The Gribov horizon and ghost interactions in Euclidean gauge theories

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The effect of the Gribov horizon in Euclidean SU(2) gauge theory is studied. Gauge fields on the Gribov horizon yield zero modes of ghosts and antighosts. We show these zero modes can produce additional ghost interactions, and the Landau gauge changes to a nonlinear gauge effectively. In the infrared limit, however, the Landau gauge is recovered, and ghost zero modes may appear again. We show ghost condensation happens in the nonlinear gauge, and the zero mode repetition is avoided.

Subject Index B05, B06

1. Introduction

A perturbative calculation in gauge theories requires gauge fixing. However, in non-Abelian gauge theories, there is a problem of gauge copies (Ref. [1]). Gribov showed that gauge-equivalent copies exist in the Landau gauge
\[ \partial_\mu A_\mu = 0. \] (1.1)

In the Coulomb gauge, it was shown that almost all gauge transformations are responsible for gauge fixing degeneracies (Ref. [2]). If gauge copies are connected by an infinitesimal gauge transformation with a gauge parameter \( \varepsilon(x) \), Eq. (1.1) gives \( \partial_\mu D_\mu \varepsilon(x) = 0 \). That is, the Faddeev–Popov (FP) operator \( -\partial_\mu D_\mu \) has zero eigenvalues. The boundary where the lowest eigenvalue of the FP operator equals zero is called the (first) Gribov horizon \( \partial \Omega_1 \). The region inside \( \partial \Omega_1 \), where eigenvalues of \( -\partial_\mu D_\mu \) are positive, is called the Gribov region \( \Omega_1 \). In general, gauge copies may exist outside \( \Omega_1 \) (Ref. [1]) and on the horizon (Ref. [3]).

There are some ideas to solve the problem. One of them is to restrict a functional integral in the Gribov region \( \Omega \) (Refs. [1,4]). (Strictly speaking, there may be some copies in \( \Omega \). Hence the more restricted region in \( \Omega \), that is called a fundamental modular region (FMR) \( \Lambda \), is considered in Ref. [5].) Another idea is to sum over all gauge copies (Refs. [6,7]). For a solvable gauge model, it was shown that correct results are obtained by collecting all gauge copies (Refs. [8,9]).

The Gribov horizon yields some effects. In the first approach, the horizon perturbs gluons into shadow particles (Refs. [4,10]). Even if the region is restricted to the FMR \( \Lambda \), there are points where the boundary of \( \Lambda \) touches the horizon \( \partial \Omega \) (Ref. [5]). These points give the singularity of the operator \( 1/\partial_\mu D_\mu \). As a result, the color Coulomb potential is enhanced and the confinement might be shown (Ref. [11]). In the second approach, gauge configurations on the Gribov horizon contribute in general,
and the FP operator has zero modes. These zero modes can cause a problem in proving the gauge equivalence (Ref. [12]). Thus physical effects of the horizon $\partial \Omega$ are worth studying.

In this paper, we study the effect of these zero modes. In the next section, we show that a pair of zero modes in the Landau gauge can yield additional ghost interactions. If we require BRST invariance, an effective Lagrangian becomes a Lagrangian in a nonlinear gauge. In Sect. 3, the gauge $\partial_\mu A_\mu \neq 0$ is considered. If there is a pair of zero modes, the nonlinear gauge is realized as well. We also show that the partition function does not vanish even if the FP operator yields a single zero mode. In Sect. 4, the effect of a single zero mode is discussed in the Landau gauge. In the low energy region, ghost condensation appears in the nonlinear gauge. The effect of the zero modes under the condensation is discussed in Sect. 5. Section 6 is devoted to a summary. In Appendix A, examples of zero modes in the Coulomb gauge are given in three dimensional space-time. In Appendix B, the effective Lagrangian becomes a Lagrangian in a nonlinear gauge. In Sect. 3, the gauge condition (Eq. (1.1)) leads to the relations

$$
\partial_\mu D_\mu = D_\mu \partial_\mu - i \bar{c} \cdot \partial_\mu D_\mu c = \int dx i \bar{c} \cdot \partial_\mu D_\mu c = \int dx i (\partial_\mu D_\mu \bar{c}) \cdot c.
$$

Namely, $\partial_\mu D_\mu$ is hermitian, and its eigenvalues are real.

The eigenfunction $u_n$ with the eigenvalue $\lambda_n$ satisfies

$$
-\partial_\mu D_\mu u_n(x) = \lambda_n u_n(x).
$$

When $A_\mu$ is on the first Gribov horizon, the lowest eigenvalue is $\lambda_0 = 0$ and $u_0(x)$ is a zero mode. If we can make $u_0(x)$ complex, as Eq. (2.4) leads to

$$
-\partial_\mu D_\mu u_0^*(x) = \lambda_n u_n^*(x),
$$

$u_0^*(x)$ is also a zero mode. We assume a pair of zero modes $(u_0(x), u_0^*(x))$ exists. Some examples of a zero-mode pair $(u_0(x), u_0^*(x))$ are presented in Appendix A. If $u_0$ is real, it may be a single zero mode. An example of such a zero mode is given in Appendix A, and its effect is discussed in Sect. 4.

Now we expand the ghost $c$ as

$$
c(x) = \xi u_0(x) + \xi^\dagger u_0^*(x) + \cdots,
$$

We assume that eigenfunctions of the FP operator form an orthonormal complete set. Strictly speaking, to ensure it, spaces and/or configurations of $A_\mu$ must be restricted. We emphasize that what is important here is that $c$ contains $\xi u_0$, $\xi^\dagger u_0^*$ and $\bar{c}$ contains $\bar{\xi} u_0$, $\bar{\xi}^\dagger u_0^*$. 

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where \( \xi \) and \( \xi^\dagger \) are independent Grassmann variables. Other modes, i.e., nonzero modes and a single zero mode, are not written explicitly. In the same way, the property Eq. (2.3) implies that the expansion

\[
\tilde{c}(x) = \tilde{\xi} u_0(x) + \tilde{\xi}^\dagger u^{\ast}_0(x) + \cdots \tag{2.7}
\]

holds. We note that if there are some pairs of zero modes \( (u^j_0(x), u^{\ast j}_0(x)) \) \( (j = 1, 2, \ldots) \), \( \xi u_0(x) + \xi^\dagger u^{\ast}_0(x) \) and \( \tilde{\xi} u_0(x) + \tilde{\xi}^\dagger u^{\ast}_0(x) \) are replaced by \( \sum_j [\xi_j u^j_0(x) + \xi^\dagger_j u^{\ast j}_0(x)] \) and \( \sum_j [\tilde{\xi}_j u^j_0(x) + \tilde{\xi}^\dagger_j u^{\ast j}_0(x)] \), respectively. However the discussion below is also applicable.

Equations (2.4) and (2.5) imply that the Lagrangian \( \int dx \bar{c} \partial_{\mu} D_{\mu} c \) does not contain the Grassmann variables \( \xi, \xi^\dagger, \tilde{\xi}, \) and \( \tilde{\xi}^\dagger \). However the measures \( Dc \) and \( D\bar{c} \) contain \( d\xi \ d\xi^\dagger \) and \( d\tilde{\xi} \ d\tilde{\xi}^\dagger \), respectively. Since a Grassmann variable \( \zeta \) satisfies

\[
\int d\zeta \ z^n = \begin{cases} 1 & (n = 1), \\ 0 & (n = 0, 2, 3, \ldots), \end{cases} \tag{2.8}
\]

the partition function vanishes:

\[
\int DcD\bar{c} \exp \left\{ - \int dx \mathcal{L}_\alpha \right\} = 0.
\]

We know that fermions in an instanton background have zero modes. These zero modes yield the additional interaction of fermions (Refs. [13, 14]). Likewise, the above ghost zero modes may produce additional ghost interactions, because

\[
\int DcD\bar{c} \xi \xi^\dagger \tilde{\xi} \tilde{\xi}^\dagger \exp \left\{ - \int dx \mathcal{L}_\alpha \right\} \neq 0. \tag{2.9}
\]

From Eqs. (2.6) and (2.7), we obtain

\[
c^A \bar{c}^B c^C \bar{c}^D = \Psi^{ABCD} \xi \xi^\dagger \tilde{\xi} \tilde{\xi}^\dagger + \cdots,
\]

where \( \Psi^{ABCD} = u^A_0 u^B_0 u^C_0 u^D_0 \), and terms denoted by \( \ldots \) lack some or all of \( \xi, \xi^\dagger, \tilde{\xi}, \) and \( \tilde{\xi}^\dagger \). Therefore Eq. (2.9) leads to

\[
\int DcD\bar{c} \sigma^{[AB][CD]} \Psi^{ABCD} \xi \xi^\dagger \tilde{\xi} \tilde{\xi}^\dagger \exp \left\{ - \int dx \mathcal{L}_\alpha \right\} \\
= \int DcD\bar{c} \sigma^{[AB][CD]} c^A \bar{c}^B c^C \bar{c}^D \exp \left\{ - \int dx \mathcal{L}_\alpha \right\}, \tag{2.10}
\]

where \( \sigma^{[AB][CD]} \) is antisymmetric with respect to \( A \) and \( B \), and \( C \) and \( D \) as well. Thus ghost zero modes produce effective ghost interactions.

Now we determine \( \sigma^{[AB][CD]} \), and construct effective Lagrangians. The first candidate is

\[
\sigma^{[AB][CD]} = f^{EAB} f^{ECD} (= \delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC}).
\]

This choice gives the term

\[
\sigma^{[AB][CD]} c^A \bar{c}^B c^C \bar{c}^D = (\bar{c} \times c) \cdot (c \times c) = -2 (\bar{c} \times c) \cdot (\bar{c} \times c),
\]

and Eq. (2.10) becomes

\[
\int DcD\bar{c} (\bar{c} \times c) \cdot (c \times c) \exp \left\{ - \int dx \mathcal{L}_\alpha \right\}. \tag{2.11}
\]
From Eq. (2.8), the equality
\[ \int d\zeta e^{\zeta} = 1 \] (2.12)
holds. Therefore, as in the instanton case (Ref. [15]), Eq. (2.11) is derived from the nonvanishing partition function
\[ \int DcD\bar{c} \exp \left\{ -\int dx \frac{K_1}{4} (i\bar{c} \times c)^2 \right\} \exp \left\{ -\int dx \mathcal{L}_a \right\}, \] (2.13)
where \( K_1 \) is a dimensionless constant.

Interaction with other fields is also possible. If we use
\[ \sigma^{[AB][CD]} = B^E B^F \left( f^{EAC} f^{FBD} - f^{EBC} f^{FAD} \right), \] (2.14)
we obtain the term
\[ \sigma^{[AB][CD]} c^A c^B \bar{c}^C \bar{c}^D = -2[B \cdot (c \times \bar{c})][B \cdot (c \times \bar{c})], \]
and Eq. (2.10) becomes
\[ \int DcD\bar{c} \left[ B \cdot (c \times \bar{c}) \right][B \cdot (c \times \bar{c})] \exp \left\{ -\int dx \mathcal{L}_a \right\}. \] (2.15)
Taking account of Eq. (2.12), we find Eq. (2.14) is derived from
\[ \int DcD\bar{c} \exp \left\{ -\int dx K_2 B \cdot (\bar{c} \times c) \right\} \exp \left\{ -\int dx \mathcal{L}_a \right\}, \] (2.16)
where \( K_2 \) is a dimensionless constant.

We can combine Eqs. (2.13) and (2.15) in a BRS-invariant form. Carrying out the BRS transformation
\[ \delta_B A_\mu = D_\mu c, \quad \delta_B c = -\frac{g}{2} c \times c, \quad \delta_B \bar{c} = iB, \] (2.17)
we obtain
\[ \delta_B \left\{ \frac{K_1}{2} (i\bar{c} \times c)^2 + K_2 [B \cdot (\bar{c} \times c)] \right\} = (-iK_1 - gK_2) \left( B \times c \right) \cdot (\bar{c} \times c). \]
If we set \( K_2 = -\frac{i}{g} K_1 = ig\alpha_2 \), we get the BRS-invariant effective Lagrangian
\[ \mathcal{L}_{\text{eff}} = -\frac{\alpha_2}{2} (ig\bar{c} \times c)^2 + \alpha_2 B \cdot (ig\bar{c} \times c) = \frac{\alpha_2}{2} B^2 - \frac{\alpha_2^2}{2} \bar{B}^2, \] (2.18)
where \( \bar{B} = -B + ig\bar{c} \times c \), and \( \alpha_2 \) is a new dimensionless constant.

Here we used the property Eq. (2.8) to derive the effective Lagrangian Eq. (2.18). In Appendix B, we derive it by using a source term.

Now we summarize the result. In the Landau gauge, when the configuration \( A_\mu \) on the Gribov horizon contributes to the partition function, the FP operator has zero modes. If a pair of zero modes

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\[ \text{\footnotesize 2} \text{ Instead of } B, \text{ we can use } A_\mu. \text{ Examples are } F_{\mu\nu} \text{ and } \partial_\nu A_\mu. \text{ However, using them, we cannot construct a Lagrangian that has mass dimension four (or lower than four) and has the off-shell BRS invariance.} \]
As \( \dot{\alpha}\) does not contain effective Lagrangian function, we must repeat the consideration in Sect. 2. Namely the zero-mode pairs give rise to the Lagrangian in the nonlinear gauge \( L_{NL} \) (Refs. [16–18]).

3. \( \alpha \neq 0 \) gauge

In the \( \alpha \neq 0 \) gauge, as \( \partial_\mu A_\mu \neq 0 \) and

\[
\int dx \, i \bar{c} \cdot \partial_\mu D_\mu c = \int dx \, i(D_\mu \partial_\mu \bar{c}) \cdot c, \quad \partial_\mu D_\mu \neq D_\mu \partial_\mu,
\]

the operator \( \partial_\mu D_\mu \) is not hermitian. We assume that the operator \( \partial_\mu D_\mu \) has a pair of zero modes \( (u_0, \bar{u}_0^* \rangle \) and a real single zero mode \( v_0 \). Then \( c \) is expanded as

\[
c(x) = \xi u_0(x) + \xi^\dagger \bar{u}_0^*(x) + \zeta v_0 + \cdots, \tag{3.1}
\]

where \( \xi, \xi^\dagger, \) and \( \zeta \) are independent Grassmann variables. Although the Lagrangian Eq. (2.2) does not contain \( \xi, \xi^\dagger, \) and \( \zeta \), the measure \( Dc \) contains \( d\xi \, d\xi^\dagger \, d\zeta \). Thus we find

\[
\int Dc \, D\bar{c} \exp \left\{ -\int dx \, L_\alpha \right\} = 0,
\]

\[
\int Dc \, D\bar{c} \exp \left\{ -\int dx \, L_\alpha \right\} \neq 0. \tag{3.2}
\]

However Eq. (3.2) contradicts ghost number conservation. To avoid this problem, a pair of zero modes \( (\bar{u}_0, \bar{u}_0^*) \) and a real single zero mode \( \bar{v}_0 \) of the operator \( D_\mu \partial_\mu \) must exist,\(^3\) and \( \bar{c} \) is expanded as

\[
\bar{c}(x) = \bar{\xi} \bar{u}_0 + \bar{\xi}^\dagger \bar{u}_0^*(x) + \bar{\zeta} \bar{v}_0(x) + \cdots. \tag{3.3}
\]

Since \( \partial_\mu D_\mu \neq D_\mu \partial_\mu \), a zero-mode pair \( (\bar{u}_0, \bar{u}_0^*) \) is different from \( (u_0, \bar{u}_0^*) \), and \( \bar{v}_0 \neq v_0 \).

Now we consider the effect of the zero-mode pairs \( (u_0, \bar{u}_0^*) \) and \( (\bar{u}_0, \bar{u}_0^*) \). Since the Lagrangian does not contain \( \xi, \xi^\dagger, \bar{\xi}, \) and \( \bar{\xi}^\dagger \), and the measure contains \( d\xi \, d\bar{\xi} \, d\xi^\dagger \, d\bar{\xi}^\dagger \), to obtain a nonzero partition function, we must repeat the consideration in Sect. 2. Namely the zero-mode pairs give rise to the effective Lagrangian \( L_{eff} \), and the nonlinear gauge is realized.

\(^3\) Let us consider a square matrix \( D \), that is not necessarily hermitian. There are eigenvectors \( V_i \) that satisfy \( DV_i = \lambda_i V_i \). Since \( \det(D - \lambda E) = \det(D - \lambda E) \), \( D \) has the same eigenvalues as \( D \). Thus we have \( \det(D - \lambda E) = \lambda_i U_i \).

As \( U_i \) satisfies \( U_i D = \lambda_i U_i \), these eigenvectors satisfy \( U_i V_i = 0 \) if \( \lambda_i \neq \lambda \) (Ref. [19]). In the present case, we assign \( D = \partial_\mu D_\mu, \partial_\mu D_\mu, V_k = (u_k, \bar{u}_k^*, v_k) \), and \( U_i = (\bar{u}_i, \bar{u}_i^*, \bar{v}_i) \).
Next we study the terms \( \zeta v_0 \) in Eq. (3.1) and \( \bar{\zeta} \bar{v}_0 \) in Eq. (3.3). The Lagrangian \( \mathcal{L}_{\text{eff}} \) has the term \( ig\alpha_2 B \cdot (\bar{c} \times c) \). Although this term is necessary to ensure BRS symmetry, as

\[
B \cdot (\bar{c} \times c) = B \cdot (\bar{\zeta} \bar{v}_0(x) \times v_0(x) + \cdots), \tag{3.4}
\]

the partition function does not vanish even if \( Dc D\bar{c} \) contains \( d\zeta d\bar{\zeta} \).

Thus, when \( \alpha \neq 0 \), the partition function changes from Eq. (2.1) to Eq. (2.18), if the FP operator \( \partial_\mu D_\mu \) has a pair of zero modes. This result is unchanged even if this operator has a single zero mode.

4. Renormalization group flow of \( \alpha \)

We return to the gauge \( \alpha = 0 \), and assume \( \partial_\mu D_\mu \) has a single zero mode \( v_0 \). Now \( \partial_\mu D_\mu = D_\mu \partial_\mu \) holds, we must set \( \bar{v}_0(x) = v_0(x) \) in Eq. (3.3), i.e.,

\[
c = \zeta v_0(x) + \cdots, \quad \bar{c} = \bar{\zeta} \bar{v}_0(x) + \cdots.
\]

Since \( v_0(x) \times v_0(x) = 0, \bar{c} \times c \) and Eq. (3.4) do not contain \( \bar{\zeta} \). Namely we cannot say that \( Z_{\alpha=0}^\text{NL} \neq 0 \) is guaranteed.

To avoid this difficulty, we first construct the partition function \( Z_{\alpha}^\text{NL} \neq 0 \), and then take the limit \( \alpha \to 0 \), i.e., \( \lim_{\alpha \to 0} Z_{\alpha}^\text{NL} \).

From the Lagrangian \( \mathcal{L}_{\text{NL}} \), the equation of motion for \( B \) is

\[
\partial_\mu A_\mu - \alpha B = -ig\alpha_2 (\bar{c} \times c).
\]

So, when \( \alpha \to 0 \), the term \(-ig\alpha_2 (\bar{c} \times c)\) must be taken into account. In this section, treating the interactions perturbatively at the one-loop level, we study the behavior of \( \alpha \).

In Appendix C, we derive the renormalization group (RG) equations

\[
\frac{\partial \alpha_1}{\partial \mu} = \frac{g^2 C_2(G)}{16\pi^2} \alpha_1 \left( \frac{13}{3} - \alpha_1 \right), \quad \frac{\partial \alpha_2}{\partial \mu} = \frac{g^2 C_2(G)}{16\pi^2} \alpha_2 \left( \frac{13}{3} - \alpha_2 \right), \tag{4.1}
\]

which coincide with the results in Refs. [20] and [21]. We emphasize that the equation for \( \alpha_1 \) does not contain \( \alpha_2 \), and vice versa. From Eq. (4.1), \( \alpha = \alpha_1 + \alpha_2 \) satisfies

\[
\frac{\partial \alpha}{\partial \mu} = \frac{g^2 C_2(G)}{16\pi^2} \left[ \frac{13}{3} \alpha - \alpha^2 + 2(\alpha - \alpha_2)\alpha_2 \right]. \tag{4.2}
\]

When \( |\alpha| \ll 1 \), Eq. (4.2) becomes

\[
\frac{\partial \alpha}{\partial \mu} \approx -\frac{g^2 C_2(G)}{8\pi^2} \alpha_2. \tag{4.3}
\]

Therefore, when \( \alpha_2 \neq 0 \), \( \alpha \) increases as \( \mu \) decreases. The quartic ghost interaction makes \( \alpha \neq 0 \), and the situation in Sect. 3 is realized. Even if a single zero mode \( v_0 \) exists, the partition function does not vanish.

Eq. (4.1) shows that \( (\alpha_1, \alpha_2) = (0, 0) \) is an infrared fixed point. Does this fact imply that the Landau gauge (Eq. (1.1)) is retrieved as \( \mu \to 0 \)? Does the process in Sect. 2 repeat again? In the next section, we show that such a problem does not happen.

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4 The parameters \( \alpha_1 \) and \( \alpha_2 \) in this article are related to the parameters in Refs. [20] and [21]:

1. after setting \( \xi = 0, \zeta = \eta \) and \( \alpha = \beta \), then \( \alpha_1 = (1 + \eta) \alpha \) and \( \alpha_2 = -\eta \alpha \) in Ref. [20];
2. \( \alpha_1 = (1 - \xi) \lambda = \alpha' + \alpha/2 \) and \( \alpha_2 = \xi \lambda = \alpha/2 \) in Ref. [21].
5. Ghost condensation

In Appendix B, we present the Lagrangian (Refs. [18,22])

\[
L_\phi = -\frac{\alpha_1}{2} B^2 + B(\partial_\mu A_\mu + \phi - w) + i\bar{c} \cdot (\partial_\mu D_\mu + g\phi \times c) c + \frac{\phi^2}{2\alpha_2}.
\] (5.1)

This Lagrangian has BRS invariance, if \(\phi\) transforms as \(\delta_B \phi = g\phi \times c\). Setting the constant \(w = 0\), and performing the \(\phi\) integration, we find \(L_\phi\) yields \(L_{NL}\). Namely, \(\phi\) is an auxiliary field that represents \(\alpha_2 \tilde{B}\).

However, in a low-energy region, \(\phi\) is not an auxiliary field. In Ref. [22], we derived another RG equation for \(\alpha_2\) given by

\[
\frac{\partial}{\partial \mu} \alpha_2 = \frac{g^2 C_2(G)}{(4\pi)^2} (\beta_0 - 2\alpha_2) \alpha_2,
\] (5.2)

which is different from Eq. (4.1). Equation (5.2) was derived by making the Wilsonian effective action for \(\phi\).5 We also showed that \(\phi\) acquires the vacuum expectation value \(\langle \phi \rangle = \phi_0\) under the energy scale

\[
\mu_0 = \Lambda \exp \left\{ -4\pi^2 / (\alpha_2 g^2) \right\},
\] (5.3)

where \(\Lambda\) is a momentum cut-off. Ghost–antighost bound states and ghost condensation appear below \(\mu_0\). We substitute \(\phi(x) = \phi_0 + \phi'(x)\) into Eq. (5.1), and choose the constant \(w = \phi_0\). This choice is necessary to maintain BRS symmetry (Ref. [23]).6 Then Eq. (5.1) becomes

\[
-\frac{\alpha_1}{2} B^2 + B(\partial_\mu A_\mu + \phi') + i\bar{c} \cdot (\partial_\mu D_\mu + g\phi' \times + g\phi_0 \times c).
\] (5.4)

Because of the dimensional transmutation (Ref. [24]), the parameter below \(\mu_0\) is not \(\alpha_2\) but \(\phi_0\).

Contrary to \(\alpha_2\), the gauge parameter \(\alpha_1\) remains in Eq. (5.4). As we explain in Appendix C.2, the RG equations (4.1) for \(\alpha_1\) persists, and \(\alpha_1 = 0\) is an infrared fixed point. So, when \(\mu \to 0\), Eq. (5.4) gives the gauge condition

\[
\partial_\mu A_\mu + \phi' \approx 0
\] (5.5)

and the ghost Lagrangian

\[
\int dx i\bar{c} \cdot (\partial_\mu D_\mu + g\phi' \times c) = \int dx i\bar{c} \cdot (D_\mu \partial_\mu)c = \int dx i(\partial_\mu D_\mu \bar{c}) \cdot c.
\]

As Eq. (5.5) means \(\partial_\mu D_\mu \neq D_\mu \partial_\mu\), we assume \(\partial_\mu D_\mu\) has a pair of zero modes \((u_0, u_0^\dagger)\) and a single zero mode \(v_0\), and \(D_\mu \partial_\mu\) has zero modes \((\tilde{u}_0, \tilde{u}_0^\dagger)\) and \(\tilde{v}_0\). Even if the measure \(DcD\bar{c}\) contains \(d\xi \; d\xi^\dagger \; d\xi \; d\xi^\dagger \; d\xi \), because the term \(i\bar{c} \cdot (g\phi_0 \times c)\) in Eq. (5.4) has

\[
-ig\phi_0 \cdot \{\tilde{\xi} \tilde{\xi} u_0(x) \times u_0(x) + \tilde{\xi}^\dagger \tilde{\xi} u^\dagger_0(x) \times u_0^\dagger(x) + \tilde{\xi} \tilde{\xi} \tilde{v}_0(x) \times v_0(x) + \cdots \},
\] (5.6)

the partition function does not vanish.

5 In Appendix C.2, we explain how to derive Eq. (5.2) from \(L_{NL}\).

6 This point is explained in Appendix D. The anti-BRS symmetry and the global gauge symmetry are also discussed.
6. Summary

In the Landau gauge $\alpha = 0$, the FP operator $-\partial_\mu D_\mu$ has zero modes on the Gribov horizon. As the ghost $c$ and the antighost $\bar{c}$ are Grassmann variables, it is natural to expect that these zero modes yield effective ghost interactions. We have shown that the quartic ghost interaction is produced by a pair of zero modes. If we impose BRS invariance, the Lagrangian in the nonlinear gauge is obtained. Thus the Landau gauge changes to the nonlinear gauge. In the $\alpha \neq 0$ gauge, the same result is obtained as well.

The effect of a single zero mode was also studied. Although there is no trouble in the $\alpha \neq 0$ gauge, the partition function $Z$ may vanish in the $\alpha = 0$ gauge. We can avoid this problem by taking the limit $\alpha \to 0$.

Usually, when $\det \partial_\mu D_\mu = 0$ for some configuration $A_\mu$, we can avoid the $Z = 0$ problem by choosing another gauge (locally) (Ref. [25]). In this paper, we have shown that such a configuration changes the gauge to the nonlinear gauge automatically.

The partition functions in the Landau gauge and the nonlinear gauge are equivalent perturbatively. In the nonlinear gauge, $(\alpha_1, \alpha_2) = (0, 0)$ is an infrared fixed point at the one-loop level. In this case, the Landau gauge is retrieved and the zero-mode problem appears again. However, this scenario is not true. The nonlinear gauge yield the ghost condensation below the energy scale $\mu_0$, and the zero-mode problem no longer happens.

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Appendix A. Examples of zero modes in the Coulomb gauge

In this appendix, choosing the gauge $\partial_j A_j = 0$, we study the eigenvalue equation

$$\begin{align*}
-\partial_j D_j u &= -(\Delta + g A_j \times \partial_j) u = \lambda u
\end{align*}$$

(A1)
in three-dimensional space-time.

A.1. A pair of zero modes

If the eigenfunction has the form $u^A = e^{is} w^A$ with $g A_j \times (\partial_j w) = 0$, Eq. (A1) becomes

$$\begin{align*}
-iH^{AB} e^{is} w^B &= (\Delta + \lambda) e^{is} w^A, \\
H^{AB} &= gf^{ACB} A^C_j (\partial_j s).
\end{align*}$$

(A2)

Since $H$ is a real antisymmetric $3 \times 3$ matrix, its eigenvalues are pure imaginary or 0, i.e.,

$$\begin{align*}
H^{AB} w^B_+ &= i h(x) w^A_+, \\
H^{AB} w^B_- &= -i h(x) w^A_-, \\
H^{AB} w^B_0 &= 0.
\end{align*}$$

(A3)

The last equation of Eq. (A3) means that the effect of $A^C_j$ disappears and $w_0$ does not become a zero mode. From Eqs. (A2) and (A3), we obtain

$$h(x) e^{\pm is} w^A_\pm = (\Delta + \lambda) e^{\pm is} w^A_\pm.$$

Thus we find the two functions $u_\pm = e^{\pm is} w_\pm$ become a zero-mode pair, if

$$h(x) u^A_\pm = \Delta u^A_\pm$$

(A4)

holds.
To give concrete examples, let us choose the Abelian configuration

\[ A_i^A(x) = a_i(x)\delta^A, \quad \partial_j a_i = 0. \]  \hspace{1cm} (A5)

\subsection{A.1. Three-torus T^3}

Gribov copies in the three-torus T^3 are studied in Ref. [5]. The constant configuration

\[ a_j(x) = \frac{C_j}{gL}, \quad C_1 = 2\pi, \quad -2\pi < C_2 < 2\pi, \quad -2\pi < C_3 < 2\pi \]

is on the first Gribov horizon, where L is the size of the torus. Setting \( s = 2\pi x_1/L \), we find Eq. (A4) is satisfied by a zero-mode pair

\[ u_\pm = \exp \{ \pm i2\pi x_1/L \} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \]

\subsection{A.1.2. Axially symmetric configuration in R^3}

Next we consider the configuration

\[ a_j(x) = \epsilon_{j3k} q(r)x_k, \]  \hspace{1cm} (A6)

where \((r, \theta, \phi)\) are spherical coordinates. Using the angular momentum operator \( \hat{L}_j = -i\epsilon_{jkl} x_k \partial_l \), we find

\[ -\Delta = -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\hat{L}_3^2}{r^2}, \quad H_{AB} = gf^{AB} q(r) i\hat{L}_3. \]

Then it is natural to set \( e^{is} = e^{im\phi} \) and

\[ w_+ = i R(r) \Theta_{lm}(\theta) \begin{pmatrix} 1 \\ i \end{pmatrix}, \]

where \( l \) and \( m \) are integers, and

\[ \Theta_{lm}(\theta) = \frac{(-1)^m}{\sqrt{2\pi}} \left( \frac{(2l+1)(l-m)!}{2(l+m)!} \right)^{1/2} P^m_l(\cos \theta). \]

We note \( e^{im\phi} \Theta_{lm}(\theta) = Y_{lm}(\theta, \phi) \) is the spherical harmonics that satisfies \( Y^{*-}_{lm} = (-1)^{-m} Y_{l,-m} \). Then Eq. (A4) becomes

\[ \left[ -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R(r)}{\partial r} + \frac{l(l+1)}{r^2} R(r) - gmq(r)R(r) \right] Y_{l,\pm m}(\theta, \phi) = 0. \]  \hspace{1cm} (A7)

Now, following Henyey (Ref. [26]), we substitute the functions

\[ R(r) = \frac{K r^\rho}{(r^2 + r_0^2)^\kappa}, \quad q(r) = \frac{d}{(r^2 + r_0^2)^\sigma} \]  \hspace{1cm} (A8)

into Eq. (A7), where \( K, r_0, d, \rho, \kappa, \) and \( \sigma \) are constants. Equation (A7) is satisfied by

\[ \sigma = 2, \quad \rho = l, \quad \kappa = l + \frac{1}{2}, \quad d = \frac{(2l+1)(2l+3)}{gm} r_0^2. \]
Thus we obtain the Abelian configuration and the corresponding zero-mode pairs as
\[
d_j = \frac{(2l + 1)(2l + 3)}{g m} \frac{r_0^2}{(r^2 + r_0^2)^2} \epsilon_{j3k} x_k,
\]
\[
u_{\pm} = i^l \frac{K r^l}{(r^2 + r_0^2)^{l+1/2}} Y_l, m(\theta, \phi) \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \quad (l \geq 1, m = 1, 2, \ldots, l).
\]
\[(A9)\]

In Ref. [26], the \(l = 1\) case is presented explicitly.

### A.2. A single zero mode

In Ref. [27], a single zero mode was found in an instanton background. Here we give an example in \(\mathbb{R}^3\). Generalizing Eqs. (A5) and (A6), we choose the configuration
\[
A_j^C(x) = \epsilon_{jCk} q(r) x_k.
\]
\[(A10)\]

Then Eq. (A1) becomes
\[
-igq(r) \Xi^{AB} u^B = (\triangle + \lambda) u^A,
\]
\[
gf^{ACB} A_j^C \partial_j = igq(r) \Xi^{AB}, \quad \Xi^{AB} = f^{ACB} \hat{L}^C.
\]
\[(A11)\]

First we solve the equation
\[
\Xi^{AB} u^B = i\alpha u^A,
\]
\[(A12)\]

where \(i\alpha\) is an eigenvalue of \(\Xi\). We substitute the expansion
\[
u^A = \sum_{m=-l}^l R_{lm}(r) Y_{lm}(\theta, \phi),
\]
and, for simplicity, choose \(l = 1\). Then we find that the eigenvalues are \(\alpha = 2, 1, \) and \(-1,\) and the numbers of eigenfunctions are \(1, 3,\) and \(5,\) respectively. We choose the real eigenfunctions
\[
u^A_\alpha = R_\alpha(r) w_\alpha^A(\theta, \phi), \quad \text{where } w_\alpha^A(\theta, \phi) \text{ are given by}
\]
\[
\alpha = 2 : \begin{pmatrix} Y_{11} - Y_{1,-1} \\ -i(Y_{11} + Y_{1,-1}) \\ -\sqrt{2} Y_{10} \end{pmatrix},
\]
\[
\alpha = 1 : \begin{pmatrix} \sqrt{2} Y_{10} \\ 0 \\ Y_{11} - Y_{1,-1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \sqrt{2} Y_{10} \\ -i(Y_{11} + Y_{1,-1}) \end{pmatrix}, \quad \begin{pmatrix} i(Y_{11} + Y_{1,-1}) \\ Y_{11} - Y_{1,-1} \\ 0 \end{pmatrix},
\]
\[
\alpha = -1 : \begin{pmatrix} \sqrt{2} Y_{10} \\ 0 \\ -(Y_{11} - Y_{1,-1}) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \sqrt{2} Y_{10} \\ i(Y_{11} + Y_{1,-1}) \end{pmatrix},
\]
\[
\begin{pmatrix} Y_{11} - Y_{1,-1} \\ i(Y_{11} + Y_{1,-1}) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} i(Y_{11} + Y_{1,-1}) \\ -(Y_{11} - Y_{1,-1}) \\ -i(Y_{11} + Y_{1,-1}) \end{pmatrix}, \quad \begin{pmatrix} Y_{11} - Y_{1,-1} \\ -i(Y_{11} + Y_{1,-1}) \\ 2\sqrt{2} Y_{10} \end{pmatrix}.
\]
Next we determine \( R_\alpha \). From Eq. (A11) with \( \lambda = 0 \) and Eq. (A12), \( R_\alpha \) satisfies

\[
- \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R_\alpha(r)}{\partial r} + \frac{l(l+1)}{r^2} R_\alpha(r) + g\alpha q(r) R_\alpha(r) = 0. \tag{A13}
\]

As in the previous subsection, we substitute Eq. (A8) into Eq. (A13). Then we find

\[
R_\alpha(r) = R(r) = \frac{K}{(r^2 + r_0^2)^{3/2}}, \quad q(r) = -\frac{15}{\alpha g} \frac{r_0^2}{(r^2 + r_0^2)^2}. \tag{A14}
\]

Two real zero modes are replaced by a pair of zero modes. So one real zero mode remains for each value of \( \alpha \).

**Appendix B. Derivation of the Lagrangians Eqs. (2.19) and (5.1) by the use of “source”**

In the instanton case, the fermion determinant does not vanish if fermion sources exist (Refs. [13,14]). Following this case, we introduce a field \( \varphi(x) \), and replace \( i \bar{c} \cdot \partial \mu D_\mu c \) with

\[
i \bar{c} \cdot [\partial \mu D_\mu + g \varphi \times] c. \tag{B1}
\]

The eigenvalue equation is

\[-[\partial \mu D_\mu + g \varphi \times] w_n = \Lambda_n w_n.\]

We treat the term \( g \varphi \times \) as perturbation, and perform the expansion

\[ w_n = w_n^{(0)} + w_n^{(1)} + \ldots, \quad \Lambda_n = \Lambda_n^{(0)} + \Lambda_n^{(1)} + \ldots, \]

where \( \Lambda_n^{(0)} = \lambda_n \), and \( w_n^{(0)} = u_n \) in Eq. (2.4) and \( w_n^{(0)} = u_n^\ast \) in Eq. (2.5). Using the normalization \( \int dx u_n^\ast \cdot u_n = 1 \) and \( \int dx u_n \cdot u_n = 0 \), we obtain

\[ \Lambda_n^{(1)} = g \int dx u_n^\ast \cdot (\varphi \times u_n) \]

for \( u_n \) and

\[ \Lambda_n^{(1)\ast} = g \int dx u_n \cdot (\varphi \times u_n^\ast) \]

for \( u_n^\ast \), where \( f^{ABC} u_A^\ast u_B \varphi^C = 0 \) has been used. Therefore, if \( \partial \mu D_\mu \) has a pair of zero modes \((u_0, u_0^\ast)\), Eq. (B1) gives rise to the determinant

\[
\det[-\partial \mu D_\mu - g \varphi \times] = \prod_n |\Lambda_n|^{k_n} \approx |\Lambda_0^{(1)}|^2 \prod_{n \neq 0} |\Lambda_n|^{k_n} \]

\[ = g^2 \int dx u_0^\ast \cdot (\varphi \times u_0)^2 \prod_{n \neq 0} |\Lambda_n|^{k_n}, \tag{B2}\]

where \( k_n \) is the number of eigenfunctions that have the eigenvalue \( \Lambda_n \) or \( \Lambda_n^\ast \). Thus, although \( \Lambda_0^{(0)} = \lambda_0 = 0 \), \( \Lambda_0^{(1)} \neq 0 \) makes the partition function nonzero.
Since
\[ g \int dx \left( \Phi \times u_0 \right) \propto \int d\xi d\bar{\xi} d\xi^\dagger d\bar{\xi}^\dagger \left[ g \int dx \bar{\xi} \cdot (\Phi \times u_0) \right] \times \left[ g \int dy \bar{\xi} u_0 \cdot (\Phi \times \xi^\dagger u_0^\dagger) \right], \]
we find
\[ DcD\bar{c} \exp \left\{ -i \int dx \left( \partial_\mu D_\mu + g \Phi \times c \right) \right\} \]
gives the determinant Eq. (B2). To derive Eq. (2.19), we multiply Eq. (B3) by
\[ \exp \left\{ -\int dx \left( \varphi + \alpha_2 B \right)^2 / (2\alpha_2) \right\}, \]
and integrate with respect to \( \varphi \):
\[ D\varphi \exp \left\{ -i \int dx \left[ \partial_\mu D_\mu + g \varphi \right] + c \right\} \cdot \int dx \left( \frac{\varphi^2}{2\alpha_2} + B \varphi + \frac{\alpha_2 B^2}{2} \right) \]
After the \( \varphi \) integration, we obtain Eq. (2.19).

We note that to derive Eq. (5.1), then Eq. (B3) must be multiplied by
\[ \exp \left\{ -\int dx \left( \varphi - w + \alpha_2 B \right)^2 / (2\alpha_2) \right\}, \]
where \( w \) is a constant determined later.

**Appendix C. Derivation of the RG equations (4.1) and (5.2)**

In Sect. C.1, using \( \mathcal{L}_{NL} \), we derive the RG equations (4.1). In Sect. C.2, the RG equation (5.2) is derived. The RG equation for \( \alpha_1 \) under the scale \( \mu_0 \) is discussed.

**C.1. The Lagrangian Eq. (2.19) and the RG equations (4.1)**

**C.1.1. Equation for \( \alpha_2 \)**

The Lagrangian \( \mathcal{L}_{NL} \) contains the quartic ghost interaction
\[ -\frac{\alpha_2}{2} (ig \bar{c} \times c)^2. \]

We define the renormalization constant \( Z_4 \) by
\[ (\alpha_2 g^2)_0 = Z_4 \tilde{Z}_3^{-2} \alpha_2 g^2, \]
where \( \tilde{c}_0 = \tilde{Z}_3^{1/2} \bar{c} \) and \( c_0 = \tilde{Z}_3^{1/2} c \). First we consider the ghost self-energy. Although \( \mathcal{L}_{NL} \) gives additional one-loop diagrams, the divergence of them cancels out. Thus we obtain, as usual, \( \tilde{Z}_3 = 1 + \tilde{Z}_3^{(1)} + \cdots \) with
\[ \tilde{Z}_3^{(1)} = \frac{2g^2}{(4\pi)^2} \frac{1}{(3 - \alpha) 4\varepsilon}, \]
where \( \varepsilon = (4-D)/2 \), and \( C_2(G) = 2 \) is inserted. We note the gauge parameter in \( \mathcal{L}_{NL} \) is \( \alpha = \alpha_1 + \alpha_2 \).

Next we study \( Z_4 \). Using the notation of Fig. C1, one-loop diagrams that contribute to \( Z_4 \) come from the diagrams in Figs. C2 and C3. However Fig. C2(b) does not yield divergence, and the divergences of Figs. C2(c1)–(c3) cancel out. Furthermore some of the diagrams derived from Fig. C3 don’t diverge.
Fig. C1. The vertex and the propagator peculiar to $\mathcal{L}_{NL}$.

Fig. C2. The diagrams that contribute to one-loop correction for the $(\bar{c} \times c)^2$ vertex.

Fig. C3. The diagrams that contribute to one-loop correction for the $(c \times c)(\bar{c} \times \bar{c})$ vertex.

Thus divergent diagrams are depicted in Fig. C4, and they give the constant

$$Z_4 = 1 + Z^{(1)}_{4a} + Z^{(1)}_{4b} + Z^{(1)}_{4c} + Z^{(1)}_{4d} + \cdots,$$

where

$$Z^{(1)}_{4a} = \frac{2g^2}{(4\pi)^2} (-\alpha_2) \frac{1}{\varepsilon}, \quad Z^{(1)}_{4b} = \frac{2g^2}{(4\pi)^2} (-\alpha_2) \frac{1}{2\varepsilon},$$

$$Z^{(1)}_{4c} = \frac{2g^2}{(4\pi)^2} (\alpha_2) \frac{1}{\varepsilon}, \quad Z^{(1)}_{4d} = \frac{2g^2}{(4\pi)^2} (-\alpha_1 - \alpha_2) \frac{1}{2\varepsilon}. \quad (C3)$$

Equation (C1) leads to

$$\mu \frac{\partial {\alpha}_2 g^2}{\partial \mu} = -\frac{\mu}{Z_4 Z_3^{-2}} \frac{\partial Z_4 Z_3^{-1}}{\partial \mu} \alpha_2 g^2,$$

$$Z_4 Z_3^{-2} = 1 + Z^{(1)}_{4a} + Z^{(1)}_{4b} + Z^{(1)}_{4c} + Z^{(1)}_{4d} - 2Z^{(1)}_3 + \cdots. \quad (C4)$$
Then performing the replacement $g \rightarrow g \mu^{-\varepsilon}$ or $1/\varepsilon \rightarrow 2 \ln \Lambda/\mu$ in Eqs. (C2) and (C3), and using the RG equation

$$\mu \frac{\partial g}{\partial \mu} = -\beta_0 \frac{g^3}{(4\pi)^2}, \quad \beta_0 = \frac{22}{3},$$

we obtain

$$\mu \frac{\partial \alpha_2}{\partial \mu} = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_2 \right) \alpha_2. \quad \text{(C5)}$$

### C.1.2. Equation for $\alpha_1$

Renormalization constants are defined as usual:

$$A_0^\mu = \sqrt{Z_3} A^\mu, \quad Z_3 = 1 + Z_3^{(1)} + \cdots, \quad B_0 = \sqrt{Z_B} B, \quad Z_B = 1 + Z_B^{(1)} + \cdots,$$

$$(\alpha_j)_0 = Z_{\alpha_j} \alpha_j, \quad Z_{\alpha_j} = 1 + Z_{\alpha_j}^{(1)} + \cdots (j = 1, 2). \quad \text{(C6)}$$

Then $\mathcal{L}_{\text{NL}}$ gives the counter terms

$$\frac{1}{2}(Z_B^{(1)} + Z_3^{(1)}) B \partial_\mu A^\mu, \quad \frac{1}{2} \left\{ (Z_B^{(1)} + Z_{\alpha_1}^{(1)}) \alpha_1 + (Z_{B_0}^{(1)} + Z_{\alpha_2}^{(1)}) \alpha_2 \right\} B^2.$$

The first counter term cancels the divergence of Fig. C5(a), and we obtain

$$Z_B^{(1)} + Z_3^{(1)} = \frac{2g^2}{(4\pi)^2} \frac{-\alpha_2}{\varepsilon}.$$

As the gauge parameter in $\mathcal{L}_{\text{NL}}$ is $\alpha$, the constant $Z_3^{(1)}$ is

$$Z_3^{(1)} = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha \right) \frac{1}{2\varepsilon}. \quad \text{(C7)}$$

as usual. Using these results, $Z_B^{(1)}$ becomes

$$Z_B^{(1)} = \frac{2g^2}{(4\pi)^2} \left( \alpha_1 - \alpha_2 - \frac{13}{3} \right) \frac{1}{2\varepsilon}. \quad \text{(C8)}$$
Fig. C5. The one-loop diagrams that contribute to the propagators for $A_\mu B$ and $B^A B^B$.

The divergence of Fig. C5(b) is canceled by the second counter term, i.e.,

$$Z^{(1)}_B(\alpha_1 + \alpha_2) + Z^{(1)}_{\alpha_1} \alpha_1 + Z^{(1)}_{\alpha_2} \alpha_2 = \frac{2g^2}{(4\pi)^2} \frac{-\alpha_2}{\epsilon}. \quad (C9)$$

From Eqs. (C5) and (C6),

$$\mu \frac{\partial}{\partial \mu} \alpha_2 = -\mu \frac{\partial Z^{(1)}_{\alpha_2}}{\partial \mu} \alpha_2 = 2g^2 \left( \frac{13}{3} - \alpha_2 \right) \frac{\alpha_2}{2\epsilon}$$

and

$$Z^{(1)}_{\alpha_2} \alpha_2 = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_2 \right) \frac{\alpha_2}{2\epsilon} \quad (C10)$$

is derived. Substituting Eqs. (C8) and (C10) into Eq. (C9), we obtain

$$Z^{(1)}_{\alpha_1} \alpha_1 = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_1 \right) \frac{\alpha_1}{2\epsilon}$$

and

$$\mu \frac{\partial}{\partial \mu} \alpha_1 = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_1 \right) \frac{\alpha_1}{\epsilon}. \quad (C11)$$

C.2. RG equations near $\mu_0$ and under $\mu_0$

C.2.1. Equation (5.2)

The RG equation (5.2) is derived from the Lagrangian $L_\phi$ (Ref. [22]). To derive it from the Lagrangian $L_{NL}$, we must replace Eq. (C4) with

$$Z_4 \tilde{Z}_3^{-2} \approx 1 + Z^{(1)}_{4a}. \quad (C12)$$

Namely, in the region $\mu_0 < \mu < \Lambda$, the interaction between $\bar{c}$ and $c$ becomes strong, and Fig. C4(a) is the main contribution. In the limit $\mu \to \mu_0$, $\bar{c}$ and $c$ make bound states and ghost condensate.

C.2.2. RG equation for $\alpha_1$

Near $\mu_0$, as we stated above, the Lagrangian Eq. (5.1) should be used. Under $\mu_0$, we must use the Lagrangian Eq. (5.4). In these Lagrangians, the gauge parameter for $A_\mu$ is not $\alpha$ but $\alpha_1$. Then, instead of Eq. (C7), we must use

$$Z^{(1)}_3 = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_1 \right) \frac{1}{2\epsilon}. \quad (C13)$$
Since the self-energies for $BA_\mu$ and $BB$ don’t have divergence now, $Z_{\alpha_1} = Z_B^{-1} = Z_3$ holds. Thus we have

$$\mu \frac{\partial}{\partial \mu} \alpha_1 = -\mu \frac{\partial Z_3}{\partial \mu} \alpha_1 = \frac{2g^2}{(4\pi)^2} \left( \frac{13}{3} - \alpha_1 \right) \alpha_1.$$

That is, the RG equation for $\alpha_1$ is unchanged.

**Appendix D. Symmetries of the Lagrangian $L_\varphi$ in Eq. (5.1)**

**D.1. BRS symmetry**

It is easy to check that $L_\varphi$ is invariant under the BRS transformation

$$\delta_B A_\mu = D_\mu c, \quad \delta_B c = -\frac{g}{2} c \times c, \quad \delta_B \bar{c} = iB, \quad \delta_B B = 0, \quad \delta_B \varphi = g\varphi \times c.$$

The constant $w$ is determined in such a way as to conserve this symmetry. From the partition function

$$Z_\varphi = \int D\mu \exp \left\{ -\int dx \left( L_{\text{inv}} + L_\varphi \right) \right\},$$

we can derive the equation of motion for $B$ as

$$\langle (-\alpha_1 B + \partial_\mu A_\mu + \varphi - w) \rangle = 0, \quad (\text{D1})$$

where

$$\langle \Phi \rangle = \frac{1}{Z_\varphi} \int D\mu \Phi \exp \left\{ -\int dx \left( L_{\text{inv}} + L_\varphi \right) \right\}.$$

Since $D\mu$ and $L_{\text{inv}} + L_\varphi$ are invariant under the BRS transformation,

$$\langle \delta_B \Phi \rangle = 0 \quad (\text{D2})$$

holds. We substitute $B = -i\delta_B \bar{c}$ and $\varphi(x) = \varphi_0 + \varphi'(x)$ into Eq. (D1), and use $\langle A_\mu \rangle = 0$, $\langle \varphi' \rangle = 0$. Then Eq. (D1) leads to $i\alpha_1 \langle \delta_B \bar{c} \rangle = w - \varphi_0$. Consistency with Eq. (D2) requires $w = \varphi_0$.

**D.2. Anti-BRS symmetry**

The anti-BRS transformation is given by

$$\tilde{\delta}_B A_\mu = D_\mu \tilde{c}, \quad \tilde{\delta}_B \tilde{c} = -\frac{g}{2} \tilde{c} \times \tilde{c}, \quad \tilde{\delta}_B c = i\tilde{B}, \quad \tilde{\delta}_B B = gB \times \tilde{c}, \quad \tilde{\delta}_B \varphi = 0.$$

When $\varphi_0 \neq 0$, from the equation of motion for $\varphi$, $\langle \alpha_2 \tilde{B} \rangle = \langle \varphi \rangle \neq 0$ holds. Therefore the anti-BRS symmetry is broken spontaneously, because

$$\langle \tilde{\delta}_B c \rangle = \langle i\tilde{B} \rangle \neq 0.$$

In addition, we must set $w = \varphi_0 \neq 0$ to maintain BRS symmetry. As $\tilde{\delta}_B L_\varphi = -g(B \times \tilde{c}) \cdot w$, the Lagrangian does not respect the anti-BRS symmetry.
Global gauge symmetry

Using the constant small parameter $\theta$, the global gauge transformation is defined by $\delta_\theta \Phi = \theta \times \Phi$, where $\Phi$ represents all the fields in $\mathcal{L}_\psi$. This symmetry breaks down just like the anti-BRS symmetry. In fact, $\varphi_0 \neq 0$ gives $\langle \delta_\theta \varphi \rangle = \theta \times \varphi_0$, and $w = \varphi_0$ brings $\delta_\theta \mathcal{L}_\psi = -w \cdot (\theta \times B)$.

Next we study the partition function $Z_\psi$. It transforms as $\delta_\theta Z_\psi \propto \langle \delta_\theta \mathcal{L}_\psi \rangle$. Using $B = -i\delta \bar{c}$ and Eq. (D2), we find

$$\delta_\theta Z_\psi \propto -i(w \times \theta) \cdot \langle \delta \bar{c} \rangle = 0.$$ 

Namely, because of BRS symmetry, $Z_\psi$ remains invariant under this symmetry.

In the same way, we can show that the breaking by $w$ cannot be observed in any function $\langle \Psi(\Phi) \rangle$, if $\Psi(\Phi)$ is BRS invariant. To show this, we consider the function

$$\langle \delta_\theta \mathcal{L}_\psi \Psi(\Phi) \rangle,$$

which appears in $\delta_\theta \langle \Psi(\Phi) \rangle$. Using $\delta_\theta \mathcal{L}_\psi = -i(w \times \theta) \cdot \delta \bar{c}$ and $\delta_B \Psi(\Phi) = 0$, we find Eq. (D3) vanishes. Thus BRS-invariant Green functions aren’t broken by $\delta_\theta \mathcal{L}_\psi$.

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