Multiplicative functions with sum zero

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Abstract

CMO functions are completely multiplicative functions \(f\) for which \(\sum_{n=1}^{\infty} f(n) = 0\). These functions were first introduced and studied by Kahane and Sa\={a}nas [5]. The main purpose of this paper is to generalise such functions to multiplicative functions and we shall call them MO functions. More precisely, we define MO functions to be multiplicative functions for which \(\sum_{n=1}^{\infty} f(n) = 0\) and \(\sum_{k=0}^{\infty} f(p^k) \neq 0\) for all \(p \in \mathbb{P}\). We give some properties and find examples of MO functions, as well as pointing out the connection between these functions and the Riemann hypothesis at the end of the paper.

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1 Introduction

An arithmetical function \(f : \mathbb{N} \rightarrow \mathbb{C}\) is called multiplicative if \(f(1) = 1\) and it satisfies \(f(mn) = f(m)f(n)\) whenever \((m, n) = 1\). We define Môbius function to be the function given by

\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^k & \text{if } n = p_{i_1}p_{i_2}\cdots p_{i_k} \text{ are distinct primes}, \\
0 & \text{otherwise},
\end{cases}
\]

or equivalently, the multiplicative function defined by \(\mu(p) = -1\) and \(\mu(p^k) = 0\) if \(k > 1\) for all primes \(p\). The partial sum of \(\mu(n)\) function not exceeding \(x\) can be defined by

\[
M(x) := \sum_{n \leq x} \mu(n)
\]
Several asymptotic formulas have been studied to be equivalent to the PNT by some scholars. For example, H. von Mangoldt 1897 [9] proved that knowing the PNT, it is easy to obtain \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \) with same elementary steps. However, E. Landau 1909 showed in [7] the converse of von Mangoldt’s result also holds. Another equivalent of the PNT, attributed to E. Landau [8], by \( M(x) = o(x) \).

In 1912, J. E. Littlewood showed in [6] that the Riemann hypothesis (RH) is equivalent to the following evaluation

\[
M(x) = \sum_{n \leq x} \mu(n) = O\left(x^{1/2+\varepsilon}\right) \quad \text{for all } \varepsilon > 0.
\]

This result have been improved by some scholars (see [7], [15] and [10]). K. Soundararajan 2009 [12] later improved it to be

\[
O\left(x^{1/4}\exp\left((\log x)^{1/2}(\log \log x)^{14}\right)\right).
\]

M. Balazard and A. de Roton [2] have slightly improved this bound by using a similar approach as K. Soundararajan. They replaced 14 by \( \frac{5}{2} + \varepsilon \) in (2). The best possible bound was conjectured by S. M. Gonek (see N. Ng [11]) to be

\[
M(x) = O\left(x^{1/8}(\log \log \log x)^{5/4}\right).
\]

That is, conjecturally, one cannot get \( M(x) \) to be \( o\left(x^{1/4}(\log \log \log x)^{5/4}\right) \).

It is also well-known that \( M(x) \) are \( \Omega(\sqrt{x}) \) since there are zeros of the Riemann zeta function \( \zeta(s) \) on the line \( \Re s = \frac{1}{2} \) (see for example [14]).

\section{CMO functions}

In this section, we introduce a class of functions which has been defined and studied by J.-P. Kahane and E. Saàs [5], called CMO functions. These are completely multiplicative \( f \) for which \( \sum_{n=1}^{\infty} f(n) = 0 \); i.e.

\[
f(mn) = f(m)f(n) \quad \text{whenever } (m, n) = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} f(n) = 0.
\]

One of their aims was to find and give necessary and/or sufficient conditions on \( f(p) \) for \( f \) being a CMO function. They also provided some examples of these functions. For instance, they discussed various examples of CMO functions including \( f(n) = \frac{\lambda(n)}{n} \), where \( \lambda(n) \) is the Liouville function and \( f(n) = \frac{\chi(n)}{n^{1/2}} \), where \( \chi \) is a non-principal Dirichlet character and \( \alpha \) is a zero of \( L_\chi \) with \( \Re \alpha > 0 \).
This study drove them to think the question of how quickly partial sums of $CMO$ functions can tend to zero. They proposed that it is always $\Omega(\frac{1}{\sqrt{x}})$ and the Generalised Riemann Hypothesis - Riemann Hypothesis (GRH-RH) would follow if their suggestion is true. This is because if GRH-RH is false then there is $\alpha$ which is a zero of $L_{\chi}$ with $\Re \alpha > \frac{1}{2}$ which means $\sum_{n \leq x} \chi(n) n^{\alpha}$ is not $\Omega(\frac{1}{\sqrt{x}})$. This suggestion is incredibly difficult to prove, but it might be easier to disprove; i.e., to find examples such that

$$\sum_{n \leq x} f(n) = O\left(\frac{1}{x^c}\right) \text{ for some } c > \frac{1}{2}, \quad (3)$$

They did not find any, so in order to find example for which $(3)$ is true we attempt to look for examples in the generalisation of $CMO$ functions.

3. **MO functions**

In this section, we introduce new functions which are a natural generalisation of $CMO$ functions. We extend the notion of $CMO$ to multiplicative functions and shall call them $MO$ functions. We would like to see how much the theory of $CMO$ functions can be generalised here. To help motivate our enquiries we consider examples of such functions and properties thereof. For example, let $f$ be a $MO$ function and $g$ a multiplicative function “close” to $f$. We shall show that $g$ is also a $MO$ function under some extra condition on $f$. We can also ask a similar question of Kahane and Sáias how quickly the partial sum of $MO$ functions up to and including $x$; (i.e. $\sum_{n \leq x} f(n)$) can tend to zero. We define these functions as follows:

**Definition 1** An arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ is called an MO function if it is multiplicative and satisfies

$$(i) \sum_{n=1}^{\infty} f(n) = 0 \quad \text{and} \quad (ii) \sum_{k=0}^{\infty} f(p^{k}) \neq 0 \text{ for all } p \in \mathbb{P}.$$ 

The extra condition $(ii)$ says the series converges but not to zero. This is needed to avoid trivial examples. For instance, let $f(1) = 1$, $f(2) = -1$ and $f(n) = 0$ for all $n > 2$. Then $\sum_{n=1}^{\infty} f(n) = 0$ but $\sum_{k=1}^{\infty} f(2^{k}) = f(1) + f(2) + f(4) + \cdots = 0$, and so does not satisfy the extra condition.

3.1 **Examples**

Like $CMO$ functions which have been studied by Kahane and Sáias \[5\], $MO$ functions are not so easy to find since these need to be conditionally convergent (as we
shall see in Proposition 7. To help the readers understanding we give three examples of MO functions. The first is based on the Möbius function, the second on the Dirichlet eta function, which corresponds to the case \( k = 2 \) in the third example.

**Example 2** The function \( \frac{\mu(n)}{n} \) is an MO function since:

(i) it is clear that \( \frac{\mu(n)}{n} \) is a multiplicative function;

(ii) it is well-known that \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \) (see for example [1]);

(iii) \( \sum_{k=0}^{\infty} \frac{\mu(p^k)}{p^k} = 1 - \frac{1}{p} \neq 0 \) for all \( p \in \mathbb{P} \).

**Example 3** Consider \( \frac{(-1)^{n-1}}{n^\alpha} \) which is multiplicative. For which values of \( \alpha \in \mathbb{C} \) with \( \Re \alpha > 0 \) is this an MO function?

(i) The series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} \) converges for \( \Re \alpha > 0 \) since \( A(x) := \sum_{n \leq x} (-1)^{n-1} = O(1) \).

Therefore, \( 0 \leq A(x) \leq 1 \) and using Abel summation, we have

\[
\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha} = \frac{A(x)}{x^\alpha} + \alpha \int_1^x \frac{A(t)}{t^{\alpha+1}} \, dt
= O\left(\frac{1}{x^{\Re \alpha}}\right) + \alpha \int_1^\infty \frac{A(t)}{t^{\alpha+1}} \, dt - \alpha \int_x^\infty \frac{O(1)}{t^{\alpha+1}} \, dt = C_\alpha + O\left(\frac{1}{x^{\Re \alpha}}\right),
\]

In particular, \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} \) converges. Now, for \( \Re \alpha > 0 \), we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} = (1 - 2^{1-\alpha})\zeta(\alpha).
\]

This is zero if and only if \( 2^\alpha = 2 \) or \( \zeta(\alpha) = 0 \) (for \( \alpha = 1 \), the sum on the left of (4) is not zero).

(ii) It remains to establish for which values of \( \alpha \) that \( \sum_{k=0}^{\infty} \frac{(-1)^{p^{k-1}}}{p^{\alpha k}} \neq 0 \) for all \( p \in \mathbb{P} \).

If \( p = 2 \), then

\[
\sum_{k=0}^{\infty} \frac{(-1)^{2^{k-1}}}{2^{\alpha k}} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{\alpha k}} = \frac{2^\alpha - 2}{2^\alpha - 1}.
\]

This is non-zero if and only if \( 2^\alpha \neq 2 \); (i.e. For \( \frac{(-1)^{n-1}}{n^\alpha} \) to be MO we therefore need \( 2^\alpha \neq 2 \)). Now if \( p \geq 3 \), then

\[
\sum_{k=0}^{\infty} \frac{(-1)^{p^{k-1}}}{p^{\alpha k}} = \sum_{k=0}^{\infty} \frac{1}{p^{\alpha k}} = \frac{1}{1 - \frac{1}{p^\alpha}}.
\]

This is non-zero for any \( \alpha \) with \( \Re \alpha > 0 \).
We see that \((-1)^{n-1}/n^\alpha\) is not an MO function if \(2^\alpha = 2\) since (ii) does not hold. Therefore we conclude that \((-1)^{n-1}/n^\alpha\) is an MO function if and only if \(\Re \alpha > 0\) and \(\zeta(\alpha) = 0\) since (i) and (ii) hold.

Furthermore, if \(\zeta(\alpha) = 0\) with \(\Re \alpha > 0\), then
\[
\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha} = O\left(\frac{1}{x^{\Re \alpha}}\right).
\]

This example can be generalised as follows:

**Example 4** Define \(g_k(n)\) as follows:

\[
g_k(n) := \begin{cases} 
1 - k & \text{if } k \text{ divides } n, \\
1 & \text{if } k \text{ does not divide } n.
\end{cases}
\]

We ask for which positive integer \(k > 1\) and \(\alpha\) with \(\Re \alpha > 0\) is the function \(\frac{g_k(n)}{n^\alpha}\) MO? When \(k = 2\) we get Example 3.

(i) We wish to find all \(k\) for which \(g_k(n)\) is a multiplicative function as follows: If \(m = n = 1\), then \(g_k(m)g_k(n) = g_k(mn)\). Now if \(k\) divides \(mn\), then we have four cases as follows: Assume \((m, n) = 1\).

(a) If \(k\) divides both \(n\) and \(m\), then \((m, n) \neq 1\). Hence we cannot have \(k\) dividing both \(m, n\) since we need \((m, n) = 1\).

(b) If \(k\) does not divide \(n\) and \(k\) divides \(m\), then \(g_k(m)g_k(n) = (1 - k)(1) = 1 - k = g_k(mn)\).

or vice versa

(c) If \(k\) does not divide \(m\) and \(k\) divides \(n\), then \(g_k(m)g_k(n) = (1)(1 - k) = 1 - k = g_k(mn)\).

(d) If \(k\) does not divide both \(n\) and \(m\), then we have two cases:

i. If \(k\) is not a prime power; (i.e. \(k = p_1^{a_1} \cdot p_2^{a_2} \cdots p_i^{a_i}\), where \(i \geq 2\) and \(a_i \geq 1\)). Then, with \(m = p_1^{a_1}\) and \(n = p_2^{a_2} \cdots p_i^{a_i}\) such that \((m, n) = 1\), we have \(g_k(m)g_k(n) = (1)(1) \neq (1 - k) = g_k(mn)\).

ii. If \(k\) is a prime power; (i.e. \(k = p^r\)). Then at least one of \(m\) or \(n\) is not a multiple of \(p\) while the other is (i.e. \(p\) does not divide \(m\), then \(p^r\) divides \(n\) or \(p\) does not divide \(n\), then \(p^r\) divides \(m\)) and \(g_k(m)g_k(n) = (1)(1 - k) = (1 - k) = g_k(mn)\) or \(g_k(m)g_k(n) = (1 - k)(1) = (1 - k) = g_k(mn)\).
However, if $k$ does not divide $mn$, then $k$ does not divide both $m$ and $n$, and $g_k(m)g_k(n) = (1)(1) = 1 = g_k(mn)$.

Thus $g_k(n)$ is multiplicative function if and only if $k$ is a prime power.

(ii) The series $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha}$ converges for $\Re \alpha > 0$ since

$$A(x) := \sum_{n \leq x} g_k(n) = \sum_{m=1}^{N} \sum_{n=(m-1)k+1}^{mk} g_k(n) + \sum_{n=Nk+1}^{x} g_k(n) = 0 + \sum_{n=Nk+1}^{x} g_k(n)$$

$$= g_k(Nk+1) + g_k(Nk+2) + \cdots + g_k(x), \quad \text{where } N = \left\lfloor \frac{x}{k} \right\rfloor$$

$$\leq k - 1 = O(1).$$

Thus $0 \leq A(x) \leq k - 1$ and using Abel summation, we have

$$\sum_{n \leq x} \frac{g_k(n)}{n^\alpha} = \frac{A(x)}{x^\alpha} + \alpha \int_{1}^{x} \frac{A(t)}{t^\alpha + 1} dt$$

$$= O\left(\frac{1}{x^\Re \alpha}\right) + \alpha \int_{1}^{\infty} \frac{A(t)}{t^\alpha + 1} dt - \alpha \int_{x}^{\infty} \frac{O(1)}{t^\alpha + 1} dt$$

$$= C_\alpha + O\left(\frac{1}{x^\Re \alpha}\right),$$

where $C_\alpha$ is a constant, as in Example 3. In particular, for $\Re \alpha > 0$, $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha}$ converges.

Now, for $\Re \alpha > 1$, we have

$$\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} - \sum_{n=1}^{\infty} \frac{k}{(kn)^\alpha} = (1 - k^{1-\alpha})\zeta(\alpha).$$

Thus $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = C_\alpha = (1 - k^{1-\alpha})\zeta(\alpha)$ for $\Re \alpha > 0$ by analytic continuation.

Also, $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = 0$ if and only if $k^\alpha = k$ or $\zeta(\alpha) = 0$.

(iii) It remains to get all $k$ and $\alpha$ for which $\sum_{m=0}^{\infty} \frac{g_k(p^m)}{p^{m\alpha}} \neq 0$ for all $p \in \mathbb{P}$. Let $k = p_0^r$, $p_0$ a prime number.

If $p_0 \neq p$, then $g_k(p^m) = 1$ for all $m \geq 0$. Hence

$$\sum_{m=0}^{\infty} \frac{g_k(p^m)}{p^{m\alpha}} = \sum_{m=0}^{\infty} \frac{1}{p^{m\alpha}} = \frac{1}{1 - \frac{1}{p^\alpha}}.$$
This is non-zero if and only if $k^\alpha \neq k$; (i.e., For $\frac{g_k(n)}{n^\alpha}$ to be MO we therefore need $k^\alpha \neq k$).

We see that $\frac{g_k(n)}{n^\alpha}$ is not an MO function if $k^\alpha = k$ since (iii) fails. Therefore, we conclude that $\frac{g_k(n)}{n^\alpha}$ is an MO function if and only if $k$ is a prime power, $\Re \alpha > 0$ and $\zeta(\alpha) = 0$ since (i), (ii) and (iii) hold.

Furthermore, if $\zeta(\alpha) = 0$ with $\Re \alpha > 0$, then

$$\sum_{n \leq x} \frac{g_k(n)}{n^\alpha} = O\left(\frac{1}{x^{\Re \alpha}}\right).$$

### 3.2 Some properties of MO functions

In this section, we establish some preliminary properties of MO functions. We shall first need the following result in the course of our discussion.

**Proposition 5** Let $f$ be a multiplicative function. Then $\sum_{n=1}^{\infty} |f(n)|$ converges, so that $f$ is absolutely convergent, if and only if $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges.

**Proof** Trivially, the series $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges if $\sum_{n=1}^{\infty} |f(n)|$ converges.

Now suppose $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges. It is follows that

$$\prod_p \left(1 + \sum_{k=1}^{\infty} |f(p^k)|\right) = \prod_p \left(\sum_{k=0}^{\infty} |f(p^k)|\right)$$

converges.

But the right hand side is at least $\prod_{p \leq x} \left\{ \sum_{k=0}^{\infty} |f(p^k)| \right\}$. Therefore, by the proof of Theorem 11.6 of [1] for any $x$, we have

$$\prod_{p \leq x} \left\{ \sum_{k=0}^{\infty} |f(p^k)| \right\} = \sum_{n \in \mathbb{N}} |f(n)| \geq \sum_{n \leq x} |f(n)|.$$

Hence $\sum_{n=1}^{\infty} |f(n)|$ converges, so that $f$ is absolutely convergent.

\[\square\]

**Proposition 6** If $f$ is a CMO function, then $f$ is an MO function; (i.e CMO $\subset$ MO).

**Proof** It is clear that $f$ is multiplicative and $\sum_{n=1}^{\infty} f(n) = 0$. It remains to show that $\sum_{k=0}^{\infty} f(p^k) \neq 0$ for all $p \in \mathbb{P}$. Now since $f$ is completely multiplicative, then $f(p^k) = f(p)^k$. Therefore

$$\sum_{k=0}^{\infty} f(p^k) = \sum_{k=0}^{\infty} f(p)^k = \frac{1}{1 - f(p)} \neq 0.$$
This series converges since $|f(p)| < 1$. Hence, by Definition 7, $f$ is an MO function.

\[\square\]

**Proposition 7** Let $f$ be an MO function. Then $\sum_{n=1}^{\infty} |f(n)|$ diverges. Indeed $\sum_{p} \sum_{k=1}^{\infty} |f(p^k)|$ diverges.

**Proof** Let us assume that the statement is false, so that

$$\sum_{n=1}^{\infty} |f(n)| \text{ converges.}$$

Then, by multiplicative property,

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \sum_{k=0}^{\infty} f(p^k) \neq 0 \text{ since } \sum_{k=0}^{\infty} f(p^k) \neq 0.$$

Yielding a contradiction since $f$ is an MO function and hence

$$\sum_{n=1}^{\infty} |f(n)| \text{ diverges.}$$

Furthermore, Proposition 5 gives $\sum_{p} \sum_{k=1}^{\infty} |f(p^k)|$ diverges, as required.

\[\square\]

### 3.2.1 Partial sums of MO functions

We know that the partial sum of an MO function not exceeding $x$ tends to zero when $x$ tends to infinity. A question raised by Kahane and Saáñas regarding CMO functions is: can one show, given $g(x)$, that there exist a CMO function $f$ with

$$\sum_{n \leq x} f(n) = \Omega(g(x))?$$

We are not considering this question, but we are interested in a related question which is: how small can we make $g(x)$, so that the above is true for all MO functions $f$? This question motivates the following propositions:

**Proposition 8** If $f$ is an MO function, then

$$\sum_{n \leq x} f(n) = \Omega\left(\frac{1}{x \log x}\right).$$
Proof  Let us assume that the statement is false, so that
\[ \sum_{n \leq x} f(n) = O \left( \frac{1}{x \log x} \right). \]

We know that for \( n \in \mathbb{N} \),
\[ f(n) = \sum_{m \leq n} f(m) - \sum_{m < n} f(m) = O \left( \frac{1}{n \log n} \right). \]

Hence
\[ f(p^k) = O \left( \frac{1}{p^k \log p^k} \right). \]

Now it follows that \( \sum_p \sum_{k=1}^\infty |f(p^k)| \) converges since
\[
\sum_p \sum_{k=1}^\infty \frac{1}{p^k \log p^k} \leq \sum_p \sum_{k=1}^\infty \frac{1}{p^k \log p} \quad (\text{since } \log p^k \geq \log p) \\
= \sum_p \frac{1}{\log p} \sum_{k=1}^\infty \frac{1}{p^k} \\
= \sum_p \frac{1}{(p-1) \log p} \quad \text{(since } p_n \log p_n \sim n(\log n)^2). 
\]

Thus
\[ \sum_p \sum_{k=1}^\infty \frac{1}{p^k \log p^k} \text{ converges.} \]

Hence, by Proposition 9, \( \sum_{n=1}^\infty |f(n)| \) converges. However, by Proposition 7, we have a contradiction, and so it follows that
\[ \sum_{n \leq x} f(n) = \Omega \left( \frac{1}{x \log x} \right). \]

\[ \square \]

Remark 9  Similarly, if \( f \) is an MO function, then
\[ \sum_{n \leq x} f(n) = \Omega \left( \frac{1}{x (\log x)^2} \right) \quad \text{for all } \varepsilon > 0. \]

We can improve Proposition 8 using the fact that \( \sum_p \frac{1}{p \log p} \) converges.

Propostion 10  If \( f \) is an MO function, then
\[ \sum_{n \leq x} f(n) = \Omega \left( \frac{1}{x (\log \log x)^2} \right). \]
Proof  Let us assume that the statement is false, so that
\[ \sum_{n \leq x} f(n) = O\left(\frac{1}{x(\log \log x)^2}\right). \]

We know that for \( n \in \mathbb{N} \),
\[ f(n) = \sum_{m \leq n} f(m) - \sum_{m < n} f(m) = O\left(\frac{1}{n(\log \log n)^2}\right). \]

Hence
\[ f(p^k) = O\left(\frac{1}{p^k(\log \log p^k)^2}\right). \]

Now it follows that \( \sum_p \sum_{k=1}^\infty |f(p^k)| \) converges since
\[ \sum_{p \geq 3} \sum_{k=1}^\infty \frac{1}{p^k(\log \log p^k)^2} \leq \sum_{p \geq 3} \sum_{k=1}^\infty \frac{1}{p^k(\log \log p)^2} \quad (since \ (\log \log p^k)^2 \geq (\log \log p)^2) \]
\[ = \sum_{p \geq 3} \frac{1}{(\log \log p)^2} \sum_{k=1}^\infty \frac{1}{p^k} \]
\[ = \sum_{p \geq 3} \frac{1}{(p-1)(\log \log p)^2} \quad (since \ (\log \log p_n)^2 \sim (\log \log n)^2). \]

For \( p = 2 \),
\[ \sum_{k=1}^\infty \frac{1}{2^k(\log \log 2)^2} \leq \frac{1}{2(\log \log 2)^2} + \frac{1}{(\log \log 4)^2} \sum_{k=2}^\infty \frac{1}{2^k} \quad \text{converges}. \]

Thus
\[ \sum_p \sum_{k=1}^\infty \frac{1}{p^k(\log \log p^k)^2} \quad \text{converges}. \]

Hence, by Proposition 5, \( \sum_{n=1}^\infty |f(n)| \) converges. However, by Proposition 7, we have a contradiction, and so it follows that
\[ \sum_{n \leq x} f(n) = \Omega\left(\frac{1}{x(\log \log x)^2}\right). \]

Remark 11  Similarly, if \( f \) is an MO function, then
\[ \sum_{n \leq x} f(n) = \Omega\left(\frac{1}{x(\log \log x)^{1+\varepsilon}}\right) \quad \text{for all } \varepsilon > 0. \quad (5) \]

Kahane and Saïas have shown that if \( f \) is a CMO function, then
\[ \sum_{n \leq x} f(n) = \Omega\left(\frac{1}{x}\right) \]

by using a deep result of D. Koukoulopoulos in [4]. We attempted to improve (5) to \( \Omega\left(\frac{1}{x}\right) \) as with the work of Kahane and Saïas, but the question is still open.
3.2.2 Closeness relation between two multiplicative functions

Let $\mathcal{M} := \{ f : \mathbb{N} \rightarrow \mathbb{C} \text{ multiplicative} \}$, and let us define an (extended) metric on $\mathcal{M}$ to be the distance function

$$D(f, g) := \sum_p \sum_{k=0}^{\infty} |g(p^k) - f(p^k)|.$$ 

Then $\mathcal{M}$ is an extended metric space since $D(f, g)$ can attain the value $\infty$. It is straightforward to check for all $f, g, h \in \mathcal{M}$

(i) $D(f, g) = 0$ if and only if $f = g$,

(ii) $D(f, g) = D(g, f)$,

(iii) $D(f, h) \leq D(f, g) + D(g, h),$

hold. We aim to extend Theorem 3 of Kahane and Saias in [5] by showing that if $f$ is an MO function and $g$ is a multiplicative function “close” to $f$, (i.e. $g$ has finite distance from $f$), then $g$ is also an MO function. We can do this under an extra condition on $f$, as the following theorem shows.

**Theorem 12** Let $f$ be an MO function for which

$$\left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right| \geq a \quad \text{for some } a > 0, \text{ for all } p \text{ and all } \Re s \geq 0, \quad (6)$$

and let $g$ be a multiplicative function such that $D(f, g)$ is finite and

$$\sum_{k=0}^{\infty} g(p^k) \neq 0 \quad \text{for all } p. \quad (7)$$

Then $g$ is an MO function.

**Proof** Let $F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. Then the series for $F(s)$ is absolutely convergent for $\Re s > 1$ and it is convergent for $\Re s > 0$ and $s = 0$ since $\sum_{n=1}^{\infty} f(n) = 0$. We note that the assumption $D(f, g)$ is finite and the fact that $f$ is an MO function imply $|g(p^k)| \rightarrow 0$ as $p^k \rightarrow \infty$. Then, by Theorem 316 of [3], $g(n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore the series for $G(s)$ converges for $\Re s > 1$ since $g$ is bounded. Therefore $F(s)$ and $G(s)$ can be written as follows:

$$F(s) = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \quad \text{and} \quad G(s) = \prod_p \sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \quad \Re s > 1.$$
Now

\[ H(s) := \prod_p \left( \sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \right) = \prod_p \left( 1 + \frac{\sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right) \]

converges absolutely for \( \Re s \geq 0 \) if and only if

\[ \sum_p \left| \frac{\sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right| \]

converges for \( \Re s \geq 0 \). But

\[ \sum_p \left| \frac{\sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right| \leq \frac{1}{a} \sum_p \sum_{k=0}^{\infty} |g(p^k) - f(p^k)| \]

by (8) so, since \( D(f, g) \) is finite, (8) converges for \( \Re s \geq 0 \) and \( H(s) \) converges absolutely to holomorphic function for \( \Re s > 0 \). However, \( H(s) = (G/F)(s) \) for \( \Re s > 1 \) then \( G(s) = F(s)H(s) \), where the series for \( F(s) \) converges for \( \Re s > 0 \) and \( s = 0 \) since \( f \) is an MO function, and \( H(s) \) converges absolutely for \( \Re s \geq 0 \). Therefore \( G(s) \) converges for \( \Re s > 0 \) and \( s = 0 \) using the extension of Theorem 1.2 of Chapter II.1. Thus we have \( G(0) = F(0)H(0) = 0 \). Hence, by assumption (7) and \( G(0) = 0 \), \( g \) is an MO function.

The proof of Theorem 12 also gives the following result.

**Corollary 13** Let \( f \) and \( g \) both be multiplicative functions such that \( D(f, g) \) is finite and satisfies

\[ \left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right| \geq a \quad \text{for some } a > 0 \text{ and all } \Re s \geq 0, \]

\[ \left| \sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \right| \geq b \quad \text{for some } b > 0 \text{ and all } \Re s \geq 0. \]

Then the following two assertions are equivalent:

\[ \sum_{n=1}^{\infty} f(n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} g(n) = 0. \]
3.3 Open problems

(i) Let $f$ be an $MO$ function. Can we show that

$$\sum_{n \leq x} f(n) = \Omega \left( \frac{1}{\sqrt{x}} \right)?$$

(ii) As pointed out earlier Kahane and Saañas suggested that for all $CMO$ functions, one has $\sum_{n \leq x} f(n) = \Omega \left( \frac{1}{\sqrt{x}} \right)$. As also mentioned, GRH-RH (Generalised Riemann Hypothesis-Riemann Hypothesis) would follow if their suggestion is correct.

In Example 2 it is known that $\sum_{n \leq x} \mu(n) = \Omega(\sqrt{x})$ since there are zeros of the Riemann zeta function $\zeta$ on the line $\Re s = \frac{1}{2}$ (see [13]). Thus, by Abel summation,

$$\sum_{n \leq x} \frac{\mu(n)}{n} = \Omega \left( \frac{1}{\sqrt{x}} \right).$$

However, for $\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha}$ and $\sum_{n \leq x} \frac{g_k(n)}{n^\alpha}$ to converge to zero in Examples 3 and 4, it is necessary that $\alpha$ be a zero of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}$ and $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha}$ with $\Re \alpha > 0$; (i.e. $\zeta(\alpha) = 0$). Suppose this is the case. We then have

$$\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha} = O \left( \frac{1}{x^{\Re \alpha}} \right) \quad \text{and} \quad \sum_{n \leq x} \frac{g_k(n)}{n^\alpha} = O \left( \frac{1}{x^{\Re \alpha}} \right),$$

and

$$\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha} = \Omega \left( \frac{1}{x^{\Re \alpha}} \right) \quad \text{and} \quad \sum_{n \leq x} \frac{g_k(n)}{n^\alpha} = \Omega \left( \frac{1}{x^{\Re \alpha}} \right).$$

In our results, we have not found any examples with $\sum_{n \leq x} f(n) = O \left( \frac{1}{x^c} \right)$ for $c > \frac{1}{2}$. This may suggest the following conjecture.

**Conjecture 14** For all multiplicative function $f$ (MO functions), we have

$$\sum_{n \leq x} f(n) = \Omega \left( \frac{1}{\sqrt{x}} \right).$$

Furthermore, the RH would follow if Conjecture 14 were true since if RH is false then there is $\alpha$ which is a zero of $\zeta$ with $\Re \alpha > \frac{1}{2}$ which means $\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha}$ and $\sum_{n \leq x} \frac{g_k(n)}{n^\alpha}$ is not $\Omega \left( \frac{1}{\sqrt{x}} \right)$. 

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