The Drinker Paradox and its Dual

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Abstract

The Drinker Paradox is as follows.

In every nonempty tavern, there is a person such that if that person is drinking, then everyone in the tavern is drinking.

Formally,

\[ \exists x (\varphi \rightarrow \forall y \varphi[x/y]) \]

Due to its counterintuitive nature it is called a paradox, even though it actually is a classical tautology. However, it is not minimally (or even intuitionistically) provable. The same can be said of its dual, which is (equivalent to) the well-known principle of independence of premise,

\[ \varphi \rightarrow \exists x \psi \vdash \exists x (\varphi \rightarrow \psi) \]

where \( x \) is not free in \( \varphi \).

In this paper we study the implications of adding these and other formula schemata to minimal logic. We show first that these principles are independent of the law of excluded middle and of each other, and second how these schemata relate to other well-known principles, such as Markov’s Principle of unbounded search, providing proofs and semantic models where appropriate.

1 Introduction

Minimal logic \([11]\) provides, as its name suggests, a minimal setting for logical investigations. Starting from minimal logic, we can get to intuitionistic logic by adding \( \textit{ex falso quodlibet} \) (EFQ), and to classical logic, by adding \( \textit{double negation elimination} \) (DNE). Therefore, every statement proven over minimal logic can also be proven in intuitionistic logic and classical logic. In addition, minimal logic has many structural advantages, and is easier to analyse. Of course, there is a price one has to pay for working within a weak framework. The price is that fewer well-known statements are provable outright, which leads to the question of how they relate. This is a very similar question to the one considered in constructive reverse mathematics (CRM; \([8, 5]\)), where the aim is to find some ordering in a multitude of principles, over intuitionistic logic. CRM has been around for some decades now, and some even trace the origins back to Brouwerian counterexamples. Most results in CRM are focused on analysis, where most theorems can be classified into being equivalent to about ten major principles. It is a natural question to ask whether we can find similar results in the absence of EFQ. Previous work by a subset of the authors \([6]\) has investigated the case of propositional schemata, but has left the predicate case

\(^1\)Either as a rule, or an axiom scheme. See below for details.
untouched. Similar work can also be found in [7, 10]. A more detailed approach, but again focused on the propositional case can be found in [9], where it was studied exactly which instances of an axiom scheme are required to prove a given instance of another axiom scheme over minimal logic. In this paper we will make first steps in the predicate case. As is so often the case, the first-order analysis is subtler and technically more difficult to deal with.

For the sake of brevity and readability we have only included non-trivial proofs. The missing proofs, which are in natural deduction style, have been put into an appendix. A version of this paper including that appendix will be made available on arxiv.org under the same title.

2 Technical Preliminaries

We will generally follow the notation and definitions found in [11].

An n-ary scheme $SCH(X_1, \ldots, X_n)$ is a formula $SCH$ containing $n$ propositional variables $X_1, \ldots, X_n$. An instance $SCH(\Phi_1, \ldots, \Phi_n)$ is obtained by replacing the variables with formulae $\Phi_1, \ldots, \Phi_n$. A scheme is derivable in a logical system if every instance is derivable in that system. A scheme is minimal (constructive) (classical) if it is derivable in minimal (intuitionistic) (classical) logic.

**Example 1.** The law of excluded middle, $LEM(\Phi) := \Phi \lor \neg \Phi$ is a classical unary scheme.

A logical system can be extended by adding that certain schemata are derivable in the system. In the case of natural deduction and minimal logic, an extension by $LEM$ is an addition of a deduction rule

$$
\frac{\alpha \lor \neg \alpha}{LEM(\alpha)}
$$

for every formula $\alpha$. This produces a subsystem of classical logic.

More general, if a formula $\Phi$ is derivable over minimal (intuitionistic) logic extended by schemata $S_0, S_1, \ldots, S_n$, then we write

$$\vdash \underline{S_0, S_1, \ldots, S_n} \Phi$$

($\vdash \underline{i, S_0, S_1, \ldots, S_n} \Phi$).

Extending a logic by a scheme differs from allowing undischarged assumptions of instances of the scheme. For example, it should follow from $LEM$ that every predicate is decidable. Consider the proof of $\vdash_{LEM} \forall x (P x \lor \neg P x)$:

$$
\frac{P x \lor \neg P x}{LEM} \\
\frac{\forall x (P x \lor \neg P x)}{\forall I}
$$

The proof uses $LEM(P x)$. However,

$$LEM(P x) \not\vdash \forall x (P x \lor \neg P x),$$

since the rule $\forall I$ requires that $x$ is not free in any open assumptions$^2$

It is trivial to check that the following holds.

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$^2$If we defined $LEM$ as the axiom scheme $\forall \bar{x} P \bar{x} \lor \neg P \bar{x}$, there would be no difference between adding it as a rule or an assumption. This trick is the same as used in [11] page 14] for EFQ and stability.
Proposition 2. Define \( \text{DNE}(\Phi) := \neg\neg\Phi \rightarrow \Phi \), and \( \text{EFQ}(\Phi) := \bot \rightarrow \Phi \). For all (finite) collections of schemata \( S \) and \( T \),

\[
S \vdash_i T \iff S \vdash_{\text{EFQ}} T
\]

and

\[
S \vdash_c T \iff S \vdash_{\text{DNE}} T
\].

A preorder \( \supset \) may be defined on finite collections of schemata by considering derivability over extensions of minimal logic.

Definition 3. For schemata \( S_0, S_1, \ldots, S_n \) and \( m \)-ary scheme \( T \), we write

\[
S_0, S_1, \ldots, S_n \supset T
\]

if

\[
\vdash s_0, s_1, \ldots s_n \; T(\alpha_0, \ldots \alpha_m)
\]

for all formulae \( \alpha_0, \ldots, \alpha_n \). We say that \( T \) is reducible to \( S_0, \ldots, S_n \). Intuitively, a proof using the scheme \( T \) can be replaced by a proof using \( S_0, S_1, \ldots, S_n \). The relation \( \supset \) extends to multiple schemata on the right-hand side in the obvious way.

To demonstrate that \( A_0, A_1, \ldots, A_n \not\supset B \), we exhibit a Kripke model (see Section 5.3 of [4] for more details on Kripke semantics\(^3\)) in which an instance of \( B \) does not hold, but where \( A_0, \ldots, A_n \) hold for every formula. A full model, as described in [6], is sufficient. A full model is one where we can freely create predicates, as long as they satisfy the usual monotonicity requirements. So it is full in the sense that everything that potentially is the interpretation of a predicate actually is one. In Section 5 we will have to consider non-full models. An intuitionistic Kripke model is one where \( \bot \) is never forced at any world. These are exactly the Kripke models that force EFQ.

In given Kripke diagrams, each state \( A \) has its labelled propositions on the right, and the domain (denoted \( T(A) \)) on the left. Where the domain is given as \( \mathbb{N} \), it should be interpreted as the countable set of constants \( \{0, 1, \ldots\} \), without the addition of any function terms.

Proposition 4. If \( \vdash B_0, \ldots B_m \; \Phi \), and \( A_0, \ldots A_n \supset B_0, \ldots, B_m \), then \( \vdash A_0, \ldots A_n \; \Phi \).

Proof. Consider a natural deduction proof of \( \vdash B_0, \ldots B_m \; \Phi \). For each \( k \), replace each instance of the rule \( B_k \) with a proof of \( \vdash A_0, \ldots A_n \; B_k \). This produces the required derivation. \( \Box \)

We examine relative strengths of a selection of schemata by considering their relations under \( \supset \). The renaming of bound variables in a scheme should not affect its strength. For simplicity of notation, it is therefore assumed that when working a predicate \( P x \), any variables other than \( x \) which appear in quantifiers are bound in \( P x \). We write \( P y \) as shorthand for \( P x[y/x] \).

3 Principles

In addition to DNE, LEM, and EFQ, which are included below for convenience, we examine the following principles as axiom schemata over minimal logic:

\[
\text{DNE}(A) := \neg\neg A \rightarrow A \quad \text{(Double Negation Elimination\(^4\))}
\]

\(^3\)While technically speaking the Kripke semantics described in [4] are for the intuitionistic case, we can use them in the minimal one, by not forcing and condition on \( \bot \)—that is treating it just like some fixed propositional symbol.

\(^4\)Also known as “Stability”.

3
EFQ(\(A\)) := \bot \to A \quad \text{(Ex Falso Quodlibet\(^5\))}
LEM(\(A\)) := A \lor \neg A \quad \text{(Law of Excluded Middle\(^6\))}
WLEM(\(A\)) := \neg A \lor \neg \neg A \quad \text{(Weak Law of Excluded Middle)}
DGP(\(A, B\)) := (A \to B) \lor (B \to A) \quad \text{(Dirk Gently’s Principle\(^7\))}
DP(\(P\)) := \exists y(P y \to \forall x P x) \quad \text{(Drinker Paradox)}
H\varepsilon(\(P\)) := \exists y(\exists x P x \to P y) \quad \text{(Schematic Form of Hilbert’s Epsilon)}
GMP(\(P\)) := \neg \forall x P x \to \exists x \neg P x \quad \text{(General Markov’s Principle)}
GLPO(\(P\)) := \forall x \neg P x \lor \exists x P x \quad \text{(General Limited Principle of Omniscience)}
GLPO’(\(P\)) := \forall x P x \lor \exists x \neg P x \quad \text{(Alternate General Principle of Omniscience)}
DNS\_\forall(\(P\)) := \forall x \neg \neg P x \to \neg \neg \forall x P x \quad \text{(Universal Double Negation Shift)}
DNS\_\exists(\(P\)) := \neg \neg \exists x P x \to \exists x \neg P x \quad \text{(Existential Double Negation Shift)}
CD(\(P, Q\)) := \forall x(P x \lor \exists x Q) \to \forall x P x \lor \exists x Q \quad \text{(Constant Domain)}
IP(\(P, Q\)) := (\exists x Q \to \exists x P x) \to \exists x(\exists x Q \to P x) \quad \text{(Independence of Premise)}

These principles are all classically derivable. That is, DNE implies all of these principles in the sense of \(\supset\).

Principles CD and IP are also stated as
\[
\begin{align*}
CD(\(P, Q\)) & \equiv \forall x(P x \lor Q) \to \forall x P x \lor Q \\
IP(\(P, Q\)) & \equiv (Q \to \exists x P x) \to \exists x(Q \to P x)
\end{align*}
\]
where \(x\) is not free in \(Q\). These forms are syntactically equivalent to the definitions above for such \(Q\), but the variable freedom condition is not convenient to work with when classifying schemata.

\(^{5}\)Also known as “explosion”.
\(^{6}\)Also known as the “principle of excluded middle” and as “tertium non datur”.
\(^{7}\)The name DGP was introduced in [6], and is a literary reference to the novel [1], whose main character believes in “the fundamental interconnectedness of all things”. DGP is otherwise also known as (weak) linearity, and is the basis for Gödel-Dummett logic [13].
4 The Drinker Paradox and Hilbert’s Epsilon

The *drinker paradox*, which was popularised by Smullyan in his book of puzzles [12], is the scheme

\[ \text{DP}(P x) := \exists y (P y \to \forall x P x) . \]

Liberally interpreted, it states that (in every nonempty tavern) there exists a person such that if that person is drinking, then everyone (in the tavern) is drinking.

Classically this is true because there is always a last person to be drinking, and it is true for that person. Due to various non-classical interpretations of “there is”, however, countermodels may be formed (see Figure 1). Notably, the constructivist may object that it is not always clear who is the last to drink—except in the case of a tavern in which the number of patrons is an enumerable positive integer amount.

The drinker paradox can alternatively be stated as

\[ \exists y \forall x (P y \to P x) . \]

The dual of the drinker paradox is the scheme

\[ \text{He}(P x) := \exists y (\exists x P x \to P y) , \]

or alternatively,

\[ \exists y \forall x (P x \to P y) . \]

He resembles an axiom scheme form of Hilbert’s Epsilon operator [2]. In particular, within a natural deduction proof, from \(\exists x P x\) it allows a temporary name for a term satisfying \(P\) to be introduced. It is equivalent to Independence of Premise

\[ \text{IP}(P x, Q) := (\exists x Q \to \exists x P x) \to \exists x (\exists x Q \to P x) . \]

This does not have the same power as Hilbert’s Epsilon operator, however.

We will now characterise (full Kripke) models in which DP and/or He hold, and use these to separate the two schemata. We will ignore models containing disconnected states (i.e. models where there are pairs of states such that every state related to one is unrelated to the other), as these can be examined by the characteristics of the individual components.

First consider a model with states \(A \preceq B\) where there is a term \(t \in T(B) \setminus T(A)\) (for example Figure 1). Create a predicate \(P x\) with \(B \models P t\) for all \(s \in T(A)\) (and take the upwards closure). Now \(B \not\models Pt\), so \(A \not\models P s \to \forall x P x\), so DP fails. Furthermore, create a predicate \(Q x\) with \(B \models Qt\) (and take the upwards closure). Then \(B \models \exists x Q x\), but \(B \not\models Q s\) for any \(s \in T(A)\). Thus He fails at \(A\). Hence any model for either DP or He must have the same terms known at every related pair of states. We will from now on consider only these models. Moreover, note that a system with only one term at each state trivially models DP and He.

Now consider a model with a branch in it, i.e. there are states \(A, B, C\) such that \(A \preceq B\), \(A \preceq C\), and \(B\) is not related to \(C\). Assume there are at least two distinct terms understood at \(A\). Let \(t\) be one such term. Then create a predicate \(P\) with \(B \models Pt\), and \(C \models Ps\) for all terms \(s \in T(A) : s \neq t\) (and any other states forcing these atomic formulae as required to maintain upwards closure). Certainly neither \(B\) nor \(C\) force \(\forall x P x\), but for every \(u \in T(A)\) either \(B\) or \(C\) forces \(Pu\), so DP fails at \(A\). Furthermore if \(u \in T(A)\) then either \(B\) or \(C\) will fail to force \(Pu\),

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8For a proof that this is actually an equivalent formulation see the appendix.

9Milly Maietti has communicated to us the—currently unpublished—result that Hilbert’s Epsilon operator implies the drinker paradox. Thus, together with our results in this paper this shows that the operator version of He is stronger than the scheme version.
but both states force $\exists_x Px$, so $H\varepsilon$ also fails at $A$. Hence any model for DP or $H\varepsilon$ with more than two terms must have no branches, i.e. be totally ordered.

Consider then a linear model with finitely many terms. Given a predicate $Qx$, if every state forces $Qt$ for every term or if every state does not force $Qt$ for any term, then both DP and $H\varepsilon$ trivially hold (by applying the classical reasoning), so we may suppose that this is not the case. For each term $t$, assign a set $U_t = \{ A \in \Sigma | A \not\models Qt \}$. By upwards closure (and the assumed linearity), if $t$ and $s$ are terms then either $U_t \subseteq U_s$ or $U_s \subseteq U_t$, meaning these sets are totally ordered with respect to the subset relation. There are finitely many of them, so there must be a maximal set $U_{t_{\text{max}}}$ with associated term $t_{\text{max}}$. Suppose a state $A$ forces $Qt_{\text{max}}$. Then $A \not\in U_{t_{\text{max}}}$, and so $A \not\in U_s$ for every term $s$. Thus $A$ forces $Qs$. Hence $Qt_{\text{max}} \rightarrow \forall_s Qx$ holds in the model, and so this is a model for DP. A similar argument shows $H\varepsilon$ also holds, using sets $V_t = \{ A \in \Sigma | A \models Qt \}$, and in particular the maximal set $V_{t_0}$, to show that $\exists_s Qx \rightarrow Q_{t_0}$ is forced everywhere.

We now know that to separate DP and $H\varepsilon$ we require linear models with infinitely many terms.

**Proposition 5.** $H\varepsilon$ does not imply DP in intuitionistic logic.

**Proof.** Consider the (intuitionistic) Kripke model with infinitely many worlds below. In general, $A_n \preceq A_{n+1}$ and $A_n \models P_0 \ldots P_n$, and the domain at every world is $\mathbb{N}$.
No state forces $\forall_x P x$, but for any term $t \in T(A_0)$ we have $A_0 \preceq A_t$ and $A_t \vDash P t$. Therefore $A_0 \not\vDash \exists_y (P y \rightarrow \forall_x P x)$, i.e. DP does not hold in this model. (In fact, this argument works for any state.)

Now consider any predicate $Q x$ in this model. If there is no state forcing $Q t$ for some $t \in \mathbb{N}$, then trivially every state forces $\exists_x Q x \rightarrow Q 0$, and it follows that $\text{He}$ is forced. On the other hand, if there are $i, t \in \mathbb{N}$ such that $A_i \vDash Q t$, then choose a pair $i, t$ with minimal $i$. Then, by upwards closure, $\exists_x Q x \rightarrow Q t$ is forced by every state. Hence every state forces $\text{He}$.

The above model is also a countermodel for DNS. As $\neg P t$ is not forced at any world for any $t$, $A_0 \vDash \forall_x (\neg \neg P x)$. However $A_0 \vDash \neg \forall_x P x$, so $A_0 \not\vDash \text{DNS}(P)$.

**Proposition 6.** DP does not imply $\text{He}$ in intuitionistic logic.

**Proof.** Consider the (intuitionistic) Kripke system with states $A_0 \succeq A_{-1} \succeq A_{-2} \succeq \ldots \succeq A_{-\infty}$.

Let $T(B) = \mathbb{N}$ for every state $B$. Set $F(A_{-\infty}) = \emptyset$, and $F(A_{-n}) = \{P n, P(n + 1), P(n + 2), \ldots\}$.

Let $t \in T(A_{-\infty})$. Then $A_{-1(t+1)} \not\vDash P t$. However, $A_{-1(t+1)} \vDash P(t + 1)$, so $A_{-1(t+1)} \vDash \exists_x P x$. Therefore $A_{-1(t+1)} \not\vDash \exists_y (\exists_x P x \rightarrow P t)$, so $\text{He}$ does not hold in this model.

Now consider any predicate $Q x$ in this model. If every state forces $\forall_x Q x$, then trivially they also force $\exists_y (Q y \rightarrow \forall_x Q x)$. On the other hand, if there are $i, t \in \mathbb{N}$ such that $A_{-i} \not\vDash Q t$ then choose a pair $i, t$ with minimal $i$ (i.e. maximal $A_{-i}$). Then by upwards closure, whenever $Q t$ is forced, $\forall_x Q x$ is also forced. Hence every state forces $Q t \rightarrow \forall_x Q x$, and so also forces DP.

In general, if a model contains an infinite sequence of states $A_0 \preceq A_1 \preceq \ldots$, then a predicate $P$ can be constructed as in Proposition 5 in order to contradict DP. On the other hand if no such sequence exists then every sequence of related states has a maximal element. Following reasoning in Proposition 6 shows that DP will hold in such a model.
Conversely, if a model contains an infinite sequence of states $B_0 \succeq B_{-1} \succeq \cdots$, along with an element $B_{-\infty}$ which precedes every state in the sequence, then $P$ may be constructed as in Proposition \footref{P} contradicting $H\varepsilon$.

If, on the other hand, no such states exist, then every set of related states either contains a minimal element or has no lower bound, i.e. every set of states contains its infimum. Let $A$ be a state in such a model. Now consider the set $S$ of states above $A$ which force $\exists_x P x$. If $S = \emptyset$, then vacuously $A \models \exists_x P x \rightarrow Pt$ for every term $t$, so $A$ forces $H\varepsilon$. Otherwise, note that $A$ is certainly a lower bound for $S$. By the above assumption, $S$ must have a minimum element $B$. Now $B \models \exists_x P x$ so $B \models Pt$ for some $t$. By upwards closure, $C \models Pt$ for every $C \succeq B$, and so specifically for all $C \in S$. Thus whenever $A \preceq C$ and $C \models \exists_x P x$, we have $C \in S$, so $C \models Pt$. Then $A \models \exists_x P x \rightarrow Pt$, and so $A$ forces $H\varepsilon$. Hence $H\varepsilon$ is forced by every state, and so holds in this model.

We now have a characterisation for models of DP and $H\varepsilon$. They are the models wherein every state has exactly one term, or otherwise,

- the model is linear, and
- all terms are known at all states (domain is constant), and
- (to model DP) every set of states has a maximal element, and/or
- (to model $H\varepsilon$) every set of states contains its infimum.

Where $T$ is the set of terms (at every state):

|                | $|T| = 1$ | $|T| \in \mathbb{N}$ | $|T| \geq |\mathbb{N}|$ |
|----------------|----------|----------------|--------------------------|
| Branched       | DP, $H\varepsilon$ | Neither | Neither |
| Linear         | DP, $H\varepsilon$ | DP, $H\varepsilon$ | Indeterminate |
| Linear, max $\Pi$ exists for all $\Pi \subseteq \Sigma$ | DP, $H\varepsilon$ | DP, $H\varepsilon$ | DP |
| Linear, inf $\Pi \in \Pi$ for all $\Pi \subseteq \Sigma$ | DP, $H\varepsilon$ | DP, $H\varepsilon$ | $H\varepsilon$ |
| Both of the two above | DP, $H\varepsilon$ | DP, $H\varepsilon$ | DP, $H\varepsilon$ |

If a model has graph-like connectedness, where all related pairs of states have finitely many states between them (and so finite paths between them), then it cannot fall under the third or fourth rows, and so cannot separate DP and $H\varepsilon$.

The models are evocative of the intuitions. For, recall the “last drinker in the tavern” reason for accepting DP as true; similarly $H\varepsilon$ can be justified by pointing to “the first person to drink”.

**Corollary 7.** DP and $H\varepsilon$ are independent of each other in minimal logic with LEM (and so certainly over decidable predicates).

**Proof.** Recall the Kripke systems in Propositions \footref{P} and \footref{P}. Considering them now as minimal Kripke systems, and forcing $\bot$ at every state forces LEM everywhere, but their respective separations still hold. \qed

## 5 Separations without full models

The **Constant Domain** principle is

$$CD(Px, Q) := \forall x (Px \lor \exists x Q) \rightarrow \forall x Px \lor \exists x Q.$$
Consider a full Kripke model in which all related worlds have the same domain. For a world $A$, if $A \vDash \forall x (Px \lor \exists xQ)$ then $A \vDash Pt \lor \exists xQ$ for all $t$ in the domain. If $A \nvdash \exists xQ$, then $A \vDash Pt$, and so $A \vDash \forall xPx$. Therefore this is a model for CD. Hence any full Kripke countermodel for CD must have related worlds with different domains, and so must also be a countermodel to $H_K$ (from the section above).

However, we cannot conclude $H_K \supset CD$, as restriction to full Kripke models does not preserve completeness of Kripke semantics. To see that $\not\vDash_{H_K} CD(Px, Q)$, we require a non-full countermodel to CD in which $H_K$ holds. Therefore, a notion of an axiom scheme holding in a non-full model is needed. For every formula $\Phi$ in the model, $H_K(\Phi)$ should be forced. Formulae in the model should be at least closed with respect to the logical operations `→', `∧', `∨', `∃', and `∀', and `⊥' must also be a formula. The constants in the domain of the root world may also appear in formulae, but no others.

Consider the following infinite model:

```
N 0, A0, A1, A2
N 0, A0, A1, A2
N 0, A0, A1, A2
N 0, A0, A1, A2
{0} 0, A0, A1, A2
```

We have $A \not\vDash CD(Px, Q)$.

$H_K$ holds trivially for propositions. It remains to confirm that $H_K$ holds for all predicates which exist in this model. Predicates are definable by combining `Px' and `Qx', with each other and with propositions, using the binary logical operations. Clearly, combining a predicate with itself in this manner is trivial. The propositions available are only $P0$, $Q0$, ⊥, since

\[
\begin{align*}
\forall x Px & \equiv \bot \\
\forall x Qx & \equiv \bot \\
\exists x Px & \equiv P0 \\
\exists x Qx & \equiv Q0
\end{align*}
\]

and $P0$, $Q0$, ⊥ are closed under the binary logical operations (with respect to equivalence in this model). First,

\[
\begin{align*}
Px \rightarrow Qx & \equiv Qx \\
Qx \rightarrow Px & \equiv P0 \\
Px \lor Qx & \equiv Px \\
Px \land Qx & \equiv Qx
\end{align*}
\]
Now, with $P_0$,

\[
\begin{align*}
&P_x \rightarrow P_0 \equiv P_0 \\
&P_0 \rightarrow P_x \equiv P_x \\
&P_x \lor P_0 \equiv P_0 \\
&P_x \land P_0 \equiv P_x \\
&Q_x \rightarrow Q_0 \equiv P_0 \\
&Q_0 \rightarrow Q_x \equiv Q_x \\
&Q_x \lor Q_0 \equiv Q_0 \\
&Q_x \land Q_0 \equiv Q_x.
\end{align*}
\]

With $Q_0$,

\[
\begin{align*}
&P_x \rightarrow Q_0 \equiv Q_0 \\
&Q_0 \rightarrow P_x \equiv Q_x \\
&P_x \lor Q_0 \equiv P_0 \\
&P_x \land Q_0 \equiv Q_x \\
&Q_x \rightarrow Q_0 \equiv P_0 \\
&Q_0 \rightarrow Q_x \equiv Q_x \\
&Q_x \lor Q_0 \equiv Q_0 \\
&Q_x \land Q_0 \equiv Q_x.
\end{align*}
\]

Finally, with $\bot$,

\[
\begin{align*}
&P_x \rightarrow \bot \equiv \bot \\
&\bot \rightarrow P_x \equiv P_0 \\
&P_x \lor \bot \equiv P_x \\
&P_x \land \bot \equiv \bot \\
&Q_x \rightarrow \bot \equiv \bot \\
&\bot \rightarrow Q_x \equiv P_0 \\
&Q_x \lor \bot \equiv Q_x \\
&Q_x \land \bot \equiv \bot.
\end{align*}
\]

Thus, $P_x$ and $Q_x$ really are the only predicates in this model. $A \vDash H \varepsilon (P), H \varepsilon (Q)$, so we have a non-full model for $H \varepsilon$ where CD fails.

### 6 From first-order to propositional schemata

Some first-order schemata are infinitary forms of propositional schemata. Viewing universal and existential generalisation as conjunction and disjunction on propositional symbols $A$ and $B$, the drinker paradox becomes

\[(A \rightarrow (A \land B)) \lor (B \rightarrow (A \land B)),\]

and so DGP follows. A formal proof requires embedding $A$ and $B$ in a single predicate. For example, over the domain of natural numbers, a predicate $P$ such that

\[P(0) \leftrightarrow A\]
$P(Sn) \leftrightarrow B$

gives $DP(Px) \vdash DGP(A, B)$. However, such an embedding is not possible if the domain contains a single element. It was shown above that DP holds in models with branches if the domain contains only one term, while in [6] it is shown that DGP holds only in $v$-free models. Therefore there can be no way of deriving instances of DGP from DP without an embedding using two or more elements in the domain.

![Figure 3: Kripke countermodel for DGP(A, B) where DP holds](image)

Domain is a semantic concept. In order to derive an instance of DGP using DP, we require syntax corresponding to the existence of more than one (distinct) term.

**Definition 8.** Natural deduction can be extended by adding term names 0 and 1, a unary predicate $D$, and the rules

- **D0:**
  
  $D0 \quad D0$

- **D1:**
  
  $\neg D1 \quad \neg D1$

- **Dx:**
  
  $\forall x (Dx \lor \neg Dx) \quad Dx$

$D$ serves to make a weak distinction between the constants named by 0 and 1.

We call minimal (intuitionistic) logic extended by these rules *two-termed* minimal (intuitionistic) logic, in which case we write $\vdash_{TT}$ in place of $\vdash$.

Semantically, an intuitionistic Kripke model for TT is one in which there are two constants 0 and 1, $D0$ holds at every world, and $Dn$ is not forced for $n \neq 0$. For minimal Kripke models, it is also possible instead that there is only one term, and $\bot$ holds everywhere.

In general, given propositional symbols $A$ and $B$, we want to define a predicate $P$ such that

\[ \forall x Px \vdash A \land B \quad \exists x Px \vdash A \lor B. \]

We recover

\[ DP(\langle Dx \to A \rangle \land \langle \neg Dx \to B \rangle) \vdash_{EFQ, TT} DGP(A, B) \]

\[ Hc(\langle Dx \to A \rangle \land \langle \neg Dx \to B \rangle) \vdash_{EFQ, TT} DGP(A, B) \]

\[ DP(\langle Dx \to \neg \neg A \rangle \land \langle \neg Dx \to \neg A \rangle) \vdash_{TT} WLEM(A) \]

\[ Hc(\langle Dx \to \neg \neg A \rangle \land \langle \neg Dx \to \neg A \rangle) \vdash_{TT} WLEM(A) \]

\[ GMP(\langle Dx \to \neg \neg A \rangle \land \langle \neg Dx \to \neg A \rangle) \vdash_{TT} WLEM(A) \]

\[ DNS3(\langle Dx \to \neg \neg A \rangle \land \langle \neg Dx \to \neg A \rangle) \vdash_{TT} WLEM(A). \]

---

10Bell in [3] suggests this “modest ‘decidability’ condition” in the form of a decidable equality for a single constant $a$, along with a constant $b \neq a$. 

11
7 Hierarchy

The preorder from ‘⊃’ produces a hierarchy. Arrows labelled with schemes indicate that those schemes must be taken together with the scheme at the tail to produce the scheme at the head.

This hierarchy is complete in the sense that no other unlabelled arrows may be added (see below). Moreover, for arrows labelled with at least one of EFQ, TT, the remaining open questions are if GMP, EFQ ⊃ CD and/or GMP, EFQ, TT ⊃ CD.

8 Semantics

In addition to the Kripke model analysis presented earlier, the following full models give all possible separations of the schemes under investigation. In cases where models should have TT, we omit labelling $D_0$ on every world for the sake of brevity.

In [6], it is shown that DGP and WLEM hold in all $v$-free models, EFQ holds in a model if and only if $\perp$ is not forced anywhere, and LEM holds if only one world does not force $\perp$. Revisiting the countermodels (and previously given reasoning) for DP and $H_\varepsilon$, we have
is a model for EFQ, TT, Hε, DGP, WLEM, CD, and a countermodel for DP, LEM, DNSₜ, while

is a model for EFQ, TT, DP, DGP, WLEM and a countermodel for Hε, LEM.

It is trivial to check that model presented in Section 5 can be modified as follows, to model both Hε and TT while still being a countermodel to CD.
It is straightforward to check whether a scheme holds or fails in a given finite full model; as only (few and) finitely many upwards closed labellings of worlds are possible, and these may be checked exhaustively. We therefore present the remaining models without comment.

\[
\{s, t\} \xrightarrow{B} \{s, t\} \xrightarrow{C} \{s, t\}
\]

is a model for EFQ, TT, DNS\(\varepsilon\), CD and a countermodel for DP, H\(\varepsilon\), DGP, WLEM, DNS\(\varepsilon\).

\[
\{s, t\} \xrightarrow{B} \perp \{s, t\} \xrightarrow{C} \perp
\]

is a model for GLPO', LEM and a countermodel for DP, H\(\varepsilon\), DGP.

\[
\{s\} \xrightarrow{B} \{s\} \xrightarrow{C}
\]

is a model for EFQ, DP, H\(\varepsilon\) and a countermodel for DGP, WLEM.

\[
\{s\} \xrightarrow{B} \perp \{s\} \xrightarrow{C} \perp
\]

is a model for TT, DP, H\(\varepsilon\), GLPO' and a countermodel for DGP.
is a model for EFQ, TT, WLEM, GMP and a countermodel for DP, $H\varepsilon$, DGP.

is a model for LEM, WLEM, DGP, GLPO', GMP, DP, $H\varepsilon$, DNS$_\forall$, DNS$_3$, CD, EFQ and a countermodel for TT.

is a model for TT, EFQ, DGP, WLEM, DNS$_\forall$ and a countermodel for DNS$_3$, CD.

is a model for TT, LEM and a countermodel for EFQ, GMP, CD, DNS$_\forall$, DP, $H\varepsilon$.

is a model for TT, DGP, GMP, GLPO' and a countermodel for EFQ, CD, $H\varepsilon$, DP.

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Appendix

**Proposition 9.** $DNE \supset LEM$

*Proof.*

\[
\begin{align*}

\neg(A \lor \neg A) & \Rightarrow \bot \quad \left(\text{DNE}\right) \\
\bot & \Rightarrow \neg A \\
\neg A & \Rightarrow \bot \quad \left(\text{I}\right) \\
\end{align*}
\]

**Proposition 10.** $DNE \supset EFQ$

*Proof.*

\[
\begin{align*}

\neg A & \Rightarrow A \\
\bot & \Rightarrow A \lor \bot \\
\end{align*}
\]

**Proposition 11.** $LEM, EFQ \supset DNE$

*Proof.*

\[
\begin{align*}

A \lor \neg A & \Rightarrow \bot \quad \left(\text{LEM}\right) \\
\bot & \Rightarrow \neg A \\
\neg A & \Rightarrow \bot \quad \left(\text{I}\right) \\
\end{align*}
\]
Proposition 12. $H \vDash IP$

Proof. 

\[
\begin{align*}
\exists x P x & \rightarrow Py \\
\exists y (\exists x P x \rightarrow Py) & \rightarrow E \\
\exists y (\exists x P x \rightarrow Py) & \rightarrow I \\
\exists x (\exists y (\exists x P x \rightarrow Py)) & \rightarrow I \\
\exists x (\exists y (\exists x P x \rightarrow Py)) & \rightarrow E
\end{align*}
\]

Proposition 13. $IP \vDash H$

Proof. 

\[
\begin{align*}
(\exists x P x \rightarrow \exists x P x) & \rightarrow E \\
\exists x (\exists x P x \rightarrow P x) & \rightarrow I \\
\exists x (\exists x P x \rightarrow P x) & \rightarrow E \\
\exists x (\exists x P x \rightarrow P x) & \rightarrow I
\end{align*}
\]

Proposition 14. $LEM \vDash GLPO$

Proof. 

\[
\begin{align*}
\neg \exists x P x & \rightarrow E \\
\neg P x & \rightarrow I \\
\exists x P x \lor \neg \exists x P x & \rightarrow E \\
\forall x \neg P x \lor \exists x P x & \rightarrow E
\end{align*}
\]

Proposition 15. $GLPO \vDash LEM$

Proof. 

\[
\begin{align*}
\forall x \neg A & \lor \exists x A \\
\forall x \neg A & \lor \exists x A \\
\forall x \neg A & \lor \exists x A
\end{align*}
\]
Proposition 16. $DNS_{\forall} \supset WGMP$

Proof.

\[
\begin{align*}
\forall x \neg P x & \rightarrow \neg \exists x P x \quad \exists I \\
\bot & \rightarrow \exists I \\

\forall x \neg P x & \rightarrow \forall I \\

\neg \forall x P x & \rightarrow \forall E \\

\exists x \neg P x & \rightarrow \exists I \\
\neg \exists x \neg P x & \rightarrow \forall E \\

\neg P x & \rightarrow \forall E
\end{align*}
\]

Proposition 17. $WGMP \supset DNS_{\forall}$

Proof.

\[
\begin{align*}
\neg \forall x P x & \rightarrow \neg \exists x \neg P x \quad WGMP \\
\neg \exists x \neg P x & \rightarrow \forall E \\

\exists x P x & \rightarrow \exists I \\
\bot & \rightarrow \exists E \\

\forall x \neg P x & \rightarrow \forall I \\

\neg \forall x P x & \rightarrow \forall E \\

\neg \exists x \neg P x & \rightarrow \forall E \\
\end{align*}
\]
Proposition 18. \( DP(Px) \) is equivalent to \( \exists_y \forall_x (Py \to Px) \)

Proof. ( \( \implies \) )

\[
\begin{align*}
\exists_y (Py \to \forall_x Px) \quad & \quad Py \to \forall_x Px \\
\forall_y (Py \to \forall_x Px) \quad & \quad Py \to E \\
\exists_y (Py \to \forall_x Px) \quad & \quad \exists_E \\
\forall_y (Py \to \forall_x Px) \quad & \quad \exists E \\
\end{align*}
\]
Proposition 19. $\exists_x (P_x)$ is equivalent to $\exists_y \forall_x (P_x \rightarrow P_y)$

Proof. $(\implies)$

\[
\begin{align*}
\exists_y (\exists_x P_x \rightarrow P_y) & \quad \exists_I \\
\exists_x P_x & \quad \exists_I \\
\forall_x (P_x \rightarrow P_y) & \quad \forall_I \\
\exists_y \forall_x (P_x \rightarrow P_y) & \quad \exists_I \\
\exists_y \forall_x (P_x \rightarrow P_y) & \quad \exists_I \\
\exists_y (\exists_x P_x \rightarrow P_y) & \quad \exists_E \\
\exists_y (\exists_x P_x \rightarrow P_y) & \quad \exists_E \\
\exists_y (\exists_x P_x \rightarrow P_y) & \quad \exists_I \\
\exists_y (\exists_x P_x \rightarrow P_y) & \quad \exists_E \\
\end{align*}
\]
**Proposition 20.** $DNE, LEM, EFQ \supset DP$

*Proof.* First

\[
\begin{align*}
\neg\neg P_x & \quad \neg P_x \quad \exists I \\
\vdash P_x & \quad \exists P_x \quad \rightarrow E \\
\neg P_x & \quad \ightarrow I \\
\neg P_x & \quad \rightarrow E \\
\neg \forall x P_x & \quad \forall I \\
\forall x P_x & \quad \forall E \\
\neg \exists x \neg P_x & \quad \neg \exists x \neg P_x \\
\vdash \exists x P_x & \quad \exists I \\
\exists x P_x & \quad \exists E
\end{align*}
\]

Now,

\[
\begin{align*}
\forall x P_x & \\
\vdash \forall x P_x & \quad \text{EFQ} \\
\neg P_x & \quad \forall P_x \quad \rightarrow I \\
\forall P_x & \quad \rightarrow E \\
\forall x P_x \lor \neg \forall x P_x & \quad \text{LEM} \\
\forall y (P_y \rightarrow \forall x P_x) & \quad \rightarrow I \\
\exists y (P_y \rightarrow \forall x P_x) & \quad \exists I \\
\exists y (P_y \rightarrow \forall x P_x) & \quad \exists E
\end{align*}
\]

\[\square\]

**Proposition 21.** $LEM \supset WLEM$

*Proof.*

\[
\begin{align*}
\neg A \lor \neg\neg A & \quad \text{LEM}
\end{align*}
\]

\[\square\]
Proposition 22. $GMP \supset WGMP$

Proof.

\[
\begin{align*}
\neg \exists x \neg P x & \quad GMP \\
\neg \exists x \neg P x & \quad \exists x \neg P x \quad \rightarrow E \\
\bot & \quad \rightarrow I \\
\neg \forall x \neg P x & \quad \neg \exists x \neg P x \quad \rightarrow I \\
\neg \forall x P x \rightarrow \neg \exists x \neg P x & \quad \rightarrow E
\end{align*}
\]

\[\square\]

Proposition 23. $DGP \supset WLEM$

Proof.

\[
\begin{align*}
A \rightarrow \neg A & \quad A \rightarrow E \\
\bot & \quad \rightarrow I \\

\neg A \quad \rightarrow I \\
\neg A \quad \rightarrow E
\end{align*}
\]

\[
\begin{align*}
(A \rightarrow \neg A) \lor (\neg A \rightarrow A) & \quad DGP \\
\neg A \lor \neg A & \quad \lor I \\
\neg A \lor \neg A & \quad \lor E
\end{align*}
\]

\[\square\]

Proposition 24. $GLPO' \supset LEM$

Proof.

\[
\begin{align*}
\forall x A & \quad \lor E \\
\exists x \neg A & \quad \lor E
\end{align*}
\]

\[
\begin{align*}
A \lor \neg A & \quad \lor I \\
\neg A & \quad \lor I
\end{align*}
\]

\[
\begin{align*}

A \lor \neg A & \quad \lor I \\
\neg A & \quad \lor I
\end{align*}
\]

\[\square\]

Proposition 25. $GLPO' \supset GMP$

Proof.

\[
\begin{align*}
\forall x P x & \quad \lor E \\
\exists x \neg P x & \quad \rightarrow I
\end{align*}
\]

\[
\begin{align*}
\forall x P x \lor \exists x \neg P x & \quad GLPOA \\
\exists x \neg P x & \quad \lor I \\
\exists x \neg P x & \quad \lor E
\end{align*}
\]

\[
\begin{align*}

\neg \forall x P x & \quad \exists x \neg P x \quad \rightarrow I \\

\neg \forall x P x \rightarrow \exists x \neg P x & \quad \rightarrow I
\end{align*}
\]

\[\square\]
Proposition 26. $DP \supset CD$

Proof.

$$
\begin{align*}
\forall x (P x \lor \exists x A) & \quad \frac{Py \lor \exists x A}{Py} \quad \forall E \quad \frac{\exists x A}{\exists x A} \quad \forall I \quad \frac{\forall x P x \lor \exists x A}{\forall x P x \lor \exists x A} \quad \exists E \\
\forall x P x \lor \exists x A & \quad \frac{\forall x P x \lor \exists x A}{\exists y (Py \rightarrow \forall x P x)} \quad DP \\
\forall x (P x \lor \exists x A) & \rightarrow (\forall x P x \lor \exists x A) \\
\end{align*}
$$

Proposition 27. $DP \supset GMP$

Proof.

$$
\begin{align*}
\neg \forall x P x \quad \frac{Py \rightarrow \forall x P x}{Py} \quad \forall E \quad \frac{\exists x P x}{\exists x P x} \quad \exists I \\
\exists y (Py \rightarrow \forall x P x) & \quad DP \\
\exists y (Py \rightarrow \forall x P x) & \quad \frac{\exists x P x \rightarrow \exists x P x}{\exists x P x} \quad \forall E \\
\neg \forall x P x & \rightarrow \exists x \neg P x \\
\end{align*}
$$

Proposition 28. $HE \supset DNS_3$

Proof.

$$
\begin{align*}
\neg Py & \quad \frac{\exists x P x \rightarrow Py}{\exists x P x} \quad \exists E \\
\neg \exists x P x & \quad \neg \neg P y \rightarrow I \\
\neg \exists x P x & \rightarrow I \\
\neg \exists x P x & \rightarrow I \\
\exists y (\exists x P x \rightarrow Py) & \quad HE \\
\neg \exists x P x & \rightarrow I \\
\end{align*}
$$
Proposition 29. \( GLPO \supset DNS_\exists \)

\[
\begin{align*}
\text{Proof.} & & \\
\forall x \neg P_x & \vdash \neg P_x & \forall E & \vdash P_x & \rightarrow E \\
\exists x P_x & \vdash \bot & \exists E & \vdash \exists P_x & \rightarrow E \\
\neg \exists x P_x & \vdash \bot & \rightarrow E & \vdash \exists x \neg P_x & \rightarrow I \\
\exists x \neg P_x & \vdash \exists x \neg P_x & \rightarrow I & \neg \neg \exists x P_x & \rightarrow \exists x \neg P_x \\
\neg x \neg P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad & \\
\exists x \neg P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad &
\end{align*}
\]

Proposition 30. \( GMP \supset DNS_\exists \)

\[
\begin{align*}
\text{Proof.} & & \\
\forall x \neg P_x & \rightarrow \exists x \neg P_x & \forall E & \vdash P_x & \rightarrow E \\
\exists x P_x & \vdash \bot & \exists E & \vdash \exists x \neg P_x & \rightarrow E \\
\neg \exists x P_x & \vdash \bot & \rightarrow E & \vdash \exists x \neg P_x & \rightarrow I \\
\exists x \neg P_x & \vdash \exists x \neg P_x & \rightarrow I & \neg \forall x P_x & \rightarrow \exists x \neg P_x \\
\neg \forall x P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad & \\
\exists x \neg P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad &
\end{align*}
\]

Proposition 31. \( GLPO' \supset WGMP \)

\[
\begin{align*}
\text{Proof.} & & \\
\forall x P_x & \rightarrow \forall x P_x & \forall E & \vdash \forall x P_x & \rightarrow E \\
\exists x \neg P_x & \rightarrow \exists x \neg P_x & \forall E & \vdash \exists x \neg P_x & \rightarrow E \\
\neg \forall x P_x & \vdash \bot & \rightarrow E & \vdash \exists x \neg P_x & \rightarrow I \\
\exists x \neg P_x & \vdash \bot & \rightarrow E & \vdash \exists x \neg P_x & \rightarrow I \\
\neg \forall x P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad & \\
\exists x \neg P_x & \rightarrow \exists x \neg P_x & \rightarrow I & \quad &
\end{align*}
\]
Proposition 32. $DP, EFQ, TT \supset DGP$

Proof. Where \( \Phi = ((Dy \rightarrow A) \land (\neg Dy \rightarrow B)) \rightarrow \forall_x ((Dx \rightarrow A) \land (\neg Dx \rightarrow B)) \),

Lemma 1:

\[
\Phi
\]

\[
\frac{\frac{A}{Dy \rightarrow A} \rightarrow I}{Dy \rightarrow A} \rightarrow I
\]

\[
\frac{\frac{\neg Dy \rightarrow B}{\neg Dy \rightarrow B} \rightarrow I}{\neg Dy \rightarrow B} \rightarrow I
\]

\[
\frac{(Dy \rightarrow A) \land (\neg Dy \rightarrow B)}{\rightarrow E}
\]

\[
\frac{\forall_x ((Dx \rightarrow A) \land (\neg Dx \rightarrow B))}{\rightarrow E}
\]

\[
\frac{(D1 \rightarrow A) \land (\neg D1 \rightarrow B)}{\rightarrow E}
\]

\[
\frac{\neg D1 \rightarrow B}{\rightarrow E}
\]

\[
\frac{\neg D1 \rightarrow D0}{\rightarrow E}
\]

\[
\frac{(A \rightarrow B) \lor (B \rightarrow A)}{\rightarrow I}
\]

Lemma 2:

\[
\Phi
\]

\[
\frac{\frac{\bot \rightarrow A}{\bot \rightarrow A} \rightarrow E}{A \rightarrow E}
\]

\[
\frac{\frac{\neg Dy \rightarrow Dy}{\neg Dy \rightarrow Dy} \rightarrow E}{Dy \rightarrow E}
\]

\[
\frac{\frac{B}{\bot \rightarrow B} \rightarrow I}{B \rightarrow B} \rightarrow I
\]

\[
\frac{(Dy \rightarrow A) \land (\neg Dy \rightarrow B)}{\rightarrow E}
\]

\[
\frac{\forall_x ((Dx \rightarrow A) \land (\neg Dx \rightarrow B))}{\rightarrow E}
\]

\[
\frac{(D0 \rightarrow A) \land (\neg D0 \rightarrow B)}{\rightarrow E}
\]

\[
\frac{D0 \rightarrow A}{\rightarrow E}
\]

\[
\frac{(A \rightarrow B) \lor (B \rightarrow A)}{\rightarrow I}
\]

\[
\frac{(A \rightarrow B) \lor (B \rightarrow A)}{A \rightarrow E}
\]

Now,

\[
\exists x \Phi
\]

\[
\frac{\forall_x (Dx \lor \neg Dx)}{\rightarrow E}
\]

\[
\frac{\frac{Dy \lor \neg Dy}{\forall E}}{\rightarrow E}
\]

\[
\frac{(A \rightarrow B) \lor (B \rightarrow A)}{\rightarrow E}
\]

\[
\frac{(A \rightarrow B) \lor (B \rightarrow A)}{\exists E}
\]
Proposition 33. $DP, TT \supset WLEM$

Proof. Where $\Phi = ((D_y \to \neg A) \land (\neg D_y \to \neg A)) \to \forall_x ((D_x \to \neg A) \land (\neg D_x \to \neg A))$,

Lemma 1:

\[
\frac{\neg A \quad A}{E} \quad \frac{\neg D_y \quad D_y}{E} \quad \frac{\neg A}{I} \quad \frac{\neg A}{I} \quad \frac{\neg D_y \to \neg A}{\land I} \quad \frac{(D_y \to \neg A) \land (\neg D_y \to \neg A)}{E} \quad \frac{\forall_x ((D_x \to \neg A) \land (\neg D_x \to \neg A))}{\forall E} \quad \frac{(D_1 \to \neg A) \land (\neg D_1 \to \neg A)}{E} \quad \frac{\neg D_1 \to \neg A \quad \neg D_1 \to \neg A}{DO} \quad \frac{\neg A}{E} \quad \frac{\neg A}{E} \quad \frac{\neg A \lor \neg A}{\lor I}
\]

Lemma 2:

\[
\frac{\neg D_y \quad D_y}{E} \quad \frac{\neg A}{I} \quad \frac{\neg A}{I} \quad \frac{\neg D_y \to \neg A}{\land I} \quad \frac{(D_y \to \neg A) \land (\neg D_y \to \neg A)}{E} \quad \frac{\forall_x ((D_x \to \neg A) \land (\neg D_x \to \neg A))}{\forall E} \quad \frac{(D_0 \to \neg A) \land (\neg D_0 \to \neg A)}{E} \quad \frac{\neg D_0 \to \neg A \quad \neg D_0 \to \neg A}{DO} \quad \frac{\neg A}{E} \quad \frac{\neg A}{E} \quad \frac{\neg A \lor \neg A}{\lor I}
\]

Now,

\[
\frac{\forall_x (D_x \lor \neg D_x)}{DX} \quad \frac{\neg A \lor \neg A}{\lor I} \quad \frac{\neg A}{E} \quad \frac{\neg A}{E} \quad \frac{\exists y \Phi \quad DP}{\forall E} \quad \frac{\neg A \lor \neg A}{\lor I} \quad \frac{\neg A \lor \neg A}{\lor I} \quad \frac{\neg A \lor \neg A}{\lor I} \quad \frac{\neg A \lor \neg A}{\lor I}
\]

$\square$
Proposition 34. \( H_e, EFQ, TT \supset DGP \)

Proof. Where \( \Phi = \exists_x ((Dx \rightarrow A) \land (\neg Dx \rightarrow B)) \rightarrow ((Dy \rightarrow A) \land (\neg Dy \rightarrow B)), \)

Lemma 1:

\[
\begin{array}{c}
\Phi \quad \exists_x ((Dx \rightarrow A) \land (\neg Dx \rightarrow B)) \\
\therefore (Dy \rightarrow A) \land (\neg Dy \rightarrow B)
\end{array}
\]

Now,

\[
\begin{array}{c}
\forall_x (Dx \lor \neg Dx) \\
\therefore (A \lor \neg A)
\end{array}
\]
Proposition 35. $H_e, TT \supset WLEM$

Proof. Where $\Phi = \exists_x ((Dx \rightarrow \neg\neg A) \land (\neg Dx \rightarrow A)) \rightarrow ((Dy \rightarrow \neg\neg A) \land (\neg Dy \rightarrow A))$,

Lemma 1:

\[
\begin{align*}
\Phi & \quad \exists x \left((Dx \rightarrow \neg\neg A) \land (\neg Dx \rightarrow A)\right) \\
& \quad \rightarrow ((Dy \rightarrow \neg\neg A) \land (\neg Dy \rightarrow A)) \\
& \quad \rightarrow E \\
& \quad \neg A \rightarrow I \\
& \quad \neg A \rightarrow \neg A \\
& \quad \land I \\
& \quad \neg A \rightarrow I \\
& \quad \neg A \rightarrow A \\
& \quad \lor I
\end{align*}
\]

Lemma 2:

\[
\begin{align*}
\Phi & \quad \exists y \left((Dx \rightarrow \neg\neg A) \land (\neg Dx \rightarrow A)\right) \\
& \quad \rightarrow ((Dy \rightarrow \neg\neg A) \land (\neg Dy \rightarrow A)) \\
& \quad \rightarrow E \\
& \quad \neg A \rightarrow I \\
& \quad \neg A \rightarrow \neg A \\
& \quad \land I \\
& \quad \neg A \rightarrow I \\
& \quad \neg A \rightarrow A \\
& \quad \lor I
\end{align*}
\]

Now,

\[
\begin{align*}
\forall x (Dx \lor \neg Dx) & \quad \rightarrow Dx \\
\forall E & \quad \neg A \land \neg A \\
\rightarrow E & \quad \exists y \Phi \\
\neg A \land \neg A & \quad \lor E
\end{align*}
\]

\qed
Proposition 36. GMP, TT ⊢ WLEM

Proof. Lemma 1:

\[
\begin{array}{c}
\forall_x ((Dx \rightarrow \neg A) \land \neg (\neg Dx \rightarrow \neg A)) \\
\text{∀E} \quad \frac{D0 \rightarrow \neg A}{\neg A} \\
\land E \quad \frac{DZ}{\neg A \rightarrow \neg A} \\
\text{⊥} \quad \frac{\neg A \land \neg A}{\rightarrow I} \\
\neg \forall_x ((Dx \rightarrow \neg A) \land (\neg Dx \rightarrow \neg A)) \rightarrow E
\end{array}
\]

Lemma 2:

\[
\begin{array}{c}
\neg ((Dx \rightarrow \neg A) \land (\neg Dx \rightarrow \neg A)) \\
\text{∀E} \quad \frac{\neg A}{\neg A \rightarrow \neg A} \\
\land I \quad \frac{Dx \rightarrow \neg A}{\neg A \land \neg A} \rightarrow E
\end{array}
\]

Lemma 3:

\[
\begin{array}{c}
\neg(Dx \rightarrow \neg A) \land (\neg Dx \rightarrow \neg A)) \\
\land I \quad \frac{\neg A}{\neg A \rightarrow \neg A} \\
\text{∀E} \quad \frac{\neg A \land \neg A}{\neg A \land \neg A} \rightarrow E
\end{array}
\]

Now, where \( \Phi = \neg \forall_x ((Dx \rightarrow \neg A) \land (\neg Dx \rightarrow \neg A)) \rightarrow \exists_x \neg ((Dx \rightarrow \neg A) \land (\neg Dx \rightarrow \neg A)) \)

\[
\begin{array}{c}
\neg \exists_x (Dx \lor \neg Dx) \rightarrow E \\
\text{∀E} \quad \frac{Dx \lor \neg Dx}{\neg A \lor \neg A} \rightarrow E
\end{array}
\]

\[\square\]
Proposition 37. \( DP, LEM \supset GLPO \)

Proof.

\[
\begin{align*}
\exists y (P y \rightarrow \forall x P x) & \quad \text{DP} \\
\exists y (P y \rightarrow \forall x P x) & \quad \text{LEM} \\
\forall x P x \lor \exists y \neg P x & \quad \forall x P x \lor \exists y \neg P x \\
\end{align*}
\]

\[ \forall x P x \lor \exists y \neg P x \]

Proposition 38. \( DNS, TT \supset WLEM \)

Proof. Lemma 1:

\[
\begin{align*}
\neg \exists x ((D x \rightarrow \neg \neg A) \land (\neg D x \rightarrow \neg A)) & \quad \exists \text{I} \\
\exists x ((D x \rightarrow \neg \neg A) \land (\neg D x \rightarrow \neg A)) & \quad \exists \text{I} \\
(D 0 \rightarrow \neg \neg A) \land (\neg D 0 \rightarrow \neg A) & \quad \exists \text{I} \\
\neg(D 0 \rightarrow \neg \neg A) \land (\neg D 0 \rightarrow \neg A) & \quad \exists \text{I} \\
\neg \neg \neg A & \quad \exists \text{I} \\
\neg \neg \neg A & \quad \exists \text{I} \\
\end{align*}
\]

Lemma 2:

\[
\begin{align*}
\neg \exists x ((D x \rightarrow \neg \neg A) \land (\neg D x \rightarrow \neg A)) & \quad \exists \text{I} \\
\exists x ((D x \rightarrow \neg \neg A) \land (\neg D x \rightarrow \neg A)) & \quad \exists \text{I} \\
(D 1 \rightarrow \neg \neg A) \land (\neg D 1 \rightarrow \neg A) & \quad \exists \text{I} \\
(D 1 \rightarrow \neg \neg A) \land (\neg D 1 \rightarrow \neg A) & \quad \exists \text{I} \\
\neg \neg A & \quad \exists \text{I} \\
\neg \neg A & \quad \exists \text{I} \\
\end{align*}
\]

Lemma 3:

\[
\begin{align*}
(D x \rightarrow \neg \neg A) \land (\neg D x \rightarrow \neg A) & \quad \exists \text{I} \\
\neg \neg A & \quad \exists \text{I} \\
\neg \neg A & \quad \exists \text{I} \\
\neg \neg A & \quad \exists \text{I} \\
\end{align*}
\]

\[ \neg \neg A \lor \neg \neg A \]
Lemma 4:

Where $\Phi := \neg\exists x ((Dx \to \neg A) \land (\neg Dx \to \neg A)) \to \exists x \neg ((Dx \to \neg A) \land (\neg Dx \to \neg A))$,