Research Article

On $k$th-Order Slant Weighted Toeplitz Operator

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1. Introduction

Toeplitz operators and slant Toeplitz operators [1] have been found immensely useful, especially in the study of prediction theory [2], wavelet analysis [3], and solution of differential equations [4]. Originally, these operators were defined and studied on the usual $H^2$ and $L^2$ spaces. During the past few decades, different generalisations of these spaces, like the weighted function spaces $H^2_w$ and $L^2_w$ and the weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ have developed. Shields [5] has made a systematic study of the multiplication operator on $L^2(\beta)$. Lauric [6] has discussed the Toeplitz operator on $H^2(\beta)$. Motivated by these studies, we introduced and studied the notion of a slant weighted Toeplitz operator [7] on $L^2(\beta)$. In this paper, we study a $k$th-order slant weighted Toeplitz operator on the space $L^2(\beta)$. We begin with the following preliminaries.

Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < \beta_n/\beta_{n+1} \leq 1$ when $n \geq 0$ and $0 < \beta_n/\beta_{n-1} \leq 1$ when $n \leq 0$.

Throughout the paper, we assume that for a fixed integer $k \geq 2$, $\beta_k/\beta_n \leq M < \infty$. Consider the following spaces [5, 6]:

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \right\}$$

$$\|f\|^2_\beta = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$ (1)

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \right\}$$

$$\|f\|^2_\beta = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$ (2)

Then, $(L^2(\beta), \| \cdot \|_\beta)$ is a Hilbert space [6] with an orthonormal basis given by $\{e_k(z) = z^k/\beta_k\}_{k \in \mathbb{Z}}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \beta_n^2.$$ (2)
Also, \( H^2(\beta) \) is a subspace of \( L^2(\beta) \). Further, the space

\[
L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid \phi L^2(\beta) \subseteq L^2(\beta), \exists c \in R \right\}
\]

such that \( \| \phi f \| \leq c \| f \| \ \forall f \in L^2(\beta) \)

is a Banach space with respect to the norm defined by

\[
\| \phi \|_{\infty} = \inf\left\{ c \mid \| \phi f \| \leq c \| f \| \ \forall f \in L^2(\beta) \right\}.
\]

The mapping \( P : L^2(\beta) \rightarrow H^2(\beta) \) is the orthogonal projection of \( L^2(\beta) \) onto \( H^2(\beta) \). For a given \( \phi \in L^\infty(\beta) \), the induced weighted multiplication operator \([5]\) is denoted by \( M_\phi \) and is given by \( M_\phi : L^2(\beta) \rightarrow L^2(\beta) \) such that

\[
M_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{n-k} \beta_n e_n(z).
\]

If we put \( \phi_1(z) = z \), then \( M_{\phi_1} = M_z \) is the operator defined as \( M_z e_k(z) = \omega_k e_{k+1}(z) \), where \( \omega_k = \beta_{k+1}/\beta_k \) for all \( k \in \mathbb{Z} \), and it is known as a weighted shift \([5]\).

The slant weighted Toeplitz operator \([7]\) \( A_\phi \) is an operator on \( L^2(\beta) \) defined as \( A_\phi : L^2(\beta) \rightarrow L^2(\beta) \) such that \( A_\phi e_k(z) = (1/\beta_k) \sum_{n=-\infty}^{\infty} a_{2n-k} \beta_n e_n(z) \).

If \( W : L^2(\beta) \rightarrow L^2(\beta) \) is such that

\[
W e_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z), \quad W e_{2n-1}(z) = 0 \ \forall n \in \mathbb{Z},
\]

then \( A_\phi \) can be alternately defined by

\[
A_\phi(f) = W M_\phi(f) = W(\phi f) \ \forall f \in L^2(\beta).
\]

### 2. kth-Order Slant Weighted Toeplitz Operator

Suppose that the operator \( W_k : L^2(\beta) \rightarrow L^2(\beta) \) is such that

\[
W_k e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_n} e_{n/k}(z) & \text{if } n \text{ is divisible by } k, \\ 0 & \text{otherwise,} \end{cases}
\]

Then the matrix of \( W_k \) is

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{\beta_1}{\beta_k} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{\beta_2}{\beta_{2k}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

The adjoint of \( W_k \) is given by

\[
W_k^* e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \ \forall n \in \mathbb{Z}.
\]

**Definition 1** (see \([8]\)). For an integer \( k \geq 2 \), the \( k \)th order slant weighted Toeplitz operator \( U_\phi : L^2(\beta) \rightarrow L^2(\beta) \) is such that \( U_\phi(f) = W_k M_\phi(f) \) for all \( f \in L^2(\beta) \). Thus, \( U_\phi e_n(z) = (1/\beta_k) \sum_{n=-\infty}^{\infty} a_{nk} \beta_n e_n(z) \).

The \((i,j)\)th entry of the matrix of \( U_\phi \) is given by

\[
\langle U_\phi e_j, e_i \rangle = \begin{cases} \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{k-1} \beta_n e_n(z), e_i(z) \end{cases}
\]

\[
= a_{k-1} \frac{\beta_i}{\beta_j}.
\]

Hence, the matrix of \( U_\phi \) with respect to this basis is as follows:

\[
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \frac{\beta_0}{\beta_0} & a_{k-1} \frac{\beta_1}{\beta_1} & a_{k-2} \frac{\beta_1}{\beta_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{\beta_0}{\beta_0} & a_{k-1} \frac{\beta_0}{\beta_1} & a_{k-2} \frac{\beta_0}{\beta_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \frac{\beta_1}{\beta_1} & a_{k-1} \frac{\beta_1}{\beta_2} & a_{k-2} \frac{\beta_1}{\beta_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \frac{\beta_2}{\beta_2} & a_{k-1} \frac{\beta_2}{\beta_2} & a_{k-2} \frac{\beta_2}{\beta_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

**Theorem 2.** The mapping \( \phi \rightarrow U_\phi \) is linear and one to one.

**Proof.** For linearity, consider

\[
U_{\alpha \phi + \beta \psi} = W_k M_{\alpha \phi + \beta \psi}
\]

\[
= W_k M_\phi + M_\psi
\]

\[
= \alpha W_k M_\phi + W_k M_\psi
\]

\[
= \alpha U_{\phi} + U_{\psi}.
\]

For checking one-to-one, let \( U_{\psi} = U_{\phi} \).

Then, \( U_{\phi} - U_{\psi} = 0 \). By linearity, we get \( U_{\phi - \psi} = 0 \). Hence,

\[
U_{\phi - \psi} e_n(z) = 0, \quad n \in \mathbb{Z},
\]

\[
\Rightarrow W_k M_{\phi - \psi} e_n(z) = 0, \quad n \in \mathbb{Z},
\]

\[
\Rightarrow W_k (\phi - \psi) e_n(z) = 0, \quad n \in \mathbb{Z}.
\]

This implies that either \( \phi - \psi = 0 \) or the degree of \((\phi - \psi)e_n(z)\) is not divisible by \( k \). But since this is true for all \( n \in \mathbb{Z} \), the second possibility is ruled out. Hence, \( \phi - \psi = 0 \) or \( \phi = \psi \). \( \square \)
Theorem 3. Consider the following: (i) \( M_zW_k = W_kM_{z^k} \); (ii) \( M_{zn}W_k = W_kM_{zn^m} \), \( m \in \mathbb{Z} \).

Proof. (i) It is sufficient to prove that
\[
M_zW_k e_n(z) = W_k M_{z^k} e_n(z) \quad \forall n \in \mathbb{Z}.
\] (15)
Suppose that \( n \) is not a multiple of \( k \). Then,
\[
M_zW_k e_n(z) = M_z 0 = 0 = W_k M_{z^k} e_n(z).
\] (16)
Now, when \( n = pk \) (multiple of \( k \)),
\[
M_zW_k e_n(z) = M_z \frac{\beta_p}{\beta_{pk}} e_{p}(z) = \frac{\beta_{p+1}}{\beta_{pk}} e_{p+1}(z).\] (17)
On the other hand,
\[
W_k M_{z^k} e_n(z) = W_k M_{z^k} e_{pk}(z)
= W_k \frac{\beta_{(p+1)k}}{\beta_{pk}} e_{(p+1)k}(z)
= \frac{\beta_{p+1}}{\beta_{pk}} e_{p+1}(z).\] (18)
Hence from (17) and (18), we get that
\[
M_zW_k e_n(z) = W_k M_{z^k} e_{n}(z) \quad \text{whenever } n \text{ is a multiple of } k.
\] (19)
We therefore conclude that for all \( n \in \mathbb{Z} \),
\[
M_zW_k e_n(z) = W_k M_{z^k} e_n(z)
\Rightarrow M_zW_k = W_k M_{z^k}.
\] (20)

(ii) We prove the result by induction on \( m \).

For \( m = 1 \), the result is true by part (i). Suppose that the result is true for \( m = l \). Then,
\[
M_zW_k = W_k M_{z^k}.
\] (21)
Now
\[
M_{z^{(l+1)k}}W_k = M_z M_{z^{lk}}W_k = M_z W_k M_{z^{lk}} = W_k M_{z^{lk}} M_{z^k}
= W_k M_{z^{(l+1)k}}.
\] (22)
Hence by induction, the result is true for all \( m \in \mathbb{Z}^+ \).

For \( m = 1 \) and \( 1 \leq j \leq k - 1 \),
\[
M_{zn}W_k e_{nk+j}(z) = M_{z^{-1}} 0 = 0 = W_k M_{z^{-1}} e_{nk+j}(z).
\] (23)
For \( m = -1 \) and \( j = nk \) (a multiple of \( k \)),
\[
M_{zn}W_k e_{j}(z) = M_{z^{-1}}W_k e_{nk}(z)
= \frac{\beta_{n}}{\beta_{nk}} e_{nk}(z)
= \frac{\beta_{n} M_{z^{-1}} e_{n}(z)}{\beta_{nk}}
= \frac{\beta_{n} e_{n}(z)}{\beta_{nk}}
= \frac{\beta_{n-1}}{\beta_{nk}} e_{n-1}(z).
\] (24)
On the other hand,
\[
W_k M_{z^{-1}} e_{nk}(z) = W_k \left( z^{-1} e^{nk} \right)
= \frac{\beta_{(n-1)k}}{\beta_{nk}} e_{(n-1)k}(z)
= \frac{\beta_{(n-1)k}}{\beta_{nk}} \frac{\beta_{n-1}}{\beta_{(n-1)k}} e_{n-1}(z)
= \frac{\beta_{n-1}}{\beta_{nk}} e_{n-1}(z).
\] (25)
From (23), (24), and (25), we conclude that
\[
M_{zn}W_k = W_k M_{zn^m}.
\] (26)
Hence, the result is true for \( m = -1 \) also. Therefore, by using induction, we can prove it for all negative integers \( m \). The case when \( m = 0 \) is trivially true.

Hence the theorem is true. \( \square \)

Corollary 4. Let \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta) \). Then,
\[
M_{\phi(z)}W_k = W_k M_{\phi(z^k)}.
\] (27)

Theorem 5. Consider the following: \( M_{\phi(z)} U_{\psi(z)} = U_{\phi(z^k)\psi(z)} \).

Proof. The proof is as follows:
\[
M_{\phi(z)} U_{\psi(z)} = M_{\phi(z)} W_k M_{\psi(z)}
= W_k M_{\phi(z^k)} M_{\psi(z)}
= W_k M_{\phi(z^k)\psi(z)}
= U_{\phi(z^k)\psi(z)}.
\] (28)\( \square \)

3. Matrix Characterisation of \( U_{\phi} \)

Definition 6. Let \( w_n = \frac{\beta_{n+1}}{\beta_n} \) for all \( n \in \mathbb{Z} \). Then, the \( k \)th order slant weighted Toeplitz matrix corresponding to the
weight sequence \( \langle w_n \rangle \) is a bilaterally infinite matrix \( \langle \lambda_{ij} \rangle \) such that
\[
\lambda_{i+1,j+k} = \frac{w_j}{w_j w_{j+1} \cdots w_{j+k-1}} \lambda_{i,j}, \quad i, j = 0, \pm 1, \pm 2, \ldots \tag{29}
\]

We have proved earlier [8] that \( U \) is a \( k \)th order slant weighted Toeplitz operator if and only if \( M_z U = U M_z \).

Next, we prove a characterisation of the \( k \)th order slant weighted Toeplitz operator in terms of the matrix previously defined.

**Theorem 7.** A necessary and sufficient condition that an operator \( U \) on \( L^2(\beta) \) is a \( k \)th order slant weighted Toeplitz operator is that its matrix with respect to the orthonormal basis \( \{ e_k(z) = z^k/\beta_k \}_{k \in \mathbb{Z}} \) is a \( k \)th order slant weighted Toeplitz matrix.

**Proof.** Suppose that \( U \) is a \( k \)th order slant weighted Toeplitz operator. Then, the corresponding matrix \( \langle \lambda_{ij} \rangle \) is given by
\[
\lambda_{ij} = \langle U e_j, e_i \rangle = a_{ki} \frac{\beta_{i-j}}{\beta_j}, \tag{30}
\]
Further,
\[
\lambda_{i+1,j+k} = \langle U e_{j+k}, e_{i+1} \rangle = a_{ki+k-j} \frac{\beta_{i+k-j}}{\beta_{j+k}} = a_{ki-j} \frac{\beta_{i+1}}{\beta_{j+k}} = \frac{w_i}{w_j w_{j+1} \cdots w_{j+k-1}} \lambda_{i,j}, \tag{31}
\]
where \( w_n = \beta_{n+1}/\beta_n \) for all \( n \in \mathbb{Z} \). Thus, the matrix of \( U \) is a \( k \)th order slant weighted Toeplitz matrix. Conversely, we assume that \( U \) is an operator on \( L^2(\beta) \) whose matrix is a \( k \)th order slant weighted Toeplitz matrix. This means that for all \( i, j \in \mathbb{Z} \), we have
\[
\langle U e_j, e_i \rangle = \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle U e_{j+k}, e_{i+1} \rangle. \tag{32}
\]

Now,
\[
\langle M_z U e_j, e_i \rangle = \langle U e_j, M_z^* e_i \rangle = \langle U e_j, w_{i-1} e_{i-1} \rangle = w_{i-1} \langle U e_j, e_{i-1} \rangle = \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_{i-1}} \langle U e_{j+k}, e_{i+1} \rangle \tag{33}
\]
\[
\times \langle U e_{j+k}, e_i \rangle \quad \text{ (using (32))} = \langle U M_z e_j, e_i \rangle, \quad \forall i, j \in \mathbb{Z}.
\]

Hence \( M_z U = U M_z \). Therefore, we conclude that \( U \) is a \( k \)th order slant weighted Toeplitz operator.

Next, consider the operator \( S : L^2(\beta) \to L^2(\beta) \) given by \( S e_j = (1/\beta_j) e_{j+1} \). Then, \( S^* e_j = (1/\beta_{j-1}) e_{j-1} \).

Now, \( S \) is bounded as \( \langle w_n \rangle \) is always positive and bounded.

**Lemma 8.** Consider the following: \( S^* = M_z^{-1} \).

**Proof.** The proof is as follows:
\[
S^* M_z e_j = S^* w_j e_{j+1} = \frac{w_j}{w_j} e_j = e_j, \tag{34}
\]
\[
M_z S^* e_j = M_z \frac{1}{w_{j-1}} e_{j-1} = \frac{w_{j-1}}{w_{j-1}} e_j = e_j, \tag{35}
\]

**Theorem 9.** A bounded operator \( U \) on \( L^2(\beta) \) is a \( k \)th order slant weighted Toeplitz operator if and only if \( U = M_z^{-1} U M_z \), where \( M_z \) and \( M_z^* \) are the multiplication operators induced by \( z \) and \( z^* \), respectively.

**Proof.** Let \( U \) be a \( k \)th order slant weighted Toeplitz operator on \( L^2(\beta) \). Then, from (32) we get that
\[
\langle U e_j, e_i \rangle = \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle U e_{j+k}, e_{i+1} \rangle = \langle U M_z e_j, e_i \rangle \tag{36}
\]
Thus, \( U = M_z^{-1} U M_z \).

For the converse part, we assume that \( U \) is a bounded operator on \( L^2(\beta) \) satisfying \( U = M_z^{-1} U M_z \) for some fixed integer \( k \geq 2 \). Then, for all \( i, j \in \mathbb{Z} \),
\[
\langle U e_j, e_i \rangle = \langle M_z^{-1} U M_z e_j, e_i \rangle = \langle S^* U M_z e_j, e_i \rangle \tag{37}
\]
\[
= \langle U M_z e_j, e_i \rangle \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle U e_{j+k}, e_{i+1} \rangle.
\]

The previous equation shows that the matrix corresponding to \( U \) is a \( k \)th order slant weighted Toeplitz matrix. Hence, by Theorem 7, \( U \) is a \( k \)th order slant weighted Toeplitz operator.

**Corollary 10.** For a fixed integer \( k \geq 2 \), the set of all \( k \)th order slant weighted Toeplitz operators is weakly closed and hence strongly closed.

**Proof.** We assume that for each positive integer \( n, U_n \) is a \( k \)th order slant weighted Toeplitz operator and \( U_n \to U \) weakly. Then, for \( f, g \in L^2(\beta) \), we get that \( \langle U_n f, g \rangle \to \langle U f, g \rangle \).
From the previous theorem, this implies that
\[
\langle M^{-1}_n U_n M_n^{\ast} f, g \rangle = \langle S^* U_n M_n^{\ast} f, g \rangle
\]
\[
= \langle U_n M_n^{\ast} f, S g \rangle
\]
\[
= \langle U_n M_n^{\ast} \cdot f, S g \rangle \rightarrow \langle U z^k \cdot f , S g \rangle
\]
\[
= \langle M^{-1}_n U_n M_n^{\ast} f, g \rangle.
\] (37)

But, for each \(n\), \(U_n = M^{-1}_n U_n M_n^{\ast}\).

Hence, \(U = M^{-1}_n U_n M_n^{\ast}\). Therefore, \(U\) is a \(k\) th order slant weighted Toeplitz operator.

4. Compression to \(H^2(\beta)\)

Definition 11. The compression of a \(k\) th order slant weighted Toeplitz operator on \(H^2(\beta)\) is denoted by \(V_{\phi}\) and defined as
\[
V_{\phi} e_j(z) = (1/\beta_j) \sum_{n=0}^{\infty} a_{nk} \beta_n e_n(z). \quad j = 0, 1, 2, \ldots
\]
Alternatively,
\[
V_{\phi} = PU_{\phi_{1/\beta^2}}
\]
\[
= PW_k M_{\phi_{1/\beta^2}}
\]
\[
= W_k PM_{\phi_{1/\beta^2}} \quad \text{(since} P \text{ reduces} W_k \text{[8])}
\]
\[
= W_k T_{\phi}
\]
where \(T_{\phi} : H^2(\beta) \rightarrow H^2(\beta)\) is the Toeplitz operator on \(H^2(\beta)\) induced by \(\phi \in L^\infty(\beta)\). The matrix of \(V_{\phi}\) is unilaterally infinite and has the form
\[
\begin{bmatrix}
\alpha_0 & \beta_0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots \\
\alpha_1 & \beta_0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots \\
\alpha_2 & \beta_1 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
\[
(39)
\]
for a given \(\phi = \sum_{n=-\infty}^{\infty} a_n z^n\).

Further, we observe the following.

(i) The mapping \(\phi \rightarrow V_{\phi}\) is linear and one to one.

(ii) \(T_{\phi} W_k = PM_{\phi} W_k = PW_k M_{\phi}(z^k) = PU_{\phi(z^k)} = V_{\phi(z^k)}\).

References

[1] M. C. Ho, “Properties of slant Toeplitz operators,” Indiana University Mathematics Journal, vol. 45, no. 3, pp. 843–862, 1996.

[2] H. Helson and G. Szegö, “A problem in prediction theory,” Annali di Matematica Pura ed Applicata, vol. 51, no. 1, pp. 107–138, 1960.

[3] L. Villemoes, “Wavelet analysis of refinement equations,” SIAM Journal on Mathematical Analysis, vol. 25, pp. 1433–1460, 1994.

[4] T. Goodman, C. Micchelli, and J. Ward, “Spectral radius formula for subdivision operators,” in Recent Advances in Wavelet Analysis, Academic Press, L. Schumaker and G. Webb, Eds., pp. 335–360, 1994.

[5] A. L. Shields, Weighted Shift Operators and Analytic Function Theory, Topics in Operator Theory, vol. 13 of Mathematical Survey, American Mathematical Society, Providence, RI, New England, 1974.

[6] V. Lauric, “On a weighted Toeplitz operator and its commutant,” International Journal of Mathematics and Mathematical Sciences, vol. 2005, no. 6, pp. 823–835, 2005.

[7] S. C. Arora and R. Kathuria, “On weighted Toeplitz operators,” Australian Journal of Mathematical Analysis and Applications, vol. 8, no. 1, article 11, 2011.

[8] S. C. Arora and R. Kathuria, “Generalised slant weighted Toeplitz operators,” World Academy of Science, Engineering and Technology, vol. 75, pp. 1103–1106, 2011.