Forecasting count data using time series model with exponentially decaying covariance structure

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Abstract

Count data appears in various disciplines. In this work, a new method to analyze time series count data has been proposed. The method assumes exponentially decaying covariance structure, a special class of the Matérn covariance function, for the latent variable in a Poisson regression model. It is implemented in a Bayesian framework, and can provide reliable estimates for covariate effects and extent of variability explained by the temporally dependent process and the white noise process. Prediction procedure has been described as well. Simulation study with four different processes show that across various scenarios the newly proposed method generally has better predictive accuracy than other popular methods. Further, two real data examples, one related to the number of dengue cases and another on the number of sales made by a retailer, are included in the paper. These two examples corroborate the earlier findings and establish that the proposed approach has good predictive abilities.

Keywords: Bayesian analysis; Count data; Dengue prediction; Exponentially decaying covariance; Retail sales prediction.
1 Introduction

Integer-valued time series or count data appears in many disciplines, ranging from economics to public health to social sciences. Popular examples of such data are the number of people affected from a virus, the number of a certain product sold per day, the number of website visits, the number of environmental events (such as tornadoes or hurricanes) at a location or the number of accidents at an intersection. Generalized linear models with Poisson or negative binomial distribution are suitable to deal with the discreteness and they can assess the effect of different regressors on the response variable. On the other hand, models like autoregressive integrated moving average (ARIMA) can analyze the covariance structure for a real-valued time series in an appropriate way. However, modelling count data demands one to consider both the discreteness and the time-dependence properties of the series. In this work, we propose a new method capable of doing that. The method assumes that the latent variable of a Poisson regression model has an exponentially decaying covariance structure. The method is implemented using a Bayesian framework and we show that it has better predictive accuracy than some other popular candidate models across various scenarios. But, before going deeper into that, let us go over some of the earlier studies who have tried to address the same problem in different ways.

Binomial thinning, and in general the class of integer-valued autoregressive (INAR) and integer-valued moving average (INMA) processes are one of the more popular methods. McKenzie [1985] is first of the most significant papers in this regard while Quddus [2008] and Weiβ [2008] are noteworthy as well. As the name suggests, these are extensions of standard real-valued AR and MA models. This family of models are further generalized to include integer valued autoregressive moving average (INARMA) processes in presence of exogenous regressors, cf. Neal and Subba Rao [2007], Enciso-Mora et al. [2009a] and Enciso-Mora et al. [2009b]. A similar flavour in the modeling approach can be found in discrete autoregressive moving average (DARMA) models. For relevant readings, refer to Jacobs and Lewis [1978] and Chang et al. [1984].

Another interesting idea is to use state-space models. Davis and Dunsmuir [2015] is a great resource for an in-depth reading on this. An earlier work by Yelland [2009] was aimed towards analyzing low-count time series data using state-space models. Yang et al. [2015], on the other hand, discussed such models for zero-inflated data and showed an application based on the evaluation of an ergonomics intervention designed to reduce workplace injuries among hospital cleaners. The study by Hostetler and Chandler [2015] is also worth mention on this note.

Furthermore, MacDonald and Zucchini [2016] proposed hidden Markov chains, which in essence is an extension of the basic Markov chain models. An application can be seen in Cooper and Lipsitch [2004]. But, there are a few issues with this model. Determining the appropriate order for the Markov chain is difficult. The number of parameters can get too big. Moreover, the results, in most applications, are not very easy to interpret. Autoregressive conditional Poisson (ACP) models aim to address some of these issues. It was first introduced in detail by Heinen [2003]. Later, Ghahramani and Thavaneswaran [2009] discussed some properties while Groß-KlußMann and Hautsch [2013] worked on an application. Chen et al. [2016], in a more recent work, developed the framework for autoregressive conditional negative binomial models. Fokianos et al. [2009], meanwhile, laid out an extensive theoretical background on Poisson autoregression.

Let us now turn our attention to generalized linear autoregressive moving average (GLARMA) methods, which have been more widely used in various applications. Theoretical results are discussed in detail in Dunsmuir [2015]. Integer-valued GARCH models are also somewhat similar in flavour and are relevant in this discussion. Negative binomial or Poisson distribution are two
most popular choices for GLARMA and INGARCH models. See Rydberg et al. [1999], Davis et al. [2005], Liesenfeld et al. [2006], Ferland et al. [2006], Fokianos [2011] and Zhu [2012] for some related works.

The rest of the paper is organized as follows. In the following section, the method, along with the Bayesian framework and implementation procedure, are described. A short simulation study is presented in Section 3. Section 4 is about data analysis and corresponding discussions. Two different types of dataset are considered in this work. In Section 4.1, our focus is on predicting weekly dengue cases for a two-month period while in Section 4.2, we will analyze retail sales data to predict number of sales in a one-week horizon. We will conclude the paper with some important remarks in Section 5. References and Tables are provided thereafter.

2 Methods

2.1 Notations

This paper discusses a forecasting method for time series count data. Let us use $t_i$, for $i \in \{1, 2, \ldots, T\}$ to denote a time point. It is worth mention that our method is flexible enough to allow any temporal resolution, and hence $t_i$ can represent hour, day, week etc, depending on the data. The following notations will be used throughout the paper:

- $\Gamma$ will denote the index set $\{t_1, t_2, \ldots, t_T\}$.
- For each $t \in \Gamma$, $Y_t$ is the value of the response variable. We would assume it to be have a Poisson distribution with parameter $\lambda_t$.
- In our modeling scheme, we would use the log-link for the Poisson parameter. Let $\mu_t = \log \lambda_t$.
- $X_t$ is the column vector for all covariates used in the trend structure of the model. $X$ is the corresponding covariate matrix such that $X' = [X_{t_1} : X_{t_2} : \ldots : X_{t_T}]$.
- In vectorized form, $\mathbf{Y}$ denotes the vector of all outcomes $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_T})$. In a similar way, we use $\mathbf{b}$ for a vector of the form $(b_{t_1}, b_{t_2}, \ldots, b_{t_T})$.
- $I_n$ denotes an identity matrix of order $n \times n$.
- For a set $A$, $|A|$ denotes the cardinality.
- Whenever used, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ denote the set of integers, the set natural numbers and the set of real numbers, respectively.

2.2 Model

The proposed model starts with the aforementioned assumption that for a latent variable $\mu_t$, $Y_t \sim \text{Poisson}(e^{\mu_t})$. $\mu_t$ depends on the covariates following an additive structure. Further, assume that, given the values of the covariates, the outcomes are temporally dependent of each other. In light of that, let us consider a hierarchical structure for $\mu_t$, as follows.

$$\mu_t = X_t' \beta + \varepsilon_t, \quad (2.1)$$
where $\beta$ is a parameter vector of appropriate order corresponding to the covariates, and $\varepsilon_t$ is a time dependent error process. The term $X_t'\beta$ captures the effects of all seasonal, continuous or discrete type covariates on the mean structure. Next, assume that the error process has the following structure:

$$\varepsilon_t = w_t + e_t. \quad (2.2)$$

Here, $(w_t)_{t \in \Gamma}$ denotes a zero-mean temporally correlated process and $(e_t)_{t \in \Gamma}$ stands for a zero-mean independent and identically distributed (iid) white noise process. Mathematically, assume that $e_t$’s are iid $N(0, \sigma^2)$. On the other hand, for $w_t$, consider a correlation structure that decays exponentially. In particular, the covariance between $w_{t_1}$ and $w_{t_2}$ is

$$\text{Cov}(w_{t_1}, w_{t_2}) = \sigma^2_w \exp(-\phi d(t_1, t_2)). \quad (2.3)$$

d$(\cdot, \cdot)$ is a distance function and is defined as the absolute difference between the two time points. Note that this covariance function is a special choice ($\nu = 0.5$) among the Matérn class of covariance functions, cf. Minasny and McBratney [2005]. Hereafter, for convenience, let us denote the covariance matrix of $w = (w_{t_1}, w_{t_2}, \ldots, w_{t_T})$ by $\sigma^2_w \Sigma_w$. Then, following the notations described in Section 2.1, the complete modeling scheme can be written as follows.

$$Y_t \sim \text{Poisson } (e^{\mu_t}), \quad (2.4)$$

$$\mu = X'\beta + w + e,$$

such that $w \sim N(0, \sigma^2_w \Sigma_w)$, $e \sim N(0, \sigma^2 I_T)$.

### 2.3 Bayesian framework

In order to implement the above modeling scheme, a Bayesian framework is used. For that, we need to assign prior distributions to the parameters in the study. Each component in $\beta$, a priori, is assumed to be independent with improper uniform prior distribution on $\mathbb{R}$. $\sigma^2$ and $\sigma^2_w$ are variance parameters and independent inverse gamma (IG) priors with parameters $a, b$, which in fact are conditionally conjugate priors (see, for example, Gelman et al. [2006]), are used for them. We take $a > 2$ to ensure finite mean and variance for the prior distribution. The parameter $\phi$ in the covariance function is considered to be fixed throughout the analysis, and we estimate it using a cross-validation scheme.

We now delve into the posterior distributions of the parameters. Throughout the discussion below, $K$ indicates a constant term, that may vary from time to time. Recall the model defined by eq. (2.4). The log-likelihood is

$$l(\beta, \sigma^2, \sigma^2_w, \mu, w; Y) = K - \sum_{t \in \Gamma} e^{\mu_t} + \sum_{t \in \Gamma} \mu_t Y_t - \frac{T}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \| \mu - X\beta - w \|^2$$

$$- \frac{T}{2} \log \sigma^2_w - \frac{1}{2\sigma^2_w} w' \Sigma_w^{-1} w. \quad (2.5)$$

Then, using the prior distributions described earlier in this section, we can write the joint posterior distribution as:

$$\log \pi(\beta, \sigma^2, \sigma^2_w, \mu, w | Y) = K - \sum_{t \in \Gamma} e^{\mu_t} + Y' \mu - \left( a + \frac{T}{2} + 1 \right) \log \sigma^2 - \frac{1}{2\sigma^2} \| \mu - X\beta - w \|^2$$

$$- \left( a + \frac{T}{2} + 1 \right) \log \sigma^2_w - \frac{1}{2\sigma^2_w} w' \Sigma_w^{-1} w - \frac{b}{\sigma^2} - \frac{b}{\sigma^2_w}. \quad (2.6)$$
Since simulating directly from the joint posterior distribution is difficult, the principles of Gibbs sampling is used. Extensive reading on Gibbs sampling can be found in Geman and Geman [1984] and Durbin and Koopman [2002]. In this method, every parameter is updated in an iterative way. For $\sigma^2$, we can get the full conditional distribution as:

$$\log \pi(\sigma^2 | Y, \mu, \beta, w) = K - (a + T + 1) \log \sigma^2 - \frac{1}{\sigma^2} \left( b + \frac{1}{2} \| \mu - X\beta - w \|^2 \right).$$

Thus,

$$\sigma^2 | Y, \mu, \beta, w \sim IG \left( a + \frac{T}{2}, b + \frac{1}{2} \| \mu - X\beta - w \|^2 \right). \quad (2.7)$$

In an exact similar way,

$$\sigma^w_2 | Y, \mu, \beta, w \sim IG \left( a + \frac{T}{2}, b + \frac{1}{2} w^T \Sigma_w^{-1} w \right). \quad (2.8)$$

For $\beta$, the conditional distribution can be written as

$$\log \pi(\beta | Y, \mu, \sigma^2, w) = K - \frac{1}{2} \beta^T \left( \frac{X'X}{\sigma^2} \right) \beta + \frac{1}{2} \beta^T \cdot \frac{X'(\mu - w)}{\sigma^2}.$$ 

Straightforward calculations then imply that the conditional posterior of $\beta$ is

$$(\beta | Y, \mu, \sigma^2, w) \sim N \left( (X'X)^{-1} X'(\mu - w), \sigma^2(X'X)^{-1} \right). \quad (2.9)$$

Using similar techniques, we can get the posterior distribution for $w$ as:

$$(w | Y, \mu, \beta, \sigma^2, \sigma^2_w) \sim N \left( (I_T + \frac{\sigma^2}{\sigma^2_w} \Sigma_w^{-1})^{-1}(\mu - X\beta), \left( \frac{1}{\sigma^2} I_T + \frac{1}{\sigma^2_w} \Sigma_w^{-1} \right)^{-1} \right). \quad (2.10)$$

Finally, we find the conditional distribution of $\mu$. Note that conditional on data and other parameters, each component of $\mu$ is independent of others. The corresponding posterior is

$$\log \pi(\mu_t | Y_t, \beta, w_t) = K - \frac{1}{2\sigma^2} (\mu_t - (X_t^T \beta + w_t + \sigma^2 Y_t))^2 - e^{\mu_t}. \quad (2.11)$$

Simulating directly from the above posterior is difficult, since the normalizing constant or the distribution function cannot be obtained in a straightforward way. Hence, we use adaptive rejection metropolis sampling (ARMS) algorithm. It is a method for efficiently sampling from complicated univariate densities, as discussed in Gilks and Wild [1992] and Gilks et al. [1995].

### 2.4 Implementation

There are a few things to consider when it comes to the implementation of the model. First one is the issue of missing data handling. We choose to remove the time points when the response variable is missing. Recall that the model does not require $\Gamma$ to be a regular set of consecutive time points. This is one of the attractive features of the model. On the other hand, a good imputation technique requires some knowledge about the missingness and the underlying true model. Further, for the dengue cases data we consider in our study, the missingness is very less (only 2.9%) whereas the retail sales data has no missing observation. Hence, our approach, formally known as listwise deletion, is suitable for this paper.
Next, as mentioned before, the parameter $\phi$ (refer to eq. (2.3)) is unknown and in order to find out the optimal value, a cross-validation scheme is used. This technique has been used in some other similar modeling works, for example Sahu and Holland [2006] and Deb and Tsay [2019]. In our case, depending on the temporal resolution of the data, a range of values for $\phi$ is chosen. Let us call it $S$. Next, we set aside a validation set ($V$), and train the model on the rest of the data for different values of $\phi$. The predictive performance of each case is then evaluated based on the mean absolute error (MAE) for the validation set. Formally, for a model trained with $\phi \in S$, let $\hat{Y}_t^{\phi}$ denote the prediction for true $Y_t$ at time point $t \in V$. The optimal choice of $\phi$ is then defined as

$$
\phi_{opt} = \arg\min_{\phi \in S} \text{MAE}_\phi = \arg\min_{\phi \in S} \frac{1}{|V|} \sum_{t \in V} \left| Y_t - \hat{Y}_t^{\phi} \right|.
$$

(2.12)

While implementing the Gibbs sampler, first 5000 observations are ignored (burn period) and then we take posterior sample of size 500 sufficiently apart from each other so as to ensure independence.

Lastly, all computations are executed in RStudio version 1.2.5033, coupled with R version 3.6.2. ARMS method was done with the help of armSpp package, (Bertolacci [2019]). For other competing methods, acp (Vasileios [2015]), forecast (Hyndman and Khandakar [2008]), glarma (Dunsmuir and Scott [2015]) and MASS (Venables and Ripley [2002]) packages were used as necessary.

### 2.5 Future prediction

The most crucial contribution of this paper is to provide a prediction strategy for $Y_{t'}$, for $t' > T$. Let $X'_t$ denote the corresponding column vector of covariates. $\mu_{t'}$ and $w_{t'}$ are defined accordingly. Note that the posterior predictive distribution $f(Y_{t'} | Y)$ can be written as

$$
f(Y_{t'} | Y) \propto \int f(Y_{t'} | \mu_{t'} \pi(\mu_{t'} | Y) \, d\mu_{t'}.
$$

(2.13)

Now, instead of solving that integral, a better and more convenient idea is to use the Gibbs sampler estimates to draw observations from the posterior predictive distribution. One can do it sequentially. At first, draw samples for $\mu, w, \beta, \sigma^2, \sigma^2_w$ using the conditional posteriors derived above. Then, we need to simulate $w_{t'}$ using the conditional distribution of $(w_{t'} | w, \sigma^2_w)$. To compute it, observe that

$$
\begin{pmatrix} w \\ w_{t'} \end{pmatrix} \sim N \left(0, \sigma^2_w \begin{bmatrix} \Sigma_w & \Sigma_{t'-t'} \\ \Sigma_{t'-t'} & 1 \end{bmatrix} \right).
$$

Here, $\Sigma_{t'-t'}$ is the column vector denoting the covariance of $w_{t'}$ with the elements of $w$. Following eq. (2.3), the $j$th element of $\Sigma_{t'-t'}$ is $\exp(-\phi d(t_j, t'))$. Thus, the principle of conditional distribution for multivariate normal distribution imply that

$$
(w_{t'} | w, \sigma^2_w) \sim N \left(\Sigma_{t'-t'}^{-1} \Sigma_w^{-1} w, \sigma^2_w \left(1 - \Sigma_{t'-t'}^{-1} \Sigma_{t'-t'}^{-1} \right) \right).
$$

(2.14)

Then, using the above realizations, an estimate for $\mu_{t'}$ can be obtained by generating samples from a normal distribution with mean $X'_{t'} \beta + w_{t'}$ and variance $\sigma^2$. Finally, predictions for $Y_{t'}$ are generated from Poisson distribution with parameter $\exp(\mu_{t'})$. A prediction interval can further be computed using many samples from the posterior predictive distribution. In all our applications, we will use 10000 samples to calculate the prediction interval.
3 Simulation study

We focus on four different toy examples and compare the predictive performance of our method with that of three other candidate models. Two of these three, namely Autoregressive conditional Poisson (ACP) method and generalized linear autoregressive moving average (GLARMA) method, were discussed in Section 1. In addition, we will also see the performance of standard real-valued ARIMA models if used in the context of count data.

Following are the four processes we use to simulate the data $Y = (Y_t)_{t=1,...,n}$. In all examples, we take $n = 200$.

(a) IID: $Y_t$’s are independent and identically distributed Poisson($\lambda$) random variables. $\lambda = 5.2$ is used.

(b) INAR: Define binomial thinning operator $\circ$ as $\alpha \circ Z = \sum_{i=1}^{Z} B_i$ where $B_i$'s, for all integer $i$, are independent and identically distributed Bernoulli($\alpha$) random variables. Let $Y$ satisfy $Y_t = \alpha \circ Y_{t-1} + \epsilon_t$, such that $\epsilon_t$’s are nonnegative integer-valued iid random variables with finite mean and variance. This is known as INAR(1) process. In this simulation study, we take $\epsilon_t \sim$ Poisson(10) and $\alpha = 0.65$.

(c) INGARCH: Let $Y_t$, conditional on $(Y_i)_{i<t}$, be distributed as Poisson($M_t$), such that $M_t = \theta + \alpha Y_{t-1} + \beta M_{t-1}$. We use $\theta = 4$, $\alpha = 0.5$ and $\beta = 0.1$. This is known as INGARCH(1,1) process.

(d) Poisson-AR: Let $Y_t \sim$ Poisson($\lambda_t$) such that $\log \lambda_t = X_t'\theta + z_t + \epsilon_t$, where $(z_t)_{t \in \mathbb{Z}}$ is an AR(3) process and $\epsilon_t$’s are iid $t$ distribution with 5 degrees of freedom. Three covariates are randomly generated to prepare $X_t$. Corresponding $\theta$ coefficients used in this study are $1.15, -0.8, 0.18$. The AR process is simulated using coefficients $0.5, -0.2, 0.1$ and normally distributed innovations with standard deviation 1.2. This is similar to ACP process but we make it slightly more complicated to match the settings of our method, albeit with different covariance structure.

The simulated series are presented in Figure 1.

For all methods, we evaluate the forecasting performance on the last eight time points. This means, a training series of 192 observations is used. Further, in case of implementing the proposed method, we need to estimate $\phi$ using a cross validation (CV) scheme (refer to Section 2.4). For that, the train series is split into a CV train series of length 184 and validation is done on the remaining part. Possible choices of $\phi$ are $0.01, 0.1, 0.25, 0.5, 1, 1.5, 3$. Based on the relationship $\exp(-\phi d) \approx 0.05$, those values depict significant dependence on past 1 to 30 time points. On the other hand, appropriate orders for ACP, GLARMA and ARIMA methods are chosen using Akaike Information Criterion (AIC). Maximum lags used in these methods are 7. For all methods, whenever appropriate, intercept and trend terms are used.

Table 1 shows the eight-step-ahead prediction accuracy for these methods. The measure we use is the mean absolute error (MAE) for the eight predictions. In the first row, we also include the results for a baseline method, where last observed value is carried forward as a prediction for the entire test set.

Interestingly, the baseline method turns out to beat GLARMA and ARIMA for the IID case, albeit only marginally. Our method shows the best results while ACP has the maximum error. For the INGARCH(1,1,) process as well, our method betters the number from the other candidate models, but ARIMA and GLARMA are closely behind. For INAR(1) process though, the newly
proposed approach has much better predictive abilities than the others. Finally, ACP is the best of the lot for the Poisson-AR model. This makes great sense since the Poisson-AR process we consider is in essence very similar to the structure of ACP. Overall, on computing the average MAE across the four different processes, we find that our method is the best of the lot. GLARMA and ARIMA are also good. But, ACP has not been up to the mark for INAR or INGARCH processes. In a nutshell, it can be said that the proposed method in this paper works generally well across various scenarios and hence, for any real-life application where the true underlying process is unknown, it will be suitable to use this method. We will further see this in the next section, where we discuss two real-life datasets of different flavors.

4 Real data analysis

4.1 Dengue cases data

This data, publicly available in GitHub repository DengAI [2017], is based on the weekly number of dengue cases from Iquitos region, Peru. It has been considered and analyzed in different capacities by several other researchers (for example, Deb et al. [2017] and Buczak et al. [2018]).

The objective here is to use our modeling technique to analyze and predict $Y_t$, the number of...
dengue cases, using a few terrain and weather-related variables as covariates. The time-span of the full data is 7th January, 2000 to 25th June, 2010. In order to evaluate the predictive performance of our model, we set aside last eight weeks (starting from 7th May, 2010) of data as the ‘test set’ and they are considered entirely unseen by the model. Further, 2.9% rows from the period before the test set have at least one missing variable and so, we remove them from our analysis. This resulted in 497 weeks worth of data and we would call it ‘train set’ hereafter.

There are 10 main covariates in this study. These are NDVI or normalized difference vegetation index for four different directions, precipitation amount (in the model, we consider a square root transformation), relative humidity, maximum air temperature, specific humidity, diurnal temperature range and maximum temperature of the stations. Brief description and summary of the response variable and these covariates in the train set are discussed in Table 2. The first row indicates that the response variable has a wide range (from 0 to 83), but a mean of only 7.54. This can be attributed to the fact that there are some rare occasions of epidemics in Iquitos. On the other hand, as Deb et al. [2017] have pointed out, both terrain and weather components from past weeks can significantly affect the number of dengue cases. This is largely dependent on the breeding period of the mosquitoes. In light of that, we plan to use the covariate values from up to lag 4, that is previous four weeks. However, that would effectively introduce 40 new covariates, thereby increasing the chance of overfitting. Thus, a variable selection method is necessary. Variance Inflation Factor (VIF) criterion, with a cutoff value of 10, is used in this purpose. That means, covariates showing VIF more than 10 are assumed to display signs of multicollinearity and thus are not going to be used in the model. This results in 36 regressors related to terrain and climate. Four NDVI covariates, precipitation amount and reanalysis diurnal temperature range are used with lagged values of the previous four time points. Station maximum temperature (up to lag 2), reanalysis maximum air temperature (up to lag 1) and reanalysis specific humidity are selected as well. In addition, to deal with seasonality, 11 monthly indicator variables (effect of December taken as 0, for the sake of identifiability issues) are used in the model.

Similar to Section 3, the optimal value of φ is chosen following a CV scheme. It is found out to be 0.5, implying that there is a significant autocorrelation of up to previous six weeks. The variance parameter for the temporal process w is estimated to be 0.42, which is much higher than the estimated white noise variance 0.03. Clearly, more variation in the data can be explained by the temporally dependent process. The estimates for the covariates in the model are presented in Table 3. Among the terrain variables, NDVI (south-east), for lag 3, has a significantly negative effect on the number of dengue cases. The weather variables are in general not significant. Only the precipitation amount (for lag 3) significantly increases the number of dengue cases. These results are very interesting in light of what we see if we fit a model without the w process. In that case, no terrain or climate variable is found out to be significant. Meanwhile, among the monthly indicators, April, May and June display significant coefficients.

Let us now turn our attention to predictive abilities, which is the main goal of this paper. As mentioned earlier, last eight weeks of data are set aside as the ‘test set’. In Figure 2, the true number of dengue cases along with the fitted values for the train set are presented. In a similar way, Figure 3 show the predictions and the confidence intervals along with the true number of cases for the test set. It is evident that the model fits the training data very well. The predictions are also fairly close, and the actual values are always inside the prediction interval. The MAE has been computed to be only 1.01. In contrast, the same is 4.82, 2.20 and 2.25 for ACP, GLARMA and ARIMA, respectively.
Figure 2: Actual and fitted values for the number of dengue cases in the train set.

Figure 3: Actual and predicted values, along with prediction interval, for the number of dengue cases in the test set.
4.2 Retail sales data

Obtained from the online available source in Kaggle [2017], this dataset contains historical sales data extracted from a Brazilian top retailer. The data is provided on daily level. It starts from 1st January, 2014 and runs until 31st July, 2016. For the sake of inventory management, the retailers want to have good prediction for the next few days, which makes it an attractive short-term forecasting problem. We set aside last week’s data, starting from 25th July, 2016, and fit our proposed method to the remaining. Thus, the training set has 930 observations whereas the test set has 7.

This problem is different from the dengue example in a couple of aspects. First, the mean of the response variable is 90.2 and the standard deviation is 80.7, whereas the range of the values is 0 to 542. This clearly shows high variability. On the other hand, this data did not have any information on possible covariates. Hence, the month and day-of-week indicators are used as the only regressors in the model.

Using a similar CV scheme as before, optimum value of $\phi$ was obtained to be 0.1. This shows very strong correlation with the last few days. Estimate of $\sigma^2_w$, the variance parameter for the temporally dependent process was greater than the estimate of white noise variance ($\sigma^2$) here as well. Thus, once again, more variation in the data was explained by the temporally dependent process.

![Figure 4: Actual and fitted values for the number of sales in the train set.](image)

Fitted values and observed number of sales for the training period are provided in Figure 4. It is evident that the model shows good fit. Estimates and standard error of the coefficients corresponding to the regressors are then presented in Table 4. Months from January to April show significantly lower sales. The same was true for August as well. In case of days of week, Monday and Tuesday have a negative effect on the response variable while Saturday sees a significantly
positive effect. It can be explained by people’s tendency to go shopping in the weekend, the shop closures on Sundays and the natural predilection to refrain from shopping on the weekdays because of work.

Finally, our proposed method shows good predictive ability in this example too. Predicted values, the prediction interval and the actual values are displayed in Figure 5. True numbers are usually inside the prediction intervals. The MAE for the next seven-day period is computed as 52.46. GLARMA also performs at par, recording 54.10 MAE for the same period. ARIMA, meanwhile, is not as good and the mean absolute error is 65.23 for it.

![Figure 5: Actual and predicted values, along with prediction interval, for the number of sales in the test set.](image)

**5 Concluding remarks**

To summarize, in this study, we have proposed a new method to analyze and forecast time series count data. Simulation study and the real life examples show that the method works well across various scenarios. It is in fact better than some other popular models when it comes to predictive accuracy. It is also precise, easy to interpret and flexible enough to include more regressors in the mean structure as necessary. Further, we have used a special class of Matérn covariance function in the dependent process. One can easily relax that assumption and consider a more general class of time-dependent processes.

Further, recall that we estimate optimal value of the exponential decay parameter $\phi$ using a cross-validation scheme. While this helps in reducing the computational burden, a complete Bayesian approach may provide more precise estimate. One can also work out the classical maximum likelihood type estimators for the model.
On a different note, we notice that the proposed method works much better than the other competing methods when the values are less (example is the dengue cases data) while the accuracy is similar to others when the numbers and the variability are higher (example is the sales data). This raises an interesting aspect, in light of the fact that zero-inflated Poisson random variables are commonly observed in count data. Thus, as a future direction to this work, it will be of prime importance to develop a similar model for zero-inflated Poisson regression setup. Further, one can extend the model to negative binomial and zero-inflated negative binomial variables as well. That would allow one to deal with count data in various capacities across different disciplines.

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### Tables

#### Table 1: Mean absolute error (MAE) of eight-step-ahead predictions for different models on four different processes.

| Method       | IID  | INAR | INGARCH | Poisson-AR | Average MAE |
|--------------|------|------|---------|------------|-------------|
| Baseline     | 1.38 | 9.25 | 4.63    | 127.38     | 35.66       |
| ACP          | 3.76 | 23.29| 13.46   | 1.78       | 9.66        |
| GLARMA       | 1.47 | 4.30 | 4.62    | 2.19       | 3.15        |
| ARIMA        | 1.49 | 4.39 | 4.60    | 2.96       | 3.36        |
| **Our method** | **1.20** | **2.96** | **4.30** | **2.63** | **2.77** |

#### Table 2: Minimum, maximum, average and standard deviation of the variables in the train set for the dengue cases data from Iquitos.

| Variable                                           | Minimum | Maximum  | Average | Std. dev. |
|----------------------------------------------------|---------|----------|---------|-----------|
| Number of dengue cases                             | 0       | 83       | 7.54    | 9.81      |
| NDVI NE                                            | 0.06    | 0.51     | 0.26    | 0.08      |
| NDVI NW                                            | 0.04    | 0.45     | 0.24    | 0.08      |
| NDVI SE                                            | 0.03    | 0.54     | 0.25    | 0.08      |
| NDVI SW                                            | 0.06    | 0.55     | 0.27    | 0.09      |
| Precipitation amount (mm)                          | 0.00    | 210.83   | 64.80   | 35.23     |
| Reanalysis relative humidity (%)                   | 57.79   | 98.61    | 88.61   | 7.61      |
| Reanalysis max air temperature (K)                 | 300.00  | 314.00   | 307.11  | 2.40      |
| Reanalysis specific humidity (g/kg)                | 12.11   | 20.46    | 17.11   | 1.45      |
| Reanalysis diurnal temp range (K)                  | 3.71    | 16.03    | 9.22    | 2.45      |
| Station Max Temp (C)                               | 30.10   | 42.20    | 34.02   | 1.32      |
Table 3: Coefficient estimate (and standard error) for the covariates in the proposed model for the dengue cases data from Iquitos. * denotes significant effect at 5% level of significance.

| Covariate   | Coef (lag 1) | Coef (lag 2) | Coef (lag 3) | Coef (lag 4) |
|-------------|--------------|--------------|--------------|--------------|
| (intercept) | 3.34 (8.31)  | 1.37 (0.94)  | 1.13 (0.92)  | 0.68 (0.90)  |
| NDVI NE     | 0.32 (0.83)  | -0.08 (0.78) | -0.13 (0.79) | 0.61 (0.72)  |
| NDVI NW     | -0.45 (0.69) | -1.37 (0.76) | -1.23 (0.77) | -1.51 (0.75)* |
| NDVI SE     | -0.73 (0.71) | -1.37 (0.76) | -1.23 (0.77) | -1.51 (0.75)* |
| NDVI SW     | 0.71 (0.73)  | -0.36 (0.86) | -0.42 (0.86) | -0.45 (0.74) |
| Precipitation | -0.01 (0.02) | 0.01 (0.02)  | 0.04 (0.02)* | 0.00 (0.02)  |
| Rean. max T | -0.02 (0.03) | 0.00 (0.005) |              |              |
| Rean. SH    | 0.07 (0.04)  |              |              |              |
| Rean. DTR   | 0.01 (0.03)  | -0.03 (0.02) | -0.02 (0.02) | -0.01 (0.02) |
| Stn. max T  | 0.04 (0.04)  | 0.02 (0.03)  | 0.01 (0.02)  |              |
| January     | 0.34 (0.21)  |              |              |              |
| February    | 0.13 (0.24)  |              |              |              |
| March       | -0.33 (0.24) |              |              |              |
| April       | -0.53 (0.25)*|              |              |              |
| May         | -0.52 (0.25)*|              |              |              |
| June        | -0.65 (0.27)*|              |              |              |
| July        | -0.46 (0.27) |              |              |              |
| August      | -0.40 (0.27) |              |              |              |
| September   | -0.10 (0.25) |              |              |              |
| October     | 0.00 (0.24)  |              |              |              |
| November    | -0.02 (0.21) |              |              |              |

Table 4: Coefficient estimate (and standard error) for the covariates in the proposed model for the sales data from Brazil. * denotes significant effect at 5% level of significance.

| Covariate  | Coef (lag 1) | Coef (lag 2) | Coef (lag 3) | Coef (lag 4) |
|------------|--------------|--------------|--------------|--------------|
| (intercept)| 4.69 (0.60)  |              |              |              |
| January    | -1.23 (0.59)*| -0.31 (0.07)*|              |              |
| February   | -1.56 (0.73)*| -0.39 (0.08)*|              |              |
| March      | -1.93 (0.69)*| -0.08 (0.10) |              |              |
| April      | -1.52 (0.64)*| -0.18 (0.09) |              |              |
| May        | -1.02 (0.64) | -0.04 (0.09) |              |              |
| June       | -0.83 (0.73) | 0.37 (0.07)* |              |              |
| July       | -0.83 (0.70) |              |              |              |
| August     | -1.42 (0.72)*|              |              |              |
| September  | -0.01 (0.85) |              |              |              |
| October    | 0.12 (0.74)  |              |              |              |
| November   | 0.00 (0.52)  |              |              |              |