The smallest singular value of random combinatorial matrices

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Abstract

Let $Q_n$ be a random $n \times n$ matrix with entries in $\{0, 1\}$ whose rows are independent vectors of exactly $n/2$ zero components. We show that the smallest singular value $s_n(Q_n)$ of $Q_n$ satisfies

$$\mathbb{P}\{s_n(Q_n) \leq \frac{\varepsilon}{\sqrt{n}}\} \leq C\varepsilon + 2e^{-cn}$$

for every $\varepsilon \geq 0$, which is optimal up to the constants $C, c > 0$. This implies earlier results of Ferber, Jain, Luh and Samotij [11] as well as Jain [15]. In particular, for $\varepsilon = 0$, we obtain the first exponential bound in dimension for the singularity probability:

$$\mathbb{P}\{Q_n \text{ is singular}\} \leq 2e^{-cn}.$$

To overcome the lack of independence between entries of $Q_n$, we introduce an arithmetic-combinatorial invariant of a pair of vectors, which we call a Combinatorial Least Common Denominator (CLCD). We prove a small ball probability inequality for the combinatorial statistic $\sum_{i=1}^{n} a_i v_{\sigma(i)}$ in terms of the CLCD of the pair $(a, v)$, where $\sigma$ is a uniformly random permutation of $\{1, 2, \ldots, n\}$ and $a := (a_1, \ldots, a_n), v := (v_1, \ldots, v_n)$ are real vectors. This inequality allows us to derive strong anti-concentration properties for the distance between a fixed row of $Q_n$ and the linear space spanned by the remaining rows, and prove the main result.

1 Introduction

Given a random $n \times n$ matrix $A$, the basic question is: how likely is $A$ to be invertible, and, more quantitatively, well conditioned? These questions can be expressed in terms of the singular values $s_k(A)$ of $A$, which are defined as the eigenvalues of $\sqrt{A^tA}$ arranged in non-decreasing order. Of particular significance are the largest and smallest singular values, which admit the following variational characterization:

$$s_1(A) = \max_{v \in \mathbb{S}^{n-1}} |Av| \quad \text{and} \quad s_n(A) = \min_{v \in \mathbb{S}^{n-1}} |Av|,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ is the unit Euclidean sphere in $\mathbb{R}^n$.

1.1 Random matrices with independent entries

The behavior of the smallest singular values of random matrices with independent entries have been intensively studied [4, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54], in part due to applications in computer science and engineering. For random matrices with independent $N(0, 1)$ Gaussian entries, Edelman [8] and Szarek [44] showed

$$\mathbb{P}\{s_n(A) \leq \frac{\varepsilon}{\sqrt{n}}\} \sim \varepsilon,$$  \quad (1)

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thereby verifying a conjecture of Smale [41] as well as a speculation of von Neumann and Goldstine [27]. Edelman’s proof relied heavily on the rotation invariance property of the Gaussian distribution. Extending Edelman’s theorem to general distributions is non-trivial. Spielman and Teng [43] conjectured that (1) should hold for random sign matrices, up to an exponentially small term that accounts for their singularity probability:

\[
P\left\{ s_n(A) \leq \frac{\varepsilon}{\sqrt{n}} \right\} \leq \varepsilon + 2e^{-cn}, \quad \varepsilon \geq 0.
\]

After early breakthroughs of Rudelson [37] and Tao and Vu [47], it was shown in a remarkable work by Rudelson and Vershynin [38] that Spielmann-Teng’s conjecture holds (up to a multiplicative constant) for all random matrices with i.i.d. centered subgaussian entries of variance 1. This result has been greatly extended and refined in subsequent works [23, 24, 36, 39, 49, 54]. In particular, Livshyts, Tikhomirov and Vershynin [24] established the same bound as Rudelson and Vershynin for random matrices whose entries are independent random variables satisfying a uniform anti-concentration estimate, and such that the expected sum of the squares of the entries is \( O(n^2) \).

1.2 Random matrices with dependent entries

The smallest singular value problem becomes significantly more difficult when one considers models of random matrices with dependencies between entries, and much less has been known [1, 3, 5, 11, 15, 18, 19, 26, 28, 29, 30, 32]. For a survey of the topic we refer the reader to [56]. Our main focus is on a simple model of random combinatorial matrices with non-independent entries. We begin with a brief discussion of relevant results.

For an even integer \( n \), let \( Q_n \) be a random \( n \times n \) matrix with entries in \( \{0, 1\} \) whose rows are independent vectors of exactly \( n/2 \) zero components. One can view \( Q_n \) as the bipartite adjacency matrix of a random \( n/2 \)-regular bipartite graph with parts of size \( n \). Prior to this paper, even the problem of singularity of \( Q_n \) was not resolved in the literature.

Define \( p_n \) to be the probability that \( Q_n \) is singular. Trivially,

\[
p_n \geq \frac{1}{\binom{n}{n/2}} = \left( \frac{1}{2} + o(1) \right)^n
\]

as the RHS is the probability that the first two rows of \( Q_n \) are equal. Nguyen [28, Conjecture 1.4] conjectured that this simple bound is also the right one.

**Conjecture 1.1.** \( p_n = \left( \frac{1}{2} + o(1) \right)^n \).

It is already non-trivial to justify that \( p_n \) approaches 0 as \( n \) tends to infinity. This was first achieved by Nguyen [28], who proved that \( p_n \) decays faster than any polynomial, using an inverse Littlewood-Offord theorem of Nguyen and Vu [31]. Recently, Ferber, Jain, Luh and Samotij [11] established a counting result for the inverse Littlewood-Offord problem to improve the bound further to \( p_n \leq \exp(-n^{0.1}) \).

The question of obtaining quantitative lower tail bounds on the least singular value of \( Q_n \) was considered by Nguyen and Vu [32], where it was shown that for any \( C > 0 \) there exists \( D > 0 \) for which

\[
P\left\{ s_n(Q_n) \leq n^{-D} \right\} \leq n^{-C}.
\]

Building on the work of Ferber et al., Jain [15] obtained a better bound:

\[
P\left\{ s_n(Q_n) \leq \frac{\varepsilon}{n^2} \right\} \leq C\varepsilon + 2e^{-n^{0.0001}}.
\]

The following is our main result.

**Theorem 1.2.** There exist constants \( C, c > 0 \) such that

\[
P\left\{ s_n(Q_n) \leq \frac{\varepsilon}{\sqrt{n}} \right\} \leq C\varepsilon + 2e^{-cn}, \quad \varepsilon \geq 0.
\]
Remark. (1) Theorem 1.2 improves on the aforementioned works of Ferber, Jain, Luh and Samotij [11], and Jain [15].

(2) For \( \varepsilon = 0 \), Theorem 1.2 yields the first exponential bound for the singularity probability:
\[
P\{Q_n \text{ is singular}\} \leq 2e^{-cn}.
\]

(3) Theorem 1.2 continues to hold in the more general case when the sum of each row is \( d \), where \( \min(d, n - d) \gtrsim n \). Here we have focused on the case \( n \) is even and \( d = n/2 \) for ease of exposition.

1.3 The main tools

The \textit{Lévy concentration function} of a random variable \( \xi \) is defined for \( \varepsilon > 0 \) as
\[
\mathcal{L}(\xi, \varepsilon) := \sup_x |P\{|\xi - x| < \varepsilon\} - 1/2|.
\]

The following theorem is our main tool in proving Theorem 1.2.

\textbf{Theorem 1.3} (Distances). Let \( R_1, \ldots, R_n \) denote the row vectors of \( Q_n \), and consider the subspace \( H_n = \text{span}(R_1, \ldots, R_{n-1}) \). Let \( \mathbf{v} \) be a random unit vector orthogonal to \( H_n \) and measurable with respect to the sigma-field generated by \( H_n \). Then
\[
\mathcal{L}(\langle R_n, \mathbf{v} \rangle, \varepsilon) \leq C\varepsilon + 2e^{-cn} \quad \text{for every} \ \varepsilon \geq 0,
\]
where \( C > 0 \) and \( c \in (0, 1) \) are constants. In particular, we have
\[
P\{\text{dist}(R_n, H_n) \leq \varepsilon\} \leq C\varepsilon + 2e^{-cn}.
\]

A version of Theorem 1.3, under the assumption that the entries are i.i.d., was obtained by Rudelson and Vershynin [38]. They quantified the amount of additive structure of a vector (in this case, a normal vector of \( H_n \)) by the Least Common Denominator (LCD). The authors of [38] proved a \textit{small ball probability} bound for weighted sums of i.i.d. random variables in terms of the LCD of the coefficient vector, and used it to estimate \( \mathcal{L}(\text{dist}(R_n, H_n), \varepsilon) \). However, in our model, the LCD is no longer applicable as the entries are not independent.

In the present paper, we develop a \textit{combinatorial} version of the least common denominator and show how it can handle the dependent coordinates.

\textbf{Definition 1.4} (Combinatorial Least Common Denominator). For a vector \( \mathbf{v} \in \mathbb{R}^n \), and parameters \( \alpha, \gamma > 0 \), define
\[
\text{CLCD}_{\alpha, \gamma} (\mathbf{v}) := \inf \left\{ \theta > 0: \text{dist} (\theta \cdot D(\mathbf{v}), \mathbb{Z}^{n/2}) < \min (\gamma|\theta \cdot D(\mathbf{v})|, \alpha) \right\}.
\]

Here by \( D(\mathbf{v}) \) we denote the vector in \( \mathbb{R}^{n/2} \) whose \((i, j)\)-coordinate is \( v_i - v_j \), for \( 1 \leq i < j \leq n \).

Remark. The requirement that the distance is smaller than \( \gamma|\theta \cdot D(\mathbf{v})| \) forces us to consider only non-trivial integer points as approximations of \( D(\mathbf{v}) \). We will use this definition with \( \gamma \) a small constant, and for \( \alpha = \mu n \) with a small constant \( \mu > 0 \). The inequality \( \text{dist} (\theta \cdot D(\mathbf{v}), \mathbb{Z}^{n/2}) < \alpha \) then yields that most coordinates of \( D(\mathbf{v}) \) are within a small distance from non-zero integers.

One new key ingredient in the proof of Theorem 1.3 is a small ball probability inequality for certain \textit{combinatorial statistics}. Given a vector \( \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \), consider the random sum
\[
W_\mathbf{v} := \eta_1 v_1 + \ldots + \eta_n v_n,
\]
where \( (\eta_1, \ldots, \eta_n) \) is taken uniformly from \( \{0, 1\}^n \) subject to \( \sum_{i=1}^n \eta_i = n/2 \). We establish the following vital relation between CLCD of \( \mathbf{v} \) and anti-concentration of \( W_\mathbf{v} \).
Theorem 1.5 (Small ball probability). For any \( b > 0 \) and \( \gamma \in (0,1) \) there exists \( C > 0 \) depending only on \( b \) and \( \gamma \) with the following property. Let \( \mathbf{v} \in \mathbb{R}^n \) such that \( |D(\mathbf{v})| \geq b\sqrt{n} \). Then for every \( \alpha > 0 \) and \( \varepsilon \geq 0 \), we have

\[
\mathcal{L}(W, \varepsilon) \leq C \varepsilon + \frac{C}{\text{CLCD}_{\alpha, \gamma}(\mathbf{v})} + C e^{-2\alpha^2/n}.
\]

Theorem 1.5 opens the door to the geometric approach of Rudelson and Vershynin [38]. We will derive Theorem 1.5 from a much more general result, namely Theorem 3.2. We also anticipate that Theorem 3.2 will have further applications.

1.4 Organization and notation

The paper is organized as follows. We prove Theorems 1.2 and 1.3 in Section 2, assuming the validity of Theorem 1.5. Section 3 is devoted to the proof of Theorem 1.5. We close, in Section 4, with some remarks and open problems.

The inner product in \( \mathbb{R}^n \) is denoted \( \langle \cdot, \cdot \rangle \), the Euclidean norm is denoted \( |\cdot| \). The Euclidean unit ball and sphere in \( \mathbb{R}^n \) are denoted \( B_2^n \) and \( S^{n-1} \), respectively. We generally use lowercase bolded letters for vectors. For a vector \( \mathbf{v} \) we denote by \( v_i \) the value of its \( i \)-th coordinate. Given \( S, P \subset \mathbb{R}^n \), we use the standard notation \( N(S, P) \) for the least number of translates of \( P \) needed to cover \( S \).

We will make use of a partition of the unit sphere \( S^{n-1} \) into a "structured" part \( \mathcal{C}(\delta, \rho) \) and a "generic" one \( \mathcal{N}(\delta, \rho) \). We carry out this step in Section 2.1 (see Definition 2.1). In Section 2.2 we show that the random variable \( |Q_n \mathbf{v}| \) satisfies a large deviation inequality (Lemma 2.3). We then use this inequality in Section 2.2 to achieve an essentially sharp upper bound for the restricted operator norm (see Proposition 2.8), and in Section 2.4 to get a good uniform lower bound for \( |Q_n \mathbf{v}| \) on the set \( \mathcal{C}(\delta, \rho) \) (see Proposition 2.10). We investigate the invertibility problem for non almost-constant vectors in Section 2.5 using a special case of our small ball probability estimate. Putting the pieces together, we deliver the proofs of Theorems 1.2 and 1.3 in Section 2.6.

2 Smallest singular value of random row-regular matrices

This section is devoted to the proofs of Theorems 1.2 and 1.3. We will follow the general strategy of Rudelson and Vershynin [38], the first step of which is to decompose the unit sphere to a "structured" part \( \mathcal{C}(\delta, \rho) \) and a "generic" one \( \mathcal{N}(\delta, \rho) \). We will make use of a partition of the unit sphere \( S^{n-1} \) into two sets of almost-constant and non almost-constant vectors. These sets were first defined in [18, 19] as follows.

Definition 2.1. Fix \( \delta, \rho \in (0,1) \) whose values will be chosen later. A vector \( \mathbf{v} \in S^{n-1} \) is called almost-constant if one can find \( \lambda \in \mathbb{R} \) such that there are at least \( (1-\delta)n \) coordinates \( i \in [n] \) satisfying \( |v_i - \lambda| \leq \frac{\rho}{\sqrt{n}} \). A vector \( \mathbf{v} \in S^{n-1} \) is called non almost-constant if it is not almost-constant. The sets of almost-constant and non almost-constant vectors will be denoted by \( \mathcal{C}(\delta, \rho) \) and \( \mathcal{N}(\delta, \rho) \), respectively.
Remark that $C(\delta', \rho') \subseteq C(\delta, \rho)$ if $\delta' \leq \delta$ and $\rho' \leq \rho$.

Using the decomposition $S^{n-1} = C(\delta, \rho) \cup N(\delta, \rho)$ of the unit sphere, we divide the invertibility problem into two subproblems, for almost-constant and non almost-constant vectors:

$$
P\left\{s_{\min}(Q_n) \leq \frac{\varepsilon}{\sqrt{n}} \right\} \leq P\left\{\inf_{v \in C(\delta, \rho)} |Q_nv| \leq \frac{\varepsilon}{\sqrt{n}} \right\} + P\left\{\inf_{v \in N(\delta, \rho)} |Q_nv| \leq \frac{\varepsilon}{\sqrt{n}} \right\}.
$$

We will deal with the former in Section 2.4, and the latter will be treated in Section 2.5.

The fact that the operator norm of $Q_n$ typically has a higher order of magnitude compared to $\sqrt{n}$ adds some complexity to the proof. To overcome this difficulty, as in [15] we exploit the presence of a “spectral gap”. Namely we show in Proposition 2.8 that, while the operator norm of $Q_n$ is $n/2$, the operator norm of $Q_n$ restricted to the hyperplane $H := \{v \in \mathbb{R}^n: v_1 + \ldots + v_n = 0\}$ is $O(\sqrt{n})$ with high probability.

The following result is a version of [19, Lemma 2.2].

**Lemma 2.2** (Non almost-constant vectors are separated). Let $\delta, \rho \in (0, 1)$. Then for any vector $v \in S^{n-1} \setminus C(\delta, \rho)$, there are disjoint subsets $\sigma_1 = \sigma_1(v)$ and $\sigma_2 = \sigma_2(v)$ of $[n]$ such that

$$
|\sigma_1|, |\sigma_2| \geq \delta n/8, \quad \text{and} \quad \frac{\rho}{2n} \leq |v_i - v_j| \leq \frac{6}{\delta n} \quad \forall i \in \sigma_1, \forall j \in \sigma_2.
$$

**Proof.** Consider the subset $\sigma \subseteq [n]$ defined as

$$
\sigma := \{k: |v_k| \leq \frac{3}{\sqrt{\delta n}}\}.
$$

As $|v| = 1$, we must have $|\sigma^c| \leq \delta n/9$.

Moreover, [19, Lemma 2.2] guarantees the existence of disjoint subsets $J$ and $Q$ of $[n]$ such that

$$
|J|, |Q| \geq \delta n/4 \quad \text{and} \quad |v_i - v_j| \geq \frac{\rho}{2n} \quad \forall i \in J, \forall j \in Q.
$$

Put $\sigma_1 = J \cap \sigma$ and $\sigma_2 = Q \cap \sigma$. It is easy to verify that $\sigma_1$ and $\sigma_2$ possess the desired properties. \qed

### 2.2 Concentration

The main result of this section, stated below, shows that for each point $v \in S^{n-1}$ the random variable $|Q_nv|^2$ is well-concentrated around its expectation.

**Lemma 2.3.** Let $M$ be a random $m \times n$ matrix, $1 \leq m \leq n$, whose rows are independent random $\{0, 1\}$-vectors of exactly $n/2$ zero components. Consider $v \in S^{n-1}$, and let $r = |v_1 + \ldots + v_n|$. Then for every $t \geq 0$, one has

$$
P\left\{|Mv|^2 - E|Mv|^2 \geq t \right\} \leq 2 \exp\left[-c_1 \min\left(\frac{t^2}{(r^2 + 1)^2 n}, \frac{t}{r^2 + 1}\right)\right]
$$

where $c_1 > 0$ is an absolute constant.

Our proof of Lemma 2.3 makes use of the following inequality, due to Kwan, Sudakov and the author [17, Lemma 2.1].

**Lemma 2.4** (Combinatorial concentration inequality). Consider $f: \{0, 1\}^n \to \mathbb{R}$ such that

$$
|f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)| \leq d_i
$$

for all $x \in \{0, 1\}^n$ and all $i \in [n]$. Then

$$
P(|f(\eta) - E f(\eta)| \geq t) \leq 2 \exp\left(-\frac{t^2}{8 \sum_{i=1}^n d_i^2}\right) \quad \text{for all } t \geq 0,
$$

where $\eta$ is taken uniformly from $\{0, 1\}^n$ subject to $\sum_{i=1}^n \eta_i = n/2$. 


In order to justify Lemma 2.3, we also require some results about random variables $X$ that satisfy $P(|X| > t) \leq 2 \exp(-O(t^\alpha))$ for all $t \geq 0$, where $\alpha > 0$ is fixed. The first is Proposition 2.5.2 in [55].

**Lemma 2.5.** Fix $\alpha > 0$. For a random variable $X$, the following properties are equivalent with parameters $K_i > 0$ differing from each other by at most an absolute constant factor:

1. Tails: $P(|X| > t) \leq 2 \exp(-t^\alpha/K_1)$ for all $t \geq 0$;
2. Moments: $(E|X|^p)^{1/p} \leq K_2p^{1/\alpha}$ for all $p \geq 1$.

The above lemma leads us to the following convenient notation.

**Definition 2.6** (Orlicz norm). Fix $\alpha > 0$. The $\psi_\alpha$-norm of $X$, denoted $|X|_{\psi_\alpha}$, is defined to be the smallest $K_2$ in the second property of Lemma 2.5. In other words,

$$|X|_{\psi_\alpha} := \sup_{p \geq 1} p^{-1/\alpha}(E|X|^p)^{1/p}.$$ 

The cases $\alpha = 2$ and $\alpha = 1$ correspond to sub-gaussian random variables and sub-exponential random variables, respectively. Sub-gaussian and sub-exponential distributions are closely related. Indeed, inspecting the definitions we quickly see that

$$|X|_{\psi_2}^2 \leq |X|^2_{\psi_1} \leq 2|X|_{\psi_2}. \quad (2)$$

We recall a concentration inequality for sums of independent sub-exponential random variables (see, e.g., [55, Theorem 2.8.1]).

**Theorem 2.7** (Bernstein’s inequality). Let $Y_1, \ldots, Y_n$ be independent, mean zero, sub-exponential random variables. Let $K = \max_i |Y_i|_{\psi_1}$. Then for every $t \geq 0$, we have

$$P\{\left|\sum_{i=1}^n Y_i\right| \geq t\} \leq 2 \exp\left[-c_2 \min\left(\frac{t^2}{K^2n}, \frac{t}{K}\right)\right]$$

where $c_2 > 0$ is an absolute constant.

We are now ready to prove Lemma 2.3.

**Proof of Lemma 2.3.** Consider the random sum $X := \eta_1v_1 + \ldots + \eta_nv_n$, where $(\eta_1, \ldots, \eta_n)$ is sampled uniformly from the set of all $\{0, 1\}$-vectors of entry sum $n/2$. Then for all $i \in [m]$, the random variables $\langle Mv, e_i \rangle$ are i.i.d. copies of $X$. Thus

$$|Mv|^2 = X_1^2 + \ldots + X_m^2,$$

where $X_1, \ldots, X_m$ are independent copies of $X$.

A simple calculation shows

$$E(X^2) = \frac{n - 2}{4(n - 1)} \chi^2 + \frac{n}{4(n - 1)} \lesssim \chi^2 + 1.$$

Moreover, applying Lemma 2.4 to the function $\eta \mapsto |\langle \eta, v \rangle|$, we get

$$|X - EY|_{\psi_2} \lesssim 1.$$

By the triangle inequality for the $\psi_2$-norm, we obtain

$$|X|_{\psi_2} \leq |X - EY|_{\psi_2} + |EY|_{\psi_2}$$
$$\leq |X - EY|_{\psi_2} + (EY^2)^{1/2} \lesssim 1 + (\chi^2 + 1)^{1/2} \lesssim \chi + 1,$$
where in the second inequality we used the bound $|EX|_{\psi_2} = |EX| \leq (EX^2)^{1/2}$. Thus
\[
|X^2 - EX^2|_{\psi_1} \leq |X^2|_{\psi_1} + |EX^2|_{\psi_1} \\
\leq 2|X|_{\psi_2} + E(X^2) \lesssim r^2 + 1,
\]
where the second inequality follows from (2). Now, noting that $m \leq n$, and applying Theorem 2.7 to $Y_i = X_i^2 - EX_i^2$ and $K \lesssim r^2 + 1$, we obtain
\[
\mathbb{P}\left\{ \left| MV - E(MV) \right|^2 \geq t \right\} \leq 2 \exp\left(-c \min\left(\frac{t^2}{(r^2 + 1)^2n}, \frac{t}{r^2 + 1}\right)\right) \quad \text{for all } t \geq 0,
\]
where $c > 0$ is an absolute constant. This completes our proof. \hfill \Box

2.3 Restricted operator norm and invertibility on a single vector

Using nets along with our concentration inequality (Lemma 2.3), one can show that the operator norm of $Q_n$ restricted to the hyperplane $H := \{v \in \mathbb{R}^n : \sum_{i=1}^n v_i = 0\}$ is typically $O(\sqrt{n})$.

**Proposition 2.8** (Restricted operator norm). There exist constants $C_3 > 0$ and $c_3 > 0$ such that the following holds. Let $M$ be a random $m \times n$ matrix, $1 \leq m \leq n$, whose rows are independent random $\{0,1\}$-vectors of exactly $n/2$ zero components. Then for all $t \geq C_3$, we have
\[
\mathbb{P}\left\{ \|M\|_{\mathcal{H}} \geq t \sqrt{n} \right\} \leq 2 \exp(-c_3 t^2 n),
\]
where $\mathcal{H} := \{v \in \mathbb{R}^n : \sum_{i=1}^n v_i = 0\}$.

**Proof.** Note that
\[
\|M\|_{\mathcal{H}} \op = \sup_{v \in \mathbb{S}^{n-1} \cap H} |Mv|.
\]
Let $\mathcal{N}$ be a $(1/2)$-net of $\mathbb{S}^{n-1} \cap H$ of cardinality at most $6^n$. Fix $v \in \mathcal{N}$. Since $v_1 + \ldots + v_n = 0$, it follows from Lemma 2.3 that for $t$ sufficiently large one has
\[
\mathbb{P}\left\{ |Mv| \geq t \sqrt{n}/2 \right\} \leq \mathbb{P}\left\{ \left| MV - E(MV) \right|^2 \geq t^2 n/8 \right\} \leq 2e^{-ct^2 n},
\]
where the first inequality holds since $E|Mv|^2 = \frac{mn}{4(n-1)} \leq n/2$. Taking the union bound yields
\[
\mathbb{P}\left\{ \|M\|_{\mathcal{H}} \geq t \sqrt{n} \right\} \leq |\mathcal{N}| \max_{v \in \mathcal{N}} \mathbb{P}\left\{ |Mv| \geq t \sqrt{n}/2 \right\} \leq 6^n \cdot 2e^{-ct^2 n},
\]
which completes the proof. \hfill \Box

Lemma 2.3 can also be used to establish the invertibility of the random matrix $Q_n$ on a single vector.

**Lemma 2.9** (Invertibility on a single vector). There exists a constant $c_4 > 0$ such that the following holds for fixed $v \in \mathbb{S}^{n-1}$. Let $M$ be a random $m \times n$ matrix, $n/2 \leq m \leq n$, whose rows are independent random $\{0,1\}$-vectors of exactly $n/2$ zero components. Then
\[
\mathbb{P}\{ |Mv| \leq \sqrt{n}/5 \} \leq 2e^{-c_4 n}.
\]

**Proof.** Let $r = |v_1 + \ldots + v_n|$. For $n \geq 10$ and for $m \geq n/2$, we have
\[
E|Mv|^2 = \frac{m(n-2)}{4(n-1)} r^2 + \frac{mn}{4(n-1)} \geq \frac{1}{9} (r^2 + 1)n.
\]
Applying Lemma 2.3 to $t = \frac{1}{18} (r^2 + 1)n$, we thus get
\[
\mathbb{P}\{ |Mv| \leq \sqrt{n}/5 \} \leq \mathbb{P}\left\{ \left| MV - E(MV) \right|^2 \geq t \right\} \leq 2e^{-cn}.
\] \hfill \Box
Remark. One can prove Lemma 2.9 in a more direct way. Indeed, let us consider the random sum $X := \eta_1v_1 + \cdots + \eta_nv_n$, where $(\eta_1, \ldots, \eta_n)$ is taken uniformly at random from the set of all $\{0, 1\}$-vectors of entry sum $n/2$. It is not difficult to derive from Lemma 2.4 that $P(|X| \leq c) \leq 1 - c$ for some constant $c > 0$. The conclusion of Lemma 2.9 then follows from a tensorization lemma (see Lemma 2.17).

2.4 Invertibility for almost-constant vectors

Here we study the invertibility problem for almost-constant vectors. The following is the main result.

Proposition 2.10 (Invertibility for almost-constant vectors). There exist constants $\delta, \rho, c_5 \in (0, 1)$ such that the following holds. Let $M$ be an $m \times n$ random matrix, $n/2 \leq m \leq n$, whose rows are independent random $\{0, 1\}$-vectors of exactly $n/2$ zero components. Then

$$
P\left\{ \inf_{v \in C(\delta,\rho)} |Mv| \leq \sqrt{n}/10 \right\} \leq 2e^{-c_5n}.
$$

We will construct a small $\varepsilon$-net $\mathcal{N}$ on $C(\delta,\rho)$ with respect to the pseudometric $d(x, y) := |M(x - y)|$. The invertibility of the random matrix $M$ over a single vector $w \in \mathcal{N}$ will follows from Lemma 2.9. Then, by a union bound, the invertibility will hold for each point in the net $\mathcal{N}$. By approximation, we will extend the invertibility to the whole $C(\delta,\rho)$.

To exploit the fact that $\|Q_n\|_{\mathcal{H}} \leq O(\sqrt{n})$, we will use the following simple result, whose proof borrows some ideas of Jain [15].

Lemma 2.11 (Rounding). Fix $\beta > 0$, and consider any $S \subset S^{n-1}$. There exists a (deterministic) net $\mathcal{N} \subset S + 2\beta B_2^n$ of cardinality at most $(2n + 2) \cdot N(S, \beta B_2^n)$ such that for every $m \in \mathbb{N}$ and for every (deterministic) $m \times n$ matrix $A$, the following holds: for every $v \in S$ one can find $w \in \mathcal{N}$ so that

$$
|A(v - w)| \leq \beta \left( 2 \|A\|_{\mathcal{H}} \|_{\text{op}} + \|A\|_{\text{op}} \frac{n}{\sqrt{n}} \right).
$$

Proof. Let $\mathcal{F} \subset \mathbb{R}^n$ be a set such that $S \subset \mathcal{F} + \beta B_2^n$ and $|\mathcal{F}| = N(S, \beta B_2^n)$. Consider the $(2n + 2)$-element subset of $\frac{\beta}{\sqrt{n}} \mathbb{Z}^n$ defined as

$$
\mathcal{Y} := \left\{ (\beta s/\sqrt{n}, \ldots, \beta s/\sqrt{n}, 0, \ldots, 0) : s = \pm 1 \right\}.
$$

We will show that the net $\mathcal{N} := \mathcal{F} + \mathcal{Y}$ has the desired properties. Indeed, we have

$$
|\mathcal{N}| \leq |\mathcal{Y}| |\mathcal{F}| = (2n + 2)|\mathcal{F}|.
$$

Fix a vector $v \in S$. It follows from the definition of $\mathcal{F}$ that $|v - x| \leq \beta$ for some $x \in \mathcal{F}$. Let $k := |\langle v - x, 1 \rangle|/\sqrt{n}/\beta$, $s := \text{sgn}(|\langle v - x, 1 \rangle|)$, and

$$
y := \left( \beta s/\sqrt{n}, \ldots, \beta s/\sqrt{n}, 0, \ldots, 0 \right) \in \mathcal{Y}.
$$

Note that $y$ is well-defined since, by the Cauchy-Schwarz inequality, we have

$$
k = |\langle v - x, 1 \rangle|/\sqrt{n}/\beta \leq |v - x|n/\beta \leq n.
$$

Let $w := x + y$. Then $w \in \mathcal{N}$. Moreover, by the triangle inequality we obtain

$$
|v - w| \leq |v - x| + |y| \leq \beta + \beta = 2\beta,
$$

and whence $w \in v + 2\beta B_2^n$. This implies

$$
\mathcal{N} \subset S + 2\beta B_2^n.
$$
We can infer from the definition of \( y \) that
\[
\langle v - w, 1 \rangle = \langle v - x, 1 \rangle - \langle y, 1 \rangle = \frac{s\beta k}{\sqrt{n}} - \frac{s\beta |k|}{\sqrt{n}} \in \left[ -\frac{\beta}{\sqrt{n}}, \frac{\beta}{\sqrt{n}} \right].
\]

Finally, writing \( v - w = \text{Proj}_H(v - w) + \frac{(v - w, 1)}{n} \), we see that
\[
|A(v - w)| \leq |A(\text{Proj}_H(v - w))| + \left| \frac{(v - w, 1)}{n} \right| |A(1)|
\leq \|A\|_H \|v - w\| + \frac{\beta}{n^{3/2}} \cdot |A|_{\text{op}} \cdot \sqrt{n}
\leq 2\beta \|A\|_H + \frac{\beta |A|_{\text{op}}}{n}.
\]

This completes our proof. \( \square \)

We next employ Lemma 2.11 to discretize the set of almost-constant vectors.

**Lemma 2.12** (Discretization of almost-constant vectors). Let \( \delta, \rho \in (0, \frac{1}{12}) \), and \( n \) is sufficiently large with respect to \( \delta \) and \( \rho \). Then there is a net \( \mathcal{N} \subset \mathbb{R}^n \) of cardinality at most \( e^{2\delta \log(5/\delta) n} \) such that

- For any \( w \in \mathcal{N} \), we have \( 1/2 \leq |w| \leq 3/2 \);
- For any \( v \in \mathcal{C}(\delta, \rho) \) there is \( w \in \mathcal{N} \) so that for any deterministic \( m \times n \) matrix \( A \) we have
\[
|A(v - w)| \leq (\delta + 2\rho) \left( 2\|A\|_H + \frac{|A|_{\text{op}}}{n} \right).
\]  

**Proof.** For a vector \( v \in \mathbb{R}^n \) and a set \( I \subseteq [n] \), we let \( v_I \) denote the vector \((v_i)_{i \in I}\).

Fix \( v \in \mathcal{C}(\delta, \rho) \). Then there exist a real number \( \lambda \) and an index set \( \sigma \subseteq [n] \) with \( |\sigma| = (1 - \delta)n \) such that \( |v_i - \lambda| \leq \frac{\rho}{\sqrt{n}} \) for all \( i \in \sigma \). To discretize the range of \( \lambda \), we observe that any \( \lambda \in [-1, 1] \) can be approximated by some
\[
\lambda_0 \in [-1, 1] \cap \frac{\rho}{\sqrt{n}} \mathbb{Z}
\]  

in the sense that \( |\lambda - \lambda_0| \leq \frac{\rho}{\sqrt{n}} \). This clearly forces
\[
|v_i - \lambda_0| \leq \frac{2\rho}{\sqrt{n}} \text{ for all } i \in \sigma.
\]

We can capture \( v_{\sigma^c} \) by quantizing its coordinates uniformly with step \( \sqrt{\frac{\rho}{n}} \). Thus there is \( \bar{u} \in \left[ \frac{\rho}{n} \mathbb{Z} \right]^{\sigma^c} \) with \( \|v_{\sigma^c} - \bar{u}\|_\infty \leq \frac{\rho}{n} \). Since \( |v| = 1 \), we find
\[
|\bar{u}| \leq |v_{\sigma^c}| + |v_{\sigma^c} - \bar{u}| \leq |v| + \sqrt{\delta n} \|v_{\sigma^c} - \bar{u}\|_\infty \leq 1 + \delta \leq 3/2.
\]  

Define a vector \( u \in \mathbb{R}^n \) by setting \( u_\sigma = (\lambda_0, \ldots, \lambda_0) \) and \( u_{\sigma^c} = \bar{u} \). It follows from (4) and (5) that \( u \in \mathcal{F} \), where
\[
\mathcal{F} := \bigcup_{|\sigma| = (1 - \delta)n} \left\{ \lambda_0 1_\sigma: \lambda_0 \in [-1, 1] \cap \frac{\rho}{\sqrt{n}} \mathbb{Z} \right\} \oplus \left\{ \bar{u} \in \left[ \frac{\rho}{n} \mathbb{Z} \right]^{\sigma^c}: |\bar{u}| \leq 3/2 \right\},
\]  

the union being over all \((1 - \delta)n\)-element subsets \( \sigma \) of \([n]\).
Lemma 2.11, we obtain a net $N \subset C$.

Letting $\beta := \delta + 2\rho \in (0, \frac{1}{4})$, we thus get $C(\delta, \rho) \subset F + \beta B_2^n$, and so $N(C(\delta, \rho), \beta B_2^n) \leq |F|$. By applying Lemma 2.11, we obtain a net $N \subset C(\delta, \rho) + 2\beta B_2^n \subset \frac{1}{2}B_2^n \setminus \frac{1}{2}B_2^n$ of cardinality at most $(2n + 2)|F|$ having property (3).

It remains to estimate the cardinality of $N$. Observe that there are $(\frac{n}{\delta n}) \leq \left(\frac{5}{\delta}\right)^\delta n$ ways to choose the subset $\sigma$ in (6). Clearly, there are at most $1 + 2\sqrt{n}/\rho$ possibilities for $\lambda_0$ in (6). Furthermore, a known volumetric argument shows that there are at most $\left(\frac{5}{\delta}\right)^\delta n$ choices for $\tilde{u}$ in (6). Therefore, we have

$$|N| \leq (2n + 2)|F| \leq (2n + 2) \cdot \left(\frac{e}{\delta}\right)^\delta n \cdot (1 + 2\sqrt{n}/\rho) \cdot \left(\frac{5}{\delta}\right)^\delta n \leq e^{2\delta \log(5/\delta)n}$$

for $n$ sufficiently large. This completes our proof. \qed

We are now ready to prove Proposition 2.10.

**Proof of Proposition 2.10.** By Proposition 2.8, there is a constant $K \geq 1$ such that

$$\mathbb{P}\{|M|_{H\cdot} > K \sqrt{n}\} \leq e^{-n}.$$ 

Take $\delta = \rho = 1/(100K)$. To complete the proof, it suffices to find a constant $c > 0$ so that the event

$$E := \left\{ \inf_{v \in C(\delta, \rho)} |Mv| \leq \sqrt{n}/10 \quad \text{and} \quad |M|_{H\cdot} \leq K \sqrt{n} \right\}$$

has probability at most $2e^{-cn}$.

To this end, let $N$ be the net constructed in Lemma 2.12. By Lemma 2.9, for each $w \in N$ we have

$$\mathbb{P}\{|Mw| \leq \sqrt{n}/5\} \leq 2e^{-cn}.$$ 

Then taking the union bound, we obtain

$$\mathbb{P}\left\{ \inf_{w \in N} |Mw| \leq \sqrt{n}/5 \right\} \leq e^{2\delta \log(5/\delta)n} \cdot 2e^{-cn} \leq 2e^{-cn/2}. \quad (7)$$

We are now in a position to bound the event $E$. Suppose that $E$ occurs, then $|M|_{H\cdot} \leq K \sqrt{n}$ and $|Mv| \leq \sqrt{n}/10$ for some $v \in C(\delta, \rho)$. We learn from the choice of $N$ that there is $w \in N$ with

$$|M(v - w)| \leq (\delta + 2\rho) \left( 2 |M|_{H\cdot} + \frac{|M|_{op}}{n} \right).$$

Since $|M|_{H\cdot} \leq K \sqrt{n}$ and $|M|_{op} \leq n$, we have

$$|Mw| \leq |Mv| + |M(v - w)| \leq \sqrt{n}/10 + (\delta + 2\rho)(2K \sqrt{n} + 1) \leq \sqrt{n}/5$$

for $\delta = \rho = 1/(100K)$. By (7), this completes our proof. \qed

### 2.5 Invertibility for non almost-constant vectors

In this section, we study the invertibility problem for non almost-constant vectors. The following is the main result.

**Proposition 2.13 (Random normal).** There exist constants $\mu, \gamma, c_0 \in (0, 1)$ such that for $n$ sufficiently large one has

$$\mathbb{P}\{ \exists v \in N(\delta, \rho) \quad \text{with} \quad Q_n^\mu v = 0 \quad \text{and} \quad \text{CLCD}_{\mu, \gamma}(v) \leq e^{c_0 n} \} \leq 2^{-n}.$$ 

In Section 2.5.1 we establish some properties of CLCD which are necessary for the proof of Proposition 2.13. The proof is then given in Section 2.5.2.
### 2.5.1 Properties of CLCD

A crucial property of the CLCD which will allow us to discretize the range of possible realizations of random normals, is stability of CLCD with respect to small perturbations.

**Lemma 2.14 (Stability of CLCD).** Consider a vector $v \in \mathbb{R}^n$, and parameters $\alpha > 0, \gamma \in (0, 1)$. Then for any $w \in \mathbb{R}^n$ with $|v - w| < \frac{\gamma |D(v)|}{5\sqrt{n}}$, we have

$$\text{CLCD}_{\alpha/2,\gamma/2}(w) \geq \min \left\{ \text{CLCD}_{\alpha,\gamma}(v), \frac{\alpha}{4\sqrt{n}|v - w|} \right\}.$$  

**Proof.** Note that $|D(x)| \leq \sqrt{n}|x|$ for every $x \in \mathbb{R}^n$. By our assumptions on $|v - w|$ and $\gamma$, we get

$$|D(v) - D(w)| = |D(v) - w| \leq \sqrt{n}|v - w| \leq \sqrt{n} \cdot \frac{\gamma |D(v)|}{5\sqrt{n}} < |D(v)|/5,$$

and hence

$$|D(v)| \leq 5|D(w)|/4.$$

Let $H := \min \{ \text{CLCD}_{\alpha,\gamma}(v), \frac{\alpha}{4\sqrt{n}|v - w|} \}$. For any $\theta \in (0, H)$, the definition of CLCD yields

$$\text{dist} \left( \theta \cdot D(v), Z^{(n)} \right) \geq \min \left( \gamma \theta |D(v)|, \alpha \right).$$

From this it follows that

$$\text{dist} \left( \theta \cdot D(w), Z^{(n)} \right) \geq \text{dist} \left( \theta \cdot D(v), Z^{(n)} \right) - |\theta \cdot D(v) - w|$$

$$\geq \min \left( \gamma \theta |D(v)|, \alpha \right) - \theta |D(v) - w|$$

$$\geq \min \left( \gamma \theta |D(w)|, \alpha \right) - (1 + \gamma)\theta |D(v) - w|$$

$$\geq \min \left( \gamma \theta |D(w)|, \alpha \right) - 2\theta \sqrt{n}|v - w|$$

$$\geq \frac{1}{2} \min \left( \gamma \theta |D(w)|, \alpha \right),$$

where the last step holds since $\theta < \frac{\gamma}{4\sqrt{n}|v - w|}$ and $4\sqrt{n}|v - w| \leq \frac{1}{5} \gamma |D(v)| \leq \gamma |D(w)|$. By definition of CLCD, this gives

$$\text{CLCD}_{\alpha/2,\gamma/2}(w) \geq \theta.$$

Since $\theta \in (0, H)$ was arbitrary, it follows that $\text{CLCD}_{\alpha/2,\gamma/2}(w) \geq H$, which proves the lemma. □

We will also need a simple result that the CLCD of any non almost-constant vector in $\mathbb{S}^{n-1}$ is $\geq \sqrt{n}$.

**Lemma 2.15 (Non almost-constant vectors have large CLCD).** Let $\delta, \rho \in (0, 1)$, and fix $v \in \mathcal{N}(\delta, \rho)$. Then for every $\alpha > 0$ and every $\gamma$ with $0 < \gamma < \frac{1}{12} \delta \rho$, we have

$$\text{CLCD}_{\alpha,\gamma}(v) \geq \frac{1}{7} \sqrt{n}.$$

**Proof.** By Lemma 2.2, there is a subset $\sigma' \subseteq \binom{[n]}{i,j}$ of cardinality

$$|\sigma'| \geq \frac{1}{64} \delta^2 n^2$$

and such that

$$\frac{\rho}{\sqrt{2n}} \leq |v_i - v_j| \leq \frac{6}{\sqrt{\delta n}}$$

for every $\{i,j\} \in \sigma'$.  

Let $H = \text{CLCD}_{\alpha,\gamma}(v)$. By definition of CLCD, one can find a vector $p = (p_{ij})_{i<j} \in \mathbb{Z}^{[n]}$ with

$$|H \cdot D(v) - p| < \gamma H |D(v)|.$$
Dividing by $H$ yields
\[ \left| D(v) - \frac{p}{H} \right| < \gamma |D(v)| \leq \gamma \sqrt{n}. \]

Then by Chebyshev inequality, there exists a subset $\sigma'' \subseteq \binom{[n]}{2}$ of cardinality
\[ |\sigma''| > \binom{n}{2} - \frac{1}{\delta^2} \delta^2 n^2 \]
and such that
\[ |v_i - v_j - \frac{p_{ij}}{H}| \leq \frac{8\gamma}{\delta \sqrt{n}} \text{ for } \{i, j\} \in \sigma''. \] (9)

As $|\sigma'| + |\sigma''| > \binom{n}{2}$, there is $\{i, j\} \in \sigma' \cap \sigma''$. Fix this pair $\{i, j\}$. It follows from (8), (9) and our assumption on $\gamma$ that
\[ \left| \frac{p_{ij}}{H} \right| \geq |v_i - v_j| - |v_i - v_j - \frac{p_{ij}}{H}| \geq \frac{\rho}{\sqrt{2n}} - \frac{8\gamma}{\delta \sqrt{n}} > 0, \]
which shows $p_{ij} \neq 0$. Similarly, we have
\[ \left| \frac{p_{ij}}{H} \right| \leq |v_i - v_j| + |v_i - v_j - \frac{p_{ij}}{H}| \leq \frac{6}{\sqrt{\delta n}} + \frac{8\gamma}{\delta \sqrt{n}} < \frac{7}{\delta \sqrt{n}}. \]

This implies $H \geq \frac{1}{7} |p_{ij}| \sqrt{\delta n} \geq \frac{1}{7} \sqrt{\delta n}$, completing our proof.

\[ \square \]

2.5.2 Level sets

In this section, we partition $\mathbb{S}^{n-1} \setminus C(\delta, \rho)$ into level sets collecting unit vectors having comparable CLCD. To show that with a high probability the normal vector does not belong to a level set with a small CLCD, we construct an approximating set whose cardinality is well controlled from above. Since CLCD is stable with respect to small perturbations, the event that the normal vector has a small CLCD is contained in the event that one of the vectors in the approximating set has a small CLCD. We then apply the small ball probability estimate for individual vectors, combined with the union bound, to show that the latter event has probability close to zero.

Unless stated otherwise, we will assume throughout this section that $\delta, \rho, \mu$ and $\gamma$ are constants with
\[ 0 < \delta, \rho \ll 1, \quad 0 < \mu \ll_{\delta, \rho} \gamma \ll_{\delta, \rho} 1. \] (10)

Let $H_0 := \frac{1}{7} \sqrt{\delta n}$. By Lemma 2.15,
\[ \text{CLCD}_{\alpha, \gamma}(v) \geq H_0 \text{ for every } v \in \mathcal{N}(\delta, \rho). \]

**Definition 2.16** (Level sets of CLCD). Let $H \geq H_0/2$. We define the level set $S_H \subseteq \mathbb{S}^{n-1}$ as
\[ S_H := \{v \in \mathcal{N}(\delta, \rho): H \leq \text{CLCD}_{\mu n, \gamma}(v) \leq 2H\}. \]

We recall the following “tensorization” lemma of Rudelson and Vershynin [38, Lemma 2.2].

**Lemma 2.17** (Tensorization lemma). Suppose that $\varepsilon_0 \in (0, 1)$, $B \geq 1$, and let $X_1, \ldots, X_m$ be independent random variables such that each $X_i$ satisfies
\[ \mathbb{P}\{|X_i| \leq \varepsilon\} \leq B \varepsilon \text{ for all } \varepsilon \geq \varepsilon_0. \]

Then
\[ \mathbb{P}\{|(X_1, X_2, \ldots, X_m)| \leq \varepsilon \sqrt{m}\} \leq (CB \varepsilon)^m \text{ for every } \varepsilon \geq \varepsilon_0, \]
where $C > 0$ is an universal constant.
One can use the tensorization lemma to control the anti-concentration of $|Q_nv|$ where $v$ is a fixed vector. Indeed, let $R_1, \ldots, R_n$ denote the (independent) rows of $Q_n$. Then $|Q_nv|^2 = \sum_{i=1}^n (R_i, v)^2$, and we can apply Lemma 2.17 to $X_i := (R_i, v)$. Moreover, we can use Theorem 1.5 to bound the Lévy concentration function of each $X_i$. This gives:

**Lemma 2.18** (Invertibility on a single vector via small ball probability). For any $b > 0$ and $\mu, \gamma \in (0, 1)$ there exist $c_7 = c_7(b, \gamma, \mu) > 0$ and $C_7 = C_7(b, \gamma) > 0$ such that the following holds. For any $v \in \mathbb{R}^n$ with $|D(v)| \geq b\sqrt{n}$ and any $\varepsilon \geq 1/\text{CLCD}_{\mu, \gamma}(v) + e^{-c_7n}$, we have

$$
P\{|Q_nv| \leq \varepsilon \sqrt{n}\} \leq (C_7\varepsilon)^n.
$$

To run the covering argument, we need the following discretization of the level set $S_H$.

**Lemma 2.19** (Discretization of level sets). Assume that the parameters $\delta, \rho, \mu$ and $\gamma$ satisfy (10). Then there exists a net $N \subset S_H + \frac{8\mu\sqrt{n}}{H}B^2_2$ of cardinality at most $(C_S H/\sqrt{n})^n$ with the following properties.

(P1) For every $w \in N$, one has $\text{CLCD}_{\alpha/2, \gamma/2}(w) \geq H/32$ and $D(w) \gtrsim_{\delta, \rho} \sqrt{n}$.

(P2) For every (deterministic) $m \times n$ matrix $A$, and for any $v \in S_H$, one can find $w \in N$ so that

$$
|A(v - w)| \leq \frac{4\mu\sqrt{n}}{H} \left(2\|A\|_{\text{op}} + \frac{\|A\|_{\text{op}}}{n}\right).
$$

**Remark.** The property (P1) is in fact redundant since every vector $w$ in $S_H + \frac{8\mu\sqrt{n}}{H}B^2_2$ satisfies (P1). However, it allows one to simplify the presentation. To prove the claim, let us consider any $w \in S_H + \frac{8\mu\sqrt{n}}{H}B^2_2$. Take $v \in S_H$ so that $|v - w| \leq 8\mu\sqrt{n}/H \lesssim \delta \mu$. Since $v \in S_H \subseteq N(\delta, \rho)$, Lemma 2.2 shows $|D(v)| \gtrsim_{\delta, \rho} \sqrt{n}$. Therefore, $|v - w| < \frac{\gamma|D(v)|}{8\sqrt{n}}$, for $\mu \ll_{\delta, \rho} \gamma$, and so Lemma 2.14 is applicable. We then get

$$
\text{CLCD}_{\mu, \gamma}(v) \geq H/32
$$

as $\text{CLCD}_{\alpha, \gamma}(v) \geq H$ and $|v - w| \leq 8\mu\sqrt{n}/H$. Moreover, the triangle inequality gives

$$
|D(w)| \geq |D(v)| - |D(v - w)| \geq |D(v)| - \sqrt{n}|v - w| \geq |D(v)|/2 \gtrsim_{\delta, \rho} \sqrt{n},
$$

where in the third inequality we used the bound $|v - w| \leq \frac{D(v)}{\delta\sqrt{n}} \leq \frac{D(v)}{\delta\sqrt{n}}$. This completes the proof of our claim.

It remains to construct a net $N \subset S_H + \frac{8\mu\sqrt{n}}{H}B^2_2$ of cardinality at most $(C_S H/\sqrt{n})^n$ satisfying (P2), a task we now begin.

**Proof of Lemma 2.19.** We will explore the additive structure of $S_H$ to construct a small net. To this end, fix $v = (v_1, \ldots, v_n) \in S_H$, and let

$$
\sigma := \left\{ i \in [n] : |v_i| \leq \sqrt{2/n} \right\}.
$$

Since $|v| = 1$, it follows from Chebyshev inequality that

$$
|\sigma| \geq n/2. \tag{11}
$$

Denote $T := \text{CLCD}_{\mu, \gamma}(v)$. By the definition of $S_H$, we have $H \leq T < 2H$.

According to the definition of CLCD, there exists an integer vector $p = (p_{ij})_{1 \leq i < j \leq n}$ such that

$$
|T : D(v) - p| < \mu n. \tag{12}
$$
For $1 \leq j \leq n$, consider the vectors $v^{(j)} \in \mathbb{R}^{n-1}$ and $p^{(j)} \in \mathbb{Z}^{n-1}$ defined as follows

$$v^{(j)} := (v_1 - v_{j}, \ldots, v_{j-1} - v_{j}, v_{j} - v_{j+1}, \ldots, v_{n} - v_{n}), \quad p^{(j)} := (p_{1j}, \ldots, p_{j-1}, p_{jj+1}, \ldots, p_{jn}).$$

It follows from (12) that

$$\sum_{j=1}^{n} |T v^{(j)} - p^{(j)}|^2 = 2|T \cdot D(v) - p|^2 < 2(\mu n)^2.$$

Noting that $|\sigma| \geq n/2$ by (11), and using the pigeonhole principle, we thus get an index $j \in \sigma$ with

$$|T v^{(j)} - p^{(j)}|^2 \leq \frac{2(\mu n)^2}{|\sigma|} \leq 4\mu^2 n.$$

Taking the square root of both sides, and dividing by $T$ gives

$$|v^{(j)} - \frac{p^{(j)}}{T}| \leq \frac{2\mu \sqrt{n}}{T} \leq \frac{2\mu \sqrt{n}}{H} \quad \text{for some } j \in [n]. \quad (13)$$

By the inequality $(x + y)^2 \leq 2x^2 + 2y^2$, we get

$$|v^{(j)}|^2 \leq 2n v_j^2 + 2(v_1^2 + \ldots + v_n^2) \leq 6,$$

where in the last step we used the bound $|v_j| \leq \sqrt{2/n}$ along with our assumption that $|v| = 1$. Combining this bound with (13) yields

$$|p^{(j)}| \leq T|v^{(j)}| + |T v^{(j)} - p^{(j)}| \leq 2H \cdot \sqrt{6} + 2\mu \sqrt{n} \leq 7H. \quad (14)$$

To locate $v$, we discretize the ranges of $v_j$ and $T$. Consider the lattice intervals

$$\Lambda := \frac{\mu}{2H} \mathbb{Z} \cap [-1,1], \quad \Theta = \frac{1}{\mu} \mathbb{Z} \cap [H,2H].$$

Then one can find $\lambda_0 \in \Lambda$ and $T_0 \in \Theta$ such that

$$|v_j - \lambda_0| \leq \frac{\mu}{2H}, \quad |T - T_0| \leq \mu/7.$$

Letting $w = (\frac{p_{1j}}{T_0} + \lambda_0, \ldots, \frac{p_{jj+1}}{T_0} + \lambda_0, \lambda_0, -\frac{p_{jj+1}}{T_0} + \lambda_0, \ldots, -\frac{p_{jn}}{T_0} + \lambda_0)$, we see that

$$|v - w|^2 = \sum_{i=1}^{j-1} (v_i - \frac{p_{ij}}{T_0} - \lambda_0)^2 + \sum_{k=j+1}^{n} (v_k + \frac{p_{jk}}{T_0} - \lambda_0)^2$$

$$\leq 3 \sum_{i=1}^{j-1} \left\{ (v_i - v_j - \frac{p_{ij}}{T})^2 + (v_j - \lambda_0)^2 + \left( \frac{p_{ij}}{T} - \frac{p_{ij}}{T_0} \right)^2 \right\} + (v_j - \lambda_0)^2$$

$$+ 3 \sum_{k=j+1}^{n} \left\{ (v_k - v_j + \frac{p_{jk}}{T})^2 + (v_j - \lambda_0)^2 + \left( \frac{p_{jk}}{T} - \frac{p_{jk}}{T_0} \right)^2 \right\}$$

$$= 3|v^{(j)} - \frac{p^{(j)}}{T}|^2 + (3n - 2)(v_j - \lambda_0)^2 + \left( \frac{1}{T} - \frac{1}{T_0} \right)^2 |p^{(j)}|^2 \leq \frac{14\mu^2 n}{H^2} \quad (15)$$

where the second line uses the inequality $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$, while the last step uses $|v^{(j)} - \frac{p^{(j)}}{T}| \leq 2\mu \sqrt{n}/H$, $|v_j - \lambda_0| \leq \mu/(2H)$, $|\frac{1}{T} - \frac{1}{T_0}| \leq \mu/(7H^2)$ and $|p^{(j)}| \leq 7H$. 

14
It follows from (14) and (15) that \( v \) is within Euclidean distance \( 4\mu\sqrt{n}/H \) from the set
\[
F_j := \left\{ \left( \frac{q_1}{T_0} + \lambda_0, \ldots, \frac{q_i}{T_0} + \lambda_0, \frac{q_{i+1}}{T_0} + \lambda_0, \ldots, \frac{q_n}{T_0} + \lambda_0 \right) : \lambda_0 \in \Lambda, T_0 \in \Theta, q \in \mathbb{Z}^n \cap \Theta \right\}.
\]
There are at most \( 1 + 4H/\mu \) choices for \( \lambda_0 \in \Lambda \), at most \( 1 + 7H/\mu \) ways to choose \( T_0 \in \Theta \), and at most \( (1 + 21H/\sqrt{n})^n \) possibilities for the integer points \( q \) in \( B(0, 7H) \). This results in
\[
|F_j| \leq (1 + 4H/\mu) \cdot (1 + 7H/\mu) \cdot (1 + 21H/\sqrt{n})^n \leq (CH/\sqrt{n})^n,
\]
where in the last inequality we used the assumption that \( H \geq \beta \sqrt{n} \).
Applying Lemma 2.11 to \( S = S_H \) and \( \beta = 4\mu \sqrt{n}/H \), we therefore obtain a net \( N \subset S_H + \frac{8\mu \sqrt{n}}{H} B_2^n \) of cardinality at most
\[
(2n + 2) \cdot |F_1 \cup \ldots \cup F_n| \leq (2n + 2) \cdot n \cdot (CH/\sqrt{n})^n \leq (C_8 H/\sqrt{n})^n
\]
with the desired properties. This completes our proof.

**Lemma 2.20 (Invertibility on a level set).** There exist constants \( \mu, \gamma, c_9 \in (0, 1) \) and \( C_9 > 0 \) such that the following holds. Suppose that \( n \geq C_9 \) and \( H_0 \leq H \leq e^{c_9 n} \). Then
\[
\mathbb{P} \left\{ \inf_{w \in S_H} |Q'_n v| \leq c_9 n/H \right\} \leq 2e^{-n}.
\]

**Proof.** By Proposition 2.8, there exists a constant \( K \geq 1 \) such that
\[
\mathbb{P} \{ \|Q'_n|_H\|_{\text{op}} > K \sqrt{n} \} \leq e^{-n}.
\]
Thus, in order to complete the proof, it suffices to find constants \( C, c > 0 \) so that for \( n \geq C \) and \( H_0 \leq H \leq e^{c n} \), the event
\[
E := \left\{ \inf_{w \in S_H} |Q'_n v| \leq \frac{c n}{2H} \text{ and } \|Q'_n|_H\|_{\text{op}} \leq K \sqrt{n} \right\}
\]
has probability at most \( e^{-n} \).
We claim that this holds with the following choice of parameters:
\[
e = \min \left\{ 1/(8C_7 C_8), C_7, 1 \right\}, \quad C = \max \{ (64/e)^2, (C_7 \gamma)^{-2} \}, \quad 0 < \delta, \gamma < 1 \text{ and } 0 < \mu < \delta, \rho, K \gamma,
\]
where \( C_7, C_7 > 0 \) are the constants in Lemma 2.18, and \( C_8 > 0 \) is the constant in Lemma 2.19. Let \( N \) be the net defined in Lemma 2.19. Fix \( w \in N \). Then we have \( \text{CLCD}_{\mu/2, \gamma/2}(w) \geq H/32 \) and \( D(w) \geq \delta, \rho \sqrt{n} \). We see that \( e := c\sqrt{n}/H \) satisfies \( e \geq \frac{1}{\text{CLCD}_{\mu/2, \gamma/2}(w)} + e^{-c_7 n} \) assuming that \( n \geq (64/e)^2 \) and \( c \leq C_7 \). Thus Lemma 2.18 applies. We then get
\[
\mathbb{P} \left\{ |Q'_n w| \leq \frac{c n}{H} \right\} \leq \left( \frac{C_7 C\sqrt{n}}{H} \right)^{n-1}.
\]
By the union bound, noting that \( n \geq (C_7 \gamma)^{-2} \) and \( H \leq e^n \), we have
\[
\mathbb{P} \left\{ \inf_{w \in N} |Q'_n w| \leq \frac{c n}{H} \right\} \leq \left( \frac{C_8 H}{\sqrt{n}} \right)^n \left( \frac{C_7 \sqrt{n}}{H} \right)^{n-1} \leq (C_8 C_7 \gamma) n \cdot H \leq 8^{-n} \cdot e^n \leq e^{-n}.
\] (16)
Assume that the event \( E \) holds. Fix \( v \in S_H \) with \( |Q'_n v| \leq \frac{c n}{2H} \). From the definition of \( N \), we see that for every fixed realization of \( Q'_n \) there exists \( w \in N \) for which
\[
|Q'_n (v - w)| \leq \frac{4\mu \sqrt{n}}{H} \left( 2 \|Q'_n|_H\|_{\text{op}} + \frac{\|Q'_n\|_{\text{op}}}{n} \right).
\]
By the triangle inequality we thus get
\[ |Q_n^t w| \leq |Q_n^t v| + \frac{4\mu \sqrt{n}}{H} \left( 2\|Q_n^t\|_{\mathcal{H}} + \frac{\|Q_n\|_{\text{op}}}{n} \right) \leq \frac{cm}{2H} + \frac{4\mu \sqrt{n}}{H} \cdot (2K \sqrt{n} + 1) \leq \frac{cn}{H} \]
as \( \mu \ll K \).

We have shown that the event \( E \) implies the event that \( \inf_{w \in N'} |Q_n^t w| \leq \frac{cn}{H} \), whose probability is at most \( e^{-n} \) due to (16). This completes our proof. \( \square \)

Now we derive Proposition 2.13 from Lemma 2.20.

**Proof of Proposition 2.13.** Let \( \mu, \gamma, c_9 \in (0, 1) \) be constants from Lemma 2.20. Consider the event
\[ E := \{ \exists v \in N(\delta, \rho) \text{ with } Q_n^t v = 0 \text{ and } \text{CLCD}_{\mu, \gamma}(v) \leq e^{c_9 n} \} . \]

By Lemma 2.15, we have \( \text{CLCD}_{\mu, \gamma}(v) \geq H_0 \) for all \( v \in N(\delta, \rho) \), and so
\[ \mathbb{P}(E) \leq \sum_{H_0/2 \leq 2^k \leq e^{c_9 n}} \mathbb{P}\left\{ \exists v \in S_{2^k} \text{ with } Q_n^t v = 0 \right\} . \]

To estimate the sum, we apply Lemma 2.20. We then get
\[ \mathbb{P}\left\{ \exists v \in S_{2^k} \text{ with } Q_n^t v = 0 \right\} \leq \mathbb{P}\left\{ \inf_{v \in S_{2^k}} |Q_n^t v| \leq c_9 n/2^k \right\} \leq 2e^{-n} \]
for \( n \) sufficiently large. Taking the union bound yields
\[ \mathbb{P}(E) \leq c_9 n \log_2 e \cdot 2e^{-n} \leq 2^{-n} . \]

\( \square \)

### 2.6 Proofs of Theorems 1.2 and 1.3

We first deduce Theorem 1.3 from our small ball probability estimate and our bound on CLCD of the random normal.

**Proof of Theorem 1.3.** Choose parameters \( \delta \) and \( \rho \) such that \( 0 < \delta, \rho \ll 1 \). It follows from Proposition 2.10 that with probability at least \( 1 - 2e^{-c_5 n} \) any unit vector orthogonal to \( H_n \) is in \( N(\delta, \rho) \). Indeed, we learn from Proposition 2.10 that
\[ \mathbb{P}\{ \exists v \in C(\delta, \rho) \text{ orthogonal to } H_n \} \leq \mathbb{P}\{ \inf_{v \in \mathcal{C}(\delta, \rho)} |Q_n^t v| \leq \sqrt{n}/10 \} \leq 2e^{-c_5 n} . \]

Applying Proposition 2.13 together with the above observation, we get
\[ \mathbf{v} \text{ is in } N(\delta, \rho) \text{ and } \text{CLCD}_{\mu, \gamma}(\mathbf{v}) \geq e^{c_6 n} \]
with probability at least \( 1 - 2e^{-c_5 n} - 2^{-n} \). Application of Theorem 1.5 finishes the proof. \( \square \)

In the rest of this section we will prove Theorem 1.2. For this purpose, fix some parameters \( \delta, \rho \in (0, 1) \) whose values will be chosen later, and define the sets of sparse, compressible, and incompressible vectors as follows:
\[ \text{Sparse}(\delta) := \{ \mathbf{x} \in \mathbb{S}^{n-1} : |\text{supp}(\mathbf{x})| \leq \delta n \} , \]
\[ \text{Comp}(\delta, \rho) := \{ \mathbf{x} \in \mathbb{S}^{n-1} : \text{dist}(\mathbf{x}, \text{Sparse}(\delta)) \leq \rho \} , \]
\[ \text{Incomp}(\delta, \rho) := \mathbb{S}^{n-1} \setminus \text{Comp}(\delta, \rho) . \]

Next we derive Theorem 1.2 from Theorem 1.3, using the “invertibility via distance” lemma from [38].
**Lemma 2.21** (Invertibility via distance). Let $M$ be any random matrix. Let $R_1, \ldots, R_n$ denote the row vectors of $M$, and let $H_k$ denote the span of all row vectors except the $k$-th. Then for every $\delta, \rho \in (0, 1)$ and every $\varepsilon \geq 0$, one has
\[
P\left\{ \inf_{x \in \text{Incomp}(\delta, \rho)} |x^\top Q_n| \leq \varepsilon \frac{\rho}{\sqrt{n}} \right\} \leq \frac{1}{\delta n} \sum_{k=1}^{n} \P\{ \text{dist}(R_k, H_k) \leq \varepsilon \}.
\]

The proof of Theorem 1.2 also makes use of the following result, which gives a good uniform lower bound for $x^\top Q_n$ on the set of compressible vectors.

**Proposition 2.22.** There exist constants $\delta, \rho, c_{10} \in (0, 1)$ such that
\[
P\left\{ \inf_{x \in \text{Comp}(\delta, \rho)} |x^\top Q_n| \leq \sqrt{n}/270 \right\} \leq 2e^{-c_{10}n}.
\]

Before proceeding with the proof of Proposition 2.22, we show how to deduce Theorem 1.2 from Lemma 2.21 and Proposition 2.22.

**Proof of Theorem 1.2.** Consider the event
\[E := \left\{ \exists v \in \mathbb{S}^{n-1} \text{ such that } |Q_n v| \leq \varepsilon \frac{\rho}{\sqrt{n}} \right\}.
\]

Fix any realization of the matrix $Q_n$ such that the event holds, i.e. there exists a vector $v \in \mathbb{S}^{n-1}$ with $|Q_n v| \leq \varepsilon \frac{\rho}{\sqrt{n}}$. Since $Q_n$ and its transpose have the same singular values, there is a vector $x \in \mathbb{S}^{n-1}$ such that $|x^\top Q_n| \leq \varepsilon \frac{\rho}{\sqrt{n}}$. From this it follows that
\[
P(E) \leq \P\left\{ \inf_{x \in \text{Comp}(\delta, \rho)} |x^\top Q_n| \leq \varepsilon \frac{\rho}{\sqrt{n}} \right\} + \P\left\{ \inf_{x \in \text{Incomp}(\delta, \rho)} |x^\top Q_n| \leq \varepsilon \frac{\rho}{\sqrt{n}} \right\}
\]
\[
\leq \P\left\{ \inf_{x \in \text{Comp}(\delta, \rho)} |x^\top Q_n| \leq \sqrt{n}/270 \right\} + \frac{1}{\delta} \P\{ \text{dist}(R_n, H_n) \leq \varepsilon \}
\]
\[
\leq 2e^{-c_{10}n} + \frac{1}{\delta} (C\varepsilon + 2e^{-cn}),
\]

where the second line follows from Lemma 2.21, and in the last passage Proposition 2.22 and Theorem 1.3 were used. This completes our proof. 

The remainder of this section is devoted to a proof of Proposition 2.22. We recall a special case of Theorem 4 from [23].

**Theorem 2.23** (Sharp net for deterministic matrices). Consider any $S \subset \mathbb{S}^{n-1}$. Pick any $\alpha \in (0, \frac{1}{2})$, $\beta \in (0, \frac{\alpha}{\alpha^2})$. Let $n \geq 1/\alpha^2$. There exists a (deterministic) net $\mathcal{N} \subset S + \frac{4\beta}{\alpha^2} B_2^n$ with
\[|\mathcal{N}| \leq N(S, \beta B_2^n) \cdot \left( \frac{C_{11}}{\alpha} \right)^{C_{12} \alpha^{0.08} n} \]
such that for every $m \in \mathbb{N}$ and for every (deterministic) $m \times n$ matrix $A$, the following holds: for every $x \in S$ there exists $y \in \mathcal{N}$ satisfying
\[|(x - y)^\top A| \leq \frac{2\beta}{\alpha \sqrt{n}} \|A\|_{\text{HS}}.
\]

Here $C_{11}$ and $C_{12}$ are absolute constants.
Remark. Theorem 2.23 can completely replace Lemma 2.11; we leave the details to the interested reader. On the other hand, Lemma 2.11 may be useful elsewhere, and its proof is much simpler than that of Theorem 2.23. For these reasons, we have decided to keep both approaches. We also embrace the following anti-concentration estimate due to Jain, Sah and Sawhney (private communication), which answered Question 4.1 in the previous version of this manuscript.

Lemma 2.24. For each \( x \in \mathbb{S}^{n-1} \), we have
\[
\mathbb{P}\{|x^T Q_n| \leq \sqrt{n/90}\} \leq e^{-n/3000}.
\]

The proof of Lemma 2.24 will be given in appendix A. We conclude this section with a proof of Proposition 2.22, employing Lemma 2.24.

Proof of Proposition 2.22. Set \( \alpha = 540(\delta + \rho) \) and \( \beta = \delta + \rho \), where \( 0 < \delta, \rho \ll 1 \). Assume that \( n \geq 1/\alpha^2 \). Observe that
\[
N(\text{Comp}(\delta, \rho), \beta B^n_2) \leq \left(\frac{n}{\delta n}\right)^{\delta n} \cdot \left(\frac{4}{\delta}\right)^{\delta n} \leq e^{2\delta \log(4/\delta)n}.
\]

Applying Theorem 2.23 together with the above observation, we get a net \( \mathcal{N} \subset 3\beta B^n_2 \setminus \beta B^n_2 \) with
\[
|\mathcal{N}| \leq e^{2\delta \log(4/\delta)n} \cdot \left(\frac{C_{11}}{\alpha}\right)^{C_{12} \alpha^{0.08} n} \leq e^{C_{\alpha^{0.08} \log(1/\alpha)n}}.
\]

For fixed \( y \in \mathcal{N} \), Lemma 2.24 tells us that
\[
\mathbb{P}\{|y^T Q_n| \leq \sqrt{n/135}\} \leq e^{-n/3000}
\]
Taking the union bound, we then get
\[
\mathbb{P}\left\{\inf_{y \in \mathcal{N}} |y^T Q_n| \leq \sqrt{n/135}\right\} \leq e^{C_{\alpha^{0.08} \log(1/\alpha)n}} \cdot e^{-n/3000} < e^{-n/4000}.
\]

Consider the event
\[
\mathcal{E} := \left\{\inf_{x \in \text{Comp}(\delta, \rho)} |x^T Q_n| \leq \sqrt{n/270}\right\}.
\]
To bound \( \mathbb{P}(\mathcal{E}) \), we suppose that \( \mathcal{E} \) occurs. Then \( |x^T Q_n| \leq \sqrt{n/270} \) for some \( x \in \mathbb{S}^{n-1} \). Since \( \|Q_n\|_{\text{HS}} \leq n \), Theorem 2.23 shows the existence of \( y \in \mathcal{N} \) with
\[
|(x - y)^T Q_n| \leq (2\beta/\alpha)\sqrt{n} \leq \sqrt{n}/270.
\]
In particular, one has
\[
|y^T Q_n| \leq |x^T Q_n| + |(x - y)^T Q_n| \leq \sqrt{n}/135.
\]
Therefore, we find
\[
\mathbb{P}(\mathcal{E}) \leq \mathbb{P}\{|y^T Q_n| \leq \sqrt{n}/135\} \leq e^{-n/4000}
\]
for \( n \) sufficiently large, which completes our proof.
3 Anti-concentration for combinatorial statistics

In this section we will derive Theorem 1.5 from a more general result, namely Theorem 3.2. We will prove Theorem 1.5 in Section 3.1 assuming the validity of Theorem 3.2. The proof of Theorem 3.2 is then spread across Sections 3.2 and 3.3, as well as appendix B.

Let \( \mathbf{a} \) and \( \mathbf{v} \) be two vectors in \( \mathbb{R}^n \). The combinatorial statistic
\[
W_{\mathbf{a}, \mathbf{v}} := a_1 v_{\sigma(1)} + \ldots + a_n v_{\sigma(n)},
\]
where \( \sigma \) is a uniformly random permutation of \([n]\), plays a fundamental role in statistics (see the book [12] for an overview) as well as probability (see e.g. [6, 10, 11, 15, 17, 18, 19, 28, 32]). In analogy with the least common denominator (LCD) developed by Rudelson and Vershynin [38], we define a combinatorial version of LCD, which will be instrumental in controlling the anti-concentration of \( W_{\mathbf{a}, \mathbf{v}} \).

**Definition 3.1** (Combinatorial least common denominator). Given two vectors \( \mathbf{a} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \), as well as parameters \( L, u > 0 \), the Combinatorial Least Common Denominator of the pair \((\mathbf{a}, \mathbf{v})\) is
\[
\text{CLCD}_{L,u}^a(\mathbf{v}) := \inf \left\{ \theta > 0 : \text{dist}(\theta \cdot D(\mathbf{a}) \otimes D(\mathbf{v}), \mathbb{Z}((2)^2)) < \min\left( u \| D(\mathbf{a}) \otimes D(\mathbf{v}) \|, L \right) \right\}.
\]
Here by \( \otimes \) we denote the tensor product.\(^1\)

The usefulness of CLCD is demonstrated in the following result, which shows how CLCD of the pair \((\mathbf{a}, \mathbf{v})\) governs the small ball probability of \( W_{\mathbf{a}, \mathbf{v}} \).

**Theorem 3.2** (Small ball probability). Let \( \mathbf{a} \) and \( \mathbf{v} \) be two vectors in \( \mathbb{R}^n \) with \( |D(\mathbf{a}) \otimes D(\mathbf{v})| \geq bn^{3/2} \) for some \( b > 0 \). Let \( L > 0 \) and \( u \in (0,1) \). Then for every \( \varepsilon \geq 0 \), we have
\[
\mathcal{L}(W_{\mathbf{a}, \mathbf{v}}, \varepsilon) \leq C\varepsilon + \frac{C}{\text{CLCD}_{L,u}^a(\mathbf{v})} + C e^{-sL^2/n^3}.
\]

The constant \( C > 0 \) here depends only on \( b \) and \( u \).

3.1 Deriving Theorem 1.5 from Theorem 3.2

In this short section we formally derive Theorem 1.5 from Theorem 3.2. For the reader’s convenience, we restate Theorem 1.5.

**Theorem 1.5** (Small ball probability). For any \( b > 0 \) and \( \gamma \in (0,1) \) there exists \( C > 0 \) depending only on \( b \) and \( \gamma \) with the following property. Let \( \mathbf{v} \in \mathbb{R}^n \) such that \( |D(\mathbf{v})| \geq b\sqrt{n} \). Then for every \( \alpha > 0 \) and \( \varepsilon \geq 0 \), we have
\[
\mathcal{L}(W_{\mathbf{v}}, \varepsilon) \leq C\varepsilon + \frac{C}{\text{CLCD}_{\alpha,\gamma}^a(\mathbf{v})} + C e^{-2\alpha^2/n}.
\]

**Proof of Theorem 1.5 assuming Theorem 3.2.** Let \( \mathbf{a} := (1, \ldots, 1, 0, \ldots, 0) \in \{0,1\}^n \), \( L := \alpha n/2 \) and \( u := \gamma \). We can interpret \( D(\mathbf{a}) \otimes D(\mathbf{v}) \) as a collection of \( n^2/4 \) copies of \( D(\mathbf{v}) \). Thus we have that
\[
|D(\mathbf{a}) \otimes D(\mathbf{v})| = \frac{1}{2n} |D(\mathbf{v})| \geq \frac{s}{8} bn^{3/2},
\]
and that
\[
\text{CLCD}_{L,u}^a(\mathbf{v}) = \text{CLCD}_{\alpha,\gamma}^a(\mathbf{v}).
\]
Hence Theorem 3.2 is applicable. Noting that \( W_{\mathbf{a}, \mathbf{v}} \) and \( W_{\mathbf{v}} \) have the same law, we then get
\[
\mathcal{L}(W_{\mathbf{v}}, \varepsilon) = \mathcal{L}(W_{\mathbf{a}, \mathbf{v}}, \varepsilon) \leq C\varepsilon + \frac{C}{\text{CLCD}_{\alpha,\gamma}^a(\mathbf{v})} + C e^{-2\alpha^2/n}.
\]

\(^1\)In particular, \( D(\mathbf{a}) \otimes D(\mathbf{v}) \) is a vector in \( \mathbb{R}(((2)^2)) \) whose \((i,j,k,\ell)\)-coordinate is \((a_i - a_j)(v_k - v_\ell)\), for \( 1 \leq i < j \leq n \) and \( 1 \leq k < \ell \leq n \).
3.2 Proof of Theorem 3.2

Recall the following anti-concentration inequality due to Essén (see e.g. [9, 38]).

Lemma 3.3. Given a random variable $\xi$ with the characteristic function $\varphi(\cdot) = \mathbb{E}\exp(2\pi i \xi)$, one has

$$\mathcal{L}(\xi, \varepsilon) \lesssim \int_{-1}^{1} \left| \varphi\left(\frac{\theta}{\varepsilon}\right) \right| d\theta, \quad \varepsilon \geq 0.$$ 

To use Essén’s lemma, we require the following critical estimate for the characteristic function of the statistic $W_{\mathbf{a}, \mathbf{v}}$.

Proposition 3.4. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$. Let $\varphi$ be the characteristic function of $W_{\mathbf{a}, \mathbf{v}}$. Then

$$|\varphi(\theta)| \leq \frac{1}{(n/2)^2} \sum_{\{s, t\}, \{p, q\} \in \binom{[n]}{2}} \cos^2 \left(\frac{\pi \theta (a_s - a_t)(v_p - v_q)}{\varepsilon}\right) \quad \left(\frac{(n-1)}{4}\right) \quad , \quad \theta \in \mathbb{R}.$$ 

The proof of Proposition 3.4 is deferred to Section 3.3 and appendix B. In this section we instead show how to deduce Theorem 3.2 from it.

Proof of Theorem 3.2. Take any $\varepsilon \geq 1/\text{CLCD}_{L,u}(\mathbf{v})$. From Proposition 3.4 we know that

$$\left| \varphi\left(\frac{\theta}{\varepsilon}\right) \right| \leq \frac{1}{(n/2)^2} \sum_{\{s, t\}, \{p, q\} \in \binom{[n]}{2}} \cos^2 \left(\frac{\pi \theta (a_s - a_t)(v_p - v_q)}{\varepsilon}\right) \quad \left(\frac{(n-1)}{4}\right).$$

By convexity, we have that $|\sin \pi z| \geq 2 \text{dist}(z, \mathbb{Z})$ for any $z \in \mathbb{R}$. Thus

$$\cos^2 \pi z = 1 - \sin^2 \pi z \leq 1 - 4 \text{dist}^2(z, \mathbb{Z}).$$

It follows that

$$\left| \varphi\left(\frac{\theta}{\varepsilon}\right) \right| \leq \left[ 1 - \frac{4}{(n/2)^2} \sum_{\{s, t\}, \{p, q\} \in \binom{[n]}{2}} \text{dist}^2\left(\frac{\theta}{\varepsilon} (a_s - a_t)(v_p - v_q), \mathbb{Z}\right) \right]^{(n-1)/4} \quad \left(\frac{(n-1)}{4}\right) \leq \exp\left( -\frac{8}{n^3} \sum_{\{s, t\}, \{p, q\} \in \binom{[n]}{2}} \text{dist}^2\left(\frac{\theta}{\varepsilon} (a_s - a_t)(v_p - v_q), \mathbb{Z}\right) \right) \quad \left(\frac{(n-1)}{4}\right) \leq \exp\left( -\frac{8}{n^3} \text{dist}^2\left(\frac{\theta}{\varepsilon} D(\mathbf{a}) \otimes D(\mathbf{v}), \mathbb{Z}^{(n/2)^2}\right) \right),$$

where in the second inequality we used the fact that $1 - z \leq e^{-z}$ for any $z \in \mathbb{R}$.

Combining this with Lemma 3.3 gives

$$\mathcal{L}(W_{\mathbf{a}, \mathbf{v}}, \varepsilon) \lesssim \int_{-1}^{1} \left| \varphi\left(\frac{\theta}{\varepsilon}\right) \right| d\theta \quad \leq \int_{-1}^{1} \exp\left(-8h^2(\theta)/n^3\right) d\theta,$$

where $h(\theta) := \text{dist}(\frac{\theta}{\varepsilon} \cdot D(\mathbf{a}) \otimes D(\mathbf{v}), \mathbb{Z}^{(n/2)^2})$.
Since $1/\varepsilon \leq \text{CLCD}_{L,u}(v)$, it follows that for any $\theta \in [-1, 1]$ we have
\[ h(\theta) \geq \min \left( \frac{\theta}{\varepsilon} \cdot D(a) \otimes D(v), L \right) \geq \min \left( \frac{ubn^{3/2} |\theta|}{\varepsilon}, L \right), \]
assuming that $|D(a) \otimes D(v)| \geq bn^{3/2}$. Therefore,
\[ \mathcal{L}(W_{a,v}, \varepsilon) \lesssim \int_{-1}^{1} \left[ \exp(-8(ub\theta/\varepsilon)^2) + \exp(-8L^2/n^3) \right] d\theta \lesssim \frac{\varepsilon}{ub} + \exp(-8L^2/n^3). \]

3.3 Proof of Proposition 3.4

Before proceeding to the proof let us fix some notation. We will use the following non-standard probability notation: $x \sim S$ indicates that a random variable $x$ is chosen uniformly from the set $S$; $x \sim S$, $y \sim T$ means that $x$ and $y$ are two independent random variables with $x \sim S$ and $y \sim T$; $x, y \sim S$ means that two independent random variables $x$ and $y$ have uniform distribution on $S$. Given a set $S$ and a positive integer $d$, let $\mathcal{M}_S^d$ denote the family of all sequences of $d$ unordered disjoint pairs of elements in $S$, i.e., $\mathcal{M}_S^d$ is the family of size-$d$ ordered matchings of $S$. We will write $S_n$ for the collection of all permutations of $[n]$. As in the previous section, we use $\varphi(\cdot)$ to denote the characteristic function of the random sum $W_{a,v} := a_1 v_{\sigma(1)} + \ldots + a_n v_{\sigma(n)}$.

The following estimate is our starting point.

**Lemma 3.5.** Let $M_1 = ((s_1, t_1), \ldots, (s_d, t_d))$ be a matching in $\mathcal{M}_n^d$. Then
\[ |\varphi(\theta)| \leq \mathbb{E}_{M_2} \prod_{i=1}^{d} \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{p_i} - v_{q_i}), \]
where $M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\})$ is drawn uniformly at random from $\mathcal{M}_n^d$.

For Lemma 3.5, we use a decoupling argument: it turns out that there is a natural way to realize the distribution of a random permutation as a mixture of a random permutation and a random Rademacher vector. The following observation appears in the proof of [10, Theorem 6] (a similar coupling also appears in [11, 17, 18]).

**Fact 3.6.** Suppose that $\sigma \sim S_n$. Let $s_1, t_1, \ldots, s_d, t_d$ be $2d$ pairwise distinct elements of $[n]$. Consider the random permutation $\tilde{\sigma}$ obtained from $\sigma$ by swapping, via a random subset $S$ of $\{(s_1, t_1), \ldots, (s_d, t_d)\}$ uniformly chosen from all $2d$ subsets.\(^2\) Then $\tilde{\sigma} \sim S_n$.

(Note that if $\sigma$ is uniformly distributed then so is $\tau \circ \sigma$ for any fixed $\tau$. Therefore, the distribution of $\tilde{\sigma}$ is a mixture of uniform distributions, which is uniform as well.)

**Proof of Lemma 3.5.** Let $\sigma, S$ and $\tilde{\sigma}$ be random variables as in Fact 3.6. Since
\[ a_{s_i} v_{\sigma(s_i)} + a_{t_i} v_{\sigma(t_i)} = -(a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + (a_{s_i} v_{\sigma(s_i)} + a_{t_i} v_{\sigma(t_i)}), \]
we see that
\[ \sum_{i=1}^{n} a_i v_{\tilde{\sigma}(i)} = -\sum_{i=1}^{d} \mathbb{1}_{\{(s_i, t_i)\in S\}} (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + \sum_{i=1}^{n} a_i v_{\sigma(i)}. \tag{17} \]
\(^2\)For example, if the subset $\{(s_i, t_i)\}$ is chosen then $\tilde{\sigma}(s_i) = \sigma(t_i)$, $\tilde{\sigma}(t_i) = \sigma(s_i)$ and $\tilde{\sigma}(k) = \sigma(k)$ for $k \in [n] \setminus \{s_i, t_i\}$. 

21
Let $\xi_1, \ldots, \xi_d$ be independent random sign variables which are independent of $\sigma$. Then (17) indicates that the random sum $\sum_{i=1}^{n} a_i \epsilon_{\sigma(i)}$ has the same distribution as the random variable

$$X := -\sum_{i=1}^{d} \frac{\xi_i + 1}{2} (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + \sum_{i=1}^{n} a_i v_{\sigma(i)}.$$

We may write

$$X = -\sum_{i=1}^{d} \frac{\xi_i}{2} (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + Y,$$

where $Y := -\frac{1}{2} \sum_{i=1}^{d} (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + \sum_{i=1}^{n} a_i v_{\sigma(i)}$ is mutually independent of $\xi_1, \ldots, \xi_d$. According to Fact 3.6, $\bar{\sigma}$ is uniformly distributed on $S_n$, so that

$$|\varphi(\theta)| = \left| \mathbb{E}_{\bar{\sigma}} \exp \left( 2\pi i \theta \sum_{i=1}^{n} a_i v_{\bar{\sigma}(i)} \right) \right|$$

$$= \left| \mathbb{E}_{\sigma, \xi_1, \ldots, \xi_d} \exp \left\{ -\pi i \theta \sum_{i=1}^{d} \xi_i (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + 2\pi i \theta Y \right\} \right|$$

$$\leq \mathbb{E}_{\sigma, \xi_1, \ldots, \xi_d} \left| \mathbb{E}_{\sigma} \exp \left\{ -\pi i \theta \sum_{i=1}^{d} \xi_i (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) + 2\pi i \theta Y \right\} \right|$$

$$= \mathbb{E}_{\sigma} \prod_{i=1}^{d} \left| \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) \right|. \quad (18)$$

We sample $M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\})$ uniformly at random from $\mathcal{M}_{[n]}^d$. Then

$$\mathbb{E}_{\bar{\sigma}} \prod_{i=1}^{d} \left| \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{\sigma(s_i)} - v_{\sigma(t_i)}) \right| = \mathbb{E}_{M_2} \prod_{i=1}^{d} \left| \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{p_i} - v_{q_i}) \right|. \quad (19)$$

From (18) and (19) we conclude that

$$|\varphi(\theta)| \leq \frac{1}{\mathbb{E}_{M_2} \prod_{i=1}^{d} \left| \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{p_i} - v_{q_i}) \right|}. \quad \square$$

Depending on our choice of $M_1$, Lemma 3.5 may give a very poor upper bound on $|\varphi(\theta)|$. For example, it delivers the trivial estimate $|\varphi(\theta)| \leq 1$ when $a_{s_1} - a_{t_1} = \cdots = a_{s_d} - a_{t_d} = 0$. To circumvent this situation we take the average over matchings $M_1$. We thus obtain:

**Lemma 3.7.** Let $M_1 = (\{s_1, t_1\}, \ldots, \{s_d, t_d\})$ and $M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\})$ be two independent random variables, each with the uniform distribution on $\mathcal{M}_{[n]}^d$. Then

$$|\varphi(\theta)| \leq \mathbb{E}_{M_1, M_2} \prod_{i=1}^{d} \left| \cos \pi \theta (a_{s_i} - a_{t_i}) (v_{p_i} - v_{q_i}) \right|.$$
The proof of Lemma 3.8 will be given in appendix B. For the rest of this section we show how Lemma 3.8 together with Lemma 3.7 implies Proposition 3.4.

Proof of Proposition 3.4. Let \( d := \lfloor n/2 \rfloor \), \( M_1 = (\{s_1, t_1\}, \ldots, \{s_d, t_d\}) \) and \( M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\}) \). By Lemma 3.7, we have

\[
|\varphi(\theta)| \leq \mathbb{E}_{M_1, M_2 \sim M_d^{[n]}} \prod_{i=1}^{d} |\cos \pi \theta(a_{s_i} - a_{t_i})(v_{p_i} - v_{q_i})|.
\] (20)

Using the Cauchy-Schwarz inequality yields

the RHS of (20) \[
= \mathbb{E}_{S_1, S_2 \sim \binom{[n]}{2d}} \left( \mathbb{E}_{M_1 \sim M_{S_1}^d, M_2 \sim M_{S_2}^d} \prod_{i=1}^{d} \cos \pi \theta(a_{s_i} - a_{t_i})(v_{p_i} - v_{q_i}) \right)
\]

\[
\leq \left( \mathbb{E}_{S_1, S_2 \sim \binom{[n]}{2d}} \left( \mathbb{E}_{M_1 \sim M_{S_1}^d, M_2 \sim M_{S_2}^d} \prod_{i=1}^{d} \cos \pi \theta(a_{s_i} - a_{t_i})(v_{p_i} - v_{q_i}) \right) \right)^{1/2}.
\] (21)

Applying Lemma 3.8 to \( d = \lfloor n/2 \rfloor \) and \( f_i(\{s, t\}, \{p, q\}) = |\cos \pi \theta(a_s - a_t)(v_p - v_q)| \) for \( i \in [d] \), we get

the RHS of (21) \[
\leq \prod_{i=1}^{d} \mathbb{E}_{\{s, t\}, \{p, q\} \sim \binom{[n]}{2}} \cos^2 \pi \theta(a_s - a_t)(v_p - v_q)
\]

\[
= \left( \mathbb{E}_{\{s, t\}, \{p, q\} \in \binom{[n]}{2}} \cos^2 \pi \theta(a_s - a_t)(v_p - v_q) \right)^{d}.
\]

Finally, combining the above inequalities, and noting that \( d \geq (n - 1)/2 \), we obtain

\[
|\varphi(\theta)| \leq \left( \mathbb{E}_{\{s, t\}, \{p, q\} \sim \binom{[n]}{2}} \cos^2 \pi \theta(a_s - a_t)(v_p - v_q) \right)^{(n-1)/4}.
\]

\( \square \)

4 Concluding remarks

In this section we highlight some possible avenues for further investigation.

4.1 Exchangeable random matrices.

Let \( M_n \) be a random \( n \times n \) matrix. One source of motivation for finding good lower bounds on the least singular value \( s_n(M_n) \) is its relation to the problem of proving the circular law for the distribution of eigenvalues of \( M_n \). A model of random matrices, which is most relevant to us, was introduced in [1]. Let \((a_{ij})_{1 \leq i, j \leq n}\) be a deterministic real matrix such that

\[
\sum_{i, j=1}^{n} a_{ij} = 0 \quad \text{and} \quad \sum_{i, j=1}^{n} a_{ij}^2 = n^2.
\]

We consider the exchangeable random matrix \( M_n \) obtained by shuffling the deterministic matrix \((a_{ij})\) using a random uniform permutation, i.e.,

\[
M_n = (a_{\sigma(i,j)})_{1 \leq i, j \leq n}, \quad \text{where} \quad \sigma \quad \text{is a uniformly random permutation of the set } \{(i, j) : 1 \leq i, j \leq n\}.
\]

23
Such random matrices have dependent entries, dependent rows, and dependent columns. In order to prove the circular law for $M_n$, Adamczak, Chafaï and Wolff [1, Theorem 1.1] established a polynomial bound on the smallest singular value of the shifted matrix $\frac{1}{\sqrt{n}}M_n - zI$:

$$
\mathbb{P}\{s_n\left(\frac{1}{\sqrt{n}}M_n - zI\right) \leq \frac{\varepsilon}{\sqrt{n}}\} \lesssim_{K,z} \varepsilon + \frac{1}{\sqrt{n}} \quad \text{for every } z \in \mathbb{C} \text{ and } \varepsilon \geq 0,
$$

where $K := \max_{i,j}|a_{ij}|$. They asked whether the factor $\frac{1}{\sqrt{n}}$ in the above estimate can be improved to $e^{-cn}$. We believe that our techniques (combined with additional twists) should allow one to solve this problem.

4.2 Inverse Littlewood-Offord theory

Given two vectors $a$ and $v$ in $\mathbb{R}^n$, we define the concentration probability as

$$
\rho_{a,v} := \sup_x \mathbb{P}(W_{a,v} = x).
$$

Söze [42, Corollary 5] showed $\rho_{a,v} \lesssim \frac{1}{n}$ assuming that $a = (1, 2, \ldots, n)$ and that $v$ is a non-constant vector, and used this estimate to bound the expected number of real roots of random polynomials with exchangeable coefficients. Motivated by their study of representations of reductive groups, Huang, McKinnon and Satriano [14] recently raised the problem of bounding $\rho_{a,v}$ when $a$ has distinct coordinates and $v$ is a non-constant vector. Under these assumptions they showed that $\rho_{a,v} \lesssim \frac{1}{n}$, which generalizes Söze’s result. Shortly after the Huang-McKinnon-Satriano paper, Pawlowski [35] gave a combinatorial proof of the refinement

$$
\rho_{a,v} \leq \frac{2\lfloor n/2 \rfloor}{n(n-1)}.
$$

(22)

This estimate is sharp, as demonstrated by $a = (1, \ldots, n)$ and $v = (-\sum_{i=2}^{n-1} i, -\sum_{i=2}^{n-1} i, n + 1, \ldots, n + 1)$. It is likely that (22) can be improved significantly by making additional assumptions about $a$ and $v$. Phrased differently, it would be very interesting to find an answer to the following basic question:

**Question 4.1.** What is the underlying reason why $\rho_{a,v}$ could be large?

We remark that Nguyen and Vu [32, Theorem 4.4] gave a partial answer to the above question when $a = (1, \ldots, 1, 0, \ldots, 0) \in \{0, 1\}^n$.

Another interesting direction is to extend Pawlowski’s result to other ambient groups. For a detailed account of the Inverse Littlewood-Offord theory, we refer the reader to an excellent survey of Nguyen and Vu [33].

4.3 Local limit theorems for combinatorial random variables

During the preparation of this manuscript we learnt of an independent, concurrent result of Sah and Sawhney [40, Lemma 2.7].

**Lemma.** For $v \in \mathbb{R}^n$, consider the random variable $W := \sum_{i=1}^{n} \eta_i v_i$, where $(\eta_1, \ldots, \eta_n)$ is drawn uniformly from $\{0, 1\}^n$ subject to $\sum_{i=1}^{n} \eta_i = s$. Let $\varphi$ be the characteristic function of $W$. Furthermore, suppose that $p = s/n$ and $\theta \in \mathbb{R}$ is such that $|v_i - v_j| \theta \leq \frac{1}{2}$ for all $1 \leq i, j \leq n$. Then

$$
|\varphi(\theta)| \leq (n + 1) \exp\left(-\frac{4p(1-p)\theta^2}{n} \sum_{1 \leq i,j \leq n} (v_i - v_j)^2\right).
$$
This estimate was critical for Sah and Sawhney’s study of local limit theorems for subgraph counts in the Erdős-Renyi random graph. Proposition 3.4 shows that one can remove the $n+1$ factor from this bound. As it turned out, the saving factor $n+1$ here plays a crucial role in our proofs of Theorems 1.2 and 1.3.

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A (by Jain, Sah and Sawhney): Proof of Lemma 2.24

We first recall the following variant of Paley-Zygmund inequality given by Litvak, Pajor, Rudelson and Tomczak-Jaegermann [20, Lemma 3.5].

Lemma A.1. Let $X$ be a nonnegative random variable with $\mathbb{E}[X^4] < \infty$. Then for $0 \leq \lambda < \sqrt{\mathbb{E}[X^2]}$ we have

$$\mathbb{P}\{X > \lambda\} \geq \frac{\left(\mathbb{E}[X^2] - \lambda^2\right)^2}{\mathbb{E}[X^4]}.$$

We also need the following general hypercontractive estimate, which is a variant of [34, Theorem 10.21].

Lemma A.2. Let $X$ be a vector of independent Bernoulli random variables with minimum atom probability lower bounded by $b$. Let $f$ be a polynomial function of $X$ with degree at most $d$. Then, for all $q \geq 2$ we have that

$$\mathbb{E}[|f(X)|^q]^{1/q} \leq (\sqrt{q - 1} b^{1/q - 1/2})^d \mathbb{E}[|f(X)|^2]^{1/2}.$$

We now prove Lemma 2.24; it is closely modelled after [20, Proposition 3.4].

Proof of Lemma 2.24. Let $\Gamma$ be the first $N = n/4$ columns of $Q_n$. Then, $\Gamma$ is an $n \times N$ matrix with independent rows, with each row distributed as the indicator of the elements in $[N]$ for a randomly chosen size $n/2$ subset of $[n]$. We also have $|x| = 1$. Let the elements of $\Gamma$ be $\Gamma_{ij}$, and the columns of $\Gamma$ be $\Gamma_j$. Let

$$y_j = (x^\top \Gamma)_j = x^\top \Gamma_j = \sum_{i=1}^{n} x_i \Gamma_{ij}.$$

For every $\tau > 0$, we have

$$\mathbb{P}\{|x^\top Q_n| \leq \tau \sqrt{N}\} \leq \mathbb{P}\{|x^\top \Gamma| \leq \tau \sqrt{N}\} = \mathbb{P}\left\{\sum_{j=1}^{N} y_j^2 \leq \tau^2 N\right\} = \mathbb{P}\left\{N - \frac{1}{\tau^2} \sum_{j=1}^{N} y_j^2 \geq 0\right\}$$

$$\leq \mathbb{E}\exp\left(\tau N - \frac{\tau^2}{2} \sum_{j=1}^{N} y_j^2\right) \leq e^{\tau N} \prod_{j=1}^{N} \sup_{\Gamma_{1}, \ldots, \Gamma_{j-1}} \mathbb{E}[\exp(-\tau y_j^2/2)]\mathbb{E}[\Gamma_1, \ldots, \Gamma_{j-1}],$$

where the last inequality is obtained by iterating the law of total expectation and since all terms considered are positive.

Now, we study $y_j$ conditional on $\Gamma_1, \ldots, \Gamma_{j-1}$. Given these values, we have $\mathbb{E}[y_j^2] \geq \mathbb{Var}[y_j] \geq \frac{2}{n}$ since conditional on the first $j-1$ columns for $j \leq n/4$, each entry in the $j^{th}$ column is distributed as $\text{Ber}(p)$ with $p$ between $1/3$ and $2/3$. Next, using hypercontractivity (Lemma A.2) with $q = 4, b = 1/3$ and $d = 1$, we find that

$$\mathbb{E}[y_j^4] \leq 27 \mathbb{E}[y_j^2]^2.$$
Thus, setting \( \lambda^2 = \mathbb{E}[y_j^2]/2 \) in Lemma A.1, we find that
\[
\mathbb{P}\{y_j > 1/3\} \geq \mathbb{P}\{y_j^2 > \mathbb{E}[y_j^2]/2\} \geq \frac{\mathbb{E}[y_j^2]^2}{4\mathbb{E}[y_j^2]} \geq \frac{1}{108}.
\]
Using this we immediately find that
\[
\mathbb{E}[\exp(-\tau y_j^2/t^2)|\Gamma_1, \ldots, \Gamma_{j-1}] \leq 107/108 + 1/108 \exp(-\tau/(9t^2)).
\]
Therefore,
\[
\mathbb{P}\{|x^\top Q_n| \leq t\sqrt{N}\} \leq e^{c_N} (107/108 + 1/108 \exp(-\tau/(9t^2)))^N,
\]
and setting \( t = 1/45 \) and \( \tau = 1/500 \) gives
\[
\mathbb{P}\{|x^\top Q_n| \leq \sqrt{n}/90\} \leq \exp(-n/3000).
\]

\[\square\]

### B Proof of Lemma 3.8

The proof of Lemma 3.8 makes use of the following Cauchy-Schwarz type inequality.

**Lemma B.1.** For positive integers \( n, k \) and \( \ell \) with \( k + \ell \leq n \), and real-valued functions \( g: \binom{n}{k} \to \mathbb{R} \) and \( h: \binom{n}{\ell} \to \mathbb{R} \), one has
\[
\mathbb{E}_{S \sim \binom{n}{k+\ell}} \left( \mathbb{E}_{I \sim \binom{k}{\ell}} g(I) h(S \setminus I) \right)^2 \leq \left( \mathbb{E}_{I \sim \binom{n}{k}} g(I)^2 \right) \left( \mathbb{E}_{J \sim \binom{n}{\ell}} h(J)^2 \right).
\]

**Proof.** Let \( \lambda_{n,k,\ell} := \frac{n \choose k+\ell} {n \choose k} {k+\ell \choose \ell} \). We can rewrite the desired inequality as
\[
\sum_{S \in \binom{n}{k+\ell}} \left( \sum_{I \in \binom{k}{\ell}} g(I) h(S \setminus I) \right)^2 \leq \lambda_{n,k,\ell} \cdot \left( \sum_{I \in \binom{n}{k}} g(I)^2 \right) \left( \sum_{J \in \binom{n}{\ell}} h(J)^2 \right).
\]

We prove (23) by induction on \( n \). If \( n = k + \ell \), then \( \lambda_{n,k,\ell} = 1 \), and so (23) becomes
\[
\left( \sum_{I \in \binom{n}{k}} g(I) h(I^c) \right)^2 \leq \left( \sum_{I \in \binom{n}{k}} g(I)^2 \right) \left( \sum_{I \in \binom{n}{\ell}} h(I^c)^2 \right),
\]
which follows from the Cauchy-Schwarz inequality. Thus we may assume not only that the statement holds for \( n - 1 \) but also that \( k + \ell \leq n - 1 \).

To use the induction hypothesis, we split the LHS of (23) into two summands depending on whether \( n \in S \in \binom{n}{k+\ell} \) with \( n \in S \) can be written as \( S = S' \cup \{n\} \) for some \( T \in \binom{n-1}{k+\ell-1} \), and using the triangle inequality, we get
\[
\sum_{n \in S \in \binom{n}{k+\ell}} \left( \sum_{I \in \binom{k}{\ell}} g(I) h(S \setminus I) \right)^2
= \sum_{S' \in \binom{n-1}{k+\ell-1}} \left( \sum_{I' \in \binom{k-1}{\ell-1}} g(I' \cup \{n\}) h(S' \setminus I') + \sum_{I \in \binom{k}{\ell}} g(I) h((S' \setminus I) \cup \{n\}) \right)^2
\leq \left( \sum_{S' \in \binom{n-1}{k+\ell-1}} \left( \sum_{I' \in \binom{k-1}{\ell-1}} g(I' \cup \{n\}) h(S' \setminus I') \right)^2 \right)^{1/2} \left( \sum_{I \in \binom{k}{\ell}} g(I)^2 \right)^{1/2}.
\]
It follows that the LHS of (23) is at most
\[
\sum_{S \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I \in \binom{S}{n-k}} g(I)h(S \setminus I) \right)^2 \\
+ \left( \sum_{S' \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I' \in \binom{S'}{n-k}} g(I' \cup \{n\})h(S' \setminus I') \right)^2 \right) \\
+ \left( \sum_{S' \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I' \in \binom{S'}{n-k}} g(I)h((S' \setminus I) \cup \{n\}) \right)^2 \right).
\]

For simplicity of notation, we define
\[
x := \sum_{I \in \binom{[n]}{k}} g(I)^2, \quad y := \sum_{J \in \binom{[n]}{n-1}} h(J)^2, \quad z := \sum_{I' \in \binom{[n]}{n-1}} g(I' \cup \{n\})^2, \quad t := \sum_{J' \in \binom{[n]}{n-1}} h(J' \cup \{n\})^2.
\]

By the induction hypothesis, we see that
\[
\sum_{S \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I \in \binom{S}{n-k}} g(I)h(S \setminus I) \right)^2 \leq \lambda_{n-1,k,\ell} \cdot xy,
\]
\[
\sum_{S' \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I' \in \binom{S'}{n-k}} g(I' \cup \{n\})h(S' \setminus I') \right)^2 \leq \lambda_{n-1,k-1,\ell} \cdot yz,
\]
\[
\sum_{S' \in \binom{[n-1]}{k+\ell-1}} \left( \sum_{I' \in \binom{S'}{n-k}} g(I)h((S' \setminus I) \cup \{n\}) \right)^2 \leq \lambda_{n-1,k+1,\ell} \cdot xt.
\]

Therefore, we can bound the LHS of (23) from above by
\[
\lambda_{n-1,k,\ell} \cdot xy + \left( \sqrt{\lambda_{n-1,k-1,\ell} \cdot yz} + \sqrt{\lambda_{n-1,k+1,\ell} \cdot xt} \right)^2
= \lambda_{n,k,\ell} \cdot (x + z)(y + t)
- \frac{\lambda_{n,k,\ell}}{(n-k)(n-\ell)} \cdot \left( \sqrt{k\ell xy} - \sqrt{(n-k)(n-\ell)zt} \right)^2
- \frac{(n-k-\ell)\lambda_{n,k,\ell}}{(k+\ell)(n-k)(n-\ell)} \cdot \left( \sqrt{k(n-\ell)xt} - \sqrt{(n-k)\ell yz} \right)^2
\leq \lambda_{n,k,\ell} \cdot (x + z)(y + t) = \text{the RHS of (23)},
\]

where in the second line we used the facts that \(\lambda_{n-1,k,\ell} = (n-k-\ell)n \) and \(\lambda_{n-1,k+1,\ell} = \frac{\ell n}{(k+\ell)(n-k)}\). This completes our proof.

We need the following consequence of Lemma B.1, whose proof uses a decoupling argument.

**Lemma B.2.** Given positive integers \(n, k\) and \(\ell\) with \(k + \ell \leq n\), and functions \(g: \binom{[n]}{k} \times \binom{[n]}{k} \to \mathbb{R}\) and \(h: \binom{[n]}{k} \times \binom{[n]}{k} \to \mathbb{R}\), we have
\[
\mathbb{E}_{S_1, S_2 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I_1 \sim \binom{[n]}{k}, I_2 \sim \binom{[n]}{k}} g(I_1, I_2)h(S_1 \setminus I_1, S_2 \setminus I_2) \right)^2 \leq \left( \mathbb{E}_{I_1, I_2 \sim \binom{[n]}{k}} g(I_1, I_2)^2 \right)^2 \left( \mathbb{E}_{J_1, J_2 \sim \binom{[n]}{k}} h(J_1, J_2)^2 \right)^2.
\]

**Proof.** For \(1 \leq i \leq 2\), let \(I_i'\) be an independent copy of \(I_i\). Then the LHS is equal to
We next bound the expression inside parentheses. It follows from the Cauchy-Schwarz inequality that

\[
\mathbb{E}_{S_1, S_2 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I_1 \sim \binom{[k]}{\ell}} \mathbb{E}_{I_2 \sim \binom{[k]}{\ell}} g(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \right) \left( \mathbb{E}_{I'_1 \sim \binom{[k]}{\ell}} \mathbb{E}_{I'_2 \sim \binom{[k]}{\ell}} g(I'_1, I'_2) h(S_1 \setminus I'_1, S_2 \setminus I'_2) \right)
\]

\[
= \mathbb{E}_{S_1 \sim \binom{[n]}{k+\ell}} \mathbb{E}_{I_1, I'_1 \sim \binom{[k]}{\ell}} \left( \mathbb{E}_{I_2 \sim \binom{[k]}{\ell}} g(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \mathbb{E}_{I'_2 \sim \binom{[k]}{\ell}} g(I'_1, I'_2) h(S_1 \setminus I'_1, S_2 \setminus I'_2) \right).
\]

We next bound the expression inside parentheses. It follows from the Cauchy-Schwarz inequality that

\[
\mathbb{E}_{S_1, S_2 \sim \binom{[n]}{k+\ell}} \mathbb{E}_{I_1, I'_1 \sim \binom{[k]}{\ell}} \left( \mathbb{E}_{I_2, I'_2 \sim \binom{[k]}{\ell}} g(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \right)^2 \leq \mathbb{E}_{S_1, S_2 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I_2, I'_2 \sim \binom{[k]}{\ell}} g(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \right)^2 \mathbb{E}_{I_2, I'_2 \sim \binom{[k]}{\ell}} g(I'_1, I'_2) h(S_1 \setminus I'_1, S_2 \setminus I'_2). \]

To bound the individual terms of the above product, we consider the functions \( \bar{g}: \binom{[n]}{k} \to \mathbb{R}_+ \) and \( \bar{h}: \binom{[n]}{k} \to \mathbb{R}_+ \) given by

\[
\bar{g}(I_1) := \mathbb{E}_{I_2 \sim \binom{[n]}{k}} g(I_1, I_2)^2, \quad \text{and} \quad \bar{h}(J_1) := \mathbb{E}_{J_2 \sim \binom{[n]}{k}} h(J_1, J_2)^2.
\]

By appealing to Lemma B.1, we get

\[
\mathbb{E}_{S_2 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I_1 \sim \binom{[k]}{\ell}} g(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \right)^2 \leq \bar{g}(I_1) \bar{h}(S_1 \setminus I_1),
\]

\[
\mathbb{E}_{S_2 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I'_1 \sim \binom{[k]}{\ell}} g(I'_1, I'_2) h(S_1 \setminus I'_1, S_2 \setminus I'_2) \right)^2 \leq \bar{g}(I'_1) \bar{h}(S_1 \setminus I'_1).
\]

Therefore, we find

\[
\text{the LHS} \leq \mathbb{E}_{S_1 \sim \binom{[n]}{k+\ell}} \left( \mathbb{E}_{I_1 \sim \binom{[k]}{\ell}} \sqrt{\bar{g}(I_1) \bar{h}(S_1 \setminus I_1)} \cdot \sqrt{\bar{g}(I'_1) \bar{h}(S_1 \setminus I'_1)} \right) = \text{the RHS},
\]

where in the third line we applied Lemma B.1 to the functions \( \sqrt{\bar{g}} \) and \( \sqrt{\bar{h}} \).

We are now ready to prove Lemma 3.8. For the reader’s convenience, we restate it here. Recall that \( \mathcal{M}_S^d \) is a shorthand for the family of size-\( d \) ordered matchings of \( S \).

**Lemma 3.8.** Let \( n \) and \( d \) be integers with \( 1 \leq d \leq n/2 \). Then for any real-valued functions \( f_1, \ldots, f_d: \binom{[n]}{2} \times \binom{[n]}{2} \to \mathbb{R} \) we have

\[
\mathbb{E}_{S_1, S_2 \sim \binom{[n]}{2d}} \left( \prod_{i=1}^d f_i(\{s_i, t_i\}, \{p_i, q_i\}) \right)^2 \leq \prod_{i=1}^d \mathbb{E}_{(s,t),(p,q) \sim \binom{[n]}{2}} f_i(\{s,t\}, \{p,q\})^2,
\]

where \( M_1 = (\{s_1, t_1\}, \ldots, \{s_d, t_d\}) \) and \( M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\}) \).
Proof. Remember that our task is to prove
\[
\mathbb{E}_{S_1, S_2 \sim \left(\frac{[n]}{2d}\right)} \left( \mathbb{E}_{M_1 \sim \mathcal{M}_{S_1}^d, M_2 \sim \mathcal{M}_{S_2}^d} \prod_{i=1}^{d} f_i(\{s_i, t_i\}, \{p_i, q_i\}) \right)^2 \leq \prod_{i=1}^{d} \mathbb{E}_{\{s, t\}, \{p, q\} \sim \left(\frac{[n]}{2}\right)} f_i(\{s, t\}, \{p, q\})^2, \tag{24}
\]
where \(M_1 = (\{s_1, t_1\}, \ldots, \{s_d, t_d\})\) and \(M_2 = (\{p_1, q_1\}, \ldots, \{p_d, q_d\})\).
We prove (24) by induction on \(d\). When \(d = 1\), both sides of (24) are equal. Thus we may assume that the statement holds for \(d - 1\) and that \(2 \leq d \leq n/2\).
Consider the real-valued function \(h : \left(\frac{[n]}{2d-2}\right) \times \left(\frac{[n]}{2d-2}\right) \rightarrow \mathbb{R}\) defined as
\[
h(J_1, J_2) = \mathbb{E}_{M'_1, M'_2} \prod_{i=2}^{d} f_i(\{s_i, t_i\}, \{p_i, q_i\})
\]
where \(M'_1 = (\{s_2, t_2\}, \ldots, \{s_d, t_d\})\) and \(M'_2 = (\{p_2, q_2\}, \ldots, \{p_d, q_d\})\) are uniformly distributed on \(\mathcal{M}_{J_i}^{d-1}\) and \(\mathcal{M}_{J_i}^{d-1}\), respectively.
Given \(S_1, S_2 \in \left(\frac{[n]}{2d}\right)\), we see that
\[
\mathbb{E}_{M_1 \sim \mathcal{M}_{S_1}^d, M_2 \sim \mathcal{M}_{S_2}^d} \prod_{i=1}^{d} f_i(\{s_i, t_i\}, \{p_i, q_i\}) = \mathbb{E}_{I_1 \sim \left(\frac{S_1}{2}\right), I_2 \sim \left(\frac{S_2}{2}\right)} f_d(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2).
\]
From this it follows that
\[
\mathbb{E}_{S_1, S_2 \sim \left(\frac{[n]}{2d}\right)} \left( \mathbb{E}_{M_1 \sim \mathcal{M}_{S_1}^d, M_2 \sim \mathcal{M}_{S_2}^d} \prod_{i=1}^{d} f_i(\{s_i, t_i\}, \{p_i, q_i\}) \right)^2
\]
\[
= \mathbb{E}_{S_1, S_2 \sim \left(\frac{[n]}{2d}\right)} \left( \mathbb{E}_{I_1 \sim \left(\frac{S_1}{2}\right), I_2 \sim \left(\frac{S_2}{2}\right)} f_d(I_1, I_2) h(S_1 \setminus I_1, S_2 \setminus I_2) \right)^2
\]
\[
\leq \left( \mathbb{E}_{I_1, I_2 \sim \left(\frac{[n]}{2d-2}\right)} f_d(I_1, I_2)^2 \right) \left( \mathbb{E}_{J_1, J_2 \sim \left(\frac{[n]}{2d-2}\right)} h(J_1, J_2)^2 \right)
\]
\[
\leq \left( \mathbb{E}_{I_1, I_2 \sim \left(\frac{[n]}{2d-2}\right)} f_d(I_1, I_2)^2 \right) \cdot \prod_{i=2}^{d} \mathbb{E}_{I_1, I_2 \sim \left(\frac{[n]}{2d-2}\right)} f_i(I_1, I_2)^2
\]
\[
= \prod_{i=1}^{d} \mathbb{E}_{\{s, t\}, \{p, q\} \sim \left(\frac{[n]}{2}\right)} f_i(\{s, t\}, \{p, q\})^2,
\]
where the third line follows from Lemma B.2, and in the fourth we used the induction hypothesis. \(\Box\)