Statistical Analysis of Parameter Estimation for 2-D Harmonics in Multiplicative and Additive Noise

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Supplementary Data

(A) The expression of $Q(\theta)$ in the proof of Theorem 2.

$$Q(\theta) = \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \left[ \mu_{sk}^{(0)} \cos \left( w_{mn}(\theta_{k}^{0}) \right) - \mu_{sk} \cos \left( w_{mn}(\theta_{k}) \right) \right] \right\}^2$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \left[ \mu_{sk}^{(0)} \sin \left( w_{mn}(\theta_{k}^{0}) \right) - \mu_{sk} \sin \left( w_{mn}(\theta_{k}) \right) \right] \right\}^2$$

$$+ 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k=1}^{p} x_r(m, n) \left[ \mu_{sk}^{(0)} \cos \left( w_{mn}(\theta_{k}^{0}) \right) - \mu_{sk} \cos \left( w_{mn}(\theta_{k}) \right) \right]$$

$$+ 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k=1}^{p} x_c(m, n) \left[ \mu_{sk}^{(0)} \sin \left( w_{mn}(\theta_{k}^{0}) \right) - \mu_{sk} \sin \left( w_{mn}(\theta_{k}) \right) \right]$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \left| x(m, n) \right|^2$$

where

$$x(m, n) = e(m, n) + \sum_{k=1}^{p} (s_k(m, n) - \mu_{sk}) e^{i w_{mn}(\theta_{k}^{0})}$$

$$w_{mn}(\theta_{k}) = m \lambda_k + n \mu_k + \phi_k$$

(B) The expressions of the elements of $Q'(\theta^0)$ in the proof of Theorem 2.

$$Q'(\theta^0) = \left( Q'_k(\theta^0) \right)_{k=1,2,...,p} = (Q'_1(\theta^0), Q'_2(\theta^0), \cdots, Q'_p(\theta^0))$$

$$Q'_k(\theta^0) = \frac{\partial Q(\theta^0)}{\partial \theta_k} = \left( \frac{\partial Q(\theta^0)}{\partial \mu_{sk}}, \frac{\partial Q(\theta^0)}{\partial \phi_k}, \frac{\partial Q(\theta^0)}{\partial \lambda_k}, \frac{\partial Q(\theta^0)}{\partial \mu_k} \right)$$

$$\frac{\partial Q(\theta^0)}{\partial \mu_{sk}} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ -x_r(m, n) \cos \left( w_{mn}(\theta_{k}^{0}) \right) - x_c(m, n) \sin \left( w_{mn}(\theta_{k}^{0}) \right) \right]$$

$$\frac{\partial Q(\theta^0)}{\partial \phi_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_{sk}^{(0)} \left[ x_r(m, n) \sin \left( w_{mn}(\theta_{k}^{0}) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_{k}^{0}) \right) \right]$$

$$\frac{\partial Q(\theta^0)}{\partial \lambda_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_{sk}^{(0)} \left[ x_r(m, n) \sin \left( w_{mn}(\theta_{k}^{0}) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_{k}^{0}) \right) \right]$$

$$\frac{\partial Q(\theta^0)}{\partial \mu_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_{sk}^{(0)} \left[ x_r(m, n) \sin \left( w_{mn}(\theta_{k}^{0}) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_{k}^{0}) \right) \right]$$
(C) The expressions of the elements of $Q''(\theta^0)$ in the proof of Theorem 2.

$$Q''(\theta^0) = \left( Q''_{kl}(\theta^0) \right)_{k,l=1,2,\ldots,p} = \begin{bmatrix}
Q''_{11}(\theta^0) & Q''_{12}(\theta^0) & \cdots & Q''_{1p}(\theta^0) \\
Q''_{21}(\theta^0) & Q''_{22}(\theta^0) & \cdots & Q''_{2p}(\theta^0) \\
\vdots & \vdots & \ddots & \vdots \\
Q''_{p1}(\theta^0) & Q''_{p2}(\theta^0) & \cdots & Q''_{pp}(\theta^0)
\end{bmatrix}
$$

$$Q''_{kl}(\theta^0) = \frac{\partial^2 Q(\theta^0)}{\partial \theta_k \partial \theta_l^*} = \begin{bmatrix}
\frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \mu_l^*} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \phi_l} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \lambda_l} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \phi_l} \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \mu_l^*} & \frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_l} & \frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \lambda_l} & \frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_l} \\
\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \mu_l^*} & \frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \phi_l} & \frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \lambda_l} & \frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \phi_l} \\
\frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \mu_l^*} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \phi_l} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \lambda_l} & \frac{\partial^2 Q(\theta^0)}{\partial \mu_k \partial \mu_l^*}
\end{bmatrix}
$$

(C1) The expression of the elements of $Q''_{kk}(\theta^0)$, for $k = 1, 2, \ldots, p$.

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_k^2} = 2MN
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k^2} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_k^0 \left[ x_r(m, n) \cos \left( w_{mn}(\theta_k^0) \right) + x_c(m, n) \sin \left( w_{mn}(\theta_k^0) \right) \right] + 2MN(\mu_k^0)^2
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k^2} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} m^2 \mu_k^0 \left[ x_r(m, n) \cos \left( w_{mn}(\theta_k^0) \right) + x_c(m, n) \sin \left( w_{mn}(\theta_k^0) \right) \right] + 2N(\mu_k^0)^2 \sum_{m=1}^{M} m^2
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \mu_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_k^0 \left[ x_r(m, n) \sin \left( w_{mn}(\theta_k^0) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_k^0) \right) \right]
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} m \left[ x_r(m, n) \sin \left( w_{mn}(\theta_k^0) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_k^0) \right) \right]
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \mu_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} n \left[ x_r(m, n) \sin \left( w_{mn}(\theta_k^0) \right) - x_c(m, n) \cos \left( w_{mn}(\theta_k^0) \right) \right]
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \lambda_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} m \mu_k^0 \left[ x_r(m, n) \cos \left( w_{mn}(\theta_k^0) \right) + x_c(m, n) \sin \left( w_{mn}(\theta_k^0) \right) \right] + 2N(\mu_k^0)^2 \sum_{m=1}^{M} m
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} m \mu_k^0 \left[ x_r(m, n) \cos \left( w_{mn}(\theta_k^0) \right) + x_c(m, n) \sin \left( w_{mn}(\theta_k^0) \right) \right] + 2M(\mu_k^0)^2 \sum_{n=1}^{N} n
$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k \partial \mu_k} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} mn \mu_k^0 \left[ x_r(m, n) \cos \left( w_{mn}(\theta_k^0) \right) + x_c(m, n) \sin \left( w_{mn}(\theta_k^0) \right) \right] + 2(\mu_k^0)^2 \sum_{m=1}^{M} \sum_{n=1}^{N} mn
$$
(C2) The expressions of the elements of $Q^p_{kl}(\theta^0)$, for $k, l = 1, 2, \ldots, p$ and $k \neq l$.

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk}\partial \mu_{sl}} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk}\partial \phi_l} = -2\mu^0_{sl} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk}\partial \lambda_l} = -2\mu^0_{sl} \sum_{m=1}^{M} \sum_{n=1}^{N} m \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sl}\partial \mu_l} = -2\mu^0_{sl} \sum_{m=1}^{M} \sum_{n=1}^{N} n \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k\partial \mu_{sl}} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k\partial \phi_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_k\partial \lambda_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} m \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \phi_l\partial \mu_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} n \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k\partial \mu_{sl}} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k\partial \phi_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} m \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k\partial \lambda_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} m^2 \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \lambda_k\partial \mu_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} mn \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_l\partial \mu_{sl}} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} n \sin \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_l\partial \phi_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_l\partial \lambda_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} mn \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right)$$

$$\frac{\partial^2 Q(\theta^0)}{\partial \mu_l\partial \mu_l} = 2\mu^0_{sk} \sum_{m=1}^{M} \sum_{n=1}^{N} n^2 \cos \left( w_{mn}(\theta^0_k) - w_{mn}(\theta^0_l) \right).$$
(D) The expression of the remainder $R_{MN}(\hat{\theta}_k, \theta^0)$ in the proof of Theorem 4.

\[
R_{MN}(\hat{\theta}_k, \theta^0) = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} s_k^2(m, n) \left[ \cos \left( 2w_{mn}(\theta_k^0) - 2w_{mn}(\hat{\theta}_k) \right) \right] - 1 \\
+ \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{j \neq k} s_j^2(m, n) \cos \left( 2w_{mn}(\theta_j^0) - 2w_{mn}(\hat{\theta}_k) \right) \\
+ \frac{2}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{j=1}^{p} e_r(m, n) s_j(m, n) \cos \left( w_{mn}(\theta_j^0) - 2w_{mn}(\hat{\theta}_k) \right) \\
+ \frac{2}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{j=1}^{p} e_r(m, n) s_j(m, n) \sin \left( 2w_{mn}(\hat{\theta}_k) - w_{mn}(\theta_j^0) \right) \\
+ \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} s_i(m, n) s_j(m, n) \cos \left( w_{mn}(\theta_i^0) + w_{mn}(\theta_j^0) - 2w_{mn}(\hat{\theta}_k) \right) \\
+ \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ e_r^2(m, n) - e_r^2(m, n) \right] \cos \left( 2w_{mn}(\hat{\theta}_k) \right) \\
+ \frac{2}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} e_r(m, n) e_r(m, n) \sin \left( -2w_{mn}(\hat{\theta}_k) \right)
\]

(E) The proof of "$f_{MN}(\hat{\theta}, \theta^0) \rightarrow 0$ a.s." in the proof of Theorem 5.

\[
\frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \left[ \mu_{sk}^0 \cos \left( w_{mn}(\theta_k^0) \right) - \hat{\mu}_{sk} \cos \left( w_{mn}(\hat{\theta}_k) \right) \right] \right\}^2 \\
\leq \frac{p}{MN} \sum_{k=1}^{p} \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ \mu_{sk}^0 \cos \left( w_{mn}(\theta_k^0) \right) - \hat{\mu}_{sk} \cos \left( w_{mn}(\hat{\theta}_k) \right) \right]^2 \\
\leq \frac{2p}{MN} \sum_{k=1}^{p} \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ (\mu_{sk}^0 - \hat{\mu}_{sk})^2 \cos^2 \left( w_{mn}(\theta_k^0) \right) + \hat{\mu}_{sk}^2 \left[ \cos \left( w_{mn}(\theta_k^0) \right) - \cos \left( w_{mn}(\hat{\theta}_k) \right) \right]^2 \right] \\
\leq \frac{2p}{MN} (\mu_{sk}^0 - \hat{\mu}_{sk})^2 + \frac{2p}{MN} \sum_{k=1}^{p} \sum_{m=1}^{M} \sum_{n=1}^{N} \hat{\mu}_{sk}^2 \left[ w_{mn}(\theta_k^0) - w_{mn}(\hat{\theta}_k) \right]^2 \\
\rightarrow 0 \quad a.s.
\]

where the first inequality follows from Cauchy-Schwarz inequality, the second inequality follows from the fact $2x^2 + 2y^2 - (x + y)^2 \geq 0$, the third inequality follows from Taylor series expansion and the last step follows from the results in Theorems 1 and 2. Along the same line, it also can be shown that

\[
\frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \sum_{k=1}^{p} \left[ \mu_{sk}^0 \sin \left( w_{mn}(\theta_k^0) \right) - \hat{\mu}_{sk} \sin \left( w_{mn}(\hat{\theta}_k) \right) \right] \right\}^2 \rightarrow 0 \quad a.s.
\]

Therefore, it can be obtained from (A.4) in the proof of Theorem 1 that $f_{MN}(\hat{\theta}, \theta^0) \rightarrow 0 \quad a.s.$