Traveling-wave solutions in continuous chains of unidirectionally coupled oscillators

S D Glyzin$^{1,2}$, A Yu Kolesov$^1$ and N Kh Rozov$^3$

$^1$ Faculty of Mathematics, Yarostavl State University, Sovetskaya str., 14, Yaroslavl, 150000, Russia
$^2$ Scientific Center in Chernogolovka RAS, Lesnaya str., 9, Chernogolovka, Moscow region, 142432, Russia
$^3$ Faculty of Mechanics and Mathematics, Moscow State University, Leninskiye Gory, Main Building, Moscow, 119991, Russia

E-mail: glyzin@uniyar.ac.ru; kolesov@uniyar.ac.ru; fpo.mgu@mail.ru

Abstract. Proposed is a mathematical model of a continuous annular chain of unidirectionally coupled generators given by certain nonlinear advection-type hyperbolic boundary value problem. Such problems are constructed by a limit transition from annular chains of unidirectionally coupled ordinary differential equations with an unbounded increase in the number of links. It is shown that any preassigned finite number of stable periodic motions of the traveling-wave type can coexist in the model.

1. Introduction

The mathematical model of an annular chain of unidirectionally coupled generators may be given by a system of ordinary differential equations of the form

$$\dot{u}_j = F(u_j) + m D (u_{j+1} - u_j), \quad j = 1, \ldots, m, \quad u_{m+1} = u_1. \quad (1)$$

where $m \geq 2$, the dot means differentiation with respect to $t$, $u \in \mathbb{R}^n$, with $n \geq 2$, the vector function $F(u)$ belongs to the class $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $D$ is an $n \times n$ square matrix. It is typically assumed that the partial system

$$\dot{u} = F(u) \quad (2)$$

corresponding to chain (1) has an equilibrium or a cycle.

Investigating chains of form (1) is relevant in view of various applications to radiophysics [1] – [5] and neurodynamics [6] – [8]. In particular, it is interesting to study the behavior of attractors of such chains as the number $m$ of links increases without bound. Here, we study this problem in the case where partial system (2) has a stable zeroth equilibrium and the coupling matrix $D$ is small in a suitable sense. For $m \gg 1$, it is totally appropriate to pass from discrete chain (1) to the corresponding continuous model. The transition procedure consists in approximating the variable $j/m$ by a continuous index $x \in [0,1](\text{mod 1})$ and replacing the term $m(u_{j+1} - u_j)$ in (1) with the derivative $\partial u / \partial x$. As a result, we obtain the boundary value problem

$$\frac{\partial u}{\partial t} = F(u) + D \frac{\partial u}{\partial x}, \quad u(t,x+1) \equiv u(t,x), \quad (3)$$
which is a mathematical model of a continuous annular chain of unidirectionally coupled generators.

We consider problem (3) as an evolutionary equation in a Banach space $E$ consisting of vector functions $u(x)$ of class $W^{1,2}([0,1];\mathbb{R}^n)$ that are periodic with period 1. We define the norm in $E$ as

$$\|u\|_E = \left( \int_0^1 \|u(x)\|^2 dx + \int_0^1 \|u'(x)\|^2 dx \right)^{1/2}.$$

Here and below, the same symbol $\|\cdot\|$ denotes the Euclidean norms in the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as the induced matrix norms corresponding to these norms. Next, we introduce the closed unbounded operator $L = Dd/dx : E \to E$ with the domain of definition $D(L)$ dense in $E$. The following statement is valid.

**Lemma 1.** The operator $L$ is a generator of the semigroup $\exp(Lt), t \geq 0$

$$\exp(Lt)u(x) = \sum_{k \in \mathbb{Z}} \exp(2\pi ikDt)u_k \exp(2\pi ikx),$$

where

$$u(x) = \sum_{k \in \mathbb{Z}} u_k \exp(2\pi ikx), \quad u_k = \int_0^1 u(x) \exp(-2\pi ikx) dx, \quad k \in \mathbb{Z}$$

of class $C_0$ of linear bounded operators in $E$ if and only if the whole spectrum of the matrix $D$ lies on the real axis and there exists a basis of its eigenvectors in the space $\mathbb{R}^n$.

This lemma indicates that as $m \to \infty$, the chain (1) by no means always generates a well-posed boundary value problem (3). This naturally provokes constraints on the matrix $D$. Everywhere in what follows, we assume that all its eigenvalues are real and simple. We are then obviously in the framework of the applicability of Lemma 1; moreover, semigroup (4), (5) also exists for small perturbations of $D$. Hence, for the vector function $v(t) = u(t, \cdot) \in E$, we have the well-defined abstract semilinear differential equation

$$\dot{v} = Lv + \mathcal{F}(v), \quad \mathcal{F}(v(t)) \overset{\text{def}}{=} F(u(t, \cdot)).$$

In addition, by virtue of the well-known regularity theorem (see [9]), the composition operator $\mathcal{F} : E \to E$ appearing in (6) is infinitely Frechet differentiable and bounded on any bounded subset $\Omega \subset E$ together with any finite number of its derivatives.

The existence of semigroup (4) allows defining the notion of a generalized solution of Eq. (6) from $E$ with an arbitrary initial condition

$$v(0) = v_0$$

Namely, a generalized solution is a continuous $E$-valued vector function $v(t)$ that satisfies the following integral equation

$$v(t) = \exp(Lt)v_0 + \int_0^t \exp \left( L(t-s) \right) \mathcal{F}(v(s)) ds.$$  

on some interval $0 \leq t \leq T$ is a generalized solution.

The general theory of abstract semilinear differential equations can be found, for example, in the monographs [10], [11]. The facts presented there indicate that the properties of these equations are largely analogous to the properties of ordinary differential equations. In particular, the following several lemmas are valid.
Lemma 2. For any bounded set $\Omega \subset E$ we can find a $t_0 = t_0(\Omega) > 0$, such that each generalized solution $v = v(t, v_0)$, $v(0, v_0) = v_0$ of Eq. (6) with the initial condition $v_0 \in \Omega$ is uniquely defined on the interval $0 \leq t \leq t_0$ and is continuous with respect to the totality of variables $(t, v_0) \in [0, t_0] \times \Omega$. If $v_0 \in \Omega \cap E(L)$ then this solution is classical for all $t \in [0, t_0]$.

Lemma 3. Let $v = v_s(t)$ be an arbitrary (generally speaking, generalized) solution to equation (6) on some interval $0 \leq t \leq \bar{t}$. Then there exists a sufficiently small $r_0 > 0$ such that any generalized solution $v = v(t, v_0)$ to this equation with an initial condition $v(0, v_0) = v_0 \in B$, $B = \{ v \in E : \| v - v_s(0) \|_E \leq r_0 \}$, is defined on the interval $0 \leq t \leq \bar{t}$ and is jointly continuous with respect to the variables $(t, v_0) \in [0, \bar{t}] \times B$. If, in addition, $v_0 \in E(L)$, then the corresponding solution $v(t, v_0)$ turns out to be classical for all $[0, \bar{t}]$.

Lemma 4. For any generalized solution $v = v_s(t)$, defined on an interval $0 \leq t \leq \bar{t}$, there exists a sequence of classical solutions $v = v_k(t)$, $k \geq 1$, $t \in [0, \bar{t}]$, such that

$$\lim_{k \to \infty} \max_{0 \leq t \leq \bar{t}} \| v_s(t) - v_k(t) \|_E = 0.$$  

Now we define the so-called maximal existence semi-interval $[0, t_{\text{max}})$, where

$$t_{\text{max}} = \sup \{ t_0 : v(t) \quad t \in [0, t_0] \}.$$  

According to Lemma 2, the set of values of $t_0$ involved in finding the least upper bound in (9) is certainly nonempty. If $t_{\text{max}} < +\infty$, then the following statement applies.

Lemma 5. We assume that for some generalized solution $v(t)$ of Eq. (6), the quantity in (9) is finite. Then the limit equality

$$\lim_{t \to t_{\text{max}}^-} \| v(t) \|_E = +\infty.$$ 

holds.

The lemmas presented here can be established in the same way as in [12], where similar results are proved. In what follows, under some additional constraints imposed on the matrix $D$ and the nonlinearity $F(u)$, we will study the attractors of the nonlinear semiflow generated by generalized solutions to equation (6) in the phase space $E$.

2. Main results

Let us specify the statement of the problem. To this end, we consider the following boundary value problem similar to (6):

$$\frac{\partial u}{\partial t} = F(u, \varepsilon) + \sqrt{\varepsilon}(D_0 + \sqrt{\varepsilon}D_1) \frac{\partial u}{\partial x}, \quad u(t, x + 1) \equiv u(t, x),$$  

where $u \in \mathbb{R}^n$, $n \geq 3$; $\varepsilon > 0$ is a small parameter, and the vector function $F(u, \varepsilon)$ is jointly infinitely differentiable with respect to the variables $(u, \varepsilon) \in \mathbb{R}^n \times [0, \varepsilon_0]$, $\varepsilon_0 > 0$, and is such that $F(0, \varepsilon) \equiv 0$. Below we will need the Taylor expansion of the vector function $F(u, \varepsilon)$ at the point $(u, \varepsilon) = (0, 0)$,

$$F(u, \varepsilon) = (A_0 + \varepsilon A_1)u + F_2(u, u) + F_3(u, u, u) + O(\varepsilon^2 \| u \| + \varepsilon \| u \|^2 + \| u \|^4),$$  

where $A_0$ and $A_1$ are $n \times n$ square matrices and $F_2$, $F_3$ are $\mathbb{R}^n$-valued quadratic and cubic symmetric forms.

Condition 1. The matrix $A_0$ in (11) has a simple pair of purely imaginary eigenvalues $\lambda = \pm i \omega_0$, $\omega_0 > 0$, and the corresponding eigenvectors $a$ and $a$ are normalized by the conditions
\( (a, b) = 1 \) and \((\overline{a}, b) = 0\) (here and below, the bar denotes complex conjugation, \((*, *)\) is the Euclidean inner product in \(\mathbb{C}^n\), and the vector \(b\) is such that \(A^*_0 b = -i\omega_0 b\)). The other eigenvalues of the matrix \(A_0\) are assumed to lie in the left complex half-plane \(\{\lambda : \text{Re}\lambda < 0\}\).

**Condition 2.** The inequalities \(\text{Re}(A_1 a, b) > 0\) and \(\text{Re}d < 0\), hold, where

\[
d = 2(F_2(\chi_0, a, b)) + 2(F_2(\chi_2, \overline{a}), b)) + 3(F_3(a, a, \overline{a}), b),
\]

\[
\chi_0 = -2A^{-1}_0 F_2(a, \overline{a}), \quad \chi_2 = [2i\omega_0 I - A_0]^{-1} F_2(a, a),
\]

and \(I\) stands for the identity matrix.

The Conditions 1 and 2, taken together, guarantee that a classical Andronov-Hopf bifurcation take place in the partial system

\[
\dot{u} = F(u, \varepsilon)
\]

Indeed, it follows from the results of \([9], [10]\) that under these conditions, for any \(0 < \varepsilon \ll 1\), system (14) has a stable cycle with amplitude of order \(\sqrt{\varepsilon}\) and with frequency close to \(\omega_0\). This cycle is simultaneously a solution to the boundary value problem (10) that does not depend on the space variable \(x\). Such a cycle will be called a spatially homogeneous (or simply homogeneous) cycle. The following constraint concerns the coupling matrices \(D_0\) and \(D_1\) that appear in (10).

**Condition 3.** The matrix \(D_1\) is arbitrary and \(D_0\) has simple eigenvalues \(\lambda_k \in \mathbb{R}, k = 1, \ldots, n\), and corresponding eigenvectors \(e_k, k = 1, \ldots, n\). These eigenvectors are assumed to be normalized by the conditions \((e_k, g_k) = 1\), where \(D_0^* g_k = \lambda_k g_k, k = 1, \ldots, n\).

Now we linearize the boundary value problem (9) at the zero equilibrium state and apply the Fourier method with respect to the system of functions \(\exp(2\pi ikx), k \in \mathbb{Z}\) to the resulting linear boundary value problem

\[
\frac{\partial u}{\partial t} = (A_0 + \varepsilon A_1) u + \sqrt{\varepsilon}(D_0 + \sqrt{\varepsilon} D_1) \frac{\partial u}{\partial x}, \quad u(t, x + 1) \equiv u(t, x).
\]

As a result, we have the following countable system of ordinary differential equations for the \(u_k\):

\[
\dot{u}_k = (A_0 + \varepsilon A_1 + iz(D_0 + \sqrt{\varepsilon} D_1)) u_k, \quad z = 2\pi k\sqrt{\varepsilon}.
\]

The stability of these equations is defined by the spectral properties of the matrix

\[
\mathcal{D}(z) = A_0 + izD_0,
\]

where we assume that the parameter \(z\) taking discrete values \(2\pi k\sqrt{\varepsilon}\) varies continuously on the half-axis \(z \geq 0\) (the case of \(z < 0\) does not need separate consideration because \(\mathcal{D}(-z) \equiv \mathcal{D}(z)\)).

Let us Conditions 1 and 3 are hold. Then in view of Condition 1 matrix (15) has eigenvalues \(\lambda_1(z)\) and \(\lambda_2(z)\) such that \(\lambda_1(z) = \overline{\lambda_2(-z)}\) and \(\lambda_1(0) = i\omega_0\), for all sufficiently small values of the parameter \(z\). Next, its easy to see that \(\lambda_1(z)\) at the point \(z = 0\) has the Taylor expansion

\[
\lambda_1(z) = i\omega_0 + \lambda_{1,1} z + \lambda_{1,2} z^2 + \ldots, \quad \lambda_{1,1} = i(D_0 a, b), \quad \lambda_{1,2} = i(D_0 a_1, b),
\]

where \(a_1\) is a solution to the linear system

\[
(A_0 - i\omega_0 I)a_1 = i(D_0 a, b)a - iD_0 a,
\]

where \((a_1, b) = 0\).
According to Condition 3, for \( z \gg 1 \), the eigenvalues \( \lambda_k(z) \), \( k = 1, \ldots, n \), of the matrix (15) satisfy the asymptotic relation
\[
\lambda_k(z) = iz \lambda_k + (A_0 e_k, g_k) + O(1/z), \quad k = 1, \ldots, n,
\]
where \( \lambda_k \) are the eigenvalues of the matrix \( D_0 \).

**Condition 4.** We assume that
\[
\text{Re} \lambda_{1,1} = 0, \quad \text{Re} \lambda_{1,2} < 0, \quad (A_0 e_k, g_k) < 0, \quad k = 1, \ldots, n,
\]
and the matrix (15) is a Hurwitz matrix for any \( z > 0 \).

Notice that for small \( z \) and for \( z \gg 1 \), the Hurwitz property of the matrix \( \mathcal{D}(z) \) follows from Conditions 1 and 3 and relations (19). In the case of \( z \sim 1 \), this property needs to be postulated. Note also that the requirement \( \text{Re} \lambda_{1,1} = 0 \) in (19) describes a certain singularity. This clarifies the role of the matrix \( D_1 \) in (10): it characterizes a deviation from the singularity.

We address the question of the existence and stability of periodic solutions of the traveling-wave type in boundary value problem (10) under Conditions 1–4. For this, we make the following substitution
\[
u(t, \tau, s, x) = u(t, \tau, s, x) + \varepsilon u_1(t, \tau, s, x) + \varepsilon^{3/2} u_2(t, \tau, s, x) + \ldots
\]
Here \( \tau = \sqrt{\varepsilon} t, \ s = \varepsilon t, \)
\[
u_0(t, \tau, s, x) = \xi(t, \tau, s, x) + \varepsilon \xi_1(t, \tau, s, x) + \varepsilon^{3/2} \xi_2(t, \tau, s, x) + \ldots
\]
\[
\xi(t, \tau, s, x), \ \xi_1(t, \tau, s, x), \ \xi_2(t, \tau, s, x) = 0, \ \xi(\tau, s, x) \text{ is a complex amplitude that (it will be specified later), and}
\]
the functions \( u_k(t, \tau, s, x), k = 1, 2 \) are some trigonometric polynomials in the variable \( \omega_0 t \) with vector coefficients depending on \( \tau, s, \) and \( x \).

To find the functions \( u_k(t, \tau, s, x), k = 1, 2 \), we substitute relations (20) and (21) together with the Taylor expansion (11) into (10) and successively equate the coefficients of \( \varepsilon \) and \( \varepsilon^{3/2} \) on the left- and right-hand sides of the expression obtained. As a result, we arrive at linear inhomogeneous equations of the form
\[
\frac{\partial u_k}{\partial t} = A_0 u_k + \gamma_k(t, \tau, s, x), \quad k = 1, 2,
\]
where \( \tau, s, x \) takes as parameters.

For \( k = 1 \), the inhomogeneity in (22) is defined by the equality
\[
\gamma_1 = \gamma_{1,0}(\tau, s, x) + \gamma_{1,1}(\tau, s, x) \exp(i \omega_0 t) + \gamma_{1,2}(\tau, s, x) \exp(-i \omega_0 t) +
\]
\[
+ \gamma_{1,2}(\tau, s, x) \exp(2 i \omega_0 t) + \gamma_{1,2}(\tau, s, x) \exp(-2 i \omega_0 t),
\]
where
\[
\gamma_{1,0} = 2|\xi|^2 F_2(a, a), \quad \gamma_{1,1} = D_0 \frac{\partial \xi}{\partial x} - a \frac{\partial \xi}{\partial \tau}, \quad \gamma_{1,2} = \xi^2 F_2(a, a).
\]
We seek the function \( u_1(t, \tau, s, x) \) just as in (23), as a sum of zero, first, and second harmonics with coefficients \( \beta_{1,r}, r = 0, 1, 2 \). In this way, we obtain the following linear inhomogeneous systems for the indicated coefficients:
\[
(i r \omega_0 I - A_0) \beta_{1,r} = \gamma_{1,r}, \quad r = 0, 1, 2.
\]
In view of Condition 1 imposed on the spectrum of the matrix \( A_0 \), these systems are uniquely solvable for \( r = 0, 2 \). The solvability condition for this system for \( r = 1 \) is
\[
(\gamma_{1,1,1,0}) = 0.
\]
Now we take into account the explicit formula for $\gamma_{1,1}$ (see (24)) and the fact that the quantity $(D_0a, b)$ is real by Condition 4. Then we obtain for the amplitude $\xi(\tau, s, x)$ the equation

$$\frac{\partial \xi}{\partial \tau} = \kappa \frac{\partial \xi}{\partial x}, \quad \kappa = (D_0a, b) \in \mathbb{R}.$$ 

This implies that

$$\xi = \xi(s, y), \quad y = \kappa \tau + x. \quad (27)$$

Then we substitute relations (27) into equalities (24). As a result, we obtain the formula

$$u_1 = -i a_1 \frac{\partial \xi}{\partial y} \exp(i\omega_0 t) + i a_1 \frac{\partial \kappa}{\partial y} \exp(-i\omega_0 t) + \chi_0 |\xi|^2 + \chi_2 \xi^2 \exp(2i\omega_0 t) + \bar{\chi}_2 \bar{\xi}^2 \exp(-2i\omega_0 t), \quad (28)$$

where $a_1$ is a solution of the system (17), and $\chi_0$ and $\chi_2$ are the vectors (13).

For $k = 2$ we obtain the equality

$$\gamma_2 = A_1 u_0 + 2F_2(u_0, u_1) + F_3(u_0, u_0, u_0) + D_0 \frac{\partial u_1}{\partial x} + D_1 \frac{\partial u_0}{\partial x} - \frac{\partial u_1}{\partial \tau} - \frac{\partial u_0}{\partial s}. \quad (29)$$

Relations (21) and (28) imply that the inhomogeneity $\gamma_2$ is a linear combination of the harmonics $\exp(\pm ri\omega_0 t)$, $r = 0, 1, 2, 3$. The function $u_2$ is sought in the same form. In this way, we obtain nondegenerate linear inhomogeneous systems for the coefficients of the harmonics with numbers $r = 0, 2, 3$ in the decomposition of this function. In the case of $r = 1$, a solvability condition similar to (26) arises and we see that the amplitude $\xi(s, y)$ is a solution to the boundary value problem

$$\frac{\partial \xi}{\partial s} = -i(D_0a_1, b) \frac{\partial^2 \xi}{\partial y^2} + (D_1a, b) \frac{\partial \xi}{\partial y} + (A_1a, b) \xi + d |\xi|^2 \xi, \quad \xi(s, y + 1) \equiv \xi(s, y), \quad (29)$$

where $d$ is the parameter (12). It is important that in view of the inequality $\text{Re}[-i(D_0a_1, b)] > 0$ (see Condition 4), this problem turns out to be parabolic.

Following the tradition, we will call the boundary value problem (29) the quasinormal form of the original problem (10). Note that for $D_0 = D_1 = 0$ the equation in (29) is a truncated normal form of (14) on a stable two-dimensional invariant manifold. If $D_0 \neq 0$, then the boundary value problem (10) does not have an analogous invariant manifold, and the derivation of problem (29) becomes a purely formal procedure. Indeed, when deriving this problem, we discarded the terms that, although being small in order of magnitude compared with $\varepsilon$, are not subordinate to the other terms. In spite of all that has been said above, the quasinormal form (29) nevertheless contains some information on the attractors of the original problem (10). Before formulating relevant rigorous results, we give two definitions.

A **traveling-wave-type self-similar cycle of problem** (29) is a periodic solution of the form

$$\xi(s, y) = \xi_0 \exp[2\pi i(\sigma s + py)], \quad (30)$$

where $\xi_0, \sigma = \text{const} \in \mathbb{R}$, $p \in \mathbb{Z}$.

A **two-dimensional self-similar torus of problem** (29) is a quasiperiodic solution of the problem of the form

$$\xi(s, y) = \xi_0(\sigma_1 s + y) \exp(2\pi i\sigma_2 s), \quad (31)$$

where $\sigma_1, \sigma_2 = \text{const} \in \mathbb{R}$ and $\xi_0(y)$ is a 1-periodic complex-valued function. We assume that the solution (31) does not reduce to the solution (30).

The following two standard assertions concern the correspondence between the solutions (30) and (31) of the quasinormal form and analogous steady-state regimes of problem (10).
Theorem 1. Suppose that the quasinormal form (29) has a self-similar cycle of the form (30) that is exponentially orbitally stable or dichotomic (in the metric of the phase space $(\text{Re}\xi, \text{Im}\xi) \in W^1_2 \times W^1_2$, where $W^1_2$ is the Sobolev space of 1-periodic functions). Then, for any positive integer $l$, there exists a sufficiently small $\varepsilon > 0$ such that for all $0 < \varepsilon \leq \varepsilon_l$ the original boundary value problem (10) admits a traveling-wave-type cycle with the same stability properties that is defined by the equalities

$$u = \sqrt{\varepsilon} \xi_0 \left[ \exp\left(2\pi i(\theta + px)\right)a + \exp\left(-2\pi i(\theta + px)\right)a \right] + \varepsilon \mathcal{U}(\theta + px, \varepsilon),$$

$$\frac{d\theta}{dt} = \frac{\omega_0}{2\pi} + \sqrt{\varepsilon} p\kappa + \varepsilon\sigma + \varepsilon^{3/2}\Delta(\varepsilon).$$

Here $\xi_0, \sigma$ and $p$ are the constants from (30), $\kappa$ is the constant from (27), $\Delta(\varepsilon)$ is a bounded function of $\varepsilon$, and the vector function $\mathcal{U}(\theta, \varepsilon)$ is continuous and bounded on the set $(\theta, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_1)$ together with its partial derivatives with respect to $\theta$ up to the order $l$ inclusive.

Theorem 2. For every self-similar torus (31) of the quasinormal form (29) that is exponentially orbitally stable or dichotomic, the original problem (10) has a two-dimensional invariant torus with the same stability properties for all sufficiently small $\varepsilon > 0$. This torus admits a parametric representation analogous to (32):

$$u = \sqrt{\varepsilon} \left[ \xi_0(\theta_1 + x) \exp(2\pi i\theta_2) a + \xi_0(\theta_1 + x) \exp(-2\pi i\theta_2) a \right] + \varepsilon \mathcal{U}(\theta_1 + x, \theta_2, \varepsilon),$$

$$\frac{d\theta_1}{dt} = \sqrt{\varepsilon} \kappa + \varepsilon\sigma_1 + \varepsilon^{3/2}\Delta_1(\theta_2, \varepsilon), \quad \frac{d\theta_2}{dt} = \frac{\omega_0}{2\pi} + \varepsilon\sigma_2 + \varepsilon^{3/2}\Delta_2(\theta_2, \varepsilon),$$

where the functions $\mathcal{U}$ and $\Delta_j, j = 1, 2$ depend periodically (with period 1) on the phase variables $\theta_1$ and $\theta_2$ and are continuous and bounded in $(\theta_1, \theta_2, \varepsilon)$ together with any fixed number of their derivatives with respect to $\theta_1$ and $\theta_2$.

Note that the special dependence on $x$ in (33) and the absence of the argument $\theta_1$ in the functions $\Delta_j, j = 1, 2$, are associated with the invariance of this torus with respect to the rotational symmetry, i.e., with respect to the changes $x \to x + c$ and $\theta_1 \to \theta_1 - c$, where $c$ is an arbitrary real number.

The theorems formulated above can be proved by the methods described in [13] and the monograph [14].

3. Attractors of the quasinormal form. Numerical results

We reduce the boundary value problem (29) to a canonical form. To this end, taking into account the inequalities $\text{Re}(A_1, b) > 0$ and $\text{Re} d < 0$ (see Condition 2), we successively make the following changes in it

$$\xi \exp(-i \text{Im}(A_1, b)s) \to \xi, \quad y + \text{Re}(D_1, a, b)s \to y, \quad \text{Re}(A_1, b)s \to s,$$

$$\xi = \sqrt{-\text{Re} \left( A_1 a, b \right) / \text{Re} d} \cdot w, \quad s \to t, \quad y \to x.$$

Then we obtain a model boundary value problem

$$\frac{\partial w}{\partial t} = \nu \left( (1 - ic_1) \frac{\partial^2 w}{\partial x^2} + ic_2 \frac{\partial w}{\partial x} \right) + w - (1 + ic_3)|w|^2w, \quad w(t, x + 1) \equiv w(t, x),$$

where

$$\nu = \frac{\text{Re} [-i(D_0 a_1, b)]}{\text{Re} (A_1, b)} > 0, \quad c_1 = -\frac{\text{Im} [-i(D_0 a_1, b)]}{\text{Re} [-i(D_0 a_1, b)]} \in \mathbb{R},$$

$$c_2 = \frac{\text{Im} (D_1 a, b)}{\text{Re} [-i(D_0 a_1, b)]} \in \mathbb{R}, \quad c_3 = \frac{\text{Im} d}{\text{Re} d} \in \mathbb{R}. $$

(35)
The problem (34) always has the homogeneous cycle
\[ w = \exp(-ic_3 t). \]  
(36)

This cycle is a particular case of a traveling wave; i.e., it is contained in the family of cycles
\[ w = w_p \exp(i\sigma_p t + 2\pi p ix), \]  
(37)

where \( w_p, \sigma_p = \text{const} \in \mathbb{R}, \ p \in \mathbb{Z}. \) Conditions for the existence and stability of such cycles are given in the following assertion.

**Theorem 3.** Suppose that
\[ 1 - 2\pi \nu p(2\pi p + c_2) > 0. \]  
(38)

for some \( p \in \mathbb{Z}. \) Then the boundary value problem (34) admits a \( p \)-th traveling wave (37) whose amplitude and frequency are defined by the formulas
\[ w_p = \sqrt{1 - 2\pi \nu p(2\pi p + c_2)}, \ \sigma_p = 4\pi^2 p^2 \nu c_1 - c_3 w_p^2. \]  
(39)

The cycle (37), (39) is exponentially orbitally stable if the infinite series of inequalities
\[ (w_p^2 + 4\nu^2 \pi^2 k^2)[2w_p^2(1 - c_1 c_3) + 4\nu^2 \pi^2 k^2(1 + c_1^2) - \nu(4\pi p + c_2)^2] > \nu(4\pi p + c_2)^2(-w_p^2 c_3 + 4\nu^2 \pi^2 k^2 c_1)^2, \]  
(40)

holds, and is unstable if at least one of these inequalities is strictly violated.

The analysis of conditions (38) and (40) for \( 0 < \nu \ll 1 \) and for fixed \( c_1, c_2, \) and \( c_3 \) shows that in this case the boundary value problem (34) has traveling waves (37), (39) with numbers \( p = 0, \pm 1, \ldots, \pm p_0, \) where \( p_0 \) is any predetermined fixed positive integer. All these cycles are simultaneously stable (unstable) if the inequality \( c_1 c_3 - 1 < 0 \ (> 0) \) holds. This and Theorem 1 imply that for \( c_1 c_3 < 1, \) as the quantity \( \text{Re}(A_1 a, b) \) increases (or, equivalently (see (35)), as the parameter \( \nu \) decreases), the original boundary value problem (10) exhibits a buffer phenomenon, i.e., an unbounded accumulation of stable cycles of the form (32).

Two conclusions hold, firstly, the homogeneous cycle (36) is stable (unstable) for \( |c_2| - 2\pi < 0 \ (> 0) \) in case, when \( \nu \gg 1 \) and \( c_j \sim 1, \ j = 1, 2, 3, \) and secondly, all traveling waves (37), (39) provided by Theorem 3 are unstable in the case of \( \nu, c_1, c_3 \sim 1, \ c_1 c_3 > 1 \) and \( |c_2| \gg 1. \) Thus, the most interesting range of the parameters \( c_1, c_2, \) and \( c_3 \) is specified by the inequalities
\[ c_1 c_3 > 1, \ |c_2| > 2\pi. \]  
(41)

Indeed, under these conditions, the boundary value problem (34) has nontrivial dynamics both for \( \nu \gg 1 \) (its homogeneous cycle (36) is certainly unstable) and for \( \nu \ll 1 \) (all existing traveling waves are unstable). Below we will show by numerical analysis that for fixed \( c_j, \ j = 1, 2, 3, \) satisfying inequalities (41) and for \( \nu \ll 1 \) problem (34) admits a chaotic attractor \( A_2, \) whose Lyapunov dimension \( d_L(A_2) \) (calculated by the well-known KaplanYorke formula [15], algorithm from [16]) tends to infinity as \( \nu \to 0. \) In a similar way, an indefinite growth of the dimension of the chaotic attractor is observed for \( \nu, c_1, c_3 \sim 1, \) and \( c_1 c_3 > 1 \) as \( |c_2| \) increases.

To describe the corresponding numerical experiments, we introduce points \( x = j/N, \) \( j = 1, \ldots, N, \) where \( N \) is an arbitrary fixed positive integer, and replace the partial derivatives with respect to \( x \) in (34) at these points with the corresponding symmetric differences. As a result, we arrive at the following finite-dimensional model for the variables \( w_j(t) = w(t, x)|_{x=j/N}: \)
\[ \dot{w}_j = \nu \left[ (1 - ic_1)N^2(w_{j+1} - 2w_j + w_{j-1}) + ic_2 N(w_{j+1} - w_{j-1})/2 \right] + w_j - (1 + ic_3)|w_j|^2 w_j, \]  
(42)

\[ j = 1, \ldots, N, \ w_0(t) = w_N(t), \ w_{N+1}(t) = w_1(t). \]
For the parameter values
\[ c_1 = 1, \quad c_2 = 8, \quad c_3 = 4, \] (43)
which satisfy inequalities (41), a chaotic attractor \( A_\nu(N) \) arises in system (42) as \( \nu \) decreases. A clear picture of this process is given by the graphs of the leading Lyapunov exponent \( \lambda_{\text{max}}(A_\nu(N)) \) and the Lyapunov dimension \( d_L(A_\nu(N)) \) of this attractor versus \( \nu \). For \( N = 30 \), the corresponding graphs drawn point-by-point with a step of 0.00005 on the interval \( 0 \leq \nu \leq 0.025 \) are shown in Figs. 1 and 2. A characteristic feature of the dependence of \( d_L(A_\nu(N)) \) on \( \nu \) is the presence of a maximum at small \( \nu \). This is associated with the fact that for a fixed \( N \), as \( \nu \to 0 \), the attractors in the difference model (42) become simpler (stable traveling waves arise that have no analogs in the distributed model (34)). Therefore, for too small values of \( \nu \), the difference model ceases to adequately describe the dynamics of problem (34). Note in addition that as \( N \) increases, the above-mentioned maximum shifts to zero and the value of \( d_L(A_\nu(N)) \) at the maximum point indefinitely increases.

At the next stage, we verify that as \( N \) increases, the attractor \( A_\nu(N) \) of the difference model (42) converges in a some sense to the attractor \( A_\nu \) of the boundary value problem (34). Indeed, numerical calculations for fixed \( \nu \) show that the quantities \( \lambda_{\text{max}}(A_\nu(N)) \) and \( d_L(A_\nu(N)) \) stabilize as \( N \) increases. Figures 3 and 4 illustrate the dependence of these quantities on \( N \) for
5 \leq N \leq 100 \text{ when } \nu = 0.007 \text{ and } \nu = 0.002 \text{ (solid lines correspond to the case of } \nu = 0.007, \\
\text{and the dashed lines, to the case of } \nu = 0.002\text{). Here it is important that smaller values of the parameter } \nu \text{ correspond to greater values of } d_L. \text{ Thus, we have every reason to expect that for the set of parameters (43) the dimension } d_L(A_\nu) \text{ of the chaotic attractor } A_\nu \text{ of the boundary value problem (34) indefinitely increases as } \nu \to 0.

A similar conclusion on the indefinite growth of the dimension of the chaotic attractor of the boundary value problem (34) is valid for \( c_1 = 1, c_3 = 4, \) and \( \nu = 1 \) as the quantity \( \alpha = 1/c_2 > 0 \) decreases. To verify this, for the indicated values of the parameters we present the results of numerical analysis of the system

\[
\dot{w}_j = i \nu N(w_{j+1} - w_{j-1})/2 - (1 + ic_3)|w_j|^2w_j + +\alpha(\nu (1 - ic_1)N^2(w_{j+1} - 2w_j + w_{j-1}) + w_j), \quad j = 1, \ldots, N,
\]

which is obtained from (42) after the substitutions \( w_j/\sqrt{c_2} \to w_j \) and \( c_2 t \to t \).

Just as in the previous case, for small \( \alpha \) system (44) admits a chaotic attractor \( A_\alpha(N) \). For \( N = 30 \), the dependence of the leading Lyapunov exponent \( \lambda_{\max} \) and the Lyapunov dimension \( d_L \) of this attractor on \( \alpha \) is demonstrated in Figs. 5 and 6. These graphs are drawn point-by-point with a step of 0.0001 on the interval 0.01 \( \leq \alpha \leq 0.08 \). The interval 0 \( \leq \alpha \leq 0.01 \) has not been considered because the difference model (44) ceases to be adequate to the continuous model (34) for fixed \( N \) and very small \( \alpha \) (chaos is destroyed, and stable traveling waves whose analogs...
in the continuous model are unstable arise). Next, Figs. 7 and 8 demonstrate the dependence of $\lambda_{\text{max}}$ and $d_L$ on $N$ for $5 \leq N \leq 100$ when $\alpha = 0.03$ and $\alpha = 0.02$ (solid and dashed lines, respectively). The information presented supports the conjecture that $d_L(A_{\alpha}) \rightarrow +\infty$ as $\alpha \rightarrow 0$, where $A_{\alpha} = \lim A_{\alpha}(N), N \rightarrow \infty$ is the attractor of the original boundary value problem (34) for $c_1 = 1, c_3 = 4, \nu = 1$, and $\alpha = 1/c_2$.

To conclude the description of the results of numerical calculation, we note that for the parameter values (43) and for all $\nu \geq 0.1$ problem (34) has a stable self-similar two-dimensional torus of the form (31). Thus, we see that the applicability domain of Theorem 2 is certainly nonempty.

4. Conclusions

The reason why the order of smallness of the coupling matrix (10) is chosen to be $\sqrt{\varepsilon}$ is associated with the spectral properties of the matrix (15). Such a coupling order has allowed us to apply the known method of quasinormal forms to the boundary value problem (10) and obtain rigorous results on the correspondence between special steady-state regimes of the boundary value problems (29) and (10). A similar correspondence between the chaotic attractors of these problems has not been established; however, there is every reason to expect that such a correspondence also takes place. Based on the analytic and numerical analysis performed above within the framework of this hypothesis, we get a sufficiently clear picture of the local dynamics of the boundary value problem (10) in the neighborhood of the zero equilibrium state. The main features of these dynamics are the following two alternatives that take place for an appropriate choice of the parameters: either the buffer phenomenon occurs or a chaotic attractor of an arbitrarily high Lyapunov dimension arises (see also [16]–[19]).

Acknowledgments

This work was supported by the Russian Science Foundation (project nos. 14-21-00158)

References

[1] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2006 Comput. Math. Math. Phys. 46 1724–1736
[2] Mishchenko E.F, Sadovnichii V.A, Kolesov A.Yu, and Rozov N.Kh 2012 Multifaceted Chaos (Moscow: Fizmatlit, Russian)
[3] Kapitaniak T and Chua L O 1994 Internat. J. Bifurcation Chaos Appl. Sci. Engrg. 4 477–482
[4] Marin L.P, Perez-Munuzuri V, Perez-Villar V, Sanchez E, and Matias MA 1999 Phys. D 128 224–235
[5] Perlikowski P, Yanchuk S, Wolf M, Stefasynski A, Mosolek P, Kapitaniak T 2010 Chaos 20 013111
[6] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2013 Theor. Math. Phys. 175 499–517
[7] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2013 Izv. Math. 77 271–312
[8] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2013 Differ. Equ. 49 1193–1210.
[9] Hassard B.D, Kazarinoff N.D, and Wan Y.H 1981 Theory and Applications of Hopf Bifurcation (London Math. Soc. Lect. Note Ser., Vol. 41, Cambridge: Cambridge Univ. Press)
[10] Kolesov A.Yu, and Rozov N.Kh 2004 Invariant Tori of Nonlinear Wave Equations (Moscow: Fizmatlit, Russian)
[11] Henry D 1981 Geometric Theory of Semilinear Parabolic Equations (Lect. Notes Math., Vol. 840, Berlin: Springer)
[12] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2009 Theor. Math. Phys. 158 246–261
[13] Kolesov Yu S 1994 Sb. Math. 78 367–78
[14] Mishchenko E.F, Sadovnichii V.A, Kolesov A.Yu, and Rozov N.Kh 2005 Autoave Processes in Nonlinear Media with Diffusion (Moscow: Fizmatlit, Russian)
[15] Frederickeon P, Kaplan J.L, Yorke E.D, and Yorke J.A 1983 J. Diff. Eqns. 49 (2) 185–207
[16] Glyzin S.D, Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2005 Differ. Eq. 41 284–289
[17] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2010 Comput. Math. Math. Phys. 50 816–830
[18] Glyzin S.D, Kolesov A.Yu, and Rozov N.Kh 2011 Comput. Math. Math. Phys. 51 1307–1324
[19] Glyzin S.D 2013 Automatic Control and Computer Sciences 47 452–469.