q-Analogue of $A_{m-1} \oplus A_{n-1} \subset A_{mn-1}$

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Abstract

A natural embedding $A_{m-1} \oplus A_{n-1} \subset A_{mn-1}$ for the corresponding quantum algebras is constructed through the appropriate comultiplication on the generators of each of the $A_{m-1}$ and $A_{n-1}$ algebras. The above embedding is proved in their $q$-boson realization by means of the isomorphism between the $A^{-q}_{(mn)} \sim \otimes^n A^{-q}_{(m)} \sim \otimes^m A^{-q}_{(n)}$ algebras.

Recently, quite a great interest has been paid to the study of quantum algebras and their applications to physical problems. Essentially quantum algebras are Hopf algebras. Hopf algebra is an algebra with additional structures: i) except the multiplication $m : A \otimes A \to A$ there is a comultiplication $\Delta : A \to A \otimes A$; ii) except the unit 1 which provides the embedding $R \to A (C \to A)$, where $R (C)$ is the real (complex) field, there is a counit $\varepsilon : A \to R (C)$. All these mappings are homomorphisms and there is an antihomomorphism $S : A \to A$ called antipode. From mathematical point of view such algebras were developed much earlier [1, 2]. The contemporary development of their theory is connected with noncommutative geometry and differential calculus [3]. In physics these new mathematical objects appear in the theory of the inverse scattering problem [4]. Later on, the quantum algebras have been applied to a number of physical regions as statistical mechanics, quantum field theory, molecular, atomic and nuclear physics.
In nuclear structure theory successful applications of models, based on algebraic chains of Lie algebras (Interaction Boson Model (IBM) [5], Two Vector Boson Model (TVBM) [6] etc.) were obtained.

It will be of interest to construct q-analogues of these chains and consider the corollary of the models based on them. The chain \( su_q(3) \oplus u_q(2) \subset u_q(6) \) was already considered by Quesne in [7].

In this paper we consider the general case of the embedding:

\[
A^q_{m-1} \oplus A^q_{n-1} \subset A^q_{mn-1}
\]  

in q-boson realization. The proper definition of the embedding (1) is a result of a careful analysis of the comultiplication structure. The present paper also provides the method of its realization, briefly described below.

As, it is well known, for any integer \( n \) the algebra \( A^q_{n-1} \) has a realization of its generators in terms of the q-boson algebra \( A^{-q}(n) \) [8, 9]. In order to obtain the realization of the generators of \( A^q_{n-1} \) in terms of the \( q \)-boson algebra \( A^{-q}(mn) \), we apply \( m - 1 \) times the comultiplication, then a q-boson realization for each term in the tensor product and finally employ the isomorphism \( A^{-q}(mn) \sim \otimes^m A^{-q}(n) \). By analogy we realize the generators of \( A^q_{mn-1} \). The generators of the \( q \)-deformed algebra \( A^q_{mn-1} \) have their realization by means of the same algebra \( A^{-q}(mn) \).

We start with the algebraic relations among the regular functionals \( l^\pm_{ij} \) of the quantum matrix group given in [11]:

\[
\sum_{m,p} R^+_{ij,mp} l^+_{mk} l^-_{pl} = \sum_{m,p} l^+_{jp} R^+_{mp,kl} l^+_{im} l^-_{pl} ; \quad \sum_{m,p} R^-_{ij,mp} l^-_{mk} l^+_{pl} = \sum_{m,p} l^-_{jp} R^-_{mp,kl} l^+_{im}
\]  

In the case of deformed \( A^q_{n-1} \) algebras the explicit form of the \( R^+ \)-matrix is given by:

\[
R^+ = q^n \{ q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{i \neq j}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j}^n e_{ij} \otimes e_{ji} \}
\]  

where \( e_{ij} \) are \( n \times n \) matrixes with elements \((e_{ij})_{km} = \delta_{ik}\delta_{jm}\).
By substituting (3) in (2) we obtain the following relations for $l_{ij}^\pm$:

\[
[l^{(e)}_{im}, l^{(e)}_{js}] = (1 - q)(l^{(e)}_{im} l^{(e)}_{js} - l^{(e)}_{js} l^{(e)}_{im}) + (q - q^{-1})(l^{(e)}_{jm} l^{(e)}_{is} - l^{(e)}_{jm} l^{(e)}_{is})
\]

\[
[l_{im}^+, l_{js}^-] = (1 - q)(l_{im}^+ l_{js}^- - l_{js}^- l_{im}^+) + (q - q^{-1})(l_{jm}^- l_{is}^+ - l_{jm}^- l_{is}^+)
\]

\[
\prod_{i=1}^{n} l_{ii}^\pm = 1 ; \quad l_{ii}^+ l_{ii}^- = 1 = l_{ii}^- l_{ii}^+
\]

\[
l_{ij}^+ = 0 \quad \text{for } i > j \quad \text{and} \quad l_{ij}^- = 0 \quad \text{for } i < j
\]

The last relations employs not only the form of $R^+$ but also some additional conditions [10].

Further by means of the substitution:

\[
H_{ij} = \tilde{H}_i - \tilde{H}_j
\]

\[
l_{ij}^\pm = \mp q^{\pm \frac{1}{2}}(q - q^{-1})Y_{ij}^\pm q^{\mp \frac{1}{2}}(H_i + H_j) \quad \text{with} \quad Y_{ii}^\pm = \mp q^{\pm \frac{1}{2}}
\]

one comes to the following relations for the Cartan-Weyl basis of q-deformed $A_n^{n-1}$ algebra:

\[
[H_{ij}, H_{km}] = 0
\]

| Borel subalgebra : $\mathcal{B}^+$ | Borel subalgebra : $\mathcal{B}^-$ |
|-----------------------------------|-----------------------------------|
| $[Y_{ik}^+, Y_{kj}^-]_{q^{-1}} = Y_{ij}^+ i < k < j$ | $[Y_{ij}^-, Y_{jk}^-]_{q^{-1}} = Y_{ik}^- i > j > k$ |
| $[Y_{ik}^-, Y_{ij}^-]_{q^{-1}} = 0 i < j < k$ | $[Y_{kj}^-, Y_{ij}^-]_{q^{-1}} = 0 i > k > j$ |
| $[Y_{kj}^+, Y_{ij}^-]_{q^{-1}} = 0 i < k < j$ | $[Y_{ik}^-, Y_{ij}^-]_{q^{-1}} = 0 i > j > k$ |
| $[Y_{ij}^+, Y_{km}^+] = 0 i < j < k < m$ | $[Y_{kj}^-, Y_{km}^-] = 0 i > j > k > m$ |
| $[Y_{ij}^+, Y_{km}^-] = 0i < k < m < j$ | $[Y_{ij}^-, Y_{km}^+] = 0 i > k > m > j$ |
| $[Y_{im}^+, Y_{km}^+] = (q - q^{-1})Y_{mj}^+Y_{im}^+$ | $[Y_{ij}^-, Y_{km}^-] = (q - q^{-1})Y_{kj}^-Y_{im}^-$ |
| $i < k < j < m$ | $i > k > j > m$ |

\[
[H_{ik}, Y_{js}^+] = (e_i - e_k, e_j - e_s)Y_{js}^+ \quad [H_{ik}, Y_{js}^-] = (e_i - e_k, e_j - e_s)Y_{js}^-
\]

Mixed commutators

\[
[Y_{ij}^+, Y_{ji}^+] = [H_{ij}]q^j < j
\]
\[ Y_{km}^+, Y_{ij}^- = (q - q^{-1}) Y_{kj}^+ Y_{im}^- q^{H_{ik}} \]
\[ j > k > i > m \]
\[ Y_{ij}^+, Y_{im}^- = 0 \]
\[ j > k > i > m \]
\[ [Y_{ij}^+, Y_{km}^-] = (q - q^{-1}) Y_{kj}^+ Y_{im}^- q^{H_{jm}} \]
\[ k > j > m > i \]
\[ [Y_{ij}^+, Y_{km}^-] = 0 \]
\[ [Y_{ij}^+, Y_{km}^-] = (q - q^{-1}) Y_{kj}^+ Y_{im}^- q^{H_{jm}} \]
\[ k > j > m > i \]

where \((e_i, e_j) = \delta_{ij}\), the \(q\)-commutator is given by \([A, B]_q = AB - qBA\) and the \(q\)-number is defined by \([x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}\). These relations are analogical to the ones obtained in \([1]\).

It should be noted here that the generators \(Y_{ij}^\pm\) can be substituted by \(\tilde{Y}_{ij}^\pm f_{ij}(q, \tilde{H})\), which will lead to modifications in the relations \((7)\) depending on the functions \(f_{ij}(q, \tilde{H})\). An example of such a mapping from \(su(2)\) to a deformed \(su_q(2)\) is given in \([12]\).

From the definition of the \(\text{Comultiplication}\) \(\Delta(l_{ij}^\pm) = \sum_{k=1}^n l_{ik}^\pm \otimes l_{kj}^\pm\) and the \(\text{Counit}\) \(\varepsilon(l_{ij}^\pm) = \delta_{ij}\) given in \([10]\) we obtain the following coalgebraic structure:

\[ \Delta H_{ij} = H_{ij} \otimes 1 + 1 \otimes H_{ij} ; \quad \varepsilon(H_{ij}) = 0 ; \quad S(H_{ij}) = -H_{ij} \]
\[ \varepsilon(Y_{ij}^\pm) = \mp \frac{q^{\pm 1}}{q - q^{-1}} \delta_{ij} ; \quad Y_{ii}^\pm = \mp \frac{q^{\pm 1}}{q - q^{-1}} ; \quad Y_{ik}^+ = 0 ; \quad Y_{ik}^- = 0 \]
\[ \Delta Y_{ij}^\pm = \mp (q - q^{-1}) q^{\pm \frac{1}{2}} \sum_{i \leq k \leq j \text{ or } (j \leq k \leq i)} Y_{ik}^\pm q^{\frac{1}{2} H_{jk}} \otimes Y_{kj}^\pm q^{\frac{1}{2} H_{ik}} \] (7)

Applying the standard definition of the antipode \(S(m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = i \circ \varepsilon)\) we deduce for the antipode of the generators \(Y_{ij}^\pm\) the following recurrent formula:

\[ S(Y_{ij}^\pm) = -q^{\mp 1} Y_{ij}^\pm \pm (q - q^{-1}) q^{\mp 1} \sum_{i < k < j \text{ or } (i > k > j)} Y_{ik}^\pm S(Y_{kj}^\pm) \] (8)

Let us introduce the \(q\)-boson algebra \(A^q_n(\mathfrak{n})\) with creation and annihilation operators \(a_i^\pm\) and their \(q\)-boson numbers \(N_i\) as in \([3, 13, 14]\).

\[ a_i^- a_i^+ - q^\mp a_i^+ a_i^- = q^{\mp N_i} \text{ and } [N_i, a_i^\pm] = \mp \delta_{ij} a_j^\pm \] (9)

The \(q\)-boson realization of the Cartan-Chevalley generators \(H_i = H_{i,i+1}\), \(Y_i^+ = Y_{i,i+1}^+\) and \(Y_i^- = Y_{i+1,i}^-\) of \(A_n^q\)-algebra given by Sun and Fu in \([3]\) is:

\[ H_i = N_i - N_{i+1} ; \quad Y_i^+ = a_{i+1}^- a_i^+ ; \quad Y_i^- = a_i^+ a_{i+1}^- \] (10)
The irreducible Fock representations $\Gamma^m_q$ with the vacuum state $|0>,$ $b^-|0>=0$, $N_i |0>=0$ is defined by the set of vectors:

$$\Gamma^m_q := \{|m> | m_1, ..., m_n> = \prod_{i=1}^n \frac{(b_i^+)^{m_i}}{\sqrt{[m_i]!}} |0> | m = \sum_{i=1}^n m_i\}$$

with the following properties:

$$dim\Gamma^m_q = \frac{(n+m-1)!}{m!(n-1)!}$$

$$N_i |m> = m_i |m>$$

where $N = \sum_{i=1}^n N_i$.

Using the definitions of $H_i$ and $N_i$ the operators $N_i$ can be expressed by

$$N_i = \frac{1}{n} N + \frac{1}{n} \sum_{s=2}^{n} \sum_{j=1}^{s-1} H_j - \sum_{j=1}^{n-1} H_j$$

The additional generators which extend $B\pm$ to the basis of Cartan-Weyl can be obtained from the Chevalley generators by means of the first relations in the Borel subalgebras $B\pm$ in (6). In this way, as in (10) we obtain the following general realization:

$$H_{ij} = N_i - N_j ; \quad Y_{ij}^\pm = a_i^+ a_j^- q^{i<k<j \text{ or } j<k<i}$$

Let us denote the generators of $A^q_{k_1 k_2}$ by $Y_{i}^{\pm}$ and $N_i$, of $A^q_{k_1}$ by $X_{\mu}^\pm$, and $N_{\mu}$, of $A^q_{k_2}$ by $Z^{\pm s}$ and $N_{s}$ and the $n$-th product of the comultiplication by:

$$\Delta^n = (id \otimes id \otimes \ldots \otimes \Delta) (id \otimes id \otimes \ldots \otimes \Delta) \ldots (id \otimes \Delta) \Delta$$

Since $\Delta$ is a homomorphism one can consider the following mapping:

$$A^q_{n-1} \rightarrow \Delta^{(n-1)} A_{m-1}^q \otimes \ldots \otimes A_{m-1}^q$$

For the sake of simplicity, the tensor product $\otimes$ will be dropped and the
index $s$ (or $\mu$) will indicate the number of the tensor space. Thus we obtain:

$$\tilde{H}_\mu = \sum_{s=1}^{k_2} H_{\mu^s}; \quad \tilde{X}_\mu^\pm = \Delta^{(k_2-1)}(X_\mu^\pm) = \frac{1}{2} \sum_{\sigma \neq s, \sigma = 1}^{k_2} \text{sign}(\sigma - s) H_\sigma^\sigma$$

$$\tilde{H}^s = \sum_{\mu=1}^{k_1} H_{\mu^s}; \quad \tilde{Z}^\pm s = \Delta^{(k_1-1)}(Z^\pm s) = \frac{1}{2} \sum_{\sigma \neq \mu, \sigma = 1}^{k_1} \text{sign}(\sigma - \mu) H_\sigma^\sigma$$

(16)

From the construction of the operators (16) and as a result of the used homomorphism $\Delta$ it is easy to prove that the generators $\tilde{X}_\mu^\pm$, $\tilde{H}_\mu$ and $\tilde{Z}^\pm s$, $\tilde{H}^s$ satisfy the commutations relations for the algebras $A_{q^2}$ and $A_{q^1}$.

Using the q-boson realization of the generators (14) we obtain:

$$\tilde{X}_\mu^+ = \sum_{s=1}^{k_2} a_{\mu^s}^+ a_{\mu+1}^- q^{\frac{1}{2}} \sum_{\sigma \neq s, \sigma = 1}^{k_2} \text{sign}(\sigma - s) (N_\sigma^\sigma - N_{\mu+1}^{\sigma+1})$$

$$\tilde{X}_\mu^- = \sum_{s=1}^{k_2} a_{\mu+1}^+ a_{\mu}^- q^{\frac{1}{2}} \sum_{\sigma \neq \mu, \sigma = 1}^{k_1} \text{sign}(\sigma - \mu) (N_\sigma^\sigma - N_\mu^\sigma)$$

$$\tilde{Z}^+ s = \sum_{\mu=1}^{k_1} a_{\mu}^+ a_{\mu+1}^- q^{\frac{1}{2}} \sum_{\sigma \neq \mu, \sigma = 1}^{k_1} \text{sign}(\sigma - \mu) (N_\sigma^\sigma - N_{\sigma+1}^s)$$

(17)

$$\tilde{Z}^- s = \sum_{\mu=1}^{k_1} a_{\mu}^+ a_{\mu+1}^- q^{\frac{1}{2}} \sum_{\sigma \neq \mu, \sigma = 1}^{k_1} \text{sign}(\sigma - \mu) (N_\sigma^\sigma - N_{\sigma+1}^s)$$

$$\tilde{H}^s = \sum_{\mu=1}^{k_1} N_{\mu}^s - N_{\mu+1}^{s+1}; \quad \tilde{H}_\mu = \sum_{s=1}^{k_2} N_{\mu}^s - N_{\mu+1}^s$$

It is correct to consider the q-bosons in $\tilde{X}$ and $\tilde{Z}$ (17) as different objects, because in $\tilde{X}$, $a_{\mu}^\pm s$ mean:

$$a_{\mu}^\pm s = \text{id} \otimes \ldots \otimes \text{id} \otimes \underbrace{a_{\mu}^\pm \otimes \text{id} \otimes \ldots \otimes \text{id}}_{k_2}$$
while in \( \tilde{Z} \):

\[
a_{\mu}^{\pm s} = id \otimes \ldots \otimes id \otimes a_{\mu}^{\pm} \otimes id \otimes \ldots \otimes id
\]

However in both cases, they satisfy the same relations:

\[
[a_{\mu}^{s}, a_{\nu}^{t}] = 0 \text{ for all } s, t, \mu, \nu \quad [a_{\mu}^{s}, a_{\nu}^{t}] = 0 \text{ for all } s \neq t; \mu \neq \nu
\]

\[
[N_{\mu}^{s}, a_{\nu}^{t}] = \pm \delta_{\mu, \nu} \delta_{s, t} a_{\mu}^{\pm t} \quad a_{\mu}^{-s} a_{\mu}^{+s} - q^{\pm 1} a_{\mu}^{+s} a_{\mu}^{-s} = q^{\pm N_{\mu}^{s}}
\] (18)

Let us define the following correspondence: \( i \leftrightarrow (\mu, s) \) \( (k_{2} \leq k_{1}) \):

\[
i \leftrightarrow (\mu, s) \quad i = 1, \ldots, k_{1}k_{2}; \quad \mu = 1, \ldots, k_{1}; \quad s = 1, \ldots, k_{2}
\]

\[
\mu = 1 + \text{int}[i/k_{2}] \text{ where int } [x] \text{ is integer part of } x
\]

\[
s = 1 + (i - 1) \text{mod}(k_{2}) \quad i = (\mu - 1)k_{2} + s
\] (19)

From the introduction of (18) in equations (9) and (18) it follows that the algebras \( \otimes k_{2} A_{q}^{-}(k_{1}) \) and \( \otimes k_{1} A_{q}^{-}(k_{2}) \) constructed by the q-bosons \( a_{\mu}^{\pm s} \) are isomorphic to the algebra \( A_{q}^{-}(k_{1}k_{2}) \) constructed by the q-bosons \( a_{i}^{\pm} \). As a result the algebras \( A_{q_{1}}^{k_{1} - 1} \) and \( A_{q_{2}}^{k_{2} - 1} \) have realization in the \( A_{q}^{-}(k_{1}k_{2}) \) algebra.

**Proposition 1.** The the generators \( \tilde{X}_{\mu}^{\pm} \), \( \tilde{H}_{\mu} \) commute with the generators \( \tilde{Z}^{\pm s} \), \( \tilde{H}^{s} \) given by (17).

**Proof.** Let us consider the commutator between the elements \( \tilde{X}_{\mu}^{+} \) and \( \tilde{Z}^{-s} \). For this purpose we define \( Q_{t, \nu} \) and \( I_{t, \nu}(\mu, s, k) \) as:

\[
Q_{t, \nu} = q^{1/2} \sum_{\sigma \neq t, \sigma = 1}^{k_{2}} \text{sign}(\sigma - t)(N_{\mu}^{\sigma} - N_{\mu+1}^{\sigma}) + \sum_{\rho \neq \nu, \rho = 1}^{k_{1}} \text{sign}(\rho - \nu)(N_{\rho}^{s} - N_{\rho+1}^{s})
\]

\[
I_{t, \nu}(\mu, s, k) = q^{1/2} \sum_{\sigma \neq t, \sigma = 1}^{k} \text{sign}(\sigma - t)(\delta_{\mu, \nu} - \delta_{\mu+1, \nu})(\delta_{\sigma, s+1} - \delta_{\sigma, s})
\]

Using (17) and (18), for the commutator we obtain:

\[
[\tilde{X}_{\mu}^{+}, \tilde{Z}^{-s}] = \sum_{t=1, \nu=1}^{k_{2}, k_{1}} \{a_{\mu}^{t} a_{\mu+1}^{s} a_{\nu}^{t+1} a_{\nu}^{s+1} I_{t, \nu}(\mu, s, k_{2}, q) - a_{\nu}^{s+1} a_{\nu}^{t} a_{\mu}^{t+1} I_{t, \nu}(s, \mu, k_{1}, q^{-1})\} Q_{t, \nu}
\] (20)
The sum over \( t \) and \( \nu \) can be represented as a sum of five terms:

(a) = \{ \nu \neq \mu, \mu + 1 \text{ and } t \neq s, s + 1 \}
(b) = \{ \nu = \mu \text{ and } t = s + 1 \}
(c) = \{ \nu = \mu + 1 \text{ and } t = s \}
(d) = \{ \nu = \mu \text{ and } t = s \}
(e) = \{ \nu = \mu + 1 \text{ and } t = s + 1 \}

In these cases we have:

\[ I_{t,\nu}(\mu, s, k_2, q) = \begin{cases} 1 & \text{in (a)} \\ q^\frac{1}{2} & \text{in (b), (d)} \\ q^{-\frac{1}{2}} & \text{in (c), (e)} \end{cases} \]

\[ I_{\nu,t}(s, \mu, k_1, q^{-1}) = \begin{cases} 1 & \text{in (a)} \\ q^\frac{1}{2} & \text{in (b), (e)} \\ q^{-\frac{1}{2}} & \text{in (c), (d)} \end{cases} \]

In the cases (a), (b) and (c) the bosons \( a_{\nu}^{+s+1}, a_{\nu}^{s}, a_{\mu}^{+t} \) and \( a_{\mu+1}^{t} \) commute and the relevant terms are equal to zero. Thus the commutator is given only by the sum of (d) and (e) i.e.

\[ [\tilde{X}_{\mu}^{+}, \tilde{Z}^{-s}] = q^{-\frac{1}{2}}a_{\mu}^{+s+1}a_{\mu+1}^{s}(q^{-N_{\mu+1}^{s+1}}Q_{s+1,\mu+1} - q^{-N_{\mu}^{s}}Q_{s,\mu}) = 0 \]

The expression \( \text{sign}(\rho - \mu) = \text{sign}(\rho - \mu - 1) \) when \( \rho < \mu \) or \( \rho > \mu + 1 \) is used essentially in the calculation of \( q^{-N_{\mu+1}^{s+1}}Q_{s+1,\mu+1} = q^{-N_{\mu}^{s}}Q_{s,\mu} \). The other commutators can be proved in the same way.

Further using (14) and the isomorphism (19) we have:

\[ a_{1+\lfloor \frac{i-1}{k_2} \rfloor}^{1+\lfloor \frac{j-1}{k_2} \rfloor} = a_{1+\lfloor \frac{i}{k_2} \rfloor}^{1+\lfloor \frac{j}{k_2} \rfloor} \pm \sum_{\sigma} N_{\sigma} \]

Finally applying (13) and (21) the generators of \( A_{k_1-1}^{q} \) and \( A_{k_2-1}^{q} \) in (17)
are expressed through the generators of $A_{q_{k_1,k_2-1}}$ in the following way:

\[
\tilde{Z}^\pm_s = \sum_{\mu=1}^{k_1} Y_{(\mu-1)k_2+s}^\pm q^{\frac{1}{2}} \sum_{\sigma \neq \mu, \sigma = 1}^{k_2} \text{sign}(\sigma - \mu) H_{(\sigma-1)k_2+s}
\]

\[
\tilde{H}^s = \sum_{\mu=1}^{k_1} H_{(\mu-1)k_2+s} ; \quad \tilde{H}_\mu = \sum_{s=(\mu-1)k_2+1}^{(\mu+1)k_2} H_{s,s+k_2}
\]

\[
\tilde{X}^+_{\mu} = \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y_{t-k_2, t}^+ q^{\frac{1}{2}} \sum_{\nu \neq t, \nu = \mu k_2+1}^{(\mu+1)k_2} \text{sign}(\nu - t) H_{\nu-k_2, \nu} + \Lambda_t^+
\]

\[
\tilde{X}^-_{\mu} = \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y_{t-k_2, t}^- q^{\frac{1}{2}} \sum_{\nu \neq t, \nu = \mu k_2+1}^{(\mu+1)k_2} \text{sign}(\nu - t) H_{\nu-k_2, \nu} + \Lambda_t^-
\]

\[
\Lambda_t^\pm = \frac{k_2-1}{k_1 k_2} (N + \sum_{\sigma=2}^{k_1 k_2} H_{1,\sigma}) \pm \sum_{\sigma=t-k_2+1}^{t-1} H_{1,\sigma}
\]

The difference $\Lambda_t^\pm$ between the expressions for $\tilde{Z}^\pm_s$ and $\tilde{X}^\pm_\mu$ is due to the ordering of indices in (19) which leads to the appearance of different terms \[ N_k \] in the $q$-boson realization (14) of the Chevalley and the additional Weyl generators. In the expression $\Lambda_t^\pm$ the operator $N$, in $q$-boson realization has the meaning of a total number of bosons operator. In general a corresponding operator may be constructed in some extension of the algebra $A_{q_{k_1,k_2-1}}$. This can be proved by induction. For $A_2^q (su_q(2))$ the operator $N$ can be obtained from the second order Casimir operator:

\[
C_2^q = X^- X^+ + [H/2]_q [H/2 + 1]_q = \frac{q^{N+1} + q^{-N-1} - q - q^{-1}}{(q - q^{-1})^2}
\]

For $n > 2$, $N^{(n)}$ – the corresponding operator $N$ for $A_n^q$, is obtained by the recurrence:

\[
N^{(n)} = \frac{n + 1}{n} \{ N^{(n-1)} + \frac{1}{n+1} \sum_{t=2}^{n+1} \sum_{p=1}^{t-1} H_p - \sum_{p=1}^{n} H_p \}
\]

Moreover in practice it is only the eigenvalues of $q^N$ which are required.
Proposition 2. The elements $\tilde{X}_\pm^\mu$, $\tilde{H}_\mu$ of $A_{k_1-1}^q$ and $\tilde{Z}_s$, $\tilde{H}_s$ of $A_{k_2-1}^q$ defined by (22) belong to the algebra $A_{k_1k_2-1}^q$ and provide an explicit embedding $A_{k_1-1}^q \oplus A_{k_2-1}^q \subset A_{k_1k_2-1}^q$ in the $q$-boson realization (14) of $A_{k_1k_2-1}^q$.

Proof. From the above it follows that the elements defined by (22) belong to the $q$-deformed $A_{k_1k_2-1}^q$ algebra. Applying the $q$-boson realization (14), the correspondence (13) and (18) we obtain the $q$-boson realization (17) of the generators $\tilde{X}_\pm^\mu$, $\tilde{H}_\mu$ and $\tilde{Z}_s$, $\tilde{H}_s$, whose commutation relations close the algebras $A_{k_1-1}^q$ and $A_{k_1k_2-1}^q$. Finally these two pairs of generators commute between themselves as proved in Proposition 1 and so they close the algebra $A_{k_1-1}^q \oplus A_{k_2-1}^q$ embedded in $A_{k_1k_2-1}$.

The results of Quesne [7] are reproduced in the case $k_1k_2 = 6$, $k_1 = 3$ and $k_2 = 2$.

In the limit $q \to 1$ we obtain the usual embedding:

$$
\tilde{H}_\mu = \sum_{s=(\mu-1)k_2+1}^{(\mu-1)k_2+k_2} H_{s,s+k_2} \quad \tilde{X}_\mu^+ = \sum_{s=1}^{k_2} Y_{(\mu-1)k_2+s,\mu k_2+s} \quad \tilde{X}_\mu^- = \sum_{s=1}^{k_2} Y_{\mu k_2+s,(\mu-1)k_2+s} \\
\tilde{H}_s = \sum_{\mu=1}^{k_1} H_{(\mu-1)k_2+s} \quad \tilde{Z}_s = \sum_{\mu=1}^{k_1} Y_{(\mu-1)k_2+s} \quad (\mu-1)k_2+s
$$

These results are obtained on the basis of the isomorphism between the algebras $A_{k_1k_2-1}^q \sim \otimes^m A_{k_1-1}^q \sim \otimes^m A_{k_2-1}^q$ and the homomorphism of the comultiplication.

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