Point-shifts of Point Processes on Topological Groups

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Abstract

This paper focuses on covariant point-shifts of point processes on topological groups, which map points of a point process to other points of the point process in a translation invariant way. Foliations and connected components generated by point-shifts are studied, and the cardinality classification of connected components, previously known on Euclidean space, is generalized to unimodular groups. An explicit counterexample is also given on a non-unimodular group. Isomodularity of a point-shift is defined and identified as a key component in generalizations of Mecke’s invariance theorem in the unimodular and non-unimodular cases. Isomodularity is also the deciding factor of when the reciprocal and reverse of a point-map corresponding to a bijective point-shift are equal in distribution. Finally, sufficient conditions for separating points of a point process are given.

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1. Introduction

1.1. Outline

A review of what is necessary from the Palm theory of stationary random measures on locally compact second-countable Hausdorff groups is given. G. Last studied the Palm theory of locally compact second-countable Hausdorff groups in [7] and [8], though only the finite intensity case is considered here. Generally speaking, the mass transport theorem is used to study point-shifts of point processes, which were also studied on \( \mathbb{R}^d \) by F. Baccelli and M.-O. Haji-Mirsadeghi in [2] and [1].

The novelty of this document is essentially summarized in the following four loosely stated results, though many results leading up to them are interesting in their own right.

Theorem 1.1.1. On unimodular groups, the cardinality classification of connected components of the graph generated by a point-shift holds. The graph is locally finite and flow-adapted. A component is of type \( F/F, I/F, \) or \( I/I \) with \( I \) (infinite) and \( F \) (finite) representing the cardinality of the component/all foils therein. A component is acyclic iff it is infinite iff there is a flow-adapted linear order isomorphic to the order of \( \mathbb{N} \) or \( \mathbb{Z} \) on the foils. There are primeval elements, i.e. elements with preimages of all orders, iff there is a finite foil in \( C \), and when they exist the primeval elements form either a unique cycle or unique bi-infinite path in \( C \).

Example 1.1.2. There exists an explicit non-unimodular group and point-shift \( \mathcal{H} \) for which the previous cardinality classification fails in many respects. The graph generated by \( \mathcal{H} \) is not locally finite. Every component consists of a single foil and is infinite (class \( I/I \)), but the components are cyclic and each component contains a unique element with preimages of all orders, which is a fixed point of \( \mathcal{H} \).

The classification theorem is precisely stated as Theorem 2.1.11 and Section 2.1 builds up to it. The counterexample showing the classification does not hold for non-unimodular groups follows in Section 2.2.

Theorem 1.1.3. On unimodular groups, Mecke’s invariance theorem holds. Point-shifts are bijective iff they preserve Palm probabilities. On possibly non-unimodular groups, the class of bijective point-shifts that preserve Palm probabilities is identified as the bijective isomodular point-shifts, the point-shifts that preserve the modular function of the group. The bijective isomodular point-shifts are also exactly the point-shifts for which the reciprocal and the reverse of the corresponding point-map are equal in distribution.
Section 3.1 and Section 3.2 cover the previous theorem. Mecke’s invariance theorem in the unimodular case is Corollary 3.1.1, and the identification of isomodular point-shifts as being the ones that preserve Palm probabilities (which may be considered a generalization of Mecke’s invariance theorem for the non-unimodular case) is Theorem 3.1.6. The study of the reciprocal and reverse of a point-map and how isomodularity plays a role culminates in Theorem 3.2.4.

**Theorem 1.1.4.** Given a measurable property $F(x, y_1, \ldots, y_n)$ of a configuration of $n+1$ points, if for every choice of distinct $y_1, \ldots, y_n$, none of which are the identity, the set of $x$ for which $F(x, y_1, \ldots, y_n) \in M$ has null Haar measure, then no stationary point process has $n+1$ distinct points $X, Y_1, \ldots, Y_n$ satisfying $F(X, Y_1, \ldots, Y_n) \in M$. This generalizes fact on $\mathbb{R}^d$ that a stationary point process has not two points $X, Y$ that are equidistant from the origin.

Section 3.3 studies when functions separate points of a point process, and the precise version of the previous theorem is Theorem 3.3.2.

### 1.2. Preliminaries

Throughout this document, let $\mathbb{X}$ be a fixed locally compact second-countable Hausdorff topological group with identity element $e \in \mathbb{X}$. It will always be assumed that $\mathbb{X}$ and $\mathbb{R}$ are equipped with their respective Borel $\sigma$-algebras $B(\mathbb{X})$ and $B(\mathbb{R})$.

Let $\lambda$ denote a fixed left-invariant [Haar measure](http://example.com) on $\mathbb{X}$, i.e. $\lambda$ is a nontrivial locally finite Borel measure satisfying $\lambda(xB) = \lambda(B)$ for all $x \in \mathbb{X}, B \in B(\mathbb{X})$. Also let $\Delta : \mathbb{X} \to (0, \infty)$ denote the [modular function](http://example.com) of $\mathbb{X}$. That is, $\lambda(xB) = \Delta(x)\lambda(B)$ for all $x \in \mathbb{X}, B \in B(\mathbb{X})$. Recall that $\Delta$ is a continuous homomorphism, i.e. $\Delta(xy) = \Delta(x)\Delta(y)$, and from this follows $\Delta(x^{-1}) = \Delta(x)^{-1}$ for all $x, y \in \mathbb{X}$.

Let $M$ be the space of all [Radon measures](http://example.com) on $(\mathbb{X}, B(\mathbb{X}))$ equipped with the Borel $\sigma$-algebra $M$ induced by the topology of vague convergence. The shift operators $\{\theta_x\}_{x \in \mathbb{X}}$ act on elements of $M$ by $\theta_x \mu(B) := \mu(x^{-1}B)$. The zero measure on $\mathbb{X}$ is denoted $0$. The support of $\mu$ is denoted $\text{supp} \mu$. Also denote by $N$ the subset of [counting measures](http://example.com). A counting measure is [simple](http://example.com) if each atom has mass 1, and a simple counting measure will be identified with its support, which is a discrete subset of $\mathbb{X}$.

A [random measure](http://example.com) is a pair $(m, P)$ where $m : (\Omega, \mathcal{A}) \to (M, M)$ is a measurable mapping and $(\Omega, \mathcal{A}, P)$ is a probability space. Often $m(\omega, B)$ is written instead of $m(\omega)(B)$ for convenience, but this is also in accordance with the viewpoint of $m$ being a transition kernel. A (simple) [point process](http://example.com) is a random measure that is almost surely a (simple) counting measure.

The underlying probability space $(\Omega, \mathcal{A}, P)$ is always assumed to be part of a [stationary framework](http://example.com) $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, P)$ and any random measure $m$ will always be assumed to be [compatible](http://example.com) with the flow $\{\theta_x\}_{x \in \mathbb{X}}$. That is, suppose $P$ is invariant with respect to a measurable flow $\{\theta_x\}_{x \in \mathbb{X}}$ that acts on the underlying measurable space $(\Omega, \mathcal{A})$:
(i) \( \theta_x : \Omega \rightarrow \Omega \) for all \( x \in \mathbb{X} \),

(ii) \( \theta_e \) is the identity on \( \Omega \),

(iii) \( \theta_{xy} = \theta_x \theta_y \) for all \( x, y \in \mathbb{X} \), in particular \( \theta_x^{-1} = \theta_{x^{-1}} \),

(iv) \( (\omega, x) \mapsto \theta_x \omega \) is \( B(\mathbb{X}) \otimes A \)-measurable, and

(v) \( P \circ \theta_x^{-1} = P \) for all \( x \in \mathbb{X} \),

and it is assumed that the random measure \( m \) on \( \mathbb{X} \) satisfies

\[
m(\theta_x \omega, B) = m(\omega, x^{-1}B), \quad x \in \mathbb{X}, \omega \in \Omega, B \in B(\mathbb{X}).
\]

Abusing notation, this is written as

\[
m(\theta_x \omega, B) = \theta_x m(\omega, B).
\]

This is an abuse of notation because on the left side the notation \( \theta_x \) means the measurable flow on \( \Omega \), and on the right side \( \theta_x \) means the shift operator on \( \mathbb{M} \). Under these assumptions any such \( m \) is stationary in the usual sense that \( m \) and \( \theta_x m \) have the same distribution for all \( x \in \mathbb{X} \).

**Example 1.2.1.** If \( m \) is a random measure on \( \mathbb{X} \) on the canonical space with \( (\Omega, A, P) := (\mathbb{M}, M, P) \) where \( m(\mu) := \mu \) is the identity map on \( \mathbb{M} \) and \( P \) is the distribution of \( m \), then \( \theta_x \mu \) defined to be the shift operator \( \theta_x \mu(B) := \mu(x^{-1}B) \) makes \( (\mathbb{M}, M, \{ \theta_x \}_{x \in \mathbb{X}}, P) \) a stationary framework when the distribution of \( m \) is shift-invariant.

It is also assumed that all random measures \( m \) that are introduced have finite and nonzero intensity, usually denoted \( \gamma \). It holds that \( \gamma = \frac{E[m(B)]}{\lambda(B)} \) for any \( B \) with finite and nonzero Haar measure.

The **Palm probability** of \( m \) is denoted \( P^m \), i.e. for all \( A \in A \),

\[
P^m(A) := \frac{1}{\gamma} E \int_{\mathbb{X}} 1_{\theta_x^{-1} \in A} w(x) m(dx)
\]

for any (and every) measurable \( w : \mathbb{X} \rightarrow \mathbb{R}_+ \) with \( \int_{\mathbb{X}} w d\lambda = 1 \). It is typical to take \( w(x) := \frac{1}{\lambda(B)} 1_{x \in B} \) for some \( B \) with \( 0 < \lambda(B) < \infty \). Expectation with respect to \( P^m \) is denoted \( E^m \).

The Palm probability \( P^m \) gives the view of the world from a typical point’s perspective. Intuitively it is the reference measure conditioned on the event that \( e \in \mathbb{m} \). Indeed, \( P^m(e \in \mathbb{m}) = 1 \) and the interpretation as a conditional expectation is exactly correct when \( \mathbb{X} \) is discrete.

The connection between \( P \) and \( P^m \) is given by the refined Campbell theorem, abbreviated to C-L-M-M for Campbell, Little, Mecke, and Matthes.

**Theorem 1.2.2 (C-L-M-M).** [7] For all \( f : \Omega \times \mathbb{X} \rightarrow \mathbb{R}_+ \) measurable,

\[
E \int_{\mathbb{X}} f(\theta_x^{-1}, x) m(dx) = \gamma E^m \int_{\mathbb{X}} f(\theta_e, x) \lambda(dx).
\]
It is also possible to convert between \( \mathbf{P} \)-a.s. and \( \mathbf{P}^m \)-a.s. events in the following manner.

**Theorem 1.2.3.** Let \( A \in \mathcal{A} \). Then the following are equivalent:

(a) \( \mathbf{P}^m(A) = 1 \),
(b) \( \mathbf{P}(m(x \in \mathbb{X} : \theta_x^{-1} \not\in A) = 0) = 1 \),
(c) \( \mathbf{P}^m(m(x \in \mathbb{X} : \theta_x^{-1} \not\in A) = 0) = 1 \).

The proof of this fact is given in the appendix. When \( m \) is a point process, Theorem 1.2.3 can be used to translate between definitions under \( \mathbf{P}^m \) and definitions under \( \mathbf{P} \). The unfamiliar reader should see Example 4.2.6 for the details of how to do this.

The primary tool that allows the study of point-shifts in this paper is the mass transport theorem.

**Theorem 1.2.4 (Mass Transport Theorem).** Suppose \( m, m' \) are compatible random measures on \( \mathbb{X} \) with respective intensities \( \gamma, \gamma' \in (0, \infty) \). Then for all diagonally invariant \( \tau \), i.e. measurable \( \tau : \Omega \times \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+ \) invariant in the sense that \( \tau(\theta_x \omega, zx, zy) = \tau(\omega, x, y) =: \tau(x, y) \), \( \omega \in \Omega \), \( x, y, z \in \mathbb{X} \),

\[
\gamma \mathbf{E}^m \int_{\mathbb{X}} \tau(e, y) m'(dy) = \gamma' \mathbf{E}^{m'} \int_{\mathbb{X}} \tau(x, e) \Delta(x^{-1}) m(dx).
\]

Interpret \( \tau(\omega, x, y) \) as the amount of mass sent from \( x \) to \( y \) on the outcome \( \omega \). Under \( \mathbf{E}^m \), \( e \) is a point of \( m \). Thus the left side of (2) is an average of mass sent out of \( e \in m \) to all points \( m' \). Under \( \mathbf{E}^{m'} \), \( e \) is a point of \( m' \). Thus the right side of (2) is a weighted average of mass received by \( e \in m' \) from all points of \( m \). If \( \Delta(x) = 1 \) for all \( x \in \mathbb{X} \), i.e. if \( \mathbb{X} \) is unimodular, then the mass transport formula is the one expected from the case of translations on \( \mathbb{R}^d \), which says that after weighting by the ratio of intensities \( \gamma/\gamma' \), the mass a typical point of \( m' \) receives is equal to the mass that a typical point of \( m \) sends, on average.

When dealing with point-shifts and point-maps, it will always be assumed that \( m \) is a simple point process. A **point-shift** on \( m \) is a covariant measurable map \( \delta : \Omega \times \mathbb{X} \to \mathbb{X} \) on the support of \( m \), i.e. for all \( x, y \in \mathbb{X}, \omega \in \Omega \), \( \delta \) satisfies \( \delta(\theta_y \omega, yx) = y \delta(\omega, x) \) and \( \mathbf{P} \)-a.e. \( \omega \in \Omega \) is such that \( \delta(\omega, X) \in m(\omega) \) for all \( X \in m(\omega) \). If unspecified, \( \delta(\omega, x) := x \) for \( x \not\in m(\omega) \). Dependence on \( \omega \) is usually dropped and \( \delta(X) \) is written instead of \( \delta(\omega, X) \). Say that \( \delta \) has a functional property, e.g. bijectivity, injectivity, surjectivity, if for \( \mathbf{P} \)-a.e. \( \omega \in \Omega \), \( \delta(\omega, \cdot) \) has the property on the support of \( m(\omega) \).

A **point-map** on \( m \) is a measurable map \( h : \Omega \to \mathbb{X} \) such that \( h(\omega) \in m(\omega) \) for \( \mathbf{P}^m \)-a.e. \( \omega \in \Omega \).

There is a natural correspondence between point-shifts and point-maps. Namely, if \( \delta \) is a point-shift, then \( h(\omega) := \delta(\omega, e) \) is a point-map, and if \( h \) is
a point-map, then \( \mathfrak{h}(\omega, X) := Xh(\theta_X^{-1}\omega) \) is a point-shift, and these operations are inverses.

To any point-shift \( \mathfrak{h} \) with corresponding point-map \( h \) on \( m \) define the functions (omitting \( \omega \) dependence):

- **Edge indicator**: \( \tau^\mathfrak{h}(x, y) := 1_{x,y \in m, \mathfrak{h}(x) = y} \) for all \( x, y \in X \).

- **Out-neighbors** and **in-neighbors** of \( e \) under \( P^m \):
  \[
  h^+ := \{ Y \in m : \tau^\mathfrak{h}(e, Y) = 1 \} = \{ h \},
  \]
  \[
  h^- := \{ X \in m : \tau^\mathfrak{h}(X, e) = 1 \} = \{ Y \in m : h(\theta_Y^{-1}\omega) = Y^{-1} \}.
  \]

- **Out-neighbors** and **in-neighbors** under \( P \) or \( P^m \):
  \[
  H^+(X) := Xh^+(\theta_X^{-1}) = \{ Y \in m : \tau^\mathfrak{h}(X, Y) = 1 \} = \{ \mathfrak{h}(X) \},
  \]
  \[
  H^-(X) := Xh^-(\theta_X^{-1}) = \{ Y \in m : \tau^\mathfrak{h}(Y, X) = 1 \} = \{ Y \in m : \mathfrak{h}(Y) = X \}
  \]
  for all \( X \in m \).

- **Preimage** of \( e \) under \( P^m \): if \( P^m \)-a.s. \( \text{card}(h^-) = 1 \), then \( h^- \) is defined to be the unique element in \( h^- \). By the upcoming Proposition 1.2.5 this is equivalent to \( \mathfrak{h} \) being bijective.

- **Reverse point-shift**: if \( P \)-a.s. \( \text{card}(H^-(X)) = 1 \) for all \( X \in m \) (equivalently \( P^m \)-a.s. \( \text{card}(h^-) = 1 \)), then \( \mathfrak{h}^-(X) \) is defined to be the unique element in \( H^- \). By the upcoming Proposition 1.2.5, \( \mathfrak{h}^- \) is defined iff \( \mathfrak{h} \) is bijective. In this case \( P \)-a.s. \( \mathfrak{h}(\mathfrak{h}^-(X)) = \mathfrak{h}^-(\mathfrak{h}(X)) = X \) for all \( X \in m \), by definition. That is, \( \mathfrak{h} \) and \( \mathfrak{h}^- \) are inverses.

With these definitions, \( \tau^\mathfrak{h} \) is diagonally invariant, \( H^+, H^-, \mathfrak{h}^- \) are covariant with the flow, and the definitions of \( h^+, h^-, h^- \) under \( P^m \) are equivalent to the definitions of \( H^+, H^-, \mathfrak{h}^- \) under \( P \) or \( P^m \) as in Example 4.2.6.

With the mass transport theorem and Theorem 1.2.3, the following can be obtained by chasing definitions:

**Proposition 1.2.5.** The following hold:

(a) \( P \)-a.s. every \( X \in m \) is the image under \( \mathfrak{h} \) of at least (resp. at most) \( k \) distinct points of \( m \) iff \( P^m \)-a.s. \( \text{card}(h^-) \geq k \) (resp. \( \leq k \)),

(b) \( P \)-a.s. every \( X \in m \) is the image under \( \mathfrak{h} \) of finitely (resp. infinitely) many distinct points of \( m \) iff \( P^m \)-a.s. \( \text{card}(h^-) < \infty \) (resp. \( = \infty \)),

(c) \( P \)-a.s. \( \mathfrak{h} \) is bijective (resp. surjective, injective) iff \( P^m \)-a.s. \( \text{card}(h^-) = 1 \) (resp. \( \geq 1, \leq 1 \)). In particular \( h^- \) and \( \mathfrak{h}^- \) are well defined iff \( \mathfrak{h} \) is bijective,

(d) For all \( f : \Omega \to \mathbb{R}_+ \) measurable,

\[
E^m[f(\theta_h^{-1})] = E^m[f \text{ card}(h^-)],
\]

(3)
(e) \(P\) - a.s. every \(X \in \mathcal{m}\) is the image under \(\mathcal{H}\) of at least (resp. at most) \(k\) points of \(\mathcal{m}\) iff for all \(f : \Omega \to \mathbb{R}_+\) measurable

\[
E^m[f(\theta^{-1}_h) \Delta(h^{-1})] \geq kE^m[f] \quad \text{(resp.} \leq kE^m[f])
\]

(f) (Test for Bijectivity)\(^1\) \(\mathcal{H}\) is bijective iff for all \(f : \Omega \to \mathbb{R}_+\) measurable

\[
E^m[f(\theta^{-1}_h) \Delta(h^{-1})] = E^m[f],
\]

(g) If \(\mathcal{H}\) is bijective, also

\[
E^m[f(\theta^{-1}_h)] = E^m[f / \Delta(h^{-1})],
\]

(h) If \(P\) - a.s. every \(X \in \mathcal{m}\) is the image under \(\mathcal{H}\) of at least (resp. at most) \(k\) points of \(\mathcal{m}\), then \(E^m[\Delta(h^{-1})] \geq k\) (resp. \(\leq k\)).

(i) If \(E^m[\Delta(h^{-1})] < \infty\), every \(X \in \mathcal{m}\) is the image of only finitely many \(Y \in \mathcal{m}\) under \(\mathcal{H}\).

(j) If \(E^m[\Delta(h^{-1})] = 1\), then \(\mathcal{H}\) is injective iff it is surjective. In particular, this is automatic if \(X\) is unimodular.

**Proof.** The proofs are sketched.

(a), (b), (c): Direct application of Theorem 1.2.3.

(d): Apply the mass transport theorem with the diagonally invariant function

\[
\tau(\omega, x, y) := f(\theta^{-1}_y \omega) 1_{x, y \in \mathcal{m}(\omega), y = \mathcal{H}(x)} \Delta(y^{-1}x).
\]

(e): Apply (a) and (d).

(f): Apply (e) with \(k := 1\).

(g): Replace \(f\) with \(\frac{f}{\Delta(h^{-1})}\) in (d) and use the fact that \(P^m\) - a.s.

\[
h^{-1}(\theta^{-1}_h) = h^{-1}(hh^{-1}(\theta^{-1}_h)) = h^{-1}\mathcal{H}^{-1}(h) = h^{-1}\mathcal{H}^{-1}(\mathcal{H}(e)) = h^{-1}.
\]

(h), (i): Take \(f := 1\) in (d) and apply (a) or (b).

(j): Take \(f := 1\) in (d), use (c) and the fact that a random variable bounded above (or below) by 1 with expectation 1 must be constant 1 a.s.

\[\square\]
1.3. Examples of Point-shifts

A few elementary examples of point-shifts on a compatible simple point process \( m \) are given. Distances in \( X \) will be measured with respect to any left-invariant metric \( d \) inducing the topology of \( X \).

1. (Closest Neighbor Shift). For \( X \in m \), let \( \mathcal{H}(X) \) be the closest \( Y \in m \) with \( Y \neq X \) (if one is uniquely determined), otherwise let \( \mathcal{H}(X) := X \).

2. (Mutual Closest Neighbor Shift). If \( X, Y \in m \) are such that \( X \) is the closest neighbor to \( Y \) and \( Y \) is the closest neighbor to \( X \), set \( \mathcal{H}(X) := Y \) and \( \mathcal{H}(Y) := X \). For \( X \in m \) not mutually closest neighbors with another point, set \( \mathcal{H}(X) := X \).

3. (Closest Neighbor from Next Generation). Suppose \( m \) is partitioned into sub-processes \( \sum_{i \in \mathbb{Z}} m_i = m \), then for \( X \in m_i \), let \( \mathcal{H}(X) \) be the closest \( Y \in m_{i+1} \) (if one is uniquely determined), otherwise \( \mathcal{H}(X) := X \).

4. (Modularity Boost). Suppose \( X \) is not unimodular. Fix a relatively compact set \( B \in B(X) \) containing the identity. For each \( X \in m \), look in the set \( XB \) and set \( \mathcal{H}(X) \) to be the \( Y \in m \cap XB \) such that \( \Delta(Y) \) is maximum (or \( \mathcal{H}(X) := X \) if such a \( Y \) is not uniquely determined). Existence of a unique maximum can be guaranteed by assuming \( \lambda(\Delta = 1) = 0 \) if \( m \) is Poisson (see Corollary 3.3.6 and use the Slivnyak-Mecke theorem).

5. (Snap-to-grid Shift on Circle Group). Let \( X := \{ z \in \mathbb{C} : |z| = 1 \} \) be the circle group, which is identified with \( [0,1) \subseteq \mathbb{R} \) via \( x \mapsto e^{2\pi ix} \). Since \( X \) is Abelian it is unimodular, and its Haar measure is normalized arc length along the circle, or Lebesgue measure on \( [0,1) \). Let \( m \) be a homogeneous Poisson point process on \( X \) with intensity \( \gamma \in (0,\infty) \). Necessarily \( m \) is stationary and simple. Imagine picking up \( X \in m \), rotating by an angle \( \theta \), and dropping \( X \) at the nearest point of \( m \). More precisely, with \( z := e^{2\pi i \theta} \in X \),

\[
\mathcal{H}(X) := \arg \min_{Y \in m} |Y - zX|, \quad X \in m, \text{P-a.s.}
\]

Such a nearest point will be uniquely determined in this case (see again Corollary 3.3.6 and use the Slivnyak-Mecke Theorem).

6. (Strip Point-shift on the \( ax+b \) Group). Recall the standard first example of a non-unimodular group: the \( ax+b \) group. Let

\[
\mathbb{X} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}
\]

with matrix multiplication and the topology inherited from \( \mathbb{R}^4 \). \( \mathbb{X} \) is identified with the right half-plane in \( \mathbb{R}^2 \) by identifying \( (a,b) \) with \( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \).
In this notation \((a, b)(c, d) = (ac, ad + b)\) and \((a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right)\). Then, cf. [5] Example 15.17 (g), \(X\) has a left-invariant Haar measure 

\[
\lambda(B) = \int \int_B \frac{1}{a^2} \, da \, db
\]

and modular function

\[
\Delta(a, b) = \frac{1}{a}.
\]

Let \(m\) be a homogeneous Poisson point process on \(X\) with intensity \(\gamma \in (0, \infty)\). Necessarily \(m\) is stationary and simple. For all \((a, b) \in X\) define the strip

\[
S(a, b) := [a, \infty) \times [b - \delta a, b + \delta a]
\]

for some fixed \(\delta > 0\). Note that the definition is chosen so \((a, b)S(1, 0) = S(a, b), \) where here \((1, 0) = e \in X\). Moreover, for any \((a, b) \in X\),

\[
\lambda(S(a, b)) = \int_{b - \delta a}^{b + \delta a} \int_a^\infty \frac{1}{x^2} \, dx \, dy = \frac{1}{a} \cdot ((b + \delta a) - (b - \delta a)) = 2\delta
\]

so in particular \(m(S(a, b)) < \infty\) a.s. By Theorem 4.2.7 \(m^1\) is Poisson under \(P^m\) with \(E^m[m^1(B)] = \gamma \lambda(B)\). Hence \(E^m[m^1(S(1, 0))] = 2\delta \gamma\) and therefore \(m(S(1, 0)) < \infty, \) \(P^m\)-a.s. Equivalently, \(P\)-a.s. \(m(S(X)) < \infty\) for all \(X \in m\) by Theorem 1.2.3. This leads to the strip point-shift \(\delta_1\) where \(\delta_1(X)\) is defined to be the right-most point of \(m\) in \(S(X)\) for each \(X \in m\). Note that there is no need to resolve ties for right-most position because \(\lambda([1] \times \mathbb{R}) = 0,\) so that \(E^m[m^1([1] \times \mathbb{R})] = 0\) and hence all \(X \in m\) will have distinct first coordinates by Proposition 3.3.5.

2. Point-shift Foliations

2.1. The Cardinality Classification of Components

In this section the cardinality classification components of point-shifts in [2] is extended to the general stationary framework for unimodular \(X\). The classification theorem is Theorem 2.1.11, and the fundamental result used in its
proof, which says it is impossible to pick out finite subsets of infinite sets in a flow-adapted manner, is Theorem 2.1.6.

Throughout this section, X is assumed to be unimodular. Fix a stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \Omega}, \mathbb{P})$, a compatible simple point process m on X with intensity $\gamma \in (0, \infty)$, and a point-map $\mathcal{H}$ on m with corresponding point-shift $\mathcal{H}$. The wording of proofs is substantially cut down by thinking of $\mathcal{H}(X)$ as the father of X. For example, the children of X are the $Y \in m$ such that $\mathcal{H}(Y) = X$. Next appear the necessary ingredients needed for the classification theorem.

**Definition 2.1.1.** The iterates $\mathcal{H}^n$ are defined by repeatedly applying the point-shift $\mathcal{H}$. That is, $\mathcal{H}^0(X) := X$ and $\mathcal{H}^{n+1}(X) := \mathcal{H}(\mathcal{H}^n(X))$ for all $X \in m$. Elements $Y \in m$ that are in the image $\mathcal{H}(m)$ for all $n \in \mathbb{N}$ are called primeval, and $\mathcal{H}^\infty(m)$ will denote the set of all primeval elements of m. Here $\mathcal{H}^n(m)$ is considered as a set, i.e. multiplicities are ignored, for all $n \leq \infty$. The set $\mathcal{H}^n(m)$ is flow-adapted for any $n \leq \infty$ because $\mathcal{H}$ is flow-adapted.

**Definition 2.1.2.** The random graph $G^\mathcal{H}$ has vertices m and directed edges from each $X \in m$ to $\mathcal{H}(X)$. The set of its undirected connected components is denoted by $C^\mathcal{H}$ and the component of $X \in m$ is denoted $C^\mathcal{H}(X)$. Then $X, Y \in m$ are in the same component iff there are $n, m \in \mathbb{N}$ such that $\mathcal{H}^n(X) = \mathcal{H}^m(Y)$. That is, $C^\mathcal{H}(X)$ is the set of all relatives of X. The graph $G^\mathcal{H}$ is flow-adapted, and hence so is $C^\mathcal{H}$.

**Definition 2.1.3.** The foliation $L^\mathcal{H}$ is defined to be the set of foils $L^\mathcal{H}(X)$ of $\mathcal{H}$ for $X \in m$, which are equivalence classes under the equivalence relation where $X, Y \in m$ are equivalent iff there is $n \in \mathbb{N}$ such that $\mathcal{H}^n(X) = \mathcal{H}^n(Y)$. That is, $L^\mathcal{H}(X)$ is the relatives of X from the same generation as X. The foliation $L^\mathcal{H}$ is flow-adapted, and $L^\mathcal{H}$ is a subdivision of $C^\mathcal{H}$. For a foil L, also denote $L_+ := L^\mathcal{H}(\mathcal{H}(X))$ for any $X \in L$. Note that if $X, X' \in L$ then $L^\mathcal{H}(\mathcal{H}(X)) = L^\mathcal{H}(\mathcal{H}(X'))$ so $L_+$ is well-defined. If there is $Y \in m$ such that $\mathcal{H}(Y) \in L$, then set $L_- := L^\mathcal{H}(Y)$. Then $L_-$ is well-defined because if $Y, Y' \in L$ are both such that $\mathcal{H}(Y), \mathcal{H}(Y') \in L$, then $L(Y) = L(Y')$. It holds that $(L_+)_- = L$ and when $L_-$ exists $(L_-)_+ = L$.

**Example 2.1.4** (Snap-to-grid Shift on Circle Group Revisited). See Figure 2 for an example component, foil, and trajectory determined by the snap-to-grid point-shift of Section 1.3.

It will be important later to know that the graph $G^\mathcal{H}$ is locally finite. The following result, generalizing one in [2], guarantees this. It crucially relies on the unimodularity of X.

**Proposition 2.1.5.** Let $D_n(X)$ denote the $n$-th order descendants of X, i.e. $D_n(X) := \{Y \in m : \mathcal{H}^n(Y) = X\}$. Also let $D(X) := \bigcup_{n=1}^{\infty} D_n(X)$. Then letting $d_n(X) := \text{card}(D_n(X))$, $d(X) := \text{card}(D(X))$, one has for every $n \geq 0$ that $E^m[d_n(0)] = 1$. In particular $d_n(0)$ is $P^m$-a.s. finite, or equivalently $P$-a.s. every $X \in m$ has $d_n(X)$ finite. If in addition, $G^\mathcal{H}$ is $P^m$-a.s. acyclic, then $E^m[d(0)] = \infty$. 

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Figure 2: Snap-to-grid point-shift with $z = i$: iterates of $e = 1$ are shown in red, and white and green dots inside a dot indicate membership in the component and foil of $e$ respectively.

**Proof.** $\mathcal{S}^n$ is a point-shift in its own right, so the mass transport theorem implies $E^m[d_n(0)] = E^m[\text{card}(D_n(0))] = 1$ since $X$ is unimodular. Thus $d_n(0) < \infty$, $P^m$-a.s., and hence $P$-a.s. $d_n(X) < \infty$ for all $X \in \mathfrak{m}$ by Theorem 1.2.3. Moreover, when $G^0$ is acyclic, the $D_n$ partition $D$ and hence $E^m[d(0)] = \sum_{n=1}^{\infty} E^m[d_n(0)] = \infty$. □

The primary tool needed to prove the classification theorem follows. It says that it is not possible to extract finite subsets of infinite subsets of $\mathfrak{m}$ in a flow-adapted way. An equivalent result for unimodular networks is Lemma 3.23 in [3] and the proof there is adapted for use here.

**Theorem 2.1.6.** Let $N$ be an $\mathbb{N} \cup \{\infty\}$-valued flow-invariant random variable and let $\mathfrak{N} = \{\mathfrak{N}_i\}_{1 \leq i \leq N}$ be a flow-adapted collection of infinite measurable subsets of $\mathfrak{m}$ and let $k$ be the number of $i$ such that $e \in \mathfrak{N}_i$. Suppose that $E^m[k] < \infty$. If $n$ is a measurable flow-adapted subset of $\mathfrak{m}$ for which $P$-a.s. $\text{card}(n \cap \mathfrak{N}_i) < \infty$ for each $i$, then $P$-a.s. $n \cap \mathfrak{N}_i = \emptyset$ for all $i$. In particular, if $n \subseteq \bigcup \mathfrak{N}$, then $P$-a.s. $n = \emptyset$.

**Proof.** Define

$$\tau(\omega, x, y) := \sum_{i=1}^{N(\omega)} 1_{x, y \in \mathfrak{N}_i(\omega)} \frac{1}{\text{card}(n(\omega) \cap \mathfrak{N}_i(\omega))}.$$
The assumptions about flow-adaptedness of $\mathcal{R}$, $n$, and $m$, and the flow-invariance of $N$ imply that $\tau$ is diagonally invariant. Then $\int_X \tau(e, y) \, m(dy) = k$ by construction since $e$ is in $k$ of the $\mathcal{R}_i$. Also $\int_X \tau(x, e) \, m(dx) = \infty$ if $e \in n \cap \mathcal{R}_i$ for some $i$ because the $\mathcal{R}_i$ are infinite. But the mass transport theorem implies

$$E^m \int_X \tau(x, e) \, m(dx) = E^m \int_X \tau(e, y) \, m(dy) = E^m[k] < \infty,$$

and thus it must be that $P^m$-a.s. $e \notin n \cap \mathcal{R}_i$ for any $i$. Equivalently, $P$-a.s. for all $X \in m$ it holds that $X \notin n \cap \mathcal{R}_i$ for any $i$. Since $n \cap \mathcal{R}_i \subseteq m$ for each $i$, it follows that $P$-a.s. $n \cap \mathcal{R}_i = \emptyset$ for all $i$. \hfill \square

More information follows about the structure of the locally finite graph $G^\mathcal{R}$. In particular, cycles in components are unique, infinite components are acyclic, foils in infinite components can be ordered like $\mathbb{N}$ or $\mathbb{Z}$ in a flow-adapted way, and $\mathcal{R}$ acts bijectively on the primeval elements.

**Lemma 2.1.7.** $P$-a.s. a connected component $C$ of $G^\mathcal{R}$ is either an infinite tree or has exactly one (directed) cycle $K(C)$ for which for all $Y \in C$ there is $n \in \mathbb{N}$ such that $\mathcal{R}^n(Y) \in K(C)$. Moreover, $P$-a.s. there are no infinite components with a cycle.

**Proof.** The fact that all elements in $C$ are connected and have out-degree 1 implies there can be at most one cycle. If there are no cycles then $C$ must be infinite since applying $\mathcal{R}$ to any element repeatedly must never repeat an element. Otherwise there is one cycle $K(C)$ and connectedness implies for every $Y \in C$ there is $n \in \mathbb{N}$ with $\mathcal{R}^n(Y) \in K(C)$.

Let $\mathcal{R}$ be the set of infinite components of $G^\mathcal{R}$ with a cycle, and let $n \subseteq \bigcup \mathcal{R}$ be the union of all the cycles of these components. Since cycles are finite, it follows that $n \cap C$ is finite for all components $C \in \mathcal{R}$. By Theorem 2.1.6 $n = \emptyset$ and hence there are no infinite components with a cycle $P$-a.s. \hfill \square

**Definition 2.1.8.** Within an infinite acyclic connected component $C \in C^\mathcal{R}$, it is possible to define an order, called the foil order, on the foils $L^\mathcal{R}(C)$ that are subsets of $C$. This is accomplished by declaring $L^\mathcal{R}(X) < L^\mathcal{R}(Y)$ for all $X \in C$. When thinking of $\mathcal{R}(X)$ as being the father of $X$, the order is that of seniority.

**Lemma 2.1.9.** The foil order on an infinite acyclic component $C$ is a total order on $C$ similar to either the order of $\mathbb{Z}$ or $\mathbb{N}$.

**Proof.** Fix any $X \in C$. Let $L_0 := L^\mathcal{R}(X)$ and recursively define $L_{n+1} := (L_n)_+$ and if it exists $L_{n-1} := (L_{n-1})_-$ for $n > 0$. Let $L$ be a foil in $C$, then it must be that $L = L_i$ for some $i$. Indeed, let $Y \in L$ and by definition of connectedness choose $n, m$ such that $\mathcal{R}^n(Y) = \mathcal{R}^m(X) \in L_m$. It then follows by induction that $Y \in L_{m-n}$, and hence $L^\mathcal{R}(Y) = L_{m-n}$. Next it is shown that $i \mapsto L_i$ is injective. Suppose for contradiction that $L_j = L_{j+N}$. Then there are $N$ pairs $(X_i, Y_{i+1})$ with $X_i \in L_i, Y_{i+1} \in L_{i+1}$ such that $\mathcal{R}(X_i) = Y_{i+1}$ for $j \leq i \leq j + N - 1$. Since $L_j = L_{j+N}$ it follows that $X_j, Y_{j+N} \in L_j$. Hence it is possible to choose $n$ such
that $\mathcal{H}^n(X_j) = \mathcal{H}^n(Y_{j+N})$ and $\mathcal{H}^n(X_i) = \mathcal{H}^n(Y_i)$ for all $j + 1 \leq i \leq j + N - 1$. Then

$$
\mathcal{H}^N(\mathcal{H}^n(X_j)) = \mathcal{H}^{N-1}(\mathcal{H}^n(Y_{j+1})) = \mathcal{H}^{N-1}(\mathcal{H}^n(X_{j+1})) = \ldots = \mathcal{H}^0(\mathcal{H}^n(Y_{j+N})) = \mathcal{H}^n(X_j)
$$

contradicts that $C$ is acyclic. Thus $i \mapsto L_i$ is injective. If there is a smallest foil $L_{i_0}$ then $i \mapsto L_{i_0+i}$ is an order isomorphism with $\mathbb{N}$, otherwise $i \mapsto L_i$ is an order isomorphism with $\mathbb{Z}$.

**Lemma 2.1.10.** P-a.s. $\mathcal{H}$ restricts to a bijective point-shift $\mathcal{H}|_n$ on the flow-adapted sub-process $\mathcal{H}^\infty(m)$ of primeval elements.

**Proof.** To emphasize that $\mathcal{H}^\infty(m)$ is a sub-process of $m$, let $n := \mathcal{H}^\infty(m)$. It was already noted that $n$ is a flow-adapted, and $\mathcal{H}$ naturally restricts to a point-shift $\mathcal{H}|_n$ on $n$ because if $X \in \mathcal{H}^\infty(m)$ then $\mathcal{H}(X) \in \mathcal{H}^\infty(m)$. By definition, primeval elements are in the image $\mathcal{H}(m)$, but moreover they are in the image $\mathcal{H}(n)$. Indeed, by Proposition 2.1.5 points in $m$ have only finitely many children. If $X \in n$ were such that none of its children were primeval, then there would be $n \in \mathbb{N}$ large enough that none of $X$’s children are in the image $\mathcal{H}(n)$. But then $X$ would not be in $\mathcal{H}^{n+1}(m)$, contradicting that $X \in \mathcal{H}^\infty(m)$. Thus the restricted point-shift $\mathcal{H}|_n$ is surjective. If $n$ is not the empty process P-a.s. then it has nonzero and finite intensity and $E^n[\Delta(h^{-1})] = 1$ by unimodiularity so that surjectivity and injectivity are equivalent by Proposition 1.2.5 (j), so $\mathcal{H}|_n$ is bijective.

The main result of this section follows.

**Theorem 2.1.11** (Cardinality Classification of a Component). P-a.s. each connected component $C$ of $G^\mathcal{H}$ is in one of the three following classes:

1. **Class $\mathcal{F}/\mathcal{F}$:** $C$ is finite, and hence so is each of its $\mathcal{H}$-foils. In this case, when denoting by $1 \leq n = n(C) < \infty$ the number of its foils:
   - $C$ has a unique cycle of length $n$;
   - $\mathcal{H}^\infty(m) \cap C$ is the set of vertices of this cycle.

2. **Class $\mathcal{I}/\mathcal{F}$:** $C$ is infinite and each of its $\mathcal{H}$-foils is finite. In this case:
   - $C$ is acyclic;
   - Each foil has a junior foil;
   - $\mathcal{H}^\infty(m) \cap C$ is a unique bi-infinite path, i.e. a sequence $\{X_n\}_{n \in \mathbb{Z}}$ of points of $m$ such that $\mathcal{H}(X_n) = X_{n+1}$ for all $n$. 

3. **Class $\mathcal{I}/\mathcal{I}$:** $C$ is infinite and all its $\mathcal{F}$-foils are infinite. In this case:

- $C$ is acyclic;
- $\mathcal{F}^\infty(m) \cap C = \emptyset$.

**Proof.** The properties of finite components $C$ are immediate, so only infinite components are considered. Recall that by Lemma 2.1.7 $\mathcal{P}$-a.s. all infinite components are acyclic. Consider the collection $\mathcal{R}$ of all infinite components that have both finite and infinite foils. Suppose $C \in \mathcal{R}$. According to Proposition 2.1.5, all $X \in m$ have only finitely many children, so that if $L$ is an infinite foil, then $L_+$ is also infinite. It follows that there is a maximum finite foil $L$ with respect to the foil order in $C$. Let $n \subseteq \bigcup \mathcal{R}$ be the union of these maximum finite foils of each $C \in \mathcal{R}$. By construction, $n \cap C$ is finite for each $C \in \mathcal{R}$, so Theorem 2.1.6 implies $n = \emptyset$ and hence $\mathcal{R} = \emptyset$, $\mathcal{P}$-a.s. Thus $\mathcal{P}$-a.s. each infinite component is either of class $\mathcal{I}/\mathcal{F}$ or $\mathcal{I}/\mathcal{I}$.

Next, redefine $\mathcal{R}$ to be the set of infinite foils $L$ of $m$, and let $n := \mathcal{F}^\infty(m)$. By construction $n \cap L$ is finite for each $L \in \mathcal{R}$ because a foil cannot have multiple primeval elements. If $X \neq Y \in L$ were both primeval, then with $n$ minimal such that $\mathcal{F}^n(X) = \mathcal{F}^n(Y)$ one finds the primeval element $\mathcal{F}^n(X)$ is the image of two distinct primeval elements $\mathcal{F}^{n-1}(X), \mathcal{F}^{n-1}(Y)$, contradicting injectivity of $\mathcal{F}|_n$ guaranteed by Lemma 2.1.10. Thus Theorem 2.1.6 implies $\mathcal{P}$-a.s. $n \cap L = \emptyset$ for all infinite foils $L$, and hence $\mathcal{P}$-a.s. $\mathcal{F}^\infty(m) \cap C \neq \emptyset$ implies $C$ is of class $\mathcal{I}/\mathcal{F}$ for all components $C$.

Conversely, it will be shown that if $C$ is class $\mathcal{I}/\mathcal{F}$, then $\mathcal{F}^\infty(m) \cap C \neq \emptyset$. Indeed, first consider the collection $\mathcal{R}$ of components $C$ of class $\mathcal{I}/\mathcal{F}$ that have a minimum foil in the foil order. Letting $n \subseteq \bigcup \mathcal{R}$ be the union of minimum foils in $C$, it holds that $n \cap C$ is the (finite) minimum foil in $C$ for each $C \in \mathcal{R}$. Thus Theorem 2.1.6 implies $n = \emptyset$ and hence $\mathcal{R} = \emptyset$, $\mathcal{P}$-a.s. Now consider a $C$ of class $\mathcal{I}/\mathcal{F}$ and an arbitrary foil $L$ of $C$. Since $L$ is finite there is a minimum $n$ such that $\mathcal{F}^n(L)$ is a single point. Let $C_0$ denote the subgraph of $G^L$ of $L$ together with all descendants of elements of $L$ and all forefathers of elements of $L$ up to $\mathcal{F}^n(L)$. Then $C_0$ is an infinite connected graph with vertices of finite degree, and hence it contains an infinite simple path $\{X_i\}_{i \leq 0}$ with $\mathcal{F}(X_i) = X_{i+1}$ for each $i < 0$ by König’s infinity lemma (c.f. Theorem 6 in [6]). For $i > 0$, define $X_i := \mathcal{F}(X_0)$. Then $\{X_i\}_{i \in \mathbb{Z}}$ is a bi-infinite path in $C$ satisfying $\mathcal{F}(X_i) = X_{i+1}$ for all $i \in \mathbb{Z}$, and thus $\{X_i\}_{i \in \mathbb{Z}} \subseteq \mathcal{F}^\infty(m) \cap C$, in particular showing $\mathcal{F}^\infty(m) \cap C \neq \emptyset$. It also holds that $\mathcal{F}^\infty(m) \cap C \subseteq \{X_i\}_{i \in \mathbb{Z}}$ since for any $X \in \mathcal{F}^\infty(m) \cap C$ it is possible to choose $n, m$ such that $\mathcal{F}^n(X) = \mathcal{F}^m(X_0) = X_m$. Uniqueness of primeval children then implies $X = X_{m-n}$. It follows that $\mathcal{F}^\infty(m) \cap C = \{X_i\}_{i \in \mathbb{Z}}$.

Thus it is shown that $\mathcal{P}$-a.s. infinite components $C$ are class $\mathcal{I}/\mathcal{F}$ if $\mathcal{F}^\infty(m) \cap C \neq \emptyset$ and in this case $\mathcal{F}^\infty(m) \cap C$ is a unique bi-infinite sequence $\{X_i\}_{i \in \mathbb{Z}}$ satisfying $\mathcal{F}(X_i) = X_{i+1}$. Since $\mathcal{I}/\mathcal{F}$ and $\mathcal{I}/\mathcal{I}$ are the only possible choices, by process of elimination it follows that $\mathcal{P}$-a.s. infinite components $C$ are of class $\mathcal{I}/\mathcal{I}$ if $\mathcal{F}^\infty(m) \cap C = \emptyset$. \qed
2.2. A Counterexample on a Non-unimodular Group

(Strip Point-shift on \( ax + b \) Group Revisited). This example serves to show that the cardinality classification (Theorem 2.1.11) does not hold for non-unimodular spaces. It is an open question whether a more general classification for such spaces exists. Consider again the strip point-shift of Section 1.3. Suppose that \( 2\delta \gamma < 1 \). It will be shown that \( P \)-a.s. \( \mathcal{S}^n(X) \) eventually becomes constant as \( n \to \infty \) for all \( X \in \mathfrak{m} \). It suffices to show that under \( P^\mathfrak{m} \) it holds that \( \mathcal{S}^n(e) \) eventually becomes constant. Indeed, with \( \mu^{(i)} \) denoting the \( i \)th factorial moment measure of \( \mu \),

\[
E^\mathfrak{m} \sum_{k=0}^{\infty} m(S(\mathcal{S}^k(e)) \setminus \{ \mathcal{S}^k(e) \})
\]

\[
= \sum_{k=0}^{\infty} E^\mathfrak{m} m(S(\mathcal{S}^k(e)) \setminus \{ \mathcal{S}^k(e) \})
\]

\[
\leq \sum_{k=0}^{\infty} E \int_{X_{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} (m^{(1)})^{(k+1)}(dx_1 \times \cdots \times dx_{k+1})
\]

\[
= \sum_{k=0}^{\infty} E \int_{X_{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} m^{(k+1)}(dx_1 \times \cdots \times dx_{k+1})
\]

\[
= \sum_{k=0}^{\infty} \int_{X_{k+1}} 1_{x_1 \in S(e)} \cdots 1_{x_{k+1} \in S(x_k)} \gamma^{k+1} \lambda(dx_{k+1}) \cdots \lambda(dx_1)
\]

\[
= \sum_{k=0}^{\infty} (2\delta)^{k+1} \gamma^{k+1}
\]

\[
< \infty,
\]

where here the Slivnyak-Mecke theorem is used along with the fact that the factorial moment measures of a Poisson point process are just powers of the intensity measure. Thus it must be that \( m(S(\mathcal{S}^k(e)) \setminus \{ \mathcal{S}^k(e) \}) = 0 \) for all \( k \) large, \( P^\mathfrak{m} \)-a.s. That is, there are no points of \( \mathfrak{m} \) in \( S(\mathcal{S}^k(e)) \) besides \( \mathcal{S}^k(e) \) itself. Consequently, \( \mathcal{S}^k(e) \) is a fixed point of \( \mathcal{S} \) and \( \mathcal{S}^k(e) \) is thus eventually constant in \( k \). Equivalently, \( P \)-a.s. for every \( X \in \mathfrak{m} \) it holds that \( \mathcal{S}^k(X) \) is eventually constant in \( k \).

Next it will be shown that every fixed point of \( \mathcal{S} \) is the image of infinitely many \( X \in \mathfrak{m} \). Again it is enough to show under \( P^\mathfrak{m} \) that if \( \mathcal{S}(e) = e \) then \( e \) is the image of infinitely many \( X \in \mathfrak{m} \). This is accomplished by finding a region of points \( (x,y) \in X \) such that

(i) \( (1,0) \in S(x,y) \), and

(ii) \( S(x,y) \cap ([1,\infty) \times \mathbb{R}) \subseteq S(1,0) \),

which implies \( \mathcal{S} \) would map a point of \( \mathfrak{m} \) at \( (x,y) \) to \( (1,0) \). The condition (i) says \( 1 \geq x \) and \( y - \delta x \leq 0 \leq y + \delta x \), i.e. \( -\delta x \leq y \leq \delta x \). Condition (ii) is
guaranteed if \([y - \delta x, y + \delta x] \subseteq [-\delta, \delta]\), i.e., if \(y \geq \delta(x - 1)\) and \(y \leq \delta(1 - x)\). The constraints

\[
0 < x \leq 1, \quad -\delta x \leq y \leq \delta x, \quad y \leq \delta(1 - x), \quad y \geq \delta(x - 1),
\]

bound a parallelogram \(D\) with corners

\[
(0, 0), \quad (1/2, \delta/2), \quad (1, 0), \quad (1/2, -\delta/2).
\]

Then

\[
\mathbb{E}^m[\mu^*(D)] = \gamma \lambda(D) \geq \gamma \int_0^{1/2} \int_{-\delta x}^{\delta x} \frac{1}{x^2} dy \, dx = \gamma \int_0^{1/2} \frac{2\delta}{x} \, dx = \infty
\]

so that the region \(D\) contains infinitely many points of \(m, \mathbb{P}^m\text{-a.s.}\). By construction, if \(\mathcal{F}(e) = e\) then every \(X \in m \cap D\) has \(\mathcal{F}(X) = e\), proving the claim.

Putting previous claims together, it holds that the foils and connected components are identical because every component contains a fixed point, and the foils and components are in bijection with the fixed points of \(\mathcal{F}\). The connected component of a fixed point \(Y\) of \(\mathcal{F}\) is all \(X \in m\) that are eventually sent to \(Y\). Thus all components and foils are infinite (class \(\mathcal{I}/\mathcal{I}\)). However, the components are not acyclic and \(\mathcal{F}^\infty(m) = \{X \in m : \mathcal{F}(X) = X\} \neq \emptyset\), contrary to what the classification theorem would suggest for unimodular \(X\). It follows that the properties of the cardinality classification cannot be extended beyond the case of unimodular \(X\).

### 3. Properties of Point-shifts

#### 3.1. Mecke’s Invariance Theorem

In the case of \(X = \mathbb{R}^d\) Mecke’s invariance theorem shows that Palm probabilities are preserved under bijective point-shifts. It will be shown in Corollary 3.1.1 that if \(X\) is unimodular then this still holds. Even stronger, a point-shift is bijective iff it preserves Palm probabilities. However, for non-unimodular \(X\) this is not so. Precisely, a notion of isomodularity will be defined and it will be proved that amongst bijective point-shifts isomodular ones are exactly those that preserve Palm probabilities (Theorem 3.1.6).

For the rest of the section, fix a stationary framework \((\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{R}}, \mathbb{P})\), a compatible simple point process \(m\) of intensity \(\gamma \in (0, \infty)\), and a point-map \(\mathcal{F}\) with associated point-shift \(\mathcal{F}\). The notation for the corresponding notation \(\tau^\mathcal{F}, h^+, h^-, H^+, H^-, h^+, \mathcal{F}^+\) mentioned in the preliminaries is retained.

The simple case of Mecke’s invariance theorem when \(X\) is unimodular follows.

**Corollary 3.1.1** (Mecke’s Invariance Theorem). *Suppose that \(X\) is unimodular. Then \(\mathcal{F}\) preserves \(\mathbb{P}^m\) iff \(\mathcal{F}\) is bijective. That is, \(\mathbb{P}^m(\theta^{-1}_\mathcal{F} \in A) = \mathbb{P}^m(A)\) for all \(A \in \mathcal{A}\) iff \(\mathcal{F}\) is bijective.*
Proof. Apply Proposition 1.2.5 (f), the test for bijectivity, and use the fact that \( \Delta(x) = 1 \) for all \( x \in X \).

With Mecke’s invariance theorem for unimodular \( X \) in place, one may ask about non-unimodular \( X \). For these \( X \), which bijective point-shifts preserve Palm probabilities? Equation (4) shows that the obstruction is the factor \( \Delta(h^{-1}) \). This motivates the definition of isomodularity, which says that a point-shift preserves the value of \( \Delta(X) \) for each \( X \in m \). Isomodularity is a special case of invariance of a subgroup under \( \mathfrak{H} \), which is defined presently.

**Definition 3.1.2.** A measurable subgroup \( G \in B(\mathbb{X}) \) of \( \mathbb{X} \) is called \( \mathfrak{H} \)-invariant if \( P \)-a.s. \( \mathfrak{H}(X) \) is in the same coset as \( X \) for all \( X \in m \). If \( \{ \Delta = 1 \} \) is \( \mathfrak{H} \)-invariant, i.e. if \( P \)-a.s. \( \Delta(\mathfrak{H}(X)) = \Delta(X) \) for all \( X \in m \), one calls \( \mathfrak{H} \) isomodular.

**Lemma 3.1.3.** If \( X \) is unimodular, then \( \mathfrak{H} \) is isomodular.

Proof. If \( X \) is unimodular, then all point-shifts \( \mathfrak{H} \) are isomodular because \( \Delta(x) = 1 \) for all \( x \in X \).

A brief detour is taken to go through the equivalent descriptions of \( \mathfrak{H} \)-invariance under \( P \) and \( P^m \).

**Proposition 3.1.4.** Let \( G \in B(\mathbb{X}) \) a measurable subgroup of \( \mathbb{X} \), and for each \( x \in \mathbb{X} \) let \( [x] := xG \) denote the coset of \( x \). Then the following are equivalent

(a) \( G \) is \( \mathfrak{H} \)-invariant, i.e. \( P \)-a.s. \([\mathfrak{H}(X)] = [X]\) for all \( X \in m \),

(b) \( P^m \)-a.s. \([h] = [e]\),

and if \( \mathfrak{H} \) is bijective, the previous statements are also equivalent to

(c) \( P \)-a.s. \([\mathfrak{H}^{-1}(X)] = [X]\) for all \( X \in m \),

(d) \( P^m \)-a.s. \([h^{-1}] = [e]\).

Proof.

\((a) \iff (b)\): The equivalence follows from Theorem 1.2.3, so that \( P^m \)-a.s. \([h] = [e]\) is equivalent to \( P \)-a.s. \([\mathfrak{H}(\theta_X^{-1})] = [e]\) for all \( X \in m \), which is the same as \([\mathfrak{H}(X)] = [X]\) after multiplying by \( X \).

\((a) \iff (c)\): Using that \( \mathfrak{H} \) and \( \mathfrak{H}^{-1} \) are inverses, replace \( X \) with \( \mathfrak{H}^{-1}(X) \) in (b) to get (c) or replace \( X \) with \( \mathfrak{H}(X) \) in (c) to get (b).

\((c) \iff (d)\): Same proof as (a) \iff (b).

\( \Box \)

Since isomodularity plays an important role in what follows, the previous result is restated for \( G := \{ \Delta = 1 \} \).

**Corollary 3.1.5.** Let \( \mathfrak{H} \) be bijective, then the following are equivalent

...
(a) $\delta$ is isomodular, i.e. $\mathbb{P}$-a.s. $\Delta(\delta(X)) = \Delta(X)$ for all $X \in \mathfrak{m}$,

(b) $\mathbb{P}_m$-a.s. $\Delta(h) = 1$,

(c) $\mathbb{P}$-a.s. $\Delta(\delta^{-}(X)) = \Delta(X)$ for all $X \in \mathfrak{m}$,

(d) $\mathbb{P}_m$-a.s. $\Delta(h^{-}) = 1$.

Now the question of which bijective point-shifts preserve Palm probabilities is answerable.

**Theorem 3.1.6.** Suppose $\delta$ is bijective. Then $\delta$ preserves $\mathbb{P}_m$ iff $\delta$ is isomodular. That is, $\mathbb{P}_m(\theta^{-}_h \in A) = \mathbb{P}_m(A)$ for all $A \in \mathcal{A}$ iff $\delta$ is isomodular.

**Proof.** Suppose $\delta$ is isomodular. Then $\Delta(h^{-}) = 1$, $\mathbb{P}_m$-a.s. by Corollary 3.1.5. Hence (5) immediately implies $\delta$ preserves $\mathbb{P}_m$. If $\delta$ not isomodular, at least one of $\mathbb{P}_m(\Delta(h^{-}) > 1)$ and $\mathbb{P}_m(\Delta(h^{-}) < 1)$ is strictly positive. The cases are nearly identical, so assume $\mathbb{P}_m(\Delta(h^{-}) > 1) > 0$ and take $A := \{\Delta(h^{-}) > 1\}$. Then take $f := 1_A$ in (5) to find

$$\mathbb{P}_m(\theta^{-}_h \in A) = \mathbb{E}_m\left[\frac{1_{\Delta(h^{-}) > 1}}{\Delta(h^{-})}\right] < \mathbb{E}_m[1_{\Delta(h^{-}) > 1}] = \mathbb{P}_m(A),$$

showing that $\mathbb{P}_m$ is not preserved. \[\square\]

3.2. **Reciprocal and Reverse of a Point-map**

A curious interplay between the reverse $h^{-}$ and the reciprocal $h^{-1}$ of a point-map is investigated, and a characterization of when the two have the same law under $\mathbb{P}_m$ is given.

The notation of the previous section is retained. That is, $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in \mathbb{X}}, \mathbb{P})$ is a stationary framework, $\mathfrak{m}$ is a compatible simple point process of intensity $\gamma \in (0, \infty)$, and $h$ is a point-map with associated point-shift $\delta$. The notation for the corresponding $\tau^h$, $h^+$, $h^-$, $H^+$, $H^-$, $h^-$, $\delta^-$ is also retained. Next follows another result along the lines of Proposition 1.2.5 (f) and (g) which sparks interest in the distributional relationship between $h^{-1}$ and $h^{-}$.  

**Corollary 3.2.1.** Suppose $\delta$ is bijective. For all $f : \mathbb{X} \to \mathbb{R}_+$ measurable it holds that

$$\mathbb{E}_m[f(h^{-1})\Delta(h^{-1})] = \mathbb{E}_m[f(h^{-})], \quad (6)$$

$$\mathbb{E}_m[f(h^{-1})] = \mathbb{E}_m\left[f(\frac{\Delta(h^-)}{h^{-}})\right]. \quad (7)$$

**Proof.** Use the fact that $\mathbb{P}_m$-a.s. $h^{-1}(\theta^{-}_h) = h^{-1} \delta^{-}(\delta(e)) = h^{-1}$ and replace $f$ by $f(h^{-})$ in each of (3) and (5). \[\square\]
One sees in (7) that non-unimodularity of \(X\) is, as usual, an obstruction. Two more results relating the distributions of \(\Delta(h^{-})\) and \(\Delta(h^{-1})\) are given. Then it is shown in Theorem 3.2.4 that amongst bijective point-shifts, the isomodular ones are precisely those for which \(h^{-1}\) and \(h^{-}\) have the same distribution under \(P^m\). Recall that this was also the class of point-shifts that preserve Palm probabilities by Theorem 3.1.6.

**Corollary 3.2.2.** Let \(\mathcal{H}\) be bijective, then for all \(r > 0\) it holds that

\[
r P^m(\Delta(h^{-1}) = r) = P^m(\Delta(h^{-}) = r),
\]

and if this number is strictly positive then for all \(A \in \mathcal{A}\)

\[
P^m(\theta_h^{-1} \in A \mid \Delta(h^{-1}) = r) = P^m(A \mid \Delta(h^{-}) = r).
\]

**Proof.** Fix \(r > 0\) and take \(f(x) := 1_{\Delta(x) = r}\) in (7). One finds

\[
P^m(\Delta(h^{-1}) = r) = \frac{1}{r} P^m(\Delta(h^{-}) = r) =: p
\]

showing the first claim. Supposing that \(p > 0\), take \(f := 1_A 1_{\Delta(h^{-}) = r}\) in (5) and use that \(P^m\)-a.s. \(h^{-1}(\theta_h^{-1}) = h^{-1}\) to find

\[
P^m(\theta_h^{-1} \in A, \Delta(h^{-1}) = r) = \frac{1}{r} P^m(A, \Delta(h^{-}) = r).
\]

Division by \(p\) finishes the proof. \(\square\)

**Lemma 3.2.3.** Let \(\mathcal{H}\) be bijective, then for all \(\alpha \in \mathbb{R}\) and \(0 \leq r \leq s \leq \infty\) it holds that

\[
E^m \left[ \Delta(h^{-1})^\alpha 1_{r \leq \Delta(h^{-1}) \leq s} \right] = E^m \left[ (h^{-})^{\alpha-1} 1_{r \leq \Delta(h^{-}) \leq s} \right].
\]  

**Proof.** Take \(f(x) := \Delta(x)^\alpha 1_{r \leq \Delta(x) \leq s}\) in (7). \(\square\)

**Theorem 3.2.4.** Let \(\mathcal{H}\) be bijective, then \(h^{-1}\) and \(h^{-}\) have the same law under \(P^m\) iff \(\mathcal{H}\) is isomodular.

**Proof.** Suppose \(\mathcal{H}\) is isomodular. Then by Corollary 3.1.5, \(P^m\)-a.s. \(\Delta(h) = \Delta(h^{-}) = 1\) and thus (7) shows that \(h^{-1}\) and \(h^{-}\) have the same law under \(P^m\).

Next suppose that \(h^{-1}\) and \(h^{-}\) have the same law under \(P^m\). Then

\[
E^m[\Delta(h^{-1})^\alpha 1_{r \leq \Delta(h^{-1}) \leq s}] = E^m[\Delta(h^{-})^{\alpha-1} 1_{r \leq \Delta(h^{-}) \leq s}]
\] (9)

for all \(\alpha \in \mathbb{R}\) and all \(0 \leq r \leq s \leq \infty\). But then for all \(\alpha \in \mathbb{R}\) and all \(0 \leq r \leq s \leq \infty\)

\[
E^m[\Delta(h^{-1})^{\alpha+1} 1_{r \leq \Delta(h^{-1}) \leq s}] = E^m[\Delta(h^{-})^{\alpha} 1_{r \leq \Delta(h^{-}) \leq s}] = E^m[\Delta(h^{-})^{\alpha} 1_{r \leq \Delta(h^{-}) \leq s}] = E^m[\Delta(h^{-})^{\alpha-1} 1_{r \leq \Delta(h^{-}) \leq s}].
\] (by (8))
Taking $\alpha := 1, r := 1, s := \infty$

\[ E^m[\Delta(h^{-1})^21_{\Delta(h^{-1})}] = E^m[\Delta(h^{-1})1_{\Delta(h^{-1})}] \]

which is absurd unless $\Delta(h^{-1}) \leq 1$, $P^m$-a.s. It also holds that with $\alpha := 1, r := 0, s := 1$,

\[ E^m[\Delta(h^{-1})^21_{\Delta(h^{-1})}] = E^m[\Delta(h^{-1})1_{\Delta(h^{-1})}] \]

which is absurd unless $\Delta(h^{-1}) \geq 1$, $P^m$-a.s. It follows that $\Delta(h^{-1}) = 1$, $P^m$-a.s.

By Corollary 3.1.5 the result follows. \qed

3.3. Separating Points of a Point Process

In this section a notion of a function separating points of a point process is introduced. As always, $m$ is a simple and compatible point process of intensity $\gamma \in (0, \infty)$ on a stationary framework $(\Omega, \mathcal{A}, \{\theta_x\}_{x \in X}, P)$.

**Definition 3.3.1.** Let $S$ be a set, $f : X \to S$, and suppose that $P$-a.s. no distinct $X,Y \in m$ have $f(X) = f(Y)$. Then say that $f$ separates points of $m$. Similarly, say that a fixed partition $\{B_i\}_{i \in J}$ of $X$ separates points of $m$ if $P$-a.s. no $B_i$ contains more than 1 point of $m$.

When separation of points occurs is studied by proving a general result concerning when there cannot be an $n$-tuple of distinct points of $m$ satisfying a given property. Recall that the $n$-th factorial moment measure of a counting measure $\mu$ with representation $\mu = \sum_i \delta_{x_i}$ is defined as $\mu^{(n)} := \sum_{i_1 \neq \cdots \neq i_n} \delta_{(x_{i_1}, \ldots, x_{i_n})}$, where the notation $i_1 \neq \cdots \neq i_n$ means that $i_1, \ldots, i_n$ are all distinct.

**Theorem 3.3.2.** Let $(S, \Sigma)$ be a measurable space and fix $M \in \Sigma$. Let $F : X \times X^n \to S$ be measurable, and suppose that for $E^m[(m^n)^{(n)}]$-a.e. $y \in X^n$,

\[ \lambda(x \in X : F(x, xy) \in M) = 0. \]

Then $P$-a.s. no $n + 1$ distinct $X,Y_1, \ldots, Y_n \in m$ have $F(X, Y_1, \ldots, Y_n) \in M$. 

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Proof. By straight calculations,

\[ P(\exists X \in \mathfrak{m}, Y \in \mathfrak{m}^{(n)} : (X, Y) \in \mathfrak{m}^{(n+1)}, F(X, Y) \in M) \leq E \int_X 1_{\exists Y \in \mathfrak{m}^{(n)} : \forall i, y_i \neq x, F(x, y) \in M} m(dx) \]

\[ \leq E \int_X m^{(n)}(\theta_c, \{ y \in \mathfrak{X}^n : \forall i, y_i \neq x, F(x, y) \in M \}) m(dx) \]

\[ = \gamma E^m \int_X m^{(n)}(\theta_c, \{ y \in \mathfrak{X}^n, \forall i, y_i \neq x, F(x, y) \in M \}) \lambda(dx) \]

\[ = \gamma E^m \int_X m^{(n)}(\theta_c, \{ x^{-1}y : y \in \mathfrak{X}^n, \forall i, y_i \neq x, F(x, y) \in M \}) \lambda(dx) \]

\[ = \gamma E^m \int_X \left( \int_{X \times X} 1_{F(x, y) \in M} (m^1)^{(n)}(dy) \lambda(dx) \right) \]

where in the third equality the C-L-M-M theorem is used. This proves the claim. \( \square \)

Theorem 3.3.2 immediately gives a condition for separating points of \( \mathfrak{m} \).

**Corollary 3.3.3 (Condition for Separating Points).** Let \((S, \Sigma)\) be a measurable space, \( f : \mathfrak{X} \to S \) measurable, and suppose for all \( y \neq e \), or more generally for \( E^m[m^1] \)-a.e. \( y \in \mathfrak{X} \),

\[ \lambda(x \in \mathfrak{X} : f(x) = f(xy)) = 0. \]

Then \( f \) separates points of \( \mathfrak{m} \). Implicit in the previous line is the assumption that the sets \( \{ x \in \mathfrak{X} : f(x) = f(xy) \} \) are measurable for all \( y \in \mathfrak{X} \). This is automatic if \((S, \Sigma)\) is a standard measurable space, or more generally if \( S \times S \) has measurable diagonal.

**Proof.** Take \( n := 1 \), \( F(x, y) := (f(x), f(y)) \) for all \( x, y \in \mathfrak{X} \), and take \( M \) to be the diagonal of \( S \times S \), then apply Theorem 3.3.2. \( \square \)

**Note 3.3.4.** Corollary 3.3.3 generalizes the well-known theorem in \( \mathfrak{X} := \mathbb{R}^d \) that a stationary point process has not two points equidistant from 0. That would be the case of \( f(x) := |x| \). Not all \( \mathfrak{X} \) have this property though. Indeed, if \( \mathfrak{X} \) is a countable group with the discrete distance \( d(x, y) := 1_{x \neq y} \), then \( \lambda(x \in \mathfrak{X} : d(x, e) = d(xy, e)) > 0 \) for all \( y \neq e \) so the result does not apply if \( \mathfrak{X} \) has more than one element.

**Proposition 3.3.5.** Let \( B \in \mathcal{B}(\mathfrak{X}) \) with \( e \in B \). If \( E^m[m^1(B)] = 0 \), then \( P \)-a.s. for all \( X \in \mathfrak{m} \) it holds that \( m(XB) = 1 \), i.e. \( X \) is the unique point of \( \mathfrak{m} \) inside \( XB \).
Proof. The hypotheses imply $P^m$-a.s. $m(B \setminus \{e\}) = 0$. By Theorem 1.2.3, $P$-a.s. all $X \in m$ are such that $\theta_X^{-1}m(B \setminus \{e\}) = 0$, i.e. $m(XB \setminus \{X\}) = 0$, and hence $m(XB) = 1$. 

Corollary 3.3.6. Let $G \in B(\mathbb{X})$ a subgroup of $\mathbb{X}$. If $E^m[m^1(G)] = 0$, then the cosets of $G$ separate points of $m$. 

Corollary 3.3.7. Let $G \in B(\mathbb{X})$ a subgroup of $\mathbb{X}$. If $E^m[m^1(G)] = 0$ but $G$ is $\mathfrak{H}$-invariant for some point-shift $\mathfrak{H}$, then $\mathfrak{H}$ is the identity point-shift $P$-a.s. 

Proof. $G$ being $\mathfrak{H}$-invariant means $\mathfrak{H}(X)$ and $X$ are in the same coset for $X \in m$, then by Corollary 3.3.6 $\mathfrak{H}$ is the identity point-shift. 

Corollary 3.3.8. Let $G \in B(\mathbb{X})$ a subgroup of $\mathbb{X}$. If $\lambda(G) = 0$ and $m$ is Poisson with intensity $\gamma \in (0, \infty)$, then the only $\mathfrak{H}$ for which $G$ is $\mathfrak{H}$-invariant is the identity. 

Proof. Theorem 4.2.7 implies that $E^m[m^1] = \gamma \lambda$ so that $E^m[m^1(G)] = \gamma \lambda(G) = 0$ and Corollary 3.3.7 applies. 

4. Concluding Remarks and Further Research

4.1. Mecke’s Invariance Theorem

In Mecke’s invariance theorem for unimodular spaces (Corollary 3.1.1), not only does bijectivity imply $\mathfrak{H}$ preserves Palm probabilities, but the converse holds as well. However, for the non-unimodular case in Theorem 3.1.6, bijectivity is required as an assumption. Are there non-bijective point-shifts that preserve Palm probabilities in non-unimodular spaces?

4.2. Point-shift Foliations

Point-shift foliations are in many ways special cases of vertex-shifts of random networks [3]. A cardinality classification is proved in [3] for unimodular random networks, and questions remain about the relationship of these two frameworks. It is shown in [3] that stationary point processes on $\mathbb{R}^d$ are embeddings of random networks. Talks with the author of [3] suggest that a certain class of unimodular networks, called Eternal Family Trees, can be viewed as stationary point processes on $\mathbb{R}^d$. Two open questions are whether stationary point processes on unimodular $\mathbb{X}$ can be viewed as unimodular random networks, and whether any unimodular random network can be viewed as a stationary point process on some unimodular $\mathbb{X}$.

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Appendix: Palm Calculus

**Theorem 4.2.1** (Inversion Formula). [7] There exists a bounded measurable $K : \Omega \times X \to \mathbb{R}$ with $K \geq 0$ such that

$$\int_X K(\theta_x, x) \, m(dx) = 1_{m \neq 0}, \quad (10)$$

and for all $K \geq 0$ (not necessarily bounded) $P$-a.s. satisfying (10), it holds that

$$E[1_{m \neq 0} f] = \gamma E^m \int_X f(\theta_x) K(\theta_x, x) \lambda(dx) \quad (11)$$

for all measurable $f : \Omega \to \mathbb{R}_+$.

**Proposition 4.2.2.** If $A \in \mathcal{A}$ is shift-invariant in the sense that $A = \theta_x^{-1} A$ for all $x \in X$, then

$$P(A) = 1 \iff P^m(A) = 1 \iff P(A \mid m \neq 0) = 1.$$ 

In particular, if $\{m = 0\} \subseteq A$ then

$$P(A) = 1 \iff P^m(A) = 1.$$ 

**Proof.** Suppose $P(A) = 1$. From the definition of Palm, for $0 < \lambda(B) < \infty$

$$P^m(A) = \frac{1}{\gamma \lambda(B)} E \int_X 1_{x \in B} 1_{\theta^{-1}_x \in A} m(dx)$$

$$\quad = \frac{1}{\gamma \lambda(B)} E \int_X 1_{x \in B} 1_A m(dx) \quad \text{(shift-invariance of } A)$$

$$\quad = \frac{1}{\gamma \lambda(B)} E[1_A m(B)]$$

$$\quad = \frac{1}{\gamma \lambda(B)} E[m(B)] \quad \text{(} P(A) = 1 \text{)}$$

$$\quad = 1.$$ 

Next suppose $P^m(A) = 1$. Then from Theorem 4.2.1 there is measurable $K : \Omega \times X \to \mathbb{R}$ such that

$$P(A \cap \{m \neq 0\}) = E[1_{m \neq 0} | A]$$

$$\quad = \gamma E^m \int_X 1_{\theta_x \in A} K(\theta_x, x) \lambda(dx) \quad \text{(inversion formula)}$$

$$\quad = \gamma E^m \left[ 1_A \int_X K(\theta_x, x) \lambda(dx) \right] \quad \text{(shift-invariance of } A)$$

$$\quad = \gamma E^m \left[ \int_X K(\theta_x, x) \lambda(dx) \right] \quad \text{(} P^m(A) = 1 \text{)}$$

$$\quad = E[1_{m \neq 0} \cdot 1] \quad \text{(inversion formula)}$$

$$\quad = P(m \neq 0).$$
Dividing by $P(m \neq 0)$ gives $P(A \mid m \neq 0) = 1$, and if $\{m = 0\} \subseteq A$, then
$$P(A) = P(A \cap \{m \neq 0\}) + P(A \cap \{m = 0\}) = P(m \neq 0) + P(m = 0) = 1.$$ 

\[\square\]

**Corollary 4.2.3.** $(m, P)$ is a (simple) point process iff $(m, P^m)$ is a (simple) point process.

**Proof.** The event that $m(B) \in \mathbb{N} \cup \{\infty\}$ for each $B \in \mathcal{B}(\mathbb{X})$ and the event that $m(\{x\}) \leq 1$ for all $x \in \mathbb{X}$ are shift-invariant, so Proposition 4.2.2 applies. Equivalence follows because 0 is a (simple) counting measure. \[\square\]

**Lemma 4.2.4.** Let $A \in \mathcal{A}$. Then
$$P^m(A) = 1 \iff P(m(x \in \mathbb{X} : \theta^{-1}_x \in A) = 0) = 1.$$ 

**Proof.** By replacing $A$ with its complement it is equivalent to show $P^m(A) = 0$ iff $P(m(x \in \mathbb{X} : \theta^{-1}_x \in A) > 0) = 0$. Note that it is the joint measurability of the action $(\omega, x) \mapsto \theta_x \omega$ that lets one conclude for $B \in \mathcal{B}(\mathbb{X})$ that sets like
$$\{m(x \in \mathbb{X} : x \in B, \theta^{-1}_x \in A) > 0\}$$
are measurable.

If $P^m(A) = 0$, from the definition of Palm, for $0 < \lambda(B) < \infty$,

$$0 = P^m(A)$$
$$= \frac{1}{\gamma \lambda(B)} \mathbb{E} \int_{\mathbb{X}} 1_{x \in B} 1_{\theta^{-1}_x \in A} m(dx)$$
$$= \frac{1}{\gamma \lambda(B)} \mathbb{E}[m(x \in \mathbb{X} : x \in B, \theta^{-1}_x \in A)].$$

Thus $\mathbb{E}[m(x \in \mathbb{X} : x \in B, \theta^{-1}_x \in A)] = 0$ and taking relatively compact $B$ increasing to $\mathbb{X}$ one finds $\mathbb{E}[m(x \in \mathbb{X} : \theta^{-1}_x \in A)] = 0$, so $P(m(x \in \mathbb{X} : \theta^{-1}_x \in A) > 0) = 0$.

Conversely, suppose $P(m(x \in \mathbb{X} : \theta^{-1}_x \in A) > 0) = 0$. Then for $B \in \mathcal{B}(\mathbb{X})$ with $0 < \lambda(B) < \infty$,

$$P^m(A) = \frac{1}{\gamma \lambda(B)} \mathbb{E} \int_{\mathbb{X}} 1_{x \in B} 1_{\theta^{-1}_x \in A} m(dx)$$
$$= \frac{1}{\gamma \lambda(B)} \mathbb{E}[m(x \in \mathbb{X} : x \in B, \theta^{-1}_x \in A)]$$
$$\leq \frac{1}{\gamma \lambda(B)} \mathbb{E}[m(x \in \mathbb{X} : \theta^{-1}_x \in A)]$$
$$= 0,$$
completing the proof. \[\square\]
Theorem 4.2.5. Let $A \in \mathcal{A}$. Then the following are equivalent:

(a) $\mathbb{P}^m(A) = 1$,
(b) $\mathbb{P}(m(x \in X : \theta_x^{-1} \not\in A) = 0) = 1$,
(c) $\mathbb{P}^m(m(x \in X : \theta_x^{-1} \not\in A) = 0) = 1$.

Proof.

(a) $\iff$ (b): This is the content of Lemma 4.2.4.

(b) $\iff$ (c): This follows from Proposition 4.2.2 and the fact that the event $\{m(x \in X : \theta_x^{-1} \not\in A) = 0\}$ contains $\{m = 0\}$ and is shift-invariant: for all $y \in X$,

$$\theta^{-1}_y \omega \in \{m(x \in X : \theta_x^{-1} \not\in A) = 0\}$$

$$\iff m(\theta^{-1}_y \omega, \{x \in X : \theta_x^{-1} \not\theta^{-1}_y \omega \not\in A\}) = 0$$

$$\iff m(\omega, \{y x : x \in X, \theta_y^{-1} x \not\in A\}) = 0$$

$$\iff \omega \in \{m(x \in X, \theta_x^{-1} \not\in A) = 0\}.$$ 

Example 4.2.6. Fix some measurable space $(S, \Sigma)$ and a measurable $f : \Omega \to S$. Define $F : \Omega \times X \to S$ by $F(\omega, x) := f(\theta_x^{-1} \omega)$ for all $\omega \in \Omega, X \in m(\omega)$, and $F(\omega, x)$ may be defined arbitrarily otherwise. It will be shown that knowing $F$ up to a $\mathbb{P}$- or $\mathbb{P}^m$-null set on the support of $m$ is equivalent to knowing $f$ up to a $\mathbb{P}^m$-null set. Indeed, suppose $f = f'$, $\mathbb{P}^m$-a.s., then it will be shown that the corresponding $F, F'$ agree $\mathbb{P}, \mathbb{P}^m$-a.s. on the support of $m$. By Theorem 1.2.3, $\mathbb{P}$- and $\mathbb{P}^m$-a.e. $\omega \in \Omega$ has for all $X \in m(\omega)$ that $f(\theta_X^{-1} \omega) = f'(\theta_X^{-1} \omega)$, i.e. $F(\omega, X) = F'(\omega, X)$. Similarly, if either $\mathbb{P}$-a.e. or $\mathbb{P}^m$-a.e. $\omega \in \Omega$ is such that $F(\omega, X) = F'(\omega, X)$ for all $X \in m(\omega)$, then

$$f(\theta_X^{-1} \omega) = F(\omega, X)$$

$$= F'(\omega, X)$$

$$= f'(\theta_X^{-1} \omega),$$

for $\mathbb{P}$-a.e. or $\mathbb{P}^m$-a.e. $\omega \in \Omega, X \in m(\omega)$, so by Theorem 1.2.3 one finds that $f = f'$, $\mathbb{P}^m$-a.s. Thus, $f$ may be defined under $\mathbb{P}^m$ or $F$ may be defined under $\mathbb{P}$ or $\mathbb{P}^m$, whichever is more convenient.

Theorem 4.2.7 (Slivnyak–Mecke Theorem). Let $m$ be a stationary random measure with intensity $\gamma \in (0, \infty)$ on $X$. Then the distribution of $m$ under $\mathbb{P}^m$ is the same as the distribution of $m + \delta_x$ under $\mathbb{P}$ if $m$ is a homogeneous Poisson point process with intensity $\gamma$ under $\mathbb{P}$.
References

[1] F. Baccelli and M.-O. Haji-Mirsadeghi. Compactification of the action of a point-map on the palm probability of a point process. arXiv preprint arXiv:1312.0287, 2013.

[2] F. Baccelli and M.-O. Haji-Mirsadeghi. Point-shift foliation of a point process. arXiv preprint arXiv:1601.03653, 2016.

[3] F. Baccelli, M.-O. Haji-Mirsadeghi, and A. Khezeli. Dynamics on unimodular random graphs. arXiv preprint arXiv:1608.05940, 2016.

[4] D. J. Daley and D. Vere-Jones. An introduction to the theory of point processes: volume II: general theory and structure. Springer Science & Business Media, 2007.

[5] E. Hewitt and K. A. Ross. Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations, volume 115. Springer Science & Business Media, 2012.

[6] D. König. Theory of finite and infinite graphs. In Theory of Finite and Infinite Graphs, pages 45–421. Springer, 1990.

[7] G. Last. Modern Random Measures: Palm Theory and Related Models. Univ. Karlsruhe, Fak. für Mathematik, 2008.

[8] G. Last. Stationary random measures on homogeneous spaces. Journal of Theoretical Probability, 23(2):478–497, 2010.