Group theoretical aspects of $L^2(\mathbb{R}^+)$, $L^2(\mathbb{R}^2)$ and associated Laguerre polynomials

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Abstract

A ladder algebraic structure for $L^2(\mathbb{R}^+)$ which closes the Lie algebra $h(1) \oplus h(1)$, where $h(1)$ is the Heisenberg-Weyl algebra, is presented in terms of a basis of associated Laguerre polynomials. Using the Schwinger method the quadratic generators that span the alternative Lie algebras $\mathfrak{so}(3)$, $\mathfrak{so}(2,1)$ and $\mathfrak{so}(3,2)$ are also constructed. These families of (pseudo) orthogonal algebras also allow to obtain unitary irreducible representations in $L^2(\mathbb{R}^2)$ similar to those of the spherical harmonics.

1 Introduction

The associated Laguerre polynomials (ALP) $^{[1]}$, $L_n^{(\alpha)}(x)$ ($x \in [0, \infty)$, $n = 0, 1, 2, \cdots$ and $\alpha$ real fixed parameter, continuous and $> -1$), are defined by the 2nd order differential equation (DE)

$$\left[ x \frac{d^2}{dx^2} + (1 + \alpha - x) \frac{d}{dx} + n \right] L_n^{(\alpha)}(x) = 0. \quad (1)$$

The ALPs reduce to the Laguerre polynomials for $\alpha = 0$. From the many recurrence relations that they verify $^{[1, 2, 3]}$, we start from the following ones

$$\left[ -\frac{d}{dx} + 1 \right] L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x), \quad \left[ x \frac{d}{dx} + \alpha \right] L_n^{(\alpha)}(x) = (n + \alpha) L_n^{(\alpha-1)}(x). \quad (2)$$

For $\alpha > -1$ and fixed, the ALP $L_n^{(\alpha)}(r)$ are orthogonal in the label $n$ with respect the weight measure $d\mu(x) = x^\alpha e^{-x} dx$

$$\int_0^\infty dx x^\alpha e^{-x} L_n^{(\alpha)}(x) L_{n'}^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nn'}.$$
For \( \alpha \) integer such that \( 0 \leq \alpha \leq n \), we have the generalization 11

\[
L_n^{(-\alpha)}(x) := \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + 1)} (-x)^\alpha L_n^{(\alpha)}(x).
\]

Hereafter we assume here \( n \in \mathbb{N}, \alpha \in \mathbb{Z}, n - \alpha \in \mathbb{N} \) and we consider \( \alpha \) as a label, like \( n \), and not a parameter fixed at the beginning.

Following the approach of previous works [4, 5, 6, 7] we introduce now a set of alternative functions including also the weight measure, in such a way to obtain the orthonormal bases we are used to in Quantum Mechanics

\[
M_n^{(\alpha)}(x) := \sqrt{\frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)}} x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x).
\]

For each fixed value of \( \alpha \geq -n \) and \( n \in \mathbb{N} \), the set of \( M_n^{(\alpha)}(x) \), is a basis of \( L^2(\mathbb{R}^+) \)

\[
\int_0^\infty M_n^{(\alpha)}(x) M_m^{(\alpha)}(x) \, dx = \delta_{nm}, \quad \sum_{n=0}^{\infty} M_n^{(\alpha)}(x) M_n^{(\alpha)}(x') = \delta(x - x').
\]

2 The symmetry algebra \( h(1)_n \oplus h(1)_p \)

The eqs. (2) rewritten in terms of \( M_n^{(\alpha)} \) take the form

\[
\begin{align*}
-\sqrt{x} \frac{d}{dx} + \frac{1}{2} \sqrt{x}(\alpha + x) & \quad M_n^{(\alpha)}(x) = \sqrt{n + \alpha + 1} M_n^{(\alpha+1)}(x), \\
\sqrt{x} \frac{d}{dx} + \frac{1}{2} \sqrt{x}(\alpha + x) & \quad M_n^{(\alpha)}(x) = \sqrt{n + \alpha} M_n^{(\alpha-1)}(x),
\end{align*}
\]

(3)

where \( p := n + \alpha \) plays, for \( n \) fixed, the role of eigenvalue of the number operator in a Heisenberg-Weyl algebra, \( h(1) \), realized on the space of functions \( M_n^{(\alpha)}(x) \). It is indeed a positive integer like \( n \), so that we can define the new functions \( \mathcal{M}_{n,p}(x) := M_n^{(p-n)}(x) \), that by inspection are symmetric in the interchange \( n \leftrightarrow p \), i.e. \( \mathcal{M}_{n,p}(x) = (-1)^{p-n} \mathcal{M}_{p,n}(x) \).

The previous recurrence relations (3) can thus be rewritten

\[
\begin{align*}
-\sqrt{x} \frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p - n}{2\sqrt{x}} & \quad \mathcal{M}_{n,p}(x) = \sqrt{p + 1} \mathcal{M}_{n,p+1}(x), \\
\sqrt{x} \frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p - n}{2\sqrt{x}} & \quad \mathcal{M}_{n,p}(x) = \sqrt{p} \mathcal{M}_{n,p-1}(x).
\end{align*}
\]

(4)

To construct the operatorial structure corresponding to the recurrence relations we define now four operators \( X, D_x, N \) and \( P \)

\[
\begin{align*}
X \mathcal{M}_{n,p}(x) &= x \mathcal{M}_{n,p}(x), & D_x \mathcal{M}_{n,p}(x) &= \frac{d \mathcal{M}_{n,p}(x)}{dx}, \\
N \mathcal{M}_{n,p}(x) &= n \mathcal{M}_{n,p}(x), & P \mathcal{M}_{n,p}(x) &= p \mathcal{M}_{n,p}(x).
\end{align*}
\]

Then, the 2nd order DE (1) becomes

\[
\mathbb{E} \mathcal{M}_{n,p}(x) = 0,
\]

(5)
where

\[ E := XD_x^2 + D_x + \frac{N + P + 1}{2} - \frac{1}{4X} (P - N)^2 - \frac{X}{4}. \]

Moreover from (4) we get the differential operators (DOs)

\[ b^\pm := \pm \sqrt{X} D_x + \sqrt{X} + \frac{1}{2X} (P - N), \quad (6) \]

that act on the functions \( \mathcal{M}_{n,p}(x) \) in such a way that \( \Delta n = 0 \) and \( \Delta p = \pm 1 \). Since \([b^-, b^+] = \mathbb{I}\) they close an \( h(1) \) algebra, \( (h(1)_p) \) with quadratic Casimir \( \mathcal{C}_p = \{b^-, b^+\} - 2(P + 1/2) \) verifying \( \mathcal{C}_p \mathcal{M}_{n,p}(x) = -2E \mathcal{M}_{n,p}(x) = 0 \).

Now taking into account the symmetry under the interchange \( n \leftrightarrow p \) of \( \mathcal{M}_{n,p}(x) \) we can define the operators \( a^\pm (N, P) := -b^\pm (P, N) \) that change the labels of \( \mathcal{M}_{n,p}(x) \) as \( \Delta p = 0 \) and \( \Delta n = \pm 1 \). Their explicit action on \( \mathcal{M}_{n,p}(x) \) is indeed

\[ a^+ \mathcal{M}_{n,p}(x) = \sqrt{n+1} \mathcal{M}_{n+1,p}(x), \quad a^- \mathcal{M}_{n,p}(x) = \sqrt{n} \mathcal{M}_{n-1,p}(x). \]

The two operators \( a^\pm \) determine thus another HW algebra, \( h(1)_n \). Since these bosonic operators \( a^\pm \) and \( b^\pm \) commute among them we have obtained in this way the global algebra \( h(1)_n \oplus h(1)_p \).

Moreover inside the Universal Enveloping Algebra \( UEA [h(1)_n \oplus h(1)_p] \) other algebras preserving the parity of \( n + p \) can be found by the Schwinger procedure [8] as we will do in the next section.

3 \( so(3), so(2, 1) \) and \( so(3, 2) \) symmetries

so(3) symmetry

We start from \( J_\pm := a_\pm b_\pm \) obtaining 2nd order DOs that, taking into account eq. (5), can be rewritten in the space \{\( \mathcal{M}_{n,p}(x) \}\) as 1st order DOs

\[ J_\pm = \pm D_x (N - P \pm 1) + \frac{1}{2X} (N - P \pm 1) (N - P) - \frac{1}{2} (N + P + 1). \quad (7) \]

Defining \( J_3 := (a_- a_+ - b_- b_+) / 2 \equiv (N - P) / 2 \) we see that \( \{J_\pm, J_3\} \) close a \( su(2) \) algebra in the space \{\( \mathcal{M}_{n,p}(x) \)\} since \([J_+, J_-] = 2J_3 - \frac{8}{X} J_3 E\). The action of \( J_\pm \) is

\[ J_+ \mathcal{M}_{n,p}(x) = \sqrt{n+1} \mathcal{M}_{n+1,p-1}(x), \quad J_- \mathcal{M}_{n,p}(x) = \sqrt{n} \mathcal{M}_{n-1,p+1}(x). \]

Also the Casimir of \( su(2) \), \( \mathcal{C}_{su(2)} = J_3^2 + \frac{1}{4} \{J_+, J_-\} \) is closely related to eq. (5) as \( \mathcal{C}_{su(2)} = J(J + 1) + \frac{1}{4} (4J_3^2 + 1) E \), where \( J \) is the diagonal operator \( J := (N + P) / 2 \).

so(2, 1) symmetry

In a similar way we can define the operators \( K_\pm := a_\pm b_\pm \), such that, like in the case of the operators \( J_\pm \), we find in the space \{\( \mathcal{M}_{n,p}(x) \)\}

\[ K_+ = XD_x + \frac{1}{2} (N + P + 2 - X), \quad K_- = -XD_x + \frac{1}{2} (N + P - X). \quad (8) \]
Both operators together with $K_3 := (a_- a_+ + b_+ b_-)/2 \equiv (N+P+1)/2$ determine a $su(1,1)$ algebra

$$[K_3, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_3,$$

since the action on the functions $\mathcal{M}_{a,p}(x)$ is

$$K_{\pm} \mathcal{M}_{a,p}(x) = \sqrt{(n+1)(p+1)} \mathcal{M}_{n+1,p+1}(x), \quad K_{-} \mathcal{M}_{a,p}(x) = \sqrt{np} \mathcal{M}_{n-1,p-1}(x).$$

The Casimir of $su(1,1)$, $\mathcal{C}_{su(1,1)} = K_3^2 - \frac{1}{2} \{ K_+, K_- \}$, is also connected with eq. (5) as $\mathcal{C}_{su(1,1)} = (M^2 - \frac{1}{4}) + X \mathcal{E}$, where $M = J_3 := (N-P)/2$.

**More $so(2,1)$ symmetries**

The commutators of $J_{\pm}$ and $K_{\pm}$ give the new operators

$$R_{\pm} := \pm [J_{\pm}, K_{\pm}], \quad S_{\pm} := \pm [J_{\pm}, K_{\pm}].$$

Provided that we define $R_3 := J + M + 1/2$ and $S_3 := J - M + 1/2$, they close two $so(2,1)$ algebras with commutators

$$[R_+, R_-] = -4R_3, \quad [R_3, R_{\pm}] = \pm 2R_{\pm},$$

and Casimir $\mathcal{C}_R = R_3^2 - \frac{1}{2} \{ R_+, R_- \} = -\frac{3}{4} + \frac{1}{2} \{ 1 + (X + 2M)^2 \} \mathcal{E}$ and similarly for $\{ S_{\pm}, S_3 \}$.

Note that under the interchange $m \leftrightarrow -m$ we have $\{ R_{\pm}, R_3 \} \leftrightarrow \{ S_{\pm}, S_3 \}$.

**$so(3,2)$ symmetry**

All the operators $\{ K_{\pm}, L_{\pm}, R_{\pm}, S_{\pm}, J, M \}$ can be written on the space $\{ \mathcal{M}_{a,p}(x) \}$ as 1st order DOs. All together they determine on $\{ \mathcal{M}_{a,p}(x) \}$ the representation of the Lie algebra $so(3,2)$ with $\mathcal{C}_{so(3,2)} = -5/4$.

### 4 Representations of $so(3)$, $so(2,1)$ and $so(3,2)$ on the plane

We introduce now the operators directly related to $so(3)$, $J := (N+P)/2$ and $J_3 \equiv M := (N-P)/2$, and define

$$\mathcal{L}_j^m(x) := \mathcal{M}_{j+m,j-m}(x) = \sqrt{\frac{(j+m)!}{(j-m)!}} x^{-m} e^{-x/2} L_{j+m}^{(-2m)}(x).$$

The operators $J_3$ and $J_{\pm}$ (7), rewritten in terms of $J$ and $M$, act on $\{ \mathcal{L}_j^m(x) \}$ as

$$J_3 \mathcal{L}_j^m(x) = m \mathcal{L}_j^m(x), \quad J_\pm \mathcal{L}_j^m(x) = \sqrt{(j+m)(j+m+1)} \mathcal{L}_{j\pm1}^m(x).$$

So, $\{ \mathcal{L}_j^m(x) \}$ with $j \in \mathbb{N}$ and $|m| \leq j$ supports the representation $\mathcal{D}_j$ of $so(3)$.

Similar results can be obtained for the other algebras $so(2,1)$ and $so(3,2)$. For instance, for the $so(2,1)$ spanned by $\{ K_{\pm}, K_3 \}$, $\{ \mathcal{L}_j^m(x) \}$ supports the irreducible representation of the discrete series with Casimir $\mathcal{C}_{su(1,1)} := m^2 - \frac{1}{4}$ with $m$ fixed and $j \geq |m|$.
On the other hand, in general these representations are not faithful because \( \mathcal{L}^m_j(x) = \mathcal{L}_{j-m}^m(x) \). The same difficulty is also present in the spherical harmonic where the associated Legendre polynomial \( P_l^m \) is related to \( P_l^{-m} \). There the degeneration was removed by introducing an angle variable. Here we follow the same procedure by considering the new functions

\[
\mathcal{X}^m_j(r, \phi) := e^{im\phi} \mathcal{L}^m_j(r^2), \quad \phi \in \mathbb{R}, -\pi \leq \phi < \pi.
\]

Under the change of variable \( x \to r^2 \), the DE (5) becomes

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{4m^2}{r^2} - r^2 + 4(j + \frac{1}{2}) \right] \mathcal{X}^m_j(r, \phi) = 0.
\]

Normalization and orthogonality of the \( \mathcal{X}^m_j(r, \phi) \) are similar to the ones of \( Y^m_j(\theta, \phi) \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_0^{\infty} 2r \, dr \, \mathcal{X}^m_j(r, \phi) \mathcal{X}^m_{j'}(r, \phi) = \delta_{jj'} \delta_{m,m'},
\]

\[
\sum_{j,m} \mathcal{X}^m_j(r, \phi) \mathcal{X}^m_{j'}(r', \phi') = \frac{\pi}{r} \delta(r - r') \delta(\phi - \phi').
\]

This means that the set \( \{ \mathcal{X}^m_j(r, \phi) \} \) is a basis in the space of square integrable functions defined on the plane, \( L^2(\mathbb{R}^2) \), like \( \{ Y^m_j(\Omega) \} \) is a basis of \( L^2(S^2) \).

Moreover, with a convenient introduction of phases we can define the operators \( \mathbb{J}_\pm := e^{\pm \phi} J_{\pm} \) and \( \mathbb{J}_3 := J_3 \), in the finite dimensional space \( \{ \mathcal{X}^m_j(r, \phi) \} \) with fixed \( j \)

\[
\mathbb{J}_\pm \mathcal{X}^m_j(r, \phi) = \sqrt{(j \mp m)(j \mp m + 1)} \mathcal{X}^{m \pm 1}_j(r, \phi), \quad \mathbb{J}_3 \mathcal{X}^m_j(r, \phi) = m \mathcal{X}^m_j(r, \phi),
\]

and analogously for the remaining operators. So \( \{ \mathcal{X}^m_j(r, \phi) \} \) support irreducible representations of \( so(3) \), \( so(2,1) \) and \( so(3,2) \) on the plane as \( \{ Y^m_j(\theta, \phi) \} \) are on the sphere. For more details see [7, 9, 10].

From the physical point of view, in spite of the analogy with the angular momentum, \( \mathbb{J}_\pm \) and \( \mathbb{J}_3 \) can be related to a one-dimensional Morse system, where \( m \) and \( j \) are connected with the potential [9].

### Conclusions

A relationship between Lie algebras and square integrable functions has been found. Indeed we need to restrict ourselves to \( L^2(\mathbb{R}^+) \) and \( L^2(\mathbb{R}^2) \), where \( E \) is identically zero, to obtain differential representations of Lie algebras in the spaces of functions defined in \( \mathbb{R}^+ \) and \( \mathbb{R}^2 \).

### Acknowledgements

This work was partially supported by the Ministerio de Economía y Competitividad of Spain (Project MTM2014-57129-C2-1-P with EU-FEDER support).
References

[1] G. Szegö, *Orthogonal Polynomials*, (Am. Math. Soc., Providence, 2003), pp. 100-105

[2] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions*, (Cambridge Univ. Press, New York, 2010)

[3] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972)

[4] E. Celeghini, M.A. del Olmo, Ann. Phys. 335 (2013) 78-85

[5] E. Celeghini, M.A. del Olmo, Ann. Phys. 333 (2013) 90-103

[6] E. Celeghini, M.A. del Olmo, M.A. Velasco, J. Phys.: Conf. Ser. 597 (2015) 012023

[7] E. Celeghini, M.A. del Olmo, arXiv: 1504.01572 [math-ph]

[8] J. Schwinger, in *Quantum Theory of Angular Momentum* (L. Biedenharn, E. van Dam, Eds.), (Academic Press, New York, 1965), pp. 229-279

[9] Y. Alhassid, F. Gürsey, F. Iachello, Ann. Phys. 148 (1983 ) 346-380

[10] J. Guerrero, V. Aldaya, J. Phys. A 39 (2006) L267-L276.