Polynomial Automorphisms, Quantization and Jacobian Conjecture Related Problems

Andrey Elishev
Alexei Kanel-Belov
Farrokh Razavinia
Jie-Tai Yu
Wenchao Zhang
Contents

Preface 5

1 Introduction 7
  1.1 Quantization and algebra problems .............................. 7
    1.1.1 Free algebras ........................................ 7
    1.1.2 Matrix representations of algebras ....................... 8
    1.1.3 Algebra of generic matrices ............................ 8
    1.1.4 The Amitsur-Levitzki theorem .......................... 9
    1.1.5 Deformation quantization .............................. 11
    1.1.6 Algebraically closed skew field ......................... 17
  1.2 Automorphisms of polynomial algebras and Kontsevich Conjecture 20
    1.2.1 Jacobian Conjecture .................................. 21
    1.2.2 Some results related to the Jacobian Conjecture ........ 22
    1.2.3 Ind-schemes and varieties of automorphisms ................ 25
    1.2.4 Conjectures of Dixmier and Kontsevich .................. 26
    1.2.5 Approximation by tame automorphisms .................... 29
    1.2.6 Holonomic D-modules, Lagrangian subvarieties and the work of Dodd ...... 31
    1.2.7 Tame automorphisms and the Kontsevich Conjecture ......... 34
    1.2.8 Homomorphism \( \phi \) and Ind-schemes .................. 36
    1.2.9 Approximation and analysis of curves in Aut .............. 39
    1.2.10 Quantization of classical algebras ..................... 41
  1.3 Torus actions on free associative algebras and the Bialynicki-Birula theorem .... 41

2 Quantization proof of Bergman’s centralizer theorem 45
  2.1 Centralizer theorems ...................................... 45
    2.1.1 Cohn’s centralizer theorem ............................ 46
    2.1.2 Bergman’s centralizer theorem ........................ 47
  2.2 Reduction to generic matrix ................................ 48
  2.3 Quantization proof of rank one .............................. 49
  2.4 Centralizers are integrally closed ........................... 52
    2.4.1 Invariant theory of generic matrices .................... 52
    2.4.2 Centralizers are integrally closed ..................... 53
3 Automorphisms, augmentation topology, and approximation

3.1 Introduction and main results

3.1.1 Automorphisms of $K[x_1, \ldots, x_n]$ and $K\langle x_1, \ldots, x_n \rangle$

3.1.2 Main results

3.2 Varieties of automorphisms

3.2.1 Elementary and tame automorphisms

3.2.2 Ind-schemes and Ind-groups

3.3 The Jacobian conjecture in any characteristic, Kanel-Belov – Kontsevich conjecture, and approximation

3.3.1 Approximation problems and Kanel-Belov – Kontsevich Conjecture

3.3.2 The Jacobian conjecture in any characteristic

3.3.3 Approximation for the automorphism group of affine spaces

3.3.4 Lifting of automorphism groups

3.4 Automorphisms of the polynomial algebra and the approach of Bodnarchuk–Rips

3.4.1 Reduction to the case when $\Psi$ is identical on $\text{SL}_n$

3.4.2 The lemma of Rips

3.4.3 Generators of the tame automorphism group

3.4.4 Aut(TAut) for general case

3.5 The approach of Bodnarchuk–Rips to automorphisms of $\text{TAut}(K\langle x_1, \ldots, x_n \rangle)$ ($n > 2$)

3.5.1 The automorphisms of the tame automorphism group of $K\langle x_1, \ldots, x_n \rangle, n \geq 4$

3.5.2 The group $\text{Aut}_{\text{Ind}}(\text{TAut}(K\langle x, y, z \rangle))$

3.6 Some open questions concerning the tame automorphism group

4 Approximation by tame automorphisms and the Kontsevich conjecture

4.1 Endomorphisms of $\mathbb{K}[X], W_n(\mathbb{K})$ and $P_n(\mathbb{K})$

4.1.1 Definitions and notation

4.1.2 Tame automorphisms

4.2 Approximation by tame automorphisms

4.3 Approximation by tame symplectomorphisms and lifting to Weyl algebra

4.4 Conclusion

4.5 Augmented Weyl algebra structure

4.5.1 Continuity of $\Phi^{hk}$ and the singularity trick

4.5.2 Lifting in the $h$-augmented and skew augmented cases

4.5.3 The lifted limit is polynomial

4.5.4 Specialization
5 Torus actions on free associative algebras, lifting and Bialynicki-Birula type theorems

5.1 Actions of algebraic tori .......................................................... 130
5.2 Maximal torus action on the free algebra .................................. 133
5.3 Discussion ............................................................................. 137

6 Jacobian conjecture, Specht and Burnside type problems

6.1 The Jacobian Conjecture and Burnside type problems, via algebras 141
   6.1.1 The Yagzhev correspondence ............................................ 142
   6.1.2 Translation of the invertibility condition to the language of identities . . . 144
   6.1.3 Lifting Yagzhev algebras ................................................. 152
   6.1.4 Inversion formulas and problems of Burnside type ............... 153
6.2 The Jacobian Conjecture for varieties, and deformations .................. 154
   6.2.1 Generalization of the Jacobian Conjecture to arbitrary varieties .... 154
   6.2.2 Deformations and the Jacobian Conjecture for free associative algebras . 156
   6.2.3 The Jacobian Conjecture for other classes of algebras ............ 160
   6.2.4 Questions related to the Jacobian Conjecture ...................... 161
   6.2.5 Reduction to simple algebras ........................................... 164
Preface

The purpose of this review is the collection and systematization of results concerning the quantization approach to the Jacobian conjecture.

The Jacobian conjecture of O.-H. Keller remains, as of the writing of this text, an open and apparently unassailable problem. Various possible approaches to the Jacobian conjecture have been explored, resulting in accumulation of a substantial bibliography, while the development of vast parts of modern algebra and algebraic geometry were in part stimulated by a search for an adequate framework in which the Jacobian conjecture could be investigated. This has engendered a situation of simultaneous existence of circumstantial evidence in favor and against the positivity of this conjecture.

One of the more established plausible approaches to the Jacobian problem concerns the study of infinite-dimensional algebraic semigroups of polynomial endomorphisms and groups of automorphisms of associative algebras, as well as mappings between those. The foundation for this approach was laid by I.R. Shafarevich. During the last several decades, the theory was developed and vastly enriched by the works of Anick, Artamonov, Bass, Bergman, Dicks, Dixmier, Lewin, Makar-Limanov, Czerniakiewicz, Shestakov, Umirbaev, Bialynicki-Birula, Asanuma, Kambayashi, Wright, and many others. In particular, the results of Anick, Makar-Limanov, Shestakov and Umirbaev established a connection between the Jacobian conjecture for the commutative polynomial algebra and its associative analogues on the one hand with combinatorial and geometric properties (stable tameness, approximation) of the spaces of polynomial automorphisms on the other.

More recently, the stable equivalence between the Jacobian conjecture and a conjecture of Dixmier on the endomorphisms of the Weyl algebra has been discovered by Kanel-Belov and Kontsevich and, independently, by Tsuchimoto. The cornerstone of this rather surprising feature is a certain mapping (sometimes referred to as the anti-quantization map) from the semigroup of Weyl algebra endomorphisms (a quantum object) to the semigroup of endomorphisms of the corresponding Poisson algebra (the appropriate classical object). In view of that, it seems reasonable to think there are insights to be gained by studying quantization of spaces of polynomial mappings and properties of the corresponding quantization morphisms.

In this direction, one of the larger milestones is given by a series of conjectures of Kontsevich concerning equivalences between polynomial symplectomorphisms, holonomic modules over alge-
bras of differential operators, and automorphisms of such algebras. Another rather non-trivial side of the quantization program rests upon the interaction with universal algebra.

In this review we present some of our progress regarding quantization, Kontsevich conjecture, as well as recall some of our recent results on the geometry of Ind-scheme automorphisms, approximation by tame automorphisms together with its symplectic version, and torus actions on free associative algebras. We also provide a review of work of Kanel-Belov, Bokut, Rowen and Yu, which sought to connect the Jacobian problem with various problems in universal algebra, as conceived by the brilliant late mathematician A.V. Yagzhev.

We have benefitted greatly from extensive and fruitful discussions with E. Aljadeff, I.V. Arzhantsev, V.A. Artamonov, E.B. Vinberg, A.E. Guterman, V.L. Dolnikov, I.Yu Zhdanovskii, A.B. Zheglov, D. Kazhdan, R.N. Karasev, I.V. Karzhemanov, V.O. Manturov, A.A. Mikhalev, S.Yu. Orevkov, E.B. Plotkin, B.I. Plotkin, A.M. Raigorodskii, E. Rips, A.L. Semenov, N.A. Vavilov, G.B. Shabat, U. Vishne and G.I. Sharygin. It is a pleasant task to express our utmost gratitude to our esteemed colleagues.

This work is supported by the Russian Science Foundation grant No. 17-11-01377.
Chapter 1

Introduction

1.1 Quantization and algebra problems

This section provides the overview of the Jacobian conjecture together with motivation for the theory of Ind-schemes and quantization, as well as some necessary preliminaries on the proof of Bergman’s centralizer theorem. Throughout this paper, all rings are associative with multiplicative identity.

1.1.1 Free algebras

A free algebra is a noncommutative analogue of a polynomial ring since its elements may be described as "polynomials" with non-commuting variables, while the free commutative algebra is the polynomial algebra. Let us first give the definition of a free monoid, which is needed in our definition of free algebras [168].

**Definition 1.1.1.** Let \( X = \{ x_i : i \in I \} \). A word is a string with elements in \( X \). A free associative monoid on a set \( X \), namely \( X^* \), is the set of words in \( X \), including the empty product to represent 1. The multiplication on \( X^* \) is given by the juxtaposition of words.

Next we can naturally give a definition of the free associative algebra respect to a generating set over a commutative ring.

**Definition 1.1.2.** Let \( C \) be a commutative ring with multiplicative identity. A free associative \( C \)-algebra \( C\langle X \rangle \) with respect to a generating set \( X = \{ x_i : i \in I \} \) is the free \( C \)-module with base \( X^* \).

**Remark 1.1.3.** This \( C \)-module becomes a \( C \)-algebra by defining a multiplication as follows: the product of two basis elements is the concatenation of the corresponding words and the product of two arbitrary \( C \)-module elements are thus uniquely determined. Note that \( C\langle X \rangle := \bigoplus_{w \in X^*} Cw \) and the elements of \( C\langle X \rangle \) are called noncommutative polynomials over \( C \) generated by \( X \).

By the same token, we can also define the free associative algebra respect to a generating set \( X = \{ x_i : i \in I \} \) over an arbitrary field \( k \), namely \( k\langle X \rangle \).
Remark 1.1.4. If $C$ is an integral domain, then the product of leading monomials of two non-commutative polynomials $f$ and $g$ in $C(X)$ is the leading monomial of $fg$. It follows that $C(X)$ is a domain (but still noncommutative) as well.

We finally remark that we will only discuss free associative $k$-algebras with respect to a generating set $X = \{x_1, \ldots, x_s\}$ (for $s \geq 2$) over a field $k$ instead of a commutative ring throughout this review.

1.1.2 Matrix representations of algebras

Let $A$ be a $k$-algebra, and let $K$ be a field extension of $k$. We talk about finite dimensional representations of $A$ in this work, so when we mention a representation, we mean it is a finite-dimensional representation.

Definition 1.1.5. An $n$-dimensional matrix representation over $K$ is a $k$-homomorphism $\rho : A \mapsto M_n(K)$ to the matrix algebra over $K$.

Remark 1.1.6. Two representations $\rho', \rho$ are equivalent if they are conjugate, namely $\rho' = \tau \rho \tau^{-1}$ for some invertible matrix $\tau \in M_n(K)$.

The representation is irreducible if the images of $A$ generates the matrix algebra as $K$-algebra, or if the map $A \otimes K \mapsto M_n(K)$ is surjective. Usually, we study the case when $K = k$. With this assumption, we call a representation $\rho : A \mapsto M_n(k)$ is irreducible if and only if it is surjective.

1.1.3 Algebra of generic matrices

In order to use the concept of generic matrices, we need to first introduce the matrix representation of free associative algebra [14]. We have introduced matrix representations of any $k$-algebras in Section 1.1.2. A matrix representation of the free associative ring $k\langle X \rangle = k\langle x_1, \ldots, x_s \rangle$ over $k$ generated by a finite set $X = \{x_1, \ldots, x_s\}$ of $s$ ($s \geq 2$) indeterminates is given by assigning arbitrary matrices as images of the variables. In itself, this is not very interesting. However, when one asks for equivalence classes of irreducible representations, the other is directed to an interesting problem in invariant theory. We will discuss this topic in the following.

Definition 1.1.7. Let $n$ be a positive integer, and let $\{x_{ij}^{(\nu)} | 1 \leq i, j \leq n, \nu \in \mathbb{N}\}$ be independent commuting indeterminates over $k$. Then

$$X_\nu := (x_{ij}^{(\nu)}) \in M_n(k[x_{ij}^{(\nu)}])$$

is called an $n \times n$ generic matrix over $k$, and the $k$-subalgebra of $M_n(k[x_{ij}^{(\nu)}])$ generated by the $X_\nu$ is called the algebra of generic matrices and will be denoted by $k\langle X_1, \ldots, X_s \rangle$ or simply $k\{X\}$. 
Chapter 1. Introduction

The algebra of generic matrices is a basic object in the study of the polynomial identities and invariants of \( n \times n \) matrices.

There is a canonical homomorphism

\[
\pi : \langle x_1, \ldots, x_s \rangle \mapsto \langle X_1, \ldots, X_s \rangle
\]

from the free associative ring on variables \( x_1, \ldots, x_s \) to this ring.

If \( u_1, \ldots, u_s \) are \( n \times n \) matrices with entries in a commutative \( k \)-algebra \( R \), then we can substitute \( u_j \) for \( X_j \), and thereby obtain a homomorphism

\[
k\langle X_1, \ldots, X_s \rangle \mapsto M_n(R).
\]

There is an important property of the homomorphism \( \pi \): an element \( f \) of the free associative algebra is in the kernel of the map \( \pi \), if and only if it vanishes identically on \( M_n(R) \) for every commutative \( k \)-algebra \( R \), and this is true if and only if \( f \) vanishes identically on \( M_n(k) \). In addition, the (irreducible) matrix representations of the free ring \( A \) of dimension \( \leq n \) correspond bijectively to the (irreducible) matrix representations of the ring of generic matrices. This result is a core tool in our proof.

Let \( u_1, \ldots, u_N \) (\( N = n^2 \)) be a basis for the matrix algebra \( M_n(\bar{K}) \), and let \( z_1, \ldots, z_N \) be indeterminates. Then the entries of the matrix \( Z = \sum z_j u_j \) are all algebraically independent. Moreover, we have the famous Amitsur’s theorem [14] as follows.

**Theorem 1.1.8** (Amitsur). The algebra \( \langle X_1, \ldots, X_s \rangle \) of generic matrices is a domain.

**Proof.** cf. [14] Theorem V.10.4 or [231] Theorem 3.2. \( \square \)

### 1.1.4 The Amitsur-Levitzki theorem

For the free associative algebra \( A = \langle X \rangle \), the *commutator* of two elements in \( A \) is defined by 

\[
[x, y] = xy - yx.
\]

The commutator has analogues for more variables, called *generalized commutators* [14] of elements \( x_1, \ldots, x_n \) of \( A \),

\[
S_n(x_1, x_2, \ldots, x_n) := \sum (-1)^{\sigma} x_{\sigma 1} \cdots x_{\sigma n},
\]

where \( \sigma \) runs over the groups of all permutations. It is clear that \( S_2(x, y) = [x, y] \). Note that the generalized commutators are multilinear and alternating polynomials in the variables. Moreover, a general multilinear polynomial in \( n \) variables has the form \( p(x_1, \ldots, x_n) = \sum c_\sigma x_{\sigma 1} \cdots x_{\sigma n} \), where the coefficients \( c_\sigma \) are elements of \( k \).

There is an important and powerful result [6] which is first proved by A. S. Amitsur and J. Levitzki in 1950.
Chapter 1. Introduction

Theorem 1.1.9 (Amitsur-Levitzki). Let $R$ be a commutative ring, and let $r$ be an integer.

(i) If $r \geq 2n$, then $\sum_r(a_1,\ldots,a_r) = 0$ for every set $a_1,\ldots,a_r$ of $n \times n$ matrices with entries in $R$.

(ii) Let $p(x_1,\ldots,x_r)$ be a nonzero multilinear polynomial. If $r < 2n$, then there exist $n \times n$ matrices $a_1,\ldots,a_r$ such that $p(a_1,\ldots,a_r) \neq 0$. In particular, $\sum_r(x_1,\ldots,x_r)$ is not identically zero.

The identity $\sum_2^n \equiv 0$ is called the standard identity of $n \times n$ matrices. Note that $\sum_2^n \equiv 0$ is the commutative law, which holds for any $1 \times 1$ matrices but not for any $n \times n$ matrices if $n > 1$.

Remark 1.1.10. The Amitsur-Levitzki theorem is quite important [14]. Suppose we study a representation $A \mapsto M_n(k)$ of a $k$-algebra $A$. Let $I \subset A$ be the ideal generated by all substitutions of elements of $A$ into $\sum_2^n$, and let $\tilde{A} = A/I$. The Amitsur-Levitzki theorem tells us that $\sum_2^n = 0$ is true in $M_d(k)$ if $d \leq n$ whereas it is not true if $d > n$. Killing $I$ has the effect of keeping the representations of dimensions $\leq n$, and cutting out all irreducible representations of higher dimension.

The original proof of the Amitsur-Levitzki theorem by Amitsur and Levitzki is a direct proof, which is quite involved. Rosset (1976) gives a short proof [167] using the exterior algebra of a vector space of dimension $2n$. This proof can be also found in [231] Theorem 1.7. Then we obtain the following proposition.

Proposition 1.1.11. Let $k\{X\}$ be the algebra of generic matrices.

a) Every minimal polynomial of $A \in k\{X\}$ is irreducible. In particular, $A$ is diagonalizable.

b) Eigenvalues of $A \in k\{X\}$ are roots of irreducible minimal polynomial of $A$, and every eigenvalue appears same amount of times.

c) The characteristic polynomial of $A$ is a power of minimal polynomial of $A$.

There is an important open problem well-known in the community.

Problem 1.1.12. Whether for big enough $n$, every non-scalar element in the algebra of generic matrices has a minimal polynomial which always coincides with its characteristic polynomial.

This is an important open problem. For small $n$, Galois group of extension quotient field of center of algebra of generic matrices might not be symmetry. But it still unknown for big enough $n$.

From above Proposition 1.1.11 c), for $n = p$, a big enough prime, we can obtain following corollary.

Corollary 1.1.13. Let $k\{X\}$ be the algebra of generic matrices of a big enough prime order $n := p$. Assume $A$ is a non-scalar element in $k\{X\}$, then the minimal polynomial of $A$ coincides with its characteristic polynomial.
Proof. Let \( m(A) \) and \( c(A) \) be the minimal polynomial and the characteristic polynomial of \( A \) respectively. Note that \( \deg c(A) = n \), and \( c(A) = (m(A))^k \). Because \( A \) is not scalar, hence \( \deg m(A) > 1 \). Since \( n \) is a prime, \( k \) divides \( n \). Hence \( k = 1 \).

Let us here remind the following fact:

**Proposition 1.1.14.** Every matrix with same eigenvectors as a matrix \( A \) commutes with \( A \).

**Proof.** Let \( A, B \in M_{n \times n}(k) \) and having \( n \) eigenvectors meaning that the have \( n \) linearly independent eigenvectors. And since \( A \) and \( B \) are \( n \times n \) matrices with \( n \) eigenvectors, then they both are diagonalizable and hence \( A = Q^{-1}D_AQ \) and \( B = P^{-1}D_BP \), for \( Q \) and \( P \) are matrices whose columns are eigenvectors of \( A \) and \( B \) associated with the eigenvalues listed in the diagonal matrices \( D_A \) and \( D_B \) respectively. But \( A \) and \( B \) according to the hypothesis, have same eigenvectors and hence \( P = Q =: S \). And hence \( A = S^{-1}D_AS \) and \( B = S^{-1}D_BS \) and so \( AB = S^{-1}D_AS S^{-1}D_BS = S^{-1}D_AD_BS \) and in a same way we have \( BA = S^{-1}D_BD_AS \) and since \( D_A \) and \( D_B \) are diagonal matrices, then commute and hence so do \( A \) and \( B \).

**Proposition 1.1.15.** If \( n \) is prime, \( A \) is a non-scalar element of the algebra of generic matrices, then all eigenvalues of \( A \) are pairwise different.

**Proof.** It follows from Proposition 1.1.11 and Corollary 1.1.13 directly.

Proposition 1.1.15 implies following results.

**Proposition 1.1.16.** The set of generic matrices commuting with \( A \) are diagonalizable with \( A \) simultaneously in the same eigenvectors basis as \( A \).

If \( A \) is a non-scalar matrix, then we have following.

**Proposition 1.1.17.** \( A \) is a non-scalar element of the algebra of generic matrices \( k\{X\} \), then every eigenvalue of \( A \) is transcendental over \( k \).

### 1.1.5 Deformation quantization

**Literature review**

In general, the “quantization problem” can be stated as following. Given a classical physical model (Hamiltonian system, Lagrange system on a Riemannian manifold etc.), quantization amounts to replacing the observable functions with operators acting on a Hilbert space, such that they satisfy some specific quantization conditions. In quantum mechanics, this quantization condition is called the *canonical commutation relation*, which is the fundamental relation between canonical conjugate quantities. For example, the commutation relation between different components of position and momentum can be expressed as \( [\hat{P}_i, \hat{Q}_j] = i\hbar\delta_{ij} \), where \( i \) is the imaginary unit and \( \delta_{ij} \) is the Kronecker delta. M. Hermann Weyl studied the Heisenberg uncertainty principle in quantum mechanics by considering the operator ring generated by \( P \) and \( Q \). For any 2n
dimensional linear space $V$, the Kronecker delta can be realized as a symplectic form $\omega$ such that $u \otimes v - v \otimes u = \omega(u, v)$ defines a Weyl algebra $W(V)$ over $V$. In this sense, classical mechanics corresponds to symmetric algebra, while the Weyl algebra is the "quantization" of symmetric algebra.

In 1940s, J. E. Moyal [156] conducted a more in-depth study of the Weyl quantization. Unlike Weyl, the object he was interested in is not operators, but the classical function space: Weyl ignores the Poisson structure of the classical function space. Instead of building a Hilbert space from a Poisson manifold and associating an algebra of operators to it, He was only concerned with the algebra. He used the star product and Moyal bracket to define a Poisson algebraic structure named Moyal algebra over the classical function space. Through the investigation of the Moyal algebra, F. Bayen [30] et al. raised that the quantum algebra can be regarded as the deformation of the classical algebra if we think of $\hbar$ as the deformation parameter. In particular, they proved that for the classical Poisson algebraic structure on the symmetric algebra over $\mathbb{R}^{2n}$, the Moyal algebra is the only possible deformation in the sense of normative equivalence. That is, quantum mechanics is the only possible "deformation" of classical mechanics.

We use the Poisson bracket to "deform" the ordinary commutative product of observables in classical mechanics, elements of our function algebra, and obtain a noncommutative product suitable for quantum mechanics. In order to make deformation, we ask that the Moyal product is not only an asymptotic expansion, but also a real analytical expansion. There is no a priori guarantee for this. From the Darboux theorem, the local Poisson algebra structure on the symplectic manifold can always be deformed into the Moyal algebra. We only need to extend this local deformation to the entire manifold after equipping a flat symplectic connection. However, for a typical Poisson manifold, the situation is much more complicated.

In mid 1970s, the existence of star-products for symplectic manifolds whose third cohomology group is trivial was proved, but this restriction turned out to be merely technical. In the early 1980s the existence of star-products for larger and larger classes of symplectic manifolds was proved, and finally it was shown that any symplectic manifold can be "quantized". A further generalization was achieved with [94] where Fedosov proved that the results about the canonical star-product on an arbitrary symplectic manifold can be used to prove that all regular Poisson manifolds can be quantized. However, in physics we sometimes require manifolds which have a degenerate Poisson bracket and so are not symplectic. Therefore all the results mentioned above provided only a partial answer to the problem of quantization.

In 1993-1994 M. Kontsevich proposed the Formality Conjecture which would imply the desired result. If the Formality Conjecture could be proved, this would infer that any finite-dimensional Poisson manifold can be canonically quantized in the sense of deformation quantization. The Formality Conjecture is proved by Kontsevich in [123]. Kontsevich then derived an explicit quantization formula which gives a formal definition of the Moyal product for any Poisson manifold. However, it is not clear whether it gives the only possible deformation quantization in the sense
Chapter 1. Introduction

of canonical equivalence.

Another direction in which research in deformation quantization has developed is strict deforma-
tion quantization in which the parameter is no longer a formal parameter, but a real one. In a way, the deformed algebras $A[\hbar]$ are identified with the original algebra $A$.

Definitions and basic results

**Definition 1.1.18** (Ring of formal power series). _Let $R$ be a commutative ring with identity, $R[X]$ is said to be a ring of formal power series in the variable $X$ over $R$ if and only if any element of $R[X]$ is of form $\sum_{i\in\mathbb{N}} a_i X^i$ and satisfying following:

\[
\sum_{i\in\mathbb{N}} a_i X^i + \sum_{i\in\mathbb{N}} b_i X^i = \sum_{i\in\mathbb{N}} (a_i + b_i) X^i \quad (1.3)
\]

\[
\sum_{i\in\mathbb{N}} a_i X^i \times \sum_{i\in\mathbb{N}} b_i X^i = \sum_{n\in\mathbb{N}} \left( \sum_{k=0}^n a_k b_{n-k} \right) X^n \quad (1.4)
\]

The above product 1.4 of coefficients is called the Cauchy product of the two sequences of coefficients, and is a sort of discrete convolution. Note that the zero element and the multiplicative identity of the ring of formal power series are the same as ring $R$’s.

**Remark 1.1.19.** _The series $A = \sum_{n\in\mathbb{N}} a_n X^n \in R[X]$ is invertible if and only if its constant coefficient $a_0$ is invertible in $R$. The inverse series of an invertible series $A$ is $B = \sum_{n\in\mathbb{N}} b_n X^n \in R[X]$ with:

\[
b_0 = \frac{1}{a_0}
\]

\[
b_n = -\frac{1}{a_0} \sum_{i=1}^n a_i b_{n-i}, \quad n \geq 1
\]

An important example is the geometric series

\[
(1 - X)^{-1} = \sum_{n=0}^{\infty} X^n.
\]

If $R$ is a field, then a series is invertible if and only if the constant term is non-zero.

**Definition 1.1.20.** _A Lie algebra is a vector space with a skew-symmetric bilinear operation $(f,g) \mapsto [f,g]$ satisfying the Jacobi identity

\[
[[f,g],h] + [[g,h],f] + [[h,f],g] = 0
\]

**Definition 1.1.21.** _A Poisson algebra is a vector space equipped with a commutative associative algebra structure $(f,g) \mapsto fg$ and a Lie algebra structure $(f,g) \mapsto \{f,g\}$ satisfying the Leibniz
rule
\[ \{fg, h\} = f\{g, h\} + \{f, h\}g \]

**Definition 1.1.22.** A Poisson manifold is a manifold \( M \) whose function space \( C^\infty(M) \) is a Poisson algebra with the pointwise multiplication as commutative product.

Let \( k \) be an arbitrary field, and \( A \) be a unitary \( k \)-algebra. Denote by \( k[\hbar] \) the ring of formal power series in an indeterminate \( \hbar \), and by \( A[\hbar] \) the \( k[\hbar] \)-module of formal power series with coefficients in \( A \).

**Definition 1.1.23.** A formal deformation or star product of the algebra \( A \) is an associative, \( \hbar \)-adic continuous, \( k[\hbar] \)-bilinear product

\[ \star : A[\hbar] \times A[\hbar] \mapsto A[\hbar] \]

satisfying the following rule on \( A \):

\[ f \star g = \sum_{n=0}^{\infty} B_n(f, g) \hbar^n = fg + \sum_{n=1}^{\infty} B_n(f, g) \hbar^n \quad \forall f, g \in A \tag{1.5} \]

where \( B_n : A \times A \mapsto A \) are bilinear operators.

**Remark 1.1.24.** We usually want the bilinear operators \( B_n \) to be bidifferential operators, i.e. bilinear maps which are differential operators with respect to each argument.

**Remark 1.1.25.** The formal deformation extends \( k[\hbar] \)-linearity in \( A[\hbar] \) with respect to:

\[ \left( \sum_{k=0}^{\infty} f_k \hbar^k \right) \star \left( \sum_{m=0}^{\infty} g_m \hbar^m \right) = \sum_{n=0}^{\infty} \left( \sum_{m+k+r=n} B_r(f_k, g_m) \right) \hbar^n. \]

There is a natural gauge group acting on star-products. This group consists of automorphisms of \( A[\hbar] \) considered as an \( k[\hbar] \)-module, of the following form:

\[ f \mapsto f + \sum_{n=0}^{\infty} D_n(f) \hbar^n, \quad \forall f \in A \subset A[\hbar] \]

\[ \sum_{n=0}^{\infty} f_n \hbar^n \mapsto \sum_{n=0}^{\infty} f_n \hbar^n + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_m(f_m) \hbar^{n+m}, \quad \forall \sum_{n=0}^{\infty} f_n \hbar^n \in A[\hbar] \]

where \( D_i : A \mapsto A \) are differential linear operators.

**Definition 1.1.26.** \( D(\hbar) \) as defined above is called a gauge transformation in \( A \). The set of such \( D(\hbar) \) is naturally a group.

If \( D(\hbar) = 1 + \sum_{m=1}^{\infty} D_m \hbar^m \) is such an automorphism, then it defines an equivalence and acts on the set of star products as

\[ \star \mapsto \star', f(\hbar) \star' g(\hbar) := D(\hbar)(D(\hbar)^{-1}(f(\hbar)) \star D(\hbar)^{-1}(g(\hbar))), \forall f(\hbar), g(\hbar) \in A[\hbar] \tag{1.6} \]
Each associative formal deformation $\star$ of the multiplication of $A$ admits a unit element $1_\star$. Moreover, such an associative formal deformation $\star$ is always equivalent to another formal deformation $\star'$ with $1_\star = 1_A$, where $1_A$ is the unit element of $A$. We are interested in star products up to gauge equivalence.

The following lemma gives a Poisson structure for an associative formal deformation of the multiplication of an associative and commutative $k$-algebra $A$.

**Lemma 1.1.27** (\cite{119}, lemma 1.1). Let $\star$ be an associative formal deformation of the multiplication of an associative and commutative $k$-algebra $A$. For $f, g \in A$, put

$$\{f, g\} := B_1(f, g) - B_1(g, f).$$

Then the map $\{,\}$ is a Poisson bracket on $A$, i.e., a $k$-linear map such that the bracket is a Lie bracket and satisfies Leibniz rule. In addition, the bracket is dependent only on the equivalence class of $\star$.

**Proof.** For simplicity, we write $[f, g]_\star$ the commutator of the star product $f \star g - g \star f$ for short. The map

$$(f, g) \mapsto \frac{1}{\hbar}[f, g]_\star$$

(1.7)

clearly defines a Lie bracket on $A[\hbar]$. The bracket $\{,\}$ equals the reduction modulo $\hbar$ of this Lie bracket, i.e. it satisfies

$$\frac{1}{\hbar}[f, g]_\star \equiv \{f, g\} \mod \hbar A[\hbar].$$

We may write it in the another form as follows

$$\{f, g\} := \left[\frac{f, g}{\hbar}\right]_{h=0} = B_1(f, g) - B_1(g, f)$$

(1.9)

Therefore the bracket $\{,\}$ is still a Lie bracket, and it also satisfies the Leibniz rule because the Lie bracket defined in 1.7 obeys the rule by associativity of the star product.

Suppose $D(\hbar)$ is an automorphism which yields equivalence of $\star$ and $\star'$, then we have

$$B_1(f, g) + D_1(fg) = B'_1(fg) + D_1(f)g + fD_1(g)$$

for all $f, g \in A$. Thus the difference $B_1(f, g) - B'_1(f, g)$ is symmetric in $f, g$ and does not contribute to $\{,\}$. \hfill $\square$

One can also decompose the operator $B_1$ into the sum of the symmetric part and of the anti-symmetric part:

$$B_1 = B_1^+ + B_1^-, B_1^+(f, g) = B_1^+(g, f), B_1^-(f, g) = -B_1^-(g, f).$$
Then gauge automorphisms affect only the symmetric part of $B_1$, i.e. $B_1^- = (B_1')^-$. The symmetric part is killed by a gauge automorphism. In this notation, we infer that

$$\{f, g\} = B_1(f, g) - B_1(g, f) = 2B_1^-(f, g).$$

Thus, gauge equivalence classes of star products modulo $\hbar^2 A[[\hbar]]$ are classified by Poisson structures. However, it is not clear whether there exists a star product for a given Poisson structure. Moreover, we may ask whether there exists a preferred choice of an equivalence class of star products. As we mentioned before, Maxim Kontsevich [123] showed that there is a canonical construction of an equivalence class of star products for any Poisson manifold.

### Formal deformation quantization

In this section, we may assume that $A$ is the algebra of smooth functions on a Poisson manifold $M$.

**Definition 1.1.28.** A deformation quantization of a Poisson manifold $M$ is a star product on $A$ such that $2B_1^- = \{ , \}$.

We will not reproduce Kontsevich’s proof here. His proof that we will not deal with is in terms of the cohomology of the Hochschild complex. From the following theorem given by M. Kontsevich [123], there is a surjection from the equivalence classes of formal deformations of $A$ onto Poisson brackets on $A$.

**Theorem 1.1.29** (Kontsevich [123]). Let $M$ be a smooth manifold and $A = C^\infty(M)$. Then there is a natural isomorphism between equivalence classes of deformations of the null Poisson structure on $M$ and equivalence classes of smooth deformations of the associative algebra $A$.

In particular, any Poisson bracket on $M$ comes from a canonically defined (modulo equivalence) star product.

Moreover, Kontsevich constructs a section of map, and his construction is canonical up to equivalence for general manifolds $M$. A later result shows that in addition to the existence of a canonical way of quantization, we can define a universal infinite-dimensional manifold parametrizing quantizations.

The simplest example of a deformation quantization is the Moyal product for the Poisson structure on $\mathbb{R}^n$. This is the first known example of a non-trivial deformation of the Poisson bracket.

**Example 1.1.30.** Let $M = \mathbb{R}^n$ and consider a Poisson structure with constant coefficients

$$\alpha = \sum_{i,j} \alpha^i_j \partial_i \wedge \partial_j, \alpha^i_j = -\alpha^j_i \in \mathbb{R}$$
where $\partial_i = \partial / \partial x^i$ is the partial derivative in the derivation of coordinate $x^i$, $i = 1, 2, \ldots, n$. In such a case, we could have
\[
\{f, g\} = \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g).
\]
The Moyal $*$-product is then given by exponentiating this Poisson operator

\[
f \ast g = e^{\hbar \alpha}(f, g) = fg + \hbar \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \alpha^{ij} \alpha^{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) + \ldots
\]
\[
= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1, \ldots, i_n; j_1, \ldots, j_n} \alpha^{i_1 j_1} \ldots \alpha^{i_n j_n} \prod_{k=1}^{n} \partial_{i_k}(f) \prod_{k=1}^{n} \partial_{j_k}(g).
\]
The Moyal product is a deformation of $(M, \alpha)$ but this formula is only valid when $\alpha$ has constant coefficients.

In particular,

**Example 1.1.31.** Let $M = \mathbb{R}^2$. Consider the Poisson bracket given by
\[
\{f, g\} = \mu \circ \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right) (f \otimes g) = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1}
\]
where $\mu$ is the multiplication of functions on $M$. Then Kontsevich’s construction yields the associative formal deformation given by
\[
f \ast g = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial x_1^{i_1} \partial x_2^{i_2}} \frac{\partial^n g}{\partial x_1^{j_1} \partial x_2^{j_2}} \frac{\hbar^n}{n!}.
\]
The explicit construction of Kontsevich’s formal quantization uses combinatorics, such as quivers. We close this section here since we do not need to construct an explicit formula of deformation quantization in our proof.

### 1.1.6 Algebraically closed skew field

The role of algebraically closed fields in commutative algebra is well known. There are some parallel generalizations of the concept of an algebraically closed skew field to non-commutative skew fields have proved useful for settling various questions in ring theory. However, there are various definitions. The diversity of definitions of algebraically closed skew fields is based on different choices of some particular characteristic of a commutative algebraically closed field. A most natural generalization is in the sense of solvability of arbitrary equations which was brought in sight by Bokut [55–57]. In [57], in particular, Bokut raises a question whether algebraically...
closed skew fields exist or not. The affirmative answer to the question is given by L. Makar-Limanov [138]. His result is one of the fundamental contributions to the theory of non-commutative algebraically closed skew fields. In [62], P. M. Cohn outlined a wide research program for skew fields that are algebraically closed in the various senses. Note that not every associative algebras can be embedded into an algebraically closed one, in the sense of solvability of arbitrary equations. For example, the “Metro-Equation” $ax - xa = 1$ (cf. [61]) is never solvable in any extension of a quaternionic skew field. In [121], P. S. Kolesnikov re-prove the Makar-Limanov theorem on the existence of an algebraically closed skew field in the sense of there being a solution for any generalized polynomial equation. He employs a simpler argument for proving that the skew field constructed is algebraically closed.

**Existence of algebraically closed skew field**

We construct a non-commutative skew field $A$ satisfying the following (cf. [121]):

**Definition 1.1.32.** A with center $F$ is said to be algebraically closed if, for any $S(x) \in A*F[x]\setminus A$, there exists an element $a \in A$ such that $S(a) = 0$; here, $*$ stands for a free product.

It is easy to see that if $A$ is a field, that is, $A = F$, then Definition 1.1.32 checks with the usual definition of an algebraically closed field.

Let $F$ be an algebraically closed field of characteristic 0 and $G$ be a commutative group generated by the elements

$$p_{1}^{\lambda_{1}}, q_{1}^{\mu_{1}}, p_{2}^{\lambda_{2}}, q_{2}^{\mu_{2}}, \ldots,$$

where $\lambda_{i}, \mu_{i} \in \mathbb{Q}$, and $p_{i}, q_{i}$ are symbols in some countable alphabet. The group is isomorphic to a direct sum of countably many additive groups $\mathbb{Q}$ of rational numbers. Then we define the lexicographic order on $G$ by setting $p_{1} \ll q_{1} \ll p_{2} \ll \cdots \ll 1$, where $a \ll b$ means that $a^{n} < b$ for all $n > 0$. Correspondingly, $p_{1}^{-1} \gg q_{1}^{-1} \gg p_{2}^{-1} \gg \cdots > 1$. Put $G_{n} = \langle p_{n}^{\lambda_{n}}, q_{n}^{\mu_{n}}, p_{n+1}^{\lambda_{n+1}}, \ldots \rangle$ and $G_{(m)} = \langle p_{1}^{\lambda_{1}}, q_{1}^{\mu_{1}}, \ldots, q_{m}^{\mu_{m}} \rangle$. Obvious, $G_{n}$ is isomorphic to $G$.

Given $G$ and $F$, we construct a set $A$ of Maltsev-Neumann series. Elements $a \in A$ has the form

$$a = \sum_{g \in H_{a}} a(g)g, \quad H_{a} \subset G \text{ is well ordered, } a(g) \in F \setminus \{0\},$$

the set $H_{a}$ is denoted by $\text{suppa}$. Choose a subset $A$ of $A$ so that

$$A = \{a \in A|\text{suppa} \subset G_{(n(a))}\}.$$ 

Accordingly, put $A_{n} = \{a \in A|\text{suppa} \subset G_{n}\}$ and $A_{(n)} = \{a \in A|\text{suppa} \subset G_{(n)}\}$. The set $A$ constructed is exactly the universe of the desired algebraic system. For the series on $A_{n}$, we define ordinary addition and multiplication, and also derivations $\left(\frac{\partial}{\partial p_{1}}, \frac{\partial}{\partial q_{1}}\right)$. Derivatives of the elements $g \in G_{n}$ w.r.t. $p_{1}$ and $q_{1}$ are elements of $A_{n}$. There are several formula related to those derivations.
which is omitted here. Following [121,138], multiplication \(*\) on \(A\) is defined thus:

\[
a, b \in A, a \ast b = \sum_{i \geq 0} \frac{1}{i!} \frac{\partial^i a}{\partial q_1^i} \frac{\partial^i b}{\partial p_1^i}
\]

The \(*\) is well defined and associative (cf. [121]). Then the system \(\langle A, +, \ast, ||\rangle\) is an associative algebra with valuation. That this is a skew field follows from the fact that \(a \ast x = 1\) has a solution in \(A\). Moreover, \(A\) does not satisfy any generalized polynomial identity, i.e. for every non-trivial generalized polynomial \(S(x) \in A \ast F[x] \not\in A\), there exists an element \(a \in A\) such that \(S(a) \neq 0\) (cf. [121] Lemma 1.3.).

We bring the following notion that generalizes the concept of an homogeneous polynomial in \(A \ast F[x]\).

**Definition 1.1.33.** An homogeneous operator over \(A_n\) is

\[
S_n(x) = \sum_{i,j} f_{i,j} x^{(i_1,j_1)} \ldots x^{(i_k,j_k)},
\]

where \(i = (i_1 \ldots i_k), j = (j_1 \ldots j_k), f_{i,j} \in A_n\), and \(x\) is a common element in \(A_n\), if the following conditions hold:

1. there exists an \(m\) such that \(f_{i,j} \in A_{(m)}\) for all \(i, j\);
2. for ant \(g \in G_n, x \in A_n\), the following inequality holds only for finitely many summands in \(S_n(x)\):

\[
|f_{i,j} x^{(i_1,j_1)} \ldots x^{(i_k,j_k)}| \leq g;
\]
3. all summands have the same degree over \(x\), denoted \(\deg S_n(x) = k\).

In [121], Kolesnikov solves \(|S(x)| = g\) and then \(S_1(x) = f_1\). In his proof, there is a modification of Makar-Limanov’s original proof which is expedient for it compensates for this loss by instilling much more simplicity in the argument for algebraic closedness.

In [122], Kolesnikov shows that every polynomial equation containing more than one homogeneous component over such a skew field has a non-zero solution necessarily. Precisely, he obtains following proposition:

**Proposition 1.1.34** (cf. [122] Theorem 1.). Let \(S_i(x), i = 1, \ldots, n\) be homogeneous operators over \(A\), where \(n \geq 1\), and \(T(x)\) be a homogeneous operator such that \(\deg S_i < \deg T\). Then the equation \(\sum_i S(x) = T(x)\) has a solution \(x \in A, x \neq 0\).

**Algebraically closed skew field in the sense of matrices**

Another conception of algebraic closedness is associated with the notion of singular eigenvalues of matrices. The definitions are given in Cohn [62].

Let \(D\) be a skew field with center \(k\). Denote by \(M_n(D)\) a ring of all \(n \times n\)-matrices over \(D\). A matrix \(A \in M_n(D)\) is said to be singular if there exists a non-zero column \(u \in D^n\) such
that \( Au = 0 \). A square matrix is singular if and only if it is not invertible. The property of being singular for a matrix is preserved under left or right multiplication by an invertible one, in particular, under elementary transformations of columns with coefficients from the skew field on the right, and of rows - on the left.

An element \( \lambda \in D \) is called a singular eigenvalue of \( A \) if \( A - \lambda I \) is a singular matrix. It is worth mentioning that singular eigenvalues of matrices are not always preserved under similarity transformations, but central eigenvalues are invariant in this sense.

The following definition of algebraically closed skew field is due to P. Cohn [62].

**Definition 1.1.35.** A skew field \( D \) is said to be algebraically closed in the sense of Cohn (written \( AC \)) if every square matrix over \( D \) has a singular eigenvalue in that skew field. \( D \) is said to be fully algebraically closed (written \( FAC \)) if every matrix \( A \in M_n(D) \), which is not similar to a triangular matrix over the center of \( D \), has a non-zero singular eigenvalue in \( D \).

The definition of \( FAC \) skew field is equivalent to the following:

**Definition 1.1.36.** A skew field \( D \) is fully algebraically closed if every matrix \( A \in M_n(D) \) which is not nilpotent has a non-zero singular eigenvalue in \( D \).

Consequently, if \( A \) is similar to a triangular matrix over the center of \( D \) then either it is nilpotent or has a non-zero eigenvalue. Conversely, if \( A \) is nilpotent then it is similar to its canonical form containing only 1 on a secondary diagonal.

**Definition 1.1.37.** We say that \( D \) is an \( AC_n \) (resp., \( FAC_n \)) skew field if every (non-nilpotent) matrix \( A \in M_{m}(D) \), \( m \leq n \), has a non-zero eigenvalue in \( D \).

**Proposition 1.1.38** (cf. [122] Theorem 2.). Let \( D \) be an \( FAC_n \) skew field and \( a_i, b_i, c \in D, i = 1, \ldots, n \). Then the equation
\[
L_n(x) = \sum_{i=1}^{n} a_i x b_i = c
\]
has a solution \( x = x_0 \in D \) if \( L_n(x) \equiv 0 \) for all \( x \in D \).

### 1.2 Automorphisms of polynomial algebras and Kontsevich Conjecture

One of the main objects of study in the theory of polynomial mappings are given by Ind-schemes, the points of which are automorphisms of various algebras with polynomial identities. The latter are normally algebras of commutative polynomials \( \mathbb{K}[x_1, \ldots, x_n] \) of \( n \) variables, the free algebras \( \mathbb{K}(x_1, \ldots, x_n) \) of \( n \) generators, some selected quotients, as well as algebras with additional structure – the prominent example is the polynomial algebra equipped with the Poisson bracket. This area of research is rooted in the widely famous Jacobian Conjecture. Thanks to the relatively
recent progress of A. Belov-Kanel and M. Kontsevich \cite{39, 40} and Y. Tsuchimoto \cite{187, 188}), as well as in connection with earlier studies, a significant place in the scientific program regarding the Jacobian Conjecture has come to be occupied by questions related to the quantization of classical algebras.

Studying the geometry and topology of Ind-schemes of automorphisms, development of approximation theory of symplectomorphisms by tame symplectomorphisms, as well as the construction of a correspondence between plane algebraic curves and holonomic modules (over the corresponding the Weyl algebra) are the basis of the approach to solving the Conjecture of A. Belov-Kanel and M. Kontsevich on automorphisms of the Weyl algebra, built by A. Elishev, A. Kanel-Belov and J.-T. Yu in articles \cite{110, 111} (cf. also \cite{84}).

1.2.1 Jacobian Conjecture

One of the most well-known unresolved problems in the theory of polynomials in several variables is the so-called the Jacobian Conjecture, formulated in 1939 by O.-H. Keller \cite{120}. Let $K$ be the main field, and for a fixed positive integer $n$ are given $n$ polynomials

$$f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$$

of $n$ variables $x_1, \ldots, x_n$. Any such system of polynomials defines a unique image endomorphism of the algebra $K[x_1, \ldots, x_n]$

$$F : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$$

$$F \leftrightarrow (F(x_1), \ldots, F(x_n)) \equiv (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n),$$

the $K$-endomorphism $F$ of polynomial algebra is determined by its action on the set of generators. Let $J(F)$ denote Jacobian (the determinant of the Jacobi matrix) of the map $F$:

$$J(F) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The Jacobian Conjecture is as follows.

**Conjecture 1.2.1** (The Jacobian Conjecture, $JC_n$). Let the characteristic of the base field $K$ be equal to zero. Then, if the Jacobian $J(F)$ of the endomorphism $F$ is equal to a nonzero constant (that is, it belongs to the set $K^\times$), then $F$ is an automorphism.

An elementary exercise is to verify the statement that automorphisms of polynomial algebra always have a nonzero Jacobian constant. The Conjecture 1.2.1 is thus partially inverse statement
of this property. It is also easy to see that if a polynomial endomorphism $F$ is invertible, then the inverse will also be a polynomial endomorphism.

The Jacobian conjecture is trivial for $n = 1$. On the other hand, when the field $\mathbb{K}$ has positive characteristic, the Jacobian Conjecture formulated as Conjecture 1.2.1 is incorrect even in the case of $n = 1$. Indeed, if char $\mathbb{K} = p$ and $n = 1$, we can take $\varphi(x) = x - x^p$; the Jacobian of such a mapping is equal to unity, but it is irreversible.

Despite the apparent simplicity of wording and context, the Jacobian Conjecture is one of the most difficult open questions of modern algebraic geometry. This problem has become the subject of numerous studies and has greatly contributed to the development of related fields of algebra, algebraic geometry and mathematical physics, which are also of independent interest.

The literature on the Jacobian Conjecture, its analogues and related problems is extensive. A detailed discussion of the results established in the context of the Jacobian Conjecture is beyond the scope of this work; Below we give a brief overview of some results directly related to the Jacobian Conjecture (i.e., for the algebra of polynomials in commuting variables). Among studies of issues similar to the Jacobian Conjecture in associative algebra, it is worth noting the work of Dicks [71] and Dicks and Levin [72] on an analogue of the Jacobian Conjecture for free associative algebras, the proof by U.U. Umirbaev [190] of an analogue of the Jacobian Conjecture for the free metabelian algebra, as well as the deep and extremely non-trivial work of A.V. Yagzhev [215–218] (see also [33]).

1.2.2 Some results related to the Jacobian Conjecture

While the general case of the Jacobian Conjecture (or even the Jacobian Conjecture on the plane) remains, at the time of writing this text, an open problem, various partial results are known. We mention but a few of them.

S.-S. Wang [213] established the Jacobian conjecture for the case of endomorphisms defined by polynomials of degree 2. Also, H. Bass, E.H. Connell, and D. Wright [19] showed that the general case of the Jacobian Conjecture would follow from the special case of the Jacobian Conjecture for the so-called endomorphisms of homogeneous cubic type, which are defined as mappings of the form

$$(x_1, \ldots, x_n) \mapsto (x_1 + H_1, \ldots, x_n + H_n)$$

where the polynomials $H_k$ are homogeneous of degree 3.

Moreover, L.M. Drużkowski [81] proved that the previous hypothesis can be weakened, by considering as $H_k$ only polynomials that are cubes of linear homogeneous polynomials.

In the works of M. de Bondt and A. van den Essen [67,68], as well as in the work of Drużkowski [82], it was shown that the Jacobian Conjecture was enough to be established for endomorphisms of homogeneous cubic type with a symmetric Jacobi matrix.
Suppose, as before, that the polynomial endomorphism $F$ is given by the set of images of the generators:

$$F \leftrightarrow (F(x_1), \ldots, F(x_n)) \equiv (F_1, \ldots, F_n).$$

Then $F$ is invertible if and only if the algebras

$$\mathbb{K}[x_1, \ldots, x_n] \text{ and } \mathbb{K}[F_1, \ldots, F_n]$$

are isomorphic to each other. Keller’s original paper [120] considered a rational analogue of the presented criterion, i.e. case of isomorphism of function fields

$$\mathbb{K}(x_1, \ldots, x_n) \text{ and } \mathbb{K}(F_1, \ldots, F_n)$$

and the invertibility following from the existence of an isomorphism is established by L.A. Campbell [59]. A generalization of Keller’s original result to the case when $\mathbb{K}(x_1, \ldots, x_n)$ is a Galois extension of the field $\mathbb{K}(F_1, \ldots, F_n)$ (see also the works of M. Razar [165] and D. Wright [214] generalizing the result mentioned).

In addition, some efforts were aimed at testing the fulfillment of the Jacobian Conjecture for all endomorphisms defined by polynomials of degree not higher than some fixed number. Moh [154, 155] performed a similar test for polynomials of two variables of degree not exceeding 100.

Despite the existence of the results described above (as well as some other similar theorems), the general case of the Jacobian Conjecture remains not only open, but, apparently, at the moment unassailable.

On the other hand, there are situations in which mappings, by their geometric properties close to polynomial endomorphisms, are nevertheless not invertible. S.Yu. Orevkov [158] points to the following reformulation of the Jacobian Conjecture, leading to a similar situation. Let $l$ be an infinitely distant line in the complex projective plane $\mathbb{C}P^2$, $U$ be its tubular neighborhood, $f_1, f_2$ are meromorphic functions on $U$, holomorphic on $U \setminus l$ and defining a locally one-to-one mapping

$$F : U \setminus l \to \mathbb{C}^2.$$  

The Jacobian conjecture is equivalent to the statement about the injectivity of mappings of this kind. S.Yu. Orevkov [158] constructed the following example.

**Theorem 1.2.2** (S.Yu. Orevkov, [158]). There is a smooth, non-compact complex analytic surface $\tilde{X}$, on which there is a smooth curve $\tilde{L}$, isomorphic to the projective line, with the self-intersection index +1, and two functions $f_1, f_2$, meromorphic on $\tilde{X}$ and holomorphic on $\tilde{X} \setminus \tilde{L}$, such that the mapping defined by

$$F : \tilde{X} \setminus \tilde{L} \to \mathbb{C}^2$$
is locally one-to-one, but not injective.

As noted in [158], if $\tilde{U}$ is a tubular neighborhood of the curve $\tilde{L}$, then the pairs $(U, l)$ (as above) and $(\tilde{U}, \tilde{L})$ are diffeomorphic, which implies the existence of a smooth immersion in a two-dimensional complex exterior space of a ball which is geometrically similar to a polynomial map and non-invertible. Also, if the pairs $(U, l)$ and $(\tilde{U}, \tilde{L})$ were biholomorphic to each other, then from the example of Orevkov one would derive the existence of a counterexample to the Jacobian Conjecture. This consideration allows one to conclude that suspicion in favor of the negativity of the Jacobian Conjecture are generally warranted.

In his classic work, D. Anick [8] developed a theory of approximation of polynomial endomorphisms by tame automorphisms (one of the main results of this paper is the proof of a symplectic analogue of Anick’s main theorem). In connection with D. Anick’s theorem on approximation, the Jacobian Conjecture can be reduced to solving the question of the invertibility of limits of sequences of tame automorphisms. Moreover, a symplectic analogue of Anick’s theorem gives a natural (albeit requiring non-trivial extensions) idea to solve the lifting problem of polynomial symplectomorphisms to automorphisms of the Weyl algebra in order to prove the Kanel-Belov – Kontsevich Conjecture (often called the Kontsevich Conjecture), an overview of which we provide in the latter section of this paper.

The central question in the approach to the Jacobian Conjecture and to the Kontsevich Conjecture based on approximation by sequences of tame automorphisms is the proof of polynomial nature of the resulting limit. While in the case of the lifting of symplectomorphisms (Kontsevich Conjecture, [111]) the proof of the correctness of the construction seems to be possible (a significant role in it is played by the invertibility of the sequence limits, which is obviously not the case for the Jacobian Conjecture), in the context of the Jacobian Conjecture there is no clarity in the matter, and considerations following from S.Yu. Orevkov [158], indicate possible significant obstacles.

The Jacobian Conjecture is studied by the methods of covering groups in S.Yu. Orevkov [157, 159] and A.G. Vitushkin [202, 203].

The Jacobian Conjecture is also the subject of highly non-trivial work of Vik S. Kulikov [129, 130].

A number of other difficult problems from the theory of polynomial automorphisms are closely connected with the Jacobian Conjecture and with affine algebraic geometry. These problems are important in the general mathematical context. For example, a special case of the classical Abyankar-Sataye Conjecture [162, 230] posits isomorphisms of all embeddings of the complex affine line into three-dimensional space (in other words, it is a conjecture about the possibilities of formally algebraic definition of the knot).
1.2.3 Ind-schemes and varieties of automorphisms

One of the essential areas of algebraic geometry, the development of which was motivated by the Jacobian Conjecture is the theory of infinite-dimensional algebraic groups. The main reference is the seminal article of I.R. Shafarevich [171], in which he defined concepts that allowed one to study questions about some natural infinite-dimensional groups – for example, the group of automorphisms of an algebra of polynomials in several variables – using tools from algebraic geometry. In particular, Shafarevich defines infinite-dimensional varieties as inductive limits of directed systems of the form

\[ \{ X_i, f_{ij}, i, j \in I \} \]

where \( X_i \) are algebraic varieties (more generally, algebraic sets) over a field \( K \), and the morphisms \( f_{ij} \) (defined for \( i \leq j \)) are closed embeddings. The inductive limit of a system of topological spaces carries a natural topology, and therefore the natural questions about connectivity and irreducibility arise, which were also studied in [171].

Following generally accepted terminology, we will call the direct limit of systems of varieties and closed embeddings an Ind-variety, and the corresponding limits of systems of schemes and morphisms of schemes an Ind-scheme.

The Jacobian Conjecture has the following elementary connection with Ind-schemes. Since the algebra of polynomials \( K[x_1, \ldots, x_n] \) can be endowed with a natural \( \mathbb{Z} \)-grading in total degree \( \deg \), which is defined as the appropriate monoid homomorphism by the requirement \( \deg x_i = 1 \), we can define the degree of endomorphism \( \varphi \): namely, if

\[ \varphi = (\varphi(x_1), \ldots, \varphi(x_n)) \]

defined by its action on algebra generators, then the degree \( \deg \varphi \) is the maximum value of \( \deg \) on the polynomials \( \varphi(x_1), \ldots, \varphi(x_n) \). It defines an increasing filtration

\[ \text{End}^{\leq N} K[x_1, \ldots, x_n], \ N \geq 0 \]

on the set \( \text{End} K[x_1, \ldots, x_n] \) of endomorphisms of the polynomial algebra; points

\[ \text{End}^{\leq N} K[x_1, \ldots, x_n] \]

are endomorphisms of degree at most \( N \). It is easy to see that the algebraic sets

\[ \text{End}^{\leq N} K[x_1, \ldots, x_n] \]

are isomorphic to affine spaces of appropriate dimension; the coordinates of the point \( \varphi \) are the coefficients of the polynomials

\[ \varphi(x_1), \ldots, \varphi(x_n) \]
the total degree filtration also enables endowing the sets of automorphisms with the Zariski topology as follows (see also [171]): if $\varphi$ is a polynomial automorphism, then consider a set of polynomials

$$(\varphi(x_1), \ldots, \varphi(x_1), \varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n))$$

- the images of generators under the action of the automorphism and its inverse. The coefficients of these polynomials serve as coordinates of $\varphi$ as a point of some affine space.

Define the subsets

$$\text{Aut}^{\leq N} K[x_1, \ldots, x_n] = \{ \varphi \in \text{Aut} K[x_1, \ldots, x_n] : \deg \varphi, \deg \varphi^{-1} \leq N \}$$

as sets of automorphisms such that all coefficients of polynomials in the presentation above for degrees greater than $n$ are zero.

The sets $\text{Aut}^{\leq N} K[x_1, \ldots, x_n]$ are algebraic sets; Indeed, the identities that define the points $\text{Aut}^{\leq N}$ are derived from the identity

$$\varphi \circ \varphi^{-1} = \text{Id}$$

and, it is easy to see, are specified by polynomials.

Now let $\mathcal{J}^{\leq N}$ denote a subset of

$$\text{End}^{\leq N} K[x_1, \ldots, x_n],$$

whose points are endomorphisms with a Jacobian equal to a nonzero constant.

Then Conjecture 1.2.1 can be clearly reformulated as follows

$$\forall \varphi \in \mathcal{J}^{\leq N} \Rightarrow \varphi \in \text{Aut} K[x_1, \ldots, x_n], \forall N, \text{ for } \text{char} K = 0.$$

### 1.2.4 Conjectures of Dixmier and Kontsevich

J. Dixmier [73] in his seminal study of Weyl algebras found a connection between the Jacobian Conjecture and the following Conjecture. Let $W_{n,K}$ denote the $n$-th Weyl algebra over the field $K$ defined as the quotient algebra of the free algebra

$$F_{2n} = K\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$$

of $2n$ generators by the two-sided ideal $I_W$, generated by polynomials

$$a_i a_j - a_j a_i, \ b_i b_j - b_j b_i, \ b_i a_j - a_j b_i - \delta_{ij} \ (1 \leq i, j \leq n)$$
(\delta_{ij} \text{ is the Kronecker symbol}). The Dixmier Conjecture states:

**Conjecture 1.2.3** (Dixmier Conjecture, \(DC_n\)). Let \( \text{char}\ K = 0 \). Then \( \text{End} \ W_{n,K} = \text{Aut} \ W_{n,K} \).

In other words, the Dixmier conjecture asks whether every endomorphism of the Weyl algebra over a field of characteristic zero is in fact an automorphism.

The Dixmier conjecture for \( n \) variables, \( DC_n \), implies the Jacobian Conjecture \( JC_n \) for \( n \) variables (see, for example, [199]). Significant progress in recent years in the study of Conjecture 1.2.1 has been achieved by A. Kanel-Belov (Belov) and M.L. Kontsevich [40] - and independently by Y. Tsuchimoto [188] (also see [187]) - in the form of the following theorem.

**Theorem 1.2.4** (A.Ya. Kanel-Belov and M.L. Kontsevich [40], Y. Tsuchimoto [188]). \( JC_{2n} \) implies \( DC_n \).

In particular, the Theorem 1.2.4 implies the stable equivalence of the Jacobian and Dixmier conjectures - i.e. the equivalence of the conjectures \( JC_\infty \) and \( DC_\infty \), where \( JC_\infty \) denotes the conjunction corresponding conjectures for all finite \( n \).

Theorem 1.2.4 laid the foundation for the research into Jacobian Conjecture based on the study of the behavior of varieties of endomorphisms and automorphisms of algebras under deformation quantization. The principal reference in this direction is given by an article by A. Kanel-Belov and M.L. Kontsevich [39]; in it several conjectures concerning Ind-varieties of automorphisms of the corresponding algebras are formulated. The main Conjecture is called the Kontsevich Conjecture and is as follows.

**Conjecture 1.2.5** (Kontsevich Conjecture, [39]). Let \( K = \mathbb{C} \) be the field of complex numbers. The automorphism group \( \text{Aut} \ W_{n,\mathbb{C}} \) of the \( n \)-th Weyl algebra over \( \mathbb{C} \) is isomorphic to the automorphism group \( \text{Aut} \ P_{n,\mathbb{C}} \) of the so-called \( n \)-th (commutative) Poisson algebra \( P_{n,\mathbb{C}} \):

\[
\text{Aut} \ W_{n,\mathbb{C}} \simeq \text{Aut} \ P_{n,\mathbb{C}}
\]

The algebra \( P_{n,\mathbb{C}} \) is by definition the polynomial algebra

\[
\mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_n]
\]

of \( 2n \) variables, equipped with the Poisson bracket - a bilinear operation \( \{ , \} \), which is a Lie bracket satisfying the Leibniz rule and acting on generators of the algebra in the following way:

\[
\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, x_j\} = \delta_{ij}.
\]

Endomorphisms of the algebra \( P_n \) are endomorphisms of the algebra of polynomials that preserve the Poisson bracket (which we sometimes call the Poisson structure in the text). Elements of \( \text{Aut} \ P_{n,\mathbb{C}} \) are called polynomial symplectomorphisms; the choice of name is due to the existence of
an (anti-) isomorphism between the group $\text{Aut} \, P_{1,\mathbb{C}}$ and the group of polynomial symplectomorphisms of the affine space $\mathbb{A}^{2n}$.

Kontsevich conjecture is true for $n = 1$. The proof of this result is a direct description of automorphism groups $\text{Aut} \, P_{1,\mathbb{C}}$ and $\text{Aut} \, W_{1,\mathbb{C}}$, contained in the classical works of Jung [99], Van der Kulk [200], Dixmier [73] and Makar-Limanov [136] (see also [135]). Namely, consider the following transformation groups: the group $G_1$ is a semidirect product

$$\text{SL}(2, \mathbb{C}) \rtimes \mathbb{C}^2$$

whose elements are called special affine transformations, and the group $G_2$ by definition consists of the following, “triangular” substitutions:

$$(x, p) \mapsto (\lambda x + F(p), \lambda^{-1} p), \quad \lambda \in \mathbb{C}^x, \quad F \in \mathbb{C}[t].$$

The automorphism group of the algebra $P_{1,\mathbb{C}}$ then [99] is isomorphic to the quotient group of the free product of the groups $G_1$ and $G_2$ by their intersection. J. Dixmier [73] and, later, L.G. Makar-Limanov [136] showed that if in the description above one replaces the commuting Poisson generators with their quantum (Weyl) analogues, one obtains a description of the group of automorphisms of the first Weyl algebra $W_{1,\mathbb{C}}$.

**Remark 1.2.6.** The theorems of Jung, van der Kulk, Dixmier and Makar-Limanov also mean that all automorphisms of the polynomial algebra of two variables and the first Weyl algebra $W_1$ are tame (definition of the concept of tame automorphism, which plays a significant role in this study, we provide in the sections below). Also, Makar-Limanov [135] and A. Czerniakiewicz [64, 65] proved that all automorphisms of the free algebra $\mathbb{K}(x, y)$ are tame.

*In view of these circumstances, the case of two variables is to be considered exceptional. However, the Jacobian Conjecture is a hard open problem even in this case.*

Recently A. Kanel-Belov, together with A. Elishev and J.-T. Yu, have suggested a proof of the general case of Kontsevich conjecture ([110,111]). An independent proof of a closely related result (based on a study of the properties of holonomic $\mathcal{D}$ - modules) was proposed by C. Dodd [74].

In contrast to the Jacobian Conjecture, which is an extremely difficult problem, in the study of the Kontsevich Conjecture there are several conceivable approaches. First of all, in the program article [39] Kanel-Belov and Kontsevich have formulated several generalizations of the Conjecture 1.2.5. In [40] and [188], which is devoted to the proof of Theorem 1.2.4, the construction of homomorphisms

$$\phi : \text{Aut} \, W_{n,\mathbb{C}} \to \text{Aut} \, P_{n,\mathbb{C}}$$

and

$$\phi : \text{End} \, W_{n,\mathbb{C}} \to \text{End} \, P_{n,\mathbb{C}},$$
involved in the construction, from a counterexample to $DC_n$, of an irreversible endomorphism with a single Jacobian, has been presented. A straightforward strengthening of Conjecture 1.2.5 is the statement that the homomorphism $\phi$ realizes the isomorphism of the Kontsevich Conjecture. Also - namely, in Chapter 8 of the article [39], an approach to resolving the problem of lifting of polynomial symplectomorphisms to automorphisms of the Weyl algebra (i.e., constructing a homomorphism inverse to $\phi$) was discussed. Conjecture 5 of the paper [39], along with Conjecture 6, which is a weaker form of Conjecture 1.2.5, make up the essential contents of the construction proposed in [39]. To solve the problem of lifting of symplectomorphisms in the sense of these conjectures, it is necessary to study the properties of $\mathcal{D}$ - modules – (left) modules over the Weyl algebra. The work of Dodd [74] is based on this approach.

1.2.5 Approximation by tame automorphisms

Tsuchimoto [187,188], and independently Kanel-Belov and Kontsevich [40], found a deep connection between the Jacobian conjecture and a celebrated conjecture of Dixmier [73] on endomorphisms of the Weyl algebra, which is stated as in Conjecture 1.2.3.

The correspondence between the two open problems, in the case of algebraically closed $K$, is based on the existence of a composition-preserving map

$$\text{End} \ W_n(K) \to \text{End} \ K[x_1, \ldots, x_{2n}]$$

which is a homomorphism for the corresponding automorphism groups. Furthermore, the mappings that belong to the image of this homomorphism preserve the canonical symplectic form on $A_{2n}^n$. In accordance with this, Kontsevich and Kanel-Belov [39] formulated several conjectures on correspondence between automorphisms of the Weyl algebra $W_n$ and the Poisson algebra $P_n$ (which is the polynomial algebra $K[x_1, \ldots, x_{2n}]$ endowed with the canonical Poisson bracket) in characteristic zero. In particular, there is a

Conjecture 1.2.7. The automorphism groups of the $n$-th Weyl algebra and the polynomial algebra in $2n$ variables with Poisson structure over the rational numbers are isomorphic:

$$\text{Aut} \ W_n(Q) \simeq \text{Aut} \ P_n(Q)$$

Relatively little is known about the case $K = Q$, and the proof techniques developed in [39] rely heavily on model-theoretic objects such as infinite prime numbers (in the sense of non-standard analysis); that in turn requires the base field $K$ to be of characteristic zero and algebraically closed (effectively $\mathbb{C}$ by the Lefschetz principle). However, even the seemingly easier analogue of the above conjecture, the case $K = \mathbb{C}$, is known (and positive) only for $n = 1$.

In the case $n = 1$, the affirmative answer to the Kontsevich conjecture, as well as positivity of several isomorphism statements for algebras of similar nature, relies on the fact that all auto-
morphisms of the algebras in question are tame (see definition below). Groups of tame automorphisms are rather interesting objects. Anick [8] has proved that the group of tame automorphisms of $\mathbb{K}[x_1, \ldots, x_N]$ is dense (in power series topology) in the subspace of all endomorphisms with non-zero constant Jacobian. This fundamental result enables one to reformulate the Jacobian conjecture as a statement on invertibility of limits of tame automorphism sequences.

Another interesting problem is to ask whether all automorphisms of a given algebra are tame [64, 65, 99, 181, 200]. For instance, it is the case [135, 139] for $\mathbb{K}[x, y]$, the free associative algebra $\mathbb{K}\langle x, y \rangle$ and the free Poisson algebra $\mathbb{K}\{x, y\}$. It is also the case for free Lie algebras (a result of P. M. Cohn). On the other hand, tameness is no longer the case for $\mathbb{K}[x, y, z]$ (the wild automorphism example is provided by the well-known Nagata automorphism, cf. [179]).

Anick’s approximation theorem was established for polynomial automorphisms in 1983. We obtain the approximation theorems for polynomial symplectomorphisms and Weyl algebra automorphisms. These new cases are established after more than 30 years. The focus of this paper is the problem of lifting of symplectomorphisms:

Can an arbitrary symplectomorphism in dimension $2n$ be lifted to an automorphism of the $n$-th Weyl algebra in characteristic zero?

The lifting problem is the milestone in the Kontsevich conjecture. The use of tame approximation is advantageous due to the fact that tame symplectomorphisms correspond to Weyl algebra automorphisms: in fact [39], the tame automorphism subgroups are isomorphic when $\mathbb{K} = \mathbb{C}$.

The problems formulated above, as well as other statements of similar flavor, outline behavior of algebra-geometric objects when subject to quantization. Conversely, quantization (and anti-quantization in the sense of Tsuchimoto) provides a new perspective for the study of various properties of classical objects; many of such properties are of distinctly K-theoretic nature. The lifting problem is a subject of a thorough study of Artamonov [9–12], one of the main results of which is the proof of an analogue of the Serre-Quillen-Suslin theorem for metabelian algebras. The possibility of lifting of (commutative) polynomial automorphisms to automorphisms of metabelian algebra is a well-known result of Umirbaev, cf. [190]; the metabelian lifting property was instrumental in Umirbaev’s resolution of the Anick’s conjecture (which says that a specific automorphism of the free algebra $\mathbb{K}\langle x, y, z \rangle$, char $\mathbb{K} = 0$ is wild). Related to that also is a series of well-known papers [179–181].

In this note we establish the approximation property for polynomial symplectomorphisms and comment on the lifting problem of polynomial symplectomorphisms and Weyl algebra automorphisms. In particular, the main results discussed here are as follows.
Theorem 1.2.8. Let \( \varphi = (\varphi(x_1), \ldots, \varphi(x_N)) \) be an automorphism of the polynomial algebra \( \mathbb{K}[x_1, \ldots, x_N] \) over a field \( \mathbb{K} \) of characteristic zero, such that its Jacobian

\[
J(\varphi) = \det \left[ \frac{\partial \varphi(x_i)}{\partial x_j} \right]
\]

is equal to 1. Then there exists a sequence \( \{\psi_k\} \subset \text{TAut} \mathbb{K}[x_1, \ldots, x_N] \) of tame automorphisms converging to \( \varphi \) in formal power series topology.

D. Anick [8] proved this tame approximation theorem for polynomial automorphisms. In this paper we get the approximation theorems for polynomial symplectomorphisms and Weyl algebra automorphisms.

Theorem 1.2.9. Let \( \sigma = (\sigma(x_1), \ldots, \sigma(x_n), \sigma(p_1), \ldots, \sigma(p_n)) \) be a symplectomorphism of \( \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \) with unit Jacobian. Then there exists a sequence \( \{\tau_k\} \subset \text{TAut} \mathcal{P}_n(\mathbb{K}) \) of tame symplectomorphisms converging to \( \sigma \) in formal power series topology.

Theorem 1.2.10. Let \( \mathbb{K} = \mathbb{C} \) and let \( \sigma : \mathcal{P}_n(\mathbb{C}) \to \mathcal{P}_n(\mathbb{C}) \) be a symplectomorphism over complex numbers. Then there exists a sequence

\[
\psi_1, \psi_2, \ldots, \psi_k, \ldots
\]

of tame automorphisms of the \( n \)-th Weyl algebra \( \mathcal{W}_n(\mathbb{C}) \), such that their images \( \sigma_k \) in \( \text{Aut} \mathcal{P}_n(\mathbb{C}) \) converge to \( \sigma \).

The last proposition is of main concern to us. As we shall see, sequences of tame symplectomorphisms lifted to automorphisms of Weyl algebra (either by means of the isomorphism of [39], or explicitly through deformation quantization \( \mathcal{P}_n(\mathbb{C}) \to \mathcal{P}_n(\mathbb{C})[[\hbar]] \)) are such that their limits may be thought of as power series in Weyl algebra generators. If we could establish that those power series were actually polynomials, then the Dixmier conjecture would imply the Kontsevich’s conjecture (with \( \mathbb{Q} \) replaced by \( \mathbb{C} \)). Conversely, approximation by tame automorphisms provides a possible means to attack the Dixmier conjecture (and, correspondingly, the Jacobian conjecture).

1.2.6 Holonomic \( \mathcal{D} \)-modules, Lagrangian subvarieties and the work of Dodd

The following general conjecture holds ([39], see also [124]).

Conjecture 1.2.11. Let \( X \) be a smooth variety. There is a one-to-one correspondence between (irreducible) holonomic \( \mathcal{D}(X) \) - modules and Lagrangian subvarieties \( T^*X \) of the corresponding dimension.
Kontsevich [124] introduces the general definition of the holonomic $\mathcal{D}$-module as follows. Let $X$ be a smooth affine algebraic variety of dimension $n$ over the field $\mathbb{K}$. Consider the $\mathbb{K}$-algebra $\mathcal{D}(X)$ of differential operators - the algebra of operators, acting on the ring $\mathcal{O}(X)$ generated by functions and $\mathbb{K}$-derivations:

$$f \mapsto gf, \quad f \mapsto \xi(f), \quad g \in \mathcal{O}(X), \quad \xi \in \Gamma(X, T_{X/\text{Spec}\mathbb{K}})$$

The natural filtration is defined on the algebra

$$\mathcal{D}(X) = \bigcup_{k \geq 0} \mathcal{D}_{\leq k}(X)$$

with respect to the order of operators, the associated graded algebra is naturally isomorphic to the algebra of functions on the cotangent bundle $T^*X$. Let $M$ be a finitely generated module over $\mathcal{D}(X)$, and $V$ be a finite-dimensional subspace of elements generating $M$. It induces a filtration

$$M_{\leq k} = \mathcal{D}_{\leq k}(X)V \subset M, \quad k \geq 0,$$

such that the associated graded module $\text{gr}(M)$ is finitely generated over $\mathcal{O}(T^*X)$. It is known (this result belongs to O. Gabber, see [124]) that its support

$$\text{supp}(\text{gr}(M)) \subset T^*X$$

is a coisotropic variety; in particular, the dimension of any of its irreducible components is not less than $n$. The support is independent of the choice of the subspace $V$ (and is denoted in the original article [124] via $\text{supp}(M)$).

A finitely generated module $M$ is called holonomic if, by definition, the dimension of its support is $n$.

Conjecture 1.2.11 (which can also be called the Kontsevich Conjecture) generalizes the Conjecture 1.2.5, as well as Conjectures 5 and 6 of the article [39] in the context of the lifting of symplectomorphisms. Namely, with any symplectomorphism one may naturally associate a Lagrangian subvariety (namely, its graph). On the other hand, holonomic $\mathcal{D}$-modules correspond to autoequivalences of Weyl algebra, from which in principle (taking into account Conjecture 5 of article [39]) one can get a correspondence with automorphisms.

In connection with these circumstances, the necessity to study the holonomic $\mathcal{D}$-modules has naturally presented itself. The problems of lifting of polynomial symplectomorphisms in the case of low dimensions - namely, for $n = 1$, which corresponds to the well-known case of Kontsevich’s conjecture, has become the prime candidate for testing these new deep insights. Some progress in this direction has been achieved in the paper [38] by Kanel-Belov and Elishev. The general case of arbitrary dimension was investigated by Kontsevich in the main article. [124] (see also [52]); significant results on the Conjecture 1.2.11 were obtained (according to our understanding) by
Dodd [74].

Namely, Dodd devised the proof of the following result.

**Theorem 1.2.12** (C. Dodd, [74]). Let $X$ be a smooth variety over $\mathbb{C}$, $L \subset T^*X$ be a Lagrangian subvariety of the cotangent space. Suppose that:

1. The projection $\pi : L \to X$ is a dominant mapping.
2. The first singular homology group $H^1_{\text{sing}}(L, \mathbb{Z})$ is trivial.
3. There exists a smooth projective compactification $\bar{L}$ of the variety $L$ with trivial $(0, 2)$-Hodge cohomology.

Then there exists a unique irreducible holonomic $\mathcal{D}(X)$-module $M$ with constant arithmetic support $^1$, equal to $L$, with multiplicity 1.

This theorem partially resolves the problem of finding sufficient conditions for the correspondence between holonomic modules and Lagrangian varieties as formulated in the Conjecture 1.2.11. Dodd also notes that in the case when $X = \mathbb{A}^n$ is an affine space, condition 2 of Theorem 1.2.12 can be dropped, in connection with which there is a corollary:

**Corollary 1.2.13** (C. Dodd, [74]). Let $L \subset T^*\mathbb{A}^m$ be a smooth Lagrangian subvariety satisfying conditions 2 and 3 of Theorem 1.2.12. Then there exists a unique irreducible holonomic $\mathcal{D}(\mathbb{A}^m)$-module $M$ whose arithmetic support is $L$, with multiplicity 1.

This result is closely related to the construction studied in [38].

As Dodd notes, Theorem 1.2.12 and Corollary 1.2.13 allow us to give a description of the Picard group $\text{Pic}(W_{n, \mathbb{C}})$ of the Weyl algebra. Recall that the Picard group of an associative algebra is defined as a group of classes (modulo isomorphism) of invertible bimodules over a given algebra, with a group operation given by the tensor product of modules.

Consider polynomial symplectomorphisms of the variety $T^*\mathbb{A}^m$. It is easy to show that the graph of any symplectomorphism $\varphi$ is a Lagrangian subvariety of $L^\varphi$ in $T^*\mathbb{A}^{2m}$, isomorphic to $\mathbb{A}^{2m}$ and, therefore, satisfies cohomological conditions of Theorem 1.2.12. Applying Corollary 1.2.13, we obtain (uniquely identified) $\mathcal{D}(\mathbb{A}^{2m}) \simeq \mathcal{D}(\mathbb{A}^m) \otimes \mathcal{D}(\mathbb{A}^m)^{\text{op}}$ - module $M^\varphi$ corresponding to $L^\varphi$. One can check [74] that the inverse symplectomorphism $\varphi^{-1}$ in such a construction corresponds to inverse bimodule.

From these considerations, Dodd obtains the following result.

**Theorem 1.2.14** (C. Dodd, [74]). There is an isomorphism of groups (over $\mathbb{C}$)

$$\text{Pic}(\mathcal{D}(\mathbb{A}^m)) \simeq \text{Symp}(T^*\mathbb{A}^m),$$

where $\text{Symp}(T^*\mathbb{A}^m)$ denotes the group of polynomial symplectomorphisms (this group is a geometric analogue of the group $\text{Aut} P_{m, \mathbb{C}}$).

$^1$For the definition of arithmetic support, see [124].
In the case $m = 1$, it is known (Dixmier, [73]) that
\[ \text{Pic}(\mathcal{D}(\mathbb{A}^1)) = \text{Aut}(\mathcal{D}(\mathbb{A}^1)), \]
and the algebra $\mathcal{D}(\mathbb{A}^1)$ is isomorphic to the first Weyl algebra $W_{1,\mathbb{C}}$. This means that we are in the situation of Conjecture 1.2.5 for $m = 1$.

### 1.2.7 Tame automorphisms and the Kontsevich Conjecture

Dodd’s constructions are deep in content and, apparently, prove the Kontsevich Conjecture on the correspondence between Lagrangian varieties and holonomic modules (more precisely, its essential part). On the other hand, starting from Theorem 1.2.14 we cannot immediately arrive at the general case of Conjecture 1.2.5 - proof of Conjecture 1 of article [39] requires a solution to the lifting problem of symplectomorphisms to automorphisms of the corresponding Weyl algebra.

One of the main results of the paper [39] was the proof of the following homomorphism properties
\[ \phi : \text{Aut} W_{n,\mathbb{C}} \to \text{Aut} P_{n,\mathbb{C}} \]
constructed in [39] and [188]. First, let $\varphi$ be an automorphism of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. We call $\varphi$ elementary if it has the form
\[ \varphi = (x_1, \ldots, x_k-1, ax_k + f(x_1, \ldots, x_k-1, x_{k+1}, \ldots, x_n), x_{k+1}, \ldots, x_n). \]
In particular, automorphisms given by linear substitutions of generators are elementary. Denote by $\text{TAut} \mathbb{K}[x_1, \ldots, x_n]$ the subgroup generated by all elementary automorphisms. Elements of this subgroup are called tame automorphisms of the algebra of polynomials; non-tame automorphisms are called wild automorphisms.

Tame automorphisms of the algebra $P_{n,\mathbb{K}}$ are, by definition, compositions of those tame elementary automorphisms which preserve the Poisson bracket. Tame automorphisms of the Weyl algebra are defined $W_{n,\mathbb{K}}$ similarly.

The following theorem is proved in [39].

**Theorem 1.2.15** (A. Kanel-Belov and M.L. Kontsevich, [39]). *The homomorphism*
\[ \phi : \text{Aut} W_{n,\mathbb{C}} \to \text{Aut} P_{n,\mathbb{C}} \]
*restricts to isomorphism*
\[ \phi|_{\text{TAut}} : \text{TAut} W_{n,\mathbb{C}} \to \text{TAut} P_{n,\mathbb{C}} \]
*between subgroups of tame automorphisms.*
In particular, due to the tame nature of automorphism groups of Weyl and Poisson algebras for \( n = 1 \), the homomorphism \( \phi \) gives an isomorphism of the Kontsevich Conjecture between \( \text{Aut} W_{1,\mathbb{C}} \) and \( \text{Aut} P_{1,\mathbb{C}} \).

It is not known whether all automorphisms of the Poisson and Weyl algebras are tame for \( n > 1 \), or even stably tame (an automorphism is called stably tame if it becomes tame after adding dummy variables and extending the action on them by means of the identity automorphism). For the algebra of polynomials in three variables, the Nagata automorphism

\[
(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z)
\]

is wild (the famous result due to I.P. Shestakov and U.U. Umirbaev, [179]).

Nevertheless, tame automorphisms turn out to play a significant role in the context of the Kontsevich Conjecture and Jacobian Conjecture, due to the following reason. D. Anick [8] showed that the set of tame automorphisms of the algebra of polynomials \( \mathbb{K}[x_1, \ldots, x_n] \) \( (n \geq 2) \) is dense in the topology of formal power series in the space \( \mathcal{J} \) of polynomial endomorphisms with nonzero constant Jacobian. In particular, for any automorphism of a polynomial algebra there exists a sequence of tame automorphisms converging to it in this topology - in other words, Anick’s theorem implies the existence of approximations of automorphisms, or approximations by tame automorphisms (and in general, endomorphisms with nonzero constant Jacobian). In view of the theorem of Anick, the Jacobian Conjecture can be formulated as a problem of invertibility of limits of sequences of tame automorphisms (this is discussed in the conclusion of the article [8]). This formulation of the Jacobian Conjecture can be directly generalized to the case of a field of arbitrary characteristic, see more below as well as in [118].

Anick’s results, together with Theorem 1.2.15, suggest the idea of solving the lifting problem of polynomial symplectomorphisms to automorphisms of the Weyl algebra, alternative to that proposed in [39] constructs. Namely, if there is a symplectic analogue of Anick’s theorem - that is, if there is an approximation of polynomial symplectomorphisms by tame symplectomorphisms, then, taking a sequence of tame symplectomorphisms converging to a given point, we can take the sequence their pre-images under the isomorphism \( \phi|_{\text{TAut}} \) and try to prove that its limit exists and is an automorphism of the Weyl algebra. The symplectic analogue of Anick’s theorem was proved in [112]. The application of approximation theory to the lifting problem constitutes the main idea of the proof of Conjecture 1.2.5 in [111].

However, the direct application of the main result of [112] to the solution of the lifting problem does not achieve the desired result, since the homomorphism \( \phi \) does not preserve the topology of formal power series (due to commutation relations in the Weyl algebra). In connection with this circumstance, the naive approximation approach needs some modification. It turns out that such a modification is possible (see [111]). The nature of this modification is significant. It is connected with the geometric properties of Ind-schemes of automorphisms of the corresponding
algebras. Therefore, the study of the geometry of Ind-schemes of automorphisms is justified in the framework of Kontsevich Conjecture.

1.2.8 Homomorphism $\phi$ and Ind-schemes

It is necessary to mention one more circumstance justifying the study of Ind-schemes in the context of Kontsevich Conjecture. In the construction of the homomorphism

$$\phi : \text{Aut} W_{n,\mathbb{C}} \rightarrow \text{Aut} P_{n,\mathbb{C}}$$

as in the papers [39] and [188], tools from model theory and non-standard analysis are used. In particular, the fixed nonprincipal ultrafilter $\mathcal{U}$ on the index set as well as a fixed infinitely large prime number $[p]$ are involved in the construction of $\phi$.

Briefly, the construction of the homomorphism $\phi$ proceeds as follows. First of all, we make the following observations (cf. [39, 40, 187, 188]).

1. Over the field $\mathbb{F}_p$ of positive characteristic $p$ the Weyl algebra $W_{n,\mathbb{F}_p}$ has a large center given by a subalgebra

$$\mathbb{F}_p[x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p]$$

($x_i, d_j$ are generators of the Weyl algebra; thus, the center is the polynomial algebra).

2. Any endomorphism $\varphi$ of the algebra $W_{n,\mathbb{F}_p}$ restricts to an endomorphism $\varphi^c$ center (i.e. maps the center into itself).

3. On the center $\mathbb{F}_p[x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p]$ there is a natural Poisson bracket which descends from the ambient algebra. Namely, we take in $\mathbb{F}_p$ the prime subfield $\mathbb{Z}_p$ and define the projection

$$\pi : W_{n,\mathbb{Z}} \rightarrow W_{n,\mathbb{Z}_p}$$

from the Weyl algebras over the ring of integers. Let $a, b$ belong to the center of $W_{n,\mathbb{Z}_p}$. Take arbitrary inverse images $a', b'$ of elements $a, b$ under reduction. Consider the expression

$$\{a, b\} = \pi \left( - \frac{[a', b']}{p} \right).$$

It can be shown that it correctly defines the Poisson bracket on the center of the algebra $W_{n,\mathbb{Z}_p}$. It can be extended linearly to the entire algebra $W_{n,\mathbb{F}_p}$. Since the Poisson bracket is induced by the commutator, for any endomorphism $\varphi$ its restriction to the center $\varphi^c$ preserves the Poisson bracket. Thus, in characteristic $p$ there is a homomorphism from the group of automorphisms of the Weyl algebra to the group of polynomial symplectomorphisms. (Strictly speaking, the result of the mapping $\varphi \mapsto \varphi^c$ has to be untwisted via the action of the inverse Frobenius automorphism; it is done in order to get rid of the $p$-th powers of the coefficients in the image).

The case of characteristic zero reduces to positive characteristic as follows. It is well known
(see [131] and [95]) that the set of minimal prime ideals of the ring

\[ A = \prod_i A_i \]

where \( A_i \) are domains and \( I \) is an arbitrary nonempty set of indices (finite or infinite), is in one-to-one correspondence with the set of all ultrafilters on the set of subsets of \( I \). Namely, if \( \mathcal{U} \) is an ultrafilter, then we put

\[ (\mathcal{U}) = \{a \in A \mid \theta(a) \in \mathcal{U}\}, \]

Where

\[ \theta(a) = \{i \in I \mid a_i = 0\}. \]

Then \( (\mathcal{U}) \) is a minimal prime ideal in \( A \), and all minimal prime ideals are of this form. Furthermore, if \( P^N \) is the set of sequences of prime numbers, and \( \mathcal{U} \) is a nonprincipal ultrafilter on subsets of natural numbers (playing here the role of the index set \( I \)), then it determines the equivalence relation on prime number sequences: two sequences are equivalent if the set of indices on which the entries of the two sequences coincide lies in \( \mathcal{U} \). Consider the set of equivalence classes:

\[ *P \equiv P^N/\sim_{\mathcal{U}}. \]

Among points in that set there are points corresponding to prime numbers (classes of stationary sequences), while there are also other classes. The latter are called infinitely large primes. Such a definition corresponds to the concept of a prime number in the ring of hyperinteger numbers (see [166] pp. 432-440).

If we fix the nonprincipal ultrafilter \( \mathcal{U} \) (on \( N \)) and consider any infinitely large prime \([p]\) represented by the sequence \((p_m)\), then we can consider the ring

\[ \prod_{m \in \mathbb{N}} F_{p_m} \]

(direct product of a countable number of algebraically closed fields of positive characteristics). The Krull dimension of the direct product (of any set) of fields is zero, therefore the minimal ideal \( (\mathcal{U}) \) will also be the maximal, so that the quotient ring is a field. When \([p]\) is infinitely large relative to \( \mathcal{U} \), this factor set has the power of the continuum (the proof can be found, for example, in [188] or in [40]). It is also easy to see that such a field is algebraically closed and of characteristic zero. Therefore, due to the well-known result from field theory, this field will be isomorphic to the field of complex numbers.

Now, consider the ultraproduct (a direct product factorized with respect to the equivalence
relation induced $\mathcal{U}$) of Weyl algebras

$$\left( \prod_{m \in \mathbb{N}} W_{n, \mathbb{F}_{p_m}} \right) / (\mathcal{U}).$$

The center of this algebra contains a (proper) subalgebra isomorphic to the algebra of polynomials

$$\mathbb{C}[\xi_1, \ldots, \xi_{2n}]$$

(it is obtained due to the isomorphism of the base field and observation 1 above). Taking an arbitrary endomorphism of algebra Weyl in characteristic zero, one can lift it to an automorphism of the ultraproduct and then apply the results of observations 1 - 3 to the endomorphisms in the ultraproduct decomposition and restrict these symplecto-endomorphisms onto the subalgebra $\mathbb{C}[\xi_1, \ldots, \xi_{2n}]$ (which is possible due to the fact that the original endomorphism is of finite degree). Thus constructed mapping defines the Belov - Kontsevich homomorphism $\phi$ (the construction given here follows Y. Tsuchimoto [188]).

The nature of infinitely large primes (as well as the nature of the ultrafilter $\mathcal{U}$) is fundamentally non-constructible. In view of that, the question of independence of the homomorphism $\phi = \phi_{[p]}$ of the choice of infinitely large prime number $[p]$ naturally arises. This issue was resolved in the paper [40]. In the proof presented in [40], the properties of the so-called loop morphism

$$\Phi : \text{Aut} W_{n, \mathbb{C}} \to \text{Aut} W_{n, \mathbb{C}},$$

obtained from $\phi_{[p]}$ and $\phi_{[p']} ([p] \neq [p'])$ under the assumption of the Conjecture 1.2.5 (which in turn is justified in the article [111]) are studied. The loop morphism provides an example of an Ind-automorphism of Ind-schemes, and some of its essential properties (for example, local unipotency) can be proved using the algebraic-geometric methods examined in the article [118]. In this sense, the situation with the Kontsevich Conjecture is similar to the investigation of Ind-schemes in [118], however, it should be noted that the geometry of Ind-schemes $\text{Aut} P_{n, \mathbb{C}}$ (and $\text{Aut} W_{n, \mathbb{C}}$) is more complicated than that of their analogues in the commutative and free associative cases.

The study of the geometry of Ind-schemes of automorphisms was conceived separately from the Jacobian Conjecture and related conjectures in the classical works of B. I. Plotkin [160, 161]. In particular, Plotkin studied the problem of the structure of sets

$$\text{Aut Aut} \quad \text{and} \quad \text{Aut End}$$

whose elements are automorphisms of automorphism groups (respectively, semigroups endomorphisms) of algebras with polynomial identities. Significant results in this direction were obtained.
by A. Kanel-Belov, R. Lipyanski and A. Berzins [37, 41, 48]. A description of the sets

\[ \text{Aut End } \mathbb{K}[x_1, \ldots, x_n] \]

and

\[ \text{Aut End } \mathbb{K}\langle x_1, \ldots, x_n \rangle \]

of automorphisms of semigroups of endomorphisms of polynomial algebra and free algebra was obtained.

Similar questions for automorphism groups can be resolved at the level of automorphisms of Ind-schemes given by

\[ \text{Aut}_{\text{Ind}} \text{Aut} \]

for the polynomial algebra and the free associative algebra. This was done by A. Kanel-Belov, A. Elishev and J.-T. Yu in the paper [118]. The fact that the study of the sets of Ind-automorphisms utilizes approximation by tame automorphisms along with techniques from algebraic geometry and topology (namely, the study of the properties of curves in Aut), which find their application in the (modified) context of the lifting problem and Kontsevich Conjecture. In this sense, the study of the geometry of Ind-schemes is necessary to prove Kontsevich Conjecture and - in the sense of the lifting problem - that study precedes it.

Generally speaking, the geometry of automorphism groups of affine varieties, going back to I. R. Shafarevich and B. I. Plotkin, is an actively developing field, regardless of progress in the work on the Jacobian Conjecture. Various topics of independent interest are discussed in the review article by J.-P. Furter and H. Kraft [97]. The study of questions similar to the results discussed in the present review was the subject of the works of T. Kambayashi [106, 107], H. Kraft and A. Regeta [127], H. Kraft and I. Stampfl [128], S. Kovalenko, A.Yu. Perepechko and M.G. Zaidenberg [126], I.V. Arzhantsev, K.G. Kuyumzhiyan and M.G. Zaidenberg [15].

Of particular interest, in the light of results obtained in [111,112], is the recent work by K. Urech and S. Zimmermann [198], which proves the following result: if an automorphism of the Cremona group of arbitrary rank is also a topology homeomorphism in either Zariski or Euclidean topology, then it is an inner automorphism (up to action of an automorphism induced by an automorphism of the base field). Also in [198] a similar result is established, obtained by replacing the Cremona group with the group of polynomial automorphisms of an affine space, thus the work of Urech and Zimmermann is a generalization of [118].

### 1.2.9 Approximation and analysis of curves in Aut

In the proof of Kontsevich conjecture in [111], namely in the proof of the correctness of the lifting of augmented symplectomorphisms, a certain kind of analysis of curves of automorphisms is
used. In essence, it is a technique that allows, by considering curves in an infinite-dimensional variety $\text{Aut}$, one to control the degrees of large-order terms in the automorphism and its image under a morphism $\Phi$, as long as the initial automorphism is close enough to the identity automorphism. Also, this procedure plays a significant role in the proof of the main results of [118].

As we mentioned before, the naive approximation by tame symplectomorphisms does not achieve the resolution of the lifting problem to the Weyl algebra. In connection with the search for a stronger approximation theory, it became necessary to introduce the deformation (or augmentation, see [111]) of algebras and, accordingly, of the power series topology. In the augmented case, the singularity analysis procedure presented below (which we often call the "singularity trick", eng. singularity trick) allows one to establish the correctness of the lifting procedure, which in turn allows one to apply the theory of approximation by tame symplectomorphisms to the construction of the lifting map acting on augmented symplectomorphisms.

This idea amounts to a very useful procedure when working with infinite-dimensional varieties. It was first described in [118] (Theorem 3.2, Lemmas 3.5, 3.6, and 3.7).

For the case of the commutative algebra of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ the singularity trick has the following form.

Let $L = L(t)$ be a curve whose points are linear automorphisms, i.e. a curve

$$L \subset \text{Aut}(K[x_1, \ldots, x_n]),$$

the points of which are given by linear changes of variables. Suppose that as $t$ tends to zero, the $i$-th eigenvalue of the matrix $L(t)$ (corresponding to linear substitutions) also tends to zero as $t^{k_i}$, $k_i \in \mathbb{N}$. Such a curve always exists.

Now note that the orders $\{k_i, i = 1, \ldots n\}$ of singularities of eigenvalues at zero are such that for any pair $(i, j)$, if $k_i \neq k_j$, there exists a positive integer $m$ what

$$\text{either } k_i m \leq k_j \text{ or } k_j m \leq k_i.$$ 

**Definition 1.2.16.** The largest such $m$ will be called the order of the curve $L(t)$ at $t = 0$.

Since all $k_i$ are natural numbers, the order equals the integer part of $\frac{k_{\max}}{k_{\min}}$.

Let $M \in \text{Aut}_0(K[x_1, \ldots, x_n])$ be a polynomial automorphism preserving the origin. The following statement holds.

**Lemma 1.2.17.** The curve $L(t)ML(t)^{-1}$ has no singularity at zero for any everywhere diagonalizable conjugating curve $L(t)$ of order $\leq N$ if and only if $M \in \hat{H}_N$, where $\hat{H}_N$ denotes the subgroup of automorphisms that are homothety\(^2\) modulo $N$ degrees of the expanding ideal $(x_1, \ldots, x_n)$.

\(^2\)Detailed definitions are given in the first section of Chapter 2.
From a topological point of view, this is a criterion for a point to belong to some sufficiently small neighborhood of the identity automorphism. In particular, variations of this singularity trick are useful in proving the continuity of mappings with sufficient regularity. The proof of Lemma 1.2.17 is given in [118].

1.2.10 Quantization of classical algebras

As already noted, the approach to the Jacobian conjecture, using techniques from the theory of deformation quantization - namely, the approach based on stable equivalence between the Jacobian conjecture and the Dixmier conjecture as well as, to a somewhat lesser extent, the Kontsevich Conjecture - is currently one of the more promising approaches to finding a possible solution to the Jacobian Conjecture. However, as in questions of the geometric theory of Ind-schemes and infinite-dimensional algebraic groups, the issues arising in connection with the application of quantization methods, due to their nontriviality and depth, is a direction whose value may well be comparable with the value of a possible solution to the original problem.

Analogues of JC and DC for algebras of quantum polynomials are not obvious and often do not admit a naive transfer of formulations (for example, E. Backelin [17] wrote about the $q$-quantum version of the Dixmier conjecture). On the other hand, the well known theorem of Umirbaev [194], showing the validity of an analogue of the Jacobian conjecture for free metabelian algebras, may be considered as an argument in favor of the validity of the Jacobian Conjecture.

Significant development of algebra and non-commutative geometry of quantum polynomials has been achieved by V.A. Artamonov [9–13]. In particular, he proved [12] the quantum-algebra analogue of the Serre conjecture (Quillen–Suslin theorem) – the result which is extremely non-trivial even in the commutative case.

In connection with the Jacobian Conjecture, we mention the works of Dicks [71], Dicks and Lewin [72] as well as Yagzhev [215–218]. In a sense, they can be interpreted as works consistent with the point of view on the Jacobian problem as a problem related to quantization. Regarding the practical benefits of studying relationships induced by quantization-type correspondences, there are known examples of application of the elements of the quantization procedure to some (previously proven by other means) problems of general algebra. An example [115, 231] is a new proof of Bergman’s centralizer theorem 2.1.4 of the free associative algebra, based on the deformation quantization procedure, which we discuss in this work.

1.3 Torus actions on free associative algebras and the Białynicki-Birula theorem

In the proof of the results concerning the geometry of Ind-schemes automorphisms, we use the famous A. Białynicki-Birula theorem [49,50] on the linearizability of regular actions of a maximal
torus on an affine space merits consists in the following.  

Let \( \mathbb{K} \) be the base field, and let \( \mathbb{K}^\times = \mathbb{K} \setminus \{0\} \) be the multiplicative group of the field, considered as an algebraic \( \mathbb{K} \)-group.

We call a \( n \)-dimensional algebraic \( \mathbb{K} \)-torus a group

\[
T_n \simeq (\mathbb{K}^\times)^n
\]

(with obviously certain multiplication).

**Definition 1.3.1.** An \( n \)-dimensional algebraic \( \mathbb{K} \)-torus is a group

\[
T_n \simeq (\mathbb{K}^\times)^n
\]

(with obvious multiplication).

Denote by \( \mathbb{A}^n \) the affine space of dimension \( n \) over \( \mathbb{K} \).

**Definition 1.3.2.** A (left, geometric) torus action is a morphism

\[
\sigma : T_n \times \mathbb{A}^n \to \mathbb{A}^n.
\]

that fulfills the usual axioms (identity and compatibility):

\[
\sigma(1, x) = x, \quad \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1t_2, x).
\]

The action \( \sigma \) is effective if for every \( t \neq 1 \) there is an element \( x \in \mathbb{A}^n \) such that \( \sigma(t, x) \neq x \).

In [49], Białynicki-Birula proved the following two theorems, for \( \mathbb{K} \) algebraically closed.

**Theorem 1.3.3.** Any regular action of \( T_n \) on \( \mathbb{A}^n \) has a fixed point.

**Theorem 1.3.4.** Any effective and regular action of \( T_n \) on \( \mathbb{A}^n \) is a representation in some coordinate system.

The notion of regular action means regularity in the sense of algebraic geometry (preservation of regular functions; Białynicki-Birula also considered birational actions in [49]). The last theorem states that any effective regular action of the maximal torus on an affine space is conjugate to a linear action (representation) - in other words, such action admits linearization.

An algebraic group action on \( \mathbb{A}^n \) is the same as the corresponding action by automorphisms on the algebra

\[
\mathbb{K}[x_1, \ldots, x_n]
\]

of coordinate functions. In other words, it is a group homomorphism

\[
\sigma : T_n \to \text{Aut} \, \mathbb{K}[x_1, \ldots, x_n].
\]
An action is effective if and only if \( \text{Ker} \sigma = \{1\} \).

The polynomial algebra is a quotient of the free associative algebra

\[
F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle
\]

by the commutator ideal \( I \) (it is the two-sided ideal generated by all elements of the form \( fg - gf \)). The definition of torus action on the free algebra is thus purely algebraic.

The following result has been established in \([85, 86]\).

**Theorem 1.3.5.** Suppose given an action \( \sigma \) of the algebraic \( n \)-torus \( \mathbb{T}_n \) on the free algebra \( F_n \). If \( \sigma \) is effective, then it is linearizable.

The theory of algebraic group actions on varieties is a substantial part of the study of Ind-varieties. Among the significant works on this subject, the reader is well advised to consult the papers of T. Kambayashi and P. Russell \([108]\), M. Koras and P. Russell \([125]\), T. Asanuma \([16]\), G. Schwartz \([170]\) and H. Bass \([18]\).

The group action linearity problem asks, generally speaking, whether any action of a given algebraic group on an affine space is linear in some suitable coordinate system (or, in other words, whether for any such action there exists an automorphism of the affine space such that it conjugates the action to a representation). This subject owes its existence largely to the classical work of A. Białynicki-Birula \([49]\), who considered regular (i.e. by polynomial mappings) actions of the \( n \)-dimensional torus on the affine space \( \mathbb{A}^n \) (over algebraically closed ground field) and proved that any faithful action is conjugate to a representation (or, as we sometimes say, linearizable). The result of Białynicki-Birula had motivated the study of various analogous instances, such as those that deal with actions of tori of dimension smaller than that of the affine space, or, alternatively, linearity conjectures that arise when the torus is replaced by a different sort of algebraic group. In particular, Białynicki-Birula himself \([50]\) had proved that any effective action of \((n-1)\)-dimensional torus on \( \mathbb{A}^n \) is linearizable, and for a while it was believed \([108]\) that the same was true for arbitrary torus and affine space dimensions. Eventually, however, the negation of this generalized linearity conjecture was established, with counter-examples due to Asanuma \([16]\).

More recently, the linearity of effective torus actions has become a stepping stone in the study of geometry of automorphism groups. In the paper \([118]\), the following result was obtained.

**Theorem 1.3.6.** Let \( \mathbb{K} \) be algebraically closed, and let \( n \geq 3 \). Then any Ind-variety automorphism \( \Phi \) of the Ind-group \( \text{Aut}(K[x_1, \ldots, x_n]) \) is inner.

The notions of Ind-variety (or Ind-group in this context) and Ind-variety morphism were introduced by Shafarevich \([171]\): an Ind-variety is the direct limit of a system whose morphisms are closed embeddings. Automorphism groups of algebras with polynomial identities, such as the (commutative) polynomial algebra and the free associative algebra, are archetypal examples; the corresponding direct systems of varieties consist of sets \( \text{Aut}^{\leq N} \) of automorphisms of total degree
less or equal to a fixed number, with the degree induced by the grading. The morphisms are inclusion maps which are obviously closed embeddings.

Theorem 1.3.6 is proved by means of tame approximation (stemming from the main result of [8]), with the following Proposition, originally due to E. Rips, constituting one of the key results.

**Proposition 1.3.7.** Let $\mathbb{K}$ be algebraically closed and $n \geq 3$ as above, and suppose that $\Phi$ preserves the standard maximal torus action on the commutative polynomial algebra\(^3\). Then $\Phi$ preserves all tame automorphisms.

The proof relies on the Białynicki-Birula theorem on the maximal torus action. In a similar fashion, the paper [118] examines the Ind-group $\text{Aut} \mathbb{K}\langle x_1, \ldots, x_n \rangle$ of automorphisms of the free associative algebra $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ in $n$ variables, and establishes results completely analogous to Theorem 1.3.6 and Proposition 1.3.7. \(^4\) In accordance with that, the free associative analogue of the Białynicki-Birula theorem was required.

Such an analogue is indeed valid, and we have established it in our notes [85,86] on the subject. We will provide the proof of this result in the sequel.

Given the existence of a free algebra version of the Białynicki-Birula theorem, one may inquire whether various other instances of the linearity problem (such as the Białynicki-Birula theorem on the action of the $(n - 1)$-dimensional torus on $\mathbb{K}[x_1, \ldots, x_n]$) can be studied. As it turns out, direct adaptation of proof techniques from the commutative realm is sometimes possible. There are certain limitations, however. For instance, Białynicki-Birula’s proof [50] of linearity of $(n - 1)$-dimensional torus actions uses commutativity in an essential way. Nevertheless, a neat workaround of that hurdle can be performed when $n = 2$, as we show in this note. Also, a special class of torus actions (positive-root actions) turns out to be linearizable. Finally, some of the proof techniques developed by Asanuma [16] admit free associative analogues; this will allow us to prove the existence of non-linearizable torus actions in positive characteristic, in complete analogy with Asanuma’s work.

---

\(^3\)That is, the action of the $n$-dimensional torus on the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$, which is dual to the action on the affine space.

\(^4\)The free associative case was amenable to the above approach when $n > 3$. 

44
Chapter 2

Quantization proof of Bergman’s centralizer theorem

We first give a brief summary of the background of two well-known centralizer theorems in the power series ring and in the free associative algebra, i.e., Cohn’s centralizer theorem and Bergman’s centralizer theorem.

2.1 Centralizer theorems

This section is a relatively independent part of the paper, and only sketches proofs with classic tools, while the following sections will focus on the new proof of Bergman’s centralizer theorem.

Throughout this section, $X$ is a finite set of noncommutative variables, and $k$ is a field. Let $X^*$ denote the free monoid generated by $X$. Let $k\langle X \rangle$ and $k\langle X \rangle$ denote the $k$-algebra of formal series and noncommutative polynomials (i.e., the free associative algebra over $k$) in $X$, respectively. Both elements of $k\langle \langle X \rangle \rangle$ and $k\langle X \rangle$ have the form $a = \sum_{\omega \in X^*} a_\omega \omega$, where $a_\omega \in k$ is the coefficient of the word $\omega$ in $a$, but they have different details inside the above formula. An element of $k\langle X \rangle$ is only a finite sum of words, while there are infinitely many terms of the sum for an element in $k\langle \langle X \rangle \rangle$. The multiplication of elements in $k\langle \langle X \rangle \rangle$ is the concatenation of words and normal multiplication of coefficients. We can only combine the coefficients which have the same corresponding words for addition. The length $|\omega|$ of a word $\omega \in X^*$ is the number of letters inside $\omega$. Now we can define the valuation

$$\nu : k\langle \langle X \rangle \rangle \mapsto \mathbb{N} \cup \{\infty\}$$

as follows: $\nu(0) = \infty$ and if $a = \sum_{\omega \in X^*} a_\omega \omega \neq 0$, then $\nu(a) = \min\{|\omega| : a_\omega \neq 0\}$. Note that if $\omega$ is constant, then $\nu(\omega) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b$ in $k\langle \langle X \rangle \rangle$.

For the words valuation, there is an easy but quite useful lemma [172].

Lemma 2.1.1 (Levi’s Lemma). Let $\omega_1, \omega_2, \omega_3, \omega_4 \in X^*$ be nonzero with $|\omega_2| \geq |\omega_4|$. If $\omega_1 \omega_2 = \omega_3 \omega_4$, then $\omega_2 = \omega \omega_4$ for some $\omega \in X^*$. 

45
The proof is trivial by backward induction on $|\omega|_2$ since $\omega_2$ has the same last letter as $\omega_4$. Next lemma extends Levi’s lemma to $k\langle X \rangle$, and we post the result as follows.

**Lemma 2.1.2** ([133], Lemma 9.1.2). Let $a, b, c, d \in k\langle X \rangle$ be nonzero. If $\nu(a) \geq \nu(c)$ and $ab = cd$, then $a = cq$ for some $q \in k\langle X \rangle$.

**Proof.** We can fix a word $u$ which appears in $b$ and $|u| = \nu(b)$. Suppose $v$ is any nonzero word appearing in $d$, then we have

$$|v| \geq \nu(d) = \nu(a) + \nu(b) - \nu(c) \geq \nu(b) = |u|.$$  \hfill (2.1)

Let $w$ be any word in $X^*$. The coefficient of $wu$ in $ab$ is $\sum_{rs=wu} a_r b_s$, where $a_r$ and $b_s$ are the coefficients of the words $r, s$ which appear in $a, b$ respectively. Similarly, the coefficient of $wu$ in $cd$ is $\sum_{yz=wu} c_y d_z$. Since $ab = cd$, we have

$$\sum_{rs=wu} a_r b_s = \sum_{yz=wu} c_y d_z. \tag{2.2}$$

By the inequality 2.1, we have $|z| \geq |u|$, and $|s| \geq |u|$ by the definition of $u$. Thus $rs = wu$ and $yz = wu$ imply $s = s_1 u$ and $z = z_1 u$ for some $s_1, z_1 \in X^*$, by Levi’s lemma. Hence $rs_1 = yz_1 = w$ and we can rewrite the formula 2.2 as

$$\sum_{rs_1=w} a_r b_{s_1} u = \sum_{yz_1=w} c_y d_{z_1} u. \tag{2.3}$$

Let $b' = \sum_{s_1 \in X} b_{s_1} u s_1$ and $d' = \sum_{z_1 \in X} b_{z_1} u z_1$. Then the equation gives $ab' = cd'$. The constant term of $b'$ is $b_u \neq 0$ and hence $b'$ is invertible in $k\langle X \rangle$. Hence if we let $q = d'b^{-1}$, then $a = cq$. \hfill \square

### 2.1.1 Cohn’s centralizer theorem

With the help of the preceding lemmas, we could post and prove this well-known centralizer theorem of $k$-algebra of formal series by P. M. Cohn.

**Theorem 2.1.3** (Cohn’s Centralizer Theorem, [60]). If $a \in k\langle X \rangle$ is not a constant, then the centralizer $C(a; k\langle X \rangle) \cong k[x]$, where $k[x]$ is the algebra of formal power series in the variable $x$.

**Proof.** Let $C := C(a; k\langle X \rangle)$. Let $a_0$ be the constant term of $a$, then it is clear that $C = C(a - a_0; k\langle X \rangle)$. So we may assume that the constant term of $a$ is zero. Thus we have a nonempty set $A = \{c \in C : \nu(c) > 0\}$ because $a \in C$ and so there exists $b \in A$ such that $\nu(b)$ is minimal. An easy observation is that $k[b] \cong k[x]$. Because suppose $\sum_{i \geq m} \beta_i b^i = 0, \beta_i \in k, \beta_m \neq 0$, then we must have $\infty = \nu(\sum_{i \geq m} \beta_i b^i) = \nu(b^n) = m \nu(b)$, which is absurd. So we just need to show that $C = k[b]$. Assume that an element $c \in C$ is not constant. Our first claim is that there exist $\beta_i \in k$ such that

$$\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geq (n + 1) \nu(b). \tag{2.4}$$
Theorem 2.1.4 helps us to finish the proof of that the centralizer is integrally closed. This will be shown in the main idea. However, we would take some necessary result in his original proof \[43\] which since \(c\) is a constant term of an element \(a \in k\langle X \rangle\) is zero and \(b, c \in C \setminus \{0\}\). If \(\nu(c) \geq \nu(b)\), then \(c = bd\) for some \(d \in C\). In fact, since the constant term of an element \(a \in k\langle X \rangle\) is zero we have \(\nu(a) \geq 1\). Thus for \(n\) large enough, we have \(\nu(a^n) = n\nu(a) \geq \nu(c)\). we also have \(a^n c = c a^n\) because \(c \in C\). Thus, by lemma 2.1.2, \(a^n = c q\) for some \(q \in k\langle X \rangle\). Hence, \(c q b = a^n b = b a^n\) and since \(\nu(c) \geq \nu(b)\), we have \(c = bd\), for some \(d \in k\langle X \rangle\), by lemma 2.1.2. Finally,

\[
bad = abd = ac = ca = bda,
\]

which gives \(ad = da\), i.e. \(d \in C\).

Now let us continue to prove the first claim. Suppose we have found \(\beta_0, \ldots, \beta_n \in k\) such that 

\[
\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geq (n + 1)\nu(b).
\]

Then since \((n + 1)\nu(b) = \nu(b^{n+1})\), we have \(c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1}d\) for some \(d \in C\), by the second claim we proved above. If \(d\) is a constant, we are done because then \(c \in k[b] \subset k[[b]]\). Otherwise, let \(\beta_{n+1}\) be the constant term of \(d\). Then \(d - \beta_{n+1} \in A\) and hence \(\nu(d - \beta_{n+1}) > \nu(b)\) by the minimality of \(b\). Therefore, by the first claim, \(d - \beta_{n+1} = bd'\) for some \(d' \in C\). Hence

\[
c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1}d = b^{n+1}(bd' + \beta_{n+1}) = b^{n+2}d' + \beta_{n+1}b^{n+1},
\]

which gives \(c - \sum_{i=0}^{n+1} \beta_i b^i = b^{n+2}d'\). Hence

\[
\nu(c - \sum_{i=0}^{n+1} \beta_i b^i) = \nu(b^{n+2}d') = (n + 2)\nu(b) + \nu(d') \geq (n + 2)\nu(b).
\]

This completes the induction, then we are done because \(\nu(c - \sum_{i \geq 0} \beta_i b^i) = \infty\) and so \(c = \sum_{i \geq 0} \beta_i b^i \in k[[b]]\).

\subsection{2.1.2 Bergman’s centralizer theorem}

Now since \(k\langle X \rangle \subset K\langle X \rangle\), it follows from the above theorem that if \(a \in k\langle X \rangle\) is not constant, then \(C(a; k\langle X \rangle)\) is commutative because \(C(a; k\langle X \rangle)\) is commutative. The next theorem is our main goal which shows that there is a similar result for \(C(a; k\langle X \rangle)\).

**Theorem 2.1.4 (Bergman’s Centralizer Theorem, [43]).** If \(a \in k\langle X \rangle\) is not constant, then the centralizer \(C(a; k\langle X \rangle) \cong k[x]\), where \(k[x]\) is the polynomial algebra in one variable \(x\).

We will not fully recover the original proof of Bergman’s centralizer theorem since this is not our main idea. However, we would take some necessary result in his original proof [43] which helps us to finish the proof of that the centralizer is integrally closed. This will be shown in the
Subsection 2.4.3.

First of all, we need to emphasize that the proof of Cohn’s centralizer theorem is included. Here is a sketch of the proof.

For simplicity, we denote by $C := C(a; k\langle X \rangle)$ the centralizer of $a$ which from now on is not a constant. Recall that the centralizer $C$ is also commutative. Moreover, $C$ is finitely generated, as module over $k[a]$ or as algebra. Then since $k\langle X \rangle$ is a 2-fir (free ideal rings, cf. Lemma 1.5 in [43]), and the center of a 2-fir is integrally closed, we obtain that the centralizer of $a$ is integrally closed in its field of fractions after using the lifting to $k\langle X \rangle \otimes k(x)$ (where $x$ is a free variable). Then our aim is to show that $C$ is a polynomial ring over $k$. In order to get this fact we shall study homomorphisms of $C$ into polynomial rings. By using “infinite” words, we obtained an embedding from $C$ into polynomial rings by lexicographically ordered semigroup algebras, which completes this sketch of the proof. Indeed, any subalgebra not equal to $k$ of a polynomial algebra $k[x]$ that is integrally closed in its own field of fractions is of form $k[y]$ (by Lüroth’s theorem).

We conclude this section by pointing out that the method of “infinite” words inspires us to find a possibility to prove Bergman’s centralizer theorem by deformation quantization. In the next section, we will establish this new approach of quantization for generic matrices.

2.2 Reduction to generic matrix

In this section, we will establish an important theorem which gives a relation of commutative subalgebras in the free associative algebra and the algebra of generic matrices. Let $k\langle X \rangle$ be the free associative algebra over a field $k$ generated by a finite set $X = \{x_1, \ldots, x_s\}$ of $s$ indeterminates, and let $k\langle X_1, \ldots, X_s \rangle$ be the algebra of $n \times n$ generic matrices generated by the matrices $X_\nu$. The canonical homomorphism $\pi : k\langle x_1, \ldots, x_s \rangle \mapsto k\langle X_1, \ldots, X_s \rangle$ shows in last section.

We claim that if we have a commutative subalgebra of rank two in the free associative algebra $k\langle X \rangle$, then we also have a commutative subalgebra of rank two if we consider a reduction to generic matrices of big enough order $n$. We also call two elements of a free algebra algebraically independent if the subalgebra generated by these two elements is a free algebra of rank two. Otherwise we will call them algebraically dependent.

In other words, if we have a commutative subalgebra $k[f, g]$ of rank two in the free associative algebra, then we have to prove that its projection to generic matrices of some order also has rank two. i.e. $\pi(f), \pi(g)$ do not have any relations.

We need following theorem:

**Theorem 2.2.1.** Let $k\langle X \rangle$ be the free associative algebra over a field $k$ generated by a finite set $X$ of indeterminates. If $k\langle X \rangle$ has a commutative subalgebra with two algebraically independent generators $f, g \in k\langle X \rangle$, then the subalgebra of $n \times n$ generic matrices generated by reduction of $f$ and $g$ in $k\langle X_1, \ldots, X_s \rangle$ also has rank two for big enough $n$. 


Proof. Assume $k[f, g]$ be a commutative subalgebra generated by $f, g \in k\langle X \rangle$ with rank two. We denote $\bar{f}, \bar{g} \in k\langle X_1, \ldots, X_s \rangle$ to be the generic matrices of $f$ and $g$ respectively after reduction 1.1 of algebra of generic matrices with $n \times n$. The rank of $k\langle \bar{f}, \bar{g} \rangle$ must be $\leq 2$ (i.e. it must be 1 or 2). Suppose the rank is 1, then for any two elements $a, b \in k\langle f, g \rangle$, there exists a minimal polynomial $P(x, y) \in k[x, y]$ ($x, y$ are two free variables) with degree $m$ such that $P(\bar{a}, \bar{b}) = 0$ because the algebra of generic matrices is a domain by Theorem 1.1.8. On the other hand, by Amitsur-Levitzki Theorem 1.1.9, there exists no polynomial with degree less than $2n$, such that $P(\bar{a}, \bar{b}) = 0$. This leads to be a contradiction if we choose $n > \lfloor m/2 \rfloor$.

Recall from the section 2.1.2 that the centralizer $C := C(a; k\langle X \rangle)$ of $a \in k\langle X \rangle \setminus k$ is a commutative subalgebra of $k\langle X \rangle$, so from the above theorem, we conclude that if the centralizer is a subalgebra in $k\langle X \rangle$ of rank two then the $\pi$-image subalgebra of $C$ has also rank two.

However, we prefer discussing this general case of subalgebras instead of just consider a centralizer subalgebra. Furthermore, we want to prove that there is no commutative subalgebras of the free associative algebra $k\langle X \rangle$ of rank greater than or equal to two.

### 2.3 Quantization proof of rank one

Up to our knowledge, there is no new proofs has been appeared after Bergman [43] for almost fifty years. We are using a method of deformation quantization presented by M. Kontsevich to give an alternative proof of Bergman’s centralizer theorem. In this section [115], we got that the centralizer is a commutative domain of transcendence degree one.

Let $k\langle X \rangle$ be the free associative algebra over a field $k$ generated by $s$ free variables $X = \{x_1, \ldots, x_s\}$. Now, we concentrate on our proof that there is no commutative subalgebras of rank greater than or equal to two. From the homomorphism $\pi : k\langle x_1, \ldots, x_s \rangle \mapsto k\langle X_1, \ldots, X_s \rangle$ and Theorem 2.2.1, we are moving our goal from the elements of $k\langle X \rangle$ to the algebra of generic matrices $k\langle X_1, \ldots, X_s \rangle$, and we consider the quantization of this algebra and its subalgebras.

Let $A, B$ be two commuting generic matrices in $k\langle X_1, \ldots, X_s \rangle$ which are algebraically independent, i.e. rank $k\langle A, B \rangle = 2$. We have the following theorem.

**Theorem 2.3.1.** Let $A, B$ be two commuting generic matrices in $k\langle X_1, \ldots, X_s \rangle$ with rank $k\langle A, B \rangle = 2$, and let $\hat{A}$ and $\hat{B}$ be quantized images (by sending multiplications to star products by means of Kontsevich’s formal quantization) of $A$ and $B$ respectively by considering lifting $A$ and $B$ in $k\langle X_1, \ldots, X_s \rangle[[\hbar]]$. Then $\hat{A}$ and $\hat{B}$ do not commute. Moreover,

$$\frac{1}{\hbar}[\hat{A}, \hat{B}]_* \equiv \begin{pmatrix}
\frac{1}{\hbar}\{\lambda_1, \mu_1\} & 0 \\
0 & \ddots \\
0 & 0 & \frac{1}{\hbar}\{\lambda_n, \mu_n\}
\end{pmatrix} \mod \hbar$$

(2.5)

where $\lambda_i$ and $\mu_i$ are eigenvalues(weights) of $A$ and $B$ respectively.
To prove this theorem, we need some preparations. It is not easy to directly compute such two generic matrices with order $n$. However, if we can diagonalize those matrices, then computation will be easier. So first of all, we should show the possibilities. Without loss of generality, we may assume that one of the generic matrices $B$ is diagonal if we have a proper choice of basis of the algebra of generic matrices. Now consider the other generic matrix $A$ which we mentioned above.

**Remark 2.3.2.** The generic matrix $A$ may not be diagonalizable over $k[x_{ij}^{(\nu)}]$, but it can be diagonalized over some integral extension of the algebra $k[x_{ij}^{(\nu)}]$ with $i, j = 1, \ldots, n; \nu = 1, \ldots, s$.

**Remark 2.3.3.** Any non-scalar element $A$ of the algebra of generic matrices must have distinct eigenvalues. In fact, by Amitsur’s Theorem 1.1.8, namely, the algebra of generic matrices is a domain, if the minimal polynomial is not a central polynomial, then the algebra can be embedded to a skew field. Hence, the minimal polynomial is irreducible, and the eigenvalues are pairwise different.

**Lemma 2.3.4.** Let $\hat{A} \equiv A_0 + \hbar A_1 (\mod \hbar^2)$ be the quantized image of a generic matrix $A \in k(X_1, \ldots, X_s)$, where $A_0$ is diagonal with distinct eigenvalues. Then, the quantized images $\hat{A}$ can be diagonalized over some finite extension of $k[x_{ij}^{(\nu)}]$.

**Proof.** Without loss of generality, suppose $A_0$ is a diagonal generic matrix with distinct eigenvalues. We want to show that there exists an invertible generic matrix $P$, such that $PAP^{-1}$ is diagonal. Now we consider their images on $k[X_1, \ldots, X_s][\hbar]$, we may assume $\hat{P} = I + \hbar T$ and the conjugation inverse $\hat{P}^{-1} = I - \hbar T \mod \hbar^2$ (where $I$ is the identity matrix). Then we have

$$(I + \hbar T)(A_0 + \hbar A_1)(I - \hbar T) = A_0 + \hbar([T, A_0] + A_1) \mod \hbar^2,$$

and we need to solve the equation $[T, A_0] = -A_1$.

This is clear since $A_0$ is diagonal. Let $A_0 = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$, $T = (t_{ij})_{n \times n}$ and $A_1 = (a_{ij})_{n \times n}$, then we have $[T, A_0] = ((\lambda_i - \lambda_j)t_{ij})_{n \times n}$. Hence,

$$T = (t_{ij})_{n \times n} = \left(-\frac{a_{ij}}{\lambda_i - \lambda_j}\right)_{n \times n}.$$

So far, we have determined the $\hbar$ term of the matrix $\hat{A}$. Hence, we may assume $\hat{A} \equiv A_0 + \hbar^2 A_2 \mod \hbar^3$, then we continue to cancel the $\hbar^2$ term. Let $\hat{P}_2 = I + \hbar^2 T_2$, and the conjugation inverse $\hat{P}_2^{-1} = I - \hbar^2 T_2$. Then, we have

$$(I + \hbar^2 T_2)(A_0 + \hbar^2 A_2)(I - \hbar^2 T_2) = A_0 + \hbar^2([T_2, A_0] + A_2) \mod \hbar^3,$$

Hence, $T_2$ is determined by equation $[T_2, A_0] = -A_2 = (a_{ij}^{(2)})_{n \times n}$. Similar computation give all entries of $T_2$, namely

$$T_2 = \left(-\frac{a_{ij}^{(2)}}{\lambda_i - \lambda_j}\right)_{n \times n}.$$
Continue this process to cancel the term of $h^3$ etc., we obtain equations $[T_i, A_0] = -A_i$ for $i = 3, 4, 5, \ldots$. This leads to the result that $A$ could be diagonalized over the extension $k[x_{ij}]_{[\frac{1}{x_i - x_j}]}$.

Now $A, B$ are two algebraically independent but commuting generic matrices in $k\langle X_1, \ldots, X_s \rangle$. From previous discussion, we may assume $A$ and $B$ can be both diagonalized over an integral extension of $k[x_{ij}]$. Consider result of diagonalization in $k\langle X_1, \ldots, X_s \rangle$ and then we compute the quantization commutator of two quantized generic matrices over $k\langle X_1, \ldots, X_s \rangle$. Now we can complete the proof of Theorem 2.3.1.

**Proof of Theorem 2.3.1.** We have shown that $A, B$ can be both diagonalized over some finite extension of $k[x_{ij}]$, then consider result of diagonalization with the quantization form in $k\langle X_1, \ldots, X_s \rangle$, i.e., we can write them into specific forms modulo $h^2$ as follows:

$$
\hat{A} \equiv \begin{pmatrix} 
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_n 
\end{pmatrix} + h \begin{pmatrix} 
\delta_1 & * \\
\vdots & \ddots \\
* & \delta_n 
\end{pmatrix} \mod h^2
$$

$$
\hat{B} \equiv \begin{pmatrix} 
\mu_1 & 0 \\
\vdots & \ddots \\
0 & \mu_n 
\end{pmatrix} + h \begin{pmatrix} 
\nu_1 & * \\
\vdots & \ddots \\
* & \nu_n 
\end{pmatrix} \mod h^2.
$$

Then we can compute the quantization commutator,

$$
[\hat{A}, \hat{B}]_* := \hat{A} \ast \hat{B} - \hat{B} \ast \hat{A} \equiv \begin{pmatrix} 
\{\lambda_1, \mu_1\} & 0 \\
\vdots & \ddots \\
0 & \{\lambda_n, \mu_n\} 
\end{pmatrix} + h \bar{\lambda} \ast \begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix}
$$

$$
- h \begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix} \ast \bar{\lambda} + h \begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix} \ast \bar{\mu} - h \bar{\mu} \ast \begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix}
$$

$$
+ h^2 \begin{Bmatrix} 
\begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix}, \begin{pmatrix} 
0 & * \\
\vdots & \ddots \\
* & 0 
\end{pmatrix} 
\end{Bmatrix} \mod h^2
$$

Note that all terms have empty diagonals except the first term, and hence the quantization commutator $[\hat{A}, \hat{B}]_* \neq 0 \mod h^4$, which completes the proof of the theorem by multiplying $\frac{1}{h}$ on two sides of above equation.

**Remark 2.3.5.** Suppose $\lambda_i$ and $\delta_i$, $i = 1, \ldots, n$ are algebraically dependent. Then there are
polynomials $P_i$ in two variables such that $P_i(\lambda_i, \delta_i) = 0$. Put

$$P(x, y) = \prod_{i=1}^{n} P_i(x, y).$$

Then $P(A, B)$ is diagonal matrix having zeros on the main diagonal, i.e. $P(A, B) = 0$. It means that if $\text{rank} k\langle A, B \rangle = 2$ then $\lambda_i, \delta_i$ are algebraically independent for some $i$.

Let us conclude this section by pointing out the whole process of this proof. Recall that we have the free associative algebra $k\langle X \rangle$ over a field $k$, if we have a commutative subalgebra of rank two generated by $a, b \in k\langle X \rangle$, then we may have a commutative subalgebra of the algebra of generic matrices $k\langle X_1, \ldots, X_s \rangle$ of rank two generated by $A, B$ (they are images of a homomorphism $\pi : k\langle X \rangle \mapsto k\langle X_1, \ldots, X_s \rangle$). Consider the element $0 = [a, b]$ of the free associative algebra $k\langle X \rangle$, homomorphism $\pi$ and canonical quantization homomorphism $q$ sending multiplications to star products, we yield that

$$0 = q\pi([a, b]) = q[A, B] = [\hat{A}, \hat{B}].$$

This leads a contradiction to Theorem 2.3.1 which shows that $[\hat{A}, \hat{B}] \neq 0$. So we obtain the following result.

**Theorem 2.3.6.** There is no commutative subalgebras of rank $\geq 2$ in the free associative algebra $k\langle X \rangle$.

The centralizer ring is commutative from our discussion in section 2.1.2, and from the above theorem, it is of rank 1. So it is a commutative subalgebra with form $k[x]$ for some $x \in k\langle X \rangle \setminus k$. We will show it implies Bergman’s centralizer theorem 2.1.4 in the next section.

### 2.4 Centralizers are integrally closed

We have shown that the centralizer $C$ is a commutative domain of transcendence degree one. For us, it was the most interesting part of the proof of the Bergman’s centralizer theorem. However, we have to prove the fact that $C$ is integrally closed in order to complete the proof of Bergman’s Centralizer Theorem. In our this work [116], our proofs are based on the characteristic free instead of very rich and advanced P. Cohn and G. Bergman’s noncommutative divisibility theorem, we use generic matrices reduction, the invariant theory of characteristic zero by C. Procesi [164] and the invariant theory of positive characteristic by A. N. Zubkov [233, 234] and S. Donkin [75, 76].

#### 2.4.1 Invariant theory of generic matrices

We will try to review some useful facts in the invariant theory of generic matrices.

Consider the algebra $A_{n,s}$ of $s$-generated generic matrices of order $n$ over the ground field $k$. Let $a_{\ell} = (a^i_{\ell})_j, 1 \leq i, j \leq n, 1 \leq \ell \leq s$ be its generators. Let $R = k[a^i_{\ell}]$ be the ring of entries
coefficients. Consider an action of matrices $M_n(k)$ on matrices in $R$ by conjugation, namely $\varphi_B : B \mapsto MBM^{-1}$. It is well-known (refer to [163, 164, 233]) that the invariant function on this matrix can be expressed as a polynomial over traces $\operatorname{tr}(a_{i1}, \ldots, a_{is})$. Any invariant on $A_{n,s}$ is a polynomial of $\operatorname{tr}(a_{i1}, \ldots, a_{in})$. Note that the conjugation on $B$ induces an automorphism $\varphi_B$ of the ring $R$. Namely, $M(a_{ij})^t M^{-1} = (a_{ij}^t)$, and $\varphi_B(M)$ of $R$ induces automorphism on $M_n(R)$. And for any $x \in A_{n,s}$, we have $\varphi_B(x) = MxM^{-1} = \operatorname{Ad}_M(x)$.

Consider $\varphi_B(x) = \operatorname{Ad}_M^{-1}\varphi_M(x)$. Then any element of the algebra of generic matrices is invariant under $\varphi_M(x)$.

When dealing with matrices in characteristic 0, it is useful to think that they form an algebra with a further unary operation the trace, $x \mapsto \operatorname{tr}(x)$. One can formalize this as follows [58]:

**Definition 2.4.1.** An algebra with trace is an algebra equipped with an additional trace structure, that is a linear map $\operatorname{tr} : R \mapsto R$ satisfying the following properties

$$
\operatorname{tr}(ab) = \operatorname{tr}(ba), a \operatorname{tr}(b) = \operatorname{tr}(b)a, \operatorname{tr}(\operatorname{tr}(a)b) = \operatorname{tr}(a)\operatorname{tr}(b) \quad \text{for all } a, b \in R.
$$

There is a well-known fact as follows.

**Theorem 2.4.2.** The algebra of generic matrices with trace is an algebra of concomitants, i.e. subalgebra of $M_n(R)$ is an invariant under the action $\varphi_M(x)$.

This theorem was first proved by C. Procesi in [164] for the ground field $k$ of characteristic zero. If $k$ is a field of positive characteristic, we have to use not only traces, but also characteristic polynomials and their linearization (refer to [75, 76]). Relations between these invariants are discovered by C. Procesi [163, 164] for characteristic zero and A. N. Zubkov [233, 234] for characteristic $p$. C. de Concini and C. Procesi also generalized a characteristic free approach to invariant theory [69].

Let us denote by $k_T\{X\}$ the algebra of generic matrices with traces. After above discussions, we have the following proposition.

**Proposition 2.4.3.** Let $n$ be a prime number, then the centralizer of $A \in k_T\{X\}$ is rationally closed in $k_T\{X\}$ and integrally closed in $k_T\{X\}$.

### 2.4.2 Centralizers are integrally closed

Let $k\langle X \rangle$ be the free associative algebra as noted. Here we will prove the following theorem.

**Theorem 2.4.4.** The centralizer $C$ of non-trivial element $f$ in the free associative algebra is integrally closed.
Let \( g, P, Q \in C := C(f; F_z) \), and suppose \( g Q^m = P^m \) for some positive integer \( m \), i.e. in localization \( g = \frac{P^m}{Q^m} \). Then there exists \( h \in C \), such that \( h^m = g \). This means that the centralizer \( C \) is integral closed.

Consider the homomorphism \( \pi \) from the free associative algebra \( F_s \) to the algebra of generic matrices with traces \( k_T\{X\} \). Let us denote by \( \bar{g} \) the image \( \pi(g) \). Then we have following proposition.

**Proposition 2.4.5.** Consider the homomorphism \( \pi : F_s \mapsto k_T\{X\} \). Let the order of matrices be a prime number \( p \gg 0 \). \( \bar{g} = \pi(g), \bar{P} = \pi(P) \) and \( \bar{Q} = \pi(Q) \). Then there exists \( \bar{h} \in k_T\{X\} \) such that

1) \( \bar{h}^m = \bar{g}, \)

2) \( \bar{h} = \bar{P} \bar{Q}, \)

3) \( \bar{h} \in \bar{C} \), where \( \bar{C} = \pi(C) \).

**Proof.** 1) and 2) follows from Proposition 2.4.3 that the algebra of generic matrices with traces of form is integral closed. We need to prove 3). Note that all eigenvalues of \( \bar{g} \) are pairwise different due to Proposition 1.1.15. So is \( \bar{f} \). Hence \( \bar{f}, \bar{g} \) are diagonalizable and \( \bar{h} \) can be diagonalized in the same eigenvectors basis. Hence, by Proposition 1.1.14, \( \bar{h} \) commutes with \( \bar{f} \), i.e. \( \bar{h} \in \bar{C} \). \( \Box \)

Now we have to prove that \( \bar{h} \) in fact belongs to the algebra of generic matrices without trace. We use the local isomorphism to get rid of traces.

**Definition 2.4.6** (Local isomorphism). Let \( \mathcal{A} \) be an algebra with generators \( a_1, \ldots, a_s \) homogeneous respect this set of generators, and let \( \mathcal{A}' \) be an algebra with generators \( a'_1, \ldots, a'_s \) homogeneous respect this set of generators. We say that \( \mathcal{A} \) and \( \mathcal{A}' \) are locally \( L \)-isomorphic if there exist a linear map \( \varphi : a_i \mapsto a'_i \) on the space of monomials of degree \( \leq 2L \), and in this case for any two elements \( b_1, b_2 \in A \) with highest term of degree \( \leq L \), we have

\[
b_i = \sum_j M_{ij}(a_1, \ldots, a_s), b'_i = \sum_j M_{ij}(a'_1, \ldots, a'_s),
\]

where \( M_{ij} \) are monomials, and for \( b = b_1 \cdot b_2, b' = b'_1 \cdot b'_2 \), we have \( \varphi(b) = b' \).

We need following lemmas, and propositions:

**Lemma 2.4.7** (Local isomorphism lemma). For any \( L \), if \( s \) is big enough prime, then the algebra of generic upper triangular matrices \( U_s \) is locally \( L \)-isomorphic to the free associative algebra. Also reduction on the algebra of generic matrices of degree \( n \) provides an isomorphism up to degree \( \leq 2s \).

Let us remind a well-known and useful fact.

**Proposition 2.4.8.** The trace of every element in \( U_s \) of any characteristic is zero.
In fact, we also proved

**Proposition 2.4.9.** If \( n > n(L) \), then the algebra of generic matrices (without traces) is \( L \)-locally integrally closed.

**Lemma 2.4.10.** Consider the projection \( \pi \) of the algebra of generic matrices with trace to \( U_s \), sending all traces to zero. Then we have

\[
\pi(h)^m = \pi(g).
\]

**Proof of Theorem 2.4.4.** Let \( p \) be a big enough prime number. For example, we can set \( p \geq 2(\deg(f) + \deg(g) + \deg(P) + \deg(Q)) \). Because space of \( k_T\{X\} \) of degree \( \leq p \) is isomorphic to space of free associative algebra. We have element \( h \) corresponding to \( \hat{h} \) up to this isomorphism. Due to local isomorphism, \( h^m = g \), \( h = P/Q \), i.e. \( hQ = P \). Also we have \( h \) commutes with \( f \), i.e. \( h \in C \).

\[\Box\]

### 2.4.3 Completion of the proof

From last two subsections, we have the following proposition:

**Proposition 2.4.11.** Let \( p \) be a big enough prime number, and \( k\{X\} \) the algebra of generic matrices of order \( p \). For any \( A \in k\{X\} \), the centralizer of \( A \) is rationally closed and integrally closed in \( k\{X\} \) over the center of \( k\{X\} \).

In our previous paper [115], we establish that the centralizer in the algebra of generic matrices is a commutative ring of transcendence degree one. According to Proposition 2.4.11, \( C(A) \) is rationally closed and integrally closed in \( k\{X\} \). If \( p \) is big enough, then \( k\{X\} \) is \( L \)-locally integrally closed.

Now we need one fact from the Bergman’s paper [43]. Let \( X \) be a totally ordered set, \( W \) be the free semigroup with identity 1 on set \( X \). We have the following lemma.

**Lemma 2.4.12 (Bergman).** Let \( u, v \in W \setminus \{1\} \). If \( u^\infty > v^\infty \), then we have \( u^\infty > (uv)^\infty > (vu)^\infty > v^\infty \).

**Proof (Bergman).** It suffices to show that the whole inequality is implied by \( (uv)^\infty > (vu)^\infty \). Suppose \( (uv)^\infty > (vu)^\infty \), then we have following

\[
(vu)^\infty = v(uv)^\infty > v(vu)^\infty = v^2(uv)^\infty > v^2(vu)^\infty = \cdots v^\infty.
\]

Similarly, we obtain \( (uv)^\infty < u^\infty \).

\[\Box\]

Similarly, we also have inequalities with “\( \geq \)” replaced by “\( = \)” or “\( \leq \)”.

55
Remark 2.4.13. Similar constructions are used in [109] for Burnside type problems or the height theorem of Shirshov.

Now let $R$ be the semigroup algebra on $W$ over field $k$, i.e. $R = F_s$ is the free associative algebra. Consider $z \in \overline{W}$ be an infinite period word, and we denote $R_2(z)$ be the $k$-subspace of $R$ generated by words $u$ such that $u = 1$ or $u^\infty \leq z$. Let $I_2(z)$ be the $k$-subspace spanned by words $u$ such that $u \neq 1$ and $u^\infty < z$. Using Lemma 2.4.12, we can get that $R_2(z)$ is a subring of $R$ and $I_z$ is a two-sided ideal in $R(z)$. It follows that $R_2(z)/I_2(z)$ will be isomorphic to a polynomial ring $k[v]$.

**Proposition 2.4.14** (Bergman). If $C \neq k$ is a finitely generated subalgebra of $F_s$, then there is a homomorphism $f$ of $C$ into polynomial algebra over $k$ in one variable, such that $f(C) \neq k$.

**Proof (Bergman).** First let us totally order $X$. Let $G$ be a finite set of generators for $C$ and let $z$ be maximum over all monomials $u \neq 1$ with nonzero coefficient in elements of $G$ of $u^\infty$. Then we have $G \subseteq R_2(z)$ and hence $C \subseteq R_2(z)$, and the quotient map $f : R_2(z) \mapsto R_2(z)/I_2(z) \cong k[v]$ is nontrivial on $C$.

Now we can finish the proof of Bergman’s centralizer theorem.

**Proof.** Consider homomorphism from the Proposition 2.4.14. Because $C$ is centralizer of $F_s$, it has transcendence degree 1. Consider homomorphism $\rho$ send $C$ to the ring of polynomial. The homomorphism has kernel zero, otherwise $\rho(C)$ will have smaller transcendence degree. Note that $C$ is integrally closed and finitely generated, hence it can be embedded into polynomial ring of one indeterminate. Since $C$ is integrally closed, it is isomorphic to polynomial ring of one indeterminate.

Consider the set of system of $C_\ell$, $\ell$-generated subring of $C$ such that $C = \bigcup_\ell C_\ell$. Let $\overline{C_\ell}$ be the integral closure of $C_\ell$. Consider set of embedding of $C_\ell$ to ring of polynomial, then $\overline{C_\ell}$ are integral closure of those images, $\overline{C_\ell} = k[z_\ell]$, where $z_\ell$ belongs to the integral closure of $C_\ell$. Consider sequence of $z_\ell$. Because $k[z_\ell] \subseteq k[z_{\ell+1}]$, and degree of $z_{\ell+1}$ is strictly less than the degree of $z_\ell$. Hence this sequence stabilizes for some element $x$. Then $k[z]$ is the needed centralizer.

### 2.4.4 On the rationality of subfields of generic matrices

We will discuss some approaches to the following open problem.

**Problem 2.4.15.** Consider the algebra of generic matrices $k\{X\}$ of order $s$. Consider $\text{Frac}(k\{X\})$, and $K$ a subfield of $\text{Frac}(k\{X\})$ of transcendence degree one over the base field $k$. Is it true that $K$ is isomorphic to a rational function over $k$, namely $K \cong k(t)$?

Let $k\{X\}$ be the algebra of generic matrices of a big enough prime order $s := p$. Let $\Lambda$ be the diagonal generic matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_s)$ in $k\{X\}$, where transcendence degrees satisfy in
Trdeg $k[\lambda_i] = 1$. Let $N$ be another generic matrix, whose coefficients are algebraically independent from $\lambda_1, \ldots, \lambda_s$. It means that if $R$ is a ring of all coefficients of $N$, with $\text{Trdeg}(R) = s^2$, then

$$\text{Trdeg } R[\lambda_1, \ldots, \lambda_s] = s^2 + \text{Trdeg } k[\lambda_1, \ldots, \lambda_s].$$

**Proposition 2.4.16.** We consider the conjugation of generic matrices $\overline{f}$ and $\overline{g}$.

a) Let $k[f_{ij}]$ be a commutative ring, and $I = \langle f_{i1} \rangle \triangleleft k[f_{ij}]$ ($i > 1$) be an ideal of $k[f_{ij}]$. Then we have $k[f_{11}] \subseteq I = 0$.

b) Let $k[f_{ij}, g_{ij}]$ be a commutative ring, and $J = \langle f_{1j}, g_{1j} \rangle \triangleleft k[f_{ij}, g_{ij}]$ ($i, j > 1$). For any algebraic function $P$ satisfies $P(f_{1j}, g_{1j}) = 0$, which means $f, g$ are algebraic dependent on the $e_1$, then $k[f_{1j}, g_{1j}] \subseteq J = 0$.

**Corollary 2.4.17.** Let $\mathcal{A}$ be an algebra of generic matrices generated by $a_1, \ldots, a_s, a_{s+1}$. Let $f \in k[a_1, \ldots, a_s]$, $\varphi = a_{s+1}f a^{-1}_{s+1}$. Let $I = \langle \varphi_{i1} \rangle \triangleleft k[a_1, \ldots, a_{s+1}]$. Then $k[\varphi_{11}] \subseteq I = 0$.

**Proof.** Note that $f = \tau \Lambda \tau^{-1}$ for some $\tau$ and a diagonal matrix $\Lambda$ by proposition 2.4.16. Then $\varphi = (a_{s+1})\Lambda(a_{s+1})^{-1}$ and we can treat $(a_{s+1})$ as a generic matrix. \hfill \Box

**Theorem 2.4.18.** Let $C := C(f; F_n)$ be the centralizer ring of $f \in F_n \setminus k$. $\overline{C}$ is the reduction of generic matrices, and $\overline{\mathcal{C}}$ is the reduction on first eigenvalue action. Then $\overline{C} \cong C$.

**Proof.** Let us recall that we already have $\overline{C} \cong C$ in [115]. If we have $P(g_1, g_2) = 0$, then clearly $P(\lambda_1(g_1), \lambda_2(g_2)) = 0$ in the reduction on first eigenvalue action. Suppose $P(g_1, g_2) = 0$. Then $P(g_1, g_2)$ is an element of generic matrices with at least one zero eigenvalue. Because minimal polynomial is irreducible, that implies that $P(g_1, g_2) = 0$. It means any reduction satisfying $\lambda_1$ satisfies completely. That what we want to prove. \hfill \Box

Consider $\overline{C}$, for any $\overline{g} = (g_{ij}) \in \overline{C}$. Investigate $g_{11}$. Suppose there is a polynomial $P$ with coefficients in $k$, such that $P(f, g) = 0$. We can make a proposition about intersection of the ideals even sharper.

Let $J = \langle f_{1j}, g_{1j} \rangle$ ($j > 1$) be an ideal of the commutative subalgebra $k[f_{ij}, g_{ij}]$, then $k[f_{1j}, g_{1j}] \subseteq J = 0$.

From the discussion above and the theorem 2.4.18, we have the following proposition.

**Proposition 2.4.19.**

$$k[f_{11}, g_{11}] \mod J \cong k[f, g]$$

**Proof.** We have mod $J$ matrices from the following form:

$$\overline{f} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \overline{g} = \begin{pmatrix} \lambda_2 & 0 & \ldots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$
Then for any $H(f, g) \mod J$, we have

$$
\overline{f} = \begin{pmatrix}
H(\lambda_1, \lambda_2) & 0 & \ldots & 0 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}
$$

Now we present an approach as follows. Consider $k(f, g)$. Let us extend the algebra of generic matrices by new matrix $T$, independent from all others. Consider conjugation of $k(f, g)$ by $T$, $Tk(f, g)T^{-1}$, and consider $\tilde{f} = TfT^{-1}$ and $\tilde{g} = TgT^{-1}$. By Corollary 2.4.17, we have

$$P(g_{11}, f_{11}) = 0 \mod J.$$

On the other hand, we have

$$k[f_{11}, g_{11}] \cap J = 0,$$

which means that

$$P(f_{11}, g_{11}) = 0 \mod J.$$ 

Put $f_{11}$ and $g_{11}$ be polynomial over commutative ring generated by all entries of $k[f, g]$ and $T$. Hence $\text{Frac}(k(f, g))$ can be embedded into fractional field of rings of polynomials. According to Lüroth theorem, $\text{Frac}(k(f, g))$ (hence $\text{Frac}(C)$) is isomorphic to fields of rational functions in one variable.

This will not guarantee rationality of our field, and there are counter examples in this situation. However, this approach seems to be useful for highest term analysis.
Chapter 3

Automorphisms, augmentation topology, and approximation

3.1 Introduction and main results

This chapter is dedicated to the review of results of Kanel-Belov, Yu and Elishev on the geometry of the Ind-schemes

\[ \text{Aut}_K K[x_1, \ldots, x_n] \]

and

\[ \text{Aut}_K \langle x_1, \ldots, x_n \rangle \]

of automorphisms of the polynomial algebra and the free associative algebra over an algebraically closed field, with the number of generators > 2. The inner character of Ind automorphisms of these Ind-schemes, together with the negative resolution of the automorphism group lifting problem, was established in [118].

3.1.1 Automorphisms of \( K[x_1, \ldots, x_n] \) and \( K\langle x_1, \ldots, x_n \rangle \)

Let \( K \) be a field. The main objects of this study are the \( K \)-algebra automorphism groups \( \text{Aut}_K K[x_1, \ldots, x_n] \) and

\[ \text{Aut}_K \langle x_1, \ldots, x_n \rangle \]

of the (commutative) polynomial algebra and the free associative algebra with \( n \) generators, respectively. The former is equivalent to the group of all polynomial one-to-one mappings of the affine space \( A^n_K \). Both groups admit a representation as a colimit of algebraic sets of automorphisms filtered by total degree (with morphisms in the direct system given by closed embeddings) which turns them into topological spaces with Zariski topology compatible with the group structure. The two groups carry a power series topology as well, since every automorphism \( \varphi \) may be identified with the \( n \)-tuple \( (\varphi(x_1), \ldots, \varphi(x_n)) \) of the images of generators. This topology plays an especially important role in the applications, and it turns out – as reflected in the main results of this study – that approximation properties arising from this topology agree well with...
properties of combinatorial nature.

Ind-groups of polynomial automorphisms play a central part in the study of the Jacobian conjecture of O. Keller as well as a number of problems of similar nature. One outstanding example is provided by a recent conjecture of Kanel-Belov and Kontsevich (B-KKC), [39, 40], which asks whether the group

\[ \text{Sympl}(\mathbb{C}^{2n}) \subset \text{Aut}(\mathbb{C}[x_1, \ldots, x_{2n}]) \]

of complex polynomial automorphisms preserving the standard Poisson bracket

\[ \{x_i, x_j\} = \delta_{i,n+j} - \delta_{i+n,j} \]

is isomorphic\(^1\) to the group of automorphisms of the \(n\)-th Weyl algebra \(W_n\)

\[ W_n(\mathbb{C}) = \mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle / I, \]

\[ I = (x_ix_j - x_jx_i, y_iy_j - y_jy_i, y_ix_j - x_jy_i - \delta_{ij}) \].

The physical meaning of Kanel-Belov and Kontsevich conjecture is the invariance of the polynomial symplectomorphism group of the phase space under the procedure of deformation quantization.

The B-KKC was conceived during a successful search for a proof of stable equivalence of the Jacobian conjecture and a well-known conjecture of Dixmier stating that \(\text{Aut}(W_n) = \text{End}(W_n)\) over any field of characteristic zero. In the papers [39, 40] a particular family of homomorphisms (in effect, monomorphisms) \(\text{Aut}(W_n(\mathbb{C})) \to \text{Sympl}(\mathbb{C}^{2n})\) was constructed, and a natural question whether those homomorphisms were in fact isomorphisms was raised. The aforementioned morphisms, independently studied by Tsuchimoto to the same end, were in actuality defined as restrictions of morphisms of the saturated model of Weyl algebra over an algebraically closed field of \(\textit{positive}\) characteristic - an object which contains \(W_n(\mathbb{C})\) as a proper subalgebra. One of the defined morphisms turned out to have a particularly simple form over the subgroup of the so-called tame automorphisms, and it was natural to assume that morphism was the desired B-KK isomorphism (at least for the case of algebraically closed base field). Central to the construction is the notion of infinitely large prime number (in the sense of hyperintegers), which arises as the sequence \((p_m)_{m \in \mathbb{N}}\) of positive characteristics of finite fields comprising the saturated model. This leads to the natural problem ([39]):

**Problem.** Prove that the B-KK morphism is independent of the choice of the infinite prime \((p_m)_{m \in \mathbb{N}}\).

A general formulation of this question in the paper [39] goes as follows:

\(^1\)In fact, the conjecture seeks to establish an isomorphism \(\text{Sympl}(K^{2n}) \simeq \text{Aut}(W_n(K))\) for any field \(K\) of characteristic zero in a functorial manner.
For a commutative ring $R$ define

$$R_\infty = \lim_{\to} \left( \prod_p R' \otimes \mathbb{Z}/p\mathbb{Z} / \bigoplus_p R' \otimes \mathbb{Z}/p\mathbb{Z} \right),$$

where the direct limit is taken over the filtered system of all finitely generated subrings $R' \subset R$ and the product and the sum are taken over all primes $p$. This larger ring possesses a unique "nonstandard Frobenius" endomorphism $Fr : R_\infty \to R_\infty$ given by

$$(a_p)_{\text{primes } p} \mapsto (a_p^p)_{\text{primes } p}.$$ 

The Kanel-Belov and Kontsevich construction returns a morphism

$$\psi_R : \text{Aut}(W_n(R)) \to \text{Sympl } R^{2n}$$

such that there exists a unique homomorphism

$$\phi_R : \text{Aut}(W_n)(R) \to \text{Aut}(P_n)(R_\infty)$$

obeying $\psi_R = Fr_* \circ \phi_R$. Here $Fr_* : \text{Aut}(P_n)(R_\infty) \to \text{Aut}(P_n)(R_\infty)$ is the Ind-group homomorphism induced by the Frobenius endomorphism of the coefficient ring, and $P_n$ is the commutative Poisson algebra, i.e. the polynomial algebra in $2n$ variables equipped with additional Poisson structure (so that $\text{Aut}(P_n(R))$ is just $\text{Sympl}(R^{2n})$ - the group of Poisson structure-preserving automorphisms).

**Question.** In the above formulation, does the image of $\phi_R$ belong to $\text{Aut}(P_n)(i(R) \otimes \mathbb{Q})$, where $i : R \to R_\infty$ is the tautological inclusion? In other words, does there exist a unique homomorphism

$$\phi^{can}_R : \text{Aut}(P_n)(R) \to \text{Aut}(P_n)(R \otimes \mathbb{Q})$$

such that $\psi_R = Fr_* \circ i_* \circ \phi^{can}_R$.

Comparing the two morphisms $\phi$ and $\varphi$ defined using two different free ultrafilters, we obtain a "loop" element $\phi\varphi^{-1}$ of $\text{Aut}_{\text{Ind}}(\text{Aut}(W_n))$, (i.e. an automorphism which preserves the structure of infinite dimensional algebraic group). Describing this group would provide a solution to this question.

In the spirit of the above we propose the following

**Conjecture.** All automorphisms of the Ind-group $\text{Sympl}(\mathbb{C}^{2n})$ are inner.

A similar conjecture may be put forward for $\text{Aut}(W_n(\mathbb{C}))$.

Automorphism groups of Weyl algebras and their generalizations, as well as automorphisms
of certain algebras of vector fields, were studied in the works of Bavula [21–23]. Reduction to positive characteristic has proven both fruitful and essential in the context of Weyl algebra. One of the precursors to the study of these algebras in characteristic \( p \) was the paper [20].

We are focused on the investigation of the group \( \text{Aut}(\text{Aut}(K[x_1, \ldots, x_n])) \) and the corresponding noncommutative (free associative algebra) case. This way of thinking has its roots in the realm of universal algebra and universal algebraic geometry and was conceived in the pioneering work of Boris Plotkin. A more detailed discussion can be found in [37].

Wild automorphisms and the lifting problem. In 2004, the celebrated Nagata conjecture over a field \( K \) of characteristic zero was proved by Shestakov and Umirbaev [178,181] and a stronger version of the conjecture was proved by Umirbaev and Yu [197]. Let \( K \) be a field of characteristic zero. Every wild \( K[z] \)-automorphism (wild \( K[z] \)-coordinate) of \( K[z][x,y] \) is wild viewed as a \( K \)-automorphism (\( K \)-coordinate) of \( K[x,y,z] \). In particular, the Nagata automorphism \( (x - 2y(y^2 + xz) - (y^2 + xz)^2z, y + (y^2 + xz)z, z) \) (Nagata coordinates \( x - 2y(y^2 + xz) - (y^2 + xz)^2z \) and \( y + (y^2 + xz)z \)) are wild. In [197], a related question was raised:

**The lifting problem.** *Can an arbitrary wild automorphism (wild coordinate) of the polynomial algebra \( K[x,y,z] \) over a field \( K \) be lifted to an automorphism (coordinate) of the free associative algebra \( K \langle x,y,z \rangle \) ?*

In the paper [36], based on the degree estimate [132,137], it was proved that any wild \( z \)-automorphism including the Nagata automorphism cannot be lifted as a \( z \)-automorphism (moreover, in [42] it is proved that every \( z \)-automorphism of \( K \langle x, y, z \rangle \) is stably tame and becomes tame after adding at most one variable). It means that if every automorphism can be lifted, then it provides an obstruction \( z' \) to \( z \)-lifting and the question to estimate such an obstruction is naturally raised.

In view of the above, we may ask the following:

**The automorphism group lifting problem.** *Is \( \text{Aut}(K[x_1, \ldots, x_n]) \) isomorphic to a subgroup of \( \text{Aut}(K \langle x_1, \ldots, x_n \rangle) \) under the natural abelianization?*

The following examples show this problem is interesting and non-trivial.

**Example 1.** There is a surjective homomorphism (taking the absolute value) from \( \mathbb{C}^* \) onto \( \mathbb{R}^+ \). But \( \mathbb{R}^+ \) is isomorphic to the subgroup \( \mathbb{R}^+ \) of \( \mathbb{C}^* \) under the homomorphism.

**Example 2.** There is a surjective homomorphism (taking the determinant) from \( GL_n(\mathbb{R}) \) onto \( \mathbb{R}^* \). But obviously \( \mathbb{R}^* \) is isomorphic to the subgroup \( \mathbb{R}^*I_n \) of \( GL_n(\mathbb{R}) \).

In this paper we prove that the automorphism group lifting problem has a negative answer.

The lifting problem and the automorphism group lifting problem are closely related to the Kanel-Belov and Kontsevich Conjecture (see Section 3.3.1).
Consider a symplectomorphism $\varphi : x_i \mapsto P_i, y_i \mapsto Q_i$. It can be lifted to some automorphism $\hat{\varphi}$ of the quantized algebra $W_\hbar[[\hbar]]$:

$$\hat{\varphi} : x_i \mapsto P_i + P_i^1\hbar + \cdots + P_i^m\hbar^m; \ y_i \mapsto Q_i + Q_i^1\hbar + \cdots + Q_i^m\hbar^m.$$ 

The point is to choose a lift $\hat{\varphi}$ in such a way that the degree of all $P_i^m, Q_i^m$ would be bounded. If that is true, then the B-KKC follows.

### 3.1.2 Main results

The main results of this paper are as follows.

**Theorem 3.1.1.** Any Ind-scheme automorphism $\varphi$ of $\text{NAut}(K[x_1, \ldots, x_n])$ for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

**Theorem 3.1.2.** Any Ind-scheme automorphism $\varphi$ of $\text{NAut}(K\langle x_1, \ldots, x_n \rangle)$ for $n \geq 3$ is semi-inner (see definition 3.1.6).

$\text{NAut}$ denotes the group of nice automorphisms, i.e. automorphisms which can be approximated by tame ones (definition 3.3.1). In characteristic zero case every automorphism is nice.

For the group of automorphisms of a semigroup a number of similar results on set-theoretical level was obtained previously by Kanel-Belov, Lipyanski and Berzins 
[37,41]. All these questions (including Aut(Aut) investigation) take root in the realm of Universal Algebraic Geometry and were proposed by Boris Plotkin. Equivalence of two algebras having the same generalized identities and isomorphism of first order means semi-inner properties of automorphisms (see [37, 41] for details).

**Automorphisms of tame automorphism groups.** Regarding the tame automorphism group, something can be done on the group-theoretic level. In the paper of H. Kraft and I. Stampfli [128] the automorphism group of the tame automorphism group of the polynomial algebra was thoroughly studied. In that paper, conjugation of elementary automorphisms via translations played an important role. The results of our study are different. We describe the group $\text{Aut}(\text{TAut}_0)$ of the group $\text{TAut}_0$ of tame automorphisms preserving the origin (i.e. taking the augmentation ideal onto an ideal which is a subset of the augmentation ideal). This is technically more difficult, and will be universally and systematically done for both commutative (polynomial algebra) case and noncommutative (free associative algebra) case. We observe a few problems in the shift conjugation approach for the noncommutative (free associative algebra) case, as it was for commutative case in [128]. Any evaluation on a ground field element can return zero, for example in Lie polynomial $[[x, y], z]$. Note that the calculations of $\text{Aut}(\text{TAut}_0)$ (resp. $\text{Aut}_{\text{ind}}(\text{TAut}_0), \text{Aut}_{\text{ind}}(\text{Aut}_0)$) imply also the same results for $\text{Aut}(\text{TAut})$ (resp. $\text{Aut}_{\text{ind}}(\text{TAut}), \text{Aut}_{\text{ind}}(\text{Aut})$) according to the approach of this article via stabilization by the torus action.
Theorem 3.1.3. Any automorphism \( \varphi \) of \( \TAut_0(K[x_1, \ldots, x_n]) \) (in the group-theoretic sense) for \( n \geq 3 \) is inner, i.e. is a conjugation via some automorphism.

Theorem 3.1.4. The group \( \TAut_0(K[x_1, \ldots, x_n]) \) is generated by the automorphism
\[
x_1 \mapsto x_1 + x_2x_3, \quad x_i \mapsto x_i, \quad i \neq 1
\]
and linear substitutions if \( \text{char}(K) \neq 2 \) and \( n > 3 \).

Let \( G_N \subset \TAut(K[x_1, \ldots, x_n]) \), \( E_N \subset \TAut(K(x_1, \ldots, x_n)) \) be tame automorphism subgroups preserving the \( N \)-th power of the augmentation ideal.

Theorem 3.1.5. Any automorphism \( \varphi \) of \( G_N \) (in the group-theoretic sense) for \( N \geq 3 \) is inner, i.e. is given by a conjugation via some automorphism.

Definition 3.1.6. An anti-automorphism \( \Psi \) of a \( K \)-algebra \( B \) is a vector space automorphism such that \( \Psi(ab) = \Psi(b)\Psi(a) \). For instance, transposition of matrices is an anti-automorphism. An anti-automorphism of the free associative algebra \( A \) is a mirror anti-automorphism if it sends \( x_ix_j \) to \( x_jx_i \) for some fixed \( i \) and \( j \). If a mirror anti-automorphism \( \theta \) acts identical on all generators \( x_i \), then for any monomial \( x_{i_1}\cdots x_{i_k} \) we have
\[
\theta(x_{i_1}\cdots x_{i_k}) = x_{i_k}\cdots x_{i_1}.
\]
Such an anti-automorphism will be generally referred to as the mirror anti-automorphism.

An automorphism of \( \Aut(A) \) is semi-inner if it can be expressed as a composition of an inner automorphism and a conjugation by a mirror anti-automorphism.

Theorem 3.1.7. a) Any automorphism \( \varphi \) of \( \TAut_0(K(x_1, \ldots, x_n)) \) and also \( \TAut(K(x_1, \ldots, x_n)) \) (in the group-theoretic sense) for \( n \geq 4 \) is semi-inner, i.e. is a conjugation via some automorphism and/or mirror anti-automorphism.

b) The same is true for \( E_n, n \geq 4 \).

The case of \( \TAut(K(x, y, z)) \) is substantially more difficult. We can treat it only on Ind-scheme level, but even then it is the most technical part of the paper (see section 3.5.2). For the two-variable case a similar proposition is probably false.

Theorem 3.1.8. a) Let \( \text{char}(K) \neq 2 \). Then \( \Aut_{\text{Ind}}(\TAut(K(x, y, z))) \) (resp. \( \Aut_{\text{Ind}}(\TAut_0(K(x, y, z))) \)) is generated by conjugation by an automorphism or a mirror anti-automorphism.

b) The same is true for \( \Aut_{\text{Ind}}(E_3) \).

By \( \TAut \) we denote the tame automorphism group, \( \Aut_{\text{Ind}} \) is the group of Ind-scheme automorphisms (see section 3.2.2).

Approximation allows us to formulate the celebrated Jacobian conjecture for any characteristic.
Lifting of the automorphism groups. In this article we prove that the automorphism group of polynomial algebra over an arbitrary field $K$ cannot be embedded into the automorphism group of free associative algebra induced by the natural abelianization.

**Theorem 3.1.9.** Let $K$ be an arbitrary field, $G = \text{Aut}_0(K[x_1, \ldots, x_n])$ and $n > 2$. Then $G$ cannot be isomorphic to any subgroup $H$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ induced by the natural abelianization. The same is true for $\text{NAut}(K[x_1, \ldots, x_n])$.

### 3.2 Varieties of automorphisms

#### 3.2.1 Elementary and tame automorphisms

Let $P$ be a polynomial that is independent of $x_i$ with $i$ fixed. An automorphism $x_i \mapsto x_i + P, x_j \mapsto x_j$ for $i \neq j$ is called elementary. The group generated by linear automorphisms and elementary ones for all possible $P$ is called the tame automorphism group (or subgroup) $\text{TAut}$ and elements of $\text{TAut}$ are tame automorphisms.

#### 3.2.2 Ind-schemes and Ind-groups

**Definition 3.2.1.** An Ind-variety $M$ is the direct limit of algebraic varieties $M = \varinjlim \{M_1 \subseteq M_2 \cdots\}$. An Ind-scheme is an Ind-variety which is a group such that the group inversion is a morphism $M_i \to M_{j(i)}$ of algebraic varieties, and the group multiplication induces a morphism from $M_i \times M_j$ to $M_{k(i,j)}$. A map $\varphi$ is a morphism of an Ind-variety $M$ to an Ind-variety $N$, if $\varphi(M_i) \subseteq N_{j(i)}$ and the restriction $\varphi$ to $M_i$ is a morphism for all $i$. Monomorphisms, epimorphisms and isomorphisms are defined similarly in a natural way.

**Example.** $M$ is the group of automorphisms of the affine space, and $M_i$ are the sets of all automorphisms in $M$ with degree $\leq j$.

There is an interesting

**Problem.** Investigate growth functions of Ind-varieties. For example, the dimension of varieties of polynomial automorphisms of degree $\leq n$.

Note that coincidence of growth functions of $\text{Aut}(W_n(\mathbb{C}))$ and $\text{Sympl}(\mathbb{C}^{2n})$ would imply the Kanel-Belov – Kontsevich conjecture [39].

**Definition 3.2.2.** The ideal $I$ generated by variables $x_i$ is called the augmentation ideal. For a fixed positive integer $N > 1$, the augmentation subgroup $H_N$ is the group of all automorphisms
\( \varphi \) such that \( \varphi(x_i) \equiv x_i \mod I^N \). The larger group \( \hat{H}_N \supset H_N \) is the group of automorphisms whose linear part is scalar, and \( \varphi(x_i) \equiv \lambda x_i \mod I^N \) (\( \lambda \) does not depend on \( i \)). We often say an arbitrary element of the group \( \hat{H}_N \) is an automorphism that is homothety modulo (the \( N \)-th power of) the augmentation ideal.

3.3 The Jacobian conjecture in any characteristic, Kanel-Belov – Kontsevich conjecture, and approximation

3.3.1 Approximation problems and Kanel-Belov – Kontsevich Conjecture

Let us give formulation of the Kanel-Belov – Kontsevich Conjecture:

\[
B - KK C_n : \text{Aut}(W_n) \simeq \text{Sympl}(\mathbb{C}^{2n}).
\]

A similar conjecture can be stated for endomorphisms

\[
B - KK C_n : \text{End}(W_n) \simeq \text{Sympl} \text{End}(\mathbb{C}^{2n}).
\]

If the Jacobian conjecture \( JC_{2n} \) is true, then the respective conjunctions over all \( n \) of the two conjectures are equivalent.

It is natural to approximate automorphisms by tame ones. There exists such an approximation up to terms of any order for polynomial automorphisms as well as Weyl algebra automorphisms, symplectomorphisms etc. However, the naive approach fails.

It is known that \( \text{Aut}(W_1) \equiv \text{Aut}_1(K[x,y]) \) where \( \text{Aut}_1 \) stands for the subgroup of automorphisms of Jacobian determinant one. However, considerations from [171] show that Lie algebra of the first group is the algebra of derivations of \( W_1 \) and thus possesses no identities apart from the ones of the free Lie algebra, another coincidence of the vector fields which diverge to zero, and has polynomial identities. These cannot be isomorphic [39, 40]. In other words, this group has two coordinate system non-smooth with respect to one another (but integral with respect to one another). One system is built from the coefficients of differential operators in a fixed basis of generators, while its counterpart is provided by the coefficients of polynomials, which are images of the basis \( \tilde{x}_i, \tilde{y}_i \).

In the paper [171] functionals on \( \mathfrak{m}/\mathfrak{m}^2 \) were considered in order to define the Lie algebra structure. In the spirit of that we have the following

**Conjecture.** The natural limit of \( \mathfrak{m}/\mathfrak{m}^2 \) is zero.

It means that the definition of the Lie algebra admits some sort of functoriality problem and it depends on the presentation of (reducible) Ind-scheme.

In his remarkable paper, Yu. Bodnarchuk [53] established Theorem 3.1.1 by using Shafarevich’s results for the tame automorphism subgroup and for the case when the Ind-scheme automorphism
is regular in the sense that it sends coordinate functions to coordinate functions. In this case the tame approximation works (as well as for the symplectic case), and the corresponding method is similar to ours. We present it here in order to make the text more self-contained, as well as for the purpose of tackling the noncommutative (that is, the free associative algebra) case. Note that in general, for regular functions, if the Shafarevich-style approximation were valid, then the Kanel-Belov – Kontsevich conjecture would follow directly, which is absurd.

In the sequel, we do not assume regularity in the sense of [53] but only assume that the restriction of a morphism on any subvariety is a morphism again. Note that morphisms of Ind-schemes \( \text{Aut}(W_n) \to \text{Sympl}(\mathbb{C}^{2n}) \) have this property, but are not regular in the sense of Bodnarchuk [53].

We use the idea of singularity which allows us to prove the augmentation subgroup structure preservation, so that the approximation works in this case.

Consider the isomorphism \( \text{Aut}(W_1) \cong \text{Aut}_1(K[x, y]) \). It has a strange property. Let us add a small parameter \( t \). Then an element arbitrary close to zero with respect to \( t^k \) does not go to zero arbitrarily, so it is impossible to make tame limit! There is a sequence of convergent product of elementary automorphisms, which is not convergent under this isomorphism. Exactly the same situation happens for \( W_n \). These effects cause problems in perturbative quantum field theory.

### 3.3.2 The Jacobian conjecture in any characteristic

Recall that the Jacobian conjecture in characteristic zero states that any polynomial endomorphism

\[
\varphi : K^n \to K^n
\]

with constant Jacobian is globally invertible.

A naive attempt to directly transfer this formulation to positive characteristic fails because of the counterexample \( x \mapsto x - x^p \) \( (p = \text{char} \ K) \), whose Jacobian is everywhere 1 but which is evidently not invertible. Approximation provides a way to formulate a suitable generalization of the Jacobian conjecture to any characteristic and put it in a framework of other questions.

**Definition 3.3.1.** An endomorphism \( \varphi \in \text{End}(K[x_1, \ldots, x_n]) \) is good if for any \( m \) there exist \( \psi_m \in \text{End}(K[x_1, \ldots, x_n]) \) and \( \phi_m \in \text{Aut}(K[x_1, \ldots, x_n]) \) such that

- \( \varphi = \psi_m \phi_m \)
- \( \psi_m(x_i) \equiv x_i \mod (x_1, \ldots, x_n)^m \).

An automorphism \( \varphi \in \text{Aut}(K[x_1, \ldots, x_n]) \) is nice if for any \( m \) there exist \( \psi_m \in \text{Aut}(K[x_1, \ldots, x_n]) \) and \( \phi_m \in \text{TAut}(K[x_1, \ldots, x_n]) \) such that

- \( \varphi = \psi_m \phi_m \)
- \( \psi_m(x_i) \equiv x_i \mod (x_1, \ldots, x_n)^m \), i.e. \( \psi_m \in H_m \).
Anick [8] has shown that if \( \text{char}(K) = 0 \), any automorphism is nice. However, this is unclear in positive characteristic.

**Question.** Is any automorphism over arbitrary field nice?

Ever good automorphism has Jacobian 1, and all such automorphisms are good - and even nice - when \( \text{char}(K) = 0 \). This observation allows for the following question to be considered a generalization of the Jacobian conjecture to positive characteristic.

The Jacobian conjecture in any characteristic: Is any good endomorphism over arbitrary field an automorphism?

Similar notions can be formulated for the free associative algebra. That justifies the following

**Question.** Is any automorphism of free associative algebra over arbitrary field nice?

**Question** (version of free associative positive characteristic case of JC). Is any good endomorphism of the free associative algebra over arbitrary field an automorphism?

### 3.3.3 Approximation for the automorphism group of affine spaces

Approximation is the most important tool utilized in this paper. In order to perform it, we have to prove that \( \varphi \in \text{Aut}_{\text{Ind}}(\text{Aut}_0(K[x_1, \ldots, x_n])) \) preserves the structure of the augmentation subgroup.

The proof method utilized in theorems below works for commutative associative and free associative case. It is a problem of considerable interest to develop similar statements for automorphisms of other associative algebras, such as the commutative Poisson algebra (for which the Aut functor returns the group of polynomial symplectomorphisms); however, the situation there is somewhat more difficult.

Suppose that \( \varphi \) is an Ind-automorphism (in either commutative of free associative case) such that it stabilizes point-wise the set \( T \) of automorphisms corresponding to the standard diagonal action of the maximal torus (in the next section we will see that this implies that \( \varphi \) also stabilizes every tame automorphism). The following two continuity theorems, for the commutative and the free associative cases, respectively, constitute the foundation of the approximation technique.

**Theorem 3.3.2.** Let \( \varphi \in \text{Aut}_{\text{Ind}}(\text{Aut}_0(K[x_1, \ldots, x_n])) \) and let \( H_N \subseteq \text{Aut}_0(K[x_1, \ldots, x_n]) \) be the subgroup of automorphisms which are identity modulo the ideal \( (x_1, \ldots, x_n)^N \) \((N > 1)\). Then \( \varphi(H_N) \subseteq H_N \).

**Theorem 3.3.3.** Let \( \varphi \in \text{Aut}_{\text{Ind}}(\text{Aut}_0(K\langle x_1, \ldots, x_n \rangle)) \) and let \( H_N \) be again the subgroup of automorphisms which are identity modulo the ideal \( (x_1, \ldots, x_n)^N \). Then \( \varphi(H_N) \subseteq H_N \).
Chapter 3. Automorphisms of Ind-schemes, augmentation topology, and approximation

Corollary 3.3.4. In both commutative and free associative cases under the assumptions above one has \( \varphi = \text{Id} \).

**Proof.** Every automorphism can be approximated via the tame ones, i.e. for any \( \psi \) and any \( N \) there exists a tame automorphism \( \psi_N' \) such that \( \psi \psi_N'^{-1} \in H_N \).

The main point therefore is why \( \varphi(H_N) \subseteq H_N \) whenever \( \varphi \) is and Ind-automorphism.

**Proof of Theorem 3.3.2.**

The method of proof is based upon the following useful fact from algebraic geometry:

**Lemma 3.3.5.** Let \( \varphi : X \to Y \) be a morphism of affine varieties, and let \( A(t) \subset X \) be a curve (or rather, a one-parameter family of points) in \( X \). Suppose that \( A(t) \) does not tend to infinity as \( t \to 0 \). Then the image \( \varphi A(t) \) under \( \varphi \) also does not tend to infinity as \( t \to 0 \).

The proof is straightforward and is left to the reader.

We now put the above fact to use. For \( t > 0 \) let \( \hat{A}(t) : \mathbb{A}_K^n \to \mathbb{A}_K^n \) be a one-parameter family of invertible linear transformations of the affine space preserving the origin. To that corresponds a curve \( A(t) \subset \text{Aut}_0(K[x_1, \ldots, x_n]) \) of polynomial automorphisms whose points are linear substitutions. Suppose that, as \( t \) tends to zero, the \( i \)-th eigenvalue of \( A(t) \) also tends to zero as \( t \to 0 \). Such a family will always exist.

Suppose now that the degrees \( \{k_i, \ i = 1, \ldots n\} \) of singularity of eigenvalues at zero are such that for every pair \( (i, j) \), if \( k_i \neq k_j \), then there exists a positive integer \( m \) such that

either \( k_im \leq k_j \) or \( k_jm \leq k_i \).

The largest such \( m \) we will call the order of \( A(t) \) at \( t = 0 \). As \( k_i \) are all set to be positive integer, the order equals the integer part of \( \frac{k_{\max}}{k_{\min}} \).

Let \( M \in \text{Aut}_0(K[x_1, \ldots, x_n]) \) be a polynomial automorphism.

**Lemma 3.3.6.** The curve \( A(t)MA(t)^{-1} \) has no singularity at zero for any diagonalizable \( A(t) \) of order \( \leq N \) if and only if \( M \in \hat{H}_N \), where \( \hat{H}_N \) is the subgroup of automorphisms which are homothety modulo the augmentation ideal.

**Proof.** The ‘If’ part is elementary, for if \( M \in \hat{H}_N \), the action of \( A(t)MA(t)^{-1} \) upon any
generator $x_i$ (with $i$ fixed) is given by

$$A(t)MA(t)^{-1}(x_i) = \lambda x_i + t^{-k_i} \sum_{l_1 + \cdots + l_n = N} a_{l_1 \ldots l_n} t^{k_1 l_1 + \cdots + k_n l_n} x_1^{l_1} \cdots x_n^{l_n} + S_i(t, x_1, \ldots, x_n),$$

where $\lambda$ is the homothety ratio of (the linear part of) $M$ and $S_i$ is polynomial in $x_1, \ldots, x_n$ of total degree greater than $N$. Now, for any choice of $l_1, \ldots, l_n$ in the sum, the expression

$$k_1 l_1 + \cdots + k_n l_n - k_i \geq k_{\min} \sum l_j - k_i = k_{\min} N - k_i \geq 0$$

for every $i$, so whenever $t$ goes to zero, the coefficient will not blow up to infinity. Obviously the same argument applies to higher-degree monomials within $S_i$.

The other direction is slightly less elementary; assuming that $M \notin \hat{H}_N$, we need to show that there is a curve $A(t)$ such that conjugation of $M$ by it produces a singularity at zero. We distinguish between two cases.

**Case 1.** The linear part $\bar{M}$ of $M$ is not a scalar matrix. Then – after a suitable basis change (see the footnote) - it is not a diagonal matrix and has a non-zero entry in the position $(i, j)$. Consider a diagonal matrix $A(t) = D(t)$ such that on all positions on the main diagonal except $j$-th it has $t^{k_i}$ and on $j$-th position it has $t^{k_j}$. Then $D(t)\bar{M}D^{-1}(t)$ has $(i, j)$ entry with the coefficient $t^{k_i - k_j}$ and if $k_j > k_i$, it has a singularity at $t = 0$.

Let also $k_i < 2k_j$. Then the non-linear part of $M$ does not produce singularities and cannot compensate the singularity of the linear part.

**Case 2.** The linear part $\bar{M}$ of $M$ is a scalar matrix. Then conjugation cannot produce singularities in the linear part and we as before are interested in the smallest non-linear term. Let $M \in H_N \setminus H_{N+1}$. Performing a basis change if necessary, we may assume that

$$\varphi(x_1) = \lambda x_1 + \delta x_2^N + S,$$

where $S$ is a sum of monomials of degree $\geq N$ with coefficients in $K$.

Let $A(t) = D(t)$ be a diagonal matrix of the form $(t^{k_1}, t^{k_2}, t^{k_3}, \ldots, t^{k_l})$ and let $(N + 1) \cdot k_2 > k_1 > N \cdot k_2$. Then in $A^{-1}MA$ the term $\delta x_2^N$ will be transformed into $\delta x_2^N t^{Nk_2-k_1}$, and all other terms are multiplied by $t^{k_2 + sk_1 - k_1}$ with $(l, s) \neq (1, 0)$ and $l, s > 0$. In this case $lk_2 + sk_1 - k_1 > 0$ and we are done with the proof of Lemma 3.3.6.

The next lemma is proved by direct computation. Recall that for $m > 1$, the group $G_m$ is defined as the group of all tame automorphisms preserving the $m$-th power of the augmentation ideal.

**Lemma 3.3.7.**

$^2$Without loss of generality we may assume that the coordinate functions $x_i$ correspond to the principal axes of $\hat{A}(t)$.
a) \([G_m, G_m] \subset H_m, m > 2\). There exist elements
\(\varphi \in H_{m+k-1}\backslash H_{m+k}, \ \psi_1 \in G_k, \ \psi_2 \in G_m\), such that \(\varphi = [\psi_1, \psi_2]\).

b) \([H_m, H_k] \subset H_{m+k-1}\).

c) Let \(\varphi \in G_m \backslash H_m, \ \psi \in H_k \backslash H_{k+1}, k > m\). Then \([\varphi, \psi] \in H_k \backslash H_{k+1}\).

**Proof.** a) Consider elementary automorphisms

\[\psi_1 : x_1 \mapsto x_1 + x_2^k, \ x_2 \mapsto x_2, \ x_i \mapsto x_i, \ i > 2;\]

\[\psi_2 : x_1 \mapsto x_1, \ x_2 \mapsto x_2 + x_1^m, \ x_i \mapsto x_i, \ i > 2.\]

Set \(\varphi = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2\).

Then

\[\varphi : x_1 \mapsto x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^m)^k,\]

\[x_2 \mapsto x_2 - (x_1 - x_2^k)^m + (x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^m)^m, \ x_i \mapsto x_i, \ i > 2.\]

It is easy to see that if either \(k\) or \(m\) is relatively prime with \(\text{char}(K)\), then not all terms of degree \(k + m - 1\) vanish. Thus \(\varphi \in H_{m+k-1}\backslash H_{m+k}\).

Now suppose that \(\text{char}(K) \nmid m\), then obviously \(m - 1\) is relatively prime with \(\text{char}(K)\). Consider the mappings

\[\psi_1 : x_1 \mapsto x_1 + x_2^k, \ x_2 \mapsto x_2, \ x_i \mapsto x_i, \ i > 2;\]

\[\psi_2 : x_1 \mapsto x_1, \ x_2 \mapsto x_2 + x_1^{m-1}x_3, \ x_i \mapsto x_i, \ i > 2.\]

Set again \(\varphi' = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2\). Then \(\varphi'\) acts as

\[x_1 \mapsto x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^{m-1}x_3)^k = x_1 - k(x_1 - x_2^k)^{m-1}x_2^{k-1}x_3 + S,\]

\[x_2 \mapsto x_2 - (x_1 - x_2^k)^{m-1}x_3 + (x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^m)x_3)^{m-1}x_3,\]

\[x_i \mapsto x_i, \ i > 2;\]

here \(S\) stands for a sum of terms of degree \(\geq m + k\). Again we see that \(\varphi \in H_{m+k-1}\backslash H_{m+k}\).

b) Let

\[\psi_1 : x_i \mapsto x_i + f_i; \ \psi_2 : x_i \mapsto x_i + g_i,\]

for \(i = 1, \ldots, n\); here \(f_i\) and \(g_i\) do not have monomials of degree less than or equal to \(m\) and \(k\), respectively. Then, modulo terms of degree \(\geq m + k\), we have \(\psi_1\psi_2 : x_i \mapsto x_i + f_i + g_i + \frac{\partial f_i}{\partial x_j}g_j\), so that modulo terms of degree \(\geq m + k - 1\) we get \(\psi_1\psi_2 : x_i \mapsto x_i + f_i + g_i\) and \(\psi_2\psi_1 : x_i \mapsto x_i + f_i + g_i\). Therefore \([\psi_1, \psi_2] \in H_{m+k-1}\).
c) If $\varphi(I^m) \subseteq I^m$ and

$$\psi : (x_1, \ldots, x_n) \mapsto (x_1 + g_1, \ldots, x_n + g_n)$$

is such that for some $i_0$ the polynomial $g_{i_0}$ contains a monomial of total degree $k$ (and all $g_i$ do not contain monomials of total degree less than $k$), then, by evaluating the composition of automorphisms directly, one sees that the commutator is given by

$$[\varphi, \psi] : (x_1, \ldots, x_n) \mapsto (x_1 + g_1 + S_1, \ldots, x_n + g_n + S_n)$$

with $S_i$ containing no monomials of total degree $< k + 1$. Then the image of $x_{i_0}$ is $x_{i_0}$ modulo polynomial of height $k$.

**Corollary 3.3.8.** Let $\Psi \in \text{Aut}_{\text{Ind}}(\text{NAut}(K[x_1, \ldots, x_n]))$. Then $\Psi(G_n) = G_n$, $\Psi(H_n) = H_n$.

Corollary 3.3.8 together with Proposition 3.4.3 of the next section imply Theorem 3.3.2, for every nice automorphism, by definition, can be approximated by tame ones. Note that in characteristic zero every automorphism is nice (Anick’s theorem).

The proof of Corollary 3.3.8 proceeds as follows (here for simplicity we put $\text{char } K = 0$, so that $\text{NAut}$ coincides with $\text{Aut}$ thanks to Anick’s theorem).

Let $\varphi$ be an Ind-automorphism which stabilizes point-wise the standard action of the maximal torus.

1. We first note (and give a proof further along the text) that in this case $\varphi$ also stabilizes point-wise the set of all tame automorphisms.

2. It follows from the singularity trick that:

$$\varphi(\hat{H}_N) \subseteq \hat{H}_N$$

(the reverse inclusion is also true due to the invertibility of $\varphi$). Namely, if $f = \varphi(g)$ is an automorphism in $\varphi(\hat{H}_N)$ but not in $\hat{H}_N$ then there is a curve $A(t)$ of order $\leq N$ such that

$$A(t) \circ f \circ A(t)^{-1}$$

admits a singularity at $t = 0$. But then

$$\varphi^{-1}(A(t) \circ f \circ A(t)^{-1}) = A(t) \circ \varphi^{-1}(f) \circ A(t)^{-1}$$

(this is thanks to the preservation of tame automorphisms) also admits a singularity at zero, which is a contradiction.
It is a fairly easy exercise to show that

$$\varphi(\hat{H}_{N+1}\backslash \hat{H}_N) = \hat{H}_{N+1}\backslash \hat{H}_N$$

for all $N$.

3. We now demonstrate that $\varphi(\hat{H}_N\backslash H_N) = \hat{H}_N\backslash H_N$ which together with the preceding results will allow us to descend from homothety to identity modulo $N$.

A. Let $N > 2$ first. Suppose $g \in \hat{H}_N\backslash H_N$. We take a tame automorphism $f$ which is given by the sum of the identity map and a non-zero term of height two. Consider the automorphism

$$g_f = f \circ g \circ f^{-1}.$$ 

It is easy to see that $g_f \in \hat{H}_2$: as the linear part of $g$ is given by a scalar matrix not equal to the identity matrix, the degree two component of $g \circ f^{-1}$ is proportional to the homothety ratio $\lambda \neq 1$, therefore the composition with $f$ cannot compensate it.

On the other hand, if $\varphi(g) \in H_N$, i.e. the linear part of $\varphi(g)$ is the identity map, then the degree two component of

$$\varphi(g_f) = f \circ \varphi(g) \circ f^{-1}$$

(this expression is again due to point-wise preservation of tame automorphisms) is equal to zero, which contradicts $\varphi(\hat{H}_{N+1}\backslash \hat{H}_N) = \hat{H}_{N+1}\backslash \hat{H}_N$.

B. Let $N = 2$. Suppose that $g \in \hat{H}_2\backslash H_2$ is a non-trivial homothety plus a term of height two. The automorphism $g$ can be approximated by tame automorphisms, in particular there exists a tame automorphism $\xi$ such that

$$\xi \circ g$$

is in $H_3$. From Case A it follows then that

$$\varphi(\xi \circ g) = \xi \circ \varphi(g)$$

is also in $H_3$. But then, since the linear part of $g$ is given by a non-trivial homothety, which means that $\xi$ scales it back to the identity matrix in order to approximate $g$ up to terms of height three, then the left action by $\xi^{-1}$ reverses the scaling, so that the linear part of

$$\xi^{-1} \circ \varphi(g) = \varphi(g)$$

is given by a non-trivial homothety, which implies $\varphi(g) \in \hat{H}_2\backslash H_2$.

4. Finally, combining all of the results, we get $\varphi(H_N) = H_N$, $N > 1$ as desired.
3.3.4 Lifting of automorphism groups

Lifting of automorphisms from $\text{Aut}(K[x_1, \ldots, x_n])$ to $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$

**Definition 3.3.9.** In the sequel, we call an action of the $n$-dimensional torus $T^n$ on $K\langle x_1, \ldots, x_n \rangle$ (the number of generators coincides with the dimension of the torus) **linearizable** if it is conjugate to the standard diagonal action given by

$$(\lambda_1, \ldots, \lambda_n) : (x_1, \ldots, x_n) \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n).$$

The following result is a direct free associative analogue of a well-known theorem of Bialynicki-Birula [49, 50]. We will make frequent reference of the classical (commutative) case as well, which appears as Theorem 3.4.1 in the text.

**Theorem 3.3.10.** Any effective action of the $n$-torus on $\mathbb{K}(x_1, \ldots, x_n)$ is linearizable.

The proof is somewhat similar to that of Theorem 3.4.1, with a few modifications. We provide it in Chapter 5.

As a corollary of the above theorem, we get

**Proposition 3.3.11.** Let $T^n$ denote the standard torus action on $K[x_1, \ldots, x_n]$. Let $\hat{T}^n$ denote its lifting to an action on the free associative algebra $K\langle x_1, \ldots, x_n \rangle$. Then $\hat{T}^n$ is also given by the standard torus action.

**Proof.** Consider the roots $\hat{x}_i$ of this action. They are liftings of the coordinates $x_i$. We have to prove that they generate the whole associative algebra.

Due to the reducibility of this action, all elements are product of eigenvalues of this action. Hence it is enough to prove that eigenvalues of this action can be presented as a linear combination of this action. This can be done along the lines of Bialynicki-Birula [49]. Note that all propositions of the previous section hold for the free associative algebra. Proof of Theorem 3.3.3 is similar. Hence we have the following

**Theorem 3.3.12.** Any Ind-scheme automorphism $\varphi$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ for $n \geq 3$ is inner, i.e. is a conjugation by some automorphism.

We therefore see that the group lifting (in the sense of isomorphism induced by the natural abelianization) implies the analogue of Theorem 3.3.2.

This also implies that any automorphism group lifting, if exists, satisfies the approximation properties.

**Proposition 3.3.13.** Suppose

$$\Psi : \text{Aut}(K[x_1, \ldots, x_n]) \to \text{Aut}(K\langle z_1, \ldots, z_n \rangle)$$
is a group homomorphism such that its composition with the natural map \( \text{Aut}(K\langle z_1, \ldots, z_n \rangle) \to \text{Aut}(K[x_1, \ldots, x_n]) \) (induced by the projection \( K\langle z_1, \ldots, z_n \rangle \to K[x_1, \ldots, x_n] \)) is the identity map. Then

1. After a coordinate change \( \Psi \) provides a correspondence between the standard torus actions \( x_i \mapsto \lambda_i x_i \) and \( z_i \mapsto \lambda_i z_i \).

2. Images of elementary automorphisms

\[
x_j \mapsto x_j, \ j \neq i, \ x_i \mapsto x_i + f(x_1, \ldots, \hat{x}_i, \ldots, x_n)
\]

are elementary automorphisms of the form

\[
z_j \mapsto z_j, \ j \neq i, \ z_i \mapsto z_i + f(z_1, \ldots, \hat{z}_i, \ldots, z_n).
\]

(Hence image of tame automorphism is tame automorphism).

3. \( \psi(H_n) = G_n \). Hence \( \psi \) induces a map between the completion of the groups of \( \text{Aut}(K[x_1, \ldots, x_n]) \) and \( \text{Aut}(K\langle z_1, \ldots, z_n \rangle) \) with respect to the augmentation subgroup structure.

**Proof of Theorem 3.1.9**

Any automorphism (including wild automorphisms such as the Nagata example) can be approximated by a product of elementary automorphisms with respect to augmentation topology. In the case of the Nagata automorphism corresponding to

\[
\text{Aut}(K\langle x_1, \ldots, x_n \rangle),
\]

all such elementary automorphisms fix all coordinates except \( x_1 \) and \( x_2 \). Because of (2) and (3) of Proposition 3.3.13, the lifted automorphism would be an automorphism induced by an automorphism of \( K\langle x_1, x_2, x_3 \rangle \) fixing \( x_3 \). However, it is impossible to lift the Nagata automorphism to such an automorphism due to the main result of [36]. Therefore, Theorem 3.1.9 is proved.

### 3.4 Automorphisms of the polynomial algebra and the approach of Bodnarchuk–Rips

Let \( \Psi \in \text{Aut}(\text{Aut}(K[x_1, \ldots, x_n])) \) (resp. \( \text{Aut}(\text{TAut}(K[x_1, \ldots, x_n])), \text{Aut}(\text{TAut}_0(K[x_1, \ldots, x_n])), \text{Aut}(\text{Aut}_0(K[x_1, \ldots, x_n])) \)).
3.4.1 Reduction to the case when $\Psi$ is identical on $\text{SL}_n$

We follow [128] and [53] using the classical theorem of Bialynicki-Birula [49, 50]:

**Theorem 3.4.1 (Bialynicki-Birula).** Any effective action of torus $T^n$ on $\mathbb{C}^n$ is linearizable (recall the definition 3.3.9).

**Remark.** An effective action of $T^{n-1}$ on $\mathbb{C}^n$ is linearizable [49, 50]. There is a conjecture whether any action of $T^{n-2}$ on $\mathbb{C}^n$ is linearizable, established for $n = 3$. For codimension $> 2$, there are positive-characteristic counterexamples [16].

**Remark.** Kraft and Stampfli [128] proved (by considering periodic elements in $T^n$) that an effective action $\varphi \in \text{Aut(IndAut)}$ preserving the Ind-group structure, consider now the standard action $x_i \mapsto \lambda_i x_i$ of the $n$-dimensional torus $T^n \subset \text{Aut}(\mathbb{C}[x_1, \ldots, x_n])$ on the affine space $\mathbb{C}^n$. Let $H$ be the image of $T^n$ under $\varphi$. Then by Theorem 3.4.1 $H$ is conjugate to the standard torus $T^n$ via some automorphism $\psi$. Composing $\varphi$ with this conjugation, we come to the case when $\varphi$ is the identity on the maximal torus. Then we have the following

**Corollary 3.4.2.** Without loss of generality, it is enough to prove Theorem 3.1.1 for the case when $\varphi|_T = \text{Id}$.  

Now we are in the situation when $\varphi$ preserves all linear mappings $x_i \mapsto \lambda_i x_i$. We have to prove that it is the identity.

**Proposition 3.4.3** (E. Rips, private communication). Let $n > 2$ and suppose $\varphi$ preserves the standard torus action on the commutative polynomial algebra. Then $\varphi$ preserves all elementary transformations.

**Corollary 3.4.4.** Let $\varphi$ satisfy the conditions of Proposition 3.4.3. Then $\varphi$ preserves all tame automorphisms.

**Proof of Proposition 3.4.3.** We state a few elementary lemmas.

**Lemma 3.4.5.** Consider the diagonal action $T^1 \subset T^n$ given by automorphisms: $\alpha : x_i \mapsto \alpha_i x_i$, $\beta : x_i \mapsto \beta_i x_i$. Let $\psi : x_i \mapsto \sum_{i,J} a_{iJ} x^J$, $i = 1, \ldots, n$, where $J = (j_1, \ldots, j_n)$ is the multi-index, $x^J = x^{j_1} \cdots x^{j_n}$. Then

$$\alpha \circ \psi \circ \beta : x_i \mapsto \sum_{i,J} \alpha_i a_{iJ} x^J \beta^J,$$

In particular,

$$\alpha \circ \psi \circ \alpha^{-1} : x_i \mapsto \sum_{i,J} \alpha_i a_{iJ} x^J \alpha^{-J}.$$
Applying Lemma 3.4.5 and comparing the coefficients we get the following

**Lemma 3.4.6.** Consider the diagonal $T^1$ action: $x_i \mapsto \lambda x_i$. Then the set of automorphisms commuting with this action is exactly the set of linear automorphisms.

Similarly (using Lemma 3.4.5) we obtain Lemmas 3.4.7, 3.4.9, 3.4.10:

**Lemma 3.4.7.**

a) Consider the following $T^2$ action:

\[ x_1 \mapsto \lambda \delta x_1, \ x_2 \mapsto \lambda x_2, \ x_3 \mapsto \delta x_3, \ x_i \mapsto \lambda x_i, \ i > 3. \]

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

\[ x_1 \mapsto x_1 + \beta x_2 x_3, \ x_i \mapsto \varepsilon_i x_i, \ i > 1, \ (\beta, \varepsilon_i \in K). \]

b) Consider the following $T^{n-1}$ action:

\[ x_1 \mapsto \lambda^I x_1, \ x_j \mapsto \lambda_j x_j, \ j > 1 \ (\lambda^I = \lambda_2^{i_2} \cdots \lambda_n^{i_n}). \]

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

\[ x_1 \mapsto x_1 + \beta \prod_{j=2}^{n} x_j^i, \ (\beta \in K). \]

**Remark.** A similar statement for the free associative case is true, but one has to consider the set $\hat{S}$ of automorphisms $x_1 \mapsto x_1 + h, \ x_i \mapsto \varepsilon_i x_i, \ i > 1, \ (\varepsilon \in K,$ and the polynomial $h \in K\langle x_2, \ldots, x_n \rangle$ has total degree $J$ - in the free associative case it is not just monomial anymore).

**Corollary 3.4.8.** Let $\varphi \in \text{Aut}(TAut(K[x_1, \ldots, x_n]))$ stabilizing all elements from $T$. Then $\varphi(S) = S$.

**Lemma 3.4.9.** Consider the following $T^1$ action:

\[ x_1 \mapsto \lambda^2 x_1, \ x_i \mapsto \lambda x_i, \ i > 1. \]

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

\[ x_1 \mapsto x_1 + \beta x_2^2, \ x_i \mapsto \lambda_i x_i, \ i > 2, \ (\beta, \lambda_i \in K). \]

**Lemma 3.4.10.** Consider the set $S$ defined in the previous lemma. Then $[S, S] = \{uvu^{-1}v^{-1}\}$ consists of the following automorphisms

\[ x_1 \mapsto x_1 + \beta x_2 x_3, \ x_2 \mapsto x_2, \ x_3 \mapsto x_3, \ (\beta \in K). \]
Lemma 3.4.11. Let \( n \geq 3 \). Consider the following set of automorphisms

\[
\psi_i : x_i \mapsto x_i + \beta_i x_{i+1} x_{i+2}, \quad \beta_i \neq 0, \quad x_k \mapsto x_k, \quad k \neq i
\]

for \( i = 1, \ldots, n-1 \). (Numeration is cyclic, so for example \( x_{n+1} = x_1 \)). Let \( \beta_i \neq 0 \) for all \( i \). Then all of \( \psi_i \) can be simultaneously conjugated by a torus action to

\[
\psi_i' : x_i \mapsto x_i + x_{i+1} x_{i+2}, \quad x_k \mapsto x_k, \quad k \neq i
\]

for \( i = 1, \ldots, n \) in a unique way.

Proof. Let \( \alpha : x_i \mapsto \alpha_i x_i \). Then by Lemma 3.4.5 we obtain

\[
\alpha \circ \psi_i \circ \alpha^{-1} : x_i \mapsto x_i + \beta_i x_{i+1} x_{i+2} \alpha_i^{-1} \alpha_{i+1}^{-1} \alpha_{i+2} \alpha_i
\]

and

\[
\alpha \circ \psi_i \circ \alpha^{-1} : x_k \mapsto x_k
\]

for \( k \neq i \).

Comparing the coefficients of the quadratic terms, we see that it is sufficient to solve the system:

\[
\beta_i \alpha_i^{-1} \alpha_{i+2}^{-1} \alpha_i = 1, \quad i = 1, \ldots, n-1.
\]

As \( \beta_i \neq 0 \) for all \( i \), this system has a unique solution.

Remark. In the free associative algebra case, instead of \( \beta x_2 x_3 \) one has to consider \( \beta x_2 x_3 + \gamma x_3 x_2 \).

3.4.2 The lemma of Rips

Lemma 3.4.12 (E. Rips). Let \( \text{char}(K) \neq 2, \ |K| = \infty \). Linear transformations and \( \psi_i' \) defined in Lemma 3.4.11 generate the whole tame automorphism group of \( K[x_1, \ldots, x_n] \).

Proposition 3.4.3 follows from Lemmas 3.4.6, 3.4.7, 3.4.9, 3.4.10, 3.4.11, 3.4.12. Note that we have proved an analogue of Theorem 3.1.1 for tame automorphisms.

Proof of Lemma 3.4.12. Let \( G \) be the group generated by elementary transformations as in Lemma 3.4.11. We have to prove that is isomorphic to the tame automorphism subgroup fixing the augmentation ideal. We are going to need some preliminaries.

Lemma 3.4.13. Linear transformations of \( K^3 \) and

\[
\psi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy
\]
generate all mappings of the form
\[ \phi^b_m(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + bx^m, \ b \in K. \]

**Proof of Lemma 3.4.13.** We proceed by induction. Suppose we have an automorphism
\[ \phi^b_{m-1}(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + bx^{m-1}. \]
Conjugating by the linear transformation \( (z \mapsto y, y \mapsto z, x \mapsto x) \), we obtain the automorphism
\[ \phi^b_{m-1}(x, z, y) : x \mapsto x, \ y \mapsto y + bx^{m-1}, \ z \mapsto z. \]
Composing this on the right by \( \psi \), we get the automorphism
\[ \varphi(x, y, z) : x \mapsto x, \ y \mapsto y + bx^{m-1}, \ z \mapsto z + yx + x^m. \]
Note that
\[ \phi_{m-1}^{-1}(x, y, z) \circ \varphi(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy + bx^m. \]
Now we see that
\[ \psi^{-1}\phi_{m-1}^{-1}(x, y, z) \circ \varphi(x, y, z) = \phi^b_m \]
and the lemma is proved.

**Corollary 3.4.14.** Let \( \text{char}(K) \nmid n \) (in particular, \( \text{char}(K) \neq 0 \)) and \( |K| = \infty \). Then \( G \) contains all the transformations
\[ z \mapsto z + bx^ky^l, \ y \mapsto y, \ x \mapsto x \]
such that \( k + l = n \).

**Proof.** For any invertible linear transformation
\[ \varphi : x \mapsto a_{11}x + a_{12}y, \ y \mapsto a_{21}x + a_{22}y, \ z \mapsto z; a_{ij} \in K \]
we have
\[ \varphi^{-1}\phi^b_m \varphi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + b(a_{11}x + a_{12}y)^m. \]
Note that sums of such expressions contain all the terms of the form \( bx^ky^l \). The corollary is proved.

### 3.4.3 Generators of the tame automorphism group

**Theorem 3.4.15.** If \( \text{char}(K) \neq 2 \) and \( |K| = \infty \), then linear transformations and
\[ \psi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy \]
generate all mappings of the form

\[ \alpha_m^b(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + by^m, \ b \in K. \]

**Proof of theorem 3.4.15.** Observe that

\[ \alpha = \beta \circ \phi_m^b(x, z, y) : x \mapsto x + by^m, \ y \mapsto y + x + by^m, \ z \mapsto z, \]

where \( \beta : x \mapsto x, \ y \mapsto x + y, \ z \mapsto z \). Then

\[ \gamma = \alpha^{-1}\psi\alpha : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy + 2bxy^m + by^{2m}. \]

Composing with \( \psi^{-1} \) and \( \phi_{2m}^{2b} \) we get the desired

\[ \alpha_{2m}^{2b}(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + 2by^m, \ b \in K. \]

**Corollary 3.4.16.** Let \( \text{char}(K) \nmid n \) and \( |K| = \infty \). Then \( G \) contains all transformations of the form

\[ z \mapsto z + bx^ky^l, \ y \mapsto y, \ x \mapsto x \]

such that \( k = n + 1 \).

The **proof** is similar to the proof of Corollary 3.4.14. Note that either \( n \) or \( n + 1 \) is not a multiple of \( \text{char}(K) \) so we have

**Lemma 3.4.17.** If \( \text{char}(K) \neq 2 \) then linear transformations and

\[ \psi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy \]

generate all mappings of the form

\[ \alpha_P : x \mapsto x, \ y \mapsto y, \ z \mapsto z + P(x, y), \ P(x, y) \in K[x, y]. \]

We have proved Lemma 3.4.12 for the three variable case. In order to treat the case \( n \geq 4 \) we need one more lemma.

**Lemma 3.4.18.** Let \( M(\vec{x}) = a \prod x_i^{k_i}, \ a \in K, \ |K| = \infty, \ \text{char}(K) \nmid k_i \) for at least one of \( k_i \)’s. Consider the linear transformations denoted by

\[ f : x_i \mapsto y_i = \sum a_{ij}x_j, \ \det(a_{ij}) \neq 0 \]

and monomials \( M_f = M(\vec{y}) \). Then the linear span of \( M_f \) for different \( f \)’s contains all homogenous polynomials of degree \( k = \sum k_i \) in \( K[x_1, \ldots, x_n] \).
Proof. It is a direct consequence of the following fact. Let \( S \) be a homogenous subspace of \( K[x_1, \ldots, x_n] \) invariant with respect to \( GL_n \) of degree \( m \). Then \( S = S^p_{m/p} ; p = \text{char}(K) \), \( S_i \) is the space of all polynomials of degree \( l \).

Lemma 3.4.12 follows from Lemma 3.4.18 in a similar way as in the proofs of Corollaries 3.4.14 and 3.4.16.

### 3.4.4 Aut(TAut) for general case

Now we consider the case when \( \text{char}(K) \) is arbitrary, i.e. the remaining case \( \text{char}(K) = 2 \). Still \( |K| = \infty \). Although we are unable to prove the analogue of Proposition 3.4.3, we can still play on the relations.

Let \( M = a \prod_{i=1}^{n-1} x_i^{k_i} \) be a monomial, \( a \in K \). For polynomial \( P(x, y) \in K[x, y] \) we define the elementary automorphism

\[
\psi_P : x_i \mapsto x_i, \quad i = 1, \ldots, n-1; \quad x_n \mapsto x_n + P(x_1, \ldots, x_{n-1}).
\]

We have \( P = \sum M_j \) and \( \psi_P \) naturally decomposes as a product of commuting \( \psi_{M_j} \). Let \( \Psi \in \text{Aut}(\text{TAut}(K[x, y, z])) \) stabilizing linear mappings and \( \phi \) (Automorphism \( \phi \) defined in Lemma 3.4.13). Then according to the corollary 3.4.8 \( \Psi(\psi_P) = \prod \Psi(\psi_{M_j}) \). If \( M = ax^n \) then due to Lemma 3.4.13

\[
\Psi(\psi_M) = \psi_M.
\]

We have to prove the same for other type of monomials:

**Lemma 3.4.19.** Let \( M \) be a monomial. Then

\[
\Psi(\psi_M) = \psi_M.
\]

**Proof.** Let \( M = a \prod_{i=1}^{n-1} x_i^{k_i} \). Consider the automorphism

\[
\alpha : x_i \mapsto x_i + x_1, \quad i = 2, \ldots, n-1; \quad x_1 \mapsto x_1, \quad x_n \mapsto x_n.
\]

Then

\[
\alpha^{-1} \psi_M \alpha = \psi_{\prod_{i=2}^{n-1} (x_i + x_1)^{k_i}} = \psi_Q \psi_{\prod_{i=2}^{n-1} x_i^{k_i}}.
\]

Here the polynomial

\[
Q = x_1^{k_1} \left( \prod_{i=2}^{n-1} (x_i + x_1)^{k_i} - ax \sum_{i=2}^{n-1} k_i \right).
\]
It has the following form

\[ Q = \sum_{i=2}^{n-1} N_i, \]

where \( N_i \) are monomials such that none of them is proportional to a power of \( x_1 \).

According to Corollary 3.4.8, \( \Psi(\psi_M) = \psi_{bM} \) for some \( b \in K \). We need only to prove that \( b = 1 \). Suppose the contrary, \( b \neq 1 \). Then

\[
\Psi(\alpha^{-1} \psi_M \alpha) = \left( \prod_{[N_i,x_1] \neq 0} \Psi(\psi_{N_i}) \right) \circ \Psi(\psi_{\alpha x_1^{n-1} k_i}) = \\
\left( \prod_{[N_i,x_1] \neq 0} \psi_{b_i N_i} \right) \circ \psi_{\alpha x_1^{n-1} k_i}
\]

for some \( b_i \in K \).

On the other hand

\[
\Psi(\alpha^{-1} \psi_M \alpha) = \alpha^{-1} \Psi(\psi_M) \alpha = \alpha^{-1} \psi_{bM} \alpha = \left( \prod_{[N_i,x_1] \neq 0} \psi_{b_i N_i} \right) \circ \psi_{\alpha x_1^{n-1} k_i}
\]

Comparing the factors \( \psi_{\alpha x_1^{n-1} k_i} \) and \( \psi_{\alpha x_1^{n-1} k_i} \) in the last two products we get \( b = 1 \). Lemma 3.4.19 and hence Proposition 3.4.3 are proved.

### 3.5 The approach of Bodnarchuk–Rips to automorphisms of \( \text{TAut}(K\langle x_1, \ldots, x_n \rangle) \) \((n > 2)\)

Now consider the free associative case. We treat the case \( n > 3 \) on group-theoretic level and the case \( n = 3 \) on Ind-scheme level. Note that if \( n = 2 \) then \( \text{Aut}_0(K[x,y]) = \text{TAut}_0(K[x,y]) \simeq \text{TAut}_0(K\langle x,y \rangle) = \text{Aut}_0(K\langle x,y \rangle) \) and description of automorphism group of such objects is known due to J. Déserti.

#### 3.5.1 The automorphisms of the tame automorphism group of \( K\langle x_1, \ldots, x_n \rangle \), \( n \geq 4 \)

**Proposition 3.5.1** (E. Rips, private communication). *Let \( n > 3 \) and let \( \varphi \) preserve the standard torus action on the free associative algebra \( K\langle x_1, \ldots, x_n \rangle \). Then \( \varphi \) preserves all elementary transformations.*

**Corollary 3.5.2.** *Let \( \varphi \) satisfy the conditions of the proposition 3.5.1. Then \( \varphi \) preserves all tame automorphisms.*
For free associative algebras, we note that any automorphism preserving the torus action preserves also the symmetric
\[ x_1 \mapsto x_1 + \beta(x_2x_3 + x_3x_2), \quad x_i \mapsto x_i, \quad i > 1 \]
and the skew symmetric
\[ x_1 \mapsto x_1 + \beta(x_2x_3 - x_3x_2), \quad x_i \mapsto x_i, \quad i > 1 \]
elementary automorphisms. The first property follows from Lemma 3.4.9. The second one follows from the fact that skew symmetric automorphisms commute with automorphisms of the following type
\[ x_2 \mapsto x_2 + x_3^2, \quad x_i \mapsto x_i, \quad i \neq 2 \]
and this property distinguishes them from elementary automorphisms of the form
\[ x_1 \mapsto x_1 + \beta x_2 x_3 + \gamma x_3 x_2, \quad x_i \mapsto x_i, \quad i > 1. \]

Theorem 3.1.2 follows from the fact that the forms \( \beta x_2 x_3 + \gamma x_3 x_2 \) corresponding to general bilinear multiplication
\[ \ast_{\beta, \gamma} : (x_2, x_3) \mapsto \beta x_2 x_3 + \gamma x_3 x_2 \]
lead to associative multiplication if and only if \( \beta = 0 \) or \( \gamma = 0 \); the approximation also applies (see section 3.3.3).

Suppose at first that \( n = 4 \) and we are dealing with \( K\langle x, y, z, t \rangle \).

**Proposition 3.5.3.** The group \( G \) containing all linear transformations and mappings
\[ x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + xy, \quad t \mapsto t \]
contains also all transformations of the form
\[ x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + P(x, y), \quad t \mapsto t. \]

**Proof.** It is enough to prove that \( G \) contains all transformations of the following form
\[ x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + aM, \quad t \mapsto t, \quad a \in K, \]
where \( M \) is a monomial.

**Step 1.** Let
\[ M = a \prod_{i=1}^{m} x^{k_i} y^{l_i} \quad \text{or} \quad M = a \prod_{i=1}^{m} y^{k_0} x^{k_i} y^{l_i}. \]
or
\[ M = a \prod_{i=1}^{m} x^{k_i}y_i \quad \text{or} \quad M = a \prod_{i=1}^{m} x^{k_i}y_i x^{k_{m+1}}. \]

Define the height of \( M \), \( H(M) \), to be the number of segments comprised of a specific generator - such as \( x^k \) - in the word \( M \). (For instance, \( H(a \prod_{i=1}^{m} x^{k_i}y_i x^{k_{m+1}}) = 2m + 1 \).) Using induction on \( H(M) \), one can reduce to the case when \( M = yx^k \). Let \( M = M'x^k \) such that \( H(M') < H(M) \). (Case when \( M = M'y^l \) is obviously similar.) Let
\[
\phi : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + M', \quad t \mapsto t.
\]

Then
\[
\alpha : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t + zx^k.
\]

The automorphism \( \phi^{-1} \circ \alpha \circ \phi : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t - M + zx^k \).

Observe that \( \beta \) is conjugate to the automorphism
\[
\beta' : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z - M, \quad t \mapsto t
\]
by a linear automorphism
\[
x \mapsto x, \quad y \mapsto y, \quad z \mapsto t, \quad t \mapsto z.
\]

Similarly, \( \gamma \) is conjugate to the automorphism
\[
\gamma' : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + yx^k, \quad t \mapsto t.
\]

We have thus reduced to the case when \( M = x^k \) or \( M = yx^k \).

**Step 2.** Consider automorphisms
\[
\alpha : x \mapsto x, \quad y \mapsto y + x^k, \quad z \mapsto z, \quad t \mapsto t
\]
and
\[
\beta : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t + az^y.
\]

Then
\[
\alpha^{-1} \circ \beta \circ \alpha : x \mapsto x, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t + azx^k + az^y.
\]
It is a composition of the automorphism

\[ \gamma : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t + azx^k \]

which is conjugate to the needed automorphism

\[ \gamma' : x \mapsto x, \ y \mapsto y, \ z \mapsto z + yx^k, \ t \mapsto t \]

and an automorphism

\[ \delta : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t + azy, \]

which is conjugate to the automorphism

\[ \delta' : x \mapsto x, \ y \mapsto y, \ z \mapsto z + axy, \ t \mapsto t \]

and then to the automorphism

\[ \delta'' : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy, \ t \mapsto t \]

(using similarities). We have reduced the problem to proving the statement

\[ G \ni \psi_M, \ M = x^k \]

for all \( k \).

**Step 3.** Obtain the automorphism

\[ x \mapsto x, \ y \mapsto y + x^n, \ z \mapsto z, \ t \mapsto t. \]

This problem is similar to the commutative case of \( K[x_1, \ldots, x_n] \) (cf. Section 3.4).

Proposition 3.5.3 is proved.

Returning to the general case \( n \geq 4 \), let us formulate the remark made after Lemma 3.4.7 as follows:

**Lemma 3.5.4.** Consider the following \( T^{n-1} \) action:

\[ x_1 \mapsto \lambda^i x_1, \ x_j \mapsto \lambda_j x_j, \ j > 1; \ \lambda^i = \lambda_2^i \cdots \lambda_n^i. \]

Then the set \( S \) of automorphisms commuting with this action is generated by the following automorphisms:

\[ x_1 \mapsto x_1 + H, \ x_i \mapsto x_i; \ i > 1, \]

where \( H \) is any homogenous polynomial of total degree \( i_2 + \cdots + i_n \).

Proposition 3.5.3 and Lemma 3.5.4 imply
Corollary 3.5.5. Let $\Psi \in \text{Aut}(\text{TAut}_0(K(\ldots, x_n)))$ stabilize all elements of torus and linear automorphisms,

$$\phi_P : x_n \mapsto x_n + P(x_1, \ldots, x_{n-1}), \quad x_i \mapsto x_i, \quad i = 1, \ldots, n-1.$$ 

Let $P = \sum P_i$, where $P_i$ is the homogenous component of $P$ of multi-degree $I$. Then

a) $\Psi(\phi_P) : x_n \mapsto x_n + P^\Psi(x_1, \ldots, x_{n-1}), \quad x_i \mapsto x_i, \quad i = 1, \ldots, n-1.$

b) $P^\Psi = \sum P_i^\Psi$; here $P_i^\Psi$ is homogenous of multi-degree $I$.

c) If $I$ has positive degree with respect to one or two variables, then $P_i^\Psi = P_i$.

Let $\Psi \in \text{Aut}(\text{TAut}_0(K(\ldots, x_n)))$ stabilize all elements of torus and linear automorphisms,

$$\phi : x_n \mapsto x_n + P(x_1, \ldots, x_{n-1}), \quad x_i \mapsto x_i, \quad i = 1, \ldots, n-1.$$ 

Let $\varphi_Q : x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_i \mapsto x_i + Q_i(x_1, x_2), \quad i = 3, \ldots, n-1, \quad x_n \mapsto x_n; \quad Q = (Q_3, \ldots, Q_{n-1}).$ Then $\Psi(\varphi_Q) = \varphi_Q$ by Proposition 3.5.3.

Lemma 3.5.6. a) $\varphi_Q^{-1} \circ \phi_P \circ \varphi_Q = \phi_{P_Q}$, where

$$P_Q(x_1, \ldots, x_{n-1}) = P(x_1, x_2, x_3 + Q_3(x_1, x_2), \ldots, x_{n-1} + Q_{n-1}(x_1, x_2)).$$

b) Let $P_Q = P_Q^{(1)} + P_Q^{(2)}$, $P_Q^{(1)}$ consist of all terms containing one of the variables $x_3, \ldots, x_{n-1}$, and let $P_Q^{(1)}$ consist of all terms containing just $x_1$ and $x_2$. Then

$$P_Q^\Psi = P_Q^\Psi = P_Q^{(1)} + P_Q^{(2)} = P_Q^{(1)} + P_Q^{(2)}.$$

Lemma 3.5.7. If $P_Q^{(2)} = R_Q^{(2)}$ for all $Q$ then $P = R$.

Proof. It is enough to prove that if $P \neq 0$ then $P_Q^{(2)} \neq 0$ for appropriate $Q = (Q_3, \ldots, Q_{n-1})$. Let $m = \deg(P)$, $Q_i = x_1^{m+1}x_2^{m+1}$. Let $\hat{P}$ be the highest-degree component of $P$, then $\hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1})$ is the highest-degree component of $P_Q^{(2)}$. It is enough to prove that

$$\hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \neq 0.$$

Let $x_1 < x_2 < x_2 < \cdots < x_{n-1}$ be the standard lexicographic order. Consider the lexicographically minimal term $M$ of $\hat{P}$. It is easy to see that the term

$$M|_{Q_i \rightarrow x_i}, \quad i = 3, \quad n-1$$

cannot cancel with any other term

$$N|_{Q_i \rightarrow x_i}, \quad i = 3, \quad n-1$$
Chapter 3. Automorphisms of Ind-schemes, augmentation topology, and approximation

of $\hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1})$. Therefore $\hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \neq 0$.

Lemmas 3.5.6 and 3.5.7 imply

**Corollary 3.5.8.** Let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilize all elements of torus and linear automorphisms. Then $P^\Psi = P$, and $\Psi$ stabilizes all elementary automorphisms and therefore the entire group $\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle)$.

We obtain the following

**Proposition 3.5.9.** Let $n \geq 4$ and let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilize all elements of torus and linear automorphisms. Then either $\Psi = \text{Id}$ or $\Psi$ acts as conjugation by the mirror anti-automorphism.

Let $n \geq 4$. Let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilize all elements of torus and linear automorphisms. Denote by $EL$ an elementary automorphism

$$EL : x_1 \mapsto x_1, \ldots, x_{n-1} \mapsto x_{n-1}, x_n \mapsto x_n + x_1 x_2$$

(all other elementary automorphisms of this form, i.e. $x_k \mapsto x_k + x_i x_j$, $x_l \mapsto x_l$ for $l \neq k$ and $k \neq i, k \neq j, i \neq j$, are conjugate to one another by permutations of generators).

We have to prove that $\Psi(EL) = EL$ or $\Psi(EL) : x_i \mapsto x_i; i = 1, \ldots, x_{n-1}, x_n \mapsto x_n + x_2 x_1$. The latter corresponds to $\Psi$ being the conjugation with the mirror anti-automorphism of $K\langle x_1, \ldots, x_n \rangle$.

Define for some $a, b \in K$

$$x *_{a,b} y = axy + byx.$$

Then, in any of the above two cases,

$$\Psi(EL) : x_i \mapsto x_i; i = 1, \ldots, x_{n-1}, x_n \mapsto x_n + x_1 *_{a,b} x_2$$

for some $a, b$.

The following lemma is elementary:

**Lemma 3.5.10.** The operation $* = *_{a,b}$ is associative if and only if $ab = 0$.

The associator of $x, y,$ and $z$ is given by

$$\{x, y, z\} = (x * y) * z - x * (y * z) =
ab(zx - xz)y + aby(xz - zx) = ab[y, [x, z]].$$

Now we are ready to prove Proposition 3.5.9. For simplicity we treat only the case $n = 4$ – the general case is dealt with analogously. Consider the automorphisms

$$\alpha : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy, \ t \mapsto t,$$
\[ \beta : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t + xz, \]
\[ h : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t - xz. \]

(Manifestly \( h = \beta^{-1} \).) Then
\[ \gamma = h\alpha^{-1}\beta\alpha = [\beta, \alpha] : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t - x^2y. \]

Note that \( \alpha \) is conjugate to \( \beta \) via a generator permutation
\[ \kappa : x \mapsto x, \ y \mapsto z, \ z \mapsto t, \ t \mapsto y, \ \kappa \circ \alpha \circ \kappa^{-1} = \beta \]
and
\[ \Psi(\gamma) : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t - x * (x * y). \]

Let \( \delta : x \mapsto x, \ y \mapsto y, \ z \mapsto z + x^2, \ t \mapsto t, \ \epsilon : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t + zy. \)

Let \( \gamma' = \epsilon^{-1}\delta^{-1}\epsilon\delta \). Then
\[ \gamma' : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t - x^2y. \]

On the other hand we have
\[ \varepsilon = \Psi(\epsilon^{-1}\delta^{-1}\epsilon\delta) : x \mapsto x, \ y \mapsto y, \ z \mapsto z, \ t \mapsto t - (x^2) * y. \]

We also have \( \gamma = \gamma' \). Equality \( \Psi(\gamma) = \Psi(\gamma') \) is equivalent to the equality \( x * (x * y) = x^2 * y \). This implies \( x * y = xy \) and we are done.

### 3.5.2 The group \( \text{Aut}_{\text{Ind}}(\text{TAut}(K\langle x, y, z \rangle)) \)

This is the most technically loaded part of the present study. At the moment we are unable to accomplish the objective of describing the entire group \( \text{Aut}_{\text{T Aut}}(K\langle x, y, z \rangle) \). In this section we will determine only its subgroup \( \text{Aut}_{\text{Ind}} \text{TAut}_0(K\langle x, y, z \rangle) \), i.e. the group of Ind-scheme automorphisms, and prove Theorem 3.1.8. We use the approximation results of Section 3.3.3. In what follows we suppose that \( \text{char}(K) \neq 2 \). As in the preceding chapter, \( \{x, y, z\}_* \) denotes the associator of \( x, y, z \) with respect to a fixed binary linear operation \( * \), i.e.
\[ \{x, y, z\}_* = (x * y) * z - x * (y * z). \]
Chapter 3. Automorphisms of Ind-schemes, augmentation topology, and approximation

Proposition 3.5.11. Let $\Psi \in \text{Aut}_{\text{Ind}}(\text{TAut}_0(K \langle x, y, z \rangle))$ stabilize all linear automorphisms. Let 

$$\phi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy.$$ 

Then either 

$$\Psi(\phi) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + axy$$ 

or 

$$\Psi(\phi) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + byx$$ 

for some $a, b \in K$.

Proof. Consider the automorphism 

$$\phi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy.$$ 

Then 

$$\Psi(\phi) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + x * y,$$ 

where $x * y = axy + byx$. Let $a \neq 0$. We can make the star product $* = *_{a,b}$ into $x * y = xy + \lambda yx$ by conjugation with the mirror anti-automorphism and appropriate linear substitution. We therefore need to prove that $\lambda = 0$, which implies $\Psi(\phi) = \phi$.

The following two lemmas are proved by straightforward computation.

Lemma 3.5.12. Let $A = K \langle x, y, z \rangle$. Let $f * g = fg + \lambda fg$. Then $\{f, g, h\} = \lambda[g, [f, h]]$.

In particular \{f, g, f\} = 0, $f * (f * g) - (f * f) * g = -\{f, f, g\} = \lambda[f, [f, g]]$.

Lemma 3.5.13. Let $\varphi_1 : x \mapsto x + yz, \ y \mapsto y, \ z \mapsto z; \varphi_2 : x \mapsto x, \ y \mapsto y, \ z \mapsto z + yx; \varphi = \varphi_2^{-1}\varphi_1^{-1}\varphi_2\varphi_1$. Then modulo terms of order $\geq 4$ we have:

$$\varphi : x \mapsto x + y^2x, \ y \mapsto y, \ z \mapsto z - y^2z$$

and

$$\Psi(\varphi) : x \mapsto x + y * (y * x), \ y \mapsto y, \ z \mapsto z - y * (y * z).$$

Lemma 3.5.14. a) Let $\phi_l : x \mapsto x, \ y \mapsto y, \ z \mapsto z + y^2x$. Then 

$$\Psi(\phi_l) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + y * (y * x).$$

b) Let $\phi_r : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy^2$. Then 

$$\Psi(\phi_r) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + (x * y) * y.$$
Proof. According to the results of the previous section we have

$$\Psi(\phi_l) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + P(y, x)$$

where $P(y, x)$ is homogenous of degree 2 with respect to $y$ and degree 1 with respect to $x$. We have to prove that $H(y, x) = P(y, x) - y * (y * x) = 0$.

Let $\tau : x \mapsto z, \ y \mapsto y, \ z \mapsto x; \ \tau = \tau^{-1}$, $\phi' = \tau \phi_l \tau^{-1} : x \mapsto x + y^2z, \ y \mapsto y, \ z \mapsto z$. Then

$$\Psi(\phi'_l) : x \mapsto x + P(y, z), \ y \mapsto y, \ z \mapsto z.$$

Let $\phi''_l = \phi_l \phi'_l : x \mapsto x + P(y, z), \ y \mapsto y, \ z \mapsto z + P(y, x)$ modulo terms of degree $\geq 4$.

Let $\tau : x \mapsto x - z, \ y \mapsto y, \ z \mapsto z$ and let $\varphi_2, \ \varphi$ be the automorphisms described in Lemma 3.5.13.

Then

$$T = \tau^{-1} \phi_l^{-1} \tau \phi''_l : x \mapsto x, \ y \mapsto y, \ z \mapsto z$$

modulo terms of order $\geq 4$.

On the other hand

$$\Psi(T) : x \mapsto x + H(y, z) - H(y, x), \ y \mapsto y, \ z \mapsto z + P$$

modulo terms of order $\geq 4$. Because $\deg_y(H(y, x)) = 2, \ deg_x(H(y, x)) = 1$ we get $H = 0$.

Proof of b) is similar.

Lemma 3.5.15. a) Let

$$\psi_1 : x \mapsto x + y^2, \ y \mapsto y, \ z \mapsto z; \ \psi_2 : x \mapsto x, \ y \mapsto y, \ z \mapsto z + x^2.$$ 

Then

$$[\psi_1, \psi_2] = \psi_2^{-1} \psi_1^{-1} \psi_2 \psi_1 : x \mapsto x, \ y \mapsto y, \ z \mapsto z + y^2x + xy^2;$$

$$\Psi([\psi_1, \psi_2]) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + (y * y) * x + x * (y * y).$$

b) \vspace{5mm}

$$\phi_l^{-1} \phi_r^{-1} [\psi_1, \psi_2] : x \mapsto x, \ y \mapsto y, \ z \mapsto z$$

modulo terms of order $\geq 4$ but

$$\Psi \left( \phi_l^{-1} \phi_r^{-1} [\psi_1, \psi_2] \right) : x \mapsto x, \ y \mapsto y,$$

$$z \mapsto z + (y * y) * x + x * (y * y) - (x * y) * y - y * (y * x) =$$

$$= z + 4\lambda[x[x, y]]$$

modulo terms of order $\geq 4$.

Proof. a) can be obtained by direct computation. b) follows from a) and the lemma 3.5.12.
Proposition 3.5.11 follows from Lemma 3.5.15.

We need a few auxiliary lemmas. The first one is an analogue of the hiking procedure from [32, 117].

**Lemma 3.5.16.** Let $K$ be algebraically closed, and let $n_1, \ldots, n_m$ be positive integers. Then there exist $k_1, \ldots, k_s \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_s \in K$ such that

- $\sum k_i = 1$ modulo $\text{char}(K)$ (if $\text{char}(K) = 0$ then $\sum k_i = 1$).
- $\sum_i k_i^{n_j} \lambda_i = 0$ for all $j = 1, \ldots, m$.

For $\lambda \in K$ we define an automorphism $\psi_\lambda : x \mapsto x$, $y \mapsto y$, $z \mapsto \lambda z$.

The next lemma provides for some translation between the language of polynomials and the group action language. It is similar to the hiking process [32, 117].

**Lemma 3.5.17.** Let $\varphi \in K\langle x, y, z \rangle$. Let $\varphi(x) = x$, $\varphi(y) = y + \sum_i R_i + R'$, $\varphi(z) = z + Q$. Let $\text{deg}(R_i) = N$, let also the degree of all monomials in $R'$ be greater than $N$, and let the degree of all monomials in $Q$ be greater than or equal to $N$. Finally, assume $\text{deg}_z(R_i) = i$ and the $z$-degree of all monomials of $R_1$ greater than $0$.

Then

a) $\psi_\lambda^{-1} \varphi \psi_\lambda : x \mapsto x$, $y \mapsto y + \sum_i \lambda^i R_i + R''$, $z \mapsto z + Q'$. Also the total degree of all monomials comprising $R'$ is greater than $N$, and the degree of all monomials of $Q$ is greater than or equal to $N$.

b) Let $\phi = \prod \left( \psi_{\lambda_i}^{-1} \varphi \psi_\lambda \right)^{k_i}$. Then

$$\phi : x \mapsto x, \ y \mapsto y + \sum_i R_i \lambda_i^{k_i} + S, \ z \mapsto z + T$$

where the degree of all monomials of $S$ is greater than $N$ and the degree of all monomials of $T$ is greater than or equal to $N$.

**Proof.** a) By direct computation. b) is a consequence of a).

**Remark.** In the case of characteristic zero, the condition of $K$ being algebraically closed can be dropped. After hiking for several steps, we need to prove just

**Lemma 3.5.18.** Let $\text{char}(K) = 0$, let $n$ be a positive integer. Then there exist $k_1, \ldots, k_s \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_s \in K$ such that

- $\sum k_i = 1$.
- $\sum_i k_i^n \lambda_i = 0$.

Using this lemma we can cancel out all the terms in the product in the Lemma 3.5.17 except for the constant one. The proof of Lemma 3.5.18 for any field of zero characteristic can be obtained through the following observation:
Chapter 3. Automorphisms of Ind-schemes, augmentation topology, and approximation

Lemma 3.5.19.
\[
\left( \sum_{i=1}^{n} \lambda_i \right)^n - \sum_j \left( \lambda_1 + \cdots + \lambda_j + \cdots + \lambda_n \right)^n + \cdots + \\
+(-1)^{n-k} \sum_{i_1 < \cdots < i_k} (x_{i_1} + \cdots + x_{i_k})^n + \cdots + (-1)^{n-1} (x_1^n + \cdots + x_n^n) = n! \prod_{i=1}^{n} x_i
\]
and if \( m < n \) then
\[
\left( \sum_{i=1}^{n} \lambda_i \right)^m - \sum_j \left( \lambda_1 + \cdots + \lambda_j + \cdots + \lambda_n \right)^m + \cdots + \\
+(-1)^{n-k} \sum_{i_1 < \cdots < i_k} (x_{i_1} + \cdots + x_{i_k})^m + \cdots + (-1)^{n-1} (x_1^m + \cdots + x_n^m) = 0.
\]

The lemma 3.5.19 allows us to replace the \( n \)-th powers by product of constants, after that the statement of Lemma 3.5.18 becomes transparent.

Lemma 3.5.20. Let \( \varphi : x \mapsto x + R_1, \ y \mapsto y + R_2, \ z \mapsto z' \), such that the total degree of all monomials in \( R_1, R_2 \) is greater than or equal to \( N \). Then for \( \Psi(\varphi) : x \mapsto x + R'_1, \ y \mapsto y + R'_2, \ z \mapsto z'' \) with the total degree of all monomials in \( R'_1, R'_2 \) also greater than or equal to \( N \).

Proof. Similar to the proof of Theorem 3.3.2.

Lemmas 3.5.20, 3.5.17, 3.5.16 imply the following statement.

Lemma 3.5.21. Let \( \varphi_j \in \operatorname{Aut}_0(K \langle x, y, z \rangle), \ j = 1, 2, \) such that
\[
\varphi_j(x) = x, \ \varphi_j(y) = y + \sum_i R_i^j + R_j^i, \ \varphi_j(z) = z + Q_j.
\]
Let \( \deg(R_i^j) = N \), and suppose that the degree of all monomials in \( R_i^j \) is greater than \( N \), while the degree of all monomials in \( Q \) is greater than or equal to \( N \); \( \deg_z(R_i) = i \), and the \( z \)-degree of all monomials in \( R_1 \) is positive. Let \( R_0^1 = 0, R_0^2 \neq 0 \).

Then \( \Psi(\varphi_1) \neq \varphi_2 \).

Consider the automorphism
\[
\phi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + P(x, y).
\]
Let \( \Psi \in \operatorname{Aut}_{\text{Ind}} \operatorname{TAut}_0(k \langle x, y, z \rangle) \) stabilize the standard torus action pointwise. Then
\[
\Psi(\phi) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + Q(x, y).
\]
We denote
\[
\bar{\Psi}(P) = Q.
\]
Our goal is to prove that \( \bar{\Psi}(P) = P \) for all \( P \) if \( \Psi \) stabilizes all linear automorphisms and \( \bar{\Psi}(xy) = xy \). We proceed by strong induction on total degree. The base case corresponds to \( k = 1 \) and \( l = 1 \) and is assumed. We then have

**Lemma 3.5.22.**

\[
\bar{\Psi}(x^k y^l) = x^k y^l
\]

provided that \( \bar{\Psi}(P) = P \) for all monomials \( P(x, y) \) of total degree \( < k + l \).

**Proof.**

Let

\[
\begin{align*}
\phi & : x \mapsto x, \ y \mapsto y, \ z \mapsto z + x^k y^l, \\
\varphi_1 & : x \mapsto x + y^l, \ y \mapsto y, \ z \mapsto z, \\
\varphi_2 & : x \mapsto x, \ y \mapsto y + x^k, \ z \mapsto z, \\
\varphi_3 & : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy, \\
h & : x \mapsto x, \ y \mapsto y, \ z \mapsto z - x^{k+1}.
\end{align*}
\]

Then, for \( k > 1 \) and \( l > 1 \)

\[
g = h \varphi_3^{-1} \varphi_1^{-1} \varphi_2^{-1} \varphi_3 \varphi_1 \varphi_2 : \\
x \mapsto x - y^l + (y - (x - y^l)^k)^l, \\
y \mapsto y - (x - y^l)^k + (x - y^l + (y - (x - y^l)^k)^l)^k, \\
z \mapsto z - xy - x^{k+1} + (x - y^l)(y - (x - y^l)^k).
\]

Observe that the height of \( g(x) - x, g(y) - y \) and \( g(z) - z \) is at least \( k + l - 1 \), when \( k > 1 \) or \( l > 1 \). We then use Theorem 3.3.2 and the induction step. Applying \( \Psi \) yields the result because \( \Psi(\varphi_i) = \varphi_i, \ i = 1, 2, 3 \) and \( \varphi(H_N) \subseteq H_N \) for all \( N \). The lemma is proved.

Let

\[
M_{k_1, \ldots, k_s} = x^{k_1} y^{k_2} \ldots y^{k_s}
\]

for even \( s \) and

\[
M_{k_1, \ldots, k_s} = x^{k_1} y^{k_2} \ldots x^{k_s}
\]

for odd \( s \), \( k = \sum_{i=1}^n k_i \). Then

\[
M_{k_1, \ldots, k_s} = M_{k_1, \ldots, k_{s-1}} y^{k_s}
\]

for even \( s \) and

\[
M_{k_1, \ldots, k_s} = M_{k_1, \ldots, k_{s-1}} x^{k_s}
\]

for odd \( s \).
We have to prove that $\bar{\Psi}(M_{k_1,\ldots,k_s}) = M_{k_1,\ldots,k_s}$. By induction we may assume that $\bar{\Psi}(M_{k_1,\ldots,k_{s-1}}) = M_{k_1,\ldots,k_{s-1}}$.

For any monomial $M = M(x,y)$ we define an automorphism

$\varphi_M : x \mapsto x, \ y \mapsto y, \ z \mapsto z + M$.

We also define the automorphisms

$\phi^e_k : x \mapsto x, \ y \mapsto y + zx^k, \ z \mapsto z$

and

$\phi^o_k : x \mapsto x + zy^k, \ y \mapsto y, \ z \mapsto z$.

We will present the case of even $s$ - the odd $s$ case is similar.

Let $D^e_{zx^k}$ be a derivation of $K\langle x, y, z \rangle$ such that $D^e_{zx^k}(x) = 0$, $D^e_{zx^k}(y) = zx^k$, $D^e_{zx^k}(z) = 0$. Similarly, let $D^o_{zy^k}$ be a derivation of $K\langle x, y, z \rangle$ such that $D^o_{zy^k}(y) = 0$, $D^o_{zy^k}(x) = zy^k$, $D^o_{zy^k}(z)^o = 0$.

The following lemma is proved by direct computation:

**Lemma 3.5.23.** Let

$u = \phi^e_k^{-1}\varphi(M_{k_1,\ldots,k_{s-1}})^{-1}\phi^e_k\varphi(M_{k_1,\ldots,k_{s-1}})$

for even $s$ and

$u = \phi^o_k^{-1}\varphi(M_{k_1,\ldots,k_{s-1}})^{-1}\phi^o_k\varphi(M_{k_1,\ldots,k_{s-1}})$

for odd $s$. Then

$u : x \mapsto x, \ y \mapsto y + M_{k_1,\ldots,k_s} + N', \ z \mapsto z + D^e_{zx^k}(M_{k_1,\ldots,k_{s-1}}) + N$

for even $s$ and

$u : x \mapsto x + M_{k_1,\ldots,k_s} + N', \ y \mapsto y, \ z \mapsto z + D^o_{zy^k}(M_{k_1,\ldots,k_{s-1}}) + N$

for odd $s$, where $N, N'$ are sums of terms of degree $> k = \sum_{i=1}^s k_i$.

Let $\psi(M_{k_1,\ldots,k_s}) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + M_{k_1,\ldots,k_s}$,

$\alpha_e : x \mapsto x, \ y \mapsto y - z, \ z \mapsto z, \alpha_o : x \mapsto x - z, \ y \mapsto y, \ z \mapsto z$.

Let $P_M = \psi(M) - M$. Our goal is to prove that $P_M = 0$.

Let

$v = \psi(M_{k_1,\ldots,k_s})^{-1}\alpha_e\psi(M_{k_1,\ldots,k_s})u\alpha_e^{-1}$

for even $s$ and

$v = \psi(M_{k_1,\ldots,k_s})^{-1}\alpha_o\psi(M_{k_1,\ldots,k_s})u\alpha_o^{-1}$
Chapter 3. Automorphisms of Ind-schemes, augmentation topology, and approximation

for odd $s$.

The next lemma is also proved by direct computation:

Lemma 3.5.24. \( a) \)

\[ v : x \mapsto x, \ y \mapsto y + H, \ z \mapsto z + H_1 + H_2 \]

for even $s$ and

\[ v : x \mapsto x + H, \ y \mapsto y, \ z \mapsto z + H_1 + H_2 \]

for odd $s$

\( b) \)

\[ \Psi(v) : x \mapsto x, \ y \mapsto y + P_{M_{k_1, \ldots, k_s}} + \tilde{H}, \ z \mapsto z + \tilde{H}_1 + \tilde{H}_2 \]

for even $s$ and

\[ \Psi(v) : x \mapsto x + P_{M_{k_1, \ldots, k_s}} + \tilde{H}, \ y \mapsto y, \ z \mapsto z + \tilde{H}_1 + \tilde{H}_2 \]

for odd $s$, where $H_2$, $\tilde{H}_2$ are sums of terms of degree greater than $k = \sum_{i=1}^s k_i$, $H$, $\tilde{H}$ are sums of terms of degree $\geq k$ and positive $z$-degree, $H_1$, $\tilde{H}_1$ are sums of terms of degree $k$ and positive $z$-degree.

Proof of Theorem 3.1.8. Part \( b) \) follows from part \( a) \). In order to prove \( a) \) we are going to show that $\bar{\Psi}(M) = M$ for any monomial $M(x, y)$ and for any $\Psi \in \text{Aut}_{\text{Ind}}(\text{TAut}(\langle x, y, z \rangle))$ stabilizing the standard torus action $T^3$ and $\phi$. The automorphism $\Psi(\Phi_M)$ has the form described in Lemma 3.5.24. But in this case Lemma 3.5.21 implies $\bar{\Psi}(M) - M = 0$.

3.6 Some open questions concerning the tame automorphism group

As the conclusion of the paper, we would like to raise the following questions.

1. Is it true that any automorphism $\varphi$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ (in the group-theoretic sense - that is, not necessarily an automorphism preserving the Ind-scheme structure) for $n = 3$ is semi-inner, i.e. is a conjugation by some automorphism or mirror anti-automorphism?

2. Is it true that $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ is generated by affine automorphisms and automorphism $x_n \mapsto x_n + x_1 x_2$, $x_i \mapsto x_i$, $i \neq n$? For $n \geq 5$ it seems to be easier and the answer is probably positive, however for $n = 3$ the answer is known to be negative, cf. Umirbaev [190] and Drensky and Yu [80]. For $n \geq 4$ we believe the answer is positive.

3. Is it true that $\text{Aut}(K[x_1, \ldots, x_n])$ is generated by linear automorphisms and automorphism $x_n \mapsto x_n + x_1 x_2$, $x_i \mapsto x_i$, $i \neq n$? For $n = 3$ the answer is negative: see the proof of the Nagata conjecture [178, 181, 197]. For $n \geq 4$ it is plausible that the answer is positive.

4. Is any automorphism $\varphi$ of $\text{Aut}(K\langle x, y, z \rangle)$ (in the group-theoretic sense) semi-inner?
5. Is it true that the conjugation in Theorems 3.1.3 and 3.1.7 can be done by some tame automorphism? Suppose $\psi^{-1}\varphi\psi$ is tame for any tame $\varphi$. Does it follow that $\psi$ is tame?

6. Prove Theorem 3.1.8 for $\text{char}(K) = 2$. Does it hold on the set-theoretic level, i.e. $\text{Aut}(\text{TAut}(K\langle x, y, z \rangle))$ are generated by conjugations by an automorphism or the mirror anti-automorphism?

Similar questions can be formulated for nice automorphisms.
Chapter 4

Approximation by tame automorphisms
and the Kontsevich conjecture

The first four sections of this chapter is based on the paper [112]. In light of this approximation method, A. Elishev, A. Kanel-Belov, and J.-T. Yu [111] proposed a proof of the Kontsevich Conjecture.

4.1 Endomorphisms of $\mathbb{K}[X]$, $W_n(\mathbb{K})$ and $P_n(\mathbb{K})$

4.1.1 Definitions and notation

The $n$-th Weyl algebra $W_n(\mathbb{K})$ over $\mathbb{K}$ is by definition the quotient of the free associative algebra

$$\mathbb{K}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$$

by the two-sided ideal generated by elements

$$b_i a_j - a_j b_i - \delta_{ij}, \ a_i a_j - a_j a_i, \ b_i b_j - b_j b_i,$$

with $1 \leq i, j \leq n$. One can think of $W_n(\mathbb{K})$ as the algebra

$$\mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

with two sets of $n$ mutually commuting generators (images of the free generators under the canonical projection) which interact according to $[y_i, x_j] = y_i x_j - x_j y_i = \delta_{ij}$, although, unless the context necessitates clarification, we would like to denote the Weyl algebra henceforth by $W_n(\mathbb{K})$ in order to avoid confusion with $\mathbb{K}[X]$ – notation reserved for the ring of polynomials in commuting variables.

The polynomial algebra $\mathbb{K}[x_1, \ldots, x_N]$ itself is the quotient of the free associative algebra by the congruence that makes all its generators commutative. When $N = 2n$ is even, the algebra
$A_{2n}$ carries an additional structure of the Poisson algebra – namely, a bilinear map

\[ \{ , \} : \mathbb{K}[x_1, \ldots, x_N] \otimes \mathbb{K}[x_1, \ldots, x_N] \to \mathbb{K}[x_1, \ldots, x_N] \]

that turns $\mathbb{K}[x_1, \ldots, x_N]$ into a Lie algebra and acts as a derivation with respect to polynomial multiplication. Under a fixed choice of generators, this map is given by the canonical Poisson bracket

\[ \{ x_i, x_j \} = \delta_{i,n+j} - \delta_{i+n,j}. \]

We denote the pair $(\mathbb{K}[x_1, \ldots, x_{2n}], \{ , \})$ by $P_n(\mathbb{K})$. In our discussion the coefficient ring $\mathbb{K}$ is a field of characteristic zero, and for later purposes (Proposition 4.3) we require $\mathbb{K}$ to be algebraically closed. Thus one may safely assume $\mathbb{K} = \mathbb{C}$ in the sequel.

Throughout we assume all homomorphisms to be unital and preserving all defining structures carried by the objects in question. Thus, by a Weyl algebra endomorphism we always mean a $\mathbb{K}$-linear ring homomorphism $W_n(\mathbb{K})$ into itself that maps 1 to 1. Similarly, the set $\text{End} \mathbb{K}[x_1, \ldots, x_n]$ consists of all $\mathbb{K}$-endomorphisms of the polynomial algebra, while $\text{End} P_n$ is the set of polynomial endomorphisms preserving the Poisson structure. We will call elements of the group $\text{Aut} P_n$ polynomial symplectomorphisms, due to the fact that they can be identified with polynomial one-to-one mappings $A_{2n}^\mathbb{K} \to A_{2n}^\mathbb{K}$ of the affine space $A_{2n}^\mathbb{K}$ which preserve the symplectic form

\[ \omega = \sum_i dp_i \wedge dx_i. \]

Any endomorphism $\varphi$ of $\mathbb{K}[x_1, \ldots, x_N]$, $P_n(\mathbb{K})$ or $W_n(\mathbb{K})$ can be identified with the ordered set

\[ (\varphi(x_1), \varphi(x_2), \ldots) \]

of images of generators of the corresponding algebra. For $\mathbb{K}[x_1, \ldots, x_N]$ and $P_n(\mathbb{K})$, the polynomials $\varphi(x_i)$ can be decomposed into sums of homogeneous components; this means that the endomorphism $\varphi$ may be written as a formal sum

\[ \varphi = \varphi_0 + \varphi_1 + \cdots, \]

where $\varphi_k$ is a string (of length $N$ and $2n$, respectively) whose entries are homogeneous polynomials of total degree $k$.\footnote{We set $\deg x_i = 1$.} Accordingly, the height $\text{ht}(\varphi)$ of the endomorphism is defined as

\[ \text{ht}(\varphi) = \inf \{ k \mid \varphi_k \neq 0 \}, \quad \text{ht}(0) = \infty. \]

This is not to be confused with the degree of endomorphism, which is defined as $\deg(\varphi) =$
The height \( \text{ht}(f) \) of a polynomial \( f \) is defined quite similarly to be the minimal number \( k \) such that the homogeneous component \( f_k \) is not zero. Evidently, for an endomorphism \( \varphi = (\varphi(x_1), \ldots, \varphi(x_N)) \) one has

\[
\text{ht}(\varphi) = \inf\{\text{ht}(\varphi(x_i)) \mid 1 \leq i \leq N\}.
\]

The function

\[
d(\varphi, \psi) = \exp(-\text{ht}(\varphi - \psi))
\]

is a metric on \( \text{End} \mathbb{K}[x_1, \ldots, x_N] \). We will refer to the corresponding topology on \( \text{End} \) (and on subspaces such as \( \text{Aut} \) and \( \text{TAut} \)) as the formal power series topology.

### 4.1.2 Tame automorphisms

We call an automorphism \( \varphi \in \text{Aut} \mathbb{K}[x_1, \ldots, x_N] \) **elementary** if it is of the form

\[
\varphi = (x_1, \ldots, x_{k-1}, ax_k + f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N), x_{k+1}, \ldots, x_N)
\]

with \( a \in \mathbb{K}^\times \). Observe that linear invertible changes of variables – that is, transformations of the form

\[
(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_N)A, \quad A \in \text{GL}(N, \mathbb{K})
\]

are realized as compositions of elementary automorphisms.

The subgroup of \( \text{Aut} \mathbb{K}[x_1, \ldots, x_N] \) generated by all elementary automorphisms is the group \( \text{TAut} \mathbb{K}[x_1, \ldots, x_N] \) of so-called **tame automorphisms**.

Let \( P_n(\mathbb{K}) = \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \) be the polynomial algebra in \( 2n \) variables with Poisson structure. It is clear that for an elementary \( \varphi \in \text{Aut} \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \) to be a symplectomorphism, it must either be a linear symplectic change of variables – that is, a transformation of the form

\[
(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n)A
\]

with \( A \in \text{Sp}(2n, \mathbb{K}) \) a symplectic matrix, or be an elementary transformation of one of two following types:

\[
(x_1, \ldots, x_{k-1}, x_k + f(p_1, \ldots, p_n), x_{k+1}, \ldots, x_n, p_1, \ldots, p_n)
\]

or

\[
(x_1, \ldots, x_n, p_1, \ldots, p_{k-1}, p_k + g(x_1, \ldots, x_n), p_{k+1}, \ldots, p_n).
\]

Note that in both cases we do not include translations of the affine space into our consideration, so we may safely assume the polynomials \( f \) and \( g \) to be at least of height one.

\[^2\text{For } W_n \text{ the degree is well defined, but the height depends on the ordering of the generators.} \]
The subgroup of Aut $P_n(\mathbb{K})$ generated by all such automorphisms is the group TAut $P_n(\mathbb{K})$ of \textbf{tame symplectomorphisms}. One similarly defines the notion of tameness for the Weyl algebra $W_n(\mathbb{K})$, with tame elementary automorphisms having the exact same form as for $P_n(\mathbb{K})$.

The automorphisms which are not tame are called \textbf{wild}. It is unknown at the time of writing whether the algebras $W_n$ and $P_n$ have any wild automorphisms in characteristic zero for $n > 1$, however for $n = 1$ all automorphisms are known to be tame [99,135,136,200]. On the other hand, the celebrated example of Nagata

$$(x + (x^2 - yz)x, y + 2(x^2 - yz)x + (x^2 - yz)^2z, z)$$

provides a wild automorphism of the polynomial algebra $\mathbb{K}[x, y, z]$.

It is known due to Kanel-Belov and Kontsevich [39, 40] that for $\mathbb{K} = \mathbb{C}$ the groups

TAut $W_n(\mathbb{C})$ and TAut $P_n(\mathbb{C})$

are isomorphic. The homomorphism between the tame subgroups is obtained by means of non-standard analysis and involves certain non-constructible entities, such as free ultrafilters and infinite prime numbers. Recent effort [38,110] has been directed to proving the homomorphism’s independence of such auxiliary objects, with limited success.

\section*{4.2 Approximation by tame automorphisms}

Let $\varphi \in \text{Aut} \mathbb{K}[x_1, \ldots, x_N]$ be a polynomial automorphism. We say that $\varphi$ is approximated by tame automorphisms if there is a sequence

$$\psi_1, \psi_2, \ldots, \psi_k, \ldots$$

of tame automorphisms such that

$$\text{ht}((\psi_k^{-1} \circ \varphi)(x_i) - x_i) \geq k$$

for $1 \leq i \leq N$ and all $k$ sufficiently large. Observe that any tame automorphism $\psi$ is approximated by itself – that is, by a stationary sequence $\psi_k = \psi$.

The following two theorems are the main results of this chapter.

\textbf{Theorem 4.2.1.} Let $\varphi = (\varphi(x_1), \ldots, \varphi(x_N))$ be an automorphism of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_N]$ over a field $\mathbb{K}$ of characteristic zero, such that its Jacobian

$$J(\varphi) = \det \left[ \frac{\partial \varphi(x_i)}{\partial x_j} \right]$$
is equal to 1. Then there exists a sequence \( \{\psi_k\} \subset \text{TAut } \mathbb{K}[x_1, \ldots, x_N] \) of tame automorphisms approximating \( \varphi \).

**Theorem 4.2.2.** Let \( \sigma = (\sigma(x_1), \ldots, \sigma(x_n), \sigma(p_1), \ldots, \sigma(p_n)) \) be a symplectomorphism of \( \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \) with unit Jacobian. Then there exists a sequence \( \{\tau_k\} \subset \text{TAut } P_n(\mathbb{K}) \) of tame symplectomorphisms approximating \( \sigma \).

Theorem 4.2.1 is a special case of a classical result of Anick [8] (Anick proved approximation for all étale maps, not just automorphisms). We give here a slightly simplified proof suitable for our context. The second theorem first appeared in [112] and is essential in our approach to the lifting problem in deformation quantization.

The proof of Theorem 4.2.1 consists of several steps each of which amounts to composing a given automorphism \( \varphi \) with a tame transformation of a specific type -- an operation which allows one to dispose in \( \varphi(x_i) \) (1 \( \leq \) i \( \leq \) N) of monomial terms of a given total degree, assuming that the lower degree terms have already been dealt with. Thus the approximating sequence of tame automorphisms is constructed by induction. As it was mentioned before, we disregard translation automorphisms completely: all automorphisms discussed here are origin-preserving, so that the polynomials \( \varphi(x_i) \) have zero free part. This of course leads to no loss of generality.

The process starts with the following straightforward observation.

**Lemma 4.2.3.** There is a linear transformation \( A \in \text{GL}(N, \mathbb{K}) \)

\[
(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_N)A
\]

such that its composition \( \varphi_A \) with \( \varphi \) fulfills

\[
\text{ht}(\varphi_A(x_i) - x_i) \geq 2
\]

for all \( i \in \{1, \ldots, N\} \).

**Proof.** Consider

\[
A_1 = \left[ \begin{array}{c} \frac{\partial \varphi(x_i)}{\partial x_j} \\ \end{array} \right] (0, \ldots, 0)
\]

– the linear part of \( \varphi \). Its determinant is equal to the value of \( J(\varphi) \) at zero, and \( J(\varphi) \) is a non-zero constant. Composing \( \varphi \) with the linear change of variables induced by \( A_1^{-1} \) (on the left) results in an automorphism \( \varphi_A \) that is identity modulo \( O(x^2) \).

Using the above lemma, we may replace \( \varphi \) with \( \varphi_A \) (and suppress the \( A \) subscript for convenience), thus considering automorphisms which are close to the identity in the formal power series topology.

The next lemma justifies the inductive step: suppose we have managed, by tame left action, to eliminate the terms of degree 2, \( \ldots, k - 1 \), then there is a sequence of elementary automorphisms
such that their left action eliminates the term of degree $k$. This statement translates into the following lemma.

**Lemma 4.2.4.** Let $\varphi$ be a polynomial automorphism such that

$$\varphi(x_1) = x_1 + f_1(x_1, \ldots, x_n) + r_1, \ldots, \varphi(x_n) = x_n + f_n(x_1, \ldots, x_n) + r_n$$

and $f_i$ are homogeneous of degree $k$ and $r_i$ are the remaining terms (thus $ht(r_i) > k$). Then one can find a sequence $\sigma_1, \ldots, \sigma_m$ of tame automorphisms whose composition with $\varphi$ is given by

$$\sigma_m \circ \ldots \circ \sigma_1 \circ \varphi : x_1 \mapsto x_1 + F_i(x_1, \ldots, x_n) + R_1, \ldots, x_n \mapsto x_n + F_n(x_1, \ldots, x_n) + R_n$$

with $F_i$ homogeneous of degree $k + 1$ and $ht(R_i) > k + 1$.

**Proof.** We will first show how to get rid of degree $k$ monomials in the images of all but one generator and then argue that the remaining image is rectified by an elementary automorphism. Let $N \leq n$ be the number of images $\varphi(x_i)$ such that $f_i \neq 0$, and let $x_1$ and $x_2$ be two generators corresponding to non-zero term of degree $k$. The image of $x_1$ admits the following presentation as an element of the polynomial ring $K[x_3, \ldots, x_n][x_1, x_2]$:

$$\varphi(x_1) = x_1 + \sum_{d} \sum_{p+q=d} \lambda_{p,q} x_1^p x_2^q + r_i$$

where the coefficients $\lambda_{p,q}$ are polynomials of the remaining variables (thus the double sum above is just a way to express $f_1$ as a polynomial in $x_1$ and $x_2$ with coefficients given by polynomials in the rest of the variables).

Consider the transformation $\Phi_{\lambda\mu}$ of the following form

$$x_1 \mapsto x_1 - \lambda(x_1 + \mu x_2)^d, \quad x_2 \mapsto x_2 - \lambda \mu^{-1}(x_1 + \mu x_2)^d, \quad x_3 \mapsto x_3, \ldots, \quad x_n \mapsto x_n,$$

with $\lambda \in K[x_3, \ldots, x_n]$ and $\mu \in K$. This mapping is equal to the composition $\psi_\mu \circ \phi_{\lambda\mu} \circ \psi_{\mu}^{-1}$ with

$$\psi_\mu : x_1 \mapsto x_1 + \mu x_2, \quad x_2 \mapsto x_2$$

and

$$\phi_{\lambda\mu} : x_1 \mapsto x_1, \quad x_2 \mapsto x_2 + \lambda \mu^{-1} x_1^d$$

and so is a tame automorphism. As the ground field $K$ has characteristic zero, it is infinite, so that we can find numbers $\mu_1, \ldots, \mu_{(d)}$ such that the polynomials

$$(x + \mu_1 y)^d, \ldots, (x + \mu_{(d)} y)^d$$

3Evidently, no loss of generality results from such explicit labelling.
form a basis of the $\mathbb{K}$-module of homogeneous polynomials in $x$ and $y$ of degree $d$ (this is an easy exercise in linear algebra). Therefore, by selecting $\Phi_{\lambda\mu}$ with appropriate polynomials $\lambda_{p,q}$ and $\mu_i$ corresponding to the basis, we eliminate, by acting with $\Phi_{\lambda\mu}$ on the left, the degree $d$ terms in the double sum. Iterating for all $d$, we dispose of $f_1$ entirely.

The above procedure yields a new automorphism $\tilde{\varphi}$ which is a composition of the initial automorphism $\varphi$ with a tame automorphism. The number $\tilde{N}$ of images of $x_i$ under $\tilde{\varphi}$ with non-zero term of degree $k$ equals $N - 1$; therefore, the procedure can be repeated a finite number of times to give an automorphism $\varphi_1$, such that the image under $\varphi_1$ of only one generator contains a non-zero term of degree $k$. Let

$$\varphi_1(x_n) = x_n + g_n(x_1, \ldots, x_n) + \tilde{r}_n$$

be the image of that generator (again, no loss of generality results from us having labelled it $x_n$). We claim now that the polynomial $g_n$ does not depend on $x_n$.

Indeed, otherwise the Jacobian of $\varphi_1$ (which must be a constant and is in fact equal to 1 in our setting) would have a degree $k - 1$ component given by

$$\partial_{x_n} g_n(x_1, \ldots, x_n) \neq 0$$

(remember that by construction $g_1 = \ldots = g_{n-1} = 0$), which yields a contradiction. Note that another way of looking at this condition is that if a polynomial mapping of the form

$$x_1 \mapsto x_1 + H_1(x_1, \ldots, x_n), \ldots, x_n \mapsto x_n + H_n(x_1, \ldots, x_n), \ ht(H_i) > 1$$

is an automorphism, the higher-degree part $(H_1, \ldots, H_n)$ must have traceless Jacobian:

$$\text{tr} \left( \frac{\partial H_i}{\partial x_j} \right) = 0.$$

Finally, since $g_n$ does not contain $x_n$, an elementary automorphism

$$x_1 \mapsto x_1, \ldots, x_{n-1} \mapsto x_{n-1}, \ x_n \mapsto x_n - g_n(x_1, \ldots, x_n)$$

eliminates this term. The lemma is proved.

The last lemma concludes the proof of Theorem 4.2.1 by induction. The proof of the inductive step is essentially a statement that a certain vector space invariant under a linear group action is, in a manner of speaking, big enough to allow for elimination by elements of the group. More precisely, let $T_{n,k}(\mathbb{K})$ be the vector space of all traceless $n$ by $n$ matrices whose entries are homogeneous of degree $k$ polynomials from $\mathbb{K}[x_1, \ldots, x_n]$, and let the group $\text{GL}(n, \mathbb{K})$ act on $T_{n,k}$ as follows: for $A \in \text{GL}(n, \mathbb{K})$ and $v \in T_{n,k}$, the image $A(v)$ is obtained by taking the product matrix $vA^{-1}$ and then performing (entry-wise in $vA^{-1}$) the linear change of variables induced by
A. Then one has the following

**Proposition 4.2.5.** If \( V \subset T_{n,k}(\mathbb{K}) \) is a \( \mathbb{K} \)-submodule invariant under the defined above action of \( \text{GL}(n, \mathbb{K}) \), then either \( V = 0 \) or \( V = T_{n,k}(\mathbb{K}) \).

Properties of similar nature played an important role in [113,114]. The invariance under linear group action will become somewhat more pronounced in the symplectomorphism case.

### 4.3 Approximation by tame symplectomorphisms and lifting to Weyl algebra

We turn to the proof of the more relevant Theorem 4.2.2 on the symplectic tame approximation. The strategy is analogous to the proof of approximation for polynomial automorphisms with unit Jacobian, with a few more elaborate details which we now consider.

The first step of the proof copies the polynomial automorphism case and takes the following form.

**Lemma 4.3.1.** There is a linear transformation \( A \in \text{Sp}(2n, \mathbb{K}) \)

\[
(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n) A
\]

such that its composition \( \sigma_A \) with \( \sigma \) fulfills

\[
\text{ht}(\sigma_A(x_i) - x_i) \geq 2, \quad \text{ht}(\sigma_A(p_i) - p_i) \geq 2
\]

for all \( i \in \{1, \ldots, n\} \).

We now proceed to formulate the inductive step in the proof as the following main lemma.

**Lemma 4.3.2.** Let \( \sigma \) be a polynomial symplectomorphism such that

\[
\sigma(x_i) = x_i + U_i, \quad \sigma(p_i) = p_i + V_i
\]

and \( U_i \) and \( V_i \) are of height at least \( k \). Then there exists a tame symplectomorphism \( \sigma_k \) such that the polynomials \( \tilde{U}_i = (\sigma_k^{-1} \circ \sigma)(x_i) - x_i \) and \( \tilde{V}_i = (\sigma_k^{-1} \circ \sigma)(p_i) - p_i \) are of height at least \( k + 1 \).

**Proof.** In order to establish the inductive step, we are going to need the following classical result (which is a particular case of Corollary 17.21 in Fulton and Harris [96]).

**Lemma 4.3.3.** Suppose \( \mathbb{K} \) is an infinite field, \( A = \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \) is the polynomial algebra with standard \( \mathbb{Z} \)-grading according to the total degree

\[
A = \bigoplus_{d \geq 0} A_d, \quad A_d = \{\text{homogeneous polynomials of total degree } d\}.
\]
Let $V$ be a $\mathbb{K}$-submodule of $A$ invariant under the action of $\text{Sp}(2n, \mathbb{K})$ (given by linear symplectic changes of variables). Suppose $V$ is contained in a given homogeneous component $A_d$. If $V \neq 0$ then $V = A_d$.

We now turn to the proof of the inductive step. Suppose that
\[
\sigma : x_i \mapsto x_i + f_i + P_i, \quad p_j \mapsto p_j + g_j + Q_j
\]
is a polynomial symplectomorphism, where $f_i$ and $g_j$ are degree $k$ components and the height of $P_i$ and $Q_j$ is greater than $k$. The preservation of the symplectic structure by $\sigma$ means that the $k$-th component obeys the following identities:
\[
\{x_i, f_j\} - \{x_j, f_i\} = 0
\]
and
\[
\{p_i, f_j\} - \{p_j, f_i\} = 0
\]
where $\{ , \}$ is the Poisson bracket corresponding to the symplectic form. In the case of standard symplectic structure these identities translate into
\[
\frac{\partial f_i}{\partial p_j} - \frac{\partial f_j}{\partial p_i} = 0, \quad \frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} = 0,
\]
in which one recognizes the condition for an appropriate differential form to be closed. The triviality of affine space cohomology then implies that there exists a polynomial $F(x_1, \ldots, x_n, p_1, \ldots, p_n)$, homogeneous of degree $k + 1$, such that
\[
\frac{\partial F}{\partial p_i} = f_i, \quad \frac{\partial F}{\partial x_i} = g_i;
\]
in this way the $k$-component of a symplectomorphism is generated by a homogeneous polynomial. The tame symplectomorphism group acts on the space of all such generating polynomials (the image of a polynomial is the polynomial corresponding to the $k$-component of the composition with the tame symplectomorphism), and the orbit of this tame action carries the structure of a $\mathbb{K}$-module (one may easily come up with a symplectomorphism corresponding to the sum of two generating polynomials). Therefore this space fulfills the conditions of the previous lemma, which in this case implies that one can, by a composition with a tame symplectomorphism, eliminate the $k$-component. The main lemma, and therefore the Theorem 3.2, is proved.

Once the approximation for the case of symplectomorphisms has been established, we can investigate the problem of lifting symplectomorphisms to Weyl algebra automorphisms. More precisely, one has the following

**Proposition 4.3.4.** Let $\mathbb{K} = \mathbb{C}$ and let $\sigma : P_n(\mathbb{C}) \to P_n(\mathbb{C})$ be a symplectomorphism over complex
numbers. Then there exists a sequence

$$\psi_1, \psi_2, \ldots, \psi_k, \ldots$$

of tame automorphisms of the $n$-th Weyl algebra $W_n(\mathbb{C})$, such that their images $\sigma_k$ in $\text{Aut} P_n(\mathbb{C})$ approximate $\sigma$.

Proof. This is an immediate corollary of Theorem 4.2.2 and the existence of tame subgroup isomorphism [39].

A few comments are in order. First, the quantization of elementary symplectomorphisms is a very simple procedure: one needs only replace the $x_i$ and $p_i$ by their counterparts $\hat{x}_i$ and $\hat{p}_i$ in the Weyl algebra $W_n$. Because the transvection polynomials $f$ and $g$ (in the expressions for elementary symplectomorphisms) depend, as it has been noted, on one type of generators (resp. $p$ and $x$), the quantization is well defined.

Second, as the tame automorphism groups $\text{TAut} W_n(\mathbb{C})$ and $\text{TAut} P_n(\mathbb{C})$ are isomorphic, the correspondence between sequence of tame symplectomorphisms converging to symplectomorphisms and sequences of tame Weyl algebra automorphisms is one to one. The main question is how one may interpret these sequences as endomorphisms of $W_n(\mathbb{C})$.

Our construction shows that these sequences of tame automorphisms may be thought of as (vectors of) power series – that is, elements of

$$\mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n]]^{2n}.$$

The main problem therefore consists in verifying that these vectors have entries polynomial in generators – that is, that the limits of lifted tame sequences are Weyl algebra endomorphisms.

One could take a more straightforward (albeit an equivalent) approach to the lifting of symplectomorphisms by following the prescription of deformation quantization: starting with a symplectic automorphism of the polynomial algebra $A = \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n]$, one constructs a map of $A[[\hbar]]$, the algebra of formal power series (in Planck’s constant $\hbar$), which preserves the star product satisfying Weyl algebra identities. The approximation theory as developed in this text is then a property of the $\hbar$-adic topology. The (algebraically closed version of) Kontsevich Conjecture would then follow if one were to establish a cutoff theorem.

Unfortunately, this naive approach is deficient in the sense that the resulting lifting is not generally canonical with respect to the choice of the converging sequence. One needs a more elaborate strategy to construct the lifting map. One such strategy will be discussed in the next chapter.
Chapter 4. Approximation by tame automorphisms

4.4 Conclusion

We have developed tame approximation theory for symplectomorphisms in formal power series topology. By virtue of the known correspondence between tame automorphisms of the even-dimensional affine space and tame automorphisms of the Weyl algebra, which is the object corresponding to the affine space in terms of deformation quantization, we have arrived at the lifting property of symplectomorphisms. This line of research may yield new insights into endomorphisms of the Weyl algebra, the Dixmier conjecture, and the Jacobian conjecture.

Inspired by this approximation idea, A. Elishev, A. Kanel-Belov and J.-T. Yu [111] provide the augmented automorphisms to prove the Belov-Kontsevich Conjecture. We will show it in the next section.

4.5 Augmented Weyl algebra structure

This section, which is credited to A. Elishev, A. Kanel-Belov and J.-T. Yu [111], is the main idea of augmented Weyl algebra structure and the solution of the Belov-Kontsevich Conjecture. We first state the following theorem.

Theorem 4.5.1. The mappings

\[ \Phi_N : \text{Aut}^\leq_N W_{n,\mathbb{C}} \to \text{Aut}^\leq_N P_{n,\mathbb{C}} \]

induced by \( \Phi \) are morphisms of algebraic varieties.

The proof can be found in [110]. This theorem has an exact (and crucial to our approach) analogue in the setting of the quantized algebras \( W_n \) and \( P_n \).

Thus, this section is devoted to the study of some conjectures arising in connection with Jacobian conjecture (namely, Kontsevich conjecture and related questions), as well as the study of geometric and topological properties of Ind-schemes of automorphisms of polynomial algebras playing a certain role in approaches to solving the above conjecture. The results of this study, in addition to their importance in the context of the recovery problem, Kontsevich conjecture and related issues, are of independent interest.

In order to resolve the symplectomorphism lifting problem and construct the inverse to the homomorphism \( \Phi \), we introduce the augmented and skew augmented Weyl and Poisson algebras.

The augmented, or \( h \)-augmented Weyl algebra \( W_{n,\mathbb{C}}^h \) is defined as the quotient of the free algebra on \( (2n+1) \) indeterminates \( \mathbb{C}\langle a_1, \ldots, a_n, b_1, \ldots, b_n, c \rangle \) by the two-sided ideal generated by elements

\[ a_i a_j - a_j a_i, \quad b_i b_j - b_j b_i, \quad b_i a_j - a_j b_i - \delta_{ij} c, \quad a_i c - c a_i, \quad b_i c - c b_i. \]

The algebra \( W_{n,\mathbb{C}}^h \), in other words, differs from \( W_{n,\mathbb{C}} \) in the form of the commutation relations – in
the case of $W_n^h$, the coordinate-momenta pairs of generators commute into $h$ (which is added as a central variable to the algebra; the variable $h$ thus somewhat resembles the Planck constant) – and one can return to the non-augmented algebra $W_{n,C}$ by specializing the augmentation parameter to $h = 1$. The augmented Poisson algebra, denoted by $P_n^h$, is defined similarly: one adds the variable $h$ to the commutative polynomial algebra of $2n$ generators and endows it with the Poisson bracket defined as:

$$\{p_i, x_j\} = h \delta_{ij}.$$

It can be verified that these new algebras behave in a way almost identical to the one we described in the prequel; in particular, the notions of tame automorphism, tame (modified) symplectomorphism and homomorphism

$$\Phi : \text{Aut}^{\leq N}(W_n^h) \to \text{Aut}^{\leq N}(P_n^h).$$

(defined for a fixed infinite prime) which is identical on the tame points, are present. We also note that the action of any $h$-augmented automorphism (or, correspondingly, symplectomorphism) on $h$ is necessarily a dilation

$$h \mapsto \lambda h$$

where $\lambda$ is a constant. Indeed, the image of $h$ cannot contain monomials proportional to $x_i$ or $p_j$ (otherwise the commutation relations will not hold), and it cannot be a polynomial in $h$ of degree greater than one, as in that case. Also, the proof of the counterpart of the Theorem 4.5.1 is established in a similar fashion.

**Theorem 4.5.2.** The mappings

$$\Phi^h_N : \text{Aut}^{\leq N} W_n^h \to \text{Aut}^{\leq N} P_n^h$$

induced by $\Phi^h$ are isomorphisms of normalized algebraic varieties.

As we shall see, one can prove the counterpart to the Conjecture 1.2.5 for these augmented algebras, and then demonstrate that the specialization to $h = 1$ yields the isomorphism between automorphism groups of the non-augmented algebras. The construction of the augmented version of the isomorphism, however, requires to further modify the algebras by making the commutators between $d_i$ and $x_j$ nonzero for $i \neq j$.

This pair of auxiliary, skew augmented algebras, denoted by $W_n^h[k_{ij}]$ and $P_n^h[k_{ij}]$ (which correspond to augmented Weyl and Poisson algebras, respectively), are defined as follows. Let the augmented Poisson generators be denoted by $\xi_i$ with $1 \leq i \leq 2n$, which we will call the main generators, (the passage from $x_i$ and $p_j$ to $\xi_i$ is made for the sake of uniformity of notation – and in fact, from the viewpoint of the singularity trick which we use below in order to establish canonicity of the lifting, all of these generators are on equal footing, unlike the standard form $x_i$ and $p_j$, for which only symplectic transformations are permitted), and let $[k_{ij}]$ be a skew-symmetric array (a
skew matrix) of central variables. The algebra $P^h_{n,\mathbb{C}}[k_{ij}]$ is generated by $2n$ commuting variables $\xi_i$, the augmentation variable $h$ and the variables $[k_{ij}]$ (thus being the polynomial algebra in these variables); the Poisson bracket is defined on the generators $\xi_i$:

$$\{\xi_i, \xi_j\} = hk_{ij}.$$ 

The bracket of any element with $h$ or with any of the $k_{ij}$ is zero.

The skew version of the algebra $W_{n,\mathbb{C}}$ is defined analogously.

It is easily seen that the new algebras essentially share the positive-characteristic properties with $W_n$ and $P_n$, from which it follows that a mapping

$$\Phi^{hk}: \text{Aut}_{W^h_{n,\mathbb{C}}[k_{ij}]} \rightarrow \text{Aut}_{P^h_{n,\mathbb{C}}[k_{ij}]}$$

analogous to $\Phi$ and $\Phi^h$ can be defined for every infinite prime $[p]$. In a manner identical to the previous section it can be established that this mapping consists of a system of morphisms of the normalized varieties $\text{Aut}^{\leq N}$, thus yielding the skew augmented analogue of Theorems 4.5.1 and 4.5.2.

**Theorem 4.5.3.** The mappings $\Phi^{hk}_N$ are morphisms of normalized varieties.

The plan of the proof of the main theorem goes as follows. Given that in all three considered cases – the non-augmented, the $h$-augmented and the skew augmented case – the Ind-morphism between (the normalizations of) Ind-varieties of automorphisms is well defined, we will examine its properties. In particular, we are going to establish the continuity of the morphism $\Phi^{hk}$ – or, to be more precise, its restriction to a certain subspace – in the power series topology (defined by the choice of grading below). That result, together with the tame approximation and $\Phi^{hk}$ being the identity map on the tame automorphisms – a property which also holds in all three considered cases – will allow us to prove that the lifted limits of tame sequences are independent of the choice of the converging sequence (canonicity of lifting) and then demonstrate that the lifted limits are given by polynomials and not power series, i.e. that the lifted limits are (skew augmented Weyl algebra) automorphisms. Effectively we will establish the skew augmented version of the Kontsevich conjecture, or rather the more relevant isomorphism between subgroups $\text{Aut}_k$ of automorphisms which act linearly on the auxiliary variables $k_{ij}$. Most of the conceptually non-trivial topological machinery – namely, the singularity trick mentioned in the introduction, are employed at this first stage. In fact, the good behavior of the skew augmented algebras with respect to the singularity trick is the sole reason for introducing these algebras in our proof.

Next, we will connect the skew augmented algebras with the $h$-augmented algebras by means of a localization argument. Once this is done, the establishing of canonicity of lifting and polynomial nature of the lifted limits in the $h$-augmented case becomes a fairly straightforward affair.

Lastly, in order to demonstrate that the results for the $h$-augmented algebras imply the Kontsevich isomorphism, we will need to specialize to $h = 1$. The procedure requires some effort,
and in fact extension of the domain for the constructed inverse morphism will be needed. The procedure will finalize the proof.

4.5.1 Continuity of $\Phi^{hk}$ and the singularity trick

We will now study the power series topology induced on subgroups of skew augmented algebra automorphisms which are linear on $k_{ij}$. As usual, the topology is metric, and it is induced by the grading specified according to the following assignment of degrees to the generators:

$$\deg h = 0,$$
$$\deg k_{ij} = 2,$$
$$\deg \xi_i = 1.$$  

Note that since $h$ and $k_{ij}$ appear as products in the commutation relations, one could assign degree two to the augmentation parameter $h$ and degree zero to the skew-form variables $k_{ij}$, in analogy with the case of augmented algebra $P_{h}^{n}$, while essentially preserving the Ind-scheme structure of Aut.

The metric which induces the power series topology is defined as

$$\rho(\varphi, \psi) = \exp(-ht(\varphi - \psi))$$

where

$$\varphi - \psi = (\varphi(\xi_1) - \psi(\xi_1), \ldots, \varphi(\xi_{2n}) - \psi(\xi_{2n}), \ldots)$$

is the algebra endomorphism defined by its images (on $\xi_i$, $h$ and $k_{ij}$), and the height $ht(\varphi)$ of an endomorphism is defined as the minimal total degree $m$ such that in one of the generator images under $\varphi$ a non-zero homogeneous component of degree $m$ exists.

Symbolically, we say that the power series topology is defined via the powers of the augmentation ideal $I$

$$I = (\xi_1, \ldots, \xi_{2n}, h, \{k_{ij}\})$$

just as it is so in the commutative case, when every variable carries degree one.

The system of neighborhoods $\{H_N\}$ of the identity automorphism in Aut $P_{n,\mathbb{C}}^{h}[k_{ij}]$ is defined by setting

$$H_N = \{g \in \text{Aut } P_{n,\mathbb{C}}^{h}[k_{ij}] : g(\eta) \equiv \eta (\text{mod } I^N)\}$$

(here $\eta$ denotes any generator in the set $\{\xi_1, \ldots, \xi_{2n}, h, \{k_{ij}\}\}$, so that elements of $H_N$ are precisely those automorphisms which are identity modulo terms which lie in $I^N$; again, the phrase "mod $I^{Nn}$" is short-hand for the distance as defined above).

Similar notions of grading, topology, and system of standard neighborhoods of a point, are valid for the algebra $W_{n,\mathbb{C}}^{h}[k_{ij}]$ (once the proper ordering of the generators in the chosen set is
The neighborhoods of the identity point for this algebra will be denoted by $G_N$.

The point of introducing the skew algebras $W^h_{n,\mathbb{C}}[k_{ij}]$ and $P^h_{n,\mathbb{C}}[k_{ij}]$ is that a certain singularity analysis procedure (the singularity tricks mentioned in the introduction) can be implemented for these algebras in full analogy with the case of the commutative polynomial algebra processed in our preceding study [118], while on the other hand there seems to be no straightforward way to execute the singularity trick for the algebras $W^h_n$ and $P^h_n$. Furthermore, after adjunction of $k^{-1}_{ij}$ (together with the entries of the inverse matrix) and extension of scalars (the localization procedure) one can embed the $h$-augmented $\mathbb{C}$-algebras $W^h_n$ and $P^h_n$ in the skew augmented algebras over the larger coefficient ring, thus connecting the $h$-augmented automorphisms with the skew augmented ones which are linear on $k_{ij}$, as we shall see below.

We will establish the continuity of the direct morphism $\Phi^h$ and perform the singularity trick in the following stable form.

Consider the algebra $P^h_{n+1,\mathbb{C}}[k_{ij}]$ with $(2n+2)$ main generators $\{\xi_1, \ldots, \xi_{2n}, u, v\}$. Let

$$\text{Aut}_{u,v,k} P^h_{n+1,\mathbb{C}}[k_{ij}]$$

denote the set of all automorphisms $\varphi$ of $P^h_{n+1,\mathbb{C}}[k_{ij}]$ such that:

1. $\varphi(\xi_i) = \xi_i + S_i$, where $S_i$ is a polynomial (in $\xi_i, u, v, h$ and $k_{ij}$) such that its height with respect to $\{\xi_1, \ldots, \xi_{2n}, u, v\}$ is at least two.
2. $\varphi(u) = u, \varphi(v) = v$.
3. $\varphi(k_{ij})$ is a $\mathbb{C}$-linear combination of $k_{ij}$, i.e. $\varphi \in \text{Aut}_k P^h_{n+1,\mathbb{C}}[k_{ij}]$.

Define the grading as before: $\xi_i, u, v$ carry degree one, $h$ carries degree zero, and $k_{ij}$ carry degree two.

Denote by $H^u_{N,k} \subseteq \text{Aut}_{u,v,k} P^h_{n+1,\mathbb{C}}[k_{ij}]$ the subgroups of $\text{Aut}_{u,v,k} P^h_{n+1,\mathbb{C}}[k_{ij}]$ consisting of elements which are the identity map modulo terms of height $N$ with respect to the grading defined above. Also, the definition is repeated for the skew augmented Weyl algebra $W^h_{n+1,\mathbb{C}}[k_{ij}]$; the resulting subgroups are denoted by $G^u_{N,k}$.

The purpose of the singularity trick set up below is the proof of the following result, which establishes continuity of the direct morphism.

**Proposition 4.5.4.** If $\Phi^h$ is the restriction of the direct morphism to $\text{Aut}_k$, then

$$\Phi^h (G^{a,v,k}_N) \subseteq H^u_{N,k}$$

for every $N$.

The singularity trick is essentially a criterion for an automorphism $\varphi$ to be an element of $H^u_{N,k}$, expressed in terms of *asymptotic behavior of certain parametric families* associated to it. The parametric families of automorphisms are constructed from $\varphi$ by conjugating it with $\mathbb{C}$-linear changes of the main generators (the latter are given by the set $\{\xi_1, \ldots, \xi_{2n}\}$). Such parameterized
variable changes are given by \((2n + 2)\) by \((2n + 2)\) matrices \(\Lambda(t)\) with

\[
(\xi_1, \ldots, \xi_{2n}, u, v) \mapsto (\xi_1, \ldots, \xi_{2n}, u, v) \Lambda(t)
\]

representing the action (such transformations of the main generators induce appropriate mappings of \([k_{ij}]\)). Note that if \(\varphi\) is in \(H^u,v,k_N\), then the conjugation by \(\Lambda(t)\) is also in \(H^u,v,k_N\), as the action upon \(u\) and \(v\) is that of \(\Lambda(t) \circ \Lambda(t)^{-1}\).

We are going to examine the behavior of such one-parameter families near singularities of \(\Lambda(t)\).

Suppose that, as \(t\) tends to zero, the \(i\)-th eigenvalue of \(\Lambda(t)\) also tends to zero as \(t^{m_i}, m_i \in \mathbb{N}\).

Let \(\{m_i, i = 1, \ldots, 2n + 2\}\) be the set of degrees of singularity of eigenvalues of \(\Lambda(t)\) at zero. Suppose that for every pair \((i, j)\) the following holds: if \(m_i \neq m_j\), then there exists a positive integer \(M\) such that

\[
either m_iM \leq m_j or m_jM \leq m_i.
\]

We will call the largest such \(M\) the order of \(\Lambda(t)\) at \(t = 0\). As \(m_i\) are all set to be positive integer, the order equals the integer part of \(\frac{\max m_i}{\min m_i}\).

We now formulate the criterion

**Proposition 4.5.5** (Singularity trick). An element \(\varphi \in Aut_{u,v,k} P^h_{n+1;\mathbb{C}[k_{ij}]}\) belongs to \(H^u,v,k_N\) if and only if for every linear matrix curve \(\Lambda(t)\) of order \(\leq N\) the curve

\[
\Lambda(t) \circ \varphi \circ \Lambda(t)^{-1}
\]

does not have a singularity (a pole) at \(t = 0\).

**Proof.** Suppose \(\varphi \in H^u,v,k_N\) and fix a one-parametric family \(\Lambda(t)\). Without loss of generality, we may assume that the first \(2n\) main generators \(\{\xi_1, \ldots, \xi_{2n}\}\) correspond to eigenvectors of \(\Lambda(t)\). If \(\xi_i\) denotes any of these main generators, then the action of \(\Lambda(t) \circ \varphi \circ \Lambda(t)^{-1}\) upon it reads

\[
\Lambda(t) \circ \varphi \circ \Lambda(t)^{-1}(\xi_i) = \xi_i + t^{-m_i} \sum_{l_1 + \cdots + l_{2n} = N} a_{l_1 \cdots l_{2n}} t^{l_1 + \cdots + m_{2n} l_{2n}} P_i(\xi_1, \ldots, \xi_{2n}, h, k_{ij}) + S_i
\]

where \(P_i\) is homogeneous of total degree \(N\) (in the previously defined grading) and the height of \(S_i\) is greater than \(N\). One sees that for any choice of \(l_1, \ldots, l_{2n}\) in the sum, the expression

\[
m_1 l_1 + \cdots + m_{2n} l_{2n} - m_i \geq m_{\min} \sum l_j - m_i = m_{\min} N - m_i \geq 0,
\]

so whenever \(t\) goes to zero, the coefficient will not go to infinity. The same argument applies to higher-degree monomials within \(S_i\).

The other direction is established by contraposition. Assuming \(\varphi \notin H^u,v,k_N\), we need to prove the existence of linear curves with suitable eigenvalue behavior near \(t = 0\) which create singularities via conjugation with the given automorphism.
Suppose first that the image of $\xi_1$ under $\varphi$ possesses a monomial which is not divisible by $\xi_1$ or any $k_{1j}$ ($j \neq 1$). Then one can take $m_1$ and $m_2 < m_1$ such that

$$(N + 1)m_2 \geq m_1 \geqNm_2$$

and set the curve $\Lambda(t)$ to be given by a diagonal matrix with entries $t^{m_1}$, $t^{m_2}$, $t^{m_2}$, ... It is easily checked that conjugation of $\varphi$ by this curve creates a pole at the coefficient of the chosen monomial.

The general case can be reduced to this special case by means of transformations of the form ($\lambda$ and $\delta$ are suitable constants)

$$\begin{align*}
\xi_1 &\mapsto \xi_1 + \lambda u + \delta v, \\
k_{ij} &\mapsto k_{ij}, \quad 1 < i, j \leq 2n, \\
k_{1j} &\mapsto k_{1j} + \lambda k_{2n+1,j} + \delta k_{2n+2,j}, \\
k_{1,2n+1} &\mapsto k_{1,2n+1} + \delta k_{2n+2,2n+1}, \\
k_{1,2n+2} &\mapsto k_{1,2n+2} + \lambda k_{2n+1,2n+2}.
\end{align*}$$

Conjugation with these transformations create in the image of $\xi_1$ under the resulting automorphism a monomial from the previous case. In order to obtain the curve $\Lambda(t)$ from the diagonal curve acting on the conjugated automorphism, one needs only conjugate it with the inverse of the above transform. The singularity trick is proved.

The skew augmented Weyl algebra counterpart of the singularity trick is valid.

**Corollary 4.5.6.** An element $\varphi \in \text{Aut}_{u,v,k} W_{n+1,C}^h [k_{ij}]$ belongs to $G^{u,v,k}_N$ if and only if for every linear matrix curve $\Lambda(t)$ of order $\leq N$ the curve

$$\Lambda(t) \circ \varphi \circ \Lambda(t)^{-1}$$

does not have a singularity at $t = 0$.

The proof of this statement is essentially the same as that of Proposition 4.5.5. Note that thanks to the choice of grading – the one in which the degree of all $k_{ij}$ is two – the reordering of the non-commuting variables in a word cannot produce monomials of smaller total degree.

The implementation of the singularity trick in the proof of Proposition 4.5.4 requires also the following general fact.

**Lemma 4.5.7.** Let

$$\Phi : X \to Y$$
be a morphism of affine algebraic sets, and let $\varphi(t)$ be a curve (more simply, a one-parameter family of points) in $X$. Suppose that $\varphi(t)$ does not tend to infinity as $t \to 0$. Then the image $\Phi \varphi(t)$ under $\Phi$ also does not tend to infinity as $t \to 0$.

The proof of the Lemma is an easy exercise and is left to the reader.

Proposition 4.5.4 is now an elementary consequence of the above Lemma together with the singularity trick (Proposition 4.5.5 and Corollary 4.5.6). Indeed, let us assume the contrary – i.e. that for some $N$

$$
\Phi^{hk}(G_N^{u,v,k}) \not\subseteq H_N^{u,v,k}.
$$

Then there exists an element $\varphi \in G_N^{u,v,k}$ such that its image $\Phi^{hk}(\varphi) \not\in H_N^{u,v,k}$. By Proposition 4.5.5, there is a linear automorphism (matrix) curve $\Lambda(t)$ of order $\leq N$ such that the curve

$$
\Lambda(t) \circ \Phi^{hk}(\varphi) \circ \Lambda(t)^{-1}
$$

has a pole at $t = 0$. Since $\Phi^{hk}$ is point-wise stable on linear variable changes, the latter curve is the image under $\Phi^{hk}$ of the curve

$$
\Lambda(t) \circ \varphi \circ \Lambda(t)^{-1}.
$$

By our assumption, $\varphi \in G_N^{u,v,k}$; therefore, by Corollary 4.5.6, the curve above has no singularity at $t = 0$. But then the statement that the curve

$$
\Lambda(t) \circ \Phi^{hk}(\varphi) \circ \Lambda(t)^{-1}
$$

– which is the image of the former curve under the morphism $\Phi^{hk}$ – has a singularity at $t = 0$ yields a contradiction with Lemma 4.5.7. Proposition 4.5.4 is proved.

The immediate consequence of Proposition 4.5.4 is the following result.

**Theorem 4.5.8.** The mapping

$$
\Phi^{hk} : \text{Aut}_{u,v,k} W^h_{n,\mathbb{C}}[k_{ij}] \to \text{Aut}_{u,v,k} P^h_{n,\mathbb{C}}[k_{ij}]
$$

is continuous in the power series topology defined at the start of the section.

This was the main objective of the singularity trick, and this result will provide the means to establish the canonicity of the symplectomorphism lifting procedure we define in the next subsection.

### 4.5.2 Lifting in the $h$-augmented and skew augmented cases

We now proceed with the resolution of the symplectomorphism lifting problem for both augmented and skew augmented algebras. More specifically, we will show how the results of the singularity analysis procedure conducted in the previous subsection provide for a way to construct the inverse to the homomorphisms $\Phi^h$ and $\Phi^{hk}$. 

114
Suppose given an automorphism \( \varphi \in \text{Aut} P_{h,n}^h \) of the \( h \)-augmented Poisson algebra. Without loss of generality, we may assume that the linear part of \( \varphi \) is the identity matrix: indeed, one can compose \( \varphi \) with tame automorphisms (tame approximation of automorphisms of \( P_{n,C}^h \) is valid according to an argument similar to that of [112]), so that the linear part of the resulting automorphism is the identity map; also the morphism \( \Phi^h \) is point-wise stable on tame automorphisms.

We add two more \( h \)-Poisson variables (and lift \( \varphi \) to an automorphism of the new algebra by demanding it be stable on the new generators) and, correspondingly, consider the skew Poisson version – the algebra \( P_{n+1,C}^h[k_{ij}] \) with the last two variables denoted by \( u \) and \( v \). Our objective is to realise the algebra \( P_{n,C}^h \) as a subalgebra in an appropriate localization of \( P_{n+1,C}^h[k_{ij}] \). To that end, we consider the algebra \( P_{n+1,C}^h[k_{ij}] \) and transform the main generators

\[
\{\xi_1, \ldots, \xi_{2n}, u, v\}
\]

to

\[
\{x_1, \ldots, x_{2n}, u, v\}
\]

with \( \{x_i, u\} = 0 \) and \( \{x_i, v\} = 0 \). The change of the generating set is required to properly define the action of \( \varphi \), so that it will be an automorphism and will be in agreement with the conditions of Proposition 4.5.4. The variable change is done according to

\[
x_i = \xi_i - \alpha_i u - \beta_i v
\]

with \( \alpha_i = k_{i,2n+2}k_{2n+1,2n+2}^{-1} \) and \( \beta_i = -k_{i,2n+1}k_{2n+1,2n+2}^{-1} \) for \( i = 1, \ldots, 2n \). We extend the coefficient ring by adding the necessary variables. The new generators \( \{x_1, \ldots, x_{2n}\} \) commute according to

\[
\{x_i, x_j\} = h(k_{ij} - \alpha_j k_{i,2n+1} + \alpha_i k_{j,2n+1} - \beta_j k_{i,2n+2} + \beta_i k_{j,2n+2} + (\alpha_i \beta_j - \alpha_j \beta_i)k_{2n+1,2n+2}) = h\tilde{k}_{ij}.
\]

Note that the new commutation relation matrix, which we denote by \( [\tilde{k}_{ij}]^4 \), is again skew-symmetric, and that its entries are \( \mathbb{C} \)-polynomial in the entries of the initial matrix and their inverses.

We now reduce the matrix \( [\tilde{k}_{ij}] \) to the standard form (corresponding to the algebra \( P_n^h \)) by transforming \( \{x_1, \ldots, x_{2n}\} \) to \( \{q_1, \ldots, q_n, p_1, \ldots, p_n\} \) with

\[
\{p_i, q_j\} = h\delta_{ij}.
\]

The new variables \( p_i \) and \( q_j \) are expressed as linear combinations of \( x_1, \ldots, x_{2n} \) with coefficients in the appropriate polynomial ring.

\(^4\)We exclude \( u \) and \( v \), so that \( i \) and \( j \) run from 1 to 2n.
The algebra $P^h_{n,C}$ is therefore a subalgebra of the algebra generated by 

$$\{q_1, \ldots, q_n, p_1, \ldots, p_n, u, v\}$$

(together with $h$ as the augmentation variable), as the Poisson bracket takes its proper form after the standard form reduction, while $C$ is a subring of the coefficient ring.

We extend our automorphism $\varphi$ to act on this algebra: on $p_i$ and $q_j$ its action is given by definition, and we impose $\varphi(u) = u$, $\varphi(v) = v$ and $\varphi(k_{ij}) = k_{ij}$. Thus, starting from $\varphi$ we have arrived at an automorphism $\tilde{\varphi}$ of the localized skew Poisson algebra.

With respect to the skew augmented algebra generator set $\{\xi_1, \ldots, \xi_{2n}, u, v\}$ this automorphism is generally not polynomial in $k_{ij}$, although it always will be polynomial in $h$. In order to construct from it an automorphism of the skew algebra, we need to get rid of the denominators first. This is accomplished by the following lemma.

**Lemma 4.5.9.** For every $\tilde{\varphi}$ constructed as above, there is a polynomial $P$ in $k_{ij}$, such that conjugation of $\tilde{\varphi}$ with the transformation

$$(\xi_1, \ldots, \xi_{2n}, u, v) \mapsto (P\xi_1, \ldots, P\xi_{2n}, Pu, Pv), \ h \mapsto P^2 h$$

is polynomial in $k_{ij}$. The polynomial $P$ depends only on the two systems of algebra generators.

**Proof.** Indeed, the denominators in the expression for $\tilde{\varphi}$ are polynomial in $k_{ij}$ coming from the separation of the $(u, v)$-plane and the standard form reduction (at which point the determinant of $[\tilde{k}_{ij}]$ makes its contribution). One can therefore find appropriate $P(k_{ij})$ to cancel these denominators. Furthermore, the polynomial $P$ depends only on the two generator systems – more specifically, on the transformation matrix between those systems.

We denote the result of the conjugation of Lemma 4.5.9 by $\varphi^P$. The images of the main generators (both in the cases of the initial – skew – generators as well as those which correspond to the standard form) under $\varphi^P$ are, by Lemma 4.5.9, polynomial in $k_{ij}$, and are also by construction polynomial in $h$.

The automorphism $\varphi^P$, when acting upon the standard form generators

$$\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$$

can be viewed as an automorphism of the $h$-augmented Poisson algebra $P^h_{n,C[\{hk_{ij}\}]}$ over the polynomial ring $C[\{hk_{ij}\}]$. The $\mathbb{Z}$-grading of this algebra is specified by assigning degree 1 to the main generators and degree 0 to $h$ and all $k_{ij}$. As an automorphism of this Poisson algebra, $\varphi^P$ admits, by an argument virtually identical to the main result of [112], a tame automorphism (symplectomorphism) sequence converging to it in the power series topology induced by the above grading. Let us fix such a sequence and denote it by $\{\psi_m\}$. 

116
Every element $\psi_k$ of the tame sequence is such that the images under $\psi_m$ of the generators $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ are polynomial in $h$ and $k_{ij}$. Importantly, the tame sequence $\{\psi_m\}$ converging to the $\mathbb{C}[\{hk_{ij}\}]$-automorphism $\varphi^P$ can be connected to the original $h$-augmented symplectomorphism $\varphi$ by inversion of the procedure which leads to the definition of $\varphi^P$. Precisely, we have the following statement.

**Proposition 4.5.10.** For every $\varphi$ and every sequence $\{\psi_m\}$ converging to $\varphi^P$ as above, there is a sequence $\{\sigma_m\}$ of tame $\mathbb{C}$-symplectomorphisms of $P_{n,\mathbb{C}}$ converging to $\varphi$ with respect to the topologies with $\deg h = 0$ and $\deg h = 2$.

**Proof.** The sequence $\{\sigma_m\}$ is constructed from $\{\psi_m\}$ by reversing the conjugation by $P$ and disposing of the stable variables $u$ and $v$. We note that the conjugation is a group homomorphism, which means that it suffices to prove that the reverse conjugation disposes of $k_{ij}$ in every elementary tame automorphism (as $\psi_m$ are composition of elementary automorphisms). The latter property, however, is obvious.

The convergence of $\{\sigma_m\}$ follows immediately from Lemma 4.5.9 and Lemma 4.5.11 below.

When acting upon the localized skew Poisson algebra generators $\{\xi_1, \ldots, \xi_{2n}\}$, the augmented symplectomorphisms $\psi_m$ need not be polynomial in $k_{ij}$, and therefore $\psi_m$ are not in general images of automorphisms of the skew Poisson algebra under localization. This is remedied by application of Lemma 4.5.9: one can find a polynomial $P_1$ in the variables $k_{ij}$, such that the conjugation of every element $\psi_m$ of the tame sequence with the mapping

$$(\xi_1, \ldots, \xi_{2n}, u, v) \mapsto (P_1\xi_1, \ldots, P_1\xi_{2n}, P_1u, P_1v), \quad h \mapsto P_1^2h$$

yields an automorphism of the localized skew Poisson algebra polynomial in $k_{ij}$. Again, as in Proposition 4.5.10, one can return to a sequence of tame symplectomorphisms of $P_{n,\mathbb{C}}$ by reversing the conjugation.

We then have the following statement.

**Lemma 4.5.11.** The sequence $\{\psi_m^{P_1}\}$ converges to the conjugated automorphism $(\varphi^P)^{P_1}$ in the power series topology with $\deg h = \deg k_{ij} = 0$ as well as in the power series topology with $\deg h = 0$, $\deg k_{ij} = 2$.

**Proof.** The first half of the statement follows from the construction of the tame sequence and from the observation that, due to the fact that the two coordinate systems are connected by a transformation that has zero free term, the height of the polynomials $P$ and $P_1$ is at least one.

One then obtains convergence in the power series topology relevant to the singularity trick (Proposition 4.5.4) from that in the approximation power series topology by noting that giving a non-zero degree to $k_{ij}$ may only make the consecutive approximations closer to the limit in the corresponding metric.

\[\square\]
The sequence \( \{ \psi_m \} \) can, due to its polynomial character with respect to \( k_{ij} \), be thought of as a sequence of tame automorphisms of the skew Poisson algebra \( P_{n+1, \mathbb{C}}^{h}[k_{ij}] \) over \( \mathbb{C} \) converging to \( (\varphi^P)_n \). We now take the pre-image of the sequence \( \{ \psi_m \} \) under the morphism \( \Phi^{hk} \) to obtain the sequence

\[
\{ \sigma'_m \} = \{ (\Phi^{hk})^{-1}(\psi_m) \}
\]

of automorphisms of the skew augmented Weyl algebra \( W_{n+1, \mathbb{C}}^{h}[k_{ij}] \). We now may take the formal limit of this sequence, which is ostensibly dependent on the choice of the convergent tame sequence \( \{ \psi_m \} \), apart from the point \( \varphi \) itself. This limit, which we denote by

\[
\Theta^{hk}_{\varphi}(\varphi, \{ \psi_m \})
\]

to reflect the dependence on the sequence, is given by formal power series in the skew augmented Weyl generators. Applying the inverse to the conjugations performed earlier and disposing of the stable variables (whose presence is justified by the form of the singularity trick and is therefore needed in the proof of independence of the choice of the convergent sequence, as we shall see below), we arrive at a vector of formal power series (the entries of which correspond to images of the generators) in the generators of the \( h \)-augmented Weyl algebra \( W_{n, \mathbb{C}}^{h} \).

The most important consequence of Theorem 4.5.8 is the independence of the lifted sequence’s formal limit of the choice of the approximating tame sequence \( \{ \psi_m \} \). We have the following proposition.

**Proposition 4.5.12.** Let \( \varphi \) be an automorphism of \( P_{n, \mathbb{C}}^{h} \) and let

\[
\psi_1, \ldots, \psi_m, \ldots
\]

and

\[
\psi'_1, \ldots, \psi'_m, \ldots
\]

be two sequences of tame automorphisms which converge to \( \varphi^P \) as in the construction above. Then the lifted sequences

\[
\{ (\Phi^{hk})^{-1}(\psi_m^P) \} \quad \text{and} \quad \{ (\Phi^{hk})^{-1}(\psi'_m^P) \}
\]

converge to the same automorphism of the power series completion of the skew augmented Weyl algebra. This means that one must have

\[
((\Phi^{hk})^{-1}(\psi_m^P))^{-1} \circ (\Phi^{hk})^{-1}(\psi'_m^P) \equiv \text{Id} \pmod{I^{N(k)}}
\]

with \( N(k) \to \infty \) as \( k \to \infty \).

**Proof.** This result follows immediately from the continuity of \( \Phi^{hk} \) established in the previous subsection. \( \square \)
The meaning of this Proposition to the symplectomorphism lifting problem is clear: in our construction, the formal limit
\[ \Theta_h^p(\varphi, \{\psi_m\}) \]
is independent of \( \{\psi_m\} \) and is therefore a well-defined function of the point \( \varphi \). Furthermore, as can be inferred directly from the tame approximation, this function is homomorphic – it preserves the group structure given by composition of automorphisms. As the conjugations are also homomorphisms, we conclude that \( h \)-augmented symplectomorphisms \( \varphi \in \text{Aut} P^h_{n,\mathbb{C}} \) are lifted homomorphically to endomorphisms of the power series completion \( \hat{W}^h_{n,\mathbb{C}} \) of the \( h \)-augmented Weyl algebra. For an augmented symplectomorphism \( \varphi \), we denote its image with respect to the lifting map by
\[ \Theta^h(\varphi). \]

Our next objective is to demonstrate that for every symplectomorphism \( \varphi \), the image \( \Theta^h(\varphi) \) under the lifting map is in fact an automorphism of the \( h \)-augmented Weyl algebra. In fact, it remains to show only that the generator images with respect to \( \Theta^h(\varphi) \) cannot be given by infinite series: indeed, that would imply that \( \Theta^h(\varphi) \) is an \( h \)-augmented Weyl endomorphism; the invertibility of the lifted mapping follows from the canonicity of lifting: indeed, \( \Theta^h \) not only preserves compositions but also maps inverses to inverses, therefore for any symplectomorphism \( \varphi \) the mapping \( \Theta^h(\varphi^{-1}) \) will be the inverse of \( \Theta^h(\varphi) \).

Alternatively, one can arrive at the invertibility after one shows the polynomial character of the lifted endomorphism: it is known \([40, 188]\) that the direct morphism \( \Phi \) – and hence, by a straightforward extension of the argument in the aforementioned work, its augmented analogue \( \Phi^h \) – distinguishes automorphisms, i.e. the image of a non-automorphism cannot be an automorphism.

The main properties of the mapping \( \Theta^h \) can now be summarized in the following way.

**Proposition 4.5.13.** 1. There exists a well-defined mapping \( \Theta^h \) whose domain is \( \text{Aut} P^h_{n,\mathbb{C}} \) and whose codomain lies in the set of automorphisms of the power series completion of \( W^h_{n,\mathbb{C}} \).

2. \( \Theta^h \) is a group homomorphism.

3. For a fixed \( \varphi \), the coordinates of \( \Theta^h(\varphi) \) – i.e. the coefficients with respect to the fixed generator basis decomposition – are given by polynomials in the coordinates of \( \varphi \).

4. The skew augmented analogue \( \Theta^hk \) of \( \Theta^h \) is continuous in the power series topology.

**Proof.** The first two statements follow immediately from the construction. The third statement follows from the fact that the lifting is independent of the approximating sequence: indeed, that implies that the coefficients of the lifted limit are read off any valid approximating sequence, or more precisely its finite subset (consisting of first several elements, as the limit symplectomorphism is polynomial). But then the coefficients are polynomial in the coordinates of the lifted tame elements, and since only a finite number of them suffices, they are also polynomial in the coordinates of the initial symplectomorphism.

The continuity of \( \Theta^hk \) is established in a manner identical to that of \( \Phi^hk \) – namely with the
help of Proposition 4.5.5 a proof similar to that of Theorem 4.5.8 can be executed. Note that it follows from the third statement of this Proposition that $\Theta^{hk}$ fulfills the conditions of Lemma 4.5.7, therefore by combining the previously proved statements with the analogous steps for the lifting map, one can show that

$$\Phi^{hk}(G^{u,v,k}_N) = H^{u,v,k}_N.$$  

\[ \square \]

4.5.3 The lifted limit is polynomial

We proceed with establishing the polynomial character of the image $\Theta^h(\varphi)$.

**Theorem 4.5.14.** Let

$$\Theta^h : \text{Aut} P^n_{n,C} \to \text{Aut} \hat{W}^n_{n,C}$$

be the lifting homomorphism constructed in the previous section and let, as before, $x_1, \ldots, x_n, d_1, \ldots, d_n, h$ denote the generators of $W^n_{n,C}$ together with its deformation parameter. Then, for every augmented symplectomorphism $\varphi \in \text{Aut} P^n_{n,C}$, the images

$$\Theta^h(\varphi)(x_1), \ldots, \Theta^h(\varphi)(d_n)$$

are polynomials in $x_i, d_i$ and $h$.

**Proof.** Suppose that, contrary to the statement of the theorem, for a fixed $\varphi$ there is an index $i$ such that, say,$^5$ $\Theta^h(\varphi)(x_i)$ is a true infinite series of Weyl monomials.

Let $\lambda$ be a parameter and let

$$\tau_\lambda : (x_1, \ldots, d_n, h) \mapsto (\lambda x_1, \ldots, \lambda d_n, \lambda^2 h)$$

denote the family of dilation transformations parameterized by $\lambda$. For fixed $\varphi$, define

$$\varphi_\lambda = \tau_\lambda^{-1} \circ \varphi \circ \tau_\lambda$$

to be the parametric family of $h$-augmented symplectomorphisms constructed by conjugating $\varphi$ with the dilations.

We introduce a pair of auxiliary variables $u$ and $v$, $\{v, u\} = h$ (their Weyl counterparts will be also denoted by $u$ and $v$ – obviously it does not create any ambiguities) and, for a fixed large enough positive integer $k$, define the following parametric family of linear transformations

$$\psi_\lambda : u \mapsto u + \lambda^k x_i, \quad p_i \mapsto p_i - \lambda^k v$$

(all other generators are unchanged). As always, we extend the action of $\varphi$ to the auxiliary

---

$^5$The case for $d_i$ is processed analogously.
variables by setting $\varphi(u) = u$ and $\varphi(v) = v$ (while the dilation extends to $(u, v) \mapsto (\lambda u, \lambda v)$). Consider the following parametric family of $h$-augmented symplectomorphisms:

$$\varphi_{t,\lambda} = \varphi_{\lambda} \circ \psi_{\lambda} \circ \varphi_{\lambda}^{-1}.$$ 

The conjugation of $\varphi$ by the inverse to the dilation $\tau_{\lambda}$ amounts to multiplying each homogeneous component of degree $m$ by $\lambda^{1-m}$. Therefore (as $\varphi$ is polynomial) for large enough $k$, the curve $\varphi_{t,\lambda}$ can be continuously extended by its limit at $\lambda = 0$ – namely, by the identity symplectomorphism. Continuity is understood in the sense of continuous dependence of coordinates of the symplectomorphism on the parameter.

Now, the image of $u$ under the lifted curve $\Theta^h(\varphi_{t,\lambda})$ is

$$u + \lambda^k \Theta^h(\varphi_{\lambda})(x_i)$$

and, as by assumption $\Theta^h(\varphi)(x_i)$ is an infinite series, is itself an infinite series. But since $\Theta^h$ is identical on linear transformations, we have

$$\Theta^h(\varphi_{\lambda}) = \tau_{\lambda}^{-1} \circ \Theta^h(\varphi) \circ \tau_{\lambda}$$

from which it follows that in the image $\Theta^h(\varphi_{t,\lambda})(u)$ there will be monomials with coefficients proportional to $\lambda^{-m}$ for $m$ greater than any fixed arbitrary natural number.

However, it follows from the third statement in Proposition 4.5.13 that the coordinates (i.e. coefficients of Weyl monomials) of the lifted symplectomorphism are continuous functions of the coordinates of the symplectomorphism, therefore since the curve $\varphi_{t,\lambda}$ is regular at $\lambda = 0$, then so must also be its image under $\Theta^h$ – a contradiction. \hfill $\square$

We can now combine this theorem with the results of the previous subsection.

**Theorem 4.5.15.** The lifting homomorphism $\Theta^h$ is the inverse to the direct homomorphism $\Phi^h$.

*Proof.* Indeed, Theorem 4.5.14 shows that the compositions $\Phi^h \circ \Theta^h$ and $\Theta^h \circ \Phi^h$ are well defined. In order to prove that, say, $\Phi^h \circ \Theta^h = \text{Id}$ one changes the basis of generators (and handles the extension of the base ring as in the prequel) to that of the skew augmented algebra and uses the fact that $\Phi^{hk} \circ \Theta^{hk}$ coincides with the identity map on the dense subset of tame symplectomorphisms and hence must be the identity map everywhere (the spaces in question are metric spaces, in particular they are Hausdorff). \hfill $\square$

This theorem has one important corollary.

**Corollary 4.5.16.** The lifting map $\Theta^h$ and the direct map $\Phi^h$ are consistent with modulo infinite prime reductions. That is, for any symplectomorphism $\varphi$, almost all its modulo $p_m$ (in the index set in the ultraproduct decomposition) reductions coincide with the (twisted by inverse Frobenius) restrictions to the center of the modulo $p_m$ reductions of its lifting $\Theta^h(\varphi)$.
Proof. This statement is essentially a reformulation of Theorem 4.5.15, if one takes into account the construction of $\Phi_h$ – the image of the Weyl algebra automorphism being reconstructed from the ultraproduct of positive characteristic automorphisms restricted to the center.

□

Remark 4.5.17. Alternatively, one could come up with a line of reasoning more conforming to the combinatorial side of the constructions employed thus far. If there were a symplectomorphism $\varphi$ which, after lifting and subsequent direct homomorphism action (ultraproduct decomposition followed by restriction to the center) produces a different symplectomorphism $\varphi'$, then one may take a sequence of elementary symplectomorphism whose total action on $\varphi$ maps it to an element which is the identity map modulo terms of degree $N$. If $\varphi' \neq \varphi$, then the action of the same elementary sequence on $\varphi'$ will produce an element which admits a term of degree $1 < M < N$. The two objects are then mapped to automorphisms of the skew augmented algebra and lifted (we also note that the mapping induced by the change of basis is continuous, as it is in essence a dilation of generators by a polynomial of height at least one). By the singularity trick (Proposition 4.5.5) such an object can be conjugated by an appropriate linear variable change in order to produce a singular curve. Now, by construction, the skew version of $\varphi$ lifts to a skew Weyl automorphism, and again by the singularity trick (essentially by the continuity of $\Theta^{hk}$) the lifting of the partially approximated automorphism (i.e. after the action of the elementary automorphisms) cannot have a singularity of order $\leq N$. However, as the skew version of $\varphi$ and $\varphi'$ (as well as its partial approximation) corresponds in the ultraproduct to restrictions to the center, the restriction of an object which is not singular of order $\leq N$ must also be non-singular of order $\leq N$, in contradiction with the existence of $\varphi'$.

The $h$-augmented counterpart of the Kontsevich conjecture follows at once from Theorem 4.5.15.

Theorem 4.5.18. The homomorphism

$$\Phi^h : \text{Aut}_{n,J} P^h_{n,J} \to \text{Aut}_{n,J} W^h_{n,J}$$

is an isomorphism.

As the map $\Phi^h$ is closely related to the morphism $\Phi^{hk}$ of the skew augmented case, and, correspondingly, as $\Theta^h$ is related to the lifting map for the skew augmented symplectomorphisms (essentially given by $\Theta^h_{P}$), we obtain another important consequence of Theorem 4.5.14.

Theorem 4.5.19. Let $\text{Aut}_{k,J} P^h_{n,J}[k_{ij}]$ and $\text{Aut}_{k,J} W^h_{n,J}[k_{ij}]$ denote the automorphism subgroups of the skew Poisson and Weyl algebras consisting of those automorphisms that map $k_{ij}$ to $C$-linear combinations of $k_{ij}$. Then the mapping

$$\Phi^{hk} : \text{Aut}_{k,J} W^h_{n,J}[k_{ij}] \to \text{Aut}_{k,J} P^h_{n,J}[k_{ij}]$$
is an isomorphism.

Theorem 4.5.19 is significant to the proof of the independence of the morphism $\Phi$ of the choice of infinite prime given in [110].

### 4.5.4 Specialization

Now that we have established the isomorphism between the automorphism groups of the $h$-augmented Weyl and Poisson algebras, the proof of the Main Theorem reduces to specializing to $h = 1$. That, however, is by no means a trivial affair, as one needs to take care not only of the automorphisms polynomial in $h$ (for which the existence of lifting has been established), but also of those which are polynomial in $h^{-1}$.

The necessity of extension of the domain of the lifting map can be seen from the following argument. Suppose $\varphi^h$ is an automorphism of the $h$-augmented algebra $P_n^h$, which acts as the identity map on $h$. Since it is stable on $h$, it corresponds to an automorphism of the $\mathbb{C}[h]$-algebra (where $h$ is a parameter and not a generator, which can be effectively adjoined to the ground field) generated by $x_i$, $p_j$ with the Poisson bracket containing $h$. This object, after appropriate localization, maps to an automorphism of the (augmented) Poisson algebra $P_n^h$ with the ground field $\mathbb{C}(h)$. On the other hand, any automorphism $\varphi$ of $P_n$, which can be made into a $\mathbb{C}(h)$-automorphism $\varphi^h$ by introducing a scalar $h$ and conjugating $\varphi$ with a mapping

$$x'_i = hx_i, \quad p'_j = p_j.$$ 

The resulting transformation will be an automorphism of the Poisson $\mathbb{C}(h)$-algebra with the bracket as in the augmented algebra $P_n^h$, however in general the images of the generators under this automorphism will contain negative powers of $h$. Its specialization to $h = 1$ returns it to $\varphi$. Therefore, every polynomial symplectomorphism has a pre-image under specialization of the $\mathbb{C}(h)$-algebra automorphisms. The conclusion is that Theorem 4.5.18 does not immediately imply that $\Phi$ is an isomorphism; rather, the domain of the lifting map $\Theta^h$ needs to be extended to the points with rational dependency on the augmentation parameter, at which point the claim that $\Phi$ has an inverse given by the specialization of the extended lifting map $\Theta^h$ becomes valid.

The extension of the domain is accomplished in the following way. For a symplectomorphism $\varphi$ which is rational in $h$, we will construct images $\Theta^h(\varphi)(x_i)$ and $\Theta^h(\varphi)(d_i)$ one by one by introducing auxiliary variables and twisting the symplectomorphism in order to create an object polynomial in $h$ – using the fact that the action of $\Theta^h$ is well defined – from which the form of the corresponding lifted generator image may be extracted. As the procedure yields not all of the images simultaneously, we will need to check its canonical nature as well as verify the commutation relations.

We fix $i$, $1 \leq i \leq n$, which corresponds to the image of $x_i$, and introduce a pair $u, v$ of auxiliary variables which are extra $x$ and $p$ with respect to the augmented Poisson bracket. We also add
the corresponding augmented Weyl variables which we denote by \( \hat{u} \) and \( \hat{v} \). Let

\[
\lambda = h^k
\]

be \( k \)-th power of the augmentation parameter, for large enough \( k \). Define the automorphism

\[
\psi_\lambda : u \mapsto u + \lambda x_i, \quad p_i \mapsto p_i - \lambda v.
\]

We extend \( \varphi \) to the new algebra by its identical action on the auxiliary variables and denote the extended map by \( \varphi_a \). Consider the following twisted automorphism:

\[
\varphi_{t,\lambda} = \varphi_a \circ \psi_\lambda \circ \varphi_a^{-1}.
\]

As \( k \) can be taken arbitrarily large, the mapping \( \varphi_{t,\lambda} \) will be polynomial in \( h \) for all \( k > k_0 \) (where \( k_0 \) depends on \( \varphi \) but is finite for the fixed automorphism). We can now read off the expression for the image of \( x_i \) under \( \varphi \) from the action of \( \varphi_{t,\lambda} \) on the auxiliary variable \( u \):

\[
\varphi_{t,\lambda}(u) = u + h^k \varphi(x_i);
\]

the expression is polynomial in \( h \), and \( \varphi_{t,\lambda} \) thus admits lifting to an automorphism of the \( h \)-augmented Weyl algebra \( W_{n,C}^h \). As we will show in a moment, the action of the lifted automorphism on \( \hat{u} \) will be given by the expression

\[
\hat{u} + h^k P_i(x_1, \ldots, d_n, h)
\]

(\( P_i \) is polynomial in \( x_1, \ldots, d_n \) and rational \– or, more precisely, Laurent-polynomial \– in \( h \)) so that one can set

\[
\hat{\varphi}(x_i) = P_i(x_1, \ldots, d_n, h)
\]

and thus, for all \( x_i \), obtain the action of the lifted symplectomorphism. Switching the roles of \( x_i \) and \( d_i \) allows for reconstruction of the images of \( d_i \). As a result, we get a mapping

\[
\hat{\varphi} : (x_1, \ldots, d_n) \mapsto (P_1(x_1, \ldots, d_n, h), \ldots, Q_n(x_1, \ldots, d_n, h)).
\]

Note, however, that as the lifting is not defined for the components in the composition (with the exception of \( \psi_\lambda \)), one cannot immediately conclude that the image of \( \hat{u} \) under the lifted map will be of the form as above, or that the parts which depend on \( x_1, \ldots, d_n \) will combine to a well defined automorphism \– these properties need to be verified.

The first step is to ensure the constructed mapping \( \hat{\varphi} \) is well defined (is canonical with respect to \( \varphi \)). This property in fact follows from the consistency with modulo infinite prime reductions given by Corollary 4.5.16, and is manifested in the form of the next two lemmas.
Lemma 4.5.20. Suppose \( \theta \) is an \( h \)-augmented polynomial symplectomorphism over \( \mathbb{C} \). Denote by \( \{\theta_p\} \) the sequence of characteristic \( p \) symplectomorphisms representing its modulo \([p]\) reduction. For a generic element \( p \) in a sequence representing \([p]\), denote the Weyl generators by \( x_1, \ldots, x_n, d_1, \ldots, d_n \) and the corresponding \( p \)-th powers generating the center of the Weyl algebra over \( \mathbb{F}_p \) by \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \). Then, for almost all \( p \) in \([p]\) (in the sense of the ultrafilter), the image under \( \theta_p \) of every central generator admits a unique pre-image Weyl polynomial \( \hat{H} \) with respect to taking the \( p \)-th power and pulling back the coefficients by the inverse Frobenius automorphism.

Proof. We prove the statement for \( H = \theta_p(\xi_i) \) – the case \( \eta_j \) is identical.

Suppose first that

\[
\theta_p(\xi_i) = \xi_i = x_i^p.
\]

Then the Newton polyhedron of the image \( \theta_p(\xi_i) \) has only one vertex, therefore – as taking the \( p \)-th power only dilates the Newton polyhedron – the polynomial \( \hat{\theta}_p(x_i) \) must be equal to \( x_i \).

The general case uses Corollary 4.5.16, which states that modulo \([p]\) reductions of \( \theta \) and its lifting \( \hat{\theta} \) are consistent – that is, for almost all \( p \) in \([p]\), the restriction of \( \hat{\theta}_p \) to the center (twisted by the inverse Frobenius acting on the coefficients) coincides with \( \theta_p \). The application is as follows. Suppose

\[
H = \theta_p(\xi_i)
\]

is the image of \( \xi_i \). From Corollary 4.5.16 we know that

\[
H = \text{Fr}_s^{-1}(\hat{\theta}_p(x_i^p))
\]

where \( \text{Fr}_s^{-1} \) is the action of the inverse Frobenius automorphism on the coefficients of the polynomial. The last equation is equivalent to

\[
\hat{\theta}_p^{-1}(\text{Fr}_s(H)) = x_i^p.
\]

By the special case above, there exists a unique Weyl polynomial \( \hat{G} \) such that

\[
\hat{G}^p = \hat{\theta}_p^{-1}(\text{Fr}_s(H)).
\]

But then

\[
H = \text{Fr}_s^{-1}(\hat{\theta}_p(\hat{G}^p))
\]

which is exactly what we wanted. \( \square \)

It will be convenient to denote the one-to-one correspondence between modulo \( p \) reductions of central polynomials coming from characteristic zero symplectomorphisms with their Weyl liftings by \( \Phi^h_p \) (for this correspondence is, as evidenced by Lemma 4.5.20, shares essential nature with the characteristic zero direct homomorphism \( \Phi^h \)).
We now apply the above lemma in order to establish the form of the pre-image Weyl polynomial in the case of auxiliary variables \( u, v \) and the central polynomial of a special type.

**Lemma 4.5.21.** Let \( u, v \) denote the extra Poisson variables, and let

\[
H = u + h^k \varphi(x_i)
\]

be the image of \( u \) under the twisted automorphism coming from \( \varphi \) as above (\( \varphi(x_i) \) is rational in \( h \) but \( h^k \varphi(x_i) \) is polynomial in \( h \)). Then the unique pre-image \( \hat{H} \) of \( H \) with respect to the correspondence \( \Phi^h_p \) of the previous lemma has the form

\[
\hat{H} = \hat{u} + h^k P_i(x_1, \ldots, d_n, h)
\]

where \( P_i \) is rational in \( h \).

**Proof.** We establish the statement in several elementary steps. Firstly, as \( H \) does not contain the auxiliary variable \( v \), \( \hat{H} \) does not contain its Weyl counterpart \( \hat{v} \): indeed, otherwise the Newton polyhedron of \( H \) would contain (in the case of \( v \) carrying great enough weight to make the corresponding monomial the highest-order term) a vertex corresponding to the monomial containing \( \hat{v} \).

Now let

\[
\hat{H} = Q(\hat{u}) + R
\]

where every monomial in \( R \) is proportional to generators other than \( u \). Then

\[
H = \Phi^h_p(\hat{H}) = \Phi^h_p(Q) + \Phi^h_p(R),
\]

as the two differential operators \( Q \) and \( R \) commute with each other and therefore taking the \( p \)-th power is executed as in the commutative case. By Lemma 4.5.20, we must have

\[
Q(\hat{u}) = \hat{u}.
\]

Finally, we show that if \( \hat{H} \) contains monomials which are products of \( \hat{u} \) with other generators, then \( \Phi^h_p(\hat{H}) \neq H \). Indeed, if such a monomial had a non-zero coefficient in \( \hat{H} \), then there would exist a grading under which this monomial would be the highest-order term (corresponding to a vertex in the Newton polyhedron). Then the image \( \Phi^h_p(\hat{H}) \) would also have a monomial corresponding to this highest-order term with non-zero coefficient, as taking the \( p \)-th power dilates the polyhedron and therefore maps the extremal points to extremal points.

The conclusion is that the polynomial \( \hat{H} \) has the form

\[
\hat{u} + \hat{P}_i(x_1, \ldots, d_n, h).
\]

\(^6\)Note that \( \Phi^h_p \) behaves toward the Newton polyhedra as the homomorphism taking the \( p \)-th power does.
Taking out $h^k$ from $\hat{\mathcal{P}}_1$ leaves us with the form we needed.

Lemma 4.5.21 provides a canonical way to relate the images $\varphi(x_i)$ and $\varphi(d_j)$ of the initial symplectomorphism with the Weyl pre-images. Therefore, as an array of differential operators, the lifting $\hat{\varphi}$ is well defined. We denote the polynomials in Weyl generators in the correspondence by

$$(\hat{\varphi}(x_1), \ldots, \hat{\varphi}(d_n)).$$

We now need to verify the commutation relations in order to establish its homomorphic character. Again we have two lemmas.

**Lemma 4.5.22.**

$$[[\hat{\varphi}(x_i), \hat{\varphi}(x_j)] = [[\hat{\varphi}(d_i), \hat{\varphi}(d_j)] = 0,$$

$$[[\hat{\varphi}(x_i), \hat{\varphi}(d_j)] = 0, \ i \neq j.$$

**Proof.** It suffices to prove $$[[\hat{\varphi}(x_1), \hat{\varphi}(x_2)] = 0$$ thanks to the variable re-labelling and the existence of the "Fourier transform" – the automorphism

$$x_i \mapsto d_i, \ d_i \mapsto -x_i.$$  

We introduce two pairs of auxiliary Poisson variables, $u_1, u_2, v_1, v_2$, and for $\lambda = h^k$ and $k$ large enough consider the automorphism $\psi$:

$$u_1 \mapsto u_1 + \lambda x_1, \ u_2 \mapsto u_2 + \lambda x_2$$

$$p_1 \mapsto p_1 - \lambda v_1, \ p_2 \mapsto p_2 - \lambda v_2$$

($\psi$ acts as the identity map on the rest of the generators).

We take the twisted automorphism

$$\varphi_{t, \lambda} = \varphi_a \circ \psi \circ \varphi_a^{-1}$$

with $\psi$ now being the chosen linear transformation and take $k$ to be large enough so that the twisted automorphism is a polynomial symplectomorphism. We then lift it with $\Theta^h$ to the $h$-augmented Weyl algebra as before.

By Lemma 4.5.21, the images of the Weyl counterparts $\hat{u}_i \ (i = 1, 2)$ of $u_1, u_2$ under the lifted twisted automorphism will have the form

$$\hat{u}_i + \lambda T_i,$$

the polynomials $T_i$ do not contain the auxiliary variables and $\Phi^h_p(T_i) = \varphi(x_i), \ i = 1, 2.
Now, as $\varphi_{t,\lambda}$ and its lifting are automorphisms, we must have

$$[\hat{u}_1 + \lambda T_1, \hat{u}_2 + \lambda T_2] = 0$$

so that

$$[\hat{u}_1, \hat{u}_2] + \lambda([\hat{u}_1, T_2] + [T_1, \hat{u}_2]) + \lambda^2[T_1, T_2] = 0$$

from which it follows immediately that

$$[T_1, T_2] = 0$$

as desired.

**Lemma 4.5.23.**

$$[\hat{\varphi}(d_i), \hat{\varphi}(x_i)] = h.$$  

**Proof.** We proceed in an manner analogous to the previous lemma: we construct the appropriate twisting from whose lifting the relevant images may be read off and then evaluate the commutator.

Let $u, v$ be auxiliary Poisson variables and let

$$\psi_1 : u \mapsto u + \lambda x_i, \quad p_i \mapsto p_i - \lambda v,$$

$$\psi_2 : v \mapsto v + \mu p_i, \quad x_i \mapsto x_i - \mu u$$

(in both cases the other generators are mapped to themselves). Consider the composition

$$\theta = \psi_1 \circ \psi_2.$$  

Then

$$\theta(u) = u + \lambda x_i, \quad \theta(v) = v + \mu p_i - \lambda \mu v$$

and

$$\theta(x_i) = x_i - \mu u - \lambda \mu x_i, \quad \theta(p_i) = p_i - \lambda v.$$  

Take

$$\varphi_{t,\lambda\mu} = \varphi_a \circ \theta \circ \varphi_a^{-1}$$

where as before $\varphi_a$ extends from $\varphi$ by the identical action on $u, v$. The images of $u, v$ under $\varphi_{t,\lambda\mu}$ read:

$$\varphi_{t,\lambda\mu}(u) = u + \lambda \varphi(x_i), \quad \varphi_{t,\lambda\mu}(v) = (1 - \lambda \mu)v + \mu \varphi(p_i).$$  

By properly selecting $\lambda$ and $\mu$ as polynomials in $h$, we can make $\varphi_{t,\lambda\mu}$ into a polynomial $h$-augmented symplectomorphism and therefore lift it with $\Theta^h$. Again, by Lemma 4.5.21, the action of the lifted automorphism on $\hat{u}$ and $\hat{v}$ will have the needed form (with the part dependent on $x_i, d_j$ given by the images under $\hat{\varphi}$). Now, the commutator of the images of $\hat{u}$ and $\hat{v}$ must be equal
to $h$. We therefore have the following:

$$h = [(1 - \lambda \mu)\hat{v} + \mu \hat{\phi}(d_i), \hat{u} + \lambda \hat{\phi}(x_i)] = h(1 - \lambda \mu) + \lambda \mu[\hat{\phi}(d_i), \hat{\phi}(x_i)]$$

from which the statement follows directly.

The conclusion is that augmented symplectomorphisms rational in $h$ are lifted to endomorphisms of the augmented Weyl algebra (also rational in $h$) by a homomorphism whose restriction to points polynomial in $h$ coincides with $\Theta^h$. What remains to show is that the lifted mappings are automorphisms, however, this is accomplished by an argument similar to that for points polynomial in $h$ (cf. discussion immediately preceding Proposition 4.5.13). Also, thanks to Lemma 4.5.21, we know that the lifting of points rational in $h$ is also the inverse mapping to the extension to these points of the direct homomorphism $\Phi^h$. The specialization to $h = 1$ may now be safely executed, and the Main Theorem follows.
Chapter 5

Torus actions on free associative algebras, lifting and Bialynicki-Birula type theorems

We first prove that every maximal torus action on the free algebra is conjugate to a linear action. This statement is the free algebra analogue of a classical theorem of A. Bialynicki-Birula. This chapter is based on two papers [85, 86].

5.1 Actions of algebraic tori

In this section we recall basic definitions of the theory of torus actions, as formulated by Bialynicki-Birula [49, 51] and others.

Let \( K \) be the ground field. Let \( I \) be a finite or a countable index set and let \( Z = \{ z_i : i \in I \} \) be the set of variables, which is sometimes referred to as the alphabet.

The free associative algebra \( F_I(K) = K \langle Z \rangle \) is the algebra generated by words in the alphabet \( Z \) (as usually, word concatenation gives the multiplication of monomials and extends linearly to define the multiplication in the algebra).

Any element of \( K \langle Z \rangle \) can be written uniquely in the form

\[
\sum_{k=0}^{\infty} \sum_{i_1, \ldots, i_k \in I} a_{i_1, i_2, \ldots, i_k} z_{i_1} z_{i_2} \cdots z_{i_k},
\]

where the coefficients \( a_{i_1, i_2, \ldots, i_k} \) are elements of the field \( K \) and all but finitely many of these elements are zero.

In our context, the alphabet \( Z \) is the same as the set of algebra generators, therefore the terms "monomial" and "word" will be used interchangeably.
In the sequel, we employ the following short-hand notation for a free algebra monomial. For an element \( z \), its powers are defined intuitively. Any monomial \( z_{i_1} z_{i_2} \ldots z_{i_k} \) can then be written in a reduced form with subwords \( z z \ldots z \) replaced by powers.

We then write

\[
z^I = z_{j_1}^{i_{j_1}} z_{j_2}^{i_{j_2}} \ldots z_{j_k}^{i_{j_k}}
\]

where by \( I \) we mean an assignment of \( i_k \) to \( j_k \) in the word \( z^I \). Sometimes we refer to \( I \) as a multi-index, although the term is not entirely accurate. If \( I \) is such a multi-index, its absolute value \( |I| \) is defined as the sum \( i_1 + \cdots + i_k \).

For a field \( \mathbb{K} \), let \( \mathbb{K}^\times = \mathbb{K}\{0\} \) denote the multiplicative group of its non-zero elements viewed.

**Definition 5.1.1.** An \( n \)-dimensional algebraic \( \mathbb{K} \)-torus is a group

\[
T_n \simeq (\mathbb{K}^\times)^n
\]

(with obvious multiplication).

Denote by \( \mathbb{A}^n \) the affine space of dimension \( n \) over \( \mathbb{K} \).

**Definition 5.1.2.** A (left, geometric) torus action is a morphism

\[
\sigma : T_n \times \mathbb{A}^n \rightarrow \mathbb{A}^n.
\]

that fulfills the usual axioms (identity and compatibility):

\[
\sigma(1, x) = x, \quad \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).
\]

The action \( \sigma \) is effective if for every \( t \neq 1 \) there is an element \( x \in \mathbb{A}^n \) such that \( \sigma(t, x) \neq x \).

In [49], Bialynicki-Birula proved the following two theorems, for \( \mathbb{K} \) algebraically closed.

**Theorem 5.1.3.** Any regular action of \( T_n \) on \( \mathbb{A}^n \) has a fixed point.

**Theorem 5.1.4.** Any effective and regular action of \( T_n \) on \( \mathbb{A}^n \) is a representation in some coordinate system.

The term "regular" is to be understood here as in the algebro-geometric context of regular function (Bialynicki-Birula also considered birational actions).

In the following section (dedicated to the proof of the free algebra version of Theorems 5.1.3 and 5.1.4), the ground field is algebraically closed.

As was mentioned in the introduction, an algebraic group action on \( \mathbb{A}^n \) is the same as the corresponding action by automorphisms on the algebra

\[
\mathbb{K}[x_1, \ldots, x_n]
\]
of coordinate functions. In other words, it is a group homomorphism

\[ \sigma : \mathbb{T}_n \rightarrow \text{Aut} \, K[x_1, \ldots, x_n]. \]

An action is effective if and only if \( \text{Ker} \, \sigma = \{1\} \).

The polynomial algebra is a quotient of the free associative algebra

\[ F_n = K\langle z_1, \ldots, z_n \rangle \]

by the commutator ideal \( I \) (it is the two-sided ideal generated by all elements of the form \( fg - gf \)). The definition of torus action on the free algebra is thus purely algebraic.

In this chapter we establish the free algebra version of the Białynicki-Birula theorem. The latter is formulated as follows.

**Theorem 5.1.5.** Suppose given an action \( \sigma \) of the algebraic \( n \)-torus \( \mathbb{T}_n \) on the free algebra \( F_n \). If \( \sigma \) is effective, then it is linearizable.

The linearity (or linearization) problem, as it has become known since Kambayashi, asks whether all (effective, regular) actions of a given type of algebraic groups on the affine space of given dimension are conjugate to representations. According to Theorem 5.1.5, the linearization problem extends to the noncommutative category. Several known results concerning the (commutative) linearization problem are summarized below.

1. Any effective regular torus action on \( \mathbb{A}^2 \) is linearizable (Gutwirth [98]).
2. Any effective regular torus action on \( \mathbb{A}^n \) has a fixed point (Białynicki-Birula [49]).
3. Any effective regular action of \( \mathbb{T}_{n-1} \) on \( \mathbb{A}^n \) is linearizable (Białynicki-Birula [50]).
4. Any (effective, regular) one-dimensional torus action (i.e., action of \( \mathbb{K}^\times \)) on \( \mathbb{A}^3 \) is linearizable (Koras and Russell [125]).
5. If the ground field is not algebraically closed, then a torus action on \( \mathbb{A}^n \) need not be linearizable. In [16], Asanuma proved that over any field \( \mathbb{K} \), if there exists a non-rectifiable closed embedding from \( \mathbb{A}^m \) into \( \mathbb{A}^n \), then there exist non-linearizable effective actions of \( (\mathbb{K}^\times)^r \) on \( \mathbb{A}^{1+n+m} \) for \( 1 \leq r \leq 1 + m \).
6. When \( \mathbb{K} \) is infinite and has positive characteristic, there are examples of non-linearizable torus actions on \( \mathbb{A}^n \) (Asanuma [16]).

**Remark 5.1.6.** A closed embedding \( \iota : \mathbb{A}^m \hookrightarrow \mathbb{A}^n \) is said to be rectifiable if it is conjugate to a linear embedding by an automorphism of \( \mathbb{A}^n \).
As can be inferred from the review above, the context of the linearization problem is rather broad, even in the case of torus actions. The regulating parameters are the dimensions of the torus and the affine space. This situation is due to the fact that the general form of the linearization conjecture (i.e., the conjecture that states that any effective regular torus action on any affine space is linearizable) has a negative answer.

Transition to the noncommutative geometry presents the inquirer with an even broader context: one now may vary the dimensions as well as impose restrictions on the action in the form of preservation of the PI-identities. Caution is well advised. Some of the results are generalized in a straightforward manner – the proof in the next section being the typical example, others require more subtlety and effort. Of some note to us, given our ongoing work in deformation quantization (see, for instance, [112]) is the following instance of the linearization problem, which we formulate as a conjecture.

**Conjecture 5.1.7.** For $n \geq 1$, let $P_n$ denote the commutative Poisson algebra, i.e. the polynomial algebra

\[ \mathbb{K}[z_1, \ldots, z_{2n}] \]

equipped with the Poisson bracket defined by

\[ \{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}. \]

Then any effective regular action of $\mathbb{T}_n$ by automorphisms of $P_n$ is linearizable.

A version of Theorem 1.3.6 for the commutative Poisson algebra is a conjecture of significant interest. It turns out that the algebra $P_n$ admits a certain augmentation by central variables which distort the Poisson structure, such that the automorphism group of the resulting algebra admits the property of Theorem 1.3.6. The case is studied in the paper [111].

### 5.2 Maximal torus action on the free algebra

In this section, we provide proof to the free algebra version (Theorem 5.1.5) of the Biały-Birula theorem [49].

The proof proceeds along the lines of the original commutative case proof of Biały-Birula. If $\sigma$ is the effective action of Theorem 5.1.5, then for each $t \in \mathbb{T}_n$ the automorphism

\[ \sigma(t) : F_n \to F_n \]

is given by the $n$-tuple of images of the generators $z_1, \ldots, z_n$ of the free algebra:

\[ (f_1(t, z_1, \ldots, z_n), \ldots, f_n(t, z_1, \ldots, z_n)). \]

Each of the $f_1, \ldots, f_n$ is a polynomial in the free variables.
Lemma 5.2.1. There is a translation of the free generators

\[(z_1, \ldots, z_n) \rightarrow (z_1 - c_1, \ldots, z_n - c_n), \quad (c_i \in \mathbb{K})\]

such that (for all \(t \in \mathbb{T}_n\)) the polynomials \(f_i(t, z_1 - c_1, \ldots, z_n - c_n)\) have zero free part.

Proof. This is a direct corollary of Theorem 5.1.3. Indeed, any action \(\sigma\) on the free algebra induces, by taking the canonical projection with respect to the commutator ideal \(I\), an action \(\overline{\sigma}\) on the commutative algebra \(\mathbb{K}[x_1, \ldots, x_n]\). If \(\sigma\) is regular, then so is \(\overline{\sigma}\). By Theorem 5.1.3, \(\overline{\sigma}\) (or rather, its geometric counterpart) has a fixed point, therefore the images of commutative generators \(x_i\) under \(\overline{\sigma}(t)\) (for every \(t\)) will be polynomials with trivial degree-zero part. Consequently, the same will hold for \(\sigma\).

We may then suppose, without loss of generality, that the polynomials \(f_i\) have the form

\[f_i(t, z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij}(t) z_j + \sum_{j,l=1}^{n} a_{ijl}(t) z_j z_l + \sum_{k=3}^{N} \sum_{|J|=k} a_{i,J}(t) z_J\]

where by \(z_J\) we denote, as in the introduction, a particular monomial

\[z_{i_1}^{k_1} z_{i_2}^{k_2} \ldots\]

(a word in the alphabet \(\{z_1, \ldots, z_n\}\) in the reduced notation; \(J\) is the multi-index in the sense described above); also, \(N\) is the degree of the automorphism (which is finite) and \(a_{ij}, a_{ijl}, \ldots\) are polynomials in \(t_1, \ldots, t_n\).

As \(\sigma_i\) is an automorphism, the matrix \([a_{ij}]\) that determines the linear part is non-singular. Therefore, without loss of generality we may assume it to be diagonal (just as in the commutative case [49]) of the form

\[\text{diag}(t_1^{m_{11}} \ldots t_n^{m_{1n}}, \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}})\].

Now, just as in [49], we have the following

Lemma 5.2.2. The power matrix \([m_{ij}]\) is non-singular.

Proof. Consider a linear action \(\tau\) defined by

\[\tau(t) : (z_1, \ldots, z_n) \mapsto (t_1^{m_{11}} \ldots t_n^{m_{1n}} z_1, \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}} z_n), \quad (t_1, \ldots, t_n) \in \mathbb{T}_n.\]

If \(T_1 \subset T_n\) is any one-dimensional torus, the restriction of \(\tau\) to \(T_1\) is non-trivial. Indeed, were it to happen that for some \(T_1\),

\[\tau(t)z = z, \quad t \in T_1, \quad (z = (z_1, \ldots, z_n))\]
then our initial action $\sigma$, whose linear part is represented by $\tau$, would be identity modulo terms of degree $> 1$:

$$\sigma(t)(z_i) = z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \cdots.$$  

Now, equality $\sigma(t^2)(z) = \sigma(t)(\sigma(t)(z))$ implies

$$\sigma(t)(\sigma(t)(z_i)) = \sigma(t) \left( z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \cdots \right)$$

$$= z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \sum_{j,l} a_{ijl}(t)(z_j + \sum_{km} a_{jkm}(t) z_k z_m + \cdots)$$

$$= z_i + \sum_{j,l} a_{ijl}(t^2) z_j z_l + \cdots$$

which means that

$$2a_{ijl}(t) = a_{ijl}(t^2)$$

and therefore $a_{ijl}(t) = 0$. The coefficients of the higher-degree terms are processed by induction (on the total degree of the monomial). Thus

$$\sigma(t)(z) = z, \ t \in T_1$$

which is a contradiction since $\sigma$ is effective. Finally, if $[m_{ij}]$ were singular, then one would easily find a one-dimensional torus such that the restriction of $\tau$ were trivial. 

Consider the action

$$\varphi(t) = \tau(t^{-1}) \circ \sigma(t).$$

The images under $\varphi(t)$ are

$$(g_1(z, t), \ldots, g_n(z, t)), \ (t = (t_1, \ldots, t_n))$$

with

$$g_i(z, t) = \sum g_{i,m_1, \ldots, m_n}(z) t_1^{m_1} \cdots t_n^{m_n}, \ m_1, \ldots, m_n \in \mathbb{Z}.$$  

Define $G_i(z) = g_{i,0, \ldots, 0}(z)$ and consider the map $\beta : F_n \to F_n$,

$$\beta : (z_1, \ldots, z_n) \mapsto (G_1(z), \ldots, G_n(z)).$$

**Lemma 5.2.3.** $\beta \in \text{Aut} F_n$ and

$$\beta = \tau(t^{-1}) \circ \beta \circ \sigma(t).$$

**Proof.** This lemma mirrors the final part in the proof in [49]. The conjugation is straightforward,
since for every $s, t \in \mathbb{T}_n$ one has
\[
\varphi(st) = \tau(t^{-1}s^{-1}) \circ \sigma(st) = \tau(t^{-1}) \circ \tau(s^{-1}) \circ \sigma(s) \circ \sigma(t) = \tau(t^{-1}) \circ \varphi(s) \circ \sigma(t).
\]

Denote by $\hat{F}_n$ the power series completion of the free algebra $F_n$, and let $\hat{\sigma}, \hat{\tau}$ and $\hat{\beta}$ denote the endomorphisms of the power series algebra induced by corresponding morphisms of $F_n$. The endomorphisms $\hat{\sigma}, \hat{\tau}, \hat{\beta}$ come from (polynomial) automorphisms and therefore are invertible.

Let
\[
\hat{\beta}^{-1}(z_i) \equiv B_i(z) = \sum_J b_{i,J} z^J
\]
(just as before, $z^J$ is the monomial with multi-index $J$). Then
\[
\hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}(z_i) = B_i(t_1^{m_{i1}} \ldots t_n^{m_{in}} G_1(z), \ldots, t_1^{m_{i1}} \ldots t_n^{m_{in}} G_n(z)).
\]

Now, from the conjugation property we must have
\[
\hat{\beta} = \hat{\sigma}(t^{-1}) \circ \hat{\beta} \circ \hat{\tau}(t),
\]
therefore $\hat{\sigma}(t) = \hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}$ and
\[
\hat{\sigma}(t)(z_i) = \sum_J b_{i,J}(t_1^{m_{i1}} \ldots t_n^{m_{in}})^{j_1} \ldots (t_1^{m_{i1}} \ldots t_n^{m_{in}})^{j_n} G(z)^J;
\]
here the notation $G(z)^J$ stands for a word in $G_i(z)$ with multi-index $J$, while the exponents $j_1, \ldots, j_n$ count how many times a given index appears in $J$ (or, equivalently, how many times a given generator $z_i$ appears in the word $z^J$).

Therefore, the coefficient of $\hat{\sigma}(t)(z_i)$ at $z^J$ has the form
\[
b_{i,J}(t_1^{m_{i1}} \ldots t_n^{m_{in}})^{j_1} \ldots (t_1^{m_{i1}} \ldots t_n^{m_{in}})^{j_n} + S
\]
with $S$ a finite sum of monomials of the form
\[
c_L(t_1^{m_{i1}} \ldots t_n^{m_{in}})^{l_1} \ldots (t_1^{m_{i1}} \ldots t_n^{m_{in}})^{l_n}
\]
with $(j_1, \ldots, j_n) \neq (l_1, \ldots, l_n)$. Since the power matrix $[m_{ij}]$ is non-singular, if $b_{i,J} \neq 0$, we can find a $t \in \mathbb{T}_n$ such that the coefficient is not zero. Since $\sigma$ is an algebraic action, the degree
\[
\sup_t \deg(\hat{\sigma})
\]
is a finite integer $N$. With the previous statement, this implies that
\[
b_{i,J} = 0, \ \text{whenever} \ |J| > N.
\]
Therefore, $B_i(z)$ are polynomials in the free variables. What remains is to notice that

$$z_i = B_i(G_1(z), \ldots, G_n(z)).$$

Thus $\beta$ is an automorphism.

From Lemma 5.2.3 it follows that

$$\tau(t) = \beta^{-1} \circ \sigma(t) \circ \beta$$

which is the linearization of $\sigma$. Theorem 5.1.5 is proved.

## 5.3 Discussion

The noncommutative toric action linearity property has several useful applications. In the work [118], it is used to investigate the properties of the group $\text{Aut} F_n$ of automorphisms of the free algebra. As a corollary of Theorem 5.1.5, one gets

**Corollary 5.3.1.** Let $\theta$ denote the standard action of $\mathbb{T}_n$ on $K[x_1, \ldots, x_n]$ – i.e., the action

$$\theta_t : (x_1, \ldots, x_n) \mapsto (t_1 x_1, \ldots, t_n x_n).$$

Let $\tilde{\theta}$ denote its lifting to an action on the free associative algebra $F_n$. Then $\tilde{\theta}$ is also given by the standard torus action.

This statement plays a part, along with a number of results concerning the induced formal power series topology on $\text{Aut} F_n$, in the establishment of the free associative analogue of Theorem 1.3.6.

The proofs in this paper, for the most part, were based upon the techniques from the commutative category. It is, however, a problem of legitimate interest to try and obtain proofs for various linearity statements using tools specific to the category of associative algebras, bypassing the known commutative results. As one outstanding example of this problem, we expect the free associative analogue of the second Bialynicki-Birula theorem to hold and formulate it here as a conjecture.

**Conjecture 5.3.2.** Any effective action of $\mathbb{T}_{n-1}$ on $F_n$ is linearizable.

Also of independent interest is the following instance of the linearity problem.

**Conjecture 5.3.3.** For $n \geq 1$, let $P_n$ denote the commutative Poisson algebra, i.e. the polynomial algebra

$$K[z_1, \ldots, z_{2n}]$$
equipped with the Poisson bracket defined by

\[ \{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}. \]

Then any effective regular action of \( \mathbb{T}_n \) by automorphisms of \( P_n \) is linearizable.

This problem is loosely analogous to the Białynicki-Birula theorem, in the sense of maximality of torus with respect to the dimension of the configurations space (spanned by \( x_i \)). There seems to be no straightforward way of finding the linearizing canonical coordinates on the phase space, however. For the Ind-variety \( \text{Aut} P_n \), a version of Theorem 1.3.6 may be stated. The geometry of \( \text{Aut} P_n \) is relevant to problems of deformation quantization.
Chapter 6

Jacobian conjecture, Specht and Burnside type problems

This chapter explores an approach to polynomial mappings and the Jacobian Conjecture and related questions, initiated by A.V. Yagzhev, whereby these questions are translated to identities of algebras, leading to a solution in [215] of the version of the Jacobian Conjecture for free associative algebras. (The first version, for two generators, was obtained by Dicks and J. Levin [71, 72], and the full version by Schofield [169].) We start by laying out the basic framework in this introduction. Next, we set up Yagzhev’s correspondence to algebras in §6.1, leading to the basic notions of weak nilpotence and Engel type. In §6.2 we discuss the Jacobian Conjecture in the context of various varieties, including the free associative algebra.

Given any polynomial endomorphism \( \phi \) of the \( n \)-dimensional affine space \( A^n_k = \text{Spec} k[x_1, \ldots, x_n] \) over a field \( k \), we define its Jacobian matrix to be the matrix

\[
\left( \frac{\partial \phi^*(x_i)}{\partial x_j} \right)_{1 \leq i, j \leq n}.
\]

The determinant of the Jacobian matrix is called the Jacobian of \( \phi \). The celebrated Jacobian Conjecture \( JC_n \) in dimension \( n \geq 1 \) asserts that for any field \( k \) of characteristic zero, any polynomial endomorphism \( \phi \) of \( A^n_k \) having Jacobian 1 is an automorphism. Equivalently, one can say that \( \phi \) preserves the standard top-degree differential form \( dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(A^n_k) \). References to this well known problem and related questions can be found in [19], [130], and [199]. By the Lefschetz principle it is sufficient to consider the case \( k = \mathbb{C} \); obviously, \( JC_n \) implies \( JC_m \) if \( n > m \).

The conjecture \( JC_n \) is obviously true in the case \( n = 1 \), and it is open for \( n \geq 2 \).

The Jacobian Conjecture, denoted as \( JC \), is the conjunction of the conjectures \( JC_n \) for all finite \( n \). The Jacobian Conjecture has many reformulations (such as the Kernel Conjecture and the Image Conjecture, cf. [87, 90, 199, 224, 225] for details) and is closely related to questions concerning quantization. It is stably equivalent to the following conjecture of Dixmier, concerning automorphisms of the Weyl algebra \( W_n \), otherwise known as the quantum affine algebra.

Dixmier Conjecture \( DC_n \): Does \( \text{End}(W_n) = \text{Aut}(W_n) \)?
The implication $DC_n \rightarrow JC_n$ is well known, and the inverse implication $JC_{2n} \rightarrow DC_n$ was recently obtained independently by Tsuchimoto [188] (using $p$-curvature) and Belov and Kontsevich [40], [39] (using Poisson brackets on the center of the Weyl algebra). Bavula [28] has obtained a shorter proof, and also obtained a positive solution of an analog of the Dixmier Conjecture for integro differential operators, cf. [26]. He also proved that every monomorphism of the Lie algebra of triangular polynomial derivations is an automorphism [27] (an analog of Dixmier’s conjecture).

The Jacobian Conjecture is closely related to many questions of affine algebraic geometry concerning affine space, such as the Cancellation Conjecture (see Section 6.2.4). If we replace the variety of commutative associative algebras (and the accompanying affine spaces) by an arbitrary algebraic variety $^1$, one easily gets a counterexample to the JC. So, strategically these questions deal with some specific properties of affine space which we do not yet understand, and for which we do not have the appropriate formulation apart from these very difficult questions.

It seems that these properties do indicate some sort of quantization. From that perspective, noncommutative analogs of these problems (in particular, the Jacobian Conjecture and the analog of the Cancellation Conjecture) become interesting for free associative algebras, and more generally, for arbitrary varieties of algebras.

We work in the language of universal algebra, in which an algebra is defined in terms of a set of operators, called its signature. This approach enhances the investigation of the Yagzhev correspondence between endomorphisms and algebras. We work with deformations and so-called packing properties to be introduced in Section 6.2 and Section 6.2.2, which denote specific noncommutative phenomena which enable one to solve the JC for the free associative algebra.

From the viewpoint of universal algebra, the Jacobian conjecture becomes a problem of “Burnside type,” by which we mean the question of whether a given finitely generated algebraic structure satisfying given periodicity conditions is necessarily finite, cf. Zelmanov [223]. Burnside originally posed the question of the finiteness of a finitely generated group satisfying the identity $x^n = 1$. (For odd $n \geq 661$, counterexamples were found by Novikov and Adian, and quite recently Adian reduced the estimate from 661 to 101). Another class of counterexamples was discovered by Ol’shanskij [145]. Kurosh posed the question of local finiteness of algebras whose elements are algebraic over the base field. For algebraicity of bounded degree, the question has a positive solution, but otherwise there are the Golod-Shafarevich counterexamples.

Burnside type problems play an important role in algebra. Their solution in the associative case is closely tied to Specht’s problem of whether any set of polynomial identities can be deduced from a finite subset. The JC can be formulated in the context of whether one system of identities implies another, which also relates to Specht’s problem.

In the Lie algebra case there is a similar notion. An element $x \in L$ is called Engel of degree $n$ if $[\ldots[[y,x],x]\ldots,x] = 0$ for any $y$ in the Lie algebra $L$. Zelmanov’s result that any finitely

---

$^1$Algebraic geometers use word variety, roughly speaking, for objects whose local structure is obtained from the solution of system of algebraic equations. In the framework of universal algebra, this notion is used for subcategories of algebras defined by a given set of identities. A deep analog of these notions is given in [32].
generated Lie algebra of bounded Engel degree is nilpotent yielded his solution of the Restricted Burnside Problem for groups. Yagzhev introduced the notion of Engelian and weakly nilpotent algebras of arbitrary signature (see Definitions 6.1.6, 6.1.4), and proved that the JC is equivalent to the question of weak nilpotence of algebras of Engel type satisfying a system of Capelli identities, thereby showing the relation of the JC with problems of Burnside type.

**A negative approach.** Let us mention a way of constructing counterexamples. This approach, developed by Gizatullin, Kulikov, Shafarevich, Vitushkin, and others, is related to decomposing polynomial mappings into the composition of $\sigma$-processes [92, 130, 171, 201–203]. It allows one to solve some polynomial automorphism problems, including tameness problems, the most famous of which is *Nagata’s Problem* concerning the wildness of Nagata’s automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z),$$

cf. [142]. Its solution by Shestakov and Umirbaev [181] is the major advance in this area in the last decade. The Nagata automorphism can be constructed as a product of automorphisms of $K(z)[x, y]$, some of them having non-polynomial coefficients (in $K(z)$). The following theorem of Abhyankar-Moh-Suzuki [2, 186] and [134] can be viewed in this context:

**AMS Theorem.** If $f$ and $g$ are polynomials in $K[z]$ of degrees $n$ and $m$ for which $K[f, g] = K[z]$, then $n$ divides $m$ or $m$ divides $n$.

Degree estimate theorems are polynomial analogs to Liouville’s approximation theorem in algebraic number theory ([47,102,132,137]). T. Kishimoto has proposed using a program of Sarkisov, in particular for Nagata’s Problem. Although difficulties remain in applying “$\sigma$-processes” (decomposition of birational mappings into standard blow-up operations) to the affine case, these may provide new insight. If we consider affine transformations of the plane, we have relatively simple singularities at infinity, although for bigger dimensions they can be more complicated. Blow-ups provide some understanding of birational mappings with singularities. Relevant information may be provided in the affine case. The paper [45] contains some deep considerations about singularities.

### 6.1 The Jacobian Conjecture and Burnside type problems, via algebras

In this section we translate the Jacobian Conjecture to the language of algebras and their identities. This can be done at two levels: At the level of the algebra obtained from a polynomial mapping, leading to the notion of weak nilpotence and Yagzhev algebras and at the level of the differential and the algebra arising from the Jacobian, leading to the notion of Engel type. The Jacobian Conjecture is the link between these two notions.
6.1.1 The Yagzhev correspondence

Polynomial mappings in universal algebra

Yagzhev’s approach is to pass from algebraic geometry to universal algebra. Accordingly, we work in the framework of a universal algebra $A$ having signature $\Omega$. $A^{(m)}$ denotes $A \times \cdots \times A$, taken $m$ times.

We fix a commutative, associative base ring $C$, and consider $C$-modules equipped with extra operators $A^{(m)} \mapsto A$, which we call $m$-ary. Often one of these operators will be (binary) multiplication. These operators will be multilinear, i.e., linear with respect to each argument. Thus, we can define the degree of an operator to be its number of arguments. We say an operator $\Psi(x_1, \ldots, x_m)$ is symmetric if $\Psi(x_1, \ldots, x_m) = \Psi(x_{\pi(1)}, \ldots, x_{\pi(m)})$ for all permutations $\pi$.

**Definition 6.1.1.** A string of operators is defined inductively. Any operator $\Psi(x_1, \ldots, x_m)$ is a string of degree $m$, and if $s_j$ are strings of degree $d_j$, then $\Psi(s_1, \ldots, s_m)$ is a string of degree $\sum_{j=1}^m d_j$. A mapping $\alpha : A^{(m)} \mapsto A$ is called polynomial if it can be expressed as a sum of strings of operators of the algebra $A$. The degree of the mapping is the maximal length of these strings.

**Example.** Suppose an algebra $A$ has two extra operators: a binary operator $\alpha(x,y)$ and a tertiary operator $\beta(x,y,z)$. The mapping $F : A \mapsto A$ given by $x \mapsto x + \alpha(x,x) + \beta(\alpha(x,x),x,x)$ is a polynomial mapping of $A$, having degree 4. Note that if $A$ is finite dimensional as a vector space, not every polynomial mapping of $A$ as an affine space is a polynomial mapping of $A$ as an algebra.

Yagzhev’s correspondence between polynomial mappings and algebras

Here we associate an algebraic structure to each polynomial map. Let $V$ be an $n$-dimensional vector space over the field $k$, and $F : V \mapsto V$ be a polynomial mapping of degree $m$. Replacing $F$ by the composite $TF$, where $T$ is a translation such that $TF(0) = 0$, we may assume that $F(0) = 0$. Given a base $\{\vec{e}_i\}_{i=1}^n$ of $V$, and for an element $v$ of $V$ written uniquely as a sum $\sum x_i \vec{e}_i$, for $x_i \in k$, the coefficients of $\vec{e}_i$ in $F(v)$ are (commutative) polynomials in the $x_i$. Then $F$ can be written in the following form:

$$x_i \mapsto F_{0i}(\vec{x}) + F_{1i}(\vec{x}) + \cdots + F_{mi}(\vec{x})$$

where each $F_{\alpha i}(\vec{x})$ is a homogeneous form of degree $\alpha$, i.e.,

$$F_{\alpha i}(\vec{x}) = \sum_{j_1 + \cdots + j_n = \alpha} \kappa_{j_1} x_1^{j_1} \cdots x_n^{j_n},$$
with $F_{0i} = 0$ for all $i$, and $F_{1i}(\vec{x}) = \sum_{k=1}^{n} \mu_{ki} x_k$.

We are interested in invertible mappings that have a nonsingular Jacobian matrix $(\mu_{ij})$. In particular, this matrix is nondegenerate at the origin. In this case $\det(\mu_{ij}) \neq 0$, and by composing $F$ with an affine transformation we arrive at the situation for which $\mu_{ki} = \delta_{ki}$. Thus, the mapping $F$ may be taken to have the following form:

$$x_i \mapsto x_i - \sum_{k=2}^{m} F_{ki}. \quad (6.1)$$

Suppose we have a mapping as in (6.1). Then the Jacobi matrix can be written as $E - G_1 - \cdots - G_{m-1}$ where $G_i$ is an $n \times n$ matrix with entries which are homogeneous polynomials of degree $i$. If the Jacobian is 1, then it is invertible with inverse a polynomial matrix (of homogeneous degree at most $(n-1)(m-1)$, obtained via the adjoint matrix).

If we write the inverse as a formal power series, we compare the homogeneous components and get:

$$\sum_{j_i, m_j = s} M_J = 0, \quad (6.2)$$

where $M_J$ is the sum of products $a_{\alpha_1} a_{\alpha_q}$ in which the factor $a_j$ occurs $m_j$ times, and $J$ denotes the multi-index $(j_1, \ldots, j_q)$.

Yagzhev considered the cubic homogeneous mapping $\vec{x} \mapsto \vec{x} + (\vec{x}, \vec{x}, \vec{x})$, whereby the Jacobian matrix becomes $E - G_3$. We return to this case in Remark 6.1.8. The slightly more general approach given here presents the Yagzhev correspondence more clearly and also provides tools for investigating deformations and packing properties (see Section 6.2.2). Thus, we consider not only the cubic case (i.e. when the mapping has the form

$$x_i \mapsto x_i + P_i(x_1, \ldots, x_n); \ i = 1, \ldots, n,$$

with $P_i$ cubic homogeneous polynomials), but the more general situation of arbitrary degree.

For any $\ell$, the set of (vector valued) forms $\{F_{\ell,i}\}_{i=1}^n$ can be interpreted as a homogeneous mapping $\Phi_{\ell} : V \mapsto V$ of degree $\ell$. When $\text{Char}(k)$ does not divide $\ell$, we take instead the polarization of this mapping, i.e. the multilinear symmetric mapping

$$\Psi_{\ell} : V^{\otimes \ell} \mapsto V$$

such that

$$(F_{\ell,i}(x_1), \ldots, F_{\ell,i}(x_n)) = \Psi_{\ell}(\vec{x}, \ldots, \vec{x}) \cdot \ell!$$

Then Equation (6.1) can be rewritten as

$$\vec{x} \mapsto \vec{x} - \sum_{\ell=2}^{m} \Psi_{\ell}(\vec{x}, \ldots, \vec{x}). \quad (6.3)$$
We define the algebra \((A, \{\Psi_\ell\})\), where \(A\) is the vector space \(V\) and the \(\Psi_\ell\) are viewed as operators \(A^\ell \mapsto A\).

**Definition 6.1.2.** The *Yagzhev correspondence* is the correspondence from the polynomial mapping \((V, F)\) to the algebra \((A, \{\Psi_\ell\})\).

### 6.1.2 Translation of the invertibility condition to the language of identities

The next step is to bring in algebraic varieties, defined in terms of identities.

**Definition 6.1.3.** A **polynomial identity** (PI) of \(A\) is a polynomial mapping of \(A\), all of whose values are identically zero.

The algebraic variety generated by an algebra \(A\), denoted as \(\text{Var}(A)\), is the class of all algebras satisfying the same PIs as \(A\).

Now we come to a crucial idea of Yagzhev:

*The invertibility of \(F\) and the invertibility of the Jacobian of \(F\) can be expressed via (2) in the language of polynomial identities.*

Namely, let \(y = F(x) = x - \sum_{\ell=2}^{m} \Psi_\ell(x)\). Then

\[
F^{-1}(x) = \sum t(x),
\]

where each \(t\) is a *term*, a formal expression in the mappings \(\{\Psi_\ell\}_{\ell=2}^{m}\) and the symbol \(x\). Note that the expressions \(\Psi_2(x, \Psi_3(x, x, x))\) and \(\Psi_2(\Psi_3(x, x, x), x)\) are different although they represent same element of the algebra. Denote by \(|t|\) the number of occurrences of variables, including multiplicity, which are included in \(t\).

The invertibility of \(F\) means that, for all \(q \geq q_0\),

\[
\sum_{|t|=q} t(a) = 0, \quad \forall a \in A.
\]

Thus we have translated invertibility of the mapping \(F\) to the language of identities. (Yagzhev had an analogous formula, where the terms only involved \(\Psi_3\).)

**Definition 6.1.4.** An element \(a \in A\) is called **nilpotent of index** \(\leq n\) if

\[
M(a, a, \ldots, a) = 0
\]

for each monomial \(M(x_1, x_2, \ldots)\) of degree \(\geq n\). \(A\) is weakly nilpotent if each element of \(A\) is nilpotent. \(A\) is weakly nilpotent of class \(k\) if each element of \(A\) is nilpotent of index \(k\). (Some authors use the terminology index instead of class.) Equation (6.5) means \(A\) is weakly nilpotent.
To stress this fundamental notion of Yagzhev, we define a Yagzhev algebra of order $q_0$ to be a weakly nilpotent algebra, i.e., satisfying the identities (6.5), also called the system of Yagzhev identities arising from $F$.

Summarizing, we get the following fundamental translation from conditions on the endomorphism $F$ to identities of algebras.

**Theorem 6.1.5.** The endomorphism $F$ is invertible if and only if the corresponding algebra is a Yagzhev algebra of high enough order.

**Algebras of Engel type**

The analogous procedure can be carried out for the differential mapping. We recall that $\Psi_\ell$ is a symmetric multilinear mapping of degree $\ell$. We denote the mapping $y \mapsto \Psi_\ell(y, x, \ldots, x)$ as $\text{Ad}_{\ell-1}(x)$.

**Definition 6.1.6.** An algebra $A$ is of Engel type $s$ if it satisfies a system of identities

$$
\sum_{\ell m_\ell = s} \sum_{\alpha_1 + \cdots + \alpha_q = m_\ell} \text{Ad}_{\alpha_1}(x) \cdots \text{Ad}_{\alpha_q}(x) = 0.
$$

(6.6)

$A$ is of Engel type if $A$ has Engel type $s$ for some $s$.

**Theorem 6.1.7.** The endomorphism $F$ has Jacobian 1 if and only if the corresponding algebra has Engel type $s$ for some $s$.

**Proof.** Let $x' = x + dx$. Then

\[
\Psi_\ell(x') = \Psi_\ell(x) + \ell \Psi_\ell(dx, x, \ldots, x) + \text{forms containing more than one occurrence of $dx$}.
\]

(6.7)

Hence the differential of the mapping

$$
F : \bar{x} \mapsto \bar{x} - \sum_{\ell=2}^m \Psi_\ell(\bar{x}, \ldots, \bar{x})
$$

is

$$
\left( E - \sum_{\ell=2}^m \ell \text{Ad}_{\ell-1}(x) \right) \cdot dx.
$$

The identities (6.2) are equivalent to the system of identities (6.6) in the signature $\Omega = (\Psi_2, \ldots, \Psi_m)$, taking $a_{\alpha_j} = \text{Ad}_{\alpha_j}$ and $m_j = \deg \Psi_\ell - 1$. 

\[\square\]
Thus, we have reformulated the condition of invertibility of the Jacobian in the language of identities.

As explained in [199], it is well known from [19] and [218] that the Jacobian Conjecture can be reduced to the cubic homogeneous case; i.e., it is enough to consider mappings of type

\[ x \mapsto x + \Psi_3(x, x, x). \]

In this case the Jacobian assumption is equivalent to the Engel condition – nilpotence of the mapping \( \text{Ad}_3(x)[y] \) (i.e. the mapping \( y \mapsto (y, x, x) \)). Invertibility, considered in [19], is equivalent to weak nilpotence, i.e., to the identity \( \sum_{|\ell|=k} \ell = 0 \) holding for all sufficiently large \( k \).

**Remark 6.1.8.** In the cubic homogeneous case, \( j = 1, \alpha_j = 2 \) and \( m_j = s \), and we define the linear map

\[ \text{Ad}_{xx} : y \mapsto (x, x, y) \]

and the index set \( T_j \subset \{1, \ldots, q\} \) such that \( i \in T_j \) if and only if \( \alpha_i = j \).

Then the equality (6.6) has the following form:

\[ \text{Ad}_{xx}^{s/2} = 0. \]

Thus, for a ternary symmetric algebra, Engel type means that the operators \( \text{Ad}_{xx} \) for all \( x \) are nilpotent. In other words, the mapping

\[ \text{Ad}_3(x) : y \mapsto (x, x, y) \]

is nilpotent. Yagzhev called this the Engel condition. (For Lie algebras the nilpotence of the operator \( \text{Ad}_x : y \mapsto (x, y) \) is the usual Engel condition. Here we have a generalization for arbitrary signature.)

Here are Yagzhev’s original definitions, for edification. A binary algebra \( A \) is Engelian if for any element \( a \in A \) the subalgebra \( \langle R_a, L_a \rangle \) of vector space endomorphisms of \( A \) generated by the left multiplication operator \( L_a \) and the right multiplication operator \( R_a \) is nilpotent, and weakly Engelian if for any element \( a \in A \) the operator \( R_a + L_a \) is nilpotent.

This leads us to the Generalized Jacobian Conjecture:

**Conjecture.** Let \( A \) be an algebra with symmetric \( k \)-linear operators \( \Psi_\ell \), for \( \ell = 1, \ldots, m \). In any variety of Engel type, \( A \) is a Yagzhev algebra.

By Theorem 6.1.7, this conjecture would yield the Jacobian Conjecture.

**The case of binary algebras**

When \( A \) is a binary algebra, Engel type means that the left and right multiplication mappings are both nilpotent.
A well-known result of S. Wang [19] shows that the Jacobian Conjecture holds for quadratic mappings

$$\vec{x} \mapsto \vec{x} + \Psi_2(\vec{x}, \vec{x}).$$

If two different points \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) of an affine space are mapped to the same point by \((f_1, \ldots, f_n)\), then the fact that the vertex of a parabola is in the middle of the interval whose endpoints are at the roots shows that all \(f_i(\vec{x})\) have gradients at this midpoint \(P = (\vec{x} + \vec{y})/2\) perpendicular to the line segment \([\vec{x}, \vec{y}]\). Hence the Jacobian is zero at the midpoint \(P\). This fact holds in any characteristic \(\neq 2\).

In Section 6.1.3 we prove the following theorem of Yagzhev, cf. Definition 6.1.12 below:

**Theorem 6.1.9** (Yagzhev). Every symmetric binary Engel type algebra of order \(k\) satisfying the system of Capelli identities of order \(n\) is weakly nilpotent, of weak nilpotence index bounded by some function \(F(k, n)\).

**Remark 6.1.10.** Yagzhev formulates his theorem in the following way:

Every binary weakly Engel algebra of order \(k\) satisfying the system of Capelli identities of order \(n\) is weakly nilpotent, of index bounded by some function \(F(k, n)\).

We obtain this reformulation, by replacing the algebra \(A\) by the algebra \(A^+\) with multiplication given by \((a, b) = ab + ba\).

The following problems may help us understand the situation:

**Problem.** Obtain a straightforward proof of this theorem and deduce from it the Jacobian Conjecture for quadratic mappings.

**Problem. (Generalized Jacobian Conjecture for quadratic mappings)** Is every symmetric binary algebra of Engel type \(k\), a Yagzhev algebra?

**The case of ternary algebras**

As we have observed, Yagzhev reduced the Jacobian Conjecture over a field of characteristic zero to the question:

Is every finite dimensional ternary Engel algebra a Yagzhev algebra?

Drużkowski [82, 83] reduced this to the case when all cubic forms \(\Psi_{3i}\) are cubes of linear forms. Van den Essen and his school reduced the JC to the symmetric case; see [88, 89] for details. Bass, Connell, and Wright [19] use other methods including inversions. Yagzhev’s approach matches that of [19], but using identities instead.
An example in nonzero characteristic of an Engel algebra that is not a Yagzhev algebra

Now we give an example, over an arbitrary field \( k \) of characteristic \( p > 3 \), of a finite dimensional Engel algebra that is not a Yagzhev algebra, i.e., not weakly nilpotent. This means that the situation for binary algebras differs intrinsically from that for ternary algebras, and it would be worthwhile to understand why.

**Theorem 6.1.11.** If \( \text{Char}(k) = p > 3 \), then there exists a finite dimensional \( k \)-algebra that is Engel but not weakly nilpotent.

**Proof.** Consider the noninvertible mapping \( F : k[x] \mapsto k[x] \) with Jacobian 1:

\[
F : x \mapsto x + x^p.
\]

We introduce new commuting indeterminates \( \{y_i\}_{i=1}^n \) and extend this mapping to \( k[x, y_1, \ldots, y_n] \) by sending \( y_i \mapsto y_i \). If \( n \) is big enough, then it is possible to find tame automorphisms \( G_1 \) and \( G_2 \) such that \( G_1 \circ F \circ G_2 \) is a cubic mapping \( \vec{x} \mapsto \vec{x} + \Psi_3(\vec{x}) \), as follows:

Suppose we have a mapping \( F : x_i \mapsto P(x) + M \)

where \( M = t_1t_2t_3t_4 \) is a monomial of degree at least 4. Introduce two new commuting indeterminates \( z, y \) and take \( F(z) = z, F(y) = y \).

Define the mapping \( G_1 \) via \( G_1(z) = z + t_1t_2, G_1(y) = y + t_3t_4 \) with \( G_1 \) fixing all other indeterminates; define \( G_2 \) via \( G_2(x) = x - yz \) with \( G_2 \) fixing all other indeterminates.

The composite mapping \( G_1 \circ F \circ G_2 \) sends \( x \) to \( P(x) - yz - yt_1t_2 - zt_3t_4 \), \( y \) to \( y + t_3t_4 \), \( z \) to \( z + t_1t_2 \), and agrees with \( F \) on all other indeterminates.

Note that we have removed the monomial \( M = t_1t_2t_3t_4 \) from the image of \( F \), but instead have obtained various monomials of smaller degree \( (t_1t_2, t_3t_4, yz, zt_3t_4, yt_1t_2) \). It is easy to see that this process terminates.

Our new mapping \( H(x) = x + \Psi_2(x) + \Psi_3(x) \) is noninvertible and has Jacobian 1. Consider its blowup

\[
R : x \mapsto x + T^2y + T\Psi_2(x), \quad y \mapsto y - \Psi_3(x), \quad T \mapsto T.
\]

This mapping \( R \) is invertible if and only if the initial mapping is invertible, and has Jacobian 1 if and only if the initial mapping has Jacobian 1, by [218, Lemma 2]. This mapping is also cubic homogeneous. The corresponding ternary algebra is Engel, but not weakly nilpotent. \( \Box \)

This example shows that a direct combinatorial approach to the Jacobian Conjecture encounters difficulties, and in working with related Burnside type problems (in the sense of Zelmanov [223], dealing with nilpotence properties of Engel algebras, as indicated in the introduction), one should take into account specific properties arising in characteristic zero.
Definition 6.1.11.1. An algebra $A$ is nilpotent of class $\leq n$ if $M(a_1, a_2, \ldots) = 0$ for each monomial $M(x_1, x_2, \ldots)$ of degree $\geq n$. An ideal $I$ of $A$ is strongly nilpotent of class $\leq n$ if $M(a_1, a_2, \ldots) = 0$ for each monomial $M(x_1, x_2, \ldots)$ in which indeterminates of total degree $\geq n$ have been substituted to elements of $I$.

Although the notions of nilpotent and strongly nilpotent coincide in the associative case, they differ for ideals of nonassociative algebras. For example, consider the following algebra suggested by Shestakov: $A$ is the algebra generated by $a, b, z$ satisfying the relations $a^2 = b, bz = a$ and all other products 0. Then $I = Fa + Fb$ is nilpotent as a subalgebra, satisfying $I^3 = 0$ but not strongly nilpotent (as an ideal), since

$$b = ((a(bz))z)a \neq 0,$$

and one can continue indefinitely in this vein. Also, [103] contains an example of a finite dimensional non-associative algebra without any ideal which is maximal with respect to being nilpotent as a subalgebra.

In connection with the Generalized Jacobian Conjecture in characteristic 0, it follows from results of Yagzhev [220], also cf. [93], that there exists a 20-dimensional Engel algebra over $\mathbb{Q}$, not weakly nilpotent, satisfying the identities

$$x^2y = -yx^2, \quad ((yx^2)x^2)x^2 = 0,$$

$$(xy + yx)y = 2y^2x, \quad x^2y^2 = 0.$$

However, this algebra can be seen to be Yagzhev (see Definition 6.1.4).

For associative algebras, one uses the term “nil” instead of “weakly nilpotent.” Any nil subalgebra of a finite dimensional associative algebra is nilpotent, by Wedderburn’s Theorem [204]). Jacobson generalized this result to other settings, cf. [153, Theorem 15.23], and Shestakov [173] generalized it to a wide class of Jordan algebras (not necessarily commutative).

Yagzhev’s investigation of weak nilpotence has applications to the Koethe Conjecture, for algebras over uncountable fields. He reproved:

* In every associative algebra over an uncountable field, the sum of every two nil right ideals is a nil right ideal [207].

(This was proved first by Amitsur [3]. Amitsur’s result is for affine algebras, but one can easily reduce to the affine case.)

Algebras satisfying systems of Capelli identities

Definition 6.1.12. The Capelli polynomial $C_k$ of order $k$ is

$$C_k := \sum_{\sigma \in S_k} (-1)^{\sigma}x_{\sigma(1)}y_1 \cdots x_{\sigma(k)}y_k.$$
It is obvious that an associative algebra satisfies the Capelli identity $c_k$ iff, for any monomial $M(x_1, \ldots, x_k, y_1, \ldots, y_r)$ multilinear in the $x_i$, the following equation holds identically in $A$:

$$\sum_{\sigma \in S_k} (-1)^\sigma M(v_{\sigma(1)}, \ldots, v_{\sigma(k)}, y_1, \ldots, y_r) = 0.$$  \hfill (6.8)

However, this does not apply to nonassociative algebras, so we need to generalize this condition.

**Definition 6.1.13.** The algebra $A$ satisfies a system of Capelli identities of order $k$, if (6.8) holds identically in $A$ for any monomial $M(x_1, \ldots, x_k, y_1, \ldots, y_r)$ multilinear in the $x_i$.

Any algebra of dimension $< k$ over a field satisfies a system of Capelli identities of order $k$. Algebras satisfying systems of Capelli identities behave much like finite dimensional algebras. They were introduced and systematically studied by Rasmynlov [149], [150].

Using Rasmynlov’s method, Zubrilin [229], also see [151, 227], proved that if $A$ is an arbitrary algebra satisfying the system of Capelli identities of order $n$, then the chain of ideals defining the solvable radical stabilizes at the $n$-th step. More precisely, we utilize a Baer-type radical, along the lines of Amitsur [4].

Given an algebra $A$, we define $\text{Solv}_1 := \text{Solv}_1(A) = \sum\{\text{Strongly nilpotent ideals of } A\}$, and inductively, given $\text{Solv}_k$, define $\text{Solv}_{k+1}$ by $\text{Solv}_{k+1}/\text{Solv}_k = \text{Solv}_1(A/\text{Solv}_k)$. For a limit ordinal $\alpha$, define

$$\text{Solv}_\alpha = \cup_{\beta < \alpha} \text{Solv}_\beta.$$  

This must stabilize at some ordinal $\alpha$, for which we define $\text{Solv}(A) = \text{Solv}_\alpha$.

Clearly $\text{Solv}(A/\text{Solv}(A)) = 0$; i.e., $A/\text{Solv}(A)$ has no nonzero strongly nilpotent ideals. Actually, Amitsur [4] defines $\zeta(A)$ as built up from ideals having trivial multiplication, and proves [4, Theorem 1.1] that $\zeta(A)$ is the intersection of the prime ideals of $A$.

We shall use the notion of sandwich, introduced by Kaplansky and Kostrikin, which is a powerful tool for Burnside type problems [223]. An ideal $I$ is called a sandwich ideal if, for any $k$,

$$M(z_1, z_2, x_1, \ldots, x_k) = 0$$

for any $z_1, z_2 \in I$, any set of elements $x_1, \ldots, x_k$, and any multilinear monomial $M$ of degree $k+2$. (Similarly, if the operations of an algebra have degree $\leq \ell$, then it is natural to use $\ell$-sandwiches, which by definition satisfy the property that

$$M(z_1, \ldots, z_\ell, x_1, \ldots, x_k) = 0$$

for any $z_1, \ldots, z_\ell \in I$, any set of elements $x_1, \ldots, x_k$, and any multilinear monomial $M$ of degree $k + \ell$.)

The next useful lemma follows from a result from [229]:

150
Lemma 6.1.14. If an ideal $I$ is strongly nilpotent of class $\ell$, then there exists a decreasing sequence of ideals $I = I_1 \supseteq \cdots \supseteq I_{\ell+1} = 0$ such that $I_s/I_{s+1}$ is a sandwich ideal in $A/I_{s+1}$ for all $s \leq \ell$.

Definition 6.1.15. An algebra $A$ is representable if it can be embedded into an algebra finite dimensional over some extension of the ground field.

Remark 6.1.16. Zubrilin [229], properly clarified, proved the more precise statement, that if an algebra $A$ of arbitrary signature satisfies a system of Capelli identities $C_{n+1}$, then there exists a sequence $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n$ of strongly nilpotent ideals such that:

- The natural projection of $B_i$ in $A/B_{i-1}$ is a strongly nilpotent ideal.
- $A/B_n$ is representable.
- If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ is any sequence of ideals of $A$ such that $I_{j+1}/I_j$ is a sandwich ideal in $A/I_j$, then $B_n \supseteq I_n$.

Such a sequence of ideals will be called a Baer-Amitsur sequence. In affine space the Zariski closure of the radical is radical, and hence the factor algebra is representable. (Although the radical coincides with the linear closure if the base field is infinite (see [35]), this assertion holds for arbitrary signatures and base fields.) Hence in representable algebras, the Baer-Amitsur sequence stabilizes after finitely many steps. Lemma 6.1.14 follows from these considerations.

Our next main goal is to prove Theorem 6.1.18 below, but first we need another notion.

The tree associated to a monomial

Effects of nilpotence have been used by different authors in another language. We associate a rooted labelled tree to any monomial: Any branching vertex indicates the symbol of an operator, whose outgoing edges are the terms in the corresponding symbol. Here is the precise definition.

Definition 6.1.17. Let $M(x_1, \ldots, x_n)$ be a monomial in an algebra $A$ of arbitrary signature. One can associate the tree $T_M$ by an inductive procedure:

- If $M$ is a single variable, then $T_M$ is just the vertex $\bullet$.
- Let $M = g(M_1, \ldots, M_k)$, where $g$ is a $k$-ary operator. We assume inductively that the trees $T_i$, $i = 1, \ldots, k$, are already defined. Then the tree $T_M$ is the disjoint union of the $T_i$, together with the root $\bullet$ and arrows starting with $\bullet$ and ending with the roots of the trees $T_i$.

Remark. Sometimes one labels $T_M$ according to the operator $g$ and the positions inside $g$.

If the outgoing degree of each vertex is 0 or 2, the tree is called binary. If the outgoing degree of each vertex is either 0 or 3, the tree is called ternary. If each operator is binary, $T_M$ will be binary; if each operator is ternary, $T_M$ will be ternary.
6.1.3 Lifting Yagzhev algebras

Recall Definitions 6.1.4 and 6.1.6.

**Theorem 6.1.18.** Suppose $A$ is an algebra of Engel type, and let $I$ be a sandwich ideal of $A$. If $A/I$ is Yagzhev, then $A$ is Yagzhev.

**Proof.** The proof follows easily from the following two propositions.

Let $k$ be the class of weak nilpotence of $A/I$. We call a branch of the tree *fat* if it has more than $k$ entries.

**Proposition 6.1.18.1.** a) The sum of all monomials of any degree $s > k$ belongs to $I$.

b) Let $x_1, \ldots, x_n$ be fixed indeterminates, and $M$ be an arbitrary monomial, with $s_1, \ldots, s_\ell > k$. Then

$$
\sum_{|t_1| = s_1, \ldots, |t_\ell| = s_\ell} M(x_1, \ldots, x_n, t_1, \ldots, t_\ell) \equiv 0. \quad (6.9)
$$

c) The sum of all monomials of degree $s$, containing at least $\ell$ non-intersecting fat branches, is zero.

**Proof.** a) is just a reformulation of the weak nilpotence of $A/I$; b) follows from a) and the sandwich property of an ideal $I$. To get c) from b), it is enough to consider the highest non-intersecting fat branches.

**Proposition 6.1.18.2 (Yagzhev).** The linearization of the sum of all terms with a fixed fat branch of length $n$ is the complete linearization of the function

$$
\sum_{\sigma \in S_n} \prod_{i=1}^{\ell} (Ad_{k_{s(i)}})(z)(t).
$$

Theorem 1.2, Lemma 6.1.14, and Zubrilin’s result give us the following major result:

**Theorem 6.1.19.** In characteristic zero, the Jacobian conjecture is equivalent to the following statement:

Any algebra of Engel type satisfying some system of Capelli identities is a Yagzhev algebra.

This theorem generalizes the following result of Yagzhev:

**Theorem 6.1.20.** The Jacobian conjecture is equivalent to the following statement:

Any ternary Engel algebra in characteristic 0 satisfying a system of Capelli identities is a Yagzhev algebra.

The Yagzhev correspondence and the results of this section (in particular, Theorem 6.1.19) yield the proof of Theorem 6.1.9.
Sparse identities

Generalizing Capelli identities, we say that an algebra satisfies a system of sparse identities when there exist \( k \) and coefficients \( \alpha_\sigma \) such that for any monomial \( M(x_1, \ldots, x_k, y_1, \ldots, y_r) \) multilinear in \( x_i \) the following equation holds:

\[
\sum_\sigma \alpha_\sigma M(c_1 v_{\sigma(1)} d_1, \ldots, c_k v_{\sigma(k)} d_k, y_1, \ldots, y_r) = 0.
\] (6.10)

Note that one need only check (6.10) for monomials. The system of Capelli identities is a special case of a system of sparse identities (when \( \alpha_\sigma = (-1)^\sigma \)). This concept ties in with the following “few long branches” lemma [228], concerning the structure of trees of monomials for algebras with sparse identities:

**Lemma 6.1.21** (Few long branches). Suppose an algebra \( A \) satisfies a system of sparse identities of order \( m \). Then any monomial is linearly representable by monomials such that the corresponding tree has not more than \( m - 1 \) disjoint branches of length \( \geq m \).

Lemma 6.1.21 may be useful in studying nilpotence of Engel algebras.

### 6.1.4 Inversion formulas and problems of Burnside type

We have seen that the JC relates to problems of “Specht type” (concerning whether one set of polynomial identities implies another), as well as problems of Burnside type.

Burnside type problems become more complicated in nonzero characteristic; cf. Zelmanov’s review article [223].

Bass, Connell, and Wright [19] attacked the JC by means of inversion formulas. D. Wright [205] wrote an inversion formula for the symmetric case and related it to a combinatorial structure called the Grossman–Larson Algebra. Namely, write \( F = X - H \), and define \( J(H) \) to be the Jacobian matrix of \( H \). Wright proved the JC for the case where \( H \) is homogeneous and \( J(H)^3 = 0 \), and also for the case where \( H \) is cubic and \( J(H)^4 = 0 \); these correspond in Yagzhev’s terminology to the cases of Engel type 3 and 4, respectively. Also, the so-called *chain vanishing theorem* in [205] follows from Engel type. Similar results were obtained earlier by Singer [183] using tree formulas for formal inverses. The inversion formula, introduced in [19], was investigated by D. Wright and his school. Many authors use the language of so-called *tree expansion* (see [183, 205] for details). In view of Theorem 6.1.11, the tree expansion technique should be highly nontrivial.

The Jacobian Conjecture can be formulated as a question of quantum field theory (see [1]), in which tree expansions are seen to correspond to Feynmann diagrams.

In the papers [183] and [205] (see also [206]), trees with one label correspond to elements of the algebra \( A \) built by Yagzhev, and 2-labelled trees correspond to the elements of the operator algebra \( D(A) \) (the algebra generated by operators \( x \mapsto M(x, \bar{y}) \), where \( M \) is some monomial). These authors deduce weak nilpotence from the Engel conditions of degree 3 and 4. The inversion
formula for automorphisms of tensor product of Weyl algebras and the ring of polynomials was studied intensively in the papers [25,28]. Using techniques from [40], this yields a slightly different proof of the equivalence between the JC and DC, by an argument similar to one given in [221]. Yagzhev’s approach makes the situation much clearer, and the known approaches to the Jacobian Conjecture using inversion formulas can be explained from this viewpoint.

Remark 6.1.22. The most recent inversion formula (and probably the most algebraically explicit one) was obtained by V. Bavula [24]. The coefficient $q_0$ can be made explicit in (6.5), by means of the Gabber Inequality, which says that if

$$f : K^n \mapsto K^n; \quad x_i \mapsto f_i(\vec{x})$$

is a polynomial automorphism, with $\deg(f) = \max_i \deg(f_i)$, then $\deg(f^{-1}) \leq \deg(f)^{n-1}$

In fact, we are working with operads, cf. the classical book [140]. A review of operad theory and its relation with physics and PI-theory in particular Burnside type problems, will appear in D. Piontkovsky [147]; see also [105, 148]. Operad theory provides a supply of natural identities and varieties, but they also correspond to geometric facts. For example, the Jacobi identity corresponds to the fact that the altitudes of a triangle are concurrent. M. Dehn’s observations that the Desargue property of a projective plane corresponds to associativity of its coordinate ring, and Pappus’ property to its commutativity, can be considered as a first step in operad theory. Operads are important in mathematical physics, and formulas for the famous Kontsevich quantization theorem resemble formulas for the inverse mapping. The operators considered here are operads.

6.2 The Jacobian Conjecture for varieties, and deformations

In this section we consider analogs of the JC for other varieties of algebras, partially with the aim on throwing light on the classical JC (for the commutative associative polynomial algebra).

6.2.1 Generalization of the Jacobian Conjecture to arbitrary varieties

J. Birman [46] already proved the JC for free groups in 1973. The JC for free associative algebras (in two generators) was established in 1982 by W. Dicks and J. Levin [71,72], utilizing Fox derivatives, which we describe later on. Their result was reproved by Yagzhev [216], whose ideas are sketched in this section. Also see Schofield [169], who proved the full version. Yagzhev then applied these ideas to other varieties of algebras [215,220] including nonassociative commutative algebras and anti-commutative algebras; U.U. Umirbaev [192] generalized these to “Schreier varieties,”
defined by the property that every subalgebra of a free algebra is free. The JC for free Lie algebras was proved by Reutenauer [152], Shpilrain [182], and Umirbaev [191].

The Jacobian Conjecture for varieties generated by finite dimensional algebras, is closely related to the Jacobian Conjecture in the usual commutative associative case, which is the most important.

Let $M$ be a variety of algebras of some signature $\Omega$ over a given field $k$ of characteristic zero, and $kM < \vec{x}$ the relatively free algebra in $M$ with generators $\vec{x} = \{x_i : i \in I\}$. We assume that $|\Omega|, |I| < \infty$, $I = 1, \ldots, n$.

Take a set $\vec{y} = \{y_i\}_{i=1}^n$ of new indeterminates. For any $f(\vec{x}) \in kM < \vec{x}$ one can define an element $\hat{f}(\vec{x}, \vec{y}) \in kM < \vec{x}, \vec{y}$ via the equation

$$f(x_1 + y_1, \ldots, x_n + y_n) = f(\vec{y}) + \hat{f}(\vec{x}, \vec{y}) + R(\vec{x}, \vec{y})$$

(6.11)

where $\hat{f}(\vec{x}, \vec{y})$ has degree 1 with respect to $\vec{x}$, and $R(\vec{x}, \vec{y})$ is the sum of monomials of degree $\geq 2$ with respect to $\vec{x}$; $\hat{f}$ is a generalization of the differential.

Let $\alpha \in \text{End}(kM < \vec{x})$, i.e.,

$$\alpha : x_i \mapsto f_i(\vec{x}); \; i = 1, \ldots, n.$$  

(6.12)

**Definition 6.2.1.** Define the Jacobi endomorphism $\hat{\alpha} \in \text{End}(kM < \vec{x}, \vec{y})$ via the equality

$$\hat{\alpha} : \left\{ \begin{array}{c} x_i \mapsto \hat{f}_i(\vec{x}), \\ y_i \mapsto y_i \end{array} \right.$$  

(6.13)

The Jacobi mapping $f \mapsto \hat{f}$ satisfies the chain rule, in the sense that it preserves composition.

**Remark 6.2.2.** It is not difficult to check (and is well known) that if $\alpha \in \text{Aut}(kM < \vec{x})$ then $\hat{\alpha} \in \text{Aut}(kM < \vec{x}, \vec{y})$.

The inverse implication is called the *Jacobian Conjecture for the variety $\mathcal{M}$. Here is an important special case.*

**Definition 6.2.3.** Let $A \in \mathcal{M}$ be a finite dimensional algebra, with base $\{\vec{e}_i\}_{i=1}^N$. Consider a set of commutative indeterminates $\vec{\nu} = \{\nu_{si} | s = 1, \ldots, n; i = 1, \ldots, N\}$. The elements

$$z_j = \sum_{i=1}^N \nu_{ji} \vec{e}_i; \; j = 1, \ldots, n$$

are called *generic elements of $A$.*

Usually in the matrix algebra $M_m(k)$, the set of matrix units $\{e_{ij}\}_{i,j=1}^m$ is taken as the base. In this case $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and $z_i = \sum_{ij} \lambda_{ij} e_{ij}$, $i = 1, \ldots, n$.

**Definition 6.2.4.** A *generic matrix* is a matrix whose entries are distinct commutative indeterminates, and the so-called algebra of generic matrices of order $m$ is generated by associative generic $m \times m$ matrices.
The algebra of generic matrices is prime, and every prime, relatively free, finitely generated associative PI-algebra is isomorphic to an algebra of generic matrices. If we include taking traces as an operator in the signature, then we get the algebra of generic matrices with trace. That algebra is a Noetherian module over its center.

Define the \( k \)-linear mappings

\[
\Omega_i : k[M < \vec{x}>] \mapsto k[\nu]; \quad i = 1, \ldots, n
\]

via the relation

\[
f(\sum_{i=1}^{N} \nu_i e_i, \ldots, \sum_{i=1}^{N} \nu_{ni} e_i) = \sum_{i=1}^{N} (f \Omega_i) e_i.
\]

It is easy to see that the polynomials \( f \Omega_i \) are uniquely determined by \( f \).

One can define the mapping

\[
\varphi_A : \text{End}(k[M < \vec{x}>]) \mapsto \text{End}(k[\vec{\nu}])
\]

as follows: If

\[
\alpha \in \text{End}(k[M < \vec{x}>]) : x_s \mapsto f_s(\vec{x}) \quad s = 1, \ldots, n
\]

then \( \varphi_A(\alpha) \in \text{End}(k[\vec{\nu}]) \) can be defined via the relation

\[
\varphi_A(\alpha) : \nu_{si} \mapsto P_{si}(\vec{\nu}); \quad s = 1, \ldots, n; \quad i = 1, \ldots, n,
\]

where \( P_{si}(\vec{\nu}) = f_s \Omega_i \).

The following proposition is well known.

**Proposition 6.2.4.1 ([220]).** Let \( A \in \mathcal{M} \) be a finite dimensional algebra, and \( \vec{x} = \{x_1, \ldots, x_n\} \) be a finite set of commutative indeterminates. Then the mapping \( \varphi_A \) is a semigroup homomorphism, sending 1 to 1, and automorphisms to automorphisms. Also the mapping \( \varphi_A \) commutes with the operation \( \hat{\circ} \) of taking the Jacobi endomorphism, in the sense that \( \hat{\varphi_A(\alpha)} = \varphi_A(\hat{\alpha}) \). If \( \varphi \) is invertible, then \( \hat{\varphi} \) is also invertible.

This proposition is important, since as noted after Remark 6.2.2, the opposite direction is the JC.

### 6.2.2 Deformations and the Jacobian Conjecture for free associative algebras

**Definition 6.2.5.** A \( T \)-ideal is a completely characteristic ideal, i.e., stable under any endomorphism.

**Proposition 6.2.5.1.** Suppose \( A \) is a relatively free algebra in the variety \( \mathcal{M} \), \( I \) is a \( T \)-ideal in \( A \), and \( \mathcal{M}' = \text{Var}(A/I) \). Any polynomial mapping \( F : A \mapsto A \) induces a natural mapping
\( F' : A/I \mapsto A/I \), as well as a mapping \( \widehat{F}' \) in \( M' \). If \( F \) is invertible, then \( F' \) is invertible; if \( \widehat{F} \) is invertible, then \( \widehat{F}' \) is also invertible.

For example, let \( F \) be a polynomial endomorphism of the free associative algebra \( k<\vec{x}> \), and \( I_n \) be the \( T \)-ideal of the algebra of generic matrices of order \( n \). Then \( F(I_n) \subseteq I_n \) for all \( n \). Hence \( F \) induces an endomorphism \( F_{I_n} \) of \( k<\vec{x}> / I_n \). In particular, this is a semigroup homomorphism. Thus, if \( F \) is invertible, then \( F_{I_n} \) is invertible, but not vice versa.

The Jacobian mapping \( \widehat{F}_{I_n} \) of the reduced endomorphism \( F_{I_n} \) is the reduction of the Jacobian mapping of \( F \).

The Jacobian Conjecture and the packing property

This subsection is based on the packing property and deformations. Let us illustrate the main idea. It is well known that the composite of ALL quadratic extensions of \( \mathbb{Q} \) is infinite dimensional over \( \mathbb{Q} \). Hence all such extensions cannot be embedded (“packed”) into a single commutative finite dimensional \( \mathbb{Q} \)-algebra. However, all of them can be packed into \( M_2(\mathbb{Q}) \). We formalize the notion of packing in §6.2.5. Moreover, for ANY elements NOT in \( \mathbb{Q} \) there is a parametric family of embeddings (because it embeds non-centrally and thus can be deformed via conjugation by a parametric set of matrices). Uniqueness thus means belonging to the center. Similarly, adjoining noncommutative coefficients allows one to decompose polynomials, as to be elaborated below.

This idea allows us to solve equations via a finite dimensional extension, and to find a parametric sets of solutions if some solution does not belong to the original algebra. That situation contradicts local invertibility.

Let \( F \) be an endomorphism of the free associative algebra having invertible Jacobian. We suppose that \( F(0) = 0 \) and

\[
F(x_i) = x_i + \sum \text{terms of order } \geq 2.
\]

We intend to show how the invertibility of the Jacobian implies invertibility of the mapping \( F \).

Let \( Y_1, \ldots, Y_k \) be generic \( m \times m \) matrices. Consider the system of equations

\[
\{ F_i(X_1, \ldots, X_n) = Y_i; \quad i = 1, \ldots, k \}.
\]

This system has a solution over some finite extension of order \( m \) of the field generated by the center of the algebra of generic matrices with trace.

Consider the set of block diagonal \( mn \times mn \) matrices:

\[
A = \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & A_n
\end{pmatrix},
\] (6.14)
where the $A_j$ are $m \times m$ matrices.

Next, we consider the system of equations

$$\{F_i(X_1, \ldots, X_n) = Y_i; \ i = 1, \ldots, k\}, \quad (6.15)$$

where the $mn \times mn$ matrices $Y_i$ have the form (6.14) with the $A_j$ generic matrices.

Any $m$-dimensional extension of the base field $k$ is embedded into $M_m(k)$. But $M_m(k) \cong M_m(k) \otimes M_n(k)$. It follows that for appropriate $m$, the system (6.15) has a unique solution in the matrix ring with traces. (Each is given by a matrix power series where the summands are matrices whose entries are homogeneous forms, seen by rewriting $Y_i = X_i + \sum$ terms of order 2 as $X_i = Y_i + \sum$ terms of order 2, and iterating.) The solution is unique since their entries are distinct commuting indeterminates.

If $F$ is invertible, then this solution must have block diagonal form. However, if $F$ is not invertible, this solution need not have block diagonal form. Now we translate invertibility of the Jacobian to the language of parametric families or deformations.

Consider the matrices

$$E^\ell_\lambda = \begin{pmatrix} E & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & 0 \\ 0 & \ldots & \lambda \cdot E & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & \ldots & \ldots & E \end{pmatrix}$$

where $E$ denotes the identity matrix. (The index $\ell$ designates the position of the block $\lambda \cdot E$.) Taking $X_j$ not to be a block diagonal matrix, then for some $\ell$ we obtain a non-constant parametric family $E^\ell_\lambda X_j(E^\ell_\lambda)^{-1}$ dependent on $\lambda$.

On the other hand, if $Y_i$ has form (6.14) then $E^\ell_\lambda Y_i(E^\ell_\lambda)^{-1} = Y_i$ for all $\lambda \neq 0; \ \ell = 1, \ldots, k$.

Hence, if $F_1$ is not an automorphism, then we have a continuous parametric set of solutions. But if the Jacobian mapping is invertible, it is locally 1:1, a contradiction. This argument yields the following result:

**Theorem 6.2.6.** For $F \in \text{End}(k<\vec{x}>)$, if the Jacobian of $F$ is invertible, then the reduction $F_1$ of $F$, modulo the $T$-ideal of the algebra of generic matrices, is invertible.

See [215] for further details of the proof. Because any relatively free affine algebra of characteristic 0 satisfies the set of identities of some matrix algebra, it is the quotient of the algebra of generic matrices by some $T$-ideal $J$. But $J$ maps into itself after any endomorphism of the algebra. We conclude:

**Corollary 6.2.7.** If $F \in \text{End}(k<\vec{x}>)$ and the Jacobian of $F$ is invertible, then the reduction $F_J$ of $F$ modulo any proper $T$-ideal $J$ is invertible.

In order to get invertibility of $\vec{F}$ itself, Yagzhev used the additional ideas:
• The block diagonal technique works equally well on skew fields.

• The above algebraic constructions can be carried out on Ore extensions, in particular for the Weyl algebras \( W_n = \mathbb{k}[x_1, \ldots, x_n; \partial_1, \ldots, \partial_n] \).

• By a result of L. Makar-Limanov, the free associative algebra can be embedded into the ring of fractions of the Weyl algebra. This provides a nice presentation for mapping the free algebra.

**Definition 6.2.8.** Let \( A \) be an algebra, \( B \subset A \) a subalgebra, and \( \alpha : A \to A \) a polynomial mapping of \( A \) (and hence \( \alpha(B) \subset B \), see Definition 6.1.1). \( B \) is a test algebra for \( \alpha \), if \( \alpha(A \setminus B) \neq A \setminus B \).

The next theorem shows the universality of the notion of a test algebra. An endomorphism is called *rationally invertible* if it is invertible over Cohn’s skew field of fractions [63] of \( \mathbb{k} < \vec{x} > \).

**Theorem 6.2.9** (Yagzhev). For any \( \alpha \in \text{End}(\mathbb{k} < \vec{x} >) \), one of the two statements holds:

- \( \alpha \) is rationally invertible, and its reduction to any finite dimensional factor also is rationally invertible.

- There exists a test algebra for some finite dimensional reduction of \( \alpha \).

This theorem implies the Jacobian conjecture for free associative algebras. We do not go into details, referring the reader to the papers [215] and [220].

**Remark.** The same idea is used in quantum physics. The polynomial \( x^2 + y^2 + z^2 \) cannot be decomposed for any commutative ring of coefficients. However, it can decomposed using noncommutative ring of coefficients (Pauli matrices). The Laplace operator in 3-dimensional space can be decomposed in such a manner.

**Reduction to nonzero characteristic**

One can work with deformations equally well in nonzero characteristic. However, the naive Jacobian condition does not give us parametric families, because of consequences of inseparability. Hence it is interesting using deformations to get a reasonable version of the JC for characteristic \( p > 0 \), especially because of recent progress in the JC related to the reduction of holonomic modules to the case of characteristic \( p \) and investigation of the \( p \)-curvature or Poisson brackets on the center [40], [39], [187].

In his very last paper [221] A.V. Yagzhev approached the JC using positive characteristics. He noticed that the existence of a counterexample is equivalent to the existence of an Engel, but not Yagzhev, finite dimensional ternary algebra in each positive characteristic \( p \gg 0 \). (This fact is also used in the papers [39, 40, 187].)

If a counterexample to the JC exists, then such an algebra \( A \) exists even over a finite field, and hence can be finite. It generates a locally finite variety of algebras that are of Engel type, but not
Yagzhev. This situation can be reduced to the case of a locally semiprime variety. Any relatively free algebra of this variety is semiprime, and the centroid of its localization is a finite direct sum of fields. The situation can be reduced to one field, and he tried to construct an embedding which is not an automorphism. This would contradict the finiteness property.

Since a reduction of an endomorphism as a mapping on points of finite height may be an automorphism, the issue of injectivity also arises. However, this approach looks promising, and may involve new ideas, such as in the papers \[39,40,187\]. Perhaps different infinitesimal conditions (like the Jacobian condition in characteristic zero) can be found.

### 6.2.3 The Jacobian Conjecture for other classes of algebras

Although the Jacobian Conjecture remains open for commutative associative algebras, it has been established for other classes of algebras, including free associative algebras, free Lie algebras, and free metabelian algebras. See §6.2.1 for further details.

An algebra is metabelian if it satisfies the identity \([x, y][z, t] = 0\).

The case of free metabelian algebras, established by Umirbaev \[190\], involves some interesting new ideas that we describe now. His method of proof is by means of co-multiplication, taken from the theory of Hopf algebras and quantization. Let \(A^\text{op}\) denote the opposite algebra of the free associative algebra \(A\), with generators \(t_i\). For \(f \in A\) we denote the corresponding element of \(A^\text{op}\) as \(f^*\). Put \(\lambda : A^\text{op} \otimes A \mapsto A\) be the mapping such that \(\lambda(\sum f_i^* \otimes g_i) = \sum f_i g_i\). \(I_A := \ker(\lambda)\) is a free \(A\)-bimodule with generators \(t_i^* \otimes 1 - 1 \otimes t_i\). The mapping \(\tilde{d}_A : A \mapsto I_A\) such that \(\tilde{d}_A(a) = a^* \otimes 1 - 1 \otimes a\) is called the universal derivation of \(A\). The Fox derivatives \(\partial a/\partial t_i \in A^\text{op} \otimes A\) \[91\] are defined via \(d_A(a) = \sum_i(t_i^* \otimes 1 - 1 \otimes t_i)\partial a/\partial t_i\), cf. \[72\] and \[190\].

Let \(C = A/\text{Id}([A, A])\), the free commutative associative algebra, and let \(B = A/\text{Id}([A, A])^2\), the free metabelian algebra. Let

\[
\partial(a) = (\partial a/\partial t_1, \ldots, \partial a/\partial t_n).
\]

One can define the natural derivations

\[
\tilde{\partial} : A \mapsto (A' \otimes A)^n \mapsto (C' \otimes C)^n,
\]

\[
\tilde{\partial} : A \mapsto (C' \otimes C)^n \mapsto C^n.
\]

where the mapping \((C' \otimes C)^n \mapsto C^n\) is induced by \(\lambda\). Then \(\ker(\tilde{\partial}) = \text{Id}([A, A])^2 + F\) and \(\tilde{\partial}\) induces a derivation \(B \mapsto (C' \otimes C)^n\), whereas \(\tilde{\partial}\) induces the usual derivation \(C \mapsto C^n\). Let \(\Delta : C \mapsto C' \otimes C\) be the mapping induced by \(d_A\), i.e., \(\Delta(f) = f^* \otimes 1 - 1 \otimes f\), and let \(z_i = \Delta(x_i)\). The Jacobi matrix is defined in the natural way, and provides the formulation of the JC for free metabelian algebras.

One of the crucial steps in proving the JC for free metabelian algebras is the following homological lemma from \[190\]:
Lemma 6.2.10. Let \( \vec{u} = (u_1, \ldots, u_n) \in (C^{op} \otimes C)^n \). Then \( \vec{u} = \vec{\partial}(\vec{w}) \) for some \( w \in \text{Id}([A, A]) \) iff
\[
\sum z_i u_i = 0.
\]

The proof also requires the following theorem:

Theorem 6.2.11. Let \( \varphi \in \text{End}(C) \). Then \( \varphi \in \text{Aut}(C) \) iff \( \text{Id}((\varphi(x_i)))_{i=1}^n = \text{Id}(z_i)_{i=1}^n \).

The paper [190] also includes the following result:

Theorem 6.2.12. Any automorphism of \( C \) can be extended to an automorphism of \( B \), using the JC for the free metabelian algebra \( B \).

This is a nontrivial result, unlike the extension of an automorphism of \( B \) to an automorphism of \( A/\text{Id}([A, A])^n \) for any \( n > 1 \).

6.2.4 Questions related to the Jacobian Conjecture

Let us turn to other interesting questions which can be linked to the Jacobian Conjecture. The quantization procedure is a bridge between the commutative and noncommutative cases and is deeply connected to the JC and related questions. Some of these questions also are discussed in the paper [79].

Relations between the free associative algebra and the classical commutative situation are very deep. In particular, Bergman’s theorem that any commutative subalgebra of the free associative algebra is isomorphic to a polynomial ring in one indeterminate is the noncommutative analog of Zak’s theorem [222] that any integrally closed subring of a polynomial ring of Krull dimension 1 is isomorphic to a polynomial ring in one indeterminate.

For example, Bergman’s theorem is used to describe the automorphism group \( \text{Aut}(\text{End}(k\langle x_1, \ldots, x_n \rangle)) \) [37]; Zak’s theorem is used in the same way to describe the group \( \text{Aut}(\text{End}(k[x_1, \ldots, x_n])) \) [41].

Question. Can one prove Bergman’s theorem via quantization?

Quantization could be a key idea for understanding Jacobian type problems in other varieties of algebras.

1. Cancellation problems.

We recall three classical problems.

1. Let \( K_1 \) and \( K_2 \) be affine domains for which \( K_1[t] \simeq K_2[t] \). Is it true that \( K_1 \simeq K_2 \)?

2. Let \( K_1 \) and \( K_2 \) be an affine fields for which \( K_1(t) \simeq K_2(t) \). Is it true that \( K_1 \simeq K_2 \)? In particular, if \( K(t) \) is a field of rational functions over the field \( k \), is it true that \( K \) is also a field of rational functions over \( k \)?

3. If \( K[t] \simeq k[x_1, \ldots, x_n] \), is it true that \( K \simeq k[x_1, \ldots, x_{n-1}] \)?
The answers to Problems 1 and 2 are ‘No’ in general (even if \( k = \mathbb{C} \)); see the fundamental paper [29], as well as [34] and the references therein. However, Problem 2 has a positive solution in low dimensions. Problem 3 is currently called the Cancellation Conjecture, although Zariski’s original cancellation conjecture was for fields (Problem 2). See ([141], [101], [66], [184]) for Zariski’s conjecture and related problems. For \( n \geq 3 \), the Cancellation Conjecture (Problem 3) remains open, to the best of our knowledge, and it is reasonable to pose the Cancellation Conjecture for free associative rings and ask the following:

**Question.** If \( K \ast k[t] \simeq k < x_1, \ldots, x_n > \), then is \( K \simeq k < x_1, \ldots, x_{n-1} > \)?

This question was solved for \( n = 2 \) by V. Drensky and J.T. Yu [77].

2. **The Tame Automorphism Problem.** Yagzhev utilized his approach to study the tame automorphism problem. Unfortunately, these papers are not preserved.

It is easy to see that every endomorphism \( \phi \) of a commutative algebra can be lifted to some endomorphism of the free associative algebra, and hence to some endomorphism of the algebra of generic matrices. However, it is not clear that any automorphism \( \phi \) can be lifted to an automorphism.

We recall that an automorphism of \( k[x_1, \ldots, x_n] \) is *elementary* if it has the form

\[
x_1 \mapsto x_1 + f(x_2, \ldots, x_n), \quad x_i \mapsto x_i, \quad \forall i \geq 2.
\]

A *tame automorphism* is a product of elementary automorphisms, and a non-tame automorphism is called *wild*. The “tame automorphism problem” asks whether any automorphism is tame. Jung [99] and van der Kulk [200] proved this for \( n = 2 \), (also see [143, 144] for free groups, [63] for free Lie algebras, and [65, 135] for free associative algebras), so one takes \( n > 2 \).

Elementary automorphisms can be lifted to automorphisms of the free associative algebra; hence every tame automorphism can be so lifted. If an automorphism \( \varphi \) cannot be lifted to an automorphism of the algebra of generic matrices, it cannot be tame. This gives us approach to the tame automorphism problem.

We can lift an automorphism of \( k[x_1, \ldots, x_n] \) to an endomorphism of \( k < x_1, \ldots, x_n > \) in many ways. Then replacing \( x_1, \ldots, x_n \) by \( N \times N \) generic matrices induces a polynomial mapping \( F_{(N)} : k^{N^2} \mapsto k^{N^2} \).

For each automorphism \( \varphi \), the invertibility of this mapping can be transformed into compatibility of some system of equations. For example, Theorem 10.5 of [146] says that the Nagata automorphism is wild, provided that a certain system of five equations in 27 unknowns has no solutions. Whether Peretz’ method can effectively attack tameness questions remains to be seen. The wildness of the Nagata automorphism was established by Shestakov and Umirbaev [181]. One important ingredient in the proof is *degree estimates* of an expression \( p(f, g) \) of algebraically independent polynomials \( f \) and \( g \) in terms of the degrees of \( f \) and \( g \), provided neither leading term
is proportional to a power of the other, initiated by Shestakov and Umirbaev [180]. An exposition based on their method is given in Kuroda [102].

One of the most important tools is the degree estimation technique, which in the multidimensional case becomes the analysis of leading terms, and is more complicated. We refer to the deep papers [47, 102, 104]. Several papers of Kishimoto contain gaps, but also provide deep insights.

One can also ask the weaker question of “coordinate tameness:” Is the image of \((x, y, z)\) under the Nagata automorphism the image under some (other) tame automorphism? This also fails, by [197].

An automorphism \(\varphi\) is called \textit{stably tame} if, when several new indeterminates \(\{t_i\}\) are adjoined, the extension of \(\varphi\) given by \(\varphi'(t_i) = t_i\) is tame; otherwise it is called \textit{stably wild}. Stable tameness of automorphisms of \(k[x, y, z]\) fixing \(z\) is proved in [45]; similar results for \(k\langle x, y, z \rangle\) are given in [42].

Yagzhev tried to construct wild automorphisms via polynomial automorphisms of the Cayley-Dickson algebra with base \(\{\vec{e}_i\}_{i=1}^8\), and the set \(\{\nu_i, \xi_i, \varsigma_i\}_{i=1}^8\) of commuting indeterminates. Let

\[
x = \sum \nu_i \vec{e}_i, \quad y = \sum \xi_i \vec{e}_i, \quad z = \sum \varsigma_i \vec{e}_i.
\]

Let \((x, y, z)\) denote the associator \((xy)z - x(yz)\) of the elements \(x, y, z\), and write

\[
(x, y, z)^2 = \sum f_i(\vec{\nu}, \vec{\xi}, \vec{\varsigma}) \vec{e}_i.
\]

Then the endomorphism \(G\) of the polynomial algebra given by

\[
G: \quad \nu_i \mapsto \nu_i + f_i(\vec{\nu}, \vec{\xi}, \vec{\varsigma}), \quad \xi_i \mapsto \xi_i, \quad \varsigma_i \mapsto \varsigma_i,
\]

is an automorphism, which likely is stably wild.

In the free associative case, perhaps it is possible to construct an example of an automorphism, the wildness of which could be proved by considering its Jacobi endomorphism (Definition 6.2.1). Yagzhev tried to construct examples of algebras \(R = A \otimes A^{op}\) over which there are invertible matrices that cannot decompose as products of elementary ones. Yagzhev conjectured that the automorphism

\[
x_1 \mapsto x_1 + y_1(x_1 y_2 - y_1 x_2), \quad x_2 \mapsto x_2 + (x_1 y_2 - y_1 x_2)y_2, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2
\]

of the free associative algebra is wild.

Umirbaev [193] proved in characteristic 0 that the \textit{Anick automorphism} \(x \mapsto x + y(xy - yz), \ y \mapsto y, \ z \mapsto z + (zy - yz)y\) is wild, by using metabelian algebras. The proof uses description of the defining relations of 3-variable automorphism groups [194–196]. Drensky and Yu [78, 80] proved in characteristic 0 that the image of \(x\) under the Anick Automorphism is not the image of any tame automorphism.

\textbf{Stable Tameness Conjecture.} \textit{Every automorphism of the polynomial algebra} \(k[x_1, \ldots, x_n]\),
resp. of the free associative algebra $k<x_1, \ldots, x_n>$, is stably tame.

Lifting in the free associative case is related to quantization. It provides some light on the similarities and differences between the commutative and noncommutative cases. Every tame automorphism of the polynomial ring can be lifted to an automorphism of the free associative algebra. There was a conjecture that any wild $z$-automorphism of $k[x,y,z]$ (i.e., fixing $z$) over an arbitrary field $k$ cannot be lifted to a $z$-automorphism of $k<x,y,z>$. In particular, the Nagata automorphism cannot be so lifted [79]. This conjecture was solved by Belov and J.-T.Yu [36] over an arbitrary field. However, the general lifting conjecture is still open. In particular, it is not known whether the Nagata automorphism can be lifted to an automorphism of the free algebra. (Such a lifting could not fix $z$).

The paper [36] describes all the $z$-automorphisms of $k<x,y,z>$ over an arbitrary field $k$. Based on that work, Belov and J.-T.Yu [42] proved that every $z$-automorphism of $k<x,y,z>$ is stably tame, for all fields $k$. A similar result in the commutative case is proved by Berson, van den Essen, and Wright [45]. These are important first steps towards solving the stable tameness conjecture in the noncommutative and commutative cases.

The free associative situation is much more rigid than the polynomial case. Degree estimates for the free associative case are the same for prime characteristic [132] as in characteristic 0 [137]. The methodology is different from the commutative case, for which degree estimates (as well as examples of wild automorphisms) are not known in prime characteristic.

J.-T.Yu found some evidence of a connection between the lifting conjecture and the Embedding Conjecture of Abhyankar and Sathaye. Lifting seems to be “easier”.

### 6.2.5 Reduction to simple algebras

This subsection is devoted to finding test algebras.

Any prime algebra $B$ satisfying a system of Capelli identities of order $n+1$ ($n$ minimal such) is said to have rank $n$. In this case, its operator algebra is PI. The localization of $B$ is a simple algebra of dimension $n$ over its centroid, which is a field. This is the famous rank theorem [150].

**Packing properties**

**Definition 6.2.13.** Let $\mathcal{M} = \{\mathfrak{M}_i : i \in I\}$ be an arbitrary set of varieties of algebras. We say that $\mathcal{M}$ satisfies the packing property, if for any $n \in \mathbb{N}$ there exists a prime algebra $A$ of rank $n$ in some $\mathfrak{M}_j$ such that any prime algebra in any $\mathfrak{M}_i$ of rank $n$ can be embedded into some central extension $K \otimes A$ of $A$.

$\mathcal{M}$ satisfies the finite packing property if, for any finite set of prime algebras $A_j \in \mathfrak{M}_i$, there exists a prime algebra $A$ in some $\mathfrak{M}_k$ such that each $A_j$ can be embedded into $A$.

The set of proper subvarieties of associative algebras satisfying a system of Capelli identities of some order $k$ satisfies the packing property (because any simple associative algebra is a matrix...
algebra over field).

However, the varieties of alternative algebras satisfying a system of Capelli identities of order $> 8$, or of Jordan algebras satisfying a system of Capelli identities of order $> 27$, do not even satisfy the finite packing property. Indeed, the matrix algebra of order 2 and the Cayley-Dickson algebra cannot be embedded into a common prime alternative algebra. Similarly, $\mathbb{H}_3$ and the Jordan algebra of symmetric matrices cannot be embedded into a common Jordan prime algebra. (Both of these assertions follow easily by considering their PIs.)

It is not known whether or not the packing property holds for Engel algebras satisfying a system of Capelli identities; knowing the answer would enable us to resolve the JC, as will be seen below.

**Theorem 6.2.14.** If the set of varieties of Engel algebras (of arbitrary fixed order) satisfying a system of Capelli identities of some order satisfies the packing property, then the Jacobian Conjecture has a positive solution.

**Theorem 6.2.15.** The set of varieties from the previous theorem satisfies the finite packing property.

Most of the remainder of this section is devoted to the proof of these two theorems.

**Problem.** Using the packing property and deformations, give a reasonable analog of the JC in nonzero characteristic. (The naive approach using only the determinant of the Jacobian does not work.)

**Construction of simple Yagzhev algebras**

Using the Yagzhev correspondence and composition of elementary automorphisms it is possible to construct a new algebra of Engel type.

**Theorem 6.2.16.** Let $A$ be an algebra of Engel type. Then $A$ can be embedded into a prime algebra of Engel type.

**Proof.** Consider the mapping $F : V \mapsto V$ (cf. (6.1)) given by

$$F : \quad x_i \mapsto x_i + \sum_j \Psi_{ij}; \quad i = 1, \ldots, n$$

(where the $\Psi_{ij}$ are forms of homogenous degree $j$). Adjoining new indeterminates $\{t_i\}_{i=0}^n$, we put $F(t_i) = t_i$ for $i = 0, \ldots, n$.

Now we take the transformation

$$G : \quad t_0 \mapsto t_0, \quad x_i \mapsto x_i, \quad t_i \mapsto t_i + t_0 x_i^2, \quad \text{for} \quad i = 1, \ldots, n.$$
The composite $F \circ G$ has invertible Jacobian (and hence the corresponding algebra has Engel type) and can be expressed as follows:

$$F \circ G : x_i \mapsto x_i + \sum_j \Psi_{ij}, \ t_0 \mapsto t_0, \ t_i \mapsto t_i + t_0 x_i^2 \text{ for } i = 1, \ldots, n.$$ 

It is easy to see that the corresponding algebra $\hat{A}$ also satisfies the following properties:

- $\hat{A}$ contains $A$ as a subalgebra (for $t_0 = 0$).
- If $A$ corresponds to a cubic homogenous mapping (and thus is Engel) then $\hat{A}$ also corresponds to a cubic homogenous mapping (and thus is Engel).
- If some of the forms $\Psi_{ij}$ are not zero, then $A$ does not have nonzero ideals with product 0, and hence is prime (but its localization need not be simple!).

Any algebra $A$ with operators can be embedded, using the previous construction, to a prime algebra with nonzero multiplication. The theorem is proved. \qed

Embedding via the previous theorem preserves the cubic homogeneous case, but does not yet give us an embedding into a simple algebra of Engel type.

**Theorem 6.2.17.** Any algebra $A$ of Engel type can be embedded into a simple algebra of Engel type.

**Proof.** We start from the following observation:

**Lemma 6.2.18.** Suppose $A$ is a finite dimensional algebra, equipped with a base $\vec{e}_1, \ldots, \vec{e}_n, \vec{e}_{n+1}$. If for any $1 \leq i, j \leq n+1$ there exist operators $\omega_{ij}$ in the signature $\Omega(A)$ such that $\omega_{ij}(\vec{e}_1, \ldots, \vec{e}_i, \vec{e}_{n+1}) = \vec{e}_j$, with all other values on the base vectors being zero, then $A$ is simple.

This lemma implies:

**Lemma 6.2.19.** Let $F$ be a polynomial endomorphism of $\mathbb{C}[x_1, \ldots, x_n; t_1, t_2]$, where

$$F(x_i) = \sum_j \Psi_{ij}.$$ 

For notational convenience we put $x_{n+1} = t_1$ and $x_{n+2} = t_2$. Let $\{k_{ij}\}^t_{i=1,j}$ be a set of natural numbers such that

- For any $x_i$ there exists $k_{ij}$ such that among all $\Psi_{ij}$ there is exactly one term of degree $k_{ij}$, and it has the form $\Psi_{i,k_{ij}} = t_1 x_j^{k_{ij}-1}$.
- For $t_2$ and any $x_i$ there exists $k_{iq}$ such that among all $\Psi_{ij}$ there is exactly one term of degree $k_{iq}$, and it has the form $\Psi_{n+2,k_{iq}} = t_1 x_j^{k_{iq}-1}$.

166
• For $t_1$ and any $x_i$ there exists $k_{iq}$ such that among all $\Psi_{ij}$ there is exactly one term of degree $k_{iq}$, and it has the form $\Psi_{n+1,k_{iq}} = t_2 x_j^{k_{iq}-1}$.

Then the corresponding algebra is simple.

**Proof.** Adjoin the term $t_\ell x_i^{k-1}$ to the $x_i$, for $\ell = 1, 2$. Let $e_i$ be the base vector corresponding to $x_i$. Take the corresponding $k_{ij}$-ary operator

$$\omega : \omega(\vec{e}_i, \ldots, \vec{e}_i, \vec{e}_{n+\ell}) = \vec{e}_j,$$

with all other products zero. Now we apply the previous lemma.

**Remark.** In order to be flexible with constructions via the Yagzhev correspondence, we are working in the general, not necessary cubic, case.

Now we can conclude the proof of Theorem 6.2.17. Let $F$ be the mapping corresponding to the algebra $A$:

$$F : x_i \mapsto x_i + \sum_j \Psi_{ij}, \quad i = 1, \ldots, n,$$

where $\Psi_{ij}$ are forms of homogeneous degree $j$. Let us adjoin new indeterminates $\{t_1, t_2\}$ and put $F(t_i) = t_i$, for $i = 1, 2$.

We choose all $k_{\alpha\beta} > \max(\deg(\Psi_{ij}))$ and assume that these numbers are sufficiently large. Then we consider the mappings

$$G_{k_{ij}} : x_i \mapsto x_i + x_j^{k_{ij}-1} t_1, \quad i \leq n; \quad t_1 \mapsto t_1; \quad t_2 \mapsto t_2; \quad x_s \mapsto x_s \text{ for } s \neq i,$$

$$G_{k_{i(n+2)}} : t_2 \mapsto x_j^{k_{ij}-1} t_1; \quad t_1 \mapsto t_1; \quad x_s \mapsto x_s \text{ for } 1 \leq s \leq n,$$

$$G_{k_{i(n+1)}} : t_1 \mapsto x_j^{k_{ij}-1} t_2; \quad t_2 \mapsto t_2; \quad x_s \mapsto x_s \text{ for } 1 \leq s \leq n.$$  

These mappings are elementary automorphisms.

Consider the mapping $H = \circ_{k_{ij}} G_{k_{ij}} \circ F$, where the composite is taken in order of ascending $k_{\alpha\beta}$, and then with $F$. If the $k_{\alpha\beta}$ grow quickly enough, then the terms obtained in the previous step do not affect the lowest term obtained at the next step, and this term will be as described in the lemma. The theorem is proved.

**Proof of Theorem 6.2.15.** The direct sum of Engel type algebras is also of Engel type, and by Theorem 6.2.17 can be embedded into a simple algebra of Engel type.

**The Yagzhev correspondence and algebraic extensions.**

For notational simplicity, we consider a cubic homogeneous mapping

$$F : x_i \mapsto x_i + \Psi_{3i}(\vec{x}).$$
We shall construct the Yagzhev correspondence of an algebraic extension.

Consider the equation
\[ t^s = \sum_{p=1}^{s} \lambda_p t^{s-p}, \]
where the \( \lambda_p \) are formal parameters. If \( m \geq s \), then for some \( \lambda_{pm} \), which can be expressed as polynomials in \( \{ \lambda_p \}_{p=1}^{s-1} \), we have
\[ t^m = \sum_{p=1}^{s} \lambda_{pm} t^{s-p}. \]

Let \( A \) be the algebra corresponding to the mapping \( F \). Consider
\[ A \otimes k[\lambda_1, \ldots, \lambda_s] \]
and its finite algebraic extension \( \hat{A} = A \otimes k[\lambda_1, \ldots, \lambda_s, t] \). Now we take the mapping corresponding (via the Yagzhev correspondence) to the ground ring \( R = k[\lambda_1, \ldots, \lambda_s] \) and algebra \( \hat{A} \).

For \( m = 1, \ldots, s-1 \), we define new formal indeterminates, denoted as \( T^m x_i \). Namely, we put \( T^0 x_i = x_i \) and for \( m \geq s \), we identify \( T^m x_i \) with \( \sum_{p=1}^{s} \lambda_{pm} T^{s-p} x_i \), where \( \{ \lambda_p \}_{p=1}^{s-1} \) are formal parameters in the centroid of some extension \( R \otimes A \). Now we extend the mapping \( F \), by putting
\[ F(T^m x_i) = T^m x_i + T^m \Psi_3(\vec{x}), \quad m = 1, \ldots, s-1. \]

We get a natural mapping corresponding to the algebraic extension.

Now we can take more symbols \( T_j, j = 1, \ldots, s \), and equations
\[ T_j^s = \sum_{p=1}^{s} \lambda_{pj} T_j^{s-p} \]
and a new set of indeterminates \( x_{ijk} = T^k_j x_i \) for \( j = 1, \ldots, s \) and \( i = 1, \ldots, n \). Then we put
\[ x_{ijm} = T^m_j x_i = \sum_{p=1}^{s} \lambda_{jpm} T_j^{s-p} x_i \]
and
\[ F(x_{ijm}) = x_{ijm} + T^m_j \Psi_3(\vec{x}), \quad m = 1, \ldots, s-1. \]

This yields an “algebraic extension” of \( A \).

**Deformations of algebraic extensions.** Let \( m = 2 \). Let us introduce new indeterminates \( y_1, y_2 \), put \( F(y_i) = y_i, \ i = 1, 2 \), and compose \( F \) with the automorphism
\[ G : T^1_1 x_i \mapsto T^1_1 x_i + y_1 x_i, \quad T^1_1 x_i \mapsto T^1_2 x_i + y_1 x_i, \quad x_i \mapsto x_i, \quad i = 1, 2, \]

\[ F(y_1) = y_1, \ i = 1, 2 \]
\[ y_1 \mapsto y_1 + y_2^2 y_1, \quad y_2 \mapsto y_2. \]

(Note that the \( T_1^1 x_i \) and \( T_2^1 x_i \) are new indeterminates and not proportional to \( x_i \)!) Then compose \( G \) with the automorphism \( H : y_2 \mapsto y_2 + y_1^2 \), where \( H \) fixes the other indeterminates. Let us call the corresponding new algebra \( \hat{A} \). It is easy to see that \( \text{Var}(A) \neq \text{Var}(\hat{A}) \).

Define an identity of the pair \((A, B)\), for \( A \subseteq B \) to be a polynomial in two sets of indeterminates \( x_i, z_j \) that vanishes whenever the \( x_i \) are evaluated in \( A \) and \( z_j \) in \( B \). The variety of the pair \((A, B)\) is the class of pairs of algebras satisfying the identities of \((A, B)\).

Recall that by the rank theorem, any prime algebra \( A \) of rank \( n \) can be embedded into an \( n \)-dimensional simple algebra \( \hat{A} \). We consider the variety of the pair \((A, \hat{A})\).

Considerations of deformations yield the following:

**Proposition 6.2.19.1.** Suppose for all simple \( n \)-dimensional pairs there exists a universal pair in which all of them can be embedded. Then the Jacobian Conjecture has a positive solution.

We see the relation with

**The Razmyslov–Kushkulei theorem [150]:** Over an algebraically closed field, any two finite dimensional simple algebras satisfying the same identities are isomorphic.

The difficulty in applying this theorem is that the identities may depend on parameters. Also, the natural generalization of the Rasmynlov–Kushkulei theorem for a variety and subvariety does not hold: Even if \( \text{Var}(B) \subseteq \text{Var}(A) \), where \( B \) and \( A \) are simple finite dimensional algebras over some algebraically closed field, \( B \) need not be embeddable to \( A \).
Conclusion

The quantization program constitutes a substantial and well designed approach to the Jacobian conjecture, as well as to various related topics in algebra and algebraic geometry. The recent developments presented in this review have been instrumental in our investigation of Kontsevich conjecture as well as the establishment of results of independent interest.

Furthermore, as can be seen from the discussion of the work of A.V. Yagzhev, there are substantial areas of the theory which require further development and which might, conceivably, hold the insights necessary for the resolution of the Jacobian conjecture.

While at present the quantization approach does not seem to be adequately developed for a successful attack on the Jacobian problem to happen (as evidenced by our discussion of Kontsevich conjecture), and the rather substantial critique of the general quantization and lifting philosophy (due to Orevkov and others) exists, further research and development of the theory is well advised.
Bibliography

[1] A. Abdesselam, The Jacobian Conjecture as a problem of Perturbative Quantum Field theory, Ann. Henri Poincare 4 (2003), no. 2, 199–215.

[2] S. Abhyankar and T. Moh, Embedding of the line in the plane, J. Reine Angew. Math. 276 (1975), 148–166.

[3] S.A. Amitsur, Algebras over infinite fields, Proc. Amer. Math. Soc. 7 (1956), 35–48.

[4] S.A. Amitsur, A general theory of radicals: III Applications, Amer. J. Math. 75 (1954), 126–136.

[5] J. Alev and L. Le Bruyn. Automorphisms of generic 2 by 2 matrices. In Perspectives in Ring Theory, pages 69–83. Springer, 1988.

[6] A. S. Amitsur and J. Levitzki. Minimal identities for algebras. Proc. Amer. Math. Soc. 1 (1950), 449-463.

[7] A. S. Amitsur and J. Levitzki. Remarks on minimal identities for algebras. Proc. Amer. Math. Soc. 2 (1951), 320-327.

[8] D. J. Anick. Limits of tame automorphisms of $k[x_1, \ldots, x_n]$. Journal of Algebra, 82(2):459–468, 1983.

[9] V. A. Artamonov. Projective metabelian groups and Lie algebras. Izvestiya: Mathematics, 12(2):213–223, 1978.

[10] V. A. Artamonov. Projective modules over universal enveloping algebras. Mathematics of the USSR-Izvestiya, 25(3):429, 1985.

[11] V. A. Artamonov. Nilpotence, projectivity, decomposability. Siberian Mathematical Journal, 32(6):901–909, 1991.

[12] V. A. Artamonov. The quantum Serre problem. Russian Math. Surveys, 53(4):3–77, 1998.

[13] V. A. Artamonov. Quantum polynomials advances in algebra and combinatorics, vol. 19-34, 2008.
[14] M. Artin. Noncommutative Rings, 1999.

[15] I. Arzhantsev, K. Kuyumzhiyan, and M. Zaidenberg. Infinite transitivity, finite generation, and Demazure roots. Advances in Mathematics, 351:1–32, 2019.

[16] T. Asamum. Non-linearizable algebraic $k^*$-actions on affine spaces. Preprint, 1996.

[17] E. Backelin. Endomorphisms of quantized Weyl algebras. Letters in Mathematical Physics, 97(3):317–338, 2011.

[18] H. Bass. A non-triangular action of $G_a$ on $A^3$. Journal of Pure and Applied Algebra, 33(1):1–5, 1984.

[19] H. Bass, E. H. Connell, and D. Wright. The Jacobian conjecture: reduction of degree and formal expansion of the inverse. Bulletin of the American Mathematical Society, 7(2):287–330, 1982.

[20] V.V. Bavula. A question of Rentschler and the Dixmier problem. (English) Zbl. 0995.16019. Ann. Math. (2) 154, No.3, 683-702 (2001).

[21] V.V. Bavula. Generalized Weyl algebras and diskew polynomial rings. arXiv: 1612.08941.

[22] V.V. Bavula. The group of automorphisms of the Lie algebra of derivations of a polynomial algebra. J. Alg. Appl. 16 (2017), arXiv: 1304.3836.

[23] V.V. Bavula. The groups of automorphisms of the Lie algebras of formally analytic vector fields with constant divergence. Comptes Rendus Mathematique, 352 (2). pp. 85-88, arXiv: 1311.2284.

[24] V.V. Bavula, The inversion formulae for automorphisms of Weyl algebras and polynomial algebras, J. Pure Appl. Algebra 210 (2007), 147–159.

[25] V.V. Bavula, The inversion formulae for automorphisms of polynomial algebras and rings of differential operators in prime characteristic, J. Pure Appl. Algebra 212 (2008), no. 10, 2320–2337.

[26] V.V. Bavula, An analogue of the conjecture of Dixmier is true for the algebra of polynomial integro-differential operators, J. of Algebra 372 (2012), 237–250.

[27] V.V. Bavula, Every monomorphism of the Lie algebra of unitriangular polynomial derivations is an authomorphism, C.R. Acad.Sci. Paris, Ser. 1, 350, (2012), no 11–12, 553–556.

[28] V.V. Bavula, The Jacobian Conjecture implies the Dixmier Problem, arXiv:math/0512250.
[29] A. Beauville, J.-L. Colliot-Thelene, J-J Sansuc, and P. Swinnerton-Dyer, *Varietes stablement rationnelles non rationnelles*, Ann. Math. 121, (1985) 283–318.

[30] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. I. Deformations of symplectic structures. Annals of Physics, 111(1):61–110, 1978.

[31] A. Belov. Linear recurrence equations on a tree. Mathematical Notes, 78(5):603–609, 2005.

[32] A. Belov. Local finite basis property and local representability of varieties of associative rings. Izv. Rus. Acad. Sci. 74 (2010) 3-134. English transl.: Izvestiya: Mathematics, 74 (2010) 1-126.

[33] A. Belov, L. Bokut, L. Rowen, and J.-T. Yu. The Jacobian conjecture, together with Specht and Burnside-type problems. In Automorphisms in Birational and Affine Geometry, pages 249–285. Springer, 2014.

[34] A. Belov, L. Makar-Limanov, and J.T. Yu, *On the Generalised Cancellation Conjecture*, J. of Algebra 281 (2004), 161–166.

[35] A. Belov, L.H. Rowen, and U. Vishne, *Structure of Zariski-closed algebras*, Trans. Amer. Math. Soc. 362 (2012), 4695–4734.

[36] A. Belov-Kanel and J.-T. Yu. On the lifting of the Nagata automorphism. Selecta Math. (N.S.) 17 (2011), 935-945.

[37] A. Kanel-Belov, A. Berzins, R. Lipyanski. Automorphisms of the semigroup of endomorphisms of free associative algebras. Int. Journ. of Algebra and Comp., Vol. 17,:5/6 (2007), 923–939, arXiv:math/0512273.

[38] A. Belov-Kanel and A. Elishev. On planar algebraic curves and holonomic D-modules in positive characteristic. Journal of Algebra and Its Applications, 15(08):1650155, 2016.

[39] A. Belov-Kanel and M. Kontsevich. Automorphisms of the Weyl algebra. Letters in Mathematical Physics, 74(2):181–199, 2005.

[40] A. Belov-Kanel and M. Kontsevich. The Jacobian conjecture is stably equivalent to the Dixmier conjecture. Moscow Mathematical Journal, 7(2):209–218, 2007.

[41] A. Belov-Kanel and R. Lipyanski. Automorphisms of the endomorphism semigroup of a polynomial algebra. Journal of Algebra, 333(1):40–54, 2011.

[42] A. Belov-Kanel and J.-T. Yu. Stable tameness of automorphisms of $F\langle x, y, z \rangle$ fixing $c$. Selecta Math. (N.S) 18 (2012), 799-802.

[43] G. M. Bergman. Centralizers in free associative algebras. Transactions of the American Mathematical Society, 137:327–344, 1969.
[44] G. M. Bergman. The diamond lemma for ring theory. Advances in Mathematics, 29(2):178–218, 1978.

[45] J. Berson, A. van den Essen, and D. Wright, *Stable tameness of two-dimensional polynomial automorphisms over a regular ring*, 2007 (rev. 2010), Advances in Mathematics 230 (2012), 2176–2197.

[46] J. Birman, *An inverse function theorem for free groups*, Proc. Amer. Math. Soc. 41 (1973), 634–638.

[47] P. Bonnet and S. Vénéreau. *Relations between the leading terms of a polynomial automorphism*, J. of Algebra 322 (2009), no. 2, 579—599. 13 Aug 2008.

[48] A. Berzins. The group of automorphisms of semigroup of endomorphisms of free commutative and free associative algebras. arXiv preprint math/0504015, 2005.

[49] A. Białynicki-Birula. Remarks on the action of an algebraic torus on $k^n$, I. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys, 14:177–181, 1966.

[50] A. Białynicki-Birula. Remarks on the action of an algebraic torus on $k^n$, II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys, 15:123–125, 1967.

[51] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Annals of mathematics*, pages 480–497, 1973.

[52] T. Bitoun. The $p$-support of a holonomic $D$-module is lagrangian, for $p$ large enough. arXiv preprint arXiv:1012.4081, 2010.

[53] Yu. Bodnarchuk. Every regular automorphism of the affine Cremona group is inner. J. Pure Appl. Algebra 157 (2001), 115-119.

[54] L. Bokut, V. Latyshev, I. Shestakov and E. Zelmanov. Selected works of A. I. Shirshov. Springer, 2009.

[55] L. A. Bokut. Embedding Lie algebras into algebraically closed Lie algebras. Algebra i Logika, 1:47–53, 1962.

[56] L. A. Bokut. Embedding of algebras into algebraically closed algebras. In Doklady Akademii Nauk, volume 145(5), pages 963–964. Russian Academy of Sciences, 1962.

[57] L. A. Bokut. Theorems of embedding in the theory of algebras. In Colloq. math, volume 14, pages 349–353, 1966.

[58] M. Bresar, C. Procesi, and S. Spenko. Functional identities on matrices and the Cayley-Hamilton polynomial. arXiv preprint arXiv:1212.4597, 2013.
[59] L. A. Campbell. A condition for a polynomial map to be invertible. Mathematische Annalen, 205(3):243–248, 1973.

[60] P. M. Cohn. Subalgebras of free associative algebras. Proceedings of the London Mathematical Society, 3(4):618–632, 1964.

[61] P. M. Cohn. Progress in free associative algebras. Israel Journal of Mathematics, 19(1-2):109–151, 1974.

[62] P. M. Cohn. A brief history of infinite-dimensional skew fields. Math. Scient, 17:1–14, 1992.

[63] P.M. Cohn. Free Rings and Their Relations. 2nd edition, Academic Press (1985).

[64] A. J. Czerniakiewicz. Automorphisms of a free associative algebra of rank 2. I. Transactions of the American Mathematical Society, 160:393–401, 1971.

[65] A. J. Czerniakiewicz. Automorphisms of a free associative algebra of rank 2. II. Transactions of the American Mathematical Society, 171:309–315, 1972.

[66] W. Danielewski, On the cancellation problem and automorphism groups of affine algebraic varieties, preprint, Warsaw 1989.

[67] M. De Bondt and A. Van den Essen. The Jacobian conjecture for symmetric Druzkowski mappings. University of Nijmegen, Department of Mathematics, 2004.

[68] M. De Bondt and A. Van den Essen. A reduction of the Jacobian conjecture to the symmetric case. Proceedings of the American Mathematical Society, 133(8):2201–2205, 2005.

[69] C. De Concini and C. Procesi. A characteristic free approach to invariant theory. In Young Tableaux in Combinatorics, Invariant Theory, and Algebra, pages 169–193. Elsevier, 1982.

[70] J. Déserti. Sur le groupe des automorphismes polynomiaux du plan affine. J. Algebra 297 (2006) 584-599.

[71] W. Dicks, Automorphisms of the free algebra of rank two. Group actions on rings (Brunswick, Maine, 1984), Contemp. Math., 43 (1985) 63-68.

[72] W. Dicks and J. Lewin. Jacobian conjecture for free associative algebras. Commun. Alg. 10, No. 12 (1982), 1285–1306.

[73] J. Dixmier. Sur les algèbres de Weyl. Bulletin de la Société mathématique de France, 96:209–242, 1968.

[74] C. Dodd. The $p$-cycle of holonomic $D$-modules and auto-equivalences of the Weyl algebra. arXiv preprint arXiv:1510.05734, 2015.

[75] S. Donkin. Invariants of several matrices. Inventiones mathematicae, 110(1):389–401, 1992.
[76] S. Donkin. Invariant functions on matrices. Mathematical Proceedings of the Cambridge Philosophical Society, 113(1):23–43, 1993.

[77] V. Drensky and J.-T. Yu. A cancellation conjecture for free associative algebras. Proceedings of the American Mathematical Society, 136(10):3391–3394, 2008.

[78] V. Drensky and J.-T. Yu, The strong Anick conjecture, Proc. Natl. Acad. Sci. USA 103 (2006), 4836–4840.

[79] V. Drensky and J.-T. Yu, Coordinates and automorphisms of polynomial and free associative algebras of rank three, Front. Math. China 2 (1) (2007), 13–46.

[80] V. Drensky and J.-T. Yu. The strong Anick conjecture is true. J. Eur. Math. Soc. (JEMS) 9 (2007), 659-679.

[81] L. Drużkowski. An effective approach to Keller’s Jacobian conjecture. Mathematische Annalen, 264(3):303–313, 1983.

[82] L. Drużkowski. The Jacobian conjecture: symmetric reduction and solution in the symmetric cubic linear case. Annales Polonici Mathematici, 1(87):83–92, 2005.

[83] L.M. Drużkowski, New reduction in the Jacobian conjecture. Effective methods in algebraic and analytic geometry, 2000 (Krakow), Univ. Iagel. Acta Math. No. 39 (2001), 203–206.

[84] A. Elishev. Automorphisms of polynomial algebras, quantization and Kontsevich conjecture. Moscow Institute of Physics and Technology, PhD Thesis, 2019.

[85] A. Elishev, A. Kanel-Belov, F. Razavinia, J.-T. Yu, and W. Zhang. Noncommutative Białyńnicki-Birula. theorem. arXiv preprint arXiv:1808.04903, 2018.

[86] A. Elishev, A. Kanel-Belov, F. Razavinia, J.-T. Yu, and W. Zhang. Torus actions on free associative algebras, lifting and Białyńnicki-Birula. type theorems. arXiv preprint arXiv:1901.01385, 2019.

[87] A. Van den Essen, The Amazing Image Conjecture, arXiv:1006.5801.

[88] A. Van den Essen and M. de Bondt, Recent progress on the Jacobian Conjecture, Proc. of the Int. Conf. Singularity Theory in honour of S. Lojajewicz, Cracow, 22-26 March 2004, in Annales Polonici Mathematici, 87 (2005), 1–11.

[89] A. Van den Essen and M. de Bondt, The Jacobian Conjecture for symmetric Drużkowski mappings, Annales Polonici Mathematici 86 No. 1 (2005), 43–46.

[90] A. Van den Essen, D. Wright, and W. Zhao, On the Image Conjecture, Journal of Algebra 340 (2011), 211—224.
[91] R.H. Fox, *Free differential calculus, I. Derivation in the free group ring*, Ann. of Math. (2) 57 (1953), 547–560.

[92] M.H. Gizatullin, *Automorphisms of affine surfaces, I, II*, Mathematics of the USSR-Izvestiya 11(1) (1977), 54–103.

[93] G. Gorni and G. Zampieri, *Yagzhev polynomial mappings: on the structure of the Taylor expansion of their local inverse*. Polon. Math. 64 (1996), 285–290.

[94] B. Fedosov. A simple geometrical construction of deformation quantization. *Journal of Differential Geometry*, 40(2):213–238, 1994.

[95] T. Frayne, A. C. Morel, and D. S. Scott. Reduced direct products. *Journal of Symbolic Logic*, 31(3):506–507, 1966.

[96] W. Fulton and J. Harris. *Representation Theory. A First Course* (2nd edition). Springer-Verlag.

[97] J.-P. Furter and H. Kraft. On the geometry of the automorphism groups of affine varieties. arXiv preprint arXiv:1809.04175, 2018.

[98] A Gutwirth. The action of an algebraic torus on the affine plane. *Transactions of the American Mathematical Society*, 105(3):407–414, 1962.

[99] H. W. E. Jung. Über ganze birationale Transformationen der Ebene. *Journal für die reine und angewandte Mathematik*, 184:161–174, 1942.

[100] S. Kaliman, M. Koras, L. Makar-Limanov, and P. Russell. C*-actions on C^3 are linearizable. *Electron. Res. Announc. Amer. Math. Soc*, 3:63–71, 1997.

[101] S. Kaliman and M. Zaidenberg, *Families of affine planes: the existence of a cylinder*, Michigan Math. J. 49 (2001), 353–367.

[102] S. Kuroda, *Shestakov-Umirbaev reductions and Nagata’s conjecture on a polynomial automorphism*, Tohoku Math. J. 62 (2010), 75–115.

[103] E. Kuzmin and I.P. Shestakov, Non-associative structures. (English), Algebra VI. Encycl. Math. Sci. 57, (1995) 197–280; translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 57 (1990), 179–266.

[104] M. Karas’, *Multidegrees of tame automorphisms of C^n*, Dissertationes Math. 477 (2011), 55 pp.

[105] A. Khoroshkin and D. Piontkovski, *On generating series of finitely presented operads*, preprint, 2012, arXiv:1202.5170.
[106] T. Kambayashi. Pro-affine algebras, Ind-affine groups and the Jacobian problem. Journal of Algebra, 185(2):481–501, 1996.

[107] T. Kambayashi. Some basic results on pro-affine algebras and Ind-affine schemes. Osaka Journal of Mathematics, 40(3):621–638, 2003.

[108] T. Kambayashi and P. Russell. On linearizing algebraic torus actions. Journal of Pure and Applied Algebra, 23(3):243–250, 1982.

[109] A. Kanel-Belov, V. Borisenko, and V. Latysev. Monomial algebras. Journal of Mathematical Sciences, 87:3463–3575, 1997.

[110] A. Kanel-Belov, A. Elishev, and J.-T. Yu. Independence of the B-KK isomorphism of infinite prime. arXiv preprint arXiv:1512.06533, 2015.

[111] A. Kanel-Belov, A. Elishev, and J.-T. Yu. Augmented polynomial symplectomorphisms and quantization. arXiv preprint arXiv:1812.02859, 2018.

[112] A. Kanel-Belov, S. Grigoriev, A. Elishev, J.-T. Yu, and W. Zhang. Lifting of polynomial symplectomorphisms and deformation quantization. Communications in Algebra, 46(9):3926–3938, 2018.

[113] A. Kanel-Belov, S. Malev and L. Rowen. The images of non-commutative polynomials evaluated on $2 \times 2$ matrices. Proc. Amer. Math. Soc. 140 (2012), 465-478.

[114] A. Kanel-Belov, S. Malev and L. Rowen. The images of multilinear polynomials evaluated on $3 \times 3$ matrices. Proc. Amer. Math. Soc. 144 (2016), 7-19.

[115] A. Kanel-Belov, F. Razavinia, and W. Zhang. Bergman’s centralizer theorem and quantization. Communications in Algebra, 46(5):2123–2129, 2018.

[116] A. Kanel-Belov, F. Razavinia, and W. Zhang. Centralizers in free associative algebras and generic matrices. arXiv preprint arXiv:1812.03307, 2018.

[117] A. Kanel-Belov, L. H. Rowen and U. Vishne. Full exposition of Specht’s problem. Serdica Math. J. 38 (2012) 313-370.

[118] A. Kanel-Belov, J.-T. Yu, and A. Elishev. On the augmentation topology of automorphism groups of affine spaces and algebras. International Journal of Algebra and Computation, 28(08):1449–1485, 2018.

[119] B. Keller. Notes for an introduction to Kontsevich’s quantization theorem, 2003.

[120] O.-H. Keller. Ganze Cremona-Transformationen. Monatshefte für Mathematik, 47(1):299–306, 1939.
[121] P. S. Kolesnikov. The Makar-Limanov algebraically closed skew field. Algebra and Logic, 39(6):378–395, 2000.

[122] P. S. Kolesnikov. Different definitions of algebraically closed skew fields. Algebra and Logic, 40(4):219–230, 2001.

[123] M. Kontsevich. Deformation quantization of Poisson manifolds. Letters in Mathematical Physics, 66(3):157–216, 2003.

[124] M. Kontsevich. Holonomic $D$-modules and positive characteristic. Japanese Journal of Mathematics, 4(1):1–25, 2009.

[125] M. Koras and P. Russell. $C^*$-actions on $C^3$: The smooth locus of the quotient is not of hyperbolic type. Journal of Algebraic Geometry, 8(4):603–694, 1999.

[126] S. Kovalenko, A. Perepechko, M. Zaidenberg, et al. On automorphism groups of affine surfaces. In Algebraic Varieties and Automorphism Groups, pages 207–286. Mathematical Society of Japan, 2017.

[127] H. Kraft and A. Regeta. Automorphisms of the Lie algebra of vector fields. J. Eur. Math. Soc.(to appear), 2015.

[128] H. Kraft and I. Stampfli. On automorphisms of the affine Cremona group. In Annales de l’Institut Fourier, volume 63(3), pages 1137–1148, 2013.

[129] V. S. Kulikov. Generalized and local Jacobian problems. Russian Academy of Sciences. Izvestiya Mathematics, 41(2):351, 1993.

[130] V. S. Kulikov. The Jacobian conjecture and nilpotent maps. Journal of Mathematical Sciences, 106(5):3312–3319, 2001.

[131] R. Levy, P. Loustaunau, and J. Shapiro. The prime spectrum of an infinite product of copies of $Z$. Fundamenta Mathematicae, 138:155–164, 1991.

[132] Y.-C. Li and J.-T. Yu. Degree estimate for subalgebras. J. Algebra 362 (2012), 92-98.

[133] M. Lothaire. Combinatorics on words, volume 17. Cambridge university press, 1997.

[134] L. Makar-Limanov A new proof of the Abhyankar-Moh-Suzuki Theorem, arXiv:1212.0163, 18 pages.

[135] L. Makar-Limanov. Automorphisms of a free algebra with two generators. Functional Analysis and Its Applications, 4(3):262–264, 1970.

[136] L. Makar-Limanov. On automorphisms of Weyl algebra. Bulletin de la Société Mathématique de France, 112:359–363, 1984.
[137] L. Makar-Limanov and J.-T. Yu. Degree estimate for subalgebras generated by two elements, J. Eur. Math. Soc. (JEMS) 10 (2008), 533-541.

[138] L. Makar-Limanov. Algebraically closed skew fields. Journal of Algebra, 93(1):117–135, 1985.

[139] L. Makar-Limanov, U. Turusbekova, and U. Umirbaev. Automorphisms and derivations of free Poisson algebras in two variables. Journal of Algebra, 322(9):3318–3330, 2009.

[140] M. Markl, S. Shnider, J. Stasheff, Operads in algebra, topology and physics, Mathematical Surveys and Monographs 96 (2002), AMS, Providence, RI.

[141] M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, J. Math. Kyoto Univ. 20 (1980), 11–42.

[142] M. Nagata, On the automorphism group of k[x, y], Department of Mathematics, Kyoto University, Lectures in Mathematics, No. 5. Kinokuniya Book-Store Co., Ltd., Tokyo, 1972. v+53 pp.

[143] J. Nielsen, Die Isomorphismen der allgemeinen, undendlichen Gruppen mit zwei Erzeugenden, Math. Ann. 78 (1918), 385–397.

[144] J. Nielsen, Die Isomorphismengruppe der freien Gruppen, Math. Ann. 91 (1924), 169–209.

[145] A.Yu. Ol’shanskij, Groups of bounded period with subgroups of prime order, Algebra i Logika 21 (1982), 553–618, translation in Algebra and Logic 21 (1983), 369–418.

[146] R. Peretz, Constructing polynomial mappings using non-commutative algebras. Aff. Algebr. geometry, 197–232, Cont. Math. 369 (2005), Amer. Math. Soc., Providence, RI.

[147] D. Piontkovski, Operads versus Varieties: a dictionary of universal algebra, preprint, 2011.

[148] D. Piontkovski, On Kurosh problem in varieties of algebras, Translated from Proceedings of Kurosh conference (Fund. Prikl. Matematika 14 (2008), 5, 171–184), Journal of Mathematical Sciences 163 No. 6 (2009), 743–750.

[149] Yu.P. Razmyslov, Algebras satisfying identity relations of Capelli type. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), 143–166, 240.

[150] Yu.P. Razmyslov, Identities of algebras and their representations. Sovremennaya Algebra. [Modern Algebra] “Nauka”, Moscow (1989), 432 pp. Translations of Mathematical Monographs 138, American Mathematical Society, Providence, RI, 1994, xiv+318 pp.

[151] Yu.P. Razmyslov and K.A. Zubrilin, Nilpotency of obstacles for the representability of algebras that satisfy Capelli identities, and representations of finite type. (Russian) Uspekhi Mat. Nauk 48 (1993), 171–172; translation in Russian Math. Surveys 48 (1993), 183–184.
[152] C. Reutenauer, *Applications of a noncommutative Jacobian matrix*, J. Pure Appl. Algebra 77 (1992), 634–638.

[153] L.H. Rowen, *Graduate algebra: Noncommutative view*, AMS Graduate Studies in Mathematics 91 (2008).

[154] T.-T. Moh. On the global Jacobian conjecture for polynomials of degree less than 100. *preprint*, 1983.

[155] T.-T. Moh. On the Jacobian conjecture and the configurations of roots. Journal für Mathematik. Band, 340:19, 1983.

[156] J. E. Moyal. Quantum mechanics as a statistical theory. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 45(1), pages 99–124. Cambridge University Press, 1949.

[157] S. Y. Orevkov. The commutant of the fundamental group of the complement of a plane algebraic curve. Russian Mathematical Surveys, 45(1):221, 1990.

[158] S. Y. Orevkov. An example in connection with the Jacobian conjecture. *Mathematical Notes*, 47(1):82–88, 1990.

[159] S. Y. Orevkov. The fundamental group of the complement of a plane algebraic curve. Mathematics of the USSR-Sbornik, 65(1):267, 1990.

[160] B Plotkin. Varieties of algebras and algebraic varieties. *Israel Journal of Mathematics*, 96(2):511–522, 1996.

[161] B. Plotkin. Algebras with the same (algebraic) geometry. *arXiv preprint math/0210194*, 2002.

[162] V. L. Popov. Around the Abhyankar-Sathaye conjecture. *arXiv preprint arXiv:1409.6330*, 2014.

[163] C. Procesi. Rings with polynomial identities, volume 17. M. Dekker, 1973.

[164] C. Procesi. The invariant theory of $n \times n$ matrices, 1976.

[165] M. Razar. Polynomial maps with constant Jacobian. *Israel Journal of Mathematics*, 32(2-3):97–106, 1979.

[166] A. Robinson. Non-standard analysis. Princeton University Press, 2016.

[167] S. Rosset. A new proof of the Amitsur-Levitzki identity. *Israel Journal of Mathematics*, 23(2):187–188, 1976.
[168] L. H. Rowen. Graduate Algebra: Noncommutative View, volume 9. American Mathematical Society, Providence, RI, 2008.

[169] A.H. Schofield. Representations of rings over skew fields. London Math. Soc. Lecture Note Series 92, Cambridge, Cambridge University Press (1985).

[170] G. Schwarz. Exotic algebraic group actions. CR Acad. Sci. Paris, 309:89–94, 1989.

[171] I. R. Shafarevich. On some infinite-dimensional groups, II. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 45(1):214–226, 1981.

[172] Y. Sharifi. Centralizers in Associative Algebras. PhD thesis, Science: Department of Mathematics, 2013.

[173] I.P. Shestakov. Finite-dimensional algebras with a nil basis. (in Russian) Algebra i Logika 10 (1971), 87–99.

[174] I.P. Shestakov. A quantization of Poisson superalgebras and a speciality of Jordan Poisson superalgebras. Algebra i Logika, 32, N 5 (1993), 572-585; English transl.: Algebra and Logic, 32, N 5 (1993), 309-317.

[175] I.P. Shestakov. A quantization of Poisson algebras and a weak speciality of related Jordan superalgebras. Doklady RAN, 334, N 1 (1994), 29-31; English transl.: Russian Acad. Sci. Dokl. Math. 49 (1994), N 1, 34-37.

[176] I.P. Shestakov. The speciality problem for Malcev algebras and deformations of Malcev Poisson algebras, in "Non-Associative Algebra and Its Applications". edited by R.Costa, A.Grishkov, H.Guzzo, and L.Peresi, Proceedings of the IV International Conference on Non-Associative Algebra and Its Applications, July 1998, Sao Paulo, 365-371, Marcel Dekker, NY, 2000.

[177] I.P. Shestakov. Speciality and Deformations of Algebras. in "Algebra: Proceedings of the International Algebraic Conference on Occasion of the 90th Birthday of A.G.Kurosh, Moscow, Russia, May 25-30, 1998" / Ed. Yu.Bahturin - Berlin; New York: de Gruyter, 2000; p. 345-356.

[178] I. P. Shestakov and U. U. Umirbaev. Degree estimate and two-generated subalgebras of rings of polynomials. J. Amer. Math. Soc. 17 (2004), 181-196.

[179] I. Shestakov and U. Umirbaev. The Nagata automorphism is wild. Proceedings of the National Academy of Sciences, 100(22):12561–12563, 2003.

[180] I. Shestakov and U. Umirbaev. Poisson brackets and two-generated subalgebras of rings of polynomials. Journal of the American Mathematical Society, 17(1):181–196, 2004.
[181] I. P. Shestakov and U. U. Umirbaev. The tame and the wild automorphisms of polynomial rings in three variables. J. Amer. Math. Soc. 17 (2004), 197-220.

[182] V. Shpilrain, On generators of $L/R^2$ Lie algebras, Proc. Amer. Math. Soc. 119 (1993), 1039–1043.

[183] D. Singer, On Catalan trees and the Jacobian conjecture, Electron. J. Combin. 8 (2001), no. 1, Research Paper 2, 35 pp. (electronic).

[184] V. Shpilrain and J.-T. Yu, Affine varieties with equivalent cylinders, J. Algebra 251 (2002), no. 1, 295–307.

[185] V. Shpilrain and J.-T. Yu, Factor algebras of free algebras: on a problem of G. Bergman, Bull. London Math. Soc. 35 (2003), 706–710.

[186] M. Suzuki, Propiétés topologiques des polynômes de deux variables complexes, et automorphismes algébrique de l’espace $C^2$, J. Math. Soc. Japan, 26 (1974), 241–257.

[187] Y. Tsuchimoto. Preliminaries on Dixmier conjecture. Mem. Fac. Sci. Kochi Univ. Ser. A Math, 24:43–59, 2003.

[188] Y. Tsuchimoto. Endomorphisms of Weyl algebra and $p$-curvatures. Osaka Journal of Mathematics, 42(2):435–452, 2005.

[189] Y. Tsuchimoto. Auslander regularity of norm based extensions of Weyl algebra. arXiv:1402.7153.

[190] U. Umirbaev. On the extension of automorphisms of polynomial rings. Siberian Mathematical Journal, 36(4):787–791, 1995.

[191] U.U. Umirbaev, On Jacobian matrices of Lie algebras, 6th All-Union Conference on Varieties of Algebraic Systems, Magnitogorsk (1990), 32–33.

[192] U.U. Umirbaev, Shreer varieties of algebras. (Russian). Algebra i Logika 33(1994), no. 3, 317–340, 343; translation in Algebra and Logie 33 (1994), 180–193.

[193] U.U. Umirbaev, Tame and wild automorphisms of polynomial algebras and free associative algebras, Max-Planck-Institute für Mathematics, Bonn, Preprint MPIM 2004–108.

[194] U. Umirbaev. The Anick automorphism of free associative algebras. Journal für die reine und angewandte Mathematik (Crelles Journal), 2007(605):165–178, 2007.

[195] U.U. Umirbaev, Defining relations of the tame automorphism group of polynomial algebras in three variables, J. Reine Angew. Math. 600 (2006), 203–235.
[196] U. U. Umirbaev, *Defining relations for automorphism groups of free algebras*, J. Algebra **314** (2007), 209–225.

[197] U. U. Umirbaev and J.-T. Yu. The strong Nagata conjecture. Proc. Natl. Acad. Sci. USA **101** (2004), 4352-4355.

[198] C. Urech and S. Zimmermann. Continuous automorphisms of Cremona groups. arXiv preprint arXiv:1909.11050, 2019.

[199] A. Van den Essen. Polynomial Automorphisms and the Jacobian Conjecture, volume 190. Birkhäuser, 2012.

[200] W. Van der Kulk. On polynomial rings in two variables. Nieuw Arch. Wisk.(3), 1:33–41, 1953.

[201] A.G. Vitushkin, *A criterion for the representability of a chain of σ-processes by a composition of triangular chains* (Russian), Mat. Zametki **65** no. 5 (1999), 643–653; transl. in Math. Notes **65** no. 5–6 (1999), 539–547.

[202] A. G. Vitushkin. On the homology of a ramified covering over $C^2$. Mathematical Notes-New York, 64(5):726–731, 1998.

[203] A. G. Vitushkin. Evaluation of the Jacobian of a rational transformation of $C^2$ and some applications. Matematicheskie Zametki, 66(2):308–312, 1999.

[204] J.H.M. Wedderburn, *Note on algebras*, Annals of Mathematics **38** (1937), 854–856.

[205] D. Wright, *The Jacobian Conjecture as a problem in combinatorics*, in the monograph Affine Algebraic Geometry, in honor of Masayoshi Miyanishi, edited by Takayuki Hibi, published by Osaka University Press 2007, ArXiv: math.Co/0511214 v2, 22 Mar 2006.

[206] D. Wright, *The Jacobian Conjecture: ideal membership questions and recent advances*, Affine algebraic geometry, 261–276, Contemp. Math. **369** (2005).

[207] A.V. Yagzhev, *On the Koethe problem* (Russian), Unpublished.

[208] A.V. Yagzhev, *Finiteness of the set of conservative polynomials of a given degree* (Russian), Mat. Zametki **41** (1987), no. 2, 148–151, 285.

[209] A.V. Yagzhev, *Nilpotency of extensions of an abelian group by an abelian group* (Russian), Mat. Zametki **43** (1988), no. 3, 424–427, 431; translation in Math. Notes **43** (1988), no. 3–4, 244–245.

[210] A.V. Yagzhev, *Locally nilpotent subgroups of the holomorph of an abelian group*. (Russian) Mat. Zametki **46** (1989), no. 6, 118.
[211] A.V. Yagzhev, *A sufficient condition for the algebraicity of an automorphism of a group.* (Russian) Algebra i Logika 28 (1989), no. 1, 117–119, 124; translation in Algebra and Logic 28 (1989), no. 1, 83–85.

[212] A.V. Yagzhev, *The generators of the group of tame automorphisms of an algebra of polynomials* (Russian), Sibirsk. Mat. Ž. 18 (1977), no. 1, 222–225, 240.

[213] S. Wang. A Jacobian criterion for separability. Journal of Algebra, 65(2):453–494, 1980.

[214] D. Wright. On the Jacobian conjecture. Illinois Journal of Mathematics, 25(3):423–440, 1981.

[215] A.V. Yagzhev. Invertibility of endomorphisms of free associative algebras (in Russian). Mat. Zametki 49 (1991), No. 4, 142–147, 160; translation in Math. Notes 49 (1991), No. 3–4, 426–430.

[216] A.V. Yagzhev. On endomorphisms of free algebras (in Russian). Sibirsk. Mat. Zh. 21 (1980), No. 1, 181–192.

[217] A.V. Yagzhev. Algorithmic problem of recognizing automorphisms among endomorphisms of free associative algebras of finite rank. Sib. Math. J. 21 (1980), 142–146.

[218] A.V. Yagzhev. Keller’s Problem. Sib. Math. J., Vol. 21 (1980), 747–754.

[219] A.V. Yagzhev, *Engel algebras satisfying Capelli identities.* (Russian) Proceedings of Shafarevich Seminar, Moscow, 2000; pages 83–88.

[220] A.V. Yagzhev, *Endomorphisms of polynomial rings and free algebras of different varieties.* (Russian) Proceedings of Shafarevich Seminar, Moscow, 2000. pages 15–47.

[221] A.V. Yagzhev, *Invertibility criteria of a polynomial mapping.* (Russian, unpublished).

[222] A. Zaks, *Dedekind subrings of $K[x_1, \ldots, x_n]$ are rings of polynomials.* Israel Journal of math 9, (1971), 285–289.

[223] E. Zelmanov, *On the nilpotence of nilalgebras*, Lect. Notes Math. 1352 (1988), 227–240.

[224] W. Zhao, *New Proofs for the Abhyankar–Gurjar Inversion Formula and the Equivalence of the Jacobian Conjecture and the Vanishing Conjecture*, Proc. Amer. Math. Soc. 139 (2011), 3141–3154.

[225] W. Zhao, *Mathieu Subspaces of Associative Algebras*, Journal of Algebra 350 (2012), 245–272, arXiv:1005.4260

[226] K.A. Zhevlakov, A.M. Slin’ko, I.P. Shestakov, and A.I. Shirshov, *Nearly Associative Rings* [in Russian], Nauka, Moscow (1978).
[227] K.A. Zubrilin, *Algebras that satisfy the Capelli identities* (Russian), Mat. Sb. 186 (1995), no. 3, 53–64; translation in Sb. Math. 186 (1995), no. 3, 359–370.

[228] K.A. Zubrilin, *On the class of nilpotence of obstruction for the representability of algebras satisfying Capelli identities* (Russian), Fundam. Prikl. Mat. 1 (1995), no. 2, 409–430.

[229] K.A. Zubrilin, *On the Baer ideal in algebras that satisfy the Capelli identities* (Russian), Mat. Sb. 189 (1998), 73–82; translation in Sb. Math. 189 (1998), 1809–1818.

[230] M. G. Zaidenberg. On exotic algebraic structures on affine spaces. In *Geometric complex analysis*, pages 691–714. World Scientific, 1996.

[231] W. Zhang. Alternative proof of Bergman’s centralizer theorem by quantization. Bar-Ilan University, Master Thesis, 2017.

[232] W. Zhang. Polynomial Automorphisms and Deformation Quantization. Bar-Ilan University, PhDThesis, 2019.

[233] A. N. Zubkov. Matrix invariants over an infinite field of finite characteristic. Siberian Mathematical Journal, 34(6):1059–1065, 1993.

[234] A. N. Zubkov. A generalization of the Razmyslov-Procesi theorem. Algebra and Logic, 35(4):241–254, 1996.