CHEM SIMONS THEORY AND THE VOLUME OF 3-MANIFOLDS

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ABSTRACT. We give some applications of the Chern Simons gauge theory to the study of the set vol(N, G) of volumes of all representations ρ: πN → G, where N is a closed oriented three-manifold and G is either Iso0 SL2(R), the isometry group of the Seifert geometry, or Iso0 H3, the orientation preserving isometry group of the hyperbolic 3-space. We focus on three natural questions arising from the definition of vol(N, G):

1. How to find non-zero values in vol(N, G)? or weakly how to find non-zero elements in vol(N, G) for some finite cover N of N?
2. Do these volumes satisfy the covering property in the sense of Thurston?
3. What kind of topological information is enclosed in the elements of vol(N, G)?

By various methods of computations, involving relations between the volume of representations and the Chern-Simons invariants of flat connections, we are able to give several meaningful results related the questions above.

We determine vol(N, G) when N supports the Seifert geometry, and we find some non-zero values in vol(N, G) for certain 3-manifolds with non-trivial geometric decomposition for either G = Iso0 H3 or Iso0 SL2(R). Moreover we will show that unlike the Gromov simplicial volume, these non-zero elements carry the gluing information between the geometric pieces of N.

For a large class 3-manifolds N, including all rational homology 3-spheres, we prove that N has a positive Gromov simplicial volume if it admits a finite covering N with vol(N, Iso0 H3) ≠ {0}. On the other hand, among such class, there are some N with positive simplicial volume but vol(N, Iso0 H3) = {0}, yielding a negative answer to question (2) for hyperbolic volume.

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1. INTRODUCTION

The volume of representations of 3-manifolds groups is a beautiful theory which has rich connections with many branches of mathematics. However the behavior of those volume functions seem still quite mysterious. To make our meaning more explicit, we first give some basic notions (which will be defined later) and properties of the volume of representations. Let \( N \) be a closed oriented 3-manifold. Let \( G \) be either \( \text{PSL}(2; \mathbb{C}) = \text{Iso}_+ \mathbb{H}^3 \), the orientation preserving isometry group of the hyperbolic 3-space, or \( \text{Iso}_e \widetilde{\text{SL}}_2(\mathbb{R}) \), the identity component of the isometry group of \( \widetilde{\text{SL}}_2(\mathbb{R}) \). For each representation \( \rho : \pi_1 N \rightarrow G \), the volume of \( \rho \) is denoted by \( \text{vol}_G(N, \rho) \). We denote by \( \text{vol}(N, G) \) the set
\[
\{ \text{vol}_G(N, \rho), \text{ when } \rho \text{ runs among the representations } \rho : \pi_1 N \rightarrow G \}
\]
Suppose \( N \) supports the hyperbolic, resp. \( \widetilde{\text{SL}}_2(\mathbb{R}) \), geometry. Then \( N \) naturally has its own hyperbolic volume \( \text{vol}_{\mathbb{H}^3}(N) \), resp. Seifert volume \( \text{vol}_{\widetilde{\text{SL}}_2(\mathbb{R})}(N) \). We denote by \( \|N\| \) the Gromov simplicial volume of \( N \), which measures, up to a multiplicative constant, the hyperbolic volume of the hyperbolic pieces of \( N \) (see [14]). The following theorem contains basic results of the theory of volume representations. For its development, see [4], [5], [30], [29] and their references.

**Theorem 1.1.** Let \( N \) be a closed oriented 3-manifold.

1. Both \( \text{vol}(N, \text{PSL}(2; \mathbb{C})) \) and \( \text{vol}(N, \text{Iso}_e \widetilde{\text{SL}}_2(\mathbb{R})) \) contain at most finitely many values and we denote by \( \text{HV}(N) \) and \( \text{SV}(N) \) the maximum value for \( \text{PSL}(2; \mathbb{C}) \) and \( \text{Iso}_e \widetilde{\text{SL}}_2(\mathbb{R}) \) respectively.

2. Suppose \( N \) supports a hyperbolic geometry. Then \( \text{vol}_{\mathbb{H}^3}(N) \) is reached by \( \text{vol}_{\text{PSL}(2; \mathbb{C})}(N, \rho) \) for some discrete and faithful representation. The similar statement is still true when \( N \) supports an \( \widetilde{\text{SL}}_2(\mathbb{R}) \)-geometry.

3. \( \text{vol}_{\text{PSL}(2; \mathbb{C})}(N, \rho) \leq \mu_3 \|N\| \), where \( \mu_3 \) denotes the volume of any ideal regular tetrahedron in \( \mathbb{H}^3 \).

4. Let \( f : M \rightarrow N \) be a map of degree \( d \) and let \( \rho : \pi_1 N \rightarrow G \) denote a representation. Then we get a representation \( \rho \circ f_* : \pi_1 M \rightarrow G \) such that \( \text{vol}_G(\rho \circ f_*, M) = d\text{vol}_G(\rho, N) \).
Accordingly this yields the classical inequalities

\[ HV(M) \geq |\text{deg} f|HV(N) \quad \text{and} \quad SV(M) \geq |\text{deg} f|SV(N) \]

**Remark 1.2.** Recall that a prime 3-manifold \( N \) admits no self-map of degree \( > 1 \) if and only if either \( N \) has a non-trivial geometric decomposition, or supports an \( \widehat{\text{SL}}_2(\mathbb{R}) \) or a hyperbolic geometry. This fact combined with Theorem 1.1 (2), (3), (4) implies that if \( \text{vol} \left( N, \text{Iso}_e \widehat{\text{SL}}_2(\mathbb{R}) \right) \neq \{0\} \) then necessarily either the geometric decomposition of \( N \) is non-trivial, or \( N \) supports an \( \widehat{\text{SL}}_2(\mathbb{R}) \) or a hyperbolic geometry and if \( \text{vol} \left( N, \text{PSL}(2; \mathbb{C}) \right) \neq \{0\} \) then necessarily \( N \) contains some hyperbolic piece(s).

Besides Theorem 1.1, Thurston pointed out the relation between Chern-Simon’s invariants and the hyperbolic volume of hyperbolic 3-manifolds for discrete and faithful representations [34]. Later such a relation is extended in [24] for cusped hyperbolic 3-manifolds and discrete and faithful representations into \( \text{PSL}(2; \mathbb{C}) \), and by [23] for closed manifolds with the group \( \widehat{\text{SL}}_2(\mathbb{R}) \) (as a subgroup of \( \text{Iso}\widehat{\text{SL}}_2(\mathbb{R}) \)).

Despite those significant results, the questions below, which is a main motivation of this paper, seems still remarkably unknown.

**Question 1.** (1) (i) How to find non-zero elements in \( \text{vol} (N, G) \)? or weakly,

(ii) How to find non-zero elements in \( \text{vol} (\widetilde{N}, G) \) for some finite cover \( \widetilde{N} \) of \( N \)?

(2) Does \( HV \) or \( SV \) satisfy the so-called ”covering property” in the sense of Thurston?

(3) What kind of topological information of \( N \) is captured by the non-zero elements in \( \text{vol} (N, G) \)?

We recall that a non-negative 3-manifolds invariant \( \eta \) satisfies the covering property, if for any finite covering \( p : \widetilde{N} \to N \), we have \( \eta(\widetilde{N}) = |\text{deg}(p)|\eta(N) \).

**Remark 1.3.** Three-manifold invariants with the covering property was first addressed by Thurston in [22, Problem 3.16]. The simplicial volume has the covering property (Gromov-Thurston-Soma). Some papers sought invariants with covering property for graph manifolds, as in [36].

So far it seems that we only know that \( HV \), resp. \( SV \), satisfies the covering property for the hyperbolic, resp. Seifert, manifolds. In hyperbolic geometry this property comes from the relation between the simplicial volume and \( HV \). In Seifert geometry one can compute \( SV \) in terms of the Euler classes of the Seifert manifold and the Euler characteristic of its orbifold and these invariants behave naturally under covering maps.

From now on the manifolds are assumed closed oriented and irreducible.

### 1.1. Volumes of Seifert manifolds.

The works of Brooks-Goldman [5], Milnor, Wood [27, 37] and Eisenbud-Hirsch-Neumann [12], allow us to describe the set \( \text{vol} \left( N, \text{Iso}_e \widehat{\text{SL}}_2(\mathbb{R}) \right) \) for each 3-manifold \( N \) supporting an \( \widehat{\text{SL}}_2(\mathbb{R}) \)-geometry.

It is known that \( N \) supports an \( \widehat{\text{SL}}_2(\mathbb{R}) \)-geometry if and only if \( N \) is a Seifert manifold with non-zero Euler number \( e(N) \) over an orbifold of negative Euler characteristic. As in [12] we use \( \lfloor a \rfloor \) and \( \lceil a \rceil \) for \( a \in \mathbb{R} \) to denote respectively, the greatest integer \( \leq a \) and the least integer \( \geq a \).
Proposition 1.4. Suppose $\hat{N}$ supports the $\tilde{\text{SL}}_2(\mathbb{R})$-geometry and that its base $2$-orbifold has a positive genus $g$. Then

$$\text{vol} \left( N, \text{Iso}\tilde{\text{SL}}_2(\mathbb{R}) \right) = \left\{ \frac{4\pi^2}{|e(N)|} \left( \sum_{i=1}^{r} \left( \frac{n_i}{a_i} \right) - n \right)^2 \right\}$$

where $n_1, \ldots, n_r, n$ are integers such that

$$\sum_{i=1}^{r} n_i/a_i - n \leq 2g - 2, \quad \sum_{i=1}^{r} n_i/a_i - n \geq 2 - 2g$$

and $a_1, \ldots, a_r$ are the indices of the singular points of the orbifold of $N$.

Remark 1.5. (1) In order to check Proposition 1.4, we will describe all representations with non-zero volume (see Proposition 4.2). They will be used in the volume computations for 3-manifolds with non-trivial geometric decompositions.

(2) Proposition 1.4 presents explicitly the rationality of the elements in $\text{vol}(N, \text{Iso}\tilde{\text{SL}}_2(\mathbb{R}))$, which was proved in [29].

1.2. Volumes of non-geometric manifolds. As a partial answer to Question 1 (1) for non-geometric manifolds, it was known only recently that each non-trivial graph manifold $N$ has a finite cover $\tilde{N}$ such that $\text{vol}(\tilde{N}, \text{Iso}\tilde{\text{SL}}_2(\mathbb{R}))$ contains non-zero elements, see [8]. Thus Question 1 (1) (ii) is reduced to the non-geometric 3-manifolds containing some hyperbolic pieces. In view of Theorem 1.1 (2) (3), as well as the result of [8], and in an attempt to seal a relation between the Gromov simplicial volume and the hyperbolic volume, Professor M. Boileau and some others wondered the following more direct version of Question 1 (1):

Question 2. Suppose $N$ has positive simplicial volume, i.e., $N$ contain some hyperbolic geometric piece(s).

(i) Is there a representation $\rho: \pi_1 N \to \text{PSL}(2, \mathbb{C})$ with positive volume?

(ii) or weakly is there a representation $\rho: \pi_1 \tilde{N} \to \text{PSL}(2, \mathbb{C})$ with positive volume for some finite covering $\tilde{N}$ of $N$?

Let $N$ be a closed irreducible non-geometric 3-manifold. Let $T_N \subset N$ denote the minimal union of disjoint essential tori and Klein bottles of $N$, unique up to isotopy, such that each piece of $N \setminus T_N$ is either Seifert or hyperbolic (see section 2.1). We say that $N$ is an one-edged manifold if $T_N$ consists of a single separating torus $T$.

The next two propositions respectively gives a negative answer to Question 2 (i) and a partially positive answer to Question 2 (ii).

Proposition 1.6. Let $N$ be a 3-manifold containing a hyperbolic piece $Q$ whose each boundary component that is non-separating in $N$ is shared by a Seifert piece of $N$. Then there is a representation $\rho: \pi_1 \tilde{N} \to \text{PSL}(2, \mathbb{C})$ with positive volume for some finite covering $\tilde{N}$ of $N$.

If each component of $T_N$ is separating in $N$, then the condition in Proposition 1.6 is automatically satisfied, and in particular

Corollary 1.7. A rational homology sphere has a positive simplicial volume iff it admits a finite covering with positive hyperbolic volume.

In the opposite direction we state:
Proposition 1.8. There are infinitely many 1-edged 3-manifolds $N$ with non-vanishing $|\|N\||$ but $\text{vol}(N, \text{PSL}(2; \mathbb{C})) = \{0\}$.

By the definition of 1-edged manifold, another immediate consequence of Proposition 1.6 and Proposition 1.8 is a negative answer of Question 1 (2) for hyperbolic volume, that is to say:

Corollary 1.9. The hyperbolic volume do not have the covering property. Namely there are finite coverings $p: \tilde{N} \to N$ such that $\text{HV}(\tilde{N}) > |\text{deg} p|\text{HV}(N) = 0$.

1.3. Volumes as Chern Simons invariants. The difficulty of Question 1 more or less can be seen from the definition: to get a non-zero element in $\text{vol}(N, G)$ we need first to find an a priori "significant" representation $\rho: \pi_1 N \to G$, and then to be able to compute its volume. Usually non of those steps are easy, especially when the manifold is not geometric. Basically in the geometric case, there is a natural significant representation given by the faithful and discrete representation of its fundamental group in the Lie group of its geometry. In the non-geometric case one can use the geometry of its pieces and try to construct "components after components" a global significant representation. However in this new situation many problems occur: First the geometric pieces have non-empty boundary and the volume of representation is not easy to manipulate in the non-closed case and moreover we must make sure that the local representations are compatible in the toral boundaries in order to be glued together. Then an other problem arises when we want to compute the volume of a global representation from the local volumes. Unlike the Gromov simplicial volume it is certainly non-additive with respect to the geometric decomposition. This latter point is a difficulty but somehow it is also a chance that this volume would take into account the way the geometric pieces are glued together. In order to prove Propositions 1.8 and 1.6 as well as Propositions 1.14 and 1.15 (see paragraph 1.4), the volume of representations will be turned into Chern Simons invariants, with certain semi-simple (non-compact) Lie groups, yielding computations that were not easy before.

Denote by $G$ the semi-simple Lie group $\text{Iso}_{e}\widetilde{\text{SL}}_2(\mathbb{R})$ or $\text{PSL}(2, \mathbb{C})$ with the associated Riemannian homogeneous spaces $X$ which is $\text{SL}_2(\mathbb{R})$ or $\mathbb{H}^3$ endowed with the closed $G$-invariant volume form $\omega_X$.

Denote by $g$ its Lie algebra. We recall (see Section 5 for more details) that the Chern Simons classes with structure group $\text{PSL}(2, \mathbb{C})$ are based on the first Pontrjagin class and in the same way we define the Chern Simons classes with structure group $\text{Iso}_{e}\widetilde{\text{SL}}_2(\mathbb{R})$ based on the invariant polynomial defined by $R(A \otimes A) = \text{Tr}(X^2) + t^2$ where $A$ is an element of the Lie algebra of $\text{Iso}_{e}\widetilde{\text{SL}}_2(\mathbb{R})$ which decomposes into $X + t$ where $X$ is in the Lie algebra of $\widetilde{\text{SL}}_2(\mathbb{R})$ and $t \in \mathbb{R}$.

Proposition 1.10. Let $\rho$ be a representation of $\pi_1 N$ into $G = \text{Iso}_{e}\text{SL}_2(\mathbb{R})$ and let $A$ be a corresponding flat $G$-connection in the principal bundle $P = N \times_\rho G$. If $P$ admits a section $\delta$ over $N$ then

\begin{equation}
\text{cs}_N(A, \delta) = \int_N \delta^* R \left( dA \wedge A + \frac{1}{3} A \wedge [A, A] \right) = \frac{2}{3} \text{vol}_G(N, \rho)
\end{equation}

In particular the Chern-Simons invariant of flat $\text{Iso}_{e}\text{SL}_2(\mathbb{R})$-connections is gauge invariant.

Remark 1.11. Assuming that $P = M \times_\rho G$ admits a section means equivalently that $\rho$ admits a lift into $\text{SL}_2(\mathbb{R})$ so that the bundle admits a reduction to an $\text{SL}_2(\mathbb{R})$-bundle and
we reckon that the correspondence in Proposition 1.10 for \( G = \text{SL}_2(\mathbb{R}) \) is pointed in [29], and verified in [23] by a long and subtle computation. However for our own understanding we reprove it in a very simple way underscoring that the correspondence is quite natural and comes directly from the structural equations of the Lie group involved (see Section 5.4).

The following correspondence is derived from [24]. Let’s denote the imaginary part of the complex number \( z \) by \( \Im(z) \).

**Proposition 1.12.** Let \( \rho \) be a representation of \( \pi_1 N \) into \( G = \text{PSL}(2; \mathbb{C}) \) and let \( A \) be a corresponding flat \( G \)-connection over \( N \). If \( N \times_\rho G \) admits a section \( \delta \) over \( N \) then

\[
\Im((\text{cs}_N(A, \delta)) = -\frac{1}{\pi^2} \text{vol}_G(N, \rho)
\]

**Remark 1.13.** The imaginary part of the Chern Simons invariants of flat \( \text{PSL}(2; \mathbb{C}) \) -connections is gauge invariant from the formula for it does not depend on the chosen section. We don’t give any geometric interpretations for the real part of \( \text{cs}_M(A, \delta) \). It is not gauge invariant and it won’t be used throughout our proofs. If the developing map \( D_\rho: \tilde{M} \rightarrow \mathbb{H}^3 \), corresponding to \( \rho: \pi_1 M \rightarrow G \), were an isometry with respect to the pull-backed metric of \( \mathbb{H}^3 \), then certainly \( \Re(\text{cs}_M(A, \delta)) \) would be the \( \mathbb{R}/\mathbb{Z} \)-valued Chern-Simons invariant of the Levi Civita connection corresponding to the Riemannian metric in \( \tilde{M} \) pull-backed from the hyperbolic metric by \( D^* \). This is correct when \( M \) is itself a complete hyperbolic manifold and when \( \rho \) is a faithful discrete representation, as it is stated in [24].

### 1.4. Complexity of the sewing involution.

Recall that each 3-manifold with non-trivial geometric decomposition is determined by the topology of its geometric pieces and the isotopy class of the gluing maps among them. At this point it is worth recalling that the simplicial volume \( \| \star \| \) tells nothing about the gluing. However both \( \text{vol}(N, \text{PSL}(2; \mathbb{C})) \) and \( \text{vol}(N, \text{IsoSL}_2(\mathbb{R})) \) somehow do contain the gluing information.

To illustrate this fact and to avoid complicated computations, below, we often focus on the simplest 3-manifolds with non-trivial geometric decomposition, namely the 1-edged manifolds. We present them as \( N = Q_- \cup_T Q_+ \), where \( \tau: \partial Q_- = T_- \rightarrow \partial Q_+ = T_+ \) is the gluing map of their two geometric pieces \( Q_- \) and \( Q_+ \).

Recall that \( N \) is termed a graph manifold if both \( Q_- \) and \( Q_+ \) are Seifert. On the other hand a finite covering \( p: \tilde{N} \rightarrow N \) is termed \( q \times q \)-characteristic, for some integer \( q \), if for any component \( \tilde{T} \) over \( T \) the map \( p \) induces the covering \( p|: \tilde{T} \rightarrow T \) corresponding to the subgroup \( q\mathbb{Z} \oplus q\mathbb{Z} \) of \( \mathbb{Z} \oplus \mathbb{Z} = \pi_1 T \).

We fix a basis \( s_\varepsilon, h_\varepsilon \) for \( H_1(\partial Q_\varepsilon; \mathbb{Z}), \varepsilon = \pm \), so that the isotopy class of \( \tau \) is given by an integral matrix \( \tau_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with determinant \(-1\) and

\[
\tau(s_-) = as_+ + ch_+ \quad \text{and} \quad \tau(h_-) = bs_+ + dh_+.
\]

Moreover, we choose the preferred basis:

(A) when \( Q_\varepsilon \) is a Seifert manifold, we require that \( T_\varepsilon(s_\varepsilon, h_\varepsilon) \) is a section-fiber basis (see section 2.3 for the definition). Notice that when both \( Q_- \) and \( Q_+ \) are Seifert manifolds, then \( b \neq 0 \), otherwise \( N \) would be a Seifert manifold.

(B) when \( Q_\varepsilon \) is a hyperbolic piece, we require that \( T_\varepsilon(s_\varepsilon, h_\varepsilon) \) is a basis that consists of the first and second shortest simple closed geodesic on its Euclidean boundary on the maximal cusp (see section 2.2). Under this basis, the norm of a curve \( as_\varepsilon + bh_\varepsilon \) defined
by $\sqrt{a^2 + b^2}$ is equivalent to the Euclidean norm in the cusp. Let us denote by $0 < k \leq K$ the real constants such that for any $a, b \in \mathbb{Z}$ then

$$k \sqrt{a^2 + b^2} \leq \text{length}(as + bh) \leq K \sqrt{a^2 + b^2}$$

See Remark 2.2 for more details about this construction.

**Proposition 1.14.** Let $N = Q_- \cup Q_+$ be an one-edged graph manifold and denote by $G$ the group $\text{Iso}_c \text{SL}_2(\mathbb{R})$. There exists a $n$-fold $q \times q$-characteristic covering $\tilde{N} \to N$, where $n, q$ depend only on $Q_-$ and $Q_+$, and a representation $\varphi : \pi_1 \tilde{N} \to G$ such that

$$\text{vol}_G(\tilde{N}, \varphi) = 8\pi^2 \frac{n}{q^2} \text{ if } a = d = 0, \quad \text{vol}_G(\tilde{N}, \varphi) = 4\pi^2 \frac{n}{q^2|b|} \text{ if } c = 0,$$

$$\text{vol}_G(\tilde{N}, \varphi) = 4\pi^2 \frac{n}{q^2|ac|} \text{ if } ac \neq 0, \quad \text{vol}_G(\tilde{N}, \varphi) = 4\pi^2 \frac{n}{q^2|cd|} \text{ if } cd \neq 0.$$

There exists a uniform constant $C = 2\pi/k$ such that for any one-cusped hyperbolic 3-manifold $Q$ then a deformation of the hyperbolic structure on $Q$ can be extended to a complete hyperbolic one in the surgered manifold $Q(a, b)$ provided $\| (a, b) \|_2 > C$ where $a, b$ are the co-prime coefficients of the curve of $\partial Q$ (in the chosen basis) identified with the meridian of the solid torus (see paragraph 2.2).

**Proposition 1.15.** Let $N = Q_- \cup Q_+$ be an one-edged 3-manifold, where $Q_-$ is Seifert and $Q_+$ is hyperbolic. Then there exists a $n$-fold $q \times q$-characteristic covering $\tilde{N} \to N$, where $n, q$ depend only on $Q_-$ and $Q_+$, and a representation $\varphi : \pi_1 \tilde{N} \to \text{PSL}(2; \mathbb{C})$ such that for any $\| (a, c) \|_2 > C$ then

$$\text{vol}_{\text{PSL}(2; \mathbb{C})}(\tilde{N}, \varphi) = n \text{vol}_{Q_+}(a, c) + \frac{\pi n(q - 1)}{2q} \text{length}(\gamma)$$

where $\gamma$ is the geodesic added to $Q_+$ to complete the cusp with respect to the $(a, c)$-Dehn filling and $\text{length}(\gamma)$ denotes its length in the complete hyperbolic structure of $Q_+, (a, c)$. The same statement is true for $(b, d)$.

**Remark 1.16.** By the computations made in [28] we get

$$\text{vol}_{Q_+}(a, c) = \text{vol}_{Q_+} - \frac{\pi}{2} \text{length}(\gamma) + O \left( \frac{1}{a^4 + c^4} \right)$$

where

$$\text{length}(\gamma) = 2\pi \frac{\Im(z_0)}{|a + z_0|c|^2} + O \left( \frac{1}{a^4 + c^4} \right)$$

where $z_0$ is a complex number with $\Im(z_0) > 0$ giving the modulus of the Euclidean structure on the torus $T$ corresponding to the cusp of $Q_+$. Substituting (1.6) and (1.7) into Proposition 1.15 we have by [28] Theorem 1A

$$\frac{\text{vol}_{\text{PSL}(2; \mathbb{C})}(\tilde{N}, \varphi)}{n} = \text{vol}_{Q_+} - \frac{\pi}{2} \frac{\Im(z_0)}{|a + z_0|c|^2} + O \left( \frac{1}{a^4 + c^4} \right)$$
1.5. Organization of the paper. The remaining of the paper is reflected from table of contents. Sections 2, 3, and 5 present necessary background and results from 3-manifold theory, volume of representations, and Chern-Simons theory respectively. The efforts are made in organizing those materials so that our results can be verified smoothly, and readers can access the topic easily. We will verify Propositions 1.4 in section 4, Proposition 1.10, 1.12 in sub-sections 5.4 and 5.5, Proposition 1.6 in Section 6, Proposition 1.8 in Sections 7, and Propositions 1.14, 1.15 in section 8 respectively.

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2. Topology and geometry of 3-manifolds

2.1. Thurston’s picture of 3-manifolds. Each closed orientable 3-manifold $N$ admits a unique prime decomposition $N_1#...#N_k$, the prime factors being unique up to the order and up to homeomorphisms.

Let $N$ be a compact orientable 3-manifold. Call an embedded surface $T$ in $N$ essential if $T$ is incompressible (see [33, page 23]) and is not parallel to any component of $\partial N$.

Let $N$ be a closed orientable prime 3-manifold. According to the theory of Thurston, Johannson and Jaco-Shalen ([33], [34], [20], [21]) combined with the geometrization of 3-manifolds achieved by G. Perelman and W. Thurston, then either

(i) $N$ supports one of the following eight geometries: $H^n, SL_2(\mathbb{R}), H^2 \times \mathbb{R}, Sol, Nil, \mathbb{R}^3, S^3$ and $S^2 \times \mathbb{R}$ (where $H^n$, $\mathbb{R}^n$ and $S^n$ are the $n$-dimensional hyperbolic space, Euclidean space and the sphere respectively); in these cases $N$ is called geometric; or

(ii) there is a minimal union $\mathcal{T}_N \subset N$ of disjoint essential tori and Klein bottles of $N$, unique up to isotopy, such that each piece of $N \setminus \mathcal{T}_N$ is either a Seifert fibered manifold, supporting the $H^3 \times \mathbb{R}$-geometry, or the $H^3$-geometry. We call the components of $N \setminus \mathcal{T}_N$ the geometric pieces (Seifert pieces and hyperbolic pieces respectively).

Call a prime closed orientable 3-manifold $N$ a (non-trivial) graph manifold if $N$ has a (non-trivial) geometric decomposition but contains no hyperbolic pieces.

2.2. Thurston Hyperbolic Dehn filling Theorem. We denote by $H^n$ the $n$-dimensional hyperbolic space that can be seen as the half-space $\mathbb{R}^{n-1} \times (0, \infty) \subset \mathbb{R}^n - \mathbb{R}$ endowed with the metric $(dx_1^2 + ... + dx_n^2)/x_n^2$. The orientations preserving isometry group $\text{Iso}_+(H^n)$ of $H^n$ can be metrically thought of as the oriented unit frame bundle over $H^n$ and algebraically it is identified with $\text{PSL}(2; \mathbb{R})$ when $n = 2$ and $\text{PSL}(2; \mathbb{C})$ when $n = 3$.

Let $M$ denote a compact, orientable 3-manifold whose boundary consists of tori $T_1, ..., T_p$ and whose interior admits a complete (finite volume) hyperbolic structure. In other words, $\text{int}M$ is isometric to $H^3 / \Gamma$, where $\Gamma$ is a discrete, torsion free subgroup of $\text{PSL}(2; \mathbb{C})$. For each $T_i \subset \partial M$ we fix a basis $\mu_i, \lambda_i$ of $H_1(T_i; \mathbb{Z})$. We define the surgered manifold $M((a_1, b_1), ..., (a_p, b_p))$ as follows:

1. If $(a_i, b_i) = (\infty, \infty)$ then $T_i$ is left unfilled.

2. If $(a_i, b_i)$ are coprime then we perform a $(a_i, b_i)$-Dehn filling on $T_i$ by gluing the solid torus $V = \mathbb{D}^2 \times S^1$, with basis $(m, l)$ so that the meridian $m$ is identified with the isotopy class of the simple closed curve $a_i \mu_i + b_i \lambda_i$ in $T_i$.

3. Otherwise, denote by $q_i > 1$ the greatest common divisor of $a_i$ and $b_i$ so that $a_i = q_i r_i$ and $b_i = q_i s_i$ with $(r_i, s_i)$ coprime. We denote by $R_{q_i}$ the rotation of order $q_i$ centered
at 0 in \( \mathbb{R}^2 \), by \( D^2 \) the closed unit 2-disk and by \( D^2/R_q \), the singular disk under the action of the group generated by \( R_q \). Then we glue the singular solid torus \( V(q_i) = D^2/R_q \times S^1 \) so that its meridian \( m \) is identified with the isotopy class of the simple closed curve \( r_i \mu_i + s_i \lambda_i \) in \( T_i \). Let’s recall the Thurston’s Hyperbolic Dehn surgery theorem, [33, 5.8.2] (see also [3] and [9]).

**Theorem 2.1.** Let \( M \) be compact oriented 3-manifold with toral boundary \( T_1 \cup \ldots \cup T_p = \partial M \) whose interior admits a complete hyperbolic structure. Then there is a real number \( C > 0 \) such that if \( \| (a_i, b_i) \| > C \) for \( i = 1, \ldots, p \), then the surgered space \( M((a_1, b_1), \ldots, (a_p, b_p)) \) is a complete hyperbolic orbifold.

The complete hyperbolic metric on \( M \) corresponds to a faithful, discrete representation \( \rho_{d_0} : \pi_1 M \to \text{PSL}(2; \mathbb{C}) \) such that, up to conjugation, \( \rho_{d_0}(\mu_i) = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) and \( \rho_{d_0}(\lambda_i) = \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) \). Theorem 2.1 claims that if each \( (a_i, b_i) \) satisfies \( \sqrt{a_i^2 + b_i^2} > C \), for \( i = 1, \ldots, p \), then the former representation may be modified such that, up to conjugation,

\[
\rho_d(\mu_i) = \begin{pmatrix} e^{2\pi \alpha_i} & 0 \\ 0 & e^{-2\pi \alpha_i} \end{pmatrix} \quad \text{and} \quad \rho_d(\lambda_i) = \begin{pmatrix} e^{2\pi \beta_i} & 0 \\ 0 & e^{-2\pi \beta_i} \end{pmatrix}
\]

and the induced structure on \( \text{int} M \) is a complete hyperbolic orbifold that can be completed in the surgered hyperbolic orbifold \( M((a_1, b_1), \ldots, (a_p, b_p)) \). Moreover Thurston’s Theorem shows that if there exists \( i \) such that \( (a_i, b_i) \neq \infty \) then

\[
\lim_{(a_1, b_1) \to \infty, \ldots, (a_p, b_p) \to \infty} \text{vol}_{\mathbb{H}^3} M((a_1, b_1), \ldots, (a_p, b_p)) = \text{vol}_{\mathbb{H}^3} M
\]

**Remark 2.2.** We denote by \( M_{\text{max}} \) the interior of \( M \) with a system of maximal cusps removed. Now we identify \( M \) with \( M_{\text{max}} \), then \( \partial M \) has a Euclidean metric induced from the hyperbolic metric and each closed Euclidean geodesic in \( \partial M \) has the induced length. The so called \( 2\pi \)-lemma claims that \( M((a_1, b_1), \ldots, (a_p, b_p)) \) is hyperbolic if the geodesic corresponding to \( (a_i, b_i) \) has length \( > 2\pi \) for \( i = 1, \ldots, p \) (see [2] for example). The first and the second shortest simple geodesics on each component \( T_i \) of \( \partial M \) must form a basis, and under this basis the norm of a curve \( (a, b) \) defined by \( \sqrt{a^2 + b^2} \) is equivalent to the Euclidean length in \( T_i \) (see page 309) for example). So under such a basis, there is a universal constant \( C \) such that, for any one cusped hyperbolic manifold, if \( \sqrt{a_i^2 + b_i^2} > C \), then \( M((a_i, b_i)) \) is hyperbolic.

### 2.3. **Seifert geometry.**

We consider the group \( \text{PSL}(2; \mathbb{R}) \) as the orientation preserving isometries of the hyperbolic 2-space \( \mathbb{H}^2 = \{ z \in \mathbb{C}, \Im(z) > 0 \} \) with \( i \) as a base point. In this way \( \widehat{\text{PSL}(2; \mathbb{R})} \) is a (topologically trivial) circle bundle over \( \mathbb{H}^2 \). Denote by \( p : \widehat{\text{SL}_2(\mathbb{R})} \to \text{PSL}(2; \mathbb{R}) \) the universal covering of \( \text{PSL}(2; \mathbb{R}) \) with the induced metric. Then \( \widehat{\text{SL}_2(\mathbb{R})} \) is a (topologically trivial) line bundle over \( \mathbb{H}^2 \). For any \( \alpha \in \mathbb{R} \), denote by \( \text{sh}(\alpha) \) the element of \( \widehat{\text{SL}_2(\mathbb{R})} \) whose projection into \( \text{PSL}(2; \mathbb{R}) \) is given by

\[
\left( \begin{array}{cc} \cos(2\pi \alpha) & \sin(2\pi \alpha) \\ -\sin(2\pi \alpha) & \cos(2\pi \alpha) \end{array} \right).
\]

Then the set \( \{ \text{sh}(\alpha), \alpha \in \mathbb{Z} \} \), is the kernel of \( p \) as well as the center of \( \widehat{\text{SL}_2(\mathbb{R})} \), acting by integral translation along the fibers of \( \widehat{\text{SL}_2(\mathbb{R})} \). By extending this \( \mathbb{Z} \)-action on the fibers to the \( \mathbb{R} \)-action we get the whole identity component of
the isometry group of $\widetilde{SL_2(\mathbb{R})}$. To summarize we have the following diagram of central extensions

$$
\begin{array}{cccc}
\{0\} & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{SL_2(\mathbb{R})} & \longrightarrow & \text{PSL}(2; \mathbb{R}) & \longrightarrow & \{1\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & \text{ISO}_e(\widetilde{SL_2(\mathbb{R})}) & \longrightarrow & \text{PSL}(2; \mathbb{R}) & \longrightarrow & \{1\}
\end{array}
$$

In particular the group $\text{Iso}_e(\widetilde{SL_2(\mathbb{R})})$ is generated by $\widetilde{SL_2(\mathbb{R})}$ and the image of $\mathbb{R}$ which intersect together in the image of $\mathbb{Z}$.

**Remark 2.3.** The Lie group $\text{Iso}_e(\widetilde{SL_2(\mathbb{R})})$ can be thought of as a quotient of the direct product $\mathbb{R} \times \widetilde{SL_2(\mathbb{R})}$ - (where each element $x$ on $\mathbb{R}$ is naturally identified with the translation $\tau_x$ of length $x$) - under the relation $(x, h) \sim (x', h')$ if and only if there exists an integer $n \in \mathbb{Z}$ such that $x' - x = n$ and $h' = \text{sh}(-n) \circ h$. In this way $\text{Iso}_e(\widetilde{SL_2(\mathbb{R})})$ can be identified with $\mathbb{R} \times \mathbb{Z} \widetilde{SL_2(\mathbb{R})}$ and it is easily checked that this group is homotopic to the circle.

Let $F_{g,n}$ be an oriented $n$-punctured surface of genus $g \geq 0$ with boundary components $s_1, \ldots, s_n$ with $n \geq 0$. Then $N' = F_{g,n} \times S^1$ is oriented if $S^1$ is oriented. Let $h_i$ be the oriented $S^1$-fiber on the torus $T_i = s_i \times h_i$. We say that $(s_i, h_i)$ is a section-fiber basis of $T_i$. Let $0 \leq s \leq n$. Now attach $s$ solid tori $V_i'$s to the boundary tori $T_i$'s of $N'$ such that the meridian of $V_i$ is identified with the slope $a_i s_i + b_i h_i$ where $a_i > 0$, $(a_i, b_i) = 1$ for $i = 1, \ldots, s$. Denote the resulting manifold by $\left(g, n - s; \frac{b_1}{a_1}, \ldots, \frac{b_s}{a_s}\right)$ which has the Seifert fiber structure extended from the circle bundle structure of $N'$. Each orientable Seifert fibered space with orientable base $F_{g,n-s}$ and with $\leq s$ exceptional fibers is obtained in such a way. If $N$ is closed, i.e. if $s = n$, then define the Euler number of the Seifert fibration by

$$e(N) = \sum_{i=1}^{s} \frac{b_i}{a_i} \in \mathbb{Q}$$

and the Euler characteristic of the orbifold $O(N)$ by

$$\chi_{O(N)} = 2 - 2g - \sum_{i=1}^{s} \left(1 - \frac{1}{a_i}\right) \in \mathbb{Q}.$$ 

From [5] we know that a closed orientable 3-manifold $N$ supports the $\widetilde{SL_2}$-geometry, i.e. there is a discrete and faithful representation $\psi : \pi_1 N \to \text{ISO}_{\text{SL}_2}$, if and only if $N$ is a Seifert manifold with non-zero Euler number $e(N)$ and negative Euler characteristic $\chi_{O(N)}$.

### 3. Volume of Representations of Closed Manifolds

Given a semi-simple, connected Lie group $G$ and a closed oriented manifold $M^n$ of the same dimension than the contractible space $X^n = G/K$, where $K$ is a maximal compact subgroup of $G$, we can associate, to each representation $\rho : \pi_1 M \to G$, a volume $\text{vol}_{\rho}(M, \rho)$ in the following ways.
3.1. Volume of representations. First of all, fix a $G$-invariant Riemannian metric $g_X$ on $X$, then denote by $\omega_X$ the corresponding $G$-invariant volume form. We think of the elements $\tilde{x}$ of $\tilde{M}$ as the homotopy classes of paths $\gamma : [0, 1] \to M$ with $\gamma(0) = x_0$ which are acted by $\pi_1(M, x_0)$ by setting $[\sigma] \tilde{x} = [\sigma \cdot \gamma]$, where $\cdot$ denotes the paths composition.

A developing map $D_\rho : \tilde{M} \to X$ associated to $\rho$ is a $\pi_1 M$-equivariant map from the universal covering $\tilde{M}$ of $M$, acted by $\pi_1 M$, to $X$, endowed with the $\pi_1 M$ action induced by $\rho : \pi_1 M \to G$. That is to say: for any $x \in \tilde{M}$ and $\alpha \in \pi_1 M$, then

$$D_\rho(\alpha \cdot x) = \rho(\alpha)^{-1} D_\rho(x)$$

Such a map does exist and can be constructed explicitly as in [1]: Fix a triangulation $\Delta_M$ of $M$. Then its lift is a triangulation $\Delta_{\tilde{M}}$ of $\tilde{M}$, which is $\pi_1 M$-equivariant. Then fix a fundamental domain $\Omega$ of $M$ in $\tilde{M}$ such that the zero skeleton $\Delta_0 \tilde{M}$ misses the frontier of $\Omega$. Let $\{x_1, ..., x_l\}$ be the vertices of $\Delta_0 \tilde{M}$ in $\Omega$, and let $\{y_1, ..., y_l\}$ be any $l$ points in $X$. We first set

$$D_\rho(x_i) = y_i, \ i = 1, ..., l.$$

Next extend $D_\rho$ in an $\pi_1 M$-equivariant way to $\Delta_0 \tilde{M}$: For any vertex $x$ in $\Delta_0 \tilde{M}$, there is a unique vertex $x_i$ in $\Omega$ and $\alpha_x \in \pi_1 M$ such that $\alpha_x \cdot x_i = x$, and we set $D_\rho(x) = \rho(\alpha_x)^{-1} D_\rho(x_i)$. Finally we extend $D_\rho$ to edges, faces, ..., and $n$-simplices of $\Delta_{\tilde{M}}$ by straightening the images to geodesics using the homogeneous metric on the contractible space $X$. This map is unique up to equivariant homotopy. Then $D_\rho^* (\omega_X)$ is a $\pi_1 M$-invariant closed $n$-form on $\tilde{M}$ and therefore can be thought of as a closed $n$-form on $M$. Thus define

$$\text{vol}_G(M, \rho) = \left| \int_M D_\rho^* (\omega_X) \right| = \left| \sum_{i=1}^s \varepsilon_i \text{vol}_X(D_\rho(\Delta_i)) \right|$$

where $\{\Delta_1, ..., \Delta_s\}$ are the $n$-simplices of $\Delta_M$, $\Delta_i$ is a lift of $\Delta_i$, and $\varepsilon_i = \pm 1$ depending on whether $D_\rho|_{\Delta_i}$ is preserving or reversing orientation.

3.2. Volume of representations as a continuous cohomology class. Let $\alpha = \{K\}$ be the base point of $X = G/K$ and for any $g_1, ..., g_l \in G$ denote by $\Delta(g_1, ..., g_l)$ the geodesic $l$-simplex of $X$ with vertices $\{o, g_1(o), ..., g_l g_1(o)\}$. There is a natural homomorphism

$$H^*(g, \mathfrak{t}; \mathbb{R}) = H^*(G - \text{invariant differential forms on } X) \to H^{\text{cont}}_c(G; \mathbb{R})$$

defined in [10] by $\eta \mapsto (g_1, ..., g_l) \mapsto f_{\Delta(g_1, ..., g_l)} \eta$ which turns out to be an isomorphism by the Van-Est Theorem [35].

Recall that for each representation $\rho : \pi_1 M \to G$ one can associate a flat bundle over $M$ with fiber $X$ and group $G$ constructed as follows: $\pi_1 M$ acts diagonally on the product $\tilde{M} \times X$ by the following formula

$$(3.1) \quad \sigma(\tilde{x}, g) = (\sigma \tilde{x}, \rho^{-1}(\sigma) g)$$

and we can form the quotient $\tilde{M} \times_\rho X = \tilde{M} \times X / \pi_1 M$ which is the flat $X$-bundle over $M$ corresponding to $\rho$.

Then for each $G$-invariant closed form $\omega$ on $X$, $q^*(\omega)$ is a $\pi_1(M)$-invariant closed form on $\tilde{M} \times X$, where $q : \tilde{M} \times X \to X$ is the projection, which induces a form $\omega'$ on $M \times \rho X$. Then $s^*(\omega')$ is a closed form on $M$, where $s : M \to M \times \rho X$ is a section.
(since $X$ is contractible, the sections exist and are all homotopic). Thus any representation $\rho: \pi_1 M \to G$ leads to a natural homomorphism
\[ \rho^*: H^*_\text{cont}(G; \mathbb{R}) = H^*(G - \text{invariant differential forms on } X) \to H^*(M; \mathbb{R}) \]
induced by $\rho^*(\omega) = s^*\omega'$. The volume of $\rho$ is therefore defined by $\text{vol}_G(M, \rho) = \int_M \rho^*(\omega_X)$.

The equivalence between the two definitions is immediate since the $\pi_1 M$-equivariant map $\text{Id} \times D\rho: \tilde{M} \to \tilde{M} \times X$ descends to a section $M \to M \times_\rho X$.

When the situation is clear from the context, we drop the reference to the structural group $G$ in the notation $\text{vol}_G(M, \rho)$ and we denote it by $\text{vol}(M, \rho)$.

3.3. Volume of representations via transversely projective foliations. Let $\bar{\mathfrak{F}}$ be a co-dimension one foliation on a closed smooth manifold $M$ determined by a 1-form $\omega$. Then by the Froebenius Theorem one has $d\omega = \omega \wedge \delta$ for some 1-form $\delta$. It was observed by Godbillon and Vey [13] that the 3-form $\delta \wedge d\delta$ is closed and the class $[\delta \wedge d\delta] \in H^3(M, \mathbb{R})$ depends only on the foliation $\bar{\mathfrak{F}}$ (and not on the chosen form $\omega$). This cohomology class is termed the Godbillon-Vey class of the foliation $\bar{\mathfrak{F}}$ and denoted by $GV(\bar{\mathfrak{F}})$.

**Proposition 3.1.** ([4] Proposition 1) Suppose $\bar{\mathfrak{F}}$ is a horizontal flat structure on a circle bundle $S^1 \to E \to M$ with structural group $\text{PSL}_2(\mathbb{R})$. Then
\[ \int_{S^1} GV(\bar{\mathfrak{F}}) = 4\pi^2 \bar{e}(E) \]
where $\int_{S^1}: H^3(E) \to H^2(M)$ denotes the integration along the fiber and $\bar{e}$ denotes the Euler class of the bundle.

Let $M$ be a closed orientable 3-manifold and $\phi: \pi_1 M \to \text{PSL}_2(\mathbb{R})$ be a representation with zero Euler class. Since $\text{PSL}_2(\mathbb{R})$ acts on $S^1$ then one can consider the corresponding flat circle bundle $M \times_\phi S^1$ over $M$ and the associated horizontal ($\text{PSL}_2(\mathbb{R}), S^1$)-foliation $\bar{\mathfrak{F}}_\phi$. Since the Euler class of $\phi$ is zero we can choose a section $\delta$ of $M \times_\phi S^1 \to M$. Brooks and Goldman (see [4] Lemma 2) showed that $\delta^* GV(\bar{\mathfrak{F}}_\phi)$ only depends on $\phi$ (and not on a chosen section $\delta$). Then they defined the Godbillon Vey invariant of $\phi$ by setting
\[ GV(\phi) = \int_M \delta^* GV(\bar{\mathfrak{F}}_\phi) \]

**Remark 3.2.** Assume that $M$ is already endowed with a $(\text{PSL}_2(\mathbb{R}), S^1)$-foliation $\bar{\mathfrak{F}}$. Then there is a canonical way to define a flat $S^1$-bundle $E$ over $M$ with structure group $\text{PSL}_2(\mathbb{R})$, a horizontal foliation $\bar{\mathfrak{F}}_\phi$ on $E$, and a section $s: M \to E$ such that $\bar{\mathfrak{F}} = s^* \bar{\mathfrak{F}}_\phi$. Then if $\phi$ denotes associated representation we get ([4] Lemma 1)
\[ GV(\phi) = \int_M GV(\bar{\mathfrak{F}}) \]

For a given representation $\phi: \pi_1 M \to \text{PSL}_2(\mathbb{R})$, $\phi$ lifts to $\bar{\phi}: \pi_1 M \to \tilde{\text{SL}_2(\mathbb{R})}$ if and only if $\bar{e}(\bar{\phi}) = 0$ in $H^2(M, \mathbb{Z})$. The following fact has been verified in [4].

**Proposition 3.3.** Let $M$ be a closed oriented 3-manifold, let $\phi: \pi_1 M \to \text{PSL}_2(\mathbb{R})$ be a representation with zero Euler class and fix a lift $\phi: \pi_1 M \to \tilde{\text{SL}_2(\mathbb{R})}$ of $\phi$. Then
\[ GV(\phi) = \text{vol}_{\tilde{\text{SL}_2(\mathbb{R})}}(M, \bar{\phi}) \]
where $\tilde{\text{SL}_2(\mathbb{R})}$ is viewed as a semi-simple Lie group acting on itself by multiplication with corresponding homogeneous space $\text{SL}_2(\mathbb{R})$. 
4. Seifert volume of representations of Seifert manifolds

This section is devoted to the proof of Proposition 1.4. Let \( N \) be a closed oriented \( SL_2(\mathbb{R}) \)-manifold whose base 2-orbifold is an orientable hyperbolic 2-orbifold \( \mathcal{O} \) with positive genus \( g \) and \( p \) singular points. Then, keeping the same notation as in section 2.3, we have a presentation

\[
\pi_1 N = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, s_1, \ldots, s_p, h : s_1^{a_1} h^{b_1} = 1, \ldots, s_p^{a_p} h^{b_p} = 1, [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = s_1 \ldots s_p \rangle
\]

with the condition \( e = \sum b_i/a_i \neq 0 \). The following result of Eisenbud-Hirsch-Neumann [12], which extends the result of Milnor-Wood’s (see [27] and [37]) from circle bundles to Seifert manifolds, is very useful for our purpose.

**Theorem 4.1.** Suppose \( N \) is a closed orientable Seifert manifold with a regular fiber \( h \) and base of genus \( g > 0 \). Then

1. ([12] Corollary 4.3) There is a \((PSL_2 \mathbb{R}, S^1)\)-horizontal foliation on \( N \) if and only if there is a representation \( \tilde{\phi} : \pi_1(N) \to SL_2 \mathbb{R} \) such that \( \tilde{\phi}(h) = \text{sh}(1) \);
2. ([12] Theorem 3.2 and Corollary 4.3) Suppose \( N = (g, 0; \alpha_1/b_1, \ldots, \alpha_n/b_n) \), then there is a \((PSL_2 \mathbb{R}, S^1)\) horizontal foliation on \( N \) if and only if

\[
\sum b_i/a_i \leq -\chi(F_g); \quad \sum r b_i/a_i - n \geq 2 - 2g
\]

In order to prove Proposition 1.4 we will check the following Proposition which describes those representations leading to a non zero volume. For each element \((a, b) \in \mathbb{R} \times SL_2(\mathbb{R})\), its image in \( \mathbb{R} \times SL_2(\mathbb{R}) \) will be denoted as \((a, b)\).

**Proposition 4.2.** A representation \( \rho : \pi_1(N) \to Iso(\widetilde{SL_2(\mathbb{R})}) = \mathbb{R} \times SL_2(\mathbb{R}) \) has non-zero volume if there are integers \( n, n_1, \ldots, n_p \) subject to the conditions

\[
\sum n_i/a_i - n \leq 2g - 2 \quad \text{and} \quad \sum r n_i/a_i - n \geq 2 - 2g
\]

such that

\[
\rho(s_i) = \left( \frac{n_i}{a_i} - \frac{b_i}{a_i} e \left( \sum_i \frac{n_i}{a_i} - n \right), g_i, \text{sh} \left( -\frac{n_i}{a_i} \right), g_i^{-1} \right)
\]

where \( g_i \) is an element of \( SL_2(\mathbb{R}) \) and

\[
\rho(h) = \left( \frac{1}{e} \left( \sum_i \left( \frac{n_i}{a_i} - n \right) \right), 1 \right)
\]

whose volume is given by

\[
\text{vol}(N, \rho) = 4\pi^2 \frac{1}{|e|} \left( \sum_i \left( \frac{n_i}{a_i} - n \right) \right)^2
\]

Moreover the \( \rho \)-image of \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \) can be chosen to lie in \( \widetilde{SL_2(\mathbb{R})} \).

**Proof.** The condition \( \text{vol}(N, \rho) \neq 0 \) implies that \( \rho(h) = (\zeta, 1) \in G = \mathbb{R} \times \widetilde{SL_2(\mathbb{R})} \) by [4] page 663] and [5] page 537], using a cohomological-dimension argument and the definition in paragraph 3.2. Suppose \( \rho(s_i) = (z_i, x_i) \). Then \( s_i^{a_i} h^{b_i} = 1 \) implies that

\[
(a_i z_i, x_i^{a_i})(b_i \zeta, 1) = (a_i z_i + b_i \zeta, x_i^{a_i}) = 1
\]
Then there is an \( n_i \in \mathbb{Z} \) such that (see Remark 2.3)

\[
(4.5) \quad a_i z_i + b_i \zeta = n_i \text{ in } \mathbb{R} \text{ and } x_i \text{ is conjugate in } \widetilde{\text{SL}}_2(\mathbb{R}) \text{ to } \text{sh} \left( -\frac{n_i}{a_i} \right)
\]

Since \([\alpha_1, \beta_1], \ldots, [\alpha_g, \beta_g] = s_1 \ldots s_p\) and since the product of commutators in \( \mathbb{R} \times \mathbb{Z} \widetilde{\text{SL}}_2(\mathbb{R}) \) must lie in \( \text{SL}_2(\mathbb{R}) \) this implies that

\[
(z_1 + \ldots + z_p, x_1 \ldots x_p) = \left( 0, \prod_{j=1}^{g} [\rho(\alpha_j), \rho(\beta_j)] \right)
\]

Then there is an \( n \in \mathbb{Z} \) such that

\[
(4.6) \quad z_1 + \ldots + z_p = n \text{ and } \prod_{j=1}^{g} [\rho(\alpha_j), \rho(\beta_j)] = x_1 \ldots x_p \text{sh}(n)
\]

Equalities (4.6) and (4.5), imply condition (4.1) in Proposition 4.2 using Theorem 4.1 and its proof in [12]. By (4.5) and (4.6), we can calculate directly

\[
(4.7) \quad z_i = \frac{n_i}{a_i} - \frac{b_i}{a_i} \zeta, \quad \zeta = \frac{1}{e} \left( \sum_{i=1}^{p} \frac{n_i}{a_i} - n \right)
\]

Plugging (4.5), (4.6) and (4.7) into \( \rho(h) = (\zeta, 1) \) and \( \rho(s_i) = (z_i, x_i) \), we obtain (4.2) and (4.3) in Proposition 4.2. Then the “moreover” part of Proposition 4.2 also follows from Theorem 4.1.

Let’s now compute the volume of such a representation. Let \( p_1 : \tilde{N} \to N \) be a covering from a circle bundle \( \tilde{N} \) over \( F \) to \( N \) so that the fiber degree is 1. Then we have

\[
\tilde{e} = e(\tilde{N}) = (\deg p_1)e
\]

Let \( \tilde{t} \) be the fiber of \( \tilde{N} \) and \( \tilde{\rho} = \rho|\pi_1 \tilde{N} \). Then \( (\tilde{t})^g = \prod_{j=1}^{g} [\tilde{\alpha}_j, \tilde{\beta}_j] \) in \( \pi_1 \tilde{N} \), and therefore \( \tilde{\rho}((\tilde{t})^g) = (e\tilde{\zeta}, 1) \in Z(G) \cap \widetilde{\text{SL}}_2(\mathbb{R}) \), since the image of the fibre must lie in the center and the image of the product of commutators must lie in \( \text{SL}_2(\mathbb{R}) \). Hence \( e\tilde{\zeta} = \tilde{n} \in \mathbb{Z} \).

Let \( p_2 : \tilde{N} \to N \) be the covering along the fiber direction of degree \( \tilde{e} \), and then \( \tilde{e} = e(\tilde{N}) = 1 \). Then \( \tilde{\rho} = \tilde{\rho} \) sends actually \( \pi_1 \tilde{N} \) into \( \widetilde{\text{SL}}_2(\mathbb{R}) \) and the fiber \( \tilde{t} \) of \( \tilde{N} \) is sent to \( \text{sh}(\tilde{n}) \). And finally there is a covering \( p_* : \tilde{N} \to N^* \) along the fiber direction of degree \( \tilde{n} \), and where \( N^* \) is a circle bundle over a hyperbolic surface \( F \) with \( e^* = e(N^*) = \tilde{n} \).

It is apparent that \( \tilde{\rho} \) descends to \( \rho^* : \pi_1 N^* \to \text{SL}_2(\mathbb{R}) \) such that \( \rho^*(h^*) = \text{sh}(1) \), where \( h^* \) denotes the \( S^1 \)-fiber of \( N^* \). According to Theorem 4.1 there is a \( (\text{PSL}_2(\mathbb{R}), S^1) \)-horizontal foliation on \( N^* \), and according to Proposition 3.1 \( \text{vol}(N^*, \rho^*) = 4\pi^2 e^* = 4\pi^2 \tilde{n}^* \), and then

\[
\text{vol}(\tilde{N}, \tilde{\rho}) = 4\pi^2 \tilde{n}^2 = 4\pi^2 e^2 \tilde{\zeta}^2.
\]

Note that

\[
\deg p_1 \deg p_2 = \frac{\tilde{e}}{e} \times \tilde{e} = \frac{\tilde{e}^2}{e}
\]

By those facts we reach (4.4) as below:

\[
\frac{\text{vol}(N, \rho)}{\deg p_1 \deg p_2} = \frac{4\pi^2 e^2 \zeta^2}{e} = 4\pi^2 e \zeta^2 = 4\pi^2 \left( \sum_{i=1}^{p} \frac{n_i}{a_i} - n \right)^2
\]

\[\square\]
Remark 4.3. Suppose in Proposition 4.2 that \( n_i = a_ik_i + r_i \), where \( 0 \leq r_i < a_i \). If we choose \( n = 2 - 2g + \sum k_i \) and \( n_i = (k_i + 1)a_i - 1 \) then the corresponding representation \( \rho_0 \) is faithful, discrete and reaches the maximal volume giving rise to the well known formula

\[
\text{vol}(N, \rho_0) = 4\pi^2 \frac{1}{|e(N)|} \chi_2(N)
\]

5. A Brief Review on the Chern-Simons Theory

Throughout this section we refer to [6] and [25]. In this part, all the objects we deal with are smooth. Let \( \pi: P \to M \) denote a principal \( G \)-bundle over a closed manifold \( M \). Suppose that \( G \) is a Lie group acting on the right on \( P \) and denote by \( R_g \) the right action

\[
P \ni x \mapsto x.g \in P
\]

where \( g \) is an element of \( G \). Denote by \( \mathfrak{g} \) the Lie algebra of \( G \). Let \( VP \) be the vertical subbundle of \( TP \).

5.1. Differential forms taking values in a Lie algebra. We denote by \( \Omega^k(P; \mathfrak{g}) \) the set of differential \( k \)-forms taking values in \( \mathfrak{g} \). We define the exterior product of \( \omega_1 \in \Omega^k(P; \mathfrak{g}) \) by \( \omega_2 \in \Omega^l(P; \mathfrak{g}) \) as an element \( \omega_1 \wedge \omega_2 \) of \( \Omega^{k+l}(P; \mathfrak{g} \otimes \mathfrak{g}) \) by setting

\[
\omega_1 \wedge \omega_2(X_1, ..., X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \text{sign}(\sigma) \omega_1(X_{\sigma(1)}, ..., X_{\sigma(k)}) \otimes \omega_2(X_{\sigma(k+1)}, ..., X_{\sigma(l)})
\]

The Lie bracket \([., .] \) in \( \mathfrak{g} \) induces a map \( \Omega^k(P; \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^k(P; \mathfrak{g}) \) and we denote by \([\omega_1, \omega_2]\) the image of \( \omega_1 \wedge \omega_2 \) under this map. Explicitly we get:

\[
[\omega_1, \omega_2](X_1, ..., X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \text{sign}(\sigma) [\omega_1(X_{\sigma(1)}, ..., X_{\sigma(k)}), \omega_2(X_{\sigma(k+1)}, ..., X_{\sigma(l)})]
\]

The differential \( d: \Omega^k(P; \mathfrak{g}) \to \Omega^{k+1}(P; \mathfrak{g}) \) is defined by the Cartan formula

\[
d\omega(X_1, ..., X_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} X_i \omega(X_1, ..., \widehat{X_i}, ..., X_{k+1}) + \\
\frac{1}{k+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_{k+1})
\]

5.2. Connections on principal bundles. The derivative at the identity 1 of \( G \) of the map

\[
G \ni g \mapsto x.g \in P
\]

induces an isomorphism \( \nu_x: \mathfrak{g} \to V_x P \subset T_x P \) and we get the exact sequence

\[
0 \to \mathfrak{g} \xrightarrow{\nu_x} T_x P \xrightarrow{d\pi_x} T_{\pi(x)} M \to 0
\]

A horizontal subbundle \( HP \) of \( TP \) is a smooth distribution such that \( T_x P = V_x P \oplus H_x P \) for any \( x \in P \) that is \( G \)-equivariant: \( H_{x.g} = dR_g(x)H_x \). This is given equivalently by the kernel of an element \( \omega \in \Omega^1(P; \mathfrak{g}) \) such that for any \( x \in P \)

1. \( \omega_x \circ \nu_x = \text{Id}_\mathfrak{g} \)
2. \( R^*_g \omega = \text{Ad}_{g^{-1}}(\omega) \).
An element of $\Omega^1(P; \mathfrak{g})$ satisfying (1) and (2) is termed a connection of $P$. Denote by $\mathcal{A}(P)$ the space of all connections on $P$. This space is naturally acted by the gauge group denoted by $G_P$ consisting of the $G$-equivariant bundle automorphisms of $P$.

The basic example is the group $G$ itself, viewed as a trivial bundle over a point or more generally the trivialized bundle $M \times G$ with the so-called Maurer-Cartan connection $\omega_{M,\text{C.}} = d(L_{g^{-1}} \circ \pi_2)$, where $L_g$ denotes the left translation in $G$ and $\pi_2$ the projection of $P$ onto $G$. This connection satisfies the Maurer Cartan equation, namely

$$d\omega_{M,\text{C.}} = -\frac{1}{2}[\omega_{M,\text{C.}}, \omega_{M,\text{C.}}]$$

Let’s make a concrete computation for $G$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Since $\mathfrak{g}$ can be thought of as the space of left invariant vector fields in $G$, its dual $\mathfrak{g}^*$ is the space of left invariant differential 1-forms on $G$. Let $\theta^1, \ldots, \theta^n$ denote the dual basis of $\mathfrak{g}^*$. Then

$$\omega_{M,\text{C.}} = \theta^1 \otimes X_1 + \cdots + \theta^n \otimes X_n$$

Let us write the constants structure of $\mathfrak{g}$ which are given by the formula

$$[X_j, X_k] = \sum_i c_{jk}^i X_i$$

Thus by the Maurer-Cartan equation we get the equalities

$$(5.1) \quad d\theta^i = -\frac{1}{2} \sum_{j, k} c_{jk}^i \theta^j \wedge \theta^k$$

In general, for a given connection $\omega$ in a bundle $P$, the element

$$(5.2) \quad F^\omega = d\omega + \frac{1}{2}[\omega, \omega]$$

is the curvature of $\omega$ lying in $\Omega^2(P; \mathfrak{g})$ and it measures the integrability of the corresponding horizontal distribution. When $F^\omega = 0$ we say that the connection is flat. Denote by $\mathcal{F}\mathcal{A}(P)$ the subset of $\mathcal{A}(P)$ which consists of flat connections on $P$. This subspace is preserved by the gauge group action.

We recall the following basic fact that will be used very often in this paper. To each flat connection $\omega$ one can associate a representation $\rho: \pi_1 M \to G$ by lifting the loops of $M$ in the leaves of the horizontal foliation given by integrating the distribution $\ker \omega$.

On the other hand $\omega$ can be recovered from $\rho$ by the following construction. The fundamental group of $M$ acts on the product $\tilde{M} \times G$ by the formula $[\sigma, ([\gamma], g)] = ([\sigma\gamma], \rho([\sigma\gamma]^{-1}) g)$ and the quotient $\tilde{M} \times_G G$ under this $\pi_1 M$-action is isomorphic to $P$ and the push forward of the vertical distribution of $\tilde{M} \times G$ in $\tilde{M} \times_G G$ corresponds to $\omega$ in $P$. We get a natural map

$$I_P: \mathcal{B}(P) = \mathcal{F}\mathcal{A}(P)/G_P \hookrightarrow \mathcal{R}(\pi_1 M, G)/\text{conjugation}$$

where $\mathcal{R}(\pi_1 M, G)$ is the set of representations of $\pi_1 M$ into $G$ acted by the conjugation in $G$. Notice that this map is usually non-surjective.

5.3. The Chern Simons classes. Given a Lie group $G$, a polynomial of degree $l$ is a symmetric linear map $f: \otimes^l \mathfrak{g} \to \mathbb{K}$, where $\mathbb{K}$ denotes either the real or the complex numbers field. The group $G$ acts on $\mathfrak{g}$ by Ad and the polynomials invariant under this action are called the invariant polynomials of degree $l$ and are denoted by $I^l(G)$ with the convention $I^0(G) = \mathbb{K}$. Denote $I(G)$ the sum $\oplus_{l \in \mathbb{N}} I^l(G)$.

The Chern Weil theory gives a correspondence $W_P$ from $I^l(G)$ to $H^{2l}(M; \mathbb{K})$ constructed in the following way. Choose a connection $\omega$ in $P$ then for any $l \geq 1$ a polynomial
are the complex valued invariant polynomials such that
that is unique up to homotopy. There exists the universal Chern Weil homomorphism
explicitly in \([6]\) by

\[
(5.3) \quad Tf(\omega) = l \int_0^1 f(\omega \wedge (\Lambda^{l-1} F^t))dt
\]

where \(F^t = tF^\omega + \frac{1}{2}(t^2 - t)[\omega, \omega]\). The form \(Tf(\omega)\) is closed when \(M\) is of dimension
\(2l - 1\). For instance when \(l = 2\) and \(M\) is a 3-manifold, plugging \(F^t\) and (5.2) into (5.3)
we get a closed 3-form on \(P\), namely

\[
(5.4) \quad Tf(\omega) = f(F^\omega \wedge \omega) - \frac{1}{6}f(\omega \wedge [\omega, \omega]) = f(d\omega \wedge \omega) + \frac{1}{3}f(\omega \wedge \omega)
\]

Considering \(G\) as a principal bundle over the point this yields to

\[
Tf(\omega_{M.C.}) = -\frac{1}{6}f(\omega_{M.C.} \wedge [\omega_{M.C.}, \omega_{M.C.}])
\]

The \((2l - 1)\)-form \(Tf(\omega_{M.C.})\) is closed, bi-invariant and defines a class in \(H^{2l-1}(G; \mathbb{R})\).
Let’s denote by

\[
I_0(G) = \{ f \in I(G), Tf(\omega_{M.C.}) \in H^{2l-1}(G; \mathbb{Z}) \}
\]

The elements of \(I_0(G)\) are termed integral polynomials. If \(f \in I_0(G)\) then there is a well
defined functional

\[
(5.5) \quad cs^*_M: \mathcal{A}_M \times G \to \mathbb{K}/\mathbb{Z}
\]

defined as follows: since \(P = M \times G\) is a trivial(ized) we can consider, for any section \(\delta\),
the Chern Simons invariant

\[
(5.6) \quad cs_M(\omega, \delta) = \int_M \delta^*Tf(\omega)
\]

Since \(f\) is an integral polynomial, the element \(cs_M(\omega, \delta)\) is well defined modulo \(\mathbb{Z}\) when
the section changes. Then define \(cs^*_M(\omega)\) to be the class of \(cs_M(\omega, \delta)\) in \(\mathbb{K}/\mathbb{Z}\).

The fundamental classical examples are \(G = SU(2; \mathbb{C})\) and \(G = SO(3; \mathbb{R})\).

The Chern Simons classes for the group \(SU(2; \mathbb{C})\) are based on the second Chern class
\(f = C_2 \in I^2_2(SL(2; \mathbb{C}))\). We recall that the Chern classes, denoted by \(C_1, C_2\) for \(SU(2; \mathbb{C})\),
are the complex valued invariant polynomials such that

\[
det \left( \lambda I_2 - \frac{1}{2i\pi} A \right) = \lambda^2 + C_1(A)\lambda + C_2(A \otimes A)
\]
when $A \in \mathfrak{sl}_2(\mathbb{C})$. Thus after developing this equality we get

$$C_2(A \otimes A) = \frac{1}{8\pi^2} \text{tr}(A^2)$$

so that we get the usual formula (using (5.4))

$$\frac{1}{8\pi^2} \text{Tr} \left( \frac{d\omega}{\omega} \right) = \frac{1}{8\pi^2} \text{Tr} \left( d\omega \right) = \frac{1}{8\pi^2} \text{Tr} \left( \frac{1}{3} \omega \wedge [\omega, \omega] \right)$$

The Chern-Simons classes of the special orthogonal group $G = SO(3; \mathbb{R})$ are based on the first Pontrjagin class $f = P_1 \in H^2(SO(3; \mathbb{R}))$ that is a real valued invariant polynomial such that

$$\det \left( \lambda J_3 - \frac{1}{2\pi} A \right) = \lambda^3 + P_1(A \otimes A) \lambda$$

when $A \in \mathfrak{so}_3(\mathbb{R})$. Thus after developing this equality we get

$$P_1(A \otimes A) = -\frac{1}{8\pi^2} \text{tr}(A^2)$$

Example 5.1. When $M$ is an oriented Riemannian closed $n$-manifold one can consider its associated $SO(n; \mathbb{R})$-bundle $SO(M)$ which consists of the positive orthonormal unit frames endowed with the Levi Civita connection. When $M$ is of dimension 3 it is well known that its is parallelizable so that there exist sections $\delta$ of $SO(M) \rightarrow M$. Therefore one can consider the Chern-Simons invariant of the Levi Civita connection on $M$ that will be denoted by $cs_{L,M}(\omega, \delta)$. 

A natural question arises in the following situation. There is an epimorphism $\pi_2: SU(2; \mathbb{C}) \rightarrow SO(3; \mathbb{R})$ that is the 2-fold universal covering. Thus any connection $\omega$ on the trivialized $SU(2; \mathbb{C})$-bundle over $M$ induces a connection $\omega'$ on the corresponding $SO(3; \mathbb{R})$-bundle over $M$. How can we compute $TP_1(\omega')$ form $TC_2(\omega)$? The answer is given in [24, pp 543, end of Section 3] by recalling that $\pi_2$ induces a homomorphism between the corresponding classifying spaces

$$\pi_2^*: H^4(BSO(3; \mathbb{R})) \rightarrow H^4(BSU(2; \mathbb{C}))$$

such that

$$\pi_2^* \tilde{W}(P_1) = -4\tilde{W}(C_2)$$

Thus using the definition and the Chern Weil universal homomorphism we get the equality

(5.8) $cs_{M}(\omega', \delta') = -4cs_{M}(\omega, \delta)$

where $\delta$ is a fixed section in the $SU(2; \mathbb{C})$-bundle over $M$ and $\delta'$ is the corresponding section in the $SO(3; \mathbb{R})$-bundle over $M$. On the other hand since $G = SO(3; \mathbb{R})$, resp. $SU(2; \mathbb{C})$, are the maximal compact subgroup of $PSL(2; \mathbb{C})$, resp. $SL(2; \mathbb{C})$, whose quotients $PSL(2; \mathbb{C})/SO(3; \mathbb{R})$, resp. $SL(2; \mathbb{C})/SU(2; \mathbb{C})$ are contractible then it follows from [17][Chapter 15, Theorem 3.1] and [11][Proposition 7.2, p. 98] that the natural inclusion gives rise to isomorphisms $H^*(BPSL(2; \mathbb{C})) \rightarrow H^*(BSO(3; \mathbb{R}))$ and
$H^*(BSL(2; \mathbb{C})) \to H^*(BSU(2; \mathbb{C}))$. We have the following commutative diagram

$$
H^*(BPSL(2; \mathbb{C})) \xrightarrow{\sim} H^*(BSO(3; \mathbb{R}))
$$

$$
H^*(BSL(2; \mathbb{C})) \xrightarrow{\sim} H^*(BSU(2; \mathbb{C}))
$$

Hence we also get (fixing a trivialization, using (5.6), (5.7), (5.8))

$$
c_{\delta M}(\omega', \delta') = -4c_{\delta M}(\omega, \delta)
$$

where $\delta$ is a fixed section in the $\text{Isom}_e(\text{SL}_2(\mathbb{R}))$-bundle over $M$ and $\delta'$ is the corresponding section in the $\text{PSL}(2; \mathbb{C})$-bundle over $M$.

5.4. **Volume and Chern Simons classes in Seifert geometry.** In this section we check Proposition 11.10 keeping the same notation as in the introduction. The proof is inspired from [5, p. 532] and we will follow faithfully their presentation. If $G = \text{Isom}_e(\text{SL}_2(\mathbb{R}))$ then the matrices

$$
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

together with the generator $T$ of $\mathbb{R}$ form a basis of the Lie algebra $\mathfrak{g}$ of $G = \text{Isom}_e(\text{SL}_2(\mathbb{R}))$. Setting $W = Z - Y - T$ we get a new basis $\{X, Y, Z, W\}$ of $\mathfrak{g}$ with commutators relations

$$
[X, Y] = -2Y, [X, Z] = 2Z,
$$

$$
[Y, Z] = [Y, W] = [Z, W] = -X, [X, W] = 2Y + 2Z
$$

which determine the coefficients in the Maurer Cartan equations. Denote by $\varphi_X, \varphi_Y, \varphi_Z, \varphi_W$ the dual basis of $\mathfrak{g}^*$. The Maurer Cartan form of $G$ is given by

$$
\omega_{\text{M.C.}} = \varphi_X \otimes X + \varphi_Y \otimes Y + \varphi_Z \otimes Z + \varphi_W \otimes W
$$

Denote by $A$ a flat connection on $M \times G$. By section 5.2, if $\tilde{M}$ denotes the universal covering and if $q: \tilde{M} \times G \to G$ denotes the projection then $A$ corresponds to the form $q^*(\omega_{\text{M.C.}})$, where $-: \tilde{M} \times G \to \tilde{M} \times G$ denotes the push-forward which makes sense since $q^*(\omega_{\text{M.C.}})$ is $\pi_1M$-invariant. The Chern Simons class of the flat connection $A$ is $\text{TR}(A) = q^*\text{TR}(\omega_{\text{M.C.}})$. Using equations (5.1) and (5.10), we calculate

$$
d\varphi_X = \varphi_Y \wedge \varphi_Z + \varphi_Y \wedge \varphi_W + \varphi_Z \wedge \varphi_W
$$

$$
d\varphi_Y = 2\varphi_X \wedge \varphi_Y - 2\varphi_X \wedge \varphi_W
$$

$$
d\varphi_Z = -2\varphi_X \wedge \varphi_Z - 2\varphi_X \wedge \varphi_W
$$

$$
d\varphi_W = 0
$$

Notice that those equations also imply that $2(\varphi_X \wedge \varphi_Y + \varphi_X \wedge \varphi_Z) = d(\varphi_Y - \varphi_Z)$ and therefore

$$
\text{TR}(\omega_{\text{M.C.}}) = \frac{2}{3} \varphi_X \wedge \varphi_Y \wedge \varphi_Z + \frac{1}{3} d(\varphi_Y \wedge \varphi_W - \varphi_Z \wedge \varphi_W)$$
The end of the proof follows from the commutativity of the diagram below and from the Stokes formula, since \( \phi_X \wedge \phi_Y \wedge \phi_Z \) represents the volume form on \( X = SL_2(\mathbb{R}) \).

\[
\begin{array}{c}
G \xrightarrow{\varphi_X} X \\
\downarrow \varphi_X & \downarrow \varphi_X \\
\tilde{M} \times G \xrightarrow{\tilde{z}} \tilde{M} \times X \\
\downarrow \pi & \downarrow \pi \\
M \times_\rho G \xrightarrow{\zeta} M \times_\rho X \\
\downarrow \delta & \downarrow \delta \\
M
\end{array}
\]

This completes the proof of the proposition.

5.5. Volume and Chern Simons classes in Hyperbolic geometry. We now check Proposition 1.12. The following construction is largely inspired from [24, p. 553-556], using a formula established by Yoshida in [38].

Denote by \( p: PSL(2; \mathbb{C}) \simeq Iso_+ \mathbb{H}^3 \to \mathbb{H}^3 \) the natural projection. For short denote \( PSL(2; \mathbb{C}) \) by \( G \). For each representation \( \rho: \pi_1 M \to G \) admitting a lift into \( SL(2; \mathbb{C}) \), we have the (trivial) principal bundle \( M \times_\rho G \) and the associated bundle \( M \times_\rho \mathbb{H}^3 \). Denote by \( A \) the flat connection over \( M \) corresponding to \( \rho \) and \( \omega_{\mathbb{H}^3} \) the \( G \)-invariant volume form on \( \mathbb{H}^3 \) corresponding to the hyperbolic metric.

The matrices \( X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) form a basis of the Lie algebra \( \mathfrak{sl}(2; \mathbb{C}) \) with commutators relations

\[
[X, Y] = -2Y, [X, Z] = 2Z, [Y, Z] = -X
\]

Denote by \( \varphi_X, \varphi_Y, \varphi_Z \) the dual basis of \( \mathfrak{sl}^*(2; \mathbb{C}) \). The Maurer Cartan form of \( G \) is

\[
\omega_{M.C.} = \varphi_X \otimes X + \varphi_Y \otimes Y + \varphi_Z \otimes Z
\]

and

\[
TP_1(\omega_{M.C.}) = \frac{1}{\pi^2} \varphi_X \wedge \varphi_Y \wedge \varphi_Z
\]

By the formula of Yoshida in [38] we know that

\[
iTP_1(\omega_{M.C.}) = \frac{1}{\pi^2} p^* \omega_{\mathbb{H}^3} + i cs_{L.C.}(\mathbb{H}^3) + d\gamma
\]

where \( p^* \omega_{\mathbb{H}^3} \) is the pull-back of \( \omega_{\mathbb{H}^3} \) under the projection \( p: PSL(2; \mathbb{C}) \to \mathbb{H}^3 \), \( cs_{L.C.}(\mathbb{H}^3) \) is the Chern Simons 3-form of the Levi Civita connection over \( \mathbb{H}^3 \) (see example 5.1) endowed with the hyperbolic metric and \( d\gamma \) is an exact real form. Let’s consider
the following commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{p} & H^3 \\
\downarrow{qG} & & \downarrow{qH^3} \\
\tilde{M} \times G & \xrightarrow{p} & \tilde{M} \times H^3 \\
\downarrow{\delta} & & \downarrow{\delta} \\
M \times_{\rho} G & \xrightarrow{p} & M \times_{\rho} H^3 \\
\end{array}
\]

Notice that the sections in the bottom triangle are obtained as follows. Since \( M \) is a 3-manifold then it follows from the obstruction theory that any principal bundle with simply connected group is trivial. Since \( p: \pi_1 M \rightarrow G \) admits a lift into \( SL(2; \mathbb{C}) \) then \( M \times_{\rho} G \) is trivial. So denote by \( \delta \) a section of \( M \times_{\rho} G \rightarrow M \). It induces, by \( p \circ \delta = s \), a section of \( M \times_{\rho} H^3 \rightarrow M \).

Since all the maps are clear from the context, in the sequel, we will drop the index in the projections \( qG \) and \( qH^3 \) and we denote them just by \( q \). Now the 3-form \( \omega_{H^3} \) induces a 3-form \( q^* \omega_{H^3} \) on \( M \times_{\rho} H^3 \) and

\[
i q^* TP_1(\omega_{M,C.}) = \frac{1}{\pi^2} q^* p^* \omega_{H^3} + i q^* cs_{L.C.}(H^3) + q^* d\gamma
\]

in \( M \times_{\rho} G \), where the push-forward operation \( q^* \) indeed makes sense since \( TP_1(\omega_{M,C.}), p^* \omega_{H^3} \) and \( cs_{L.C.}(H^3) \) are left invariant forms in \( G \). Then

\[
ics_M(A, \delta) = \frac{1}{\pi^2} \int_M \delta^* q^* p^* \omega_{H^3} + i \int_M \delta^* q^* cs_{L.C.}(H^3) + \int_M \delta^* q^* d\gamma
\]

Since \( \delta^* q^* p^* \omega_{H^3} = \delta^* p^* q^* \omega_{H^3} = s^* q^* \omega_{H^3} \) and \( \int_M \delta^* q^* d\gamma = 0 \) by the Stokes formula, we have

\[
ics_M(A, \delta) = \frac{1}{\pi^2} \int_M s^* q^* \omega_{H^3} + i \int_M \delta^* q^* cs_{L.C.}(H^3) = \frac{1}{\pi^2} \text{vol}(M, \rho) + i cs(M_\rho; \delta),
\]

where we denote \( \int_M \delta^* q^* cs(H^3) \) by \( i cs(M_\rho; \delta) \). We get eventually

\[
ics_M(A, \delta) = cs(M_\rho; \delta) - \frac{i}{\pi^2} \text{vol}(M, \rho)
\]

5.6. Normal form near toral boundary of 3-Manifolds. In this part we recall the machinery developed in [24]. Let \( M \) be a compact oriented 3-manifold with connected and toral boundary \( T = \partial M \) endowed with a basis \( s, h \) of \( H_1(\partial M; \mathbb{Z}) \). Notice that we study the case of manifold with connected toral boundary only to simplify the notations, for all the results stated in this section extend naturally to compact 3-manifolds with non-connected toral boundary. Let \( \rho: \pi_1 M \rightarrow G \) be a representation where \( G \) is either \( PSL(2; \mathbb{C}) \) or \( SL(2; \mathbb{R}) \). We consider the space of flat connections \( \mathcal{F}(A(P)) \) where \( P \) is the trivialized bundle \( M \times G \). For representations into \( SL(2; \mathbb{R}) \) the corresponding principal bundles are always trivial whereas the representations \( \rho \) into \( PSL(2; \mathbb{C}) \) leading to a trivial bundle are precisely those who admit a lift \( \pi \) into \( SL(2; \mathbb{C}) \). Moreover if follows from [24] and [23] that after a conjugation, the representation \( \rho|\pi T \) can be put in normal
form, which either hyperbolic, elliptic or parabolic. Since the parabolic form won’t be used in the explicit way we only recall the definitions of those representations which are elliptic/hyperbolic in the \( \text{PSL}(2; \mathbb{C}) \)-case and elliptic in the \( \widetilde{\text{SL}}(2; \mathbb{R}) \)-case in the boundary of \( M \). By [24], when \( G = \text{PSL}(2; \mathbb{C}) \), we may assume, after conjugation, that there exist \( \alpha, \beta \in \mathbb{C} \) such that

\[
\rho(s) = \begin{pmatrix} e^{2i\pi \alpha} & 0 \\ 0 & e^{-2i\pi \alpha} \end{pmatrix} \quad \text{and} \quad \rho(h) = \begin{pmatrix} e^{2i\pi \beta} & 0 \\ 0 & e^{-2i\pi \beta} \end{pmatrix}
\]

When \( G = \widetilde{\text{SL}}(2; \mathbb{R}) \) we may assume that after projecting into \( \text{PSL}(2; \mathbb{R}) \) there exist \( \alpha, \beta \in \mathbb{R} \) such that, up to conjugation,

\[
s \mapsto \begin{pmatrix} \cos(2\pi \alpha) & \sin(2\pi \alpha) \\ -\sin(2\pi \alpha) & \cos(2\pi \alpha) \end{pmatrix} \quad \text{and} \quad h \mapsto \begin{pmatrix} \cos(2\pi \beta) & \sin(2\pi \beta) \\ -\sin(2\pi \beta) & \cos(2\pi \beta) \end{pmatrix}
\]

In either case, if \( A \) denotes a connection on \( P \) corresponding to \( \rho \) then after a gauge transformation \( g \) the connection \( g \ast A \) is in normal form:

\[
g \ast A[T \times [0, 1]] = (i \alpha dx + i \beta dy) \otimes X
\]

We quote the following result stated in [24, Lemma 3.3] with \( G = \text{SL}(2; \mathbb{C}) \) and in [23, Theorem 4.2] with \( G = \widetilde{\text{SL}}(2; \mathbb{R}) \), that will be used latter :

**Proposition 5.2.** Let \( A \) and \( B \) denote two flat connections in normal form over an oriented 3-manifold with toral boundary. If \( A \) and \( B \) are equal near the boundary and if they are gauge equivalent then

i) \( \varsigma_M(A, \delta) = \varsigma_M(B, \delta') \) when \( G = \widetilde{\text{SL}}(2; \mathbb{R}) \) and,

ii) \( \varsigma_M(A, \delta) - \varsigma_M(B, \delta') \in \mathbb{Z} \) when \( G = \text{PSL}(2; \mathbb{C}) \).

The second statement follows from [24, Lemma 3.3] using identity (5.9).

**Remark 5.3.** As a consequence of Proposition 5.2, if \( A \) and \( B \) are flat connections on a solid torus that are equal near the boundary then the associated representations are automatically conjugated so that the conclusion of the proposition applies.

Since the complex function \( z \mapsto e^{2i\pi z} \) identifies \( \mathbb{C}/\mathbb{Z} \) with \( \mathbb{C}^* \), it will be convenient to use rather the invariant (see also (5.5))

\[
\varsigma_M^*(A) = e^{2i\pi \varsigma_M(A, \delta)}
\]

when \( G = \text{PSL}(2; \mathbb{C}) \) and \( A \) is a connection in normal form, than \( \varsigma_M(A, \delta) \). Denote by \( \mathcal{E} \mathcal{R}_{\partial M}^0(\text{PSL}(2; \mathbb{C})) \) the image, up to conjugation, of the elliptic/hyperbolic representations of \( \pi_1 M \) into \( \text{SL}(2; \mathbb{C}) \) induced by the projection \( \text{SL}(2; \mathbb{C}) \to \text{PSL}(2; \mathbb{C}) \).

There is a natural map \( t: \mathbb{C}^2 \to \mathcal{E} \mathcal{R}_{\partial M}^0(\text{PSL}(2; \mathbb{C})) \) which sends \( (\alpha, \beta) \in \mathbb{C}^2 \) to the conjugation class of the representation inducing the homomorphism \( \mathbb{Z} \times \mathbb{Z} \to \text{PSL}(2; \mathbb{C}) \) defined by

\[
s \mapsto \begin{pmatrix} e^{2i\pi \alpha} & 0 \\ 0 & e^{-2i\pi \alpha} \end{pmatrix} \quad \text{and} \quad h \mapsto \begin{pmatrix} e^{2i\pi \beta} & 0 \\ 0 & e^{-2i\pi \beta} \end{pmatrix}
\]

on the boundary. Let \( \rho \) be an element of \( \mathcal{E} \mathcal{R}_{\partial M}^0(\text{PSL}(2; \mathbb{C})) \). Then the map \( \rho \mapsto (\alpha, \beta) \) is a lifting \( L \) of the restriction map \( r: \mathcal{E} \mathcal{R}_{\partial M}^0(\text{PSL}(2; \mathbb{C})) \to \mathcal{E} \mathcal{R}_{\partial M}^0(\text{PSL}(2; \mathbb{C})) \) such that the
following diagram commutes

$$L\quad\rightarrow\quad C^2$$

$$\mathcal{E}R^0_M(\text{PSL}(2;\mathbb{C}))\quad\stackrel{r}{\longrightarrow}\quad\mathcal{E}R^0_{\partial M}(\text{PSL}(2;\mathbb{C}))$$

To any $\rho$ in $\mathcal{E}R^0_M(\text{PSL}(2;\mathbb{C}))$ we associate, after fixing a lift $L$ of $r$, the triple

$$(\alpha, \beta, \text{cs}_M^*(A)) \in C^2 \times C^*$$

where $A$ is the connection over $M$ corresponding to $L(\rho)$ in normal form

$$A[\mathcal{I}] \times [0, 1] = (i\alpha dx + i\beta dy) \otimes X$$

near the boundary. Using [24 Theorem 2.5] with the group $\text{PSL}(2;\mathbb{C})$ instead of $\text{SU}(2;\mathbb{C})$ it turns out that if we replace $A$ by $B$ with a lift $(\alpha + 1/2, \beta, \text{cs}_M^*(B))$ or by $C$ with a lift $(\alpha, \beta + 1/2, \text{cs}_M^*(C))$ then we get the following equalities:

$$\text{cs}_M^*(B) = \text{cs}_M^*(A)e^{-4i\pi\beta} \quad \text{and} \quad \text{cs}_M^*(C) = \text{cs}_M^*(A)e^{4i\pi\alpha}$$

whereas $t(\alpha, \beta) = t(\alpha + 1/2, \beta) = t(\alpha, \beta + 1/2)$. We end this section by quoting the following result established in [24 Theorem 2.7] for the group $\text{SU}(2;\mathbb{C})$.

**Proposition 5.4.** Let $M$ be an oriented compact 3-manifold with toral boundary $\partial M = T$. Let $\rho_1 : \pi_1 M \rightarrow \text{PSL}(2;\mathbb{C})$ be a path of elliptic/hyperbolic representations in $\mathcal{E}R^0_M(\text{PSL}(2;\mathbb{C}))$. Choose a lifting $L(\rho_1) = (\alpha(t), \beta(t))$ of the restriction $r : \mathcal{E}R^0_M(\text{PSL}(2;\mathbb{C})) \rightarrow \mathcal{E}R^0_{\partial M}(\text{PSL}(2;\mathbb{C}))$ and denote by $A_t$ the corresponding path of flat connections. Then

$$\text{cs}_M^*(A_1) = \text{cs}_M^*(A_0) \exp\left(-8i\pi \int_0^1 (\alpha(t)\beta'(t) - \beta(t)\alpha'(t))dt\right)$$

As a corollary we quote the formula stated in [24 end of p. 555]:

**Corollary 5.5.** Let $V = D^2 \times S^1$ denote the solid torus and let $\rho : \pi_1 V \rightarrow \text{PSL}(2;\mathbb{C})$ denotes a representation. If $L(\rho) = (1/2, \beta)$ with respect to the meridian-longitude basis on $\partial V = \partial D^2 \times S^1$ then

$$\text{cs}_M^*(B) = \exp(-4i\pi\beta)$$

where $B$ is the flat connection over $V$ corresponding to the lifting $L(\rho)$.

**Proof.** We begin by computing $\text{cs}_M^*(A)$ where $A$ is a connection in normal form corresponding to the lift $(0, \beta) \in C^2$. To this purpose we consider the path of flat connections $\omega_t$ given by $(0, t\beta)$. Since $\omega_0$ is the trivial connections then $\text{cs}_M^*(\omega_0) = 1$ and Proposition 5.4 implies $\text{cs}_M^*(\omega_1) = 1$. Applying formula (5.16) once gives rise to $(1/2, \beta)$ with the corresponding connections whose $\text{cs}_M^*$ is equal to $e^{-4i\pi\beta}$. This proves the formula. □

5.7. **Additivity principle.** Fix a closed oriented 3-manifold $M$ and denote by $[M]$ its orientation class. Let $T$ be a separating torus cutting $M$ into $M_1$ and $M_2$. Denote by $[M_1, \partial M_1]$ and $[M_2, \partial M_2]$ the induced orientations classes so that the induced orientations on $\partial M_1$ and $\partial M_2$ are opposite on $T$, and we have $[M] = [(M_1, \partial M_1)] + [(M_2, \partial M_2)]$.

Fix a regular neighbourhood $W(T) = [0, 1] \times T$ such that $T = \{1/2\} \times T$, $M_1 \cap W(T) = [0, 1/2] \times T$ and $M_2 \cap W(T) = [1/2, 1] \times T$. Let $A$ denote a flat connection over $M$. Applying the same arguments as in [24] we may assume that $A|W(T)$ is in normal form. Then by linearity of the integration

$$\text{cs}_M^*(A) = \text{cs}_M^*(A|M_1)\text{cs}_M^*(A|M_2)$$
Denote by $V$ the solid torus $D^2 \times S^1$ with meridian $m$. Denote by $c$ a slope in $T$ and for each $i = 1, 2$ we perform a Dehn filling to $M_i$ identifying $c$ with $m$ and denote by $M_i = M_i \cup V$ the resulting closed oriented manifold. Suppose both $A|M_1$ and $A|M_2$ smoothly extend to flat connections over $\tilde{M}_1$ and $\tilde{M}_2$, respectively denoted by $\hat{A}_1$ and $\hat{A}_2$. This is to say that for any representation $\rho$ corresponding to $A$ then $[c] \in \ker \rho$. By the linearity we have

$$c_{\hat{A}_1}^* (A|\tilde{M}_1) c_{\hat{A}_2}^* (A|\tilde{M}_2) = c_{\hat{A}_1}^* (A|M_1) c_{\hat{A}_2}^* (A|M_2) c_{\hat{A}_1}^* (A|V).$$

Since the extensions from $M_i$, $i = 1, 2$, to $V$, based on the normal form on $[0, 1] \times T$, are the same on the $T$ direction but opposite on the $[0, 1]$ direction, then using Proposition 5.2 and Remark 5.3,

$$c_{\hat{A}_1}^* (A|V) c_{\hat{A}_2}^* (A|V) = 1$$

Then applying equalities (1.2) of Proposition 1.10 and (1.3) in Proposition 1.12 to the former equality we get

$$(5.17) \quad \text{vol}_G(M, \rho) = \text{vol}_G(M_1(c), \hat{\rho}_1) + \text{vol}_G(M_2(c), \hat{\rho}_2)$$

for $G = \text{SL}_2(\mathbb{R})$ or $\text{PSL}(2; \mathbb{C})$, and where $\hat{\rho}_i$ is the extension of $\rho|\pi_1 M_i$ to $\pi_1 \tilde{M}_i$, $i = 1, 2$.

### 6. Manifolds with virtually positive hyperbolic volumes

This section is devoted to the proof of Proposition 1.6. To this purpose we still need to make some technical statements. Consider a compact oriented 3-manifold $Q$ with connected, toral boundary. Denote by $i : \partial Q \to Q$ the natural inclusion. Applying the homology exact sequence to the pair $(Q, \partial Q)$ with real coefficients, we get immediately

$$\text{Rank} \left( H_1 (\partial Q; \mathbb{R}) \xrightarrow{i_*} H_1 (Q; \mathbb{R}) \right) = 1$$

Therefore we may choose a meridian-longitude basis $(\mu, \lambda)$ of $H_1 (\partial Q; \mathbb{Z})$ such that $i_*(\lambda)$ has infinite order whereas $i_*(\mu)$ is a torsion element in $H_1 (Q; \mathbb{Z})$. When $Q$ has non-connected toral boundary we write $\partial Q = T_1 \cup \ldots \cup T_r$ and we fix a basis $(\lambda_i, \mu_i)$ on each component of $T_i$ as in Theorem 2.1.

In the sequel we will use the following covering result which was first proved in [16] for the hyperbolic case and then extended in [20].

**Lemma 6.1.** Let $Q$ be a compact, oriented and irreducible 3-manifold with toral boundary. Then there exists a prime number $q_0$, depending only on $Q$, such that for any prime number $q \geq q_0$ there exists a finite covering $p : \tilde{Q} \to Q$ inducing the $q \times q$-characteristic covering over $\partial Q$ such that each Seifert piece of $\tilde{Q}$ is a product of a surface with positive genus by the circle.

We next state

**Lemma 6.2.** Let $Q$ be a compact oriented 3-manifold with toral boundary whose interior admits a complete (finite volume) hyperbolic metric. Then there exists a prime number $p_0$, depending only on $Q$, such that for any family of slopes $(m_1, \ldots, m_r)$ in $\partial Q$ with $m_i \subset T_i$ for $i = 1, \ldots, r$ and for any prime number $q \geq p_0$ there exists a finite covering $p : \tilde{Q} \to Q$ inducing the $q \times q$-characteristic covering over $\partial Q$ and a representation $\rho : \pi_1 Q(p^{-1}(m_1 \cup \ldots \cup m_r)) \to \text{PSL}(2; \mathbb{C})$ of positive volume, where $Q(p^{-1}(m_1 \cup \ldots \cup m_r))$ denotes the closed surgered manifold obtained from $Q$ after performing a Dehn filling on each component $U_i^j$ of $p^{-1}(T_i)$ identifying the meridian of a solid torus with a component of $p^{-1}(m_i) \cap U_i^j$ for any $i = 1, \ldots, r$. 
Proof. Choose a prime number $p_0$ such that $p_0 > \max\{C, q_0\}$, where $C$ is given in Theorem 2.1 and $q_0$ is the prime number given in Lemma 6.1 applied to $Q$. Thus given a slope $\rho = a_i \lambda_i + b_i \mu_i$ in $T_i$, with $(a_i, b_i)$ co-prime, then for any prime number $q \geq p_0$ we have $|(qa_i, qb_i)|_2 > C$ for $i = 1, \ldots, r$. Therefore we can apply Theorem 2.1 implying that $Q((qa_1, qb_1), \ldots, (qa_r, qb_r))$ is a hyperbolic orbifold. In particular there exists a representation $\rho : \pi_1(Q((qa_1, qb_1), \ldots, (qa_r, qb_r)) \to \text{PSL}(2; \mathbb{C})$ with positive volume and such that for each $i = 1, \ldots, r$ the element $\rho(m_i)$ is conjugated to \begin{pmatrix} e^{i\pi/q} & 0 \\ 0 & e^{-i\pi/q} \end{pmatrix} and $\rho(l_i)$ is conjugated to \begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix} for some $x_i \in \mathbb{C}^*$, where $l_i$ is the slope $c_i \lambda_i + d_i \mu_i$ in $T_i$ with $a_i d_i - b_i c_i = 1$. On the other hand, Lemma 6.1 applies to $Q$ for any prime number $q \geq p_0$ which gives rise to a finite covering $\rho : Q \to \tilde{Q}$. Since it induces the $q \times q$-characteristic covering on the boundary then it induces an orbifold finite covering $\tilde{\rho} : \tilde{Q}(p^{-1}(m_1 \cup \ldots \cup m_r)) \to Q((qa_1, qb_1), \ldots, (qa_r, qb_r))$. Accordingly the composition $\rho \circ \tilde{\rho}_* : \pi_1(\tilde{Q}(p^{-1}(m_1 \cup \ldots \cup m_r)))$ whose volume is positive. This completes the proof of the lemma.

Proof of Proposition 1.6. Suppose that $N$ has a hyperbolic piece $Q$ such that each non-separating component of $\partial Q$ in $N$ is shared by a Seifert piece. Let $N = Q \cup (Q_1 \cup \ldots \cup Q_l)$, where each $Q_i$ is a component of $N \setminus Q$. For each component $Q_i$, with connected boundary we fix its meridian-longitude basis $(\mu_i, \lambda_i)$; and for each component $Q_j$ with non-connected boundary $T^1_j, \ldots, T^d_j$, by the condition posed on $Q$ we denote by $h^l_j, l = 1, \ldots, l_j$ the regular fiber of the Seifert piece adjacent to $Q$ represented in $T^l_j$.

For each component $Q_i$ denote by $q_i$ the prime number such that for any prime number $q \geq q_i$ there exists a $q \times q$-characteristic finite covering map $\tilde{Q}_i \to Q_i$ satisfying the conclusion of Lemma 6.1. Notice that when $\partial Q_i$ is connected, for each component $T^k_i \subset \partial \tilde{Q}_i$, $k = 1, \ldots, r_i$, over $\partial Q_i$, then each component $\mu^k_i$ of $p^{-1}(\mu_i) \cap T^k_i$ and $\lambda^k_i$ of $p^{-1}(\lambda_i) \cap T^k_i$ is a basis of $H_1(T^k_i; \mathbb{Z})$. When $\partial Q_i$ is non-connected then we fix a trivialization of the Seifert pieces adjacent to $\partial \tilde{Q}_i$ providing a section-fiber basis of $H_1(T^j_i; \mathbb{Z})$ for each component of $\partial \tilde{Q}_i$.

We apply Lemma 6.2 to $Q$ with the family of slopes $\{\mu_i\}_i$ and $\{h^l_j\}_{i,j}$ and we denote by $q$ a prime number such that $q > \max\{p_0, q_1, \ldots, q_l\}$ with the corresponding covering $\tilde{Q} \to Q$. We denote by $p : M \to N$ a finite covering such that each component of $p^{-1}(Q_i)$, resp. $p^{-1}(Q)$ is homeomorphic to $Q_i$ for $i = 1, \ldots, l$, resp. to $Q$. Such a covering can be constructed following the arguments of [26].

For each component $\tilde{Q}$ of $p^{-1}(Q)$, let $\tilde{Q} = \tilde{Q}(\cup_i p^{-1}(\mu_i) \cup_{i,j} p^{-1}(h^l_j))$. By Lemma 6.2 there exists a representation $\rho : \pi_1(\tilde{Q}) \to \text{PSL}(2; \mathbb{C})$ such that

$$\text{vol} \left( \tilde{Q}, \rho \right) > 0$$

and satisfying the following conditions:

(1) when $\partial Q_i$ is connected $\rho(\mu^k_i)$ is trivial and $\rho(\lambda^k_i), k = 1, \ldots, r_i$ are all conjugated to the same element of type \begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix} where $x_i \in \mathbb{C}^*$;

(2) when $\partial Q_i$ is non-connected then the fibers of the Seifert pieces adjacent to $\partial \tilde{Q}_i$ over $\partial Q_i$, are sent to the trivial element under $\rho$.
Remark 6.3. Notice that even though the representations \( \rho \) should be indexed by the components of \( p^{-1}(Q) \) we keep the same notations for all of them for the sake of simplicity.

When \( \partial Q_i \) is connected we choose a basis \( e_1^i, \ldots, e_n^i \) of the torsion-free submodule of \( H_1(Q_i; \mathbb{Z}) \) so that \( \lambda_i \in \langle e_1^i \rangle \). Denote by \( k_i \) the non-trivial integer such that \( \lambda_i = k_i e_1^i \) and choose a complex number \( \xi_i = \omega_i e^{i\theta_i} \in \mathbb{C}^* \) such that \( e^{i\theta_i} = x_i \) and consider the homomorphism \( \eta_i : \langle e_1^i \rangle \to \text{PSL}(2; \mathbb{C}) \) sending \( e_1^i \) to \( \begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i^{-1} \end{pmatrix} \). For each component \( \tilde{Q}_i \) of \( p^{-1}(Q_i) \), consider the representation given by the composition

\[
\pi_1 \tilde{Q}_i \to H_1(\tilde{Q}_i; \mathbb{Z}) \xrightarrow{\rho_i} H_1(Q_i; \mathbb{Z}) \to \langle e_1^i \rangle \xrightarrow{\eta_i} \text{PSL}(2; \mathbb{C})
\]

where \( H_1(Q_i; \mathbb{Z}) \to \langle e_1^i \rangle \) denotes the natural projection onto the \( \mathbb{Z} \)-factor spanned by \( e_1^i \). Note the composition of the above homomorphisms sends \( \mu_i^k \) to the unit of \( \text{PSL}(2; \mathbb{C}) \), since \( \mu_i^k \) was sent to the torsion part of \( H_1(Q_i, \mathbb{Z}) \) first, and then was sent to the unit under the natural projection. Therefore the composition of the above homomorphisms gives rise to a cyclic representation

\[
\eta : \pi_1 \tilde{Q}_i \to \text{PSL}(2; \mathbb{C})
\]

where \( \tilde{Q}_i = \tilde{Q}_i(\cup_{j} \mu_i^k), \) such that

\[
\eta(\lambda_i^k) = \begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix},
\]

and

\[
\text{vol}(\tilde{Q}_i, \eta) = 0
\]

Indeed, the former equality can be verified in the following way. The path

\[
(\eta_i)_{t}(e_1^i) = \begin{pmatrix} \omega_i e^{i\theta_i} & 0 \\ 0 & \omega_i e^{-i\theta_i} \end{pmatrix}
\]

of representations \( < e_1^i > \to \text{PSL}(2; \mathbb{C}) \), where \( \omega_i = t a_i + (1 - t) \), provides a path of representations

\[
\eta_t : \pi_1 \tilde{Q}_i \to \text{PSL}(2; \mathbb{C})
\]

such that \( \eta_1 = \eta \) and \( \eta_0 \) is the trivial representation.

Consider the associated path of flat connections \( A_t \). This path defines a connection \( A \) on the product \( \tilde{Q}_i \times [0, 1] \) that is no longer flat but whose curvature \( F_A \) satisfies the equation \( F_A^k \wedge F_A^k = 0 \) (this latter point follows from the fact that \( F_A^{A'} = 0 \) for any \( t \)). Hence it follows from the construction of the Chern Simons invariant (paragraph 5.3) combined with the Stokes formula that \( \text{cs}_{\tilde{Q}_i}^\rho (A_0) = \text{cs}_{\tilde{Q}_i}^\rho (A_1) = 1 \). Therefore \( \text{vol}(\tilde{Q}_i, \eta) = 0 \). Mind that Remark 6.3 still applies to \( \eta \).

Suppose now that \( \partial Q_j \) isn’t connected and denote by \( \tilde{Q}_j \) a component over \( Q_j \). Let \( \{S_j^i\}_l \) denote the Seifert pieces of \( \tilde{Q}_j \) adjacent to \( \partial \tilde{Q}_j \) with a fixed trivialization \( F_j^i \times S^1 \). Denote by \( \{s_{k,l}\}_{k,l} \) the components of \( \partial F_j^i \cap \partial \tilde{Q}_j \), by \( \{s'_{k,l}\}_{k,l} \) the components of \( \partial F_j^i \setminus \partial F_j^i \cap \partial \tilde{Q}_j \) and by \( \tilde{Q}_l \) its \( S^1 \)-fiber. Denote by \( F_j^l \) the surface obtained from \( F_j^i \) after crunching each boundary component \( s'_{k,l} \) into a point and denote by \( A \) the set of “singular points” obtained by crunching each components of \( \cup_l (\partial F_j^l \setminus \partial F_j^i \cap \partial \tilde{Q}_j) \) to a point. Consider the following relation on \( \tilde{Q}_j \): \( x \sim y \) iff either \( x \) and \( y \) lie on \( \tilde{Q}_j \setminus \cup_l S_j^i \) or \( x \) and \( y \) lie on the same \( S^1 \)-fibre of a Seifert piece \( F_j^l \times S^1 \) for some \( l \). Consider the \text{crunching} map \( \xi : \tilde{Q}_j \to \tilde{Q}_j/R \). The quotient space \( \tilde{Q}_j/R \) is homeomorphic to
via the continuous cohomology (see paragraph 3.2) the induced homomorphism leads to an isomorphism Such a representation can be defined since \( \pi_1 \tilde{Q}_j/\mathcal{R} \) has a positive genus and since each element of PSL(2; \( \mathbb{C} \)) is a commutator by \([32]\). Denote by \( \tilde{Q}_j \) the closed manifold \( \tilde{Q}_j \) (when \( \partial Q_j \) is connected) and \( \tilde{Q}_j \) is non-connected), in normal form and corresponding to the representations \( \rho|\pi_1 Q_j \), \( \eta|\pi_1 \tilde{Q}_j \) and \( \phi|\pi_1 \tilde{Q}_j \) over each component \( Q, \tilde{Q}_i \) and \( \tilde{Q}_j \) of \( p^{-1}(Q) \). By our construction they can be glued together in a smooth and flat way giving rise to a global representation \( \psi : \pi_1 M \to \text{PSL}(2; \mathbb{C}) \). By the additivity principle (a general for \( m \) of (5.17)) we know that \( \text{vol}(M,\psi) = d\text{vol}(\tilde{Q},\rho) > 0 \) where \( d \) denotes the number of components of \( p^{-1}(Q) \). This completes the proof of Proposition 1.6.\[\square\]

7. MANIFOLDS WITH POSITIVE GROMOV SIMPLICIAL VOLUME BUT VANISHING HYPERBOLIC VOLUMES

Proof of Proposition 7.3] We first begin by constructing an 1-edged manifold with a hyperbolic piece adjacent to a Seifert piece whose hyperbolic volume vanishes. Let \( M_1 \) denote \( F \times S^1 \) where \( F \) is a surface with positive genus and connected boundary. There is a natural section-fiber basis \( (s, h) \subset \partial M_1 \). On the other hand, it follows from [18] that there are infinitely many one cusped, complete, finite volume hyperbolic manifolds \( M_2 \) endowed with a basis \( (\mu, \lambda) \subset \partial M_2 \) such that both \( M_2(\lambda) \) and \( M_2(\mu) \) have zero simplicial volume (because they are actually connected sums of lens spaces). Denote by \( \varphi : \partial M_1 \to \partial M_2 \) the homeomorphism defined by \( \varphi(s) = \mu \) and \( \varphi(h) = \lambda^{-1} \). Let \( M_\varphi = M_1 \cup_\varphi M_2 \). Then \( M_\varphi \) is an one-edged Haken manifold with positive simplicial volume. Denote \( T_{M_\varphi} \) by \( T \).
Let $\rho : \pi_1 M_\varphi \to \text{PSL}(2; \mathbb{C})$ be any representation and denote by $A$ the resulting connection over $M_\varphi$. Notice that either $\rho(s)$ or $\rho(h)$ is trivial. Indeed if $\rho(h) \neq 1$, its centralizer $Z(\rho(h))$ must be abelian in PSL(2; $\mathbb{C}$). Since $h$ is central in $\pi_1 M_1$, this means that $\rho(\pi_1 M_1)$ is abelian. Since $s$ is homologically zero in $M_1$, then $\rho(s) = 1$.

Let $\zeta$ be either $s$ or $h$ so that $\rho(\zeta) = 1$. After putting $A$ in normal form with respect to $T$, denote by $A_1$ and $A_2$ the flat connections over $M_1$ and $M_2$ respectively. Since $\rho(\zeta)$ is trivial then $A_1$ and $A_2$ do extend over $M_1(\zeta)$ and $M_2(\zeta)$ to flat connections $\hat{A}_1$ and $\hat{A}_2$, and thus
\[ cs_{M_\varphi}(A) = cs_{M_1(\zeta)}(\hat{A}_1) \times cs_{M_2(\zeta)}(\hat{A}_2) \]

Eventually taking the imaginary part we get
\[ \text{vol}(M_\varphi, \rho) = \text{vol}(M_1(\zeta), \hat{\rho}_1) + \text{vol}(M_2(\zeta), \hat{\rho}_2) \]

where $\hat{\rho}_1$ denotes the extension of $\rho|\pi_1 M_1$ to $\pi_1 M_1(\zeta)$. Since both $\text{vol}(M_1(\zeta), \hat{\rho}_1)$ and $\text{vol}(M_2(\zeta), \hat{\rho}_2)$ do vanish, the proof of Proposition 1.1 is complete. \hfill $\blacksquare$

**Remark 7.1.** It is worth mentioning that inequality (7.1) still holds replacing $M_2$ by any one cusped oriented complete hyperbolic manifold. In this more general case we have $\text{vol}(M_2(\zeta), \hat{\rho}_2) < \text{vol}M_2$ and therefore, the volume of the hyperbolic piece of $M_\varphi$ is never reached by the representations of $\pi_1 M_\varphi$ into PSL(2; $\mathbb{C}$).

### 8. Volumes of Representations of 1-Edged 3-Manifolds

In this section we verify Propositions [1.14] and [1.15]. Let $N = Q_- \cup_\tau Q_+$ be a one-edged 3-manifold, where the gluing map $\tau : \partial Q_- = T_- \to \partial Q_+ = T_+$ is an orientation reversing homeomorphism. Recall that on each $T_\varepsilon$ we fix a basis $T_\varepsilon(s_\varepsilon, h_\varepsilon)$ as in paragraph 1.4, (A) and (B). Denote by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the integral matrix of $\tau$ under the basis $(s_-, h_-)$ and $(s_+, h_+)$, such that $\det A = -1$, and
\[ \tau(s_-) = as_+ + ch_+, \quad \tau(h_-) = bs_+ + dh_+ \]

It follows from Lemmas [6.1] and [7] that $N$ admits a $n$-sheeted covering $\tilde{N}$, where $n$ depends only on the pieces $Q_-$ and $Q_+$, that induces the (say) $q \times q$-characteristic covering over $T = T_N$, such that:

1. $\tilde{N} = \tilde{Q}_- \cup_\tilde{\tau} \tilde{Q}_+$, where $\tilde{Q}_\varepsilon$ covers $Q_\varepsilon$ for $\varepsilon = \pm$ and $\tilde{Q}_\varepsilon$ is a product of a surface of genus $\geq p + 2$ with $S^1$ if $Q_\varepsilon$ is a Seifert piece,
2. when $Q_-$ and $Q_+$ are both Seifert manifolds then $\tilde{Q}_-$ and $\tilde{Q}_+$ can be chosen connected so that $\tilde{N}$ is a $p$-edged manifold with $p \geq 2$ (A $p$-edged 3-manifold is a manifold whose dual graph consists of 2 vertices and $p$ edges and each edge is shared by the two vertices),
3. The basis $T_\varepsilon(s_\varepsilon, h_\varepsilon)$ can be lifted, and denoted by $T_\varepsilon^j(s_\varepsilon^j, h_\varepsilon^j)$ for $j = 1, \ldots, p$, and the matrix of the gluing $\tilde{\tau} : T_\varepsilon^j \to T_\varepsilon^{j'}$ is the same than $A$ for all $j$ under the lifted basis, where the $T_\varepsilon^j$'s denote the components of $\partial \tilde{Q}_\varepsilon$.

#### 8.1. Both pieces are Seifert

This subsection is devoted to the proof of Proposition [1.14]. It follows from the paragraph above that it is sufficient to check the following (simply remember $p = \frac{4\pi^2}{\sigma^2}$).

**Lemma 8.1.** Let $N$ denote a closed oriented graph manifold satisfying conditions (1)-(3).

Denote by $G$ the group $\mathbb{R} \times_\mathbb{Z} \text{SL}_2(\mathbb{R})$.

(i) if $a = d = 0$ then $8\pi^2 p \in \text{vol}(N, G)$. 

(ii) if $ac \neq 0$ then $4\pi^2 p/|ac| \in \text{vol}(N,G)$,
(iii) if $cd \neq 0$ then $4\pi^2 p/|cd| \in \text{vol}(N,G)$,
(iv) if $c = 0$ then $4\pi^2 p/|b| \in \text{vol}(N,G)$.

Proof. Recall first that in this case $b \neq 0$ since $N$ is not a Seifert manifold.

Denote by $Q_- = F_- \times S^1$ and by $Q_+ = F_+ \times S^1$ the two (connected) Seifert pieces, and recall that $(s_i^-, h_i^-)$ and $(s_i^+, h_i^+)$ are section-fiber basis of $\partial Q_-$ and $\partial Q_+$ respectively.

Pick some base points $x_- \in \text{int}Q_-$, $x_+ \in \text{int}Q_+$ and choose $p$ arcs connecting $x_-$ with $h_i^+ \cap s_i^-$, resp. $x_+$ with $h_i^+ \cap s_i^+$, to see these elements in $\pi_1Q_-$, resp. in $\pi_1Q_+$.

(i) Suppose first $a = d = 0$. Then $b = c = \pm 1$. We get directly a representation

$$\rho: \pi_1N \to \text{SL}_2(\mathbb{R})$$

defined by $\rho(s_i^-) = \text{sh}(1)$ and $\rho(h_i^+) = \text{sh}(1)$. Such a representation does exist since the genus of $F_+$ and $F_- \geq p + 2$. The additivity principle gives

$$\text{vol}(N, \rho) = \text{vol}(Q_+(-1,1), \rho) + \text{vol}(Q_-(1,1), \rho)$$

By Proposition 4.2 we know that $\text{vol}(Q_+(-1,1), \rho) = \text{vol}(Q_+(1,1), \rho) = 4\pi^2 p$ hence the proof of point (i) follows.

(ii) Let’s now assume $ac \neq 0$. Then the closed manifold $Q_+((a,c),..., (a,c))$ is still a Seifert manifold with Euler number $\pm pc/a$.

In Proposition 4.2 by choosing $z_1 = ... = z_p = n = 0$ and $u_1 = ... = u_p = 1$ we have a representation $\rho_+: \pi_1Q_+ \to \mathbb{R} \times_{\mathbb{Z}} \text{SL}_2(\mathbb{R})$ such that

$$\rho_+(s_i^+) = \left(0, \text{sh}\left(-\frac{1}{a}\right)\right), \rho_+(h_i^+) = \left(\frac{1}{c}, 1\right)$$

since $g(F_+) > p$, the condition (4.1) is clearly satisfied. According to (4.2) and (4.3), we have

$$\rho_+(s_i^-) = (0,\text{sh}(0)), \rho_+(h_i^-) = \left(\frac{d}{c}\text{sh}\left(-\frac{b}{a}\right)\right)$$

Now we can extend $\rho_+$ to $\rho: \pi_1N \to \mathbb{R} \times_{\mathbb{Z}} \text{SL}_2(\mathbb{R})$ merely by sending the whole subgroup $\pi_1F_-$ of $\pi_1Q_-$ to the unit of $G$. To apply the additive principle, we need further to construct a $c$-fold cyclic covering $\tilde{q}: \tilde{N}_c \to N$ so that the induced representation $\tilde{\rho} = \rho \circ q_+: \pi_1\tilde{N}_c \to \mathbb{R} \times_{\mathbb{Z}} \text{SL}_2(\mathbb{R})$ has image in $\text{SL}_2(\mathbb{R})$. This covering $\tilde{q}: \tilde{N}_c \to N$ can be obtained by combining the covering $p_+: Q_+ \to Q_+$ defined by $\varphi_+: \pi_1Q_+ \to \mathbb{Z}/c\mathbb{Z}$ with $\varphi_+(s_i^+) = \overline{0}$, $\varphi_+(h_i^+) = \overline{1}$; and the covering $p_-: Q_- \to Q_-$ defined by $\varphi_-: \pi_1Q_- \to \mathbb{Z}/c\mathbb{Z}$ with $\varphi_-(s_i^-) = \overline{0}$, $\varphi_-(h_i^-) = \overline{1}$.

It is easy to verify that

$$\tilde{\rho}(s_i^+) = 0, \text{sh}\left(-\frac{1}{a}\right), \tilde{\rho}(h_i^+) = (0,\text{sh}(1))$$

$$\tilde{\rho}(s_i^-) = (0,\text{sh}(0)), \tilde{\rho}(h_i^-) = 0, \text{sh}\left(-\frac{1}{a}\right)$$

Where the $(\tilde{s}_i^+, \tilde{h}_i^+)$’s are the lifts of the $(s_i^+, h_i^+)$’s into $\tilde{N}_c$. Hence indeed $\tilde{\rho}$ takes its values in $\text{SL}_2(\mathbb{R})$. Since $s_i^- = as_i^+ + ch_i^+$ then we get $\tilde{s}_i^- = a\tilde{s}_i^+ + \tilde{h}_i^+$. So the Euler number $e(Q_+((a,1),..., (a,1)) = p/a$. Now according to (4.4) and $|e| = |p/a|$, we have

$$\text{vol}(Q_+((a,1),..., (a,1), \tilde{\rho}) = 4\pi^2 \left|\frac{1}{|e|}\right|e^2 = 4\pi^2|e| = 4\pi^2|p/a|$$

On the other hand it is clear that $\text{vol}\left(Q_-((1,0),..., (1,0)), \tilde{\rho}\right) = 0$. 
Since \( \widetilde{SL_2(\mathbb{R})} \) is contractible, as we did in [8], we can apply the additive principle to compute \( \text{vol}(\widetilde{N}, \rho) \). Precisely by (5.17) we have
\[
\text{vol}(\widetilde{N}, \rho) = \text{vol}(\widetilde{Q} + ((a, 1), ..., (a, 1)), \rho) + \text{vol}(\widetilde{Q} + ((1, 0), ..., (1, 0)), \rho) = 4\pi^2 |p/a|
\]
Since \( \text{vol}(\widetilde{N}, \rho) = |c| \text{vol}(N, \rho) \), we have \( \text{vol}(N, \rho) = 4\pi^2 p/|ac| \). This proves point (ii).

(iii) The proof is the same as that of (ii) just by replacing \( A \) by \( A^{-1} \).

(iv) We will get directly a representation \( \rho: \pi_1 N \to \widetilde{SL_2(\mathbb{R})} \) by first setting \( \rho(h_0^+) = \text{sh}(1), \rho(s_i^+) = \text{sh}(\varepsilon_i/b) \) for \( i = 1, ..., p \), where \( \varepsilon_i \) denotes the sign of \( a \). Since \( |p/b| \leq p < 2g - 2 \) such a representation exists. On the other hand, we get a representation \( \psi: \pi_1 Q + \to \widetilde{SL_2(\mathbb{R})} \) by setting \( \psi(h_0^+) = 1 \) and \( \psi(s_i^+) = \text{sh}(1/b) \). Again such a representation does exist. Then we can use the additive principle to get \( \text{vol}(\rho) = 4\pi^2 |p/b| \).

This completes the proof of Lemma 8.1 and therefore the proof of Proposition 1.14. \( \square \)

8.2. Both Seifert and hyperbolic pieces do appear. This subsection is devoted to the proof of Proposition 1.15. Let’s say that \( Q_+ \) is hyperbolic and \( Q_- \) is Seifert. The proof below is divided in two cases, which involve quite different arguments in certain stages.

As in [24] denote by \( D \) the space of deformations of hyperbolic structure on \( \text{int} Q_+ \) near the complete one \( d_0 \in D \). Since \( Q_+ \) has only one cusp there are functions
\[
(\alpha, \beta): \ D^* = D \setminus \{d_0\} \to \mathbb{C}^2
\]
such that for each \( d \in D^* \) there exists a representation \( \rho_d^+ : \pi_1 Q_+ \to \text{PSL}(2; \mathbb{C}) \) induced on the boundary by the representation
\[
(8.5) \quad s_+ \mapsto \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{-2i\pi\alpha} \end{pmatrix} \quad \text{and} \quad h_+ \mapsto \begin{pmatrix} e^{2i\pi\beta} & 0 \\ 0 & e^{-2i\pi\beta} \end{pmatrix}
\]
In this situation the map \( D^* \ni d \mapsto (\alpha, \beta) \in \mathbb{C}^2 \) is a lifting of the composition map
\[
D^* \to \mathcal{E}R^0_M(\text{PSL}(2; \mathbb{C})) \to \mathcal{E}R^0_{\partial M}(\text{PSL}(2; \mathbb{C}))
\]
By the Thurston Hyperbolic Dehn filling Theorem there is a constant \( C > 0 \) such that if (say) \( \| (a, c) \|_2 > C \) then there exists \( d \in D^* \) such that
\[
\alpha a + c\beta = 1/2
\]
Let \( V = D^2 \times S^1 \) be a solid torus endowed with the standard meridian-parallel basis \((m, l)\). The representation \( \rho_d^+ \) extends to a complete and faithful representation
\[
\tilde{\rho}_d^+: \pi_1 Q_+(a, c) \to \text{PSL}(2; \mathbb{C})
\]
where \( Q_+(a, c) \) is obtained by gluing \( \partial V \) to \( \partial Q_+ \) identifying the meridian of \( V \) with the curve \( as_+ + ch_+ \). Let \( A_{d}^+ \) denote the connection over \( Q_+(a, c) \) in normal hyperbolic form over \( \partial Q_+ \) which decomposes into \( A_0^+ \cup A_1^+ \) over \( Q_+ \cup V \). We choose a lifting \( L(\tilde{\rho}_d^+) \) such that
\[
A_0^+|\partial Q_+ = (i\alpha dx + i\beta dy) \otimes X
\]
in the basis \((s_+, h_+)\). By (8.5) and (8.6) this means that
\[
A_0^1|\partial V = \left( \frac{1}{2} i dx + i(b\alpha + d\beta) dy \right) \otimes X
\]
in the basis \((m, l)\). The similar construction can be done replacing \((a, c)\) by \((b, d)\).

1. Pinching the section \( s_- = \partial F_- \). Denote by \( \rho_+: \pi_1 \tilde{Q}_+ \to \text{PSL}(2; \mathbb{C}) \) the representation defined by the composition \( \rho_d^+ \circ p_* \), where \( p: \tilde{Q}_+ \to Q_+ \) is the \( q \times q \)-characteristic.
covering map defined above. This representation induces the following relations by (8.1) and (8.5): \( \rho_+(s_\rho) \) is the trivial element and \( \rho_+(h^J) \) is sent to
\[
\begin{pmatrix}
  e^{2i\pi q(b\alpha + d\beta)} & 0 \\
  0 & e^{-2i\pi q(b\alpha + d\beta)}
\end{pmatrix}
\]
in PSL(2; \mathbb{C}). Thus there exists a global representation \( \rho: \pi_1\tilde{N} \to \text{PSL}(2; \mathbb{C}) \) such that \( \rho|\pi_1\tilde{Q}_+ = \rho_+ \). Denote \( \rho_- = \rho|\pi_1\tilde{Q}_- \). Let \( A \) be a flat connection in hyperbolic normal form with respect to \( T_{\tilde{S}} \) such that \( A = A_- \cup A_+ \) where \( A_- \), resp. \( A_+ \), is the restriction of \( A \) over \( \tilde{Q}_- \), resp. \( \tilde{Q}_+ \). Notice that \( A_+ = (p|\tilde{Q}_+)^*(A^+_d) \).

For each \( j \in \{1, \ldots, p\} \) we identify the meridian of a solid torus \( V_j^\pm = \mathbb{D}^2 \times S^1 \) with \( s_\rho \) and with \( as_\rho^j + ch_\rho^j \) and then we get closed manifolds \( \tilde{Q}_-(-1,0) \) and \( \tilde{Q}_+(a,c) \) where \( A_- \) and \( A_+ \), extend to flat connections \( \tilde{A}_- \) and \( \tilde{A}_+ \) such that
\[
\cs_{\tilde{N}}(A) = \cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) \times \cs_{\tilde{Q}_-(-1,0)}(\tilde{A}_-)
\]
by the additivity principle.

**Remark 8.2.** Again \( \tilde{A}_+|V_j^\pm = (p|V_j^\pm)^*(A^0_d) \) but mind that \( p|V_j^\pm: V_j^\pm \to \mathcal{V} \) is a \( q \times q \)-characteristic covering branched along the core of the solid torus. Therefore the lifting of the representations induced on the \( V_j^\pm \)'s is \( (1/2, q(b\alpha + d\beta)) \).

Denote by \( \tilde{\rho}_+ \), \( \tilde{\rho}_- \) the extension of \( \rho_+ \) and \( \rho_- \) to \( \pi_1\tilde{Q}_+(a,c) \) and \( \pi_1\tilde{Q}_-(-1,0) \). Splitting the former equality into real and imaginary parts according to (5.17) we get then
\[
\text{vol}(\tilde{N}, \rho) = \text{vol}(\tilde{Q}_+(a,c), \tilde{\rho}_+) + \text{vol}(\tilde{Q}_-(-1,0), \tilde{\rho}_-)
\]
Since \( \tilde{Q}_- \) is by construction a product of a surface with the circle then
\[
\text{vol}(\tilde{Q}_-(-1,0), \tilde{\rho}_-) = 0
\]
On the other hand, notice that, a priori, the representation \( \rho_+ \) lies by no means in the deformation space of hyperbolic structures of \( \tilde{Q}_+ \). Nevertheless we need a geometric interpretation of \( \text{vol}(\tilde{N}, \rho) = \text{vol}(\tilde{Q}_+(a,c), \tilde{\rho}_+) \). It follows from Remark 8.2 and from Corollary 5.5 applied \( p \) times that
\[
\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) = \cs_{\tilde{Q}_+(a,c)}(A_+) \exp(-4i\pi pq(b\alpha + d\beta))
\]
Since \( \rho_+ \) is induced by restriction from \( \rho_d^+ \) then \( \cs_{\tilde{Q}_+(a,c)}(A_+) = (\cs_{\tilde{Q}_+(a,c)}(A^+_d))^p q^2 \) therefore
\[
\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) = (\cs_{\tilde{Q}_+(a,c)}(A^+_d))^p q^2 \exp(-4i\pi pq(b\alpha + d\beta))
\]
Again by Corollary 5.5
\[
\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) = \cs_{\tilde{Q}_+(a,c)}(A^+_d) \exp(-4i\pi (b\alpha + d\beta))
\]
This leads to
\[
\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) = (\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_d^+))^p q^2 \exp(4i\pi pq(q-1)(b\alpha + d\beta))
\]
and accordingly by equality (5.15) and Remark 11.3 that can be applied to \( \cs_{\tilde{Q}_+(a,c)}(\tilde{A}_d^+) \) for \( \rho_d^+ \) is a faithful and discrete representation
\[
\cs_{\tilde{Q}_+(a,c)}(\tilde{A}_+) = \exp(2i\pi \cs_{\tilde{Q}_+(a,c)}(\tilde{A}_d^+)^p q^2 \times \\
\exp \left( \frac{2pq}{\pi} \text{vol}Q_+(a,c) + 4i\pi pq(q-1)(b\alpha + d\beta) \right)
\]
Again using (5.17) and splitting the former equality into real and imaginary parts we get eventually
\[
\text{vol}(\tilde{N}, \rho) = \text{vol}(\tilde{Q}_+(a, c), \tilde{\rho}_+) = pq^2 \text{vol}Q_+(a, c) + \frac{\pi pq(q - 1)}{2} \text{length}(\gamma)
\]
where \(\gamma\) is the geodesic added to \(Q_+\) to complete the cusp with respect to the \((a, c)\)-Dehn filling.

2. Pinching the fiber \(h_-\). By the Thurston Hyperbolic Dehn filling Theorem there is a constant \(C > 0\) such that if \(\|(b, d)\|_2 > C\) then there exists \(d \in D^*\) such that (i)
\[(8.7) \quad b\alpha + d\beta = 1/2\]
Let \(V = D^2 \times S^1\) be a solid torus endowed with the standard meridian-parallel basis \((m, l)\). The representation \(\rho_d\) extends to a complete and faithful representation \(\hat{\rho}_d : \pi_1Q_+(b, d) \to \text{PSL}(2; \mathbb{C})\), where \(Q_+(b, d)\) is obtained by gluing \(\partial V\) to \(\partial Q_+\) identifying the meridian of \(V\) with the curve \(bs_+ + dh_+\). Let \(A^+_d\) denote the connection over \(Q_+(b, d)\) in normal hyperbolic form over \(\partial Q_+\) which decomposes into \(A^+_d \cup A^0_d\) over \(Q_+ \cup V\). We choose a lifting \(L(\rho^+_d)\) such that
\[
A^+_d|\partial Q_+ = (i\alpha dx + i\beta dy) \otimes X
\]
in the basis \((s_+, h_+)\). By equation (8.7) this means that
\[
A^0_d|\partial V = \left(\begin{array}{cc}
\frac{1}{2}\alpha dx + (\alpha + c\beta)dy & 0 \\
0 & e^{-2\pi i q(a\alpha + c\beta)}
\end{array}\right) \otimes X
\]
in the \((m, l)\) basis.

Denote by \(\rho_+ : \pi_1\tilde{Q}_+ \to \text{PSL}(2; \mathbb{C})\) the representation defined by the composition \(\rho^+_d \circ p_+\), where \(p : \tilde{Q}_+ \to Q_+\) is the \(q \times q\)-characteristic covering map defined above. This representation induces the following relations: \(\rho_+(h^j_+)\) is the trivial element and \(\rho_+(s^j_-)\) is sent to
\[
\left(\begin{array}{cc}
e^{2\pi i q(a\alpha + c\beta)} & 0 \\
0 & e^{-2\pi i q(a\alpha + c\beta)}
\end{array}\right)
\]
in \(\text{PSL}(2; \mathbb{C})\). Again since by [2] each element of \(\text{PSL}(2; \mathbb{C})\) is a commutator, there exists a global representation \(\rho : \pi_1\tilde{N} \to \text{PSL}(2; \mathbb{C})\) such that \(\rho|\pi_1\tilde{Q}_+ = \rho_+\). Denote \(\rho_- = \rho|\pi_1\tilde{Q}_-\). Let \(A\) be a flat connection in hyperbolic normal form with respect to \(\tilde{F}_N\) such that \(A = A_+ \cup A_-\) where \(A_+,\) resp. \(A_-\), is the restriction of \(A\) over \(\tilde{Q}_-,\) resp. \(\tilde{Q}_+\). For each \(j = 1, \ldots, p\) we identify the meridian of a solid torus \(D^2 \times S^1\) with \(h^j_-\) and with \(bs^j_+ + dh^j_-\) then we get closed manifolds \(\tilde{Q}_-(0, 1)\) and \(\tilde{Q}_+(b, d)\) where \(A_+\) and \(A_-\) extend to flat connections \(\hat{A}_+\) and \(\hat{A}_-\) such that
\[
\text{cs}_{\hat{N}}^\ast(A) = \text{cs}_{\tilde{Q}_+(b, d)}^\ast((\hat{A}_+) \times \text{cs}_{\tilde{Q}_-(0, 1)}^\ast((\hat{A}_-))
\]
Keeping the same notation as in the previous section, by splitting the former equality into real and imaginary parts according to (5.17), we get
\[
\text{vol}(\tilde{N}, \rho) = \text{vol}(\tilde{Q}_+(b, d), \tilde{\rho}_+) + \text{vol}(\tilde{Q}_-(0, 1), \tilde{\rho}_-)
\]
Since \(\tilde{Q}_-\) is by construction a connected sum of \(S^2 \times S^1\)-factors, then \(\text{vol}(\tilde{Q}_-(0, 1), \tilde{\rho}_-) = 0\). Applying the same arguments as in the former section we get
\[
\text{vol}(\tilde{N}, \rho) = pq^2 \text{vol}Q_+(b, d) + \frac{\pi pq(q - 1)}{2} \text{length}(\gamma)
\]
where \( \gamma \) is the geodesic added to \( Q_+ \) to complete the cusp with respect to the \((b, d)\)-Dehn filling. This proves Proposition 1.15.

REFERENCES

[1] G. Besson, G. Courtois, S. Gallot, Inégalités de Milnor Wood géométriques, Comment. Math. Helv. 82 (2007), 753–803.
[2] S. Bleiler, C. Hodgson, Spherical space forms and Dehn filling. Topology 35 (1996), no. 3, 809C833.
[3] M. Boileau; S. Maillot; J. Porti, Three-dimensional orbifolds and their geometric structures, Panoramas et Synthèses, 15. Société Mathématique de France, Paris, 2003. viii+167 pp.
[4] R. Brooks, W. Goldman, The Godbillon-Vey invariant of a transversely homogeneous foliation, Trans. Amer. Math. Soc. 286 (1984), no. 2, 651–664.
[5] R. Brooks, W. Goldman, Volumes in Seifert space, Duke Math. J. 51 (1984), no. 3, 529–545.
[6] S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. of Math. (2) 99 (1974), 4869.
[7] P. Derbez, S. Wang, Finiteness of mapping degrees and \( \text{PSL}(2, \mathbb{R}) \)-volume on graph manifolds, Algebraic and Geometric Topology 9 (2009) 1727–1749.
[8] P. Derbez, S. Wang, Graph manifolds have virtually positive Seifert volume, to appear in J. London Math. Soc.
[9] W. Dunbar; G. Robert Meyerhoff, Volumes of hyperbolic \( 3 \)-orbifolds, Indiana Univ. Math. J. 43 (1994), no. 2, 611-637.
[10] J.L. Dupont, Simplicial de Rham cohomology and characteristic classes of flat bundles, Topology 15 (1976), no. 3, 233245.
[11] J.L. Dupont, Curvature and Characteristic classes, Lecture Notes in Mathematics, Vol. 640. Springer-Verlag, Berlin-New York, 1978. viii+175 pp.
[12] D. Eisenbud, U. Hirsch, W. Neumann, Transverse foliations of Seifert bundles and self homeomorphism of the circle, Comment. Math. Helv. 56 (1981), no. 4, 638–660.
[13] C. Godbillon, J. Vey, Un invariant des feuilletages de codimension 1. (French) C. R. Acad. Sci. Paris Sér. A-B 273 1971 A92-A95.
[14] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. No. 56, (1982), 5–99.
[15] J. Hass, H.J. Rubinstein, S.C. Wang, Boundary slopes of immersed surfaces in 3-manifolds. J. Differential Geom. 52 (1999), no. 2, 303-325.
[16] J. Hempel, Residual finiteness for 3-manifolds. Combinatorial group theory and topology (Alta, Utah, 1984), 379C396, Ann. of Math. Stud., 111, Princeton Univ. 1987.
[17] G. Hochschild, The structure of Lie groups, Holden-Day, Inc., San Francisco-London-Amsterdam 1965 ix+230 pp.
[18] J. A. Hoffman; D. Matignon, Examples of bireducible Dehn fillings, Pacific Journal of Math., Vol. 209, No. 1 (2003), 67-83.
[19] J. Jaco, W. H. Lectures on three-manifold topology, Regional Conference Series in Mathematics 43, Amer. Math. Soc., Providence, RI, 1980.
[20] W. Jaco, P.B. Shalen, Seifert fibered space in 3-manifolds, Mem. Amer. Math. Soc. 21 (1979).
[21] Johannson, K. Homotopy equivalence of 3-manifolds with boundary, Lecture Notes in Math. 761, Springer-Verlag, Berlin, 1979.
[22] R. Kirby, Problems in low-dimensional topology, Geometric topology, Edited by H.Kazeez, AMS/IP, Vol. 2, International Press, 1997.
[23] Vu The Khoi, A cut-and-paste method for computing the Seifert volumes, Math. Ann. 326 (2003), no. 4, 759–801.
[24] P. Kirk, E. Klassen, Chern-Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of \( T^2 \), Comm. Math. Phys. 153 (1993), no. 3, 521–557.
[25] S. Kobayashi; K. Nomizu, Foundations of differential geometry, Vol. I. and II. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996. xii+329 pp.
[26] J. Luecke, Finite covers of \( 3 \)-manifolds containing essential tori, Trans. Amer. Math. Soc. 310 (1988), no. 1, 381–391.
[27] J. Milnor On the existence of a connection with curvature zero. Comment. Math. Helv. 32 1958 215–223.
[28] W. Neumann; D. Zagier Volume of hyperbolic three-manifolds Topology Vol.24, no. 3, 1985, 307-332.
[29] A. Reznikov, Volumes of discrete groups and topological complexity of homology spheres, Math. Ann. 306 (1996), no. 3, 547–554.
[30] A. Reznikov, Rationality of secondary classes, J. Differential Geom. 43 (1996), no. 3, 674–692.
[31] A. Resnikov, Analytic topology, Progress in Math. Vol. 201, Birkhauser, 2002, 519-532.
[32] P. Samuel; H.-C. Wang, Commutators in a semi-simple Lie group Proc. Amer. Math. Soc. 13, 1962, 907-913.
[33] W. P. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Princeton 1977.
[34] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 316, 1982, 357–381.
[35] W. Van Est, Une application d’une méthode de Cartan-Leray. (French) Indag. Math. 17 (1955), 542–544.
[36] S.C. Wang, Y.Q Wu, Covering invariants and co-Hopficity of 33-manifold groups. Proc. London Math. Soc. (3) 68 (1994), no. 1, 203C224.
[37] J. Wood, Bundles with totally disconnected structure group. Comment. Math. Helv. 46 (1971), 257–273.
[38] T. Yoshida, The η-invariant of hyperbolic 3-manifolds, Invent. Math. 81 (1985), no. 3, 473–514.

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