The Curse of Dimensionality for Numerical Integration of Smooth Functions

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Abstract

We prove the curse of dimensionality for multivariate integration of $C^r$ functions: The number of needed function values to achieve an error $\varepsilon$ is larger than $c_r(1 + \gamma)^d$ for $\varepsilon \leq \varepsilon_0$, where $c_r, \gamma > 0$. The proofs are based on volume estimates for $r = 1$ together with smoothing by convolution. This allows us to obtain smooth fooling functions for $r > 1$.

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1 Introduction

We study multivariate integration for different classes $F_d$ of smooth functions $f: \mathbb{R}^d \to \mathbb{R}$. Our emphasis is on large values of $d \in \mathbb{N}$. We want to approximate

$$S_d(f) = \int_{D_d} f(x) \, dx \quad \text{for } f \in F_d$$

up to some error $\varepsilon > 0$, where $D_d \subset \mathbb{R}^d$ has Lebesgue measure 1. The results in this paper hold for arbitrary sets $D_d$, the standard example of course is $D_d = [0, 1]^d$.

We consider (deterministic) algorithms that use only function values, and classes $F_d$ of functions bounded in absolute value by 1 and containing all constant functions $f(x) \equiv c$ with $|c| \leq 1$. An algorithm that uses no function value at all must be a constant, $A_0(f) \equiv b$, and its error is at least

$$\max_{f \in F_d} |S_d(f)| = 1.$$ 

We call this the initial error of the problem, it does not depend on $d$. Hence multivariate integration is well scaled and that is why we consider $\varepsilon < 1$.

Let $n(\varepsilon, F_d)$ denote the minimal number of function values needed for this task in the worst case setting. By the curse of dimensionality we mean that $n(\varepsilon, F_d)$ is exponentially large in $d$. That is, there are positive numbers $c$, $\varepsilon_0$ and $\gamma$ such that

$$n(\varepsilon, F_d) \geq c (1 + \gamma)^d \quad \text{for all } \varepsilon \leq \varepsilon_0 \quad \text{and infinitely many } d \in \mathbb{N}.$$  

\footnote{We add that $n(\varepsilon, F_d)$ is the information complexity of multivariate integration over $F_d$ and is proportional to the (total) complexity as long as $F_d$ is convex and symmetric. The last two assumptions are needed to guarantee that a linear algorithm is optimal and its implementation cost is linear in $n(\varepsilon, F_d)$.}
For many natural classes $F_d$ the bound in (2) will hold for all $d \in \mathbb{N}$. This applies in particular to the classes considered in this paper.

There are many classes $F_d$ for which the curse of dimensionality has been proved, see [5, 7] for such examples. However, it has not been known if the curse of dimensionality occurs for probably the most natural class which is the unit ball of $r$ times continuously differentiable functions,

$$
C^r_d = \{ f \in C^r(\mathbb{R}^d) \mid \| D^\beta f \| \leq 1 \text{ for all } |\beta| \leq r \},
$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, with non-negative integers $\beta_j$, $|\beta| = \sum_{j=1}^{d} \beta_j$, and $D^\beta$ denotes the operator of $\beta_j$ times differentiation with respect to the $j$th variable for $j = 1, 2, \ldots, d$. By $\| \cdot \|$ we mean the sup norm, $\| D^\beta f \| = \sup_{x \in \mathbb{R}^d} |(D^\beta f)(x)|$.

For $r = 0$, we obviously have $n(\varepsilon, C^0_d) = \infty$ for all $\varepsilon < 1$ and all $d \in \mathbb{N}$. Therefore from now on we always assume that $r \geq 1$. For $r = 1$, the curse of dimensionality for $C^1_d$ follows from the results of Sukharev [8]. Whether the curse holds for $r \geq 2$ has been an open problem for many years.

The class $C^r_d$ for $D_d = [0,1]^d$ (and functions and norms restricted to $D_d$) was already studied in 1959 by Bakhvalov [2], see also [4]. He proved that there are two positive numbers $a_{d,r}$ and $A_{d,r}$ such that

$$
a_{d,r} \varepsilon^{-d/r} \leq n(\varepsilon, C^0_d) \leq A_{d,r} \varepsilon^{-d/r} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1). \tag{3}
$$

This means that for a fixed $d$ and for $\varepsilon$ tending to zero, we know that $n(\varepsilon, C^0_d)$ is of order $\varepsilon^{-d/r}$ and the exponent of $\varepsilon^{-1}$ grows linearly in $d$. Unfortunately, Bakhvalov’s result does not allow us to conclude whether the curse of dimensionality holds for the class $C^r_d$. In fact, if we reverse the roles of $d$ and $\varepsilon$, and consider a fixed $\varepsilon$ and $d$ tending to infinity, the bound (3) on $n(\varepsilon, C^r_d)$ is useless. We prove the following result and hereby solve Open Problem 1 from [5]:

**Main Theorem.** The curse of dimensionality holds for the classes $C^r_d$ with the super-exponential lower bound

$$
n(\varepsilon, C^r_d) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1),
$$

where $c_r \in (0, 1]$ depends only on $r$.

We also prove that the curse of dimensionality holds for even smaller classes of functions $F_d$ for which the norms of arbitrary directional derivatives are bounded proportionally to $1/\sqrt{d}$.

We now discuss how we obtain lower bounds on $n(\varepsilon, F_d)$ for numerical integration defined on convex and symmetric classes $F_d$. The standard proof technique is to find a fooling
function \( f \in F_d \) that vanishes at the points \( \mathcal{P} = \{ x_1, x_2, \ldots, x_n \} \) at which we sample functions from \( F_d \), and the integral of \( f \) is as large as possible. All algorithms that use function values at \( x_j \)'s must give the same approximation of the integral for \( f \) and \(-f\). Thus, each such algorithm makes an error of at least \( |S_d(f) - S_d(-f)|/2 = |S_d(f)| \) for one of the functions. That is why the integral of \( f \) is a lower bound on the worst case error of all algorithms using function values at \( x_j \)'s. If, for all choices of \( x_1, x_2, \ldots, x_n \), there are functions \( f \in F_d \) vanishing at \( x_j \)'s with integrals larger than \( \varepsilon \) then \( n(\varepsilon, F_d) \geq n \).

We start with the fooling function
\[
f_0(x) = \min \left\{ 1, \frac{1}{\delta \sqrt{d}} \text{dist}(x, \mathcal{P}_\delta) \right\} \quad \text{for all} \quad x \in \mathbb{R}^d,
\]
where
\[
\mathcal{P}_\delta = \bigcup_{i=1}^{n} B^d_\delta(x_i)
\]
and \( B^d_\delta(x_i) \) is the ball with center \( x_i \) and radius \( \delta \sqrt{d} \). The function \( f_0 \) is Lipschitz. By a suitable smoothing via convolution we construct a fooling function \( f_r \in C^r_d \) with \( f_r|_\mathcal{P} = 0 \).

## 2 Preliminaries

In this section, we precisely define our problem. Let \( F_d \) be a class of continuous functions \( f : \mathbb{R}^d \to \mathbb{R} \) such that \( S_d(f) \), see (1), exists for every \( f \in F_d \). We approximate the integral \( S_d(f), f \in F_d \), by algorithms
\[
A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \ldots, f(x_n)),
\]
where \( x_j \in \mathbb{R}^d \) can be chosen adaptively and \( \phi_{n,d} : \mathbb{R}^n \to \mathbb{R} \) is an arbitrary mapping. Adaption means that the selection of \( x_j \) may depend on the already computed values \( f(x_1), f(x_2), \ldots, f(x_{j-1}) \). The (worst case) error of the algorithm \( A_{n,d} \) is defined as
\[
e(A_{n,d}) = \sup_{f \in F_d} |S_d(f) - A_{n,d}(f)|.
\]
The minimal number of function values to guarantee that the error is at most \( \varepsilon \) is defined as
\[
n(\varepsilon, F_d) = \min\{ n \in \mathbb{N} \mid \exists A_{n,d} \text{ such that } e(A_{n,d}) \leq \varepsilon \}.
\]
Hence we minimize \( n \) over all choices of adaptive sample points \( x_j \) and mappings \( \phi_{n,d} \). It is well known that, as long as the class \( F_d \) is convex and symmetric, we may restrict the
minimization of \( n \) by considering only nonadaptive choices of \( x_j \) and linear mappings \( \phi_{n,d} \). Furthermore,

\[
    n(\varepsilon, F_d) = \min \left\{ n \in \mathbb{N} \mid \inf_{P \subset \mathbb{R}^d, \#P = n} \sup_{f \in F_d, f|_{P=0}} |S_d(f)| \leq \varepsilon \right\},
\]

see [4, Prop. 1.2.6] or [9, Theorem 5.5.1]. In this paper, we always consider convex and symmetric \( F_d \) so that we can use the last formula for \( n(\varepsilon, F_d) \). For more details see, e.g., Chapter 4 in [5].

As already mentioned, our lower bounds are based on a volume estimate of a neighborhood of certain sets in \( \mathbb{R}^d \), see also [3]. In the following, we denote by \( A_\delta \) the \((\delta \sqrt{d})\)-neighborhood of \( A \subset \mathbb{R}^d \), which is defined by

\[
    A_\delta = \{ x \in \mathbb{R}^d \mid \text{dist}(x, A) \leq \delta \sqrt{d} \},
\]

where \( \text{dist}(x, A) = \inf_{a \in A} \| x - a \|_2 \) denotes the Euclidean distance of \( x \) from \( A \).

Furthermore, we denote by \( B^d_\delta(x) \) the \( d \)-dimensional ball with center \( x \in \mathbb{R}^d \) and radius \( \delta \sqrt{d} \), i.e.,

\[
    B^d_\delta(x) = \{ y \in \mathbb{R}^d \mid \| x - y \|_2 \leq \delta \sqrt{d} \}.
\]

We will need some standard volume estimates for Euclidean balls. Recall that the volume of a Euclidean ball of radius 1 is given by

\[
    V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.
\]

From Stirling’s formula for the \( \Gamma \) function, we have

\[
    \Gamma(x + 1) = \sqrt{2\pi} x^x e^{-x + \frac{\theta_x}{12x}} \quad \text{for all} \quad x > 0,
\]

where \( \theta_x \in (0, 1) \), see [11, p. 257]. This leads to the estimate

\[
    \Gamma(x + 1) > 2\pi e \left( \frac{x}{e} \right)^x \quad \text{for all} \quad x > 0.
\]

Combining this estimate with the volume formula for the ball, we obtain for all \( d \in \mathbb{N} \),

\[
    \lambda_d(B^d_\delta(x)) = (\delta \sqrt{d})^d V_d < (\delta \sqrt{d})^d \left( \frac{2\pi e}{d} \right)^{d/2} \frac{d}{\sqrt{\pi d}} = \left( \frac{\delta \sqrt{2\pi e}}{\sqrt{d}} \right)^d < \left( \delta \sqrt{2\pi e} \right)^d,
\]

where \( \lambda_d \) is the Lebesgue measure. The volume formula for the Euclidean unit ball also shows the recurrence relation

\[
    \frac{V_{d-1}}{V_d} = \frac{d}{d-1} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all} \quad d \geq 4.
\]
This easily implies
\[ \frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} < \frac{2}{\sqrt{d - 2}} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all} \quad d \geq 4. \]

The last inequality can be used in an inductive argument leading to
\[ \frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} \leq 1 \quad \text{for all} \quad d \geq 2. \]  

(7)

This will be needed later.

3 Convolution

In this section, we fix \( k \in \mathbb{N} \) and study the convolution
\[ f_k := f * g_1 * \ldots * g_k \]

of a function \( f \) defined on \( \mathbb{R}^d \) with (normalized) indicator functions \( g_j \). We are interested in properties of \( f_k \) in terms of the properties of the initial function \( f \). Recall that the convolution of two functions \( f \) and \( g \) on \( \mathbb{R}^d \) is defined by
\[ (f * g)(x) = \int_{\mathbb{R}^d} f(x - t) g(t) \, dt \quad \text{for all} \quad x \in \mathbb{R}^d. \]

Fix a number \( \delta > 0 \) and a sequence \((\alpha_j)_{j=1}^k\) with \( \alpha_j > 0 \) such that
\[ \sum_{j=1}^k \alpha_j \leq 1. \]

For example, we may take \( \alpha_j = 1/k \) for \( j = 1, 2 \ldots, k \). For \( j = 1, \ldots, k \), we define the ball
\[ B_j = \left\{ x \in \mathbb{R}^d \mid \|x\|_2 \leq \alpha_j \delta \sqrt{d} \right\} \]

and the function \( g_j : \mathbb{R}^d \to \mathbb{R} \) by
\[ g_j(x) = \frac{1_{B_j}(x)}{\lambda_d(B_j)} = \frac{1}{\lambda_d(B_j)} \begin{cases} 1 & \text{if } x \in B_j, \\ 0 & \text{otherwise.} \end{cases} \]  

(8)

Thus, the convolution of a function \( f \) with \( g_j \) can be written as
\[ (f * g_j)(x) = \frac{1}{\lambda_d(B_j)} \int_{B_j} f(x + t) \, dt \quad \text{for all} \quad x \in \mathbb{R}^d. \]
We will frequently use the following probabilistic interpretation. Let $Y_j$ be a random variable that is uniformly distributed on $B_j$. Then the convolution of $f$ with $g_j$ can be written as the expected value

$$(f * g_j)(x) = \mathbb{E}[f(x + Y_j)].$$

The next theorem is the basis for the induction steps of the proofs of our main results. For $f: \mathbb{R}^d \to \mathbb{R}$, we use the Lipschitz constant

$$\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_2}.$$ 

Define

$$C^r = \{f: \mathbb{R}^d \to \mathbb{R} \mid D^{\theta_\ell} \ldots D^{\theta_1} f \text{ is continuous for all } \ell \leq r \text{ and all } \theta_1, \ldots, \theta_r \in S^{d-1}\},$$

where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $D^{\theta_\ell} f(x) = \lim_{h \to 0} \frac{1}{h} (f(x + h\theta_\ell) - f(x))$ is the derivative in the direction of $\theta_\ell$.

**Theorem 1.** For $k \in \mathbb{N}$ and $f \in C^r$, define

$$f_k = f * g_1 * \ldots * g_k$$

with $g_j$ from (8). For $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ and let $\Omega_\delta$ be its neighborhood defined as in (5). Then

1. if $f(x) = 0$ for all $x \in \Omega_\delta$ then $f_k(x) = 0$ for all $x \in \Omega$,
2. $\text{Lip}(f_k) \leq \text{Lip}(f)$,
3. if $\int_{\Omega_\delta} f(x + t) \, dx \geq \varepsilon$ for all $t \in \mathbb{R}^d$ with $\|t\|_2 \leq \delta \sqrt{d}$ then $\int_{\Omega_\delta} f_k(x) \, dx \geq \varepsilon$,
4. for all $\ell \leq r$ and all $\theta_1, \theta_2, \ldots, \theta_{\ell} \in S^{d-1}$,
   $$\text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f_k) \leq \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f),$$
5. $f_k \in C^{r+k}$, and for all $\ell \leq r$, all $j = 1, \ldots, k$ and all $\theta_1, \theta_2, \ldots, \theta_{\ell+j} \in S^{d-1}$,
   $$\text{Lip}(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \ldots D^{\theta_1} f_k) \leq \left(\prod_{i=1}^{j} \frac{1}{\delta \alpha_i}\right) \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f).$$

The parts (i)–(iv) of this theorem show that some properties of the initial function $f$ are preserved by convolutions. Part (v) states that we gain one “degree of smoothness” with every convolution, losing only a multiplicative constant for its Lipschitz constant.
Proof. First note that we can write $f_k$ as

$$f_k(x) = \mathbb{E}[f(x + Y)], \quad \text{for all } x \in \mathbb{R}^d,$$

where $Y$ is a random variable with probability density function $g_1 \ast \ldots \ast g_k$. By construction of $g_i$’s which are the indicator functions of the balls whose sum of the radii is at most $\delta \sqrt{d}$, we have

$$\{t \in \mathbb{R}^d \mid g_1 \ast \ldots \ast g_k(t) > 0\} \subset \{t \in \mathbb{R}^d \mid \|t\|_2 \leq \delta \sqrt{d}\},$$

which implies that $x + Y \in \Omega_\delta$ almost surely for every $x \in \Omega$. Thus, $f(x) = 0$ for all $x \in \Omega_\delta$ implies that $f_k(x) = 0$ for all $x \in \Omega$, which is property (i).

Property (ii) is proven by

$$|f_k(x) - f_k(y)| = |\mathbb{E}[f(x + Y) - f(y + Y)]| \leq \mathbb{E}[|f(x + Y) - f(y + Y)|]$$

$$\leq \text{Lip}(f) \mathbb{E}[\|x + Y - (y + Y)\|_2] = \text{Lip}(f) \|x - y\|_2.$$

To prove (iii), we use Fubini’s theorem and we obtain

$$\int_{\mathbb{R}^d} f_k(x) \, dx = \int_{\mathbb{R}^d} \mathbb{E}[f(x + Y)] \, dx = \mathbb{E} \left[ \int_{\mathbb{R}^d} f(x + Y) \, dx \right] \geq \varepsilon$$

by assumption.

For the proof of properties (iv) and (v), let $\theta = (\theta_1, \ldots, \theta_\ell) \in (\mathbb{S}^{d-1})^\ell$. We write $D^\theta$ for $D^{\theta_1} \ldots D^{\theta_\ell}$. Clearly, $f \in C^r$ and $\ell \leq r$ implies that $D^\theta f \in C^{r-\ell} \subseteq C$. Since $f_k$ is at least as smooth as $f$, both $D^\theta f$ and $D^\theta f_k$ are well defined.

We need the well-known fact that $D^\theta(f \ast g) = (D^\theta f) \ast g$ if $f \in C^\ell$ and $g$ has compact support. For $g = g_1 \ast \ldots \ast g_k$, we have

$$|D^\theta f_k(x) - D^\theta f_k(y)| = |((D^\theta f) \ast g)(x) - ((D^\theta f) \ast g)(y)|$$

$$= \left| \int_{\mathbb{R}^d} [(D^\theta f(x + t) - D^\theta f(y + t)] \, g(t) \, dt \right|$$

$$\leq \text{Lip}(D^\theta f) \|x - y\|_2 \int_{\mathbb{R}^d} g(t) \, dt$$

$$= \text{Lip}(D^\theta f) \|x - y\|_2$$

for all $x, y \in \mathbb{R}^d$. The last equality follows since the $g_k$ is normalized. This proves (iv).

For (v), we need to prove that $f_k \in C^{r+k}$ with $f_0 = f \in C^r$ by assumption, and then it is enough to show that for all $m \leq r + k$ and all $\theta = (\theta_m, \ldots, \theta_1) \in (\mathbb{S}^{d-1})^m$,

$$\text{Lip}(D^\theta f_k) \leq \frac{1}{\delta \alpha_k} \text{Lip}(D^\theta f_{k-1}),$$

8
where \( \bar{\theta} = (\theta_{m-1}, \ldots, \theta_1) \in (\mathbb{S}^{d-1})^{m-1} \).

Assume inductively that \( f_{k-1} \in C^{m-1} \), which holds for \( k = 1 \). This implies \( D^{\bar{\theta}}(f_{k-1} \ast g_k) = (D^{\bar{\theta}} f_{k-1}) \ast g_k \), and

\[
D^{\bar{\theta}} f_k(x) = D^{\theta_m}((D^{\bar{\theta}} f_{k-1}) \ast g_k)(x) = D^{\theta_m} \left( \frac{1}{\lambda_d(B_k)} \int_{\mathbb{R}^d} D^{\bar{\theta}} f_{k-1}(x + t) 1_{B_k}(t) \, dt \right) = \frac{1}{\lambda_d(B_k)} D^{\theta_m} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} D^{\bar{\theta}} f_{k-1}(x + s + h \theta_m) 1_{B_k}(s + h \theta_m) \, dh \, ds \right) = \frac{1}{\lambda_d(B_k)} \int_{\theta^\perp_m} D^{\theta_m} \left( \int_{\mathbb{R}} D^{\bar{\theta}} f_{k-1}(x + s + h \theta_m) 1_{B_k}(s + h \theta_m) \, dh \right) \, ds,
\]

where \( \theta^\perp_m \) is the hyperplane orthogonal to \( \theta_m \). For any function \( f \) on \( \mathbb{R} \) of the form

\[ f(x) = \int_{x-a}^{x+a} g(y) \, dy \]

with some continuous function \( g \) we have

\[ f'(x) = g(x + a) - g(x - a). \]

Therefore, we obtain

\[
D^{\bar{\theta}} f_k(x) = \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta^\perp_m} \left[ D^{\bar{\theta}} f_{k-1}(x + s + h_{\max}(s) \theta_m) - D^{\bar{\theta}} f_{k-1}(x + s - h_{\max}(s) \theta_m) \right] \, ds
\]

with

\[ h_{\max}(s) = \max\{ h \geq 0 \mid s + h \theta_m \in B_k \}. \]

For each \( s \in B_k \cap \theta^\perp_m \), define the points \( s_1 = s + h_{\max}(s) \theta_m \in B_k \) and
\(s_2 = s - h_{\max}(s) \theta_m \in B_k\). Then

\[
|D^\theta f_k(x) - D^\theta f_k(y)| \leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m} \left[ \left| D^\theta f_{k-1}(x + s_1) - D^\theta f_{k-1}(x + s_2) \right| - D^\theta f_{k-1}(y + s_1) + D^\theta f_{k-1}(y + s_2) \right] ds
\]

\[
\leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m} \left[ \left| D^\theta f_{k-1}(x + s_1) - D^\theta f_{k-1}(y + s_1) \right| + \left| D^\theta f_{k-1}(x + s_2) - D^\theta f_{k-1}(y + s_2) \right| \right] ds
\]

\[
\leq \frac{2 \lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} \text{Lip}(D^\theta f_{k-1}) \|x - y\|_2.
\]

In particular, this shows the implication

\(f_{k-1} \in C^{m-1} \implies f_k \in C^m\)

for all \(k \in \mathbb{N}\). Taking \(m = r + k\) we have \(f_k \in C^{r+k}\), as claimed.

For \(m \leq r + k\), it remains to bound \(2\lambda_{d-1}(B_k \cap \theta_m^\perp)/\lambda_d(B_k)\). Recall that \(B_k\) is a ball with radius \(\delta\alpha_k \sqrt{d}\) and that \(V_d\) is the volume of the Euclidean unit ball in \(\mathbb{R}^d\). We obtain from (7) that

\[
\frac{2 \lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} = \frac{2(\delta\alpha_k \sqrt{d})^{d-1}}{(\delta\alpha_k \sqrt{d})^d} \frac{V_{d-1}}{V_d} = \frac{2}{\delta\alpha_k \sqrt{d}} \frac{V_{d-1}}{V_d} \leq \frac{1}{\delta\alpha_k}.
\]

This concludes the proof that

\[
\text{Lip}\left(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \ldots D^{\theta_1} f_k\right) \leq \left(\prod_{i=1}^j \frac{1}{\delta\alpha_{k+1-i}}\right) \text{Lip}\left(D^{\theta_{\ell}} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f\right),
\]

but since the order of convolution is arbitrary, we obtain in the same way

\[
\text{Lip}\left(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \ldots D^{\theta_1} f_k\right) \leq \left(\prod_{i \in J} \frac{1}{\delta\alpha_i}\right) \text{Lip}\left(D^{\theta_{\ell}} D^{\theta_{\ell-1}} \ldots D^{\theta_1} f\right)
\]

for all \(J \subset \{1, \ldots, k\}\) with \(#J = j\). In particular, this implies (v). \(\square\)
4 Main Results

Let $\mathcal{P} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ be a collection of $n$ points. As pointed out in the introduction, we want to construct functions that vanish at $\mathcal{P}$ and have a large integral. For this, we choose

$$f_0(x) = \min \left\{1, \frac{1}{\delta \sqrt{d}} \text{dist}(x, \mathcal{P}_\delta)\right\} \quad \text{for all } x \in \mathbb{R}^d,$$

where

$$\mathcal{P}_\delta = \bigcup_{i=1}^n B^d_\delta(x_i)$$

and $B^d_\delta(x_i)$ is the ball with center $x_i$ and radius $\delta \sqrt{d}$.

The function $\text{dist}(\cdot, \mathcal{P}_\delta)$ is Lipschitz with constant 1. Hence, for $\delta \leq 1$,

$$\text{Lip}(f_0) = \frac{1}{\delta \sqrt{d}}. \quad (9)$$

Additionally, $f_0(x) = 0$ for all $x \in \mathcal{P}_\delta$ by definition.

Using these facts we can apply Theorem 1 to prove the curse of dimensionality for the following class of functions that are defined on $\mathbb{R}^d$. For a fixed $r \in \mathbb{N}$, we now take $\alpha_1 = \cdots = \alpha_r = \frac{1}{r}$ and define

$$F_{d,r,\delta} = \{f : \mathbb{R}^d \to \mathbb{R} \mid f \in C^r \text{ satisfies } (10)-(12)\},$$

where

$$\|f\| \leq 1, \quad (10)$$

$$\text{Lip}(f) \leq \frac{1}{\delta \sqrt{d}}, \quad (11)$$

$$\forall k \leq r : \max_{\theta_1, \ldots, \theta_k \in S^{d-1}} \text{Lip}(D^{\theta_1} \cdots D^{\theta_k} f) \leq \frac{1}{\delta \sqrt{d}} \left(\frac{r}{\delta}\right)^k. \quad (12)$$

**Theorem 2.** For any $r \in \mathbb{N}$ and $\delta \in (0, 1]$,

$$n(\varepsilon, F_{d,r,\delta}) \geq (1 - \varepsilon) \begin{cases} 1 & \text{for } d = 1, \\ \left(\delta \sqrt{18\varepsilon \pi}\right)^{-d} & \text{for } d \geq 2, \end{cases} \quad \text{for all } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $F_{d,r,\delta}$ for $\delta < 1/\sqrt{18\varepsilon \pi}$. 
This result shows that the growth rate of \( n(\varepsilon, F_{d,r,\delta}) \) in \( d \) can be arbitrarily large if we choose \( \delta \) small enough.

**Proof.** Since the initial error for the classes \( F_{d,r,\delta} \) is 1, we obtain \( n(\varepsilon, F_{d,r,\delta}) \geq 1 \) for all \( \varepsilon \in (0,1) \). This proves the statement for \( d = 1 \).

For \( d \geq 2 \), we use Theorem 1 with \( k = r, \Omega = \mathcal{P} \) and \( f_r(x) = f_0 * g_1 * \ldots * g_r(x) \). Here, the \( g_j \)'s are as in Theorem 1. Recall that we have chosen \( \alpha_1 = \ldots = \alpha_r = 1/r \) and \( \alpha_j = 0 \) for \( j > r \). The properties of the initial function \( f_0 \) and Theorem 1 immediately imply that \( f_r \) satisfies (10)–(12). It remains to bound its integral. Note that \( f_0(x) = 1 \) for all \( x \notin \mathcal{P}_{3\delta} \). Clearly, \( f_r(x) \geq 0 \) for all \( x \in \mathbb{R}^d \). Since \( f_r(x) \) depends only on the values \( f_0(x + t) \) for \( t \in \mathbb{R}^d \) with \( \|t\|_2 \leq \delta \sqrt{d} \), it follows that \( f_r(x) = 1 \) for \( x \notin \mathcal{P}_{3\delta} \). We thus obtain

\[
\int_{D_d} f_r(x) \, dx \geq \int_{D_d \setminus \mathcal{P}_{3\delta}} f_r(x) \, dx = 1 - \lambda_d(\mathcal{P}_{3\delta} \cap D_d) \\
\geq 1 - \lambda_d(\mathcal{P}_{3\delta}) \geq 1 - n\lambda_d(B_{3\delta}^d) \\
\geq 1 - \frac{n(3\delta \sqrt{2e\pi})^d}{\sqrt{\pi d}} \\
> 1 - n\left(3\delta \sqrt{2e\pi}\right)^d,
\]

where the next to last inequality follows from the bound in (6). Hence \( \int_{D_d} f_r(x) \, dx \leq \varepsilon \) implies that

\[
n \geq (1 - \varepsilon) (\delta \sqrt{18e\pi})^{-d}.
\]

Since this holds for arbitrary \( \mathcal{P} \), the result follows.

By Theorem 2 we know how the parameter \( \delta \) comes into play. For \( p > 0 \), let

\[
\delta = \frac{1}{\sqrt{18e\pi}} d^{-p/(r+1)}.
\]

For this \( \delta \), we obtain a somehow stronger form of the curse of dimensionality for the class

\[
\widetilde{F}_{d,r,p} = \{ f : \mathbb{R}^d \to \mathbb{R} \mid f \in C^r \text{ satisfies (13)–(15)} \},
\]

where

\[
\|f\| \leq 1, \quad \text{(13)}
\]

\[
\text{Lip}(f) \leq d^{-\frac{1}{2} + \frac{p}{r+1}} \sqrt{18e\pi}, \quad \text{(14)}
\]

\[
\forall k \leq r : \max_{\theta_1, \ldots, \theta_k \in \mathbb{S}^{d-1}} \text{Lip}(D^{\theta_1} \ldots D^{\theta_k} f) \leq d^{-\frac{1}{2} + \frac{p(k+1)}{r+1}} r^k (\sqrt{18e\pi})^{k+1}. \quad \text{(15)}
\]
Theorem 3. For any \( r \in \mathbb{N} \) and \( p > 0 \),
\[
n(\varepsilon, \tilde{F}_{d,r,p}) \geq (1 - \varepsilon) d^{p d/(r+1)} \quad \text{for all} \quad d \in \mathbb{N} \quad \text{and} \quad \varepsilon \in (0,1).
\]
Hence the curse of dimensionality holds for the class \( \tilde{F}_{d,r,p} \).

Note that the classes \( \tilde{F}_{d,r,p} \) are contained in the classes
\[
\mathcal{C}_d^r = \{ f \in C^r \mid \|D^\beta f\| \leq 1 \quad \text{for all} \quad |\beta| \leq r \},
\]
if \( p < 1/2 \) and \( d \) is large enough. This holds if
\[
d \geq \left( r^r (18e\pi)^{(r+1)/2} \right)^{1/(1/2-p)}.
\]
(16)
From this we easily obtain the main result already stated in the introduction.

Main Theorem. For any \( r \in \mathbb{N} \), there exists a constant \( c_r \in (0,1] \) such that
\[
n(\varepsilon, \mathcal{C}_d^r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all} \quad d \in \mathbb{N} \quad \text{and} \quad \varepsilon \in (0,1).
\]
Hence the curse of dimensionality holds for the class \( \mathcal{C}_d^r \).

Proof. The case \( d = 1 \) is trivial since the initial error for the classes \( \mathcal{C}_d^r \) is again 1.

For \( d \geq 2 \), we know from Theorem 3 and the discussion thereafter that \( n(\varepsilon, \mathcal{C}_d^r) \geq (1 - \varepsilon) d^{p d/(r+1)} \) for all \( p < 1/2 \) if \( d \geq d_0 \), where \( d_0 = d_0(r, p) \) is the right hand side of (16). This implies
\[
n(\varepsilon, \mathcal{C}_d^r) \geq \tilde{c}_{r,p} (1 - \varepsilon) d^{p d/(r+1)} \quad \text{for all} \quad d \geq 2.
\]
with
\[
\tilde{c}_{r,p} = d_0^{-p d_0/(r+1)},
\]
which depends only on \( r \) and \( p \). The choice \( p^* = (r + 1)/(2r + 3) \) yields the result with \( c_r = \tilde{c}_{r,p^*} \).

Note that \( c_r \) in the last theorem is super-exponentially small in \( r \).

Remark 1. The reader might find it more natural to define classes of functions \( F_{d,r}(D_d) \) that are defined only on \( D_d \subset \mathbb{R}^d \). Not all such functions can be extended to smooth functions on \( \mathbb{R}^d \), and even if they can be extended then the norm of the extended function could be much larger. Our lower bound results for functions defined on \( \mathbb{R}^d \) can be also applied for functions defined on \( D_d \subset \mathbb{R}^d \) and this makes them even stronger.
Remark 2. Note that the possibility of super-exponential lower bounds on the complexity depends on the definition of the Lipschitz constant. For the class

$$F_d = \left\{ f : [0,1]^d \rightarrow \mathbb{R} \mid \sup_{x,y \in [0,1]^d} \frac{|f(x) - f(y)|}{\|x - y\|_\infty} \leq 1 \right\},$$

Sukharev [8] proved that the product mid-point rule is optimal with error $e_n = \frac{d}{2d+2} n^{-1/d}$ for $n = m^d$. Hence, roughly, $n(\varepsilon, F_d) \approx 2^{-d\varepsilon^{-d}}$ and the complexity is “only” exponential in $d$ for $\varepsilon < 1/2$.

Remark 3. We mention two results for the very small class

$$F_d = C^\infty_d = \{ f \in C^\infty([0,1]^d) \mid \|D^\beta f\| \leq 1 \text{ for all } \beta \in \mathbb{N}^d \}.$$ 

O. Wojtaszczyk [10] proved that $\lim_{d \to \infty} n(\varepsilon, F_d) = \infty$ for every $\varepsilon < 1$, hence the problem is not strongly polynomially tractable. It is still open whether the curse of dimensionality holds for this class $F_d$. The same class $F_d$ was studied for the approximation problem in [6]. For this problem the curse of dimensionality is present even if we allow algorithms that use arbitrary linear functionals.

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