\chi^N(x) := \frac{1}{r_N^d} \chi \left( \frac{x}{r_N} \right).$$

(14)

This rescaled form-factor then satisfies:

i) $\text{supp}(\chi^N) \subseteq B(r_N; 0)$.

ii) $\|\chi^N\|_\infty = r_N^{-d}$.

iii) $\|\chi^N\|_1 = \int \chi^N(x)dx = 1$.

Given a microscopic density $\mu = \frac{1}{N} \sum_{i=1}^{N} \delta(x - q_i)\delta(\xi - p_i)$, we define

$$\hat{\mu}^N := \chi^N * x \mu = \frac{1}{N} \sum_{i=1}^{N} \chi^N(x - q_i)\delta(\xi - p_i),$$

(15)

corresponding to the empirical charge/momentum distribution of $N$ rigid charges with centers of mass $q_1, ..., q_N$ and momenta $p_1, ..., p_N$.

Note that the total charge $Q := \int \int d\hat{\mu}^N(x, \xi)$ remains normalized to 1 for all $N \in \mathbb{N}$ and any rescaling sequence $(r_N)_{N}$.

**Lemma 3.2.**

Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $(r_N)_{N}$ a rescaling sequence and $\chi^N$ the rescaled
form-factor as defined above. Let $\nu \in \Omega$ and $\hat{\nu} = \chi^N \ast \nu$. Then, we have for any $1 \leq p < \infty$:

$$W_p(\hat{\nu}, \nu) \leq r_N. \tag{16}$$

In particular, a sequence $(\nu_n)_{n \in \mathbb{N}}$ converges to some $\nu \in \mathcal{P}$ if and only if $(\hat{\nu}_n)_{n}$ does.

Proof. Define $\pi(x, y) := \nu(x)\chi^N(x - y)$ and observe that $\int \chi(x) \pi(x, y) = \hat{\nu}(y)$, $\int \chi(y) \pi(x, y) = \nu(x)$, hence $\pi \in \Pi(\hat{\nu}, \nu)$. Now observe that $\pi$ has support in $\{|x - y| < r_N\}$. Thus, we have

$$W_p(\hat{\nu}, \nu) = \inf_{\pi \in \Pi(\hat{\nu}, \nu)} \left( \int_{\Omega \times \Omega} |x - y|^p d\pi(x, y) \right)^{1/p} \leq \left( \int_{\Omega \times \Omega} |x - y|^p d\pi(x, y) \right)^{1/p} \leq r_N.

$$

Lemma 3.3. For all $\mu, \nu \in \mathcal{P}$ and $1 \leq p < \infty$:

$$W_p(\hat{\mu}, \hat{\nu}) \leq W_p(\mu, \nu). \tag{17}$$

Proof. In view of the dual Kantorovich problem (6), we find for $(\Phi_1, \Phi_2) \in L^1(\mu) \times L^1(\nu)$ with $\Phi_1(y) - \Phi_2(x) \leq |x - y|^p$:

$$\int \Phi_1 d\hat{\mu}(x) - \int \Phi_2 d\hat{\nu}(y) = \int (\chi \ast \Phi_1)(x) d\mu(x) - \int (\chi \ast \Phi_2)(y) d\nu(y)

$$

Now we observe that

$$|\chi \ast \Phi_1 - \chi \ast \Phi_2(y)| = \left| \int \chi(z)\Phi_1(x - z) dz - \int \chi(z)\Phi_2(y - z) dz \right| \leq \int \chi(z) |\Phi_1(x - z) - \Phi_2(y - z)| dz \leq \int \chi(z) |x - y|^p dy = |x - y|^p.

$$

Hence, we have

$$\int \Phi_1 d\hat{\mu} - \int \Phi_2 d\hat{\nu} \leq W_p(\mu, \nu),$$

and taking the supremum over all $(\Phi_1, \Phi_2)$ yields the desired inequality. \qed
The microscopic model

Now we have to state what the equations of motion for the rigid charges actually are. As before,

\[ k : \mathbb{R}^d \to \mathbb{R}^d, q \mapsto \frac{q}{|q|^d}, \]

denotes the Coulomb kernel. That is, if \( \Psi : \mathbb{R}^d \to \mathbb{R} \) is a solution of Poisson’s equation

\[ \Delta \Psi = \mp c \rho, \lim_{|q| \to +\infty} \Psi(q) = 0 \]

in the sense of

\[ \Psi(q) = \int \frac{\sigma}{|q' - q|^{d-2}} \rho(q') \, dq', \quad d \geq 3, \]

or \( \Psi(q) = -\sigma \int \ln(q' - q) \rho(q') \) for \( d = 2 \), then

\[ \nabla \Psi(q) = k * \rho(q) = \sigma \int \frac{q' - q}{|q' - q|^d} \rho(q') \, dq'. \]

We consider a system of \( N \) rigid charges with form-factor \( \chi^N \) and radius \( r_N \). Its configuration is given by \( X(t) = (q_1, \ldots, q_N; p_1, \ldots, p_N) \), where \( q_i(t) \) is the center of mass of particle \( i \), and \( p_i(t) \) the corresponding momentum at time \( t \). The microscopic equations of motion, in the so called mean-field scaling, are given by

\[
\begin{aligned}
\dot{q}_i(t) &= p_i(t) \\
\dot{p}_i(t) &= K_N(q_i; q_1, \ldots, q_N)
\end{aligned}
\]

with

\[
K_N(q_i; q_1, \ldots, q_N) := \frac{1}{N} \sum_{j=1}^N \int \int \chi^N(q_j - y) k(z - y) \chi^N(q_i - z) \, dy \, dz.
\]

These equations give rise to a Hamiltonian flow \( \Phi_{t,s} \) on \( \mathbb{R}^{dN} \times \mathbb{R}^{dN} \) satisfying \( \Phi_{t,s}^N(X(s)) = X(t) \) and \( \Phi_{t,s}^{N_s}(X) = X \).

From a purely formal point of view, regularizing the interaction kernel by convoluting it \textit{twice} with the mollifier \( \chi \) may seem strange. However, this microscopic model has a straight-forward physical interpretation in terms of rigid charges. Moreover, one should note that (unlike for the corresponding equation with only one mollifier) total energy is conserved in this system.
Lemma 4.1. Let \( X(t) = (q_1, ..., q_N; p_1, ..., p_n)(t) \) a solution of (18). Then, the total energy
\[
E = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2N} \sum_{i,j} \int \int \chi(y - q_i) \chi(z - q_j) \sigma_{|z-y|} dy dz
\] (20)
is a constant of motion.

Note that \( E \) includes the self energy \( \frac{1}{N} \int \int \chi(y - q_i) \chi(z - q_i) \sigma_{|z-y|} dy dz \).

Proof. Check that the system (18) is canonical with Hamiltonian \( H(q_i, p_i) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2N} \int \int \chi(y - q_i) \chi(z - q_j) \sigma_{|z-y|} dy dz \).

4.1 The regularized Vlasov-Poisson equation

For the microscopic model described above, we introduce a corresponding mean-field equation:
\[
\partial_t f + p \cdot \nabla_q f + k^N[\rho_t] \cdot \nabla_p f = 0,
\]
\[
k^N[\rho_t](q) := \chi^N * \chi^N * k * \rho_t(q),
\]
\[
\rho_t(q) = \rho[f_t](q) = \int f(t, q, p) \, d^d p
\] (23)
we call this the regularized Vlasov-Poisson system with cut-off parameter \( r_N \). Note that with the notation of Sec. 3, \( k^N[\rho] = k * \hat{\rho} = k * \check{\rho} \).

4.2 The method of characteristics

Let \( \nu = (\nu_t)_{t \in [0,T]} \) a continuous family of probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \). Let \( \rho_t[\nu] = \int \nu(q, p) \, d^d p \) the induced (charge-)distribution on \( \mathbb{R}^d \). We denote by \( \varphi_{t,s}^\nu = (Q^\nu(t, s, q_0, p_0), P^\nu(t, s, q_0, p_0)) \) the one-particle flow on \( \mathbb{R}^d \times \mathbb{R}^d \) solving:
\[
\begin{aligned}
\frac{d}{dt} Q &= P \\
\frac{d}{dt} P &= \chi^N * \chi^N * k * \rho(Q) \\
Q(s, s, q_0, p_0) &= q_0 \\
P(s, s, q_0, p_0) &= p_0
\end{aligned}
\] (24)
This flow exists and is well-defined since the vector-field is Lipschitz for all \( N \). Now, if \( f^N(t, q, p) \) is a solution of (21), it is straight-forward to check that for any \( t, s \geq 0 \):
\[
f^N_{t} = \varphi_{t,s}^N \# f_s.
\] (25)
Here, \( \varphi(\cdot)\#f \) denotes the image-measure of \( f \) under \( \varphi \), defined by 
\[ \varphi\#f(A) = f(\varphi^{-1}(A)) \]
for any Borel set \( A \subseteq \mathbb{R}^6 \).

Conversely, if \( f_t \) is a fixed-point of \( (\nu_t) \rightarrow \varphi_{\nu_t}\#f_0 \), it is a solution \( (21) \) with initial datum \( f_0 \). In particular, these results give rise to the following important observation:

**Lemma 4.2.** For \( N \in \mathbb{N} \), let \( X = (q_1, ..., q_N; p_1, ..., p_N) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN} \) and 
\[ \mu_0^N[X] = \frac{1}{N} \sum_{i=1}^{N} \delta(q - q_i)\delta(p - p_i). \]

Now, for once, we can let the measure \( \mu_0^N[X] \) evolve according to the regularized Vlasov-Poisson equation \( (21) \), yielding a trajectory \( \mu_t^N \) in \( \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \). On the other hand, we can let the configuration \( X \) evolve according to \( (18) \), yielding a trajectory \( X_t = \Phi_t^N(X) \in \mathbb{R}^d \times \mathbb{R}^d \). These two evolutions actually agree, in the sense that 
\[ \mu_t^N = \varphi_t^\mu \# \mu_t^N[X] = \mu_t^N[X]. \]

**Proof.** By integrating over a test function, we see that 
\[ \varphi_t^\mu \# \mu_t^N[X] = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i(t)}(t)\delta_{p_i(t)}(t), \]
where \((q_i(t), p_i(t))_{i=1,...,N}\) solve
\[ \begin{aligned}
    \dot{q}_i(t) &= p_i(t) \\
    \dot{p}_i(t) &= \chi^N(\cdot - q_i(t))k^*\rho_t^N(q_i) 
\end{aligned} \]
and compare \( \chi^N \) with \( (19) \).

For the (unregularized) Vlasov-Poisson equation, the corresponding vector-field is not Lipschitz. However, if we assume the existence of a solution \( f_t \) with bounded density \( \rho_t \), the mean-field force \( k \ast \rho_t \) does satisfy a Log-Lip bound of the form \( |k \ast \rho_t(x) - k \ast \rho_t(y)| \leq C|x - y| \log^{-}(|x - y|) \), where \( \log^{-}(x) = \max\{0, -\log(x)\} \). This is sufficient to ensure the existence of a characteristic flow \( \psi_{t,s} = (Q_{t,s}, P_{t,s}) \) solving
\[ \begin{aligned}
    \frac{d}{dt}Q_{t,s} &= P_{t,s} \\
    \frac{d}{dt}P_{t,s} &= k \ast \rho(f_t)[Q_{t,s}] \\
    Q(s, s, q_0, p_0) &= q_0 \\
    P(s, s, q_0, p_0) &= p_0
\end{aligned} \] (27)
such that \( f_t = \psi_{t,s}\#f_s \), for all \( t, s \geq 0 \).
4.3 Existence of Solutions

For the regularized Vlasov-Poisson equations (21), all forces are Lipschitz and the solution theory is fairly standard [7, 4]. In the Coulomb case the issue is more subtle. Fortunately, at least in the 3-dimensional case, we can rely on various results establishing global existence and uniqueness of (strong) solutions under reasonable conditions on the initial distribution \( f_0 \) (Pfaffelmoser, 1990 [20], Schaeffer, 1991 [22], Lions and Perthame, 1991 [14], Horst, 1993 [12], Loeper, 2006 [16]).

For the rest of the paper, we shall work under the following assumption:

**Assumption 4.3.** For \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^+_0) \) there exists a \( T^* > 0 \) such that the Vlasov-Poisson system (1–3) has a unique solution \( f(t, x, p) \) for \( 0 \leq t < T^* \) with \( f(0, \cdot, \cdot) = f_0 \). Moreover, as we consider the sequence of solutions of the regularized equations, the charge density remains bounded uniformly in \( N \) and \( t \), i.e. \( \exists C_0 < +\infty \) such that

\[
\|\rho[f^N_t]\|_\infty < C_0, \quad \forall t \in [0, T], T < T^*, \forall N \in \mathbb{N} \cup \{\infty\},
\]

where we introduce the notation \( f^\infty_t := f_t \).

The theorem of Lions and Perthame ensures that this assumption is satisfied for a reasonable class of initial distributions and \( T^* = +\infty \).

**Theorem 4.4** (Lions and Perthame).

Let \( f_0 \geq 0, f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) satisfy

\[
\int |p|^m f_0(q, p) \, dq \, dp,
\]

for all \( m < m_0 \) and some \( m_0 > 3 \).

1) Then, the Vlasov-Poisson system defined by equations (1–3) has a continuous, bounded solution \( f(t, \cdot, \cdot) \in C(\mathbb{R}^+; L^p(\mathbb{R}^d \times \mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)) \) (for \( 1 \leq p < \infty \)) satisfying

\[
\sup_{t \in [0, T]} \int |p|^m f(t, q, p) \, dp < +\infty,
\]

for all \( T < \infty, m < m_0 \).

2) If, in fact, \( m_0 > 6 \) and we assume that \( f_0 \) satisfies

\[
\sup_{t \in [0, T]} \int |p|^m f(t, q, p) \, dp < +\infty,
\]

for all \( R > 0 \) and \( T > 0 \), then there exists \( C > 0 \) such that

\[
\|\rho[f^N_t]\|_\infty < C, \quad \forall t > 0 \forall N \in \mathbb{N} \cup \{\infty\}.
\]
Note that Lions/Perthame formulate eq. (32) only for $f_t$, though they note that their proof actually yields an upper bound on the charge densities $\rho[f^N_t]$ as one considers a sequence of regularized time-evolutions.

5 Statement of the results

Our derivation of the Vlasov-Poisson dynamics, respectively the proof of propagation of molecular chaos, consists of three parts.

1) We perform the mean-field limit for the (regularized) Vlasov-Poisson dynamics under the assumption of a sufficiently fast convergence of the empirical distributions $\mu^N_0$ against the macroscopic profile $f_0$. Concretely, we show that if

$$\lim_{N \to \infty} r_N^{-(1+\frac{d}{p}+\epsilon)} W_p(\mu^N_0, f_0) = 0, \quad (33)$$

for $p \in [2, \infty)$ and some $\epsilon > 0$, then for all $t \leq T < T^*$

$$\lim_{N \to \infty} r_N^{-(1+\frac{d}{p})} W_p(\mu^N_t, f^N_t) \to 0. \quad (34)$$

This means, in particular, the required rate of convergence depends on how fast the electron radius (i.e. the cut-off parameter) decreases as $N \to \infty$.

2) We show that for given $f_0$, the sequence of solutions $f^N_t$ of the regularized Vlasov-Poisson equations converges weakly to the solution $f_t$ of the proper Vlasov-Poisson equation in the limit $r_N \to 0$.

3) We establish that the initial conditions for which (33) is satisfied are in fact typical if they are chosen randomly according to the law $f_0$. To this end, we rely on a recent theorem of Fournier and Guillin [8], providing quantitative estimates for the convergence in Wasserstein distance. These concentration inequalities will finally determine the lower bound on the cut-off parameter $r_N$.

Putting everything together, we find that if the initial configurations are distributed according to $f_0$, the time-evolved empirical density $\mu^N_t[X]$ converges in law to the corresponding solutions $f_t$ of the Vlasov-Poisson equation. By a well-known result of probability theory, this is equivalent to molecular chaos in the following sense: Let the product measure $F^N_0(X) = \prod_{i=1}^N f_0(q, p)$ on $\mathbb{R}^{6N}$ evolve according to the N-particle flow defined by (18). Then, at time $t > 0$, we have $F^N_t(X) = \Phi^N_t \# F^N_0(X) \approx \prod_{i=1}^N f_t(p, q)$, where the approximation is understood in
terms of the convergence if marginals. That is, writing \( x_i = (q_i, p_i) \):

\[
(k) F^N_t(x_1, ..., x_k) := \int F^N_t(X) \, dx_{k+1}...dx_N - \prod_{i=1}^{k} f_i(x_i), \quad N \to \infty.
\]

(E.g. Kac, 1956 [13], Grünbaum, 1971 [10], Sznitman, 1991, Prop.2.2 [23], see Mischler and Hauray, 2014 [18] for recent quantitative results).

We state our precise result in the following theorem.

**Theorem 5.1.**

Let \( f_0 \) be a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) such that the Vlasov-Poisson equation (1) has a unique solution on \([0, T^*), T^* \in \mathbb{R}^+ \cup \{\infty\}\), with \( f(0, \cdot, \cdot) = f_0 \). Let \((r_N)_{N \in \mathbb{N}}\) be a rescaling sequence and \( f^N_t \) the unique strong solution of the regularized Vlasov-Poisson equation (21) with \( f^N(0, q, p) = f_0(q, p) \). Assume that the sequence \((f^N_N)_{N}\) satisfies the uniform bound (28) for the corresponding charge-densities.

Let \( p \geq 2 \) and assume that there exists \( k > 2p \) such that

\[
M_k(f_0) := \int (|q| + |p|)^k f_0(q, p) \, dq \, dp < +\infty.
\]

Suppose that for \( \epsilon > 0 \) it holds that \( r_N \geq N^{-m} \) with

\[
m = \frac{1 - \epsilon}{p (1 + \epsilon) + d}, \quad \frac{p}{2d} \quad \text{if} \quad p \in [2, d), \quad \frac{1}{2} \quad \text{if} \quad p \geq d.
\]

Then there exist constants \( C_1, C_2, C_3 \) such that for all \( T < T^* \) and \( N \) large enough that \( r_N \leq \exp\left(-\frac{2C_1 T}{\epsilon^2}\right) \), we have

\[
\mathbb{P}_0(\sup_{t \in [0, T]} W_p(\mu^N_t[X], f_t) > r_N^{-1-\epsilon}) \leq C_2 (a(N, \epsilon) + b(N, p)),
\]

with

\[
a(N, \epsilon) = \begin{cases} e^{-C_3 N} & , \quad p \neq d, \\ e^{-C_1 \frac{N^{\alpha}}{1+2(N^{1/2})^2}} & , \quad p = d \\ \end{cases}, \quad b(N, p) = \begin{cases} N^{1-p+\frac{p}{2}} & , \quad p \in [2, d) \\ N^{1-p} & , \quad p \geq d \end{cases},
\]

where the probability \( \mathbb{P}_0 \) is understood in terms of the product measure \( F_0[X] = \prod_{i=1}^{N} f_0(x_i) \) on \((\mathbb{R}^d \times \mathbb{R}^d)^N\).

**Remark 5.2.**

1. To minimize the cut-off, the optimal choice is \( p = d \), for which \( m \) can be taken arbitrary close to \( \frac{1}{3d} \). In the physically most relevant case \( d = 3 \), the minimal cut-off width is thus of order \( N^{-\frac{1}{12}+\epsilon} \).

2. If the finite moment condition (35) is replaced by the assumption of a finite exponential moment \( \int e^{\gamma |x|^\alpha} \, d(f_0(x)) \), the rate of convergence becomes exponential, as well.
6 A Gronwall-type Argument

Our mean-field limit is based on the following stability result by Loeper [16, Thm. 2.9], which is proven by methods from the theory of optimal transportation.

**Proposition 6.1** (Loeper).
Let \( k(q) \) the Coulomb kernel and \( \rho_1, \rho_2 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) two (probability) densities.

\[
\| k * \rho_1 - k * \rho_2 \|_{L^2(\mathbb{R}^d)} \leq \left[ \max\{\| \rho_1 \|_\infty, \| \rho_2 \|_\infty \} \right]^{1/2} W_2(\rho_1, \rho_2) \tag{36}
\]

Moreover, we require the following estimates for the mean-field force with Coulomb kernel:

**Lemma 6.2.**
Let \( k \) as before and \( \rho \in L^1(\mathbb{R}^d) \) a (probability) density. It holds that

i) \( \| k * \rho \|_\infty \leq |S^{d-1}| \| \rho \|_\infty + \| \rho \|_1 \).

ii) \( \| \chi_N * k * \rho \|_{Lip} \leq C_L |\log(r_N)| \left( \| \rho \|_1 + \| \rho \|_\infty \right) \)

where \( C_L \) is a constant depending on \( \chi \).

**Proof.**

i) For the first inequality, we compute

\[
\| k * \rho \|_\infty \leq \left\| \int k(y) \rho_1(x-y) \, dy \right\|_\infty + \left\| \int k(y) \rho_2(x-y) \, dy \right\|_\infty \\
\leq \| \rho \|_\infty \int_{|y| < 1} \frac{1}{|y|^{d-1}} |y| \, dy + \| \rho \|_1 = |S^{d-1}| \| \rho \|_\infty + 1
\]

ii) We split the expression as

\[
\| \nabla (\chi_N * k * \rho) \|_\infty \leq \| \nabla (\chi_N * k |_{x \geq r_N^{d+1}} * \rho) \|_\infty + \| \nabla (\chi_N * k |_{x < r_N^{d+1}} * \rho) \|_\infty \\
\leq \| \chi_N \|_1 \| \nabla k |_{x \geq r_N^{d+1}} \|_\infty + \| \nabla \chi_N \|_\infty \| k |_{x < r_N^{d+1}} \|_1 \| \rho \|_1
\]

Now, we have:

\[
| \nabla k |_{x \geq r_N^{d+1}} * \rho (x) | \leq \int_{|y| \geq r_N^{d+1}} \frac{1}{|y|^d} \rho(x-y) \, dy \\
\leq \int_{r_N^{d+1} \leq |y| \leq 1} \frac{1}{|y|^d} \rho(x-y) \, dy + \int_{|y| > 1} \frac{1}{|y|^d} \rho(x-y) \, dy \\
\leq 4C \| \rho \|_\infty \log(r_N) + \| \rho \|_1.
\]
Furthermore:
\[ \| \nabla \chi^N \|_\infty = r_N^{-(d+1)} \| \nabla \chi \|_\infty \]
and
\[ \|k|_{x < r^N_N}\|_1 = \int_{|y| < r^N_N} \frac{1}{|y|^{d-1}} = C r^d_N \]
Putting everything together, the statement follows. \(\square\)

For the macroscopic profile \(f^N_t\) solving the (regularized) Vlasov-Poisson equation, the charge-density \(\rho_t = \rho[f^N_t]\) remains bounded by assumption. The problem is thus to control the charge-density of the microscopic system uniformly in \(N\), i.e. as the electron radius goes to zero and the forces become more singular. The idea here is to show that as long as \(\mu^N_t\) and \(f^N_t\) are close (as probability measures), the boundedness of \(\rho[f^N_t]\) implies the boundedness of \(\rho[\mu^N_t]\). A simple such estimate can be obtained as follows (c.f. [3, Prop. 2.1]).

**Lemma 6.3.** There exists a constant \(C\) depending on \(\chi\) such that
\[ \|\dot{\rho}[\mu^N_t]\|_\infty = \|\dot{\rho}[f^N_t]\|_\infty \leq \|\rho[f_t]\|_\infty + C r_N^{-(d+1)} W_1(\mu^N_t, f^N_t). \]  

**Proof.** Observing that \(\rho[\mu^N_t] = \chi^N * \mu^N_t(x)\) and \(\rho[f^N_t] = \chi^N * f^N_t(x)\) for all \(t \in [0, T]\), we find for all \(x \in \mathbb{R}^d\):
\[ |(\rho[\mu^N_t] - \rho[f^N_t])(x)| = \|\chi^N * (\mu^N_t - f^N_t)(x)\| \leq \|\chi^N\|_{Lip} W_1(\mu^N_t, f^N_t). \]
Since \(\|\chi^N\|_{Lip} \leq r_N^{-(d+1)} \|\nabla \chi(x)\|_\infty\) and \(\|\rho[f_t]\|_\infty = \|\chi * \rho[f]\|_\infty \leq \|\chi\|_1 \|\rho[f]\|_\infty\), the lemma follows. \(\square\)

In view of the general Kantorovich-Rubinstein duality, we can generalize this to Wasserstein distances of higher order.

**Lemma 6.4.** There exists a constant \(C\) depending only on the dimension \(d\) such that
\[ \|\dot{\rho}[\mu^N_t]\|_\infty \leq C \|\rho[f^N_t]\|_\infty + r_N^{-(p+d)} W_p(\mu^N_t, f^N_t). \]  

**Proof.** Observe that \(\dot{\rho}[\mu^N_t] = \rho[\dot{\mu}^N_t] = \chi^N * \mu^N_t(x)\) for all \(t \in [0, T]\). For any integrable function \(\Phi\), we consider its \(d\)-conjugate \(\Phi^c\)
\[ \Phi^c(y) := \sup_x (\Phi(x) - |x - y|^p). \]
Note that \(\Phi^c(y) \geq \Phi(y)\) and \(\Phi(x) - \Phi^c(y) \leq |x - y|^p, \forall x, y \in \mathbb{R}^d\).
Now, we write
\[
\rho(\mu_t^N)(x) = r_N^{-(d+p)} \int r_N^{d+p} \chi^N(x-y) \, d\mu(y,\nu) - \int (r_N^{d+p} \chi^N(x-\cdot))^c(z) \, df(z,\xi) \\
+ \int (r_N^{d+p} \chi^N(x-\cdot))^c(z) \, df(z,\xi)
\]

By the Kantorovich duality theorem, we have
\[
\int r_N^{d+p} \chi^N(x-y) \, d\mu(y,\nu) - \int (r_N^{d+p} \chi^N(x-\cdot))^c(z) \, df(z,\xi) < W_p^p(\mu_t^N, f_t^N).
\]

It remains to estimate
\[
\int (r_N^{d+p} \chi^N(x-\cdot))^c(z) \, df(z,\xi) = \int (r_N^{d+p} \chi^N(x-\cdot))^c \rho^f(z) \, dz.
\]

Recalling that \(\|\chi^N\|_\infty = r_N^{-d}\) and \(\text{supp} \{\chi^N(x)\} \subseteq B(0, r_N^d)\), we find
\[
(r_N^{d+p} \chi^N(x-\cdot))^c(z) = \sup_y \{r_N^{d+p} \chi^N(x-y) - |y-z|^p\} \leq r_N^{d+p} \|\chi^N\|_\infty = r_N^p.
\]

Moreover, we observe that \(\text{supp} \{(r_N^{d+p} \chi^N(x-\cdot))^c(z)\} \subseteq B(x; r_N + r_N^d)\), since \(|z-x| > r_N + r_N^d|\) implies that \(\chi^N(x-y) = 0\), unless \(|y-z| \geq r_N\). But then: \(r_N^{d+p} \chi^N(x-y) - |y-z|^p\leq r_N^{d+p} r_N^{-d} - r_N^p = 0\).

Hence,
\[
\int (r_N^{d+p} \chi^N(x-\cdot))^c(z) \, df(z,\xi) = \int (r_N^{d+p} \chi^N(x-\cdot))^c \rho^f(z) \, dz \\
\leq \|\rho[f_t^N]\|_\infty r_N^p |B(x; r_N + r_N^d)| \leq \frac{4}{3} \pi r_N^p (r_N + r_N^d)^d \|\rho[f_t^N]\|_\infty \\
\leq 2^{d^2} |B^d(1)| \|\rho[f_t^N]\|_\infty \ r_N^{d+p}.
\]

In total:
\[
\|\rho(\mu_t^N)\|_\infty \leq r_N^{-(p+d)} W_p^p(\mu_t^N, f_t^N) + 2^{d^2} |B^d(1)| \|\rho[f_t^N]\|_\infty.
\]

Hence, the desired bound holds.

As we want to establish a Gronwall estimate for the “distance” between empirical density and Vlasov density, we aim for a bound of the following type:
\[
\text{dist}(\mu_{t+\Delta t}^N, f_{t+\Delta t}^N) - \text{dist}(\mu_t^N, f_t^N) \propto \text{dist}(\mu_t^N, f_t^N) \Delta t + o(\Delta t).
\]

Hence, the choice of an appropriate metric, giving precise meaning to \(\text{dist}(\mu_t^N, f_t^N)\), is a balancing act. While a stronger metric is, in general,
more difficult to control, it also yields stronger bounds as it appears on the right hand side of the Gronwall inequality.

Now, if we compare the time-evolution of a particle evolving according to the mean-field dynamics with a particle evolving according to the “true” microscopic dynamics, the growth of their spatial distance is trivially bounded by the difference of their respective momenta. The only challenge lies in controlling fluctuations in the force, i.e., the growth of their distance in moment space. Therefore, the idea, first employed in [14], is the following: to be a little more rigid on deviations in the spatial coordinates and use this to obtain a better control on the particle momenta, respectively the forces.

**Definition 6.5.** Let \((r_N)_{N \in \mathbb{N}}\) be a rescaling sequence. On \(\mathbb{R}^d \times \mathbb{R}^d\) we introduce the following (\(N\)-dependent) metric:

\[
d_N((q_1, p_1), (q_2, p_2)) := \sqrt{\log(r_N)} |q_1 - q_2| + |p_1 - p_2|.
\]

(39)

Now let \(W^p_N(\cdot, \cdot)\) be the \(p\)'th Wasserstein distance on \(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\) with respect to \(d_N\), i.e.:

\[
W^p_N(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} d_N(x, y)^p \, d\pi(x, y) \right)^{1/p}.
\]

(40)

Note that \(W_p(\mu, \nu) \leq W^p_N(\mu, \nu) \leq \sqrt{\log(r_N)} W_p(\mu, \nu), \forall \mu, \nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\). Finally, we define

\[
W^*_p(\mu, \nu) := \min \{ r_N^{-(1+\frac{d}{p})} W^p_N(\mu, \nu), 1 \}
\]

(41)

for all \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\) and \(N \in \mathbb{N}\).

Obviously, convergence with respect to \(W^*_p\) is much stronger than convergence with respect to \(W_p\). Concretely, we have for any sequence \((\nu_N)_{N \in \mathbb{N}}\) and any \(\nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\):

\[
W^*_p(\mu_N, \mu) \to 0 \Rightarrow W_p(\mu_N, \mu) = o(r_N^{1+\frac{d}{p}})
\]

We come to the central part of our argument:

**Proposition 6.6.**

Let \(f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\) satisfy the assumptions of Thm. [4.3]. Let \((\mu_N^0)_{N \in \mathbb{N}}\) be a sequence of empirical distributions s.t. \(\mu_0^N \to f_0\) and assume that, in fact, this convergence is fast enough so that \(W^*_p(\mu_0^N, f_0) = o(r_N^{1+\frac{d}{p}})\) for \(p \in [2, \infty)\) and some \(\epsilon > 0\). Then \(\forall 0 \leq t < T^*:\)

\[
W^*_p(\mu_t^N, f_t^N) \leq W^*_p(\mu_0^N, f_0) e^{C_1(\sqrt{\log(r_N)})+1} \to 0, N \to \infty
\]

(42)

where \(C_1\) is a constant depending on \(\sup_{N, s \leq t} \|\rho[f_s^N]\|_\infty\).
Proof. Let \( N \in \mathbb{N} \) and \( \pi_0 \in \Pi(\mu_0^N, f_0) \). Let \( \varphi^\mu_t = (Q^\mu_t, P^\mu_t) \) and \( \varphi^f_t = (Q^f_t, P^f_t) \) be the flow induced by the characteristic equation \([24]\) for \( \mu^N_t \) and \( f^N_t \), respectively. For any \( t \in [0, T], T < T^* \), define the (again \( N \)-dependent) measure \( \pi_t \) on \( \mathbb{R}^{6N} \times \mathbb{R}^{6N} \) by \( \pi_t = (\varphi^\mu_t, \varphi^f_t) \# \pi_0 \). Then \( \pi_t \in \Pi(f^N_t, f_t), \forall t \in [0, T] \). Set

\[
D_p(t) := \left[ \int_{\mathbb{R}^{6N} \times \mathbb{R}^{6N}} d^N(x, y)^p \, d\pi_t(x, y) \right]^{1/p}
\]

\[
= \left[ \int_{\mathbb{R}^{6N} \times \mathbb{R}^{6N}} (\sqrt{\log(r_N)} |x^1 - y^1| + |x^2 - y^2|)^p \, d\pi_t(x, y) \right]^{1/p}
\]

\[
= \left[ \int_{\mathbb{R}^{6N} \times \mathbb{R}^{6N}} (\sqrt{\log(r_N)} |Q^\mu(t, x) - Q^f(t, y)| + |P^\mu(t, x) - P^f(t, y)|)^p \, d\pi_0(x, y) \right]^{1/p}
\]

Note that \( W_p^N(\mu^N_t, f^N_t) < D_p(t) \) for any \( \pi_0 \in \Pi(f_0, f_0) \).

Now we set:

\[
D_p^*(t) := \min \left\{ r_N^{-(1+\frac{d}{p})} D(t), 1 \right\}.
\]

(43)

Obviously, \( \frac{d}{dt} D_p^*(t) \leq 0 \) whenever \( D(t) \geq r_N^{1+\frac{d}{p}} \). For \( D(t) < r_N^{1+\frac{d}{p}} \), we compute:

\[
\frac{d}{dt} D_p^*(t) \leq p \int \left( \sqrt{\log(r_N)} |Q^\mu(t, x) - Q^f(t, y)| + |P^\mu(t, x) - P^f(t, y)| \right)^{p-1} \left( \sqrt{\log(r_N)} |Q^\mu(t, x) - P^f(t, y)| + |Q^f(t, x) - P^\mu(t, y)| \right) \, d\pi_0(x, y)
\]

The interesting term to control is the interaction term

\[
\int |\hat{k} \ast \hat{\rho}^\mu_t(Q^\mu_t(x)) - \hat{k} \ast \hat{\rho}^f_t(Q^f_t(y))| \, d\pi_0(x, y)
\]

\[
\leq \int |\hat{k} \ast \hat{\rho}^\mu_t(Q^\mu_t(x)) - \hat{k} \ast \hat{\rho}^\mu_t(Q^f_t(y))| \, d\pi_0(x, y) + \int |\hat{k} \ast \hat{\rho}^f_t(Q^f_t(y)) - \hat{k} \ast \hat{\rho}^f_t(Q^\mu_t(x))| \, d\pi_0(x, y)
\]

(44)

(45)

We begin with \([41]\) and find:

\[
\int \left| (\hat{k} \ast \hat{\rho}^\mu_t)(Q^\mu_t(x) - Q^f_t(y)) \right| \, d\pi_0(x, y)
\]

\[
\leq C_L |\log(r_N)| (1 + \|\hat{\rho}^\mu_t\|_\infty) \int |Q^\mu_t(x) - Q^f_t(y)| \, d\pi_0(x, y).
\]

(46)
Now, from Lemma 6.4 we know that \(\|\mu^p\|_\infty\) is bounded by
\[
C\|\mu[f_i^N]\|_\infty + r_N^{-(3+p)} W_p^p(\mu^N_i, f_i^N)
\]
\[
\leq C\|\mu[f_i^N]\|_\infty + r_N^{-(3+p)} D_p(t)
\]
\[
\leq C \sup_{N \in \mathbb{N}} \|\mu[f_i^N]\|_\infty + 1 \leq C C_0 + 1 =: C_p,
\]
as long as \(D_p(t) \leq r_N^{1+\frac{4}{p}}\), i.e. \(D_p(t) \leq 1\). Note that this bound holds independent of \(N\). For the second term \([43]\), we estimate:
\[
\int |\hat{k} \ast (\hat{\rho}^p_t - \hat{\rho}^\mu_t)(Q(\varphi_t^f))| \, d\pi_0(x,y)
\]
\[
\leq \left[ \int |\hat{k} \ast (\hat{\rho}^p_t - \hat{\rho}^\mu_t)(Q(\varphi_t^f))|^2 \, d\pi_0(x,y) \right]^{1/2}
\]
\[
\leq \left( \left( \int (\hat{k} \ast \rho_t^p - \hat{k} \ast \rho_t^\mu)^2 f(t,y) \, dy \right) \right)^{1/2} = \left( \int (\hat{k} \ast \rho_t^p - \hat{k} \ast \rho_t^\mu)^2 \rho_t(q) \, d\rho(q) \right)^{1/2}
\]
\[
\leq \|\rho_t^p\|_\infty^{1/2} \|k \ast (\hat{\rho}_t^p - \hat{\rho}_t^\mu)\|_2 \leq C_p^{1/2} \|k \ast (\hat{\rho}_t^p - \hat{\rho}_t^\mu)\|_2
\]
And according to Proposition [6.1]
\[
\|k \ast (\hat{\rho}_t^p - \hat{\rho}_t^\mu)\|_2 \leq \left[ \max\{\|\hat{\rho}_t^\mu\|_\infty, \|\hat{\rho}_t^\mu\|_\infty\} \right]^{1/2} W_2(\hat{\rho}_t^p, \hat{\rho}_t^\mu) \leq C_p^{1/2} D_p(t).
\]
Here, we used that for \(p \geq 2\):
\[
W_2(\hat{\rho}_t^p, \hat{\rho}_t^\mu) \leq W_2(\rho_t^p, \rho_t^\mu) \leq W_2(\mu_t^N, f_t^N) \leq W_p(\mu_t^N, f_t^N) \leq D_p(t).
\]
Putting everything together and setting \(C_1 := C_p C_L > C_p\), we have
\[
\frac{d}{dt} D_p(t) \leq p C_1 (\sqrt{\log(r_N)}) + 1) D_p(t)
\]
or, after deviating by \(D_p^{-1}\) and multiplying both sides by \(r_N^{-(1+\frac{2}{p})}\),
\[
\frac{d}{dt} D_p(t) \leq C_1 (\sqrt{\log(r_N)}) + 1) D_p(t),
\]
By an application of Gronwall’s Lemma, we conclude that:
\[
D_p^*(t) \leq D_p^*(0) e^{t \sqrt{\log(r_N)} + 1}
\]
Finally, taking on the right hand side the infimum over all \(\pi_0 \in \Pi(\mu_0^N, f_0)\), \(D_p^*(0)\) becomes \(W_p^*(0)\) and we get for all \(t \in T\):
\[
W_p^*(\mu_t^N, f_t^N) \leq W_p^*(\mu_0^N, f_0) e^{t \sqrt{\log(r_N)} + 1}
\]
If there exists an $\varepsilon > 0$ such that $\lim_{N \to \infty} \frac{W_p(\mu_N^N, f_0)}{r_N^{-d/p + \varepsilon}} = 0$, the right hand side converges to 0.

\[ \square \]

**Proposition 6.7.** Let $f_0$ satisfy the assumptions of Theorem 5.7. Let $f_t^N$ and $f_t$ be the solution of the regularized, respectively the proper Vlasov-Poisson equation with initial $f_0$. Then, for any $p \in [1, \infty)$:

\[ W_p(f_t^N, f_t) \leq 2C_0 r_N e^{|C'_1| \sqrt{|\log(r_N)|} + 1}, \tag{50} \]

where $C'_1$ is another constant depending on $\sup_{N,t \leq T} \|\rho[f_t^N]\|_\infty$.

**Proof.** Let $\rho_t^N := \rho[f_t^N]$ and $\rho_t^\infty := [f_t^N]$ be the charge density induced by $f_t^N$ and $f_t$, respectively. Let $\varphi_t^N = (Q_t^N, P_t^N)$ the characteristic flow of $f_t^N$ and $\psi_t = (Q_t, P_t)$ the characteristic flow of $f_t$ such that $f_t^N = \varphi_t^N \# f_0 = \psi_t \# f_0$. We consider $\pi_0(x, y) := f_0(x)\delta(x - y) \in \Pi(f_0, f_0)$, which is already the minimizer yielding $W_p^N(f_0, f_0) = 0$ and set $\pi_t = (\varphi_t^N, \psi_t) \# \pi_0 \in \Pi(f_t^N, f_t)$.

As above, we define for $p \geq 2$:

\[ D_p(t) := \left[ \int_{\mathbb{R}^6 \times \mathbb{R}^6} \left( \sqrt{|\log(r_N)|} \right)^p \left( |x^1 - y^1| + |x^2 - y^2| \right)^p \, d\pi_t(x, y) \right]^{1/p} \tag{51} \]

and compute

\[ \frac{d}{dt} D_p(t) \leq p \int \left( \sqrt{|\log(r_N)|} \right)^p \left( |Q^N(t, x) - Q(t, y)|^p + |P^N(t, x) - P(t, y)|^p \right)^{p-1} \]

\[ \left( \sqrt{|\log(r_N)|} \right)^p \left( |P^N(t, x) - P(t, y)|^p + |k * \rho_t^N(Q^N(x)) - k * \rho_t^\infty(Q^\infty(x))| \right) \, d\pi_0(x, y) \]

The proof proceeds analogous to Prop. 6.6, simplified by the fact that the charge densities remain bounded by assumption. The only noteworthy difference is in eq. (48), where we use Lemma 3.2 to conclude:

\[ W_2(\rho_t^N, \rho_t) \leq W_2(\rho_t^N, \rho_t) + 2r_N \leq W_p(f_t^N, f_t) + 2r_N \leq D_p(t) + 2r_N \tag{52} \]

In total, we find with $C'_1 := C_0 C_L$:

\[ \frac{d}{dt} D_p(t) \leq p C_0(C_L \sqrt{|\log(r_N)|} + 1) D_p^p(t) \]

\[ + 2p C_0 r_N \int \left( \sqrt{|\log(r_N)|} \right)^p \left( |Q^N(t, x) - Q^\infty(t, y)|^p + |P^N(t, x) - P^\infty(t, y)|^p \right)^{p-1} \]

\[ \leq p C'_1 \left( \sqrt{|\log(r_N)|} + 1 \right) D_p^p(t) + 2p C_0 r_N (D_p(t))^{\frac{p-1}{p}} \]

20
or
\[ D_p(t + \Delta t) - D_p(t) \leq \left( C_1' \sqrt{|\log r_N|} + 1 \right) D_p(t) + 2 C_0 r_N \Delta t + o(\Delta t). \]

Using Gronwall’s inequality and the fact that \( D_p(0) = 0 \), we have
\[ W_p(f_i^N, f_t) \leq D_p(t) \leq 2 C_0 r_N e^{C_1' \sqrt{|\log r_N|} + 1}, \]
from which the desired statement follows. The case \( p = 1 \) is included since \( W_1(\cdot, \cdot) \leq W_2(\cdot, \cdot) \).

\[ \square \]

7 Typicality

In the previous section, we performed the mean-field limit under the assumption of a sufficiently fast convergence of the initial distribution. How string this result is, now depends on two questions: 1) how fast can we let the electron radius (i.e. the cut-off parameter) \( r_N \) go to zero? 2) How restrictive is the assumption \( W_p(\mu_0^N, f_0) = o(r_N^{1+\frac{1}{p}+\epsilon}) \)?

If we found that only very special sequences of initial distributions \( \mu_0^N \) achieve the desired rate of convergence, our result would not be very satisfying from a physical point of view. If, on the other hand, we can show that the admissible initial configurations are \emph{typical}, it would mean that, on the contrary, the mean-field approximation fails only for very special (in this sense “conspiratorial”) initial conditions, leading to clustering and/or strong correlations among the particles.

It is a classical result - known as the empirical law of large numbers or Varadarajan’s theorem - that if \( x_1, \ldots, x_N \) are i.i.d. with law \( f \), their empirical density \( \mu^N[X] = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \) goes to \( f \) in probability.

Establishing quantitative bounds (“concentration estimates”) on large deviations is, however, an active field of research and various authors have achieved various results under different regularity assumptions on \( f \). (E.g. \[3\], \[2\], \[6\]). We cite here a recent, particularly strong result due to Fournier and Guillin \[8\], on which our final conclusion, formulated in Theorem 5.1, is based.

**Theorem 7.1** (Fournier and Guillin).

Let \( f \) be a probability measure on \( \mathbb{R}^k \) such that \( \exists q > 2p \):
\[ M_q(f) := \int_{\mathbb{R}^k} |x|^q df(x) < +\infty \]
Let \((X_i)_{i=1,...,N}\) be a sample of independent variables, distributed according to the law \(f\) and \(\mu^N[X] := \sum_{i=1}^{N} \delta_{X_i}\). Then, for any \(s > 0\) there exist constants \(c, C\) depending only on \(q, M_q(f)\), \(s\) such that for all \(N \geq 1\) and \(\xi > 0\):

\[
\mathbb{P} \left( W_p^N(\mu^N, f) > \xi \right) \leq C N (N \xi)^{-\frac{q}{p}} + C 1_{\xi \leq 1} a(N, \xi)
\]

with

\[
a(N, \xi) := \begin{cases} 
\exp(-cN\xi^2) & \text{if } p > k/2 \\
\exp(-cN(\frac{\xi}{\ln(2+1/\xi)})^2) & \text{if } p = k/2 \\
\exp(-cN\xi^{k/p}) & \text{if } p \in [1, k/2) 
\end{cases}
\]

(53)

7.1 Proof of the Theorem

Proof of Theorem 5.1.

Let \(p \geq 2\) and \(\epsilon > 0\). Let \(A \subseteq \mathbb{R}^6\) be the \((N\text{-dependent})\) set defined by

\[
X \in A \iff W_p(\mu^N_0[X], f_0) > r_N^{1+\frac{d}{p}+\epsilon}.
\]

(54)

Applying Theorem 7.1 in 2\(d\)-dimensions with \(\xi = r_N^{p(1+\frac{d}{p}+\epsilon)}\) and \(r_N = N^{-m}\) we find with regard to (53) that \(\lim_{N \to \infty} \mathbb{P}_0(A) = 0\) if there exists \(\gamma \in (0, 1)\) with

\[
m = \gamma \frac{p}{p + d + p \epsilon} \begin{cases} 
\frac{p}{2d} & \text{if } p \in [2, 3) \\
\frac{1}{2} & \text{if } p \geq 3 
\end{cases}
\]

in which case we get:

\[
\mathbb{P}_0(A) \leq C \begin{cases} 
\exp(-cN^{1-\gamma}) & \text{if } p \neq d \\
\exp(-cN^{1-\gamma}(2+N/2)) & \text{if } p = d 
\end{cases} + CN \begin{cases} 
N^{-\frac{d}{p}+\frac{d}{p}} & \text{if } p \in [2, d) \\
N^{-\frac{d}{p}} & \text{if } p \geq d 
\end{cases}.
\]

(55)

For simplicity, we set \(\gamma = 1 - \epsilon\) in the formulation of Theorem 5.1

Now for the typical initial conditions, i.e. \(X \in A^c\), we have according to Proposition 6.6 for all \(t \leq T\):

\[
W_p^* (\mu^N_0, f^N_0) \leq W_p^* (\mu^N_0, f_0) e^{t C_1(\sqrt{\log(r_N)}+1)} \leq \sqrt{\ln(r_N)} r_N^{-(1+\frac{d}{p})} W_p(\mu^N_0, f_0) e^{t C_1(\sqrt{\log(r_N)}+1)} \leq \sqrt{\ln(r_N)} r_N^* e^{T C_1(\sqrt{\log(r_N)}+1)}
\]

(55)
Now, note that $e^{\sqrt{\ln r_N}} = (e^{\ln r_N})^{\frac{1}{\sqrt{\ln r_N}}} = r_N^{\frac{1}{\sqrt{\ln r_N}}}$. Hence, there exists $N_0 \in \mathbb{N}$ such that (55) < 1, $\forall N \geq N_0$. More precisely, it suffices to choose $N_0$ so that $r_{N_0} < e^{-\left(\frac{2C_1 T}{\epsilon}\right)^2}$. But now,

$$W^*_p(\mu^N_t, f^N_t) < 1 \Rightarrow W^*_p(\mu^N_t, f^N_t) < r_N^{1+\frac{d}{p}} W_p(\mu^N_t, f^N_t) < r_N^{1+\frac{d}{p}}. \quad (56)$$

We conclude the proof of part 1) by using Proposition 6.7 (and the fact that $C_1 > C_2 > 2C_0$) to deduce:

$$W_p(\mu^N_t[X], f_t) \leq W_p(\mu^N_t, f^N_t) + W_p(f^N_t, f_t) \leq r_N^{1+\frac{d}{p}} + \frac{1}{2} r_N^{1-\epsilon} \leq r_N^{1-\epsilon},$$

for all $X \in \mathcal{A}^c$ and $t \in [0, T]$.

\[\square\]
References

[1] N. Boers and P. Pickl. On mean field limits for dynamical systems. Preprint: arXiv:1307.2099, 2015.

[2] E. Boissard. Simple bounds for the convergence of empirical and occupation measures in 1-Wasserstein distance. *Electronic Journal of Probability*, 16:2296–2333, 2011.

[3] F. Bolley, A. Guillin, and C. Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probability Theory and Related Fields*, 137(3):541–593, 2007.

[4] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles. *Comm. Math. Phys.*, 56(2):101–113, 1977.

[5] H. Dehmelt. A single atomic particle forever floating at rest in free space: New value for electron radius. *Physica Scripta*, 1988(T22):102–110, 1988.

[6] S. Dereich, M. Scheutzow, and R. Schottstedt. Constructive quantization: approximation by empirical measures. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 49, pages 1183–1203. Institut Henri Poincaré, 2013.

[7] R. L. Dobrushin. Vlasov equations. *Functional Analysis and Its Applications*, 13(2):115–123, 1979.

[8] N. Fournier and A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, pages 1–32, 2014.

[9] F. Golse. The mean-field limit for a regularized Vlasov-Maxwell dynamics. *Commun. Math. Phys.*, 310(3):789–816, 2012.

[10] F.A. Grünbaum. Propagation of chaos for the Boltzmann equation. *Arch. Rational Mech. Anal.*, 42:323–345, 1971.

[11] M. Hauray and P.-E. Jabin. Particles approximations of Vlasov equations with singular forces: Propagation of chaos. To appear in *Ann. Sci. Éc. Norm. Supér*, 2013.

[12] E. Horst. On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Math. Methods Appl. Sci.*, 16:75–85, 1993.

[13] M. Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-1955*, volume vol III, pages 171–197. University of California Press, 1956.

[14] D. Lazarovici and P. Pickl. A mean-field limit for the Vlasov-Poisson system. Preprint: arXiv:1502.04608, 2015.
[15] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Inventiones Mathematicae*, 105:415–430, 1991.

[16] G. Loeper. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *J. Math. Pures Appl.*, 86:68–79, 2006.

[17] S. Lyle. *Self-Force and Inertia: Old Light on New Ideas*, volume 796 of *Lecture Notes in Physics*. Springer, Berlin, 2010.

[18] S. Mischler and M. Hauray. On Kac’s chaos and related problems. *Journal of Functional Analysis*, 266(10):6055–6157, 2014.

[19] H. Neunzert. An introduction to the nonlinear Boltzmann-Vlasov equation. In C. Cercignani, editor, *Kinetic Theories and the Boltzmann Equation*, volume 1048 of *Lecture Notes in Mathematics*, pages 60–110. Springer, Berlin, 1984.

[20] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2):281–303, 1990.

[21] G. Rein. Global weak solutions of the relativistic Vlasov-Maxwell system revisited. *Comm. in Math. Sci.*, 2:145–158, 2004.

[22] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[23] A.-S. Sznitman. Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX – 1989*, volume 1464 of *Lecture Notes in Mathematics*, pages 165–251. Springer, Berlin, 1991.

[24] C. Villani. *Optimal Transport Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 2009.