VORTEX STRUCTURES FOR SOME GEOMETRIC FLOWS FROM PSEUDO-EUCLIDEAN SPACES

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Abstract. For some geometric flows (such as wave map equations, Schrödinger flows) from pseudo-Euclidean spaces to a unit sphere contained in a three-dimensional Euclidean space, we construct solutions with various vortex structures (vortex pairs, elliptic/hyperbolic vortex circles, and also elliptic vortex helices). The approaches base on the transformations associated with the symmetries of the nonlinear problems, which will lead to two-dimensional elliptic problems with resolution theory given by the finite-dimensional Lyapunov-Schmidt reduction method in nonlinear analysis.

1. Introduction.

1.1. The problems from Pseudo-Euclidean spaces. We will concern the following Schrödinger type problems for the unknown maps $m(t, \tau, s) : \mathbb{R} \times \mathbb{R}^{K,N} \to \mathbb{S}^2 \subset \mathbb{R}^3$

$$\partial_t m = m \times (\Delta_{K,N} m + |\nabla_{K,N} m|^2 m), \quad \partial_t m = m \times (\Delta_{K,N} m + |\nabla_{K,N} m|^2 m - m_3 \vec{e}_3 + m_3^2 m),$$

and their stationary cases for the unknown maps $m(\tau, s) : \mathbb{R}^{K,N} \to \mathbb{S}^2 \subset \mathbb{R}^3$

$$\Delta_{K,N} m + |\nabla_{K,N} m|^2 m = 0,$$

$$\Delta_{K,N} m + |\nabla_{K,N} m|^2 m - m_3 \vec{e}_3 + m_3^2 m = 0.$$

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Note that for the convenience of future use of notation, we here denote \( \tau \) and \( s \) the variables in pseudo-Euclidean spaces. In the above and in the sequel, the differential operators on \( \mathbb{R}^{K,N} \) are defined by

\[
\triangle_{K,N} m \equiv -\triangle_\tau m + \triangle_s m = -\sum_{i=1}^{K} \frac{\partial^2 m}{\partial \tau_i^2} + \sum_{j=1}^{N} \frac{\partial^2 m}{\partial s_j^2},
\]

(5)

\[
\nabla_{K,N} m = \left( -\frac{\partial m}{\partial \tau_1}, \ldots, -\frac{\partial m}{\partial \tau_K}, \frac{\partial m}{\partial s_1}, \ldots, \frac{\partial m}{\partial s_N} \right),
\]

(6)

\[
|\nabla_{K,N} m|^2 \equiv -|\nabla_\tau m|^2 + |\nabla_s m|^2 = -\sum_{i=1}^{K} \left| \frac{\partial m}{\partial \tau_i} \right|^2 + \sum_{j=1}^{N} \left| \frac{\partial m}{\partial s_j} \right|^2.
\]

(7)

Problem (1) can be regarded as the Schrödinger flows for maps from pseudo-Euclidean spaces into \( S^2 \) in the sense of [3] (also see [14]). While (3) can be viewed as a generalization of the \( \sigma \)-model with field \( m : \mathbb{R}^{K,N} \to S^2 \). In fact, the solutions \( m : \mathbb{R}^{K,N} \to S^2 \) to (3) are always called as generalized wave maps or harmonic maps from pseudo-Euclidean spaces.

Setting \( K = 0 \) in (2), the equation is (sometimes called Schrödinger type flow)

\[
\frac{\partial}{\partial t} m = m \times \left( \triangle m - m^3 \vec{e}_3 \right) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^{0,N}.
\]

(8)

From the physical side of view, one expects that topological solitons, which are half magnetic bubbles, exist in solutions of (8) (see [8] and [13]). Indeed, in [6]-[7], F. Hang and F. Lin have established the corresponding static theory for such magnetic vortices. Applying a reduction method, F. Lin and J. Wei [11] looked for a solution of the traveling wave \( m(s', s_N - Ct) \) (i.e., travel in the \( s_N \)-direction with the speed \( C = \epsilon \) > 0) of the equation (8). Then \( m \) must be a solution of

\[
-\epsilon \frac{\partial m}{\partial s_N} = m \times \left( \triangle m - m_3 \vec{e}_3 \right).
\]

(9)

After a proper scaling in the space, (9) becomes

\[
-\frac{\epsilon}{c} \frac{\partial m}{\partial s_N} = m \times \left( \triangle m - m_3 \frac{\epsilon^2}{c^2} \vec{e}_3 \right), \quad s \in \mathbb{R}^N.
\]

(10)

The main results of the paper [11] are the following: Let \( N \geq 2 \) and \( \epsilon \) sufficiently small, there is an axially symmetric solution \( m = m(|s'|, s_N) \in C^\infty(\mathbb{R}^N, S^2) \) of (10) such that

\[
E_\epsilon(m) = \int_{\mathbb{R}^N} \frac{1}{2} \left( |\nabla m|^2 + \left| m_3 \right|^2 \frac{\epsilon^2}{c^2} \right) \, ds < \infty,
\]

and that \( m \) has exactly one vortex at \((|s'|, s_N) = (a_\epsilon, 0)\) of degree +1, where \( a_\epsilon \approx \frac{1}{2} \).

The traveling velocity \( C \) obeys

\[
C = \epsilon \quad \text{if} \quad N = 2, \quad C = (N - 2) \epsilon |\log \epsilon| \quad \text{if} \quad N \geq 3.
\]

Naturally such solution \( m \) gives rise to a nontrivial (two-dimensional) traveling wave solution of (8) with a pair of vortex and antivortex which undergoes the Kelvin Motion for the case \( N = 2 \). Solutions constructed of this form are called traveling vortex rings for the case of the dimension \( N \geq 3 \). Later on, J. Wei and J. Yang [16] concerned the existence of traveling wave solutions with invariance under skew motions and possessing vortex helices for (8).

Under the stereographic projection, setting

\[
W = \frac{m_1 + im_2}{1 + m_3} \in \mathbb{C},
\]
then (3) and (4) can be written as

\begin{equation}
\Delta_{K,N} W - \frac{2\overline{W}}{1 + |W|^2} \nabla_{K,N} W \cdot \nabla_{K,N} W = 0 \quad \text{on } \mathbb{R}^{K,N},
\end{equation}

\begin{equation}
\Delta_{K,N} W - \frac{2\overline{W}}{1 + |W|^2} \nabla_{K,N} W \cdot \nabla_{K,N} W + \frac{1 - |W|^2}{1 + |W|^2} W = 0 \quad \text{on } \mathbb{R}^{K,N},
\end{equation}

while (1) and (2) become, on \( \mathbb{R} \times \mathbb{R}^{K,N} \),

\begin{equation}
i \partial_t W + \Delta_{K,N} W - \frac{2\overline{W}}{1 + |W|^2} \nabla_{K,N} W \cdot \nabla_{K,N} W = 0,
\end{equation}

\begin{equation}
i \partial_t W + \Delta_{K,N} W - \frac{2\overline{W}}{1 + |W|^2} \nabla_{K,N} W \cdot \nabla_{K,N} W + \frac{1 - |W|^2}{1 + |W|^2} W = 0.
\end{equation}

Here and throughout the paper, we use \( \overline{W} \) to denote the complex conjugate of a function \( W \).

Before going further, we recall that in [4] W. Ding and H. Yin proposed to study the periodic solutions of the Schrödinger flow in the case where the target manifold \( N \) is a Kähler-Einstein manifold with positive scalar curvature. Such a class of special solutions can be viewed as a sort of geometric solitary wave solutions ([14]). Recently, from the viewpoint of differential geometry C. Song, X. Sun and Y. Wang [15] propose a notion “geometric soliton”, which is concerning special solutions for some geometric flows and can be regarded as a geometric generalization of the classical solitary wave solutions. In the present paper, we also use the similar method to study (1)-(4). In other words, we shall make use of the isometry groups in both the target manifold \( \mathbb{C} \) and the pseudo-Euclidean space to transform (11)-(14) into elliptic problems and then try to construct solutions with various vortex structures of 0-dimension (single point vortices or vortex pairs) or 1-dimension (vortex helices or vortex rings).

1.2. Geometric notions. For the statements of the results, some notions in pseudo-Euclidean spaces will be reviewed. The Poincaré group is the group of isometries of Minkowski spacetime \( \mathbb{R}^{1,3} \). It is a 10-dimensional noncompact Lie group and includes

- translations (i.e., displacements) in time and space, with generators \( P \). These form the Abelian Lie group of translations on space-time.
- rotations in space (this forms the non-Abelian Lie group of 3-dimensional rotations, with generators \( J \)).
- boosts, i.e., transformations connecting two uniformly moving bodies, with generators \( K \) in the forms

\[
\begin{pmatrix}
\cosh \theta & \sinh \theta & 0 & 0 \\
\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
\cosh \theta & 0 & \sinh \theta & 0 \\
0 & 1 & 0 & 0 \\
\sinh \theta & 0 & \cosh \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \theta & 0 & 0 & \cosh \theta
\end{pmatrix}.
\]

The last two symmetries, \( J \) and \( K \), can also be considered as hyperbolic rotations of Minkowski space and together make up the Lorentz group. Objects which are invariant under this group are said to possess Poincaré invariance or relativistic invariance.

In the space \( \mathbb{R}^{K+N} \) with signature \((-,-,\cdots,-,+,\cdots,+)\), a vector \( v \neq 0 \) is called spacelike (resp. timelike or null) if \( (v,v) > 0 \) (resp. \( (v,v) < 0 \) or \( (v,v) = 0 \)). A
curve $\ell$ in $\mathbb{R}^{K+N}$ is said to be spacelike if all of its velocity vectors are spacelike; similarly for timelike and null. The definition of circles in $\mathbb{R}^{1,2}$ can be found in [12]:

**Definition 1.1.** A circle in $\mathbb{R}^{1,2}$ is the orbit of a point $p$ out a straight line $\ell$ under the action of the group of hyperbolic rotations in $\mathbb{R}^{1,2}$ that leave $\ell$ pointwise fixed.

By taking an orthogonal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ with $\vec{e}_1$ timelike in $\mathbb{R}^{1,2}$, the circles can be described as

(1). $\ell$ is a time like line: The group of the Lorentz motions $\{R_\theta : \theta \in \mathbb{R}\}$ that fix $\ell$ pointwise is given by the matrix

$$R_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}$$

where $\vec{e}_1$ spans the line $\ell$. The circles of $\mathbb{R}^{1,2}$ corresponding with this case are

$$\alpha(s) = c + r(\cos s \vec{e}_2 + \sin s \vec{e}_3), \quad r \neq 0, \ c \in \ell.$$

These are the standard circles in Euclidean spaces and will be called \textbf{elliptic circles} in Minkowski spaces $\mathbb{R}^{1,2}$.

(2). $\ell$ is a spacelike line: Suppose that $\ell$ is generated by the vector, say $\vec{e}_3$. The group of rotations determined by $\ell$ is

$$R_\theta = \begin{pmatrix}
\cosh \theta & \sinh \theta & 0 \\
\sinh \theta & \cosh \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  

The circles are obtained

$$\alpha(s) = c + r(\cosh s \vec{e}_1 + \sinh s \vec{e}_2), \quad r \neq 0, \ c \in \ell.$$

These curves are hyperbolas in Euclidean spaces and will be named \textbf{hyperbolic circles} in Minkowski spaces $\mathbb{R}^{1,2}$.

(3). $\ell$ is a null line: By considering $\ell$ spanned by $\vec{e}_1 + \vec{e}_2$, the group of rotations associated to $\ell$ is

$$R_\theta = \begin{pmatrix}
1 + \frac{1}{2} \theta^2 & -\frac{1}{2} \theta^2 & -\theta \\
\frac{1}{2} \theta^2 & 1 - \frac{1}{2} \theta^2 & -\theta \\
-\theta & \theta & 1
\end{pmatrix}.$$  

The circles are the curves in the form

$$\alpha(s) = c + \frac{1}{2r}(\vec{e}_1 - \vec{e}_2) + s \vec{e}_3 + \frac{1}{2} rs^2(\vec{e}_2 + \vec{e}_1), \quad r \neq 0, \ c \in \ell.$$

We call these Euclidean parabolaos as \textbf{parabolic circles} in $\mathbb{R}^{1,2}$.

Finally, we here also mention two types of helices in Minkowski spaces:

elliptic helices: \hspace{1cm} $\alpha(s) = \lambda s \vec{e}_1 + r(\cos s \vec{e}_2 + \sin s \vec{e}_3)$,

hyperbolic helices: \hspace{1cm} $\alpha(s) = r(\cosh s \vec{e}_1 + \sinh s \vec{e}_2) + \lambda s \vec{e}_3$. 

1.3. Main theorems. The main objective is to concern (11)-(14) and show the existence of vortex phenomena:

- \( K = 1, N = 2 \): pairs of point vortices;
- \( K = 1, N = 3 \): elliptic vortex circles/helices;
- \( K = 2, N = 2 \): hyperbolic vortex circles.

We first give some observations of the construction of vortex phenomena for the wave map equations (11). If \( K = 1, N = 2 \), by setting

\[
W(\tau, s_1, s_2) = U(s_1, s_2)e^{i\omega \tau},
\]

in (11), we get that \( U \) will satisfy

\[
\triangle U - \frac{2\dot{U}}{1 + |U|^2} \nabla U \cdot \nabla U + |\omega|^2 \frac{1 - |U|^2}{1 + |U|^2} U = 0 \quad \text{on} \quad \mathbb{R}^2.
\]

If \( \omega \neq 0 \), rescaling variables as follows

\[
U(s_1, s_2) = u(\hat{s}_1, \hat{s}_2) \quad \text{with} \quad \hat{s}_1 = \omega s_1, \quad \hat{s}_2 = \omega s_2,
\]

will give that

\[
\triangle u - \frac{2\dot{u}}{1 + |u|^2} \nabla u \cdot \nabla u + \frac{1 - |u|^2}{1 + |u|^2} u = 0 \quad \text{on} \quad \mathbb{R}^2.
\]

It is the local form of a harmonic map with potential from \( \mathbb{R}^2 \) into \( S^2 \subset \mathbb{R}^3 \). The existence of \( u \) with point vortices was given by F. Hang and F. Lin in [6].

The above solution with single vortex in two-dimensional space can be considered as a solution in higher dimensional space. On the other hand, it is well known that (11) can be transformed into the same type problems on lower spatial dimensions by the Galilean transformations. For the convenience of readers, we here state the details. For \( K = 1, N \geq 3 \), the first trial of solutions to (11) in the form

\[
W(\tau, s_1, \ldots, s_N) = U(\hat{s}_1, \ldots, \hat{s}_N),
\]

with the Galilean transformation

\[
\hat{s}_1 = s_1, \ldots, \hat{s}_{N-1} = s_{N-1}, \quad \hat{s}_N = s_N - c\tau,
\]

will lead to

\[
\triangle_{N-1} U - \frac{2\dot{U}}{1 + |U|^2} \nabla_{N-1} U \cdot \nabla_{N-1} U

+ (1 - |c|^2) \left[ \frac{\partial^2 U}{\partial s_N^2} - \frac{2\dot{U}}{1 + |U|^2} \frac{\partial U}{\partial s_N} \frac{\partial U}{\partial \hat{s}_N} \right] = 0 \quad \text{on} \quad \mathbb{R}^N.
\]

Here and in the sequel, we will denote \( \triangle_l \) and \( \nabla_l \) the Laplace operator and gradient operator on \( l \)-dimensional spaces respectively. Rescaling variables as follows

\[
U(\hat{s}_1, \ldots, \hat{s}_N) = \hat{U}(\hat{s}_1, \ldots, \hat{s}_N),
\]

\[
\hat{s}_1 = \hat{s}_1, \ldots, \hat{s}_{N-1} = \hat{s}_{N-1}, \quad \hat{s}_N = \hat{s}_N / \sqrt{|1 - c^2|},
\]
equation (16) becomes, on $\mathbb{R}^N$,

$$\nabla_{N-1} \hat{U} - \frac{2 \hat{U}}{1 + |\hat{U}|^2} \nabla_{N-1} \hat{U} \cdot \nabla_{N-1} \hat{U} + \text{sign} \left[ \frac{\partial^2 \hat{U}}{\partial s_N^2} - \frac{2 \hat{U}}{1 + |\hat{U}|^2} \frac{\partial \hat{U}}{\partial s_N} \frac{\partial \hat{U}}{\partial s_N} \right] = 0,$$  \hspace{1cm} (17)

where the constant $\text{sign}$ is given by

$$\text{sign} = \begin{cases} 1, & \text{if } 1 - c^2 > 0; \\ 0, & \text{if } 1 - c^2 = 0; \\ -1, & \text{if } 1 - c^2 < 0. \end{cases}$$

If $1 - c^2 \geq 0$, (17) can be considered as an equation of harmonic maps from $N$ (or $N - 1$)-dimensional Euclidean space into $\mathbb{S}^2 \subset \mathbb{R}^3$. On the other hand, if $1 - c^2 < 0$, (17) is still a wave map equation from a lower dimensional space $\mathbb{R}^{1,N-1}$ into $\mathbb{S}^2 \subset \mathbb{R}^3$. If applicable, we can iterate the above transformations $N - 2$ times and get an equation for a wave map starting from $\mathbb{R}^{1,2}$. As a summary, here are the results for the existence of single point vortices for wave maps. Let $c, \omega$ be two constants with $1 - c^2 < 0$ and $\omega \neq 0$. If $K = 1$, $N = 2$, there is a solution to (11) in the following form

$$W(t, \tau, s_1, s_2) = U_1(s_1, s_2) e^{i\omega \tau}.$$  \hspace{1cm} (18)

In the case of $K = 1$, $N = 3$, there is a solution to (11) in the form

$$W(t, \tau, s_1, s_2, s_3) = U_2(s_1, s_2) e^{i\omega (s_3 - c \tau)}.$$  \hspace{1cm} (19)

Moreover, $U_1$ and $U_2$ possess single point vortices of degree $+1$ or $-1$.

Hence, we will use the point vortex solutions of (15) to construct solutions with other vortex structures to (11)-(12) in the following forms

$$W(t, \tau, s) = U\left(\tau', s' \mid s_N = \tau_1 \tau_K \right) e^{i\omega_1 \tau K} \quad \text{(Type A)}$$  \hspace{1cm} (20)

or

$$W(t, \tau, s) = U\left(\tau', s \mid s_N = \tau_1 \tau_3 \right) e^{i\omega_2 \tau K + i\omega_3 s N} \quad \text{(Type B)},$$  \hspace{1cm} (21)

while to (13)-(14) in the following forms

$$W(t, \tau, s) = U\left(\tau', s' \mid s_N = \tau_2 \tau_K - \tau_3 t \right) e^{i\omega_4 \tau K} \quad \text{(Type C)}$$  \hspace{1cm} (22)

or

$$W(t, \tau, s) = U\left(\tau', s' \mid s_N = \tau_4 t \right) e^{i\omega_5 \tau K + i\omega_6 s N} \quad \text{(Type D)},$$  \hspace{1cm} (23)

where $c_i, \omega_j \in \mathbb{R}$, $i = 1, \ldots, 4$, $j = 1, \ldots, 6$ and

$$\tau = (\tau_1, \ldots, \tau_K), \quad s = (s_1, \ldots, s_N),$$

$$\tau' = (\tau_1, \ldots, \tau_{K-1}), \quad s' = (s_1, \ldots, s_{N-1}).$$

Then after profile function $U(\tau', s)$ takes proper scaling on space $\mathbb{R}^{K-1,N}$, equations (11) and (12) become

$$i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U + \triangle_{K-1,N} U - \frac{2 \hat{U}}{1 + |\hat{U}|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \quad \text{on } \mathbb{R}^{K-1,N},$$  \hspace{1cm} (24)
while (13) and (14) reduce to

\[-i\kappa\mu \frac{\partial U}{\partial s} + i\mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s} + \frac{1 - |U|^2}{1 + |U|^2} U + \triangle K_{-1,N} U - \frac{2\bar{U}}{1 + |U|^2} \nabla K_{-1,N} U \cdot \nabla K_{-1,N} U = 0 \quad \text{on} \quad \mathbb{R}^{K_{-1,N}}, \quad (21)\]

where real constants \(\kappa = \kappa(c_i,\omega_j)\) and \(\mu = \mu(c_i,\omega_j)\). For more details of the computation the reader can refer to Section 2.1, and also concrete expressions of the parameters \(\kappa\) and \(\mu\) in Table 1. For the sake of convenience, we call equation (20) Reduced Wave Map Type equation and equation (21) Reduced Schrödinger Map Type equation.

Then we will look for solutions with various vortex structures to equations (20) and (21) in Section 2.2. Let us state the main theorem.

**Theorem 1.2.** There exist \(\kappa \in \mathbb{R}\) and small constant \(\mu > 0\) such that (20) and (21) possess the following four types of vortex solutions on the domain \(\Omega_{K,N}\) which is defined by

\[\Omega_{K,N} = \begin{cases} \mathbb{R}^2, & K = 1, N = 2, \\ \mathbb{R}^3, & K = 1, N = 3, \\ \{(\tau_1,s_1,s_2) \in \mathbb{R}^{1,2} : |\tau_1| < |s_1|\}, & K = 2, N = 2. \end{cases}\]

- **\(K = 1, N = 2\) (Vortex Pairs):** \(U(s_1,s_2)\) possess a pair of vortices of degree \(\pm 1\) at \((\pm d, 0)\) with

\[\frac{1}{d} \sim \mu \quad \text{in} \quad (20) \quad \text{or} \quad \frac{1}{d} \sim (1 - 2\kappa)\mu \quad \text{in} \quad (21);\]

- **\(K = 1, N = 3\) (Elliptic Vortex Circles):** \(U(s_1,s_2,s_3)\) possess vortex structure directed along the elliptic circles

\[\Psi \in \mathbb{R} \mapsto (d \cos \Psi, d \sin \Psi, 0) \in \mathbb{R}^3,\]

with

\[\frac{1}{d} \log d \sim \mu \quad \text{in} \quad (20) \quad \text{or} \quad \frac{1}{d} \log d \sim (1 - 2\kappa)\mu \quad \text{in} \quad (21).\]

Moreover, \(U\) are also invariant under the rotation expressed by cylinder coordinates

\[\Sigma : (r,\Psi,s_3) \mapsto (r,\Psi + \alpha, s_3), \quad \forall \alpha \in \mathbb{R}.\]

- **\(K = 1, N = 3\) (Elliptic Vortex Helices):** \(U(s_1,s_2,s_3)\) possess vortex structures directed along the elliptic helices

\[\Psi \in \mathbb{R} \mapsto (d \cos \Psi, d \sin \Psi, \lambda \Psi) \in \mathbb{R}^3,\]

with

\[\frac{1}{\sqrt{d^2 + \lambda^2}} \log d \sim \mu \quad \text{in} \quad (20) \quad \text{or} \quad \frac{1}{\sqrt{d^2 + \lambda^2}} \log d \sim (1 - 2\kappa)\mu \quad \text{in} \quad (21).\]

Moreover, \(U\) are also invariant under the skew motion expressed in the cylinder coordinates

\[\Sigma : (r,\Psi,s_3) \mapsto (r,\Psi + \alpha, s_3 + \lambda \alpha), \quad \forall \alpha \in \mathbb{R}.\]
• \( K = 2, N = 2 \) (Hyperbolic Vortex Circles): \( U(\tau_1, s_1, s_2) \) possess vortex structures directed along the hyperbolic circles

\[
\Psi \in \mathbb{R} \mapsto (d \sinh \Psi, d \cosh \Psi, 0) \in \mathbb{R}^{1,2},
\]

with

\[
\frac{1}{d} \log d \sim \mu \text{ in } (20) \quad \text{or} \quad \frac{1}{d} \log d \sim (1 - 2\kappa)\mu \text{ in } (21).
\]

Moreover, \( U \) are also invariant under the hyperbolic rotation expressed in the cylinder coordinates

\[
\Sigma: (r, \Psi, s_2) \mapsto (r, \Psi + \alpha, s_2), \quad \forall \alpha \in \mathbb{R}.
\]

**Remark 1.** More words are in order to explain the results in the above theorem.

1. In the proof of above theorem, we need that the positive parameter \( \mu \) is small enough, see Remark 4. Actually we can take suitable constants \( c_i, \omega_i \) such that \( \mu > 0 \) and \( (1 - 2\kappa)\mu > 0 \) are both small enough. So the parameter \( d \) of the locations of vortex structures is large.

2. In Euclidean space \( \mathbb{R}^3 \), the cylinder coordinates take

\[
s_1 = r \cos \Psi, \quad s_2 = r \sin \Psi, \quad s_3 = s_3,
\]

while in Minkowski space \( \mathbb{R}^{1,2} \) the cylinder coordinates are

\[
\tau_1 = r \sinh \Psi, \quad s_1 = r \cosh \Psi, \quad s_2 = s_2, \quad \forall |s_1| > |\tau_1|,
\]

\[
\tau_1 = r \cosh \Psi, \quad s_1 = r \sinh \Psi, \quad s_2 = s_2, \quad \forall |s_1| < |\tau_1|.
\]

These coordinates will be helpful for us to transform the problems to elliptic cases of 2D and to find symmetric solutions in Section 2.

3. For more details of the solutions, the reader can refer to Section 2. We have constructed vortex structures of 0-dimension (vortex pairs of degrees \( \pm 1 \)) for \( K = 1, N = 2 \) and 1-dimension (elliptic vortex circles/helices) for \( K = 1, N = 3 \). Moreover, the existence of vortex structures of elliptic vortex circles/helices can be extended to higher dimensions, see results in [11] and [16]. For the case of \( K = 2, N = 2 \), we only find solutions with hyperbolic vortex circles, other than hyperbolic vortex helices due to technical reasons, see Remark 3. It is an interesting problem to find vortex phenomena with more complex structures.

**Remark 2.** We would like to explain the results in the hyperbolic vortex circle case with an example. Without loss of generality, let us consider the Type C solution with Hyperbolic Vortex Circles to (14). From the transformations in Section 2.1, we know that by taking the Type C function

\[
W(t, \tau, s) = U\left(\sqrt{1 + |\omega_4|^2}, (\tau_1, s_1, \frac{s_2 - c_2 \tau_2 - c_3 t}{1 - |c_4|^2})e^{i\omega_4 \tau_2}\right),
\]

problem (14) reduces to (21) with

\[
\kappa = \frac{c_3}{2c_2 \omega_4} \quad \text{and} \quad \mu = \frac{2c_2 \omega_4}{\sqrt{(1 - c_2^2)(1 + \omega_4^2)}},
\]
According to Remark 1 and Theorem 1.2, we can choose suitable parameters $c_2, c_3, \omega_4$ to find a solution $U$ with the Hyperbolic Vortex Circle to (21) on the region

$$\Omega_{2,2} = \{(\tau_1, s_1, s_2) \in \mathbb{R}^{1,2} : |\tau_1| < |s_1|\}.$$  

It means that we only find a solution $W(t, \tau_1, \tau_2, s_1, s_2)$ with Hyperbolic Vortex Circle solves (14) on the region

$$\{(t, \tau_1, \tau_2, s_1, s_2) \in \mathbb{R} \times \mathbb{R}^{2,2} : |\tau_1| < |s_1|\},$$

which only covers one half of the space $\mathbb{R} \times \mathbb{R}^{2,2}$.

Here it should be mentioned that we can obtain a similar solution $\tilde{W}$ to (14) on the other half region of the space $\mathbb{R} \times \mathbb{R}^{2,2}$ by considering the following type of the solution

$$\tilde{W}(t, \tau_1, \tau_2, s_1, s_2) = \tilde{U}(s_1, \tau_1, \tau_2 - \tilde{c}_2 s_2 - \tilde{c}_3 t) e^{i\tilde{\omega}_4 s_2},$$

which is given by interchanging variables $(\tau_1, \tau_2)$ with $(s_1, s_2)$ of the Type C solution in (19). The same transformations in Section 2.1 will give that $\tilde{U}$ also satisfies (21) with

$$\kappa = -\frac{\tilde{c}_3}{2\tilde{c}_2 \tilde{\omega}_4} \quad \text{and} \quad \mu = \frac{2\tilde{c}_2 \tilde{\omega}_4}{\sqrt{(1 - \tilde{c}_2^2)(\tilde{\omega}_4^2 - 1)}}.$$  

Though finding two solutions with Hyperbolic Vortex Circles solve (14) on disjoint half regions of the space $\mathbb{R} \times \mathbb{R}^{2,2}$, we can not simply take these two solutions as a global solution on the space $\mathbb{R} \times \mathbb{R}^{2,2}$ since these two solutions do not match on the common boundary.

On the other hand, it is instructive to consider a related problem. C. Kenig, G. Ponce and L. Vega in [9] have studied the following Schrödinger equation which is analogous to (1):

$$\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} = i \mathcal{L} u + P(u, \nabla u, \bar{u}, \nabla \bar{u}), \\
u(0, x) = u_0(x),
\end{cases}
\end{aligned}$$

where $u = u(t, x)$ is a complex valued function from $\mathbb{R} \times \mathbb{R}^N$, $\mathcal{L}$ is a non-degenerate differential operator of second-order

$$\mathcal{L} = \sum_{j \leq K} \partial_j^2 - \sum_{j > K} \partial_j^2,$$

for some $K \in \{1, \cdots, n\}$, and $P : \mathbb{C}^{2n+2} \to \mathbb{C}$ is a polynomial satisfying certain constraints. They proved the local well-posedness of the above initial value problem in appropriate Sobolev spaces. No doubt, it is a hard task to settle the general existence and uniqueness results of such a class of Cauchy problems of (1).

The organization of the paper is as follows: In Section 2.1, by looking for special solutions to (11)-(14), we convert them into simpler problems which are defined on $\mathbb{R}^{K-1,N}$. In Section 2.2, we take advantage of the isometries of $\mathbb{R}^{K-1,N}$ to introduce various vortex structures (vortex pairs, vortex circles and vortex helices), and then transform the above derived problems into two-dimensional elliptic cases. Some preliminaries on block vortex solutions are prepared in Section 3.1. An outline of the strategies of the resolution theory for the mentioned two-dimensional elliptic problems will be provided in Section 3.2. Section 4 is devoted to the details of fulfilling the strategies and construction of solutions with elliptic vortex helices,
which will exemplify the proofs for Theorem 1.2. The finite-dimensional Lyapunov-Schmidt reduction method from nonlinear analysis will play an important role. Some detailed computations are provided in Appendix A.

2. Formulations of the problems.

2.1. Transform methods and special solutions. In this section and Appendix A, we will show how to convert (11)-(12) and (13)-(14) into (20) and (21) respectively. At the same time, we will give the explicit expressions of the parameters $\mu$ and $\kappa$ in (20)-(21) from case to case.

Recall (5)-(6) for the definitions of the differential operators $\Delta_{K,N}$ and $\nabla_{K,N}$ in $\mathbb{R}^{K,N}$. For simplicity, we will show the transform procedure of the Schrödinger map equations with anisotropic terms, i.e., the equation (14) to (21). Other cases are similar (even simpler) as (14), and the readers can refer to Appendix A.

Case 1. Taking the special solution of (14) in the form

$$W(t, \tau, s) = U\left(\tau', s', s_N - c_2 \tau_K - c_3 t\right)e^{i \omega_4 \tau K} \quad \text{with} \quad 1 - |c_2|^2 > 0,$$

this leads to the problem

$$-ic_4 \frac{\partial U}{\partial s_N} + i \frac{1 - |U|^2}{1 + |U|^2} 2c_2 \omega_4 \frac{\partial U}{\partial s_N} - |c_2|^2 \frac{\partial^2 U}{\partial s_N^2} + \frac{2U}{1 + |U|^2} \nabla_{K-N} U - \frac{2U}{1 + |U|^2} \nabla_{K-N} U \cdot \nabla_{K-N} U$$

$$+ (1 + |\omega_4|^2) \frac{1 - |U|^2}{1 + |U|^2} U = 0 \quad \text{on} \quad \mathbb{R}^{K-N}.$$

Rescaling variables as follows, for $j = 1, \cdots, K-1$ and $l = 1, \cdots, N-1$,

$$\tau_j \to \sqrt{1 + |\omega_4|^2} \tau_j, \quad s_l \to \sqrt{1 + |\omega_4|^2} s_l, \quad s_N \to \sqrt{1 - |c_2|^2} s_N,$$

the above equation becomes

$$-i \kappa \mu \frac{\partial U}{\partial s_N} + i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U$$

$$+ \Delta_{K-N} U - \frac{2U}{1 + |U|^2} \nabla_{K-N} U \cdot \nabla_{K-N} U = 0 \quad \text{on} \quad \mathbb{R}^{K-N},$$

where

$$\kappa = \frac{c_4}{2c_2 \omega_4} \quad \text{and} \quad \mu = \frac{2c_2 \omega_4}{\sqrt{(1 - |c_2|^2)(1 + |\omega_4|^2)}}. \quad (22)$$

That’s to say, if we take a solution of the following form

$$W(t, \tau, s) = U\left(\sqrt{1 + |\omega_4|^2} \left(\tau', s', s_N - c_2 \tau_K - c_3 t\right)\right)e^{i \omega_4 \tau K} \quad (23)$$

with $1 - |c_2|^2 > 0$, (14) becomes (21) with delicate expressions (22) of parameters $\kappa$ and $\mu$.

Case 2. We also look for a solution to (14) in the form,

$$W(t, \tau, s) = U\left(\tau', s', s_N - c_4 t\right)e^{i \omega_5 \tau K + i \omega_6 s_N} \quad \text{with} \quad 1 + |\omega_5|^2 - |\omega_6|^2 > 0. \quad (24)$$
This gives that \( U \) will satisfy
\[
-ic_4 \frac{\partial U}{\partial s_N} + i \frac{1 - |U|^2}{1 + |U|^2} 2 \omega_6 \frac{\partial U}{\partial s_N} + (1 + |\omega_5|^2 - |\omega_6|^2) \frac{1 - |U|^2}{1 + |U|^2} U \\
+ \triangle_{K-1,N} U - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \quad \text{on} \quad \mathbb{R}^{K-1,N}.
\]
Rescaling variables as follows, for \( j = 1, \cdots K-1 \) and \( l = 1, \cdots N \),
\[
\tau_j \rightarrow \sqrt{1 + |\omega_5|^2 - |\omega_6|^2} \tau_j, \quad s_l \rightarrow \sqrt{1 + |\omega_5|^2 - |\omega_6|^2} s_l,
\]
we derive that
\[
-ik \mu \frac{\partial U}{\partial s_N} + ik \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{2\omega_6}{1 + |\omega_5|^2 - |\omega_6|^2} U \\
+ \triangle_{K-1,N} U - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \quad \text{on} \quad \mathbb{R}^{K-1,N},
\]
where
\[
\kappa = \frac{c_4}{2\omega_6} \quad \text{and} \quad \mu = \frac{2\omega_6}{\sqrt{1 + |\omega_5|^2 - |\omega_6|^2}}.
\]
In other words, if we take a solution of the following form
\[
W(t, \tau, s) = U \left( \sqrt{1 + |\omega_5|^2 - |\omega_6|^2} (\tau', s'; (s_N - c_4 t)) \right) e^{i\omega_5 \tau_k + i\omega_6 \cdot s_N},
\]
with \( 1 + |\omega_5|^2 - |\omega_6|^2 > 0 \), (14) becomes (21) with parameters \( \kappa \) and \( \mu \) in (25).

For the convenience of readers, we give the delicate expressions of parameters \( \kappa \) and \( \mu \) for other cases in Table 1 (refer to Appendix A for computations). For the meanings of Type A, Type B, Type C, Type D in Table 1, the reader can refer to (18) and (19).

### 2.2. Various vortex structures.

In order to construct various vortex structures and prove Theorem 1.2, we would take advantage of symmetries of Reduced Wave Map Type equation (20) and Reduced Schrödinger Map Type equation (21) to convert them into two-dimensional elliptic equations with singular terms. For simplicity, we only consider solutions with elliptic vortex helices for Reduced Wave Map Type equation (20) and Reduced Schrödinger Map Type equation (21) in the case of \( K = 1 \) and \( N = 3 \). Other vortex structures in Theorem 1.2 can be described in a similar and even simpler way.

Taking \( K = 1 \) and \( N = 3 \) and then using the cylinder coordinates in \( \mathbb{R}^3 \)
\[
s_1 = r \cos \Psi, \quad s_2 = r \sin \Psi, \quad s_3 = s,
\]
equation (20) is transformed to
\[
i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_3} + \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \Psi^2} + \frac{\partial^2 U}{\partial s_3^2} + \frac{1 - |U|^2}{1 + |U|^2} U \\
= \frac{2U}{1 + |U|^2} \left[ \left( \frac{\partial U}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial U}{\partial \Psi} \right)^2 + \left( \frac{\partial U}{\partial s_3} \right)^2 \right] \quad \text{in} \quad \mathbb{R}^3.
\]
For (27), we want to find a solution \( u \) which has a vortex helix directed along the curve in the form
\[
\Psi \in \mathbb{R} \mapsto (d \cos \Psi, d \sin \Psi, \lambda \Psi) \in \mathbb{R}^3, \quad d = \dot{d}/\varepsilon, \quad \lambda = \dot{\lambda}/\varepsilon,
\]
Table 1. The Expressions of Parameters $\kappa$ and $\mu$

| Equations | Sol. Type | $\kappa$ | $\mu$ |
|-----------|-----------|---------|-------|
| (11)      | Type A    | null    | $\frac{2c_1}{\sqrt{1-|c_1|^2}}$ |
|           | Type B    | null    | $\frac{2\omega_3}{\sqrt{|\omega_2|^2-|\omega_3|^2}}$ |
| (12)      | Type A    | null    | $\frac{2c_1 \omega_1}{\sqrt{(1-|c_1|^2)(1+|\omega_1|^2)}}$ |
|           | Type B    | null    | $\frac{2\omega_3}{\sqrt{1+|\omega_2|^2-|\omega_3|^2}}$ |
| (13)      | Type C    | $\frac{c_3}{2c_2 \omega_4}$ | $\frac{2c_2}{\sqrt{1-|c_2|^2}}$ |
|           | Type D    | $\frac{c_4}{2\omega_6}$ | $\frac{2\omega_6}{\sqrt{|\omega_5|^2-|\omega_6|^2}}$ |
| (14)      | Type C    | $\frac{c_3}{2c_2 \omega_4}$ | $\frac{2c_2 \omega_4}{\sqrt{(1-|c_2|^2)(1+|\omega_4|^2)}}$ |
|           | Type D    | $\frac{c_4}{2\omega_6}$ | $\frac{2\omega_6}{\sqrt{1+|\omega_5|^2-|\omega_6|^2}}$ |

where $\hat{d}$ and $\hat{\lambda}$ are universal constants. Thus let $u$ be invariant under the skew motion

$$\Sigma: (r, \Psi, s_3) \mapsto (r, \Psi + \alpha, s_3 + \lambda \alpha), \; \forall \alpha \in \mathbb{R},$$

i.e., $u$ has the symmetry

$$U(r, \Psi, s_3) = U(r, \Psi + (-\Psi), s_3 + \lambda (-\Psi)) = U(r, 0, s_3 - \lambda \Psi).$$

Then we derive that

$$i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_3} + \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \left(1 + \frac{\lambda^2}{r^2}\right) \frac{\partial^2 U}{\partial s_3^2} + \frac{1 - |U|^2}{1 + |U|^2} U$$

$$= \frac{2U}{1 + |U|^2} \left[ \left(\frac{\partial U}{\partial r}\right)^2 + \left(1 + \frac{\lambda^2}{r^2}\right) \left(\frac{\partial U}{\partial s_3}\right)^2 \right],$$

which will be defined on the region $\{(r, s_3) \in [0, \infty) \times (-\lambda \pi, \lambda \pi)\}$.

Set

$$\sigma = \frac{\lambda}{\hat{d}}, \; \gamma = \sqrt{1 + \sigma^2}.$$ (30)

It is worth mentioning that these two positive parameters are of independence of $\varepsilon$. To uniformly deal with equations in Section 4, we introduce new notation

$$U(r, s_3) = u(x_1, x_2), \; (x_1, x_2) = (r, \gamma^{-1} s_3) \quad \text{and} \quad z = x_1 + ix_2, \; \mu = \varepsilon |\log \varepsilon|. $$
Hence, (29) becomes

\[ i\varepsilon |\log \varepsilon|^\gamma - 1 \frac{1 - |u|^2}{1 + |u|^2} \frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{1}{1 + \lambda^2} \frac{\partial u}{\partial x_1} + \left( 1 + \lambda^2 \right) \frac{\partial^2 u}{\partial x_1^2} + \left( 1 + \lambda^2 \right) \frac{\partial^2 u}{\partial x_2^2} \]

\[ + \frac{1 - |u|^2}{1 + |u|^2} = \frac{2\tilde{u}}{1 + |u|^2} \left[ \frac{\partial u}{\partial x_1} + \left( 1 + \lambda^2 \right) \frac{\partial^2 u}{\partial x_2^2} \right]. \tag{31} \]

As we have done in the above (i.e., using the cylinder coordinates in $\mathbb{R}^3$ and taking the new notation), for the existence of solutions with vortex helices, equation (21) is transformed to

\[ i\varepsilon |\log \varepsilon|^\gamma - 1 \frac{1 - |u|^2}{1 + |u|^2} \frac{\partial u}{\partial x_2} - i\kappa \varepsilon |\log \varepsilon|^\gamma - 1 \frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{1 + \lambda^2} \frac{\partial u}{\partial x_1} + \frac{1 - |u|^2}{1 + |u|^2} \]

\[ + \left( 1 + \lambda^2 \right) \frac{\partial^2 u}{\partial x_2^2} = \frac{2\tilde{u}}{1 + |u|^2} \left[ \frac{\partial u}{\partial x_1} + \left( 1 + \lambda^2 \right) \frac{\partial^2 u}{\partial x_2^2} \right]. \tag{32} \]

We shall consider (31) and (32) on the region

\[ \mathcal{G} = \mathbb{R} \times (-\lambda\pi/\gamma, \lambda\pi/\gamma) \]

and then impose the boundary conditions

\[ |u(z)| \to 1 \quad \text{as} \quad |x_1| \to +\infty, \]

\[ \frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad \forall \, x_2 \in (-\lambda\pi/\gamma, \lambda\pi/\gamma), \]

\[ u(x_1, -\lambda\pi/\gamma) = u(x_1, \lambda\pi/\gamma), \quad \forall \, x_1 \in \mathbb{R}, \]

\[ u_{x_1}(x_1, -\lambda\pi/\gamma) = u_{x_1}(x_1, \lambda\pi/\gamma), \quad \forall \, x_1 \in \mathbb{R}. \] \tag{34}

In fact, it is natural for us to impose the last three conditions in (34) because we would like to construct solutions which are symmetric with respect to the $x_2$-axis and periodic with respect to variable $x_2$. Moreover, it is easy to see that (31) and (32) are also invariant under the following two transformations

\[ u(z) \to u(\bar{z}), \quad u(z) \to u(-\bar{z}). \tag{35} \]

Whence, if we write the solutions $u$ to (31) and (32) in the form

\[ u(x_1, x_2) = u_1(x_1, x_2) + iu_2(x_1, x_2), \]

then $u_1$ and $u_2$ satisfy the following conditions:

\[ |u(z)| \to 1 \quad \text{as} \quad |x_1| \to +\infty, \]

\[ u_1(x_1, x_2) = u_1(-x_1, x_2), \quad u_2(x_1, x_2) = u_2(-x_1, x_2), \]

\[ u_1(x_1, -x_2) = u_1(x_1, x_2), \quad u_2(x_1, -x_2) = u_2(x_1, x_2), \]

\[ \frac{\partial u_1}{\partial x_1}(0, x_2) = 0, \quad \frac{\partial u_2}{\partial x_1}(0, x_2) = 0, \] \tag{36}

\[ u_1(x_1, -\lambda\pi/\gamma) = u_1(x_1, \lambda\pi/\gamma), \quad u_2(x_1, -\lambda\pi/\gamma) = u_2(x_1, \lambda\pi/\gamma), \]

\[ \frac{\partial u_1}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial u_1}{\partial x_2}(x_1, \lambda\pi/\gamma), \quad \frac{\partial u_2}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial u_2}{\partial x_2}(x_1, \lambda\pi/\gamma). \]

We will use these conditions to construct solutions to (31) and (32). We pause here to give a remark.
Remark 3. For the case of \( K = 2 \) and \( N = 2 \), we consider solutions with hyperbolic vortex helices for Reduced Wave Map Type equation (20) and Reduced Schrödinger Map Type equation (21) on the domain 
\[
\Omega_{2,2} = \{(\tau_1, s_1, s_2) \in \mathbb{R}^{1,2} : |\tau_1| < |s_1| \}.
\]
Taking the cylinder coordinates in \( \Omega_{2,2} \)
\[
\tau_1 = r \sinh \Psi, \quad s_1 = r \cosh \Psi, \quad s_2 = s_2,
\]
equation (20) is transformed to
\[
i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_2} + \frac{\partial^2 U}{\partial \tau^2} + \frac{1}{r} \frac{\partial U}{\partial \tau} - \frac{1}{r^2} \frac{\partial^2 U}{\partial \Psi^2} + \frac{\partial^2 U}{\partial s_2^2} + \frac{1}{1 + |U|^2} U = 2 \bar{U}_1 + \frac{1}{1 + |U|^2} \left[ \left( \frac{\partial U}{\partial r} \right)^2 - \frac{1}{r^2} \left( \frac{\partial U}{\partial \Psi} \right)^2 + \left( \frac{\partial U}{\partial s_2} \right)^2 \right].
\] (38)

For (38), we want to find a solution \( U \) which has a vortex structure directed along the hyperbolic circle in the form
\[
\Psi \in \mathbb{R} \rightarrow (d \sinh \Psi, d \cosh \Psi, 0),
\]
with a parameter \( d \). Thus let \( U \) be invariant under rotation
\[
\Sigma : (r, \Psi, s_2) \mapsto (r, \Psi + \alpha, s_2), \quad \forall \alpha \in \mathbb{R}, \quad \text{i.e.} \quad U(r, \Psi, s_2) \equiv U(r, s_2).
\]
Then \( U \) satisfies the following elliptic problem
\[
i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_2} + \frac{\partial^2 U}{\partial \tau^2} + \frac{1}{r} \frac{\partial U}{\partial \tau} - \frac{1}{r^2} \frac{\partial^2 U}{\partial \Psi^2} + \frac{1}{1 + |U|^2} U
\]
\[
= \frac{2\bar{U}}{1 + |U|^2} \left[ \left( \frac{\partial U}{\partial r} \right)^2 - \frac{1}{r^2} \left( \frac{\partial U}{\partial \Psi} \right)^2 + \left( \frac{\partial U}{\partial s_2} \right)^2 \right].
\]

On the other hand, if we make a trial of a solution to (38) with a vortex structure directed along the hyperbolic circle in the form
\[
\Psi \in \mathbb{R} \rightarrow (d \sinh \Psi, d \cosh \Psi, \lambda \Psi),
\]
with two parameters \( d \) and \( \lambda \), in such a way that \( U \) is also invariant skew rotation
\[
\Sigma : (r, \Psi, s_2) \mapsto (r, \Psi + \alpha, s_2 + \lambda \alpha), \quad \forall \alpha \in \mathbb{R},
\]
i.e.,
\[
U(r, \Psi, s_2) = U(r, \Psi - \lambda(-\Psi), s_2 + \lambda(-\Psi)) = U(r, 0, s_2 - \lambda \Psi).
\]
Then \( U \) satisfies the problem
\[
i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_2} + \frac{\partial^2 U}{\partial \tau^2} + \frac{1}{r} \frac{\partial U}{\partial \tau} + \left( 1 - \frac{\lambda^2}{r^2} \right) \frac{\partial^2 U}{\partial s_2^2} + \frac{1}{1 + |U|^2} U
\]
\[
= \frac{2\bar{U}}{1 + |U|^2} \left[ \left( \frac{\partial U}{\partial r} \right)^2 + \left( 1 - \frac{\lambda^2}{r^2} \right) \left( \frac{\partial U}{\partial s_2} \right)^2 \right].
\]

This is not an elliptic equation and out of the range of the present paper. \( \square \)

Finally, we draw a conclusion in the following remark.
Remark 4. We have transformed all problems to two-dimensional elliptic cases which are suitable variances of (15). Furthermore, we can choose real constants $c_i$'s and $\omega_i$'s such that $\mu$ is a small positive constant, which gives a small parameter $\varepsilon > 0$ by the relation $\mu = \varepsilon$ or $\mu = |\varepsilon| \log |\varepsilon|$. For the proof of Theorems 1.2, we shall construct solutions to these two-dimensional elliptic problems by the standard Lyapunov-Schmidt reduction methods in a way such that each possesses a vortex of degree +1 at $(d, 0)$ and its antipair of degree −1 at $(-d, 0)$. Additional to the computations for standard vortices in two-dimensional case, there are two extra derivative terms

$$
\frac{1}{x_1} \frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2} \left(1 + \frac{\lambda^2}{x_1^2} \right) \left[ \frac{\partial^2 u}{\partial x_1^2} - \frac{2u}{1 + \left| \frac{\partial u}{\partial x_2} \right|^2} \right].
$$

(39)

It is obvious that the first singular term can be handled by the second requirement in (34), while more details for the second one will be showed in Sections 4.1-4.2. For simplicity, in the rest part of the present paper, we only sketch the resolution and arguments for (32), which basically follow the methods in [16] for the existence of elliptic vortex helices. From the viewpoint of analysis, we only have one more term than equation (3.12) in [16].

3. Some preliminaries and outline of the proof. In Section 3.1, we collect some important facts which will be used later. These include the asymptotic behaviors and nondegeneracy of the solution with degree one vortex. For the convenience of readers, we also provide an outline of the proof of Theorem 1.2 in Section 3.2.

3.1. Preliminaries. For (15), if we look for a solution with a standard vortex of degree +1 in the form

$$
u = w^+ := \rho(\ell)e^{i\theta},
$$

in polar coordinate $(\ell, \theta)$ in $\mathbb{R}^2$, then $\rho$ satisfies

$$
\rho'' + \frac{\rho'}{\ell} - \frac{2\rho(\rho')^2}{1 + \rho^2} + \left(1 - \frac{1}{\ell^2}\right) \frac{1 - \rho^2}{1 + \rho^2} \rho = 0.
$$

(40)

Another solution $w^- := \rho(\ell)e^{-i\theta}$ will be of vortex of degree −1. These two functions will be our block elements for future construction of approximate solutions. The existence and the following properties of $\rho$ are proved in [6].

Lemma 3.1. There hold the asymptotic behaviors:

(1). $\rho(0) = 0, \quad 0 < \rho(\ell) < 1, \quad \rho'(\ell) > 0$ for $\ell > 0$,

(2). $\rho(\ell) = 1 - c_0 \frac{e^{-\ell}}{\sqrt{\ell}} + O(\ell^{-3/2} e^{-\ell})$ as $\ell \to +\infty$, where $c_0 > 0$. □

Notation. For simplicity, from now on, we use $w = \rho(\ell)e^{i\theta}$ to denote the degree +1 vortex.

Setting $w = w_1 + iw_2$ and $z = x_1 + ix_2$, then we consider the following linearized operator of (15) around the standard profile $w$:

$$
\mathcal{L}_0(\phi) = \nabla^2 \phi - \frac{4(w_1 \nabla w_1 + w_2 \nabla w_2)}{1 + |w|^2} \nabla \phi - \frac{4 \nabla (w, \phi)}{1 + |w|^2} \nabla w + \frac{4(\nabla w, \nabla \phi)}{1 + |w|^2} w
\begin{align*}
&+ \frac{8 \langle w, \phi \rangle (w_1 \nabla w_1 + w_2 \nabla w_2)}{(1 + |w|^2)^2} \nabla w - \frac{4(1 + |\nabla w|^2) \langle w, \phi \rangle}{(1 + |w|^2)^2} w
\end{align*}
\begin{align*}
&+ \frac{2 |\nabla w|^2}{1 + |w|^2} \phi + \frac{1 - |w|^2}{1 + |w|^2} \phi.
\end{align*}

|
The nondegeneracy of $w$ is contained in the following lemma (see Appendix in [11] for the proof).

**Lemma 3.2.** Suppose that

\[ L_0[\phi] = 0, \]

where $\phi = iw\psi$, and $\psi = \psi_1 + i\psi_2$ satisfies the following decaying estimates

\[ |\psi_1| + |z|\|\nabla \psi_1\| \leq C(1 + |z|)^{-\varrho}, \quad |\psi_2| + |z|\|\nabla \psi_2\| \leq C(1 + |z|)^{-1-\varrho}, \]

for some $0 < \varrho < 1$. Then

\[ \phi = \hat{c}_1 \frac{\partial w}{\partial x_1} + \hat{c}_2 \frac{\partial w}{\partial x_2} \]

for certain real constants $\hat{c}_1, \hat{c}_2$. \[\square\]

3.2. **Outline of the proof.** To prove Theorem 1.2 we will use the finite-dimensional Lyapunov-Schmidt reduction method to find solutions to problems such as (32) in Section 2, see also Remark 4. The finite-dimensional Lyapunov-Schmidt reduction procedure has been used in many other problems. See [1], [5], [10] and the references therein. M. del Pino, M. Kowalczyk and M. Musso [2] were the first to use this procedure to study the Ginzburg-Landau equation for the existence of vortices in a bounded domain. The methods in the present paper basically follow those in dealing with vortex phenomena for Schrödinger map equations in [11] and [16]. Here are the main steps of the approach.

**Step 1.** Constructions of approximate solutions

To construct a real solution, the first step is to construct an approximate solution, denoted by $V_d$ in (77), possessing a vortex located at $(d, 0)$ and its antipair located at $(-d, 0)$. Here $d$ is a parameter to be determined in the reduction procedure. In the construction of approximation to solutions to (32), there are singularities caused by the application of the terms in (39) to the phase terms of the standard vortices, which will be handled in Sections 4.1 and 4.2.

The approximate solution $V_d$ has the symmetry

\[ u_2(x_1, x_2) = u_2(x_1, -x_2), \quad u_2(x_1, x_2) = u_2(-x_1, x_2). \]  

By substituting $V_d$ into (32), we can derive the estimations of the error in suitable weighted norms. The reader can refer to the papers [2] and [11].

**Step 2.** Finding a perturbation

We intend to look for real solutions to (32) by adding a perturbation term, say $\psi$, to the approximation $V_d$ where the perturbation term is small in suitable norms. More precisely, for the perturbation $\psi = \psi_1 + i\psi_2$ with conditions in (83), we take the solution $u$ in the form (cf. (81))

\[ u(y) = \eta(V_d + iV_d\psi) + (1 - \eta)V_d e^{iy}. \]

This perturbation method near the vortices was introduced in [2].

For given parameters $d$ and small $\varepsilon$, instead of considering (32) we look for a $\psi$ to the projected form (89). By writing the projected problem in the form of the perturbation term $\psi$ (with a linear part and a nonlinear part) and then using Lemma 3.2, we can find the perturbation term $\psi$ through \textit{a priori estimates} and the contraction mapping theorem.

**Step 3.** Adjusting the parameters

Note that the perturbation term $\psi$ and the Lagrange multipliers $C$ in (89) are functions of the parameters $d$. To get real solutions to (32), we shall choose suitable
parameter \( d \) such that \( C \) is zero. It is equivalent to solving a reduced algebraic equation for the Lagrange multiplier

\[
C_\varepsilon(d) = 0.
\]

We can derive the equation in (90) by the standard reduction procedure. This will be done in Section 4.5. In other words, we achieve the balance between the vortex-antivortex interaction and the effect of motion of vortices by adjusting the locations of the vortices.

4. The construction of vortex helices. In this section, we will construct solutions with vortices to (32), which will provide elliptic vortex helices to (1). Before going further, some notation is provided.

Notation. To the end of constructing vortex pairs locating at \((d, 0)\) and \((-d, 0)\), we write

\[
\frac{\partial^2 u}{\partial x_1^2} + \left(1 + \frac{\lambda^2}{x_1^2}\right) \gamma^{-2} \frac{\partial^2 u}{\partial x_2^2} = \triangle u + \gamma^{-2} \left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{\bar{d}^2}\right] \frac{\partial^2 u}{\partial x_2^2},
\]

and

\[
\frac{2\bar{u}}{1 + |u|^2} \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(1 + \frac{\lambda^2}{x_1^2}\right) \gamma^{-2} \left(\frac{\partial u}{\partial x_2}\right)^2
\]

\[
= \frac{2\bar{u}}{1 + |u|^2} \nabla u \cdot \nabla u + \frac{2\bar{u}}{1 + |u|^2} \gamma^{-2} \left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{\bar{d}^2}\right] \left(\frac{\partial u}{\partial x_2}\right)^2,
\]

where the parameters \( d \), \( \lambda \), and \( \gamma \) are given in (28) and (30), and then set

\[
S_0[u] := \triangle u - \frac{2\bar{u}}{1 + |u|^2} \nabla u \cdot \nabla u + F(u) \quad \text{with} \quad F(u) = \frac{1 - |u|^2}{1 + |u|^2} u,
\]

\[
S_1[u] := -\frac{2\bar{u}}{1 + |u|^2} \gamma^{-2} \left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{\bar{d}^2}\right] \left(\frac{\partial u}{\partial x_2}\right)^2,
\]

\[
S_2[u] := \gamma^{-2} \left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{\bar{d}^2}\right] \frac{\partial^2 u}{\partial x_2^2}, \quad S_3[u] := \frac{1}{x_1} \frac{\partial u}{\partial x_1}, \quad S_4[u] := -i\kappa\gamma^{-1} |\log \varepsilon| \frac{\partial u}{\partial x_2}, \quad S_5[u] := i\gamma^{-1} |\log \varepsilon| \frac{1 - |u|^2}{1 + |u|^2} \frac{\partial u}{\partial x_2},
\]

\[
S[u] := S_0[u] + S_1[u] + S_2[u] + S_3[u] + S_4[u] + S_5[u].
\]

It is worth mentioning that (32) is also singular when \( x_1 = 0 \) due to the terms \( S_1[u] \) and \( S_2[u] \). We need more analysis to handle the singularity as \( x_1 \to 0 \). Whence, for the convenience of further careful analysis, we divide the region \( \mathcal{S} \) (cf. (33)) into three parts:

\[
\mathcal{S}_+ = \left\{ (x_1, x_2) : x_1 > 2\bar{\omega}/\varepsilon, -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma \right\},
\]

\[
\mathcal{S}_0 = \left\{ (x_1, x_2) : -2\bar{\omega}/\varepsilon < x_1 < 2\bar{\omega}/\varepsilon, -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma \right\}, \quad \text{(43)}
\]

\[
\mathcal{S}_- = \left\{ (x_1, x_2) : x_1 < -2\bar{\omega}/\varepsilon, -\lambda\pi/\gamma < x_2 < \lambda\pi/\gamma \right\}.
\]

Here \( \bar{\omega} \) is a small positive constant less than 1/1000.
For \( \vec{e}_1 = (1, 0) \) and any given \((x_1, x_2) \in \mathbb{R}^2\), let \( \rho_{\vec{e}_1} \) and \( \rho_{-\vec{e}_1} \) be respectively the angle arguments of the vectors \( x - d\vec{e}_1 = (x_1 - d, x_2) \) and \( x + d\vec{e}_1 = (x_1 + d, x_2) \) in the \((x_1, x_2)\) plane. We also let

\[
\ell_1(x_1, x_2) = \sqrt{(x_1 - d)^2 + x_2^2}, \quad \ell_2(x_1, x_2) = \sqrt{(x_1 + d)^2 + x_2^2},
\]

be the distance functions between the point \((x_1, x_2)\) and the pair of vortices locating at the points \( d\vec{e}_1 \) and \(-d\vec{e}_1\).

### 4.1. First approximate solution and its error.

Recall the vortex model solutions \( w^+ \) and \( w^- \) with the function \( \rho \) given in (40), and also the parameters in (28). Define the smooth cut-off function \( \tilde{\gamma} \) in the form

\[
\tilde{\gamma}(s) = 1 \text{ for } |s| \leq 1, \quad \tilde{\gamma}(s) = 0 \text{ for } |s| \geq 2.
\]

By writing

\[
\tilde{\rho} = \rho(\ell_1)\rho(\ell_2), \quad \varphi_0 = \rho_{\vec{e}_1} - \rho_{-\vec{e}_1},
\]

for each fixed \( d := \tilde{d}/\varepsilon \) with \( \tilde{d} \in [1/100, 100] \), we define the first approximate solution

\[
v_0(z) := \eta_e(x_1) e^{i\varphi_0} + (1 - \eta_e(x_1))\tilde{\rho} e^{i\varphi_0},
\]

where \( \eta_e(x_1) = \tilde{\eta}(\varepsilon|x_1|/\varepsilon) \). As in (43), here \( \varepsilon \) is the small positive constant less than \( 1/1000 \).

One of the main objectives of this section is to compute the error \( \mathcal{S}[v_0] \), as those done in [16]. For the convenience of reader, we here sketch the computations and only show the method for dealing with the singular terms.

The calculation for the terms \( \mathcal{S}_1[v_0] \) and \( \mathcal{S}_2[v_0] \) are proceeded as

\[
\mathcal{S}_1[v_0] = -\frac{2\tilde{\rho}}{1+|\tilde{\rho}|^2} \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left( \frac{\partial^2 \tilde{\rho}}{\partial x_2^2} - |\tilde{\rho}|^2 \frac{\partial \varphi_0}{\partial x_2} \right)^2 + 2i\tilde{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial^2 \varphi_0}{\partial x_2^2} e^{i\varphi_0},
\]

and

\[
\mathcal{S}_2[v_0] = \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{\partial^2 \varphi_0}{\partial x_2^2} + 2i\tilde{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial^2 \varphi_0}{\partial x_2^2} - \tilde{\rho} \frac{\partial \varphi_0}{\partial x_2} \left( \frac{\partial^2 \varphi_0}{\partial x_2^2} \right)^2 e^{i\varphi_0} + i\nu_0 \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{\partial^2 \varphi_0}{\partial x_2^2}.
\]

Whence, there holds

\[
\mathcal{S}_1[v_0] + \mathcal{S}_2[v_0] = \Lambda_{11} e^{i\varphi_0} + i\nu_0 \mathcal{S}_2[\varphi_0],
\]

where

\[
\Lambda_{11} = \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left( -\frac{2\tilde{\rho}}{1+|\tilde{\rho}|^2} \frac{\partial \tilde{\rho}}{\partial x_2} \right)^2 + \frac{\partial^2 \tilde{\rho}}{\partial x_2^2} + \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{1-|\tilde{\rho}|^2}{1+|\tilde{\rho}|^2} \left( -\tilde{\rho} \frac{\partial \varphi_0}{\partial x_2} \right)^2 + 2i\tilde{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial^2 \varphi_0}{\partial x_2^2}.
\]

We here need more analysis on the last term \( i\nu_0 \mathcal{S}_2[\varphi_0] \) in the above formula. Note that

\[
\frac{\partial^2 \varphi_0}{\partial x_2^2} = \frac{\partial^2 \theta_{\vec{e}_1}}{\partial x_2^2} - \frac{\partial^2 \theta_{-\vec{e}_1}}{\partial x_2^2} = \frac{-2(x_1 - d)x_2}{\ell_1^2} - \frac{-2(x_1 + d)x_2}{\ell_2^2}.
\]
In the neighborhood of \( d\vec{c}_1 \) there holds
\[
\gamma^{-2} \left( \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right) = -\frac{2\sigma^2}{d\gamma} (x_1 - d) + \frac{\sigma^2}{d\gamma^2} \frac{3d(x_1 - d)^2 + 2(x_1 - d)^3}{x_1^2} \tag{49}
\]
whence there is a singularity in \( S_2[\varphi_0] \) of the form
\[
-\frac{2\sigma^2}{d\gamma^2} (x_1 - d) \frac{\partial^2 \varphi_1}{\partial x_2^2} = -\frac{2\sigma^2}{d\gamma^2} y_1 \frac{\partial^2 \varphi_1}{\partial y_2^2} = 4\sigma^2 y_1^2 \frac{\partial^2 \varphi_1}{\partial y_2^2} \text{ with } y = x - d\vec{c}_1. \tag{50}
\]
A similar singularity exists in the neighborhood of \(-d\vec{c}_1\)
\[
\frac{2\sigma^2}{d\gamma^2} (x_1 + d) \frac{\partial^2 \varphi_1}{\partial x_2^2} = \frac{2\sigma^2}{d\gamma^2} \tilde{y}_1 \frac{\partial^2 \varphi_1}{\partial \tilde{y}_2^2} = 4\sigma^2 \tilde{y}_1^2 \frac{\partial^2 \varphi_1}{\partial \tilde{y}_2^2} \text{ with } y = x + d\vec{c}_1. \tag{51}
\]
The term \( S_3[\varphi_0] \) obeys the following asymptotic behavior
\[
S_3[\varphi_0] = \frac{x_1 - d}{x_1 \ell_1} \frac{\rho'(\ell_1)}{\rho(\ell_1)} v_0 + \frac{x_1 + d}{x_1 \ell_2} \frac{\rho'(\ell_2)}{\rho(\ell_2)} v_0 + i v_0 \frac{1}{x_1} \frac{\partial \varphi_0}{\partial x_1} \equiv \Lambda_{12} e^{i\varphi_0} + i v_0 S_3[\varphi_0],
\]
where
\[
\Lambda_{12} = \frac{x_1 - d}{x_1 \ell_1} \frac{\rho'(\ell_1)}{\rho(\ell_1)} + \frac{x_1 + d}{x_1 \ell_2} \frac{\rho'(\ell_2)}{\rho(\ell_2)}. \tag{52}
\]
By the computation
\[
S_3[\varphi_0] = \frac{1}{x_1} \frac{\partial \varphi_0}{\partial x_1} = \frac{1}{x_1} \frac{\partial^2 \varphi_1}{\partial x_2^2} - \frac{1}{x_1} \frac{\partial \varphi_1}{\partial x_1} = \frac{1}{x_1} \left( \frac{-x_2}{\ell_1^2} + \frac{x_2}{\ell_2^2} \right), \tag{53}
\]
we find that it is a singular term. More precisely, in the neighborhood of \( d\vec{c}_1 \) with variable \( y = x - d\vec{c}_1 \), there is a singularity in the form
\[
\frac{1}{x_1} \frac{\partial \varphi_1}{\partial x_1} = \left[ \frac{1}{d} - \frac{y_1}{d(d + y_1)} \right] \frac{\partial \varphi_1}{\partial y_1} \text{ with } \frac{1}{x_1} \frac{\partial \varphi_1}{\partial x_1} = -\frac{1}{d} \frac{y_1}{|y|^2}. \tag{54}
\]
A similar singularity exists in the neighborhood of \(-d\vec{c}_1\) with the variable \( \tilde{y} = x + d\vec{c}_1 \)
\[
-\frac{1}{x_1} \frac{\partial \varphi_1}{\partial x_1} = \left[ \frac{1}{d} - \frac{\tilde{y}_1}{d(\tilde{y}_1 - d)} \right] \frac{\partial \varphi_1}{\partial \tilde{y}_1} \text{ with } \frac{1}{x_1} \frac{\partial \varphi_1}{\partial x_1} = -\frac{1}{d} \frac{\tilde{y}_1}{|\tilde{y}|^2}. \tag{55}
\]
The term \( S_5[\varphi_0] \) can be estimated as follows
\[
S_5[\varphi_0] = i\gamma^{-1} \epsilon \log \frac{1 + |\rho|^2}{1 + \epsilon |\rho|^2} \frac{\partial \rho(\ell_1)}{\partial x_2} e^{i\varphi_0} + O(e^{-d/2 - \ell_1/2}) = \Lambda_{14} e^{i\varphi_0},
\]
where
\[
\Lambda_{14} \equiv i\gamma^{-1} \epsilon \log \frac{1 + |\rho|^2}{1 + \epsilon |\rho|^2} \left[ \frac{\partial \rho(\ell_1)}{\partial x_2} + i \rho(\ell_1) \frac{\partial \varphi_0}{\partial x_2} \right] + O(\epsilon \log \frac{1}{\epsilon} e^{-d/2 - \ell_1/2}). \tag{56}
\]
In summary, following the computations in [16], we obtain for \( \epsilon \in S_+ \)
\[
S[\varphi_0] = S_0[\varphi_0] + S_1[\varphi_0] + S_2[\varphi_0] + S_3[\varphi_0] + S_4[\varphi_0] + S_5[\varphi_0]
\]
\[
= \sum_{i=0}^{4} \Lambda_{1i} e^{i\varphi_0} + i v_0 S_2[\varphi_0] + i v_0 S_3[\varphi_0], \tag{57}
\]
where we have denoted
\[
\Lambda_{10} = - \frac{2\rho(\rho^2 - 1)}{\rho^2 + 1} \nabla \theta_{\vec{d} \vec{e}_1} \cdot \nabla \theta_{-\vec{d} \vec{e}_1} + \frac{\rho(\rho^2 - 1)}{\rho^2 + 1} \nabla \theta_{-\vec{d} \vec{e}_1} \cdot \nabla \theta_{\vec{d} \vec{e}_1}
\]
and
\[
\Lambda_{13} = - i \kappa \gamma^{-1} \log \frac{1}{\varepsilon} \left[ \frac{\partial \phi(\ell_1)}{\partial x_2} + i \rho(\ell_1) \frac{\partial \varphi_0}{\partial x_2} \right] + O\left( \varepsilon \log \frac{1}{\varepsilon} e^{-d/2-\ell_1/2} \right). \tag{59}
\]
A similar (and almost identical) estimate also holds in the region \( \mathcal{S}_- \). For the convenience of later use, we denote that for \( z \in \mathcal{S}_+ \cup \mathcal{S}_- \)
\[
\hat{E} = S_0[v_0] - i \nu_0 S_2[\varphi_0] - i \nu_0 S_3[\varphi_0] = \sum_{i=0}^{4} \Lambda_{1i} e^{i \varphi_0}. \tag{60}
\]
We now compute the error on \( \mathcal{S}_0 \). In this region, we only do the computation on the region \( \{ -\omega/\varepsilon < x_1 < \omega/\varepsilon \} \), where \( \eta = 1 \) and \( \nu_0 = e^{i \varphi_0} \). Note that \( S_0[v_0] = 0 \) and \( S_5[v_0] = 0 \). On the other hand
\[
S_1[v_0] + S_2[v_0] = \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left( - \frac{2\nu_0}{1 + |v_0|^2} \left( \frac{\partial \nu_0}{\partial x_1} \right)^2 + \frac{\partial^2 \nu_0}{\partial x_2^2} \right) = i \nu_0 \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{\partial^2 \varphi_0}{\partial x_2^2} = i \nu_0 S_2[\varphi_0].
\]
More explicit computations are provided
\[
\frac{\partial^2 \varphi_0}{\partial x_2^2} = \frac{-2(x_1 - d)x_2}{\ell_1^4} - \frac{2(x_1 + d)x_2}{\ell_2^4} = \frac{4dx_2}{(d^2 + x_2^2)^2} - \frac{8x_2 x_2^2 d(\ell_1^4 + \ell_2^4)}{\ell_1^4 \ell_2^4} - \frac{2x_2 x_2^3 d}{\ell_1^4 \ell_2^4 (d^2 + x_2^2)^2} \left( \ell_1^4 (d^2 + x_2^2 + x_2^2) + \ell_2^4 (d^2 + x_2^2 + x_2^2) \right) + \frac{16x_2 x_2^4 d^3}{\ell_1^4 \ell_2^4 (d^2 + x_2^2)^2} \left( \ell_1^4 + \ell_2^4 \right) (d^2 + x_2^2) + \ell_2^4 \ell_1^4 .
\]
We now write
\[
i \nu_0 S_2[\varphi_0] \equiv \Lambda_4 + i \nu_0 S_2[-2 \arctan(x_2/d)], \tag{61}
\]
due to the relation
\[
i \nu_0 S_2[-2 \arctan(x_2/d)] = i \nu_0 \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{4dx_2}{(d^2 + x_2^2)^2}. \tag{62}
\]
Note that \( i \nu_0 S_2[-2 \arctan(x_2/d)] \) is a singular term as \( x_1 \) approaches 0. The term \( S_3[v_0] \) on \( \mathcal{S}_0 \) obeys the following asymptotic behavior
\[
S_3[v_0] = \frac{1}{x_1} \frac{\partial \nu_0}{\partial x_1} = i \nu_0 \frac{1}{x_1} \frac{\partial \varphi_0}{\partial x_1} = i \nu_0 S_3[\varphi_0],
\]
with the expression
\[
\frac{1}{x_1} \frac{\partial \varphi_0}{\partial x_1} = \frac{1}{x_1} \frac{\partial \theta_{d \vec{e}_1}}{\partial x_1} - \frac{1}{x_1} \frac{\partial \theta_{-d \vec{e}_1}}{\partial x_1} = \frac{1}{x_1} \left( \frac{\partial}{\partial x_1} \left( \frac{x_2}{\ell_1^4} + \frac{x_2}{\ell_2^4} \right) = \frac{4dx_2}{\ell_1^4 \ell_2^4} . \tag{63}
\]
Hence, due to the fact that \( v_0 \) is an even term on the variable \( x_1 \), one can check that \( S_3[v_0] \) does not contain singular components as \( x_1 \to 0 \).

Adding all terms gives that, for \( z \in \mathcal{S}_0 \)

\[
S[v_0] = i v_0 S_2[\varphi_0] + i v_0 S_3[\varphi_0] + \kappa \gamma^{-1} \varepsilon \log \frac{1}{\varepsilon} v_0 \frac{\partial \varphi_0}{\partial x_2} \]

\[- \gamma^{-1} \varepsilon \log \frac{1 - |v_0|^2}{\varepsilon (1 + |v_0|^2)} v_0 \frac{\partial \varphi_0}{\partial x_2} \]  

(64)

Here we also denote for \( z \in \mathcal{S}_0 \)

\[
\hat{E} \equiv S[v_0] - i v_0 S_2[\varphi_0] - i v_0 S_3[\varphi_0].
\]

(65)

As a conclusion, a correction in the phase term will be introduced to get rid of the mentioned singularities, while a direct application of the above computations yields the decay estimates for the error \( \hat{E} \).

**Lemma 4.1.** It holds that for \( z \in (B_2(d\varepsilon_1) \cup B_2(-d\varepsilon_1))^\varepsilon \cap \mathcal{S}_0 \)

\[
\left| \text{Re} \left( \frac{\hat{E}}{(iv_0)} \right) \right| \leq \frac{C_\varepsilon^{1-e}}{(1 + \ell_1)^3} + \frac{C_\varepsilon^{1-e}}{(1 + \ell_2)^3},
\]

(66)

\[
\left| \text{Im} \left( \frac{\hat{E}}{(iv_0)} \right) \right| \leq \frac{C_\varepsilon^{1-e}}{(1 + \ell_1)^{1+e}} + \frac{C_\varepsilon^{1-e}}{(1 + \ell_2)^{1+e}},
\]

(67)

where \( \varrho \in (0, 1) \) is a constant. 

\[ \square \]

4.2. **Further improvement of the approximation.** Following the method in [16], we now define a new correction \( \varphi_d \) in the phase term. Then we will define an improved approximation and estimate its error by substituting it into (32).

To cancel the singularities in (54) and (50) rewritten in the form

\[
\frac{y^2}{d|y|^2} - \frac{4\sigma^2 y_1^2 y_2}{d\gamma^2 |y|^4}
\]

we want to find a function \( \Phi(y_1, y_2) \) by solving the problem in the translated coordinates \((y_1, y_2)\)

\[
\frac{\partial^2 \Phi}{\partial y_1^2} + \frac{\partial^2 \Phi}{\partial y_2^2} = \frac{y_2}{d|y|^2} - \frac{4\sigma^2 y_1^2 y_2}{d\gamma^2 |y|^4} \quad \text{in } \mathbb{R}^2.
\]

(68)

In fact, we can solve this problem by separation of variables and then obtain

\[
\Phi(y_1, y_2) = \frac{1}{4d\gamma^2} y_2 \log |y|^2 - \frac{\sigma^2 y_3^3}{2d\gamma^2 |y|^2} + \frac{3\sigma^2}{8d\gamma^2} y_2.
\]

(69)

Let \( \chi \) be a smooth cut-off function such that \( \chi(\vartheta) = 1 \) for \( \vartheta < d/10 \) and \( \chi(\vartheta) = 0 \) for \( \vartheta > d/5 \). We formally choose the correction term in the form

\[
\varphi_d(z) = \varphi_s(z) + \varphi_1(z) + \varphi_2(z).
\]

More precisely, by recalling the cut-off function \( \eta_r(x_1) \) in (47) we here add the last term

\[
\varphi_2 = 2 \eta_r(x_1) \arctan(x_2/d)
\]

(70)

in the above formula to cancel the singular term in (62) due to the fact

\[
\Delta \varphi_2 = 2 \eta_r(x_1) \frac{\partial^2}{\partial x_2^2} \arctan(x_2/d) + O(\varepsilon^2)
\]

\[
= \eta_r(x_1) - \frac{4}{d} \frac{d x_2}{(d^2 + x_2^2)^2} + O(\varepsilon^2).
\]

(71)
According to (69), the singular part is defined by
\[
\varphi_s(z) = \chi(\varepsilon_1) \left( \frac{x_2}{4d\gamma^2} \log \frac{\ell_1^2}{\ell_2^2} - \frac{\sigma^2}{2d\gamma^2} \frac{x_2^3}{\ell_1^4} + \frac{3\sigma^2}{8d\gamma^2} x_2 \right) + \chi(\varepsilon_2) \left( \frac{x_2}{4d\gamma^2} \log \frac{\ell_2^2}{\ell_1^2} - \frac{\sigma^2}{2d\gamma^2} \frac{x_2^3}{\ell_2^4} + \frac{3\sigma^2}{8d\gamma^2} x_2 \right).
\] (72)

While by recalling the operators \(S_2, S_3\) in (42), the region \(\mathcal{G}\) in (33) and its decompositions in (43), we find the term \(\varphi_1(z)\) by solving the problem
\[
\left[ \triangle + S_2 + S_3 \right] \varphi_1 = -\left[ \triangle + S_2 + S_3 \right] (\varphi_0 + \varphi_s + \varphi_r) \quad \text{in} \quad \mathcal{G},
\]
\[
\varphi_1 = -\varphi_0 - \varphi_s - \varphi_r \quad \text{on} \quad \partial\mathcal{G}.
\] (73) (74)

We can derive the estimate of \(\varphi_1\) by computing the right hand side of (73). In other words, going back to the original variable \((r, \bar{s}_3)\) in (26) and letting
\[\hat{\varphi}(r, \bar{s}_3) = \varphi_1(z)\]
we see that
\[
\left| \Delta_{r, s_3} \hat{\varphi} + S_2[\hat{\varphi}] + S_3[\hat{\varphi}] \right| \leq \frac{C}{\left( \sqrt{1 + r^2 + |s_3|^2} \right)^3}.
\]

Thus we can choose \(\varphi_1\) such that
\[\hat{\varphi} = O\left( \frac{1}{\sqrt{1 + r^2 + |s_3|^2}} \right).\]

The regular term \(\varphi_1\) is \(C^1\) in the original variable \((r, \bar{s}_3)\).

We observe also that by our definition, the function
\[\hat{\varphi} := \varphi_0 + \varphi_d,\] (75)

satisfies
\[\left[ \triangle + S_2 + S_3 \right] \hat{\varphi} = 0 \quad \text{on} \quad \mathcal{G}, \quad \hat{\varphi} = 0 \quad \text{on} \quad \partial\mathcal{G}.\] (76)

From the decomposition of \(\varphi_d\), we see that the singular term contains \(x_2 \log \ell_1\) which becomes dominant when we calculate the speed.

Finally, we define an improved approximate solution
\[V_d(z) := \eta_\varepsilon(x_1) e^{i(\varphi_0 + \varphi_d)} + (1 - \eta_\varepsilon(x_1)) \rho(\ell_1) \rho(\ell_2) e^{i(\varphi_0 + \varphi_d)}.\] (77)

4.3. Error estimates. Recall the notation in (46) and (75). We write \(V_d\) in (77) in the form
\[V_d(z) = \eta_\varepsilon(x_1) e^{i\hat{\varphi}} + (1 - \eta_\varepsilon(x_1)) \hat{\rho} e^{i\hat{\varphi}}, \quad \forall z = x_1 + ix_2 \in \mathcal{G}.
\]

We can follow the method in [16] to check that \(V_d\) is a good approximate solution in the sense that it satisfies the conditions in (36) and has a small error. More precisely, it is easy to show that
\[V_d(z) = \overline{V_d}(\bar{z}), \quad V_d(z) = V_d(-\bar{z}), \quad \partial V_d / \partial x_1 (0, x_2) = 0,
\]
and
\[V_d(x_1, \frac{\lambda \pi}{\gamma}) = V_d(x_1, -\frac{\lambda \pi}{\gamma}), \quad \forall x_1 \in \mathbb{R}.
\]

Recall the boundary condition in (76). It is obvious that
\[\text{Im} \ V_d = \left[ \eta_\varepsilon(x_1) + (1 - \eta_\varepsilon(x_1)) \hat{\rho} \right] \sin \hat{\varphi} = 0 \quad \text{on} \quad \partial\mathcal{G},\] (78)
and on $\partial S$

$$\frac{\partial \text{Re} V_d}{\partial x_2} = \frac{\partial}{\partial x_2} \left[ \eta_\varepsilon(x_1) \cos \hat{\varphi} + (1 - \eta_\varepsilon(x_1)) \hat{\rho} \cos \hat{\varphi} \right] = \frac{\partial \hat{\rho}}{\partial x_2} (1 - \eta_\varepsilon(x_1)). \quad (79)$$

It can be checked that $V_d$ satisfies the conditions in (36) except

$$\frac{\partial V_d}{\partial x_2}(x_1, \frac{\lambda \pi}{\gamma}) \neq \frac{\partial V_d}{\partial x_2}(x_1, -\frac{\lambda \pi}{\gamma}), \quad \forall x_1 \in \mathbb{R}.$$ 

Recall all the components of $S[v_0]$ in (57) and $\hat{E}$ in (60). Note that

$$\triangle \varphi_d + S_2[\varphi_d] + S_3[\varphi_d] = -S_2[\varphi_0] - S_3[\varphi_0]. \quad (80)$$

Here we have used the relation $\triangle \varphi = 0$ and the equation in (76). The error on $S_+\varepsilon$ is

$$S[V_d] = S[v_0] + i v_0 (\triangle \varphi_d + S_2[\varphi_d] + S_3[\varphi_d]) e^{i \hat{\varphi}} + \sum_{i=0}^{4} \Lambda_{2i} e^{i \hat{\varphi}}.$$ 

Note that $\Lambda_{ij}$’s are defined in (58), (48), (52), (56), and other terms are given by

$$\Lambda_{20} = \frac{\hat{\rho} - 1}{\hat{\rho}^2 + 1} |\nabla \varphi_d|^2 - 2 \left( 1 - \hat{\rho}^2 \right) \hat{\rho} \nabla \varphi_0 \cdot \nabla \varphi_d + 2i \left( 1 - \hat{\rho}^2 \right) \hat{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial \varphi_d}{\partial x_2},$$

$$\Lambda_{21} = \gamma^{-2} \left( \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right) \left[ \left( 1 - \hat{\rho}^2 \right) \hat{\rho} \frac{\partial \varphi_d}{\partial x_2} \right]^2 - 2 \left( 1 - \hat{\rho}^2 \right) \hat{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial \varphi_d}{\partial x_2} + 2i \left( 1 - \hat{\rho}^2 \right) \hat{\rho} \frac{\partial \varphi_0}{\partial x_2} \frac{\partial \varphi_d}{\partial x_2},$$

$$\Lambda_{22} = \kappa \gamma^{-1} \varepsilon \log \frac{1}{\varepsilon} \frac{\partial \varphi_d}{\partial x_2}, \quad \Lambda_{23} = -\gamma^{-1} \varepsilon \log \frac{1}{\varepsilon} \frac{1 - |\hat{\rho}|^2}{1 + |\hat{\rho}|^2} \frac{\partial \varphi_d}{\partial x_2}.$$ 

For the convenience of notation, we will set $\Lambda_{22} = 0$ in the sequel. A similar formula also holds on $S_-\varepsilon$. Setting $z = d\bar{e}_1 + y$ in $S_+\varepsilon$, we then have

$$\nabla \varphi_d = \nabla \varphi_\varepsilon + O(\varepsilon) = -\frac{1}{2d \gamma^2} \log d \nabla y_2 + O(\varepsilon \log |y|).$$

Thus

$$\nabla \hat{\rho} \cdot \nabla \hat{\varphi} = O(\varepsilon \log \varepsilon |\rho'| + O(\varepsilon |\rho| \log |y|).$$

These asymptotic expressions will play an important role in the reduction part. The error $S[V_d]$ on $S_0\varepsilon$ has the form

$$S[V_d] = i V_d (\triangle \hat{\varphi} + S_2[\hat{\varphi}] + S_3[\hat{\varphi}]) + V_d O(\varepsilon^2 \log \varepsilon) = V_d O(\varepsilon^2 \log |\varepsilon|).$$
4.4. Setting up of the problem. Following the method in [16], we introduce the set-up of the reduction procedure. We look for solutions to (32) with boundary conditions in (34) in the form

\[ u(z) = \eta(V_d + iV_d\psi) + (1 - \eta)V_d e^{i\psi}. \]  

(81)

Here \( \eta \) is a function such that

\[ \eta = \hat{\eta}(|z - d\epsilon|) + \hat{\eta}(|z + d\epsilon|), \]  

(82)

where \( \hat{\eta} \) is defined in (45).

The conditions imposed on \( u \) in (34) and (35) can be transmitted to \( \psi \)

\[ \psi(z) = \psi(-z), \quad \psi(z) = -\overline{\psi(z)}, \]

\[ \frac{\partial \psi}{\partial x_1}(0, x_2) = 0, \quad \psi(x_1, -\lambda\pi/\gamma) = \psi(x_1, \lambda\pi/\gamma), \]

(83)

\[ \left[ \frac{\partial V_d}{\partial x_2} + iV_d \psi_{x_2} \right]_{(x_1, -\lambda\pi/\gamma)} = \left[ \frac{\partial V_d}{\partial x_2} + iV_d \psi_{x_2} \right]_{(x_1, \lambda\pi/\gamma)}. \]

In fact, the property \( \psi(z) = \psi(-z) \) implies that

\[ \frac{\partial \psi}{\partial x_1}(0, x_2) = 0. \]

More precisely, by direct computation, for \( \psi = \psi_1 + i\psi_2 \) with \( \psi_1 \) and \( \psi_2 \) real-valued, there hold the conditions

\[ \psi_1(x_1, x_2) = \psi_1(-x_1, x_2), \quad \psi_1(x_1, x_2) = -\psi_1(x_1, -x_2), \]

\[ \psi_2(x_1, x_2) = \psi_2(-x_1, x_2), \quad \psi_2(x_1, x_2) = \psi_2(x_1, -x_2), \]

\[ \frac{\partial \psi_1}{\partial x_1}(0, x_2) = 0, \quad \frac{\partial \psi_2}{\partial x_1}(0, x_2) = 0, \]

\[ \psi_1(x_1, -\lambda\pi/\gamma) = \psi_1(x_1, \lambda\pi/\gamma) = 0, \quad \psi_2(x_1, -\lambda\pi/\gamma) = \psi_2(x_1, \lambda\pi/\gamma), \]  

(84)

\[ \frac{\partial \psi_1}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial \psi_1}{\partial x_2}(x_1, \lambda\pi/\gamma), \]

\[ \frac{\partial \psi_2}{\partial x_2}(x_1, \lambda\pi/\gamma) = -\frac{\partial \psi_2}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{1 - \eta_2(x_1)}{\eta_1(x_1) + (1 - \eta_2(x_1))}\frac{\partial \tilde{\rho}}{\partial x_2} |_{(x_1, \lambda\pi/\gamma)}. \]

These conditions will be of importance in solving the linear problems in that it excludes all but one kernel. We may write \( \psi = \psi_1 + i\psi_2 \) with \( \psi_1, \psi_2 \) real-valued and then set

\[ u = V_d + \phi, \quad \phi = \eta_iV_d\psi + (1 - \eta)V_d(e^{i\psi} - 1). \]

In the sequel, we will derive the explicit local forms of the equation for the perturbation term \( \psi \).

Let \( R > 1 \) be a fixed constant. In the inner region

\[ \mathcal{G}_1 = \{ z \in \mathcal{G} | z \in B_{9R}(d\epsilon_1) \cup B_{9R}(-d\epsilon_1) \}, \]

(85)

we have

\[ u = V_d + \phi, \]

and the equation for \( \phi \) becomes

\[ \mathbb{L}_1[\phi] + \mathcal{N}_1[\phi] = -S[V_d]. \]
In the above, we have denoted the linear operator by

\[
\mathbb{L}_1[\phi] = \Delta \phi + \frac{1}{x_1} \frac{\partial \phi}{\partial x_1} - \frac{4V_d}{1 + |V_d|^2} \nabla V_d \cdot \nabla \phi - \frac{2\bar{\psi}}{1 + |V_d|^2} \nabla V_d \cdot \nabla V_d \\
+ \frac{2\bar{V}_d (V_d \phi + \bar{V}_d \phi)}{(1 + |V_d|^2)^2} \nabla V_d \cdot \nabla V_d - \frac{4\bar{V}_d}{1 + |V_d|^2} \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{\partial V_d}{\partial x_2} \cdot \frac{\partial \phi}{\partial x_2} \\
- \frac{2\bar{\phi}}{1 + |V_d|^2} \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left( \frac{\partial V_d}{\partial x_2} \right)^2 \\
+ \frac{2\bar{V}_d (V_d \phi + \bar{V}_d \phi)}{(1 + |V_d|^2)^2} \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left( \frac{\partial V_d}{\partial x_2} \right)^2 \\
+ \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \frac{\partial^2 \phi}{\partial x_2^2} + F'(V_d) \phi.
\]

The nonlinear operator is

\[
\mathbb{N}_1[\phi] = F(V_d + \phi) - F(V_d) - F'(V_d) \phi + O\left(1 + |\phi||\nabla \phi|^2\right) \\
+ \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] O\left(1 + |\phi||\nabla \phi|^2\right) - i\kappa \gamma^{-1} \varepsilon \log \varepsilon \left| \frac{\partial \phi}{\partial x_2} \right|
\]

\[
+ i \gamma^{-1} \varepsilon \log \varepsilon \frac{1 - |V_d + \phi|^2}{1 + |V_d + \phi|^2} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial V_d}{\partial x_2} \right)
\]

\[
- i \gamma^{-1} \varepsilon \log \varepsilon \frac{1 - |V_d|^2}{1 + |V_d|^2} \frac{\partial V_d}{\partial x_2}
\]

(86)

where the definition of \( F \) is given in (42).

In the outer region

\[
\mathcal{S}_2 = \left\{ z \in \mathcal{S} \mid z \in (B_{4R}(d\bar{e}_1) \cup B_{4R}(-d\bar{e}_1))^c \right\}
\]

(87)

we have \( u = V_d e^{i\psi} \). By simple computations we obtain

\[
\frac{\mathcal{S}[V_d e^{i\psi}]}{iV_d e^{i\psi}}
\]

\[
= \Delta \psi + \frac{1}{x_1} \frac{\partial \psi}{\partial x_1} + 2 \nabla \frac{V_d}{V_d} \cdot \nabla \psi \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} + i \left( \nabla \psi \right)^2 \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} \\
+ \gamma^{-2} \left[ \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{d^2} \right] \left\{ \frac{\partial^2 \psi}{\partial x_2^2} + 2 \frac{\partial V_d}{V_d} \frac{\partial \psi}{\partial x_2} \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} \right. \\
\left. + \frac{i (\partial V_d)}{\partial x_2} \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} - i \frac{2|V_d|^2 (1 - e^{-2\psi})}{(1 + |V_d|^2)(1 + |V_d|^2 e^{-2\psi})} \left( \frac{1}{V_d} \frac{\partial V_d}{\partial x_2} \right)^2 \right\} \\
- i \gamma^{-1} \kappa \varepsilon \log \varepsilon \frac{1 - |\partial \psi|}{\varepsilon} \frac{\partial \psi}{\partial x_2} + \gamma^{-1} \varepsilon \log \varepsilon \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} \frac{1}{V_d} \frac{\partial V_d}{\partial x_2} \\
+ \frac{\mathcal{S}[V_d]}{iV_d} + i \gamma^{-1} \varepsilon \log \varepsilon \frac{1 - |V_d|^2 e^{-2\psi}}{1 + |V_d|^2 e^{-2\psi}} \frac{\partial \psi}{\partial x_2}
\]
We can also write the problem as an equation of $\psi = \psi_1 + i\psi_2$

$L_2[\psi] + N_2[\psi] = -S[V_d]/iV_d,$

with conditions in (84). In the above, we have denoted

\[
L_2[\psi] = \Delta \psi + 2 \frac{1}{x_1 \partial x_1} + 2 \frac{1 - |V_d|^2}{1 + |V_d|^2} \nabla V_d \cdot \nabla \psi - i \frac{4|V_d|^2 \psi_2}{(1 + |V_d|^2)^2} \left( \left( \frac{\nabla V_d}{V_d} \right)^2 + 1 \right).
\]

$$
N_2[\psi] = \frac{1}{V_d} \nabla \psi \cdot \nabla V_d O(\psi) + O(|V_d|^2 - 1) + |\psi_2|) \nabla \psi \cdot \nabla \psi + O(|e^{-\psi_2} - 1 + \psi_2|)
$$

Recall that $\psi = \psi_1 + i\psi_2$. Then setting $z = d\bar{c}_1 + y$, we have for $z \in \mathbb{R}^2_+$

$$
L_2[\psi] = \begin{bmatrix}
\Delta \psi_1 + \frac{1}{x_1} \frac{\partial \psi_1}{\partial x_1} + \gamma^{-2} \left( \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{x_2^2} \right) \frac{\partial^2 \psi_1}{\partial x_1^2} + O(e^{-|\psi_1|})|\nabla \psi_1| + |\psi_2|

\Delta \psi_2 + \frac{1}{x_1} \frac{\partial \psi_2}{\partial x_1} + \gamma^{-2} \left( \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{x_2^2} \right) \frac{\partial^2 \psi_2}{\partial x_1^2} - \frac{4|V_d|^2 \epsilon_2}{1 + |V_d|^2} \frac{\partial^2 \psi_2}{\partial x_2^2} + O(e^{-\frac{1}{2}|\psi_1|})|\nabla \psi_2| + O(\frac{1}{1 + |\psi_1|})|\psi_2|
\end{bmatrix},
$$

$$
N_2[\psi] = \begin{bmatrix}
O(e^{-|\psi_1|}|\nabla \psi_1|^2 + |\psi_2| - \frac{1}{1 + |\psi_1|})|\nabla \psi_1| + |\psi_2|, e^{-|\psi_1|}) \nabla \psi_1| + |\psi_2|)

O(e^{-|\psi_1|}|\nabla \psi_2|^2 + |\psi_2| - \frac{1}{1 + |\psi_1|})|\nabla \psi_2| + |\psi_2|)

O(e^{-|\psi_1|}|\nabla \psi_1|^2 + |\psi_2| - \frac{1}{1 + |\psi_1|})|\nabla \psi_1| + |\psi_2|)

O(e^{-|\psi_1|}|\nabla \psi_2|^2 + |\psi_2| - \frac{1}{1 + |\psi_1|})|\nabla \psi_2| + |\psi_2|)
\end{bmatrix}.
$$

Note that

$$
\Delta + \frac{1}{x_1} \frac{\partial \psi}{\partial x_1} + \gamma^{-2} \left( \frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{x_2^2} \right) \frac{\partial^2 \psi}{\partial x_1^2} = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{x_1} \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{x_1} \frac{\partial \psi}{\partial x_1} + \frac{\lambda^2}{\gamma^2} \frac{1}{x_1^2} \frac{\partial^2 \psi}{\partial x_2^2}.
$$

Let us remark that the explicit forms of all the linear and nonlinear terms will be very useful for later analysis in resolution theory.

Let us fix

$$
E = -S[V_d], \quad \tilde{E} = \frac{E}{iV_d}, \quad p > 13, \quad 0 < q < 1.
$$

Recall that $\phi = iV_d \psi$, $\psi = \psi_1 + i\psi_2$. Recall $\ell_1 = |z - d\bar{c}_1|$ and $\ell_2 = |z + d\bar{c}_1|$, and define

$$
||h||_{s+} = ||iV_d h||_{L^p(\mathbb{R}_+)} + \sum_{j=1}^2 \left[ ||\epsilon_j^{l_1+q}h_1||_{L^\infty(\mathbb{R}_+)} + ||\epsilon_j^{l_1+q}h_2||_{L^\infty(\mathbb{R}_+)} \right].
$$
\[ ||\psi||_* = ||\phi||_{W^{2,p}(S_1)} + \sum_{j=1}^2 \left[ ||\ell_j^{1+\varphi_1}\psi_1||_{L^\infty(S_2)} + ||\ell_j^{1+\varphi_2}\nabla\psi_1||_{L^\infty(S_2)} \right] \]
\[ + \sum_{j=1}^2 \left[ ||\ell_j^{1+\varphi_3}\psi_2||_{L^\infty(S_2)} + ||\ell_j^{2+\varphi_3}\nabla\psi_2||_{L^\infty(S_2)} \right]. \]

In the above, \( S_1 \) and \( S_2 \) are given in (85) and (87). We remark that we use the norm \( L^p_{\text{loc}} \) (or \( W^{2,p}_{\text{loc}} \)) in the inner part due to the fact that the error term contains terms like \( \varepsilon \log |z - \bar{d}e_1| \) which is not \( L^\infty \)-bounded. Using the norms defined above, we can have the following error estimates.

**Lemma 4.2.** It holds that for \( z \in S_2 \)
\[ ||\text{Re}(\tilde{E})|| \leq C\varepsilon^{1-\varrho} + C\varepsilon^{1-\varrho}, \]
\[ ||\text{Im}(\tilde{E})|| \leq C\varepsilon^{1-\varrho} + C\varepsilon^{1-\varrho}, \]
and also
\[ ||\tilde{E}||_{L^p(S_1)} \leq C\varepsilon|\log \varepsilon|, \]
where \( \varrho \in (0,1) \) is a constant. As a consequence, there holds
\[ ||\tilde{E}||_{**} \leq C\varepsilon^{1-\varrho}. \]

### 4.5. Projected nonlinear problems and reduced equations.

Recall \( S_1 \) and \( S_2 \) given in (85) and (87) and other notation in Section 4.4. Let the co-kernel be
\[ Z_d := \frac{\partial V_d}{\partial d} \left[ \tilde{\eta}\left(\frac{|z-d\bar{e}_1|}{R}\right) + \tilde{\eta}\left(\frac{|z+d\bar{e}_1|}{R}\right) \right], \]
(88)
where \( \tilde{\eta} \) is defined at (45) as before. Then \( Z_d \) satisfies the requirements in (35). We consider now the full nonlinear projected problem
\[ L[\psi] + N[\psi] = E + C Z_d, \]
(89)
where we have denoted that
\[ L = L_1, \quad N = N_1, \quad E = E, \quad \text{in} \ S_1; \]
\[ L = L_2, \quad N = N_2, \quad E = \hat{E}, \quad \text{in} \ S_2. \]

Note that in the above we have used the relation \( \phi = iV_d\psi \) in \( S_1 \). By contraction mapping theorem, we make a conclusion by the following the resolution theory

**Proposition 1.** There exists a constant \( C \), depending on \( p, \varrho \) only such that for all \( \varepsilon \) sufficiently small, \( d \) large, the following holds: there exists a unique solution \( \psi_{\varepsilon,d} \) to (89) and \( \psi_{\varepsilon,d} \) satisfies
\[ ||\psi_{\varepsilon,d}||_* \leq C\varepsilon^{1-\varrho}. \]
Furthermore, \( \psi_{\varepsilon,d} \) is continuous in \( d \).

Here, we do not give the proof to the last proposition. The reader can refer to the arguments in [11] for similar details.

From Proposition 1, we deduce the existence of a solution \( (\psi, C) \) to (89). To find a solution to (32), we shall choose suitable \( d \) such that \( C \) is zero. This can be realized by the standard reduction procedure in the sequel.
Multiplying (89) by \( \frac{1}{1 + |V_d|^2} Z_d \) and integrating, we obtain

\[
C \Re \left( \int_{\mathbb{R}^2} \frac{1}{(1 + |V_d|^2)^2} Z_d Z_d^* \right) = - \Re \left( \int_{\mathbb{R}^2} \frac{1}{(1 + |V_d|^2)^2} Z_d \mathcal{E} \right) + \Re \left[ \int_{\mathbb{R}^2} \frac{1}{(1 + |V_d|^2)^2} Z_d \left( \mathcal{L}[\psi] + \mathcal{N}[\psi] \right) \right]
\]

\[
= - \Re \left( \int_{\mathbb{R}^2} \frac{1}{(1 + |V_d|^2)^2} Z_d \mathcal{E} \right) + \Re \left[ \int_{\mathbb{R}^2} \frac{1}{(1 + |V_d|^2)^2} Z_d \left( \mathcal{L}_1[\phi] + \mathcal{N}_1[\phi] \right) \right].
\]

Following similar estimates in [16], we get

\[
C(d) = c_0 \pi \gamma \left[ - \frac{1}{4d\gamma} \log d + \frac{1 - 2\kappa}{4} \varepsilon \log \varepsilon \right] + O(\varepsilon),
\]

where \( c_0 \neq 0 \). Therefore, by taking \( 1 - 2\kappa > 0 \), we obtain a solution to \( C(d) = 0 \) with the following asymptotic behavior:

\[
\frac{1}{d} \log d \sim (1 - 2\kappa) \varepsilon \log \varepsilon.
\]

Hence, we have solved (32) and found solutions with elliptic vortex helices to (13) and (14). This, together with the following remark, will complete the proof of the elliptic vortex helix case in Theorem 1.2.

**Remark 5.** Recall the parameters \( \kappa \) and \( \mu = \varepsilon \) or \( \mu = \varepsilon |\log \varepsilon| \) are given in Table 1, while \( \gamma \) in (30). We now can determine the relations between the traveling velocities of the vortex structures and their geometric parameters due to (91). More precisely,

- For the existence of solutions of the Type C form in (19) with elliptic vortex helix structures to (14) (i.e., (2)) in Theorem 1.2, we shall consider (32) with constants,

\[
\kappa = \frac{c_3}{2c_2 \omega_2}, \quad \mu = \varepsilon |\log \varepsilon| = \frac{2c_2 \omega_4}{\sqrt{(1 - |c_2|^2)(1 + |\omega_4|^2)}}.
\]

This implies that the vortex helix travels along \( s_3 \) axis when its speed is sufficiently small with relation between its geometric parameters and the traveling velocity

\[
\frac{1}{\gamma} \frac{1}{d} \log d = \frac{1}{\sqrt{d^2 + \lambda^2}} \log d \sim (1 - 2\kappa) \mu = \frac{2(c_2 \omega_4 - c_3)}{\sqrt{(1 - |c_2|^2)(1 + |\omega_4|^2)}}.
\]

- Other detailed relations in Theorems 1.2 can be derived similarly.

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**Appendix A. The transform procedure of the equations.** In this section we will show how to convert equations (11)-(12), (13) to (20), (21) by choosing special forms of solutions respectively.
A.1. Generalized wave map equation: Equation (11). We concern the special solution to (11) in the form (18).

Case 1. Taking the special solution to (11) in the form

\[ W(\tau, s) = U(\tau', s', s_N - c_1 \tau_K) e^{i \omega_1 \tau_K}, \] with \( 1 - |c_1|^2 > 0, \)

this leads to the problem, on \( \mathbb{R}^{K-1,N}, \)

\[
\begin{align*}
& i \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} - |c_1|^2 \frac{\partial^2 U}{\partial s_N^2} + \frac{2U}{1 + |U|^2} |c_1|^2 \frac{\partial U}{\partial s_N} \frac{\partial U}{\partial s_N} \\
& + \Delta_{K-1,N} U - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U + |\omega_1|^2 \frac{1 - |U|^2}{1 + |U|^2} U = 0.
\end{align*}
\]

Rescaling variables with \( \omega_1 \neq 0 \) as follows, for \( j = 1, \cdots, K - 1 \) and \( l = 1, \cdots, N - 1, \)

\[ \tau_j \to \Delta \tau_j, \quad s_l \to \omega_1 s_l, \quad s_N \to \frac{\omega_1}{\sqrt{1 - |c_1|^2}} s_N, \]

it becomes that

\[
\begin{align*}
i \mu \frac{1 - |U|^2}{1 + |U|^2} & \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U + \Delta_{K-1,N} U \\
& - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0
\end{align*}
\]

where

\[ \mu = \frac{2c_1}{\sqrt{1 - |c_1|^2}}. \] (92)

That’s to say, if we take a solution of the following form

\[ W(\tau, s) = U \left( \omega_1 \tau', \omega_1 s', \frac{\omega_1}{\sqrt{1 - |c_1|^2}} (s_N - c_1 \tau_K) \right) e^{i \omega_1 \tau_K} \]

with \( 1 - |c_1|^2 > 0 \) and \( \omega_1 \neq 0, \) (11) becomes (20) with parameters \( \kappa \) and \( \mu \) in (92).

Case 2. We also look for a solution to (11) in the form

\[ W(\tau, s) = U(\tau', s) e^{i \omega_2 \tau_K + i \omega_3 s_N} \] with \( |\omega_2|^2 - |\omega_3|^2 > 0. \)

This gives that \( U \) will satisfy

\[
\begin{align*}
i \frac{1 - |U|^2}{1 + |U|^2} & \frac{\partial U}{\partial s_N} + (|\omega_2|^2 - |\omega_3|^2) \frac{1 - |U|^2}{1 + |U|^2} U \\
& + \Delta_{K-1,N} U - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \quad \text{on} \quad \mathbb{R}^{K-1,N}.
\end{align*}
\]

Rescaling variables as follows, for all \( j = 1, \cdots, K - 1 \) and \( l = 1, \cdots, N, \)

\[ \tau_j \to \sqrt{|\omega_2|^2 - |\omega_3|^2} \tau_j, \quad s_l \to \sqrt{|\omega_2|^2 - |\omega_3|^2} s_l, \]

we derive that

\[
\begin{align*}
i \mu & \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U + \Delta_{K-1,N} U \\
& - \frac{2U}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0
\end{align*}
\]

where

\[ \mu = \frac{2\omega_3}{\sqrt{|\omega_2|^2 - |\omega_3|^2}}. \] (93)
In other words, if we take a solution of the following form
\[ W(t, \tau, s) = U \left( \sqrt{|\omega_2|^2 - |\omega_3|^2} \tau', \sqrt{|\omega_2|^2 - |\omega_3|^2} s \right) e^{i \omega_2 \tau_K + i \omega_3 s N} \]
with \(|\omega_2|^2 - |\omega_3|^2 > 0\), (11) becomes (20) with parameters \(\kappa\) and \(\mu\) in (93).

A.2. Generalized wave map equation with anisotropic term: Equation (12). This section is devoted to the special solution to (12) in the form (18).

Case 1. Taking the special solution to (12) in the form
\[ W(\tau, s) = U \left( \tau', s', s_N - c_1 \tau_K \right) e^{i \omega_1 \tau_K} \quad \text{with} \quad 1 - |c_1|^2 > 0, \]
this leads to the problem, on \(\mathbb{R}^{K-1,N}\),
\[
i \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} - |c_1|^2 \frac{\partial^2 U}{\partial s_N^2} + \frac{2 \dot{U}}{1 + |U|^2} |c_1|^2 \frac{\partial U}{\partial s_N} \frac{\partial U}{\partial s_N} \\
+ \triangle_{K-1,N} U - \frac{2 \dot{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U + (1 + |\omega_1|^2) \frac{1 - |U|^2}{1 + |U|^2} U = 0.
\]
Rescaling variables as follows, for \(j = 1, \cdots, K - 1\) and \(l = 1, \cdots, N - 1\),
\[
\tau_j \to \sqrt{1 + |\omega_1|^2} \tau_j, \quad s_l \to \sqrt{1 + |\omega_1|^2} s_l, \quad s_N \to \sqrt{1 + |\omega_1|^2} \sqrt{1 - |c_1|^2} s_N,
\]
it becomes that
\[
i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U + \triangle_{K-1,N} U \\
- \frac{2 \dot{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0,
\]
where
\[
\mu = \frac{2 c_1 \omega_1}{\sqrt{(1 - |c_1|^2)(1 + |\omega_1|^2)}}.
\]
Hence, if we take a solution of the following form
\[ W(\tau, s) = U \left( \sqrt{1 + |\omega_1|^2} \tau', \sqrt{1 + |\omega_1|^2} s', \sqrt{1 + |\omega_1|^2} \frac{s_N - c_1 \tau_K}{\sqrt{1 - |c_1|^2}} \right) e^{i \omega_1 \tau_K} \]
with \(1 - |c_1|^2 > 0\), (12) becomes (20) with delicate expressions (94) of parameters \(\kappa\) and \(\mu\).

Case 2. We also look for a solution of (12) in the form
\[ W(\tau, s) = U \left( \tau', s \right) e^{i \omega_2 \tau_K + i \omega_3 s N} \quad \text{with} \quad 1 + |\omega_2|^2 - |\omega_3|^2 > 0. \]
This gives that \(U\) will satisfy
\[
i \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + (1 + |\omega_2|^2 - |\omega_3|^2) \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} \\
+ \triangle_{K-1,N} U - \frac{2 \dot{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \quad \text{on} \quad \mathbb{R}^{K-1,N}.
\]
Rescaling variables as follows, for all \(j = 1, \cdots, K - 1\) and \(l = 1, \cdots, N\),
\[
\tau_j \to \sqrt{1 + |\omega_2|^2 - |\omega_3|^2} \tau_j, \quad s_l \to \sqrt{1 + |\omega_2|^2 - |\omega_3|^2} s_l,
\]
we derive that
\[ i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U + \Delta_{K-1,N} U \]
\[ - \frac{2\hat{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0, \]
where
\[ \mu = \frac{2\omega_3}{\sqrt{1 + |\omega_2|^2 - |\omega_3|^2}}. \] (95)

If we take a solution of the following form
\[ W(t, \tau, s) = U \left( \sqrt{1 + |\omega_2|^2 - |\omega_3|^2} \tau', \sqrt{1 + |\omega_2|^2 - |\omega_3|^2} s \right) e^{i \omega_2 \tau + i \omega_3 s} \]
with \( 1 + |\omega_2|^2 - |\omega_3|^2 > 0 \), (12) becomes (20) with parameters \( \kappa \) and \( \mu \) in (95).

A.3. **Generalized Schrödinger map equation:** **Equation (13).** We consider the special solution to (13) in the form (19).

**Case 1.** Taking the special solution to (13) in the form
\[ W(t, \tau, s) = U \left( \tau', s_N - c_2 \tau_K - c_3 t \right) e^{i \omega_4 \tau_K} \quad \text{with} \quad 1 - |c_2|^2 > 0, \]
this leads to the problem, on \( \mathbb{R}^{K-1,N} \),
\[ - i c_3 \frac{\partial U}{\partial s_N} + i \frac{1 - |U|^2}{1 + |U|^2} 2c_2 \omega_4 \frac{\partial U}{\partial s_N} - |c_2|^2 \frac{\partial^2 U}{\partial s_N^2} + \frac{2\hat{U}}{1 + |U|^2} |c_2|^2 \frac{\partial U}{\partial s_N} \frac{\partial U}{\partial s_N} \]
\[ + \Delta_{K-1,N} U - \frac{2\hat{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U + |\omega_4|^2 \frac{1 - |U|^2}{1 + |U|^2} U = 0. \]
Rescaling variables with \( \omega_4 \neq 0 \) as follows, for \( j = 1, \cdots, K-1 \) and \( l = 1, \cdots, N-1 \)
\( \tau_j \rightarrow \omega_4 \tau_j, \quad s_l \rightarrow \omega_4 s_l, \quad s_N \rightarrow \frac{\omega_4}{\sqrt{1 - |c_2|^2}} s_N, \)
it becomes that
\[ - i \kappa \mu \frac{\partial U}{\partial s_N} + i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \frac{1 - |U|^2}{1 + |U|^2} U \]
\[ + \Delta_{K-1,N} U - \frac{2\hat{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0, \]
where
\[ \kappa = \frac{c_3}{2c_2 \omega_4} \quad \text{and} \quad \mu = \frac{2c_2}{\sqrt{1 - |c_2|^2}}. \] (96)
So, if we take a solutions of the following form
\[ W(t, \tau, s) = U \left( \omega_4 \tau', \omega_4 s', \frac{\omega_4}{\sqrt{1 - |c_2|^2}} (s_N - c_2 \tau_K - c_3 t) \right) e^{i \omega_4 \tau_K} \]
with \( 1 - |c_2|^2 > 0 \) and \( \omega_4 \neq 0 \), (13) becomes (21) with parameters \( \kappa \) and \( \mu \) in (96).

**Case 2.** We also look for a solution to (13) in the form
\[ W(t, \tau, s) = U \left( \tau', s_N - c_4 t \right) e^{i \omega_5 \tau_K + i \omega_6 s_N} \quad \text{with} \quad |\omega_5|^2 - |\omega_6|^2 > 0. \]
This gives that $U$ will satisfy

\[-i\kappa \frac{\partial U}{\partial s_N} + i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \left( |\omega_5|^2 - |\omega_6|^2 \right) \frac{1 - |U|^2}{1 + |U|^2} U + \frac{\Delta_{K-1,N} U - 2\bar{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \text{ on } \mathbb{R}^{K-1,N}.
\]

Rescaling variables as follows, for all $j = 1, \cdots K - 1$ and $l = 1, \cdots N$,

\[
\tau_j \to \sqrt{|\omega_5|^2 - |\omega_6|^2} \tau_j, \quad s_l \to \sqrt{|\omega_5|^2 - |\omega_6|^2} s_l,
\]

we derive that

\[-i\kappa \mu \frac{\partial U}{\partial s_N} + i \mu \frac{1 - |U|^2}{1 + |U|^2} \frac{\partial U}{\partial s_N} + \left( |\omega_5|^2 - |\omega_6|^2 \right) \frac{1 - |U|^2}{1 + |U|^2} U + \frac{\Delta_{K-1,N} U - 2\bar{U}}{1 + |U|^2} \nabla_{K-1,N} U \cdot \nabla_{K-1,N} U = 0 \text{ on } \mathbb{R}^{K-1,N},
\]

where

\[
\kappa = \frac{c_4}{2\omega_6} \quad \text{and} \quad \mu = \frac{2\omega_6}{\sqrt{|\omega_5|^2 - |\omega_6|^2}}.
\]

Hence, if we take a solutions of the following form

\[
W(t, \tau, s) = U \left( \sqrt{|\omega_5|^2 - |\omega_6|^2} (\tau', s', (s_N - c_4 t)) \right) e^{i\omega_5 \tau K + i\omega_6 s N}
\]

with $|\omega_5|^2 - |\omega_6|^2 > 0$, (13) becomes (21) with parameters $\kappa$ and $\mu$ in (97).

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