Three- and four-body kaonic nuclear states: Binding energies and widths

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Abstract

Using separable potentials for $\bar{K}N - \pi \Sigma$ interaction, we investigated four-body kaonic nuclear systems such as $K^-ppn$ and $K^-K^-pp$, with the Faddeev AGS method in the momentum representation. The Faddeev calculations are based on the quasi-particle method and the method of the energy dependent pole expansion was used to obtain the separable representation for the integral kernels in the three- and four-body equations. Different types of $\bar{K}N - \pi \Sigma$ potentials based on phenomenological and chiral SU(3) approach are used and it was shown that the kaonic nuclear systems under consideration are tightly bound.

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I. INTRODUCTION

The $\bar{K}N$ interaction, which is affected by $\Lambda(1405)$ resonance, plays an important role in the exotic systems, including the antikaon particle [1–5]. Thus, to study the kaonic systems, it is necessary to know the $\bar{K}N$ interaction. The first prediction of a quasi-bound state in kaonic nuclear systems was made in [3, 6–8], showing that these systems could be strongly bound. For the past two decades, many theoretical calculations were performed, focusing on the three- and four-body kaonic systems [9–20].

Alongside theoretical studies, many experimental searches have been also carried out to investigate the possible existence of the quasi-bound state in the kaonic systems (especially $K^-pp$ system). The investigations for the $K^-pp$ quasi-bound state have been explored by FINUDA experiment at the DAPhNE collider [21] and also by OBELIX at CERN [22] and DISTO at SATURNE [23]. Further experimental results were obtained by E15 and E27 groups at J-PARC [24, 25]. However, the possible existence of the quasi-bound state in the $K^-pp$ systems is still highly uncertain and there are some doubts in the extracted experimental results. The new planned experiments by HADES [26] and LEPS [27] Collaborations, and also by J-PARC [24, 25] experiments may unravel this problem.

The purpose of the present paper is to explore the binding energy and width of four-body kaonic nuclear systems including one or two antikaon particle. The problem can be solved using methods developed within four-body theories. To reduce the four-body Faddeev equations to a set of single-variable integral equations, one can employ different methods [28, 29]. One can do the reduction procedure numerically by making use of the so-called HSE method proposed by Narodetsky [28] and also by using the energy-dependent pole expansion method which developed by Sofianos et al [29]. In Refs [16] and [17], the HSE method was employed to solve the Faddeev equations of $K^-ppn$ and $K^-K^-pp$ systems, respectively. One can also perform the four-body calculation using the energy-dependent pole expansion method [29] or the so-called EDPE method. In EDPE method the form factors are energy dependent. In the present study, $K^-ppn$ and $K^-K^-pp$ quasi-bound state positions were calculated. Using the EDPE method, we found the separable expressions for the [3+1] and [2+2] subsystems. At the same time, the obtained results for EDPE method can be compared with those by Hilbert-Schmidt pole expansion methods and also study the behavior of the binding energy and width of kaonic systems under these situations. The dependence of the pole energy on different models of $\bar{K}N - \pi\Sigma$ interaction will be studied.
There is an opinion that the $K^-pp$ system has a two-pole structure similar to the $KN$ system [30]. To study this issue, the Faddeev amplitudes for $KN$ and $KNN$ systems were calculated in the complex energy plane. With this method, we investigated how the pole energy manifest itself in two- and three-body scattering amplitudes. We examined whether the first and the second pole of these systems can be seen in the corresponding scattering amplitudes. Different models of interactions, which are derived chirally and phenomenologically, will be included in our calculations for $KN - \pi\Sigma$ system [31, 32].

The paper is organized as follows: in Sect. II, we will explain the formalism used for the four-body $\bar{K}NNN$ and $\bar{K}\bar{K}NN$ systems and give a brief description of the quasi-particle method and separable representation of the Faddeev amplitudes by EDPE method. The two-body inputs of the calculations and the computed binding energies and widths are presented in Sect. III and in Section IV, we give conclusions.

II. THREE- AND FOUR-BODY CALCULATIONS

In the present work, the possible existence of a quasi-bound state in the $K^-ppn$ and $K^-K^-pp$ four-body systems was studied. We used the quasi-particle method to solve the four-body Faddeev equations. The key point of the quasi-particle method is the separable representation of the off-shell scattering amplitudes in two- and three-body subsystems [28, 33, 34]. Using properly symmetrized and antisymmetrized states with respect to identical kaons and nucleons, we will have the following subsystems of the $\bar{K}NNN$ four-body systems, without defining the interacting pairs.

$$\alpha = 1 : \bar{K} + (NNN), \quad \alpha = 2 : N + (\bar{K}NN),$$
$$\alpha = 3 : (\bar{K}N) + (NN),$$

The quantum numbers of the $\bar{K}NNN$ are $I = 0$ and $s = \frac{1}{2}$, in actual calculations, when we include isospin and spin indexis the number of configurations is equal to twelve, corresponding to different possible two-quasi-particle partitions.

$$\bar{K}(NN[NN]_{s=0,1}), \quad (\bar{K}[NN]_{s=0,1})N,$$
$$([\bar{K}N]_{I=0,1}N)_{s=0,1}N,$$
$$[\bar{K}N]_{I=0,1} + NN, \quad [NN]_{s=0,1} + \bar{K}N.$$
In the case of $\bar{K}KN$ system, we have one pair of identical kaon and one pair identical nucleon. Therefore, we will have four different subsystems, which are given by

\begin{align}
\alpha &= 1 : \bar{K} + (\bar{K}NN), \quad \alpha = 2 : N + (\bar{K}K), \\
\alpha &= 3 : (\bar{K}N) + (\bar{K}N), \quad \alpha = 4 : (\bar{K}\bar{K}) + (NN).
\end{align}

The quantum numbers of the $\bar{K}KN$ system are $I = 0$ and $s = 0$. Therefore, the number of configurations will be ten when we add the isospin and spin index

\begin{align}
\bar{K}(\bar{K}[NN]_{s=0}), & \quad \bar{K}([\bar{K}N]_{I=0,1}N), \\
([\bar{K}\bar{K}]_{I=1}N), & \quad ([\bar{K}K]_{I=0,1})N, \\
[\bar{K}K]_{I=1} + NN, & \quad \bar{K}\bar{K} + [NN]_{s=0}, \\
([\bar{K}N]_{I=0,1} + \bar{K}N).
\end{align}

The whole dynamics of $\bar{K}NNN$ system is described in terms of the transition amplitudes $A_{\alpha\beta}$ which connect the quasi-two-body channels characterized by Eqs. (1) and (3). In Fig. 1, the four different rearrangement channels of the $\bar{K}NNN$ and six rearrangement channels of the $\bar{K}\bar{K}NN$ four-body system including the K- and H-type diagrams are represented. In Fig. 1, the partitions defined in 1 and 3 are depicted, including the two-quasi-particles in the subsystems. Antisymmetrization of nucleons and symmetrization of the kaons to be made within each channel. The Faddeev equations for kaonic systems under consideration can be expressed by \[35, 36\]

\[
A^{I_{i,j},s's'}_{\alpha(i)\beta(j),nn'}(p, p', E) = \mathcal{R}^{I_{i,j},s's'}_{\alpha(i)\beta(j),nn'}(p, p', E) \\
+ \sum_{\gamma;l,m} \sum_{I_{k},s''} \int dp'' \mathcal{R}^{I_{k}I_{i,j},s's''}_{\alpha(i)\beta(k),nl}(p, p'', E) \theta_{lm}^{s's''} \\
\times A^{I_{k}I_{j},s''s'}_{\gamma(k)\beta(j),nn''}(p'', p', E).
\] (5)

Here, the operators $A^{I_{i,j},s's'}_{\alpha(i)\beta(j),nn'}$ are the four-body transition amplitudes, which describe the dynamics of the four-body $\bar{K}NNN$ and $\bar{K}\bar{K}NN$ systems and the $\theta_{lm}^{s's''}$-functions are the effective propagators. The total energy of the four-body system and the momentum of the spectator particle is defined by $E$ and $p$, respectively. To define the spectator particle or interacting particles in each two- and three-body subsystem, we used the $i$, $j$ and $k$ indices and the isospin of the interacting particles are defined by $I_i$. The indices $n, l, m$ are used for defining which term of the separable expansion of the subamplitudes is used. The operators $\mathcal{R}^{I_{i,j},s's'}_{\alpha(i)\beta(j),nn'}$ are driving terms, which describe the effective particle-exchange potential realized by the exchanged particle between the quasi-particles in channels $\alpha$ and $\beta$, which can be written as
\[ R^{I, \alpha, s}_{i,j} (p, p', E) = \frac{\Omega^{I, l_j}_{ss'}}{2} \int_{-1}^{+1} d(\hat{p} \cdot \hat{p}') \times u^{\alpha, s}_{n_i, l_i}(q, \epsilon_{\alpha}) \tau(z) u^{\beta, s'}_{n'_j, l'_j}(q', \epsilon_{\beta}) \]  

where the symbols \( \Omega^{I, l_j}_{ss'} \) are the Clebsch-Gordan coefficients, the functions \( u^{\alpha, s}_{n_i, l_i} \) are the form factors that generated by the separable representation of the sub-amplitudes appearing in the channels (1 and 3) and \( z \) is given as \( z = E - \frac{p^2}{2M_i^*} - \frac{p'^2}{2M_j^*} - \frac{\vec{p} \cdot \vec{p}'}{m} \). The subsystem energies are defined by \( \epsilon_{\alpha} \). Finally, the momenta \( \vec{q}(\vec{p}, \vec{p}') \) and \( \vec{q}'(\vec{p}', \vec{p}) \) are given in terms of \( \vec{p} \) and \( \vec{p}' \). We use the relations

\[ \vec{q} = \vec{p}' + \frac{M^\alpha}{m} \vec{p}, \quad \vec{q}' = \vec{p} + \frac{M^\beta}{m} \vec{p}' \]  

where \( m \) is exchanged particle or quasi-particle mass and the reduced masses \( M_{\alpha} \) and \( M_{\alpha} \) in the channel \( \alpha \) of the [3+1] subsystem are defined by

\[ M^\alpha_i = \frac{m^\alpha (m^\alpha + m^\alpha_k + m^\alpha_l)}{(m^\alpha + m^\alpha_j + m^\alpha_k + m^\alpha_l)}, \]

\[ M^\alpha_j = \frac{m^\alpha_j (m^\alpha_k + m^\alpha_l)}{(m^\alpha_j + m^\alpha_k + m^\alpha_l)}, \]  

FIG. 1. (Color online) Diagrammatic representation of different partitions of the \( \bar{K}NNN \) (up) and \( \bar{K}KN \) (down) systems without including the particle, spin and isospin labels. The anti-kaons are defined by turquoise circles and the nucleons by brown circles.

Before solving the four-body equations, one should solve the bound state problem for the two- and three-body subsystems that are specified in the partitions (1 and 3). The three-body Faddeev equations [10] in the AGS take the form

\[ \mathcal{K}^{I, l_i}_{i,j} = \mathcal{M}^{I, l_i}_{i,j} + \sum_{k,l_k} \mathcal{M}^{I, l_i}_{i,j} \tau^{I, l_k}_{i,k} \mathcal{K}^{I, l_k}_{i,k} \]  

5
The inputs for the AGS system of equations (9) are the two-body operators $\tau_{k,k}^{\alpha,s}$, evaluated in the presence of a spectator particle. The operators $K_{ij,IiIj}^{\alpha,s}$ are the usual transition amplitudes between Faddeev channels [10] and the operators $M_{ij,IiIj}^{\alpha,s}$ are the corresponding Born terms. Faddeev partition indices $i, j, k = 1, 2, 3$ denote simultaneously an interacting pair and a spectator particle.

To take the coupling between $\bar{K}N$ and $\pi\Sigma$ channels directly into account, the formalism of Faddeev equations should be extended to include the particle channels [10, 11]. Thus, all three-body operators should have particle indices for each state in addition to the Faddeev indices. In the present calculations, the $\pi\Sigma N$ channel of the $\bar{K}NN$ system and $\pi\bar{K}\Sigma$ channel of the $\bar{K}\bar{K}N$ system have not been included directly and one-channel Faddeev AGS equations are solved for the $\bar{K}NN$ and the $\bar{K}\bar{K}N$ systems. We approximated the full coupled-channel interaction by constructing the so-called exact optical $\bar{K}N - \pi\Sigma$ potential [37]. The exact optical potential provides exactly the same elastic $\bar{K}N$ scattering amplitude as the coupled-channel model of interaction. Thus, our coupled-channels four-body calculations with coupled-channel $\bar{K}N - \pi\Sigma$ interaction is equivalent to the one-channel four-body calculation using the so-called exact optical $\bar{K}N(-\pi\Sigma)$ potential. The decaying to the $\pi\Sigma N$ and $\pi\bar{K}\Sigma$ channels is taken into account through the imaginary part of the optical $\bar{K}N(-\pi\Sigma)$ potential. Since, we do not include the $\pi\Sigma N$ and $\pi\bar{K}\Sigma$ channels directly into our calculations, in Eq. (9) we neglected the particle indices of the operators.

We have to introduce a separable representation for the three-body amplitudes and driving terms, which will be necessary to find the solution of three-body subsystems. In the present work, for this purpose we apply the EDPE expansion method [28, 35]. The separable form of the Faddeev transition amplitudes is given by

$$K_{ij,IiIj}^{\alpha,s}(q, q', \epsilon) = \sum_{n,m} N_r \sum_{iIj} u_{n,iIi}^{\alpha,s}(q, \epsilon) \theta_{nm}(\epsilon) u_{m,jIj}^{\alpha,s}(q', \epsilon).$$

The starting point of the energy dependent pole expansion method (EDPE) is the eigenvalue equations for the vertex functions $u_{n,iIi}^{\alpha,s}(q, B_{\alpha})$

$$u_{n,iIi}^{\alpha,s}(q, B_{\alpha}) = \frac{1}{\lambda_n^{\alpha}} \sum_{jIj} \int M_{ij,IiIj}^{\alpha,s}(q, q'; B_{\alpha})$$

$$\times \tau_{ij}^{\alpha,s}(B_{\alpha} - \frac{q'^2}{2M_j}) u_{n,jIj}^{\alpha,s}(q', B_{\alpha}) dq'.$$
By solving equations (9), we can define the binding energy and width of $K^{-}pp$, $K^{-}K^{-}p$ and $^3He$ systems. Since, the $K^{-}d$ is not bound, in Eq. (9), we will put $B_{K^{-}d} = -\epsilon_d$ (the deuteron binding energy). Therefore, by taking $B_{K^{-}pp} = -\epsilon_{K^{-}pp}$ (the $K^{-}pp$ binding energy), $B_{K^{-}K^{-}p} = -\epsilon_{K^{-}K^{-}p}$ (the $K^{-}K^{-}p$ binding energy) and $B_{NNN} = -\epsilon_{^3He}$ (the triton binding energy), we can define the form factors $u_{n,li}^{\alpha,s}(q, B_\alpha)$ for each state. The extrapolation of the vertices $u_{n,li}^{\alpha,s}$ onto the whole energy axes is achieved according to the following expression

$$u_{n,li}^{\alpha,s}(q, \epsilon) = \frac{1}{\lambda_\alpha^2} \sum_{jI_j} \int \mathcal{M}_{ij,li}^{\alpha,s}(q, q'; \epsilon) \times \tau_{jI_j}^{\alpha,s}(B_\alpha - \epsilon_{q'^2/2M_j^2}) u_{n,li}^{\alpha,s}(q', B_\alpha) dq'.$$

(12)

After finding the vertex functions $u_{n,li}^{\alpha,s}(q, \epsilon)$, we can define the effective EDPE propagators $\theta^{(s)}_\alpha(\epsilon)$ in Eqs. 5 and 10 by

$$\left(\rho^{(s)}_\alpha^{-1}(\epsilon)\right)_{mn} = \sum_{jI_j} \int \left[ u_{m,jI_j}^{\alpha,s}(q, B_\alpha) \tau_{jI_j}^{\alpha,s}(B_\alpha - \epsilon_{q'^2/2M_j^2}) - u_{m,jI_j}^{\alpha,s}(q, \epsilon) \tau_{jI_j}^{\alpha,s}(\epsilon - \epsilon_{q'^2/2M_j^2})\right] u_{n,jI_j}^{\alpha,s}(q, \epsilon) dq'.$$

(13)

Before we proceed to solve the four-body equations, we also need as input the equations describing two independent pairs of interacting particles such as $(\bar{K}N)(NN)$, $(\bar{K}\bar{K})(NN)$ and $(\bar{K}N)(\bar{K}N)$ [16, 17]. Thus, one should define the vertex functions and the EDPE propagators for each isospin state of these subsystems. We have taken $B_{(\bar{K}N)_{I=0}(NN)} = -\epsilon_{(\bar{K}N)_{I=0}}$ (the $\Lambda(1405)$ resonance mass and width), $B_{(\bar{K}N)_{I=1}(NN)} = -\epsilon_d$, $B_{(\bar{K}\bar{K})_{I=1}(NN)} = -\epsilon_d$, $B_{(\bar{K}N)_{I=0}(K\bar{N})_{I=0}} = -\epsilon_{(\bar{K}N)_{I=0}}$ and finally $B_{(\bar{K}N)_{I=1}(\bar{K}N)_{I=1}} = 0$.

III. RESULTS AND DISCUSSION

Before we proceed to represent the obtained results, we will have a survey on the two-body interactions. The two-body interactions are the central input to our few-body calculations. The orbital angular momentum of all interactions is taken to be zero. We used separable potentials in momentum representation in the form

$$V_I^{\alpha\beta}(k^\alpha, k^\beta, E) = g_I^{\alpha}(k^\alpha) \lambda_I^{\alpha\beta} g_I^{\beta}(k^\beta),$$

(14)

where $g_I^{\alpha}(k^\alpha)$ is the form factor of the interacting two-body system with relative momentum $k^\alpha$ and isospin $I$. Here, $\lambda_I^{\alpha\beta}$ is the strength parameter of the interaction. The interactions are further
labeled with the \( \alpha \) values to take the \( \bar{K}N - \pi\Sigma \) coupling directly into account. Using separable potentials in the form \( \mathbf{14} \) for two-body interaction, we can define the two-body t-matrices in the form

\[
T_{I}^{\alpha\beta}(k_{\alpha}, k_{\beta}; E) = g_{I}^{\beta}(k_{\alpha})\tau_{I}^{\alpha\beta}(E)g_{I}^{\alpha}(k_{\beta}),
\]

(15)

where the operator \( \tau_{I}^{\alpha\beta}(E) \) is the usual two-body propagator. To describe the \( \bar{K}N - \pi\Sigma \) interaction, which plays a crucial role in the present three- and four-body calculations, we considered three different phenomenological and chiral potentials \([31, 32]\). The potentials have one- and two-pole structure of the \( \Lambda(1405) \). The parameters of the \( \bar{K}N - \pi\Sigma \) phenomenological potentials, are given in Ref. \([32]\). These potentials are adjusted to reproduce the SIDDHARTA experiment results \([38]\).

Thus, depending on a pole structure of the \( \Lambda(1405) \), we refer these potentials as “SIDD-1” and “SIDD-2” potential. The parameters of the \( \bar{K}N - \pi\Sigma \) chiral potential, are given in Ref. \([31]\) which is an energy-dependent potential. Another important interaction in our few-body calculations is the nucleon-nucleon interaction. The potential that we considered here is the one-term PEST potential from Ref. \([39]\), which is a separable approximation of the Paris model of \( NN \) interaction. The parameters of the PEST potential are given in Ref. \([39]\).

The experimental information on \( \bar{K}-\bar{K} \) interaction is poor. We used a separable potential for the \( \bar{K}\bar{K} \) with \( I = 1 \), in a Yamaguchi form

\[
V_{\bar{K}\bar{K}}^{I=1}(k, k') = \lambda_{\bar{K}\bar{K}}^{I=1}g_{\bar{K}\bar{K}}(k)g_{\bar{K}\bar{K}}(k'),
\]

\[
g_{\bar{K}\bar{K}}(k) = \frac{1}{k^{2} + \Lambda_{\bar{K}\bar{K}}^{2}}.
\]

(16)

The range parameter value 3.9 fm\(^{-1} \) is adopted for \( \bar{K}\bar{K} \) interaction to represent the exchange of heavy mesons and the strength parameter \( \lambda_{\bar{K}\bar{K}}^{I=1} \) is adjusted to reproduce the \( K^{+}K^{+} \) scattering length, for which we used as a guideline the result of lattice QCD calculation as \( a_{K^{+}K^{+}} = 0.141 \) fm \([40]\).

It has been suggested in Ref. \([30]\), that the \( K^{-}pp \) system might exhibit a double-pole structure similar to \( \Lambda(1405) \). Based on the their calculations, such double poles of the \( K^{-}pp \) system are related to the experimental results. The observed signal close to the \( \pi\Sigma N \) threshold in DISTO and J-PARC E27 experiments, which indicate a deeply bound \( K^{-}pp \) state are regarded as the second pole of the \( K^{-}pp \) system, while the observed signal close to the \( \bar{K}NN \) threshold in J-PARC E15 experiment is considered as the first pole. The position of a quasi-bound state in the three-body problem is usually defined by solving the homogeneous integral equations \((11)\). To find the resonance energy of the three-body system using these equations, one should search for a complex
energy at which the first eigenvalue of the kernel matrix becomes equal to one. The essence of the calculation scheme is the integration in the complex plane [11]. In the present work, we used another way to find the $K^{-pp}$ pole position(s) without integration in the complex momentum plane. The signal of the quasi-bound state would be observed in the Faddeev amplitudes.

We studied how the signature of the $K^{-pp}$ system shows up in the three-body scattering amplitudes by using coupled-channel Faddeev AGS equations. To achieve this goal, we must solve the inhomogeneous integral equations for the amplitudes defined in Eq. (9). Since the input energy of AGS equations is complex the standard moving singularities that are caused by the opened channel $\pi\Sigma N$, will not appear. With this method, we computed the scattering amplitudes at complex energies. The calculated resonance energies that have presented in Table I, give pole positions of the $[\bar{K}N]_{I=0}$ and $K^{-pp}$ system and the results for Faddeev amplitudes are depicted in Figs. 2 and 3. Using Eq. 9, the amplitude $|K^{-2,0}_{N,N,00}(q,q',\epsilon)|$ for $(\bar{K}NN)_{s=0}$ system is calculated. The operator $|K^{-2,0}_{N,N,00}(q,q',\epsilon)|$ is the usual Faddeev amplitude, describing the elastic process $[(\bar{K}N)_{I=0} + N]_{s=0} \rightarrow [(\bar{K}N)_{I=0} + N]_{s=0}$. In the present calculations, the momentums $q$ and $q'$ were taken to be 150MeV/c. The real part of the three-body energy, $\epsilon$, changes from 2270 MeV to 2370 MeV and the imaginary part changes from -100 to 0 MeV. In Fig. 2, we used the energy-dependent chiral potential to calculate the scattering amplitudes and in Fig. 3, we used one- and two-pole version of the SIDD potential. As one can see, both of the poles related to the structure of $\Lambda(1405)$ resonance can be seen in two-body scattering amplitudes. One close to the $\bar{K}N$ threshold with small width and the other close to the $\pi\Sigma$ threshold with large width, while the second pole in the $K^{-pp}$ system cannot be seen for all models of $\bar{K}N$ interaction.

Starting from Faddeev AGS equations 5 and using different versions of the $\bar{K}N - \pi\Sigma$ potentials, the binding energy and width of the $K^{-ppn}$ and $K^{-K^-pp}$ quasi-bound state were evaluated. The dependence of the pole energy on different models of $\bar{K}N - \pi\Sigma$ interaction was studied. The separable expansion of the Faddeev amplitudes plays an important role, which enable us to reduce the four-body Faddeev amplitudes to a single variable integral equation. An important parameter in the separable expansion of the Faddeev amplitudes is the number of terms $(N_r)$ in Eq. 10. In Fig. 4, the sensitivity of the binding energy and width of four-body systems to the number of terms $N_r$ is investigated. The rate of convergence of $K^{-ppn}$ and $K^{-K^-pp}$ binding energies is investigated. One can see that the choice $N_r = 15$ provides rather satisfactory accuracy.
FIG. 2. (Color online) Global view of the calculated scattering amplitudes for the two-body $\bar{K}N$ and three-body $[\bar{K}NN]_{s=0}$ systems, where $\bar{K}NN$ is the subsystem of $\bar{K}NNN$ in $\alpha = 2$ channel. The upper diagram shows the results for $T_{I=0}^{\bar{K}N}$ and the lower shows the results for $|K_{NN,00}^{2,0}(q,q',\epsilon)|$ using the same potential, where $|K_{NN,00}^{2,0}(q,q',\epsilon)|$ describes the elastic process $[(\bar{K}N)_{I=0} + N]_{s=0} \rightarrow [(\bar{K}N)_{I=0} + N]_{s=0}$. In few-body calculations, we used energy-dependent chiral potential for $\bar{K}N - \pi\Sigma$ interaction which reproduces the two-pole structure of $\Lambda(1405)$ resonance.

TABLE I. Pole position(s) (in MeV) are extracted from the scattering amplitudes. The pole position is related to a quasi-bound states in the $\bar{K}N$. The calculations are performed with the SIDDHARTA and chiral energy-dependent potential.

| Potential          | First pole         | Second pole        |
|--------------------|--------------------|--------------------|
| $V_{SIDD-1}^{\bar{K}N-\pi\Sigma}$ | 1428.6 $- i46.5$  |                    |
| $V_{SIDD-2}^{\bar{K}N-\pi\Sigma}$ | 1419.6 $- i56.0$  | 1380.1 $- i104.5$ |
| $V_{chiral}^{\bar{K}N-\pi\Sigma}$ | 1420.6 $- i20.3$  | 1343.0 $- i72.5$  |

In Table III, the pole position of the quasi-bound states in the $\bar{K}NNN$ and $\bar{K}\bar{K}NN$ systems are presented for one- and two-pole version of the SIDDHARTA model of the $\bar{K}N - \pi\Sigma$ interaction. The pole energies of the $K^-ppn$ and $K^-\bar{K}^-pp$ systems are calculated with respect to the threshold of the corresponding four-body system and by keeping 15 terms in the energy-dependent pole expansion of the amplitudes (10).
TABLE II. Pole position(s) (in MeV) of the scattering amplitudes, which is related to a quasi-bound state in the $\bar{K}NN$ system. The Faddeev AGS calculations for $\bar{K}NN$ system performed with the SIDDHARTA and chiral energy-dependent potential. The potentials produce the one- and two-pole structure of the $\Lambda(1405)$ resonance. To extract the results in the second column of the table (Direct pole search), we solved Eq. 12 and for driving the results in the third column (Faddeev amplitudes), we solved the inhomogeneous Faddeev equations 9.

|                     | Direct pole search | Faddeev amplitudes |
|---------------------|-------------------|--------------------|
| $V_{\bar{K}N-\pi\Sigma}^{SIDD-1}$ | 2326.0 $- i34.2$ | 2326.1 $- i34.2$  |
| $V_{\bar{K}N-\pi\Sigma}^{SIDD-2}$ | 2325.0 $- i24.1$ | 2324.5 $- i24.5$  |
| $V_{\bar{K}N-\pi\Sigma}^{chiral}$   | 2346.5 $- i22.0$ | 2346.3 $- i22.0$  |

FIG. 3. (Color online) Same as Fig.2, but in the present calculations, we used the one- and two-pole version of the SIDD potential $V_{\bar{K}N-\pi\Sigma}^{SIDD}$. For diagrams on the left side, we used the one-pole version and for diagrams on the right side, the two-pole version of the SIDD potential was used. The two-body and three-body results are represented in the upper and lower row, respectively.

The $\bar{K}N$ channel is strongly coupled to the $\pi\Sigma$ channel. Therefore, in actual calculation the $\bar{K}NNN$ and $\bar{K}\bar{K}NN$ four-body equations should be generalized to include the coupled channels $\bar{K}NNN - \pi\Sigma NN$ and $\bar{K}\bar{K}NN - \pi\bar{K}\Sigma N - \pi\pi\Sigma\Sigma$, respectively. Plus the $\bar{K}N - \pi\Sigma$ and
nucleon-nucleon interactions, there are other interactions in the lower-lying four-body channels, namely $\pi\pi$, $\pi K$, $\pi N$, $\Sigma K$, $\Sigma\Sigma$ and $\Sigma N$ interactions. There is scarce information about some of these interactions and also when we include these remaining interactions the number of channels will increase rapidly and the treatment of the four-body turns out to be very complicated. These computational costs can be reduced by using an effective single-channel $\bar{K}N(-\pi\Sigma)$ potential. Therefore, in our calculations the lower-lying four-body channels are included effectively and consequently the remaining interactions in the lower four-body channels are neglected for the systems under consideration. Using Eqs. 14 and 15, we can define the optical t-matrices in the form

$$T_{\alpha\alpha}^I = \frac{1}{1 - \lambda_{I,\text{opt}}^I G_{\alpha}^I} \lambda_{I,\text{opt}}^{I,\alpha},$$

where the operator $G_{\alpha}$ is the Green’s function in $\alpha$ channel and the operator $\lambda_{I,\text{opt}}^{I,\alpha}$ can be defined by

$$\lambda_{I,\text{opt}}^{I,\alpha} = \lambda_{I,\alpha}^{I} + \lambda_{I,\beta}^{I} \frac{G_{\beta}^{I}}{1 - \lambda_{I,\beta}^{I} G_{\beta}^{I}} \lambda_{I,\alpha}^{I}.$$

A definitive study of the $K^-ppn$ and $K^-K^-pp$ bound states could be performed using standard energy-dependent $\bar{K}N$ input potential, too [31]. The energy-dependent potentials provide a weaker $\bar{K}N$ attraction for lower energies than the energy-independent potentials. Therefore, one expects that, the quasi-bound states resulting from the energy-dependent potential happen to be shallower. The comparison of the obtained results for the chiral $\bar{K}N - \pi\Sigma$ interaction with the calculated binding energies for phenomenological $\bar{K}N$ interaction shows that energy-independent...
potentials produce much deeper bound state for four-body kaonic systems under consideration in the present work.

It was shown in Ref. [37], that exact optical potential approximation is more accurate for the one-pole version of the $\bar{K}N$ interaction than for the two-pole model of interaction. In other words, the optical approximation provides exactly the same elastic $\bar{K}N$ amplitude as the coupled-channel model of interaction for one-pole potential while for two-pole model, we cannot see this behavior. In present calculations, we can expect that the binding energies resulting from the optical approximation of the $V_{2,SIDD}^{\bar{K}N-\pi\Sigma}$ potential be different from the full coupled channel calculations of the $\bar{K}NNN - \pi\Sigma NN$ and $\bar{K}\bar{K}NN - \pi\pi\Sigma\Sigma$ systems. Comparing the results in Tables II and III, we can see that the binding energy of the $K^-pp$ system resulting from both one- and two-pole models of the SIDD potential are close to each other while in four-body calculations, the one-pole potential reproduce a deeper bound state for $K^-ppn$ and $K^-K^-pp$ systems. This difference may come from the fact that the optical approximation is not very appropriate for two-pole potentials. The fully coupled-channel calculations of the systems under consideration in the future may help us to make a better judgment.

The calculated binding energy and width values of the $K^-ppn$ and $K^-K^-pp$ quasi-bound state are compared in Table III with other theoretical results. The Faddeev calculations of the $K^-ppn$ and $K^-K^-pp$ systems were also carried out in Refs. [16, 17] using the same $V_{KNN}$ potentials. The HSE method was used there to find the separable expression of the Faddeev amplitudes in (2+2) and (3+1) subsystems. Comparing the present results for $K^-ppn$ and $K^-K^-pp$ systems with those in [16] and [17] shows that the obtained binding energies within the EDPE method are deeper than those resulting from the HSE method.

Faddeev-Yakubowsky equations were solved in [42] with phenomenological energy independent $\bar{K}N$ potentials. Therefore, in principle, their calculation with the energy independent version of the $\bar{K}N$ potential should give a result, which are close to ours with a phenomenological model of interaction. The variational calculations using AY potential were also carried out in [43]. It is seen, however, that only their binding energies are comparable to ours, while the widths obtained in [43] are much bigger than ours. Variational calculations using chiral energy independent $\bar{K}N$ potential were also done in Refs. [20, 43]. The obtained binding energies are shallower than those calculated in the present work. It is caused by the relative weakness of the chiral $\bar{K}N$ interaction as compared to phenomenological $\bar{K}N$. 

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TABLE III. The dependence of the pole position(s) (in MeV) of the $K^-ppn$ and $K^-K^-pp$ systems on the different models of the $KN - \pi \Sigma$ interactions is investigated. $V^{SIDD-1}_{KN-\pi\Sigma}$ and $V^{SIDD-2}_{KN-\pi\Sigma}$ standing for a one-pole and a two-pole structure of the $\Lambda(1405)$ resonance, which are produced phenomenologically and $V^{chiral}_{KN-\pi\Sigma}$ is used for energy-dependent chiral potential. The calculated binding energies and widths are also compared with other theoretical results. It was referred to the potentials in Refs. [3] and [41] as the Akaishi-Yamazaki (AY) and HW potentials, respectively. The first one is based on chiral dynamics and the second one is constructed phenomenologically.

|                | $B_{K^-ppn}$ | $\Gamma_{K^-ppn}$ | $B_{K^-K^-pp}$ | $\Gamma_{K^-K^-pp}$ |
|----------------|--------------|-------------------|----------------|-------------------|
| **Present AGS (EDPE):** |              |                   |                |                   |
| with $V^{1,SIDD}_{KN-\pi\Sigma}$ | 73.5         | 22.0              | 99.2           | 11.4              |
| with $V^{2,SIDD}_{KN-\pi\Sigma}$ | 58.5         | 27.0              | 89.0           | 11.4              |
| with $V^{chiral}_{KN-\pi\Sigma-\pi\Lambda}$ | 41.4         | 31.5              | 60.9           | 65.0              |
| **Previous AGS (HSE):** |              |                   |                |                   |
| with $V^{1,SIDD}_{KN-\pi\Sigma}$ | 68.8 [16]    | 22.0 [16]         | 93.7 [17]      | 30.6 [17]         |
| with $V^{2,SIDD}_{KN-\pi\Sigma}$ | 55.9 [16]    | 17.6 [16]         | 84.2 [17]      | 7.8 [17]          |
| with $V^{1,KEK}_{KN-\pi\Sigma}$ [17] | –            | –                 | 84.6           | 24.2              |
| with $V^{2,KEK}_{KN-\pi\Sigma}$ [17] | –            | –                 | 81.8           | 4.6               |
| **Faddeev-Yakubowsky:** |              |                   |                |                   |
| MAY [42] | 74            | –                 | 104            | –                 |
| **Variational:** |              |                   |                |                   |
| BGL [20] | 29.3          | 32.9              | 32.1           | 80.5              |
| KTT with AY [3] | 92 – 98       | $\sim$ 83        | $\sim$ 92     | $\sim$ 73        |
| KTT with HW [41] | $\sim$ 29     | $\sim$ 30        | $\sim$ 32     | $\sim$ 79        |
IV. CONCLUSION

In summary, the Faddeev-type calculations of $\bar{K}NN$ and $\bar{K}\bar{K}NN$ systems were performed. We have calculated the binding energy and width of these kaonic systems. To investigate the dependence of the resulting binding energies and widths on models of $\bar{K}N - \pi\Sigma$ interaction, different versions of $\bar{K}N - \pi\Sigma$ potentials, which produce the one- or two-pole structure of $\Lambda(1405)$ resonance, were used. In the present calculations, we approximated the full coupled-channel one- and two-pole models of interaction by constructing the exact optical $\bar{K}N - \pi\Sigma$ potential. Therefore, one-channel Faddeev AGS equations are solved for the $\bar{K}NN$ and $\bar{K}\bar{K}NN$ systems and the decaying to the $\pi\Sigma NN$ and $\pi\pi\Sigma\Sigma$ channels is taken into account through the imaginary part of the optical $\bar{K}N(-\pi\Sigma)$ potential. For $K^-ppn$ system, we obtained binding energy $\sim 41$ MeV using the chiral and $58 - 73$ MeV for the SIDD $\bar{K}N$ potentials. The width is about $\sim 30$ MeV for chiral potential, while the SIDD potentials give $\sim 22 - 27$ MeV. The calculations yielded binding energy $B_{\text{chiral}} \sim 61$ and $B_{\text{pheno.}} \sim 90-100$ MeV for $K^-K^-pp$ system. The obtained widths for $K^-K^-pp$ are $\Gamma_{\text{chiral}} \sim 65$ and $\Gamma_{\text{pheno.}} = 11$ MeV. It is expected that the omission of the lower-lying four-body channels may have an important effect on the width of the state specially in the case of $\bar{K}\bar{K}NN$ system. The $\pi\pi\Sigma\Sigma$ threshold is much lower in comparison to the bound state as the one for $\pi\Sigma NN$ in the case of $\bar{K}NNN$ system. Thus, there is more phase space available for the decay and that again could lead to a larger width. Therefore, the full coupled-channel calculations of the systems under consideration in the future may help us to make a better judgment about the effects of the lower-lying channels. For the four-body systems, the $\bar{K}NN$ system was also studied. The Faddeev amplitudes for this system were calculated and it was shown that double-pole structure cannot be seen in the Faddeev amplitudes.

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