Fermionic spectra in integrable models

Rinat Kedem

Abstract. This is a brief review of several algebraic constructions related to generalized fermionic spectra, of the type which appear in integrable quantum spin chains and integrable quantum field theories. We discuss the connection between fermionic formulas for the graded dimensions of the spaces of conformal blocks of WZW theories, quantum cluster algebras, discrete integrable noncommutative evolutions and difference equations.

Mathematics Subject Classification (2010). Primary 81R10; Secondary 82B23, 05E10

Keywords. Fermionic character formulas, Fusion products, discrete integrable systems

1. Partition functions in statistical mechanics and conformal field theory

In statistical mechanics, a fundamental object of interest is the partition function, the sum over the space of configurations $C$ of the Boltzmann weight $e^{-E/kT}$, where $E$ is the energy of a configuration:

$$Z = \sum_{c \in C} e^{-E(c)/kT}.$$

If the lattice is two-dimensional, the standard test for integrability is the existence of a commuting family of transfer matrices. For a system with periodic boundary conditions, the partition function can be written as the trace of the product of transfer matrices. These are operators on the Hilbert space of a one-dimensional slice of the lattice, which depend on a spectral parameter. The coefficients in expansion of this operator as a series in the spectral parameter gives commuting integrals of the motion, hence integrability.

The Hamiltonian associated with the one-dimensional system is one of those integrals. For example, the six-vertex model transfer matrix is associated with the $XXZ$ Heisenberg spin chain Hamiltonian $H$.

The two-dimensional integrable lattice model may undergo a second-order phase transition at certain critical points, in the infinite-lattice limit. At the critical point, the behavior of the model may be described by an effective conformal field theory. The correspondence includes the identification of the critical exponents, given by the conformal dimensions, and the specific heat, given by the central charge of the
family of Virasoro representations which make up the Hilbert space of the quantum field theory. It was shown in [30, 26] that the massless part of the spectrum – that is, order 1 excitations in the statistical model, and quasi-particles in the quantum field theory – are also related, and an identification can be made via the partition functions.

For the lattice model the spectrum is computed from the Bethe ansatz. The “order one”, or massless, excitations, which contribute to the conformal partition function, have a “quasi-particle-like” behavior. For small momenta, their energy is a linear function of the momentum. We call this the linearized spectrum.

In conformal field theory, the chiral part of the partition function is given by the specialized characters of certain (not necessarily irreducible) Virasoro modules. The full partition function is a modular invariant sesquilinear combination of these and includes both the chiral and anti-chiral parts.

In the original work of the author and collaborators [30, 26], it was shown that, starting from the Bethe ansatz, linearizing the spectrum and considering only massless excitations above the vacuum, the resulting partition function is equal to the chiral part of the partition function in the conformal field theory, given by Virasoro characters.

The spectrum obtained from the Bethe ansatz is invariably of fermionic nature. At the time when this work was done, few fermionic constructions of Virasoro modules were known. For example, the Feigin-Fuks construction of the most interesting Virasoro modules involves a resolution of the Verma module using the singular vectors, and is thus given by an inclusion-exclusion principle, or (in general) an infinite alternating sum.

A fermionic construction is a basis of the representation given by the action of skew-commuting operators on the vacuum. This gives rise to fermionic statistics: Identical fermions cannot occupy the same point in phase space. One type of generalization of fermionic statistics will be given below. These rules are combinatorial and this is reflected in the expression for the partition function.

There are various ways of constructing bases for any given Virasoro module. The idea of fermionic constructions is that physically meaningful ones reflect the spectrum away from criticality of the integrable quantum field theory. The particle content is some reflection of the form of the primary fields of the conformal field theory. These fields are the generalized fermions. See [24] for a recent example of this.

This note is organized as follows. In Section 2 we will give a few examples of fermionic partition functions related to WZW models. In Section 3, we will relate the general fermionic formulas for graded dimensions of the space of conformal blocks to cluster algebras and quantum cluster algebras. In Section 4, we will show how the integrability of the resulting discrete difference equations (Q-systems and their quantized version) can be used to give difference equations satisfied by generating functions for the graded dimensions of the space of conformal blocks. These are variants of quantum difference Toda equations. These dimensions are the dimensions of the moduli space of holomorphic vector bundles on the sphere with prescribed punctures, and their graded analogs.
1.1. Acknowledgements. The author would like to express her gratitude to her advisor B.M. McCoy, with whom the original formulation of the physical interpretation of conformal partition functions and their fermionic expressions was made; To M. Jimbo, T. Miwa for their patient mentorship over many years; to them as well as B. Feigin and E. Ardonne, with whom she first worked on fusion products; and most especially to P. Di Francesco for an ongoing illuminating collaboration. The author thanks S. Fomin, H. Nakajima, N. Reshetikhin for their kindness and for helpful discussions related to this work over the years. This work has been supported by the National Science foundation through several grants, most recently NSF DMS grant 1100929.

2. Generalized fermionic formulas

Let us be specific about what we mean by a generalized fermion and the resulting fermionic formula for the partition function. This phenomenon occurs in finite or infinite systems. The natural finite-dimensional system to start from is a solvable model on the finite lattice with a spectrum governed by the statistics of the Bethe ansatz equations. The eigenstates of the Hamiltonian with a Bethe ansatz solution are in bijection with solutions of a coupled set of algebraic equations. The solutions are specified by a set of integers chosen distinctly on certain finite intervals. We interpret a choice of one integer as a quasi-particle, and a choice of \(m\) integers as \(m\) quasi-particles. The corresponding choice of integers is proportional to their momentum, one of the conserved quantities. The fact that the integers should be distinct is what gives them a fermionic nature.

The resulting combinatorics is as follows. We approximate the energy of each quasi-particle as a linear function of the momentum (they are massless) and hence the Bethe integers. This is a reasonable assumption in the conformal, infinite-size limit. Suppose the Hilbert space with \(m\) quasi-particles has \(m\) integers chosen distinctly from the set \([1, p + m]\) for some integer \(p \geq 0\). Let \(q = e^{-\alpha}\), where \(\alpha\) is the proportionality constant between the energy and the Bethe integers. Then the partition function of \(m\) quasi-particles is

\[
Z_p(q) = \sum_{m \geq 0} q^{m(m+1)/2} \left[ \begin{array}{c} p + m \\ m \end{array} \right]_q.
\]

where the \(q\)-binomial coefficient is defined as

\[
\left[ \begin{array}{c} p + m \\ m \end{array} \right]_q = \prod_{j=1}^{m} \frac{(1 - q^{p+j})}{(1 - q^j)}, \quad p \geq m,
\]

and is defined to be zero if \(p < m\). The partition function of fermions on the interval \([1, p + m]\) is

\[
Z_p(q) = \sum_{m \geq 0} q^{m(m+1)/2} \left[ \begin{array}{c} p + m \\ m \end{array} \right]_q.
\]
In the limit $p \to \infty$, this formula becomes

$$Z(q) = \sum_{m \geq 0} q^{m(m+1)/2} \frac{1}{\prod_{j=1}^{m} (1 - q^j)}.$$ 

The important characteristics to note are

1. There is a quadratic function of the particle number $m$ in the exponent. This is the “ground state energy” of a fermionic system with $m$ particles.

2. There is a $q$-binomial coefficient, or its $p \to \infty$ limit, which is just the weighted sum over configurations above the ground state of $m$ fermions.

A slight generalization of fermionic statistics always occurs in the Bethe ansatz solution: The integer $p$ is a linear function of $m$ itself, in addition to the external parameters of the system (such as size).

Moreover, there is in general more than one “color” of quasi-particle, and these have available energy ranges for each color separately. Again, these are free fermions, except for the generalized statistic which hides in the integers $p_i$ for each color: Each $p_i$ is a function of the number $m_j$ of quasi-particles of type $j$ in the system.

Thus a fermionic formula for the (conformal, linearized version of the) partition on the finite lattice might have the form

$$Z(q) = \sum_{m}^{(1)} q^{Q(m)} \prod_{i} \left[ \frac{p_i + m_i}{m_i} \right]_q .$$

(1)

Here, $m \in \mathbb{Z}^k_+$ for some $k$. The ground state energy $Q(m)$ is a quadratic function of the particle content $m$ which depends on the model, as are $p_i$, which are in general linear functions of $m$, and may tend to infinity as the size of the system becomes infinite. Here, the superscript $(1)$ on the summation indicates possible restrictions on the summation variables corresponding to symmetry sectors of the Hamiltonian. A finite system will have only a finite number of terms in the summation. Moreover there may be several different symmetry sectors of the Hamiltonian, in which case the partition function can be projected to the different sectors separately.

If the model has a conformal limit (the size of the system is infinite while the spectrum remains linearized, that is, the system remains critical), the partition function – properly normalized and restricted – tends in the limit to the graded character of some Virasoro module. That is, $Z(q)$ is proportional to the trace of $q^{L_0}$ over the module, where $L_0$ is the grading element of the Virasoro algebra. A conformal field theory is built out of such modules. This gives a direct connection between the spectrum of the lattice model and the conformal field theory in certain cases.

2.1. The Fock space as a limit of the reduced wedge product.

It is well known that the basic representation of the affine algebra $\widehat{sl}_n$ can be realized as a quotient of the Fock space of free fermions by a Heisenberg algebra.
It is possible to give a finite-dimensional version of this construction \[27\]. As it is closely connected to the graded tensor product construction introduced below in Section 2.4, we briefly summarize it. This finite-dimensional fermionic space gives – in the inductive limit – the Frenkel Kac construction of the level-1 modules.

Let \( V = V(\omega_1) \cong \mathbb{C}^n \) be the defining representation of \( g = \mathfrak{sl}_n \), and \( V(z) = V \otimes \mathbb{C}[z] \) a representation on which \( g^- := g \otimes \mathbb{C}[t^{-1}] \subset \mathfrak{sl}_n \) acts as \( x \otimes f(t^{-1})v = f(z)xv \) with \( x \in g, f(t) \in \mathbb{C}[t], \) and \( v \in V(z) \).

Consider the \( N \)-fold tensor product \( V_N(z_1, \ldots, z_N) = V(z_1) \otimes \cdots \otimes V(z_N) \cong V^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N] \) on which \( g^- \) acts by the usual co-product:

\[
\Delta_x(x \otimes f(t)) = \sum_{i=1}^N x^{(i)}f(z_i^{-1})
\]

where \( x^{(i)} \) indicates \( x \) acting on the \( i \)th factor in the tensor product. Obviously, this action commutes with the diagonal action of the symmetric group \( S_N \), simultaneously permuting factors in the tensor product and variables \( z_i \).

It also commutes with the action of the negative part of the Heisenberg algebra \( \mathcal{H}_- \), acting on the space by multiplication by symmetric polynomials in \( z_1, \ldots, z_N \). (Operators of the form \( \text{id} \otimes t^{-n}, n > 0 \).) Thus, we have three commuting actions. We quotient by the action of the Heisenberg, and project onto the alternating representation of \( S_N \), and the result is called the reduced wedge space. It is a finite dimensional space described explicitly as follows.

The quotient by the Heisenberg action is the quotient of \( \mathbb{C}[z_1, \ldots, z_N] \) by symmetric polynomials of positive degree \( I_N \). That is,

\[
V_N[\mathbb{C}]/\text{Im}\mathcal{H}^- = V^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N]/I_N := V^\otimes N \otimes R_N
\]

The space \( R_N \) is isomorphic to the cohomology ring of the Flag variety and to the regular representation of \( S_N \). In particular, it is finite-dimensional. It is graded by the homogeneous degree in \( z_i \) and the action of the symmetric group preserves the graded components. Thus,

\[
R_N \cong \bigoplus_{\lambda \vdash N} W_\lambda \otimes M_{\lambda, N},
\]

where \( W_\lambda \) are the irreducible representations of \( S_N \) and \( M_{\lambda, N} \) is a graded multiplicity space. We also have the decomposition

\[
V^\otimes N \cong \bigoplus_{\nu \vdash N} \bigoplus_{l(\nu) \leq n} V(\varpi) \otimes W_\nu
\]

where \( \varpi \) is the partition \( \nu \) stripped of its columns of length \( n \), and \( V(\lambda) \) are irreducible finite-dimensional representations of \( g \).
Taking the tensor product with $R_N$ and projecting onto the alternating representation with respect to the diagonal action of $S_N$, we identify $\nu = \lambda^t$. Thus, the reduced wedge space is isomorphic to

$$F_N \simeq \oplus M_{\mu^t, N},$$

where $\lambda^t$ is the transpose of $\lambda$. The Hilbert polynomial of $M_{\lambda^t, N}$ is a Kostka polynomial. In the limit as $N \to \infty$, the properly normalized coefficient of $V(\lambda)$ is a character of the $W$-algebra, which is the centralizer of $\mathfrak{g}$ acting on the level-1 module of $\hat{\mathfrak{g}}$, and the character of $F_N$ tends to the character of the basic representation of the affine algebra. Thus the reduced wedge product is a truncation of this space, a Demazure module.

We will give fermionic formulas for the generalizations of this Kostka polynomial below.

### 2.2. The Hilbert space of the generalized Heisenberg model.

We now give a very general setting which gives rise to fermionic partition functions. The wedge space in the previous section is a special case of this construction.

The fermionic formula of type (1) appears in particular in the generalized Heisenberg spin chain with periodic boundary conditions. This is a quantum spin chain, whose Hamiltonian is derived via the $R$-matrix which intertwines tensor products of Yangian modules $Y(\mathfrak{g})$. The simplest case of this is known as the XXX spin chain, which was the subject of Bethe’s original ansatz [4].

To define this spin chain, choose a the following data:

1. Any finite-dimensional Yangian module $V_0$. This is known as the auxiliary space.

2. A sequence of $N$ Yangian modules $\{V_1, ..., V_N\}$ of KR-type (see below).

The choice of non-isomorphic representations $V_i, i > 0$ is the anisotropy of the model.

Let $R_{ij} : V_i \otimes V_j \mapsto V_j \otimes V_i$ be the intertwiner of finite-dimensional representations, known as (a rational) $R$-matrix. For generic spectral parameters, the tensor product is irreducible and $R$ is unique, up to scalar multiple. The transfer matrix of the generalized anisotropic Heisenberg model with periodic boundary conditions is the trace over $V_0$ of the matrix $M = R_{0,1}R_{0,2} \cdots R_{0,N}$. The transfer matrix $T_{V_0}$ is an operator on the space $V_1 \otimes V_2 \otimes \cdots \otimes V_N$, which is the Hilbert space of the spin chain, also known as the quantum space.

Since the $R$-matrix satisfies the Yang-Baxter equation, it follows easily that the transfer matrices corresponding to different auxiliary spaces commute. Expanding the transfer matrix as a series in the spectral parameter of $V_0$, each of the coefficients in the expansion – an element in an algebra acting on the Hilbert space – commutes with the other coefficients. These coefficients therefore form a family of commuting integrals of motion. The spin chain is a quantum integrable system. The quantum spin chain Hamiltonian is one of the integrals.

This model has a Bethe ansatz solution, at least when the modules $\{V_i\}$ are of Kirillov-Reshetikhin (KR)-type [31] [31] [5]. Such modules are parameterized
by a highest weight with respect to the Cartan subalgebra of $g \subset Y(g)$ and a spectral parameter. The highest weight of a KR-module is a multiple of one of the fundamental weights of $g$.

The eigenvectors and eigenvalues of the Hamiltonian are given by solutions of the Bethe equations. Solutions are parameterized in terms of sets of distinct integers in the same manner described above. The linearized spectrum is proportional to the sum of these integers.

For this particular model, there is an arbitrary number of quasi-particle species or “colors” for each root of the Lie algebra $g$, which obey generalized fermionic statistics. The statistics depends only on the Cartan matrix and the highest weights of $\{V_i\}$. We will write down this function explicitly, as it is key to the rest of the paper (we restrict our attention here to simply-laced $g$ here for simplicity; The other cases are explained in [23, 1, 18]).

The Hilbert space of the model is $V_1 \otimes \cdots \otimes V_N$, so its dimension is $\prod_{i=1}^N |V_i|$. (The ordering of these representations does not effect the spectrum.) The partition function of the linearized spectrum gives a graded version of this dimension, and we will provide a representation theoretical interpretation of this grading.

Let $\lambda_1, \ldots, \lambda_N$ be the highest weights of $V_1, \ldots, V_N$ respectively. Each $\lambda_i$ is a multiple of one of the fundamental weights, and therefore the choice of highest weights is parameterized by a multi-partition

$$\nu = (\nu^{(1)}, \ldots, \nu^{(r)}), \quad \nu^{(a)} \vdash n_a,$$

where the non-negative integers $n = (n_1, \ldots, n_r)$ are defined by $\sum_{i=1}^N \lambda_i = \sum_{a=1}^r n_a \omega_a$ and $r$ is the rank of the algebra.

Define a set of integers $m = (m_1, \ldots, m_r)$ as follows:

$$C m = n - \ell, \quad \ell_a = \langle \alpha_a, \lambda \rangle.$$

for any choice of a dominant integral weight $\lambda$ such that $m \in \mathbb{Z}^r_+$. The evaluation of the character $\chi_z V(\lambda)$ of the irreducible $g$-module $V(\lambda)$ at $z = (1, \ldots, 1)$ is the dimension of $V(\lambda)$.

**Theorem 2.1.** The linearized partition function $Z_{\nu}(q)$ is the evaluation at $z = (1, \ldots, 1)$ of

$$M_{\nu}(q; z) = \sum_{\lambda} M_{\nu, \lambda}(q) \chi_{z}(V_{\lambda}),$$

where

$$M_{\nu, \lambda}(q) = \sum_{\mu \vdash m} q^{Q(\mu)} \prod_{a=1}^r \prod_{j \geq 1} \left[ \begin{array}{c} p_{j}^{(a)} + \mu_j^{(a)} - \mu_{j+1}^{(a)} \\ \mu_j^{(a)} - \mu_{j+1}^{(a)} \end{array} \right]_q.$$

The sum extends over all multipartitions $\mu^{(a)}$ of $m_a$, and

- The integers $p_{a,j}$ are the sum over the first $j$ rows of the integer sequence $\pi^{(a)} = \nu^{(a)} - \sum_b C_{a,b} \mu^{(b)}$;
The quadratic function in the exponent is

\[ Q(\mu) = \frac{1}{2} \sum_{a,b=1}^{r} \sum_{i \geq 1} \mu_i^{(a)} C_{a,b} \mu_i^{(b)} . \]

**Theorem 2.2** (Combinatorial Kirillov-Reshetikhin conjecture, [18]). *The sets of Bethe ansatz integers correctly count the dimension of the Hilbert space of the anisotropic Heisenberg model.*

That is, when evaluated at \( q = 1 \), Equation (3) gives an expression for the dimension of the space of \( g \)-linear homomorphisms from the tensor product of KR-modules to the irreducible representation \( V(\lambda) \). This was known as the Kirillov-Reshetikhin conjecture [23].

**Remark 2.3.** The sum in (3) is known as the “\( M \)-sum” in the language of [23]. There is a similar sum called the “\( N \)-sum”, where the definition of the \( q \)-binomial coefficient is continued to values of \( p < 0 \) by

\[ \left[ \begin{array}{c} p+m \\ m \end{array} \right]_q = \frac{(q^{p+1}; q)_\infty (q^{m+1}; q)_\infty}{(q; q)_\infty (q^{p+m+1}; q)_\infty}, \quad (a; q)_\infty = \prod_{i \geq 0} (1 - aq^i). \]

The fact that \( N(q) = M(q) \) is highly non-trivial; it was first conjectured by [23], who showed that the \( N \)-sum gave the correct dimension of the tensor product. It was later proven in [18, 19] and shown to be closely tied with the Laurent property [15] of the quantum cluster algebra [3] associated with the \( Q \)-system, defined below.

**Remark 2.4.** The sum (3) is a generating function for certain Betti numbers of quiver varieties in special cases [34], see also more recent work giving a geometric context [32].

2.3. Space of conformal blocks in WZW theory. The formula for the linearized partition function of the Heisenberg spin chain is of interest for several reasons.

First, it is known that, in special stabilized infinite limits, its conformal limit is the Wess-Zumino-Witten model at a level which depends on the representations \( V_i \).

**Example 2.5.** Let \( g = \mathfrak{sl}_2 \), set \( V_i = V(k\omega_1) \) for all \( i = 1, ..., N \), and consider the limit as the number of representations, \( N = 2M \) becomes infinite. Then the limit \( M \to \infty \) of the normalized partition function \( \lim_{M \to \infty} \mathcal{Z}_{2M}(q; z) \) is the character of the level-\( k \) module of the affine Lie algebra \( \widehat{\mathfrak{sl}}_2 \) with highest weight \( k\Lambda_0 \).

**Example 2.6.** Let \( g = \mathfrak{sl}_n \) and \( V_i = V(\omega_1) \simeq \mathbb{C}^n \). Then in the limit \( N \to \infty \) the normalized, linearized partition function (3) is a Kostka polynomial [27]. In the conformal limit, this gives a character of the \( W_n \)-algebra which centralizes the action of \( g \) when acting on the level-1 modules.
Another important role of the linearized partition function of the Heisenberg model is that it gives the dimension (at $q = 1$) of the space of conformal blocks of WZW theory (when $k \gg 1$ is an integer). This is the dimension of the moduli space of holomorphic vector bundles on a Riemann surface with $N$ punctures, with specified monodromy given by the representations $V_i$, which are taken to be arbitrary $\hat{g}$-modules induced from KR-modules, localized at distinct points. It is also known as the space of coinvariants.

**Remark 2.7.** The reason we take $k$ to be integer is that the integrality property of the representations is used in the proof of the statement. The reason we require $k \gg 1$ is that for finite $k$, one has the Verlinde coefficients rather than the Littlewood Richardson coefficients for multiplicities of the irreducibles in the tensor product of integrable modules affine algebra modules. We did not take this into account in [3]. A separate conjecture for the fermionic formula of the linearized partition function of this space can be found in [11]. If $k$ is sufficiently large, the multiplicity is just as a sum of products of Littlewood Richardson coefficients or their generalization.

We have a graded version of the dimension of the moduli space, meaning we keep track of a certain grading or a refinement of the space. It is known that, in special cases, this corresponds to keeping track of the Betti numbers for a certain quiver variety, giving a geometric meaning to the graded dimensions.

**2.4. A grading on the tensor product.** It is known that the Hilbert space of the Heisenberg model, together with the linearized spectrum of the Hamiltonian, in the limit when the number of representations $V_i$ becomes infinite (taking all $V_i \simeq V$, the defining representation, for example), gives the characters of affine algebras in the limit as the (chiral) conformal partition function. The relevant conformal field theory is the WZW model at level 1.

**Remark 2.8.** There is also an explicit construction of this infinite-dimensional Hilbert space for the XXZ model, using the quantum affine algebra, using a stabilized semi-infinite tensor product [7]. In this case it is possible to construct the transfer matrix in terms of intertwining operators which gives a direct connection with the deformed primary fields of the conformal field theory.

Moreover, we identify the dimension of the Hilbert space of the finite, inhomogeneous Heisenberg model with dimension of the space of conformal blocks (for level $k$ sufficiently large).

These two facts form the motivation for the following definition of a graded tensor product [11]. Whereas there are other definitions of an “energy function” on the tensor product which defines a grading on the tensor product in the case of quantum affine algebras (these correspond to the XXZ model, or the limit $q \to 0$ in the case of the crystal basis), the definition here refers only to the undeformed current algebra.

**Remark 2.9.** KR-modules are defined for three algebras: For the quantum affine algebra $U_q(\hat{g})$, the Yangian $Y(g)$, and the current algebra $\hat{g}$. [31] [5]. One of the
consequences of the theorems of [18] is that these all have the same structure under restriction to the underlying finite dimensional algebra, $\mathfrak{g}$ or $U_q(\mathfrak{g})$ [29]. Here we use only the current algebra version.

**Definition 2.10.** Let $V$ be a cyclic $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$-module, defined by the representation $\pi$. We define the representation $\pi_\zeta$ on $V$ as follows. Given $x \otimes f(t) \in \mathfrak{g}[t]$ and $w \in V$, $\pi_\zeta(x \otimes f(t))w = \pi(x \otimes f(t + \zeta))w$, for some $\zeta \in \mathbb{C}^*$.

That is, the localization takes place at $\zeta$. We use the shorthand $V(\zeta)$ for the module with the action $\pi_\zeta$, even though the vector space itself is simply $V$.

Now pick $V_i(\zeta_i)$ to be KR-modules of $\mathfrak{g}[t]$, with $1 \leq i \leq N$, with $\zeta_i \neq \zeta_j$ for all $i \neq j$. Let $v_i$ be the cyclic, highest weight vector of $V_i$. We have $V_i(\zeta_i) = U(\mathfrak{g}[t])v_i$, and the tensor product is also cyclic (as long as the localization parameters are distinct):

$$V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N) = U(\mathfrak{g}[t])v_1 \otimes \cdots \otimes v_N.$$

(The assumption that we have KR-modules is not essential at this point, only that each of the modules $V_i$ is cyclic.)

The algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded by degree in $t$, and so is its universal enveloping algebra. Let $U_i$ denote the graded component. The action of $U_i$ on the tensor product of cyclic vectors inherits this filtration, and therefore we have a filtration of the tensor product itself. The associated graded space of this filtered space is called the Feigin-Loktev “fusion” product, $\mathcal{F}_V$.

**Theorem 2.11** ([1, 18, 29]). The associated graded space is isomorphic to the tensor product of KR-modules as a $\mathfrak{g}$-module. That is, it is independent of the localization parameters $\zeta_i$.

For the proof of this theorem, it is essential that $V_i$ are of KR-type. The graded $\mathcal{F}_V$ is defined as a quotient space, so in general, its dimension may be greater than the dimension of the tensor product itself. It corresponds to the “collision” of all the points $\zeta_i$.

The theorem about the dimension of this space was proven using a function space realization for the space of conformal blocks (coinvariants), and the use of the Kirillov-Reshetikhin conjecture about the explicit fermionic formula for the dimension of this space [1]. The final step in this proof uses a theorem of [23] and the proof of the “$M = N$” conjecture at $q = 1$ in [18].

Let $\nu$ be the parameterization of the collection of the KR-modules as in (2), and let $\mathcal{F}_{\nu,\lambda}[n] = \text{Hom}_g(\mathcal{F}_{\nu}[n], V(\lambda))$. and consider the Hilbert polynomial

$$\sum_{n \geq 0} q^n \dim \mathcal{F}_{\nu,\lambda}[n].$$

The following strong version of Theorem 2.11 is proven in [19]:

**Theorem 2.12.** ([19]) The Hilbert polynomial of the Feigin-Loktev graded tensor product is equal to the conformal partition function (3).

We will introduce an expression for the partition function (3) as a constant term in the product of solutions of the $Q$-system, a discrete recursion relation, in the next section. At the same time, we will identify the $Q$-system as a mutation in a cluster algebra, which therefore has a natural $q$-deformation. The proof of Theorem 2.12 will uses the methods of [18] applied to this quantum cluster algebra.
Remark 2.13. In special cases, the FL graded tensor product is an affine Demazure module \([17]\), which has a grading by the Cartan element \(d\) of the affine algebra. This grading is essentially the same as the FL-grading. Therefore we are guaranteed that the appropriate semi-infinite graded tensor product is the full affine algebra module.

By definition \([1]\), the idea of an associated graded space is equivalent to taking all the spectral parameters \(\zeta_i \to 0\). The sum over the multipartitions \(\mu\) in equation (3) can be viewed as a sum over all possible desingularizations of this degeneracy (this is evident from the derivation using functional space realization in \([1]\); see also \([36]\)).

3. Difference equations and the fermionic formulas

It was originally observed in the context of the completeness conjecture of the Bethe ansatz, and later by the original attempt at proving the combinatorial Kirillov-Reshetikhin conjecture \([23]\), the fermionic sum \(M_\mu(q; z)\) in Equation (3) is closely related to a difference equation called the \(Q\)-system.

Let \(\chi_{a,k}\) be the character of the KR-module with highest weight \(k\omega_a\), restricted to \(\mathfrak{g} \subset \mathfrak{g}^{[t]}\).

Example 3.1. If \(\mathfrak{g} = \mathfrak{sl}_n\) then the KR-modules are irreducible under the restriction to \(\mathfrak{g}\), and are the modules with “rectangular highest weights”. In that case, \(\chi_{a,k}\) is a Schur function \(S_{(a)^k}(z_1, \ldots, z_n)\) with \(\prod z_i = 1\).

For any Lie algebra, the functions \(\chi_{a,k}\) satisfy a simple difference equation: In the case where \(\mathfrak{g}\) is simply-laced, this is a two-step recursion relation. Consider the system

\[
Q_{k+1}^{(a)}Q_{k-1}^{(a)} = (Q_k^{(a)})^2 - \prod_{b \neq a} (Q_k^{(b)})^{-C_{ab}},
\]

(4)

(The relation is only slightly more cumbersome for non-simply laced algebras, and has a generalization for \(\mathfrak{g}\) an affine algebra.)

A two-step recursion relation has a unique solution given initial data. The natural initial data for the \(Q\)-system is

1. The character of the trivial representation is equal to 1, \(\chi_{a,0} = 1\). Therefore, set \(Q_0^{(a)} = 1\) for all \(1 \leq a \leq r\) where \(r\) is the rank of the algebra.

2. Identify \(Q_1^{(a)}\) with the character of the fundamental KR-modules, \(\chi_{a,1}\).

Theorem 3.2 (\([35]\)). The characters of the Kirillov-Reshetikhin are solutions of the \(Q\)-system (4) with the initial data (1) and (2).

The \(Q\)-system is a specialization of the \(T\)-system, satisfied by the transfer matrices of the XXZ spin chain, or by the \(q\)-characters \([20]\) of the KR-modules.
Remark 3.3. For any given quantum spin chain, one can derive Bethe ansatz equations from different functional relations, obtaining a different set of coupled algebraic equations (the Bethe equations). Although it is standard procedure to use Baxter’s equation to derive Bethe equations, it is also possible to use the $T$-system, see e.g. [6]. The resulting equations, their solution and linearized spectrum, take a different form depending on the original functional relation. This reflects the fact that in the degenerate case of a massless spectrum (the critical point) there may be several different descriptions of the spectrum as a quasi-particle spectrum. This degeneracy is resolved when a massive integrable perturbation is considered.

The transfer matrices satisfy $T$-system relation, a conjecture of Kirillov and Reshetikhin proved (for finite, simply-laced Lie algebras) by Nakajima [35], using the realization of the representation theory of the quantum affine algebra in terms of his quiver varieties. The $T$-system is satisfied by $q$-characters of the KR-modules [20]. This is the algebraic Kirillov-Reshetikhin conjecture. Nakajima even proved a deformed version of the $T$-system which holds for twisted tensor products of KR-modules. Theorem 3.2 follows from this work.

The relation of the $Q$-system to the multiplicity formulas starts as follows. Recall the definition of the “$N$-sum” in Remark 2.3. Then there is a constant term identity for the $N$-sum in terms of solutions of the $Q$-system. Define $Z_{\nu,\lambda}(Q_0, Q_1)^{(k)} = \prod_{a=1}^{r} Q_1^{(a)}(Q_0^{(a)})^{-1} \left( \prod_{i \geq 1}^{(a)}(Q_i^{(a)})^{\nu_i^{(a)} - \nu_i^{(a)} + 1} \right) (Q_k^{(a)}(Q_{k+1}^{(a)})^{-1})^{(\alpha, \lambda)+1}$, where $Q_i^{(a)}$ are solutions of the $Q$-system [4]. Define $\langle Z \rangle$ to be the constant term of $Z$ in $\{Q_1^{(a)}\}_a$, evaluated at $\{Q_0^{(a)} = 1\}_a$.

Theorem 3.4. Let $N_{\nu,\lambda}^{(k)}(1) = \langle Z_{\nu,\lambda}(Q_0, Q_1)^{(k)} \rangle$. Then there exists an integer $J$ such that whenever $k > J$, $N_{\nu,\lambda}^{(k)}(1)$ is independent of $k$, and is equal to $N_{\nu,\lambda}(1)$.

The proof is by induction, using direct computation, starting from the fermionic formula [3].

We still need to show that $N = M$, however. Moreover, we need an identity for the $q$-multiplicities themselves. For this, we do not need to use the representation theoretical interpretation of the $Q$-system. Instead, we will use the Laurent property of cluster algebras.

3.1. $Q$-systems as mutations in a cluster algebra. Here, we give an interpretation of the variables $Q_i^{(a)}$ as cluster variables in a cluster algebra.

A cluster algebra is the commutative algebra generated by the union of cluster variables, defined recursively. It was originally introduced by Fomin and Zelevinsky [15] in a representation theoretical context, but has been shown to have applications
far beyond the original motivation. We refer to Fomin’s ICM lecture notes for a

good overview [14].

We use only the simplest version. Let $B$ be a skew symmetric $n \times n$ integer

matrix (equivalently, a quiver with no 1- or 2- cycles), called the exchange matrix.

Vertices of the quiver are numbered from 1 to $n$ and the integer $B_{ij}$ is the number

of arrows from $j$ to $i$. Let $x = (x_1, \ldots, x_n)$ be formal (commutative) variables

associated with the vertices. Fix $1 \leq j \leq n$ and define $x'_j$

$$x'_j = \prod_{i:j \to i} x_i + \prod_{i:i \to j} x_i.$$  \hfill (6)

This is called a mutation of $x$ in the direction $j$, denoted by the operation $\mu_j$.

If $i \neq j$, $\mu_j(x_i) = x_i$. However, the quiver itself changes under the mutation as

follows:

- For any sequence $i \to j \to k$, add an arrow $i \to k$.
- Reverse any arrows incident to $i$.
- Erase any resulting 2-cycles.

The collection of generators of the cluster algebra is the result of all possible

sequences of mutations of $x$. The pair $(B, x)$ is called the seed data.

Any $Q$-system (that is, Equation (4) and its generalizations), can be shown to

be a mutation in a cluster algebra [28]. In the case of (4), it is the clust er algebra

defined by the seed data $(x_0, B)$ where

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}, \quad x_0 = (Q_0^{(1)}, \ldots, Q_0^{(r)}; Q_1^{(1)}, \ldots, Q_1^{(r)}).$$  \hfill (7)

Note that we do not impose $Q_0^{(a)} = 1$ at this stage.

The $Q$-system equations are a special subset of the mutations of the cluster algebra [1]. They can be shown to be the equations a discrete integrable system [9], with the integrals of motion given by those of the Toda system [22]. This is due to

the existence of an integrable Poisson structure compatible with the cluster algebra

structure. Such a Poisson structure can always be deformed to give a quantum system, which in the case of cluster algebras is called a quantum cluster algebra [3, 13] (see below).

Any cluster algebra (and a much larger class of discrete rational evolution equations) can be shown to have a Laurent property. The transformation (6) is a

rational transformation. Although it is obvious after a single mutation, it is not at

all obvious after several steps of mutations that the rational function is, in fact, a

Laurent polynomial in the seed data, because the term in the denominator is itself

a polynomial in the initial seed data.

\footnote{Although traditionally, the coefficients in a cluster algebra are taken to be +1, we keep the minus sign in the current context. This can be dealt with by (1) renormalizing the $Q$-variables or (2) introducing coefficients [8]. However this is irrelevant in the current context.}
Theorem 3.5 (Laurent property [16]). Any cluster variable in a cluster algebra is a Laurent polynomial in the cluster variables of any other seed in the cluster algebra.

Taking the $Q$-system with the initial seed data consistent with the character interpretation, this implies the following:

Theorem 3.6 ([8]). Any cluster variable (not just solutions of the $Q$-system) in the cluster algebra with seed data (7) is a polynomial in the variables $(Q_1^{(1)}, \ldots, Q_1^{(r)})$ after evaluation at $Q_0^{(a)} = 1$.

**Proof.** This is a consequence of the Laurent phenomenon and the fact that the right hand side (the numerator in the exchange relation) of (4) vanishes at $k = 0$.

We illustrate this for the case of $g = \mathfrak{sl}_2$. The generalization is clear. Let $x$ be a cluster variable in the cluster algebra. Then $x(Q_{-1}; Q_0) = Q_0^{-m} \sum_{n \in \mathbb{Z}} p_n(Q_0) Q_1^n$, where $p_n$ is a polynomial. Performing the exchange $Q_1 = N(Q_0)/Q_{-1}$, we have

$$x = Q_0^{-m} \sum_{n \in \mathbb{Z}} p_n(Q_0) Q_1^{-n} N(Q_0)^n.$$ 

If $n < 0$, since $N(Q_0)^n$ in the denominator is a polynomial, it must cancel with the term $p_n(Q_0)$ in the numerator, because the result must be a Laurent polynomial. That is, for any $n < 0$, $p_n(Q_0)$ is divisible by $N(Q_0)$. (Up until this point, the argument holds for any bi-partite cluster algebra of any rank.) Therefore, $p_n(1) = 0$ for any $n < 0$. Therefore, the cluster variable $x$ under this evaluation has only terms $Q_1^n$ with $n \geq 0$. The generalization to arbitrary rank relies on the identical argument: All variables and indices should be changed to multi-variables and multi-indices.

When applied to the $Q$-system, the theorem implies that the KR-modules are generated as Groethendieck ring by the fundamental KR-modules.

The importance of polynomiality is in the proof of the “$M = N$ conjecture” [23] which is the final step in the proof of the combinatorial KR-conjecture [18, 29] and hence the Feigin-Loktev conjectures [1].

Theorem 3.7 ([18]). The constant term in $Q_1^{(a)}$ of $Z_{\nu, \lambda}(Q_0 = 1, Q_1)$ has no contributions from terms in the summation in which any of the integers $p_{a,i} < 0$. That is, $M_{\nu, \lambda}(1) = N_{\nu, \lambda}(1)$.

The Laurent phenomenon and the polynomiality theorem generalize to the quantum $Q$-system.

### 3.2. The quantum $Q$-system from quantum cluster algebras.

We are interested in the $q$-graded version of Theorem [57]. This is obtained by using a $q$-deformation of the $Q$-system. There is a constant term identity for the graded partition function [3] in terms of the solutions of the quantum $Q$-system. Aside from enabling us to prove that “$N(q) = M(q)$”, it gives yet another interpretation of the grading of the multiplicity.

Given any skew-symmetric exchange matrix, one can define a quantum cluster algebra [3, 13], a deformation of the compatible Poisson structure of the cluster algebra [21]. A quantum cluster algebra is a non-commutative algebra generated
by the seed data obeying $q$-commutation relations, together with all its mutations. The combinatorial data is the same as in the classical case, and the exchange matrix is still the same matrix $B$.

Performing this deformation for the cluster algebra of the $Q$-system, one obtains a quantum $Q$-system:

$$t^{\Lambda_{a,b}} Q_{k+1}^{(a)} Q_{k-1}^{(a)} = (Q_k^{(a)})^2 - \prod_{b \neq a} (Q_k^{(a)}) - C_{a,b},$$

where $Q_k^{(a)}$ generate a non-commutative algebra defined by $[8]$ and the commutation relations

$$Q_n^{(a)} Q_{n+1}^{(b)} = t^{\Lambda_{a,b}} Q_{n+1}^{(b)} Q_n^{(a)},$$

with $\Lambda = |C|^{-1}$. The variables $\{Q_n^{(1)}, \ldots, Q_n^{(r)}\}$ commute.

We will eventually identify $q = t^{-|C|}$ in our derivation of the $M$-sums below.

Given initial seed data $x_0 = (Q_0^{(a)}, Q_1^{(a)})_{a \in [1,r]}$, any cluster variable can be expressed as a Laurent polynomial in the initial seed data, with coefficients in $\mathbb{Z}[t, t^{-1}]$ (the Laurent phenomenon for quantum cluster algebras was proven in [3]). Therefore, any Laurent polynomial $M$ of cluster variables can be expressed as a Laurent polynomial in terms of any initial cluster seed, for example, $\{Q_0^{(a)}, Q_1^{(a)}\}$. By using the commutation relations [3], this Laurent polynomial can be written in a normal ordered form, as a finite sum

$$M = \sum_{n,m \in \mathbb{Z}} \prod_{a=1}^r (Q_0^{(a)})^{m_a} \prod_{b=1}^r (Q_1^{(b)})^{n_b} f_{m,n}(t)$$

where $f_{m,n}(t) \in \mathbb{Z}[t, t^{-1}]$.

We define the analogue of “a constant term” identity in the quantum case by taking the constant term of this expression in $Q_1^{(a)}$, and by evaluating at $Q_0^{(a)} = 1$.

**Definition 3.8.** Given a Laurent polynomial $M$ in $\{Q_0^{(a)}, Q_1^{(b)}\}_{a,b}$, define its constant term evaluated at $Q_0^{(a)} = 1$ by first, defining the coefficients $f_{m,n}(t) \in \mathbb{Z}[t, t^{-1}]$ as in (10), then defining

$$(M) = \sum_m f_{m,0}(t).$$

(Note that it is important to perform the evaluation after normal ordering the expression, otherwise, we miss out on the $t$-grading.)

The quantum Laurent property can be shown to imply that for the quantum $Q$-system, any cluster variable, after evaluation at $Q_0^{(a)} = 1$, is in fact a polynomial in $\{Q_1^{(a)}\}_a$ with coefficients in $\mathbb{Z}[t, t^{-1}]$ (the analog of theorem 3.6).

For a given finite sequence $\nu$ and a fixed $k$, define

$$M^{(k)}_{\nu,\lambda} = \prod_{a=1}^r (Q_1^{(a)} (Q_0^{(b)})^{-1}) \prod_{i \geq 1} \prod_{a=1}^r (Q_i^{(a)})^{\nu_i^{(a)} - i^{(a)}} \prod_{a=1}^r (Q_k^{(a)} (Q_{k+1}^{(a)})^{-1})^{(\omega + \lambda)+1}$$

(11)
Again, when \( k \) is sufficiently large, \( \langle M_{\nu, \lambda}^{(k)} \rangle \) is independent of \( k \).

Upon multiplying by an appropriate power of \( q \) and identifying the deformation parameter \( t \) of the cluster algebra as \( q = t^{-|C|} \), we have

**Theorem 3.9** (Constant term identity [19]).

\[
M_{\nu, \lambda}(q^{-1}) = q^{h(\nu, \lambda)} \langle M_{\nu, \lambda} \rangle.
\]

for \( k \) sufficiently large. Here the normalization factor is

\[
h(\nu, \lambda) = -\frac{1}{2} \sum_{a,b=1}^{\infty} \sum_{i \geq 1} C_{ab}^{-1} \nu_i^{(a)} \nu_i^{(b)} - \frac{1}{2} \sum_{a=1}^{r} \sum_{b=1}^{r} C_{aa}^{-1} \ell_a - \sum_{a,b=1}^{r} C_{ab}^{-1} \nu_i^{(b)}.
\]

The polynomiality property, which follows from the Laurent property for the quantum \( Q \)-system, implies

**Lemma 3.10** ([19]). The cluster variables in the quantum cluster algebra corresponding to the \( Q \)-system, after normal ordering and evaluation at \( Q_0^{(a)} = 1 \) for all \( a \), are polynomials in \( \{Q_0^{(a)}\}_a \).

Thus, we have the graded version of the \( M = N \) identity:

**Theorem 3.11** ([19]). In the summation in Equation (3), terms with \( p_{a,i} < 0 \) do not contribute to the sum in the \( q \)-graded version of the identity. That is, \( M_{\nu, \lambda}(q) = N_{\nu, \lambda}(q) \).

### 4. Difference equations

So far, we have said nothing about the integrability of the \( Q \)-system and its \( Q \)-deformed version. But in fact, this is a two-step recursion relation of rank \( r \), and it has \( r \) integrals of the discrete evolution (which are in involution with each other with respect to the Poisson structure of the cluster algebra, or the commutation relations of the quantum cluster algebra). In type \( A \) for example, the solutions \( Q_k^{(a)} \) satisfy linear recursion relations with \( r + 2 \) terms, and with coefficients which are integrals of the motion (or constants).

These integrals of the motion can be used to find differential/difference equations satisfied by generating functions for partition functions (characters of graded tensor products). This derivation is analogous to the construction of the Whittaker functions, which are solutions of the quantum Toda equations in the case of classical Lie groups, where the integrals of the motion are the Casimir elements of the algebra [33]. More recently there has been a certain interest in the so-called Gaiotto vector, which is the analog of the Whittaker vector for Virasoro algebras, or some degenerate version thereof.

Since certain stabilized limits of the graded tensor products tend to various Virasoro modules or integrable affine algebra modules, it is useful to first write these equations for the finite tensor product. The result are Toda-like equations
satisfied by the generating function (the relation of fermionic character formulas and Toda equations was noted in, e.g. [12]). This can be used to derive difference equations satisfied by the stabilized limits of the graded tensor products, and even solve them in special cases. One can obtain the character formulas of Feigin and Stoyanovskii, or of spinon type, by analyzing these difference equations.

The analog of the Whittaker function in the case of the graded tensor product is the generating function [10]

\[ G(q; z, y) = \sum_{\nu, \lambda} q^{f_1(\nu)} ch_z V(\lambda) M_{\nu, \lambda}(q) \prod_{a,i} (y_{a,i})^{\nu_{a,i} - \nu_{a,i+1}} \]

with \( f_1(\nu) = \frac{1}{2} \sum_{a,b,i} \nu_{a,i} C_{a,b}^{-1} \nu_{b,i} + \sum_{a,b} C_{ab}^{-1} \nu_{b,i} \). Using the factorization formula of Section 3.2 this has a very simple form:

\[ G(q; z, y) = \left\{ \prod_{a=1}^r Q_1^{(a)} (Q_0^{(a)})^{-1} \prod_{a=1}^r \prod_{j \geq 1} (1 - y_j^{(a)} Q_j^{(a)})^{-1} \tau(z) \right\} \]

where

\[ \tau(z) = \lim_{k \to \infty} \sum_{\lambda} q^{f_2(\lambda)} ch_z V(\lambda) \prod_{a=1}^r (Q_k^{(a)} (Q_{k+1}^{(a)})^{-1})^{(\alpha_a, \lambda)+1} \]

where \( f_2(\lambda) = -\frac{1}{2} \sum_a C_{a,a}^{-1} \ell_a \).

We claim that \( \tau(z) \) plays the role of the Whittaker vector, with the role of the Casimir elements played by the discrete integrals of motion of the \( Q \)-system. These act by scalars on this function, whereas they act as \( q \)-difference operators on the product of \( Q \)'s to the left. This is the origin of the difference equations satisfied by the partition functions.

5. Summary

We reviewed here the role played by the fermionic formulas for the characters of graded tensor products of current algebra modules, their close connection with discrete integrable equations called \( Q \)-systems, and their \( q \)-deformations. In the process, we used the formulation of these systems in terms of (quantum) cluster algebras, and found that the grading coming from the affine algebra action on the tensor product can be identified with the grading coming from the \( q \)-deformation of the cluster algebra, hence from the natural Poisson structure satisfied by the cluster algebra variables. The resulting graded tensor products, in the stabilized, semi-infinite limit, give a construction of affine algebra or Virasoro modules. The integrability of the quantum \( Q \)-system is closely connected with difference equations satisfied by the characters of these modules.
6. References

References

[1] Eddy Ardonne and Rinat Kedem. Fusion products of Kirillov-Reshetikhin modules and fermionic multiplicity formulas. *J. Algebra*, 308(1):270–294, 2007.

[2] Rodney J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989. Reprint of the 1982 original.

[3] Arkady Berenstein and Andrei Zelevinsky. Quantum cluster algebras. *Adv. Math.*, 195(2):405–455, 2005.

[4] H. Bethe. Theorie der metalle i. eigenwerte und eigenfunktionen der linearen atomkette. *Zeitschrift fr Physik*, (71):2005–6, 1931.

[5] Vyjayanthi Chari. On the fermionic formula and the Kirillov-Reshetikhin conjecture. *Internat. Math. Res. Notices*, (12):629–654, 2001.

[6] Srinandan Dasmahapatra, Rinat Kedem, and Barry M. McCoy. Spectrum and completeness of the three state superintegrable chiral potts model. *Nucl.Phys.*, B396:506–540, 1993.

[7] Brian Davies, Omar Foda, Michio Jimbo, Tetsuji Miwa, and Atsushi Nakayashiki. Diagonalization of the XXZ Hamiltonian by vertex operators. *Comm. Math. Phys.*, 151(1):89–153, 1993.

[8] Philippe Di Francesco and Rinat Kedem. Q-systems as cluster algebras. II. Cartan matrix of finite type and the polynomial property. *Lett. Math. Phys.*, 89(3):183–216, 2009.

[9] Philippe Di Francesco and Rinat Kedem. Q-system cluster algebras, paths and total positivity. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 6:Paper 014, 36, 2010.

[10] Philippe Di Francesco and Rinat Kedem. unpublished, 2014.

[11] B. Feigin and S. Loktev. On generalized Kostka polynomials and the quantum Verlinde rule. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 61–79. Amer. Math. Soc., Providence, RI, 1999.

[12] Boris Feigin, Evgeny Feigin, Michio Jimbo, Tetsuji Miwa, and Evgeny Mukhin. Fermionic formulas for eigenfunctions of the difference Toda Hamiltonian. *Lett. Math. Phys.*, 88(1-3):39–77, 2009.

[13] V. V. Fock and A. B. Goncharov. Cluster X-varieties, amalgamation, and Poisson-Lie groups. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 27–68. Birkhäuser Boston, Boston, MA, 2006.

[14] Sergey Fomin. Total positivity and cluster algebras. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 125–145. Hindustan Book Agency, New Delhi, 2010.

[15] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[16] Sergey Fomin and Andrei Zelevinsky. The Laurent phenomenon. *Adv. in Appl. Math.*, 28(2):119–144, 2002.
Fermionic spectra in integrable models

[17] G. Fourier and P. Littelmann. Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Adv. Math.*, 211(2):566–593, 2007.

[18] Philippe Di Francesco and Rinat Kedem. Proof of the combinatorial Kirillov-Reshetikhin conjecture. *Int. Math. Res. Not. IMRN*, (7):Art. ID rnm006, 57, 2008.

[19] Philippe Di Francesco and Rinat Kedem. Quantum cluster algebras and fusion products. *Int. Math. Res. Not. IMRN*, (doi: 10.1093/imrn/rnt004), 2013.

[20] Edward Frenkel and Nicolai Reshetikhin. The \(q\)-characters of representations of quantum affine algebras and deformations of \(w\)-algebras. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 163–205. Amer. Math. Soc., Providence, RI, 1999.

[21] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. Cluster algebras and Poisson geometry. *Mosc. Math. J.*, 3(3):899–934, 1199, 2003. {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.

[22] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. *Cluster algebras and Poisson geometry*, volume 167 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[23] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[24] Michio Jimbo, Tetsuji Miwa, and Feodor Smirnov. Fermions acting on quasi-local operators in the XXZ model. In *Symmetries, integrable systems and representations*, volume 40 of *Springer Proc. Math. Stat.*, pages 243–261. Springer, Heidelberg, 2013.

[25] V. G. Kac and A. K. Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of *Advanced Series in Mathematical Physics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[26] R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer. Fermionic sum representations for conformal field theory characters. *Phys. Lett. B*, 307(1-2):68–76, 1993.

[27] Rinat Kedem. Fusion products, cohomology of \(GL_N\) flag manifolds, and Kostka polynomials. *Int. Math. Res. Not.*, (25):1273–1298, 2004.

[28] Rinat Kedem. \(Q\)-systems as cluster algebras. *J. Phys. A*, 41(19):194011, 14, 2008.

[29] Rinat Kedem. A pentagon of identities, graded tensor products, and the Kirillov-Reshetikhin conjecture. In *New trends in quantum integrable systems*, pages 173–193. World Sci. Publ., Hackensack, NJ, 2011.

[30] Rinat Kedem and Barry M. McCoy. Construction of modular branching functions from Bethe’s equations in the 3-state Potts chain. *J. Statist. Phys.*, 71(5-6):865–901, 1993.

[31] A. N. Kirillov and N. Yu. Reshetikhin. Formulas for the multiplicities of the occurrence of irreducible components in the tensor product of representations of simple Lie algebras. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 205(Differentialnye Geom. Gruppy Li i Mekh. 13):30–37, 179, 1993.

[32] Ryosuke Kodera and Katsuyuki Naoi. Loewy series of Weyl modules and the Poincaré polynomials of quiver varieties. *Publ. Res. Inst. Math. Sci.*, 48(3):477–500, 2012.
[33] Bertram Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

[34] G. Lusztig. Fermionic form and betti numbers. [ArXiv:math/0005010](https://arxiv.org/abs/math/0005010).

[35] Hiraku Nakajima. $t$-analogs of $q$-characters of Kirillov-Reshetikhin modules of quantum affine algebras. *Represent. Theory*, 7:259–274 (electronic), 2003.

[36] A. V. Stoyanovskii and B. L. Feigin. Functional models of the representations of current algebras, and semi-infinite Schubert cells. *Funktsional. Anal. i Prilozhen.*, 28(1):68–90, 96, 1994.

Department of Mathematics MC-382, University of Illinois, Urbana, IL 61801 USA
E-mail: rinat@illinois.edu