Étale Homotopy Obstructions of Arithmetic Spheres

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March 2017

Abstract

Let $K$ be a field of characteristic $\neq 2$ and let $X$ be the affine variety over $K$ defined by the equation

$$X: a_0x_0^2 + \cdots + a_nx_n^2 = 1$$

where $n \geq 0$ and $a_i \in K$. In this paper we compute the lowest mod 2 étale homological obstruction class to the existence of a $K$-rational point on $X$, and show that it is the cup product of the form

$$o_{n+1} = [a_0] \cup \cdots \cup [a_n].$$

Our computation is an Étale-homotopy analogue of the topological fact that Stiefel-Whitney classes are the homological obstructions to find a section to the unit sphere bundle of a real vector bundle.

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1 Introduction

The arithmetic of quadratic forms is a well-established subject. In 1987 Hilbert introduced the quadratic symbol over a local field $K$, defined by

$$(a, b) = \begin{cases} 1 & \text{if } ax^2 + by^2 = 1 \text{ is solvable over } K \\ -1 & \text{else} \end{cases}$$

and its generalization to number fields using the various completions at the places of the field.

Later on, those symbols recognized as a special case of a general Hasse-Witt invariant $HW_2(B) \in Br(K)$, attached to a quadratic form $B$ over a field $K$ of characteristic $\neq 2$. Those classes are arithmetic analogues of the Stiefel-Whitney classes in algebraic topology, and form part of a sequence $HW_k(B) \in H^k_{\text{ét}}(K, \mu_2)$ of classes, satisfying the Whitney product formula analogous to the product formula for Stiefel-Whitney classes from classical topology.

Applying the product formula to the top Hasse-Witt invariant, one immediately see that $HW_{\text{rank}(B)}(B)$ is an obstruction to a solution of the equation $B(v, v) = 1$ over $K$, just like the original Hilbert symbol is defined as an obstruction to solution of such equation for $\text{rank}(B) = 2$.

In [1] (and expended in [2]), Y.Harpaz and the third author introduced a general obstruction theory for rational points on algebraic varieties based on the notion of relative étale homotopy type.

In this paper, we shall link the classical theory of Hasse-Witt classes and the quadratic symbol with the general étale obstruction theory. Specifically, for a quadratic form $B$, we show that the mod-2 homological obstruction for a solution of the equation $B(v, v) = 1$ coincides with the top Hasse-Witt class of $B$. 
1.1 Topological Motivation and Outline of the Proof

We present below the outline of the proof and the topological origin of the argument.

1.1.1 Obstructions for Unit Sections

Classically, given an \( n \)-dimensional topological vector bundle \( p: E \to Y \), there is an associated cohomology class \( SW_n(p) \in H^n(Y; \mathbb{Z}/2) \), the Stiefel-Whitney class, which serves as an obstruction to the existence of a global non-zero section for \( p \). Vector bundles are classified by maps to the classifying space

\[
Gr(n, \infty) \cong BGL_n \cong BO_n,
\]

by pulling back the universal vector bundle \( \tilde{p}: \tilde{E} \to BO_n \). The Stiefel-Whitney class of the vector bundle associated with \( f: Y \to BO_n \) can be computed as the pullback

\[
SW_n(f^*(\tilde{p})) = f^*(SW_n(\tilde{p})).
\]

In the arithmetic setting, we cannot use the topological space \( BO_n \) so we consider the analogous stack. Specifically, given a field \( K \) of characteristic different then 2 we denote by \( BO_n,K \) the stack classifying algebraic vector bundles over \( K \)-schemes equipped with a non-degenerate quadratic form. We thus have a universal sphere bundle

\[
\tilde{S} \to \tilde{E} \to BO_n,K
\]

which is the variety of all vectors of norm 1 with respect to the universal quadratic form. Analogously to the topological case, the étale \( \mathbb{Z}/2 \)-cohomology ring of \( BO_n,K \) is a polynomial ring freely generated by classes \( HW_1, \ldots, HW_n \) of degrees \( \deg(HW_i) = i \) over \( H_{\text{ét}}^*(K, \mathbb{Z}/2) \).

In this paper we prove that given a quadratic bundle \( p: E \to Y \) classified by \( f: Y \to BO_n,K \), the class \( SW(p) = f^*(HW_n) \) is an obstruction to the existence of a section for the sphere bundle \( f^*(\tilde{S}) \to Y \). Furthermore, this is precisely the obstruction class as defined in [2].

1.1.2 The Computation of the Obstruction for the Sphere

We study the variety \( X: \sum_{i=0}^n a_i x_i^2 = 1 \), which is properly thought of as the unit sphere in an \( n+1 \) dimensional linear space equipped with the quadratic form defined by \( \sum_{i=0}^n a_i x_i^2 \). Hence, it corresponds to a (rational) point on \( BO_{n+1} \) so that there is a pullback diagram

\[
\begin{array}{ccc}
X & \to & \tilde{S}_{n+1} \\
p & & \downarrow \tilde{p} \\
\text{Spec } K & \to & BO_{n+1}.
\end{array}
\]
The sphere $X$ is the unit sphere in a quadratic space which can be decomposed as a direct product of 1 dimensional quadratic spaces of the form $a_i x^2$. This means we have a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \tilde{S}^{n+1} \\
p \downarrow & & \downarrow \tilde{p} \\
\text{Spec } K & \times_{f_*} & BO^{n+1} \\
\end{array}
$$

The obstruction class $SW(p) = f^*(HW_n)$ can now be computed using Whitney’s product formula

$$SW(p) = f_0^*(HW_1) \cup \cdots \cup f_n^*(HW_1).$$

In the 0-dimensional case, $X: ax^2 = 1$ and the obstruction class is

$$[a] \in H^1(K; \mathbb{Z}/2) \cong K^\times/(K^\times)^2$$

In general, we have

$$SW(p) = [a_0] \cup \cdots \cup [a_n].$$

### 1.2 Organization of the Paper

In Section 2 we introduce relative étale homotopy type in the case of a morphism of $\infty$-stacks. We also prove compatibilities of the relative homotopy and homology types of such morphisms, and derive basic properties of those, such as smooth base-change and behavior with respect to colimits. In Section 3 obstruction theory is introduced in the context of $\infty$-topoi. These two sections serve as the theoretical foundation needed for the computation, and fix the required notations.

Quadratic bundles and their classifying stack are introduced and examined in Section 4, and the relative étale topological type is computed for the universal quadratic bundle over these classifying stacks. Finally, Section 5 contains the computation of the obstruction class for arithmetic spheres.

### 1.3 Acknowledgements

Shachar Carmeli is supported by the Adams Fellowship Program of the Israel Academy of Sciences and Humanities. Tomer Schlank is supported by the Alon Fellowship and ISF1588/18.

### 2 Étale Homotopy Type

In this section we shall recall the definition of the relative homotopy type as defined in [2]. We use the formalism of $\infty$-stacks and $\infty$-topoi and the machinery developed there.
We work with the $\infty$-category $\sTop_{\infty}$ of $\infty$-topoi, as in [3, Section 6].

Let $K$ field of characteristic $\neq 2$, fixed throughout the paper and let $Sch_{/K}$ denote the essentially small site of schemes of finite type over $K$ endowed with the étale topology. For a site $C$ we denote by $Shv_{\infty}(C)$ the $\infty$-topos of sheaves of spaces over $C$.

**Definition 2.0.1.** An $\infty$-stack over $K$ is an object $X \in Shv_{\infty}(Sch_{/K})$.

By [4, Lemma 2.11], there is a colimit preserving functor

$$Shv_{\infty}(\bullet_{\text{et}}): Shv_{\infty}(Sch_{/K}) \to \sTop_{\infty}.$$ 

Hence, we can functorially assign to every $\infty$-stack $X$ an $\infty$-topos $Shv_{\infty}(\mathcal{X}_{\text{et}})$, which we think of as étale sheaves over $X$. For an $\infty$-category $C$, the category of pro-objects is defined in [3, 7.1.6.1]. It is the category of finite limits preserving functors $C \to S$ into the $\infty$-category of spaces, considered as a full subcategory of $Fun(C, S)$. The main property of the pro-category we use is the following

**Proposition 2.0.2.** [4, Proposition 2.3] Let $F: C \to D$ be a functor preserving finite limits. Then the induced functor $Pro(F): Pro(C) \to Pro(D)$ admits a left adjoint.

From now on we shall abuse notation by denoting $Pro(F)$ just by $F$.

Let $f: T \to U$ be a geometric morphism of $\infty$-topoi. By definition, the morphism $f^*: U \to T$ preserves finite limits, hence $Pro(f^*)$ admits a left adjoint which we denote $f_*$. Hence, we get functors

$$f_*, f^*: Pro(T) \rightleftarrows Pro(U): f^*$$

fitting into a pair of adjunctions

$$f_! \dashv f^* \dashv f_*.$$

Recall the definition of the relative homotopy type of a geometric morphism $f$

**Definition 2.0.3.** Let $f: T \to U$ be a geometric morphism of $\infty$-topoi. The relative topological realization of $T$ over $U$ is

$$\text{Ét}(f) = f_!(\ast_T) \in Pro(U),$$

where $\ast_T$ is the terminal object of $T$, considered as a constant pro-object.

**Remark 2.0.4.** In the case where $T$ is discrete, the above definition coincides with the definition given in [2], as the $\infty$-category $Pro T$ is equivalent to the $\infty$-category which results from the Barnea-Schlank model structure, as is proven in [5].
In [6], Carchedi defines the étale profinite homotopy type of an \(\infty\)-stack. In [7], Cough defines a relative version of the topological type for a morphism of an \(\infty\)-stack to a scheme. For our application, we need a relative version of the étale topological type for a morphism of \(\infty\)-stacks. Namely, given a morphism of \(\infty\)-stacks \(f: X \to Y\), we wish to define its relative topological realization.

By the functoriality of \(\mathbf{Shv}_\infty(\bullet_{et})\), we have a geometric morphism, denoted abusively by

\[
\mathbf{f}: \mathbf{Shv}_\infty(X) \to \mathbf{Shv}_\infty(Y).
\]

Applying the general theory of relative topological realization, we obtain a pro-sheaf \(\mathbf{\acute{E}t}(\mathbf{f}) = \mathbf{f}^\#(\ast_X) \in \mathbf{ProShv}_\infty(Y_{et})\).

Our aim now is to give a formula for the relative topological realization of a colimit of a morphism of \(\infty\)-stacks in terms of the topological realizations of the components. To do this we shall recall first some general categorical constructions associated with diagrams of \(\infty\)-topoi and \(\infty\)-categories in general.

2.1 Adjunctions and Limits of Infinity Categories

Here we recall and expand the results of [8], which we use in our computation of relative topological type of morphisms of stacks.

Let \(I\) be a simplicial set. Consider a functor \(C \bullet: I \to \mathbf{Cat}_\infty\), i.e. a diagram of \(\infty\)-categories of shape \(I\). In this case, one can form the inverse limit \(\lim C \bullet\) and the lax limit \(\mathbf{Lax} C \bullet\) (see e.g. [8, Definition 4.1]). There is also a notion of op-lax limit for this diagram, which is just

\[
\mathbf{OpLax} C \bullet : = (\mathbf{Lax} C \bullet)^{op}.
\]

Both limits comes equipped with fully faithful embeddings \(\lim C \bullet \to \mathbf{Lax} C \bullet\) and \(\lim C \bullet \to \mathbf{OpLax} C \bullet\).

**Proposition 2.1.1.** [8, Proposition 5.1] Let \(\phi \bullet: C \bullet \to D \bullet\) be a morphism of \(I\) diagrams in \(\mathbf{Cat}_\infty\). If for every \(i \in I\) the functor \(\phi_i: C_i \to D_i\) admits a right (resp. left) adjoint, then the induced functor \(\mathbf{Lax}(\phi): \mathbf{Lax} C \bullet \to \mathbf{Lax} D \bullet\) (resp. \(\mathbf{OpLax}(\phi): \mathbf{OpLax} C \bullet \to \mathbf{OpLax} D \bullet\)) admits a right (resp. left) adjoint.

**Remark 2.1.2.** The result in [8] is stated only for right adjoints. The case of left adjoints follows in a similar (but dual) manner.

As in [8, Remark 5.2], even if \(f \bullet: C \bullet \to D \bullet\) admits right (or left) adjoints level-wise, then the induced functor \(\lim(\phi): \lim C \bullet \to \lim D \bullet\) might not admit a right (or left) adjoint. However, under some extra assumptions it does.

**Definition 2.1.3.** Let

\[
A \xrightarrow{f} B \xleftarrow{u} \xrightarrow{v} C \xrightarrow{g} D
\]

\[
\begin{align*}
\text{Proposition 2.1.1.} & \\
\text{Remark 2.1.2.} & \\
\text{Definition 2.1.3.} & \\
\end{align*}
\]
be a commutative square in $\text{Cat}_\infty$ (i.e. a natural isomorphism $gu \sim vf$).
Suppose that $u$ and $v$ admit left adjoints $L_u$ and $L_v$. Let $BC: L_u g \to f L_u$ denote the Beck-Chevalley natural transformation. If $BC$ is an equivalence, we say that the commutative square satisfies the left Beck-Chevalley condition (or, in short, left BC-condition).

One can easily define the dual notion of right BC-condition, in case where $u, v$ admit right adjoints. As a matter of convention, the BC-condition shall always refer to the left (or right) adjoints of the vertical maps.

**Lemma 2.1.4.** Let $F: C \rightleftarrows D: G$ be an adjunction between $\infty$-categories. Let $C' \subseteq C$ and $D' \subseteq D$ be full subcategories, such that $F|_{C'}$ factors through $D'$ and $G|_{D'}$ factors through $C'$. Then $G|_{D'}$ is a right adjoint to $F|_{C'}$ and the square

\[
\begin{array}{ccc}
C' & \rightarrow & C \\
\downarrow F|_{C'} & & \downarrow F \\
D' & \rightarrow & D
\end{array}
\]

satisfies the right Beck-Chevalley condition. Similarly, the canonical square

\[
\begin{array}{ccc}
D' & \rightarrow & D \\
\downarrow G|_{D'} & & \downarrow G \\
C' & \rightarrow & C
\end{array}
\]

satisfies the left Beck-Chevalley condition.

**Proof.** First, clearly $G|_{D'}$ is right adjoint to $F|_{C'}$ (compare [8, Lemma 5.4]). We shall show that the first square satisfies the Beck-Chevalley conditions, as the second follows analogously. Let $u: id_C \to GF$ and $c: FG \to id_D$ denote the unit and counit of the adjunction respectively. Let $i_C: C' \hookrightarrow C$ and $i_D: D' \hookrightarrow D$ denote the fully faithful embeddings. The Beck-Chevalley map of the first square is, by definition, the composition

\[
Gi_D (G_{C'}G|_{D'}) \cong (GF)i_C G|_{D'} \rightarrow i_C G|_{D'}
\]

When applied to an object $d \in D' \subseteq D$, up to identifications of objects in a full subcategory with their image in the ambient category, this is just the composition $G(d) \xrightarrow{\cong} GFG(d) = GFG(d) \xrightarrow{\sim} G(d)$. This composition is an equivalence by the zygzag identity for the adjunction $F \dashv G$.

**Remark 2.1.5.** Note that, clearly, the Beck-Chevalley map of the first square is precisely the natural transformation rendering the second diagram commutative and vice versa.

**Definition 2.1.6.** Let $f_\bullet: C_\bullet \to D_\bullet$ be a natural transformation of $I$-shaped diagram of $\infty$-categories. We say that $f_\bullet$ satisfies the right (resp. left) BC-condition, if for every $i \in I$ the functor $f_i$ admits right (resp. left) adjoint and
for every $e : i \to j$ in $I$ the resulting commutative square

\[
\begin{array}{ccc}
  C_i & \xrightarrow{e_*} & C_j \\
  \downarrow f_i & & \downarrow f_j \\
  D_i & \xrightarrow{e_*} & D_j
\end{array}
\]

satisfies the right (resp. left) Beck-Chevalley condition.

Before we state our main usage of Beck-Chevalley transformations, let us discuss a little bit the construction of the right (or left) adjoint on the level of lax limits. As usual we shall discuss the case where the transformation has right adjoint, leaving the dual verification to the reader. If $f_\bullet : C_\bullet \to D_\bullet$ is a natural transformation of $I$-shaped diagrams of $\infty$-categories, and if $f_i : C_i \to D_i$ has a right adjoint $g_i$ for every $i \in I$, then $f_{lax}$ has a right adjoint $g_{lax}$. Let $\int_I C_\bullet \to I$ and $\int_I D_\bullet \to I$ denote the respective coCartesian fibrations over $I$. Then $g_{lax}$ is obtained from a functor $G : \int_I D_\bullet \to \int_I C_\bullet$ (over $I$) by applying $G$ to sections of the structure map $\int_I D_\bullet \to I$. This functor restricts for every $i \in I$ to a functor $G_i : D_i \to C_i$ which is naturally identified with $g_i$, see [8, Proposition 5.1]. By inspection of the construction of $G$ as in [8, Section 5], one can identify also how $G$ acts on morphisms. Every morphism in $\int_I D_\bullet$ canonically factors as a composition of a coCartesian morphism and a morphism lying in a single $D_i$. Hence, given the equivalence $G|_{D_i} \simeq g_i$, it suffices to describe the application of $G$ on coCartesian edges. Let $i, j \in I$ and $e : i \to j$ be an edge. Then we have induced functors $e_* : D_i \to D_j$ and $e_* : C_i \to C_j$. The fact that $f$ is a natural transformation gives us a commutative square

\[
\begin{array}{ccc}
  C_i & \xrightarrow{e_*} & C_j \\
  \downarrow f_i & & \downarrow f_j \\
  D_i & \xrightarrow{e_*} & D_j
\end{array}
\]

For every $d \in D_i \subseteq \int_I D_\bullet$ we have an essentially unique coCartesian edge $\psi : d \to e_*d$. Then, $G(\psi)$ is the composition of the unique coCartesian edge $\psi' : g_i(d) \overset{\simeq}{\to} G(d) \to e_*G(d) \overset{\simeq}{\to} e_*g_i(d)$ with the Beck-Chevalley map of the square above, namely the map

$$BC : e_*g_i(d) \to g_je_*(d) \overset{\simeq}{\to} G(e_*(d)).$$

**Proposition 2.1.7.** Let $f_* : C_\bullet \to D_\bullet$ be a morphism of diagrams of $\infty$-categories. If $f_\bullet$ satisfies the right (resp. left) Beck-Chevalley condition, then the induced functor $\lim f_* : \lim C_\bullet \to \lim D_\bullet$ admit a right (resp. left) adjoint. Moreover, the canonical commutative square

\[
\begin{array}{ccc}
  \lim C_\bullet & \xleftarrow{\lim f_*} & \text{Lax } C_\bullet \\
  \downarrow \lim f_* & & \downarrow \text{Lax } f_* \\
  \lim D_\bullet & \xleftarrow{\lim f_*} & \text{Lax } D_\bullet
\end{array}
\]
(resp. the square
\[
\begin{array}{c c c}
\lim C \leftarrow & \text{OpLax} C
\end{array}
\]
\[
\begin{array}{c c}
\lim f & \to \text{OpLax} f
\end{array}
\]
\[
\begin{array}{c c c}
\lim D & \leftarrow & \text{OpLax} D
\end{array}
\]
) satisfies the right (resp. left) Beck-Chevalley condition.

Proof. We do the right case, as the left case is completely analogous. Let \( g_i \) denote the right adjoint to \( f_i \). By Lemma 2.1.4, it suffices to prove that the restriction to \( \lim D \) of the right adjoint to \( \text{Lax} f \) factors through \( \lim C \). Namely, it would suffice to show that, in the notation of the discussion above this proposition, \( G \) sends coCartesian edges to coCartesian edges. If \( \psi : d \to d' \) is coCartesian, then \( G(\psi) \) is a composition of a coCartesian edge with the Beck-Chevalley map. By the assumption, those Beck-Chevalley maps are isomorphisms, so \( G \) send coCartesian edges to coCartesian edges.

We now turn to discuss descent properties of the Beck-Chevalley condition from diagrams to their limits.

**Proposition 2.1.8.** Let \( I \) be a small \( \infty \)-category. Let

\[
\begin{array}{c c c}
A & \xrightarrow{f} & B
\end{array}
\]
\[
\begin{array}{c c}
\downarrow u & & \downarrow v
\end{array}
\]
\[
\begin{array}{c c c}
C & \xrightarrow{g} & D
\end{array}
\]

be a commutative square of \( I \)-shaped diagrams of \( \infty \)-categories. If for every \( i \in I \) the square

\[
\begin{array}{c c c}
A_i & \xrightarrow{f_i} & B_i
\end{array}
\]
\[
\begin{array}{c c}
\downarrow u_i & & \downarrow v_i
\end{array}
\]
\[
\begin{array}{c c c}
C_i & \xrightarrow{g_i} & D_i
\end{array}
\]

satisfies the right (resp. left) Beck-Chevalley condition, then the square

\[
\begin{array}{c c c}
\text{Lax} A & \xrightarrow{\text{Lax} f} & \text{Lax} B
\end{array}
\]
\[
\begin{array}{c c}
\downarrow \text{Lax} u & & \downarrow \text{Lax} v
\end{array}
\]
\[
\begin{array}{c c c}
\text{Lax} C & \xrightarrow{\text{Lax} g} & \text{Lax} D
\end{array}
\]

satisfies the right (resp. left) Beck-Chevalley condition.
(resp. the square

\[
\begin{array}{ccc}
\text{OpLax} A & \xrightarrow{\text{OpLax} f} & \text{OpLax} B \\
\downarrow \text{OpLax} u & & \downarrow \text{OpLax} v \\
\text{OpLax} C & \xrightarrow{\text{OpLax} g} & \text{OpLax} D
\end{array}
\]

) satisfies the right (resp. left) Beck-Chevalley condition.

Proof. It is easy to see that the Beck-Chevalley map of the square of lax limits restricts at every \( i \in I \) to the Beck-Chevalley map of the square

\[
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_i \\
\downarrow u_i & & \downarrow v_i \\
C_i & \xrightarrow{g_i} & D_i
\end{array}
\]

The result now follows from the fact that equivalences in \( \text{Lax} B \) and \( \text{OpLax} B \) are those maps which restrict to an equivalence at every \( i \in I \).

Before we state the descent result of BC-conditions, let us recall some pasting conditions for BC-squares. Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C \\
\downarrow f & & \downarrow g & & \downarrow h \\
A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C'
\end{array}
\]

of \( \infty \)-categories for which all the vertical functors admit right adjoints. Then, the BC-map of the outer square is the composition of the BC-maps of the two smaller squares, namely the right BC map of the outer square is given by

\[
baR_f b_{BC} b_{Co} \rightarrow R_h b' a'.
\]

In particular, if the two small squares satisfies the right BC-condition so is the outer square. Moreover, if \( b \) is conservative, the outer and the left squares satisfies the right BC condition, then so does the right square. We call those two properties horizontal pasting of Beck-Chevalley squares. (The case for the left BC condition is similar).

Theorem 2.1.9. Let \( I \) be a small \( \infty \)-category and let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow u & & \downarrow v \\
C & \xrightarrow{g} & D
\end{array}
\]
be a commutative square of $I$-shaped diagrams of $\infty$-categories. Suppose that $u_\bullet$ and $v_\bullet$ satisfy the right (resp. left) Beck-Chevalley condition, and for every $i \in I$ the restricted square

\[
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_i \\
\downarrow{u_i} & & \downarrow{v_i} \\
C_i & \xrightarrow{g_i} & D_i
\end{array}
\]

satisfies the right (resp. left) Beck-Chevalley condition. Then the square

\[
\begin{array}{ccc}
\lim A_\bullet & \xrightarrow{\lim f_\bullet} & \lim B_\bullet \\
\downarrow{\lim u_\bullet} & & \downarrow{\lim v_\bullet} \\
\lim C_\bullet & \xrightarrow{\lim g_\bullet} & \lim D_\bullet
\end{array}
\]

satisfies the right (resp. left) Beck-Chevalley condition.

Proof. We do the right case, the left follows analogously. Consider the commutative cube

\[
\begin{array}{ccc}
\lim A_\bullet & \xrightarrow{\lim f_\bullet} & \lim B_\bullet \\
\downarrow{\lim u_\bullet} & & \downarrow{\lim v_\bullet} \\
\lim C_\bullet & \xrightarrow{\lim g_\bullet} & \lim D_\bullet
\end{array}
\]

The right and left faces satisfies the right Beck-Chevalley condition by Proposition 2.0.2. The front face satisfies the right Beck-Chevalley condition by Proposition 2.1.8. Since the inclusion of the lax limit to the limit is fully faithful, to show that the back-face satisfies the right BC-condition, its enough to show that its horizontal pasting with the right face satisfies the right BC-condition. By the commutativity of the diagram, this is the same as the BC-condition for the horizontal pasting of the left face and the front face. Since these two satisfy the right Beck-Chevalley condition, we are done.

\[\square\]

2.2 Relative Homotopy Type of Colimits of Infinity Topoi

Let $I$ be a small $\infty$-category and let $f_\bullet : \mathcal{T}_\bullet \to \mathcal{U}_\bullet$ be a morphism of $I$-shaped diagrams in $\mathcal{Top}_\infty$. By definition of colimits in $\mathcal{Top}_\infty$, the colimits $\lim_{\to} \mathcal{T}_\bullet$ and
The diagrams $\mathcal{T}_\bullet$ and $\mathcal{U}_\bullet$ can be considered as $I^{op}$-shaped diagrams in $Cat_\infty$ via the forgetful functor $\mathcal{S}op^{op}_{\infty} \to Cat_\infty$ [3, Definition 6.3.1.5]. Denote by $(\mathcal{T}_\bullet)^*$ and $(\mathcal{U}_\bullet)^*$ the resulting $I^{op}$-shaped diagrams in $Cat_\infty$. The geometric natural transformation $f^*_\bullet: \mathcal{T}_\bullet \to \mathcal{U}_\bullet$ induces a natural transformation $f^*_{lax}: (\mathcal{U}_\bullet)^* \to (\mathcal{T}_\bullet)^*$.

Let $f = \lim_{\rightarrow} f_\bullet$, $\mathcal{T} = \lim_{\rightarrow} T_\bullet = \lim_{\rightarrow} (\mathcal{T}_\bullet)^*$, and $\mathcal{U} = \lim_{\rightarrow} U_\bullet = \lim_{\rightarrow} (\mathcal{U}_\bullet)^*$. In this case, the functors $f^{*}_{lax}: Pro(\mathcal{U}) \to Pro(\mathcal{T})$ admits left adjoints, and hence by Proposition 2.1.1 the op-lax functor $f^*_{lax}$ associated with $f^*_\bullet$ admits a left adjoint.

Let $\rho_\bullet: \mathcal{T}_\bullet \to \mathcal{T}$ and $\nu_\bullet: \mathcal{U}_\bullet \to \mathcal{U}$ denote the comparison geometric morphisms, where the targets are considered as constant $I$-shaped diagrams. We can now form the following natural commutative diagram

$$
\begin{array}{c}
\text{OpLax Pro(}\mathcal{T}_\bullet\text{)}^* \\
\downarrow f^*_{lax}
\end{array}
\begin{array}{c}
\text{Pro(}\mathcal{T}\text{)} \\
\downarrow (f^*)^!
\end{array}
\begin{array}{c}
\text{Pro(}\mathcal{U}_\bullet\text{)}^* \\
\downarrow \nu^*_{lax}
\end{array}
\begin{array}{c}
\text{Pro(}\mathcal{U}\text{)} \\
\downarrow \pi^*
\end{array}
$$

where $\pi: I^{op} \to \ast$ denote the projection, and $\pi^*$ is pre-composition with $\pi$. Let $\tilde{\rho}^* = \rho^*_{lax} \pi^*$ and $\tilde{\nu}^* = \nu^*_{lax} \pi^*$. Let $\tilde{\rho}_2$ and $\tilde{\nu}_2$ denote the respective left adjoints.

Hence, we have a commutative diagram

$$
\begin{array}{c}
\text{OpLax Pro(}\mathcal{T}_\bullet\text{)}^* \\
\downarrow f^*_{lax}
\end{array}
\begin{array}{c}
\text{Pro(}\mathcal{T}\text{)} \\
\downarrow f^*
\end{array}
\begin{array}{c}
\text{OpLax Pro(}\mathcal{U}_\bullet\text{)}^* \\
\downarrow \tilde{\rho}^*
\end{array}
\begin{array}{c}
\text{Pro(}\mathcal{U}\text{)} \\
\downarrow \tilde{\nu}^*
\end{array}
$$

and as a result we obtain a Beck-Chevalley map $BC^\sharp_2: f_{lax}^* \tilde{\rho}^* \tilde{\nu}^* f_2$. Taking the mate of this map we obtain a natural transformation $\mu: \tilde{\nu}_2 f_{lax}^* \tilde{\rho}^* f_2^* \to f_2^*$. 

**Theorem 2.2.1.** The natural transformation $\mu$ is an equivalence at every object of the full subcategory $\mathcal{T} \subseteq Pro(\mathcal{T})$.

**Proof.** Let $x \in \mathcal{T}$. We have to show that

$$
\mu_x: \tilde{\nu}_2 f_{lax}^* \tilde{\rho}^* (x) \to f_2^* (x)
$$
is an equivalence. Since the objects of \( \mathcal{U} \) co-generate the pro-category \( \text{Pro}(\mathcal{U}) \), it suffices to prove that for every \( y \in \mathcal{U} \), the induced map

\[
\mu_x^*: \text{Hom}_{\text{Pro}(\mathcal{U})}(f^x, y) \to \text{Hom}_{\text{Pro}(\mathcal{U})}(\check{\nu} f_{lax, x} \check{\rho}^x, y).
\]

is an equivalence. Consider the following diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{Pro}(\mathcal{T})}(x, f^* y) & \xrightarrow{(\check{\rho}^x)} & \text{Hom}_{\text{OpLax}}(T^x, \check{\rho}^x f^* y) \\
\sim & & \sim \\
\text{Hom}_{\text{Pro}(\mathcal{T})}(x, f^* y) & \xrightarrow{(\check{\rho}^x)} & \text{Hom}_{\text{OpLax}}(T^x, \check{\rho}^x f^* y) \\
\sim & & \sim \\
\text{Hom}_{\text{Pro}(\mathcal{T})}(x, f^* y) & \xrightarrow{(\check{\rho}^x)} & \text{Hom}_{\text{OpLax}}(T^x, \check{\rho}^x f^* y) \\
\sim & & \sim \\
\text{Hom}_{\text{Pro}(\mathcal{T})}(x, f^* y) & \xrightarrow{(\check{\rho}^x)} & \text{Hom}_{\text{OpLax}}(T^x, \check{\rho}^x f^* y) \\
\end{array}
\]

in which the arrows labeled \( \alpha \) and \( \beta \) come from the natural fully faithful embeddings \( \text{Id}_{\text{Cat}_{\infty}} \to \text{Pro} \) and \( \text{lim} \to \text{OpLax} \), the horizontal unlabeled arrows arises from the commutativity of the diagram (2), and the vertical equivalences all come from the various adjunctions.

The upper big rectangle commute by the definition of \( \mu \). The lower left trapezoid and the two lower right rectangles commute by the naturality of \( \alpha \) and \( \beta \). Finally, the triangle at the middle-left commute by the definition of \( \check{\rho}^x \) (it factors uniquely through the actual limit).

Hence, the big outer square commute up to homotopy, and \( \mu_x^* \) is an equivalence if and only if the bottom left horizontal map

\[
\text{Hom}_{\mathcal{T}}(x, f^* y) \xrightarrow{\check{\rho}^x} \text{Hom}_{\text{lim}}(\mathcal{T}^x, \check{\rho}^x f^* y)
\]

is an equivalence. But \( \check{\rho}^x \) is induced from the comparison map \( \mathcal{T} \to \text{lim} \mathcal{T}^x \) which is an equivalence by the assumption.

From this result, one immediately get as a corollary a formula for the relative homotopy type of a colimit of morphisms of \( \infty \)-topoi. Let \( f_* : \mathcal{T}_* \to \mathcal{U}_* \) be a morphism of \( \infty \)-topoi, with colimit \( f : \mathcal{T} \to \mathcal{U} \). Then, with the notation as above, we have a well defined element \( f_{lax, \check{\rho}^x} \in \text{OpLax}(\mathcal{U}_*) \). Denote this element by \( |f| \). Then, by Theorem 2.2.1 we obtain:

**Corollary 2.2.2.**

\[
\check{\nu} f_* |f| \cong |f|
\]

Informally, this means that the relative homotopy type of the colimit is the colimit of the relative homotopy types.
2.3 The Relative Étale Homological Type

In this section we shall recall the theory of sheaves of modules over a ring in a general ∞-topos. Note that the case of classical topos and in particular of classical stacks is well studied, and in particular all the results of this section are well known for the case of a classical topoi and stacks. The goal is therefore to promote parts of the classical theory to the ∞-categorical and higher stacks settings.

For a ring Λ, let \( D(Λ) \) denote the stable ∞-category of complexes of Λ-modules. Let \( S \) be the ∞-category of spaces. There is an adjunction

\[
\begin{array}{rcl}
I_Λ: S & \rightleftarrows & D(Λ): R_Λ
\end{array}
\]

where \( R_Λ \) denotes the functor \( \text{Hom}_{D(Λ)}(Λ, \bullet) \) and we have \( \pi_n(I_Λ(X)) = H_n(X, Λ) \).

Let \( \mathcal{P} \) be the ∞-category of presentable ∞-categories with colimit preserving functors between them. Then \( \mathcal{P} \) is a symmetric monoidal ∞-category as in [9, Section 4.8]. In fact, the symmetric monoidal structure satisfies \( C \otimes D \cong \text{RFun}(C^{\text{op}}, D) \), the limits preserving functors from \( C^{\text{op}} \) to \( D \), by [9, Proposition 4.8.1.17]. The adjunction (3) can be seen as a morphism \( I_Λ: S \rightarrow D(Λ) \) in \( \mathcal{P} \).

We shall now define the notion of relative homological type of a geometric morphism of ∞-topoi. For an ∞-topos \( T \) and a ring \( Λ \) let \( Λ_T = I_Λ(∗_T) \).

**Definition 2.3.1.** Let \( f: T \rightarrow U \) be a geometric morphism of ∞-topoi. Define the relative homological type of \( f \) to be

\[
|f|_Λ = f_∗Λ_T \in \text{Pro}(\text{Shv}_∞(U, Λ)).
\]

We would like to compare now the relative homological and the homotopy types of a geometric morphism. Before we do that, we shall discuss some basic properties of the symmetric monoidal structure on \( \mathcal{P} \).

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2.3.1 Tensor Products of Presentable Infinity Categories and Beck-Chevalley Conditions

Here we shall recall and discuss some basic properties of the tensor product of presentable \(\infty\)-categories. Let \(C, D \in \Pr^L\) be two presentable \(\infty\)-categories. In [9, Section 4.8] the equivalence of \(\text{RFun}(C^{\text{op}}, D)\) and \(C \otimes D\) is proved by comparing the co-represented functors. Directly from this identification, it is easy to deduce part of the functoriality of of the right hand side as a functor \(\Pr^L \times \Pr^L \to \Pr^L\).

For \(u : C \to C'\) a morphism in \(\Pr^L\) and \(D \in \Pr^L\), then the right adjoint to the composition

\[
\text{RFun}(C^{\text{op}}, D) \simeq C \otimes D \xrightarrow{u \otimes \text{id}} D' \otimes D \simeq \text{RFun}((C')^{\text{op}}, D)
\]

is easily identified with the morphism of pre-composition with \(u^{\text{op}}\). Similarly, the right adjoint to the composition

\[
\text{RFun}(D^{\text{op}}, C) \simeq D \otimes C \xrightarrow{\text{id} \otimes u} D \otimes C' \simeq \text{RFun}(D^{\text{op}}, C')
\]

is homotopic to the morphism of post-composition with the right adjoint to \(u\).

If \(C\) is a presentable \(\infty\)-category, then we denote by \(C^\omega\) the full subcategory of compact objects in \(C\). We have a fully faithful embedding \(\text{Ind}(C^\omega) \hookrightarrow C\). The \(\infty\)-category \(C\) is called compactly generated if this embedding is essentially surjective, and hence \(C \cong \text{Ind}(C^\omega)\). If \(C\) and \(D\) are \(\infty\)-categories we denote by \(\text{Fun}^{\text{lex}}(C, D)\) the full subcategory of \(\text{Fun}(C, D)\) spanned by the functors that commute with finite limits.

**Proposition 2.3.2.** Let \(C\) be a compactly generated presentable \(\infty\)-category and let \(D\) be a presentable \(\infty\)-category. Then restriction to the compact objects induces an equivalence

\[
C \otimes D \cong \text{Fun}^{\text{lex}}((C^\omega)^{\text{op}}, D)
\]

**Proof.** Recall that for presentable \(\infty\)-categories \(A\) and \(B\), restriction along the inclusion \(A \to \text{Pro}A\) induces an equivalence \(\text{RFun}(\text{Pro}(A), B) \cong \text{Fun}^{\text{lex}}(A, B)\). Hence, we have

\[
C \otimes D \cong \text{RFun}(C^{\text{op}}, D) \cong \text{RFun}((\text{Ind}(C^\omega))^{\text{op}}, D) \cong \text{RFun}((C^\omega)^{\text{op}}, D) \cong \text{Fun}^{\text{lex}}((C^\omega)^{\text{op}}, D)
\]

and the composition of those equivalences easily seen to be the one induced from the restriction along the inclusion \(C^\omega \hookrightarrow C\). \(\square\)

We shall now discuss Beck-Chevalley conditions for the exterior tensor product of morphisms in \(\Pr^L\). For this we need the following lemma.

**Lemma 2.3.3.** Let \(F : C \rightleftarrows C' : G\) be an adjunction and let \(H : D \to D'\) be a functor. Then post-composition with the adjunction datum of \(F\) and \(G\) give rise to an adjunction

\[
F \circ : \text{Fun}(D, C) \rightleftarrows \text{Fun}(D, C') : G \circ.
\]

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Moreover, the square

\[
\begin{array}{ccc}
\text{Fun}(D', C) & \xrightarrow{\cdot H} & \text{Fun}(D, C) \\
F \circ & & F \circ \\
\text{Fun}(D', C') & \xrightarrow{\cdot H} & \text{Fun}(D, C')
\end{array}
\]

satisfies the right Beck-Chevalley condition, and the square

\[
\begin{array}{ccc}
\text{Fun}(D', C') & \xrightarrow{\cdot H} & \text{Fun}(D, C') \\
G \circ & & G \circ \\
\text{Fun}(D', C) & \xrightarrow{\cdot H} & \text{Fun}(D, C)
\end{array}
\]

satisfies the left Beck-Chevalley condition.

**Proof.** It is clear that \(F \circ\) and \(G \circ\) are adjoint by post-composing with the adjunction datum. We shall prove the Beck-Chevalley condition for the first square, as the second is completely analogous. Let \(c: FG \to \text{id}_{C'}\) and \(u: \text{id}_{C} \to GF\) denote the unit and the counit of the adjunction \(F \dashv G\). The Beck-Chevalley map of the first square has value on \(\phi: D' \to D'\) is the composition

\[
G(\phi H) \xrightarrow{G(u(\phi H))} (GF)(\phi H) \cong (G((FG)(\phi))H) \xrightarrow{G((c\phi)H)} (G\phi)H.
\]

By the zygzag identities and the associativity of composition this map is homotopic to the associativity isomorphism \(G(\phi H) \cong (G\phi)H\) and hence it is an equivalence. \(\Box\)

Here again, the Beck-Chevalle map of one square is the homotopy rendering the second commutative and vice versa.

**Proposition 2.3.4.** Let \(u: C \to C'\) and \(v: D \to D'\) be morphisms in \(\mathcal{P}r^L\). Assume that \(C\) and \(C'\) are compactly generated and that \(u\) maps compact objects to compact objects. Suppose further that \(v\) is left exact. Then the square

\[
\begin{array}{ccc}
C \otimes D & \xrightarrow{u \otimes \text{id}_D} & C' \otimes D \\
\text{id}_C \otimes v & & \text{id}_{C'} \otimes v \\
C \otimes D' & \xrightarrow{u \otimes \text{id}_{D'}} & C' \otimes D'
\end{array}
\]

satisfies the right Beck-Chevalley condition.

**Proof.** Let \(R_u\) be the right adjoint to \(v\). It would suffice to prove that the square of right adjoints satisfies the left Beck-Chevalley condition. The square of right adjoints is canonically equivalent to the square

\[
\begin{array}{ccc}
\text{RFun}((C')^{op}, D') & \xrightarrow{u^{op} \circ} & \text{RFun}((C)^{op}, D') \\
\circ \circ R_v & & \circ \circ R_v \\
\text{RFun}((C')^{op}, D) & \xrightarrow{u^{op} \circ} & \text{RFun}(C^{op}, D)
\end{array}
\]
By Proposition 2.3.2, this square is equivalent to the square

\[
\begin{array}{c}
\text{Fun}^{lex}(\mathcal{C}^{\omega}_{\text{op}}, D') \\
\downarrow \circ R_v \\
\text{Fun}^{lex}(\mathcal{C}^{\omega}_{\text{op}}, D)
\end{array}
\begin{array}{c}
\text{Fun}^{lex}(\mathcal{C}^{\omega}_{\text{op}}, D') \\
\downarrow \circ R_v \\
\text{Fun}^{lex}(\mathcal{C}^{\omega}_{\text{op}}, D)
\end{array}
\]

\[
\text{(4)}
\]

The adjunction \(v: D \rightleftarrows D': R_v\) gives rise to an adjunction between functors \(\text{Cat}_{\infty} \to \text{Cat}_{\infty}\) of the form

\[
v \circ : \text{Fun}(\bullet, D) \rightleftarrows \text{Fun}(\bullet, D'): R_v \circ .
\]

Since \(v\) is left exact the restriction of \(v \circ\) to \(\text{Fun}^{lex}(\bullet, D)\) has its image in \(\text{Fun}^{lex}(\bullet, D')\), and hence by Lemma 2.1.4 we get a restricted adjunction

\[
v \circ : \text{Fun}^{lex}(\bullet, D) \rightleftarrows \text{Fun}^{lex}(\bullet, D'): \circ R_v,
\]

and the square

\[
\begin{array}{c}
\text{Fun}^{lex}(\bullet, D) \\
\downarrow \circ R_v \\
\text{Fun}^{lex}(\bullet, D')
\end{array}
\begin{array}{c}
\text{Fun}(\bullet, D) \\
\downarrow \circ R_v \\
\text{Fun}(\bullet, D')
\end{array}
\]

satisfies the left Beck-Chevalley condition. Hence, the left Beck-Chevalley condition of the square (4) follows from the left Beck-Chevalley condition for the square

\[
\begin{array}{c}
\text{Fun}((\mathcal{C}^{\omega}_{\text{op}}, D') \\
\downarrow \circ R_v \\
\text{Fun}((\mathcal{C}^{\omega}_{\text{op}}, D)
\end{array}
\begin{array}{c}
\text{Fun}((\mathcal{C}^{\omega}_{\text{op}}, D') \\
\downarrow \circ R_v \\
\text{Fun}((\mathcal{C}^{\omega}_{\text{op}}, D)
\end{array}
\]

(compare the proof of Theorem 2.1.9). This square satisfies the left Beck-Chevalley condition by Lemma 2.3.3.

As a corollary, we get

**Corollary 2.3.5.** Let \(f: \mathcal{T} \to \mathcal{U}\) be a geometric morphism of \(\infty\)-topoi and let \(u: C \rightleftarrows D: v\) be a morphism in \(\Pr^L\). Assume that \(C\) and \(D\) are compactly generated and that \(u\) preserves compact objects. Then the commutative square

\[
\begin{array}{ccc}
\mathcal{T} \otimes C & \overset{id_T \otimes u}{\longrightarrow} & \mathcal{T} \otimes D \\
\downarrow f^* \otimes id_C & & \downarrow f^* \otimes id_D \\
\mathcal{U} \otimes C & \overset{id_U \otimes u}{\longrightarrow} & \mathcal{U} \otimes D
\end{array}
\]

satisfies the right Beck-Chevalley condition.
2.3.2 Comparison of the Relative Homological type and Homotopy Type

We shall now discuss the compatibility of the relative homological and homotopical types for geometric morphisms.

**Proposition 2.3.6.** For every morphism of $\infty$-topoi $f: T \to U$, the diagram

$$
\begin{array}{ccc}
\text{Pro}(U) & \xrightarrow{I_\Lambda(U)} & \text{Pro}(\text{Shv}_\infty(U, \Lambda)) \\
|f| & & |f| \\
\downarrow f_! & & \downarrow f_! \\
\text{Pro}(T) & \xrightarrow{I_\Lambda(T)} & \text{Pro}(\text{Shv}_\infty(T, \Lambda))
\end{array}
$$

commutes up to homotopy.

**Proof.** Passing to the right adjoints, it would suffice to prove that the diagram

$$
\begin{array}{ccc}
\text{Pro}(U) & \xleftarrow{R_\Lambda(U)} & \text{Pro}(\text{Shv}_\infty(U, \Lambda)) \\
\uparrow f^* & & \uparrow f^* \\
\text{Pro}(T) & \xleftarrow{R_\Lambda(T)} & \text{Pro}(\text{Shv}_\infty(T, \Lambda))
\end{array}
$$

commutes up to homotopy. This diagram is the prolongation of

$$
\begin{array}{ccc}
U & \xleftarrow{R_\Lambda(U)} & \text{Shv}_\infty(U, \Lambda) \\
\uparrow f^* & & \uparrow f^* \\
T & \xleftarrow{R_\Lambda(T)} & \text{Shv}_\infty(T, \Lambda)
\end{array}
$$

hence it suffices to prove that this square commutes up to homotopy.

The square

$$
\begin{array}{ccc}
U & \xleftarrow{f^*} & T \\
\downarrow I_\Lambda(U) & & \downarrow I_\Lambda(T) \\
\text{Shv}_\infty(U, \Lambda) & \xleftarrow{f} & \text{Shv}_\infty(T, \Lambda)
\end{array}
$$

commutes by the naturality of $I_\Lambda$ and it would suffice to show that it satisfies the right Beck-Chevalley condition. This in turn follows from Corollary 2.3.5 since $I_\Lambda: D(\Lambda) \to S$ map compact objects to compact objects and $S$ and $D(\Lambda)$ are compactly generated. \(\square\)

If we denote $\Lambda \otimes x = I_\Lambda(x)$ then the following

**Corollary 2.3.7.** Let $f: T \to U$ be a geometric morphism, and let $\Lambda$ be a ring. Then

$$
\Lambda \otimes f_!(x) \cong f_!(\Lambda \otimes x).
$$

In particular,

$$
|f|_\Lambda \cong |f| \otimes \Lambda.
$$
2.3.3 Cohomological Remark

We conclude with a remark regarding various cohomological invariants. If $\mathcal{T}$ is an $\infty$-topos and $x \in \mathcal{T}$ we can define the cohomology of $x$ in a priori two different ways. On one hand, we can set

$$H^n(x, \Lambda) = \pi_0 \text{Hom}_{\text{Shv}_{\infty}(\mathcal{T}, \Lambda)}(x \otimes \Lambda, \Lambda_{\mathcal{T}}[n]),$$

which is analogous to the definition using singular cochains of a space. On the other hand, we can define the cohomology using maps to Eilenberg-McLane spaces, by

$$H^n(x, \Lambda) = \text{Hom}_{\mathcal{T}}(x, \Gamma^*_T K(\Lambda, n))$$

where $\Gamma_T : \mathcal{S} \to \mathcal{T}$ is the unique geometric morphism. However, since

$$\Gamma^*_T K(\Lambda, n) = \Gamma^*_\mathcal{U} R\Lambda(\Lambda[n]) \cong R\Lambda \Gamma^*_\mathcal{T} \Lambda[n]$$

the two definitions agree by the adjunction between $I_\Lambda$ and $R\Lambda$. Moreover, if $f : \mathcal{T} \to \mathcal{U}$ is a geometric morphism then for $x \in \mathcal{T}$ we have

$$H^n(f_!(x), \Lambda) = \text{Hom}_\mathcal{U}(f_! x, \Gamma^*_\mathcal{U} K(\Lambda, n)) \cong \text{Hom}_\mathcal{T}(x, f^* \Gamma^*_\mathcal{T} K(\Lambda, n)) \cong$$

$$\cong \text{Hom}_\mathcal{T}(x, \Gamma^*_\mathcal{T} K(\Lambda, n)) = H^n(x, \Lambda)$$

so the functor $f_!$ preserves the cohomology.

2.4 Smooth Base Change for Higher Stacks

If $X$ is a scheme of finite type over $K$, then $h\text{Shv}_{\infty}(X, \Lambda)$ is the derived category of sheaves of $\Lambda$-modules over $X$. In particular, every statement on $\text{Shv}_{\infty}(X, \Lambda)$ that can be tested on the homotopy category can be approached using classical sheaf theory and étale cohomology. The most important for us is the smooth base-change theorem.

**Proposition 2.4.1** (Smooth Base-Change for Schemes). Let $\Lambda$ be a finite ring of size prime to $\text{char}(K)$. Let

$$\begin{array}{ccc}
X & \xrightarrow{g'} & Y \\
\downarrow{f'} & & \downarrow{f} \\
Z & \xrightarrow{g} & W
\end{array}$$

be a pullback diagram of schemes of finite type over $K$, such that $f$ (and hence $f'$) is smooth. Then the diagram

$$\begin{array}{ccc}
\text{Shv}_{\infty}(X, \Lambda) & \xrightarrow{g'_*} & \text{Shv}_{\infty}(Y, \Lambda) \\
\downarrow{f'_*} & & \downarrow{f_*} \\
\text{Shv}_{\infty}(Z, \Lambda) & \xrightarrow{g_*} & \text{Shv}_{\infty}(W, \Lambda)
\end{array}$$

satisfies the left BC-condition. Namely, the Beck-Chevalley map for this square, denoted $BC_* : f^* g_* \to g'_*(f')^*$, is an equivalence.
Remark 2.4.2. Note that this is the same as the right Beck-Chevalley condition for the square

\[
\begin{array}{ccc}
Shv_\infty(X, \Lambda) & \xleftarrow{g^*} & Shv_\infty(Y, \Lambda) \\
\downarrow{f^*} & & \downarrow{f^*} \\
Shv_\infty(Z, \Lambda) & \xleftarrow{g^*} & Shv_\infty(W, \Lambda).
\end{array}
\]  

After prolongation, all the functors \(f^*, g^*, g'_*, (f')^*\) admit left adjoints and \(BC_*\) induces a natural transformation \(BC'_*: f'^* g'^* \to g^* f_*\). This map is easily identified with the left Beck-Chevalley map of the square

\[
\begin{array}{ccc}
\text{Pro } Shv_\infty(X, \Lambda) & \xleftarrow{g^*} & \text{Pro } Shv_\infty(Y, \Lambda) \\
\downarrow{f^*} & & \downarrow{f^*} \\
\text{Pro } Shv_\infty(Z, \Lambda) & \xleftarrow{g^*} & \text{Pro } Shv_\infty(W, \Lambda)
\end{array}
\]  

As an immediate result, we get the following

**Corollary 2.4.3.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{g'} & Y \\
\downarrow{f'} & & \downarrow{f} \\
Z & \xrightarrow{g} & W
\end{array}
\]

be a pullback diagram of schemes of finite type over \(K\) with \(f\) smooth, and \(\Lambda\) be a finite ring of order prime to the characteristic of \(K\). Then the left Beck-Chevalley map \(BC'_*: f'^* g'^* \to g^* f_*\) associated with the square (5) is an equivalence.

We shall now show that the conclusion of Corollary 2.4.3 holds for stacks as well.

**Definition 2.4.4.** Let \(f: \mathcal{X} \to \mathcal{Y}\) be a morphism of stacks. We say that \(f\) is smooth if for every morphism \(Y \to \mathcal{Y}\) for a scheme \(Y\) of finite type over \(K\), the pull-back map \(f': Y \times_{\mathcal{Y}} \mathcal{X} \to Y\) is a smooth morphism of schemes.

**Theorem 2.4.5** (Smooth base-change for \(\infty\)-stacks). Let

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{g'} & \mathcal{Y} \\
\downarrow{f'} & & \downarrow{f} \\
\mathcal{Z} & \xrightarrow{g} & \mathcal{W}
\end{array}
\]

be a pullback square of stacks, with \(f\) smooth and \(g\) schematic. Let \(\Lambda\) be a finite
ring of order prime to char(\mathcal{K}). Then the two squares
\[
\begin{array}{c}
\text{Shv}_\infty(\mathcal{X}, \Lambda) \xrightarrow{g'_*} \text{Shv}_\infty(\mathcal{Y}, \Lambda) \\
\downarrow f'_* \quad \quad \downarrow f_*
\end{array}
\]
\[
\begin{array}{c}
\text{Shv}_\infty(\mathcal{Z}, \Lambda) \xrightarrow{g_*} \text{Shv}_\infty(\mathcal{W}, \Lambda)
\end{array}
\]
and
\[
\begin{array}{c}
\text{Pro Shv}_\infty(\mathcal{X}, \Lambda) \xleftarrow{g'^*} \text{Pro Shv}_\infty(\mathcal{Y}, \Lambda) \\
\downarrow f'^* \quad \quad \downarrow f^*
\end{array}
\]
\[
\begin{array}{c}
\text{Pro Shv}_\infty(\mathcal{Z}, \Lambda) \xleftarrow{g^*} \text{Pro Shv}_\infty(\mathcal{W}, \Lambda)
\end{array}
\]
satisfy the left BC-condition.

Proof. We shall prove for the first square, as the second square follows from the first. Let \( I \) be a small infinity category and \( W \xrightarrow{\rho} \mathcal{W} \) an \( I \)-shaped diagram exhibiting \( \mathcal{W} \) as a colimit of schemes of finite type. Pulling back this presentation to \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{Z} \) along the morphisms in the pullback square, and using the fact that colimits in a topos are universal, we get a commutative cube of \( I \)-shaped diagrams in which all the faces are pullback squares and the front face consist of constant diagrams

![Diagram](https://via.placeholder.com/150)

Applying \( \text{Shv}_\infty(-, \Lambda) \) to the back-face we get a commutative square of \( I^{op} \)-shaped diagrams
\[
\begin{array}{c}
\text{Shv}_\infty(X_*, \Lambda) \xleftarrow{g'_*} \text{Shv}_\infty(Y_*, \Lambda) \\
\downarrow f'_* \quad \quad \downarrow f_*
\end{array}
\]
\[
\begin{array}{c}
\text{Shv}_\infty(Z_*, \Lambda) \xleftarrow{g_*} \text{Shv}_\infty(W_*, \Lambda)
\end{array}
\]
with limit the square

\[
\begin{array}{c}
Shv_\infty(X, \Lambda) \xrightarrow{g^*} Shv_\infty(Y, \Lambda) \\
\downarrow f^* \quad \quad \quad \downarrow f^*
\end{array}
\]

\[
\begin{array}{c}
Shv_\infty(Z, \Lambda) \xrightarrow{g^*} Shv_\infty(W, \Lambda) \\
\downarrow f^* \quad \quad \quad \downarrow f^*
\end{array}
\]

Hence, in view of Theorem 2.1.9, it would suffice to show that the maps \(f^*\) and \(f'^*\) satisfy the right Beck-Chevalley condition, and that for each \(i \in I\) the square

\[
\begin{array}{c}
Shv_\infty(X_i, \Lambda) \xrightarrow{g_i^*} Shv_\infty(Y_i, \Lambda) \\
\downarrow f_i^* \quad \quad \quad \downarrow f_i^*
\end{array}
\]

\[
\begin{array}{c}
Shv_\infty(Z_i, \Lambda) \xrightarrow{g_i^*} Shv_\infty(W_i, \Lambda) \\
\downarrow f_i^* \quad \quad \quad \downarrow f_i^*
\end{array}
\]

satisfies the right Beck-Chevalley condition. Both the statements follows from the Smooth Base-change Theorem for schemes. \qed

As an immediate consequence, we get

**Corollary 2.4.6.** Let

\[
\begin{array}{c}
X \xrightarrow{g'} Y \\
\downarrow f' \\
Z \xrightarrow{g} W
\end{array}
\]

be a pullback square of stacks, with \(f\) smooth and \(g\) schematic. Let \(\Lambda\) be a finite ring of order prime to \(\text{char}(K)\). Then

\[g^*|f|_\Lambda \cong |f'|_\Lambda\]

### 3 Higher Obstruction Theory

Let \(f: X \to Y\) be a morphism of schemes. If \(f\) has a section, then the relative étale topological type \(\text{Ét}(f)\) has a global section. Thus, obstructions to global sections of \(\text{Ét}(f)\) give rise to obstructions for sections of \(f\).

We will present obstruction theory in the context of higher topoi, starting with obstruction to a global section inside a given topos and then extending it to the case of an obstruction for the existence of a section for a morphism of topos. We describe the formalism of obstruction theory using the theory of gerbes, this will make it easy to understand how the obstruction behaves via pullback.
Remark 3.0.1. Throughout this section we only state theorems for dimensions $\geq 2$, as the theory of gerbes requires some refinement to work for lower dimension. This is due to the need for $\pi_n$ to be an abelian group.

We shall mention, however, that for the homological obstructions the theory naturally extend to those cases as well.

### 3.1 Obstruction Theory for Global Sections of Sheaves in $\infty$-topoi

Let $\mathcal{T}$ be an $\infty$-topos, and let $t$ be an object of $\mathcal{T}$. We ask whether $t$ has a global section, i.e. whether the terminal map $t \to *_{\mathcal{T}}$ has a section. Let’s consider the Postnikov tower (as in [3, def 5.5.6.23]) of $t$:

$$t \to \cdots \to P_{\leq n}t \to \cdots \to P_{\leq -1}t$$

from which we see that if there is a global section for $t$ then there is a global section for each $P_{\leq n}t$. As in the usual construction in topological obstruction theory, we assume that we have found a global section $\sigma_{n-1}: * \to P_{\leq n-1}t$ and we will give an obstruction for the existence of a compatible global section $\sigma_n$ making the following diagram commutative:

\[
\begin{array}{ccc}
P_{\leq n}t & \xrightarrow{\sigma_n} & * \\
\downarrow & & \downarrow \\
P_{\leq n-1}t & \xrightarrow{\sigma_{n-1}} & P_{\leq n-1}t.
\end{array}
\]

We do this as follows: Let $G_n$ be the pullback of the above diagram. The existence of a lift as above is equivalent to the existence of a section for the left downward arrow:

\[
\begin{array}{ccc}
G_n & \xrightarrow{\delta} & P_{\leq n}t \\
\downarrow & & \downarrow \\
* & \xrightarrow{\sigma_{n-1}} & P_{\leq n-1}t.
\end{array}
\]

The pullback $G_n$ is an $n$-gerbe banded over $\pi_n t$, as defined in [3, 7.2.2.20]. The picture of $G_n$ to keep in mind is as a sheaf which is locally an Eilenberg-MacLane sheaf, with homotopy group $\pi_n t$ at level $n$. Readers familiar with obstruction theory of topological spaces should be quick to recognize the above generalization.

By [3, 7.2.2.28], for $n \geq 2$ there is an equivalence between the set of equivalence classes of $n$-gerbes banded over a discrete abelian group object $A$ and the cohomology group $H^{n+1}_F(A)$. If $G$ is an $n$-gerbe banded by $A$, we denote by $[G]$ its corresponding cohomology class.

The association of a gerbe to a cohomology class can be described as follows. A class $\sigma \in H^{n+1}_F(A)$ is defined by a map

$$*_{\mathcal{T}} \xrightarrow{\sigma} K(A, n+1).$$
The fiber of the above map is the corresponding \( n \)-gerbe. Conversely, any \( \sigma \) can be fit uniquely as such a fiber

\[
\begin{array}{ccc}
G & \rightarrow & *_{\tau} \\
\downarrow & & \downarrow 0 \\
*_{\tau} & \xrightarrow{\alpha} & K(A, n + 1).
\end{array}
\]

**Definition 3.1.1.** Let \( n \geq 2 \), let \( t \in \mathcal{T} \) be an element of an \( \infty \)-topoi, and let \( \sigma_{n-1}: *_{\tau} \rightarrow P_{n-1}(t) \) be a section of the \( n-1 \)-st Postnikov filtration of \( t \). Let \( G_n \) denote the pull-back \( P_n \times P_{n-1} *_{\tau} \). Then we define the \( n+1 \)-st obstruction class for extending \( \sigma_{n-1} \), denoted by \( o_{n+1}(t, \sigma_{n-1}) \), as the class \([G_n]\) in \( H_{\tau}^{n+1}(\pi_n(t)) \).

**Warning 3.1.2.** Note the numbering conventions:

- The \( n + 1 \)-st homotopy obstruction corresponds to a \( n + 1 \)-th cohomology class.
- The \( n \) gerbe is \( n \)-truncated and \( n \)-connective.
- The gerbe is a pullback over a global section of the \( n - 1 \) postnikov truncation.

**Remark 3.1.3.** This construction of obstruction theory is consistent with the classical obstruction theory for lifting problems from classical topology.

**Remark 3.1.4.** We shall always consider a gerbe banded by an abelian group object of the topos of discrete sheaves, \( A \in Disc(\mathcal{T}) \), as a pair \((G, \phi)\) such that \( G \) is an \( n \)-gerbe and \( \phi: \pi_n(G) \xrightarrow{\sim} A \) is an isomorphism.

### 3.2 Functoriality Properties of the Obstructions

In this sub-section we wish to show that the obstruction \( o_{n+1}(t, \sigma_{n-1}) \) is functorial in \( t \), and also in \( \mathcal{T} \) w.r.t. geometric morphisms.

Let \( A, B \) be two abelian group objects in \( Disc(\mathcal{T}) \). Let \( f: A \rightarrow B \) be a homomorphism. Then we get an induced map \( f_*: H^k_T(A) \rightarrow H^k_T(B) \). This is evident from the description of the cohomology as spaces of maps to Eilenberg-MacLane objects, but we would like to describe this map in the language of gerbes.

**Proposition 3.2.1.** Let \( n \geq 2 \). Let \((G, \phi)\) be an \( n \)-gerbe banded by \( A \) and \((G', \phi')\) an \( n \)-gerbe banded by \( A' \). Let \( f: A \rightarrow A' \) be a map of group objects. Then there is a map \( g: G \rightarrow G' \) inducing \( f \) on \( \pi_n \), if and only if \( f_*[G] = [G'] \) in \( H_{\tau}^{n+1}(A') \).

**Proof.** If \( f_*[G] = [G'] \) then we have a commutative diagram

\[
\begin{array}{ccc}
\sigma & \xrightarrow{*_{\tau}} & \sigma' \\
\downarrow & & \downarrow \scriptstyle{K(f,n+1)} \\
K(A, n + 1) & \xrightarrow{K(f,n+1)} & K(A', n + 1)
\end{array}
\]
such that the fiber of \( \sigma \) is \( G \) and the fiber of \( \sigma' \) is \( G' \). Then we have an induced map on the fibers which clearly induce the map \( f \) on \( \pi_n \), by comparing the long exact sequences of homotopy groups associated to the morphism \( K(f, n + 1) \).

We shall prove the converse by passing to the associated complexes. Let \( C = P_{\leq n}C_*(G, \mathbb{Z}) \) and \( C' = P_{\leq n}C_*(G', \mathbb{Z}) \). The map \( g : G \to G' \) induces a map \( P_{\leq n}C_*(g) : C \to C' \), commuting with the augmentations. In particular, we get a morphism of exact triangles

\[
\begin{array}{ccc}
K & \longrightarrow & C \\
\downarrow & & \downarrow \\
K' & \longrightarrow & C' \\
\end{array}
\]

where \( K \) and \( K' \) are the fibers of the augmentations of \( C \) and \( C' \) respectively. The banding maps \( \phi : \pi_n G \to A \) and \( \phi' : \pi_n G' \to A' \) induces equivalences \( K \cong A[n] \) and \( K' \cong A'[n] \), via the Hurewicz isomorphism.

Shifting the exact triangles above and identifying \( K \) and \( K' \) with \( A[n] \) and \( A'[n] \) we get a commutative square, who’s commutativity follows from the assumption that \( g : G \to G' \) induces the map \( f \) on homotopy groups:

\[
\begin{array}{ccc}
Z & \longrightarrow & A[n + 1] \\
\downarrow & & \downarrow \\
Z & \longrightarrow & A'[n + 1].
\end{array}
\]

It remains to show that the morphisms \( \alpha \) and \( \alpha' \) correspond via the functor \( M \) to the morphisms \( \sigma : * \to K(A, n + 1) \) and \( \sigma' : * \to K(A, n + 1) \) classifying the cocycles \([G]\) and \([G']\) and thus we have \( \sigma' = f \circ \sigma \) so \( f_*[G] = [G'] \). The proof of this correspondence is postponed to the proof of Proposition 3.4.2.

We now use the preceding proposition to prove the functoriality of the obstruction class with respect to a morphism in \( \mathcal{T} \).

Let \( f : t \to t' \) be a morphism in an \( \infty \)-topos \( \mathcal{T} \). Consider an \( n - 1 \) section

\[
\sigma_{n-1}(t) : * T \to P_{\leq n-1}(t).
\]

Composing with \( P_{\leq n-1}(f) \) we get a section

\[
P_{\leq n-1}(f)_* \sigma_{n-1}(t) : * T \to P_{\leq n-1}(t').
\]

Let us denote \( \sigma_{n-1}(t') = P_{\leq n-1}(f)_* \sigma_{n-1}(t) \). Note that the fiber \( G_n \) of the map \( P_{\leq n}t \to P_{\leq n-1}t \) is an \( n \)-gerbe. We get a commutative diagram of Cartesian squares
Proposition 3.2.2. Let $f: t \to t'$ be a morphism in an $\infty$-topos $T$. Then

$$f_* o_{n+1}(t, \sigma_{n-1}) = o_{n+1}(t', P_{\leq n-1}(f)(\sigma_{n-1}))$$

Proof. This follows from Proposition 3.2.1 and the existence of the map $G_n(t) \to G_n(t')$.

Going on to explain the functoriality of the obstruction with respect to a geometric morphism, we shall first say a word about the map on cohomology induced by that geometric morphism. Indeed, if $f: T' \to T$ is a geometric morphism, and $\phi: *_{T'} \to K(A, n+1)$ represent some element in $H^{n+1}_T(A)$ then we have $f^* K(A, n+1) \cong K(f^* A, n+1)$ and $f^* *_{T'} \cong *_{T'}$, so we can regard $f^* \phi$ as representing a class in $H^{n+1}_T(f^* A)$. This construction gives us a pull-back map on cohomology associated with a geometric morphism.

The functoriality of the obstruction itself can now be checked easily. Since $f^*$ preserve finite limits, on the level of gerbes we obviously have $f^*[G] = [f^*G]$ (just compare the canonical fibrations of the corresponding Eilenberg-Maclane objects). Since $f^*$ also preserve Postnikov filtrations, we deduce the following:

Proposition 3.2.3. $f^* o_{n+1}(t, \sigma_{n-1}) = o_{n+1}(f^* t, f^* \sigma_{n-1})$.

3.3 Obstruction Theory for a Geometric Morphism

Let $f: T \to T'$ be a morphism of $\infty$-topoi. The relative topological type for $f$ is defined as an object of of $\text{Pro}(T')$, given by $f_* *_{T'}$. It is easy to extend the definition of the obstruction for sections to pro-objects. Indeed, the Postnikov tower of a pro-object is computed object-wise, so the fiber of the maps between
consecutive filtra is a pro-gerbe, hence defines a cohomology class with coefficients in the corresponding pro-homotopy group in the obvious way. Namely, if \( \{G_i\}_{i \in I} \) is a system of gerbes banded over \( \{A_i\} \) with compatibility maps between them, they corresponds to a compatible system of maps \( \{\tau \to K(A_i, n + 1)\} \) which is a pro-cohomology class by definition.

**Definition 3.3.1.** Let \( f: T \to T' \) be a geometric morphism of \( \infty \)-topoi. Let \( \sigma_{n-1}: \tau \to P_{\leq n-1}(f \tau) \). Then we denote by \( o_{n+1}(f, \sigma_{n-1}) \) the obstruction \( o_{n+1}(f \tau, \sigma_{n-1}). \)

**Remark 3.3.2.** The higher obstructions might depend on the chosen section, and the possible options are governed by the differentials of the Bausfield-Kan spectral sequence associated to the postnikov filtration of \( t \). We will be mostly interested in \( o_{n+1}(f, \sigma_{n-1}) \) in cases where \( f \tau \) is \( n \)-connected, and then the obstruction does not depend on the choice of section. In this case we shorten the notation and write \( o_{n+1}(f) \), omitting the section from the notation.

**Proposition 3.3.3.** As mentioned in the beginning of this section, if the morphism \( f: T \to T' \) admits a section then all the obstructions vanish.

**Proof.** Let \( s: T' \to T \) be a section, so that \( f \circ s = \text{id}_{T'} \). The terminal map \( s_\sharp(\tau) \to \tau \) gives a global section
\[
\tau = \text{id}_{\tau} = f s_\sharp(\tau) \to f_\sharp(\tau)
\]
and thus all the obstruction classes vanishes. \( \square \)

**3.4 Homological Obstruction Theory and Extensions of Sheaves**

Let \( T \) be an \( \infty \)-topos and \( \Lambda \) be a ring. Let \( \Lambda_T = I_\Lambda(\tau) \) denote the constant sheaf with value \( \Lambda \) in \( Shv_\infty(T, \Lambda) \). Similarly to the homotopical obstruction, we can define the homological obstruction to a section

**Definition 3.4.1.** For \( n \geq 2 \), let \( t \) be an object in \( T \). Consider \( t_\Lambda = R_\Lambda I_\Lambda t \), and let \( \sigma_{n-1}: \tau \to t_\Lambda \). We define the \( n+1 \)-st homological obstruction class for extending \( \sigma_{n-1} \) as the homotopical obstruction class \( o_{n+1}(t_\Lambda, \sigma_{n-1}) \).

It turns out that the lowest homological obstruction is given by an extension class in \( Ext^{n+1}_{Shv_\infty(T, \Lambda)}(\Lambda_T, H_n(\text{Et}(f))) \), which is the main claim of this subsection.

Let \( H \in Disc(Shv_\infty(T, \Lambda)) \) be a discrete sheaf of \( \Lambda \) modules over \( T \). Let \( \alpha \in Ext^{n+1}_{Shv_\infty(T, \Lambda)}(\Lambda_T, H) \). To \( \alpha \) we can attach a fiber sequence
\[
H[n] \to M_\alpha \to \Lambda_T,
\]
unique up to equivalence of fiber sequences.

The map \( R_\Lambda(M_\alpha) \to R_\Lambda(\Lambda_T) \) is an equivalence after applying the Postnikov truncation \( P_{n-1} \), as \( R_\Lambda \) preserves fibre sequences and because \( R_\Lambda H[n] \)
is $n-1$-connected. On the other hand, $P_{n-1}(R\Lambda(\Lambda_T))$ has a canonical section corresponding to the element $1 \in \Lambda$ hence inducing a section $\sigma_{n-1}: *_T \to P_{n-1}(R\Lambda(M_\alpha))$. Thus, we can define the obstruction class

$$o_n^\alpha := o_n(R\Lambda(M_\alpha), \sigma_{n-1}) \in H^{n+1}_T(\pi_0(R\Lambda(M_\alpha))) = H^{n+1}_T(H).$$

**Proposition 3.4.2.** In the notation above, for $n \geq 2$, the canonical isomorphism

$$H^{n+1}_T(H) \cong \Ext^{n+1}_{\Shv{T,\Lambda}}(\Lambda_T, H)$$

(6)

sends $o_\alpha$ to $\alpha$.

The proof is based on the following observation.

**Lemma 3.4.3.** Let $f: \Lambda \to H[n+1]$ be a map, classified by cohomology class $\beta \in \Ext^{n+1}_{\Shv{T,\Lambda}}(\Lambda_T, H)$. Then the composition

$$*_T \xrightarrow{1_\Lambda} R\Lambda(\Lambda_T) \xrightarrow{R\Lambda(f)} R\Lambda(H[n+1]) \cong K(H, n+1)$$

is classified by $\beta$, under the equivalence (6).

**Proof.** The identification (6) of $\pi_0(\Hom_T(*_T, K(H, n+1)))$ with $\Ext^{n+1}_{\Shv{T,\Lambda}}(\Lambda_T, H)$, is given by the adjunction isomorphism

$$\Hom_T(*_T, R\Lambda(H[n+1])) \cong \Hom_{\Shv{T,\Lambda}}(\Lambda \otimes *_T, H[n+1])$$

using the identifications $R\Lambda(H[n+1]) \cong K(H, n+1)$ and $\Lambda \otimes *_T = \Lambda_T$. The unit map for the above adjunction can be computed on the terminal object directly to be $*_T \xrightarrow{1_\Lambda} R\Lambda(\Lambda \otimes *_T) = R\Lambda(\Lambda_T)$. The composition

$$*_T \xrightarrow{1_\Lambda} R\Lambda(\Lambda) \xrightarrow{R\Lambda(f)} K(H, n+1)$$

can therefore be identified with the composition

$$*_T \xrightarrow{1_\Lambda} R\Lambda(\Lambda \otimes *_T) \xrightarrow{R\Lambda(f)} R\Lambda(H[n+1]),$$

which is exactly the morphism corresponding to

$$f \in \Hom_{\Shv{T,\Lambda}}(\Lambda \otimes *_T, H[n+1])$$

via the adjunction $I_\Lambda \dashv R\Lambda$, and thus to the class $\beta$. \qed

**Proof of Proposition 3.4.2.** First of all, the fiber sequence

$$H[n] \to M_\alpha \to \Lambda_T$$

induces by shifting a fiber sequence

$$M_\alpha \to \Lambda_T \to H[n+1].$$
The homotopy class of the map $\Lambda T \to H[n + 1]$ above is classified by $\alpha$.

Applying the functor $R_\Lambda$ to this fiber sequence we get a pullback diagram

$$
\begin{array}{ccc}
R_\Lambda(M_\alpha) & \longrightarrow & K(\Lambda, 0) \\
\downarrow & & \downarrow \\
* \tau & \longrightarrow & K(H, n + 1)
\end{array}
$$

Note that the map $* \tau \to K(H, n + 1)$ in the diagram is the canonical base-point, i.e. classified by the trivial cocycle, since it is the $R_\Lambda$ of the zero map $0 \to H[n + 1]$.

Consider now the map $1_\Lambda : * \tau \to R_\Lambda(\Lambda T)$, picking the point 1. We can extend the pullback square above to the diagram

$$
\begin{array}{ccc}
G_\alpha & \longrightarrow & * \tau \\
\downarrow & & \downarrow 1_\Lambda \\
R_\Lambda(M_\alpha) & \longrightarrow & K(\Lambda, 0) \\
\downarrow & & \downarrow \\
* \tau & \longrightarrow & K(H, n + 1)
\end{array}
$$

in which both of the squares are pullback squares. By the definition of the cocycle $o_\alpha^n$, it remains to show that the composition

$$
* \tau \xrightarrow{1_\Lambda} K(\Lambda, 0) \rightarrow K(H, n + 1)
$$

is classified by the cocycle $\alpha$. Indeed, in this case $G_\alpha$, which is the gerbe classified by $o_\alpha^n$, is the equalizer of the canonical section of $K(H, n + 1)$ and the section classified by $\alpha$, hence classified by $\alpha$ by [3, Proposition 7.2.2.8]. This last result follows directly from Lemma 3.4.3. \qed

4 Quadratic Bundles

In this section we calculate and recall the properties of the universal quadratic bundle that are needed for the final computation of the obstruction. The notation and definitions will be the same as in [10].

4.1 The Universal Quadratic Bundle

Let $BG$ denote the simplicial variety defined over a field $K$ of characteristic $\neq 2$ given by the standard Milnor realization, namely $BG_k = G^k$ with the usual face and degeneracy maps. It is known that $BG$ classifies étale -locally trivial principle $G$ bundles. In our case, for $G = O_n$, we claim that $BG$ presents the stack classifying quadratic bundles.
Definition 4.1.1. Let $X$ be a scheme over $K$. A quadratic bundle of rank $n$ over $X$ is a locally free sheaf $\mathcal{E}$ over $X$ of rank $n$ together with a non-degenerate pairing $B: \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$ of sheaves of $\mathcal{O}_X$ modules.

A quadratic bundle of rank $d$ is called trivial if it is isomorphic to the quadratic bundle $(\mathcal{O}_X^n, \sum_i x_i^2)$. The stack of locally trivial quadratic bundle is represented by the simplicial scheme $BO_n$. To see this, note that there is an equivalence between the groupoid of locally trivial quadratic bundles on a scheme $S$ and principal $BO_n$-bundles on $S$. Indeed, let $V_n$ be the standard quadratic representation of $BO_n$. If $P$ is a principal $O_n$ bundle then $P \times_{\mathcal{O}_n} V$ is a quadratic bundle. Conversely, if $(\mathcal{E}, B)$ is a quadratic bundle, then the collection of orthonormal frames of $\mathcal{E}$ is a principal $O_n$-bundle, locally trivial if $\mathcal{E}$ is. These two constructions give an equivalence of the functors on schemes given by locally trivial quadratic bundles and $BO_n$.

Lemma 4.1.2. Every quadratic bundle is locally trivial in the étale topology.

Proof. Let $(\mathcal{E}, B)$ be a quadratic bundle over $X$. Then Zariski locally, $\mathcal{E}$ is a trivial vector bundle. Thus, we may assume without lost of generality that $\mathcal{E} \cong \mathcal{O}_X^n$ and $B$ is some symmetric matrix of functions, which is everywhere non-degenerate. Choose a closed point $x \in X$. The reduction to $k(x)$ of $B$ can be diagonalized. Choose a matrix $\tilde{A}$ such that $\tilde{A}B|_{k(x)}\tilde{A}^T$ is diagonal, and lift $\tilde{A}$ arbitrarily to a neighborhood of $x$. We may assume that $A$ is everywhere defined, by shrinking $X$. Then, since $B_{1,1}$ is invertible at $x$ and by shrinking $X$ further, we may assume that $B_{1,1}$ is invertible. By applying row and column operations we can now eliminate all the other entries in the first row and column of $B$.

Applying the same procedure to the first minor of $B$, and inductively to all matrices $\{B_{i,j}\}_{i,j \geq k}$ we can find some Zariski Neighborhood $X'$ and invertible matrix $A'$ of functions on $X'$ such that $A'B(A')^T$ is diagonal, say $A'B(A')^T = \text{diag}(f_1, ..., f_n)$. Hence, we can assume that $B$ itself is the diagonal matrix $\text{diag}(f_1, ..., f_n)$. Since $B$ is non-degenerate, the $f_i$-s are invertible. Since $\text{char}(K) \neq 2$, we can add $\sqrt{f_i}$ to $\mathcal{O}_X X$ to get an étale cover of $X$. Formally, we consider $\Gamma = \{(x, y_1, ..., y_n) \in X \times K^n : y_i^2 = f_i(x)\}$ and the projection $p_X: \Gamma \rightarrow X$ is an étale cover. Moreover, $p_X^*(B) = \text{diag}(y_1^2, ..., y_n^2) \equiv \text{diag}(1, ..., 1)$ via the matrix $\text{diag}(1/y_1, ..., 1/y_n)$. \hfill $\square$

Corollary 4.1.3. $BO_n$ is the stack classifying quadratic bundles.

Corollary 4.1.4. $\text{Hom}(\text{Spec}(K), BO_n)$ is equivalent to the groupoid of non-degenerate quadratic forms of rank $n$ over $K$.

Let $S^n$ denote the standard $n$-sphere over $K$. If $B$ is a quadratic form, let $S_B$ denote the unit sphere of $B$, given by

$$S_B = \{v \in \mathbb{A}^{n+1} : B(v, v) = 1\}.$$ 

Then $S^n = S_{x_0^2 + \cdots + x_n^2}$.
A similar argument to the proof of Lemma 4.1.2 shows that the quotient stack \( O_{n+1} \backslash S^n \) classifies quadratic bundles with an orthonormal section. The groupoid of such data, \((\mathcal{E}, B, \sigma)\), is canonically equivalent to the groupoid of quadratic bundles of rank \( n \), via the rank \( n \) quadratic bundle \((\mathcal{E}, B, \sigma) \mapsto (\sigma^\perp, B|_{\sigma^\perp})\). Thus, we have an equivalence

\[
O_{n+1} \backslash S^n \cong BO_n.
\]

To distinguish the quotient stacks from the objects they classify, we denote by \( Q_{n+1} \) the stack of quadratic rank \( n + 1 \) bundles, and by \( Q_{n+1}^S \cong Q_n \) the stack of quadratic bundles with a length 1 section. They are indeed stacks, as they are equivalent to stacks represented by \( BO_n \), as displayed in the following commutative diagram

\[
\begin{array}{ccc}
Q_n & \cong & Q_{n+1}^S \\
\downarrow & & \downarrow \\
O_{n+1} \backslash S^n & \cong & BO_n \\
\downarrow & & \downarrow \\
Q_{n+1} & \cong & BO_{n+1}
\end{array}
\]  

(7)

**Proposition 4.1.5.** Let \( X \overset{f}{\to} Q_n \) be a morphism classifying a quadratic bundle \((\mathcal{E}, B)\). Then the pullback \( f^* Q_{n+1}^S \) is the sphere bundle of \((\mathcal{E}, B)\) over \( X \). In particular, the morphism \( Q_{n+1}^S \to Q_n \) is schematic.

**Proof.** A morphism \( g : Y \to S_B(\mathcal{E}) \) determines a morphism \( \bar{g} : Y \to S_B(\mathcal{E}) \to X \). Since \( \mathcal{E} \) admits a canonical unit section over \( S_B(\mathcal{E}) \), its pullback along \( g \) determines a unit section of \( E \). Conversely, if \( \bar{g} : Y \to X \) is a morphism and \( \sigma \) is a unit section of \( \bar{g}^* \mathcal{E} \), then we get a map \( g : Y \to S_B(\mathcal{E}) \), since pullback of vector bundles commutes with formation of total space in a way compatible with unit sections.

Hence, \( S_B(\mathcal{E}) \) classifies pairs of a morphism to \( X \) and a unit section of the pullback of \( \mathcal{E} \). This is by definition the groupoid classified by the pullback \( X \times_{Q_n} Q_{n+1}^S \).

\( \square \)

**Corollary 4.1.6.** Let \( B \) be a quadratic form over \( K \), then we have a pullback diagram

\[
\begin{array}{ccc}
S_B & \to & Q_{n+1}^S \\
\downarrow & & \downarrow \\
\text{spec}(K) & \to & Q_n
\end{array}
\]

### 4.2 Stiefel-Whitney Classes and the Cohomology of \( BO_n \)

Here we recall the cohomological properties of \( BO_n \) without proofs, as computed in [11].

The étale cohomology of \( BO_n \) with \( \mathbb{Z}/2 \) coefficients is the tensor product of the geometric and the arithmetic cohomologies.
Theorem 4.2.1 ([11, Thm. 1]). Let $K$ be a field of characteristic $\neq 2$. And let $A$ denote the mod 2 Galois cohomology ring $\mathbb{H}_c^*(K, \mathbb{Z}/2)$ of $K$. Then there is an isomorphism of graded algebras of the form

$$H^*_c(BO_n, \mathbb{Z}/2) \cong A[HW_1, \ldots, HW_n]$$

where the polynomial generator $HW_i$ has degree $i$.

Remark 4.2.2. The classes $HW_i$ get their notation from ”Hasse-Witt” invariants, which are the arithmetic analogue of the Stiefel Whitney classes, associated with bilinear forms.

Furthermore, Whitney’s product formula holds. In this paper we will only need the special case of the product formula associated with the diagonal embedding $\Delta: BO_1^n \to BO_n$, which is a consequence of the proof of the above theorem. Using the Kunneth formula, it is easy to show that

$$H^*_c(BO_1^n, \mathbb{Z}/2) = A[HW_1^{(0)}, \ldots, HW_1^{(n-1)}]$$

where $HW_1^{(i)}$ are all in degree 1, the pullbacks of $HW_i$ under the projections $BO_1^n \to BO_1$, and $A$ is the mod 2 cohomology of $K$, as above.

Theorem 4.2.3 ([11, Thm. 2.13]). In the notations above,

$$\Delta^*(HW_i) = \sigma_i(HW_1^{(0)}, \ldots, HW_1^{(n-1)})$$

where $\sigma_i$ is the $i$’th symmetric polynomial in $n$ variables.

Corollary 4.2.4. In particular, $\Delta^*(HW_n) = HW_1^{(0)} \cup \cdots \cup HW_1^{(n-1)}$.

The cohomology groups behave in the expected way for the canonical morphism $BO_m \to BO_n$.

Theorem 4.2.5. Let $m < n$, and consider the map $BO_m \to BO_n$, given by the standard embedding of the orthogonal groups. Then the corresponding map of cohomologies is given by

$$A[HW_1, \ldots, HW_n] \to A[HW_1, \ldots, HW_m]$$

$\forall i \leq m. \ HW_i \to HW_i$

$\forall i > m. \ HW_i \to 0$.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
H^*_c(BO_n, \mathbb{Z}/2) & \rightarrow & H^*_c(BO_1^n, \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
H^*_c(BO_m, \mathbb{Z}/2) & \rightarrow & H^*_c(BO_1^m, \mathbb{Z}/2).
\end{array}
$$
The right downward arrow, defined by the embedding of \( BO_1^m \) to the first \( m \) factors of \( BO_1^n \), is the morphism of polynomial rings

\[
A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_m]
\]

given by sending \( x_i \) to \( x_i \) if \( i \leq m \) or to 0 otherwise. It remains to observe that this projection sends \( \sigma_i(x_1, \ldots, x_n) \) to \( \sigma_i(x_1, \ldots, x_m) \) if \( i \leq m \) and to 0 otherwise.

### 4.3 The Relative Homological Type of a Sphere Bundle

The aim of this subsection is to calculate the relative étale homological type for the projection \( p_n : Q^n_S \to Q^n \) of stacks over \( K \).

We will show the following arithmetic version of classical obstruction theoretic interpretation of the Stiefel-Whitney classes:

**Theorem 4.3.1.** Let \( p_n : Q^n_S \to Q^n \) denote the natural projection. Then the sheaf of \( \mathbb{Z}/2 \) modules \( \mathbb{Z}/2 \otimes \text{Et}(p_n) \) is the extension of \( \mathbb{Z}/2 \) by \( \mathbb{Z}/2[n-1] \) classified by the Hasse-Witt class

\[
HW_n \in H^0_{\text{ét}}(Q^n, \mathbb{Z}/2) \cong \text{Hom}_{\text{Sh}_{\infty}}((Q^n)_\alpha, \mathbb{Z}/2)(\mathbb{Z}/2, \mathbb{Z}/2[n]) \cong \text{Ext}^1_{\text{Sh}_{\infty}}((Q^n)_\alpha, \mathbb{Z}/2)(\mathbb{Z}/2, \mathbb{Z}/2[n-1]).
\]

The proof of the first part of this result, that \( \mathbb{Z}/2 \otimes \text{Et}(p_n) \) is an extension as above, is done by a dévissage argument starting from the homotopy type of the standard sphere, going through a trivial bundle over a scheme, and then proven for \( Q^n \) by writing it as a colimit of schemes with a trivial quadratic bundle.

We start by analyzing the case of the trivial sphere bundle over the point.

**Proposition 4.3.2.** The fibre of the terminal map from the relative homological realization over \( K \) of the standard sphere \( \text{Et}_{/K}(S^{n-1}) \) to \( \mathbb{Z}/2 \) is

\[
\mathbb{Z}/2[n-1] \to \text{Et}_{/K}(\mathbb{Z}/2 \otimes S^{n-1}) \to \mathbb{Z}/2.
\]

**Proof.** Consider the fibre \( \mathcal{G} \) of the terminal map \( \text{Et}_{/K}(\mathbb{Z}/2 \otimes S^{n-1}) \to \mathbb{Z}/2 \).

We shall show that \( \mathcal{G} \) is isomorphic to \( \mathbb{Z}/2[n-1] \). Since the group \( \mathbb{Z}/2 \) has no non trivial automorphisms, it suffices to check that \( \mathcal{G} \cong \mathbb{Z}/2[n-1] \) after base change to the separable closure \( \overline{K} \). This result will follow if we can show that the étale homology of \( \overline{\mathcal{G}} \) is concentrated in degree \( n-1 \), and equal to \( \mathbb{Z}/2 \) at this degree.

For the calculation of the étale homology groups over \( \overline{K} \), it is enough to calculate the étale cohomology groups of the sphere \( S^{n-1} \), by the universal coefficient theorem. In [12, Exp. XII] it is proven that the reduced étale cohomology groups of the spheres with coefficients in \( \mathbb{Z}_2 \) consists of one copy of \( \mathbb{Z}/2 \) concentrated in degree \( n-1 \), and thus it also follows for étale homology with coefficients in \( \mathbb{Z}/2 \) by the universal coefficients theorem.

The next case is for trivial bundles over \( K \)-schemes.
Proposition 4.3.3. Let $X$ be a scheme over $K$, and let $(E, B)$ be a trivial quadratic bundle of rank $n$ on $X$. Let $p_B : S_B(E) \to X$ denote the projection from the sphere bundle of $B$ to the base $X$. Then the fiber of the natural map $\mathbb{Z}/2 \otimes \text{Ét}(p_B) \to \mathbb{Z}/2$ is equivalent to the constant sheaf $\mathbb{Z}/2[n - 1]$.

Proof. In this case $S_B(E) \cong X \times S^{n-1}$ as a scheme over $X$. Thus, the problem reduces to Proposition 4.3.2 via the smooth base change theorem 2.4.1, as we now explain. Consider the diagram

\[
\begin{array}{ccc}
X \times S^{n-1} & \xrightarrow{g} & S^{n-1} \\
\downarrow q & & \downarrow p \\
X & \xrightarrow{f} & *.
\end{array}
\]

Note that the map $p$ is smooth, as $\text{char}(K) \neq 2$, and thus the smooth base change theorem holds in this case.

By Corollary 2.4.6 we get

$$\text{Ét}(q) \otimes \mathbb{Z}/2 \cong f^*\text{Ét}(p) \otimes \mathbb{Z}/2.$$ 

This concludes the reduction, as $f^*$ preserves fibre sequences and sends constant sheaves to constant sheaves.

Finally, we are ready to consider the relative homological type of the universal sphere bundle. First, we show that the sheaf $\mathbb{Z}/2 \otimes \text{Ét}(p_n)$ is indeed an extension of $\mathbb{Z}/2[n - 1]$ by $\mathbb{Z}/2$.

Proposition 4.3.4. Let $p_n : Q^S_n \to Q_n$ denote the natural projection. Then $\mathbb{Z}/2 \otimes \text{Ét}(p_n)$ is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2[n - 1]$.

Proof. Let $O^\bullet_n : \Delta^{op} \to \text{Sch}/K$ denote the simplicial diagram giving the Milnor realization of $BO_n$. Let $p_n^\bullet : O^\bullet_{n-1} \to O^\bullet_n$ denote the standard inclusions. Then $p_n : Q^S_n \to Q_n$ is isomorphic to the colimit $\varinjlim p_n^\bullet$.

Let $\rho_n^\bullet : O^\bullet_n \to Q_n$ and $\rho^S_n : O^\bullet_{n-1} \to Q^S_n$ denote the comparison maps. Let $\tilde{\rho} : \text{Shv}_\infty((Q_n)_{et}) \rightleftarrows \text{OpLaxShv}_\infty(O^\bullet_n) : \tilde{\rho}^\natural$ be the induced map on sheaves (as in the proof of Theorem 2.2.1). Similarly, for $\rho^S_n$ let $(\tilde{\rho}^S_n)^\ast$ and $\tilde{\rho}^S_n$ be the corresponding adjoint functors.

It follows from Theorem 2.2.1 that the natural map $\mu : \tilde{\rho}_T(p_n)_{lax, \sharp}(\tilde{\rho}^S_n)^\ast \to (p_n)_{lax, \sharp}$ is an equivalence at $\ast Q^\natural_n$, namely

$$\mu : \tilde{\rho}_T(p_n)_{lax, \sharp}(\tilde{\rho}^S_n)^\ast \ast Q^\natural_n \cong (p_n)_{lax, \sharp} = \text{Ét}(Q^S_n).$$

The object $\tilde{\rho}^\ast Q_n$ is final in $\text{OpLaxProShv}_\infty(O^\bullet_n)$ and hence there is a unique map $\text{Aug} : (p_n)_{lax, \sharp}(\tilde{\rho}^S_n)^\ast \ast Q^\natural_n \to \tilde{\rho}^\ast Q_n$ in $\text{OpLaxProShv}_\infty(O^\bullet_n)$ which restrict to the terminal map $\text{Ét}(p_n[k]) \to *_{Q^\natural_n[k]}$ at every $[k] \in \Delta^{op}$. Torsoring $\text{Aug}$ with $\mathbb{Z}/2$ we obtain a map

$$\text{Aug}_{\mathbb{Z}/2} : (p_n)_{lax, \sharp}(\tilde{\rho}^S_n)^\ast \mathbb{Z}/2 \cong \mathbb{Z}/2 \otimes (p_n)_{lax, \sharp}(\tilde{\rho}^S_n)^\ast \ast Q^\natural_n \cong \tilde{\rho}^\ast \mathbb{Z}/2$$

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where the first and last equivalences are the ones in 2.3.7. Let $F$ denote the fiber of $\text{Aug}_{\mathbb{Z}/2}$, so we have a fiber sequence

$$F \longrightarrow (p_n)_{\text{lax},Z}((\hat{\rho}^S)^*\mathbb{Z}/2)^{\text{Aug}_{\mathbb{Z}/2}} \longrightarrow \hat{\rho}^*\mathbb{Z}/2.$$ 

Applying $\hat{\rho}_2$ to this sequence we get a fiber sequence

$$\hat{\rho}_2F \longrightarrow \hat{\rho}_2(p_n)_{\text{lax},Z}((\hat{\rho}^S)^*\mathbb{Z}/2)^{\text{Aug}_{\mathbb{Z}/2}} \longrightarrow \hat{\rho}_2\hat{\rho}^*\mathbb{Z}/2.$$ 

The result will now follow from this fiber sequence once we show that $\hat{\rho}_2\hat{\rho}^*\mathbb{Z}/2 \cong \mathbb{Z}/2$ in $\text{Pro} \mathcal{S}h_{\infty}(Q_n, \mathbb{Z}/2)$ and that $\hat{\rho}_2F \cong \mathbb{Z}/2[n-1]$.

For the first, let $\text{id} : Q_n \rightarrow Q_n$ denote the identity map, we get from Theorem 2.2.1 that

$$\mu : \text{id} = \text{id} \rightarrow \hat{\rho}_2\hat{\rho}_2\hat{\rho}^* \cong \hat{\rho}_2\hat{\rho}^*$$

is an equivalence at $*_{Q_n}$ and hence it is also an equivalence after taking the tensor product with $\mathbb{Z}/2$.

We pass the the second equivalence $\hat{\rho}_2F \cong \mathbb{Z}/2[n-1]$. Note that by the Smooth Base-change Theorem (2.4.5), after tensoring with $\mathbb{Z}/2$ the object $(p_n)_{\text{lax},Z}((\hat{\rho}^S)^*\mathbb{Z}/2)_{\text{lim}}$ lies in the limit. In other words, by Theorem 2.1.7, this object lies in the full-subcategory $\lim \text{Pro} \mathcal{S}h_{\infty}(O_n^*, \mathbb{Z}/2)$ of $\text{OpLax Pro} \mathcal{S}h_{\infty}(O_n^*, \mathbb{Z}/2)$.

Since the same is true for $\hat{\rho}^*\mathbb{Z}/2$, we see that $F$ lies also in $\lim \text{Pro} \mathcal{S}h_{\infty}(O_n^*, \mathbb{Z}/2)$. Moreover, by Proposition 4.3.3, the value of $F$ at every $O_n^{[k]}$ is equivalent to $\mathbb{Z}/2[n-1]$. In particular, $F[1-n]$ has value $\mathbb{Z}/2$ at every object $O_n^{[k]}$, and hence it is equivalent to $\hat{\rho}^*\mathbb{Z}/2$ since $\mathbb{Z}/2$ has no nontrivial automorphisms. Hence

$$\hat{\rho}_2F \cong \hat{\rho}_2\hat{\rho}^*\mathbb{Z}/2[n-1] \cong \mathbb{Z}/2[n-1],$$

using again the equivalence $\hat{\rho}_2\hat{\rho}^* \cong \text{id}$ at objects of $\mathcal{S}h_{\infty}((Q_n)_{\text{et}})$.

To conclude the proof of Theorem 4.3.1, we will see that this extension is classified by $\text{HW}_n$. Consider the fiber sequence

$$\mathbb{Z}/2[n-1] \rightarrow \mathbb{Z}/2 \otimes \text{Et}(p_n) \rightarrow \mathbb{Z}/2.$$ (8)

Recall that sheaf cohomology with coefficients in $\mathbb{Z}/2$ is computed by applying $\text{Hom}_{\mathcal{S}h_{\infty}((Q_n)_{\text{et}}, \mathbb{Z}/2)}(\_ , \mathbb{Z}/2)$, abbreviated by $\text{Hom}_{Q_n}(\_ , \mathbb{Z}/2)$, on the abelian sheaf and taking the corresponding homotopy group ($\mathbb{Z}/2$-modules) of the resulting Hom-spaces. Applying the above to the fiber sequence 8, we get a long exact sequence of the form

$$\cdots \rightarrow \pi_{n-1}(\text{Hom}_{Q_n}(\mathbb{Z}/2[n-1], \mathbb{Z}/2)) \rightarrow \pi_n(\text{Hom}_{Q_n}(\mathbb{Z}/2, \mathbb{Z}/2)) \rightarrow \pi_n(\text{Hom}_{Q_n}(\mathbb{Z}/2 \otimes \text{Et}(p_n), \mathbb{Z}/2)) \rightarrow \cdots$$

However, by adjunction we have

$$\text{Hom}_{Q_n}(\mathbb{Z}/2 \otimes \text{Et}(p_n), \mathbb{Z}/2) \cong \text{Hom}_{Q_n}(\mathbb{Z}/2, \mathbb{Z}/2)$$

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and the resulting map

\[ H^n_{\text{ét}}(\mathbb{Q}^n, \mathbb{Z}/2) = \pi_n(\text{Hom}_{\mathbb{Q}^n}(\mathbb{Z}/2, \mathbb{Z}/2)) \to \]

\[ \to \pi_n(\text{Hom}_{\mathbb{Q}^n}(\mathbb{Z}/2 \otimes \text{Ét}(p_n), \mathbb{Z}/2)) \cong \]

\[ \cong \pi_n(\text{Hom}_{\mathbb{Q}^n}(\mathbb{Z}/2, \mathbb{Z}/2)) \cong \]

\[ \cong H^0_{\text{ét}}(Q^S_n, \mathbb{Z}/2) \]

corresponds to the pullback map induced on cohomology by \( p_n \).

Recall that

\[ H^*_{\text{ét}}(\mathbb{Q}^n, \mathbb{Z}/2) \cong A[H_{W1}, \ldots, H_{Wn}] \]

while

\[ H^*_{\text{ét}}(\mathbb{Q}^S_n, \mathbb{Z}/2) \cong H^*_{\text{ét}}(\mathbb{Q}^{n-1}, \mathbb{Z}/2) \cong A[H_{W1}, \ldots, H_{Wn-1}] \]

Moreover, the induced map on cohomology, \( p_n^* \), is given by reduction modulo the ideal generated by \( HW_n \). In particular, the kernel of \( p_n^* \) in degree \( n \) is one dimensional and generated by \( HW_n \) as a subspace.

Also, we clearly have

\[ \pi_{n-1}(\text{Hom}_{\mathbb{Q}^n}(\mathbb{Z}/2[n-1], \mathbb{Z}/2)) \cong \pi_0(\text{Hom}_{\mathbb{Q}^n}(\mathbb{Z}/2, \mathbb{Z}/2)) \cong \mathbb{Z}/2 \]

since \( Q_n \) is connected.

Thus, the portion on the long exact sequence (9) presented above is isomorphic to

\[ \ldots \to \mathbb{Z}/2 \xrightarrow{\delta} (A[H_{W1}, \ldots, H_{Wn}])^n \mod HW_n \to (A[H_{W1}, \ldots, H_{Wn-1}])^n \to \ldots \]

and exactness of the sequence now forces the map \( \delta \) to be given by \( \delta(1) = HW_n \).

It follows that the extension (8) is classified by \( HW_n \), proving Theorem 4.3.1.

5 Obstructions for Arithmetic Spheres

Carrying on to compute the obstruction classes, we shall start with dimensions 0,1. These obstructions are computed directly using simple computations interesting in their own right, and can be read independently of the rest of this paper.

After that the computation for dimensions \( \geq 2 \) is carried on using the full machinery developed in the sections above. It is interesting to note that the theorems proved earlier in this paper could be extended to allow for a proof of low dimensions, as the theory of gerbes behaves properly in low dimensions for abelian sheaves and we are dealing with the homological obstruction.

5.1 Obstruction Theory for Sphere Bundles of Dimensions 0,1

The computations below only rely on the definitions given in [1]. Rigour will be diminished for the benefit of clarity and brevity.
Proposition 5.1.1 (Case of 0-dimensional sphere). Let $a \in \mathbb{K}^\times$. The first étale $\mathbb{Z}/2$-homology obstruction for the existence of a rational point on $ax^2 = 1$ is precisely the class $[a]$.

Proof. As in [1], the relative étale homotopy type is given by taking the absolute étale homotopy type after base change to an algebraic closure, and remembering the Galois action. Then, the homotopy obstruction is obtained by the usual obstruction for the existence of a fixed point for the Galois action.

In our case, the étale homotopy type is equivalent to 2 points with a Galois action similar to the action on $\{\sqrt{a}, -\sqrt{a}\}$. Thus, the homotopy obstruction is given by arbitrarily selecting a point and considering the map

$$\Gamma_K \to \{\sqrt{a}, -\sqrt{a}\}$$

$$\sigma \mapsto \sigma \sqrt{a},$$

which is indeed an element in $H^1(\text{Spec} \mathbb{K}, \{\sqrt{a}, -\sqrt{a}\})$. Therefore, after passing to the homology obstruction, we get exactly the morphism $\Gamma_K \to \mathbb{Z}/2$ corresponding to the class $[a]$. \hfill \square

Proposition 5.1.2 (Case of 1-dimensional sphere). Let $a, b \in \mathbb{K}^\times$. The second étale $\mathbb{Z}/2$-homology obstruction for the existence of a rational point on the sphere $X$ defined by the quadratic equation $ax^2 + by^2 = 1$ is precisely the class $[a] \cup [b]$.

Proof. In [1], the relative étale homotopy type of $X$ over $K$ is defined by the inverse system over all hypercovers $U_\bullet \to X$ of the component simplicial sets $\pi_0 U_\bullet$, obtained by base-change to the separable closure and taking the $\Gamma_K$ set of connected components levelwise. In [2], the relationship between this definition of the relative étale homotopy type and the one given in this paper is discussed. Homotopical questions will get the same answer in both cases, and in particular the obstruction class over a field can be computed by choosing appropriate hypercovers.

Let $U_\bullet \to X$ be a hypercover and denote by $\overline{U_\bullet}$ its base-change to the separable closure. Considering the Galois action on the set of connected components, we have a $\Gamma_K$-equivariant map $\text{Et}/K(X) \to \pi_0(\overline{U_\bullet}) = |U_\bullet|$, and in particular it induces a map $\pi_n \text{Et}/K(X) \to \pi_n |U_\bullet|$. Moreover, the induced map

$$H^{n+1}(\Gamma_K, \pi_n \text{Et}/K(X)) \to H^{n+1}(\Gamma_K, \pi_n |U_\bullet|)$$

maps the obstructions for rational point of $X$, to the obstructions for homotopy fixed point in $|U_\bullet|$. For the special case $n = 1$, the map $\pi_1 \text{Et}/K(X) \to \pi_1 |U_\bullet|$ should be interpreted as morphism of fundamental groupoids.

We will pick a specific hypercover $U_\bullet$ and compute the fundamental groupoid of $\overline{U_\bullet}$ together with the $\Gamma_K$-action, and show that it is a $\mathbb{Z}/2\mathbb{Z}$-gerbe, in the sense that it is a groupoid with $\Gamma_K$-action, which is non-equivariantly equivalent to $B\mathbb{Z}/2\mathbb{Z}$. Hence, by the simple observation that the morphism $H_1(\text{Et}/K(X), \mathbb{Z}/2\mathbb{Z}) \to H_1(|U_\bullet|, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, we see that we have an isomorphism

$$H^2(\Gamma_K, \pi_1(\mathbb{Z}/2 \otimes \text{Et}/K(X))) \cong H^2(\Gamma_K, \pi_1 |U_\bullet|) \cong H^2(\Gamma_K, \mathbb{Z}/2).$$
It remains to show that this gerbe is classified by the cocycle \([a] \cup [b]\). Consider the field extension \(L = K[\sqrt{a}, \sqrt{-b}]\) of \(K\). We’ll assume throughout the remainder of this proof that this extension is of degree 4, as well as assuming \(i = \sqrt{-1} \in K\), both assumptions are not needed for the conclusion to hold but simplify the proof while the proofs for the general case only require small adjustments from the proof below and are left to the interested reader. This extension is Galois, with Galois group \(\Gamma_{L/K} = \mathbb{Z}/2 \oplus \mathbb{Z}/2\). We shall build the hypercover with respect to this field, and the obstruction class will be seen to reside in \(H^2(\Gamma_{L/K}, \mathbb{Z}/2) \subset H^2(\Gamma_{K}, \mathbb{Z}/2)\).

Observe that \(X\) is a twist of \(\mathbb{G}_m\), trivialized over \(L\). We construct a degree 8 cover \(U \to X\) by taking the Weil restriction from \(L\) to \(K\) of the double cover \(\mathbb{G}_m\) \(z \mapsto z^2 \to \mathbb{G}_m\).

Concretely, we can write
\[
(z, s, t) \quad U = \text{Spec } K[z, s, t]/(s^2 - a, t^2 + b)
\]
\[
(z^2 + z^{-2}, \frac{z^2 - z^{-2}}{2}) \quad X = \text{Spec } K[x, y]/(ax^2 + by^2 - 1).
\]

This cover can be extended to a hypercover \(U_\bullet \to X\) by taking the Čech nerve, so that \(U_\bullet\) is a simplicial variety given by
\[
U_n = \mathcal{U} \times_X \cdots \times_X \mathcal{U} = \text{Spec } K[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}, s_1, t_1, \ldots, s_n, t_n]/(s^2 - a, t^2 + b, z_i^2 - (\frac{s_i}{s_j} + \frac{t_i}{t_j}) \frac{z^2}{2} - (\frac{s_i}{s_j} - \frac{t_i}{t_j}) \frac{z^{-2}}{2}).
\]

The 0 dimensional skeleton of \([U_\bullet]\) consists of 4 points, each point is indexed by one solution of \(s, t\) to \(s^2 = a\) and \(t^2 = -b\). The Galois group \(\Gamma_{L/K}\) acts on the set of points naturally.

The edges of \([U_\bullet]\) (the connected components of \(\mathcal{U} \times_X \mathcal{U} = \mathcal{U} \times_X \mathcal{U}\)) are grouped into pairs, so that any two vertices have two edges among them for each direction. For two points indexed by \(s_1, t_1\) and \(s_2, t_2\), the two morphisms from the former to the latter are given by the two solution to
\[
z_1^2 = \frac{s_1 + t_1}{s_2 + t_2} \frac{z^2}{2} + \frac{s_1 - t_1}{s_2 - t_2} \frac{z^{-2}}{2},
\]
so that
\[
z_1 \in \{\pm \frac{z^1}{s_2}, \pm iz \pm 1\}
\].

For example, if \(s_2 = -s_1\) and \(t_2 = t_1\) then we have the two morphisms indexed by \(z_1 = \pm iz_2^{-1}\), which we think of as the morphisms \(z \mapsto \pm iz^{-1}\). The Galois group acts on the edges by taking morphisms to morphisms with the same "name" (the same choice of solution \(z_1 = \pm \sqrt{\mp 1} \pm i\)). Note that the Galois group does indeed preserve \(\frac{z_1}{s_2} = \frac{z_2}{\sigma(s_2)}\).
The 2 dimensional faces are thought of as homotopies between the morphisms, so that there is a homotopy iff the diagram is commutative, as can be computed directly. From all of the above, it is easy to calculate that \( \pi_1(U_\bullet) = \mathbb{Z}/2 \).

Now we can carry on to calculate the obstruction class. In general, if \( \Gamma \) is a group acting on a topological space \( X \), we can use the skeleton filtration of \( E\Gamma \), instead of the Postnikov filtration of \( X \), in order to compute the obstructions. In the case of the first obstruction class of a connected space \( X \), the formula takes the following form. Pick any \( x \in X \), and for each \( \sigma \in \Gamma \), choose a path \( \gamma_\sigma \) connecting \( x \) to \( \sigma(x) \). Then, for each simplex \([1, \sigma, \tau] \in E\Gamma_2\), we get a map \( \partial \Delta^2 \to X \) given by the following triangle in the space \( X \):

\[
\begin{array}{ccc}
& x \\
\sigma(x) & \searrow & \gamma_\sigma \\
\sigma(\gamma_{\sigma^{-1}}) & \longrightarrow & \tau(x)
\end{array}
\]

This triangle represents a class in \( \pi_1(X, x) \), so we get a map \( E\Gamma_2 \to X \) which is easily seen to be a 2-cocycle. This cocycle is exactly the obstruction to a homotopy fixed point of \( \Gamma \) in \( X \), or in other words, the obstruction for section to the map \( X//\Gamma \to B\Gamma \).

This cocycle is indeed \([a] \cup [b] \), as we can explicitly calculate. We exemplify this calculation on one value of the cocycle. Let \( \sigma_a, \sigma_b \in \Gamma_{L/K} \) be the automorphisms satisfying

\[
\begin{align*}
\sigma_a(\sqrt{a}) &= -\sqrt{a}, \quad \sigma_a(\sqrt{-b}) = \sqrt{-b}, \\
\sigma_b(\sqrt{-b}) &= -\sqrt{-b}, \quad \sigma_b(\sqrt{a}) = \sqrt{a}.
\end{align*}
\]

If we pick the base point defined by \( s = \sqrt{a}, t = \sqrt{-b} \) and edges

\[
\begin{align*}
\gamma_1 &= z, \\
\gamma_{\sigma_a} &= iz^{-1}, \\
\gamma_{\sigma_b} &= z^{-1}, \\
\gamma_{\sigma_a \sigma_b} &= iz,
\end{align*}
\]

then we can calculate the cocycle on \( \sigma_a, \sigma_b \) as

\[
\gamma_{\sigma_a}^{-1} \circ \sigma_a(\gamma_{\sigma_b}) \circ \gamma_{\sigma_a \sigma_b} = (iz^{-1}) \circ (z^{-1}) \circ (iz) = -z
\]

which is not the chosen map \( \gamma_1 \), and hence results in a nontrivial element. Carrying on this calculations to other pairs, this identifies the cocycle with the cup product.

5.2 Obstruction Theory for Sphere Bundles of Higher Dimensions

In this subsection we finally relate the Stiefel-Whitney classes to the relative obstruction theory of sphere bundles. By Proposition 3.4.2 and the fact that
\(\mathbb{Z}/2 \otimes \hat{\text{Et}}(\pi_n)\) is an extension of \(\mathbb{Z}/2\) by \(\mathbb{Z}/2[n-1]\) classified by \(HW_n\) (Theorem 4.3.1), we deduce that the \(n\)-th homological obstruction for section of \(\pi_n : Q_n^S \to Q_n\) is exactly \(HW_n\). Using this and a base-change result, we will show that for every morphism \(f : X \to Q_n\) classified by a quadratic bundle \((\mathcal{E}, B)\) over \(X\), the extension \(\mathbb{Z}/2[n-1] \to \mathbb{Z}/2 \otimes \hat{\text{Et}}(\pi_B) \to \mathbb{Z}/2\) is classified by the pullback \(f^*(HW_n)\).

**Proposition 5.2.1.** Consider the pullback diagram

\[
\begin{array}{ccc}
S_B(\mathcal{E}) & \xrightarrow{f'} & Q_n^S \\
\downarrow {\pi_B} & & \downarrow {\pi_n} \\
X & \xrightarrow{f} & Q_n
\end{array}
\]

The canonical comparison map

\[\mathbb{Z}/2 \otimes \hat{\text{Et}}(\pi_B) \to \mathbb{Z}/2 \otimes f^* \hat{\text{Et}}(\pi_n)\]

is an isomorphism.

**Proof.** The map \(Q_n^S \to Q_n\) is a smooth map of stacks, in the sense of Definition 2.4.4, as it is a schematic map presented by a simplicial map \(O_{n-1}^\bullet \to O_n^\bullet\) which is level-wise smooth. Hence, by theorem 2.4.5, the diagram

\[
\begin{array}{ccc}
\text{Pro Shv}_\infty(S_B(\mathcal{E}), \mathbb{Z}/2) & \xleftarrow{(f')^*} & \text{Pro Shv}_\infty(Q_n^S, \mathbb{Z}/2) \\
\uparrow {\pi_B} & & \uparrow {\pi_n^*} \\
\text{Pro Shv}_\infty(X, \mathbb{Z}/2) & \xrightarrow{f^*} & \text{Pro Shv}_\infty(Q_n, \mathbb{Z}/2)
\end{array}
\]

satisfies the left BC-condition, which precisely means that the map above is in fact an isomorphism.

**Theorem 5.2.2.** Let \((\mathcal{E}, B)\) be a quadratic bundle of rank \(n\) over a scheme \(X\) over \(K\) classified by a map \(f_{\mathcal{E},B} : X \to Q_n\). Let \(\pi_B : S_B(\mathcal{E}) \to X\) denote the projection of the sphere bundle to the base \(X\). Then, the mod \(2\) obstruction for section of \(\pi_B\) is the \(n\)-th Stiefel Whitney class of \((\mathcal{E}, B)\), namely

\[o_n(\pi_B, \mathbb{Z}/2) = f_{\mathcal{E},B}^* HW_n\]

**Proof.** Recall that the obstruction is compatible with morphisms of sheaves and with pullbacks by morphisms of schemes. Since \(\mathbb{Z}/2 \otimes \hat{\text{Et}}\pi_B \cong f_{\mathcal{E},B}^* \mathbb{Z}/2 \otimes \hat{\text{Et}}\pi_n\), we get

\[o_n(\mathbb{Z}/2 \otimes \hat{\text{Et}}\pi_B) = f_{\mathcal{E},B}^* o_n(\mathbb{Z}/2 \otimes \hat{\text{Et}}\pi_n) = f_{\mathcal{E},B}^* (HW_n).\]
5.3 Application to Unit Spheres

Let $B$ be a quadratic form of an $n+1$ dimensional vector space over a field $K$ of characteristic $\neq 2$. We can diagonalize $B$ to a form of the form $a_0x_0^2 + \cdots + a_nx_n^2$.

The theory developed in Section 4 implies a map $\text{Spec } K \xrightarrow{f_B} Q_{n+1}$, and by Theorem 5.2.2, the obstruction for a rational point of $S_B$ is given by $f_B^* \text{HW}_{n+1}$.

Moreover, we can write $V$ as a product of 1 dimensional quadratic spaces

$$V = V_0 \times \cdots \times V_n$$

where each space $V_i$ is equipped with the bilinear form $B_i(x) = a_i x^2$. Each $V_i$ is classified by a map $\text{Spec } K \xrightarrow{f_{B_i}} Q_1$, and thus we can factor $f_B$ as

$$\text{Spec } K \xrightarrow{f_{B_0} \times \cdots \times f_{B_n}} Q_{n+1}^1 \xrightarrow{\Delta} Q_{n+1}.$$

Recall that, for a field $K$, $H^1(B, \mathbb{Z}/2) \cong K^\times/(K^\times)^2$. We denote by $[a]$ the cohomology class corresponding to $a \in K^\times$.

**Lemma 5.3.1.** Let $X$ be the 0 dimensional sphere given by $ax^2 = 1$, and let $f: \text{Spec } K \to Q_1$ be the map classifying $X$. We have

$$f^*(\text{HW}_1) = [a].$$

**Proof.** Note that $Q_1 \cong B\mathbb{Z}/2$ and that $Q_1^2 \cong E\mathbb{Z}/2 \cong \text{Spec } K$.

It is well known that 1-cocycles classify Galois covers. In our case, the class identified with $\text{HW}_1$ in $H^1(B\mathbb{Z}/2, \mathbb{Z}/2)$ classifies the cover $E\mathbb{Z}/2 \to B\mathbb{Z}/2$. Therefore, $f^*(\text{HW}_1)$ classifies the cover obtained via pullback, $X \to \text{Spec } K$.

This cocycle can be directly computed to be $[a]$. \hfill $\Box$

**Proposition 5.3.2.** If $B$ is a quadratic form with diagonal entries $a_0, \ldots, a_n$, then $f_B^* \text{HW}_n = [a_0] \cup \cdots \cup [a_n]$.

**Proof.** The factorization $f_B = \Delta \circ (f_{B_0} \times \cdots \times f_{B_n})$ gives

$$f_B^* \text{HW}_n = (f_{B_0}^* \times \cdots \times f_{B_n}^*) \circ \Delta^* \text{HW}_n = (f_{B_0}^* \times \cdots \times f_{B_n}^*)(\text{HW}_1^{(0)} \cup \cdots \cup \text{HW}_1^{(n)}) = f_{B_0}^* \text{HW}_1^{(0)} \cup \cdots \cup f_{B_n}^* \text{HW}_1^{(n)} = [a_0] \cup \cdots \cup [a_n].$$

The equality (10) is true due to the Whitney formula (4.2.4). Equality (11) is true by general properties of the cup product. The last equality, (12), follows from Lemma 5.3.1. \hfill $\Box$

**Corollary 5.3.3.** Let $B$ be the quadratic form defined over $K$ given by

$$B(x_0, \ldots, x_n) = \sum_i a_i x_i^2.$$

The $n$-th mod 2 obstruction for a rational point on the sphere $S_B$ is

$$\text{HW}_n(B) = [a_0] \cup \cdots \cup [a_n]$$
Proof. This follows from the previous proposition by applying Theorem 5.2.2.

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