Floating Bodies of Equilibrium in 2D, 
the Tire Track Problem and 
Electrons in a Parabolic Magnetic Field

Franz Wegner, Institut für Theoretische Physik 
Ruprecht-Karls-Universität Heidelberg 
Philosophenweg 19, D-69120 Heidelberg 
Email: wegner@tphys.uni-heidelberg.de

Abstract Explicit solutions of the two-dimensional floating body problem (bodies that can float in all positions) for relative density $\rho \neq \frac{1}{2}$ and of the tire track problem (tire tracks of a bicycle, which do not allow to determine, which way the bicycle went) are given, which differ from circles. Starting point is the differential equation given by the author in [12, 8]. The curves are also trajectories of charges in a perpendicular parabolic magnetic field.

1 Introduction

In this paper we consider a class of curves, which are solutions to three problems: 
(i) the two-dimensional version of the floating body problem asked by Stanislaw Ulam in the Scottish Book[1] (problem 19): Is a sphere the only solid of uniform density which will float in water in any position? Thus we ask for the non-circular cross-section of a long cylindrical log, which can float in all orientations with the cross-section perpendicular to the surface of the fluid.
(ii) It also deals with the tire track problem, which asks whether it is possible, to have tire tracks of the front and the rear wheel of a bicycle, which do not allow to determine which way the bicycle went.
(iii) Finally the boundary of the cross-section describes the trajectory of a charge moving in a perpendicular parabolic magnetic field.

This paper is an improved and extended version of the solution given in [2]. It is extended insofar as figures for several curves are given. The discussion of the tire track problem is extended and explicit results for the cross-sections are given. Moreover the solution is formulated in terms of the four extreme radii, which makes the solution and symmetries more transparent.
Two observations\[3, 4, 5, 6, 7, 8\] are on the basis of the floating body problem: 
(i) Let \(h\) be the distance between the centers of gravity of the parts of the body above and below the water-line. Then the potential energy of the body is a constant times this height. Thus \(h\) has to be constant.
(ii) If one rotates the body infinitesimally, then the line between the two centers of gravity will only remain perpendicular to the water-line, if the length \(2\ell\) of this water-line obeys
\[
\frac{2}{3} \beta^3 \left( \frac{1}{A_1} + \frac{1}{A_2} \right) = h,
\]
where \(A_1\) and \(A_2\) are the cross-sections above and below the water-line, which of course are constant. Thus the length \(2\ell\) of the chord across the cross-section along the water-line is constant. Moreover since the areas above and below the water-line stay constant, the envelope \(\gamma\) of the chords are touched in the middle of the water-lines. Since for relative density (density of body divided by the density of the fluid) \(\rho = 1/2\), the water-line is the same, when the body is rotated by \(180^\circ\). Auerbach\[3\] could give the solution for this case. Then the boundary of the cross-section is obtained by starting from a Zindler curve \(\gamma\) for the envelope, that is a curve which closes after the direction of the tangent to the curve has increased by \(\pi\) (such curves have typically an odd number of cusps, for examples see red curves and water-lines in figs. 10 to 12 and 18). If one adds tangents of length \(\ell\) in both directions to all points of the curve \(\gamma\), then these end points yield the boundary \(\Gamma\) of such a cross-section, provided this boundary is sufficiently convex. (By sufficiently convex it is meant, that all water-lines cross the boundary twice, but not more often). In the following the case \(\rho \neq \frac{1}{2}\) will be mainly considered.

Another important property is obtained as follows: Consider two close-by water-lines \(A_1A_2\) and \(B_1B_2\) through the cross-section (Fig. 1). The body is kept fixed and the direction of the gravitational force is rotated.) Put the \(x\)-axis parallel to \(A_1A_2\). Then the vector \(A_1B_1\) is given by \((dx_1, -\ell_1d\phi)\) and \(A_2B_2\) by \((dx_2, \ell_2d\phi)\). Constant length \(d\ell = 0\) implies \(dx_2 = dx_1\). Constant areas \(A_1\) and \(A_2\) imply \(df_1 = \frac{1}{2}\ell_1^2d\phi = df_2 = \frac{1}{2}\ell_2^2d\phi\). Thus \(\ell_1 = \ell_2 = \ell\). This implies, that the infinitesimal arcs at the perimeter \(du_1 = \sqrt{(dx_1)^2 + \ell_1^2(d\phi)^2}\) and \(du_2 = \sqrt{(dx_2)^2 + \ell_2^2(d\phi)^2}\) are equal. Thus the part of the perimeter below the water-line is constant. Of course the same is also true for the part above the water-line.

One can conclude the other way round: If a curve has the property that if one moves from two fixed points \(A_1\) and \(A_2\) by constant arcs \(u\) along the perimeter and the length of the chord remains fixed, then also the areas separated by the chord stay constant. One argues that since \(\ell\) stays constant one has \(dx_1 = dx_2\).
Since \( du_1 = du_2 \) also \( \ell_1 = \ell_2 \) and thus \( df_1 = df_2 \). In the main part of the paper this property will be used. By measuring the distance along the arc from a fixed point of the boundary one introduces the arc parameter \( u \). It will be shown that for certain differences \( 2\delta u \) of the arc parameter the length of the chord is constant, that is the distance between the point at arc parameter \( u - \delta u \) and at arc parameter \( u + \delta u \) does not depend on \( u \). I call this the property of constant chord length.

In the case of \( \rho = 1/2 \) the ends of the water-lines differ by an arc parameter which is one half of the circumference. Similarly one may look for solutions where the difference \( 2\delta u \) of the arc parameter is a rational fraction of the circumference. Since however the area bounded by these water-lines and the part of the circumference is constant (the total cross-section is constant and the parts below the water also), the shape of such an equilateral triangle or quadrangle is constant and does not yield interesting results. An equilateral pentagon, however, of constant area and constant side length can vary its shape. Montejano et al. \[5, 4\] have used this property to obtain solutions for these special cases. Unfortunately there was no sufficiently convex boundary among the solutions.

Another interesting property is that since \( \ell_1 = \ell_2 \), one has \( dy_1 = -dy_2 \), which implies that the angles \( \delta \) between the tangents and the chord are equal, \( \delta_1 = \delta_2 \).

The floating body problem is related to the tire track problem \((6, 9, 10\) and papers cited therein). It goes back to a criticism of the discussion between Sherlock Holmes and Watson in The Adventure of the Priory School \[11\] on which way a bicycle went whose tires’ traces are observed. Let the distance between the front and the rear wheel of the bicycle be \( \ell \). The end points of the tangent lines of length \( \ell \) to the trace of the rear wheel in the direction the bicycle went yields the points of the traces of the front wheel. Thus if the tangent lines in both directions end at the trace of the front wheel, it is open which way the bicycle went. Thus curves \( \gamma \) for the rear wheel and \( \Gamma \) for the front wheel are solutions for such an ambiguous direction of the bicycle. The tire track problem consists in finding such curves \( \Gamma \) and \( \gamma \) different from circles and straight lines. It is equivalent to the two-dimensional floating body problem with the exception that the tire tracks need not close or may wind several times around some point.

Starting from \( r = r_0 (1 + \epsilon \cos(n\psi)) + O(\epsilon^2) \) for the boundary of the floating body and carrying the expansion in \( \epsilon \) through to seventh order the author conjectured that one obtains solutions for \( n - 2 \) different densities \( \rho \) for the same boundary \[\text{(12)}\text{[8].} \) This means that one and the same boundary \( \Gamma \) has the above mentioned property for several different chord lengths \( 2\ell \). Assuming now that this is also true, if the curve does not close as it is allowed in the tire track problem, one may assume that it also holds if the curve closes apart from an infinitesimal small angle difference \( \delta\chi \) with chords of infinitesimal length \( 2\ell \). Then a differential equation of third order for this curve can be derived. After one integration this differential equation yields
\[
\kappa(r) := \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} = 4ar^2 + 2b,
\]

in polar coordinates \((r, \psi)\) with \(r\) function of \(\psi\) and the prime indicating differentiation with respect to \(\psi\). \(\kappa\) is the curvature of the curve. \(a\) is given by the limit \(\frac{4a}{\ell^3} = \lim_{\delta \chi \to 0} \ell^3/\delta \chi\) for vanishing \(\ell\) and \(\delta \chi\) (defined by \(\chi - \hat{\chi}\) of sect. 6.1). \(b\) is the integration constant for this first integral. Thus these curves describe the trajectories of particles of mass \(m\), charge \(e\) and velocity \(w\) in a perpendicular magnetic field, which depends quadratically on \(r\),

\[
B(r) = \frac{mw}{e}(4ar^2 + 2b).
\]

Another integral of the third-order differential equation is

\[
\frac{r + r''}{(r^2 + r'^2)^{3/2}} = -2ar + \frac{2c}{r^3}.
\]

If we introduce the inverse radius \(r = 1/\bar{r}\), then this equation can be rewritten

\[
\bar{\kappa}(\bar{r}) = \frac{\bar{r}^2 + 2\bar{r}'^2 - \bar{r}\bar{r}''}{(\bar{r}^2 + \bar{r}'^2)^{3/2}} = -\frac{2a}{\bar{r}^4} + 2c.
\]

Thus \(\bar{r} = 1/r\) describes the trajectory of a charge in a perpendicular magnetic field

\[
\bar{B}(\bar{r}) = \frac{mw}{e}\left(\frac{2a}{\bar{r}^4} - 2c\right).
\]

Elimination of \(r''\) from eqs. (2, 4) yields our basic differential equation

\[
\frac{1}{\sqrt{r^2 + r'^2}} = ar^2 + b + cr^{-2}.
\]

It will be shown that the resulting curves solve the floating body problem, provided they are sufficiently convex and closed. Indeed there are such solutions. For the tire track problem the restrictions are less rigorous. But among them there are solutions, which require artistic mastery of the bicyclist, since he/she has to move back and forth. In more difficult cases the bicycle has to allow the steerer to be rotated by more than 180° degrees.

Quite generally the following remarkable property of these curves will be shown: Consider two copies of these curves (not necessarily closed). Choose an arbitrary point on each curve. Then there exists always an angle by which the two curves can be rotated against each other, so that the distance between the points on the two curves stays constant, if one moves from the given points on both curves by the same arc distance. I call this the \textit{property of constant distance}. This remarkable property made it possible to obtain the above-mentioned differential equation by choosing the two points infinitesimally close.

In [12] the following necessary condition was shown: If for some chord the angles \(\delta\) between the chord and the tangents on the curve obey \(\delta_1 = \delta_2\) at
\[ r_1 \neq r_2 \] then also the derivative \( \frac{d(\delta_1 - \delta_2)}{du} \) vanishes. This is a necessary condition, but it is not sufficient. I am indebted to Serge Tabachnikov for pointing this out to me. Here the differential equation (7) will be solved explicitly as a function of the arc parameter \( u \), and it will be shown that the curve has the desired property.

The outline of the paper is as follows: In sect. 2 an appropriate parametrization is introduced. Depending on the parameters \( a, b, \) and \( c \) one may have no or one or two (real) branches. Two branches means that the curves are not congruent. The equations are given in terms of the arc parameter. In sect. 3 the radius \( r \) as a function of the arc parameter \( u \) is determined. It is given in terms of Weierstrass \( \wp \)-functions. The invariants \( g_2 \) and \( g_3 \) are real. In the case of one branch the discriminant is negative, otherwise positive. This is discussed in sect. 4. In the following section the polar angle and thus the whole curve is determined as function of \( u \). Starting from the differential equation (7) the solution can be written

\[
z(u) := x + iy = Ce^{i\chi} \frac{\sigma(u - 3v)}{\sigma(u + v)} e^{2\zeta(2v)u},
\]

where \( u \) is the (real) arc parameter. The purely imaginary \( v \) and the real invariants \( g_2 \) and \( g_3 \) of the Weierstrass integrals \( \zeta \) and \( \sigma \), and the constant \( C \) depend on the coefficients \( a, b, \) and \( c \). \( \chi \) is an arbitrary angle which defines the orientation of the curve around the origin and can be chosen arbitrarily.

Instead of solving (7) one could immediately start with the curve defined by (8) and first show that the arc-parameter \( u \) indeed measures the distance along the curve, which is done in subsect. 5.3.

Secondly the above mentioned property of constant distance is shown in sect. 6. Given a difference \( \delta u \), there exists a difference of angles, by which two copies of the curve described by \( z \) and \( \hat{z} \) have to be rotated against each other, so that the distance \( 2\ell = |z(u + \delta u) - \hat{z}(u - \delta u)| \) does not depend on \( u \) (eq. 141). If the difference of the angles is such that the two curves fall onto each other, then one obtains the desired property of constant chord length for floating bodies and tire tracks. The property of constant distance holds even between the two different branches in the case of positive discriminant. A remark on the expression of Finn for the curvatures of the curves is added in subsection 6.3. Finally the area above and below the water-line is determined and it is shown that the centers of gravity of the parts of the cross-section above and below the water-line lie on circles (compare Auerbach[3]).

In the following section the periodicity of the curves is discussed and figures of several curves are shown. In addition a number of these examples are shown as animations on the internet[13]. These examples are indicated by an asterisk at the figure caption.

The limit case, where the discriminant vanishes, is considered in section 8. In this case the curves can be expressed in terms of trigonometric and exponential functions. Moreover there is a simple construction principle of these curves explained in subsection 8.4. In section 9 the limit is considered, in which the curve oscillates around a straight line. (This does not constitute a solution of
the floating body problem, but of the tire-track problem.) There one finds an 
infinite set of \( \delta u \)s, for which the length of the chord between the points of arc 
parameter \( u - \delta u \) and \( u + \delta u \) is independent of \( u \) and where \( 2 \delta u \) is not a period 
of the curve. The curves found in this section are also trajectories of electrons 
in a linearly varying magnetic field as considered by Evers et al.\cite{14}.

In the following section the \textit{carousel}s by Bracho, Montejano and Oliveros 
\cite{5, 4} are discussed, where five chords or line segments close to an equilateral 
pentagon. The solution given here allows to reconstruct these curves. Some of 
them will be shown and discussed.

I cannot claim that the solutions given here are the only solutions to the 
floating body problem, but I am not aware of any other solutions. There remains 
the question whether starting from the known solutions small deformations can 
yield new solutions similarly as a class of solutions were found from deforming 
the circle \cite{7, 8}. The solutions of the tire track problem are not restricted to 
closed and not terminating curves. In view of the paper by Finn \cite{9, 10} who 
describes a procedure to continue from an initial back-tire track segment one 
expects a large manifold of solutions for this latter problem.

2 Basic Equations

2.1 A first discussion

The curves described by the differential equation \cite{7} will now be considered quite 
generally. For this purpose the angle \( \phi \) is introduced indicating the direction of 
the tangent to the curve with respect to the radial vector by

\[
\frac{dr}{d\psi} = r \tan \phi \tag{9}
\]

as shown in fig. \ref{fig:2}. Then eq. \cite{7} can be written

\[
aq^2 + bq + c = \sqrt{q} \cos \phi, \tag{10}
\]

where the square \( q = r^2 \) of the radius is introduced. Thus the curves obey

\[
- \sqrt{q} \leq aq^2 + bq + c \leq \sqrt{q}. \tag{11}
\]
Whenever
\[ aq^2 + bq + c = \pm \sqrt{q} \] (12)
is reached, the curve has found an extreme distance from the origin.

Depending on the constants \( a, b, \) and \( c \) the inequality (11) allows for zero, one or two curves. This can be seen by plotting the parabola \(< \) depicting \( \pm \sqrt{q} \) and the parabola \( \triangledown \) showing \( aq^2 + bq + c \) versus \( q \). To the extend segments of the parabola \( \triangledown \) lie inside the parabola \( < \) one obtains curves. These segments are indicated by thick lines. Various possible cases are shown in fig. 3.

\[ \begin{align*}
&\text{Fig. 3a} & &\text{Fig. 3b} & &\text{Fig. 3c} \\
&\text{Fig. 3d} & &\text{Fig. 3e} & &\text{Fig. 3f} \\
&\text{Fig. 3g} & &\text{Fig. 3h} & &\text{Fig. 3i}
\end{align*} \]

Fig. 3 Various cases of the relative position of the two parabola

does not allow for any real curves, since the parabola \( \triangledown \) lies completely outside \(< \). In fig. 3b, the parabola \( \triangledown \) touches \(< \). Only one radius is allowed, which corresponds to a circle. Fig. 3c allows for some variation of the radius. It yields a curve oscillating between the two extreme radii. From this case one obtains good solutions for the floating body and the tire track problem. In fig. 3d, the parabola \( \triangledown \) touches the \( q \)-axis. Thus the curve shows a tangent in radial direction. In fig. 3e, the curve goes backward in \( \psi \)-direction close to the smallest
radius. Fig. 3 allows for a small circle plus a curve. Fig. 3k is the limit case where the curve approaches asymptotically the circle. Fig. 3i allows for two curves both of which approach asymptotically a common circle, one from outside, the other from inside. Fig. 3 is the generic case of two curves.

The extreme distances from the origin are reached, when \( \cos \phi = \pm 1 \). This yields

\[
ar^4 + br^2 + c - r = 0 \tag{13}
\]

where for real \( r_i \) the \( |r_i| \) are the extreme radii of the curves. \( r_i \) is chosen positive, if \( \cos \phi = +1 \), and negative, if \( \cos \phi = -1 \). The polynomial (13) can be expressed in terms of these extreme radii \( r_i \),

\[
ar^4 + br^2 + c - r = a \prod_{i=1}^{4} (r - r_i). \tag{14}
\]

Comparing the coefficients of the polynomial one obtains

\[
0 = \sum_i r_i, \tag{15}
\]

\[
b = a \sum_{i<j} r_i r_j = \frac{a}{2} (\sum_i r_i)^2 - \frac{a}{2} \sum_i r_i^2 = -\frac{a}{2} \sum_i q_i, \tag{16}
\]

\[-1 = -a(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4) = -aP_r, \tag{17}
\]

\[c = ar_1 r_2 r_3 r_4 = a\hat{P}, \tag{18}
\]

which allows the coefficients \( a, b, c \) to be expressed by the \( r_i \). Thus the sum of the four \( r_i \) vanishes. Here

\[q_i = r_i^2 \tag{19}\]

is introduced and use is made of the polynomials \( P_r \) and \( \hat{P} \) defined in appendix B in eqs. (357) and (365) and of eq. (359). In particular one obtains

\[a = \frac{1}{P_r}. \tag{20}\]

In the following \( a \) is assumed to be positive. Reversing simultaneously the signs of \( a, b, \) and \( c \) corresponds to reversing all \( r_i \), which yields the same curves, but a different orientation. If \( a = 0 \), then according to (2) the curve is a circle \( (b \neq 0) \) or a straight line \( (b = 0) \).

### 2.2 Parametrization according to the shape

Multiplication of all \( r_i \) by the same factor yields similar curves, but no new shape. Since the four \( r_i \) obey the restriction \( \sum_i r_i = 0 \), only two parameters are left, which determine the shape of the curves. Thus the parametrization

\[r_{4,3} = r_0(1 \pm \epsilon), \quad r_{1,2} = r_0(-1 \pm \hat{\epsilon}) \tag{21}\]

is introduced, where \( r_0 \) sets the scale of the curve, whereas \( \epsilon \) and \( \hat{\epsilon} \) determine its shape. \( \epsilon \) and \( \hat{\epsilon} \) can always be chosen either real or purely imaginary. A sign
change of $\epsilon$ and/or $\dot{\epsilon}$ corresponds to a permutation of the $r_i$. Therefore the $\epsilon^2, \dot{\epsilon}^2$ plane is plotted in figure. Both $\epsilon^2$ and $\dot{\epsilon}^2$ are real. $r_0$ is assumed to be real.

Consider the four quadrants in fig. $\epsilon$ is real and $\dot{\epsilon} = -i\mu$ is imaginary in the left upper quadrant A. Here one has one real curve. In the right upper quadrant B both $\epsilon$ and $\dot{\epsilon}$ are real. There one obtains two branches. In the right lower quadrant C $\epsilon$ is imaginary and $\dot{\epsilon}$ is real. By exchanging $\epsilon$ and $\dot{\epsilon}$ one comes back to quadrant A. Finally in the lower left quadrant D both $\epsilon$ and $\dot{\epsilon}$ are imaginary. This does not correspond to any real curve. Thus it is sufficient to consider the quadrants A and B.

Without restriction of the general case

\[ \epsilon - 2 > \dot{\epsilon} > 0. \]  

(22)

can be required in region B. This corresponds to the ordering $q_4 > q_3 > q_2 > q_1$ and is covered by the region $B_1$. Fig. 5 repeats fig. 3 with the assignments of the $r_i$.

The parabola in figure

\[ [(\epsilon - 2)^2 - \dot{\epsilon}^2][(\epsilon + 2)^2 - \dot{\epsilon}^2] = (\epsilon^2 - \dot{\epsilon}^2)^2 - 8(\epsilon^2 + \dot{\epsilon}^2 - 2) = 0 \]  

(23)
and the line $\hat{\epsilon}^2 = \epsilon^2$ decomposes the whole quadrant $B \hat{\epsilon}^2 > 0$, $\epsilon^2 > 0$ into six regions. If one allows an arbitrary assignment of the $r_i$ to $r_0(1 \pm \epsilon)$ and $r_0(-1 \pm \hat{\epsilon})$ then one obtains 24 equivalent permutations. Since only the squares of $\epsilon$ and $\hat{\epsilon}$ are plotted, this reduces to six sets of $(\epsilon^2, \hat{\epsilon}^2)$, which are represented by the six regions $B_1$ to $B_6$.

### 2.3 Equations to be solved

From equation (7) one obtains

$$\left(\frac{dr}{d\psi}\right)^2 = \frac{r^4}{(ar^4 + br^2 + c)^2} - r^2$$

and thus

$$\frac{d\psi}{dq} = \frac{aq^2 + bq + c}{2q\sqrt{q - (aq^2 + bq + c)^2}}$$

with $q = r^2$. If one denotes the distance on the curve from some fixed point (arc-parameter) by $u$ then one obtains

$$\frac{du}{d\psi} = \sqrt{r^2 + r'^2} = \frac{1}{ar^2 + b + cr^2} = \frac{1}{aq + b + cq^{-1}}$$

and

$$\frac{du}{dq} = \frac{1}{2\sqrt{q - (aq^2 + bq + c)^2}}$$

First (sect. 3) $q(u)$ will be calculated from

$$\left(\frac{dq}{da}\right)^2 = -4a^2 \prod_{i=1}^{4}(q - q_i),$$

$$u = \int \frac{dq}{2a \sqrt{-\prod_{i=1}^{4}(q - q_i)},}$$

since

$$(aq^2 + bq + c)^2 - q = (ar^4 + br^2 + c - r)(ar^4 + br^2 + c + r)$$

$$= a^2 \prod_{i}(r - r_i) \prod_{i}(r + r_i) = a^2 \prod_{i}(q - q_i),$$

Secondly $\psi(u)$ (sect. 5) is determined from

$$\psi(u) = \int du(aq + b + cq^{-1}).$$

Using eqs. 21 16 the curvature is given by

$$\kappa = 4ar^2 + 2b = a(4r^2 - \sum r_i^2).$$
One concludes that a convex curve oscillating between \( r_0(1 + \epsilon) \) and \( r_0(1 - \epsilon) \) has to fulfill
\[
4\epsilon < \epsilon^2 - \hat{\epsilon}^2. \quad (33)
\]
This condition guarantees for positive \( \epsilon \) that the curvature does not change sign.

3 Radius as Function of Arc Parameter

The solution of the differential equation (28) reads
\[
q = q_4 \left\{ \frac{\wp(u) - p_3}{\wp(u) - p_1} \right\} \quad (34)
\]
with the Weierstrass \( \wp \)-function. To show this, one calculates the derivative
\[
\frac{dq}{du} = \frac{q_4(p_1 - p_3)}{(\wp(u) - p_1)^2} \frac{\wp'(u)}{(\wp(u) - p_1)} \quad (35)
\]
\[
= -\frac{2q_4(p_1 - p_3)}{(\wp(u) - p_1)^2} \sqrt{(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)}. \quad (36)
\]
From eq. (34) one obtains
\[
\frac{1}{\wp(u) - p_1} = \frac{1}{p_1 - p_3} \left( \frac{q}{q_4} - 1 \right), \quad \frac{\wp(u) - e_i}{\wp(u) - p_1} = \frac{p_1 - e_i}{p_1 - p_3} \left( \frac{q}{q_4} - \frac{p_3 - e_i}{p_1 - e_i} \right). \quad (37)
\]
Eq. (28) is fulfilled provided the relations
\[
-4a^2 = \frac{4\prod_{i=1}^{3}(p_1 - e_i)}{r_4^2(p_1 - p_3)^2}, \quad q_i = q_4 \frac{p_3 - e_i}{p_1 - e_i}, \quad i \neq 4 \quad (38)
\]
hold. The last eqs. and \( \sum_{i=1}^{3} e_i = 0 \) yield four linear homogeneous eqs. for \( p_1, p_3 \) and the \( e_i \). Together with the first equation one obtains the solutions
\[
p_1 = a^2 q_4(-q_4 + q_1 + q_2 + q_3) - \frac{a^2}{3} P_m, \quad (39)
\]
\[
p_3 = a^2\left(\frac{q_1 q_2 q_3}{q_4} + q_1 q_2 + q_1 q_3 + q_2 q_3\right) - \frac{a^2}{3} P_m, \quad (40)
\]
\[
e_1 = a^2(q_1 q_4 + q_2 q_3) - \frac{a^2}{3} P_m, \quad (41)
\]
\[
e_2 = a^2(q_2 q_4 + q_1 q_3) - \frac{a^2}{3} P_m, \quad (42)
\]
\[
e_3 = a^2(q_3 q_4 + q_1 q_2) - \frac{a^2}{3} P_m \quad (43)
\]
with the polynomial \( P_m \) as defined in eq. (35S). Permutations of the \( r_i \) yield permutations of the \( e_i \). Therefore the invariants \( g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3) \) and \( g_3 = 4e_1 e_2 e_3 \) and also \( \wp(u) \) are invariant under all permutations of the \( r_i \).
For future purposes the parameter $v$ is introduced by

$$p_1 = \varphi(v),$$

which yields

$$\varphi'(v) = \mp 2ia^3(q_4 - q_1)(q_4 - q_2)(q_4 - q_3),$$

$$\varphi''(v) = 2a^4(q_4 - q_1)(q_4 - q_2)(q_4 - q_3)(3q_4 - q_1 - q_2 - q_3).$$

From these expressions one concludes

$$\varphi(2v) = -\frac{a^2}{4}(q_1^2 + q_2^2 + q_3^2 + q_4^2) + \frac{a^2}{6}P_m,$$  \hfill (47)

$$\varphi'(2v) = \mp 2ia^3P_q = \mp \frac{2i}{P_r} = \mp 2ia$$  \hfill (48)

with the polynomial $P_q$ defined in eq. (356) and by use of eqs. (20, 361). These expressions are invariant under all permutations of the $r_i$.

Using the condition $\sum_i r_i = 0$ one obtains

$$p_3 = \varphi(3v).$$

To show this one starts from

$$\varphi(3v) = -\varphi(v) - \varphi(2v) + \frac{(\varphi'(v) - \varphi'(2v))^2}{4(\varphi(v) - \varphi(2v))^2},$$

$$\varphi(v) = -\varphi(v) - \varphi(2v) + \frac{(\varphi'(v) + \varphi'(2v))^2}{4(\varphi(v) - \varphi(2v))^2},$$

from which one concludes

$$\varphi(3v) = \varphi(v) - \frac{\varphi'(v)\varphi'(2v)}{(\varphi(v) - \varphi(2v))^2}. \quad (52)$$

Comparison shows

$$\varphi(v) - \varphi(2v) = a^2(-2\hat{P} + q_4(q_1 + q_2 + q_3 - q_4)).$$ \hfill (53)

with the polynomial $\hat{P}$ defined in eq. (365). By means of eqs. (362, 366) one finds

$$\varphi(v) - \varphi(2v) = -2a^2r_4P_r.$$ \hfill (54)

and

$$(\varphi(v) - \varphi(2v))^2 = 4a^4q_4P_q.$$ \hfill (55)

Insertion into eq. (52) yields eq. (49). One obtains the derivative $\varphi'(3v)$ by means of

$$\varphi'(3v) = -\frac{r_1r_3r_4}{r_4^3}\varphi'(v).$$ \hfill (56)
which by use of eq. \(366\) can be written
\[
\wp'(3v) = -\frac{P_r}{q_4^2} \wp'(v).
\] (57)

The solution \(34\) can be written
\[
q(u) = q_4 E_0 \left( u \left| \frac{3v, -3v}{v, -v} \right. \right) = q_4 \frac{\sigma(u + 3v)\sigma(u - 3v)}{\sigma(u + v)\sigma(u - v)} \frac{\sigma^2(v)}{\sigma^2(3v)},
\] (58)
where the arc parameter \(u\) is measured from the extreme radius \(r = r_4\). The notation \(E_n \left( u \left| \left\{ u_i \right\} \right. \right)\) for double-periodic functions of \(u\) is explained in appendix \(A.2\). Apart from the origin it has zeroes at \(u_i\) and poles at \(v_i\) and behaves like \(u^n\) at the origin. Correspondingly the radius reads
\[
r(u) = C \left( \frac{\sigma(u + 3v)\sigma(3v - u)}{\sigma(u + v)\sigma(v - u)} \right)^{1/2}, \quad C = r_4 \frac{\sigma(v)}{\sigma(3v)}
\] (59)
The prefactor \(C\) is rewritten by means of the identity
\[
\frac{\sigma(3v)}{\sigma(v)\sigma^2(2v)} = \wp(v) - \wp(2v),
\] (60)
which can be seen by realizing that both the left and right hand-side of this equation is double-periodic and can be written
\[
\frac{3}{4} E^{-2} \left( v \left| \left[ \frac{2u_1, -2u_1, \omega_1, -\omega_1, \omega_2, -\omega_2, \omega_1 + \omega_2, -\omega_1 - \omega_2, \omega_1 - 2\omega_2, 2\omega_2 - 2\omega_1, 2\omega_2, 3 - 2\omega_1}{3} \right] \right. \right).
\] (61)
One obtains
\[
C = \frac{r_4}{\sigma^2(2v)(\wp(v) - \wp(2v))} = -\frac{P_r}{2\sigma^2(2v)} = \frac{i}{\wp'(2v)\sigma^2(2v)} = \frac{-i\sigma^2(2v)}{\sigma(4v)},
\] (62)
where eqs. \(48\) and \(34\) have been used. Thus the factor \(r_4\) drops out and the prefactor \(C\) is invariant against any permutations of the \(r_i\).

4 Single Branches and Pairs of Branches

In the preceding section the radius square \(q\) has been obtained as a function of the arc parameter \(u\) in eq. \(34\). At \(u = 0\) the function \(\wp(u)\) diverges and \(q\) approaches \(q_4 = r_4^2\). As \(u\) is increased up to \(\omega_3\) one obtains \(\wp(\omega_3) = e_3\) and \(q\) approaches its minimum \(q = q_4(e_3 - p_3)/(e_3 - p_1) = q_3\) along the curve. Then \(q\) increases until at \(u = 2\omega_3\) it returns to \(q = q_4\). Instead starting at \(u = 0\) with \(q = q_4\) one can start at \(u = \omega_3\) with \(q = q_3\). Therefore consider
\[
q(u + \omega_i) = q_4 \frac{\wp(u + \omega_i) - \wp(3v)}{\wp(u + \omega_i) - \wp(v)},
\] (63)
with half-periods $\omega_i, \ i = 1, 2, 3$. $\omega$ is called a half-period, if it is not a period of $\wp$, but $2\omega$ is a period. The expression (63) can be rewritten

$$q(u + \omega_i) = q_4E_0 \left( \frac{u + \omega_i}{v, -v} \right)$$

$$= \frac{\varphi(\omega_i) - \varphi(3v)}{\varphi(\omega_i) - \varphi(v)} E_0 \left( \frac{3v - \omega_i, -3v - \omega_i}{v - \omega_i, -v - \omega_i} \right)$$

$$= q_4 \frac{e_i - p_3 \wp(u) - \wp(3v - \omega_i)}{e_i - p_1 \wp(u) - \wp(v - \omega_i)}$$

$$= q_4 \frac{\wp(u) - p^{(i)}_1}{\wp(u) - p^{(i)}_3}$$  \hspace{1cm} (64)$$

with

$$p^{(i)}_1 = \wp(v^{(i)}), \quad p^{(i)}_3 = \wp(3v^{(i)}), \quad v^{(i)} = v - \omega_i.$$  \hspace{1cm} (65)$$

Explicit calculation by means of

$$\wp(v - \omega_i) = \frac{e_j^2 + e_j e_k + e_i \wp(v)}{\wp(v) - e_i},$$  \hspace{1cm} (66)$$

where $j$ and $k$ are the elements out of (1,2,3) different from $i$, shows that $p^{(i)}_1$ and $p^{(i)}_3$ are obtained from $p_1$ and $p_3$ by exchanging $q_i$ and $q_4$ in the expressions (39) and (40).

The expression (47) for $\wp(2v)$ is invariant under permutations of the $q_i$. Solving for $v$ from this equation yields besides $v$ from (39) also the $v^{(i)}$, $i = 1, 2, 3$. In the following $v$ will be chosen, so that $3\wp'(v) < 0$. Then the upper sign in eqs. (45, 48) apply. Since from (340) one obtains with $u = -\omega_i$

$$\wp'(v^{(i)}) = -\frac{\wp(v - \omega_i) - e_i}{\wp(v) - e_i} \wp'(v),$$  \hspace{1cm} (67)$$

one finds that $\wp'(v^{(i)})$ is obtained from $\wp'(v)$ by exchanging $q_i$ and $q_4$ and one has also to take the upper sign.

4.1 Single branch

If two of the $r_i$ are complex and two are real, then one (real) curve is obtained. The choice of complex $r_2 = r_1^*$ and of real $r_4$ and $r_3$ yields

$$q_2 = q_1^*, \quad q_4 > q_3.$$  \hspace{1cm} (68)$$

Then $e_3$ is real, whereas $e_1$ and $e_2$ constitute a conjugate complex pair

$$e_2 = e_1^*, \quad e_3 = e_3^*.$$  \hspace{1cm} (69)$$

Consequently the discriminant

$$D = 16(e_3 - e_1)^2 (e_3 - e_2)^2 (e_2 - e_1)^2 < 0$$  \hspace{1cm} (70)$$
Fig. 6 Two elementary cells of periodicity for negative discriminant inside the rectangle.

is negative. For negative discriminant $D$ the periods and half-periods are located in the complex plane as shown in fig. 6.

The rectangle $0 \leq \Re z < 2\omega_3, 0 \leq \Im z < 2\omega'/i$ covers two elementary cells for the double-periodic function $\wp(z)$. The function is real along the straight lines drawn. For constant $\Im z$ (horizontal lines) it varies between $+\infty$ and $e_3$. For constant $\Re z$ (vertical lines) it varies between $-\infty$ and $e_3$. The singularities are at the points indicated by •, $\wp(\bullet) = \infty$. The derivative of the function $\wp(z)$ vanishes at half-periods $\omega$. There are three different half-periods which do not differ by periods. They are indicated by •, +, and ■. Thus one has $\wp'(\bullet) = \wp'(+) = \wp'(■) = 0$. At these points the function approaches $\wp(\omega_i) = e_i$. Examples are $\wp(\omega_1) = \wp(-\omega_1) = \wp(\bullet) = e_1$ with $2\omega_1 = \omega' - \omega_3$, $\wp(\omega_2) = \wp(-\omega_2) = \wp(+) = e_2$ with $2\omega_2 = -\omega' - \omega_3$, $\wp(\omega_3) = \wp(\omega') = \wp(■) = e_3$.

Next consider the location of $p_1$. In section 3 the $p_1$ related to $q_4$ was considered. To indicate this connection one may write $p_1 = p_1^{(4)}$ and generally introduce

$$p_1^{(i)} = a^2 q_i (-2q_i + \sum_{j=1}^{4} q_j) - \frac{a^2}{3} P_m.$$  \hspace{1cm} (71)

Here one is interested in the two real $p_1^{(4)}$ and $p_1^{(3)}$. One obtains

$$p_1^{(4)} - e_3 = -a^2(q_4 - q_1)(q_4 - q_2) < 0, \hspace{1cm} (72)$$

$$p_1^{(3)} - e_3 = -a^2(q_3 - q_1)(q_3 - q_2) < 0. \hspace{1cm} (73)$$
Thus one has
\[ p_1^{(4)}, p_1^{(3)} < e_3. \]  
(74)

The real part of \( v^{(i)} \) given by
\[ \wp(v^{(i)}) = p_1^{(i)} \]  
(75)
lies on the vertical lines in fig. 6
\[ \Re v^{(4)}, \Re v^{(3)} = n\omega \]  
(76)
with integer \( n \). In the following the main choice for \( v^{(4)} = v \) will be the one with
\[ \Re v = 0, \quad 0 < \Im v < \omega', \]  
(77)
so that
\[ \Re \wp'(v) = 0, \quad \Im \wp'(v) < 0. \]  
(78)
in agreement with the requirement introduced before. As mentioned before adding \( \omega_3 \) to \( u \) is equivalent to exchanging \( q_4 \) and \( q_3 \). Both choices yield the same curve. To exchange \( q_4 \) with \( q_1 \) or \( q_2 \) does not yield a positive \( q \) and thus a real curve.

The \( \omega_i \) are obtained from
\[ \omega_3 = \frac{K(\frac{1}{2} - \frac{3e_3}{4H_3})}{\sqrt{H_3}}, \quad \omega' = \frac{K(\frac{1}{2} + \frac{3e_3}{4H_3})}{\sqrt{H_3}}, \]  
\[ H_3 = \sqrt{(e_3 - e_1)(e_3 - e_2)}, \]  
(79)
where for the elliptic integral \( K(m) \) the notation of Abramowitz and Stegun [15] is used. The reader is reffered to chapter 17 of this handbook for elliptic integrals and chapter 18 for Weierstrass elliptic functions (watch permutations of indices \( i \) of \( e_i \) and \( \omega_i \) in comparison to the use here), and to appendix A of the present paper.

### 4.2 Pair of branches

If all four \( r_i \) are real, then one obtains two branches. The \( r_i \) should be ordered according to
\[ q_4 > q_3 > q_2 > q_1. \]  
(81)

Then the differences of the \( e_1, e_2, e_3 \) obey
\[ e_3 - e_2 = a^2(q_3 - q_2)(q_4 - q_1) > 0, \]  
(82)
\[ e_2 - e_1 = a^2(q_4 - q_3)(q_2 - q_1) > 0 \]  
(83)
and thus
\[ e_3 > e_2 > e_1. \]  
(84)
In this case, where all three $e_i$ are real, the discriminant $D$, eq. (70), is positive. Then the periods and half-periods are located in the complex plane as shown in fig. 7.

One elementary cell of periodicity is given by the rectangle $0 \leq \Re z < 2\omega_3$, $0 \leq \Im z < 2\omega_1/i$. Again the function is real along the straight lines drawn. The function approaches infinity at $\wp(\bullet) = \infty$. The half-periods are indicated by $\blacksquare$, $\blacklozenge$ and $\blacktriangle$. The derivatives vanish at these half-periods, $\wp'(\blacksquare) = \wp'(\blacklozenge) = \wp'(\blacktriangle) = 0$. The values of $\wp$ at the half-periods are denoted by $\wp(\omega_1) = e_1$, $\wp(\omega_2) = \wp(-\omega_1) = e_2$, $\wp(\omega_3) = \wp(-\omega_2) = e_3$.

One obtains the inequalities for the $p_i^{(i)}$ defined in (74)

\begin{align*}
p_1^{(4)} - e_1 &= -a^2(q_4 - q_2)(q_4 - q_3) < 0, \quad (85) \\
p_1^{(1)} - e_1 &= -a^2(q_2 - q_1)(q_3 - q_4) < 0, \quad (86) \\
p_1^{(3)} - e_2 &= a^2(q_3 - q_2)(q_4 - q_3) > 0, \quad (87) \\
p_1^{(3)} - e_3 &= -a^2(q_3 - q_1)(q_3 - q_2) < 0, \quad (88) \\
p_1^{(2)} - e_2 &= a^2(q_2 - q_1)(q_3 - q_4) > 0, \quad (89) \\
p_1^{(2)} - e_3 &= -a^2(q_3 - q_2)(q_4 - q_2) < 0. \quad (90)
\end{align*}

Thus one has

\begin{align*}
p_1^{(4)}, p_1^{(1)} < e_1 < e_2 < p_1^{(3)}, p_1^{(2)} < e_3. \quad (91)
\end{align*}
Consequently one finds for the real part of $v^{(i)}$ defined in (75)

$$
\Re v^{(4)}, \Re v^{(1)} = 2n\omega_3, \quad \Re v^{(3)}, \Re v^{(2)} = (2n + 1)\omega_3
$$

(92)

with integer $n$. In the following the main choice for $v^{(4)} = v$ will be the one with

$$
\Re v = 0, \quad 0 < \Im v < \omega_1.
$$

(93)

so that as in the case of one curve eq. (78) and the upper signs in eqs. (45, 48) apply. Adding $\omega_1$ or $\omega_2$ yields a different variation of the radius between $\sqrt{q_1}$ and $\sqrt{q_1}$. However, adding $\omega_1$ or $\omega_2$ yields a different variation of the radius between $\sqrt{q_1}$ and $\sqrt{q_2}$. It corresponds to $u$ running along a horizontal line in fig. 7 through $\diamondsuit$ and $\clubsuit$, where $\varphi$ is again real. This is equivalent to subtracting $\omega_1$ or $\omega_2$ from $v$. Thus in this case one obtains two branches.

The $\omega_i$ are given by

$$
\omega_3 = \frac{K(e_3 - e_1)}{\sqrt{e_3 - e_1}}, \quad \omega_1 = i\frac{K(e_3 - e_2)}{\sqrt{e_3 - e_1}}
$$

(94)

There are the special cases where two of the radii $r_i$ coincide. In these cases the discriminant vanishes. In fig. 3f one has $r_1 = r_2$. This case yields a small circle plus a curve with varying distance from the origin. In this case the imaginary period goes to infinity. In fig. 3h, where $r_2 = r_3$ there are two curves asymptotically approaching the same circle. There the real period approaches infinity. The limit case is shown in fig. 3g, where $r_1 = r_2 = r_3$. In this case both periods approach infinity. These limit cases are considered in section 8.

5 Angle and Curve as Function of the Arc Parameter

5.1 The angle

The angle $\psi$ as function of $u$ is determined from eq. (41)

$$
\psi(u) = \int du (aq + b + c) = \int du \left( aq_4 \frac{\varphi(u) - p_3}{\varphi(u) - p_1} + b + \frac{c}{q_4} \frac{\varphi(u) - p_1}{\varphi(u) - p_3} \right).
$$

(95)

With

$$
\int du \frac{1}{\varphi(u) - \varphi(v)} = \frac{1}{\varphi'(v)} \left( 2u \zeta'(v) + \ln \left( \frac{\sigma(u - v)}{\sigma(u + v)} \right) \right)
$$

(96)

one obtains

$$
\psi(u) = \chi + c_0 u + c_1 \ln \left( \frac{\sigma(v - u)}{\sigma(v + u)} \right) + c_3 \ln \left( \frac{\sigma(3v - u)}{\sigma(3v + u)} \right)
$$

(97)

18
with the real integration constant $\chi$ and

$$c_0 = aq_4 + b + \frac{c}{q_4} + 2c_1\zeta(v) + 2c_3\zeta(3v), \quad (98)$$

$$c_1 = \frac{aq_4(p_1 - p_3)}{\wp'(v)}, \quad (99)$$

$$c_3 = \frac{c(p_3 - p_1)}{q_4\wp'(3v)}. \quad (100)$$

Evaluation of $c_1$ and $c_3$ yields

$$c_1 = c_3 = -\frac{i}{2}, \quad (101)$$

where the sign convention (78) is applied. Further one obtains from (13)

$$aq_4 + b + \frac{c}{q_4} = \frac{1}{r_4}. \quad (102)$$

This yields

$$\psi(u) = \chi + c_0u - \frac{i}{2} \ln \left( \frac{\sigma(v - u)}{\sigma(v + u)} \right) - \frac{i}{2} \ln \left( \frac{\sigma(3v - u)}{\sigma(3v + u)} \right) \quad (103)$$

with

$$c_0 = \frac{1}{r_4} - i(\zeta(v) + \zeta(3v)). \quad (104)$$

Using

$$\zeta(2v \pm v) = \zeta(2v) \pm \zeta(v) + \frac{\wp'(2v) \mp \wp'(v)}{2(\wp(2v) - \wp(v))} \quad (105)$$

one obtains

$$\zeta(3v) + \zeta(v) = 2\zeta(2v) + \frac{\wp'(2v)}{\wp(2v) - \wp(v)} = 2\zeta(2v) - \frac{i2a^3P_q}{2a^2P_1r_4} = 2\zeta(2v) - \frac{i}{r_4} \quad (106)$$

and thus

$$c_0 = -2i\zeta(2v), \quad (107)$$

which yields

$$e^{i\psi(u)} = e^{i\chi} \left( \frac{\sigma(v - u)\sigma(3v - u)}{\sigma(v + u)\sigma(3v + u)} \right)^{1/2} e^{2u\zeta(2v)}. \quad (108)$$

### 5.2 The curve

Combining the result for the radius and the angle one obtains the curve as a function of the arc parameter $u$

$$z(u) := x + iy = r(u)e^{i\psi(u)} = e^{i\chi} \frac{P_r}{2\sigma^2(2v)} \frac{\sigma(u - 3v)}{\sigma(u + v)} e^{2u\zeta(2v)} \quad (109)$$
as claimed in (8). Changing simultaneously the sign of $u$ and $v$ does not change $z$,

$$z(u, -v) = z(-u, v).$$

(110)
Thus changing the sign of $v$ is equivalent to changing the orientation of measuring the arc-parameter.

Taking the complex conjugate of $z$ yields

$$(e^{-ix} z(u, v))^* = e^{-ix} z(u, -v) = e^{-ix} z(-u, v).$$

(111)
Thus the curve has a mirror axis which runs through the origin and $e^{ix}$.

By means of eqs. (329, 330) one finds that adding a period $2\omega$ to $v$ does not alter $z$,

$$z(u, v + 2\omega) = z(u, v).$$

(112)
One obtains

$$e^{i\psi(u+2\omega)} = e^{i\psi(u)} e^{i\psi_{\text{per}}},$$

(113)
$$e^{i\psi_{\text{per}}} = e^{-8i\zeta(\omega_3) + 4i\omega_3(2v)}.$$  

(114)
Thus the curve is periodic under rotations by $\psi_{\text{per}}$. This equation determines the angle $\psi_{\text{per}}$ only modulo $2\pi$. Define

$$\psi_c := 4\pi + 8i\zeta(\omega_3) - 4i\omega_3(2v).$$

(115)
Then $e^{i\psi_{\text{per}}} = e^{i\psi_c}$. In sect. 7 it will be shown that this angle equals $\psi_{\text{per}}$ in the region A' of fig. 4. There it will be discussed, how $\psi_{\text{per}}$ and $\psi_c$ are related in the other regions.

Adding a half period to $v$ yields

$$e^{i\psi_{\text{per}}(v+\omega)} = e^{i\psi_{\text{per}}(v)} e^{-8i\zeta(\omega_3) + 8i\omega_3(\omega)} = e^{i\psi_{\text{per}}(v)},$$

(116)
where use has been made of the Legendre-relation (334). Thus two curves with $v$ differing by a half-period have the same $e^{i\psi_{\text{per}}}$.

### 5.3 More on the curve

Instead of verifying that (109) obeys the differential equation (7) one may immediately start with the expression

$$z = x + iy = C\frac{\sigma(u + v')}{\sigma(u + v)} e^{ic_0 u}$$

(117)
and demonstrate that it is solution to the problem by first showing that $u$ is really the arc parameter provided one has the appropriate relations between $v$, $v'$, and $c_0$, which requires

$$\left| \frac{dz}{du} \right| = 1$$

(118)
and secondly show that it is possible to find differences $2\delta u$ of the arc parameter, so that the distance $|z(u + \delta u) - z(u - \delta u)|$ does not depend on $u$. Here it will be shown that (118) is fulfilled with appropriate choices of $v'$, $c_0$, and $C$ for given real $g_2, g_3$ and imaginary $v$. The investigation of the property of constant distance is postponed to sect. 6. One obtains

$$\frac{dz}{du} = z\Phi(u), \quad \Phi(u) = \zeta(u + v') - \zeta(u + v) + ic_0,$$

(119)

$$\frac{dz^*}{du} = z^*\Phi^*(u), \quad \Phi^*(u) = \zeta(u + v'^*) - \zeta(u + v^*) + ic_0^*$$

(120)

for real $u$. Thus

$$\frac{dz}{du} \frac{dz^*}{du} = \Phi(u)\Phi^*(u)zz^* = 1$$

(121)

has to hold for real $u$ with

$$zz^* = CC^* \frac{\sigma(u + v')\sigma(u + v'^*)}{\sigma(u + v)\sigma(u + v^*)} e^{i(co - c_0^*)u}.$$ 

(122)

Since in this representation $\Phi(u)\Phi^*(u)zz^*$ is holomorphic in $u$, it has to be constant for complex $u$. Since $\Phi(u)$ and $\Phi^*(u)$ are periodic functions in $u$, also $zz^*$ has to have this property, which yields

$$\Phi(u) = c_u \mathcal{E}_0 \left( u \left| \begin{array}{c} u_1, u_2 \\ -v, -v' \end{array} \right. \right), \quad \Phi^*(u) = c_u^* \mathcal{E}_0 \left( u \left| \begin{array}{c} u_1^*, u_2^* \\ -v^*, -v'^* \end{array} \right. \right),$$

(123)

$$zz^* = c_z \mathcal{E}_0 \left( u \left| \begin{array}{c} -v', -v'^* \\ -v, -v^* \end{array} \right. \right),$$

where $u_1, u_2$ depend on the choice of $c_0$, and $c_u$ and $c_z$ are constant prefactors. Further $v' + v'^*$ can differ from $v + v^*$ only by an integer multiple of $2\omega_3$. The product of the three functions in (123) yields

$$\mathcal{E}_0 \left( u \left| \begin{array}{c} u_1, u_2, u_1^*, u_2^* \\ -v, -v, -v^*, -v'^* \end{array} \right. \right).$$

(124)

In order that this function is constant one chooses modulo periods $2\omega$

$$u_1 = u_2 = -v^*.$$ 

(125)

(The choice $u_1 = -v$ or $u_2 = -v$ would yield infinite $c_u$.) Thus

$$\Phi(u) \propto \mathcal{E}_0 \left( u \left| \begin{array}{c} -v^*, -v^* \\ -v, -v^* \end{array} \right. \right),$$

(126)

which requires

$$v' = -v + 2v^* + 2\omega.$$ 

(127)

Thus one concludes

$$\Re v' = \Re v + 2\Re \omega, \quad \Im v' = -33v + 23\omega.$$ 

(128)
The real part of \(v\) can be absorbed into \(u\). Thus one may choose \(v\) purely imaginary. Since the functions \(E\) are invariant against addition of a period to any of its arguments, one may choose \(\omega = 0\), which yields \(v' = -3v\). Since \(\Phi(u)\) has to vanish for \(u = -v^*\), one obtains

\[
ic_0 = 2\zeta(2v). \tag{129}
\]

\(\Phi(u)\) can be written

\[
\Phi(u) = \zeta(u - 3v) - \zeta(u + v) + 2\zeta(2v) = -\frac{\psi'(2v)}{\psi(2v) - \psi(u - v)}, \tag{130}
\]

where eq. (333) has been used. The prefactors in eqs. (123) are

\[
c_u = -\zeta(3v) + 2\zeta(2v) - \zeta(v) = -\frac{\sigma(v)\sigma(4v)}{\sigma^2(2v)\sigma(3v)}, \quad c_z = C^2 \frac{\sigma^2(3v)}{\sigma^2(v)}. \tag{131}
\]

Thus one has the condition

\[
c_u c_u^* c_z = -C^2 \frac{\sigma^2(4v)}{\sigma^4(2v)} = 1 \tag{132}
\]

in agreement with eq. (62).

## 6 Line Segments of Constant Length

### 6.1 Property of constant distance

In this section the distance \(2\ell\) between points at \(u + \delta u\) and \(u - \delta u\) on the same or on different curves of equal or different branch is considered

\[
(2\ell)^2 = (z^*(u + \delta u) - \hat{z}^*(u - \delta u)) (z(u + \delta u) - \hat{z}(u - \delta u)) \tag{133}
\]

in order to find line segments of constant length \(2\ell\) independent of \(u\),

\[
z(u) = C e^{i\chi} \frac{\sigma(u - 3v)}{\sigma(u + v)} e^{2u\zeta(2v)}, \tag{134}
\]

\[
\hat{z}(u) = \hat{C} e^{i\chi} \frac{\sigma(u - 3\hat{v})}{\sigma(u + \hat{v})} e^{2u\zeta(2\hat{v})}. \tag{135}
\]

For simplicities sake assume that \(v\) and \(\hat{v}\) are purely imaginary. Then the conjugate complex of \(z\) and \(\hat{z}\) is obtained simply by reversing the sign of \(v\), \(\hat{v}\), \(\chi\) and \(\hat{\chi}\). Since \(u\) and \(\delta u\) are real, they are left unchanged upon determining the conjugate complex of \(z\) and \(\hat{z}\) and only after this conjugation they are extended into the complex plane. Thus one has

\[
z^*(u) = C e^{-i\chi} \frac{\sigma(u + 3v)}{\sigma(u - v)} e^{-2u\zeta(2v)}, \tag{136}
\]

\[
\hat{z}^*(u) = \hat{C} e^{-i\chi} \frac{\sigma(u + 3\hat{v})}{\sigma(u - \hat{v})} e^{-2u\zeta(2\hat{v})}. \tag{137}
\]
The expression $(2\ell)^2$ becomes an analytic function in $u$ and $\delta u$. Choosing

$$\hat{v} = v + \nu \omega_1$$

allows not only to consider line segments between copies of one and the same curve (even $\nu$), but also to connect two curves of different branches (odd $\nu$).

Similar to eq. (143) addition of $2\omega$ to the argument of $u$ multiplies both $z$ and $\hat{z}$ by the same factor $e^{8\nu \zeta(\omega) - 4\nu \zeta(2v)}$ and divides both $z^*$ and $\hat{z}^*$ by this factor. Thus $4\ell^2$ is a periodic function in $u$.

Apparently $z^*(u + \delta u)$ has a pole at $u = v - \delta u$ and $\hat{z}^*(u - \delta u)$ a pole at $u = \hat{v} + \delta u$. In order that $(2\ell)^2$ is constant, the other factor $z(u + \delta u) - \hat{z}(u - \delta u)$ has to vanish for these values of $u$,

$$z(v) - \hat{z}(v - 2\delta u) = -Ce^{i\chi(2v)} - C' e^{i\chi'} \frac{\sigma(-2\delta u + v - 3\hat{v})}{\sigma(-2\delta u + v + \hat{v})} e^{-4\delta u \zeta(2v) + 2v \zeta(2v)} = 0, \quad (139)$$

$$z(\hat{v} + 2\delta u) - \hat{z}(\hat{v}) = C e^{i\chi} \frac{\sigma(2\delta u + v - 3\hat{v})}{\sigma(2\delta u + v + \hat{v})} e^{4\delta u \zeta(2v) + 2v \zeta(2v)} + C' e^{i\chi'} e^{2v \zeta(2v)} = 0. \quad (140)$$

Both eqs. yield

$$e^{i(\chi - \hat{\chi})} = (1 - 1) e^{-2\delta u (\zeta(2v) + \zeta(2v))} \frac{\sigma(2\delta u + v + \hat{v})}{\sigma(2\delta u - v - \hat{v})}. \quad (141)$$

Similarly $z(u + \delta u)$ has a pole at $u = -v - \delta u$ and $\hat{z}(u - \delta u)$ has a pole at $u = -\hat{v} + \delta u$. It turns out that $z^*(u + \delta u) - z^*(u - \delta u)$ also vanish, if (141) is fulfilled. Thus if condition (144) between $\chi - \hat{\chi}$ and $\delta u$ is fulfilled, then the distance $2\ell$ does not depend on the arc parameter $u$. This proves the property of constant distance.

Next the number of solutions will be determined. First one realizes that $\chi - \hat{\chi}$ is a monotonic increasing function of $\delta u$, which can be seen as follows. The derivative

$$\frac{d(\chi - \hat{\chi})}{d\delta u} = 2i \left( \zeta(2v) + \zeta(2\hat{v}) - \zeta(2\delta u + v + \hat{v}) + \zeta(2\delta u - v - \hat{v}) \right) \quad (142)$$

can be rewritten

$$\frac{d(\chi - \hat{\chi})}{d\delta u} = \left\{ \begin{array}{ll}
\frac{2i\nu(2v)}{\nu(2v) - \nu(2u_1) - \nu(2\delta u)} & \text{for } \hat{v} = v, \\
\frac{-2i\nu(2v + \omega_1)}{\nu(2v + \omega_1) - \nu(2\delta u)} & \text{for } \hat{v} = v + \omega_1.
\end{array} \right. \quad (143)$$

One convinces oneself that all numerators and denominators are positive. Thus $\chi - \hat{\chi}$ is a monotonic increasing function of $\delta u$. Next consider the change of $\chi - \hat{\chi}$ during one period, that is while increasing $\delta u$ by $\omega_3$,

$$\Delta \chi = \Delta \arg \sigma(2\delta u + v + \hat{v}) - \Delta \arg \sigma(2\delta u - v - \hat{v}) + 2i\omega_3(\zeta(2v) + \zeta(2\hat{v})). \quad (144)$$
Since $0 < \Im(2v) < \omega'/2$ and $0 < \Im(2\hat{v}) < \omega_1$ one obtains from eqs. (331, 332)
\[
\Delta \chi = -2\pi - 4i(v + \hat{v})\zeta(\omega_3) + 2i\omega_3(\zeta(2v) + \zeta(2\hat{v})) = 2\pi - \frac{\psi_c(v) + \psi_c(\hat{v})}{2}. \tag{145}
\]
If curves are considered, which close after $n$ periods, that is for which
\[
\psi_c(v) = 2\pi \frac{m}{n}, \tag{146}
\]
with $m, n$ coprime holds, then $\chi - \hat{\chi}$ increases by
\[
n\Delta \chi = 2\pi \times \begin{cases} n - m & \text{for } \hat{v} = v, \\ 2n - m & \text{for } \hat{v} = v + \omega_1, \end{cases} \tag{147}
\]
since from eqs. (115) and (334) one obtains
\[
\psi_c(v + \omega_1) = \psi_c(v) - 4\pi. \tag{148}
\]
The numbers $n - m$ and $2n - m$ are the number of solutions $\delta u$ for given $\chi - \hat{\chi}$
after they repeat, since with $\delta u$ also $\delta u + n\omega_3$ is a solution, which yields line
segments between the same points of curves. If chords on one and the same
curve are considered ($\hat{v} = v, \hat{\chi} = \chi$), which constitutes the original problem,
then one solution is the trivial solution $\delta u = 0$. Thus one is left with only
$n - m - 1$ non-trivial solutions, which moreover appear pairwise, since with $\delta u$
also $-\delta u$ is a solution. For $m/n = 1/2$, for example one has only the trivial
solution. If however two copies of the curve are rotated against each other,
then a set of line segments of constant length is obtained. Examples are shown
in figures 8 and 9. The asterisk at the figure number indicates that there is
a corresponding animation on the internet\cite{13}, where one can see the chord or
segment of constant length moving.

\[\text{Fig.}^* \text{ 8  } m/n = 1/2, \epsilon = 0.2\]

\[\text{Fig.}^* \text{ 9  } m/n = 1/2, \epsilon = 0.2\]

Here and in the following the curves $\Gamma$ are shown in black and blue, the
chords and line-segments are shown in green and cyan. The color switches in
the middle, where the lines touch the red envelopes $\gamma$. 

24
6.2 Length of the line segments

Next a simple expression for the length of the line segments is derived assuming that \( \delta u \) has been determined. Since \( 4\ell^2 \) is independent of \( u \), one considers the expression in the vicinity of diverging \( z^*(u + \delta u) \), that is around \( u = -\delta u + v \). There the residuum is given by

\[
\text{res} = C e^{-i\chi \sigma(4v)} e^{-2v\zeta(2v)}.
\]

(149)

This divergence is compensated by the zero of \( z(u + \delta u) - \hat{z}(u - \delta u) \). In order to obtain \( 4\ell^2 \) the residuum has to be multiplied by the derivate

\[
\left| \frac{dz(u + \delta u)}{du} \right|_{u=\delta u + v} - \left| \frac{d\hat{z}(u - \delta u)}{du} \right|_{u=-\delta u + v} = \Phi(v)z(v) - \hat{\Phi}(-2\delta u + v)\hat{z}(-2\delta u + v)
\]

(150)

Using (130) one obtains

\[
\Phi(v) = 0, \quad \hat{\Phi}(-2\delta u + v) = \frac{-\varphi'(2\hat{v})}{\varphi(2\hat{v}) - \varphi(-2\delta u + v - \hat{v})}.
\]

(152)

Thus the residuum has to be multiplied by \( -\hat{\Phi}(-2\delta u + v)\hat{z}(-2\delta u + v) \), and since \( \hat{z}(-2\delta u + v) = z(v) \), one obtains

\[
4\ell^2 = \text{res} (-z(v)\hat{\Phi}(-2\delta u + v)) = -C^2\sigma(4v) \frac{\varphi'(2\hat{v})}{\varphi(2\hat{v}) - \varphi(-2\delta u + v - \hat{v})}.
\]

(153)

Making use of eq. (62) and that \( 2v \) and \( 2\hat{v} \) differ by a period \( 2\omega \), one obtains finally

\[
4\ell^2 = \frac{1}{\varphi(2\delta u + v - \hat{v}) - \varphi(2v)}.
\]

(154)

This expression yields bounds on the length of the line segment. Within one and the same curve one uses

\[
e_3 \leq \varphi(2\delta u) \leq \infty
\]

(155)

For \( \varphi(2\delta u) = \infty \) one has \( 4\ell^2 = 0 \). In the other limit case one finds

\[
e_3 - \varphi(2v) = \frac{a^2}{4}(q_4 + q_3 - q_2 - q_1)^2 = \frac{1}{(r_1 + r_2)^2} = \frac{1}{(r_3 + r_4)^2},
\]

(156)

where eqs. (20), (357), and (360) have been used. This yields the bounds

\[
0 \leq 2\ell \leq |r_1 + r_2| = r_3 + r_4.
\]

(157)

In the case of two branches being connected by a line segment one has

\[
e_1 \leq \varphi(2\delta u + \omega_1) \leq e_2.
\]

(158)
Then the bounds read

$$\lvert r_1 + r_3 \rvert = r_2 + r_4 \leq 2\ell \leq \lvert r_2 + r_3 \rvert = r_1 + r_4. \quad (159)$$

There are obvious geometric bounds. For line segments between points on the same curve one has the upper bounds $r_4 + |r_3|$ and if there is a second branch also $|r_2| + |r_1|$. Lower bound is 0. For line segments between different branches one has upper bounds $r_4 + |r_3|$ and $|r_2|$ and lower bounds $r_4 - |r_2|$ and $|r_3| - |r_1|$. It turns out that the limits given by eq. (154) are in some cases more restrictive. They are given in the tables (160) and (161).

| Region | Sign of Same curve | $2\ell_{\text{min}}$ | $2\ell_{\text{max}}$ |
|--------|--------------------|----------------------|----------------------|
| $A'$   | + +                | 0                    | $r_4 + r_3$          |
| $A''$  | + -                | 0                    | $r_4 - |r_3|$         |
| $B'$   | + - - -            | 0                    | $r_4 - |r_3| = |r_1| + |r_2|$ |
| $B''$  | + - - +            | 0                    | $r_4 - |r_3| = |r_2| - r_1$ |

| Region | Different curves | $2\ell_{\text{min}}$ | $2\ell_{\text{max}}$ |
|--------|------------------|----------------------|----------------------|
| $B'$   | $r_4 - |r_2| = |r_1| + |r_3|$ | $r_4 - |r_1| = |r_3| + |r_2|$ |
| $B''$  | $r_4 - |r_2| = |r_3| - r_1$ | $r_4 + r_1 = |r_3| + |r_2|$ |

6.3 Remark on the approach by Finn

David Finn considers the bicycle problem in [9, 10]. Denoting the curvature of trace of the rear wheel by $\kappa_\beta$ he obtains for the curvature of the trace of the front wheel

$$\kappa(u \pm \delta u) = \frac{\kappa_\beta}{(1 + (\ell \kappa_\beta)^2)^{1/2}} \pm \ell \frac{\kappa_\beta}{1 + (\ell \kappa_\beta)^2} \frac{ds_\beta}{du}, \quad (162)$$

where the two signs apply to the bicycle riding in both directions. He gives $\kappa_\beta$ for an interval $2 \delta u$ and continues so that this equation is fulfilled. If one eliminates $\kappa_\beta$ then one obtains

$$\frac{d(\kappa(u + \delta u) + \kappa(u - \delta u))}{du} = (\kappa(u + \delta u) - \kappa(u - \delta u))$$

$$\times \sqrt{\frac{1}{\ell^2} - \frac{1}{4}(\kappa(u + \delta u) + \kappa(u - \delta u))^2} \quad (163)$$

Performing some algebra one sees that this equation is fulfilled by our solution. To do this one may express $\kappa$ and $\ell$ in terms of $\varphi(u)$, $\varphi(\delta u)$ and $\varphi(v)$. 
6.4 The area

Next the area $A_2$ below the water-line is determined, which is the area of the sector $A_s$ from the origin to the end points of the water-line with arc parameters $u - \delta u$ and $u + \delta u$ minus the area $A_t$ of the triangle with corners at the origin and at the end-points of the water-line,
\[ A_2 = A_s - A_t. \]  (164)

The area of the sector is obtained by integrating
\[ dA_s = \frac{1}{2}(x\,dy - y\,dx) = \frac{1}{4i}(z^*dz - zd^*). \]  (165)

Use of $dz = z\Phi(u)du$, $dz^* = z^*\Phi^*(u)du$, $z^*\Phi(u)\Phi^*(u) = 1$ from section 5.3 yields
\[ dA_s = \frac{1}{4i}z^*(\Phi(u) - \Phi^*(u))du = \frac{1}{4i}\left(\frac{1}{\Phi^*(u)} - \frac{1}{\Phi(u)}\right)du. \]  (166)

Insertion of eq. (130) and use of eq. (48) yields
\[ A_s = \int_{u-\delta u}^{u+\delta u} \frac{du}{8a} (2\varphi(2v) - \varphi(u+v) - \varphi(u-v)) \]  (167)
\[ = \frac{1}{8a} (4\delta u \varphi(2v) + \zeta(u + \delta u + v) + \zeta(u + \delta u - v) - \zeta(u - \delta u + v) - \zeta(u - \delta u - v)) \]  (168)
\[ = \frac{1}{8a} (4\delta u \varphi(2v) + 2\zeta(\delta u + v) + 2\zeta(\delta u - v) \]  (169)
\[ + \frac{\varphi'(\delta u + v)}{\varphi(\delta u + v) - \varphi(u)} + \frac{\varphi'(\delta u - v)}{\varphi(\delta u - v) - \varphi(u)}) \]

The area $A_{per}$ of the sector by increasing $u$ by $2\omega_3$, that is $\delta u = \omega_3$ is given by
\[ A_{per} = \frac{1}{2a}(\omega_3\varphi(2v) + \zeta(\omega_3)). \]  (170)

Then the density $\rho$ for a cross-section with $\psi_{per} = 2\pi/n$ is given by
\[ \rho = \frac{A_2}{nA_{per}}. \]  (171)

The area of the triangle spanned by the origin and the points on the curve at arc parameters $u + \delta u$ and $u - \delta u$ is given by
\[ A_t = \frac{1}{2}(x(u - \delta u)y(u + \delta u) - x(u + \delta u)y(u - \delta u)) \]  (172)
\[ = \frac{1}{4\pi i} (z^*(u - \delta u)z(u + \delta u) - z^*(u + \delta u)z(u - \delta u)) \]  (173)
\[ = \frac{C^2}{4i} \left( \frac{\sigma(u - \delta u + 3v)\sigma(u + \delta u - 3v)}{\sigma(u - \delta u - v)\sigma(u + \delta u + v)} e^{4\delta u \zeta(2v)} \right) \]  (174)
A water-line is then given by the chord of constant length. This was expected. The constant a below the water-line and eqs. (327, 344) have been used. Use of eqs. (48, 62) yields
\[
\frac{\sigma^2(\delta u + 3v)}{\sigma^2(\delta u - v)} = \frac{\sigma(2\delta u - 3v)}{\sigma(2\delta u - v)} \frac{\varphi'(\delta u + 2v)}{\varphi(\delta u + v) - \varphi(\delta u - v)} e^{-4\delta u \zeta(2v)}.
\] (174)

In the last expression one rewrites
\[
\frac{\sigma^2(\delta u + 3v)}{\sigma^2(\delta u - v)} = \frac{\sigma(2\delta u + 2v)}{\sigma(2\delta u + v)} \frac{\varphi'(\delta u + 2v)}{\varphi(\delta u + v) - \varphi(\delta u - v)} e^{4\delta u \zeta(2v)}.
\] (175)

where numerator and denominator on the left hand side have been multiplied by \(\sigma^2(\delta u + v)\) and eqs. \((18, 22)\) have been used. Use of eqs. \((18, 22)\) yields \(\mathcal{A}_2 = \mathcal{A}_0 + \frac{\varphi'(\delta u + v)}{8\sigma(\varphi(\delta u + v) - \varphi(u))} (1 + f_+) + \frac{\varphi'(\delta u - v)}{8\sigma(\varphi(\delta u - v) - \varphi(u))} (1 + f_-)\)

with
\[
\mathcal{A}_0 = \frac{1}{8\sigma} (4\delta u \varphi(2v) + 2\zeta(\delta u + v) + 2\zeta(\delta u - v) + f_+ \frac{\varphi'(\delta u + v)}{\varphi(\delta u + v) - \varphi(\delta u + v)} + f_- \frac{\varphi'(\delta u - v)}{\varphi(\delta u - v) - \varphi(\delta u - v)}).
\] (178)

In order that the area \(\mathcal{A}_2\) does not depend on \(u\), the factors \(1 + f_\pm\) have to vanish. This yields exactly the condition \((141)\) with \(\chi = \hat{\chi}, \nu = 0, \hat{v} = v\) for the chord of constant length. This was expected. The constant area below the water-line is then given by \(\mathcal{A}_2 = \mathcal{A}_0\) with \(f_\pm = -1\). The expression for the area \(\mathcal{A}_0\) can be simplified. One obtains
\[
\frac{\varphi'(\delta u \pm v)}{\varphi(\delta u \mp 2v) - \varphi(\delta u \pm v)} = 2\zeta(\delta u \pm v) - \zeta(2\delta u \mp 2v) \mp \zeta(4v)
\] (180)

by using theorem I of appendix \((A.2)\). The functions on both sides of the equation are periodic in \(\delta u\), their singularities within an elementary cell are simple poles at \(\delta u = \mp v\) with residua 2, and at \(\delta u = \pm v\) and \(\delta u = \pm v + \omega_i\) with residua \(-1/2\). The constant \(\mp \zeta(4v)\) is obtained by evaluating both sides at \(\delta u = \mp v + \omega_i\). Thus one obtains
\[
\mathcal{A}_2 = \mathcal{A}_0 = \frac{1}{8\sigma} (4\delta u \varphi(2v) + \zeta(2\delta u - 2v) + \zeta(2\delta u + 2v)).
\] (181)

6.5 Centers of gravity

Let \(\zeta_{\phi}\) denote the center of gravity of the area below the water-line and \(\phi\) denote the angle of \(z(u + \delta u) - z(u - \delta u)\) against the real axis in the complex
Thus
\[ z(u + \delta u) - z(u - du) = 2\ell e^{i\phi}. \]  
(182)

An infinitesimal change of \( \phi \) yields
\[ A_2 dz_g = \frac{2}{3} (z(u + \delta u) - z(u - du)) df, \]
(183)

which with \( df = \ell^2/2 \) and eq. (182) yields
\[ A_2 dz_g = \frac{2}{3} \ell^3 e^{i\phi} d\phi \]
(184)
and thus
\[ z_g = \frac{2\ell^3}{3A_2} (-ie^{i\phi} + \text{const}_2). \]
(185)

Similarly one obtains for the center of gravity \( z_{g1} \) of the area above the water-line
\[ z_{g1} = \frac{2\ell^3}{3A_1} (ie^{i\phi} + \text{const}_1). \]
(186)

Thus both centers of gravity move on circles. This is a necessary and sufficient condition for the solution (see Auerbach[3] and references therein). Increasing \( u \) by \( 2\omega_3 \) will increase \( \phi \) by \( \psi_{\text{per}} \), and the center of gravity is rotated by \( \psi_{\text{per}} \). If it is not an integer multiple of \( 2\pi \), then both constants vanish. If it is such a multiple, then continuity normally shows that it will vanish, too. We realize that then the line between the centers of gravity is perpendicular to the water-line and the distance \( h \) between both centers agrees with the expression (1). This derivation did not use the solution of (7).

7 Periodicity and Figures

7.1 Limit of small \( \epsilon \)

In the limit of small \( \epsilon \) one obtains (in this subsection we put \( r_0 = 1 \))
\[ e_{1,2} = \frac{4 + \mu^2}{12\mu^2} \pm \frac{2\ell}{\mu^2} + O(\epsilon^2), \quad e_3 = \frac{4 + \mu^2}{6\mu^2} + O(\epsilon^2). \]
(187)
Thus for \( \epsilon = 0 \) the limiting case of vanishing discriminant is approached with \( \omega' \) tending to \( i\infty \) and
\[ \omega_3 = \frac{\pi\mu}{\sqrt{4 + \mu^2}} \]
(188)
from
\[ e_3 = \left( \frac{\pi}{2\omega_3} \right)^2 \frac{2}{3}. \]
(189)
In order to determine $\omega'$ one may use the representation of the Weierstrass functions in appendix A.3. It is sufficient to use the terms up to linear order in $q$. With

$$q = e^{2\pi i \omega_1/\omega_3} = -e^{\pi i \omega'/\omega_3} = -\hat{q}^2, \quad \hat{q} = e^{\pi i \omega'/2\omega_3}$$

(190)

one obtains

$$\wp(z) = \left(\frac{\pi}{2\omega_3}\right)^2 \left(\frac{2}{1 - \cos u} + 8q(1 - \cos u) - \frac{1}{3}\right), \quad u = \frac{\pi z}{\omega_3}$$

(191)

For argument $\omega' + \delta z$ and small imaginary part of $\delta z$ one obtains

$$\wp\left(\frac{\omega'}{2} + \delta z\right) = \left(\frac{\pi}{2\omega_3}\right)^2 \left(-\frac{1}{3} - 8i\hat{q}\sin\frac{\pi \delta z}{\omega_3}\right).$$

(192)

One concludes from $e_1 = \wp\left(\frac{\omega'}{2}\right)$

$$\hat{q} = \frac{e}{\mu(4 + \mu^2)}.$$  

(193)

Next $v = \frac{\omega'}{2} + \delta v$ is determined from

$$p_1 = \wp(v) = -\frac{\mu^2 + 4}{12\mu^2} - \frac{e}{\mu^2}$$

(194)

and yields

$$\sin\left(\frac{\pi \delta v}{\omega_3}\right) = -\frac{i\mu}{2}.$$  

(195)

Now $\psi_c$, eq. (115) can be evaluated with $\omega' = \omega_3 + 2\omega_1$

$$\psi_c = 4\pi + 8vi\zeta(\omega_3) - 4i\omega_3\zeta(2v)$$

$$= 4\pi + i(4\omega' + 8\delta v)\zeta(\omega_3) - 4i\omega_3\zeta(\omega' + 2\delta v)$$

$$= 4\pi + i(4\omega_3 + 8\omega_1 + 8\delta v)\zeta(\omega_3) - 4i\omega_3\zeta(\omega_3 + 2\omega_1 + 2\delta v)$$

$$= 4\pi + 8i(\omega_1\zeta(\omega_3) - \omega_3\zeta(\omega_1)) + i(4\omega_3 + 8\delta v)\zeta(\omega_3) - 4i\omega_3\zeta(\omega_3 + 2\delta v).$$

(196)

The term $8i(\omega_1\zeta(\omega_3) - \omega_3\zeta(\omega_1))$ in the last line yields $-4\pi$ due to Legendre’s relation (334), the second one can be evaluated by means of eq. (339), where $q$ is now negligible and yields

$$\psi_c = -2\pi i \cot\left(\frac{\pi}{2} + \frac{\pi \delta v}{\omega_3}\right) = \pi i \tan\left(\frac{\pi \delta v}{\omega_3}\right) = 2\pi \frac{\mu}{\sqrt{4 + \mu^2}}.$$  

(197)

Thus one obtains

$$\psi_{per} = 2\omega_3 = 2\pi \mu / \sqrt{4 + \mu^2} = \psi_c$$

(198)

in the limit of small $\epsilon$.  

30
Two dimensional bodies which can float in all directions are given by \( \psi_{\text{per}} = 2\pi/n \), thus for \( m = 1 \) and sufficiently small \( \epsilon \). In this limit the \( \delta u \) can be determined from eq. (141) with

\[
\zeta(\omega' + \delta z) = \frac{n^2}{12}(\omega' + \delta z) - ni - \frac{n}{2}\tan\left(\frac{n\delta z}{2}\right) + O(q), \quad (199)
\]

\[
\sigma(\omega' + \delta z) = 2i\frac{n}{nq}e^{-in\delta z} = \frac{n^2}{2}\left(\omega' + \delta z\right)^2/24 + O(q), \quad (200)
\]

where eqs. (349) and (355) and \( \frac{\pi}{\omega'} = n \), \( \frac{m}{\omega'} = \frac{n^2}{12} \) have been used. Then eq. (141) yields

\[
\tan(n \delta u) = n \tan(\delta u), \quad (201)
\]

in agreement with the results obtained in refs. [7, 12, 8], where \( \delta u \) corresponds to \( \frac{\pi}{2} - \delta_0 \) and in ref. [6], where \( \delta u \) corresponds to \( \pi \rho \).

A few cross-sections of the bodies are shown in figs. 10 to 23. For odd \( n \) the innermost envelope corresponds to density \( \rho = 1/2 \).
\[ \psi_{\text{per}} = \frac{\Delta \psi}{\text{sign} \left( \frac{\partial \psi}{\partial u} \right)_{r=r_i}}, \tag{202} \]
\[ \Delta \psi = \psi(u + 2\omega_3) - \psi(u) \tag{203} \]
Since due to eqs. (26, 13)
\[
\frac{d\psi}{du}
\bigg|_{r=r_i}
= aq_i + b + \frac{c}{q_i} = \frac{1}{r_i}
\] (204)
one obtains
\[
\psi_{\text{per}} = \text{sign } r_i \Delta \psi.
\] (205)
\(\Delta \psi\) is obtained from eq. (109) by the increase of
\[
\arg \sigma(u - 3v) - \arg \sigma(u + v) + 2u3\zeta(2v)
\] (206)as \(u\) is increased by \(2\omega_3\). As shown in (331, 332) this depends on the imaginary part of the argument of the functions \(\sigma\). Thus in addition to
\[
\Delta \psi_0 = 8vi\zeta(\omega_3) - 4i\omega_3\zeta(2v)
\] (207)one obtains extra multiple of \(2\pi\). In region \(A'\) one obtains \(3\pi\) from \(\sigma(u - 3v)\) and \(\pi\) from \(\sigma(u + v)\). This changes as one crosses to \(A''\), since there \(3v < \omega'/(3i)\). This can be seen from the sign change of \(\psi'/3v\), eq. (60) due to the sign change of \(r_i\). One observes, that now the curve passes at its minimum radius on the other side of the origin. In the table (208) the ranges of \(v\) are listed in the various regions of fig. 4 and the corresponding \(\psi_{\text{per}}\) are given:

| region | inequality | \(\psi_{\text{per}}\) | \(\psi_{\text{per}}\) | \(\psi_{\text{per}}\) | \(\psi_{\text{per}}\) |
|--------|------------|-----------------|-----------------|-----------------|-----------------|
| \(A'\) | \(\frac{2}{3} \leq 3v < \frac{\pi}{3}\) | \(\psi_c\) | \(\psi_c\) | \(\psi_c\) | \(\psi_c\) |
| \(A''\) | \(0 < 3v < \frac{2}{3}\) | \(\psi_c - 2\pi\) | \(2\pi - \psi_c\) | \(2\pi - \psi_c\) | \(2\pi - \psi_c\) |
| \(B'\) | \(0 < 3v < \frac{2}{3}\) | \(\psi_c - 2\pi\) | \(2\pi - \psi_c\) | \(4\pi - \psi_c\) | \(4\pi - \psi_c\) |
| \(B''\) | \(\frac{2}{3} < 3v < \frac{\pi}{3}\) | \(\psi_c - 2\pi\) | \(2\pi - \psi_c\) | \(2\pi - \psi_c\) | \(2\pi - \psi_c\) |

(208)

As examples figures out of the various regions are shown for \(\psi_{\text{per}} = 0\). One of them looks like an eight.

\[\infty\]

**Fig. 24** \(m/n = 0/1, \quad \epsilon = 0.5, \quad \text{region } A'\)**

**Fig. 25** \(m/n = 0/1, \quad \epsilon = 1.5, \quad \text{region } A''\)**

**Fig. 26** \(m/n = 0/1, \quad \epsilon = 2.5, \quad \text{region } B''\)**
Here and in the following the parametrization (146) $\psi_c = 2\pi \frac{m}{n}$ is used. It should be noted, that $\psi$, $\varphi(v)$, and $\psi_c$ vary continuously across all the borders between the regions $A'$, $A''$, $B'$, and $B''$. $\psi_c$ approaches for fixed $\epsilon$ the limits

$$\lim_{\mu \to \infty} \psi_c = 2\pi, \quad \lim_{\mu \to 0, \epsilon \leq 2} \psi_c = -\infty, \quad \lim_{\tilde{\epsilon} \to \epsilon - 2, \epsilon \geq 2} \psi_c = -\infty. \quad (209)$$

Probably (I did not really check this) $\psi_c$ is a monotonic decreasing function of $\tilde{\epsilon}^2$ for fixed $\epsilon$.

7.3 More figures

In the following more curves are shown. Now larger values of $\epsilon$ are chosen as already done for the case $m/n = 0/1$. The next examples are for $m/n = 1/3, 1/4, 1/5,$ and $1/6$. One observes that one obtains two branches for $\epsilon > 5/2$ and $m/n = 1/3$ (figs. 33 - 39).
Fig. 30 \( m/n = 1/3, \epsilon = 1.1 \)

Fig. 31 \( m/n = 1/3, \epsilon = 1.1 \)

Fig. 32 \( m/n = 1/3, \epsilon = 1.5 \)

Fig. 33 \( m/n = 1/3, \epsilon = 2.6 \)

Fig. 34 \( m/n = 1/3, \epsilon = 3.0 \)

Fig.* 35 \( m/n = 1/3, \epsilon = 3.0 \)
Fig. 36 $m/n = 1/3$, $\epsilon = 3.0$

Fig. 37 $m/n = 1/3$, $\epsilon = 3.0$

Fig. 38 $m/n = 1/3$, $\epsilon = 3.0$

Fig. 39 $m/n = 1/3$, $\epsilon = 3.5$

Fig. 40 $m/n = 1/4$, $\epsilon = 0.5$

Fig. 41 $m/n = 1/4$, $\epsilon = 0.9$
Fig. 42 $m/n = 1/4$, $\epsilon = 1.0$

Fig. 43 $m/n = 1/4$, $\epsilon = 1.5$

Fig. 44 $m/n = 1/4$, $\epsilon = 2.0$

Fig. 45 $m/n = 1/4$, $\epsilon = 2.5$

Fig.* 46 $m/n = 1/5$, $\epsilon = 0.5$

Fig.* 47 $m/n = 1/5$, $\epsilon = 0.9$
Fig. 48 $m/n = 1/5$, $\epsilon = 1.0$

Fig. 49 $m/n = 1/5$, $\epsilon = 1.5$

Fig. 50 $m/n = 1/5$, $\epsilon = 2.0$

Fig. 51 $m/n = 1/5$, $\epsilon = 2.5$

Fig. 52 $m/n = 1/6$, $\epsilon = 4.0$

Examples for $m/n = 2/5, 3/5, 4/5$ are shown in figs. 53 to 58.
Note that the curve in fig. 58 obeys the convexity condition \( \hat{\epsilon}^2 = -6.032657 \). There are no non-trivial solutions for the chord for \( m/n = 4/5 \). Some examples for negative \( m/n \) are:
Fig. 59 \( m/n = -1/3, \epsilon = 1.0 \)

Fig. 60 \( m/n = -2/5, \epsilon = 1.0 \)

Fig. 61 \( m/n = -2/5, \epsilon = 1.0 \)

Fig. 62 \( m/n = -1/5, \epsilon = 0.2 \)

Fig. 63 \( m/n = -1/5, \epsilon = 0.2 \)

Fig. 64 \( m/n = -1/5, \epsilon = 0.5 \)

Fig. 65 \( m/n = -3/7, \epsilon = 0.4 \)
For $m/n = 0/1$ one obtains the $Eight$, which was already shown in figs. 24 to 27. Here some of the envelopes $\gamma$ are shown.

\[ m/n = -1/2, \epsilon = 0.4 \]

\[ m/n = -3/2, \epsilon = 0.4 \]
Fig. 74 $m/n = 0/1$, $\epsilon = 2.5$

Fig. 75 $m/n = 0/1$, $\epsilon = 2.5$  

Fig. 76 $m/n = 0/1$, $\epsilon = 2.5$

Fig. 77 $m/n = 0/1$, $\epsilon = 2.5$

Fig. 78 $m/n = 0/1$, $\epsilon = 2.5$

Fig. 79 $m/n = 0/1$, $\epsilon = 2.5$

Another example is shown in fig. 92.

8 The Limit Case $\mu = 0$

The discriminant $D$ vanishes in the limit $\mu = 0$ and the elliptic functions degenerate to trigonometric and exponential functions. In this limit two of the radii $r_i$ coincide.
8.1 The case $\epsilon > 2$

In this region one obtains with

$$r_4 = r_0(1 + \epsilon), \quad r_3 = r_0(1 - \epsilon), \quad r_2 = r_1 = -r_0$$  \hspace{1cm} (210)

the expressions for $e_i$

$$e_1 = e_2 = \frac{4 - \epsilon^2}{12\epsilon^2 r_0^2} < 0, \quad e_3 = \frac{\epsilon^2 - 4}{6\epsilon^2 r_0^2} > 0,$$  \hspace{1cm} (211)

and the functions

$$\varphi(u) = \frac{2\lambda^2}{3} + \lambda^2 \cot^2(\lambda u),$$  \hspace{1cm} (212)

$$\zeta(u) = \frac{\lambda^2 u}{3} + \lambda \cot(\lambda u),$$  \hspace{1cm} (213)

$$\sigma(u) = \frac{1}{\lambda} e^{\frac{u^2}{6} - \frac{u}{2}} \sin(\lambda u),$$  \hspace{1cm} (214)

$$\lambda = \frac{\sqrt{\epsilon^2 - 4}}{2\epsilon r_0}. \hspace{1cm} (215)$$

In this degenerate case one has

$$\omega = \frac{\pi r_0}{2\lambda} = \frac{\pi \epsilon r_0}{\sqrt{\epsilon^2 - 4}}, \quad \omega' = \infty.$$  \hspace{1cm} (216)

From

$$\varphi(v) = -\frac{(\epsilon + 2)(\epsilon + 10)}{12\epsilon^2 r_0^2}$$  \hspace{1cm} (217)

one concludes

$$\cot(\lambda v) = \frac{1}{\sqrt{\epsilon^2 - 4}}, \quad \cos(2\lambda v) = \frac{\epsilon}{2}, \quad \sin(2\lambda v) = \frac{i\sqrt{\epsilon^2 - 4}}{2}$$  \hspace{1cm} (218)

and

$$\psi_c = 2\pi \left(2 - \frac{\epsilon}{\sqrt{\epsilon^2 - 4}}\right).$$  \hspace{1cm} (219)

Evaluation of eq. (109) yields in a first step

$$z = \frac{P_r}{2\sigma^2(2v)} \frac{\sigma(u - 3v)}{\sigma(u + v)} e^{2\zeta(2v)u}$$  \hspace{1cm} (220)

$$= -\frac{r_0 \sin(\lambda(u - 3v))}{\sin(\lambda(u + v))} e^{2u \lambda \cot(2\lambda v)}. \hspace{1cm} (221)$$

Multiplication of numerator and denominator by $\sin(\lambda u)$ and use of $\sin a \sin b = \frac{1}{2}$$ \sin(a - b) - \cos(a + b)$ yields

$$z = -r_0 \frac{\cos(2\lambda v) - \cos(\lambda(2u - 4v))}{\cos(2\lambda v) - \cos(2\lambda u)} e^{2u \lambda \cot(2\lambda v)}. \hspace{1cm} (222)$$
From eqs. (218) one obtains
\[ \cos(4\lambda v) = \frac{\epsilon^2}{2} - 1, \quad \sin(4\lambda v) = i\frac{\sqrt{\epsilon^2 - 4}}{2}. \] (223)

This yields
\[ z = r_0 \frac{(\epsilon^2 - 2) \cos(2\lambda u) + i\epsilon\sqrt{\epsilon^2 - 4} \sin(2\lambda u) - \epsilon}{\epsilon - 2 \cos(2\lambda u)} e^{-iu/r_0}. \] (224)

Starting from eqs. (141) and (154) and using the expressions (212) to (214) one obtains the condition for the line segment of constant distance and its length
\[ \tan(2\lambda \delta u) = 2\lambda r_0 \tan\left(\frac{\delta u}{r_0} - \frac{\chi - \tilde{\chi}}{2}\right), \] (225)
\[ \ell^2 = \frac{r_0^2}{1 + \frac{\epsilon^2}{\epsilon^2 - 4} \cot^2(2\lambda \delta u)}. \] (226)

Examples of the curves \( \Gamma \) are shown in figs. 80 to 83.

![Fig. 80](image1)

![Fig. 81](image2)

Fig. 80 \( m/n = 1/3, \ \epsilon = 2.5 \)

Fig. 81 \( m/n = -1/3, \ \epsilon = 2.213594 \)
8.2 The case $\epsilon < 2$

In the case of $\epsilon < 2$ one has

$$r_4 = r_0(1 + \epsilon), \quad r_3 = r_2 = -r_0, \quad r_1 = r_0(1 - \epsilon).$$

(227)

Then one obtains

$$e_1 = \frac{\epsilon^2 - 4}{6\epsilon^2 r_0^2} < 0, \quad e_2 = e_3 = \frac{4 - \epsilon^2}{12\epsilon^2 r_0^2} > 0,$$

(228)

and the functions

$$\wp(u) = \frac{-2\lambda^2}{3} + \lambda^2 \coth^2(\lambda u),$$

(229)

$$\zeta(u) = -\frac{\lambda^2 u}{3} + \lambda \coth(\lambda u),$$

(230)

$$\sigma(u) = \frac{1}{\lambda} e^{-\lambda^2 u^2/6} \sinh(\lambda u),$$

(231)

$$\lambda = \frac{\sqrt{4 - \epsilon^2}}{2\epsilon r_0}.$$  

(232)

In this degenerate case the half-periods approach

$$\omega_3 = \infty, \quad \omega_1 = \frac{\pi i}{2\lambda} = \frac{\pi i r_0}{\sqrt{4 - \epsilon^2}}.$$  

(233)

Again one obtains (217) and concludes

$$\coth(\lambda v) = -i \sqrt{2 + \frac{\epsilon}{2 - \epsilon}}, \quad \cosh(2\lambda v) = \frac{\epsilon}{2}, \quad \sinh(2\lambda v) = i \frac{\sqrt{4 - \epsilon^2}}{2}.$$  

(234)
Eq. (109) is evaluated in a first step

\[ z = \frac{P_r}{2\sigma^2(2\nu)} \sigma(u + v) e^{2\zeta(2\nu)u} \]

\[ = -r_0 \frac{\sinh(\lambda(u - 3v))}{\sinh(\lambda(u + v))} e^{2u\lambda \coth(2\lambda v)}. \]

Multiplication of numerator and denominator by \( \sinh(\lambda(u - v)) \) and use of \( \sinh a \sinh b = \frac{1}{2} (\cosh(a + b) - \cosh(a - b)) \) yields

\[ z = r_0 \frac{\cosh(\lambda(2u - 4v)) - \cosh(2\lambda v)}{\cosh(2\lambda u) - \cosh(2\lambda v)} e^{2u\lambda \coth(2\lambda v)}. \]

From eqs. (234) one obtains

\[ \cosh(4\lambda v) = \frac{\epsilon^2}{2} - 1, \quad \sinh(4\lambda v) = \frac{i\epsilon \sqrt{4 - \epsilon^2}}{2}. \]

This yields

\[ z = r_0 \frac{(2 - \epsilon^2) \cosh(2\lambda u) + i\epsilon \sqrt{4 - \epsilon^2} \sinh(2\lambda u) + s\epsilon}{2 \cosh(2\lambda u) - s\epsilon} e^{-i\nu/r_0} \]

(239)

with \( s = +1 \). Whereas for \( \epsilon > 2 \) the radius of the curve oscillates between \( |r_3| \) and \( r_4 \), there are two curves for \( \epsilon < 2 \), one with largest distance \( r_4 = r_0(1 + \epsilon) \), the other with minimal distance \( |r_1| = r_0|\epsilon - 1| \) from the origin. This second curve is obtained by adding \( \omega_1 \) to \( v \), or equivalently \( \pi i/2 \) to \( \lambda v \). Thus \( \cosh(4\lambda v) \), \( \sinh(4\lambda v) \), \( \coth(2\lambda v) \) are unchanged, but \( \cosh(2\lambda v) \) changes sign. Starting from eq. (237) one finds that for the second curve one has to choose \( s = -1 \) in eq. (239). Both these curves approach asymptotically the circle with radius \( r_0 \).

The condition for the line segment of constant length and its length read

\[ \tanh(2\lambda \delta u) = 2\lambda r_0 \tan(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}), \]

\[ 0 \leq \frac{2\ell}{r_0} = \frac{2\epsilon}{\sqrt{(4 - \epsilon^2) \coth^2(2\lambda \delta u) + \epsilon^2}} \leq \epsilon, \]

(241)

and that for the line segment connecting both curves

\[ \tanh(2\lambda \delta u) = \frac{1}{2\lambda r_0 \cot(\frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2}), \]

\[ \epsilon \leq \frac{2\ell}{r_0} = \frac{2\epsilon}{\sqrt{(4 - \epsilon^2) \tanh^2(2\lambda \delta u) + \epsilon^2}} \leq 2. \]

Curves for \( \epsilon = 1.6 \) are shown in figs. S4 to S7. Figure S4 is shown in the movie[13] for a different solution \( \delta u \).
Fig. 84 $m/n = 1/0, \epsilon = 1.6$

Fig. 85 $m/n = 1/0, \epsilon = 1.6$

Fig. 86 $m/n = 1/0, \epsilon = 1.6$

Fig. 87 $m/n = 1/0, \epsilon = 1.6$

8.3 The limit $\epsilon = 2$

In the limit $\epsilon = 2$ one has

$$r_4 = 3r_0, \quad r_3 = r_2 = r_1 = -r_0.$$  \hfill (244)

This yields

$$e_1 = e_2 = e_3 = 0.$$  \hfill (245)

The Weierstrass-functions are degenerate to

$$\wp(u) = \frac{1}{u^2}, \quad \zeta(u) = \frac{1}{u}, \quad \sigma(u) = u$$  \hfill (246)

and there is no periodicity left, $\omega = \omega' = \infty$. With $\wp(u) = -1/r_0^2$ one obtains

$$v = \frac{1}{r_0}$$  \hfill (247)

and

$$z = \frac{u - 3\text{i}r_0}{u/r_0 + 1} e^{-\text{i}u/r_0} = \frac{3r_0 + \text{i}u}{1 - \text{i}u/r_0} e^{-\text{i}u/r_0}.$$  \hfill (248)

Again this curve approaches asymptotically the unit circle. The condition for the line segment of constant length reduces to

$$e^{i(-2\delta u/r_0 + \chi - \hat{\chi})} = \frac{1 - \text{i} \delta u / r_0}{1 + \text{i} \delta u / r_0},$$  \hfill (249)
equivalently this may be written
\[ \frac{\delta u}{r_0} = \tan \left( \frac{\delta u}{r_0} - \frac{\chi - \hat{\chi}}{2} \right) \]  
(250)
and \( \ell \) is given by
\[ \ell^2 = \frac{r_0^2 \delta u^2}{\delta u^2 + \delta u^2}. \]  
(251)

Fig. 88 \( m/n = 1/0, \epsilon = 2 \)  
Fig. 89 \( m/n = 1/0, \epsilon = 2 \)

### 8.4 Elementary construction

The circle
\[ \hat{z} = r_0 e^{-iu/r_0} \]  
(252)
is a second solution for \( \epsilon > 2 \). For \( \epsilon \leq 2 \) this circle is approached asymptotically for \( u \to \pm \infty \). In all three cases the length of the line segment between this circle and the non-circular curves given above is
\[ 2\ell = \epsilon r_0. \]  
(253)

This property allows an interesting construction of these curves. Consider a straight line (indicated in the figures by cyan and green) of length \( 2\ell = \epsilon r_0 \). One end of it moves along the (magenta) circle of radius \( r_0 \) in figs. [90](#) to [93](#). The middle point of this line moves always in direction of the line itself, as if it were carried by a wheel aligned in this direction, creating the red envelope. Then the other end-point moves along the curve \( \Gamma \).

If \( \epsilon < 2 \), then there are two such curves, one outside and one inside the circle. Both approach asymptotically the circle for \( u \to \pm \infty \). If \( \epsilon \geq 2 \), then there is only one curve, which for \( \epsilon = 2 \) still approaches asymptotically the circle with \( r - r_0 \propto 1/u^2 \), whereas for \( \epsilon < 2 \) the difference \( r - r_0 \) decays exponentially with \( u \). For \( \epsilon > 2 \) there is only one curve, which oscillates between radii \( r_4 \) and \( r_3 \), which allows for periodically closed curves.

Examples for \( \epsilon > 2 \) are shown in figs. [90](#) to [93](#) below.
Fig. 90 $m/n = 1/3, \epsilon = 2.5$

Fig. 91 $m/n = -1/3, \epsilon = 2.213594$

Fig. 92 $m/n = 0/1, \epsilon = 2.309401$

Fig. 93 $m/n = -2/1, \epsilon = 2.06559$

Examples for $\epsilon < 2$ are given in figs. 94 and 95 showing the outer and inner curve.

Fig. 94 $m/n = 1/0, \epsilon = 1.6$

Fig. 95 $m/n = 1/0, \epsilon = 1.6$

Finally the construction for $\epsilon = 2$ is shown.

Fig. 96 $m/n = 1/0, \epsilon = 2$
9 Linear Case

In this section the limit is considered, in which \( r_0 \) tends to infinity, but the differences between the radii \( r_i \) is kept finite. In subtracting this average radius by taking the limit

\[
\lim_{r_0 \to \infty} (z(u) - r_0)
\]

one obtains curves, which repeat under translation in \( y \)-direction. Therefore this is called the linear case. One can distinguish three cases.

9.1 Two branches

The choice

\[
r_4 = r_0 + d, \quad r_3 = -r_0 - \hat{d}, \quad r_2 = -r_0 + \hat{d}, \quad r_1 = r_0 - d, \quad d > \hat{d} > 0
\]

yields

\[
a = \frac{1}{2r_0(d^2 - \hat{d}^2)},
\]

\[
e_3 = \frac{d^2 + d^2 + 6d\hat{d}}{3(d^2 - \hat{d}^2)^2} - \frac{1}{12r_0^2},
\]

\[
e_2 = \frac{d^2 + d^2 - 6d\hat{d}}{3(d^2 - \hat{d}^2)^2} - \frac{1}{12r_0^2},
\]

\[
e_1 = \frac{-2(d^2 + d^2)}{3(d^2 - \hat{d}^2)^2} + \frac{1}{6r_0^2},
\]

\[
\wp(2v) = e_1 - \frac{1}{4r_0^2}.
\]

\( v \) lies infinitesimal close to \( \omega_1/2 \),

\[
\wp\left(\frac{\omega_1}{2}\right) = \frac{d^2 - 5d^2}{3(d^2 - \hat{d}^2)^2} + O(1/r_0^2),
\]

\[
\wp'\left(\frac{\omega_1}{2}\right) = -\frac{4d\hat{d}}{(d^2 - \hat{d}^2)^2} + O(1/r_0^2),
\]

\[
\wp(v) = \wp\left(\frac{\omega_1}{2}\right) - \frac{d}{r_0(d^2 - \hat{d}^2)} + O(1/r_0^2).
\]

This yields

\[
v = \frac{\omega_1}{2} + \delta v, \quad \delta v = -\frac{1}{4r_0}(d^2 - \hat{d}^2).
\]

One starts from

\[
z = r_1 \frac{\sigma(u - 3v)\sigma(v)}{\sigma(u + v)\sigma(-3v)} e^{2\wp(2v)}.
\]
Due to the factor $r_4$ in front which is of order $r_0$, one has to take into account all terms of order $1/r_0$ in the fraction and in the exponential on the right hand side of this equation. With

$$\sigma(u - 3v) = \sigma(u - \frac{3}{2}\omega_1 - 3\delta v) = -\sigma(u + \frac{1}{2}\omega_1 - 3\delta v)e^{2(\frac{1}{2}\omega_1 - u + 3\delta v)\zeta(\omega_1)}$$

(266)

and

$$\sigma(u + \frac{1}{2}\omega_1 - 3\delta v) = \sigma(u + v - 4\delta v) = \sigma(u + v)e^{-4\delta v\zeta(u+v)}$$

(267)

one obtains

$$z = r_4e^{2u(\zeta(2\omega_1) - \zeta(\omega_1)) + 4\delta v(\zeta(\frac{1}{2}\omega_1) - \zeta(u + \frac{1}{2}\omega_1))}.$$ 

(268)

Further

$$\zeta(2\omega_1) - \zeta(\omega_1) = \zeta(\omega_1 + 2\delta v) - \zeta(\omega_1) = -2\delta v \varphi(\omega_1) = -2\delta ve_1$$

(269)

and

$$\zeta(u - v) - \zeta(u + v) + 2\zeta(v) = \varphi'(v)\mathcal{E}_2\left(u \left| \varnothing, v, -v \right. \right) = \frac{\varphi'(v)}{\varphi(u) - \varphi(v)}$$

(270)

yields

$$z = r_4^{e^{4\delta v}}(-e, u + \zeta(\frac{1}{2}\omega_1) - \zeta(u + \frac{1}{2}\omega_1)) \quad (271)$$

$$z = r_0 + d - 2r_0\delta v\left(\zeta(u - v) + \zeta(u + v) + \frac{\varphi'(v)}{\varphi(u) - \varphi(v)} + 2e_1u\right).$$

(272)

Adding a constant $y_0$ to $y$, which corresponds to the angle $\chi$ with $r_0\chi = y_0$ the final result is

$$x - r_0 = \pm d \mp \frac{2d(d^2 - \hat{d}^2)}{(d^2 - \hat{d}^2)^2\varphi(u) - \frac{1}{3}(d^2 - 5\hat{d}^2)}$$

(273)

$$y = y_0 + \frac{d^2 - \hat{d}^2}{2}\left(\zeta(u - \frac{1}{2}\omega_1) + \zeta(u + \frac{1}{2}\omega_1) - \frac{4(d^2 + \hat{d}^2)}{3(d^2 - \hat{d}^2)^2}u\right)$$

(274)

where the upper signs apply.

Instead of starting from $r_4$ one can start from $r_1$. Then the signs for the expression for $x - r_0$ have to be changed, as indicated by the lower signs in equation (273).

By inserting the extreme values of $\varphi(u)$ one finds that $x - r_0$ oscillates between $d$ and $\hat{d}$ for one curve and between $-d$ and $-\hat{d}$ for the other curve. This yields the two branches. Shifting the arc parameter by $2\omega_3$ shifts the curve by $y_{\text{per}}$ in the $y$-direction and leaves $x$ invariant,

$$x(u + 2\omega_3) = x(u), \quad y(u + 2\omega_3) = y(u) + y_{\text{per}},$$

(275)

$$y_{\text{per}} = -\frac{4(d^2 + \hat{d}^2)}{3(d^2 - \hat{d}^2)}\omega_3 + 2(d^2 - \hat{d}^2)\zeta(\omega_3).$$

(276)
In order to obtain the condition for the line segment of constant length one starts out again from eq. (141) and expands both sides up to order $O(1/r_0)$

\[
e^{i(x-\chi)} = 1 + i\frac{y_0 - \hat{y}_0}{r_0},
\]

\[
\zeta(2v) + \zeta(2\hat{v}) = \zeta(\omega_1 + 2\delta v) + \zeta(\omega_1 + 2n\omega_1 + 2\delta v)
\]

\[
= 2(n+1)\zeta(\omega_1) - 4e_1\delta v,
\]

\[
\zeta(2v) + \zeta(2\hat{v}) = 2\left(\delta u + \frac{\zeta(\omega_1)}{2}\right)\zeta(\omega_1) - 4e_1\delta v,
\]

\[
\frac{\frac{\sigma(2\delta u + v + \hat{v})}{\sigma(2\delta u - v - \hat{v})}}{\frac{\sigma(2\delta u + (n+1)\omega_1 + 2\delta v)}{\sigma(2\delta u + (n+1)\omega_1 - 2\delta v)}} = e^{2(1+n)(2\delta u - 2\delta v)\zeta(\omega_1)}.
\]

Then the terms in leading order cancel and the terms of order $O(1/r_0)$ yield the condition for $\delta u$

\[
y_0 - \hat{y}_0 = \frac{4(d^2 + \hat{d}^2)}{3(d^2 - d^2)}\delta u - (d^2 - \hat{d}^2)(\zeta(2\delta u + (1+n)\omega_1) - (1+n)\zeta(\omega_1))
\]

which for one and the same branch (even $n$) yields

\[
y_0 - \hat{y}_0 = \frac{4(d^2 + \hat{d}^2)}{3(d^2 - d^2)}\delta u - (d^2 - \hat{d}^2)(\zeta(2\delta u + \omega_1) - \zeta(\omega_1))
\]

and for a line segment between the two different branches (odd $n$)

\[
y_0 - \hat{y}_0 = \frac{4(d^2 + \hat{d}^2)}{3(d^2 - d^2)}\delta u - (d^2 - \hat{d}^2)\zeta(2\delta u).
\]

From eq. (157) one learns that the length of the line segment of constant length for the same curve shifted by some $y_0 - \hat{y}_0$ cannot exceed $d - \hat{d}$. Examples are shown in figs. 97 to 99.

These line segments are not the trivial ones parallel to the $y$-axis of length $y_{per}$ and integer multiples. The length of a line segment of constant length between two different curves has a lower bound $d + \hat{d}$, but no upper bound. Indeed one obtains an infinity of solutions. A pair of curves is shown in figs. 100 to 103 with three different lengths of the line segment.

The curves shown here are also trajectories of massive charges in a perpendicular magnetic field. In this linear limit one obtains from eqs. (3, 32, 255, 256) that $B$ increases linearly

\[
B = -\frac{4mw}{e(d^2 - d^2)}(x - r_0).
\]
Note that on both trajectories in figs. 100 to 103 the charges move in the same direction. Such trajectories as well as those described in the following subsections have been obtained by Evers et al [14]. An example of a pair of drifting orbits is shown in fig. 1 of their paper.

9.2 The limit $\dot{d} = 0$

In the limit $\dot{d} = 0$ one obtains

$$e_2 = e_3 = \frac{1}{3d^2}.$$  \hfill (285)
\[ e_1 = -\frac{2}{3d^2}, \quad (286) \]
\[ \varphi(u) = -\frac{2}{3d^2} + \frac{1}{d^2} \coth^2 \left( \frac{u}{d} \right), \quad (287) \]
\[ \zeta(u) = -\frac{u}{3d^2} + \frac{1}{d} \coth \left( \frac{u}{d} \right), \quad (288) \]
\[ x - r_0 = \pm \frac{\hat{d}}{\cosh(2u/d)}, \quad (289) \]
\[ y = y_0 - u + d \tanh(2u/d), \quad (290) \]
\[ \omega_1 = \frac{d\pi}{2}, \quad \omega_3 = \infty. \quad (291) \]

The curve is no longer periodic. The conditions for line segments of constant length are

\[ y_0 - \hat{y}_0 - 2\delta u = \begin{cases} 
- d \tanh(2\delta u/d), \quad \text{same sign of } x - r_0, \\
- d \coth(2\delta u/d), \quad \text{different sign of } x - r_0.
\end{cases} \quad (292) \]

Thus there is no chord of constant length for one and the same curve. The length of the line segments is given by

\[ 2\ell = |y_0 - \hat{y}_0 + 2\delta u|. \quad (293) \]

There are always two solutions for line segments between pairs of curves.

A similar elementary construction as in subsect. 8.4 is possible. Now one end of the straight line of length \(2\ell\) moves along the straight line \(x - r_0 = 0\) as shown in fig. 107. Auerbach\[^3\] has composed his examples for \(\rho = 1/2\) from several copies of the central part of the curve and the corresponding piece of the straight line of fig. 107.
Evers et al. [14] show in the upper fig. 2 the variation of the classical trajectory of an electron as the gradient of $B$ is varied, which corresponds here to a variation of $\hat{d}^2/d^2$. The drifting orbit transforms into snake states, which will be considered in the next subsection. The limit case is the state $\hat{d} = 0$ shown here.

### 9.3 One branch

In the case of one branch (two $r_i$ are real, two $r_i$ are complex) the choice is

$$r_4 = r_0 + d, \quad r_3 = r_0 - d, \quad r_2 = -r_0 + is, \quad r_1 = -r_0 - is.$$  

(294)

The calculation is rather similar to that in subsection 9.1. Indices 1 and 3 have to be exchanged and $\hat{d}$ becomes $i\hat{s}$. This yields

$$a = \frac{1}{2r_0(d^2 + s^2)^2},$$  

(295)

$$e_1 = \frac{d^2 - s^2 + 6ids}{3(d^2 + s^2)^2} - \frac{1}{12r_0^2},$$  

(296)

$$e_2 = \frac{d^2 - s^2 - 6ids}{3(d^2 + s^2)^2} - \frac{1}{12r_0^2},$$  

(297)

$$e_3 = \frac{2(s^2 - d^2)}{3(d^2 + s^2)^2} + \frac{1}{6r_0^2},$$  

(298)

$$\wp(2v) = e_3 - \frac{1}{4r_0^2}.$$  

(299)

Now $v$ lies infinitesimal close to $\omega'/2$. One determines

$$\wp(\omega'/2) = \frac{-s^2 - 5d^2}{3(d^2 + s^2)^2} + O(1/r_0^2),$$  

(300)

$$\wp'(\omega'/2) = -\frac{4d}{(d^2 + s^2)^2} + O(1/r_0^2),$$  

(301)

$$\wp(v) = \wp(\omega'/2) - \frac{d}{r_0(d^2 + s^2)} + O(1/r_0^2).$$  

(302)

From this one concludes

$$v = \frac{\omega'}{2} + \delta v, \quad \delta v = -i\frac{d^2 + s^2}{4r_0}. $$  

(303)

In the following calculation one has to replace $\omega_1$ by $\omega'$ and $e_1$ by $e_3$ and one obtains in analogy to (272)

$$z = r_0 + d - 2r_0 \delta v \left( \zeta(u - v) + \zeta(u + v) - \frac{\wp'(v)}{\wp(u) - \wp(v)} + 2e_3 u \right).$$  

(304)
Thus the final result is

\[
x - r_0 = d - \frac{2d(d^2 + s^2)}{(d^2 + s^2)^2 \varphi(u) + \frac{1}{3}(s^2 + 5d^2)},
\]

\[
y = y_0 + \frac{d^2 + s^2}{2} \left( \zeta(u - \frac{1}{2} \omega') + \zeta(u + \frac{1}{2} \omega') - \frac{4(d^2 - s^2)}{3(d^2 + s^2)^2} u \right)
\]
$x - r_0$ oscillates between $d$ and $-d$. Moreover one has
\begin{align*}
x(u + n\omega_3) - r_0 &= (-n)^n(x(u) - r_0), \\
y(u + n\omega_3) &= y(u) + \frac{n}{2}y_{\text{per}},
\end{align*}
(307)
\begin{equation}
y_{\text{per}} = -\frac{4(d^2 - s^2)}{3(d^2 + s^2)}\omega_3 + 2(d^2 + s^2)\zeta(\omega_3).
\end{equation}
(309)

The condition for the line segment of constant length can be obtained similarly as for the case of two curves. One obtains
\begin{equation}
y_0 - \tilde{y}_0 = -\frac{4(d^2 - s^2)}{3(d^2 + s^2)}\delta u + (d^2 + s^2)(\zeta(2\delta u + \omega_3) - \zeta(\omega_3)),
\end{equation}
(310)
where one replaces $\zeta(2\delta u + \omega') - \zeta(\omega') = \zeta(2\delta u + \omega_3) - \zeta(\omega_3)$, since $\omega'$ and $\omega_3$ differ by a period.

The examples of figs. 108 to 111 show curves along which a bicyclist could drive. One and the same curve $\Gamma$ for the front tire corresponds to different traces of the rear tire for bicycles of different lengths.

In figs. 112 to 114 the traces are more artistic, since periodically the bicyclist has to move back and forth with the rear tire, but always forward with the front tire.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig112.png}
\caption{Fig. 112 $s/d = 0.7$  \hspace{1em} Fig. 113 $s/d = 0.7$  \hspace{1em} Fig. 114 $s/d = 0.7$}
\end{figure}

If $s/d$ decreases further the loops overlap stronger until at $s/d = 0.458786$ the charge moves in an eight shown in fig. 116. For this value of $s/d$ one has $y_{\text{per}} = 0$. $y_{\text{per}}$ becomes negative for smaller values of $s/d$. The expression (283) for the magnetic field reads now
\begin{equation}
B = -\frac{4mw}{e(d^2 + s^2)}(x - r_0).
\end{equation}
(311)
These eights have still the property that between two of them one has line segments of constant length. When $s/d$ becomes even smaller then the charges move into the opposite direction. In fig. 1 of ref. [14] two snake state trajectories are shown moving in opposite directions. In fig. 2 of this paper the eight can be seen where the electron is reflected in the magnetic bottle neck.

10 The Carousel by Montejano, Oliveros and Bracho

Oliveros, Montejano and Bracho[5, 4] have considered special cases of these curves. They consider a carousel, which is a dynamical equilateral (not equiangular) polygon in which the midpoint of each edge travels parallel to it. Then the trace of the vertices constitute the curves $\Gamma$ whereas the midpoints outline the envelopes $\gamma$. They perform explicitly calculations for pentagons. In general they allow for five different curves for the vertices.

Denoting the curves by indices $j$ one obtains solutions for the carousel from the curves introduced here by defining

$$z_j(u) := e^{2j\delta \chi} z(u), \quad (312)$$

and requiring

$$|z_j(u + \delta u) - z_{j-1}(u - \delta u)| = 2\ell, \quad z_5(u + 10\delta u) = z_0(u). \quad (313)$$

The first condition is fulfilled with $e^{2j\delta \chi} = \text{r.h.s. of} \ [14]$. The second condition which guarantees that the pentagon is closed yields

$$z_5(u + 10\delta u) = e^{10j\delta \chi} z_0(u + 10\delta u) = z_0(u). \quad (314)$$

Thus $10\delta u$ has to be an integer multiple of $2\omega_3$,

$$\delta u = \frac{k}{5} \omega_3 \quad (315)$$
with integer $k$. Then the condition reduces to
\[ e^{i(10\delta\chi+k\psi_{2\ell})} = 1. \] (316)

An example is fig. 6 of ref. [5]. Since the curves go through the origin, one has $\epsilon = 1$. One realizes that the curves seem to be close to those for $m/n = -1/3$, $\epsilon = 1$ in fig. 59 which yields $\hat{\epsilon}^2 = -0.034862$. One finds that condition (316) is fulfilled for $\hat{\epsilon}^2 = -0.025835$ and apparently for two different $\kappa$ and two different lengths $2\ell$ of the line segments. This is shown in figs. 119 and 120. The second one seems to reproduce fig. 6 of ref. [5].

A solution for five 'Eights' is obtained for $m/n = 0/1$, $\epsilon = 0.689454$ in figs. 121 and 122. The ratio of the arc length of the outer and the inner loop of the eights is 2/3. Fig. 8 in [5] shows the same eights (to the extend one can judge this with the eyes), but the pentagon shown there is misleading. It should be as in fig. 122.

The corresponding carousel for the linear case is shown in fig. 123 and 124. This solution looks like that of fig. 2 of [4].
Of interest is the case where all five figures fall onto each other and the curve is closed. Then

$$\psi_{\text{per}} = \frac{2\pi}{n}, \quad \delta u = \frac{k' n \omega_3}{5}$$  \hspace{1cm} (317)

with integer $k'$. One with $n = 7$-fold symmetry is obtained for $\epsilon = 0.459104$ in figs. 125, 126 and should be compared with fig. 9 of [5] and fig. 3 of [4], the other one with $n = 12$-fold symmetry for $\epsilon = 0.5727853$ in figs. 127, 128 and is to be compared with fig. 1 of [4].
\section*{A Weierstrass Functions}

\subsection*{A.1 Definitions and relations of Weierstrass functions}

The Weierstrass function $\wp$ is defined by

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$  \hspace{1cm} (318)

with the requirement that one of the singularities is at $z = 0$ and

$$e_1 + e_2 + e_3 = 0.$$ \hspace{1cm} (319)

Commonly the two integrals are defined

$$\zeta(a) = \frac{1}{a} - \int_0^a (\wp(z) - \frac{1}{z^2})dz,$$ \hspace{1cm} (320)

$$\sigma(a) = a \exp \left( \int_0^a (\zeta(z) - \frac{1}{z})dz \right).$$ \hspace{1cm} (321)
The function $\wp$ is an even function of its argument, $\zeta$ and $\sigma$ are odd functions. The Laurent and Taylor expansions start with

\[
\begin{align*}
\wp(a) &= \frac{1}{a^2} + \frac{g_2}{20} a^2 + \frac{g_3}{28} a^4 + \ldots \quad (322) \\
\zeta(a) &= \frac{1}{a} - \frac{g_2}{60} a^3 - \frac{g_3}{140} a^5 - \ldots \quad (323) \\
\sigma(a) &= a - \frac{g_2}{240} a^5 - \frac{g_3}{840} a^7 - \ldots \quad (324)
\end{align*}
\]

There exist addition theorems

\[
\begin{align*}
\wp(a + b) &= -\wp(a) - \wp(b) + \frac{(\wp'(a) - \wp'(b))^2}{4(\wp(a) - \wp(b))^2}, \quad (325) \\
\zeta(a + b) &= \zeta(a) + \zeta(b) + \frac{1}{2} \frac{\wp'(a) - \wp'(b)}{\wp(a) - \wp(b)}, \quad (326) \\
\sigma(a + b)\sigma(a - b) &= -\sigma^2(a)\sigma^2(b)(\wp(a) - \wp(b)). \quad (327)
\end{align*}
\]

If $\omega$ is a half-period, that is $\omega$ itself is not a period of $\wp$, but $2\omega$ is, then the following relations hold for integer $n$

\[
\begin{align*}
\wp(a + 2n\omega) &= \wp(a), \quad (328) \\
\zeta(a + 2n\omega) &= \zeta(a) + 2n\zeta(\omega), \quad (329) \\
\sigma(a + 2n\omega) &= (-)^n \sigma(a)e^{2n(a+n\omega)\zeta(\omega)}. \quad (330)
\end{align*}
\]

The increase of the argument of $\sigma(a)$, if the real part of $a$ is increased by $2\omega_3$, is obtained from eq. (331)

\[
\arg \sigma(a + 2\omega_3) - \arg \sigma(a) = -\pi(2m + 1) + 2\Im a\zeta(\omega_3) \quad (331)
\]

with integer $m$, where $-\pi(2m + 1)$ comes from the minus-sign in (330) and $m$ has to be determined. If $a$ is real, then $\sigma$ changes sign, whenever $a$ is an integer multiple of $2\omega_3$. If $a$ has a small positive (negative) imaginary part, then $a$ passes the zero clockwise (counter-clockwise) in the complex plane, yielding a decrease (an increase) of the argument by $\pi$. Further jumps by $2\pi$ occur, whenever the imaginary part of $a$ crosses an integer multiple of $3\omega_3$. Thus $m$ in eq. (331) is given by

\[
2m\Im \omega_1 < \Im a < 2(m + 1)\Im \omega_1. \quad (332)
\]

From (326) one obtains

\[
\zeta(a + b) + \zeta(a - b) = 2\zeta(a) + \frac{\wp'(a)}{\wp(a) - \wp(b)}. \quad (333)
\]

Legendre’s relation reads

\[
\eta \omega' - \eta' \omega = \frac{\pi i}{2} \quad \text{for} \quad \frac{\omega'}{\omega} > 0, \quad (334)
\]

where $2\omega$ and $2\omega'$ span one elementary cell.
A.2 Double-periodic functions

Two basic theorems on double-periodic functions are

**Theorem I** Two elliptic functions which have the same period, the same poles, and the same principal parts at each pole differ by a constant.

**Theorem II** The quotient of two elliptic functions whose periods, poles, and zeroes (and multiplicities of poles and zeroes) are the same, is a constant. Due to this second theorem the elliptic function

\[ E_{n-m}(u) = \left( \frac{g_{n-m}(u)}{g_{n-m}'(u)} \right) \]

is defined as the periodic function with zeroes at \( u_1, u_2, ... u_m \) and poles at \( v_1, v_2, ... v_n \) in an elementary cell. Zeroes and poles at the origin are counted separately. If \( n > m \), then the function has an \( n - m \)-fold zero at \( u = 0 \). If \( n < m \), then the function has instead a pole of order \( m - n \) at \( u = 0 \). The function \( E \) is normalized so that

\[ E_{n-m}(u) = u^{n-m} + O(u^{n-m+1}). \]

Moreover one has to require that the sums \( \sum_i u_i \) and \( \sum_i v_i \) differ only by a period \( 2\omega \)

\[ \sum_{i=1}^m u_i = \sum_{i=1}^n v_i + 2 \sum_j k_j \omega_j \]

with integer \( k_j \). The functions \( E \) can explicitly be expressed in terms of the functions \( \sigma \)

\[ E_{n-m}(u) = \sigma_{n-m}(u) \prod_{i=1}^m \frac{\sigma(u-u_i)}{\sigma(-u_i)} \prod_{i=1}^n \frac{\sigma(-v_i)}{\sigma(u-v_i)} e^{2u \sum_j k_j \zeta(\omega_j)}. \]

Using eqs. (330) and (331) one finds that this function is double-periodic. \( g_2 \) and \( g_3 \) are not explicitly indicated. In these eqs. it is always assumed that they are the same in all functions.

Multiplication yields

\[ E_{n-m}(u) E_{n'-m'}(u) = E_{n+n'-m-m'}(u) \]

If locations of zeroes and poles coincide they have to be cancelled

\[ E_{n-m}(u) = E_{n-m}(u) \]
Examples are
\[
\varphi(u) - \varphi(v) = \mathcal{E}_2 \begin{pmatrix} u & v, -v \end{pmatrix},
\]
(342)
\[
\frac{\varphi(u) - \varphi(v)}{\varphi(u) - \varphi(v_3)} = \mathcal{E}_0 \begin{pmatrix} v_3, -v_3 \end{pmatrix},
\]
(343)
\[
\varphi'(u) = -2\mathcal{E}_{-3} \begin{pmatrix} \omega_1, \omega_2, \omega_3 \end{pmatrix} = -\frac{\sigma(2u)}{\sigma'(u)}.
\]
(344)

and
\[
\begin{vmatrix}
1 & \varphi(u) & \varphi'(u) \\
1 & \varphi(v) & \varphi'(v) \\
1 & \varphi(w) & \varphi'(w)
\end{vmatrix} = -2\mathcal{E}_{-3} \begin{pmatrix} u, v, w, -v - w \end{pmatrix} \mathcal{E}_{-2} \begin{pmatrix} v, w \end{pmatrix} = -2\sigma(u - v)\sigma(u - w)\sigma(v - w)\sigma(u + v + w) \sigma^3(u)\sigma^3(v)\sigma^3(w).
\]
(345)

from which one concludes that the determinant vanishes, if \( u + v + w = 0 \),
\[
\begin{vmatrix}
1 & \varphi(u) & \varphi'(u) \\
1 & \varphi(v) & \varphi'(v) \\
1 & \varphi(u + v) & -\varphi'(u + v)
\end{vmatrix} = 0.
\]
(346)

In the limit \( u \to v \) one obtains
\[
\begin{vmatrix}
1 & \varphi(v) & \varphi'(v) \\
0 & \varphi'(v) & \varphi''(v) \\
1 & \varphi(2v) & -\varphi'(2v)
\end{vmatrix} = 0.
\]
(347)

### A.3 Representations of Weierstrass functions

The Weierstrass function \( \zeta(z) \) with elementary periods \( 2\omega \) and \( 2\omega' \) has simple poles with residues 1 at
\[
z = 2m\omega + 2n\omega'
\]
for integer \( m \) and \( n \). It reads
\[
\zeta(z) = \frac{n}{\omega} z + \frac{\pi}{2\omega} \left( \cot\left( \frac{u}{2}\right) + \sum_{n=1}^{\infty} \left( \cot\left( \frac{u}{2} + \frac{n\pi\omega'}{\omega}\right) + \cot\left( \frac{u}{2} - \frac{n\pi\omega'}{\omega}\right) \right) \right)
\]
\[
= \frac{n}{\omega} z + \frac{\pi}{2\omega} \sin u \left( \frac{1}{1 - \cos u} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - 2q^n \cos u + q^{2n}} \right)
\]
(349)

with
\[
u = \frac{\pi z}{\omega},
\]
\[
q = \exp(2i\pi\omega'/\omega),
\]
(350)
Finally the integral of \( \zeta(\omega) = \omega \) yields
\[
\eta = \zeta(\omega) = \omega \left( \frac{\pi}{2\omega} \right)^2 \left( \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{\sin^2 \left( \frac{n\pi\omega}{\omega} \right)} \right)
\]
\[
= \omega \left( \frac{\pi}{2\omega} \right)^2 \left( \frac{1}{3} - 8 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \right).
\] (352)

The derivative of the \( \zeta \)-function yields the Weierstrass \( \wp \)-function
\[
\wp(z) = -\frac{d\zeta(z)}{dz}
\]
\[
= -\frac{\eta}{\omega} + \left( \frac{\pi}{2\omega} \right)^2 \left( \frac{1}{\sin^2 \left( \frac{\pi}{2} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{\sin^2 \left( \frac{\pi}{2} + \frac{n\pi\omega}{\omega} \right)} + \frac{1}{\sin^2 \left( \frac{\pi}{2} - \frac{n\pi\omega}{\omega} \right)} \right) \right)
\]
\[
= -\frac{\eta}{\omega} + 2 \left( \frac{\pi}{2\omega} \right)^2 \left( \frac{1}{1 - \cos u} - 4 \sum_{n=1}^{\infty} \frac{q^n(\cos u - 2q^n + q^{2n} \cos u)}{(1 - 2q^n \cos u + q^{2n})^2} \right).
\] (353)

The constant term to \( \wp(z) \) fixes the constant term in the Laurent expansion of \( \wp(z) = \frac{1}{z^2} + 0z^3 + \ldots \) to vanish. The derivative of the Weierstrass function reads
\[
\wp'(z) = -4 \left( \frac{\pi}{2\omega} \right)^3 \sin u \left( \frac{1}{(1 - \cos u)^2} \right.
\]
\[
-4 \sum_{n=1}^{\infty} \frac{q^n(1 + 2q^n \cos u - 6q^{2n} + 2q^{3n} \cos u + q^{4n})}{(1 - 2q^n \cos u + q^{2n})^3} \right) \). (354)

Finally the integral of \( \zeta \) is obtained
\[
\sigma(z) = z \exp \left( \int_0^z (\wp'(z') - \frac{1}{z'})dz' \right)
\]
\[
= \frac{2\omega}{\pi} \sin \frac{u}{2} \exp \left( \frac{\eta z^2}{2\omega} \right) \prod_{n=1}^{\infty} \frac{1 - 2q^n \cos u + q^{2n}}{1 - 2q^n + q^{2n}}.
\] (355)

These representations of the Weierstrass functions have been used for the calculation of the figures.

**B Symmetric Polynomials**

Four symmetric polynomials \( P_q, P_r, P_m \) and \( \hat{P} \) are introduced. Symmetric means that they are invariant under permutation of the \( r_i \) and \( q_i \), resp. Three of them are defined by
\[
P_q = \frac{1}{8}(q_1 + q_2 - q_3 - q_4)(q_1 - q_2 + q_3 - q_4)(q_1 - q_2 - q_3 + q_4),
\] (356)
\[
P_r = \frac{1}{8}(r_1 + r_2 - r_3 - r_4)(r_1 - r_2 + r_3 - r_4)(r_1 - r_2 - r_3 + r_4)
\]
\[
= -(r_1 + r_2)(r_1 + r_3)(r_2 + r_3) = (r_4 + r_1)(r_4 + r_2)(r_4 + r_3),
\] (357)
\[
P_m = \sum_{i<j} q_i q_j = \frac{1}{2}(q_1 + q_2 + q_3 + q_4)^2 - \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2).
\] (358)
The identity for $P_r$ is derived from $\sum_i r_i = 0$. Expanding the last expression in polynomial and subtracting $r_4(r_4 + r_1 + r_2 + r_3)$, which vanishes one obtains

$$P_r = r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4.$$  \hspace{1cm} (359)

Realizing that

$$q_1 + q_2 - q_3 - q_4 = r_1^2 + r_2^2 - r_3^2 - (r_1 + r_2 + r_3)^2 = -2(r_3 + r_1)(r_3 + r_2) \hspace{1cm} (360)$$

and similarly for permutations one finds

$$P_r^2 = P_q.$$ \hspace{1cm} (361)

Consider

$$r_4 P_r = r_4 (r_4 + r_1)(r_4 + r_2)(r_4 + r_3)$$
$$= r_4 (r_4^3 + r_4^2 r_1 + r_4^2 r_2 + r_4 r_1 r_2 + r_4 r_1 r_3 + r_4 r_2 r_3) + r_1 r_2 r_3$$
$$= \frac{1}{2} q_4 (q_4 - q_1 - q_2 - q_3) + r_1 r_2 r_3 r_4,$$ \hspace{1cm} (362)

where use is made of

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{1}{2} (r_1 + r_2 + r_3)^2 - r_1^2 - r_2^2 - r_3^2 = \frac{1}{2} (q_4 - q_1 - q_2 - q_3). \hspace{1cm} (363)$$

Using also permutations one obtains

$$0 = (r_1 + r_2 + r_3 + r_4) P_r = 4 r_1 r_2 r_3 r_4 - 4 \hat{P}$$ \hspace{1cm} (364)

with

$$\hat{P} = -\frac{1}{8} (q_1^2 + q_2^2 + q_3^2 + q_4^2) + \frac{1}{4} P_m$$
$$= \frac{1}{8} (q_1 + q_2 + q_3 + q_4)^2 - \frac{1}{4} (q_1^2 + q_2^2 + q_3^2 + q_4^2).$$ \hspace{1cm} (365)

This yields

$$r_1 r_2 r_3 r_4 = \hat{P}$$ \hspace{1cm} (366)

in terms of the squares $q_i = r_i^2$.

**References**

[1] R.D. Mauldin (ed.), *The Scottish Book*, Birkhäuser Boston 1981

[2] F. Wegner, *Floating Bodies of Equilibrium. Explicit Solution*, e-Print archive physics/0603160

[3] H. Auerbach, *Sur un problème de M. Ulam concernant l’équilibre des corps flottant*, Studia Math. 7 (1938) 121-142
[4] J. Bracho, L. Montejano, D. Oliveros, Carousels, Zindler curves and the floating body problem, Per. Math. Hung. 49 (2004) 9-23

[5] D. Oliveros and L. Montejano, De volantines, espirógrafos y la flotación de los cuerpos, Revista Ciencias 55-56 (1999) 46-53

[6] S. Tabachnikov Tire track geometry: variations on a theme Israel J. of Math. 151 (2006) 1-28 archive math.DG/0405445

[7] F. Wegner, Floating Bodies of Equilibrium I, e-Print archive physics/0203061

[8] F. Wegner, Floating Bodies of Equilibrium Studies in Appl. Math. 111 (2003) 167-183

[9] D.L. Finn, Which way did you say that bicycle went? Math. Mag. 77 (2004) 357-367

[10] D.L. Finn, Which way did you say that bicycle went? http://www.rose-hulman.edu/~finn/research/bicycle/tracks.html

[11] A.C. Doyle, The Adventure of the Priory School in The Return of Sherlock Holmes

[12] F. Wegner, Floating Bodies of Equilibrium II, e-Print archive physics/0205059

[13] F. Wegner, Three Problems - One Solution, http://www.tphys.uni-heidelberg.de/~wegner/Fl2mvs/Movies.html 
Drei Probleme - Eine Lösung, http://www.tphys.uni-heidelberg.de/~wegner/Fl2mvs/Filme.html

[14] F. Evers, A.D. Mirlin, D.G. Polyakov, P. Woelfle, Semiclassical theory of transport in a random magnetic field, cond-mat/9901070, Phys. Rev. B 60 (1999) 8951

[15] M. Abramowitz and I. A. Stegun, eds. Handbook of Mathematical Functions Dover Publ., New York