Factorization dynamics and Coxeter-Toda lattices

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Abstract

It is shown that the factorization relation on simple Lie groups with standard Poisson Lie structure restricted to Coxeter symplectic leaves gives an integrable dynamical system. This system can be regarded as a discretization of the Toda flow. In case of $SL_n$ the integrals of the factorization dynamics are integrals of the relativistic Toda system. A substantial part of the paper is devoted to the study of symplectic leaves in simple complex Lie groups, its Borel subgroups and their doubles.

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Introduction

An integrable Hamiltonian system on a symplectic manifold consists of a Hamiltonian that generates the dynamics together with a Lagrangian fibration on the manifold such that the flow lines generated by the Hamiltonian are parallel to the fibers. Usually, the fibers are level surfaces of functions called higher integrals. The fibration by level surfaces is Lagrangian when the integrals Poisson commute and the flow lines are parallel to the fibers when the integrals Poisson commute with the Hamiltonian.

The level surfaces of the integrals are equipped with natural affine coordinates in which the dynamics is linear \cite{Arn89, HZ94}. Integrable systems on Poisson Lie groups have the following characteristic features:

- The phase space of such system is a symplectic manifold which is a symplectic leaf of a factorizable Poisson Lie group $G$.
- The level surfaces of integrals are $G$-orbits with respect to the adjoint action of the group on itself.

One should notice that for some symplectic leaves the $G$-invariant functions do not form complete set of Poisson commuting integrals (their level sets are not Lagrangian submanifolds, but only co-isotropic). In such cases still there is a complete system of integrals, but the complimentary integrals may have singularities. An example is so-called full Toda system \cite{Kos79, DLNT86}.

Since symplectic leaves of Poisson Lie group $G$ are connected components of orbits of the dressing action of the dual Poisson Lie group $G^*$ on $G$, the invariant tori of such systems lie in the intersection of $Ad_G$ and $G^*$-orbits in $G$.

Surprisingly enough, most of the known integrable systems on Poisson Lie groups are of this type. Such integrable systems have a Lax representation. Systematic treatment of such integrable systems was done by Semenov-Tian-Shanskii \cite{STS85}. Linearization of this construction in a neighborhood of identity gives the similar construction based on Lie algebras which has been pioneered by Kostant \cite{Kos79} on the example of Toda lattices and by Adler \cite{Adl79} on the example of KdV equation.

An integrable discrete dynamical system on a symplectic manifold is a symplectomorphism which acts parallel to fibers of a Lagrangian fibration given by level surfaces of integrals. More generally, it can be a Poisson relation preserving the fibration, for details see \cite{Ves91}.
In this paper we derive integrable systems related to Toda models \([\text{Tod88}]\) (and references therein). We show that for simple Lie groups the factorization relation restricted to symplectic leaves that are associated with a Coxeter element in the Weyl group yields a discrete integrable evolution. Such a dynamical system will be called \textit{Coxeter-Toda lattice} and the dynamics \textit{factorization dynamics}. It turns out that different choices of Coxeter element produce isomorphic integrable systems. The integrals for the factorization dynamics are in case of \(G = SL_n\) the integrals of so-called relativistic Toda lattice introduced in \([\text{Rui90}]\). (Since we will deal only with simple Lie groups with standard Poisson Lie structure we can avoid going into the general discussion of factorizable Poisson Lie groups.) The phase space of the Coxeter-Toda lattice is the symplectic leaf mentioned above. On a Zariski open subset of such a leaf which is isomorphic to \(\mathbb{C}^{2r}\) one can introduce coordinates \(\chi_i^\pm\), \(i = 1, \ldots, r = \text{rank}G\) with the following Poisson brackets:

\[
\{\chi_i^+, \chi_j^+\} = 0, \\
\{\chi_i^+, \chi_j^-\} = -2d_iC_{ij}\chi_i^+\chi_j^-.
\]

Here \(C_{ij}\) is the Cartan matrix of \(G\) and the \(d_i\) co-prime positive integers symmetrizing it.

The factorization relation restricted to a Coxeter symplectic leaf gives a symplectomorphism which acts on coordinates \(\chi_i^\pm\) as follows

\[
\alpha(\chi_i^+) = \chi_i^- \\
\alpha(\chi_i^-) = \frac{\chi_i^+}{\chi_i^-} \prod_{j=1}^r (1 - \chi_j^-)^{-C_{ji}}.
\]

This symplectomorphism is integrable. We will call it the discrete Toda evolution. Its integrals have the following description in terms of characters of finite dimensional representations of \(G\).

Let \(x_i^-, h_i, x_i^+\) be Chevalley generators of the Lie algebra \(g = Lie(G)\) and \(\varphi_i : SL_2(i) \subset G\) be the natural embedding of the \(SL_2\) subgroup generated by the elements \(x_i^-, h_i, x_i^+\) corresponding to the simple root \(\alpha_i\). For a Coxeter element \(w\) of the Weyl group \(W\) of \(G\) fix a reduced decomposition \(w = s_{i_1} \cdots s_{i_r}\) where \(r = \text{rank}(g)\) and define the element of \(G\)

\[
g = \prod_{j=1}^r \left(\frac{\chi_j^+}{\chi_j^-}\right)^{h_j} \exp(-\chi_i^+x_i^+)\exp(x_i^-)\cdots\exp(-\chi_i^+x_i^+)\exp(x_i^-)
\]

Here \(\{h_j\}_{j=1}^r\) are elements of the Cartan subalgebra of \(g\) corresponding to fundamental weights, \(h_i = \sum_{j=1}^r C_{ij}h^j\).
The functions

\[ Ch_V(\chi^+, \chi^-) = Tr_V(g) \]  

(1)

where \( V \) is a finite dimensional representation of \( G \) form Poisson commutative subalgebra in the algebra of functions the phase space. They are the integrals of the map \( \alpha \). The characters of fundamental irreducible representations of \( G \) generate the subalgebra of integrals.

Consider the function

\[ H_d(\chi^\pm) = \frac{1}{2}(\xi, \xi) \]

where \( g = exp(\xi) \) and \((.,.)\) is the Killing form on Lie \( G \). The Hamiltonian flow generated by this function interpolates the map \( \alpha \).

For \( G = SL_n \) the integrals (1) are the integrals of so-called relativistic Toda lattice [Rui90]. In a neighborhood of the identity these integrals turn into the integrals of the (usual) Toda lattice. In the same sense as a Lie algebra can be regarded as a linearization of a Lie group, the usual Toda lattices are linearizations of Coxeter-Toda lattices.

Integrable discretizations of Toda lattices have been discovered by Hirota [Hir77] who studied solitonic aspects of them (see also [DJM82]). Later they were re-derived in [Sur90, Sur91b] from discrete time version of a Lax pair. The Hamiltonian interpretation based on classical \( r \)-matrices was derived in [Sur91a] and generalized to Toda systems related to all classical Lie groups (and their affine extensions). In [KR97] a discrete version of Toda field theory was described together with the Hamiltonian structure and its quantization.

The role of matrix factorization in discrete integrable systems was noticed quite some time ago. The references include [Sym82, QNCvdL84, MV91, DLT89].

The primary goal of this article is not to produce new discrete integrable systems (although those related to exceptional Lie groups are new) but rather to demonstrate how the discrete Toda evolution together with its integrals (1) can be derived in a systematic way from the geometry of Poisson Lie groups, and from the factorization relation.

A large part of this paper is devoted to the study of the phase space of these systems. This requires the careful study of symplectic leaves of \( B \) (a Borel subgroup in a simple algebraic Lie group \( G \) with the standard Poisson Lie structure) and of its double.

In section 1 we remind basic facts about Poisson Lie groups and describe the factorization dynamics on factorizable Poisson Lie groups. Section 2 contains the analysis of symplectic leaves of simple complex algebraic groups \( G \) with a standard Poisson Lie structure. In section 3 we describe symplectic
leaves of the Borel subgroup \( B \) of a simple Poisson Lie group \( G \). Section 4 contains the description of symplectic leaves of the double of \( B \) and of how they are related to symplectic leaves of \( B \) and of \( G \). The factorization dynamics on Coxeter symplectic leaves is studied in section 5. The interpolating flow and the relation to the (usual) Toda lattices is described in section 6. In the conclusion we point out what may be done next in this direction.

1 Basic facts about simple Poisson Lie groups

1.1 Basic facts about Poisson Lie groups

A Poisson Lie group is a Lie group equipped with a Poisson structure which is compatible with the group multiplication.

There is a functorial correspondence between connected, simply connected Poisson Lie groups and Lie bialgebras [Dri87]. The Lie bialgebra corresponding to a given Poisson Lie group is called tangent Lie bialgebra.

The dual of a Lie bialgebra \( p \) is the dual vector space \( p^* \) equipped with the Lie bracket dual to Lie cobracket of \( p \) and with the Lie cobracket dual to Lie bracket on \( p \). The dual \( P^* \) of a Poisson Lie group \( P \) is, by definition, the connected, simply connected Poisson Lie group having the dual \( p^* \) of the Lie bialgebra \( p \) corresponding to \( P \) as Lie bialgebra. Denote by \( p^{*\text{op}} \) the Lie bialgebra \( p^* \) with opposite cobracket (which is minus the original cobracket).

The double \( D(p) \) of \( p \) is the direct sum \( p \oplus p^{*\text{op}} \) as a Lie coalgebra and its Lie bracket is determined uniquely by the requirement that the natural inclusions \( i : p \to D(p) \) and \( j : p^{*\text{op}} \to D(p) \) (into the first and second summand, respectively) are Lie bialgebra homomorphisms.

The double \( D(P) \) of \( P \) is the connected, simply connected Poisson Lie group having \( D(p) \) as its Lie bialgebra. The maps \( i \) and \( j \) lift to injective Poisson maps \( i : P \to D(P) \), \( j : P^{*\text{op}} \to D(P) \) and consequently to a map \( \mu \circ (i \times j) : P \times P^{*\text{op}} \to D(P) \) : \( (x, y) \mapsto i(x)j(y) \) which is also a local Poisson isomorphism. By a local isomorphism we mean an isomorphism between neighborhoods of the identity.

A symplectic leaf of a Poisson manifold is an equivalence class of points which can be joined by piecewise Hamiltonian flow lines. When the Poisson manifold is a Poisson Lie group \( P \), there is another description of these leaves which involves the dressing action of the dual Poisson Lie group on \( P \).

The Poisson Lie group \( P^* \) acts on \( D(P) \) via left multiplication, \( y \cdot x := j(y)x \). We also have a map \( \varphi : P \to D(P)/j(P^{*\text{op}}) \) which is the composition of \( i \) with the natural projection. In a neighborhood of the identity this map \( \varphi \) is a Poisson isomorphism and induces dressing action of \( P^* \) on \( P \) [STS85]. The
map \( \varphi \) is a finite cover and has open dense range.

The symplectic leaves of \( P \) are orbits of dressing action of \( G^{\text{op}} \) and are connected components of preimages of left \( P^\ast \)-orbits in \( D(P)/j(P^\ast) \).

Among the cases which have been investigated we point out the following three, \( P = G \) (a complex connected and simply connected simple Lie group with standard Poisson structure), \( P = B \) a Borel-(Poisson)-subgroup of \( G \), and \( P = K \) the compact real form of \( G \).

For \( P = K \), the double, which can be identified with \( G \) as a real group, is globally isomorphic to \( K \times K^{\text{op}} \) as a real manifold via Iwasawa factorization. The map \( \varphi \) in this case is a global Poisson isomorphism [LW90]. There is particular simple relation between the Bruhat decomposition of \( K \) and its symplectic leaves [Soi90, LW90]. It is worth noticing that as the double of \( K \) the complex simple Lie group is equipped with real Poisson structure which is different from the standard Lie Poisson structure on \( G \).

In the first two cases, which are the ones we shall consider in detail below, the double is only locally isomorphic to \( P \times P^{\text{op}} \). The symplectic leaves of \( G \) have been studied in [HL93]. Symplectic leaves for \( B \) were described in [DCKP95]. We reproduce the results of [HL93] [DCKP95] below but will describe symplectic leaves in \( G \) and \( B \) more explicitly.

### 1.2 Standard Poisson structure on a simple Lie group

Let \( G \) be a simple complex Lie group. Fix a labeling of the nodes on the Dynkin diagram associated with the Lie algebra \( \text{Lie} \, G \) by integers \( i = 1, \ldots, r = \text{rank}(G) \). Assign the simple root \( \alpha_i \) to the node labeled by \( i \). Let \( C \) be the Cartan matrix, that is,

\[
C_{ji} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}.
\]

Denote by \( d_i \) the length of \( i \)-th simple root, then \( d_i C_{ij} = d_j C_{ji} \).

Fix a Borel subgroup \( B \subset G \). This fixes the polarization of the root system and together with the enumeration of nodes of Dynkin diagram fixes the generators of the Lie algebra \( \text{Lie} \, G \) \( \{ h_i, x_i^\pm \}_{i=1,\ldots,r} \) corresponding to simple roots of \( \text{Lie} \, G \). The determining relations for these generators are:

\[
\begin{align*}
[h_i, h_j] &= 0, \\
[x_i^+, x_j^-] &= \delta_{ij} h_i, \\
[h_i, x_j^\pm] &= \pm C_{ij} x_j^\pm \\
ad(x_i^\pm)^{1-C_{ij}} x_j^\pm &= 0, \quad i \neq j.
\end{align*}
\]
The standard Lie bialgebra structure on Lie $G$ compatible with the chosen Borel subgroup $B$ is given by the cobracket acting on generators as follows

$$\delta(h_i) = 0, \quad \delta(x_i^\pm) = d_i x_i^\pm \wedge h_i.$$ 

This induces the Poisson Lie structure on $G$ for which the Lie bialgebra described above is the tangent Lie bialgebra. The Borel subgroup $B$ and its opposite $B^-$ are Poisson Lie subgroups.

The Lie bialgebra $\text{Lie}(G)$ is isomorphic to the double of the Lie bialgebra $\text{Lie}(B)$ quotiented by the diagonally embedded Cartan subalgebra [Dri87].

We denote by $N$ and $N^-$ the nilpotent subgroups of $B$ and $B^-$, respectively. Since $H = B \cap B^-$ we have two natural projections and isomorphisms $\theta : B \to B/N \cong H$ and $\theta^- : B^- \to B^-/N^- \cong H$. We shall also write $B^+$ and $N^+$ for $B$ and $N$, respectively.

2 Symplectic leaves of $G$

2.1 Bruhat decomposition of the double of $G$

A simple Lie group $G$ with fixed Borel subgroup $B$ admits Bruhat decomposition with respect to $B$:

$$G = \bigcup_{w \in W} BwB$$

Here $BwB \overset{\text{def}}{=} B\tilde{w}B$ where $\tilde{w}$ is a representative of $w \in N_G(H)/H$ in $N_G(H)$ (clearly $B\tilde{w}B$ depends only on the class $w \in N_G(H)/H$).

There is also a Bruhat decomposition of $G$ with respect to $B^-$:

$$G = \bigcup_{w} B^- w B^- .$$

Recall [KS98] that the double $D(G)$ is, as a group, isomorphic to $G \times G$. The cell decompositions of $G$ therefore give the Bruhat decomposition of $D(G)$ with respect to $D^- = B^- \times B$:

$$D(G) = \bigcup_{(w_1, w_2) \in W \times W} D^-(w_1, w_2)D^-,$$

$D^-(w_1, w_2)D^- = B^- w_1 B^- \times B w_2 B$, where $W \times W = N_{D(G)}(H \times H)/H \times H$ is the Weyl group of $D(G)$. We can also represent $D^- \subset D(G)$ as

$$D^- = (H \times H)(N^- \times N^+) = (N^- \times N^+)(H \times H) .$$
Then for the Bruhat cell $D^{-}(w_1, w_2)D^{-}$ we can write
\[
D^{-}(w_1, w_2)D^{-} = (N_{w_1}^{-} \times N_{w_2}^{+})(H \times H)(\dot{w}_1, \dot{w}_2)D^{-}
\] (2)
where $N_{w}^{\pm} = \{n \in N_{w_{1}}^{\pm} | \dot{w}^{-1}n\dot{w} \in N_{w}^{\pm}\}$ (clearly this definition of $N_{w}^{\pm}$ does not depend on the choice of $\dot{w}$).

2.2 Left cosets $D(G)/j(G^{-})$

Let $G^{-} = G^{op}$ which may be identified with $\{(b^{-}, b) \in B^{-} \times B | \theta^{-}(b^{-}) = \theta(b)^{-1}\}$, a subgroup of $B^{-} \times B$. We write $j : G^{-} \hookrightarrow B^{-} \times B$ for this identification. There is a natural isomorphism:
\[
D^{-}/j(G^{-}) \simeq H.
\] (3)
The group $H \times H$ acts on cosets $(\dot{w}_1, \dot{w}_2)D^{-}/j(G^{-})$ by left multiplication:
\[
(h, h')(\dot{w}_1, \dot{w}_2)(b^{-}, b)j(G^{-}) = (\dot{w}_1, \dot{w}_2)(h_{w_1}b^{-}, h'_{w_2}b)j(G^{-}) = (\dot{w}_1, \dot{w}_2)(Ad_{h_{w_1}}b^{-}, h'_{w_2}h_{w_1}(Ad_{h_{w_1}^{-1}}b)j(G^{-})
\]
Here $(b^{-}, b) \in B^{-} \times B$ and we write $h_{w} = \dot{w}^{-1}h\dot{w}$. Using also (3) we conclude that this action has stationary subgroup
\[
H^{w_1, w_2} = \{(h, h') \in H \times H | h_{w_1} = h'_{w_2}^{-1}\}.
\] (4)
Thus, we have an isomorphism $D^{-}(w_1, w_2)D^{-}/j(G^{-}) \cong N_{w_1}^{-} \times N_{w_2}^{+} \times H$ and, in particular, $\dim(D^{-}(w_1, w_2)D^{-}/j(G^{-})) = l(w_1) + l(w_2) + r$.

2.3 Double cosets $j(G^{-})\backslash D(G)/j(G^{-})$

For double cosets, we have
\[
j(G^{-})(n_{w_1}^{-}, n_{w_2}^{+})(\tilde{h}_1, \tilde{h}_2)(\dot{w}_1, \dot{w}_2)(b^{-}, b)j(G^{-}) = j(G^{-})(\tilde{h}_1, \tilde{h}_2)(\dot{w}_1, \dot{w}_2)j(G^{-})
\]
where $\tilde{h}_1 = h_1\theta^{-}(b^{-})_{w_1}^{-1}, \tilde{h}_2 = h_2\theta^{-}(b^{-})_{w_2}^{-1}$. The set of such double cosets is according to (4) isomorphic to
\[
j(H)\backslash H \times H / j_{w_1w_2}(H)
\] (5)
where $j(H) \subset H \times H$ is the subgroup that consists of elements $(h, h^{-1})$, $h \in H$ and $j_{w_1w_2}(h) = (h_{w_1}, h_{w_2}^{-1})$. The coset of $(h_1, h_2) \in H \times H$ in (5) is the
set \{(h, h^{-1})(1, h_1h_2h''h''^{-1}_{w^{-1}_2w_1})| h, h'' \in H\}. Thus (3) is isomorphic to \(H_{w_2^{-1}w_1}\), where \(H_w\) is the space of \(H\)-orbits on \(H\) with respect to the action

\[ h : h' \rightarrow h'h^{-1}. \]  

(6)

All orbits are naturally isomorphic and we denote the one through 1 by \(H^w\). Furthermore, \(H_w\) is isomorphic to \(\ker (w_2^{-1}w_1 - id) = \{h \in H|h_{w_2^{-1}w_1} = h\}\). Thus, we proved:

**Proposition 1** We have an isomorphism

\[ j(G^-)\backslash D^-(w_1, w_2)D^-/j(G^-) \simeq H_{w_2^{-1}w_1}. \]

(7)

Each \(j(G^-)\) orbit corresponding to an element of this set is isomorphic to

\[ N^-_{w_1} \times N^+_w \times H_{w_2^{-1}w_1} \]

(8)

In particular, each such orbit has the dimension

\[ \ell(w_1) + \ell(w_2) + \dim(\coker (w_2^{-1}w_1 - 1)). \]

Notice that the isomorphism (7) and the isomorphisms between \(j(G^-)\)-orbits and sets (8) are not canonical but depend on the choice of representatives \(\dot{w}_1, \dot{w}_2\). What we really have here is a fiber bundle

\[ D^-(w_1, w_2)D^-/j(G^-) \rightarrow j(G^-)\backslash D^-(w_1, w_2)D^-/j(G^-) \]

(9)

over the torus \(H_{w_2^{-1}w_1}\) whose fibers are \(j(G^-)\)-orbits.

### 2.4 Symplectic leaves of \(G\) and double Bruhat cells

Double Bruhat cells are defined as intersections of \(B\)-Bruhat cells and \(B^-\)-Bruhat cells:

\[ G^{w_1,w_2} = B^-w_1B^- \cap Bw_2B. \]

It is known that \(\dim(G^{w_1,w_2}) = l(w_1) + l(w_2) + r\) (for example \([FZ99]\)).

Let \(\varphi : G \rightarrow D \rightarrow D/j(G^-)\) be the composition of diagonal embedding with the natural projection. According to (2) we have

\[ D^-(w_1, w_2)D^-/j(G^-) \cong (N^-_{w_1} \times N^+_w)(\dot{w}_1, \dot{w}_2)i(H). \]

Define

\[ \Gamma := \{\epsilon \in H|\epsilon^2 = 1\} \]
Theorem 1 The restriction of $\varphi$ to $G^{w_1,w_2} \to D^-(w_1,w_2)D^-/j(G^-)$ is a cover map with group of deck transformations $\Gamma$. Its image is a dense open subset.

Here is the outline of the proof. Let $g = n_{w_1}^- w_1 b^- = n_{w_2}^+ w_2 b^+ \in B^- w_1 B^- \cap Bw_2 B$ where $n_{w_1}^- \in N_{w_1}^-$, $n_{w_2}^+ \in N_{w_2}^+$ and $b^\pm \in B^\pm$. Then we have

$$\varphi(g) = (g, g) j(G^-) = (n_{w_1}^- w_1 b^-, n_{w_2}^+ w_2 b^+) j(G^-).$$

Therefore $\varphi(g)$ is an element of $D^-(w_1, w_2)D^-/j(G^-)$.

Conversely, assume $x_1 = n_{w_1}^- w_1 b^-$ and $x_2 = n_{w_2}^+ w_2 b^+$ then $(x_1, x_2) j(G)$. This class has a representative of the form $(g, g) j(G)$ if and only if there exists $(\eta_+, \eta_-) \in G_-$ such that $n_{w_1}^- w_1 \eta_- = n_{w_2}^+ w_2 \eta_+$. According to [FZ99] such elements exist on a dense open subset of $N_{w_1}^- \times N_{w_2}^+$. Therefore the image of $\varphi$ is open dense in $D^-(w_1, w_2)D^-/j(G^-)$. Furthermore

$$\varphi(g \varepsilon) = (g \varepsilon, g \varepsilon^{-1}) j(G^-) = \varphi(g)$$

for each $\varepsilon \in \Gamma$. This shows that $\Gamma$ acts fixed point freely on the preimages of points. Since $i(\Gamma) = i(H) \cap j(H)$ is the kernel of $\varphi$, $\Gamma$ is the group of deck transformations.

Since the symplectic leaves in $G$ are connected components of preimages of $j(G^-)$-orbits in $D(G)/j(G^-)$ we obtain the following description of leaves.

Corollary 1 The double Bruhat cell $G^{w_1,w_2}$ is a collection of symplectic leaves of $G$ each one being a connected component of the preimage of a double coset $j(G^-) \backslash D^-(w_1, w_2)D^-/j(G^-)$.

We will describe symplectic leaves which belong to $G^{w_1,w_2}$ more explicitly later using Hamiltonian reduction.

3 Symplectic Leaves of $B$

3.1 $B^-$ double cosets in $D(B)$

The double of a Borel subgroup $B$ of $G$ is isomorphic to $G \times H$ as a group [KS98]. Furthermore, $B^{\text{op}} \cong B^-$, sitting inside $G \times H$ as $j : B^- \to G \times H : j(b^-) = (b^-, \Theta^-(b^-))$. In particular, $D(B)$ has the following cell decompositions

$$D(B) = G \times H = \bigcup_{w \in W} B^- w B^- \times H = \bigcup_{w \in W} BwB \times H .$$

(10)
Denote $D(B)_w = B^- w B^- \times H$ and $D(B)^w = B w B \times H$. For the quotient $D(B)/\underline{j}(B^-)$ we have

$$D(B)/\underline{j}(B^-) = \bigcup_{w \in W} (B^- w B^- \times H)/\underline{j}(B^-) \cong \bigcup_{w \in W} N^- w \times H.$$  

Let us compute double cosets:

$$\underline{j}(B^-)(b^- \hat{w}^-, h)\underline{j}(B^-) = \underline{j}(B^-)(h b^- \hat{w}^-, 1)\underline{j}(B^-) = \underline{j}(B^-)(h h^- \hat{w}^-, 1)\underline{j}(B^-) = \underline{j}(B^-)(\hat{w} h', 1)\underline{j}(B^-) \cong \underline{j}(H)(\hat{w} h', 1)\underline{j}(H).$$

Clearly $\underline{j}(h) = (h, h^{-1}) \in H \times H \subset G \times H$. Therefore we have the isomorphism

$$\underline{j}(B^-)\backslash D(B)_w/\underline{j}(B^-) \cong H_w$$

where, we recall, $H_w$ is the space of $H$-orbits on $H$ for the action $\underline{j}(B^-)$. In particular, $\dim(H_w) = \dim(\ker(w - id))$.

Choose a point $(n^- w \hat{w}, h)$ in $D(B)$ representing an equivalence class in $D(B)/\underline{j}(B^-)$. The left $\underline{j}(B^-)$ orbit passing through this point is the set of elements

$$\{(b^- n^- w \hat{w}, \theta(b^-)^{-1} h)\underline{j}(B^-), b^- \in B^- \}$$

$$= \{(\tilde{n}^- w \hat{w} \theta(b^-) w, \theta(b^-)^{-1} h)\underline{j}(B^-) \mid b^- \in B^-, b^- n^- w = \tilde{n}^- w \theta(b^-) \}$$

$$= \{(\tilde{n}^- w \hat{w}, \theta(b^-)^{-1} \theta(b^-) w h)\underline{j}(B^-) \mid b^- \in B^-, b^- n^- w = \tilde{n}^- w \theta(b^-) \}$$

Thus, this orbit is isomorphic to $N^- w \times H^w$.

Finally, since these orbits are isomorphic to $(\mathbb{C}^{\ell(w)} \times (\mathbb{C}^\times)^{\dim(\ker(w - id))})$ we proved the theorem:

**Theorem 2** Each $\underline{j}(B^-)$-orbit in $D(B)_w/\underline{j}(B^-)$ is isomorphic to $(\mathbb{C}^{\ell(w)} \times (\mathbb{C}^\times)^{\dim(\ker(w - id))})$.

Similar to the case of $G$, the isomorphisms are not canonical but we have a fiber bundle $D(B)_w/\underline{j}(B^-) \to \underline{j}(B^-)\backslash D(B)/\underline{j}(B^-)$ whose fibers are the $\underline{j}(B^-)$ orbits.
3.2 Factorization of left cosets

For \( w \in W \) define the subset \( B_w = B \cap B^- w B^- \). Fix a reduced decomposition

\[
w = s_{i_1} \ldots s_{i_{\ell(w)}}
\]

where \( \ell(w) \) is the length of \( w \). Consider the subset

\[
B_{i_1, \ldots, i_{\ell(w)}} = B_{s_{i_1}} \ldots B_{s_{i_{\ell(w)}}} \subset B
\]

where \( B_{s_i} = B(i) \cap B(i)^- s_i B^-(i) \) and \( B(i) = B \cap SL_2(i) \) is the intersection of the Borel subgroup in \( G \) and of the \( SL_2 \)-subgroup generated by the \( i \)-th simple root. The set \( B_{i_1, \ldots, i_{\ell(w)}} \) is the image of \( B_{i_1} \times \ldots \times B_{i_{\ell(w)}} \) under the multiplication in \( G \).

For \( w \in W \) define numbers of “repetitions”

\[
n_i = \{ \# \text{ of } i \text{ in the sequence } \{i_1, \ldots, i_{\ell(w)}\} \}
\]

and define the support of \( w \) as \( I(w) = \{ i \mid 1 \leq i \leq r, \ n_i \neq 0 \} \).

If \( n_i \geq 1 \) consider the following action of \((\mathbb{C}^\times)^{n_i-1}\) on \( B_{i_1} \times \ldots \times B_{i_{\ell(w)}}\):

\[
(x_1, \ldots, x_{n_i-1}) : (b_{i_1}, \ldots, b_{i_{\ell(w)}}) \mapsto (\ldots, b_i \varphi_i(x_1), \ldots, \text{Ad}_{\varphi_i(x_1)}(b_j), \ldots, \varphi_i(x_1)^{-1} b_i \varphi_i(x_2), \ldots, \text{Ad}_{\varphi_i(x_2)}(b_k), \ldots, \varphi_i(x_2)^{-1} b_i \varphi_i(x_3), \ldots)
\]

Here \( \varphi_i : \mathbb{C}^\times \hookrightarrow SL_2 \hookrightarrow G \) is the composition of embedding, \( \mathbb{C}^\times \) into \( SL_2 \) as the (complex) Cartan subgroup and \( SL_2 \) into \( G \) as the \( i \)-th \( SL_2 \)-triple. It is clear that for different \( n_i, n_j \), both greater than 1, the corresponding actions commute so that \( w \) gives rise to an action of the torus \( J \), the product of all \((\mathbb{C}^\times)^{n_i-1}\), over \( i \) with \( n_i > 1 \).

**Proposition 2** The multiplication map \( B_{s_{i_1}} \times \ldots \times B_{s_{i_{\ell(w)}}} \to B_{i_1, \ldots, i_{\ell(w)}} \) commutes with the \( J \)-action, assuming \( J \) acts trivially on \( B_{i_1, \ldots, i_{\ell(w)}} \) and establishes a birational isomorphism \( B_{i_1, \ldots, i_{\ell(w)}} \simeq (B_{s_{i_1}} \times \ldots \times B_{s_{i_{\ell(w)}}})/J \).

Here is the outline of the proof. We can choose the elements for \((\mathbb{C}^\times)^{n_i-1}\) in such a way, that the Cartan parts of the elements \( b_i \) of \( (b_{i_1}, \ldots, b_{i_{\ell(w)}}) \) will all be trivial, all except one. If will do this for each \( i \in I(w) \) we will have cross-section of the action of \( J \). Then it quickly follows that this cross-section is a birational isomorphism.

The support \( I(w) \) of \( w \) defines naturally a sub-diagram of the Dynkin diagram of \( G \) (by deleting all nodes not in \( I(w) \)) and hence a subgroup of \( G \). Let \( B'_w \) be the image in \( G \) of the Bruhat cell corresponding to \( w \) in this subgroup. Then multiplication provides an isomorphism between \( B'_w \times H(w) \) and \( B_w \) where \( H(w) \) is the subgroup of \( H \) corresponding to the simple roots \( \alpha_i \) with \( i \notin I(w) \). The following is known (see for example [FZ99]).
Theorem 3  
• For each \( w \in W \) with fixed reduced decomposition the set \( B_{i_1, \ldots, i_{\ell(w)}} \) is Zariski open in \( B_w' \).

• For each two reduced decompositions \( w = s_{i_1} \cdots s_{i_{\ell(w)}} \) and \( w = s_{j_1} \cdots s_{j_{\ell(w)}} \) there is a birational isomorphism between \( B_{i_1, \ldots, i_{\ell(w)}} \) and \( B_{j_1, \ldots, j_{\ell(w)}} \).

3.3 Symplectic leaves of \( B \)

According to the general theory, symplectic leaves of \( B \) are connected components of preimages of \( j(B^-) \)-orbits in \( D(B)/j(B^-) \) with respect to the map \( \varphi : B \subset D(B) \rightarrow D(B)/j(B^-) \).

Proposition 3 The restriction of \( \varphi \) to \( B_w \) is a covering map \( B_w \rightarrow D(B)_w/j(B^-) \). The corresponding group of deck transformations is isomorphic to \( \Gamma \).

Let us describe the symplectic leaves of \( B_w \) more explicitly, using results of previous subsection.

There is a natural coordinate system in a neighborhood of the identity of the subgroup \( B(i) \) in which the groups elements are written as

\[
\exp (a_i h_i + b_i x_i^+) = \exp(a_i h_i) \exp(b_i' x_i^+),
\]

where \( b_i' = e^{-\alpha_i \frac{b_i}{a_i}} \sinh(a_i) \).

The corresponding global coordinates on \( B(i) \) are \( A_i = e^{a_i}, \quad B_i = b_i \frac{\sinh(a_i)}{a_i} \).

In these coordinates the above element is represented by the \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
A_i & B_i \\
0 & A_i^{-1}
\end{pmatrix}
\]

in two dimensional representation of \( SL_2 \).

The subgroup \( B(i) \) is a Poisson Lie subgroup in \( SL_2(i) \) with the following Poisson brackets between coordinate functions

\[
\{A_i, B_i\} = -d_i A_i B_i.
\]

Here and below we will abuse notations and will denote coordinates and coordinate functions by the same letters.

The symplectic leaves of \( B(i) \) are, one 2-dimensional leaf \( B_{s_i} = \{A_i, B_i \mid A_i \in \mathbb{C}^\times, B_i \in \mathbb{C}^\times\} \), and a 1-dimensional family of zero-dimensional leaves \( \{A_i = t, B_i = 0\} \).

The product \( B_{s_{i_1}} \times \cdots \times B_{s_{i_{\ell(w)}}} \) carries natural product symplectic structure. Since the multiplication map is Poisson, the sub-manifold \( B_{i_1, \ldots, i_{\ell(w)}} \subset B_w' \) is a Poisson sub-manifold. According to Theorem 3, \( B_{i_1, \ldots, i_{\ell(w)}} \) is Zariski open in \( B_w' \) which implies that the symplectic leaves of \( B_{i_1, \ldots, i_{\ell(w)}} \) are Zariski open sub-varieties in the symplectic leaves of \( B \).
The following result, combined with the product Poisson structure on $B_{i_1} \times \ldots \times B_{i_\ell(w)}$ allows to describe symplectic leaves of $B_{i_1,\ldots,i_\ell(w)}$ explicitly via Hamiltonian reduction.

**Proposition 4** The action [11] of $J = (\mathbb{C}^\times)^{\ell(w) - |I(w)|}$ on $B_{i_1} \times \ldots \times B_{i_\ell(w)}$ is Hamiltonian.

Here $|I(w)|$ is the cardinality of the support of $w$. The Hamiltonians generating this action do not commute. This means that the pull-back of the moment map maps the Poisson algebra of functions on $B_{i_1} \times \ldots \times B_{i_\ell(w)}$ to the Poisson algebra of functions on a hyper-plane in the vector space dual to a central extension of Lie algebra $j$. Therefore symplectic leaves of the quotient space $(B_{s_{i_1}} \times \ldots \times B_{s_{i_\ell(w)}})/J$ are preimages of the corresponding coadjoint orbits with respect to the moment map. In other words the symplectic leaves of these quotient spaces can be obtained via Hamiltonian reduction. We will leave the details of this Hamiltonian reduction to separate publication.

As a corollary of this construction one can derive "product formulae" for symplectic leaves similar to those derived in [Soi90] for compact Lie groups.

Below we will consider symplectic leaves corresponding to Coxeter elements. In this case all $n_i = 1$ and so $J$ is trivial. It follows that the product formula doesn’t require the Hamiltonian reduction and a parametrization of the leaf can be given by the coordinates $A_i, B_i$, $i = 1, \ldots, r$.

### 3.4 Coxeter symplectic leaves of $B$

An element of the Weyl groups $W$ is called Coxeter element if its reduced decomposition into the product of simple reflections $w = s_{i_1} \ldots s_{i_\ell(w)}$ does not have repetitions in the sequence of sub-indices and if $l(w) = r$ (i.e. in this product each generator of $W$ appear exactly once). It is not difficult to see that if $w$ is a Coxeter element, $\dim(\text{coker}(w - id))$ is $r$ and therefore the subset $B_w$ is a symplectic leaf of $B$. We will call them Coxeter symplectic leaves.

Let $U_i : B_{s_i} \hookrightarrow B$ be the natural inclusion of $B_{s_i} \subset B(i)$ into $B$. Then any element of $B_{i_1,\ldots,i_\ell(w)}$ can be written as

$U_{i_1}(A_{i_1}, B_{i_1}) \ldots U_{i_r}(A_{i_r}, B_{i_r}).$

Thus, for Coxeter symplectic leaves $A_i, B_i$ (more precisely, their logarithms) are Darboux coordinates.
3.5 Symplectic leaves of $B^-$

Symplectic leaves of $B^-$ can be described similarly to how it was done for $B$. They also can obtained from the ones for $B$ since $B$ is anti-isomorphic to $B^-$ as a Poisson manifold (there is an isomorphism of groups, which maps one Poisson tensor to the negative to the other). Let $C_i, D_i, be coordinates on the lower triangular part of $SL_2(i)$ in which group elements are represented by matrices $L_i(D_i, C_i) = \begin{pmatrix} D_i & 0 \\ C_i & D_i^{-1} \end{pmatrix}$ in the two dimensional irreducible representation of $SL_2$. These coordinate functions have the following Poisson brackets:

$$\{D_i, C_i\} = d_i D_i C_i.$$

Denote by $B^-_{s_i}$ the sub-variety of the lower triangular part of $SL_2(i)$ where $C_i \neq 0$. Fix the Coxeter element $w \in W$ and its reduced decomposition $w = s_{i_1} \ldots s_{i_r}$. On a Zariski open subset of the Coxeter symplectic leaf $B^-_w$ one can introduce the natural coordinates $C_{i}, D_{i}, i = 1, \ldots, r$. Every element of this subset can be written as:

$$L_{i_r}(D_{i_r}, C_{i_r}) \ldots L_{i_1}(D_{i_1}, C_{i_1})$$

where $L_i : B^-_{s_i} \subset B^-$ are natural inclusions.

4 Symplectic leaves of $D(B)$

4.1 Symplectic leaves of $D(B)$

As above, let us identify $D(B)$ with $G \times H$ as a group. The Poisson structure on $D(B) = G \times H$ is not the product structure. Symplectic leaves of $D(B)$ can be described similarly to how it was done for $G$. Since $D(B)$ is a factorizable Poisson Lie group

$$D(D(B)) \simeq D(B) \times D(B).$$

Fix this isomorphism together with the identification $D(B) = G \times H$. This gives the following cell decomposition for $D(D(B))$:

$$D(B) \times D(B) = \bigsqcup_{w'1, w'2} (B^- w_1 B^- \times H) \times (B w_2 B \times H).$$

The Poisson Lie group $D^-(B) = D(B)^{\text{op}}$ can naturally be identified with $B^- \times B$. 
Let $D(B)_w$ and $D(B)^w$ be Bruhat cells of $D(B)$ defined in (10). The double cosets $j(D^-(B)) \setminus D(B)_{w_1} \times D(B)^{w_2}/j(D^-(B))$ can be computed similarly to Proposition 1:

$$j(D^-(B)) \setminus D(B)_{w_1} \times D(B)^{w_2}/j(D^-(B)) \simeq H_{w_1} \times H_{w_2}.$$ (12)

The $D^-(B)$-orbit passing through the coset class of $((\check{w}_1, h_1), (\check{w}_2, h_2)) \in D(B)_{w_1} \times D(B)^{w_2}$ is isomorphic to

$$(N^-_{w_1} \times H^{w_1}) \times (N^+_{w_2} \times H^{w_2}).$$ (13)

Notice that $j(D^-(B))$-orbits in $D(B)_{w_1} \times D(B)^{w_2}/j(D^-(B))$ are isomorphic to the product of corresponding orbits for $B$ and for $B^-$. Again we have a natural fiber bundle

$$(D(B)_{w_1} \times D(B)^{w_2})/j(D^-(B)) \to j(D^-(B)) \setminus D(B)_{w_1} \times D(B)^{w_2}/j(D^-(B))$$

and $j(D^-(B))$-orbits are fibers of this bundle. The connected components of preimages of these fibers under the map $\varphi : D(B) \to (D(B) \times D(B))/j(D^-(B))$ are the symplectic leaves of $D(B)$. Symplectic leaves whose image are orbits in $D(B)_{w_1} \times D(B)^{w_2}/j(D^-(B))$ will be denoted as $S_{w_1,w_2}$.

**4.2 Relation between symplectic leaves of $B$ and $D(B)$**

Embeddings $i : B \hookrightarrow D(B)$ and $j : B^- \hookrightarrow D(B)$ combined with the multiplication and inversion in $D(B)$ give rise to the map

$$I : B \times B^- \to D(B) = G \times H , \quad I(b, b^-) = (b(b^-)^{-1}, \theta(b)\theta^-(b^-))$$ (13)

which is most important to define the factorization relation, see below. The image of this map is Zariski open in $D(B)$. This map is also a Poisson map and therefore maps symplectic leaves of $B \times B^-$ to symplectic leaves of $D(B)$. The intersection of the image of $I$ any of the symplectic leaves is Zariski open in this leaf. This “explains” the formula (12).

**4.3 Relation between symplectic leaves of $D(B)$ and $G$**

The Cartan subgroup $H$ acts naturally on $B \times B^-$ by diagonal multiplication from the right,

$$h(b, b^-) = (bh, b^- h).$$ (14)

The following is clear.
Lemma 1 The map $I$ commutes with the $H$-action

$$I(bh, b^- h) = I(b, b^-)(1, h^2)$$

and induces a Poisson map $\tilde{I}$ between corresponding cosets:

$$\frac{(B \times B^-)}{H} \xrightarrow{\tilde{I}} G \cong \frac{D(B)}{H}$$

Here the coset is taken with respect to the action of $H$ on $D(B)$ by the multiplication by $(1, h^2)$ from the right.

It is also clear that the image of $\tilde{I}$ is Zariski open in $G$ and that $\tilde{I}$ is a birational isomorphism. Since the action (14) is Hamiltonian, symplectic leaves of $(B \times B^-)/H$ can be obtained via Hamiltonian reduction from symplectic leaves of $B \times B^-$. Therefore, symplectic leaves of $G$ can be obtained via Hamiltonian reduction from symplectic leaves of $D(B)$.

Symplectic leaves of $G$ can be also described via Hamiltonian reduction similarly to how it was done for symplectic leaves of $B$. For this consider two elements $u, v \in W$ and fix their reduced decomposition $u = s_{i_1} \ldots s_{i_l}, v = s_{j_1} \ldots s_{j_m}$. Consider the image of

$$B_{s_{i_1}} \times \ldots \times B_{s_{i_l}} \times B_{s_{j_m}}^- \times \ldots \times B_{s_{j_1}}^-$$

under the multiplication and inverse map:

$$G_{i_1, \ldots, i_l, j_1, \ldots, j_m} = B_{s_{i_1}} \ldots B_{s_{i_l}} B_{s_{j_1}}^- \ldots B_{s_{j_m}}^- .$$

The double Bruhat cell $G^{u,v}$ has natural decomposition $G^{u,v} = G^{u,v} \times H(u, v)$ where $H(u, v)$ is the subgroup of $H$ generated by elements corresponding to simple roots which do not belong to $I(u) \cup I(v)$. It follows from [FZ99] that the variety $G_{i_1, \ldots, i_l, j_1, \ldots, j_m}$ is birationally isomorphic to $G^{u,v}$. On the other hand it is also birationally isomorphic to the quotient of

$$B_{s_{i_1}} \times \ldots \times B_{s_{i_l}} \times B_{s_{j_1}}^- \times \ldots \times B_{s_{j_1}}^-$$

with respect to the appropriate Hamiltonian toric action. This allows to construct all symplectic leaves of $G$ via Hamiltonian reduction. We will leave the details of this construction for another publication.

5 Factorization dynamics on Poisson Lie groups
5.1 Dynamics of Poisson relations

Here we will remind basic facts about Poisson relations and their dynamics. Let \((M, p)\) be a Poisson manifold with the Poisson tensor \(p \in \wedge^2 TM\). Denote by \(p^{(2)} \in \wedge^2 T(M \times M)\) the Poisson tensor corresponding to the following product of Poisson manifolds: \((M, -p) \times (M, p)\).

A smooth relation of finite type on a manifold \(M\) is a submanifold \(R \subset M \times M\), such that natural projections \(\pi_1, \pi_2 : M \times M \to M\), \(\pi_1(x, y) = x\), \(\pi_2(x, y) = y\) have a finite number of preimages.

Denote by \(T_{\perp} \mathcal{R}\) the forms on \(M \times M\) which vanish on \(T \mathcal{R} \subset T(M \times M)\).

A smooth relation on \(M\) is called a Poisson relation if \(p^{(2)} \mid_{T_{\perp} \mathcal{R}} = 0\) and \(\dim(R) = \dim(M)\).

If a relation \(R = \{(x, \phi(x)) \mid x \in M\}\) is a graph of a map \(\phi : M \to M\) it is Poisson if and only if \(\phi\) is a Poisson map.

An \(n\)-th iteration of a relation \(R\) on \(M\) is a submanifold \(R^{(n)} \subset M^{\times(n+1)}\) such that

\[
R^{(n)} = \{(x_1, \ldots, x_{n+1}) \mid x_i \in M, (x_i, x_{i+1}) \in R \subset M \times M\}.
\]

A function \(F \in C^\infty(M)\) is called an integral of a smooth relation \(R \subset M \times M\) if

\[
F(x) = F(y) \quad \text{for all} \quad (x, y) \in R.
\]

A smooth relation on a symplectic manifold is Poisson if and only if it is a Lagrangian submanifold in \(M \times M\) (equipped with the product symplectic structure). It is called integrable if there exists \(n\) independent Poisson commuting functions \(I_1, \ldots, I_n\) which are integrals of \(R\).

Similarly one can define Poisson and symplectic relations in an algebro-geometric setting. For more details about the dynamics of symplectic relations see [Ves91].

5.2 Factorization relations on Poisson Lie groups

We will study very specific Poisson relations on Poisson Lie groups which we will call factorization relations.

Let \(P\) be a Poisson Lie group and \(D(P)\) be its double. A factorization relation on \(P \times P^{op}\) is a sub-variety \(\mathcal{F} \subset (P \times P^{op}) \times (P \times P^{op})\), defined as

\[
\mathcal{F} = \{(g^+, g^-), (h^+, h^-) \mid i(g^+)j(g^-)^{-1} = j(h^-)^{-1}i(h^+)\}
\]

where \(i : P \hookrightarrow D(P)\) and \(j : P^{op} \hookrightarrow D(P)\) are the natural inclusions of Poisson Lie groups.
Proposition 5  
• Functions on $D(P)$ which are invariant with respect to the adjoint action of $D(P)$ form a Poisson commutative subalgebra in the Poisson algebra of functions on $D(P)$.

• A function on $P \times P^{op}$ which is the composition of the map $M(i \times j) : P \times P^{op} \to D(P)$, $(g^+, g^-) \mapsto i(g^+)j(g^-)^{-1}$ and of an $Ad$-invariant function on $D(P)$ is an integral of the factorization map.

Part 1 of this proposition is well known [STS85]; part 2 is obvious:

$$f(i(g^+)j(g^-)^{-1}) = f(j(h^-)^{-1}i(h^+)) = f(i(h^+)j(h^-)^{-1})$$

Let $\Sigma_1$ and $\Sigma_2$ be symplectic leaves in $P$ and $P^{op}$ respectively. Restricting the relation $F$ to the symplectic leaf $\Sigma = \Sigma_1 \times \Sigma_2 \subset P \times P^{op}$ we obtain a Poisson relation on $\Sigma$ and central $Ad$-invariant functions on $D(P)$ will produce the integrals of this relation.

It may happen that one can make a Hamiltonian reduction of $\Sigma$ in such a way that on the reduced space we have enough central functions, in a sense that their level surfaces are half of the dimension of the reduced symplectic manifold. In this case the factorization dynamics on $\Sigma$ or on the reduced space will be integrable.

In the next sections we will show that this is exactly what happens with symplectic leaves corresponding to the Coxeter elements. As we will see this gives an integrable system which is a “nonlinear” version of an open Toda system corresponding to the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. It becomes the usual Toda system in a neighborhood of the identity.

Remark One can argue that factorization dynamics is integrable on all (appropriately reduced) symplectic leaves of $P \times P^{op}$ when $P$ is a Borel subgroup of simple Lie group $G$. In a neighborhood of the identity such systems become “complete” Toda systems (corresponding to parabolic subgroups in $G$) [Kos79, DLT89]. But this will be the subject for a separate publication.

6 Factorization dynamics on Coxeter symplectic leaves

6.1 Integrals

Consider a Coxeter symplectic leaf $S_{w,w}$ of $D(B)$ corresponding to the Coxeter element $w$. Fix reduced decomposition $w = s_{i_1} \ldots s_{i_r}$. On Zariski open subset $S'_{w,w}$ of $S_{w,w}$ each element of $G \cap S_{w,w}$ (provided that $G$ is embedded
to $D(B) = G \times H$ as $(G, e))$ can be represented by the product

$$UL^{-1} = U_{i_1} \ldots U_{i_r} L_{i_1}^{-1} \ldots L_{i_r}^{-1}. \quad (15)$$

Here we abbreviated $U_i \equiv U_i(A_i, B_i)$, $L_i = L_i(D_i, C_i)$. This subset depends on the choice of reduced decomposition of $w$. We will suppress this dependence since different reduced decompositions give birationally isomorphic subsets.

For each $i = 1, \ldots, r$ and given reduced decomposition $w = s_{i_1} \ldots s_{i_r}$ define $\{i\} = \{i_\alpha = 1, \ldots r \mid \alpha > \beta, i = i_\beta\}$ and $\{i\} = \{i_\alpha = 1, \ldots r \mid \alpha < \beta, i = i_\beta\}$.

**Proposition 6** The following identities hold

$$UL^{-1} = \prod_i A_i^h U_i(1, \tilde{V}_i) L_{i_1}^{-1}(1, \tilde{W}_{i_1}) \ldots U_{i_r}(1, \tilde{V}_{i_r}) \cdot L_{i_r}^{-1}(1, \tilde{W}_{i_r}) \prod_i D_i^{-h_i},$$

$$UL^{-1} = U_i(1, V_i) \ldots U_{i_r}(1, V_{i_r}) \prod_i (\frac{A_i}{D_i})^h D_i L_{i_1}^{-1}(1, W_{i_1}) \ldots L_{i_r}^{-1}(1, W_{i_r}) \quad (16)$$

where

$$V_i = B_i A_i \prod_{j \in \{i\}} A_j^{C_{ji}}, \quad \tilde{V}_i = B_i A_i^{-1} \prod_{j \in \{i\}} A_j^{-C_{ji}},$$

$$W_i = C_i D_i^{-1} \prod_{j \in \{i\}} D_j^{-C_{ji}}, \quad \tilde{W}_i = C_i D_i \prod_{j \in \{i\}} D_j^{C_{ji}}.$$

The proof of this proposition and of the next lemma is a simple exercise.

**Lemma 2** \(\tilde{V}_i = V_i \prod_j A_j^{-C_{ji}}, \quad \tilde{W}_i = W_i \prod_j D_j^{C_{ji}}.\)

Define variables $\chi_i^\pm, G_i, F_i$ as

$$\chi_i^+ = V_i W_i, \quad \chi_i^- = \chi_i^+ \prod_j (\frac{A_j}{D_j})^{-C_{ji}},$$

$$G_i = \frac{B_i}{C_i} A_i D_i \prod_{j \in \{i\}} A_j^{C_{ji}} \prod_{j \in \{i\}} D_j^{C_{ji}}, \quad F_i = A_i D_i.$$

**Proposition 7** Considered as functions on $S'_{w,w}$, $\chi_i^\pm, F_i,$ and $G_i$ have the following Poisson brackets:

$$\{\chi_i^+, \chi_j^+\} = \{\chi_i^-, \chi_j^-\} = 0,$$

$$\{\chi_i^+, \chi_j^-\} = -2d_i C_{ij} \chi_i^+ \chi_j^-,$$

$$\{\chi_i^+, F_j\} = \{\chi_i^+, G_j\} = 0,$$

$$\{F_i, G_j\} = -2d_i F_i G_j \delta_{i,j}. $$
The proof is a straightforward computation based on the definition of $\chi^\pm_i$, $F_i$, and $G_i$ and on the Poisson brackets between $A_i$, $B_i$, $C_i$, $D_i$:

\[
\{A_i, B_j\} = -d_i \delta_{ij} A_i B_j ,
\{D_i, C_j\} = d_i \delta_{ij} D_i C_j ,
\{A_i, A_j\} = \{A_i, C_j\} = \{A_i, D_j\} = \{B_i, C_j\} = \{B_i, D_j\} = \{D_i, D_j\} = 0 ,
\{B_i, B_j\} = \{C_i, C_j\} = 0 .
\]

Using Proposition 3, the definition of $\chi^\pm_i$ and elementary algebra we arrive at the following

**Proposition 8** Let $V$ be a finite-dimensional representation of $G$ and $\text{Ch}_V$ be its character. Then

\[
\text{Ch}_V(UL^{-1}) = \text{Ch}_V(\prod_{j=1}^{r} (\chi_j^+)^{h_j} \phi_{i_1}(g_{i_1}) \ldots \phi_{i_r}(g_{i_r}))
\]

Here $\phi_i : SL_r(i) \hookrightarrow G$ is the embedding of $SL_2$ generated by $x^+_i$, $h_i$, $x^-_i$ into $G$, $g_i$ and $\bar{g}_i$ are elements of $SL_2$ whose image in 2-dimensional irreducible representation is given by the following weight basis of 2-dimensional irreducible representation:

\[
g_i = \begin{pmatrix} 1 & -\chi^-_i \\ -1 & \chi^-_i \end{pmatrix} , \quad \bar{g}_i = \begin{pmatrix} 1 & \chi^+_i \\ -1 & 1-\chi^+_i \end{pmatrix} .
\]

The element $\{h^i\}$ forms the basis in $\mathfrak{h} \subset \mathfrak{g}$ corresponding to fundamental weights: $h_j = \sum_i C_{ji} h^i$. Observe that $[h^i, X^\pm_j] = \pm \delta_{ij} X^\pm_1$, hence by conjugating $UL^{-1}$ with an element $\exp ah^i$ of $H$ one can alter the off-diagonals of the $g_i's$. This was used in the proof of Proposition 3.

Now let us interpret these two propositions form the point of view of Hamiltonian reduction.

**Proposition 9** (1) Functions $\log G_i$ generate $H$ action $[\mathcal{A}]$ on $S_{w,w}'$.

(2) Functions $\log F_i$ generate the adjoint action of $H$ on $S_{w,w}' \subset D(B)$, $h : (g, h') \mapsto (hgh^{-1}, h')$.

This proposition can be derived immediately from formulae $[\mathcal{B}]$ and from the explicit form of Poisson brackets in terms of coordinates $A_i, B_i, C_i, D_i$. 

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Characters as functions on the group are invariant with respect to the adjoint action. Therefore Proposition \[3\] implies that characters, computed on \(G \cap S_{w,w}'\), do not depend on \(F_i, G_i\) which can be seen also by direct computation (proposition \[5\]).

As it follows from 4.2 we can naturally identify
\[
G \cap S_{w,w}' = S_{w,w}' / H
\]
where the \(H\) action is generated by \(\log F_i\). Level surfaces of functions \(G_i\) are symplectic leaves of \(G \cap S_{w,w}'\) and \(\log \chi^\pm\) are Darboux coordinates on these symplectic leaves. All this is clear from the structure of Poisson brackets in Proposition \[4\].

### 6.2 Factorization map

Consider the map \(\alpha: (\mathbb{C}^\times)^{2r} \to (\mathbb{C}^\times)^{2r}\),
\[
\begin{align*}
\alpha(\chi_i^+) &= \chi_i^- \\
\alpha(\chi_i^-) &= \frac{(\chi_i^-)^2}{\chi_i^+} \prod_j (1 - \chi_j^-)^{-C_{ji}} \\
\alpha(F_i) &= F_i \\
\alpha(G_i) &= G_i \prod_j F_j^{C_{ij}}
\end{align*}
\]
defined outside of the hyper-planes \((\chi_j^- = 1, \chi_i^+ = 0)\). Since we are interested in integrable systems whose Hamiltonians are given by functions on \(G\) invariant under conjugations and since these functions restricted to a Coxeter orbit do not depend on \(F\) and \(G\) variables we will focus on the action of the factorization dynamics on \(\chi^\pm\). Here we will continue the practice of abusing notation and will denote coordinates and coordinate functions by the same letter.

Let \(\text{Ch}_V(\chi^+, \chi^-)\) be functions on \((\mathbb{C}^\times)^{2r}\) as defined in Proposition \[8\].

**Theorem 4** \(\text{Ch}_V(\alpha(\chi^+), \alpha(\chi^-)) = \text{Ch}_V(\chi^+, \chi^-)\).

*Proof:* We will use two formulae for these functions derived in Proposition \[8\].

\[
\text{Ch}_V(\alpha(\chi^+), \alpha(\chi^-)) = \text{Ch}_V\left(\prod_{j=1}^r \left(\frac{\alpha(\chi_j^+)}{\alpha(\chi_j^-)}\right)^{h_j} \prod_i \phi_i \left(\begin{array}{cc}
1 & \alpha(\chi_i^+) \\
-1 & 1 - \alpha(\chi_i^+)
\end{array}\right)\right)
\]
\[
\begin{align*}
\text{Ch}_V \left( \prod_{j=1}^r \left( \frac{X_j^+}{X_j} \right) \prod_{j=1}^r \left( 1 - \chi_j^+ \right) \prod_i \phi_i \left( \begin{array}{cc}
1 & \chi_i^- \\
-1 & 1 - \chi_i^-
\end{array} \right) \right) \\
= \text{Ch}_V \left( \prod_{j=1}^r \left( \frac{X_j^+}{X_j} \right) \prod_i \phi_i \left( \begin{array}{cc}
1 - \chi_i^- & \chi_i^- \\
-1 & 1
\end{array} \right) \right)
\end{align*}
\]

Here the product is taken in the order \((i_1, \ldots, i_r)\). q.e.d.

**Proposition 10** Let \( F \subset S'_w \times S'_w \) be the factorization relation restricted to Coxeter symplectic leaves of \( D(B) \). The diagram

\[
\begin{array}{ccc}
F & \chi_L \searrow \alpha \swarrow \chi_R \\
(\mathbb{C}^*)^{2r} & \to & (\mathbb{C}^*)^{2r}
\end{array}
\]

is commutative. Here \( \chi_L \) is the composition of the projection to the first component in \( S'_w \times S'_w \) and the map \( \chi : (A_i, B_i, C_i, D_i) \mapsto (\chi^+_i) \) and \( \chi_R \) is the composition of the projection to the right component and \( \chi \).

**Proof:** On the image of the factorization map \( I : B \times B \to D(B) \), elements of \( S_{w,w^{-1}} \subset D(B) \) can be represented as

\[
(UL^{-1}, \text{diag}(U)\text{diag}(L))
\]

where \( U \) and \( L \) are as above. The factorization relation \( F \subset S'_w \times S'_w \) consists of points

\[
(UL^{-1}, \text{diag}(U)\text{diag}(L)), (\bar{U}L^{-1}, \text{diag}(\bar{U})\text{diag}(\bar{L}))
\]

satisfying conditions

\[
UL^{-1} = \bar{L}^{-1}\bar{U} \\
\text{diag}(U)\text{diag}(L)) = \text{diag}(\bar{U})\text{diag}(\bar{L})
\]

Let \( U_i, U'_i, U''_i, \bar{U}_i, L_i, L'_i, L''_i, \bar{L}_i \) be factors of \( U, L, \ldots \) satisfying relations

\[
UL^{-1} = U_i \ldots U_i L_i^{-1} \ldots L_i^{-1} = U'_i L'_i^{-1} \ldots U_r' L_r'^{-1}
\]

\[
= L''_{r-1} U''_1 \ldots L''_{r-1} U''_r = \bar{L}^{-1}_1 \ldots \bar{L}^{-1}_r \bar{U}_i \ldots \bar{U}_r
\]
Then the coordinates $A$, $B$, $C$, $D$ of these elements have to satisfy the relations

\[
\begin{align*}
A' &= A_i, \\
B'_i &= B_i \prod_{j \in \{i\}_-} D_{ji}^{c_{ji}} \quad C'_i &= C_i \prod_{j \in \{i\}_+} A_j^{-c_{ji}} \\
A'_i D'_i - B'_i C'_i &= A''_i D''_i - B''_i C''_i \\
C'_i A'_i^{-1} &= C''_i A''_i \\
A'_i D'_i &= A''_i D''_i \\
\tilde{B}_i &= B''_i \prod_{j \in \{i\}_+} D_j^{c_{ji}} \\
\tilde{C}_i &= C''_i \prod_{j \in \{i\}_-} A_j^{-c_{ji}} \\
\tilde{A}_i &= A''_i \\
\tilde{D}_i &= D''_i
\end{align*}
\]

Let us find $\ddot{x}_i^+$ from these relations:

\[
\begin{align*}
\ddot{x}_i^+ &= B_i C_i \frac{\tilde{A}_i}{D_i} \prod_{j \in \{i\}_-} A_j^{-c_{ji}} \prod_{j \in \{i\}_+} D_j^{-c_{ji}} \\
&= B''_i C''_i \prod_{j \in \{i\}_+} D_j^{c_{ji}} \prod_{j \in \{i\}_-} A_j^{-c_{ji}} \frac{A''_i}{D''_i} \prod_{j \in \{i\}_-} A_j^{c_{ji}} \prod_{j \in \{i\}_+} D_j^{-c_{ji}} \\
&= B''_i C''_i \frac{A''_i}{D''_i} B'_i C'_i \frac{D'_i}{A'_i} \\
&= B_i C_i \frac{D_i}{A_i} \prod_{j \in \{i\}_-} D_j^{c_{ji}} \prod_{j \in \{i\}_+} A_j^{-c_{ji}} \\
&= x_i^+ \frac{\tilde{x}_i^-}{\tilde{x}_i^+} = \chi_i^-
\end{align*}
\]

Similarly,

\[
\begin{align*}
\ddot{x}_i^- &= \ddot{x}_i^+ \prod_{j} \left( \frac{D'_j}{A'_j} \right)^{c_{ji}} = \chi_i^- \prod_{j} \left( \frac{D''_j}{A''_j} \right)^{c_{ji}} \\
&= \chi_i^- \prod_{j} \left( \frac{D'_j}{A'_j}(1 - B'_j C'_j \frac{D'_i}{A'_i}) \right)^{-c_{ji}} = \frac{(\chi_i^-)^2}{\chi_i^+} \prod_{j} (1 - \chi_i^-)^{-c_{ji}}
\end{align*}
\]

Here we used the identities $\prod_j (D'_j/A'_j)^{C_{ji}} = \chi_i^-/\chi_i^+$ and $B'_j C'_j D'_j/A'_j = \chi_j^-$. This proves the Proposition.

**Corollary 2** The map $\alpha$ is Poisson.
This can also be checked by direct calculation using Poisson brackets between $\chi_i^\pm$.

Thus, we have a Poisson map $\alpha : ((\mathbb{C}^\times)^{2r}) \rightarrow ((\mathbb{C}^\times)^{2r})$ defined outside of hyper-planes $\chi_i^- = 1, \chi_i^+ = 0$, which preserves functions $\text{Ch}_V(\chi^+, \chi^-)$.

**Proposition 11**

1. $\{\text{Ch}_V, \text{Ch}_W\} = 0$ for every pair of finite dimensional representations $V$ and $W$.
2. $\text{Ch}_V$, as a function of the $\chi_i^\pm$, is independent of the choice of the Coxeter element $w$.

**Proof:** The first part of this proposition is a general fact about factorizable Poisson Lie groups.

For the second we have to show that $\text{Ch}_V(\chi^+, \chi^-)$ does not depend on the order $(i_1, \ldots, i_r)$ of the indices. Clearly $\text{Ch}_V(\chi^+, \chi^-)$ doesn’t change if we change the order by an elementary transposition (exchange of two consecutive indices) of two indices which are not linked in the Coxeter diagramm. Let us call these transpositions free elementary transpositions. Furthermore, $\text{Ch}_V(\chi^+, \chi^-)$ is also invariant under a cyclic permutation as may be seen using the observation made after Proposition 8. Thus the proposition follows from the easily established fact that every elementary transposition can be obtained by successive applications of cyclic permutations and free elementary transpositions.

q.e.d.

To summarize, with each Coxeter symplectic leaf of $G$ we associated a (complex holomorphic, algebraic) integrable system on $((\mathbb{C}^\times)^{2r})$ for which the integrals are given by characters (there are exactly $r$ independent of them) but all these systems are trivially isomorphic. The coordinates $\chi_i^\pm$ simply describe different points in the group if one changes the Coxeter element. The factorization relation restricted to a Coxeter symplectic leaf gives a discrete-time evolution preserving these integrals.

### 6.3 Real positive form

Consider the real form $G_\mathbb{R}$ of the complex algebraic group $G$. Introduce variables $\chi_i^\pm = -u_i^\pm$. The domain $u_i^+ > 0$ we will call positive domain. The following is clear.

**Proposition 12** Functions $\text{Ch}_V(u^+, u^-)$ are positive for $u_i^+ , u_i^- > 0$ and

$$\text{Ch}_V(u^+, u^-) = \text{Tr}_V \left( \prod_{j=1}^{r} \left( \frac{u_j^+}{u_j^0} \right)^{h_j} \phi_i \left( \begin{smallmatrix} 1 & u_i^+ \\ 1 & 1 + u_i^+ \end{smallmatrix} \right) \ldots \phi_i \left( \begin{smallmatrix} 1 & u_r^+ \\ 1 & 1 + u_r^+ \end{smallmatrix} \right) \right)$$
\[ = \text{Tr}_V \left( \prod_{j=1}^{r} \left( \frac{u^+_j}{u^-_j} \right)^{h_j} \phi_i \left( \begin{array}{cc} 1 + u^-_{i_i} & u^-_{i_i} \\ 1 & 1 \end{array} \right) \right) \]

It is also clear that the map \(\alpha\) is defined globally on positive domain:

\[ \alpha(u^+_i) = u^+_i, \quad \alpha(u^-_i) = \frac{(u^-_i)^2}{u^+_i} \prod_j (1 + u^-_j)^{-c_{j_i}}. \]

Let \(G_{>0}\) be positive part of \(G_R\) (see [Lus94] and [FZ98] for definitions). For \(SL(n)\) the positive part consists of all real unimodular \(n \times n\) matrices with positive principal minors.

**Lemma 3** On \(G_{>0}\) there exists unique factorization

\[ g = g_+(g_-)^{-1} \]

where \(g^\pm_+ \in B_{>0}^\pm = B^\pm \cap G_{>0}\) and \(\theta(g_+) = \theta^-(g_-)^{-1}\).

Let \(S_{w,w}^+\) be the positive Coxeter symplectic leaf of \(G_R\). It is the connected component of \(\varphi^{-1}\) of the corresponding orbit in \(D(G_R)/j(G_{R,-})\) which lies in \(G_{>0}\). The positive domain described above is essentially a positive symplectic leaf and thus, on the positive domain the factorization map \(\alpha\) is the restriction of the factorization map \(g = g_+ g_-^{-1} \mapsto \tilde{g} = g_-^{-1} g_+\).

### 7 The interpolating flow and continuous time nonlinear Toda lattices

#### 7.1 Interpolating flow

From now on we consider the factorization dynamics in positive real domain. As it was already pointed out the factorization dynamics on the positive real domain is a graph of a Poisson map. The trajectory of this map is defined recursively as \(x(n+1) = x_-(n)^{-1} x_+(n)\) for \(x(n) = x_+(n) x_-(n)^{-1}\).

**Proposition 13** The trajectory of the factorization map restricted to the positive real domain which starts at \(x(0)\) has the form:

\[ x(n) = g_+(n)^{-1} x(0) g_+(n), \]

\[ g(n) = x(0)^n = g_+(n) g_-(n)^{-1}. \]
Proof: \( x(0)^n = x_+(0)x(1)^{n-1}x_-(0)^{-1} = x_+(0) \ldots x_+(n)x_-(n)^{-1} \ldots x_-(0)^{-1} \) shows that \( g_+(n) = x_+(0) \ldots x_+(n) \) which quickly leads to the statement. q.e.d.

This proposition is a discrete analogue of the following theorem of Semenov-Tian-Shansky [STS85] for continuous time systems which describes the trajectories of Hamiltonian systems on Poisson Lie groups generated by Ad-invariant functions. Define the Lie \( G \)-valued gradient \( \nabla f \) of a function \( f : G \to \mathbb{R} \) by \( (\nabla f(g), \eta) := < df(g), (X_\eta) > \) where we write \( X_\eta \) for the left invariant vector field on \( G \) corresponding to \( \eta \) and \( < \omega, X > \) is the value of the form \( \omega \) on the vector field \( X \).

**Theorem 5** Let \( H \) be an \( \text{Ad}_G \)-invariant function on \( G \). The trajectory \( x(t) \) of the Hamiltonian equations of motion generated by \( H \) is given by

\[
x(t) = g_+(t)^{-1}x(0)g_+(t)
\]

where \( g(t) = \exp(t\nabla H(x(0))) = g_+(t)g_-(t)^{-1} \).

Now we are in a position to derive a Hamiltonian flow which interpolates the factorization dynamics. Obviously, a Hamiltonian \( H_d \) which has a flow whose time 1 map is given by factorization as above has to solve the equation

\[
g = \exp(\nabla H_d).
\]

Thus, for \( g = e^\xi \) and \( \xi \in \text{Lie} \ G \) we should have \( \xi = \nabla H_d(e^\xi) \).

**Proposition 14** In a neighborhood of the identity all \( \text{Ad}_G \)-invariant solutions of the equation

\[
\xi = \nabla H(e^\xi)
\]

have the form

\[
H_d(e^\xi) = \frac{1}{2}(
\xi, \xi) + \text{const}.
\]

**Proof:**

Let \( H \) an \( \text{Ad}_G \)-invariant solution of the above equation and \( \tilde{H} = H \circ \exp. \) Then \( \tilde{H} \) is \( \text{ad}_g \)-invariant and hence \( dH|_\xi(ad_\eta(\xi)) = 0 \) for all \( \xi, \eta \in \text{Lie} \ G. \)

By (17), \( (\xi, \eta) = < dH(e^\xi), X_\eta > = < d\tilde{H}(\xi), \eta >. \) Here we trivialized the tangent bundle on \( G \) by left translations. Thus, for \( \tilde{H} \) we have the equation \( (\xi, \eta) = d\tilde{H}|_\xi(\eta) \). Integration yields now the statement of the proposition. q.e.d.

If \( G = SL(n, \mathbb{R}) \) then \( H_d(g) = \frac{1}{2}\text{tr}((\log^2(g))) \) in a sufficiently small neighborhood of the identity [Sur91a].

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The Hamiltonian $H_d$ is quite remarkable since it gives the so-called classical quantum $R$-matrix [WX92, Res92]. The function $H_d$ is the most singular part of the quantum $R$-matrix in the appropriate semi-classical limit [Skl82, Res95, Res96]. The map $\alpha$ generated by time 1 flow of $H_d$ is the classical quantum $R$-matrix in the sense of [WX92] restricted to the product of Coxeter symplectic leaves and reduced by Hamiltonian reduction.

### 7.2 Linearization in a neighborhood of 1

Consider the family of diffeomorphisms of $\mathbb{R}^{2n}$ to $\mathbb{R}_{+}^{2n}$:

$$\beta : (0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}_{+}^{2n}$$

$$\beta_\epsilon(\pi_i) = \epsilon^2 e^{\phi_i + \epsilon \pi_i}$$

$$\beta_\epsilon(\phi_i) = \epsilon^2 e^{\phi_i}$$

Here $(\pi_i, \phi_i)$ are coordinates in $\mathbb{R}^{2n}$ such that $(\beta_\epsilon(\pi_i), \beta_\epsilon(\pi_i))$ are the coordinates $(u_i^+, u_i^-)$ which were used above. Then, assuming a symplectic structure on $\mathbb{R}^{2n}$ s.t.

$$\{\phi_i, \phi_j\} = \{\pi_i, \pi_j\} = 0, \quad \{\phi_i, \pi_j\} = d_i C_{ij}$$

the maps $\beta_\epsilon$ are symplectomorphisms.

Define maps $\alpha_\epsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \epsilon \in (0, 1]$ as $\alpha_\epsilon = \beta_\epsilon^{-1} \circ \alpha \circ \beta_\epsilon$. They act on coordinates $(\phi_i, \pi_i)$ as

$$\alpha_\epsilon(\pi_i) = \pi_i + \sum_{j=1}^{r} C_{ji} \frac{1}{\epsilon} \ln(1 + \epsilon^2 e^{\phi_j})$$

$$\alpha_\epsilon(\phi_i) = \phi_i + \epsilon \pi_i + \sum_{j=1}^{r} C_{ji} \frac{1}{\epsilon} \ln(1 + \epsilon^2 e^{\phi_j}).$$

By construction these maps are symplectomorphisms for (18).

In the limit $\epsilon \to 0$ equation (19) defines a vector field on $\mathbb{R}^{2n}$ with coordinates

$$\dot{\phi}_i = \lim \frac{\alpha_\epsilon(\phi_i) - \phi_i}{\epsilon} = \pi_i$$

$$\dot{\pi}_i = \lim \frac{\alpha_\epsilon(\pi_i) - \pi_i}{\epsilon} = \sum_{j} C_{ji} e^{\phi_j}$$

This is the Hamiltonian vector field (assuming symplectic structure (18)) generated by the (usual) Toda Hamiltonian

$$H_{\text{Toda}} = \frac{1}{2}(\xi_0, \xi_0)$$
where
\[ \xi_0 = \sum_{i=1}^{r} (\pi_i h^i + e^{\phi_i} x_i^- + x_i^-) . \]

Thus the family of maps (13) “retracts” to the Toda Hamiltonian flow in the neighborhood of the identity.

Equivalently, we have:
\[ \lim_{n \to \infty} \alpha^n_\varepsilon(\phi, \pi) = (\phi(t), \pi(t)) \]
where \( t = n\varepsilon \) is fixed and \( \phi(t), \pi(t) \) is the Hamiltonian flow generated by \( H_{\text{Toda}} \) passing through \( (\phi, \pi) \) at \( t = 0 \).

It is easy to find the leading terms of the asymptotic expansion of the integrals in the limit \( \varepsilon \to 0 \). Indeed, composing map \( \beta_\varepsilon \) with functions \( Ch_V \) and \( H_d \) we have:
\[ H_V(\phi, \pi) = (Ch_V \circ \beta_\varepsilon)(\phi, \pi) = Tr_V(\exp(\xi_\varepsilon)) \]
\[ H_d(\phi, \pi) = \frac{1}{2}(\xi_\varepsilon, \xi_\varepsilon) \]
where
\[ \exp(\xi_\varepsilon) = \prod_{j=1}^{r} \exp(\varepsilon \pi_j h_j) \prod_{i} \exp(\varepsilon e^{\phi_i} x_i^-) \exp(\varepsilon x_i^-) \]

As \( \varepsilon \to 0 \),
\[ \xi_\varepsilon = \varepsilon \xi_0 + O(\varepsilon^2) , \]
Thus, for \( H_V \) and \( H_d \) we have
\[ H_V = \dim V (1 + \varepsilon^2 \frac{c_V}{\dim(g)} H_{\text{Toda}} + O(\varepsilon^3)) \]
\[ H_d = \varepsilon^2 H_{\text{Toda}} + O(\varepsilon^3) \]

Here we assumed that \( V \) is irreducible and \( c_V \) is the value of the Casimir operator action on \( V \). Higher Toda Hamiltonians can be obtained from higher order terms of \( \varepsilon \)-expansion of \( H_V \).

8 Conclusion

As it was mentioned in the introduction, the main goal of this paper was systematic derivation of Coxeter-Toda systems from the symplectic geometry of Poisson Lie groups. Naturally, such analysis can be done for loop groups as well. The corresponding models will be affine versions of Coxeter-Toda
systems. For $A_n$ root system this will give the relativistic Toda chain first described by Ruijsenaars $^{[Rui90]}$. In a similar way one can construct discrete versions of Toda field theories. For $A_n$-case it has been done in $^{[KR97]}$.

Notice also that somewhat unexpectedly the same Hirota equations appear as a system of equations for transfer-matrices of some solvable models in statistical mechanics $^{[BR90][KNS94]}$. Although it is clear that the explanation of this coincidence lies in the theory of $q-W$-algebra $^{[ER97]}$, the complete picture is still missing.

The factorization dynamics restricted to other symplectic leaves will give ”nonlinear” Toda-Kostant systems which are related to general coadjoint orbits.

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