NUMBER OF POINTS OF FUNCTION FIELDS OVER
FINITE FIELDS

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INTRODUCTION

Let $k$ be a field; if $\sim$ is an adequate equivalence relation on algebraic
cycles, we denote by $\text{Mot}_\sim(k)$ or simply $\text{Mot}_\sim$ the category of motives
modulo $\sim$ with rational coefficients, and by $\text{Mot}^\text{eff}_\sim$ its full subcategory
consisting of effective motives $[15]^1$. We use the convention that the
functor $X \mapsto h(X)$ from smooth projective $k$-varieties to $\text{Mot}^\text{eff}_\sim$ is
covariant. We shall in fact only consider the two extreme cases: rational
equivalence (rat) and numerical equivalence (num).

Using the point of view of birational motives (developed jointly with
Sujatha [9]), we give a proof almost without cohomology (see proof
of Lemma 2) of a recent result of Esnault on the existence of ratio-
nal points for smooth projective varieties with “trivial” Chow group
of zero-cycles over a finite field [4]. We also prove that the number
of rational points modulo $q$ is a stable birational invariant of smooth
projective varieties over $\mathbb{F}_q$: the idea of considering effective motives
and their divisibility by the Lefschetz motive was anticipated by Serre
[16]. This answers a question of Kollár; the 3-dimensional case had
been dealt with by Lachaud and Perret earlier [13]. However, as was
pointed out by Chambert-Loir, this birational invariance in fact follows
from much earlier work of Ekedahl [3], who does not use any form of
resolution of singularities!

1. Birational motives

Definition 1. The category $\text{Mot}_\sim^\circ$ is the Karoubian envelope (or idem-
potent completion) of the quotient of $\text{Mot}^\text{eff}_\sim$ by the ideal $J$ consisting of
morphisms factoring through an object of the form $M \otimes L$, where $L$ is
the Lefschetz motive. This is a tensor additive category. If $M \in \text{Mot}^\text{eff}_\sim$,
we denote by $\bar{M}$ its image in $\text{Mot}_\sim^\circ$.

$^1$With notation as in [15], an object of $\text{Mot}_\sim$ is effective if it is isomorphic to an
triple $(X, p, n)$ with $n \leq 0$; one may then find such a triple with $n = 0$. 
Lemma 1 ([9, Lemmas 5.3 and 5.4]). Let $X, Y$ be two smooth projective irreducible $k$-varieties. Then, in $\text{Mot}^0_{\text{rat}}$, we have

$$\text{Hom}(\tilde{h}(X), \tilde{h}(Y)) = CH_0(Y_{k(X)}) \otimes \mathbb{Q}.$$ 

(Let us briefly recall the proof: for $X, Y$ smooth projective, let $I(X, Y)$ be the subgroup of $CH^\dim Y(X \times Y) \otimes \mathbb{Q}$ formed of those correspondences which vanish on $U \times Y$ for some dense open subset $U$ of $X$. Then $I$ is an ideal in the category of rational Chow correspondences: the proof [9, Lemma 5.3] is a slight generalisation of the argument in [5, Ex. 16.1.11]. It is even monoidal, and extends to a monoidal ideal $I$ in $\text{Mot}^0_{\text{eff}}$, which obviously contains $J$. Using de Jong’s theorem [7, Th. 4.1] and Chow’s moving lemma, one sees that $I \otimes \mathbb{Q} = J \otimes \mathbb{Q}$ [9, Lemma 5.4]. In characteristic 0, one may remove the coefficients $\mathbb{Q}$ by using Hironaka’s resolution of singularities.)

Example 1. Let $X$ be smooth and projective over $k$. Then $\tilde{h}(X) \simeq 1$ in $\text{Mot}^0_{\text{rat}}$ if and only if $CH_0(X_{k(X)}) \otimes \mathbb{Q} \simeq \mathbb{Q}$ (write $\tilde{h}(X) \simeq 1 \oplus \tilde{h}(X) \geq 1$ in $\text{Mot}^0_{\text{eff}}$).

Remark 1. If $K$ is the function field of a smooth projective variety $X$, we may define a motive $\tilde{h}(K) \in \text{Mot}^0_{\text{rat}}$ as follows. If $Y$ is another smooth projective model of $K$, then [the closure of] the graph of a birational isomorphism from $X$ to $Y$ defines an isomorphism $\tilde{h}(X) \sim \tilde{h}(Y)$. If there is a third model $Z$, then the system of these isomorphisms is transitive, so defining $\tilde{h}(K)$ as the direct limit of the $\tilde{h}(X)$ for this type of isomorphisms makes sense and is (canonically) isomorphic to any of the $\tilde{h}(X)$. This construction is functorial for inclusions of fields. If char $k = 0$, it is even functorial for $k$-places by [9, Lemma 5.6], although we won’t use this. (Extending it to arbitrary function fields in characteristic $p$ would demand more work.)

Note that if $K \subseteq L$, then $\tilde{h}(K)$ is a direct summand of $\tilde{h}(L)$: to see this, consider smooth projective models $X, Y$ of $K$ and $L$. Let $f : U \to X$ be a corresponding dominant morphism, where $U$ is an open subset of $Y$. By Noether’s normalisation theorem, we may find an affine open subset $V \subset U$ such that the restriction of $f$ to $V$ factors through $X \times \mathbb{P}^n$, where $n = \dim Y - \dim X$. Since $\tilde{h}(X \times \mathbb{P}^n) \sim \tilde{h}(X)$, we are reduced to the case where $L/K$ is finite, and then it follows from a transfer argument. In particular, $\tilde{h}(X) \simeq 1$ if $X$ is unirational, as expected in [16]. The converse is not true: an Enriques surface $X$ verifies $\tilde{h}(X) \simeq 1$ by [2] (see example 1), but is not rational, hence not unirational over a field of characteristic 0 because $\text{Pic}(X)$ contains a $\mathbb{Z}/2$ summand (this counterexample was explained by Jean-Louis Colliot-Thélène).
For \( \sim = \text{num} \), the category \( \text{Mot}^\circ \sim \) is abelian semi-simple [8]. From [1, Prop. 2.1.7], we therefore get:

**Proposition 1.** a) The projection functor \( \pi : \text{Mot}^\text{eff}_\text{num} \to \text{Mot}^\circ \text{num} \) is essentially surjective (i.e. taking the karoubian envelope is irrelevant in the definition of \( \text{Mot}^\circ \text{num} \)).  
b) \( \pi \) has a section \( i \) which is also a left and right adjoint.  
c) The category \( \text{Mot}^\text{eff}_\text{num} \) is the coproduct of \( \text{Mot}^\text{eff}_\text{num} \otimes L \) and \( i(\text{Mot}^\circ \text{num}) \), i.e. any object of \( \text{Mot}^\text{eff}_\text{num} \) can be uniquely written as a direct sum of objects of these two subcategories.  
d) The sequence  
\[
0 \to K_0(\text{Mot}^\text{eff}_\text{num}) \to L \to K_0(\text{Mot}^\circ \text{num}) \to 0
\]
is split exact. \( \square \)

(In d), the injectivity on the left corresponds to the fact that the functor \( - \otimes L \) is fully faithful.)

**Remark 2.** a) In \( \text{Mot}^\circ \text{num} \), we can extend the end of Remark 1 as follows: let \( K, L \) two function fields of smooth projective varieties such that \( K \hookrightarrow L(t_1, \ldots, t_m) \) and \( L \hookrightarrow K(t_1, \ldots, t_n) \) for some \( m, n \). Then \( \bar{h}(K) \simeq \bar{h}(L) \). To get such a result in \( \text{Mot}^\circ \text{rat} \), one would need to have enough information on the algebra \( \text{End}(\bar{h}(K)) \).

b) Proposition 1 b) shows via Remark 1 that to any function field \( K/k \) one may canonically associate an effective numerical motive \( h(K) \in \text{Mot}^\circ \text{num} \), which is a direct summand of \( h(X) \) for any smooth projective model \( X \) of \( K \) (if any).

The following conjecture was suggested by Luca Barbieri-Viale:

**Conjecture 1** (cf. [17, Conj. 0.0.11]). For any field \( k \), the projection functor \( \text{Mot}^\text{eff}_\text{rat} \to \text{Mot}^\circ \text{rat} \) has a right adjoint.

2. Number of rational points modulo \( q \)

From now on, \( k = F_q \) is a field with \( q \) elements. Then, for all \( n \geq 1 \), the assignment  
\[
X \mapsto |X(F_q^n)| = \deg(\Delta_X \cdot F_X^n)
\]
for a smooth projective variety \( X \), where \( \Delta_X \) is the class of the diagonal and \( F_X \) is the Frobenius endomorphism viewed as a correspondence, extends to a ring homomorphism  
\[
\#_n : K_0(\text{Mot}^\text{eff}_\text{num}) \to \mathbb{Q}
\]
by the rule \( \#_n(X, p) = \deg(p \cdot F_X^n) \) if \( p = p^2 \in \text{End}(h(X)) \), cf. [12, p. 80].

**Lemma 2.** The homomorphisms \( \#_n \) take their values in \( \mathbb{Z} \).
Proof. It is enough to prove this for \( n = 1 \). More conceptually, we have \( \deg(t \cdot F_X) = \text{Tr}(p \circ F_X) \) in the rigid tensor category \( \text{Mot}_{\text{rat}} \). We may compute this trace after applying a Weil cohomology \( H \), e.g. \( l \)-adic cohomology. (We then have to consider \( H(X) \) as a \( \mathbb{Z}/2 \)-graded vector space and compute a super-trace.) Let \( H(X) = V \oplus W \), with \( V = \text{Ker}(H(p) - 1) \) and \( W = \text{Ker}(H(p)) \). Since \( F_X \) is a central correspondence, it commutes with \( p \), hence \( H(F_X) \) respects \( V \) and \( W \) and
\[
\text{Tr}(p F_X) = \text{Tr}(H(p F_X)) = \text{Tr}(H((p F_X)|_V) + \text{Tr}(H((p F_X)|_W)
= \text{Tr}(H(F_X)|_V).
\]

Since the minimum polynomial of \( H(F_X) \) kills \( H((F_X)|_V) \), the eigenvalues of the latter are algebraic integers. Hence \( \text{Tr}(p F_X) \) is an algebraic integer and therefore is in \( \mathbb{Z} \).

\[ \Box \]

Theorem 1. The homomorphism (1) induces a ring homomorphism
\[ \#_n : K_0(\text{Mot}^0_{\text{num}}) \to \mathbb{Z}/q^n. \]

This follows from Lemma 2 and Proposition 1 d) (note that \( \#_n(L) = q^n \)).

\[ \Box \]

Corollary 1 (Esnault [4]). Let \( X \) be a smooth projective variety over \( \mathbb{F}_q \) such that \( CH_0(X_{\mathbb{F}_q(X)}) \otimes \mathbb{Q} = \mathbb{Q} \). Then \( |X(\mathbb{F}_q)| \equiv 1 \pmod{q} \).

Proof. By Example 1, one has \( \bar{h}(X) \simeq 1 \) in \( \text{Mot}^0_{\text{rat}} \), hence a fortiori in \( \text{Mot}^0_{\text{num}} \). \[ \Box \]

Corollary 2 (cf. [3, Th. 4], [13]). The number of rational points modulo \( q \) is a stable birational invariant of smooth projective \( \mathbb{F}_q \)-varieties.

Indeed, two stably birationally isomorphic varieties have isomorphic motives in \( \text{Mot}^0_{\text{rat}} \). \[ \Box \]

Remarks 3. a) Using Remark 2 a) we could strengthen Corollary 2 as follows: for two smooth projective varieties \( X, Y \), \( |X(\mathbb{F}_q)| \equiv |Y(\mathbb{F}_q)| \pmod{q} \) provided there exist \( m, n \) and dominant rational maps \( X \times \mathbb{P}^m \to Y, Y \times \mathbb{P}^n \to X \). However, this also follows from [3].

b) Using Remark 2 b) we may canonically associate to any function field \( K/\mathbb{F}_q \) a series of integers \( (a_n)_{n \geq 1} \) such that, for all \( n, \#_n(h(K)) = a_n \pmod{q^n} \) (see Remark 1 for the definition of \( h(K) \)). Naturally \( a_n \) is not necessarily positive in general. More conceptually, we may associate to \( K \) its zeta function, defined as the zeta function of the motive \( i(h(K)) \).

c) Killing \( L^c \) instead of \( L \) would yield congruences modulo \( q^c \) rather than modulo \( q \), cf. [4, §3]; compare also [18]. But one would lose the fact that function fields have motives as in the previous remarks.

d) Unfortunately the proof of Lemma 2 uses cohomology, hence the proof of Theorem 1 is not completely cohomology-free.
3. A conjectural converse

Note that the functions \( \#_n \) of (1) extend to ring homomorphisms \( K_0(\text{Mot}_{\text{num}}) \to \mathbb{Z}[1/q] \), still denoted by \( \#_n \).

**Theorem 2.** Assume that the Tate conjecture holds. Let \( M \in K_0(\text{Mot}_{\text{num}}) \) be such that \( \#_n(M) \in \mathbb{Z} \) for all \( n \geq 1 \). Then \( M \in K_0(\text{Mot}_{\text{eff}}) \). Conversely, if this implication holds for any \( M \in K_0(\text{Mot}_{\text{num}}) \), then the Tate conjecture holds.

**Proof.** Write \( M = \sum m_i[S_i] \), where \( m_i \in \mathbb{Z} \setminus \{0\} \) and the \( S_i \) are simple pairwise non-isomorphic motives. For each \( i \), let \( w_i \) be a Weil number of \( S_i \), that is, a root of the minimum polynomial of \( F_{S_i} \), and \( K_i = \mathbb{Q}(w_i) \). Then \( \#_n(M) = \sum m_i Tr_{K_i/\mathbb{Q}}(w_i^n) \). It follows from the assumption and from [11, Lemma 2.8] that \( w_i \) is an algebraic integer for all \( i \). (To apply loc. cit., compute in a Galois extension of \( \mathbb{Q} \) containing all \( K_i \) and observe that the Tate conjecture implies that no \( w_i \) is equal to a conjugate of \( w_j \) for \( i \neq j \) [14, proof of Prop. 2.6].)

By Honda’s theorem, for each \( i \) there is an abelian variety \( A_i \) over \( \mathbb{F}_q \) and a simple direct summand \( T_i \) of \( h(A_i) \) whose Weil numbers are the Galois orbit of \( w_i \) (ibid.). Reapplying Tate’s conjecture, we get that \( S_i \simeq T_i \), hence \( S_i \) is effective for all \( i \).

To prove the converse, let \( M \in \text{Mot}_{\text{num}} \) be simple and such that \( F_M = 1 \). Then \( \#_n(M) = 1 \) for all \( n \). Therefore \( M \) is effective. Writing \( M \) as \( (X, p) \) for some smooth projective variety \( X \), we have that \( M \) is a direct summand of \( h^0(X) \) for weight reasons. It follows easily that \( M \simeq 1 \). By [6, Th. 2.7], this implies the Tate conjecture.

**Remark 4.** Unfortunately we have to apply the Tate conjecture to \( S_i \otimes T_i^* \) in the proof, hence cannot provide a hypothesis only involving \( M \).

**Corollary 3.** The ring homomorphism

\[
(\overline{\#}_n)_{n \geq 1} : K_0(\text{Mot}_{\text{num}}) \to \prod_{n=1}^{\infty} \mathbb{Z}/q^n
\]

is injective if and only if the Tate conjecture is true. \( \square \)

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