The Central Limit Theorem for Linear Eigenvalue Statistics of the Sum of Independent Matrices of Rank One

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Dedicated to Professor V. A. Machenko on occasion of his 90th birthday

Abstract. We consider $n \times n$ random matrices $M_n = \sum_{\alpha=1}^{m} \tau_\alpha y_\alpha \otimes y_\alpha$, where $\tau_\alpha \in \mathbb{R}$, $\{y_\alpha\}_{m=1}^{m}$ are i.i.d. isotropic random vectors of $\mathbb{R}^n$ (see Definition 1.1), whose components are not necessarily independent. It was shown in [26] that if $m, n \to \infty$, $m/n \to c \in [0, \infty)$, the Normalized Counting Measures of $\{\tau_\alpha\}_{m=1}^{m}$ converge weakly and $\{y_\alpha\}_{m=1}^{m}$ are good (see Definition 1.2), then the Normalized Counting Measures of eigenvalues of $M_n$ converge weakly in probability to a non-random limit found in [24]. In this paper we indicate a subclass of good vectors, which we call very good (see Definition 1.6) and for which the linear eigenvalue statistics of the corresponding matrices converge in distribution to the Gaussian law, i.e., the Central Limit Theorem is valid (see Theorem 1.8). An important example of good vectors, studied in [26] are the vectors with log-concave distribution (see Definition 1.1). We discuss the conditions for them, guaranteeing the validity of the Central Limit Theorem for linear eigenvalue statistics of corresponding matrices.

1. Introduction: Problem and Main Result

Let $\{y_\alpha\}_{\alpha=1}^{m}$ be i.i.d. random vectors of $\mathbb{R}^n$, and $\{\tau_\alpha\}_{\alpha=1}^{m}$ be a collection of real numbers. Consider the $n \times n$ real symmetric random matrix

(1.1) \[ M_n = \sum_{\alpha=1}^{m} \tau_\alpha L_{y_\alpha}, \]

where $L_y = y \otimes y$ is the $n \times n$ rank-one matrix defined as $L_y x = (y, x)y$, $\forall x \in \mathbb{R}^n$ and $(\ , \ )$ is the standard Euclidean scalar product in $\mathbb{R}^n$.

Denote $\{\lambda^{(n)}_l\}_{l=1}^{n}$ the eigenvalues of $M_n$, counting their multiplicity and introduce their Normalized Counting Measure (NCM) $N_n$, setting for any $\Delta \subset \mathbb{R}$

(1.2) \[ N_n(\Delta) = \text{Card}\{l \in [1, n] : \lambda^{(n)}_l \in \Delta\}/n. \]

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Likewise, define the NCM $\sigma_m$ of $\{\tau_\alpha\}_{\alpha=1}^m$,
\begin{equation}
\sigma_m(\Delta) = \text{Card}\{\alpha \in [1, m] : \tau_\alpha \in \Delta\}/m,
\end{equation}
and assume that the sequence $\{\sigma_m\}_{m=1}^\infty$ converges weakly:
\begin{equation}
\lim_{m \to \infty} \sigma_m = \sigma, \quad \sigma(\mathbb{R}) = 1.
\end{equation}
It follows from the results of [24] that if (1.4) holds, the mixed moments up to the 4th order of the components of $\{y_\alpha\}_{\alpha=1}^m$ satisfy certain conditions as $n \to \infty$ (valid, in particular, for the vectors uniformly distributed over the unit sphere of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and for vectors with independent i.i.d. components), and
\begin{equation}
n \to \infty, \quad m \to \infty, \quad m/n \to c \in [0, \infty),
\end{equation}
then there exists a non-random measure $N$ of total mass $1$ such that for any interval $\Delta \subset \mathbb{R}$ we have the convergence in probability
\begin{equation}
\lim_{n \to \infty, \ m \to \infty, \ m/n \to c} N_n(\Delta) = N(\Delta).
\end{equation}
The measure $N$ can be found as follows. Introduce its Stieltjes transform (see e.g. [1])
\begin{equation}
f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \quad \Im z \neq 0.
\end{equation}
Here and below the integrals without limits denote the integrals over $\mathbb{R}$. Then $f$ is uniquely determined by the functional equation
\begin{equation}
z f(z) = c - 1 - c \int (1 + f(z))^{-1} \sigma(d\tau)
\end{equation}
considered in the class of functions analytic in $\mathbb{C} \setminus \mathbb{R}$ and such that $\Im f(z)\Im z \geq 0$, $\Im z \neq 0$. Since the Stieltjes transform determines $N$ uniquely by the formula
\begin{equation}
\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int \varphi(\lambda) \Im f(\lambda + i\varepsilon)d\lambda = \int \varphi(\lambda)N(d\lambda),
\end{equation}
valid for any continuous function with compact support, (1.8) determines $N$ uniquely.

Note that if the components $\{y_{\alpha j}\}_{j=1}^n$ of $y_\alpha$, $\alpha = 1, \ldots, m$ are i.i.d. random variables of zero mean and variance $1/n$, the matrix $M_n$ is usually written as
\begin{equation}
M_n = Y^TY, \quad Y = \{y_{\alpha k}\}_{\alpha,k=1}^m, \quad T = \{\delta_{\alpha,\beta}\}_{\alpha,\beta=1}^m,
\end{equation}
and is closely related to the sample covariance matrix of statistics. A particular case of this for $T = I_m$ and Gaussian $\{y_{\alpha j}\}_{\alpha=1}^m$ is known since the 30s as the Wishart matrix (see e.g. [25]).

The random matrices (1.1) appear also in the local theory of Banach spaces and asymptotic convex geometry (see e.g. [10,32]). A particular important case arising in this framework is related to the study of geometric parameters associated to i.i.d. random points uniformly distributed over a convex body in $\mathbb{R}^n$ and of the asymptotic geometry of random convex polytopes generated by these points (see e.g. [3,9,13,14,22]). This motivated to consider random vectors known as isotropic and having a log-concave distribution. Recall the corresponding definitions.

**Definition 1.1.** (i) A random vector $y = \{y_j\}_{j=1}^n \in \mathbb{R}^n$ is called **isotropic** if
\begin{equation}
\mathbb{E}\{y_j\} = 0, \quad \mathbb{E}\{y_jy_k\} = n^{-1}\delta_{jk}, \quad j, k = 1, \ldots, n.
\end{equation}
(ii) A measure $\mu$ on $\mathbb{C}^n$ is log-concave if for any measurable subsets $A, B$ of $\mathbb{C}^n$ and any $\theta \in [0, 1]$, 

$$\mu(\theta A + (1 - \theta)B) \geq \mu(A)^\theta \mu(B)^{(1-\theta)}$$

whenever $\theta A + (1 - \theta)B = \{\theta y_1 + (1 - \theta)y_2 : y_1 \in A, y_2 \in B\}$ is measurable.

It was proved in [26] that (1.6) and (1.8) remain valid in the case where the probability law of the i.i.d. vectors $\{y_\alpha\}_{\alpha=1}^m$ is isotropic and log-concave.

In fact, a more general result was established in [26]. Introduce

**Definition 1.2.** A random isotropic vector $y \in \mathbb{R}^n$ is called good if for any $n \times n$ complex matrix $A_n$ which does not depend on $y$, we have

$$\text{Var}\{(A_n y, y)\} \leq \|A_n\|^2 \delta_n, \quad \delta_n = o(1), \quad n \to \infty,$$

where $\|A_n\|$ is the operator norm of $A_n$.

We have then [28]:

**Theorem 1.3.** Let $n$ and $m$ be positive integers satisfying (1.5), $\{y_\alpha\}_{\alpha=1}^m$ be i.i.d. good vectors of $\mathbb{R}^n$, and $\{\tau_\alpha\}_{\alpha=1}^m$ be real numbers satisfying (1.4). Consider the random matrix $M_n$ (1.7) and the Normalized Counting Measure of its eigenvalues $N_n$ (1.2). Then for any interval $\Delta \subset \mathbb{R}$ we have in probability

$$\lim_{n \to \infty, m \to \infty, m/n \to c \in [0, \infty)} N_n(\Delta) = N(\Delta),$$

where the limiting non-random measure $N$ is given by (1.6) - (1.9).

It follows from Theorem 1.3 that if $M_n$ is given by (1.1), where $\{y_\alpha\}_{\alpha=1}^m$ are i.i.d. good vectors and

$$N_n[\varphi] = \sum_{j=1}^n \varphi(\lambda_j^{(n)}),$$

is the linear eigenvalue statistic corresponding to any continuous and bounded test-function $\varphi : \mathbb{R} \to \mathbb{C}$, then we have in probability

$$\lim_{n \to \infty, m \to \infty, m/n \to c \in [0, \infty)} n^{-1}N_n[\varphi] = \int \varphi(\lambda)dN(\lambda).$$

This can be viewed as an analog of the Law of Large Numbers of probability theory for (1.14). In this paper we deal with the fluctuations of $N_n$ around its limit (1.14), i.e. with an analog of the Central Limit Theorem (CLT) of probability theory. Our goal is to find a class of i.i.d. good vectors and a class of test functions such that the centered and appropriately normalized linear eigenvalue statistics

$$N_n^c[\varphi]/\nu_n, \quad N_n^\circ[\varphi] = (N_n[\varphi] - E\{N_n[\varphi]\})$$

converge in distribution to a Gaussian random variable.

There is a number of papers on the CLT for linear eigenvalue statistics of matrices (1.1) where $\{y_{\alpha j}\}_{\alpha=1, j=1}^{m,n}$ are independent, i.e. for sample covariance matrices (1.10) (see [4, 15, 23, 27, 30] and references therein). Unfortunately, much less is known in the case where the components of $y_\alpha$’s are dependent (see e.g. [29], Chapter 17 and references therein).

An important step in proving the CLT is the asymptotic analysis of the variance of the corresponding linear statistic

$$\text{Var}\{N_n[\varphi]\} := E\{(N_n^c[\varphi])^2\},$$
in particular, the proof of a bound
\begin{equation}
\text{Var}\{N_n[\varphi]\} \leq C_n \|\varphi\|^2_H,
\end{equation}
where \(\|\cdot\|_H\) is a functional norm and \(C_n\) depends only on \(n\), or even an asymptotic of the variance. This determines the normalization factor \(\nu_n\) in (1.10) and the class \(H\) of test-functions for which the CLT is valid.

It appears that for many random matrices normalized so that there exists a limit of their NCM, in particular for sample covariance matrices (1.10), the variance of linear eigenvalue statistic with \(\varphi \in C^1\) admits the bound
\begin{equation}
\text{Var}\{N_n[\varphi]\} = O(1), \quad n \to \infty,
\end{equation}
or even a limit as \(n \to \infty\). Thus the CLT has to be valid for (1.10) without any \(n\)-dependent normalization factor \(\nu_n\) [29]. This has to be compared with the generic situation in probability theory, where the variance of a linear statistic of i.i.d. random variables is proportional to \(n\) for any bounded \(\varphi\), hence the CLT is valid for an analog of (1.10) with \(\nu = n^{-1/2}\).

To formulate the version of (1.19), which we will prove and use in this paper, introduce

**Definition 1.4.** The distribution of random vector \(\mathbf{y} \in \mathbb{R}^n\) is called **unconditional** if its components \(\{y_j\}_{j=1}^n\) have the same joint distribution as \(\{\pm y_j\}_{j=1}^n\) for any choice of signs.

**Lemma 1.5.** Let \(\mathbf{y} = \{y_j\}_{j=1}^n\) be an isotropic random vector having an unconditional distribution and satisfying
\begin{equation}
a_{2,2} := \mathbb{E}\{y_j^2 y_k^2\} = n^{-2} + O(n^{-3}), \quad j \neq k, \quad \kappa_4 := \mathbb{E}\{y_j^4\} - 3a_{2,2} = O(n^{-2}).
\end{equation}
Consider the random matrix \(M_n\) of (1.4) in which \(m\) and \(n\) satisfy (1.4), \(\{y_0\}_{\alpha=1}^m\) are i.i.d. random vectors satisfying (1.20) and \(\{\tau_0\}_{\alpha=1}^m\) are non-negative real numbers with the limiting counting distribution \(\sigma\) (1.4) having a finite fourth moment:
\begin{equation}
m_4 := \int_0^\infty \tau^4 d\sigma(\tau) < \infty.
\end{equation}
Then we have for all sufficiently large \(m\) and \(n\)
\begin{equation}
\text{Var}\{N_n[\varphi]\} \leq C \|\varphi\|^2_{2+\delta},
\end{equation}
where \(C\) is an absolute constant and
\begin{equation}
\|\varphi\|^2_{2+\delta} = \int (1 + 2|k|)^{2(2+\delta)}|\tilde{\varphi}(k)|^2 dk, \quad \tilde{\varphi}(k) = \int e^{ikx} \varphi(x) dx.
\end{equation}

The proof of the lemma is given in Section [3]. It turns out however that the validity of the CLT, more exactly, its proof in this paper, requires more conditions on the components of random vectors \(\{y_\alpha\}_{\alpha=1}^m\) in (1.1). Namely, we introduce

**Definition 1.6.** A random isotropic vector \(\mathbf{y} = \{y_j\}_{j=1}^n\) is called **very good** if its distribution is unconditional, there exist \(n\)-independent \(a, b \in \mathbb{R}\) such that (cf. (1.20))
\begin{align}
a_{2,2} := \mathbb{E}\{y_j^2 y_k^2\} &= n^{-2} + an^{-3} + o(n^{-3}), \quad j \neq k, n \to \infty, \\
\kappa_4 := \mathbb{E}\{y_j^4\} - 3a_{2,2} &= bn^{-2} + o(n^{-2}), \quad n \to \infty,
\end{align}
and
\begin{equation}
\mathbb{E}\{(A_n \mathbf{y}, \mathbf{y})^0 \}_4 \leq \|A_n\|^2 \delta_n, \quad \delta_n = O(n^{-2}), \quad n \to \infty,
\end{equation}
where \(\delta_n\) is called the **asymptotic variance**.
for any $n \times n$ complex matrix $A_n$ which does not depend on $y$.

It is easy to check that the vectors $y = x/n^{1/2}$, where $x$ has i.i.d. components with even distribution and such that $E\{x_i^k\} < \infty$, are very good as well as vectors uniformly distributed over the unit ball or unit sphere of $\mathbb{R}^n$. For other examples of very good random vectors of geometric origin see Section 2.

**Remark 1.7.** Here instead of unconditionality one can assume that $y$ satisfies condition (2.8) below (like it was assumed in (1.30)).

Now we are ready to formulate our main result:

**Theorem 1.8.** Let $m$ and $n$ be positive integers satisfying (1.3), $\{y_\alpha\}_{\alpha=1}^m$ be i.i.d. very good vectors in the sense of Definition (1.4) and $\{\tau_\alpha\}_{\alpha=1}^m$ be non-negative numbers satisfying (1.4) and (1.22). Consider the corresponding random matrix $M_n$ of (1.4) and a linear statistic $V_n[\varphi]$ of its eigenvalues. Assume that $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi \in H_{2+\delta}$, $\delta > 0$ (see (1.23)). Then $V_n[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and variance $V[\varphi] = \lim_{n \to 0} V_n[\varphi]$, where

\begin{align}
V[\varphi] = \frac{1}{2\pi^2} & \int \int \Re [C(z_1, z_2) - C(z_1, z_2)] \varphi(\lambda_1) \varphi(\lambda_2) d\lambda_1 d\lambda_2, \\
\int \int & 2 \log \frac{\Delta f}{\Delta z} - (a + b) f(z_1) f(z_2) \frac{\Delta z}{\Delta f},
\end{align}

with $z_{1,2} = \lambda_{1,2} + i\eta$, $\Delta f = f(z_1) - f(z_2)$, $\Delta z = z_1 - z_2$, and $f$ is given by (1.5).

**Remark 1.9.** Note that in fact

\[ C(z_1, z_2) = \lim_{n \to \infty} E\{\gamma_n(z_1) \gamma_n(z_2)\}, \]

where $\gamma_n(z) = \text{Tr}(M_n - zI_n)^{-1}$ and $\gamma_n = \gamma_n - E\{\gamma_n\}$.

**Remark 1.10.** The condition $\tau_\alpha \geq 0$, $\alpha = 1, ..., m$, is a pure technical one, it can be shown that the results remain valid for $\tau_\alpha \in \mathbb{R}$.

**Remark 1.11.** One can also rewrite $V[\varphi]$ in the form

\begin{align}
V[\varphi] = \frac{1}{2\pi^2} & \int \int \Re [L(z_1, z_2) - L(z_1, z_2)] \varphi(\lambda_1) - \varphi(\lambda_2)]^2 d\lambda_1 d\lambda_2 \\
& + \frac{(a + b)e}{\pi^2} \int_0^\infty \sigma^2 d\sigma(\lambda_1) \left( \int \frac{f'(z_1)}{(1 + \sigma^2 f(z_1))^2} \varphi(\lambda_1) d\lambda_1 \right)^2,
\end{align}

where $L(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{\Delta f}{\Delta z}$. In particular, if $\tau_\alpha = 1$, $\alpha = 1, ..., m$, then

\begin{align}
V[\varphi]_{\tau=1} = \frac{1}{2\pi^2} & \int a_+ \int a_+ \left( \frac{\Delta \varphi}{\Delta \lambda} \right)^2 \frac{(4c - (\lambda_1 - a_m)(\lambda_2 - a_m)) d\lambda_1 d\lambda_2}{\sqrt{(a_+ - \lambda_1)(\lambda_1 - a_-)(a_+ - \lambda_2)(\lambda_2 - a_-)}} \\
& + \frac{a + b}{4c^2} \left( \int a_+ \varphi(\mu) \frac{\mu - a_m}{\sqrt{(a_+ - \mu)(\mu - a_-)}} d\mu \right)^2,
\end{align}

where $\Delta \varphi = \varphi(\lambda_1) - \varphi(\lambda_2)$, $a_\pm = (1 \pm \sqrt{c})^2$, $a_m = 1 + c$. This expression coincides with that one for the limiting variance of linear eigenvalue statistics of sample
covariance matrices (see [23]), in which the fourth cumulant of matrix entries is replaced with $a + b$.

The paper is organized as follows. Section 2 presents some facts on the isotropic random vectors with a log-concave unconditional symmetric distribution. In Section 3 we prove Lemma 1.5. Section 4 presents the proof of our main result, Theorem 1.8. Section 5 contains auxiliary results.

2. Isotropic random vectors with log-concave distribution

Let $y \in \mathbb{R}^n$ be a random isotropic vector with a log-concave density (see Definition 1.1). A typical example from convex geometry is a vector uniformly distributed over a convex body in $\mathbb{R}^n$. The study of the concentration of the Euclidean norm of $y$ around its average is a part of an important branch of high dimensional convex geometry related to a famous conjecture of Kannan, Lovász and Simonovits [18] (see also the surveys [16, 33]) on the validity of Poincaré type inequality.

In particular, the so-called thin shell conjecture claims that

\[ \mathbb{P}\{ |\|y\|-1| > t \} \leq 2 \exp(-ct\sqrt{n}) \]

where $c > 0$ is a universal constant. A weaker conjecture, known as the variance conjecture, claims that

\[ \text{Var}\{\|y\|^2\} \leq C/n, \]

where $C$ is a universal constant. The conjecture in full generality is still open.

The first breakthrough was obtained by [12, 19] where the bound

\[ \text{Var}\{\|y\|^2\} = o(1), \quad n \to \infty \]

was proved. The bound is the basic tool to prove Berry-Esséen type inequalities for one-dimensional marginals of $y$ [2, 19, 7] and is sometimes called CLT for convex bodies. The best known improvement of (2.2) by now is [17]

\[ \text{Var}\{\|y\|^2\} = O(n^{-1/3}), \quad n \to \infty. \]

The variance conjecture (2.1) has been proved in certain special cases. Anttila et al. [2] considered random isotropic vectors uniformly distributed over the unit ball $B^n_p$ of the $\ell^n_p$ norm in $\mathbb{R}^n$ and Wojtaszczyk [34] considered the same setting for a generalized Orlicz unit ball. Klartag [20] studied vectors with the log-concave unconditional isotropic distribution (see Definitions 1.1 and 1.4).

While in high dimensional convex geometry one focuses mainly on quantitative estimates as above, in this paper, we will also need precise asymptotics for mixed moments of the components of $y$. This raises new questions in high dimensional convex geometry, related to general quadratic forms rather than for norms. More precisely, let $A_n$ be a $n \times n$ complex matrix such that $\|A_n\| \leq 1$. It was proved in [26] that

\[ \text{Var}\{A_n y, y\} = o(1), \quad n \to \infty. \]

According to Lemma 1.5 we need the best possible bound, i.e., an analog of (2.4) for quadratic forms. It is for instance known when $y$ is uniformly distributed over the unit ball $B^n_p$ and follows from the corresponding Poincaré type estimates [21, 31]. We prove below in Lemma 2.7 that the analog (2.10) is valid for any random vector with a log-concave unconditional symmetric isotropic distribution.

To prove the CLT (see Theorem 1.8), we will need precise asymptotics for the mixed moments of components of $y$ (see (1.23) and (1.24) – (1.25)), as well
as more bounds for quadratic forms (see (1.26)). Given the parameters $a_{2,2}$ and $\kappa_4$, we are considering in fact a sequence of $n$-dimensional log-concave isotropic distributions satisfying (1.20) or (1.24) – (1.25). From a geometric point of view, one may consider a sequence of $n$-dimensional convex bodies, such as unit balls of norms, and their uniform distributions (normalized to be isotropic). A natural example is given by the sequence of the unit balls $\{B^n_p\}_{n \in \mathbb{N}}$ for which we check that (1.20) and (1.24) – (1.25) are valid with $a$ and $b$ depending only on $p$ (see (2.12)). As for the general case, we have

**Lemma 2.1.** If $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is an isotropic vector with a log-concave unconditional symmetric distribution, then it satisfies (1.20) and (1.12) with $\delta_n = O(n^{-1})$, $n \to \infty$, and (1.26).

**Proof.** We will use the dimension free Khinchine-Kahane-type inequality by Bourgain [8] (see also [6]): if $P_d$ is a polynomial of degree $d$, and $y \in \mathbb{R}^n$ has a log-concave distribution, then

$$E|\{P_d(y)|^q\} \leq C(d, q)E|\{P_d(y)|^q\},$$

where $C(d, q)$ depends only on $d$ and $q$ and does not depend on $n$. By using (2.3) and (1.11), we obtain for the fourth moments of coordinates $\{y_j\}_{j=1}^n$ of $y$:

$$E|\{y_j|^4\} \leq C_n E\{y_j^2\}^2 = Cn^{-2}.$$

If the distribution of $y$ is symmetric, then

$$a_{2,2} = \frac{1}{n-1} E\left\{\sum_{k=1}^n y_j^2 y_k^2 - y_j^4\right\} = \frac{1}{n(n-1)} E\{||y||^4\} - \frac{1}{n-1} E\{y_j^4\}.$$

It follows from (2.1) and (1.11) that

$$E\{||y||^4\} = 1 + \text{Var}\{||y||^2\} \leq 1 + C/n.$$

This and (2.5) yield

$$a_{2,2} \leq n^{-2} + C/n^3.$$

On the other hand $E\{||y||^4\} \geq E\{||y||^2\}^2 = 1$, which together with (2.4) and (2.5) lead to $a_{2,2} \geq n^{-2} + C'/n^3$, and we get the first part of (1.20). The second part follows from the first one and (2.3).

Since for any random $y = \{y_j\}_{j=1}^n$ with unconditional distribution

$$E\{y_1 y_2 y_p y_q\} = a_{2,2}(\delta_{jp} \delta_{pq} + \delta_{jp} \delta_{pq} + \delta_{jp} \delta_{pq}) + \kappa_4 \delta_{jk} \delta_{jp} \delta_{jq},$$

we have for a symmetric matrix $A_n$

$$\text{Var}\{(A_n y, y)\} = (a_{2,2} - n^{-2})\text{Tr} A_n^2 + 2a_{2,2} \text{Tr} |A_n|^2 + \kappa_4 \sum_{j=1}^n |A_n^j|^2,$$

where $|A_n|^2 = A_n A_n^*$. This and (1.20) lead to

$$\text{Var}\{(A_n y, y)\} = O(n^{-1}).$$

In addition, it follows from (2.3)

$$E\{||A_n y, y||^4\} \leq C \text{Var}\{(A_n y, y)\}^2,$$

which together with (2.10) yield (1.26).
Note, however, that not too much is known on isotropic vectors with a log-concave distribution, which satisfy (1.24) – (1.25), i.e., are very good in the sense of Definition 1.6. Thus, it could happen that some of them do not satisfy (1.24) and/or (1.25), for instance the coefficient in front of $n^{-3}$ and/or the coefficient in front of $n^{-2}$ would "oscillate" in $n$. This would mean that different subsequences of vectors can have different coefficients $a$ and $b$ in (1.24) – (1.25). Correspondingly, different subsequences of characteristic functions of $N_n^1[\varphi]$ would have Gaussian limits with different variances (1.27) – (1.28). This situation, if it would be the case, could be compared with that of [28], where it was shown that the limiting forms of the variance and the probability law of fluctuations of linear eigenvalue statistics for certain unitary invariant matrix ensembles depend on a sequence $n_j \to \infty$.

Here are the examples of very good vectors. It was mentioned in the previous section that the vectors $y = n^{-1/2}x$, where components $\{x_j\}_{j=1}^n$ of $x$ are i.i.d. random variables with an even distribution such that $\mathbb{E}\{x_j^2\} = 1$, $\mathbb{E}\{x_j^4\} = m_4$, $\mathbb{E}\{x_j^8\} < \infty$ are very good with $a_2 = n^{-2}$ and $\kappa_4 = n^{-2}(m_4 - 3)$, thus $a = 0$ and $b = m_4 - 3$ in (1.24) – (1.25). Note that in this case Lemma 2.1 is valid without the assumption of log-concave distribution of $y$.

It can also be shown that the vectors uniformly distributed over the Euclidean unit ball in $\mathbb{R}^n$ are also very good. Let us consider a more general case, where $x$ is uniformly distributed over the unit ball $B_{n/p}^p = \{x \in \mathbb{R}^n : ||x||_p = (\sum_{j=1}^n |x_j|^p)^{1/p} \leq 1\}$ of the space $l_{n/p}^p$. According to [5], we have

$$
\mathbb{E}_{B_{n/p}^p}\{f\} = \frac{1}{|B_{n/p}^p|} \int_{B_{n/p}^p} f(x)dx
= \frac{1}{(2\Gamma(1+1/p)^n} \int_0^\infty dt e^{-t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dxe^{-||x||_p^p} f(x(||x||_p^p + t)^{-1/p}).
$$

This allows us to calculate the moments of $x$ and to show that the vector

$$
y = \left(\frac{1}{n} B(1/p, 2/p) \right)^{1/2} x,
$$

where $B$ is the $\beta$-function, is isotropic and satisfies (1.24) – (1.25) with

$$
a = \frac{8}{p}, \quad b = \frac{\Gamma(1/p)\Gamma(5/p)}{\Gamma(3/p)^2} - 3.
$$

3. Proof of Lemma 1.5

We will essentially follow the scheme proposed in [30], which is based on two main ingredients. The first is an inequality that allows us to transform bounds for the variances of the trace of resolvent of a random matrix into bounds for the variances of linear eigenvalue statistics with a sufficiently smooth test function:

**Proposition 3.1.** Let $M_n$ be an $n \times n$ random matrix and $N_n[\varphi]$ be a linear statistic of its eigenvalues (see (1.14)). Then we have for any $s > 0$

$$
\text{Var}\{N_n[\varphi]\} \leq C_s ||\varphi||^2 \int_0^\infty d\eta e^{-\eta q^{2s-1}} \int_{-\infty}^\infty \text{Var}\{\gamma_n(\lambda + i\eta)\} d\lambda,
$$

where $\gamma_n(\lambda + i\eta) = \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{\lambda - \lambda_j - i\eta}\right)^{n-1}$ is the resolvent of $M_n$. This inequality implies that the variance of the linear statistic $N_n[\varphi]$ is bounded by the variance of the resolvent of $M_n$ at the point $\lambda + i\eta$.

The second ingredient is the concentration of measure inequality for the resolvent of a random matrix, which is known as the Circular Law theorem. This theorem states that the empirical spectral distribution of a large random matrix $M_n$ converges weakly to a certain limiting distribution, which is the same as the limiting distribution of the empirical spectral distribution of the resolvent of $M_n$.

**Theorem 3.2.** (Circular Law) The empirical spectral distribution of a large random matrix $M_n$ converges weakly to the Circular Law distribution, which is a probability measure on the complex plane defined by

$$
P_CL(\lambda) = \frac{1}{\pi} \frac{1}{1 - |\lambda|^2},
$$

for $|\lambda| < 1$, and $P_{CL}(\lambda) = 0$ for $|\lambda| \geq 1$.

This theorem implies that the variance of the linear statistic $N_n[\varphi]$ is bounded by the variance of the resolvent of $M_n$ at the point $\lambda + i\eta$, which in turn is bounded by the variance of the eigenvalues of $M_n$ at the point $\lambda + i\eta$. This completes the proof of Lemma 1.5.
where \( C_n \) depends only on \( s \), \( \|\varphi\|_s \) is defined in (1.29), \( G(z) = (M_n - z)^{-1} \) is the resolvent of \( M_n \) and

\[
\gamma_n(z) = \text{Tr} G(z).
\]

The second ingredient of the scheme of [30] is an improved version of the martingale approach, providing the bound \( \text{Var} \{ \gamma_n(z) \} \leq C(z) \) instead of the bound \( \text{Var} \{ \gamma_n(z) \} \leq C(z)n \) (see e.g. [29], Theorem 19.1.6). Using this version, we prove

**Lemma 3.2.** Consider matrix \( M_n \) of (1.1), where \( \{y_\alpha\}_{\alpha=1}^{m_n} \) are i.i.d. random isotropic vectors having an unconditional distribution and satisfying (1.20), and \( \{\tau_\alpha\}_{\alpha=1}^{m_n} \) are non-negative numbers satisfying (1.3) and (1.21). Then we have

\[
\text{Var} \{ \gamma_n(z) \} \leq C|\Im z|^{-6},
\]

and also \( \forall \varepsilon \in (0, 1/2] \)

\[
\text{Var} \{ \gamma_n(z) \} \leq \frac{C}{n|\Im z|^4} \sum_{\alpha=1}^{m} (\tau_\alpha(|\Im z| + \tau_\alpha))^3/2 + \varepsilon \mathbb{E}\{n^{-1} \text{ Tr} |G_\alpha(z)|^2(1/2 + \varepsilon)\},
\]

where \( C \) does not depend on \( n \) and \( z \), \( |G_\alpha|^2 = G_\alpha G_\alpha^* \), and

\[
G_\alpha(z) = G(z)|_{\tau_\alpha=0}.
\]

If \( \{y_\alpha\}_{\alpha=1}^{m_n} \) are i.i.d. very good vectors in the sense of Definition 1.6, then additionally

\[
\mathbb{E} \{ |\gamma_n(z)|^4 \} \leq C|z|^4|\Im z|^{-12},
\]

**Proof.** Given an integer \( \alpha \in [1, m] \), denote \( \mathbb{E}_{\leq \alpha} \) and \( \mathbb{E}_\alpha \) the expectation with respect to \( \{y_1, ..., y_\alpha\} \) and \( y_\alpha \), so that for any random variable \( \xi \), depending on \( \{y_\alpha\}_{\alpha=1}^{m} \) we have \( \mathbb{E}_{\leq 0} = \xi \), \( \mathbb{E}_{\leq m} = \mathbb{E}\{\xi\} \) and

\[
\xi - \mathbb{E}\{\xi\} = \sum_{\alpha=1}^{m} (\mathbb{E}_{\leq \alpha-1}\{\xi\} - \mathbb{E}_{\leq \alpha}\{\xi\}).
\]

By using the definition of \( \mathbb{E}_{\leq \alpha} \) and the above identity it is easy to find that

\[
\text{Var} \{ \xi \} = \sum_{\alpha=1}^{m} \mathbb{E}\{|\mathbb{E}_{\leq \alpha-1}\{\xi\} - \mathbb{E}_{\leq \alpha}\{\xi\}|^2\}. \tag{3.7}
\]

Denote also

\[
M^\alpha_n = M_n|_{\tau_\alpha=0}, \quad \gamma^\alpha_n(z) = \text{Tr} G^\alpha(z), \quad \xi^\alpha = \xi - \mathbb{E}\{\xi\},
\]

where \( G^\alpha(z) \) is defined in (3.3). Applying (3.7) to \( \xi = \gamma_n \) (see (3.2)) and using the Schwarz inequality, we get

\[
\text{Var} \{ \gamma_n \} \leq \sum_{\alpha=1}^{m} \mathbb{E}\{|\gamma^\alpha_n(z)|^2\}. \tag{3.9}
\]

Furthermore, since \( M_n - M^\alpha_n = \tau_\alpha L_\alpha \) is the rank one matrix (see 1.11 and 3.8), we can write the formula

\[
G - G^\alpha = -\frac{\tau_\alpha G^\alpha L_\alpha G^\alpha}{1 + \tau_\alpha (G^\alpha y_\alpha, y_\alpha)}, \tag{3.10}
\]

implying for \( \gamma_n \) and \( \gamma^\alpha_n \) of (3.5) and (3.8)

\[
\gamma_n - \gamma^\alpha_n = -\frac{\tau_\alpha ((G^\alpha)^2 y_\alpha, y_\alpha)}{1 + \tau_\alpha (G^\alpha y_\alpha, y_\alpha)} = -\frac{B_{\alpha n}}{A_{\alpha n}}, \tag{3.11}
\]
where
\begin{align}
A_{\alpha n} &= 1 + \tau_\alpha (G^\alpha y_\alpha, y_\alpha), \\
B_{\alpha n} &= \frac{d}{dz}A_{\alpha n} = \tau_\alpha ((G^\alpha)^2 y_\alpha, y_\alpha).
\end{align}

It follows from the spectral theorem for real symmetric matrices that there exists a non-negative measure \(m^\alpha\) such that
\begin{equation}
(G^\alpha y_\alpha, y_\alpha) = \int_0^\infty \frac{m^\alpha(d\lambda)}{\lambda - z}.
\end{equation}

This and (3.12) – (3.13) yield
\begin{equation}
|A_\alpha| \geq |3A_\alpha| = \tau_\alpha|3(G^\alpha y_\alpha, y_\alpha)| = \tau_\alpha|3z| \int_0^\infty \frac{m^\alpha(d\lambda)}{|\lambda - z|^2},
\end{equation}
and
\begin{equation}
|B_\alpha| \leq |\tau_\alpha| \int_0^\infty \frac{m^\alpha(d\lambda)}{|\lambda - z|^2},
\end{equation}
implying
\begin{equation}
|B_{\alpha n}/A_{\alpha n}| \leq 1/|3z|.
\end{equation}

This and the identity
\begin{equation}
\frac{1}{A} = \frac{1}{E\{A\}} - \frac{A^\circ}{A E\{A\}}, \quad A^\circ = A - E\{A\},
\end{equation}
allows us to write
\begin{equation}
E\{(\gamma_n)^{\circ}\} = E\{(\gamma_n - \gamma_n^{\circ} - E\{\gamma_n - \gamma_n^{\circ}\})^2\}
\leq E\left\{\left|\frac{B_{\alpha n}}{A_{\alpha n}} - \frac{E\{B_{\alpha n}\}}{E\{A_{\alpha n}\}}\right|^2\right\} = E\left\{\left|\frac{(B_{\alpha n})^{\circ}_{\alpha n}}{E\{A_{\alpha n}\}} - \frac{B_{\alpha n}}{A_{\alpha n}} - \frac{(A_{\alpha n})^{\circ}_{\alpha n}}{E\{A_{\alpha n}\}}\right|^2\right\}
\leq 2E\left\{\left|\frac{(B_{\alpha n})^{\circ}_{\alpha n}}{E\{A_{\alpha n}\}}\right|^2\right\} + \frac{2}{|3z|^2}E\left\{\left|\frac{(A_{\alpha n})^{\circ}_{\alpha n}}{E\{A_{\alpha n}\}}\right|^2\right\}.
\end{equation}

Let us estimate the second term on the r.h.s. of (3.19). Since \((A_{\alpha n})^{\circ}_{\alpha n} = \tau_\alpha(G^\alpha y_\alpha, y_\alpha)^{\circ}_{\alpha n}\), then in view of definitions of (1.20)
\begin{equation}
\frac{1}{\tau_2^2}E\{(A_{\alpha n})^{\circ}_{\alpha n}\} = (a_{2,2} - \frac{1}{n^2})|\text{Tr} G^\alpha|^2 + 2a_{2,2} |\text{Tr} G^\alpha|^2 + \kappa_4 \sum_{j=1}^n |G_{jj}^\alpha|^2.
\end{equation}

This, (1.20) and the bound \(\|G^\alpha\| \leq |3z|^{-1}\) yield
\begin{align}
E\{(A_{\alpha n})^{\circ}_{\alpha n}\} &\leq \frac{C\tau_2^2}{n^2} |\text{Tr} G^\alpha|^2, \\
E\{(B_{\alpha n})^{\circ}_{\alpha n}\} &\leq \frac{C\tau_2^2}{n^2} |\text{Tr} G^\alpha|^4 \leq \frac{C\tau_2^2}{n^2|3z|^2} |\text{Tr} G^\alpha|^2.
\end{align}

It follows then from (3.19), (3.21), and (3.21) – (3.22) that
\begin{equation}
\text{Var}\{\gamma_n(z)\} \leq \frac{C}{n|3z|^2} \sum_{\alpha=1}^m \tau_\alpha^2 E\left\{\frac{n^{-1} |\text{Tr} G^\alpha(z)|^2}{|E\{A_{\alpha n}\}|^2}\right\}.
\end{equation}
Let $N_n^\alpha$ be the normalized counting measure of $M_n^\alpha$. Then we have in view of (3.8), (cf. (3.11))

$$\gamma_n^\alpha(z) = \int_0^\infty \frac{N_n^\alpha(d\lambda)}{\lambda - z}.$$  

(3.24)

In addition, (1.1) and (3.8) imply

$$E_\alpha \{A_{an} \} = 1 + \tau_\alpha n^{\frac{1}{3}} \gamma_n^\alpha.$$  

(3.25)

This and an argument similar to that leading to (3.17) yield

$$n^{-1} \text{Tr} |G^\alpha(z)|^2 \leq \frac{1}{\tau_\alpha |3z|}.$$  

(3.26)

It also follows from (3.10) that

$$A_{an}^{-1} = 1 + \tau_\alpha (Gy_\alpha \cdot y_\alpha)$$  

implying, together with (1.11) and the Jensen inequality

$$|E_\alpha \{A_{an} \}|^{-1} \leq E_\alpha \{|A_{an}^{-1} \} \leq 1 + \tau_\alpha |3z|^{-1}. $$  

(3.27)

Now (3.28), (3.29) and (1.21) yield (3.3).

To prove (3.6) we will use an analog of (3.7) for the 4th moment of $\gamma_n^\alpha = \gamma_n - E \{\gamma_n \}$ (see e.g. (11) and (29), Section 18.1.2), which together with an argument analogous to that leading to (3.19), yields

$$E \{|\gamma_n^\alpha|^{4} \} \leq C m \sum_{\alpha=1}^{m} E \{|(B_{an})_\alpha^\circ|^{4}\} \leq 1 \sum_{\alpha=1}^{m} E \{|(B_{an})_\alpha^\circ|^{4}\} + 1 \sum_{\alpha=1}^{m} E \{|(A_{an})_\alpha^\circ|^{4}\} \leq C.$$

(3.29)

where (see (1.26))

$$E_\alpha \{|(A_{an})_\alpha^\circ|^{4} \} \leq \tau_n^4 / |3z|^4 n^2, \quad E_\alpha \{|(B_{an})_\alpha^\circ|^{4} \} \leq \tau_n^4 / |3z|^8 n^2.$$  

(3.30)

Since the matrix $M_n$ with non-negative $\tau_\alpha, \quad \alpha = 1, \ldots, m$ is positive definite, it follows from (3.11) that

$$\Im(z(G^\alpha \cdot y_\alpha, y_\alpha)) = \Im \int_0^\infty \frac{\lambda m^\alpha(d\lambda)}{\lambda - z} = \Im \int_0^\infty \frac{\lambda m^\alpha(d\lambda)}{|\lambda - z|^2}.$$  

This yields the inequality $\Im \cdot \Im(z(G^\alpha \cdot y_\alpha, y_\alpha)) \geq 0$, so that

$$|A_{an}|^{-1} \leq \frac{z}{|3z + \tau_\alpha \Im(z(G^\alpha \cdot y_\alpha, y_\alpha))|} \leq |z||3z|^{-1},$$  

(3.31)

and by the Jensen inequality

$$|E_\alpha \{A_{an} \}|^{-1} \leq |z||3z|^{-1}.$$  

(3.32)

Now (3.6) follows from (3.29) and (3.30) and (3.32). □
Proof of Lemma 1.5. It follows from (3.1) and (3.4) that

\[
\text{Var}\{N_n[\varphi]\} \leq C_s||\varphi||^2 \int_0^\infty d\eta e^{-\eta \eta^2} \frac{1}{n} \sum_{n=1}^m (\tau_\alpha(\eta + \tau_\alpha))^{3/2 + \varepsilon} 
\]

(3.33)

\[
\times \int_{-\infty}^{\infty} E\{n^{-1} \text{Tr} |G^\alpha(z)|^{1+2\varepsilon}\} d\lambda,
\]

and we have for \( z = \lambda + i\eta \) (cf. (3.24))

\[
\int_{-\infty}^{\infty} E\{n^{-1} \text{Tr} |G^\alpha(z)|^{1+2\varepsilon}\} d\lambda = \int_{-\infty}^{\infty} E\{N_n^n(d\lambda)\} \int_{-\infty}^{\infty} \frac{d\mu}{((\lambda - \mu)^2 + \eta^2)^{1/2 + \varepsilon}} \leq \frac{C}{\eta^{\varepsilon}}.
\]

Thus, for any \( s = 2 + \delta > \varepsilon \), the integral over \( \eta \) in (3.33) converges. Lemma 1.5 is proved.

4. Proof of Theorem 1.8

It suffices to show that if

(4.1)

\[ Z_n(x) = E\{e_n(x)\}, \quad e_n(x) = e^{ixN_n^n[\varphi]}, \]

then we have uniformly in \( |x| \leq C \)

(4.2)

\[ \lim_{n \to \infty} Z_n(x) = \exp\{-x^2V[\varphi]/2\} \]

with \( V[\varphi] \) of (1.27). Define for any test-functions \( \varphi \in H_{2+\delta} \)

(4.3)

\[ \varphi_y = P_y \ast \varphi, \]

where \( P_y \) is the Poisson kernel

(4.4)

\[ P_y(x) = \frac{y}{\pi(x^2 + y^2)}, \]

and "\( \ast \)" denotes the convolution. We have

(4.5)

\[ \lim_{y \downarrow 0} ||\varphi - \varphi_y||_{2+\delta} = 0. \]

Denote for the moment the characteristic function (4.1) by \( Z_n[\varphi] \), to make explicit its dependence on the test function. We have then for any converging subsequence \( \{Z_{n_j}[\varphi]\}_j \)

\[ \lim_{n_j \to \infty} Z_{n_j}[\varphi] = \lim_{y \downarrow 0} \lim_{n_j \to \infty} (Z_{n_j}[\varphi] - Z_{n_j}[\varphi_y]) + \lim_{y \downarrow 0} \lim_{n_j \to \infty} Z_{n_j}[\varphi_y]. \]

Since by (1.22) and (4.5)

\[ |Z_{n_j}[\varphi] - Z_{n_j}[\varphi_y]| \leq |x|\text{Var}\{N_{n_j}[\varphi] - N_{n_j}[\varphi_y]\}^{1/2} \leq C|x||\varphi - \varphi_y||_{2+\delta} \to 0, \]

as \( y \downarrow 0 \), then

(4.6)

\[ \lim_{n_j \to \infty} Z_{n_j}[\varphi] = \lim_{y \downarrow 0} \lim_{n_j \to \infty} Z_{n_j}[\varphi_y]. \]

Hence it suffices to find the limit of \( Z_{yn} := Z_n[\varphi_y] = E\{e_{y,n}(x)\} \) with \( e_{y,n}(x) = e^{ixN_n^n[\varphi_y]}, \) as \( n \to \infty \).

It follows from (4.3) – (4.4) that

(4.7)

\[ N_n[\varphi_y] = \frac{1}{\pi} \int \varphi(\mu)3\gamma_n(z)d\mu, \quad z = \mu + iy. \]
This allows us to write
\[ \frac{d}{dx} Z_{yn}(x) = \frac{1}{2\pi} \int \varphi(\mu)(Y_n(z, x) - Y_n(\overline{z}, x))d\mu, \]
where \( Y_n(z, x) = \mathbb{E}\{\gamma_n(z)e_{yn}(x)\} \). Now the first bound in (3.3) yields
\[ |Y_n(z, x)| \leq 2\mathbb{V}ar\{\gamma_n(z)\}^{1/2} \leq C|z|^{-6}. \]
This and the dominated convergence theorem imply that if \( \varphi \in L^1 \), then the limit of the integral in (4.8) as \( n \to \infty \) can be obtained from that of \( Y_n \) for any fixed non-real \( z \).

We have from the resolvent identity and (3.2)
\[ \gamma_n(z) = -\frac{n}{z} + \frac{1}{z} \text{Tr} M_n G(z) = \frac{-n + m}{z} - \frac{1}{z} \sum_{\alpha=1}^m A_{\alpha n}^{-1}(z), \]
where \( A_{\alpha n} \) is defined in (3.12). This implies
\[ Y_n(z, x) = -\frac{1}{z} \sum_{\alpha=1}^m \mathbb{E}\{A_{\alpha n}^{-1}(z)(e_{\alpha n}^\circ(x))^\circ\} - \frac{1}{z} \sum_{\alpha=1}^m \mathbb{E}\{A_{\alpha n}^{-1}(z)(e_n(x) - e_{\alpha n})^\circ\} \]
\[ =: T_1^{(n)} + T_2^{(n)}. \]

Iterating (3.13) three times, we get for \( T_1^{(n)} \) of (4.9):
\[ T_1^{(n)} = \frac{1}{z} \sum_{\alpha=1}^m \mathbb{E}\{A_{\alpha n}^{-1}(z)(e_{\alpha n}^\circ(x))^\circ\} - \frac{1}{z} \sum_{\alpha=1}^m \mathbb{E}\{A_{\alpha n}^{-1}(z)2(e_{\alpha n}^\circ(x))^\circ\} \]
\[ + \frac{1}{z} \sum_{\alpha=1}^m \frac{\mathbb{E}\{(A_{\alpha n}^{-1}(z))^3A_{\alpha n}^{-1}(z)(e_{\alpha n}^\circ(x))^\circ\}}{\mathbb{E}\{A_{\alpha n}^{-1}(z)\}^3} \]
\[ =: T_{11}^{(n)} - T_{12}^{(n)} + T_{13}^{(n)}. \]
It follows from (5.3) and (3.31) - (3.32) that
\[ T_{13}^{(n)} = O(n^{-1/2}), \quad n \to \infty. \]
Consider now \( T_{11}^{(n)} \). Since \( e_{\alpha n}^\circ \) does not depend on \( \gamma_n \), (5.21) implies
\[ \mathbb{E}\{A_{\alpha n}(z)(e_{\alpha n}^\circ(x))^\circ\} = \mathbb{E}\{\mathbb{E}_\alpha\{A_{\alpha n}(z)\}(e_{\alpha n}^\circ(x))^\circ\} \]
\[ =: \tau_{\alpha n} \mathbb{E}\{\gamma_n(z)(e_{yn}^\circ(x))^\circ\} = \tau_{\alpha n}^{-1}Y_n(z, x) + R_n, \]
where \( R_n = \tau_{\alpha n}^{-1}\mathbb{E}\{\gamma_n(z)(e_{yn}^\circ(x))^\circ\} \). Applying consequently (4.14), the Schwarz inequality and then (3.3), (5.2) and (4.3), we get
\[ \tau_{\alpha n}^{-1}n|R_n| \leq 2\mathbb{E}\{|(\gamma_n - \gamma_{\alpha n})^\circ(z)|\} + \frac{|x|}{2\pi} \int |\varphi(\lambda_2)|\mathbb{E}\{|\gamma_n(z)|(|\gamma_n - \gamma_{\alpha n})^\circ(z_2)|\}d\lambda_2 \]
\[ \leq C_2\tau_{\alpha n}^{-1/2}. \]
Here and below we denote by \( C_2 \) any positive quantity depending only on \( |3z| \) and \( |z| \). It follows then from (4.12) - (4.13) that
\[ \mathbb{E}\{A_{\alpha n}(z)(e_{\alpha n}^\circ(x))^\circ\} = \tau_{\alpha n}^{-1}Y_n(z, x) + O(n^{-3/2}). \]
Plugging this and (5.21) in \( T_{11}^{(n)} \) of (4.10), we get
\[ T_{11}^{(n)} = Y_n(z, x)\frac{1}{nz} \sum_{\alpha=1}^m \frac{\tau_{\alpha}}{(1 + \tau_{\alpha}f(z))^2} + o(1), \quad n \to \infty, \]
and by (1.4) and (1.5)

\[ T_{11}^{(n)} = Y_n(z, x) \int_{\mathbb{R}} \frac{\tau d\sigma(\tau)}{(1 + \tau f(z))^2} + o(1), \quad n \to \infty. \] 

Next, it follows from (5.7) – (5.8), the Schwarz inequality and (5.5) that

\[ |E\{(A_{an}^o)^2(e_{yn}^o)^o\}| = |E\{E_{\alpha}\{(A_{an}^o)^2\}(e_{yn}^o)^o\} + \tau_{an}^2 n^{-2}E\{((\gamma_{an}^o)^2(e_{yn}^o)^o)\}| = o(n^{-1}), \]

hence,

\[ T_{12}^{(n)} = o(1), \quad n \to \infty. \]

Now (4.11), (4.14) and (4.16) yield for (4.10):

\[ T_{1}^{(n)} = Y_n(z, x) \int_{\mathbb{R}} \frac{\tau d\sigma(\tau)}{(1 + \tau f(z))^2} + o(1), \quad n \to \infty. \]

Consider (4.11) of (3.9). Since by (4.7)

\[ e_{yn} - e_{yn}^o = \frac{ix e_{yn}}{\pi} \int \varphi(\lambda_1) \Im(\gamma_n - \gamma_n^o)^o(z_1)d\lambda_1 \]

\[ + O(\left| \int \varphi(\lambda) \Im(\gamma_n - \gamma_n^o)^o(z_1)d\lambda_1 \right|), \]

then in view of (3.11)

\[ E\{A_{an}^{-1}(z)(e_{yn} - e_{yn}^o)^o(x)\} = \frac{ix}{\pi} \int \varphi(\lambda_1) E\{e_{yn}(x)(A_{an}^{-1})^o(z)\Im(B_{an}A_{an}^{-1}^o(z_1))d\lambda_1 \]

\[ + \int \int O(R_1^{(1)})\varphi(\lambda_1)\varphi(\lambda_2)d\lambda_1d\lambda_2, \]

where

\[ R_1^{(1)} = E\{(A_{an}^{-1})^o(z)\Im(B_{an}A_{an}^{-1}^o(z_1))\Im(B_{an}A_{an}^{-1}^o(z_2))\}. \]

Applying (3.18) to $A_{an}^{-1}(z)$, $A_{an}^{-1}(z_1)$, $A_{an}^{-1}(z_2)$, we get

\[ E\{(A_{an}^{-1})^o(z)(B_{an}A_{an}^{-1}^o(z_1))\Im(B_{an}A_{an}^{-1}^o(z_2))\}
\]

\[ = \frac{E\{(A_{an}^oA_{an}^{-1})^o(z)(B_{an} - A_{an}^oA_{an}^{-1}B_{an})^o(z_1)(B_{an} - A_{an}^oA_{an}^{-1}B_{an})^o(z_2))\}}{E\{A_{an}(z)\}E\{A_{an}(z_1)\}E\{A_{an}(z_2)\}}. \]

This and (5.3) yield

\[ R_1^{(1)} = O(n^{-1/2}), \quad n \to \infty. \]

Similarly, iterating twice (3.18), applying (5.3) and (5.5) and taking into account that only linear terms in $A_{an}^o$ and $B_{an}^o$ give non-vanishing contribution, we get

\[ E\{e_{yn}(x)(A_{an}^{-1})^o(z)(B_{an}A_{an}^{-1}^o(z_1))\}
\]

\[ = - \frac{E\{e_{yn}(x)A_{an}^o(z)B_{an}^o(z_1)\}}{E\{A_{an}(z)\}^2E\{A_{an}(z_1)\}^2} + \frac{E\{e_{yn}(x)A_{an}^o(z)A_{an}^o(z_1)B_{an}(z_1)\}}{E\{A_{an}(z)\}^2E\{A_{an}(z_1)\}^2} + O(n^{-3/2}) \]

\[ = E\{e_{yn}(x)\} \left[ - \frac{E\{A_{an}^o(z)B_{an}^o(z_1)\}}{E\{A_{an}(z)\}^2E\{A_{an}(z_1)\}^2} + \frac{E\{A_{an}^o(z)A_{an}^o(z_1)\}E\{B_{an}(z_1)\}}{E\{A_{an}(z)\}^2E\{A_{an}(z_1)\}^2} \right] + O(n^{-3/2}). \]
This, the bound $|Z_{yn} - E\{e_{yn}^n\}| = O(n^{-1/2})$ (cf. (4.13)), and (3.13) imply

$$
E\{e_{yn}^n(z)(A_{\alpha n}^{-1})^\circ(z)(B_{\alpha n}A_{\alpha n}^{-1})^\circ(z_1)\}
$$

$$
= -Z_{yn}(x) \frac{1}{E\{A_{\alpha n}(z)A_{\alpha n}(z_1)\}^2} \frac{\partial}{\partial z_1} E\{A_{\alpha n}(z)A_{\alpha n}(z_1)\} + O(n^{-3/2})
$$

$$
= -Z_{yn}(x) \frac{\tau_2^2 D(z, z_1)}{(1 + \tau_2 f(z))} + O(n^{-3/2}),
$$

where

$$
D_{\tau_2}(z, z_1) = \frac{\partial}{\partial z_1} \frac{2\Delta f/\Delta z + (a + b)f(z_1)}{1 + \tau_2 f(z)}
$$

and we used (5.4) and (5.6). Plugging (4.19) – (4.21) in (4.18) and applying (1.4) – (1.5), we get for $T_2(n)$ of (4.9):

$$
T_2(n) = -xZ_{yn}(x) \phi(\lambda_1) c \int_0^\infty \int_0^\infty \frac{\tau^2 d\sigma(\tau)}{(1 + \tau f(z))} [D_r(z, z_1) - D_r(z, \overline{z_1})] + o(1).
$$

This and (3.9) – (3.10) yield

$$
Y_n(z, x) = \left( c \int_0^\infty \frac{\tau d\sigma(\tau)}{(1 + \tau f(z))} - z \right)^{-1}
$$

$$
\times \frac{xZ_{yn}(x)}{2\pi} \int d\lambda_1 \phi(\lambda_1) c \int_0^\infty \int_0^\infty \frac{\tau^2 d\sigma(\tau)}{(1 + \tau f(z))} [D_r(z, z_1) - D_r(z, \overline{z_1})] + o(1),
$$

and after some calculations based on (1.7) we finally get

$$
Y_n(z, x) = \frac{xZ_{yn}(x)}{2\pi} \int d\lambda_1 \phi(\lambda_1) [C(z, z_1) - C(z, \overline{z_1})] + o(1),
$$

where $C(z, z_1)$ is defined in (1.28). Now it follows from (4.8) and (4.22) that

$$
\frac{d}{dx} Z_{yn}(x) = -x V_y[\varphi] Z_{yn}(x) + o(1),
$$

where $V_y[\varphi]$ is given in (1.27). If we consider $\overline{Z}_{yn}(x) = e^{x^2 V_y[\varphi]} Z_{yn}(x)$, then (4.23) yields

$$
d\overline{Z}_{yn}(x)/dx = o(1), \quad n \to \infty
$$

for any $|x| \leq C$, and since $\overline{Z}_{yn}(0) = Z_{yn}(0) = 1$, we obtain $\overline{Z}_{yn}(x) = 1 + o(1)$ uniformly in $|x| \leq C$. Hence,

$$
\lim_{n \to \infty} Z_{yn}(x) = \exp\{-x^2 V_y[\varphi]\}.
$$

Now we take into account (4.6), allowing us to pass to the limit $y \downarrow 0$, and obtain (1.27). The theorem is proved.
5. Auxiliary results

**Lemma 5.1.** Under conditions of Theorem 1.8 we have:

\[(5.1)\] \(\lim_{n \to \infty} E\{n^{-1} \gamma_n^0 \} = f, \quad |\gamma_n - \gamma_n^0| = O(1), \quad n \to \infty,\)

\[(5.2)\] \(\text{Var}\{\gamma_n - \gamma_n^0\} = O(n^{-1}), \quad n \to \infty,\)

\[(5.3)\] \(E\{|A_{\alpha n}^0|^p\} \leq \frac{C r_n^p}{n^{p/2}|\zeta|^p}, \quad E\{|B_{\alpha n}^0|^p\} \leq \frac{C r_n^p}{n^{p/2}|\zeta|^p}, \quad p = 1, 2, 3, 4,\)

\[(5.4)\] \(E\{A_{\alpha n}\}^{-1} = (1 + \tau_{\alpha} f)^{-1} + o(1), \quad n \to \infty,\)

\[(5.5)\] \(\text{Var}\{n E_\alpha\{A_{\alpha n}(z_1) A_{\alpha n}(z_2)\}\} = o(1), \quad n \to \infty,\)

\[(5.6)\] \(\lim_{n \to \infty} \frac{\tau_{\alpha}^2}{\tau} E\{A_{\alpha n}^0(z_1) A_{\alpha n}(z_2)\} = (a + b) f(z_1) f(z_2) + 2 \Delta f / \Delta z,\)

where \(\gamma_n^0, A_{\alpha n}, B_{\alpha n}\) are defined in (3.8) and (3.12) – (3.13), and \(f\) is a unique solution of (1.8).

**Proof.** (i) It follows from (3.14) and (3.17) that \(|\gamma_n - \gamma_n^0| \leq |\zeta|^{-1}\). On the other hand, Theorem 1.3 implies \(\lim_{n \to \infty} E\{n^{-1} \gamma_n\} = f\). This leads to (5.1).

(ii) Consider \(V = \text{Var}\{\Delta \gamma_n\}, \Delta \gamma_n = \gamma_n - \gamma_n^0\). By (3.14) and (3.18) we have

\[
V = E\{\Delta \gamma_n \Delta \gamma_n^0\} = -E\{(B_{\alpha n}/A_{\alpha n}) \Delta \gamma_n^0\}
\]

\[
= -E\{B_{\alpha n} \Delta \gamma_n^0\}/E\{A_{\alpha n}\} - E\{A_{\alpha n}(B_{\alpha n}/A_{\alpha n}) \Delta \gamma_n^0\}/E\{A_{\alpha n}\},
\]

and by (3.17), (3.28), the Schwarz inequality, and (3.21) – (3.22)

\[
V \leq CV^{1/2}(\text{Var}\{B_{\alpha n}\}^{1/2} + \text{Var}\{A_{\alpha n}\}^{1/2} / |\zeta|) \leq CV^{1/2} n^{-1/2}.
\]

This yields (5.2).

(iii) Note that

\[(5.7)\] \(A_{\alpha n}^0 = (A_{\alpha n})^0 + \tau_{\alpha} n^{-1} (\gamma_n^0)^0,\)

and that (3.3) and (3.6) imply

\[(5.8)\] \(\text{Var}\{\gamma_n^0\}, \quad E\{|(\gamma_n^0)^0|^4\} = O(1), \quad n \to \infty,\)

This allows us to replace \(A_{\alpha n}^0\) by \((A_{\alpha n})^0\) as \(n \to \infty\) (see e.g. (4.16)). In view of (3.21) and (5.7) – (5.8) we have

\[(5.9)\] \(E\{|A_{\alpha n}^0|^2\} = E\{|(A_{\alpha n})^0|^2\} + \tau_{\alpha}^2 n^{-2} \text{Var}\{\gamma_n^0\} = O(n^{-1}).\)

This yields (5.3) for \(p = 2\). Similarly, (5.3) for \(p = 3, 4\) follows from (1.26) and (5.7) – (5.8).

(iv) We have \(E\{A_{\alpha n}\}^{-1} = (1 + \tau_{\alpha} f)^{-1} + r_n\), where

\[
r_n = \tau_{\alpha} (f - E\{n^{-1} \gamma_n^0\}) E\{A_{\alpha n}\}^{-1} (1 + \tau_{\alpha} f)^{-1}.
\]

The bound \(|(1 + \tau_{\alpha} f)^{-1}| \leq |z / \zeta z|\), (3.32) and (5.1) imply \(r_n = o(1)\), hence (5.3).
(v) It follows from (2.8) – (2.9) and (1.24) – (1.25) that

\[(5.10)\]

\[n\tau_\alpha^{-2}E_\alpha\{(A_{an})_\alpha^o(z_1)A_{an}(z_2)\} = n\left[(a_{2,2} - n^{-2}) \text{Tr} G^\alpha(z_1) \text{Tr} G^\alpha(z_2) \right.\]
\[+ 2a_{2,2} \text{Tr} G^\alpha(z_1)G^\alpha(z_2) + \kappa_n \sum_{j=1}^{n} G_{jj}^\alpha(z_1)G_{jj}^\alpha(z_2)\]
\[= an^{-2}\gamma_n^\alpha(z_1)\gamma_n^\alpha(z_2) + 2n^{-1}\gamma_n^\alpha(z_1) - \gamma_n^\alpha(z_2) + bg_n^\alpha(z_1, z_2) + O(n^{-1}),\]

where \(g_n^\alpha\) is defined in (5.19) (see Lemma 5.2). Now \((5.8)\) follows from \((5.8)\) – \((5.9)\) and \((5.13)\).

(vi) The relation \((5.9)\) follows from \((5.1)\), \((5.10)\) and \((5.12)\). \(\Box\)

**Lemma 5.2.** Consider

\[(5.11)\]

\[g_n(z_1, z_2) = n^{-1} \sum_{j=1}^{n} G_{jj}(z_1)G_{jj}(z_2).\]

Then under conditions of Theorem 1.8 we have as \(n \to \infty\)

\[(5.12)\]

\[E\{g_n(z_1, z_2)\} = f(z_1)f(z_2) + o(1),\]
\[(5.13)\]

\[\text{Var}\{g_n(z_1, z_2)\} = o(1),\]

where \(f\) is given by \((1.8)\).

**Proof.** Since by the resolvent identity

\[G_{jj}(z_1) = -\frac{1}{z_1} + \frac{1}{z_1}(MG)_{jj}(z_1) = -\frac{1}{z_1} + \frac{1}{z_1} \sum_{\alpha=1}^{m} \tau_\alpha y_\alpha(Gy_\alpha)_{jj}(z_1),\]

then

\[(5.14)\]

\[E\{g_n(z_1, z_2)\} = -E\{n^{-1}\gamma_n(z_2)\}\frac{1}{z_1} + \frac{1}{z_1} \sum_{\alpha=1}^{m} \tau_\alpha E\{y_\alpha(Gy_\alpha)_{jj}(z_1)G_{jj}(z_2)\}\frac{1}{z_1} + \frac{1}{z_1} \sum_{\alpha=1}^{m} \tau_\alpha E\{y_\alpha(Gy_\alpha)_{jj}(z_1)G_{jj}(z_2)\}\frac{1}{z_1}.

It follows from \((3.10)\) that \((Gy_\alpha)_{jj} = A_{an}^{-1}(G^\alpha y_\alpha)_{jj}, G_{jj} = G_{jj}^\alpha - \tau_\alpha A_{an}^{-1}(G^\alpha y_\alpha)_{jj}^2\), where \(A_{an}\) is defined in \((3.12)\). Applying \((3.10)\), we obtain

\[(5.15)\]

\[E\{y_\alpha(Gy_\alpha)_{jj}(z_1)G_{jj}(z_2)\} = E\{y_\alpha(A_{an}^{-1}(G^\alpha y_\alpha)_{jj}(z_1)G_{jj}^\alpha(z_2))\}
\[+ \tau_\alpha E\{y_\alpha(A_{an}^{-1}(G^\alpha y_\alpha)_{jj}(z_1)(A_{an}^{-1}(G^\alpha y_\alpha)_{jj}^2(z_2))\}
\[= E\{A_{an}(z_1)\}^{-1}E\{y_\alpha(Gy_\alpha)_{jj}(z_1)G_{jj}^\alpha(z_2)\} + r_n,\]

where

\[(5.16)\]

\[r_n = E\{y_\alpha(A_{an}^{-1}A_{an}^{-1}(G^\alpha y_\alpha)_{jj}(z_1)G_{jj}^\alpha(z_2))\}
\[E\{A_{an}(z_1)\}^{-1}E\{y_\alpha((1 + A_{an}^{-1}A_{an}^{-1}(G^\alpha y_\alpha)_{jj}(z_1)(1 + A_{an}^{-1}A_{an}^{-1})^{-1}((G^\alpha y_\alpha)_{jj}(z_2))^2))\}
\[E\{A_{an}(z_1)\}E\{A_{an}(z_2)\}^{-1}.\]
Since \(|(G^a y_\alpha)_j| \leq |3z|^{-2}||y_\alpha||^2\), then by the Schwarz inequality, (5.3), and (1.24) the first term in the r.h.s. of (5.16) is less than
\[
\text{CVar}\{A_{\alpha n}\}^{1/2} \mathbb{E}\{\gamma_{\alpha j}^2 |(G^a y_\alpha)_j|^2\} = O(n^{-3/2}).
\]
By the same reason, all the terms in the numerator of the second term on the r.h.s. of (5.16), which contain \(A_{\alpha n}\) are of the order \(O(n^{-3/2})\). For the only term that does not contain \(A_{\alpha n}\) we have
\[
\mathbb{E}\{y_{\alpha j} (G^a y_\alpha)_j (z_1) (G^a y_\alpha)_j^2 (z_2)\} = \sum_{p,q,s=1}^n G_{jp}^\alpha (z_1) G_{jq}^\alpha (z_2) G_{js}^\alpha (z_2) \mathbb{E}\{y_{\alpha p} y_{\alpha q} y_{\alpha s}\}
\]
\[
= 3 \alpha_{2,2} G_{jj}^\alpha (G^a G^a)_{jj} + \kappa_4 G_{jj}^\alpha G_{jj}^\alpha = O(n^{-2}),
\]
where we used (2.8) and (1.20). Hence
\[
(5.17)
\]
\[
r_n = O(n^{-3/2}), \quad n \to \infty.
\]
Besides, it follows from (1.11) that
\[
(5.18)
\]
Plugging (5.17) – (5.18) in (5.15) and then in (5.14), we get
\[
E\{g_n(z_1, z_2)\} = \frac{1}{z_1} f(z_2) + \frac{1}{n z_1} \sum_{\alpha,j=1}^{m,n} \tau_\alpha \frac{E\{g_\alpha^a(z_1, z_2)\}}{1 + \tau_\alpha f(z_1)} + o(1), \quad n \to \infty,
\]
where
\[
(5.19)
\]
\[
g_\alpha^a(z_1, z_2) = n^{-1} \sum_{j=1}^n G_{jj}^\alpha (z_1) G_{jj}^\alpha (z_2),
\]
and we took into account (5.1) - (5.4). Using the argument similar to that leading to (5.17), we obtain \(E\{g_n - g_\alpha^a\} = o(1), n \to \infty\). This and (1.14) show that
\[
\lim_{n \to \infty} E\{g_n(z_1, z_2)\} = \left( c \int \frac{\tau d\sigma(\tau)}{1 + \tau f(z_1)} - z_1 \right)^{-1} f(z_2) = f(z_1) f(z_2),
\]
and we get (5.12). The proof of (5.13) follows the scheme of proof in Lemma 3.2. □

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