n-dimensional PDM-damped harmonic oscillators: Linearizability, and exact solvability

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Abstract: We consider position-dependent mass (PDM) Lagrangians/Hamiltonians in their standard textbook form, where the long-standing gain-loss balance between the kinetic and potential energies is kept intact to allow conservation of total energy (i.e., \(L = T - V, H = T + V,\) and \(dH/dt = dE/dt = 0\)). Under such standard settings, we discuss and report on \(n\)-dimensional PDM damped harmonic oscillators (DHO). We use some \(n\)-dimensional point canonical transformation to facilitate the linearizability of their \(n\)-PDM dynamical equations into some \(n\)-linear DHOs’ dynamical equations for constant mass setting. Consequently, the well know exact solutions for the linear DHOs are mapped, with ease, onto the exact solutions for PDM DHOs. A set of one-dimensional and a set of \(n\)-dimensional PDM-DHO illustrative examples are reported along with their phase-space trajectories.

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I. INTRODUCTION

A large number of research on classical and quantum particles endowed with "effective" position-dependent mass (PDM) has been carried out [1-48] ever since the introduction of the Mathews-Lakshmanan (M-L) oscillator [1]. Such PDM concept manifestly introduces a mathematically challenging problem in both classical and quantum mechanics. On the quantum mechanical side, the PDM von Roos Hamiltonian [21] is indulged with a parametric ordering ambiguity problem, infected by the non-unique representation of the said Hamiltonian (c.f., e.g. [25, 33, 34, 36] and references cited therein). Only very recently that a proper definition for the position-dependent mass momentum operator is introduced by Mustafa and AlGadhi [29], resolving hereby the ordering ambiguity problem. Moreover, using the von Roos Hamiltonian [21], Mustafa [25] has shown that the traditional textbook commutation relation between the PDM creation \(\hat{A}^+\) and annihilation \(\hat{A}\) operators, \([\hat{A}, \hat{A}^+] = 1\), offers a unique parametric ordering (known in the literature as MM-ordering [34]) for the PDM von Roos Hamiltonian. On the classical mechanical side, however, the mean stream of research focuses on a set of interesting non-standard/non-traditional PDM Lagrangian/Hamiltonian settings [1,20], where the intimate long-standing gain-loss balance between the kinetic and potential energies is ignored in the process, and consequently the total energy of the dynamical system is no longer a conserved quantity [7, 24].
The position-dependent mass terminology, nevertheless, should be understood as a metaphorical consequential manifestation of some position-dependent deformation in the coordinate system or, equivalently, a position-dependent deformation in the velocity of the particle at hand. Yet, it has been reported that a point mass moving within the curved coordinates/space transforms, effectively, into a PDM in Euclidean coordinates/space (c.f., e.g., [12, 16, 23, 24, 29] and references cited therein). This understanding leads, in fact, to an interesting set of standard/traditional PDM Lagrangians/Hamiltonians with their exact solutions are inferred from the well known exactly solvable dynamical systems (e.g., Mustafa [47]). That is, a standard Lagrangian/Hamiltonian (i.e., \( L = T - V, \ H = T + V \)) in the generalized coordinates

\[
L (q, \dot{q}, t) = \frac{1}{2} m_o \dot{q}^2 - V (q) \iff H (q, p_q, t) = \frac{p_q^2}{2m_o} + V (q) ; \ p_q = m_o \dot{q} \tag{1}
\]

would transform into

\[
L (x, \dot{x}, t) = \frac{1}{2} m_o m (x) \dot{x}^2 - V (q (x)) \iff H (x, p_x, t) = \frac{p_x^2}{2m_o m (x)} + V (q (x)) ; \ p_x = m (x) \dot{x}, \tag{2}
\]

under the point canonical transformation

\[
q \rightarrow q (x) = \int \sqrt{m (x)} dx = \sqrt{Q (x)} x \iff \sqrt{m (x)} = \sqrt{Q (x)} \left( 1 + \frac{Q'(x)}{2Q(x)} x \right). \tag{3}
\]

This would in turn yield the Euler-Lagrange dynamical invariance

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \tag{4}
\]

between the two dynamical systems. For example, the harmonic oscillator Lagrangian

\[
L (q, \dot{q}, t) = \frac{1}{2} m_o \dot{q}^2 - \frac{1}{2} m_o \omega^2 q^2 \iff L (x, \dot{x}, t) = \frac{1}{2} m_o m (x) \dot{x}^2 - \frac{1}{2} m_o \omega^2 Q (x) x^2 \tag{5}
\]

would yield the Euler-Lagrange dynamical invariance

\[
\ddot{q} + \omega^2 q = 0 = \ddot{x} + \frac{m' (x)}{2m (x)} \dot{x}^2 + \sqrt{Q (x)} m^2 (x) \omega^2 x \tag{6}
\]

where

\[
q = A \cos (\omega t + \varphi) = \sqrt{Q (x)} x. \tag{7}
\]

Moreover, the long-standing gain-loss balance between the kinetic and potential energies is kept intact (through the correlation in (3) between the two dimensionless scalar multipliers \( \sqrt{m (x)} \) and \( \sqrt{Q (x)} \)). This would, in turn, facilitate exact solvability of the dynamical system at hand. For more details on this issue the reader may refer to Mustafa [47] for illustrative examples on isochronous PDM oscillators in one- and \( n \)-dimensions. Moreover, equation (6) offers a straightforward linearization of the quadratic Liénard-type differential equation \( \ddot{x} + f (x) \dot{x}^2 + g (x) = 0 \), with \( f (x) = m' (x) / 2m (x) \) and \( g (x) = \sqrt{Q (x)} / m (x) \omega^2 x \), into a simple linear oscillator \( \ddot{q} + \omega^2 q = 0 \) and inherits its exact solution.

It would be interesting, therefore, to study PDM classical particles moving under the influence of a conservative oscillator potential force field \( V (q) = m_o \omega^2 q^2 / 2 \) and a non-conservative Rayleigh dissipative force field \( R (\dot{q}) = b \dot{q}^2 \)
(an elegant and convenient tool to accommodate dissipative forces) and described by the damped harmonic oscillator (DHO) dynamical equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = 0 \iff \ddot{q} + 2\eta \omega \dot{q} + \omega^2 q = 0; \quad \eta = \frac{b}{2m_\omega \omega}, \tag{8}
\]

where \( \eta > 0 \) is the damping ratio and \( \eta = 0 \) corresponds to the linear harmonic oscillator. Apparently, this equation of motion is a linear Liénard-type differential equation (i.e., \( \ddot{q} + h(q) \dot{q} + g(q) = 0 \), \( h(q) = 2\eta \omega \) and \( g(q) = \omega^2 q \)) that admits an exact solution of the form

\[
q(t) = e^{-\eta t} [A \cosh(\beta t) + B \sinh(\beta t)]; \quad \beta = \sqrt{\eta^2 - 1}, \tag{9}
\]

and subjected to the initial values of the problem at hand. Moreover, it is worthwhile to mention that the values of the damping ratio \( \eta \) determine the nature of damping. Namely, \( \eta > 1 \) addresses "over-damping", \( \eta = 1 \) addresses "critical damping", and \( \eta < 1 \) addresses "under-damping". Nevertheless, it is obvious that the DHO equation of (8) cannot be obtained from a standard Hamiltonian with a conserved total energy. This should, in turn, not only document a linearization of a mixed Liénard-type differential equation (i.e., \( \ddot{x} + f(x) \dot{x}^2 + h(x) \dot{x} + g(x) = 0; \ f(x) = m'(x)/2m(x), h(x) = 2\eta \omega, \) and \( g(x) = \sqrt{Q(x)/m(x)} \omega^2 x \)) into a linear Liénard-type differential equation (i.e., \( \ddot{q} + \tilde{h}(q) \dot{q} + \tilde{g}(q) = 0, \ \tilde{h}(q) = 2\eta \omega \) and \( \tilde{g}(q) = \omega^2 q \)), but also give the corresponding PDM-deformed DHO’s dynamical system that inherits the exact solvability of the DHO in (9) (to be exemplified in the current methodical proposal below). To the best of our knowledge, such methodical proposal/approach has never been reported elsewhere.

In the current study, we consider (in section II) the \( n \)-dimensional PDM damped harmonic oscillators within the standard textbook format, where the long-standing gain-loss balance between the kinetic and potential energies is kept intact to allow conservation of total energy (i.e., \( L = T - V, \ H = T + V \), and \( dH/dt = dE/dt = 0 \)). Therein, we use some \( n \)-dimensional point canonical transformation to facilitate the linearizability of their \( n \)-PDM dynamical equations into some \( n \)-linear DHOs’ dynamical equations for constant mass setting. Hence, the well know exact solutions for the linear DHOs are mapped, with ease, onto the exact solutions for PDM DHOs. A set of one-dimensional and a set of \( n \)-dimensional PDM-DHO illustrative examples are reported, along with their phase-space trajectories, in section III. Our concluding remarks are given in section IV.
II. PDM DAMPED HARMONIC OSCILLATORS IN $n$-DIMENSIONS

Consider a classical particle of mass $m_o$ moving, within the generalized $n$-dimensional coordinates, under the influence of the $n$-dimensional harmonic oscillator potential $V(q)$ (i.e., conservative force field) and the $n$-dimensional Rayleigh dissipative force field $R(q)$ (i.e., non-conservative force field) given, respectively, by

$$V(q) = \frac{1}{2} m_o \omega_o^2 \sum_{j=1}^{n} q_j^2,$$

and

$$R(q) = b \sum_{j=1}^{n} \dot{q}_j^2,$$

where $\omega_o = \sqrt{k/m_o}$ is the undamped angular frequency and $b$ is the damping coefficient. Then, the corresponding Lagrangian reads

$$L(q, \dot{q}, t) = \frac{1}{2} m_o \sum_{j=1}^{n} \dot{q}_j^2 - \frac{1}{2} m_o \omega_o^2 \sum_{j=1}^{n} q_j^2,$$

and the $n$-Euler-Lagrange equations of motion become

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = 0; \quad i = 1, 2, \ldots, n \in \mathbb{N}.$$

Consequently, in a straightforward manner, one obtains the $n$-equations of motion

$$\ddot{q}_i + 2\eta \omega_o \dot{q}_i + \omega_o^2 q_i = 0; \quad \eta = \frac{b}{2 m_o \omega_o}, \quad i = 1, 2, \ldots, n \in \mathbb{N},$$

which are known as the damped harmonic oscillators equations of motion, where $\eta$ identifies the damping ratio and the second term represents the effect of damping. In the literature, moreover, this equation is also known to belong to the class of the linear-type Liénard oscillators. Equation (16) on the other hand, admits exact solutions in the forms of

$$q_i(t) = e^{-\eta \omega_o t} \left[ A_i \cosh(\beta t) + B_i \sinh(\beta t) \right]; \quad \beta = \omega_o \sqrt{\eta^2 - 1}.$$

This solution is to satisfy and suit the initial boundary conditions, of course. Throughout the current proposal, we require that $q_i(0) \neq 0$ and $\dot{q}_i(0) = 0$ so that the solutions in (17) reduce to

$$q_i(t) = A_i e^{-\eta \omega_o t} \cosh(\beta t); \quad \beta = \omega_o \sqrt{\eta^2 - 1}.$$

Obviously, the values of the damping ratio $\eta$ in (18) determine the nature/behavior of the solution and suggest the following classifications of the system. That is, for $\eta > 1$ the system is classified as an "Over-damped system" that exponentially decays to its steady/equilibrium state without oscillating (the larger the values of $\eta$ the slower the decay is). For $\eta = 1$ the system is a "Critically-damped system" and decays to its steady/equilibrium state as fast as possible. Finally, for $\eta < 1$ the system is an "Under-damped system" and oscillates with the amplitude gradually decreases to zero.

We may now follow the canonical point transformation introduced by Mustafa [23, 25, 47] and define

$$dq_i = \delta_{ij} \sqrt{m(r)} \, dx_j \iff \frac{\partial q_i}{\partial x_j} = \delta_{ij} \sqrt{m(r)} \iff q_i(r) = \int \sqrt{m(r)} \, dx_i = \sqrt{Q(r)} x_i,$$
This would, with \( m(r) = m(r) \), \( Q(r) = Q(r) \) (to avoid complexity of calculations) and
\[
\hat{q}_i = \frac{\sum_{k=1}^{n} \left( \frac{\partial x_k}{\partial q_i} \right) \hat{x}_k}{\sqrt{\sum_{k=1}^{n} \left( \frac{\partial x_k}{\partial q_i} \right)^2}} \iff \hat{q}_i = \hat{x}_i,
\]
immediately imply, using (19) along with \( \hat{q}_i = \hat{x}_i \), that
\[
\dot{q}(r) = \sqrt{Q(r)r},
\]
and, by using (19),
\[
\dot{q}_i (r) = \sum_{k=1}^{n} \partial_{x_k} q_i (r) \hat{x}_k = \dot{x}_i \sqrt{m(r)} \iff \dot{q}(r) = \sqrt{m(r)} \hat{r}.
\]

It is obvious that the relation between \( m(r) \) and \( Q(r) \) is determined, in a straightforward manner, by (20) and (21) as
\[
\sqrt{m(r)} = \sqrt{Q(r)} \left( 1 + \frac{Q'(r)}{2Q(r)} r \right); r = \sqrt{\sum_{j=1}^{n} x_j^2}.
\]

Here, we have used the assumption that \( r \parallel \hat{r} \iff (\hat{r} \cdot r) r = (\hat{r} \cdot r) \hat{r} = r^2 \hat{r} \) (i.e., no rotational effects are involved in the problem at hand). Consequently, the \( n \) Euler-Lagrange equations (15) would result
\[
\ddot{x}_i + \frac{m'(r)}{2rm(r)} (\hat{r} \cdot r) \dot{x}_i = \dot{\omega}_\sigma \dot{x}_i + \sqrt{Q(r)} \frac{Q(r)}{m(r)} \omega_\sigma^2 x_i = 0.
\]

On the other hand, let us consider a PDM-particle moving in a conservative PDM-deformed oscillator potential force field
\[
V(r) = \frac{1}{2} m_\sigma \omega_\sigma^2 Q(r) \sum_{j=1}^{n} x_j^2
\]
and subjected to a non-conservative PDM Rayleigh dissipation function
\[
\mathcal{R}(r, \dot{r}) = bm(r) \sum_{j=1}^{n} \dot{x}_j^2.
\]

Then the standard PDM-Lagrangian describing this particle reads
\[
L(r, \dot{r}, t) = \frac{1}{2} m_\sigma m(r) \sum_{j=1}^{n} \dot{x}_j^2 - \frac{1}{2} m_\sigma \omega_\sigma^2 Q(r) \sum_{j=1}^{n} x_j^2,
\]
and the \( n \)-PDM Euler-Lagrange equations are given by
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} + \frac{\partial \mathcal{R}}{\partial \dot{x}_i} = 0
\]
to imply the \( n \)-PDM equations of motion
\[
\ddot{x}_i + \frac{m'(r)}{rm(r)} (\hat{r} \cdot r) \dot{x}_i = \frac{m'(r)}{2rm(r)} (\hat{r} \cdot \hat{r}) x_i + 2\eta \omega_\sigma \dot{x}_i + \sqrt{Q(r)} \frac{Q(r)}{m(r)} \omega_\sigma^2 x_i = 0.
\]
Apparently, the Euler-Lagrange equations of (23) (manifestly introduced by the canonical point transformation (19)) and the PDM ones of (28) are still far from securing invariance. We, therefore, appeal to the recently introduced total vector Newtonian invariance amendment, i.e.,

\[ \sum_{i=1}^{n} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i}ight) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i}\right) \dot{x}_i = 0 = \sum_{i=1}^{n} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i}ight) - \frac{\partial L}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i}\right) \dot{x}_i \] (29)

by Mustafa [15, 47], and multiply both equations (23) and (28), by the corresponding unit vectors \( \hat{x}_i \) and sum over \( i = 1, 2, \cdots, n \) to obtain (from both equations)

\[ a + \frac{m'(r)}{2rm(r)} v^2 r + 2 \eta \omega \phi + \sqrt{\frac{Q(r)}{m(r)}} \omega^2 x = 0, \] (30)

where

\[ v^2 = \sum_{j=1}^{n} \dot{x}_j^2, \quad r = \sum_{j=1}^{n} x_j \hat{x}_j. \]

At this point, however, it is obvious that equation (29) is nothings but the total vector presentation of the Euler-Lagrange vector-component dynamical equations. In fact, this procedure identifies the textbook mapping between Euler-Lagrange and Newtonian dynamics. Moreover, both equations, (23) and (28), lead to the same Newtonian dynamical equation (30). Hence, we may now safely decompose equation (30) into its components to read

\[ \ddot{x}_i + \frac{m'(r)}{2rm(r)} x_i \left(\sum_{j=1}^{n} \dot{x}_j^2\right) + 2 \eta \omega \phi \dot{x}_i + \sqrt{\frac{Q(r)}{m(r)}} \omega^2 x_i = 0; \quad i = 1, 2, \cdots, n \in \mathbb{N}. \] (31)

These equations represent the equations of motion for a PDM damped oscillator described by the standard n-dimensional PDM-Lagrangian (26) for a PDM-particle subjected to a PDM Rayleigh dissipative force field (25).

Furthermore, it is unavoidable to mention the manifestly user friendly transition procedure between the Liénard-type differential equations. Namely, in the one-dimension, equation (31) represent a subset of the mixed-type Liénard equation

\[ \ddot{x} + f(x) \dot{x}^2 + h(x) \dot{x} + g(x) = 0 \] (32)

where

\[ f(x) = \frac{m'(x)}{2m(x)}, \quad h(x) = 2 \eta \omega, \quad g(x) = \sqrt{\frac{Q(x)}{m(x)}} \omega^2 x. \] (33)

Which admits linearizability into the one dimensional linear Liénard-type oscillator (16) (i.e., the damped harmonic oscillator equation) through some reverse engineering of our canonical point transformation above. Yet, for \( \eta = 0 \) equation (31) reads the quadratic Liénard-type oscillator which is obviously linearizable into a simple harmonic oscillator equation \( \ddot{q} + \omega_s^2 q = 0; \quad q = A \cos(\omega_s t + \varphi). \)

Under such PDM settings, one would use (26) and obtain

\[ p_i = \frac{\partial L}{\partial \dot{x}_i} = m(r) \dot{x}_i \quad \iff \quad \dot{p}_i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} - \frac{\partial R}{\partial \dot{x}_i}. \] (34)

In what follows we discuss some illustrative examples in one- and n-dimensions.
FIG. 1: For the PDM \( m(x) \) in (35), we show the effect of (a) the damping ratio \( \eta \) (at \( \omega = 1, \lambda = 2, A = 1 \)), (b) the amplitude \( A \) (at \( \omega = 1, \lambda = 0.5, \eta = 0.05 \), under-damped), and (c) the PDM parameter \( \lambda \) (at \( \omega = 1, A = 0.9, \eta = 0.05 \), under-damped), on the coordinate \( x(t) \) of (38) as it evolves in time. The phase trajectories \( p(t) \) vs \( x(t) \) (at \( \omega = 1, A = 0.9, \eta = 0.05 \), under-damped) are shown in (d) for \( \lambda = 1, 2, 3 \), (e) \( \lambda = 0.5 \), and (f) \( \lambda = 4 \).

III. ILLUSTRATIVE EXAMPLES

A. One-dimensional PDM damped oscillators:

1. A velocity deformation \( m(x) = 1/\left(1 + \lambda^2 x^2\right) \)

A PDM of the form

\[
m(x) = \frac{1}{1 + \lambda^2 x^2}
\]  

(35)

would lead, through (22), to

\[
\sqrt{Q(x)}x = \frac{1}{\lambda} \ln \left(\lambda x + \sqrt{1 + \lambda^2 x^2}\right).
\]  

(36)

Using (19) and (18), one would write

\[
q(x) = Ae^{-\omega_x \eta t} \cosh (\beta t) = \frac{1}{\lambda} \ln \left(\lambda x + \sqrt{1 + \lambda^2 x^2}\right),
\]  

(37)

to imply

\[
x(t) = \frac{1}{\lambda} \sinh \left(A\lambda e^{-\omega_x \eta t} \cosh (\beta t)\right),
\]  

(38)
as an exact solution for the one-dimensional form of Eq. (23) that reads the dynamical equation of motion
\[ \ddot{x} - \frac{\lambda^2 x}{1 + \lambda^2 x^2} \dot{x}^2 + \frac{\omega_0^2}{\lambda} \dot{x} + \frac{\omega_0^2}{\lambda} \sqrt{1 + \lambda^2 x^2} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right) = 0. \] (39)

This equation of motion corresponds to the PDM-particle (35) moving in the vicinity of a conservative potential force field
\[ V(x) = \frac{1}{2} \omega_0^2 \left( \frac{1}{\lambda} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right) \right)^2, \] (40)

and feels a non-conservative Rayleigh dissipative force field
\[ R(x, \dot{x}) = \frac{b}{1 + \lambda^2 x^2} \dot{x}^2. \] (41)

Moreover, the corresponding PDM-standard Lagrangian and PDM-Hamiltonian are given by
\[ L = \frac{1}{2} \frac{\dot{x}^2}{1 + \lambda^2 x^2} - \frac{1}{2} \omega_0^2 \left( \frac{1}{\lambda} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right) \right)^2, \] (42)

and
\[ H = \left( 1 + \lambda^2 x^2 \right) \frac{p_x^2}{2} + \frac{1}{2} \omega_0^2 \left( \frac{1}{\lambda} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right) \right)^2. \] (43)

In Figure 1(a) we show the response of the PDM system of (38) to the values of the damping ratio \( \eta \). It is clear that our PDM system of (39) (a Mathews-Lakshmanan type PDM) inherits the traditional/standard constant mass behaviour as it evolves with time. We observe that in the under-damped case, \( \eta < 1 \) (blue curve), the system oscillates with a slightly different frequency than the undamped case, \( \eta = 0 \) (red curve), with its amplitude decreasing to zero. Moreover, for the critical-damping case, \( \eta = 1 \) (green curve), the system exponentially returns to the steady state (i.e., \( x(t) = 0 \)) faster than the over-damping case, \( \eta > 1 \) (purple curve). In Figures 1(b) and 1(c), we show the effects of the amplitude \( A \) and the mass parameter \( \lambda \), respectively, on the behaviour of the under-damping case. Both figures clearly indicate that the frequencies of oscillations are not affected by the values of the amplitude or the mass parameter. Such a behaviour is a signature for isochronous damped harmonic oscillators. In Figures 1(d), 1(e), and 1(f), it is obvious that the PDM phase-space trajectories of the under-damped oscillators shrinks as time goes on and tends to wards the origin (the steady/equilibrium state).

2. A coordinate deformation with singularity: \( Q(x) = 1 / (1 - \lambda x) \)

Such coordinate deformation would result in a PDM-function
\[ m(x) = \frac{1}{4} \frac{(2 - \lambda x)^2}{(1 - \lambda x)^3}. \] (44)

In this case, equations (19) and (18) would result
\[ q^2 = \frac{x^2}{1 - \lambda x} \implies x = x_\pm(t) = \frac{q}{2} \left( -\lambda q \pm \sqrt{\lambda^2 q^2 + 4} \right); \ q = Ae^{-\omega_1 \eta t} \cosh(\beta t), \] (45)

as the exact solution for the one-dimensional form of Eq. (23) that reads the dynamical equation of motion
\[ \ddot{x} - \frac{\lambda (\lambda x - 4)}{2 (\lambda x - 1) (\lambda x - 2)} \dot{x}^2 + \frac{\omega_0^2}{\lambda} \dot{x} + \frac{2 (\lambda x - 1)}{(\lambda x - 2) \omega_0^2} x = 0. \] (46)
Which describes a PDM particle with a deformation in its velocity moving in the vicinity of a conservative PDM-deformed potential force field

\[ V(x) = \frac{\omega_o^2}{2(1 - \lambda x)} x^2, \]  

and feels a non-conservative PDM-deformed Rayleigh dissipative force field

\[ R(x, \dot{x}) = \frac{1}{4} \frac{(\lambda x - 2)^2}{(1 - \lambda x)^3} \dot{x}^2. \]  

Then the standard PDM-Lagrangian and PDM-Hamiltonian that describe this PDM-particle in read

\[ L = \frac{1}{8} \frac{(2 - \lambda x)^2}{(1 - \lambda x)^3} \dot{x}^2 - \frac{\omega_o^2}{2(1 - \lambda x)} x^2 \Rightarrow H = \frac{1}{2} \frac{(1 - \lambda x)^3}{(2 - \lambda x)^3} p_x^2 + \frac{\omega_o^2}{2} \frac{1}{(1 - \lambda x)} x^2. \]  

In Figures 2(a) and 2(b) we observe similar trends of response of the PDM system of \([45]\) to the values of the damping ratio \(\eta\). The PDM phase-space trajectories of the PDM under-damped oscillators, Figures 2(c) and 2(d), are observed to shrink with time.

3. A damped Morse-type PDM-oscillator: \(m(x) = e^{2\lambda x}\)

A velocity deformation of an exponential form \(m(x) = e^{2\lambda x}\) would imply that

\[ Q(x) = \frac{e^{2\lambda x}}{\lambda^2 x^2} (1 - e^{-\lambda x})^2 \Rightarrow q = \frac{1}{\lambda} (e^{\lambda x} - 1) = Ae^{-\omega_c \eta t} \cosh(\beta t). \]  

Which results

\[ x = \frac{1}{\lambda} \ln(\lambda q + 1) \Rightarrow x(t) = \frac{1}{\lambda} \ln(\lambda Ae^{-\omega_c \eta t} \cosh(\beta t)), \]  

as an exact solution that satisfies the corresponding equation of motion

\[ \ddot{x} + \lambda \dot{x}^2 + 2\eta \omega_c \dot{x} + \frac{\omega_o^2}{\lambda} (1 - e^{-\lambda x}) = 0. \]  

This equation belongs to a PDM-particle moving under the influence of a conservative PDM-deformed potential force field

\[ V(x) = \frac{1}{2} \omega_o^2 \frac{e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2, \]  

and a non-conservative PDM-deformed Rayleigh dissipative force field

\[ R(x, \dot{x}) = be^{2\lambda x} \dot{x}^2. \]  

Moreover, the standard PDM-Lagrangian and PDM-Hamiltonian that describe this PDM-particle are given by

\[ L = \frac{1}{2} e^{2\lambda x} \dot{x}^2 - \frac{1}{2} \omega_o^2 \frac{e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2 \Rightarrow H = \frac{1}{2} e^{-2\lambda x} p_x^2 + \frac{1}{2} \frac{\omega_o^2 e^{2\lambda x}}{\lambda^2} (1 - e^{-\lambda x})^2. \]  

In Fig. 3(a), we observe similar trends of response of the PDM system of \([51]\) to the values of the damping ratio, as those of the examples reported above. So are the phase-space trajectories, Fig. 3(b).
FIG. 2: For the PDM (44) we plot the coordinates \( x_\pm(t) \) of the solution (45) at \( A = 1, \omega = 1, \lambda = 1, \) and \( \eta = 0, 0.2, 1.0, 1.5 \), where (a) is for \( x_+(t) \) and (b) for \( x_-(t) \). The phase trajectories at \( A = 1, \omega = 1, \eta = 0.05 \) (under-damped) are plotted for (c) \( p_+(t) \) vs \( x_+(t) \) and for (d) \( p_-(t) \) vs \( x_-(t) \) at \( \lambda = 0.1, 0.3, 0.5 \).

B. \( n \)-dimensional PDM damped oscillators

1. Coordinates deformed by \( Q(r) = 1/(1 + \lambda^2 r^2) \)

A deformation in the coordinates in the form of

\[
Q(r) = \frac{1}{1 + \lambda^2 r^2}; \quad r^2 = \sum_{j=1}^{n} x_j^2, 
\]

would result a PDM of the form

\[
m(r) = \frac{1}{(1 + \lambda^2 r^2)^3}. 
\]
FIG. 3: For the damped Morse-type PDM-oscillator with \( m(x) = e^{2\lambda x} \) we plot (a) the coordinate \( x(t) \) of (51) as it evolves with time at \( A = 1, \omega = 1, \lambda = 1 \) and \( \eta = 0.0, 0.2, 1.0, 1.5 \). The corresponding phase trajectories \( p(t) \) vs \( x(t) \) at \( A = 1, \omega = 1, \eta = 0.05 \) (under-damped) are plotted in (d) for \( \lambda = 1, 3, 5 \).

FIG. 4: For the PDM (57) we plot in (a) the \( i \)th coordinate \( x_i(t) \) in (58) as it evolves with time at \( A = 1, \omega = 1, \lambda = 1 \) for \( \eta = 0.0, 0.2, 1.0, 1.5 \). The corresponding phase trajectories \( p_i(t) \) vs \( x_i(t) \) with \( A = 1, \omega = 1, \eta = 0.05 \) (under-damped) are shown in (b) for \( \lambda = 1, 3, 5 \).

Using the point transformation (19) would lead, with \( q^2 = \sum_{j=1}^{n} q_{j}^2 \), to

\[
q^2 = Q(r) \Rightarrow r^2 = \frac{q^2}{1 - \lambda^2 q^2} \Rightarrow r = \frac{1}{1 - \lambda^2 q^2} q \Rightarrow x_i = \frac{1}{1 - \lambda^2 q^2} q_i.
\]
Consequently, with \( q_i = A_i e^{-\omega n_\alpha t} \cosh (\beta t) \) and \( A^2 = \sum_{j=1}^{n} A_j^2 \),
\[
x_i (t) = \frac{A_i \cosh (\beta t)}{\sqrt{e^{2\omega n_\alpha t} - \lambda^2 A^2 \cosh^2 (\beta t)}}; \quad \lambda < \frac{1}{A^1}
\] (58)
which serve as the \( n \)-dimensional exact solutions for the \( n \) dynamical standard PDM damped oscillators equations
\[
\ddot{x}_i - \frac{3\lambda^2}{1 + \lambda^2 r^2} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + 2\eta r \omega \sigma \dot{x}_i + \omega_n^2 (1 + \lambda^2 r^2) x_i = 0.
\] (59)

These dynamical equations belong to a PDM particle, \( m (r) = (1 + \lambda^2 r^2)^{-3} \), described by the \( n \)-dimensional PDM-Lagrangian and PDM-Hamiltonian
\[
L = \frac{1}{2} \left( \sum_{j=1}^{n} x_j^2 - \frac{\omega_n^2}{2 (1 + \lambda^2 r^2)} \right) \Rightarrow H = (1 + \lambda^2 r^2)^{3} \frac{p^2}{2} + \frac{\omega_n^2}{2 (1 + \lambda^2 r^2)} \sum_{j=1}^{n} x_j^2
\] (60)
and feels a non-conservative \( n \)-dimensional PDM-deformed Rayleigh dissipative force field
\[
R (r, \dot{r}) = \frac{b}{1 + \lambda^2 r^2)^3} \sum_{j=1}^{n} \dot{x}_j^2.
\] (61)

In Figure 4(a), for the PDM in (57) we show the system response to the damping ratio \( \eta \) and plot the corresponding phase-space trajectories at different PDM parameter \( \lambda \)-values. Similar trends are observed for both the \( i \)th coordinate and the corresponding phase space trajectories.

2. Power-law type PDM deformed damped oscillators

A power-law type coordinates deformation in the form of
\[
Q (r) = \alpha r^\sigma
\] (62)
would imply a PDM function
\[
m (r) = \alpha^2 (\sigma + 1)^2 r^{2\sigma}
\] (63)
In this case, with \( q_i = A_i e^{-\omega n_\alpha t} \cosh (\beta t) \), \( q^2 = \sum_{j=1}^{n} q_j^2 \) and \( A^2 = \sum_{j=1}^{n} A_j^2 \),
\[
q = \alpha r^\sigma \Rightarrow \quad q \equiv \left( \frac{q^-\sigma}{\alpha} \right)^{1/(\sigma + 1)}
\] (64)
which satisfies the \( n \)-equations of motion for the damped PDM oscillators
\[
\ddot{x}_i + \frac{\sigma}{r^2} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + 2\eta r \sigma \dot{x}_i + \frac{\omega_n^2}{\sigma + 1} x_i = 0.
\] (65)

These \( n \)-PDM dynamical equations of motion for a PDM particle \( m (r) = \alpha^2 (\sigma + 1)^2 r^{2\sigma} \) described by the \( n \)-dimensional standard PDM-Lagrangian and PDM-Hamiltonian
\[
L = \frac{\alpha^2 (\sigma + 1)^2 r^{2\sigma}}{2} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) - \frac{\omega_n^2}{2} \alpha r^\sigma \sum_{j=1}^{n} x_j^2 \Rightarrow H = \frac{\alpha^2 (\sigma + 1)^{-2} r^{-2\sigma}}{2} p^2 + \frac{\omega_n^2}{2} \alpha r^\sigma \sum_{j=1}^{n} x_j^2
\] (66)
FIG. 5: For the power-law type PDM of (63) we plot the coordinate \(x(t)\) in (64) at \(A = 1, \omega = 1, \alpha = 1\) and \(\eta = 0\) in (a) for \(\sigma = -0.5\) and in (b) for \(\sigma = 2\). The effect of the PDM parameter \(\alpha\) on the phase trajectories are plotted at \(A = 1, \omega = 1, \eta = 0.05\) (under-damped), where for \(\sigma = 0.5\) we plot in (c) for \(\alpha = 1\) and in (d) for \(\alpha = 3\). For \(\sigma = -0.5\) we plot (e) for \(\alpha = 1\) and (f) for \(\alpha = 3\).

In Figures 5(a) and 5(b) we show the dynamical system (64) response, for the PDM in (63), to the damping ratio for \(\sigma = -0.5\) and \(\sigma = 2\), respectively. The phase-space trajectories at \(\sigma = 0.5\) are shown in 5(c) for the PDM parametric value \(\alpha = 1\), and in 5(d) for \(\alpha = 3\). At \(\sigma = -0.5\) we use in 5(e) the PDM parametric value \(\alpha = 1\), and in 5(f) \(\alpha = 3\).

and feels a non-conservative \(n\)-dimensional PDM-deformed Rayleigh dissipative force field

\[
\mathcal{R}(\mathbf{r}, \dot{\mathbf{r}}) = b\alpha^2 (\sigma + 1)^2 r^{2\sigma} \sum_{j=1}^{n} \dot{x}_j^2, \tag{67}
\]

In Figures 5(a) and 5(b) we show the dynamical system (64) response, for the PDM in (63), to the damping ratio for \(\sigma = -0.5\) and \(\sigma = 2\), respectively. The phase-space trajectories at \(\sigma = 0.5\) are shown in 5(c) for the PDM parametric value \(\alpha = 1\), and in 5(d) for \(\alpha = 3\). At \(\sigma = -0.5\) we use in 5(e) the PDM parametric value \(\alpha = 1\), and in 5(f) \(\alpha = 3\).

IV. CONCLUDING REMARKS

In this work, we have considered standard PDM-Lagrangians/PDM-Hamiltonians, for which the *gain-loss balance* correlation between the kinetic and potential energies is preserved to secure conservation of the total energy (i.e., \(L = T - V, H = T + V\), and \(dH/dt = dE/dt = 0\)). Within such standard settings, we have discussed and reported on the \(n\)-dimensional PDM damped harmonic oscillators subjected to a conservative PDM-harmonic oscillator force field \(24\) and a non-conservative PDM-Rayleigh dissipative force field \(25\). We have used the \(n\)-dimensional point canonical transformation \(19\) and \(20\), along with the total vector *Newtonian invariance amendment* in \(29\) (which
has been recently introduced by Mustafa [15, 47], to facilitate the linearizability of their $n$-PDM dynamical equations [31] into some $n$-linear DHOs’ dynamical equations [16] for constant mass setting. Consequently, the well know exact solutions for the linear DHOs [18] are mapped, with ease, onto the exact solutions for PDM DHOs. This is documented in the set of one-dimensional and the set of $n$-dimensional PDM-DHO illustrative examples of section III. Their phase-space trajectories are also reported in Figures 1-5. To the best of our knowledge, neither the current methodical approach nor the current results have been reported elsewhere.

In the light of our experience above, our observations are in order.

In connection with the prominent nonlinear dynamical systems of Liénard-type (c.f., e.g., Ref[4]), we have effectively shown that a mixed Liénard-type differential equation

\[ \ddot{x} + f(x) \dot{x}^2 + h(x) \dot{x} + g(x) = 0; \quad f(x) = \frac{m'(x)}{2m(x)}, \quad h(x) = 2\eta\omega, \quad g(x) = \sqrt{\frac{Q(x)}{m(x)}}\omega^2 x \]

is linearizable into a linear Liénard-type differential equation

\[ \ddot{q} + \tilde{h}(q) \dot{q} + \tilde{g}(q) = 0; \quad \tilde{h}(q) = 2\eta\omega, \quad \tilde{g}(q) = \omega^2 q. \]

Moreover, if the damping ratio is switched off, $\eta = 0$, then $h(x) = 0$ and the mixed Liénard-type differential equation reduces into a quadratic Liénard-type. Consequently, the later is linearizable into a simple harmonic oscillator one. All of which admit exact solvability within the lines discussed above. The current methodical proposal offers not only a user friendly and a straightforward approach that allows transition between the three Liénard-type dynamical equations, but also provides their PDM damped and/or undamped harmonic oscillator counterparts and facilitates their exact solvability.

Yet, it should be made clear that, the Lagrangian/Hamiltonian structures for dissipative damped harmonic oscillator systems used early on by Pradeep et al. [5], Chandrasekar et al. [6] and the related comment of Bender et al. [7] do not belong to the class of the standard textbook PDM-Lagrangians/PDM-Hamiltonians used in the current methodical proposal. The two completely different approaches should not be confused with each other, therefore.

On the feasibly viable applicability of the above methodical proposal, it would be interesting to study particles subjected to drag (of fluid, say) and random forces (effect of Brownian motion) to yield Langevin equation. Where the classical description is connected with a probabilistic one, which would suggest that the Fokker-Plank equation (FPE) inhomogeneous medium with variable diffusion PDM-coefficient is equivalent with deformed FPE with constant mass and constant diffusion coefficient. For more details on this issue the reader may refer to (c.f., e.g. da Costa et al. [44]). Among other possible applications are the Killing vector fields and Noether momenta approach for quantum PDM Hamiltonians of Carinena et al. [48], the generation of integrable nonlinear dynamical systems of Lakshmanan-Chandrasekar [4], etc.

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