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Published in:
Astronomy & Astrophysics

DOI:
10.1051/0004-6361/201935012

Published: 24/04/2020

Please cite the original version:
Käpylä, P. J., Rheinhardt, M., Brandenburg, A., & Käpylä, M. J. (2020). Turbulent viscosity and magnetic Prandtl number from simulations of isotropically forced turbulence. Astronomy & Astrophysics, 636, Article 93. https://doi.org/10.1051/0004-6361/201935012
Turbulent viscosity and magnetic Prandtl number from simulations of isotonically forced turbulence

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Received 3 January 2019 / Accepted 25 January 2020

ABSTRACT

Context. Turbulent diffusion of large-scale flows and magnetic fields plays a major role in many astrophysical systems, such as stellar convection zones and accretion discs.

Aims. Our goal is to compute turbulent viscosity and magnetic diffusivity which are relevant for diffusing large-scale flows and magnetic fields, respectively. We also aim to compute their ratio, which is the turbulent magnetic Prandtl number, Pm, for isotropically forced homogeneous turbulence.

Methods. We used simulations of forced turbulence in fully periodic cubes composed of isothermal gas with an imposed large-scale sinusoidal shear flow. Turbulent viscosity was computed either from the resulting Reynolds stress or from the decay rate of the large-scale flow. Turbulent magnetic diffusivity was computed using the test-field method for a microphysical magnetic Prandtl number of unity. The scale dependence of the coefficients was studied by varying the wavenumber of the imposed sinusoidal shear and test fields.

Results. We find that turbulent viscosity and magnetic diffusivity are in general of the same order of magnitude. Furthermore, the turbulent viscosity depends on the fluid Reynolds number (Re) and scale separation ratio of turbulence. The scale dependence of the turbulent viscosity is found to be well approximated by a Lorentzian. These results are similar to those obtained earlier for the turbulent magnetic diffusivity. The results for the turbulent transport coefficients appear to converge at sufficiently high values of Re and the scale separation ratio. However, a weak trend is found even at the largest values of Re, suggesting that the turbulence is not in the fully developed regime. The turbulent magnetic Prandtl number converges to a value that is slightly below unity for large Re. For small Re we find values between 0.5 and 0.6 but the data are insufficient to draw conclusions regarding asymptotics. We demonstrate that our results are independent of the correlation time of the forcing function.

Conclusions. The turbulent magnetic diffusivity is, in general, consistently higher than the turbulent viscosity, which is in qualitative agreement with analytic theories. However, the actual value of Pm found from the simulations (≈0.9−0.95) at large Re and large scale separation ratio is higher than any of the analytic predictions (0.4−0.8).

Key words. turbulence – Sun: rotation – stars: rotation

1. Introduction

Turbulent transport is often invoked to explain phenomena in astrophysical systems such as accretion (e.g. Shakura & Sunyaev 1973; Frank et al. 2002), maintenance of stellar differential rotation (Rüdiger 1980, 1989; Rüdiger et al. 2013), and large-scale magnetic field generation (Moffatt 1978; Krause & Rädler 1980). Turbulence is typically thought to diffuse large-scale structures analogously to molecular diffusion but at a rate that is several orders of magnitude higher (e.g. Väisälä et al. 2014).

Turbulent diffusion coefficients, such as turbulent viscosity (νt) and magnetic diffusivity (ηm), are often estimated using arguments from the mixing length theory (MLT) according to which \( ν_t ≈ η_m ≈ ul/3 \), where \( u \) and \( l \) are the characteristic velocity and length scale of the turbulence. Such estimates yield values of the order of \( 10^5−10^6 \) m² s⁻¹ for the solar convection zone, which coincide with values estimated for the turbulent magnetic diffusivity \( η_l \) from sunspot decay in the quenched case (Krause & Rüdiger 1975; Petrovay & van Driel-Gesztesy 1997; Rüdiger & Kitchatinov 2000) and from cross helicity measurements in the unquenched (quiet Sun) case (Rüdiger et al. 2011). With the advent of the test-field method (Schrijver et al. 2005, 2007), it has become possible to measure turbulent transport coefficients that are relevant for the electromotive force (e.g., the turbulent magnetic diffusivity) from simulations. Detailed studies using this method indicate that the MLT estimate yields the correct order of magnitude in the kinematic regime (e.g. Sur et al. 2008; Käpylä et al. 2009a), provided that \( l \) is identified with the inverse of the wavenumber \( k_l \) of the energy-carrying eddies. This result can further be affected by other physical properties, such as the presence of kinetic helicity in the flow, which can reduce the value of \( η_l \) (Brandenburg et al. 2017). The test-field method also revealed an approximately Lorentzian dependence on the wavenumber of the mean field (Brandenburg et al. 2008a).

In the absence of a corresponding test-field method for hydrodynamics, the estimates of \( ν_t \) are typically much less...
accurate than those obtained for $\eta$ from such methods. Estimates of turbulent viscosity from shearing box simulations, however, also indicate a value of the order of the MLT estimate (e.g. Snellman et al. 2009; Käpylä et al. 2010). Computing $v_1$ from simulations with imposed linear shear flows is problematic due to hydrodynamical instabilities that can be excited (e.g. Elperin et al. 2003; Käpylä et al. 2009b). Furthermore, also non-diffusive contributions to the turbulent stress exist. First, the anisotropic kinetic alpha (AKA) effect can occur in the presence of Galilean non-invariant flows\(^1\) and can give rise to hydrodynamic instabilities analogous to the electromagnetic dynamo (e.g. Frisch et al. 1987; Brandenburg & von Rekowski 2001; Käpylä et al. 2018)\(^2\). Second, anisotropic turbulence with global rotation leads to a $\Lambda$ effect, which is relevant for causing differential rotation (e.g. Rüdiger 1989; Kitchatinov & Rüdiger 2005; Käpylä & Brandenburg 2008; Käpylä 2019a). Typically, these effects cannot easily be disentangled from the contribution of turbulent viscosity. Additionally, a spatially non-uniform kinetic helicity Yokoi & Brandenburg (2016) in rotating non-mirror symmetric flows leads to the generation of large-scale flows.

Contrary to the micropysical magnetic Prandtl number, which can vary over tens of orders of magnitude in the astrophysical context, depending on the physical characteristics of the system under study (e.g. Brandenburg & Subramanian 2005), the ratio of $v_1$ to $\eta$, that is the turbulent magnetic Prandtl number $Pm_t$, is thought to be of the order of unity in the astrophysically relevant regime of high Reynolds numbers. Nevertheless, astrophysical applications of the possibility of $Pm_t$ being different from unity have been discussed. These include both accretion disc turbulence and solar convection. In the context of accretion onto a magnetised star, one often assumes that the field lines of the star’s magnetic field are being dragged with the flow towards the star, so as to achieve a pitch angle suitable for jet launching (Blandford & Payne 1982). This requires the turbulent magnetic diffusivity to be small (Elstner & Rüdiger 2000), while subsequent work has shown that $Pm_t$ has only a weak influence on the pitch angle (Rüdiger & Shalybkov 2002).

Another application has been suggested in the context of the solar convection zone. For flux transport dynamos to explain the equatorward migration of the sunspot belts, one must assume the turbulent magnetic diffusivity to be of the order of \(10^7 m^2 s^{-1}\) (Chatterjee et al. 2004). On the other hand, to prevent the contours of constant angular velocity from being constant on cylinders, the turbulent viscosity must be around \(10^9 m^2 s^{-1}\), or even larger (Brandenburg et al. 1990). Thus, again, a turbulent magnetic Prandtl number in excess of unity is required for this model to be successful. A large turbulent viscosity is sometimes argued to be a consequence of the magnetic stress from small-scale dynamo action (Karak et al. 2018). Whether this idea has a solid foundation remains open, however.

The analytic estimates of the turbulent magnetic Prandtl number range between 0.4 up to the first-order smoothing approximation (FOSA) to 0.8 under various versions of the $\tau$ approximation (Yousef et al. 2003; Kitchatinov et al. 1994), of which the spectral minimal $\tau$ approximation (MTA) applied to fully developed turbulent convection yields values in the range 0.23–0.46 (Rogachevskii & Kleoerin 2006). Different renormalisation group analyses yield $Pm_t \approx 0.42–0.79$ (e.g. Fournier et al. 1982; Kleoerin & Rogachevskii 1994; Verma 2001; Jurčišinová et al. 2011). Furthermore, the turbulent magnetic Prandtl number has been studied from simulations of forced turbulence with a decaying large-scale field component by Yousef et al. (2003) who found that $Pm_t$ is approximately unity irrespective of the microphysical magnetic Prandtl and Reynolds numbers. However, their dataset is limited to a few representative cases that do not probe the Reynolds number or scale dependences systematically.

Our aim is to compute the turbulent viscosity and turbulent magnetic Prandtl number from direct simulations of homogeneous isotropically forced turbulence where we systematically vary the Reynolds number and scale separation ratio and compare the obtained results with analytic ones. To achieve this, we impose a large-scale shear flow with a harmonic profile on the (non-rotating) flow and determine the turbulent viscosity either from the generated Reynolds stresses or from the decay rate of the large-scale flow. For obtaining the turbulent magnetic diffusivity we employ the test-field method.

### 2. Model

#### 2.1. Basic equations

We model a compressible gas in a triply periodic cube with edge length $L$. It obeys an isothermal equation of state defined by

\[ p = \frac{\rho c_s^2}{\gamma}, \]

with pressure $p$, density $\rho$ and constant speed of sound $c_s$. Hence, we solve the continuity and Navier–Stokes equations with both an imposed random and large-scale shear forcing

\[ \frac{D\ln \rho}{Dt} = -\nabla \cdot \mathbf{U}, \]

\[ \frac{D\mathbf{U}}{Dt} = -c_s^2 \nabla \ln \rho + \frac{1}{\rho} \nabla \cdot (2\nu \mathbf{S}) + f - \frac{1}{\tau} \left( \mathbf{U}_g - \mathbf{U}^{(0)}_g \right) \mathbf{e}_y, \]

where $D/Dt = \partial/\partial t + \mathbf{U} \cdot \nabla$ is the advective time derivative, $\mathbf{U}$ is the velocity, $\nu$ is the constant kinematic viscosity, $\mathbf{S}_i = \frac{1}{2} (\mathbf{U}_{ij} + \mathbf{U}_{ji}) - \frac{1}{3} \delta_{ij} \mathbf{U} \cdot \mathbf{U}$ is the traceless rate of strain tensor, and the commas denote spatial derivatives. The forcing function $f$ is given by

\[ f = f_0 N(t) f_\mathbf{k}(t) \text{Re} \left\{ \exp \left[ i(\mathbf{k} \cdot \mathbf{x} + \phi(t)) \right] \right\}, \]

where $\mathbf{k}(t)$ is a random wavevector and

\[ f_\mathbf{k}(t) = (\mathbf{k} \times \mathbf{e}(t)) / \sqrt{k^2 - (\mathbf{k} \cdot \mathbf{e}(t))^2}, \]

is used to produce a nonhelical transversal sinusoidal $f$, where $\mathbf{e}(t)$ is an arbitrary random unit vector, not aligned with $\mathbf{k}$, and $\phi(t)$ is a random phase. $N(t) = c_s^{2/3} (\delta t)^{1/3}$ is a normalisation factor, $\mathbf{k} = |\mathbf{k}|$, $\delta t$ is the length of the integration time step and $f_0$ is a constant dimensionless scaling factor. The quantities $\mathbf{k}$, $\mathbf{e}$, and $\phi$ change at every time step, so that the external force is delta-correlated (white) in time. Numerically, we integrate the forcing term by using the Euler–Maruyama scheme (Higham 2001). We consider models where $\mathbf{k}$ is within a narrow shell of wavevectors with $k$ close to a chosen $k_c$, and determined such that the forcing always obeys the periodic boundary conditions.

The last term in Eq. (2) maintains a large-scale shear flow on top of the forced background turbulence via relaxing the horizontally $(xy)$ averaged part of the $y$ velocity, indicated by the overbar, towards the temporally constant profile $\mathbf{U}^{(0)}_g$; $\mathbf{e}_y$ is the unit vector in the $y$-direction. The relaxation time scale $\tau$ is chosen to match the turnover time $(\Delta t_{\text{rms}} k_c)^{-1}$ of the turbulence, where $\Delta t_{\text{rms}}$ is the rms value of the fluctuating velocity.
$u_{ms} = \langle (U - \overline{U})^2 \rangle^{1/2}$, with the average taken over the full volume as indicated by the angle brackets, and over the statistically steady part of the simulations, indicated by the subscript $t$. Our results are not sensitive to the relaxation time $\tau$ in the range $0.1 < \tau u_{ms} k_1 < 10$ so the (arbitrary) choice $\tau u_{ms} k_1 = 1$ is justified. We choose a simple harmonic form for the shear flow according to

$$\overline{U}_y = U_0 \cos(k_1 z),$$

where $U_0$ is the flow amplitude, and $k_U = k_1, 2k_1$,$\ldots$, $k_U^\text{max}$, $k_1 = 2\pi/L$.

### 2.2. Input and output quantities

We measure density in terms of its initially uniform value $\rho_0$, velocity in units of the sound speed $c_s$, and length in terms of $k_1^{-1}$. Furthermore, in the cases with the test-field method employed, we choose a system of electromagnetic units in which $\mu_0 = 1$, where $\mu_0$ is the permeability of vacuum. The simulations are fully defined by choosing the forcing amplitude $f_0$ and scale $k_U/k_1$, kinematic viscosity $v$, microscopic magnetic Prandtl number

$$Pm = \frac{v}{\eta},$$

where $\eta$ is the microscopic magnetic diffusivity in the test-field method, and the shear parameter

$$Sh_c = \frac{U_0 k_U}{c_s k_1}.$$  

We further assume that the scale of the test fields always equals that of the imposed large-scale flow, that is $k_0 = k_U$, and that the value of $Pm$ for the test-field simulations equals unity. For the scale separation ratio $\mathcal{S}$ we employ the definition

$$\mathcal{S} = k_0 / k_U.$$  

The following quantities are used as diagnostics of our models. We quantify the level of turbulence in the simulations by the fluid and magnetic Reynolds numbers

$$Re = \frac{u_{ms}}{v k_1}, \quad Rm = \frac{u_{ms}}{\eta k_1} = Pm \Re.$$  

The strength of the imposed shear is measured by the dynamic shear number

$$Sh = \frac{U_0 k_U}{\eta k_1}.$$  

Guided by MLT and FOSA, we normalise both the turbulent viscosity and magnetic diffusivity by

$$\nu_0 = \eta_0 = u_{ms}/3k_1,$$

while the turbulent magnetic Prandtl number is given by

$$Pm_t = \frac{\nu_t}{\eta_t}.$$  

### 3. Computation of $\nu_t$ and $\eta_t$

#### 3.1. Mean-field analysis

In what follows, we rely upon Reynolds averaging, specifically defining the mean quantities as averages over $x$ and $y$. Hence, they can only depend on $z$ and time. Averages are indicated by overbars and fluctuations by lowercase or primed quantities, thus $U = \overline{U} + u$, $\rho = \overline{\rho} + \rho'$ etc.

#### 3.1.1. Hydrodynamics

In the incompressible case all turbulent effects can besubsumed in the Reynolds stress tensor $Q_{ij} = \overline{uu_i u_j}$ whose divergence appears in the evolution equation of the mean flow. Including compressibility and starting from

$$\partial_t (\rho U_i) + \partial_j (\rho U_i U_j) = -\partial_i P + \cdots,$$

where the dots stand for viscous and external forces, one obtains after averaging

$$\partial_t (\rho u_i u_j + \overline{U}_i \overline{u}_j + \rho' u_i u_j + \rho' U_i \overline{U}_j)$$

$$= -\partial_i P + \cdots.$$  

The contributions proportional to the Reynolds stresses, $\overline{p u_i u_j}$, no longer cover all turbulent effects originating from the inertial terms. However, in our weakly compressible setups with $Ma \approx 0.1$ the difference between, for example $|\langle \rho u_i u_j \rangle | / \rho_0 u_{ms}^2$ and $\overline{\rho u_i u_j} / \rho_0 u_{ms}^2$ is $O(10^{-2})$. Thus, we will consider, as in the incompressible case, only the Reynolds stresses. When restricting to first order in the mean quantities, they can be decomposed into three contributions,

$$Q_{ij} = Q^{(0)}_{ij} + Q^{(ff)}_{ij} + Q^{(T)}_{ij}$$

where $Q_{ij}^{(0)}$ is already present in the absence of both a mean flow $\overline{U}$ and a gradient of $\ln \rho$, $Q_{ij}^{(ff)}$ is occurring due to the presence of $\nabla \ln \rho$, and $Q_{ij}^{(T)}$ occurs due to the presence of $\overline{U}$ (for a justification see Appendix A). As in our simulations no significant $\nabla \ln \rho$ occurs, we disregard $Q_{ij}^{(ff)}$. Further, as the fluctuations are isotropically forced, the only non-zero components of $Q_{ij}^{(T)}$ are $Q_{xx} = Q_{yy} = Q_{zz}$. Apart from small fluctuations, they do not depend on $z$ and thus do not act onto the mean flow. Note that due to the absence of a global rotation there is also no contribution of the A effect in $Q_{ij}$. In what follows we drop the superscript $(\overline{U})$ for brevity.

For sufficiently slowly varying mean flows and sufficient scale separation, $Q_{ij}$ can be approximately represented by the truncated Taylor expansion

$$Q_{ij} = A_{ijk} \overline{U}_k + N_{ij,k} \overline{U}_k,$$

with the symmetry requirements

$$A_{ijk} = A_{jik}, \quad N_{ij,k} = N_{ij,k}.$$  

Here, $A_{ijk}$ describes the AKA effect, while $N_{ij,k}$ comprises turbulent viscosity (amongst other effects). For isotropic (and hence

3 Neglecting density fluctuations may not be rigorously justified, given that the variety of potentially new effects owing to compressibility has not yet been fully explored, but see the recent studies by Rogachevskii et al. (2018) and Yokoi (2018) for the electromotive force. However, recent hydrodynamic results for the A effect suggest that the effect of compressibility is weak up to $Ma \approx 0.8$ (Käpylä 2019b).

4 Further turbulence effects result from the term $\mathbf{S} \cdot \nabla \ln \rho$, but are not considered here either because of our assumption of weak compressibility. We recall that $\mathbf{S}$ is the traceless rate of strain tensor used in Eq. (2).

5 Note that relation (16) is yielding the stresses without truncation when interpreted to be a representation of the Fourier-transformed kernel of a general convolution-like relationship between $Q_{ij}$ and $\overline{U}_j$ (cf. Brandenburg et al. 2008a).
homogeneous) fluctuations, that is, in the kinematic limit \( \mathbf{U} \to 0 \), \( A_{ijk} = 0 \), and \( N_{ijkl} \) must have the form

\[ N_{ijkl} = -\nu_\delta (\delta_i \delta_j + \delta_i \delta_k - \delta_j \delta_k) - \zeta_i \delta_j \delta_k, \]

where the constants \( \nu_\delta \) and \( \zeta_i \) are the turbulent shear and bulk viscosities, respectively. The Reynolds stresses appear then correspondingly as

\[ Q_{ij} = -\nu_\delta (\mathbf{U}_{ij} + \mathbf{U}_{ji}) - \sigma_i \delta_j \mathbf{\nabla} \cdot \mathbf{U}, \]

with the first term reproducing the Boussinesq ansatz. Although our turbulence is isotropically forced, the presence of finite shear causes it to be anisotropic with preferred directions given by the direction of the mean flow \( \mathbf{U} \) and, say, its curl, \( \mathbf{W} = \mathbf{\nabla} \times \mathbf{U} \). Given that it is the divergence of \( Q_{ij} \) which enters the mean momentum equation and mean quantities depend only on \( z \), merely the components \( A_{33k} \) and \( N_{333k} \) matter in (16). As \( \mathbf{U} \) needs not to be solenoidal, \( \mathbf{U} \) might in general depend on \( z \) and the turbulent bulk viscosity is then of interest.

Further simplification is obtained when assuming that the mean velocity has only one component. In our setup, the mean electromotive force can be represented by two rank–2 tensors with the first term reproducing the Boussinesq ansatz. Although our turbulence is isotropically forced, the presence of finite shear causes it to be anisotropic with preferred directions given by the direction of the mean flow \( \mathbf{U} \) and, say, its curl, \( \mathbf{W} = \mathbf{\nabla} \times \mathbf{U} \). Given that it is the divergence of \( Q_{ij} \) which enters the mean momentum equation and mean quantities depend only on \( z \), merely the components \( A_{33k} \) and \( N_{333k} \) matter in (16). As \( \mathbf{U} \) needs not to be solenoidal, \( \mathbf{U} \) might in general depend on \( z \) and the turbulent bulk viscosity is then of interest.

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are evolved with the flow taken from the simulation by as in Brandenburg et al. (2008b). The fluctuating magnetic fields a one out of a set of large-scale test fields. Neither the fluctuating coeﬃcients also anisotropic and nor the test fields can be unambiguously inverted from Eq. (30) by choosing an appropriate number of independent test fields.

We use four stationary test fields

\begin{align*}
\mathbf{B}^\text{lc} &= B_0(\cos k_B z, 0, 0), \\
\mathbf{B}^\text{ls} &= B_0(0, \cos k_B z, 0),
\end{align*}

where \( k_B \) is a wavenumber. As explained in Sect. 3.1.2, Eq. (30) simplifies here to Eq. (21) with \( \eta_1 = \beta_{23} \) and \( \eta_2 = -\beta_{13} \). Because of \( a_{11,22} = 0 \), we neglect the latter for weak imposed shear flows and simplify Eq. (21) further to

\[ \bar{\mathbf{E}}_z = -\eta_{ij} \bar{\mathbf{J}}_j. \]

We are interested in the diagonal components of \( \eta_{ij} \) which we represent in terms of the turbulent diﬀusivity by

\[ \eta = \frac{1}{2}(\eta_{11} + \eta_{22}). \]

In the case of homogeneous isotropic turbulence, the turbulent transport coeﬃcients are uniform across the system and volume averages are appropriate. In the present case, however, the turbulence can neither be considered fully isotropic nor homogeneous due to the imposed \( z \) dependent shear ﬂow, which makes the coeﬃcients also anisotropic and \( z \) dependent. Both effects are weak though in the computed \( \eta_{ij} \); see Sect. 4.2 for the eﬀect of anisotropy.

Exponential growth of the test solutions \( b^T \) at high \( \text{Rm} \) is a known issue in the test-ﬁeld method (Sur et al. 2008). To circumvent it, we reset the \( b^T \) periodically to zero with a resetting time that is roughly inversely proportional to the magnetic Reynolds number.

The error of the turbulent magnetic Prandtl number is computed from

\[ \delta \text{Pr}_m \approx \text{Pr}_m \left( \frac{\delta \nu}{\nu} + \frac{\delta \eta}{\eta} \right), \]

where \( \delta \nu \) and \( \delta \eta \) are the errors of turbulent viscosity and diﬀusivity, respectively.

### 4. Results

We perform several sets of simulations where we vary the forcing wavenumber \( k_{i1} \), determining the scale separation ratio, fluid and magnetic Reynolds numbers \( \text{Re} \) and \( \text{Rm} \), respectively, and the wavenumber of the large-scale flow \( k_U \). Representative examples of the flow patterns realised in runs with small, medium, and high Reynolds numbers (from left to right) and forcing wavenumbers \( k_1 = (3, 5, 10, 30) \) (from top to bottom, Sets D–G) are shown in Fig. 2. We also typically evolve the test-ﬁeld equations in our runs so the results pertaining to \( \nu_{ij} \) and \( \eta_{ij} \) are always obtained from the same simulation. All of our runs are listed in Tables 1–3.

In Fig. 3 we show representative results for \( a_3 \) and \( \nu_{ij} \) obtained with the methods M1 and M2a from Sets A–C (see Table 1).
Fig. 2. Normalised streamwise velocity component $\tilde{U}_y = U_y/c_s$ at the periphery of the computational domain for increasing scale separation ratio (from top to bottom), and increasing Reynolds number (left to right) for selected runs from D1 to G6.

with forcing wavenumbers 3, 5, and 10. The coefficient $a_2$, corresponding to the AKA-effect, is consistent with zero for all values of shear and with both methods that can detect it. This conclusion applies to all of our models and is consistent with the Galilean invariance of our forcing.

We note that, as $a_2$ from method M2a is always very small, it has a negligible effect on the quality of the fit and the value of $\nu_1$ in comparison to method M2b. For simplicity, we present results obtained using M2b in what follows. Overall, no statistically significant values were obtained for the coefficients $a_1$, $a_3$, $n_1$, and $n_3$.

4.1. Turbulent viscosity

4.1.1. Dependence on position and sensitivity to shear

The turbulent viscosities obtained for the two zeros employed in M1 (M1₁ and M1₂ in Fig. 3) agree within error estimates and agree also with those obtained from M2a and M2b. This suggests
The turbulence. However, if the shear is sup

Fig. 3. AKA–effect coefficient $a_2$ (top) and turbulent viscosity $\nu_t$ (bottom) as functions of $Sh$ for three scale separation ratios, $\mathcal{S} = 3$ (Set A, left), 5 (Set B, middle), and 10 (Set C, right). The colours refer to methods M1 (blue and red), and M2a (black). $M_1$ and $M_2$ refer to the two zeros employed in Eqs. (23) and (24).

### Table 1. Summary of the runs with varying shear.

| Run | Re | Sh | $Sh_{[10^{-5}]}$ | $k_t$ | $Ma$ | $\nu_t$ | $\eta_t$ | $\rho_n$ | Grid |
|-----|----|----|----------------|------|-----|--------|--------|--------|------|
| A1  | 21 | 0.015 | 160 | 3 | 0.105 | 1.509 | 1.899 | 0.795 | 72 |
| A2  | 21 | 0.030 | 319 | 3 | 0.106 | 1.566 | 1.873 | 0.836 | 72 |
| A3  | 21 | 0.060 | 638 | 3 | 0.106 | 1.575 | 1.912 | 0.824 | 72 |
| A4  | 22 | 0.146 | 1595 | 3 | 0.109 | 1.452 | 1.900 | 0.764 | 72 |
| A5  | 25 | 0.264 | 3191 | 3 | 0.121 | 1.496 | 2.098 | 0.713 | 72 |
| A6  | 35 | 0.369 | 6381 | 3 | 0.173 | 1.618 | 2.192 | 0.738 | 72 |
| B1  | 21 | 0.009 | 98 | 5 | 0.105 | 1.768 | 2.018 | 0.876 | 144 |
| B2  | 21 | 0.019 | 196 | 5 | 0.105 | 1.699 | 2.031 | 0.836 | 144 |
| B3  | 21 | 0.037 | 392 | 5 | 0.106 | 1.769 | 2.097 | 0.844 | 144 |
| B4  | 21 | 0.091 | 981 | 5 | 0.107 | 1.760 | 2.083 | 0.845 | 144 |
| B5  | 23 | 0.171 | 1962 | 5 | 0.115 | 1.818 | 2.361 | 0.770 | 144 |
| B6  | 33 | 0.235 | 3923 | 5 | 0.167 | 2.180 | 4.048 | 0.538 | 144 |
| C1  | 21 | 0.005 | 50 | 10 | 0.106 | 1.858 | 2.110 | 0.880 | 144 |
| C2  | 21 | 0.009 | 100 | 10 | 0.106 | 1.905 | 2.113 | 0.901 | 144 |
| C3  | 21 | 0.019 | 199 | 10 | 0.106 | 1.916 | 2.136 | 0.897 | 144 |
| C4  | 21 | 0.047 | 499 | 10 | 0.106 | 1.926 | 2.216 | 0.869 | 144 |
| C5  | 22 | 0.092 | 997 | 10 | 0.109 | 1.969 | 2.450 | 0.804 | 144 |
| C6  | 23 | 0.131 | 1496 | 10 | 0.115 | 2.063 | 2.811 | 0.734 | 144 |
| C7  | 29 | 0.136 | 1994 | 10 | 0.147 | 3.140 | 5.037 | 0.623 | 144 |

Notes. $Ma = \frac{u_{rms}/c_s}$ is the Mach number, $\nu_t = \nu_t/\nu_0$, and $\eta_t = \eta_t/\eta_0$. Furthermore, $\tilde{k}_t = k_t = 1$ in all runs yielding $\mathcal{S} = \mathcal{S}_t$. Here and in subsequent tables, $\nu_t$ is measured with method M2b and $\eta_t$ with the test-field method.

that $\nu_t$ has only a weak dependence on $z$ or that its spatial profile is such that it is not captured by this method.

When the amplitude of the shear flow is varied, the values of $\nu_t$ start to increase rapidly at the largest values of $Sh$; see Fig. 3. This is because the Navier–Stokes equations are inherently nonlinear. Therefore, imposing a large-scale flow has an impact on the turbulence. However, if the shear is sufficiently weak, such feedback is small and reliable results for $\nu_t$ can be obtained. To assess this question, we perform simulations at fixed kinematic viscosity and given forcing wavenumber $k_t$ while varying the shear systematically. With the other quantities held unchanged, the fluid Reynolds number is a measure of the rms–velocity of the turbulence. In Fig. 4, we show the Reynolds numbers realised in the same sets as in Fig. 3. We find that $Re$ increases mildly as a function of $Sh$ for weak shear ($Sh \lesssim 0.1$) and starts to increase sharply at higher values while the location of the transition depends weakly on the forcing wavenumber such that the larger the $k_t$, the smaller $Sh$ is needed for the increase to occur.

The increase of $Re$ is due to the fact that the turbulence becomes increasingly affected by the imposed shear and attains significant anisotropy. In some cases with the highest values of $Sh$, we also see large-scale vorticity generation, which is likely related to what is known as vorticity dynamo (e.g. Käpylä et al. 2009b). Such hydrodynamic instability can be excited by the off-diagonal components of the turbulent viscosity tensor in anisotropic turbulence in the presence of shear (Elperin et al. 2003, 2007).

These tests suggest that values of $Sh$ below 0.1 are needed for the influence of the shear on the turbulence to remain weak. However, the excitation condition of the vorticity dynamo manifestly depends on the scale separation ratio and likely also on the Reynolds number. In our runs, we choose a constant value of $Sh$ for which $Sh$ remains clearly below the excitation threshold (note that $Sh = Sh Ma$). Another factor supposedly contributing at large Reynolds numbers is shear-produced turbulence – possibly through some sort of finite amplitude instability. Given that the shear strengths (in terms of $Sh$) considered here are relatively small, this effect is likely to be weak in comparison to the turbulence production due to the applied forcing.

### 4.1.2. Dependence on $Re$

Results for the turbulent viscosity as a function of the fluid Reynolds number are shown in Fig. 5 for Sets D–G (see Table 2). Here the value of the shear parameter $Sh_c$ is constant in each set.
Additionally, the normalized relaxation time $\tau_{\text{norm}} = 1$ is kept fixed by adjusting $r$, and $k_f$ is varied between 3 (Set D) and 30 (Set G). Furthermore, these runs use $k_0 = k_B = 1$.

We find that for low Re and poor scale separation the signal is noisy and produces large errors in $v_1$ unless very long time series are produced. The runs with $k_f \approx 3$ and Re $\approx 1$ were in all sets typically run for several thousand turnover times whereas for larger Reynolds numbers and scale separations the integration times can be an order of magnitude shorter. The results in the low Reynolds number regime are in agreement with $v_1 \propto \text{Re}$ as expected from analytic studies using FOSA (Krause & Rüdiger 1974). The value of $v_1$ increases until Re $\approx 10$ after which it saturates roughly to a constant between 10–20%.

4.1.3. Results from M3

As a representative example, Fig. 6 shows the decay of the $k = k_1$ constituent of $\overline{U_y}$ for ten independent initial conditions derived from Run F8. We compare the results for $v_1$ from methods M2b and M3 in Fig. 7 for Sets F and Fd. The runs of the latter were set up such that $N = 10$ snapshots from each of the runs in Set F with imposed shear were used as initial conditions. Thus each run in Set F works as a progenitor to ten decay experiments with the same system parameters in Set Fd. We find that the results from methods M2b and M3 coincide within the error estimates for low and intermediate Reynolds numbers (Re $\leq 20$). However, there is a systematic tendency for the $v_1$ from the decay experiments to exceed the value from the Reynolds stress method for Re $\gtrsim 30$ by 10–20%.

4.1.4. Dependence on scale separation ratio

The dependence of $v_1$ on the scale separation ratio $\mathcal{S}$ for four different forcing scales is given in Table 3 (Sets H–K). We fit the data to a Lorentzian as a function of the inverse scale separation ratio $\mathcal{S}^{-1}$.
In various cases ranging from magnetic diffusion to passive scalar diffusion, the results obtained so far differ from the predictions of theoretical models. For instance, in Table 4, there is no obvious correlation approximation (SOCA), but the reason for this deviation is unclear. Also, looking at Table 4, there is no obvious correlation between the values of $\nu_t$ and $\sigma_t$, which is another fit parameter:

$$\nu_t(k_U/k_d) = \frac{\nu_t(0)}{1 + \sigma_t(k_U/k_d)^c} = \frac{\nu_t(0)}{1 + \sigma_t^{c/2}}.$$  

The inverse relative errors $\nu_t/\delta\nu_t$ are used as weights in the fit. The data and the fits are shown in Fig. 8. While the data is in reasonable agreement with a Lorentzian with $\nu_t(0) = 1.90$ and $\sigma_t = 1.80$, the more general function with $\nu_t(0) = 2.00$, $\sigma_t = 1.61$, and $c = 1.44$ yields a somewhat better fit. Data consistent with Lorentzian behaviour has been found earlier for $\eta_t$ in low Reynolds number turbulence (Brandenburg et al. 2008a); see Table 4 for an overview of the values found previously in various cases ranging from magnetic diffusion in isotropic turbulence to passive scalar diffusion in shear flows, in which $\sigma_t$ was typically below unity. However, a value of $c$ close to 4/3, as found here, is indeed expected for fully developed Kolmogorov turbulence; see the discussion in the conclusions of Madarassy & Brandenburg (2010).

Table 3. Summary of the runs with varying scale separation ratio $\mathcal{S} = k_t/k_U$.

| Run | Re | Sh | $\text{Sh}_{[10^{-1}]}$ | $k_t$ | $\Delta t$ | $\nu_t$ | $\eta_t$ | $Pm_t$ | Grid |
|-----|----|----|----------------|------|--------|-------|--------|-------|-----|
| H1  | 20 | 20 | 3.0 | 0.106 | 1.517 | 1.894 | 0.801 | 144 |
| H2  | 20 | 153 | 3.0 | 0.106 | 1.101 | 1.477 | 0.746 | 144 |
| H3  | 20 | 20 | 3.0 | 0.106 | 0.784 | 1.168 | 0.671 | 144 |
| I1  | 20 | 0.047 | 5.0 | 0.106 | 1.705 | 2.037 | 0.837 | 144 |
| I2  | 20 | 0.094 | 10.0 | 0.106 | 1.436 | 1.782 | 0.806 | 144 |
| I3  | 20 | 0.140 | 15.0 | 0.106 | 1.120 | 1.538 | 0.728 | 144 |
| I4  | 20 | 0.187 | 20.0 | 0.106 | 0.942 | 1.337 | 0.704 | 144 |
| J1  | 21 | 0.024 | 25.0 | 0.106 | 1.883 | 2.142 | 0.879 | 144 |
| J2  | 21 | 0.048 | 50.0 | 0.106 | 1.790 | 2.009 | 0.891 | 144 |
| J3  | 21 | 0.072 | 75.0 | 0.106 | 1.584 | 1.899 | 0.834 | 144 |
| J4  | 21 | 0.095 | 100.0 | 0.106 | 1.421 | 2.111 | 0.673 | 144 |
| J5  | 21 | 0.119 | 125.0 | 0.106 | 1.265 | 1.635 | 0.744 | 144 |
| J6  | 21 | 0.166 | 175.0 | 0.106 | 1.015 | 1.396 | 0.727 | 144 |
| K1  | 16 | 0.008 | 8.4 | 0.106 | 1.930 | 2.215 | 0.871 | 144 |
| K2  | 17 | 0.016 | 17.0 | 0.106 | 1.790 | 2.009 | 0.891 | 144 |
| K3  | 16 | 0.025 | 25.0 | 0.106 | 1.584 | 1.899 | 0.834 | 144 |
| K4  | 16 | 0.042 | 42.0 | 0.106 | 1.421 | 2.111 | 0.673 | 144 |
| K5  | 17 | 0.083 | 85.0 | 0.106 | 1.265 | 1.635 | 0.744 | 144 |
| K6  | 17 | 0.123 | 125.0 | 0.106 | 1.015 | 1.396 | 0.727 | 144 |
| K7  | 17 | 0.164 | 175.0 | 0.106 | 1.015 | 1.396 | 0.727 | 144 |

Notes. All quantities have the same meanings as in Table 1.

$$\nu_t(k_U/k_d) = \frac{\nu_t(0)}{1 + \sigma_t(k_U/k_d)^c} = \frac{\nu_t(0)}{1 + \sigma_t^{c/2}},$$

where $\nu_t(0)$ and $\sigma_t$ are fit parameters. We also fit the data to a more general function where the exponent $c$ of $k_U/k_d$ is another fit parameter:

$$\nu_t(k_U/k_d) = \frac{\nu_t(0)}{1 + \sigma_t^{k_U/k_d}} = \frac{\nu_t(0)}{1 + \sigma_t^{c/2}}.$$ 

The inverse relative errors $\nu_t/\delta\nu_t$ are used as weights in the fit. The data and the fits are shown in Fig. 8. While the data is in reasonable agreement with a Lorentzian with $\nu_t(0) = 1.90$ and $\sigma_t = 1.80$, the more general function with $\nu_t(0) = 2.00$, $\sigma_t = 1.61$, and $c = 1.44$ yields a somewhat better fit. Data consistent with Lorentzian behaviour has been found earlier for $\eta_t$ in low Reynolds number turbulence (Brandenburg et al. 2008a); see Table 4 for an overview of the values found previously in various cases ranging from magnetic diffusion in isotropic turbulence to passive scalar diffusion in shear flows, in which $\sigma_t$ was typically below unity. However, a value of $c$ close to 4/3, as found here, is indeed expected for fully developed Kolmogorov turbulence; see the discussion in the conclusions of Madarassy & Brandenburg (2010).

Note that $\sigma_t$ was found to be larger under the second-order correlation approximation (SOCA). Also, looking at Table 4, there is no obvious connection between the values of $\sigma_t$ in different physical circumstances. More analyses are needed to assess the robustness of the results obtained so far.

Knowing the value of $\sigma_t$ is important for more accurate mean-field modelling. In physical space, a prescription like Eq. (35) corresponds to a convolution, which makes the Reynolds stress at a given position dependent on the mean velocity within a certain neighbourhood. In that way, non-locality is modelled.

$\sigma_t$ is determined by the intersection point of the asymptotics $\nu_t(0)$ and $\nu_t(0)^{c/2}/\sigma_t$, where $\nu_t$ becomes $\nu_t(0)/2$, as $\mathcal{S} = \sqrt{\sigma_t}$. This is generally ignored in the common use of turbulent viscosity, although some attempts have been made to include such affects to leading order (Rüdiger 1982). Ignoring non-locality corresponds to the limit $\sigma_t \to 0$ or $k_U/k_d \to 0$, which is often a questionable assumption; see Brandenburg & Chatterjee (2018) for a discussion in the context of spherical mean-field dynamos.

4.2. Turbulent diffusivity $\eta_t$.

The turbulent magnetic diffusivity $\eta_t$ from Sets D–G is shown in Fig. 9. We find a similar qualitative behaviour as for $\nu_t$, so that for small magnetic Reynolds numbers the value of $\eta_t$ is proportional to $Rm$ and the results converge when the scale separation ratio is small enough. As in the case of the turbulent viscosity, we find a weak declining trend as a function of $Rm$ at its highest values which was neither observed by Sur et al. (2008) in similar simulations without shear nor by Brandenburg et al. (2008b) and Mitra et al. (2009) in runs where the large-scale flow was imposed via the shearing-box approximation. However, the error estimates in the aforementioned studies are clearly greater than in the present one and thus a weak decreasing trend as a function of $Rm$ cannot be
ruled out. Furthermore, the shear flows in the present simulations are significantly weaker than in the cases of Brandenburg et al. (2008b) and Mitra et al. (2009), such that their influence on the turbulent transport coefficients is also weaker.

We assess the effect of the shear flow on the results by performing an additional set of simulations in which it is omitted, but otherwise the same parameters as in Set F are employed. We show the results for the difference of $\eta$ in these sets in Fig. 10. The difference is typically of the order of a few per cent such that in most cases the value from the case with shear is greater. This is of the same order of magnitude as the error estimates for $\eta$. Thus we conclude that the systematic error due to the large-scale anisotropy induced by the shear flow is insignificant in the determination of the turbulent diffusivity.

4.3. Turbulent magnetic Prandtl number

Our results for $\mathrm{Pm}_t$, as a function of Reynolds number and scale separation ratio $\mathcal{S}$ are shown in Fig. 11. We find that $\mathrm{Pm}_t$ for $Re \lesssim 20$ is roughly a constant for each value of $k_l$ while increasing from roughly 0.8 for $k_l = 3$ to 0.95 for $k_l = 10$. Especially at low $Re$, the convergence with respect to the scale separation is not as clear as for $\nu$ and $\eta$, individually. With respect to low Reynolds numbers, we see an increasing trend starting from values between 0.55 and 0.65 at $Re \approx 5$ until $Re \approx 20$. At even lower $Re$ the uncertainty in the determination of $\nu$ becomes larger and the values of $\mathrm{Pm}_t$ have substantial error margins.

The turbulent magnetic Prandtl number has been computed with various analytical techniques, see Table 5. Considering the limit of $\nu \to 0$ or $Re \to \infty$, different flavours of FOSA yield either $\mathrm{Pm}_t = 0.8$ (Kitchatinov et al. 1994) or 0.4 (Yousef et al. 2003), whereas MTA results for fully developed turbulent convection favours lower values (Rogachevskii & Kleedorl 2006). A similar spread of values from $\mathrm{Pm}_t \approx 0.42$ (Verma 2001) to $\approx 0.7$–0.8 (Fournier et al. 1982; Kleedorl & Rogachevskii 1994; Jurčíšinová et al. 2011) has been reported using renormalisation group methods for the case of three spatial dimensions and weak magnetic fields.

Particularly at high scale separation, our results are not compatible with any of the analytic results but indicate a higher value than all of the theories. This can be due to the fact that the turbulence in the simulations is not in the fully developed regime and because the scale separation achieved is still insufficient. Furthermore, analytic theories may have to simplify the approximations that cannot be justified in high-Reynolds number turbulence.

4.4. Dependence on forcing time scale

The bulk of the simulations considered here use a $\delta$ correlated random (white) forcing, see Eq. (3), such that a new wavevector is chosen at every time step. It cannot, however, be ruled out a priori that the results depend on the correlation time of the forcing. Here we test the sensitivity of the results with respect to this correlation time by comparing the default case of white–in–time
forcing with cases where the forcing wavevector is held constant for a time $\delta t_f$. We take Run C4 as our fiducial model where $\delta t_f k_h u_{\text{rms}} \approx 0.02$ and $\delta t_f$ is equal to the time step of the simulation. We increased $\delta t_f k_h u_{\text{rms}}$ in steps by a factor of 50 altogether and computed turbulent viscosity, magnetic diffusivity, and magnetic Prandtl number; see Fig. 12. We find that $\eta_t$, $\eta_m$, and $Pm$. are essentially constant in this range of parameters. Our method of switching the forcing with a period $\delta t_f$ is crude because it induces discontinuities, but the approximate constancy of the coefficients suggests that robust results for the turbulent transport coefficients are obtained nevertheless.

5. Conclusions

We have computed the turbulent viscosity ($\nu_t$) and magnetic diffusivity ($\eta_m$) from simulations of forced turbulence using imposed shear flows and the test-field method, respectively. As expected, $\nu_t$ and $\eta_m$ are found to be proportional to the respective Reynolds number at low $Re$ and $Rm$. With increasing values of $Re$ and $Rm$, the turbulent transport coefficients saturate at around $Re \approx Rm \approx 10$, but show a weakly decreasing trend beyond. The value of the turbulent viscosity estimated from the Reynolds stress, which is interpreted to reflect the response of the system to a large-scale flow, and from the decay of a mean flow in the presence of turbulence are in fair agreement. However, the latter yields systematically slightly higher values for high Reynolds numbers by less than 10%.

The turbulent magnetic Prandtl number $Pm_\tau$ saturates between 0.8 and 0.95 for $Re \gtrsim 10$ depending on the scale separation ratio. We note that these values are somewhat higher than those from the renormalisation group approach and, especially, the first-order smoothing approach. The value of $Pm_\tau$ computed here corresponds to the kinematic case where the magnetic field is weak, which is often not the case in astrophysical systems. Analytic studies predict quenching of turbulent viscosity and magnetic diffusivity when the magnetic fields are dynamically significant (e.g. Kitchatinov et al. 1994). The quenching of $\eta_m$ has also been computed from numerical simulations (e.g. Käpylä & Brandenburg 2009; Brandenburg et al. 2008c; Karak & Brandenburg 2009). Similar studies for turbulent viscosity are so far lacking. Such results will be reported elsewhere.

One of the other remaining issues to be addressed in the future is the role of compressibility effects, in particular that of fluctuations of $\rho$. In addition to making analytic progress by identifying potentially new effects owing to their presence, it would be useful to extend our simulations to the regime of larger Mach numbers. Another possible extension of our work is to study the potential of a small-scale dynamo to give rise of magnetic stresses that could enhance the turbulent viscosity, as has been suggested in the solar context to alleviate the discrepancies between observations and simulations of differential rotation and convective velocities (Karak et al. 2018).

Acknowledgements. We thank the anonymous referee for critical and constructive comments on the manuscript. The simulations were performed using the supercomputers hosted by CSC – IT Center for Science Ltd. in Espoo, Finland, who are administered by the Finnish Ministry of Education. This work was supported by the Deutsche Forschungsgemeinschaft Heisenberg programme (grant No. KA 4825/2-1; PJK), the Academy of Finland ReSoLVE Centre of Excellence (Grant No. 307411; PIK, MR, MIK), the NSF Astronomy and Astrophysics Grants Program (Grant 1615100), and the University of Colorado through its support of the George Ellery Hale visiting faculty appointment. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (project “UniSDyn”, Grant agreement No. 818665).

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in $\vec{H}$ and free of triple correlations of fluctuating quantities. We denote the turbulence present under the condition $\vec{U} = \nabla \vec{H} = 0$ by $u^{(0)}$. If $u^{(0)}$ describes isotropic turbulence, only the diagonal components $Q_{ii}$ exist. Proceeding to the situation $\vec{U} = 0$, $\nabla \vec{H} \neq 0$, indicated by the superscript $(0)$, the fluctuations $u$, $h$ can be written as $u = u^{(0)} + u^{(0)1}$, $h = h^{(0)} + h^{(0)1}$. Correspondingly, the mean force is

$$F = 2\nu \nabla \cdot \nabla h - \nabla \cdot \nabla u = F^{(0)} + F^{(0)1}, \quad (A.1)$$
with $s = \vec{S} - \vec{S}$, and

$$F^{(0)} = 2\nu \nabla (s^{(0)} \cdot \nabla h^{(0)}) - \nabla u^{(0)1} \cdot \nabla u^{(0)}, \quad (A.2)$$

$$F^{(0)1} = 2\nu (s^{(0)} \cdot \nabla h^{(0)} + h^{(0)} - u^{(0)} \cdot \nabla u^{(0)} - u^{(0)1} \cdot \nabla u^{(0)}), \quad (A.3)$$

Restricting to first order in the mean quantities, we get

$$F^{(0)1} = 2\nu (s^{(0)} \cdot \nabla h^{(0)} + u^{(0)} \cdot \nabla h^{(0)})$$

$$- u^{(0)} \cdot \nabla u^{(0)} - u^{(0)1} \cdot \nabla u^{(0)}, \quad (A.4)$$

Analogously, for $\vec{U} \neq 0$, $\nabla \vec{H} = 0$, we have

$$F = F^{(0)} + F^{(0)1}, \quad (A.5)$$

with

$$F^{(0)1} = 2\nu (s^{(0)} \cdot \nabla h^{(0)} + u^{(0)} \cdot \nabla h^{(0)})$$

$$- u^{(0)} \cdot \nabla u^{(0)} - u^{(0)1} \cdot \nabla u^{(0)}, \quad (A.6)$$

in first order. If now both $\vec{U}$ and $\nabla \vec{H}$ do not vanish, one can see from the equations for the fluctuating quantities, again restricted to first order in the mean quantities,

$$\delta h^{(0)} = -\nabla \cdot \nabla h^{(0)} - u^{(0)} \cdot \nabla \cdot \nabla h^{(0)}$$

$$- u^{(0)} \cdot \nabla u^{(0)} + u^{(0)} \cdot \nabla u^{(0)}, \quad (A.7)$$

$$\delta u^{(0)} = -c_1 \nabla \cdot \nabla u^{(0)} + \nu \left( \nabla \cdot u^{(0)} \right), \quad (A.8)$$

that due to their linearity in $h^{(0)}$, $u^{(0)}$, and the additivity of their inhomogeneities depending on $\vec{U}$ and $\vec{H}$, respectively,

$$h^{(0)} = h^{(0)} + h^{(0)}$$

$$u^{(0)} = u^{(0)} + u^{(0)}, \quad (A.10)$$

holds and hence

$$F = F^{(0)} + F^{(0)1} + F^{(0)1}, \quad (A.12)$$
to first order in $\vec{U}$ and $\vec{H}$. Given that the major part of $F$ is $\nabla \cdot Q$, an equivalent relationship can be assumed for $Q$. 

Appendix A: Justification for Eq. (15)

In the isothermal case, it is convenient to consider the quantity $H = \ln \rho$ instead of the density itself resulting in equations linear