Ward Identities for Interacting Electronic Systems

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Abstract

A Ward-Takahashi identity, as a consequence of gauge invariance and in a form that relates self-energy to the two-particle Bethe-Salpeter scattering kernel, was first derived by Vollhardt and Wölfle for a system of independent particles moving in a random medium. This is generalized to a class of interacting electronic systems in materials with or without random impurities, following a procedure previously used for classical waves transport in disordered media. This class of systems also possesses other symmetry properties such as invariance under time translations and local spin rotations, which imply local conservation laws for energy and spin current. They imply additional Vollhardt-Wölfle type identities. We present non-perturbative derivations of these identities, and consider the constraints they impose on the relationship between the self-energy and the two-particle scattering kernel.

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Ward identities are statements of local conservation laws, and reflect the corresponding symmetry properties of the system. The original Ward identity [1] and its generalization by Takahashi [2] were derived for quantum electrodynamics, as a consequence of local current conservation, and relate the current vertex function to the electron propagator. In investigating the problem of the Anderson localization, Vollhardt and Wölfle [3] considered a system of independent electrons moving in a random potential, and derived a Ward-Takahashi identity in a form that relates the electron self-energy to a two-particle Bethe-Salpeter kernel induced by statistical randomness of the medium. More recently, similar identities were derived for classical waves transport in disordered media [4], for which, however, energy conservation underlies the physics involved. It is clear from the latter derivations, which make use of a Bethe-Salpeter type integral equation, that Vollhardt and Wölfle’s version of the Ward identity is valid for a considerably wider class of systems, such as interacting electrons moving in materials with or without random impurities. For this general class of electronic systems, energy and spin, in addition to charge, are also conserved, reflecting invariance properties of the system under time translation and spin rotation, and implying additional Vollhardt-Wölfle type identities. We consider in this paper the derivations of these identities, and their implications on the self-energy.

The model Hamiltonian we consider is of the form,

\[ H = \int d^3x \left[ \frac{1}{2m} \overrightarrow{\nabla} \Psi(\vec{x}, t) \cdot \overrightarrow{\nabla} \Psi(\vec{x}, t) + V_1(\vec{x}) \Psi(\vec{x}, t) \Psi(\vec{x}, t) \right] + \frac{1}{2} \int d^3xd^3x' \Psi(\vec{x}, t) \Psi(\vec{x}, t) V_2(\vec{x} - \vec{x}') \Psi(\vec{x'}, t) \Psi(\vec{x'}, t), \]

where \( \Psi \) is the two-component electron field, quantized according to the usual equal-time anti-commutation relations, \( V_1 \) the potential provided by atoms in the material, including randomly distributed static impurities, and \( V_2 \) the interaction potential between any two electrons with the property

\[ V_2(\vec{x} - \vec{x}') = V_2(\vec{x'} - \vec{x}). \]
\[
L = i \overline{\Psi}(x) \Psi(x) - \frac{1}{2m} \overrightarrow{\nabla} \overline{\Psi}(x) \cdot \overrightarrow{\nabla} \Psi(x) - V_1(\vec{x}) \overline{\Psi}(x) \Psi(x) \\
- \frac{1}{2} \int d^3x' \overline{\Psi}(\vec{x}, t) \Psi(\vec{x}, t) V_2(\vec{x} - \vec{x}') \overline{\Psi}(\vec{x'}, t) \Psi(\vec{x'}, t).
\]

For convenience, we use the four-dimensional notations with \( x^\mu = (t, x, y, z) \) and \( g^{\mu\nu} = (1, -1, -1, -1) \) for \( \mu = 0, 1, 2, 3 \). Due to the presence of the non-local interaction term \( V_2 \) in the Lagrangian density, one should exercise care in deriving the equation of motion. Instead of naively applying the Euler-Lagrange equation, one considers the minimization of the action, \( \delta \int L d^4x \), to derive the expected equation of motion:

\[
i \frac{\partial}{\partial t} \Psi = \left( -\frac{1}{2m} \nabla^2 + V_1 \right) \Psi + \int d^3x' \overline{\Psi}(\vec{x'}, t) \Psi(\vec{x'}, t) V_2(\vec{x} - \vec{x}') \overline{\Psi}(\vec{x}, t) \Psi(x).
\]

The Lagrangian density is invariant under the gauge transformation,

\[
\Psi(x) \rightarrow e^{i\lambda} \Psi(x),
\]

\[
\overline{\Psi}(x) \rightarrow e^{-i\lambda} \overline{\Psi}(x),
\]

the time translation,

\[
t \rightarrow t' = t + \varepsilon,
\]

\[
\Psi(\vec{x}, t) \rightarrow \Psi(\vec{x}, t' - \varepsilon),
\]

\[
\overline{\Psi}(\vec{x}, t) \rightarrow \overline{\Psi}(\vec{x}, t' - \varepsilon),
\]

and the spin rotation

\[
\Psi(x) \rightarrow e^{i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}} \Psi(x),
\]

\[
\overline{\Psi}(x) \rightarrow \overline{\Psi}(x) e^{-i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}},
\]

where \( \vec{\sigma} \) are the Pauli spin matrices. The corresponding conserved quantities are charge, energy, and spin, respectively.

In disordered media, there is a random distribution of impurities. We need to consider a statistical ensemble of impurity configurations and average over the ensemble. Under rather general assumptions on the statistical ensemble, as has been fully discussed by, e.g., Frisch.
statistical averaging results in an effective electron-electron interaction, which adds to the real electron-electron interaction potential $V_2$.

Local current conservation as a consequence of gauge invariance is of the form

$$\partial_\mu j^\mu = 0,$$

where

$$j^0 = \overline{\Psi} \Psi, \quad j^k = \frac{i}{2m}[\overline{\Psi} \partial^k \Psi - (\partial^k \overline{\Psi}) \Psi].$$

It implies that

$$\partial_x^\mu \langle 0\{ j^\mu(x) \Psi(y) \overline{\Psi}(z) \} 0 \rangle$$

$$= -\delta^4(x-y) \langle 0\{ \Psi(x) \overline{\Psi}(z) \} 0 \rangle + \delta^4(x-z) \langle 0\{ \Psi(y) \overline{\Psi}(x) \} 0 \rangle$$

$$= -\delta^4(x-y) \frac{1}{i} G(x,z) + \delta^4(x-z) \frac{1}{i} G(y,x).$$

The above equation holds for any single configuration of the impurities. When statistical averaging on an ensemble of impurity configurations is applied to it, an additional statistical interaction infuses itself into the structure of the expressions on both sides of the equation. An irreducible current vertex function is defined by

$$\langle 0\{ j^\mu(x) \Psi(y) \overline{\Psi}(z) \} 0 \rangle^{av} = \int d^4\eta d^4z \frac{1}{i} G^{(e)}(y-\eta) \Gamma_\mu(\eta|x|\zeta) \frac{1}{i} G^{(e)}(\zeta-z),$$

where expressions on both sides of the equation refer to configurationally averaged quantities, and use has been made of the fact that averaging restores translational invariance and, as a consequence, the averaged propagation function, $G^{(e)}$, depends only on the difference of the two coordinates. Averaging applied to both sides of (10) leads to

$$\partial_x^\mu \int d^4\eta d^4\zeta \frac{1}{i} G^{(e)}(y-\eta) \Gamma_\mu(\eta|x|\zeta) \frac{1}{i} G^{(e)}(\zeta-z)$$

$$= -\delta^4(x-y) \frac{1}{i} G^{(e)}(x-z) + \delta^4(x-z) \frac{1}{i} G^{(e)}(y-x),$$

which implies the Ward-Takahashi identity:

$$\partial_x^\mu \Gamma_\mu(\eta|x|\zeta) = G^{(e)-1}(\eta-x) \frac{1}{i} \delta^4(x-\zeta) - \delta^4(\eta-x) \frac{1}{i} G^{(e)-1}(x-\zeta).$$
For a system of non-interacting electrons moving in disordered media, Vollhardt and Wölffe \[2\], using diagram considerations, derived their version of the Ward-Takahashi identity that relates the electron self-energy with the two-particle Bethe-Salpeter kernel instead of the current vertex function. Rather than following their analysis in terms of diagrams, we here adopt the non-perturbative procedure previously used for the derivation of energy-conservation Ward-Takahashi identities for classical waves transport in disordered media \[3\].

A crucial input in the non-perturbative derivation is the following integral equation for the vertex function $\Gamma_\mu$,

$$
\Gamma_\mu(\eta|x|\zeta) - \Gamma_\mu^{(0)}(\eta|x|\zeta) = \int d^4y d^4y' d^4z d^4z' \left[ \frac{1}{i} g^{(e)}(y' - y) \Gamma_\mu(y|x|z) \frac{1}{i} g^{(e)}(z - z') K(\eta \zeta, y' z') \right],
$$

where $K$ is the Bethe-Salpeter kernel, and $\Gamma_\mu^{(0)}$ denotes the vertex function corresponding to a reference homogeneous medium and in the absence of the interaction $V_2$. It is of identical form as the one in quantum electrodynamics \[4\]. The validity of this integral equation, schematically illustrated in Fig. 1, is on the same footing as the Bethe-Salpeter equation. We note that all quantities in the above integral equation refer to statistically averaged ones. Combining the two equations (13) and (14) results in the Ward identity in the Vollhardt-Wölffe form:

$$
\Sigma(\eta - x) \delta^4(x - \zeta) - \Sigma(x - \zeta) \delta^4(\eta - x) = \int d^4y d^4z \left[ \delta^4(y - x) g^{(e)}(x - z) - \delta^4(x - z) g^{(e)}(y - x) \right] U(z - \zeta, \eta - y, \tilde{y} - \tilde{z}),
$$

where the self-energy $\Sigma$ is defined by

$$
G^{(e)-1} = g^{(0)-1} - \Sigma
$$

and $U$ is an alternative expression for $K$, with specified coordinate dependence. The dependence of $K$ on the coordinates is inferred from the following two properties. First, because of the restored translational invariance due to averaging, only three of the four coordinates
in $K$ are independent, and are chosen to be $z - \zeta, \eta - y,$ and $y - z$. Second, since the two channels are mediated by the effective statistical interaction and $V_2$, both of which are static, $(y - z)$ can actually be replaced by $(\vec{y} - \vec{z})$. In momentum space, the identity takes the form

$$\Sigma(q + p) - \Sigma(q) = \int \frac{d^3\omega}{(2\pi)^3} [G^{(e)}(q + \omega) - G^{(e)}(q + p + \omega)]U(q, q + p, -\omega).$$

(17)

This is the Ward identity derived by Vollhardt and Wölfle [3], but here extended to include the non-local interaction $V_2$ between any two electrons.

The Lagrangian density is invariant under the time translation. This implies the local conservation law

$$\partial^\mu T_{\mu 0} = 0,$$

(18)

where

$$T_{00}(x) = \frac{1}{2m} \vec{\nabla} \Psi(x) \cdot \vec{\nabla} \Psi(x) + V_1(x)\overline{\Psi}(x)\Psi(x)$$

$$+ \frac{1}{2} \int d^3x \overline{\Psi}(x, t)\Psi(x, t)V_2(x - x')\overline{\Psi}(x)\Psi(x),$$

(19)

$$T_{k0}(x) = \frac{1}{2m} [\partial_k \overline{\Psi}(x)\partial_0 \Psi(x) + \partial_0 \overline{\Psi}(x)\partial_k \Psi(x)].$$

In particular, the energy density is $T_{00}$. Similar to the case of current conservation, we consider in the present case the divergence of the three-point function $\langle 0|T\{T_{\mu 0}(x)\Psi(y)\overline{\Psi}(z)\}|0\rangle$. To proceed, we need to know the equal-time commutation relations $[T_{00}(x), \Psi(y)],$ etc.. Due to the presence of the non-local interaction $V_2$, we consider the commutation relation with the integrated total energy:

$$[\int d^3x T_{00}(\vec{x}, t), \Psi(\vec{y}, t)]$$

$$= -\int d^3x \delta^3(\vec{x} - \vec{y}) \left\{ \left[ \frac{1}{2m} \nabla^2 + V_1(\vec{x}) \right] \Psi + \int d^3x' \overline{\Psi}(\vec{x}', t)\Psi(\vec{x}', t)V_2(\vec{x} - \vec{x}')\Psi(x) \right\}$$

$$- \frac{1}{2m} \int d^3x \vec{\nabla} \cdot [\delta^3(\vec{x} - \vec{y}) \vec{\nabla} \Psi(x)]$$

(20)

There is some uncertainty as to what to do with the 3-dim divergence term in the above equation. We have considered various arguments for or against its inclusion in the local equal-time commutation relation. The most convincing argument against its inclusion comes
from an exercise in deriving the Ward identities by using the path-integral formalism. We thus conclude that, on account of (4),

\[
[T_{00}(\vec{x}, t), \Psi(\vec{y}, t)] = -\delta^3(\vec{x} - \vec{y}) \Psi(\vec{x}, t),
\]

(21)

and similarly for \( \overline{\Psi} \). Consequently,

\[
\partial_\mu x \langle 0 | T \{ T_{\mu 0}(x) \Psi(y) \overline{\Psi}(z) \} | 0 \rangle = -\delta^4(x - y) \frac{\partial}{\partial t_x} G(x, z) - \delta^4(x - z) \frac{\partial}{\partial t_z} G(y, x).
\]

(22)

Defining the energy vertex function \( \Gamma^{(E)}_\mu(\eta | x | \zeta) \) according to

\[
\langle 0 | T \{ T_{\mu 0}(x) \Psi(y) \overline{\Psi}(z) \} | 0 \rangle^{av} = \int d^4\eta d^4\zeta \frac{1}{i} G^{(e)}(y' - \eta) \Gamma^{(E)}_\mu(\eta | x | \zeta) \frac{1}{i} G^{(e)}(\zeta - z)
\]

(23)

we obtain from (22) the relation

\[
\partial_\mu \Gamma^{(E)}_\mu(\eta | x | \zeta) = G^{(e)}(\eta - x) \frac{\partial}{\partial t_x} \delta^4(x - \zeta) + G^{(e)}(x - \zeta) \frac{\partial}{\partial t_\zeta} \delta^4(\eta - x).
\]

(24)

which is the Ward-Takahashi identity for energy conservation. Compared with the current-conservation Ward-Takahashi identity (13), there are here in (24) extra linear time derivatives, which are the unmistakable signatures for energy. The two identities describe charge flow and energy flow, respectively.

To derive the corresponding Vollhardt-Wölfle type identity, we again make use of an integral equation for the energy vertex function, which is identical in structure to the charge counterpart,

\[
\Gamma^{(E)}_\mu(\eta | x | \zeta) - \Gamma^{(E)(0)}_\mu(\eta | x | \zeta) = \int d^4 y' d^4 z' d^4 z G^{(e)}(y' - y) \Gamma^{(E)}_\mu(\eta | x | z) \frac{1}{i} G^{(e)}(z - z') K(\eta \zeta; y' z').
\]

(25)

Combining (24) and (25), we obtain the desired identity

\[
\Sigma(\eta - x) \frac{\partial}{\partial t_x} \delta^4(x - \zeta) + \Sigma(x - \zeta) \frac{\partial}{\partial t_\zeta} \delta^4(\eta - x)
\]

(26)

\[
= \int d^4 y d^4 z [\delta^4(y - x) \frac{\partial}{\partial t_x} G^{(e)}(x - z) + \delta^4(x - z) \frac{\partial}{\partial t_z} G^{(e)}(y - x)] U(z - \zeta, \eta - y, \vec{y} - \vec{z})
\]

In momentum space, it is of the form
\[ q_0 \Sigma(q + p) - (q_0 + p_0) \Sigma(q) \]
\[ = \int \frac{d^3 \omega}{(2\pi)^3} [q_0 G^{(e)}(q + \vec{\omega}) - (q_0 + p_0) G^{(e)}(q + p + \vec{\omega})] U(q, q + p, -\vec{\omega}). \quad (27) \]

The Lagrangian density (3) is also invariant under local spin rotations, for which
\[ \Psi(x) \rightarrow e^{i\vec{\omega} \cdot \vec{\sigma}} \Psi(x), \quad \overline{\Psi}(x) \rightarrow \overline{\Psi}(x) e^{-i\vec{\omega} \cdot \vec{\sigma}} \quad (28) \]

This symmetry property implies the local conservation law
\[ \partial_\mu S^\mu_a = 0, \quad a = 1, 2, 3, \quad (29) \]
where the spin current is given by
\[ S^0_a = \overline{\Psi} \sigma_a \frac{\sigma_a}{2} \Psi, \quad (30) \]
\[ \overrightarrow{S}_a = \frac{i}{2m} (\overrightarrow{\nabla} \frac{\sigma_a}{2} \Psi - \overrightarrow{\nabla} \frac{\sigma_a}{2} \overline{\Psi}). \]

Using a procedure in complete parallel to the cases of gauge and time-translation symmetries, we obtain the following Vollhardt-Wölfle type identity for \( a = 1, 2, 3, \)
\[ \Sigma(\eta - x) \frac{\sigma_a}{2} \delta^4(x - \zeta) - \delta^4(\eta - x) \frac{\sigma_a}{2} \Sigma(x - \zeta) \]
\[ = \int d^4 y d^4 z [\delta^4(y - x) \frac{\sigma_a}{2} G^{(e)}(x - z) - \delta^4(x - z) G^{(e)}(y - x) \frac{\sigma_a}{2}] U(z - \zeta, \eta - y, \vec{y} - \vec{z}). \quad (31) \]

In momentum-space representation, it is of the form
\[ \Sigma(q + p) \frac{\sigma_a}{2} - \frac{\sigma_a}{2} \Sigma(q) = \int \frac{d^3 \omega}{(2\pi)^3} [\frac{\sigma_a}{2} G^{(e)}(q + \vec{\omega}) - G^{(e)}(q + p + \vec{\omega}) \frac{\sigma_a}{2}] U(q, q + p, -\vec{\omega}). \quad (32) \]

The collection of the three Vollhardt-Wölfle identities (15), (26), and (31), or their momentum-space counterparts (17), (27), and (32) set fairly stringent conditions on the relationship between the self-energy and the two-particle scattering kernel. Combining (17) and (27) yields the surprisingly simple but strong relation
\[ \Sigma(q) = \int \frac{d^3 \omega}{(2\pi)^3} G^{(e)}(q + p + \vec{\omega}) U(q, q + p, -\vec{\omega}). \quad (33) \]

In configuration space, it is of the form
\[ \delta^4(\eta - x)\Sigma(\eta - \zeta) = \int d^4y d^4z \delta^4(x - z)G^{(e)}(y - z)K(\eta\zeta; yz). \]  

(34)

We have checked and verified this relation in terms of low-order diagrams. According to (33) or (34), all self-energy diagrams can be generated from the corresponding irreducible kernel diagrams. For example, the self-energy diagram in Fig.2a can be generated from the irreducible kernel diagram in Fig.2b by connecting with a propagator as shown in Fig.2c.

In the verification of the relation (34), we have disposed of the tadpole type self-energy diagrams, such as in Fig.3a, which can not be generated according to (34). The disposal of these tadpole type diagrams is based on the following considerations. First, in the traditional formulation of the Lagrangian field theory, the Lagrangian density is defined in terms of normal ordered products of the field operators. All tadpole diagrams, by the very definition of the normal ordering, do not appear at all. Second, if for any reason, normal ordering of the products is not adopted, as is effectively the case in the path-integral formulation of quantum field theory, tadpole diagrams do appear. However, their contribution to the self-energy is “eliminated” order by order by the “mass” insertion diagram in Fig.3b, due to the presence of the mass counter term, actually the chemical-potential counter term in the present context, which is introduced in the Lagrangian density to achieve renormalization.

We note that renormalization counter terms are introduced in either of the above two formulations, although details of their functions differ somewhat. For example, in the case on hand, the “mass” counter term absorbs the (infinite) tadpole contribution to the self-energy in the path-integral approach, while in the traditional approach of Lagrangian field theory, because the tadpole diagrams do not appear, there is no need for the “mass” counter term to absorb them.

The strong relationship (34) perhaps reflects the fact that both the charge and the energy are exclusively carried by the same carriers, the electrons. It would be of interest to investigate additional implications of the set of three Vollhardt-Wölfle identities discussed in this note.
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Fig. 1. Schematic content of Eq. (14) and (25).
FIG. 2. Schematic relationship between $\Sigma$ and $K$. 
FIG. 3. "Tadpole" self-energy.