Gravitational acceleration and tidal effects in spherical-symmetric density profiles

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Abstract

Pure power-law density profiles, $\rho(r) \propto r^{b-3}$, are classified in connection with the following reference cases: (i) isodensity, $b = 3$, $\rho = \text{const}$; (ii) isogravity, $b = 2$, $g = \text{const}$; (iii) isothermal, $b = 1$, $v = [GM(r)/r]^{1/2} = \text{const}$; (iv) isomass, $b = 0$, $M = \text{const}$. A restricted number of different families of density profiles including, in addition, cored power-law, generalized power-law, polytropes, are studied in detail with regard to both one-component and two-component systems. Considerable effort is devoted to the existence of an extremum point (maximum absolute value) in the gravitational acceleration within the matter distribution. Predicted velocity curves are compared to the data inferred from observations. Tidal effects on an inner subsystem are investigated and an application is made to globular clusters within the Galaxy. To this aim, the tidal radius is defined by balancing the opposite gravitational forces from the Galaxy and the selected cluster on a special point of the cluster boundary, lying between related centres of mass. The position of 17 globular clusters with respect to the stability region, where the tidal radius exceeds the observed radius, is shown for assigned dark-to-visible mass ratios and density profiles, among those considered, which are currently used for the description of galaxies and/or dark matter haloes.

keywords - cosmology: dark matter - galaxies: structure - globular clusters: general.
1 Introduction

Typical galaxies, such as spirals and giant ellipticals, are multi-component systems mainly substructured as (1) nonbaryonic dark halo; (2) baryonic halo; (3) disk; (4) bulge; (5) inner accretion disk; (6) central accreting supermassive black hole, in brief hole. For sufficiently extended subsystems i.e. (1)-(4), a description in terms of density profiles enables the determination of the theoretical circular velocity profile and the comparison with its empirical counterpart inferred from observations (e.g., Haud and Einasto 1989).

Strictly speaking, density profiles should relate to equilibrium equations (e.g., Jaffe, 1983; Hernquist, 1990; Dehnen, 1993; Stone and Ostriker, 2015) but an acceptable fit to light distribution, via a mass-luminosity relation, could be sufficient for the description of related subsystems (e.g., De Vaucouleurs, 1948; Sersic, 1963; Kormendy, 1977).

In most cases, each subsystem is treated separately and the model is a simple superposition of the various components. More self-consistent models should account for the fact that each subsystem ought to readjust under the tidal action from the remaining subsystems. On the other hand, a simple superposition model may be adequate in deriving the general properties of a mass distribution, provided the number of free parameters remains reasonably low (e.g., Carignan 1985).

Though a quantitative (i.e. with regard to parameter values) classification of density profiles for one-component (e.g., Zhao 1996; An and Zhao, 2013) and two-component (e.g., Ciotti and Pellegrini 1992; Ciotti, 1999) is long dating, still less effort has been devoted to a qualitative (i.e. with regard to intrinsic properties) classification. To this aim, the current paper takes into consideration a restricted number of density profiles, part obeying an equilibrium equation and part being purely descriptive. For sake of simplicity, attention shall be restricted to spherical-symmetric configurations, even if the formulation maintains for homeoidally striated ellipsoidal configurations as far as radial properties are concerned (e.g., Caimmi 1993, 2003; Caimmi and Marmo 2003). To emphasize the above mentioned property, spherical-symmetric density profiles shall be quoted henceforth as homeoidally striated spherical density profiles.

Pure power-law density profiles, \( \rho(r) \propto r^{b-3}, 0 \leq b \leq 3 \), are used to perform a qualitative classification characterized by the following reference cases: (i) isodensity, \( b = 3 \), \( \rho = \text{const} \); (ii) isogravity, \( b = 2 \), \( g = \text{const} \); (iii) isothermal, \( b = 1 \), \( v = [GM(r)/r]^{1/2} = \text{const} \); (iv) isomass, \( b = 0 \), \( M = \text{const} \).

In the last case, \( \rho(r) = m_0/r^3 \), where \( m_0 \) is a proportionality constant dimensioned as a mass which, in addition, has to be infinitesimal of higher
order with respect to $1/\log r$ to avoid a logarithmic divergence in the mass, $M(r)$. More specifically, $m_0$ has to be infinitesimal of equal order with respect to $r^3$ to ensure a finite nonzero mass at the centre, $0 < M(0) = M(r) < +\infty$.

A selected density profile can be classified as pseudo isodense, pseudo isogravity, pseudo isothermal, pseudo isomass, according if it is sufficiently close to the appropriate reference case. In particular, density profiles where $3 \geq b > 2$ in the inner regions and $2 > b \geq 0$ in the outer regions, could exhibit a nonmonotonic gravitational acceleration with the occurrence of an extremum point (maximum absolute value) inside the domain. Accordingly, further classification of density profiles should include the presence or the absence of a maximum in $|g|$ within a selected matter distribution.

Additional investigation, conceived as an application of the general theory, can be devoted to the stability of globular clusters against the tidal action of the Galaxy via a simple formulation of tidal radius, involving balance of opposite gravitational forces exerted from a selected cluster and the Galaxy on a special point on the cluster surface lying between related centres of mass (e.g., Von Hoerner 1958; Brosche et al. 1999). For a fixed density profile, clusters where the tidal radius exceeds the observed radius are considered bound, if otherwise (partially) unbound.

The position of globular clusters can be plotted together with the stability region, and the number of bound and (partially) unbound globular clusters can be determined for selected Galactic density profiles. Finally, Galactic density profiles can be constrained according if globular clusters showing no sign of tidal tails or tidal streams lie within the instability region and vice versa. Though a more accurate definition of tidal radius should be used to infer quantitative conclusions, still the trend is expected to be similar and the validity of the method is left unchanged.

The current paper is structured in the following way. One-component, homeoidally striated spherical density profiles are described in Section 2, where further attention is devoted to a few families of density profiles, namely (a) pure power-law; (b) cored power-law; (c) polytropes; (d) Plummer (1911); (e) Hernquist (1990); (f) Begeman et al. (1991); (g) Spano et al. (2008); (h) Burkert (1995). Two-component, concentric, homeoidally striated spherical density profiles are described in Section 3, where further attention is devoted to a few combinations of density profiles, namely (i) Plummer-Begeman et al.; (j) Hernquist-Begeman et al; (k) Plummer-Spano et al.; (l) Hernquist-Spano et al. Tidal effects on embedded subsystems are analysed in Section 4, where the tidal radius is defined via gravitational balance on a special point and an application to globular clusters within galaxies is performed, with regard to density profiles currently used for the description of galaxies and/or dark matter haloes, among those considered. The results are discussed in Section
where the particularization to a sample of Galactic globular clusters is also shown. Finally, the conclusion is drawn in Section 6. Further details on specific arguments are shown in the Appendix.

2 Homeoidally striated spherical density profiles

With respect to a reference frame where the origin coincides with the centre of mass, let $\rho(r)$ be a generic, homeoidally striated spherical density profile. Accordingly, the equipotential surfaces are concentric spheres centered on the origin. The mass, the squared circular velocity, and the gravitational acceleration profile, respectively, are:

$M(r) = 4\pi \int_0^r \rho(r) r^2 \, dr$; \hspace{1cm} (1)

$v^2(r) = \frac{GM(r)}{r}$; \hspace{1cm} (2)

$g(r) = -\frac{GM(r)}{r^2}$; \hspace{1cm} (3)

and the local slope of the gravitational acceleration profile reads:

$$\frac{dg}{dr} = G \left[ \frac{2M(r)}{r^3} - \frac{1}{r^2} \frac{dM}{dr} \right];$$ \hspace{1cm} (4)

$$\frac{dM}{dr} = 4\pi \rho(r) r^2;$$ \hspace{1cm} (5)

where $G$ is the constant of gravitation.

Extremum points of the gravitational acceleration profile must fulfill the condition, $dg/dr = 0$, or:

$$M(r) = 2\pi r^3 \rho(r);$$ \hspace{1cm} (6)

which, in terms of the global density, translates into:

$$\bar{\rho}(r) = \frac{3}{4\pi} \frac{M(r)}{r^3} = \frac{3}{2} \rho(r);$$ \hspace{1cm} (7)

where the radial coordinate, satisfying Eq. (7), is the extremum point of the gravitational acceleration profile i.e. the first derivative of the gravitational potential. Keeping in mind no gravitational force is exerted at the origin, $r = 0$, the extremum point has to be a minimum (maximum in absolute
value). For an extension to homeoidally striated ellipsoidal density profiles, an interested reader is addressed to Appendix A.

In dimensionless coordinates, a generic density profile reads (e.g., Caimmi and Marmo 2003):

\[ \rho(r) = \rho^\dagger f(\xi) ; \quad 0 \leq \xi \leq \Xi ; \quad (8) \]

\[ \xi = \frac{r}{r^\dagger} ; \quad \Xi = \frac{R}{r^\dagger} ; \quad f(1) = 1 ; \quad (9) \]

where \( \rho^\dagger, r^\dagger \) are a scaling density and a scaling radius, respectively, \( R \) is the truncation radius, and \( \rho^\dagger = \rho(r^\dagger) \) via Eq. (9). The logarithmic slope at the dimensionless scaling radius, hereafter quoted as the scaling logarithmic slope, is \( (\frac{d \log f}{d \log \xi})_{\log \xi=0} \) (Caimmi et al. 2005). It shall be intended any density profile may be extended outside the truncation radius, putting \( \rho(r) = 0, \ r > R \).

The related mass profile reads (e.g., Caimmi and Marmo 2003):

\[ M(r) = M^\dagger \nu_{\text{mas}}(\xi) ; \quad (10) \]

\[ M^\dagger = \frac{4\pi}{3} \rho^\dagger (r^\dagger)^3 ; \quad \nu_{\text{mas}}(\xi) = \frac{3}{2} \left[ \int_0^\xi F(\xi) d\xi - \xi F(\xi) \right] ; \quad (11) \]

where \( M^\dagger \) is a scaling mass, \( \nu_{\text{mas}} \) is a profile factor i.e. depending on \( \xi \) only, and the integrand is defined as:

\[ F(\xi) = 2 \int_\xi^\Xi f(\xi) \xi d\xi ; \quad F(\Xi) = 0 ; \quad \frac{dF}{d\xi} = -2\xi f(\xi) ; \quad (12) \]

for further details, an interested reader is addressed to the parent paper (Roberts 1962).

The related global density, by definition, is:

\[ \bar{\rho}(r) = \frac{3}{4\pi} \frac{M(r)}{r^3} = \frac{3}{4\pi} \frac{M^\dagger \nu_{\text{mas}}(\xi)}{\xi^3} = \rho^\dagger \frac{\nu_{\text{mas}}(\xi)}{\xi^3} ; \quad (13) \]

and the ratio of global to local density reads:

\[ \frac{\bar{\rho}(r)}{\rho(r)} = \frac{\nu_{\text{mas}}(\xi)}{\xi^3 f(\xi)} ; \quad (14) \]

the substitution of Eqs. (11) and (10) into (2) and (3) yields the squared circular velocity and gravitational acceleration profiles in dimensionless coordinates, as:

\[ v^2(r) = (v^\dagger)^2 \frac{\nu_{\text{mas}}(\xi)}{\xi} ; \quad (15) \]
$$(v^\dagger)^2 = \frac{GM^\dagger}{r^\dagger};$$  \hspace{1cm} (16)  
$$g(r) = g^\dagger \frac{\nu_{\text{mas}}(\xi)}{\xi^2};$$  \hspace{1cm} (17)  
$$g^\dagger = -\frac{GM^\dagger}{(r^\dagger)^2};$$  \hspace{1cm} (18)  

where $v^\dagger$ and $g^\dagger$ are a scaling circular velocity and a scaling gravitational acceleration, respectively.

The combination of Eqs. (10), (11), (15)-(18), yields:

$$\nu_{\text{mas}}(\xi) = \frac{M(r)}{M^\dagger} = \left[ \frac{v(r)}{v^\dagger} \right]^2 \xi = \frac{g(r)}{g^\dagger} \xi^2;$$  \hspace{1cm} (19)  

and the combination of Eqs. (7) and (14) yields:

$$\frac{\nu_{\text{mas}}(\xi)}{\xi^3 f(\xi)} = \frac{3}{2};$$  \hspace{1cm} (20)  

the solution of which is the extremum point of the gravitational acceleration profile in dimensionless coordinates.

Let the isodensity surface where the gravitational acceleration attains the extremum point be defined as effective surface of the system, and the related radius as effective radius. In absence of an extremum point, the effective radius necessarily takes place on the boundary of the domain, either $r_{\text{eff}} = 0$ or $r_{\text{eff}} = R$, in dimensionless coordinates either $\xi_{\text{eff}} = 0$ or $\xi_{\text{eff}} = \Xi$.

The particularization of the above results to some special families of density profiles shall be performed in the following subsections.

### 2.1 Pure power-law density profiles

Pure power-law density profiles are defined as (e.g., Caimmi 2008):

$$f(\xi) = \xi^{b-3}; \quad 0 \leq \xi \leq \Xi; \quad 0 \leq b \leq 3;$$  \hspace{1cm} (21)  

where $b > 3$ implies increasing density for increasing radial coordinate, and $b < 0$ implies infinite central mass. The scaling logarithmic slope is:

$$\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi = 0} = b - 3;$$  \hspace{1cm} (22)  

which is constant in the case under consideration.
The substitution of Eq. (21) into (10)-(18) yields after some algebra:

$$F(\xi) = \frac{2}{b-1}(\Xi^{b-1} - \xi^{b-1}) ; \quad (23)$$

$$\nu_{\text{mas}}(\xi) = \frac{3}{b} \xi^b ; \quad (24)$$

$$\frac{\tilde{\rho}(r)}{\rho^i} = \frac{3}{b} \xi^{b-3} ; \quad (25)$$

$$\frac{\tilde{\rho}(r)}{\rho(r)} = \frac{3}{b} ; \quad (26)$$

$$\frac{v^2(r)}{(v^\dagger)^2} = \frac{3}{b} \xi^{b-1} ; \quad (27)$$

$$\frac{g}{g^i} = \frac{3}{b} \xi^{b-2} ; \quad (28)$$

where $b = 3$ corresponds to the isodensity ($\rho = \text{const}$) sphere, $b = 2$ to the isogravity ($g = \text{const}$) sphere, $b = 1$ to the isothermal ($v^2 = GM(r)/r = \text{const}$) sphere, $b = 0$ to the isomass ($M = \text{const}$) sphere, according to Eqs. (8) and (21).

In the last case, $b = 0$ via Eq. (26) implies either infinite mass for finite local density, or finite mass for null local density outside the centre of mass. Restricting to the latter alternative (isomass sphere), it can be seen that $\rho^i \to 0$, $M^i \to 0$, $\nu_{\text{mas}}(\xi) \to +\infty$, $M(\xi) = M(\Xi) = M$, and the density profile represents a mass point surrounded by a massless atmosphere i.e. a Roche system.

According to Eq. (28), power-law density profiles exhibit no (absolute) extremum point of the gravitational acceleration.

### 2.2 Cored power-law density profiles

Aiming to eliminate the central density cusp exhibited by pure power-law density profiles, cored power-law density profiles are defined as (e.g., Secco 2005; Caimmi 2008):

$$f(\xi) = \begin{cases} 
1 ; & 0 \leq \xi \leq 1 ; \\
\xi^{b-3} ; & 1 \leq \xi \leq \Xi ;
\end{cases} \quad (29)$$

where the inner part of related power-law density profile is replaced by a homogeneous sphere of density, $\rho^i = \rho(r^i)$. The scaling logarithmic slope is discontinuous with a null value at $\log \xi \to 0^-$ and a value expressed by Eq. (22) at $\log \xi \to 0^+$. 
The substitution of Eq. (29) into (10)-(18) yields after some algebra:

\[ F(\xi) = \begin{cases} 
(1 - \xi^2) + \frac{2}{b-1}(\Xi^{b-1} - 1) & ; \quad 0 \leq \xi \leq 1 \\
\frac{2}{b-1}(\Xi^{b-1} - \xi^{b-1}) & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (30) \]

\[ \nu_{\text{max}}(\xi) = \begin{cases} 
\xi^3 & ; \quad 0 \leq \xi \leq 1 \\
1 + \frac{3}{b}(\xi - 1) & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (31) \]

\[ \bar{\rho}(r) = \begin{cases} 
1 & ; \quad 0 \leq \xi \leq 1 \\
\frac{1}{\xi} + \frac{3}{b}\xi^{b-1} & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (32) \]

\[ \bar{\rho}(r) = \begin{cases} 
1 & ; \quad 0 \leq \xi \leq 1 \\
\frac{1}{\xi} + \frac{3}{b}\xi^{b-1} & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (33) \]

\[ \frac{v^2(r)}{(v^1)^2} = \begin{cases} 
\xi^2 & ; \quad 0 \leq \xi \leq 1 \\
\frac{1}{\xi} + \frac{3}{b}\xi^{b-1} & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (34) \]

\[ \frac{g}{g^1} = \begin{cases} 
\xi & ; \quad 0 \leq \xi \leq 1 \\
\frac{1}{\xi^2} + \frac{3}{b}\xi^{b-2} & ; \quad 1 \leq \xi \leq \Xi
\end{cases}; \quad (35) \]

where \( b = 3 \) corresponds to the isodensity (\( \rho = \text{const} \)) sphere. The substitution of Eqs. (29) and (31) into (20) yields after some algebra:

\[ 3 \left[ 1 - \frac{3 - b}{3} \xi^{-b} \right] = \frac{3}{2}; \quad (36) \]

and the extremum point of the gravitational acceleration is defined by the solution of Eq. (36) as:

\[ \xi = \left( \frac{2 \ 3 - b}{3 \ 2 - b} \right)^{1/b}; \quad (37) \]

where \( \xi \geq 1 \) in that no extremum point exists for the inner homogeneous sphere, \( 0 \leq \xi \leq 1 \), according to Eq. (28), and, on the other hand, \( \xi \leq \Xi \), according to Eq. (29), which implies the following:

\[ 1 \leq \phi(b) = \left( \frac{2 \ 3 - b}{3 \ 2 - b} \right)^{1/b} \leq \Xi; \quad (38) \]

\[ \lim_{b \to 0} \left( \frac{2 \ 3 - b}{3 \ 2 - b} \right)^{1/b} = \exp \left( \frac{1}{6} \right) \]; \quad (39) \]

where \( \phi(b) \) is defined within the domain, \( 0 \leq b < 2 \), \( b = 3 \), as powers with real exponents must necessarily exhibit nonnegative basis, and is monotonically increasing.

In conclusion, an extremum point is attained by the gravitational acceleration for sufficiently steep (\( 0 \leq b < 2 \)) cored power-law density profiles, provided Eq. (38) is satisfied. In particular, \( \Xi \geq \exp(1/6) \) for \( b = 0 \) i.e. a generalized Roche system.
2.3 Polytropic density profiles

Polytropic spheres obey the Lane-Emden equation (e.g., Chandrasekhar 1939, Chap. IV):

\[
\frac{1}{\xi_{LE}^2} \frac{d}{d\xi_{LE}} \left( \xi_{LE}^2 \frac{d\theta}{d\xi_{LE}} \right) = -\theta^n ; \tag{40}
\]

\[
\theta(0) = 1 ; \quad \theta(\Xi_{LE}) = 0 ; \quad \left( \frac{d\theta}{d\xi_{LE}} \right)_0 = 0 ; \tag{41}
\]

\[
\theta(\xi_{LE}) = 1 - \frac{1}{6} \xi_{LE}^2 + \frac{n}{120} \xi_{LE}^4 - \ldots ; \quad 0 \leq \xi_{LE} < 1 ; \tag{42}
\]

where \( n \) is the polytropic index and the dimensionless radial coordinate, \( \xi_{LE} \), and the dimensionless density, \( \theta^n \), are related to their dimensional counterparts as:

\[
r = \alpha_{LE} \xi_{LE} ; \quad R = \alpha_{LE} \Xi_{LE} ; \quad \alpha_{LE} = \left[ \frac{(n + 1) K \lambda_{LE}^{1+1/n}}{4 \pi G \lambda_{LE}^3} \right]^{1/2} ; \tag{43}
\]

\[
\rho(r) = \lambda_{LE} \theta^n(\xi_{LE}) ; \quad \rho(0) = \lambda_{LE} ; \quad \rho(R) = 0 ; \tag{44}
\]

where \( \lambda_{LE} \) is the central density, \( K \lambda_{LE}^{1+1/n} \) the central pressure, and \( \alpha_{LE} \) the polytropic scaling radius. Polytropic density profiles lie between the limiting cases of homogeneous configurations (\( n = 0 \)) and Roche systems or Plummer systems (\( n = 5 \)).

The relations between current and polytropic dimensionless density and radial coordinate, may be deduced by comparison of Eqs. (8) and (9) with (44) and (43), respectively. The result is:

\[
f(\xi) = \frac{\lambda_{LE} \theta^n(\xi_{LE})}{\rho^i} ; \tag{45}
\]

\[
\xi = \frac{\alpha_{LE}}{r^i} \xi_{LE} ; \quad \Xi = \frac{\alpha_{LE}}{r^i} \Xi_{LE} ; \tag{46}
\]

and the boundary condition, \( f(1) = 1 \), translates into:

\[
\theta \left( \frac{r^i}{\alpha_{LE}} \right) = \left( \frac{\rho^i}{\lambda_{LE}} \right)^{1/n} ; \tag{47}
\]

where, in particular:

\[
\lim_{n \to 0} \left( \frac{\rho^i}{\lambda_{LE}} \right)^{1/n} = 1 - \frac{1}{6} \left( \frac{r^i}{\alpha_{LE}} \right)^2 ; \tag{48}
\]
for homogeneous configurations, using next Eq. (60).

As the Lane-Emden equation, Eq. (40), and related solutions, have widely
been studied in literature (e.g., Chandrasekhar 1939, Chap. IV; Horedt 2004),
the quantities of interest for polytropic density profiles shall be expressed in
terms of the dimensionless density, \( \theta^n \), and the dimensionless radial coordi-
nate, \( \xi_{LE} \).

Accordingly, the scaling logarithmic slope is:

\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi=0} = \frac{n}{\log \xi = 0} \left( \frac{d \log \theta}{d \log \xi_{LE}} \frac{d \log \xi_{LE}}{d \log \xi} \right) \tag{49}
\]

and Eqs. (10), (11), (13), (14), (15), (17) and (20), take the equivalent form (e.g., Chandrasekhar 1939, Chap. IV, §6):

\[
M(r) = -M_{LE} \xi_{LE}^2 \frac{d \theta}{\xi_{LE}} ; \tag{50}
\]

\[
M_{LE} = 4\pi \alpha_{LE}^3 \lambda_{LE} ; \tag{51}
\]

\[
\bar{\rho}(r) = -3 \frac{d \theta}{\xi_{LE}} ; \tag{52}
\]

\[
\bar{\rho}(r) = -3 \frac{d \theta / d \xi_{LE}}{\xi_{LE}} ; \tag{53}
\]

\[
\left[ \frac{v(r)}{v_{LE}} \right]^2 = \frac{GM(r)}{v_{LE}^2 r^2} = -\xi_{LE} \frac{d \theta}{d \xi_{LE}} ; \tag{54}
\]

\[
v_{LE}^2 = \frac{GM_{LE}}{\alpha_{LE}} ; \tag{55}
\]

\[
g(r) = -\frac{GM(r)}{g_{LE} r^2} = - \frac{d \theta}{d \xi_{LE}} ; \tag{56}
\]

\[
g_{LE} = -\frac{GM_{LE}}{\xi_{LE}^2} ; \tag{57}
\]

\[
\frac{1}{\xi_{LE}} \frac{d \theta}{d \xi_{LE}} = -\frac{1}{2} \theta^n (\xi_{LE}) ; \tag{58}
\]

where the last result follows from Eq. (11), implying the existence of a so-
lution to Eq. (58) for \( n > 0 \), as \( \theta \) is continuous together with its first and
second derivates via Eq. (40).

The substitution of Eq. (10) into (58) yields after some algebra:

\[
\frac{d^2 \theta}{d \xi_{LE}^2} = 0 ; \tag{59}
\]
which shows that, for polytropic density profiles, the extremum point of the gravitational acceleration coincides with the extremum point of the first derivative of $\theta$, as expected, the gravitational potential inside polytropic spheres being proportional to $\theta$ (e.g., Chandrasekhar 1939, Chap. IV, §7).

In the special cases, $n = 0, 1, 5$, Eq. (40) can analytically be integrated. The result is (e.g., Chandrasekhar 1939, Chap. IV, §4):

$$\theta(\xi_{LE}) = 1 - \frac{1}{6} \xi_{LE}^2 ; \quad \Xi_{LE} = \sqrt{6} ; \quad n = 0 ; \quad (60)$$

$$\theta(\xi_{LE}) = \frac{\sin \xi_{LE}}{\xi_{LE}} ; \quad \Xi_{LE} = \pi ; \quad n = 1 ; \quad (61)$$

$$\theta(\xi_{LE}) = \left(1 + \frac{1}{3} \xi_{LE}^2\right)^{-1/2} ; \quad \Xi_{LE} \rightarrow +\infty ; \quad n = 5 ; \quad (62)$$

accordingly, Eq. (58) has no solution for $n = 0$, as expected for homogeneous configurations, and:

$$\frac{\xi_{LE} \cos \xi_{LE} - \sin \xi_{LE}}{\xi_{LE}^2} = -\frac{1}{2} \sin \xi_{LE} ; \quad (63)$$

for $n = 1$, which has a solution, $\xi_{LE} \approx 2.08157599$, and:

$$-\frac{1}{2} \left(1 + \frac{1}{3} \xi_{LE}^2\right)^{-3/2} 2 \frac{2}{3} \xi_{LE} = -\frac{1}{2} \xi_{LE} \left(1 + \frac{1}{3} \xi_{LE}^2\right)^{-5/2} ; \quad (64)$$

for $n = 5$, which has a solution, $\xi_{LE} = \sqrt{3/2} \approx 1.224744871$.

In conclusion, an extremum point is exhibited by the gravitational acceleration for inhomogeneous polytropic density profiles ($0 < n_{\text{min}} \leq n \leq 5$), where the solution of related transcendental equation can readily be determined in the special cases, $n = 1, 5$, via Eqs. (63), (64), respectively. The threshold, $n_{\text{min}}$, $0 < n_{\text{min}} < 1$, is defined by equating the solution of Eq. (58) to the dimensionless radius, $\xi_{LE}(n_{\text{min}}) = \Xi_{LE}(n_{\text{min}})$, as $n < n_{\text{min}}$ would imply $\xi_{LE}(n_{\text{min}}) > \Xi_{LE}(n_{\text{min}})$, which lies outside the domain, $0 \leq \xi_{LE} \leq \Xi_{LE}$.

### 2.4 Plummer density profiles

Plummer density profiles (Plummer 1911), hereafter quoted as P density profiles, are nothing but $n = 5$ polytropes represented by Eqs. (8), (9), via (45)-(47), where parameters are interrelated as (Caimmi and Valentinuzzi 2008):

$$\xi = \frac{\xi_{LE}}{\sqrt{3}} ; \quad r^i = \sqrt{3} \alpha_{LE} ; \quad \rho^i = \frac{\lambda_{LE}}{2^{5/2}} ; \quad (65)$$
the result is (Caimmi and Valentinuzzi 2008):

\[ f(\xi) = \frac{2^{5/2}}{(1 + \xi^2)^{5/2}} ; \]  

(66)

where the scaling logarithmic slope reads:

\[ \left( \frac{d \log f}{d \log \xi} \right)_{\log \xi=0} = \frac{5}{2} ; \]  

(67)

and the substitution of Eq. (66) into (10)-(18) yields after some algebra:

\[ F(\xi) = \frac{2^{7/2}}{3} \left[ \frac{1}{(1 + \xi^2)^{3/2}} - \frac{1}{(1 + \Xi^2)^{3/2}} \right] ; \]  

(68)

\[ \nu_{\text{mas}}(\xi) = \frac{2^{5/2} \xi^3}{(1 + \xi^2)^{3/2}} ; \]  

(69)

\[ \frac{\bar{\rho}(r)}{\rho^1} = \frac{2^{5/2}}{(1 + \xi^2)^{3/2}} ; \]  

(70)

\[ \frac{\bar{\rho}(r)}{\rho(r)} = 1 + \xi^2 ; \]  

(71)

\[ \left( \frac{v(r)}{v^1} \right)^2 = \frac{2^{5/2} \xi^2}{(1 + \xi^2)^{3/2}} ; \]  

(72)

\[ \frac{g(r)}{g^1} = \frac{2^{5/2} \xi}{(1 + \xi^2)^{3/2}} ; \]  

(73)

finally, the particularization of Eq. (20) to the case under consideration reads:

\[ 1 + \xi^2 = \frac{3}{2} ; \]  

(74)

which shows an extremum point of the gravitational acceleration at \( \xi = 1/\sqrt{2} \approx 0.707106781 \), in agreement with Eqs. (64) and (65).

### 2.5 Hernquist density profiles

Hernquist density profiles (Hernquist 1990), hereafter quoted as H density profiles, are defined as (e.g., Caimmi and Valentinuzzi 2008):

\[ f(\xi) = \frac{8}{\xi(1 + \xi)^3} ; \]  

(75)
where the scaling logarithmic slope reads:

\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi = 0} = -\frac{5}{2}; \tag{76}
\]

and the substitution of Eq. (75) into (10)-(18) yields after some algebra:

\[
F(\xi) = 8 \left[ \frac{1}{(1 + \xi)^2} - \frac{1}{(1 + \Xi)^2} \right]; \tag{77}
\]

\[
\nu_{\text{max}}(\xi) = \frac{12\xi^2}{(1 + \xi)^2}; \tag{78}
\]

\[
\frac{\bar{\rho}(r)}{\rho^i} = \frac{12}{\xi(1 + \xi)^2}; \tag{79}
\]

\[
\frac{\bar{\rho}(r)}{\rho(r)} = \frac{3}{2}(1 + \xi); \tag{80}
\]

\[
\left[ \frac{v(r)}{v^i} \right]^2 = \frac{12\xi}{(1 + \xi)^2}; \tag{81}
\]

\[
\frac{g(r)}{g^i} = \frac{12}{(1 + \xi)^2}; \tag{82}
\]

finally, the particularization of Eq. (20) to the case under consideration reads:

\[
\frac{3}{2}(1 + \xi) = \frac{3}{2}; \tag{83}
\]

which implies no extremum point for the gravitational acceleration inside the domain. In addition, Eq. (82) discloses that a finite gravitational acceleration occurs even if a test particle lies infinitely close to the centre.

### 2.6 Generalized power-law density profiles

Generalized power-law density profiles are defined as (e.g., Bottazzi 2011):

\[
f(\xi) = \frac{C_\gamma + 1}{C_\gamma + \xi^\gamma (C_\alpha + \xi^\alpha)^\chi} ; \tag{84}
\]

where the following families are worth of note.

- \( C_\gamma = 0; C_\alpha = 0 \); generalized power-law density profiles reduce to pure power-law density profiles with exponent, \( \beta \), expressed as:

\[
\beta = \gamma + \alpha \chi ; \quad \chi = \frac{\beta - \gamma}{\alpha}; \tag{85}
\]

which shows the meaning of the exponent, \( \chi \), appearing in Eq. (84).
• $C_\gamma = 0; C_\alpha = 1$; generalized power-law density profiles reduce to a subclass analysed in earlier attempts (Hernquist 1990; Zhao 1996) hereafter quoted as Z density profiles. Special cases are P density profiles, $(\alpha, \chi, \gamma) = (2, 5/2, 0)$; H density profiles, $(\alpha, \chi, \gamma) = (1, 3, 1)$; pseudo isothermal density profiles (e.g., Begeman et al. 1991), $(\alpha, \chi, \gamma) = (2, 1, 0)$, hereafter quoted as I density profiles; generalized pseudo isothermal density profiles (Spano et al. 2008), $(\alpha, \chi, \gamma) = (2, 3/2, 0)$, hereafter quoted as S density profiles.

• $C_\gamma = 1; C_\alpha = 0$; generalized power-law density profiles reduce to a subclass of Z density profiles, $(\alpha, \chi, \gamma) = (\gamma, 1, \alpha \chi)$, including I density profiles.

• $C_\gamma = 1; C_\alpha = 1$; generalized power-law density profiles reduce to a subclass of density profiles including a special case analysed in an earlier attempt (Burkert 1995), $(\alpha, \chi, \gamma) = (2, 1, 1)$, hereafter quoted as B density profiles.

For generalized power-law density profiles, the scaling logarithmic slope reads:

\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi=0} = -\frac{\gamma}{C_\gamma + 1} - \frac{\beta - \gamma}{C_\alpha + 1}; \quad (86)
\]

which, for Z density profiles, reduces to:

\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi=0} = -\frac{\beta + \gamma}{2}; \quad (87)
\]

the geometrical meaning of the dimensionless scaling radius is analysed in Appendix B.

The substitution of Eq. (84) into (10)-(18) yields, in general, integrals which cannot be analytically calculated. For this reason, further considerations shall be restricted to the above mentioned special cases, where the primitive functions can be explicitly expressed.

### 2.7 Pseudo isothermal density profiles

For I density profiles, Eq. (84) reduces to:

\[
f(\xi) = \frac{2}{1 + \xi^2}; \quad (88)
\]

where the scaling logarithmic slope reads:

\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi=0} = -1; \quad (89)
\]
according to Eq. (87).

The substitution of Eq. (88) into (10)-(18) yields after some algebra:

\begin{align*}
F(\xi) &= 2 \ln \frac{1 + \xi^2}{1 + \xi^2}; \quad (90) \\
\nu_{\text{mas}}(\xi) &= 6(\xi - \arctan \xi); \quad (91) \\
\frac{\rho(r)}{\rho^*} &= \frac{6}{\xi^3}(\xi - \arctan \xi); \quad (92) \\
\frac{\rho(r)}{\rho^*} &= \frac{3}{\xi^3}(1 + \xi^2)(\xi - \arctan \xi); \quad (93) \\
\left[ \frac{v(r)}{v^*} \right]^2 &= \frac{6}{\xi}(\xi - \arctan \xi); \quad (94) \\
\frac{g(r)}{g^*} &= \frac{6}{\xi^2}(\xi - \arctan \xi); \quad (95)
\end{align*}

Finally, the particularization of Eq. (20) to the case under consideration reads:

\[ \arctan \xi = \frac{\xi(2 + \xi^2)}{2(1 + \xi^2)}; \quad (96) \]

and the extremum point for the gravitational acceleration is defined by the solution of Eq. (96), as \( \xi \approx 1.514994606 \).

### 2.8 Spano et al. density profiles

For S density profiles, Eq. (84) reduces to:

\[ f(\xi) = \frac{2^{3/2}}{(1 + \xi^2)^{3/2}}; \quad (97) \]

where the scaling logarithmic slope reads:

\[ \left( \frac{d \log f}{d \log \xi} \right)_{\log \xi = 0} = -\frac{3}{2}; \quad (98) \]

according to Eq. (87).

The substitution of Eq. (97) into (10)-(18) yields after some algebra:

\begin{align*}
F(\xi) &= 2^{5/2} \left( \frac{1}{\sqrt{1 + \xi^2}} - \frac{1}{\sqrt{1 + \xi^2}} \right); \quad (99) \\
\nu_{\text{mas}}(\xi) &= 6\sqrt{2} \left( \arcsinh \xi - \frac{\xi}{\sqrt{1 + \xi^2}} \right); \quad (100)
\end{align*}
\[
\rho(r) = 6\sqrt{2} \left( \frac{\arcsinh \xi - \frac{\xi}{\sqrt{1 + \xi^2}}}{\xi^3} \right) ; \quad (101)
\]
\[
\rho(r) = \frac{3\sqrt{2}}{\xi^3} (1 + \xi^2)^{3/2} \left( \arcsinh \xi - \frac{\xi}{\sqrt{1 + \xi^2}} \right) ; \quad (102)
\]
\[
\left[ \frac{v(r)}{v^\dagger} \right]^2 = \frac{6\sqrt{2}}{\xi} \left( \arcsinh \xi - \frac{\xi}{\sqrt{1 + \xi^2}} \right) ; \quad (103)
\]
\[
\frac{g(r)}{g^\dagger} = \frac{6\sqrt{2}}{\xi^2} \left( \arcsinh \xi - \frac{\xi}{\sqrt{1 + \xi^2}} \right) ; \quad (104)
\]
finally, the particularization of Eq. (20) to the case under consideration reads:
\[
\arcsinh \xi = \frac{\xi(2 + 3\xi^2)}{2(1 + \xi^2)^{3/2}} ; \quad (105)
\]
and the extremum point for the gravitational acceleration is defined by the solution of Eq. (105), as \( \xi \approx 1.027115657 \).

2.9 Burkert density profiles

For B density profiles, Eq. (84) reduces to:
\[
f(\xi) = \frac{2}{\xi} \left( \frac{2}{1 + \xi + \xi^2} \right) ; \quad (106)
\]
where the scaling logarithmic slope reads:
\[
\left( \frac{d \log f}{d \log \xi} \right)_{\log \xi = 0} = -\frac{3}{2} ; \quad (107)
\]
according to Eq. (87).

The substitution of Eq. (106) into (10)- (18) yields after some algebra:
\[
F(\xi) = -2 \ln \frac{1 + \xi^2}{1 + \Xi^2} + 4 \ln \frac{1 + \xi}{1 + \Xi} + 4(\arctan \Xi - \arctan \xi) ; \quad (108)
\]
\[
\nu_{\text{max}}(\xi) = 3[\ln(1 + \xi^2) + 2 \ln(1 + \xi) - 2 \arctan \xi] ; \quad (109)
\]
\[
\frac{\bar{\rho}(r)}{\rho^\dagger} = \frac{3}{\xi^3} [\ln(1 + \xi^2) + 2 \ln(1 + \xi) - 2 \arctan \xi] ; \quad (110)
\]
\[
\frac{\bar{\rho}(r)}{\rho(r)} = \frac{3}{4 \xi^3} [\ln(1 + \xi^2) + 2 \ln(1 + \Xi) - 2 \arctan \xi] ; \quad (111)
\]
\[
\left[ \frac{v(r)}{v^\dagger} \right]^2 = \frac{3}{\xi} \left[ \ln(1 + \xi^2) + 2 \ln(1 + \xi) - 2 \arctan \xi \right] ; \quad (112)
\]
\[
\frac{g(r)}{g^\dagger} = \frac{3}{\xi^2} \left[ \ln(1 + \xi^2) + 2 \ln(1 + \xi) - 2 \arctan \xi \right] ; \quad (113)
\]
finally, the particularization of Eq. (20) to the case under consideration reads:

\[
\ln(1 + \xi^2) + 2 \ln(1 + \xi) - 2 \arctan \xi = \frac{2\xi^3}{(1 + \xi)(1 + \xi^2)} \ ; \quad (114)
\]

and the extremum point for the gravitational acceleration is defined by the solution of Eq. (114), as \( \xi \approx 0.963398283 \).

2.10 Gravitational acceleration vs. density profile

The dependence of the gravitational acceleration on the density profile can be related to the dimensionless effective radius, \( \xi_{\text{eff}} \), where the maximum absolute value is attained. For the density profiles considered in the current Section, the results are listed in Table 1 together with values of related parameters. An inspection of Table 1 shows the occurrence of an extremum point (maximum absolute value) in the gravitational acceleration distribution, with the exception of pure power-law, cored power-law (2 \( \leq b \leq 3 \)), H density profiles, and the possible exception of cored power-law (0 \( \leq b < 2 \)) density profiles, according to the selected dimensionless truncation radius, \( \Xi \).

3 Two-component density profiles

Let \( \rho_i(r) \), \( \rho_j(r) \), be generic but concentric homeoidally striated spherical density profiles, and \( M_i(r) \), \( M_j(r) \), related masses enclosed within the distance, \( r \), from the centre. The total mass, the squared circular velocity, and the gravitational acceleration are expressed by Eqs. (1)-(3) where \( \rho(r) = \rho_i(r) + \rho_j(r) \), 
\( M(r) = M_i(r) + M_j(r) \), which implies \( \bar{\rho}(r) = \bar{\rho}_i(r) + \bar{\rho}_j(r) \).

It can be seen the validity of Eq. (7) for each subsystem i.e. the existence of an extremum point of related gravitational acceleration, \( g_u = -GM_u(r)/r^2, \ u = i, j \), is a necessary condition for the validity of Eq. (7) for the whole system i.e. the existence of an extremum point of related gravitational acceleration, \( g = g_i + g_j \).

In dimensionless coordinates, generic density profiles are expressed by Eqs. (8)-(20), provided each quantity is indexed by \( i \) or \( j \), according to the selected subsystem. In addition, the following relations hold (e.g., Caimmi and Valentinnuzzi 2008):

\[
\begin{align*}
\xi_i &= y^i \xi_j \ ; \quad \Xi_j = \frac{y_j}{y} \ ; \quad \frac{\nu_{i,\text{max}}(r)}{\nu_{i,\text{max}}(r)} = \frac{m(r)}{m^\dagger} \ ; \quad (115a) \\
y &= \frac{R_i}{R_i} \ ; \quad y^\dagger = \frac{r^i}{r^i} \ ; \quad m(r) = \frac{M_j(r)}{M_i(r)} \ ; \quad m^\dagger = \frac{M_j^\dagger}{M_j^\dagger} \ ; \quad (115b)
\end{align*}
\]
Table 1: Parameters of one-component density profiles considered in the text: polytropic index, \( n \), for polytropes; constants, \( C_\gamma, C_\alpha \), and exponents, \( \alpha, \chi, \gamma, \beta \), for generalized power-law density profiles, Eqs. (84)-(85); and dimensionless effective radius, \( \xi_{\text{eff}} \). An extremum point for the gravitational acceleration occurs only if \( 0 < \xi_{\text{eff}} < \Xi \). For polytropic density profiles, \( \xi_{\text{eff}} = \Xi, \ n = 0, \) and \( \xi_{\text{eff}} = \sqrt{3/2}, \ n = 5, \) respectively. For Plummer density profile, \( \xi_{\text{eff}} = 1/\sqrt{2}. \) The notations specifying power-law density profiles, (0-2) and (2-3), are to be intended as \( (0 \leq b \leq 2) \) and \( (2 \leq b \leq 3) \), respectively. See text for further details.

| density profile        | \( n \) | \( C_\gamma \) | \( C_\alpha \) | \( \alpha \) | \( \chi \) | \( \gamma \) | \( \beta \) | \( \xi_{\text{eff}} \) |
|------------------------|--------|----------------|----------------|------------|----------|----------|--------|----------------|
| pure power-law (0-2)   | 0      | 0              | \( \alpha \)  | \( \frac{\beta - \gamma}{\alpha} \) | \( \gamma \) | \( \beta \) | 0      | \( \Xi \)       |
| pure power-law (2-3)   | 0      | 0              | \( \alpha \)  | \( \frac{\beta - \gamma}{\alpha} \) | \( \gamma \) | \( \beta \) | \( \Xi \) | \( \Xi \)       |
| cored power-law (0-2)  |        |                | \( \alpha \)  | \( \frac{\beta - \gamma}{\alpha} \) | \( \gamma \) | \( \beta \) | \( \Xi \) | \( \Xi \)       |
| cored power-law (2-3)  |        |                | \( \alpha \)  | \( \frac{\beta - \gamma}{\alpha} \) | \( \gamma \) | \( \beta \) | \( \Xi \) | \( \Xi \)       |
| polytropic             | 0      |                |                | \( 2.449489743 \) | \( 2.08157599 \) | \( 1.224744871 \) | \( 0.707106781 \) | \( 2.08157599 \) |
| polytropic             | 1      |                |                | \( 5.08157599 \) | \( 1.224744871 \) | \( 2.08157599 \) | \( 3.00000000 \) | \( 0.707106781 \) |
| polytropic             | 5      |                |                | \( 1.224744871 \) | \( 2.08157599 \) | \( 0.707106781 \) | \( 2.08157599 \) | \( 1.224744871 \) |
| Plummer (1911)         | 0      | 1              | 2.5            | 0          | 5        | 0        | 5      | 0              |
| Hernquist (1990)       | 0      | 1              | 3              | 1          | 4        | 0        | 0      | 1              |
| Begeman et al. (1991)  | 0      | 1              | 1.0            | 0          | 2        | 1.5      | 0.2    | 1.538461534    |
| Spano et al. (2008)    | 0      | 1              | 1.5            | 0          | 3        | 1.5      | 0.3    | 1.027115657    |
| Burkert (1995)         | 1      | 1              | 1.0            | 1          | 3        | 0.963398283 | | 0.963398283 |
\[
\frac{\rho_j}{\rho_i} = \frac{m_j}{(y^i)^3};
\]  

(115c)

where \( y \geq 1 \) without loss of generality, which implies \( i \) is the inner subsystem and \( j \) the outer unless their surfaces coincide.

Using Eqs. (10)-(18) and (115), the mass enclosed within the radius, \( r \), and related squared circular velocity and gravitational acceleration are expressed as:

\[
M(r) = \nu_i,\text{mas}(\xi_i) + \nu_j,\text{mas}(\xi_j); \quad (116)
\]

\[
v^2(r) = \left( v_{i,\text{mas}}(\xi_i) \right)^2 + \left( v_{j,\text{mas}}(\xi_j) \right)^2; \quad (117)
\]

\[
g(r) = \frac{g_i,\text{mas}(\xi_i)}{\xi_i^2} + \frac{g_j,\text{mas}(\xi_j)}{\xi_j^2}; \quad (118)
\]

and the extremum point of the gravitational acceleration, via Eqs. (7)-(20) and (115), is the solution of the following equation:

\[
\rho_i \left[ \frac{\nu_i,\text{mas}(\xi_i)}{\xi_i^3} - \frac{3}{2} f_i(\xi_i) \right] + \rho_j \left[ \frac{\nu_j,\text{mas}(\xi_j)}{\xi_j^3} - \frac{3}{2} f_j(\xi_j) \right] = 0; \quad (119)
\]

where \( \xi_i \) and \( \xi_j \) are related via Eqs. (115a)-(115b).

Some special density distributions shall be considered in the following, and Eq. (119) shall be written in terms of the additional parameters, \( m^i \) and \( y^i \), from which the extremum points of the gravitational acceleration also depend.

### 3.1 PI density profiles

In the special case of an inner P density profile, \( i = P \), and an outer I density profile, \( j = I \), the substitution of Eqs. (66)-(73) and (88)-(95) into (115)-(118) yields:

\[
\xi_P = y^i \xi_I; \quad (120)
\]

\[
m(r) = m^i \frac{6(\xi_i - \arctan \xi_i)}{2^{5/2}\xi_i^3(1 + \xi_i^2)^{-3/2}}; \quad (121)
\]

\[
M(r) = M_i^i \frac{2^{5/2}\xi_P^3}{(1 + \xi_P^2)^{3/2}} + M_j^i 6(\xi_i - \arctan \xi_i); \quad (122)
\]

\[
v^2(r) = \left( v_{i,\text{mas}}(\xi_i) \right)^2 \frac{2^{5/2}\xi_P^2}{(1 + \xi_P^2)^{3/2}} + \left( v_{j,\text{mas}}(\xi_j) \right)^2 \frac{6(\xi_i - \arctan \xi_i)}{\xi_i}; \quad (123)
\]

\[
g(r) = g_i^i \frac{2^{5/2}\xi_P}{(1 + \xi_P^2)^{3/2}} + g_j^i \frac{6(\xi_i - \arctan \xi_i)}{\xi_i^2}; \quad (124)
\]
and the extremum point of the gravitational acceleration is the solution of Eq. (119) particularized to the case under consideration, which after some algebra reads:

\[
\frac{2^{3/2} \rho_p^\dagger}{(1 + \xi_p^2)^{3/2}} \frac{2 \xi_p^2 - 1}{1 + \xi_p^6} + 3 \rho_i^\dagger \left[ \frac{2 + \xi_i^3}{\xi_i^3 (1 + \xi_i^2)} \frac{2 \arctan \xi_i}{\xi_i^3} \right] = 0 ;
\]

(125)

that shows an additional dependence on the parameters, \(m^\dagger, y^\dagger\), via Eqs. (115c) and (120), respectively.

### 3.2 HI density profiles

In the special case of an inner H density profile, \(i = H\), and an outer I density profile, \(j = I\), the substitution of Eqs. (75)-(82) and (88)-(95) into (115)-(118) yields:

\[
\xi_H = y^\dagger \xi_I ;
\]

(126)

\[
m(r) = m^\dagger \frac{\xi_I - \arctan \xi_I}{2 \xi_H^2 (1 + \xi_H^2)^{-2}} ;
\]

(127)

\[
M(r) = M_H^\dagger \frac{12 \xi_H^2}{(1 + \xi_H^2)^2} + M_I^\dagger 6(\xi_I - \arctan \xi_I) ;
\]

(128)

\[
v^2(r) = (v_H^\dagger)^2 \frac{12 \xi_H}{(1 + \xi_H^2)^2} + (v_I^\dagger)^2 \frac{6(\xi_I - \arctan \xi_I)}{\xi_I} ;
\]

(129)

\[
g(r) = g_H^\dagger \frac{12}{(1 + \xi_H^2)^2} + g_I^\dagger \frac{6(\xi_I - \arctan \xi_I)}{\xi_I^2} ;
\]

(130)

and the extremum point of the gravitational acceleration is the solution of Eq. (119) particularized to the case under consideration, which after some algebra reads:

\[
\frac{12 \rho_H^\dagger}{(1 + \xi_H^2)^2} + 3 \rho_i^\dagger \left[ \frac{2 + \xi_i^3}{\xi_i^3 (1 + \xi_i^2)} \frac{2 \arctan \xi_i}{\xi_i^3} \right] = 0 ;
\]

(131)

that shows an additional dependence on the parameters, \(m^\dagger, y^\dagger\), via Eqs. (115c) and (126), respectively.

### 3.3 PS density profiles

In the special case of an inner P density profile, \(i = P\), and an outer S density profile, \(j = S\), the substitution of Eqs. (66)-(73) and (97)-(104) into (115)-(118) yields:

\[
\xi_P = y^\dagger \xi_S ;
\]

(132)
\[ m(r) = m^\dagger \frac{6\sqrt{2} [\text{arcsinh} \xi_S - \xi_S (1 + \xi_S^2)^{-1/2}]}{2^{5/2} \xi_S^2 (1 + \xi_S^2)^{-3/2}}; \quad (133) \]

\[ M(r) = M^\dagger_P \frac{2^{5/2} \xi_P^3}{(1 + \xi_P^2)^{3/2}} + M^\dagger_S 6\sqrt{2} \left( \text{arcsinh} \xi_S - \frac{\xi_S}{\sqrt{1 + \xi_S^2}} \right); \quad (134) \]

\[ v^2(r) = (v^\dagger_P)^2 \frac{2^{5/2} \xi_P^2}{(1 + \xi_P^2)^{3/2}} + (v^\dagger_S)^2 6\sqrt{2} \left( \frac{\text{arcsinh} \xi_S}{\xi_S} - \frac{1}{\sqrt{1 + \xi_S^2}} \right); \quad (135) \]

\[ g(r) = g^\dagger_P \frac{2^{5/2} \xi_P^3}{(1 + \xi_P^2)^{3/2}} + g^\dagger_S 6\sqrt{2} \left( \frac{\text{arcsinh} \xi_S}{\xi_S} - \frac{1}{\sqrt{1 + \xi_S^2}} \right); \quad (136) \]

and the extremum point of the gravitational acceleration is the solution of Eq. (119) particularized to the case under consideration, which after some algebra reads:

\[ \frac{2^{3/2} \rho^\dagger_P}{(1 + \xi_P^2)^{3/2}} \frac{2 \xi_P^2 - 1}{1 + \xi_P^2} + \frac{3 \sqrt{2} \rho^\dagger_S}{\xi_S^2} \left[ 2 \frac{\text{arcsinh} \xi_S}{\xi_S} - \frac{2 + 3 \xi_S^2}{(1 + \xi_S^2)^{3/2}} \right] = 0; \quad (137) \]

that shows an additional dependence on the parameters, \( m^\dagger, y^\dagger \), via Eqs. (115c) and (132), respectively.

### 3.4 HS density profiles

In the special case of an inner H density profile, \( i = \text{H} \), and an outer S density profile, \( j = \text{S} \), the substitution of Eqs. (75)-(82) and (97)-(104) into (115)-(118) yields:

\[ \xi_H = y^\dagger_S; \quad (138) \]

\[ m(r) = m^\dagger \frac{6\sqrt{2} [\text{arcsinh} \xi_S - \xi_S (1 + \xi_S^2)^{-1/2}]}{12 \xi_H^2 (1 + \xi_H)^2}; \quad (139) \]

\[ M(r) = M^\dagger_P \frac{12 \xi_H^2}{(1 + \xi_H)^2} + M^\dagger_S 6\sqrt{2} \left( \text{arcsinh} \xi_S - \frac{\xi_S}{\sqrt{1 + \xi_S^2}} \right); \quad (140) \]

\[ v^2(r) = (v^\dagger_H)^2 \frac{12 \xi_H^2}{(1 + \xi_H)^2} + (v^\dagger_S)^2 6\sqrt{2} \left( \frac{\text{arcsinh} \xi_S}{\xi_S} - \frac{1}{\sqrt{1 + \xi_S^2}} \right); \quad (141) \]

\[ g(r) = g^\dagger_P \frac{12}{(1 + \xi_H)^2} + g^\dagger_S \frac{6\sqrt{2}}{\xi_S} \left( \frac{\text{arcsinh} \xi_S}{\xi_S} - \frac{1}{\sqrt{1 + \xi_S^2}} \right); \quad (142) \]

and the extremum point of the gravitational acceleration is the solution of Eq. (119) particularized to the case under consideration, which after some
algebra reads:

\[
\frac{12\rho_{H}^{\dagger}}{(1 + \xi_{H})^{3}} + \frac{3\sqrt{2}\rho_{S}^{\dagger}}{\xi_{S}^{2}} \left[ 2\frac{\text{arcsinh} \xi_{S}}{\xi_{S}} - \frac{2 + 3\xi_{S}^{2}}{(1 + \xi_{S}^{2})^{3/2}} \right] = 0 ;
\tag{143}
\]

that shows an additional dependence on the parameters, \( m^{\dagger}, y^{\dagger} \), via Eqs. (115c) and (138), respectively.

4 Tidal effects on embedded subsystems

Let a subsystem be completely embedded within another one. Let the former and the latter be hereafter quoted as the embedded and the embedding sphere, respectively, owing to the assumption of spherical symmetry. Accordingly, related centres of mass shall be hereafter quoted as centres, keeping in mind the centre of mass coincides with the geometrical centre in the case under consideration. The dependence of the tidal radius of the embedded sphere on the density profile of the embedding sphere shall be exploited below for both one-component and two-component systems, having in mind globular clusters within galaxies and dark matter haloes.

To this aim, the tidal radius must be clearly defined and considerations be restricted to density profiles (among those listed in Table 1) which satisfactorily fit to observed or inferred matter distributions in galaxies and/or dark matter haloes, namely \( P, H, I, S, B \) for one-component systems while, on the other hand, \( PI, HI, PS, HS \) are acceptable for two-component systems.

4.1 Tidal radius

Let the embedded and the embedding sphere be conceived as a globular cluster and a galaxy, respectively, both assumed spherical-symmetric. The tidal radius of the embedded sphere can be defined by use of an either local (e.g., von Hoerner 1958; Vesperini 1997; Brosche et al. 1999; Gajda and Lokas 2015) or global (e.g., Caimmi and Secco 2003) criterion, where the former shall be preferred here in that it involves a single test particle instead of the embedded sphere as a whole.

Let \( M_{G}(R), a_{G} \), be the mass profile and the truncation radius, respectively, of the embedding sphere, and \( M_{C}, a_{C} \), the mass and the truncation radius, respectively, of the embedded sphere. Let \( P \) be the intersection point between the surface of the embedded sphere and the segment, \( \overline{OO'} \), joining the centre of the embedded and the embedding sphere, respectively. A test
particle of unit mass placed on $P$ is subjected to a gravitational force:

$$ F_C(P) = \frac{GM_C}{a_C^2}; \quad (144) $$

due to the embedded sphere, and:

$$ F_G(P) = -\frac{GM_G(R_C - a_C)}{(R_C - a_C)^2}; \quad (145) $$

due to the embedding sphere, where $R_C, a_C \leq R_C \leq a_G - a_C$ is the distance between the centre of the embedded and the embedding sphere, respectively, as depicted in Fig. 1.

Let the tidal radius of the embedded sphere, $a_C^*$, be defined as related to a null resulting gravitational attraction, $F_G(P) + F_C(P) = 0$, or more explicitly:

$$ \frac{GM_C}{a_C^2} = \frac{GM_G(R_C - a_C)}{(R_C - a_C)^2}; \quad (146) $$

which, using Eqs. (10) and (11), is equivalent to:

$$ \left( \frac{1 - \gamma}{\gamma} \right)^2 = \frac{\nu_{\text{mas}}(\xi_C - \delta_C)}{(\mu^\dagger)^2}; \quad (147) $$

$$ \xi_C = \frac{R_C}{r_G}; \quad \delta_C = \frac{a_C}{r_G}; \quad \gamma = \frac{a_C}{R_C} = \frac{\delta_C}{\xi_C}; \quad \mu^\dagger = \left( \frac{M_C}{M_G} \right)^{1/2}; \quad (148) $$

where the validity of Eq. (147) implies $a_C = a_C^*$ or $\gamma = \gamma^*$.

With regard to the embedding sphere, a nonmonotonic trend of the gravitational acceleration, characterized by a maximum absolute value within the truncation radius, is a necessary condition for the occurrence of two different values of the tidal radius of the embedded sphere: one, sufficiently close to the centre of the embedding sphere, and one other, sufficiently close to the truncation radius. Then the embedded sphere can be considered bound for $1 \geq \gamma > (\gamma^*)^-, \; 0 \leq \gamma < (\gamma^*)^+$, and (partially) unbound for $(\gamma^*)^+ \leq \gamma \leq (\gamma^*)^-$, where the apices, $-$ and $+$, relate to the lower and higher value of the galactocentric distance, respectively.

An instantaneous tidal radius can be defined for the embedded sphere for fixed mass, $M_C$, and distance, $R_C - a_C$, or $\xi_C - \delta_C$ in dimensionless coordinates. Accordingly, $\gamma$ varies via $a_C$ instead of $R_C$ as in the former alternative, see also Fig. 1. The instantaneous tidal radius, $\eta(\xi_C - \delta_C)$, can be inferred from Eq. (147) as solution of a second degree equation in $\gamma$. The
result is:

\[
\eta (\xi - \delta C) = \frac{\mu (\xi - \delta C)}{\mu (\xi - \delta C) + 1} ;
\]

\[
\mu (\xi - \delta C) = \left[ \frac{M}{M_G (\xi - \delta C)} \right]^{1/2} = \left[ \frac{(\mu^*)^2}{\nu_{\text{mas}} (\xi - \delta C)} \right]^{1/2} ;
\]

where \( \eta (\xi - \delta C) = \gamma^* (\xi - \delta C) > 1 \), which has no physical meaning, as shown below. Then the acceptable solution is \( \eta^+ (\xi - \delta C) = \gamma^* (\xi - \delta C) < 1 \) which, for sake of simplicity, shall be denoted as \( \eta \) in the following. By definition, \( \gamma/\eta = a_C/a_C^* \).

The above results can be extended to two-component embedding spheres by use of Eq. (116), keeping in mind \( M_G (R_C - a_C) = M_i, G (R_C - a_C) + M_j, G (R_C - a_C) \). Accordingly, the combination of Eqs. (116) and (146) yields:

\[
\left( \frac{1 - \gamma}{\gamma} \right)^2 = \frac{\nu_{i, \text{mas}} (\xi_i, C - \delta_i, C)}{(\mu_i^*)^2} + \frac{\nu_{j, \text{mas}} (\xi_j, C - \delta_j, C)}{(\mu_j^*)^2} ;
\]

\[
\xi_{u, C} = \frac{R_C}{r_{u, G}} ; \quad \delta_{u, C} = \frac{a_C}{r_{u, G}} ; \quad \gamma = \frac{a_C}{\xi_{u, C}} = \frac{\delta_{u, C}}{\xi_{u, C}} ; \quad \mu_{u}^* = \left( \frac{M}{M_{u, G}} \right)^{1/2} ;
\]

\[
u_{i, \text{mas}} (\xi_i, C - \delta_i, C) + (\mu_i^*)^2 \]

\[
u_{j, \text{mas}} (\xi_j, C - \delta_j, C) + (\mu_j^*)^2
\]

\[
\text{in addition, the combination of Eqs. (116) and (149) produces:}
\]

\[
\eta^+ (\xi_{u, C} - \delta_{u, C}) = \frac{\mu (\xi_{u, C} - \delta_{u, C})}{\mu (\xi_{u, C} - \delta_{u, C}) + 1} ;
\]

\[
\mu (\xi_{u, C} - \delta_{u, C}) = \left[ \frac{M}{M_i, G (\xi_i, C - \delta_i, C) + M_j, G (\xi_j, C - \delta_j, C)} \right]^{1/2} = \left[ \frac{(\mu_i^*)^2}{\nu_{i, \text{mas}} (\xi_i, C - \delta_i, C)} + \frac{(\mu_j^*)^2}{\nu_{j, \text{mas}} (\xi_j, C - \delta_j, C)} \right]^{1/2} ;
\]

where the acceptable solution, for reasons mentioned above, is \( \eta^+ (\xi_{u, C} - \delta_{u, C}) = \gamma^*_+ (\xi_{u, C} - \delta_{u, C}) < 1 \) which, for sake of simplicity, shall be denoted as \( \eta \) in the following.

With regard to a generic configuration, as the one depicted in Fig.1, it can be seen \( \gamma < \eta \) and \( \gamma > \eta \) for bound and (partially) unbound embedded sphere, respectively. The point, \( (\gamma, \eta) = (1, 1) \), relates to the configuration where the centre of the embedding sphere lies on the surface of the embedded sphere, as depicted in Fig.2. The point, \( (\gamma, \eta) = (0, 0) \), relates to the configuration where the embedding sphere and the (no longer) embedded sphere are infinitely distant.
The subdomain, \(0 \leq \gamma < 1\), makes a necessary condition for the occurrence of the tidal radius, in that the gravitational force exerted on the point, \(P\), from either subsystem acts along the same direction but on opposite sides, as depicted in Fig. 1. The special case, \(\gamma = 1\), relates to a null gravitational force on \(P\) from the embedding sphere. The subdomain, \(\gamma > 1\), implies \(R_C - a_C < 0\) and, in consequence, a mass enclosed within a negative volume. Accordingly, further considerations shall be restricted to the subdomain, \(0 \leq \gamma \leq 1\), where the physical meaning is preserved, which implies \(0 \leq \eta \leq 1\).

### 4.2 Application to globular clusters within galaxies

#### 4.2.1 One-component embedding density profiles

As a guiding example, let the embedding sphere be considered with fixed truncation radius, \(a_G = 100\ \text{kpc}\), mass, \(M_G = M_G(a_G) = 10^{12} \text{m}_\odot\), concentration, \(\Xi = 10\), and density profiles selected among those listed in Table 1 which are currently used for fitting to galaxies and/or dark matter haloes. Similarly, let the embedded sphere be considered with fixed truncation radius, \(a_C = 30\ \text{pc}\), and mass, \(M_C = M_C(a_C) = 10^5 \text{m}_\odot\), regardless of the density profile. The values of fractional mass, \(\nu_{\text{mas}} = M_G/M_G^\dagger\), scaling mass, \(M_G^\dagger\), scaling density, \(\rho_G^\dagger\), fractional distance related to tidal radius, \((\xi)^\dagger = (\xi_C)^\dagger - (a_C^\dagger)^\dagger = [(R_C^\dagger)^\dagger - (a_C^\dagger)^\dagger]/r_C^\dagger\), are listed in Table 2. For H density profiles \((\xi^*)^- = 0\), which implies \((\gamma^*)^- = 1\), hence \((R_C^*^-)^\dagger \to +\infty\), \((a_C^*^-)^\dagger \to +\infty\).

The trend of \(\eta\) vs. \(\gamma\) for the cases listed in Table 2 is plotted in Fig. 3, while the neighbourhood of the origin is zoomed in Fig. 4, where density profiles are continued outside the truncation radius.

An inspection of Figs. 3 and 4 shows that, in the light of the assumed formulation of tidal radius, the embedded sphere is bound provided close enough to \((\gamma \approx 0.01-0.2)\) and far enough from \((\gamma \approx 0.0003)\) the centre of the embedding sphere. Conversely, the embedded sphere is (partially) unbound provided sufficiently far from both the centre and the surface \((0.0003 \approx \gamma \approx 0.01-0.2)\) of the embedding sphere. The sole exception is from H density profiles, which implies (partially) unbound embedded spheres up to \(\gamma \approx 0.0003\) i.e. close to the truncation radius.

#### 4.2.2 Two-component embedding density profiles

As a guiding example, let the embedding two-component sphere be considered with fixed truncation radius, \(a_{i,G} = 30\ \text{kpc}\), \(a_{j,G} = 100\ \text{kpc}\); mass, \(M_G = \)}
Table 2: Fractional mass, $\nu_{\text{mas}} = M_G/M_G^\dagger$, scaling mass, $M_G^\dagger$ (unit $10^{10}m_\odot$), scaling density, $\rho_G^\dagger$ (unit $10^{10}m_\odot$/kpc$^3$), fractional distance related to tidal radius, $(\xi^*)^\mp = (\xi_C^*)^\mp - (\delta_C^*)^\mp = [(R_C^*)^\mp - (a_C^*)^\mp]/r_C^\dagger$, for embedding one-component density profiles ($p$) listed in Table 1 and currently used for the description of galaxies and/or dark matter haloes. In any case, the following parameters remain unchanged: truncation radius, $a_G = 100$ kpc; total mass, $M_G = 10^{12}m_\odot$; concentration, $\Xi = 10$. With regard to the embedded sphere, the truncation radius and the total mass are kept fixed to $a_C = 30$ pc and $M_C = 10^5m_\odot$, respectively. For H density profiles $(\xi^*)^- = 0$, which implies both $(R_C^-)^-$ and $(a_C^-)^-$ extend to infinite. See text for further details.

| $p$ | $\nu_{\text{mas}}$ | $M_G^\dagger$ | $\rho_G^\dagger$ | $(\xi^*)^-$ | $(\xi^*)^+$ |
|-----|------------------|----------------|----------------|-------------|-------------|
| P   | 5.5730D+0        | 1.7943D+1      | 4.2837D-3      | 1.0949D-2   | 9.4789D+0   |
| H   | 9.9174D+0        | 1.0083D+1      | 2.4072D-3      | 0           | 9.4355D+0   |
| I   | 5.1173D+0        | 1.9541D+0      | 2.9944D-1      | 8.8055D+0   |
| S   | 1.6998D+1        | 5.8832D+0      | 6.7043D-3      | 9.3217D+0   |
| B   | 1.9406D+1        | 5.1531D+0      | 1.2302D-3      | 9.2917D+0   |

$M_{i,G} = 10^{11}m_\odot$, $M_{j,G} = 10^{12}m_\odot$; concentration, $\Xi_i = \Xi_j = 10$; and density profiles as specified in Section 3. Similarly, let the embedded sphere be considered with fixed truncation radius, $a_C = 30$ pc, and mass, $M_C = 10^5m_\odot$, regardless of the density profile. The values of fractional mass, $\nu_{u,\text{mas}} = M_{u,G}/M_{u,G}^\dagger$, scaling mass, $M_{u,G}^\dagger$, scaling density, $\rho_{u,G}^\dagger$, fractional distance related to tidal radius, $(\xi_u^*)^\mp = (\xi_{u,C}^*)^\mp - (\delta_{u,C}^*)^\mp = [(R_{u,C}^*)^\mp - (a_{u,C}^*)^\mp]/r_{u,G}^\dagger$, $u = i, j$, are listed in Table 3. For H density profiles $(\xi^*)^- = 0$, which implies $(\gamma^*)^- = 1$, hence $(R_C^-)^- \rightarrow +\infty$, $(a_C^-)^- \rightarrow +\infty$, and, in addition, $(\xi_j)^+ > \Xi_j$, i.e. $a_C = a_j^G$ provided $R_C \approx a_j^G$.

The trend of $\eta$ vs. $\gamma$ for the cases listed in Table 2 is plotted in Fig. 5, while the neighbourhood of the origin is zoomed in Fig. 6, where density profiles end at the truncation radius of the outer embedding sphere, related to $\gamma = 0.0003$.

An inspection of Figs. 5 and 6 shows that, in the light of the assumed formulation of tidal radius, the embedded sphere is bound for PI and PS density profiles, provided it is close enough to $(\gamma \approx 0.5)$ and far enough from $(\gamma \approx 0.0003)$ the centre of the embedding sphere. Conversely, the embedded sphere is (partially) unbound provided sufficiently far from both the centre and the surface $(0.0003 \approx \gamma \approx 0.5)$ of the outer embedding sphere. On the
Table 3: Fractional mass, \( \nu_{u,\text{mas}} = M_{u,G}/M_{u,G}^\dagger \), scaling mass, \( M_{u,G}^\dagger \) (unit \( 10^{10} m_\odot \)), scaling density, \( \rho_{u,G}^\dagger \) (unit \( 10^{10} m_\odot/\text{kpc}^3 \)), fractional distance related to tidal radius, \((\xi_u^*)^\mp = (\xi_{u,C}^*)^\mp - (\delta_{u,C}^*)^\mp = [(R_{u}^*)^\mp - (a_{C}^*)^\mp]/r_{u,G}^\dagger \), for embedding two-component density profiles \((ij)\) mentioned in Section 3 and usable for the description of galaxies within dark matter haloes. In any case, the following parameters remain unchanged: truncation radius, \( a_{i,G} = 30 \) kpc, \( a_{j,G} = 100 \) kpc; total mass, \( M_{i,G} = 10^{11} m_\odot \), \( M_{j,G} = 10^{12} m_\odot \); concentration, \( \Xi_i = \Xi_j = 10 \). With regard to the embedded sphere, the truncation radius and the total mass are kept fixed to \( a_C = 30 \) pc and \( M_C = 10^5 m_\odot \), respectively. For each case, upper and lower lines relate to the inner \((u = i, \text{baryonic})\) and outer \((u = j, \text{nonbaryonic})\) embedding subsystem. See text for further details.

| \( u \) | \( \nu_{u,\text{mas}} \) | \( M_{u,G}^\dagger \) | \( \rho_{u,G}^\dagger \) | \( (\xi_u^*)^- \) | \( (\xi_u^*)^+ \) |
|---|---|---|---|---|---|
| P  | 5.5730D+0 | 1.7943D+0 | 1.5866D-2 | 9.7519D-3 | 3.3024D+1 |
| I  | 5.1173D+1 | 1.9541D+0 | 4.6652D-4 | 2.9256D-3 | 9.9073D+0 |
| H  | 9.9174D+0 | 1.0083D+0 | 8.9156D-3 | 0 | 3.3426D+1 |
| I  | 5.1173D+1 | 1.9541D+0 | 4.6652D-4 | 0 | 1.0028D+1 |
| P  | 5.5730D+0 | 1.7943D+0 | 1.5866D-2 | 9.4355D-3 | 3.3145D+1 |
| S  | 1.6998D+1 | 5.8832D+0 | 1.4045D-3 | 2.8306D-3 | 9.9434D+0 |
| H  | 9.9174D+0 | 1.0083D+0 | 8.9156D-3 | 0 | 3.3390D+1 |
| S  | 1.6998D+1 | 5.8832D+0 | 1.4045D-3 | 0 | 1.0017D+1 |
other hand, HI and HS density profiles imply (partially) unbound embedded spheres up to $\gamma \gtrsim 0.0003$ i.e. for galactocentric distances slightly exceeding the truncation radius of the outer embedding sphere, $R_G \gtrsim a_{j,G}$. The curves are mainly depending on the inner density profile ($i = P,H$), while the effect of the outer density profile ($j = I,S$) can be neglected to a good extent.

The occurrence of an extremum point (maximum absolute value) in the gravitational acceleration profile is also determined by the inner subsystem for the cases under discussion. With regard to PI and PS density profiles, the dimensionless effective radius, via Eqs. (125) and (137), respectively, takes place at $(\xi_{\text{eff}})_P = 0.716584739$ or equivalently $(\xi_{\text{eff}})_I = 0.214975422$ and $(\xi_{\text{eff}})_S = 0.663475075$ or equivalently $(\xi_{\text{eff}})_S = 0.199042522$, respectively. With regard to HI and HS density profiles, Eqs. (131) and (143) show no solution, which implies no extremum point in the gravitational acceleration.

5 Discussion

With regard to tidal effects, the above results are grounded on the definition of tidal radius, which has been selected for reasons of simplicity. Though using different definitions implies different results (e.g., Brosche et al. 1999; Caimmi and Secco 2003), a similar trend is expected. According to the formulation of the current paper, a test particle on the surface of the embedded sphere remains no longer bound if the gravitational force from the embedded sphere is counterbalanced by the gravitational force from the embedding sphere. To this respect, small perturbations suffice to make a test particle be lost from the embedded sphere. To gain more insight, let the embedding sphere, the embedded sphere, the test particle, be conceived as a galaxy (in particular, the Galaxy), a globular cluster (GC), a (long-lived) star, respectively.

By definition, all stars lying on GC boundaries necessarily exhibit null radial velocity or, in other words, the stars under consideration are at the apocentre of their orbits. Among elliptic orbits with different eccentricities and equal major axis, the pendulum orbit has the lowest energy, due to a null tangential velocity on the apocentre\footnote{It is worth noticing the major axis of the pendulum orbit doubles the major axis of the Keplerian orbit with unit eccentricity, for fixed apocentre.}. Stars with nonzero tangential velocity on GC boundaries should be less bound as in presence of centrifugal force. On the other hand the centrifugal force, due to GC orbital motion within the embedding galaxy, has no influence on the above mentioned gravitational balance.
Then the definition of tidal radius, assumed in the current paper, relates to a necessary condition. More specifically, bound GCs imply $\gamma < \eta$ but the reverse could not hold. With this caveat in mind, it shall be supposed in the following the condition is also sufficient i.e. $\gamma < \eta$ implies bound GCs.

For a galaxy of assigned mass, truncation radius, concentration, and for a specified GC, the trend of $\eta$ vs. $\gamma$ depends on the galactic density profile. An inspection of Figs. 3-4 shows a far more extended stability region, $\gamma < \eta$, for density profiles (P, I, S, B) where the gravitational acceleration has an extremum point, with respect to density profiles (H) where the gravitational acceleration is monotonically increasing in absolute value. Different GCs imply different masses, $M_C$, and truncation radii, $a_C$, which translates into different curves on the ($O\gamma \eta$) plane.

Restricting to the Galaxy with GC subsystem included, let the following values be assumed: mass, $M_G = 5 \cdot 10^{10} m_\odot$; truncation radius, $a_G = 125$ kpc, concentration, $\Xi = 10$. Let a GC sample ($N = 16$) studied in an earlier attempt (Brosche et al. 1999) be considered, with the addition of a further element (Pal5) for which different masses can be inferred (e.g., Caimmi and Secco, 2003). The GC subsystem (thick disk, old halo, young halo), S, observed radius, $r_C$, galactocentric distance, $R_C$, mass logarithm, log($M_C/m_\odot$), taken from the parent paper (Brosche et al. 1999), are listed in Table 4 together with the inferred inverse fractional distance, $\gamma = r_C/R_C$. The inferred fractional instantaneous tidal radius, $\eta = \mu/(1+\mu)$, for different density profiles among those listed in Table 1 is shown in Table 5 where the inverse fractional distance is repeated for better comparison with related plots. Concerning Pal5, calculations were performed with regard to four possible mass values: for further details and references, an interested reader is addressed to the parent paper (Caimmi and Secco 2003).

The location of GCs listed in Table 5 on the ($O\gamma \eta$) plane is shown in Fig. 7 for different Galactic density profiles among those listed in Table 1. The region where GCs are bound, according to the assumed formulation of tidal radius, lies above the dotted straight line, $\eta = \gamma$. The inverse fractional distance, $\gamma$, related to NGC 5466 and Pal5, is marked by a dotted vertical line.

The above mentioned GCs appear (partially) unbound for all considered density profiles, with the possible exception of I. In fact, Pal5 is known to be experiencing progressive disruption via tidal shocks during disk passages (e.g., Odenkirchen et al. 2002; Dehnen et al. 2004) and NGC 5466 shows a tidal stream (Grillmair and Johnson 2006), as inferred in an earlier attempt for different formulations of tidal radius (Caimmi and Secco 2003). Remaining GCs are placed inside the stability region, $\gamma < \eta$, with a restricted number close to the boundary, $\gamma = \eta$, for all considered density profiles.
Table 4: Parameters of globular clusters studied in an earlier paper (Brosche et al. 1999). An additional cluster, Pal5, considered in a subsequent paper (Caimmi and Secco 2003) is also included for different mass values. Column caption: 1 - name (NGC or Pal); 2 - subsystem (A - [Fe/H] > −1, thick disk; B - old halo; C - young halo); 3 - observed radius, $r_C$/pc; 4 - galactocentric distance, $R_C$/kpc; 5 - decimal logarithm of mass, $\log \phi = \log(M_C/m_\odot)$; 6 - inverse fractional distance, $\gamma = r_C/R_C$.

| NGC | S | $r_C$/pc | $R_C$/kpc | $\log \phi$ | $10^4 \gamma$ |
|-----|---|----------|-----------|-----------|-------------|
| 0104 | A | 50.7 | 7.4 | 6.16 | 6.85135 |
| 0362 | C | 35.7 | 9.3 | 5.75 | 3.83871 |
| 4147 | C | 34.5 | 21.3 | 4.85 | 1.61972 |
| 5024 | C | 119.3 | 18.8 | 5.91 | 6.34574 |
| 5272 | C | 103.0 | 12.2 | 5.95 | 8.44262 |
| 5466 | C | 101.4 | 17.2 | 5.23 | 5.89535 |
| 5904 | C | 63.0 | 6.2 | 5.91 | 10.16129 |
| 6205 | B | 55.4 | 8.7 | 5.81 | 6.36782 |
| 6218 | B | 21.6 | 4.5 | 5.32 | 4.80000 |
| 6254 | B | 27.0 | 4.6 | 5.38 | 5.86957 |
| 6341 | B | 35.0 | 9.6 | 5.67 | 3.64583 |
| 6779 | B | 25.0 | 9.7 | 5.34 | 2.57732 |
| 6838 | A | 10.1 | 6.7 | 4.61 | 1.50746 |
| 6934 | C | 37.5 | 14.3 | 5.39 | 2.62238 |
| 7078 | C | 65.7 | 10.4 | 6.05 | 6.31731 |
| 7089 | B | 71.1 | 10.4 | 6.00 | 6.83654 |
| Pal5 | C | 20 | 18.6 | 3.78 | 1.07527 |
| &nbsp; | &nbsp; | 20 | 18.6 | 3.65 | 1.07527 |
| &nbsp; | &nbsp; | 20 | 18.6 | 3.15 | 1.07527 |
| &nbsp; | &nbsp; | 20 | 18.6 | 2.98 | 1.07527 |
Table 5: Fractional instantaneous tidal radius, $\eta = \mu/(1 + \mu)$, inferred for globular clusters listed in Table 4, embedded within a one-component sphere of mass, $M_G = 5 \cdot 10^{10}m_\odot$; truncation radius; $a_G = 125$ kpc; concentration, $\Xi = 10$; one-component density profiles among those listed in Table 1. The inverse fractional distance, $\gamma = r_C/R_C$, is also repeated for better comparison with related plots.

| NGC  | $10^3\gamma$  | P    | I    | H    | S    | B    |
|------|---------------|------|------|------|------|------|
| 0104 | 6.85135       | 14.57470 | 61.99897 | 13.02944 | 32.40667 | 33.18122 |
| 0362 | 3.83871       | 7.19259 | 29.56671 | 7.11140 | 15.52356 | 16.11275 |
| 4147 | 1.61972       | 1.47333 | 4.25334 | 1.71465 | 2.53435 | 2.68024 |
| 5024 | 6.34574       | 5.25419 | 16.16833 | 6.08087 | 9.41380 | 9.95776 |
| 5272 | 8.44262       | 7.17412 | 26.91316 | 7.74374 | 14.58642 | 15.31472 |
| 5466 | 5.89535       | 2.51561 | 8.15645 | 2.89185 | 4.64889 | 4.91754 |
| 5904 | 10.16129      | 13.49411 | 59.55797 | 11.00866 | 30.72825 | 31.10789 |
| 6205 | 6.36782       | 8.24594 | 34.44840 | 7.92721 | 18.00489 | 18.61784 |
| 6218 | 4.80000       | 10.25214 | 47.63253 | 6.99596 | 24.08752 | 23.88570 |
| 6254 | 5.86957       | 10.68040 | 49.44294 | 7.37929 | 25.04429 | 24.86855 |
| 6341 | 3.64583       | 6.36749 | 25.99165 | 6.37330 | 13.66944 | 14.21220 |
| 6779 | 2.57732       | 4.31703 | 17.67155 | 4.33957 | 9.26801 | 9.64317 |
| 6838 | 1.50746       | 2.75645 | 12.44064 | 2.34830 | 6.30028 | 6.41645 |
| 6934 | 2.62238       | 3.36082 | 11.87662 | 3.76543 | 6.55913 | 6.92043 |
| 7078 | 6.31731       | 9.15972 | 36.16933 | 9.42575 | 19.28644 | 20.12065 |
| 7089 | 6.83654       | 8.65576 | 34.23661 | 8.90570 | 18.23766 | 19.02693 |

| Pal5 | $10^3\eta$    | P    | I    | H    | S    | B    |
|------|---------------|------|------|------|------|------|
| 1.07527 | 0.04547 | 1.41764 | 0.52651 | 0.81934 | 0.86712 |
| 1.07527 | 0.03938 | 1.22795 | 0.45600 | 0.70965 | 0.75104 |
| 1.07527 | 0.02205 | 0.68773 | 0.25530 | 0.39736 | 0.42054 |
| 1.07527 | 0.01811 | 0.56487 | 0.20968 | 0.32636 | 0.34540 |
The key parameter, on which the above results mainly depend, is the mass of the Galaxy. In fact, a longer Galactic truncation radius implies less amount of mass acting on a selected GC and vice versa. Accordingly, the tidal force is proportional to the Galactic mass and (for unchanged Galactic mass) inversely proportional to the Galactic truncation radius, as expected.

With regard to the results listed in Table 5 and plotted in Fig. 7, the Galactic mass is restricted to stars and gas leaving aside nonbaryonic dark matter. To this respect, the presence of additional mass makes the inverse fractional distance, \( \gamma \), unchanged by definition via Eq. (118), while the fractional tidal radius, \( \eta = \eta^+ \), by definition via Eq. (149), maintains the same formal expression where the mass ratio, \( \mu (\xi_C - \delta_C) \), reads:

\[
\mu (\xi_C - \delta_C) = \left[ \frac{1 + \zeta_C}{1 + \zeta_G (\xi_C - \delta_C)} \right]^{1/2} \left[ \frac{M_C}{M_G (\xi_C - \delta_C)} \right]^{1/2}; \quad (155)
\]

where masses are still restricted to baryonic matter and \( \zeta \) is the ratio between nonbaryonic and baryonic mass for a selected subsystem. A constant ratio, \( \zeta_G (\xi_C - \delta_C) = \zeta_C = \zeta \), implies the results are left unchanged in that Eq. (155) reduces to (150).

The above procedure is repeated for two-component embedding spheres considered in Subsection 4.2.2 and the results are listed in Table 6 and plotted in Fig. 8.

An inspection of Table 6 and Fig. 8 discloses that GCs are (partially) unbound, owing to (i) a less extended inner sphere, and (ii) a massive outer sphere. In this view, the presence of tidal tails or tidal streams detected in e.g., NGC 362, NGC 6934, NGC 7078, NGC 7089, (Grillmair et al. 1995); NGC 5904, NGC 6205, (Leon et al. 2000); and the above mentioned NGC 5466 (Grillmair and Johnson 2006); Pal5 (Odenkirchen et al. 2002; Dehnen et al. 2004); should be predicted for the whole amount of Galactic GCs, regardless of tidal shocks during disk passages.

The formulation used for the tidal radius relates to a necessary condition, implying GCs above the straight line, \( \eta = \gamma \), plotted in Figs. 7-8, could also be (partially) unbound.

If a similar amount of fractional dark matter with respect to the Galaxy, \( M_{j,C}/M_{i,C} = 10 \) in the case under discussion, is considered for each GC and calculations are repeated, the position on the \( (O \gamma \eta) \) plane looks similar to Fig. 7. In other words, a comparable amount of dark-to-visible mass ratio within GCs and the Galaxy makes GCs lie inside or near the stability region, as expected from Eq. (155).

For a generic density profile, \( g(R) \propto v^2(R)/R \) via Eqs. (2) and (3) where, in general, \( v(R) \propto R^\beta \) hence \( g(R) \propto R^{2\beta-1} \) locally. Then no extremum point
Table 6: Fractional instantaneous tidal radius, $\eta = \mu/(1 + \mu)$, inferred for globular clusters listed in Table 4, embedded within a two-component ($ij$) sphere of mass, $M_i, G = 10^{11} m_\odot$, $M_j, G = 10^{12} m_\odot$; truncation radius, $a_i, G = 30$ kpc, $a_j, G = 100$ kpc; concentration, $\Xi_i = \Xi_j = 10$; two-component density profiles considered in Subsection 4.2.2. The inverse fractional distance, $\gamma = r_C/R_C$, is also repeated for better comparison with related plots.

| NGC | $10^3 \gamma$ | $10^3 \eta$ | PI | PS | HI | HS |
|-----|----------------|---------------|-----|-----|-----|-----|
| 0104 | 6.85135        | 3.93932       | 3.39026 | 4.43380 | 3.69007 |
| 0362 | 3.83871        | 2.27233       | 1.86155 | 2.49265 | 1.97695 |
| 4147 | 1.61972        | 0.57317       | 0.42316 | 0.58079 | 0.42620 |
| 5024 | 6.34574        | 2.06309       | 1.52597 | 2.10705 | 1.54349 |
| 5272 | 8.44262        | 2.59695       | 2.01639 | 2.76012 | 2.08992 |
| 5466 | 5.89535        | 0.98317       | 0.73032 | 1.01065 | 0.74137 |
| 5904 | 10.16129       | 3.15739       | 2.81089 | 3.60758 | 3.11436 |
| 6205 | 6.36782        | 2.49305       | 2.07314 | 2.75719 | 2.21766 |
| 6218 | 4.80000        | 1.84201       | 1.71409 | 2.11444 | 1.92698 |
| 6254 | 5.86957        | 1.95273       | 1.81290 | 2.24338 | 2.03818 |
| 6341 | 3.64583        | 2.04998       | 1.66786 | 2.24019 | 1.76554 |
| 6779 | 2.57732        | 1.39733       | 1.13396 | 1.52498 | 1.19904 |
| 6838 | 1.50746        | 0.68612       | 0.60120 | 0.77926 | 0.66092 |
| 6934 | 2.62238        | 1.27816       | 0.96695 | 1.33509 | 0.99089 |
| 7078 | 6.31731        | 3.08644       | 2.47106 | 3.34126 | 2.59605 |
| 7089 | 6.83654        | 2.91484       | 2.33834 | 3.15573 | 2.45203 |
| Pal5 | 1.07527        | 0.17804       | 0.13165 | 0.18190 | 0.13319 |
|     | 1.07527        | 0.15419       | 0.11401 | 0.15754 | 0.11535 |
|     | 1.07527        | 0.08631       | 0.06382 | 0.08819 | 0.06457 |
|     | 1.07527        | 0.07089       | 0.05242 | 0.07243 | 0.05303 |

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for the gravitational acceleration is expected if $\beta > 1/2$, $0 \leq R \leq a_G$, while the contrary holds if $\beta > 1/2$, $R \sim 0$, and $\beta < 1/2$, $R \sim a_G$. For comparison with the trend inferred from observations, a sample ($N = 37$) of empirical rotation curves, $v(R)$, used in a recent investigation (Marr 2015), is considered. Incomplete rotation curves lacking data for large ($R > 10kpc; N_{10} = 13$) and short ($R < 2kpc; N_2 = 4$) galactocentric distances, are excluded. The resulting subsample ($N_0 = N - N_{10} - N_2 = 20$) shows rotation curve slopes substantially larger than unity for short galactocentric distances and nearly flat for large galactocentric distances. If inferred velocities are related to stable circular orbits, the gravitational acceleration must necessarily exhibit an extremum point (maximum absolute value), which excludes density profiles where no extremum point is present, such as H, HI, HS.

6 Conclusion

Homeoidally striated spherical-symmetric density profiles have been classified with reference to four basic matter distributions, namely (i) isodensity i.e. $\rho(r) = \text{const}$; (ii) isogravity i.e. $g(r) = \text{const}$; (iii) isothermal i.e. $v(r) = \left[\frac{GM(r)}{r}\right]^{1/2} = \text{const}$; (iv) isomass i.e. $M(r) = \text{const}$. In particular, the following density profiles have been included: pure power-law; cored power-law; polytropic; Plummer (1911), shortened as P; Hernquist (1990), shortened as H; Begeman et al. (1991), shortened as I; Spano et al. (2008), shortened as S; Burkert (1995), shortened as B; for one-component systems, and a few combinations of the above mentioned ones, namely PI, HI, PS, HS, for two-component systems. Special effort has been devoted to the occurrence of an extremum point in $g(r)$, where the gravitational attraction on a test particle of unit mass attains the maximum absolute value.

Tidal effects on subsystems have been considered using a definition of tidal radius which is related to a necessary condition. More specifically, given an embedded sphere (subsystem) within an embedding sphere (one-component or two-component system), the embedded sphere attains the tidal radius when the gravitational force from the embedding and the embedded sphere, on the point placed on the boundary of the latter and lying between related centres, are equal in strength but act on opposite sides.

With regard to one-component systems, density profiles currently used for representing galaxies and/or dark matter haloes, among those listed in Table 4, have been considered for the embedding sphere. The trend of the fractional instantaneous tidal radius, $\eta$, vs. the inverse fractional distance, $\gamma$, has shown the stability region, $\gamma < \eta$, is far more extended for density profiles (P, I, S, B) where the gravitational acceleration attains a maximum absolute
value, with respect to density profiles (H) where the gravitational acceleration
is monotonically increasing in absolute value. In the former alternative, the
tidal radius takes place ($\gamma = \eta$) for two distinct configurations, while in the
latter alternative the same holds for a single configuration.

The above mentioned trend is exacerbated for considered two-component
systems, where PI and PS density profiles exhibit a restricted stability region
for embedded spheres, while HI and HS density profiles show no stability
region. The gravitational acceleration attains a maximum absolute value in
the former case, while a monotonic behaviour occurs in the latter. The main
features of two-component embedding spheres appear to depend on the inner
subsystem, with little effects arising from the outer.

The location of 17 globular clusters studied in earlier attempts (Brosche
et al 1999; Caimmi and Secco 2003), for which the radius, the mass, and the
Galactocentric distance are known, has been shown on the ($O \gamma \eta$) plane
for assigned Galactic truncation radius, mass, concentration, and density pro-
files among those listed in Table I. Restricting to star and gas subsystem,
sample globular clusters have been found to lie within the stability region,
$\gamma < \eta$, except Pal5 and NGC 5466, which exhibit noticeable tidal effects (e.g.,
Odenkirchen et al. 2002; Grillmair and Johnson 2006). Taking nonbaryonic
dark matter into consideration, it has been shown the results remain un-
changed in the special case where the mass ratio between non baryonic and
baryonic matter within globular clusters and the Galaxy, at any distance
from the centre, attains a constant value.

On the other hand, all sample globular clusters have been found to lie
outside the stability region if the Galaxy ($M_{G} = 10^{11}m_{\odot}, a_{G} = 30$ kpc)
is embedded within a nonbaryonic dark matter halo ($M_{j,G} = 10^{12}m_{\odot}, a_{j,G} =
100$ kpc), for PI, PS, HI, HS, two-component density profiles, unless the
dark-to-visible mass ratio within single globular clusters is comparable to its
counterpart within the Galaxy.

A comparison of predicted rotation curves, $v(R)$, with a subsample ($N_{0} =
20$) of empirical rotation curves has shown the occurrence of an extremum
point (maximum absolute value) in the gravitational acceleration. Accord-
ingly, density profiles where no extremum point takes place, such as H, HI,
HS, have necessarily been excluded.

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Appendix

A  Homeoidally striated ellipsoidal density profiles

Density profiles in dimensionless coordinates, expressed by Eqs. (8)-(9), may be extended to the case where the isopycnic i.e. constant density surfaces are similar and similarly placed ellipsoids, which implies the dimensionless radial coordinate, \( \xi = r(\mu)/r^l(\mu) \), maintains unchanged on an arbitrary isopycnic surface, \( r = r(\mu) \), \( \mu \) polar angle (e.g., Caimmi and Marmo 2003). In the special case of homogeneous ellipsoids, the gravitational potential and force on points not outside the ellipsoids are (e.g., Caimmi and Secco 1992):

\[
\begin{align*}
V(x_1, x_2, x_3) &= \pi G \rho \sum_{r=1}^{3} A_r (a_r^2 - x_r^2) ; \\
\frac{\partial V}{\partial x_r} &= -2\pi G \rho A_r x_r ;
\end{align*}
\]

where \( a_r \) are the semiaxes of the ellipsoid and \( A_r \) are shape factors for which the following inequalities hold (e.g., MacMillan 1930, Chap. II, §33; Caimmi 1996):

\[
\begin{align*}
A_1 &\leq A_2 \leq A_3 ; & a_1 &\geq a_2 \geq a_3 ; \\
a_1 A_1 &\leq a_2 A_2 \leq a_3 A_3 ; & a_1 &\geq a_2 \geq a_3 ;
\end{align*}
\]

accordingly, the largest gravitational force on the boundary of a homogeneous ellipsoid is exerted at the top of the minor axis, \( P_\mp \equiv (0, 0, \mp a_3) \). Using Newton’s theorem, the above result can be extended to a generic inner ellipsoid with similar and similarly placed boundary.

Turning to the general case of homeoidally striated ellipsoids, let \( P \equiv (x_1, x_2, x_3) \) be a generic point within the ellipsoid, as represented in Fig. 9.

Owing to Newton’s theorem, no attraction on \( P \) is exerted by the homeoid bounded by the external isopycnic surface, \( \Sigma(R) \), and the one where \( P \) lies, \( \Sigma(r) \). Then only the ellipsoid bounded by \( \Sigma(r) \) has to be considered.

Let \( \Sigma(r) \) together with all the enclosed isopycnic surfaces be “compressed” along \( x_1 \) and \( x_2 \) directions to attain a spherical shape with radius equal to the minor axis of \( \Sigma(r) \), as depicted in Fig. 9. The result is a homeoidally striated sphere with mass equal to the mass bounded by \( \Sigma(r) \). It is suggested from geometrical considerations that the attraction exerted on \( P \) by the homeoidally striated sphere is larger than the attraction exerted on \( P \) by the homeoidally striated ellipsoid.
Let all the isopycnic surfaces within $\Sigma(r)$ be “stretched” along $x_1$ and $x_2$ directions, with the minor axis left unchanged, to attain a confocal ellipsoidal shape with respect to $\Sigma(r)$, as depicted in Fig.9. The result is a focaloidally striated ellipsoid with mass equal to the mass bounded by $\Sigma(r)$ and boundary $\Sigma(r)$. Let $a'_1, a''_2, a''_3$, be the semi-axes of a generic isopycnic surface within $\Sigma(r)$, and $a'_1, a'_2, a'_3$, the semi-axes of $\Sigma(r)$. The semi-axes of an ellipsoid internal and confocal to $\Sigma(r)$ are $c''_r = \sqrt{(a'_r)^2 - \lambda}$, where $\lambda$ is determined by the boundary condition, $c''_3 = a''_3$, as $\lambda = (a'_3)^2 - (a''_3)^2$. Accordingly, the equatorial semi-axes of the confocal ellipsoid are:

\[
(c'_r)^2 = (a'_r)^2 - \lambda = (\epsilon_{r3}a'_3)^2 - (a'_3)^2 ;
\]

which is equivalent to:

\[
(c''_r)^2 = [\epsilon_{r3}^2 - 1][(a'_3)^2 - (a''_3)^2] + (a''_r)^2 ;
\]

where $\epsilon_{r3} = a_r/a_3 = a'_r/a'_3 = a''_r/a''_3$, $r = 1, 2$, are the axis ratios of the isopycnic surface. As $\epsilon_{r3} \geq 1$, $a'_3 \geq a''_3$, Eq. (161) discloses that, in fact, the equatorial semi-axes of the confocal ellipsoid are stretched with respect to the equatorial semi-axes of the related isopycnic surface, $c''_r \geq a''_r$, $r = 1, 2$.

It is suggested from geometrical considerations that the attraction exerted on $P$ by the focaloidally striated ellipsoid is lower than the attraction exerted on $P$ by the homeoidally striated ellipsoid. Owing to MacLaurin’s theorem, the attraction exerted by a focaloidally striated ellipsoid on a surface point, $P$, equals the attraction exerted on the same point by a homogeneous ellipsoid of equal mass and boundary.

A homeoidally striated ellipsoid may be conceived as a homogeneous ellipsoid with equal boundary and density as at the surface isopycnic, to which is superimposed an infinity of homogeneous ellipsoids bounded by isopycnic surfaces, $\Sigma(r''\rho)$, with infinitesimal density, $d\rho = \rho(r'') - \rho(r'' + d\rho)$. Owing to MacLaurin’s theorem, the attraction exerted by the generic homogeneous ellipsoid on a surface point, $P$, equals the attraction exerted by a homogeneous ellipsoid with equal mass, confocal and confocally placed with respect to the one under consideration, with $P$ on its boundary. The largest attraction exerted on the boundary of a homogeneous ellipsoid is at the top of the minor axis, which implies the largest attraction exerted on the boundary of a homeoidally striated ellipsoid is also at the top of the minor axis.

For the homeoidally striated sphere, the gravitational attraction along the minor axis, via Eq. (157) keeping in mind $A_r = 2/3$ for spherical configurations, reads:

\[
\frac{\partial V}{\partial x_3} = -\frac{Gm(x_3)}{x_3^2} ;
\]
where \(m(x_3)\) is the mass of the homeoidally striated ellipsoid enclosed within the isopycnic surface with minor axis equal to \(x_3\).

For the focaloidally striated ellipsoid, the gravitational attraction along the minor axis, via Eq. (157) reads:

\[
\frac{\partial V}{\partial x_3} = -\frac{3}{2} G m(x_3) \frac{\varepsilon_{31} \varepsilon_{32} A_3}{x_3^2}; \quad (163)
\]

which is different from its counterpart related to the homeoidally striated sphere, Eq. (162), by a shape factor, \((3/2)\varepsilon_{31} \varepsilon_{32} A_3\), as expected.

Accordingly, the extremum point of the radial profile (along the minor axis) of the attraction i.e. the gravitational acceleration is the same for both the homeoidally striated sphere and the focaloidally striated ellipsoid, as the two profiles are vertically shifted one with respect to the other, by an amount equal to the shape factor. Then it may safely be expected the extremum point of the gravitational acceleration, related to the homeoidally striated ellipsoid, is close to its counterpart related to the homeoidally striated sphere and the focaloidally striated ellipsoid.

**B  Geometrical interpretation of the dimensionless scaling radius for generalized power law density profiles**

Dimensionless generalized power law density profiles, defined by Eq. (84), in decimal logarithmic scale are expressed as:

\[
\log f = \log(C_\gamma + 1) + \chi \log(C_\alpha + 1) - \log(C_\gamma + \xi^\gamma) - \chi \log(C_\alpha + \xi^\alpha); \quad (164)
\]

where all parameters cannot be negative.

The logarithmic derivatives, \(d^n \log f / d(\log \xi)^n\), can be calculated as \([d^n \log f / d(\log \xi)^n][d\xi / d(\log \xi)]^n = \xi^n[d^n \log f / d\xi^n]\). For the first, second, third, and fourth derivative, the result is:

\[
\frac{d \log f}{d \log \xi} = -\frac{\gamma \xi^\gamma}{C_\gamma + \xi^\gamma} - \frac{\chi \alpha \xi^\alpha}{C_\alpha + \xi^\alpha}; \quad (165)
\]

\[
\frac{d^2 \log f}{d(\log \xi)^2} = -\frac{C_\gamma \gamma \xi^\gamma}{(C_\gamma + \xi^\gamma)^2} - \frac{\chi C_\alpha \alpha^2 \xi^\alpha}{(C_\alpha + \xi^\alpha)^2}; \quad (166)
\]

\[
\frac{d^3 \log f}{d(\log \xi)^3} = -\frac{C_\gamma \gamma^3 \xi^\gamma}{(C_\gamma + \xi^\gamma)^3} (C_\gamma - \xi^\gamma) - \frac{\chi C_\alpha \alpha^3 \xi^\alpha}{(C_\alpha + \xi^\alpha)^3} (C_\alpha - \xi^\alpha); \quad (167)
\]

\[
\frac{d^4 \log f}{d(\log \xi)^4} = -C_\gamma \gamma^4 \xi^\gamma \frac{(C_\gamma - \xi^\gamma)^2 - 2C_\gamma \xi^\gamma}{(C_\gamma + \xi^\gamma)^4} - \chi C_\alpha \alpha^4 \xi^\alpha \frac{(C_\alpha - \xi^\alpha)^2}{(C_\alpha + \xi^\alpha)^4}; \quad (168)
\]
and the particularization to the dimensionless scaling radius, $\xi = 1$, $\log \xi = 0$, yields:

\[
\begin{align*}
\left[ \frac{d \log f}{d \log \xi} \right]_{\log \xi=0} &= -\frac{\gamma}{C_\gamma + 1} - \frac{\chi \alpha}{C_\alpha + 1} ; \\
\left[ \frac{d^2 \log f}{d (\log \xi)^2} \right]_{\log \xi=0} &= -\frac{C_\gamma \gamma^2}{(C_\gamma + 1)^2} - \frac{\chi C_\alpha \alpha^2}{(C_\alpha + 1)^2} ; \\
\left[ \frac{d^3 \log f}{d (\log \xi)^3} \right]_{\log \xi=0} &= -\frac{C_\gamma \gamma^3 (C_\gamma - 1)}{(C_\gamma + 1)^3} - \frac{\chi C_\alpha \alpha^3 (C_\alpha - 1)}{(C_\alpha + 1)^3} ; \\
\left[ \frac{d^4 \log f}{d (\log \xi)^4} \right]_{\log \xi=0} &= -\frac{C_\gamma \gamma^4 [(C_\gamma - 1)^2 - 2C_\gamma]}{(C_\gamma + 1)^4} - \frac{\chi C_\alpha \alpha^4 (C_\alpha - 1)^2}{(C_\alpha + 1)^4} ;
\end{align*}
\]

where Eq. (169) coincides with (86), as $\chi \alpha = \beta - \gamma$ by definition.

Turning to the general case and keeping in mind all parameters cannot be negative, the first and the second derivative are always negative while the third derivative is null provided the following relation:

\[
\frac{C_\gamma \gamma^3 \xi^\gamma (C_\gamma - \xi^\gamma)}{(C_\gamma + \xi)^3} + \frac{\chi C_\alpha \alpha^3 \xi^\alpha (C_\alpha - \xi^\alpha)}{(C_\alpha + \xi)^3} = 0 ;
\]

is satisfied. In particular, Eq. (173) holds at the dimensionless scaling radius, $\xi = 1$, for $C_\gamma = C_\alpha = 1$ (B density profiles); $C_\gamma = 1, C_\alpha = 0$ (Z density profiles); $C_\gamma = 0, C_\alpha = 1$ (Z density profiles). Accordingly, the slope variation rate, $d^2 \log f / d (\log \xi)^2$, has an extremum point at $\xi = 1$ where, in addition, the fourth derivative turns out to be positive, which implies the slope variation rate attains a minimum (a maximum in absolute value) at $\xi = 1$. 

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Figure 1: The embedding and the embedded sphere, centred in $O'$ and $O$, respectively. The minimum gravitational force exerted on the surface of the embedded sphere takes place on $P$, where the attraction due to each subsystem acts along the same direction but on opposite sides. A null gravitational force on $P$ defines $a_C$ as the tidal radius of the embedded sphere, $a^*_C$, where $O$ moves to $O^*$ while $P$ remains unchanged. Points mentioned above are represented as saltires. The figure is not in scale for sake of clarity: the embedded sphere, even if the radius equals the tidal radius (dotted), is completely lying within the embedding sphere. See text for further details.
Figure 2: Mutual positions of the embedding and the embedded sphere for different values of the inverse fractional distance, $\gamma = a_C/R_C$. Related centres are represented as saltires.
Figure 3: The instantaneous tidal radius, $\eta = \mu/(\mu + 1)$, vs. the inverse fractional distance, $\gamma = a_C/R_C$, for an embedded sphere of mass, $M_C = 10^5 m_\odot$, truncation radius, $a_C = 30$ pc, and an embedding sphere of mass, $M_G = 10^{12} m_\odot$, truncation radius, $a_G = 100$ kpc, concentration, $\Xi = 10$, density profiles listed in Table 2 and captioned as: B - Burkert (1995); H - Hernquist (1990); I - Begemann et al. (1991); P - Plummer (1911); S - Spano et al. (2008). The locus, $\eta = \gamma$, is also shown as a dotted line, above which ($\gamma < \eta$) the embedded sphere is considered bound according to the assumed formulation of tidal radius. See text for further details.
Figure 4: Zoom of Fig. 3 near the origin. The truncation radius of the embedding sphere is attained just after the intersection of related curve with the dotted straight line, $\eta = \gamma$, i.e. just after the tidal radius and the truncation radius of the embedded sphere coincide. The continuation of each curve outside the truncation radius of the embedding sphere, assuming the continuation of the density profile, is also shown.
Figure 5: The instantaneous tidal radius, $\eta = \mu/(\mu + 1)$, vs. the inverse fractional distance, $\gamma = a_C/R_C$, for an embedded sphere of mass, $M_C = 10^5 m_\odot$, truncation radius, $a_C = 30$ pc, and an embedding two-component sphere of mass, $M_{i,G} = 10^{11} m_\odot$, $M_{j,G} = 10^{12} m_\odot$, truncation radius, $a_{i,G} = 30$ kpc, $a_{j,G} = 100$ kpc, concentration, $\Xi_i = \Xi_j = 10$, density profiles, $ij = PI$, $PS$, $HI$, $HS$, captioned as in Fig. 3. Curves related to PI, PS; HI, HS; density profiles, respectively, are nearly coincident and cannot be resolved on the plane of the figure. The locus, $\eta = \gamma$, is also shown as a dotted line, above which $(\gamma < \eta)$ the embedded sphere is considered bound according to the assumed formulation of tidal radius. See text for further details.
Figure 6: Zoom of Fig. 5 near the origin. The ending point of each curve (left) relates to the truncation radius of the outer embedding sphere \((j = I, S)\), where \(\gamma = 0.0003\). The scale has been left unchanged with respect to Fig. 4 for better comparison.
Figure 7: Location of globular clusters, listed in Table 5, on the \((O\gamma\eta)\) plane, for an embedding sphere of mass, \(M_G = 5 \cdot 10^{10} m_\odot\), truncation radius, \(a_G = 125\) kpc, concentration, \(\Xi = 10\), density profiles among those listed in Table 1, captioned as in Fig. 3. The locus, \(\eta = \gamma\), is also shown as an inclined dotted line, above which \((\gamma < \eta)\) globular clusters are bound according to the assumed formulation of tidal radius. The position of NGC 5466 and Pal5, which exhibit noticeable tidal effects, are marked by vertical dotted lines. See text for further details.
Figure 8: Location of globular clusters, listed in Table 6, on the $(O \gamma \eta)$ plane, for a two-component embedding sphere of mass, $M_{i,G} = 10^{11} m_\odot$, $M_{j,G} = 10^{12} m_\odot$; truncation radius, $a_{i,G} = 30 \text{ kpc}$, $a_{j,G} = 100 \text{ kpc}$; concentration, $\Xi_i = \Xi_j = 10$; two-component density profiles considered in Subsection 4.2.2 captioned as in Fig. 5. The locus, $\eta = \gamma$, is also shown as an inclined dotted line, above which ($\gamma < \eta$) globular clusters are bound according to the assumed formulation of tidal radius. The position of NGC 5466 and Pal5, which exhibit noticeable tidal effects, are marked by vertical dotted lines. See text for further details.
Figure 9: The attraction exerted by a homeoidally striated ellipsoid on an internal or surface point, \( P \equiv (x_1, x_2, x_3) \), lies between its counterparts related to a homeoidally striated sphere and to a focaloidally striated (with respect to the isopycnic surface passing through \( P \)) ellipsoid where, in both cases, isopycnic surfaces tangent at the minor axis enclose an equal mass, \( m(x_3) \). See text for further details. The outer ellipsoid represents the boundary of the homeoidally striated ellipsoid, \( \Sigma(R) \). The inner ellipsoid represents the isopycnic surface of the homeoidally striated ellipsoid, \( \Sigma(r) \), passing through \( P \), also shown (cross). The sphere represents the isopycnic surface of the homeoidally striated sphere, enclosing the same mass as \( \Sigma(r) \). The horizontal bar represents the focal ellipse with respect to the focaloidally striated ellipsoid, enclosing the same mass as the central isopycnic surface of the homeoidally striated ellipsoid (saltire). The coordinate axes, \( x_1 \) and \( x_2 \), are not drawn to avoid confusion.