Exponential operator method for finding exact solutions of the propagator equation in the presence of a magnetic field

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Abstract. A general method of obtaining the exact analytical solutions of the propagator equations for charged particles in the presence of a constant magnetic field is developed. The method is applied to calculate the exact electron propagator in a strong magnetic field.

1. Introduction

Analyzes of elementary particle loop processes in extreme astrophysical conditions, such as strong magnetic fields, require a knowledge of exact particle propagators. There are known expressions for propagators of scalar, Dirac and massive vector fields in the presence of a constant magnetic field both in the coordinate and in the momentum spaces. In general, they are obtained either following the tedious Fock—Schwinger procedure or by first getting the exact solutions of the corresponding wave equation of interest followed by summation over the allowed quantum numbers. In this paper, we present a general method of obtaining the exact analytical solutions of the propagator equation based on the appropriate decomposition of the delta function into the sum/integral of the Hamiltonian-like operator eigenstates with the subsequent integration of the corresponding operator exponent in the proper time domain. When the parts of the operator exponent commute, one can decouple them from each other and apply each part separately to the delta function decomposition series/integral. This approach doesn’t require neither the preliminary steps of normalization and orthogonalization of the wave equation’s solutions nor their summation over the allowed quantum numbers, therefore, straightforwardly leading to the correct expression for the density matrix summed over polarization states.

2. Outline of the Fock—Schwinger approach

To obtain the expression of the propagator satisfying the equation:

\[ H\left(\partial_x, x\right)G(x, x') = \delta^{(4)}(x - x'), \]  

using the Fock—Schwinger method (see, e.g., Ref. [1]) one should stick to the following steps. First, the propagator \( G(x, x') \) is represented as an integral:

\[ G(x, x') = (-i) \int_{-\infty}^{0} d\tau U(x, x'; \tau), \]
where $\tau$ is called the Fock proper time. Considering $U(x, x'; \tau)$ as some sort of an evolution operator satisfying a Schrödinger-type equation

$$i \partial_\tau U(x, x'; \tau) = H(\partial_x, x) U(x, x'; \tau),$$

with the appropriate boundary conditions

$$U(x, x'; -\infty) = 0, \quad U(x, x'; 0) = \delta^{(4)}(x - x'),$$

one obtains the following expression:

$$U(x, x'; \tau) = \exp \left[ -i \tau H(\partial_x, x) \right] \delta^{(4)}(x - x').$$

Next, we represent (3) using (5) and the common notation for the bra-ket product in the coordinate space:

$$\langle x | A | x' \rangle = \int d^4 X \delta^{(4)}(X - x) A(X) \delta^{(4)}(X - x').$$

From that we get the following equation:

$$i \partial_\tau \langle x | e^{-i\tau H} | x' \rangle = \langle x | H e^{-i\tau H} | x' \rangle.$$

In some special cases it is possible to factor out a scalar function $F(x, x'; \tau)$:

$$i \partial_\tau \langle x | e^{-i\tau H} | x' \rangle = F(x, x'; \tau) \langle x | e^{-i\tau H} | x' \rangle,$$

making the equation easy to integrate:

$$U(x, x'; \tau) = \exp \left[ -i \int^\tau d\tau' F(x, x'; \tau') \right] C(x, x').$$

In general this factorization is possible if one is able to make use of the commutation relations in the Heisenberg picture. It is worth noting that for the problem of charged particles in the presence of a constant electromagnetic field it is definitely the case. The final expression for $G(x, x')$ will be a function in the coordinate space but often it is needed to consider propagators in the momentum space as a Fourier decomposition. At least two possible scenarios were found so far. First, we could make a Fourier transformation of the expression $G(x, x')$, which itself is a challenging task. To learn about existing techniques one could refer to Ref. [2] where the cases of scalar, massive vector and fermion fields were considered.

Another possibility, more straightforward and with clear physical meaning, is to construct the propagator from the exact solutions of the corresponding wave equation. Being computationally trivial in the scalar case this task becomes more time consuming when considering vector and fermion fields which have additional degrees of freedom. Complexity arises when one needs to evaluate wave function normalization, find the orthogonal set of solutions and to sum over the possible polarization states to obtain density matrix. If the wave function itself is not of a great interest, but rather the propagator, it is possible to skip those steps by going directly to the expression with no sign of individual polarization states left.

### 3. Exponential operator method

Looking at the expression (5) it becomes clear that the exchange of the operator-valued expression to the c-number function in the right-hand side is possible if one achieves to decompose $\delta$-function as a series/integral of the eigenstates of the operator $H(\partial_x, x)$, hence the name of the approach. Replacing $H$ in the exponent with its eigenvalue will result in the series/integral of
the same structure but with different coefficients (which are, obviously, the eigenvalues of $H$ for each eigenstate in the decomposition).

Let us consider a toy example, i.e. a free scalar field:

$(-\partial^2 - m^2)G(x, x') = \delta^{(4)}(x - x')$.  \hspace{1cm} (10)

Normally it is solved considering the translational symmetry, i.e. $G(x, x') = G(x - x')$, with the subsequent Fourier decomposition of both $G$ and the $\delta$-function. But actually this knowledge is not required if we apply the proposed approach with just decomposing the $\delta$-function and applying the operator exponent:

$G(x, x') = (-i) \int_{-\infty}^{0} d\tau e^{-i\tau(-\partial^2 - m^2 + i\epsilon)} \int \frac{d^4p}{(2\pi)^4} e^{-i(p(x-x'))}$. \hspace{1cm} (11)

From equation (11), it is clear that substitution $-\partial^2 \rightarrow p^2$ is possible. Finally, integration with respect to $\tau$ yields the well-known expression:

$G(x, x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i(p(x-x'))}}{p^2 - m^2 + i\epsilon}$. \hspace{1cm} (12)

From this example, a general idea of the exponential operator method could be seen: it is only required to know the form of the wave equation’s solution $\phi(x)$, in the above case, $\phi(x) \sim \exp(-i(px))$. The rest is achieved by appropriately choosing the $\delta$-function decomposition.

4. Electron in a magnetic field

The real utility of the method could be seen if we consider a somewhat harder problem, e.g. a charged fermion in the presence of a constant magnetic field. One could write the equation for the propagator of a fermion with charge $eQ$ ($e$ is an elementary charge) in the form:

$[(i\partial_{\mu} - eQA_{\mu})\gamma^\mu - m]G(x, x') = I\delta^{(4)}(x - x')$. \hspace{1cm} (13)

One of the standard procedures of manipulating the Dirac equation is to make it a second order equation. We apply the same trick here by representing $G$ as:

$G(x, x') = [(i\partial_{\nu} - eQA_{\nu})\gamma^\nu + m] S(x, x')$. \hspace{1cm} (14)

Using the properties of gamma matrices, we arrive at:

$H(\partial_x, x) S(x, x') = I \delta^{(4)}(x - x')$, \hspace{1cm} (15)

with the $H$ operator having the form:

$H(\partial_x, x) = (p_0^2 - p_z^2 - m^2 + \beta(d_2^2 - \xi^2)) I + Q\beta \Sigma_3$. \hspace{1cm} (16)

Here, the Landau gauge $A^\mu = (0, 0, Bx, 0)$ is chosen, and the standard notations are used:

$\beta = eB$, \hspace{1cm} $\xi = \sqrt{\beta} \left(x - Q \frac{p_y}{\beta}\right)$, \hspace{1cm} $\Sigma_3 = \text{diag}(+1, -1, +1, -1)$. \hspace{1cm} (17)

The following substitutions justified by the appropriate $\delta$-function decomposition (19) are also made: $i\partial_0 \rightarrow p_0$, $-i\partial_y \rightarrow p_y$, $-i\partial_z \rightarrow p_z$. We see that expression (16) is nothing but an operator, describing a set of four harmonic oscillators with energies satisfying the relation:

$p_0^2 = p_z^2 + m^2 + \beta(2n + 1) \pm Q\beta$. \hspace{1cm} (18)
Having the information about the form of the solution of the wave equation as an input to the exponential operator method, we therefore can choose the decomposition of the $\delta$-function as:

$$\delta^{(4)}(x - x') = \sqrt{3} \sum_{n=0}^{\infty} \int \frac{d^2 p_x dp_y}{(2\pi)^3} e^{-i(p(x-x'))_{n,y}} V_n(\xi)V_n(\xi') ,$$  \hfill (19)

where $\| \|$ stands for $t$ and $z$ components, and $V_n(\xi)$ are the harmonic oscillator eigenfunctions. The propagator reads then:

$$S(x, x') = (-1)\sqrt{3} \sum_{n=0}^{\infty} \int \frac{d^2 p_x dp_y}{(2\pi)^3} \int_{-\infty}^{0} d\tau e^{-ir\left[(p^2 - m^2 + i\varepsilon - \beta(2n+1)) + Q\beta \Sigma_3\right]} e^{-i(p(x-x'))_{n,y}} V_n(\xi)V_n(\xi').$$  \hfill (20)

In the exponent, the eigenvalue relation is used: $(d^2 - \xi^2)V_n(\xi) = -(2n+1)V_n(\xi)$. The identity matrix $I$ commutes with $\Sigma_3$, and we could split $e^{-ir\left[-\cdot\right]}$ into two exponents and evaluate them separately. In order to have the same expression in the exponent for all values of the propagator matrix, we should shift the summation for some elements. Finally, integrating out the exponent (assuming $Q = -1$) we get:

$$G(x, x') = \sqrt{3} \left( -i\partial_\nu + eA_\nu \right) \gamma^\nu + m \sum_{n=0}^{\infty} \int \frac{d^2 p_x dp_y}{(2\pi)^3} \frac{e^{-i(p(x-x'))_{n,y}}}{p^2 - m^2 - 2n\beta + i\varepsilon} V(\xi, \xi').$$  \hfill (21)

The matrix $V(\xi, \xi') = \text{diag}(V_{n-1}(\xi)V_{n-1}(\xi'), V_n(\xi)V_n(\xi'), V_{n-1}(\xi)V_n(\xi'), V_n(\xi)V_{n+1}(\xi'))$.

The rest of calculations consists in applying the $\left( -i\partial_\nu + eA_\nu \right) \gamma^\nu + m$ operator, carrying out the integration over $dp_y$ and computing the inverse Fourier transform. These steps are the same as if we were constructing the propagator from the exact solutions, see e.g. Ref. [2]. The main point here is that by using the exponential operator method we skipped the orthogonalization and normalization steps along with summation over polarizations and went straight to the correct expression of the density matrix.

5. Conclusion
This method is by no means a conceptually new way to obtain propagators. It is better to be considered as a shortcut to the whole computation procedure when the knowledge of the exact wave equation’s solutions is not necessary, so that one could skip orthogonalization and normalization procedures along with the further summation over the polarization states to get the density matrix. It is worth noting that this approach heavily relies on the knowledge of the appropriate $\delta$-function decomposition. The method could be of great benefit if one would apply it to problems with even higher number of degrees of freedom, therefore greatly reducing the computation time. We plan to demonstrate this in an extended paper, which is under preparation.

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References
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