Doubly structured mapping problems of the form $\Delta x = y$ and $\Delta^* z = w$

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Abstract

For a given class of structured matrices $S$, we find necessary and sufficient conditions on vectors $x, w \in \mathbb{C}^{n+m}$ and $y, z \in \mathbb{C}^n$ for which there exists $\Delta = [\Delta_1 \; \Delta_2]$ with $\Delta_1 \in S$ and $\Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta x = y$ and $\Delta^* z = w$. We also characterize the set of all such mappings $\Delta$ and provide sufficient conditions on vectors $x, y, z$, and $w$ to investigate a $\Delta$ with minimal Frobenius norm. The structured classes $S$ we consider include (skew)-Hermitian, (skew)-symmetric, pseudo (skew)-symmetric, $J$-(skew)-symmetric, pseudo (skew)-Hermitian, positive (semi)definite, and dissipative matrices. These mappings are then used in computing the structured eigenvalue/eigenpair backward errors of matrix pencils arising in optimal control.

Keywords: structured matrix, backward error, minimal Frobenius norm, Hermitian, positive definite, positive semidefinite, Hamiltonian, dissipative matrix

AMS subject classification. 15A04, 15A60, 15A63, 65F20, 65F35,

1. Introduction

Problem 1 (Doubly structured mapping problem). For a given class of structured matrices $S \subseteq \mathbb{C}^{n,n}$, and vectors $x, w \in \mathbb{C}^{n+m}$ and $y, z \in \mathbb{C}^n$, we consider the following mapping problem:

- **Existence:** Find necessary and sufficient conditions on vectors $x, y, z$, and $w$ for the existence of $\Delta = [\Delta_1 \; \Delta_2]$, where $\Delta_1 \in S$ and $\Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta x = y$ and $\Delta^* z = w$.

We call such a mapping $\Delta$ as doubly structured mapping (DSM) as it has two structures defined on it; (i) conjugate transpose of $\Delta$ satisfies $\Delta^* z = w$, and (ii) $\Delta$ has the form $\Delta = [\Delta_1 \; \Delta_2]$ with $\Delta_1 \in S$.

- **Characterization:** Determine the set

\[
S_d^S := \{ \Delta : \Delta = [\Delta_1 \; \Delta_2], \; \Delta_1 \in S, \Delta_2 \in \mathbb{C}^{n,m}, \; \Delta x = y, \; \Delta^* z = w \}
\]  

(1) of all such doubly structured mappings.

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Minimal Frobenius norm: Characterize all solutions to the doubly structured mapping problem that have minimal Frobenius norm.

The structures we consider on $\Delta_1$ in a DSM problem include symmetric, skew-symmetric, pseudosymmetric, pseudoskew-symmetric, Hermitian, skew-Hermitian, pseudo-Hermitian, pseudoskew-Hermitian, positive (negative) semidefinite, and dissipative matrices.

The minimal norm solutions to such doubly structured mappings can be very handy in the perturbation analysis of matrix pencils arising in control systems [14]. In particular, for the computation of structured eigenvalue/eigenpair backward errors of matrix pencils $L(z)$ of the form

$$L(z) = M + zN := \begin{bmatrix} 0 & J - R & B \\ (J - R)^* & 0 & 0 \\ B^* & 0 & S \end{bmatrix} + z \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $J, R, E, Q \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$ and $S \in \mathbb{C}^{n,m}$ satisfy $J^* = -J$, $R^* = R$ is positive semidefinite, $E^* = E$, and $S^* = S$ is positive definite. The pencil $L(z)$ arises in $H_\infty$ control problems and in the passivity analysis of dynamical systems [10, 16]. One example of such a system is a port-Hamiltonian descriptor system [5, 18]. Our work is motivated by [14], where the eigenpair backward errors have been computed while preserving the block and symmetry structures of pencils of the form $L(z)$, where only the Hermitian structure was considered on $R$. The definiteness structure on $R$ describes the energy dissipation in the system and guarantees the stability of the underlying port-Hamiltonian system [15, 9]. This makes it essential to preserve the definiteness of $R$ to preserve the system’s port-Hamiltonian structure.

The standard mapping problem for matrices is to find $\Delta \in \mathbb{C}^{n,n}$ for given vectors $x, y \in \mathbb{C}^n$, such that $\Delta x = y$. Such mapping problems have been well studied in [11], where authors provide complete, unified, and explicit solutions for structured mappings from Lie and Jordan algebras associated with orthosymmetric scalar products. The minimal norm solutions to the structured mappings provide an important tool in solving nearness problems for control systems, e.g. [6, 7, 12, 13, 4]. The DSMs extend the mapping problem of finding $\Delta \in \mathbb{C}^{n,n}$ for given vectors $x, y, z, w \in \mathbb{C}^n$ such that $\Delta x = y$ and $\Delta^* z = w$ [12, 13].

This paper is organized as follows: In Section 2 we review some preliminary results on mapping problems. In Section 3 we present necessary and sufficient conditions for the existence of DSMs with structures belonging to a Jordan or a Lie algebra. In particular, we consider doubly structured Hermitian, skew-Hermitian, symmetric, and skew-symmetric mapping problems. We provide solutions to the doubly structured semidefinite mapping problem in Section 4. In Section 5 we introduce two types of doubly structured dissipative mappings. The minimal norm solutions to the DSM problems are then used in estimating various structured eigenpair backward errors for the pencil $L(z)$ arising in control systems, see Section 6.
Notation. In the following, we denote the identity matrix of size $n \times n$ by $I_n$, the spectral norm of a matrix or a vector by $\| \cdot \|$ and the Frobenius norm by $\| \cdot \|_F$. The Moore-Penrose pseudoinverse of a matrix or a vector $X$ is denoted by $X^\dagger$ and $P_X = I_n - XX^\dagger$ denotes the orthogonal projection onto the null space of $n \times n$ matrix $X^\ast$. For a square matrix $A$, its Hermitian and skew-Hermitian parts are respectively denoted by $A_H = \frac{A + A^*}{2}$ and $A_S = \frac{A - A^*}{2}$. For $A = A^* \in \mathbb{F}^{n,n}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we denote $A \succ 0$ ($A \prec 0$) and $A \succeq 0$ ($A \preceq 0$) if $A$ is Hermitian positive definite (negative definite) and Hermitian positive semidefinite (negative semidefinite). $\Lambda(A)$ denotes the set of all eigenvalues of the matrix $A$. $\text{Herm}(n)$, $\text{SHerm}(n)$, $\text{Sym}(n)$, and $\text{SSym}(n)$ stand respectively for the set of $n \times n$ Hermitian, skew-Hermitian, symmetric, and skew-symmetric matrices.

2. Preliminaries

In this section, we state some elementary lemmas and recall some mapping results that will be necessary to solve the DSM problem.

**Lemma 1.** \[3\] Let the integer $s$ be such that $0 < s < n$, and $R = R^* \in \mathbb{C}^{n,n}$ be partitioned as $R = \begin{bmatrix} B & C^* \\ C & D \end{bmatrix}$ with $B \in \mathbb{C}^{s,s}$, $C \in \mathbb{C}^{n-s,s}$ and $D \in \mathbb{C}^{n-s,n-s}$. Then $R \succeq 0$ if and only if

1. $B \succeq 0$,

2. $\ker(B) \subseteq \ker(C)$, and

3. $D - CB^\dagger C^* \succeq 0$, where $B^\dagger$ denotes the Moore-Penrose pseudoinverse of $B$.

**Lemma 2.** \([4]\) Let $X, Y \in \mathbb{C}^{n,m}$. Suppose that $\text{rank}(X) = r$ and consider the reduced singular value decomposition $X = U_1 \Sigma_1 V_1^*$ with $U_1 \in \mathbb{C}^{n,r}$, $\Sigma_1 \in \mathbb{C}^{r,r}$ and $V_1 \in \mathbb{C}^{m,r}$. If $X^*Y + Y^*X \succeq 0$, then $U_1^\dagger (YX^\dagger + (YX^\dagger)^*) U_1 \succeq 0$.

**Lemma 3.** Let $X, Y, Z, W \in \mathbb{C}^{n,m}$ with $X^*W = Y^*Z$. Suppose that $\text{rank}(X) = \text{rank}(Z) = r$ and consider the reduced singular value decompositions $X = U_1 \Sigma_1 V_1^*$ and $Z = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^*$, where $U_1, \tilde{U}_1 \in \mathbb{C}^{n,r}$, $V_1, \tilde{V}_1 \in \mathbb{C}^{m,r}$, and $\Sigma_1, \tilde{\Sigma}_1$ are the diagonal matrices containing the nonzero singular values of $X$ and $Z$, respectively. If $U_1 = \tilde{U}_1$, then

$$U_1^\dagger (YX^\dagger + (YX^\dagger)^*) U_1 = U_1^\dagger (YX^\dagger \pm WZ^\dagger) U_1.$$  

(3)
Moreover, theorem 1.

Theorem 1. Let \( x, y \in \mathbb{C}^{n \times n} \) and define \( S := \{ \Delta \in \mathbb{C}^{n \times n} : \Delta x = y \} \). Then \( S \neq \emptyset \) and

\[
S = \{ xy^\dagger + Z P_x : Z \in \mathbb{C}^{n \times n} \}.
\]

Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F = \frac{\| y \|_2}{\| x \|_2} \), where the infimum is attained by \( \Delta = xy^\dagger \).

We next recall a few mapping results from [1] that give a characterization and minimal Frobenius norm solution for the Hermitian, skew-Hermitian, complex symmetric, and complex skew-symmetric mapping problems.

Theorem 2. Let \( x, y \in \mathbb{C}^{n \times n} \) and define \( S := \{ \Delta \in \mathbb{C}^{n \times n} : \Delta^* = \Delta, \Delta x = y \} \). Then \( S \neq \emptyset \) if and only if \( x^* y \in \mathbb{R} \). If the later condition holds, then

\[
S = \{ xy^\dagger + (xy^\dagger)^* - (x^\dagger y)xx^\dagger + P_x H P_x : H \in \mathbb{C}^{n \times n}, H^* = H \}.
\]

Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = 2 \| xy^\dagger \|_F^2 - \text{trace}((x^\dagger y)^* xx^\dagger) \), where the infimum is uniquely attained by \( \Delta = xy^\dagger + (xy^\dagger)^* - (x^\dagger y)xx^\dagger \) which is obtained by setting \( H = 0 \) in [1].

Theorem 3. Let \( x, y \in \mathbb{C}^{n \times n} \) and define \( S := \{ \Delta \in \mathbb{C}^{n \times n} : \Delta x = y, \Delta^* = -\Delta \} \). Then \( S \neq \emptyset \) if and only if \( x^* y \in i\mathbb{R} \). If the later condition holds, then

\[
S = \{ xy^\dagger - (xy^\dagger)^* + (x^\dagger y)xx^\dagger + P_x S P_x : S \in \mathbb{C}^{n \times n}, S^* = -S \}.
\]
Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = 2\|yx^\dagger\|_F^2 - \text{trace}((x^\dagger y)^* xx^\dagger) \), where the infimum is uniquely attained by \( \hat{\Delta} = yx^\dagger - (yx^\dagger)^* + (x^\dagger y)xx^\dagger \).

**Theorem 4.** \([\text{13}]\) Let \( x, y \in \mathbb{C}^n \setminus \{0\} \) and let \( S := \{ \Delta \in \mathbb{C}^{n,n} : \Delta^T = \Delta, \Delta x = y \} \). Then \( S \not= \emptyset \) and
\[
S = \left\{ yx^\dagger + (yx^\dagger)^* - (yx^\dagger)^* xx^\dagger + (P_x)^T H P_x : H \in \mathbb{C}^{n,n}, H^T = H \right\}.
\]
Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = 2\|yx^\dagger\|_F^2 - \text{trace}(yx^\dagger(yx^\dagger)^* (xx^\dagger)^T) \), where the infimum is uniquely attained by \( \hat{\Delta} = yx^\dagger + (yx^\dagger)^* - (yx^\dagger)^* xx^\dagger \), which is obtained by setting \( H = 0 \) in (5).

**Theorem 5.** \([\text{13}]\) Let \( x, y \in \mathbb{C}^n \setminus \{0\} \) and let \( S := \{ \Delta \in \mathbb{C}^{n,n} : \Delta^T = -\Delta, \Delta x = y \} \). Then \( S \not= \emptyset \) if and only if \( \mathcal{I}^T y = 0 \). If the later condition holds, then
\[
S = \left\{ yx^\dagger - (yx^\dagger)^* (xx^\dagger)^T yx^\dagger + (P_x)^T H P_x : H \in \mathbb{C}^{n,n}, H^T = -H \right\}.
\]
Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = 2\|yx^\dagger\|_F^2 - \text{trace}(yx^\dagger(yx^\dagger)^* (xx^\dagger)^T) \), where the infimum is uniquely attained by \( \hat{\Delta} = yx^\dagger - (yx^\dagger)^* + (xx^\dagger)^T yx^\dagger \) which is obtained by setting \( H = 0 \) in (6).

The next result from \([\text{13}]\) solves the mapping problem of finding \( \Delta \in \mathbb{C}^{n,n} \) for given vectors \( x, y, z, w \in \mathbb{C}^n \) such that \( \Delta x = y \) and \( \Delta^* z = w \).

**Theorem 6.** \([\text{13}]\) Let \( x, w \in \mathbb{C}^n \setminus \{0\}, y, z \in \mathbb{C}^n \setminus \{0\} \) and let
\[
S := \{ \Delta \in \mathbb{C}^{n,m} : \Delta x = y, \Delta^* z = w \}.
\]
Then \( S \not= \emptyset \) if and only if \( x^* w = y^* z \). If the later condition holds, then
\[
S = \left\{ yx^\dagger + (wz^\dagger)^* - (wz^\dagger)^* xx^\dagger + P_z R P_x : R \in \mathbb{C}^{n,m} \right\}.
\]
Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = \|yx^\dagger\|_F^2 + \|wz^\dagger\|_F^2 - \text{trace}(wz^\dagger(wz^\dagger)^* xx^\dagger) \), where the infimum is uniquely attained by \( \hat{\Delta} = yx^\dagger + (wz^\dagger)^* - (wz^\dagger)^* xx^\dagger \).

We close this subsection by stating two results that provide solutions to the positive semidefinite mapping problem and dissipative mapping problem, respectively.

**Theorem 7.** \([\text{12}] \) Let \( x, y \in \mathbb{C}^n \setminus \{0\} \) and let \( S := \{ \Delta \in \mathbb{C}^{n,n} : \Delta x = y, \Delta = \Delta^* \geq 0 \} \). Then \( S \not= \emptyset \) if and only if \( x^* y \) is strictly positive. If the later condition holds, then
\[
S = \left\{ yx^\dagger + \frac{yx^\dagger}{x^* y} (K P x) : K \in \mathbb{C}^{n,m}, K \geq 0 \right\}.
\]
Moreover, \( \inf_{\Delta \in S} \| \Delta \|_F^2 = \frac{\|yx^\dagger\|_F^2}{x^* y} \) and infimum is uniquely attained by the rank one matrix \( \hat{\Delta} = \frac{yx^\dagger}{x^* y} \).

**Theorem 8.** \([\text{4}] \) Let \( x, y \in \mathbb{C}^n \setminus \{0\} \) and let \( S := \{ \Delta \in \mathbb{C}^{n,n} : \Delta + \Delta^* \geq 0, \Delta x = y \} \). Then \( S \not= \emptyset \) if and only if \( \text{Re}(x^* y) \geq 0 \). Moreover, if \( \text{Re}(x^* y) > 0 \), then
\[
S = \left\{ yx^\dagger + (yx^\dagger)^* (Z P x + P_x K P x + P_x G P x) : Z, K, G \in \mathbb{C}^{n,n} \text{ satisfy (5)} \right\},
\]
(7)
where
\[ G^* = -G, \quad K \succeq 0, \quad \text{and} \quad K - \frac{1}{4 \Re (x^* y)} (2y + Z^* x) (2y + Z^* x)^* . \] (8)

Further,
\[ \inf_{\Delta \in S} \| \Delta \|_F^2 = 2 \| y \|_2^2 - \| x^* y \|_2^2 \| x \|_4^4, \]
where the infimum is uniquely attained by \( \mathcal{H} := y x^\dagger - (y x^\dagger)^* P_x \), which is obtained by setting \( K = 0, \ G = 0, \) and \( Z = -2(y x^\dagger)^* \) in (7).

3. DSM problems with structures belonging to a Jordan or a Lie algebra

In this section, we define the structures on \( \Delta_1 \) in a DSM problem that are associated with some orthosymmetric scalar product. Let \( M \in \mathbb{C}^{n,n} \) be unitary such that \( M \) is either symmetric or skew-symmetric or Hermitian or skew-Hermitian. Define the scalar product \( \langle \cdot, \cdot \rangle_M : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C} \) by
\[
\langle x, y \rangle_M := \begin{cases} 
 y^T M x, & \text{bilinear form} \\
 y^* M x, & \text{sesquilinear.}
\end{cases}
\]

Then the adjoint of a matrix \( A \in \mathbb{C}^{n,n} \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_M \) is denoted by \( A^* \) and defined by
\[
A^* = \begin{cases} 
 M^{-1} A^T M, & \text{for bilinear form} \\
 M^{-1} A^* M, & \text{for sesquilinear.}
\end{cases}
\]

We also have a Lie algebra \( \mathbb{L} \) and a Jordan algebra \( \mathbb{J} \), associated with \( \langle \cdot, \cdot \rangle_M \) defined by
\[
\mathbb{L} := \{ A \in \mathbb{C}^{n,n} : A^* = -A \} \quad \text{and} \quad \mathbb{J} := \{ A \in \mathbb{C}^{n,n} : A^* = A \},
\]
respectively, see [11] for more details. We refer to [11, Table 2.1] for some known structured matrices in some \( \mathbb{L} \) or \( \mathbb{J} \) associated with a scalar product. This include symmetric, skew-symmetric, pseudosymmetric, pseudoskew-symmetric, Hermitian, skew-Hermitian, pseudo-Hermitian, pseudoskew-Hermitian, etc.

If \( \mathbb{S} \in \{ \mathbb{L}, \mathbb{J} \} \) and if we define \( M \mathbb{S} := \{ MA : A \in \mathbb{S} \}, \) then it is easy to check that
\[
A \in \mathbb{S} \iff MA \in \{ \text{Herm}(n), \text{SHerm}(n), \text{Sym}(n), \text{SSym}(n) \}.
\] (9)

In view of (9), the following result shows that the doubly structured mapping problems with \( \Delta_1 \in \{ \text{Herm}(n), \text{SHerm}(n), \text{Sym}(n), \text{SSym}(n) \} \) are prototypes of more general structured matrices belonging to Jordan and Lie algebras [11, 2].

**Theorem 9.** Consider a Lie algebra \( \mathbb{L} \) and a Jordan algebra \( \mathbb{J} \), associated with a scalar product \( \langle \cdot, \cdot \rangle_M \) and let \( \mathbb{S} \in \{ \mathbb{L}, \mathbb{J} \} \). Then for given vectors \( x, w \in \mathbb{C}^{n+m} \) and \( y, z \in \mathbb{C}^n \), \( \mathbb{S}_2 \not= \emptyset \) if and
only if $S''_d \neq \emptyset$ for some $S' \in \{\text{Herm}(n), \text{SHerm}(n), \text{Sym}(n), \text{SSym}(n)\}$. Further, $\hat{\Delta}$ is of minimal Frobenius norm in $S''_d$ if and only if $M\hat{\Delta}$ is of minimal Frobenius norm in $S''_d$.

Given Theorem 9 the DSM problem with structures belonging to $L$ or $\parallel$ can be solved by reducing it to the DSM problem for $S' \in \{\text{Herm}(n), \text{SHerm}(n), \text{Sym}(n), \text{SSym}(n)\}$. Thus, in the following, we only consider the DSMs with structures $S' \in \{\text{Herm}(n), \text{SHerm}(n), \text{Sym}(n), \text{SSym}(n)\}$.

3.1. Doubly structured Hermitian mappings

This section considers the doubly structured Hermitian mapping (DSHM) problem, i.e., when $S = \text{Herm}(n)$ in Problem 7. We have the following result that completely solves the existence and characterization problem for DSHMs and provides sufficient conditions for the minimal norm solution to the DSHM problem.

**Theorem 10.** Given $x = [x_1^T \ x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m$, $y, z \in \mathbb{C}^n$, and $w = [w_1^T \ w_2^T]^T$ with $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^m$. Define

$$S''_d := \{\Delta : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = \Delta_1, \Delta x = y, \Delta^* z = w\}.\ (10)$$

Then $S''_d \neq \emptyset$ if and only if $x^*w = y^*z$ and $z^*w_1 \in \mathbb{R}$. If the later conditions hold true, then

$$S''_d = \{H + \widetilde{H}(K, R) : K \in \mathbb{C}^{n,n}, R \in \mathbb{C}^{n,m}, K^* = K\},\ (11)$$

where $H = [H_1 \ H_2]$ and $\widetilde{H}(K, R) = [\widetilde{H}_1(K) \ \widetilde{H}_2(K, R)]$ with $H_1, H_2, \widetilde{H}_1(K), \widetilde{H}_2(K, R)$ given by

$$H_1 = w_1z^* + (z^*w_1)^* - (z^*w_1)zz^*,\ (12)$$

$$H_2 = yx_2^* + (x_2^*w_2)^* - (x_2^*w_2)x_2x_2^* + (w_2z_2^*)^*P_{x_2},\ (13)$$

$$\widetilde{H}_1(K) = P_zK P_z,\ (14)$$

$$\widetilde{H}_2(K, R) = P_zRP_{x_2} - P_zKP_zx_2x_2^*,\ (15)$$

and

$$\inf_{\Delta \in S''_d} ||\Delta||_F^2 \geq ||H_1||_F^2 + \inf_{K \in \mathbb{C}^{n,n}} ||H_2 + \widetilde{H}_2(K, 0)||^2_F.\ (16)$$

Moreover, if $x_1 = \alpha z$ for some nonzero $\alpha \in \mathbb{C}$, then equality holds in (10) and we have

$$\inf_{\Delta \in S''_d} ||\Delta||_F^2 = ||H_1||_F^2 + ||H_2||_F^2,$$

where infimum is uniquely attained by the matrix $H$.

**Proof.** Let us suppose that $S''_d$ is nonempty. Then there exists $\Delta = [\Delta_1 \ \Delta_2]$ with $\Delta_1 \in \mathbb{C}^{n,n}$ and $\Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta_1^* = \Delta_1, \Delta x = y,$ and $\Delta^* z = w$. This implies that $y^*z = (\Delta x)^*z = x^*\Delta^* z = x^*w$.

Also, $z^*w_1 = z^*\Delta_1^* z = z^*\Delta_1 z = (\Delta_1^* z)^*z = w_1^*z$, since $\Delta_1^* z = w_1$ and $\Delta_1^* = \Delta_1$. This implies that
H is easy to check that \( \Delta_1, \Delta_2 \). Therefore, from Theorem 6, \( \Delta_0 \), \( \Delta_1 = [\Delta_1 \Delta_2] \), such that \( \Delta x = y, \Delta^* z = w, \) and \( \Delta_0 = \Delta_1 \). This implies that
\[
\Delta_1 x_1 + \Delta_2 x_2 = y, \quad \Delta_1 z = w_1 \quad \text{and} \quad \Delta_2^* z = w_2.
\]
(17)

Since \( \Delta_1 \) is a Hermitian matrix taking \( z \) to \( w_1 \), from Theorem 2 \( \Delta_1 \) has the form
\[
\Delta_1 = w_1 z^\dagger + (w_1 z^\dagger)^* - (z^\dagger w_1) zz^\dagger + \mathcal{P}_2 K \mathcal{P}_2
\]
for some Hermitian matrix \( K \in \mathbb{C}^{n,n} \). By substituting \( \Delta_1 \) from (18) in (17), we get
\[
\Delta_2 x_2 = y - (w_1 z^\dagger + (w_1 z^\dagger)^*) - (z^\dagger w_1) zz^\dagger + \mathcal{P}_2 K \mathcal{P}_2 x_1 \quad \text{and} \quad \Delta_2^* z = w_2,
\]
i.e., a mapping of the form \( \Delta_2 x_2 = \tilde{y} \) and \( \Delta_2^* z = w_2 \), where \( \tilde{y} = y - (w_1 z^\dagger + (w_1 z^\dagger)^*) - (z^\dagger w_1) zz^\dagger + \mathcal{P}_2 K \mathcal{P}_2 x_1 \). The vectors \( x_2, \tilde{y}, z, \) and \( w_2 \) satisfy
\[
\tilde{y}^\dagger z = (y - (w_1 z^\dagger + (w_1 z^\dagger)^*) - (z^\dagger w_1) zz^\dagger + \mathcal{P}_2 K \mathcal{P}_2 x_1)^* z
\]
\[
= y^\dagger z - x_1^\dagger w_1 \quad \text{and} \quad \Delta_1 z = w_1
\]
\[
= x_2^\dagger w_2 \quad \text{and} \quad x^\dagger w = y^\dagger z.
\]
Therefore, from Theorem 6 \( \Delta_2 \) can be written as
\[
\Delta_2 = \tilde{y} x_2^\dagger + (w_2 z^\dagger)^* x_2 x_2^\dagger + \mathcal{P}_2 R \mathcal{P}_2 x_2,
\]
(20)
for some \( R \in \mathbb{C}^{n,m} \).

Thus, in view of (18) and (20), we have

\[
[\Delta_1 \Delta_2] = [w_1 z^\dagger + (w_1 z^\dagger)^* - (z^\dagger w_1) zz^\dagger + \mathcal{P}_2 K \mathcal{P}_2 \tilde{y} x_2^\dagger + (w_2 z^\dagger)^* x_2 x_2^\dagger + \mathcal{P}_2 R \mathcal{P}_2 x_2] = \left[ H_1 + \tilde{H}_1(K) \ H_2 + \tilde{H}_2(K,R) \right]
\]
\[
= H + \tilde{H}(K,R).
\]
(21)

This proves \( \subseteq \) in (11).

Conversely, let \( [\Delta_1 \Delta_2] = [H_1 + \tilde{H}_1(K) \ H_2 + \tilde{H}_2(K,R)] \), where \( H_1, \tilde{H}_1(K) \), \( H_2, \) and \( \tilde{H}_2(K,R) \) are defined by (12) and (15) for some matrices \( R \in \mathbb{C}^{n,m} \) and \( K \in \mathbb{C}^{n,n} \) such that \( K^* = K \). Then it is easy to check that \( [\Delta_1 \Delta_2] x = y \) and \( [\Delta_1 \Delta_2]^* z = w \) since \( x^* w = y^* z \). Also \( (H_1 + \tilde{H}_1(K))^* = H_1 + \tilde{H}_1(K) \) since \( z^* w_1 \in \mathbb{R} \) and \( K^* = K \). Hence \( [\Delta_1 \Delta_2] \in \mathcal{S}_d^{Herm} \). This shows \( \supseteq \) in (11).
In view of (11), we have
\[
\inf_{\Delta \in S_d^{\text{Herm}}} \| \Delta \|_F^2 = \inf_{K \in \mathbb{C}^{n \times n}, R \in \mathbb{C}^{n \times m}, K^* = K} \left\| H + \tilde{H}(K,R) \right\|_F^2
\]
\[= \inf_{K \in \mathbb{C}^{n \times n}, R \in \mathbb{C}^{n \times m}, K^* = K} \left( \left\| [H_1 H_2] + [\tilde{H}_1(K) \tilde{H}_2(K,R)] \right\|_F^2 \right)\]
\[= \inf_{K \in \mathbb{C}^{n \times n}, R \in \mathbb{C}^{n \times m}, K^* = K} \left( \left\| H_1 + \tilde{H}_1(K) \right\|_F^2 + \left\| H_2 + \tilde{H}_2(K,R) \right\|_F^2 \right)\]
\[\geq \inf_{K \in \mathbb{C}^{n \times n}, K^* = K} \left\| H_1 + \tilde{H}_1(K) \right\|_F^2 + \inf_{K \in \mathbb{C}^{n \times n}, R \in \mathbb{C}^{n \times m}, K^* = K} \left\| H_2 + \tilde{H}_2(K,R) \right\|_F^2\]
\[\tag{22}
= \left\| H_1 \right\|_F^2 + \inf_{K \in \mathbb{C}^{n \times n}, K^* = K} \left\| H_2 + \tilde{H}_2(K,R) \right\|_F^2 \]
\[\tag{23}
= \left\| H_1 \right\|_F^2 + \inf_{K \in \mathbb{C}^{n \times n}, K^* = K} \left( \inf_{R \in \mathbb{C}^{n \times m}} \left\| H_2 + \tilde{H}_2(K,R) \right\|_F^2 \right)\]
\[\tag{24}
= \left\| H_1 \right\|_F^2 + \inf_{K \in \mathbb{C}^{n \times n}, K^* = K} \left\| H_2 + \tilde{H}_2(K,0) \right\|_F^2,\]
where the first inequality in (22) follows due to the fact that for any two real valued functions \(f\) and \(g\) defined on the same domain, \(\inf(f + g) \geq \inf f + \inf g\). Also equality in (23) follows since the infimum in the first term is attained when \(K = 0\). In fact, for any \(K \in \mathbb{C}^{n \times n}\) such that \(K^* = K\), we have \((H_1 + \tilde{H}_1(K))z = w_1\), which implies from Theorem 2 that the minimum of \(\|H_1 + \tilde{H}_1(K)\|_F\) is attained when \(K = 0\). Further, for a fixed \(K\) and for any \(R \in \mathbb{C}^{n \times m}\), \(H_2 + \tilde{H}_2(K,R)\) is a matrix satisfying \((H_2 + \tilde{H}_2(K,R))x_2 = \tilde{y}\) and \((H_2 + \tilde{H}_2(K,R))^* z = w_2\). This implies from Theorem 6 that for any fixed \(K\), the minimum of \(\|H_2 + \tilde{H}_2(K,R)\|_F\) over \(R\) is attained when \(R = 0\), which yields (24). This proves (11).

Next suppose if \(x_1 = \alpha z\) for some nonzero \(\alpha \in \mathbb{C}\), then \(\tilde{H}_2(K,0) = 0\) for every \(K \in \mathbb{C}^{n \times n}\). This implies from (24) that
\[
\inf_{\Delta \in S_d^{\text{Herm}}} \| \Delta \|_F^2 \geq \|H_1\|_F^2 + \|H_2\|_F^2 = \|H\|_F^2, \tag{25}
\]
and in this case the lower bound is attained since \(H \in S_d^{\text{Herm}}\). This completes the proof. \(\square\)

### 3.2. Doubly structured skew-Hermitian mappings

A result analogous to Theorem 10 can be obtained for doubly structured skew-Hermitian mappings (DSSHMs), i.e., when \(S = \text{S Herm}(n)\) in Problem 1. In the following, we state the result for DSSHMs and skip its proof as it is similar to the proof of Theorem 10.

**Theorem 11.** Given \(x = [x_1^T \ x_2^T]^T\) with \(x_1 \in \mathbb{C}^n\) and \(x_2 \in \mathbb{C}^m\), \(y, z \in \mathbb{C}^n\), and \(w = [w_1^T \ w_2^T]^T\) with \(w_1 \in \mathbb{C}^n\) and \(w_2 \in \mathbb{C}^m\). Define
\[
S_d^{\text{S Herm}} := \{ \Delta : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n \times n}, \Delta_2 \in \mathbb{C}^{n \times m}, \Delta_1^* = -\Delta_1, \Delta x = y, \Delta^* z = w \}.
\]
Then \(S_d^{\text{S Herm}} \neq \emptyset\) if and only if \(x^* w = y^* z\) and \(z^* w_1 \in i\mathbb{R}\). If the later conditions hold true, then
\[
S_d^{\text{S Herm}} = \left\{ H + \tilde{H}(K,R) : K \in \mathbb{C}^{n \times n}, R \in \mathbb{C}^{n \times m}, K^* = -K \right\},
\]
where $H = [H_1 \ H_2]$ and $\tilde{H}(K, R) = [\tilde{H}_1(K) \ \tilde{H}_2(K, R)]$ with $H_1, H_2, \tilde{H}_1(K), \tilde{H}_2(K, R)$ given by

\begin{align*}
H_1 &= -w_1 z^\dagger + (w_1 z^\dagger)^* + (z^\dagger w_1) z z^\dagger, \\
H_2 &= y x_2^\dagger - (-w_1 z^\dagger + (w_1 z^\dagger)^* + (z^\dagger w_1) z z^\dagger) x_1 x_2^\dagger + (w_2 z^\dagger)^* P_{x_2}, \\
\tilde{H}_1(K) &= P_z K P_z, \\
\tilde{H}_2(K, R) &= P_z R P_{x_2} - P_z K P_z x_1 x_2^\dagger,
\end{align*}

and

$$\inf_{\Delta \in S_d^{\text{sym}}} \|\Delta\|^2_F \geq \|H_1\|^2_F + \inf_{K \in \mathbb{C}^{n,n}} \|H_2 + \tilde{H}_2(K, 0)\|^2_F.$$  \quad (26)

Moreover, if $x_1 = az$ for some nonzero $a \in \mathbb{C}$, then equality holds in (26) and we have

$$\inf_{\Delta \in S_d^{\text{sym}}} \|\Delta\|^2_F = \|H_1\|^2_F + \|H_2\|^2_F,$$

where infimum is uniquely attained by the matrix $H$.

3.3. Doubly structured complex symmetric/skew-symmetric mappings

In this section, we consider the doubly structured symmetric mapping (DSSM) problem, i.e., when $S = \text{Sym}(n)$ in Problem [1]. We have the following result for DSSMs, proof of which is kept in Appendix A.

**Theorem 12.** Given $x = [x_1^T \ x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m$, $y, z \in \mathbb{C}^n$, and $w = [w_1^T \ w_2^T]^T$ with $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^m$. Define

$$S_d^{\text{Sym}} := \{\Delta : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^T = \Delta_1, \Delta x = y, \Delta^* z = w\}.$$  \quad (27)

Then $S_d^{\text{Sym}} \neq \emptyset$ if and only if $x^* w = y^* z$. If the later conditions hold true, then

$$S_d^{\text{Sym}} = \left\{H + \tilde{H}(K, R) : K \in \mathbb{C}^{n,n}, R \in \mathbb{C}^{n,m}, K^T = K\right\},$$  \quad (28)

where $H = [H_1 \ H_2]$ and $\tilde{H}(K, R) = [\tilde{H}_1(K) \ \tilde{H}_2(K, R)]$ with $H_1, H_2, \tilde{H}_1(K), \tilde{H}_2(K, R)$ given by

\begin{align*}
H_1 &= \bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T - \bar{z}^\dagger \bar{z} T \bar{w}_1 \bar{z}^\dagger, \\
H_2 &= y x_2^\dagger - \left(\bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T - \bar{z}^\dagger \bar{z} T \bar{w}_1 \bar{z}^\dagger\right) x_1 x_2^\dagger + (w_2 z^\dagger)^* P_{x_2}, \\
\tilde{H}_1(K) &= P_z^T K P_z, \\
\tilde{H}_2(K, R) &= P_z R P_{x_2} - (P_z)^T K P_z x_1 x_2^\dagger,
\end{align*}

and

$$\inf_{\Delta \in S_d^{\text{sym}}} \|\Delta\|^2_F \geq \|H_1\|^2_F + \inf_{K \in \mathbb{C}^{n,n}} \|H_2 + \tilde{H}_2(K, 0)\|^2_F.$$  \quad (33)

Moreover, if $x_1 = \alpha \bar{z}$ for some nonzero $\alpha \in \mathbb{C}$, then equality holds in (33) and we have

$$\inf_{\Delta \in S_d^{\text{sym}}} \|\Delta\|^2_F = \|H_1\|^2_F + \|H_2\|^2_F,$$

where infimum is uniquely attained by the matrix $H$. 

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Proof. See Appendix A.

A result analogous to Theorem 12 can be obtained for doubly structured skew-symmetric mappings, i.e., when $S = \text{SSym}(n)$ in Problem 1 as follows.

**Theorem 13.** Given $x = [x^T \ x^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m$, $y, z \in \mathbb{C}^n$, and $w = [w^T \ w^T]^T$ with $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^m$. Define

$$S_d^{\text{SSym}} := \{ \Delta : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^T = -\Delta_1, \Delta x = y, \Delta^* z = w \}. \quad (34)$$

Then $S_d^{\text{SSym}} \neq \emptyset$ if and only if $x^* w = y^* z$ and $z^T w_1 = 0$. If the later conditions hold true, then

$$S_d^{\text{SSym}} = \{ \bar{H} + \bar{H}(K, R) : K \in \mathbb{C}^{n,n}, R \in \mathbb{C}^{m,m}, K^T = -K \}, \quad (35)$$

where $H = [H_1 \ H_2]$ and $\bar{H}(K, R) = [\bar{H}_1(K) \ \bar{H}_2(K, R)]$ with $H_1, H_2, \bar{H}_1(K), \bar{H}_2(K, R)$ given by

$$H_1 = -\bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T + \bar{z}^\dagger \bar{T} \bar{z} \bar{w}_1 \bar{z}^\dagger, \quad (36)$$

$$H_2 = yx_2^\dagger - ( -\bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T + \bar{z}^\dagger \bar{T} \bar{z} \bar{w}_1 \bar{z}^\dagger ) x_1 x_2^\dagger + (\bar{w}_2 \bar{z}^\dagger)^* \mathcal{P}_{x_2}, \quad (37)$$

$$\bar{H}_1(K) = \mathcal{P}_z^T K \mathcal{P}_z, \quad (38)$$

$$\bar{H}_2(K, R) = \mathcal{P}_z R \mathcal{P}_{x_2} - (\mathcal{P}_z)^T K \mathcal{P}_z x_1 x_2^\dagger, \quad (39)$$

and

$$\inf_{\Delta \in S_d^{\text{SSym}}} \| \Delta \|^2_F \geq \| H_1 \|^2_F + \inf_{K \in \mathbb{C}^{n,n}} \| H_2 + \bar{H}_2(K, 0) \|^2_F. \quad (40)$$

Moreover, if $x_1 = \alpha \bar{z}$ for some nonzero $\alpha \in \mathbb{C}$, then equality holds in (40) and we have

$$\inf_{\Delta \in S_d^{\text{SSym}}} \| \Delta \|^2_F = \| H_1 \|^2_F + \| H_2 \|^2_F,$$

where infimum is uniquely attained by the matrix $H$.

**Proof.** The proof follows on the lines of the proof of Theorem 12 (see Appendix A) by using Theorem 5 in place of Theorem 4.

---

### 4. Solution to the doubly structured semidefinite mapping problem

This section considers the doubly structured positive semidefinite mapping (DSPSDM) problem, i.e., when $S$ is the set of all $n \times n$ positive semidefinite matrices in Problem 1. We first prove a lemma that will be needed in characterizing the set of all solutions to the DSPSD mapping problem.

**Lemma 5.** Let $A, B \in \mathbb{C}^{n,n}$ such that $B \succeq 0$. If $\lambda \in \Lambda(A)$ implies that $\text{Re}(\lambda) \leq 0$, then $\text{Re}(\text{trace}(AB)) \leq 0$. 


Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. Then by assumption $\text{Re}(\lambda_i) \leq 0$ for all $i$. Also, $B \geq 0$ implies that there exists a unitary matrix $U \in \mathbb{C}^{n,n}$ such that $U^*BU = D$, where $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i \geq 0$ for all $i$. Thus we have

$$\text{trace}(AB) = \text{trace}(AUDU^*) = \text{trace}(U^*AUD) = \sum_{j=1}^{n} \alpha_{jj}d_j,$$

(41)

where $U^*AU = (\tilde{\alpha}_{ij})$. This implies that

$$\text{Re}(\text{trace}(AB)) = \sum_{j=1}^{n} \text{Re}(\tilde{\alpha}_{jj})d_j \leq \max_{j} \tilde{\alpha}_{jj} \cdot \sum_{j=1}^{n} \text{Re}(\tilde{\alpha}_{jj}) = \max_{j} \tilde{\alpha}_{jj} \cdot \sum_{j=1}^{n} \text{Re} \lambda_j \leq 0,$$

since $A$ and $U^*AU$ are unitary similar and have the same eigenvalues, and $d_i \geq 0$ and $\text{Re}(\lambda_i) \leq 0$ for all $i = 1, \ldots, n$. \qed

We have the following result that completely solves the existence and characterization problem for DSPSDMs and provides sufficient conditions for the minimal norm solution to the DSPSDM problem.

**Theorem 14.** Given $x = [x_1^T \ x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m \setminus \{0\}$, $y, z \in \mathbb{C}^n$, and $w = [w_1^T \ w_2^T]^T$ with $w_1 \in \mathbb{C}^n \setminus \{0\}$ and $w_2 \in \mathbb{C}^m \setminus \{0\}$. Define

$$S_d^\text{PSD} := \{ \Delta : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1 \succeq 0, \Delta x = y, \Delta^*z = w \}.$$  

(42)

Then $S_d^\text{PSD} \neq \emptyset$ if and only if $x^*w = y^*z$ and $z^*w_1 > 0$. If the later conditions hold true, then

$$S_d^\text{PSD} = \left\{ H + \tilde{H}(K, R) : K \in \mathbb{C}^{n,n}, R \in \mathbb{C}^{n,m}, K \succeq 0 \right\},$$

(43)

where $H = [H_1 \ H_2]$ and $\tilde{H}(K, R) = [\tilde{H}_1(K) \ \tilde{H}_2(K, R)]$ with $H_1, H_2, \tilde{H}_1(K), \tilde{H}_2(K, R)$ given by

$$H_1 = \frac{w_1w_2^*}{z^*w_1},$$

(44)

$$H_2 = yx_2^* \frac{x_1^*}{\|x_2\|^2} - \frac{w_1w_2^*x_1x_2^*}{(z^*w_1)^2} + \frac{zw_2^*}{\|z\|^2} - \frac{(w_2^*x_2)z}{\|z\|^2},$$

(45)

$$\tilde{H}_1(K) = \mathcal{P}_zK\mathcal{P}_z,$$

(46)

$$\tilde{H}_2(K, R) = \mathcal{P}_zRP_{x_2} - \mathcal{P}_zK\mathcal{P}_zx_2^*,$$

(47)

and

$$\inf_{\Delta \in S_d^\text{PSD}} \|\Delta\|_F^2 \geq \|H_1\|_F^2 + \inf_{K \in \mathbb{C}^{n,n}} \|H_2 + \tilde{H}_2(K, 0)\|_F^2.$$  

(48)

Moreover, if $x_1 = \alpha z$ for some nonzero $\alpha \in \mathbb{C}$, or, if all eigenvalues of the matrix $M := yx_1^* - \frac{w_1w_2^*}{z^*w_1}x_1^*$ lie in the left half of the complex plane, i.e., $\lambda \in \Lambda(M)$ implies that $\text{Re} (\lambda) \leq 0$, then we have

$$\inf_{\Delta \in S_d^\text{PSD}} \|\Delta\|_F^2 = \|H_1\|_F^2 + \|H_2\|_F^2,$$

where the infimum is uniquely attained by the matrix $H$.  

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Proof. Suppose that $\mathcal{S}_d^{\text{PSD}} \neq \emptyset$ and let $\Delta \in \mathcal{S}_d^{\text{PSD}}$. Then $\Delta = [\Delta_1 \Delta_2]$ with $\Delta_1 \in \mathbb{C}^{n,n}$, $\Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta_1 \succeq 0$, $\Delta x = y$, and $\Delta^* z = w$. This implies that $y^* z = (\Delta x)^* z = x^* \Delta^* z = x^* w$. Also, $z^* w_1 = z^* \Delta_1^* z = z^* \Delta_1 z \succeq 0$, since $\Delta_1^* z = w_1$ and $\Delta_1^* \cdot \Delta_1 \succeq 0$. In fact, we have $z^* w_1 > 0$, since if $z^* w_1 = 0$, then this implies that $z^* \Delta_1^* z = 0$ and hence $w_1 = \Delta_1 z = 0$, which is a contradiction as $w_1 \neq 0$. Conversely, let $x^* w = y^* z$ and $z^* w_1 > 0$. Then it is easy to see that the matrix $H = [H_1 \ H_2]$ satisfies $H x = y$, $H^* z = w$. Also $H_1 \succeq 0$ for being a rank one symmetric matrix, which implies that $H \in \mathcal{S}_d^{\text{PSD}}$.

Next, we prove (43). For this, let $\Delta \in \mathcal{S}_d^{\text{PSD}}$, i.e., $\Delta = [\Delta_1 \Delta_2]$ such that $\Delta x = y$, $\Delta^* z = w$, and $\Delta_1 \succeq 0$. This implies that

$$\Delta_1 x_1 + \Delta_2 x_2 = y, \quad \Delta_1 z = w_1 \quad \text{and} \quad \Delta_2^* z = w_2.$$  \hspace{1cm} (49)

Since $\Delta_1 \succeq 0$ taking $z$ to $w_1$, from Theorem 7 $\Delta_1$ has the form

$$\Delta_1 = \frac{w_1 w_1^*}{z^* w_1} + P_z K P_z,$$  \hspace{1cm} (50)

for some positive semidefinite matrix $K \in \mathbb{C}^{n,n}$. By substituting $\Delta_1$ from (50) in (49), we get

$$\Delta_2 x_2 = \tilde{y} \quad \text{and} \quad \Delta_2^* z = w_2,$$  \hspace{1cm} (51)

where $\tilde{y} = y - \left(\frac{w_1 w_1^*}{z^* w_1} + P_z K P_z\right) x_1$. Again since $\tilde{y}^* z = x_2^* w_2$, in view of Theorem 6 $\Delta_2$ has the form

$$\Delta_2 = \tilde{y} x_2^* + (w_2 z^*)^* - (w_2 z^*) x_2^* x_2^* + P_z R P_{x_2},$$  \hspace{1cm} (52)

for some $R \in \mathbb{C}^{n,m}$. Thus in view of (50) and (52), we have

$$[\Delta_1 \ \Delta_2] = \begin{bmatrix} \frac{w_1 w_1^*}{z^* w_1} + P_z K P_z & \tilde{y} x_2^* + (w_2 z^*)^* - (w_2 z^*) x_2^* x_2^* + P_z R P_{x_2} \\ \tilde{y} x_2^* - (w_1^* x_1) w_1 x_2^* + (w_2 z^*)^* - (w_2 z^*) x_2^* x_2^* + P_z R P_{x_2} \end{bmatrix} + \begin{bmatrix} P_z K P_z & P_z R P_{x_2} - P_z K P_z x_2 x_2^* \end{bmatrix} = [H_1 \ H_2] + [\tilde{H}_1(K) \ \tilde{H}_2(K, R)] = H + \tilde{H}(K, R).$$  \hspace{1cm} (53)

This proves “$\subseteq$” in (43).

Conversely, let $x^* w = y^* z$ and $z^* w_1 > 0$ and consider $[\Delta_1 \Delta_2] = [H_1 + \tilde{H}_1(K) \ H_2 + \tilde{H}_2(K, R)]$, where $H_1, \tilde{H}_1(K), H_2,$ and $\tilde{H}_2(K, R)$ given by (44)-(47) for some matrices $R \in \mathbb{C}^{n,m}$ and $K \in \mathbb{C}^{n,n}$ such that $K \succeq 0$. Then, it is easy to verify that $[\Delta_1 \Delta_2] x = y$ and $[\Delta_1 \Delta_2]^* z = w$. Also, $\Delta_1 = H_1 + \tilde{H}_1(K) \succeq 0$, being the sum of two positive semidefinite matrices. This implies $[\Delta_1 \Delta_2] \in \mathcal{S}_d^{\text{PSD}}$ and hence shows “$\supseteq$” in (43).

In view of (43) and by following the arguments similar to the proof of (10) in Theorem 10 we have that

$$\inf_{\Delta \in \mathcal{S}_d^{\text{PSD}}} \|\Delta\|^2_F \geq \|H_1\|^2_F + \inf_{K \in \mathbb{C}^{n,n}, K \succeq 0} \|H_2 + \tilde{H}_2(K, 0)\|^2_F.$$  \hspace{1cm} (54)
Next we show that equality holds in \((54)\) for two cases. First suppose that \(x_1 = \alpha z\) for some nonzero \(\alpha \in \mathbb{C}\). Then \(P_z x_1 = 0\) and thus \(\tilde{H}_2(K, 0) = 0\) for any \(K \geq 0\). This implies from \((54)\) and \((57)\), we have that \(x, w, y, z\) \((\text{DSDM})\) problems: (i) for given vectors \(H\) and again the lower bound is uniquely attained since \(H \in S_d^{\text{PSD}}\). Now suppose that \(\lambda \in \Lambda(M)\), where \(M := yx_1^* - \frac{w_1 x_1}{z^* w_1} w_1 x_1^*\) implies that \(\text{Re}(\lambda) \leq 0\). Then for any \(K \in \mathbb{C}^{n,m}\) such that \(K \geq 0\), we have

\[
\|H_2 + \tilde{H}_2(K, 0)\|_F^2
\]

\[
= \|H_2\|_F^2 + \|\tilde{H}_2(K, 0)\|_F^2 + 2 \text{Re} \left( \text{trace} \left( \frac{yx_1^*}{|x_1|^2} - \frac{w_1 x_1}{|w_1|^2} x_1^* \right) + \text{Re} \left( \text{trace} \left( \frac{y}{|x_1|^2} x_1 \right) \right) \right)
\]

\[
\geq \|H_2\|_F^2 + \|\tilde{H}_2(K, 0)\|_F^2 - 2 \text{Re} \left( \text{trace} \left( \frac{y}{|x_1|^2} x_1 \right) \right)
\]

(55)

where \((55)\) follows by repeated use of the identity \(\text{trace}(AB) = \text{trace}(BA)\) for matrix \(A, B \in \mathbb{C}^{n,m}\), and the fact that \(P_z z = 0\). The last inequality \((56)\) follows because of the fact that \(\text{Re}(\text{trace}(MP_z K P_z)) \leq 0\), this is due to Lemma \(5\) since \(P_z K P_z \geq 0\) as \(K \geq 0\), and by assumption that \(\lambda \in \Lambda(M)\) implies \(\text{Re}(\lambda) \leq 0\). This implies from \((56)\) that

\[
\inf_{K \in \mathbb{C}^{n,m}, K \geq 0} \|H_2 + \tilde{H}_2(K, 0)\|_F^2 \geq \|H_2\|_F^2.
\]

(57)

Thus from \((54)\) and \((57)\), we have that

\[
\inf_{\Delta \in S_d^{\text{PSD}}} \|\Delta\|_F^2 \geq \|H_1\|_F^2 + \|H_2\|_F^2 = \|H\|_F^2,
\]

and again the lower bound is uniquely attained since \(H \in S_d^{\text{PSD}}\). This completes the proof. \(\square\)

**Remark 1.** We note that although in Theorem \(14\) we considered only the DSPSDM problem, there is a corresponding result for the doubly structured negative semidefinite mapping (DSNSDM) problem, i.e., when \(S\) is the set of \(n \times n\) negative semidefinite matrices in Problem \(1\). The corresponding result for DSNSDMs follows from Theorem \(14\) by replacing \(w_1\) with \(-w_1\) and \(x_1\) with \(-x_1\).

### 5. Solution to the doubly structured dissipative mapping problem

Let \(\text{Diss}(n)\) denote the set of all \(n \times n\) dissipative matrices, i.e., \(A \in \text{Diss}(n)\) implies that \(A + A^* \succeq 0\). In this section, we consider two types of doubly structured dissipative mapping (DSDM) problems: (i) for given vectors \(x, w, y, z \in \mathbb{C}^n\), find \(\Delta \in \text{Diss}(n)\) such that \(\Delta x = y\), and \(\Delta^* z = w\). We call this mapping problem as **Type-1 DSDM problem**; (ii) for given vectors \(x, w \in \mathbb{C}^{n+m}\), and \(y, z \in \mathbb{C}^n\), find \(\Delta = [\Delta_1 \Delta_2]\) with \(\Delta_1 \in \text{Diss}(n)\) and \(\Delta_2 \in \mathbb{C}^{n,m}\) such that \(\Delta x = y\) and \(\Delta^* z = w\). We call this mapping problem a **Type-2 DSDM problem**.
5.0.1. Type-1 doubly structured dissipative mappings

In the following, we tackle the type-1 DSDM problem in a general case when \( X, Y, Z, \) and \( W \) are matrices of size \( n \times m \). The result provides a complete, unified, and explicit solution to the Type-1 DSDM problem when \( X \) and \( Z \) share the same range space.

**Theorem 15.** Let \( X, Y, Z, W \in \mathbb{C}^{n,m} \) with \( \text{rank}(X) = \text{rank}(Z) = r \), and let \( X = U \Sigma V^*, Z = \tilde{U} \Sigma \tilde{V}^* \) be the singular value decompositions of \( X \) and \( Z \) with \( U = [U_1 \ U_2], \tilde{U} = [\tilde{U}_1 \ \tilde{U}_2], \) where \( U_1, \tilde{U}_1 \in \mathbb{C}^{n,r} \). Define \( S_{d_1}^{\text{Diss}} := \{ \Delta \in \mathbb{C}^{n,n} : \Delta + \Delta^* \succeq 0, \Delta X = Y, \Delta^* Z = W \} \). Suppose that \( U_1 = \tilde{U}_1 \) and \( \ker(U_1^*(YX^† + (YX^†)^*)U_1) \subseteq \ker(U_2^*(YX^† + WZ^†)U_1) \). Then \( S_{d_1}^{\text{Diss}} \neq \emptyset \) if and only if

\[
YX^†X = Y, \quad WZ^†Z = W, \quad X^*W = Y^*Z, \quad X^*Y + Y^*X \succeq 0.
\]

Moreover, if \( S_{d_1}^{\text{Diss}} \neq \emptyset \), then

1. Characterization:

\[
S_{d_1}^{\text{Diss}} = \left\{ U \begin{bmatrix} U_1^*YX^†U_1 & U_1^*(WZ^†)^*U_2 \\ U_2^*YX^†U_1 & U_2^*(K + G)U_2 \end{bmatrix} : K, G \in \mathbb{C}^{n,n} \text{ satisfy (60)} \right\},
\]

where

\[
G^* = -G, \quad K \succeq 0, \quad K - U_2JU_2^* \succeq 0,
\]

with \( J = \frac{1}{2}U_2^*(YX^† + WZ^†)(YX^† + (YX^†)^*)(YX^† + WZ^†)^*U_2 \).

2. Minimal norm mapping:

\[
\inf_{A \in S_{d_1}^{\text{Diss}}} \| A \|_F^2 = \|YX^†\|_F^2 + \|WZ^†\|_F^2 - \text{trace} \left( (WZ^†(WZ^†)^*)XX^† \right) + J \|_F^2,
\]

where the infimum is uniquely attained by the matrix

\[
\mathcal{H} := YX^† + (WZ^†)^* - (WZ^†)^*XX^† + \mathcal{P}_ZU_2JU_2^* \mathcal{P}_X = U \begin{bmatrix} U_1^*YX^†U_1 & U_1^*(WZ^†)^*U_2 \\ U_2^*YX^†U_1 & J \end{bmatrix} U^*,
\]

which is obtained by setting \( K = U_2JU_2^* \) and \( G = 0 \) in (59).

**Proof.** First suppose that \( \Delta \in S_{d_1}^{\text{Diss}}, \) i.e., \( \Delta + \Delta^* \succeq 0, \Delta X = Y, \) and \( \Delta^* Z = W \). Then clearly \( YX^†X = \Delta X(X^†X) = \Delta X = Y \) and \( WZ^†Z = \Delta^* Z(Z^†Z) = \Delta^* Z = W \). Also, \( X^*W = X^*\Delta^*Z = (\Delta X)^*Z = Y^*Z \) and \( X^*Y + Y^*X = X^*\Delta X + X^*\Delta^*X = X^*(\Delta + \Delta^*)X \succeq 0 \). Conversely, suppose that \( X, Y, Z, \) and \( W \) satisfy (58). Then the matrix \( \mathcal{H} \) defined in (59) satisfies \( \mathcal{H}X = Y \) and \( \mathcal{H}^*Z = W \). Further, we have

\[
\mathcal{H} + \mathcal{H}^* = U \begin{bmatrix} U_1^* (YX^† + (YX^†)^*) U_1 & U_1^* ((WZ^†)^* + (YX^†)^*) U_2 \\ U_2^* (WZ^† + YX^†) U_1 & 2J \end{bmatrix} U^*.
\]
with \( J = \frac{1}{2} \bar{U}_2^* (YX^\dagger + WZ^\dagger)(YX^\dagger + (YX^\dagger)^*)((YX^\dagger)^* + (WZ^\dagger)^*) \bar{U}_2 \). Clearly \( J = J^* \) and in fact \( J \succeq 0 \) from Lemma 2, since \( X^* Y + Y^* X \succeq 0 \). Thus in view of Lemma 1 we have \( \mathcal{H} + \mathcal{H}^* \succeq 0 \), since from Lemma 2 \( \bar{U}_1^* (YX^\dagger + (YX^\dagger)^*) \bar{U}_1 \succeq 0 \), by assumption \( \ker(U_1^* (YX^\dagger + (YX^\dagger)^*) \bar{U}_1) \subseteq \ker(U_2^* (YX^\dagger + WZ^\dagger) \bar{U}_1) \), and \( J \succeq 0 \). This implies that \( \mathcal{H} \in S_{d_1^{\text{Diss}}} \).

Next, we prove (59). First suppose that \( \Delta \in S_{d_1^{\text{Diss}}} \), i.e., \( \Delta + \Delta^* \succeq 0 \), \( \Delta X = Y \), and \( \Delta^* Z = W \).

Let \( \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \), \( \tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix} \), \( V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \), \( \tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix} \), where \( V_1, \tilde{V}_1 \in \mathbb{C}^{m,r} \), \( V_2, \tilde{V}_2 \in \mathbb{C}^{m,m-r} \), and \( \Sigma_1, \tilde{\Sigma}_1 \in \mathbb{C}^{r,r} \) such that \( X = U_1 \Sigma_1 V_1^* \) and \( Z = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^* \) become the reduced SVDs of \( X \) and \( Z \), respectively. Now consider

\[
\hat{\Delta} = U^* \Delta U = \hat{\Delta}_H + \hat{\Delta}_S, \tag{62}
\]

where

\[
\hat{\Delta}_H = U^* \Delta H U = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \quad \text{and} \quad \hat{\Delta}_S = U^* \Delta S U = \begin{bmatrix} S_{11} & S_{12} \\ -S_{12}^* & S_{22} \end{bmatrix}.
\]

Clearly, \( \|\Delta\|_F = \|\hat{\Delta}\|_F \), since Frobenius norm is unitarily invariant, and also \( \Delta H \succeq 0 \) if and only if \( \hat{\Delta}_H \succeq 0 \). As \( \Delta X = Y \), we have \( U^* \Delta U U^* X = U^* Y \) which implies that

\[
\hat{\Delta} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} X = \begin{bmatrix} U_1^* Y \\ U_2^* Y \end{bmatrix} \implies \begin{bmatrix} H_{11} + S_{11} & H_{12} + S_{12} \\ H_{12}^* - S_{12}^* & H_{22} + S_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^* Y \\ U_2^* Y \end{bmatrix}. \tag{63}
\]

This implies that

\[
(H_{11} + S_{11}) \Sigma_1 V_1^* = U_1^* Y \quad \text{and} \quad (H_{12}^* - S_{12}^*) \Sigma_1 V_1^* = U_2^* Y. \tag{64}
\]

Thus from (64), we have

\[
H_{11} + S_{11} = U_1^* Y V_1 \Sigma_1^{-1} = U_1^* Y X^\dagger U_1 \tag{65}
\]

and

\[
H_{12}^* - S_{12}^* = U_2^* Y V_1 \Sigma_1^{-1} = U_2^* Y X^\dagger U_1, \tag{66}
\]

since \( X^\dagger = V_1 \Sigma_1^{-1} U_1^* \) and \( X^\dagger U_1 = V_1 \Sigma_1^{-1} \). Similarly, \( \Delta^* Z = W \) implies that \( U^* \Delta^* U U^* Z = U^* W \) and we have

\[
\hat{\Delta}^* \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} Z = \begin{bmatrix} U_1^* W \\ U_2^* W \end{bmatrix} \implies \begin{bmatrix} H_{11} - S_{11} & H_{12} - S_{12} \\ H_{12}^* + S_{12}^* & H_{22} - S_{22} \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_1 \tilde{V}_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^* W \\ U_2^* W \end{bmatrix},
\]

since \( \tilde{U}_1 = U_1 \) and \( Z = U_1 \tilde{\Sigma}_1 \tilde{V}_1^* \). This implies that

\[
(H_{11} - S_{11}) \tilde{\Sigma}_1 \tilde{V}_1^* = U_1^* W \quad \text{and} \quad (H_{12}^* + S_{12}^*) \tilde{\Sigma}_1 \tilde{V}_1^* = U_2^* W. \tag{67}
\]

Thus, we have

\[
H_{11} - S_{11} = U_1^* W \tilde{V}_1 \tilde{\Sigma}_1^{-1} = U_1^* W Z^\dagger U_1 \tag{67}
\]
and
\[ H_{12} + S_{12} = U_2^* W \tilde{V}_1 \tilde{\Sigma}_1^{-1} U_1 = U_2^* W Z \dagger U_1, \]
(68)
since \( Z \dagger = \tilde{V}_1 \tilde{\Sigma}_1^{-1} U_1^\dagger \) and \( Z^\dagger U_1 = \tilde{V}_1 \tilde{\Sigma}_1^{-1} \). Thus from (65) and (67),
\[ H_{11} = U_1^\dagger \left( \frac{Y X \dagger + W Z \dagger}{2} \right) U_1 = U_1^\dagger \left( \frac{Y X \dagger + (Y X \dagger)^*}{2} \right) U_1 \]
(69)
and
\[ S_{11} = U_1^\dagger \left( \frac{Y X \dagger - W Z \dagger}{2} \right) U_1 = U_1^\dagger \left( \frac{Y X \dagger - (Y X \dagger)^*}{2} \right) U_1, \]
(70)
where in (69) and (70), we have used Lemma 3. Similarly, from (66) and (68),
\[ H_{12}^* = U_2^* \left( \frac{Y X \dagger + W Z \dagger}{2} \right) U_1 \quad \text{and} \quad S_{12}^* = U_2^* \left( \frac{W Z \dagger - Y X \dagger}{2} \right) U_1. \]
(71)
Note that since \( X^\dagger Y + Y^\dagger X \preceq 0 \), in view of Lemma 2, we have that \( H_{11} \succeq 0 \). Therefore
\[ \hat{\Delta} = \begin{bmatrix} U_1^\dagger Y X \dagger U_1 & U_1^\dagger (W Z \dagger)^* U_2 \\ U_2^\dagger Y X \dagger U_1 & H_{22} + S_{22} \end{bmatrix}, \]
(72)
where \( H_{22}, S_{22} \in \mathbb{C}^{n-r,n-r} \) are such that \( \hat{\Delta}_H \succeq 0 \) and \( \hat{\Delta}_S = -\hat{\Delta}_S \). That means, \( S_{22} \) satisfies that \( S_{22} = -S_{22} \), and in view of Lemma 1, \( H_{22} \) satisfies the constraints \( H_{22} \succeq 0 \) and \( H_{22} - J \preceq 0 \) with \( J = \frac{1}{2} U_2^\dagger (Y X \dagger + W Z \dagger)(Y X \dagger + (Y X \dagger)^*)(Y X \dagger + W Z \dagger)^* U_2 \). Thus from (72), we have
\[ \Delta = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} U_1^\dagger Y X \dagger U_1 & U_1^\dagger (W Z \dagger)^* U_2 \\ U_2^\dagger Y X \dagger U_1 & H_{22} + S_{22} \end{bmatrix} \begin{bmatrix} U_1^\dagger \\ U_2^\dagger \end{bmatrix}. \]
(73)
By setting \( K = U_2 H_{22} U_2^\dagger \) and \( G = U_2 S_{22} U_2^\dagger \) in (73), we obtain that
\[ \Delta = U \begin{bmatrix} U_1^\dagger Y X \dagger U_1 & U_1^\dagger (W Z \dagger)^* U_2 \\ U_2^\dagger Y X \dagger U_1 & U_2^\dagger (K + G) U_2 \end{bmatrix} U^*, \]
(74)
where \( G \) and \( K \) satisfy the conditions (60). This proves \( \Delta^* \subseteq \Delta \) in (59).

For the other side inclusion in (59), let \( A \) be any matrix of the form
\[ A = \begin{bmatrix} U_1^\dagger Y X \dagger U_1 & U_1^\dagger (W Z \dagger)^* U_2 \\ U_2^\dagger Y X \dagger U_1 & U_2^\dagger (K + G) U_2 \end{bmatrix} U^*, \]
where \( K \) and \( G \) satisfy the conditions (60). Then using the fact that \( U_1 U_1^\dagger + U_2 U_2^\dagger = UU^* = I_n \), \( U_1 U_1^\dagger = X X \dagger = Z Z \dagger \), \( U_1 U_1^\dagger = I_r \), and \( U_2 U_2^\dagger = I_{n-r} \), \( A \) can be written as
\[ A = Y X \dagger + (W Z \dagger)^* X X \dagger + \mathcal{P}_2(K + G) \mathcal{P}_X. \]
(75)
Clearly \( A \) satisfies that \( AX = Y \) and \( A^* Z = W \), since \( Y X \dagger X = Y \), \( W Z \dagger Z = W \), \( \mathcal{P}_X X = 0 \) and \( \mathcal{P}_Z Z = 0 \). Further, in view of (60) and Lemma 1, we have that
\[ U^*(A + A^*) U = \begin{bmatrix} U_1^\dagger (Y X \dagger + (Y X \dagger)^*) U_1 & U_1^\dagger (Y X \dagger + W Z \dagger)^* U_2 \\ U_2^\dagger (Y X \dagger + W Z \dagger)^* U_1 & U_2^\dagger (2K) U_2 \end{bmatrix} \succeq 0, \]
(76)
since \( U_1^* (YX^\dagger + (YX^\dagger)^*) U_1 \geq 0 \) from Lemma 2 as \( X^* Y + Y^* X \geq 0 \), by assumption ker\((U_1^* (YX^\dagger + (YX^\dagger)^*) U_1) \subseteq \) ker\((U_2^*(YX^\dagger + WZ^\dagger) U_1) \), and \( U_2^* KU_2 - J \geq 0 \), since \( K \) satisfies that \( K - U_2^* J U_2^* \geq 0 \). This implies that \( A + A^* \geq 0 \) and hence \( A \in S_{\text{diss}}^2 \). This proves “\( \geq \)” in (59).

Suppose that \( S_{\text{diss}}^2 \neq \emptyset \) and let \( \Delta \in S_{\text{diss}}^2 \), then from (59) we have that

\[
\| \Delta \|^2_F = \left\| \begin{bmatrix} U_1^* YX^\dagger U_1 & U_1^* (WZ^\dagger) U_2 \\ U_2^* YX^\dagger U_1 & U_2^* (K + G) U_2 \end{bmatrix} \right\|_F^2
\]

\[
= \| U^* YX^\dagger U_1 \|^2_F + \| U_1^* (WZ^\dagger) U_2 \|^2_F + \| U_2^* (K + G) U_2 \|^2_F
\]

\[
= \| YX^\dagger U_1 \|^2_F + \| U_1^* (WZ^\dagger) U_2 \|^2_F + \| U_2^* (K + G) U_2 \|^2_F
\]

\[
= \| YX^\dagger U_1 \|^2_F + \| U_1^* (WZ^\dagger) U_2 \|^2_F + \| U_2^* (K + G) U_2 \|^2_F - \| YX^\dagger U_2 \|^2_F - \| U_1^* (WZ^\dagger) U_1 \|^2_F
\]

\[
= \| YX^\dagger U_1 \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| U_2^* (K + G) U_2 \|^2_F - \| YX^\dagger U_2 \|^2_F - \| U_1^* (WZ^\dagger) U_1 \|^2_F
\]

\[
= \| YX^\dagger U_1 \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| U_2^* (K + G) U_2 \|^2_F - \| YX^\dagger U_2 \|^2_F - \| U_1^* (WZ^\dagger) U_1 \|^2_F
\]

\[
= \| YX^\dagger U_1 \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| U_2^* (K + G) U_2 \|^2_F - \| YX^\dagger U_2 \|^2_F - \| U_1^* (WZ^\dagger) U_1 \|^2_F
\]

where the last equality follows as for any square matrix \( A = AH + AS \) we have \( \| A \|^2_F = \| AH \|^2_F + \| AS \|^2_F \). This implies that

\[
\inf_{\Delta \in S_{\text{diss}}^2} \| \Delta \|^2_F
\]

\[
= \inf_{K,G \in \mathbb{C}^{n,n},G^* = -G,K - U_2^* J U_2^* \geq 0} \| YX^\dagger \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| U_2^* KU_2 \|^2_F + \| U_2^* GU_2 \|^2_F - \text{trace}(WZ^\dagger (WZ^\dagger)^\ast XX^\dagger)
\]

\[
\geq \inf_{K \in \mathbb{C}^{n,n},K - U_2^* J U_2^* \geq 0} \| YX^\dagger \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| U_2^* KU_2 \|^2_F - \text{trace}(WZ^\dagger (WZ^\dagger)^\ast XX^\dagger)
\]

\[
\geq \| YX^\dagger \|^2_F + \| (WZ^\dagger)^\ast \|^2_F + \| J \|^2_F - \text{trace}(WZ^\dagger (WZ^\dagger)^\ast XX^\dagger)
\]

(76)

(77)

where the first inequality in (76) is obvious since for any \( G \in \mathbb{C}^{n,n} \| U_2^* GU_2 \|^2_F \geq 0 \) and the second inequality is due to Lemma 3 since \( \cdot \|_F \) is unitarily invariant and \( J \geq 0 \) implies that for any \( K \in \mathbb{C}^{n,n} \) such that \( K - U_2^* J U_2^* \geq 0 \) we have \( \| K \|^2_F \geq \| U_2^* J U_2^* \|^2_F \). Thus by setting \( K = U_2^* J U_2^* \) and \( G = 0 \), we obtain a unique matrix \( \mathcal{H} \) that attains the lower bound in (77), i.e.,

\[
\inf_{\Delta \in S_{\text{diss}}^2} \| \Delta \|^2_F = \| \mathcal{H} \|^2_F = \| YX^\dagger \|^2_F + \| (WZ^\dagger)^\ast \|^2_F - \text{trace}(WZ^\dagger (WZ^\dagger)^\ast XX^\dagger) + \| J \|^2_F.
\]

This completes the proof. \( \square \)

The vector case Type-1 DSDMs, i.e., when \( m = 1 \) in Theorem 15 is particularly interesting. This is because (i) the conditions on the free matrices in (59) are more simplified, and (ii) it will be used in computing the structured eigenpair backward errors for pencil \( L(z) \) defined in (2), see Section 6. For future reference, we state the vector case separately.
5.0.2. Type-2 doubly structured dissipative mappings

In this section, we consider the Type-2 DSDM problem, i.e. when $S = \text{Diss}(n)$ in Problem 1. We have the following result that completely solves the existence and characterization problem of the Type-2 DSDM problem and derives sufficient conditions for computing the minimal norm solution to the Type-2 DSDM problem.

**Theorem 17.** Given $x = [x_1^T \ x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^m \setminus \{0\}$, $y, z \in \mathbb{C}^n$, and $w = [w_1^T \ w_2^T]^T$ with $w_1 \in \mathbb{C}^n \setminus \{0\}$ and $w_2 \in \mathbb{C}^m \setminus \{0\}$. Define

$\mathcal{S}_{d_2}^{\text{Diss}} := \{ \Delta : [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{m,m}, \Delta_1 + \Delta_1^* \succeq 0, \Delta x = y, \Delta^* z = w \}. \quad (80)$

Then $\mathcal{S}_{d_2}^{\text{Diss}} \neq \emptyset$ if and only if $x^* w = y^* z$ and $\text{Re}(z^* w_1) \geq 0$. If $x^* w = y^* z$ and $\text{Re}(z^* w_1) > 0$, then

$\mathcal{S}_{d_2}^{\text{Diss}} = \left\{ H + \bar{H}(Z, K, G, R) : R \in \mathbb{C}^{n,m}, K, G, Z \in \mathbb{C}^{n,n} \text{ satisfy } (82) \right\}, \quad (81)$

where

$G^* = -G, \ K \succeq 0, \ \text{and } K - \frac{1}{4 \text{Re}(z^* w_1)} (2w_1 + Z^* z) (2w_1 + Z^* z)^* \succeq 0, \quad (82)$
\( H = [H_1 \ H_2] \) and \( \bar{H}(Z,K,G,R) = [\bar{H}_1(Z,K,G) \ \bar{H}_2(Z,K,G,R)] \) with \( H_1, \ H_2, \ \bar{H}_1(Z,K,G), \ \bar{H}_2(Z,K,G,R) \) given by

\[
\begin{align*}
H_1 &= (w_1 z^\dagger)^* + \mathcal{P}_z w_1 z^\dagger \\
H_2 &= y x_2^\dagger - (w_1 z^\dagger)^* x_1 x_2^\dagger - \mathcal{P}_z w_1 z^\dagger x_1 x_2^\dagger + (w_2 z^\dagger)^* x_2 x_2^\dagger \\
\bar{H}_1(Z,K,G) &= \mathcal{P}_z Z^* z z^\dagger + \mathcal{P}_z K \mathcal{P}_z - \mathcal{P}_z G \mathcal{P}_z \\
\bar{H}_2(Z,K,G,R) &= -\mathcal{P}_z Z^* z z^\dagger x_1 x_2^\dagger - \mathcal{P}_z K \mathcal{P}_z x_1 x_2^\dagger + \mathcal{P}_z G \mathcal{P}_z x_1 x_2^\dagger + \mathcal{P}_z R \mathcal{P}_x z,
\end{align*}
\]

and

\[
\inf_{\Delta \in \mathcal{S}_{d_2}^{\text{Diss}}} \|\Delta\|_F^2 \geq \|\bar{H}_1\|_F^2 + \inf_{Z,K,G \text{ satisfying } (83)} \|H_2 + \bar{H}_2(Z,K,G,0)\|_F^2,
\]

where

\[
\hat{H}_1 := H_1 - 2 \mathcal{P}_z w_1 z^\dagger.
\]

Moreover, if \( y = \beta z \) for some \( \beta \in \mathbb{C} \) and \( z \) is orthogonal to \( x_1 \), then

\[
\inf_{\Delta \in \mathcal{S}_{d_2}^{\text{Diss}}} \|\Delta\|_F^2 = \|\hat{H}_1\|_F^2 + \|\hat{H}_2\|_F^2,
\]

where \( \hat{H}_1 \) is defined by \( \hat{H}_1 := H_1 - 2 \mathcal{P}_z w_1 z^\dagger \) and \( \hat{H}_2 := y x_2^\dagger - (w_1 z^\dagger)^* x_1 x_2^\dagger + (w_2 z^\dagger)^* x_2 x_2^\dagger \), and the infimum in \( (83) \) is uniquely attained by the matrix \( \hat{H} = [\hat{H}_1 \ \hat{H}_2] \).

**Proof.** First suppose that \( \mathcal{S}_{d_2}^{\text{Diss}} \neq \emptyset \) and let \( \Delta \in \mathcal{S}_{d_2}^{\text{Diss}} \). Then \( \Delta = [\Delta_1 \ \Delta_2] \) with \( \Delta_1 \in \mathbb{C}^{n,n} \), \( \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta_1 + \Delta_1^* \geq 0 \), \( \Delta x = y \), and \( \Delta^* z = w \). This implies that \( y^* z = (\Delta x)^* z = x^* \Delta^* z = x^* w \). Since \( \Delta^* z = w \), we have \( \Delta_1^* z = w_1 \) which implies that

\[
2 \text{Re}(z^* w_1) = z^* w_1 + w_1^* z = z^* \Delta_1^* z + z^* \Delta_1 z = z^*(\Delta_1 + \Delta_1^*) z \geq 0,
\]

since \( \Delta_1 + \Delta_1^* \geq 0 \). Conversely, let \( x^* w = y^* z \) and \( \text{Re}(z^* w_1) \geq 0 \). Then it is easy to check that \( \hat{H} = [\hat{H}_1 \ \hat{H}_2] \) with \( \hat{H}_1 := H_1 - 2 \mathcal{P}_z w_1 z^\dagger \) and \( \hat{H}_2 := H_2 + 2 \mathcal{P}_z w_1 z^\dagger x_1 x_2^\dagger \) satisfies \( \hat{H} x = y \) and \( \hat{H}^* z = w \). Further,

\[
\hat{H}_1 + \hat{H}_1^* = (w_1 z^\dagger)^* - \mathcal{P}_z w_1 z^\dagger + (w_1 z^\dagger) - (w_1 z^\dagger)^* \mathcal{P}_z = (z^* w_1 + w_1^* z) z z^\dagger \geq 0,
\]

since \( \text{Re}(z^* w_1) \geq 0 \) and \( z z^\dagger \geq 0 \) being a rank one symmetric matrix. This implies that \( \hat{H} \in \mathcal{S}_{d_2}^{\text{Diss}} \).

Next we prove \( (83) \). For this, let \( \Delta \in \mathcal{S}_{d_2}^{\text{Diss}} \), i.e., \( \Delta = [\Delta_1 \ \Delta_2] \) such that \( \Delta x = y \), \( \Delta^* z = w \), and \( \Delta_1 + \Delta_1^* \geq 0 \). This implies that

\[
\Delta_1 x_1 + \Delta_2 x_2 = y, \quad \Delta_1^* z = w_1, \quad \text{and} \quad \Delta_2^* z = w_2.
\]

Since \( \Delta_1 + \Delta^*_1 \geq 0 \) with \( \Delta_1^* \) taking \( z \) to \( w_1 \), from Theorem\( \mathcal{S} \)

\[
\Delta_1^* = w_1 z^\dagger + (w_1 z^\dagger)^* \mathcal{P}_z + z z^\dagger Z \mathcal{P}_z + \mathcal{P}_z K \mathcal{P}_z + \mathcal{P}_z G \mathcal{P}_z,
\]
or, equivalently,

$$
\Delta_1 = (w_1 z_1)^* + \mathcal{P}_z w_1 z_1^\dagger + \mathcal{P}_z Z z z_1^\dagger + \mathcal{P}_z K \mathcal{P}_z - \mathcal{P}_z \mathcal{G} \mathcal{P}_z,
$$

(91)

for some $Z, K, G \in \mathbb{C}^{n,n}$ such that

$$
G^* = -G, \quad K \succeq 0, \quad \text{and} \quad K - \frac{1}{4 \text{Re} (z^* w_1)} (2 w_1 + Z^* z) (2 w_1 + Z^* z)^* \succeq 0.
$$

By substituting $\Delta_1$ from (91) in (90), we get

$$
\Delta_2 x_2 = \tilde{y} \quad \text{and} \quad \Delta_2^* z = w_2,
$$

(92)

where $\tilde{y} = y - \Delta_1 x_1 = y - ((w_1 z_1)^* + \mathcal{P}_z w_1 z_1^\dagger + \mathcal{P}_z Z z z_1^\dagger + \mathcal{P}_z K \mathcal{P}_z - \mathcal{P}_z \mathcal{G} \mathcal{P}_z) x_1$. Again since $\tilde{y}^* z = x_2^* w_2$, in view of Theorem 3 $\Delta_2$ has the form

$$
\Delta_2 = \tilde{y}^* x_2^\dagger + (w_2 z_1^\dagger)^* - (w_2 z_1^\dagger)^* x_2 x_1^\dagger + \mathcal{P}_z \mathcal{R} \mathcal{P}_x x_2,
$$

(93)

for some $R \in \mathbb{C}^{n,m}$. In view of (91) and (93), we have

$$
[\Delta_1 \Delta_2] = [H_1 \ H_2] + [\tilde{H}_1(Z, K, G) \ \tilde{H}_2(Z, K, G, R)] = H + \tilde{H}(Z, K, G, R). \quad (94)
$$

This proves “$\supset$” in (81).

Conversely, let $x^* w = y^* z$ and $\text{Re} (z^* w_1) \geq 0$, consider $[\Delta_1 \Delta_2] = [H_1 + \tilde{H}_1(Z, K, G) \ H_2 + \tilde{H}_2(Z, K, G, R)]$, where $H_1, H_2, \tilde{H}_1(Z, K, G)$, and $\tilde{H}_2(Z, K, G, R)$ be given by (83)–(86) for some matrices $R \in \mathbb{C}^{n,m}$ and $Z, K, G \in \mathbb{C}^{n,n}$ satisfying (82). Then a straightforward calculation shows that $[\Delta_1 \Delta_2] x = y$ and $[\Delta_1 \Delta_2]^* z = w$. Also $\Delta_1 + \Delta_1^* \succeq 0$. To see this, let $u_1 = \frac{w_1}{\|z\|}$ and $U_2 \in \mathbb{C}^{n,n-1}$ be such that $U = [u_1 \ U_2]$ becomes unitary. This implies that

$$
\Delta_1^* = (H_1 + \tilde{H}_1(Z, K, G))^* = w_1 z_1^\dagger + (w_1 z_1^\dagger)^* \mathcal{P}_z + z z_1^\dagger Z \mathcal{P}_z + \mathcal{P}_z K \mathcal{P}_z + \mathcal{P}_z \mathcal{G} \mathcal{P}_z
$$

$$
= w_1 z_1^\dagger + (w_1 z_1^\dagger)^* U_2 U_2^* + z z_1^\dagger Z U_2 U_2^* + U_2 U_2^* K U_2 U_2^* + U_2 U_2^* G U_2 U_2^*
$$

$$
= (u_1 u_1^* + U_2 U_2^*)(w_1 z_1^\dagger) u_1 u_1^* + u_1 u_1^* (w_1 z_1^\dagger)^* U_2 U_2^* + U_2 U_2^* Z U_2 U_2^* + U_2 U_2^* (K + G) U_2 U_2^*
$$

$$
= U \begin{bmatrix} u_1^*(w_1 z_1^\dagger) u_1 & u_1^*(w_1 z_1^\dagger)^* U_2 + u_1^* Z U_2 \\ U_2^*(w_1 z_1^\dagger) u_1 & U_2^*(K + G) U_2 \end{bmatrix} U^*,
$$

(95)

where we have used the fact that $UU^* = I_n$ and $u_1 = \frac{z}{\|z\|}$. Thus in view of Lemma 1 and 95, we have that $\Delta_1 + \Delta_1^* \succeq 0$, since $Z, K, G$ satisfy (82). This proves “$\supset$” in (81).
In view of (81), we have

\[
\inf_{\Delta \in \mathcal{S}^{n \times n}_2} \| \Delta \|_F^2 = \inf_{R \in \mathbb{C}^{n \times m}, K, G, Z \in \mathbb{C}^{n \times n}} \| H + \tilde{H}(Z, K, G, R) \|_F^2
\]

\[
= \inf_{R \in \mathbb{C}^{n \times m}, K, G, Z \in \mathbb{C}^{n \times n}} \left( \left\| \left[ H_1 \ H_2 \right] + \left[ \tilde{H}_1(Z, K, G) \ \tilde{H}_2(Z, K, G, R) \right] \right\|_F^2 \right)
\]

\[
= \inf_{R \in \mathbb{C}^{n \times m}, K, G, Z \in \mathbb{C}^{n \times n}} \left( \left\| H_1 + \tilde{H}_1(Z, K, G) \right\|_F^2 + \left\| H_2 + \tilde{H}_2(Z, K, G, R) \right\|_F^2 \right)
\]

\[
\geq \inf_{K, G, Z \in \mathbb{C}^{n \times n} \text{ satisfying } \text{ (82)}} \left\| H_1 + \tilde{H}_1(Z, K, G) \right\|_F^2 + \inf_{R \in \mathbb{C}^{n \times m}, K, G, Z \in \mathbb{C}^{n \times n} \text{ satisfying } \text{ (82)}} \left\| H_2 + \tilde{H}_2(Z, K, G, R) \right\|_F^2
\]

(96)

\[
= \left\| H_1 \right\|_F^2 + \inf_{K, G, Z \in \mathbb{C}^{n \times n} \text{ satisfying } \text{ (82)}} \left( \inf_{R \in \mathbb{C}^{n \times m}} \left\| H_2 + \tilde{H}_2(Z, K, G, R) \right\|_F^2 \right)
\]

(97)

\[
= \left\| \tilde{H}_1 \right\|_F^2 + \inf_{K, G, Z \in \mathbb{C}^{n \times n} \text{ satisfying } \text{ (82)}} \left( \inf_{R \in \mathbb{C}^{n \times m}} \left\| H_2 + \tilde{H}_2(Z, K, G, R) \right\|_F^2 \right)
\]

(98)

where the first inequality in (96) follows due to the fact that for any two real valued functions \( f \) and \( g \) defined on the same domain, \( \inf(f + g) \geq \inf f + \inf g \). Also equality in (97) follows since the infimum in the first term is attained when \( K = 0, G = 0 \), and \( Z = -2(w_1 z^1)^* \). In fact, for any \( K, G, Z \in \mathbb{C}^{n \times n} \) satisfying (82), the matrix \( H_1 + \tilde{H}_1(Z, K, G) \) is a dissipative map taking \( z \) to \( w_1 \), which implies from Theorem 5 that the minimum of \( \| H_1 + \tilde{H}_1(Z, K, G) \|_F \) is attained when \( K = 0, G = 0 \), and \( Z = -2(w_1 z^1)^* \), i.e., \( \tilde{H}_1 = H_1 + \tilde{H}_1(-2(w_1 z^1)^*, 0, 0) = H_1 - 2P_2 w_1 z^1 \). Similarly, for a fixed \( K, G, Z \in \mathbb{C}^{n \times n} \) satisfying (82) and for any \( R \in \mathbb{C}^{n \times m} \), the matrix \( H_2 + \tilde{H}_2(Z, K, G, R) \) satisfies that \( (H_2 + \tilde{H}_2(Z, K, G, R)) x_2 = y \) and \( (H_2 + \tilde{H}_2(Z, K, G, R))^* z = w_2 \). This implies from Theorem 6 that for any fixed \( Z, K, G \), the minimum of \( \| H_2 + \tilde{H}_2(Z, K, G, R) \|_F \) over \( R \) is attained when \( R = 0 \). This justifies (98) and hence proves (87).

Next we prove (99) under the assumption that \( y = \beta z \) for some \( \beta \in \mathbb{C} \) and \( z \) is orthogonal to \( x_1 \). For this, let us first estimate the infimum in the right hand side of (98). For any \( Z, K, G \in \mathbb{C}^{n \times n} \),
satisfying (82), we have
\[
\|H_2 + \tilde{H}_2(Z, K, G, 0)\|_F^2 = \|H_2\|_F^2 + \|\tilde{H}_2(Z, K, G, 0)\|_F^2 + 2 \Re \left( \text{trace} \left( (H_2)(\tilde{H}_2(Z, K, G, 0))^* \right) \right) \\
= \|\tilde{H}_2\|_F^2 + \|\tilde{H}_2(Z, K, G, 0)\|_F^2 - 2 \Re \left( \text{trace} \left( (P_2(yx_2^\dagger - (w_1 z^\dagger)x_1 x_2^\dagger + (w_2 z^\dagger)x_2 x_2^\dagger)^* P_2) \right) \right) \\
= \|\tilde{H}_2\|_F^2 + \|\tilde{H}_2(Z, K, G, 0)\|_F^2 - 2 \Re \left( \text{trace} \left( (P_2(yx_2^\dagger - (w_1 z^\dagger)x_1 x_2^\dagger + (w_2 z^\dagger)x_2 x_2^\dagger)^* (P_2 K + P_2 G) \right) \right) \\
\geq \|\tilde{H}_2\|_F^2,
\]
where the equality in (99) follows from (86) and (84) using the fact that \(z\) is orthogonal to \(x_1\) and by setting \(\tilde{H}_2 := yx_2^\dagger - (w_1 z^\dagger)x_1 x_2^\dagger + (w_2 z^\dagger)x_2 x_2^\dagger P_2\). This implies that
\[
\inf_{K, G, Z \in \mathbb{C}^{n \times n} \text{ satisfying } (82)} \left\| H_2 + \tilde{H}_2(Z, K, G, 0) \right\|_F^2 \geq \|\tilde{H}_2\|_F^2 \quad (100)
\]
In view of (98) and (100), when \(y = \beta z\) for some \(\beta \in \mathbb{C}\) and \(z \perp x_1\), we have that
\[
\inf_{\Delta \in \mathcal{S}_2^{\text{sym}}} \|\Delta\|_F^2 \geq \|\tilde{H}_1\|_F^2 + \|\tilde{H}_2\|_F^2. \quad (101)
\]
Notice that the lower bound in (101) is uniquely attained for \(\Delta = [\tilde{H}_1 \; \tilde{H}_2]\) which is obtained by taking \(Z = -2(w_1 z^\dagger)^*, \; K = 0, \; G = 0,\) and \(R = 0\) in (81). This completes the proof. \(\square\)

**Remark 3.** A remark similar to Remark 2 also holds for Type-2 DSDMs.

### 6. DSM’s in computing structured eigenpair backward errors of pencils \(L(z)\)

Consider the pencil \(L(z)\) in the form (2) that arises in passivity analysis of port-Hamiltonian systems. In this section, we exploit the minimal-norm DSMs from Section 5 to develop eigenpair backward error estimates under block- and symmetry-structure-preserving perturbations. These results extend the work done in [14], where the pencil \(L(z)\) was considered without semidefinite structure on the block \(R\). Let us introduce the perturbation \(\Delta_M + z \Delta_N\) of the pencil \(L(z) = M + z N\), where
\[
\Delta_M = \begin{bmatrix} 0 & \Delta_J - \Delta_R & \Delta_B \\ \Delta_J^* - \Delta_R^* & 0 & 0 \\ \Delta_B^* & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta_N = \begin{bmatrix} 0 & \Delta_E & 0 \\ -\Delta_E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (102)
\]
for \(\Delta_J, \Delta_R, \Delta_E \in \mathbb{C}^{n \times n}\) and \(\Delta_B \in \mathbb{C}^{n \times m}\), that affect the blocks \(J, R, E, B\) of \(L(z)\). Thus motivated by [14], we define various structured eigenpair backward errors of \(L(z)\) for a given pair \((\lambda, u) \in \mathbb{C} \times \mathbb{C}^{2n+m} \setminus \{0\}\) as follows:
1. **the block-structure-preserving** eigenpair backward error $\eta^B(J, R, E, B, \lambda, u)$ of $L(z)$ with respect to perturbations from the set

\[
B(J, R, E, B) := \{ \Delta_M + z\Delta_N : \Delta_M, \Delta_N \text{ defined by (102) for } \\
\Delta_J, \Delta_R, \Delta_E, \Delta_B \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m} \} \tag{103}
\]

is defined by

\[
\eta^B(J, R, E, B, \lambda, u) = \{ \| [\Delta_M \, \Delta_N] \|_F : ((M - \Delta_M) + \lambda(N - \Delta_N)) u = 0, \\
\Delta_M + z\Delta_N \in B(J, R, E, B) \}; \tag{104}
\]

2. **the symmetry-structure-preserving** eigenpair backward error $\eta^S(J, R, E, B, \lambda, u)$ of $L(z)$ with respect to perturbations from the set

\[
S(J, R, E, B) := \{ \Delta_M + z\Delta_N : \Delta_M, \Delta_N \text{ defined by (102) for } \\
\Delta_J, \Delta_R, \Delta_E \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_J^* = -\Delta_J, \Delta_R^* = \Delta_R, \Delta_E^* = \Delta_E \} \tag{105}
\]

is defined by

\[
\eta^S(J, R, E, B, \lambda, u) = \{ \| [\Delta_M \, \Delta_N] \|_F : ((M - \Delta_M) + \lambda(N - \Delta_N)) u = 0, \\
\Delta_M + z\Delta_N \in S(J, R, E, B) \}; \tag{106}
\]

3. **the semidefinite-structure-preserving** eigenpair backward error $\eta^{S_d}(J, R, E, B, \lambda, u)$ of $L(z)$ with respect to perturbations from the set

\[
S_d(J, R, E, B) := \{ \Delta_M + z\Delta_N : \Delta_M, \Delta_N \text{ defined by (102) for } \\
\Delta_J, \Delta_R, \Delta_E \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_J^* = -\Delta_J, \Delta_R^* = \Delta_R \geq 0, \Delta_E^* = \Delta_E \} \tag{107}
\]

is defined by

\[
\eta^{S_d}(J, R, E, B, \lambda, u) = \{ \| [\Delta_M \, \Delta_N] \|_F : ((M - \Delta_M) + \lambda(N - \Delta_N)) u = 0, \\
\Delta_M + z\Delta_N \in S_d(J, R, E, B) \}; \tag{108}
\]

We note that by choosing different perturbation sets in (103), (105), and (107), the corresponding backward errors can be defined by allowing perturbations only to specific blocks $J, R, E, B$ of $L(z)$. For example, if we chose $B(J, R) := B(J, R, 0, 0)$, where $\Delta_E = 0$ and $\Delta_B = 0$ in (103) to allow perturbation only in blocks $J$ and $R$ of $L(z)$, then the corresponding backward error is given by $\eta^B(J, R, \lambda, u) := \eta^B(J, R, 0, 0, \lambda, u)$. Similarly, the backward errors $\eta^S(J, R, \lambda, u)$ and $\eta^{S_d}(J, R, \lambda, u)$ can be defined by restricting the perturbation sets as $S(J, R) := S(J, R, 0, 0)$ and $S_d(J, R) := S_d(J, R, 0, 0)$, where $\Delta_E = 0$ and $\Delta_B = 0$ in (105) and (107), respectively.
Let Remark 4. \( \eta \) of \( (1, 0) \) since \( \lambda \) if and only if \( J \) on \( \Delta_1 \) 0 is equivalent to solving the doubly structured mapping defined by (109)-(110), where the structure on \( \Delta \) is useful in preserving the semidefinite structure on \( \mathbb{R} \). Let (110) = \[ u \in \mathbb{C}^n, \ u_2 \in \mathbb{C}^n, \ u_3 \in \mathbb{C}^m \] Then for any \( \Delta L(z) = \Delta M + z\Delta N \), where \( \Delta M \) and \( \Delta N \) are defined by (102) for \( \Delta J, \Delta R, \Delta E \in \mathbb{C}^{n,n} \) and \( \Delta B \in \mathbb{C}^{n,m} \), we have \( (L(\lambda) - \Delta L(\lambda)) u = 0 \) if and only if

\[
(\Delta J - \Delta R + \lambda \Delta E)u_2 + \Delta Bu_3 = (J - R + \lambda \Delta E)u_2 + Bu_3 \\
(\Delta J^* - \Delta R^* - \lambda \Delta E^*)u_1 = ((J - R)^* - \lambda \Delta E^*)u_1 \\
\Delta B^*u_1 = B^*u_1 + Su_3
\]

if and only if

\[
\begin{pmatrix}
\Delta J - \Delta R + \lambda \Delta E & \Delta B \\
\end{pmatrix}
\begin{pmatrix}
u_2 \\
u_3 \\
\end{pmatrix} = \begin{pmatrix}
(J - R + \lambda \Delta E)u_2 + Bu_3 \\
\end{pmatrix}
\]

and \( \Delta E \) are defined \( \cdot \) by \( \mathbb{R} \). Thus for any \( \Delta L(z) = \Delta M + z\Delta N \) and \( (\lambda, u) \in \mathbb{R} \times \mathbb{C}^{2n+m} \setminus \{0\} \), \( (L(\lambda) - \Delta L(\lambda)) u = 0 \) is equivalent to solving the doubly structured mapping defined by (109)-(110), where the structure on \( \Delta_1 \) depends on the structures imposed on the perturbations \( \Delta J, \Delta R, \) and \( \Delta E \).

In view of (109) and (110), the following lemma is analogous to [14, Lemma 6.2] that will be useful in preserving the semidefinite structure on \( R \) in the backward error \( \eta^{\text{SS}}(J, R, E, B, \lambda, u) \).

**Lemma 6.** Let \( L(z) \) be a pencil as in (2), and let \( \lambda \in \mathbb{R} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1 \ T \ u_2 \ T \ u_3 \ T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^m \), and let \( x, y, z \) and \( w \) be defined by (109) and (110). Then the following statements are equivalent.

1. There exists \( \Delta J, \Delta R, \Delta E \in \mathbb{C}^{n,n} \) and \( \Delta B \in \mathbb{C}^{n,m} \) such that \( \Delta J \in \text{SParam}(n), \Delta R \geq 0, \) and \( \Delta E \in \text{SParam}(n) \) satisfying (109) and (110).

2. There exists \( \Delta = [\Delta_1 \ \Delta_2] \), \( \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta_1 + \Delta_2 \leq 0, \Delta x = y, \) and \( \Delta^* z = w \).

3. \( u_3 = 0. \)
Moreover, we have

\[
\inf \left\{ \left\| \begin{bmatrix} \Delta_J & \Delta_R & \Delta_E & \Delta_B \end{bmatrix} \right\|_F^2 : \Delta_J \in \text{S Herm}(n), \Delta_R, \Delta_E \in \text{Herm}(n), \Delta_R \succeq 0, \right\}
\]

\[
\Delta_B \in \mathbb{C}^{n,m} \text{ satisfying } (109) \text{ and } (110)
\]

\[
= \inf \left\{ \left\| \frac{\Delta_1 + \Delta_1^*}{2} \right\|_F^2 + \frac{1}{1+|\lambda|^2} \left\| \frac{\Delta_1 - \Delta_1^*}{2} \right\|_F^2 + \left\| \Delta_2 \right\|_F^2 : \Delta = [\Delta_1 \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1 + \Delta_1^* \preceq 0, \Delta x = y, \Delta^* z = w \right\}.
\]

Proof. The proof is similar to the proof of Lemma 6.2 due to Type-2 doubly structured dissipative mapping from Theorem 17.

**Theorem 18.** Let \( L(z) \) be a pencil as in (2), let \( \lambda \in i \mathbb{R} \setminus \{0\} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^n \). Set \( \tilde{y} = (J - R + \lambda E)u_2 \) and \( w_1 = -(J + R + \lambda E)u_1 \). Then \( \eta^{\mathcal{S}(J, R, E, B, \lambda, u)} \) is finite if and only if \( u_3 = 0 \). If the later condition holds and if \( u_2 \) satisfies that \( Ru_2 \neq 0 \) and \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \), then

\[
\frac{1}{1+|\lambda|^2} \left\| H_1 \right\|_F^2 + \left\| H_2 \right\|_F^2 \leq \left( \eta^{\mathcal{S}(J, R, E, B, \lambda, u)} \right)^2 \leq \left\| H_1 \right\|_F^2 + \left\| H_2 \right\|_F^2,
\]

where

\[
H_1 = \tilde{y} u_2 + (w_1 u_1^*)^* P_{u_2} P_{u_2}^* \quad \text{with} \quad J = \frac{1}{4 \text{ Re} (u_2^* \tilde{y})} \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right) \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right)^* \quad (114)
\]

and \( H_2 = u_1^* B \).

Proof. In view of Remark 4 and Lemma 6, we obtain that \( \eta^{\mathcal{S}(J, R, E, B, \lambda, u)} \) is finite if and only if \( u_3 = 0 \). Thus by substituting \( u_3 = 0 \) in (109) and (110), and using Lemma 6 in (108), we have that

\[
\left( \eta^{\mathcal{S}(J, R, E, B, \lambda, u)} \right)^2 = \inf \left\{ \left\| \frac{\Delta_1 + \Delta_1^*}{2} \right\|_F^2 + \frac{1}{1+|\lambda|^2} \left\| \frac{\Delta_1 - \Delta_1^*}{2} \right\|_F^2 + \left\| \Delta_2 \right\|_F^2 : \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1 + \Delta_1^* \preceq 0, \Delta_1 u_2 = \tilde{y}, \Delta_1^* u_1 = w_1, \Delta_2 u_1 = B^* u_1 \right\}.
\]

(115)

If \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \) and \( u_2 \notin \ker(R) \), then from Theorem 15 and Remark 2 there always exists a Type-1 doubly structured dissipative mapping \( \Delta_1 \) such that \( \Delta_1 + \Delta_1^* \preceq 0 \) and

\[
\Delta_1 u_2 = \tilde{y}, \quad \text{and} \quad \Delta_1^* u_1 = w_1.
\]

(116)

This is because of Theorem 16 as the necessary and sufficient conditions \( u_1^* u_1 = \tilde{y} u_2 \) and \( \text{Re} (u_2^* \tilde{y}) \leq 0 \) for the existence of such a \( \Delta_1 \) are satisfied, since \( R \succeq 0 \), \( J^* = -J \), \( E^* = E \), and \( \lambda \in i \mathbb{R} \). Further, the minimal Frobenius norm of such a \( \Delta_1 \) is attained by the unique matrix \( H_1 \) defined in (114). Similarly, from Theorem 1 for any \( u_1 \) there always exists \( \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta_2^* u_1 = B^* u_1 \) and the minimal Frobenius norm of such a \( \Delta_2 \) is attained by \( H_2 := u_1 u_1^* B \).
Next, observe that for any \( \Delta_1 \in \mathbb{C}^{n,n} \), we have \( \| \Delta_1 \|_F^2 = \left\| \frac{\Delta_1 + \Delta_1^*}{2} \right\|_F^2 + \left\| \frac{\Delta_1 - \Delta_1^*}{2} \right\|_F^2 \). This implies that for any \( \Delta_1 \in \mathbb{C}^{n,n} \) and \( \Delta_2 \in \mathbb{C}^{n,m} \), we have
\[
\frac{1}{1 + |\lambda|^2} \| \Delta_1 \|_F^2 + \| \Delta_2 \|_F^2 \leq \left\| \frac{\Delta_1 + \Delta_1^*}{2} \right\|_F^2 + \frac{1}{1 + |\lambda|^2} \left\| \frac{\Delta_1 - \Delta_1^*}{2} \right\|_F^2 + \| \Delta_2 \|_F^2 \leq \| \Delta_1 \|_F^2 + \| \Delta_2 \|_F^2.
\]
(117)
Thus by taking the infimum over all \( \Delta_1 \in \mathbb{C}^{n,n} \) and \( \Delta_2 \in \mathbb{C}^{n,m} \) satisfying the mappings in the right hand side of (116), and by using the minimal Frobenius norm mappings \( H_1 \) and \( H_2 \), we obtain (113). This completes the proof. \( \Box \)

6.1. Numerical experiments

In Table 1 we present some numerical experiments to illustrate the results of this section. We generate a random pencil \( L(z) \) of the form (2) with no eigenvalues on the imaginary axis and compare the various eigenpair backward errors for perturbations to all the blocks \( J, R, E, \) and \( B \) of \( L(z) \). The \( \lambda \)-values are chosen randomly on the imaginary axis, and \( u \in \mathbb{C}^{2n+m} \) is chosen to satisfy the conditions of Theorem 18. The block structured backward error \( \eta^B(J, R, E, B, \lambda, u) \) and the symmetry structured eigenpair backward error \( \eta^S(J, R, E, B, \lambda, u) \) were obtained in [14, Theorem 6.3]. The semidefinite structure-preserving backward error \( \eta^{Sd}(J, R, E, B, \lambda, u) \) is obtained in Theorem 18. We observe that the eigenpair backward error is significantly larger when semidefinite structure-preserving perturbations are considered instead of block structure-preserving ones or symmetry structure-preserving ones. The tightness of the lower and upper bounds for \( \eta^{Sd}(J, R, E, B, \lambda, u) \) depends on the value of \( \lambda \), as shown in Theorem 18.

Table 1: Comparison of various block-/symmetry-/semidefinite-structure-preserving eigenpair backward errors of \( L(z) \) under perturbations to the blocks \( J, R, E, \) and \( B \) of \( L(z) \). Here, l.b. and u.b. respectively stand for the terms lower and upper bound.

| \( \lambda \) | \( \eta^B \) [14] | l.b. of \( \eta^S \) [14, Theorem 6.3] | u.b. of \( \eta^S \) [14, Theorem 6.3] | l.b. of \( \eta^{Sd} \) Theorem 18 | u.b. of \( \eta^{Sd} \) Theorem 18 |
|-----------|----------------|----------------|----------------|----------------|----------------|
| 0.1380i   | 23.9305        | 28.2248        | 28.4919        | 30.6476        | 30.9366        |
| 0.5100i   | 23.5909        | 25.7498        | 29.0013        | 27.9347        | 31.3382        |
| 0.8950i   | 23.0355        | 22.2134        | 20.7630        | 24.0542        | 32.2223        |
| 1.0480i   | 22.8160        | 20.9155        | 32.0553        | 42.2684        | 32.6987        |
| 1.3210i   | 22.4764        | 18.9091        | 35.5143        | 20.4225        | 33.7134        |
| 1.9080i   | 22.0725        | 15.8640        | 48.9597        | 17.0707        | 36.5371        |
| 2.5080i   | 22.1087        | 13.9975        | 71.4942        | 15.0184        | 40.1794        |

The eigenpair backward errors of \( L(z) \) when only specific blocks in the pencil \( L(z) \) are perturbed also follow similar lines and have been kept in Appendix B for future reference. In Table 2 we
summarize the results for symmetry and semidefinite structure-preserving backward errors with respect to other combinations of the perturbation blocks $J$, $R$, $E$, and $B$ of $L(z)$. Table 2 also covers the cases of symmetry structure-preserving backward errors left open in [14].

Table 2: An overview of the results for the symmetry- or semidefinite-structure-preserving eigenpair backward error when only specific blocks in the pencil $L(z)$ are perturbed.

| perturbation blocks | $\eta^S(\cdot,\cdot,\cdot,\lambda, u)$ | $\eta^{S_d}(\cdot,\cdot,\cdot,\lambda, u)$ |
|---------------------|-------------------------------------|-------------------------------------|
| J and R             | [14, Theorem 4.14]                  | Theorem 19                          |
| J and E             | [14, Theorem 4.6]                  | [14, Theorem 4.6]                  |
| J and B             | Theorem 20                          | Theorem 20                          |
| R and E             | [14, Theorem 4.10]                  | Theorem 24                          |
| R and B             | Theorem 21                          | Theorem 22                          |
| E and B             | Theorem 23                          | Theorem 23                          |
| J, R and E          | [15, Theorem 5.11]                  | Theorem 25                          |
| J, R and B          | [14, Theorem 5.4]                  | Theorem 28                          |
| R, E and B          | [15, Theorem 5.7]                  | Theorem 27                          |
| J, E and B          | Theorem 26                          | Theorem 26                          |
| J, R, E and B       | [15, Theorem 6.3]                  | Theorem 18                          |

References

[1] Bibhas Adhikari, *Backward Perturbation and Sensitivity analysis of Structured polynomial Eigenomial Eigenvalue Problem*, PhD thesis, Department of Mathematics, IIT Guwahati, Assam, India, 2008.

[2] Bibhas Adhikari and Rafikul Alam, *Structured procrustes problem*, Linear Algebra and its Applications, 490 (2016), pp. 145–161.

[3] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudoinverses*, SIAM Journal on Applied Mathematics, 17 (1969), pp. 434–440.

[4] Mohit Kumar Baghel, Nicolas Gillis, and Punit Sharma, *Characterization of the dissipative mappings and their application to perturbations of dissipative-hamiltonian systems*, Numerical Linear Algebra with Applications, (2021).
[5] Christopher Beattie, Volker Mehrmann, Hongguo Xu, and Hans Zwart, \textit{Linear port-hamiltonian descriptor systems}, Mathematics of Control, Signals, and Systems, 30 (2018), p. 17.

[6] Shreemayee Bora, Michael Karow, Christian Mehl, and Punit Sharma, \textit{Structured eigenvalue backward errors of matrix pencils and polynomials with hermitian and related structures}, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 453–475.

[7] \textit{Structured eigenvalue backward errors of matrix pencils and polynomials with palindromic structures}, SIAM Journal on Matrix Analysis and Applications, 36 (2015), pp. 393–416.

[8] Philippe G Ciarlet, Philippe Gaston Ciarlet, Bernadette Miara, and Jean-Marie Thomas, \textit{Introduction to numerical linear algebra and optimisation}, Cambridge University Press, 1989.

[9] Nicolas Gillis and Punit Sharma, \textit{Finding the nearest positive-real system}, SIAM Journal on Numerical Analysis, 56 (2018), pp. 1022–1047.

[10] P. Kunkel and V. Mehrmann, \textit{Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index}, Math. Control Signals Systems, 20 (2008), pp. 227–269.

[11] D S. Mackey, N. Mackey, and F. Tisseur, \textit{Structured mapping problems for matrices associated with scalar products. part i: Lie and jordan algebras}, SIAM Journal on Matrix Analysis and Applications, 29 (2008), pp. 1389–1410.

[12] C. Mehl, V. Mehrmann, and P. Sharma, \textit{Stability radii for linear Hamiltonian systems with dissipation under structure-preserving perturbations}, SIAM Journal on Matrix Analysis and Applications, 37 (2016), pp. 1625–1654.

[13] \textit{Stability radii for real linear Hamiltonian systems with perturbed dissipation}, BIT Numerical Mathematics, 57 (2017), pp. 811–843.

[14] Christian Mehl, Volker Mehrmann, and Punit Sharma, \textit{Structured eigenvalue/eigenvector backward errors of matrix pencils arising in optimal control}, Electronic Journal of Linear Algebra, 34 (2018).

[15] Christian Mehl, Volker Mehrmann, and Marek Wojtylak, \textit{Linear algebra properties of dissipative hamiltonian descriptor systems}, SIAM Journal on Matrix Analysis and Applications, 39 (2018), pp. 1489–1519.
Appendix A. Proof of Theorem 12

Proof. Let us suppose that \( S_d^{\text{Sym}} \neq \emptyset \). Then there exists \( \Delta = [\Delta_1 \; \Delta_2] \) with \( \Delta_1 \in \mathbb{C}^{n,n} \) and \( \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta^T \Delta = \Delta \), and \( \Delta \ast = w \). This implies that \( y \ast z = (\Delta x)^* = x^* \Delta^* z = x^* w \).

Conversely, if \( x^* w = y^* z, z^T w_1 \in \mathbb{R} \), then \( H = [H_1 \; H_2] \) satisfies that \( H x = y, H^* z = w \), and \( H_1^T = H_1 \), which implies that \( H \in S_d^{\text{Sym}} \).

Next, we prove (28). First suppose that \( \Delta \in S_d^{\text{Sym}} \), i.e., \( \Delta = [\Delta_1 \; \Delta_2] \), such that \( \Delta x = y, \Delta^* z = w \), and \( \Delta_1^T = \Delta_1 \). This implies that

\[
\Delta_1 x_1 + \Delta_2 x_2 = y, \quad \bar{\Delta}_1 z = w_1 \quad \text{and} \quad \bar{\Delta}_2^* z = w_2. \tag{A.1}
\]

Since \( \Delta_1 \) is a Complex-Symmetric matrix taking \( \bar{z} \) to \( \bar{w}_1 \), from Theorem 3 \( \Delta_1 \) has the form

\[
\Delta_1 = \bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T - \bar{z}^T \bar{w}_1 \bar{z}^\dagger + (P_{\bar{z}})^T K P_{\bar{z}} \tag{A.2}
\]

for some Complex-symmetric matrix \( K \in \mathbb{C}^{n,n} \). By substituting \( \Delta_1 \) from (A.2) in (A.1), we get

\[
\Delta_2 x_2 = \bar{y} \quad \text{and} \quad \Delta_2^* z = w_2, \tag{A.3}
\]

i.e., a mapping of the form \( \Delta_2 x_2 = \bar{y} \) and \( \Delta_2^* z = w_2 \), where

\[
\bar{y} = \left( y - (\bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T - \bar{z}^T \bar{w}_1 \bar{z}^\dagger + (P_{\bar{z}})^T K P_{\bar{z}}) x_1 \right). \]

The vectors \( x_2, \bar{y}, z, \) and \( w_2 \) satisfy

\[
\bar{y}^* z &= \left( y - (\bar{w}_1 \bar{z}^\dagger + (\bar{w}_1 \bar{z}^\dagger)^T - \bar{z}^T \bar{w}_1 \bar{z}^\dagger + (P_{\bar{z}})^T K P_{\bar{z}}) x_1 \right)^* z \\
&= \left( y^* - x_1^* (w_1 z^\dagger + (w_1 z^\dagger)^T - z^T w_1 z^\dagger + P_z K P_z)^T \right) z \\
&= \left( y^* - x_1^* (w_1 z^\dagger + (w_1 z^\dagger)^T - z^T w_1 z^\dagger + P_z K P_z)^T \right) z \quad (\because K^T = K) \\
&= y^* z - x_1^* w_1 \quad (\because (w_1 z^\dagger + (w_1 z^\dagger)^T - z^T w_1 z^\dagger + P_z K P_z) z = w_1) \\
&= x_2^* w_2 \quad (\because x^* w = x_1^* w_1 + x_2^* w_2 \text{ and } x^* w = y^* z).}

\[30\]
Therefore, from Theorem 6, \( \Delta_2 \) can be written as

\[
\Delta_2 = \tilde{y}x_2^\dagger + (w_2z^\dagger)^* - (w_2z^\dagger)^*x_2x_2^\dagger + \mathcal{P}_2R\mathcal{P}_{x_2},
\]

(A.4)

for some \( R \in \mathbb{C}^{n,m} \).

Thus, in view of (A.2) and (A.4), we have

\[
[\Delta_1 \Delta_2] = \begin{bmatrix} \tilde{w}z^\dagger + (\tilde{w}z^\dagger)^T - \tilde{z}^T \tilde{w}z^\dagger + \mathcal{P}_2^T K \mathcal{P}\tilde{z} & \tilde{y}x_2^\dagger + (w_2z^\dagger)^*x_2x_2^\dagger + \mathcal{P}_2R\mathcal{P}_{x_2} \\
H_1 + \tilde{H}_1(K) & H_2 + \tilde{H}_2(K, R) \end{bmatrix}
\]

\[
= H + \bar{H}(K, R).
\]

(A.5)

This proves “\( \subseteq \)” in (28).

Conversely, let \( [\Delta_1 \Delta_2] = [H_1 + \tilde{H}_1(K) \ H_2 + \tilde{H}_2(K, R)] \), where \( H_1, \tilde{H}_1(K), H_2, \) and \( \tilde{H}_2(K, R) \) are defined by (29)-(32), for some matrices \( R \in \mathbb{C}^{n,m} \) and \( K \in \mathbb{C}^{n,n} \) such that \( K^T = K \). Then it is easy to check that \( [\Delta_1 \Delta_2]x = y \) and \( [\Delta_1 \Delta_2]^*z = w \) since \( x^*w = y^*z \). Also \( (H_1 + \tilde{H}_1(K))^T = H_1 + \tilde{H}_1(K) \) since \( K^T = K \). Hence \( [\Delta_1 \Delta_2] \in \mathcal{S}^{\text{Sym},K} \). This shows “\( \supseteq \)” in (28).

In view of (28), we have

\[
\inf_{\Delta \in \mathcal{S}^{\text{Sym},K}} \|\Delta\|_F^2 = \inf_{K \in \mathbb{C}^{n,n},R \in \mathbb{C}^{n,m},K^T = K} \left( \|H_1\|^2_F + \inf_{\Delta \in \mathcal{S}^{\text{Sym},K}} \|H + \tilde{H}(K, R)\|_F^2 \right)
\]

(A.6)

\[
= \|H_1\|^2_F + \inf_{K \in \mathbb{C}^{n,n},R \in \mathbb{C}^{n,m},K^T = K} \|H_2 + \tilde{H}_2(K, R)\|_F^2
\]

(A.7)

\[
= \|H_1\|^2_F + \inf_{K \in \mathbb{C}^{n,n},K^T = K} \left( \inf_{R \in \mathbb{C}^{n,m}} \|H_2 + \tilde{H}_2(K, R)\|_F^2 \right)
\]

(A.8)

where the first inequality in (A.6) follows due to the fact that for any two real valued functions \( f \) and \( g \) defined on the same domain, \( \inf(f+g) \geq \inf f + \inf g \). Also equality in (A.7) follows since the infimum in the first term is attained when \( K = 0 \). In fact, for any \( K \in \mathbb{C}^{n,n} \) such that \( K^T = K \), we have \( (H_1 + \tilde{H}_1(K))z = w_1 \), which implies from Theorem 6 that the minimum of \( \|H_1 + \tilde{H}_1(K)\|_F \) is attained when \( K = 0 \). Further, for a fixed \( K \) and for any \( R \in \mathbb{C}^{n,m} \), \( H_2 + \tilde{H}_2(K, R) \) is a matrix satisfying \( (H_2 + \tilde{H}_2(K, R))x_2 = \tilde{y} \) and \( (H_2 + \tilde{H}_2(K, R))^*z = w_2 \). This implies from Theorem 6 that for any fixed \( K \), the minimum of \( \|H_2 + \tilde{H}_2(K, R)\|_F \) over \( R \) is attained when \( R = 0 \), which yields (A.8). This proves (33).

Next suppose if \( x_1 = \alpha z \) for some nonzero \( \alpha \in \mathbb{C} \), then \( \tilde{H}_2(K, 0) = 0 \) for every \( K \in \mathbb{C}^{n,n} \). This
obtain \( \eta \) structured backward error \( \eta \) and (108), the corresponding backward errors are denoted by \( J \).

Appendix B. Estimation of \( \eta^S(\cdot, \cdot, \cdot, \lambda, u) \) and \( \eta^Sd(\cdot, \cdot, \cdot, \lambda, u) \) when perturbing any two/three of the blocks \( J, R, E \) and \( B \) of the pencil \( L(z) \)

Let \( L(z) \) be a pencil of the form (2), \( \lambda \in i\mathbb{R} \) and \( u = [u_1^T \ u_2^T \ u_3^T]^T \) with \( u_1, u_2 \in \mathbb{C}^n \setminus \{0\} \) and \( u_3 \in \mathbb{C}^m \).

Appendix B.1. Perturbing only \( J \) and \( R \)

Suppose that only \( J \) and \( R \) blocks of \( L(z) \) are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by \( \eta^B(J, R, \lambda, u) := \eta^B(J, R, 0, 0, \lambda, u) \), \( \eta^S(J, R, \lambda, u) := \eta^S(J, R, 0, 0, \lambda, u) \), and \( \eta^Sd(J, R, \lambda, u) := \eta^Sd(J, R, 0, 0, \lambda, u) \). In this case, the block-structured and the symmetry-structured backward errors \( \eta^B(J, R, \lambda, u) \) and \( \eta^S(J, R, \lambda, u) \) were obtained in [14, Theorem 4.14]. Thus, we provide estimation only for the semidefinite-structured backward error \( \eta^Sd(J, R, \lambda, u) \). In view of (109) and (110), when \( \Delta_E = 0, \Delta_B = 0 \) we obtain

\[
\begin{align*}
(\Delta_J - \Delta_R)^* u_1 & = - (J + R + \lambda E) u_1 \quad \text{for any } y, \\
B^* u_1 + S u_3 & = 0.
\end{align*}
\]

This gives us the following lemma which is analogous to Lemma 6.

Lemma 7. Let \( L(z) \) be a pencil as in (2), and let \( \lambda \in i\mathbb{R} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^m \), and let \( x, y, z \) and \( w \) be defined as in (B.1) and (B.2). If \( z = \alpha x \), then the following statements are equivalent.

1. There exists \( \Delta_J, \Delta_R \in \mathbb{C}^{n,n} \) such that \( \Delta_J \in \text{SHer}(n), \Delta_R \succeq 0 \) satisfying (B.1) and (B.2).

2. There exists \( \Delta \in \mathbb{C}^{n,n} \) such that \( \Delta + \Delta^* \preceq 0, \Delta x = y, \Delta^* z = w \).

3. \( u_3^* B^* u_1 = 0 \).

Moreover, we have

\[
\inf \left\{ \|\Delta_J \|_F^2 : \Delta_J \in \text{SHer}(n), \Delta_R \in \text{Herm}(n), \Delta_R \succeq 0, \text{ satisfying (B.1) and (B.2)} \right\} = \inf \left\{ \|\Delta \|_F^2 : \Delta \in \mathbb{C}^{n,n}, \Delta + \Delta^* \preceq 0, \Delta x = y, \Delta^* z = w \right\}.
\]
Proof. The proof is analogous to \[14\,\text{Lemma 4.13}\] due to Type-1 doubly structured dissipative mapping from Theorem\[16\].

**Theorem 19.** Let $L(z)$ be a pencil as in \[2\], let $\lambda \in i\mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T, u_2^T, u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\check{y} = (J - R + \lambda E)u_2$ and $w_1 = -(J + R + \lambda E)u_1$. Let $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$. Then $\eta^{S_d}(J, R, \lambda, u)$ is finite if and only if $u_3 = 0$ and $B^*u_1 = 0$. If the later condition holds and if $u_2$ satisfies that $R\check{u}_2 \neq 0$, then

$$\eta^{S_d}(J, R, \lambda, u) = \|H\|_F,$$

where

$$H = \check{y}u_2^* + (w_1u_1^*)^*P_{u_2} + P_{u_2}JP_{u_2}, \quad \text{and} \quad J = \frac{1}{4\Re (u_2^*\check{y})}(\check{y} + \frac{\alpha}{|\alpha|^2}w_1)(\check{y} + \frac{\alpha}{|\alpha|^2}w_1)^*.$$

**Proof.** The proof is analogous to the proof of \[14\,\text{Theorem 4.14}\] due to Lemma\[7\] and Theorem\[16\].

**Appendix B.2. Perturbing only $J$ and $B$**

In this section, suppose that only $J$ and $B$ blocks of $L(z)$ are subject to perturbation. Then in view of \[104\], \[106\] and \[108\], the corresponding backward errors are denoted by $\eta^B(J, B, \lambda, u) := \eta^B(J, 0, 0, B, \lambda, u)$, $\eta^S(J, B, \lambda, u) := \eta^S(J, 0, 0, B, \lambda, u)$, and $\eta^{S_d}(J, B, \lambda, u) := \eta^{S_d}(J, 0, 0, B, \lambda, u)$. Note that the block-structured backward error $\eta^B(J, B, \lambda, u)$ was obtained in \[14\,\text{Theorem 4.17}\], but the symmetry-structured backward error $\eta^S(J, B, \lambda, u)$ were not known in \[14\] due to unavailability of the doubly structured skew-Hermitian mappings. Also note that $\eta^{S_d}(J, B, \lambda, u) = \eta^S(J, B, \lambda, u)$, because we are not perturbing $R$ and there is no semidefinite structure on $J$ and $B$. To estimate $\eta^S(J, B, \lambda, u)$ from \[109\] and \[110\], when $\Delta_R = 0$ and $\Delta_E = 0$, we obtain

$$\begin{bmatrix}
\Delta_J
\frac{\Delta B}{:= \Delta_1}
\end{bmatrix}
\begin{bmatrix}
u_2
u_3
\end{bmatrix}
= 
\begin{bmatrix}
(J - R + \lambda E)u_2 + Bu_3
\end{bmatrix}
\begin{bmatrix}
\alpha
\end{bmatrix}
\begin{bmatrix}
\Delta J
\frac{\Delta B}{:= \Delta_1}
\end{bmatrix}^*
\begin{bmatrix}
u_1
\end{bmatrix}
= 
\begin{bmatrix}
-(J + R + \lambda E)u_1 =: w_1
\end{bmatrix}
\begin{bmatrix}
B^*u_1 + Su_3 =: w_2
\end{bmatrix}
\begin{bmatrix}
\alpha
\end{bmatrix}.
$$

This leads to the following lemma.

**Lemma 8.** Let $L(z)$ be a pencil as in \[2\], and let $\lambda \in i\mathbb{R}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T, u_2^T, u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$ and $u_3 \in \mathbb{C}^m$, and let $x, y, z$ and $w$ be defined as in \[B.5\] and \[B.6\]. Then the following statements are equivalent.

1. There exists $\Delta_J \in \mathbb{C}^{n,n}$ and $\Delta_B \in \mathbb{C}^{n,m}$ such that $\Delta_J \in \text{SHerm}(n)$ satisfying \[B.5\] and \[B.6\].
2. $u_3 = 0$ and $Ru_1 = 0$.

Proof. The proof is immediate from Theorem 11 since $J^* = -J$, $R \succeq 0$, and $E^* = E$. □

Theorem 20. Let $L(z)$ be a pencil as in (2), let $\lambda \in i\mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T \; u_2^T \; u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\tilde{y} = (J - R + \lambda E)u_2$ and $w_1 = -(J + R + \lambda E)u_1$, $X = [u_2 \; u_1]$, $Y = [\tilde{y} \; -w_1]$. Then $\eta^S(J, B, \lambda, u)$ is finite if and only if $u_3 = 0$ and $Ru_1 = 0$. If the later condition holds and if $YX^\dagger X = Y$, $Y^*X = -X^*Y$ and if $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then

$$\eta^S(J, B, \lambda, u) = \sqrt{\|H_1\|_F^2 + \|H_2\|_F^2},$$

(B.7)

where

$$H_1 = YX^\dagger - (YX^\dagger)^* - XX^\dagger XX^\dagger^* \quad \text{and} \quad H_2 = u_1u_1^*B.$$ 

(B.8)

Proof. In view of Remark 4 and Lemma 8, we obtain that $\eta^S(J, B, \lambda, u)$ is finite if and only if $u_3 = 0$ and $Ru_1 = 0$. Thus by using $u_3 = 0$ and $Ru_1 = 0$ in (B.5) and (B.6), and using Lemma 8 in the definition of $\eta^S(J, B, \lambda, u)$ from (100), we have that

$$\eta^S(J, B, \lambda, u) = \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_J & \Delta_B \\ \Delta_J^* & -\Delta_J \end{bmatrix}, \Delta_J \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_J^* = -\Delta_J, \Delta_Ju_2 = \tilde{y}, \Delta_J^*u_1 = w_1, \Delta_B^*u_1 = B^*u_1 \right\}$$

$$= \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_J & \Delta_B \\ \Delta_J^* & -\Delta_J \end{bmatrix}, \Delta_J \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_J^* = -\Delta_J, \Delta_J[u_2 \; u_1] = [\tilde{y} \; -w_1], \Delta_B^*u_1 = B^*u_1 \right\}$$

$$= \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_J & \Delta_B \\ \Delta_J^* & -\Delta_J \end{bmatrix}, \Delta_J \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_J^* = -\Delta_J, \Delta_JX = Y, \Delta_B^*u_1 = B^*u_1 \right\}. \quad \text{(B.9)}$$

If $YX^\dagger X = Y$, $Y^*X = -X^*Y$ and $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then from [1, Theorem 2.2.3], there always exists a skew-Hermitian mapping $\Delta_J$ such that $\Delta_J^* = -\Delta_J$ and $\Delta_JX = Y$. The minimal Frobenius norm of such a $\Delta_J$ is attained by the unique matrix $H_1$ defined in (B.8). Similarly, from Theorem 1 for any $u_1$ there always exists $\Delta_B \in \mathbb{C}^{n,m}$ such that $\Delta_B^*u_1 = B^*u_1$ and the minimal Frobenius norm of such a $\Delta_B$ is attained by $H_2 := u_1u_1^*B$. Using the minimal Frobenius norm mappings $H_1$ and $H_2$, we obtain (1.7). This completes the proof.

Appendix B.3. Perturbing only $R$ and $B$

Here, suppose that only $R$ and $B$ blocks of $L(z)$ are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by $\eta^B(R, B, \lambda, u) := \eta^B(0, R, 0, B, \lambda, u)$, $\eta^S(R, B, \lambda, u) := \eta^S(0, R, 0, B, \lambda, u)$, and $\eta^{S*}(R, B, \lambda, u) := \eta^{S*}(0, R, 0, B, \lambda, u)$. Note that the backward error $\eta^B(R, B, \lambda, u)$ was obtained in [14, Remark 4.18]. In this section, we compute the eigenpair backward errors $\eta^S(R, B, \lambda, u)$ and $\eta^{S*}(R, B, \lambda, u)$. 

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In view of Remark 4 and (B.10)-(B.11), we obtain that

\[\begin{bmatrix}
  -\Delta_R \\
  \Delta_B
\end{bmatrix}
\begin{bmatrix}
  u_2 \\
  u_3
\end{bmatrix}
= (J - R + \lambda E)u_2 + Bu_3
\]  \hspace{1cm} (B.10)

\[\begin{bmatrix}
  -\Delta_R \\
  \Delta_B
\end{bmatrix}^* u_1
= \begin{bmatrix}
  -(J + R + \lambda E)u_1 =: w_1 \\
  B^*u_1 + Su_3 =: w_2
\end{bmatrix}
\]  \hspace{1cm} (B.11)

In view of Theorem 10 there exists \(\Delta_R \in \mathbb{C}^{n,n}\) and \(\Delta_B \in \mathbb{C}^{n,m}\) such that \(\Delta_R \in \text{Herm}(n)\) satisfying (B.10) and (B.11) if and only if \(u_3 = 0\) and \(u_1^*(J + \lambda E)u_1 = 0\). We have the following result for \(\eta^S(R, B, \lambda, u)\).

**Theorem 21.** Let \(L(z)\) be a pencil as in (2), let \(\lambda \in \mathbb{R} \setminus \{0\}\) and \(u \in \mathbb{C}^{2n+m} \setminus \{0\}\). Partition \(u = [u_1^T \ u_2^T \ u_3^T]^T\) such that \(u_1, u_2 \in \mathbb{C}^n\), and \(u_3 \in \mathbb{C}^m\). Set \(\tilde{y} = (J - R + \lambda E)u_2\) and \(w_1 = -\tilde{y} - (J + R + \lambda E)u_1\), \(X = [u_2 \ u_1] \), \(Y = [\tilde{y} \ w_1]\). Then \(\eta^S(R, B, \lambda, u)\) is finite if and only if \(u_3 = 0\) and \(u_1^*(J + \lambda E)u_1 = 0\). If the later condition holds and if \(YX^1X = Y, Y^*X = X^*Y\) and if \(u_2 = \alpha u_1\) for some nonzero \(\alpha \in \mathbb{C}\), then

\[\eta^S(R, B, \lambda, u) = \sqrt{\|H_1\|_F^2 + \|H_2\|_F^2},\]  \hspace{1cm} (B.12)

where

\[H_1 = YX^1 + (YX^1)^* - XX^1YX^1 \quad \text{and} \quad H_2 = u_1u_1^TB.\]  \hspace{1cm} (B.13)

**Proof.** In view of Remark 4 and (B.10)-(B.11), we obtain that \(\eta^S(R, B, \lambda, u)\) is finite if and only if \(u_3 = 0\) and \(u_1^*(J + \lambda E)u_1 = 0\). Thus by using \(u_3 = 0\) (B.10) and (B.11), we have from (109) that

\[\eta^S(R, B, \lambda, u) = \inf \left\{ \|\Delta\|_F^2 : \begin{array}{l}
  \Delta = [\Delta_R \ \Delta_B], \Delta_R \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_R^* = \Delta_R, \\
  \Delta_Ru_2 = \tilde{y}, \Delta_R^*u_1 = u_1, \Delta_B^*u_1 = B^*u_1
\end{array} \right\}
= \inf \left\{ \|\Delta\|_F^2 : \begin{array}{l}
  \Delta = [\Delta_R \ \Delta_B], \Delta_R \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_R^* = \Delta_R, \\
  \Delta_R[u_2 \ u_1] = [\tilde{y} \ w_1], \Delta_B^*u_1 = B^*u_1
\end{array} \right\}
= \inf \left\{ \|\Delta\|_F^2 : \begin{array}{l}
  \Delta = [\Delta_R \ \Delta_B], \Delta_R \in \mathbb{C}^{n,n}, \Delta_B \in \mathbb{C}^{n,m}, \Delta_R^* = \Delta_R, \\
  \Delta_RX = Y, \Delta_B^*u_1 = B^*u_1
\end{array} \right\}
\]  \hspace{1cm} (B.14)

If \(YX^1X = Y, Y^*X = X^*Y\) and \(u_2 = \alpha u_1\) for some nonzero \(\alpha \in \mathbb{C}\), then from 4, Theorem 2.2.3[1], there always exists a Hermitian mapping \(\Delta_R\) such that \(\Delta_R^* = \Delta_R\) and \(\Delta_RX = Y\). From 4, Theorem 2.2.3[1], the minimal Frobenius norm of such a \(\Delta_R\) is attained by the unique matrix \(H_1\) defined in (B.13). Similarly, from Theorem 4 for any \(u_1\) there always exists \(\Delta_B \in \mathbb{C}^{n,m}\) such that \(\Delta_B^*u_1 = B^*u_1\) and the minimal Frobenius norm of such a \(\Delta_B\) is attained by \(H_2 := u_1u_1^TB\). Using minimal Frobenius norm mappings \(H_1\) and \(H_2\), we obtain (B.12). This completes the proof.
Next, we estimate the semidefinite structured backward error $\eta^S_d(R, B, \lambda, u)$. For this, we need the following lemma.

**Lemma 9.** Let $L(z)$ be a pencil as in (2), and let $\lambda \in i\mathbb{R}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T, u_2^T, u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n \setminus \{0\}$ and $u_3 \in \mathbb{C}^m$, and let $x, y, z$ and $w$ be defined as in (B.10) and (B.11). Then the following statements are equivalent.

1. There exists $\Delta_R \in \mathbb{C}^{n,n}$ and $\Delta_B \in \mathbb{C}^{n,m}$ such that $\Delta_R \succeq 0$ satisfying (B.10) and (B.11).

2. There exists $\Delta = [\Delta_1 \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta_1 \preceq 0, \Delta x = y$, and $\Delta^* z = w$.

3. $u_3 = 0, u_1^*(J + \lambda E)u_1 = 0$, and $Ru_1 \neq 0$.

Moreover, we have

$$
\inf \left\{ \left\| [\Delta_R \Delta_B] \right\|_F^2 : \Delta_R \in \text{Herm}(n), \Delta_R \succeq 0 \Delta_B \in \mathbb{C}^{n,m} \text{ satisfying (B.10) and (B.11)} \right\} = \inf \left\{ \left\| \Delta \right\|_F^2 : \Delta = [\Delta_1 \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = \Delta_1 \preceq 0, \Delta x = y, \Delta^* z = w \right\}.
$$

**Proof.** The proof is immediate from the doubly structured semidefinite mapping from Theorem 13. □

**Theorem 22.** Let $L(z)$ be a pencil as in (2), let $\lambda \in i\mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T, u_2^T, u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\hat{y} = (J - R + \lambda E)u_2$ and $w_1 = -(J + R + \lambda E)u_1, X = [u_2 \ u_1], Y = [\hat{y} \ w_1]$. Then $\eta^S_d(R, B, \lambda, u)$ is finite if and only if $u_3 = 0, u_1^*(J + \lambda E)u_1 = 0$, and $Ru_1 \neq 0$. If the later condition holds and if $YX^*X = Y, X^*Y < 0$ and if $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then

$$
\eta^S_d(R, B, \lambda, u) = \sqrt{\|H_1\|_F^2 + \|H_2\|_F^2}, \quad (B.15)
$$

where

$$
H_1 = Y(Y^*X)^{-1}Y^* \quad \text{and} \quad H_2 = u_1u_1^*B. \quad (B.16)
$$

**Proof.** In view of Remark 4 and Lemma 9 we obtain that $\eta^S_d(R, B, \lambda, u)$ is finite if and only if $u_3 = 0, u_1^*(J + \lambda E)u_1 = 0$, and $Ru_1 \neq 0$. Thus by using $u_3 = 0$ in (B.12) and (B.13), and using...
Lemma 9 in (100), we have that
\[
\eta^S_d(R, B, \lambda, u) = \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}, \Delta_1 \in \mathbb{C}^{n \times n}, \Delta_2 \in \mathbb{C}^{n \times m}, \Delta_1^* = \Delta_1 \preceq 0, \\
\Delta_1 u_2 = \tilde{y}, \Delta_1^* u_1 = w_1, \Delta_2^* u_1 = B^* u_1 \right\}
\]
\[
= \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}, \Delta_1 \in \mathbb{C}^{n \times n}, \Delta_2 \in \mathbb{C}^{n \times m}, \Delta_1^* = \Delta_1 \preceq 0, \\
\Delta_1 [u_2 u_1] = [\tilde{y} w_1], \Delta_2^* u_1 = B^* u_1 \right\}
\]
\[
= \inf \left\{ \|\Delta\|_F^2 : \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}, \Delta_1 \in \mathbb{C}^{n \times n}, \Delta_2 \in \mathbb{C}^{n \times m}, \Delta_1^* = \Delta_1 \preceq 0, \\
\Delta_1 X = Y, \Delta_2^* u_1 = B^* u_1 \right\}. 
\]
(B.17)

If \( Y X^\dagger X = Y, X^* Y < 0 \), and \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \), then from [13, Theorem 2.2], there always exists a negative definite mapping \( \Delta_1 \) such that \( \Delta_1^* = \Delta_1 \preceq 0 \) and \( \Delta_1 X = Y \). From [13, Theorem 2.2] the minimal Frobenius norm of such a \( \Delta_1 \) is attained by the unique matrix \( H_1 \) defined in [3,10]. Similarly, from Theorem 1 for any \( u_1 \) there always exists \( \Delta_2 \in \mathbb{C}^{n \times m} \) such that \( \Delta_2^* u_1 = B^* u_1 \) and the minimal Frobenius norm of such a \( \Delta_2 \) is attained by \( H_2 := u_1 u_1^\dagger B \).

Thus using the minimal Frobenius norm mappings \( H_1 \) and \( H_2 \), we obtain (B.15).

Appendix B.4. Perturbing only \( E \) and \( B \)

In this section, suppose that only \( E \) and \( B \) blocks of \( L(z) \) are subject to perturbation.

Then in view of (103), (104) and (105), the corresponding backward errors are denoted by \( \eta^E_d(E, B, \lambda, u) := \eta^E(0, 0, E, B, \lambda, u) \), \( \eta^S_d(E, B, \lambda, u) := \eta^S(0, 0, E, B, \lambda, u) \), and \( \eta^S_d(E, B, \lambda, u) := \eta^S(0, 0, E, B, \lambda, u) \). Again note that \( \eta^E_d(E, B, \lambda, u) \) was obtained in [14, Theorem 4.19], and we have \( \eta^S_d(E, B, \lambda, u) = \eta^S(E, B, \lambda, u) \) because we are not perturbing \( R \) and there is no semidefinite structure on \( E \) or \( B \).

In view of (109) and (110), when \( \Delta_J = 0 \) and \( \Delta_R = 0 \), we have
\[
\begin{bmatrix}
\lambda \Delta_E & \Delta_B \\
\vspace{1em}
= : \Delta_3 & = : \Delta_2
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
J - R \lambda E u_2 + B u_3 \\
\vspace{1em}
= : \gamma
\end{bmatrix}
\]
(B.18)
\[
\begin{bmatrix}
\lambda \Delta_E & \Delta_B \\
\vspace{1em}
\begin{bmatrix}
u_1 \\
\vspace{1em}
= : \xi
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vspace{1em}
\begin{bmatrix}
u_2 \\
\vspace{1em}
\begin{bmatrix}
u_3 \\
\vspace{1em}
\begin{bmatrix}
u_1 \\
\vspace{1em}
\begin{bmatrix}
u_2 \\
\vspace{1em}
\begin{bmatrix}
u_3 \\
\vspace{1em}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
-(J + R \lambda E) u_1 = : w_1 \\
\vspace{1em}
B^* u_1 + S u_3 = : w_2 \\
\vspace{1em}
= : w
\end{bmatrix}
\]
(B.19)

As \( \lambda \in \mathbb{R} \) and \( \Delta_E^* = \Delta_E \), we have that \( \Delta_1 = \lambda \Delta_E \) is skew-Hermitian. Then a direct application of the doubly structured skew-Hermitian mapping from Theorem 10 yields the following lemma.

Lemma 10. Let \( L(z) \) be a pencil as in (2), and let \( \lambda \in \mathbb{R} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^m \), and let \( x, y, z \) and \( w \) be defined as in (B.18) and (B.19). Then the following statements are equivalent.
1. There exists \( \Delta_E \in \mathbb{C}^{n,n} \) and \( \Delta_B \in \mathbb{C}^{n,m} \) such that \( \Delta_E \in \text{Herm}(n) \) satisfying (B.18) and (B.19).

2. There exists \( \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta_1^* = -\Delta_1, \ \Delta x = y, \ \text{and} \ \Delta^* z = w. \)

3. \( u_3 = 0 \) and \( Ru_1 = 0. \)

Moreover, we have

\[
\inf \left\{ \| [\Delta_E \ \Delta_B] \|^2_F : \Delta_E \in \text{Herm}(n), \Delta_B \in \mathbb{C}^{n,m} \text{ satisfying (B.18) and (B.19)} \right\} 
= \inf \left\{ \frac{1}{|\lambda|^2} \| \Delta_1 \|^2_F + \| \Delta_2 \|^2_F : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = -\Delta_1, \ \Delta x = y, \ \Delta^* z = w \right\}. 
\]

The following result provides bounds for the backward error \( \eta^S(E, B, \lambda, u) \).

**Theorem 23.** Let \( L(z) \) be a pencil as in (2), let \( \lambda \in i\mathbb{R} \setminus \{0\} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^{n}, \) and \( u_3 \in \mathbb{C}^{m} \). Set \( \tilde{y} = (J - R + \lambda E)u_2 + w_1 = -(J + R + \lambda E)u_1, \ X = [u_2 \ u_1], \ Y = [\tilde{y} \ -w_1] \). Then \( \eta^S(E, B, \lambda, u) \) is finite if and only if \( u_3 = 0 \) and \( Ru_1 = 0 \). If the latter conditions holds and if \( YX^\dagger X = Y, \ Y^*X = -X^*Y \) and if \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \), then

\[
\eta^S(E, B, \lambda, u) = \sqrt{\frac{1}{|\lambda|^2} \| H_1 \|^2_F + \| H_2 \|^2_F}, \tag{B.20}
\]

where

\[
H_1 = YX^\dagger - (YX^\dagger)^* - XX^\dagger YX^\dagger, \quad H_2 = u_1^* B. \tag{B.21}
\]

**Proof.** In view of Lemma 110 the proof is similar to the proof of Theorem 20. \( \square \)

**Appendix B.5. Perturbing only \( R \) and \( E \)**

Here suppose that only \( R \) and \( E \) blocks of \( L(z) \) are subject to perturbation. Then in view of (104), (106), and (108), the corresponding backward errors are denoted by \( \eta^S(R, E, \lambda, u) := \eta^S(0, R, E, 0, \lambda, u), \eta^S(R, E, \lambda, u) := \eta^S(0, R, E, 0, \lambda, u), \) and \( \eta^{S+}(R, E, \lambda, u) := \eta^{S+}(0, R, E, 0, \lambda, u). \)

We note that the backward errors \( \eta^S(R, E, \lambda, u) \) and \( \eta^{S+}(R, E, \lambda, u) \) were considered in [14, Theorem 4.10]. Thus, in this section, we consider only \( \eta^{S+}(R, E, \lambda, u) \). From (109) and (110), when \( \Delta_J = 0 \) and \( \Delta_B = 0 \) we get

\[
(-\Delta R + \lambda \Delta_E) u_2 := x = (J - R + \lambda E)u_2 + Bu_3 \tag{B.22}
\]

\[
(-\Delta R + \lambda \Delta_E)^* u_1 := z = -(J + R + \lambda E)u_1 \tag{B.23}
\]

\[
B^* u_1 + Su_3 = 0. \tag{B.24}
\]

This leads to the following lemma which will be useful in estimating \( \eta^{S+}(R, E, \lambda, u) \).
Lemma 11. Let $L(z)$ be a pencil as in (2), and let $\lambda \in i \mathbb{R}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T \ u_2^T \ u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$ and $u_3 \in \mathbb{C}^m$, and let $x, y, z$ and $w$ be defined as in (B.22) and (B.23). If $z = \alpha x$ for some nonzero $\alpha \in \mathbb{C}$, then the following statements are equivalent.

1. There exists $\Delta_R, \Delta_E \in \mathbb{C}^{n,n}$ such that $\Delta_E \in \text{Herm}(n)$ and $\Delta_R \succeq 0$ satisfying (B.22) and (B.23).

2. There exists $\Delta \in \mathbb{C}^{n,n}$ such that $\Delta + \Delta^* \preceq 0$, $\Delta x = y$, $\Delta^* z = w$.

3. $u_3^* B^* u_1 = 0$.

Moreover, we have

$$\inf \left\{ \left\| \Delta_R \Delta_E \right\|_F^2 : \Delta_E \in \text{Herm}(n), \Delta_R \succeq 0 \text{ satisfying (B.22) and (B.23)} \right\}$$

$$= \inf \left\{ \left\| \Delta + \Delta^* \right\|_F^2 + \frac{1}{|\lambda|^2} \left\| \Delta - \Delta^* \right\|_F^2 : \Delta \in \mathbb{C}^{n,n}, \Delta + \Delta^* \preceq 0, \Delta x = y, \Delta^* z = w \right\}.$$  

Proof. The proof is analogous to [14, Lemma 4.9], due to Type-1 doubly structured dissipative mapping from Theorem 16.

Theorem 24. Let $L(z)$ be a pencil as in (2), let $\lambda \in i \mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T \ u_2^T \ u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\tilde{y} = (J - R + \lambda E) u_2$ and $w_1 = -(J + R + \lambda E) u_1$. If $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then $\eta^{S_4}(R, E, \lambda, u)$ is finite if and only if $u_3 = 0$ and $B^* u_1 = 0$. If the later condition holds and if $u_2$ satisfies that $Ru_2 \neq 0$, then

$$\frac{\|H\|_F}{|\lambda|} \leq \eta^{S_4}(R, E, \lambda, u) \leq \sqrt{\frac{\|H + H^*\|_F^2}{2} + \frac{1}{|\lambda|^2} \left\| \frac{H - H^*}{2} \right\|_F^2}, \quad \text{if } |\lambda| \geq 1,$$

and

$$\|H\|_F \leq \eta^{S_4}(R, E, \lambda, u) \leq \sqrt{\frac{\|H + H^*\|_F^2}{2} + \frac{1}{|\lambda|^2} \left\| \frac{H - H^*}{2} \right\|_F^2}, \quad \text{if } |\lambda| \leq 1,$$

where

$$H = \tilde{y} u_2^\dagger + (w_1^\dagger u_1^\dagger)^* P u_2 + P u_2 J P u_2 \quad \text{and} \quad J = \frac{1}{4 \text{Re}(u_2^\dagger \tilde{y})} \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right) \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right)^*.$$  

(B.25)

Proof. The proof is analogous to the proof of [14, Theorem 4.10] using Lemma 11 and Theorem 16.

Appendix B.6. Perturbing only $J$, $R$ and $E$

Suppose that the blocks $J$, $R$ and $E$ of $L(z)$ are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by $\eta^B(J, R, E, \lambda, u) := \eta^B(J, R, E, 0, \lambda, u)$, $\eta^S(J, R, E, \lambda, u) := \eta^S(J, R, E, 0, \lambda, u)$, and $\eta^{S_4}(J, R, E, \lambda, u) := \eta^{S_4}(J, R, E, 0, \lambda, u)$.

The block- and symmetry-structured backward errors $\eta^B(J, R, E, \lambda, u)$ and $\eta^S(J, R, E, \lambda, u)$ were
obtained in [14, Theorem 5.11]. In this section, we focus on estimating the backward error \( \eta^{S_d}(J, R, E, \lambda, u) \).

From [109] and [110], when \( \Delta_B = 0 \) we have

\[
(\Delta_J - \Delta_R + \lambda \Delta_E) u_2 = (J - R + \lambda E) u_2 + Bu_3
\]

(B.26)

\[
(\Delta_J - \Delta_R + \lambda \Delta_E)^* u_1 = -(J + R + \lambda E) u_1
\]

(B.27)

\[ B^* u_1 + S u_3 = 0. \]

(B.28)

In view of (B.26) and (B.28), we have the following lemma which is analogous to [6] for estimating \( \eta^{S_d}(J, R, E, \lambda, u) \).

Lemma 12. Let \( L(z) \) be a pencil as in (2), and let \( \lambda \in i \mathbb{R} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^m \), and let \( x, y, z \) and \( w \) be defined as in (B.26) and (B.27). If \( z = \alpha x \), then the following statements are equivalent.

1. There exists \( \Delta_J, \Delta_R, \Delta_E \in \mathbb{C}^{n,n} \) such that \( \Delta_J \in \text{SHer}(n) \), \( \Delta_E \in \text{Herm}(n) \), and \( \Delta_R \geq 0 \) satisfying (B.26) and (B.27).

2. There exists \( \Delta \in \mathbb{C}^{n,n} \) such that \( \Delta + \Delta^* \leq 0 \), \( \Delta x = y \), \( \Delta^* z = w \).

3. \( u_3^* B^* u_1 = 0 \).

Moreover, we have

\[
\inf \left\{ \| \Delta_J, \Delta_R, \Delta_E \|_F^2 : \Delta_J \in \text{SHer}(n), \Delta_E \in \text{Herm}(n), \Delta_R \geq 0, \right. \]

\[
\text{satisfying (B.26) and (B.27)} \right\}

\[
= \inf \left\{ \left\| \frac{\Delta + \Delta^*}{2} \right\|_F^2 + \frac{1}{1 + |\lambda|^2} \left\| \frac{\Delta - \Delta^*}{2} \right\|_F^2 : \Delta \in \mathbb{C}^{n,n}, \Delta + \Delta^* \leq 0, \Delta x = y, \Delta^* z = w \right\}.
\]

Proof. The proof is analogous to [14, Lemma 5.10], due to Type-1 doubly structured dissipative mapping from Theorem [16].

Theorem 25. Let \( L(z) \) be a pencil as in (2), let \( \lambda \in i \mathbb{R} \setminus \{0\} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \), and \( u_3 \in \mathbb{C}^m \). Set \( \tilde{y} = (J - R + \lambda E) u_2 \) and \( w_1 = -(J + R + \lambda E) u_1 \). If \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \), then \( \eta^{S_d}(J, R, \lambda, u) \) is finite if and only if \( u_3 = 0 \) and \( B^* u_1 = 0 \). If the later condition holds and if \( u_2 \) satisfies that \( R u_2 \neq 0 \), then

\[
\frac{\|H\|_F}{\sqrt{1 + |\lambda|^2}} \leq \eta^{S_d}(J, R, E, \lambda, u) \leq \sqrt{\left\| H + H^* \right\|_F^2 + \frac{1}{1 + |\lambda|^2} \left\| H - H^* \right\|_F^2},
\]

where

\[
H = \tilde{y} u_2^* + (w_1 u_1^*)^* P_{u_2} + P_{u_2} J P_{u_2} \quad \text{and} \quad J = \frac{1}{4 \Re \{u_2^* \tilde{y}\}} (\tilde{y} + \frac{\alpha}{|\alpha|^2} w_1) (\tilde{y} + \frac{\alpha}{|\alpha|^2} w_1)^*.
\]
Proof. The proof is analogous to the proof of [14, Theorem 5.11] due to Lemma 12 and Theorem 16.

Appendix B.7. Perturbing only J, E and B

In this section, suppose that the blocks \( J \), \( E \) and \( B \) of \( L(z) \) are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by
\[
\eta_B(J, E, B, \lambda, u) := \eta_B(J, 0, E, B, \lambda, u), \quad \eta_S(J, E, B, \lambda, u) := \eta_S(J, 0, E, B, \lambda, u), \quad \eta_{Sd}(J, E, B, \lambda, u) := \eta_{Sd}(J, E, B, \lambda, u).
\]
The backward error \( \eta_B(J, E, B, \lambda, u) \) was given in [14, Remark 5.8], and we have \( \eta_{Sd}(J, E, B, \lambda, u) = \eta_S(J, E, B, \lambda, u) \) because there is no semidefinite structure on \( J \) or \( E \) or \( B \).

Thus we focus on computing \( \eta_{Sd}(J, E, B, \lambda, u) \).

From (109) and (110), when \( \Delta_R = 0 \) we have
\[
\begin{bmatrix}
\Delta_J + \lambda \Delta_E & \Delta_B
\end{bmatrix}_{\Delta_1} \begin{bmatrix}
u_2 \\
u_3
\end{bmatrix} = \begin{cases}
(J - R + \lambda E)u_2 + Bu_3
\end{cases}
\]
\[=: y \]
\[\begin{bmatrix}
\Delta_J + \lambda \Delta_E & \Delta_B
\end{bmatrix}^* \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = \begin{cases}
-(J - R + \lambda E)u_1 =: w_1
\end{cases}
\]
\[=: w \]

Thus using doubly structured skew-Hermitian mapping from Theorem 11 in (B.29) and (B.30) gives the following lemma.

Lemma 13. Let \( L(z) \) be a pencil as in (2), and let \( \lambda \in i\mathbb{R} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^m \), and let \( x, y, z \) and \( w \) be defined as in (B.29) and (B.30). Then the following statements are equivalent.

1. There exists \( \Delta_J \in \mathbb{C}^{n,n} \) and \( \Delta_E \in \mathbb{C}^{n,m} \) such that \( \Delta_J \in \text{SHerm}(n) \), \( \Delta_E \in \text{Herm}(n) \) satisfying (B.29) and (B.30).

2. There exists \( \Delta = [\Delta_1 \ \Delta_2] \), \( \Delta_1 \in \mathbb{C}^{n,n} \), \( \Delta_2 \in \mathbb{C}^{n,m} \) such that \( \Delta_1^* = -\Delta_1 \), \( \Delta x = y \), and \( \Delta^* z = w \).

3. \( u_3 = 0 \) and \( Ru_1 = 0 \).

Moreover, we have
\[
\inf \left\{ \left\| \begin{bmatrix} \Delta_J & \Delta_E & \Delta_B \end{bmatrix} \right\|^2_F : \Delta_J \in \text{SHerm}(n), \Delta_E \in \text{Herm}(n), \Delta_B \in \mathbb{C}^{n,m} \text{ satisfying (B.29) and (B.30)} \right\}
\]
\[= \inf \left\{ \frac{1}{\|\lambda\|_1} \left[ \left\| \Delta_1 \right\|_F^2 + \left\| \Delta_2 \right\|_F^2 \right] : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = -\Delta_1, \Delta x = y, \Delta^* z = w \right\}.
\]

Proof. The proof is similar to [14, Lemma 5.6] due to doubly structured skew Hermitian mapping from Theorem 11.

\textbf{Theorem 26.} Let $L(z)$ be a pencil as in \cite{2}, let $\lambda \in i\mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T, u_2^T, u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\bar{y} = (J - R + \lambda \mathbf{E})u_2$ and $w_1 = -(J + R + \lambda \mathbf{E})u_1$, $X = [u_2 \ u_1]$, $Y = [\bar{y} \ -w_1]$. Then $\eta^S(E,B,\lambda,u)$ is finite if and only if $u_3 = 0$ and $Ru_1 = 0$. If the later condition holds and if $Y^\dagger X = Y$, $Y^*X = -X^*Y$, if $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then

$$
\eta^S(J,E,B,\lambda,u) = \sqrt{\frac{1}{1 + |\lambda|^2} \|H_1\|_F^2 + \|H_2\|_F^2},
$$

(B.31)

where

$$
H_1 = YX^\dagger - (YX^\dagger)^* - XX^\dagger YX^\dagger \quad \text{and} \quad H_2 = u_1u_1^TB.
$$

(B.32)

\textbf{Proof.} In view of Remark 4 and Lemma 13 we have that $\eta^S(J,E,B,\lambda,u)$ is finite if and only if $u_3 = 0$ and $Ru_1 = 0$. Thus by using $u_3 = 0$ in (B.29) and (B.30), and using Lemma 13 in (106), we have that

$$
\eta^S(J,E,B,\lambda,u)^2 = \inf \left\{ \frac{1}{\bar{1} + |\lambda|^2} \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 : \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = -\Delta_1, \right. \\
\Delta_1 u_2 = \bar{y}, \Delta_1 u_1 = w_1, \Delta_2^* u_1 = B^*u_1 \left. \right\}
$$

$$
= \inf \left\{ \frac{1}{\bar{1} + |\lambda|^2} \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 : \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = -\Delta_1, \right. \\
\Delta_1[u_2 \ u_1] = [\bar{y} \ -w_1], \Delta_2^* u_1 = B^*u_1 \left. \right\}
$$

$$
= \inf \left\{ \frac{1}{\bar{1} + |\lambda|^2} \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 : \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1^* = -\Delta_1, \right. \\
\Delta_1 X = Y, \Delta_2^* u_1 = B^*u_1 \left. \right\}.
$$

(B.33)

If $YX^\dagger X = Y$, $Y^*X = -X^*Y$ and $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then from \cite{1} Theorem 2.2.3], there always exists a skew-Hermitian mapping $\Delta_1$ such that $\Delta_1^* = -\Delta_1$ and $\Delta_1 X = Y$. The minimal Frobenius norm of such a $\Delta_1$ is attained by the unique matrix $H_1$ defined in (133).

Similarly, from Theorem 1 for any $u_1$ there always exists $\Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta_2^* u_1 = B^*u_1$ and the minimal Frobenius norm of such a $\Delta_2$ is attained by $H_2 := u_1u_1^TB$. Thus using the minimal Frobenius norm mappings $H_1$ and $H_2$, we obtain (B.31). This completes the proof. \hfill \Box

\textbf{Appendix B.8. Perturbing only $R,E$ and $B$}

Here, suppose that the blocks $R$, $E$ and $B$ of $L(z)$ are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by $\eta^B(R,E,B,\lambda,u)$ := $\eta^B(0,R,E,B,\lambda,u)$, $\eta^S(R,E,B,\lambda,u)$ := $\eta^S(0,R,E,B,\lambda,u)$, and $\eta^{Sd}(R,E,B,\lambda,u)$ := $\eta^{Sd}(0,R,E,B,\lambda,u)$. The block and symmetry structured backward errors $\eta^B(R,E,B,\lambda,u)$ and $\eta^S(R,E,B,\lambda,u)$ were obtained in \cite{14} Theorem 5.7]. In this section we compute bounds for semidefinite structured backward error $\eta^{Sd}(R,E,B,\lambda,u)$. 

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Lemma 14. Let $L(z)$ be a pencil as in (2), and let $\lambda \in i\mathbb{R}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T \ u_2^T \ u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$ and $u_3 \in \mathbb{C}^m$, and let $x, y, z$ and $w$ be defined as in (B.34) and (B.35). Then the following statements are equivalent.

1. There exists $\Delta_R, \Delta_E, \Delta_F \subseteq \mathbb{C}^{n,n}$ and $\Delta_B \subseteq \mathbb{C}^{n,m}$ such that $\Delta_R \succ 0$, and $\Delta_E \in \text{Herm}(n)$ satisfying (B.34) and (B.35).

2. There exists $\Delta = [\Delta_1 \ \Delta_2], \Delta_1 \subseteq \mathbb{C}^{n,n}, \Delta_2 \subseteq \mathbb{C}^{m,m}$ such that $\Delta_1 + \Delta_1^* \preceq 0$, $\Delta x = y$, and $\Delta^* \Delta = w$.

3. $u_3 = 0$.

Moreover, we have

$$\inf \left\{ \| [\Delta_R \ \Delta_E \ \Delta_B] \|_F^2 : \Delta_R, \Delta_E, \Delta_B \in \text{Herm}(n), \Delta_R \succ 0, \Delta_B \in \mathbb{C}^{n,m} \text{ satisfying (B.34) and (B.35)} \right\} = \inf \left\{ \frac{\| \Delta_1 + \Delta_1^* \|^2_F}{2} \left[ \frac{1}{|\lambda|^2} \left\| \frac{\Delta_1 - \Delta_1^*}{2} \right\|_F^2 + \| \Delta_2 \|_F^2 \right] : \Delta = [\Delta_1 \ \Delta_2], \Delta_1 \subseteq \mathbb{C}^{n,n}, \Delta_2 \subseteq \mathbb{C}^{m,m}, \Delta_1 + \Delta_1^* \preceq 0, \Delta x = y, \Delta^* \Delta = w \right\}.$$ 

Proof. The proof is similar to the proof of [14, Lemma 5.6] due to Type-2 doubly structured dissipative mapping from Theorem 17.

Theorem 27. Let $L(z)$ be a pencil as in (2), let $\lambda \in i\mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $u = [u_1^T \ u_2^T \ u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$, and $u_3 \in \mathbb{C}^m$. Set $\tilde{y} = (J - R + \lambda E)u_2$ and $w_1 = -(J + R + \lambda E)u_1$. Then $\eta^S(J, R, B, \lambda, u)$ is finite if and only if $u_3 = 0$. If the later condition holds and if $u_2$ satisfies that $R u_2 \neq 0$ and $u_2 = \alpha u_1$ for some nonzero $\alpha \in \mathbb{C}$, then

$$\sqrt{\| H_1 \|_F^2 + \| H_2 \|_F^2} \leq \eta^S(R, E, B, \lambda, u) \leq \sqrt{\| H_1 + H_1^* \|_F^2 + \| H_2 \|_F^2},$$

when $|\lambda| \leq 1$, and

$$\sqrt{\frac{1}{|\lambda|^2} \| H_1 \|_F^2 + \| H_2 \|_F^2} \leq \eta^S(R, E, B, \lambda, u) \leq \sqrt{\| H_1 + H_1^* \|_F^2 + \| H_2 \|_F^2 + \| H_2 \|_F^2},$$

when $|\lambda| > 1$.
when $|\lambda| > 1$, where

$$H_1 = \tilde{y}u_2^\dagger + (w_1u_1^\dagger)^*P_{u_2} + P_{u_2}JF_{u_2}$$

with $J = \frac{1}{4\text{Re}(u_2^*\tilde{y})}(\tilde{y} + \frac{\alpha}{|\alpha|^2}w_1)(\tilde{y} + \frac{\alpha}{|\alpha|^2}w_1)^*$,

where and $H_2 = u_1u_1^\dagger B$.

Proof. In view of Remark 4 and Lemma 14, the proof is similar to the proof of [14, Theorem 5.7].

Appendix B.9. Perturbing only $J, R$ and $B$

Finally, suppose that the blocks $J, R$ and $B$ of $L(z)$ are subject to perturbation. Then in view of (104), (106) and (108), the corresponding backward errors are denoted by

$$\eta_B(J, R, B, \lambda, u) := \eta_B(J, R, 0, B, \lambda, u), \quad \eta_S(J, R, B, \lambda, u) := \eta_S(J, R, 0, B, \lambda, u).$$

Again note that the block and symmetry structured backward errors $\eta_B(J, R, B, \lambda, u)$ and $\eta_S(J, R, B, \lambda, u)$ were respectively obtained in [14, Theorem 5.3] and [14, Theorem 5.4]. Thus, in this section, we focus only on computing the semidefinite structured backward error $\eta_S(J, R, B, \lambda, u)$.

From (109) and (110), when $\Delta E = 0$ we have

$$\begin{bmatrix} \Delta J - \Delta R & \Delta B \\ \Delta J - \Delta R & \Delta B \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} (J - R + \lambda E)u_2 + Bu_3 \\ (J + R + \lambda E)u_1 =: w_1 \end{bmatrix} =: y$$

$$\begin{bmatrix} -\Delta J - \Delta R & \Delta B \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} -(J + R + \lambda E)u_1 =: w_1 \\ B^*u_1 + Su_3 =: w_2 \end{bmatrix} =: z.$$

Then a direct use of Type-2 doubly structured dissipative mapping from Theorem 17 in (B.36) and (B.37), gives the following lemma.

Lemma 15. Let $L(z)$ be a pencil as in (2), and let $\lambda \in i\mathbb{R}$ and $u \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition

$u = [u_1^T \ u_2^T \ u_3^T]^T$ such that $u_1, u_2 \in \mathbb{C}^n$ and $u_3 \in \mathbb{C}^m$, and let $x, y, z$ and $w$ be defined as in (B.36) and (B.37). Then the following statements are equivalent.

1. There exists $\Delta J, \Delta R \in \mathbb{C}^{n,n}$ and $\Delta B \in \mathbb{C}^{n,m}$ such that $\Delta J \in \text{SHerm}(n), \Delta R \succeq 0$ satisfying (B.36) and (B.37).

2. There exists $\Delta = [\Delta_1 \ \Delta_2], \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}$ such that $\Delta_1 + \Delta_1^* \preceq 0, \Delta x = y$, and $\Delta^*z = w$.

3. $u_3 = 0.$
Moreover, we have

\[ \inf \left\{ \left\| \begin{bmatrix} \Delta_J & \Delta_R & \Delta_B \end{bmatrix} \right\|^2_F : \Delta_J \in \text{SHerm}(n), \Delta_R \in \text{Herm}(n), \Delta_B \in \mathbb{C}^{n,m}, \right. \]

satisfying \((B.36)\) and \((B.37)\)

\[ = \inf \left\{ \left\| \Delta_1 \right\|^2_F + \left\| \Delta_2 \right\|^2_F : \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}, \Delta_1 \in \mathbb{C}^{n,n}, \Delta_2 \in \mathbb{C}^{n,m}, \Delta_1 + \Delta_1^* \preceq 0, \Delta x = y, \Delta^* z = w \right\}. \]

**Theorem 28.** Let \( L(z) \) be a pencil as in \((2)\), let \( \lambda \in \mathbb{iR} \setminus \{0\} \) and \( u \in \mathbb{C}^{2n+m} \setminus \{0\} \). Partition \( u = [u_1^T \ u_2^T \ u_3^T]^T \) such that \( u_1, u_2 \in \mathbb{C}^n \), and \( u_3 \in \mathbb{C}^m \). Set \( \tilde{y} = (J - R + \lambda E)u_2 \) and \( w_1 = -(J + R + \lambda E)u_1 \). Then \( \eta^{S_i}(J, R, B, \lambda, u) \) is finite if and only if \( u_3 = 0 \). If the later condition holds and if \( u_2 \) satisfies that \( Ru_2 \neq 0 \) and \( u_2 = \alpha u_1 \) for some nonzero \( \alpha \in \mathbb{C} \), then

\[ \eta^{S_i}(J, R, B, \lambda, u) = \sqrt{\left\| H_1 \right\|^2_F + \left\| H_2 \right\|^2_F}, \tag{B.38} \]

where

\[ H_1 = \tilde{y}u_2^* + (w_1 u_1^*)^* \mathcal{P}_{u_2} + \mathcal{P}_{u_2} J \mathcal{P}_{u_2} \text{ with } J = \frac{1}{4 \text{Re} \left( u_2^* \tilde{y} \right)} \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right) \left( \tilde{y} + \frac{\alpha}{|\alpha|^2} w_1 \right)^*, \tag{B.39} \]

and \( H_2 = u_1 u_1^* B \).

**Proof.** In view of Remark 4 and Lemma 15, the proof is analogous to the proof of Theorem 18 using Type-1 doubly structured dissipative mapping from Theorem 16. \( \Box \)