On the number of edges in a graph with no \((k+1)\)-connected subgraphs

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Abstract

Mader proved that for \(k \geq 1\) and \(n \geq 2k\), every \(n\)-vertex graph with no \((k+1)\)-connected subgraphs has at most \((1+\sqrt{k^2}) (n-k)\) edges. He also conjectured that for \(n\) large with respect to \(k\), every such graph has at most \(\frac{3}{2} (k - \frac{1}{2}) (n-k)\) edges. Yuster improved Mader’s upper bound to \(\frac{19}{12k} k(n-k)\) for \(n \geq \frac{9k}{4}\). In this note, we make the next step towards Mader’s Conjecture: we improve Yuster’s bound to \(\frac{19}{12k} k(n-k)\) for \(n \geq \frac{5k}{2}\).

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1 Introduction

All graphs considered here are finite, undirected, and simple. For a graph \(G\), \(V(G)\) and \(E(G)\) denote its vertex set and edge set respectively. If \(U \subseteq V(G)\), then \(G[U]\) denotes the induced subgraph of \(G\) whose vertex set is \(U\), and \(G - U := G[V(G) \setminus U]\). For \(v \in V(G)\), \(N(v) := \{u \in V(G) : uv \in E(G)\}\) denotes the neighborhood of \(v\) in \(G\).

Let \(k \in \mathbb{N}\). Recall that a graph \(G\) is \((k+1)\)-connected if, for every set \(S \subseteq V(G)\) of size \(k\), the graph \(G[V(G) \setminus S]\) is connected and contains at least two vertices (so \(|V(G)| \geq k + 2\). Mader [1] posed the following question:

What is the maximum possible number of edges in an \(n\)-vertex graph that does not contain a \((k+1)\)-connected subgraph?

It is easy to see that for \(k = 1\) the answer is \(n - 1\): every tree on \(n\) vertices contains \(n - 1\) edges and no 2-connected subgraphs, whereas every graph on \(n\) vertices with at least \(n\) edges contains a cycle, and cycles are 2-connected. Thus for the rest of the note we will assume \(k \geq 2\).

The following construction due to Mader [2] gives an example of a graph with no \((k+1)\)-connected subgraphs and a large number of edges. Fix \(k\) and \(n\), and suppose that \(n = kq + r\), where \(1 \leq r \leq k\). The graph \(G_{n,k}\) has vertex set \(\bigcup_{i=0}^{q} V_i\), where the sets \(V_0, \ldots, V_q\) are pairwise disjoint and satisfy the following contitions.

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1. \(|V_0| = \ldots = |V_{q-1}| = k\), while \(|V_q| = r\).
2. \(V_0\) is an independent set in \(G_{n,k}\).
3. For \(1 \leq i \leq q\), \(V_i\) is a clique in \(G_{n,k}\).
4. Every vertex in \(V_0\) is adjacent to every vertex in \(\bigcup_{i=1}^{q} V_i\).
5. \(G_{n,k}\) has no other edges.

Note that \(V_0\) is a separating set of size \(k\) and every component of \(G_{n,k} - V_0\) has at most \(k\) vertices. It follows that \(G_{n,k}\) has no \((k+1)\)-connected subgraphs. A direct calculation shows that \(G_{n,k}\) has at most \(\frac{3}{2} \left(k - \frac{1}{3}\right)(n-k)\) edges, where the equality holds if \(n\) is a multiple of \(k\). Mader [2] conjectured that this example is, in fact, best possible.

**Conjecture 1** (Mader [2]). Let \(k \geq 2\). Then for \(n\) sufficiently large, the number of edges in an \(n\)-vertex graph without a \((k+1)\)-connected subgraph cannot exceed \(\frac{3}{2} \left(k - \frac{1}{3}\right)(n-k)\).

Mader himself proved Conjecture 1 for \(k \leq 6\). Moreover, he showed that for all \(k\), the weaker version of the conjecture, where the coefficient \(\frac{3}{2}\) is replaced by \(1 + \frac{1}{\sqrt{2}}\), holds. Yuster [4] improved this result by showing that the coefficient can be taken to be \(\frac{103}{120}\).

**Theorem 2** (Yuster [4]). Let \(k \geq 2\) and \(n \geq \frac{2k}{\sqrt{2}}\). Then every \(n\)-vertex graph \(G\) with \(|E(G)| > \frac{103}{120}k(n-k)\) contains a \((k+1)\)-connected subgraph.

Here we improve Yuster’s bound, obtaining the value \(\frac{19}{12}\) for the coefficient.

It turns out that for this problem, computations work out nicer if we “normalize” vertex and edge counts by assigning a weight \(\frac{1}{k}\) to each vertex and a weight \(\frac{1}{k^2}\) to each edge in a graph. Using this terminology, we can restate Conjecture 1 in the following way.

**Conjecture 1**. Let \(k \geq 2\). Then for \(\gamma\) sufficiently large, every graph \(G\) with \(\frac{1}{k}|V(G)| = \gamma\) and \(\frac{1}{k^2}|E(G)| > \frac{1}{2}(\gamma - 1)\) contains a \((k+1)\)-connected subgraph.

Our main result in these terms is as follows.

**Theorem 3**. Let \(k \geq 2\). Then every graph \(G\) with \(\frac{1}{k}|V(G)| = \gamma \geq \frac{5}{2}\) and \(\frac{1}{k^2}|E(G)| > \frac{10}{12}(\gamma - 1)\) contains a \((k+1)\)-connected subgraph.

We follow the ideas of Mader and Yuster: Use induction on the number of vertices for graphs with at least \(\frac{5}{2}k\) vertices. The hardest part is to prove the case when after deleting a separating set of size \(k\), exactly one of the components of the remaining graph has fewer than \(\frac{3}{2}k\) vertices, since the induction assumption does not hold for \(n < \frac{5}{2}k\). New ideas in the proof are in Lemmas 8 and 9 below.

### 2 Proof of Theorem 3

We want to derive a linear in \((n-k)\) bound on the number of edges in a graph that does not contain \((k+1)\)-connected subgraphs. But the bound becomes linear only for graphs with large number of vertices; while for small graphs the dependency is quadratic in \(n - k\). The main difficulties we encounter are around the transition between the quadratic and linear regimes. To deal with small \(n\), we use the following lemma due to Matula [3], whose bound is asymptotically exact for \(n < 2k\).
Lemma 4 (Matula [3]). Let \( k \geq 2 \). Then every graph \( G \) with \( |V(G)| = n \geq k + 1 \) and \( |E(G)| > \binom{n}{2} - \frac{1}{3}(n-k)^2 - 1 \) contains a \((k+1)\)-connected subgraph.

We will use the following “normalized” version of this lemma.

Lemma 4. Let \( k \geq 2 \). Then every graph \( G \) with \( \frac{1}{k}|V(G)| = \gamma > 1 \), and

\[
\frac{1}{k^2} |E(G)| > \frac{1}{6} (\gamma^2 + 4\gamma - 2)
\]

contains a \((k+1)\)-connected subgraph.

Proof. Indeed, (1) yields

\[
|E(G)| > \frac{k^2}{6} (\gamma^2 + 4\gamma - 2)
\]

\[
= \left( \frac{\gamma k}{2} \right)^2 - \frac{1}{3} (\gamma k - k)^2 - 1 + \frac{\gamma k}{2} - \frac{1}{3}
\]

\[
> \left( \frac{\gamma k}{2} \right)^2 - \frac{1}{3} (\gamma k - k)^2 - 1,
\]

and we are done by original Matula’s lemma.

From now on, fix a graph \( G \) with \( \frac{1}{k}|V(G)| = \gamma \geq \frac{5}{2} \) and \( \frac{1}{k^2} |E(G)| > \frac{19}{12} (\gamma - 1) \), and suppose for contradiction that \( G \) does not contain a \((k+1)\)-connected subgraph. Choose \( G \) to have the least possible number of vertices (so we can apply induction hypothesis for subgraphs of \( G \)). Since \( G \) is not \((k+1)\)-connected, it contains a separating set \( S \subset V(G) \) of size \( k \). Let \( A \subset V(G) \setminus S \) be such that \( G[A] \) is a smallest connected component of \( G - S \), and let \( B := V(G) \setminus (S \cup A) \). Let \( \alpha := \frac{1}{k}|A| \) and \( \beta := \frac{1}{k}|B| \).

We start by showing that the graph \( G \) cannot be too small, using Matula’s Lemma.

Lemma 5. \( \gamma > 3 \).

Proof. Suppose that \( \gamma \leq 3 \). Then, by Lemma 4,

\[
0 \leq \frac{1}{k^2} |E(G)| - \frac{19}{12} (\gamma - 1) \leq \frac{1}{6} (\gamma^2 + 4\gamma - 2) - \frac{19}{12} (\gamma - 1) = \frac{1}{12} (2\gamma^2 - 11\gamma + 15).
\]

The function \( g(\gamma) = 2\gamma^2 - 11\gamma + 15 \) on the right-hand side of (2) is convex in \( \gamma \). Hence it is maximized on the boundary of the interval \([\frac{5}{2}; 3]\). But it is easy to check that \( g(\frac{5}{2}) = g(3) = 0 \), hence it is nonpositive on the whole interval. Therefore, \( \gamma > 3 \).

All the edges in \( G \) either belong to the graph \( G[S \cup B] \), or are incident to the vertices in \( A \). The number of edges in \( G[S \cup B] \) can be bounded either using Matula’s lemma (which is efficient for \( \beta \leq \frac{3}{2} \)) or using the induction hypothesis (which can be applied if \( \beta > \frac{3}{2} \)). Hence the difficulty is in bounding the number of edges incident to the vertices in \( A \).

The first step is to show that \( A \) cannot be too large, because otherwise we can use induction.

Lemma 6. \( \alpha < \frac{3}{2} \).
Proof. If \( \alpha \geq \frac{3}{2} \), then we can apply the induction hypothesis both for \( G[S \cup A] \) and for \( G[S \cup B] \), and thus obtain
\[
\frac{1}{k^2} E(G) \leq \frac{19}{12} \alpha + \frac{19}{12} \beta = \frac{19}{12} (\alpha + \beta) = \frac{19}{12} (\gamma - 1).
\]

The next lemma shows that \( A \) cannot be too small either, since otherwise the total number of edges between the vertices in \( A \) and the vertices in \( S \cup A \) is small.

**Lemma 7.** \( \alpha > 1 \).

**Proof.** Suppose that \( \alpha \leq 1 \). Then \( \beta > 1 \), since \( \alpha + \beta + 1 = \gamma > 3 \). If \( \beta \geq \frac{3}{2} \), then using the induction hypothesis for \( G[S \cup B] \), we get
\[
\frac{1}{k^2} E(G) \leq \frac{1}{2} \alpha^2 + \frac{19}{12} \beta \leq \frac{3}{2} \alpha + \frac{19}{12} \beta < \frac{19}{12} (\alpha + \beta) = \frac{19}{12} (\gamma - 1).
\]

Thus \( \beta < \frac{3}{2} \). Therefore, \( \alpha > \frac{1}{2} \). In this case, applying Lemma 4 to \( G[S \cup B] \) reduces the problem to proving the inequality
\[
\frac{1}{2} \alpha^2 + \alpha + \frac{1}{6} \left( (\beta + 1)^2 + 4(\beta + 1) - 2 \right) \leq \frac{19}{12} (\alpha + \beta),
\]
which is equivalent to
\[
6 \alpha^2 + 2 \beta^2 - 7 \alpha - 7 \beta + 6 \leq 0. \tag{3}
\]

For \( \alpha \) fixed, the left-hand side of (3) is monotone decreasing in \( \beta \) when \( \beta < \frac{7}{4} \), so its maximum is attained at \( \beta = 1 \). Thus (3) will hold if the function \( g_1(\alpha) = 6 \alpha^2 - 7 \alpha + 1 \) is nonpositive. Since \( g_1(\alpha) \) is a convex function, its maximum on the interval \([\frac{1}{2}; 1]\) is attained at one of the boundary points. We have
\[
g_1 \left( \frac{1}{2} \right) = 6 \cdot \left( \frac{1}{2} \right)^2 - 7 \cdot \frac{1}{2} + 1 = -1 < 0, \quad \text{and} \quad g_1(1) = 6 \cdot 1^2 - 7 \cdot 1 + 1 = 0. \quad \blacksquare
\]

So we know that \( 1 < \alpha < \frac{3}{2} \). How can we bound the number of edges incident to the vertices in \( A \)? The ideas of Lemma 7 and of Lemma 4 are not sufficient here. The solution is to combine them by applying Lemma 4 only to the graph \( G[A \cup (S \setminus S')] \), where \( S' \) is a subset of \( S \) with relatively few edges between \( A \) and \( S' \). To obtain such set \( S' \), we will use Lemma 8 below, which asserts that there are many vertices in \( S \) that have not too many neighbors in \( A \).

**Lemma 8.** Let \( S_1 := \{ v \in S : \frac{1}{k} |N(v) \cap A| \leq \frac{1}{2} (\alpha + 1) \} \). Then \( \frac{1}{k} |S_1| > \frac{1}{3} \).

**Proof.** Suppose that \( \frac{1}{k} |S_1| = \sigma \leq \frac{1}{3} \). Let \( G_1 := G[A \cup (S \setminus S_1)] \). Since \( G_1 \) is not \((k + 1)\)-connected, it has a separating set \( T \subset V(G_1) \) of size \( k \). Let \( X \) and \( Y \) form a partition of \( V(G_1) \setminus T \) and be separated by \( T \) in \( G_1 \). Without loss of generality assume that \( |X \cap A| \geq |Y \cap A| \). Then
\[
\frac{1}{k} |X \cap A| \geq \frac{1}{2} \cdot \frac{1}{k} |A \setminus T| \geq \frac{1}{2} (\alpha - 1).
\]

Hence if \( v \in Y \cap S \), then
\[
\frac{1}{k} |N(v) \cap A| \leq \frac{1}{k} (|A| - |X \cap A|) \leq \frac{1}{2} (\alpha + 1),
\]
which means that \( v \in S_1 \). But that is impossible, since \( S_1 \cap V(G_1) = \emptyset \). Thus \( Y \cap S = \emptyset \), i.e. \( Y \subseteq A \). In particular, since \( |X \cap A| \geq |Y \cap A| = |Y| \), we have \( \frac{1}{k}|Y| \leq \frac{1}{2} \alpha \). Then
\[
\frac{1}{k}|V(G) \setminus Y| = \alpha + \beta + 1 - \frac{1}{k}|Y| \geq \frac{1}{2} \alpha + \beta + 1 \geq \frac{5}{2},
\]
so the induction hypothesis holds for \( G - Y \), and
\[
\frac{1}{k^2}|E(G - Y)| \leq \frac{19}{12} \left( \frac{1}{k}|V(G - Y)| - 1 \right).
\]
Hence we are done if
\[
\frac{1}{k^2}(|E(G)| - |E(G - Y)|) \leq \frac{19}{12} \cdot \frac{1}{k}|Y|,
\]
so assume that that is not the case. Let \( \mu := \frac{1}{k}|Y| \). Then
\[
\frac{1}{2} \mu^2 + \mu(1 + \sigma) > \frac{19}{12} \mu,
\]
so
\[
\mu > \frac{7}{6} - 2\sigma.
\]

We consider two cases.

**Case 1:** \( X \cap S \neq \emptyset \). Let \( v \in X \cap S \). Then \( v \) has more than \( k \cdot \frac{1}{2}(\alpha + 1) \) neighbors in \( A \), none of which belong to \( Y \). Hence \( \mu < \frac{1}{2} (\alpha - 1) \), and so
\[
\frac{1}{2}(\alpha - 1) > \frac{7}{6} - 2\sigma.
\]
Therefore,
\[
\alpha > \frac{10}{3} - 4\sigma \geq \frac{10}{3} - 4 \cdot \frac{1}{3} = 2;
\]
a contradiction.

**Case 2:** \( X \cap S = \emptyset \). Then \( S \setminus S_1 \subseteq T \), and the set \( T \cap A \) separates \( X \) and \( Y \) in \( G[A] \) and satisfies \( \frac{1}{k}|T \cap A| = \frac{1}{k}(|T| - |T \cap S|) = 1 - (1 - \sigma) = \sigma \). Note that since \( |Y| \leq |X| \), we have
\[
\frac{7}{6} - 2\sigma < \mu \leq \frac{1}{2}(\alpha - \sigma),
\]
so
\[
\sigma > \frac{7}{9} - \frac{1}{3} \alpha > \frac{7}{9} - \frac{1}{3} \cdot \frac{3}{2} = \frac{5}{18}.
\]
Now observe that
\[
\frac{1}{k^2}|E(G[A])| \leq \frac{1}{2} \alpha^2 - \mu(\alpha - \sigma - \mu),
\]
Since \( \frac{7}{6} - 2\sigma < \mu \leq \frac{1}{2} (\alpha - \sigma) \), the latter expression is less than
\[
\frac{1}{2} \alpha^2 - \left( \frac{7}{6} - 2\sigma \right) \cdot \left( \alpha + \sigma - \frac{7}{6} \right).
\]
Hence \( \frac{1}{k^2}(|E(G)| - |E(G - A)|) \) is less than
\[
\frac{1}{2} (\alpha + 1) \sigma + \alpha (1 - \sigma) + \frac{1}{2} \alpha^2 - \left( \frac{7}{6} - 2\sigma \right) \cdot \left( \alpha + \sigma - \frac{7}{6} \right).
\]
Case 2.1. \( \beta \leq \frac{3}{2} \). Then, after adding Matula’s estimate for the number of edges in \( G[S \cup B] \) and subtracting \( \frac{19}{12}(\alpha + \beta) \), it is enough to prove that the following quantity is nonpositive:

\[
\frac{1}{2}(\alpha + 1)\sigma + \alpha(1 - \sigma) + \frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right)
+ \frac{1}{6}((\beta + 1)^2 + 4(\beta + 1) - 2) - \frac{19}{12}(\alpha + \beta),
\]

which in equal to

\[
\frac{1}{36}(18\alpha^2 + 54\alpha\sigma - 63\alpha + 6\beta^2 - 21\beta + 72\sigma^2 - 108\sigma + 67).
\]

Note that for \( \alpha \) and \( \sigma \) fixed, the last expression is monotone decreasing in \( \beta \) (recall that \( \beta \leq \frac{3}{2} \), while the minimum is attained at \( \beta = \frac{7}{4} \)), so its maximum is attained when \( \beta = \alpha \), where it turns into

\[
\varphi_1(\alpha, \sigma) = \frac{1}{36}(24\alpha^2 + 54\alpha\sigma - 84\alpha + 72\sigma^2 - 108\sigma + 67).
\]

Since \( \varphi_1(\alpha, \sigma) \) is convex in both \( \alpha \) and \( \sigma \), it attains its maximum at some point \( (\alpha_0, \sigma_0) \), where \( \alpha_0 \in \{1, \frac{3}{2}\} \) and \( \sigma_0 \in \{\frac{1}{18}, \frac{1}{2}\} \). It remains to check the four possibilities:

\[
\varphi_1 \left( 1, \frac{5}{18} \right) = -\frac{11}{162} < 0, \quad \varphi_1 \left( 1, \frac{1}{3} \right) = -\frac{1}{12} < 0,
\]

\[
\varphi_1 \left( \frac{3}{2}, \frac{5}{18} \right) = -\frac{125}{648} < 0, \quad \text{and} \quad \varphi_1 \left( \frac{3}{2}, \frac{1}{3} \right) = -\frac{1}{6} < 0.
\]

Case 2.2. \( \beta > \frac{3}{2} \). Then, instead of using Matula’s bound for \( G[S \cup B] \), we can apply the induction hypothesis, so it is enough to prove that the function

\[
\varphi_2(\alpha, \sigma) = \frac{1}{2}(\alpha + 1)\sigma + \alpha(1 - \sigma) + \frac{1}{2}\alpha^2 - \left(\frac{7}{6} - 2\sigma\right) \cdot \left(\alpha + \sigma - \frac{7}{6}\right) - \frac{19}{12}\alpha
\]

is nonpositive. Again, we only have to check the boundary values:

\[
\varphi_2 \left( 1, \frac{5}{18} \right) = -\frac{49}{324} < 0, \quad \varphi_2 \left( 1, \frac{1}{3} \right) = -\frac{1}{6} < 0,
\]

\[
\varphi_2 \left( 3, \frac{5}{18} \right) = -\frac{125}{648} < 0, \quad \text{and} \quad \varphi_2 \left( 3, \frac{1}{3} \right) = -\frac{1}{6} < 0.
\]

This finishes the proof.

Now we can simply try to use as the set \( S' \) the set \( S_1 \) itself. This choice indeed gives a good bound if \( A \) is large, as the next lemma shows.

Lemma 9. \( \alpha < \frac{4}{3} \).
Proof. Suppose that \( \alpha \geq \frac{4}{3} \). Recall that \( \sigma = |S_1| > \frac{1}{3} \). Using Lemma 4 for \( G[A \cup (S \setminus S_1)] \), we get that

\[
\frac{1}{k^2}(|E(G)| - |E(G - A)|) \\
\leq \frac{1}{6} \left( \alpha + 1 - \sigma \right)^2 + 4(\alpha + 1 - \sigma) - 2 + \frac{1}{2}(\alpha + 1)\sigma \\
= \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3).
\]

Case 1: \( \beta \leq \frac{3}{2} \). Adding Matula’s estimate for \( G[B \cup S] \) and subtracting \( \frac{19}{12}(\alpha + \beta) \), we get

\[
\frac{1}{k^2}|E(G)| - \frac{19}{12}(\alpha + \beta) \\
\leq \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3) + \frac{1}{6} \left( (\beta + 1)^2 + 4(\beta + 1) - 2 \right) - \frac{19}{12}(\alpha + \beta) \\
= \frac{1}{12}(2\alpha^2 + 2\alpha\sigma - 7\alpha + 2\sigma^2 - 7\beta + 2\sigma^2 - 6\sigma + 12).
\]

Again, the maximum is attained when \( \beta = \alpha \), so we should consider the expression

\[
\varphi_3(\alpha, \sigma) = \frac{1}{6}(2\alpha^2 + \alpha\sigma - 7\alpha + \sigma^2 - 3\sigma + 6).
\]

It is convex in both \( \alpha \) and \( \sigma \), so again it is enough to check the boundary points:

\[
\varphi_3\left(\frac{4}{3}, \frac{1}{3}\right) = -\frac{1}{27} < 0, \quad \varphi_3\left(\frac{4}{3}, 1\right) = -\frac{2}{27} < 0,
\]

\[
\varphi_3\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{7}{108} < 0, \quad \text{and} \quad \varphi_3\left(\frac{3}{2}, 1\right) = -\frac{1}{12} < 0.
\]

Case 2: \( \beta > \frac{3}{2} \). Then, instead of using Matula’s bound for \( G[S \cup B] \), we can apply the induction hypothesis, so it is enough to prove that the function

\[
\varphi_4(\alpha, \sigma) = \frac{1}{6}(\alpha^2 + \alpha\sigma + 6\alpha + \sigma^2 - 3\sigma + 3) - \frac{19}{12}\alpha = \frac{1}{12}(2\alpha^2 + 2\alpha\sigma - 7\alpha + 2\sigma^2 - 6\sigma + 6)
\]

is nonpositive. The function is convex in both \( \alpha \) and \( \sigma \), so we check the boundary points:

\[
\varphi_4\left(\frac{4}{3}, \frac{1}{3}\right) = -\frac{1}{18} < 0, \quad \varphi_4\left(\frac{4}{3}, 1\right) = -\frac{5}{54} < 0,
\]

\[
\varphi_4\left(\frac{3}{2}, \frac{1}{3}\right) = -\frac{7}{108} < 0, \quad \text{and} \quad \varphi_4\left(\frac{3}{2}, 1\right) = -\frac{1}{12} < 0.
\]

This finishes the proof.

The next lemma is the final piece of the jigsaw. It shows that if \( A \) is small, we can still obtain the desired bound if we take the set \( S' \) to be slightly bigger than \( S_1 \).

Lemma 10. \( \alpha > \frac{4}{3} \).
Proof. Suppose that $\alpha \leq \frac{4}{3}$. Then $1 - 2(\alpha - 1) \geq \frac{1}{2}$. Let $S'$ be a subset of $S$ with $\frac{1}{k}|S'| = 1 - 2(\alpha - 1)$ such that $\frac{1}{k}|S' \cap S_1| \geq \frac{1}{3}$. Observe that the normalized number of edges between $A$ and $S'$ is at most

$$\frac{1}{k^2} |A| \cdot |S'| - \frac{1}{2}(\alpha - 1) \cdot \frac{1}{3},$$

by the definition of $S_1$. Hence, using Lemma 4 for $G[A \cup (S \setminus S')]$, we get that

$$\frac{1}{k^2} (|E(G)| - |E(G - A)|) \leq \frac{1}{6} \left( (3\alpha - 2)^2 + 4(3\alpha - 2) - 2 \right) + \alpha(3 - 2\alpha) - \frac{1}{6}(\alpha - 1)$$

$$= \frac{1}{6}(-3\alpha^2 + 17\alpha - 5).$$

Case 1: $\beta \leq \frac{3}{2}$. Adding Matula’s estimate for $G[B \cup S]$ and subtracting $\frac{19}{12}(\alpha + \beta)$, we get

$$\frac{1}{k^2} |E(G)| - \frac{19}{12}(\alpha + \beta) \leq \frac{1}{6}(-3\alpha^2 + 17\alpha - 5) + \frac{1}{6}((\beta + 1)^2 + 4(\beta + 1) - 2) - \frac{19}{12}(\alpha + \beta)$$

$$= \frac{1}{12}(-6\alpha^2 + 15\alpha + 2\beta^2 - 7\beta - 4).$$

Since $\alpha \leq \beta \leq \frac{3}{2}$, the maximum is attained when $\beta = \alpha$, in which case the last expression turns into $-\frac{1}{3}(\alpha - 1)^2 \leq 0$.

Case 2: $\beta > \frac{3}{2}$. Then, instead of using Matula’s bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that

$$\frac{1}{6}(-3\alpha^2 + 17\alpha - 5) - \frac{19}{12}\alpha = -\frac{1}{12}(6\alpha^2 - 15\alpha + 10) \leq 0. \tag{4}$$

Since the discriminant of the quadratic $6\alpha^2 - 15\alpha + 10$ is negative, $(4)$ holds for all $\alpha$, and we are done.

Since Lemmas 9 and 10 contradict each other, we conclude that such graph $G$ does not exist. This completes the proof of the theorem.

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References

[1] W. Mader. Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großen Kantendichten, Abh. Math. Sem. Univ. Hamburg Volume 37, 1972. Pages 86–97.

[2] W. Mader. Connectivity and edge-connectivity in finite graphs. Surveys in Combinatorics, B. Bollobás, Ed., Cambridge University Press, London, 1979. Pages 66–95.

[3] D.W. Matula. Ramsey theory for graph connectivity. J. Graph Theory, Volume 7, 1983. Pages 95–105.

[4] R. Yuster. A note on graphs without $k$-connected subgraphs. Ars Combinatoria, Volume 67, 2003. Pages 231–235.