TIGHT CLOSURE OF PARAMETER IDEALS IN LOCAL RINGS
F-RATIONAL ON THE PUNCTURED SPECTRUM

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Abstract. Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring of characteristic \(p > 0\). The aim of this paper is to show that \(\ell_R(q^*/q)\) does not depend on the choice of parameter ideal \(q\) provided \(R\) is an \(F\)-injective local ring that is \(F\)-rational on the punctured spectrum.

1. Introduction

Throughout this paper, let \((R, \mathfrak{m}, k)\) be a local ring of dimension \(d\) and \(q\) a parameter ideal of \(R\). The motivation of this paper comes from the theory of Buchsbaum rings. Recall that the length \(\ell_R(R/q)\) is always greater than or equal to the multiplicity \(e(q)\) for all parameter ideals \(q\). Furthermore, \(R\) is Cohen-Macaulay if and only if \(\ell_R(R/q) = e(q)\) for some (and hence for all) \(q\). The ring \(R\) is called generalized Cohen-Macaulay if the difference \(\ell_R(R/q) - e(q)\) is bounded above for all \(q\). More precisely, if \(R\) is generalized Cohen-Macaulay then

\[
\ell_R(R/q) - e(q) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H^i_m(R))
\]

for all parameter ideals \(q\), and the equality occurs for all parameter ideals \(q\) contained in a large enough power of \(m\). The ring \(R\) is said to be Buchsbaum if \(\ell_R(R/q) - e(q)\) does not depend on the choice of parameter ideal \(q\).

Suppose \((R, \mathfrak{m}, k)\) is an equidimensional excellent local ring of characteristic \(p > 0\). A classical result of Kunz [17] says that \(R\) is regular if and only if the Frobenius endomorphism \(F : R \rightarrow R, x \mapsto x^p\) is flat. Kunz’s theorem is the starting point to study the singularities of \(R\) in terms of Frobenius homomorphism, say \(F\)-singularities. \(F\)-singularities appear in the theory of tight closure (cf. [13] for its introduction), which was systematically introduced by Hochster and Huneke around the mid 80’s [11]. The main objects of \(F\)-singularities are \(F\)-regularity, \(F\)-rationality, \(F\)-purity, and \(F\)-injectivity. Recall that \(R\) is said to be \(F\)-rational if \(q^* = q\) for all parameter ideals \(q\), where \(q^*\) is the tight closure of \(q\). It should be noted that if \(R\) is \(F\)-rational then it is Cohen-Macaulay and normal (here we assume that \(R\) is excellent and equidimensional). In [9] Goto and Nakamura considered rings satisfying that \(\ell_R(q^*/q)\) is bounded above. They proved that \(\ell_R(q^*/q)\) is bounded above for every parameter ideal \(q\) if and only if \(R\) is \(F\)-rational on the punctured spectrum \(\text{Spec}^o(R) = \text{Spec}(R) \setminus \{\mathfrak{m}\}\). It is worth to noting that these conditions imply the generalized Cohen-Macaulay property of \(R\). In the present paper, we study the counterpart of Buchsbaum
rings in the $F$-singularities realm.

Recall that the Frobenius endomorphism yields the (natural) Frobenius action on local cohomology $F : H^i_m(R) \to H^i_m(R)$ for all $i \geq 0$. We say $R$ is $F$-injective if the Frobenius action on all local cohomology modules $H^i_m(R)$ are injective (cf. [6]). Ma [21] showed that if an $F$-injective ring is generalized Cohen-Macaulay, then it is Buchsbaum. Therefore if $R$ is an $F$-injective local ring that is $F$-rational on the punctured spectrum, it is Buchsbaum. As the main result of this paper, we prove the following.

**Main Theorem.** Let $(R, \mathfrak{m}, k)$ be an equidimensional excellent local ring of characteristic $p > 0$. Suppose $R$ is an $F$-injective local ring that is $F$-rational on the punctured spectrum. Then $\ell_R(\mathfrak{q}^t/\mathfrak{q})$ does not depend on the choice of parameter ideal $\mathfrak{q}$.

The paper is organized as follows. In the next section we collect results of generalized Cohen-Macaulay rings and of $F$-singularities used in this paper. Section 3 is devoted for a proof of the main theorem in the $F$-finite case. We prove the main theorem for any equidimensional excellent local ring in Section 4. In the last section we state several open questions for future research.

**Acknowledgement.** The author is deeply grateful to Linquan Ma for his discussion about the $\Gamma$-construction. He is also grateful to Nguyen Cong Minh and Kei-ichi Watanabe for their useful comments on this work. The author is grateful to the referee for his/her careful reading and useful comments.

2. Preliminary

2.1. Buchsbaum and generalized Cohen-Macaulay modules. Let us recall the definition of Buchsbaum and generalized Cohen-Macaulay modules (cf. [26], [27]). Let $M$ be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ and let $\mathfrak{q}$ be a parameter ideal of $M$. We denote by $e(\mathfrak{q}, M)$ the multiplicity of $M$ with respect to $\mathfrak{q}$ (cf. [2] for details).

**Definition 2.1.** Let $M$ be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, k)$ such that $t = \dim M > 0$. Then $M$ is called generalized Cohen-Macaulay, if the difference

$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M)$$

is bounded above, where $\mathfrak{q}$ ranges over the set of all parameter ideals of $M$.

**Remark 2.2.** Let the notation be as in Definition 2.1.

1. It is well-known that $M$ is Cohen-Macaulay if and only if $H^i_m(M) = 0$ for all $i < t$. Similarly, $M$ is generalized Cohen-Macaulay if and only if $H^i_m(M)$ is a finitely generated $R$-module for all $i < t$.

2. Suppose $R$ is equidimensional and is an image of a Cohen-Macaulay local ring. Then $M$ is generalized Cohen-Macaulay if and only if the non-Cohen-Macaulay locus of $M$ is isolated.

3. Let $M$ be a generalized Cohen-Macaulay $R$-module over $(R, \mathfrak{m}, k)$ such that $t = \dim M > 0$. Then

$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M) \leq \sum_{i=0}^{t-1} \binom{t-1}{i} \ell_R(H^i_m(M))$$

for every parameter ideal $\mathfrak{q}$ of $M$. 
Definition 2.3 (cf. [27]). Let $M$ be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m}, k)$ such that $t = \dim M > 0$. A parameter ideal $\mathfrak{q}$ of $M$ is called standard if

$$\ell_R(M/\mathfrak{q}M) - e(\mathfrak{q}, M) = \sum_{i=0}^{t-1} \binom{t-1}{i} \ell_R(H^i_{\mathfrak{m}}(M)).$$

An $\mathfrak{m}$-primary ideal $\mathfrak{a}$ is said to be standard if every parameter ideal contained in $\mathfrak{a}$ is standard.

We say that $M$ is Buchsbaum if every parameter ideal of $M$ is standard, i.e. $\mathfrak{m}$ is a standard ideal of $M$.

Standard ideals admit a cohomological characterization as follows.

Remark 2.4. (1) The parameter ideal $\mathfrak{q}$ of $M$ is standard if and only if the canonical homomorphism $H^i(\mathfrak{q}; M) \to H^i_{\mathfrak{m}}(M)$ is surjective for all $i < t$. Moreover $M$ is Buchsbaum if and only if the canonical homomorphism $H^i(\mathfrak{m}; M) \to H^i_{\mathfrak{m}}(M)$ is surjective for all $i < t$, where the Koszul cohomology can be defined by any set of generators of $\mathfrak{m}$ ([27, Theorem 3.4]).

(2) Let $M$ be a generalized Cohen-Macaulay module over $(R, \mathfrak{m}, k)$ such that $d = \dim M > 0$ and let $n \in \mathbb{N}$ be a positive integer such that $\mathfrak{m}^n H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$. Then every parameter element $x \in \mathfrak{m}^{2n}$ of $M$ admits the splitting property, i.e. $H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(M)$ for all $i < d-1$. Furthermore, every parameter ideal contained in $\mathfrak{m}^{2n}$ is standard (cf. [31]).

We also need the notion of limit closure of parameter ideal in the sequel.

Definition 2.5. Let $(R, \mathfrak{m}, k)$ be a local ring, let $M$ be a finitely generated module with $t = \dim M$ and let $\underline{x} = x_1, \ldots, x_t$ be a of system of parameters of $M$. The limit closure of $\underline{x}$ in $M$ is defined as a submodule of $M$:

$$(\underline{x})_{\lim}^M = \bigcup_{n>0} ((x_1^{n+1}, \ldots, x_t^{n+1})M : M) (x_1 \cdots x_t)^n)$$

with the convention that $(\underline{x})_{\lim}^M = 0$ when $t = 0$. If $M = R$, then we simply write $(\underline{x})_{\lim}$.

From the definition, it is clear that $(\underline{x})M \subseteq (\underline{x})_{\lim}^M$.

Remark 2.6. Let the notation be as in Definition 2.5

(1) The quotient $(\underline{x})_{\lim}^M / (\underline{x})M$ is the kernel of the canonical map

$$H^t(\underline{x}; M) \cong M / (\underline{x})M \to H^t_{\mathfrak{m}}(M).$$

This implies the following fact. Let $\mathfrak{q} = (x_1, \ldots, x_t)$ and put $q_{\lim}^M := (\underline{x})_{\lim}^M$. Hence the notation $q_{\lim}^M$ is independent of the choice of $x_1, \ldots, x_t$ which generate $\mathfrak{q}$.

(2) It is known that $(\underline{x})M = (\underline{x})_{\lim}^M$ if and only if $\underline{x}$ forms an $M$-regular sequence.

(3) It is shown that the Hochster’s monomial conjecture is equivalent to the claim that $q_{\lim} \neq R$ for every parameter ideal $\mathfrak{q}$ of $R$.

(4) If $M$ is a generalized Cohen-Macaulay module of dimension $t > 0$, then for every parameter ideal $\mathfrak{q}$ we have

$$\ell_R(q_{\lim}^M / qM) \leq \sum_{i=0}^{t-1} \binom{t-1}{i} \ell_R(H^i_{\mathfrak{m}}(M)),$$

and the equality occurs if and only if $\mathfrak{q}$ is standard by Definition 2.3 and [3] Theorem 5.1 (see also [3] Proposition 3.6).
2. Tight closure and F-singularities. In the rest of this paper, we always assume that $(R, m, k)$ is an excellent equidimensional local ring of characteristic $p > 0$ and of dimension $d > 0$. If we want to notationally distinguish the source and target of the $e$-th Frobenius endomorphism $F^e : R \xrightarrow{p} R$, we will use $F^e(R)$ to denote the target. $F^e(R)$ is an $R$-bimodule, which is the same as $R$ as an abelian group and as a right $R$-module, that acquires its left $R$-module structure via the $e$-th Frobenius endomorphism $F^e$. By definition the $e$-th Frobenius endomorphism $F^e : R \to F^e(R)$ sending $x$ to $F^e(x^{p^e}) = x \cdot F^e(1)$ is an $R$-homomorphism. We say $R$ is $F$-finite if $F_*(R)$ is a finite $R$-module. When $R$ is reduced, we will use $R^{1/p^e}$ to denote the ring whose elements are $p^e$-th roots of elements of $R$. Note that these notations (when $R$ is reduced) $F^e(R)$ and $R^{1/p^e}$ are used interchangeably in the literature.

**Definition 2.7** ([11] [12] [13]). Let $R^c = R \setminus \cup_{p \in \text{Min} R} p$. Then for any ideal $I$ of $R$ we define

1. The **Frobenius closure** of $I$, $I^F = \{x \mid x^q \in I^{[q]}$ for some $q = p^e$, where $I^{[q]} = (x^q \mid x \in I)$.
2. The **tight closure** of $I$, $I^* = \{x \mid cx^q \in I^{[q]}$ for some $c \in R^c$ and for all $q \gg 0$.

**Remark 2.8.** An element $x \in I^F$ if it is contained in the kernel of the composition

$$R \to R/I \xrightarrow{\text{id} \otimes F^e} R/I \otimes F^e(R)$$

for some $e \geq 0$. Similarly, an element $x \in I^*$ if it is contained in the kernel of the composition

$$R \to R/I \to R/I \otimes F^e(R) \xrightarrow{\text{id} \otimes F^e(c)} R/I \otimes F^e(R)$$

for some $c \in R^c$ and for all $e \gg 0$. In general, let $N$ be a submodule of an $R$-module $M$. The tight closure of $N$ in $M$, denoted by $N^*_M$, consists elements that are contained in the kernel of the composition

$$M \to M/N \to M/N \otimes F^e(R) \xrightarrow{\text{id} \otimes F^e(c)} M/N \otimes F^e(R)$$

for some $c \in R^c$ and for all $e \gg 0$.

Let $x_1, \ldots, x_d$ be a system of parameters of $R$. Recall that local cohomology $H^i_m(R)$ may be computed as the homology of the Čech complex

$$0 \to R \to \bigoplus_{i=1}^t R_{x_i} \to \cdots \to R_{x_1 \ldots x_d} \to 0.$$  

Then the Frobenius endomorphism $F : R \to R$ induces a natural Frobenius action $F : H^i_m(R) \to H^i_{m[p]}(R) \cong H^i_m(R)$. There is a very useful way of describing the top local cohomology. It can be given as the direct limit of Koszul cohomologies

$$H^d_m(R) \cong \lim_{\longrightarrow} R/(x_1^n, \ldots, x_d^n).$$

Then for each $\overline{a} \in H^d_m(R)$, which is the canonical image of $a + (x_1^n, \ldots, x_d^n)$, we find that $F(\overline{a})$ is the canonical image of $a^p + (x_1^pn, \ldots, x_d^pn)$ (see [25]).

**Remark 2.9.** Recall that we always assume $R$ is excellent and equidimensional.

1. An element $x \in 0^*_d(R)$ if there exists $c \in R^c$ such that $c F^e(x) = 0$ for all $e \gg 0$. Let $x_1, \ldots, x_d$ be a system of parameters of $R$. The direct system $\lim_{\longrightarrow} R/(x_1^n, \ldots, x_d^n) \cong H^d_m(R)$ induces the direct system of tight closures

$$\lim_{\longrightarrow} (x_1^n, \ldots, x_d^n)^*/(x_1^n, \ldots, x_d^n) \cong 0^*_d(R).$$
By [4] Remark 5.4 we have \((x_1,\ldots,x_d)_{\lim} \subseteq (x_1,\ldots,x_d)^*\) for all system of parameters \(x_1,\ldots,x_d\). By Remark 2.6 (1) we obtain the direct system

\[ \lim \frac{(x_1^n,\ldots,x_d^n)^*}{(x_1^n,\ldots,x_d^n)_{\lim}} \cong \mathcal{O}_{\mathcal{H}^d(R)} \]

with all maps in the direct system are injective. As a consequence, \(\ell_R(\mathcal{O}_{\mathcal{H}^d(R)}) \leq \infty\). In this case \(\max_q \{\ell_R(\mathcal{O}_{\mathcal{H}^d(R)})\} = \ell_R(0_{\mathcal{H}_{\mathfrak{m}}^d(R)})\), and \(\ell_R(\mathcal{O}_{\mathcal{H}^d(R)}) = \ell_R(0_{\mathcal{H}_{\mathfrak{m}}^d(R)})\) for all parameter ideals \(q\) contained in a large enough power of maximal ideal.

(2) If we consider the target of the Frobenius endomorphism \(F : R \to R\) as an \(R\)-module \(F_\ast(R)\) via the Frobenius endomorphism, then the Frobenius action on \(H^i_{\mathfrak{m}}(R)\) becomes an \(R\)-homomorphism \(F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F_\ast(R))\) for all \(i \geq 0\).

We now present \(F\)-singularities used in this paper.

**Definition 2.10.** A local ring \((R, \mathfrak{m}, k)\) is called \(F\)-rational if every parameter ideal is tight closed, i.e. \(q^* = q\) for all \(q\).

**Remark 2.11.** It is well known that an equidimensional excellent local ring \(R\) is \(F\)-rational if and only if \(R\) is Cohen-Macaulay and \(0_{\mathcal{H}^d_{\mathfrak{m}}(R)} = 0\). Furthermore, \(R\) is normal provided it is \(F\)-rational.

The following is the main result of Goto and Nakamura [9 Theorem 1.1].

**Theorem 2.12.** Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring. Then the following are equivalent.

1. \(\ell_R(q^*/q)\) is bounded above;
2. \(R\) is \(F\)-rational on the punctured spectrum \(\text{Spec}^\circ(R) = \text{Spec}(R) \setminus \{\mathfrak{m}\}\);
3. \(R\) is generalized Cohen-Macaulay and \(\ell_R(0_{\mathcal{H}^d_{\mathfrak{m}}(R)}) < \infty\).

In more detail, we have the following.

**Proposition 2.13.** Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring of characteristic \(p > 0\) and of dimension \(d > 0\) that is \(F\)-rational on the punctured spectrum. Then for every parameter ideal \(q\) of \(R\) we have

\[ \ell_R(q^*/q) \leq \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_{\mathfrak{m}}(R)) + \ell_R(0_{\mathcal{H}^d_{\mathfrak{m}}(R)}). \]

Moreover, the equality occurs if and only if \(q\) is a standard parameter ideal satisfying one of the following condition

1. The canonical map \(q^*/q \to 0_{\mathcal{H}^d_{\mathfrak{m}}(R)}\) is surjective;
2. The canonical map \(q^*/q_{\lim} \to 0_{\mathcal{H}^d_{\mathfrak{m}}(R)}\) is an isomorphism.

Furthermore, we have the equality for all parameter ideals \(q\) contained in a large enough power of \(\mathfrak{m}\).

**Proof.** Note that \(q \subseteq q_{\lim} \subseteq q^*\), so \(\ell_R(q^*/q) = \ell_R(q^*/q_{\lim}) + \ell_R(q_{\lim}/q)\). The assertion now follows from Remarks 2.6 (4) and 2.9 (1). \(\square\)

Since every parameter ideal of a Buchsbaum ring is standard, we have

**Proposition 2.14.** Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring of characteristic \(p > 0\) and of dimension \(d > 0\). Suppose \(R\) is a Buchsbaum ring that is \(F\)-rational on the punctured spectrum. Then the following are equivalent.
(1) \( \ell_R(q^*/q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_m(R)) + \ell_R(0^{*}_{H^i_m(R)}) \) for all parameter ideals \( q \);  
(2) The canonical map \( q^*/q \to 0^{*}_{H^i_m(R)} \) is surjective for all parameter ideals \( q \);  
(3) The canonical map \( q^*/q^{\lim} \to 0^{*}_{H^i_m(R)} \) is an isomorphism for all parameter ideals \( q \).

The relation between Buchsbaum rings and \( F \)-singularities appears explicit in \cite{21}.  

**Definition 2.15.** A local ring \((R, m, k)\) is \( F \)-injective if the Frobenius action on \( H^i_m(R) \) is injective for all \( i \geq 0 \).

**Remark 2.16.** If \( R \) is \( F \)-injective then it is reduced (cf. \cite{23} Lemma 3.11]). Conversely, a reduced ring \( R \) is \( F \)-injective if and only if the inclusion \( R \hookrightarrow F^e_s(R) \) induces injective \( R \)-homomorphisms \( H^i_m(R) \to H^i_m(F^e_s(R)) \) for all \( e, i \geq 0 \).

**Remark 2.17.** Recently, Ma \cite{21} Corollary 3.5] showed that an \( F \)-injective generalized Cohen-Macaulay ring is Buchsbaum. Thus we have the following implications of singularities in this paper

\[
\begin{array}{ccc}
\text{F-rational} & \longrightarrow & \text{F-rational/Spec}^c(R) & \& \text{F-injective} & \longrightarrow & \text{F-rational/Spec}^c(R) \\
\downarrow & \smile & \downarrow & \smile & \downarrow & \smile & \downarrow \\
\text{CM} \& \text{normal} & \longrightarrow & \text{Buchsbaum} \& \text{reduced} & \longrightarrow & \text{Generalized CM}.
\end{array}
\]

### 3. Proof of the main theorem in the \( F \)-finite case

We prove the main theorem for \( F \)-finite rings. Our method is inspired by the proof of Theorem 3.7 of \cite{11} Arxiv: 1512.05374, Version 1. Note that any \( F \)-finite ring is excellent by Kunz \cite{18}.

**Theorem 3.1.** Let \((R, m)\) be an equidimensional \( F \)-finite local ring of dimension \( d > 0 \). Suppose \( R \) is an \( F \)-injective local ring that is \( F \)-rational on the punctured spectrum. Then \( \ell_R(q^*/q) \) does not depend on the choice of parameter ideal \( q \).

**Proof.** By \cite{21} Corollary 3.5] we have \( R \) is Buchsbaum. By Proposition 2.14] we need only to show that the canonical map \( q^*/q^{\lim} \to 0^{*}_{H^i_m(R)} \) is an isomorphism for any parameter ideal \( q \). Let \( e \) be a large enough positive integer such that canonical map \( (q^{[e]})^*/(q^{[e]})^{\lim} \to 0^{*}_{H^i_m(R)} \) is an isomorphism (cf. Remark \cite{29]). Since the role of \( q^{[e]} \) in \( R \) is the same as that of \( q \) in \( F^e_s(R) \), so the canonical map

\[
q^{[e]}_{F^e_s(R)}/q^{[e]}_{F^e_s(R)} \to 0^{*}_{H^i_m(F^e_s(R))}
\]

is an isomorphism, where \( q^{[e]}_{F^e_s(R)} \) is the tight closure of \( q \) \( F^e_s(R) \) as a submodule of \( F^e_s(R) \). Since \( R \) is reduced, we have the short exact sequence of \( R \)-module

\[
0 \to R \to F^e_s(R) \to F^e_s(R)/R \to 0,
\]

where both \( F^e_s(R) \) and \( F^e_s(R)/R \) are Buchsbaum as \( R \)-modules (see the proofs of \cite{21} Theorem 3.4] and \cite{23} Theorem 4.17]). Set \( S = F^e_s(R) \). By Remark 2.16] we have the short exact sequence of local cohomology

\[
0 \to H^i_m(R) \to H^i_m(S) \to H^i_m(S/R) \to 0
\]

for all \( i \geq 0 \). Thus

\[
\ell_R(H^i_m(S)) = \ell_R(H^i_m(R)) + \ell_R(H^i_m(S/R))
\]

for all \( 0 \leq i \leq d - 1 \). By the proof of \cite{23} Theorem 4.17] we have the following short exact sequence

\[
0 \to R/q \to S/qS \to (S/R)/q(S/R) \to 0.
\]
Therefore we have the following commutative diagram
\[
\begin{array}{c}
0 \longrightarrow R/\mathfrak{q} \longrightarrow S/\mathfrak{q}S \longrightarrow (S/R)/\mathfrak{q}(S/R) \longrightarrow 0 \\
\beta \downarrow \quad \alpha \downarrow \quad \varphi \downarrow \\
0 \longrightarrow H^d_m(R) \longrightarrow H^i_m(S) \longrightarrow H^i_m(S/R) \longrightarrow 0.
\end{array}
\]
By Remark 2.9 (1), \( \ker(\beta) = \mathfrak{q}^{\lim}/\mathfrak{q} \). Since \( R \) is Buchsbaum,
\[
\ell_R(\ker(\beta)) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_m(R)).
\]
Similarly, we have
\[
\ell_R(\ker(\alpha)) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_m(S)),
\]
and
\[
\ell_R(\ker(\varphi)) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_m(S/R)).
\]
Notice that \( \ell_R(H^i_m(S)) = \ell_R(H^i_m(R)) + \ell_R(H^i_m(S/R)) \) for all \( 0 \leq i \leq d - 1 \), so
\[
\ell_R(\ker(\beta)) = \ell_R(\ker(\alpha)) + \ell_R(\ker(\varphi)).
\]
Therefore the above commutative diagram induces the following commutative diagram
\[
\begin{array}{c}
0 \longrightarrow R/\mathfrak{q}^{\lim} \longrightarrow S/\mathfrak{q}^{\lim}_S \longrightarrow (S/R)/\mathfrak{q}^{\lim}_{(S/R)} \longrightarrow 0 \\
\overline{\pi} \downarrow \quad \pi \downarrow \quad \varphi \downarrow \\
0 \longrightarrow H^d_m(R) \longrightarrow H^i_m(S) \longrightarrow H^i_m(S/R) \longrightarrow 0,
\end{array}
\]
where \( \overline{\alpha}, \overline{\beta} \) and \( \overline{\varphi} \) are injective. As above we know that the restriction of \( \overline{\alpha} \) on the tight closures \( \mathfrak{q}^*_S/\mathfrak{q}^*_S \rightarrow 0^*_{\mathfrak{q}^*_m(S)} \) is an isomorphism. We claim the restriction map
\[
\overline{\beta} : \mathfrak{q}^*/\mathfrak{q}^{\lim} \rightarrow 0^*_{H^d_m(R)}
\]
is an isomorphism. It is enough to show this map is surjective. Indeed, let \( y \) be any element in \( 0^*_{H^d_m(R)} \). We have \( \mu(y) \in 0^*_{\mathfrak{q}^*_m(S)} \). Thus there exists \( z \in \mathfrak{q}^*_S \) such that \( \overline{\alpha}(\overline{z}) = \mu(y) \). We have
\[
\overline{\varphi}(\overline{\pi}(\overline{z})) = \theta(\overline{\alpha}(\overline{z})) = \theta(\mu(y)) = 0.
\]
Thus \( \pi(\overline{z}) = 0 \) since \( \overline{\varphi} \) is injective. Therefore we have \( x \in R \) such that \( \nu(\overline{z}) = \overline{z} \in \mathfrak{q}^*_S/\mathfrak{q}^{\lim}_S \). Note that the role of \( \mathfrak{q} \) in \( F_*(R) \) is the same that of \( \mathfrak{q}^{[p]} \) in \( R \). Now the condition of \( x \) means \( x^{p^e} \in (\mathfrak{q}^{[p]} \mathfrak{q}^{[p^e]})^* \). Thus we have some \( c \in R^o \) such that \( c x^{p^e+e'} \in \mathfrak{q}^{[p^e+e']} \) for all \( e' \gg 0 \). Hence \( x \in \mathfrak{q}^* \). Moreover \( \mu(\overline{\beta}(\overline{z})) = \overline{\alpha}(\nu(\overline{x})) = \mu(y) \). Thus \( \overline{\beta}(\overline{z}) = y \) since \( \mu \) is injective, and so the canonical map \( \mathfrak{q}^*/\mathfrak{q}^{\lim} \rightarrow 0^*_{H^d_m(R)} \) is surjective as desired. The proof is complete.

We do not have the converse of the previous theorem by the following.

**Example 3.2.** Let \( R = k[X,Y,Z]/(X^3 + Y^3 + Z^3) \), where \( k \) is a perfect field of characteristic \( 0 < p \neq 3 \). The ring \( R \) is a Gorenstein ring with isolated singular. The test ideal of \( R \) is the maximal ideal (cf. [14] Example 4.8). We have \( \ell_R(\mathfrak{q}^*/\mathfrak{q}) = 1 \) for all parameter ideals \( \mathfrak{q} \). However \( R \) is \( F \)-injective if and only if \( p \equiv 1 \mod 3 \).
properties of $\mathcal{R}$ is

parameters $x_k$ put $A$ maximal ideal of $\mathcal{R}$ inseparable. The maximal ideal of $\mathcal{R}$ $A$ Then the natural map $\mathcal{A}$ by $\Gamma \subset k$ coefficient field $\mathcal{A}$ to reduce the problem to the case of $F$-finite rings of

the previous section. We briefly recall the construction. Let $(\mathcal{R}, \mathfrak{m}, k)$ be a complete local ring with coefficient field $k$ of characteristic $p > 0$ and of dimension $d$. Let us fix a $p$-basis $\Lambda$ of $k \subset \mathcal{R}$ and let $\Gamma \subset \Lambda$ be a cofinite subset (we refer the reader to [22] for the definition of a $p$-basis). We denote by $k_e$ (or $k_{\Gamma, e}$ to signify the dependence on the choice of $\Gamma$) the purely inseparable extension field $k[\Gamma^{1/p^e}]$ of $k$, which is obtained by adjoining $p^e$-th roots of all elements in $\Gamma$. Next, fix a system of parameters $x_1, \ldots, x_d$ of $\mathcal{R}$. Then the natural map $A := k[[x_1, \ldots, x_d]] \to \mathcal{R}$ is module-finite. Let us put

$$A^\Gamma := \bigcup_{e > 0} k_e[[x_1, \ldots, x_d]].$$

Then the natural map $A \to A^\Gamma$ is faithfully flat and purely inseparable and $\mathfrak{m}_A A^\Gamma$ is the unique maximal ideal of $A^\Gamma$. Now we set $R^\Gamma := A^\Gamma \otimes_A \mathcal{R}$. Then $R \to R^\Gamma$ is faithfully flat and purely inseparable. The maximal ideal of $\mathcal{R}$ expands to the maximal ideal of $R^\Gamma$. The crucial fact about $R^\Gamma$ is that it is an $F$-finite local ring (see [12] (6.6 Lemma)). Moreover, we can preserve some good properties of $\mathcal{R}$ if we choose a sufficiently small cofinite subset $\Gamma$. For example, if $\mathcal{R}$ is reduced, so is $R^\Gamma$ for any sufficiently small choice of $\Gamma$ cofinite in $\Lambda$. For each prime ideal $p$ of $\mathcal{R}$, $\sqrt[p]{R^\Gamma}$ is a prime ideal of $R^\Gamma$. Furthermore if we choose $\Gamma$ small enough, $pR^\Gamma$ is also prime. We need several lemmas.

**Lemma 4.1** ([5], Lemma 2.9). Let $(\mathcal{R}, \mathfrak{m}, k)$ be a complete local ring that is $F$-injective. Then for any sufficiently small choice of $\Gamma$ cofinite in $\Lambda$, $R^\Gamma$ is $F$-injective.

The next result can be proven by the same method used in [20] Proposition 5.6, so we omit the detail proof. Note that we use [28, Lemma 2.3, Theorem 3.5] to replace the roles of [20] Proposition 5.4, Lemma 5.5 in the proof of [20] Proposition 5.6.

**Lemma 4.2.** Let $(\mathcal{R}, \mathfrak{m}, k)$ be a complete local ring that is $F$-rational on the punctured spectrum. Then for any sufficiently small choice of $\Gamma$ cofinite in $\Lambda$, $R^\Gamma$ is $F$-rational on the punctured spectrum.

Let $M$ be an $\mathcal{R}$-module with a Frobenius action $F$. A submodule $N$ of $M$ is called $F$-compatible if $F(N) \subset N$. An $\mathcal{R}$-module $M$ with a Frobenius action is said to be simple if it has no nontrivial $F$-compatible submodules. If $\mathcal{R}$ is a complete local domain, $0_{H^d_{\mathfrak{m}}(R)}$ is the unique maximal proper $F$-compatible submodule of $H^d_{\mathfrak{m}}(R)$ by Smith [25]. Hence $H^d_{\mathfrak{m}}(R)/0_{H^d_{\mathfrak{m}}(R)}^*$ is simple. In general, for reduced equidimensional excellent local rings Smith [25, Proposition 2.5] showed that $0_{H^d_{\mathfrak{m}}(R)}^*$ is the unique maximal proper $F$-compatible submodule of $H^d_{\mathfrak{m}}(R)$ which is annihilated by an element $c \in R^\circ$.

**Lemma 4.3.** Let $(\mathcal{R}, \mathfrak{m}, k)$ be an equidimensional complete local ring. Then for any sufficiently small choice of $\Gamma$ cofinite in $\Lambda$, $0_{H^d_{\mathfrak{m}}(R^\Gamma)}^* = 0_{H^d_{\mathfrak{m}}(R)}^* \otimes_R R^\Gamma$.

**Proof.** It is obvious that we can assume $\mathcal{R}$ is reduced. Set $(0) = p_1 \cap \cdots \cap p_n$ the primary decomposition of $(0)$. We prove by induction on $n$ the following claim.

**Claim.** $H^d_{\mathfrak{m}}(R)/0_{H^d_{\mathfrak{m}}(R)}^* \cong H^d_{\mathfrak{m}}(R_1)/0_{H^d_{\mathfrak{m}}(R_1)}^* \oplus \cdots \oplus H^d_{\mathfrak{m}}(R_n)/0_{H^d_{\mathfrak{m}}(R_n)}^*$ as $\mathcal{R}$-modules with Frobenius actions, where $R_i = R/p_i$ for all $1 \leq i \leq n$.

Indeed, we have nothing to do when $n = 1$. Suppose $n > 1$ and set $\mathcal{R}' = R/(p_1 \cap \cdots \cap p_{n-1})$. We
have the following short exact sequence
\[ 0 \to R \to R' \oplus R_n \to R'' \to 0, \]
where \( R'' = R/(p_1 \cap \cdots \cap p_{n-1} + p_n) \). This short exact sequence induces the following exact sequence of local cohomology with homomorphisms are compatible with Frobenius actions
\[ \cdots \to H_{m}^{d-1}(R'') \xrightarrow{\delta} H_{m}^{d}(R) \to H_{m}^{d}(R') \oplus H_{m}^{d}(R_n) \to H_{m}^{d}(R'') = 0. \]
Note that \( \dim R'' < d \), so \( \text{Im}(\delta) \) is an \( F \)-compatible submodule of \( H_{m}^{d}(R) \) of dimension less than \( d \), and so that \( \text{Im}(\delta) \subseteq 0_{H_{m}^{d}(R)^{\ast}} \). Therefore we obtain the following short exact sequence by restricting on tight closures
\[ 0 \to \text{Im}(\delta) \to 0_{H_{m}^{d}(R)}^{\ast} \oplus 0_{H_{m}^{d}(R_n)}^{\ast} \to 0. \]
Thus we have the following commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Im}(\delta) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H_{m}^{d}(R) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H_{m}^{d}(R') \\ \\
\end{array}
\]
Hence \( H_{m}^{d}(R)/0_{H_{m}^{d}(R)}^{\ast} \cong H_{m}^{d}(R')/0_{H_{m}^{d}(R')}^{\ast} \oplus H_{m}^{d}(R_n)/0_{H_{m}^{d}(R_n)}^{\ast} \). The Claim now follows from the inductive hypothesis.

We continue to prove the lemma. Notice that for all \( i \leq n \), \( H_{m}^{d}(R_i)/0_{H_{m}^{d}(R_i)}^{\ast} \) is simple by Smith’s result. For any sufficiently small choice of \( \Gamma \) cofinite in \( \Lambda \) we have \( R_{\Gamma} \) is reduced (so \( p_i R_{\Gamma} \) is prime for all \( i \leq n \)). We can even assume that \( (H_{m}^{d}(R_i)/0_{H_{m}^{d}(R_i)}^{\ast}) \otimes_{R} R_{\Gamma} \) is simple as an \( R_{\Gamma} \)-module with a Frobenius action for all \( i \leq n \) by [20 Lemma 5.8]. Thus
\[
(H_{m}^{d}(R)/0_{H_{m}^{d}(R)}^{\ast}) \otimes_{R} R_{\Gamma} \cong H_{m}^{d}(R_{\Gamma})/(0_{H_{m}^{d}(R)}^{\ast}) \otimes_{R} R_{\Gamma}
\]
is a direct sum of \( n \) simple \( R_{\Gamma} \)-modules with Frobenius actions. On the other hand \( 0_{H_{m}^{d}(R')}^{\ast} \otimes_{R} R_{\Gamma} \subseteq 0_{H_{m}^{d}(R')}^{\ast} \) and \( H_{m}^{d}(R_{\Gamma})/(0_{H_{m}^{d}(R')}^{\ast}) \) is also a direct sum of \( n \) simple \( R_{\Gamma} \)-modules with Frobenius actions by the Claim. Therefore \( 0_{H_{m}^{d}(R')}^{\ast} = 0_{H_{m}^{d}(R)}^{\ast} \otimes_{R} R_{\Gamma} \). The proof is complete. \( \square \)

**Lemma 4.4.** Let \((R, m, k)\) be an equidimensional complete local ring that is \( F \)-rational on the punctured spectrum. Then for any sufficiently small choice of \( \Gamma\) cofinite in \( \Lambda\), \((qR_{\Gamma})^{\ast} = q^{\ast}R_{\Gamma}\) for all parameter ideals \( q \) of \( R \).

**Proof.** By Lemmas 4.2 and 4.3, \( R_{\Gamma} \) is \( F \)-rational on the punctured spectrum and \( 0_{H_{m}^{d}(R_{\Gamma})}^{\ast} = 0_{H_{m}^{d}(R)}^{\ast} \otimes_{R} R_{\Gamma} \) for any sufficiently small choice of \( \Gamma \) cofinite in \( \Lambda \). Since \( R \to R_{\Gamma} \) is faithfully flat and \( mR_{\Gamma} \) is the maximal ideal of \( R_{\Gamma} \), we have \( \ell_{R}(H_{m}^{i}(R)) = \ell_{R_{\Gamma}}(H_{m}^{i}(R_{\Gamma})) \) for all \( i \leq d - 1 \), and
\[
\ell_{R}(0_{H_{m}^{d}(R)}^{\ast}) = \ell_{R_{\Gamma}}(0_{H_{m}^{d}(R_{\Gamma})}^{\ast}).
\]
Let \( q = (x_1, \ldots, x_d) \) be any parameter ideal of \( R \). By Proposition 2.18 there exists a positive integer \( k \) such that
\[
\ell_{R}(q_{k}^{d}/q_{k}) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_{R}(H_{m}^{i}(R)) + \ell_{R}(0_{H_{m}^{d}(R)}^{\ast}),
\]
where \( q_k = (x_1^k, \ldots, x_d^k) \) is a parameter ideal of \( R \). Since \( q_k^*R^\Gamma \subseteq (q_kR^\Gamma)^* \) and \( R \to R^\Gamma \) is faithfully flat, we have \( \ell_{R^\Gamma}((q_kR^\Gamma)^*/q_kR^\Gamma) \geq \ell_R(q_k^*/q_k) \). Therefore

\[
\ell_{R^\Gamma}((q_kR^\Gamma)^*/q_kR^\Gamma) \geq \sum_{i=0}^{d-1} \binom{d}{i} \ell_{R^\Gamma}(H^i_{m}(R^\Gamma)) + \ell_{R^\Gamma}(0^*_{H^d_{m}(R^\Gamma)}).
\]

By Proposition 2.13 again we have

\[
\ell_{R^\Gamma}((q_kR^\Gamma)^*/q_kR^\Gamma) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_{R^\Gamma}(H^i_{m}(R^\Gamma)) + \ell_{R^\Gamma}(0^*_{H^d_{m}(R^\Gamma)}).
\]

Hence \( \ell_{R^\Gamma}((q_kR^\Gamma)^*/q_kR^\Gamma) = \ell_R(q_k^*/q_k) \), and so \( (q_kR^\Gamma)^* = q_k^*R^\Gamma \).

We next show that \( (qR^\Gamma)^* = q^*R^\Gamma \). Set \( x = x_1 \ldots x_d \). By [17 Proposition 3.3] we have \( q^* = q_k^* : R \ x^{k-1} \) and \( (qR^\Gamma)^* = (q_kR^\Gamma)^* : R^\Gamma \ x^{k-1} \). Moreover \( (q_kR^\Gamma)^* = q_k^*R^\Gamma \) and \( R \to R^\Gamma \) is a flat extension, so we obtain \( (qR^\Gamma)^* = q^*R^\Gamma \). This completes the proof. \( \square \)

We now prove the main result of this paper.

**Proof of the main theorem.** Suppose \( R \) is an \( F \)-injective local ring that is \( F \)-rational on the punctured spectrum. We will show that \( \ell_R(q^*/q) \) does not depend on the choice of parameter ideal \( q \). It is not hard to see that the \( m \)-adic complement \( \hat{R} \) is also an \( F \)-injective local ring that is \( F \)-rational on the punctured spectrum. Moreover \( \ell_R(q^*/q) = \ell_{\hat{R}}((q\hat{R})^*/q\hat{R}) \) (cf. [13 Exercise 4.1]). Therefore we can assume henceforth that \( R \) is complete. Let \( q \) be any parameter ideal of \( R \). By Proposition 2.13 we need only to show that

\[
\ell_R(q^*/q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_{m}(R)) + \ell_R(0^*_{H^d_{m}(R)}).
\]

Taking a sufficiently small choice \( \Gamma \) cofinite in \( \Lambda \) satisfying Lemmas 4.1, 4.2, 4.3, and 4.4 we have that \( R^\Gamma \) is an \( F \)-injective local ring that is \( F \)-rational on the punctured spectrum and \( q^*R^\Gamma = (qR^\Gamma)^* \) for all parameter ideals \( q \). Since \( R^\Gamma \) is \( F \)-finite,

\[
\ell_{R^\Gamma}((qR^\Gamma)^*/qR^\Gamma) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_{R^\Gamma}(H^i_{m}(R^\Gamma)) + \ell_{R^\Gamma}(0^*_{H^d_{m}(R^\Gamma)}).
\]

by Theorem 3.1. However \( \ell_{R^\Gamma}((qR^\Gamma)^*/qR^\Gamma) = \ell_R(q^*/q) \), so

\[
\ell_R(q^*/q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_{m}(R)) + \ell_R(0^*_{H^d_{m}(R)}).
\]

The proof is complete. \( \square \)

5. **Open Questions**

Recall that the parameter test ideal of \( R \) is \( \tau^R = \cap_q (q : R q^*) \), where \( q \) runs over all parameter ideals. Notice that \( \tau^R \) is an \( m \)-primary ideal if and only if \( R \) is \( F \)-rational on the punctured spectrum (so \( R \) is generalized Cohen-Macaulay). Moreover any parameter ideal contained in \( \tau^R \) is standard by [13 Remark 5.11]. Based on Proposition 2.13 we have the following natural question.  

\[ \text{I thank Professor Kei-ichi Watanabe for this question.} \]
Question 1. Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring that is \(F\)-rational on the punctured spectrum. Is it true that for every parameter ideal \(q\) is contained in \(F^R\), we have

\[
\ell_R(q^e / q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H^i_{\mathfrak{m}}(R)) + \ell_R(0^e_{H^i_{\mathfrak{m}}(R)}).
\]

Let \(A\) be an Artinian \(R\)-module with a Frobenius action \(F : A \to A\). Then we define the Frobenius closure \(0^F_A\) of the zero submodule of \(A\) is the submodule of \(A\) consisting all element \(z\) such that \(F^e(z) = 0\) for some \(e \geq 0\). \(0^F_A\) is the nilpotent part of \(A\) by the Frobenius action. By [10] Proposition 1.11 and [19] Proposition 4.4 there exists a non-negative integer \(e\) such that \(0^F_A = \text{Ker}(A \xrightarrow{F^e} A)\) for all \(i \geq 0\) (see also [24]). The smallest of such integers is called the Harshorne-Speiser-Lyubeznik number of \(A\) and denoted by \(HSL(A)\). We define the Harshorne-Speiser-Lyubeznik number of a local ring \((R, \mathfrak{m})\) as follows.

\[
HSL(R) := \min\{e \mid 0^F_{H^i_{\mathfrak{m}}(R)} = \text{Ker}(H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} H^i_{\mathfrak{m}}(R)) \text{ for all } i = 0, \ldots, d\}.
\]

The \(HSL(R)\) is closely related with the Frobenius test exponent of parameter ideals (see [15] [16]). Recall that the Frobenius test exponent of \(R\), here we denote by \(Fte(R)\), is the smallest non-negative integer \(e\) such that \((q^F)^{[p^e]} = q^{[p^e]}\) for all parameter ideals \(q\), and \(Fte(R) = \infty\) if we have no such \(e\).\(^2\) The relation between \(0^F_{H^i_{\mathfrak{m}}(R)}\) and \(q^F\) appears explicit in [23]. In more details, if \(q^F = q\) for all parameter ideals \(q\) then \(0^F_{H^i_{\mathfrak{m}}(R)} = 0\) for all \(i \geq 0\), i.e. \(R\) is \(F\)-injective. Although the converse is not true for non-equidimensional local rings, we believe the following question has an affirmative answer.

Question 2. Let \((R, \mathfrak{m}, k)\) be an equidimensional excellent local ring of characteristic \(p > 0\). Then is it true that \(q^F = q\) for all parameter ideals \(q\), i.e. \(Fte(R) = 0\), if and only if \(0^F_{H^i_{\mathfrak{m}}(R)} = 0\) for all \(i \geq 0\), i.e. \(HSL(R) = 0\).

Notice that Ma [21] gave a positive answer for the above question for generalized Cohen-Macaulay rings. Suppose \(R\) is an excellent equidimensional local ring that is \(F\)-injective on the punctured spectrum. Then we can check that \(\ell_R(0^F_{H^i_{\mathfrak{m}}(R)}) < \infty\) for all \(i \geq 0\). Inspired by the main result of this paper, we ask the following question.

Question 3. Let \((R, \mathfrak{m}, k)\) be an excellent generalized Cohen-Macaulay local ring that is \(F\)-injective on the punctured spectrum. Is it true that

\[
\ell_R(q^F / q) \leq \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(0^F_{H^i_{\mathfrak{m}}(R)}).
\]

for all parameter ideals \(q\).

If \(R\) is a generalized Cohen-Macaulay local ring (of characteristic \(p > 0\)), then some power of \(\mathfrak{m}\) is a standard ideal of \(R\) (cf. Remark 2.4 (2)). Hence there exists a non-negative integer \(e\) such that

\(^2\)The Harshorne-Speiser-Lyubeznik number of \(R\) is often defined in terms only the top local cohomology module \(H^d_{\mathfrak{m}}(R)\). However, we think that it should be defined by all local cohomology modules. Moreover, our’s definition is more suitable with \(F\)-singularities. Indeed, it is clear that \(R\) is \(F\)-injective if and only if \(HSL(R) = 0\).

\(^3\)If \(R\) is Cohen-Macaulay, then Katman and Sharp [16] showed that \(Fte(R)\) is just \(HSL(R)\). In this paper, they posed the question that whether \(Fte(R)\) is an integer for any local ring \(R\). This question holds true for generalized Cohen-Macaulay rings, but it is still open in general. Recently, the author proved that \(Fte(R) \geq HLS(R)\) for any local ring \(R\).
every parameter ideal of $m^{[p^e]}$ is standard. This condition is equivalent to the condition that $F^e(R)$ is a Buchsbaum $R$-module provided $R$ is $F$-finite. I thank Nguyen Cong Minh for the following question.

**Question 4.** Let $(R, m, k)$ be a generalized Cohen-Macaulay local ring of characteristic $p > 0$. Does there exist an integer $e$ (that is bounded above by the Frobenius invariants of $R$ such as $HSL(R)$ and $Fte(R)$) such that $m^{[p^e]}$ is a standard ideal of $R$.

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