Higher bottomonia

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We show the results of an unquenched quark model calculation of the bottomonium spectrum with self energy corrections due to the coupling to the meson-meson continuum. We also give results for the open bottom strong decay amplitudes of higher bottomonia in a \( 3P_0 \) model.

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I. INTRODUCTION

Several studies on meson spectroscopy have been published in the years, based on different pictures for mesons; they include \( qq \) mesons \[1, 2, 3\] (see also the unquenched lattice QCD calculations of Refs. [10, 11]), meson-meson molecules [21, 58], tetraquarks [31, 33] and quarkonium hybrids [34, 36] and references can be found in review papers like [37].

Many of these studies investigate hadron properties within the quark model (QM). The QM [1, 3–9, 16, 38–46] can reproduce the behavior of observables such as the spectrum and the magnetic moments, but it neglects pair-creation effects, which are manifest as a coupling to meson-meson (meson-baryon) channels. Above threshold, this coupling leads to strong decays; below threshold, it leads to virtual \( qq \) – \( \bar{q}ar{q} \) (\( qqq \) – \( qar{q} \)) components in the hadron wave function and shifts of the physical mass with respect to the bare mass, as already shown by several authors in the baryon [47–53] and meson [14, 54, 64] sectors. Indeed, since the earliest days of hadron spectroscopy, it has been recognized that the properties of a level can be strongly influenced by the closest channels \[54, 66\].

An early example was the resonance \( \Lambda(1405) \), decaying into \( \Sigma \pi \) but strongly influenced by the nearby \( \bar{K}N \) threshold \[63\], or the \( f_0(980) \), decaying into \( \pi\pi \) but behaving remarkably as a \( K\bar{K} \) meson-meson molecule \[65\]. The continuum coupling effects can be seen, not only in the spectrum, but also in other observables, like the importance of the orbital angular momentum in the spin of the proton, as explicitly calculated for the first time with a quark model in Ref. [63], or the calculation of the flavor asymmetry of the proton \[71\] and the strange content of the electromagnetic form factors of the nucleon \[72\]; all of those predictions were obtained within the unquenched quark model (UQM) for baryons of Refs. [63, 71].

Another important piece of information on a meson has to do with its possible decay modes, including strong, electromagnetic and weak decays. The dominant strong decays are transitions to open flavor final states, where the initial \( bb \) (or \( c\bar{c} \)) meson decays by producing a light \( qq \) quark-antiquark pair (\( q = u, d \) or \( s \)) followed by separation into two open-bottom (or open-charm) mesons. Since the QCD mechanism underlying this dominant decay process is still poorly understood, several phenomenological models have been developed in order to carry on this type of studies. They include pair-creation models [1, 22, 70], elementary meson emission models [72, 73] and effective Lagrangian approaches [16]. The simplest of these approaches is that of the \( 3P_0 \) pair-creation model, in all its possible variants [72, 73, 75, 83]. See also Refs. [84, 90].

In this paper, we present an UQM [64, 69, 71, 91] calculation of the bottomonium spectrum (\( 1S, 2S, 3S, 1P, 2P, 3P \) and \( 1D \) states) with self energies corrections due to the coupling to the meson-meson continuum. These loop corrections are computed considering a complete set of accessible \( SU_f(5) \otimes SU_{spin}(2) \) ground-state (i.e. \( 1S \)) mesons. \( 1S \) intermediate states, being at lower energies with respect to \( P \)-wave and \( D \)-wave intermediate meson states, give the main contribution to the self energies of the \( bb \) resonances that we are going to study. Our results are fitted to the experimental data, so that the calculated masses of the mesons of interest are the sum of a bare energy term, computed within Godfrey and Isgur’s relativized quark model \[8\], and a self energy correction, computed within the UQM formalism for mesons \[64, 91\].

We could have used another model to calculate the bare meson masses, instead of the Godfrey and Isgur’s one \[8\]. It is chosen because it is reliable and provides a good starting point. This model is very successful and able to describe many data with a single set of universal parameters. In this paper, we do not want to extend or substitute it. We intend to do something completely different, that makes it possible to explore the emerging of new physics in proximity of the meson-meson decay thresholds; this is where a naive quark model without continuum coupling effects will eventually fail, but it is still to be demonstrated. Thus, we will show the predictions for \( \chi_b(3P) \) states obtained within the UQM for mesons \[64, 91\], that, in the future, can be tested by BaBar, Belle, CDF, D0 and LHCb Collaborations.

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UQM predictions for the splittings between the members of the $\chi_b(3P)$ multiplet differ from those obtained within quark models, such as Godfrey and Isgur’s [3]. This is due to important continuum coupling effects, the $\chi_b(3P)$ resonances being close to the first open bottom decay thresholds. The unquenching of another quark model will be the subject of a subsequent article [92].

Finally, we present our results for the strong decay widths of higher bottomonia. The widths are computed within a modified version of the $^3P_0$ pair-creation model [64, 91]. In particular, the changes we consider include the substitution of the pair-creation strength, $\gamma_0$, with an effective one, $\gamma_0^{\text{eff}}$, to suppress unphysical heavy quark pair-creation, and the introduction of a quark form factor into the model’s transition operator, to take the non point-like nature of the created pair of constituent quarks into account. The spectrum of $B\bar{B}$ mesons, which we need in this calculation, is computed within the relativized QM by Godfrey and Isgur [3], whose results we have re-fitted to get a better reproduction of the higher part of the bottomonium spectrum and extended up to higher excitations. These results for the higher excitations, never published before, only represent a subproduct of our article, since the main focus is on the construction of a new model for the $b\bar{b}$ states, the UQM, and on the open bottom strong decay widths of $b\bar{b}$ resonances.

We think that the present article may be helpful to the experimentalists in their search for new $b\bar{b}$ states as well as in clarifying the properties of the already observed ones.

II. FORMALISM

A. Self energies

The Hamiltonian we consider,

$$H = H_0 + V,$$  \hspace{1cm} (1)

is the sum of a first part, $H_0$, acting only in the bare meson space, and a second part, $V$, which can couple a meson state $|A\rangle$ to the meson-meson continuum $|BC\rangle$.

The dispersive equation, resulting from a nonrelativistic Schrödinger equation, is

$$\Sigma(E_a) = \sum_{BC} \int_{0}^{\infty} q^2 dq \frac{|V_{a,bc}(q)|^2}{E_a - E_{bc}},$$  \hspace{1cm} (2)

where the bare energy $E_a$ satisfies:

$$M_a = E_a + \Sigma(E_a).$$  \hspace{1cm} (3)

$M_a$ is the physical mass of the meson $A$, with self energy $\Sigma(E_a)$. In Eq. (2) one has to take the contributions from various meson-meson intermediate states $|BC\rangle$ into account. These channels, with relative momentum $q$ between $B$ and $C$, have quantum numbers $J_{bc}$ and $\ell$ coupled to the total angular momentum of the initial state $|A\rangle$. $V_{a,bc}$ stands for the coupling due to the operator $V$ between the intermediate state $|BC\rangle$ and the unperturbed quark-antiquark wave function of the meson $A$; $E_{bc} = E_b + E_c$ is the total energy of the channel $BC$, calculated in the rest frame. Finally, if the bare energy of the meson $A$, $E_a$, is greater the threshold $E_{bc}$, the self energy of Eq. (2) contains poles and is a complex number: in this case one has real loops instead of virtual ones.

Since the physics of the dynamics depends on the matrix elements $V_{a,bc}(q)$, one has to choose a precise form for the transition operator, $V$, responsible for the creation of $q\bar{q}$ pairs. Our choice is that of the unquenched quark model (UQM) of Refs. 64, 91.

B. An unquenched quark model for bottomonia

In the unquenched quark model for mesons [64, 91], the effects of quark-antiquark pairs are introduced explicitly into the quark model through a QCD-inspired $^3P_0$ pair-creation mechanism. This approach, which is a generalization of the unitarized quark model by Törnqvist and Zenczykowski [37] (see also Ref. [93]) is based on a QM, to which $q\bar{q}$ pairs with vacuum quantum numbers are added as a perturbation and where the pair-creation mechanism is inserted at the quark level.

Under these assumptions, the meson wave function is made up of a zeroth order quark-antiquark configuration plus a sum over the possible higher Fock components, due to the creation of $^3P_0$ $q\bar{q}$ pairs. To leading order in pair-creation, one has

$$|\psi_A\rangle = N \left[ |A\rangle + \sum_{BC\ell J} \int d\bar{q} \left| BC\bar{q}\ell J \right\rangle \frac{\langle BC\bar{q}\ell J | T^\dagger | A\rangle}{E_a - E_b - E_c} \right],$$  \hspace{1cm} (4)

where $T^\dagger$ stands for the $^3P_0$ quark-antiquark pair-creation operator [64, 91], depending on an effective pair-creation strength $\gamma_0^{\text{eff}}$, $A$ is the meson, $B$ and $C$ represent the intermediate state mesons, and $E_a$, $E_b = \sqrt{M_b^2 + q^2}$ and $E_c = \sqrt{M_c^2 + q^2}$ are the corresponding energies, $\bar{q}$ and $\ell$ the relative radial momentum and orbital angular momentum between $B$ and $C$ and $\bar{J} = \bar{J}_b + \bar{J}_c + \bar{\ell}$ is the total angular momentum. The wave functions of the mesons $A$, $B$ and $C$ can be written as harmonic oscillator wave functions, depending on a single oscillator parameter $\alpha = 0.5$ GeV.

The $^3P_0$ quark-antiquark pair-creation operator, $T^\dagger$, is given by [64, 91]

$$T^\dagger = -3\gamma_0^{\text{eff}} \int d\bar{p}_3 d\bar{p}_4 \delta(\bar{p}_3 + \bar{p}_4) C_{34} F_{34} e^{-r_3^2(\bar{p}_3 - \bar{p}_4)^2/\alpha} \left[ \chi_{34} \times \mathcal{Y}_1(\bar{p}_3 - \bar{p}_4) \right]\frac{b_3^{\dagger} (\bar{p}_3) d_4^{\dagger} (\bar{p}_4)}{6},$$  \hspace{1cm} (5)
where \( b_i^\dagger (\vec{p}_3) \) and \( d_i^\dagger (\vec{p}_4) \) are the creation operators for a quark and an antiquark with momenta \( \vec{p}_3 \) and \( \vec{p}_4 \), respectively. The \( q\bar{q} \) pair is characterized by a color singlet wave function \( C_{34} \), a flavor singlet wave function \( F_{34} \), a spin triplet wave function \( \chi_{34} \) with spin \( S = 1 \) and a solid spherical harmonic \( \varPsi_i(\vec{p}_3 - \vec{p}_4) \), which indicates that the quark and antiquark are in a relative wave. Since the operator \( T \) creates a pair of constituent quarks with an effective size, the pair-creation point has to be smeared out by a Gaussian factor, whose width \( r_q \) was determined from meson decays to be in the range \( 0.25 - 0.35 \) fm \[51,93,94\]. In our calculation, we take the value \( r_q = 0.335 \) fm \[64\]. The pair-creation strength, \( \gamma_0^{\text{eff}} = \frac{m_q}{m_n} \gamma_0 \), is fitted to the strong decay \( T(4S) \to B\bar{B} \), and the value for \( \gamma_0 \) is extracted.

In short, the two main differences with the old \( ^3P_0 \) model are the introduction of a quark form factor, as already done by many authors like Törnqvist and Zenczykowski \[47\], Silvestre-Brac and Gignoux \[51\] and Geiger and Isgur \[93,94\], and the use of the effective strength \( \gamma_0^{\text{eff}} = \frac{m_q}{m_n} \gamma_0 \), since it is well known that heavy flavor pair-creation is suppressed. We think that both the improvements, i.e. the introduction of the quark form factor and the effective strength \( \gamma_0^{\text{eff}} \), already used in Refs. \[64,91\], can make the model more realistic.

The matrix elements of the pair-creation operator \( T^\dagger \) were derived in explicit form in the harmonic oscillator basis as in Ref. \[72\], using standard Jacobi coordinates.

In the UQM, the coupling \( V_{a,bc} \) between the meson-meson continuum, \( BC \), and the unperturbed wave function of the meson \( A \) can be written as

\[
V_{a,bc}(q) = \sum_{i\ell J} \langle BC | q \ell J | T^\dagger | A \rangle .
\]

In general, two different diagrams can contribute to the transition matrix element \( \langle BC | q \ell J | T^\dagger | A \rangle \) (see Fig. \[1\]): in the first one, the quark in \( A \) ends up in \( B \), while in the second one it ends up in \( C \). In the majority of cases, one of these two diagrams vanishes; however, for some matrix elements, both must be taken into account \[91\], as for example, this is the case of the coupling \( \eta_b \to \Upsilon \Upsilon \), where the initial \( | \bar{b}b \rangle \) state is coupled to the final state \( | \bar{b}b; \bar{b}b \rangle \) and the created pair is a \( \bar{b}b \) one.

Finally, by substituting Eq. \[6\] into Eq. \[2\], we have:

\[
\Sigma(E_a) = \sum_{BC|J} \int_0^\infty q^2 dq \, \frac{|\langle BC | q \ell J | T^\dagger | A \rangle|^2}{E_a - E_b - E_c} .
\]

The values of the pair-creation model's parameters, used to compute the strong decays of Sec. \[71,91\] and the vertices \( \langle BC | q \ell J | T^\dagger | A \rangle \) of Eq. \[7\], are reported in Table \[I\].

| Parameter | Value |
|-----------|-------|
| \( \gamma_0 \) | 0.732 |
| \( \alpha \) | 0.500 GeV |
| \( r_q \) | 0.335 fm |
| \( m_n \) | 0.330 GeV |
| \( m_s \) | 0.550 GeV |
| \( m_c \) | 1.50 GeV |
| \( m_b \) | 4.70 GeV |

**TABLE I: Pair-creation model parameters.**

C. \( ^3P_0 \) pair-creation model

In the \( ^3P_0 \) pair-creation model \[72\], the open flavor strong decays of \( \bar{b}b \) mesons take place via the production of a light \( q\bar{q} \) pair (i.e. \( q = u, d \) or \( s \)), with vacuum, i.e. \( ^3P_0 \), quantum numbers, followed by the separation of the initial meson into two open-bottom mesons.

More recent variants of the \( ^3P_0 \) model include a quark form factor in the transition operator \[64,69-71,91\], that takes the non point-like nature of the constituent quarks into account, and an effective pair-production strength \( \gamma_0^{\text{eff}} \), that suppresses unphysical heavy \( q\bar{q} \) pair-creation \[58,64,91\].

In particular, in Ref. \[58\] it is stated that in the \( ^3P_0 \) model approach the pair-creation is flavor-independent, which implies an enhancement of heavy quarks creation compared to light quarks one, without a fundamental reason for that. Thus, an effective pair-creation strength \( \gamma_0^{\text{eff}} \) \[58,64,91\], defined as

\[
\gamma_0^{\text{eff}} = \frac{m_n}{m_i} \gamma_0 ,
\]

is introduced, with \( i = n \) (i.e. \( u \) or \( d \)), \( s \), \( c \) and \( b \) (see Table \[I\]). This problem was already recognized and corrected by several authors \[58,64,91\]. The same mechanism of Eq.
D. Godfrey and Isgur’s relativized quark model

The relativized QM \cite{5} is a potential model for q\bar{q} meson spectroscopy, developed in 1985 by Godfrey and Isgur (see also Ref. \cite{12}).

The starting Hamiltonian of the model \cite{5} is given by
\[
H = \sqrt{q^2 + m_1^2} + \sqrt{q^2 + m_2^2} + V_{\text{conf}} + V_{\text{hyp}} + V_{\text{so}},
\]
where \(m_1\) and \(m_2\) are the masses of the constituent quark and antiquark inside the meson, \(q\) is their relative momentum (with conjugate coordinate \(r\)), \(V_{\text{conf}}\), \(V_{\text{hyp}}\) and \(V_{\text{so}}\) are the confining, hyperfine and spin-orbit potentials, respectively.

The confining potential \cite{5},
\[
V_{\text{conf}} = - \left( \frac{3}{4} c + \frac{3}{4} br - \frac{\alpha_s(r)}{r} \right) F_1 \cdot F_2,
\]
contains a constant, \(c\), a linear confining term and a Coulomb-like interaction, depending on a QCD-motivated running coupling constant \(\alpha_s(r)\).

The hyperfine interaction is written as \cite{5}
\[
V_{\text{hyp}} = - \frac{\alpha_s(r)}{m_1 m_2} \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \left( \frac{\vec{s}_i}{m_i} + \frac{\vec{s}_j}{m_j} \right) \cdot \vec{L} \cdot \vec{F}_i \cdot \vec{F}_j,
\]
\[
(11)
\]

The spin-orbit potential \cite{5},
\[
V_{\text{so}} = V_{\text{so,cm}} + V_{\text{so,tp}},
\]
is the sum of two contributions, where
\[
V_{\text{so,cm}} = - \frac{\alpha_s(r)}{r^3} \left( \frac{1}{m_i} + \frac{1}{m_j} \right) \left( \frac{\vec{s}_i}{m_i} + \frac{\vec{s}_j}{m_j} \right) \cdot \vec{L} \cdot \vec{F}_i \cdot \vec{F}_j,
\]
is the color-magnetic term and
\[
V_{\text{so,tp}} = - \frac{1}{2r} \frac{\partial H_{ij}^{\text{conf}}}{\partial r} \left( \frac{\vec{s}_i}{m_i} + \frac{\vec{s}_j}{m_j} \right) \cdot \vec{L},
\]
is the Thomas-precession one.

What is known as Godfrey and Isgur’s model \cite{5} is the Hamiltonian of Eq. \cite{5}, plus some relativistic effects. These effects include the introduction of a potential smearing and the replacement of factors of quark mass with quark kinetic energy. This is exactly the model used in the present paper for the bare energy calculation of Sec. III.B.

III. RESULTS

A. Open bottom strong decays in the \(^3P_0\) pair-creation model

In this section, we show our calculation of the open bottom strong decay widths of higher bottomonia (see table IV).

To get results for the masses of the higher lying \(b\bar{b}\) resonances, we use the relativized QM of Ref. \cite{5}, whose mass formula we have re-fitted to the most recent experimental data (see Table III). Something similar was done in Ref. \cite{88} for charmonia.

This re-fit is necessary to compute the strong decays, which require precise values for the masses of the decaying mesons, also for the higher lying states. Godfrey and Isgur’s results \cite{5} show a deviation from the most recent experimental data of the order of 50 MeV in the case of 4S states. In particular, one can notice that Godfrey and Isgur’s prediction for \(\Upsilon(4S)\)’s mass (10.63 GeV \cite{5}) is approximately 50 MeV higher than the corresponding experimental data (10579.4 \pm 1.2 MeV \cite{92}) and, moreover, their theoretical prediction for \(\eta_b(4S)\)’s mass (10.62 GeV) is 40 MeV higher than \(\Upsilon(4S)\)’s observed mass, while on contrary an \(\eta_b(4S)\) state should be lower in energy. We extract the value of \(\eta_b(4S)\)’s mass, absent in the original paper of 1985 \cite{5}, by running a numerical program that calculates Godfrey and Isgur model’s spectrum for \(b\bar{b}\) states with the value of the model parameters reported in Ref. \cite{5}. 4S resonances are important, being the lowest energy \(b\bar{b}\) states decaying into two open-bottom mesons. Since we are interested in calculating observables (the strong decay widths) that have a strong dependence on the masses of the mesons involved in the calculation, we think that it is important to update 1985 Godfrey and Isgur’s results in the \(b\bar{b}\) sector. At that time, many \(b\bar{b}\) states were still unobserved. Moreover, since Godfrey and Isgur’s results differ from the experimental data in the 4S case, we think that this may also be the case of other higher lying radial excitations, such as \(4P\). In our fit, we preferred to get a better reproduction of the radial excitations instead of the low-lying ones, because the latter are useless in computing the decays.

\[
\begin{align*}
& m_b = 5.024 \text{ GeV} \\
& \Lambda = 0.156 \text{ GeV} \\
& \epsilon_c = -0.242 \\
& \epsilon_t = 0.030 \\
& \epsilon_{\text{so}(V)} = -0.053 \\
& \epsilon_{\text{so}(S)} = 0.019
\end{align*}
\]

\|TABLE II: Resulting values of Godfrey and Isgur’s model \cite{5} parameters, obtained by re-fitting the mass formula of Eq. \cite{5} to the most recent experimental data \cite{92}.\|

The decay widths are calculated within the \(^3P_0\) model.
\( \Phi_{A\to BC} = \Phi_{A\to BC}(q_0) \sum_{\ell,J} |\langle BCq_0^J| T^\dagger |A \rangle|^2 \). \hfill (14)

Here, \( \Phi_{A\to BC}(q_0) \) is the standard relativistic phase space factor \(^{64,91} \). \( \Phi_{A\to BC} = 2\pi q_0 \frac{E_b(q_0) E_c(q_0)}{M_a} \), \hfill (15)

depending on the relative momentum \( q_0 \) between \( B \) and \( C \) and on the energies of the two intermediate state mesons, \( E_b = \sqrt{M_b^2 + q_0^2} \) and \( E_c = \sqrt{M_c^2 + q_0^2} \). The values of the masses \( M_a, M_b \) and \( M_c \), used in the calculation, are taken from the PDG \(^{95} \) and Ref. \(^{96} \), and in the case of still unobserved states we use Godfrey and Isgur’s model predictions, obtained with the values of the model’s parameters of Table \( \text{II} \) (see Tables \( \text{III} \) and \( \text{IV} \), second column). In our calculation, we use a variational basis of 200 harmonic oscillator shells, so that our results converge very well.

| State | \( J^P \) | \( M \) [GeV] | Source |
|-------|---------|----------|--------|
| \( B \) | \( 0^- \) | 5.279 | \(^{95} \) |
| \( B^* \) | \( 1^- \) | 5.325 | \(^{95} \) |
| \( B_0 \) | \( 0^- \) | 5.366 | \(^{95} \) |
| \( B^*_0 \) | \( 1^- \) | 5.416 | \(^{95} \) |
| \( B_c \) | \( 0^- \) | 6.277 | \(^{95} \) |
| \( B^*_c \) | \( 1^- \) | 6.340 | \(^{9} \) |

TABLE III: Masses of open bottom mesons used in the calculations.

The operator \( T^\dagger \) inside the \( 3P_0 \) amplitudes \( \langle BCq_0^J| T^\dagger |A \rangle \) is that of Eq. \(^8 \), which also contains the quark form factor of Refs. \(^{93,94} \). The quark form factor, that takes the non point-like nature of the constituent quarks into account, is not included in the original formulation of the \( 3P_0 \) model \(^7 \).

Another difference between our calculation and those of Refs. \(^{72,74,82,84} \) is the substitution of the pair-creation strength \( \gamma_0 \) with the effective strength \( \gamma_0^{\text{eff}} \) of Eq. \(^8 \). The introduction of this effective mechanism suppresses those diagrams in which a heavy \( q\bar{q} \) pair is created, as discussed in Sec. \( \text{III} \). More details on this mechanism can be found in Refs. \(^{69,64,91} \).

Finally, the results of our calculation, obtained with the values of the model parameters of Table \( \text{II} \) are reported in Table \( \text{IV} \). See also Table \( \text{V} \) where our results are compared to the existing experimental data \(^{97,97} \). Other results for the strong decays of bottomonia can be found in Ref. \(^{99} \), where the authors calculated the decay widths of open charm, charmonium and \( b\bar{b} \) \( \Upsilon \) states [\( \Upsilon(4S), \Upsilon(10860) \) and \( \Upsilon(11020) \)] within the \( 3P_0 \) model.

### B. Bare and self energy calculation of \( b\bar{b} \) states

The relativized QM \(^2 \) is now used to compute the bare energies of the \( b\bar{b} \) mesons, \( E_a \), that we need in the self energy calculation of Table \( \text{VII} \). In our study, we compute the bare energies \( E_a \)'s as the eigenvalues of Eq. \(^9 \), with the values of the model parameters of Table \( \text{VI} \). At variance with QM calculations, such as those of Ref. \(^9 \) and Table \( \text{V} \) second column, we do not fit the eigenvalues of Eq. \(^9 \), i.e. the \( E_a \)'s, to the experimental data \(^{95} \). In our case, the quantities fitted to the spectrum of bottomonia \(^{95,96} \) are the physical masses \( M_a \)'s of Eq. \(^8 \) and therefore the fitting procedure is an iterative one.

Once the values of the bare energies are known, it is possible to calculate the self energies \( \Sigma(E_a) \)'s of the \( b\bar{b} \) states through Eq. \(^7 \), summing over a complete set of accessible \( SU_f(5) \otimes SU_{spins}(2) \) 1S intermediate states. If the bare energy of the initial meson \( A \) is above the threshold \( BC \), i.e. \( E_a > M_b + M_c \), the self energy contribution due to the meson-meson channel \( BC \) is computed as

\[
\Sigma(E_a)(BC) = \mathcal{P} \int_{M_b+M_c}^{\infty} \frac{dE_a - E_a - E_{bc}}{E_a} \frac{q_{E_a} E_a}{E_{bc}} \left| \langle BCq^J_0 | T^\dagger | A \rangle \right|^2 
+ 2\pi i \left\{ \frac{4\pi q_{E_a} E_a}{E_{bc}} |\langle BCq^J_0 | T^\dagger | A \rangle|^2 \right\}_{E_{bc}=E_a},
\]

where the symbol \( \mathcal{P} \) indicates a principal part integral, calculated numerically, and

\[
2\pi i \left\{ \frac{4\pi q_{E_a} E_a}{E_{bc}} |\langle BCq^J_0 | T^\dagger | A \rangle|^2 \right\}_{E_{bc}=E_a}
\]

is the imaginary part of the self energy.

Finally, the results of our calculation, obtained with the set of parameters of Table \( \text{II} \) and the effective pair-creation strength of Eq. \(^8 \), are given in Table \( \text{VIII} \) and Fig. \( \text{2} \). This means that the vertices \( \langle BCq^J_0 | T^\dagger | A \rangle \) of Eqs. \(^7 \) and \(^8 \) are computed with the same set.
TABLE IV: Strong decay widths (in MeV) in heavy meson pairs for higher bottomonium states. Column 2 gives the values of the masses of the decaying $b\bar{b}$ states: when available, we use the experimental values from PDG\[95\], obtained with the values of the model parameters of Table II. Columns 3-8 show the BC decay width contributions from various channels $BC$, such as $BB$, $BB^*$ and so on. The values of the model parameters are given in table III. The symbol – in the table means that a certain decay is forbidden by selection rules or that the decay cannot take place because it is below the threshold.

| State     | $\Gamma_{\text{theor}}$ ($^3P_0$) | $\Gamma_{\text{exp}}$ [95] | $\Gamma_{\text{exp}}$ [97] |
|-----------|-----------------------------------|-----------------------------|-----------------------------|
| $\Upsilon(4^3S_1)$ | 21                               | 21 ± 3                      | -                           |
| $\Upsilon(10860)$ | 71                               | 55 ± 28                     | 74 ± 4                      |
| $\Upsilon(4^3S_1)$ | 36                               | 79 ± 16                     | 37 ± 3                      |

TABLE V: Our results for the open bottom strong decay widths of Table IV are compared to the existing experimental data [95, 97].

$$m_b = 4.568 \text{ GeV} \quad b = 0.1986 \text{ GeV}^2 \quad \alpha_s^f = 0.600$$

$$\Lambda = 0.200 \text{ GeV} \quad c = 0.628 \text{ GeV} \quad \sigma_0 = 0.0127 \text{ GeV}$$

$$s = 2.655 \quad \epsilon_c = -0.2948 \quad \epsilon_t = 0.0129$$

$$\epsilon_{so(V)} = -0.0715 \quad \epsilon_{so(S)} = 0.0573$$

TABLE VI: Values of Godfrey and Isgur's model parameters, obtained by fitting the results of Eq. (3) to the experimental data [95, 97].

| State     | $J^{PC}$ | $BB$ | $BB^*$ | $B^+B^*$ | $B_sB^*$ | $B_s^*B^*_s$ |
|-----------|----------|------|--------|----------|----------|--------------|
| $\Upsilon(4^3S_1)$ | 1$^{--}$ | 21   | -      | -        | -        | -            |
TABLE VII: Self energies, $\Sigma(E_a)$ (in MeV, see column 15), for 1S, 2S, 3S, 1P, 2P, 3P and 1D bottomonium states due to coupling to the meson-meson continuum, calculated with the effective pair-creation strength of Eq. (9) and the values of the UQM parameters of Table [9]. Columns 3-14 show the contributions to $\Sigma(E_a)$ from various channels $BC$, such as $BB$, $BB^*$ and so on. In column 16 are reported the values of the bare energies, $E_a$, calculated within the relativized QM [8], with the values of the model parameters of Table [9]. In column 17 are reported the theoretical estimations $M_a$ of the masses of the $b\bar{b}$ states, which are the sum of the self energies $\Sigma(E_a)$ and the bare energies $E_a$ (see also Fig. 2). Finally, in column 18 are reported the experimental values of the masses of the $b\bar{b}$ states, as from the PDG [97, 98]. The symbol $\ldots$ means that the contribution from a channel is suppressed by selection rules (spins, G-parity, ...).

| State | $J^{PC}$ | $BB$ | $BB^*$ | $B^+B^*$ | $B_sB_s$ | $B_cB_c$ | $B_sB_c$ | $B_s^*B_c^*$ | $B_c^*B_c^*$ | $\eta_0\eta_0$ | $\eta_0\eta_0^*$ | $\Sigma(E_a)$ | $E_a$ | $M_a$ | $M_{exp.}$ |
|-------|----------|-------|--------|-----------|-----------|-----------|-----------|---------------|--------------|-------------|-------------|----------------|------|-----|--------|
| $\eta_0(1^1S_0)$ | $0^{-+}$ | -26 | -26 | -5 | -5 | -1 | -1 | -1 | -1 | 0 | -64 | 9455 | 9391 | 9391 |
| $\Upsilon(1^3S_1)$ | $1^{-+}$ | -5 | -19 | -32 | -1 | -4 | -7 | 0 | 0 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | -69 | 9558 | 9489 | 9460 |
| $\eta_0(2^1S_0)$ | $0^{-+}$ | -43 | -41 | -8 | -7 | -1 | -1 | -1 | -1 | -1 | 0 | -101 | 10081 | 9980 | 9999 |
| $\Upsilon(2^3S_1)$ | $1^{-+}$ | -8 | -31 | -51 | -2 | -6 | -9 | 0 | 0 | -1 | -1 | -1 | 0 | -108 | 10130 | 10022 | 10023 |
| $\eta_0(3^1S_0)$ | $0^{-+}$ | -59 | -52 | -8 | -8 | -1 | -1 | -1 | -1 | -1 | 0 | -103 | 10467 | 10338 | - |
| $\Upsilon(3^3S_1)$ | $1^{-+}$ | -14 | -45 | -68 | -2 | -6 | -10 | 0 | 0 | -1 | -1 | -1 | 0 | 146 | 10504 | 10358 | 10355 |
| $h_0(1^3P_0)$ | $0^{++}$ | -22 | -69 | -3 | -13 | 0 | -1 | 0 | 0 | -1 | 0 | -115 | 10000 | 9885 | 9899 |
| $\chi_{b0}(1^3P_0)$ | $1^{++}$ | -49 | -47 | -9 | -8 | -1 | -1 | -1 | -1 | 0 | -114 | 9993 | 9879 | 9893 |
| $\chi_{b2}(1^3P_2)$ | $2^{++}$ | -11 | -32 | -55 | -2 | -6 | -9 | 0 | 0 | -1 | -1 | -1 | 0 | -117 | 10017 | 9900 | 9912 |
| $h_0(2^3P_0)$ | $0^{++}$ | -14 | 0 | -1 | -1 | 0 | 0 | -1 | 0 | 0 | -103 | 10363 | 10226 | 10233 |
| $\chi_{b0}(2^3P_0)$ | $1^{++}$ | -66 | -59 | -10 | -9 | -1 | -1 | -1 | -1 | 0 | -114 | 10393 | 10247 | 10260 |
| $\chi_{b2}(2^3P_2)$ | $2^{++}$ | -16 | -42 | -72 | -2 | -6 | -10 | 0 | 0 | -1 | -1 | -1 | 0 | -114 | 10406 | 10257 | 10269 |
| $h_0(3^3P_0)$ | $0^{++}$ | -4 | 0 | -1 | -1 | 0 | 0 | -1 | 0 | 0 | -186 | 10681 | 10495 | - |
| $\chi_{b0}(3^3P_0)$ | $1^{++}$ | -25 | -74 | -11 | -10 | -1 | -1 | -1 | -1 | 0 | -121 | 10701 | 10580 | - |
| $\chi_{b2}(3^3P_2)$ | $2^{++}$ | -19 | -16 | -79 | -3 | -8 | -12 | 0 | 0 | -1 | -1 | -1 | 0 | -138 | 10716 | 10578 | - |
| $\Upsilon(1^1D_2)$ | $2^{++}$ | -72 | -66 | -11 | -10 | -1 | -1 | -1 | -1 | 0 | -161 | 10283 | 10122 | - |
| $\Upsilon(1^3D_1)$ | $1^{++}$ | -24 | -22 | -90 | -3 | -3 | -16 | 0 | 0 | -1 | -1 | -1 | 0 | -159 | 10271 | 10112 | - |
| $\Upsilon(1^3D_2)$ | $2^{++}$ | -70 | -68 | -10 | -11 | -1 | -1 | -1 | -1 | 0 | -161 | 10282 | 10121 | 10164 |
| $\Upsilon(3^1D_3)$ | $3^{++}$ | -43 | -43 | -78 | -3 | -8 | -11 | 0 | 0 | -1 | -1 | -1 | 0 | -163 | 10290 | 10127 | - |

TABLE VIII: Mass barycenters of $\chi_b(1P)$, $\chi_b(2P)$ and $\chi_b(3P)$ systems (column 1) and mass splittings between the members of the $\chi_b(1P)$, $\chi_b(2P)$ and $\chi_b(3P)$ multiplets (column 2 and 3), from Table [9]. These are the results of our UQM calculation of the $b\bar{b}$ spectrum with self energy corrections of Table [9]. The results are expressed in MeV. The notation $\Delta M_{21}(2P)$ stands for the mass difference between the $\chi_{b2}(1P)$ and $\chi_{b1}(1P)$ resonances, $\Delta M_{10}(1P)$ for the mass difference between the $\chi_{b1}(1P)$ and $\chi_{b0}(1P)$ resonances, and so on.

| System | $M_{th}^{(1P)}$ | $\Delta M_{21}(1P)$ | $\Delta M_{10}(1P)$ |
|--------|----------------|-----------------|-----------------|
| $\chi_b(1P)$ | 9876 | 21 | 30 |
| $\chi_b(2P)$ | 10242 | 13 | 18 |
| $\chi_b(3P)$ | 10551 | -2 | 85 |

FIG. 3: Mass barycenter (in GeV) of the $\chi_b(3P)$ system in our UQM calculation.
Distinguish between quark model or unquenched quark at ATLAS and D0 are performed, they will be able to cause of important threshold effects in the UQM case.

\[
\chi_{1P} \quad \Delta M_{21}(1P) \quad \Delta M_{10}(1P) \\
9894 \quad 21 \quad 30 \\
M_{\chi_b(2P)} \quad \Delta M_{21}(2P) \quad \Delta M_{10}(2P) \\
10241 \quad 15 \quad 21 \\
M_{\chi_b(3P)} \quad \Delta M_{21}(1P) \quad \Delta M_{10}(1P) \\
10510 \quad 17 \quad 13 \\
\]

TABLE IX: Mass barycenters of \(\chi_b(1P)\), \(\chi_b(2P)\) and \(\chi_b(3P)\) systems (column 1) and mass splittings between the members of the \(\chi_b(1P)\), \(\chi_b(2P)\) and \(\chi_b(3P)\) multiplets (column 2 and 3), from Table VIII. These are the results of our re-fit of Godfrey and Isgur model’s spectrum for \(b\bar{b}\) states with the model parameters of Table VII. The results are expressed in MeV.

\[
\chi_{1P} \quad \Delta M_{21}(1P) \quad \Delta M_{10}(1P) \\
9872 \quad 21 \quad 30 \\
M_{\chi_b(2P)} \quad \Delta M_{21}(2P) \quad \Delta M_{10}(2P) \\
10244 \quad 15 \quad 21 \\
M_{\chi_b(3P)} \quad \Delta M_{21}(1P) \quad \Delta M_{10}(1P) \\
10536 \quad 12 \quad 16 \\
\]

TABLE X: Mass barycenters of \(\chi_b(1P)\), \(\chi_b(2P)\) and \(\chi_b(3P)\) systems (column 1) and mass splittings between the members of the \(\chi_b(1P)\), \(\chi_b(2P)\) and \(\chi_b(3P)\) multiplets (column 2 and 3) within the relativized QM of Sec. III A. Our results of Tables VIII and IX are substantially equivalent for the \(\chi_b(1P)\) and \(\chi_b(2P)\) systems and the mass splittings between the members of the three multiplets, from our re-fit of the relativized QM. These results are extracted by running a numerical program that calculates Godfrey and Isgur model’s spectrum for \(b\bar{b}\) states with the values of the model parameters reported in the original paper of 1985. The results are expressed in MeV.

\[
M_{\chi_b(3P)} = 10.551 \pm 0.014 \text{(stat.)} \pm 0.017 \text{(syst.)} \text{GeV} \\
\]

and \(M_{\chi_b(3P)}\) is the mass of the \(\chi_b(3P)\) state. It is interesting to observe that, in the case of the \(\chi_b(3P)\) system, important threshold effects break the scheme for the splittings between \(\chi_b(3P)\) and \(\chi_b(3P)\) resonances, that holds in the \(\chi_b(1P)\) and \(\chi_b(2P)\) cases. See also Table IX where we give results for the mass barycenters of \(\chi_b(1P)\), \(\chi_b(2P)\) and \(\chi_b(3P)\) systems and the mass splittings between the members of the three multiplets, from our re-fit of the relativized QM of Sec. III A. Our results of Tables VIII and IX are substantially equivalent for the \(\chi_b(1P)\) and \(\chi_b(2P)\) systems and different for the \(\chi_b(3P)\) system, important threshold effects break the scheme in the \(\chi_b(3P)\) case.

We think that, when the high-statistics experiments at ATLAS and D0 are performed, they will be able to distinguish between quark model or unquenched quark model predictions for the masses of the states belonging to this multiplet. Indeed, our idea is that the QM can give a good reproduction of the experimental data, except in the proximity of the thresholds, when the results should be corrected by unquenching the quark model.

Up to now, for the bottomonium case we have investigated states that are far away from meson-meson decay thresholds, with the exception of \(3P\) states, due to the complexity of the calculations as one goes up with the energies and the shells; nevertheless, the study of these higher excitations will be the next step.

D. Discussion of the results

In this paper, we computed the bottomonium spectrum with self-energy corrections. In the UQM formalism of Refs. [69, 91], the effects of \(q\bar{q}\) sea pairs are introduced explicitly into the QM through a QCD-inspired \(3P_0\) pair-creation mechanism.

The self energies, we studied in this paper, are corrections to the meson masses arising from the coupling to the meson-meson continuum. Neglected in naive QM’s, these loop effects provide an indication of the quality of the quenched approximation used in QM’s calculations, where only valence quarks are taken into account. It is thus worthwhile seeing what happens when these pair-creation effects are introduced into the quark model, similarly to what is done in unquenched lattice QCD calculations [19, 23]. Therefore, we could say that these kind of studies can also be seen as inspections of the QM, of its power in predicting the properties of hadrons and of its range of applicability: if the departure from QM’s results is important, one can see new physics emerging or better extra degrees of freedom.

Several studies on the goodness of the quenched approximation in the QM have already been done, such as those of Refs. [69, 71, 78, 91, 93, 94]. Many of them show that the quark model can predict several hadron properties with a quite high level of accuracy. Nevertheless, there are observables whose expectation value on the valence component of a certain hadron is null, even if the expectation value on the sea component of the same hadron is nonzero: for example, this occurs in the case of the flavor asymmetry of the nucleon [70], where one has to incorporate loop effects into the QM in order to carry out this kind of calculation. This is also the case of the self energy calculation of Ref. [69] and of the present paper.

Our results for the self energies of bottomonia show that the loop corrections to the spectrum of \(b\bar{b}\) mesons (see Table VII) are relatively small. Specifically for bottomonium states, they are approximately 1–2% of the corresponding meson mass, while we have shown in Ref. [64] that the charmonium mass shifts induced by loop corrections are in the order of 2–6%. The relative mass shifts, namely the difference between the self energies of two meson states, are in the order of a few tens of MeV, but can become qualitatively important in proximity of a threshold. Nevertheless, these continuum-coupling effects can become qualitatively important because of the high precision predictions of meson masses one can ob-
tain in the heavy-quark sector.

In Ref. [91], a similar approach was already applied to lower bottomonia. Nevertheless, the results of Ref. [91] were only preliminary, not only because in the present paper the method was applied to the spectrum up to the excited states and in Ref. [91] only to the low-lying ones. Indeed, one can notice that in Ref. [91] the effective pair-creation strength $\gamma_0$ was fitted to the strong decay $\psi(3770) \to DD$, calculated in SU$_f(5)$, while in the present paper it was fitted to $\Upsilon(4S) \to BB$. We think that this is more correct: 1) because a calculation of the properties of $b\bar{b}$ mesons, based on a $^3P_0$-type model for the decay vertices, has parameters that should be fitted in the most appropriate sector, thus the $b\bar{b}$ one in this case, considering the decay(s) of a $b\bar{b}$ meson(s); 2) because $b\bar{b}$ mesons have open bottom decay thresholds that are located at high energies with respect to the masses of the mesons belonging to the $c\bar{c}$ sector, i.e. the first $b\bar{b}$ meson decaying into a $B\bar{B}$ pair is a 4$S$ one. Thus, if one wants to get reliable results for the open bottom strong decays of higher bottomonia, one should fit the parameters in such a way to get a good reproduction of the widths of these high radial (and orbital) excitations. In the charmonium case, the lowest energy state decaying into an open charm $DD$ pair is the $\psi(3770)$, i.e. a 1$D$ state. In general, 1$D$ states lie at lower energies with respect to 4$S$ ones. Moreover, in Ref. [91] the bare energies were taken as free parameters, and the sum of the bare and self energies was fitted to the experimental data. In the present paper, the bare energies were calculated within Godfrey and Isgur’s relativized QM [5] and the sum of the bare and self energies was fitted to the experimental data. This is more consistent and much more elegant and, above all, increases the predictive power of the model; thus, we think that it constitutes an improvement to the previous calculation.

These continuum coupling effects are particularly important in the case of suspected non $q\bar{q}$ states, such as the $X(3872)$ [100]. Indeed, it is true that, in general, the relativized QM [5] can give a more precise overall description of the data. Nevertheless, as shown in Ref. [64] in the case of the $X(3872)$, the relativized QM may have problems when one considers states that are close to a meson-meson decay threshold. In this case, we think that it is necessary to introduce continuum coupling corrections. At the moment, there are two possible interpretations for the $X(3872)$ [101]: a weakly bound $1^{++} DD^*$ molecule [28, 30] or a $c\bar{c}$ state [14, 63, 102, 104], with $1^{++}$ quantum numbers. In particular, in Ref. [64] it is shown that the continuum coupling effects of the $X(3872)$ can give rise to $D\bar{D}^*$ and $D^*\bar{D}^*$ components in addition to the $c\bar{c}$ core and determine a downward energy shift, which is necessary to obtain a better reproduction of the experimental data. Perhaps, this may also be the case of the $\chi_b(3P)$ resonances (or at least of one of them), that lie quite close to $B\bar{B}$, $B\bar{B}^*$ and $B^*\bar{B}^*$ decay thresholds. We think that the present experimental data [98, 99] cannot exclude this possibility.

Our work is not an extension of Godfrey and Isgur’s relativized QM [5], that we used to compute the bare energies of the $b\bar{b}$ states, this is the unquenching of the quark model [64]. In Ref. [63], the authors did something similar. Nevertheless: a) they used the non relativistic potential model to compute the bare energies instead of the relativized Godfrey and Isgur’s one [5], as we did; b) they used the $^3P_0$ model to compute the self energy corrections, as we did, but they used a standard $^3P_0$ transition operator. So their results are biased by the fact that they did not take the suppression of heavy quark pair production into account, as we did. For example, in the case of $\chi_b(3P)$ states, their results are different from ours.

In this paper, we also calculated the strong decay widths of $b\bar{b}$ states within a modified version of the $^3P_0$ model of hadron decays [64, 91]. The corrections we introduced into the transition operator of the model include: 1) the use of a quark form factor, to take the effective size of the constituent quarks into account [61, 64, 91, 93, 94]; 2) the replacement of the pair-creation strength $\gamma_0$ with the effective strength of Eq. (5), that suppresses heavy $q\bar{q}$ pair-creation [58, 64, 61]. The introduction of this effective mechanism is necessary, since in the original formulation of the $^3P_0$ model the flavor-independent pair-creation implies an unphysical enhancement of heavy quarks creation compared to light quarks one, without a fundamental reason for that [58]. These results for the strong decay widths, that required the refit of Godfrey and Isgur mass formula to take the latest experimental data into account, may be particularly useful to the experimentalists. Indeed, while the knowledge of the $T$ states and their decay modes is relatively good, this is not true for $\eta_b$ and $\chi_b$ states and, above all, for all the other states, such as the $D, F$ and $G$-wave ones, that we analysed in the present paper.

Finally, we think that the present paper may be a useful help to the experimentalists in their search for new $b\bar{b}$ states. In the last few years the interest in heavy quarkonium physics has increased enormously, and also the number of collaborations devoted to the topic because of the development of new B factories. In particular, BaBar [108, 109], Belle [110], CDF [111] and D0 [90] have produced many interesting results. Moreover, all four detectors at LHC (Alise, Atlas, CMS and LHCb) have the capacity to study charmonia and bottomonia and have already produced some new results [112], such as the discovery of a new $\chi_b(3P)$ system [98]. There are also approved proposals for new experiments, such as Belle II [113]. Therefore, we think it is important to study the properties of heavy mesons in order to provide (updated) informations to the experimentalists about spectra, strong decay widths, helicity amplitudes, and so on.
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Appendix A: SUf(5) couplings

The SUf(5) flavor couplings that we have to calculate in the \( 3^3P_0 \) model are \( \langle F_B(14)F_C(32)|F_A(12)F_0(34) \rangle \) for the first diagram of Fig. 4 and \( \langle F_B(32)F_C(14)|F_A(12)F_0(34) \rangle \) for the second diagram, where \( F_X(ij) \) represents the flavor wave function for the meson \( X \) (i.e., the initial meson \( A \), the final mesons \( B \) and \( C \) or the \( 3^3P_0 \) created pair 0) made up of the quarks \( i \) and \( j \). These overlaps can be easily calculated if we adopt a matrix representation of the mesons. In this case, the two diagrams become, respectively,

\[
\langle F_B(14)F_C(32)|F_A(12)F_0(34) \rangle = \frac{1}{\sqrt{5}} Tr[F_A F_B^T F_0 F_C^T],
\]

\[
\langle F_B(32)F_C(14)|F_A(12)F_0(34) \rangle = \frac{1}{\sqrt{5}} Tr[F_A F_C^T F_0 F_B^T].
\]

(App1)

Appendix B: Parameters of the \( 3^3P_0 \) pair-creation model

The value of the width of the constituent quark form factor, \( r_q = 0.335 \text{ fm} \), is taken from Ref. 64. The value of the harmonic oscillator parameter is taken as \( \alpha = 0.5 \text{ GeV} \). Finally, the value of the pair-creation strength, \( \gamma_0 \), has to be fitted to the reproduction of experimental strong decay widths. We have chosen to fit \( \gamma_0 \) to the experimental strong decay width \( \Upsilon(4S) \rightarrow BB \) \( \text{(55)} \). In this case, since the created pair \( q\bar{q} \) is \( uu \) or \( dd \), the effective pair creation strength \( \gamma_0 \) coincides with \( \gamma_0 \) [see Eq. (3)].

The decay width is calculated within the \( 3^3P_0 \) model \( \text{(51, 57)} \) as

\[
\Gamma_{\Upsilon(4S) \rightarrow BB} = 2\Phi_{A \rightarrow BC} | \langle BC\bar{q}_0 | T^\dagger | A \rangle |^2 = 2\Phi_{\Upsilon(4S) \rightarrow BB} | \langle BB\bar{q}_0 | T^\dagger | \Upsilon(4S) \rangle |^2 = 21 \text{ MeV},
\]

where the factor of 2 is introduced since \( \Upsilon(4S) \) decays into \( B^0\bar{B}^0 \) or \( B^+B^- \), \( \langle BC\bar{q}_0 | T^\dagger | A \rangle \) is the \( 3^3P_0 \) amplitude describing the coupling between the meson \( |A\rangle = |\Upsilon(4S)\rangle \) and the final state \( |BC\rangle = |BB\rangle \) and

\[
\Phi_{A \rightarrow BC} = 2\pi q_0 \frac{E_b E_c}{\mathcal{M}_a}
\]

is the standard relativistic phase space factor \( \text{(87)} \), with \( E_b = \sqrt{M_b^2 + q_0^2} \) and \( E_c = \sqrt{M_c^2 + q_0^2} \).

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