DISCRETE QUANTUM DRINFELD-SOKOLOV CORRESPONDENCE

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ABSTRACT. We construct a discrete quantum version of the Drinfeld-Sokolov correspondence for the sine-Gordon system. The classical version of this correspondence is a birational Poisson morphism between the phase space of the discrete sine-Gordon system and a Poisson homogeneous space. Under this correspondence, the commuting higher mKdV vector fields correspond to the action of an Abelian Lie algebra. We quantize this picture (1) by quantizing this Poisson homogeneous space, together with the action of the Abelian Lie algebra, (2) by quantizing the sine-Gordon phase space, (3) by computing the quantum analogues of the integrals of motion generating the mKdV vector fields, and (4) by constructing an algebra morphism taking one commuting family of derivations to the other one.

1. Introduction

1.1. Background. The link between integrable systems and quantum groups has been intensively studied during the last few years from several viewpoints. The goal of this article is to present a discrete and quantum version of a natural construction occurring in the theory of integrable systems, namely the Drinfeld-Sokolov correspondence \cite{DS}. At the classical level, this correspondence was discovered by Drinfeld and Sokolov in the eighties, by using the dressing method of Zakharov and Shabat \cite{SZ}. It is a bijective map between phase spaces of certain evolution equations (such as the KdV, mKdV, or Toda hierarchy) and homogeneous spaces. Each phase space is equipped with an infinite commuting family of vector fields: the Hamiltonian fields generated by the integrals of motion of the KdV, mKdV, or Toda hierarchy. One of the main properties of the Drinfeld-Sokolov correspondence is that it leads to a geometric interpretation of these commutative families: they correspond to the action of an Abelian Lie algebra on a double coset space. After Drinfeld and Sokolov, Feigin and Frenkel developed a cohomological approach based on the fact that screening operators of the Toda theory satisfy the Serre relations \cite{FF2} and \cite{FF3}. This allowed them (1) to prove the existence of a commutative family of integrals of motion in the quantum case and (2) to suggest a possible discretization of the problem, generalizing those introduced much earlier by Izergin and Korepin (see \cite{IK1}, \cite{IK2}). At the semi-classical level, the discretized Toda system has been studied by Enriquez and Feigin in the case when the Lie algebra is $\widehat{\mathfrak{sl}}_2$ \cite{EFe}. This is the discrete sine-Gordon theory. By imitating the cohomological approach of Feigin and Frenkel, the authors (1) proved the existence of a classical family of integrals of motion in involution and (2) constructed a Drinfeld-Sokolov correspondence between phase spaces equipped with the Hamiltonian action of the integrals of motion, and homogeneous spaces equipped with the action of an Abelian Lie algebra. Moreover, the phase spaces are endowed with a natural structure of Poisson manifold, the homogeneous spaces are Poisson homogeneous spaces and the correspondence is a Poisson isomorphism. The
aim of our present work is to quantify this result. So, this article fills in the discrete-quantum square of the following array.

|                | classical | quantum |
|----------------|-----------|---------|
| continuous     | Drinfeld-Sokolov (1981) | ? |
| discrete       | Enriquez-Feigin (1995) | this article (2000) |

1.2. The classical Drinfeld-Sokolov correspondence. The Drinfeld-Sokolov correspondence is inspired by the application of dressing methods developed by Zakharov and Shabat in the theory of integrable systems [SZ]. The integrable systems studied by Drinfeld and Sokolov by this method are Korteweg-de Vries hierarchies (KdV) or modified Korteweg-de Vries hierarchies (mKdV), associated with an affine Kac-Moody algebra. For example, in the case when the Kac-Moody algebra $\hat{g}$ is $\hat{sl}_2$, the second equation of the mKdV hierarchy is (the first one being $\partial_z u = u_z$)

$$u_t = u_{zzz} + 6u^2 u_z.$$  \(1\)

The main achievement of Drinfeld and Sokolov was (1) to associate Lax pairs $(A(u), L(u))$ to these equations taking values in affine Kac-Moody Lie algebras and then (2) by assigning to a point of the phase space, the matrix conjugating its Lax matrix to a prescribed form, to set up a bijection between the phase space and a coset space (3) to show that the corresponding system on the coset space is "linear". This way, Drinfeld and Sokolov achieved the linearization of their system. More precisely, if $u$ belongs to the phase space, the matrices $K(u)$ conjugating the matrix $L(u)$ into a standard form belong to a pro-algebraic pro-unipotent subgroup $N_+$ of the Kac-Moody group $G$ associated to $\hat{g}$. Moreover, such a matrix $K(u)$ is determined uniquely up to a multiplication by an element of a commutative subgroup $A_+$ in $N_+$, and all the coefficients of the class of $K(u)$ in $N_+/A_+$ are differential polynomials in $u$. As a result, one gets a map from the phase space of the hierarchy to $N_+/A_+$. The Drinfeld-Sokolov theorem asserts that this map is bijective. Moreover, in the corresponding bijection between the rings of functions, the hierarchy equations viewed as commutative flows on the phase space correspond to the right action of the Lie algebra of the normalizer of $A_+$ in $G$, $N_+$ being embedded into the flag variety $B_- \backslash G$. The Hamiltonian structure (which one can associate to these hierarchies) were studied by Gelfand, Dickey and Dorfman ([GDi1], [GDi2], [GDo]).

1.3. The viewpoint of Feigin and Frenkel. Feigin and Frenkel reformulated the Drinfeld-Sokolov correspondence in a cohomological language. This allowed them to identify the action of $\mathfrak{n}_+$ on the phase space $U$ (which is, according to the Drinfeld-Sokolov correspondence, the same as the left action by vector fields of $\mathfrak{n}_+$ on the homogeneous space $N_+/A_+$) with the Hamiltonian action of screening charges of the Toda system associated with the Lie algebra $\hat{g}$. Besides, their formalism led to a quantization as well as a discretization of the Toda system. Precisely, let $\mathfrak{g}$ be a semi-simple Lie algebra and $\hat{g}$ be the affine Kac-Moody algebra built from $\mathfrak{g}$. The Toda
system associated with the Lie algebra \( \mathfrak{g} \) is the following system of differential equations:

\[
\partial_z \partial_\tau \phi_i(z, \tau) = \sum_{j=0}^{l} (\alpha_i, \alpha_j) \exp(-\phi_j(z, \tau)), \quad i = 1, \ldots, l,
\]

(2)

where \( \alpha_0, \ldots, \alpha_l \) are simple roots of \( \mathfrak{g} \). Each function \( \phi_i(z, \tau) \) depends on \( z \) as well as the time variable \( \tau \), and \( \phi_0(z, \tau) = -\frac{1}{a_0} \sum_{i=1}^{l} a_i \phi_i(z, \tau) \), where \( a_0, \ldots, a_l \) are labels of the Dynkin diagram.

In the case when \( \mathfrak{g} = \mathfrak{so}_2 \), the system reduces to the sine-Gordon equation:

\[
\partial_\tau \partial_z \phi(z, \tau) = \exp(\phi(z, \tau)) - \exp(-\phi(z, \tau)).
\]

(3)

Let \( \pi_0 \) be the ring of functions on \( U \). We have \( \pi_0 = \mathbb{C}[u_i^{(n)}]_{1 \leq i \leq l; 0 < n} \). This is a differential ring equipped with the derivation \( \partial \) defined by \( \partial u_i^{(n)} = u_i^{(n+1)} \). Because of the presence of \( \exp(-\phi_i) \) (and contrary to the mKdV case) one cannot view the Toda equations as derivations of the evolution operator (i.e., a linear map commuting with the action of derivation \( \partial \)),

\[
\tilde{\mathcal{H}} = \sum_{i \in J} \tilde{Q}_i : \pi_0 \rightarrow \bigoplus_{i \in J} \pi_{-\alpha_i}
\]

(4)

for some differential modules \( \pi_{-\alpha_i} = \pi_0 \otimes \exp(-\phi_i) \) equipped with derivations \( \partial \) defined by \( \partial (u \otimes \exp(-\phi_i)) = (\partial u) \otimes \exp(-\phi_i) - (uu_i^{(0)}) \otimes \exp(-\phi_i) \). One of the results of Feigin and Frenkel shows that \( \tilde{Q}_i = -T_i e_i^G \), where \( e_i^G \) denotes the image under the Drinfeld-Sokolov correspondence of the left action by the vector field \( e_i \) on \( N_+/A_+ \), and \( T_i \) denotes the multiplication by \( \exp(-\phi_i) \) sending \( \pi_0 \) to \( \pi_{-\alpha_i} \). In the Hamiltonian formalism of Feigin and Frenkel, one defines \( \mathcal{F}_{-\alpha_i} := \pi_{-\alpha_i}/\mathrm{Im}\partial \). This is the space of functionals of the form

\[
u \mapsto \int_{|z|=1} P(u(z), \partial_z u(z), \ldots) \exp(-\phi_i(z)) dz
\]

(5)

with \( u(z) = (u_i^{(0)}, \ldots, u_i^{(0)}) : S^1 \rightarrow \mathfrak{h} \), where \( S^1 \) is the unit circle and \( \phi_i \) is an anti-derivative of \( u_i^{(0)} \). The space \( \mathcal{F}_0 \) is the local functionals space of the Toda system (one uses only derivatives of \( u \)). It was shown in [GDi, GDo] that it may be equipped with the structure of a "vertex Poisson" algebra (this notion was first introduced in these papers, and developed in [BD, EF]). Such vertex Poisson structures are classical limits of families of VOA structure (as opposed to classical limits of associative algebra structures in the case of Poisson structures). It is possible to extend the Poisson bracket to bilinear maps:

\[
\mathcal{F}_0 \times \mathcal{F}_{-\alpha_i} \rightarrow \mathcal{F}_{-\alpha_i}
\]

(6)

satisfying the Jacobi identity on \( \mathcal{F}_0 \). In other words, the \( \mathcal{F}_{-\alpha_i} \) are \( \mathcal{F}_0 \)-modules. By passing to the quotients, the morphisms \( \tilde{Q}_i \) define screening operators \( Q_i : \mathcal{F}_0 ightarrow \mathcal{F}_{-\alpha_i} \). Feigin and Frenkel showed that \( \tilde{Q}_i = \{ \cdot, \int \exp(-\phi_i) \} \), where \( \int \exp(-\phi_i) \) is the projection of \( \exp(-\phi_i) \) onto \( \mathcal{F}_{-\alpha_i} \). This projection is called the classical screening charge. Then, we define a Hamiltonian

\[
\mathcal{H} = \sum_{i \in J} \tilde{Q}_i : \mathcal{F}_0 \rightarrow \bigoplus_{i \in J} \mathcal{F}_{-\alpha_i}.
\]

(7)
Indeed, in this formalism, we can write Toda equations as

$$\partial_\tau u(z) = \{u(z), \mathcal{H}\}. \quad (8)$$

The integrals of motion of the Toda system are the local functionals which commute with all the screening charges. On the other hand, it was known [BMP] that the $Q_i$ satisfy the Serre relations for $\mathfrak{g}$. This gives an action of the nilpotent Lie algebra $\mathfrak{n}_+$ on $\pi_0$ and allowed Feigin and Frenkel to interpret the space of integrals of motion as the first cohomology group of $\mathfrak{n}_+$ with coefficients in $\pi_0$. Using resolutions of BGG type, the authors managed to compute this cohomology, yet without giving explicit formulas for the integrals of motions (except for some particular cases). They showed that the space of integrals of motion forms a graded one-dimensional Lie subalgebra of $\mathfrak{F}_0$ (for the gradation defined by $\partial^0 u^{(k)}_i = -k$ on $\pi_0$), generated by integrals of motion $I_m$, $m \in \mathbb{Z}$, and $\partial^0 I_m = m$. This Lie subalgebra is the classical $\mathcal{W}$-algebra of $\mathfrak{g}$. Moreover, the Hamiltonian flow of $I_m$ corresponds to the mKdV hierarchy equations.

1.4. The quantum sine-Gordon system. The interpretation of the Poisson structure on $\mathfrak{F}_0$ in terms of Kirillov-Kostant structures allows the following quantum deformation ([FF3]).

1.4.1. The continuous quantum model. Consider the quantum Heisenberg algebra of $\widehat{\mathfrak{sl}}_2$, generated by $I, q_i, b^+_i, i \in \{0, 1\}, n \in \mathbb{Z}$, satisfying the following relations:

$$[I, q_i] = [I, b^+_i] = 0$$
$$[q_i, b^+_j] = (\alpha_i, \alpha_j) \delta_{n, 0} \beta^2 I$$
$$[b^+_i, b^-_n] = n(\alpha_i, \alpha_j) \delta_{n+m, 0} \beta^2 I,$$

where $\delta_{i,j}$ denotes the Kronecker symbol, $\beta^2$ is a deformation parameter ($q = \exp(i\pi \beta^2)$), and $(\alpha_i, \alpha_j)$ denotes the scalar product of two roots $\alpha_i$ and $\alpha_j$ in $\widehat{\mathfrak{sl}}_2$.

This algebra acts on the direct sum of Fock modules in such a way that the vertex operators:

$$V_i(z) = \exp(\phi_i(z)) = \exp(\phi_i^-(z)) \exp(\phi_i^+(z)),$$

satisfy the following commutation relations:

$$V_j(w)V_i(z) = \exp(i\pi \beta^2(\alpha_i, \alpha_j))V_i(z)V_j(w) \quad (9)$$

in the domain $|z| > |w|$. Here, $\phi_i(z)$ is the free field:

$$\phi_i(z) = \sum_{n \neq 0} -\frac{1}{n} b^+_n z^{-n} - b^-_0 \ln(z) - q_i,$$

and

$$\phi_i^+(z) = \sum_{n > 0} -\frac{1}{n} b^+_n z^{-n} - b^+_0 \ln(z)$$

$$\phi_i^-(z) = \sum_{n < 0} -\frac{1}{n} b^-_n z^{-n} - b^-_0 \ln(z).$$
The screening charges are defined by:

$$S_i = \int_{|z|=1} V_i(z)dz = \int_{|z|=1} :\exp(\phi_i(z)) : dz$$  \hspace{1cm} (10)

for $i \in \{0, 1\}$. The integrals of motion of the system are expressions of the form

$$\int_{|z|=1} P(\partial_k^k \phi_i)_{k \geq 0, i \in \{0, 1\}}(z)dz$$

which commute with $S_0$ and $S_1$. Feigin and Frenkel showed that the screening charges $S_0$ and $S_1$ satisfy quantum Serre relations of $\hat{sl}_2$. This allowed them – as in the classical case – to interpret the space of quantum integrals of motion as the first cohomology group of a certain complex. They proved that the integrals of motion commute with each other and that the space of integrals of motion generates a quantum deformation of the classical $W$-algebra.

1.4.2. The discrete quantum sine-Gordon model. The first model of a discrete integrable system was introduced by Izergin and Korepin in 1982 for the sine-Gordon system in order to resolve ultraviolet divergence problems occurring in the continuous theory. The $q$-commutation relations between vertex operators naturally lead to the aforementioned discretization, which is adopted by Izergin and Korepin.

Set $q = \exp(i\pi\beta^2)$, and replace the complex numbers $z$ by relative integers $k \in \mathbb{Z}$. The vertex operators $V_0(z)$ and $V_1(z)$ are replaced by variables $y_k$ and $x_k$ satisfying the relations:

$$\forall k < l, \quad x_kx_l = qx_lx_k$$  \hspace{1cm} (11)

$$ykyl = qylyk$$  \hspace{1cm} (12)

$$ykx_l = q^{-1}x_lyk$$  \hspace{1cm} (13)

$$\forall k \leq l, \quad x_kyl = q^{-1}ylx_k,$$  \hspace{1cm} (14)

coming from (9). The analogues of screening charges are $S_0$ and $S_1$ defined by $S_0 = \sum_{k=-\infty}^{+\infty} y_k$ and $S_1 = \sum_{k=-\infty}^{+\infty} x_k$ in a certain completion of the algebra of variables on the given lattice. The Hamiltonian of the system is $H = S_0 + S_1$. The integrals of motion correspond to expressions of the form

$$\sum_{i=-\infty}^{+\infty} P(x_i^{\pm 1}, x_{i+1}^{\pm 1}, \ldots, x_{i+k}^{\pm 1}, y_i^{\pm 1})$$

which commute with $S_0$ and $S_1$, where $P(X_1^{\pm 1}, Y_1^{\pm 1}, \ldots, X_k^{\pm 1}, Y_k^{\pm 1})$ denotes a polynomial in the variables $X_1^{\pm 1}, \ldots, Y_k^{\pm 1}$, these variables being $q$-commutative.

As in the continuous case, Enriquez and Feigin showed that the screening charges satisfy the Serre relations (for $\hat{sl}_2$), which gives a cohomological interpretation for the integrals of motion. By means of Demazure desingularization, they managed to compute this cohomology in the classical limit $q \to 1$, to give formulae for densities of integrals of motion, and to prove that the integrals of motion are in involution. This justifies calling the system a discrete integrable system. Moreover, Enriquez and Feigin identified the phase space of this system with the homogeneous space $H_- \backslash B_-$, where $B_-$ is a Borel subgroup of the loop group of $SL_2$, and $H_-$ is a subgroup of $B_-$ consisting of diagonal matrices, and established a discrete version of the Drinfeld-Sokolov correspondence. The Hamiltonian action by integrals of motions corresponds to a (left) action.
of a commutative Lie algebra $\mathfrak{h}_+$ on the homogeneous space, which is embedded into $H_-\backslash G/N_+$. Moreover, the correspondence is a Poisson morphism.

2. Main results

In this section, we present our main results which deal with the discrete sine-Gordon theory. Proposition 3 is proved in [2]. In Proposition 3, we construct a quantization of the Poisson homogeneous space considered by Enriquez and Feigin [EF]. Theorem 1 is a discrete and quantum version of the Drinfeld-Sokolov correspondence. It generalizes a theorem of Enriquez and Feigin to the quantum case.

2.1. Some basic definitions. The following notation will be used throughout the article. Let $q$ be a formal variable.

The quantum phase space of the discrete sine-Gordon system. Let $A_q$ be the algebra generated over $\mathbb{Q}[q,q^{-1}]$ by the non-commutative variables $x_i^{\pm 1}$ and $y_i^{\pm 1}$, $i \in \mathbb{Z}$, subject to relations (11), . . . , (14). This is the quantum phase space of our system. It can be shown that a basis for $A_q$ is given by the family $\prod_{i=-\infty}^{+\infty} x_i^{\alpha_i} y_i^{\beta_i}$, where $(\alpha_i)$ and $(\beta_i)$ are two sequences in $\mathbb{Z}^{\mathbb{Z}}$ which have only a finite number of non-zero elements. At the semi-classical limit, $A_q$ defines a Poisson structure on $A_{cl} = \mathbb{Q}[x_i^{\pm 1}, y_i^{\pm 1}, i \in \mathbb{Z}]$. There is a gradation deg on $A_q$ defined by $\text{deg}(x_i) = -\text{deg}(y_i) = 1$ for all $i \in \mathbb{Z}$. Let $T^{\pm 1}_2$ denote the half-translation antigraded automorphism defined on $A_q$ by $T^{\pm 1}_2(x_i) = y_i$ and $T^{\pm 1}_2(y_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. We shall set $T = (T^1_2)^2$. For $n \in \mathbb{N}$, let $A_q[n]$ be the submodule of $A_q$ of all elements of degree $n$. The discrete analogue of $\pi_0$ is $A_q[0]$.

The functional spaces and integrals of motion. By definition, the functional spaces are $\mathcal{F}_n$, $n \in \mathbb{N}$ defined by $\mathcal{F}_n = A_q[n]/\text{Im}(T - \text{Id})$. If $P \in A_q[n]$, one notes $I(P)$ its class in $\mathcal{F}_n$. The discrete analogues of the screening charges $S_0$ and $S_1$ seen in (10), are $\Sigma_+ = I(0)$ and $\Sigma_- = I(0)$, $\Sigma_+ \in \mathcal{F}_1$ and $\Sigma_- = I(0)$. The Hamiltonian of the system is $H = \Sigma_+ + \Sigma_- \in \mathcal{F}_{-1} \oplus \mathcal{F}_1$. We set $\mathcal{I} = \text{Ker}[\Sigma_-] \cap \text{Ker}[\Sigma_+] \subset \mathcal{F}_0$. It is the space of all local integrals of motion of the system. If $P \in A_q[0]$ and if $I(P) \in \mathcal{I}$, we say that $P$ is a density of an integral of motion.

The homogeneous space of Enriquez and Feigin. We set $G = \text{SL}_2(\mathbb{C}(\langle \lambda^{-1} \rangle))$, $B_- = \pi^{-1}(\bar{B}_-)$, and $N_+ = p^{-1}(\bar{N}_+)$, where $\bar{B}_-$ (resp. $\bar{N}_+$) is the subgroup of $\text{SL}_2(\mathbb{C})$ defined by all lower triangular (resp. upper unipotent) matrices, $\pi$ (resp. $p$) is the induced map from $\text{SL}_2(\mathbb{C}[\langle \lambda^{-1} \rangle])$ (resp. $\text{SL}_2(\mathbb{C}[\lambda])$) to $\text{SL}_2(\mathbb{C})$ obtained by sending $\lambda^{-1}$ (resp. $\lambda$) to 0. We also denote by $H_-$ the subgroup of $B_-$ given by all the diagonal matrices of the form $\text{diag}(a, a^{-1})$, $a \in \mathbb{C}[\langle \lambda^{-1} \rangle]^*$. The Poisson homogeneous space considered by Enriquez and Feigin is $H_--\bar{B}_-$ endowed with the Poisson structure induced by the Poisson bivector $P_\infty = r^L - r^R$ on $G$ such that the map $H_--\bar{B}_- \rightarrow H_--\bar{G}/\bar{N}_+$ is a Poisson morphism. Here, $r$ is the standard $r$-matrix on $G$, $r_\infty$ corresponds to the conjugation of $r$ with an element of the Weyl group with an infinite length (it is the $r$-matrix associated with the “new realizations” of the Drinfeld) and $r^L$ (resp. $r^R$) is the left (resp. right) translation of $r$ (resp. $r_\infty$) on $G$. The homogeneous space $H_--\bar{B}_-$ is endowed with an action from the left by the Abelian Lie algebra $\mathfrak{h}_+ := \{\text{diag}(a, -a), a \in \lambda \mathbb{C}[\lambda]\}$. Enriquez and Feigin identified this action with the Hamiltonian action of the integrals of motion on $A_{cl}$. Theorem 1 generalizes this result.
2.2. The results. The first few results deal with the integrals of motion of the discrete sine-Gordon system.

**Proposition 1.** Let \( n \) be an integer. For any \( F \in \mathcal{F}_0, G \in \mathcal{F}_n, P \in I^{-1}(F) \subset A_q[0] \) and \( Q \in I^{-1}(G) \subset A_q[n] \), the two sums \( \sum_{i=\infty}^{-\infty} [T^i P, Q] \) and \( \sum_{i=\infty}^{-\infty} [P, T^i Q] \) are equal and do not depend on \( P \) nor \( Q \) but only of \( F \) and \( G \). Moreover, the bilinear map:

\[
\mathcal{F}_0 \times \mathcal{F}_n \rightarrow \mathcal{F}_n
\]

\[
(F, G) \mapsto [F, G]
\]

(15)

with \( F = I(P), G = I(Q) \), and

\[
[I, J] := \frac{1}{q-1} \sum_{i=\infty}^{\infty} [T^i P, Q]
\]

(16)

\[
= \frac{1}{q-1} \sum_{i=\infty}^{\infty} [P, T^i Q]
\]

(17)

satisfies the Jacobi identity on \( \mathcal{F}_0 \). In other words, the space \( \mathcal{F}_0 \) of all local functionals is a Lie algebra, and \( \mathcal{F}_n \) is a \( \mathcal{F}_0 \)-module.

The following proposition shows that the local functionals act on \( A_q \) by adjoint action.

**Proposition 2.** For any \( F \in \mathcal{F}_0 \) and \( x \in A_q \), the sum \( \sum_{k=\infty}^{\infty} [T^k P, x] \) does not depend of the representative \( P \in I^{-1}(F) \subset A_q[0] \) chosen for \( F \) and defines a derivation on \( A_q \) which commutes with the automorphism \( T \). This derivation is the adjoint action of \( F \) on \( A_q \) and is denoted by \( \text{ad}(F) \). Moreover, if \( \text{Der}_T(A_q) \) denotes the Lie algebra of all derivations on \( A_q \) which commute with \( T \), we have a Lie algebra morphism:

\[
\text{ad} : \quad \mathcal{F}_0 \rightarrow \text{Der}_T(A_q)
\]

\[
F \mapsto \frac{1}{q-1} \sum_{k=\infty}^{\infty} [T^k P, .]
\]

(18)

and the kernel of this morphism is equal to the class of constants in \( \mathcal{F}_0 \).

Proposition 3 gives an explicit basis for the space \( I \) of local integrals of motion. This should be compared to the formulas involving quantum trace identities of Izergin and Korepin [IK2] (see also [FTT] and [H]) at least in the classical case. Indeed, in the quantum case, the Izergin and Korepin formulas for the sine-Gordon system are no longer local.

**Proposition 3.** The local classical integrals of motion of the discrete sine-Gordon system can be deformed in the quantum case. A basis for \( I \) is given by the family \( I_n = I(\psi_n), n > 0 \). The generating function of the densities of integrals of motion \( \psi_n \) is given by

\[
\ln_q U + \ln_q V = \sum_{p=1}^{\infty} \frac{1}{[p]} \psi_p \lambda^{-p}
\]

(19)

where for all integers \( p, [p] := \frac{q^p - 1}{q - 1} \), \( U = \lim_{N \rightarrow \infty} U_N \), \( V = \lim_{N \rightarrow \infty} V_N \),
\[ U_N := \frac{1}{1 - \frac{(\lambda x_1 y_1)^{-1}}{(\lambda y_1 x_2)^{-1}} - \cdots \frac{(\lambda x_{N-1} y_{N-1})^{-1}}{1 - (\lambda y_{N-1} x_N)^{-1}}} \tag{20} \]

\[ V_N := T^\dagger U_N \tag{21} \]

The limits being taken in the sense of the \( \lambda^{-1} \)-topology. By convention, we have set \( \frac{a}{b} := ab^{-1} \), and for all power series \( f \) in \( \lambda^{-1} \) with non-zero constant term,

\[ \ln_q f := \sum_{p=1}^{\infty} \frac{1}{p} (1 - f^{-1})^p. \]

The following proposition is proved in section \[3\].

**Proposition 4.** The space \( I \) is a commutative Lie subalgebra in \( F_0 \).

Proposition \[5\] gives a quantization of the homogeneous space considered by Enriquez and Feigin.

**Proposition 5.** The algebra given by generators : \( u_i, m_i, i > 0 \) and (quadratic) relations

\[ (\lambda^{-1} u(\lambda) - \mu^{-1} u(\mu))(u(\lambda) - u(\mu)) = q(u(\lambda) - u(\mu))(\lambda^{-1} u(\lambda) - \mu^{-1} u(\mu)) \tag{22} \]

\[ (\lambda^{-1} m(\lambda) - \mu^{-1} m(\mu))(m(\lambda) - m(\mu)) = q^{-1}(m(\lambda) - m(\mu))(\lambda^{-1} m(\lambda) - \mu^{-1} m(\mu)) \tag{23} \]

\[ u(\lambda)m(\mu) = q^{-1}m(\mu)u(\lambda) \tag{24} \]

with \( u(\lambda) = \sum_{i=0}^{\infty} (-1)^i u_{i+1} \lambda^{-i} \) and \( m(\lambda) = \sum_{i=0}^{\infty} (-1)^i m_{i+1} \lambda^{-i} \) is a quantum deformation of the algebra of functions over the Poisson homogeneous space \( (H_+ \backslash B_-, P_\infty) \).

The following proposition shows that the action by vector fields of the Abelian Lie subalgebra \( h_+ \) on \( H_+ \backslash B_- \) can also be quantized.

**Proposition 6.** There is \( (H_n)_{n \in \mathbb{N}^*} \), a commutative family of derivations on \( \mathbb{C}[H_+ \backslash B_-]_q \) defined by the formulas:

\[ H_\mu(u(\lambda)) = \frac{1}{\lambda^{-1} - \mu^{-1}} (\lambda^{-1} u(\lambda) - \mu^{-1} u(\mu))v(\mu)u(\lambda) - \frac{\mu^{-1}}{\lambda^{-1} - \mu^{-1}} (u(\lambda) - u(\mu))(1 + v(\mu)u(\mu)); \tag{25} \]

\[ H_\mu(m(\lambda)) = \frac{\mu^{-1}}{\lambda^{-1} - \mu^{-1}} (1 + m(\mu)w(\mu))(m(\lambda) - m(\mu)) - \frac{1}{\lambda^{-1} - \mu^{-1}} m(\lambda)w(\mu)(\lambda^{-1} m(\lambda) - \mu^{-1} m(\mu)) \tag{26} \]

with \( H(\mu) = \sum_{k=1}^{\infty} (-1)^k H_k \mu^{-k} \), \( v(\mu) = -(u(\mu) + \mu m(\mu)^{-1})^{-1} \) and \( w(\mu) = -(m(\mu) + \mu u(\mu)^{-1})^{-1} \).

This family of derivations deforms the classical action by vector fields of \( h_+ \) on \( H_+ \backslash B_- \).

The following theorem is a quantum version of the Drinfeld-Sokolov correspondence. It shows that the quantization of the action of \( h_+ \) on \( H_+ \backslash B_- \) considered in Proposition \[6\] can be identified with the adjoint action of the integrals of motion on the phase space \( A_q \).
Theorem 1. There is an injective and birational map \( DS_q \) from \( C[H_\cdot \setminus B_\cdot]_q \) to \( A_q \) defined by

\[
DS_q(u(\lambda)) = \lim_{N \to \infty} \frac{y_0^{-1}}{1 + \frac{(\lambda x_0 y_0)^{-1}}{1 + \frac{(\lambda y_0 x_0)^{-1}}{1 + \cdots \frac{(\lambda y_N x_{N+2})^{-1}}{1 + (\lambda x_N y_{N+1})^{-1}}}}}
\]

and

\[
DS_q(m(\lambda)) = \lim_{N \to \infty} \frac{x_1^{-1}}{1 + \frac{(\lambda x_1 y_1)^{-1}}{1 + \frac{(\lambda y_1 x_2)^{-1}}{1 + \cdots \frac{(\lambda y_N x_N)^{-1}}{1 + (\lambda x_N y_N)^{-1}}}}}
\]

the limit being taken with respect to the \( \lambda^{-1} \)-adic topology, and where, in the first case, \( \frac{a}{b} := b^{-1}a \), and in the second case, \( \frac{a}{b} := ab^{-1} \). We have : \( DS_q \circ H_n = \text{ad}(I_n) \circ DS_q \) for all integers \( n \).

The algebras \( C[H_\cdot \setminus B_\cdot]_q \) and \( A_q \) have fraction fields, and the phrase "birational" means that \( DS_q \) induces an isomorphism of these fraction fields.

3. Commutativity of the local integrals of motion of the discrete sine-Gordon system

We give here a (new) proof of the commutativity of the quantum local integrals of motion \( [K2] \) (see also \( [FTT], [V], [H] \)). This constitutes Proposition \( [\ ] \). Our proof is based on the explicit form taken by the elements \( I_n \) of the basis of \( \mathcal{I} \) given in the Proposition \( [\ ] \). First, we note that, as a consequence of Proposition \( [\ ] \), \( \mathcal{I} \) is a Lie subalgebra of \( \mathcal{F}_0 \). We shall use Proposition \( [\ ] \) given below, proved in \( [GI] \) and which is equivalent to Proposition \( [\ ] \). If \( a \) and \( b \) are two integers, we set:

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} 1, & \text{si } b = 0; \\ 0, & \text{si } b < 0 \text{ ou } b > \text{Max}(0,a); \\ [a]! \frac{[b]!}{[a-b]!}, & \text{si } 0 \leq b \leq a; \end{cases}
\]

with \( [n]! = \prod_{i=1}^{n} [i] \) and \( [i] = \frac{q^i - 1}{q - 1} \).

Also, for any relative integers \( N, a_1, \ldots, a_N \), we set \( F_q(a_1, \ldots, a_N) = \prod_{i=1}^{N} \left[ a_i + a_{i+1} - 1 \right]^{-1} \).

Proposition 7. A basis for \( \mathcal{I} \) is given by the family \( I_n = I(\psi_n) \), with \( \psi_n = A_n + B_n \), \( B_n = T^{\frac{1}{2}} A_n \), and

\[
A_n = \sum_{[\alpha]} \frac{[\alpha]}{[\alpha_1]} F_q(\alpha_1, \ldots, \alpha_{2N-2})(x_1 y_1)^{-\alpha_1} (y_1 x_2)^{-\alpha_2} \cdots (y_{N-1} x_N)^{-\alpha_{2N-2}}
\]
the sum being taken on the set of indices \(\alpha_1, \ldots, \alpha_{2N-2}\) such that \(\alpha_i \in \mathbb{N}, \alpha_1 \in \mathbb{N}^*, \) and \(\alpha_1 + \ldots + \alpha_{2N-2} = n.\) Here, \(N\) is any integer such that \(n \leq 2(N-1)\).

One will achieve the proof of the commutativity of \(I\) in several steps. The key step is the existence of a filtration on a Lie subalgebra \(\mathcal{F}_0'\) which contains \(I\).

3.1. Gradation on \(\mathcal{F}_0\). We denote \(\deg_p\) the principal gradation on \(A_q\) defined by \(\deg_p(x_i) = \deg_p(y_i) = 1\) for all \(i \in \mathbb{Z}\). Also, we set \(e_{2i-1} = (x_iy_i)^{-1}\) and \(e_{2i} = (y_ix_{i+1})^{-1}\). The elements \(e_{j,1}, j \in \mathbb{Z}\) generate \(A_q[0]\). So, any element \(u \in \mathcal{F}_0\) can be represented by a sum of terms \(P_k, k \in \mathbb{Z}\) with \(\deg_p(P_k) = 2k \in 2\mathbb{Z}\). Therefore, the gradation \(\deg_p\) on \(A_q\) induces a gradation on the module \(\mathcal{F}_0\) by the following way : an element \(u \in \mathcal{F}_0\) is said to be homogeneous of degree \(k\) if there exists \(P \in A_q[0]\) such that \(I(P) = u\) and \(\deg_p(P) = 2k\). The formula (16) shows that \((I, \deg)\) is a Lie subalgebra of \(\mathcal{F}_0\) generated by \(I_n = I(\psi_n), n \in \mathbb{N}^*\) with \(\deg(I_n) = -n\).

3.2. The subalgebra \(B_q[0]\) of \(A_q[0]\). Let \(B_q[0]\) be the subalgebra (without unit) of \(A_q[0]\) generated by the \(e_i, i \in \mathbb{Z}\). For all \(i, j \in \mathbb{Z}\), we have \(e_i e_{i+1} = q^{-1} e_{i+1} e_i\) and \(e_i e_j = e_j e_i\) if \(|i-j| \geq 2\). A basis for \(B_q[0]\) is given by the family \(\varepsilon_\alpha = \prod_{i \in \mathbb{Z}} e_i^{\alpha_i}\), where \(\alpha\) is a non-zero sequence in \(\mathbb{N}^2\) such that all almost all elements are zeros (except for a finite number of them). We define a function \(l\) on \(B_q[0]\) by the following way. If \(P = \sum_{\alpha} \lambda_\alpha \varepsilon_\alpha\) is a non-zero element in \(B_q[0]\) we set \(l(P) := \text{Inf}\{l(\alpha)/\lambda_\alpha \neq 0\} \) with \(l(\alpha) := \text{card}\{i \in \mathbb{Z}/ \alpha_i \neq 0\}\). We have \(l(P) \in \mathbb{N}^*\). By convention, we consider that \(l(0) = +\infty\). Obviously, \(B_q[0]\) is invariant by the translation automorphism \(T\) defined in \(3.1\) and \(l \circ T = l\). Moreover, if \(P \in B_q[0]\), then \(l(PQ) \geq \text{Max}(l(P), l(Q))\) and \(l(P + Q) \geq \text{Inf}(l(P), l(Q))\) with equality in the last inequality if \(l(P) = l(Q)\).

3.3. The Lie subalgebra \(\mathcal{F}_0'\) of \(\mathcal{F}_0\). We note \(\mathcal{F}_0'\) the quotient module \(B_q[0]/\text{Im}(T - \text{Id})\). A basis for \(\mathcal{F}_0'\) is given by the elements \(I(\varepsilon_\alpha)\) where \(\alpha\) is an almost zero sequence satisfying the property \(\alpha_i = 0\) if \(i \leq 0\), and \(\alpha_1 \neq 0\) or \(\alpha_2 \neq 0\). The module \(\mathcal{F}_0'\) is a Lie subalgebra by taking the formula (16) as a definition. Furthermore, we have a natural injective map of Lie algebras \(\mathcal{F}_0' \hookrightarrow \mathcal{F}_0\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
B_q[0] & \hookrightarrow & A_q[0] \\
\downarrow & & \downarrow \\
\mathcal{F}_0' & \hookrightarrow & \mathcal{F}_0.
\end{array}
\]

The first horizontal map is the canonical injection of \(B_q[0]\) in \(A_q[0]\). The vertical maps are the canonical projections. Note that \(\mathcal{F}_0'\) is a graded Lie subalgebra of \(\mathcal{F}_0\) with respect to gradation \(\deg\). Also, according to Proposition \(\mathcal{F}_0'\) contains \(I\).

3.4. Filtration on \(\mathcal{F}_0'\). Let us start with the following lemma.

Lemma 1. Let \(u \in \mathcal{F}_0'\) with \(u \neq 0\). Then \(L_u := \{l(P) / P \in B_q[0]\) and \(l(P) = u\}\) is a bounded non-empty set in \(\mathbb{N}^*\).

Proof. Obviously, \(L_u \neq \emptyset\). Consider \(Q \in B_q[0]\) such that \(Q\) is generated by \(e_i, n_1 \leq i \leq n_2\). Let \(V_1\) (resp. \(V_2\)) be the submodule generated by the monomials \(\varepsilon_\alpha\) with \(l(\alpha) \leq n_2 - n_1 + 1\) (resp. \(l(\alpha) > n_2 - n_1 + 1\)). We have \(B_q[0] = V_1 \oplus V_2\), \(Q \in V_1\), and \(V_i (i = 1, 2)\) is invariant by \(T\). Assume that there exists \(P \in B_q[0]\) such that \(I(P) = u\) and \(l(P) > n_2 - n_1 + 1\). Then,
there exists $R$ such that $P = Q + T(R) - R$. We set $R = R_1 + R_2$ with $R_i \in V_i$ $(i = 1, 2)$. By projecting on $V_2$, we obtain $P = T(R_2) - R_2$. So, $u = 0$. But this contradicts our hypothesis. 

Lemma $3$ allows us to define a length function on $F'_0$.

**Definition.** Let $u \in F'_0$, with $u \neq 0$. We set $l(u) = \text{Max } L_u$. By convention, $l(0) = +\infty$.

Thanks to section 3.3, the following lemma allows us to compute lengths in $F'_0$ explicitly.

**Lemma 2.** Let $x \in B_q[0]$ be a non-zero element such that $x$ is a linear combination of monomials of the form $\epsilon_{\alpha}$, with $\alpha_i = 0$ if $i \leq 0$ and $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Then, $l(I(x)) = l(x)$.

**Proof.** Set $l(x) = k$. Let $V_1$ (resp. $V_2$) denotes the submodule of $B_q[0]$ generated by all monomials $\epsilon_{\alpha}$ with length $k$ (resp. with a length different from $k$). We proceed as in the proof of the Lemma $3$. 

We are now able to show that $F'_0$ is a filtered Lie algebra.

**Lemma 3.** Let $u$ and $v$ be two elements of $F'_0$. Then, $l([u,v]) \geq \text{Max } (l(u), l(v))$.

**Proof.** Set $j = l(u)$, $k = l(v)$ and $n = \text{Max } (j, k)$. There exist $P$ and $Q$ in $B_q[0]$ such that $I(P) = u$, $I(Q) = v$, $l(P) = j$ and $l(Q) = k$. For all $k \in \mathbb{Z}$, we have $l(T^\alpha(P)) = l(P)$. So, $l((T^\alpha(P))Q) \geq n$ and $l(Q(T^\alpha(P))) \geq n$. So, $l([T^\alpha(P), Q]) \geq n$. Hence, $l(\sum_{\alpha = -\infty}^{\infty} [T^\alpha(P), Q]) \geq n$ and $l([u,v]) \geq n$. 

3.5. End of the proof. For $n \in \mathbb{N}^*$, we set $u_n = e_1^n$ and $v_n = e_2^n$. Thanks to Proposition $3$, there exists $w_n \in B_q[0]$ such that $l(w_n) \geq 2$ and $\psi_n = u_n + v_n + w_n$. Clearly, we have $[I(u_n), I(u_p)] = [I(v_n), I(v_p)] = 0$ for all $n, p \in \mathbb{N}$. On the other hand, by using Lemma 3, the computation shows that $l([I(u_n), I(v_p)]) = 2$. So, we deduce from Lemma 3 and the bilinearity of the Lie bracket that $l([I_n, I_p]) \geq 2$ for all $n, p \in \mathbb{N}$. But, for degree reasons, $[I_n, I_p]$ is proportional to $I_{n+p}$ and $l(I_{n+p}) = 1$. Hence, $[I_n, I_p] = 0$.

4. **Quantization of the Poisson homogeneous space $(H_-\backslash B_-, P_\infty)$**

The aim of this section is to prove Proposition $3$. As shown in the subsection 3.3, at the semi-classical level, the generators $u_i$ and $m_i$, $i > 0$, give a natural system of coordinate functions on $H_-\backslash B_-$ which satisfy the Poisson relations obtained by taking the limit $q \to 1$ in (22), (23), (24). Therefore, to prove Proposition $3$, it is enough to show that $\mathbb{C}[H_-\backslash B_-]_q$ is a flat deformation of $\mathbb{C}[H_-\backslash B_-]$. The idea is to obtain a realization of the algebra $\mathbb{C}[H_-\backslash B_-]_q$ in $\mathcal{A}_q$ by using Lemma 3 which asserts that the finite screening charges of the discrete sine-Gordon system satisfy the quantum Serre relations. In all the following, the ground ring is no longer $\mathbb{Q}[q, q^{-1}]$ but $\mathbb{C}[q, q^{-1}]$. 
4.1. The Poisson homogeneous space \((H_- \setminus B_-, P_\infty)\). The Poisson manifold \((H_- \setminus B_-, P_\infty)\) was defined in \([2]\). Any element \(\bar{x} \in H_- \setminus B_-\) can be expressed uniquely in the following form:

\[
\bar{x} = \text{cl}_{H_-} \begin{pmatrix}
ds_{\text{cl}}(\lambda)(\bar{x}) & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

with \(u_{\text{cl}}(\lambda) \in \mathbb{C}[H_- \setminus B_-][[\lambda^{-1}]]\) and \(v_{\text{cl}}(\lambda) \in \lambda^{-1}\mathbb{C}[H_- \setminus B_-][[\lambda^{-1}]]\). For \(i \in \mathbb{N}^*\), we define coordinate functions \(u_{i,\text{cl}}\) and \(m_{i,\text{cl}}\) on \(H_- \setminus B_-\) by:

\[
u_{\text{cl}}(\lambda) = \sum_{i=0}^{\infty} (-1)^i u_{i+1,\text{cl}} \lambda^{-i}
\]

\[
m_{\text{cl}}(\lambda) = \sum_{i=0}^{\infty} (-1)^i m_{i+1,\text{cl}} \lambda^{-i}
\]

and \(m_{\text{cl}}(\lambda) := -\lambda v_{\text{cl}}(\lambda) (1 + u_{\text{cl}} v_{\text{cl}}(\lambda))^{-1} \in \mathbb{C}[H_- \setminus B_-][[\lambda^{-1}]]\).

The functions \(u_{i,\text{cl}}\) and \(m_{i,\text{cl}}\) are algebraically independent and \(\mathbb{C}[H_- \setminus B_-] = \mathbb{C}[u_{i,\text{cl}}, m_{i,\text{cl}}, i > 0]\). Moreover, computation shows that the Poisson relations between these functions (the Poisson structure is induced by the field of bivectors \(P_\infty\)) are precisely the ones we get from (22), (23), (24) when \(q \to 1\).

4.2. The Enriquez-Feigin morphism. Let \(n_-\) be a nilpotent subalgebra of \(\widehat{sl}_2\) and \(U_q n_-\) be the quantum algebra generated by the generators \(f_+\) and \(f_-\) subject to the quantum Serre relations:

\[
f_+^2 f_+ - (q + 1 + q^{-1})(f_+^2 f_+ f_- - f_- f_+ f_+^2) - f_+ f_- = 0.
\]

Let \(\text{deg}\) be the gradation on \(U_q n_-\) defined by \(\text{deg} f_\pm = \pm 1\). In the sequel of the article, if \((A, \text{deg}_A)\) and \((B, \text{deg}_B)\) are two graded algebras, we define the twisted tensor product \(A \hat{\otimes} B\) by the formula:

\[
(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = q^{-\text{deg}_A(a_2) \text{deg}_B(b_1)}(a_1 a_2 \otimes b_1 b_2)
\]

for homogeneous elements \(a_1, a_2, b_1, b_2\) in \(A\) and \(B\). There is a unique graded algebra morphism \(\Delta\) from \(U_q n_-\) to \(U_q n_- \hat{\otimes} U_q n_-\) given by \(\Delta(f_\pm) = f_\pm \otimes 1 + 1 \otimes f_\pm\). This morphism is called the twisted comultiplication on \(U_q n_-\).

Lemma 4 ([EF]). Let \(n \in \mathbb{N}\). Then \(\Sigma_{+,n} := \sum_{i=1}^{n} x_i\) and \(\Sigma_{-,n} := \sum_{i=1}^{n} y_i\) satisfy the quantum Serre relations. In other words, there exists a graded algebra morphism \(f_n\) defined by

\[
\begin{align*}
f_n : & \ U_q n_- \rightarrow A_q \\
& f_+ \mapsto x_i \\
& f_- \mapsto 0
\end{align*}
\]

Proof. For \(1 \leq i \leq n\), we define two graded algebra morphisms \(\varphi_i\) and \(\psi_i\) by:

\[
\varphi_i : U_q n_- \rightarrow \mathbb{C}[x_i, x_i^{-1}]
\]

\[
\begin{align*}
& f_+ \mapsto x_i \\
& f_- \mapsto 0
\end{align*}
\]

and

\[
\psi_i : U_q n_- \rightarrow \mathbb{C}[y_i, y_i^{-1}]
\]

\[
\begin{align*}
& f_+ \mapsto 0 \\
& f_- \mapsto y_i
\end{align*}
\]
with the convention \( \deg x_i = -\deg y_i = 1 \). Then, it appears that the map \( f_n \) defined in [34] is equal to \( \text{mult}_{2n} \circ (\varphi_1 \otimes \psi_1 \otimes \ldots \otimes \varphi_n \otimes \psi_n) \circ \Delta^{(2n)} \) where \( \text{mult}_{2n} \) is the algebras monomorphism

\[
\text{mult}_{2n} : C[x_1, x_1^{-1}] \otimes \cdots \otimes C[y_n, y_n^{-1}] \rightarrow A_q,
\]

\[
u_1 \otimes v_1 \otimes \cdots \otimes v_n v_n \mapsto u_1 v_1 \cdots u_n v_n.
\]

So as to realize \( C[H_\text{\tiny\{.B\}_-)] \) in \( A_q \), it is useful to give another expression of the quantum algebra \( U_q \text{\tiny\{.N\}_-} \).

**Definition.** Let \( q^\frac{1}{2} \) be an indeterminate, \( q^\frac{1}{2} = (q^\frac{1}{2})^2 \), \( q = (q^\frac{1}{2})^2 \), \( K_0 = C[q^\frac{1}{2}, q^{-\frac{1}{2}}] \), \( K = C[q, q^{-1}] \).

\[
H = \begin{pmatrix}
q^{-\frac{1}{2}} & 0 & 0 & 0 \\
0 & q^\frac{1}{2} & 0 & 0 \\
0 & 0 & q^\frac{1}{2} & 0 \\
0 & 0 & 0 & q^{-\frac{1}{2}}
\end{pmatrix} \in M_4(K_0) \simeq M_2(K_0)^{\otimes 2},
\]

(35)

and \( R(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\lambda-\mu}{q^{-\frac{1}{2}} - q^\frac{1}{2} \mu} & \frac{1}{(q^\frac{1}{2} - q^{-\frac{1}{2}}) \mu} & 0 \\
0 & \frac{\lambda-\mu}{q^{-\frac{1}{2}} - q^\frac{1}{2} \mu} & \frac{1}{(q^\frac{1}{2} - q^{-\frac{1}{2}}) \mu} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in M_4(K_0) \simeq M_2(K_0)^{\otimes 2}.
\]

(36)

By definition, \( C[N_+ \text{\tiny\{.q\}] \) is the algebra generated over \( K \) by the \( a^{(r)}_{i,j} \) for \( i, j \in \{1, 2\} \) \( r \in \mathbb{N} \) and the relations:

\[
a_{2,1}^{(0)} = 0, a_{1,1}^{(0)} = a_{2,2}^{(0)} = 1,
\]

(37)

\[
R(\lambda, \mu)L^1(\lambda)HL^2(\mu) = L^2(\mu)HL^1(\lambda)R(\lambda, \mu),
\]

(38)

\[
a_{1,1}(q\lambda) [a_{2,2}(\lambda) - a_{2,1}(\lambda)a_{1,1}(\lambda)^{-1}a_{1,2}(\lambda)] = 1,
\]

(39)

Relations (38) and (39) have coefficients in \( K \). So, the above definition makes sense. The relation (39) is the quantum determinant relation. It can be shown [33] that \( C[N_+ \text{\tiny\{.q\}] \) is a quantum deformation of the algebra of functions on the Poisson manifold \( N_+ \) defined in [2.1] equipped with the Poisson bivector \( P = r^G - r^D + \frac{1}{4}(h^G \otimes h^D - h^D \otimes h^G) \) where \( r \) denotes the \( r \)-trigonometric matrix:

\[
r(\lambda, \mu) = \frac{-1}{4} \frac{\lambda + \mu}{\lambda - \mu} h \otimes h - \frac{1}{\lambda - \mu} (\lambda e \otimes f + \mu f \otimes e)
\]

with \( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). We can define a gradation \( \deg \) on \( C[N_+ \text{\tiny\{.q\}] \) by \( \deg(a^{(k)}_{i,j}) = i - j \) for integers \( i, j, k \) as well as a twisted comultiplication \( \bar{\Delta} \) given by \( \bar{\Delta}(L(\lambda)) = \)

Moreover, the map
\[
\Phi : \quad U_q \事情 \to \mathbb{C}[N_+]_q
\]
\[
f_+ \mapsto a_{1,2}^{(1)}
\]
\[
f_- \mapsto a_{1,2}^{(0)}
\]
is an algebra morphism such that \(\tilde{\Delta} \circ \Phi = (\Phi \otimes \Phi) \circ \tilde{\Delta}\). Let us consider the morphism \(f_n\) defined in Lemma 4 above. Thanks to the proof of Lemma 3 there exists a graded algebra morphism
\[
g_n : \quad \mathbb{C}[N_+]_q \to A_q
\]
\[
\mathcal{L}(\lambda) \mapsto \prod_{i=1}^n \left( \begin{array}{cc} 1 & 0 \\ \lambda x_i & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
(40)
It is clear that for \(r > n\), \(a_{1,1}^{(r-1)}, a_{1,2}^{(r-1)}, a_{2,1}^{(r)}, a_{2,2}^{(r)} \in \text{Ker} g_n\). Moreover, \(a_{1,1}^{(n-1)}, a_{1,2}^{(n-1)}, a_{2,1}^{(n)}\) and \(a_{2,2}^{(n)}\) are invertible. This leads to the study of the quantum algebra \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\) given below.

4.3. The quantum Schubert cell \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\). The interest in studying the algebra \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\) stems from the fact that the generating series of certain functions defined on this quantum algebra satisfies the same relation (22) as the generators \(u_i, i \in \mathbb{N}^r\) in \(\mathbb{C}[H_- \setminus B_-]_q\), and that we can deduce from (10), the existence of an algebra morphism from \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\) to \(A_q\).

**Definition.** The algebra \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\) is given by generators \(a_{i,j}^{(k)} (i, j \in \{1, 2\}; k \in \{0, \ldots, n\})\), \(a_{1,1}^{(n-1)}, a_{1,2}^{(n-1)}, a_{2,1}^{(n)}, a_{2,2}^{(n)}\), and relations (37), (38), (39) with \(a_{i,j}^{(1)}(\lambda) = \sum_{k=0}^{n} a_{i,j}^{(k)} \lambda^k\) and \(a_{1,1}^{(n)} = a_{1,2}^{(n)} = 0\) as well as relations which express the fact that \(a_{1,1}^{(n-1)}\) (resp. \(a_{1,2}^{(n-1)}, a_{2,1}^{(n-1)}, a_{2,2}^{(n-1)}\)) is an inverse for \(a_{1,1}^{(n-1)}\) (resp. \(a_{1,2}^{(n-1)}, a_{2,1}^{(n)}, a_{2,2}^{(n)}\)).

At the semi-classical limit, for \(q \to 1\), we get the algebra of functions on the Schubert cell \((B_- w_n B_- \cap N_+, P)\) with \(w_n = \text{diag}(\lambda^{-n}, \lambda^n)\) with a Poisson structure given by the fact that \((B_- w_n B_- \cap N_+, P)\) may be viewed as a symplectic leaf of the Poisson manifold \((N_+, P)\). The algebra \(\mathbb{C}[B_- w_n B_- \cap N_+]_q\) is just a rough quantum deformation of \((B_- w_n B_- \cap N_+, P)\). To obtain an exact quantum deformation, one should impose relations between \(a_{1,1}^{(n-1)}, a_{1,2}^{(n-1)}, a_{2,1}^{(n)}, a_{2,2}^{(n)}\) and all the \(a_{i,j}^{(k)}\) on the definition. However, we don’t need to be so precise and our definition will suffice. There is a natural morphism \(p : \mathbb{C}[N_+]_q \to \mathbb{C}[B_- w_n B_- \cap N_+]_q\). Moreover, if \(C\) is an algebra and if \(f : \mathbb{C}[N_+]_q \to C\) is an algebra morphism such that \(a_{1,1}^{(r-1)}, a_{1,2}^{(r-1)}, a_{2,1}^{(r)}, a_{2,2}^{(r)} \in \text{Ker} f\) for \(r > n\), and \(f(a_{1,1}^{(n-1)}), f(a_{1,2}^{(n-1)}), f(a_{2,1}^{(n)}), f(a_{2,2}^{(n)})\) are invertible in \(C\), then there exists \(g : \mathbb{C}[B_- w_n B_- \cap N_+]_q \to C\) an algebra morphism such that the following diagram is commutative:
\[
\begin{array}{ccc}
\mathbb{C}[N_+]_q & \xrightarrow{p} & \mathbb{C}[B_- w_n B_- \cap N_+]_q \\
\downarrow f & & \downarrow g \\
C & & C
\end{array}
\]
By virtue of (41), it follows that there exists an algebra morphism \( h_n : \mathbb{C}[B_w B \cap N_+]_q \rightarrow A_q \) such that:

\[
h_n(L(\lambda)) = \prod_{i=1}^{n} \left( \begin{array}{ccc} 1 & 0 \\ \lambda x_i & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 1 \\ 0 & 1 \end{array} \right) \tag{41}
\]

with \( L(\lambda) = [a_{i,j}(\lambda)]_{ij \in \{1,2\}} \) and \( a_{i,j}(\lambda) = \sum_{k=0}^{n} a_{i,j}^{(k)} \lambda^k \). The element \( a_{2,2}(\lambda) \) is invertible in the ring \( \mathbb{C}[B_{-w} B \cap N_+]_q((\lambda^{-1})) \). We set \( \alpha(\lambda) := a_{2,2}(\lambda)^{-1}a_{2,1}(\lambda) \).

**Lemma 5.** The function \( \alpha(\lambda) \) satisfies the same relation (22) as the function \( u(\lambda) \). We have:

\[
(\lambda^{-1}\alpha(\lambda) - \mu^{-1}\alpha(\mu)) (\alpha(\lambda) - \alpha(\mu)) = q(\alpha(\lambda) - \alpha(\mu)) (\lambda^{-1}\alpha(\lambda) - \mu^{-1}\alpha(\mu)).
\]

**Proof.** By definition, for two elements \( a \) and \( b \) of an algebra, We denote by \([a,b]\) the commutator \( ab - ba \). Then,

\[
\begin{align*}
[\alpha(\lambda), \alpha(\mu)] &= a_{2,2}(\lambda)^{-1} [a_{2,1}(\lambda), a_{2,2}(\mu)^{-1}] a_{2,1}(\mu) + [a_{2,2}(\lambda)^{-1}, a_{2,2}(\mu)^{-1}] a_{2,1}(\lambda) a_{2,1}(\mu) \\
&\quad + a_{2,2}(\mu)^{-1} [a_{2,2}(\lambda)^{-1}, a_{2,1}(\lambda), a_{2,1}(\mu)] + a_{2,2}(\mu)^{-1} [a_{2,2}(\lambda)^{-1}, a_{2,1}(\mu)] a_{2,1}(\lambda).
\end{align*}
\]

Relation (38) shows that

\[
\forall i, j \in \{1,2\}, \quad [a_{i,j}(\lambda), a_{i,j}(\mu)] = 0 \tag{42}
\]

and

\[
[a_{2,1}(\lambda), a_{2,2}(\mu)] = [a_{2,1}(\mu), a_{2,2}(\lambda)] \tag{43}
\]

\[
= (1 - q^{-1}) (\mu a_{2,2}(\mu) a_{2,1}(\lambda) - \lambda a_{2,2}(\lambda) a_{2,1}(\mu)). \tag{44}
\]

So, thanks to (42) and (43), we get

\[
\begin{align*}
[\alpha(\lambda), \alpha(\mu)] &= a_{2,2}(\lambda)^{-1} a_{2,2}(\mu)^{-1} [a_{2,1}(\lambda), a_{2,2}(\mu)] (a_{2,2}(\lambda)^{-1} a_{2,2}(\mu)^{-1} a_{2,1}(\lambda) - a_{2,2}(\mu)^{-1} a_{2,1}(\mu)) \\
&= (1 - q^{-1}) (\mu\alpha(\lambda) - \lambda\alpha(\mu)) (\alpha(\lambda) - \alpha(\mu)). \tag{45}
\end{align*}
\]

The result follows from this last equality.

Let us see now what is the image of \( \alpha(\lambda) \) by the map \( h_n \).

**Lemma 6.** We have \( h_n(\alpha(\lambda)) = \frac{y_n^{-1}}{1 + \frac{(\lambda x_n y_n)^{-1}}{1 + \frac{(\lambda y_n^{-1} x_n)^{-1}}{1 + \frac{\cdot \cdot \cdot (\lambda y_1 x_2)^{-1}}{1 + (\lambda x_1 y_1)^{-1}}}}} \), with \( \frac{a}{b} := b^{-1}a \) for two elements \( a \) and \( b \) such that \( b \) is invertible.

**Proof.** Let us define elements \( a_n, b_n, c_n, d_n \) in \( A_q[\lambda] \) such that \( h_n(L(\lambda)) = \left( \begin{array}{ccc} a_n & b_n \\ c_n & d_n \end{array} \right) \). Then, thanks to (41), we have \( h_n(a_{2,1}(\lambda)) = c_n \), \( h_n(a_{2,2}(\lambda)) = d_n \) and \( h_n(\alpha(\lambda)) = d_n^{-1} c_n \). If \( n = 1 \) then \( d_1 = 1 + \lambda(x_1 y_1) \) is invertible in \( A_q((\lambda^{-1})) \) and \( h_1(\alpha(\lambda)) = (1 + (\lambda x_1 y_1)^{-1})^{-1} y_1^{-1} \). Let us make the hypothesis that \( n > 1 \). We have:

\[
\begin{align*}
c_n &= c_{n-1} + \lambda d_{n-1} x_n \\
d_n &= c_{n-1} y_n + d_{n-1} (\lambda x_n y_n + 1). \tag{47}
\end{align*}
\]
Therefore, the computation shows that

\[ h_n(\alpha(\lambda)) = \left(1 + (\lambda x_n y_n)^{-1} \right) \left(1 + q^{-1}(d_n^{-1} c_{n-1} y_{n-1}) (\lambda y_n x_n)^{-1}\right)^{-1} y_n^{-1} \]

Therefore, if we set \( V_n := h_n(\alpha(\lambda)) y_n \) and \( W_n^{(k)} := \left(1 + q^k V_{n-1} (\lambda y_{n-1} x_n)^{-1}\right)^{-1} \), we see that

\[ V_n = \left(1 + (\lambda x_n y_n)^{-1} W_n^{(-1)}\right)^{-1}. \]

By induction on \( n \), we deduce that for all integers \( k \) with \( k > n \), \((x_k y_k) \) and \( V_n \) commute. Hence, by induction on \( p \),

\[ \forall p \in \mathbb{N}, \quad (\lambda x_n y_n)^{-p} W_n^{(k)} = W_n^{(k+p)} (\lambda x_n y_n)^{-p}. \]

So,

\[ V_n = \left(1 + W_n^{(0)} (\lambda x_n y_n)^{-1}\right)^{-1} \]

\[ = \left(1 + (1 + V_{n-1} (\lambda y_{n-1} x_n)^{-1}) (\lambda x_n y_n)^{-1}\right)^{-1}. \]

The result follows from this by induction on \( n \).

Set \( \alpha(\lambda) = \sum_{i=0}^{\infty} (-1)^i \alpha_{i+1} \lambda^{-i} \) and let us see what the images of \( \alpha_i \) in \( A_q \) are. For that, we shall need the following proposition.

**Proposition 8.** Let \( N \) be an integer and \( I_N \) the ideal in \( A_N := \mathbb{C}[q,q^{-1}]\{\{t_1, \ldots, t_N\}\} \) generated by elements \( t_it_{i+1} - qt_{i+1}t_i \) for \( i \in \{1, \ldots, N\} \) and \( t_i t_j - t_j t_i \) for \( |i - j| \geq 2 \). Then, in the ring \( A_N/I_N \), we have:

\[ \left(1 - \left(1 - (1 - t_N)^{-1} t_{N-1}\right)^{-1} t_2\right)^{-1} t_1^{-1} = \sum_{\alpha_1, \ldots, \alpha_N} F_q(\alpha_1, \ldots, \alpha_N) t_N^{\alpha_N} \ldots t_1^{\alpha_1}. \]

We recall that the function \( F_q \) has been defined in section \[ . \]

**Proof.** We note \( F_N \) the quantum fraction in the left hand side, \( v_N \) the valuation on \( A_N \) corresponding to the gradation given by \( \deg t_j \) for all \( j \), and \( i_N \) the valued injection from \( A_N \) to \( A_{N+1} \) given by \( i_N(t_j) = t_{j+1} \). If \( N = 1 \), the result is obvious. Let us assume that the property is true until the rank \( N \). Then, \( v_{N+1}(i_N(F_N) t_1) \geq 1 \) for \( v_N(F_N) \geq 0 \). So, \( 1 - i_N(F_N) t_1 \) is invertible in \( A_{N+1} \) and \( F_{N+1} \) exists. Set \( t'_1 = t_1, \ldots, t'_{N-1} = t_{N-1} \) and \( t'_N = (1 - t_{N+1})^{-1} t_N \). Then, the \( t'_j \) satisfy the same relations as the generators \( t_j \) in \( A_N/I_N \). Moreover,

\[ F_{N+1} = \left(1 - \left(1 - (1 - t_{N+1})^{-1} t_{N-1}\right)^{-1} t_2\right)^{-1} t_1^{-1}. \]

So, the induction hypothesis implies that \( F_{N+1} = \sum_{\alpha_1, \ldots, \alpha_N} F_q(\alpha_1, \ldots, \alpha_N) t_N^{\alpha_N} \ldots t_1^{\alpha_1} \). On the other hand, an induction on \( k \) shows that:

\[ [(1 - t_{N+1})^{-1} t_N]^k = (1 - t_{N+1})^{-1}(1 - q^{k-1} t_{N+1})^{-1} t_N^k. \]

Therefore,

\[ F_{N+1} = \sum_{\alpha_1, \ldots, \alpha_N} F_q(\alpha_1, \ldots, \alpha_N)(1 - t_{N+1})^{-1}(1 - q^{\alpha_N-1} t_{N+1})^{-1} t_N^{\alpha_N} \ldots t_1^{\alpha_1}. \]
The result follows from the classical relation :
\[
(1-q^st)^{-1} = \sum_{k \geq 0} \left[ \frac{N+k-1}{k} \right] t^k.
\]

Lemma 6 and Proposition 8 allow us to obtain explicitly images of \( \alpha_i \) by \( h_n \).

**Corollary 1.** We have \( h_n(\alpha_i) = \sum F_q(\alpha_1, \ldots, \alpha_{2n-1})(x_1y_1)^{-\alpha_2n-1} \cdots (x_ny_n)^{-\alpha_1y_n^{-1}} \), the sum being taken on all integers \( \alpha_1, \ldots, \alpha_{2n-1} \) such that \( \alpha_1 + \cdots + \alpha_{2n-1} = i - 1 \).

The fact that the \( \alpha_i \) satisfy relation (22) leads to the study of the following quantum algebra.

### 4.4. The quantum homogeneous space \( \mathbb{C}[S_\infty \backslash B_-]_q \).

Let \( S_\infty \) be the sub-group of \( B_- \) constituted by all lower triangular matrices of the form \( \begin{pmatrix} a & \lambda^{-1}b \\ 0 & a^{-1} \end{pmatrix} \).

**Definition.** We denote by \( \mathbb{C}[S_\infty \backslash B_-]_q \) the algebra given by generators \( u_i \), \( i \in \mathbb{N}^* \), and relation (22) :
\[(\lambda^{-1}u(\lambda) - \mu^{-1}u(\mu))(u(\lambda) - u(\mu)) = q(u(\lambda) - u(\mu))(\lambda^{-1}u(\lambda) - \mu^{-1}u(\mu)) \text{ with } u(\lambda) = \sum_{i=0}^{\infty} (-1)^i u_{i+1} \lambda^{-i}.
\]

We can check that relations coming from (22) are equivalent to the equalities
\[\forall i < j, \quad [u_i, u_j] = (1-q^{-1}) \sum_{k=i}^{i+j-1} u_k u_{i+j-k}. \tag{49}\]

The algebra \( \mathbb{C}[S_\infty \backslash B_-]_q \) is a graded algebra with the gradation given by \( \deg u_i = 1 \) for all \( i \). Thanks to Lemma 6, there exists a specialization morphism :
\[r : \mathbb{C}[S_\infty \backslash B_-]_q \rightarrow \mathbb{C}[B_- w_n B_- \cap N_+]_+. \tag{50}\]

We set \( h_n' = T^{-n} \circ h_n \circ r \) where \( T \) is the translation automorphism on \( A_q \). Thanks to Corollary 1, for all integers \( i, j, m \) with \( i \leq 2n \) and \( i \leq 2m \) we have \( h_n'(u_i) = h_n'(u_j) \). Now, if we take into account Lemma 3 we deduce the existence of a graded algebra morphism
\[h : \mathbb{C}[S_\infty \backslash B_-]_q \rightarrow A_q \]
\[u(\lambda) \mapsto \lim_{N \rightarrow \infty} \frac{y_0^{-1}}{1 + \frac{(\lambda x_0 y_0)^{-1}}{1 + \frac{(\lambda y^{-1} x_0)^{-1}}{1 + \frac{\cdots (\lambda y_{-N+1} x_{-N+2})^{-1}}{1 + (\lambda x_{-N+1} y_{-N+1})^{-1}}}}} \tag{51}\]

with the convention that \( \frac{a}{b} = b^{-1}a \) if \( b \) is invertible. Explicitly, the image of \( u_i \) by \( h \) is given by the formula :
\[h(u_i) = \sum F_q(a_1, a_2, \ldots) (x_k y_{-k})^{-a_{2k+1}/2} (y_{-k} x_{-k+1})^{-a_{2k}} \cdots (x_0 y_0)^{-\alpha_1} y_0^{-1}, \tag{52}\]
the sum being taken on all integers \( a_i \) such that \( \sum_k a_k = i - 1 \). For example, \( h(u_1) = y_0^{-1} \) and \( h(u_2) = (x_0 y_0)^{-1} y_0^{-1} \). We note that for all integers \( i \) and \( n \) with \( i \leq 2(n - 1) \) we have \( h_n(\alpha_i) = T^m h(u_i) \).

**Proposition 9.** A basis for \( \mathbb{C}[S_\infty \setminus B_-]_q \) is given by the family \( \xi_a := \prod_{i=1}^{\infty} u_i^{a_i} \) where \( a = (a_i)_{i \in \mathbb{N}} \).\( \square \)

**Proof.** Thanks to (43), \( \mathbb{C}[S_\infty \setminus B_-]_q \) is spanned by the family \( \xi_a \). But this set of vectors is also free. Indeed, this is a consequence of (1) the existence of \( h \) given above, (2) the fact that in each new element of the sequence \( h(u_i) \) occurs one and only new element of the form \( x_m \) or \( y_m \) (according to the parity of \( i \)) and (3) the fact that the family \( \prod x_i^{a_i} y_i^{b_i} \) forms a basis of \( A_q \) where \( (a_i) \) and \((b_i)\) are almost zero sequences in \( \mathbb{Z}^\mathbb{Z} \).

Note that the proof of Proposition 9 shows the following result.

**Corollary 2.** The algebra morphism \( h \) is injective.

In the classical case, when \( q \to 1 \), we see from (52), that \( h \) is a birational map. We can also deduce from Proposition 9 that \( \mathbb{C}[S_\infty \setminus B_-]_q \) is a quantum deformation of the algebra of functions on the Poisson manifold \( S_\infty \setminus B_- \) equipped with the Poisson structure induced by the field of bivectors \( r^L - r^R \) (52).

**4.5. End of the proof.** It is based on Proposition 9 and Lemma 7 which show together that \( \mathbb{C}[H_- \setminus B_-]_q \) is in a way the quantum “double” of \( \mathbb{C}[S_\infty \setminus B_-]_q \).

**Definition.** We note \( \mathbb{C}[S_\infty \setminus B_-]_q^+ \) the algebra given by generators : \( m_i, i \in \mathbb{N}^* \) and relations : (23), i.e., \( (\lambda^{-1} m(\lambda) - \mu^{-1} m(\mu)) (m(\lambda) - m(\mu)) = q^{-1} (m(\lambda) - m(\mu)) (\lambda^{-1} m(\lambda) - \mu^{-1} m(\mu)) \) with \( m(\lambda) = \sum_{i=q}^{\infty} (-1)^i m_i \lambda^{-i} \). We note also by \( \deg \) the gradation on \( \mathbb{C}[S_\infty \setminus B_-]_q^+ \) defined by \( \deg m_i = -1 \) for all \( i \) and by \( \varphi^+ \) the anti-isomorphism of algebras :

\[
\varphi^+ : \mathbb{C}[S_\infty \setminus B_-]_q \longrightarrow \mathbb{C}[S_\infty \setminus B_-]_q^+ \forall i \in \mathbb{N}^*, \quad u_i \mapsto m_i.
\]

The map \( \varphi^+ \) is an anti-graded involution. On the other hand, there exists also an involution \( \varphi \) on \( A_q \) which is an algebras anti-graded anti-automorphism defined by :

\[
\varphi : A_q \longrightarrow A_q \quad \forall i \in \mathbb{Z}, \quad x_i \mapsto y_{1-i}, \quad y_i \mapsto x_{1-i}.
\]

By considering the map \( \varphi \circ h \circ \varphi^+ \), we deduce the existence of a graded algebra morphism

\[
h_+ : \mathbb{C}[S_\infty \setminus B_-]_q^+ \longrightarrow A_q \quad m(\lambda) \mapsto \lim_{N \to \infty} \frac{x_1^{-1}}{1 + \frac{(\lambda x_1 y_1)^{-1}}{1 + \frac{(\lambda y_1 x_2)^{-1}}{1 + \frac{(\lambda x_2 y_2)^{-1}}{1 + \frac{(\lambda y_2 x_3)^{-1}}{1 + \ldots}}}}}
\]

(55)
with the convention that \( \frac{a}{b} = ab^{-1} \). Explicitly, thanks to Proposition 3, we get
\[
h_+(m_i) = \sum F_q(\alpha_1, \alpha_2, \ldots) x_1^{-1}(x_1y_1)^{-\alpha_1} \ldots (x_ky_k)^{-\alpha_{2k-1}}(y_kx_{k+1})^{-\alpha_{2k}} \ldots
\]
As usual, the sum is taken on all almost zero sequences \((a_k)\) such that \(\sum_k a_k = i - 1\). Here also, in each new term of the sequence \(m_i\) occurs one and only one new variable of the form \(x_k\) or \(y_k\) (according to the parity of \(i\)). Therefore, the same argument as before shows the two following results.

**Proposition 10.** A basis for \(\mathbb{C}[S_\infty B_-]^+_q\) is given by the family \(\eta_a := \prod_{i=1}^\infty m_i^{a_i}\) where \(a = (a_i)_{i \in \mathbb{N}^*}\) denotes any almost zero sequence of integers.

**Corollary 3.** The map \(h_+\) is injective.

In the classical case, \(h_+\) is also a birational isomorphism from \(\mathbb{C}[S_\infty B_-]^+_q\) to \(\mathbb{C}[x_1^{\pm 1}, y_1^{\pm 1}, i > 0]\). Let us consider again the quantum algebra \(\mathbb{C}[H_- B_-]_q\) defined in Section 4.

**Lemma 7.** The natural map
\[
\mathbb{C}[H_- B_-]_q \rightarrow \mathbb{C}[S_\infty B_-]_q \otimes \mathbb{C}[S_\infty B_-]_q^+
\forall i, \quad u_i \mapsto u_i \otimes 1
\]
\[
m_i \mapsto 1 \otimes m_i
\]
is a graded algebra isomorphism.

**Proof.** It is enough to construct the inverse. But, there exist natural morphisms \(f\) and \(f^+\) from \(\mathbb{C}[S_\infty B_-]_q\) and \(\mathbb{C}[S_\infty B_-]_q^+\) to \(\mathbb{C}[H_- B_-]_q\). These morphisms \(q^{-1}\)-commute. So, there exists \(f \otimes f^+ : \mathbb{C}[S_\infty B_-]_q \otimes \mathbb{C}[S_\infty B_-]_q^+ \rightarrow \mathbb{C}[H_- B_-]_q\). One can check that this gives an inverse for the studied map.

Hence, by virtue of Proposition 3 and 4, we deduce that \(\mathbb{C}[H_- B_-]_q\) is indeed a flat deformation of \(\mathbb{C}[H_- B_-]\). This completes the proof of Proposition 3. Note that \(\mathbb{C}[H_- B_-]_q\) is a graded algebra with the gradation given by \(\text{deg } u_i = -\text{deg } m_i\) for all \(i\) and that a basis of \(\mathbb{C}[H_- B_-]_q\) is given by the family \(\prod_{i=1}^\infty u_i^{a_i} \prod_{j=1}^\infty m_j^{b_j}\) where \((a_i)\) and \((\beta_j)\) are two almost zero sequences of integers.

5. **Quantum Drinfeld-Sokolov Correspondence**

The aim of this section is to prove Theorem 1. As a result of the previous section, we have already proved the existence of \(DS_q\). Indeed, it suffices to consider the morphisms \(h\) and \(h_+\) seen in (51) and (52), to note that \(\text{Im } h \subset \mathbb{C}[x_i^{\pm 1}, y_i^{\pm 1}, i \leq 0]_q\), \(\text{Im } h_+ \subset \mathbb{C}[x_i^{\pm 1}, y_i^{\pm 1}, i > 0]_q\) and to take into account Lemma 3 together with the isomorphism \(A_q \simeq \mathbb{C}[x_i^{\pm 1}, y_i^{\pm 1}, i \leq 0]_q \otimes \mathbb{C}[x_i^{\pm 1}, y_i^{\pm 1}, i > 0]_q\). Note that the injectivity of \(h\) and of \(h_+\) imply the one of \(DS_q\). It remains to prove the equality \(DS_q \circ H_\mu = \text{ad}(I_\mu) \circ DS_q\). For that, the idea is first to prove the existence of \(H_n\) (this will be achieved in subsection 5.4) and then, using the embedding of \(\mathbb{C}[H_- B_-]_q\) into \(A_q\), to extend \(H_n\) not only on \(A_q\) but also on \(A_q\) the algebra obtained from \(A_q\) by adding the two half screening charges \(\sum_\pm\) of the discrete sine-Gordon system. The interest in considering this algebra is that
it is endowed with an $U_q\widehat{\mathfrak{b}}_-$-module-algebra structure, where $\widehat{\mathfrak{b}}_-$ is a Borel subalgebra of $\widehat{\mathfrak{sl}}_2$.
Moreover, the adjoint actions of integrals of motion extend to $A_q$ and commute with the action of $U_q\widehat{\mathfrak{b}}_-$. Conversely, each derivation which commutes with the action of $U_q\widehat{\mathfrak{b}}_-$ is the adjoint action of an integral of motion. We shall use this fact to complete the proof. First, we start by giving precise definitions of the quantum group $U_q\widehat{\mathfrak{b}}_-$ and algebras $\overline{A}_q$ and $\mathbb{C}[H_- \setminus B_- N_+]_q$.

**Definition.** Let $\widehat{\mathfrak{b}}_-$ be a Borel subalgebra of $\widehat{\mathfrak{sl}}_2$. We note $U_q\widehat{\mathfrak{b}}_-$ the quantum group given by generators $:k^\pm_\varepsilon, f_{\varepsilon',\varepsilon}:$, with $\varepsilon, \varepsilon' \in \{+, -\}$ and relations:

$$k^\pm_\varepsilon k^\mp_\varepsilon = 1 \quad (57)$$

$$k^\pm_\varepsilon k^\pm_\varepsilon = k^\pm_\varepsilon \quad (58)$$

$$k^\pm_\varepsilon f_{\varepsilon',\varepsilon} k^\mp_\varepsilon = q^{\alpha_\varepsilon,\varepsilon'} f_{\varepsilon',\varepsilon} \quad (59)$$

together with the quantum Serre relations between $f_\pm$ and $f_\mp$:

$$f_\pm^2 f_\mp - (q + 1 + q^{-1})(f_\pm^2 f_\mp f_\pm - f_\pm f_\mp f_\pm^2) - f_\mp f_\pm^3 = 0 \quad (60)$$

**We shall use neither the antipode nor the co-unit in this article.**

5.1. **The extended phase space $A_q$.** If we consider only a finite number of sites $x^\pm_i, y^\pm_i, i \in \{1, \ldots, n\}$, it can be shown from Lemma 4 that we get a $U_q\widehat{\mathfrak{b}}_-$-module-algebra. For all $x$ homogeneous with respect to $\deg$, the formulas are the following:

$$f_+ x = \sum_{i=1}^n [x_i, x]_q \quad (63)$$

$$f_- x = \sum_{i=1}^n [y_i, x]_q \quad (64)$$

$$k_\pm x = q^{\pm \deg x} x \quad (65)$$

If we consider now an infinite number of sites at the left of an arbitrary site $x^\pm_i, y^\pm_i, i \leq N$, we also obtain a $U_q\widehat{\mathfrak{b}}_-$-module-algebra. For that, we set:

$$f_+ x = \sum_{i \leq N} [x_i, x]_q \quad (66)$$

$$f_- x = \sum_{i \leq N} [y_i, x]_q \quad (67)$$

$$k_\pm x = q^{\pm \deg x} x \quad (68)$$

for any $x$ homogeneous with respect to $\deg$. This follows from the fact that for all $x \in A_q, x_i$ and $x$ $q$-commute provided that $i$ is small enough. However, if we consider the whole algebra $A_q$, there is no longer a $U_q\widehat{\mathfrak{b}}_-$-module-algebra structure on it. For that, it is necessary to add the half screening charges $\Sigma_+$ and $\Sigma_-$ which correspond heuristically to $\sum_{i>0} x_i$ and $\sum_{i>0} y_i$. 
Definition. We note $\tilde{A}_q$ the algebra given by generators: $\Sigma_+, \Sigma_-, x_i^\pm, y_i^\pm, i \in \mathbb{Z}$ and relations:

$$\forall i < j, \quad x_ix_j = qx_jx_i$$
$$y_iy_j = qy_jy_i$$
$$y_ix_j = q^{-1}x_jy_i$$

$$\forall i \leq j, \quad x_iy_j = q^{-1}y_jx_i$$

$$\forall i \in \mathbb{Z}, \quad x_i\Sigma_+ - q\Sigma_+x_i = \sum_{j=1}^{i}[x_i, x_j]_q$$
$$x_i\Sigma_- - q^{-1}\Sigma_-x_i = \sum_{j=1}^{i}[x_i, y_j]_q$$
$$y_i\Sigma_+ - q^{-1}\Sigma_+y_i = \sum_{j=1}^{i}[y_i, x_j]_q$$
$$y_i\Sigma_- - q\Sigma_-y_i = \sum_{j=1}^{i}[y_i, y_j]_q$$

(69)

$$\Sigma_+^{\pm}\Sigma_+ = (q + 1 + q^{-1}) (\Sigma_+^{\pm}\Sigma_-^{\pm} - \Sigma_+^{\pm}\Sigma_-^{\pm}) - \Sigma_+^{\pm}\Sigma_+^{\pm} = 0.$$  

(70)

As usual, for two elements $a$ and $b$, $[a, b]_q$ denotes the $q$-commutator of $a$ and $b$. The gradation $\deg$ on $\tilde{A}_q$ is given by:

$$\forall i \in \mathbb{Z}, \quad \deg x_i = \deg \Sigma_+ = -\deg y_i = -\deg \Sigma_- = 1.$$ 

The following result can be proved easily.

Lemma 8. A basis for $\tilde{A}_q$ is given by the family $\prod_{i=-\infty}^{\infty} x_i^{\alpha_i} y_i^{\beta_i} u$ where $u$ belongs to a basis $B$ of $C[\Sigma_+, \Sigma_-]_q \simeq U_q \hat{\mathfrak{b}}_-$ and $(\alpha_i), (\beta_i)$ are two almost zero sequences in $\mathbb{Z}^\mathbb{Z}$.

Hence, thanks to subsection 2.1, we get the following lemma.

Lemma 9. There is a natural graded algebra embedding $A_q \hookrightarrow \tilde{A}_q$. This embedding identifies generators $x_i^\pm, y_i^\pm, i \in \mathbb{Z}$ with the ones of $\tilde{A}_q$.

The semi-classical limit of $\tilde{A}_q$ is $C\{x_i^\pm, y_i^\pm, i \in \mathbb{Z}, \Sigma_0, \{\Sigma_0, \Sigma_1\}, \{\Sigma_0, \Sigma_2\}, \ldots, \epsilon_k \in \{+, -\}\}$. We note this algebra $A_{cl}$. Let us remark that it is possible to extend the half-translation automorphism $T^{\pm}\Sigma_+^{\pm}$ on $A_{cl}$ by setting $T^{\pm}\Sigma_+^{\pm} = \Sigma_-^{\pm}$ and $T^{\pm}\Sigma_-^{\pm} = \Sigma_+^{\pm} - x_1$. It can be shown that $A_{cl}$ is the localized of a subalgebra of a projective limit of algebras. Explicitly, these algebras are the ones generated by variables $x_i$ and $y_i$ for $i \leq n$ with obvious projection morphisms. The considered subalgebra is the one generated by $x_i, y_i, i \in \mathbb{Z}$ and the half-screening charges $\Sigma_+$ and $\Sigma_-$ identified with $(x_1, x_1 + x_2, \ldots)$ and $(y_1, y_1 + y_2, \ldots)$. The multiplicative set is generated by elements $x_i^{-1}$ and $y_i^{-1}$ for $i \in \mathbb{Z}$. It satisfies the Ore relation $[D]$. Therefore, we can deduce from formulas (66), (67), (68) that there exists a $U_q \hat{\mathfrak{b}}_-$-module-algebra structure on $\tilde{A}_q$ given by:

$$f_+.x = [\Sigma_+, x]_q = \sum_{i=-\infty}^{0} [x_i, x]_q + [\Sigma_+, x]_q$$  

(71)

$$f_- . x = [\Sigma_-, x]_q := \sum_{i=-\infty}^{0} [y_i, x]_q + [\Sigma_-, x]_q$$

(72)

$$k_\pm . x = q^{\pm \deg x} x.$$  

(73)
At the semi-classical limit, it also gives a $U\hat{\mathfrak{b}}_-$-module-algebra structure on $\hat{A}_{cl}$.

5.2. The quantum homogeneous space $\mathbb{C}[H_- \setminus B_- N_+]_q$. Geometrically, at the classical level, adding half screening charges is the same as studying the Schubert cell $B_- N_+$ of $G$ instead of its Borel group $B_-$.  

**Definition.** We denote by $\mathbb{C}[H_- \setminus B_- N_+]_q$ the quantum algebra given by generators : $\Sigma_+, \Sigma_-, u_i, m_i$, $i \in \mathbb{Z}$ and relations : 

$$
\begin{align}
(\lambda^{-1}u(\lambda) - \mu^{-1}u(\mu))(u(\lambda) - u(\mu)) = q(u(\lambda) - u(\mu))(\lambda^{-1}u(\lambda) - \mu^{-1}u(\mu)) \\
(\lambda^{-1}m(\lambda) - \mu^{-1}m(\mu))(m(\lambda) - m(\mu)) = q^{-1}(m(\lambda) - m(\mu))(\lambda^{-1}m(\lambda) - \mu^{-1}m(\mu))
\end{align}
$$

$$u(\lambda)m(\mu) = q^{-1}m(\mu)u(\lambda)$$

$$u(\lambda)\Sigma_\pm = q^{\pm 1}\Sigma_\pm u(\lambda)$$

$$m(\lambda)\Sigma_\pm = q^{-\pm 1}\Sigma_\pm m(\lambda) = 1 - q^{-1}$$

$$m(\lambda)\Sigma_\mp - q\Sigma_\mp m(\lambda) = -(q - 1)\lambda^{-1}m(\lambda)^2$$

$$\Sigma_\pm^3 - (q + 1 + q^{-1})\Sigma_\mp\Sigma_\pm^2 - \Sigma_\pm\Sigma_\mp\Sigma_\pm - \Sigma_\pm^2\Sigma_\pm = 0$$

with the same notation as before i.e., $u(\lambda) = \sum_{i=0}^\infty (-1)^iu_{i+1}\lambda^{-i}$ and $m(\lambda) = \sum_{i=0}^\infty (-1)^im_{i+1}\lambda^{-i}$.

The algebra $\mathbb{C}[H_- \setminus B_- N_+]_q$ is graded with the gradation given by:

$$\forall i \in \mathbb{Z}, \quad \text{deg } u_i = \text{deg } \Sigma_\pm = -\text{deg } m_i = -\text{deg } \Sigma_\mp = 1.$$  

The relations between $u(\lambda)$, $m(\lambda)$ and $\Sigma_\pm$ will appear to be natural when we prove the following proposition which claims the existence of the morphism $DS_q$.

**Proposition 11.** The map

$$DS_q : \mathbb{C}[H_- \setminus B_- N_+]_q \rightarrow \hat{A}_{cl}$$

$$\forall i \in \mathbb{N}^*, \quad u_i \mapsto DS_q(u_i)$$

$$m_i \mapsto DS_q(m_i)$$

$$\Sigma_\pm \mapsto \Sigma_\pm$$

exists and defines a graded algebra morphism.

**Proof.** We need to prove some compatibility relations. The ones dealing with $DS_q(u_k)$, $k \in \mathbb{N}^*$ and $\Sigma_\pm$ follow from the fact that all the terms in $DS_q(u_k)$ are sums and products of $x_i^{\pm 1}$ and $y_i^{\pm 1}$ for $i \leq 0$. The ones dealing with $DS_q(m_k)$ and $\Sigma_\pm$ can be handled in the following way. If we take again the involution $\varphi$ defined in (54), we have $DS_q(m_k) = \varphi(DS_q(u_k))$. So, for any integer $n$ large enough,

$$[DS_q(m_k), \Sigma_+]_q = [DS_q(m_k), \sum_{i=1}^n x_i]_q = \varphi\left(\sum_{i=-n+1}^0 y_i, DS_q(u_k)\right)$$

$$= (T^n \circ \varphi)(\sum_{i=1}^n y_i, T^n(DS_q(u_k)))_q.$$. 
Recall the notation of subsection 4.3 and in particular the morphism $h_n : \mathbb{C}[B_-w_nB_- \cap N_+]_q \rightarrow A_q$, we have $T^n DS_q(u_k) = h_n(\alpha_k)$ and $h_n(a^{(0)}_{1,2}) = \sum_{i=1}^{n} y_i$. Then the result comes from commuting relations in $\mathbb{C}[B_-w_nB_- \cap N_+]_q$. We prove the compatibility relation between $DS_q(m_k)$ and $\sum_-$ using a similar method.

From the commuting relations in $\mathbb{C}[H_- \setminus B_-N_+]_q$ together with the results of section 4, we can deduce the following corollary.

**Corollary 4.** If $\mathcal{B}$ denotes a basis for $\mathbb{C}[\sum_-, \sum_+]_q \simeq \mathcal{U}_q \mathcal{h}_-$, then the family $\prod_{i=1}^{\infty} u_i^{\alpha_i} \prod_{j=1}^{\infty} m_j^{\beta_j} u$ where $(\alpha_i)$ and $(\beta_j)$ are two almost zero sequences of integers and $u \in \mathcal{B}$ is a basis for $\mathbb{C}[H_- \setminus B_-N_+]_q$. We also obtain the following result.

**Corollary 5.** The morphism $\overline{DS}_q$ is injective.

**Proof.** This is a consequence of Corollary 4, Corollary 8 and of the already seen fact that each new term of the sequence $u_k$ (resp. $m_k$) gives a new variable $x_{-i}$ (resp. $x_i$) or $y_{-i}$ (resp. $y_i$), $i \in \mathbb{N}$ according to the parity of $k$. □

Corollary 4 also shows that $\mathbb{C}[H_- \setminus B_-N_+]_q$ is a flat deformation of the function algebra of the Poisson manifold $(H_- \setminus B_-N_+, P_\infty)$. Poisson relations on this manifold show that we obtain a quantum deformation. Moreover, the map:

$$F : \mathbb{C}[\lambda^{-1}] \times \mathbb{C}[\lambda^{-1}] \times N_+ \rightarrow (H_- \setminus B_-, N_+),$$

$$(u_{cl}(\lambda), v_{cl}(\lambda), n_+) \mapsto (1, \lambda, v_{cl}(\lambda)) = \left(1, \frac{1}{v_{cl}(\lambda)}, 0\right), n_+)$$

is a bijection and the elements $\sum_+$ and $\sum_-$ correspond classically to the functions $a_{2,1}^{(1)}$ and $a_{1,2}^{(0)}$ on $N_+$. On the other hand, by virtue of subsection 4.1, classical limits of $u_i$ and $m_i$ correspond to coordinate functions with generating functions $u_{cl}(\lambda)$ and $m_{cl}(\lambda)$. Note that Corollary 4 also implies the following lemma.

**Lemma 10.** There is a natural graded algebra embedding $\mathbb{C}[H_- \setminus B_-]_q \hookrightarrow \mathbb{C}[H_- \setminus B_-N_+]_q$. This embedding identifies generators $u_i$ (resp. $m_i$), $i \in \mathbb{Z}$ of $\mathbb{C}[H_- \setminus B_-]_q$ with the ones of $\mathbb{C}[H_- \setminus B_-N_+]_q$. Moreover, we have the following commutative diagram where all maps are graded algebras embedding:

$$\begin{array}{ccc}
\mathbb{C}[H_- \setminus B_-]_q & \hookrightarrow & \mathbb{C}[H_- \setminus B_-N_+]_q \\
DS_q \downarrow & & \downarrow \overline{DS}_q \\
A_q & \hookrightarrow & A_q.
\end{array}$$

Proposition 11 together with Lemma 10 lead to the following result.
Corollary 6. There is a $U_q \mathfrak{b}_-$-module-algebra structure on $\mathbb{C}[H_- \setminus B_- N_+]_q$ given by:

\[
\begin{align*}
    f_+(u(\lambda)) &= -(q-1)\lambda^{-1}u(\lambda)^2 - (q-q^{-1})u(\lambda)\Sigma_+ \\
    f_-(u(\lambda)) &= (1-q^{-1}) + (q-q^{-1})u(\lambda)\Sigma_- \\
    k_+(u(\lambda)) &= q^{-1}u(\lambda) \\
    f_+(m(\lambda)) &= 1 - q + (q-q^{-1})m(\lambda)\Sigma_+ \\
    f_-(m(\lambda)) &= (1-q^{-1})\lambda^{-1}m(\lambda)^2 - (q-q^{-1})m(\lambda)\Sigma_- \\
    k_+(m(\lambda)) &= q^{-1}m(\lambda) \\
    f_{\varepsilon, \Sigma} &= [\Sigma, \Sigma]_q.
\end{align*}
\]

The morphism $\overline{\mathbb{D}S_q}$ defined above is a $U_q \mathfrak{b}_-$-module-algebra morphism.

Proof. This comes from the fact that the morphism $\mathbb{D}S_q$ is injective and from the computation of $f_\pm \overline{\mathbb{D}S_q}(x)$ for $x \in \mathbb{C}[H_- \setminus B_- N_+]_q$. For example, according to formula (71), the computation of $f_+ \overline{\mathbb{D}S_q}(u_k)$ leads to the computation of $\sum_{i=-\infty}^{0} [x_i, \overline{\mathbb{D}S_q}(u_k)]_q$. This sum is finite. So, recalling the involution $\varphi$ on $A_q$ defined in (74), we have:

\[
\sum_{i=-\infty}^{0} [x_i, \overline{\mathbb{D}S_q}(u_k)]_q = \sum_{i=1}^{\infty} \varphi([\overline{\mathbb{D}S_q}(m_k), y_i]) = \varphi([\overline{\mathbb{D}S_q}(m_k), \Sigma_-])_q
\]

Then it suffices to use (79) to get the expression of $f_+ \overline{\mathbb{D}S_q}(u_k)$. \(\square\)

5.3. Adjoint action of integrals of motion on $\tilde{A}_q$. Let $I$ be an integral of motion. By using the definition seen in (15) of the adjoint action of $I$ on $A_q$ together with the equality between (16) and (17), it can be shown that there exists an unique homogeneous element $R_+(I) \in A_q[1]$ without constant term such that $\text{ad}(I)(x_1) = T(R_+(I)) - R_+(I)$. For instance, if $I = I_1$ is the first integral of motion with respect to the basis $(I_k)$ of $\mathbb{I}$ seen in Proposition 3, then $R_+(I_1) = -y_0^{-1}$. It follows that for any integer $n$, we have $\text{ad}(I)(x_1 + \ldots + x_n) = T^n(R_+(I)) - R_+(I)$.

On the other hand, thanks to the form taken by the $I_k$, we have $\text{ad}(I) \circ T^\frac{1}{2} = T^\frac{1}{2} \circ \text{ad}(I)$ for all $I \in \mathbb{I}$. So, there exists also $R_-(I) \in A_q[-1]$ with $R_-(I) = T^\frac{1}{2}(R_-(I))$ such that $\text{ad}(I)(y_1) = T(R_-(I)) - R_-(I)$. So, for all $n$, $\text{ad}(I)(y_1 + \ldots + y_n) = T^n(R_-(I)) - R_-(I)$. This leads to extend the derivation $\text{ad}(I)$ on $A_q$ as explained in the following proposition.

Proposition 12. Let $\text{Der}(\tilde{A}_q)$ be the Lie algebra of derivations on $\tilde{A}_q$. For $I \in \mathbb{I}$, there is a unique derivation $\text{ad}(I)$ on $\tilde{A}_q$ which satisfies formula (15) if $x$ belongs to $A_q$ and $\text{ad}(I)(\Sigma_\pm) = -R_{\pm}(I)$. Moreover, the kernel of the Lie algebra morphism:

\[
\begin{align*}
    \text{ad} : \mathbb{I} &\rightarrow \text{Der}(\tilde{A}_q) \\
    I &\rightarrow \text{ad}(I)
\end{align*}
\]

is $\mathbb{C}[q, q^{-1}]$ i.e., the one-dimensional Lie subalgebra of all constant integrals of motion. Its image is $\text{Der}_{U_q \mathfrak{b}_-}(\tilde{A}_q)$ the Lie subalgebra of all derivations which commute with the action of $U_q \mathfrak{b}_-$. 

Proof. Let \( I \in \mathcal{I} \). So as to prove the existence of \( \text{ad}(I) \) on \( \tilde{A}_q \), it is necessary to show some compatibility relations like:

\[
\forall j \in \mathbb{Z}, \quad [\text{ad}(I)(x_j), \Sigma_+]_q + [x_j, -R_+]_q = \sum_{k=1}^{j} \text{ad}(I)([x_j, x_k]_q).
\]

(90)

But, according to Proposition \( \Box \), \( \text{ad}(I) \) is well defined on \( A_q \). So, for any fixed integer \( j \), we have:

\[
\forall n \in \mathbb{N}, \quad [\text{ad}(I)(x_j), \Sigma_{+,n}]_q + [x_j, T^n(R_+) - R_+]_q = \sum_{k=1}^{j} \text{ad}(I)([x_j, x_k]_q)
\]

with \( \Sigma_{+,n} = \sum_{k=1}^{n} x_k \). This equality leads to (90) by taking \( n \) large enough. The other relations except those coming from the quantum Serre relations between \( \Sigma_{\pm} \) and \( \Sigma_{\mp} \) can be proved in the same way. The unicity of \( \text{ad}(I) \) is obvious.

To prove that \( \text{ad}(I) \) and \( f_\pm \) commute, we set \( C_\pm = \{ x \in \tilde{A}_q/ \text{ad}(I) \circ f_\pm(x) = f_\pm \circ \text{ad}(I)(x) \} \). We remark that for all \( x \) homogeneous with respect to \( \text{deg} \), we have \( \text{deg}(\text{ad}(I)(x)) = \text{deg}(x) \). Hence, \( C_\pm \) is a graded subalgebra of \( \tilde{A}_q \). Then, computation shows that \( A_q \cup \{ \Sigma_+, \Sigma_- \} \subset C_\pm \).

It follows that \( C_\pm = \tilde{A}_q \).

Conversely, let us fix \( D \in \text{Der}_{U_q \hat{b}_-}(\tilde{A}_q) \). Using the fact that the result is true at the classical level (\( \mathcal{EFd} \)) and the fact that classical integrals of motions can be quantized, first, we show the following result :

\[
\forall \delta \in \text{Der}_{U_q \hat{b}_-}(\tilde{A}_q) \forall n \in \mathbb{N} \exists I \in \mathcal{I} \exists \delta' \in \text{Der}_{U_q \hat{b}_-}(\tilde{A}_q), \forall x \in \tilde{A}_q, \delta(x) = \text{ad}(I)(x) + (q - 1)^n \delta'(x).
\]

(91)

Then, we can deduce from (91) and from the explicit form of the basis \( (I_k) \) of \( \mathcal{I} \) that \( D \) is a graded derivation (it means that if \( x \in \tilde{A}_q \) is homogeneous with respect to \( \text{deg} \), then \( D(x) \) is also homogeneous with respect to \( \text{deg} \) and \( \text{deg} D(x) = \text{deg} x \)), \( [D, T^\pm] = 0 \) or in other words, \( D \) and \( T^\pm \) commute, and \( A_q \) is invariant by \( D \). Hence, we show that \( D \) is entirely defined by \( D(x_0) \) : the natural map coming from the foregoing,

\[
\text{Der}_{U_q \hat{b}_-}(\tilde{A}_q) \rightarrow \text{Der}_{T^\pm}(A_q) \quad \delta \rightarrow \delta|_{\tilde{A}_q}
\]

(92)

is injective. Let us give some definitions. Let \( V_p \) be the free sub-module of \( A_q \) of all homogeneous elements of degree \( p \) with respect to the principal gradation \( \text{deg}_p \) (see subsection \( \Box \)). Let \( B_p \) be a basis for \( V_p \). If there exists \( q \in \mathbb{Z} \) such that \( p = -2q + 1 \), then \( B_p \) can be chosen such that \( \text{ad}(I_q)(x_0) \) belongs to \( B_p \). There is \( N \in \mathbb{N} \) and \( \alpha_p \in V_p, p \in \{-N, \ldots , N\} \) such that

\[
D(x_0) = \sum_{p=-N}^{N} \alpha_p.
\]

Let \( p \in \{-N, \ldots , N\} \). By projecting (91) on \( V_p \) with \( \delta = D \) and \( x = x_0 \), we see that the valuations in \( q-1 \) of all coefficients of \( \alpha_p \) on basis \( B_p \) (except perhaps the element \( \text{ad}(I_q)(x_0) \) of \( B_p \) if there is an integer \( q \) such that \( p = -2q + 1 \) are arbitrarily large. Hence, \( \alpha_0 = 0 \) or \( \alpha_p \) is proportional to \( \text{ad}(I_q)(x_0) \). Thus, there is \( I \in \mathcal{I} \) such that \( D(x_0) = \text{ad}(I)(x_0) \). Then, the injectivity of the map (92) shows that \( D = \text{ad}(I) \).
5.4. Existence of $H_n$. The aim of this section is to prove Proposition 13. In fact, we shall prove the existence of $H_n$ not only on $\mathbb{C}[H_\lambda \setminus B_-]_q$ but also on $\mathbb{C}[H_\lambda \setminus B_-N_+q]$. This will imply Proposition 6.

**Proposition 13.** There is a commutative family of derivations $(H_n)_{n \in \mathbb{N}^*}$ on $\mathbb{C}[H_\lambda \setminus B_-N_+]_q$ which quantizes the classical action by vector fields of $\mathfrak{h}_+$ on $H_\lambda \setminus B_-N_+$ and which commute with the action of $U_q\mathfrak{h}_-$ on $\mathbb{C}[H_\lambda \setminus B_-N_+]_q$ (for the definition of $\mathfrak{h}_+$, see subsection 2.4). If $H(\mu) = \sum_{k=1}^{\infty}(-1)^kH_k\mu^{-k}$ denotes the generating function of $(H_n)_{n \in \mathbb{N}^*}$, then, the derivations $H_n$ are defined by formulas:

$$H_\mu(u(\lambda)) = \frac{1}{\lambda - \mu}(\lambda^{-1}u(\lambda) - \mu^{-1}u(\mu))v(\mu)u(\lambda) - \mu^{-1}(u(\lambda) - u(\mu))(1 + v(\mu)u(\mu))$$

$$H_\mu(m(\lambda)) = \frac{\mu^{-1}}{\lambda - \mu}(1 + m(\mu)w(\mu))(m(\lambda) - m(\mu)) - \frac{1}{\lambda - \mu}m(\mu)w(\mu)(\lambda^{-1}m(\lambda) - \mu^{-1}m(\mu))$$

$$H(\mu)_{\Sigma_+} = -\mu^{-1}(u(\mu) + u(\mu)v(\mu)u(\mu))$$

$$H(\mu)_{\Sigma_-} = v(\mu)$$

with $v(\mu) = -(u(\mu) + \mu m(\mu)^{-1})^{-1}$ and $w(\mu) = -(m(\mu) + \mu u(\mu)^{-1})^{-1}$.

5.5. The classical case. For $n \in \mathbb{N}$, set $h_n = \text{diag}(\lambda^n, -\lambda^n)$. With the notation of subsection 5.2 and in particular of the map $F$ defined in (81), we show that the left translation action of $h_n$ on generating series $u_{cl}(\lambda)$ and $v_{cl}(\lambda)$ of coordinate functions $u_i$ and $m_i$ is given by formulas:

$$h_n.u_{cl} = [(1 + 2u_{cl}v_{cl})\lambda^n]_{\leq}u_{cl} - 2[u_{cl}(1 + u_{cl}v_{cl})\lambda^n]_{\leq} - [2v_{cl}\lambda^n]_{\leq}u_{cl}^2 + [(1 + 2u_{cl}v_{cl})\lambda^n]_{\leq}u_{cl}$$

$$h_n.v_{cl} = 2[v_{cl}\lambda^n]_{\leq}(1 + 2u_{cl}v_{cl}) - 2[(1 + 2u_{cl}v_{cl})\lambda^n]_{\leq}$$

where for $x \in \mathbb{C}([\lambda^{-1}])$, $x_{\leq}$ (resp. $x_{\geq}$) denotes the part of $x$ in $\mathbb{C}(\lambda^{-1})$] (resp. $\lambda^{-1}\mathbb{C}(\lambda^{-1})$). Let $h(\mu)$ be the generating series of $h_n$. From (97) and (98), it is possible to compute the action of $\frac{1}{2}h(\mu)$ on $u_{cl}(\lambda)$ and $m_{cl}(\lambda)$. It can be checked that these relations are precisely the ones we get from (93) and (94) when $q \to 1$. In the same way, with the identification of $\Sigma_+$ and $\Sigma_-$ with $1 \otimes a_{1,2}^{(1)}$ and $1 \otimes a_{1,2}^{(0)}$, it can be shown that we have $\frac{1}{2}h(\mu)\Sigma_+ = -\mu^{-1}(u(\mu) + u(\mu)^2v(\mu))$ and $\frac{1}{2}h(\mu)\Sigma_- = v(\mu)$. Hence, it is clear that if derivation $H_n$ exists then it deforms the classical action of $\frac{1}{2}h_n$ on the homogeneous space $H_\lambda \setminus B_-N_+$.

5.6. The algebra $U_{\lambda,\mu}$. To obtain algebra $U_{\lambda,\mu}$, we just need to replace generating series $u(\lambda)$ and $u(\mu)$ by variables $u_{\lambda}$ and $u_{\mu}$.

**Definition.** We denote by $U_{\lambda,\mu}$ the algebra over the ground ring $\mathbb{C}[\lambda^{-1}, \mu^{-1}]$ given by generators $u_{\lambda}$ and $u_{\mu}$ and relation : $(\lambda^{-1}u_{\lambda} - \mu^{-1}u_{\mu})(u_{\lambda} - u_{\mu}) = q(u_{\lambda} - u_{\mu})(\lambda^{-1}u_{\lambda} - \mu^{-1}u_{\mu})$.

Note that $U_{\lambda,\mu}$ is a graded algebra with respect to the gradation given by $\text{deg } u_{\lambda} = \text{deg } u_{\mu} = 1$. It can be proved that $U_{\lambda,\mu}$ does not have any torsion of zero divisor, and that a basis is given
by the family \((u_\alpha^\lambda u_\mu^\beta)\) with \((\alpha, \beta) \in \mathbb{N}^2\). Thanks to definition of \(U_{\lambda,\mu}\) and \(\mathbb{C}[S_\infty \backslash B_-]_q\), there is an algebra morphism:

\[
\begin{align*}
U_{\lambda,\mu} & \quad \longrightarrow \quad \mathbb{C}[S_\infty \backslash B_-]_q[[\lambda^{-1}, \mu^{-1}]] \\
u_\lambda & \quad \longmapsto \quad u(\lambda) \\
u_\mu & \quad \longmapsto \quad u(\mu).
\end{align*}
\]

(99)

It can be shown that this morphism is injective. There are unique coefficients \(c^{k,l}_{\alpha,\beta}\) satisfying

\[
\forall k, l \in \mathbb{N}, \quad (u_\mu)^k (u_\lambda)^l = \sum_{\alpha + \beta = k + l} c^{k,l}_{\alpha,\beta} (u_\lambda)^\alpha (u_\mu)^\beta.
\]

(100)

We give below formulas which deal with cases \(k\) or \(l\) equal to 1 or 2. These formulas will be useful because the relations between \(u_\lambda\) and \(u_\mu\) are quadratic as well as the right side of (93).

**Proposition 14.** There are coefficients \(c_{\alpha,\beta}, d_{\alpha,\beta}, c^{(2)}_{\alpha,\beta}, d^{(2)}_{\alpha,\beta}\) such that:

\[
\forall n \in \mathbb{N}, \quad u_\mu(u_\lambda)^n = \sum_{\alpha + \beta = n+1} c_{\alpha,\beta} (u_\lambda)^\alpha (u_\mu)^\beta.
\]

(101)

\[
(u_\mu)^n u_\lambda = \sum_{\alpha + \beta = n+1} d_{\alpha,\beta} (u_\lambda)^\alpha (u_\mu)^\beta.
\]

(102)

\[
(u_\mu)^2(u_\lambda)^n = \sum_{\alpha + \beta = n+1} c^{(2)}_{\alpha,\beta} (u_\lambda)^\alpha (u_\mu)^\beta.
\]

(103)

\[
(u_\mu)^n(u_\lambda)^2 = \sum_{\alpha + \beta = n+1} d^{(2)}_{\alpha,\beta} (u_\lambda)^\alpha (u_\mu)^\beta.
\]

(104)

For all \(\alpha \geq 0\) and \(b \geq 1\), we have:

\[
c_{\alpha,0} = \frac{(q^{\alpha-1} - 1)\lambda^{-1}}{q^{\alpha-1}\lambda^{-1} - \mu^{-1}}
\]

(105)

\[
\forall \beta \neq 0, \quad c_{\alpha,\beta} = q^{\alpha-1} \prod_{j=\alpha+1}^{\alpha+\beta-1} (q^j - 1) \left(\frac{(\lambda^{-1} - \mu^{-1})}\prod_{j=\alpha}^{\alpha+\beta-1}(q^j\lambda^{-1} - \mu^{-1})\right).
\]

(106)

Also, for all \(\alpha \geq 0\) and \(\beta \geq 2\),

\[
c^{(2)}_{\alpha,0} = \frac{(q^{\alpha-2} - 1)(q^{\alpha-1} - 1)\lambda^{-2}}{(q^{\alpha-2}\lambda^{-1} - \mu^{-1})(q^{\alpha-1}\lambda^{-1} - \mu^{-1})}
\]

(107)

\[
c^{(2)}_{\alpha,1} = q^{\alpha-2}(q^{\alpha-1} - 1)[2] \left(\frac{(\lambda^{-1} - \mu^{-1})}{(q^{\alpha-2}\lambda^{-1} - \mu^{-1})(q^{\alpha-1}\lambda^{-1} - \mu^{-1})}\right)
\]

(108)

\[
c^{(2)}_{\alpha,\beta} = q^{\alpha-2} \prod_{j=\alpha+1}^{\alpha+\beta-2} (q^j - 1) \left(\frac{(\lambda^{-1} - \mu^{-1})}{\prod_{j=\alpha}^{\alpha+\beta-2}(q^j\lambda^{-1} - \mu^{-1})}\right)
\]

(109)

with:

\[
P^{(2)}_{\alpha,\beta}(\lambda^{-1}, \mu^{-1}) = q^{[\alpha + \beta - 1]}(q\lambda^{-1} - \mu^{-1})(q^{\alpha-2}\lambda^{-1} - \mu^{-1})
\]

(110)

\[- [\alpha](\lambda^{-1} - \mu^{-1})(q^{\alpha+\beta-1}\lambda^{-1} - \mu^{-1}).
\]
Coefficients $d_{\alpha,\beta}$ (resp. $d_{\alpha,\beta}^{(2)}$) are obtained from $c_{\alpha,\beta}$ (resp. $c_{\alpha,\beta}^{(2)}$) by:

$$\forall \alpha, \beta, \quad \lambda^{-(\beta-1)} c_{\alpha,\beta} = \mu^{-(\beta-1)} d_{\beta,\alpha}$$  \hspace{1cm} (111)

$$\lambda^{-(\beta-2)} c_{\alpha,\beta}^{(2)} = \mu^{-(\beta-2)} d_{\beta,\alpha}^{(2)}$$  \hspace{1cm} (112)

**Proof.** Coefficients $c_{\alpha,\beta}^{(2)}$ and $d_{\alpha,\beta}^{(2)}$ can be obtained by computation from $c_{\alpha,\beta}$ and $d_{\alpha,\beta}$. To prove (105) and (106), we define $c_{\alpha,0}$ and $c_{\alpha,\beta}$ by these formulas, and we try to prove (101). For that, we express $u_\lambda$ and $u_\mu$ in terms of variables $v := u_\lambda - u_\mu$ and $v' := \lambda^{-1} u_\lambda - \mu^{-1} u_\mu$. These variables being $q$-commuting, we expand the two expressions $\sum_{\alpha+\beta+1} c_{\alpha,\beta} (u_\lambda)^{\alpha} (u_\mu)^{\beta}$ and $u_\mu (u_\lambda)^n$ as a sum of terms in $v^i v'^j$. Then, we fix $i$ and $j$ and we want to identify coefficients in $v^i v'^j$. This leads to prove an equality between polynomials in $\frac{\lambda^{-1}}{\mu^{-1}}$ which reduces as a relation between $q$-integers. In the same manner, we prove (102). \(\square\)

5.7. **Proof of Proposition 13.** First, we define derivations $H_n$ on the free algebra $A$ generated by $u_i$, $m_i$, $i > 0$, $\sum_+$ and $\sum_-$. To prove that the $H_n$ give derivations on $\mathbb{C}[H_\cdot \setminus B_\cdot N_+]_q$, we have to prove several relations. The most complicated one is

$$\pi \circ H(\nu). (\text{Relation between } u(\lambda) \text{ and } u(\mu)) = 0$$  \hspace{1cm} (113)

where $\pi$ denotes the projection of $A$ onto $\mathbb{C}[H_\cdot \setminus B_\cdot N_+]_q$. To prove (113), we decompose $H(\nu).u(\lambda)$ and $H(\nu).u(\mu)$ according to the relation:

$$H(\mu).u(\lambda) = \mu^{-1} (u(\mu) - u(\lambda)) + \sum_{k=0}^{+\infty} (-1)^{k+1} q^{k+1} \times$$

$$\times (\lambda^{-1} u(\lambda) u(\mu)^k u(\lambda) + \mu^{-1} u(\mu)^{k+2} - \mu^{-1} u(\mu)^{k+1} u(\lambda) - \mu^{-1} u(\lambda) u(\mu)^{k+1}) m(\mu)^{k+1} \mu^{-(k+1)}$$  \hspace{1cm} (114)

which comes from (93) and (76) by decomposing $v(\nu)$ into a generating series in $u(\mu)^k m(\mu) \mu^{-k}$. Thus, the left side of (113) is of the form $\sum_{k=0}^{+\infty} P_k (u(\lambda), u(\mu), u(\nu)) m(\nu)^k \nu^{-k}$ where $P_k (u(\lambda), u(\mu), u(\nu))$ is a polynomial in non-commutative variables $u(\lambda), u(\mu)$ and $u(\nu)$.

Let $k \in \mathbb{N}$. According to the fact that the relations between $u(\lambda)$ and $u(\mu)$ are quadratic and that the only terms in which appear in (93) are also quadratic, it is possible to reorganize the terms of polynomial $P_k (u(\lambda), u(\mu), u(\nu))$ by using Proposition 14 and the morphism defined in (93) so as to obtain a sum of monomials of the form $(u(\lambda))^{\alpha} (u(\mu))^{\beta} (u(\nu))^\gamma$. Then, we fix $\alpha, \beta, \gamma$ and we show that the coefficient of $(u(\lambda))^{\alpha} (u(\mu))^{\beta} (u(\nu))^\gamma$ in $P_k$ is equal to 0. Thus, $P_k = 0$, and (113) is true. In the same way, we prove all other relations. Thus, $H_n$ exists. To prove the commutativity, we deduce from formulas (93) and (94) that

$$H(\mu)(v(\lambda)) = \frac{1}{\lambda^{-1} - \mu^{-1}} (\mu^{-1} v(\lambda) - \lambda^{-1} v(\mu)) + \frac{1}{\lambda^{-1} - \mu^{-1}} v(\lambda) (\mu^{-1} u(\mu) - \lambda^{-1} u(\lambda)) v(\mu)$$  \hspace{1cm} (115)

$$+ \frac{1}{\lambda^{-1} - \mu^{-1}} v(\mu) (\mu^{-1} u(\mu) - \lambda^{-1} u(\lambda)) v(\lambda).$$
Then, the computation shows that \( H(\mu) \circ H(\nu) = H(\nu) \circ H(\mu) \) (it is not necessary for that to decompose in generating series). On the other hand, with the help of the formulas coming from Corollary \[3\] the computation shows that \( H_\mu \circ f_\pm = f_\pm \circ H_\mu \). This completes the proof of Proposition \[13\], and also of Proposition \[3\].

5.8. End of the proof of Theorem \[1\]. We are going to prove that for any integer \( n \), \( \text{ad}(I_n) \circ \mathbb{D}S_q = \mathbb{D}S_q \circ H_n \). In other words, \( \mathbb{D}S_q \) set up a Drinfeld-Sokolov correspondence for the extended phase space \( \tilde{A}_q \) and the quantum homogeneous space \( \mathbb{C}[H_\lambda \mathcal{B}_\nu]_q \). Thanks to Lemma \[10\], the result will follow. To simplify, we note \( U_q = \mathbb{C}[H_\lambda \mathcal{B}_\nu]_q \) and \( \tilde{U}_q = \mathbb{C}[H_\lambda \mathcal{B}_\nu N_+]_q \). In the following of the article, we shall identify the elements of \( \tilde{U}_q \) with their images in \( \tilde{A}_q \) by the algebra monomorphism \( \mathbb{D}S_q \). The following lemma will be useful.

**Lemma 11.** Let \( D_1 \) and \( D_2 \) be two derivations defined on \( A_q \) (resp. \( \tilde{A}_q \)) such that \( D_1(x) = D_2(x) \) for all \( x \in U_q \) (resp. \( \tilde{U}_q \)). Then, \( D_1 = D_2 \).

**Proof.** Let us denote by \( C \) the subalgebra of \( A_q \) (resp. \( \tilde{A}_q \)) of all elements \( x \) such that \( D_1(x) = D_2(x) \). If \( x \in C \), is invertible in \( A_q \) (resp. \( \tilde{A}_q \)) then \( x^{-1} \in C \). By induction, using the explicit forms of \( u_n \) and \( m_n \) given in (52) and (54), we show that for all \( n \in \mathbb{Z} \), \( x_n^{-1}, y_n^\pm \in C \). For example, \( y_0^{-1} \in C \), \( x_0^{-1} = q^{-1}w_2y_0^2 \in C \). Hence, we get the result. \( \square \)

Let \( n \in \mathbb{N}^* \). Then \( H_n \) is a derivation on \( \tilde{U}_q \subset \tilde{A}_q \). We have to prove that \( H_n \) extends as a derivation on \( \tilde{A}_q \) and that \( H_n = \text{ad}(I_n) \).

5.8.1. First, let us assume that \( H_n \) has an extension to \( A_q \). Then, \( H_n \) has also an extension on \( A_q \). Moreover, thanks to the definition of \( H_n \) together with the relations (24) and (28) which give expressions for \( u(\lambda) \) and \( m(\lambda) \) as quantum continued fractions, we show that the image of \( \tilde{U}_q \) by \( T^{-\frac{1}{2}} \) is generated by \( \tilde{U}_q \) and \( u_1^{-1} \) and that \( H_n \circ T^{-\frac{1}{2}}(x) = T^{-\frac{1}{2}} \circ H_n(x) \) for all \( x \in \tilde{U}_q \).

Lemma \[11\] ensures that this relation is also true on \( \tilde{A}_q \). On the other hand, computation shows that for all \( x \in U_q \), \( H_n \circ \varphi(x) = \varphi \circ H_n(x) \) where \( \varphi \) is the involution on \( A_q \) defined in (54). Using again Lemma \[11\], this last equality extends on \( A_q \). So, by using the relation \( \varphi \circ T^\frac{1}{2} = T^{-\frac{1}{2}} \circ \varphi \), we deduce that \( H_n \circ T^\frac{1}{2}(x) = T^{\frac{1}{2}} \circ H_n(x) \) for all \( x \in A_q \). It can be shown that this relation is also true for \( x = \sum_\pm \). It follows that \( H_n \circ T^{\pm \frac{1}{2}} = T^{\mp \frac{1}{2}} \circ H_n \) on \( \tilde{A}_q \). On the other hand, by virtue of Proposition \[13\], we have \( H_n \circ f_\pm(y_0^{-1}) = f_\pm \circ H_n(y_0^{-1}) \) for all \( u_0 = y_0^{-1} \). From the equality \( f_\pm \circ T^{-\pm \frac{1}{2}} = T^{-\mp \frac{1}{2}} \circ f_\pm \), we deduce that \( H_n \circ f_\pm(x) = f_\pm \circ H_n(x) \) for all \( x \in B_q \), where \( B_q \) denotes the subalgebra of \( A_q \) generated by \( x_i^{-1} \) and \( y_i^\pm \), for \( i \in \mathbb{Z} \). Then, the \( U_q \)-\( \mathfrak{b}_\nu \)-module-algebra structure of \( \tilde{A}_q \) implies that \( H_n \circ f_\pm(x) = f_\pm \circ H_n(x) \) for all \( x \in A_q \) and also on \( \tilde{A}_q \) for this equality is also true if \( x = \sum_\varepsilon, \varepsilon \in \{+, -\} \). Thus, \( H_n \in \text{Der}_{U_q \tilde{\mathfrak{b}}}(\tilde{A}_q) \), and thanks to Proposition \[12\], we see that \( H_n \) is a linear combination of \( \text{ad}(I_k) \). But, the same proposition also shows that there is a gradation \( \text{deg} \) on \( \text{Der}_{U_q \tilde{\mathfrak{b}}}(\tilde{A}_q) \) given by \( \text{deg} \delta = n \) for \( \delta \in \text{Der}_{U_q \tilde{\mathfrak{b}}}(\tilde{A}_q) \) if there is a homogeneous element \( \alpha \in \tilde{A}_q \) with respect to the principal gradation \( \text{deg}_p \) on \( \tilde{A}_q \) defined on \( A_q \) in subsection \[3.4\] and extended on \( \tilde{A}_q \) by \( \text{deg}_p \Sigma_\pm = 1 \), such that \( \delta(\alpha) \) is also homogeneous with respect to \( \text{deg}_p \) and \( \text{deg}_p(\delta(\alpha)) = n + \text{deg}_p(\alpha) \). Note that if \( \delta \) is homogeneous, this last property occurs not only for one special homogeneous element \( \alpha \) with respect to the gradation \( \text{deg}_p \) but also for all homogeneous elements \( x \in \tilde{A}_q \) with respect to \( \text{deg}_p \). Now, from the definition of \( H_n \),
it is easy to see that \( \deg H_n = \deg I_n \). Then, the computation of both \( H_n(y_0^{-1}) \) and \( \text{ad}(I_n)(y_0^{-1}) \) on the basis element \( y_k x_k^2 y_{-k+1} x_0^{-2} y_0^{-1} \) or \( x_k y_{-k+1} x_0^{-2} y_0^{-1} \) of the basis \( \prod x_i^{\alpha_i} y_i^{\beta_i} \) (with \( k \) such that \( n = 2k \) or \( n = 2k + 1 \) according to the parity of \( n \)) shows that \( H_n = \text{ad}(I_n) \). Thus, to conclude, it suffices to extend \( H_n \) on \( A_q \).

5.8.2. **Proof of the existence of an extension.** In the classical case, we use the fact that the classical limit \( U_{q, \text{cl}} \) of \( U_q \) possesses the same field of fraction \( K \) as \( A_{q, \text{cl}} \) to extend \( H_{n, \text{cl}} \) in a derivation of \( K \). The extension is unique. So, the relation \( H_{n, \text{cl}} \circ T^{-\frac{1}{2}} = T^{-\frac{1}{2}} \circ H_{n, \text{cl}} \) true on \( A_{q, \text{cl}} \) is also true on \( K \). The same argument as above with the involution \( \varphi \) shows that \( H_{n, \text{cl}} \circ T^{\pm \frac{1}{2}} = T^{\pm \frac{1}{2}} \circ H_{n, \text{cl}} \).

But, \( H_{n, \text{cl}}(y_0^{-1}) = H_{n, \text{cl}}(y_1) \in U_{q, \text{cl}} \subset A_{q, \text{cl}} \). Hence, we get the result.

In the classical case, it is a little bit more complicated. The problem comes from the fact that it is not obvious that non-zero elements of \( A_q \) as well as of \( \mathbb{C}[H \setminus B_\ast]_q \) satisfy Ore conditions. Nevertheless, we show that this is true when \( q \) is a formal variable “close to 1”. For that, we develop the notion of extended Ore conditions.

**Definition.** Let \( A \) be an algebra without any zero divisors over a field \( k \) and \( (A[[t]], \ast) \) a formal deformation of the multiplication on \( A \). For all \( n \), we note by \( \pi_n \) the natural projection of \( A[[t]] \) on \( A_n := A[[t]]/(t^n) \). A multiplicative set \( S \) in \( A[[t]] \) is said to satisfy the extended Ore conditions if \( \pi_n(S) \) satisfy the Ore conditions in \( A_n \) equipped with the natural non-commutative product induced by \( \ast \).

If \( S \) is a multiplicative set which satisfies the extended Ore conditions in \( A[[t]] \), then there are natural morphisms : \( (A_n)_{S_n} \longrightarrow (A_p)_{S_p} \) pour \( n > p \). We note \( A[[t]]_S \) the projective limit of \( (A_n)_{S_n} \).

**Examples.**

1. Let \( B = \mathbb{C}[X_i^{-1}, Y_i^{-1}, i \in \mathbb{Z}] \), and \( K \) be the field of fractions of \( B \). Let us consider the isomorphism of free modules :

\[
\prod_{i=1}^{\infty} X_i^{-\alpha_i} \prod_{j=1}^{\infty} Y_j^{-\beta_j} \longrightarrow \prod_{i=1}^{\infty} x_i^{-\alpha_i} \prod_{j=1}^{\infty} y_j^{-\beta_j}
\]

where \( (\alpha_i) \) and \( (\beta_j) \) are two almost zero sequences in \( \mathbb{N}^\mathbb{Z} \) and \( B'_q \) is the \( (q - 1) \)-adic completion of the subalgebra \( B_q \) aforementioned. This isomorphism leads to a formal deformation \( \ast' \) of the multiplication on \( B[[q - 1]] \). We show that the multiplicative set \( S' \) of all elements non divisible by \( q - 1 \) satisfy the extended Ore conditions. Moreover, \( \ast' \) is a star-product, i.e., there are \( B_n \) bidifferential operators such that \( \ast' = \sum B_n(q - 1)^n \). Hence, we get a non-commutative structure on \( (K[[q - 1]], \ast') \) which contains \( (B[[q - 1]], \ast') \) and it can be proved that \( (B[[q - 1]], \ast') \simeq (K[[q - 1]], \ast') \).

2. One can also define a structure of non-commutative algebra \( (\mathbb{C}[U_i, M_i, i > 0][[q - 1]], \ast) \) thanks to the isomorphism of free modules :

\[
\prod_{i=1}^{\infty} U_i^{\alpha_i} M_i^{\beta_i} \longrightarrow \prod_{i=1}^{\infty} u_i^{\alpha_i} m_i^{\beta_i}
\]
where $U_q'$ denotes the $(q-1)$-adic completion of $U_q$. Contrary to the previous case, it is not so easy to check that the non-commutative product $\ast$ is a star product. So, there is a priori no reason for $(\mathbb{C}[[U_i,M_i,i>0]][[q-1]],\ast)$ to exist. However, we prove the following result.

**Lemma 12.** For all $x \in A_q$, there is $\omega$ a monomial in $x_i^{-1}, y_i^{-1}$ such that $\omega \in U_q$ and $\omega x \in U_q$.

**Proof.** It suffices to prove the lemma for $x = x_i^{-1}$ or $x = y_i^{-1}$ with $i \in \mathbb{Z}$. Thanks to the symmetry relation between $u_n$ and $m_n$ i.e., $m_n = \varphi(u_n)$ and $u_n \in \mathbb{C}[x_j^{-1}, y_j^{-1}, j < 0]_q$, we can assume that $i < 0$. Then, we prove the result by induction. For example, $1.y_0^{-1} = u_1$, $q^{-1}.y_0^{-2}x_0^{-1} = u_2$ and $y_0^{-1} = u_1$, etc. □

Lemma [T2] implies that the multiplicative set $S$ of all elements in $\mathbb{C}[[U_i,M_i,i>0]][[q-1]]$ which are non-divisible by $q-1$ satisfy the extended Ore conditions and that we have an isomorphism $(\mathbb{C}[[U_i,M_i,i>0]][[q-1]]_S,\ast) \simeq (B[[q-1]]_S,\ast')$. Now, to conclude, we say that the $(q-1)$-adic completion of $H_n$ defined on $\mathbb{C}[[U_i,M_i,i>0]][[q-1]]$ has naturally an extension on $(K[[q-1]],\ast')$. Moreover, this extension is unique. The same arguments as above with the involution $\varphi$ and the half-translation automorphism $T^\pm$ show that $H_n$ and $T^\pm$ commute on $K[[q-1]]$. But $H_n(y_0^{-1}) \subset A_q$. It follows that $A_q$ is invariant by $H_n$.

6. **Conclusion and outlooks**

First of all, it would be interesting to see whether if it is possible to extend our result to the more general case of an arbitrary non-twisted Lie algebra and to study other possible models of discretization proposed by Enriquez and Feigin [EF]. It would be also interesting to study in details the case when $q$ is a root of the unity.

6.1. **Affine Poisson homogeneous space.** Theorem I leaves us with the feeling that there is a general Drinfeld-Sokolov correspondence for the discrete Toda theory and that the homogeneous space of the correspondence is a Poisson homogeneous space equipped with a Poisson structure induced by a Poisson bivector $\pi$ of the form $\pi = r^L - r^R$, where $r$ and $r'$ are two $r$-matrices such that their Schouten bracket $[r,r]$ and $[r',r']$ are equal and invariant by the adjoint action of the Lie group $G$ on $\wedge^3 \text{Lie}(G)$ and where $r^L$ (resp. $r^R$) is the left (resp. right) translation of $r$ (resp. $r'$) on $G$. A group $G$ endowed with such a Poisson structure is a particular case of an affine Poisson homogeneous space (APHS), according to the terminology introduced by Dazord and Sondaz [DaSo] (see also [I] and [K]). By definition, an APHS is a Poisson manifold which is a principal homogeneous space under the action of a Poisson-Lie group and that if it is the case, then there are on $G$ two commuting actions by Poisson-Lie groups.

6.2. **Links with Parmentier’s work.** Our method of quantizing the Poisson manifold $H_\ast \setminus B_\ast$ equipped with the Poisson structure induced by the field of bivectors $P_\infty = r^L - r^R_\infty$ (which is truly a quotient of an APHS) lays on the study of the classical case and on the fact that the phase space of the discrete sine-Gordon system had a natural quantization. But, it is perhaps not the easiest way to quantize $(H_\ast \setminus B_\ast, P_\infty)$. Indeed, in the case we dealt with, $r$ denotes the standard $r$-matrix and $r_\infty$ denotes the $r$-matrix corresponding to Drinfeld new realizations. So, according to Parmentier works, to get a quantization of the APHS $G$ with Poisson structure
given by the field of bivectors $P_\infty$, it suffices to have a twist relying the two Hopf algebra structures $(U_q\mathfrak{g}, \Delta)$ and $(U_q\mathfrak{g}, \Delta_{nr})$ where, in the first case, the comultiplication is the “Drinfeld-Jimbo” comultiplication, and in the second case, it is the one corresponding to the Drinfeld new realizations. But such a twist appears in the paper [KT]. We need to apply this method and to investigate further about how derivations $H_n$ appear in it. We plan to study this question elsewhere.

6.3. **The continuous case.** It would be also interesting to see whether it is possible to deduce from our results solutions to problems of continuous Toda theory, to compute explicitly integrals of motion, to quantize in terms of Vertex Operator Algebra the Vertex Poisson Algebra shown by Enriquez and Frenkel on homogeneous spaces [EFr2], and to obtain a quantum version of Drinfeld-Sokolov correspondence in terms of V.O.A. in the continuous case.

6.4. **The Drinfled-Sokolov reduction.** Finally, we indicate that there exists another correspondence, close to the one we discussed here, which is called the Drinfeld-Sokolov reduction [DS]. This correspondence allows to construct $\mathcal{W}$-algebras from Kac-Moody algebras. There is a Poisson isomorphism between the manifold of scalar differential operators of order $n$ with the second Gelfand-Dickey bracket on one hand, and the manifold of matrix differential operators of order $1$ viewed as a subspace of $\widehat{\mathfrak{sl}}_n^*$, with Kirillov-Kostant bracket on the other hand. The quantization of this correspondence is studied in [FF1]. A $q$-deformed version of this correspondence, in which manifolds of differential operators are replaced by manifolds of $q$-difference operators is proposed in [FRS] and [SS]. Quantization of this correspondence leads to $q$-deformed $\mathcal{W}$-algebra.

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