AUTOMORPHIC-TWISTED SUMMATION FORMULAE FOR PAIRS OF QUADRATIC SPACES

Abstract. Motivated by the conjectures of Braverman-Kazhdan, Lafforgue, Ngô and Sakellaridis, we prove a summation formula for certain spaces of test functions on the zero locus of a quadratic form. The functions are built from the Whittaker coefficients of automorphic representations on $GL_n$. We also give an expression of the local factors where all the data are unramified.

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2020 Mathematics Subject Classification. Primary: 11F70, Secondary: 11F66.

Key words and phrases. Poisson summation conjecture, Eisenstein series, theta series.
1. Introduction

In this paper, we prove a summation formula for a family of test functions on the zero locus of a quadratic form. The test functions are built out of Whittaker coefficients of an automorphic representation of $GL_n(\mathbb{A}_F)$. We first put our work in context and then state our results precisely.

1.1. Generalized Poisson summation formulae for spherical varieties. Conjectures of Braverman-Kazhdan [BK00], Lafforgue [Laf14], Ngô [Ngô14, Ngô20] and Sakellaridis [Sak12] suggest that every affine spherical variety admits a generalized Poisson summation formula. We refer to this conjecture as the Poisson summation conjecture. The Poisson summation conjecture implies the functional equation and meromorphic continuation for fairly general Langlands $L$-functions, which by the converse theorem, implies Langlands functoriality in great generality.

In [GL19], such a summation formula is proved where the underlying scheme is built out of a triple of quadratic spaces. This setting is of particular interest because it is the first case in which the Poisson summation conjecture is known where the underlying affine spherical variety is not a torus bundle over a flag variety. Their method of proof involves replacing the cuspidal representation of $SL_2^3(\mathbb{A}_F)$ appearing in Garrett’s integral representation of the Rankin triple product $L$-function [Gar87, PSR87] with a $\theta$-function.

This suggests that new summation formulae can be obtained by replacing cusp forms on symplectic groups appearing in known integral representations with restrictions of $\theta$-functions on metaplectic groups. In this paper, we take another step towards this general program. In more detail, we use the exceptional isogeny $SL_2 \times SL_2 \to SO_4$ to substitute two $\theta$-functions into the Rankin-Selberg integral for $SO_{2\ell} \times GL_n$ constructed in [Kap12] in the special case $\ell = 2$. In the next subsection, we state our formula precisely and then give a representation-theoretic interpretation.

Remark. More generally, one may consider substituting restrictions of minimal representations (in the sense of [GS05]) into integral representations of $L$-functions.

The Rankin-Selberg integral in [Kap12] represents a Langlands-Shahidi $L$-function, and it is illuminating to consider our procedure from the point of view of the Langlands-Shahidi method. It is well-known that Langlands-Shahidi $L$-functions can be roughly enumerated by root systems together with a simple root. The Dynkin diagram that remains after deleting the simple root is the Dynkin diagram of a Levi subgroup. Our construction corresponds to the Dynkin diagram $D_{n+2}$ with the unique simple root such that the complement of the root is the Dynkin diagram for $SL_2 \times SL_2 \times SL_n$.

The summation formula we prove in this paper enlarges the collection of cases in which we know the Poisson summation conjecture. At present the set of cases is very small ([BK02, GL19, GH20]). This provides crucial test cases to examine for insight into the general
picture. Moreover, since the ultimate goal is to study higher rank automorphic \(L\)-functions that are currently not understood, incorporating a cusp form of arbitrarily high rank from the outset is a step in the right direction. We also point out that there are methods of building up new summation formulae from old ones modeled on the manner that the summation formula for the basic affine space of [BK99] is built up out of a family of Poisson summation formulae for a two-dimensional symplectic vector space. Thus it is of interest to have various families of summation formulae to serve as building blocks.

1.2. Main Theorem. Let us make the summation formula precise. Let \(F\) be a number field and \(\mathbb{A}_F\) be its Adele ring. Let \(d_1, d_2\) be two even positive integers, and \(V_1 = \mathbb{G}_a^{d_1}, V_2 = \mathbb{G}_a^{d_2}\) be a pair of affine spaces over \(F\) equipped with non-degenerate quadratic forms \(Q\) and \(Q'\) respectively. Let \(V := V_1 \oplus V_2\).

Let \(Y \subset V\) be the closed subscheme whose points in an \(F\)-algebra \(R\) are
\[
Y(R) := \{y = (y_1, y_2) \in V(R) : Q(y_1) = 2Q'(y_2)\}.
\]
Below we will use \(R\) to denote a “test” \(F\)-algebra, sometimes without further comment.

Let
\[
V' := \{\gamma = (\gamma_1, \gamma_2) \in V(R) : \gamma_i \neq 0\}.
\]
We let \(\mathbb{P}Y' \subseteq \mathbb{P}V\) be the corresponding quasi-projective scheme. This is the scheme attached to the pair of quadratic spaces mentioned above.

Our summation formula will involve functions on \(Y\) twisted by Whittaker functions attached to a higher rank cuspidal automorphic representation. Let \(n\) be a positive integer. Let \(\tau\) be a cuspidal automorphic representation of \(\text{GL}_n(\mathbb{A}_F)\). Let \(H\) be the split orthogonal group \(\text{SO}_{2n+1}\). Let
\[
G(R) := \{g = (g_1, g_2) \in \text{GL}_2^2(R) : \det g_1 = \det g_2^{-1}\},
\]
and let \(\xi_s\) be a smooth holomorphic section from the space
\[
\text{Ind}_{Q_n(\mathbb{A})}^{H(\mathbb{A})}(\tau \otimes |\det|^{s-\frac{1}{2}}).
\]
Here \(Q_n \leq H\) is a parabolic subgroup with Levi \(\text{GL}_n\). In Section 3, following [Sou93, Kap12], we construct a family of inductions of Whittaker functions (lying in \(\text{Ind}_{Q_n}^{H}(\mathcal{W}_{\tau,s})\), where \(\mathcal{W}_{\tau,s} = \mathcal{W}_{\tau} \otimes |\det|^{s-\frac{1}{2}}\) and \(\mathcal{W}_{\tau}\) is the Whittaker model for \(\tau\))
\[
H(\mathbb{A}_F) \times \mathbb{C} \longrightarrow \mathbb{C}
\]
\[(h, s) \longmapsto W_{\xi_s}(h, 1)\].
Here \(W\) is an indication that this is a Whittaker function on \(\text{GL}_n(\mathbb{A}_F)\) for \(\tau\) when restricted to an appropriate Levi subgroup of \(H\).

**Remark.** The spaces \(V, V_1, V_2\) have no relationship with the groups \(G, H\) and the split \(2n + 1\)-dimensional space \(H\) is acting on. We merely require that \(n \geq \ell = 2\).
We extend the Weil representation of $\text{SL}_2(\mathbb{A}_F)$ on $\mathcal{S}(V(\mathbb{A}_F)) = \mathcal{S}(V_1(\mathbb{A}_F) \times V_2(\mathbb{A}_F))$ to a representation $\rho$ of $G(\mathbb{A}_F)$ on $\mathcal{S}(V_1(\mathbb{A}_F) \times V_2(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ in Section 3 via a standard procedure.

We then define for $y \in Y'(\mathbb{A}_F)$ the global integral
\[(1.2.2) \quad I(f, W_{\xi_s})(y) = \int_{U_2(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} \rho(g) f(y, 1) \int_{N^0(\mathbb{A}_F)} W_{\xi_s}(wut(g), a_y) \psi_1(u) \, du \, dg.
\]
Here
\[(1.2.3) \quad U_2 \subset G
\]
is a maximal unipotent subgroup, $N^0$ is a certain unipotent subgroup of $H$ (see (2.1.3)), and $\iota$ is an embedding map from $G$ to $H$ (see (2.2.2)). Also,
\[(1.2.4) \quad a_y = \left( -4Q'(y_2) I_{n-1} \right) \in \text{GL}_n(\mathbb{A}_F)
\]
encodes the value of the quadratic form. We point out that the integral $I(f, W_{\xi})(y)$ is only well-defined for $y \in Y'(\mathbb{A}_F)$, not for $y \in V(\mathbb{A}_F)$, due to the invariance properties of the Weil representation.

Thus we have a map
\[(1.2.5) \quad \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times) \otimes \text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau,s}) \longrightarrow C^\infty(\mathcal{Y}'(\mathbb{A}_F)).
\]
We define $\mathcal{S}(Y(\mathbb{A}_F), \tau, s)$ to be the image of (1.2.5). We view this as a Schwartz space of functions on $\mathcal{Y}'(\mathbb{A}_F)$ twisted by $\tau \otimes | \det |^{s-\frac{1}{2}}$. One has a restricted direct product decomposition
\[(1.2.6) \quad \mathcal{S}(V(\mathbb{A}_F)) \otimes \text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau,s}) = \otimes_v \mathcal{S}(V(F_v)) \otimes \text{Ind}_{Q_n(F_v)}^H(\mathcal{W}_{\tau_v,s}).
\]
Here the restricted direct product is with respect to the basic vectors $\mathbb{1}_{V(\mathcal{O})} \otimes W_{\rho_{\tau,s}}$, where $\mathcal{O}$ denotes the ring of integers of $F$, and $W_{\rho_{\tau,s}}$ is the unique normalized spherical vector in $\text{Ind}_{Q_n(F_v)}^H(\mathcal{W}_{\tau_v,s})$ at unramified places. Thus one has a restricted direct product decomposition
\[
\mathcal{S}(Y'(\mathbb{A}_F), \tau, s) = \otimes_v \mathcal{S}(Y'(F_v), \tau_v, s)
\]
with respect to the vectors $\mathbb{1}_{V(\mathcal{O})} \otimes W_{\rho_{\tau,s}}$ at unramified places. We refer to $I(\mathbb{1}_{V(\mathcal{O})} \otimes O^\times, W_{\rho_{\tau,s}})$ as the basic function. We give an expression of it in Theorem 1.2 below.

The space $\mathcal{S}(Y'(\mathbb{A}_F), \tau, s)$ comes equipped with a correspondence on the last horizontal line of the following diagram:
\[
\begin{array}{ccc}
\mathcal{S}(V(\mathbb{A}_F)) \otimes \text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau,s}) & \xrightarrow{M(\tau,s)} & \mathcal{S}(V(\mathbb{A}_F)) \otimes \text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau^\vee,1-s}) \\
\downarrow I & & \downarrow I \\
\mathcal{S}(Y'(\mathbb{A}_F), \tau, s) & \xrightarrow{\text{M}^\vee} & \mathcal{S}(Y'(\mathbb{A}_F), \tau^\vee, 1-s)
\end{array}
\]
Here $M(\tau, s)$ is the usual intertwining operator from $\text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau,s})$ to $\text{Ind}_{Q_n(\mathbb{A}_F)}^H(\mathcal{W}_{\tau^\vee,1-s})$.

Let
\[(1.2.7) \quad V''(R) = \{ (\gamma_1, \gamma_2) \in V(R) : Q(\gamma_1) = Q'(\gamma_2) = 0 \}.
\]
Our summation formula follows:
Theorem 1.1. For \( g \in \text{O}(V_1) \times \text{O}(V_2) \), the sum \( \sum_{y \in \mathcal{Y}^*(F)} I(f, W_{\xi_e})(gy) \) admits a meromorphic continuation to the whole s-plane. It satisfies a functional equation

\[
\sum_{y \in \mathcal{Y}^*(F)} I(f, W_{\xi_e})(gy) = \sum_{y \in \mathcal{Y}^*(F)} I(f, M(\tau, s)W_{\xi_e})(gy).
\]

Here \( f(y, 1) \in \mathcal{S}(\text{V}(A_F) \times A_F^n) \) is a Schwartz-Bruhat function such that \( \rho(g)f(\gamma, u) = 0 \) for all \( (g, \gamma, u) \in G(F) \times \mathcal{V}^*(F) \times F^\times \).

Theorem 1.1 has a different form than predicted by the papers mentioned at the beginning of the paper. It seems reasonable to expect that there exists a spherical variety \( Z \) for \( \text{O}(V_1) \times \text{O}(V_2) \times \text{GL}_n \) equipped with a Schwartz space \( \mathcal{S}(Z(A_F)) \) with Fourier transform \( \mathcal{F} \) and Poisson summation formula

\[
\sum_{z \in Z(F)} f(z) = \sum_{z \in Z(F)} \mathcal{F}(f)(z),
\]

such that

\[
\int_{[\text{GL}_n]} \xi_s(1, g') \sum_{z \in Z(F)} f(z(g, g')) dg' = \sum_{y \in \mathcal{Y}^*(F)} I(f, W_{\xi_e})(gy)
\]

and

\[
\int_{[\text{GL}_n]} \xi_s(1, g') \sum_{z \in Z(F)} \mathcal{F}(f)(z(g, g')) dg' = \sum_{y \in \mathcal{Y}^*(F)} I(f, M(\tau, s)W_{\xi_e})(gy)
\]

for \( g \in \text{O}(V_1) \times \text{O}(V_2) \) and \( g' \in \text{GL}_n(A_F) \).

The integral \( I(f, W_{\xi_e})(y) \) mixes the arithmetic of the quadratic forms \( \mathcal{Q} \) and \( \mathcal{Q}' \) and the cuspidal automorphic representation \( \tau \). It is Eulerian for each \( y \) (see the discussion around (3.0.8)). Ideally, one would like an expression for the unramified local factors in terms of a suitable local model for \( \tau \) and the point \( y \). We achieve this in Theorem 1.2 below. This is far more difficult than the corresponding calculation in [GL19]. To execute it, we adapt an argument appearing in [Kap12], which ultimately relates the integral to the Bessel model of \( \text{Ind}^H_{Q_k}(W_{\tau, s}) \) attached to a character on a unipotent subgroup of \( H \) and a character on \( SO_2 \).

Theorem 1.2. For all the data unramified, \( \Re(s) \) large, and \( d_2 > d_1 \), we have

\[
I(\mathbb{I}_{V(\mathcal{O})} \times \mathcal{O}_s, W_{\rho_{r,s}})(y) = \alpha(y_1, y_2)|4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{d_2}{2}} \sum_{k=\text{val}(4\mathcal{Q}'(y_2))} q^{(n-2\frac{d_1}{2})k} C_{k,s}(y).
\]

Here \( \alpha(y_1, y_2) \) is as defined in Eq. (7.0.3), and

\[
C_{k,s}(y) = \int_{iF+\sigma_1} \int_{iF+\sigma_2} \left( 1 - q^{s_2} + (q - 1)q^{-\text{val}(y_2)s_2} \right) \zeta_v(-s_2 - d_1/2 + 2) \zeta_v(-s_2)^2 (\log q)^2
\]

\[
\times q^{-(s_1+s_2)k} B_{s_1, s_1} \left( -4\mathcal{Q}'(y_2)^{\omega^k} I_{2n-1}(-4\mathcal{Q}'(y_2)^{\omega^k})^{-1} \right) ds_1 ds_2,
\]

where \( \chi' \otimes \tau \) are characters of \( \text{GL}_2 \), \( \zeta_v \) is the Dedekind zeta function, \( \text{val} \) is the valuation, \( \mathcal{Q}' \) is the quadratic form, and \( \omega^k \) are certain powers of \( \omega \).

Theorem 1.2 provides a powerful tool for studying the analytic properties of automorphic forms associated with quadratic spaces. It generalizes and extends the results of [GL19] and [Kap12], offering new insights into the behavior of these forms in the unramified setting.
which is the product of the \(-k\)-th coefficient in \(q^{s_1}\) and the \(k\)-th coefficient in \(q^{s_2}\) of a product of Laurent series in \(q^{s_1}\) and \(q^{s_2}\), where \(\gamma\) represents gamma factor for \(GL_1 \times GL_n\), and \(B_{\psi_1,s_1}\) is the normalized unramified Bessel function defined in 7.0.11.

**Remarks.**

- Let \(b_Z\) be the basic function at unramified places in the conjectural Schwartz space \(S(Z(\mathbb{A}_F))\). Then the quantity computed in Theorem 1.2 should correspond to

\[
\int_{GL_n(F)} b_Z(z,g')\xi_s(1,g')dg.
\]

- As mentioned before, the key difference between the results in this paper with the work of Kaplan and Soudry is that we substitute the theta functions on \(G(\mathbb{A})\) for cusp forms in their work. In the unramified setting, we must relate \(\rho(g)f(y,1)\) to functions lying in an appropriate Whittaker model in order to apply Kaplan and Soudry’s methods.

Let us indicate how our constructions are related to the \(\theta\)-correspondence and Rankin-Selberg \(L\)-functions. Let \(\pi_i\) be a cuspidal automorphic representation of \(O(V_i)(F)\backslash O(V_i)(\mathbb{A}_F)\) for \(i = 1,2\). Assume that \(\pi_i\) is the \(\theta\)-lift of a cuspidal automorphic representation \(\sigma_i\) of \(SL_2(\mathbb{A}_F)\). Assume moreover that the central character of \(\sigma_1 \otimes \sigma_2\) is trivial when restricted to the diagonal copy of \(\pm I_2\). Then using the isomorphism

\[
\pm(I_2,I_2)\backslash SL_2 \times SL_2 \rightarrow SO_4
\]

the representation \(\sigma_1 \otimes \sigma_2\) defines a cuspidal automorphic representation \(\sigma\) of \(SO_4\). Let

\[
r : L(SO_4 \times GL_n) \rightarrow \mathbb{G}_a^{4n}
\]

be the tensor product of the two standard representations. Finally, \(\phi_i\) is a cusp form in the space of \(\pi_i\).

Then it follows from [Kap12] that the integral

\[
\int_{O(V)(F)\backslash O(V)(\mathbb{A}_F)} \int_{U_2(HF)^{\mathbb{A}_F}\backslash G(HF)} \sum_{y \in \mathbb{P}Y'(F)} \rho(g,h)f(h^{-1}y,1)\phi_1(h_1)\phi_2(h_2)
\]

\[
\times \int_{N^0(HF)} W_{\xi_s}(wu_1(g),aw)\psi_1(u)du dg dh
\]

is Eulerian, with unramified local factors equal to

\[
\frac{L(s,\sigma \times \tau,r)}{L(2s,\pi,\text{Sym}^2)}.
\]

**1.3. Representation theoretic interpretation.** We provide an interpretation of our summation formula from a representation-theoretic perspective.

In (2.1.3) and (3.0.2) we define a unipotent subgroup \(N^0 \subset SO_{2n+1}\) and a character \(\psi_1 : N^0(F)\backslash N^0(\mathbb{A}_F) \rightarrow \mathbb{C}\). In particular, \(N^0(\mathbb{A}_F)\) acts on

\[
\text{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s})
\]
via $\psi_1$ (where $W$ denotes the Whittaker model) and we can consider the coinvariants

$$\text{Ind}_{Q_n(k_F)}^H(W_{r,s})N^\circ(k_F),\psi_1.$$ 

There is an embedding $\iota: G \to SO_{2n+1}$. The image normalizes $N^\circ$ and stabilizes $\psi_1$, and we can consider the coinvariants

$$(\text{Ind}_{Q_n(k_F)}^H(W_{r,s})N^\circ(k_F),\psi_1 \otimes S(V(k_F)))_{G(k_F)}.$$ 

The integral $I(f, \xi, s)$ may be viewed as a functional

$$(1.3.1) \quad (\text{Ind}_{Q_n(k_F)}^H(W_{r,s})N^\circ(k_F),\psi_1 \otimes S(V(k_F)))_{G(k_F)} \longrightarrow \mathcal{O}^\infty(Y'((k_F))).$$

The functional equation in the main theorem (Theorem 1.1) ultimately is a consequence of the existence of the intertwining operator on $\text{Ind}_{Q_n}^H(W_{r,s})$ associated with the longest Weyl element and the functional equation of the corresponding Eisenstein series defined in Eq. (3.0.1) with respect to the intertwining operator.

1.4. Outline of the paper. We set up the notation for the various algebraic groups in Section 2. In Section 3, we establish our summation formula assuming various quantities converge. The main theorem is made rigorous by showing the absolute convergence of the sum of the global integrals in Section 4, Section 5, and Section 6. We give a list of symbols at the end of the paper.

In Section 7, we give the computation of the local integral when all the data are unramified. We justify in Section 8 the absolute convergence of various integrals and the final result in Section 7.

Acknowledgements

I would like to thank my advisor Jayce Getz for suggesting this problem and for providing relentless support and valuable advice. I am also grateful to Eyal Kaplan for answering two questions related to his thesis and pointing out several typos in an earlier version of this paper, and Spencer Leslie for helpful discussions and comments. The interpretation of Section 1.3 is my understanding of comments of Yiannis Sakellaridis, which are greatly appreciated. I also want to thank Orsola Capovilla-Searle, Huajie Li, Stephen Mckean, Aaron Pollack, and Jiandi Zou for helpful comments. This work was partially supported by Jayce Getz’s NSF grant DMS-1901883. Finally, I want to thank the anonymous referee for a thorough reading and helpful comments.

2. Preliminaries

2.1. Groups. For this section we let $F$ be a field of characteristic zero. Let $O$ be the ring of integers of $F$ and $\varpi$ be the uniformizer of $O$. To define points of $F$-schemes we let $R$ denote an $F$-algebra. All algebraic groups we define below are affine algebraic groups over $F$. 
Let
\[ J_k = \begin{pmatrix} 0 & 1 \\ \cdot & \cdot \\ 1 & 0 \end{pmatrix} \in \text{GL}_k(F) \]
for \( k \) a positive integer. Let \( \text{SO}_k \) be the special orthogonal group with respect to \( J_k \).

We say that a parabolic subgroup of \( \text{SO}_k \) is standard if it contains the Borel subgroup of upper triangular matrices. Let
\[
G(R) := \{ (g_1, g_2) \in \text{GL}_2^2(R) : \det g_1 = \det g_2^{-1} \} \quad \text{and} \quad H := \text{SO}_{2n+1}.
\]

Let
\[
G' := \text{SO}_4.
\]

We denote by \( T_{G'} \) and \( T_H \) the corresponding maximal split tori consisting of diagonal matrices.

2.1.1. Subgroups of \( H \). Let \( Q_H \) be the standard parabolic subgroup with Levi subgroup whose points in an \( F \)-algebra \( R \) are
\[
M_H(R) := \{ (x, c) \in H : (x, c) \in \text{GL}_{n-2}(R) \times \text{SO}_5(R), x^* = J_{n-2}(t^{-1}x)J_{n-2} \}.
\]

Let \( N_H \) be the unipotent subgroup whose points in an \( F \)-algebra \( R \) are
\[
N_H(R) := \left\{ \begin{pmatrix} z & x & y \\ I_5 & x' \\ z^* \end{pmatrix} : x \in M_{(n-2) \times 5}(R), y \in M_{n-2}(R), z \in Z_H(R) \right\}.
\]

Here \( z^* = J_{n-2}(t^{-1}z)J_{n-2}, x' = -J_5(t^1x)J_{n-2}z^* \), and \( Z_H \) is the unipotent radical of the Borel subgroup of upper triangular matrices of \( \text{GL}_{n-2} \).

We let \( Y_H \) be the subgroup of \( N_H \) whose points in an \( F \)-algebra \( R \) are
\[
Y_H(R) := \left\{ \begin{pmatrix} z & x & 0 & 0 \\ I_2 & 0 & 0 & x' \\ 1 & 0 & 0 & z^* \end{pmatrix} : z \in Z_H(R), z^* = J_{n-2}(t^{-1}z)J_{n-2}, x' = -J_2(t^1x)J_{n-2}z^* \right\},
\]
and we denote \( N^o \) the subgroup of \( N_H \) whose points in an \( F \)-algebra \( R \) are
\[
N^o(R) := \left\{ \begin{pmatrix} I_{n-2} & x & y & 0 \\ I_2 & 0 & 0 & z \\ 1 & 0 & y' \\ I_2 & x' \\ I_{n-2} \end{pmatrix} : x' = -J_2(t^1x)J_{n-2}, y' = -J_1(t^1y)J_{n-2} \right\}.
\]

such that \( N^o \) is isomorphic to \( Y_H \backslash N_H \).
For $x \in \text{GL}_n(R)$ let
\begin{equation}
(2.1.4) \quad v(x) := \begin{pmatrix} x & 1 \\ J_n(x^{-1})J_n & 1 \end{pmatrix} \in \text{GL}_{2n+1}(R).
\end{equation}

Let $Q_n$ be the standard parabolic subgroup with Levi subgroup $M_n$ whose points in an $F$-algebra $R$ are
\begin{equation}
(2.1.5) \quad M_n(R) := \{v(x) : x \in \text{GL}_n(R)\}.
\end{equation}

Let $N_n$ be the unipotent subgroup whose points in an $F$-algebra $R$ are
\begin{equation}
(2.1.6) \quad N_n(R) := \left\{ \begin{pmatrix} z & x & y \\ 1 & x' & z^* \\ x & 1 & z \end{pmatrix} : z \in Z_n, z^* = J_n(tz^{-1})J_n, x' = -J_1(tx)J_nz^* \right\},
\end{equation}
where $Z_n$ is the unipotent radical of the Borel subgroup of upper triangular matrices of $\text{GL}_n$.

Accordingly, we denote $Q_n \subset H$ as the opposite parabolic subgroup with Levi subgroup $M_n$, and we let $N_n$ be the corresponding unipotent radical of $Q_n$.

Let
\begin{equation}
(2.1.7) \quad w = \begin{pmatrix} \frac{1}{2}I_2 & I_{n-2} \\ I_{n-2} & (-1)^{n-2} \end{pmatrix}
\end{equation}
be a Weyl group element in $H$.

Let $Q_{G'}$ be a subgroup of $H$ whose points in an $F$-algebra $R$ are
\begin{equation}
(2.2.1) \quad Q_{G'}(R) = \left\{ \begin{pmatrix} a & b & c & -2b & d \\ 1 & 0 & 0 & -2b' \\ 1 & 0 & c' & b' \end{pmatrix}, \begin{pmatrix} a^{-1} \\ 0 \\ 0 \end{pmatrix} : a \in R^\times, c \in R, b' = -ba^{-1}, c' = -ca^{-1}, d = -\frac{4b'^2 + c'^2}{2a^{-1}} \right\}.
\end{equation}

2.2. Embedding of the groups. For the construction of the global integral, we use two embeddings of groups. Here we give the explicit maps we use in our integral.

We have a sporadic isogeny between the algebraic groups $\text{SL}_2 \times \text{SL}_2$ and $G' = \text{SO}_4$. It induces a surjection $G \to G'$ given on points in an $F$-algebra by
\begin{equation}
(2.2.1) \quad \iota_1 : G(R) \longrightarrow G'(R)
\end{equation}
\begin{equation}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} aa' & -ab' & ba' & bb' \\ -ac' & ad' & -bc' & -bd' \\ ca' & -cb' & da' & db' \\ cc' & -cd' & dc' & dd' \end{pmatrix}.
\end{equation}
Lemma 2.1. Let \( H \) naturally embeds in \( G' \). Let \( \text{Image of the maps} \). Using the map from \( G \) to \( G' = SO_4 \) and \( G' \) in \( SO_5 \) (which naturally embeds in \( H \)), we make the image of subgroups of \( G \) in \( SO_5 \) precise.

Let \( M_1 \) be the subgroup of the maximal torus of \( G \) whose points in an \( F \)-algebra \( R \) are

\[
M_1(R) := \left\{ (\begin{pmatrix} 1 & 0 \\ 0 & m^{-1} \end{pmatrix}), (\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}) : m \in R^\times \right\}.
\]

Lemma 2.1. Let \( M'_1 = \mathcal{I}(M_1) \subset SO_5 \). Then

\[
M'_1(R) = \left\{ (\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}) : m \in R^\times \right\}.
\]

Let \( G_1 \) be a subgroup of the maximal torus of \( G \) whose points in an \( F \)-algebra \( R \) are

\[
G_1(R) := \left\{ (\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}), (\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}) : b \in R^\times \right\}.
\]

Lemma 2.2. Let \( G'_1 = \mathcal{I}(G_1) < SO_5 \). Then

\[
G'_1(R) = \left\{ \begin{pmatrix} 1 & \frac{1}{2} + \frac{1}{4}(b + b^{-1}) & \frac{1}{2}(b - b^{-1}) & 2(b - b^{-1}) \\ \frac{1}{2} + \frac{1}{4}(b + b^{-1}) & 1 & \frac{1}{2}(b - b^{-1}) & -\frac{1}{2}(b - b^{-1}) \\ \frac{1}{2}(b - b^{-1}) & \frac{1}{2}(b + b^{-1}) & 1 & \frac{1}{2}(b - b^{-1}) \\ \frac{1}{2}(b - b^{-1}) & \frac{1}{2}(b + b^{-1}) & -\frac{1}{2}(b - b^{-1}) & 1 \end{pmatrix} : b \in R^\times \right\}.
\]

Let \( A_1 \) be a subgroup of \( T_G \) whose points in \( F \)-algebra \( R \) are

\[
A_1(R) := \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : a_1 \in R^\times \right\}.
\]
**Lemma 2.3.** Let \( A'_1 = \iota(A_1) \). Then

\[
A'_1(R) = \left\{ \begin{pmatrix} a_1 & \frac{1}{2} + \frac{1}{4}(a_1^2 + a_1^{-2}) & \frac{1}{2}(a_1^2 - a_1^{-2}) & 2(a_1^2 - a_1^{-2}) \\ \frac{1}{2}(a_1^2 + a_1^{-2}) & \frac{1}{2}(a_1^2 + a_1^{-2}) & -\frac{1}{2}(a_1^2 - a_1^{-2}) \\ \frac{1}{2}(a_1^2 + a_1^{-2}) & \frac{1}{2}(a_1^2 + a_1^{-2}) & \frac{1}{3}(a_1^2 - a_1^{-2}) & \frac{1}{3}(a_1^2 + a_1^{-2}) \\ a_1^{-1} \end{pmatrix} : a_1 \in R^\times \right\}.
\]

Let \( A_2 \) be the subgroup of \( T_G \) whose points in \( F \)-algebra \( R \) are

\[
A_2(R) := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} : a_2 \in R^\times \right\}.
\]

**Lemma 2.4.** Let \( A'_2 = \iota(A_2) \). Then

\[
A'_2(R) = \left\{ \begin{pmatrix} a_2 & \frac{1}{2} + \frac{1}{4}(a_2^2 + a_2^{-2}) & \frac{1}{2}(-a_2^2 + a_2^{-2}) & 2(-a_2^2 + a_2^{-2}) \\ -\frac{1}{2}(a_2^2 + a_2^{-2}) & \frac{1}{2}(a_2^2 + a_2^{-2}) & -\frac{1}{2}(a_2^2 - a_2^{-2}) \\ \frac{1}{2}(a_2^2 + a_2^{-2}) & \frac{1}{2}(a_2^2 + a_2^{-2}) & \frac{1}{3}(-a_2^2 + a_2^{-2}) & \frac{1}{3}(a_2^2 + a_2^{-2}) \\ a_2^{-1} \end{pmatrix} : a_2 \in R^\times \right\}.
\]

Note that we have \( T_G = A_1A_2G_1 \).

Let \( U_2 \) be the maximal unipotent radical of the Borel subgroup of upper triangular matrices of \( G \). Let \( N_2 \) be a subgroup of the \( U_2 \) whose points in an \( F \)-algebra \( R \) are

\[
N_1(R) := \left\{ \begin{pmatrix} 1 & \frac{c}{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} : c \in R \right\}.
\]

**Lemma 2.5.** Let \( N'_1 = \iota(N_1) < SO_5 \). Then

\[
N'_1(R) = \left\{ \begin{pmatrix} 1 & 0 & c & -\frac{1}{2}c^2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -c & 1 \\ 1 & 0 & 1 \\ c \in R \right\}.
\]

Let \( N_2 \) be a subgroup of \( U_2 \) whose points in an \( F \)-algebra \( R \) are

\[
N_2(R) := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} : b \in R \right\}.
\]

**Lemma 2.6.** Let \( N'_2 = \iota(N_2) < SO_5 \)

\[
N'_2(R) = \left\{ \begin{pmatrix} 1 & b & 0 & -2b \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2b \\ 1 & 0 & -b & 1 \\ b \in R \right\}.
\]

Let \( M_{SL_2} \) be a subgroup of \( SL_2 \times SL_2 \) whose points in an \( F \)-algebra \( R \) are
\[
M_{SL_2}(R) := \left\{ \left( \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \right) : m \in R^\times \right\}.
\]

\[\text{Lemma 2.7.} \quad \text{Let } Q_G = M_{SL_2}N_1N_2. \text{ Then } \iota(Q_G)(R) \text{ is}
\[
\left\{ \begin{pmatrix} a^2 & b & c & -2b \\ 1 & 0 & 0 & -2b' \\ 1 & 0 & c' & 1 \\ b' & 1 & c' & a' \end{pmatrix} : a \in R^\times, c \in R, b' = -ba^{-1}, c' = -ca^{-1}, d = \frac{-4b^2 + c'^2}{2a^{-1}} \right\}.
\]

2.4. \textbf{Summary.} We have given a Levi decomposition
\[
A_1A_2G_1N_1N_2 = A_1G_1M_1N_1N_2
\]
of the Borel of upper triangular matrices in \( G(R) := \{ (g_1, g_2) \in GL_2(R) : \det g_1 = \det g_2^{-1} \} \).

We let \( G' := SO_4 \). Moreover, we have a commutative diagram
\[
\begin{array}{ccc}
A_1A_2G_1N_1N_2 & \longrightarrow & G \\
\downarrow \iota_1 & & \downarrow \iota_1 \\
G'' := SO_4 & \longrightarrow & SO_5 \\
\end{array}
\]

Our main theorems are stated without the use of \( G' \), but we require it in the proofs.

2.5. \textbf{Notations for local fields.} Let \( F \) be a global field and \( v \) a place of \( F \). We denote by \( \mathcal{O} \) the ring of integers of \( F \) and \( \mathcal{O}_v \) the ring of integers of \( F_v \) for nonarchimedean \( v \). We denote by \( \varpi_v \) a uniformizer for \( \mathcal{O}_v \) and \( q_v := |\mathcal{O}_v/\varpi_v| \) the residual characteristic. The idelic norm is denoted by \(| \cdot |\) and the local norm on \( F_v \) (normalized in the usual manner) is denoted by \(| \cdot |_v\).

2.6. \textbf{Measures.} We fix a nontrivial character \( \psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times \) and then choose Haar measures on \( F_v \) for all places \( v \) that are self-dual with respect to \( \psi \). This yields a measure on \( \mathbb{A}_F \). We let
\[
d^x v := \zeta_v(1) \frac{dx_v}{|x|_v}.
\]
This is a Haar measure on \( F_v^\times \).

For every split reductive group \( G \) we fix a maximal compact subgroup \( K \leq G(\mathbb{A}_F) \) that is hyperspecial at all finite places and we normalize the Haar measure on \( G(F_v) \) so that \( K_v \) has measure 1. For the \( F_v \)-points of unipotent subgroups we normalize the Haar measure by transporting measures from \( F_v \) to the root subgroups in the usual manner.
3. The Summation Formula

In this section, we use the Rankin-Selberg integral for $\text{SO}_{2\ell} \times \text{GL}_n$ developed in [Kap12] to deduce the expression of our global integral when we take $\ell = 2$. We state and prove our main theorem of this paper in Theorem 3.2 assuming the absolute convergence statement.

The main theorem will be made rigorous by showing the absolute convergence of the sum of the global integrals in Section 5.

Let $F$ be a number field. We first briefly recall the construction of the Rankin-Selberg integral in [Kap12, Section 3]. Let $\tau$ be an irreducible automorphic representation for $\text{GL}_n(\mathbb{A}_F)$.

Let $\xi_s$ be a smooth holomorphic section from the (normalized induction) space $\text{Ind}^{H(\mathbb{A}_F)}_{Q_n(\mathbb{A}_F)}(\tau \otimes |\det|^{s-\frac{1}{2}})$.

We have the Eisenstein series
\begin{equation}
E_{\xi_s}(h) := \sum_{y \in Q_n(F) \setminus H(F)} \xi_s(yh, 1),
\end{equation}
where the first variable of $\xi_s$ is on $H$, and the second variable of $\xi_s$ is on $\text{GL}_n$.

Let $\psi$ be a non-trivial additive character of $F \setminus \mathbb{A}_F$. For $u \in N_H(\mathbb{A}_F)$, let
\begin{equation}
\psi_1(u) = \psi \left( \sum_{i=1}^{n-3} u_{i,i+1} + u_{n-2,n} + \frac{1}{2} u_{n-2,n+2} \right)
\end{equation}
be a character of $N_H(\mathbb{A}_F)$, trivial on $N_H(F)$.

Then the $\psi_1$-coefficient of $E_{\xi_s}$ with respect to $N_H(\mathbb{A}_F)$ is
\[ E_{\xi_s}^{\psi_1}(h) = \int_{N_H(F) \setminus N_H(\mathbb{A}_F)} E_{\xi_s}(uh) \psi_1(u) du. \]

Let $\varphi$ be a cusp form on $G(\mathbb{A}_F)$. The Rankin-Selberg integral in this case is
\[ I_\varphi = \int_{G'(F) \setminus G'(\mathbb{A}_F)} \varphi(g) E_{\xi_s}^{\psi_1}(g) dg. \]

This global integral converges absolutely in the whole $s$-plane except at the poles of the Eisenstein series $E_{\xi_s}(h)$, and the absolute convergence follows from the rapid decay of the cusp form $\varphi$ and the moderate growth of the Eisenstein series $E_{\xi_s}(h)$.

Let $d_1, d_2$ be two even positive integers, and $V_1 = \mathbb{G}_d^{d_1}$, $V_2 = \mathbb{G}_d^{d_2}$ be a pair of affine spaces over $F$ equipped with non-degenerate quadratic forms $Q$ and $Q'$ respectively. Let $V := V_1 \oplus V_2$. Let
\begin{equation}
Y(R) := \{ y = (y_1, y_2) \in V(R) : Q(y_1) = 2Q'(y_2) \},
\end{equation}
and let
\[ V' := \{ \gamma = (\gamma_1, \gamma_2) \in V(R) : \gamma_i \neq 0 \}. \]
We let $P \subset PV$ be the corresponding quasi-projective scheme. This is the scheme attached to the pair of quadratic spaces mentioned above.

For a fixed $u \in F^\times$, let $\rho_u$ be the usual Weil representation on $\text{SL}_2^2(\mathbb{A}_F)$ with quadratic forms $uQ(\gamma)$. Let $f \in \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$.

We construct the Weil representation $\rho$ for $G(\mathbb{A}_F)$ following [YZZ13], which extends the usual Weil representation of $\text{SL}_2^2(\mathbb{A}_F)$ as follows:

\[
\rho(g)f(\gamma, u) = \rho_u(g)f(\gamma, u), \quad g \in \text{SL}_2^2(\mathbb{A}_F)
\]

\[
\rho(((1_a), (1_{a-1})))f(\gamma, u) = f(\gamma, a^{-1}u)a^{-\frac{\dim V}{2}}, \quad a \in \mathbb{A}_F^\times
\]

Here $\mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ is the usual Schwartz space for vector spaces.

Let $f \in \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ be a Schwartz-Bruhat function such that $\rho(g)f(\gamma, u) = 0$ for all $g \in G(\mathbb{A}_F)$ and $(\gamma, u) \in V''(F) \times F^\times$, where $V''(R) = \{(\gamma_1, \gamma_2) \in V(R) : \mathcal{Q}(\gamma_1) = \mathcal{Q}'(\gamma_2) = 0\}$.

We let

\[
\Theta_f(g) = \sum_{(\gamma, u) \in V(\gamma) \times F^\times} \rho(g)f(\gamma, u)
\]

be the theta function on $G(\mathbb{A}_F)$.

Using the formula of $I(\varphi, \xi, s)$, we define a global integral as

\[
I_{\Theta_f} = \int_{G(F) \backslash G(\mathbb{A}_F)} \Theta_f(g)E_{\xi, s}(g)dg.
\]

Since we take $\rho(g)f(0) = 0$ for all $g \in G(\mathbb{A}_F)$, $\theta_f(g)$ is cuspidal. Thus $I(\Theta_f, \xi, s)$ converges absolutely in the whole $s$-plane except at the poles of the Eisenstein series $E_{\xi, s}^{\psi_1}(h)$ similar as the Rankin-Selberg integral $I(\varphi, \xi, s)$.

By the action of Weil representation, we have

\[
I_{\Theta_f} = \int_{\text{SL}_2^2(F) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in V(F)} \rho(g)f(\gamma, 1)E_{\xi, s}^{\psi_1}(g)dg.
\]

We first unfold the Eisenstein series $E_{\xi, s}$ for $\Re(s)$ large.

**Lemma 3.1.** For $\Re(s)$ large, we have

\[
I_{\Theta_f} = \int_{\mathcal{Q}(\gamma) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in V(F)} \rho(g)f(\gamma, 1) \int_{Y_{\mathbb{H}}(F) \backslash N_{\mathbb{H}}(\mathbb{A})} \xi(w_0u_{\varphi}(g), 1)\psi_1(u)dudg.
\]

Here

\[
w_0 = \begin{pmatrix}
I_2 & I_{n-2} \\
(-1)^{n-2} & I_2 \\
I_{n-2} & I_2
\end{pmatrix} \in H.
Proof. By the embedding map $\iota_1$ from $G$ to $G'$ (see Eq. (2.2.1)), we have a long exact sequence

$$1 \to \{\pm I_2\} \to \SL_2^0(F) \xrightarrow{\iota} G'(F) \xrightarrow{\text{sn}} H^1(F, \pm I_2) \cong F^\times/(F^\times)^2 \to 1,$$

where the map $\text{sn}$ denotes the spinor norm.

It follows that

$$\iota_2(G'(F)) = \bigcup_{\epsilon \in F^\times/(F^\times)^2} \left( I_{n-2} \epsilon I_3 \epsilon^{-1} I_{n-2} \right) \iota(\SL_2^2(F)), \quad Q_{G'}(F) = \bigcup_{\epsilon \in F^\times/(F^\times)^2} \left( I_{n-2} \epsilon I_3 \epsilon^{-1} I_{n-2} \right) \iota(Q_G)(F).$$

Then

$$Q_{G'}(F) \setminus \SL_2^2(F) \cong \iota(Q_G)(F) \setminus \iota(\SL_2^2(F)) \cong Q_{G'}(F) \setminus \iota_2(G'(F)).$$

By [Kap12, Proof of Proposition 3.1, Page 151-154], after unfolding the Eisenstein series, the only non-vanishing contribution of $E_{\xi_s}^{\psi_1}$ in $I(\Theta_f, \xi, s)$ for $\Re(s)$ large is

$$\sum_{y \in Q_{G'}(F) \setminus \iota_2(G'(F))} \int_{Y_H(F) \setminus N_H(A_F)} \xi(w_0 y u \iota(g), 1) \psi_1(u) du,$$

which is equivalent to

$$\sum_{y \in Q_{G}(F) \setminus \SL_2^2(F)} \int_{Y_H(F) \setminus N_H(A_F)} \xi(w_0 y u \iota(g), 1) \psi_1(u) du.$$

Then we have

$$I_{\Theta_f} = \int_{\SL_2^2(F) \setminus G(A_F)} \sum_{\gamma \in V(F)} \rho(g) f(\gamma, 1) \times \sum_{y \in Q_G(F) \setminus \SL_2^2(F)} \int_{Y^n(F) \setminus N_H(A_F)} \xi(w_0 y u \iota(g), 1) \psi_1(u) dudg$$

$$= \int_{Q_G(F) \setminus G(A_F)} \sum_{\gamma \in V(F)} \rho(g) f(\gamma, 1) \int_{Y^0(F) \setminus N_H(A_F)} \xi(w_0 y u \iota(g), 1) \psi_1(u) dudg. \quad \square$$

The main theorem of the paper is:

**Theorem 3.2.** Let

$$I(f, \xi_s, s)(y) = \int_{U_2(A_F) \setminus G(A_F)} \rho(g) f(y, 1) \int_{N^2(A_F)} W_{\xi_s}(w u u \iota(g), a_g) \psi_1(u) du dg.$$

Let

$$(3.0.6) \quad W_{\xi_s}(w u u \iota(g), 1) = \int_{Z_n(F) \setminus Z_n(\mathbb{A})} \xi_s(w u u \iota(g), z) \psi_0^{-1}(z) dz,$$
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where \( w \) is defined in Eq. (2.1.7) and \( f \) satisfies Eq. (3.0.4). For \( g \in O(V_1)(\mathbb{A}_F) \times O(V_2)(\mathbb{A}_F) \), the sum \( \sum_{y \in \mathcal{P}^y(F)} I(f, W_{\xi_y})(gy) \) admits a meromorphic continuation to the whole \( s \)-plane which satisfies a functional equation

\[
\sum_{y \in \mathcal{P}^y(F)} I(f, W_{\xi_y})(gy) = \sum_{y \in \mathcal{P}^y(F)} I(f, M(\tau, s)W_{\xi_y})(gy).
\]

Proof. We use the defining property of the action of the Weil representation on \( f \), and Lemma 2.1 to Lemma 2.6 to unfold the integral.

Firstly, using Lemma 2.1, Lemma 2.5, Lemma 2.6, and the action of \( N_1 \) on \( f \) we have

\[
I_{\Theta_f} = \int_{Q_1(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V_1(F)} \rho(g)f(\gamma, 1) \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(g), 1)\psi_1(u)dudg
\]

\[
= \int_{N_1(\mathbb{A}_F)N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \sum_{\gamma \in V_1(F)} \rho(r'g)f(\gamma, 1)
\]

\[
\times \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(g), 1)\psi_1(u)dudr'dg
\]

\[
= \int_{N_1(\mathbb{A}_F)N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{F\backslash \mathbb{A}_F} \sum_{\gamma \in V_1(F)} \rho(g)\psi(\frac{c}{2}Q(\gamma_1) - cQ'(\gamma_2))f(\gamma, 1)
\]

\[
\times \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(g), 1)\psi_1(u)dudc dg
\]

\[
= \int_{N_1(\mathbb{A}_F)N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V_1'(F)} \rho(g)f(\gamma, 1)
\]

\[
\times \left( \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(g), 1)\psi_1(u)du \right) dg.
\]

Here the last line holds since by [Kap12, Proof of Propostion 3.1], the function

\[
g \mapsto \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(g), 1)\psi_1(u)du
\]

is invariant on the left for \( r' \in N_1(\mathbb{A}_F) \).

Using the action of \( N_2 \) on \( f \) we have

\[
I_{\Theta_f} = \int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)M_1(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V_1'(F)} \rho(zg)f(\gamma, 1)
\]

\[
\times \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(zg), 1)\psi_1(u)dudzdg
\]

\[
= \int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)M_1(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V_1'(F)} \rho(g)
\]

\[
\times \left( \int_{Y_1(F)\backslash N_1(\mathbb{A}_F)} \xi_s(w_0wu(zg), 1)\psi_1(u)du \right) dg.
\]
Thus we can interchange $uz$

The mapping on $N$ is well-defined since the elements of $N$ is well-defined since the elements of $N$

Using the action of $M$ on $f$ we have

As in [Kap13, Page 42], for fixed $u$ and $g$, the function on $N_2(\mathbb{A}_F)$

is well-defined since the elements of $N_2(\mathbb{A}_F)$ and $Y_H(\mathbb{A}_F)$ commute. Also, since $z$ normalizes $Y_H(\mathbb{A}_F)$, stabilizes $\psi_1(y)$ and $\xi_s(w_0z, 1) = \xi_s(w_0, 1)$, the function is left-invariant on $N_2(F)$. The mapping on $N_H(\mathbb{A}_F)$

is left-invariant by $Y_H(\mathbb{A}_F)$. Also, $z$ in the integral normalizes $N_H(\mathbb{A}_F)$ and stabilizes $\psi_1$. Thus we can interchange $uz$ to $zu$ in the integral. We have

$$I_{\Theta_f} = \int_{N_2(\mathbb{A}_F)N_2^*(\mathbb{A}_F), G(\mathbb{A}_F)} \rho(g) f(y, 1)$$

$$\times \int_{Y_H(\mathbb{A}_F)\setminus N_H(\mathbb{A}_F)} \xi_s((w_0zw_0^{-1})w_0u(y)\psi_1(yu)\psi(4bQ'(y_2))dndudzdg.$$
Let
\[ w' = \begin{pmatrix} I_2 & 0 \\ I_{n-2} & 0 \end{pmatrix}. \]

As in [Kap13, Page 43], the double integral \( \int_{N_2^*} \int_{\mathcal{U}(F) \backslash \mathcal{H}(\mathbb{A}_F)} \) can be written as \( \int_{\tilde{Z}_n / Z_n} \) where \( w' \tilde{Z}_n w'^{-1} = Z_n. \)

We note that now the character on the group \( Z_n \) is
\[ \psi_1'(z) = \psi(-4Q'(y_2)z_{1,2} + \frac{1}{2}z_{2,3} + \sum_{i=3}^{n-1} z_{i,i+1}) \]
for \( z \in Z_n(\mathbb{A}_F). \)

We use a conjugation by
\[ \begin{pmatrix} \frac{1}{2}I_2 & 0 \\ 0 & I_{n-2} \end{pmatrix} \]
to replace the character \( \psi_1' \) to a character \( \psi_{y,Q'} \), where \( \psi_{y,Q'} \) is the generic character of \( Z_n \) such that
\[ \psi_{y,Q'}(z) = \psi(-4Q'(y_2)z_{1,2} + z_{2,3} + \cdots + z_{n-1,n}). \]

Then we have our sum
\[ I_{\Theta_f} = \int_{N_2(\mathbb{A}_F) / N_2(\mathbb{A}_F)} \sum_{g \in \mathcal{Y}'(F)} \rho(g) f(y,1) \int_{\mathcal{U}_H(\mathbb{A}_F) \backslash \mathcal{H}(\mathbb{A}_F)} W_{\xi_s,Q'}(wu(g),1) \psi_1(u) du dg \]
\[ = \sum_{g \in \mathcal{Y}'(F)} \int_{U_2(\mathbb{A}_F) / G(\mathbb{A}_F)} \rho(g) f(y,1) \int_{\mathcal{U}_H(\mathbb{A}_F) \backslash \mathcal{H}(\mathbb{A}_F)} W_{\xi_s,Q'}(wu(g),1) \psi_1(u) du dg, \]
where
\[ W_{\xi_s,Q'}(wu,1) = \int_{Z_n(\mathbb{F}) \backslash Z_n(\mathbb{A})} \xi_s(wu(g), z) \psi_0^{-1}(z) dz. \]

We recall that \( W_{\xi_s,Q'} \) may be factored into a product of local Whittaker functions if \( \xi_s \) is a pure tensor, and hence the integral
\[ \int_{U_2(\mathbb{A}_F) / G(\mathbb{A}_F)} \rho(g) f(y,1) \int_{\mathcal{U}_H(\mathbb{A}_F) \backslash \mathcal{H}(\mathbb{A}_F)} W_{\xi_s,Q'}(wu(g),1) \psi_1(u) du dg \]
is Eulerian. Thus \( I_{\Theta_f} \) is a sum of Eulerian integrals.

**Lemma 3.3.** We have
\[ W_{\xi_s,Q'}(g,1) = W_{\xi_s}(g,a_y), \]
where
\[ W_{\xi_s}(g,a_y) = \int_{Z_n(\mathbb{F}) \backslash Z_n(\mathbb{A})} \xi_s(wu(g), a_y z) \psi_0^{-1}(z) dz. \]
Here \( \psi_0 \) is the standard character on \( Z_n(\mathbb{A}). \)
Proof. We have
\[ \psi_0(a_yza_y^{-1}) = \psi_{y,Q'}(z) \]
for \( z \in \mathbb{Z}_n \).

Thus we have the function
\[ g \mapsto W_{\xi_\iota}(g, a_y) \]
lies in \( \text{Ind}^H_{Q_n}(W_{r,s,y,Q'}) \) since
\[ W_{\xi_\iota}(g, a_yz') = W_{\xi_\iota}(g, (a_yz' a_y^{-1})a_y) = \psi_0(a_yz' a_y^{-1})W_{\xi_\iota}(g, a_y) = \psi_{y,Q'}(z')W_{\xi_\iota}(g, a_y) \]
for \( z' \in \mathbb{Z}_n(A) \).

Thus we have
\[ I_\Theta f = \sum_{y \in \mathbb{P}Y'_{r,s}(F)} \left( \int_{U_2(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} \rho(g)f(y, 1) \int_{Y_{H_\iota}(\mathbb{A}_F) \setminus N_{H_\iota}(\mathbb{A}_F)} W_{\xi_\iota}(wu_{\iota}(g), a_y) \psi_1(u) \, dudg. \]

The manipulations of the integral will be justified in Section 6 by showing the sum converges absolutely for \( \Re(s) \) large. Then for \( \Re(s) \) large, we have
(3.0.9) \[ I_\Theta f = \sum_{y \in \mathbb{P}Y'_{r,s}(F)} I(f, W_{\xi_\iota})(y). \]

Thus we have that \( \sum_{y \in \mathbb{P}Y'_{r,s}(F)} I(f, W_{\xi_\iota})(y) \) admits a meromorphic continuation to all \( s \)-plane.

By the functional equation of the Eisenstein series \( E_{\xi_\iota} \), we obtain the desired functional equation for the sum of the global integral. Also, the poles of our sum of integrals come from the poles of \( E_{\xi_\iota} \) [GPSR87, BG92]. \( \square \)

4. Bounds of the local integrals in the non-Archimedean case

Let \( F \) be a non-Archimedean local field of characteristic zero. Let \( K_G = G(\mathcal{O}) \), and let \( K_H = \text{SO}_{2n+1}(\mathcal{O}) \). In this section we give bounds for the local factors of the global integral \( I(f, W_{\xi_\iota}) \) in the non-Archimedean case.

In Lemma 4.1 and Lemma 4.2 we first bound the inner integral of our local integral using some techniques from the proof of convergence of non-Archimedean Rankin-Selberg integral for \( \text{SO}_{2t+1} \times \text{GL}_n \) in [Sou93, Section 4].

We give a bound for our local integral in the general case in Lemma 4.3, and then in the unramified case in Lemma 4.5.

The local integral is
\[ I_s(y) := I(f, W_{\rho_{r,s}})(y) = \int_{U_2(F) \setminus G(F)} \rho(g)f(y, 1) \int_{N_{\mathcal{O}}(F)} W_{\rho_{r,s}}(wu_{\iota}(g, a_y) \psi_1(u) \, dudg. \]

For \( t = \text{diag}(t_1, \ldots, t_n) \in \text{GL}_n(F) \), let
(4.0.1) \[ t' := \begin{pmatrix} t_1 & & \\ & w_0 t_1^{-1} & \\ & & w_0 \end{pmatrix} \in T_H(F), \]
where \( w_0 \in \text{GL}_n(F) \) is the antidiagonal matrix.
For a quasi-character $\eta : F^\times \rightarrow \mathbb{C}^\times$ there is a unique real number $\Re(\eta)$ such that $|\eta| \cdot |^{-\Re(\eta)}$ is unitary. We say $\eta$ is positive if $\Re(\eta) > 0$.

One can prove the following lemma using the same argument as that proving [Sou93, Lemma 4.4]:

**Lemma 4.1.** Let $(n, t', k) \in N_n(F) \times T_H(F) \times K_H$, we have

$$|W_{\rho, \tau}(nt'k, a_y)| \leq |\det t|^\Re(s) + \frac{n-1}{2} \sum_{j=1}^t \overline{c_{j,s}} \eta_j(a_y t).$$

Here $c_{j,s} \in \mathbb{C}$ and $\eta_j$ are positive quasi-characters of $T_H(F)$ which depend on $\tau$.

The points of the group $w N^0 w^{-1}$ in an $F$-algebra $R$ are

$$w N^0(R)w^{-1} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & v_2 \\ v_1 & v_2 & 1 \end{pmatrix} : v_1, v_2, v_3 \in R^{n-2}, v'_1 = -t v_1 J_{n-2}, T \in M_{n-2}(R) \right\}.$$ 

Let $v \in H(F)$ be a unipotent element of $H$ of the form

$$v = \begin{pmatrix} 1 & 1 & \frac{c}{2} \\ 0 & 0 & -\frac{c^2}{2} \\ v_1 & v_2 & 1 \end{pmatrix}$$

for some $c \in F$.

**Lemma 4.2.** For $\Re(s)$ large, the integral

$$\int_{w N^0(F)w^{-1}} |W_{\rho, \tau}(uv, 1)|du$$

converges.

**Proof.** For $u \in w N^0(F)w^{-1}$, we have

$$uv = \begin{pmatrix} 1 & 1 & c \\ 0 & 0 & -\frac{c^2}{2} \\ v_1 & v_2 & 1 \end{pmatrix}.$$ 

We denote the Iwasawa decomposition of $uv$ as $uv = nt'k$ where $(n, t', k) \in N_n(F) \times T_H(F) \times K_H$, and we denote the $i$-th line of $uv$ as $(uv)_i$.

By Lemma 4.1, the integral is majorized by

$$\sum_{j=1}^\nu c_{j,s} \int_{w N^0(F)w^{-1}} [D(uv)]^{\Re(s) + \frac{n-1}{2}} E_j(uv)du,$$
where
\[ D(nt'k) = |\det t|, \]
\[ E_j(nt'k) = \eta_j(t). \]

We use techniques following [Sou93, Lemma 1, Section 11.15]. Let \( \{e_1, \ldots, e_{2n+1}\} \) be the standard basis of \( F^{2n+1} \). We take the sup-norm on \( \wedge^p F^{2n+1} \) according to the basis \( \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq n\} \). \( K_H \) preserves this norm. We have
\[ \|v_1 \wedge v_2 \wedge \cdots \wedge v_p\| \leq \|v_1\| \cdot \|v_2\| \cdots \|v_p\|, \quad v_j \in F^{2n+1}. \]

Let \( e_{n+1+j} = e_{-n+j-1} \) for \( j = 1, \ldots, n \), we have
\[ |t_{j+1} \cdots t_1| = \|(e_{-(j+1)}uv) \wedge \cdots (e_{-1}uv)\| \]
\[ = \|(e_{-(j+1)} + (uv)_{2n+1-j}) \wedge \cdots (e_{-1} + (uv)_{2n+1})\| \]
\[ \leq \prod_{i=0}^j \max\{1, \|(uv)_{2n+1-j}\|\} = \prod_{i=0}^j \|(uv)_{2n+1-j}\|. \]

Here \( uv \) is a matrix, \( e_{-j+1}uv \) is a vector, and \( \|(uv)_{2n+1-j}\| = \max\{1, \|(uv)_{2n+1-j}\| \} \) and \( \|\cdot\| \) denotes the sup-norm. For
\[ h = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n+1} \end{pmatrix} \in H(F), \]
we let
\[ \mathcal{L}(h) = \begin{pmatrix} h_{n+2} \\ \vdots \\ h_{2n+1} \end{pmatrix} \]
be the bottom \( n \) rows of \( h \).

Since the coordinates of \( \mathcal{L}(uv) \) appear in the coefficients of
\[ |t_{j+1} \cdots t_1| = \|(e_{-(j+1)}uv) \wedge \cdots e_{-1}uv\|, \]
we have
\[ |t_{j+1} \cdots t_1|^{-1} \geq \max\{1, \|\mathcal{L}(uv)\|, \ldots, \|\mathcal{L}(uv)\|\}. \]

We denote
\[ [\mathcal{L}(uv)] = \max\{1, \|\mathcal{L}(uv)\|\}, \]
where \( \|\cdot\| \) is the sup-norm.

Then we have
\[ [\mathcal{L}(uv)]^{-2j} \leq \frac{t_j}{t_{j+1}} \leq [\mathcal{L}(uv)]^{2j}, \quad j = 1, \ldots, n-1, \]
and
\[ [\mathcal{L}(uv)]^{-n} \leq D(u'v) \leq [\mathcal{L}(uv)]^{-1}. \]
Since $[L(uv)] \leq [u][v]$, we have

$$(4.0.9) \quad E_j(uv) \leq [L(uv)]^C \leq [u]^C[v]^C$$

for some positive constant $C$ which depends only on $\tau$.

By the structure of $L(uv)$, we have

$$[L(uv)]^{-1} = \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|, \|\frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\}^{-1}$$

$$\leq \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|\}^{-1}$$

$$= [u]^{-1}.$$

Here the middle line holds since if $\max\{1, \frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\} = 1$, we have equality, and we have

$$\max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|, \|\frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\} > \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|\}$$

otherwise.

Thus we have

$$(4.0.10) \quad D(uv) \leq [L(uv)]^{-1} \leq [u]^{-1}.$$

By Eq. (4.0.9) and Eq. (4.0.10), for $\Re(s) + \frac{n-1}{2} - C > 0$, Eq. (4.0.4) is bounded by

$$(4.0.11) \quad \sum_{j=1}^{\nu} c_{j,s}[v]^C \int_{wN^0(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+C} du$$

which converges absolutely.

Now we proceed to bound our local integral in the general case.

**Lemma 4.3.** For $\Re(s)$ large enough, we have

$$I_s(y) \ll \int_{F^\times} \int_{F^\times} f'(a_1y_1, a_2y_2) |a_1|^{\Re(s)+\frac{d_1-n-1}{2}} |a_2|^{\Re(s)+\frac{d_2-n-1}{2}} d^\times a_1 d^\times a_2.$$

Here $f' \in \mathcal{S}(V(F))$ and $\ll$ is synonymous with the big O notation.

**Proof.** We apply the Iwasawa decomposition of $U_2(F)\backslash G(F)$ with respect to the usual Borel subgroup of $G(F)$. Since $W_{\rho, s}$ is smooth, it suffices to bound

$$\int_{T_G(F)} \int_{K_G} |\rho(ak)f(y, 1)||\delta_{BG}^{-1}(a) \int_{N^0(F)} |W_{\rho, s}(wut(ak), a_y)| duda dk.$$

By the defining property of the Weil representation, the above integral is

$$\int_{G_1} \int_{A_1(F)} \int_{A_2(F)} \int_{K_G} |\rho(xa_1a_2k)f(y, 1)||\delta_{BG}^{-1}(a_1a_2)|$$
\[
\times \int_{N^\circ(F)} |W_{\psi,\iota}(wu(xa_1a_2k), a_y)dua_1da_2dk
\]
\[
= \int_{F^x} \int_{F^x} \int_{F^x} \int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{-2}
\times \int_{N^\circ(F)} |W_{s,\phi}((a_1^{-1}b_1^{-1}), (a_2^{-1}b_2))| \iota(k), a_y|du^xbd^x a_1d^x a_2dk.
\]
Here \( \iota ((a_1^{-1}b_1^{-1}), (a_2^{-1}b_2)) \) is
\[
\begin{pmatrix}
I_{n-1} & a_1a_2 \\
\frac{1}{2} + \frac{1}{4}(b^{-1}a_1a_2^{-1} + ba_1^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1} - ba_1^{-1}a_2) & 2(b^{-1}a_1a_2^{-1} - ba_1^{-1}a_2) \\
\frac{1}{2}(b^{-1}a_1a_2^{-1} - ba_1^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1} + ba_1^{-1}a_2) & -\frac{1}{2}(b^{-1}a_1a_2^{-1} - ba_1^{-1}a_2) \\
\frac{1}{4}b^{-1}a_1a_2^{-1} + ba_1^{-1}a_2) & -\frac{1}{4}(b^{-1}a_1a_2^{-1} - ba_1^{-1}a_2) & \frac{1}{4}(b^{-1}a_1a_2^{-1} + ba_1^{-1}a_2)
\end{pmatrix}_{(a_1a_2)^{-1}}
\]
Apply the Iwasawa decomposition to \( SO_3(F) \) we have for \( b \in F^x \),
\[
\begin{pmatrix}
\frac{1}{2} + \frac{1}{4}(b+b^{-1}) & \frac{1}{2}(b-b^{-1}) & 2(b-b^{-1}) \\
\frac{1}{2}(b-b^{-1}) & \frac{1}{2}(b+b^{-1}) & -\frac{1}{2}(b-b^{-1}) \\
2(\frac{1}{2}b+b^{-1}) & -\frac{1}{2}(b-b^{-1}) & \frac{1}{4}(b+b^{-1})
\end{pmatrix}
= \begin{pmatrix}
1 & \frac{c}{2} & -\frac{1}{4}c^2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
b \\
0 \\
0
\end{pmatrix}
\]
Here \( k' \in SO_3(O) \subset K_H \), and
\[
[b] = \begin{cases}
 b \text{ if } |b| \leq 1, \\
 b^{-1} \text{ if } |b| > 1,
\end{cases}
\]
\[
c = \begin{cases}
 -2 \text{ if } |b| = b, \\
 2 \text{ if } |b| = b^{-1}.
\end{cases}
\]
We also notice that
\[
\begin{pmatrix}
1 & \frac{c}{2} & -\frac{1}{4}c^2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
|b|^2 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
|b| & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
|b|^{-1} & -\frac{1}{4}c^2|b|^{-2} \\
0 & 1 & -c|b|^{-1} \\
0 & 0 & 1
\end{pmatrix}
\]
Thus our local integral is majorized by
\[
\int_{F^x} \int_{F^x} \int_{F^x} \int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{-2}
\times \int_{N^\circ(F)} |W_{\psi,\iota}(wudiag(I_{n-2}, a_1a_2, I_3, (a_1a_2)^{-1}, I_{n-2})tn'k''\iota(k), a_y)|du^xbd^x a_1d^x a_2dk.
\]
Here
\[
t = \begin{pmatrix}
I_{n-1} & b^{-1}a_1a_2^{-1} \\
|ba_1^{-1}a_2| & 1
\end{pmatrix}_{(a_1a_2)^{-1}}
\]
(4.0.15) \[ n' = \begin{pmatrix} I_{n-1} & 1 & c[b_{1}^{-1}a_2] & -\frac{1}{2}c^2[b_{1}^{-1}a_2]^2 \\ 1 & \frac{1}{2}c & \end{pmatrix}, \]

(4.0.16) \[ k'' = \begin{pmatrix} I_{n-1} \\ k' \end{pmatrix}. \]

We have

Then by the property of $W_{\rho, s}$ Eq. (4.0.13) is

\[
\int_{F} \int_{F} \int_{F} \int_{K_{G}} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1-1}{2}}|a_2|^{\frac{d_2-1}{2}}|a_1a_2|^{|\Re(s)-\frac{n+1}{2}|}|b^{-1}a_1a_2^{-1}|^{\Re(s)-\frac{n-3}{2}}
\times \int_{N^{\times}(F)} |W_{\rho,s}|(|w'u'k''(k), a_9\text{diag}(a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2})|dud^\times ba^\times a_1d^\times a_2dk.
\]

This is

\[
\int_{F} \int_{F} \int_{F} \int_{K_{G}} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1-1}{2}}|a_2|^{\frac{d_2-1}{2}}|a_1a_2|^{|\Re(s)-\frac{n+1}{2}|}|b^{-1}a_1a_2^{-1}|^{\Re(s)-\frac{n-3}{2}}
\times \int_{N^{\times}(F)} |W_{\rho,s}|(|wuw^{-1})(wn'u^{-1})wk''(k), a_9\text{diag}(a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2})|dud^\times ba^\times a_1d^\times a_2dk.
\]

Since

\[ wn'u^{-1} = \begin{pmatrix} 1 & -\frac{1}{2}c[b_{1}^{-1}a_2]^{-1} & -\frac{1}{2}c^2[b_{1}^{-1}a_2]^2 \\ 1 & \frac{1}{2}c & \end{pmatrix}, \]

we have

\[ |wn'u^{-1}| \ll |[b^{-1}a_1a_2^{-1}]|^2. \]

Then the above integral is majorized by

\[
\int_{F} \int_{F} \int_{F} \int_{K_{G}} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1-1}{2}}|a_2|^{\frac{d_2-1}{2}}|a_1a_2|^{|\Re(s)-\frac{n+1}{2}|}|b^{-1}a_1a_2^{-1}|^{\Re(s)-\frac{n-3}{2}}
\times \int_{wN^{\times}(F)w^{-1}} |W_{\rho,s}|(|u(wn'u^{-1})wk''(k), a_9\text{diag}(a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2})|dud^\times ba^\times a_1d^\times a_2dk.
Since \( w k'' k \in K_H \), we apply Lemma 4.1 and Lemma 4.2. Then the local integral is majorized by

\[
\sum_{j=1}^{\nu} c_{j,s} \int_{F^s} \int_{F^t} \int_{F^s} \int_{K_G} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{d_1} |a_2|^{d_2} |a_1 a_2|^{|\Re(s) - \frac{n+1}{2}}
\]

(4.0.17)

\[
\times \left| \int_{w N^0(F) w^{-1}} \left[ u \right]^{-\Re(s) - \frac{n-1}{2} + C} d\nu \right| a_1 d^x a_2 dk.
\]

Here \( \eta_j \) are positive quasi-characters that depend only on \( \tau \).

Also, for \( u \in w N^0(F) w^{-1} \), if we denote the Iwasawa decomposition of \( u(w n' w^{-1}) \) by \( u(w n' w^{-1}) = n t' k \) (using notations as in Lemma 4.1), then \( a_y \text{diag}(a_1 a_2, |b^{-1} a_1 a_2^{-1}|, I_{n-2}) t \) lies in the support of a gauge on \( \text{GL}_n(F) \). Also, there are constants \( c_1 \) that depends only on \( \tau \) such that

\[
\left| \frac{-4 Q'(y_2) a_1 a_2 t_1}{|b^{-1} a_1 a_2^{-1}| t_2} \right| \leq c_1.
\]

Thus by Lemma 4.2 we have

(4.0.18)

\[
| -4 Q'(y_2) a_1 a_2 | \leq c_1 |u| |b^{-1} a_1 a_2^{-1}|^{-2}.
\]

Therefore,

\[
|\eta_j|(a_y \text{diag}(a_1 a_2, |b^{-1} a_1 a_2^{-1}|, I_{n-2})) \leq c_1 |u| |b^{-1} a_1 a_2^{-1}|^{-2c_2 - c_3}
\]

for some positive integers \( c_2, c_3 \) that depend only on \( \tau \) for \( j = 1, \ldots, \nu \).

Thus the integral is majorized by

\[
\sum_{j=1}^{\nu} c_{j,s} \int_{F^s} \int_{F^t} \int_{F^s} \int_{K_G} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{d_1} |a_2|^{d_2} |a_1 a_2|^{|\Re(s) - \frac{n+1}{2}}
\]

\[
\times \left| \int_{w N^0(F) w^{-1}} \left[ u \right]^{-\Re(s) - \frac{n-1}{2} + C + 2c_2} d\nu \right| a_1 d^x a_2 dk.
\]

Then by definition of the symbol \(|\cdot|\) (see Eq. (4.0.12)) and \([\cdot]\) (as in Eq. (4.0.6)), for \( \Re(s) \) large, the above sum of integrals is majorized by a constant times

\[
\int_{F^s} \int_{F^t} \int_{K_G} |\rho(k) \tilde{f}(a_1 y_1, a_2 y_2)||a_1|^{d_1} |a_2|^{d_2} |a_1 a_2|^{|\Re(s) - \frac{n+1}{2}} d^x a_1 a_2 dk,
\]

where \( \tilde{f} \in S(V(F)) \).

Let \( f(v) = \int_{K_G} \rho(k) f(v) \) for \( v \in V(F) \), we have \( \tilde{f} \in S(V(F)) \). The integral is equal to

\[
\int_{F^s} \int_{F^t} \left| \tilde{f}(a_1 y_1, a_2 y_2)||a_1|^{d_1} |a_2|^{d_2} |a_1 a_2|^{|\Re(s) - \frac{n+1}{2}} d^x a_1 a_2,
\]

which converges for \( \Re(s) \) large enough. \( \square \)
Now we give a bound for the local integral in the unramified case. Suppose the data are unramified as in Section 7. Suppose \( F \) is such that \( q \geq n \), and \( |Q'(y_2)| = 1 \). Let \( f = \Pi_{1}^{n}(\mathcal{O}) \times \mathcal{O}^{*} \).

In the unramified case, \( \rho(g)f(y, 1) \) and \( W_{\rho, s} \) are right invariant by \( K_{G} \) and \( K_{H} \), so after applying the Iwasawa decomposition, we have that

\[
I_{s}(y) = \int_{M_{1}(F)} \int_{G_{1}(F)} \rho(mx)f(y)\delta_{BG}^{-1}(mx) \int_{N^{0}(F)} W_{\rho, s}(wu(mx), a_{y}) \psi(u) dudmdx.
\]

**Lemma 4.4.** Let \( a = (a_{1}, \ldots, a_{n}) \in \text{GL}_{n}(F) \) be such that

\[
|a_{1}| = q^{-k_{1}} \geq |a_{2}| = q^{-k_{2}} \geq \ldots \geq |a_{n}| = q^{-k_{n}},
\]

where \( k_{n} \leq 0 \) (i.e. the positive cone). Let \( W_{\tau} \) be the unramified Whittaker function for \( \tau \). There exists a positive integer \( c_{0} \) which depends on \( \tau \) such that

\[
|W_{\tau}(a)| \leq \delta_{ BG}^{\frac{s}{2}}(a)|q^{-k_{n}c_{0}}|\det(a)|^{-c_{0}}.
\]

**Proof.** The result follows from arguments in [JPSS79, Section 2.4] for \( k_{n} = 0 \) by twisting the corresponding rational representation of \( \text{GL}_{n}(\mathbb{C}) \) in the explicit formula of \( W_{\tau}(a) \) (see [CS80]).

**Lemma 4.5.** For \( \Re(s) \) large enough, we have

\[
I_{s}(y) \leq \zeta_{\nu}(\Re(s)+c_{\tau}) \int_{F^{x}} \int_{F^{x}} |\Pi_{1}^{n}(\mathcal{O})^{*}(a_{1}y_{1}, a_{2}y_{2})||a_{1}|^{\Re(s)+\frac{d_{1}-n-1}{2}c_{0}}|a_{2}|^{\Re(s)+\frac{d_{2}-n-1}{2}c_{0}} d^{x}a_{1}d^{x}a_{2},
\]

where \( c_{0} > 0 \), \( c_{\tau} \) are integers depend only on \( \tau \).

**Proof.** As in Lemma 4.3, and since in the unramified case \( |2| = 1 \), \( \rho(k)f(y) = f(y) \) for \( k \in K_{G} \), we have

\[
I_{s} \leq \int_{F^{x}} \int_{F^{x}} \int_{F^{x}} |f(a_{1}y_{1}, a_{2}y_{2}, b)||a_{1}|^{\frac{d_{1}}{2}}|a_{2}|^{\frac{d_{2}}{2}}|a_{1}a_{2}|^{\Re(s)-\frac{n+1}{2}}||b^{-1}a_{1}a_{2}^{-1}||^{\Re(s)-\frac{n-3}{2}}
\]

\[
\times \int_{w_{N^{0}(F)}w^{-1}} |W_{\rho, s}(wu(w'n'w^{-1}), a_{y}\text{diag}(a_{1}a_{2}, |b^{-1}a_{1}a_{2}^{-1}|, I_{n-2}))dudbd^{x}a_{1}d^{x}a_{2}.
\]

Using the notation as in Lemma 4.1 and Lemma 4.2, we write \( u(w'n'w^{-1}) = nt'k \), by Eq. (4.0.7) and Eq. (4.0.8) we have

\[
|t_{n}| \leq [B(u(w'n'w^{-1}))]^{-2} \leq ([u]|b^{-1}a_{1}a_{2}^{-1}|)^{-2}n^{-2}.
\]

Also, by the property of \( W_{\rho, s} \), we have

\[
|W_{\rho, s}(u(w'n'w^{-1}), a_{y}\text{diag}(a_{1}a_{2}, |b^{-1}a_{1}a_{2}^{-1}|, I_{n-2}))
\]

\[
=|D(u(w'n'w^{-1}))|^\Re(s)+\frac{n+1}{2}|W_{\tau}(a_{y}\text{diag}(a_{1}a_{2}, |b^{-1}a_{1}a_{2}^{-1}|, I_{n-2})t)|.
\]

Thus by Lemma 4.4 and Eq. (4.0.8), for \( \Re(s) \) large we have

\[
|W_{\rho, s}(u(w'n'w^{-1}), a_{y}\text{diag}(a_{1}a_{2}, |b^{-1}a_{1}a_{2}^{-1}|, I_{n-2}))
\]

\[
=|D(u(w'n'w^{-1}))|^\Re(s)+\frac{n+1}{2}|W_{\tau}(a_{y}\text{diag}(a_{1}a_{2}, |b^{-1}a_{1}a_{2}^{-1}|, I_{n-2})t)|.
\]
By Eq. (4.0.8) again, we have

\[
\det(a_y) \leq (|u||b^{-1}a_1a_2^{-1}|)^{-2c_0}.
\]

Thus, we obtain

\[
|\det t|^{-c_0} \leq (|u||b^{-1}a_1a_2^{-1}|)^{-2c_0}.
\]

Therefore,

\[
I_s(y) \leq \int_{F^*} \int_{F^*} \int_{F^*} \left| \Pi_{1}^{1/2} a_1 \left| a_2 \right| |a_1|^{\frac{d_1}{2}} \left| a_2 \right|^{\frac{d_2}{2}} |a_1 a_2|^{\Re(s) - \frac{n-1}{2} - c_0}
\]

\[
\times \left[ |b^{-1}a_1a_2^{-1}| \right]^{\Re(s) - \frac{n-1}{2} - c_0} \int_{\mathbb{N}^2(F)w^{-1}} [u]^{\Re(s) - \frac{n-1}{2} - c_0} dudx a_1 d^x a_2.
\]

Then, for \( \Re(s) \) large, we have

\[
I_s(y) \leq \int_{F^*} \int_{F^*} \int_{F^*} \left| \Pi_{1}^{1/2} a_1 \left| a_2 \right| |a_1|^{\frac{d_1}{2}} \left| a_2 \right|^{\frac{d_2}{2}} |a_1 a_2|^{\Re(s) - \frac{n-1}{2} - c_0}
\]

\[
\times \int_{\mathbb{N}^2(F)w^{-1}} [u]^{\Re(s) - \frac{n-1}{2} - c_0} dudx bd^x a_1 d^x a_2
\]

\[
= \int_{F^*} \int_{F^*} \left| \Pi_{1}^{1/2} a_1 \left| a_2 \right| |a_1|^{\frac{d_1}{2}} \left| a_2 \right|^{\frac{d_2}{2}} |a_1 a_2|^{\Re(s) - \frac{n-1}{2} - c_0}
\]

\[
\times \int_{\mathbb{N}^2(F)w^{-1}} [u]^{\Re(s) - \frac{n-1}{2} - c_0} dudx a_1 d^x a_2.
\]

We have

\[
\int_{\mathbb{N}^2(F)w^{-1}} [u]^{\Re(s) - \frac{n-1}{2} + 2c_0(n-2) - c_0} du \leq 1 + \sum_{k=1}^{\infty} \int_{[u]=q^k} q^{-(\Re(s) + n + \frac{1}{2} - 2c_0(n-2) + c_0)k} du
\]

\[
\leq 1 + \frac{q^{-(\Re(s) + (c_0 + 1)n^2 - (c_0 + 2)n - c_0 - \frac{5}{2})}}{1 - q^{-(\Re(s) + (c_0 + 1)n^2 - (c_0 + 2)n - c_0 - \frac{5}{2})}}
\]

\[
= \zeta_a(\Re(s) - (c_0 + 1)n^2 + (c_0 + 2)n + c_0 + \frac{5}{2}).
\]

Therefore, we obtain

\[
I_s(y) \leq \zeta_a(\Re(s) - (c_0 + 1)n^2 + (c_0 + 2)n + c_0 + \frac{5}{2})
\]

\[
\times \int_{F^*} \int_{F^*} \left| \Pi_{1}^{1/2} a_1 \left| a_2 \right| |a_1|^{\frac{d_1}{2}} \left| a_2 \right|^{\frac{d_2}{2}} |a_1 a_2|^{\Re(s) - \frac{n-1}{2} - c_0} d^x a_1 d^x a_2
\]

\[
= \zeta_a(\Re(s) + c_\tau) \zeta_a(\Re(s) - (c_0 + 1)n^2 + (c_0 + 2)n + c_0 + \frac{5}{2})
\]

\[
\times \int_{F^*} \int_{F^*} \left| \Pi_{1}^{1/2} a_1 \left| a_2 \right| |a_1|^{\frac{d_1}{2}} \left| a_2 \right|^{\frac{d_2}{2}} |a_1 a_2|^{\Re(s) - \frac{n-1}{2} - c_0} d^x a_1 d^x a_2,
\]

where \( c_\tau \) is some integer depending only on \( \tau \). \( \square \)
5. Convergence of the local integral in the Archimedean case

In this section we give bounds for the local integrals in the Archimedean case. Let \( F \) be an Archimedean local field. Let \( K_G \) be the maximal compact subgroup of \( G(F) \), and let \( K_H \) be the maximal compact subgroup of \( H(F) \).

The local integral in the Archimedean case is

\[
I_s(y) := I(f, W_{\rho_{\tau,s}}) = \int_{U_2(F) \backslash G(F)} \rho(g)f(y,1) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuw(g), a_y) \psi_1(u) du dg.
\]

As in the non-Archimedean case, in Lemma 5.1 and Lemma 5.2 we first use techniques similar to those employed in the proof of absolute convergence of the Archimedean local integral for the Rankin-Selberg integral for \( SO_{2\ell+1} \times GL_n \) in [Sou93, Section 5] to bound our inner integral.

For \( t \in GL_n(F) \) let

\[
t' := \begin{pmatrix} t_1 & & w_0 \\ & \ddots & \\ w_0^{-1} & & t_n \end{pmatrix},
\]

(5.0.1)

where \( w_0 \in GL_n(F) \) is the antidiagonal matrix.

Arguing analogously as in [Sou93, Lemma 5.2], we have

**Lemma 5.1.** Let \((n,t',k) \in N_n(F) \times T_H(F) \times K_H\), there is a positive integer \( N \) and \( c_s \in \mathbb{R} > 0 \) depending on \( W_{\rho_{\tau,s}} \) such that

\[
|W_{\rho_{\tau,s}}(ntk,1)| \leq c_s |\det t|^{\Re(s) + \frac{n-1}{2}} \|t\|^N
\]

for \( t = \text{diag}(t_1,t_2,\ldots,t_{n-1},1) \), where

\[
\|t\|^2 = 1 + \sum_{i=1}^{n-1} |t_i|^2 + \sum_{i=1}^{n-1} |t_i|^{-2}.
\]

□

**Lemma 5.2.** For \( v \in H(F) \) as defined in Eq. (4.0.2) and \( \Re(s) \) large,

\[
\int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}(uv,1)| du
\]

converges.

**Proof.** As in Lemma 4.2, for \( u \in wN^\circ(F)w^{-1} \) and \( v \) a unipotent element of the form Eq. (4.0.2), we denote the Iwasawa decomposition of \( uv \) as \( uv = nt'k \) where \((n,t',k) \in N_n(F) \times T_H(F) \times K_H\), and we denote the \( i \)-th line of \( uv \) as \((uv)_i\).

By Lemma 5.1, there is a positive integer \( N \) such that for \( \xi_{\tau,s} \) there is \( c_s \in \mathbb{C} \) for \( \xi_{\tau,s} \) such that

\[
|W_{\rho_{\tau,s}}(nt'k,1)| \leq c_s |\det t|^{\Re(s) + \frac{n-1}{2}} |w_\tau(t_n)||t||^N.
\]

(5.0.2)

Here \( t = \text{diag}(t_1t_2\ldots t_n, t_2 \ldots t_n), \|t\| = \|t_n^{-1}t\| \), and \( w_\tau \) is the central character of \( \tau \). We assume \( N \) is even, then \( \|a\|^N \) is a sum of positive quasicharacters.
As in the non-Archimedean case, we denote \( \mathcal{L}(uv) = \left( \begin{array}{c} (uv)_{n+2} \\ \vdots \\ (uv)_{2n+1} \end{array} \right) \).

Using the technique as in [Sou93, Section 7.3, Lemma 3] we get

\[
(1 + \|\mathcal{L}(uv)\|^2)^{-\frac{n}{2}} \leq \det(t) \leq (1 + \|\mathcal{L}(uv)\|^2)^{-\frac{1}{2}}.
\]

Here \( \|\mathcal{L}(uv)\| \) denotes the standard norm on \( M_{n \times (2n+1)}(F) \), and

\[
(5.0.3) \quad \max\{ \frac{t_j}{t_{j+1}}, \frac{t_{j+1}}{t_j} \} \leq (1 + \|\mathcal{L}(uv)\|^2)^{n}, \quad j = 1, \ldots, n - 1.
\]

Similar to the non-Archimedean case, we have

\[
(1 + \|\mathcal{L}(uv)\|)^{\frac{1}{2}} = (1 + \sup\{\|v_1\|, \|v_2\|, \|v_3\|, \|T\|, \|\frac{1}{2}v_2c^2\|, \|v_3c\|\})^{-\frac{1}{2}}
\]

\[
\leq (1 + \sup\{\|v_1\|, \|v_2\|, \|v_3\|, \|T\|\})^{-\frac{1}{2}}
\]

\[
= (1 + \|u\|)^{-\frac{1}{2}},
\]

where \( \|\cdot\| \) denotes the standard matrix norms.

By Eq. (5.0.2) we have

\[
(5.0.4) \quad |W_{\rho, s}(uv, 1)| \leq \sum_j c_s(1 + \|u\|)^{-\frac{\Re(s)}{2} - \frac{n-2}{4}}|\chi_j(t)|.
\]

Here the sum is finite and the \( \chi_j \) are positive quasi-characters which depend on \( \tau \).

By Eq. (5.0.3), we have

\[
|\chi_j(t)| \leq (1 + \|\mathcal{L}(uv)\|^2)^C \leq (1 + \|u\|^2)^C(1 + \|v\|^4)^C
\]

for some positive constant \( C \) which depends on \( \tau \).

Thus we have

\[
(5.0.5) \quad \int_{wN^\circ(F)w^{-1}} |W_{\rho, s}(uv, 1)| du \leq \sum_j c_s(1 + \|v\|^4)^C \int_{wN^\circ(F)w^{-1}} (1 + \|u\|)^{-\frac{\Re(s)}{2} - \frac{n-2}{4} + C} du.
\]

The sum over \( j \) is finite. Since by definition \( \|\cdot\| \) is positive, the integral converges for \( \Re(s) \) large.

We now proceed to bound our local integral \( I_s(y) \).

**Lemma 5.3.** For \( y = (y_1, y_2) \in \mathbb{P}Y'(F) \), we have that

\[
I_s(y) \ll \int_{F^\times} \int_{F^\times} f'(a_1y_1, a_2y_2)|a_1|^\Re(s) + \frac{d_1-n-1}{2} - c_0|a_2|^\Re(s) + \frac{d_2-n-1}{2} - c_0 \, dx_1 \, dx_2
\]

for \( \Re(s) \) large enough where \( f' \in \mathcal{S}(V(F)) \) is nonnegative and \( c_0 \in \mathbb{R}_{\geq 0} \) depends only on \( \tau \) as in Lemma 4.5.
Proof. Applying the Iwasawa decomposition of $G(F)$ with respect to the standard Borel subgroup, we have

$$I_s(y) = \int_{T_G(F)} \int_{K_G} \rho(ak) f(y,1) \delta_{B_G}^{-1}(a) \int_{\mathbb{A}^2} W_{\rho_{\tau,s}}(wuu(ak), a_y) \psi_1(u) dud^\times a_1 d^\times a_2 dk.$$  

By the action of Weil representation it suffices to bound

$$\int_{F^\times} \int_{F^\times} \int_{F^\times} \int_{K_G} \rho(k) f(a_1y_1, a_2y_2, b) ||a_1|^{\frac{d_1}{2}} |a_2|^{\frac{d_2}{2}} |a_1a_2|^{-2} \times \int_{N^\circ(F)} |W_{\rho_{\tau,s}}|(wuu \left( \begin{array}{c} a_1 \\ a_1^{-1}b^{-1} \end{array} \right), \left( \begin{array}{c} a_2 \\ a_2^{-1}b \end{array} \right)) \iota(k), a_y) dud^\times bd^\times a_1 d^\times a_2 dk.$$

Here $t \left( \left( \begin{array}{c} a_1 \\ a_1^{-1}b^{-1} \end{array} \right), \left( \begin{array}{c} a_2 \\ a_2^{-1}b \end{array} \right) \right)$ is

\[
\begin{pmatrix}
I_{n-1} & a_1 a_2 & \frac{1}{2}+\frac{1}{4}(b^{-1}a_1a_2^{-1}+b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & 2(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) \\
\frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & -\frac{1}{4}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) \\
\frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) \\
\frac{1}{4}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{4}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{4}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) & \frac{1}{4}(b^{-1}a_1a_2^{-1}b_{a_2}^{-1}a_2) \\
\end{pmatrix}^{(a_1a_2)^{-1}} I_{n-1}.
\]

As in the non-Archimedean case, applying the Iwasawa decomposition of $SO_3(F)$ we have for $b \in F^\times$

$$\left( \begin{array}{c}
\frac{1}{2}+\frac{1}{4}(b+b^{-1}) \\
\frac{1}{2}(b+b^{-1}) \\
\frac{1}{4}(1-b^{-1}) \\
\frac{1}{4}(b+1)
\end{array} \right) = \left( \begin{array}{c}
1-c-\frac{1}{2}c^2 \\
0 \\
1-c-a^{-1}c \\
0
\end{array} \right) \left( \begin{array}{c}
k'
\end{array} \right).$$

where $k' \in K_{SO_3} \subset K_H$.

For $F = \mathbb{R}$, let $K' = \text{diag}(SO(2,\mathbb{R}),1)$ and $S$ be such that $^tSJ'S = J_3$, where $J' = \text{diag}(-I_2,1)$. We have $K_{SO_3} = ^tSK'S$. Let $||\cdot||$ denote the Euclidean vector norm. Using the above decomposition, we have

$$||\left( \frac{1}{2} - \frac{1}{4}(b+b^{-1}), -\frac{1}{4}(b-b^{-1}), \frac{1}{2} + \frac{1}{4}(b+b^{-1}) \right)|| = ||(0,0,a^{-1})k||,$$

$$||(b-b^{-1}), \frac{1}{2}(b+b^{-1}), -\frac{1}{2}(b-b^{-1})|| = ||(0,1,-ca^{-1})k'||.$$  

Since the action of $K_{SO_3}$ preserve the Euclidean norm, we get

$$|a^{-1}| = ||(0,0,a^{-1})|| = ||\left( \frac{1}{2} - \frac{1}{4}(b+b^{-1}), -\frac{1}{4}(b-b^{-1}), \frac{1}{2} + \frac{1}{4}(b+b^{-1}) \right)||,$$

$$|ca^{-1}| \leq ||(0,1,-ca^{-1})|| = ||(b-b^{-1}), \frac{1}{2}(b+b^{-1}), -\frac{1}{2}(b-b^{-1})||.$$  

For $F = \mathbb{C}$, $K_{SO_3} \cong SO(3,\mathbb{R})$. Similar as in the real case, the action of $K_{SO_3}$ preserve $||\cdot||$. We have

$$|a^{-1}| = ||(0,0,a^{-1})||^2 = ||\left( \frac{1}{2} - \frac{1}{4}(b+b^{-1}), -\frac{1}{4}(b-b^{-1}), \frac{1}{2} + \frac{1}{4}(b+b^{-1}) \right)||^2,$$

$$|ca^{-1}| \leq ||(b-b^{-1}), \frac{1}{2}(b+b^{-1}), -\frac{1}{2}(b-b^{-1})||^2.$$  

Therefore in both cases, we have

$$|a| \ll \max(|b|, |b^{-1}|)^{-1} = \min(|b|, |b^{-1}|),$$
\[ |ca^{-1}| \ll \max(|b|, |b^{-1}|). \]

We denote \(|b| = b|b| 1\) and \(|b| = b^{-1}\) otherwise. Since
\[
\begin{pmatrix} 1 + \frac{1}{2}c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 + \frac{1}{2}c^2 & -\frac{1}{2}c^2a^{-2} \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
the local integral is majorized by
\[
\int_{F^c \times \mathbb{F}^c \times \mathbb{F}^c \times \mathbb{Q}^c} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{\frac{1}{2}d_2 - 2}|a_2|^{\frac{1}{2}d_2 - 2}|a_1 a_2||\mathbb{N}(s) - \frac{n+1}{2}||b^{-1} a_1 a_2^{-1}||\mathbb{N}(s) - \frac{n-3}{2}
\times \int_{\mathbb{Q}^c(F)} |W_{\rho, s'}|(w n w^{-1}) \chi_a \text{diag}(a_1 a_2, [b^{-1} a_1 a_2^{-1}], I_{n-2}) dudx a_1 d^\times a_2 dk.
\]
This is
\[
\int_{F^c \times \mathbb{F}^c \times \mathbb{F}^c \times \mathbb{Q}^c} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{\frac{1}{2}d_2 - 2}|a_2|^{\frac{1}{2}d_2 - 2}|a_1 a_2||\mathbb{N}(s) - \frac{n+1}{2}||b^{-1} a_1 a_2^{-1}||\mathbb{N}(s) - \frac{n-3}{2}
\times \int_{\mathbb{Q}^c(F)w^{-1}} |W_{\rho, s'}|(u (w n w^{-1}) w k'' \chi(k), a_y \text{diag}(a_1 a_2, [b^{-1} a_1 a_2^{-1}], I_{n-2}) dudx bdx a_1 d^\times a_2 dk,
\]
where
\[
\begin{pmatrix} I_{n-1} \\ 1 \\ -\frac{1}{2c^2a^{-2} - 2a^{-1}} \\ 1 \end{pmatrix}, \quad k'' = \begin{pmatrix} I_{n-1} \\ k' \\ I_{n-1} \end{pmatrix}.
\]

Since
\[
wnw^{-1} = \begin{pmatrix} 1 \\ I_{n-2} \\ 1 \\ -\frac{1}{2c^2a^{-2} - 2a^{-1}} \\ 1 \end{pmatrix},
\]
by Eq. (5.0.4) and Eq. (5.0.5) in Lemma 5.2, the local integral is majorized by
\[
\sum_j c_j \int_{F^c \times \mathbb{F}^c \times \mathbb{F}^c \times \mathbb{Q}^c} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{\frac{1}{2}d_2 - 2}|a_2|^{\frac{1}{2}d_2 - 2}|a_1 a_2||\mathbb{N}(s) - \frac{n+1}{2}
\times |b^{-1} a_1 a_2^{-1}||\mathbb{N}(s) - \frac{n-3}{2} \chi_1 \text{diag}(a_1 a_2, [b^{-1} a_1 a_2^{-1}], I_{n-2}) (1 + ||u|| - \frac{1}{2c^2a^{-1}}) \text{d}x \text{d}k \text{d}x a_1 d^\times a_2 dk.
\]
Since \(|ca^{-1}| \ll |b^{-1}|^{-1}\), the above sum is majorized by
\[
\sum_j c_j \int_{F^c \times \mathbb{F}^c \times \mathbb{F}^c \times \mathbb{Q}^c} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{\frac{1}{2}d_2 - 2}|a_2|^{\frac{1}{2}d_2 - 2}|a_1 a_2||\mathbb{N}(s) - \frac{n+1}{2}
\times |b^{-1} a_1 a_2^{-1}||\mathbb{N}(s) - \frac{n-3}{2} \chi_1 j (-4Q'(y_2)a_1 a_2)|\chi_2 \text{d}x \text{d}k \text{d}x a_1 d^\times a_2 dk.
\]
Here \(\chi_1, \chi_2\) are positive quasi-characters, and \(C'\) is some positive integer depends only on \(\tau\).
Thus for $\Re(s)$ large, our sum is majorized by
\[
\sum_j c_j \int_{F^*} \int_{F^*} \int_{F^*} \int_{F^*} \int_{K^G} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{d_1-2} |a_2|^{d_2-2}
\times |a_1 a_2|^{\Re(s) - \frac{n+1}{2}} | - 4Q'(y_2) |c_j| a_1 a_2 |c_j| d^x b d^x a_1 d^x a_2 dk,
\]
where $c_j$ is some integer depends on $\tau$. This is majorized by
\[
\sum_j c_j \int_{F^*} \int_{F^*} \int_{F^*} \int_{F^*} \int_{K^G} |\rho(k) f(a_1 y_1, a_2 y_2, b)||a_1|^{d_1-2} |a_2|^{d_2-2}
\times |a_1 a_2|^{\Re(s) - \frac{n+1}{2}} |a_1 y_1|^{c_j} |a_2 y_2|^{c_j} d^x b d^x a_1 d^x a_2 dk.
\]

Then for $\Re(s)$ large, the local integral is majorized by a constant times a finite sum of integrals of the form
\[
\int_{F^*} \int_{F^*} \int_{K^G} |\rho(k) \tilde{f}(a_1 y_1, a_2 y_2)||a_1|^{\Re(s) + \frac{d_1 - n - 1}{2}} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2}} d^x a_1 d^x a_2,
\]
where $\tilde{f} \in S(V(\mathcal{F}))$ and $\tilde{f}(v) = \int_{K^G} \rho(k) f(v)$ for $v \in V(\mathcal{F})$. Then we deduce the lemma. \(\square\)

6. Absolute Convergence

In this section we handle the absolute convergence of the sum of the global integral in the main theorem (Theorem 3.2), which will make the proof of Theorem 3.2 rigorous. The proof follows from the absolute convergence of the local integrals in the Archimedean case and the non-Archimedean case.

Lemma 6.1. The sum of the global integral
\[
\sum_{y \in \mathbb{P}^n(\mathcal{F})} I(f, W_{\xi_y})(y) = \sum_{y \in \mathbb{P}^n(\mathcal{F})} \int_{U_2(\mathcal{A}) \setminus G(\mathcal{A})} \rho(g) f(y, 1) \int_{N^0(\mathcal{A})} W_{\xi_y}(wu(g), 1) \psi(u) du dg
\]
converges absolutely for $\Re(s)$ large enough.

Proof. Let $y = (y_1, y_2) \in \mathbb{P}^n(\mathcal{F})$. Let $S$ be a finite set of places of $\mathcal{F}$ which includes the infinite places and all the finite places such that $q_v < n$, $f^S = 1_{V(\hat{\mathcal{O}})^s} \prod_{v \in S} \mathfrak{o}_v^{\tau_v}$, $\tau_v$ unramified, $|Q'(y_2)|_v = 1$ and $\rho_v(k)f_v = f_v$ for $k \in G(\mathcal{O}_v)$ for $v \not\in S$.

Using the results and notations of Lemma 4.3, Lemma 4.5 and Lemma 5.3, for $\Re(s)$ large we have
\[
I(f, W_{\xi_y})(y) = \prod_{v \mid \infty} I_v(f_v, W_{\rho_v})(y) \prod_{v \not\in S} I_v(f_v, W_{\rho_v})(y) \prod_{v \not\in S} I_v(f_v, W_{\rho_v})(y)
\ll \prod_{v \mid \infty} \int_{F_v^c} \int_{F_v^c} |f_v'(a_1 y_1, a_2 y_2)||a_1|^{\Re(s) + \frac{d_1 - n - 1}{2}} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2}} d^x a_1 d^x a_2
\]
This is a product of mirabolic Eisenstein series which converge absolutely for \( \Re \) which, when \( G \) and that the residual characteristic is not 2. We also assume that \( d \) \( \text{GL} \) the character \( \psi \) \( I \) \( \text{integral} \) \( □ \) enought (see \[ JS81a \]).

\[ f \] Here \( ( \times = y \)

\[ f, W \]

We assume all data are unramified, i.e. the local field \( F \) is absolutely unramified, and the character \( \psi \) is unramified. Let \( \tau \) be an irreducible unramified generic representation of \( \text{GL}_n(F) \). We denote \( K_G = G(\mathcal{O}) \). We assume that the matrices of \( Q \) and \( Q' \) are invertible and that the residual characteristic is not 2. We also assume that \( d_2 > d_1 \).

Let \( f \in S(V(F) \times F^\times) \) be \( f(v, u) = 1_{V(\mathcal{O}) \times O^\times} (v, u) \) for \( v \in V(F) \), \( u \in F^\times \). Let \( \rho \) denote the local Weil representation of \( G(F) \). Then \( \rho(k)f = f \) for \( k \in K_G \).

Let

\[ W_{\rho, s} \in \rho_{\tau, s} = \text{Ind}_{Q_n(F)}^{H(F)} (\mathcal{W}(\tau, \psi_0) \otimes | \det |^{\frac{s}{2}}) \]

Here \( f' \in S(V(\mathbb{A}_F)) \), and \( c_0 > 0 \) is a constant depends only on \( \tau \).

Thus the sum of the global integral is majorized by a finite sum of a sum of integrals of the form

\[ \sum_{y \in \mathcal{F}(\mathbb{F}(F))} \zeta_F(\Re(s) + c_\tau) \int_{\mathbb{A}_F} \int_{\mathbb{A}_F} |f''(a_1 y_1, a_2 y_2)||a_1|^{\Re(s) + \frac{d_1 - n - 1}{2} - c_0} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2} - c_0} d^\times a_1 d^\times a_2, \]

which, when \( \Re(s) \) is large, is majorized by

\[ \sum_{y \in \mathcal{F}(\mathbb{F}(F))} \zeta_F(\Re(s) + c_\tau) \int_{\mathbb{A}_F} \int_{\mathbb{A}_F} |f'(a_1 y_1, a_2 y_2)||a_1|^{\Re(s) + \frac{d_1 - n - 1}{2} - c_0} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2} - c_0} d^\times a_1 d^\times a_2 \]

\[ = \sum_{y \in \mathcal{F}(\mathbb{F}(F))} \zeta_F(\Re(s) + c_\tau) \int_{F^\times \backslash \mathbb{A}_F} \int_{F^\times \backslash \mathbb{A}_F} |f'(a_1 y_1, a_2 y_2)||a_1|^{\Re(s) + \frac{d_1 - n - 1}{2} - c_0} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2} - c_0} \times d^\times a_1 d^\times a_2. \]

This is a product of mirabolic Eisenstein series which converge absolutely for \( \Re(s) \) large enough (see \[ JS81a \]).

\[ \Box \]

### 7. Unramified Computation

In this section we give the computation for the unramified local factor of the global integral \( I(f, W_{\rho, s}) \). The results of this section shed light on the nature of the local integrals \( I(f, W_{\rho, s}) \).

We assume all data are unramified, i.e. the local field \( F \) is absolutely unramified, and the character \( \psi \) is unramified. Let \( \tau \) be an irreducible unramified generic representation of \( \text{GL}_n(F) \). We denote \( K_G = G(\mathcal{O}) \). We assume that the matrices of \( Q \) and \( Q' \) are invertible and that the residual characteristic is not 2. We also assume that \( d_2 > d_1 \).

Let \( f \in S(V(F) \times F^\times) \) be \( f(v, u) = 1_{V(\mathcal{O}) \times O^\times} (v, u) \) for \( v \in V(F) \), \( u \in F^\times \). Let \( \rho \) denote the local Weil representation of \( G(F) \). Then \( \rho(k)f = f \) for \( k \in K_G \).

Let

\[ W_{\rho, s} \in \rho_{\tau, s} = \text{Ind}_{Q_n(F)}^{H(F)} (\mathcal{W}(\tau, \psi_0) \otimes | \det |^{\frac{s}{2}}) \]
be the unique spherical vector satisfying $W_{\rho_{r,s}}(1,1) = 1$. Here, as the notation indicates, we are viewing the induced representation as a space of smooth functions taking values in the Whittaker model.

The Satake parameter for $\rho_{r,s}$ is defined (up to a permutation) by

$$
t_{\rho_{r,s}} = \text{diag}(\chi_{1,s}(\varpi), \ldots, \chi_{n,s}(\varpi), \chi_{n,s}(\varpi)^{-1}, \ldots, \chi_{1,s}(\varpi)^{-1}) \in \text{Sp}_{2n}(\mathbb{C}).$$

Here each $\chi_i : F^\times \to \mathbb{C}^\times$ is an unramified character and

$$\chi_i := \chi_i \cdot |\cdot|^{s-\frac{1}{2}}.$$

We set

$$\chi_s := \chi_{1,s} \otimes \cdots \otimes \chi_{n,s} : (F^\times)^n \to \mathbb{C}^\times.$$ 

For an unramified character $\mu$ of split $\text{SO}_2(F)$, the Satake parameter is

$$t_\mu = \text{diag}(\mu(\varpi), \mu(\varpi)^{-1}) \in \text{SO}_2(\mathbb{C}).$$

We then have the local integral

$$I_s(y) := I(f, W_{\rho_{r,s}})(y) = \int_{U_2(F) \backslash G(F)} \rho(g)f(y,1) \int_{N^\times(F)} W_{\rho_{r,s}}(wu(g), a_y)\psi_1(u)dudg,$$

as the unramified local component of the global integral $I(f, W_{\rho_{r,s}})(y)$ by construction. We assume $y$ is integral. In order to compute the local integral $I_s$ we will adapt a procedure in [Kap12]. This requires us to write the function $\rho(g)f(y,1)$ in terms of functions lying in an appropriate Whittaker model.

Let

$$\alpha(y_1, y_2) = \prod_{i=1}^2 (2 + \sum_{k'=1}^{\text{val}(y_i)} q^{k'} (1 - (q - 1) \sum_{k=1}^{\text{val}(y_i)} q^{\frac{d_i - 2k}{2}}))^{-1}.$$

Let $H_{1,y}, H_{2,y} \in C^\infty(T_G(F))$ be such that

$$H_{1,y} \left( \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \right) = |a_1|^{\frac{d_1}{2} - 2} |a_2|^{\frac{d_2}{2} - 2},$$

$$H_{1,y} \left( \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \right) = \alpha(y_1, y_2),$$

where $a_i, d \in F^\times$.

Using the Iwasawa decomposition with respect to the Borel subgroup of $G(F)$ consisting of lower triangular matrices in $G(F)$, we define functions $\Phi_{1,y}, \Phi_{2,y} \in C^\infty(U_2(F) \backslash G(F)/K_G)$ (where $U_2(F)$ is the unipotent radical of the lower Borel subgroup of $G(F)$) by

$$\Phi_{1,y}(g) : G(F) \to \mathbb{C}$$

$$(\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} a & d \\ 0 & a d^{-1} \end{pmatrix}) \kappa_1, \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (a'd')^{-1} \end{pmatrix} \kappa_2) \mapsto H_{1,y} \left( \begin{pmatrix} a & d \\ 0 & a d^{-1} \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (a'd')^{-1} \end{pmatrix} \right),$$

and

$$\Phi_{2,y}(g) : G(F) \to \mathbb{C}$$

$$(\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} a & d \\ 0 & a d^{-1} \end{pmatrix}) \kappa_1, \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (a'd')^{-1} \end{pmatrix} \kappa_2) \mapsto H_{2,y} \left( \begin{pmatrix} a & d \\ 0 & a d^{-1} \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (a'd')^{-1} \end{pmatrix} \right).$$
Here $a, a', d \in F^\times$, $t_i \in F$, and $(\kappa_1, \kappa_2) \in K_G$.

We let $\Phi_y = \Phi_{1,y} \Phi_{2,y}$, and $H_y = H_{1,y} H_{2,y}$. We first show that an appropriate integral of $\Phi_y$ is $\rho(g) f(y,1)$ in the following lemma:

**Lemma 7.1.** We have

\[
(7.0.4) \quad \rho(g) f(y,1) = \int_{U_2(F)} \Phi_y(n g) \psi_{U_2,Q}(n)^{-1} d n,
\]

where $\psi_{U_2,Q}(n) = \psi(n_1 Q(y_1) + n_2 Q'(y_2))$ for $n = ((\begin{smallmatrix} 1 & n_1 \\ 0 & 1 \end{smallmatrix}))$, $((\begin{smallmatrix} n_2 \\ 0 \end{smallmatrix})) \in U_2(F)$.

**Proof.** As a function of $g$, both sides of the equality are invariant under $K_G$ on the right and both transform via the same character under $U_2(F)$ on the left. Thus it suffices to verify the equality Eq. (7.0.4) for

\[
\begin{align*}
g_1 &= \left( \begin{smallmatrix} a_1 & \nu_1 \\ \nu_1 & a_1^{-1} \end{smallmatrix} \right), \\
g_2 &= \left( \begin{smallmatrix} 1 & d \\ d & 1 \end{smallmatrix} \right).
\end{align*}
\]

We have

\[
\rho \left( \left( \begin{smallmatrix} a_1 & \nu_1 \\ \nu_1 & a_1^{-1} \end{smallmatrix} \right), \left( \begin{smallmatrix} a_2 & \nu_2 \\ \nu_2 & a_2^{-1} \end{smallmatrix} \right) \right) f(y,1) = |a_1|^d |a_2|^f \mathbb{I}_{V(\mathcal{O})}(a_1 y_1, a_2 y_2)
\]

\[
= |a_1|^d |a_2|^f \mathbb{I}_{\mathcal{O}^{0}}(a_1) \mathbb{I}_{\mathcal{O}^{(1)}}(a_2),
\]

\[
\rho \left( \left( \begin{smallmatrix} 1 & d \\ d & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & d^{-1} \\ d^{-1} & 1 \end{smallmatrix} \right) \right) f(y,1) = \mathbb{I}_{\mathcal{O}^{0}}(d) \mathbb{I}_{\mathcal{O}^{(d)}}(d^{-1}) |d|^d.
\]

On the other hand, the right hand side of Eq. (7.0.4) in the two cases are

\[
\begin{align*}
\int_{F^2} \Phi_y \left( \left( \begin{smallmatrix} n_1 & 1 \\ 1 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} a_1 & \nu_1 \\ \nu_1 & a_1^{-1} \end{smallmatrix} \right) \right) \psi(n_1 Q(y_1) + n_2 Q'(y_2)) d n_1 d n_2,
\int_{F^2} \Phi_y \left( \left( \begin{smallmatrix} n_1 & 1 \\ 1 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} d & \nu_1 \\ \nu_1 & d^{-1} \end{smallmatrix} \right) \right) \psi(n_1 Q(y_1) + n_2 Q'(y_2)) d n_1 d n_2.
\end{align*}
\]

Let $\Phi_y(g_1, g_2) = \Phi_y'(g_1) \Phi_y''(g_2)$ where $(g_1, g_2) \in G(F)$, and

$\Phi_y', \Phi_y'' \in C^\infty(\overline{U}_2(F) \setminus SL_2(F)/SL_2(\mathcal{O}_F))$.

By symmetry, it suffices to verify the equalities

\[
\begin{align*}
\mathbb{I}_{\mathcal{O}^{(1)}}(\mathcal{O}_F)(a) |a|^d = & \int_{F} \Phi_y (\left( \begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix} \right)) \psi(n Q(y_1)) d t, \\
\mathbb{I}_{\mathcal{O}^{0}}(\mathcal{O}_F)(d) = & \int_{F} \Phi_y (\left( \begin{smallmatrix} 1 & 1 \\ d & 1 \end{smallmatrix} \right)) \psi(n Q(y_1)) d t.
\end{align*}
\]

for $a, d \in F^\times$. 
Applying the Iwasawa decomposition with respect to the lower Borel subgroup of $\text{SL}_2(F)$ to \((1 \, ^{n}_{1} \, ^{a}_{-1})\) and \((1 \, ^{n}_{1} \, ^{1}_{d})\), we have

\[
\begin{align*}
(1 \, ^{n}_{1} \, ^{a}_{-1}) &= \begin{cases} 
( ^{a}_{-1}) & \text{for } |a^{-2}n| \leq 1 \\
( ^{1}_{n-1} \, ^{a^{-1}n}_{-1}) & \text{for } |a^{-2}n| > 1 
\end{cases} \\
(1 \, ^{n}_{1} \, ^{1}_{d}) &= \begin{cases} 
( ^{1}_{d}) & \text{for } |dn| \leq 1 \\
( ^{1}_{n-1} \, ^{dn}_{-1}) & \text{for } |dn| > 1 
\end{cases}
\end{align*}
\]

Then it suffices to verify the equalities

\[
\mathbb{I}_{\varphi_{\text{val}(y_{1})}\mathcal{O}_{F}}(a)|a|^{\frac{d_{1}}{2}} = \int_{|n| \leq |a|^{2}} H_{y}(( ^{a}_{-1}))\psi(n\mathcal{Q}(y_{1}))^{-1}dn
\]

\[
+ \int_{|a|^{2} < |n|} H_{y}(( ^{a^{-1}n}_{-1}))\psi(n\mathcal{Q}(y_{1}))^{-1}dn,
\]

\[
\mathbb{I}_{\varphi_{\text{val}(y_{1})}\mathcal{O}_{F}}(d) = \int_{|n| \leq |d|^{-1}} H_{y}(( ^{1}_{d}))\psi(n\mathcal{Q}(y_{1}))^{-1}dn
\]

\[
+ \int_{|d|^{-1} < |n|} H_{y}(( ^{dn}_{-1}))\psi(n\mathcal{Q}(y_{1}))^{-1}dn.
\]

We first verify Eq. (7.0.5).

For \(|a| > q^{\text{val}(y_{1})}\), \(H_{y}(a) = 0\) and \(H_{y}(a^{-1}n) = 0\) when \(|a|^{2} < n\), thus the right hand side of Eq. (7.0.5) is 0, while the left hand side is also 0. In particular, we have the equality for \(|a| > q^{\text{val}(y_{1})}\).

For \(|a| = q^{\text{val}(y_{1})}\), \(H_{y}(a^{-1}n) = 0\) when \(|a|^{2} < n\), thus we have that the right hand side of Eq. (7.0.5) is

\[
\int_{|n| \leq |a|^{2}} H_{y}(a)\psi(n\mathcal{Q}(y_{1}))^{-1}dn = H_{y}(a) \int_{|n| \leq q^{2\text{val}(y_{1})}} \psi(n\mathcal{Q}(y_{1}))^{-1}dn
\]

\[
= q^{2\text{val}(y_{1})}H_{y}(a) \int_{\mathcal{O}_{F}} \psi(n\varphi^{-2\text{val}(y_{1})}\mathcal{Q}(y_{1}))^{-1}dn
\]

\[
= q^{2\text{val}(y_{1})}H_{y}(a)
\]

\[
= q^{2\text{val}(y_{1})}|a|^{\frac{d_{1}}{2}}\mathbb{I}_{\varphi_{\text{val}(y_{1})}\mathcal{O}_{F}}(a)
\]

\[
= q^{\frac{d_{1}}{2}\text{val}(y_{1})},
\]

so Eq. (7.0.5) is valid in this case.

For \(|a| < q^{\text{val}(y_{1})}\), we have that the left hand side of Eq. (7.0.5) is

\[
H_{y}(a) \int_{|n| \leq |a|^{2}} \psi(n\mathcal{Q}(y_{1}))^{-1}dn + \int_{|a|^{2} < |n|} H_{y}(a^{-1}n)\psi(n\mathcal{Q}(y_{1}))^{-1}dn
\]

\[
= q^{-2\text{val}(a)}H_{y}(a) + \int_{q^{-\text{val}(a)} < |n| \leq q^{\text{val}(y_{1})}} H_{y}(n)\psi(n\mathcal{Q}(y_{1}))^{-1}q^{-\text{val}(a)}dn
\]
We justify this step by showing the above integral converges absolutely for $\Re(s)$ large.

This exhausts the cases and proves the lemma. □

For Eq. (7.0.6), the right hand side is

$$
\int_{|n| \leq |d|^{-1}} H'_{y}((1_d)) \psi(nQ(y_1))^{-1}dn + \int_{|d|^{-1} < |n|} H'_{y}((d_n^{-1})) \psi(nQ(y_1))^{-1}dn
$$

$$
= H'_{y}((1_d)) \int_{|n| \leq |d|^{-1}} \psi(nQ(y_1))^{-1}dn + \int_{|d|^{-1} < |n|} H'_{y}((d_n^{-1})) \psi(nQ(y_1))^{-1}dn.
$$

When $d \in \mathcal{O}^\times$, the above expression is

$$
\frac{1}{2 + \sum_{k=1}^{\val(y_1)} q^{k'}(1 - (q - 1) \sum_{k=1}^{\val(y_1)} q^{(\frac{d^2}{2} - 2)k})} (1 + \int_{|dn| > 1} H'_{y}((d_n^{-1})) \psi(nQ(y_1))^{-1}dn) = 1.
$$

This is equal to the left hand side of Eq. (7.0.6).

Finally, when $d \not\in \mathcal{O}^\times$, both the left hand side and right hand side of Eq. (7.0.6) are 0. This exhausts the cases and proves the lemma.

Inserting the result of Lemma 7.1 to the local integral $I$, we obtain

$$
I_s(y) = \int_{U_2(F) \setminus G(F)} \int_{U_2(F)} \Phi_y(n|g) \psi_{U_2,Q}(n)^{-1}dn \int_{N^\circ(F)} W_{\rho_{r,s}}(wu(g), a_y) \psi_1(u)dudg.
$$

Then by collapsing the integrals we get

$$
(7.0.7) \quad I_s(y) = \int_{G(F)} \Phi_y(g) \int_{N^\circ(F)} W_{\rho_{r,s}}(wu(g), a_y) \psi_1(u)dudg.
$$

We justify this step by showing the above integral converges absolutely for $\Re(s)$ large in Lemma 8.2 below.

Since $\Phi_y \in C^\infty(\mathcal{U}_2(F) \setminus G(F)/K_G)$, applying the Iwasawa decomposition with respect to the lower Borel subgroup $B_G(F)$ of $G(F)$ we get

$$
I_s(y) = \int_{T_G(F)} \Phi_y(g) \delta_{B_G}^{-1}(g) \int_{U_2(F)} \int_{N^\circ(F)} W_{\rho_{r,s}}(wu(g), a_y) \psi_1(u)dudydg
$$

$$
= \int_{A_1(F)} \int_{M_1(F)} \int_{G_1(F)} \Phi_y(AMx) \delta_{B_G}^{-1}(AMx)
$$

$$
\times \int_{F^\times} \int_{F^\times} \int_{F^\times} W_{\rho_{r,s}}(wu(AMx), a_y) \psi_1(u)dudyadm dx
$$

$$
= \int_{F^\times} \int_{F^\times} \int_{F^\times} \Phi_y\left(\begin{pmatrix} a & -1 \cr -1 & (MB)^{-1} \end{pmatrix}, \left(\begin{pmatrix} m & b \\ (am)^{-1} & (am)^{-1} \end{pmatrix} \right)\right) |am|^2
$$

$$
\times \int_{F^\times} \int_{F^\times} \int_{F^\times} W_{\rho_{r,s}}\left(\begin{pmatrix} am & ab \cr (am)^{-1} & (am)^{-1} \end{pmatrix}, a_y \right) \psi_1(u)dudyadm dxb
$$
\[ I_F = \left[ -\frac{2\pi i}{\log q} \right], \sigma_1, \sigma_2 \in \mathbb{R}, \text{ and} \]

\[ c_q = \left( \frac{\log q}{2\pi i} \right)^\sigma. \]

Let \( a_i \in F^\times \), and

\[ H_{y,s_2}(1) = \int_{F^\times} H_{2,y} \left( \left( \frac{1}{a_1} \right), \left( \frac{a_2}{(a_1 a_2)_{a-1}} \right) \right) \chi_{s_1,s_2}^{-1} \left( \left( \frac{1}{a_1} \right), \left( \frac{a_2}{(a_1 a_2)_{a-1}} \right) \right) d^x a_1 d^x a_2. \]

By Mellin inversion, for \( \left( \left( \frac{1}{(m b)_{a-1}} \right), \left( \frac{m}{b} \right) \right) \in M_1(F)G_1(F) \), we have

\[ H_{2,y} \left( \left( \frac{1}{(m b)_{a-1}} \right), \left( \frac{m}{b} \right) \right) = c_q \int \int_{iF+\sigma_1} H_{y,s_2}(1) \chi_{s_1,s_2} \left( \left( \frac{1}{m_{-1} b} \right), \left( \frac{m}{b} \right) \right) ds_1 ds_2. \]

Let

\[ c(q^{s_2}, y_2) = 1 - q^{s_2} + (q - 1)q^{-\text{val}(y_2)s_2}. \]

We have

\[ H_{y,s_2}(1) = \int_{F^\times} H_{2,y} \left( \left( \frac{1}{a_1} \right), \left( \frac{a_2}{(a_1 a_2)_{a-1}} \right) \right) |a_1|^{-s_1} |a_2|^{-s_2} d^x a_1 d^x a_2 \]

\[ = \int_{F^\times} 1/_{\mathcal{O}_F}(a_1)1/_{\mathcal{O}_F}(a_2) - \sum_{k=1}^{\infty} q^{\frac{1}{2}k-2}k(q-1)1/_{\mathcal{O}_F}(a_2) |a_2|^{-s_2} d^x a_2 \]

\[ = \int_{F^\times} 1/_{\mathcal{O}_F}(a_2) |a_2|^{-s_2} d^x a_2 + (q-1) \sum_{k=1}^{\infty} q^{\frac{1}{2}k-2}k \int_{F^\times} 1/_{\mathcal{O}_F}(a_2) |a_2|^{-s_2} d^x a_2 \]

\[ = \sum_{i=-\text{val}(y_2)}^{\infty} q^{is_2} + (q-1) \sum_{k=1}^{\infty} \sum_{i=k-\text{val}(y_2)}^{\infty} q^{is_2} \]

\[ = c(q^{s_2}, y_2) \zeta_q(-s_2)^2. \]

We denote \( \chi'_ {s_1,s_2} = \chi_{s_1,s_2}^{-1}H_{1,y}^{-1}_{BG} \). Let

\[ I_{s_1,s_2}(y) = \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1} \]

\[ \times \int_{\mathcal{U}_2(F)} \int_{N^\times(F)} W_{\rho,s} \left( wuy_2 \left( \frac{m}{b_{-1} m} \right), a_y \right) \psi_1(u) dudy d^x m d^x b. \]
Then
\[ I_s(y) = c_q \int_{F^\times} H'_y \left( \frac{a_1}{a_1^{-1}} \right) |a_1|^{-2} \int_{F^\times} \int_{I_{F^+} + \sigma} \int_{iI_{F^+} + \sigma} H_{y,s_2}(1) \chi_{s_1,s_2} \left( \left( \frac{1}{(mb)^{-1}} \right), \left( \frac{m}{b} \right) \right) \]
\[ \times \int_{\mathbb{U}(2)} \int_{N^\circ(F)} W_{\rho,s} \left( wuy \left( m \left( \frac{a_1}{a_1^{-1}} \right) \right), a_y \right) \]
\[ \times \psi_1(u)dudyd^xmd^xb s_1ds_2 \]
\[ = c_q \alpha(y_1,y_2) \int_{iI_{F^+} + \sigma} \int_{iI_{F^+} + \sigma} H_{y,s_2}(1) \int_{F^\times} H'_y \left( \frac{a_1}{a_1^{-1}} \right) |a_1|^{-s_2} d^x a_1 \int_{F^\times} \int_{F^\times} |m|^{-s_1 + s_2 + \frac{d_2}{2}} \]
\[ \times |b|^{-s_1} \int_{\mathbb{U}(2)} \int_{N^\circ(F)} W_{\rho,s} \left( wuy \left( m \left( \frac{b_1}{b_1^{-1}} \right) \right), a_y \right) \psi_1(u)dudyd^xmd^xb s_1ds_2 \]
\[ \times \psi_1(u)dudyd^xmd^xb s_1ds_2. \]

We show in Lemma 8.3 that \( I_{s_1,s_2} \) converges when \( \Re(s_1), \Re(-s_2) \) large and \( s \) in some vertical strip in the right-half plane depends on \( \Re(s_1), \Re(-s_2) \).

We have now written \( I_s(y) \) in terms of Whittaker functions as mentioned below Eq. (7.0.2). We now make use of the local functional equation for the local Rankin-Selberg integral on \( \text{GL}_1 \times \text{GL}_n \) to simplify \( I_s(y) \). This requires the introduction of certain dual local integrals. Let
\[ r(a) = \left( 0 \begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right) \in \text{GL}_n(F). \]

We construct the dual integral
\[ I'_s(y) = \int_{\mathbb{U}(2)} \int_{N^\circ(F)} W_{\rho,s} \left( wuy \left( m \left( \frac{b_1}{b_1^{-1}} \right) \right), r(a)y \right) \psi_1(u)dudydg. \]

Applying the same process as for \( I_s \), we have that
\[ I'_s(y) = c_q \alpha(y_1,y_2) \int_{iI_{F^+} + \sigma} \int_{iI_{F^+} + \sigma} \int_{F^\times} H'_y \left( \frac{a_1}{a_1^{-1}} \right) |a_1|^{-s_2} d^x a_1 \int_{F^\times} \int_{F^\times} |m|^{-s_1 + s_2 + \frac{d_2}{2}} |b|^{-s_1} \]
\[ \times \int_{\mathbb{U}(2)} \int_{N^\circ(F)} W_{\rho,s} \left( wuy \left( m \left( \frac{b_1}{b_1^{-1}} \right) \right), r(a)y \right) \]
\[ \times \psi_1(u)dudyd^xmd^xb s_1ds_2. \]

We denote
\[ I'_{s_1,s_2}(y) = \int_{F^\times} \int_{F^\times} |m|^{-s_1 + s_2 + \frac{d_2}{2}} |b|^{-s_1} \]
\[ \times \int_{\mathbb{U}(2)} \int_{N^\circ(F)} W_{\rho,s} \left( wuy \left( m \left( \frac{b_1}{b_1^{-1}} \right) \right), r(a)y \right) \]
\[ \times \psi_1(u)dudyd^xmd^xb. \]

**Remark.** We show in Lemma 8.4 that \( I'_{s_1,s_2} \) converges when
\[ (7.0.10) \quad \Re(s_1) + \Re(-s_2) \gg 1 \text{ and } A(s_1,s_2) \geq \Re(s) \geq B(s_1,s_2) \]
for some $A(s_1, s_2) > B(s_1, s_2)$. The convergence region of $I'_{s, s_1, s_2}$ intersects the convergence region of $I_{s, s_1, s_2}$ when $\Re(-s_2)$ in some vertical strip.

We now relate $I'_{s, s_1, s_2}$ to our integral $I_{s, s_1, s_2}$.

**Lemma 7.2.** In the region of convergence of $I_{s, s_1, s_2}$ (see Lemma 8.3), we have

$$I_{s, s_1, s_2}(y) = \gamma(s - s_1 + s_2 + \frac{d_2 + 1}{2}, \tau)^{-1} I'_{s, s_1, s_2}(y),$$

where $\gamma(s - s_1 + s_2 + \frac{d_2 + 1}{2}, \tau)$ is the $GL_1 \times GL_n$ gamma factor defined in [JPSS83].

**Proof.** We let $\chi_1, \chi_2$ be characters on $F^\times$ such that $\chi_1 = \cdot | -s_1 + s_2 + \frac{d_2}{2}$, $\chi_2 = \cdot | -s_1$.

Let $\mu$ be a character of $SO_2(F)$, and let $\pi_\xi = Ind_{B'G'}^{G'}(\chi \otimes \mu)$. Let $\varphi_\xi(g, m, I_2) \in V_{\pi_\xi}$. Let

$$I_1 = \int_{GL_1(F) \setminus G'} \int_{N^\circ(F)} \int_{GL_1(F)} \varphi_\xi(g, m, I_2) \times W_{\rho_{r,s}}(wuy(g, \text{diag}(m, I_{n-1}))) \det m |_{s - \xi - \frac{n-1}{2}}^{-\frac{1}{2}} \psi_1(u) dmdudg,$$

and

$$I_2 = \int_{GL_1(F) \setminus G'} \int_{N^\circ(F)} \int_{GL_1(F)} \varphi_\xi(g, m, I_2) \times \int_{M_1 \times (n-2)(F)} W_{\rho_{r,s}}(wuy(g, r(a))) \det m ^{s - \xi - \frac{n-1}{2}} \psi_1(u) dadmdudg.$$ 

By [Kap12, Claim 4.1], one has

$$I_1 = \gamma(s - \xi, \chi_1 \otimes \tau) I_2.$$

Let

$$I'_1 = \int_{G'} \varphi_\xi(g, I_1, I_2) \int_{N^\circ(F)} W_{\rho_{r,s}}(wuy(g, 1)) \psi_1(u) dudydg,$$

and

$$I'_2 = \int_{G'} \varphi_\xi(g, I_1, I_2) \int_{N^\circ(F)} \int_{M_1 \times (n-2)(F)} W_{\rho_{r,s}}(wuy(g, r(a))) \psi_1(u) dadudydg.$$ 

By [Kap12, Proof of Lemma 4.1], one has

$$I_1 = I'_1, I_2 = I'_2.$$

By the definition of $\varphi_\xi$ as element of an induced representation and since we are in the unramified case, we have

$$I'_1 = \int_{G'} \chi(g) \int_{U_2(F)} \int_{N^\circ(F)} W_{\rho_{r,s}}(wuy(g, 1)) \psi_1(u) dudydg$$

$$= \int_{F^\times} \int_{F^\times} \chi_1(m) \chi_2(b) |m|^{-\frac{1}{2}} \int_{U_2(F)} \int_{N^\circ(F)} W_{\rho_{r,s}}(wuy_2 \left( \frac{m}{b} \frac{b^{-1}}{m} \right), 1)$$

$$\times \psi_1(u) dudyd^\times m d^\times b.$$
and
\[ I'_2 = \int_{T_{Gr}(F)} \chi(g) \int_{U^2(F)} \int_{N^0} \int_{M_{1\times(n-2)}(F)} W_{\rho',s}(wuy, r(a)) \psi_1(u) \, da \, dy \, dg \]
\[ = \int_{F^\times} \int_{F^\times} \chi_1(m) \chi_2(b) |m|^{-\frac{1}{2}} \]
\[ \times \int_{U^2(F)} \int_{N^0(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho',s} \left( wuy_2 \left( \begin{array}{c} m \\ b \end{array} \right), r(a) \right) \psi_1(u) \, da \, dy \, db. \]

Here
\[ \chi \left( \begin{array}{c} m \\ b \end{array} \right) = \chi_1(m) \chi_2(b) |m|^{-\frac{1}{2}}, \]
where \( \chi_1, \chi_2 \) are characters on \( F^\times \).

Then we have
\[ I'_1 = \gamma(s - \zeta, \chi_1 \otimes \tau) I'_2. \]

Inserting the result to our local integral \( I_s \) we get
\[ I_s(y) = c_q \alpha(y_1, y_2) \int_{iF^+} \int_{iF^+} H_{y,s_1}(1) \int_{F^\times} H_y' \left( a_1 \left( \begin{array}{c} a_1 \\ a_1 \end{array} \right) \right) |a_1|^{-s_2} a_1 \]
\[ \times \gamma(s - s_1 + s_2, \chi \otimes \tau)^{-1} I'_{s,s_1,s_2} ds_1 ds_2 \]
\[ = c_q \alpha(y_1, y_2) \int_{iF^+} \int_{iF^+} \frac{c(q^2, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2) H_{y,s_2}(1)}{\gamma(s - s_1 + s_2, \chi \otimes \tau)} I'_{s,s_1,s_2} ds_1 ds_2. \]

**Remark.** Since \( c(q^2, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2) H_{y,s_2}(1) \) converges when
\[ \Re(-s_2) > \frac{d_1}{2} + 3, \]
\[ c(q^2, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2) H_{y,s_2}(1) \gamma(s - s_1 + s_2, \chi \otimes \tau)^{-1} I'_{s,s_1,s_2} \]
converges when
\[ \max \left( \frac{d_1}{2} + 3, -n + \frac{d_2}{2} + 8 + c_1 \right) < \Re(-s_2) < n + \frac{d_2}{2} + 2 + c_1 + c_2, \]
\( \Re(s_1) + \Re(-s_2) \) large, and \( \Re(s) \) bounded in some vertical strip depends on \( \Re(s_1) + \Re(-s_2) \) as in Lemma 8.3. Since we set \( d_2 > d_1 \) the region for \( \Re(-s_2) \) is non-empty.

To compute our final result, we remain to compute \( I'_{s,s_1,s_2}(y) \). By the properties of \( W_{\rho',s} \), we have
\[ I'_{s,s_1,s_2}(y) = |4 Q'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{1}{2}} \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1} \]
\[ \times \int_{U^2(F)} \int_{N^0(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho',s} \left( r(a) wuy_2 \left( \begin{array}{c} m \\ b \end{array} \right),1 \right) \psi_1(u) \, da \, dy \, db. \]
Since $\overline{U}_2(F)$ normalizes $N^0(F)$ and $\psi_1$, we can interchange $uy$ to $yu$ in the integral. Denoting $(r(a)w_0)^{-1}(yu) = (r(a)w_0)(yu)(r(a)w_0)^{-1}$, we have that

$$I'_{s,s_1,s_2}(y)$$

$$= |4Q'(y_2)|^{-R(s)+\frac{1}{2}-\frac{n}{2}} \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1}$$

$$\times \int_{\overline{U}_2(F)} \int_{N^0(F)} \int_{M_1 \times (n-2)(F)} W_{p',s}(r(a)w_0^{-1}yu) W_{s'}(r(a)w_0^{-1}yu)$$

$$\times \psi_1(u) \overline{\psi} V d\nu d\tau d\lambda m^\times d^\times b.$$
\[ \times \mu_{s_1}(x)\psi_1(u)dvdxda. \]

We regard the \( dvdx \) integral over \( \tilde{w}_0 V(F) \times w'\tilde{w}_0 SO_2(F) \) as a function on \( H(F) \):

\[ B_{\psi'_1,s_1}(h) := \int_{w'\tilde{w}_0 SO_2(F)} \int_{\tilde{w}_0 V(F)} W_{\rho,s}(vxh,1)\psi_1(u)\mu_{s_1}(x)dvdxda. \]  

(7.0.11)

Let

\[ \psi'_1 = \psi \left( \sum_{i=1}^{n-1} v_{i,i+1} + \frac{1}{2} v_{2n,1} \right) \quad (v \in \tilde{w}_0 Z(F) \ltimes \tilde{w}_0 V(F)) \]

be a character.

Similar to the Bessel function in [Kap12, Page 162], our function \( B_{\psi'_1,s_1} \) is an unramified Bessel function which corresponds to the Bessel functional defined for the subgroup

\[ \left( \tilde{w}_0 Z(F) \ltimes \tilde{w}_0 V(F) \right) \times w'\tilde{w}_0 SO_2(F) = \tilde{w}_0 R^o(F) \]

(for \( w'\tilde{w}_0 SO_2(F) \) split) and representations \( \rho_{\tau,a} \), \( \psi'_1 \) and \( \mu \).

For \( \delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n \), we denote

\[ \varpi_H^\delta = \text{diag}(\varpi_1^{\delta_1}, \ldots, \varpi_n^{\delta_n}, 1, \varpi_1^{-\delta_1}, \ldots, \varpi_1^{-\delta_n}) \in T_H(F). \]

Substituting the function to our local integral we get

\[ I'_{s_1,s_2}(y) = \sum_{k \in \mathbb{Z}} |\varpi|^k \int_{M_1 \times (n-2)(F)} B_{\psi'_1,s_1}(\tilde{w}_0 u(a)w\varpi_H^k)da, \]

where \( \tilde{k} = (0, \ldots, 0) \in \mathbb{Z}^n \).

We have the following equality using the same argument as that proving [Kap12, Claim 4.3]:

**Lemma 7.3.**

\[ \int_{M_1 \times (n-2)(F)} B_{\psi'_1,s_1}(\tilde{w}_0 u(a)w\varpi_H^k)da = B_{\psi'_1,s_1}(\varpi_H^k)|\varpi^k|^{n-2}, \]

where \( \delta_k = (k, 0_{n-1}) \in \mathbb{Z}^n \).

Thus we get

\[ I'_{s_1,s_2}(y) = |4 \mathcal{Q}'(y_2)|^{-R(s)+\frac{1}{2}} \sum_{k \in \mathbb{Z}} q^{-(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{\psi'_1,s_1}(\varpi_H^k). \]

Since \( B_{\psi'_1,s_1} \) is an unramified Bessel function, by the vanishing condition of unramified Bessel function, \( B_{\psi'_1,s_1}(\varpi_H^k) = 0 \) unless \( k \leq -\text{val}(4 \mathcal{Q}'(y_2)) \). We have

\[ I'_{s_1,s_2}(y) = |4 \mathcal{Q}'(y_2)|^{-R(s)+\frac{1}{2}} \sum_{k \leq -\text{val}(4 \mathcal{Q}'(y_2))} q^{-(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{\psi'_1,s_1}(\varpi_H^k). \]
We proceed to state the main theorem of this section. Consider

\[ c(q^{s_2}, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2 \] 

\[ \frac{\gamma(s - s_1 + s_2, \chi' \otimes \tau)}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} B_{\psi', s_1}^{\delta_{k-\text{val}(\varpi')}}(\varpi_H^{\delta_{k-\text{val}(\varpi')}}) \]

as a product of Laurent series in \( q^{s_1} \) and \( q^{s_2} \), where \( c(q^{s_2}, y_2) \) is as defined in Eq. (7.0.9). Let

\[ C_{k,s}(y) := c_q \int_{iI_F + \sigma_1} \int_{iI_F + \sigma_2} q^{(-s_1+s_2)k} c(q^{s_2}, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2 \] 

\[ \frac{\gamma(s - s_1 + s_2, \chi' \otimes \tau)}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} B_{\psi', s_1}^{\delta_{k-\text{val}(\varpi')}}(\varpi_H^{\delta_{k-\text{val}(\varpi')}}), \]

where \( c_q \) is as defined in Eq. (7.0.8). This is nothing but the product of the \(-k\)-th coefficient in \( q^{s_1} \) and the \( k\)-th coefficient in \( q^{s_2} \) of (7.0.12).

**Theorem 7.4.** For all the data unramified and \( \Re(s) \) large, we have

\[ I_s(y) = \alpha(y_1, y_2)|4Q'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{3}{2}} \sum_{k=\text{val}(4Q'(y_2))}^{\infty} q^{(n-2+\frac{d_2}{2})k} C_{k,s}(y), \]

where \( \alpha(y_1, y_2) \) is as defined in Eq. (7.0.3).

**Proof.** Combining all the above results we get

\[ I_s(y) = c_q \alpha(y_1, y_2) \int_{iI_F + \sigma_1} \int_{iI_F + \sigma_2} \frac{c(q^{s_2}, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} I_{s, s_1, s_2}^{\gamma_2} \] 

\[ \times \sum_{k=\text{val}(4Q'(y_2))}^{\infty} q^{(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{\psi', s_1}^{\varpi_1}(\varpi_H^{\delta_{k-\text{val}(\varpi')}})(ds_1)ds_2 \] 

\[ = c_q \alpha(y_1, y_2) \sum_{k=\text{val}(4Q'(y_2))}^{\infty} q^{(n-2+\frac{d_2}{2})k} \int_{iI_F + \sigma_1} \int_{iI_F + \sigma_2} \frac{c(q^{s_2}, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} \] 

\[ \times q^{(-s_1+s_2)k} B_{\psi', s_1}^{\varpi_1}(\varpi_H^{\delta_{k-\text{val}(\varpi')}})(ds_1)ds_2. \]

We will make the above manipulations rigorous in Lemma 8.5 by showing that the infinite sum

\[ \frac{c(q^{s_2}, y_2) \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} \sum_{k=\text{val}(4Q'(y_2))}^{\infty} q^{(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{\psi', s_1}^{\varpi_1}(\varpi_H^{\delta_{k-\text{val}(\varpi')}}) \]

converges absolutely for \( \Re(s_1), \Re(-s_2) \) large and \( \Re(s) \) lies in some region in the right plane that depends on \( \Re(s_1), \Re(-s_2) \). \( \square \)

We give an explicit expression for the unramified Bessel function following [Kap12] below. We first define two functions in the formula of unramified Bessel function using Satake parameters \( \chi_{i,s} \) and the character \( \mu_{s_1} : \nabla (\chi) = (-1)^n \det(\chi_{i,s}(\varpi)^{n-i+1} - \chi_{i,s}(\varpi)^{-(n-i+1)}),\)
\[ D(\chi_s, \mu_{s_1}) = \prod_{i=1}^{n} \chi_{i,s}(\varpi)^{-n+1-i} \prod_{i=1}^{n-1} (1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi)q^{-\frac{1}{2}})(1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi)^{-1}q^{-\frac{1}{2}}). \]

**Lemma 7.5.** Let \( W \) be the Weyl group of \( H \). Let \( t_\tau \) be the Satake parameter for \( \tau \). We denote the local \( L \)-function for \( \mu_{s_1} \times \tau \) and symmetric square \( L \)-function for \( \tau \) as

\[ L(s, \mu_{s_1} \times \tau) = \det(1 - (t_\mu \otimes t_\tau)q^{-s})^{-1} = \prod_{i=1}^{n} (1 - \chi_i(\varpi)\mu_{s_1}(\varpi)q^{-s})^{-1}(1 - \chi_i(\varpi)\mu_{s_1}(\varpi)^{-1}q^{-s})^{-1}, \]

\[ L(2s, \tau, \text{Sym}^2) = \prod_{1 \leq i < j \leq n} (1 - \chi_i(\varpi)\chi_j(\varpi)q^{-2s})^{-1} \prod_{i=1}^{n} (1 - \chi_i(\varpi)q^{-2s})^{-1}. \]

For \( t = \varpi_H^{\delta - \text{val}(4\mathcal{Q}'(y_2))} \in T_H(F) \) such that \(-k - \text{val}(4\mathcal{Q}'(y_2)) \geq 0\), we have

\[ B_{\psi_1,s_1}(t) = \frac{L(s, \mu_{s_1} \times \tau)}{L(2s, \tau, \text{Sym}^2)} \delta_H^\frac{1}{2}(t) S(\chi_s, \mu_{s_1}, t) \]

Here

\[ S(\chi_s, \mu_{s_1}, t) = \frac{1}{\Delta(\chi_s)} \sum_{w \in W} \text{sgn}(w) D(\chi_s, \mu_{s_1}) w \chi_s(t)^{-1}. \]

**Proof.** We denote by \( B^0 \) the normalized Bessel function \( B^0 = B_{\psi_1,s_1}/B_{\psi_1,s_1}(1) \).

Let \( f_0 \in \text{Ind}_H^G(\chi_s) \) be such that \( f_0(K) = 1 \). Since we normalize \( W_{\rho_{\tau,s}} \) by \( W_{\rho_{\tau,s}}(1,1) = 1 \), we have \( B_{\psi_1,s_1} = W_{\tau,s}(1)^{-1} B_{f_0}(1) B^0 \), where

\[ W_{\tau,s}(1) = \sum_{1 \leq i < j \leq n} (1 - \chi_{i,s}(\varpi)\chi_{j,s}(\varpi)^{-1}q^{-1}) \]

(see [CS80]), and

\[ B_{f_0}(1) = \frac{\prod_{1 \leq i < j \leq n} (1 - \chi_{i,s}(\varpi)\chi_{j,s}(\varpi)q^{-1})(1 - \chi_{i,s}(\varpi)\chi_{j,s}(\varpi)^{-1}q^{-1}) \prod_{i=1}^{n} (1 - \chi_{i,s}(\varpi)q^{-1})}{\prod_{i=1}^{n} (1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi)q^{-\frac{1}{2}})(1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi)^{-1}q^{-\frac{1}{2}})}. \]

Using the explicit formula for the unramified Bessel functional (see [BFF97], [Kap12, Section 2.5]), we deduce the result using similar notations as in [Kap12, Section 2.5].

8. **Convergence of local integrals in unramified computation**

Let \( F \) be a non-Archimedean local field. In this section we prove absolute convergence for various local integrals that appear in the unramified computations in Section 7. We point out that the bounds we prove in this section are not used in our global considerations.

The points of the group \( wN^o U_2 w^{-1} \) in an \( F \)-algebra \( R \) are

\[ wN^o U_2^o(R)w^{-1} \]
\[
\begin{pmatrix}
\frac{1}{c_1} & 1 & I_{n-2} \\
-c_2 & 0 & v_3 \\
-v_1 & v_2 & T
\end{pmatrix}
\begin{pmatrix}
\frac{1}{c_2} & 0 & v_3 \\
-c_1 & 0 & v_2 \\
\alpha c_1 - \frac{1}{2} v_2 \alpha c_1 & v_1 & \alpha c_2 - c_1 \cdot 1
\end{pmatrix}
: c_1, c_2 \in \mathbb{R}, v_1, v_2, v_3 \in \mathbb{R}^{n-2}, T \in M_{(n-2)}(\mathbb{R})
\]

Here \( \mathcal{U}'_2 = i(\mathcal{U}_2) \) and \( \alpha = \frac{1}{2} \).

**Lemma 8.1.** For \( \Re(s) \) large, the integral

\[(8.0.1) \quad \int_{wN^0(F)\mathcal{U}'_2(F)w^{-1}} W_{\rho,\epsilon}(uv, 1)du\]

converges, where \( v \) is a unipotent element of \( H(F) \) defined as in Eq. (4.0.2).

**Remark.** We remark that this is different from Lemma 4.1. The integral is over the group \( wN^0(F)\mathcal{U}'_2(F)w^{-1} \) whereas the integral in Lemma 4.1 is over \( wN^0(F)w^{-1} \).

**Proof.** For \( u \in wN^0(F)\mathcal{U}'_2(F)w^{-1} \), we have

\[uv = \begin{pmatrix}
1 & c & -\frac{1}{2} c^2 \\
-c_2 & 0 & v_3 \\
-v_1 & v_2 & T
\end{pmatrix},
\]

where \( c \in F \).

Considering the Iwasawa decomposition of \( u \in wN^0(F)\mathcal{U}'_2(F)w^{-1} \) in \( H \) (using the notation as in Lemma 4.1), we have

\[uv = n a' k.\]

Here \( (n, a', k) \in N_n(F) \times T_H(F) \times K_H \) and \( a' = \text{diag}(a, 1, w_0 a^{-1} w_0) \) for \( a \in \text{GL}_n(F) \). We denote the \( i \)-th line of \( uv \) as \((uv)_i\). By a similar argument as in Lemma 4.2, we have

\[ [\mathcal{L}(uv)]^{-2j} \leq \left| \frac{a_j}{a_{j+1}} \right| \leq [\mathcal{L}(uv)]^{2j}. \]

Here \( j = 1, \ldots, n \) and \( [\mathcal{B}(uv)] = \max\{1, \|\mathcal{B}(uv)\|\} \) and \( \|\cdot\| \) is the sup-norm, and

\[ [\mathcal{L}(uv)]^{-n} \leq D(uv) = |\text{det}(a)| \leq [\mathcal{L}(uv)]^{-1}. \]

Note that by arguing as in Lemma 4.1 and Lemma 4.2, the integral Eq. (8.0.1) is majorized by

\[(8.0.2) \quad \sum_{j=1}^n c_{j,s}(v)^C \int_{wN^0(\mathcal{U}'_2(F))w^{-1}} [u]^{-\Re(s) - \frac{n-1}{2} + C}du,
\]

where \( C \) is a positive constant which depends only on \( \tau \).
Lemma 8.2. The integral

\begin{equation}
I_s(y) = \int_{G(F)} \Phi_y(g) \int_{N^\circ(F)} W_{\rho,s}(wuv(g), a_y) \psi_1(u) dudg
\end{equation}

converges absolutely for $\Re(s)$ large enough.

Proof. By the Iwasawa decomposition with respect to the lower Borel subgroup of $G(F)$, we have

\[
I_s(y) = \int_{U_2(F)} \Phi_y(vg) \delta^{-1}_{B_G}(g) \int_{N^\circ(F)} W_{\rho,s}(wuv(g), a_y) \psi_1(u) dudvdg.
\]

Since $\Phi_y$ is invariant under $U_2(F)$, we have

\[
I_s(y) = \int_{G_1(F)} \Phi_y(xa_1a_2) \delta^{-1}_{B_G}(xa_1a_2)
\]

\[
\times \int_{U_2(F)} \int_{N^\circ(F)} W_{\rho,s}(wuv(xa_1a_2), a_y) \psi_1(u) dudvdxda_1da_2.
\]

Using the Iwasawa decomposition of $SO_3(F)$ as in Lemma 4.3, we have

\[
|I_s(y)| \leq \int_{F^\times} \int_{F^\times} |\Phi_y| \left( \left( \begin{array}{c} a_1 \\ b \end{array} \right)^{-1}, \left( \begin{array}{c} a_2 \\ b^{-1}a_2 \end{array} \right)^{-1} \right) |a_1a_2|^{|\Re(s)|-n+5} |[b_1a_2^{-1}]|^{\Re(s)-n+3}
\]

\[
\times \int_{U_2(F)} \int_{N^\circ(F)} |W_{\rho,s}|((wuvn', \text{diag}(\-4\mathcal{Q}'(y_2)a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2})) dudvdg a_1d^x a_2
\]

\[
= \int_{F^\times} \int_{F^\times} \int_{F^\times} |\Phi_y| \left( \left( \begin{array}{c} a_1 \\ b \end{array} \right)^{-1}, \left( \begin{array}{c} a_2 \\ b^{-1}a_2 \end{array} \right)^{-1} \right) |a_1a_2|^{|\Re(s)|-n+5} |[b_1a_2^{-1}]|^{\Re(s)-n+3}
\]

\[
\times \int_{U_2(F)} \int_{N^\circ(F)} |W_{\rho,s}|((wuvw^{-1})(wn'w^{-1}), \text{diag}(-4\mathcal{Q}'(y_2)a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2}))
\]

\[
\times dudvdg a_1d^x a_2
\]

\[
= \int_{F^\times} \int_{F^\times} \int_{F^\times} |\Phi_y| \left( \left( \begin{array}{c} a_1 \\ b \end{array} \right)^{-1}, \left( \begin{array}{c} a_2 \\ b^{-1}a_2 \end{array} \right)^{-1} \right) |a_1a_2|^{|\Re(s)|-n+5} |[b_1a_2^{-1}]|^{\Re(s)-n+3}
\]

\[
\times \int_{wN^\circ U_2(F)w^{-1}} |W_{\rho,s}|((uwn'w^{-1}), \text{diag}(-4\mathcal{Q}'(y_2)a_1a_2, [b^{-1}a_1a_2^{-1}], I_{n-2}))
\]

\[
\times dudg a_1d^x a_2,
\]

where $n'$ is as defined in Eq. (4.0.15).

By Eq. (8.0.2) in Lemma 8.1 and a similar argument as in Lemma 4.3, the above integral is majorized by a finite sum of integrals of the form

\[
\int_{F^\times} \int_{F^\times} \int_{F^\times} |\Phi_y| \left( \left( \begin{array}{c} a_1 \\ b \end{array} \right)^{-1}, \left( \begin{array}{c} a_2 \\ b^{-1}a_2 \end{array} \right)^{-1} \right) |a_1a_2|^{|\Re(s)|-n+5} |[b_1a_2^{-1}]|^{\Re(s)-n+3-c_1}
\]

\[
\times \left( \int_{wN^\circ U_2(F)w^{-1}} [u]^{-\Re(s)-n+5} \left( c_2 + du \right) d^x b d^x a_1 d^x a_2,
\]

\[
\times \left( \int_{wN^\circ U_2(F)w^{-1}} [u]^{-\Re(s)-n+5} \left( c_2 + du \right) d^x b d^x a_1 d^x a_2,
\]

\[
\times \left( \int_{wN^\circ U_2(F)w^{-1}} [u]^{-\Re(s)-n+5} \left( c_2 + du \right) d^x b d^x a_1 d^x a_2.
\]
where $c_1, c_2 > 0$ are constants depend only on $\tau$.

Substituting the above result into our local integral while using the explicit formula for $\Phi_y$, we have that for $\Re(s)$ large the above integral is majorized by

$$
\int_{F^*} \int_{F^*} \int_{F^*} \left| H_{1,y} \left( \left( \frac{a_1}{b_{a_1}} \right), \left( \frac{a_2}{b_{a_2}} \right) \right) \right| |H_{2,y}\left( \left( \frac{a_1}{b_{a_1}} \right), \left( \frac{a_2}{b_{a_2}} \right) \right)| \times \left| a_1a_2 \right|^{\frac{-1}{n} + \frac{3}{2}} d^X b d^X a_1 \times |a_2|^{\frac{-1}{n} + \frac{1}{2}} d^X a_2
$$

which converges absolutely for $\Re(s)$ large by a similar computation as for $H_{y,s_1,s_2}(1)$ in Section 7.

Lemma 8.3. There exists positive integers $C_1 < C_2, C_3, C_4$ which depend on $(\tau, d_1, d_2, n)$ such that

$$
I_{s,s_1,s_2}(y) = \int_{F^*} \int_{F^*} \chi_{s_1,s_2} \left( \left( \frac{1}{m'b} \right)^{-1}, (m'b) \right) \int_{\mathbb{R}^2(F)} \int_{\mathcal{N}^2(F)} W_{\rho_{\tau,s}} \left( \left( \frac{m}{b}, b^{-1} \right) \right), a_1 \right) \psi_1(u) dudyd^x md^x b
$$

converges absolutely for

$$
\Re(s_1) + \Re(-s_2) + C_1 < \Re(s) < \Re(s_1) + 2\Re(-s_2) + C_2, \\
C_3 < \Re(s_1) + \Re(-s_2), C_4 < \Re(-s_2).
$$

Proof. As in Lemma 8.2, $I_{s,s_1,s_2}$ is majorized by a finite sum of integrals of the form

$$
\int_{F^*} \int_{F^*} \left| m \right|^{-s_1+s_2+\frac{d_2}{2}} \left| b \right|^{-s_1} \left| m \right|^{\Re(s) - n + 5} \left| b \right|^{\Re(s) - n + 3 - c_1} \int_{wN^O\mathbb{U}_{\tau}^2(F)w^{-1}} \left[ u \right]^{-\Re(s) - \frac{n+1}{2} + c_2} d^x m d^x b,
$$

where $c_1, c_2 > 0$ are constants depend only on $\tau$.

Also, as in Lemma 4.3, we have

$$
\left| m \right| \leq c'[u] \left| b \right|^{-2}.
$$

Here $c'$ is a constant depends only on $\tau$.

Thus the integral is majorized by a finite sum of integrals of the form

$$
\int_{F^*} \left| b \right|^{-s_1} \left| b \right|^{\Re(s) - n + 3 - c_1} d^x b \int_{wN^O\mathbb{U}_{\tau}^2(F)w^{-1}} \int_{\left| m \right| \leq c'[u] \left| b \right|^{-2}} \left| m \right|^{-s_1+s_2+\frac{d_2}{2} + \Re(s) - n + 5} \times \left[ u \right]^{-\Re(s) - \frac{n+1}{2} + c_2} d^x m d u.
$$
The integral
\[ \int \frac{|m|^{-s_1 + s_2 + d_2/2 + \Re(s) - n + 5\cdot d^x \cdot m}}{|m| \leq c'|u||b|^2} \]
converges absolutely when
\[-\Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) - n + 5 > 0.\]

Thus when \( \Re(s) > \Re(s_1) - \Re(s_2) - \frac{d_2}{2} + n - 5 \), the integral \( I'_{s,s_1,s_2} \) is majorized by a finite sum of integrals of the form
\[ \int_{F^\times} |b|^{-s_1} [b^2]^{-\Re(s) + 2s_1 - 2s_2 + d_2 + n - 7 - c_1} d^x b \int_{wN^sU_2(F)w^{-1}} [u]^{-s_1 + s_2 + \frac{d_2 - 3n + 11}{2}} + c_2 du. \]
This integral converges absolutely when
\[-\Re(s_1) + \Re(s_2) + \frac{d_2 - 3n + 11}{2} + c_2 < -C,\]
\[-\Re(s) + 2\Re(s_1) - 2\Re(s_2) + d_2 + n - 7 - c_1 - \Re(s_1) > 0,\]
\[\Re(s) - 2\Re(s_1) + 2\Re(s_2) - d_2 - n + 7 + c_1 - \Re(s_1) < 0.\]

Here \( C \) is a positive integers which depends on \( \tau \).

Summarizing, the convergence region is equivalent to
\[\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} + n - 5 < \Re(s) < \min(\Re(s_1), 3\Re(s_1)) + 2\Re(-s_2) + d_2 + n - 7 - c_1,\]
\[\Re(s_1) + \Re(-s_2) > \frac{d_2 - 3n + 11}{2} + c_2 + C.\]

This is non-empty when
\[\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} + n - 5 < \min(\Re(s_1), 3\Re(s_1)) + 2\Re(-s_2) + d_2 + n - 7 - c_1,\]
which is equivalent to
\[\Re(-s_2) > -\frac{3d_2}{2} + 2 + c_1.\]
Thus we get the non-empty region in the statement of the lemma.

Let \( X_1 \subset H \) be the unipotent subgroup of \( H \) whose points in an \( F \)-algebra \( R \) are
\[ X_1(R) = \left\{ \begin{pmatrix} 1 & I_{n-2} \\ 0 & v_1^\alpha \\ -ac_1 & v_1^\alpha & \alpha c_2^1 \\ v_2 & T & v_1 & v_3 \\ 0 & v_2 & -ac_1 & 1 \end{pmatrix} \right\} : c_1, c_2 \in F, v_1, v_2, v_3 \in F^{n-2}, T \in M_{n-2}(F) \}, \]
where \( \alpha = \frac{1}{2} \).

**Lemma 8.4.** There are constants \( C_0, C''_0, c_2 > 0, c_1 \) which depend on \( (\tau, d_1, d_2, n) \) such that
\[ I'_{s,s_1,s_2}(y) = \int_{F^\times} \int_{F^\times} x'_{s_1,s_2} \left( \begin{pmatrix} 1 & (mb)^{-1} \\ (m) \end{pmatrix} \right), \left( \begin{pmatrix} m \\ b \end{pmatrix} \right) \]
converges absolutely for

\[ \Re(s_1) + \Re(-s_2) - \frac{d_1}{2} - 5 - c_1 > \Re(s) \]

\[ \text{where} \]

\[ \Re(s_1) + \frac{2C_0}{2C_0 - 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n - 3 + c_2}{2C_0 - 1}, \]

\[ \Re(s_1) + \frac{2C_0}{2C_0 + 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n - 3 + c_2}{2C_0 + 1}, \]

\[ - n + \frac{d_2}{2} + 8 + c_1 < \Re(-s_2) < - n + \frac{d_2}{2} + 2 + c_1 + c_2, \]

\[ c_0(-n + 3 - c_2) + C'' < \Re(s_1) + \Re(-s_2). \]

**Proof.** We use the approach outlined in the proof of [Sou93, Proposition 11.16].

As in Lemma 8.3, we have

\[ I_{s_1,s_2}' \leq \int_{F^s} \int_{F^s} |m|^{-s_1 + s_2 + \frac{d_2}{2} + \Re(s) + 5}|b|^{-s_1} \Re(s)^{-n+3} \]

\[ \times \int_{\mathcal{U}_2(F)} \int_{\mathcal{N}^0(F)} \int_{\mathcal{M}_{1 \times (n-2)}(F)} |W_{\rho_{r,s}}(wuy, r(e) \text{diag}(-4Q'(y_2) m, [b], I_{n-2})) \text{det} yd^{\times} md^{\times} b, \]

where

\[ r(e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \\ 1 & 0 & e \end{pmatrix}. \]

Since

\[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \\ 1 & 0 & e \end{pmatrix} \begin{pmatrix} m \\ [b] \\ I_{n-2} \end{pmatrix} = \begin{pmatrix} b \\ I_{n-2} \\ m \end{pmatrix}, \]

we have

\[ I_{s_1,s_2}'(y) \leq | - 4Q'(y_2)|^{s_1 - s_2 - \frac{d_2}{2} - \Re(s) - 5} \int_{F^s} \int_{F^s} |m|^{-s_1 + s_2 + \frac{d_2}{2} + \Re(s) + 5}|b|^{-s_1} \Re(s)^{-n+3} \]

\[ \times \int_{\mathcal{U}_2(F)} \int_{\mathcal{N}^0(F)} \int_{\mathcal{M}_{1 \times (n-2)}(F)} |W_{\rho_{r,s}}(v(r(m^{-2}e)) \text{diag}([b], I_{n-2}, m)) \text{det} yd^{\times} md^{\times} b. \]

Here the symbol \( v(r(m^{-2}e)) \in M_n(F) \) is as defined in Eq. (2.1.4).

By a change of variable \( e \mapsto m^2 e \), we have that

\[ I_{s_1,s_2}'(y) \leq | - 4Q'(y_2)|^{s_1 - s_2 - \frac{d_2}{2} - \Re(s) - 5} \int_{F^s} \int_{F^s} |m|^{-s_1 + s_2 + \frac{d_2}{2} + \Re(s) + 5}|b|^{-s_1} \Re(s)^{-n+3}|m|^n \]
\[ \times \int_{U_2(F)} \int_{N_0(F)} \int_{M_{1 \times (n-2)}(F)} |W_{\rho_{r,s}}(v(r(e))wuyn', \text{diag}([b], I_{n-2}, m))| \text{d}u \text{d}y \text{d}m \text{d}x \text{d}b \]

\[ = -4Q'(y_2)^{s_1-s_2-\frac{d_2}{2}-R(s)-5} \int_{F \times F} |m|^{-s_1+s_2+\frac{d_2}{2}+R(s)+5} |b|^{-s_1} |[b]|^{R(s)-n+3} \]

\[ \times \int_{N_0(F)} \int_{M_{1 \times (n-2)}(F)} |W_{\rho_{r,s}}(v(r(e))(wuw^{-1})(wyw^{-1})(wn'w^{-1}), \text{diag}([b], I_{n-2}, m))| \text{d}u \text{d}y \text{d}m \text{d}x \text{d}b. \]

Since for \( u \in N_0(F), \)

\[ wuw^{-1} \in \begin{pmatrix} 1 & I_{n-2} & v_1 & v_2 & v_3 \ 0 & v_1' & v_2' & v_3' \ 0 & 0 & v_1'' & 1 \end{pmatrix} \]

and

\[ v(r(e)) \begin{pmatrix} 1 & I_{n-2} & v_1 & v_2 & v_3 \ 0 & v_1' & v_2' & v_3' \ 0 & 0 & v_1'' & 1 \end{pmatrix} = \begin{pmatrix} 1 & I_{n-2} & v_1 & v_2 & v_3 \ 0 & v_1' & v_2' & v_3' \ 0 & 0 & v_1'' & 1 \end{pmatrix} v(r(e)), \]

we have

\[ |W_{\rho_{r,s}}(v(r(e))(wuw^{-1})(wyw^{-1})(wn'w^{-1}), \text{diag}([b], I_{n-2}, m))| = |W_{\rho_{r,s}}(u'v(r(e))(wuw^{-1})(wyw^{-1}), \text{diag}([b], I_{n-2}, m))|. \]

Here

\[ u' = \begin{pmatrix} 1 & I_{n-2} & v_1 & v_2 & v_3 \ 0 & v_1' & v_2' & v_3' \ 0 & 0 & v_1'' & 1 \end{pmatrix}. \]

We observe that

\[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \ 1 & 0 & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \ 0 & 0 & I_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & -c_1 e & c_1 \\ 0 & I_{n-2} & 0 \ 0 & 0 & I_{n-2} \end{pmatrix}. \]

Also, for \( y \in U_2(F), \)

\[ wyw^{-1} = \begin{pmatrix} 1 & c_1 & I_{n-2} \\ -a_2 c_1 & 0 & 0 \ -a_2 c_1 & 0 & 0 \ \frac{1}{2} c_2 & \frac{1}{2} a_2 c_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & I_{n-2} \\ -a_2 c_1 & 0 & 0 \ -a_2 c_1 & 0 & 0 \ \frac{1}{2} c_2 & \frac{1}{2} a_2 c_1 & 0 \end{pmatrix}, \]

\[ = v \left( \begin{pmatrix} 1 & c_1 & I_{n-2} \end{pmatrix} \right) \begin{pmatrix} 1 & 1 & I_{n-2} \\ -a_2 c_1 & 0 & 0 \ -a_2 c_1 & 0 & 0 \ \frac{1}{2} c_2 & \frac{1}{2} a_2 c_1 & 0 \end{pmatrix}, \]
where \( c_1, c_2 \in F, \alpha = \frac{1}{2} \). This implies that

\[
|W_{\rho, \tau, s}|(u'v(r(e))(\omega w w^{-1})(\omega n w^{-1}), \text{diag}([b], I_{n-2}, m))
\]

\[
=|W_{\rho, \tau, s}| \left( u' v \left( \begin{pmatrix} 0 & -c_1 e \\ I_{n-2} & 1 \end{pmatrix} \right) y' (\omega n w^{-1}) v(r(e)), \text{diag}([b], I_{n-2}, m) \right).
\]

Here

\[
y' = \begin{pmatrix} 1 & -c_1 e \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ -c_1 e' e' & 1 \\ -c_1 e' e' & 1 \\ -c_1 e' e' & 1 \\ 0 & 1 \end{pmatrix},
\]

where \( e' = -J_{n-2}^{1} e \).

Since

\[
\begin{pmatrix} 1 & I_{n-2} \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} v' \begin{pmatrix} 0 & -c_1 e \\ I_{n-2} & 1 \end{pmatrix} v \left( \begin{pmatrix} 1 & -c_1 e \\ 0 & 1 \end{pmatrix} \right)
\]

\[
= v \left( \begin{pmatrix} 1 & -c_1 e \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & I_{n-2} \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} v' \begin{pmatrix} 0 & -c_1 e \\ I_{n-2} & 1 \end{pmatrix} v \left( \begin{pmatrix} 1 & -c_1 e \\ 0 & 1 \end{pmatrix} \right)
\]

we have

\[
|W_{\rho, \tau, s}|(u' v \left( \begin{pmatrix} 1 & -c_1 e \\ 0 & 1 \end{pmatrix} \right) y' (\omega n w^{-1}) v(r(e)), \text{diag}([b], I_{n-2}, m))
\]

\[
=|W_{\rho, \tau, s}| \left( v'' y' (\omega n w^{-1}) v(r(e)), \text{diag}([b], I_{n-2}, m) \right)
\]

Here

\[
u'' = \begin{pmatrix} 1 & I_{n-2} \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & I_{n-2} \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)
\]

Thus, using the following change of variables

\[
v_1 \mapsto v_1 - c_1 v_2 - \frac{1}{2} c_2 e' \\
v_2 \mapsto v_2 - \alpha c_1 e \\
v_3 \mapsto v_3 - \alpha c_2 e' \\
T \mapsto T - e' v_1 + ev_1 - \frac{1}{2} c_2 e',
\]

we obtain the integral \( I_{s, s_1, s_2} \) is bounded by
\[
\int_{F^{2}} \int_{F^{2}} |m|^{-s_{1} + s_{2} + \frac{d}{2} + \Re(s) + 5} |b|^{-s_{1}} |[b]|^{\Re(s) - n + 3} \\
\times \int_{X_{1}(F)} \int_{M_{1} \times (n-2)(F)} |W_{\rho_{s,t}}|(x_{1}(wnw^{-1})v(r(e)), \text{diag}([b], I_{n-2}, m))dxdydmdb.
\]

Now we proceed to decompose \(v(r(e))\).

We have
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & I_{n-2} \\
1 & 0 & e
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & I_{n-2} & 0 \\
0 & e & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & I_{n-2} & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Then we apply the Iwasawa decomposition with respect to the standard Borel subgroup of \(\text{GL}_{n}(F)\)
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & I_{n-2} & 0 \\
0 & e & 1
\end{pmatrix} = n_e t_e k_e.
\]

Here \(n_e = \text{diag}(I_{2}, n'_e)\) where \(n'_e\) lies in the unipotent radical of the standard Borel subgroup of \(\text{GL}_{n-2}(F)\), \(t_e = (t_1, \ldots, t_n)\), where \(t_1 = t_2 = 1\), \(k_e \in \text{GL}_{n}(O)\).

By the structure of this decomposition, we have
\[
[e] \leq |t_n| = |t_3 \cdots t_{n-1}|^{-1}.
\]

since \(\text{det}(t_e) = 1\).

The integral \(I_{s_{1}, s_{2}}\) is majorized by
\[
\int_{F^{2}} \int_{F^{2}} |m|^{-s_{1} + s_{2} + \frac{d}{2} + \Re(s) + 5} |b|^{-s_{1}} |[b]|^{\Re(s) - n + 3} \\
\times \int_{X_{1}(F)} \int_{M_{1} \times (n-2)(F)} |W_{\rho_{s,t}}|(x_{1}(wnw^{-1})v(n_e), \text{diag}([b], I_{n-2}, m)t_e)dxdydmdb.
\]

By change of variables \(x_1 \mapsto v(n_e)(x_1(wnw^{-1}))v(n_e)^{-1}\) the above integral is
\[
\int_{F^{2}} \int_{F^{2}} |m|^{-s_{1} + s_{2} + \frac{d}{2} + \Re(s) + 5} |b|^{-s_{1}} |[b]|^{\Re(s) - n + 3} \\
\times \int_{X_{1}(F)} \int_{M_{1} \times (n-2)(F)} |W_{\rho_{s,t}}|(x_{1}(wnw^{-1}), \text{diag}([b^2], I_{n-2}, m^2)t_e)dxdydmdb.
\]

Thus by similar arguments as in Lemma 8.1 the integral is majorized by a finite sum of integrals of the form
\[
\int_{F^{2}} \int_{F^{2}} |m|^{-s_{1} + s_{2} + \frac{d}{2} + \Re(s) + 5} |b|^{-s_{1}} |[b]|^{\Re(s) - n + 3 - 2C} \\
\times \int_{X_{1}(F)} \int_{M_{1} \times (n-2)(F)} [x_{1}]^{-\Re(s) - \frac{n-1}{2}} + C \eta_j(\text{diag}([b^2], I_{n-2}, m^2)t_e)dedudydmdb.
\]

Here \(\eta_j\) is some positive quasi-character depends on \(\tau\) and \(C\) is some positive integer which depends on \(\tau\).
Using the notation as in Lemma 8.1, we denote the Iwasawa decomposition of $x_1 \in X_1(F)$ as $x_1 = na^l k$. Then we have that $\text{diag}([b], I_{n-2}, m)at_e$ lies in the support of a gauge on $\text{GL}_n(F)$, we have

$$\left| \frac{a_i}{a_3 t_3} \right| \leq 1, \quad \left| \frac{a_i t_i}{a_i + t_{i+1}} \right| \leq 1, \quad \left| \frac{t_{n-1}}{m t_n} \right| \leq 1,$$

where $i = 3, \ldots, n - 2$.

Thus, by similar arguments as in Lemma 8.1 we have

$$[\epsilon] \leq |t_3 \cdots t_{n-1}|^{-1} \leq [x_1]^{C'} |[b]|^{-2C'},$$

where $C'$ is some positive integer. By [Sou93, Proposition 11.15, Lemma 2] we have

$$\max \left\{ \left| \frac{t_i}{t_{i+1}} \right|, \left| \frac{t_{i+1}}{t_1} \right| \right\} \leq [\epsilon]^{2n} \leq [x_1]^{2nC'} |[b]|^{-4nC'}.$$

Thus we have

$$|m| \geq [x_1]^{-C_0} |[b]|^{2C_0},$$

where $C_0$ is a positive integer depends on $\tau$.

Then, the integral $I'_{s_1, s_2}$ is majorized by a finite sum of integrals of the form

$$\int_{F^\times} |b|^{-s_1} |[b]|^{\Re(s) - n + 3 - c_2}
\times \int_{X_1(F)} \int_{m \geq [x_1]^{-C_0} |[b]|^{2C_0}} |m|^{-s_1 + 2 + \Re(s) + 5 + c_1} [x_1]^{-\Re(s) - \frac{n-1}{2} + c_3} dx_1 d^\times m d^\times b,$$

where $c_2, c_3 > 0, c_1$ are constants depend on $\tau$.

Similar as in Lemma 8.3, the above integral converges absolutely when

$$- \Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) + 5 + c_1 < 0,$$

$$- C_0 (-\Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) + 5 + c_1) - \Re(s) - \frac{n-1}{2} + c_3 < -C''$$

$$2C_0 (-\Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) + 5 + c_1) + \Re(s) - n + 3 - c_2 - \Re(s_1) > 0,$$

$$- 2C_0 (-\Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) + 5 + c_1) + \Re(s) - n + 3 - c_2 - \Re(s_1) < 0,$$

where $C''$ is a positive integer which depends on $\tau$. Thus we deduce the lemma.

Then above region (8.0.4) is simplified to

$$\Re(s) < \Re(s_1) + \Re(-s_2) - \frac{d_1}{2} - 5 - c_1$$

$$\Re(s) > \max \left( \frac{C_0}{C_0 + 1} (\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-1 - C''}{C_0 + 1},
\Re(s_1) + \frac{2C_0}{2C_0 + 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3 + c_2}{2C_0 + 1},
\Re(s_1) + \frac{2C_0}{2C_0 - 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3 + c_2}{2C_0 - 1} \right).$$
For this to be non-empty we need
\[ \Re(s_1) - \Re(s_2) - \frac{d_1}{2} - 5 - c_1 > \max\left( \frac{C_0}{C_0 + 1} (\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n - 1 - C''}{C_0 + 1}, \right. \\
\Re(s_1) + \frac{2C_0}{2C_0 + 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n - 3 + c_2}{2C_0 + 1}, \]
\[ \left. \Re(s_1) + \frac{2C_0}{2C_0 - 1} (\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n - 3 + c_2}{2C_0 - 1} \right). \]

For the above to be valid we need
\[ -n + \frac{d_2}{2} + 8 + c_1 < \Re(-s_2) < n + \frac{d_2}{2} + 2 + c_1 + c_2, \]
\[ c_0(-n + 3 - c_2) + C'' < \Re(s_1) + \Re(-s_2). \]

Since \( n \geq 3, -n + \frac{d_2}{2} + 8 + c_1 < n + \frac{d_2}{2} + 2 + c_1 + c_2, \) then the above inequalities are valid. Thus region (8.0.5) is non-empty.

\[ \text{Lemma 8.5.} \quad \text{The infinite sum} \]
\[ \sum_{k=\text{val}(4\Q'(y_2))}^{\infty} \frac{c(q^2, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2\zeta_v(-s_2)^2 q^{-s_1 + s_2 + n - 2 + \frac{d_1}{2} k}}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} \frac{1}{B_{\psi_1, s_2}(\varpi_H^{\delta_k, \text{val}(4\Q'(y_2))})} \]
\[ \text{converges absolutely when} \]
\[ (8.0.6) \quad \Re(s_1) - \Re(s_2) - 2 - \frac{d_1}{2} \leq \Re(s) \leq \Re(s_1) - \Re(s_2) + n + 1 - \frac{d_1}{2}, \]
\[ (8.0.7) \quad \Re(s_1) > C_1, \Re(-s_2) > C_2, \]
where \( C_1, C_2 \) are constants depends on \((n, d_1, d_2)\).

\[ \text{Proof.} \quad \text{By the formula of} \ c_{s_2} \text{ and} \ B_{\psi_1, s_1, s_2}, \text{it suffices to show} \]
\[ \sum_{k=\text{val}(4\Q'(y_2))}^{\infty} q^{-(s_1 + s_2 - n - 1 + \frac{d_1}{2}) k} \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2 \gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1} \sum_{w \in W} w \chi_s(\varpi^{\delta_k})^{-1} \]
\[ \text{converges absolutely in the region given by 8.0.6 and 8.0.7.} \]

Note that \( \zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2 \) converges when
\[ \Re(-s_2) > 0, \Re(-s_2 - \frac{d_1}{2} + 2) > 0, \]
and \( \gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1} \) converges when
\[ \Re(s - s_1 + s_2 + \frac{d_1 + 5}{2}) \geq \frac{1}{2}, \]
which lies in the given region. It remains to show the sum
\[ \sum_{k=0}^{\infty} q^{-(s_1 - s_2 + 1 - \frac{d_1}{2}) k} \sum_{w \in W} w \chi_s(\varpi^{\delta_k})^{-1} \]
\[ \text{converges absolutely in the region.} \]
We have
\[ |\chi_s(\varpi^{\delta_k})^{-1}| = q^{(s-\frac{1}{2})k}|\chi_1(\varpi^{-k})|. \]
By [JS81b, Corollary 2.5] we have
\[ |w\chi_1(\varpi^{-k})| < q^{\frac{k}{2}} \]
for any \( w \in W \) and \( k > 0 \). Then it suffices to observe that
\[ \sum_{k=0}^{\infty} q^{-(s_1-s_2+n+1-\frac{d_1}{2})k} q^{sk} \]
and
\[ \sum_{k=0}^{\infty} q^{-(s_1-s_2+n+1-\frac{d_1}{2})k} q^{-(s-1)k} \]
converges in the given region because they are convergent geometric series. The convergence region is obviously non-empty since \( n \) is positive. \( \Box \)
**List of symbols**

| Symbol | Description |
|--------|-------------|
| $A_1$ | subgroup of $T_G$ (2.3.4) |
| $A_2$ | subgroup of $T_G$ (2.3.5) |
| $a_y$ | $\text{diag}(-4Q'(y_2), I_{n-1}) \in \text{GL}_n$ (1.2.4) |
| $B_{\psi_1, s_1}'$ | unramified Bessel function 7.0.11 |
| $G$ | $\{g = (g_1, g_2) \in \text{GL}_2(R) : \det g_1 = \det g_2^{-1}\}$ (2.1.1) |
| $G'$ | $\text{SO}_4$ (2.1.2) |
| $G_1$ | subgroup of $T_G$ (2.3.3) |
| $H$ | $\text{SO}_{2n+1}$ (2.1.1) |
| $\iota$ | embedding map from $G$ to $H$ (2.2.2) |
| $I(f, W_{\xi_s})$ | global integral 1.2.2 |
| $M_1$ | subgroup of $T_G$ (2.3.1) |
| $M_{\text{SL}_2}$ | subgroup of maximal torus of $\text{SL}_2 \times \text{SL}_2$ (2.3.8) |
| $M_n$ | Levi subgroup of $Q_n$ (2.1.5) |
| $\mu$ | irreducible unramified character of $\text{SO}_2$ 7 |
| $N_1$ | subgroup of $U_2$ 2.3.6 |
| $N_2$ | subgroup of $U_2$ 2.3.7 |
| $N^o$ | unipotent subgroup of $H$ (2.1.3) |
| $\overline{N}_n$ | opposite unipotent radical of $Q_n$ (2.1.6) |
| $\mathbb{P}Y'$ | quasi-projective subscheme of $Y'$ 1 |
| $Q$ | quadratic form on $V_1$ 3 |
| $Q'$ | quadratic form on $V_2$ 3 |
| $Q_n$ | standard parabolic subgroup of $H$ 2.1.1 |
| $\overline{Q}_n$ | opposite parabolic subgroup of $H$ 2.1.1 |
| $w$ | Weyl group element of $H$ (2.1.7) |
| $\rho$ | Weil representation 3 |
| $\tau$ | irreducible cuspidal representation of $\text{GL}_n$ 3 |
| $T_G$ | maximal torus of $G$ 2.1 |
| $T_H$ | maximal torus of $H$ 2.1 |
| $\Theta_f$ | Theta function 3 |
| $U_2$ | maximal unipotent subgroup of $G$ (1.2.3) |
| $\overline{U}_2$ | opposite of $U_2$ 7 |
| $V$ | $V_1 \times V_2$ 1 |
| $V_i$ | quadratic space of even dimension 1 |
| $W_{\xi_s}$ | Whittaker function on $\text{GL}_n$ when restricted to a Levi subgroup of $H$ 3.0.6 |
| $W_{\rho_{r,s}}$ | local vector in $\text{Ind}^H_{Q_n}(\mathcal{W}(\tau, \psi_0) \otimes |\det|^{s-\frac{1}{2}})$ 7.0.1 |
| $\xi_s$ | global smooth holomorphic section in the space $\text{Ind}^H_{Q_n}(\tau \otimes |\det|^{s-\frac{1}{2}})$ 3 |
| $Y$ | $\{v \in V(R) : Q(v_1) = 2Q'(V_2)\}$ (3.0.3) |
| $Y'$ | subscheme of $Y$ such that no $y_i = 0$ 1 |
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