Stability for a class of semilinear fractional stochastic integral equations

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Abstract

In this paper we study some stability criteria for some semilinear integral equations with a function as initial condition and with additive noise, which is a Young integral that could be a functional of fractional Brownian motion. Namely, we consider stability in the mean, asymptotic stability, stability, global stability and Mittag-Leffler stability. To do so, we use comparison results for fractional equations and an equation (in terms of Mittag-Leffler functions) whose family of solutions includes those of the underlying equation.

Keywords: Comparison results for fractional differential equations, fractional Brownian motion, Mittag-Leffler function, stability criteria, Young integral for Hölder continuous functions

Subject classifications: 34A08, 60G22, 26A33, 93D99

1 Introduction

Currently fractional systems are of great interest because of the applications they have in several areas of science and technology, such as engineering, physics, chemistry, mechanics, etc. (see, e.g. [4], [15], [18], [36] and the references therein). Particularly we can mention system identification [4], robotics [26], control [4, 36], electromagnetic theory [14], chaotic dynamics and synchronization [12, 13, 42, 44], applications on viscoelasticity [2], analysis of electrode processes [16], Lorenz systems [12], systems with retards [6], quantic evolution of complex systems [19], numerical methods for fractional partial differential equations [5, 30, 31], among other. A nice survey of basic properties of deterministic fractional differential equations is in Lakshmikantham and Vatsala [20]. Also, many researchers have established stability criteria of mild solutions of stochastic fractional differential equations using different techniques.

For deterministic systems, the stability of fractional linear equations has been analyzed by Matignon [27] and Radwan et al. [38]. Besides, several authors have studied non-linear cases using Lyapunov method (see, e.g. Li et al. [23] and its references). In particular, non-linear fractional systems with a function as initial condition using also the Lyapunov technique have been considered in the Ph.D. Thesis of Martínez-Martínez [28]. Moreover, in the work of Junsheng et al. [17] the form of the solution for a linear fractional equation with a constant initial condition in terms of Mittag-Leffler function is given by means of the Adomian decomposition method. Wen et al. [41] have established stability results for fractional non-linear equations via the Gronwall inequality. Lemma 2.2 below can be seen as an extension of the results in [41] and the Gronwall inequality stated in [41]. In [41], the stability is used to obtain synchronization of fractional systems.

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On the other hand, a process used frequently in literature is fractional Brownian motion \( B^H = \{ B^H_t, t \geq 0 \} \) due to the wide range of properties it has, such as long range memory (when the Hurst parameter \( H \) is greater than one half) and intermittency (when \( H < 1/2 \)). Unfortunately, in general, it is not a semimartingale (the exception is \( H = 1/2 \)). Thus, we cannot use classical Itô calculus in order to integrate processes with respect to \( B^H \) when \( H \neq 1/2 \), but we may use another approaches such as Young integration (see Gubinelli [11], Young [45], Zähle [49], Dudley and Norvaiša [8], Lyons [25]). The reader can also see Nualart [33], and Russo and Vallois [39] for other types of integrals. As a consequence, an important application is the analysis of stochastic integral equations driven by fractional Brownian motion that has been considered by several authors these days for different interpretations of stochastic integrals (see, e.g. Lyons [25], Quer-Sardanyons and Tindel [37], León and Tindel [21], Nualart [33], Friz and Hairer [9], Lin [24] and Nualart and Răşcanu [34]).

Stability of stochastic systems driven by Brownian motion has been also studied. Some authors use fundamental solution of these equations in order to investigate the stability of random systems. An example of this is the paper of Applebay and Freeman [1], who give the solution in terms of the principal matrix of integrodifferential equations with an Itô integral noise and find the equivalence between almost sure exponential convergence and the p-th mean exponential convergence to zero for these systems. Bao [3] uses Gronwall inequality to state the mean square stability for Volterra-Itô equations with a function as initial condition and bounded kernels. Several researchers have studied stability of stochastic systems via Lyapunov function techniques. An example of this is the paper of Li et al. [23], who proves stability in probability for Itô-Volterra integral equation, also Zhang and Li [47] have stated a stochastic type stability criteria for stochastic integro-differential equations with infinite retard, and, Zhang and Zhang [48] have dealt with conditional stability of Skorohod Volterra type equations with anticipative kernel. Nguyen [32] present the solution via the fundamental solution for linear stochastic differential equations with time-varying delays to obtain the exponential stability of these systems. The noise is an additive one and has form \( \int_0^t \sigma(s)dW_s^H \). Here

\[
W_t^H = \int_0^t (t-s)^{H-1/2}dW_s, \quad H \in (1/2, 1),
\]

\( W \) is a Brownian motion and \( \sigma \) is a deterministic function such that

\[
\int_0^\infty \sigma^2(s)e^{2\lambda s}ds < \infty,
\]

for some \( \lambda > 0 \). Also, Zeng et al. [50] utilize the Lyapunov function techniques to prove stability in probability and moment exponential stability for stochastic differential equation driven by fractional Brownian motion with parameter \( H > 1/2 \). Yan and Zhang [43] proved sufficient conditions for the asymptotical stability in \( p \)-th moment for the closed form of the solution to a fractional impulsive partial neutral stochastic integro-differential equation with state dependent retard in Hilbert space. In the linear case, Fiel et al. [10] have used the Adomian decomposition method to find the mild solution of an stochastic fractional integral equation with a function as initial condition driven by a Hölder continuous process in terms of Young or Skorohod integrals. This closed form is given in terms of Mittag-Leffler functions. Thus, these results can be seen as extensions of the results given in [10] and [41]. Finally, the stability of equation (1.1) in the case that \( \theta \) is either a Hölder continuous process, or a functional of fractional Brownian motion is considered in Section [4].

\[
X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[AX(s) + h(X(s))]ds + Z_t, \quad t \geq 0.
\]

The initial condition \( \xi = \{ \xi_t, t \geq 0 \} \) is a function, \( h \) is a \( O(x) \) as \( x \to 0 \) (i.e. there are \( C > 0 \) and \( \delta > 0 \) such that \( |h(x)| \leq C|x| \) for \( |x| < \delta \), \( \beta \in (0, 1) \), \( A < 0 \) and \( Z \) is a Young integral of the form

\[
Z_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}d\theta_s.
\]

Here, \( \theta = \{ \theta_s, s \geq 0 \} \) is a \( \gamma \)-Hölder continuous function that may represent the paths of a functional of fractional Brownian motion, where \( \gamma \in (0, 1), \alpha \in (1,2) \) and \( \alpha + \gamma > 2 \). Unlike other papers where the involved kernels are bounded functions we consider the case that kernels are not bounded, and use comparison results as a main tool. It is worth mentioning that it is considered the stability of the solution to (1.1) in [41] with \( Z \equiv 0 \) and \( \xi \) a constant.

This work is organized as follows. In Section [2] we introduce a fractional integral equation, whose family of solutions include those of (1.1). Also, in Section [3] we state a comparison result for fractional systems that becomes the main tool for our results. In Section [4] we study some stability criteria for equation (1.1) in the case that \( Z \equiv 0 \). These results can be seen as extensions of the results given in [10] and [41]. Finally, the stability of equation (1.1) in the case that \( \theta \) is either a Hölder continuous process, or a functional of fractional Brownian motion is considered in Section [4].
2 Preliminaries

In this section we introduce the framework and the definitions that we use to prove our results. Although some results are well-known, we give them here for the convenience of the reader. Part of the main tool that we need is the stability of some fractional linear systems as it was presented by Fiel et al. [10] and a comparison result (see Lemma 2.3 below).

2.1 The Mittag-Leffler function

The Mittag-Leffler function is an important tool of fractional calculus due to its properties and applications. As we can see in Lemma 2.2 below, solutions of semilinear fractional integral equations depend on it. In order to see a more detailed exposition on this function, the reader is referred to the book of Podlubny [36]. For such that, the seminorm $\Gamma$ where $\Gamma$ is the Gamma function. A function $f$ defined on $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $f^{(n)}$ for all $n \in \mathbb{N} \cup \{0\}$, and if

$(-1)^n f^{(n)}(t) \geq 0$

for all $t > 0$. In particular, we have that each completely monotonic function on $(0, \infty)$ is positive, decreasing and convex, with concave first derivative (see, e.g. [29] and [40]). Moreover, for $a < 2$, there is a positive constant $C_{a,b}$ such that

$|E_{a,b}(z)| \leq \frac{C_{a,b}}{1 + |z|}, \quad z \leq 0,$

(2.1)

(see [36], Theorem 1.6). For $a, b > 0$ and $\lambda \in \mathbb{R}$, this function satisfies the following (see (1.83) in [36]):

$\frac{d}{dz} \left( z^{b-1} E_{a,b}(\lambda z^a) \right) = z^{b-2} E_{a,b-1}(\lambda z^a).$

(2.2)

Also, for $b > 0$ (see equality (1.99) in [36]), we have

$\int_0^z s^{b-1} E_{a,b}(\lambda s^a) ds = z^b E_{a,b+1}(\lambda z^a).$

(2.3)

2.2 The Young integral

Here we introduce the Young integral, which is an integral with respect to Hölder continuous functions. This was initially defined for functions with $p$-variation in Young [45].

For $T > 0$ and $\gamma \in (0, 1)$, let $C^\gamma_1([0, T]; \mathbb{R})$ be the set of $\gamma$-Hölder continuous functions $g : [0, T] \to \mathbb{R}$ of one variable such that, the seminorm

$||g||_{\gamma,[0,T]} := \sup_{r,t \in [0,T], r \neq t} \frac{|g_t - g_r|}{|t - r|^\gamma},$

is finite. Also by $||g||_{\infty,[0,T]}$, we denote the supremum norm of $g$.

Using the following result, we can understand easily the basic properties of Young integral for Hölder continuous functions. The proof of this theorem can be found in [11] (see also [21]). Sometimes we write $\mathcal{J}_{st}(f dg)$ instead of $\int_s^t f_a d_g u_a$.

**Theorem 2.1** Let $f \in C^\kappa_1([0,T]; \mathbb{R})$ and $g \in C^\gamma_1([0,T]; \mathbb{R})$, with $\kappa + \gamma > 1$. Then:

1. $\mathcal{J}_{st}(f dg)$ coincides with the usual Riemann integral if $f$ and $g$ are smooth functions.

2. We have, for $s \leq t \leq T$,

$\mathcal{J}_{st}(f dg) = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} f_{t_i}(g_{t_{i+1}} - g_{t_i}),$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \ldots, t_n = t\}$ of $[s, t]$, whose mesh tends to zero. In particular, $\mathcal{J}_{st}(f dg)$ coincides with the Young integral as defined in [45].

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3. The integral \( J(f) \) satisfies:

\[
|J_{s,t}(f)g| \leq ||f||_{\infty}||g||_{\Gamma}|t-s|^\gamma + c_{\gamma,k}||f||_{\infty}||g||_{\Gamma}|t-s|^\gamma+k,
\]

where \( c_{\gamma,k} = (2^{\gamma+k} - 2)^{-1} \).

We observe that this integral has been extended by Zähle [49], Gubinelli [11], Lyons [25], among others. For a more detailed exposition on the Young integral the reader is referred to the paper of Dudley and Norvaiša [8] (see also Gubinelli [11], and León and Tindel [21]).

2.3 Semilinear Volterra integral equations with additive noise

Here we consider the Volterra integral equation

\[
X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}AX(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}d\theta_s, \quad t \geq 0,
\]

(2.4)

where the initial condition \( \xi = \{\xi_t, t \geq 0\} \) is bounded on compact sets and measurable, \( \beta \in (0,1), \ A \in \mathbb{R}, \ \alpha \in (1,2), \ \theta = \{\theta_s, s \geq 0\} \) is a \( \gamma \)-Hölder continuous function with \( \gamma \in (0,1) \) and \( \Gamma \) is the Gamma function. The second integral in (2.4) is a Young one and is well-defined if \( \alpha - 1 + \gamma > 1 \), because \( s \mapsto (t-s)^{\alpha-1} \) is \( (\alpha - 1) \)-Hölder continuous on \([0,t]\).

Now we give two lemmas that we need in the remaining of the paper. The following result provides a closed form for the solution to equation (2.4) and its proof is in [10].

Lemma 2.2 Let \( \alpha + \gamma > 2 \) and \( A \in \mathbb{R} \). Then, the solution to (2.4) has the form

\[
X(t) = \xi_t + A \int_0^t (t-s)^{\beta-1}E_{\beta,\gamma}(A(t-s)^{\beta})\xi_s ds + \int_0^t (t-s)^{\alpha-1}E_{\beta,\alpha}(A(t-s)^{\beta})d\theta_s, \quad t \geq 0.
\]

(2.5)

Moreover, if

\[
\xi_t = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1}g(s)ds, \quad t \geq 0,
\]

where \( \eta > 0 \) and \( g \in L^1([0,M]) \) for each \( M > 0 \), then we have

\[
X(t) = \int_0^t (t-s)^{\eta-1}E_{\beta,\eta}(A(t-s)^{\beta})g(s)ds + \int_0^t (t-s)^{\alpha-1}E_{\beta,\alpha}(A(t-s)^{\beta})d\theta_s, \quad t \geq 0.
\]

In this work we use comparison methods in order to obtain the stability of some fractional systems. We can find comparison theorems in the literature for fractional evolution equations (see, e.g. Theorem 4.2 in [20]), but, unfortunately this results are not suitable for our purpose. Thus, we give the following lemma, that is a version of Theorem 2.2.5 in Pachpatte [35] and allows us to prove stability for the semi-linear equations that we study. Hence, this result is a fundamental tool in the development of this paper.

Lemma 2.3 Let \( k : [0,T] \times \mathbb{R} \rightarrow \mathbb{R} \) be a function such that:

i) \( k(\cdot, x) \) is measurable on \([0,T]\) for each \( x \in \mathbb{R} \).

ii) There is a constant \( M > 0 \) such that \( |k(s,x) - k(s,y)| \leq M|x - y| \), for any \( s \in [0,T] \) and \( x, y \in \mathbb{R} \).

iii) \( k \) is bounded on bounded sets of \([0,T] \times \mathbb{R} \).

iv) \( k(s, \cdot) \) is non-decreasing for any \( s \in [0,T] \).

Also, let \( B \in \mathbb{R}, \beta \in (0,1), \) and \( x \) and \( y \) two continuous functions on \([0,T]\) such that

\[
x(t) \leq y(t) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(B(t-s)^{\beta})k(s,x(s))ds, \quad t \in [0,T].
\]

(2.6)

Then \( x \leq u \) on \([0,T]\), where \( u \) is the solution to the equation

\[
u(t) = y(t) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(B(t-s)^{\beta})k(s,u(s))ds, \quad t \in [0,T].
\]

(2.7)
Remark. The assumptions of $k$ yield that equation (2.7) has a unique continuous solution.

Proof. Denote by $C([0, T])$ the family of continuous functions on $[0, T]$. Let $G : C([0, T]) \to C([0, T])$ be given by

$$(Gz)(t) = y(t) + \int_0^t (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, z(s)) ds, \quad t \in [0, T].$$

It is not difficult to see that Hypotheses i) and iii) imply that $G$ is well-defined. It means, $G(z)$ is a continuous function for each $z \in C([0, T])$. Remember that we denote $\sup_{t \in [a, b]} |z(t)|$ by $||z||_{\infty, [a, b]}$ for any function $z \in C([a, b])$. Then, from the continuity of $E_{\beta, \beta}$ and hypothesis ii), there is a constant $M > 0$ such that, for every $z, \tilde{z} \in C([0, T])$, we have

$$||Gz - \tilde{G}z||_{\infty, [0, T]} \leq M \int_0^t (t - s)^{\beta - 1} |k(s, z(s)) - k(s, \tilde{z}(s))| ds$$

$$\leq M M \int_0^t (t - s)^{\beta - 1} |z(s) - \tilde{z}(s)| ds$$

$$\leq \frac{M M}{\beta} T^{\beta} ||z - \tilde{z}||_{\infty, [0, T]}, \quad \text{for } t \in [0, T].$$

Similarly, for $\bar{T} \leq T$, we are able to see that

$$||Gz - G\tilde{z}||_{\infty, [0, T]} \leq \frac{M M}{\beta} ||z - \tilde{z}||_{\infty, [0, T]} \bar{T}^{\beta}.$$

Consequently, if $\frac{M M}{\beta} \bar{T}^{\beta} < 1$, $G$ is a contraction on $C([0, \bar{T}])$. Therefore the sequence $v_{n+1} = Gv_n$, with $v_0 = x$, is such that $v_n(t) \to u(t)$ and $v_n(t) \leq v_{n+1}(t)$ for $t \in [0, \bar{T}]$, due to Hypothesis iv), (2.6), and $E_{\beta, \beta}$ being a completely monotonic function. Thus the result is true if we write $\bar{T}$ instead of $T$.

Now, suppose the lemma holds for the interval $[0, n\bar{T}]$, $n \in \mathbb{N}$. Then, by (2.6) we can write

$$x(t) \leq y(t) + \int_0^{n\bar{T}} (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, x(s)) ds + \int_{n\bar{T}}^t (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, x(s)) ds$$

$$\leq \bar{y}(t) + \int_{n\bar{T}}^t (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, x(s)) ds, \quad t \in [n\bar{T}, (n + 1)\bar{T}],$$

where

$$\bar{y}(t) = y(t) + \int_0^{n\bar{T}} (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, u(s)) ds.$$ 

Finally, defining $G^{(n)} : C([n\bar{T}, (n + 1)\bar{T}]) \to C([n\bar{T}, (n + 1)\bar{T}])$ by

$$(G^{(n)}z)(t) = \bar{y}(t) + \int_{n\bar{T}}^t (t - s)^{\beta - 1} E_{\beta, \beta} (B(t - s)^{\beta}) k(s, z(s)) ds,$$

and using the fact that equation (2.7) has a unique solution due to Hypothesis ii), we can proceed as in the first part of this proof to see that $x \leq u$ on $[0, (n + 1)\bar{T}]$. Thus, the result follows using induction on $n$.

\[ \square \]

3 A class of nonlinear fractional-order systems

In this section we establish two sufficient conditions for the stability of a deterministic semilinear Volterra integral equation. Thus, we improve the results in [10] for this kind of systems when the noise is null (i.e., $\mathcal{Z}$ in (1.1) is equal to zero).

3.1 A constant as initial condition

This part is devoted to refine Theorem 1 of [41] in the one-dimensional case. Toward this end, in this section, we suppose that the initial condition is a constant. That is, we first consider the fractional equation

$$X(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} AX(s) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} h(X(s)) ds, \quad t \geq 0,$$

with $x_0 \in \mathbb{R}$, $\beta \in (0, 1)$, $A < 0$ and $h : \mathbb{R} \to \mathbb{R}$ a measurable function.

In the remaining of this paper we deal with the following hypotheses.
(H1) There is a constant $C > 0$ such that $A + C < 0$ and $|h(x)| \leq C|x|$, for all $x \in \mathbb{R}$.

(H2) There are $\delta_0 > 0$ and $C > 0$ such that $A + C < 0$ and $|h(x)| \leq C|x|$, for $|x| < \delta_0$.

Now, we consider several definitions of stability.

**Definition 3.1** Any solution $X$ to equation (3.7) is said to be:

i) globally stable in the large if $X(t)$ goes to zero as $t$ tends to infinity, for all $x_0 \in \mathbb{R}$.

ii) Mittag-Leffler stable if there is $\delta > 0$ such that $|x_0| < \delta$ implies

$$|X(t)| \leq \left[ m(x_0) E_{\beta,1}(Bt^\beta) \right]^b, \quad t \geq 0,$$

where $\beta \in (0, 1)$, $B < 0$, $b > 0$ and $m$ is a positive and locally Lipschitz function with $m(0) = 0$.

iii) stable if for $\varepsilon > 0$, there is $\delta > 0$ such that $|x_0| < \delta$ implies $|X(t)| < \varepsilon$, for all $t \geq 0$.

iv) stable in the large if there is $\delta > 0$ such that $|x_0| < \delta$ implies $\lim_{t \to \infty} X(t) = 0$.

v) asymptotically stable if it is stable and stable in the large.

**Remark 3.2** Observe that, under the assumptions that $h$ is continuous and satisfies (H1), equation (3.1) has at least one solution on $[0, \infty)$ because of [20] (Theorems 3.1 and 4.2). Indeed, in [20] (Theorem 4.2) we can consider

$$g(t, x) = \begin{cases} 0 & \text{if } x \leq 0, \\ (|A| + C)x & \text{if } x > 0. \end{cases}$$

Similarly, for a continuous function $h$ satisfying (H2), we introduce the function

$$\phi(x) = \begin{cases} x & \text{if } |x| \leq \delta_0/2, \\ \delta_0 & \text{if } x > \delta_0/2, \\ -\delta_0 & \text{if } x < -\delta_0/2. \end{cases}$$

Then, using [20] (Theorems 3.1 and 4.2) again, the equation

$$X(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (AX(s) + h(\phi(X(s))))ds$$

(3.2)

has at least one solution defined on $[0, \infty)$ due to $|Ax + h(\phi(x))| \leq |Ax| + C|\phi(x)| \leq (|A| + C)|x|$. Hence equation (3.1) has at least one continuous solution on $[0, \infty)$ if (3.2) is stable and $x_0$ is small enough because, in this case, the solution of (3.2) is also a solution of equation (3.1) and $h \circ \phi$ is bounded. So, without loss of generality we can assume that (3.1) has at least one continuous solution because one of the main purposes of the paper is to deal with the stability of (1.1).

We need the following lemma to prove some of our results. The main idea of its proof is in the paper of Martínez-Martínez et al. [28]. Here we give an sketch of the proof for the convenience of the reader.

**Lemma 3.3** Let $h$ be as in (H2) (resp. (H1)). Then, for $0 < x_0 < \delta_0$ (resp. $x_0 > 0$), any continuous solution $X$ of (3.1) satisfies $X(t) > 0$ for all $t \geq 0$.

**Proof** (An idea). Let (H2) (resp. (H1)) be true and $x_0 \in (0, \delta_0)$ (resp. $x_0 > 0$). Then the continuity of $X$ implies that there is $\tau > 0$ such that $X(t) \in (0, \delta_0)$ (resp. $X(t) > 0$) for all $t \in [0, \tau]$. Consequently

$$0 < X(t) \leq x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [A + C]X(s)ds < x_0, \quad t \in [0, \tau].$$

(3.3)

In other words, we have proved that $X(t)$ is less than $x_0$ if $X > 0$ on $[0, t]$. Now suppose that $\tau_0 = \inf\{ t > 0 : X(t) = 0 \}$ is finite. Hence, from (3.1) we deduce

$$x_0 = -\frac{1}{\Gamma(\beta)} \int_0^{\tau_0} (\tau_0 - s)^{\beta-1} [AX(s) + h(X(s))]ds.$$
Therefore, using (3.1) again, we have

\[
X(t) = \frac{1}{\Gamma(\beta)} \left( \int_0^t (t-s)^{\beta-1}[AX(s) + h(X(s))]ds - \int_0^{\tau_0} (\tau_0 - s)^{\beta-1}[AX(s) + h(X(s))]ds \right), \quad t \leq \tau_0. \tag{3.4}
\]

Since \(AX(s) + h(X(s)) \leq [A + C]X(s) \leq 0\) on \([0, \tau_0]\) due to the hypothesis of this result, we are able to write

\[
X(t) \leq \frac{1}{\Gamma(\beta)} (|A| + C) \int_0^{\tau_0} (\tau_0 - s)^{\beta-1}X(s)ds, \quad t \in [0, \tau_0]. \tag{3.5}
\]

So, as \((\tau_0 - s)^{-1} \leq (t - s)^{-1}\), \(X(\tau_0) = 0\) and the continuity of the solution of equation (3.1), iterating inequality (3.5) we can find a positive constant \(\tilde{C}\) such that

\[
X(t) \leq \tilde{C} \left( \frac{(|A| + C)(\tau_0 - t)^\beta}{\Gamma(\beta + 1)} \right)^{n-1}.
\]

Thus, taking \(t \in (0, \tau_0)\) such that \(\frac{|A| + C(\tau_0 - t)^\beta}{\Gamma(\beta + 1)} < 1\) we deduce \(X(t) = 0\), which is a contradiction. \(\square\)

**Remark.** As it was pointed out in [28], if the initial condition in equation (3.1) is a non-decreasing, continuous and non-negative function instead of a constant, we can repeat the procedure in this proof in order to obtain the same result. Indeed, suppose that \(0 < \xi_t < \delta_0\) (resp. \(\xi_t > 0\)) for all \(t \geq 0\), first of all (3.3) becomes

\[
0 < X(t) \leq \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[A + C]X(s)ds < \xi_t.
\]

Secondly, instead of equality (3.4) we have

\[
X(t) = \xi_t - \xi_{\tau_0} + \frac{1}{\Gamma(\beta)} \left( \int_0^t (t-s)^{\beta-1}[AX(s) + h(X(s))]ds - \int_0^{\tau_0} (\tau_0 - s)^{\beta-1}[AX(s) + h(X(s))]ds \right)
\]

\[
\leq \frac{1}{\Gamma(\beta)} \left( \int_0^t (t-s)^{\beta-1}[AX(s) + h(X(s))]ds - \int_0^{\tau_0} (\tau_0 - s)^{\beta-1}[AX(s) + h(X(s))]ds \right), \quad t \leq \tau_0,
\]

due to \(\xi\) being non-decreasing. Hence, it is not difficult to see that (3.5) is still satisfied.

An immediate consequence of the first part of the proof of Lemma 3.3 is the following.

**Corollary 3.4** Assume either (H2), or (H1) is satisfied. Then, any continuous solution to equation (3.1) is stable.

**Proof.** If \(x_0 > 0\), the result follows from (3.3).

For \(x_0 < 0\) and \(X\) a solution of (3.1), we have \(-X\) is a solution of

\[
Y(t) = -x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[AY(s) + \hat{h}(Y(s))]ds, \quad t \geq 0,
\]

with \(\hat{h}(x) = -h(-x)\). \(\square\)

Now we establish the main result of this subsection.

**Proposition 3.5** Let \(h\) be a function satisfying (H2) (resp. (H1)). Then, any continuous solution of equation (3.1) is Mittag-Leffler stable and therefore is also asymptotically stable (resp. globally stable in the large).

**Proof.** Let (H2) (resp. (H1)) be satisfied and \(0 < x_0 < \delta_0\) (resp. \(x_0 > 0\)). Then \(0 < X(t) < \delta_0\) (resp. \(X(t) > 0\)) by Lemma 3.3 and its proof (see 3.3).

On the other hand, consider the solution \(Z\) of the following linear fractional equation

\[
Z(t) = 2x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[A + C]Z(s)ds, \quad t \geq 0.
\]

Then by the continuity of the solutions \(X\) and \(Z\), there exists \(\tau > 0\) such that, for all \(t \in (0, \tau)\), we have \(0 < X(t) < Z(t)\). If this inequality is satisfied for any \(t > 0\), we can ensure that \(X\) is asymptotically stable (resp. and globally stable in the
large), and that this solution also is Mittag-Leffler stable because the solution to $Z$ of last equation is given by (see [17] or Lemma 2.2)

$$Z(t) = 2x_0 E_{\beta, 1}((|A + C|t)^\beta), \quad t \geq 0.$$  

We now suppose that there exists $t_0 > 0$ such that $X(t_0) = Z(t_0)$ and $X(t) < Z(t)$, for $t < t_0$. Set $Y = X - Z$, then

$$Y(t) = -x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} AY(s) \, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} [h(X(s)) - CZ(s)] \, ds, \quad t \geq t_0.$$

From (2.5) (see also [17]) we observe that $Y$ also satisfies the equality

$$Y(t) = -x_0 E_{\beta, 1}((At)^\beta) + \int_0^t (t - s)^{\beta - 1} E_{\beta, \beta}(A(t - s)^\beta) [h(X(s)) - CZ(s)] \, ds, \quad t \geq 0.$$  

For $s \in (0, t_0)$, we have $[h(X(s))] \leq CX(s) < CZ(s)$. Thus $h(X(s)) - CZ(s) < 0$. Consequently, by the completely monotonic property of $E_{\beta, \beta}$ we have $Y(t_0) < 0$, and this is a contradiction because it is supposed that $Y(t_0) = 0$. Now we can conclude that $X$ is Mittag-Leffler stable.

Finally we consider the case that $-\delta_0 < x_0 < 0$ (resp. $x_0 < 0$). Note that $\hat{X} = -X$ is such that

$$\hat{X}(t) = -x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} A\hat{X}(s) \, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \tilde{h}(\hat{X}(s)) \, ds, \quad t \geq 0,$$

with $\tilde{h}(x) = -h(-x)$. Hence, by the first part of this proof and the fact that $\tilde{h}$ satisfies (H2) (resp. (H1)), we have that the proof is complete.

\[\Box\]

**Remark.** Let $X$ be a solution to equation (3.1). Wen et al. [41] (Theorem 1) have proved that the solution to equation (3.1) is stable if $\lim_{|x| \to 0} \frac{|h(x)|}{|x|^\beta} \to 0$. Also, Zhang and Li [46] have used a result similar to Lemma 2.2 to prove that $X$ is asymptotically stable for the case that $\lim_{|x| \to 0} \frac{|h(x)|}{|x|^\beta} = 0, \beta \in (1, 2)$ and $\beta + \frac{1}{|A|} < 2$. Proposition 3.5 establishes that $X$ is asymptotically stable under a weaker condition. Namely (H2). This is possible because we use a comparison type result and the fact that this solution does not change sign.

### 3.2 A function as initial condition

Here we treat the case that the initial condition is a function satisfying some suitable conditions.

Consider the following deterministic Volterra integral equation

$$X(t) = \xi + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} A X(s) \, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} h(X(s)) \, ds, \quad t \geq 0. \quad (3.6)$$

Here $\beta \in (0, 1), A < 0,$ and $h : \mathbb{R} \mapsto \mathbb{R}$ and $\xi : \mathbb{R} \mapsto \mathbb{R}$ are two measurable functions.

Concerning the existence of a continuous solution of equation (3.6) we remark the following. For a continuous function $h$ as in (H1) and $\xi$ continuous, we can consider the equation

$$Z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s, Z(s)) \, ds,$$

where $f(s, x) = A(x + \xi_s) + h(x + \xi_s)$, which has a solution $Z$ due to Theorem 4.2 in [20] (with $g(s, x) = (|A| + C)(x + |\xi_s|)$) and Lemma 2.2. Therefore $Z + \xi$ is a solution of (3.6). Similarly if $\xi$ is “small enough” and $h$ is either a continuous Lipschitz function on a neighbourhood of zero, or as in (H2), then we can proceed as in Remark 3.2 to see that (3.6) has at least one solution in this case. Therefore, as in Remark 3.2 we can assume that (3.6) has at least one continuous solution.

On the other hand, in this paper we analyze several stability criteria for different classes $E$ of initial conditions. Sometimes $E$ is a subset of a normed linear space $X$ of continuous functions endowed with the norm $|| \cdot ||_X$. In other words we consider normed linear spaces $(X, || \cdot ||_X)$. Mainly, in the remaining of this paper, we deal with the following classes of initial conditions.

**Definition 3.6** We have the following assumptions on $\xi$:

1. If the initial condition $\xi$ is continuous on $[0, \infty)$ and there is $\xi_\infty \in \mathbb{R}$ such that, given $\varepsilon > 0$, there exists $t_0 > 0$ such that $|\xi_s - \xi_\infty| \leq \varepsilon$ for any $s \geq t_0$, we say that $\xi$ belongs to the family $E^1$. 

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2. $\mathcal{E}^2$ is the set of all functions $\xi$ of class $C^1(\mathbb{R}_+)$ (i.e., $\xi$ has a continuous derivative on $\mathbb{R}_+$) such that
\[
\lim_{t \to \infty} |\xi_t|/t^\beta = 0 \quad \text{and} \quad |\xi'_t| \leq \frac{\tilde{C}}{t^{1-\eta}}, \quad \text{for some } \eta \in (0, \beta) \text{ and } \tilde{C} \in \mathbb{R}.
\]

3. $\mathcal{E}^3$ is the space of continuous functions of the form
\[
\xi_t = \frac{1}{1(\eta)} \int_0^t (t-s)^{\eta-1} g(s)ds,
\]
with $g \in L^1([0, \infty)) \cap L^p([0, \infty))$, $\eta \in (0, \beta + 1)$ and $p > \frac{1}{\eta} \lor 1$.

The stability concepts that we develop in this section are the following.

**Definition 3.7** Let $\mathcal{E} \subset X$. A solution $X$ of (3.6) is said to be:

i) globally stable in the large for the class $\mathcal{E}$ (or globally $\mathcal{E}$-stable in the large) if $X(t)$ tends to zero as $t \to \infty$, for every $\xi \in \mathcal{E}$.

ii) $\mathcal{E}$-stable if for $\varepsilon > 0$, there is $\delta > 0$ such that $\|X\|_{0, t, \infty} < \varepsilon$ for every $\xi \in \mathcal{E}$ satisfying $\|\xi\|_X < \delta$.

iii) asymptotically $\mathcal{E}$-stable if it is $\mathcal{E}$-stable and there is $\delta > 0$ such that $\lim_{t \to \infty} X(t) = 0$ for any $\xi \in \mathcal{E}$ such that $\|\xi\|_X < \delta$.

In the following auxiliary result, $\mathcal{E}^4$ is the family of functions $\xi$ having the form (3.7) with $\eta = \beta$ and $g$ is a continuous function such that $\lim_{t \to \infty} g(t) = 0$. In this case, the involved norm is $\|\xi\|_X = \|g\|_{0, \infty}$.

**Lemma 3.8** Let $B < 0$ and $\xi \in \mathcal{E}^4$. Then the solution to the equation
\[
Y(t) = \xi_t + \frac{1}{1(\beta)} \int_0^t (t-s)^{\beta-1} BY(s)ds, \quad t \geq 0,
\]
is $\mathcal{E}^4$-stable and globally $\mathcal{E}^4$-stable in the large.

**Proof.** We observe that, by Lemma 2.7, we have
\[
Y(t) = \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(B(t-s)^{\beta})g(s)ds = \int_0^t s^{\beta-1} E_{\beta, \beta}(Bs^{\beta})g(t-s)ds, \quad t \geq 0.
\]
So, the completely monotone property of $E_{\beta, \beta}$, (2.1) and (2.3) lead us to establish
\[
|Y(t)| \leq \left( \sup_{s \geq 0} |g(s)| \right) \int_0^t s^{\beta-1} E_{\beta, \beta}(Bs^{\beta})ds = \left( \sup_{s \geq 0} |g(s)| \right) t^\beta E_{\beta, \beta+1}(Bt^\beta) \\
\leq \frac{C_{\beta, \beta+1}}{|B|} \|g\|_{0, \infty}.
\]
Thus, $Y$ is $\mathcal{E}^4$-stable.

Also, by (2.3), we are able to write
\[
Y(t) = \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(B(t-s)^{\beta})g(s)ds = g(t)t^\beta E_{\beta, \beta+1}(Bt^\beta) + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(B(t-s)^{\beta})[g(s) - g(t)]ds, \quad t \geq 0.
\]
Therefore, using (2.1), and the proof of Proposition 3.3.1 in [10] again, together with the facts that $B < 0$ and $g$ is a continuous function such that $\lim_{t \to \infty} g(t) = 0$, we obtain $Y(t) \to 0$ as $t \to \infty$.

Now we give a general result.
**Theorem 3.9** Let (H2) (resp. (H1)) be true, and \( \mathcal{E} \) a family of continuous functions of a normed linear space \( X \) such that the solution of the equation
\[
Y(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AY(s) ds, \quad t \geq 0, 
\]  
(3.8)
is asymptotically \( \mathcal{E} \)-stable (resp. globally \( \mathcal{E} \)-stable in the large). Then any continuous solution of equation (3.6) is also asymptotically \( \mathcal{E} \)-stable (resp. globally \( \mathcal{E} \)-stable in the large).

**Proof.** Suppose that (H1) (resp. (H2)) is true. Let \( X \) be a continuous solution to equation (3.6). Take \( Z = X - Y \), then we have
\[
Z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |AZ(s) + h(X(s))| ds, \quad t \geq 0. 
\]
Thus, Lemma 2.2 allows us to write
\[
Z(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) h(X(s)) ds, \quad t \geq 0. 
\]
Hence, for \( \xi \in \mathcal{E} \) we have (resp. for \( \xi \in \mathcal{E} \) such that \( \|Y\|_{\infty,[0,\infty)} < \delta_0 \), which gives \( |\xi_0| = |Y(0)| < \delta_0 \), the continuity of \( X \) implies that there is \( \xi_t \) such that \( \|X\|_{\infty,[0,\xi_t]} < \delta_0 \) and
\[
|Z(t)| \leq C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) X(s) ds 
\]
\[
\leq C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) Z(s) ds + C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) Y(s) ds, \quad t \geq 0 \text{ (resp. } t \leq \xi_t \text{)}, 
\]
where we make use of the completely monotonic property of \( E_{\beta,\beta} \). Invoking Lemma 2.3 and the uniqueness of the solutions for the involved equations, \( |Z(t)| \leq u(t) \) for all \( t \geq 0 \) (resp. \( t \leq \xi_t \)), where \( u \) is the solution to
\[
u(t) = \frac{C}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|Y(s)\| ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|A + C\| u(s) ds, \quad t \geq 0. 
\]
Finally observe that \( \|X(t)\| \leq u(t) + |Y(t)| \) for \( t \geq 0 \) (resp. \( t \leq \xi_t \) such that \( \|X\|_{\infty,[0,\xi_t]} < \delta_0 \)). Thus Lemma 3.8 implies that \( u \) is globally \( \mathcal{E}^4 \)-stable in the large (resp. \( u \) is \( \mathcal{E}^4 \)-stable and globally \( \mathcal{E}^4 \)-stable in the large), which gives that the proof is complete. \( \square \)

**Remark.** For each \( i \in \{1, \ldots, n\} \) let \( X^i \) be a normed linear space of functions. Note that if \( \xi = \sum_{i=1}^n \xi^{(i)} \), where \( \xi^{(i)} \in \mathcal{E}^i \subset X^i \) and (3.8) is \( \mathcal{E}^{(i)} \)-stable for each \( i \in \{1, \ldots, n\} \), then (3.8) is also \( \mathcal{E} \)-stable, where \( \mathcal{E} \) is the family of functions of the form \( \sum_{i=1}^n s^{(i)} \) and the involved seminorm is \( \|\xi\|_X = \sum_{i=1}^n \|s^{(i)}\|_{X^i} \). Indeed, by Lemma 2.2 we have that the solution \( Y \) is given by
\[
Y(t) = \sum_{i=1}^n Y^{(i)}(t) = \sum_{i=1}^n \left( \xi^{(i)}_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) s^{(i)} ds \right), \quad t \geq 0, 
\]
where, for each \( i \in \{1, \ldots, n\} \), \( Y^{(i)} \) is the unique solution to the linear equation
\[
Y^{(i)}(t) = \xi^{(i)}_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AY^{(i)}(s) ds, \quad t \geq 0. 
\]
}

In the following result we see that the family \( \mathcal{E} : = \{ \xi \in C([0,\infty)) : \xi = \sum_{i=1}^n \xi^{(i)}, \xi^{(i)} \in \mathcal{E}^i \} \) is an example of a family of functions for which the assumptions of Theorem 3.9 is satisfied. Here, \( \|\cdot\|_{X^i} = \|\cdot\|_{\infty,[0,\infty)} \), \( \|\xi^{(i)}\|_{X^i} = \|\xi^{(i)}\|_{X^i} \) and \( \|\xi^{(i)}\|_{X^i} = \|\xi^{(i)}\|_{L^1([0,\infty))} + \|\xi^{(i)}\|_{L^p([0,\infty))} \) where \( \xi^{(i)} \) denotes \( s \mapsto s^{1-v} \xi^{(i)} \) and \( \xi^{(3)} \) is given by the right-hand side of (3.7). Thus, in this case \( \|\xi\|_X = \sum_{i=1}^n \|\xi^{(i)}\|_{X^i} \).

**Proposition 3.10** Let \( A < 0 \) and \( \beta \in (0,1) \). Then any solution to (3.3) is \( \mathcal{E} \)-stable and \( \mathcal{E} \)-stable in the large.

**Proof.** By previous remark we only need that equation (3.3) is \( \mathcal{E}^i \)-stable and \( \mathcal{E}^i \)-stable in the large, for \( i = 1, 2, 3 \). To prove this, let \( Y \) be the solution to equation (3.3). The global \( \mathcal{E}^2 \)-stability in the large has already been considered in 10 (Theorem 3.3). Now we divide the proof in three steps. 

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Step 1. Here we consider the case $i = 1$. Then Lemma 3.2 and 3.3 give that, for $t \geq 0$,

$$
|Y(t)| \leq |\xi_t^{(1)}| + |A| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^{\beta}) |\xi_s^{(1)}| ds \\
\leq ||\xi_t^{(1)}||_{\infty,[0,\infty)} \left(1 + |A| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^{\beta}) ds \right) \\
= ||\xi_t^{(1)}||_{\infty,[0,\infty)} (1 + |A| t^{\beta} E_{\beta,\beta+1}(A t^{\beta})) \\
\leq ||\xi_t^{(1)}||_{\infty,[0,\infty)} (1 + C_{\beta,\beta+1}),
$$

which implies that the solution of (3.8) is $\xi^{(1)}$-stable.

Step 2. For $i = 2$, we get

$$
|Y(t)| \leq |\xi_t^{(2)} E_{\beta,1}(A t^{\beta})| + |A| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^{\beta}) (\xi_s^{(2)} - \xi_t^{(2)}) ds \\
\leq ||\xi_t^{(2)}||_{\chi^{(2)}} \left(1 + |A| \int_0^t (t-s)^{\beta} E_{\beta,\beta}(A(t-s)^{\beta}) s^{\beta-1} ds \right), \quad t \geq 0.
$$

Consequently, (10) (proof of Theorem 3.2.2) yields

$$
|Y(t)| \leq ||\xi_t^{(2)}||_{\chi^{(2)}} (1 + t^\nu \Gamma(\nu) [v E_{\beta,\nu+1}(A t^{\beta}) - E_{\beta,\nu}(A t^{\beta})]) \\
\leq C||\xi_t^{(2)}||_{\chi^{(2)}}, \quad t \geq 0,
$$

where $C > 0$ is a constant and we have utilized that $v < \beta$.

Step 3. Finally we consider the case $i = 3$. In this scenario, from Lemma 3.2, we obtain

$$
|Y(t)| = \left| \int_0^t (t-s)^{\eta-1} E_{\beta,\eta}(A(t-s)^{\beta}) g(s) ds \right| \\
= \left| \int_0^t s^{\eta-1} E_{\beta,\eta}(A s^{\beta}) g(t-s) ds \right| \\
\leq \left| \int_0^{t/1} s^{\eta-1} E_{\beta,\eta}(A s^{\beta}) g(t-s) ds \right| + \left| \int_{t/1}^t s^{\eta-1} E_{\beta,\eta}(A s^{\beta}) g(t-s) ds \right| \\
= I_1^{(3)}(t) + I_2^{(3)}(t), \quad t \geq 0.
$$

For $I_1^{(3)}$ we can apply Hölder inequality to write, for $q^{-1} = 1 - p^{-1}$ and $C > 0$

$$
I_1^{(3)}(t) \leq C_{\beta,\eta} \left[ \int_0^1 s^{q(\eta^{-1})} ds \right]^{1/q} \left[ \int_0^{t/1} |g(t-s)|^p ds \right]^{1/p} \\
\leq C||g||_{L^p([0,\infty))}, \quad t \geq 0,
$$

and for $I_2^{(3)}$ we use the fact that $\eta - 1 - \beta < 0$. Thus

$$
|I_2^{(3)}(t)| \leq \frac{C_{\beta,\eta}}{|A|} ||g||_{L^1([0,\infty))}.
$$

Remark. Observe that $\mathcal{E}^1$ contains the bounded variation functions on compact sets of $\mathbb{R}_+$ of the form $\xi = \xi^{(1)} - \xi^{(2)}$, where $\xi^{(1)}$ and $\xi^{(2)}$ are two non-decreasing and bounded functions on $\mathbb{R}_+$.

The following result is an immediate consequence of Theorem 3.9 and Proposition 3.10.

**Theorem 3.11** Suppose that (H2) (resp. (H1)) holds. Let $\xi$ be as in Proposition 3.10. Then, any continuous solution to (3.5) is asymptotically $\mathcal{E}$-stable (resp. globally $\mathcal{E}$-stable in the large).
4 Semilinear integral equations with additive noise

In this section we consider the equation

\[ X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[AX(s) + h(X(s))]ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)d\theta_s, \quad t \geq 0. \]  

(4.1)

Here \( \xi, \beta, A \) and \( h \) are as in equation (3.6). Henceforth we assume that \( \alpha \in (1, 2), \theta = \{\theta(s, s \geq 0)\} \) is a \( \gamma \)-Hölder continuous function with \( \gamma \in (0, 1) \) such that \( \theta_0 = 0 \) and \( \gamma + \alpha > 2 \), and \( f \) is a \( \tau \)-Hölder continuous function in \( C^1(\mathbb{R}_+) \), with \( \tau + \gamma > 1 \). Note that, in this case, the Young integral in the right-hand side of (4.1) is equal to \( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}d\theta_s \), where \( \tilde{\theta}_s = \int_0^s f(r)dr \) due to (10) (Lemma 2.4). Thus, Lemma 2.2 is still true for (4.1) and (10) (Lemma 2.7) implies

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)d\theta_s = \alpha - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2}ds. \]

Hence, the existence of a continuous solution to (4.1) can be considered as in Section 3.2.

**Definition 4.1** Let \( E \subset X \) be a family of continuous functions. We say that a solution \( X \) of (4.1) is

1. \( (E, p) \)-stable if for \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \|X\|_{L^\infty([0, \infty))} < \varepsilon \) for any \( (\xi, f, \theta) \) such that

\[ \|\xi\|_X + \|f\theta\|_{L^p([0, \infty))} + \|f\theta\|_{L^{p'}([0, \infty))} + \|f\theta\|_{L^1([0, \infty))} < \delta. \]  

(4.2)

2. asymptotically \( (E, p) \)-stable if it is \( (E, p) \)-stable and there is \( \delta > 0 \) such that \( \lim_{t \to \infty} X(t) = 0 \) for any \( (\xi, f, \theta) \) satisfying (4.2).

An extension of Theorem 3.9 is the following.

**Theorem 4.2** Let \( H(2) \) (resp. \( H(1) \)) be satisfied and \( E \) a class of continuous functions such that the solution of the equation

\[ Y(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}AY(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)d\theta_s, \quad t \geq 0, \]  

(4.3)

is asymptotically \( (E, p) \)-stable (resp. globally \( E \)-stable in the large). Then, any continuous solution of (4.1) is also asymptotically \( (E, p) \)-stable (resp. globally \( E \)-stable in the large).

**Proof.** Observe \( X(0) = \xi_0 \). Consequently the proof is similar to that of Theorem 3.9.

Now we state a consequence of Theorem 4.2.

**Theorem 4.3** Assume \( H(2) \) (resp. \( H(1) \)) holds. Let \( \xi \) be as in Proposition 3.10 \( f \in C^1([0, \infty)) \) such that \( \tilde{f} \in L^1([0, \infty)) \) and \( f \theta \in L^1([0, \infty)) \cap L^{p'}([0, \infty)) \) for some \( p > \frac{1}{\beta - 1} \), and \( \beta + 1 > \alpha \). Then, any continuous solution to (4.1) is asymptotically \( (E, p) \)-stable (resp. globally \( E \)-stable in the large).

**Proof.** Suppose that \( H(2) \) (resp. \( H(1) \)) is satisfied. By Theorem 4.2, we only need to see that the solution \( Y \) of equation (4.3) is asymptotically \( (E, p) \)-stable (resp. globally \( E \)-stable in the large). Towards this end, we invoke Lemma 2.2 and 10 (Lemma 2.4) to establish

\[ Y(t) = \xi_t + A \int_0^t (t-s)^{\beta-1}E_{\beta, \beta}(A(t-s)^{\beta})\xi ds + \int_0^t (t-s)^{\alpha-1}E_{\beta, \alpha}(A(t-s)^{\beta})f(ds)d\theta_s. \]

Thus, \( I_4(t) = \int_0^t (t-s)^{\alpha-2}E_{\beta, \alpha-1}(A(t-s)^{\beta})\theta f(ds)ds \) and \( I_4(t) = \int_0^t (t-s)^{\alpha-1}E_{\beta, \alpha}(A(t-s)^{\beta})\theta f(ds)ds \) imply

\[ I_3(t) = \int_0^t (t-s)^{\alpha-2}E_{\beta, \alpha-1}(A(t-s)^{\beta})\theta f(ds)ds + \int_0^t (t-s)^{\alpha-1}E_{\beta, \alpha}(A(t-s)^{\beta})\theta f(ds)ds = I_{3,1}(t) + I_{3,2}(t), \quad t \geq 0. \]  

(4.4)
For $I_{3.1}$ we have, from (2.1) and $q^{-1} = 1 - p^{-1}$,

$$
|I_{3.1}(t)| \leq \int_0^{1 \wedge t} s^{\alpha - 2}|E_{\beta, \alpha - 1}(A s^\beta)||\theta_{t-s} f(t-s)ds + \int_{1 \wedge t}^t s^{\alpha - 2}|E_{\beta, \alpha - 1}(A s^\beta)||\theta_{t-s} f(t-s)ds
$$

$$
\leq C_{\beta, \alpha - 1} \left( \int_0^1 s^{\alpha - 2}ds \right)^{1/q} \left( \int_0^{1 \wedge t} |\theta_{t-s} f(t-s)|^p ds \right)^{1/p} + C_{\beta, \alpha - 1} \int_0^t |\theta_{t-s} f(t-s)ds
$$

$$
\leq C \left( ||\theta f||_{L^p([0, \infty))} + ||\theta f||_{L^1([0, \infty))} \right), \quad t \geq 0.
$$

Finally, using (2.1) again and the fact that $\beta + 1 > \alpha$,

$$
I_{3.2}(t) \leq \int_0^{1 \wedge t} s^{\alpha - 1}|E_{\beta, \alpha}(A s^\beta)||\theta_{t-s} \dot{f}(t-s)ds + \int_{1 \wedge t}^t s^{\alpha - 1}|E_{\beta, \alpha}(A s^\beta)||\theta_{t-s} \dot{f}(t-s)ds
$$

$$
\leq C_{\beta, \alpha} \int_0^t |\theta_{t-s} \dot{f}(t-s)|ds + C_{\beta, \alpha} \frac{\int_0^t |\theta_{t-s} \dot{f}(t-s)|ds}{|A|} \int_0^t |\theta_{t-s} \dot{f}(t-s)|ds
$$

$$
\leq C \int_0^\infty |\theta_{t-s} \dot{f}(t-s)|ds, \quad t \geq 0.
$$

Hence (4.4) and (4.5) yield that the proof is complete. \hfill \square

Observe that, in the previous proof, the inequality

$$
I_{3.2}(t) \leq C \int_0^\infty |\theta_{t-s} \dot{f}(t-s)|ds, \quad t \geq 0,
$$

is still true for $\beta + 1 \geq \alpha$, wich is used in the proof of Theorem 4.9 below.

4.1 Stochastic integral equations with additive noise

In the remaining of this paper we suppose that all the introduced random variables are defined on a complete probability space $(\Omega, F, P)$.

Remark 4.4 Note that, in equation (4.1), we can consider a random variable $A : \Omega \rightarrow (-\infty, 0)$, stochastic processes $\xi$, $\theta$ and $f$, and a random field $h$ such that for almost all $\omega$, $A(\omega)$, $\xi(\omega)$, $\theta(\omega)$, $f(\omega, \cdot)$ and $h(\omega, \cdot)$ satisfy the hypotheses of Theorem 4.3 (or Theorem 4.2), then we can analyze stability for equation (4.1) $\omega$ by $\omega$ (i.e., with probability one). An example for the process $\theta$ is fractional Brownian motion $B^H$ with Hurst parameter $H \in (0, 1)$. Fractional Brownian motion is a centered Gaussian process with covariance

$$
R_H(s, t) = E(B^H_s B^H_t) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \geq 0.
$$

It is well-known that $B^H$ has $\gamma$-Hölder continuous paths on compact sets, for any exponent $\gamma < H$ due to Kolmogorov continuity theorem (see Decreusefond and Üstünel [7]).

The last remark motivates the following:

Definition 4.5 A continuous solution $X$ to equation (4.1) is said to be globally $E$-stable in the mean if $E|X(t)| \rightarrow 0$ as $t \rightarrow \infty$ for any proces $\xi \in E$.

An immediate consequence of the proof of Theorem 4.9 we can state the following extension of Theorem 4.2.

Theorem 4.6 Let $h$ satisfy (H1), $A < 0$, $E$ a family of continuous processes and $f, \theta$ as in Remark 4.4 such that the solution to equation (4.3) is stable in the mean. Then, any continuous solution to equation (4.7) is also $E$-stable in the mean.

Remark. In [10] (Theorem 4.3) we can find examples of families of processes for which the solution of (4.3) is $E$-stable in the mean.

Other definition motivated by Remark 4.4 is the following:

Definition 4.7 Let $E \subset X$ be a family of continuous functions. We say that a continuous process $\xi$ belongs to $E$ in the mean ($\xi \in E_m$ for short) if $E(\xi) \in E$. 

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Now we consider the stochastic integral equation

\[ X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [AX(s) + h(X(s))] ds + \frac{1}{\Gamma(\beta + 1)} \int_0^t (t-s)^\beta f(s) dB_t^\gamma, \quad t \geq 0. \]  

(4.6)

Here, in order to finish the paper, \( A, h, \beta, \gamma \) and \( f \) are as in equation (4.1) such that \( \beta + \gamma > 1 \), and \( \xi \) is a continuous stochastic process. We remark that we interpret equation (4.6) path by path (i.e. \( \omega \) by \( \omega \)).

The following definition is also inspired by Remark 4.4.

**Definition 4.8** Let \( \mathcal{E} \subset X \) be a family of continuous functions. We say that a continuous solution to equation (4.6) \((\mathcal{E}, p)\)-stable in the mean if for a given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( ||E||_{X \subset \mathcal{E}} < \varepsilon \) for any \( \xi \in \mathcal{E}_m \) such that

\[ ||E||_{X} + ||f(\cdot, \gamma)||_{L^1([0, \infty))} + ||f(\cdot, \gamma)||_{L^p([0, \infty))} \leq \delta. \]

**Remark.** In this definition, if \( \xi = \sum_{i=1}^n \xi^{(i)} \), with \( \xi^{(i)} \in \mathcal{E}_m \), then we set \( ||\xi||_X = \sum_{i=1}^n ||\xi^{(i)}||_X \).

**Theorem 4.9** Let (H2) be true, \( \xi \) as in Proposition 3.7, \( p > \frac{1}{\beta} \) and \( f \in C^1((0, \infty)) \) a positive function with negative derivative such that \( (r \mapsto r^\gamma \cdot \int f(r)) \in L^1((0, \infty)) \) and \( (r \mapsto r^\gamma \cdot \int f(r)) \in L^1((0, \infty)) \cap L^p((0, \infty)) \). Moreover, let \( h \) be a non-decreasing and locally Lipschitz function, which is concave on \( \mathbb{R}_+ \) and convex on \( \mathbb{R}_- \cup \{0\} \). Then, the solution to equation (4.6) \((\mathcal{E}, p)\)-stable in the mean, where \( \xi \in \mathcal{E} \) if and only if \( \xi = \xi^{(1)} - \xi^{(2)} \) with \( \xi^1, \xi^2 \) two non-negative, non-decreasing and continuous processes in \( \mathcal{E}_m \).

**Proof.** Let \( X \) be the continuous solution to equation (4.6). Then Lemma 2.3 implies

\[
X(t) = \xi_t E_{\beta, 1}(At^\beta) + A \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) (\xi_s - \xi_t) ds + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) h(X(s)) ds
+ \int_0^t (t-s)^{\beta-1} E_{\beta, \beta+1}(A(t-s)^\beta) f(s) dB_t^\gamma, \quad t \geq 0,
\]

where the last inequality follows from the facts that \( 0 \leq \xi^{(1)}, \xi^{(2)} \) are two non-decreasing processes, \( f, (-\hat{f}) \geq 0 \) and from (10) (Lemma 2.7). Therefore, we can state, by Lemma 2.3, that \( X \leq X^{(1)} \) where \( X^{(1)} \) is the solution to

\[
X^{(1)}(t) = \xi^{(1)}_t E_{\beta, 1}(At^\beta) + A \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) (\xi^{(1)}_s - \xi^{(1)}_t) ds
+ \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) h(X^{(1)}(s)) ds + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) |B^\gamma_t| f(s) ds
- \int_0^t (t-s)^{\beta} E_{\beta, \beta+1}(A(t-s)^\beta) \hat{f}(s) B^\gamma_t ds, \quad t \geq 0.
\]

(4.7)

Observe that we also have \( X^{(1)}(t) \geq 0 \) due to \( h(0) = 0 \), Lemma 2.3 and

\[-X^{(1)}(t) \leq \int_0^t (t-s)^{\beta-1} (A(t-s)^\beta) \hat{h}(-X^{(1)}(s)) ds, \quad t \geq 0,
\]

with \( \hat{h}(x) = -h(-x), x \in \mathbb{R} \). Proceeding similarly we have \( -X(t) \leq X^{(2)}(t) \), with \( X^{(2)}(t) > 0 \) and

\[
X^{(2)}(t) = \xi^{(2)}_t E_{\beta, 1}(At^\beta) + A \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) (\xi^{(2)}_s - \xi^{(2)}_t) ds
+ \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) \hat{h}(X^{(2)}(s)) ds + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(A(t-s)^\beta) |B^\gamma_t| f(s) ds
- \int_0^t (t-s)^{\beta} E_{\beta, \beta+1}(A(t-s)^\beta) \hat{f}(s) B^\gamma_t ds, \quad t \geq 0.
\]

(4.8)
In other words, we have
\[
E \left( |X(t)| \right) \leq E \left( X^{(1)}(t) \right) + E \left( X^{(2)}(t) \right), \quad t \geq 0. \tag{4.9}
\]

Finally, observe that (4.7), (4.8), the fact that \( A \) is a negative number and Jensen inequality give, for \( \theta_s = s^\gamma \),
\[
E \left( X^{(1)}(t) \right) \leq E(\xi_{t}^{(1)})E_{\beta,1}(A^{\beta}) + A \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^{\beta})E(\xi_{s}^{(1)} - \xi_{t}^{(1)}))ds
+ \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^{\beta})h(E[X^{(1)}(s)])ds + \int_{0}^{t} (t-s)^{\beta}E_{\beta,\beta+1}(A(t-s)^{\beta})f(s)d\theta_s, \quad t \geq 0,
\]
and
\[
E \left( X^{(2)}(t) \right) \leq E(\xi_{t}^{(2)})E_{\beta,1}(A^{\beta}) + A \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^{\beta})E(\xi_{s}^{(2)} - \xi_{t}^{(2)}))ds
+ \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^{\beta})h(E[X^{(2)}(s)])ds + \int_{0}^{t} (t-s)^{\beta}E_{\beta,\beta+1}(A(t-s)^{\beta})f(s)d\theta_s, \quad t \geq 0,
\]

Hence by (4.9), Lemma 2.2, Hypothesis (H2), and the proofs of Proposition 3.10 and Theorem 4.3 we get that the result holds. Indeed, for \( i = 1, 2 \),
\[
E(X^{(i)}(t)) \leq u^{(i)}(t), \quad t \geq 0,
\]
where \( u^{(i)} \) is the unique solution to the equation
\[
u^{(i)} = E(\xi_{i}^{(i)}) + \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1}|A + C|u^{(i)}(s)ds + \frac{1}{\Gamma(\beta+1)} \int_{0}^{t} (t-s)^{\beta}f(s)d\theta_s, \quad t \geq 0.
\]

\[
\Box
\]

**Example 4.10** A function \( h \) that satisfies the conditions of Theorem 4.9 is
\[
h(x) = \begin{cases} 1 - e^{-Cx}, & \text{if } x \geq 0 \\ e^{Cx} - 1, & \text{if } x < 0, \end{cases}
\]
where \( C > 0 \). Indeed, we have that
\[
h'(x) = \begin{cases} Ce^{-Cx}, & \text{if } x \geq 0 \\ Ce^{Cx}, & \text{if } x < 0. \end{cases}
\]
Thus, given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that
\[
|h(x)| \leq (C + \varepsilon)|x| \quad \text{for} \quad |x| \leq \delta.
\]

**Example 4.11** Here we give a function that satisfies Assumption 2 on Definition 3.6. Let \( \xi_{t} = g(t) \sin \frac{1}{t}, \ t \geq 0 \). The function \( g \) is bounded and satisfies \( g(t) = \psi(t)c_{0}t^{\beta-\gamma} + \varphi(t)t^{\frac{\gamma-1}{\gamma}}, \) where \( \psi, \varphi \in C^\infty(\mathbb{R}+) \) are such that
\[
\psi(t) = \begin{cases} 1 & \text{if } t \in [0, 1]; \\ 0 & \text{if } t \geq 2, \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} 0 & \text{if } t \in [0, 1]; \\ 1 & \text{if } t \geq 2. \end{cases}
\]
Thus
\[
\xi_{t}' = g'(t) \sin \frac{1}{t} - g(t)t^{-2} \cos \frac{1}{t}, \quad t \geq 0.
\]
Now it is easy to verify our claim is true using straightforward calculations.

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