Multiparticle extension of the higher spin algebra

M A Vasiliev

I.E.Tamm Department of Theoretical Physics, Lebedev Physical Institute, Leninsky prospect 53, 119991, Moscow, Russia

E-mail: vasiliev@lpi.ru

Received 2 January 2013, in final form 8 January 2013
Published 2 May 2013
Online at stacks.iop.org/CQG/30/104006

Abstract

Multiparticle extension of a higher spin algebra \( l \) is introduced as the Lie superalgebra associated with the universal enveloping algebra \( U(l) \). While conventional higher spin symmetry does not mix \( n \)-particle states with different \( n \), multiparticle symmetries do so. Quotients of multiparticle algebras are considered, that act on the space of \( n \)-particle states with \( 0 \leq n \leq k \) analogous to the space of first \( k \) Regge trajectories of string theory. Original higher spin algebra is reproduced at \( k = 1 \). Full multiparticle algebras are conjectured to describe vacuum symmetries of string-like extensions of higher spin gauge theories. The relation of the multiparticle algebras with 3D current operator algebras is described. The central charge parameter, to be related to the parameter \( \mathcal{N} \) in AdS/CFT correspondence, enters via the definition of supertrace. Extension to higher \( p \)-brane-like symmetries is introduced inductively.

PACS numbers: 11.25.Hf, 04.60.-m, 11.90.+t

1. Introduction

Higher spin (HS) gauge theories describe interactions of massless fields of all spins. The first example of full nonlinear HS theory was given in the 4D case [1], while its modern formulation was worked out in [2] (see [3] for a review). HS gauge theories involve infinite towers of massless (gauge) fields of HSs. In this respect, HS gauge theory is analogous to string theory, which also describes interactions of excitations of states of all spins. These two classes of theories are however different in several respects.

Known HS gauge theories only involve totally symmetric fields, while string theory contains HS fields of various symmetry types. Field spectra of HS gauge theories are somewhat analogous to the first Regge trajectory of string theory, though describing only massless (gauge) fields of HSs. On the other hand, string theory describes only massive HSs. Another distinction is that HS theories admit consistent interactions only in a curved background which is \((\text{AdS})_d\) in the most symmetric case, while fully consistent formulation of string theory is available in a Ricci flat background.

It was anticipated for a long time that HS theory and string theory should be related and, eventually, string theory should be understood as some HS theory where masses are generated
via spontaneous breakdown of HS symmetries (for example, this was conjectured in [4]). Although this conjecture is supported by the analysis of high-energy limit of string amplitudes [5] and passed some nontrivial checks [6–9], no satisfactory understanding of this relation beyond the free-field sector of the tensionless limit of string theory [10–12] was available. An interesting idea of singleton string whose spectrum is represented by multiple tensor products of singletons was put forward in [13, 14]. Somewhat similarly, it was recently conjectured [15] that string theory should admit an interpretation of a theory of bound states of HS gauge theory. Consideration of this paper agrees with these conjectures, specifying a symmetry underlying a string-like extension of HS gauge theory.

In more detail, it is anticipated that there should exist a string-like HS gauge theory that describes at least as many degrees of freedom as string theory. However, as long as HS symmetries are unbroken, this hypothetical theory will be free of an $\alpha'$-like mass-scale parameter, describing massless fields along with massive fields whose masses are scaled in units of inverse AdS radius. Similar to usual HS gauge theory, if exists, such string-like HS gauge theory could only be formulated in a curved background which is (A)dS in the most symmetric case. Upon HS symmetries are spontaneously broken, the theory should acquire an independent mass-scale parameter giving mass to HS fields, and admit formulation in flat space. Involving infinite tower of massive HS fields, in this case it should reduce to one or another version of string theory. As in usual HS theory, the key step consists of identification of a global symmetry, gauging of which underlies a string-like HS gauge theory.

As discussed below, such a symmetry has to obey a number of nontrivial conditions. The multiparticle algebra analyzed in this paper will be argued to pass these conditions, hence providing a promising candidate for the HS algebra of a string-like HS theory. Being broken in a vacuum associated with usual string theory, multiparticle symmetry can hardly be seen in this framework since most of related transformations will have a Stueckelberg form. In other words, to uncover hidden symmetries of string theory, one has to find a string-like HS gauge model where these symmetries are unbroken (and, hence, related states are massless). The identification of multiparticle algebra as a candidate for such a symmetry is one of the main outputs of this paper.

Naively, the difference between the two classes of theories is minor. HS theories are formulated in terms of fields $B(Y|X)$ that depend on spacetime coordinates $X$ and auxiliary variables $Y^A$. The latter, depending on a model, can be either spinors [16, 2] or a pair of vectors [17]. The variables $Y^A$ are noncommutative, obeying commutation relations

$$[Y^A, Y^B] = 2C^{AB},$$

(1.1)

with some non-degenerate antisymmetric matrix $C^{AB}$ which is either the charge conjugation matrix with spinor indices $A, B$ or has the form $C^{AB} = \epsilon^{\alpha\beta}\eta^{ab}$ with $A = (a, \alpha)$ where $\alpha = 1, 2$, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \neq 0$, $a$ is the vector index of $o(d-1, 2)$ and $\eta^{ab}$ is an $o(d-1, 2)$-invariant metric. These oscillators are analogous to a pair of string oscillators, say $x^1_i$ and $x^{m-1}_i$, that satisfy

$$[x^1_i, x^{m-1}_i] = \eta^{im},$$

(1.2)

where $\eta^{im}$ is Minkowski metric in $d$ dimensions.

It looks like it is enough to let more species of oscillators $Y^A \rightarrow Y^A_i (i = 1, \ldots, r)$ be present to get HS theory closer to string theory picture which should emerge in the $r \rightarrow \infty$ limit. This idea is supported by the analysis of unfolded formulation of free mixed symmetry HS gauge fields which were shown [18–21] to be naturally described in terms of differential forms $\omega(Y_i|x)$ of various degrees, valued in appropriate tensor $o(d-1, 2)$-modules realized by polynomial functions $\omega(Y_i|x)$ of oscillators $Y^A_i$. However, to go beyond the free-field level, it is necessary to find such a non-Abelian algebra $hs$ that fields of the unfolded formulation...
of the theory fit into $hs$-modules. In particular, 1-forms among $\omega(Y|s)$ should be valued in its adjoint representation. A strong criterion, called admissibility condition \cite{22}, requires an HS algebra to admit a unitary module that decomposes into direct sum of unitary modules of the spacetime symmetry algebra $s$, whose pattern matches the list of relativistic fields associated with the list of forms $\omega(Y|s)$. For symmetric HS fields, this is indeed the case \cite{22, 23} due to Flato–Fronsdal theorem \cite{24} and its higher dimensional generalization \cite{25} which relates the tensor product of a pair of scalar and/or spinor unitary modules of the conformal algebra $o(d − 1, 2)$ in $d − 1$ dimensions to the towers of massless fields in $d$ dimensions. In this realization, the AdS$_d$ HS algebra is identified with the algebra of endomorphisms of the space of single-particle states of conformal fields in $d − 1$ dimension. However, no analogue of this construction appropriate for the description of mixed symmetry fields of general type is available. The problem is most obvious for odd $d$ where conformal scalar and spinor fields are the only unitary propagating conformal fields. This indicates that generalization of HS theory to mixed symmetry fields and, eventually, to string theory, may require some deviation from the standard constructions of HS theory.

$1$-forms $\omega(Y|s)$ valued in $s \subset hs$ describe vielbein and connection of the spin-2 gravitational field. In the usual HS theory of symmetric fields, the corresponding gravitational fields were associated with the (sub)algebra of bilinears of the oscillators

$$T^{AB} = \frac{1}{2} \{Y^A, Y^B\}. \quad (1.3)$$

For instance, in the 4D spinor realization where $A, B = 1, \ldots, 4$, $T^{AB}$ are generators of the AdS$_4$ algebra $sp(4) \sim o(3, 2)$. For arbitrary dimension $d$, generators of $o(d − 1, 2)$ were identified \cite{17} with the subalgebra of $sp(2(d + 1))$ spanned by those $T^{AB}$ that are invariant under the $sp(2)$ subalgebra rotating indices $a$ of $Y^a$.

However, the straightforward extension of this construction to any number $r$ of oscillators

$$T^{AB} = \frac{1}{2} \sum_{i=1}^{r} \{Y_i^A, Y_i^B\} \quad (1.4)$$

does not respect the admissibility condition facing the following problem. A natural framework for unitary $hs$-modules is provided by tensor products of Fock modules where the oscillators $Y^a$ act. Let $E$ be the energy operator among $T^{AB}$. If the lowest energies for spin $s$ fields were $E(s)$ in the $r = 1$ $hs$-module (recall that in the absence of a free mass parameter, lowest energies in AdS$_4$ are scaled in units of the inverse AdS radius), in the tensor product of $r$ such modules energies will increase like $rE(s)$. Since the lowest energy determines the mass of a particle, if it described a massless symmetric field in the $r = 1$ case, it will correspond to certain massive (and hence non-gauge) field at higher $r$. In particular, spin-1 and spin-2 fields become massive at $r > 1$, i.e. the resulting theory can neither contain Yang–Mills theory nor gravity.

So far, no non-Abelian HS algebra appropriate for the description of general mixed symmetry fields was available, though some particular mixed symmetry fields result from gauging of the HS algebra associated with the tensor product of fermions in any dimension \cite{25} as well as with 4D conformal HS algebras of \cite{26, 27} and their further 4D \cite{28–30} (for the respective Flato–Fronsdal like theorem see \cite{31}) and higher dimensional \cite{32, 33} extensions.

A well-known feature of the conventional formulation of string theory, which seems to be closely related to the above discussion of HS theory, is that its consistent generalization to the AdS background is far from being trivial. Indeed, as in any relativistic theory, the Lorentz symmetry acts on all spacetime (spinor) tensors both in HS theory and in string theory. Hence, Lorentz generators should have a form (1.4) where summation is over all modes that carry spacetime indices. The fact that the commutator of translations (transvections) in the AdS
algebra gives Lorentz generators requires the AdS translation generators to be built from all modes. However, as in HS theory, this would immediately lead to wrong (infinite) vacuum energy of graviton. Hence, translation generators in string theory are built solely from zero modes, whose construction admits no AdS deformation (see however interesting work [34] where an extended formulation of string theory, that avoids this problem, was proposed). This feature of string theory indicates that the straightforward construction via tensoring of oscillators is too naive in the both cases.

In this paper, we propose a class of algebras that extend usual HS algebras in a string theory fashion, avoiding the most obvious problems mentioned above. The proposed construction was deduced from the analysis of [35] of the current operator algebra of 3D massless free theory, which generates symmetries of the space of multiparticle states of the AdS4 HS theory and its boundary image. Hence, we call them multiparticle algebras. The purely algebraic approach of this paper provides an efficient tool for the description of the current operator algebra of [35], leading to manifest formulæ for the current OPE in section 4. One of the surprising outputs of this construction is that OPEs of currents with a different number $N$ of constituent-free fields are described by different basis choices in the same multiparticle algebra. This may look surprising, since this property has no analogue in two-dimensional conformal field theory where $N$ affects the central charge of the Virasoro algebra. Indeed, that the central extension in the Virasoro algebra is nontrivial literally means that models with different central charges do not correspond to different bases of the Virasoro algebra. As follows from our consideration, the situation in higher dimensional conformal theories is different. In particular, multiparticle algebras admit no nontrivial central extension. Practically, the respective bases are uniquely fixed by the conditions that (i) Wick theorem is respected as the characteristic property of free-field theory and (ii) the central term properly depends on $N$. Since different free theories are described by the same multiparticle algebra, it is appealing to speculate that, using the same multiparticle algebra, it may even be possible to formulate nonlinear conformal systems not respecting Wick theorem (see also section 6).

It should be stressed that our construction applies to a very general class of theories including the 4D $N = 4$ SYM boundary theory closely related to conventional superstring theory. Similarly, the analysis of current operator algebra of [35] goes beyond the 3D case, allowing in particular to evaluate $n$-point functions of 4D conformal currents.

The formal definition is simple. Let an HS algebra $h_u(V)$ be the Lie algebra of maximal symmetries of $V$, i.e. of the free theory of fields $\Phi$ that have $V$ as the space of single-particle states. Multiparticle algebras $m_m(V)$ are appropriate real forms of the complex Lie superalgebras associated with the universal enveloping algebras $U(h_u(V))$. The multiparticle algebra acts on the space of all multiparticle states in a theory where $V$ is the space of single-particle states.

Algebras $m_m^k(V)$ act on the space of $r$-particle states with $r \leq k$, $h_u(V) \subset m_m^k(V)$ for any $k \geq 1$. In particular, $m_m^1(V) = h_u(V) \oplus u(1)$, where $u(1)$ represents the symmetry of the physical vacuum which is the space of 0-particle states. Algebras $m_m^k(V)$ are certain quotients of $m_0(V)$ and should be associated with a theory which, roughly speaking, describes first $k$ Regge trajectories of string theory. $m_m^k(V)$ should be associated with the full-fledged string-like extension of HS theories. We believe that the proposed scheme has a potential to unify string theory and HS theory within a theory which contains both of them as different particular cases and/or limits.

The paper is organized as follows. In section 2, a general structure of known HS algebras is recalled. In section 3, we present the construction of associative multiparticle algebra which is illustrated in section 3.6 by the example of Weyl algebra. It is applied to the description of current operator algebras of [35] in section 4 and to the extension of each HS algebra.
2. Higher spin algebras

From the perspective of bulk HS gauge theories in AdS, HS algebras represent a global symmetry of a maximally symmetric vacuum solution of the nonlinear HS gauge theory in question. They should be distinguished from local HS symmetries of HS gauge theories, resulting from gauging (localization) of global HS symmetries along with their further field-dependent deformation\(^1\). In this paper, we focus on the global multiparticle algebras, which provide the starting point for the search of the full-fledged nonlinear multiparticle gauge theories.

All HS algebras underlying known nonlinear HS gauge theories admit the following realization. Let \( V_{\Phi} \) be the space of single-particle states of a set of free unitary conformal fields \( \Phi \). We will use notation \( H(V_{\Phi}) \) for the complex associative algebra of endomorphisms \( \text{End}(V_{\Phi}; \mathbb{C}) \). As such, \( H(V_{\Phi}) \) is closely related to the algebra of all symmetries of the free-field theory of \( \Phi \) as is most easily seen from the unfolded dynamics approach (see e.g. [41]).

For an algebra \( A \) with the product law \( \star \) (from now on by algebra we mean associative algebra if not specified otherwise), \( l(A) \) denotes the associated Lie (super)algebra with the (graded) commutator

\[
[a, b], = a \star b - b \star a \quad \forall a, b \in A
\]

as a Lie product. Then the HS algebra \( h_{\alpha}(V_{\Phi}) \) is the real form of \( l(H(V_{\Phi})) \) singled out by the conditions

\[
\sigma(a) = a, \quad (2.2)
\]

with such conjugation \( \sigma \) of \( l(H(V_{\Phi})) \) \( \sigma(ia) = -i\sigma(a) \), \( \sigma^2 = \text{Id} \) that the corresponding symmetry transformations of \( V_{\Phi} \) are unitary\(^2\). Given conformal fields \( \Phi \) in \( d \) dimensions, \( h_{\alpha}(V_{\Phi}) \) can be interpreted either as a conformal HS algebra in \( d \) dimensions or as an AdS\(_{d+1}\) HS algebra.

\( h_{\alpha}(V_{\Phi}) \) admits further truncations of the orthogonal and symplectic types induced by an involutive anti-automorphism \( \rho \) of \( H(V_{\Phi}) \):

\[
\rho(ab) = \rho(b)\rho(a), \quad \rho^2 = \text{Id}. \quad (2.3)
\]

For the \( \Phi^i \) carrying color index \( i = 1, 2, \ldots, n \), there are two options for the extension of \( \rho \) that lead to two types of HS algebras. Namely, if \( \rho \) was an anti-automorphism of the model with a single field \( \Phi \), its color extension \( \rho^{\text{col}} \) is

\[
\rho^{\text{col}}(\Phi^i) = \eta_{ij}\rho(\Phi^j), \quad (2.4)
\]

where \( \eta_{ij} \) is some nondegenerate matrix. (Note that \( \rho^{\text{col}} \) maps left and right \( H(V_{\Phi}) \)-modules to each other.) The truncation condition is

\[
\rho^{\text{col}}(a) = -i\rho(a) a, \quad (2.5)
\]

\(^1\) The latter phenomenon is typical for any theory of gravity where diffeomorphisms can be interpreted as a deformation of localized Poincaré or (A)dS transformations by curvature-dependent terms [36–40].

\(^2\) Note that the anti-Hermiticity condition implying unitarity of symmetry transformations requires that the conjugation \( \sigma \) of \( l(H_{\Phi}) \) be generated by an involution \( \dagger \) of \( H_{\Phi} \) via \( \sigma(a) := -a^\dagger \) in the Lie algebra (i.e. bosonic) case. Since involution reverses the order of product factors, \( (ab)^\dagger = b^\dagger a^\dagger \), the anti-Hermiticity condition does not allow us to start with a real algebra \( H_{\Phi} \) (anti-Hermitian operators do not form an associative algebra).
where \( p(a) = 0 \) or 1 is the boson–fermion parity of \( a \). Depending on whether \( \eta_{ij} \) is symmetric or antisymmetric, this gives the algebras \( h_{ij}(V_\Phi) \) or \( h_{\text{asp}}(V_\Phi) \), respectively. In the case of a single field \( \Phi (n = 1) \), \( h_{ij}(V_\Phi) \) is the minimal HS algebra. For HS algebras associated with symmetric fields of integer spins, \( h_{ij}(V_\Phi) \) contains even spins. Note that if \( \eta_{ij} \) has no definite symmetry, then the subalgebra singled out by (2.5) is a direct sum of algebras \( h_{ij}(V_\Phi) \) and \( h_{\text{asp}}(V_\Phi) \) with smaller \( n \). (For more details, we refer the reader to [16, 23] where this construction was originally applied to \( \text{AdS}_4 \) HS algebras.)

HS algebras available in the literature belong to the three classes \( h_{ij}(n_{s1}, n_{s2}, \ldots |d) \), \( h_{ij}(n_{s1}, n_{s2}, \ldots |d) \) and \( h_{\text{asp}}(n_{s1}, n_{s2}, \ldots |d) \) with \( 0 \leq s_1 < s_2 < s_3 \ldots \). Here, \( d \) is the dimension of spacetime where a set of conformal fields \( \Phi' \) contains \( n_{s1} \) fields of spin \( s_1 \), \( n_{s2} \) fields of spin \( s_2 \), etc. \( h_{ij}(n_{s1}, n_{s2}, \ldots |d) \) is the algebra \( h_{ij}(V_\Phi) \) for the corresponding set of fields \( \Phi' \), while \( h_{\text{asp}}(n_{s1}, n_{s2}, \ldots |d) \) are its subalgebras singled out by condition (2.4).

In principle, one can also consider the case where different fields \( \Phi' \) live in spacetimes of different dimensions. Although no algebras of this type were so far considered in the literature, recent results of [42], where it was shown that current interactions of 4D massless fields acquire natural interpretation in terms of a mixed system of 4D and 6D conformal fields, suggest that they may also be of interest. For such algebras, one can use notation \( h \cdot \cdot \cdot (n_{s1}, n_{s2}, \ldots) \).

The list of ghost-free propagating conformal fields in \( d \) dimensions depends on whether \( d \) is even or odd. As shown in [43, 44], apart from the massless scalar and spinor of any dimension, only mixed symmetry fields with field strengths described by rectangular Young diagrams of height \( d/2 \) in even spacetime dimension correspond to unitary theories. Hence, for even \( d \), conformal massless fields are characterized by a single-spin parameter \( s \) associated with a length \( s \) or \( s = \frac{1}{2} \) of the tensor or spinor–tensor Young diagram, respectively. The conformal scalar and spinor correspond to Young diagrams of zero length, hence making sense for odd \( d \) as well.

The presented construction of HS algebras is closely related to Flato–Fronsdal-type theorems on the relation between the tensor product of conformal fields in \( d \) dimensions and massless fields in \( \text{AdS}_{d+1} \). Indeed, HS gauge fields associated with the \( \text{AdS}_{d+1} \) HS algebra are valued in the algebra of operators that act in \( V_\Phi \), which is \( V_\Phi^d \otimes V_\Phi \) as a linear space. An important feature of HS gauge theories is [16, 17] that Weyl 0-forms, which contain all degrees of freedom of the \( \text{AdS}_{d+1} \) system, are valued in the so-called twisted adjoint module which is isomorphic to \( V_\Phi^d \otimes V_\Phi \) as a linear space. This implies that degrees of freedom of the \( \text{AdS}_{d+1} \) HS theory belong to the module equivalent to the tensor square of the conformal module \( V_\Phi \) in \( d \) dimensions (here we do not distinguish between vector spaces \( V_\Phi^d \) and \( V_\Phi \) which are isomorphic in the unitary case). Hence, the construction of HS algebras is such that the spectrum of fields of the bulk HS theory is designed to result from the tensor product of boundary conformal fields. This is in kinematical agreement with the idea of \( \text{AdS}/\text{CFT} \) correspondence because \( V_\Phi \otimes V_\Phi \) is the space of conformal conserved currents of the theory of free fields \( \Phi \) as is most directly seen in the unfolded dynamics approach [45], which in fact is of course not surprising given that \( V_\Phi \otimes V_\Phi \) is the space of conformal HS symmetries. Hence HS algebras \( h \cdot \cdot \cdot (n_{s1}, n_{s2}, \ldots |d) \) are properly designed to support \( \text{AdS}/\text{CFT} \) correspondence between boundary conformal theories and bulk HS theories, the issue which acquired a lot of interest in recent years. (See, e.g. [46–48, 28, 49–78, 15, 79, 80]. For reviews and more references see also [81–83].) Of course, being applicable to free fields, the above consideration has to be reanalyzed at the interaction level. For example, as shown in [74], except for two particular reductions of the HS gauge theory which, in accordance with the theorem of Maldacena and Zhiboedov [72], are dual to the boundary theory of free currents, all other nonlinear \( \text{AdS}_4 \) HS gauge theories turn out to be dual to a 3D conformal HS gauge theory of interacting currents.
Although original AdS$_4$ HS algebras were obtained in [84–86] from different arguments [87] aimed at reformulation of HS theory in terms of differential forms, that eventually led to its unfolded formulation [16, 1, 2], they belong to the class of HS algebras discussed above. Even with no reference to symmetries of 3D unitary conformal fields, original AdS$_4$ HS algebras were interpreted as algebras of 3D conformal HS gauge theory by Fradkin and Linetsky in [91].

Specifically, the original bosonic AdS$_4$ HS algebra found in [84] is $h_0(1_0|3)$. Its extension to $h_0(1_0, 1_{1/2}|3)$ was proposed in [86] and to $h \cdots (n_0, m_{1/2}|3)$ in [23] where they were called $h \cdots (n, m|4)$. Other way around, 4D conformal HS algebras introduced by Fradkin and Linetsky in [27] in the context of 4D conformal HS gauge theory were later interpreted as AdS$_4$ HS algebras in [88, 89, 29, 30]. All these algebras are $h \cdots (n_0, n_{1/2}, n_1|4)$ where $n_0, n_{1/2}$ and $n_1$ are the numbers of fields of respective spins in a supermultiplet of 4D N-extended conformal superalgebra with $N = 1, 2, 4$ including the case of $N = 4$ SYM multiplet $h_0(n_0, n_{1/2}, n_1|4)$ with $n_0 = 6n_1$, $n_{1/2} = 4n_1$. Their realization in terms of 4D boundary fields was considered in [28] including the generalization to $h \cdots (n_j, n_j, \ldots |4)$ with nonzero $n_j$ at $s > 1$. Extension to $h \cdots (n_j|d)$ with any even $d \geq 4$ was given in [33].

Algebra $h_0(1_0|6)$ interpreted as the minimal AdS$_7$ HS algebra was considered in [32]. In [28], algebras $h \cdots (1_0, 1_1, 1_2 \ldots \infty|4)$ and $h \cdots (1_0, 1_{1/2}, 1_1 \ldots \infty|4)$ (the case of $M = 4$ in notations of [28]) and $h \cdots (1_0, 1_1, 1_2 \ldots \infty|6)$ and $h \cdots (1_0, 1_{1/2}, 1_1 \ldots \infty|6)$ (the case of $M = 8$ in notations of [28]) were identified with $I(M)_4$, where the Weyl algebra $A_M$ is the algebra of various polynomials of $M$ pairs of oscillators. $h_0(1_0|d)$ was identified as the algebra of conformal HS symmetries of a massless scalar by Eastwood in [90] and was used for the construction of HS gauge theories in AdS$_{d+1}$ in [17] where it was also extended to $h_0(n_0|d)$. HS superalgebras $h_0(n_0, n_{1/2}|d)$ were introduced in [25].

All HS algebras listed above admit realizations in terms of Weyl algebras, which are particularly useful for the formulation of nonlinear HS gauge theories of [2, 17]. There are two types of constructions mentioned in the introduction. The one with elementary spinor oscillators was used in [85, 86, 91, 26, 23, 88, 89, 28–30]. Another one with elementary oscillators carrying vector indices used in [17, 25, 33] applies to HS models in any dimension. Being closely related to twistor theory, the spinor realization is likely to be both simpler and deeper.

For example, HS algebras $h_0(n_0, m_{1/2}|3)$ were shown in [23] (where they were denoted $hu(n, m|4)$) to be realized by matrices

$$
P_{ij}^k(Y) = \begin{pmatrix} P_{ij}^k(Y) \\ P_{ij}^m(Y) \end{pmatrix}$$

\begin{pmatrix} n \\ m \end{pmatrix}

(2.6)

where matrix-valued polynomials $P^E(Y)$ and $P^O(Y)$ are respectively the even and odd functions of the oscillators $Y_{ij} (A = 1, 2, 3, 4$ is the 4D Majorana spinor index) that obey the star-product commutation relations (1.1) where $C_{AB}$ is the 4D charge conjugation matrix.

The space $V_0$ of single-particle states of $n$ massless scalars and $m$ massless spinors in three dimensions is realized as the direct sum of $n$ even subspaces $F_0$ and $m$ odd subspaces $F_1$ of the Fock module $F$:

$$
F : f(Y_a)|0\rangle = Y_a|0\rangle = 0,
$$

(2.7)

where $Y_a$ is a pair of mutually conjugated canonical oscillators in the set $Y^A$. $F_0$ and $F_1$ are spanned, respectively, by even and odd functions $f(Y_a)$.
The fact that HS algebras are naturally realized in terms of the Weyl algebra is not accidental. As an algebra of endomorphisms of a single-particle space \( V_{\Phi} \) of conformal fields, the HS algebra can be represented by differential operators of various degrees acting on \( V_{\Phi} \). The latter belong to the Weyl algebra which is the algebra of various differential operators with polynomial coefficients. Then \( V_{\Phi} \) is represented as a Fock module of the star-product algebra or some of its quotient.

The list of full nonlinear HS gauge theories with propagating HS gauge fields known so far, which admit HS algebras as algebras of symmetries of their maximally symmetric vacua, is much shorter than that of HS algebras given above. It includes \( \text{AdS}_d \) theories based on \( h \cdots (n_0, m_{1/2})^3 \) [1, 2] and \( \text{AdS}_{d+1} \) theories based on \( h \cdots (n_0)d + 1 \) [17]. (For HS theories in \( \text{AdS}_{d+1} \) with \( d \leq 2 \) not considered in this paper, where HS gauge fields carry no degrees of freedom, see [3] and references therein.) The problem of the construction of full nonlinear HS theories associated with other HS algebras remains open, though some partial results on the construction of cubic interactions in the respective theories were obtained in [89, 30, 92].

Although the construction of HS algebras sketched above can be used for the description of particular mixed symmetry fields (see also [31]), it can unlikely be applied to the generic case rich enough to incorporate string theory. Hence, some strategy change is needed. Before going into technical detail in the following section, we comment on the general idea.

In the literature (see, e.g. [20, 31, 93]), the construction of HS algebras is often related to the universal enveloping algebra \( U(s) \) of the spacetime (conformal) symmetry (super)algebra \( s \). If the \( s \)-module \( V_{\Phi} \) is irreducible, then \( H(V_{\Phi}) \) is isomorphic to \( U(s)/I_{V_{\Phi}} \), where \( I_{V_{\Phi}} \) is the ideal of \( U(s) \) which consists of those of its elements that annihilate \( V_{\Phi} \). This is tautologically the case for a single-field \( \Phi \), since this just means that its single-particle states form an irreducible \( s \)-module. However, the identification of HS algebras with quotients of \( U(s) \) may be misleading because the algebras \( H(V_{\Phi}) \) differ from \( U(s)/I_{V_{\Phi}} \) for reducible \( V_{\Phi} \). Indeed, since \( H(V_{\Phi}) \) is the maximal algebra acting on \( V_{\Phi} \),

\[
U(s)/I_{V_{\Phi}} \subset H(V_{\Phi}).
\]

Isomorphism \( H(n_0, n_{1/2}, \ldots) \sim U(s)/I_{V_{\Phi}} \) takes place only for irreducible \( V_{\Phi} \), i.e. iff \( n_{0} = 1 \) and \( n_{s} = 0 \) at \( s \neq s_{0} \) for some \( s_{0} \).

Given algebra \( A \), we introduce the associative multiparticle algebra \( M(A) \) as \( U(l(A)) \). The algebra \( H(V_{\Phi}) \) of endomorphisms of \( V_{\Phi} \), which underlies the construction of the HS algebra \( h_{\Phi}(V_{\Phi}) \), gives rise to \( M(H(V_{\Phi})) \). Being defined as universal enveloping of \( h_{\Phi}(V_{\Phi}) \), \( M(H(V_{\Phi})) \) acts on each \( h_{\Phi}(V_{\Phi}) \)-module. In particular, it acts on the space

\[
V_{\Phi} = \bigoplus_{n=0}^{\infty} \bigoplus V_{\Phi}^n, \quad V_{\Phi}^n = \text{Sym} V_{\Phi} \otimes \cdots \otimes V_{\Phi}
\]

which is nothing else but the space of all multiparticle states of the fields \( \Phi \). The multiparticle algebra associated with the fields \( \Phi \) will be identified with the appropriate real form of the Lie (super)algebra \( l(M(H(V_{\Phi}))) \) or some of its quotients considered in section 3.

As discussed in section 3.5, apart from the simplest possibility where \( l(M(H(V_{\Phi}))) \) acts independently on every \( V_{\Phi}^n \), multiparticle algebras admit representations mixing \( V_{\Phi}^n \) with different \( n \). Relating multiparticle states of the field theory of \( \Phi \) such as, e.g. \( N = 4 \) SYM theory, multiparticle algebras look particularly appealing in the string theory context. The problem of increase of lowest energies discussed in the introduction is avoided because a multiparticle algebra contains \( h_{\Phi}(V_{\Phi}) \) as subalgebra that acts on \( V_{\Phi} \) as in the original HS theory, hence having the same weights (in particular, energies) in this sector. At the same time, the fact that the multiparticle algebra is much smaller than the maximal symmetry algebra \( h_{\Phi}(V_{\Phi}) \) acting on the space of all multiparticle states of \( \Phi \) should leave enough flexibility for the description of interacting fields \( \Phi \).
3. Associative multiparticle algebra

3.1. Definition

Let \( A \) be an algebra with the product law \( \star \) and basis elements \( t_i \) obeying

\[
t_i \star t_j = f^k_{ij}t_k.
\]  (3.1)

Associativity implies

\[
f^m_{ik}f^k_{ij} = f^m_{ij}f^k_{nj}.
\]  (3.2)

In HS algebras, \( t_i \) denotes the infinite set of elements \( t_i = (1, Y^A, Y^AY^B, \ldots) \), while \( \star \) is the star product on functions of \( Y \).

Algebra \( M(A) \) is defined as follows. As a linear space, it is isomorphic to the direct sum of all symmetric tensor degrees of \( A \):

\[
M(A) = \bigoplus_{n=0}^{\infty} \text{Sym} A \otimes \cdots \otimes A.
\]  (3.3)

A natural basis of \( M(A) \) is provided by symmetrized tensor product monomials

\[
T_{i_1 \ldots i_n} = T_{j_1 \ldots j_n} = T_{k_1 \ldots k_n} = T_{\ldots k_n} \quad \forall j, k.
\]  (3.4)

Let \( A^\ast \) be the space of linear functionals on \( A \), i.e.

\[
\alpha \in A^\ast: \quad \alpha = \sum_i \alpha_i t^i, \quad t^i(t_j) = \delta^i_j,
\]  (3.5)

where \( \{t^i\} \) is the basis of \( A^\ast \) dual to \( \{t_i\} \). \( M(A) \) is the algebra of functions \( F(\alpha) \) on \( A^\ast \) with the product law

\[
F(\alpha) \circ G(\alpha) = F(\alpha) \exp \left( \frac{\partial}{\partial \alpha_j} f^n_{ij} \frac{\partial}{\partial \alpha_i} \right) G(\alpha),
\]  (3.6)

where the derivatives \( \frac{\partial}{\partial \alpha_i} \) and \( \frac{\partial}{\partial \alpha_j} \) act on \( F \) and \( G \), respectively. An elementary computation gives

\[
(F_1 \circ (F_2 \circ F_3))(\alpha) = \exp \left( f^n_{ij} \sum_{y < \beta = 1, 2, 3} \frac{\partial^2}{\partial \alpha_y \partial \alpha_{\beta}} \right) F_1(\alpha) F_2(\alpha) F_3(\alpha),
\]  (3.7)

\[
(F_1 \circ (F_2 \circ F_3))(\alpha) = \exp \left( f^n_{ij} \sum_{y < \beta = 1, 2, 3} \frac{\partial^2}{\partial \alpha_y \partial \alpha_{\beta}} \right) F_1(\alpha) F_2(\alpha) F_3(\alpha),
\]  (3.8)

where \( \frac{\partial}{\partial \alpha_i} \) acts on \( F_\beta(\alpha) \) (\( \beta = 1, 2, 3 \)). Hence, the \( A \)-associativity (3.2) implies the associativity of the product \( \circ \) of \( M(A) \).\(^3\) Note that \( A \subset M(A) \) is represented by linear functions on \( A^\ast \). Hence, the \( \star \) product acts on linear functions on \( A^\ast \) according to

\[
\alpha_j \star \alpha_k = f^m_{jk} \alpha_m.
\]  (3.9)

Note however that \( A \circ A \) does not belong to \( A \).

\(^3\) Note that somewhat similar (though different in some important details) algebras of oscillators were mentioned in \([13]\) in the context of singleton strings aimed at the description of multisingleton states which is another name for multiparticle states.
Algebra $M(A)$ is unital, with the unit element $Id$ identified with $F(\alpha) = 1$. Hence

$$M(A) = \mathbb{K} \oplus M'(A),$$

where $\mathbb{K}$ is the field over which $A$ and $M(A)$ were defined. (In HS context, the most important case is $\mathbb{K} = \mathbb{C}$.) Indeed, from (3.6) it follows that the unit element never appears on the rhs of $F \circ G$ if $F(\alpha)$ and/or $G(\alpha)$ is a homogeneous monomial of non-zero degree. The $\mathbb{Z}_2$ grading of $M(A)$ is induced by that of $A$:

$$F((-1)^{\pi(\alpha)}\alpha) = (-1)^{\pi(F)}F(\alpha).$$

$M(A)$ is isomorphic to the universal enveloping algebra $U(I(A))$,

$$M(A) \sim U(I(A)).$$

Indeed, by the definition of $I(A)$,

$$[t_i, t_j] = g_{ij} t_k, \quad g_{ij} = f^k_{ij} - f^k_{ji}. \tag{3.13}$$

On the other hand, from (3.6) it follows that

$$\alpha_i \circ \alpha_j - \alpha_j \circ \alpha_i = g_{ij}^{\star} \alpha_k. \tag{3.14}$$

Along with the associativity of $M(A)$, equation (3.10) and the fact that $M(A)$ is isomorphic to $U(I(A))$ as a linear space, equation (3.14) proves (3.12). The concise form of the product law (3.6) is specific to the case where a Lie algebra $I$ of $U(I)$ is associated with an associative algebra $A$, i.e. $I = I(A)$.

The following useful property of $M(A)$ is a simple consequence of equation (3.6):

$$\forall f, g \in A: \quad \exp f(\alpha) \circ \exp g(\alpha) = \exp(f \bullet g)(\alpha), \tag{3.15}$$

where

$$f \bullet g := f + g + f \star g = (f + e_\ast) \star (g + e_\ast) - e_\ast \in A \tag{3.16}$$

and $e_\ast$ denotes the unit element of $A$ if the latter is unital (recall that $f, g \in A$ implies that $f(\alpha)$ and $g(\alpha)$ are linear in $\alpha$). The associativity of $\star$ implies the associativity of $\bullet$:

$$(f \bullet g) \bullet h = f \bullet (g \bullet h) = (f + e_\ast) \star (g + e_\ast) \star (h + e_\ast) - e_\ast. \tag{3.17}$$

Note that the product $\bullet$ is associative even if $A$ is not unital.

Let

$$G_v = \exp(v) \in M(A), \quad v = v^i \alpha_i, \tag{3.18}$$

where $v^i \in \mathbb{K}$ are free parameters. Equation (3.15) gives

$$G_v \circ G_u = G_{v \circ u}. \tag{3.19}$$

This formula is convenient for practical computations with $G_v$ used as the generating function for elements of $M(A)$ resulting from differentiation over $v^i$.

### 3.2. Linear maps

Algebra $M(A)$ is double filtered in the following sense. Let $V_n$ be the linear space of order $n$ polynomials of $\alpha_i$. From equations (3.6) and (3.14), it follows that for any $F_n \in V_n$ and $F_m \in V_m$,

$$F_n \circ F_m \in V_{n+m}, \quad F_n \circ F_m - F_m \circ F_n \in V_{n+m-1}. \tag{3.20}$$

This property holds for any universal enveloping algebra (see, e.g. [94]).
A linear map of \( M(A) \) to itself is represented by

\[
U(\alpha, a) = \sum_{m,n=0}^{\infty} U^{i_1...i_n}_{j_1...j_m} \alpha_{i_1} \ldots \alpha_{i_n} a^{j_1} \ldots a^{j_m} \tag{3.21}
\]

with

\[
U(\alpha, a)[F] = U \left( \alpha, \frac{\partial}{\partial \alpha} \right) F(\alpha), \tag{3.22}
\]

where derivatives \( \frac{\partial}{\partial \alpha} \) act on \( F(\alpha) \). This formula can be interpreted as representing the action of the normal-ordered oscillator algebra with the generating elements \( \alpha_i \) and \( a^i \) acting on the Fock module spanned by \( F(\alpha)(0) \) with \( a^i(0) = 0 \). To respect the double filtration property, mapping order-\( n \) polynomials to order-\( n \) polynomials, \( U(\alpha, a) \) should obey

\[
U^{i_1...i_n}_{j_1...j_m} = 0 \quad \text{at} \quad m > n. \tag{3.23}
\]

Maps of this class, which we call filtered, are of most interest in this paper.

In these terms, the unit map is \( \text{Id} = 1 \). The map induced by a linear map \( u(t_i) = u_i t_j \) of \( A \) is represented by

\[
U(\alpha, a) = \exp(\alpha a^i a^j - \alpha_i \alpha_j). \tag{3.24}
\]

Consider maps of the form

\[
U(f) \equiv U(\alpha, af) = \phi \exp(\alpha f^i(a)) \tag{3.25}
\]

with some \( \alpha \)-independent coefficients \( f^i(a) \) and constant \( \phi \). The map \( U(f) \) is filtered provided that \( f^i(a) \) is at least linear in \( a \), i.e.

\[
f^i(0) = 0. \tag{3.26}
\]

Interpreting \( a \) as parameters, we can identify any \( f(\alpha) = \sum_i f^i \alpha_i \in M(A) \) with \( f(t) = \sum_i f^i t_i \in A \). For \( U(f) \) (3.25) acting on \( G_v \) (3.18), we obtain

\[
U(f)(G_v) = \exp(\tilde{f}(v) \alpha_i), \quad \tilde{f}(v) = v^i + f^i(v). \tag{3.27}
\]

Hence, equation (3.15) gives

\[
U(f)(G_v) \circ U(g)(G_\mu) = \exp((\tilde{f}(v) \bullet \tilde{g}(\mu))(\alpha)), \tag{3.28}
\]

where \( \tilde{f}(v) \) and \( \tilde{g}(\mu) \) are now interpreted as elements of \( A \), i.e. \( \tilde{f}(v) = \tilde{f}(v) \alpha_i \).

Important classes of linear maps \( U \) of \( M(A) \) onto itself are represented by automorphisms

\[
T(G_1 \circ G_2) = T(G_1) \circ T(G_2) \tag{3.29}
\]

and anti-automorphisms

\[
\mathcal{R}(G_1 \circ G_2) = \mathcal{R}(G_2) \circ \mathcal{R}(G_1) \tag{3.30}
\]

for \( \forall G_{1,2} \in M(A) \). To see whether or not \( T \) and \( \mathcal{R} \) are, respectively, automorphism and anti-automorphism of \( M(A) \), it is enough to check these properties for \( G_1 = G_\nu \) and \( G_2 = G_\mu \) with arbitrary \( \nu \) and \( \mu \), hence solving the equations

\[
T(G_\nu) \circ T(G_\mu) = T(G_{\nu \mu}), \tag{3.31}
\]

\[
\mathcal{R}(G_\mu) \circ \mathcal{R}(G_\nu) = \mathcal{R}(G_{\nu \mu}). \tag{3.32}
\]

Let \( \tau \) and \( \rho \) be, respectively, an automorphism and anti-automorphism of \( A \), i.e.

\[
\tau(a \star b) = \tau(a) \star \tau(b), \quad \rho(a \star b) = \rho(b) \star \rho(a) \quad \forall a, b \in A. \tag{3.33}
\]
In terms of basis elements $t_i$ and structure coefficients $f^k_{ij}$, this means that matrices $\tau_i^j$ and $\rho_i^j$ defined via

$$
\tau(t_i) = \tau_i^j t_j, \quad \rho(t_i) = \rho_i^j t_j
$$

(3.34)

obey

$$
\tau_i^j f^k_{ij} = f^k_{ij} \tau_i^j, \quad \rho_i^j f^k_{ij} = f^k_{ij} \rho_i^j.
$$

(3.35)

Equation (3.6) implies that $\tau$ and $\rho$ induce automorphism $T$ and anti-automorphism $R$ of $M(A)$,

$$
T(F(\alpha)) = F(\tau(\alpha)), \quad R(F(\alpha)) = F(\rho(\alpha)).
$$

(3.36)

These maps are described by $U(a, \alpha)$ (3.24) with $u_i^j$ identified either with $\tau_i^j$ or with $\rho_i^j$.

Analogously, one proceeds for conjugation $\sigma$ and involution $\dagger$ which are antilinear (i.e. conjugating complex numbers) counterparts of automorphism and anti-automorphism, respectively,

$$
S(F(\alpha)) = \bar{F}(\sigma(\alpha)), \quad (F(\alpha))^\dagger = \bar{F}(\alpha^\dagger),
$$

(3.37)

where $\bar{F}$ is complex conjugated to $F$, i.e. the coefficients of the expansion in powers of $\sigma(\alpha)$ and $\alpha^\dagger$ are complex conjugated to those of the expansion in powers of $\alpha$.

Consider maps (3.25) with

$$
f^i(a)t_i = f(a_n), \quad a_n = a \star \cdots \star a, \quad a_1 = a = t_i^1, \quad a_0 = e_\star,
$$

(3.38)

where $f(a_n)$ is a linear function of $a_n(n \geq 1)$. Such maps have the form (3.25) since $a_n \in A$.

A particularly important subclass of maps (3.25) is represented by $U_u$ of the form

$$
U_u(a) = \exp[u(a) - a],
$$

(3.39)

$a \in A$ and

$$
u(a) = (u_1^1 a + u_1^2 e_\star) \star (u_2^1 a + u_2^2 e_\star)^{-1}, \quad (e_\star + \beta a)^{-1} := \sum_{n=0}^{\infty} (-\beta)^n a_n
$$

(3.40)

with $u_i^j \in \mathbb{K}$. Composition of such maps gives a map of the same class

$$
U_u U_v = U_{uv},
$$

(3.41)

where $(uv)^i_j = u_i^k v_k^j$ is the matrix product in $\text{Mat}_2(\mathbb{K})$. The maps $U_u$ with $\det|u| \neq 0$ are invertible and form a usual Mobius group.

From equation (3.27), it follows that

$$
U_u(G_{\nu}) = G_{u(\nu)}.
$$

(3.42)

Consider the composition law of $M(A)$ in the basis associated with $G_{u(\nu)}$, assuming that new basis elements, that replace (3.4), are

$$
T_{\nu_{i_1} \cdots \nu_{i_n}} = \frac{\partial^\nu}{\partial v^{i_1} \cdots \partial v^{i_n}} G_{u(\nu)} \bigg|_{\nu=0}.
$$

(3.43)

To this end, we have to compute

$$
G_\nu \circ G_\mu = U_{\nu}^{-1}(G_{u(\nu)} \circ G_{u(\mu)}).
$$

(3.44)

Equation (3.16) gives

$$
G_\nu \circ G_\mu = G_{u(\nu) \star u(\mu)}.
$$

(3.45)
Generally, maps (3.39) are not filtered, not respecting condition (3.26). The subgroup $P$ of filtered maps (3.39) is represented by lower triangular matrices

$$u_{\alpha,\beta}(f) = bf \ast (e\ast + \beta f)^{-1}$$

with the composition law

$$b_{1,2} = b_1 b_2, \quad \beta_{1,2} = \beta_2 + \beta_1 b_2.$$

(3.47)

Clearly, $P$ is isomorphic to the affine group of translations and dilatations of $\mathbb{R}^1$.

For affine transformations (3.46), we will use notation $U_{b,\beta}$ instead of $U_{u}$.

In these terms, the unit element is

$$\text{Id} = U_{1,0}$$

(3.48)

and

$$U_{b,\beta}^{-1} = U_{b^{-1}, -\beta b^{-1}}.$$  

(3.49)

The map

$$R = U_{-1,1}$$

(3.50)

is involutive

$$R^2 = \text{Id}$$

(3.51)

and describes an anti-automorphism of $M(A)$. Indeed, one can check (3.32) using that

$$R(G_\nu) \circ R(G_\mu) = \exp[(e\ast + \nu)^{-1} - e\ast] \ast ((e\ast + \mu)^{-1} - e\ast)]$$

(3.52)

and, by (3.19),

$$R(G_\mu \circ G_\nu) = \exp((e\ast + \mu \ast \nu)^{-1} - e\ast).$$

(3.53)

Equation (3.42) gives

$$R(G_\nu) = G_{-\ast \ast(e\ast + \nu)^{-1}}.$$  

(3.54)

Differentiation over $\nu^i$ gives, in particular,

$$R(\text{Id}) = \text{Id},$$

(3.55)

$$R(\alpha_i) = -\alpha_i,$$  

(3.56)

$$R(\alpha_i \alpha_j) = \alpha_i \alpha_j + \{\alpha_i, \alpha_j\},$$  

(3.57)

where, for simplicity, we consider the even case with $\pi(\alpha_i) = 0$.

The anti-automorphism $R$ of $M(A)$ exists independent of the specific structure of $A$ and is called principal anti-automorphism of $U(l(A))$ [94]. Note that the form of $R$ (3.39) and (3.50) is specific for the universal enveloping algebra of a Lie algebra associated with an algebra $A$.

Given algebra $A$, the opposite algebra $\tilde{A}$ is isomorphic to $A$ as a linear space and has the product law \tilde{\circ}:

$$a \tilde{\circ} b = b \circ a.$$  

(3.58)

An anti-automorphism $\rho$ of $A$ can be interpreted as the homomorphism between $A$ and $\tilde{A}$. If $\rho$ is invertible, $\tilde{A}$ is isomorphic to $A$. Hence, equation (3.51) proves

$$M(\tilde{A}) \sim M(A), \quad \forall A,$$

(3.59)

which is of course in agreement with the realization of $M(A)$ as $U(l(A))$ [94].

For affine maps, the composition law (3.45) takes the form

$$G_{\nu} \circ G_{\mu} = G_{\nu \ast (\mu \ast \nu)},$$

(3.60)
where
\[
\sigma_{\nu, \beta}(v, \mu) = -\beta^{-1}(e_* - (e_* + \beta \mu) \star (e_* - \beta(b + \beta)v \star \mu)^{-1} \star (e_* + \beta v)).
\]
(3.61)

In particular, this formula gives
\[
\begin{align*}
\sigma_{1,0}(v, \mu) &= v + \mu + v \star \mu = v \star \mu, \\
\sigma_{-1,1}(v, \mu) &= v + \mu + \mu \star v = \mu \star v, \\
\sigma_{1, -1}(v, \mu) &= 2(e_* - (2e_* - \mu) \star (4e_* + v \star \mu)^{-1} \star (2e_* - v)).
\end{align*}
\]
(3.62) (3.63) (3.64)

Here, \(\sigma_{1,0}(v, \mu)\) corresponds to the unit map, \(\sigma_{-1,1}(v, \mu)\) corresponds to the anti-automorphism \(R\), while \(\sigma_{1, -1}(v, \mu)\) describes the map reproducing the \(N \to 0\) limit of the \(F\)-current operator algebra of [35].

### 3.3. Supertrace and central charge

Let \(A\) possess a (super)trace \(\text{tr}\) obeying
\[
\text{tr}(a \star b) = (-1)^{(\pi(a)\pi(b))}\text{tr}(b \star a), \quad \forall a, b \in A
\]
(3.65)

(\(\pi(a) = 0\) or 1 is the \(\mathbb{Z}_2\) grading of \(a\); usual trace is a particular case with \(\pi(a) \equiv 0\).) Let \(A\) admit such a basis \(t_i\) that
\[
\text{tr}\left(\sum_i a^i t_i\right) = a^0,
\]
(3.66)
i.e. \(\text{tr}(t_i) = \delta^0_i\). Then
\[
g_{ij} = f_{ij}^0
\]
(3.67)
is (graded)symmetric
\[
g_{ij} = (-1)^{\pi_i \pi_j} g_{ji}.
\]
(3.68)

Note that \(\text{tr}\) is supposed to be even, i.e. \(I_0\) is even which implies that \(g_{ij}\) is nonzero if \(\pi_i = \pi_j\). If the bilinear form \(\text{tr}(a \star b)\) is non-degenerate, which is necessarily true if \(A\) is simple since zeros of \(\text{tr}(a \star b)\) form a two-sided ideal of \(A\), \(g_{ij}\) can be interpreted as a non-degenerate metric. The associativity of \(A\) implies via \(\text{tr}(t_i \star t_j) = (-1)^{\pi_i} \text{tr}(t_j \star t_i)\) graded the cyclicity of the structure coefficients
\[
f_{ijk} = (-1)^{\pi_i} f_{jki}, \quad f_{ijk} = f_{jik} g_{ik}.
\]
(3.69)

For the unital algebra \(A\), it is convenient to set \(I_0 = e_*\) that is reachable via rescaling of \(\text{tr}\) in the non-degenerate case with \(\text{tr}(e_*) \neq 0\). In the degenerate case with \(\text{tr}(e_*) = 0\), a basis element supporting trace differs from \(e_*\) analogously to the case of \(psu(2,2|4)\) familiar from \(N = 4\) SYM. In that case, \(I(A)\) acquires an ideal associated with \(e_*\) to that associated with \(e_*\). Note that, being related to \(N = 4\) SYM, the degenerate case may be of primary importance in the multiparticle extension of HS theory.

Trace of \(A\) induces a family of traces of \(M(A)\). Indeed, for exponentials \(G_\nu\) (3.18) define the trace as
\[
\text{Tr}_{\Phi, \phi_\nu}(G_\nu) = \Phi(\text{tr}(\phi_\nu(v))), \quad \phi_\nu(v) = \sum_{n=0}^{\infty} \phi_n v \star \cdots \star v.
\]
(3.70)

with any star-product function \(\phi_\nu(v)\) and the usual function \(\Phi(x)\). From (3.19), it follows that
\[
\text{Tr}_{\Phi, \phi_\nu}(G_\nu \circ G_\mu) = \Phi(\text{tr}(\phi_\nu(v \star \mu))).
\]
(3.71)
Using (3.65) and (3.16), it is easy to see that
\[ \text{tr}(\phi_*(v \bullet \mu)) = \text{tr}(\phi_*(\mu \bullet v)) \]
and, hence,
\[ \text{Tr}_{\Phi_1,\phi^*}(G_\nu \circ G_\mu) = \text{Tr}_{\Phi_1,\phi^*}(G_\mu \circ G_\nu). \] (3.72)
Since \( G_\nu \) is the generating function for any element of \( M(A) \), formula (3.70) defines a trace of \( M(A) \). Thus the space of traces of \( M(A) \) admits at least a freedom in two functions of one variable. Note that the freedom in the definition of trace in \( M(A) \) reflects the fact that \( M(A) \) is not simple as is the case for every universal enveloping algebra.

Remarkably, the freedom of the definition of \( \text{Tr} \) of \( M(A) \) is closely related to the freedom in the central charge of the current operator algebra. To reproduce the dependence on the central charge, the basis has to be modified further by virtue of a field redefinition of the form
\[ G_\nu \rightarrow \eta(\nu) G_\nu, \] (3.73)
where \( \eta(\nu) \) is some map from \( A \) to \( K \). This map modifies the product law (3.60) to
\[ \tilde{G}_\nu \circ \tilde{G}_\mu = \frac{\bar{\eta}(\mu)\bar{\eta}(v)}{\bar{\eta}(\sigma_{b,\beta}(v,\mu))} \tilde{G}_{\sigma_{b,\beta}(v,\mu)}, \quad \bar{\eta}(v) = \eta(v_{b,\beta}(v)). \] (3.74)
The form of the current OPE gets modified since the basis is still defined by the formula analogous to (3.43) with respect to \( \tilde{G}_\nu \).

As shown in section 4, in the case of the \( F \)-current algebra, the map (3.73) is defined, so that
\[ \text{Tr}_{\Phi_1,\phi^*}(\tilde{G}_\nu) = 1 \] for certain \( \Phi_1(x) \) and \( \phi_* \). In the case of the \( A \)-current algebra, the appropriate field redefinition is still of the form (3.73), but it is not directly related to the rescaling of some trace of \( M(A) \).

3.4. Ideals and quotients

3.4.1. Ideals induced by (anti)automorphism. Let \( \tau \) be an automorphism of \( A \). A set of elements that obey
\[ \tau(a) = a \] (3.75)
forms a subalgebra \( A_{\tau} \) of \( A \). Suppose that \( \tau \) is involutive, i.e. \( \tau^2 = \text{Id} \). Then \( A_{\tau} \) consists of \( \tau \)-even elements
\[ a = \frac{1}{2}(a + \tau(a)). \] (3.76)
Let some \( a \in A_{\tau} \) have the form
\[ a = (b - \tau(b)) \star c, \quad a \in A_{\tau}, \quad c \in A. \] (3.77)
The fact that \( a \in A_{\tau} \) implies
\[ a = \frac{1}{2}(b - \tau(b)) \star (c - \tau(c)). \] (3.78)
Elements (3.77) form a two-sided ideal \( I_{\tau} \) of \( A_{\tau} \). Indeed,
\[ a \star (b - \tau(b)) = a \star b - \tau(a \star b) \quad \forall a \in A_{\tau}. \] (3.79)
Equation (3.78) implies that elements \( a = b \star (c - \tau(c)) \) form the same ideal \( I_{\tau} \) of \( A_{\tau} \). The algebra
\[ A^\tau = A_{\tau}/I_{\tau} \] (3.80)
is spanned by those elements of \( A_{\tau} \) that cannot be represented as a product of \( \tau \)-odd elements of \( A \).

In many cases, including usual HS algebras with nontrivial \( \tau \), the latter condition turns out to be too strong implying \( A^\tau = 0 \). For example, this is true for \( A \) generated by oscillators \( Y^A \) treated as odd elements of the automorphism \( \tau \) (which is the boson–fermion automorphism for spinorial \( Y^A \)). However, in the case of \( M(A) \), this construction leads to a nontrivial result.
Let $\rho$ be an involutive anti-automorphism of $A$. As explained in section 2, depending on a particular choice of $\rho$, condition (2.5) singles out the subalgebras $h_{\rho}(V)$ or $h_{\text{asp}}(V)$ of the HS Lie algebra $h_{\rho}(V)$. In the general case, let us call them $l_{\rho}(V)$. Let $R$ be the anti-automorphism of $M(A)$ associated with $\rho$ via (3.36). Then

$$T = R R,$$

(3.81)

where $R$ is the principal anti-automorphism (3.50), is an involutive automorphism of $M(A)$. It is not difficult to see that $M^T(A) \sim U(l_{\rho}(V))$. Indeed, using (3.56), the condition $T(\alpha_i) = \alpha_i$ implies in particular

$$\rho(\alpha_i) = -\alpha_i,$$

(3.82)

which is just equation (2.5). As a result, all $\rho$-even elements of $A$ do not belong to $M^T(A)$. The factorization of the ideal $I_T(A)$ takes away the dependence on all $\rho$-even elements of $A$. As the algebra of functions of $\rho$-odd elements of $A$, $M^T(A) \sim U(l_{\rho}(A))$.

Using the approach of section 3.2, it is not difficult to obtain explicit formulae for the composition law of $M^T(A)$ in the form analogous to (3.60). Indeed, consider basis (3.42) associated with $G_{\sigma_{1,1/2}(v)}$. Impose the condition

$$\rho(v) = -v,$$

(3.83)

which is nothing else but the factorization condition removing dependence on $\rho$-even elements. It is not difficult (but fun) to see that the so-defined elements $G_{\sigma_{1,1/2}(v)}$ are $T$ invariant, i.e.

$$TG_{\sigma_{1,1/2}(v)} = G_{\sigma_{1,1/2}(v)},$$

(3.84)

whose property relies on the identity $\frac{2r}{(1+r)} = -1 + \frac{1+\frac{1}{r}}{1+\frac{1}{r}}$. This allows us to use $\tilde{G}_{v} = G_{\sigma_{1,1/2}(v)}$ as the generating function for elements of $M^T(A)$. The composition law (3.60) gives

$$\tilde{G}_{v} \circ \tilde{G}_{\mu} = \tilde{G}_{\sigma_{1,1/2}(v,\mu)},$$

(3.85)

Remarkably, $\sigma_{1,1/2}(v,\mu)$ obeys (3.83) provided that $v$ and $\mu$ do, as one can easily see expanding $\sigma_{1,1/2}(v,\mu)$ (3.64) in power series. Hence, the composition law (3.60), (3.64) gives directly the composition law in $M^T(A)$. (A priori, it could happen that the composition of two generating functions gives a generating function that does not respect (3.83), hence requiring the factorization of elements of the ideal that might complicate the problem enormously.) Note that, in particular, these formulae provide a simple realization of the universal enveloping algebras of orthogonal and symplectic Lie algebras since the latter are subalgebras of $gl_n$ extracted by the anti-automorphisms $\rho$ generated by symmetric and antisymmetric bilinear forms, respectively.

In fact, the current operator algebra of [35] is associated with $M^T(A)$ where the anti-automorphism $\rho$ of $A$ is defined as $\rho(f(Y)) = i^T f(i(Y))$. Indeed, it is well known that nontrivial conserved currents $J^i_j$ of (odd)even spins are (anti)symmetric in their color indices $i, j$. As explained in [35], this happens just because they obey the condition $\rho(J^i_j) = -J^i_j$.

3.4.2. Ideals induced by central elements. Let $C_{\nu}$ be a basis of the centrum $C(A)$ of $A$ that forms a subset of $\nu$. In terms of structure coefficients (3.1), this implies

$$f_{ij}^{\nu} = f_{ji}^{\nu},$$

(3.86)

By virtue of (3.6), elements $h(C, Id) \in M(A)$ are central in $M(A)$. Any $h(C, Id)$ generates a two-sided ideal of $M(A)$. In particular, ideals $I_{c_{\alpha}}$,

$$F(t) \in I_{c_{\alpha}} : \quad F(t) = \prod \left(G_{\alpha} - c_{\alpha}Id\right) \circ G(t), \quad G(t) \subset M(A),$$

(3.87)

as well as the quotient algebras $M_{c_{\alpha}}(A) = M(A)/I_{c_{\alpha}}$, are parametrized by $c_{\alpha} \in \mathbb{K}$.  


A particularly important case is where $A$ is a unital algebra and $\mathcal{C} = e_\ast \in A$. Then $M_r(A)$ is parametrized by a single parameter $c$ resulting from the factorization of elements proportional to $e_\ast - cI_d$:

$$M_r : e_\ast - cI_d \sim 0.$$  \hspace{1cm} (3.88)

In the case where trace is supported by $e_\ast$, the shift $e_\ast \rightarrow e_\ast + cI_d$ gives the composition law

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp \left( \frac{\partial}{\partial \alpha_i} (f^a_j \alpha^a_j + c g_{ij}) \right) G(\alpha).$$  \hspace{1cm} (3.89)

In the shifted variables, $M_r(A)$ results from dropping $e_\ast$ in all formulæ. As a result, equation (3.89) where $F$ and $G$ depend only on traceless $\alpha$, describes the composition law in $M_r(A)$.

In the context of HS theories, the relevance of algebras $M_r(A)$ is not clear, however, since, as explained in section 3.5, the factorization (3.88) identifies the physical vacuum (no particles) with the lowest energy state of the space of single-particle states.

3.4.3. Finite-order quotients. Within infinite zoo of ideals of $M(A)$, we will be particularly interested in those that lead to quotient algebras realized by a finite number of tensor products of $A$. Given the function $\Phi(f_n, \text{tr}(f_m))$, the span of elements of the form

$$\sum_{\alpha} \Phi(f^a_n, \text{tr}(f^a_m)) \circ G^\alpha, \quad \forall f^a \in A, G^\alpha \in M(A)$$  \hspace{1cm} (3.90)

forms a two-sided ideal $\mathcal{I}_\Phi$ of $M(A)$. Indeed, since various $h \in A$ generate $M(A)$, it suffices to show that

$$h \circ \mathcal{I}_\Phi \in \mathcal{I}_\Phi, \quad h \in A.$$  \hspace{1cm} (3.91)

By virtue of (3.6), we observe

$$h \circ \Phi(f_n, \text{tr}(f_m)) = (-1)^{\pi(h)\pi(\Phi)} \Phi(f_n, \text{tr}(f_m)) \circ h = \sum_k [h, f_k] \frac{\partial}{\partial f_k} \Phi(f_n, \text{tr}(f_m)).$$  \hspace{1cm} (3.92)

Using that $[h, \ldots]_\ast$ is a derivation and that $\text{tr}(h, \ldots)_\ast = 0$, we obtain

$$h \circ \Phi(f_n, \text{tr}(f_m)) = (-1)^{\pi(h)\pi(\Phi)} \Phi(f_n, \text{tr}(f_m)) \circ h + \frac{\partial}{\partial \lambda} \Phi(f_n(\lambda), \text{tr}(f_m(\lambda))) \bigg|_{\lambda=0}.$$  \hspace{1cm} (3.93)

where

$$f(\lambda) = f + \lambda [h, f]_\ast, \quad f_n(\lambda) = f(\lambda) \ast \cdots \ast f(\lambda).$$  \hspace{1cm} (3.94)

This proves (3.91) since the rhs of equation (3.93) can be represented as a linear combination of polynomials $\Phi(f_n, \text{tr}(f_m))$ with different $f$.

Naively, factorization over order $n + 1$ polynomials $\Phi(f) = (f)^{n+1} + \cdots$ should give an algebra spanned by order $n$ polynomials. However, in most cases, this is not true because the ideal $\mathcal{I}_\Phi$ turns out to be much larger, coinciding with $M(A)$. Indeed, consider for example bilinear $\Phi_{\gamma}(f_n)$,

$$\Phi_{\gamma}(f_n) = f^2 + 2\gamma f \ast f$$  \hspace{1cm} (3.95)

with an arbitrary parameter $\gamma$. In this case, elements

$$(fg + \gamma (f, g)_\ast) \circ h = fgh + f(g \ast h) + g(f \ast h) + \gamma (f, g)_\ast h + \gamma (f, g)_\ast \ast h$$  \hspace{1cm} (3.96)
belong to $\mathcal{I}_{\Phi_\gamma}$. Obviously,
\begin{equation}
(f * g) \star h + g(f * h) \sim -\gamma (f * g * h + g * f * h + f * h * g),
\end{equation}
where equivalence is up to terms that belong to $\mathcal{I}_{\Phi_\gamma}$. This gives
\begin{equation}
(fg + \gamma \{fg\})h \sim fg - \gamma (f * g * h + f * h * g) \sim \gamma (f * g \star h + f * h \star g).
\end{equation}

Antisymmetrization of this expression with respect to $h$ and $g$ gives
\begin{equation}
\gamma (1 - \gamma) \{f, [h, g]\} \star \in \mathcal{I}_{\Phi_{\gamma_1}}.
\end{equation}

Hence, except for $\gamma = 0$ or $\gamma = 1$, $\mathcal{I}_{\Phi_{\gamma}}$ contains all elements of $A$ that can be represented as $[f, \{h, g\}]_\star$, i.e. belong to the ideal $\mathfrak{l}_2(A)$ of $l(A)$. In the relevant cases where $\mathfrak{l}_1(A) \cong (f, g)\star \in \mathcal{I}_{\Phi_{\gamma}}$, for some $f, g \in l(A)$, and hence $\mathfrak{l}_2(A)$, is simple, it coincides with almost all $l(A)$. Namely, in the cases of interest $l(A) = l_0(A) \oplus l_1(A)$, where $l_0(A)$ is the Abelian algebra spanned by central elements of $A$. (This is fully analogous to the relation $l(\text{Mat}_n(\mathbb{C})) \cong \text{gl}_n(\mathbb{C}) = \text{sl}_n(\mathbb{C}) \oplus \mathbb{C}$, where $l_1(\text{Mat}_n(\mathbb{C})) = \text{sl}_n(\mathbb{C})$.) Since $\mathcal{I}_{\Phi_{\gamma}}$ is too large for generic $\gamma$, we consider the special cases of $\gamma = 0$ or $\gamma = 1$.

Obviously, $M(A)/\mathcal{I}_{\Phi_0} \cong A \oplus \mathbb{K}$. On the other hand, from (3.57) it follows that the case of $\gamma = 1$ is related to $\gamma = 0$ by the principal anti-automorphism $\mathbf{R}$, i.e. $M(A)/\mathcal{I}_{\Phi_{\gamma}} = A \oplus \mathbb{K}$. Hence the cases of $\gamma = 0$ and $\gamma = 1$ are exchanged via exchange of $A$ with $\tilde{A}$,
\begin{equation}
M(A)/\mathcal{I}_{\Phi_{\gamma}} \sim M(\tilde{A})/\mathcal{I}_{\Phi_{1-\gamma}}, \quad \gamma = 0, 1.
\end{equation}

As explained in section 3.4.2, the unit elements $e_* \in A$ and $Id \in M(A)$ can be identified via factorization of the ideal $I_*$. However, in the quotient algebras $M(A)/\mathcal{I}_{\Phi_{\gamma}}$, the parameter $c$ is no longer arbitrary. Indeed,
\begin{equation}
e_* * f = e_* f + e_* * f \sim f - 2\gamma f = (1 - 2\gamma)f = (1 - 2\gamma)Id \circ f.
\end{equation}

Hence, the factorization over both $I_*$ and $\mathcal{I}_{\Phi_{\gamma}}$ is possible at $c(\gamma) = (1 - 2\gamma)$, i.e. $c(0) = 1$ and $c(1) = -1$.

The example of $\mathcal{I}_{\Phi_0}$ admits natural generalization to the ideals $\mathcal{I}^{N+1}$ generated by equation (3.90) with
\begin{equation}
\Phi = f^{N+1}.
\end{equation}
As a linear space, the quotient algebra $M_N(A) := M(A)/\mathcal{I}^{N+1}$ is
\begin{equation}
M_N(A) = \bigoplus_{n=0}^N \text{Sym} A \otimes \cdots \otimes A.
\end{equation}
Here, the only possible value of $c$ is
\begin{equation}
c_N = N.
\end{equation}

Indeed,
\begin{equation}
e_* * f^N = e_* f^N + N(e_* * f) f^{N-1} \sim N f^N
\end{equation}
since $e_* f^N \in \mathcal{I}^{N+1}$. Now one can consider quotient algebras $M_N(A) = M_N(A)/\mathcal{I}_{\Phi_{N-1}Id}$. Note that $M^I(A) = A$. Similarly, the generalization of $\mathcal{I}_{\Phi_0}$ to higher $N$ leads to ideals $\mathcal{I}^{N+1}$ of $M(A)$ and quotients $M_N(\tilde{A})$ and $M_N^I(\tilde{A})$.
3.5. Modules

Since $M(A) \sim U(l(A))$, any $A$-module generates an $M(A)$-module. The tensor product of any number of $A$-modules forms an $l(A)$-module and, hence, $M(A)$-module. Beyond that, $M(A)$ admits less trivial modules which may be relevant in the context of multiparticle HS theories.

Let $V$ be an $A$-module. Recall that in the HS context $A = H_{\Phi}$ and $V = V_{\Phi}$ is the space of single-particle states of some fields $\Phi$. The space of all multiparticle states is

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \otimes V^n, \quad V^n = \text{Sym} V \otimes \cdots \otimes V. \quad (3.107)$$

When referring to a particular field theory associated with $V_{\Phi}$, we will use notation $\mathcal{V}_{\Phi}$.

Let $t_n$ be a basis of $V$,

$$v \in V : \quad v = \sum_{\alpha} v^\alpha t_n, \quad (3.108)$$

and

$$t_i(t_n) = T_i^\alpha t_n, \quad (t_i \bullet t_j)(t_n) = T_{ij}^\gamma T_{\gamma n}^\beta \implies T_{ij}^\gamma T_{\gamma n}^\beta = f_{ij}^k T_{kn}^\beta. \quad (3.109)$$

Let $V^*$ be dual to $V$,

$$\lambda \in V^* : \quad \lambda = \sum_{\beta} \lambda^\beta \tau^* \beta. \quad (3.110)$$

Similar to the realization of $M(A)$ in terms of functions $F(\alpha)$, elements of $\mathcal{V}$ can be represented by functions $\phi(\lambda)$ on $V^*$:

$$\phi(\lambda) = \sum_{n=0}^{\infty} \phi^{\alpha_1 \cdots \alpha_n} \lambda_{\alpha_1} \cdots \lambda_{\alpha_n}. \quad (3.111)$$

Let $F(\alpha) \in M(A)$, $\phi(\lambda) \in \mathcal{V}$. $\mathcal{V}$ can be endowed with the structure of $M(A)$-module by setting

$$F(\alpha)(\phi(\lambda)) = F(\alpha) \exp \left( \frac{\partial}{\partial \alpha_i} t_i(\lambda, \beta) \frac{\partial}{\partial \lambda_\beta} \phi(\lambda) \right) \bigg|_{\alpha_i = t_i(\lambda)}. \quad (3.112)$$

where $t_i(\lambda)$ is some linear function of $\lambda_\alpha$ that obeys the condition

$$t_i(\lambda, \beta) \frac{\partial t_i(\lambda)}{\partial \lambda_\beta} = f_i^j \lambda_j. \quad (3.113)$$

The fact that equation (3.112) defines an $M(A)$-module, i.e.

$$F(t)(G(t)(\phi(\lambda)) = (F \circ G)(\phi(\lambda)) \quad (3.114)$$

is easy to see using equations (3.109) and (3.113).

Let $\mathcal{V}(V, t)$ be the $M(A)$-module $\mathcal{V}$ (3.112) determined by an $A$-module $V$ and $t_i(\lambda)$ solving (3.113). The $M(A)$-module $\mathcal{V}(V, 0)$ is associated with $t_i = 0$. A less trivial option of $t_i(\lambda)$ is

$$t_i(\lambda) = \delta(\lambda, \alpha) = \delta^p T_{\alpha \beta} \lambda_\beta. \quad (3.115)$$

is parametrized by a vector $\delta^p \in V$. Indeed, in this case (3.113) holds by virtue of (3.109).

The module $\mathcal{V}(V, 0)$ is infinitely reducible. Indeed, from equation (3.112) with $t_i = 0$, it follows that the subspace $V^p$ of homogeneous polynomials $\phi(\lambda)$ of degree $p$ remains invariant under the action of $M(A)$. (Note that the existence of the ideals $I_{N+1}$ (3.103) is closely related to the fact of reducibility of $\mathcal{V}(V, 0)$.) Clearly, $V^p$ are canonical $U(l(A))$-modules associated with symmetrized tensor products of the $l(A)$-module $V$, i.e. spaces of $p$-particle states in their multiparticle interpretation.

For $t_i \neq 0$ (3.115), the action of $M(A)$ (3.112) is not homogeneous, mixing $V^n$ with different $n$. Suppose that $V$ is induced from the vacuum vector $\alpha$, i.e. $A(\alpha) = V$. Let $V^*$ be
the right $A$-module and $o^* \in V^*$ be normalized so that $o^*(o) = 1$. From equations (3.112) and (3.115), it follows that, in this case, the $M(A)$-module $\mathcal{V}(V, t)$ is induced from the vacuum element $O \in \mathcal{V}$ identified with $\phi(\lambda) = 1$. Modules of this type are somewhat analogous to Fock modules of the oscillator algebra (as illustrated in the following section by the example of Weyl algebra) and are expected to play an important role in multiparticle theories.

$M_c(A)$ (3.88) results from $M(A)$ via the factorization of elements proportional to $e_* - cId$. Similarly, an $M_c(A)$-module $\mathcal{V}_c(V, t)$ results from $\mathcal{V}(V, t)$ via the factorization of elements induced from

$$o - c O \sim 0.$$  \hfill (3.116)

Indeed, according to (3.112), the action of $e_* - cId$ on $O$ gives

$$(e_* - cId)(O) = o(\lambda) - cO.$$  \hfill (3.117)

To reduce the $M(A)$-module $\mathcal{V}(V, t)$ to the $M_c(A)$-module $\mathcal{V}_c(V, t)$ using relation (3.116), one has to remove the dependence on $\lambda_o$ from $\lambda = \lambda_o o^* + \sum_{\beta} \lambda_{\beta}\beta^\beta$. In other words, $\mathcal{V}_c(V, t)$ consists of functions of all elements of $\mathcal{V}^*$ except for the vacuum $o^*$. Since such a factorization identifies the vacuum $o$ of the space of single-particle states, usually describing the lowest energy state of one or another particle, with the physical vacuum $O$ with no particles, its physical meaning is however obscure. In fact, the difficulties of the naive extension of HS algebras discussed in the introduction resulted just from the consideration of $M_c(A)$-modules instead of $M(A)$-modules.

Another construction applicable to a Lie algebra $l(A)$ with any $A$, which is particularly useful in the context of HS theory, is that of twisted adjoint modules. Let $\tau$ be some automorphism of $A$. The $\tau$-twisted adjoint $l_A$-module $A_{\tau}$ has $A$ as a linear space where $l_A$ acts as follows:

$$a(b) = a \star b - b \star \tau(a), \quad a \in l_A, \quad b \in A_{\tau}.$$  \hfill (3.118)

Any $\tau$-twisted adjoint module $A_{\tau}$ of $l(A)$ admits the straightforward extension to $T$-twisted adjoint module of $l(M(A))$. This simple observation is expected to play a key rôle for the formulation of a multiparticle generalization of HS gauge theory.

### 3.6. Weyl algebra and Fock module

The Weyl algebra $A_M$, which underlies the construction of most of HS algebras, is the unital algebra generated by $2M$ elements $Y_A$ satisfying (1.1). Remarkably, it can itself be interpreted as the quotient of a multiparticle algebra $M(a_M)$. Here, $a_M$ is the algebra with the generating elements $Y_{\Omega}$ and $h$ obeying relations

$$Y_A \star Y_B = K_{AB}h, \quad Y_A \star h = h \star Y_A = 0, \quad h \star h = 0,$$  \hfill (3.119)

where $A, B = 1, \ldots, M$, and $K_{AB}$ is some matrix with the nondegenerate antisymmetric part

$$C_{AB} = (K_{AB} - K_{BA}).$$  \hfill (3.120)

Algebra $a_M$ is obviously associative since any triple product of its elements vanishes.

$M(a_M)$ is spanned by functions $f(Y_A, h)$. This is not yet Weyl algebra, but rather the algebra of quantum operators in the deformation quantization framework with $h$ interpreted as a deformation parameter. The Weyl algebra $A_M = M_h(a_M)$ results from $M(a_M)$ via the factorization of the ideal generated by $h - hId$ with parameter $h$. Note that $M_h(a_M)$ with various $h \neq 0$ are pairwise isomorphic. The ‘classical’ case of $h = 0$ is degenerate.

Different choices of the symmetric part of $K_{\Omega A}$ lead to different product laws (3.6) which correspond to different star products for the same Weyl algebra. Indeed, it is well known that different choices of $K_{\Omega A}$ with the same $C_{\Omega A}$ (3.120) encode different ordering prescriptions.
Let us now explain how Fock module of Weyl algebra results from the construction of section 3.5. Consider for simplicity the case of \( a_1 \) with the defining relations

\[
Y_+ Y_+ = h, \quad Y_+ Y_- = 0, \quad Y_- Y_+ = 0, \quad Y_\pm h = h Y_\pm = 0, \quad h h = 0. \tag{3.121}
\]

A left \( a_1 \)-module in a two-dimensional vector space \( V \) with the basis elements \( v \) and \( v_+ \), which can be realized as a quotient of the left adjoint \( a_1 \)-module, is

\[
Y_- v = 0, \quad Y_- v_+ = v, \quad Y_+ v = 0, \quad Y_+ v_+ = 0, \quad hv = 0, \quad hv_+ = 0. \tag{3.122}
\]

It is easy to see that

\[
t_-(v) = 0, \quad t_+(v) = v_+, \quad t_0(v) = v \tag{3.123}
\]

solve equation (3.113). With this substitution, equation (3.112) gives the \( M(a_1) \)-module realized by functions \( \Phi(v_+, v) \) with

\[
\alpha_-(\Phi) = v \frac{\partial}{\partial v_+} \Phi, \quad \alpha_+ (\Phi) = v_+ \Phi, \quad \alpha_h (\Phi) = v \Phi. \tag{3.124}
\]

The factorization of the ideal of \( M(a_1) \) generated by \( h = h \text{Id} \) along with its image in the constructed \( M(a_1) \)-module implies the substitution \( v \rightarrow h \), giving the Fock module of the Weyl algebra \( A_1 \).

4. Current operator algebra

In this section, we show how the current operator algebra in the twistor space results from our construction. The spacetime current operator algebra, which results from the twistor one via unfolded formulation of the current conservation equations, is not considered in this paper. We refer the reader to [35] for details on this relation.

The dictionary between notations of this paper and [35] is as follows. Free currents \( \mathcal{J}^2(\mathcal{U}, \mathcal{V}) \), where \( \mathcal{U} \) and \( \mathcal{V} \) denote the twistor variables used in [35], identify with the generators \( t_i \) or, equivalently, with the basis \( a_i \) of \( A^* \). The normal ordered product \( \mathcal{J}^2(U_1, V_1) \ldots \mathcal{J}^2(U_b, V_b) \) is constructed symmetric with respect to the permutation of arguments \( (U_b, V_b) \) with different \( b \), is represented by the usual product \( t_{i_1} \ldots t_{i_b} \).

Parameters \( v^i \) have to be identified with the parameters of currents called \( g(W_1, W_2) \) in [35] and usual powers \( (v^i)^n \) represent \( \mathcal{J}^{2n}(\mathcal{U}, \mathcal{V})^n \).

The currents considered in [35] are invariant under certain involutive operation \( \mu \) as a consequence of the construction of currents in terms of bilinears of free fields. In the setup of this paper \( \mu = -\rho \), where \( \rho \) is the anti-automorphism of section 3. The current algebra of [35] is nothing else but the quotient algebra \( M^T(A) \) introduced in section 3.4.1, where \( T \) is the anti-automorphism of \( M(A) \) generated by \( \rho \). As explained in section 3.4.1, the specific form of the composition law associated with \( \sigma_{1, -\frac{1}{2}} \) is compatible with the (factorization) condition \( \rho(v) = -v \) imposed in [35].

4.1. \( F \)-current algebra

As mentioned in the end of section 3.4.1, the \( F \)-current algebra at \( \mathcal{N} = 0 \) is described by the composition law (3.85). To describe \( \mathcal{N} \)-dependent terms, one should generalize it using (3.73) with an appropriate function \( \eta(v) \). For

\[
\eta(v) = \text{Tr}_{\Phi, \phirouch}(G_{\phirouch,\phirouch}(v)), \quad \Phi(x) = \exp -x \tag{4.1}
\]

with some \( \phirouch(v) \), the factor on the rhs of (3.74) is

\[
\exp \text{tr}(\phirouch(u_{1,\beta}(v)) \star u_{1,\beta}(\mu) + u_{1,\beta}(v) + u_{1,\beta}(\mu) - \phirouch(u_{1,\beta}(\mu)) - \phirouch(u_{1,\beta}(v))). \tag{4.2}
\]
The characteristic property of the $F$-current operator algebra is that the trace-dependent part of the OPE for both right and left multiplication with the bilinear current $f^2$ only involves the trace $\text{tr}(v \star \mu)$ between parameters of two elementary currents $J^2_v$ and $J^2_\mu$. This imposes the condition that the part of (4.2) linear either in $\mu$ or in $v$ should have the form $\frac{1}{2} N \text{tr}(v \star \mu)$. This gives the differential equation

$$\phi_1'(u_1, \beta(v)) \star (u_1, \beta(v) + e) = \frac{1}{8} N v$$

solved by

$$\phi_1(u_1, \beta(v)) = \frac{1}{8} N (\beta^{-1} \ln(\phi + \beta v) - (1 + \beta)^{-1} \ln(\phi + (1 + \beta)v)).$$

For $\beta = -\frac{1}{2}$, this gives

$$\phi_1(u_1, -\frac{1}{2}(v)) = -\frac{1}{4} N \ln(\phi - \frac{1}{2} v \star v).$$

Introducing

$$\tilde{G}_v = \eta(v) G_{u_{-\frac{1}{2}}(v)},$$

the trace-dependent version of formula (3.60) at $b = 1$, $\beta = -1/2$ takes the form

$$\tilde{G}_v \circ \tilde{G}_\mu = \left( \frac{\det e_0 - \frac{1}{2} v \star v | \det e_0 - \frac{1}{2} \mu \star \mu |}{\det e_0 - \frac{1}{2} \sigma_{-\frac{1}{2}}(v, \mu) \star \sigma_{-\frac{1}{2}}(v, \mu)} \right) \frac{N}{2} \tilde{G}_{u_{-\frac{1}{2}}(v, \mu)},$$

(4.7)

where, as usual,

$$\det |A| = \exp \text{tr}(\ln(A)).$$

(4.8)

(Of course, the $\star$-determinant possesses the multiplication property $\det |A \star B| = \det |A| \det |B|.$)

Formula (4.7) gives the generating function for the $F$-current operator algebra of [35]. To see this, it remains to check the parts of $\sigma_{-\frac{1}{2}}(v, \mu)$ linear either in $v$ or in $\mu$ which describe left and right multiplication with $J^2_v$ and $J^2_\mu$, respectively. Once, formula (4.7) correctly reproduces this part of the algebra, associativity implies that it describes the full operator algebra.

For example, denoting by $\tilde{G}_v^2$, the part of $\tilde{G}_v$ linear in $\mu$, we obtain from (4.7),

$$\tilde{G}_v \circ \tilde{G}_\mu^2 = \tilde{G}_v (\mu + \frac{1}{2} (v \star \mu - \mu \star v) - \frac{1}{2} v \star \mu \star v + \frac{1}{2} N \text{tr}(v \star \mu) Id).$$

(4.9)

These terms reproduce the OPE of $\mathcal{J}^{2n}_v \mathcal{J}^{2m}_\mu$. Indeed, the first term is the regular one. The second results from single contractions of the constituent fields. The third term results from double contractions of the constituent fields of $\mathcal{J}^2_\mu$ with two different $\mathcal{J}^2_v$, while the last one comes from the double contraction of the constituent fields of $\mathcal{J}^2_\mu$ with those of some $\mathcal{J}^2_v$.

Formula (4.7) represents the OPE of $\mathcal{J}^{2n}_v \mathcal{J}^{2m}_\mu$ by the $n$- and $m$-linear terms in $v$ and $\mu$. Note that the $\mathcal{N}$-dependent central term does not contribute to the commutator $\mathcal{J}^{2n}_v \mathcal{J}^{2m}_\mu = \mathcal{J}^{2m}_\mu \mathcal{J}^{2n}_v$. This is because the dependence on $\mathcal{N}$ was introduced in (4.1) in terms of a trace of $M(A)$.

4.2. $A$-current algebra

The $A$-current algebra is the algebra of currents with stripped indices $a, b = +, -$ associated with the creation and annihilation parts of the constituent fields. This algebra is anticipated to play a role in the analysis of amplitudes (hence $A$-current algebra). Its structure differs from that of the $F$-current algebra in several respects. In particular, the part of OPE associated with unity does contribute to the operator commutator. Hence it cannot be derived via a field redefinition (3.73) where $\eta$ is expressed in terms of some trace of $M(A)$ as in (4.1), requiring
\[\eta\] of some other form. Another novelty is that the \(A\)-current algebra involves two associative products instead of one in the \(F\)-current case. This bi-associative structure underlies the construction of the butterfly product of \([35]\). In this section, we first describe the relevant bi-associative structure and then present explicit formulae for the \(A\)-current operator algebra.

### 4.2.1. Bi-associative algebra

By bi-associative algebra we mean a linear space \(A\) over a field \(\mathbb{K}\) endowed with two mutually associative products \(\star\) and \(\cdot\). This implies that \(A\) is an associative algebra with respect to the product \(\alpha \star \beta\) with any \(\alpha, \beta \in \mathbb{K}\). Equivalently, in addition to associativity of \(\star\) and \(\cdot\),

\[(a \star b) \cdot c = a \star (b \cdot c), \quad (a \cdot b) \star c = a \cdot (b \star c).\]  

(4.10)

Biassociative algebras can be easily introduced as follows. Choose two elements \(\tau_1, \tau_2\) of some associative algebra \(A\) with a product law \(\ast\). Then the two products \(a \ast b := a \ast \tau_1 \ast b\) and \(a \cdot b := a \ast \tau_2 \ast b\) are associative along with their linear combination, endowing \(A\) with any of their bi-associative structure.

\(A\) is assumed to be unital with respect to both \(\star\) and \(\cdot\). However, the respective units \(e_\star\) and \(e\) may be different. They satisfy the following obvious relations:

\[e \cdot e = e, \quad e_\star e_\star = e_\star, \quad e \cdot e = e_\star \cdot e = e, \quad e_\star e = e_\star \cdot e = e.\]  

(4.11)

One can see that, in addition, via relative rescaling of the two products (and, hence, respective units), it is possible to achieve

\[e_\star e_\star = e, \quad e_\star e = e.\]  

(4.12)

In what follows, (4.12) is assumed to be true. These relations imply in particular

\[a \cdot b = a \ast e_\star b, \quad a \ast b = a \ast e \cdot b\]  

(4.13)

and that

\[\Pi_{\pm} = \frac{1}{2}(e_\star \pm e)\]  

(4.14)

are projectors with respect to each of the products

\[\Pi_{\pm} \ast \Pi_{\pm} = \Pi_{\pm} \cdot \Pi_{\pm} = \Pi_{\pm}, \quad \Pi_{\pm} \ast \Pi_{\mp} = \Pi_{\pm} \cdot \Pi_{\mp} = 0.\]  

(4.15)

Inverse elements are defined as usual

\[a^{-1}_\star a = a \ast a^{-1}_\star = e_\star, \quad a^{-1} \cdot a = a \cdot a^{-1} = e.\]  

(4.16)

Property (4.12) has the following nice consequences:

\[a^{-1}_\star = e_\star \cdot a^{-1} \cdot e_\star, \quad a^{-1} = e \ast a^{-1}_\star \ast e,\]  

(4.17)

\[(a \ast b)_\star^{-1} = b^{-1}_\star \cdot a^{-1}_\star, \quad (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}.\]  

(4.18)

Note that from here it follows that \(e_\star\) and \(e\) coincide with their inverses with respect to each of the products

\[(e_\star)^{-1} = e_\star, \quad (e)^{-1} = e.\]  

(4.19)

In \(A\)-current operator algebra, an important role is played by an involutive antiautomorphism \(\rho\) obeying

\[\rho(a \ast b) = \rho(b) \ast \rho(a), \quad \rho^2 = Id, \quad \rho(e_\star) = e, \quad \rho(e) = -e, \quad \rho(\Pi_{\pm}) = \Pi_{\mp}.\]  

(4.20)

By virtue of (4.13), this implies

\[\rho(a \cdot b) = -\rho(b) \cdot \rho(a).\]  

(4.21)

The products \(\triangleright\) and \(\triangleleft\) of \([35]\) are

\[a \triangleright b = -a \ast \Pi_{\mp} \ast b, \quad a \triangleleft b = a \ast \Pi_{\mp} \ast b.\]  

(4.22)
4.2.2. Bi-maps and OPE. Starting from the algebra $A$ endowed with the product law $\star$, we can now apply the linear map (3.39) with $u(f)$ built with the aid of $\cdot$. For maps analogous to (3.46),

$$u_{b,\beta}(f) = bf \cdot (e + \beta f)^{-1} = \frac{b}{\beta} (e - (e + \beta f)^{-1}),$$  

(4.23)

an elementary computation gives again formula (3.60), where

$$\sigma_{b,\beta}(v, \mu) = -\beta^{-1}(e - (e + \beta \mu) \bullet [e - (\beta v - 2\Pi_-)] \star (e + \beta v)).$$  

(4.24)

At $\cdot = \star, e = e_{\star}$, this formula reproduces (3.61). The map, that leads to the basis corresponding to the $A$-current OPE, is a composition of some maps (3.42) and (4.23).

Indeed, from general properties of the current operator algebra, it follows that it should be isomorphic to the algebra $M^T(A)$ associated with an appropriate anti-automorphism $\rho$. Hence, the product law of [35] should result from (3.85) via some field redefinition analogous to (3.39). However, now we should confine ourselves to such field redefinitions that leave $M^T(A)$ invariant. This condition rules out field redefinitions (3.39) and (3.40) constructed in terms of the product $\star$ because even $\star$-degrees of elements $\nu$ obeying (3.83) do not obey this condition, hence not belonging to $M^T(A)$. The trick is that it is possible to use the product law $\cdot$, which is $\rho$-odd itself obeying (4.21), to perform a change of variables (4.23) that maps $M^T(A)$ to itself.

Thus, the map $U^*_{1, -\frac{1}{2}}$ with respect to the product $\star$ should be followed by some map $U_{b, \beta}$ with respect to $\cdot$. It turns out that the appropriate form of the current operator algebra results from the map $U_{1, -\frac{1}{2}}$. Practically, it is most convenient to apply the map

$$U^*_{1, -\frac{1}{2}}(G_v) = U^*_{1, -\frac{1}{2}}(U_{1, -\frac{1}{2}}(G_v))$$  

(4.25)

directly to (3.19) rather than to apply $U_{1, -\frac{1}{2}}$ to the product law (3.85).

Using formulae of section 4.2.1, it is not difficult to obtain that

$$U^*_{1, -\frac{1}{2}}(G_v) = G_{\nu}(e_{\star} - \Pi_{\nu})_{\nu}^{-1}$$  

(4.26)

with $\Pi_+$ (4.14). The product law in the new basis is

$$\tilde{G}_{\nu} \star \tilde{G}_{\mu} = \tilde{G}_{\sigma_{\nu}^*_{1, -\frac{1}{2}}(v, \mu)},$$  

(4.27)

where

$$\sigma_{\nu}^*_{1, -\frac{1}{2}}(v, \mu) = (e_{\star} + v \bullet \Pi_{\nu}) \star (e_{\star} + \mu \bullet \Pi_{\nu})_{\nu}^{-1} \star (e_{\star} - \mu \bullet \Pi_{\nu}) \star (e_{\star} + v \bullet \Pi_{\nu} \bullet \mu \bullet \Pi_{\nu})_{\nu}^{-1} \star v.$$  

(4.28)

To reproduce the trace-dependent terms, we proceed as in the $F$-current algebra case, requiring that if such terms are linear in $\mu$, they should also be linear in $v$ having the form

$$-N \operatorname{tr}(\Pi_+ \star v \bullet \Pi_{\nu} \bullet \mu).$$  

(4.29)

This is achieved via (3.74) with

$$\tilde{\eta}(x) = \exp[-N \operatorname{tr}(\ln(e_{\star} - \Pi_{\nu} + x))],$$  

(4.30)

giving the composition law

$$\tilde{G}_{\nu} \circ \tilde{G}_{\mu} = \left(\frac{\det(e_{\star} - \Pi_{\nu})}{\det(e_{\star} - \Pi_{\nu} \bullet v \bullet e_{\star} - \Pi_{\nu} \bullet \mu)}\right)^{\frac{N}{2}} \tilde{G}_{\sigma_{\nu}^*_{1, -\frac{1}{2}}(v, \mu)}.$$

(4.31)
This formula encodes in a concise form the full operator algebra of free 3D conserved $A$-currents. The prefactor on its rhs provides the generating function for two-point functions $\langle J^{2n}_\nu J^{2m}_{\nu'} \rangle$.

To compare this product with the $A$-current operator algebra of [35], it remains to consider the part linear in $\mu$:

$$\tilde{G}_\nu \circ \tilde{G}^\mu_{\nu} = \tilde{G}_\nu (\mu + v \star \Pi - \mu \star \Pi + v)$$

$$- v \star \Pi \mu \star \Pi + v = \mathcal{N} tr (\Pi_+ \star v \star \Pi_- \mu = (\mu_\nu \star /\Phi_1 \star \nu^{22}2)), \text{this formula just reproduces OPE for } J^\nu_\nu J^\mu_{\nu} \text{ of [35]}. \tag{4.32}$$

Using (4.22), this formula just reproduces OPE for $J^2_\nu J^2_{\nu}$ of [35].

5. Multiparticle Lie (super)algebras and further extensions

As explained in section 2, HS algebras in their conformal interpretation describe maximal symmetries of the space of single-particle states of free conformal field theory of some fields $\Phi$. We propose that a multiparticle extension of HS symmetry, that can serve as the symmetry algebra of the space $V_\Phi$ of all multiparticle states of fields $\Phi$, is provided by an appropriate real form of the Lie (super)algebra $l(M(H_{\Phi}))$. Specifically, we consider the Lie algebra $m_\nu(V_\Phi)$ which is the real form of $l(M(H_{\Phi}))$ singled out by the condition

$$\mathcal{S}(f) = f, \tag{5.1}$$

where $\mathcal{S}$ is the conjugation of $l(M(H_{\Phi}))$ induced via (3.37) by the conjugation $\sigma$ in the reality conditions (2.2) for $h_\nu(V_\Phi)$ (see [23, 3, 17]).

Indeed, in accordance with equation (3.112), $M(H_{\Phi})$ acts on the space $V_\Phi$ (3.107) of all multiparticle states of the field $\Phi$. Hence, the Lie algebra $l(M(H_{\Phi}))$ is the complexified symmetry algebra of $V_\Phi$, while $m_\nu(V_\Phi)$ represents the appropriate real symmetry of $V_\Phi$.

Real algebras $m_\nu(V_\Phi)$ and $m_{\nu/}(V_\Phi)$ are singled out by the same condition (5.1) from the complex Lie algebras $M^T(H_{\Phi})$ introduced in section 3.4.1 by virtue of the anti-automorphism $\rho$ used in (2.5) to single out subalgebras $h_\nu(V_\Phi)$ or $h_{\nu/}(V_\Phi)$ from $h_\nu(V_\Phi)$.

As a linear space,

$$H_V = \sum_{m,n=0}^\infty (V^m)^* \otimes V^n, \tag{5.2}$$

while

$$M(H_V) = m_\nu(V) = \sum_{n=0}^\infty \oplus \text{Sym} (V^* \otimes V) \otimes \ldots \otimes (V^* \otimes V). \tag{5.3}$$

$m_\nu(V_\Phi)$ and $m_{\nu/}(V_\Phi)$ are represented by the further (anti)symmetrization $(V^* \otimes V)_\rho$ of $V^* \otimes V$

$$m_\rho(V) = \sum_{n=0}^\infty \oplus \text{Sym} (V^* \otimes V)_\rho \otimes \ldots \otimes (V^* \otimes V)_\rho \tag{5.4}$$

upon the identification $V^* = \rho(V)$ expressing condition (2.4). Obviously, multiparticle extension of an HS algebra is not itself an HS algebra.

It is useful to use the following realization of multiparticle algebras. Consider $N$ copies of generators $t^\alpha_j$ of $A (\alpha = 1, \ldots, N)$ that satisfy

$$t^\alpha_i \star t^\beta_j = t^\beta_j \star t^\alpha_i, \quad t^\alpha_i \star t^\alpha_i = t^\alpha_i \star t^\alpha_i \quad \alpha \neq \beta. \tag{5.5}$$

Consider the subalgebra of the enveloping algebra of these relations spanned by polynomials $P(t^\alpha_I)$ that are symmetric under the group $S_N$ permuting different species $t^\alpha_I$, i.e.

$$P(t^\alpha_I) \in M_N : \quad T_{\alpha\beta} P(t) = P(t) T_{\alpha\beta} \tag{5.6}$$
where $T_{\alpha \beta}$ are generators of $S_N$, which exchange species $\alpha$ and $\beta$,
\begin{equation}
T_{\alpha \beta} = T_{\beta \alpha}, \quad T_{\alpha \beta} t^\beta_i = t^\alpha_i T_{\alpha \beta}
\end{equation}
(no summation over repeated indices). This algebra is isomorphic to $M_N(A)$ via the following identification:
\begin{align*}
\alpha_i & \in M_N(A): \sum_{\delta = 1}^N t^\delta_i, \\
\alpha_i \alpha_j & \in M_N(A): \frac{1}{2} \sum_{\delta \neq \beta = 1}^N t^\delta_i \bullet t^\beta_j, \\
\alpha_i \alpha_j \alpha_k & \in M_N(A): \frac{1}{6} \sum_{\delta \neq \beta \neq \gamma \neq \delta = 1}^N t^\delta_i \bullet t^\beta_j \bullet t^\gamma_k,
\end{align*}
etc. Being isomorphic to $M_N(A)$ as a linear space and having proper adjoint action of $l(A)$ generated by $t_i$, the resulting algebra is isomorphic to $M_N(A)$. For $N = \infty$, this construction gives $M(A)$. Obviously, $M_1(A) = A \oplus \mathbb{R}$.

Algebras $M_N(A)$ result from $M_N(A)$ via the factorization of the ideal (3.88). The unit element $e_c$ of $A$ is realized as $e_c = \sum e^\alpha_i$. In accordance with section 3.4, for $M_N(A)$ with finite $N$, the only possible value of $c$ is $N$ since, for any element $f_N$ of maximal degree $N$, $e_c \bullet f_N = N f_N$. As a result, the naive $N \to \infty$ limit of $M_N(A)$, via extension of the number of species of $t^\alpha_i$ to infinity may be problematic leading to divergent $c$. Again, this indicates that algebras $M_N(A)$ may have no direct physical application.

In fact, the difficulties with the extension of HS algebras discussed in the introduction resulted just from the condition that all species of oscillators in (1.4) have a common unit element. In other words, the difficulties were due to an attempt to use algebras $M_N(A)$. In the framework of algebras $M_N(A)$ and $M(A)$, oscillators of any sort (index $\alpha$) have their own unit elements $e^\alpha_i$. Single-particle states are realized as $N^{-1/2} \sum_{\alpha = 1}^N |\psi_\alpha\rangle$. In particular, the lowest energy single-particle state is represented by
\begin{equation}
N^{-1/2} \sum_{\alpha = 1}^N |0_\alpha\rangle,
\end{equation}
where $|0_\alpha\rangle$ represents the lowest energy state in the respective sector. It suffices to require
\begin{equation}
e^\beta_i |0_\alpha\rangle = \delta^\beta_\alpha |0_\alpha\rangle
\end{equation}
to achieve that the action of generators (1.4) on the state (5.9) remains the same as in the original $N = 1$ module, hence escaping the problem with lowest energies.

The linear space of $M(H_V)$ (5.3) represents the space of multiparticle states of the bulk HS theory which is different from the space of multiparticle states $\sum_{n=0}^\infty \mathbb{R} V^n$ of the boundary theory. In field-theoretical terms, algebra $M(H_V)$ and its associated Lie (super)algebra are well suited for the description of the bulk multiparticle theory that does not include boundary fields. It may, however, be useful to unify both types of fields in the same framework. For example, such a generalization should underlie the extension of the analysis of the AdS$_4$/CFT$_3$ HS correspondence of [74] to the case where boundary currents are built from the boundary conformal fields. This can be achieved via the following generalization of the proposed construction.

Consider the algebra $H_{H_V}$ spanned by elements
\begin{equation}
F \in H_{H_V}: \quad F = (f, |v\rangle, \langle v|, \phi) \quad f \in H_V, \quad |v\rangle \in V, \quad \langle v| \in V^*, \quad \phi \in \mathbb{R}
\end{equation}
with the product law
\[ F_1 * F_2 = (f_1 * f_2 + |v_1\rangle \langle v_2|, f_1 |v_2\rangle + \phi_2 |v_1\rangle, \langle v_1| f_2 + \phi_1 |v_2\rangle, \phi_1 \phi_2 + \langle v_1| v_2\rangle). \] (5.12)

\( H_{V'} \) can be interpreted as the algebras of endomorphisms of the space \( V' = V \oplus \mathbb{K} \).

The supertrace of \( H_V \) generates the supertrace of \( H_{V'} \):
\[ \text{str}_s F = \text{str}_s f + \phi \] (5.13)

provided that the inner product \( \langle | \rangle \) is defined via
\[ \langle v_1 | v_2 \rangle = (-1)^{p_1 + p_2} \text{str}_s (|v_2\rangle \langle v_1|). \] (5.14)

Note that such HS algebras were used in [95] for the description of 2D HS gauge theory.

Clearly, \( M(H_{V'}) \) contains all the combinations of multiparticle and multi-antiparticle states of the original boundary theory where \( V \) was the space of single-particle states. Since single-particle boundary states can be interpreted as singletons, the resulting construction is analogous to that of singleton strings discussed in [13, 14]. It seems to be most appropriate for the analysis of multiparticle amplitudes in the boundary theory.

Finally, multiparticle algebras admit further extension to brane-like symmetries via algebras \( M^n(A) \) defined inductively,
\[ M^{n+1}(A) = M(M^n(A)) \] (5.15)

with \( M^0(A) = A \) and \( M^1(A) = M(A) \). For the oscillator realization of \( A \) by functions of oscillators \( Y_A \), elements of \( M^n(A) \) are represented by functions \( f(Y) \) of \( Y_{i_1}^{\alpha_1}...i_n^{\alpha_n} \), endowed with \( p \) copies of indices \( i_k \) running from 0 to \( \infty \), such that \( f(Y) \) is symmetric with respect to permutation of \( Y_{i_1}^{\alpha_1}...i_n^{\alpha_n} \) for any \( k \). As the variables \( Y_{i_1}^{\alpha_1}...i_n^{\alpha_n} \) are reminiscent of the modes on a \( p \)-dimensional surface, the algebras \( M^n(A) \) are expected to be related to brane-like theories.

Note that, in the HS setup, the continuous spectrum difficulty in brane theory discovered in [96] is likely to be resolved in units of the background curvature. The \( p \)-brane algebras acting on hypothetical generalized membrane theories, introduced analogously to \( m_o(V) \), \( m_s(V) \) and \( m_{usp}(V) \), we call \( m'_0(V) \), \( m'_s(V) \) and \( m'_{usp}(V) \), respectively.

6. Conclusion

Extensions of HS algebras suggested in this paper are anticipated to underlie multiparticle extensions of HS gauge theories containing mixed symmetry fields associated with higher Regge trajectories of string theory. An important feature of multiparticle algebras, that differs them from the known HS algebras, is that they are not Lie (super)algebras of endomorphisms of unitary modules associated with one or another set of relativistic particles. This modification makes it possible to avoid difficulties of the naive extension of the construction of HS algebras to mixed symmetry HS fields.

Known HS algebras \( h_o(V) \) can be realized as the (matrix-valued) Weyl algebra for some set of oscillators \( Y_A \) and unit element \( e_s \) or a quotient of some of its subalgebra. The multiparticle algebra \( m_o(V) \) is realized by symmetric functions \( f_o(\tilde{Y}_1, \ldots, \tilde{Y}_n) \) of any number \( n = 0, 1, 2 \ldots \) of variables \( \tilde{Y}_a = (Y_A, e_s) \). The spaces \( F^n \) of functions \( f_o(\tilde{Y}_1, \ldots, \tilde{Y}_n) \) with various \( n \) are reminiscent of \( n \)th higher Regge trajectories in string theory. A spacetime symmetry algebra \( s \) belongs to both \( h_o(V) \) and \( m_o(V) \). Following [2], the idea is to try to formulate a multiparticle HS theory in terms of differential forms \( \Omega_s(\tilde{Y}_1, \ldots, \tilde{Y}_n) \) of various degrees \( p \geq 0 \).

Gravitational field is associated with the 1-forms valued in \( s \). The spacetime symmetry algebra \( s \subset h_o(V) \) can be extended to a larger finite-dimensional subalgebra of \( m_o(V) \). For
example, in the case where $H_V$ is Weyl algebra, one can consider the algebra spanned by two types of bilinears

$$f_1(Y) = f_{1AB} Y^A \star Y^B, \quad f_2(Y) = f_{2AB} Y^A Y^B,$$

where the symmetrized tensor product is replaced by the usual product. Various $f_1(Y)$ span $sp(2M)$, while $f_2(Y)$ extends it to $sp(2M) \oplus sp(2M)$ where the $sp(2M)$ spanned by $f_1(Y)$ is embedded diagonally. Hence there is more room for the choice of background fields in $m_\mu(V)$ than in $h_\mu(V)$.

The problem of increase of vacuum energies mentioned in the introduction does not occur because the $h_\mu(V)$-module $V \otimes V$, which describes symmetric massless fields, belongs to the $h_\mu(V)$-module $V$ which is also an $m_\mu(V)$-module.

In HS gauge theory, free massless fields are formulated in terms of gauge 1-forms valued in the adjoint representation of the HS algebra and 0-forms $C$ valued in the module often called the Weyl module since it contains the Weyl tensor in the spin 2 sector along with its HS generalizations. In the unfolded formulation, all degrees of freedom in the system are represented by 0-forms. Hence, the Weyl module treated as a module of the spacetime symmetry algebra $s$ is complex equivalent to a unitary module of single-particle states in the system.

The Weyl module is realized as the twisted adjoint module with respect to automorphism $\tau$ that changes the sign of translations in the AdS algebra, i.e.

$$\tau(L^{ab}) = L^{ab}, \quad \tau(P^a) = -P^a,$$

where $L^{ab}$ and $P^a$ are generators of Lorentz transformations and AdS translations, respectively. As explained in section 3.3, $\tau$ induces the $T$-twisted adjoint $m_\mu(V)$-module. Let us call it $C$.

As a linear space, it is isomorphic to the sum of all symmetrized tensor products of $C$:

$$C = \sum_{n=0}^{\infty} \oplus \text{Sym} C \otimes \cdots \otimes C.$$

As such, it should be complex equivalent to the space of all multiparticle states of the AdS HS theory (not to be confused with the space of multiparticle states of the boundary conformal theory). This implies that the extension of the HS theory based on $m_\mu(V)$ should describe all multiparticle states of the original HS theory, while $m_\mu(V)$ is a symmetry that acts on these states. Since $C$ is complex equivalent to a unitary module of the AdS$_d$ algebra $s$, its symmetrized tensor products and hence $C$ also do. This suggests that the algebra $m_\mu(V)$ respects the admissibility condition of [22].

To extend the construction to the full nonlinear multiparticle HS theory, it is necessary to extend the construction of [2] to various algebras $m_\mu(V)$ associated with HS algebras $h_\mu(V)$. Hopefully, solution to this problem can drive us to new understanding of a fundamental theory underlying both string theory and HS theory.

Another application of the multiparticle algebras is that they are expected to fix unambiguously the form of all correlators of conserved conformal currents of all spins. Indeed, as shown in [72, 75], the form of the current operator algebra for conserved conformal HS currents, and hence correlators, is unique for $d > 2$. This fact has been used in [80], where connected parts of $n$-particle correlators were found with the essential use of their covariance under HS symmetry which determines each of them up to a factor. The multiparticle algebra proposed in this paper relates $n$-particle correlators with different $n$. Hence, it should determine all $n$-particle correlators up to an overall coefficient.

One consequence of the analysis of this paper is that the number of constituent conformal fields $N$ is not an essential parameter of the multiparticle algebra. In other words, current
operator algebras with different \( \mathcal{N} \) are all equivalent, being associated with different basis choices in the same multiparticle algebra. This phenomenon is closely related to the enormous ambiguity in the choice of trace operation in the multiparticle algebra: the same multiparticle algebra can give rise to inequivalent \( \mathcal{N} \)-dependent \( n \)-point functions once the latter are expressed in terms of different \( \mathcal{N} \)-dependent trace operations. Surprisingly, conclusions of the analysis of multiparticle algebras in this paper differ essentially from what we used to in 2D conformal theory where \( \mathcal{N} \) does contribute to the central charge and cannot be removed by a basis change since central extension of the Virasoro algebra is nontrivial. The related point is that multiparticle algebras considered in this paper possess no nontrivial central extension. In fact, this raises an interesting problem of reformulation of 2D conformal field theory within the scheme proposed in this paper. At the present stage, this problem is not quite straightforward since the unfolded machinery, which maps the spacetime description to the twistor one used in this paper, has not been yet developed far enough for 2D conformal models (see, however, [95]).

Another interesting point is that, beyond a few distinguished bases leading to operator algebra of free currents with different \( \mathcal{N} \), there exist infinitely many bases where the form of current operator algebra and \( n \)-point functions do not respect the Wick theorem. This raises a question whether or not this opens a way toward the construction of non-free theories. Indeed, once the basis changes within \( M(A) \) relate free theories with different \( \mathcal{N} \), which are not equivalent as field theories, more general basis changes, most of which do not respect the Wick theorem, may generate non-free models. (Recall that unfolded equations map operators in the twistor space to conserved spacetime currents independently of their construction in terms of free fields; see [35].) The nontrivial part of the story is to check which of the resulting nonlinear theories are standard conformal theories in the sense that stress tensor (i.e. spin two current) has a standard OPE with other primary currents. We hope to consider this interesting issue elsewhere.

Acknowledgments

I am grateful to O Gelfond for stimulating discussions and useful comments and to P Sundell for the correspondence. This research was supported in part by RFBR grant no 11-02-00814-a.

References

[1] Vasiliev M A 1990 Phys. Lett. B 243 378
[2] Vasiliev M A 1992 Phys. Lett. B 285 225
[3] Vasiliev M A 1999 arXiv:hep-th/9910096
[4] Vasiliev M A Proc. Theory of Elementary Particles (Sellin, 1987) pp 234–52
[5] Gross D J 1986 Phys. Rev. Lett. 56 1229
[6] Bianchi M, Morales J F and Samtleben H 2003 J. High Energy Phys. JHEP07(2003)062 (arXiv:hep-th/0305052)
[7] Beisert N, Bianchi M, Morales J F and Samtleben H 2004 J. High Energy Phys. JHEP07(2004)058 (arXiv:hep-th/0405057)
[8] Bianchi M 2004 C. R. Phys. 5 1091 (arXiv:hep-th/0409292)
[9] Bianchi M and Didenko V 2005 arXiv:hep-th/0502220
[10] Lindstrom U and Zabzine M 2004 Phys. Lett. B 584 178 (arXiv:hep-th/0305098)
[11] Bonelli G 2003 Nucl. Phys. B 669 159 (arXiv:hep-th/0305155)
[12] Sagnotti A and Tsulaia M 2004 Nucl. Phys. B 682 83 (arXiv:hep-th/0311257)
[13] Engquist J and Sundell P 2006 Nucl. Phys. B 752 206 (arXiv:hep-th/0508124)
[14] Engquist J, Sundell P and Tamassia L 2007 J. High Energy Phys. JHEP02(2007)097 (arXiv:hep-th/0701051)
[15] Chang C-M, Minwalla S, Sharma T and Yin X 2012 (arXiv:1207.4485 [hep-th])
[16] Vasiliev M A 1989 Ann. Phys., NY 190 59
[64] Gaberdiel M R, Gopakumar R, Hartman T and Raju S 2011 J. High Energy Phys. JHEP08(2011)077 (arXiv:1106.1897 [hep-th])
[65] Chang C-M and Yin X 2012 J. High Energy Phys. JHEP10(2012)024 (arXiv:1106.2580 [hep-th])
[66] Gaberdiel M R and Vollenweider C 2011 J. High Energy Phys. JHEP08(2011)104 (arXiv:1106.2634 [hep-th])
[67] Jevicki A, Jin K and Ye Q 2011 J. Phys. A: Math. Theor. A 44 465402 (arXiv:1106.3983 [hep-th])
[68] Anninos D, Hartman T and Strominger A 2011 arXiv:1108.5735 [hep-th]
[69] Campoleoni A, Fredenhagen S and Pfenninger S 2011 J. High Energy Phys. JHEP09(2011)113 (arXiv:1107.0290 [hep-th])
[70] Kraus P and Perlmutter E 2011 J. High Energy Phys. JHEP11(2011)061 (arXiv:1108.2567 [hep-th])
[71] Anninos M, Kraus P and Perlmutter E 2012 J. High Energy Phys. JHEP07(2012)113 (arXiv:1111.3926 [hep-th])
[72] Maldacena J and Zhiboedov A 2011 arXiv:1112.1016 [hep-th]
[73] Gaberdiel M R, Hartman T and Jin K 2012 J. High Energy Phys. JHEP04(2012)103 (arXiv:1203.0015 [hep-th])
[74] Vasiliev M A 2012 arXiv:1203.5554 [hep-th]
[75] Maldacena J and Zhiboedov A 2012 arXiv:1204.3882 [hep-th]
[76] Gupta R K and Lal S 2012 J. High Energy Phys. JHEP07(2012)071 (arXiv:1205.1130 [hep-th])
[77] Alkalaev K B 2012 arXiv:1207.1079 [hep-th]
[78] Didenko V E and Skvortsov E D 2012 arXiv:1207.6786 [hep-th]
[79] Colombo N and Sundell P 2012 arXiv:1208.3880 [hep-th]
[80] Didenko V E and Skvortsov E D 2012 arXiv:1210.7963 [hep-th]
[81] Gaberdiel M R and Gopakumar R 2012 arXiv:1207.6697 [hep-th]
[82] Giombi S and Yin X 2012 arXiv:1208.4036 [hep-th]
[83] Jevicki A, Jin K and Ye Q 2012 arXiv:1212.5215 [hep-th]
[84] Fradkin E S and Vasiliev M A 1987 Ann. Phys. 177 63
[85] Vasiliev M A 1988 Fortschr. Phys. 36 33
[86] Fradkin E S and Vasiliev M A 1988 Int. J. Mod. Phys. A 3 2983
[87] Vasiliev M A 1987 Fortschr. Phys. 35 741
[88] Vasiliev M A 1987 Yad. Fiz. 45 1784
[89] Sezgin E and Sundell P 2001 J. High Energy Phys. JHEP09(2001)036 (arXiv:hep-th/0105001)
[90] Vasiliev M A 2001 Nucl. Phys. B 616 106 (arXiv:hep-th/0106200)
[91] Vasiliev M A 2003 Nucl. Phys. B 652 407 (erratum)
[92] Eastwood M G 2005 Ann. Math. 161 1645 (arXiv:hep-th/0206233)
[93] Fradkin E S and Linetshy V Y 1989 Mod. Phys. Lett. A 4 731
[94] Alkalaev K 2011 J. High Energy Phys. JHEP03(2011)031 (arXiv:1011.6109 [hep-th])
[95] Bekkaet X 2011 arXiv:1111.4554 [math-ph]
[96] Dixmier J 1974 Algebres enveloppantes (Paris: Gauthier-Villars)
[97] Vasiliev M A 1995 Phys. Lett. B 363 51 (arXiv:hep-th/9511063)
[98] de Wit B, Luscher M and Nicolai H 1989 Nucl. Phys. B 320 135