A GEOMETRIC REALIZATION OF SOCLE-PROJECTIVE CATEGORIES FOR POSETS OF TYPE A

RALF SCHIFFLER AND ROBINSON-JULIAN SERNA

To the memory of A.G. Zavadskij (1946–2012)

Abstract. This paper establishes a link between the theory of cluster algebras and the theory of representations of partially ordered sets. We introduce a class of posets by requiring avoidance of certain types of peak-subposets and show that these posets can be realized as the posets of quivers of type $A$ with certain additional arrows. This class of posets is therefore called posets of type $A$. We then give a geometric realization of the category of finitely generated socle-projective modules over the incidence algebra of a poset of type $A$ as a combinatorial category of certain diagonals of a regular polygon. This construction is inspired by the realization of the cluster category of type $A$ as the category of all diagonals by Caldero, Chapoton and the first author [10].

We also study the subalgebra of the cluster algebra generated by those cluster variables that correspond to the socle-projectives under the above construction. We give a sufficient condition for when this subalgebra is equal to the whole cluster algebra.

Contents

1. Introduction 2
2. Preliminaries 3
   2.1. Category of diagonals $\mathcal{C}_T$ 3
   2.2. Socle-projective modules over incidence algebras 5
3. Posets of type $A$ 10
4. Category of sp-diagonals 15
   4.1. The functor $\Omega$ 18
5. Associated subalgebra of the cluster algebra 25
Acknowledgements 26
References 26

Key words and phrases. category of diagonals, cluster category, cluster algebra, poset of type $A$, socle-projective representation, Auslander Reiten quiver.

The first author was supported by NSF CAREER Grant DMS-1254567, NSF Grant DMS-1800860 and by the University of Connecticut.

The second author was supported in part by Colciencias Conv. 727. He would like to thank Department of Mathematics at University of Connecticut for hospitality and support during his visit in Fall 2018.
1. Introduction

Geometric realizations of algebraic structures using the combinatorial geometry of surfaces have been developed by different authors in recent years (for instance see [2, 6, 10, 14–16, 26, 28]). This approach provides geometric and combinatorial tools for the study of the objects and morphisms in the category. It plays an important role in cluster-tilting theory and in representation theory in general. For example, the category $C$ of diagonals (not including boundary edges) in a regular polygon $\Pi_{n+3}$ with $n + 3$ vertices introduced by Caldero, Chapoton and the first author [10] is a geometric realization of the cluster category of type $A_n$; which, in greater generality, was defined simultaneously by Buan-Marsh-Reiten-Reineke-Todorov [9]. They defined cluster categories as orbit categories of the bounded derived category of hereditary algebras. As an application in [10], the module category of a cluster-tilted algebra of type $A_n$ is described by a category of diagonals $C_T$ in $\Pi_{n+3}$, where $T$ is a triangulation of $\Pi_{n+3}$.

The present work links the theory of cluster algebras [18] and cluster categories with the theory of representations of partially ordered sets (in short, posets) through a geometric realization inspired by the one in [10].

The representation theory of posets was established parallel to the development of the representation theory of Artin algebras; the notion of a matrix representation of a poset $\mathcal{P}$ over an algebraically closed field $k$ was introduced in the 1970s by Nazarova and Roiter [25]. Aside from matrix representations of a poset $\mathcal{P}$, the concept of $\mathcal{P}$-space (or representation of $\mathcal{P}$) over a field $k$ was introduced by Gabriel [19] in connection with the investigation of representations of quivers. If $(\mathcal{P}, \preceq)$ is a finite poset, the category of $\mathcal{P}$-spaces of the poset $\mathcal{P}$ is nothing else than the category of socle-projective modules of the incidence algebra $k\mathcal{P}^*$ of the enlarged poset $\mathcal{P}^* = \mathcal{P} \cup \{\star\}$ such that $x \prec \star$ for each $x \in \mathcal{P}$; however, there are genuine methods in representation theory of posets such as the differentiation algorithms [11,12,32,35]. In a more general situation, Simson studied the category of peak $\mathcal{P}$-spaces which is identified with the category of socle-projective $k\mathcal{P}$-modules, where $k\mathcal{P}$ is the incidence algebra of a poset $\mathcal{P}$ [31,33]. He gave the finiteness criterion for those categories while his student J. Kosakowska classified the sincere posets of finite representation type [20–22]. Moreover, the tameness criterion was given by Kasjan and Simson in [23]. In general, the theory of representations of posets plays an important role in the study of lattices over orders, in the classification of indecomposable lattices over some simple curve singularities and in the classification of abelian groups of finite rank (see [1,32]).

In this paper, we introduce a class of posets which we call posets of type $A$. Roughly speaking, they are posets with $n \geq 1$ elements whose category of socle-projective representations is embedded in the category of representations of a Dynkin quiver of type $A_n$. We characterize these posets as those not allowing a peak-subposet of one of four types, see Definition 3.1. Then, we define a subcategory $C_{(T,F)}$ of the category $C_T$ of diagonals of a triangulated polygon $\Pi_{n+3}$ with $n + 3$ vertices to give a geometric realization of the category of socle-projective representations $\text{mod}_{sp}(k\mathcal{P})$ of posets $\mathcal{P}$ of type $A$, where $T$ is a triangulation of $\Pi_{n+3}$ associated to a Dynkin quiver $Q$ of type $A_n$ and $F$ is a set of additional arrows for $Q$. We
show that there is an equivalence of categories $\mathcal{C}(T,F) \to \text{mod}_{sp}(kP)$ in Theorem 1.4. Moreover, we define a subalgebra $A(P)$ of the cluster algebra $A = A(x,Q)$ generated by the cluster variables associated to diagonals in $\mathcal{C}(T,F)$ and diagonals in $T$; then, we establish that if $P$ is the poset whose Hasse quiver is a Dynkin quiver $Q$ of type $A_n$ then $A = A(P)$ in Theorem 5.2.

The paper is organized as follows: In section 2, we recall some notation and results about categories of diagonals in regular polygons and categories of socle-projective representations of posets. In section 3, we define and study posets of type $A$. Section 4 is devoted to proving our main result, Theorem 4.4. Finally, the last section deals with the subalgebras $A(P)$ of the cluster algebra $A$.

2. Preliminaries

2.1. Category of diagonals $\mathcal{C}_T$. We recall some results and notation of [10] (see also Chapter 3 in [29]) which are used in this work. A diagonal in a regular polygon is a straight line segment that joins two of the vertices and goes through the interior of the polygon. A triangulation of the polygon is a maximal set of non-crossing diagonals. Such a triangulation cuts the polygon into triangles.

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of a regular polygon $\Pi_{n+3}$ (or $(n+3)$-gon) with $n+3$ vertices and let $\gamma$ and $\gamma'$ be diagonals that are not in $T$. The diagonal $\gamma$ is related to the diagonal $\gamma'$ by a pivoting elementary move if they share a vertex on the boundary (this vertex is called pivot), the other vertices of $\gamma$ and $\gamma'$ are the vertices of a boundary edge of the polygon and the rotation around the pivot is positive (for the trigonometric direction) from $\gamma$ to $\gamma'$. Let $P_v : \gamma \to \gamma'$ denote the pivoting elementary move from $\gamma$ to $\gamma'$ with pivot $v$. Compositions of pivoting elementary moves are called pivoting paths.

The combinatorial $k$-linear additive category $\mathcal{C}_T$ of diagonals is defined as follows: The objects are positive integral linear combinations of diagonals that are not in $T$. By additivity, it is enough define morphisms between diagonals. To do that, we recall that the mesh relations are the equivalence relation between pivoting paths induced by identifying every couple of pivoting paths of the form

$$\gamma \xrightarrow{P_{v_1}} \beta \xrightarrow{P_{v_2}} \gamma' = \gamma \xrightarrow{P_{v_2}} \beta' \xrightarrow{P_{v_1}'} \gamma'$$

where $v_1 \neq v_2'$ and $v_2 \neq v_1'$ (see Figure 1). In these relations, diagonals in $T$ or boundary edges are allowed with the following convention: If one of the intermediate edges ($\beta$ or $\beta'$) is either boundary edge or diagonal in $T$, the corresponding term in the mesh relation is replaced by zero. Thus, the space of morphisms from a diagonal $\gamma \notin T$ to a diagonal $\gamma' \notin T$ is the quotient of the vector space over $k$ spanned by pivoting paths from $\gamma$ to $\gamma'$ modulo the mesh relations.

The following lemma describes the relative positions of diagonals $\gamma$ and $\gamma'$, when there exist a nonzero morphism between them.

**Lemma 2.1.** [10] Lemma 2.1] The vector space $\text{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$ is nonzero if and only if there exists a diagonal $\tau_i \in T$ such that $\tau_i$ crosses the diagonals $\gamma$ and $\gamma'$ and the relative positions of them are as in Figure 2. That is, let $v_1, v_2$ be the endpoints of $\tau_i$ and $u_1, u_2$ (respectively $u'_1, u'_2$) be the endpoints of $\gamma$ (respectively $\gamma'$). Then
ordering the vertices of the polygon in the positive trigonometric direction starting at \( v_1 \), we have \( v_1 < u_1 \leq u'_1 < v_2 < u_2 \leq u'_2 \). In this case, \( \text{Hom}_{C_T}(\gamma, \gamma') \) is of dimension one.

A triangulation \( T \) of the \((n + 3)\)-gon is said to be triangulation without internal triangles if each triangle has at least one side on the boundary of the polygon. It is important to recall that every triangulation \( T \) of the \((n + 3)\)-gon gives rise to a cluster-tilted algebra \( kQ_T/I \) of type \( A_n \), where \( I \) is the two-sided ideal generated by all length two subpath of oriented 3-cycles in \( Q_T \), and every cluster-tilted algebra is of this form. In particular, every Dynkin quiver of type \( A_n \) corresponds to a triangulation without internal triangles. The map associates a quiver \( Q_T \) to the triangulation \( T = \{\tau_1, \ldots, \tau_n\} \) of \( \Pi_{n+3} \) as follows: The vertices of \( Q_T \) are \((Q_T)_0 = \{1, 2, \ldots, n\}\) and there is an arrow \( x \to y \) in \((Q_T)_1\) precisely if the diagonals \( \tau_x \) and \( \tau_y \) bound a triangle in which \( \tau_y \) lies counter-clockwise from \( \tau_x \) (see Figure 3 and Example 4.2).

A vertex \( x \in (Q_T)_0 \) belongs to the support \( \text{supp} \gamma \) of a diagonal \( \gamma \notin T \) if the diagonal \( \tau_x \in T \) crosses \( \gamma \). The following lemma permits to see diagonals that are not in
Let $\gamma$ be a diagonal which does not belong to $T$. The set $\text{supp} \gamma$ is connected as a subset of the quiver $Q_T$.

In [10], the authors defined a $k$-linear additive functor $\Theta$ from $C_T$ to the category $\text{mod} kQ_T/I$ of finitely generated $kQ_T/I$-modules. The image of a diagonal $\gamma \notin T$ is the representation $M^\gamma = (M^\gamma_x, f^\gamma_\alpha)$ defined as follows: For each vertex $x$ in $Q_T$,

$$M^\gamma_x = \begin{cases} k & \text{if } x \in \text{supp} \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For any arrow $\alpha : x \to y$ in $Q_T$,

$$f^\gamma_\alpha = \begin{cases} \text{id}_k & \text{if } M^\gamma_x = M^\gamma_y = k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any pivoting elementary move $P : \gamma \to \gamma'$ they defined the morphism $\Theta(P)$ from $(M^\gamma_x, f^\gamma_\alpha)$ to $(M'^\gamma_x, f'^\gamma_\alpha)$ to be $\text{id}_k$ whenever possible and 0 otherwise.

The category $C_T$ of diagonals gives a geometric realization of the category of finitely generated $kQ_T/I$-modules in the following sense.

**Theorem 2.3.** [10] Theorems 4.4 and 5.1

(a) The functor $\Theta$ is an equivalence of categories.

(b) The irreducible morphisms of $C_T$ are direct sums of the generating morphisms given by pivoting elementary moves.

(c) The mesh relations of $C_T$ are the mesh relations of the AR-quiver of $C_T$.

(d) The AR-translation is given on diagonals by $r^-$. Here, $r^-$ (respectively $r^+$) denotes the clockwise (respectively counter-clockwise) elementary rotation of the regular polygon $\Pi_{n+3}$.

(e) The projective indecomposable objects of $C_T$ are the diagonals in $r^+(T)$.

(f) The injective indecomposable objects of $C_T$ are the diagonals in $r^-(T)$.

2.2. Socle-projective modules over incidence algebras. In this section, we recall some of the main results regarding socle-projective modules over incidence algebras of posets due to Simson [31, 33].

We denote by $(\mathcal{P}, \preceq)$ a finite partially ordered set (in short, poset) with respect to the partial order $\preceq$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. For the sake of simplicity we write $\mathcal{P}$ instead of $(\mathcal{P}, \preceq)$. Let $\max \mathcal{P}$ (respectively $\min \mathcal{P}$) be the set of all maximal (respectively minimal) points of $\mathcal{P}$. A poset $\mathcal{P}$ is called an $r$-peak poset if $|\max \mathcal{P}| = r$. We recall that a full subposet $\mathcal{P}'$ of $\mathcal{P}$ is said to be a peak-subposet if $\max \mathcal{P}' \subseteq \max \mathcal{P}$. In the sequel, we denote $\mathcal{P}^- = \mathcal{P} \setminus \max \mathcal{P}$.

The Hasse diagram of $\mathcal{P}$ is obtained as follows: One represents each element of $\mathcal{P}$ as a vertex in the plane and draws a line segment or curve that goes upward from $x$ to $y$ whenever $y$ covers $x$, that is, whenever $x \prec y$ and there is no $z$ such that $x \prec z \prec y$. These lines may cross each other but must not touch any vertices other than their endpoints. Such a diagram, with labeled vertices, uniquely determines its partial order.

**Example 2.4.** Given the one-peak poset $\mathcal{P}$ whose Hasse diagram is
the subposet \{2, 4, 5, 6\} is not a peak-subposet of \(\mathcal{P}\), whereas \{1, 6, 3\} is a peak-subposet of \(\mathcal{P}\).

For a point \(a \in \mathcal{P}\), the subposets of \(\mathcal{P}\)

\[ a^\triangledown = \{ x \in \mathcal{P} \mid a \leq x \}\]

\[ a_\Delta = \{ x \in \mathcal{P} \mid x \leq a \}\]

are called up-cone and down-cone respectively. In the literature, the up-cone of \(a\) is also called the principal filter of \(a\) and its down-cone is its principal ideal. A poset \(\mathcal{P}\) is called a chain (or a totally ordered set or a linearly ordered set) if and only if for all \(x, y \in \mathcal{P}\) we have \(x \leq y\) or \(y \leq x\). On the other hand, an ordered set \(\mathcal{P}\) is called an antichain if and only if for all \(x, y \in \mathcal{P}\) we have \(x \leq y\) in \(\mathcal{P}\) only if \(x = y\). The cardinality of a maximal antichain in a poset \(\mathcal{P}\) is called the width \(w(\mathcal{P})\) of \(\mathcal{P}\). If some subsets \(X_1, \ldots, X_n\) of \(\mathcal{P}\) do not intersect mutually (but may have comparable points), then their union \(X_1 \cup \cdots \cup X_n\) is called a sum and is denoted by \(X_1 + \cdots + X_n\). We recall that according to the Dilworth’s theorem if the width \(w(\mathcal{P}) = n\) then \(\mathcal{P}\) is a sum of \(n\) chains. For details on posets, we refer to [13, 34].

Given a finite poset \(\mathcal{P}\), by \(k\mathcal{P}\) we mean the incidence algebra of the poset \(\mathcal{P}\). \(k\mathcal{P}\) can be described as a bound quiver algebra \(kQ/I\) induced by the Hasse quiver \(Q\) of \(\mathcal{P}\) whose vertices are the points of \(\mathcal{P}\) and there is an arrow \(\alpha : x \rightarrow y\) for each pair \(x, y \in \mathcal{P}\) such that \(y\) covers \(x\). The ideal \(I\) is generated by all the commutativity relations \(\gamma - \gamma'\) with \(\gamma\) and \(\gamma'\) parallel paths in \(Q\). In this case, the category \(mod(k\mathcal{P})\) of the finitely generated \(k\mathcal{P}\)-modules is identified with the well known category \(\text{rep}(Q, I)\) of representations of the bound quiver \((Q, I)\).

Example 2.5. The incidence algebra \(k\mathcal{P}\) of the poset \(\mathcal{P}\) defined in Example 2.4 is the bound quiver algebra \(kQ/I\), where \(Q\) is the is the quiver

![Quiver Diagram](attachment:image.png)

and the ideal \(I\) is generated by the relation \(\alpha \beta - \delta \xi\).

Recall that the socle \(\text{soc}\) \(M\) of a module \(M\) is the semisimple submodule generated by all simple submodules of \(M\). A module \(M\) is called socle-projective if \(\text{soc}\) \(M\) is a projective module. We denote by \(\text{mod}_{sp}(k\mathcal{P})\) the full subcategory of \(\text{mod}(k\mathcal{P})\) whose objects are the socle-projective \(k\mathcal{P}\)-modules. We have an explicit description of the objects in \(\text{mod}_{sp}(k\mathcal{P})\) as follows.

Proposition 2.6. [30, Section 3] Each \(k\mathcal{P}\)-module \(M\) in \(\text{mod}(k\mathcal{P})\) is identified with a collection \(M = (M_x, yh_x)_{x,y \in \mathcal{P}}\) of finite-dimensional \(k\)-vector spaces \(M_x\), one for
each point \(x \in \mathcal{P}\), and a collection of \(k\)-linear maps \(y_h : M_x \to M_y\), one for each relation \(x \preceq y\) in \(\mathcal{P}\), such that

(a) \(y_h\) is the identity of \(M_x\) for all \(x \in \mathcal{P}\) and \(w \cdot y_h = w\) for all \(x \preceq y \preceq w\) in \(\mathcal{P}\).

Furthermore, \(M = (M_x, y_h)_{x,y \in \mathcal{P}}\) is a socle-projective module if it also holds that

(b) \(\bigcap_{z \in \max \mathcal{P}} \ker z_h = 0\) for all \(x \in \mathcal{P}^-\) such that \(x \prec z\).

Note that it is enough to define the linear maps \(y_h\) when \(y\) covers \(x\), that is, one for each arrow in the Hasse quiver of \(\mathcal{P}\) because if \(x \prec y\) but \(y\) does not cover \(x\) then for any chain \(x = x_0 \prec x_1 \prec \cdots \prec x_l = y\) in \(\mathcal{P}\) such that \(x_i+1\) covers \(x_i\) we have that \(y_h = y_{x_{i-1}} \cdots y_{x_0}\). The condition (a) implies that it is well defined.

**Example 2.7.** Let \(\mathcal{P}\) be the one-peak poset given in Example 2.4 whose Hasse quiver is shown in Example 2.5. The \(k\mathcal{P}\)-module \(M\) given by the system

\[
\begin{array}{c}
1 \\
\downarrow \\
0
\end{array}
\quad \begin{array}{c}
k \\
\downarrow \\
(1,0)
\end{array}
\quad \begin{array}{c}
k^2 \\
\downarrow \\
k^2 \\
\downarrow \\
k^2 \\
\downarrow \\
(0,1)
\end{array}
\]

is socle-projective. Indeed, since \(\mathcal{P}\) has a unique maximal point \(z = 3\) we have that \(\bigcap_{z \in \max \mathcal{P}} \ker z_h = \ker z_h\) for all \(x \in \mathcal{P}^-\). Note that, the kernel \(\ker z_h\) of the map \(z_h\) from \(M_x\) to \(M_3\) is zero for each \(x = 1, 2, 4, 5, 6\).

For example, replacing \(\begin{array}{c}1 \\
\end{array}\) on arrow \(\gamma\) by \(\begin{array}{c}0 \\
\end{array}\) would resulting a representation that is not socle-projective.

Let \(M = (M_x, y_h)_{x,y \in \mathcal{P}}\) and \(N = (N_x, y'_h)_{x,y \in \mathcal{P}}\) be two objects in \(\mathrm{mod}(k\mathcal{P})\). A morphism of \(k\mathcal{P}\)-modules \(f : M \to N\) is a collection \(f = (f_x)_{x \in \mathcal{P}}\) of linear maps

\[f_x : M_x \to N_x\]

such that for each relation \(x \preceq y\) in \(\mathcal{P}\) the diagram

\[
\begin{array}{ccc}
M_x & \xrightarrow{y_h} & M_y \\
\downarrow{f_x} & & \downarrow{f_y} \\
N_x & \xrightarrow{y'_h} & N_y
\end{array}
\]

commutes, that is,

\[f_y \circ y_h = y_h' \circ f_x.\]

On the other hand, from the point of view of the representations of posets introduced by Nazarova and Roiter \[25\], the category \(\mathrm{mod}_\mathcal{P}(k\mathcal{P})\) is identified with the category \(\mathcal{P}\)-spr of peak \(\mathcal{P}\)-spaces (or socle-projective representations of \(\mathcal{P}\)) over the field \(k\) defined by Simson \[33\]. The objects of \(\mathcal{P}\)-spr are systems

\[M = (M_x)_{x \in \mathcal{P}}\]

of finite dimensional \(k\)-vector spaces \(M_x\) satisfying the following conditions:
(a) For each \( x \in \mathcal{P} \), \( M_x \) is a \( k \)-subspace of the space
\[
M^\bullet = \bigoplus_{z \in \text{max} \mathcal{P}} M_z.
\]
(b) The inclusion \( M_z \hookrightarrow M^\bullet \) is defined as usual for each \( z \in \text{max} \mathcal{P} \).
(c) For each \( x \prec y \) in \( \mathcal{P} \) it holds that \( \pi_y(M_x) \subseteq M_y \), where \( \pi_y \in \text{End } M^\bullet \) is the
composition of the direct summand projection of \( M^\bullet \) on
\[
M^\bullet_y = \bigoplus_{y \leq z \in \text{max} \mathcal{P}} M_z
\]
with the natural embedding homomorphism \( M^\bullet_y \hookrightarrow M^\bullet \).
(d) If \( z \in \text{max} \mathcal{P} \) and \( x \notin z \_ \) then \( \pi_z(M_x) = 0 \).

A morphism \( f : M \to L \) between two peak \( \mathcal{P} \)-spaces \( M \) and \( L \) is a collection of
\( k \)-linear maps \( f = (f_z : M_z \to L_z)_{z \in \text{max} \mathcal{P}} \) such that for all \( x \in \mathcal{P} \)
\[
\left( \bigoplus_{z \in \text{max} \mathcal{P}} f_z \right)(M_x) \subseteq L_x.
\]

The category of peak \( \mathcal{P} \)-spaces over \( k \) is denoted by \( \mathcal{P} \)-spr. Following \([27, 31, 33]\),
\( \mathcal{P} \)-spr is an additive Krull–Schmidt category of finite global dimension which is
closed under taking kernels and extensions. Furthermore, it has enough projective
objects, AR-sequences, source maps, and sink maps.

The coordinate vector of a peak \( \mathcal{P} \)-space \( M \) is the vector
\[
d = \text{cdim } M = (d_x)_{x \in \mathcal{P}} \in \mathbb{Z}^\mathcal{P}
\]
such that \( d_x = \dim_k M_x \) if \( x \in \text{max} \mathcal{P} \) and \( d_x = \dim_k(M_x/M_x) \) otherwise, where \( M_x = \sum_{y < z} \pi_x(M_y) \). This vector allows us to define the coordinate support \( \text{csupp } M \)
of \( M \) given by the peak-subposet \( \text{csupp} M = \{ x \in \mathcal{P} \mid (\text{cdim } M)_x \neq 0 \} \) of \( \mathcal{P} \).

Example 2.8. Let \( \mathcal{P} \) be the one-peak poset given in Example 2.4. The system
\( M = (M_1, M_2, M_3, M_4, M_5, M_6) = (k \oplus 0, k^2, k^2, k^2, k \oplus 0) \) is a peak \( \mathcal{P} \)-space,
whereas the system \((0, 0, k, k, k, k)\) does not satisfy condition (c) because \( M_5 \) is not
a \( k \)-subspace of \( M_2 \). In this case, the coordinate vector is \( \text{cdim } M = (1, 1, 2, 0, 1, 1) \)
and the coordinate support is the peak-subposet \( \text{csupp } M = \{ 1, 2, 3, 5, 6 \} \) of \( \mathcal{P} \). For
greater clarity, note that \( M_2 = \pi_2(M_1) = M_1 = k \oplus 0 \), where \( \pi_2 \) is the identity
of \( k^2 \), that is, \( \text{dim}_k(M_2/M_2) = 1 \). On the other hand, given the peak \( \mathcal{P} \)-space
\( L = (k, k, k, k, k, k) \), the map \( f : k^2 \xrightarrow{(1 \ 1)} k \) is a morphism from \( M \) to \( L \).

A peak \( \mathcal{P} \)-space \( M \) is said to be sincere if it is indecomposable and \( \text{csupp } M = \mathcal{P} \).
Furthermore, if there exists a sincere peak \( \mathcal{P} \)-space we say that \( \mathcal{P} \) is a sincere poset.

The category \( \mathcal{P} \)-spr (or the poset \( \mathcal{P} \)) is said to be of finite representation type if it has
only a finite number of nonisomorphic indecomposable peak \( \mathcal{P} \)-spaces, otherwise,
it is of infinite representation type. The classification of all sincere \( r \)-peak posets
of representation finite type was given by M. Kleiner \([24]\), for the case \( r = 1 \), and
by J. Kosakowska \([20, 22]\), in the remaining cases. Moreover, they gave the list of
the sincere peak \( \mathcal{P} \)-spaces, where \( \mathcal{P} \) is a sincere poset of finite representation type.
Such lists are important because we can get all indecomposable peak \( \mathcal{P} \)-spaces of
a given poset \( \mathcal{P} \) lifting all sincere \( \mathcal{S} \)-spaces of all sincere peak-subposets \( \mathcal{S} \) of \( \mathcal{P} \) via the well-known subposet induced functor \(|\mathcal{S}| \rightarrow \mathcal{P} \)

\[(2.1) \quad \mathcal{S} : \mathcal{S}\text{-spr} \rightarrow \mathcal{P}\text{-spr}\]

that assigns to the peak \( \mathcal{S} \)-space \((M_x)_{x \in \mathcal{S}}\) the peak \( \mathcal{P} \)-space \((\hat{M}_x)_{x \in \mathcal{P}}\), where \( \hat{M}_x \) is defined by

\[
\hat{M}_x = \begin{cases} 
M_x, & \text{if } x \in \text{max } \mathcal{S}, \\
0, & \text{if } x \notin (\text{max } \mathcal{S})_{\Delta}, \\
\sum_{y \leq x} \pi_x(M_y), & \text{if } x \in (\text{max } \mathcal{S})_{\Delta}.
\end{cases}
\]

**Proposition 2.9.** [32, Proposition 5.14] Up to isomorphism, any indecomposable object \( M \) in \( \mathcal{P}\text{-spr} \) is the image \( \mathcal{T}_{\mathcal{S}}(L) \) of a sincere peak \( \mathcal{S} \)-space \( L \in \mathcal{S}\text{-spr} \), where \( \mathcal{S} \) is a sincere peak-subposet of \( \mathcal{P} \). As a consequence, \( \text{csupp } M = \mathcal{S} \).

For a finite poset \( \mathcal{P} \), two functors play an important role in the proof of the equivalence of the categories \( \mathcal{P}\text{-spr} \) and \( \text{mod}_{\text{sp}}(k\mathcal{P}) \); one of them is called the embedding functor

\[(2.2) \quad \rho : \mathcal{P}\text{-spr} \rightarrow \text{mod}(k\mathcal{P}) \]

defined by \( \rho(U) = (U_y, y\pi_x)_{x, y \in \mathcal{P}} \), where \( y\pi_x : U_x \rightarrow U_y \) is the unique \( k \)-linear map making the diagram

\[
\begin{array}{ccc}
U_x & \xrightarrow{y\pi_x} & U_y \\
\downarrow & & \downarrow \\
U_x & \xrightarrow{\pi_y} & U_y
\end{array}
\]

commutative. The other functor is the called adjustment functor \([32,33]\)

\[(2.3) \quad \theta : \text{mod}(k\mathcal{P}) \rightarrow \mathcal{P}\text{-spr} \]

given by \( \theta(M_x, y_h x)_{x, y \in \mathcal{P}} = (U_x)_{x \in \mathcal{P}} \), such that

\[
U_x = \begin{cases} 
M_x, & \text{if } x \in \text{max } \mathcal{P}, \\
\text{Im}(f : M_x \rightarrow \bigoplus_{z \in \text{max } \mathcal{P}} M_z) & \text{otherwise},
\end{cases}
\]

where \( f = (z_h x)_{z \in \text{max } \mathcal{P}} \) and

\[
z_h x = \begin{cases} 
h_x & \text{if } x \prec z \text{ in } \mathcal{P}, \\
0 & \text{otherwise}.
\end{cases}
\]

**Example 2.10.** The peak \( \mathcal{P} \)-space \( M \) given in Example [2.8] is the image by \( \theta \) of the socle-projective \( k\mathcal{P} \)-module defined in Example [2.7]. Moreover, the functor \( \rho \) sends the peak \( \mathcal{P} \)-space \( M \) to the socle-projective representation mentioned.

Then, we have the following equivalence of categories given by Simson.

**Lemma 2.11.** [33, Lemma 2.1] The functor \( \rho \) induces an equivalence of categories \( \rho : \mathcal{P}\text{-spr} \rightarrow \text{mod}_{\text{sp}}(k\mathcal{P}) \) and the inverse of \( \rho \) is the restriction of \( \theta \) to \( \text{mod}_{\text{sp}}(k\mathcal{P}) \).
in $\text{mod}(kP)$, where $P$ is projective and $P'$ is semisimple projective. The full subcategory $\text{prin}(kP)$ of $\text{mod}(kP)$ generated by the prinjective modules is closely related to the category $\mathcal{P}\text{-spr}$ because the functor $\theta$ induces a full dense additive functor

$$\theta_P : \text{prin}(kP) \to \mathcal{P}\text{-spr}$$

such that $\ker\theta_P$ consist of all maps in $\text{prin}(kP)$ having a factorization through a direct sum of copies of the projective $kP^-$-modules. Furthermore, the functor $\theta_P$ preserves and reflects finite representation type, and induces an equivalence of categories

$$\text{prin}(kP)/\ker\theta_P \cong \mathcal{P}\text{-spr}.$$ 

Thus, we can formulate the following criterion of finite representation type due Simson.

**Theorem 2.12.** [33, Theorem 3.1] The following conditions are equivalent

(a) The category $\mathcal{P}\text{-spr}$ is of finite representation type.

(b) The category $\text{prin}(kP)$ is of finite representation type.

(c) The poset $\mathcal{P}$ does not contain as a peak-subposet any of the posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ presented in [33, Section 5].

### 3. Posets of type $\mathbf{A}$

In this section, we introduce a family of posets which we call posets of type $\mathbf{A}$ because of a characterization using a type $\mathbf{A}$ quiver given in Proposition 3.8.

**Definition 3.1.** A finite connected poset $\mathcal{P}$ is said to be poset of type $\mathbf{A}$ if $\mathcal{P}$ does not contain as a peak-subposet any of the following posets:

\[
\begin{array}{cccc}
\mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_{4,n}, n \geq 0 \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

**Example 3.2.** The poset $\mathcal{P}$ given in Example 2.4 is a poset of type $\mathbf{A}$ because although $\{2, 4, 5, 6\}$ is a subposet of type $\mathcal{R}_2$ it is not a peak-subposet of $\mathcal{P}$. On the other hand, the poset

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

is a three-peak poset of type $\mathbf{A}$ which can be viewed as a Dynkin quiver of type $\mathbb{E}_7$. We say that two maximal points $z$ and $z'$ in a poset $\mathcal{P}$ are neighbors if $z_\Delta \cap z'_\Delta \neq \emptyset$. Then, we describe this notion when $\mathcal{P}$ is a poset of type $\mathbf{A}$ as follows.

**Lemma 3.3.** Let $\mathcal{P}$ be an $r$-peak poset of type $\mathbf{A}$ with $r \geq 2$. The following statements hold:

(a) The points $z, z' \in \max \mathcal{P}$ are neighbors if and only if $z_\Delta \cap z'_\Delta = \{x\}$, for some $x \in \min \mathcal{P}$.

(b) There exists a point $z \in \max \mathcal{P}$ such that $z$ has a unique neighbor.
Proof. Since \( R_2 \) is not peak-subposet of \( \mathcal{P} \) then \( x \in z_\delta \cap z_\delta' \) implies that \( x \in \text{min} \mathcal{P} \) because otherwise there exists \( y < x \) and then the subposet \( \{y, x, z, z'\} \) is of the form \( R_4 \). Now, if \( x \neq x' \in z_\delta \cap z_\delta' \) then \( R_{4,0} \) is peak-subposet of \( \mathcal{P} \) which is a contradiction. Thus, the set \( z_\delta \cap z_\delta' \) is a singleton. Clearly the converse implication is true. On other hand, since \( \mathcal{P} \) is a connected poset then each maximal point \( z \) has at least one neighbor, but \( z \) does not have three neighbor points. Indeed, if \( z_1, z_2, z_3 \) are distinct neighbors of \( z \) with \( x_i \in z_\delta \cap (z_i)_\delta \) then we have the subposet

\[
\begin{array}{c}
\downarrow \\
1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow \\
2 & 3 & 1 \\
\end{array}
\]

If \( x_1, x_2, x_3 \) are three distinct points then \( \{z, x_1, x_2, x_3\} \) is a peak-subposet of type \( R_1 \), a contradiction. Suppose that two of the \( x_i \) are equal, for instance \( x_1 = x_2 \). Then \( \{z_1, z, z_2, x_1\} \) is a peak-subposet of type \( R_3 \), a contradiction. Thus \( z \) has at most two neighbors. Finally, if each maximal point has exactly two neighbor points then \( R_{4,n} \) is peak-subposet of \( \mathcal{P} \) for some \( n \geq 0 \), which is contradictory, and we are done. \( \square \)

Actually, the posets of type \( \Delta \) can be viewed as posets associated to certain quivers which are obtained from Dynkin quivers of type \( \Delta \) by adding some new arrows. To explain this, we need the following definitions:

Let \( Q \) be an acyclic quiver and let \( \mathcal{P}_Q = Q_0 \) be its set of vertices. We define an order on \( \mathcal{P}_Q \) by \( x \preceq y \) if and only if there exists a path from \( x \) to \( y \) in \( Q \). We say that \( \mathcal{P}_Q \) is the \textit{poset associated to the quiver} \( Q \). Note that there is a unique poset associated to a finite acyclic quiver, but the converse is false in general. As an example, the poset associated to the quiver

\[
\begin{array}{c}
\downarrow \\
1 & 2 & 3 \\
\end{array}
\]

is \( \{1 < 3 < 2\} \). However, the Hasse quiver of this poset is \( 1 \rightarrow 3 \rightarrow 2 \). Thus, the two quivers have the same associated poset. As another example, corresponding to the poset \( \mathcal{P} = \{1, 2\} \) together with the usual ordering \( 1 < 2 \), we get countably many quivers with \( n \) arrows from 1 to 2 for any natural number \( n \in \mathbb{N} \).

Recall that a vertex \( x \in Q_0 \) is said to be a \textit{sink vertex} (respectively \textit{source vertex}) if there is no arrow \( \alpha \) in \( Q_1 \) such that \( s(\alpha) = x \) (respectively \( t(\alpha) = x \)), where \( s(\alpha) \) is the starting vertex and \( t(\alpha) \) is the target vertex of the arrow \( \alpha \).

Let \( Q \) be a Dynkin quiver of type \( \Delta \) and let \( z \in Q_0 \) be a sink vertex. The maximal full subquiver \( Q^{(z)} \) of \( Q \) with \( z \) as the unique sink is called the \( z \)-\textit{subquiver} of \( Q \). In other words, the vertices of \( Q^{(z)} \) are the vertices in the support \( \text{Supp} \ I(z) \) of the indecomposable injective representation \( I(z) \) at vertex \( z \).

Example 3.4. The quiver \( Q = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \) of type \( \Delta_7 \) contains the 2-subquiver \( 1 \rightarrow 2 \rightarrow 3 \), the 5-subquiver \( 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \), and the 7-subquiver \( 6 \rightarrow 7 \).

We will now add new arrows to our quiver \( Q \) as follows.
Definition 3.5. A set \( F = \{\alpha_1, \ldots, \alpha_t\} \) of new arrows for \( Q \) is called an alien set for \( Q \) if the following conditions hold.

(a) For each \( \alpha \in F \) there exists a sink vertex \( z \) in \( Q \) such that \( s(\alpha), t(\alpha) \in \text{Supp} \ I(z) \).
(b) \( t(\alpha) \) is not a source vertex in \( Q \) unless it is an extremal vertex in \( Q \).
(c) For all \( \alpha \in F \), the arrow \( \alpha \) is the unique path from \( s(\alpha) \) to \( t(\alpha) \) in \( Q^F \), where \( Q^F \) is the quiver such that \( Q^F_0 = Q_0 \) and \( Q^F_1 = Q_1 \cup F \).
(d) The quiver \( Q^F \) is acyclic.

The arrows in an alien set for \( Q \) will be called alien arrows.

Example 3.6. If \( Q \) is the quiver \( 1 \to 2 \to 3 \to 4 \to 5 \to 6 \) of type \( A_6 \) then \( F = \{\alpha : 5 \to 2\} \) is an alien set for \( Q \) and \( Q^F \) is the quiver in Example 2.5. Note that the poset \( \mathcal{P}_{Q^F} \) is the one-peak poset of type \( A \) defined in Example 2.2.

Example 3.7. Let \( Q \) be the quiver in Example 3.3. The set

\[
F = \{\alpha : 3 \to 1, \beta : 6 \to 4\}
\]

is an alien set for \( Q \). Moreover, the quiver \( Q^F \) is equal to

\[
\begin{array}{c}
1 \\
\alpha \\
\downarrow \alpha \\
2 \\
\beta \\
\downarrow \beta \\
3 \\
\alpha \\
\downarrow \alpha \\
4 \\
\beta \\
\downarrow \beta \\
5 \\
\alpha \\
\downarrow \alpha \\
6 \\
\beta \\
\downarrow \beta \\
7
\end{array}
\]

Note that the poset \( \mathcal{P}_{Q^F} \) associated to \( Q^F \) is the three-peak poset of type \( A \) defined in Example 3.2.

The following proposition characterizes posets of type \( A \).

Proposition 3.8. A poset \( \mathcal{P} \) is of type \( A \) if and only if there exists a Dynkin quiver \( Q \) of type \( A \) and an alien set \( F \) for \( Q \) such that \( \mathcal{P} = \mathcal{P}_{Q^F} \) is the poset associated to the quiver \( Q^F \).

Proof. In order to prove the necessary condition we proceed by induction on the number \( r \) of peaks in \( \mathcal{P} \). First we suppose that \( \mathcal{P} \) is a one-peak poset with a maximal point \( z \). Since \( \mathcal{R}_1 \) is not peak-subposet of \( \mathcal{P} \) we conclude that \( w(\mathcal{P}) \leq 2 \). Thus, if \( w(\mathcal{P}) = 1 \) then \( \mathcal{P} \) is a chain and it can be viewed as a linearly oriented quiver \( Q \) of type \( A \). Clearly, if \( F = \emptyset \) then \( \mathcal{P} \) is the poset associated to the quiver \( Q^F \). On the other hand, if \( w(\mathcal{P}) = 2 \) then by Dilworth’s theorem \( \mathcal{P}^- \) is a sum of two chains \( \mathcal{P}_1 = \{x_1 \prec \cdots \prec x_s\} \) and \( \mathcal{P}_2 = \{y_1 \prec \cdots \prec y_t\} \). Given the quiver

\[
Q = x_1 \cdots x_s \overset{z}{\longrightarrow} y_t \cdots y_1,
\]

the set \( F = F_1 \cup F_2 \) such that \( F_1 = \{\alpha : x \to y \mid y \in \mathcal{P}_2 \text{ covers } x \in \mathcal{P}_1\} \) and \( F_2 = \{\beta : y \to x \mid x \in \mathcal{P}_1 \text{ covers } y \in \mathcal{P}_2\} \) is an alien set for \( Q \). Let \( \alpha : x \to y \) be an alien arrow in \( F \). We suppose that there is another path from \( x \) to \( y \) in \( Q^F \), then there exists an alien arrow \( \alpha' : x' \to y' \) in \( Q^F \) such that \( x \leq x', y' \leq y \) and \( x \neq x' \) or \( y \neq y' \). However, in this case, \( y \) does not cover \( x \) which is a contradiction. Thus, \( F \) is an alien set for \( Q \) and \( \mathcal{P} \) is the poset \( \mathcal{P}_{Q^F} \) associated to the quiver \( Q^F \).

Now, we suppose that the assertion is true for any \( h \)-peak poset of type \( A \), for all \( 1 \leq h \leq r-1 \). Let \( \mathcal{P} \) be a \( r \)-peak poset of type \( A \). By Lemma 3.3 part (b)
we can choose a point \( z \in \max \mathcal{P} \) such that \( z \) has a unique neighbor. The peak-subposets \( \mathcal{P} = \{ z_1, \ldots, z_{r-1}, z \} \) and \( \mathcal{P}_z = z_h \) of \( \mathcal{P} \) are two posets of type \( \Lambda \), where \( \max \mathcal{P} = \{ z_1, \ldots, z_{r-1}, z \} \). By induction there are two Dynkin quivers \( Q' \) and \( Q'' \) of type \( \Lambda \) and two alien sets \( F' \) and \( F'' \) for \( Q' \) and \( Q'' \) respectively such that \( \mathcal{P} \) is the poset associated to the quiver \( Q'F'' \) and \( \mathcal{P}_z \) is the poset associated to the quiver \( Q''F' \). We suppose that \( z' \in (\max \mathcal{P}) \setminus \{ z \} \) is the neighbor of the point \( z \). By Lemma 3.3 part (a) we conclude that \( z_h \cap z_{h'} = \{ x \} \), where \( x \in \min \mathcal{P} \), in other words, \( x \) is a source vertex in \( Q' \) and \( Q'' \). Clearly \( \mathcal{P} \cap \mathcal{P}_z = \{ x \} \), otherwise \( z \) would have two neighbors. Now we are going to prove that the point \( x \) is an extremal vertex of both quivers \( Q' \) and \( Q'' \). Since \( Q'' \) has a unique sink vertex \( z \) and \( x \) is a source vertex in \( Q'' \) then \( x \) is an extremal vertex in \( Q'' \). Moreover, if \( x \) is a source vertex which is not an extremal vertex in \( Q' \) then \( \mathcal{R}_3 \) would be a peak-subposet of \( \mathcal{P} \) and in this way we get a contradiction. Then the quiver \( Q = (Q_0, Q_1) \) such that \( Q_0 = Q'_0 \cup Q''_0 \) and \( Q_1 = Q'_1 \cup Q''_1 \) is a Dynkin quiver of type \( \Lambda \). Note also that \( F = F' \cup F'' \) is an alien set for \( Q \) because there is no alien arrow ending at \( x \), otherwise \( \mathcal{R}_3 \) would be a peak-subposet of \( \mathcal{P} \). Furthermore, \( \mathcal{P} \) is the poset associated to the quiver \( QF' \).

The sufficiency of the assertion is proved as follows; let us suppose that \( \mathcal{P} \) is the poset \( \mathcal{P}_{QF} \) associated to a quiver \( QF' \), where \( Q \) is a Dynkin quiver of type \( \Lambda \) and \( F \) is an alien set for \( Q \), we shall prove that \( \mathcal{P} \) is of type \( \Lambda \). Locally an alien arrow \( \alpha \in F \) with \( s(\alpha), t(\alpha) \in \supp I(z) \), where \( z \) is a sink vertex in \( Q \) is such that \( s(\alpha) \neq z \), otherwise the quiver \( QF' \) would be cyclic. Then the maximal points in \( \mathcal{P} \) are exactly the sink vertices in the quiver \( Q \). Since the subposet \( z_h = Q_0^{(z)} \) of \( \mathcal{P} \) is a poset of width at most two then \( \mathcal{R}_1 \) is not a peak-subposet of \( \mathcal{P} \). On the other hand, by Lemma 3.3 part (b), if \( z, z' \in \max \mathcal{P} \) are neighbors and \( x \in z_h \cap z'_{h'} \) then \( x \in \min \mathcal{P} \), thus \( x \in \min \mathcal{P}_Q \). Since \( Q \) is a Dynkin quiver of type \( \Lambda \), then \( x \) is a source vertex in \( Q \). However, \( x \) is a non extremal vertex in \( Q \) because an alien arrow always connects two vertices in the same \( z \)-subquiver. Definition 3.3 part (b) implies that \( \mathcal{P} \) does not contain \( \mathcal{R}_2 \) as peak-subposet. Now, we suppose that \( \mathcal{P} \) contains \( \mathcal{R}_3 \) as peak-subposet, that is, there are three maximal points \( z, z', z'' \) in \( \mathcal{P} \) and a point \( x \in \mathcal{P} \) such that \( x \in z_h \cap z'_{h'} \cap z''_{h''} \). Thus, by the same arguments as above, \( z, z' \) and \( z'' \) are sink vertices in \( Q \). Moreover, since \( \mathcal{R}_3 \) is not a peak-subposet of \( \mathcal{P} \), then \( x \) is a minimal point in \( \mathcal{P} \) which implies that \( x \) is a source vertex in the quiver \( Q \). Moreover, since \( Q \) is a Dynkin quiver of type \( \Lambda \), we can suppose that there is no path in \( Q \) from \( x \) to \( z'' \); thus, Definition 3.3 implies that \( x = z'' \) in \( \mathcal{P} \), a contradiction. These arguments allow us to conclude that \( \mathcal{R}_3 \) is not peak-subposet of \( \mathcal{P} \). In the same way, we can see that for all \( n \geq 0 \), \( \mathcal{R}_{4,n} \) is not a peak-subposet of \( \mathcal{P} \).

A poset \( \mathcal{P} \) is said to be locally of width \( n \) or have local width \( n \) if \( n \) is the minimum integer such that for each \( z \in \max \mathcal{P} \) it holds that \( w(z_h) \leq n \). Clearly a poset of type \( \Lambda \) has local width less than or equal to two. The following lemma describes sincere posets of type \( \Lambda \) and their socle-projective indecomposable modules.

**Lemma 3.9.** Let \( \mathcal{P} \) be a poset of type \( \Lambda \). Then

(a) mod\( \Lambda_k \mathcal{P} \) is of finite representation type.
(b) \( \mathcal{P} \) is a sincere poset if and only if \( \mathcal{P} \) is isomorphic to one of the following posets:
For some $r \geq 1$. Furthermore, the module $M = (M_{x,y}h_x)_{x,y \in \mathcal{P}}$ in $\text{mod}_{sp}(k\mathcal{P})$ such that $M_x = k$ for all $x \in \mathcal{P}$ and $y h_x = id_k$ for each $x \leq y$ in $\mathcal{P}$ is the unique sincere indecomposable object in $\text{mod}_{sp}(k\mathcal{P})$.

Proof. According to Theorem 2.12 part (c), to prove the part (a) is enough to observe that no poset listed in [33, section 5] is a peak-subposet of $\mathcal{P}$. Indeed, the posets of the series $\mathcal{P}_{2,n+1}$, $\mathcal{P}_{2,n}$, $\mathcal{P}_{3,n}$, $n \geq 0$ and the poset $\mathcal{P}_{2,0}$ contain $\mathcal{R}_3$ as peak-subposet. Moreover, the posets of the series $\mathcal{P}_{2,n+1}$, $\mathcal{P}_{3,n}$, $n \geq 0$ contain $\mathcal{R}_1$ as peak-subposet and the posets of the series $\mathcal{P}_{3,n}$, $n \geq 0$ contain $\mathcal{R}_2$ as peak-subposet. Note that by definition $\mathcal{P}$ does not contain as a peak-subposet a poset of the series $\mathcal{P}_{1,n}$, $n \geq 0$. Moreover, we note that any poset of the form $\{\mathcal{P}_4, \ldots, \mathcal{P}_{110}\}$ contains as peak-subposet to $\mathcal{R}_i$ for some $i = 1, 2, 3$.

In order to prove (b), first we consider that $\mathcal{P}$ is one-peak poset. In this case, according to the list of sincere one-peak-posets (see [21]) we have that $\mathcal{P} = S^{(1)}_i$ for some $i = 1, 2, 3$. Moreover, we observe in the known lists of sincere $r$-peak posets of finite type that $\mathcal{F}_{1}^{(2)} = S^{(2)}_1$, $\mathcal{F}_{2}^{(2)} = S^{(2)}_2$, $\mathcal{F}_{3}^{(2)} = S^{(2)}_3$ are the sincere two-peak posets of type $\mathcal{A}$ (see [20]), $\mathcal{F}_{4}^{(3)} = S^{(3)}_1$, $\mathcal{F}_{5}^{(3)} = S^{(3)}_2$, $\mathcal{F}_{6}^{(3)} = S^{(3)}_3$, $\mathcal{F}_{7}^{(3)} = S^{(3)}_4$ are the sincere three-peak posets of type $\mathcal{A}$ (see [21]) and $\mathcal{F}_{8}^{(r)} = S^{(3)}_1$, $\mathcal{F}_{9}^{(r)} = S^{(3)}_2$, $\mathcal{F}_{10}^{(r)} = S^{(3)}_3$, $\mathcal{F}_{11}^{(r)} = S^{(3)}_4$ are the sincere $r$-peak posets of type $\mathcal{A}$, with $r \geq 4$ (see [22]). Thus, the first part of (b) is true. Now, we observe in the mentioned lists that for each $i = 1, 2, 3$, and for each $r \geq 1$ the sincere $r$-peak poset $S^{(r)}_i$ has only one sincere prinjective indecomposable $kS^{(r)}_i$-module $M = (M_{x,y}h_x)_{x,y \in S^{(r)}_i}$ such that $M_x = k$ and $y h_x = id_k$ for each $x \leq y$. In this way, Lemma 3.9 part (b) implies that the second part of (b) is true.

The following lemma will be used to prove the categorical equivalence proposed in Theorem 3.3.

Lemma 3.10. Let $\mathcal{P}$ be a poset of type $\mathcal{A}$ associated to the quiver $Q_\mathcal{P}$ as in Proposition 3.3. Then

(a) Up to isomorphism, any indecomposable $k\mathcal{P}$-module $M = (M_{x,y}h_x)_{x,y \in \mathcal{P}}$ in $\text{mod}_{sp}(k\mathcal{P})$ is such that $M_x = k$ and $y h_x = id_k$ for all $x \leq y$ in $\text{supp} M$.

(b) The support $\text{supp} M$ of an indecomposable object in the category $\text{mod}_{sp} k\mathcal{P}$ is connected as a subset of the quiver $Q$.

Proof. Let $M = (M_{x,y}h_x)_{x,y \in \mathcal{P}}$ be an indecomposable object in $\text{mod}_{sp}(k\mathcal{P})$. Then the image $\theta(M) = (\theta(M)_x)_{x \in \mathcal{P}}$ of $M$ by the adjustment functor $\theta$ defined in Equation (2.14) is an indecomposable object in $\mathcal{P}$-spr. Let $S = \text{csupp}(\theta(M))$ be the coordinate support of $\theta(M)$. Then the poset $S$ is a peak-subposet of $\mathcal{P}$. Thus, Definition 3.1 implies that $S$ is a poset of type $\mathcal{A}$. Indeed, if $R \in \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5\}$ is peak-subposet of $S$ then $R$ is peak-subposet of $\mathcal{P}$, a contradiction. Also, $S$ is a sincere poset because the peak $S$-space $(\theta(M)_x)_{x \in S}$ is sincere. Lemma 3.9 part (b) implies that for some $r \geq 1$ and $i = 1, 2, 3$, we have $S = S^{(r)}_i$. Moreover, by Proposition 2.9 we have that $\theta(M)$ is isomorphic to $T_S(L)$ where $L = (L_x)_{x \in S}$ is a sincere peak $S$-space in $S$-spr and $T_S$ is the subposet induced functor defined in
Thus, for each $z \in \mathcal{P}$ a peak-subposet of $\{x \in \mathbb{S} \mid y \leq x \in \mathcal{S}\}$ is a sincere module in $\text{mod}_y k\mathcal{S}$, and Lemma 3.9 then proves that $N_x = k$, for all $x \in \mathcal{S}$ and $y g_x = \text{id}_x$ for each $x \not\leq y$ in $\mathcal{S}$. Hence, Lemma 3.11 implies that $L = \theta(N)$. Now, by definition of $\theta$ we have that $L_z = k$ for all $z \in \max \mathcal{S}$. Let $\{e_z \mid z \in \max \mathcal{S}\}$ be the standard basis of the space $L^\bullet$, for then $x \in \mathcal{S}^-$ we have that $L_x$ is the subspace of $L^\bullet$ generated by the vector $w_x = \sum_{z \not\leq x} e_z$. Moreover, let $\hat{L} = (\hat{L}_x)_{x \in \mathcal{P}} = T \in (\mathcal{S}^\bullet)$ be the image of $L$ by the functor $T \in (\mathcal{S}^\bullet)$ then

$$L_x = \begin{cases} (w_x), & \text{if } x \in (\max \mathcal{S})_\triangle \cap \mathcal{S}^\vee, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the image of $\hat{L}$ by the functor $\rho$ is the $k\mathcal{P}$-module $M = \hat{L}_x = (\hat{L}_{x,y})_{x,y \in \mathcal{P}}$ where $y g_x : \hat{L}_x \to \hat{L}_y$ is such that $\lambda w_x \mapsto \lambda w_y$ if $x \leq y$ in $(\max \mathcal{S})_\triangle \cap \mathcal{S}^\vee$ and $y g_x = 0$ if $x \not\leq y$ and either $x$ or $y$ is not in $(\max \mathcal{S})_\triangle \cap \mathcal{S}^\vee$. It is easy to see that there is a natural isomorphism between $\hat{L}_x$ and the representation described in (a).

To prove (b) it is enough to see that the set $(\max \mathcal{S})_\triangle \cap \mathcal{S}^\vee$ is connected as a subset of the quiver $Q$ for any sincere peak-subposet $\mathcal{S}$ of $\mathcal{P}$. Note that the poset $\mathcal{S}^{\mathcal{P}}_1$ is a peak-subposet of $\mathcal{S} = \mathcal{S}^{(r)}$ for all $i = 1, 2, 3$. We suppose that

$$\mathcal{S}^{(r)}_1 = \{z_1 \triangleright x_2 \triangleright z_2 \triangleright z_3 \triangleright \cdots \triangleright z_{r-1} \triangleright x_r \triangleright z_r\}$$

then $\{z_1, \ldots, z_r\} \subseteq \max \mathcal{P}$ and since $R_2 \not\subseteq \mathcal{P}$ we have that $\{x_2, \ldots, x_r\} \subseteq \min \mathcal{P}$. Thus, for each $z_i$, with $2 \leq i \leq r - 1$ the $z_i$-subquiver $Q^{(z_i)}_0$ of $Q$ has the form $x_i \rightarrow \cdots \rightarrow z_i \rightarrow \cdots \rightarrow x_{i+1}$. Since each vertex in $Q^{(z_i)}_0$ belongs to the set $\{z_i\}_\triangle \cap \{x_i, x_{i+1}\}_\vee$, then $Q^{(z_i)}_0 \subset \text{supp} M$ for each $2 \leq i \leq r - 1$. Let $w'$ (respectively $w''$) be the left (respectively right) extremal vertex of the quiver associated to $\mathcal{S}$ and let $x$ (respectively $y$) be minimal element in $(x_z \cap (\mathcal{P} \setminus \mathcal{S})) \cup \{w\}$ (respectively $(x_y \cap (\mathcal{P} \setminus \mathcal{S})) \cup \{w''\}$) then it is easy to see that $\text{supp} M = [x, y]_Q$, where $[x, y]_Q$ denote a interval of $Q$ which is a connected subset of $Q$.\]

4. Category of sp-diagonals

In this section, we define a category $\mathcal{C}_T(F)$ of diagonals associated to a poset $\mathcal{P}$ of type $A$ and we prove in Theorem 4.3 and Corollary 4.5 that this category gives a geometric realization of the category of finitely generated socle-projective modules over the incidence $k$-algebra $k\mathcal{P}$.

Let $\mathcal{P}$ be a poset of type $A$ associated to the quiver $Q^F$ as in Proposition 3.8. Thus, $Q$ is a Dynkin quiver of type $A_n$ and $F$ is an alien set for $Q$. Let $T = \{\tau_1, \ldots, \tau_n\}$ be the triangulation of a $(n + 3)$-gon $\Pi_{n+3}$ such that $Q_T = Q$. A fan in $T$ is a maximal subset $\Sigma_\rho \subseteq T$ of at least two diagonals such that all the diagonals in $\Sigma_\rho$ share the vertex $v$ of $\Pi_{n+3}$. A diagonal $\tau \in \Sigma_\rho$ is said to be the peak-diagonal of $\Sigma_\rho$ if it is maximal in $\Sigma_\rho$ in accordance with the order $\tau_x < \tau_y$ if and only if there is a path from the vertex $x$ to the vertex $y$ in the quiver $Q$. Geometrically, the peak-diagonal of a fan $\Sigma_\rho$ is the diagonal that can be obtained from each other diagonal in $\Sigma_\rho$ by a clockwise rotation around the vertex $v$ (see Figure 2).

**Definition 4.1.** A diagonal $\gamma \not\in T$ is an sp-diagonal if it satisfies the following conditions:
(a) If \( \gamma \) crosses \( \tau \in T \) then \( \gamma \) crosses the peak-diagonal of a fan \( \Sigma \) in \( T \) such that \( \tau \in \Sigma \). Henceforth, any diagonal \( \gamma \notin T \) satisfying this condition will be called a \( \star \)-diagonal.

(b) For all alien arrows \( \alpha \in F \) with \( s(\alpha), t(\alpha) \in \text{supp} \ I(z) \), if \( \gamma \) crosses \( \tau_{s(\alpha)} \) and \( \tau \) then \( \gamma \) also crosses \( \tau_{t(\alpha)} \). Diagonals \( \gamma \notin T \) satisfying this condition will be called non-frozen diagonals.

**Example 4.2.** Let \( Q \) be the quiver in Example 3.4 then \( Q = Q_T \), where \( T \) is the following triangulation

![Fan of a triangulation](image)

In this case, the sets \( \{ \tau_1, \tau_2 \}, \{ \tau_2, \tau_3 \}, \{ \tau_3, \tau_4, \tau_5 \}, \{ \tau_5, \tau_6 \} \) and \( \{ \tau_6, \tau_7 \} \) are fans of \( T \). We have used bold font for the peak-diagonal of each fan. Note that, the peak-diagonal corresponds to a sink vertex in the quiver \( Q_T \). Moreover, let \( Q^k \) be the quiver in the Example 3.7. Then, the diagonals

\[
\gamma_1, \quad \gamma_2
\]

are such that \( \text{supp} \gamma_1 = \{3, 4\} \) and \( \text{supp} \gamma_2 = \{1, 2, 3\} \). Thus, \( \gamma_1 \) is not a \( \star \)-diagonal because it crosses \( \tau_4 \) but it does not cross the peak-diagonal \( \tau_5 \) in the unique fan \( \{ \tau_3, \tau_4, \tau_5 \} \) of \( \tau_4 \), whereas \( \gamma_2 \) is a \( \star \)-diagonal because it crosses \( \tau_2 \) which is the peak-diagonal in the fans \( \{ \tau_1, \tau_2 \} \) and \( \{ \tau_2, \tau_3 \} \) for \( \tau_1, \tau_2 \) and \( \tau_3 \).

Given the alien arrows \( \alpha : 3 \to 1 \) and \( \beta : 6 \to 4 \) (see Example 3.7), a diagonal \( \gamma \) is frozen by \( \alpha \) if \( \gamma \) crosses \( \tau_3 \) and \( \tau_2 \) but not \( \tau_1 \); whereas the diagonals frozen by \( \beta \) cross \( \tau_6 \) and \( \tau_5 \) but not \( \tau_4 \) (see Figure 5).

Note that, \( \gamma_2 \) is an sp-diagonal because it is non-frozen and \( \star \)-diagonal.

The following lemma describes the relation between \( \star \)-diagonals and socle-projective modules in \( \text{mod} \ kQ_T \).

**Lemma 4.3.** Let \( \Theta : \mathcal{C}_T \to \text{mod} \ kQ_T \) be the equivalence of categories of Theorem 2.3, where \( Q_T \) is a Dynkin quiver of type \( \Lambda \). Then \( \gamma \) is a \( \star \)-diagonal if and only if \( \Theta(\gamma) \) is socle-projective.
Proof. Since $Q_T$ is a Dynkin quiver of type $A$, then $T = \{\tau_x \mid x \in (Q_T)_0\}$ is a triangulation without internal triangles. First, we suppose that $\gamma$ is a $\star$-diagonal. Let $x$ be a vertex in $Q_T$ such that the indecomposable simple $kQ_T$-module $S(x)$ at vertex $x$ is a submodule of $\Theta(\gamma) = M^\gamma$. We shall prove that $S(x)$ is a projective $kQ_T$-module. Since $\text{Hom}(S(x), M^\gamma) \neq 0$, then $M^\gamma_x = k$, that is, $\tau_x$ crosses $\gamma$. By hypothesis, there exists a fan $\Sigma$ containing $\tau_x$ such that $\gamma$ crosses the peak-diagonal $\tau_z$ of $\Sigma$. If $x \neq z$ then $\tau_x < \tau_z$, that is, there is a path $p$ in $Q_T$ from $x$ to $z$ whose vertices are in $\text{supp} \gamma$. Moreover, a nonzero morphism $f = (f_x)_{x \in (Q_T)_0}$ of representations from $S(x)$ to $M^\gamma$ is such that $f_t = 0$ for all $t \neq x$ because $S(x)$ is the simple representation at vertex $x$. Let $x \to y$ be the arrow in $p$ starting in $x$, then the diagram

$$
\begin{array}{c}
S(x)_x \\
\downarrow f_x \\
M^\gamma_x \\
\end{array} \quad \longrightarrow \quad 
\begin{array}{c}
S(x)_y \\
\downarrow 0 \\
M^\gamma_y \\
\end{array}
$$

commutes because $f$ is a morphism of representations of the quiver $Q_T$. Since $S(x)_y$ is zero and $M^\gamma_x = M^\gamma_y = k$, then $f_x = 0$. Therefore, the morphism $f$ is zero, a contradiction. Thus, we conclude that $x = z$, that is, $\tau_x$ is a peak-diagonal. In other words, $x$ is a sink vertex in $Q_T$ and then $S(x)$ is projective. Since all simple submodules of $M^\gamma$ are projectives, we have that $\text{soc} M$ is projective.

In the other direction, we have that $\Theta(\gamma)$ is socle-projective. Let $\tau_x$ be a diagonal in $T$ crossing $\gamma$. If $\tau_x$ is a peak-diagonal then the definition of $\star$-diagonal is trivially satisfied. If $\tau_x$ is not a peak-diagonal, we suppose that for all fans $\Sigma$ containing $\tau_x$, $\gamma$ does not cross the peak-diagonal in $\Sigma$. We have that the number $s$ of fans containing $\tau_x$ is either one or two. In the case $s = 1$, let $\tau_y$ be the maximal diagonal in the fan $\Sigma$ which crosses $\gamma$. Then $\tau_x \leq \tau_y < \tau_z$, where $\tau_z$ is the peak-diagonal in $\Sigma$. In other words, there is a path $p$ from $x$ to $z$ in $Q_T$ passing by $y$, such that the vertices $x, \ldots, y$ in $p$ belong to $\text{supp} \gamma$, whereas the others vertices in $p$ are not in $\text{supp} \gamma$. In particular, $M^\gamma_x = M^\gamma_y = k$ and $M^\gamma_z = 0$. Let $S(y)$ be the simple representation of $Q_T$ at vertex $y$. Because the diagram

$$
\begin{array}{c}
S(y)_x \\
\downarrow 0 \\
M^\gamma_x \\
\end{array} \quad \longrightarrow \quad 
\begin{array}{c}
S(y)_y \\
\downarrow \lambda \\
M^\gamma_y \\
\end{array} \quad \longrightarrow \quad 
\begin{array}{c}
S(y)_z \\
\downarrow 0 \\
M^\gamma_z \\
\end{array}
$$



Figure 5. Diagonals frozen by $\alpha$ (left) and by $\beta$ (right).
commutes, we conclude that there is a nonzero injective morphism from $S(y)$ to $M'$. Therefore, $S(y)$ is a non-projective module which is a submodule of $M'$, a contradiction to the hypothesis. In the case $s = 2$, if $\tau_y$ (respectively $\tau_{y'}$) is the maximal diagonal in $\Sigma$ (respectively $\Sigma'$) crossing $\gamma$. By the above arguments, we conclude that $S(y)$ and $S(y')$ are non-projective summands of $\text{soc} M'$, a contradiction to the hypothesis. Therefore, $\gamma$ is a $*$-diagonal. \hfill $\Box$

Let $\mathcal{C}(T,F)$ be the full subcategory of the category of diagonals $\mathcal{C}_T$ generated by all sp-diagonals in $\mathcal{C}_T$. We denote by $E(T,F)$ the set whose elements are the sp-diagonals in $\mathcal{C}(T,F)$, the diagonals in $T$, and the boundary edges in $\Pi_{n+3}$.

Since the irreducible morphisms in $\mathcal{C}(T,F)$ cannot be factorized through sp-diagonals, we introduce the notion of a pivoting sp-move from $\gamma \in E(T,F)$ to $\gamma' \in E(T,F)$, that is, a composition of pivoting elementary moves of the form

$$P : \gamma = \gamma_0 \xrightarrow{P_1} \gamma_1 \xrightarrow{P_2} \ldots \xrightarrow{P_s} \gamma_s = \gamma'$$

with the same pivot $v$ such that $\gamma_1, \ldots, \gamma_{s-1}$ are not sp-diagonals in $\Pi_{n+3}$. Note that the irreducible morphisms in $\mathcal{C}(T,F)$ are precisely the pivoting sp-moves between sp-diagonals.

Next, we analyze the relations in the category $\mathcal{C}(T,F)$. These come from the mesh relations in $\mathcal{C}_T$. We suppose that $\gamma$ and $\gamma'$ are sp-diagonals and that the compositions $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'\gamma$ and $\gamma \xrightarrow{P_1} \beta' \xrightarrow{P_2} \gamma'$ of two pivoting sp-moves are as in Figure 6. We have that

$$\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma' = \gamma \xrightarrow{P_1} \beta' \xrightarrow{P_2} \gamma'',$$

taking into account the following convention: If one of the intermediate edges ($\beta$ or $\beta'$) is either a boundary edge or a diagonal in $T$, the corresponding term in the identity is replaced by zero.

![Figure 6. Mesh relations in $\mathcal{C}(T,F)$.](image)

4.1. **The functor $\Omega$.** Let $\mathcal{P}$ be the poset of type $\mathbb{A}$ associated to the quiver $Q^F$, where $Q$ is a quiver of Dynkin type $\mathbb{A}$ and $F$ an alien set for $Q$ and denote by $T$ a triangulation associated to $Q$. Let us define a $k$-linear additive functor

$$\Omega : \mathcal{C}(T,F) \rightarrow \text{mod}_{\text{sp}}(k\mathcal{P})$$

from the category of sp-diagonals to the category of finitely generated socle-projective $k\mathcal{P}$-modules such that for any sp-diagonal $\gamma$ we have $\Omega(\gamma) = M^\gamma = (M^\gamma_x, yh^\gamma_x)$ where
$M^\gamma$ is defined by the following identities:

\[
M^\gamma_x = \begin{cases} 
k & \text{if } x \in \text{supp } \gamma, \\
0 & \text{otherwise.}
\end{cases}
\text{ and if } x \preceq y \in \mathcal{P} \text{ then } y h_x^\gamma = \begin{cases} 
\text{id}_k & \text{if } x, y \in \text{supp } \gamma, \\
0 & \text{otherwise.}
\end{cases}
\]

Now, we define the functor $\Omega$ on morphisms. By additivity, it is sufficient to define the functor on morphisms between sp-diagonals. Our strategy is to define the functor on morphisms between sp-moves and then check that the mesh relations in $\mathcal{C}(T,F)$ hold. For any pivoting sp-move $P : \gamma \to \gamma'$, we define the morphism

\[
\Omega(P) = (\Omega(P)_x)_{x \in \mathcal{P}} : (M^\gamma_x, y h_x^\gamma) \to (M'^\gamma_x, y h_x'^\gamma)
\]

by the formula

\[
\Omega(P)_x = \begin{cases} 
\text{id}_k, & \text{if } M^\gamma_x = M'^\gamma_x = k, \\
0, & \text{otherwise.}
\end{cases}
\]

By definition, $\Omega$ maps compositions of pivoting sp-moves to compositions of the images of the pivoting sp-moves. Note that if $\mathcal{P}$ is the poset $\mathcal{P}_Q$ associated to a Dynkin quiver $Q$ of type $A$ (without alien arrows) then the functor $\Omega$ is the restriction of the functor $\Theta$ defined in section 2.1 to the full subcategory of $\mathcal{C}_T$ generated by the $\star$-diagonals in $\mathcal{C}_T$.

Now, we prove that the functor $\Omega$ is well-defined and that it is an equivalence of categories.

**Theorem 4.4.** $\Omega$ is an equivalence of categories.

**Proof.** Recall here that $\mathcal{P}$ and $\mathcal{P}_Q$ are two different posets, that they have the same vertices and that $\mathcal{P}$ is obtained from $\mathcal{P}_Q$ by adding edges to the Hasse diagram corresponding to the alien arrows in $F$. In particular, $x \prec y$ in $\mathcal{P}_Q$ implies $x \prec y$ in $\mathcal{P}$.

In order to prove that $M^\gamma \in \text{mod}_g(k\mathcal{P})$ we have to prove the conditions (a) and (b) in Proposition 2.6. To prove (a) it is enough to consider the non trivial situation when $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \text{supp } \gamma$ and $y \notin \text{supp } \gamma$. First we note that, by Lemma 2.2, if $x \prec y \prec w$ in $\mathcal{P}_Q$ then $y \in \text{supp } \gamma$ which is contradictory. Therefore, we have that $x \nleq y$ or $y \nleq w$ in $\mathcal{P}_Q$. We consider the following cases (recall that we always suppose $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \text{supp } \gamma$ and $y \notin \text{supp } \gamma$).

**Case 1** $y \prec w$ in $\mathcal{P}_Q$ and $x \nleq y$ in $\mathcal{P}_Q$. In this case, there exists an alien arrow $\alpha : x' \to y'$ on vertices of a $g$-subquiver $Q^{(z)}$ of $Q$ such that $x \preceq x' \prec z$ and $y \preceq y' \prec w \prec z$ in $\mathcal{P}_Q$. Since $w \in \text{supp } \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$, then $\gamma$ crosses $\tau_z$. Thus, since $x \in \text{supp } \gamma$ Lemma 2.2 implies that $\gamma$ crosses $\tau_{x'}$, and since $\gamma$ is non-frozen, then $\gamma$ crosses $\tau_{y'}$. Again using Lemma 2.2 we obtain that $\gamma$ crosses $\tau_y$, that is, $y \in \text{supp } \gamma$ which is contradictory.

**Case 2** $x \prec y$ in $\mathcal{P}_Q$ and $y \nleq w$ in $\mathcal{P}_Q$. In this case, there exists an alien arrow $\alpha : y' \to w'$ on vertices of a $g$-subquiver $Q^{(z)}$ such that $x \prec y' \prec z$ and $w' \preceq w \prec z$ in $\mathcal{P}_Q$. Since $w \in \text{supp } \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$, then $\gamma$ crosses $\tau_z$. Thus, since $x \in \text{supp } \gamma$ Lemma 2.2 implies that $\gamma$ crosses $\tau_y$, in other words, $y \in \text{supp } \gamma$ which cannot be.
Case 3 $x \neq y$ and $y \neq w$ in $P_Q$. In this case, there exist two alien arrows $\alpha : x' \to y'$ and $\alpha : y'' \to w'$ on vertices of a $z$-subquiver $Q^{(z)}$ of $Q$ such that $x \preceq x' \prec w' \preceq w \prec z$ and $y \preceq y'' \prec z$ in $P_Q$. Since $w \in \text{supp} \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$, then $\gamma$ crosses $\tau_z$. Thus, Lemma 2.2 implies that $\gamma$ crosses $\tau_{y'}$, and since $\gamma$ is non frozen we conclude that $\gamma$ crosses $\tau_{y'}$.

Again using Lemma 2.2 we obtain that $\gamma$ crosses $\tau_y$, that is, $y \in \text{supp} \gamma$ which is contradictory.

We have shown that if $x \prec y \prec w$ in $P = P_{Q^F}$ such that $x, w \in \text{supp} \gamma$ then $y \in \text{supp} \gamma$. Thus, $w h_{k}^\gamma = y h_{k}^\gamma = w h_{k}^\gamma = id_k$ and condition (a) holds. To prove condition (b), let $x$ be an element of $P^- = P \setminus \text{max} P$. If $x \notin \text{supp} \gamma$ then clearly $z h_{k}^\gamma = 0$ for all $z \in \text{max} P$ such that $x \prec z$. If $x \in \text{supp} \gamma$ then $z h_{k}^\gamma = id_k$ for some $z' \in \text{max} P$, where $\tau_{z'}$ is the peak-diagonal in some fan containing $\tau_x$. Thus, $\bigcap_{z' \in \text{max} P} \ker z h_{k} = 0$ for all $x \in P^-$ such that $x \prec z$. This shows that $\Omega(\gamma) = M^\gamma$ is indeed an object in $\text{mod}_{sp}(k P)$.

Let us now check that $\Omega(P)$ is well defined for every pivoting sp-move $P : \gamma \to \gamma'$. Indeed, it is enough to show that for any relation $x \prec y$ such that $y$ covers $x$ in $P$ the diagram

$$
\begin{array}{ccc}
M^\gamma_x & \xrightarrow{h_{k}^\gamma} & M^\gamma_y \\
\downarrow{\Omega(P)_x} & & \downarrow{\Omega(P)_y} \\
M^{\gamma'}_x & \xrightarrow{h_{k}^\gamma} & M^{\gamma'}_y
\end{array}
$$

commutes. Note that the result holds if $M^\gamma_x = 0$ or $M^{\gamma'}_y = 0$ and also if both $M^\gamma_y$ and $M^{\gamma'}_x$ are null spaces. Suppose now that $M^\gamma_x = M^{\gamma'}_y = k$. If $M^\gamma_y = M^{\gamma'}_x = k$, then all four maps are $id_k$ and the diagram commutes. The only remaining case is if exactly one of $M^\gamma_y$, $M^{\gamma'}_x$ is nonzero. We will show that this cannot happen. Suppose that $M^\gamma_y = 0$ and $M^{\gamma'}_x = k$, that is, $x, y \in \text{supp} \gamma$, $y \in \text{supp} \gamma'$ and $x \notin \text{supp} \gamma'$. Since $y$ covers $x$ in $P$, there exists an arrow $\alpha : x \to y$ in $Q^F$. If $x \prec y$ in $P_Q$ then $\alpha$ is an arrow in $Q$, that is, $\tau_x$ and $\tau_y$ share a vertex of the polygon and are connected by a pivoting elementary move. Since $P : \gamma \to \gamma'$ is a pivoting sp-move we get that $\tau_x$ crosses $\gamma$, that $\tau_x$ and $\gamma'$ have a common point on the boundary of the polygon and that $\tau_y$ crosses $\gamma$ and $\gamma'$. This implies that $\tau_y$ is clockwise from $\tau_x$ and that contradicts the orientation $x \to y$ in the quiver $Q$ (see Figure 3). Next, we suppose that $x \not\prec y$ in $P_Q$, then $\alpha : x \to y$ is an alien arrow in $F$ with $x$ and $y$ in $\text{Supp} I(z)$ for some sink vertex $z$ in $Q$. Now, by Definition 3.3 part (b), $y$ is not a source vertex in $Q$ unless $y$ is an extremal vertex in $Q$. Thus, there is at most one arrow in $Q$ with starting point $y$, and therefore there is exactly on fan $\Sigma$ containing $\tau_y$ and $\tau_x$ is its peak-diagonal. By Definition 4.1 both $\gamma$ and $\gamma'$ cross $\tau_z$, because they are $\star$-diagonals crossing $\tau_y$. 


On the other hand, there is a pivoting path from τ_{z} to τ_{x} in Π_{n+3}, since x belongs to Supp I(z). But this is impossible, because if τ → τ_{x} is a pivot, then τ does not cross γ'. The other case where M_{x}^{γ'} = k and M_{y}^{γ} = 0 is proved in a similar way.

To show that the functor Ω is well defined, it only remains to check the mesh relations. Indeed, let γ \xrightarrow{P_{1}} β, β \xrightarrow{P_{2}} γ', γ \xrightarrow{P_{3}} β', β' \xrightarrow{P_{4}} γ' be pivoting sp-moves as in Figure 6 with γ, γ' sp-diagonals and β \neq β' sp-diagonals, diagonals in T or boundary edges. Note that, we can exclude the case where β and β' are both diagonals in the triangulation T or both boundary edges because in this case either γ or γ' is a diagonal in T. Without loss of generality, we may assume from now on that β is an sp-diagonal. Suppose first that β' is an sp-diagonal; then one has to check the commutativity of the following diagram

\[ M_{x}^{γ} \xrightarrow{Ω(P_{1})_{x}} M_{x}^{β} \xrightarrow{Ω(P_{2})_{x}} M_{x}^{γ'} \]

for all x ∈ P. Again, the only non trivial case happens when M_{x}^{γ} = M_{x}^{γ'} = k. In this case we also have M_{x}^{β} = M_{x}^{β'} = k because any diagonal crossing both γ and γ' must also crosses β and β'. Thus all maps are id_{k} and the diagram commutes.

Suppose now that β' is a boundary edge or diagonal in T. Then we have to show that the composition M_{x}^{γ} \xrightarrow{Ω(P_{1})_{x}} M_{x}^{β} \xrightarrow{Ω(P_{2})_{x}} M_{x}^{γ'} is zero for all x ∈ P. Clearly if β' is a boundary edge or diagonal in T then no diagonal τ ∈ T can cross both γ and γ' then Hom(Ω(γ), Ω(γ')) = 0.

In order to prove that Ω is dense we fix an indecomposable M ∈ mod_{sp}(kP). Then by Lemma 3.10 part (b), Lemma 2.2 and Theorem 2.3 part (a) there exists a diagonal γ \notin T such that supp γ = supp M. We show that γ is an sp-diagonal. Indeed, since the socle of M is projective, Lemma 4.3 implies that γ is a ⋆-diagonal. Moreover, given an alien arrow α : x → y in F, with x and y in Supp I(z) for some sink vertex z in Q_{0} such that x, z ∈ supp M then zh_{x} = id_{k}. By Proposition 2.6 part (a), we have that zh_{x} = zh_{y} · yh_{x}, thus y ∈ supp M. Therefore γ crosses τ_{y} and thus γ is a non-frozen diagonal. We conclude that γ is an sp-diagonal and that Ω(γ) = M.

To show that Ω is faithful, it is enough to prove that the image of a nonzero morphism between sp-diagonals is a nonzero morphism in mod_{sp}(kP). Indeed, let
$P \in \text{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$ be a nonzero morphism in $\mathcal{C}_{(T, F)}$. Then $P$ also is a nonzero morphism in $\mathcal{C}_T$. Lemma 4.4 implies that there exists a diagonal $\tau_x \in T$ crossing $\gamma$ and $\gamma'$ as in Figure 2. In particular, $M_x = M_x' = k$, and therefore $\Omega(P)_x = \text{id}_k \neq 0$.

Finally, we show that functor $\Omega$ is full. To do so, let $\Omega(\gamma) \xrightarrow{g} \Omega(\gamma')$ be a nonzero morphism in $\text{mod}_{sp}(k\mathcal{P})$. Then $g = (g_x)_{x \in Q_0}$, where $g_x$ is a linear map from $\Omega(\gamma)_x$ to $\Omega(\gamma')_x$. The map $\hat{g} = (\hat{g}_x)_{x \in Q_0}$ from $\Theta(\gamma)$ to $\Theta(\gamma')$ such that $\hat{g}_x = g_x$ is a morphism of representations in $\text{mod } k\mathcal{Q}$. Indeed, for each arrow $\alpha : x \to y$ in $Q_1$, we have $x \prec y \in \mathcal{P}$. Since $g$ is morphism in $\text{mod}_{sp} k\mathcal{P}$, then the diagram

$$
\begin{array}{ccc}
\Omega(\gamma)_x & \xrightarrow{y h^2_x} & \Omega(\gamma)_y \\
\downarrow g_x & & \downarrow g_y \\
\Omega(\gamma')_x & \xrightarrow{y h^2'_x} & \Omega(\gamma')_y
\end{array}
$$

commutes. Note that the elements in $\mathcal{P}$ are the vertices in $Q_0$. Moreover, if $\gamma$ is an sp-diagonal then the representations $\Theta(\gamma) = (\Theta(\gamma)_x, f^\gamma_x)$ in $\text{mod } k\mathcal{Q}$ and $\Omega(\gamma) = (\Omega(\gamma)_x, y h^2_x)$ in $\text{mod}_{sp} (k\mathcal{P})$ have the same $k$-vector spaces $\Omega(\gamma)_x = \Theta(\gamma)_x$ for all $x \in \mathcal{P}$ and the same maps $y h^2_x = f^\gamma_x$ for each $\alpha : x \to y$ in $Q_1$ (the map $f^\gamma_x$ is not defined when $\alpha$ is an alien arrow for $Q$). Thus we have a commutative diagram

$$
\begin{array}{ccc}
\Theta(\gamma)_x & \xrightarrow{f^\gamma_x} & \Theta(\gamma)_y \\
\downarrow \hat{g}_x & & \downarrow \hat{g}_y \\
\Theta(\gamma')_x & \xrightarrow{f'^\gamma_x} & \Theta(\gamma')_y
\end{array}
$$

and hence the map $\hat{g}$ is a morphism in $\text{mod } k\mathcal{Q}$. Under the equivalence of categories $\Theta : \mathcal{C}_T \to \text{mod } k\mathcal{Q}_T$ of Theorem 2.3, the morphism $\hat{g}$ corresponds to a morphism $P \in \text{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$, with $\Theta(P) = \hat{g}$. Since $\gamma$ and $\gamma'$ are sp-diagonals in $\mathcal{C}_T$, $P$ also is a morphism in the full subcategory $\mathcal{C}_{(T, F)}$ of $\mathcal{C}_T$. The definition of the functors $\Theta$ and $\Omega$ on morphisms implies that $\Omega(P) = g$.

The following corollary is an direct consequence of the arguments used in Theorem 4.3 and section 4.

**Corollary 4.5.** Let $\mathcal{P}$ be a poset of type $A$ associated to the quiver $Q^F$ as in Proposition 3.8 and let $\mathcal{C}_{(T, F)}$ be the corresponding category of sp-diagonals. Then

(a) The irreducible morphisms of $\mathcal{C}_{(T, F)}$ are direct sums of the generating morphisms given by pivoting sp-moves.

(b) Let $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ be a composition of two pivoting sp-moves as in Figure 2 where $\gamma$, $\gamma'$, and $\beta$ are sp-diagonals. Then

(i) The sequence $0 \to \gamma \to \beta \to \beta' \to \gamma' \to 0$ is an AR-sequence if $\beta'$ is a sp-diagonal.

(ii) The sequence $0 \to \gamma \to \beta \to \gamma' \to 0$ is an AR-sequence if $\beta'$ is either a boundary edge or a diagonal in $T$.

(iii) If $\beta' \notin E(T, F)$ then $\gamma'$ is an indecomposable projective in $\mathcal{C}_{(T, F)}$ and $\gamma$ is an indecomposable injective in $\mathcal{C}_{(T, F)}$. 

Example 4.6. Let $Q$ and $F$ as in Example 3.6. Then the triangulation $T$ associated to $Q$ has the form

The AR-quiver $\Gamma(C_T)$ of the category $C_T$ has the shape

Here, we have drawn the polygons with sp-diagonals using red color, that is, the diagonals $\gamma$ such that $\gamma$ crosses $\tau_3$ and if $\gamma$ crosses $\tau_5$ then $\gamma$ crosses $\tau_2$. Hence, the AR-quiver $\Gamma(C_{(T,F)})$ of the category $C_{(T,F)}$ is the red part of $\Gamma(C_T)$, where dotted lines have been drawn to describe the action of the AR-translation in $\Gamma(C_{(T,F)})$.

Example 4.7. Let $\mathcal{P} = \mathcal{P}_{Q^F}$ be the three-peak poset of type $A$ defined in Example 3.2 which is the poset associated to the quiver $Q^F$ in Example 3.7. Recall that, $\mathcal{P}$ can be viewed as a Dynkin quiver of type $E_7$. Thus, the AR-quiver $\Gamma(\text{mod}(k\mathcal{P}))$ of the module category $\text{mod}(k\mathcal{P})$ can be built using the knitting algorithm (see [29]) and it has the form
In the diagram, we have drawn the dimensions of indecomposable socle-projective modules with red color. Hence, the AR-quiver $\Gamma(\text{mod}_{sp}(k\mathcal{P}))$ of the category of finitely generated socle-projective modules $\text{mod}_{sp}(k\mathcal{P})$ has the form

On the other hand, since the triangulation $T$ associated to the quiver $Q$ was described in Example 4.2, the AR-quiver $\Gamma(C_T)$ of the category of diagonals $C_T$ has the form

Here, we have drawn the polygons with sp-diagonals using red color. Hence, the AR-quiver $\Gamma(C_{(T,F)})$ of the category $C_{(T,F)}$ is identified with the AR-quiver $\Gamma(\text{mod}_{sp}(k\mathcal{P}))$. 
5. Associated subalgebra of the cluster algebra

Let \( \mathcal{P} \) a poset of type \( A \) and let \( Q^F \) be the quiver associated to \( \mathcal{P} \) as in Proposition 3.8. We denote by \( \mathcal{A} = \mathcal{A}(x, Q) \) the cluster algebra associated to the initial seed \( (x, Q) \) [18]. It is well-known that the initial cluster variables in \( x \) correspond to the shift of indecomposable projectives in the cluster category (see [9]). Let \( \mathcal{A}(\mathcal{P}) \) be the subalgebra of \( \mathcal{A} \) generated by the cluster variables \( x_i \) such that \( \gamma \) is an sp-diagonal in the category \( \mathcal{C}(T, F) \) together with the cluster variables in the initial cluster \( x \). It is a natural to ask under which conditions we have \( \mathcal{A}(\mathcal{P}) = \mathcal{A} \). A partial answer is given in Theorem 5.2.

**Lemma 5.1.** Let \( Q \) be a quiver of tree type with \( n \) vertices and let \( \mathcal{A} = \mathcal{A}(x, Q) \) be the cluster algebra associated to \( Q \) with initial cluster \( x = \{x_1, \ldots, x_n\} \). If \( \mathcal{A}' \) is the subalgebra of \( \mathcal{A} \) generated by the cluster variables \( x_1, \ldots, x_n, x_{P_1}, \ldots, x_{P_r} \), where for all \( i = 1, \ldots, n \), \( x_{P_i} \) is the cluster variable associated to the indecomposable projective \( kQ \), then \( \mathcal{A}' = \mathcal{A} \).

**Proof.** Because of [7, Corollary 1.21] it suffices to show that \( \mathcal{A}' \) contains the initial cluster \( x_1, \ldots, x_n \) as well as the \( n \) cluster variables \( x'_1, \ldots, x'_n \) obtained from the initial cluster by a single mutation. We proceed by induction on the number \( n \) of vertices in \( Q \). The case \( n = 1 \) is trivial. Now, let us consider \( Q \) a tree with \( n \) vertices, then \( Q \) has \( n-1 \) arrows. Let \( w \) be a leaf of \( Q \) and define \( Q' \) to be the full subquiver of \( Q \) whose vertices are \( Q_0 \setminus \{w\} \). Then \( Q \) is obtained from \( Q' \) by adding one vertex \( w \) and one arrow \( \alpha_w \) that starts or ends at \( w \). We have two cases: either (i) \( \alpha_w : t \to w \) or (ii) \( \alpha_w : w \to t \) for some \( t \in Q_0 \). We recall the so-called exchange relation

\[
x'_k x_k = p_k^- + p_k^+,
\]

defined for any vertex \( k \) in \( Q \), where \( p_k^- = \prod_{\alpha: r \to k} x_r \) and \( p_k^+ = \prod_{\beta: k \to r} x_r \). Here the product \( p_k^- \) (respectively \( p_k^+ \)) is taken over all arrows \( \alpha \in Q_1 \) (respectively \( \beta \in Q_1 \)) that terminate (respectively start) in vertex \( k \). We shall prove that the variables \( x'_j \) and \( x'_l \) belong to \( \mathcal{A}' \). In case (i), \( w \) is a sink vertex and then \( x'_w = x_{P_w} \). Hence, \( x'_w \in \mathcal{A}' \). Additionally, following the knitting algorithm, we have that

\[
x_{P_i} x_t = 1 + p_i \prod_{\beta: \beta \to r} x_{P_r},\tag{5.2}
\]

where the product is taken over all arrows \( \beta \in Q_1 \) that start in vertex \( t \). We multiply (5.2) by \( p_i \) and we obtain \( x_{P_i} x_t p_i^+ = p_i^+ + p_i \prod_{\beta: \beta \to r} x_{P_r} \). Since \( \alpha_w \) is an arrow from \( t \) to \( w \), then \( x_{P_i} x_t p_i^+ = p_i^+ + p_i x_{w=1} x_{P_w} \delta \) where the product \( \delta = \prod_{\beta: \beta \to r \neq w} x_{P_r} \) is taken over all arrows \( \beta \in Q_1 \) that start in vertex \( t \) and terminate in a vertex \( r \neq w \). Also, \( x_{P_i} x_t p_i^+ = p_i^+ + p_i (1 + x_t) \delta \) because \( x_w x_{P_w} = x'_w x_w = 1 + x_t \). Since \( x_{P_i} p_i^+ \in \mathcal{A}' \) we have

\[
x_{P_i} p_i^+ = \frac{p_i^+ + p_i (1 + x_t) \delta}{x_t} = \frac{p_i^+ + p_i^- \delta}{x_t} \in \mathcal{A}' .
\]

Since \( p_i^- \delta \) belongs to \( \mathcal{A}' \), equation (5.1) implies \( x'_t \in \mathcal{A}' \). In case (ii), we have

\[
x_{w=1} x_{P_w} = 1 + x_{P_i} .
\]
Multiplying (5.3) by $x_t$ and using (5.2) we deduce that
\[ x_w x_P x_t = x_t + 1 + p_t^{-1} \prod_{\beta : t \to r} x_{P_{\beta}}. \]
Since $x_P, x_t \in A'$ then
\[ x_P x_t = \frac{x_t + 1 + p_t^{-1} \prod_{\beta : t \to r} x_{P_{\beta}}}{x_w} \in A'. \]
Moreover, since there is an arrow $\alpha_w$ from $w$ to $t$ then $x_w$ is a factor of $p_t^{-1}$; thus, $x'_w = \frac{1 + p_t^{-1}}{x_w} \in A'$. Analogous to the proof of the case (i), we can prove that $x'_t \in A'$.

As a consequence of the hypothesis of induction on the quiver $Q'$ the variables $x'_s$ with vertex $s \neq t$ in $Q'_0$ belong to $A'$. Thus, [7, Corollary 1.21] implies the result. □

**Theorem 5.2.** Let $\mathcal{P}$ be a poset of type $A$ associated to the quiver $Q^0$ as in Proposition 3.8 and let $A(\mathcal{P})$ be the subalgebra of $A$ associated to $\mathcal{P}$. Then $A(\mathcal{P}) = A$.

**Proof.** In this case, the poset $\mathcal{P}$ is viewed as the quiver $Q$ of type $A$. Then, the subcategory $C_{(T,F)}$ of $C_T$ is given by $\ast$-diagonals because $F = \emptyset$ and it is equivalent to the category $\mod_{sp} kQ$ of socle-projective $kQ$-modules (see Theorem 2.3). By Theorem 2.3 part (e) the indecomposable projectives in $\mod kQ$ can be identified with diagonals $r^+(T)$ in the category $C_T$ which are clearly $\ast$-diagonals. Hence, the category $A(\mathcal{P})$ contains the clusters variables described in the hypothesis of the above Lemma. Moreover, $Q$ is a tree quiver. As a consequence, $A(\mathcal{P}) = A$. □

**Acknowledgements**

The second author would like to express his sincere gratitude to his advisors Agustín Moreno Cañadas and Hernán Giraldo for helpful suggestions and valuable discussions on the topic.

**References**

[1] D.M. Arnold, *Abelian Groups and Representations of Finite Partially Ordered Sets*, CMS Books in Mathematics, vol. 2, Springer, 2000.
[2] E. Barnard, E. Gunawan, E. Meehan, and R. Schiffler, *Cambrian combinatorics on quiver representations (type A)*, arXiv:1912.02840 (2019).
[3] K. Baur and R. Marsh, *A geometric description of the $M$-cluster categories of type $D_n$*, Int. Math. Res. Not. (2007), 1–19.
[4] K. Baur and R.J. Marsh, *A geometric description of the $M$-cluster categories*, Trans. Amer. Math. Soc. 360 (2008), no. 11, 5789–5803.
[5] K. Baur and H. Torkildsen, *A Geometric Interpretation of Categories of Type $\tilde{A}$ and of Morphisms in the Infinite Radical*, Alg. Represent. Theory (2019), 1–36.
[6] K. Baur and R.C. Simões, *A geometric model for the module category of a gentle algebra*, arXiv:1803.05802 (2018).
[7] A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), no. 1, 1–52.
[8] T. Brüstle and J. Zhang, *On the Cluster Category of a Marked Surface without punctures*, Algebra Number Theory 5 (2011), no. 4, 529-566.
[9] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572–618.
[10] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 (2006), no. 3, 1347–1364.
[11] A.M. Cañadas and A.G. Zavadskij, *Categorical description of some differentiation algorithms*, J Algebra Appl. 5 (2006), no. 5, 629-652.
[12] A.M. Cañadas and V.C. Vargas, *On the apparatus of differentiation DI–DV for posets*, São Paulo J. Math. Sci. (2019).

[13] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 2002.

[14] L. David-Roesler and R. Schiffler, *Algebras from surfaces without punctures*, J. Algebra 350 (2012), 218–244.

[15] L. Demonet and X. Luo, *Ice quivers with potential associated with triangulations and Cohen-Macaulay modules over orders*, Trans. Amer. Math. Soc. 368 (2016), 4257–4293.

[16] L. Demonet and S. Taniguchi, *Ice quivers with potential arising from once-punctured polygons and Cohen-Macaulay modules*, J. Eur. Math. Soc. 52 (2016), 141–205.

[17] P. Dowbor and S. Kasjan, *Galais covering technique and tame non-simply connected posets of polynomial growth*, J. Pure Appl. Alg. 147 (2000), 1–24.

[18] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[19] P. Gabriel, *Indecomposable representations I*, Manuscripta Math. 6 (1972), 71–103.

[20] J. Kosakowska, *Classification of sincere two-peak posets of finite prinjective type and their sincere prinjective representations*, Colloq. Math. 86 (2001), 27–77.

[21] ______, *Sincere posets of finite prinjective type with three maximal elements and their sincere prinjective representations*, Colloq. Math. 93 (2002), 155–208.

[22] ______, *Indecomposable sincere prinjective modules over multiple sincere posets of finite prinjective type with at least four maximal elements*, Representations of algebras II (Beijing, 2000), Vol. 2, Beijing Normal University Press, 2002, pp. 253–291.

[23] S. Kasjan and D. Simson, *Varieties of poset representations and minimal posets of wild prinjective type*, Representations of algebras (Ottawa, ON, 1992), CMS Conf. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1993, pp. 245–284. MR1206948

[24] M. Kleiner, *On the exact representations of partially ordered sets of finite type*, J. Soviet Math. 3 (1975), no. 5, 616–628.

[25] L.A. Nazarova and A.V. Roiter, *Representations of partially ordered sets*, Zap. Nauchn. Sem. LOMI 28 (1972), 5–31; English transl. in J. Soviet Math. 3 (1975).

[26] S. Opper, P-G. Plamondon, and S. Schroll, *A geometric model for the derived category of gentle algebras*, arXiv:1801.09659v5 (2018).

[27] J.A. de la Peña and D. Simson, *Prinjective modules, reflection functors, quadratic forms and Auslander-Reiten sequences*, Trans. Amer. Math. Soc. 329 (1992), no. 2, 733–753.

[28] R. Schiffler, *A geometric model for cluster categories of type D_n*, J. Algebraic Combin. 27 (2008), no. 1, 1–21.

[29] ______, *Quiver Representations*, CMS Books in Mathematics, Springer International Publishing, Switzerland, 2014.

[30] D. Simson, *Peak reductions and waist reflection functors*, Fund. Math. 137, 115–144.

[31] ______, *Two-peak posets of finite prinjective type*, Representations of Finite Dimensional algebras (Tsukuba University, Japan, 1990), CMS Conf. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 1991, pp. 287–298.

[32] ______, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, London, London, 1992.

[33] ______, *Posets of finite prinjective type and a class of orders*, J. Pure Appl. Algebra 90 (1993), 77–103.

[34] R. Stanley, *Enumerative Combinatorics, Second Edition*, Vol. I, Cambridge University Press, 2012.

[35] A.G. Zavadskij, *On two-point differentiation and its generalization*, Contemp. Math. 376 (2005), 413.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269-3009, USA

E-mail address: schiffler@math.uconn.edu

SCHOOL OF MATHEMATICS AND STATISTICS, PEDAGOGICAL AND TECHNOLOGICAL UNIVERSITY OF COLOMBIA, TUNJA, BOYACÁ 150001

E-mail address: robinson.serna@uptc.edu.co