Sampling Parts of Random Integer Partitions: A Probabilistic and Asymptotic Analysis

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Abstract

Let \( \lambda \) be a partition of the positive integer \( n \), selected uniformly at random among all such partitions. Corteel et al. (1999) proposed three different procedures of sampling parts of \( \lambda \) at random. They obtained limiting distributions of the multiplicity \( \mu_n = \mu_n(\lambda) \) of the randomly-chosen part as \( n \to \infty \). The asymptotic behavior of the part size \( \sigma_n = \sigma_n(\lambda) \), under these sampling conditions was found by Fristedt (1993) and Mutafchiev (2014). All these results motivated us to study the relationship between the size and the multiplicity of a randomly-selected part of a random partition. We describe it obtaining the joint limiting distributions of \((\mu_n, \sigma_n)\), as \( n \to \infty \), for all these three sampling procedures. It turns out that different sampling plans lead to different limiting distributions for \((\mu_n, \sigma_n)\). Our results generalize those obtained earlier and confirm the known expressions for the marginal limiting distributions of \( \mu_n \) and \( \sigma_n \).

Key words: integer partitions, part sizes, random sampling, limiting distributions

Mathematics Subject classifications: 05A17, 60C05, 60F05

1 Introduction

Partitioning integers into summands (parts) is a subject of intensive research in combinatorics, number theory and statistical physics. If \( n \) is a positive integer, then by a partition, \( \lambda \), of \( n \), we mean a representation

\[
\lambda : \quad n = \sum_{j=1}^{n} jm_j, \quad (1.1)
\]
in which \( m_j \), called multiplicities of parts \( j, j = 1, 2, ..., n \), are non-negative integers. We use \( \Lambda(n) \) to denote the set of all partitions of \( n \) and let \( p(n) = |\Lambda(n)| \). The number \( p(n) \) is determined asymptotically by the famous partition formula of Hardy and Ramanujan [9]:

\[
p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right), \quad n \to \infty.
\]

A precise asymptotic expansion for \( p(n) \) was found later by Rademacher [14] (more details may be also found in [2; Chapter 5]). For instance, Rademacher’s result implies that

\[
p(n) = \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) - \frac{1}{4\pi \sqrt{2} n^{3/2}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) + O \left( \exp \left( \frac{\pi}{2} \sqrt{\frac{2n}{3}} \right) \right), \quad n \to \infty.
\]

Further on, we assume that, for fixed integer \( n \geq 1 \), a partition \( \lambda \in \Lambda(n) \) is selected uniformly at random (uar), i.e. with probability \( 1/p(n) \). In this way, each numerical characteristic of \( \lambda \) can be regarded as a random variable defined on the space \( \Lambda(n) \).

Corteel et al. [3] proposed and studied three procedures of sampling parts of a random partition \( \lambda \in \Lambda(n) \). Basic statistics of a randomly selected part are the part size and its multiplicity. Corteel et al. [3] focused on the multiplicity \( \mu_{n,j} = \mu_{n,j}(\lambda) \) \( (j = 1, 2, 3) \) of the randomly-selected part and found limiting distributions for \( \mu_{n,j} \), as \( n \to \infty \), in these three cases of sampling (here the subscript \( j \) specifies the concrete sampling procedure that is followed; the definitions of these three sampling procedures will be given in the next section). In the same way, let \( \sigma_{n,j} = \sigma_{n,j}(\lambda) \) \( (j = 1, 2, 3) \) be the size of the randomly-selected part. Limit theorems for \( \sigma_{n,j} \) were obtained in [6] and [13]. All these results motivated us to study the relationship between the size and the multiplicity of a randomly-selected part of a random integer partition. We describe it obtaining the joint limiting distributions of \( \mu_{n,j} \) and \( \sigma_{n,j} \) \( (j=1,2,3) \) as \( n \to \infty \). Our results generalize those obtained earlier in [6,3,13] and confirm the known expressions for the marginal limiting distributions of \( \mu_{n,j} \) and \( \sigma_{n,j} \).

We organize our paper as follows. In Section 2 we describe the sampling procedures proposed by Corteel et al. [3]. The main results of this paper are stated in Section 3. The method of proof is also briefly described there. Section 4 contains some auxiliary facts on generating functions and some asymptotics that we need further. We present the proofs of our limit theorems in Sections 5 - 7.
2 Basic Random Variables and Definitions of the Sampling Procedures

For any \( \lambda \in \Lambda(n) \) selected uar, we define the random variables
\[
\alpha_j^{(n)}(\lambda) = \text{the number of parts of size } j \text{ in } \lambda.
\]

By \( I_A \) we denote the indicator of an event \( A \) and, for any two real numbers \( d, s \geq 1 \) and integer \( m \geq 1 \), we set
\[
Z_{d,s}^{(n)} = \sum_{1 \leq j \leq s} \alpha_j^{(n)} I\{\alpha_j^{(n)} \leq d\}, \quad (2.1)
\]
\[
Y_{m,s}^{(n)} = \sum_{1 \leq j \leq s} I\{\alpha_j^{(n)} = m\}. \quad (2.2)
\]

\( Z_{d,s}^{(n)} \) counts the number of parts of size not greater than \( s \) and multiplicity not greater than \( d \) in a randomly-chosen partition \( \lambda \), while \( Y_{m,s}^{(n)} \) is the number of distinct parts with multiplicity \( m \) and size not greater than \( s \). Obviously,
\[
Z_n = \sum_{j=1}^n \alpha_j^{(n)} \quad (2.3)
\]
equals the total number of parts and
\[
Y_n = \sum_{j=1}^n I\{\alpha_j^{(n)} > 0\} \quad (2.4)
\]
- the number of distinct parts in \( \lambda \in \Lambda(n) \).

To describe the sampling procedures introduced by Corteel et al. [3] we notice that they are two-step procedures that combine the outcomes of two experiments. Therefore, they lead to three different product probability spaces. Since in each procedure we first sample uar a partition \( \lambda \in \Lambda(n) \), the probability space on \( \Lambda(n) \), equipped with the uniform probability measure \( Pr(\lambda \in \Lambda(n)) = \frac{1}{p(n)} \), is included in each product space. The second steps of sampling are, however, different and therefore, for each different procedure we obtain a different product space and different product probability measure. In what follows next, we adopt the common notation \( \mathbb{P}(\cdot) \) for the product probability measure of each sampling procedure and follow the concept of a product space developed in [8; Chapter 1.6]. By \( \mathbb{E}(X) \) we denote the expected value of the random variable \( X \) defined on the integer partition space \( \Lambda(n) \).

Procedure 1. Given a partition \( \lambda \in \Lambda(n) \) chosen uar (step 1), we select a part uar among all \( Z_n \) parts of \( \lambda \) (without any bias, step 2). By the product measure formula [8; Chapter 1.6], \( \mathbb{P}(\cdot) \) and \( \mathbb{E}(X) \)
\[
\mathbb{P}(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,1} \leq d, \sigma_{n,1} \leq s\})
= Pr(\lambda \in \Lambda(n))\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = \left(\frac{1}{p(n)}\right) \left(\frac{Z_{d,s}^{(n)}}{Z_n}\right).
\]
Summation over all $\lambda \in \Lambda(n)$ yields
\[ P(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = E \left( \frac{Z_{d,s}^{(n)}}{Z_n} \right). \tag{2.5} \]

**Procedure 2.** Given a partition $\lambda \in \Lambda(n)$ chosen uar (step 1), we select a part among all $Y_n$ different parts (step 2). Recalling definitions (2.2) and (2.4) of the random variables $Y_{m,s}^{(n)}$ and $Y_n$, respectively, we obtain in a similar way that
\[ P(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,2} = m, \sigma_{n,2} \leq s\}) = \left( \frac{1}{p(n)} \right) \left( \frac{Y_{m,s}^{(n)}}{Y_n} \right), \tag{2.6} \]
and
\[ P(\mu_{n,2} = m, \sigma_{n,2} \leq s) = E \left( \frac{Y_{m,s}^{(n)}}{Y_n} \right). \]

**Procedure 3.** Given a partition $\lambda \in \Lambda(n)$ chosen uar (step 1), we select a part of $\lambda$ with the probability proportional to its size and multiplicity (step 2). Thus we set
\[ P(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,3} = m, \sigma_{n,3} \leq s\}) = \left( \frac{1}{p(n)} \right) \left( \frac{m}{n} \right) \sum_{1 \leq j \leq s} jI\{\alpha_j^{(n)} = m\}, \tag{2.7} \]
which in turn implies that
\[ P(\mu_{n,3} = m, \sigma_{n,3} \leq s) = \frac{m}{n} \sum_{1 \leq j \leq s} jPr(\alpha_j^{(n)} = m). \tag{2.8} \]

**Remark.** Sampling procedure 3 can be interpreted in terms of Ferrers diagrams - the graphical representations of the integer partitions $\lambda \in \Lambda(n)$ [2; Chapter 1.3]. It is obtained as follows. We use the notation $\lambda_k$ to denote the $k$th largest part of $\lambda$ for $k$ a positive integer; if the number of parts $Z_n$ of $\lambda$ is $< k$, then $\lambda_k = 0$. The Ferrers diagram illustrates $\lambda$ by a two-dimensional array of dots, composed by $\lambda_1$ dots in the first (most left) row, $\lambda_2$ dots in the second row, ..., $\lambda_{Z_n}$ dots in the last $Z_n$th row. Therefore, a Ferrers diagram may be considered as a union of disjoint blocks (rectangles) of dots with base $j$ and height $\alpha_j^{(n)}$ (the multiplicity of part $j$). So, (2.7) and (2.8) imply that the sampling probability in Procedure 3 is proportional to the area of the block to which the chosen part belongs.

### 3 Statement of the Main Results and Brief Description of the Method of Proof

For sampling procedures 1 - 3, we have proved the following limit theorems.
Theorem 1 For the reals $u$ and $v$, we let

$$F(u, v) = \begin{cases} 
    0 & \text{if } \min \{u, v\} \leq 0 \\
    0 & \text{if } \min \{u, v\} > 0 \text{ but } u + v \leq 1, \\
    u + v - 1 & \text{if } 0 < u \leq 1, 0 < v \leq 1 \text{ and } u + v > 1, \\
    \min \{1, v\} & \text{if } u > 1 \text{ and } 0 < v \leq 1, \\
    \min \{1, u\} & \text{if } v > 1 \text{ and } 0 < u \leq 1, \\
    1 & \text{if } u > 1 \text{ and } v > 1. 
\end{cases}$$

Then, we have

$$\lim_{n \to \infty} P\left(\frac{2 \log \mu_{n,1}}{\log n} \leq u, \frac{2 \log \sigma_{n,1}}{\log n} \leq v\right) = F(u, v).$$

Theorem 2 Let $0 < t < \infty$. Then, for any positive integer $m$, we have

$$\lim_{n \to \infty} P\left(\mu_{n,2} = m, \frac{\pi \sigma_{n,2}}{\sqrt{6n}} \leq t\right) = \int_0^t e^{-my}(1 - e^{-y})dy.$$

Theorem 3 Let $0 < t < \infty$. Then, for any positive integer $m$, we have

$$\lim_{n \to \infty} P\left(\mu_{n,3} = m, \frac{\pi \sigma_{n,3}}{\sqrt{6n}} \leq t\right) = \frac{6m}{\pi^2} \int_0^t y(1 - e^{-y})e^{-my}dy.$$

Remark 1. Since the inequalities $\frac{2 \log \mu_{n,1}}{\log n} \leq u, \frac{2 \log \sigma_{n,1}}{\log n} \leq v$ are equivalent to $\mu_{n,1} \leq n^{u/2}, \sigma_{n,1} \leq n^{v/2}$, respectively, Theorem 1 implies that the proportion of parts of size $\leq n^{v/2}$ and multiplicity $\leq n^{u/2}, 0 < u, v < 1$, is approximately equal to $u + v - 1$ if $u + v > 1$; if $u + v \leq 1$ this proportion approaches zero as $n \to \infty$. For the other two sampling procedures, Theorems 2 and 3 show that typically chosen part sizes are of order $\text{const} \sqrt{n}$, while their multiplicities are finite - both converge weakly to discrete random variables whose support is the set $\{1, 2, ...\}$.

Remark 2. For the sake of completeness, we present here a list of the known marginal limiting distributions for the size and multiplicity of the randomly-chosen part. They can be obtained as corollaries of Theorems 1-3. Proper references are also given.

$$\lim_{n \to \infty} P\left(\frac{2 \log \mu_{n,1}}{\log n} \leq t\right) = t, \quad 0 < t < 1$$

[3; p.195];

$$\lim_{n \to \infty} P\left(\frac{2 \log \sigma_{n,1}}{\log n} \leq t\right) = t, \quad 0 < t < 1$$

[6; p.712];

$$\lim_{n \to \infty} P(\mu_{n,2} = m) = \frac{1}{m(m+1)}, \quad m = 1, 2, ..$$
\[ \lim_{n \to \infty} P\left( \frac{\pi \sigma_{n,2}}{\sqrt{6n}} \leq t \right) = 1 - e^{-t}, \quad 0 < t < \infty \]  
[13; Theorem 2];

\[ \lim_{n \to \infty} P(\mu_{n,2} = m) = \frac{6(2m + 1)}{\pi^2 m(m + 1)^2}, \quad m = 1, 2, \ldots \]  
[3; p. 195];

\[ \lim_{n \to \infty} P\left( \frac{\pi \sigma_{n,3}}{\sqrt{6n}} \leq t \right) = \frac{6}{\pi^2} \int_{0}^{t} \frac{y}{e^{y} - 1} dy, \quad 0 < t < \infty \]  
[13; Theorem 3].

We conclude this section with a description of our method of proof. It combines probabilistic with analytical tools. We employ Fristedt’s conditioning device [6], which allows to transfer probability distributions of linear combinations of the multiplicities \( \alpha^{(n)}_j \) into conditional distributions of the corresponding linear combinations of independent and geometrically distributed random variables. Using this method, we show that, as \( n \to \infty \), the expected values in (2.5) and (2.6) are close to the ratios of the expectations of the random variables that are involved there. The asymptotic behavior of the expectations of \( Y_n \) and \( Z_n \), defined by (2.4) and (2.3), respectively, is well known:

\[ \mathbb{E}(Y_n) \sim \sqrt{6n}/\pi, \]  
(3.1)

\[ \mathbb{E}(Z_n) \sim (\sqrt{6n}/2\pi) \log n \]  
(3.2)

(see [16] and [3], respectively). We use combinatorial enumeration identities for generating functions, Cauchy coefficient formula and the saddle-point method in terms of Hayman admissibility theory [10] (see also [5; Chapter VIII.5]) to obtain the asymptotic behavior of \( \mathbb{E}(Z_{d,s}^{(n)}) \) (see (2.5)). Finally, (2.6) and (2.8) are analyzed using an approach developed by Corteel et al. [3] and based on Euler-MacLaurin sum formula.

4 Generating Functions and the Analytical Background of the Proofs

We start with the notation \( g(x) \) for the generating function of the sequence \( \{p(n)\}_{n \geq 1} \). For \( |x| < 1 \), \( g(x) \) admits the well known representation

\[ g(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-1} \]  
(4.1)

(see e.g. [2; Theorem 1.1]). Our first lemma is related to the probability generating function and the expectation of the random variable \( Z_{d,s}^{(n)} \), defined by (2.5).
Lemma 1  For any reals $d, s \geq 1$ and complex variables $x$ and $z$, satisfying $|x| < 1$ and $|z| < 1$, we have

\[
1 + \sum_{n=1}^{\infty} p(n)x^n \mathbb{E}(z^{Y_{d,s}^{(n)}}) = g(x) \prod_{1 \leq j \leq s} \frac{1 - (zx^j)^{d+1}(1-x^j)}{1-zx^j}.
\]  

(4.2)

Moreover

\[
1 + \sum_{n=1}^{\infty} p(n)x^n \mathbb{E}(Z_{d,s}^{(n)})
\]  

(4.3)

\[= g(x) \left( \sum_{1 \leq j \leq s} \frac{x^j}{1-x^j} - (d+1) \sum_{1 \leq j \leq s} \frac{x^{j(d+1)}}{1-x^{j(d+1)}} \right) \prod_{1 \leq j \leq s} (1 - x^{j(d+1)}).
\]

Proof. The generating function identity (4.2) follows from a more general argument developed in [15; Chapter V.5]. To state it we need some preliminary notations. We let $B \subseteq \{1, 2, \ldots\}$ and let $\Omega_j \subseteq \mathbb{N}_0 = \{0, 1, \ldots\}$, $j \geq 1$, be a sequence of sets. By $\sum$ we denote a sum over all $j \in B$, satisfying (1.1) with $m_j \in \Omega_j$, $j \geq 1$. Then, we have

\[
\prod_{j \in B} \sum_{m_j \in \Omega_j} (z_jx^j)^{m_j} = 1 + \sum_{n \geq 1} x^n \sum_{j_1} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n},
\]  

(4.4)

where $x, z_1, z_2, \ldots$ are formal variables. In (4.4) we set $B = \{1, 2, \ldots, [s]\}$,

\[
\Omega_j = \begin{cases} 
\{0, 1, \ldots, [d]\} & \text{if } j \leq s, \\
\mathbb{N}_0 & \text{if } j > s
\end{cases}
\]

and

\[
x_j = \begin{cases} 
x & \text{if } j \leq s, \\
1 & \text{if } j > s.
\end{cases}
\]

(Here $[s]$ and $[d]$ denote the integer parts of $s$ and $d$, respectively.) The required identity (4.2) now follows from (2.1) and (1.1). A differentiation with respect to $z$ in (4.2) leads to the the expectations of $Z_{d,s}^{(n)}$ and identity (4.3).

The next lemma establishes a similar generating function identity for the random variable $Y_{m,s}^{(n)}$ defined by (2.2). It can be proved repeating the argument from [3; Theorem 1].

Lemma 2  For any real number $s \geq 1$, positive integer $m$ and complex variables $x$ and $z$, satisfying $|x| < 1$ and $|z| < 1$, we have

\[
1 + \sum_{n \geq 1, j \geq 0} x^n \mathbb{E}(z^{Y_{m,s}^{(n)}}) = g(x) \prod_{1 \leq k \leq s} (1 + (z-1)x^{mk}(1-x^k)).
\]

This in turn implies that

\[
\mathbb{E}(Y_{m,s}^{(n)}) = \sum_{1 \leq k \leq s} (p(n - mk) - p(n - (m+1)k)).
\]  

(4.5)

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Further on, for the sake of simplicity, we let
\[ c = \frac{\pi}{\sqrt{6}} \] (4.6)

We notice that Hardy-Ramanujan-Rademacher’s formula in its form implies that
\[ p(n) = \frac{e^{2cn^{1/2}}}{4\sqrt{3n}}(1 + O(n^{-1/2})), \quad n \to \infty. \]

Using this expression, Corteel et al. [3; p. 190] have obtained the following asymptotic estimates.

**Lemma 3** For enough large \( n \), we have
\[
\frac{p(n - mj)}{p(n)} = \left(1 + O\left(\frac{mj}{n^{3/2}}\right) + O((n - mj)^{-1/2})\right)e^{-cmj/n^{1/2}}
\]
\[
= \begin{cases} 
(1 + O(n^{-1/2}))e^{-cmj/n^{1/2}} & \text{if } mj \leq n/2, \\
O(e^{-cn^{1/2}/2}) & \text{if } mj > n/2.
\end{cases}
\] (4.7)

Lemma 3 enables us to interpret the sum in \( \sum \) as a Riemann integral sum.

Our next preliminary fact is related to Hardy-Ramanujan formula (1.2). We shall present it into a slightly different form, which will be used further to find the asymptotic of \( \mathbb{E}(Z_{d,s}(n)) \). To introduce the reader into the subject, we notice that Hardy-Ramanujan formula has been subsequently generalized in various directions most notably by Meinardus [11] (see also [2; Chapter 6]). Meinardus obtained the asymptotic of the Taylor coefficients of infinite products of the form
\[
\prod_{k=1}^{\infty}(1 - x^k)^{-b_k}
\] (4.8)
under certain general assumptions on the sequence of non-negative numbers \( \{b_k\}_{k \geq 1} \). Meinardus approach is based on considering the Dirichlet generating series
\[
D(z) = \sum_{k=1}^{\infty} b_k k^{-z}, \quad z = u + iv.
\] (4.9)

Since we shall use this result, below we briefly describe Meinardus assumptions avoiding their precise statements as well as some extra notations and concepts.

The first Meinardus assumption \((M_1)\) specifies the domain \( \mathcal{H} = \{z : \Re(z) = u \geq -C_0\}, 0 < C_0 < 1 \), in the complex plane, in which \( D(z) \) has an analytic continuation. The second one \((M_2)\) is related to the asymptotic behavior of \( D(z) \), whenever \( |\Im(z)| = v \to \infty \). A function of the complex variable \( z \) which is bounded by \( O(|\Im(z)|^{C_1}) \), \( 0 < C_1 < \infty \), in certain domain of the complex plane is called function of finite order. Meinardus second condition \((M_2)\) requires that \( D(z) \) is of finite order in the whole domain \( \mathcal{H} \). Finally, the Meinardus third condition \((M_3)\) implies a bound on the ordinary generating
function of the sequence \( \{b_k\}_{k \geq 1} \). It can be stated in a way simpler than the Meinardus original expression by the inequality
\[
\sum_{k=1}^{\infty} b_k e^{-k\omega} \sin^2(\pi ku) \geq C_2 \omega^{-\epsilon_1}, \quad 0 < \frac{\omega}{2\pi} < |u| < \frac{1}{2}.
\]
for sufficiently small \( \omega \) and some constants \( C_2, \epsilon_1 > 0 \) \( (C_2 = C_2(\epsilon_1)) \) (see [7; p. 310]).

It is known that Euler partition generating function \( g(x) \) (which is obviously of the form (4.8)) satisfies the Meinardus scheme of conditions \((M_1)-(M_3)\) (see e.g. [2; Theorem 6.3]).

The proof of our Theorem 1 will be based on an asymptotic analysis of a Cauchy integral stemming from (4.3). We shall apply there the saddle-point method in the sense of Hayman [10] (see also [5; Chapter VIII.5]). In [10] Hayman studied a wide class of power series satisfying a set of relatively mild conditions and established general formulas for the asymptotic order of their coefficients. In the proof of Theorem 1 we shall essentially use that the generating function \( g(x) \) is admissible in the sense of Hayman. To present Hayman’s idea and show how it can be applied, we need to introduce some auxiliary notations.

We consider here a function \( G(x) = \sum_{n=1}^{\infty} G_n x^n \) that is analytic for \(|x| < \rho, 0 < \rho < \infty \). For \( 0 < r < \rho \), we let
\[
a(r) = r \frac{G'(r)}{G(r)}, \quad \text{(4.10)}
\]
\[
b(r) = r \frac{G'(r)}{G(r)} + r^2 \frac{G''(r)}{G(r)} - r^2 \left( \frac{G'(r)}{G(r)} \right). \quad \text{(4.11)}
\]
In the statement of Hayman’s result we use the terminology given in [5; Chapter VIII.5]. We assume that \( G(x) > 0 \) for \( x \in (R_0, \rho) \subset (0, \rho) \) and satisfies the following three conditions.

- **Capture condition.** \( \lim_{r \to \rho} a(r) = \infty \) and \( \lim_{r \to \rho} b(r) = \infty \).
- **Locality condition.** For some function \( \delta = \delta(r) \) defined over \((R_0, \rho)\) and satisfying \( 0 < \delta < \pi \), one has
  \[
  G(re^{i\theta}) \sim G(r)e^{i\theta a(r)-\theta^2 b(r)/2}
  \]
as \( r \to \rho \), uniformly for \( |\theta| \leq \delta(r) \).

- **Decay condition.**
  \[
  G(re^{i\theta}) = o \left( \frac{G(r)}{\sqrt{b(r)}} \right)
  \]
as \( r \to \rho \), uniformly for \( \delta(r) \leq \theta < \pi \).

**Hayman Theorem.** Let \( G(x) \) be Hayman admissible function and \( r = r_n \) be the unique solution in the interval \((R_0, \rho)\) of the equation
\[
a(r) = n. \quad \text{(4.12)}
\]
Then the Taylor coefficients of $G(x)$ satisfy, as $n \to \infty$,

$$G_n \sim \frac{G(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}$$  \hspace{1cm} (4.13)

with $b(r_n)$ given by (4.11).

The next lemma presents an alternative formula for the partition function $p(n)$.

**Lemma 4** If $r = r_n$ satisfies (4.12) for sufficiently large $n$, then

$$p(n) \sim \frac{r_n^n a(r_n)}{\sqrt{2\pi b(r_n)}}, \quad n \to \infty,$$

where $a(r_n)$ and $b(r_n)$ are given by (4.10) and (4.11) with $G(x) \equiv g(x)$.

**Proof.** Since in (4.1) we have $b_k = 1, k \geq 1$, the Dirichlet generating series (4.9) is $D(z) = \zeta(z)$, where $\zeta$ denotes the Riemann zeta function. We set in (4.10) and (4.11) $r = e^{-h_n}, h_n > 0$, where $h_n$ is the unique solution of the equation

$$a(e^{-h_n}) = n.$$  \hspace{1cm} (4.14)

(4.14) is an obvious modification of (4.12). Granovsky et al. [7] showed that the first two Meinardus conditions imply that the unique solution of (4.14) has the following asymptotic expansion:

$$h_n = \sqrt{\frac{\zeta(2)}{n}} + \frac{\zeta(0)}{2n} + O(n^{-1-\beta}) = \frac{\pi}{\sqrt{6n}} - \frac{1}{4n} + O(n^{-1-\beta}),$$  \hspace{1cm} (4.15)

where $\beta > 0$ is a fixed constant (here we have also used that $\zeta(0) = -\frac{1}{2}$; see [1; Chapter 23.2]). We also notice that (4.11) and (4.15) imply that

$$b(e^{-h_n}) = 2\zeta(2)h_n^{-3} + O(h_n^{-2}) \sim \frac{\pi^2}{3} h_n^{-3} \sim \frac{2\sqrt{6}}{\pi} n^{3/2}$$  \hspace{1cm} (4.16)

(see [12; Lemma 2.2] with $D(z) = \zeta(z)$). Hence, by (4.14) and (4.16), $a(e^{-h_n}) \to \infty$ and $b(e^{-h_n}) \to \infty$ as $n \to \infty$, that is, Hayman’s “capture” condition is satisfied with $r = r_n = e^{-h_n}$. To show next that Hayman’s “decay” condition is satisfied by $g(x)$ we set

$$\delta_n = \frac{h_n^{4/3}}{\Omega(n)} = \frac{\pi^{4/3}}{(6n)^{2/3}\Omega(n)} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$  \hspace{1cm} (4.17)

with $h_n$ given by (4.15), where $\Omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. We can apply now an estimate for $|g(e^{-h_n+\theta})|$ established in a general form in [12; Lemma 2.4] using all three Meinardus conditions. It states that there are two positive constants $c_0$ and $\epsilon_0$, such that, for sufficiently large $n$,

$$|g(e^{-h_n+\theta})| \leq g(e^{-h_n}) e^{-c_0 h_n^{-\epsilon_0}}$$  \hspace{1cm} (4.18)
uniformly for \( \delta_n \leq |\theta| < \pi \). This, in combination with (4.16), implies that
\[
|g(e^{-h_n+i\theta})| = o(g(e^{-h_n})/\sqrt{b(e^{-h_n})})
\]
uniformly in the same range for \( \theta \), which is just Hayman’s “decay” condition. Finally, by Lemma 2.3 of [12], established using Meinardus conditions \((M_1)\) and \((M_2)\), Hayman’s “locality” condition is also satisfied by \( g(x) \). In fact, this lemma implies in the particular case \( D(z) = \zeta(z) \) that
\[
e^{-i\theta n} \frac{g(e^{-h_n+i\theta})}{g(e^{-h_n})} = e^{-\theta^2 b(e^{-h_n})/2} \left(1 + O(1/\Omega^2(n))\right)
\]
uniformly for \(|\theta| \leq \delta_n\), where \( b(e^{-h_n}) \) and \( \delta_n \) are determined by (4.16) and (4.17), respectively. Hence all conditions of Hayman’s theorem hold and we can apply it with \( G_n = n, G(x) = g(x), r_n = e^{-h_n} \) and \( \rho = 1 \) to find that
\[
p(n) \sim \frac{e^{nh_n}g(e^{-h_n})}{\sqrt{2\pi b(e^{-h_n})}}, \quad n \to \infty,
\]
which completes the proof. ■

Remark. To show that formula (4.20) yields (1.2), one has to replace (4.15) and (4.16) in the right-hand side of (4.20). The asymptotic of \( g(e^{-h_n}) \) is determined by a general lemma due to Meinardus [11] (see also [2; Lemma 6.1]). Since \( \zeta(0) = -\frac{1}{2} \) and \( \zeta'(0) = -\frac{1}{4} \log (2\pi) \) (see [1; Chapter 23.2]), in the particular case of \( g(e^{-h_n}) \) this lemma implies that
\[
g(e^{-h_n}) = \exp \left(\zeta(2)h_n^{-1} - \zeta(0) \log h_n + \zeta'(0) + O(h_n^{c_1})\right)
\]
\[
= \exp \left(\frac{\pi^2}{6h_n} + \frac{1}{2} \log h_n - \frac{1}{2} \log (2\pi) + O(h_n^{c_1})\right), \quad n \to \infty,
\]
where \( 0 < c_1 < 1 \). The rest of the computation leading to (1.2) is based on simple algebraic manipulations and cancellations.

5 Proof of Theorem 1

We base our proof on the definition of Sampling Procedure 1 and eq. (2.5). We want to replace the expected value in its right-hand side by the ratio \( E(Z_{d,s}^{n})/E(Z_n) \). So, we notice first that Erdős and Lehner [4] proved that, in probability, the total number of parts \( Z_n \) is asymptotic to \( E(Z_n) \) as \( n \to \infty \). Hence, for any \( \epsilon > 0 \), the probability of the event
\[
A_n = \left\{ \lambda \in A(n) : \left| \frac{Z_n}{E(Z_n)} - 1 \right| > \epsilon \right\}
\]
tends to 0 as \( n \to \infty \). Further, we rewrite (2.5) in the following way:
\[
P(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = E \left( \frac{Z_{d,s}^{n}}{Z_n} I_{A_n} \right) + E \left( \frac{Z_{d,s}^{n}}{Z_n} I_{A_n} \right),
\]

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For \( \lambda \in A_n \) and \( 0 < \epsilon < 1 \), we have \((1 - \epsilon)\mathbb{E}(Z_n) \leq Z_n \leq (1 + \epsilon)\mathbb{E}(Z_n)\) and therefore,

\[
\frac{\mathbb{E}(Z_{d,s}^{(n)})}{(1 + \epsilon)\mathbb{E}(Z_n)} \leq \mathbb{E}\left(\frac{Z_{d,s}^{(n)}}{Z_n} I_{A_n}\right) \leq \frac{\mathbb{E}(Z_{d,s}^{(n)})}{(1 - \epsilon)\mathbb{E}(Z_n)}.
\] (5.2)

Since \( Z_{d,s}^{(n)} \leq Z_n \), the second summand in (5.1) is not greater than \( \mathbb{P}(A_n) \). Hence, combining (5.1) and (5.2), we obtain

\[
\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = (1 + O(\epsilon)) \frac{\mathbb{E}(Z_{d,s}^{(n)})}{\mathbb{E}(Z_n)} + \mathbb{P}(A_n).
\]

Letting \( n \to \infty \) and then \( \epsilon \to 0 \) and replacing \( \mathbb{E}(Z_n) \) by the right-hand side of (3.2), uniformly for \( d, s \geq 1 \), we finally get

\[
\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) \sim \frac{2\pi \mathbb{E}(Z_{d,s}^{(n)})}{\sqrt{6n \log n}} = \frac{2c\mathbb{E}(Z_{d,s}^{(n)})}{\sqrt{n \log n}},
\] (5.3)

where \( c \) is the constant from (4.6).

Our proof continues with an application of Cauchy coefficient formula to (4.3). We use the circle \( x = e^{h_n+it}, -\pi < t \leq \pi \), as a contour of integration and the notation

\[
\varphi_{d,s}(x) = \left(\sum_{1 \leq j \leq s} \frac{x^j}{1 - x^j} \right) - (d + 1) \sum_{1 \leq j \leq s} \frac{x^{(d+1)j}}{1 - x^{(d+1)}j} \prod_{1 \leq j \leq s} (1 - x^{(d+1)j})
\] (5.4)

to obtain

\[
p(n)\mathbb{E}(Z_{d,s}^{(n)}) = \frac{e^{nh_n}}{2\pi} \int_{-\pi}^{\pi} g(e^{-h_n+it})\varphi_{d,s}(e^{-h_n+it})e^{-itn}d\theta.
\]

Then, we break up the range of integration as follows:

\[
p(n)\mathbb{E}(Z_{d,s}^{(n)}) = J_1(d, s, n) + J_2(d, s, n),
\] (5.5)

where

\[
J_1(d, s, n) = \frac{e^{nh_n}}{2\pi} \int_{-\delta_n}^{\delta_n} g(e^{-h_n+it})\varphi_{d,s}(e^{-h_n+it})d\theta,
\] (5.6)

\[
J_2(d, s, n) = \frac{e^{nh_n}}{2\pi} \int_{\delta_n < |t| \leq \pi} g(e^{-h_n+it})\varphi_{d,s}(e^{-h_n+it})d\theta
\] (5.7)

and \( \delta_n \) is defined by (4.17).

In our next step we set

\[
d = n^{u/2}, \quad s = n^{v/2}, \quad 0 \leq u, v \leq 1
\] (5.8)

and obtain estimates for the sums:

\[
S_1 = \sum_{1 \leq j \leq s} \frac{e^{-jh_n}}{1 - e^{-jh_n}},
\] (5.9)
\[ S_2 = \sum_{1 \leq j \leq s} \frac{e^{-j(d+1)h_n}}{1 - e^{-j(d+1)h_n}}. \]  

Here the sequence \( \{h_n\}_{n \geq 1} \) is defined by (4.15).

Using the approximation of a Riemann sum by an integral, (4.15), (5.8) and (4.10), for \( S_1 \) we get

\[ S_1 = \left(1 + O\left(\frac{1}{n}\right)\right) \sqrt{n} \sum_{1 \leq j \leq n^{v/2}} \frac{e^{-c j / \sqrt{n}}}{1 - e^{-c j / \sqrt{n}}} \frac{1}{\sqrt{n}} \]

\[ \sim \sqrt{n} \int_{1/\sqrt{n}}^{n^{v/2}} e^{-cz} \frac{1}{1 - e^{-cz}} \frac{1}{\sqrt{n}} dz = \frac{\sqrt{n}}{c} \int_{e^{-cz}}^{e^{-c \sqrt{n}}} \frac{1}{1 - e^{-z}} dz \]

\[ = \frac{\sqrt{n}}{c} \log \left(1 - e^{-cn \sqrt{n}}\right) = \frac{\sqrt{n}}{2e} \log n + O(n^{v/2}). \]  

In the same way one can show that

\[ S_2 = \begin{cases} \frac{1-u}{v} \log n + O(n^{\frac{1-u}{v}}) & \text{if } v + u \geq 1, \\ \frac{1-u}{v} \log n + O(n^{\frac{1-u}{2}}) & \text{if } v + u < 1. \end{cases} \]  

We are now ready to find an estimate for the second integral in (5.5) (see (5.7)). First, we have

\[ | \prod_{1 \leq j \leq s} (1 - e^{-h_n j(d+1) + ij\theta(d+1)}) | \]

\[ \leq \prod_{1 \leq j \leq s} (1 - e^{-h_n j(d+1)}) + \prod_{1 \leq j \leq s} e^{-h_n j(d+1)} | 1 - e^{ij\theta(d+1)} | \]

\[ \leq 1 + e^{-h_n (d+1)s} (1 + \prod_{1 \leq j \leq s} | e^{ij\theta(d+1)} |) = 1 + 2e^{-h_n (d+1)s} \leq 3. \]  

Hence, in terms of notations (5.4), (5.9) and (5.10), by (5.8), (5.11) and (5.12),

\[ | \varphi_{d,s}(e^{-h_n + \theta}) | = O((S_1 + (d+1)S_2) = O(\sqrt{n} \log n), \quad -\pi \leq \theta \leq \pi. \]

Replacing this estimate and applying inequality (4.18) to the integrand of (5.7), we obtain

\[ | J_2(d, s, n) | = O(e^{nh_n}(e^{-h_n})\sqrt{n}(\log n)e^{-c_0 h_n^{-10}}). \]

The required estimate now follows from (4.18) and (4.20) in the following way:

\[ | J_2(d, s, n) | = O\left(\frac{e^{nh_n}(e^{-h_n})n^{1/2+3/4} (\log n)e^{-c_0 h_n^{-10}}}{\sqrt{b(e^{-h_n})}}\right) \]

\[ = O(p(n)n^{5/4}(\log n)e^{-c_0 h_n^{-10}}) = O(p(n)n^{5/4}(\log n)e^{-c_0 p(n)/2}) \]

\[ = o(\sqrt{n} (\log n)p(n)), \]  

(5.14)
where $\epsilon'_0 > 0$.

The estimate for $J_1(d, s, n)$ follows from Hayman’s ”locality” condition (4.19). First, we need to expand $\varphi_{d,s}$ by Taylor formula. We have

$$
\varphi_{d,s}(e^{-h_n + i\theta}) = \varphi_{d,s}(e^{-h_n}) + O\left( |\theta| \frac{d}{dx} \varphi_{d,s}(x) \mid_{x=e^{-h_n}} \right)
$$

$$
= \varphi_{d,s}(e^{-h_n}) + O\left( \delta_n \frac{d}{dx} \varphi_{d,s}(x) \mid_{x=e^{-h_n}} \right).
$$

(5.15)

To find the asymptotic of $\varphi_{d,s}(e^{-h_n})$, in addition to (5.11) and (5.12), we also need the limit of

$$
\prod_{1 \leq j \leq s} (1 - e^{-j(d+1)h_n})
$$

as $n \to \infty$, whenever $d$ and $s$ satisfy (5.8) (see (5.4)). Using approximations by Riemann integrals as in the analysis of $S_1$ and $S_2$, it is easy to show that

$$
\prod_{1 \leq j \leq s} (1 - e^{-j(d+1)h_n}) = \exp\left( \sum_{1 \leq j \leq s} \log (1 - e^{-j(d+1)h_n}) \right) \to \begin{cases} 1 & \text{if } v + u \geq 1, \\ 0 & \text{if } v + u < 1. \end{cases}
$$

Hence, from (5.8)-(5.12) it follows that

$$
\varphi_{d,s}(e^{-h_n}) \begin{cases} \sim \frac{u+v-1}{2\sqrt{n}} \sqrt{n} \log n & \text{if } v + u \geq 1, \\ = o(\sqrt{n}) & \text{if } v + u < 1. \end{cases}
$$

(5.16)

The estimate of the error term in (5.15) is tedious and follows the same line of reasoning. We have

$$
\frac{d}{dx} \varphi_{d,s}(x) \mid_{x=e^{-h_n}}
$$

$$
= \left( \sum_{1 \leq j \leq s} \frac{j x^{j-1}}{(1 - x)^2} - (d + 1) \sum_{1 \leq j \leq s} \frac{j x^{j(d+1)}}{(1 - x)^{(d+1)}(d+1)} \right)
$$

$$
\times \left( \prod_{1 \leq j \leq s} (1 - x^{j(d+1)}) \right) \mid_{x=e^{-h_n}} + \left( \prod_{1 \leq j \leq s} (1 - x^{j(d+1)}) \right)
$$

$$
\times \exp\left( -(d + 1) \sum_{1 \leq j \leq s} \frac{j x^{j(d+1)-1}}{1 - x^{j(d+1)}} \right) \mid_{x=e^{-h_n}}.
$$

(5.17)

It can be seen that the first two sums in the right-hand side of (5.17) are of order $O(n \log n)$, while the first product factor is estimated by (5.13). Hence, the first summand in (5.17) is of order $O(n \log n)$. For the sum in the exponent of the second summand of the right-hand side of (5.17), one can show that there exists a constant $C > 0$ such that

$$
(d + 1) \sum_{1 \leq j \leq s} \frac{j x^{j(d+1)-1}}{1 - x^{j(d+1)}} \mid_{x=e^{-h_n}} \geq C \sqrt{n} \log n.
$$
Therefore the second summand in (5.17) is $O(e^{-C \sqrt{n \log n}})$. Hence

$$\frac{d}{dx} \varphi_{d,s}(x) \bigg|_{x=e^{-h_n}} = O(n \log n)$$

and by (4.17) and (5.16), the expansion in (5.15) becomes

$$\varphi_{d,s}(e^{-h_n+i\theta}) = \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right), \quad (5.18)$$

where $\Omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Inserting this estimate and (4.19) into (5.6) and applying the asymptotic for the partition function $p(n)$ from (4.20), we obtain

$$J_1(d, s, n) = e^{n h_n} g(e^{-h_n}) \left( \int_{-\delta_n}^{\delta_n} e^{-y^2 b(e^{-h_n})/2} \left( 1 + O(1/\Omega(n)) \right) dy \right)$$

$$\times \left( \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right) \right)$$

$$\sim \frac{e^{n h_n} g(e^{-h_n})}{\sqrt{b(e^{-h_n})} 2\pi} \left( \int_{-\delta_n}^{\delta_n} e^{-y^2/2} dy \right) \left( \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right) \right)$$

$$\sim \frac{e^{n h_n} g(e^{-h_n})}{\sqrt{b(e^{-h_n})} 2\pi} \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right)$$

$$\sim \varphi_{d,s}(e^{-h_n}) \left( \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right) \right), \quad (5.19)$$

where for the second asymptotic equivalence we have used (4.16) and (4.17) in order to get

$$\delta_n \sqrt{b(e^{-h_n})} \sim \frac{\pi^{5/6} \sqrt{2}}{6^{1/6} \Omega(n)^{1/12}} \to \infty$$

if $\Omega(n) \to \infty$ as $n \to \infty$ not too fast, so that $\Omega(n)^{1/12} \to \infty$. It is now clear that (5.5)-(5.7), (5.14) and (5.19) yield

$$p(n) E(Z^{(n)}_{d,s}) = p(n) \varphi_{d,s}(e^{-h_n}) + o(\sqrt{n \log n})$$

and therefore

$$E(Z^{(n)}_{d,s}) = \varphi_{d,s}(e^{-h_n}) + o(\sqrt{n \log n}).$$

The result of Theorem 1 now follows from (5.3), (3.2), (4.6) and (5.16).
6 Proof of Theorem 2

We base our proof on (2.6), Lemmas 2 and 3 and asymptotic equivalence (3.1). To replace the expectation in the right hand side of (2.6) by the ratio \( \frac{\mathbb{E}(Y_{m,s}(n))}{\mathbb{E}(Y_{n})} \), similarly to what we did in the proof of Theorem 1, we shall study how unlikely is the event

\[
B_n = \left\{ \lambda \in \Lambda(n) : \frac{cY_n(\lambda)}{\sqrt{n}} - 1 > \epsilon \right\}, \quad \epsilon > 0,
\]

where \( c \) is constant from (4.6). Using Fristedt’s method [6], Corteel et al. [3] showed that

\[
\mathbb{P}(B_n) \leq e^{-c_2 \sqrt{n}}, \quad c_2 = c_2(\epsilon) > 0.
\]

(6.1)

Remark. Fristedt’s approach [6] is based on the identity

\[
\mathbb{P}(\alpha_j^{(n)} = m_j, j = 1, ..., n) = \mathbb{P}(\gamma_j = m_j, j = 1, ..., n \mid \sum_{j \geq 1} j \gamma_j = n),
\]

(6.2)

where \( \gamma_j \) is a sequence of independent geometrically distributed random variables, whose distribution is given by

\[
\mathbb{P}(\gamma_j = k) = (1 - q)^{j+k}, \quad k = 0, 1, ...
\]

and \( \{m_j\}_{j \geq 1} \) are non-negative integers. Eq. (6.2) holds for every fixed \( q \in (0, 1) \).

It is natural to take \( q \) so that \( \mathbb{P}(\sum_{j \geq 1} j \gamma_j = n) \) is as large as possible. Fristedt’s almost optimal choice for \( q \) is \( q = e^{-c/\sqrt{n}} \). Then, the bound in (6.1) is easily obtained using this value of \( q \).

Next, we represent the probability in (2.6) in the following way

\[
\mathbb{P}(\mu_{n,2} = m, \sigma_{n,2} \leq s) = \mathbb{E} \left( \frac{Y_{m,s}(n)}{Y_n} I_{B_n^c} \right) + \mathbb{E} \left( \frac{Y_{m,s}(n)}{Y_n} I_{B_n} \right),
\]

(6.3)

where \( I_{B_n} \) and \( I_{B_n^c} \) denote the indicators of events \( B_n \) and \( B_n^c \), respectively. Since, for any \( \lambda \in B_n^c \),

\[
\frac{c}{\sqrt{n}(1 + \epsilon)} < \frac{1}{Y_n} < \frac{c}{\sqrt{n}(1 - \epsilon)}
\]

if \( 0 < \epsilon < 1 \), the first summand in (6.3) is estimated by

\[
\mathbb{E} \left( \frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n^c} \right) = \frac{c}{\sqrt{n}} (1 + O(\epsilon)) \mathbb{E}(Y_{m,s}^{(n)} I_{B_n^c})
\]

\[
= \frac{c}{\sqrt{n}} (1 + O(\epsilon)) (\mathbb{E}(Y_{m,s}^{(n)}) - \mathbb{E}(Y_{m,s}^{(n)} I_{B_n})).
\]

(6.4)

Clearly, with probability 1, \( Y_{m,s}^{(n)} \leq n \). Hence, using (6.1), we obtain

\[
\mathbb{E}(Y_{m,s}^{(n)} I_{B_n}) = O(n \mathbb{P}(B_n)) = O(ne^{-c_2 \sqrt{n}})
\]

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and (6.4) becomes
\[
E \left( \frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n} \right) = \frac{c}{\sqrt{n}} (1 + O(\epsilon)) E(Y_{m,s}^{(n)}) + O(ne^{-c_2\sqrt{n}}).
\]

The second term in the right hand side of (6.3) is easily estimated using (6.1) since it is not greater than \( Pr(B_n) \). Consequently,
\[
P(\mu_n = m, \sigma_n \leq s) = \frac{c}{\sqrt{n}} (1 + O(\epsilon)) E(Y_{m,s}^{(n)}) + O(ne^{-c_2\sqrt{n}})
\]
uniformly for any fixed integer \( m \geq 1 \) and real \( s \geq 1 \). Hence, our next task is to obtain an estimate for \( E(Y_{m,s}^{(n)}) \), as \( n \to \infty \), whenever \( s = tsqrt{n}/c, m \geq 1 \) is fixed integer and \( t \in (0, \infty) \) is also fixed. Combining results of (4.5) and (4.7) of Lemmas 2 and 3, respectively, and approximating the sum by the corresponding Riemann integral, we get
\[
E(Y_{m,s}^{(n)}) = (1 + O(1/\sqrt{n}))
\times \sum_{1 \leq k \leq c^{-1}\sqrt{n}} \left( \exp(-cmk/\sqrt{n}) - \exp(-c(m+1)k/\sqrt{n}) \right)
\sim \frac{c}{\sqrt{n}} \int_0^{c^{-1}t} (e^{-cmy} - e^{-c(m+1)y})dy.
\]

Replacing this expression into (6.5) and letting first \( n \to \infty \) and then \( \epsilon \to 0 \), we obtain
\[
P(\mu_n = m, \sigma_n \leq t) \to c \int_0^{c^{-1}t} (e^{-cmy} - e^{-c(m+1)y})dy
= \int_0^{t} e^{-my}(1 - e^{-y})dy,
\]
which completes the proof of Theorem 2.

7 Proof of Theorem 3

The proof will be based on an asymptotic analysis of formula (2.8), setting there \( s = c^{-1}\sqrt{n}t \) as \( n \to \infty \) (see again (6.6)) and assuming that \( m \) is fixed positive integer. First, we let \( \Lambda_k(n) \) to denote the set of partitions of \( n \) with no part equal to \( k \). Also, let \( P_k(n) = | \Lambda_k(n) | \). In [3; p. 189] Corteel et al. give a combinatorial proof of the following identity:
\[
Pr(\alpha_j^{(n)} = m) = \frac{P_j(n - mj)}{p(n)} = \frac{p(n - jm) - p(n - j(m + 1))}{p(n)}.
\]
Replacing this expression into the right hand side of (2.8) and applying (1.7), as in the proof of Theorem 2, we obtain

$$
\mathbb{P}\left(\mu_{n,3} = m, \frac{c\sigma_{n,3}}{\sqrt{n}} \leq t\right) \sim m \sum_{1 \leq j \leq c-1} \frac{j}{\sqrt{n}} \left( e^{-cmj/\sqrt{n}} - e^{c(m+1)j/\sqrt{n}} \right) \frac{1}{\sqrt{n}}
$$

$$
\rightarrow m \int_{0}^{c-1} y(e^{-cmy} - e^{-c(m+1)y})dy = \frac{m}{c^2} \int_{0}^{t} y(1 - e^{-y})e^{-my}dy.
$$

This completes the proof of Theorem 3.

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