Nonlocality breaks the relations between measures of quantum objectivity

Dario A. Chisholm,1 Luca Innocenti,1 and G. Massimo Palma1,2

1 Università degli Studi di Palermo, Dipartimento di Fisica e Chimica – Emilio Segrè, via Archirafi 36, I-90123 Palermo, Italy
2 NEST, Istituto Nanoscienze-CNR, Piazza S. Silvestro 12, 56127 Pisa, Italy

We show the existence of two different aspects of quantum objectivity, “redundancy” and “consensus”. Though used as synonyms in this context, we prove that they quantify different features of the emergence of classicality from quantum mechanics. We show that the two main frameworks to measure quantum objectivity, namely spectrum broadcast structure and quantum Darwinism, naturally emerge from these two notions. Furthermore, by analyzing explicit examples of nonlocal states, we highlight the potentially stark difference between the degrees of redundancy and consensus. In particular, this causes a break in the hierarchical relations between spectrum broadcast structure and quantum Darwinism. Our framework provides a new perspective to interpret known and future results in the context of quantum objectivity, which paves the way for a deeper understanding of the emergence of classicality from the quantum realm.

INTRODUCTION

Introduction — The emergence of classicality is arguably one of the oldest foundational open questions in quantum mechanics, and the quantum-to-classical transition still the subject of active research. Decoherence [1, 2] is often considered a promising mechanism to explain such transition [3, 4]: as quantum systems are never really isolated [1, 2], their surrounding environment “reads” the system, causing coherent superpositions to be replaced by statistical mixtures, and the corresponding loss of quantum features [4]. However, the theory of decoherence is not per se sufficient to directly explain some pivotal markers of classicality, such as the emergence of objectivity in quantum systems.

While an intrinsic property of classical systems, “objectivity” is usually understood in the quantum domain as consensus between observers. More specifically, given a quantum system \( S \) interacting with an environment \( E \) composed of a number of constituents \( E_i \), we say that the system is objective when observers performing independent measurements on different \( E_i \) reach a consensus about some property of the system [5]. This is only possible if, due to prior interactions between system and environment, the relevant information was encoded with high redundancy into the environment [6–9].

Summary of results — The notion of “consensus” is also often found in the literature, and is usually used as a synonym of redundancy. However, we show that in several situations these concepts ought to be carefully distinguished, and that redundancy is a necessary, but not sufficient, condition to achieve consensus. In particular, we show that in the presence of nonlocal information encoding, consensus and redundancy quantify significantly different features. In fact, even though some information about \( S \) might be redundantly encoded into the environment, it is possible that due to the nonlocality of such encoding, the observers might not be able to access it, and thus the redundancy might fail to realize into a corresponding amount of consensus. Such situations arise naturally whenever there are inter-environmental interactions [10–14]. While redundancy is an intrinsic property of a state, consensus is also a function of the particular scenario, and in particular of how the environment is distributed between observers.

To clarify the distinction between the superficially similar notions of redundancy and consensus, we propose a natural way to quantify them. We find that, remarkably, this approach leads naturally to the two main quantifiers of “quantum objectivity”, namely, spectrum broadcast structures (SBS) [15] and quantum Darwinism (QD) [16, 17]. More precisely, we find QD and SBS to quantify consensus and redundancy, respectively.

Finally, we focus on the established hierarchy between SBS and QD: while SBS implies QD [15, 18], the opposite is only true under the additional assumption of vanishing discord and strong independence [18, 19]. We show that this hierarchy does not hold in scenarios where consensus and redundancy are distinct, a notable example being highly nonlocal states. In particular, SBS states need not satisfy the QD condition.

MAIN RESULTS

Redundancy and consensus — To clarify the differences between the different aspects involved in the notion of quantum objectivity, we introduce here operative definitions of “redundancy” and “consensus” in quantum systems. Even though directly verifying these conditions might be hard in practice, we will show that they naturally reduce to QD and SBS in specific appropriate approximation regimes.

Definition of redundancy — The idea of redundancy is that information about the system is encoded into several independent environmental fractions. More precisely, we thus say that a state \( \rho \) has redundancy \( n \), and write \( \text{Red}(\rho) = n \), if \( n \) is the largest integer such that there is a
partitions $\mathcal{E} = \bigotimes_{i=1}^n \mathcal{E}_i$ such that $I_{\text{acc}}(S : \mathcal{E}_i) \simeq S(\rho_S)$ for all $i = 1, \ldots, n$, where $S(\rho_S)$ is the von Neumann entropy of $\rho_S$, and $I_{\text{acc}}(S : \mathcal{E}_i)$ is the accessible mutual information [20] between system and $i$-th environmental fraction $\mathcal{E}_i$. The accessible information is here defined as the mutual information between the probability distributions resulting from measuring $S$ and $\mathcal{E}_i$, maximized over all possible choices of measurements. We use the accessible, rather than the full quantum mutual information, to ensure that correlations are actually observable from outcome probabilities.

What redundancy does not quantify — This notion of redundancy is however unconcerned with the practical retrievability of the information. Even though information about a system can be encoded into its environment with high redundancy, actually accessing this information might require a careful partitioning of the environment, to avoid information being hidden in the correlations between different observers. To quantify the likelihood of actually observing information redundantly encoded into the environment, one ought to introduce the notion of consensus. While the redundancy is concerned with the maximum number of environmental fractions encoding information about $S$, the notion of consensus is also concerned with how hard it is to find partitions realizing a set amount of rendancy. More precisely, we want to define the consensus of a state $\rho$ as the largest number of observers that can, with high probability, retrieve information about the system from their measurement outcomes.

Definition of consensus — To this end, consider the probability $P(\rho, n)$ of finding a partition $\mathcal{E} = \bigotimes_{i=1}^n \mathcal{E}_i$ with redundancy $n$:

$$P(\rho, n) \equiv \text{Prob}(I_{\text{acc}}(S : \mathcal{E}_i) \simeq S(\rho_S), \ i = 1, \ldots, n),$$

where the probability effectively counts the number of partitions $\mathcal{E} = \bigotimes_{i=1}^n \mathcal{E}_i$ such that $\dim(\mathcal{E}_i) = \dim(\mathcal{E}_j)$ for all $i, j$ — and thus such that $\dim(\mathcal{E}_i) = f \dim(\mathcal{E})$ for some $f \in (0, 1)$. While in principle one could consider partitions with uneven fractions, we will focus on this simplest case, as it is the one usually considered in the topical literature. We can now define the “consensus” of $\rho$ as the largest $n$ such that $P(\rho, n) \simeq 1$. Thus $\rho$ has a degree of consensus $n$ iff, for any partition of the form $\mathcal{E} = \bigotimes_{i=1}^n \mathcal{E}_i$, each $\mathcal{E}_i$ is maximally correlated with $S$, and the correlation is fully accessible [21].

RELATIONS WITH KNOWN MEASURES OF OBJECTIVITY

Non-equivalence of redundancy and consensus — This formalization of the notions of redundancy and consensus makes it clear that they quantify distinct features of the emergence of quantum objectivity. In particular, it is possible to have high redundancy and low consensus, in situations where the redundant information encoding cannot be easily accessed by the observers without taking great care in how the environmental fractions are distributed. Furthermore, we will show here that the two main quantifiers of quantum objectivity used in the literature, SBS and QD, directly correspond to redundancy and consensus, respectively.

Redundancy vs SBS — An SBS state is one that admits a decomposition of the form

$$\rho_{\text{SBS}} = \sum_i p_i |i\rangle\langle i| \bigotimes_{j=1}^N R_j^i,$$

for some collection of states $R_j^i$ — referred to in this context as macrofractions [15] — such that $R_j^i R_j^i = \delta_{ii'}(R_j^i)^2$ for all $j$. Such a decomposition is defined with respect to a specific partition $\mathcal{E} = \bigotimes_{j=1}^N \mathcal{E}_j$, of the environment, $R_j^i \in \mathcal{E}_j$, and the conditions on $R_j^i$ ensure that different states of the system can be recovered from measurements in each $\mathcal{E}_j$. SBS states are thus always objective, provided observers are able to measure environmental fractions compatibly with the partitioning corresponding to the SBS structure. It follows that the redundancy of such a $\rho_{\text{SBS}}$ is precisely $N$, the largest number of macrofractions that allow to write the state as an SBS.

Consensus vs QD — QD defines “objectivity” via the quantum mutual information (QMI) between system $S$ and an environment fraction $\mathcal{E}_f$ of size $f \dim(\mathcal{E})$, where $f \in (0, 1)$ [16, 17]. For non-uniform environments, this QMI will depend on the choice of environmental fraction. In such instances, it is therefore warranted to consider the average QMI, defined as $\bar{I}(S : \mathcal{E}_f) \equiv \langle I(S : \mathcal{E}_f) \rangle$, where the average is taken with respect to all environmental fractions $\mathcal{E}_f$ [9, 22–24]. A state is then said to be objective according to QD if there is $f \in (0, 1)$ such that

$$\bar{I}(S : \mathcal{E}_f) \simeq S(\rho_S).$$
If $f$ satisfies eq. (3), then there are at least $1/f$ observers that can simultaneously agree on some property of the system. We can then quantify QD-objectivity as $1/f$ for the smallest such $f$. One further possible issue arising in the context of QD is the presence of quantum discord [25, 26] in these QMI. Namely, in some scenarios computing $I(S : E_f)$ might falsely overestimate the actual accessible correlations between system and environment, which are the correlations that can be observed via some suitable choice of measurement basis on the environmental fractions. These arguably are, for the purpose of objectivity, the correlations one is actually interested in, and therefore the use of accessible mutual informations might more accurately quantify the sought-after objectivity of states [19, 27].

We can now observe that QD-objectivity precisely corresponds to the degree of consensus previously introduced. In fact, if $P(\rho, n) \simeq 1$, then any macrofractioning of the environment of the form $E = \bigotimes_{i=1}^{N} E_i$ gives $I_{\text{acc}}(S : E_i) \simeq S(\rho S)$, and therefore the average accessible mutual information also satisfies $\bar{I}_{\text{acc}}(S : E_f) \simeq S(\rho S)$ with $1/f = n$. Vice versa, if $\bar{I}_{\text{acc}}(S : E_f) \simeq S(\rho S)$, then each $E_i$ gives $I_{\text{acc}}(S : E_i) \simeq S(\rho S)$ —note that $S(\rho S)$ is an upper bound for $I_{\text{acc}}(S : E_f)$— and thus $P(\rho, n) \simeq 1$.

**STATES HIGHLIGHTING THE DIFFERENCE BETWEEN CONSENSUS AND REDUNDANCY**

**nonlocal information encoding** — A notable class of states highlighting the differences between redundancy and consensus are highly nonlocal states. Nonlocal states are understood here as those in which some information about the system can only be recovered by means of nonlocal measurements on several environmental constituents, as pictorially represented in fig. 1. While it is usually the case that a single element does not hold enough information about the system, we define nonlocality in terms of fragility of such information encodings with respect to loss of pieces of the environment, and show that it is tightly related with the departures between redundancy and consensus. We focus in particular on the extreme cases, where losing even a single environmental constituent results in a severe degradation of the correlations.

We focus here on many-qubit states for simplicity. To have both high redundancy and maximal nonlocality of the information encoding, we require the state of the system to be maximally correlated with a number of maximally nonlocal environmental states of the form

$$|\text{GHZ}^{(k)}_{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes k} \pm |1\rangle^{\otimes k}) \in (\mathbb{C}^2)^{\otimes k}. \quad (4)$$

While these two states are fully distinguishable when all $k$ qubits are measured, they become completely indistinguishable if even only a single qubit is lost, as

$$\text{tr}_i(\text{GHZ}^{(k)}_{+}) = \text{tr}_i(\text{GHZ}^{(k)}_{-}), \quad \forall i = 1, \ldots, k. \quad (5)$$

The integer $k$ governs the degree of nonlocality of the states. Note that GHZ-like states are characterized by not being determined by their reduced density matrices. More explicitly, this means that for any pair of orthogonal states $|\psi\rangle, |\phi\rangle \in (\mathbb{C}^2)^{\otimes k}$ such that $\text{tr}_i(|\psi\rangle\langle\psi|) = \text{tr}_i(|\phi\rangle\langle\phi|)$ for all $i$, $|\psi\rangle$ and $|\phi\rangle$ are local unitary equivalent to $|\text{GHZ}^{(k)}_{\pm}\rangle$ [28, 29].

We thus consider system-environmental states $|\Psi\rangle_{k,N} \in (\mathbb{C}^2)^{\otimes Nk+1}$ of the form

$$|\Psi\rangle_{k,N} = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\text{GHZ}^{(k)}_{+}\rangle^{\otimes N} + |1\rangle \otimes |\text{GHZ}^{(k)}_{-}\rangle^{\otimes N}). \quad (6)$$

These states have degree of redundancy $N$. If furthermore $k = 1$, the information encoding is local, and we also have a degree of consensus $N$. In fact, in this case, there is only one possible macrofractioning of the environment — which is composed of $Nk = N$ qubits — into $N$ constituents, so that $P(\rho, N) = 1$.

However, as soon as $k > 1$, the two aspects of objectivity diverge. In such instances, the QMI depends on the way in which the $kN$ environmental qubits are partitioned between observers. We recognize in particular
two conditions that are sufficient to determine the QMI $I(S : E_i)$ corresponding to a given environmental fraction $\mathcal{E}_i$. The fist condition is whether at least one full GHZ state is in $\mathcal{E}_i$. The second condition is whether at least one qubit from each GHZ state is in $\mathcal{E}_i$. If neither condition is satisfied $I(S : E_i) = 0$; if either condition is satisfied $I(S : E_i) = S(\rho_S)$; finally, if both conditions are satisfied, then $I(S : E_i) = 2S(\rho_S)$. The corresponding four possible scenarios are pictorially represented in fig. 2. We refer to the SM for the full derivation of these quantities, where we also show that performing the same calculations for the accessible mutual information gives $I_{\text{acc}}(S : E_i) = S(\rho_S)$ in all cases except when neither condition is satisfied. With these results, computing average QMI and accessible mutual information reduces to a combinatorial problem.

In fig. 3 we give the values of $I_{\text{acc}}(S : E_f)$ as a function of $f$ for different values of $N$, for $k = 4$. As clear from these results, a large degree of redundancy does not guarantee an equal amount of consensus. Therefore, it could be possible that, on average, observers will not measure any intact macrofraction, and will be unable to recover information about the system, rendering the redundant information inaccessible. In particular, redundancy values of 40, 160 and 1280 correspond to consensus values of 1, 2 and 4. This choice of states thus highlights how nonlocal information encoding results in non-equivalence between redundancy and consensus. We also see that the consensus increases with the size of the environment, which is a consequence of the fact that for large $N$ the probability of selecting at least $k \ll N$ qubits belonging to the same macrofraction tends to unity. In the macroscopic limit, the system exhibits objectivity even in the worst-case scenario of maximally nonlocal information encoding. As discussed previously, it is possible to recover information about the system by measuring a single element from each GHZ — that is, reading each element of a column in fig. 2. However, in the macroscopic limit of $N \gg k$, the probability of picking such partition goes to zero.

**Different information encoding: W states.** — We have shown that maximally nonlocal information encoding is only possible for states LU equivalent to the GHZ state. Nonetheless, many entangled state also feature a high degree of nonlocality. One such standard example of three-qubit nonlocal states are W states [30]. These can be seen as an intermediate case between maximally local and maximally nonlocal information encoding. We then consider system-environment states of the form

$$\frac{1}{2} (|0\rangle \otimes |W_{+}^{(k)}\rangle^{\otimes N} + |1\rangle \otimes |W_{-}^{(k)}\rangle^{\otimes N}),$$

with $|W_{\pm}^{(k)}\rangle$ a pair of W-like orthogonal many-qubit states. In the case of $k = 3$, these read

$$|W_{+}^{(3)}\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

$$|W_{-}^{(3)}\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + \omega_3 |010\rangle + \omega_3^2 |100\rangle),$$

with $\omega_3 \equiv e^{2\pi i/3}$. These definitions are straightforwardly generalized to general $k$.

In this case, the mutual information between system and environmental fraction does not reduce to the four possible cases as with the GHZ state structure, as now it becomes relevant how many qubits are selected from each macrofraction. In any case, it is still possible to reduce the QMI calculation to a combinatorial, albeit more complex, problem. See the SM for the full calculation.

While the individual qubits still carry no information about the system, a partially surviving $|W_{+}^{(k)}\rangle$ state is slightly distinguishable from the $|W_{-}^{(k)}\rangle$ counterpart, meaning that an observer may be able to determine the state of the system without having access to a full macrofraction. For this reason we expect it to be easier to achieve state objectivity in this case compared to the maximally nonlocal one.

This is confirmed by fig. 4, where we compare the average QMI between W-state and GHZ-state encodings for a nonlocality degree of $k = 3$. We can see that the W-state encoding always results in a higher QMI compared with the GHZ case. In particular, redundancy values of 8, 32 and 128 correspond to consensus values of 2, 3 and 6 for W-states and 1, 2 and 3 for GHZ-states. In any case, even though a W-state encoding allows to reach state objectivity with a smaller environmental size, the qualitative behaviour and relative implications are the same as with the GHZ case.
CONCLUSIONS

We have shown that objectivity in quantum systems can be quantified as redundancy or consensus, two different notions of which we have given operative definitions. We further showed that these two notions can be quantified by means of SBS and QD, respectively. While redundancy and consensus often reduce to the same quantity, they can be very different, in particular in highly nonlocal scenarios, such as when the information is encoded in GHZ or W states. Our results highlight the role of nonlocality in the emergence of objectivity, and show that in general nonlocally encoded information tends to hinder the emergence of consensus. This is due to the fact that, even if redundancy is high, it will be difficult for observers to extract relevant information about the system. On the other hand, based on probabilistic considerations, the hindering effects of nonlocality is compensated by large environments, which suggests that, in the limit of macroscopic environments, high values of consensus are reached even with nonlocal information encoding.

Finally, we discussed how the difference between redundancy and consensus breaks the previously established relations between SBS and QD, since there are wide classes of states where being SBS does not necessarily imply QD. We note however, that said relation could be reestablished, provided that some extra care is taken whenever redundancy and consensus do not coincide: if a state is SBS for (on average) all environmental macrofractionings of a certain minimum size, then SBS would imply QD even in the nonlocal scenario. Our framework allows for a clearer interpretation of results obtained using quantifiers such as SBS and QD, thus paving the way for a more thorough understanding of the different aspects of quantum objectivity, and thus the emergence of classicality from the quantum world.

[1] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2007).
[2] A. Rivas and S. F. Huelga, Open quantum systems, Vol. 10 (Springer, 2012).
[3] M. Schlosshauer, Decoherence, the measurement problem, and interpretations of quantum mechanics, Reviews of Modern Physics 76, 1267 (2005).
[4] W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Reviews of Modern Physics 75, 715 (2003).
[5] H. Ollivier, D. Poulin, and W. H. Zurek, Objective Properties from Subjective Quantum States: Environment as a Witness, Physical Review Letters 93, 220401 (2004).
[6] M. A. Ciampini, G. Pinna, P. Mataloni, and M. Paternostro, Experimental signature of quantum Darwinism in photonic cluster states, Physical Review A 98, 020101 (2018).
[7] M.-C. Chen, H.-S. Zhong, Y. Li, D. Wu, X.-L. Wang, L. Li, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, Emergence of classical objectivity of quantum Darwinism in a photonic quantum simulator, Science Bulletin 64, 580 (2019).
[8] T. Unden, D. Louzon, M. Zwołak, W. Zurek, and F. Jelezko, Revealing the Emergence of Classicality Using Nitrogen-Vacancy Centers, Physical Review Letters 123, 140402 (2019).
[9] D. A. Chisholm, G. García-Pérez, M. A. C. Rossí, S. Mancis, and G. M. Palma, Witnessing objectivity on a quantum computer, Quantum Science and Technology 7, 015022 (2021).
[10] E. Ryan, M. Paternostro, and S. Campbell, Quantum Darwinism in a structured spin environment, Physics Letters A 416, 127675 (2021).
[11] G. L. Giorgi, F. Galve, and R. Zambrini, Quantum Darwinism and non-Markovian dissipative dynamics from quantum phases of the spin-1/2 XXS model, Physical Review A 92, 022105 (2015).
[12] N. Mirkin and D. A. Wisniacki, Many-Body Localization and the Emergence of Quantum Darwinism, Entropy 23, 1377 (2021).
[13] N. Milazzo, S. Lorenzo, M. Paternostro, and G. M. Palma,
Role of information backflow in the emergence of quantum Darwinism, Physical Review A 100, 012101 (2019).

[14] T. P. Le, A. Winter, and G. Adesso, Thermality versus Objectivity: Can They Peacefully Coexist?, Entropy 23, 1506 (2021).

[15] R. Horodecki, J. K. Korbicz, and P. Horodecki, Quantum origins of objectivity, Physical Review A 91, 032122 (2015).

[16] R. Blume-Kohout and W. H. Zurek, Quantum Darwinism: Entanglement, branches, and the emergent classicality of redundantly stored quantum information, Physical Review A 73, 062310 (2006).

[17] W. H. Zurek, Quantum Darwinism, Nature Physics 5, 181 (2009).

[18] J. K. Korbicz, Roads to objectivity: Quantum Darwinism, Spectrum Broadcast Structures, and Strong quantum Darwinism – a review, Quantum 5, 571 (2021).

[19] T. P. Le and A. Olaya-Castro, Strong Quantum Darwinism and Strong Independence are Equivalent to Spectrum Broadcast Structure, Physical Review Letters 122, 010403 (2019).

[20] J. Preskill, Quantum shannon theory, arXiv preprint arXiv:1604.07450 (2016).

[21] It is possible to make the definitions of consensus and redundancy more mathematically rigorous by interpreting the equations $I_{\text{acc}}(S : E_i) \simeq S(\rho_S)$ and $P(\rho, n) \simeq 1$ as standing for $I_{\text{acc}}(S : E_i) \geq S(\rho_S) - \epsilon$ and $P(\rho, n) \geq 1 - \delta$, respectively, for suitable choices of thresholds $\epsilon$ and $\delta$.

[22] R. Blume-Kohout and W. H. Zurek, A Simple Example of “Quantum Darwinism”: Redundant Information Storage in Many-Spin Environments, Foundations of Physics 35, 1857 (2005).

[23] T. P. Le and A. Olaya-Castro, Objectivity (or lack thereof): Comparison between predictions of quantum Darwinism and spectrum broadcast structure, Physical Review A 98, 032103 (2018).

[24] G. García-Pérez, D. A. Chisholm, M. A. C. Rossi, G. M. Palma, and S. Maniscalco, Decoherence without entanglement and quantum Darwinism, Physical Review Research 2, 012061 (2020).

[25] H. Ollivier and W. H. Zurek, Quantum Discord: A Measure of the Quantumness of Correlations, Physical Review Letters 88, 017901 (2001).

[26] H. M. Wiseman, Quantum discord is Bohr’s notion of non-mechanical disturbance introduced to counter the Einstein–Podolsky–Rosen argument, Annals of Physics 338, 361 (2013).

[27] T. P. Le and A. Olaya-Castro, Witnessing non-objectivity in the framework of strong quantum Darwinism, Quantum Science and Technology 5, 045012 (2020).

[28] S. N. Walck and D. W. Lyons, Only $n$-Qubit Greenberger-Horne-Zeilinger States Are Undetermined by Their Reduced Density Matrices, Physical Review Letters 100, 050501 (2008).

[29] S. N. Walck and D. W. Lyons, Only $n$-qubit Greenberger-Horne-Zeilinger states contain $n$-partite information, Physical Review A 79, 032326 (2009).

[30] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A 62, 062314 (2000).
SUPPLEMENTARY CALCULATIONS

Here we show how to obtain the average mutual information between system \( S \) and environmental fraction \( E_f \), when the global system-environment state is the one in eq. (6) (information encoded in GHZ-like states) and eq. (7) (information encoded in W-like states). In both cases, we will first have to compute all possible values of \( I(S : E_f) \), which depend on the environmental constituents making up \( E_f \). We will then make combinatorial considerations to compute the probability associated with each \( I(S : E_f) \) value, thus computing the averaged mutual information. The obtained results correspond to the ones shown in fig. 3 and fig. 4.

MUTUAL INFORMATION FOR GHZ ENCODINGS

Here we consider all possible states resulting from tracing out environmental components from the global state, when information about the system is encoded in GHZ-like states, and compute the QMI between the system and the resulting environmental fraction. We are interested in computing the accessible information, but we will see how, in this specific case, we can easily recover the accessible information by computing the QMI first. The global state is the same as in eq. (6)

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |+\rangle^{\otimes N} + |1\rangle \otimes |\rangle^{\otimes N} \right),
\]

(S.1)

where \( |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes k \pm |1\rangle \otimes k) \) are the same as in eq. (4), the GHZ states into which the information is encoded.

Depending on which qubits are traced out from the global state, four different scenarios may arise, which we will analyse individually.

First case

So long as we trace out entire GHZ states, so that only \( fN \) remain \((f \in [0,1])\), the resulting state is the following:

\[
\rho_{SE_f^{(0)}} = \frac{1}{2} \left( |0\rangle \langle 0| \otimes |+\rangle^{\otimes fN} + |1\rangle \langle 1| \otimes |\rangle^{\otimes fN} \right),
\]

(S.2)

whose entropy is 1 bit. If we trace out the system as well we obtain the following

\[
\rho_{E_f^{(0)}} = \frac{1}{2} \left( |+\rangle \langle +|^{\otimes fN} + |\rangle \langle |^{\otimes fN} \right),
\]

(S.3)

whose entropy is also 1. Hence the mutual information is \( I(S : E_f^{(0)}) = 1 \).

Let’s consider the case in which we trace out only some qubits from a GHZ state. We can rewrite eq. (S.2) as follows

\[
\rho_{SE_f^{(0)}} = \frac{1}{2} \left( |0\rangle \langle 0| \otimes |+\rangle^{\otimes fN-1} \otimes |\rangle^{\otimes fN} + |1\rangle \langle 1| \otimes |\rangle^{\otimes fN} \right),
\]

(S.4)

and after tracing out \( m \) ancillae from the same group \((m < k)\) we obtain

\[
\rho_{SE_f^{(0)}} = \frac{1}{4} \left( |0\rangle \langle 0| \otimes |+\rangle^{\otimes fN-1} + |1\rangle \langle 1| \otimes |\rangle^{\otimes fN-1} \right) \otimes \left( |0\rangle \langle 0| \otimes k-m + |1\rangle \langle 1| \otimes k-m \right).
\]

(S.5)

This is the tensor product of two states of entropy 1, meaning it itself has entropy 2. Similarly after tracing out the system we have

\[
\rho_{E_f^{(0)}} = \frac{1}{4} \left( |+\rangle \langle +|^{\otimes fN-1} + |\rangle \langle |^{\otimes fN-1} \right) \otimes \left( |0\rangle \langle 0| \otimes k-m + |1\rangle \langle 1| \otimes k-m \right),
\]

(S.6)

also of entropy 2. The procedure can be generalized: so long as we trace individual qubits from a complete GHZ state, we increase the entropy of \( SE_f^{(0)} \) and \( E_f^{(0)} \) by one. The mutual information is \( I(S : E_f^{(0)}) = 1 \) as long as there is still an intact GHZ state remaining.
Second case

The previous results correspond to the case where at least one GHZ state is entirely traced out, and that at least one GHZ state remained intact. Let’s now consider the case where we trace out a single qubit from the global (pure) state. The initial state is:

\[
\rho_{SE} = \frac{1}{2} \left[ \left| 0 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-1} \otimes \left| \bar{+} \right\rangle + \left| 1 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-1} \otimes \left| \bar{-} \right\rangle \right] \left[ \left| 0 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-1} \otimes \left| \bar{+} \right\rangle + \left| 1 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-1} \otimes \left| \bar{-} \right\rangle \right],
\]

(S.7)

by tracing out a single environmental qubit we obtain

\[
\rho_{SE}^{(1)} = \frac{1}{2} \left[ \left| 0 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-1} + \left| 1 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-1} \right] \left[ \left| 0 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-1} + \left| 1 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-1} \right] \left| 0 \right\rangle \left\langle 0 \right| \left( \rho_{k-1} \right),
\]

(S.8)

this is a state on entropy 1. If we trace out the system as well we obtain

\[
\rho_{SE}^{(1)} = \frac{1}{4} \left[ \left| \bar{+} \right\rangle \left\langle \bar{+} \right|^{\otimes N-1} + \left| \bar{-} \right\rangle \left\langle \bar{-} \right|^{\otimes N-1} \right] \left[ \left| 0 \right\rangle \left\langle 0 \right| \left( \rho_{k-1} \right) + \left| 1 \right\rangle \left\langle 1 \right| \left( \rho_{k-1} \right) \right],
\]

(S.9)

that instead has entropy 2, meaning that the mutual information is \( I(S : E_f^{(1)}) = 2 \) even if the global state is not pure.

We write the generic state in the case where \( n \) qubits, pertaining to different GHZ groups, have been traced out

\[
\rho_{SE}^{(1)} = \frac{1}{2^n} \sum_{a_1=0}^{1} \cdots \sum_{a_n=0}^{1} \left[ \left| 0 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-n} + \left( - \sum a_i \right) \left| 1 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-n} \right] \left[ \left| 0 \right\rangle \otimes \left| \bar{-} \right\rangle^{\otimes N-n} + \left( - \sum a_i \right) \left| 1 \right\rangle \otimes \left| \bar{+} \right\rangle^{\otimes N-n} \right] \left( a_i \right) \left( a_i \right).
\]

(S.10)

Since all the terms in the sum are orthogonal with one another, we have a statistical mixture of \( 2^n \) states, all with equal probability, so that the entropy of this state is \( n \). By tracing out the system we obtain

\[
\rho_{SE}^{(1)} = \frac{1}{2^{n+1}} \left[ \left| \bar{+} \right\rangle \left\langle \bar{+} \right|^{\otimes N-n} + \left| \bar{-} \right\rangle \left\langle \bar{-} \right|^{\otimes N-n} \right] \sum_{a_1=0}^{1} \cdots \sum_{a_n=0}^{1} \left( a_i \right) \left( a_i \right),
\]

(S.11)

that instead has entropy \( n + 1 \). This means that even as we trace out environmental qubits, as long as we make sure that no GHZ state is traced out entirely, the mutual information between system and environment is \( I(S : E_f^{(1)}) = 2 \).

Third case

We can also consider the scenario when qubits are traced out from all GHZ states but no GHZ state is traced out entirely, in this case the resulting state is

\[
\rho_{SE}^{(2)} = \frac{1}{2^n} \sum_{a_1=0}^{1} \cdots \sum_{a_n=0}^{1} \left[ \left| 0 \right\rangle + \left( - \sum a_i \right) \left| 1 \right\rangle \right] \left[ \left| 0 \right\rangle + \left( - \sum a_i \right) \left| 1 \right\rangle \right] \left( a_i \right) \left( a_i \right),
\]

(S.12)

of entropy \( N \), and when tracing out the system we obtain

\[
\rho_{SE}^{(2)} = \frac{1}{2^{N+1}} \sum_{a_1=0}^{1} \cdots \sum_{a_n=0}^{1} \left( a_i \right) \left( a_i \right),
\]

(S.13)

Also of entropy \( N \), meaning that in this case the mutual information is \( I(S : E_f^{(2)}) = 1 \).

Fourth case

There is only one scenario we have yet not considered, the case in which at least one qubit has been traced out from each GHZ state, and at least one GHZ state has been traced out entirely. in this case system and environment are in a tensor product, meaning that there are no correlations and that the mutual information is \( I(S : E_f^{(3)}) = 0 \):

\[
\rho_{SE}^{(3)} = \frac{1}{2^{n+1}} \left( \left| 0 \right\rangle \left\langle 0 \right| + \left| 1 \right\rangle \left\langle 1 \right| \otimes \left( \left| 0 \right\rangle^{\otimes k-m} + \left| 1 \right\rangle^{\otimes k-m} \right)^{\otimes n}.
\]

(S.14)
For simplicity, we have considered the case where the global state is in an equal weight superposition between $|0\rangle \otimes |\tilde{\psi}\rangle^\otimes N$ and $|1\rangle \otimes |\tilde{\psi}\rangle^\otimes N$. All the calculations remain correct even for different superposition weights, as long as the resulting mutual information is not measured in bits, but in units of the system entropy.

Here we have computed the quantum mutual information. If one is interested in the accessible mutual information, it is worth noting how, whenever the quantum mutual information between system and environment is 1, the corresponding states are classical-classical, meaning that the accessible information is also 1. This also implies that when the quantum mutual information is 2, the accessible information is 1.

**COMBINATORIAL COMPUTATION FOR GHZ ENCODINGS**

Following the considerations in the previous section, we can see that, given a specific environmental fraction $E_f$, there are two conditions that define the mutual information $I(S : E_f)$.

- **Condition A:** $E_f$ contains at least one complete GHZ state
- **Condition B:** $E_f$ contains at least one qubit from each GHZ state

If only one of the two conditions is satisfied, then $I(S : E_f) = S(\rho_S)$. If both conditions are satisfied then $I(S : E_f) = 2S(\rho_S)$. If neither condition is satisfied, then $I(S : E_f) = 0$. The probabilities associated with the two conditions are independent from one another, and can thus be computed separately.

We will think of the environment as being made of boxes, each box can contain at most $k$ qubits, and the total number of boxes is $N$. We randomly fill the boxes with $m$ qubits, where $m$ is the size of the environmental fraction. Condition A corresponds to the case where at least one box is completely full. Condition B correspond to the case where no box is empty. The issue now is understanding the probabilities of the boxes being empty or full when we randomly fill them with qubits.

We wish to know how many configurations correspond to having at least one box full, $N_A$. We assume that we entirely fill one box beforehand, and distribute the remaining qubits in the remaining free slots. The number of possible combinations is $\binom{Nk}{m}$, but we must also take into account that the initial choice of the full box was arbitrary, so that the total number of combinations is $N \binom{Nk}{m}$. However, this corresponds to an over-counting, because we are counting multiple times the cases where more than one box is completely filled. We correct this over-counting by subtracting all the possible combinations where two boxes are completely filled beforehand, keeping in mind that this also corresponds to an over-counting of the number of events with more than two boxes completely filled, and so on. The total number of configurations that fulfill condition A results in a sum of alternating sign, that reads

$$N_A = \sum_{j=1}^{N} (-1)^{1+j} \binom{N}{j} \binom{Nk - jk}{m - jk}.$$  \hfill (S.15)

The total number of possible configurations is simply $N = \binom{Nk}{m}$, so that the probability that condition A is fulfilled is $p_A = \frac{N_A}{N}$.

Instead of computing the number of configurations where none of the boxes are empty, $N_B$, it’s easier to compute the number of configurations where there is at least an empty box, $N_{\overline{B}}$, by following a similar reasoning to the previous case: we keep an entire box empty, and then distribute the qubits in the remaining boxes, also in this case we need to take care of the resulting over-counting, leading to

$$N_{\overline{B}} = \sum_{j=1}^{N} (-1)^{1+j} \binom{N}{j} \binom{Nk - jk}{m}.$$  \hfill (S.16)

Since we are interested in the number of configurations where none of the boxes are empty, we have that $N_{\overline{B}} = N - N_{\overline{B}}$, and the probability that condition B is fulfilled is $p_B = \frac{N_{\overline{B}}}{N}$.

Based on these results we can finally compute the averaged mutual information between the system $S$ and an environmental fraction $E_f$ as $\bar{I}(S : E_f) = (p_A + p_B)S(\rho_S)$.

If we want to compute the averaged accessible information $I_{\text{acc}}(S : E_f)$, we should take into account that when both conditions A and B are satisfied, the accessible information is only $I_{\text{acc}}(S : E_f) = S(\rho_S)$, therefore

$$\bar{I}_{\text{acc}}(S : E_f) = (p_A + p_B - p_B p_A)S(\rho_S).$$  \hfill (S.17)
MUTUAL INFORMATION FOR W ENCODINGS

Here we consider all possible states resulting from tracing out environmental components from the global state, when information about the system is encoded in W-like states, and compute the QMI between the system and the resulting environmental fraction. In the case of W encodings, it is not trivial to compute the accessible information, we will therefore compute the QMI, using it as an upper bound for the accessible information.

For simplicity, we will focus on the case of nonlocality degree $k = 3$. The global state (eq. (7) in the main text) is

\[ |\Psi\rangle = \sqrt{p} |0\rangle \otimes |w^+\rangle^\otimes N + \sqrt{1 - p} |1\rangle \otimes |w^-\rangle^\otimes N. \]  

(S.18)

Where $|w^+\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $|w^-\rangle = \frac{1}{\sqrt{3}}(|001\rangle + \omega_3 |010\rangle + \omega_3^2 |100\rangle)$ are the states in eq. (8), with $\omega = e^{2\pi i/3}$. We will use a compact notation with $|0\rangle^k_N = |0\rangle \otimes |w^+\rangle^\otimes N$ and $|1\rangle^k_N = |1\rangle \otimes |w^-\rangle^\otimes N$.

As with the GHZ-like encoding, depending on which qubits are traced out, we will compute the resulting QMI for four different cases.

First case

We will first consider the case where at least one W state has been fully traced out. In this case, if two qubits are traced out from a W state, we create a mixture between two pure state. More specifically, if we trace out the $i$-th qubit the resulting state is:

\[ \tr_i \{|w^\pm\rangle \langle w^\pm|\} = \rho_\perp^{(i)} = 1/k |0\rangle \langle 0| \otimes |w^+\rangle^\otimes (k-1) + \frac{k-1}{k} |w^+_i\rangle \langle w^+_i| + |w^-_i\rangle \langle w^-_i|, \]

with $|0\rangle \otimes |w^+\rangle^\otimes (k-1)$ and $|w^+_i\rangle \langle w^+_i| + |w^-_i\rangle \langle w^-_i|$ orthogonal to one another.

When we trace out one W state from the global state, but at least one W state remains intact, the states are the following:

\[ \rho_{SE}^{(n)} = p |0\rangle \langle 0| \otimes n \otimes \rho_\perp^{(i)} \otimes m + (1-p) |1\rangle \langle 1| \otimes n \otimes \rho_\perp^{(i)} \otimes m, \]  

(S.19)

\[ \rho_{E_{f}}^{(n)} = p |w^+\rangle \langle w^+| \otimes n \otimes \rho_\perp^{(i)} \otimes m + (1-p) |w^-\rangle \langle w^-| \otimes n \otimes \rho_\perp^{(i)} \otimes m, \]  

(S.20)

It is actually not needed to compute the entropies of the states above. The two states clearly have the same entropy, so the mutual information is $I(S : E_{f}^{(0)}) = S(\rho_S)$.

Second case

If at least one group is traced out entirely, and all other groups have been traced out partially, the states are the following:

\[ \rho_{SE}^{(i)} = p |0\rangle \langle 0| \otimes n \otimes \rho_\perp^{(i)} \otimes m + (1-p) |1\rangle \langle 1| \otimes n \otimes \rho_\perp^{(i)} \otimes m, \]  

(S.21)

\[ \rho_{E_{f}}^{(i)} = p |0\rangle \langle 0| \otimes n \otimes \rho_\perp^{(i)} \otimes m + (1-p) |1\rangle \langle 1| \otimes n \otimes \rho_\perp^{(i)} \otimes m, \]  

(S.22)

We can explicitly rewrite eq. (S.21) in the following way

\[ \rho_{SE}^{(i)} = p |0\rangle \langle 0| \otimes n \otimes \left(1/k |0\rangle \langle 0| \otimes (k-1) + (k-1)/k |w^+_i\rangle \langle w^+_i| \right)^\otimes m + (1-p) |1\rangle \langle 1| \otimes n \otimes \left(1/k |0\rangle \langle 0| \otimes (k-1) + (k-1)/k |w^-_i\rangle \langle w^-_i| \right)^\otimes m, \]  

(S.23)

all the terms in the above states are pure and orthogonal to one another, so the resulting entropy is only due to classical mixing. The entropy of this state results from the probability vector $(p, 1-p) \otimes (1/k, (k-1)/k)^\otimes m$, whose entropy is straightforward to compute.

In the case of eq. (S.22) we end up having states of the form $p |w^+_i\rangle \langle w^+_i| \otimes m + (1-p) |w^-_i\rangle \langle w^-_i| \otimes m$, the eigenvalues of such states are $\lambda = (1 \pm \sqrt{1 + 4p(1-p)(f-1)})/2$ with $f = |\langle w^+_i| \otimes m |w^-_i\rangle \otimes m|^2$, in the case of $p = 1/2$ they reduce to $\lambda = (1 \pm \sqrt{f})/2$. 


The entropy for the state in eq. (S.22) is thus
\[
S[(1/3, 2/3)^{\otimes n}] + \sum_{j=0}^{m} (1/3)^{m-j}(2/3)^j \left(\frac{m}{j}\right) S(p_{w_i^j}^+ | w_i^j)^{\otimes j} + (1 - p_{w_i^j}) | w_i^-)^{\otimes j})
\] (S.24)
where \(S[(1/3, 2/3)^{\otimes n}]\) is the entropy of the probability vector \((1/3, 2/3)^{\otimes n}\). The QMI is thus
\[
I(S : E^{(1)}_f) = H'(m) = \sum_{j=0}^{m} (1/3)^{m-j}(2/3)^j \left(\frac{m}{j}\right) S(p_{w_i^j}^+ | w_i^j)^{\otimes j} + (1 - p_{w_i^j}) | w_i^-)^{\otimes j})
\] (S.25)
Notice how it is necessary to know the number of W states from which only one qubit has been traced out.

**Third case**

Let’s now focus on the alternate scenario, where we never entirely trace out a W-group. We start once again from the global state eq. (S.18), assuming \(p = \frac{1}{2}\)
\[
\frac{1}{2} |\tilde{0}\rangle_{N-1} w^+ + |\tilde{1}\rangle_{N-1} w^- \rangle \langle \tilde{0}\rangle_{N-1} w^+ + \langle \tilde{1}\rangle_{N-1} w^-].
\] (S.26)
After tracing out one qubit we obtain the following state:
\[
\frac{1}{3} |\tilde{0}\rangle_{N-1} + |\tilde{1}\rangle_{N-1} \rangle \langle \tilde{0}\rangle_{N-1} w^+ + |\tilde{1}\rangle_{N-1} w^- \rangle \langle \tilde{0}\rangle_{N-1} w^+ + \langle \tilde{1}\rangle_{N-1} w^-],
\] (S.27)
so we have a statistical mix of pure distinguishable states. So long as we trace out at most one qubit from each W-group, the procedure can be easily generalised and we end up with a sum of pure distinguishable states where \((1/3, 2/3)^{\otimes n}\) is the resulting probability vector from which we can compute the entropies. To be precise, relative phases may come up depending on which qubit is traced out. As we keep tracing out qubits, these phases may build up, making the exact writing of the states non-trivial. However, this relative phases are of no consequence for us as we are always left with distinguishable states (notice that in the previous cases such phases did not exist, since we always had at least one fully traced-out W state). This is no longer true when we trace out more than one qubit from each W-group, as in this case such relative phases influence the entropy of the resulting states.

To illustrate this, we start from the initial state
\[
|\Psi\rangle = |\tilde{0}\rangle_N + |\tilde{1}\rangle_N,
\] (S.28)
after tracing out one qubit we have (from now on, we will use \(P(|\psi\rangle)\) as a compact notation for \(|\psi\rangle\langle\psi|\))
\[
\frac{1}{3} P(|\tilde{0}\rangle_{fN} |0\rangle + \phi_i^0 |\tilde{1}\rangle_{fN} |0\rangle) + \frac{2}{3} P(|\tilde{0}\rangle_{fN} w_i^+ + |\tilde{1}\rangle_{fN} w_i^-).
\] (S.29)
\(\phi_i^0\) is the relative phase of the \(i\)-th qubit of the \(|w^-\rangle\) state, the superscript 0 indicates that it refers to the first W state we are tracing out. To be precise, \([\phi_0^0, \phi_0^1, \phi_0^2]=[1, e^{+i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}]\). If we trace out another qubit from another W state we have
\[
\frac{1}{9} P(|\tilde{0}\rangle_{fN} |0\rangle + \phi_0^0 \phi_1^0 |\tilde{1}\rangle_{fN} |0\rangle) + \frac{2}{9} P(|\tilde{0}\rangle_{fN} |0w_j^+\rangle + \phi_0^0 |\tilde{1}\rangle_{fN} |0w_j^+\rangle) + \frac{2}{9} P(|\tilde{0}\rangle_{fN} |w_i^+ 0\rangle + \phi_1^0 |\tilde{1}\rangle_{fN} |w_i^+ 0\rangle)
\] (S.30)
\[
\frac{2}{9} P(|\tilde{0}\rangle_{fN} |w_i^+ 0\rangle + \phi_1^0 |\tilde{1}\rangle_{fN} |w_i^+ 0\rangle) + \frac{4}{9} P(|\tilde{0}\rangle_{fN} |w_i^+ w_j^-\rangle + |\tilde{1}\rangle_{fN} |w_i^+ w_j^-\rangle).
\]
See how this procedure always keeps the resulting states distinguishable, so that the entropy of the resulting state is only due to the relative probabilities. Now we trace another qubit from one of the surviving pairs, resulting in
\[
\frac{1}{9} P(|\tilde{0}\rangle_{fN} |0\rangle + \phi_0^0 \phi_1^0 |\tilde{1}\rangle_{fN} |0\rangle) + \frac{1}{9} P(|\tilde{0}\rangle_{fN} |0\rangle + \phi_0^0 \phi_1^1 |\tilde{1}\rangle_{fN} |0\rangle) + \frac{1}{9} P(|\tilde{0}\rangle_{fN} |01\rangle + \phi_0^0 \phi_1^1 |\tilde{1}\rangle_{fN} |01\rangle)
\] (S.31)
In this case some of the resulting projectors are not distinguishable from one another anymore, resulting in mixed states. The coherence factor of the resulting mixed projector will be $\frac{1}{2}(\phi_j^i + \phi_j^{i'})$, whose absolute value is $\frac{1}{2} \forall j \neq j'$. By tracing out a qubit from the other W state we have

$$\frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |00\rangle + \phi_j^i \phi_{j'}^i |\tilde{1}\rangle_{jN} |00\rangle \right) + \frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |00\rangle + \phi_j^{i'} \phi_{j'}^{i'} |\tilde{1}\rangle_{jN} |00\rangle \right) + \frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |01\rangle + \phi_j^i \phi_{j'}^{i'} |\tilde{1}\rangle_{jN} |01\rangle \right)$$

$$+ \frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |10\rangle + \phi_j^i \phi_{j'}^{i'} |\tilde{1}\rangle_{jN} |10\rangle \right) + \frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |10\rangle + \phi_j^{i'} \phi_{j'}^{i'} |\tilde{1}\rangle_{jN} |10\rangle \right) + \frac{1}{9} P \left( |\tilde{0}\rangle_{jN} |11\rangle + \phi_j^i \phi_{j'}^{i'} |\tilde{1}\rangle_{jN} |11\rangle \right).$$

(S.32)

Once again some projectors are not distinguishable from one another, resulting in mixed states. The coherence factor of the resulting projectors is either $\frac{1}{2}(\phi_j^i + \phi_j^{i'})$ or $\frac{1}{2}(\phi_j^i + \phi_j^{i'})$. In general the absolute value of the resulting coherence factor will be $\frac{1}{2}$, where $l$ is the number of $|0\rangle$ states corresponding to W states were two qubits have been traced out. Interestingly enough, the entropy of such states is the same one of the states of the form $\frac{1}{2} P \left( |\psi^{-}\rangle \right) + \frac{l}{2} P \left( |\psi^{-}\rangle \right)$.

To sum things up, whenever we have a state where we have not traced out more than one qubit per pair, then the probability vector of the resulting state is $(1/3, 2/3)^{\otimes m}$. All the states are pure and distinguishable, so the entropy is only the entropy resulting from the probability. If we further trace out qubits from the surviving W states, we end up with a probability vector of the form $(2/3, 1/3)^{\otimes k} \otimes (1/3, 2/3)^{\otimes m-k}$, where $k$ is the number of W states from which we have traced out two qubits, and $m-k$ is the number of W states from which we have traced out only one qubit. So actually, while the individual probabilities have changed, the entropy of the probability vector remains the same. However, now the state are still distinguishable but not necessarily pure. The mixed states are mixtures with a coherence factor of $\frac{1}{2}$ with $0 \leq l \leq k$. The entropy of the state $\rho_{SE_j^{(2)}}$ is thus

$$S(\rho_{SE_j^{(2)}}) = S[(1/3, 2/3)^{\otimes n}] + \sum_{i=0}^{m-k} (1/3)^i (2/3)^{m-k-i} m-k \sum_{j=0}^{k} (2/3)^k (1/3)^{k-j} \frac{1}{2} P \left( |\psi^{+}\rangle \right) + \frac{1}{2} P \left( |\psi^{-}\rangle \right).$$

(S.33)

We will refer to the second term of eq. (S.33) as $\tilde{H}$. When we trace out the system as well, all the relative phases are be lost, and we end up once again with a collection of pure distinguishable states with a resulting probability vector of $(p, 1-p) \otimes (1/3, 2/3)^{\otimes n}$, this leads to a mutual information of $I(S : E_j^{(2)}) = 2S(\rho_S) - \tilde{H}$.

**Fourth case**

In the fourth and final case, at least one ancilla has been traced out from all groups, but no group has been entirely traced out. The state structure of the resulting state is exactly the same as in the previous scenario, so that the entropy is also the same. Instead, when tracing out the system as well the state structure is the same one as eq. (S.22). Combining our previous results, the mutual information in this case is $I(S : E_j^{(3)}) = S(\rho_S) + H' - \tilde{H}$, where we need to keep track of the number of surviving pairs as well as the number of surviving singlets.

**COMBINATORIAL COMPUTATION FOR W ENCODINGS**

In the case of the W states, the combinatorial calculation is now more complicated compared to the GHZ encoding. Using the parallelism introduced earlier, it is no longer sufficient to know the probabilities for the boxes to be empty or full, but we also need to know the probabilities for a certain number of boxes to have exactly one qubit or two qubits, as this now influences the corresponding mutual information.

We can think the combinatorial calculation in this way: once we fix the number of qubits of the environmental fraction $m$, we can go through all the possible integer partitions of that specific number (for example, two integer partitions of 5 are $1^22^1$ and $1^5$), these partitions correspond to the possible ways in which we can distribute the qubits in different boxes. Some partitions however are not compatible with the physical problem at hand: for example, we cannot have integers larger than $k$, and the total number of integers making up our partitions cannot be larger than $N$. We may also wish to impose extra conditions: for example, if we wish that no W state is fully selected, we can impose
that no integers in the partitions should be larger than $k - 1$, and if we wish that no W state is entirely traced out, we can impose that the total number of integers in the partition is exactly $N$. If a partition $i$ is $1^{i_1}2^{i_2}...$ the number of possible combinations is given by the polynomial $\left(\sum_j i_j\right)$, if we also consider the degeneracy resulting from shuffling the qubits inside a box, we have that for every integer partition $i$ of $m$, the number of possible configurations is

$$N_i = \left(\sum_j i_j\right)\prod_j \binom{k}{j}^{i_j}, \quad (S.34)$$

with an associated probability $p_i = \frac{N_i}{N}$.