The Relativistic framework of Positioning systems

J.-F. Pascual-Sánchez

Dept. Matemática Aplicada, Facultad de Ciencias,
Universidad de Valladolid, Valladolid, 47005, Spain, EU

Abstract

Emission relativistic coordinates are a class of spacetime coordinates defined and generated by four emitters (satellites, pulsars) broadcasting their proper time by radio signals. They are the main ingredient of the simplest conceivable relativistic positioning system. The emission coordinates are independent of any observer. Receiving directly the proper time at emission of four satellites, any user or observer can measure the values of the emission coordinates, from which he/she can obtain his trajectory and hence, in particular, his position. Moreover, if and only if the four satellites also broadcast to the users the proper times they are receiving by cross-link autonavigation from the other emitters, the positioning system is called autolocated or autonomous. In an autolocated positioning system the trajectories of the satellites of the constellation can also be known by the users and they can also obtain the metric of the spacetime (the gravitational field) on the constellation.

The study of autolocated relativistic positioning systems has been initiated by Coll and collaborators several years ago and it has been aimed for developing an exact fully relativistic theory of positioning systems and gravimetry, based on the framework and concepts of General Relativity. This exact relativistic framework is the alternative to considering post-newtonian relativistic corrections in a classical Newtonian framework, which is the customary approach yet now used in the GPS and the GLONASS.

*Electronic address: e-mail: jfpascua@maf.uva.es
I. INTRODUCTION

Practically all observations and experiments in General Relativity are interpreted in a classical Newtonian conceptual framework. In this talk I will focus on the essential differences between a Newtonian plus post-newtonian relativistic corrections framework, as that of the current Global Navigation Satellite Systems (GNSS), and a fully relativistic framework which would be desirable to implement in the future Galileo system due to its theoretical and practical advantages. The Newtonian conceptual framework uses a 3-dim spatial reference system and a time reference, separately, as still appears in the Resolution 3 of the XXVI General Assembly of the International Astronomical Union, I.A.U., held in Prague last year [1]. In this framework, the relativistic effects are added with the same status as any non-desired perturbation (gravitational influence of other planets, effects of the atmosphere,...). This is made with corrections, coming from general relativity when compared with newtonian gravity, in weak gravitational fields and with small velocities (post-newtonian formalism).

Typically, this is what is done nowadays in all the current GNSS, where the primary relativistic effects, i.e., some post-newtonian corrections of second order $1/c^2$, coming from both Special and General Relativity must be taken into account. In general, the satellites of the GNSS are affected by Relativity in three different ways: in the equations of motion, in the signal propagation and in the beat rate of the satellite clocks.

In the first part of the talk I will review the main clock effects at second order, because they are the only measurable ones in the current GNSS due to the accuracy of nanoseconds of the satellite clocks of the GPS and GLONASS systems. At present, the above approach is perfectly justified from a practical and numerical point of view and it has been successful applied to the GPS, see [2]. However, if the time resolutions are increased (more accurate clocks), it will be necessary in the future to consider other effects of order $1/c^2$ and also it will be necessary to consider other effects of third (as the Shapiro delay) or fourth order (as the Lense-Thirring effect), see [3].

In this situation, it can be wondered if it would not be more convenient to change the present newtonian framework to an exact formulation in full General Relativity. This would imply to abandon the classical post-newtonian framework.

However, a project for the Galileo system is aimed to develop a theory of positioning systems in the framework of general relativity. This project, which is still in a state of the-
oretical construction, is called SYPOR (a French acronym for SYstme de POsitionnemnet Relativiste). This project is based in the following mathematical result obtained by Coll and Morales some time ago [4] which is almost unknown by the navigation and timing scientific communities, in spite of its fundamental importance in the construction of positioning systems: In the 4-dim Newtonian spacetime there exists 4, and only 4, causal classes of reference frames, whereas in the relativistic 4-dim Lorentzian spacetime, due to the freedom introduced by the finite propagation of light, there exists 199, and only 199, causal classes of reference frames.

In general, a causal class is defined by a spacetime frame, a dual coframe and the 2-dim surfaces generated by the vectors of the frame. Only a causal class, among the 199 Lorentzian ones, is privileged to construct a generic (valid for a wide class of spacetimes), gravity free (the previous knowledge of the gravitational field is not necessary) and immediate (non retarded) positioning system, this is the causal class of the emission coordinates.

As a natural frame is the set of derivations along the parameterized lines of a coordinate system, the definition of a determinate causal class extends to the coordinate system itself. However, because the causal class of a coordinate system is the causal class of its natural frame at every point and hence local, a coordinate system may present different causal classes at different points of its domain of definition and it is said to be inhomogeneous.

The generalized use so far of coordinate systems of the standard Newtonian causal class, with a timelike coordinate parameter and three spacelike ones, exaggerates the partial interest of the newtonian evolution vision 3+1 à la ADM, i.e. the time evolution of spatial 3-hypersurfaces, or with the splitting 1+3 by timelike congruences of observers, in the description of the coupled spacetime changes of the physical systems.

On the other hand, emission coordinates belong to a causal class of spacetime coordinates defined and generated by four emitters (satellites) broadcasting their proper time by means of radio signals. In other words, the broadcasted signals are the emission coordinates by themselves. These emission coordinates are covariant (frame independent) and completely independent of any observer or user. In principle as they define a positioning system, no observers are necessary at all and hence there is no necessity of any synchronization procedure. However, any observer can measure the values of the emission coordinates which can give his position in the $R^4$ grid of emission coordinates.

Moreover, if and only if the emitters also broadcast to the users the proper times they
are receiving by cross-link autonavigation from the other emitters, the system is called autolocated and the trajectories of the emitters can also be known by the users and they can also obtain the metric of the spacetime (the gravitational field) acting on the constellation.

The study of autolocated positioning systems has been initiated by Coll and collaborators several years ago and it has been aimed for developing an exact fully relativistic theory of positioning systems and gravimetry, based on the framework and concepts of General Relativity. This is completely different to considering post-newtonian “relativistic effects” in a classical Newtonian framework, which is the customary approach yet now used in the present GNSS.

Just now important results have been obtained by Coll and collaborators in 2 spacetime dimensions, and some very interesting properties have been already obtained for 3 and 4 spacetime dimensions. I will review them in the second part of the talk. All the exposition will be mainly based on [5].

II. THE NEWTONIAN FRAMEWORK: RELATIVISTIC EFFECTS IN GNSS

In the GPS system the satellite clocks keep time to an accuracy of about 4 ns per day. This means once an atomic clock of the satellites vehicles (SV) is set, then after one day it should be correct to within about 4 nanoseconds. Therefore, the errors introduced by the relativistic effects at post-newtonian order $c^{-2}$ were crucial in the conception of the current GNSS and in its correct operation nowadays because these effects are of the order of hundred or thousand nanoseconds.

To describe these errors we could start from a gravitational metric near the Earth that comes from the Equivalence Principle (local equivalence of inertia and gravity for experiments not sensitive to tidal forces), that reads:

$$ds^2 = c^2 dr^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - dr^2 - r^2 d\psi^2,$$

and it is not necessary to begin from the full nonlinear Schwarzschild metric or even from its linear approximation. Let us consider the special case of circular orbits for the satellites of the constellation. Thus the clock on the Earth and the satellite clock travel at constant distance around the Earth center, therefore $dr = 0$ for each clock. For both clocks, one
obtains:

\[
\frac{1}{c^2} \left( \frac{ds}{dt} \right)^2 = \left( 1 + \frac{2\phi}{c^2} \right) - \frac{v^2}{c^2},
\]

(2)

where \(\phi\) is the Newtonian potential and \(v = r \frac{d\phi}{dt}\) is the coordinate tangential speed along the circular equatorial orbit, measured by using the far-away coordinate time \(t\).

Now, let us apply (1) twice, first to the clock in a satellite (S), (using \(r = r_S\), \(v = v_S\) and proper time \(d\tau_S = ds/c\)) and secondly to a fixed clock in the Earth’s equator and turning with it (E) (using, \(r = r_E\), \(v = v_E\) and proper time \(d\tau_E = ds/c\)), with the same elapsed coordinate time, \(dt\), corresponding to an inertial observer at spatial infinity. Then, dividing both expressions one obtains at linear order:

\[
\frac{d\tau_E}{d\tau_S} = 1 - \frac{GM}{r_E c^2} - \frac{v_E^2}{2c^2} + \frac{GM}{r_S c^2} + \frac{v_S^2}{2c^2}.
\]

(3)

In (3) two main effects appear: 1.-First: Einstein gravitational shift effect. The clocks go forward or their time run faster when they are far away from a center of gravitational attraction hence the time scale is slower for the Earth bound clock. 2.-Second: Special relativistic Doppler time dilation. As the satellite clocks move faster than the clocks on the Earth’s surface this gives rise to the special relativistic Doppler effect of second order in speed. This effect gives place to a red shift in frequency for a signal sent downward from the satellites to Earth.

Therefore, this kinematic Doppler effect of second order “works against” the Einstein gravitational violet shift effect. To the altitude of the satellites of the GPS, \(H_{GPS} = 20,183\) km, the Einstein gravitational shift effect is greater than special relativistic Doppler one.

The main relativistic effects we are considering are of order \(O(c^{-2})\). This is the order of approximation used in the GPS. So, why is this order of approximation good enough? Because, as said above, the accuracy of the SVs atomic clocks is nanoseconds and only these two effects are of higher magnitude.

Introducing numerical values in (3), the net result for the GPS system is of the order of 39,000 ns per day when measured in a laboratory in the rotating Earth. This figure is tantamount to an error in distance of 11,700 meters after a day of operation. As one will exposed in the talk, the previous net result, in which only local proper times appear, must also be corrected by using the so-called global GPS coordinate time and a more realistic gravitational potential for the Earth. To do this, we must restart with a more realistic
metric in the weak external gravity field of the Earth which is obtained from the linear Schwarzschild metric including the quadrupole moment in the gravitational potential, the centrifugal potential and the kinematical Sagnac effect due to Earth rotation, \[2, 5\].

On the other hand, it is important to point out that the necessity to consider other smaller relativistic effects at the order \(O(c^{-3})\) when laser cooled atomic clocks of last generation are used, with measurable errors of order of picoseconds, as in the ACES (Atomic Clock Ensemble in Space) mission of the ESA for the International Space Station, ISS, and those of the order \(O(c^{-4})\) \[3\], when one works with intrinsic errors of femtoseconds in the near future.

### III. THE RELATIVISTIC FRAMEWORK OF POSITIONING SYSTEMS

#### A. Reference systems and positioning systems

Let us define location systems as the physical realizations of some coordinate systems. Location systems are of two different types: reference systems and positioning systems. The first ones are 4-dimensional reference systems which allow one observer, considered at the origin, to assign four coordinates to the events of his neighborhood by means of a radio signal. Due to the finite speed of light, this assignment is retarded with a time delay. The second ones are 4-dimensional positioning systems (as intended to be used in the SYPOR project) which allow to every event of a given domain to know its proper coordinates in an immediate way.

In Relativity, a (retarded) reference system can be constructed starting from an (immediate) positioning system (it is sufficient that each event sends its coordinates to the observer at the origin) but not the other way around. In contrast, in Newtonian theory, 3-dimensional reference and positioning systems are interchangeable and as the velocity of transmission of information, the speed of light, is supposed to be infinite, the Newtonian reference systems are not retarded but immediate.

The reference and positioning systems defined here are 4-dimensional objects, including time location.

Following \[4\], the best way to visualize and characterize a spacetime coordinate system is to start from four families of coordinate 3-surfaces, then, their mutual intersections give
six families of coordinate 2-surfaces and four congruences of coordinate lines. Alternatively, one can use the related covectors or 1-forms \( \{ \theta^i \} \), \( i = 1, 2, 3, 4 \), instead of the 3-surfaces, and the vectors of a coordinate tangent frame \( \{ e_i \} \), \( i = 1, 2, 3, 4 \), instead of four congruences of coordinate lines which are their integral curves. In this way, for a specific domain of a Lorentzian or Newtonian spacetime, each coordinate system is fully characterized by its causal class, which is defined by a set of 14 characters:

\[
\{ c_1, c_2, c_3, c_4, C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}, c_1, c_2, c_3, c_4 \},
\]

being \( c_i \) the Lorentzian causal character of the vector \( e_i \), i.e. if it is spacelike, timelike or lightlike; \( C_{ij} \) the causal character of the adjoint 2-plane \( \{ e_i \wedge e_j \} \) and, finally, \( c_i \) the causal character of the covectors \( \theta^i \) of the dual coframe, \( \theta^i(e_j) = \delta_j^i \). The covector \( \theta^i \) is timelike (resp. spacelike) iff the 3-plane generated by the three vectors \( \{ e_j \}_{j \neq i} \) is spacelike (resp. timelike). This applies for both Newtonian and Lorentzian spacetimes. In addition, for the latter, the covector \( \theta^i \) is lightlike iff the 3-plane generated by \( \{ e_j \}_{j \neq i} \) is lightlike or null.

This new degree of freedom (lightlike) in the causal character, which is proper of Lorentzian relativistic spacetimes but which does not exist in Newtonian spacetimes, allows to obtain in [4], as it has been commented in the introduction, the following conclusion:

In the 4-dimensional Newtonian spacetime there exist four, and only four, causal classes of frames, whereas in the relativistic 4-dimensional Lorentzian spacetime, due to the freedom commented above, there exists 199, and only 199, causal classes of frames.

As notation for the causal characters, we will use lower case roman types \((s,t,l)\) to represent the causal character of vectors (resp. spacelike, timelike, lightlike), and capital types \((S,T,L)\) and lower case italic types \((s,t,l)\) to denote the causal character of 2-planes and covectors, respectively.

In Relativity, a specific causal class, among the 199 ones, can be assigned to any of the different coordinate systems used in all the solutions of the Einstein equations. However, for the same coordinate system and the same solution, the causal class can change depending on the region of the spacetime considered and the coordinate system is said inhomogeneous.

A coordinate system can be constructed starting from a tangent frame or its coordinate lines. But in this way, in general, there are obstructions to obtain positioning systems that are generic, i.e., valid for many different spacetimes. On the other hand, generic positioning systems can be constructed from four one-parameter families of 3-surfaces or, equivalently,
from a cotangent frame. One clock broadcasting its proper time is described in the spacetime by a timelike line in which each event is the vertex of a future light cone. The set of these light cones of a emitter constitutes a one-parameter (proper time) family of null hypersurfaces. So, four clocks broadcasting their proper times determine four one-parameter families of null or lightlike 3-surfaces.

In a relativistic spacetime, the wave fronts of those signals parameterized by the proper time of the clocks, define four families of light cones (3-dimensional null hypersurfaces) which contain all the lightlike geodesics of four emitters and making a contravariant null (or light) coordinate system [4], see Fig. 1. Such a coordinate system does not exist in a Newtonian spacetime where the light travels at infinite speed.

B. Coll positioning systems

Among the 199 Lorentzian causal classes, only one is privileged to construct a generic (valid for a wide class of spacetimes), gravity free (the previous knowledge of the gravitational field is not necessary) and immediate positioning system. This is the causal class \{ssss, SSSSSS, llll\} of the Coll-Morales homogeneous coordinate system [4, 5, 6]. In this class the emission coordinates of the Coll positioning systems are included. These emission coordinates have been also studied in [7, 8] in the special case of a flat Minkowski spacetime without gravity.

The coordinate system of this causal class is always homogeneous and it has associated four families of null 3-surfaces or a real null coframe, whose mutual intersections give six families of spacelike 2-surfaces and four congruences of spacelike lines.

In this primary positioning system, an user at any event in a given spacetime region can know its proper coordinates. The four proper times of four satellites (\{\tau^A\}; A = 1, 2, 3, 4) read at an event by a receiver or user constitute the null (or light) proper emission coordinates or user positioning data of this event, with respect to four SVs, see Fig. 2. These four numbers can be understood as the “distances” between the event and the four satellites.

In a certain domain \(\Omega \subset \mathbb{R}^4\) of the grid of parameters (\{\tau^A\}; A = 1, 2, 3, 4), any user receiving continuously his emission coordinates may know his trajectory. If the observer has his own clock, with proper time denoted by \(\sigma\), then he can know his trajectory, \(\tau^A = \tau^A(\sigma)\), and his four-velocity, \(u^A(\sigma) = d\tau^A/d\sigma\). There is not space-time asymmetry like in the
standard Newtonian coframe \((t \, s \, s \, s)\) (one timelike \(t\) and three spacelike \(s\)). In emission coordinates obtained from a general real null coframe \((l \, l \, l \, l) = \{d\tau^1, d\tau^2, d\tau^3, d\tau^4\}\), which is neither orthogonal nor normalized, the contravariant spacetime metric is symmetric with null diagonal elements and it has the general expression \[12\]:

\[
(g^{AB}) = (d\tau^A \cdot d\tau^B) = \begin{pmatrix}
0 & g^{12} & g^{13} & g^{14} \\
g^{12} & 0 & g^{23} & g^{24} \\
g^{13} & g^{23} & 0 & g^{34} \\
g^{14} & g^{24} & g^{34} & 0
\end{pmatrix},
\]  

(5)

where \(g^{AB} > 0\) for \(A \neq B\). Four null covectors can be linearly dependent although none of them is proportional to another. To ensure that the four null covectors are linearly independent and span a 4-dim spacetime, it is sufficient that \(\det(g^{AB}) \neq 0\). Finally, this metric has a Lorentzian signature \((+, -, -, -)\) iff \(\det(g^{AB}) < 0\). The expression \[5\] of the metric is observer independent and has six degrees of freedom.

A splitting of this metric can be considered, changing from the six independent components (ten components minus four gauge degrees of freedom of passive coordinate transformations) of \(g^{AB}\) to a more convenient set, which neatly separates two shape parameters depending only on the direction of the covectors \(d\tau^A\) or equivalently depending exclusively on the trajectories of the emitters, from other four scaling parameters depending on the length of the covectors or depending on the proper time of each satellite. Four satellites
emitting, without the necessity of a synchronization convention, not only their proper times \( \tau^A \), but also the proper times \( \tau^{AB} \) of three close satellites received by the satellite \( A \) in \( \tau^A \) (in total \( \{ \tau^A, \tau^{AB} \} ; A \neq B ; A, B = 1, 2, 3, 4 \) ) and transmitted, constitute an autonomous or autolocated positioning system. This is because the three proper times, together with its proper time, that a satellite clock receives from other SVs constitute its proper emission coordinates.

The sixteen data \( \{ \tau^A, \tau^{AB} \} \) or emitter positioning data received by a user, allows the user to know his/her trajectory, the trajectories of four SVs and the metric of the spacetime (the gravitational field) acting on the constellation.

Coll positioning systems are yet now quite well developed for two-dimensional spacetimes \cite{9,10} for arbitrary and special observers in Minkowski and Schwarzschild spacetimes. However, the known results for the two dimensional case are not trivially generalizable for the realistic four dimensional one \cite{11,12} and much work remains to be done.

Finally it is to be pointed out that, although the main idea of positioning systems based on emission coordinates is simple, our partial lack of intuition at present about the general use of emission coordinates is usually accompanied by many prejudices coming from the newtonian framework, basically from the newtonian evolution vision 3+1 à la ADM.
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