An iterative algorithm for reconstructing a 2D vector field by its limited-angle ray transform

S V Maltseva\textsuperscript{1*}, I E Svetov\textsuperscript{1} and A K Louis\textsuperscript{2}

\textsuperscript{1}Sobolev Institute of Mathematics, Novosibirsk, Russia
\textsuperscript{2}Saarland University, Saarbrucken, Germany
E-mail: maltsevasv@math.nsc.ru

Abstract. In this paper, we consider problem of reconstructing a vector field in the unit disc by using the ray transform in a limited angular domain. We propose an approach to solving this problem. The approach is based on the Papoulis-Gerchberg algorithm. It is an iterative algorithm including iterations in the main space of vector fields and in the frequency domain. The results of numerical simulation are demonstrated.

1. Introduction

The classical tomography problem is the problem of restoring a function by its known values of the Radon transform by the complete data. To solve the problem, there are a number of methods and algorithms that have well proven themselves. First of all, these are algorithms based on the inversion formulas for the Radon transform (see, for example, [1], [2]). In addition, there are the singular value decompositions of the Radon transform operator [2], [3], with which the approximation of a desired function is presented in the form of a series. In practice, the least squares method with bases based on polynomials [4] and B-splines has well proven itself [5]. We also note the approximate inverse method [6], [7], [8], which allows one to restore the function by its Radon transform with the usage of calculating the inner products.

The tomography problems with complete data are incorrect (see, for example, [9]), and the tomography problems with limited data are more incorrect. In [10], in the frame of the tomography problem with limited data, a singular value decomposition was constructed and stability issues were also considered. Incompleteness of data (in comparison with the classical problem of computerized tomography) leads to the need for additional definition, extrapolation of data. So in [11], extrapolation of data was proposed, which was obtained as the sum of the Legendre polynomials with certain coefficients, and given an estimate of stability. Another method for solving the problem is the Papoulis-Gerchberg iterative algorithm, originally developed for the extrapolation of band-limited signals [12], [13]. This algorithm is based on the projection-slice theorem [2] and is a multiple application of the direct and inverse Fourier transforms taking into account a priori information in the space of the sought for function and its Fourier image. The projection-slice theorem is formulated for a parallel data collection scheme. For the case of a fan-beam data collection scheme, this theorem cannot be used. In [15], [14], a projective transform was constructed and a projection-slice theorem for a fan-beam scheme was formulated. Based on this theorem, the Papoulis-Gerchberg method was constructed for reconstructing a function by its Radon transform with the fan-beam data collection.
This paper discusses the problem of reconstructing a vector field in $\mathbb{R}^2$ from its ray transform with limited data. For the full reconstruction of a two-dimensional vector field, it is necessary to know the values of the two ray transforms: longitudinal and transverse. We list several methods for the inversion of these ray transforms that have well proven themselves when solving problems with complete data. First of all, these are inversion formulas, which allow one to restore either components of the vector field itself or the potentials that generate the field [16]. The least squares method has been modified for reconstructing vector fields. There are options with a basis constructed on polynomials [17] and constructed on $B$-splines [18]. In addition, there are singular value decompositions of the operators of longitudinal and transverse ray transforms acting on vector fields [19], [20]. The approximate inversion method was also used to solve the vector tomography problems [21]–[23].

For reconstructing the two-dimensional vector fields from incomplete data, we propose a method based on a modification of the Papoulis-Gerchberg method that was used to solve the scalar tomography problem [24]. A study of the developed algorithm by the methods of numerical experiment is carried out. The results of numerical calculations are presented.

2. The Papoulis-Gerchberg algorithm for the scalar tomography problem

In this section, we describe one of the versions of reconstructing a function (a scalar field) by its Radon transform with limited data [24]. Let a function $f(x_1, x_2)$ be defined in the unit disk $B = \{x = (x_1, x_2)|x_1^2 + x_2^2 \leq 1\}$. The Radon transform operator acts on $f$ according to the rule

$$\left(\mathcal{R}f\right) (s, \varphi) = \int_{-1}^{1} f(s \cos \varphi - t \sin \varphi, s \sin \varphi + t \cos \varphi) dt,$$  

where the vector $(-\sin \varphi, \cos \varphi)$ (defined by the variable $\varphi$) sets the direction of a beam of parallel straight lines along which the integration holds. The variable $s$ defines a distance between the origin and the straight line. The problem of reconstructing the function $f(x), x \in B$, by a set of integrals (1) at $s \in [-1, 1], \varphi \in [-\pi/2, \pi/2]$, is called the classical problem of scalar tomography. By the scalar tomography problem with limited data we mean the problem of reconstructing $f(x), x \in B$, by a set of integrals (1), where $s \in [-1, 1], \varphi \in [-\varphi_{\text{max}}, \varphi_{\text{max}}], \varphi_{\text{max}} \leq \pi/2$. At $\varphi_{\text{max}} = \pi/2$, this problem becomes the classical scalar tomography problem (with complete data).

The back-projection operator with angular limitation acts on the function $g(s, \varphi)$ by the formula

$$\left(\mathcal{R}^\#g\right) (x) = \frac{1}{2\varphi_{\text{max}}} \int_{-\varphi_{\text{max}}}^{\varphi_{\text{max}}} g(x_1 \cos \varphi + x_2 \sin \varphi, \varphi) d\varphi.$$  

The operator $\mathcal{R}^\#$ is the averaging of $g(s, \varphi)$ along all the straight lines passing through the point $x$ and allows us to go from the variables $(s, \varphi)$ to the variables $x$.

We denote the application of the two-dimensional Fourier transform by $\mathcal{F}_2$ and $\hat{\cdot}$. The problem of reconstructing $f$ leads to the following equation in the Fourier space

$$\mathcal{F}_2 \left(\mathcal{R}^\#g\right) (\nu) = \chi_\Omega \frac{1}{|\nu|} (\mathcal{F}_2 f) (\nu), \quad g = \mathcal{R} f,$$  

where $\nu = (\nu_1, \nu_2)$ are variables in the Fourier space, $|\nu| = \sqrt{\nu_1^2 + \nu_2^2}$, $\chi_\Omega$ is the characteristic function of the angular sector $\Omega$ defined by the inequality $|\tan \varphi| \leq \tan \varphi_{\text{max}}$. A part of the plane which is complementary to $\Omega$ is denoted by $\bar{\Omega}$. The function $\chi_\Omega = 1 - \chi_{\bar{\Omega}}$ is a characteristic function of $\bar{\Omega}$. If we know $(\mathcal{F}_2 f)(\nu)$ for all $\nu$, then we can apply the inverse Fourier transform...
\( \mathcal{F}^{-1}_2 \) and obtain \( f(x) \). However, equation (3) allows us to reconstruct \( (\mathcal{F}_2 f)(\nu) \) only for \( \nu \in \Omega \). Thus, we come to the necessity of redefining \( (\mathcal{F}_2 f)(\nu) \) at \( \nu \in \bar{\Omega} \). The Papoulis-Gerchberg method allows us to extrapolate the Radon transform with limited data on \( \bar{\Omega} \).

The Papoulis-Gerchberg algorithm is iterative, and operations hold in the Fourier space and in the main space (the space of the sought for function) alternately. Every step of the algorithm in the approximation of the function \( f \) or its Fourier transform \( \hat{f} \) takes into account a priori information (for example, about the support boundedness). Let \( X \) be the support of \( f \), then \( \chi_X \) is a characteristic function of \( X \). Thus, for \( f \) we have \( \chi_X f = f \). Define a band-limiting operator \( B_X \) in the Fourier space

\[
B_X \hat{f} = \mathcal{F}_2 \chi_X \mathcal{F}^{-1}_2 \hat{f}.
\]

A part of \( (\mathcal{F}_2 f)(\nu) \) at \( \nu \in \Omega \) is obtained from formula (3) and \( \hat{f}(\nu) = |\nu| \mathcal{F}_2(\mathcal{R}^# g)(\nu) \), \( g = \mathcal{R} f \).

Based on this formula, an initial approximation of the iterative process may be chosen as

\[
\hat{f}_0 = \chi_{\Omega} (|\nu| \cdot \mathcal{F}_2(\mathcal{R}^# g))(\nu).
\]

Further steps are carried out in the Fourier space and in the main space alternately by the following scheme

\[
\hat{f}_0 \rightarrow \mathcal{F}_2^{-1} \hat{f}_0 \rightarrow \chi_X \left( \mathcal{F}_2^{-1} \hat{f}_0 \right) \rightarrow \mathcal{F}_2 \chi_X \left( \mathcal{F}_2^{-1} \hat{f}_0 \right) = B_X \hat{f}_0.
\]

The next approximation of \( \hat{f} \) (with extrapolation of the data on \( \bar{\Omega} \)) is

\[
\hat{f}_1 = \begin{cases} 
B_X \hat{f}_0, & \nu \notin \Omega, \\
\hat{f}_0, & \nu \in \Omega.
\end{cases}
\]

Further, the process continues according to the formula

\[
\hat{f}_n = \hat{f}_0 + \chi_{\bar{\Omega}} B_X \hat{f}_{n-1}.
\]

As a stop of the process we may choose the inequality execution

\[
\frac{\|f_n - f_{n-1}\|_{L_2(B)}}{\|f_n\|_{L_2(B)}} < \varepsilon,
\]

where \( \varepsilon \) is small.

3. The Statement of the vector tomography problem with limited data

Let us introduce some definitions for the statement of the vector tomography problems. Let the function \( \psi(x) \) be defined in \( B \). The inner derivation operator \( d \cdot \) acts on \( \psi \) and maps it into a vector field with the components

\[
(d\psi)^i = \frac{\partial \psi}{\partial x_i}, \quad i = 1, 2.
\]

A vector field \( v \) is called potential if \( v = d\psi \) for some function \( \psi \). In this case, we mean that the potential vector field \( v \) is generated by the potential \( \psi \) using the operator \( d \).

The divergence operator acts on the vector field \( v(x) = (v^1(x), v^2(x)) \) and transforms it to the scalar field defined by the equality

\[
\text{div} v = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2}.
\]
A vector field \( v \) is called solenoidal if \( \text{div} v = 0 \). Components of the solenoidal field can be expressed using one function \( \psi \) \[ v^i = (d^\perp \psi)^i = (-1)^i \frac{\partial \psi}{\partial x_{3-i}}. \] (4)

Here the operator \( d^\perp \cdot \) is called the \( \perp \)-inner derivation operator. In this case, we mean that the solenoidal vector field \( v \) is generated by the potential \( \psi \) using the operator \( d^\perp \cdot \).

It is known [26] that every vector field \( v \) vanishing on the boundary of the domain \( \partial B \) may be uniquely decomposed in the sum \[ v = s v + d \psi, \]
where \( \text{div}^* v = 0, \psi|_{\partial B} = 0 \).

Consider a vector field \( v(x) = (v^1(x_1, x_2), v^2(x_1, x_2)), x \in B \). Denote by \( \xi = (\cos \varphi, \sin \varphi) \) the vector orthogonal to the beam of straight lines, and by \( \xi^\perp = (-\sin \varphi, \cos \varphi) \) — the directional vector for the beam. The longitudinal ray transform operator \( P \) acts on the vector field \( v(x) \) according to the rule
\[ (Pv)(s, \varphi) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left( (\xi^\perp)^1 v^1(s \xi + t \xi^\perp) + (\xi^\perp)^2 v^2(s \xi + t \xi^\perp) \right) dt. \] (5)

The transverse ray transform operator \( P^\perp \) acts on the vector field \( v(x) \) according to the rule
\[ (P^\perp v)(s, \varphi) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left( \xi^1 v^1(s \xi + t \xi^\perp) + \xi^2 v^2(s \xi + t \xi^\perp) \right) dt. \]

Unlike the Radon transform operator \( R \), the operators of the ray transforms of the vector fields \( P \) and \( P^\perp \) have non-trivial kernels (see, for example, [16]). Namely, the kernel of \( P \) (\( P^\perp \)) consists of all potential (solenoidal) vector fields with potentials vanishing on the boundary of the domain. In this way, the whole field may be reconstructed only by the values of both transforms. If we know \( P \) (\( P^\perp \)), then we can reconstruct only the solenoidal (potential) part of the field.

There are [18] connections between the longitudinal and transverse ray transforms:
\[ [P(d^\perp \psi)] = [P^\perp (d \psi)]. \]

Therefore, we can consider, for example, only the problem of reconstructing the solenoidal part of \( v(x), x \in B \). If a set of integrals (5) is given for \( s \in [-1, 1], \varphi \in [-\pi/2, \pi/2] \), then it is the problem with complete data. In this study we consider the problem of reconstructing the solenoidal part of \( v(x), x \in B \), — by its longitudinal ray transform \( (Pv)(s, \varphi) \) if \( s \in [-1, 1], \varphi \in [-\varphi_{\text{max}}, \varphi_{\text{max}}], \varphi_{\text{max}} \leq \pi/2 \). This problem is a problem with limited data.

Further we deal with reconstructing the solenoidal field \( v \) by the values of its longitudinal ray transform \( Pv \).

4. The Papoulis-Gerchberg algorithm for the vector tomography problem
In this section, we describe our scheme of the Papoulis-Gerchberg algorithm designed for the solving the problem of reconstructing the solenoidal part of a vector field by its known values of the longitudinal ray transform with limited data, i.e. for the approximate inversion of the integral operator \( P \).
The back-projection operator $\mathcal{P}^#$ acts on the function $g(s, \varphi)$ and maps it onto the vector field $((\mathcal{P}^# g)^1(x), (\mathcal{P}^# g)^2(x))$, defined by the formulas

\begin{align*}
(\mathcal{P}^# g)^1(x) &= \frac{1}{2\varphi_{\text{max}}} \int_{-\varphi_{\text{max}}}^{\varphi_{\text{max}}} g(x_1 \cos \varphi + x_2 \sin \varphi, \varphi) \cos \varphi d\varphi, \\
(\mathcal{P}^# g)^2(x) &= \frac{1}{2\varphi_{\text{max}}} \int_{-\varphi_{\text{max}}}^{\varphi_{\text{max}}} g(x_1 \cos \varphi + x_2 \sin \varphi, \varphi) \sin \varphi d\varphi.
\end{align*}

(6)

Consider the formula of reconstructing the solenoidal part of a vector field by its longitudinal ray transform with complete data, see [16]

$$v^i = \frac{1}{2} \mathcal{I}^{-1}(\mathcal{P}^# g)^i, \quad g = \mathcal{P} v, i = 1, 2.$$ 

An action of the Rietz potential $\mathcal{I}^{-1}$ [2] on the function $h(x)$ is defined using the Fourier transform

$$\mathcal{F}_2(\mathcal{I}^{-1} h)(\nu) = |\nu| (\mathcal{F}_2 h)(\nu).$$

Thus, we have the following equations in the Fourier space

$$\mathcal{F}_2(\mathcal{P}^# g)^i(\nu) = \chi_{\Omega} \frac{1}{|\nu|} (\mathcal{F}_2 v^i)(\nu), \quad i = 1, 2,$$

which are similar to (3). Hence we design the initial approximation for the Fourier images of components of the sought for field

$$\hat{\vartheta}^i_0 = D_{\Omega} [\nu \cdot (\mathcal{F}_2 (\mathcal{P}^# g)^i)](\nu), \quad i = 1, 2.$$

Further steps are carried out in the Fourier space and in, alternately, the main space by the following scheme

$$\hat{\vartheta}^i_0 \implies \mathcal{F}_2^{-1} \hat{\vartheta}^i_0 \implies D_X (\mathcal{F}_2^{-1} \hat{\vartheta}^i_0) \implies \mathcal{F}_2 D_X (\mathcal{F}_2^{-1} \hat{\vartheta}^i_0) = B_X \hat{\vartheta}^i_0, \quad i = 1, 2.$$

The next approximation for $\hat{\vartheta}^i$ is

$$\hat{\vartheta}^i = \begin{cases} B_X \hat{\vartheta}^i_0, & \nu \notin \Omega, \\
\hat{\vartheta}^i_0, & \nu \in \Omega, \end{cases} \quad i = 1, 2.$$

Further, the process continues according to the formula

$$\hat{\vartheta}^i_n = \hat{\vartheta}^i_0 + \chi_{\Omega} B_X \hat{\vartheta}^i_{n-1}, \quad i = 1, 2.$$

As a stop of the process, we may choose as an example execution of the inequality

$$\frac{\|v_n - v_{n-1}\|_{L^2(S^1(B))}}{\|v_n\|_{L^2(S^1(B))}} < \varepsilon,$$

where $\varepsilon$ is small.
5. Simulation

In this section, we demonstrate the results of numerical tests aimed at reconstructing vector fields using our approach. All the test fields are solenoidal, the potential part is absent. Data for the problem are values of the longitudinal ray transform. Test fields are reconstructed at the square $[-1, 1]^2$ with uniform grid with the step $1/32$ on each axis. The back-projection is calculated at the square $[-A, A]^2$ with the step $1/32$ on each axis. The Fourier transform is realized using the procedure of the fast Fourier transform. Thus, $A$ must take values equal to the power of 2, we use $A = 1, 2, 4, 8$.

Numerical experiments are aimed at the investigation of the influence of the parameters of problem (a number of iterations, $\varphi_{\text{max}}$, a number of directions $N_\varphi$, smoothness of the sought for field, the parameter $A$) on the reconstruction error. The mean-square reconstruction error is calculated by the formula

$$\delta = \sqrt{\frac{\sum_{i=0}^{63} \sum_{j=0}^{63} (v_{ij}^1 - u_{ij}^1)^2 + (v_{ij}^2 - u_{ij}^2)^2}{\sum_{i=0}^{63} \sum_{j=0}^{63} (u_{ij}^1)^2 + (u_{ij}^2)^2}},$$

where $u = (u^1, u^2)$ is the original vector field, $v = (v^1, v^2)$ is the reconstructed vector field, $v_{ij}^k$ ($u_{ij}^k$) is a value of the $k$-th component of reconstruction (the original field) at point the $(x_i^1, x_j^2)$, $x_i^1 = -1 + i/32$, $x_j^2 = -1 + j/32$, $i, j = 0, \ldots, 63$.

The test solenoidal vector fields are generated, using formula (4), by the following potentials

$$\psi_0(x, y) = \exp \left\{ -24(x^2 + y^2) \right\},$$

$$\psi_k(x, y) = \begin{cases} 4 \left( 0.25 - x^2 - y^2 \right)^{6-k}, & x^2 + y^2 < 0.25, \\ 0, & \text{otherwise}, \end{cases} \quad k = 1, \ldots, 5.$$

Here $\psi_0(x, y)$ generates the vector field of infinite smoothness, $\psi_1(x, y) - \psi_5(x, y)$ generate the vector fields of smoothness from $C^3$ to those discontinued, respectively.

**Test 1.** We investigate the dependence of $A$ on the reconstruction error $\delta$. The aim of the test is to find the optimal value for the parameter $A$. The test vector field is generated by the potential $\psi_4$, the discretization of the ray transform $P$ with respect to the variables $s, \varphi$ is $300 \times 300$, $\varphi_{\text{max}} = 1.4137$. In the first row of Table 1, there are values of $A$, in the second row there are values of $\delta$. Table 1 shows that an increase of $A$ from 1 to 2 leads to a significant decrease of $\delta$. However, further increase of $A$ leads to an insignificant decrease of $\delta$ with increasing time costs for the calculation of the back-projection in a larger area. We choose the value $A = 2$ as optimal for our algorithm.

| $A$ | 1   | 2   | 4   | 8   | 16  |
|-----|-----|-----|-----|-----|-----|
| $\delta$ | 0.3506 | 0.0719 | 0.0704 | 0.0629 | 0.0624 |

**Test 2.** We investigate the influence of smoothness of the sought for field and the number of iterations at $\delta$. The discretization of the ray transform with respect to the variables $s, \varphi$ is $300 \times 300$, $\varphi_{\text{max}} = 1.4$, $A = 2$. As test fields, we take six vector fields generated by the
potentials $\psi_0(x,y) - \psi_5(x,y)$. Fig. 1 shows the dependence of $\delta$ on the number of iterations of the Papoulis-Gerchberg method. The horizontal axis corresponds to the number of iterations, the vertical axis corresponds to the value of $\delta$. The correspondence between the color of the lines and the potentials is the following: red — for $\psi_0$, orange — for $\psi_1$, yellow — for $\psi_2$, green — for $\psi_3$, blue — for $\psi_4$ and purple — for $\psi_5$.

![Figure 1. The dependence of $\delta$ on the number of iterations (test 2).](image1)

![Figure 2. The dependence of $\delta$ on $\varphi_{\text{max}}$ (test 3).](image2)

We detect the following behavior of a relative error for all the fields: the value of $\delta$ decreases to a certain value with increasing the number of iterations, after this we see almost imperceptible increase of $\delta$. Wherein the number of iteration at which $\delta$ reaches its minimum values increases with increasing the smoothness of the field (apart from a discontinuous field). We consider four decimal places in the decimal notation of $\delta$. By $N_{\text{opt}}$ we denote the first number of iterations at which $\delta$ reaches its minimum. Table 2 shows the dependence of $N_{\text{opt}}$ and $\delta$ on the smoothness of a test field.

| $\psi$   | $N_{\text{opt}}$ | $\delta$   |
|----------|------------------|------------|
| $\psi_5$ | 37               | 0.1409     |
| $\psi_4$ | 23               | 0.1212     |
| $\psi_3$ | 30               | 0.1406     |
| $\psi_2$ | 37               | 0.1575     |
| $\psi_1$ | 45               | 0.1699     |
| $\psi_0$ | 53               | 0.1686     |

**Test 3.** We investigate the influence of $\varphi_{\text{max}}$ at $\delta$. The test field is generated by the potential $\psi_4$ and has continuous components. The discretization of the ray transform with respect to the variables $s$, $\varphi$ is $300 \times 300$, $A = 2$. Fig. 2 shows the dependence of $\delta$ on the value of $\varphi_{\text{max}}$. The horizontal axis corresponds to $\varphi_{\text{max}}$ from $\pi/2.5 \approx 1.2566$ to $\pi/2 \approx 1.5708$ (the case of complete data). The vertical axis corresponds to the value of $\delta$. We see decreasing $\delta$ with increasing $\varphi_{\text{max}}$, as expected.

Fig. 3 (a, b) shows components of the test vector field, its longitudinal ray transform with limited data (c) and a component of the reconstruction obtained using the Papoulis-Gerchberg method (d, e).

**Test 4.** In the last test, we show the influence of the number of directions $N_{\varphi}$ at $\delta$. The test field is generated by the potential $\psi_4$, $\varphi_{\text{max}} = \pi/2.22 \approx 1.4137$, $A = 2$. The dependence of $\delta$ on $N_{\varphi}$ is shown in Table 3. The first row of the Table corresponds to the values of $N_{\varphi}$, the second
Figure 3. The components of a solenoidal vector field with the potential $\psi_4$ (a, b), the values of its ray transform (c) and its reconstruction (d, e).

row corresponds to the values of $\delta$. The minimum value of $\delta$ is reached at $N_\phi = 25$. An increase of $N_\phi$ leads to increasing $\delta$. This circumstance may be associated with the overdeterminedness of the problem with complete data and the error accumulation of the numerical calculation. The value $N_\phi = 25$ is considered to be optimal for the discretization of the ray transform.

| $N_\phi$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 300 |
|---------|----|----|----|----|----|----|----|----|----|-----|
| $\delta$ | 0.5494 | 0.2921 | 0.1790 | 0.0629 | 0.1260 | 0.0994 | 0.1506 | 0.0931 | 0.0811 | 0.100 |

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