1 Introduction

This article is about a class of special functions that cannot be expressed in terms of the classical elementary functions. Nevertheless, they arise in a wide range of applications and have surprisingly rich mathematical properties.

In this article, we describe some of these properties, which lie at the intersection of many directions in mathematics, including dynamical systems theory, differential or difference Galois theory and algebraic geometry. These intersections are expanded in further detail in lectures delivered at UCLA and interested readers may find it helpful to view the videos of the lectures at https://www.math.ucla.edu/dls/nalini-joshi.

Imagine two rigid rulers, one placed on top of the other. (See Figure 1) The height of the combined object is the sum of the heights of each individual ruler. This is an example of a linear system.

But whether we are observing the heights of colliding waves near a beach, the ebb and flow of infective cells in a patient, or massive fluctuations in gravitational fields arising from colliding black holes, the corresponding mathematical models turn out not to be linear — for example, the sum of travelling waves near a beach before collision is not their combined height at the time of collision. This article is about transcendental functions that solve nonlinear mathematical models arising in such applications.

The history behind these functions starts, like mathematics, with counting. Addition and subtraction of counting numbers lead to the integers. Multiplication and division lead to rational numbers. Solving polynomial equations with integer coefficients leads to algebraic numbers. But there are numbers called transcendental numbers that escape algebraic construction.

Transcendental numbers lie in places on the num-
ber line that are like regions in old maps filled with
drawings of fabulous monsters. Although the name
imbues them with a mystical aura, they arise every-
where in real life, as we know from the presence of the
number $\pi$ in almost every part of mathematics. This
article is about functions that play a similar role.

| Polynomial | $1/x$ | $x$ | $x^2$ |
|------------|-------|-----|-------|
| Rational   | $\sqrt{x}$ | $\frac{3+2x}{x}$ | $\sqrt{x}$ |
| Algebraic  | $\Gamma(x)$ | $J_\nu(x)$ | $\frac{1}{\sqrt{x}}$ |
| Transcendental | $e^x$ | | |

Figure 3: Function classes

To get to such functions, imagine replacing count-
ing numbers with polynomials. Multiplication and
division of polynomials lead to rational functions.
Solving polynomials of more than one variable — for
example, finding solutions $y(x)$ of

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x) = 0,$$

where $a_k(x)$ are polynomials of $x$ with integer coeffi-
cients — leads to algebraic functions. Functions that
are not algebraic are called transcendental functions.

Within this class, there are further levels of tran-
scedentality. At the base level sit familiar transcen-
dental functions, such as exponential, trigonometric
and logarithmic functions. But there are functions
that escape any algebraic operations applied to these
or their compositions.

The exploration of transcendental functions has a
long history, reaching back at least to Euler in 1755.
One enduring question is whether a newly discovered
function can be expressed in terms of earlier known
transcendental functions. As for transcendental num-
bers, the main question here is to ask whether there
exist algebraic relations between them. Derivatives
or differences of transcendental functions also play a
crucial role in answering such questions.

Each new level of transcendentality aims to capture
functions that escape the previous steps. There is a
hierarchy of known cases, which arise as solutions of
differential or difference equations. Functions in the
hierarchy are categorized according to whether or not
they result from operations called classical operations
applied to functions at a previous step.

This article focuses on currently known transcen-
dental functions that are furthest away from polyno-
imals, in particular, those at the top of the known
hierarchy of functions solving polynomial differential
or difference equations. They are solutions of certain
nonlinear difference or differential equations, called
the discrete Painlevé equations or Painlevé equations
[Inc56,Jos19].

To describe examples of such equations, we start
with basic notation. Difference equations act on the
ring $R$ of sequences $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ equipped with an
iteration operator $\sigma : R \to R$. For example, Euler’s
Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,$$

satisfies a difference equation, given by $\sigma(w(z)) =
z w(z)$, where $\sigma(w(z)) = w(z+1)$.

Each application of the iteration operator $\sigma$
corres-
dons to a shift $z \mapsto z + 1$ on a line. Given $a \in \mathbb{C}$,
more general shifts $z \mapsto z + a$ occur on a line and
we continue to denote the corresponding iteration by
$\sigma(w(z)) = w(z+a)$. But, there are more complicated
curves on which iterations can be also defined. Two
such types of curves and their associated iteration
operators have led to extensions of the Gamma func-
tion. One of these is $\sigma_q(w(z)) = w(qz)$, with $q \neq 0, 1,$
$q \in \mathbb{C}$, which iterates points on a spiral. The other
is $\sigma_{ell}$, which iterates points on a curve parametrized
by elliptic functions, called an elliptic curve [Cas91].
Given $a, b \in \mathbb{C}$, elliptic curves have a canonical cubic
form \( y^2 = 4x^3 - ax - b \). Assume the curve is nonsingular, i.e., \( a^3 \neq 27b^2 \). On such a curve, the iteration proceeds as follows: start with two points \( P_0, P_1 \) on the curve. Then the line passing through them intersects the curve again, producing a third point. The iteration of \( \sigma \) intersects the curve again, producing a third point. \( P_2 \) is the reflection of this third point across the horizontal axis, given by \( 2P_1 - P_0 \).

discrete Painlevé equations. There are 22 classes of such equations described by Sakai [Sak01] through an algebro-geometric study of dynamical systems. A concise list of canonical cases can be found in [Jos19, Appendix D]. Discrete Painlevé equations have striking mathematical properties. We illustrate them here for selected examples, which are listed below.

For conciseness, we write \( \sigma(w) = w, \sigma^{-1}(w) = w \), for all three types of iterations.

\[
\begin{align*}
dP_1 : & \quad w (w + w + w) = a n + b + c w, \\
qP_1 : & \quad w w = \frac{1}{w} - \frac{1}{a q^n w^2}, \\
cRCG : & \quad cn(\gamma_n)dn(\gamma_n)(1 - k^2 sn^2(z_n))wn(wn+1 + wn-1) \\
& \quad - cn(z_n)dn(z_n)(1 - k^2 sn^2(z_n)sn^2(\gamma_n)) \\
& \quad \cdot (wn+1wn-1 + wn^2) \\
& \quad + (cn^2(z_n) - cn^2(\gamma_n))cn(z_n)dn(z_n) \\
& \quad \cdot (1 + wn+2wn+1 + wn-1) = 0,
\end{align*}
\]

where

\[
z_n = (\gamma_c + \gamma_o)n + \omega, \quad \gamma_n = \begin{cases} \gamma_c, & \text{for } n = 2j, \\
\gamma_o, & \text{for } n = 2j + 1, \end{cases}
\]

with \( \gamma_c, \gamma_o, a, b, c, d, q \) being constants, with \( q \neq 0, 1 \) and where \( cn, dn, sn \) denote Jacobi elliptic functions, i.e., doubly periodic, meromorphic functions that parametrize biquadratic curves.

Consider, for example, the autonomous case of \( qP_1 \), which arises in the limit \( |q^n| \to \infty \):

\[
|\gamma w w| = 1. \quad (4)
\]

This equation has an invariant defined by

\[
K(x, y) = \frac{x^2 y^2 + x + y}{xy}, \quad (5)
\]

which satisfies \( K(\gamma w, w) - K(w, w) = 0 \) when \( w \) satisfies Equation (4). Given an initial value \( (w_0, w_1) \), the solution orbit, or trajectory, \( \{w_n\}_{n=0}^{\infty} \) lies on the curve defined by \( K(x, y) = \kappa := K(w_0, w_1) \). This leads to a one-parameter family of curves defined as the zero set of

\[
f(x, y) = x^2 y^2 + x + y - \kappa xy, \quad (6)
\]
which is called a pencil of curves [Gri89]. Each curve in this pencil can be mapped to an elliptic curve [Cas91]. A two-dimensional real slice of the pencil is illustrated in Figure 5.

The function $K$ can be defined by Equation (5) even when $z = q^n$ is not at infinity, but it is no longer invariant on solution trajectories. Instead, we have

$$K(w, w) - K(w, w) = -\frac{1}{z} \frac{w}{ \left( w - w \right) \left( w - 1/z \right)},$$

and so for arbitrarily large $|z|$, and bounded values of $w, w, w$, $K$ is slowly varying with $z$. A solution trajectory of $qP_1$ then moves from a point on a contour shown in Figure 5 to a point on another contour as $n$ is iterated. Such trajectories are like threads that link one contour diagram to another, as shown in Figure 6.

![Figure 6: A real solution trajectory of $qP_1$ threading through contour lines of $K$ defined in Equation (5).](image)

In the case of discrete Painlevé equations, each fibre of this foliated vector bundle is an elliptic surface, because the invariant contours of the corresponding autonomous system are elliptic curves. Given an initial value, the corresponding solution follows a trajectory that pierces each fibre. While such trajectories may be locally well defined, this is not guaranteed everywhere.

We illustrate this for the example of $qP_1$, which in system form is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} zx - 1 \\ z^2 y \end{pmatrix},$$

(7)

where $x = w, y = w$, and $z = aq^n$. When $x$ or $y$ are arbitrarily small, their iterates become unbounded, but this can be handled by embedding the system into two-dimensional projective space $\mathbb{P}^1 \times \mathbb{P}^1$. However, this is not sufficient to handle all problematic initial values, such as the point $b_0: (x, y) = (1/z, 0)$ where the right side of Equation (7) becomes undefined.

For the autonomous case (4), this is a base point, i.e., a point where all curves of the elliptic pencil intersect. An initial point on each such curve is transported along it with constant speed, like a car travelling on a road. But because all the curves intersect at the origin, there is a gridlock or traffic jam that prevents the cars on each line from moving through this point. The operation of blow-up or resolution, which was known to Newton [Abh76], redefines the roads through that impasse, so that each car can travel continuously through it on its journey. See Section 4 for further detail.

When all base points are resolved, the resulting space is called a space of initial values. There is a way to characterize this space uniquely for each Painlevé and discrete Painlevé equation. To do so, we have to understand the self-intersection numbers of lines. Each time a point is blown up on a line, the self-intersection number of that line decreases by 1. For example, a vertical line, denoted by $H_x$ (for constant $x$-value) does not intersect itself in $\mathbb{P}^1 \times \mathbb{P}^1$.

---

We have only written the forward iteration here for conciseness, but to be complete we need to consider both forward and backward iterations of $(x, y)$.
However, when a base point lying on $H_x$ is blown up, the self-intersection number of this line becomes $-1$. Two separate blow-ups on $H_x$ would lead to a self-intersection number of $-2$.

The resulting resolved initial-value space can be characterised by a reflection group, through an isomorphism known as Duval correspondence or McKay correspondence. Here lines of self-intersection number $-2$ play a special role - they are mapped to simple roots of reflection groups, with their intersections being mapped to inner products. An example of reflection groups is given in Section 3 and shown to give rise to $dP_1$ through a translation operator on the lattice generated by such a group.

There are many more properties than we can cover in this article. Each discrete Painlevé equation is a compatibility condition for a pair of associated linear problems (called Lax pairs) and Riemann–Hilbert-type methods applied to these linear problems provide information about the solutions. And, as remarked above, their generic solutions are “more transcendental” than any previously known special functions. (See Section 5.) In the following, we describe some of these distinctive properties, using the examples listed above for illustration. Further properties and examples are discussed in detail in [Jos19].

2 The beginnings

In 1939, Shohat [Sho39] discovered a curious relationship between Equation (1) and orthogonal polynomials.

Consider the monic Hermite polynomials $\Phi_n(x)$, where $\Phi_0(x) = 1$, $\Phi_1(x) = x$, $\Phi_2(x) = x^2 - 1$, ... From the orthogonality relations

$$\int_{-\infty}^{\infty} \Phi_n(x)\Phi_m(x)e^{-x^2/2}dx = \sqrt{2\pi n!}\delta_{nm},$$

where $\delta_{nm}$ is the Kronecker delta, we can deduce a difference equation for $\Phi_n(x)$ (commonly known as a 3-term recurrence relation)

$$\Phi_{n+1}(x) - x\Phi_n(x) + n\Phi_{n-1} = 0.$$

Shohat extended this class to include weight functions of the form $p(x) = \exp(-x^4/4)$ on $\mathbb{R}$ and obtained the recurrence relation

$$\Phi_n(x) - (x - c_n)\Phi_{n-1}(x) + \lambda_n\Phi_{n-2}(x) = 0,$$

for $n \geq 2$, where $c_n$ are independent of $x$, and deduced a nonlinear difference equation for $\lambda_n$. Extending the weight function further to $p(x) = \exp(-x^4/4 + tx^2)$, this equation for $\lambda_n$ becomes

$$\lambda_{n+2}(\lambda_{n+1} + \lambda_{n+2} + \lambda_{n+3}) = n + 2t\lambda_n.$$  

This difference equation is a special case of the discrete Painlevé equation [1] and is called the string equation in physics, due to its appearance in the Hermitian random matrix model of quantum gravity. This is one instance of many applications related to random matrix theory in which solutions of the Painlevé and discrete Painlevé equations play critical roles [For10].

Denote the solution of Equation (9) as $\lambda_n(t)$. We can also deduce a differential equation for $\lambda_n(t)$ as a function of $t$. It turns out to be one of the classical Painlevé equations [Inc50], which had an entirely different beginning.

In the late 19th century, Picard, Painlevé and other mathematicians were engaged in a search for new functions with properties that generalized those of elliptic functions. Their results led to six canonical classes of second-order nonlinear ODEs now called the Painlevé equations:

$$P_1: \quad w'' = 6w^2 + t,$$
$$P_2: \quad w'' = 2w^3 + tw + \alpha,$$
$$P_3: \quad w'' = \frac{w^2}{w} - \frac{w}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma w^3 + \delta w,$$
$$P_4: \quad w'' = \frac{3w^3}{2w} + 4tw^2 + 2(t^2 - \alpha)w - \beta^2/2w,$$
$$P_5: \quad w'' = \left( \frac{1}{2w} + \frac{1}{w-1} \right)w^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2w}((\alpha w^2 + \beta) + \gamma w^3 + \delta w + \delta w(w-1)w-1),$$
$$P_6: \quad w'' = \frac{1}{2}\left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t} \right)w^2.$$
The first difficulty in the search mounted by Painlevé was to overcome possible obstructions to analytic continuability created by movable singularities. To illustrate, consider

\[ y' + y^2 = 0 , \]

which has explicit solutions \( y(x) = 1/(x-x_0) \), for arbitrary \( x_0 \). (It also has an identically zero solution.) Such poles are said to be movable, because their locations are not fixed by the coefficients of the differential equations they satisfy, but instead they move as initial conditions change. To allow analytic continuation of all locally defined solutions, Painlevé imposed a necessary condition that all solutions should be single-valued around all movable singularities. In the modern literature, this condition has been restricted further to require that all movable singularities should be poles and the restricted definition is now widely called the Painlevé property \([Jos19]\).

Painlevé devised an ingenious idea to drive the search for ODEs that satisfy his eponymous property. This idea is based on creating a transformation of variables for a given ODE in the class

\[ y'' = F(y', y, t), \]

where \( F \) is rational in \( y' \) and \( y \) and analytic in a domain in \( t \). The transformation incorporates a parameter \( \alpha \in \mathbb{C} \) analytically in a domain around \( \alpha = 0 \). Necessary conditions then arise from investigating the limiting case \( |\alpha| \to 0 \) and identifying those cases that fail to have the Painlevé property. Because the transformed equation is analytic in \( \alpha \), this necessarily implies that the full equation \([10]\) also fails to have the required property.

But Painlevé’s first attempt at classifying ODEs had gaps that were identified and later filled by his student Gambier. Gambier also pointed out that an ODE identified independently by R. Fuchs was the most general case possible in the classification. Painlevé also gave sufficient conditions to prove that solutions of the ODEs in each class are globally analytically continuable \([4]\).

The proof that generic solutions are “new” transcendental functions is non-trivial. An essential step is to show that they cannot be expressed as solutions of algebraic equations with previously known functions as coefficients. Painlevé tackled this step by showing that a solution \( y \) and its derivative \( y' \) cannot satisfy an algebraic equation. We explain these arguments in Section 5. But further steps needed an extension of Galois theory for number fields to function fields involving derivatives or differences. There are now several proofs; see \([Ume07]\).

The geometric study of initial value spaces of the Painlevé equations began in the 1970s, in the French school led by Raymond Gérard in Strasbourg. This culminated in Okamoto’s construction of the compactified and resolved spaces of initial values \([Oka79]\) of the six Painlevé equations. In 1992, researchers recognized a difference equation arising in a model of quantum gravity as a discrete Painlevé equation. This stimulated developments \([Jos19]\) leading to Sakai’s geometric approach \([Sak01]\) resulting in 22 classes of discrete Painlevé equations.

## 3 Affine reflection groups

In this section, we describe an algebraic view of discrete Painlevé equations that provides a useful perspective of the structure and dynamics of their solutions. For any given differential (or difference) equation, an enduring question in mathematics is to search for its symmetries – i.e., transformations under which the equations remain invariant. For the

---

3 This is not necessarily a parameter in the Painlevé equations listed above.

4 There are now differing opinions about the validity of his method of proof \([Jos19] \S 1.2\).
discrete Painlevé equations, symmetries are given by affine reflection groups that are closely related to the structure of their initial value spaces.

When a differential (or difference) equation includes parameters, a symmetry has the following realization. Given a solution corresponding to one choice of parameters, can it be transformed to a solution of another copy of the equation with a different choice of parameters? We illustrate this idea here for $P_{IV}$. Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) denote sequences of parameters and \( w_n \) the corresponding solution of $P_{IV}$. Classical results show that there exist transformations:

\[
P_{IV}(w_n; t; \alpha_n, \beta_n) \mapsto P_{IV}(w_{n \pm 1}, t; \alpha_{n \pm 1}, \beta_{n \pm 1}),
\]

where

\[
\alpha_n = -\frac{n}{2} - \frac{c_0}{2} + \frac{3}{2} c_1 (-1)^n, \quad (11a)
\]

\[
\beta_n = n + c_0 + c_1 (-1)^n, \quad (11b)
\]

c_0, c_1 \text{ are constants, and the solutions are related by}

\[
\begin{align*}
2w_n w_{n+1} &= -w'_n - w_n^2 - 2tw_n + \beta_n, \quad (12a) \\
2w_n w_{n-1} &= w'_n - w_n^2 - 2tw_n + \beta_n. \quad (12b)
\end{align*}
\]

Adding the two equations (12) we get

\[
2w_n (w_{n+1} + w_n + w_{n-1}) = -4tw_n + 2\beta_n,
\]

a generalization of $dP_1$ with \( c = -2t, \ a = 1, \) and \( b = c_0 \). Okamoto showed that such transformations form an affine reflection group of type $A_2^{(1)}$, described below. Moreover, the iteration \( n \mapsto n \pm 1 \) in Equations (12) correspond to translations on the $A_2^{(1)}$ root lattice, as indicated in Figure 7.

To see how affine reflection groups arise, consider a symmetric system equivalent to $P_{IV}$:[4]

\[
\begin{align*}
f_0' &= f_0(f_1 - f_2) + \gamma_0, \\
f_1' &= f_1(f_2 - f_0) + \gamma_1, \\
f_2' &= f_2(f_0 - f_1) + \gamma_2,
\end{align*}
\]

where \( f_0 + f_1 + f_2 = t, \ \gamma_0 + \gamma_1 + \gamma_2 = 1, \) and primes indicate differentiation in \( t \). By eliminating \( f_0, \ f_2, \) it can be shown that $P_{IV}$ holds for $f_1$ with \( f_1 = -w/\sqrt{2}, \ t \mapsto \sqrt{2}t, \ \alpha = \gamma_0 - \gamma_2 \) and \( \beta = \pm \gamma_1 \).

Define operations \( s_0, \ s_1, \ s_2 \) and \( \pi \) on Equations (13) by

\[
\begin{align*}
s_i(\gamma_i) &= -\gamma_i, \quad s_i(\gamma_j) = \gamma_j + \gamma_i, \quad j = i \pm 1, \\
s_i(f_i) &= f_i, \quad s_i(f_j) = f_j \pm \gamma_i f_i, \quad j = i \pm 1, \\
\pi(\gamma_j) &= \gamma_{j+1}, \quad \pi(f_j) = f_{j+1},
\end{align*}
\]

for \( i, j \in \mathbb{N} \) (mod3). The operators \( s_i \) satisfy the Coxeter relations

\[
s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad i \in \mathbb{Z}/3\mathbb{Z},
\]

which generate the affine reflection group $W = \langle s_0, s_1, s_2 \rangle = A_2^{(1)}$. Each \( s_j, \ j \in \mathbb{Z}/3\mathbb{Z}, \) represents a reflection across a line in the lattice shown in Figure 7. The operator \( \pi \), called a diagram automorphism, acts in the following way:

\[
\pi^3 = 1, \quad \pi s_i = s_{i+1} \pi, \quad i = 0, 1, 2.
\]

Augmenting $W$ by $\pi$ leads to the extended group

\[
\tilde{W} = \langle s_0, s_1, s_2, \pi \rangle.
\]

These transformations are collectively called Bäcklund transformations of $P_{IV}$.
Given a triangle in the triangular lattice, there are three fundamental lines $\ell_0, \ell_1, \ell_2$ given by extending its edges. Given a point in the lattice, let $\alpha_j$ be given by the orthogonal distance from that point to $\ell_j, j = 0, 1, 2$. This leads to a natural coordinate system $(\alpha_0, \alpha_1, \alpha_2)$ for points on the lattice.

The lines $\ell_0, \ell_1, \ell_2$ also provide directions along which translations act. So there are three translations on the triangular lattice in Figure 7. One of these is given by $T = e^{\pi s_2 s_1}$. The remaining translations are equivalent to this one under conjugation by group operations.

The actions of $s_i, i = 0, 1, 2$, and $\pi$ on the coordinates $(\alpha_0, \alpha_1, \alpha_2)$ can be found explicitly by using Euclidean geometry. We find

$$T(\alpha_0) = \alpha_0 + 1, \ T(\alpha_1) = \alpha_1 - 1, \ T(\alpha_2) = \alpha_2,$$
and

$$T(f_1) = t - f_0 - f_1 - \frac{\alpha_0}{f_0}, \ T^{-1}(f_0) = t - f_0 - f_1 + \frac{\alpha_1}{f_1}.$$ Setting $x_n = T^n(f_1), y_n = T^n(f_0)$, we obtain a system of difference equations:

$$x_{n+1} = t - y_n - x_n - \frac{\alpha_0 + n}{y_n}, \quad (14)$$
$$y_{n-1} = t - y_n - x_n + \frac{\alpha_1 + n}{x_n}, \quad (15)$$

which contains dP1. This result illustrates a crucial point in the construction of discrete Painlevé equations: their iteration is given by translations on lattices generated by affine reflection groups.

### 4 Initial-Value Spaces

Solutions of dynamical systems follow curves in the space of all possible initial values. But the continuation of a given trajectory is not always guaranteed. In particular, trajectories can all intersect at problematic points called base points (shown on the horizontal plane in Figure 8). We explain how to disentangle them by using a procedure called blow-up or resolution in algebraic geometry.

Figure 8: The lines in the horizontal $(x, y)$-plane denote local solution trajectories all intersecting at the origin. The vertical axis represents the exceptional line $e$, which replaces the origin after a resolution (i.e., after the transformation in Equation (16) is applied). The bold lines intersecting with the vertical axis represent the resolution of the original solution trajectories.

Metaphorically, the solution trajectories are like roads that have a gridlock at the origin. Cars traveling on each road are stuck at the base point because their equations of motion are not defined at the intersection. The mathematical resolution of the gridlock acts by lifting each road to a different vertical height on $e$, before returning the car to the continuation of the road it was on originally.

Consider the case of trajectories on the horizontal plane in Figure 8. This plane represents $\mathbb{R}^2$ containing a one-parameter family of lines intersecting at the origin. A blow up of the origin $(x, y) = (0, 0)$ is achieved by taking new coordinates:

$$\begin{align*}
  x_1 &= \frac{x}{y}, \\
  y_1 &= y.
\end{align*} \quad (16)$$

The line $e = \{y_1 = 0\} \cup \{x_2 = 0\}$ plays a distinguished role and is called an exceptional line. It is given by the vertical axis in Figure 8.
Take, for example, an autonomous version of the first Painlevé equation, for a given constant $g_2$, in the form $y'' = 6y^2 - g_2/2$. Integration yields

$$f(x, y) = (y')^2 - 4y^3 - g_2y - g_3,$$

where $g_3$ is an arbitrary constant. The zeroes of $f$ form a Weierstrass cubic pencil, well known in the theory of elliptic curves [Cas91]. In homogeneous coordinates $[u : v : w] \in \mathbb{P}^2$, we have the homogeneous pencil

$$F[u, v, w; g_3] = wv^2 - 4u^3 - g_2uw^2 - g_3w^3 = 0,$$

where $y = u/w$, $y' = v/w$ are affine coordinates in the finite domain. All curves in the pencil intersect at $[u, v, w] = [0, 1, 0]$, which lies at infinity. Note that this is precisely where the solution $y(x)$ has a pole.

For the Weierstrass pencil, nine blow-ups are needed to regularize the space. Let the sequence of blow-ups be $\pi_i : X_i \to X_{i-1}$ of $p_i \in X_{i-1}$, with $X = X_9 \to X_8 \to \ldots \to X_0 = \mathbb{P}^2$. Each blow-up $\pi_i$ replaces a base point $p_i$ with an exceptional line $L_i$. At the end of the resolution procedure, we are left with divisor classes $L_0, \ldots, L_9$, which form a free $\mathbb{Z}$-module basis of the Picard group $\text{Pic}(X)$, equipped with an intersection form.

A parallel construction can be carried out for the Painlevé and discrete Painlevé equations. Each point in the regularized space is given by initial values of the equation at some time: $(w, w')$ for the case of ODEs or $(w, w)$ for difference equations. For this reason, Okamoto [Oka79] called the resulting regularized space $X$ for the Painlevé equations a space of initial values.

The case of $P_1$ leads to exceptional lines $L_0, \ldots, L_9$ that intersect pairwise as shown in Figure 9. Under the Duval or McKay correspondence, each $L_0, \ldots, L_8$ in Figure 9 is mapped to a node, with an edge joining a pair of nodes if the corresponding lines intersect. The resulting graph is the Dynkin diagram of $E_8^{(1)}$ shown in Figure 10.

![Figure 9: The intersection diagram of exceptional divisors arising from resolutions of $P_1$. The solid lines $L_0, \ldots, L_8$ are lines of self-intersection $-2$, while the dashed line $L_9$ is a line of self-intersection $-1$.](image1)

![Figure 10: The Dynkin diagram of $E_8^{(1)}$.](image2)

5 Ladder of transcendentality

In this section, we describe some of the arguments that led to a proof that solutions of Painlevé or discrete Painlevé equations are indeed new transcendental functions, leading to their position at the top of a ladder of transcendentality.

Earlier known transcendental functions that are used widely in applied mathematics include the exponential, Airy, Bessel, parabolic cylinder, hypergeometric, or elliptic functions. These are called classical special functions. Elliptic functions also satisfy addition formulas (see Figure 4(c)), which are nonlinear difference equations, while the Gamma function solves a linear difference equation. All equations
satisfied by such functions are polynomial in the dependent variable and its derivatives or differences.

Such examples laid the foundations of the theory of differential or difference polynomials. Consider any function \( y \in \mathbb{C}^n \) for any positive integer \( n > 2 \). An \((n+2)\)-variable polynomial

\[
P(x, y, y', \ldots, y^{(n)}) ,
\]

where \( y^{(k)} \) is the \( k \)-th derivative of \( y \), is called a differential polynomial. The solutions of \( P = 0 \) are said to be differentially algebraic. There is a natural extension to difference polynomials

\[
P(n, y_n, y_{n+1}, \ldots, y_{n+k}) ,
\]

where \( y_{n+j} \) is the \( j \)-th iteration of \( y_n \).

For example, \( y : x \mapsto e^x \) is a root of the differential polynomial \( P_0(x, y, y') = y' - y \) and \( y : z \mapsto \Gamma(z) \) is a root of the difference polynomial \( P_1(z, y, \sigma(y)) = \sigma(y) - zy \). Weierstrass’ elliptic function \( y = \wp(x) \), with \( y' = \wp'(x) \), is a root of the differential polynomial \( f \) given in Equation \((17)\).

Various approaches have been developed to extend Galois theory, used for number systems, to these settings. The technical nature of these developments put them outside the scope of this short article, but there are simpler heuristic arguments to show why general solutions of discrete Painlevé equations or Painlevé equations cannot be rational, algebraic or depend algebraically on classical transcendental functions.

We outline such an argument for solutions of the first Painlevé equation. Consider solutions \( w(t) \) of \( P_1 \) as functions of initial values. For any nonzero constant \( a \), replace \( w \) by \( u(z) = a^2 w(t) \), where \( z = at \), to get

\[
u_{zz} = 6u^2 + a^5 z .
\]

If \( w(t) \) were rational in its initial values, then \( u \) would also be rational. But, in the limit \( a \to 0 \), the solution becomes \( \wp(t - t_0, 0, g_3) \), for arbitrary \( t_0 \), which is not rational in \( t_0 \) and \( g_3 \). Therefore, \( w(t) \) cannot be a rational function of its initial values.

For the remainder of the argument, we need the fact that solutions of \( P_1 \) are meromorphic functions. Consider the possibility that \( P_1 \) admits a first integral, which is a polynomial in \( w \) and \( w' \). In other words, we have a polynomial

\[
P = w^n + Q_1(t, w)w^{n-1} + \ldots + Q_n(t, w) = 0 , \tag{18}
\]

where \( Q_i \), with \( i = 1, \ldots, n \), are polynomial in \( w \). This equation should hold in a neighbourhood of each pole \( t_0 \) of \( w \), and transforming to \( w = U^{-1} \), \( w' = -U'/U^2 = -V/U^2 \), in a sufficiently small such domain, we find that the equation becomes

\[
V^n - U^2Q_1(t, U^{-1}) V^{n-1} + \ldots + (-1)^n Q_n(t, U^{-1}) U^{2n} = 0 .
\]

The coefficients should be polynomial in \( U \). This means that \( \deg_u Q_j(t, w) \) should be at most \( 2j \).

On the other hand, replacing \( w \) and \( w' \) in Equation \((18)\) by \( w = a^{-2} u(z) \), \( w' = a^{-3} u_z(z) \), where \( z = z_0 + at \) and \( a \) is a constant, we obtain an equation of the form

\[
P = a^{-k} P_0(u, u_z) + O(a^{-k+1}) ,
\]

for some integer \( k \), where \( P_0 \) is a first integral of the equation \( u_{zz} = 6u^2 \). Therefore, it must have the form

\[
P_0 = b(u_z^2 - 4u^3)^d ,
\]

where \( b \) and \( d \) are constants. Putting this together with the above scaling, we find that \( k = 6d = 3n \).

The argument is completed by using the fact that the solutions of \( P_1 \) have movable double poles. Substituting the Laurent expansions of \( w \) and \( w' \) into \( P \), we find a contradiction to the requirement that \( P \) be polynomial in \( w \). Therefore, \( P \) cannot exist in the form assumed.

Now assume that \( w \) is an algebraic function of \( t \). Then it can be expanded in a Puiseux series

\[
w \sim \sum_{n=0}^{\infty} a_n t^{\rho - n} ,
\]

for some sector in the domain \( |t| \gg 1 \). If \( \rho \leq 0 \), then \( w, w', w'' \) would be finite at \( z = \infty \), which contradicts the fact that \( w \) solves \( P_1 \). On the other hand, if \( \rho > 0 \), then it follows from the equation \( P_1 \) that it
must be integer, but the term $w^2$ in the equation then introduces a larger term of order $O(z^{2\rho})$, which is not balanced by any other term. So the equation cannot be satisfied and, therefore, $w$ cannot be algebraic.

Finally, consider the possibility that $w$ is a classical transcendent, which satisfies an algebraic differential equation. Then by further differentiation if necessary, we can eliminate $w'$ and any higher derivatives to obtain a polynomial equation

$$P(t, w, w') = 0.$$ 

But this was shown to be impossible above. Therefore, $w$ cannot be a known transcendental function.

References

[Abh76] S. S. Abhyankar, *Historical ramblings in algebraic geometry and related algebra*, The American Mathematical Monthly 83 (1976), 409–448.

[Cas91] J.W.S. Cassels, *Lectures on Elliptic Curves*, London Mathematical Society Student Texts, vol. 24, Cambridge University Press, 1991.

[Fuc05] Richard Fuchs, *Sur quelques équations différentielles linéaires du second ordre*, Comptes Rendus de l’Academie des Sciences Paris 141 (1905), 555–558.

[For10] Peter J. Forrester, *Log-Gases and Random Matrices*, London Mathematical Society Student Texts, vol. 24, Cambridge University Press, 1991.

[Gam10] Bertrand Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes*, Acta Mathematica 33 (1910), 1–55.

[Gri89] Phillip Griffiths, *Introduction to algebraic curves*, Vol. 76, American Mathematical Society, 1989.

[Hö87] Otto Hölder, *Über die eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen*, Math. Ann. 28 (1887), 113.

[Inc56] Edward L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1956.

[Jos19] Nalini Joshi, *Discrete Painlevé Equations*, CBMS Regional Conference Series in Mathematics, vol. 131, American Mathematical Society, Providence, Rhode Island, 2019.

[Mil] John Milnor, *Foliations and foliated vector bundles*, Collected papers of John Milnor (John McCleary, ed.), Vol. IV, American Mathematical Society, Providence, Rhode Island, 2009, pp. 279–320.

[MS74] John Milnor and James D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, Princeton, NJ, 1974.

[NY99] Masatoshi Noumi and Yasuhiro Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*, Nagoya Mathematical Journal 153 (1999), 53–86.

[Oka79] Kazuo Okamoto, *Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé. espaces des conditions initiales*, Japanese Journal of Mathematics. New series 5 (1979), 1–79.

[Oka86] Kazuo Okamoto, *Studies on the Painlevé equations III Second and fourth Painlevé equations PII and PIV*, Math. Ann. 275 (1986), 221–255.

[Pai02] Paul Painlevé, *Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme*, Acta Mathematica 25 (1902), 1–85.

[Rub89] Lee A. Rubel, *A survey of transcendentally transcendental functions*, The American Mathematical Monthly 96 (1989), 777–788.

[Sak01] Hidetaka Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Communications in Mathematical Physics 220 (2001), 165–229.

[Sho39] Jacques Shohat, *A differential equation for orthogonal polynomials*, Duke Mathematical Journal 5 (1939), 401–417.

[Ume07] Hiroshi Umemura, *Invitation to Galois theory, Differential equations and quantum groups*, 2007, pp. 269–289.