SEMI-MARKOV APPROACH TO CONTINUOUS TIME RANDOM WALK LIMIT PROCESSES

BY MARK M. MEERSCHAERT\textsuperscript{1,2} AND PETER STRAKA\textsuperscript{2}

\textit{Michigan State University and UNSW Australia}

Continuous time random walks (CTRWs) are versatile models for anomalous diffusion processes that have found widespread application in the quantitative sciences. Their scaling limits are typically non-Markovian, and the computation of their finite-dimensional distributions is an important open problem. This paper develops a general semi-Markov theory for CTRW limit processes in $\mathbb{R}^d$ with infinitely many particle jumps (renewals) in finite time intervals. The particle jumps and waiting times can be coupled and vary with space and time. By augmenting the state space to include the scaling limits of renewal times, a CTRW limit process can be embedded in a Markov process. Explicit analytic expressions for the transition kernels of these Markov processes are then derived, which allow the computation of all finite dimensional distributions for CTRW limits. Two examples illustrate the proposed method.

1. Introduction. Continuous time random walks (CTRWs) assume a random waiting time between each successive jump. They are used in physics to model a variety of anomalous diffusion processes (see Metzler and Klafter [34]), and have found applications in numerous other fields (see, e.g., [6, 17, 37, 38]). The scaling limit of the CTRW is a time-changed Markov process in $\mathbb{R}^d$ [31]. The clock process is the hitting time of an increasing Lévy process, which is non-Markovian. The distribution of the scaling limit at one fixed time $t$ is then usually calculated by solving a fractional Fokker–Planck equation [34], that is, a governing equation that involves a fractional derivative in time. The analysis of the joint laws at multiple times, however, becomes much more complicated, since the limit process is not Markovian. In fact, the joint distribution of the CTRW limit at two or more different times has yet to be explicitly calculated, even in the simplest cases; see Baule and Friedrich [4] for further discussion.

The main motivation of this paper is to resolve this problem, and our approach is to develop the semi-Markov theory for CTRW scaling limits. CTRWs are renewed after every jump. As it turns out, the discrete set of renewal times of CTRWs converges to a “regenerative set” in the scaling limit, which is not discrete and can be a

\textsuperscript{1}Supported in part by NSF Grants DMS-10-25486 and DMS-08-03360.
\textsuperscript{2}Supported in part by NIH Grant R01-EB012079.

MSC2010 subject classifications. Primary 60K15, 60F17; secondary 60K20.

Key words and phrases. Continuous time random walk, semi-Markov process, functional limit theorem, renewal theory, anomalous diffusion, time-change, Lévy process.
random fractal or a random set of positive Lebesgue measure. This regenerative set allows for the definition of the scaling limit of the previous and next renewal time after a time $t$. By incorporating these times into the state space, a CTRW limit can become Markovian. Although CTRW scaling limits have appeared in many applications throughout the literature, to our knowledge the renewal property has only been studied for a discrete CTRW. Moreover, CTRW limits are examples for possibly discontinuous semi-Markov processes with infinitely many renewals in finite time, and hence the development here complements the literature on continuous semi-Markov processes [15].

It is known [25] that semi-Markov processes can be constructed by assuming a Markov additive process $(A_u, D_u)$ and defining $X_t = A(E_t)$, where $E_t$ is the hitting time of the level $t$ by the process $D_u$. With this procedure, one also constructs CTRW limit processes. However, such CTRW limits are homogeneous in time, and several applications require time-inhomogeneous CTRW limit processes [16, 27]. Hence, we will assume that $(A_u, D_u)$ is a diffusion process with jumps (such that $D_u$ is strictly increasing), modeling the cumulative sum of non-i.i.d. jumps and waiting times (see Section 2) which vary with time and space. In this setting, we develop a semi-Markov theory for time-inhomogeneous CTRW limits.

Coupled CTRW limits, for which waiting times and jumps are not independent, turn out to be particularly interesting. As recently discovered [21, 40], switching the order of waiting time and jump (i.e., jumps precede waiting times) yields a different scaling limit called the overshooting CTRW limit (OCTRW limit). The two processes can have completely different tail behavior [21], and hence provide versatile models for a variety of relaxation behaviors in statistical physics [42]. Both CTRW and OCTRW limit turn out to be semi-Markov processes; however, incorporating the previous renewal time only renders the CTRW limit Markovian and not the OCTRW limit, and the opposite is true when the following renewal time is incorporated. In the uncoupled case, CTRW and OCTRW have the same limit, and hence both approaches yield Markov processes.

This paper gives explicit formulae for the joint transition probabilities of the CTRW limit (resp., OCTRW limit), together with its previous renewal time (resp., following renewal time, see Section 3). These formulae facilitate the calculation of all finite-dimensional distributions for CTRW (and OCTRW) limits. The time-homogeneous case is discussed in Section 4. Finally, Section 5 provides some explicit examples, for problems of current interest in the physics literature.

2. Random walks in space–time. A continuous time random walk (CTRW) is a random walk in space–time, with positive jumps in time. Let $c > 0$ be a scaling parameter, and let

$$(S_n^c, T_n^c) = (A_0^c, D_0^c) + \sum_{k=1}^{n} (J_k^c, W_k^c)$$
denote a Markov chain on $\mathbb{R}^d \times [0, \infty)$ that tracks the position $S^c_n$ of a randomly selected particle after $n$ jumps, and the time $T^c_n$ the particle arrives at this position. The particle starts at position $A^c_0$ at time $D^c_0$, $N^c_t = \max\{k \geq 0 : T^c_k \leq t\}$ counts the number of jumps by time $t$, and the CTRW

$$X^c_t = S^c_{N^c_t}$$

is the particle location at time $t$. The waiting times $W^c_k$ are assumed positive, and when $t < T^c_0$ we define $N^c_t = 0$. The process $N^c_t$ is inverse to $T^c_n$, in the sense that $N^c_{T^c_n} = n$. Often the sequence $(J^c_k, W^c_k)$ is assumed to be independent and identically distributed, which is the appropriate statistical physics model for particle motions in a heterogeneous medium whose properties are invariant over space and time. The dependence on the time scale $c > 0$ facilitates triangular array convergence schemes, which lead to a variety of interesting limit processes [2, 3, 22, 32]. The CTRW is called uncoupled if the waiting time $W^c_k$ is independent of the jump $J^c_k$; see, for example, [5]. Coupled CTRW models have been applied in physics [34, 39] and finance [29, 36]. If the waiting times are i.i.d. and the jump distribution depends on the current position in space and time, the CTRW limit is a time-changed Markov process governed by a fractional Fokker–Planck equation [16]. A closely related model called the overshooting CTRW (OCTRW) is

$$Y^c_t = S^c_{N^c+1},$$

a particle model for which $J^c_1$ is the random initial location, and each jump $J^c_k$ is followed by the waiting time $W^c_k$. See [23] for applications of OCTRW in finance, where $Y_t$ represents the price at the next available trading time. See [42] for an application of OCTRW to relaxation problems in physics.

In statistical physics applications, it is useful to consider the diffusion limit of the (O)CTRW as the time scale $c \to \infty$. To make this mathematically rigorous, let $\mathcal{D}([0, \infty), \mathbb{R}^{d+1})$ denote the space of càdlàg functions $f : [0, \infty) \to \mathbb{R}^{d+1}$ with the Skorokhod $J_1$ topology, and suppose

$$\left( S^c_{\lfloor cu \rfloor}, T^c_{\lfloor cu \rfloor} \right) = (A^c_0, D^c_0) + \sum_{k=1}^{\lfloor cu \rfloor} (J^c_k, W^c_k) \Rightarrow (A_u, D_u),$$

where “$\Rightarrow$” denotes the weak convergence of probability measures on $\mathcal{D}([0, \infty), \mathbb{R}^{d+1})$ as $c \to \infty$. Suppose the limit process $(A_u, D_u)$ is a canonical Feller process with state space $\mathbb{R}^{d+1}$, in the sense of [35], III Section 2. That is, we assume a stochastic basis $(\Omega, \mathcal{F}_\infty, \mathcal{F}_u, \mathbb{P}^{\chi, \tau})$ in which $\Omega$ is the set of right-continuous paths in $\mathbb{R}^{d+1}$ with left-limits and $(A_u(\omega), D_u(\omega)) = \omega(u)$ for all $\omega \in \Omega$. The filtration $\mathcal{F} = \{\mathcal{F}_u\}_{u \geq 0}$ is right continuous and $(A_u, D_u)$ is $\mathcal{F}$-adapted. The laws $\{\mathbb{P}^{\chi, \tau}\}_{(\chi, \tau) \in \mathbb{R}^{d+1}}$ are determined by a Feller semigroup of transition operators $(T_u)_{u \geq 0}$ and are such that $(A_0, D_0) = (\chi, \tau), \mathbb{P}^{\chi, \tau}$-a.s. The $\sigma$-fields $\mathcal{F}_\infty$ and $\mathcal{F}_0$ are augmented by the $\mathbb{P}^{\chi, \tau}$-null sets. Expectation with respect to $\mathbb{P}^{\chi, \tau}$ is denoted by $E^{\chi, \tau}$. The map $(\chi, \tau) \mapsto E^{\chi, \tau}[Z]$ is Borel-measurable for every $\mathcal{F}_\infty$-measurable
random variable \( Z \). If the space–time jumps form an infinitesimal triangular array ([30], Definition 3.2.1), then \((A_u, D_u) - (A_0, D_0)\) is a Lévy process [32]. In the uncoupled case, \( A_u - A_0 \) and \( D_u - D_0 \) are independent Lévy processes [31]. If the space–time jump distribution depends on the current position, it was argued in [16, 41] that the limiting process \((A_u, D_u)\) is a jump-diffusion in \( \mathbb{R}^{d+1} \).

If (2.1) holds, and if

\[
\text{the sample paths } u \mapsto D_u \text{ are } \mathbb{P}^{\mathcal{X}_t}\text{-a.s. strictly increasing and}
\]

then [40], Theorem 3.6, implies that

\[
X^c_t \Rightarrow X_t := (A_{E_t-})^+ \quad \text{and} \quad Y^c_t \Rightarrow Y_t := A_{E_t}
\]

in \( \mathcal{D}([0, \infty), \mathbb{R}^d) \) as \( c \to \infty \),

where

\[
E_t = \inf\{u > 0 : D_u > t\}
\]

is the first passage time of \( D_u \), so that \( E_{D_u} = u \). Then the inverse process \((2.4)\) is defined on all of \( \mathbb{R} \) and has a.s. continuous sample paths. The CTRW limit (CTRWL) process \( X_t \) in (2.3) is obtained by evaluating the left-hand limit of the outer process \( A_{u-} \) at the point \( u = E_t \), and then modifying this process to be right-continuous. This changes the value of the process at time points \( t > 0 \) such that \( u = E_t \) is a jump point of the outer process \( A_u \), and \( E_{t+\varepsilon} > E_t \) for all \( \varepsilon > 0 \). If \( A_u \) and \( D_u \) have no simultaneous jumps, then the CTRW limit \( X_t \) equals the OCTRW limit \( Y_t \) ([40], Lemma 3.9). However, these two processes can be quite different in the coupled case. For example, if \( J^c_k = W^c_k \) form a triangular array in the domain of attraction of a stable subordinator \( D_u \), and if \( A_0 = D_0 = 0 \), then \( A_u = D_u \), and \( X_t = D_{E_t-} < t < D_{E_t} = Y_t \) almost surely [7], Theorem III.4. See Example 5.4 for more details.

We assume the Feller semigroup \( T_u \) that governs the process \((A_u, D_u)\) acts on the space \( C_0(\mathbb{R}^{d+1}) \) of continuous real-valued functions on \( \mathbb{R}^{d+1} \) that vanish at \( \infty \), and that it admits an infinitesimal generator \( \mathcal{A} \) of jump-diffusion form [1], equation (6.42). In light of (2.2), this generator takes the form

\[
\mathcal{A}f(x, t) = \sum_{i=1}^{d} b_i(x, t) \partial_{x_i} f(x, t) + \gamma(x, t) \partial_t f(x, t) + \frac{1}{2} \sum_{1 \leq i, j \leq d} a_{ij}(x, t) \partial_{x_i}^2 f(x, t)
\]

\[
+ \int \left[ f(x + y, t + w) - f(x, t) - \sum_{i=1}^{d} h_i(y, w) \partial_{x_i} f(x, t) \right] K(x, t; dy, dw),
\]

(2.5)
where \((x, t) \in \mathbb{R}^{d+1}\), \(b_i\) and \(\gamma\) are real-valued functions, and \(A = (a_{ij})\) is a function taking values in the nonnegative definite \(d \times d\)-matrices. Here, \(K(x, t; dy, dw)\) is a jump-kernel from \(\mathbb{R}^{d+1}\) to itself, so that for every \((x, t) \in \mathbb{R}^{d+1}\), \(C \mapsto K(x, t; C)\) is a measure on \(\mathbb{R}^{d+1}\) that is finite on sets bounded away from the origin, and \((x, t) \mapsto K(x, t; C)\) is a measurable function for every Borel set \(C \subset \mathbb{R}^{d+1}\). The truncation function \(h_i(x, t) = x_i 1\{ (x, t) \in [-1, 1]^{d+1}\}\). Since the sample paths of \(Du\) are strictly increasing, \(\gamma \geq 0\), the diffusive component of \(Du\) is zero, and the measures \(K(x, t; dy, dw)\) are supported on \((dy, dw) \in \mathbb{R}^d \times [0, \infty)\). Instead of assuming that \(K(x, t; dy, dw)\) integrates \(1 \wedge \| (y, w) \|^2\), it then suffices to assume

\[
\int [1 \wedge (\| y \|^2 + |w|)] K(x, t; dy, dw) < \infty \quad \forall (x, t) \in \mathbb{R}^{d+1}.
\]

The space–time jump kernel \(K\) can be interpreted as the joint intensity measure for the long jumps and long waiting times which do not rescale to 0 as \(c \to \infty\). If the measures \((dy, dw) \mapsto K(x, t; dy, dw)\) are supported on “the coordinate axes” \((\mathbb{R}^d \times \{0\}) \cup (\{0\} \times [0, \infty))\), then large jumps occur independently of long waiting times, and the CTRWL and OCTRWL are identical ([40], Lemma 3.9). We refer to this as the uncoupled case, and to the opposite case as the coupled case.

Finally, we assume that the coefficients \(b_i\), \(\gamma\), \(a_{ij}\) and \(K\) satisfy Lipschitz and growth conditions as in [1], Section 6.2, so that \((Au, Du)\) has an interpretation as the solution to a stochastic differential equation, as well as a semimartingale [19], Section III.2. Then for any canonical Feller process \((Au, Du)\) on \(\mathbb{R}^{d+1}\), we define the CTRWL process \(X_t = (AE_t)^{+}\), and the OCTRWL process \(Y_t = AE_t\), where \(E_t\) is given by (2.4). If we set \(A_{0-} = A_0\), then \(E_t\), \(X_t\) and \(Y_t\) are defined for all \(t \in \mathbb{R}\).

2.1. Forward and backward renewal times. Although the (O)CTRWL is not Markovian, it turns out that it can be embedded in a Markov process on a higher dimensional state space, by incorporating information on the forward/backward renewal times. Define the regenerative set

\[
M = \{(t, \omega) \subset \mathbb{R} \times \Omega : t = Du(\omega) \text{ for some } u \geq 0\},
\]

the random set of image points of \(Du\). These will turn out to be the renewal points of the inverse process \(E_t\) defined in (2.4). Since \(Du\) is càdlàg and has a.s. increasing sample paths, for almost all \(\omega\) the complement of the \(\omega\)-slice \(M(\omega) := \{ t \in \mathbb{R} : (t, \omega) \in M\}\) in \(\mathbb{R}\) is a countable union of intervals of the form \([Du_-(\omega), Du(\omega))\), where \(u \geq 0\) ranges over the jump epochs of the process \(Du\). For example, if \(Du\) is compound Poisson with positive drift, then \(M\) is a.s. a union of intervals \([a, b)\) of positive length. If \(Du\) is a \(\beta\)-stable subordinator with no drift, then \(M\) is a.s. a fractal of dimension \(\beta\) [8].

For any \(t \geq 0\), we write \(G_t\), the last time of regeneration before \(t\), and \(H_t\), the next time of regeneration after \(t\), as

\[
G_t(\omega) := \sup \{ s \leq t : s \in M(\omega) \} \leq t \leq \inf \{ s > t : t \in M(\omega) \} =: H_t(\omega),
\]
where for convenience we set $G_t(\omega) = \inf M(\omega) = \tau$, $\mathbb{P}^{X,\tau}$-a.s. whenever the supremum is taken over the empty set. In terms of the CTRW model, the particle has been resting at its current location since time $G_t$, and will become mobile again at time $H_t$. It will become clear in the sequel that the future evolution of $X_t$ and $Y_t$ on the time interval $[H_t, \infty)$ depends only the position $Y_t$ at time $t = H_t$, meaning that $H_t$ is a Markov time for $X_t$ and $Y_t$.

Note that $G_t$ and $H_t$ are a.s. defined for all $t \in \mathbb{R}$ and their sample paths are càdlàg. By our assumptions on $Du$ and the definition (2.4), it is easy to see that $G_t = DE_t$ and $H_t = DE_t$, $\mathbb{P}^{X,\tau}$-a.s.

The age process $V_t$ and the remaining lifetime $R_t$ from renewal theory can be defined by

\begin{equation}
V_t := t - G_t \quad \text{and} \quad R_t := H_t - t \quad \text{for all } t \in \mathbb{R}.
\end{equation}

At any time $t > 0$, the particle has been resting at its current location for an interval of time of length $V_t$, and will move again after an additional time interval of length $R_t$. We will show below that the processes $(X_{t-}, V_{t-})$ and $(Y_t, R_t)$ are Markov, and we will compute the joint distribution of these $\mathbb{R}^{d+1}$-valued processes at multiple time points, using the Chapman–Kolmogorov equations. The joint laws of $(X_{t-}, Y_t, V_{t-}, R_t)$ were first calculated in [13, 28], but only in the case where the space–time process $(Au, Du)$ is Markov additive (see Section 4) and only for Lebesgue-almost all $t \geq 0$. We now calculate this joint law in our more general time-inhomogeneous setting, for all $t \geq 0$. We need the following additional definitions: Let

$$
C = \{(t, \omega) \subset \mathbb{R} \times \Omega : Du^- (\omega) = t = Du(\omega) \text{ for some } u > 0\} \subset M
$$

be the random set of points traversed continuously by $Du$. The set $C$ is obtained by removing from the set $M$ of regenerative points all points $t$ which satisfy $t = Du > Du^-$ for some $u > 0$ (i.e., the right end points of all contiguous intervals). Moreover, since $(Au, Du)$ visits each point in $\mathbb{R}^{d+1}$ at most once, it admits a 0-potential, or mean occupation measure, $U^{X,\tau}$ defined via

\begin{align*}
\int f(x, t) U^{X,\tau} (dx, dt) &= \mathbb{E}^{X,\tau} \left[ \int_0^\infty f(Au, Du) du \right] = \int_0^\infty T_u f(\chi, \tau) du \\
&= \mathbb{E}^{X,\tau} \left[ \int_0^\infty f(Au^-, Du^-) du \right]
\end{align*}

for any nonnegative measurable function $f : \mathbb{R}^{d+1} \to [0, \infty)$. The last equality holds because $(Au, Du)$ only jumps countably many times. Since $(Au, Du)$ has infinite lifetime, $U^{X,\tau}$ is an infinite measure. We assume that $Du$ is transient [11], so that $U^{X,\tau}(\mathbb{R}^d \times I) < \infty$ for any compact interval $I \subset [0, \infty)$. For instance, any subordinator is transient [8].

Next we derive the joint law of the Markov process $(X_{t-}, Y_t, V_{t-}, R_t)$. The proof uses sample path arguments, and we consider two cases, starting with the case $\{t \notin C\}$:
Proposition 2.1. Fix \((x, \tau) \in \mathbb{R}^{d+1}\) and \(t \geq \tau\). Then
\[
\mathbb{E}^{X,\tau}[f(X_{t-}, Y_t, V_{t-}, R_t)1\{t \notin C\}] = \int_{x \in \mathbb{R}^d} \int_{s \in [\tau, t]} U^{X,\tau}(dx, ds) \times \int_{y \in \mathbb{R}^d} \int_{w \in [t-s, \infty)} K(x, s; dy, dw) f(x, x+y, t-s, w-(t-s))
\]
for all nonnegative measurable \(f\) defined on \(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}\).

Proof. The complement of the section set \(C(\omega)\) in \(\mathbb{R}\) is a.s. a countable union of closed intervals \([Du^-, Du]\), where \(u\) is a jump epoch of \(Du\). Hence, for \(t \notin C\) we have \(G_{t-} \leq t \leq H_t\) and \(G_{t-} < H_t\), hence \(\Delta D_{E_t} = H_t - G_{t-} > 0\). In the complementary case \(\{t \in C\}\), the sample path of \(E_t\) is left-increasing at \(t\), and hence the \(\mathcal{F}\)-optional time \(E_t\) is announced by the optional times \(E_{t-1/n}\). Hence, \(E^*_t := E_t \cdot 1\{t \in C\} + \infty \cdot 1\{t \notin C\}\) is \(\mathcal{F}\)-predictable ([24], page 410), and since in our setting \((Au, Du)\) is a canonical Feller process, it is quasi left-continuous ([24], Proposition 22.20), and \(\Delta(A, D)E_t = (0, 0)\) a.s. Writing \(J = \{(u, \omega) \in \mathbb{R}^+ \times \Omega: \Delta(A, D)u \neq (0, 0)\}\) for the random set of jump epochs of \((Au, Du)\), we hence find that
\[
f(X_{t-}, Y_t, V_{t-}, R_t)1\{t \notin C\}
= \sum_{u \in J} f(A_{u-}, A_u, t - D_{u-}, D_u - t)1\{D_{u-} \leq t \leq D_u\}
= \sum_{u \in J} f(A_{u-} + \Delta A_u, t - D_{u-} + \Delta D_u - t)
\times 1\{D_{u-} \leq t \leq D_{u-} + \Delta D_u\},
\]
noting that all members of the sum except exactly one \((u = E_t)\) equal 0. The last expression equals \(\int W(\omega, u; x, s)\mu(\omega, du; dy, dw)\) for the optional random measure
\[
(2.10) \quad \mu(\omega, du; dy, dw) = \sum_{v \geq 0} 1_{J}(v, \omega)\delta_{(v, \Delta(A_v(\omega), D_v(\omega))}(du; dy, dw)
\]
on \(du \times (dy, dw) \in \mathbb{R}^+ \times \mathbb{R}^{d+1}\) associated with the jumps of \((Au, Du)\), and the predictable integrand
\[
W(\omega, u; y, w) := f(A_{u-}(\omega), A_{u-}(\omega) + y, t - D_{u-}(\omega), D_{u-}(\omega) + w - t)
\times 1\{D_{u-}(\omega) \leq t \leq D_{u-}(\omega) + w\}.
\]
The compensator \(\mu^p\) of \(\mu\) equals [19], page 155
\[
(2.11) \quad \mu^p(\omega; du; dy, dw) = K(A_{u-}(\omega), D_{u-}(\omega); dy, dw) du.
\]
Then the compensation formula [19], II.1.8, implies that

\[
\mathbb{E}^{X,\tau}[f(X_t-\Delta Y,t;\Delta V,t\wedge \Delta R)1_{\{t \notin C\}}] = \mathbb{E}^{\chi,\tau}[\int_0^\infty \int_{y \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} f(x+y,t-s,s) \mu^{\tau}(\omega,du;dy,dw)ds \wedge dw] = \mathbb{E}^{\chi,\tau}[\int_0^\infty \int_{y \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} f(x,t-s,s+t) \times 1_{\{s \leq t \leq s+w\}} K(x,t;dy,dw)U^{\chi,\tau}(dx,dt),
\]

which is equivalent to (2.9). □

The following proposition handles the case \( \{t \in C\} \).

**PROPOSITION 2.2.** Fix \((\chi, \tau) \in \mathbb{R}^{d+1}\) and \(t \geq \tau\). Suppose that the temporal drift \(\gamma\) is bounded and continuous, and assume that the mean occupation measure \(U^{\chi,\tau}(dx,dt)\) is Lebesgue-absolutely continuous with a continuous density \(u^{\chi,\tau}(x,t)\). Then

\[
\mathbb{E}^{\chi,\tau}[f(Y_t)1_{\{t \in C\}}] = \int_{x \in \mathbb{R}^d} f(x) \gamma(x,t) u^{\chi,\tau}(x,t) dx
\]

for all bounded measurable \(f\). Also (2.12) remains true if \(Y_t\) is replaced by \(X_t-\Delta Y_t, Y_t-\Delta V_t\) or \(X_t\).

**PROOF.** Similarly to the proof in [28], \(D_u\) admits a decomposition into a continuous and a discontinuous part via

\[
D^c_u = \int_0^t \gamma(A_s, D_s) ds, \quad D^d_u = \sum_{0 \leq s \leq u} \Delta D_s, \quad t \geq 0.
\]

To see this, we first note that \((A_u, D_u)\) is a semimartingale, and hence \(D_u\) allows the decomposition

\[
D_u = \sum_{s \leq u} \Delta D_s 1_{\{\Delta D_s > 1\}} + B_u + M_u,
\]

where \(B_u\) is a predictable process of finite variation (the first characteristic of \(D_u\)) and \(M_u\) is a local martingale. Due to [19], IX Section 4a, and (2.5), \(B_u = \)
\[ \int_0^u \tilde{\gamma}(A_s, D_s) \, ds \]

where \( \tilde{\gamma}(x, t) = \gamma(x, t) + \int s 1 \{\|y, s\| \leq 1\} K(x, t; dy, ds) \).

Since \( D_u \) has no diffusive part, \( M_u \) is purely discontinuous and equal to

\[ M_u = \sum_{s \leq u} \Delta D_s 1\{\Delta D_s \leq 1\} - \int_0^u \int w 1\{\|y, w\| \leq 1\} K(s, D_s; dy, dw) \, ds. \]

But then (2.13) reads \( D_u = D_u^d + D_u^c \).

For fixed \( \omega \), the paths of \( D, D^c \) and \( D^d \) are nondecreasing and define Lebesgue–Stieltjes measures \( dD, dD^c \) and \( dD^d \) on \([0, \infty)\). Then for any bounded measurable \( f \) and \( g \), we have

\[ \int_0^\infty f(A_u)g(D_u)\gamma(A_u, D_u) \, du = \int_0^\infty f(A_u)g(D_u) \, dD^c_u. \]  

The continuous measure \( dD^c \) does not charge the countable set \( \{u : \Delta D_u \neq 0\} \) of discontinuities of \( D_u \) and coincides with \( dD \) on the complement \( \{u : \Delta D_u = 0\} \). Hence the right-hand side of (2.14) can be written as

\[ \int_0^\infty f(A_u)g(D_u)1\{u : \Delta D_u = 0\} \, dD_u. \]  

The following substitution formula holds for all right-continuous, unbounded and strictly increasing \( F : [0, \infty) \to [0, \infty) \), the inverse \( F^{-1}(t) = \inf\{u : F(u) > t\} \) and measurable \( h : [0, \infty) \to [0, \infty) \):

\[ \int_0^\infty h(u) dF_u = \int_0^\infty h(F^{-1}(t)) \, dt. \]

To see this, first show the statement for \( h \) an indicator function of an interval \((a, b] \subset [0, \infty)\) and then for a function taking finitely many values. The statement for positive \( h \) then follows by approximation via a sequence of finitely valued functions from below, and for general \( h \) by a decomposition into positive and negative part. Applying the substitution formula to (2.15) with \( F(u) = D_u \), the right-hand side of (2.14) reduces to

\[ \int_0^\infty f(Y_t)g(H_t)1\{t : \Delta D_{E_t} = 0\} \, dt. \]

Now note that \( \Delta D_{E_t} = 0 \) is equivalent to \( t \in C \) and implies \( H_t = t \). Hence, the above lines show that the left-hand side of (2.14) equals

\[ \int_0^\infty f(Y_t)g(t)1\{t \in C\} \, dt. \]

Take expectations and apply Tonelli’s theorem to get

\[ \int_{\mathbb{R}^{d+1}} f(x)\gamma(x, t)g(t)u^{X, t}(x, t) \, dx \, dt = \int_0^\infty \mathbb{E}^{X, t}[f(Y_t)1\{t \in C\}]g(t) \, dt. \]

Since \( g \) is an arbitrary nonnegative bounded measurable function, this yields (2.12) for almost every \( t \). By our assumption that \( D_u \) is transient, \( U^{X, t}(\mathbb{R}^d \times I) < \infty \) for compact \( I \subset [0, \infty) \), and then it can be seen that the continuous function \( u^{X, t}(x, t) \) must be bounded on \( \mathbb{R}^d \times I \). Let \( I \) contain \( t \) and apply dominated convergence to
see that the right-hand side of (2.12) is continuous in \( t \). We have already noted in the proof of Proposition 2.1 that \( \Delta(A, D) E_t = (0, 0) \) on \( [t \in C] \), which shows the continuity of the left-hand side. This shows the equality for all \( t \geq 0 \), and also that \( X_t - X_t^- = 0 = Y_t - Y_t^- \) on \( [t \in C] \). \( \square \)

We can now characterize the joint law of \( (X_t-, Y_t, V_t-, R_t) \):

**Theorem 2.3.** Fix \( (\chi, \tau) \in \mathbb{R}^{d+1} \) and \( t \geq \tau \). If \( \gamma \) does not vanish, then suppose that the mean occupation measure \( U_{\chi, \tau}(dx, dt) \) has a continuous Lebesgue density \( u_{\chi, \tau}(x, t) \), and if \( \gamma \equiv 0 \), let \( u_{\chi, \tau}(x, t) \equiv 0 \). Then

\[
\mathbb{E}^{X, \tau}\left[ f(X_t-, Y_t, V_t-, R_t) \right] = \int_{x \in \mathbb{R}^d} f(x, x, 0, 0)\gamma(x, t)u_{\chi, \tau}(x, t) \, dx
\]

\[
+ \int_{x \in \mathbb{R}^d} \int_{s \in [\tau, t]} U^{X, \tau}(dx, ds)
\times \int_{y \in \mathbb{R}^d} \int_{w \in [t-s, \infty)} K(x, s; dy, dw) f\left(x, x + y, t-s, w - (t-s)\right)
\]

for all bounded measurable \( f \). Moreover, \( X_\tau = Y_\tau = \chi \) and \( V_\tau = R_\tau = 0 \), \( \mathbb{P}^{X, \tau} \)-almost surely.

**Proof.** On the set \( [t \in C] \), \( V_t- = 0 = R_t \). The above formula then follows from Propositions 2.2 and 2.1. Assumption (2.2) and the right-continuity of \( D \) yields \( V_\tau = R_\tau = 0 \). The sample paths of \( E \) are then seen to be right-increasing at \( \tau \) and \( E_t > E_\tau = 0 \) for \( t > \tau \). The right-continuity of \( A \) together with \( A_0 = \chi \), \( \mathbb{P}^{X, \tau} \)-a.s. yields \( X_\tau = Y_\tau = \chi \). \( \square \)

### 3. The Markov Embedding

In this section, we establish the Markov property of the processes \((Y_t, R_t)\) and \((X_t-, V_t-)\). Since \( \{E_t \leq u\} = \{D_u \geq t\} \), \( \mathbb{P}^{X, \tau} \)-a.s. for every \((\chi, \tau) \in \mathbb{R}^{d+1}[31] \), equation (3.2), we see that \( E_t \) is an \( \mathcal{F} \)-optional time for every \( t \). We introduce the filtration \( \mathcal{H} = \{\mathcal{H}_t\}_{t \in \mathbb{R}} \) where \( \mathcal{H}_t = \mathcal{F}_E \) and note that \((Y_t, R_t)\) is adapted to \( \mathcal{H} \). Moreover, if \( T \in \mathcal{H} \)-optional, then \( E_T : \omega \mapsto E_T(\omega) \) is \( \mathcal{F} \)-optional (see Lemma A.1). We define the family of operators \( \{Q_{s,t}\}_{s \leq t} \) acting on the space \( B_0(\mathbb{R}^d \times [0, \infty)) \) of real-valued bounded measurable functions \( f \) defined on \( \mathbb{R}^d \times [0, \infty) \) as follows:

\[
Q_{s,t} f(y, 0) = \mathbb{E}^{y, s}[f(Y_t, R_t)]
\]

\[
Q_{s,t} f(y, r) = 1[r > t-s] f(y, r-(t-s))
+ 1[0 \leq r \leq t-s] Q_{s+r,t} f(y, 0).
\]

The dynamics of \( Q_{s,t} \) can be interpreted as follows: If the process \((Y_t, R_t)\) starts at \((y, r)\), the position in space \( y \) does not change while the remaining lifetime \( R_t \) decreases linearly to \( 0 \). When \( r = 0 \), the process continues with the dynamics given
by \((Y_t, R_t)\) started at location \(y\) at time \(s + r\). Note that \(Q_{s,t} f(y,r)\) is measurable in \((s,t,y,r)\), for every bounded measurable \(f\), by the construction of the probability measures \(\mathbb{P}^{x,r}\). We can now state the strong Markov property of \((Y_t, R_t)\) with respect to \(\mathcal{H}\) and \(Q_{s,t}\).

**THEOREM 3.1.** Suppose that the operators \(Q_{s,t}\) are given by (3.1). Then:

(i) The operators \(Q_{s,t}\) satisfy the Chapman–Kolmogorov equations:
\[
Q_{q,s} Q_{s,t} f = Q_{q,t} f, \quad q \leq s \leq t,
\]
and moreover, \(Q_{s,t} 1 = 1\).

(ii) Let \((\chi, \tau) \in \mathbb{R}^{d+1}, t \geq 0\) and let \(T\) be a \(\mathcal{H}\)-optional time. Then
\[
\mathbb{P}^{x,\tau}[f(Y_{T+t}, R_{T+t}) | \mathcal{H}_T] = Q_{T,T+t} f(Y_T, R_T), \quad \mathbb{P}^{x,\tau}\text{-almost surely}
\]
for every real-valued bounded measurable \(f\).

(iii) The process \(t \mapsto (Y_t, R_t)\) is quasi-left-continuous with respect to \(\mathcal{H}\).

Hence, \((Y_t, R_t)\) is a Hunt process with respect to \(\mathcal{H}\) and transition operators \(Q_{s,t}\).

**PROOF.** A proof is given in the Appendix. \(\square\)

We define the filtration \(\mathcal{G} = \{\mathcal{G}_t\}_{t \in \mathbb{R}}\) via \(\mathcal{G}_t = \mathcal{F}_{E_t}\), the \(\sigma\)-field of all \(\mathcal{F}\)-events strictly before \(E_t\). Evidently, the left-continuous process \((X_{t-}, V_{t-})\) is adapted to \(\mathcal{G}\). The main idea behind the Markov property of \((X_{t-}, V_{t-})\) is that, knowing the current state \((x,v) = (X_{t-}, V_{t-})\) and the joint distribution of the next space–time increment given by the kernel \(K(x,v; dy, dw)\) in (2.5), one can calculate the distribution of the next renewal time \(H_t\) and the position \(Y_t\) at that time. Then the probability of events after the renewal point \(H_t\) can be calculated starting at the point \((Y_t, H_t)\) in space–time. We introduce the following notation: Define the family of probability kernels \(\{K_v\}_{v \geq 0}\) on \(\mathbb{R}^{d+1}\)
\[
K_v(x,t; C) = \frac{K(x,t; C \cap (\mathbb{R}^d \times [v, \infty)))}{K(x,t; \mathbb{R}^d \times [v, \infty))},
\]
(3.2)
\[
K_0(x,t; C) = \delta_{(0,0)}(C),
\]
where \(C\) is a Borel set. For \(v > 0\), \(K_v(x,t; dy, dw)\) is the conditional probability distribution of a space–time jump \((y,w)\) (a jump–waiting time pair), given that a time-jump (a waiting time) greater than or equal to \(v\) occurs. Should the denominator \(K(x,t; \mathbb{R}^d \times [v, \infty))\) equal 0, we set \(K_v(x,t; C) = 0\). If \(v = 0\), then \(K_0\) is the Dirac-measure concentrated at \((0,0)\) in \(\mathbb{R}^{d+1}\). Since \(v \mapsto K(x,t; C \cap \mathbb{R}^d \times [v, \infty))\) is decreasing, and hence measurable, it follows that \(v \mapsto K_v(x,t; C)\) is measurable for every \((x,t) \in \mathbb{R}^{d+1}\) and Borel \(C \subset \mathbb{R}^{d+1}\).
We now define the family of operators \( \{P_{s,t}\}_{s \leq t} \) acting on the space \( B_b(\mathbb{R}^d \times [0, \infty)) \) of real-valued bounded measurable functions defined on \( \mathbb{R}^d \times [0, \infty) \):

\[
\begin{align*}
P_{s,t} f(x, 0) &= \mathbb{E}^{x,s}[f(X_{t-}, V_{t-})], \\
P_{s,t} f(x, v) &= f(x, v + t - s) K_v(x, s - v; \mathbb{R}^d \times [v + t - s, \infty)) \\
&\quad + \int_{y \in \mathbb{R}^d} \int_{w \in [v, v + t - s)} P_{s+w-v, t} f(x + y, 0) K_v(x, s - v; dy, dw).
\end{align*}
\]

The dynamics given by \( P_{s,t} \) can be interpreted as follows. With probability \( K_v(x, s - v; \mathbb{R}^d \times (v + t - s, \infty)) \), the process remains at \( x \) and the age increases by \( t - s \). This is the probability that the size of a jump of \( D \) whose base point is at \( (x, s - v) \) exceeds \( v + t - s \), given that it exceeds \( v \). The remaining probability mass for the jump of \( (A, D) \) is spread on the set \( (y, w) \in \mathbb{R}^d \times [v, v + t - s) \), and the starting point is updated from \( x \) to \( x + y \) at the time \( s - v + w \).

**Theorem 3.2.** Let \( P_{s,t} \) be the operators defined by (3.3). Then:

(i) The operators \( (P_{s,t}) \) satisfy the Chapman–Kolmogorov property:

\[
P_{q,s} P_{s,t} f = P_{q,t} f, \quad q \leq s \leq t,
\]

and moreover, \( P_{s,t} 1 = 1 \).

(ii) The process \( (X_{t-}, V_{t-}) \) satisfies the simple Markov property with respect to \( G \) and \( P_{s,t} \):

\[
\mathbb{E}^{x,\tau}[f(X_{t-}, V_{t-}) | G_s] = P_{s,t} f(X_{s-}, V_{s-}), \quad \mathbb{P}^{x,\tau}-a.s.
\]

for all \((\chi, \tau) \in \mathbb{R}^{d+1}, \tau \leq s \leq t \) and real-valued bounded measurable \( f \).

**Proof.** A proof is given in the Appendix. \( \square \)

**Remark 3.3.** It would be interesting to investigate whether the moderate Markov property (e.g., see Chung and Glover [10]) holds for \( (X_{t-}, V_{t-}) \). An application of the compensation formula to the process \( (X_{t-}, G_{t-}) \) might yield a proof, but this would require the semimartingale characteristics of \( (X_{t-}, G_{t-}) \), which we have not been able to calculate.

4. The time-homogeneous case. If the coefficients \( b(x, t), \gamma(x, t), a(x, t) \) and \( K(x, t; dy, dw) \) of the generator \( A \) in (2.5) do not depend on \( t \in \mathbb{R} \), then we say that \( (A_u, D_u) \) is a Markov additive process. This means that the future of \( (A_u, D_u) \) only depends on the current state of \( A_u \); see, for example, [12].
THEOREM 4.1. If the space–time random walk limit process \((A_u, D_u)\) in (2.1) is Markov additive, then the Markov processes \((X_{t-}, V_{t-})\) and \((Y_t, R_t)\) are time-homogeneous. Writing \(K_r(x; dy, ds) := K_r(x, t; dy, ds)\) and \(\mathbb{P}^x = \mathbb{P}^{x,0}\), the transition semigroup \(Q_{t-s} := Q_{s,t}\) of the Markov process \((Y_t, R_t)\) is given by

\[
Q_t f(y, 0) = \mathbb{E}^y[f(Y_t, R_t)],
\]

\[
Q_t f(y, r) = \begin{cases} 
0 & 0 \leq t < r \\
1 & 0 \leq r \leq t \\
Q_{t-r} f(y, 0) & r > t
\end{cases}
\]

and the transition semigroup \(P_{t-s} := P_{s,t}\) of the Markov process \((X_{t-}, V_{t-})\) is given by

\[
P_t f(x, 0) = \mathbb{E}^x[f(X_{t-}, V_{t-})],
\]

\[
P_t f(x, v) = f(x, v + t) K_v(x; \mathbb{R}^d \times [v + t, \infty)) + \int_{\mathbb{R}^d \times [v + t]} P_{v + t - w} f(x + y, 0) K_v(x; dw, dy),
\]

acting on the bounded measurable functions defined on \([0, \infty) \times \mathbb{R}^d\).

PROOF. Since \((A_u, D_u)\) is Markov additive, we have \(\partial_s A f = A \partial_s f\) for all \(s \in \mathbb{R}\), where the shift operator \(\partial_s f(x, t) = f(x, t + s)\). It follows that the resolvents \((\lambda - A)^{-1}\), the semigroup \(T_u\) and the kernel \(U_f(\chi, \tau) = U^{\chi,\tau}(f)\) commute with \(\partial_s\). Then \(\mathbb{E}^{\chi,\tau} f(A_u, D_u) = \mathbb{E}^{\chi,0} f(A_u, \tau + D_u)\) for all \(u\) and measurable \(f\), and hence it suffices to work with the laws \(\mathbb{P}^{x,0}\). Now in Theorem 2.3, writing \(U^{\chi,\tau}(dx, dt) = U^{\chi}(dx, dt - \tau)\), we have

\[
\mathbb{E}^{\chi,\tau} f(X_{t-}, Y_t, V_{t-}, R_t) = \mathbb{E}^{\chi,0} f(X_{(t-\tau)-}, Y_{(t-\tau)-}, V_{(t-\tau)-}, R_{(t-\tau)-})
\]

where \(\tau = 0\) without loss of generality. It follows that (4.1) and (4.2) are semigroups acting on the bounded measurable functions defined on \([0, \infty) \times \mathbb{R}^d\), compare [18], equations (19) and (31). □

REMARK 4.2. Under the assumptions of Theorem 2.3, a simple substitution yields the formulation of \(P_t\) and \(Q_t\) in terms of transition probabilities: For \(P_t\), we find

\[
P_t(x_0, 0; dx, dv)
\]

\[= \gamma(x, t) u_{x_0}(x, t) dx \delta_0(dv)
\]

\[+ K(x_0; \mathbb{R}^d \times [v, \infty)) U^{x_0}(dx, t - dv) 1[0 \leq v \leq t],
\]

\[
(4.3)
\]

\[
P_t(x_0, v_0; dx, dv)
\]

\[= \delta_{x_0}(dx) \delta_{v_0 + t}(dv) K_{v_0}(x_0; \mathbb{R}^d \times [v_0 + t, \infty))
\]

\[+ \int_{y \in \mathbb{R}^d} \int_{w \in [v_0, v_0 + t]} P_{v_0 + t - w}(x_0 + y, 0; dx, dv) K_{v_0}(x_0; dy, dw),
\]
and for $Q_t$ we have

$$Q_t(y_0, 0; dy, dr) = \gamma(y, t) u(t) dy \delta_0(dr)$$

$$+ \int_{x \in \mathbb{R}} \int_{w \in [0, t]} U(y, d) K(x; dy - x, dr + t - w),$$

(4.4)

$$Q_t(y_0, r_0; dy, dr) = \mathbf{1}\{0 < t < r_0\} \delta_{y_0}(dy) \delta_{r_0-}(dr) + \mathbf{1}\{0 \leq r_0 \leq t\} Q_{t-r_0}(y_0, 0; dy, dr).$$

5. Finite-dimensional distributions. In this section, we provide two examples to illustrate the explicit computation of finite dimensional distributions for the CTRWL process $X_t$ and the OCTRWL process $Y_t$.

**EXAMPLE 5.1 (The inverse stable subordinator).** A very simple CTRW model takes deterministic jumps $J^c_{n} = c^{-1}$ and waiting times $W^c_k$ in the domain of attraction of a standard $\beta$-stable subordinator $D_u$ such that $\mathbb{E}[e^{-s D_u}] = e^{-us^\beta}$. Setting $(A_0, D_0) = (\chi, \tau)$, (2.1) holds with $(A_u, D_u) = (\chi + u, \tau + \bar{D}_u)$, where $\bar{D}_u$ is a $\beta$-stable subordinator. Here, the CTRWL and the OCTRWL coincide, since $A_u$ has no jumps. If $(\chi, \tau) = (0, 0)$, then in (2.3) we have $X_t = Y_t = E_t$, the inverse $\beta$-stable subordinator. Now we will compute the joint distributions of this first passage time process. The joint Laplace transform of these finite-dimensional distributions was computed by Bingham [9] but to our knowledge, the distributions themselves have not been reported in the literature.

The space–time limit $(A_u, D_u)$ is a canonical Feller process on $\mathbb{R}^{d+1}$ with generator $A$ given by (2.5) with $d = 1$, $b_1 \equiv 1$, $\gamma \equiv 0$, $a_{11} \equiv 0$, and jump kernel $K(x, t; dy, dw) = \delta_0(dy) \Phi(w) dw$ by [33], Proposition 3.10, where the Lévy measure $\Phi(w) dw = \beta w^{-\beta-1} dw / \Gamma(1 - \beta)$. The stable Lévy process $\bar{D}_u$ has a smooth density $g(t, u)$ so that $\mathbb{P}^0, 0(D_u \in dt) = g(t, u) dt$ for every $u > 0$ by [20], Theorem 4.10.2. The underlying process $(A_u, D_u)$ is Markov additive, hence $(X_{t-}, V_{t-})$ and $(Y_t, R_t)$ are time-homogeneous Markov processes. In [40], Lemma 4.2, it was shown that $(X_t, V_t)$, has no fixed discontinuities, hence $(X_{t-}, V_{t-})$ has the same law as $(X_t, V_t)$. One checks that the 0-potential of $(A_u, D_u)$ is absolutely continuous with density

$$u^{\chi\tau}(x, t) = g(t - \tau, x - \chi) \mathbf{1}\{t > \tau, x > \chi\}.$$

(5.1)

Then it follows from (4.3) that the transition semigroup of $(X_{t-}, V_{t-})$ is given by

$$P_t(x_0, 0; dx, dv) = g(t - v, x - x_0) \Phi(v, \infty) dx dv,$$

(5.2)
\[ P_t(x_0, v_0; dx, dv) \]
\[ = \delta_{x_0}(dx)\delta_{v_0+\tau}(dv) \left( \frac{v_0 + t}{v_0} \right)^{-\beta} 1\{v_0 > 0\} \]
\[ + \left( \frac{v_0}{v} \right)^{\beta} \int_{s=v_0}^{v_0 + t - v} g((t - v) - (s - v_0), x - x_0) \]
\[ \times \frac{\beta s^{-1-\beta} ds}{\Gamma(1-\beta)} 1\{x > x_0, 0 < v < t\} dx dv. \]

Hence, for \(0 < t_1 < t_2\), the joint distribution of \((E_{t_1}, V_{t_1}, E_{t_2}, V_{t_2})\) is
\[ \mathbb{P}^{0,0}(E_{t_1} \in dx, V_{t_1} \in dv, E_{t_2} \in dy, V_{t_2} \in dw) \]
\[ = P_{t_2-t_1}(x, v; dy, dw) P_{t_1}(0, 0; dx, dv) \]
\[ = g(t_1 - v, x) \Phi(v, \infty) 1\{x > 0, 0 < v < t_1\} dx dv \]
\[ \times \left[ \delta_x(dy)\delta_{v+t_2-t_1}(dw) \left( \frac{v + t_2 - t_1}{v} \right)^{-\beta} \right. \]
\[ + \int_{s=v_0}^{v_0 + t - v - w} g((t_2 - t_1 - w) - (s - v), y - x) \]
\[ \times \frac{\beta s^{-\beta-1} ds}{\Gamma(1-\beta)} \left( \frac{v}{w} \right)^{\beta} dy dw 1\{y > x, 0 < w < t_2 - t_1\} \]
\[ \text{since } E_0 = V_0 = 0 \text{ for the physical starting point } (0, 0). \]

Integrating out the backward renewal times \(V_{t_1}\) and \(V_{t_2}\), it follows that the joint distribution of \((E_{t_1}, E_{t_2})\) is
\[ \mathbb{P}(E_{t_1} \in dx, E_{t_2} \in dy) \]
\[ (5.3) = 1\{x > 0\} \delta_x(dy) \int_{v=0}^{t_1} g(t_1 - v, x) \frac{(v + t_2 - t_1)^{-\beta}}{\Gamma(1-\beta)} dv \]
\[ + \int_{v=0}^{t_1} \int_{w=0}^{t_2-t_1} \int_{s=v_0}^{v_0 + t - v - w} g((t_2 - t_1 - w) - (s - v), y - x) dy 1\{y > x\} \]
\[ \times \frac{\beta s^{-\beta-1} ds}{\Gamma(1-\beta)} \left( \frac{v}{w} \right)^{\beta} dw dv. \]

**Remark 5.2.** The joint distribution of \((E_{t_1}, E_{t_2})\) can also be computed from the OCTRW embedding, but the computation appears to be simpler using the CTRWL embedding.

**Remark 5.3.** Baule and Friedrich [4] compute the Laplace transform of the joint distribution function \(H(x, y, s, t)\) of \(x = E_s\) and \(y = E_t\) and show that
\[ (\partial_x + \partial_y) H(x, y, s, t) = -(\partial_x + \partial_y)^{\beta} H(x, y, s, t) \]
on $0 < s < t$ and $0 < x < y$. Equation (5.3) provides an explicit solution to this governing equation, which solves an open problem in [4]. The finite dimensional laws of any uncoupled CTRW limit can easily be calculated from the finite dimensional laws of $E_t$, given the law of the process $A_u$. This follows from a simple conditioning argument; see, for example, [31].

**Example 5.4.** Kotulski [26] considered a CTRW with jumps equal to the waiting times $J^n_c = W^n_c$, in the domain of attraction of a standard $\beta$-stable subordinator $\bar{D}_u$ such that $\mathbb{E}[e^{-s\bar{D}_u}] = e^{-us^\beta}$. Equation (2.1) holds with $(A_u, D_u) = (A_0 + \bar{D}_u, D_0 + \bar{D}_u)$. The space–time limit $(A_u, D_u)$ is a canonical Feller process on $\mathbb{R}^{d+1}$ with generator $A$ given by (2.5) with $d = 1$, $\gamma \equiv 0$ and $K(x, t; dy, dw) = \delta_w(dy)\Phi(dw)$, where $\Phi(dw) = \phi(w)dw = \beta w^{\beta-1}dw/\Gamma(1-\beta)$. The stable Lévy process $\bar{D}_u$ has a smooth density $g(t, u)$ so that $P_0, 0, (\bar{D}_u \in dt) = g(t, u)dt$ for every $u > 0$. Since the Markov process $(A_u, D_u)$ is Markov additive, we need only compute the potential for $\tau = 0$:

\begin{equation}
U^{X, 0}(dx, dt) = \delta_{x+t}(dx) \int_0^\infty g(t, u)du dt.
\end{equation}

Next, one sees that

\begin{equation}
\int_0^\infty g(t, u)du = \frac{t^{\beta-1}}{\Gamma(\beta)}
\end{equation}

by taking Laplace transforms on both sides (also see [33], Example 2.9). The 0-potential hence equals

\begin{equation}
U^{X, 0}(dx, dt) = \delta_{x+t}(dx) \frac{t^{\beta-1}}{\Gamma(\beta)} dt.
\end{equation}

With $\Phi([v, \infty)) = v^{-\beta}/\Gamma(1-\beta)$, (4.3) reads

\begin{align*}
P_t(x_0, 0; dx, dv) &= \frac{v^{-\beta}}{\Gamma(1-\beta)} \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \delta_{x_0+t-v}(dx) dv \mathbb{1}\{0 < v < t\}, \\
P_t(x_0, v_0; dx, dv) &= \delta_{x_0}(dx) \delta_{v_0+t}(dv) \left(\frac{v_0 + t}{v_0}\right)^{-\beta} \\
&+ \int_{s=v_0}^{v_0+t} \left(\frac{v}{v_0}\right)^{-\beta} \delta_{x_0+v_0+t-v}(dx) \frac{(v_0 + t - s - v)^{\beta-1}}{\Gamma(\beta)} ds dv.
\end{align*}

Note that the above formulae extend Example 5.5 in [5], which calculates the law of $X_{t-}$. The joint distribution of $\{(X_{t_i-}, V_{t_i-}) : 0 \leq i \leq n\}$ can now be computed.
by a simple conditioning argument. Similarly, the semigroup for \((Y_t, R_t)\) reads

\[
Q_t(y_0, r_0; dy, dr) = \delta_{y_0}(dy)\delta_{r_0-t}(dr)1\{r_0 \geq t\} + Q_{t-r_0}(dy-y_0, dr)1\{0 < r_0 < t\}
\]

\[
= \delta_{y_0}(dy)\delta_{r_0-t}(dr)1\{r_0 \geq t\} + 1\{0 < r_0 < t\}\delta_{t-r_0}(dy)1\{r_0 \geq t\} + 1\{0 < r_0 < t\}\delta_{r_0}(dy) + \int_{w=0}^{t-r_0} w^{-\beta-1} \frac{\beta(t-r_0 + r - w)^{-\beta-1}}{\Gamma(1-\beta)} dw dr.
\]

The joint distributions of \(X_{t-}, Y_t\) lead directly to the joint distribution of CTRWL, OCTRWL, respectively, for a wide variety of coupled models; see [21].

**APPENDIX: PROOFS**

**LEMMA A.1.** Let \(T > 0\) be \(\mathcal{H}\)-optional. Then \(E_T : \omega \mapsto E_T(\omega)(\omega)\) is \(\mathcal{F}\)-optional.

**PROOF.** We first assume that \(T\) is single valued. That is, fix \(t > 0\) and \(U \in \mathcal{H}_t\), and let \(T(\omega) = t \cdot 1\{\omega \in U\} + \infty \cdot 1\{\omega \notin U\}\). It is easy to check that \(T\) is indeed \(\mathcal{H}\)-optional. Now \(\{E_T \leq u\} = \{E_t \leq u\} \cap U\), and the right-hand side lies in \(\mathcal{F}_u\), which follows from \(U \in \mathcal{H}_t = \mathcal{F}_{E(t)}\) and the definition of the stopped \(\sigma\)-algebra \(\mathcal{F}_{E(t)}\). Now consider an \(\mathcal{H}\)-optional time \(T\) with countably many values \(t_n\), so that \(\Omega = \bigcup_{n \in \mathbb{N}} \{\omega : T(\omega) = t_n\}\). Then due to the a.s. nondecreasing sample paths of \(E\), we have \(E(\inf T_n) = \inf E(T_n)\), and an application of [24], Lemma 6.3/4, together with the right-continuity of the filtrations \(\mathcal{F}\) and \(\mathcal{H}\) shows that \(E_T\) is \(\mathcal{H}\)-optional.

Stopping times allows for a decomposition into a predictable and totally inaccessible part [24]. The following lemma gives an interpretation for stopping times of the form \(E_T\).

**LEMMA A.2.** Let \(T > 0\) be an \(\mathcal{H}\)-predictable stopping time. Then the \(\mathcal{F}\)-stopping time \(E_T\) is predictable on the set \(\{\omega : E_{T-\varepsilon}(\omega) < E_T(\omega) \forall \varepsilon > 0\} = \{V_{T-} = 0\}\) and totally inaccessible on the complement \(\{\omega : \exists \varepsilon > 0, E_{T-\varepsilon}(\omega) = E_T(\omega)\} = \{V_{T-} > 0\}\). Moreover, \(\Delta(A, D)_{E_T} = (0, 0)\) on \(\{V_{T-} = 0\}\) and \(\Delta D_{E_T} > 0\) on \(\{V_{T-} > 0\}\), \(\mathbb{P}^{\chi,\tau}\)-a.s.

**PROOF.** Let \(T_n\) be an announcing sequence ([24], page 410), for \(T\), that is \(T_n\) are \(\mathcal{H}\)-stopping times, \(T_n < T, T_n \uparrow T\) a.s. Then due to the a.s. continuity of sample paths of \(E\), the sequence \(E_{T_n}\) announces \(E_T\) on the set \(\{V_{T-} = 0\}\), that is \(E_T\) is predictable on this set. As a canonical Feller process, \((A, D)\) is quasi-left-continuous, and all its jump times are totally inaccessible ([24], Proposition 22.20),
hence $\Delta(A, D)_{E_T} = (0, 0)$, $P^{X,T}$-a.s. on $\{V_T^- = 0\}$. On the complementary set $\{V_T^- > 0\}$, we have $0 < H_T - G_T^- = \Delta D_{E_T}$, and hence the process $D$ jumps at $E_T$. □

**Proof of Theorem 3.1.** We first prove (ii). Consider the set of $\omega$ such that $H_T(\omega) > t$. In this case, $M_\omega \cap (T, t) = \emptyset$, and hence $E_T = E_t$, so $(Y_t, H_t) = (Y_T, H_T)$, which implies that

$$
\mathbb{E}^{X,T}[f(Y_t, R_t)1_{\{H_T > t\}}|H_T] = f(Y_T, H_T - t)1_{\{H_T > t\}}
$$

(A.1)

This corresponds to the first case in (3.1). Turning to the second case, $H_T(\omega) \leq t$, consider the shift operators $\theta_t$ acting on $\Omega$, which are defined as usually by $(\theta_t \omega)(u) = \omega(t + u)$, or equivalently

$$
(A, D)_u(\theta_t \omega) = (A, D)_{t+u}(\omega),
$$

(A.2)

since $(A, D)$ is canonical for $\Omega$. Then from the definition of the inverse process $E$, we find

$$
E_t(\theta_{E_T} \omega) = \inf\{u \geq 0 : D_u(\theta_{E_T} \omega) > t\} = \inf\{u \geq 0 : D_{u+E_T}(\omega) > t\}
$$

(A.3)

$$
= \inf\{u : u - E_T(\omega) \geq 0, D_u(\omega) > t\} - E_T(\omega)
$$

where $\theta_{E_T} \omega = \theta_u \omega$ if $E_T(\omega) = u$. Now observe that $(A, D)_{E_t}$ is the point in $\mathbb{R}^{d+1}$ where the process $(A, D)$ enters the set $\mathbb{R}^d \times (t, \infty)$. This point will be the same for the space–time path started at the earlier time $E_T$, that is,

$$
(A, D)_{E_t} \circ \theta_{E_T} = (A, D)_{E_t}.
$$

(A.4)

In fact, using (A.2) and (A.3) we find

$$
(A, D)_{E_t}(\theta_{E_T} \omega) = (A, D)_{E_t(\theta_{E_T} \omega)(\theta_{E_T} \omega)} = (A, D)_{E_T(\omega) + E_t(\theta_{E_T} \omega)}(\omega)
$$

$$
= (A, D)_{E_t(\omega)}(\omega) = (A, D)_{E_t}(\omega)
$$

for all $\omega \in \Omega$. Hence, we have shown that

$$
H_t(\theta_{E_T} \omega) = H_t(\omega), \quad Y_t(\theta_{E_T} \omega) = Y_t(\omega)
$$

on the set $\{H_T \leq t\}$. This yields

$$
\mathbb{E}^{X,T}[f(Y_t, R_t)1_{\{H_T \leq t\}}|H_T] = \mathbb{E}^{X,T}[f(Y_t, R_t) \circ \theta_{E_T} 1_{\{H_T \leq t\}}|H_T]
$$

$$
= \mathbb{E}^{X,T}[f(Y_t, R_t) \circ \theta_{E_T} |\mathcal{F}_{E_T}] 1_{\{H_T \leq t\}}
$$

(A.5)

$$
= \mathbb{E}^{(A,D)_{E_T}}[f(Y_t, R_t)1_{\{H_T \leq t\}}]
$$

$$
= \mathbb{E}_{Y_T,H_T}[f(Y_t, R_t)1_{\{H_T \leq t\}}]
$$
\[ P^{X,\tau} \text{-almost surely, using the strong Markov property of } (A, D) \text{ at the stopping time } E_T. \text{ Then (ii) follows by adding equations (A.1) and (A.5).} \]

As for (i), let \( (y_0, r_0) \in \mathbb{R}^d \times [0, \infty). \) Then \( P^{y_0, q + r_0} (Y = y_0, R_q = r_0) = 1, \) and hence by nested conditional expectations and the above calculations we have

\[
Q_{q,t} f(y_0, r_0) = \mathbb{E}^{y_0, q + r_0} \left[ Q_{q,t} f(Y_q, R_q) \right]
\]

\[
= \mathbb{E}^{y_0, q + r_0} \left[ \mathbb{E}^{y_0, q + r_0} \left[ f(Y_t, R_t) \mid \mathcal{H}_q \right] \right]
\]

\[
= \mathbb{E}^{y_0, q + r_0} \left[ \mathbb{E}^{y_0, q + r_0} \left[ f(Y_t, R_t) \mid \mathcal{H}_s \right] \mid \mathcal{H}_q \right]
\]

\[
= \mathbb{E}^{y_0, q + r_0} \left[ Q_{s,t} f(Y_q, R_q) \right] = Q_{q,s} Q_{s,t} f(y_0, r_0).
\]

We turn to the remaining case (iii). By definition of \( R_t, \) it suffices to show that if \( T \) is a \( \mathcal{H} \)-predictable time, then \( (Y, H)_{T-} = (Y, H)_T, \) \( P^{X,\tau} \)-a.s. for every \( (\chi, \tau) \in \mathbb{R}^d \times \mathbb{R}^d. \) Hence let \( T_n < T, T_n \nearrow T \) be a sequence of \( \mathcal{H} \)-optional times announcing \( T. \) As in Lemma A.2, we check the two cases in which the \( \mathcal{F} \)-stopping time \( E_T \) is predictable or totally inaccessible.

On the set \( \{ \omega: E_T^\varepsilon(\omega) < E_T(\omega), \forall \varepsilon > 0 \} \), the process \( E \) is left-increasing at \( T, \) continuous, and \( E_{T_n} \uparrow E_T, E_{T_n} < E_T \) if \( T_n \uparrow T, T_n < T. \) Moreover, \( \Delta(A, D)_{E_T} = (0, 0) \) a.s. (Lemma A.2). Hence,

\[
(H, Y)_{T-} = (A, D)_{E_T-} = (A, D)_{E_T} = (H, Y)_T.
\]

On the set \( \{ \omega: \exists \varepsilon > 0: E_T^\varepsilon(\omega) = E_T(\omega) \}, E \) is left-constant at \( T. \) Hence, \( E_{T_n} = E_T \) for large \( n, \) and

\[
(H, Y)_{T-} = \lim(H, Y)_{T_n} = \lim(A, D)_{E_{T_n}} = (A, D)_{E_T} = (H, Y)_T.
\]

The two cases together imply that \( (H, Y)_{T-} = (H, Y)_T \) a.s. □

For the proof of Theorem 3.2, we will need the following lemma.

**Lemma A.3.** Let \( (\chi, \tau) \in \mathbb{R}^d, \) and let \( t \geq \tau. \) Then for every bounded measurable \( f \) defined on \( \mathbb{R} \times \mathbb{R}^d, \) we have \( P^{X,\tau} \)-a.s.:

\[
\mathbb{E}^{X,\tau} \left[ f(X_{t-}, G_{t-}; \Delta(A, D)_{E(t)}) \mid \mathcal{G}_t \right] = \int_{\mathbb{R}^d} K_{V_{t-}}(X_{t-}, G_{t-}; dx \times dz) f(X_{t-}, G_{t-}; x, z).
\]

**Proof.** Since \( (X_{t-}, G_{t-}) \) are \( \mathcal{G}_t \)-measurable, by a monotone class argument and dominated convergence, it suffices to prove the formula

\[
(A.6) \quad \mathbb{E}^{X,\tau} \left[ f(\Delta(A, D)_{E(t)}) \mid \mathcal{G}_t \right] = \int_{\mathbb{R}^d} K_{V_{t-}}(X_{t-}, G_{t-}; f)
\]
for all bounded measurable $f$ defined on $\mathbb{R}^{d+1}$. As in Lemma A.2, we consider the two cases $\{V_t^- = 0\}$ and $\{V_t^- > 0\}$. On $\{V_t^- = 0\}$, we have $\Delta(A, D)_{E_t} = (0, 0)$, $\mathbb{P}^{\mathcal{X}_t \tau}$-a.s., and hence

$$\mathbb{E}^{\mathcal{X}_t \tau}[f(\Delta(A, D)_{E(t)})1\{V_t^- = 0\}]$$

(A.7)

$$= f(0, 0) = \delta_{(0,0)}(f) = K_{V_t^-}(X_t^-, G_t^-; f)1\{V_t^- = 0\}.$$ 

On $\{V_t^- > 0\}$, the process $D$ jumps at $E_t$ (Lemma A.2), and since $D$ has increasing sample paths this is equivalent to

(A.8) “there exists a unique number $s > 0$ such that $D_s^- < t \leq D_s$.”

We rewrite the restriction of (A.6) to $\{V_t^- > 0\}$ in integral form:

$$\mathbb{E}^{\mathcal{X}_t \tau}[f(\Delta(A, D)_{E(t)})1_{C}1\{V_t^- > 0\}]$$

$$= \mathbb{E}^{\mathcal{X}_t \tau}[K_{V_t^-}(X_t^-, G_t^-; f)1_{C}1\{V_t^- > 0\}], \quad C \in \mathcal{G}_t,$$

where $1_C(\omega) = 1$ iff $\omega \in C$. Now we invoke [14], Theorem IV.67(b), which says that there exists an $\mathcal{F}$-adapted predictable process $Z$ such that $1_C = Z_{E(t)}$. Then it suffices to show that for every $\mathcal{F}$-adapted predictable process $Z$, the following two random variables have the same expectation with respect to $\mathbb{P}^{\mathcal{X}_t \tau}$:

$$f(\Delta(A, D)_{E(t)})Z_{E(t)}1\{V_t^- > 0\},$$

(A.9)

$$K_{V_t^-}f(X_t^-, G_t^-)Z_{E(t)}1\{V_t^- > 0\}.$$

We begin on the right-hand side and find, using (A.8) and $X_t^- = A_{E_t}^-, G_t^- = D_{E_t}^-$

$$\mathbb{E}^{\mathcal{X}_t \tau}[K_{V_t^-}(X_t^-, G_t^-; f)Z_{E_t}1\{V_t^- > 0\}]$$

$$= \mathbb{E}^{\mathcal{X}_t \tau}[K_{t-D_{E_t}^-}(A_{E_t}^-, D_{E_t}^-; f)Z_{E_t}1\{D_{E_t}^- < t\}]$$

$$= \mathbb{E}^{\mathcal{X}_t \tau}\left[\sum_{s > 0} K_{t-D_{E_t}^-}(A_{s}^-, D_{s}^-; f)Z_s1\{D_{s}^- < t \leq D_{s}^-\}\right]$$

$$= \mathbb{E}^{\mathcal{X}_t \tau}\left[\sum_{s > 0} K_{t-D_{E_t}^-}(A_{s}^-, D_{s}^-; f)Z_s1\{D_{s}^- < t\}1\{\Delta D_s \geq t - D_{s}^-\}\right]$$

$$= \mathbb{E}^{\mathcal{X}_t \tau}[W(\cdot, s; y, w)\mu(\cdot, ds; dy, dw)] = \cdots,$$

where the optional random measure $\mu$ is as in (2.10) and

$$W(\omega, s; y, w) = K_{t-D_{s}^-}(\omega)(A_{s}^-(\omega), D_{s}^-; f)Z(s, \omega)$$

$$\times 1\{D_{s}^- < t\}1\{w \geq t - D_{s}^-; \omega\}.$$
is a predictable integrand. The compensation formula [19], II.1.8, and (2.11) then yield

\[
\cdots = \mathbb{E}^{X,\tau}\left[ W(\cdot, s; y, w) \mu^P(\cdot, ds; dy, dw) \right]
= \mathbb{E}^{X,\tau}\left[ \int_0^\infty K_{t-D_s}(A_{s-}, D_{s-}; f) Z_s \mathbf{1}\{D_s < t\} \times K(A_{s-}, D_{s-}; \mathbb{R}^d \times (t - D_s, \infty)) \right] ds
\]

Using the definition of \( K_v \) (3.2), this equals

\[
\mathbb{E}^{X,\tau}\left[ \int_0^\infty \int_{\mathbb{R}^d \times [t-D_s, \infty)} K(A_{s-}, D_{s-}; dy, dw) f(y, w) Z_s \times \mathbf{1}\{D_s < t\} ds \right]
\]

(A.10)

Proceeding similarly with the left-hand side of (A.9), we find

\[
\mathbb{E}^{X,\tau}\left[ f(\Delta(A, D) E(t)) Z_{E(t)} \mathbf{1}\{V_{t-} > 0\} \right]
= \mathbb{E}^{X,\tau}\left[ \sum_{s > 0} f(\Delta(A, D)_s) Z_s \mathbf{1}\{D_s < t \leq D_s + \Delta D_s\} \right]
\]

(A.11)

\[
= \mathbb{E}^{X,\tau}\left[ \tilde{W}(\cdot, s; y, w) \mu(\cdot, ds; dy, dw) \right]
= \mathbb{E}^{X,\tau}\left[ \tilde{W}(\cdot, s; y, w) \mu^P(\cdot, ds; dy, dw) \right]
\]

where \( \tilde{W}(\omega, s; y, w) = f(y, w) Z_s(\omega) \mathbf{1}\{D_s(\omega) < t \leq D_s(\omega) + w\} \). We check that (A.11) and (A.10) are equal. Hence, we have shown

\[
\mathbb{E}^{X,\tau}\left[ f(\Delta(A, D) E(t)) \mathbf{1}\{V_{t-} > 0\} | G_t \right]
= K_{V_{t-}} f(X_{t-}, G_{t-}) \mathbf{1}\{V_{t-} > 0\},
\]

(A.12)

and adding equations (A.7) and (A.12) yields (A.6). \[\square\]

For later use, we note the formula

\[
K_{v+t}(x, z; C) K_v(x, z; \mathbb{R}^d \times [v+t, \infty))
\]

\[
= K_v(x, z; C) \quad (x, z) \in \mathbb{R}^{d+1}, v, t \geq 0,
\]

valid for all Borel-sets \( C \subset \mathbb{R}^d \times [v+t, \infty) \).
PROOF OF THEOREM 3.2. We begin with statement (ii). We consider the two cases $H_s \geq t$ and $H_s < t$. On the set $\{H_s \geq t\}$, $E$ is constant on the interval $[s, t]$, and hence we have $(G, X)_t = (G, X)_s$. Using $G_s + \Delta D_E_s = H_s$ and Lemma A.3, we calculate

$$
\mathbb{E}^{X,\tau}[f(X_t, V_t)1\{H_s \geq t\} | G_s]
= f(X_s, t - G_s)\mathbb{P}^{X,\tau}(H_s \geq t | G_s)
$$

(A.14) $$
= f(X_s, t - s + V_s)\mathbb{P}^{X,\tau}(\Delta D_E_s \geq t - G_s | G_s)
= f(X_s, t - s + V_s)K_{V_s}(X_s, G_s; [t - G_s, \infty) \times \mathbb{R}^d)
= f(X_s, V_s + t - s)K_{V_s}(X_s, s - V_s; [t - s + V_s, \infty) \times \mathbb{R}^d),
$$
which corresponds to the first summand in (3.3).

We now turn to the case $H_s < t$, and recall the shift operators $\theta_t$ from (A.2). For the left-continuous version of $(A, D)$, we can write

$$(A, D)_{t-} \circ \theta_E = (A, D)_{s+t-}(\omega), \quad s \geq 0, t > 0.
$$

Note that we had to assume $t > 0$ above, for the left-hand limit to be defined. We find now, similarly to (A.4),

$$(A, D)_{E_t-} \circ \theta_E_s = (A, D)_{E_t-}
on \{H_s < t\}. \quad \text{Indeed, by (A.3), } E_t(\omega) = E_s(\omega) + E_t(\theta_E_s \omega), \text{ and so}

$$(A, D)_{E_t-}(\theta_E_s \omega) = (A, D)_{E_t}(\theta_E_s \omega) - (\theta_E_s \omega) = (A, D)_{E_t(\theta_E_s \omega)} - (\omega)
= (A, D)_{E_t(\theta_E_s \omega) - (\omega)} = (A, D)_{E_t-}(\omega).
$$

If $t > 0$ and $H_s(\omega) < t$, then by (A.3) $E_t(\theta_E_s \omega) = E_t(\omega) - E_s(\omega) > 0$, and the left-hand limit is well defined. Thus, we have shown that on the set $\{H_s < t\}$ we have $(X_{t-}, V_{t-}) = (X_{t-}, V_{t-}) \circ \theta_E_s$. We will use the strong Markov property of $(A, D)$ in the following form:

$$
\mathbb{E}^{X,\tau}[F \circ \theta_T | \mathcal{F}_T] = \mathbb{E}^{A_T, DT}[F], \quad \mathbb{P}^{X,\tau}\text{-a.s.},
$$
valid for all $\mathcal{F}$-stopping times $T$ and random variables $F$ on $(\Omega, \mathcal{F}_\infty, \mathbb{P}^{X,\tau})$. Using Lemma A.3 and the strong Markov property at $E_s$, we then calculate

$$
\mathbb{E}^{X,\tau}[f(X_{t-}, V_{t-})1\{H_s < t\} | G_s]
= \mathbb{E}^{X,\tau}\left[\mathbb{E}^{X,\tau}[f(X_{t-}, V_{t-}) \circ \theta_E_s | H_s]1\{H_s < t\} | G_s]\right]
= \mathbb{E}^{X,\tau}\left[\mathbb{E}^{Y_s,H_s}[f(X_{t-}, V_{t-})]1\{H_s < t\} | G_s]\right]
= \mathbb{E}^{X,\tau}\left[\mathbb{E}^{(X,G)_{s-}+\Delta(A,D)_{E_t}}[f(X_{t-}, V_{t-})]1\{G_s + \Delta D_{E_s} < t\} | G_s]\right]
$$

(A.15) $$
= \int_{\mathbb{R}^{d+1}} K_{V_s}(X_{s-}, G_{s-}; dy \times dw)
$$
\[ \times \mathbb{E}(X,G)_{t-}(y,w) \left[ f(X_{t-}, V_{t-}) \right] \mathbf{1}\{G_s + w < t\} \]
\[ = \int_{\mathbb{R}^d \times [V_s, V_s + t - s]} K_{V_s}(X_{s-}, s - V_{s-}; dy \times dw) \times \mathbb{E}_{X_s + y, s-V_s}^{s-v+w} \left[ f(X_{t-}, V_{t-}) \right], \]

which corresponds to the second summand in (3.3). Adding equations (A.14) and (A.15) yields statement (ii). For statement (i), we calculate

\[ P_{r,s} P_{s,t} f(x, v) \]
\[ = P_{s,t} f(x, v + s - r) K_v(x, r - v; \mathbb{R}^d \times [v + s - r, \infty)) \]
\[ + \int_{\mathbb{R}^d \times [v + s - r, v + t - r]} K_v(x, r - v; dy \times dw) \mathbb{E}^{x+y, r-v+w} \left[ P_{s,t} f(X_{s-}, V_{s-}) \right] \]
\[ = K_v(x, r - v; \mathbb{R}^d \times [v + s - r, \infty)) \]
\[ \times \left\{ f(x, v + t - r) K_{v+s-r}(x, r - s; \mathbb{R}^d \times [v + t - r, \infty)) \right\} \]
\[ + \int_{\mathbb{R}^d \times [v + s - r, v + t - r]} K_{v+s-r}(x, r - v; dy \times dw) \]
\[ \times \mathbb{E}^{x+y, r-v+w} \left[ f(X_{t-}, V_{t-}) \right] \]
\[ + \int_{\mathbb{R}^d \times [v + s - r, v + t - r]} K_v(x, r - v; dy \times dw) \mathbb{E}^{x+y, r-v+w} \left[ P_{s,t} f(X_{s-}, V_{s-}) \right] \]
\[ = \cdots. \]

Using (A.13) and applying the statement (ii) with \((\chi, \tau) = (x + y, r - v + w)\) yields

\[ \cdots = f(x, v + t - r) K_v(x, r - v; \mathbb{R}^d \times [v + t - r, \infty)) \]
\[ + \int_{\mathbb{R}^d \times [v + s - r, v + t - r]} K_v(x, r - v; dy \times dw) \mathbb{E}^{x+y, r-v+w} \left[ f(X_{t-}, V_{t-}) \right] \]
\[ + \int_{\mathbb{R}^d \times [v + s - r, v + t - r]} K_v(x, r - v; dy \times dw) \]
\[ \times \mathbb{E}^{x+y, r-v+w} \left[ \mathbb{E}^{x+y, r-v+w} \left[ f(X_{t-}, V_{t-}) | G_s \right] \right] \]
\[ = P_{r,t} f(x, v), \]

which is statement (i). \(\square\)

REFERENCES

[1] Applebaum, D. (2009). Lévy Processes and Stochastic Calculus, 2nd ed. Cambridge Studies in Advanced Mathematics 116. Cambridge Univ. Press, Cambridge. MR2512800
[2] BAEUMER, B., BENSON, D. A. and MEERSCHAERT, M. M. (2005). Advection and dispersion in time and space. Phys. A 350 245–262.

[3] BAEUMER, B., MEERSCHAERT, M. M. and NANE, E. (2009). Space–time duality for fractional diffusion. J. Appl. Probab. 46 1100–1115. MR2582709

[4] BAULE, A. and FRIEDRICH, R. (2007). A fractional diffusion equation for two-point probability distributions of a continuous-time random walk. Europhys. Lett. 77 1002. MR2295888

[5] BECKER-KERN, P., MEERSCHAERT, M. M. and NANE, E. (2009). Space–time duality for fractional diffusion. J. Appl. Probab. 46 1100–1115. MR2582709

[6] BAULE, A. and FRIEDRICH, R. (2007). A fractional diffusion equation for two-point probability distributions of a continuous-time random walk. Europhys. Lett. 77 10002. MR2295888

[7] BERTOIN, J. (1996). Lévy Processes. Cambridge Tracts in Mathematics 121. Cambridge Univ. Press, Cambridge. MR1406564

[8] BERTOIN, J. (1999). Subordinators: Examples and applications. In Lectures on Probability Theory and Statistics (Saint-Flour, 1997). Lecture Notes in Math. 1717 1–91. Springer, Berlin. MR1746300

[9] BINGHAM, N. (1971). Limit theorems for occupation times of Markov processes. Z. Wahrsch. Verw. Gebiete 14 694–722. MR0281255

[10] CHUNG, K. L. and GLOVER, J. (1979). Left continuous moderate Markov processes. Z. Wahrsch. Verw. Gebiete 49 237–248. MR0547825

[11] CHUNG, K. L. and WALSH, J. B. (2005). Markov Processes, Brownian Motion, and Time Symmetry, 2nd ed. Grundlehren der Mathematischen Wissenschaften 249. Springer, New York. MR2152573

[12] ÇINLAR, E. (1972). Markov additive processes. I. Z. Wahrsch. Verw. Gebiete 24 85–93. MR0329047

[13] ÇINLAR, E. (1976). Entrance–exit distributions for Markov additive processes. In Stochastic Systems: Modeling, Identification and Optimization, Vol. I (R. J.-B. Wets, ed.) 22–38. Springer, Berlin. MR0445621

[14] DELLAHERIE, C. and MEYER, P. A. (1978). Probabilities and Potential. North-Holland Mathematics Studies 29. North-Holland, Amsterdam. MR0521810

[15] HARLAMOV, B. (2008). Continuous Semi-Markov Processes. ISTE, London. MR2376500

[16] HENRY, B. I., LANGLANDS, T. A. M. and STRAKA, P. (2010). Fractional Fokker–Planck equations for subdiffusion with space- and time-dependent forces. Phys. Rev. Lett. 105 170602.

[17] HENRY, B. I. and WEARNE, S. L. (2000). Fractional reaction–diffusion. Phys. A 276 448–455. MR1780373

[18] JACOD, J. (1973). Semi-groupes et mesures invariantes pour les processus semi-markoviens a espace d’état quelconque. Ann. Inst. Henri Poincare 1 77–112.

[19] JACOD, J. and SHIRYAEV, A. N. (1987). Limit Theorems for Stochastic Processes. Grundlehren der Mathematischen Wissenschaften 293. Springer, Berlin. MR0959133

[20] JUREK, Z. J. and MASON, J. D. (1993). Operator-Limit Distributions in Probability Theory. Wiley, New York. MR1243181

[21] JURLEWICZ, A., KERN, P., MEERSCHAERT, M. M. and SCHEFFLER, H. P. (2012). Fractional governing equations for coupled random walks. Comput. Math. Appl. 64 3021–3036. MR2989331

[22] JURLEWICZ, A., MEERSCHAERT, M. M. and SCHEFFLER, H.-P. (2011). Cluster continuous time random walks. Studia Math. 205 13–30. MR2822501

[23] JURLEWICZ, A., WYLOMANSKA, A. and ZEBROWSKI, P. (2009). Coupled continuous-time random walk approach to the Rachev–Rüschendorf model for financial data. Phys. A 388 407–418.
[24] Kaltenberg, O. (2002). Foundations of Modern Probability, 2nd ed. Springer, New York.
MR1876169

[25] Kaspi, H. and Maisonneuve, B. (1988). Regenerative systems on the real line. Ann. Probab. 16 1306–1332. MR0942771

[26] Kotulski, M. (1995). Asymptotic distributions of continuous-time random-walks—a probabilistic approach. J. Stat. Phys. 81 777–792.

[27] Magdziarz, M., Weron, A. and Klafter, J. (2008). Equivalence of the fractional Fokker–Planck and subordinated Langevin equations: The case of a time-dependent force. Phys. Rev. Lett. 101 1–4.

[28] Maisonneuve, B. (1977). Changement de temps d’un processus markovien additif. In Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976). Lecture Notes in Math. 581 529–538. Springer, Berlin. MR0488326

[29] Meerschaert, M. M. and Scalas, E. (2006). Coupled continuous time random walks in finance. Phys. A 370 114–118. MR2263769

[30] Meerschaert, M. M. and Scheffler, H.-P. (2001). Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice. Wiley, New York. MR1840531

[31] Meerschaert, M. M. and Scheffler, H.-P. (2004). Limit theorems for continuous-time random walks with infinite mean waiting times. J. Appl. Probab. 41 623–638. MR2074812

[32] Meerschaert, M. M. and Scheffler, H.-P. (2008). Triangular array limits for continuous time random walks. Stochastic Process. Appl. 118 1606–1633. MR2442372

[33] Meerschaert, M. M. and Sikorskii, A. (2012). Stochastic Models for Fractional Calculus. de Gruyter Studies in Mathematics 43. de Gruyter, Berlin. MR2884383

[34] Metzler, R. and Klafter, J. (2000). The random walk’s guide to anomalous diffusion: A fractional dynamics approach. Phys. Rep. 339 77. MR1809268

[35] Revuz, D. and Yor, M. (1999). Continuous Martingales and Brownian Motion, 3rd ed. Grundlehren der Mathematischen Wissenschaften 293. Springer, Berlin. MR1725357

[36] Scalas, E. (2006). Five years of continuous-time random walks in econophysics. In The Complex Networks of Economic Interactions. Lecture Notes in Econ. and Math. Systems 567 3–16. Springer, Berlin. MR2198447

[37] Scalas, E., Gorenflo, R. and Mainardi, F. (2000). Fractional calculus and continuous-time finance. Phys. A 284 376–384. MR1773804

[38] Schumer, R., Benson, D. A., Meerschaert, M. M. and Baeumer, B. (2003). Fractal mobile/immobile solute transport. Water Resour. Res. 39 1–12.

[39] Shlesinger, M. F., Klafter, J. and Wong, Y. M. (1982). Random walks with infinite spatial and temporal moments. J. Stat. Phys. 27 499–512. MR0659807

[40] Straka, P. and Henry, B. I. (2011). Lagging and leading coupled continuous time random walks, renewal times and their joint limits. Stochastic Process. Appl. 121 324–336. MR2746178

[41] Weron, A. and Magdziarz, M. (2008). Modeling of subdiffusion in space–time-dependent force fields beyond the fractional Fokker–Planck equation. Phys. Rev. E (3) 77 1–6.

[42] Weron, K., Jurlewicz, A., Magdziarz, M., Weron, A. and Trzmiel, J. (2010). Overshooting and undershooting subordination scenario for fractional two-power-law relaxation responses. Phys. Rev. E (3) 81 1–7.

DEPARTMENT OF STATISTICS AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824
USA
E-MAIL: mcubed@stt.msu.edu

SCHOOL OF MATHEMATICS AND STATISTICS
UNSW AUSTRALIA
SYDNEY, NSW 2052
AUSTRALIA
E-MAIL: p.straka@unsw.edu.au