Weak Gödel’s incompleteness property for some decidable versions of the calculus of relations

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Abstract

Relativization is one of the central topics in the study of algebras of relations. Some relativized relation algebras behave much nicer than the original relation algebras. In this paper, we study the atomicity of the finitely generated free algebras of these nice classes of relativized relation algebras. In particular, we give an answer for the open problem, posed by I. Németi in 1985, which asks whether the finitely generated free algebras of the class of the weak associative relation algebras $WA$ are atomic or not.

1 Introduction

The first order predicate calculus has its origins in the calculus of relations. The calculus of relations was created and developed in the second half of the nineteenth century by De Morgan, Peirce and Schröder. The creation of this calculus was the result of the continuous efforts searching for a “good general algebra of logic”. But these efforts took place decades before the emergence of first order calculus. The early notation for quantifiers originates with Peirce. Löwenheim’s original version of the Löwenheim-Skolem theorem, c.f. (Löwenheim, 1915), is not a theorem about first order logic but about the calculus of relations.

In 1940, Tarski proposed an axiomatization for a large part of the calculus of relations. In the next decades Tarski’s axiomatization led to the creation of the theory of relation algebras which was shown to be incomplete by Lyndon’s discovery of nonrepresentable relation algebras. Alfred Tarski showed that mathematics can be built up in the equational theory of the relation algebras, hence it is undecidable (Tarski and Givant, 1987). He raised the problem “how much associativity of relation composition is needed for this result”. Roger
Maddux defined the class of weakly-associative relation algebra, $\mathcal{WA}$, by weakening the associativity of the relation composition. István Németi showed that the equational theory of $\mathcal{WA}$ is decidable.

This class also can be seen as a class of relativized relation algebras. Indeed, it is proved in [Maddux 1982] that an algebra is a $\mathcal{WA}$ if and only if it is isomorphic to the concrete algebra of some subrelations of a symmetric and reflexive relation. The notion of a relativized algebra has been introduced in the theory of Boolean algebras and then it was extended to algebras of logics by Leon Henkin. Relativization in algebraic logic started as a technique for generalizing representations of algebras of logics. Relativization of an algebra amounts to intersecting all its elements with a fixed set (usually a subset of the unit) and to defining the new operations as the restrictions of the old operations on this set. Relativized algebras were not really studied in their own right, but as tools to obtain results for the standard algebras. At the end of the twentieth century, Andréka, van Benthem, Monk and Németi started promoting relativized algebras as structures which are interesting independently of their classical versions, see e.g. [Henkin et al. 1981]. Indeed, relativization in many cases turns the negative results into positive ones. Several relativized versions of algebras do have most of the nice properties which their standard counterparts lack.

From the universal algebra, the free algebras of a variety play an essential role in understanding this variety. They show, in some sense, the structure of the different “concepts” (represented by terms) of the variety. The intrinsic structures of the finitely generated free relativized relation algebras are very involved. Some problems concerning these algebras are still open. For example, the problem addressing the atomicity of these algebras has not been solved yet. The non-atomicity of the free algebras of logics is equivalent to weak Gödel’s incompleteness property of the corresponding logic. See [Németi 1985, proposition 8] and [Gyenis, 2011].

In the present paper, we study the atomicity of the free algebras of some interesting classes of relativized relation algebras. In particular, we give an answer for the atomicity problem of the free algebras of the class $\mathcal{WA}$. This problem goes back to 1985 when I. Németi posed it in his Academic Doctoral Dissertation [Németi, 1986]. In 1991, Németi posed the same problem again in [Andréka et al. 1991, Problem 38]. This problem was posed again as an open problem in 2013 in the most recent book in algebraic logic [Andréka et al., 2013, Problem 1.3.3]. Solving this problem, we show that the free algebras of $\mathcal{WA}$ generated by at least one generator are not atomic but the 0-generated free algebra of $\mathcal{WA}$ is finite, hence atomic.

One of the logics corresponding to relativized relation algebras is called arrow logic. It is a two-dimensional modal logic and it has various applications, e.g., in linguistics (dynamic semantics of natural language, relational semantics of Lambek Calculus), and in computer science (dynamic algebra, dynamic logic). For more about relativized relation algebras and arrow logic as modal logic see [van Benthem, 1991, 1994, 1996; Gabbay et al., 2003; Hirsch and Hodkinson, 2002; Marx et al., 1996a; Marx and Venema, 1997; Mikulás, 1993].
We need to recall some notions from universal algebra. For the definitions of these notions, one can look up any book in universal algebra (e.g., Burris and Sankappanavar, 1981). Let \( t \) be any algebraic type and let \( K \) be a class of algebras of type \( t \). Let \( m \) be any cardinal. \( \mathfrak{T}_{m,t} \) denotes the term algebra of type \( t \) generated by \( m \)-many free variables. \( \mathfrak{Fr}_{m} K \) denotes the free algebra of the class \( K \) generated by \( m \)-many generators.

2 Interesting relativized relation algebras

Let \( W \) be an arbitrary set of ordered pairs. Set \( \text{Id}^W = \{ (r, s) \in W : r = s \} \).
Define a unary operation \( \text{˘}^W \) and a binary operation \( ;^W \) on \( \mathcal{P}(W) \) as follows.
For any \( R, S \subseteq W \),
\[
\text{˘}^W R = \{ (r, s) \in W : (s, r) \in R \},
\]
\[
R^W S = \{ (r, s) \in W : (\exists u)(r, u) \in R \land (u, s) \in S \}.
\]
When no confusion is likely, we merely omit the superscript \( W \) from the above defined objects. A relativized relation set algebra is an algebra of the form
\[
\mathfrak{A} = \langle A, \cap, \cup, \setminus, \emptyset, W, ;^W, \text{˘}^W, \text{Id}^W \rangle,
\]
where \( W \) is an arbitrary set of ordered pairs and \( A \subseteq \mathcal{P}(W) \) is a family of subsets of \( W \) that is closed under the Boolean set theoretic operations \( \cap, \cup, \setminus \), closed under the relativized operations \( ;^W, \text{˘}^W \) and contains the following sets \( \emptyset, W, \text{Id}^W \). Suppose that \( U \) is the smallest set such that \( W \subseteq U \times U \). Then \( W \) and \( U \) are called the unit and the base of \( \mathfrak{A} \), respectively. The class of relativized relation set algebras is denoted by \( \mathcal{FRA}_0 \). Let \( rl \) denote the algebraic type of \( \mathcal{FRA}_0 \) where \( l' \) denotes the constant \( \text{Id} \) called identity, its complement is denoted by \( 0' \) and is called diversity.

Definition 2.1. Let \( H \subseteq \{ R, S, T \} \), where \( R, S \) and \( T \) stand for “Reflexive”, “Symmetric” and “Transitive” respectively. A relation is said to be an \( H \)-relation if it satisfies the properties in \( H \). The class of \( H \)-relativized relation set algebras, \( \mathcal{RRA}_H \), is the subclass of \( \mathcal{FRA}_0 \) which contains all of those algebras whose units are \( H \)-relations on their bases, \( \mathcal{RRA}_H \) denotes the class of the algebras isomorphic to \( H \)-relativized representable relation set algebras.

The notion of \( H \)-relativization with \( H \subseteq \{ R, S, T \} \) we use here was suggested by M. Marx. We note that \( \mathcal{RRA}_{\{R,S,T\}} \) is the class of the standard representable relation algebras. We also note that the class of the weak associative relation algebras \( \mathcal{WRA} \) coincides with the class \( \mathcal{RRA}_{\{R,S\}} \). In Andréka, 1991, Andréka et al., 1994, Andréka et al., 1996, Kryszewski, 1991, Marx, 1999, Marx et al., 1996a, Maddux, 1982 and Németi, 1987, it was shown that, for arbitrary \( H \subseteq \{ R, S, T \} \), the class \( \mathcal{RRA}_H \) enjoys any of the following properties if and only if \( T \not\in H \): finite axiomatizability, decidability, finite algebra property, finite base property, weak and strong interpolation, Beth definability and super amalgamation property.
In this paper, we concentrate only on those classes $RRA_H$, where $H \subseteq \{S, R\}$. Our main aim is to prove that, for every $m \in \omega$ and every $H \subseteq \{R, S\}$, the $m$-generated free algebra of the class $RRA_H$ is atomic if and only if $m = 0$ and $H = \{S, R\}$. We concentrate on finite numbers $m$ only because it is known that all the infinitely-generated free $RRA_H$ algebras are atomless.

3 Normal forms for relation type

In this section, we give a disjunctive normal form for any term in the signature of relation algebras. Disjunctive normal forms can provide elegant and constructive proofs of many standard results, c.f., [Anderson 1954] and [Fine 1975].

Throughout this section, we fix $m \in \omega$. For every $n$, we define a set $F^m_n \subseteq T^{m,rl}$ of normal forms of degree $n$ such that every member of $F^m_n$ contains complete information about the normal forms of the smaller degrees. Then, we write up each term in $T^{m,rl}$ as a disjunction of normal forms of the same degree.

We need to set up some notation and definitions. Let $n \in \omega$ and let $\tau_0, \ldots, \tau_n \in T^{m,rl}$. Define $\prod_{i=0}^0 \tau_i := \tau_0$ and inductively define $\prod_{i=0}^k \tau_i := (\prod_{i=0}^{k-1} \tau_i) \cdot \tau_k$, for every $1 \leq k \leq n$. Let $Y \subseteq T^{m,rl}$ be a finite set of terms. If $Y = \emptyset$, then define $\prod Y = 1$. Assume that $|Y| = n$, for some $n \geq 1$. Pick any bijection $f : n \to Y$ and let

$$\prod Y := \prod_{i=0}^{n-1} f(i).$$

This is ambiguous, but in this paper we deal with classes of Boolean algebras with operators, so it doesn’t matter which bijection is taken for the above product.

Definition 3.1. Let $\emptyset \neq Y \subseteq T^{m,rl}$ be finite and let $\alpha \in Y \{-1, 1\}$. Define

1. $CY = \{x, y : x, y \in Y\} \cup \{\bar{x} : x \in Y\}$, the one-step closure of $Y$ by the modal operations $;\text{ and}^*;$.  
2. $Y^\alpha = \prod\{x^\alpha : x \in Y\}$, where for every $x \in Y$, $x^\alpha = x$ if $\alpha(x) = 1$ and $x^\alpha = -x$ if $\alpha(x) = -1$.

Definition 3.2. Let $D_m = \{1', x_0, \ldots, x_{m-1}\}$, where $x_0, \ldots, x_{m-1}$ are the $m$ free variables that generate $T^{m,rl}$. For every $n \in \omega$, we define the followings inductively.

- The normal forms of degree 0, $F^m_0 = \{D^\beta_m : \beta \in D_m \{-1, 1\}\}$.

- The set of normal forms of degree $n+1$,

$$F^m_{n+1} = \{D^\beta_m \cdot (CF^m_n)^\alpha : \beta \in D_m \{-1, 1\} \text{ and } \alpha \in CF^m_n \{-1, 1\}\}.$$

- The set of all forms, $F^m = \bigcup_{k \in \omega} F^m_k$.  

4
Every form in $F^m_0$ is determined by the information telling whether it is below the identity or the diversity and whether it is below any free variable or its complement. Every form of degree $n + 1$, $n \in \omega$, is determined by the same information plus information telling whether this term is below (or below the complement of) the composition of any couple of forms in $F^m_n$, and whether this form is below (or below the complement of) the converse of any form in $F^m_n$.

Let $K$ be the class of all Boolean algebras with operators of type $rl$. The next theorem says that, for every $n \in \omega$, the normal forms of degree $n$ form a partition of the unit (inside $K$). It also indicates that every term in $\mathfrak{T}m_{m,rl}$ can be rewritten in the form of disjunctions of some normal forms of the same degree.

**Theorem 3.1.** Let $n \in \omega$. Then the followings are true:

(i). $K \models \Sigma F^m_n = 1$.

(ii). For every $\tau, \sigma \in F^m_n$, if $\tau \neq \sigma$ then $K \models \tau \cdot \sigma = 0$.

(iii). There exists an effective method to find, for every $\tau \in F^m_n$, a finite $S \subseteq F^m_{n+1}$ such that $K \models \tau = \Sigma S$.

(iv). There exists an effective method to find, for every $\tau \in \mathfrak{T}m_{m,rl}$, an $k \in \omega$ and a finite $S_\tau \subseteq F^m_k$ such that $K \models \tau = \Sigma S_\tau$.

**Proof.**

(i). Since the Boolean reduct of every member of $K$ is Boolean algebra, we have $(\forall S \subseteq \mathfrak{T}m_{m,rl}) K \models \Sigma \{S^\beta : \beta \in S \{-1, 1\}\} = 1$. In particular, for every $n \geq 1$, we should have

$$
K \models \Sigma F^m_0 = \Sigma \{D^\alpha_m : \alpha \in D^m_m \{-1, 1\}\} = 1, \quad \text{and}
$$

$$
K \models \Sigma F^m_n = \Sigma \{D^\alpha_m : \Sigma \{C(F^m_{n-1}) : \beta \in F^m_{n-1} \{-1, 1\}\} : \alpha \in D^m_m \{-1, 1\}\}
$$

$$
= \Sigma \{D^\alpha_m : \alpha \in D^m_m \{-1, 1\}\}
$$

$$
= 1.
$$

(ii). Let $\alpha_1, \alpha_2 \in D^m_m \{-1, 1\}$ be such that $\alpha_1 \neq \alpha_2$. Then there exists $x \in D_m$ such that, without loss of generality, $\alpha_1(x) = 1$ and $\alpha_2(x) = -1$. Therefore, $K \models D^\alpha_m \leq x$, $K \models D^\alpha_m \leq -x$ and $K \models D^\alpha_1 \cdot D^\alpha_2 = 0$. Let $n \geq 1$, and let $\tau, \sigma \in F^m_n$. Similarly, if $\tau \neq \sigma$, then without loss of generality we can assume that there exists $y \in D_m \cup CF^m_{n-1}$ such that $K \models \tau \leq y$ and $K \models \sigma \leq -y$. Therefore, $K \models \tau \cdot \sigma = 0$, as desired.

(iii). For every $\alpha \in D^m_m \{-1, 1\}$, we have $K \models D^\alpha_m = \Sigma \{D^\alpha_m \cdot (CF^m_0)^\beta : \beta \in F^m_0 \{-1, 1\}\}$. Inductively, let $n \geq 1$ and assume that for every $\sigma \in F^m_{n-1}$ there exists $S_\sigma \subseteq F^m_n$ such that $K \models \sigma = \Sigma S_\sigma$. For every $\sigma_1, \sigma_2 \in F^m_{n-1}$, define $S_{\sigma_1, \sigma_2} = \{\gamma_1 : \gamma_1 \in S_{\sigma_1} \text{ and } \gamma_2 \in S_{\sigma_2}\}$ and $S_{\sigma_1} = \{\gamma : \gamma \in S_{\sigma_1}\}$. 


Let \( \tau = D_m^\alpha \cdot (CF_{n-1}^m)^\beta \in F_n^m \). For every \( \beta' \in CF_{n-1}^m \{ -1, 1 \} \), we say that \( \beta' \) is compatible with \( \beta \), in symbols \( \beta' \sim \beta \), if for every \( \sigma \in CF_{n-1}^m \),

\[
\beta(\sigma) = 1 \iff (\exists \gamma \in S_\tau) \beta'(\gamma) = 1.
\]

Let \( S_\tau = \{ D_m^\alpha \cdot (CF_{n-1}^m)^\beta : \beta' \in CF_{n-1}^m \{ -1, 1 \}, \beta' \sim \beta \} \subseteq F_{n+1}^m \). Recall that \( K \) is a class of Boolean algebras with operators, therefore \( K \models \tau = \Sigma S_\tau \).

(iv). By induction on terms. For every \( \tau \in D_m \), we have

\[
K \models \tau = \Sigma \{ D_m^\alpha : \alpha \in D_m \{ -1, 1 \}, \alpha(\tau) = 1 \}.
\]

Let \( \sigma_1, \sigma_2 \in \Sigma_{m,rl} \) be such that there is an effective method to find \( n_1, n_2 \) and finite \( S_1 \subseteq F_n^m \) and \( S_2 \subseteq F_n^m \) such that \( K \models \sigma_1 = \Sigma S_1 \) and \( K \models \sigma_2 = \Sigma S_2 \). By item (iii) we may assume that \( n_1 = n_2 =: n \).

- If \( \tau = \sigma_1 + \sigma_2 \) then \( K \models \tau = \Sigma (S_1 \cup S_2) \).
- If \( \tau = \sigma_1 \cdot \sigma_2 \) then \( K \models \tau = \Sigma \{ x \cdot y : x \in S_1, y \in S_2 \} \subseteq F_n^m \).
- If \( \tau = -\sigma_1 \), then \( K \models \tau = \Sigma (F_n^m \setminus S_1) \).
- If \( \tau = \sigma_1 ; \sigma_2 \), then, for every \( w \in S := \{ y ; z : y \in S_1, z \in S_2 \} \), let

\[
S_w = \{ D_m^\alpha \cdot (CF_{n-1}^m)^\beta : \alpha \in D_m \{ -1, 1 \}, \beta \in F_n^m \{ -1, 1 \}, \beta(w) = 1 \}.
\]

Therefore, \( K \models \tau = \Sigma \bigcup \{ S_w : w \in S \} \).

- If \( \tau = \sigma_1 \), then, for every \( w \in S := \{ \bar{y} : y \in S_1 \} \), let

\[
S_w = \{ D_m^\alpha \cdot (CF_{n-1}^m)^\beta : \alpha \in D_m \{ -1, 1 \}, \beta \in F_n^m \{ -1, 1 \}, \beta(w) = 1 \}.
\]

Therefore, \( K \models \tau = \Sigma \bigcup \{ S_w : w \in S \} \).

\( \square \)

Back to the relativized relation algebras. Let \( k \in \omega \). Theorem 3.1 (i), (ii) can also be used to label the elements of the unit of any relativized relation algebra with normal forms from \( F_k^m \). Let \( \mathfrak{A} \in RRA_\emptyset \) and \( ev \) be some evaluation of \( x_0, \ldots, x_{m-1} \) into \( A \). Suppose that \( V \) is the unit of \( \mathfrak{A} \). Let \( (r, s) \in V \), define \( D_k^{\mathfrak{A}}(r, s) := \tau \), where \( \tau \) is the unique form in \( F_k^m \) such that \( (\mathfrak{A}, \iota, (r, s)) \models \tau \), where, for every term \( \sigma \), we write \( (\mathfrak{A}, ev, (r, s)) \models \sigma \) if and only if \( (r, s) \in [\sigma]_0^A \).

**Remark 3.1.** Suppose that \( k \geq 1 \). Let \( (r, s) \in V \) and suppose that \( U \) is the base of \( \mathfrak{A} \). Note that, in order to determine the normal form in \( F_k^m \) that \( (r, s) \) satisfies in \( (\mathfrak{A}, \iota) \), it is necessary and sufficient to determine the terms from \( F_{k-1}^m \) which the neighbors of \( (r, s) \) satisfy, where the neighbors of \( (r, s) \) is defined as

\[
\text{ngbr}(r, s) = \{ (r, s), (s, r), (r, r), (s, s) \} \cap V \\
\cup \{ (x, w) : (w, y) \in V, w \in U, \{ x, y \} \subseteq \{ r, s \} \} \\
\cup \{ (w, y) : (x, w) \in V, w \in U, \{ x, y \} \subseteq \{ r, s \} \}.
\]
This is so by the definition of normal forms. Indeed, every normal form was determined by the information on the forms of the first smaller degree.

The following definition focuses on how to obtain the information that the normal forms carry from their syntactical construction.

Definition 3.3. Define \( \text{color}_m : F^m \to F^m \) as follows. For every \( k \in \omega \), every \( \beta \in D_m \{-1, 1\} \) and every \( \alpha \in CF_k \{-1, 1\} \), define

\[
\text{color}_m(D_m^\beta \cdot (CF_k)^\alpha) = \text{color}_m(D_m^\beta) = \{ y \in D_m : \beta(y) = 1 \}.
\]

For every term \( \tau \in F^m \), \( \tau \) is said to be white if \( 1' \in \text{color}_m(\tau) \) and is said to be black otherwise.

Definition 3.4. Define \( \text{sub}_m : F^m \to P(2F^m_k) \) as follows:

1. For every \( \tau \in F^m_0 \), \( \text{sub}_m(\tau) = \emptyset \).
2. Let \( k \in \omega \), \( \beta \in D_m \{-1, 1\} \), \( \alpha \in CF_m \{-1, 1\} \) and \( \tau = D_m^\beta \cdot (CF_m^\alpha) \in F^m_{k+1} \). Define \( \text{sub}_m(\tau) = \{ (\sigma_1, \sigma_2) \in 2F^m_k : \beta(\sigma_1; \sigma_2) = 1 \} \).

Definition 3.5. Define the following partial functions \( f_m, R_m, L_m : F^m \to F^m \) as follows:

1. For every \( \tau \in F^m_0 \), \( \tau \not\in \text{dom}(f_m) \cup \text{dom}(R_m) \cup \text{dom}(L_m) \).
2. Let \( k \in \omega \), \( \beta \in D_m \{-1, 1\} \), \( \alpha \in CF_m \{-1, 1\} \) and \( \tau = D_m^\beta \cdot (CF_m^\alpha) \in F^m_{k+1} \).
   (a) If there exists a unique \( \sigma \in F^m_k \) such that \( \alpha(\sigma) = 1 \) then \( \tau \in \text{dom}(f_m) \) and \( f_m(\tau) = \sigma \). Otherwise, \( \tau \not\in \text{dom}(f_m) \).
   (b) If there exists a unique white \( \lambda \in F^m_k \) such that \( \alpha(\sigma; \lambda) = 1 \) for some \( \sigma \in F^m_k \) then \( \tau \in \text{dom}(R_m) \) and \( R_m(\tau) = \lambda \). Otherwise, \( \tau \not\in \text{dom}(R_m) \).
   (c) If there exists a unique white \( \lambda \in F^m_k \) such that \( \alpha(\lambda; \sigma) = 1 \) for some \( \sigma \in F^m_k \) then \( \tau \in \text{dom}(L_m) \) and \( L_m(\tau) = \lambda \). Otherwise, \( \tau \not\in \text{dom}(L_m) \).

4 The atomicity of the free relativized relation algebras

Throughout this section, let \( m \in \omega \) and \( H \subseteq \{ R, S \} \) be arbitrary but fixed. Recall that we are searching for the atoms in \( \mathfrak{F}(m,RRA_H) \). By theorem 3.1 (iv), it is enough to search for the atoms among the normal forms. Our main aim is to prove the following.

Theorem 4.1. \( \mathfrak{F}(m,RRA_H) \) is atomic if and only if \( m = 0 \) and \( H = \{ R, S \} \).

We prove some propositions considering some special cases of the above theorem. The next proposition proves one direction of theorem 4.1.
Proposition 4.1. The free algebra \( \mathfrak{fr}_0 \{ R, S \} \) is finite, hence atomic.

Proof. Let \( H = \{ R, S \} \) and let \( Y = \{ e_1, e_2, m_2, m_3 \} \), where

\[
\begin{align*}
e_1 &= 1^{'} \cdot - (0^{'} ; 0^{'}), \\
e_2 &= 1^{'} \cdot (0^{'} ; 0^{'}), \\
m_2 &= 0^{'} \cdot -(0^{'} ; 0^{'}), \\
m_3 &= 0^{'} \cdot (0^{'} ; 0^{'}). \
\end{align*}
\]

Clearly, for every \( \tau \in F_0^0 \) there exists \( \sigma \in Y \) such that \( \mathfrak{fr}_0 RRA_H | \tau = \sigma \). It is easy to check the following.

\[
\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{m}_2 = m_2 \text{ and } \tilde{m}_3 = m_3.
\]

Also, in \( \mathfrak{fr}_0 RRA_H \) we have

| \( ; \) | \( e_1 \) | \( e_2 \) | \( m_2 \) | \( m_3 \) |
|-----|-----|-----|-----|-----|
| \( e_1 \) | \( e_1 \) | 0 | 0 | 0 |
| \( e_2 \) | 0 | \( e_2 \) | \( m_2 \) | \( m_3 \) |
| \( m_2 \) | 0 | \( m_2 \) | 0 | 0 |
| \( m_3 \) | 0 | \( m_3 \) | 0 | \( e_2 + m_3 \) |

Therefore, for every \( \tau \in F_1^0 \) there exists \( \sigma \in Y \) such that \( \mathfrak{fr}_0 RRA_H | \tau = \sigma \). Hence, theorem 3.1, (iv), implies that \( \mathfrak{fr}_0 RRA_H \) is \( + \)-generated by \( \{ e_1, e_2, m_2, m_3 \} \).

Hence \( \mathfrak{fr}_0 RRA_H \) is finite, and consequently atomic. In fact, \( Y \) is the set of atoms of \( \mathfrak{fr}_0 RRA_H \), thus the above give a complete description of this free algebra.

To prove the other direction of the theorem, suppose that \( H \neq \{ R, S \} \) or \( m > 0 \). Let \( t = 0^{'} \cdot (0^{'} ; 0^{'}) \cdot ((0^{'} ; 0^{'}); (0^{'} ; 0^{'})) \), we show that there is no atom below \( t \) in \( \mathfrak{fr}_m RRA_H \). To this end, fix a finite number \( q \geq 1 \) and a satisfiable normal form \( \tau \in F_q^m \) such that \( \mathfrak{fr}_m RRA_H | \tau \neq 0 \) \( \tau \leq t \). Our strategy goes through the following steps.

**Step 1** Construct an algebra \( \mathfrak{g}^{\tau, H} \in RRA_H \) and an evaluation \( \iota^{\tau, H} \).

**Step 2** Prove that \( (\mathfrak{g}^{\tau, H}, \iota^{\tau, H}) \) witnesses the satisfiability of the form \( \tau \).

**Step 3** Select a special sequence of edges \( e_q, \ldots, e_0 \in E^{\tau, H} \) such that \( \iota^{\tau, H}(e_q) = \tau \).

**Step 4** Use this sequence to extend \( (\mathfrak{g}^{\tau, H}, \iota^{\tau, H}) \) to \( (\mathfrak{g}^{\tau, H}_+, \iota^{\tau, H}_+) \).

**Step 5** Prove that every element of the above sequence satisfies a term in \( (\mathfrak{g}^{\tau, H}_+, \iota^{\tau, H}_+) \) and another one in \( (\mathfrak{g}^{\tau, H}_+^*, \iota^{\tau, H}_+^*) \) such that both of them are disjoint and each of which is below its labels.

Hence the labels of the sequence selected in Step 2 are not atoms in \( \mathfrak{fr}_m RRA_H \). Therefore, \( \tau \) is not an atom in \( \mathfrak{fr}_m RRA_H \).
Step 1: We first construct a graph $G^*$ that acts as the unit of $\mathfrak{G}^*$. The construction of $G^*$ goes through countably many rounds. At round $n \in \omega$, we construct a graph $G_n = (V_n, E_n)$ with a labeling function $l_n : E_n \to \bigcup \{ F_j^m : 0 \leq j \leq q \}$ and with a depth function $d_n : E_n \cup Id_{V_n} \to \{ 0, \ldots, q \}$. To achieve our purpose, we require our constructed graphs to obey the following “consistency” conditions. For every $n \in \omega$ and every $u, v, w \in V_n$ such that $(u, v) \in E_n$ and $d_n(u, v) = k$ for some $k \leq q$, we have the followings.

1. $E_n$ is an $H$-relation on $V$, $V_n \subseteq V_{n+1}$, $E_n \subseteq E_{n+1}$, $l_n \subseteq l_{n+1}$ and $d_n \subseteq d_{n+1}$.

2. $l_n(u, v) \in F_k^m$ and $1' \in \text{color}_m(l_n(u, v))$ if and only if $u = v$.

3. If $(v, u) \in E_n$ then $d_n(v, u) \in \{ k - 1, k, k + 1 \}$.

4. $d_n(u, u) \in \{ k - 1, k, k + 1 \}$ and $d_n(v, v) \in \{ k - 1, k, k + 1 \}$.

5. If $\{ (u, w), (w, v) \} \subseteq E_n$, then $\{ d_n(u, w), d_n(w, v) \} \subseteq \{ k - 1, k, k + 1 \}$.

6. Suppose that $k \geq 1$, $(v, u) \in E_n$ and $d_n(v, u) = k - 1$. Then either $d_n(u, u) = k$ and $d_n(v, v) = k - 1$, or, $d_n(u, u) = k - 1$ and $d_n(v, v) = k$.

7. Suppose that $(v, u) \notin E_n$ or $(v, u) \in E_n$ but $d_n(v, u) = k$. Then either $d_n(u, u) \neq k + 1$ or $d_n(v, v) \neq k + 1$.

8. If $\{ (u, w), (w, v) \} \subseteq E_n$, then $\mathfrak{G} \models \text{WA} | l_n(u, v) \cdot (l_n(u, w); l_n(w, v)) \neq 0$.

9. $\text{RRA}_H \models J(u, v) := \zeta(u, v) \cdot \zeta(u, u) \cdot \zeta(v, v) \neq 0$. Where $\zeta(u, v)$ and $\zeta(v, v)$ are given as follows. If $(v, u) \in E_n$ then $\zeta(u, v) = l_n(u, v) \cdot l_n(v, u)$, otherwise $\zeta(u, v) = l_n(u, v) \cdot 1$. If $(u, u) \in E_n$ then $\zeta(u, u) = l_n(u, u); l_n(u, v)$, otherwise $\zeta(u, u) = -1 \cdot l_n(u, v)$). If $(v, v) \in E_n$ then $\zeta(v, v) = l_n(u, v); l_n(v, v)$, otherwise $\zeta(v, v) = -l_n(u, v)\cdot 1$.

Conditions (2), (3), (4), (5) are used to show that the edges carry labels that don’t interrupt the desired consistency in the sense of remark 3.1

We construct the graph $G^*$ inductively, conditions (6) and (7) are used for the induction step together with conditions (8) and (9) which allow us to give labels of the degrees we need in a consistent way with our purpose as we shall see.

For constructing $G_0$, pick two different nodes $u, v$. Define, $V_0 = \{ u, v \}$ and $E_0 = \{ (u, v), (v, u) \} \cup \{ (u, u) : \tau \in \text{dom}(L_m) \} \cup \{ (v, v) : \tau \in \text{dom}(R_m) \}$. Define, $d_0(u, v) = d_0^H(u, u) = q$ and $d_0(v, u) = d_0(v, v) = q - 1$. Remember that $\tau$ is satisfiable form, then there exists an algebra $\mathfrak{B} \in \text{RRA}_H$, an evaluation $\iota$ and $(r, s)$ in the unit of $\mathfrak{B}$ such that $(\mathfrak{B}, \iota, (r, s)) \models \tau$. Define, $\iota^H_0(u, v) = \tau$, $\iota^H_0(u, u) = f_0^H(\tau)$, $\iota^H_0(v, u) = D_m^\mathfrak{B}(r, s)$ only if $\tau \in \text{dom}(L_m)$ and $\iota^H_0(v, v) = R_m(\tau)$ only if $\tau \in \text{dom}(R_m)$. We note that $f_0^H(\tau)$ exists because $\text{RRA}_H \models 0 \neq \tau \leq t$.

We have to check that $G_0$ satisfies the consistency conditions. Here, only conditions (8), (9) need a little thought: they are satisfied because $\tau$ is
a nonzero normal form and the labels are given by the algebra $\mathcal{B}$ and the evaluation $\iota$. We need to extend our piece of $G^{r,H}$ in a way that guarantees that $(u,v)$ satisfies $\tau$ at the end of the construction. Whence, we need to add decompositions for all the edges according to the information given by $\text{sub}_m$ of their labels. We don't care about any other information because our strategy goes as follows. Simultaneously with constructing a new edge, we add its converse and its loops according to the information given by $f_m(\tau)$, $L_m(\tau)$ and $R_m(\tau)$.

More generally, let $n \in \omega$ and suppose that $G_n = (V_n, E_n), l_n, d_n$ have been constructed. In the round $n + 1$, let $W_n = \{e \in E_n : d_n(e) \geq 1, \text{ and } e \notin E_{n-1} \text{ if } n \geq 1\}$, i.e., the set of the edges that we have to consider the information given by the black couples in their $\text{sub}_m's$. Let $Y_n = \{(a,b) \in W_n : b, a \notin E_n, \text{ or, } (b,a) \in E_n \text{ and } d_n(a,b) \geq d_n(b,a)\}$. In fact, for every edge $(a,b) \in W_n \setminus Y_n$, we don't need to consider the black forms $(\sigma_1, \sigma_2) \in \text{sub}_m(l_n(a,b)) \cap \{\text{dom}(f_m)\}$. Indeed, $(b,a) \in Y_n$ and we add edges according to the information given by $\text{sub}_m(l_n(b,a))$. The converses of some of these newly added edges are also added and they are enough to carry the information given by $\text{sub}_m(l_n(a,b))$. Let $U$ be an infinite set disjoint from $V_n$. For every $e \in Y_n$, create an injective function

$$g_e : \{(\sigma_1, \sigma_2) \in \text{sub}_m(l_n(e)) : \sigma_1, \sigma_2 \text{ are both black}\} \to U,$$

and for every $e \in W_n \setminus Y_n$, create an injective function

$$g_e : \{(\sigma_1, \sigma_2) \in \text{sub}_m(l_n(e)) \setminus \{\text{dom}(f_m)\} : \sigma_1, \sigma_2 \text{ are both black}\} \to U,$$

such that the ranges of the functions $g_e$$'s are pairwise disjoint. Let $e = (a,b) \in W_n$ and let $w \in \text{Rng}(g_e)$. Then there exists a pair of black forms $(\sigma_1, \sigma_2) \in \text{sub}_m(l_n(e)) \cap \text{dom}(g_e)$ such that $g_e((\sigma_1, \sigma_2)) = w$. Remember that $G_n$ satisfies the consistency condition $(\mathbf{S})$. Therefore, there exists an algebra $\mathcal{B} \in RRA_H$, an evaluation $\iota$ and $(r,s)$ in the unit of $\mathcal{B}$ such that $(\mathcal{B}, \iota, (r,s)) \models l_n^{r,H}(e)$. Since both $\sigma_1, \sigma_2$ are black, then there exists $p$ in the base of $\mathcal{B}$ different from $r$ and $s$ such that $(r,p), (p,s)$ are in the unit of $\mathcal{B}$, $(\mathcal{B}, \iota, (r,p)) \models \sigma_1$ and $(\mathcal{B}, \iota, (p,s)) \models \sigma_2$. If $S \in H$, then $(p,r), (s,p)$ are both in the unit of $\mathcal{B}$. If $R \in H$, then $(p,r)$ is in the unit of $\mathcal{B}$. If $\sigma_1 \in \text{dom}(f_m)$, then $(p,r)$ is in the unit of $\mathcal{B}$ and $(\mathcal{B}, \iota, (p,r)) \models f_m(\sigma_1)$. If $\sigma_2 \in \text{dom}(f_m)$, then $(s,p)$ is in the unit of $\mathcal{B}$ and $(\mathcal{B}, \iota, (s,p)) \models f_m(\sigma_2)$. If $\sigma_1 \in \text{dom}(R_m)$ (hence $\sigma_2 \in \text{dom}(L_m)$), then $(p,p)$ is in the unit of $\mathcal{B}$ and $(\mathcal{B}, \iota, (p,p)) \models R_m(\sigma_1)(= L_m(\sigma_2))$. Suppose that $d_n(e) = k$, for some $k \geq 1$.

**Case 1** Suppose that $(h,a) \in E_n$ and $d_n(h,a) = k - 1$. We define $E_w$ as follows.

$$E_w : = \{(a,w), (w,b)\} \cup \{(w,a) : \sigma_1 \in \text{dom}(f_m) \text{ or } S \in H\} \cup \{(b,w) : \sigma_2 \in \text{dom}(f_m) \text{ or } S \in H\} \cup \{(w,w) : \sigma_1 \in \text{dom}(L_m) \text{ or } R \in H\}.$$
Suppose that \( d_n(a, a) = k \) and \( d_n(b, b) = k - 1 \). Define,

\[
l_w(a, w) = \sigma_1, \quad l_w(w, b) = \sigma_2,
\]

\[
l_w(w, a) = \begin{cases} 
D_{k-1}^{m.a}(p, r) & \sigma_1 \in \text{dom}(f_m) \text{ or } S \in H \\
\text{not defined} & \text{otherwise}.
\end{cases}
\]

\[
l_w(b, w) = \begin{cases} 
f_m(\sigma_2) & \sigma_2 \in \text{dom}(f_m) \\
D_0^{m.a}(s, p) & \sigma_2 \notin \text{dom}(f_m) \text{ and } S \in H \text{ and}
\text{not defined} & \text{otherwise}.
\end{cases}
\]

\[
l_w(w, w) = \begin{cases} 
L_m(\sigma_1) & \sigma_1 \in \text{dom}(L_m) \\
D_0^{m.a}(p, p) & \sigma_1 \notin \text{dom}(L_m) \text{ and } R \in H
\text{not defined} & \text{otherwise}.
\end{cases}
\]

For the other case when \( d_n(a, a) = k - 1 \) and \( d_n(b, b) = k \), we define \( E_w \) and \( l_w \) in a symmetric way to above. For the depths, define \( d_w(w, w) = k - 2 \), if \( k \geq 2 \), and \( d_w(w, w) = 0 \) otherwise. For every edge \( e \in E_w \), \( d_w(e) \) is defined to be the degree of the normal form \( l_w(e) \).

![Diagram](image.png)

**Figure 1**: Case 1

**Case 2** Suppose that \((b, a) \notin E_n\), or, \((b, a) \in E_n\) and \(d_n(b, a) \geq k\). We have one of the following cases.
(i) Suppose that \( d_n(a,a) \neq k + 1 \) and \( d_n(b,b) \neq k + 1 \). Let
\[
E_w := \{(a,w),(w,b)\}
\]
\[
\cup \{(w,a) : \sigma_1 \in \text{dom}(f_m) \text{ or } S \in H\}
\]
\[
\cup \{(b,w) : \sigma_2 \in \text{dom}(f_m) \text{ or } S \in H\}
\]
\[
\cup \{(w,w) : \sigma_1 \in \text{dom}(L_m) \text{ or } R \in H\}.
\]

We define \( l_w \) as follows: \( l_w(a,w) = \sigma_1, l_w(w,b) = \sigma_2 \),
\[
l_w(w,a) = D_{k-1}^{3,i}(p,r) \text{ if and only if } \sigma_1 \in \text{dom}(f_m) \text{ or } S \in H,
\]
\[
l_w(b,w) = D_{k-1}^{3,i}(s,p) \text{ if and only if } \sigma_2 \in \text{dom}(f_m) \text{ or } S \in H, \text{ and,}
\]
\[
l_w(w,w) = \begin{cases} L_m(\sigma_1) & \sigma_1 \in \text{dom}(L_m) \\ D_0^{3,i}(p,p) & \sigma_1 \notin \text{dom}(L_m) \text{ and } R \in H \\ \text{not defined} & \text{otherwise.} \end{cases}
\]

Define \( d_w(w,w) = k - 2 \), if \( k \geq 2 \), and \( d_w(w,w) = 0 \) otherwise. For every edge \( e \in E_w \), define \( d_w(e) \) is defined to be the degree of the normal form \( l_w(e) \).

(ii) Suppose that \( d_n(a,a) = k + 1 \), then \( d_n(b,b) \neq k + 1 \). Let
\[
E_w := \{(a,w),(w,b)\}
\]
\[
\cup \{(w,a) : D_{k-1}^{3,i}(p,r) \in \text{dom}(f_m)\}
\]
\[
\cup \{(b,w) : \sigma_2 \in \text{dom}(f_m) \text{ or } S \in H\}
\]
\[
\cup \{(w,w) : D_{k-1}^{3,i}(p,r) \in \text{dom}(L_m) \text{ or } R \in H\}.
\]

Define, \( l_w(a,w) = D_k^{3,i}(r,p), l_w(w,b) = \sigma_2 \),
\[
l_w(w,a) = D_k^{3,i}(p,r) \text{ if and only if } D_k^{3,i}(r,p) \in \text{dom}(f_m),
\]
\[
l_w(b,w) = D_k^{3,i}(s,p) \text{ if and only if } \sigma_2 \in \text{dom}(f_m) \text{ or } S \in H, \text{ and,}
\]
\[
l_w(w,w) = L_m(D_k^{3,i}(r,p)) \text{ if and only if } D_k^{3,i}(r,p) \in \text{dom}(L_m).\]

Similarly, for the case \( d_n(b,b) = k + 1 \), we define \( E_w \) and \( l_w \) in the same spirit of the above item. Define \( d_w(w,w) = k - 1 \) and, for every edge \( e \in E_w \), \( d_w(e) \) is defined as expected.

Define the graph \( G_{n+1} := (V_{n+1}, E_{n+1}) \), the labeling \( l_{n+1} \) and the depth \( d_{n+1} \) as follows.

\[
V_{n+1} = V_n \cup \{\text{Rng}(g_e) : e \in W_n\}, \quad E_{n+1} = E_n \cup \{E_w : w \in V_{n+1} \setminus V_n\},
\]
\[
l_{n+1} = l_n \cup \{l_w : w \in V_{n+1} \setminus V_n\}, \quad d_{n+1} = d_n \cup \{d_w : w \in V_{n+1} \setminus V_n\}.
\]

Note that, the special choices of the labels and the depths of the new edges guarantee that \( G_{n+1} \) is subjected to the consistency conditions listed above.
In fact, we used the assumption that $G_n$ satisfies these conditions to get some algebras which satisfy some pieces of $G_n$ then we used these algebras to label the extra edges added to these pieces to get $G_{n+1}$ satisfying the required conditions. We continue building the graph $G^{τ,H}$ following the same argument by considering the information given by $sub_m$ of the labels of the edges. So $G^{τ,H}$ is constructed, basically, by knitting particular pieces of some members of $RRA_H$ and by assigning compatible depths caring remark 3.3. This is what we targeted by the consistency conditions. We note that these choices of the depths are not the only possible choices, but with these choices we could reach our aim. Also, one may notice that the depths of the loops are the keys we use to follow remark 3.3. We note that the resulting graph might be infinite independently from the choices of the depths.

Let $G^{τ,H} = (V^{τ,H}, E^{τ,H})$, where $V^{τ,H} := \bigcup \{V_n : n \in \omega \}$ and $E^{τ,H} := \bigcup \{E_n : n \in \omega \}$. We still need the depths and the labels, let $d^{τ,H} = \bigcup \{d_n : n \in \omega \}$ and $l^{τ,H} = \bigcup \{l_n : n \in \omega \}$. Now, define $\mathfrak{G}^{τ,H}$ as the full $RRA_H$ algebra with unit $E^{τ,H}$,

$$
\mathfrak{G}^{τ,H} = \langle \mathcal{P}(E^{τ,H}), \cap, \cup, \emptyset, E^{τ,H}, \{E^{τ,H}\}, \cdot(\cdot^{τ,H}) \rangle.
$$

By the consistency conditions every $E_m$ is an $H$-relation on $V_n$. Consequently, $\mathfrak{G}^{τ,H} \in RRA_H$ as desired. Define an evaluation of the free variables $x_0, \ldots, x_{m-1}$ as follows. For every $i < m$, define $\iota^{τ,H}(x_i) = \{ e \in E^{τ,H} : x_i \in color_m(l^{τ,H}(e)) \}$. Now, we need to prove the following proposition.

**Step 2:** In the next proposition, we prove that every edge in $E^{τ}$ satisfies its label in $(\mathfrak{G}^{τ,H}, \iota^{τ,H})$. Therefore, in particular, The unique edge $(u, v)$ satisfies $τ$ in $(\mathfrak{G}^{τ,H}, \iota^{τ,H})$.

**Proposition 4.2.** For every edge $e \in E^{τ,H}$, $(\mathfrak{G}^{τ,H}, \iota^{τ,H}, e) \models l^{τ,H}(e)$.

**Proof.** We make use of the consistency conditions 2)-(9). Let $e = (e_0, e_1) \in E^{τ,H}$. Then, by condition 9, we have $\mathfrak{G}_m RRA_H \models l^{τ,H}(e) \neq 0$. For every $0 \leq k \leq d^{τ,H}(e)$, let $tag^k_H(e)$ be the unique term in $F_{k}^{m}$ such that $\mathfrak{G}_m RRA_H = l^{τ,H}(e) \leq tag^k_H(e)$. For every $d^{τ,H}(e) \leq k \leq q$, let $tag^k_H(e) := l^{τ,H}(e)$. It suffices to prove that

$$
(\forall e \in E^{τ,H}) (\forall 0 \leq k \leq q) (\mathfrak{G}^{τ,H}, \iota^{τ,H}, e) \models tag^k_H(e).
$$

For this, we use induction on $k$. Condition 2) and the special choice of $\iota^{τ,H}$ ensure that $(\mathfrak{G}^{τ,H}, \iota^{τ,H}, e) \models tag^0_H(e)$, for every $e \in E^{τ,H}$. Suppose that, for some $0 \leq k \leq q - 1$, $(\mathfrak{G}^{τ,H}, \iota^{τ,H}, e) \models tag^k_H(e)$ for every $e \in E^{τ,H}$. Let $(e_0, e_1) \in E^{τ,H}$, if $k \geq d^{τ,H}(e)$ then $(\mathfrak{G}^{τ,H}, \iota^{τ,H}, e) \models tag^k_H(e) = l^{τ,H}(e)$. So, we can assume that $k < d^{τ,H}(e)$ (then $j := d^{τ,H}(e) \geq 1$). Let $α = D_m \{-1, 1\}$, $β = CF^{m}_{k} \{-1, 1\}$ and $β' = CF^{-1}_{j-1} \{-1, 1\}$ be such that

$$
tag^k_{k+1}(e) := D_m^α \cdot (CF^{m}_{k})^β \text{ and } σ := D_m^α \cdot (CF^{-1}_{j-1})^{β'}.
$$
By condition \([2]\) and the special choice of the evaluation \(e^{\tau,H}\), we have
\[
(\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models D_m^\alpha.
\]
Let \(\gamma \in F_m^m\). Suppose that \(\beta(\bar{\gamma}) = 1\). Then there exists \(\gamma_1 \in F_{m-1}^m\) such that \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\gamma_1 \leq \gamma)\) and \(\beta(\gamma_1) = 1\), i.e., \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_1) \neq 0\).

By conditions \([9]\) and \([3]\), there exists \(l \geq j - 1\), \(\gamma_2 \in F_m^m\) and an edge such that \((e_1, e_0) \in E^{\tau,H}, \tau^{\tau,H}(e_1, e_0) = \gamma_2\) and \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_2) \neq 0\).

Let \(\gamma_3\) be the unique term in \(F_{m-1}^m\) such that \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_3) \neq 0\) and, consequently, \(\beta(\gamma_3) = 1\). Recall that \(\beta(\bar{\gamma}_3) = 1\), then \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_3 \cdot \gamma_1) = \sigma \cdot \gamma_3 \cdot \gamma_1 \neq 0\). By theorem \([3,1][1]\) this happens only if \(\gamma_1 = \gamma_3\). Hence, \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_3) \neq 0\).

By induction hypothesis, we have \((\mathcal{G}^{\tau,H}, e^{\tau,H}, (e_1, e_0)) \models \text{tag}_{g_k}^{\tau,H}, e \models \bar{\gamma}\) and \((\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \gamma\).

Conversely, suppose that \(\beta(\bar{\gamma}) = -1\). Assume toward a contradiction that there exist \(l \geq j - 1\), \(\gamma_1 \in F_m^m\) such that \((e_1, e_0) \in E^{\tau,H}, \tau^{\tau,H}(e_1, e_0) = \gamma_1\) and \((\mathcal{G}^{\tau,H}, e^{\tau,H}, (e_1, e_0)) \models \gamma\).

By the induction hypothesis and theorem \([3,1][1]\), we should have \(\text{tag}_{g_k}^{\tau,H}, e \models \gamma\). By condition \([9]\), \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_1) \neq 0\). Hence, \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \gamma_1 \cdot \gamma_3) \neq 0\). which makes a contradiction. Therefore,
\[
(\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \gamma \iff \beta(\bar{\gamma}) = 1.
\]

Let \(\lambda \in F_m^m\) be a white term and \(\gamma \in F_m^m\) be a black term. Suppose that \(\beta(\lambda; \gamma) = 1\). Then there exist a white \(\lambda_1 \in F_{m-1}^m\) and a black \(\gamma_1 \in F_{m-1}^m\) such that \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_1 \leq \lambda)\) and \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\gamma_1 \leq \gamma)\) and \(\beta(\lambda_1; \gamma_1) = 1\).

Hence, \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \lambda_1 \cdot \gamma_1) \neq 0\). But \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_1; \gamma_1 \leq \gamma_1)\) and \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \lambda_1; \gamma_1) \neq 0\).

By conditions \([9]\) and \([4]\) there exist \(l \geq j - 1 \geq k\) and \(\lambda_2 \in F_m^m\) such that \((e_1, e_0) \in E^{\tau,H}, \tau^{\tau,H}(e_0, e_0) = \lambda_2\) and \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_2; \sigma) \neq 0\). Let \(\lambda_3\) be the unique term in \(F_{m-1}^m\) such that \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_2 \leq \lambda_3)\). Hence, \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_3; \gamma_1) \neq 0\) and \(\beta(\lambda_3; \gamma_1) = 1\) recalling \(\beta(\lambda_3; \gamma_1) = 1\), then \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\lambda_3 \cdot \gamma_1) \neq 0\).

By theorem \([3,1][1]\), this means that \(\lambda_1 = \lambda_3\) and \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \lambda_3) \neq 0\).

By induction, we have \((\mathcal{G}^{\tau,H}, e^{\tau,H}, (e_1, e_0)) \models \text{tag}_{g_k}^{\tau,H}, e \models \gamma\). Therefore, \((\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \gamma\).

Conversely, suppose that \(\beta(\lambda; \gamma) = -1\) and \((\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \gamma\). Assume toward a contradiction that there exist \(l \geq j - 1\), \(\lambda_1 \in F_m^m\) such that \((e_0, e_0) \in E^{\tau,H}, \tau^{\tau,H}(e_0, e_0) = \lambda_1\) and \((\mathcal{G}^{\tau,H}, e^{\tau,H}, (e_0, e_0)) \models \lambda\).

By the induction hypothesis and theorem \([3,1][1]\), we should have \(\text{tag}_{g_k}^{\tau,H}, e \models \gamma\). By condition \([9]\), \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \lambda_1; \sigma) \neq 0\). Hence, \(\mathfrak{f}\tau \mathcal{t}_m RRA_H \models (\sigma \cdot \lambda_1; \sigma) \neq 0\), which makes a contradiction. Therefore,
\[
(\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \lambda; \gamma \iff \beta(\lambda; \gamma) = 1.
\]

Similarly,
\[
(\mathcal{G}^{\tau,H}, e^{\tau,H}, e) \models \gamma; \lambda \iff \beta(\gamma; \lambda) = 1.
\]
\(\exists w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that \(\exists w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that 

By the construction and condition \([5]\), there exist a node \(w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that 

\[\exists w \in g_e, j_1, j_2 \geq l - 1\] and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that 

there exist a node \(w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that 

By induction, we have \((\Theta^{\tau,H}, \tau, H, (e_0, w)) \models \text{tag}_{k-1}^{\tau,H}((e_0, w)) = \gamma_1\) and \((\Theta^{\tau,H}, \tau, H, (w, e_1)) \models \text{tag}_{k-1}^{\tau,H}((w, e_1)) = \gamma_2\). Therefore, \((\Theta^{\tau,H}, \tau, H, e) \models \gamma_1; \gamma_2\) Conversely, Suppose that \(\beta(\gamma_1; \gamma_2) = -1\) and assume toward a contradiction that there exists a node \(w \in V^{\tau,H}, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) and \(\gamma_3 \in F^m\) such that both \((e_0, w), (w, e_1) \in E^{\tau,H}, \tau, H, (e_0, w) = \gamma_1, \tau, H, (w, e_1) = \gamma_2\) \((\Theta^{\tau,H}, \tau, H, (e_0, w)) \models \gamma_1\) and \((\Theta^{\tau,H}, \tau, H, (w, e_1)) \models \gamma_2\). By a similar argument, using condition \([8]\), 

one can show a contradiction. Therefore, 

\[(\Theta^{\tau,H}, \tau, H, e) \models \gamma_1; \gamma_2 \iff \beta(\gamma_1; \gamma_2) = 1. \quad (5)\]

By \([1]\), \([2]\), \([3]\), \([4]\) and \([5]\), it follows that \((\Theta^{\tau,H}, \tau, H, e) \models \text{tag}_{k-1}^{\tau,H}(e)\), as desired. 

\[\square\]

The algebra \(\Theta^{\tau,H}\) and the evaluation \(\tau, H\) have been constructed and were shown to witness satisfiability of \(\tau\). Now, we carry on with our plan and move to the next step.

**Step 3:** We need to find a sequence (zigzag!) of special edges.

**Definition 4.1.** An edge \((a, b) \in E^{\tau,H}\) is said to be

- a useful edge if \((b, a) \in E^{\tau,H}, d^{\tau,H}(b, a) < d(a, b)\), and either \((a, b)\) is the unique edge in \(E^{\tau,H}\) whose depth is \(q\) and \(a \neq b\), or, there exists a unique node \(w \in V^{\tau,H}\) such that \(\{(a, w), (w, b), (w, a), (b, w)\} \subseteq E^{\tau,H}\) and \(d^{\tau,H}(a, w), d^{\tau,H}(w, a), d^{\tau,H}(w, b)\) are all greater than or equal to \(d^{\tau,H}(b, w)\).

- a side edge if \(d^{\tau,H}(a, b) = 0, (b, a) \in E^{\tau,H}\) if and only if \(S \in H, \{(a, a), (b, b)\} \subseteq E^{\tau,H}\) if and only if \(R \in H\) and \(d(a, a) = d(b, b) = 0\).

By the above definition, the unique edge \((u_q, v_q) := (u, v) \in E_0\) with label \(\tau\) and depth \(q\) is useful edge. Recall that \(\exists w \in V^{\tau,H}\) such that \(\{(u_q, u_q), (u_q, v_q), (v_q, v_q)\} \subseteq E_1\). By the construction (Case 1), there exists an edge, \(e_{q-1} = (u_q, v_q) \subseteq E_1\) that is useful and \(d^{\tau,H}(e_{q-1}) = q-1\). Continuing in the same manner, we get a sequence of useful edges \(e_q = (u_q, v_q), \ldots, e_1 = (u_1, v_1)\) in \(E^{\tau,H}\), a side edge \(e_0 = (u_0, v_0)\) and a sequence of nodes \(w_q, \ldots, w_1 \in V^{\tau,H}\) such that

(a) \(\tau, H(e_q) = \tau\) and \(d^{\tau,H}(e_j) = j\) for every \(0 \leq j \leq q\).

(b) \(e_k \in \{(u_{k+1}, w_{k+1}), (w_{k+1}, v_{k+1})\}\) for every \(0 \leq k < q\).

(c) For every \(0 < k \leq q\) and every node \(y \in V^{\tau,H} \setminus \{w_k\}\), if \(\exists w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) such that 

\(\exists w \in g_e, j_1, j_2 \geq l - 1\) and \(\gamma'' \in F^m\) such that 

then \(z\) is the unique node for the useful edge \(e_k\) mentioned in the above definition.
Step 4: We note that the selection of the sequence $e_q, \ldots, e_0$ is not unique. The idea of showing that $\tau$ is not an atom of $\mathfrak{r}_m RRA_H$ is as follows. We extend $G^{\tau,H}$ to $G^{\tau,H}_+$ as follows. Recall that we have either $m \neq 0$ or $H \neq \{R, S\}$.

Suppose that $m \geq 1$ Pick a brand new node $h$ and let $V^{\tau,H}_+ = V^{\tau,H} \cup \{h\}$.
Define $E^{\tau,H}_+$ as follows.

$$E^{\tau,H}_+ = E^{\tau,H} \cup \{(u_0, h), (h, v_0)\} \cup \{(h, h) : R \in H\} \cup \{(h, u_0) : S \in H\} \cup \{(v_0, h) : S \in H\}$$

We extend the labels as follows. For every $e \in E^{\tau,H}_+$, let $l^{\tau,H}_+(e) = l^{\tau,H}(e)$. If there is no $Z \in V^{\tau,H}$ with $\{(u_0, z), (z, v_0)\} \subseteq E^{\tau,H}$ (in other words, if $S \not\in H$), define

$$l^{\tau,H}_+(u_0, h) = l^{\tau,H}(h, u_1) = 0' \cdot \prod_{i \in m} x_i.$$ 

If there is $Z \in V^{\tau,H}$ with $\{(u_0, z), (z, v_0)\} \subseteq E^{\tau,H}$, then $Z$ is unique because $(u_0, v_0)$ is a side edge of $G^{\tau,H}$. Since $m \geq 1$, then there exists $\gamma_1, \gamma_2 \in F_0^m$ such that $\text{color}_m(\gamma_1) \neq \text{color}_m(\tau^{+H}(u_0, z))$ and $\text{color}_m(\gamma_2) \neq \text{color}_m(\tau^{+H}(u, z, u_1))$. Define,

$$l^{\tau,H}_+(u_0, h) = \gamma_1 \text{ and } l^{\tau,H}_+(h, u_1) = \gamma_2.$$

Define,

$$l^{\tau,H}_+(h, u_0) = 0' \cdot \prod_{i \in m} x_i \text{ if and only if } S \in H,$n

$$l^{\tau,H}_+(u_1, h) = 0' \cdot \prod_{i \in m} x_i \text{ if and only if } S \in H$$
and

$$l^{\tau,H}_+(h, h) = 1' \cdot \prod_{i \in m} x_i \text{ if and only if } R \in H.$$ 

The depths are extended in the natural way, $d^{\tau,H}_+(e) = d^{\tau,H}(e)$, for every $e \in \text{dom}(d^{\tau,H})$, $d^{\tau,H}_+(h, h) = 0$ and $d^{\tau,H}_+(e) = d$ if and only if $l^{\tau,H}_+(e) \in F_0^m$, for every $e \in E^{\tau,H}_+ \setminus E^{\tau,H}$ and every $d \leq q$.

Suppose $m = 0$ and $S \not\in H$. Hence, extend $G^{\tau,H}$ by adding the converse of $e_0$. Let $V^{\tau,H}_+ := V^{\tau,H}$ and $E^{\tau,H}_+ = E^{\tau,H} \cup \{(v_0, u_0)\}$. Define $l^{\tau,H}_+(v_0, u_0) := l^{\tau,H}(e)$, for every $e \in E^{\tau,H}$, and $l^{\tau,H}_+(v_0, u_0) = 0' \cdot \prod_{i \in m} x_i$. The depths are given by $d^{\tau,H}_+ = d^{\tau,H} \cup \{(v_0, u_0, 0)\}$.

Suppose $m = 0$ and $R \not\in H$. We extend $G^{\tau,H}$ by adding the loop of $(w_1, w_1)$.
Let $V^{\tau,H}_+ := V^{\tau,H}$ and $E^{\tau,H}_+ = E^{\tau,H} \cup \{(w_1, w_1)\}$. Define $l^{\tau,H}_+(w_1, w_1) := l^{\tau,H}(e)$, for every $e \in E^{\tau,H}$, and $l^{\tau,H}_+(w_1, w_1) = 1' \cdot \prod_{i \in m} x_i$. The depths remain as they were, i.e., $d^{\tau,H}_+ = d^{\tau,H}$.
Let $\mathcal{G}_{+}^{\tau,H} = \langle \mathcal{P}(E_{+}^{\tau,H}), \cap, \cup, \setminus, \emptyset, E_{+}^{\tau,H}, \{E_{+}^{\tau,H}, \setminus(E_{+}^{\tau,H}) \rangle$. Clearly $\mathcal{G}_{+}^{\tau,H} \in RRA_{H}$.

Define the evaluation $\iota_{+}^{\tau,H} : \{x_{0}, \ldots, x_{m-1}\} \rightarrow \mathcal{P}(E_{+}^{\tau,H})$ as follows. For every $i \in m$, $\iota_{+}^{\tau,H}(x_{i}) = \{e \in E_{+}^{\tau,H} : x_{i} \in color_{m}(\iota_{+}^{\tau,H}(e))\}$. The same argument used in proposition 4.2 can be used to prove the following.

$$\forall e \in E_{+}^{\tau,H}, (\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, e) \models \iota_{+}^{\tau,H}(e). \quad (6)$$

**Step 5:** Recall the useful edges $e_{0}, \ldots, e_{q}$ and the side edge $e_{0}$. For every $0 \leq j \leq q$, let $son(e_{j})$ and $daughter(e_{j})$ be the unique terms in $E_{+}^{\tau,H}$ such that

$$(\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, e_{j}) \models son(e_{j}) \text{ and } (\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, e_{j}) \models daughter(e_{j}).$$

Hence, for every $0 \leq j \leq q$, $son(e_{j})$ and $daughter(e_{j})$ are non-zero forms and, by lemma 1.2 and equation (6) above, each of which is below the label of $e_{j}$ in $\mathfrak{A}_{m}RRA_{H}$. Now, we are ready to show that $\tau$ is not an atom in $\mathfrak{A}_{m}RRA_{H}$.

**Proposition 4.3.** For every $0 \leq j \leq q$,

$$\mathfrak{A}_{m}RRA_{H} \models son(e_{j}) \cdot daughter(e_{j}) = 0.$$  

**Proof.** By induction on $j$. By the special construction of $G_{+}^{\tau,H}$, it is clear that $\mathfrak{A}_{m}RRA_{H} \models son(e_{0}) \cdot daughter(e_{0}) = 0$. Suppose that, for some $j \leq q - 1$, $\mathfrak{A}_{m}RRA_{H} \models son(e_{j}) \cdot daughter(e_{j}) = 0$. We may assume that $e_{j} = (u_{j+1}, w_{j+1})$. Let $\sigma = D_{+}^{\tau,H}(w_{j+1}, v_{j+1})$. By condition (c) and without loss of generality, we may assume that there is NO node $y \in V_{+}^{\tau,H} \setminus \{w_{j+1}\}$ with $(\{u_{j+1}, y\}, \{y, v_{j+1}\}) \subseteq E_{+}^{\tau,H}$ and

$$(\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (u_{j+1}, y)) \models son(e_{j}), \quad (\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (y, v_{j+1})) \models \sigma,$$

$$(\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (u_{j+1}, y)) \models son(e_{j}), \quad (\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (y, v_{j+1})) \models \sigma.$$  

Hence,

$$(\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (u_{j+1}, v_{j+1})) \models son(e_{j}) \cdot \sigma \text{ but } (\mathcal{G}_{+}^{\tau,H}, \iota_{+}^{\tau,H}, (u_{j+1}, v_{j+1})) \models -(son(e_{j}) \cdot \sigma).$$

Therefore, $\mathfrak{A}_{m}RRA_{H} \models son(e_{j+1}) \cdot daughter(e_{j+1}) = 0$, as desired.  

In particular, $\mathfrak{A}_{m}RRA_{H} \models son(e_{q}) \cdot daughter(e_{q}) = 0$ and, hence, $\tau$ is not an atom in the free algebra $\mathfrak{A}_{m}RRA_{H}$. Recall that $\tau$ was an arbitrary normal form with $\mathfrak{A}_{m}RRA_{H} \models 0 \neq \tau \leq t$. Hence, proving that $\tau$ is not an atom in $\mathfrak{A}_{m}RRA_{H}$ yields to the following proposition.

**Proposition 4.4.** Suppose that $m \neq 0$ or $H \neq \{R, S\}$. There is no atom in the free algebra $\mathfrak{A}_{m}RRA_{H}$ that is below $t$. Therefore, $\mathfrak{A}_{m}RRA_{H}$ is not atomic.

The remaining direction of theorem 4.1 follows from proposition 4.4. It would be nice to list all the atoms of the non atomic $\mathfrak{A}_{m}RRA_{H}$’s. However, we believe that the free algebra $\mathfrak{A}_{m}RRA_{H}$ contains finitely many atoms if and only if $S \in H$. 

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