Casimir energy density in closed hyperbolic universes

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Abstract
The original Casimir effect results from the difference in the vacuum energies of the electromagnetic field, between that in a region of space with boundary conditions and that in the same region without boundary conditions. In this paper we develop the theory of a similar situation, involving a scalar field in spacetimes with negative spatial curvature.

1 INTRODUCTION

In a previous work [1] the Casimir energy density was obtained for a Robertson-Walker (RW) cosmological model with constant, negative spatial curvature.

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Its spatial section was Weeks manifold, which is the hyperbolic 3-manifold with the smallest volume (normalized to $K = -1$ curvature) in the SnapPea census [3].

Here we further develop and clarify the theoretical formalism of that paper.

Our sign conventions for general relativity are those of Birrell and Davies [3]: metric signature $(+ − − −)$, Riemann tensor $R_{\alpha \beta \gamma \delta} = \partial_{\delta} \Gamma_{\alpha \beta \gamma} - \ldots$, Ricci tensor $R_{\mu \nu} = R^{\alpha}_{\mu \alpha \nu}$.

2 THE ORIGINAL CASIMIR EFFECT

The original effect was calculated by Casimir [4]. Briefly, one sets two metallic, uncharged parallel plates, separated by a small distance $a$. Between them the electromagnetic field wavenumbers normal to the plates are constrained by the boundaries. So there is a difference $\delta E$ between the vacuum energy for this configuration and the vacuum energy for unbounded space. If $A$ is the area of each plate, one has (see, for example, [5], [6], [7])

$$\delta E \approx \frac{\hbar c}{2} \int \int \frac{dk_x dk_y}{(2\pi)^2} \left[ \sum_{n \in \mathbb{Z}} \sqrt{k_x^2 + k_y^2 + (\pi n/a)^2} - 2a \int \frac{dk_z}{2\pi} \sqrt{k_x^2 + k_y^2 + k_z^2} \right],$$

where we omitted damping factors needed to avoid infinities. The results is

$$\delta E(a) = -\frac{\pi^2 \hbar c}{720 a^3} A$$

for the energy difference, and

$$F(a) = -\frac{\pi^2}{240a^4} A$$

for the attractive force between the plates.

3 CASIMIR ENERGY (CE) IN COSMOLOGY WITH NONTRIVIAL TOPOLOGY

There is no boundary for a universe model with closed (i.e., compact and boundless) spatial sections. But a field in these models has periodicities,
which leads to an effect similar to the above one, that may also be called a Casimir effect.

A simple example, taken from Birrell and Davis [3], is that of a scalar field \( \phi(t, x) \) in spacetime \( \mathbb{R}^1 \times S^1 \), with one closed space direction. If \( S^1 \) has length \( L \) then

\[
\phi(t, x + L) = \phi(t, x),
\]

and the vacuum energy density is

\[
\rho = -\pi \hbar c/6L^2.
\]

An analytical expression for the CE in a class of closed hyperbolic universes (CHUs) was obtained by Goncharov and Bytsenko [8].

Here we develop a formalism succinctly described in [1], for the numerical calculation of the CE density of closed hyperbolic universes.

Our notation: \( i, j, ... = 1 - 3; \alpha, \mu = 0 - 3; \mathbf{x} = (x^i); \mathbf{x} = (x^\mu) = (t, \mathbf{x}) \).

Sign conventions are those of [3]: metric signature (+ − − −), Riemann tensor \( \mathcal{R}^{\alpha}_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha_{\beta\gamma} - ... \), Ricci tensor: \( R_{\mu\nu} = \mathcal{R}^{\alpha}_{\mu\alpha\nu} \).

4 SCALAR FIELD \( \phi(x) \) IN CURVED SPACE-TIME

The action for a scalar field in a curved spacetime of metric \( g_{\mu\nu} \) and mass \( m \) is

\[
S = \int \mathcal{L}(\mathbf{x}) d^4\mathbf{x},
\]

with

\[
\mathcal{L} = \frac{1}{2} \sqrt{-g} \left[ g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - (m^2 + \xi R) \phi^2 \right],
\]

where \( R \) is scalar curvature of spacetime, \( g = \det(g_{\mu\nu}) \), and \( \xi \) is a constant.
With $\xi = 1/6$ ("conformal" value) we get the equation for $\phi(x)$:

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow (\Box + m^2 + \frac{1}{6}R)\phi = 0,$$

where $\Box$ is the generalized d’Alembertian:

$$\Box \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = (-g)^{-1/2} \partial_\mu \left[ (-g)^{1/2} g^{\mu \nu} \partial_\nu \phi \right].$$

The energy-momentum tensor is (cf. [3])

$$T_{\mu \nu} = 2(-g)^{-1/2} \frac{\delta S}{\delta g^{\mu \nu}} = \frac{2}{3} \phi^{,\mu} \phi^{,\nu} + \frac{1}{6} g_{\mu \nu} \phi^{, \sigma} \phi^{, \sigma} - \frac{1}{3} \phi^{, \mu \nu} + \frac{1}{12} g_{\mu \nu} \phi \Box \phi - \frac{1}{6} R_{\mu \nu} \phi^2 + \frac{1}{24} g_{\mu \nu} R \phi^2 + \frac{1}{4} g_{\mu \nu} m^2 \phi^2.$$

**5 COORDINATES IN $H^3$**

The hyperbolic (or Bólyai-Lobachevsky) space $H^3$ is isometric to the hypersurface

$$(x^4)^2 - x^2 = 1, \quad x^4 \geq 1,$$

imbedded in an abstract Minkowski space $(\mathbb{R}^4, diag(1, 1, 1, -1))$.

This upper branch of a hyperboloid is similar to the mass shell of particle physics,

$$E^2 - p^2 = m^2, \quad E \geq m.$$

Hence a point in $H^3$ may be represented by the Minkowski coordinates $x^b$, $b = 1 - 4$, subject to constraints (1), and rigid motions in $H^3$ are proper, orthochronous Lorentz transformations.

We relate the spherical coordinates $(\chi, \theta, \varphi)$ to the displaced Minkowski ones $x^b - x'^b$, $b = 1 - 4$:

$$x^1 - x'^1 = \sinh \chi \sin \theta \cos \varphi,$$
$$x^2 - x'^2 = \sinh \chi \sin \theta \sin \varphi,$$
$$x^3 - x'^3 = \sinh \chi \cos \theta,$$
$$x^4 - x'^4 = \cosh \chi.$$

Note that $\chi(x, x') = \sinh^{-1} |x - x'|.$
6 STATIC MODELS OF NEGATIVE SPATIAL CURVATURE

The Robertson-Walker metric for spatial curvature $K = -1/a^2$ is

$$ds^2 = dt^2 - a^2(d\chi^2 + \sinh^2 \chi d\Omega^2)$$

$$= dt^2 - a^2\left(\delta_{ij} - \frac{x^i x^j}{1 + x^2}\right)dx^i dx^j,$$

where in general $a = a(t)$.

Einstein’s equations give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^2} + \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$

$$\frac{3\ddot{a}}{a} = -4\pi G(\rho + 3P) + \Lambda.$$ 

Assuming $\dot{a} = \ddot{a} = 0$ and $P = \rho/3$ we get $a^2 = -3/2\Lambda$, hence $\Lambda < 0$, and

$$a = \sqrt{3/2|\Lambda|},$$

$$\rho = \Lambda/8\pi G < 0.$$ 

We will comment below on this negative energy density. These models are stable (!) under curvature fluctuations:

$$a \rightarrow a + \varepsilon(t) \implies \ddot{\varepsilon} + |\Lambda|\varepsilon = 0.$$ 

7 CLOSED HYPERBOLIC 3-MANIFOLDS (CHMs)

A CHM is obtained by a pairwise identification of the $n$ faces of a fundamental polyhedron (FP), or Dirichlet domain, in hyperbolic space. It is isometric to the quotient space $H^3/\Gamma$, where $\Gamma$ is a discrete group of isometries of $H^3$, defined by generators and relations, which acts on $H^3$ so as to produce the tesselation

$$H^3 = \bigcup_{\gamma \in \Gamma} \gamma(FP).$$
Each cell $\gamma(FP)$ is a copy of $FP$, hence we have periodicity of functions on a CHM, and the possibility of a cosmological Casimir effect. 

Face-pairing generators $\gamma_k$, $k = 1 - n$, satisfy

$$FP \cap \gamma_k(FP) = \text{face } k \text{ of } FP.$$ 

With these generators the relations also have a clear geometrical meaning: they correspond to the cycles of cells around the edges of $FP$.

The software SNAPPEA \cite{2} includes a “census” of about 11,000 orientable CHMs, with normalized volumes from 0.94270736 to 6.45352885. For each of these the $FP$ centered on a special basepoint $O$ is given, as well as the face-pairing generators in both the $SL(2, \mathbb{C})$ and the $SO(1, 3)$ representations. 

An algorithm \cite{12} to find a set of cells $\gamma(FP)$ that completely cover a ball of radius $r$ reduces this problem to one of finding all motions $\gamma \in \Gamma$, such that

$$\text{distance}[O, \gamma(O)] < r + \text{(radius of } FP\text{'s circumscribing sphere)}.$$ 

For a study of CHMs from a cosmological viewpoint, see for example \cite{9} and references therein. For numerical data on a couple of them, see \cite{10}, \cite{11}.

8 CLOSED HYPERBOLIC UNIVERSES

We are considering static CHUs. As obtained in Sec. 6, the metric is

$$ds^2 = dt^2 - \frac{3}{2|\Lambda|} \left( \delta_{ij} - \frac{x^i x^j}{1 + x^2} \right) dx^i dx^j.$$ 

The spacetimes have nontrivial topology:

$$M^4 = \mathbb{R}^1 \times \Sigma,$$

where $\mathbb{R}^1$ is the time axis and $\Sigma = H^3/\Gamma$ is a CHM.

As found above, these models have negative energy density, $\rho = \Lambda / 8\pi G$, which has no obvious physical meaning, and violates the energy condition $T_{\mu\nu}u^\mu u^\nu \geq 0$. But we are dealing with the very early universe, where one feels freer to speculate. And a recent paper by Olum \cite{13} casts doubt on the universality of this condition.
Our original motivation was the possibility of preinflationary homoge-
nization through chaotic mixing, leading to $\Omega_0 < 1$ inflation (cf. Cornish et
al. [14]).

Another guess is that these models might have a place in the path integrals
for quantum cosmology.

9 THE ENERGY-MOMENTUM OPERATOR

If we use the equation for $T_{\mu\nu}$ in Sec. 4 to calculate $< 0 | T_{\mu\nu} | 0 >$ we get terms like

$$< 0 | \phi(x) \phi(x) | 0 > ,$$

which lead to infinities.

To avoid this one replaces $x$ by $x'$ in the first factor, then in the second
factor, and average the result. Thus the above expectation value becomes
one-half Hadamard’s function $G^{(1)}$:  

$$G^{(1)}(x, x') = < 0 | [\phi(x), \phi(x')]_+ | 0 > ,$$

and we obtain (cf. Christensen [15], with our signs)

$$< 0 | T_{\mu\nu}(x, x') | 0 > = \hat{T}_{\mu\nu}(x, x') G^{(1)}(x, x') ,$$

with the operator

$$\hat{T}_{\mu\nu}(x, x') = \frac{1}{6}(\nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu') + \frac{1}{12} g_{\mu\nu}(x) \nabla_\rho \nabla_\rho'$$

$$- \frac{1}{12} (\nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu') + \frac{1}{48} g_{\mu\nu}(x) (\nabla_\rho \nabla_\rho + \nabla_\rho' \nabla_\rho')$$

$$- \frac{1}{12} \left[ R_{\mu\nu}(x) - \frac{1}{4} g_{\mu\nu}(x) R(x) \right] + \frac{1}{8} m^2 g_{\mu\nu}(x) ,$$

where $\nabla_\alpha$ and $\nabla_\alpha'$ are covariant derivatives with respect to $x^\alpha$ and $x'^\alpha$, respectively.

Eventually one takes the limit $x \to x'$ to get the CE density. But first we
have to investigate $G^{(1)}(x, x')$. 
10 FEYNMAN’S PROPAGATOR

$G^{(1)}(x,x')$ will be obtained from Feynman’s propagator for a scalar field $G_F(x,x')$.

In an $\mathbb{R}^1 \times H^3$ universe, $G_F$ gets an extra factor $(\chi / \sinh \chi)$, where $\chi = \sinh^{-1}|x-x'|$, with respect to its flat spacetime counterpart; and the squared interval $(x-x')^2$ in the latter becomes $2 \sigma = (t-t')^2 - a^2 \chi^2$, which is the squared geodesic distance between $x$ and $x'$. The derivation of the following expression (with opposite sign because of a different metric signature) is outlined in [1]:

\[
G_F(x,x') = \frac{m^2}{8\pi \sinh \chi} \frac{H^{(2)}_1(m\sqrt{2\sigma})}{m\sqrt{2\sigma}},
\]

where $H^{(2)}_1$ is Hankel’s function of second kind and degree one.

For our spacetime $\mathbb{R}^1 \times H^3/\Gamma$, point $x$ may be reached by the projections of all geodesics that link $x'$ to $\gamma x$ in the covering space $H^3$ [in Minkowski coordinates $(\gamma x)^i = \sum_{b=1}^4 \gamma^i_b x^b$, $i = 1-3$].

Therefore our propagator is

\[
G_F(x,x') = \frac{m^2}{8\pi} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\sinh \chi(\gamma)} \frac{H^{(2)}_1(m\sqrt{2\sigma(\gamma)})}{m\sqrt{2\sigma(\gamma)}},
\]

with $\chi(\gamma) = \sinh^{-1}|\gamma x - x'|$ and $2 \sigma(\gamma) = (t-t')^2 - a^2 \chi^2(\gamma)$.

11 HADAMARD’S FUNCTION

Hadamard’s function is related to $G_F$ and the principal value Green’s function $\tilde{G}$ by

\[
G_F(x,x') = -\tilde{G}(x,x') - \frac{i}{2} G^{(1)}(x,x').
\]

In our problem, both $\tilde{G}$ and $G^{(1)}$ are real, so that

\[
G^{(1)}(x,x') = -2 \text{Im } G_F(x,x').
\]
We need $G^{(1)}(x, x')$ for $x'$ near $x$, hence when $2\sigma(\gamma)$ is near $-a^2 \chi^2 \leq 0$. So we write the argument of $H_1^{(2)}$ as $iu_\gamma$, with $u_\gamma = m\sqrt{2|\sigma(\gamma)|}$. From the properties of Bessel functions,

$$-2 \text{Im} \left[ (iu_\gamma)^{-1}H_1^{(2)}(iu_\gamma) \right] = (4/\pi)u_\gamma^{-1}K_1(u_\gamma),$$

where $K_1$ is a modified Bessel function of degree one.

Hadamard’s function for a universe $\mathbf{R}^1 \times \mathbf{H}^3/\Gamma$ is then

$$G^{(1)}_\Gamma(x, x') = \frac{m^2}{2\pi^2} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\sinh \chi(\gamma)} \frac{K_1(u_\gamma)}{u_\gamma},$$

The $\gamma = 1$ term in this sum corresponds to the infinite $\mathbf{R}^1 \times \mathbf{H}^3$ universe. Similarly to what was done for the two-plate Casimir effect in Sec. 2, we subtract it out to get a finite energy density. Therefore the expression in Sec. 9 for $<0|T_{\mu\nu}(x, x')|0>$ leads to

$$<0|T_{\mu\nu}(x, x')|0> = \hat{T}(x, x')G_C(x, x'),$$

where $G_C = G^{(1)}_\Gamma - G^{(1)}_{\{1\}}$.

### 12 THE CASIMIR ENERGY DENSITY

Finally, the CE density is given by

$$<0|T_{00}(x)|0> = \lim_{x' \to x} \hat{T}_{00}(x, x') G_C(x, x'),$$

where

$$G_C(x, x') = \frac{m^2}{2\pi^2} \sum_{\gamma \in \Gamma \setminus \{1\}} \sinh^{-1} |\gamma x - x'| \frac{K_1(m\sqrt{-2\sigma(\gamma)})}{m\sqrt{-2\sigma(\gamma)}},$$

with $-2\sigma(\gamma) = a^2(\sinh^{-1} |\gamma x - x'|)^2 - (t - t')^2$.

Looking at the expressions for $\hat{T}_{00}(x, x')$ and $G_C(x, x')$, one sees they are pretty complicated.

Now enters the power of computers!
Calculations were performed by one of us (DM), for a grid of points $(\theta, \varphi)$ on a sphere of radius $r$ inside the FP, for a number of static CHUs. In [1] the parameters are, in Planckian units, $m = 0.5$, $a = 10$, and $r = 0.6$, and the summation for $G_C(x, x')$ contains a few thousand terms; the obtained density values oscillate around $-2.65 \times 10^{-6}$. New results will be published elsewhere.

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