Quantum Spheres for $OSp_q(1/2)$

N. Aizawa

Department of Applied Mathematics,
Osaka Women’s University,
Sakai, Osaka 590-0035, Japan

and

R. Chakrabarti

Department of Theoretical Physics,
University of Madras,
Guindy Campus, Chennai 600 025, India

Abstract

Using the corepresentation of the quantum supergroup $OSp_q(1/2)$ a general method for constructing noncommutative spaces covariant under its coaction is developed. In particular, a one-parameter family of covariant algebras, which may be interpreted as noncommutative superspheres, is constructed. It is observed that embedding of the supersphere in the $OSp_q(1/2)$ algebra is possible. This realization admits infinitesimal characterization à la Koornwinder. A covariant oscillator realization of the supersphere is also presented.
I. INTRODUCTION

Lie supergroups and superalgebras have been used as basic tools in various fields of theoretical physics. Supersymmetry in quantum field theories and string field theory is the most well-known example of application of Lie superalgebras. Other examples are found in exactly solvable lattice models, interacting boson-fermion models in nuclear physics, extended t-J models in condensed matter physics, and so on. On the other hand, importance of noncommutative geometry in theoretical physics, especially in string theory and quantum gravity, has come into focus recently. Therefore if these two notions are combined to form a noncommutative geometry of supersymmetric nature, we can expect that the combination will play important roles in various fields in physics. An attempt to introduce a noncommutative superspace was made by Manin in the context of quantum supergroup. Then differential calculus on the noncommutative superspace was developed by two different approaches. The present authors extended the notions of noncommutative differential geometry such as connection and curvature to the supersymmetric case and investigated the superspace for super-Jordanian deformed OSp(1/2) group.

A supersphere having bosonic and fermionic coordinates is defined as an algebra whose defining relations are covariant under the coaction of the quantum supergroup OSp_q(1/2). In this paper, we construct a one-parameter family of superspheres by developing a general procedure based on the representation theory of OSp_q(1/2) and its dual U_q[osp(1/2)]. By this method, quantum superspaces and superspheres are described in a unified way. Furthermore, the method can be used to find higher dimensional noncommutative superspaces covariant under OSp_q(1/2). Our work is motivated by two reasons: (1) In order to investigate noncommutative geometry, it is important to have explicit examples of noncommutative superspaces. Manin’s work is an analogue of flat space, while here we consider an analogue of curved space. (2) There exist some models of integrable quantum field theories with OSp(m/2n) symmetries where superspheres appear. Consideration of quantum group extensions of such models will require quantum superspheres. We start with the simplest and the most important group OSp_q(1/2) to construct quantum superspheres.

Let us briefly recall the studies of noncommutative sphere based on quantum groups, since supersymmetric counterparts of some of them are considered in this paper. Podleś introduced an algebra which is covariant under the adjoint corepresentation of quantum group SU_q(2). The algebra is interpreted as a noncommutative version of two-sphere, and called q-sphere. The q-sphere has one more parameter in addition to its radius and the deformation parameter q. Thus what Podleś constructed is a one-parameter family of noncommutative two-spheres. The parameter is specific to q-sphere and does not have commutative counterpart. Differential calculus on the q-sphere was initiated by Podleś, then classification of differential structures on q-sphere was made in Refs. 9, 10, 11. An interesting relation of q-sphere to q-hypergeometric functions is discussed in Ref. 12, where orthogonal bases on q-sphere are explicitly determined in terms of big q-Jacobi polynomials. The q-sphere can be realized by embedding it in SU_q(2). This embedding admits an elegant description of q-sphere as an algebra which is invariant under left and right actions of a twisted primitive element of the quantum algebra U_q[su(2)]. Podleś q-sphere has been generalized in two different directions: higher dimensional, and Jorda-
nian $SU(2)$. Higher dimensional $q$-spheres, more precisely, noncommutative analogue of $(2n + 1)$-spheres were constructed in a similar way by replacing $SU_q(2)$ with $SU_q(n + 1)$. Furthermore, an invariant integral on quantum $(2n + 1)$-sphere is obtained in Ref. 14. Another family of quantum two-spheres is obtained 15 by using Jordanian deformation of $SL(2)$. One of its distinctions from Podleś $q$-sphere is that the Jordanian quantum sphere requires different twisted primitive elements for left and right invariances.

Throughout this paper, the quantum superalgebra $U_q[osp(1/2)]$ and the quantum supergroup $OSp_q(1/2)$ are denoted by $U$ and $A$, respectively. We assume that $q$ is generic in this paper. The plan of this paper is as follows. In the first two preliminary sections we fix our notations and conventions, and list formulae used in subsequent sections. Section II is a summary of definitions and representation theory of $U$. For computational purpose, we give all the defining relations of $A$ explicitly in Sec. III. A relation between representations of $U$ and corepresentations of $A$ is given in Sec. IV. A product law of two corepresentations, which is a quantum supergroup analogue of Wigner’s product law in the quantum theory of angular momentum, is also derived in Sec. IV. A general prescription to find an algebra covariant under the coaction of $A$ is given in Sec. V. As a simple application of the method, the most general form of quantum superspaces is derived. The method is applied to construct a one-parameter family of quantum superspheres in Sec. VI. Properties of the quantum supersphere are examined and the similarities to those for $q$-sphere are pointed out.

II. $U_q[osp(1/2)]$ AND ITS REPRESENTATIONS

A. Definition and representations

The universal enveloping algebra $U = U_q[osp(1/2)]$ is generated by the two even $K^\pm$, and the two odd elements $v^\pm$ satisfying the commutation properties

\[
KK^{-1} = K^{-1}K = 1, \quad Kv^\pm = q^{\pm 1/2}v^\pm K, \\
{\{v_+, v_\}} = -\frac{K^2 - K^{-2}}{q^4 - q^{-4}}.
\]

(2.1)

The Casimir element is given by

\[
C = \left(\frac{q^{1/2}K^2 - q^{-1/2}K^{-2}}{q^4 - q^{-4}}\right)^2 - \frac{qK^2 + q^{-1}K^{-2}}{(q + q^{-1})(q^2 + q^{-2})} v^- v_+ - (q^{1/2} + q^{-1/2})^2 v_- v_+^2.
\]

(2.2)

The coproduct ($\Delta$), the counit ($\epsilon$), and the antipode ($S$) maps read

\[
\Delta(K^\pm) = K^\pm \otimes K^\pm, \quad \Delta(v^\pm) = v^\pm \otimes K^{-1} + K \otimes v^\pm, \\
\epsilon(K^\pm) = 1, \quad \epsilon(v^\pm) = 0, \\
S(K^\pm) = K^{\mp 1}, \quad S(v^\pm) = -q^{\mp 1/2}v^\pm.
\]

(2.3)

The finite dimensional irreducible representations of the $U$ algebra are said to be of the grade-star 17 type. Each irreducible representation is specified by a nonnegative integer $\ell$ and the corresponding $(2\ell + 1)$ dimensional graded vector space $V^{(\ell)}$ that admits a
nondegenerate Hermitian bilinear form denoted by $\langle \cdot , \cdot \rangle$. The subspaces of $V^{(\ell)}$ having different parities are orthogonal with respect to the bilinear form. The graded adjoint operation ($\ast$) is defined by

$$ (A^* f, g) = (-1)^{\hat{A}} (f, Ag), \quad A \in \mathcal{U}; \ f, g \in V^{(\ell)}, $$

where $\hat{A}$ denotes the parity of $A$. The $\ast$-operation is assumed to be an algebra anti-isomorphism and coalgebra isomorphism:

$$ (A_1 A_2)^* = (-1)^{\hat{A}_1 \hat{A}_2} A_2^* A_1^*, \quad (A_1 \otimes A_2)^* = A_1^* \otimes A_2^*. $$

The grade-star representation of $\mathcal{U}$ is characterized by

$$ K^* = K, \quad v_{\pm}^* = \pm (-1)^{\varepsilon} v_{\mp}, \quad \varepsilon = 0, 1, $$

where $\varepsilon$ refers to the class of the representation.

Let $\{ e_m^\ell(\lambda) \mid m = \ell, \ell - 1, \ldots, -\ell \}$ be a basis of $V^{(\ell)}$, where each basis vector has a definite parity. The index $\lambda = 0, 1$ specifies the parity of the highest weight vector $e_{\ell}^\ell(\lambda)$. The parity of $e_m^\ell(\lambda)$ equals $\ell - m + \lambda$, as it is obtained by the application of $v_{-m}^{\ell}$ on $e_{\ell}^\ell(\lambda)$. For the superalgebras the norm of the representation basis need not be chosen positive definite. In this work, however, we assume the positive definiteness of the basis elements:

$$ (e_m^\ell(\lambda), e_{m'}^\ell(\lambda)) = \delta_{\ell\ell'} \delta_{mm'}. $$

It turns out that this convention relates the parity $\lambda$ and the class $\varepsilon$ as follows:

$$ \lambda = \varepsilon + 1 \ (\text{mod} \ 2). $$

With these settings, the irreducible representations of $\mathcal{U}$ are given by

$$ Ke_m^\ell(\lambda) = q^{m/2} e_m^\ell(\lambda), $$

$$ v_+^\ell e_m^\ell(\lambda) = \sqrt{[\ell - m][\ell + m + 1]\varrho} e_{m+1}^\ell(\lambda), $$

$$ v_-^\ell e_m^\ell(\lambda) = (-1)^{\ell - m - 1} \sqrt{[\ell + m][\ell - m + 1]\varrho} e_{m-1}^\ell(\lambda), $$

where $[n]$ and $\varrho$ are defined by

$$ [n] = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{1/2} + q^{-1/2}}, \quad \varrho = \frac{q^{-1/2} + q^{1/2}}{q^{-1} - q}. $$

Our phase convention for $v_\pm$ agrees with that of Ref. 16, but it differs from that of Ref. 17. For later convenience, the representation matrices for $\ell = 1, 2$ cases are given explicitly. The generators in the $\ell = 1$ representation read

$$ K = \text{diag}(q^{1/2}, 1, q^{-1/2}), $$

$$ v_+ = \sqrt{2}[\varrho] \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_- = \sqrt{2}[\varrho] \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. $$
and for the $\ell = 2$ case they are given by

$$K = \text{diag}(q, q^{1/2}, 1, q^{-1/2}, q^{-1}),$$

$$v_+ = \begin{pmatrix}
0 & \sqrt{[4]q} & 0 & 0 & 0 \\
0 & 0 & \sqrt{[3]q} & 0 & 0 \\
0 & 0 & 0 & \sqrt{[3]q} & 0 \\
0 & 0 & 0 & 0 & \sqrt{[4]q} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$v_- = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-\sqrt{[4]q} & 0 & 0 & 0 & 0 \\
0 & \sqrt{[3]q} & 0 & 0 & 0 \\
0 & 0 & -\sqrt{[3]q} & 0 & 0 \\
0 & 0 & 0 & \sqrt{[4]q} & 0
\end{pmatrix}. \tag{2.14}$$

The eigenvalue of the Casimir element depends only on the highest weight:

$$Ce_\ell^{\ell}(\lambda) = \left(\frac{q^{\ell+1/2} - q^{-\ell-1/2}}{q^4 - q^{-4}}\right)^2 e_\ell^{\ell}(\lambda). \tag{2.15}$$

The quantity $[n]$, known as Kulish symbol, plays a role similar to the $q$-number. For a positive integer $n$ its factorial is defined as $[n]! \equiv [n][n-1] \cdots [1]$.

### B. Tensor product representations

Tensor product of the irreducible representations of $\mathcal{U}$ has been discussed in Ref. 16, 17. The decomposition of tensor product in the irreducible representations is identical to the classical case

$$V^{(\ell_1)} \otimes V^{(\ell_2)} = V^{(\ell_1 + \ell_2)} \oplus V^{(\ell_1 + \ell_2 - 1)} \oplus \cdots \oplus V^{(|\ell_1 - \ell_2|)}.$$ 

The explicit formulae of the Clebsch-Gordan coefficients (CGC) are obtained in Ref. 17. We list below the relations which will be used in the later sections.

In spite of our assumption (2.9) of the positivity of the basis states, the norm of the tensor product of the bases is not always positive definite. For instance, the following norm can be negative for some combinations of $\ell_a, m_a$ ($a = 1, 2$) and $\lambda$:

$$(e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda), e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda))$$

$$= (-1)^{(\ell_1-m_1+\lambda)(\ell_2-m_2+\lambda)}(e_{m_1}^{\ell_1}(\lambda), e_{m_1}^{\ell_1}(\lambda))(e_{m_2}^{\ell_2}(\lambda), e_{m_2}^{\ell_2}(\lambda)). \tag{2.16}$$

The irreducible basis of the tensor product representations is obtained by using the CGC:

$$e_m^{\ell}(\ell_1, \ell_2, \Lambda) = \sum_{m_1, m_2} C_{m_1 m_2}^{\ell_1 \ell_2} e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda), \tag{2.17}$$

where $m = m_1 + m_2$, and $\Lambda = \ell_1 + \ell_2 + \ell (\text{mod} 2)$ is the parity of the highest weight vector $e_{\ell_1}^{\ell}(\Lambda, \ell_2, \Lambda)$. Since our phase convention for representations of $v_\pm$ differs from that of Ref.
17, we can not use the expression of CGC given therein. The CGC in our convention is explicitly given by

\[ C_{\ell_1 \ell_2 \ell}^{m_1 m_2 m} = (-1)^{(\ell_1 - \ell_2 + m_1)(\ell_1 - m_1 + \lambda + (\ell_1 - \ell_2 + m_2 + 1)/2} \times q^{m_2(m+1)/2 + (\ell_1 - \ell_2) (\ell_1 + \ell_2 + 1)/4 - \ell(\ell + 1)/4} \times \left( [2\ell + 1] \frac{[\ell_1 + \ell_2 - \ell]! [\ell + m]! [\ell - m]! [\ell - m_1]! [\ell_2 - m_2]!}{[k]! [\ell - m - k]! [\ell_1 - \ell + m_2 + k]! [\ell_2 - m_2 - k]!} \right)^{1/2} \times \sum_k (-1)^{k(\ell - 1)/2 + k(\ell_1 + \ell_2 - m)} q^{k(\ell + m + 1)/2} \]

where the index \( k \) runs over all non-negative integers maintaining the arguments of \([x]\) non-negative. The derivation of (2.18) is described in Appendix A. All the CGC are, we note, of parity zero. The basis (2.17) is pseudo orthogonal:

\[ (e^\ell_m (\ell_1, \ell_2, \Lambda), e^\ell_m (\ell_1, \ell_2, \Lambda)) = (-1)^{\ell-m+\lambda}(\ell_1+\ell_2+\ell+\lambda) \delta_\ell_\ell \delta_m_m. \]  

The CGC satisfies two pseudo orthogonality relations

\[ \sum_{m_1,m_2} (-1)^{(\ell_1-m_1+\lambda)(\ell_2-m_2+\lambda)} C_{\ell_1 \ell_2 \ell}^{m_1 m_2 m'} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell} = (-1)^{(\ell-m+\lambda)(\ell_1+\ell_2+\ell+\lambda)} \delta_\ell_\ell \delta_m_m', \]  

\[ \sum_{\ell,m} (-1)^{(\ell-m+\lambda)(\ell_1+\ell_2+\ell+\lambda)} C_{m_1 \ell_2 \ell}^{\ell_1 \ell_2 \ell} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell} = (-1)^{(\ell-m+\lambda)(\ell_2-m_2+\lambda)} \delta_m_m' \delta_m_m''. \]  

Using (2.21), the construction (2.17) is readily inverted:

\[ e^\ell_{m_1} (\Lambda) \otimes e^\ell_{m_2} (\Lambda) = (-1)^{(\ell_1-m_1)(\ell_2-m_2)} \sum_{\ell,m} (-1)^{(\ell-m)(\ell_1+\ell_2+\ell)} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell} e^\ell_m (\ell_1, \ell_2, \Lambda). \]  

## III. QUANTUM SUPERGROUP \( OSpq(1/2) \)

The quantum supergroup \( \mathcal{A} = OSpq(1/2) \) is defined as a Hopf dual to the universal enveloping algebra \( \mathcal{U} \). In this section, all defining relations of \( \mathcal{A} \) will be given explicitly. The universal \( \mathcal{R} \)-matrix of \( \mathcal{U} \) is given in Ref. 16. For the defining \( \ell = 1 \) representation, it reads

\[ \mathcal{R}^{(\ell=1)} = \begin{pmatrix} q & \cdot & \cdot & \cdot \\ \cdot & 1 & \omega & \cdot \\ \cdot & q^{-1} & \lambda & \rho \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \]

\[ \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \]

\[ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \]

\[ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ q^{-1} & \cdot & \cdot & \cdot \end{pmatrix} \]

\[ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & q \end{pmatrix} \]
where
\[ \omega = q - q^{-1}, \quad \lambda = -q^{-1/2} \omega, \quad \rho = (1 + q^{-1}) \omega, \] (3.2)
and the dot is used instead of zero for better readability. Let \( P \) be a permutation operator: \( P(v \otimes w) = (-1)^{\tilde{\omega}} w \otimes v \). Standard FRT\(^{18} \) construction is obtained via the matrix \( R \):
\[
R = P R^{(\ell=1)} P = \begin{pmatrix}
q & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & q^{-1} & \cdot & \cdot & \cdot \\
\cdot & \omega & 1 & \cdot & \cdot \\
\cdot & -\lambda & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \rho & -\lambda & q^{-1} \\
\cdot & \cdot & \cdot & \omega & 1 \\
\cdot & \cdot & \cdot & \cdot & q
\end{pmatrix}.
\] (3.3)

The quantum \( T \)-matrix, whose elements generate the algebra \( A \) is given by
\[
T = (t_{ij}) = \begin{pmatrix}
a & \alpha & b \\
\gamma & e & \beta \\
c & \delta & d
\end{pmatrix},
\] (3.4)
where the entries in latin (greek) characters are of even (odd) parity. The parity of the supermatrix \( T \) is zero, i.e., \( t_{ij} = \hat{i} + \hat{j} \). The RTT-relation describes the exchange properties on the entries of \( T \). The \( q \)-orthosymplectic condition reads
\[
T^{st} C T = D C, \quad T C^{-1} T^{st} = D C^{-1},
\] (3.5)
where
\[
C = \begin{pmatrix}
0 & 0 & -q^{-1/2} \\
0 & 1 & 0 \\
q^{1/2} & 0 & 0
\end{pmatrix}, \quad C^{-1} = \begin{pmatrix}
0 & 0 & q^{-1/2} \\
0 & 1 & 0 \\
-q^{1/2} & 0 & 0
\end{pmatrix},
\] (3.6)
and the super-determinant \( D \) is given by
\[
D = ad - qbc - q^{1/2} \alpha \delta.
\] (3.7)

The \( T^{st} \) denotes the super-transpose of \( T \). The super-transpose of an arbitrary matrix is given as \( A^{st}_{ij} = (-1)^{\hat{i}+\hat{j}} A_{ji} \). The RTT-relations require \( D \) to be central. The coproduct and counit of \( T \) are given as usual
\[
\Delta(T) = T \otimes T, \quad \epsilon(T) = \text{diag}(1, 1, 1).
\] (3.8)
The group-like property of the super-determinant \( \Delta(D) = D \otimes D \) is obtained by taking the coproduct of both sides of the relation (3.5). This allows us to set the constraint
\[
D = ad - qbc - q^{1/2} \alpha \delta = 1.
\] (3.9)
The antipode of $T$ satisfies $S(T)T = TS(T) = 1$, and it explicitly reads

$$S(T) = C^{-1}T^*C = \begin{pmatrix} d & q^{-1/2}b & -q^{-1}b \\ -q^{1/2} & e & q^{-1/2}a \\ -qc & -q^{1/2} & a \end{pmatrix}. \quad (3.10)$$

The RTT-relations reveal that not all the entries of $T$ are independent. We express the elements $e, \beta$ and $\gamma$ in terms of the rest. Using (2,2) component of the first relation in (3.5)

$$e^2 = 1 - 2q^{-1/2}\alpha\delta + \omega bc,$$  \quad (3.11)

we solve for $e$ and its inverse $e^{-1}$:

$$e = 1 - q^{-1/2}\alpha\delta + (q - 1)bc,$$

$$e^{-1} = (1 + q^{-1/2}\alpha\delta - (q - 1)bc)(1 + q^{-1}(q - 1)^2bc)^{-1}. \quad (3.12)$$

Inclusion of the element $e^{-1}$ allows us to solve for $\beta$ and $\gamma$:

$$\beta = q^{-3/2}b\delta - q^{1/2}d\alpha,$$

$$\gamma = q^{-1/2}a\delta - q^{3/2}c\alpha, \quad \alpha\delta = \gamma/\beta. \quad (3.13)$$

In summary, the algebra $\mathcal{A}$ is generated by $a, b, c, d, \alpha$ and $\delta$. The generators satisfy the relations

\begin{align*}
ab &= q^2ba & ac &= q^2ca, & [a, d] &= -\lambda\alpha\delta + \rho bc, \\
ax &= qa, & a\delta &= q\delta a + q\omega ca, & [b, c] &= 0, \\
bd &= q^2db, & b\alpha &= q^{-1}ab, & b\delta &= q\delta b, \\
cd &= q^2dc, & c\alpha &= q^{-1}c\alpha, & c\delta &= q\delta c, \\
da &= q^{-1}ad - \omega\delta b, & d\delta &= q^{-1}\delta d, & \alpha\delta &= -q\delta\alpha - q\lambda bc, \\
\alpha^2 &= -q^{-1}[2]ab, & \delta^2 &= -q^{-1}[2]cd.
\end{align*} \quad (3.14)

For later convenience, the commutation relations involving $e, \beta$ and $\gamma$ are listed below:

\begin{align*}
[a, e] &= \omega\gamma\alpha, & a\beta &= q\beta a + q\omega b\gamma, & a\gamma &= q\gamma a, \\
[b, e] &= 0, & b\beta &= q\beta b, & b\gamma &= q^{-1}\gamma b, \\
[c, e] &= 0, & c\beta &= q\beta c, & c\gamma &= q^{-1}\gamma c, \\
[d, e] &= \omega\delta\beta, & d\beta &= q^{-1}\beta d, & d\gamma &= q^{-1}\gamma d - \omega\beta c, \\
[e, \alpha] &= -\lambda\gamma b, & [e, \beta] &= \lambda b\delta, & [e, \gamma] &= \lambda\alpha c, \\
[e, \delta] &= -\lambda c\beta, & \{\alpha, \beta\} &= -\omega eb, & \{\alpha, \gamma\} &= 0, \\
\beta\gamma &= -q^{-1}\gamma\beta - \lambda bc, & \{\beta, \delta\} &= 0, & \{\gamma, \delta\} &= \omega ce, \\
\beta^2 &= -q^{-1}[2]bd, & \gamma^2 &= -q^{-1}[2]ac.
\end{align*} \quad (3.15)

Additional relations may also be proved:

\begin{align*}
e\alpha &= q^{1/2}(b\gamma - \beta a), & e\beta &= q^{1/2}\delta b - q^{-1/2}\alpha d, \\
e\gamma &= q^{1/2}(\delta a - \alpha c), & e\delta &= q^{-1/2}\gamma d - q^{1/2}\beta c.
\end{align*} \quad (3.16)
The duality between $\mathcal{U}$ and $\mathcal{A}$ is given by a nondegenerate pairing $\langle \cdot, \cdot \rangle$

$$\langle K, T \rangle = \text{diag}(q^{1/2}, 1, q^{-1/2}),$$
$$\langle v_+, T \rangle = \sqrt{2q} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \langle v_-, T \rangle = \sqrt{2q} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ (3.17)

The pairing can be extended to tensor product algebras by setting

$$\langle X_1 \otimes X_2, a_1 \otimes a_2 \rangle = (-1)^{\tilde{X}_a^m} \langle X_1, a_1 \rangle \langle X_2, a_2 \rangle.$$ (3.18)

IV. COREPRESENTATIONS OF $\mathcal{A}$

A vector space $V$ is called a right $\mathcal{A}$-comodule if there exists a linear mapping $\varphi_R : V \to V \otimes \mathcal{A}$ satisfying

$$(\varphi_R \otimes \text{id}) \circ \varphi_R = (\text{id} \otimes \Delta) \circ \varphi_R, \quad (\text{id} \otimes \epsilon) \circ \varphi_R = \text{id}. \quad (4.1)$$

Similarly, the left $\mathcal{A}$-comodule is defined as a vector space $V$ equipped with a linear mapping $\varphi_L : V \to \mathcal{A} \otimes V$ such that

$$(\text{id} \otimes \varphi_L) \circ \varphi_L = (\Delta \otimes \text{id}) \circ \varphi_L, \quad (\epsilon \otimes \text{id}) \circ \varphi_L = \text{id}. \quad (4.2)$$

The mapping $\varphi_R$ ($\varphi_L$) is called a corepresentation, or, equivalently, a right (left) coaction of $\mathcal{A}$ on $V$.

Employing the duality of the algebras $\mathcal{U}$ and $\mathcal{A}$, we may follow the standard construction of the action of $\mathcal{U}$ on a $\mathcal{A}$-comodule $V$. Namely, starting from the corepresentations of $\mathcal{A}$, we may obtain representations of $\mathcal{U}$. Reversing the argument, we now obtain the hitherto unknown corepresentations of $\mathcal{A}$ from the already known irreducible representations of $\mathcal{U}$.

Let an arbitrary element $X \in \mathcal{U}$ act on the vector space $V^{(\ell)}$ described in §II.A. The matrix representation of $X$ on $V^{(\ell)}$ is denoted by $D^\ell(X; \lambda)$:

$$X e_m^\ell(\lambda) = \sum_{m'} e_{m'}^\ell(\lambda) D^\ell_{m'm}(X; \lambda). \quad (4.3)$$

The parity of the element $T^\ell_{m'm}(\lambda)$ of $\mathcal{A}$, defined via the duality relation

$$D^\ell_{m'm}(X; \lambda) = (-1)^{X^\ell(m' + \lambda)} \langle X, T^\ell_{m'm}(\lambda) \rangle,$$ (4.4)

may be assigned as

$$\widehat{T^\ell_{m'm}(\lambda)} = m' + m \pmod{2}. \quad (4.5)$$

To prove this, we compare the parity of the both sides of (4.3). The resultant equation $\widehat{D^\ell_{m'm}(X; \lambda)} = \tilde{X} + m' + m \pmod{2}$ establishes the relation (4.5). Though we know, via (2.11), that all nonvanishing entries of the representation matrices are even, we should
keep the parity of $D^\ell(X; \lambda)$ in the present formulation. Next we obtain the coproduct and counit maps for $T_{m'm}(\lambda)$:

$$\Delta(T_{m'm}(\lambda)) = \sum_{m''} T_{m'm''}(\lambda) \otimes T_{m''m}(\lambda), \quad \epsilon(T_{m'm}(\lambda)) = \delta_{m'm}. \quad (4.6)$$

The matrix $D^\ell(X; \lambda)$, being a representation of $X \in \mathcal{U}$, obeys the rule

$$D_{m'm}(XY; \lambda) = \sum_{m''} D_{m'm''}(X; \lambda) D_{m''m}(Y; \lambda) \quad \forall X, Y \in \mathcal{U}. \quad (4.7)$$

Both sides of (4.7) may be rewritten by using (4.4):

\[
\begin{align*}
\text{l.h.s.} & = (1)^{\sum_{m''} \langle X, T_{m'm''}(\lambda) \rangle} \langle X, T_{m'm}(\lambda) \rangle \\
& = (1)^{\sum_{m''} \langle X \otimes Y, \Delta(T_{m'm}(\lambda)) \rangle}, \\
\text{r.h.s.} & = \sum_{m''} (1)^{\sum_{m''} \langle X, T_{m'm''}(\lambda) \rangle} \langle Y, T_{m''m}(\lambda) \rangle \\
& = \sum_{m''} (1)^{\sum_{m''} \langle X \otimes Y, T_{m'm''}(\lambda) \otimes T_{m''m}(\lambda) \rangle}.
\end{align*}
\]

The property (4.7) now assumes the form

$$\left\langle X \otimes Y, \Delta(T_{m'm}(\lambda)) - \sum_{m''} T_{m'm''}(\lambda) \otimes T_{m''m}(\lambda) \right\rangle = 0. \quad (4.8)$$

Since the relation (4.8) is true for arbitrary $X, Y$, and the pairing is nondegenerate, we obtain the coproduct map in (4.6). To obtain the counit map, we consider

$$D_{m'm}(1; \lambda) = \langle 1, T_{m'm}(\lambda) \rangle \equiv \epsilon(T_{m'm}(\lambda)), \quad (4.9)$$

and use the fact that the unit element of $\mathcal{U}$ is always represented by the identity matrix $(\delta_{m'm})$. This completes the proof of (4.6).

Defining the map $\varphi_R : V^{(\ell)} \rightarrow V^{(\ell)} \otimes \mathcal{A}$ by

$$\varphi_R(e^\ell_m(\lambda)) = \sum_{m'} e^\ell_{m'}(\lambda) \otimes T_{m'm}(\lambda), \quad (4.10)$$

it is easy to show, via (4.6), that $V^{(\ell)}$ equipped with $\varphi_R$ is a right $\mathcal{A}$-comodule. Thus the quantum supermatrix $T^\ell(\lambda)$ provides the $(2\ell + 1)$-dimensional corepresentation of $\mathcal{A}$ on $V^{(\ell)}$. The relation between representations and corepresentations is summarized in the same form as non-super case

$$X e^\ell_m(\lambda) = (\text{id} \otimes X) \circ \varphi_R(e^\ell_m(\lambda)), \quad (4.11)$$

where the action of $\mathcal{U}$ on $\mathcal{A}$ is defined by the nondegenerate pairing. It is easy to find $T^\ell(\lambda)$ for $\ell = 0, 1$ from (4.4):

$$T^0_{00}(\lambda) = 1 \quad \text{for } \lambda = 0, 1, \quad (4.12)$$
\[ T^1(0) = T = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix}, \quad T^1(1) = \begin{pmatrix} a & -\alpha & b \\ -\gamma & e & -\beta \\ c & -\delta & d \end{pmatrix}, \] (4.13)

where the indices of rows and columns of \( T^1(\lambda) \) run over 1, 0 and -1.

One of the important properties of the corepresentations \( T^\ell(\lambda) \) is that they satisfy the product law which gives a rule to combine two corepresentations to get the third one:

\[ \delta_{\nu\mu} T^\ell_{m'm}(\Lambda) = \sum_{m_1,m_2} (-1)^p C^\ell_1_{m_1} \ell_2 m_2 \ell' C^\ell_1_{m_1} \ell_2 m_2 \ell' T^\ell_{m_1} T^\ell_{m_2} (\lambda), \] (4.14)

\[ p = (m'_1 + m_1)(\ell_2 - m'_2 + \lambda) + (\ell_1 - m'_1)(\ell_2 - m'_2) + (\ell' - m')(\ell_1 + \ell_2 + \ell'). \]

To prove the above product law, we first derive the fusion rule of representation matrices \( D^\ell(X;\lambda) \) of \( \mathcal{U} \), and then apply the duality argument à la the proof of (4.6).

Denoting the coproduct of \( X \in \mathcal{U} \) as \( \Delta(X) = \sum_a X_a \otimes X^a \), we use the projection relation (2.17) to obtain

\[ X e^\ell_m (\ell_1, \ell_2, \Lambda) = \sum_{m_1,m_2,a} (-1)^a (\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda) \times C^\ell_1 m_1 m_2 \ell_2 m_2 \ell' \delta^\ell_1 m_1 \delta^\ell_2 m_2 \delta^\ell_1 m_2 \delta^\ell_2 m_2 \lambda, \] (4.15)

We employ (4.3) to compute both sides of (4.15):

- **r.h.s.**

\[ \sum_{m_1,m_2} \sum_a (-1)^a (\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda) \times C^\ell_1 m_1 m_2 \ell_2 m_2 \ell' \delta^\ell_1 m_1 \delta^\ell_2 m_2 \delta^\ell_1 m_2 \delta^\ell_2 m_2 \lambda \]

- **l.h.s.**

\[ \sum_{m'} e^\ell_{m'} (\ell_1, \ell_2, \Lambda) D^\ell m'm \lambda \]

yielding the product law of \( D^\ell(X,\lambda) \):

\[ \delta_{\nu\mu} D^\ell_{m'm}(X,\Lambda) = \sum_{m_1,m_2} \sum_a (-1)^a (\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda) + p \times C^\ell_1 m_1 m_2 \ell_2 m_2 \ell' C^\ell_1 m_1 m_2 \ell_2 m_2 \ell' \delta^\ell_1 m_1 \delta^\ell_2 m_2 \delta^\ell_1 m_2 \delta^\ell_2 m_2 \lambda, \] (4.16)

To derive the fusion rule for \( T^\ell(\lambda) \), we consider the dual pairing

\[ \langle X, T^\ell_{m_1} (\lambda) T^\ell_{m_2} (\lambda) \rangle \]

\[ = \langle \Delta(X), T^\ell_{m_1} (\lambda) \otimes T^\ell_{m_2} (\lambda) \rangle \]

\[ = \sum_a (-1)^a (\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda) \times C^\ell_1 m_1 m_2 \ell_2 m_2 \ell' \delta^\ell_1 m_1 \delta^\ell_2 m_2 \lambda, \]
which, in turn, allows us to evaluate the following sum:

\[
\sum_{m_1', m_2'} (-1)^p C_{m_1'}^{\ell_1} C_{m_2'}^{\ell_2} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} \left\langle X, T_{m_1' m_1}^{\ell_1} (\lambda) T_{m_2' m_2}^{\ell_2} (\lambda) \right\rangle
\]

\[
= (-1)^{\lambda (m' - m + \Lambda)} \sum_{m_1', m_2'} (-1)^{\lambda (m - m_1 + \Lambda) + \lambda (m_2 - m_2' + \Lambda) + p} \times C_{m_1'}^{\ell_1} C_{m_2'}^{\ell_2} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} D_{m_1 m_1'} (X_a; \lambda) D_{m_2 m_2'} (X_a; \lambda)
\]

\[
= \delta_{\ell, \ell'} (-1)^{\lambda (m' - m + \Lambda)} D_{m' m} (X; \Lambda)
\]

\[
= \left\langle X, \delta_{\ell, \ell'} T_{m' m}^{\ell} (\lambda) \right\rangle.
\]

The second equality is due to (4.16). Invoking the arbitrariness of \( X \in \mathcal{U} \), and the nondegeneracy of dual pairing \( \langle \, , \, \rangle \), we obtain the fusion rule (4.14).

V. **A-COVARIENT ALGEBRAS**

A. **General prescription**

In this section, we will give a general prescription to find \( A \)-covariant algebras. By \( A \)-covariant algebras, we mean algebras whose defining relations are covariant under the right coaction \( \varphi_R \) of \( A \) defined by (4.10). Probably, the simplest way to find such an algebra is to introduce an algebraic structure on the representation space \( V^{(\ell)} \). Assuming \( \mu \) to be a multiplication map in \( V^{(\ell)} \), \textit{i.e.}, \( \mu(f \otimes g) = fg; f, g \in V^{(\ell)} \), we specifically construct the following composite object:

\[
E_L^L (\Lambda) \equiv \mu (e^L_M (\ell, \ell, \Lambda)) = \sum_{m_1, m_2} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} e^L_{m_1} (\lambda) e^L_{m_2} (\lambda), \quad (5.1)
\]

where \( \Lambda = L \mod 2 \). The right coaction on \( E_L^L (\Lambda) \) is shown to be

\[
\varphi_R (E_L^L (\Lambda)) = \sum_{M'} E_{M'}^L (\Lambda) \otimes T_{M' M}^L (\Lambda). \quad (5.2)
\]

The proof may be done in a straightforward way by inverting the relation (5.1)

\[
e^L_{m_1} (\lambda) e^L_{m_2} (\lambda) = (-1)^{(\ell - m_1)(\ell - m_2)} \sum_{L, M} (-1)^{(L - M) L} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} E_L^L (\Lambda), \quad (5.3)
\]

and subsequently using the product law (4.14)

\[
\varphi_R (E_L^L (\Lambda)) = \sum_{m_1, m_2} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} \varphi_R (e^L_{m_1} (\lambda)) \varphi_R (e^L_{m_2} (\lambda))
\]

\[
= \sum_{m_1', m_2'} (-1)^{(m_1' + m_1)(\ell - m_2' + \Lambda)} C_{m_1}^{\ell_1} C_{m_2}^{\ell_2} e^L_{m_1} (\lambda) e^L_{m_2} (\lambda) \otimes T_{m_1' m_1}^{\ell_1} (\lambda) T_{m_2' m_2}^{\ell_2} (\lambda)
\]
where $p' = (m_1' + m_1)(\ell - m_2') + (\ell - m_1')(\ell - m_2) + (L' - M' - L')$.

Employing (5.2) we now extract a set of relations which are covariant under $\varphi_R$. The $L = 0$ relation $\varphi_R(E^0_0(0)) = E^0_0(0)$ signifies that $E^0_0(0)$ is a scalar under the right coaction. It may be equated to a constant parameter $r$:

$$E^0_0(0) = \sum_{m_1, m_2} C^\ell_{m_1 m_2} e^\ell_{m_1}(\lambda)e^\ell_{m_2}(\lambda) = r. \tag{5.4}$$

If $L = \ell$ and $\lambda = \ell \pmod 2$, then $E^\ell_m(\lambda)$ and $e^\ell_m(\lambda)$ have the same parity, and they transform identically under $\varphi_R$. Therefore $E^\ell_m(\lambda)$ is, in general, proportional to $e^\ell_m(\lambda)$. It may be noted that the following relations are covariant

$$E^\ell_m(\lambda) = \sum_{m_1, m_2} C^\ell_{m_1 m_2} e^\ell_{m_1}(\lambda)e^\ell_{m_2}(\lambda) = \xi e^\ell_m(\lambda), \tag{5.5}$$

where the proportionality constant $\xi \to 0$ as $q \to 1$. For the case $\lambda \neq \ell \pmod 2$, $E^\ell_m(\ell)$ has different parity from $e^\ell_m(\lambda)$, even though they transform identically. In this case, the constant $\xi$ in (5.5) is regarded as a Grassmann number that also vanish at $q = 1$. For $L \neq 0, \ell$, the element $E^\ell_m(\ell)$ can not be proportional to $e^\ell_m(\lambda)$ as they transform differently. The relevant covariant relations are, therefore, of the form

$$E^L_M(\Lambda) = \sum_{m_1, m_2} C^\ell_{m_1 m_2} T^\ell_{m_1} T^\ell_{m_2}(\lambda) = 0. \tag{5.6}$$

As will be seen in the subsequent sections, the simultaneous use of all relations from (5.4) to (5.6) gives an inconsistent result, since some of them do not have correct classical limits. In order to obtain a consistent covariant algebra, we have to make a choice regarding the relations to be used for defining the algebra. Then the consistency has to be verified. As it is clear from the above discussion, the covariant algebras can have at most two more parameters ($r, \xi$) in addition to the deformation parameter $q$. It is emphasised that the origin of the parameters is clearly explained in the framework of the representation theory.

We have formulated a method to construct $\mathcal{A}$-covariant algebras with respect to the right coaction. It is possible to repeat the same discussion for the left coaction.

**B. Quantum superspace ($\ell = 1$, $\lambda = 0$)**

Let us investigate the covariant algebra for $\ell = 1$ case, where the relevant tensor product decomposition is given by $1 \otimes 1 = 2 \oplus 1 \oplus 0$. We assume that $\lambda = 0$, and denote the basis of $V^{(1)}$ by $z_m = e^1_m(0)$. Thus $z_{\pm 1}$ are parity even and $z_0$ is parity odd. The CGC for the decomposition is given in Appendix C. For $L = 0$, we obtain from (5.4)

$$q^{1/2} z_{-1} z_1 + z_0^2 - q^{-1/2} z_1 z_{-1} = r. \tag{5.7}$$
For $L = 1$, we have $\Lambda \neq \lambda$, and, therefore, the parameter $\xi$ is a Grassmann number:

\[
-q^{1/2}z_0z_1 + q^{-1/2}z_1z_0 = \xi z_1,
\]
\[
z_{-1}z_1 + (q^{-1/2} + q^{1/2})z_0^2 - z_1z_{-1} = \xi z_0,
\]
\[
q^{1/2}z_{-1}z_0 - q^{-1/2}z_0z_{-1} = \xi z_{-1}.
\]

(5.8)

For $L = 2$, we obtain, using (5.6), unacceptable relations such as

\[
z_1^2 = 0, \quad q^{-1/2}z_0z_1 + q^{1/2}z_1z_0 = 0.
\]

Thus we take (5.7) and (5.8) as defining relations of our covariant algebra. We need to check the following conditions in order to verify whether or not the algebra is well-defined:

(a) The constant $r$ commutes with all generators

(b) Product of three generators, say $z_1z_0z_{-1}$, has two ways of reversing its ordering:

These two ways give the same result.

It is straightforward to verify that the condition (a) is satisfied. The condition (b), however, requires setting $\xi = 0$.

Therefore, we define our covariant algebra by combining relations (5.7) and (5.8), while maintaining $\xi = 0$:

\[
z_1z_0 = qz_0z_1, \quad z_0z_{-1} = qz_{-1}z_0,
\]
\[
z_1z_{-1} = q^2z_{-1}z_1 - q(q^{-1/2} + q^{1/2})r,
\]
\[
z_0^2 = -q^{-1}[2]z_1z_{-1} - q^{-1}r.
\]

(5.9)

This may be interpreted as the most general form of a quantum superspace. The simplest quantum superspace corresponds to the choice of $r = 0$.

C. Quantum superspace ($\ell = 1, \lambda = 1$)

In this subsection, we study another quantum superspace where the parity is opposite to the previous example. Setting $\theta_m = e_1^m(1)$, we note that $\theta_{\pm 1}$ have odd parity, while $\theta_0$ has even parity. Following (5.4), we obtain

\[
q^{1/2}\theta_{-1}\theta_1 - \theta_0^2 - q^{-1/2}\theta_1\theta_{-1} = r.
\]

(5.10)

The $L = 1$ relations may be obtained from (5.5) except that in this case, as $\Lambda = \lambda$ holds, the parameter $\xi$ is not a Grassmann number. However, these relations such as

\[
q^{1/2}\theta_0\theta_1 + q^{-1/2}\theta_{-1}\theta_0 = \xi \theta_1.
\]
are unacceptable as they include, in the \( q \to 1 \) limit, anticommutators for the product of even and odd elements. On the other hand, \( L = 2 \) relations have proper classical limits:

\[
\begin{align*}
\theta_{\pm 1}^2 &= 0, \\
q^{-1/2}\theta_0\theta_1 - q^{1/2}\theta_1\theta_0 &= 0, \\
q^{-1}\theta_{-1}\theta_1 - [2]\theta_0^2 + q\theta_1\theta_{-1} &= 0, \\
-q^{-1/2}\theta_{-1}\theta_0 + q^{1/2}\theta_0\theta_{-1} &= 0.
\end{align*}
\] (5.11)

An interesting observation for this case is that we have two kinds of quantum superspaces. Firstly we note that the relations (5.11) are enough to define a covariant algebra, since it may be shown that the condition (b) of Sec. V.B is satisfied. Thus the relations (5.11) define a quantum superspace. Alternately, combining (5.11) with (5.10) we obtain another set of covariant relations

\[
\begin{align*}
\theta_2^\pm 1 &= 0, \\
q\theta_0\theta_1 &= \theta_0\theta_1, \\
q\theta_0\theta_{-1} &= \theta_{-1}\theta_0, \\
\theta_1\theta_{-1} + \theta_{-1}\theta_1 &= -\frac{[2]}{3}r, \\
\theta_0^2 &= -(q^{-1/2} + q^{1/2})\theta_1\theta_{-1} - q^{1/2}\frac{[2]}{3}r,
\end{align*}
\] (5.12)

which have correct classical limit for an arbitrary value of \( r \). With these relations, it may be checked that the conditions (a) and (b) of Sec V.B are satisfied. Thus we define the second quantum superspace by (5.12).

The above quantum superspaces are covariant under the coaction

\[
\varphi_R(\theta_m) = \sum_{m'} \theta_{m'} \otimes T^{1}_{m'm}(1),
\] (5.13)

where \( T^{1}(1) \) is given by (4.13). Defining a new basis of the quantum superspace by

\[
\theta'_0 = \theta_0, \quad \theta'_{\pm 1} = -\theta_{\pm 1},
\] (5.14)

we observe that the \( \theta'_m \) are covariant under the same corepresentation matrix as the case of \( \lambda = 0 \):

\[
\theta'_m = \sum_{m'} \theta'_{m'} \otimes T^{1}_{m'm}(0).
\] (5.15)

\[\text{VI. } \mathcal{A}\text{-COVARIANT SPHERE}\]

\[\text{A. Construction}\]

In this section, we investigate an algebra covariant under the coaction of \( T^2(\lambda) \), i.e. the adjoint corepresentation of \( \mathcal{A} \). This may be interpreted as a supersymmetric extension of a noncommutative sphere. The corepresentation matrices \( T^2(\lambda) \) are found from (2.14) and (4.4), or, alternately, by coupling the elements of two \( T^1(\lambda) \) matrices via the product law (4.14). We restrict ourselves to the case of \( \lambda = 0 \):

\[
T^2(0) = \begin{pmatrix}
a^2 & a_1a_\alpha & a_1\gamma & a\alpha & a_\beta & a_{-1}^2 & a_{-1}\gamma & a_{-1}\alpha & a_{-1}\beta & a_{-1}^2 \\
\gamma_1a_\gamma & \alpha + q^{-1}\gamma_1a_\alpha & \kappa_2(a_\beta + q^{-1}\gamma_1a_\beta) & -\alpha_\beta + q^{-1}\alpha_\beta & \kappa_1\gamma_1a_\beta & \kappa_1a_\beta & \kappa_2(a_\beta + q^{-1}\gamma_1a_\beta) & \kappa_1\gamma_1a_\beta & \kappa_1a_\beta & \kappa_1a_\beta \\
\kappa_3a_c & \kappa_2(a_\beta + q^{-1}\gamma_1a_\beta) & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c & \kappa_3a_c \\
\gamma_1\delta & q^{-1}\gamma_1\delta & \kappa_2(\gamma_1c_\gamma + q^{-1}\gamma_1c_\delta) & \kappa_2(\gamma_1c_\gamma + q^{-1}\gamma_1c_\delta) & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta \\
q^{-1} & \kappa_1\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta & \kappa_3\gamma_1c_\delta \\
\end{pmatrix}
\] (6.1)
where

\[ \kappa_1 = \sqrt{\frac{[4]}{q[2]}}, \quad \kappa_2 = \sqrt{q^{-[3]}}, \quad \kappa_3 = \kappa_1 \kappa_2. \quad (6.2) \]

The basis of \( V^{(2)} \) is denoted by \( Y_m = e^m_0(0) \), where \( m = 0, \pm 1, \pm 2 \). Here \( Y_0, Y_{\pm 2} \) are even, and \( Y_{\pm 1} \) are odd. Following the prescription in Sec.V.A, we seek a covariant algebra under the right coaction of \( T^2(0) \). The CGC for \( \ell = 2 \) are found in Appendix D. The relation for \( L = 0 \) is obtained via (5.4):

\[ q^{-1}Y_2Y_{-2} - q^{-1/2}Y_1Y_{-1} - Y_0^2 + q^{1/2}Y_{-1}Y_1 + qY_{-2}Y_2 = r, \quad (6.3) \]

where \( r \) is a constant. Equation (6.3) may be regarded as the radius relation of the quantum supersphere. Explicit constructions for the \( L = 2 \) case are obtained from (5.5):

\[ q^{-3/2}Y_2Y_0 - \left(\frac{[3]!}{[4]}\right)^{1/2} Y_1^2 - q^{3/2}Y_0Y_2 = \xi Y_2, \]

\[ q^{-1/2} \left(\frac{[3]!}{[4]}\right)^{1/2} Y_2Y_{-1} + q^{-1/2}\mu_21Y_1Y_0 - q^{1/2}\mu_22Y_0Y_1 - q^{1/2} \left(\frac{[3]!}{[4]}\right)^{1/2} Y_{-1}Y_2 = \xi Y_1, \]

\[ q^{1/2}Y_2Y_{-2} - \mu_22Y_1Y_{-1} + \mu_23Y_0^2 - \mu_23Y_{-1}Y_1 - q^{-1/2}Y_{-2}Y_2 = \xi Y_0, \quad (6.4) \]

\[ q^{-1/2} \left(\frac{[3]!}{[4]}\right)^{1/2} Y_1Y_{-2} + q^{-1/2}\mu_21Y_0Y_{-1} - q^{1/2}\mu_22Y_{-1}Y_0 - q^{1/2} \left(\frac{[3]!}{[4]}\right)^{1/2} Y_{-2}Y_1 = \xi Y_{-1}, \]

\[ q^{-3/2}Y_0Y_{-2} - \left(\frac{[3]!}{[4]}\right)^{1/2} Y_{-1}^2 - q^{3/2}Y_{-2}Y_0 = \xi Y_{-2}, \]

where \( \xi \) is a constant vanishing in the classical limit, and

\[ \mu_{21} = \frac{2}{[4]}(q^2 + q^{3/2}[2]), \quad \mu_{22} = \frac{2}{[4]}(q^2 - q^{-3/2}[2]), \quad \mu_{23} = \frac{2}{[4]} \left(\frac{[8]}{[4]} + 2\frac{[4]}{[2]} + 1\right). \quad (6.5) \]

The construction (5.6) may be applied to the remaining values of \( L \). The relations for the \( L = 3 \) case read

\[ q^{-1}Y_2Y_1 - qY_1Y_2 = 0, \]

\[ Y_2Y_0 - \mu_{31}Y_1^2 - Y_0Y_2 = 0, \]

\[ qY_2Y_{-1} - \mu_{32}Y_1Y_0 + \mu_{33}Y_0Y_1 - q^{-1}Y_{-1}Y_2 = 0, \quad (6.6) \]

\[ -q^2Y_2Y_{-2} + q^{1/2}(3 + q^2)Y_1Y_{-1} + [3]\omega Y_0^2 + q^{-1/2}(3 + q^{-2})Y_{-1}Y_1 + q^{-2}Y_{-2}Y_2 = 0, \]

\[ qY_1Y_{-2} - \mu_{32}Y_0Y_{-1} + \mu_{33}Y_{-1}Y_0 - q^{-1}Y_{-2}Y_1 = 0, \]

\[ Y_0Y_{-2} - \mu_{31}Y_{-1}^2 - Y_{-2}Y_0 = 0, \]

\[ q^{-1}Y_{-1}Y_{-2} - qY_{-2}Y_{-1} = 0, \]

where

\[ \mu_{31} = -\sqrt{\frac{[4]!}{[2]^2}} \omega, \quad \mu_{32} = \left(\frac{[4]}{[3]!}\right)^{1/2} (q^2 + q^{-1/2}[2]), \quad \mu_{33} = \left(\frac{[4]}{[3]!}\right)^{1/2} (q^{-2} - q^{1/2}[2]). \quad (6.7) \]
We observe that the relations (6.4) together with (6.6) have the correct classical limits. We have obtained a set of twelve relations for five generators. To test whether they consistently define an algebra, we need to check for the conditions (a) and (b) mentioned in Sec.V.B. It may be proved by direct computation that the said conditions are, however, not satisfied.

In order to make the algebra well-defined, we incorporate the $L = 1$ relations listed below:

$$q^{-3/2}Y_2Y_{-1} - q^{-1/2}\left(\frac{[3]!}{[4]}\right)^{1/2}Y_1Y_0 - q^{1/2}\left(\frac{[3]!}{[4]}\right)^{1/2}Y_0Y_1 + q^{3/2}Y_{-1}Y_2 = 0,$$

$$-q^{-1/2}Y_2Y_{-2} + q^{-1}\mu_{11}Y_1Y_{-1} + \varpi\left[\frac{3}{4}\right]^2Y_0 - q\mu_{12}Y_1Y_{-1} - q^{1/2}Y_{-2}Y_2 = 0,$$

$$-q^{-3/2}Y_1Y_{-2} + q^{-1/2}\left(\frac{[3]!}{[4]}\right)^{1/2}Y_0Y_{-1} + q^{1/2}\left(\frac{[3]!}{[4]}\right)^{1/2}Y_{-1}Y_0 - q^{3/2}Y_{-2}Y_1 = 0,$$

where

$$\varpi = q^{1/2} + q^{-1/2}, \quad \mu_{11} = q + q^{-1}\left[\frac{2}{4}\right], \quad \mu_{12} = q^{-1} + q\left[\frac{2}{4}\right].$$

The remaining $L = 4$ relations can not be incorporated, because they contain unacceptable equations such as $Y_{\pm 2}^2 = 0$.

As all the relations in (6.4), (6.6), and (6.8) are covariant by construction, their linear combinations are also covariant. Taking linear combination of the fifteen relations, twelve "commutation relations" of generators and three constraints are obtained. The commutation relations read

$$Y_2Y_1 = q^2Y_1Y_2, \quad Y_{-1}Y_{-2} = q^2Y_{-2}Y_{-1},$$

$$q^{-2}Y_2Y_0 = q^2Y_0Y_2 + \varpi\left[\frac{4}{6}\right]^3\xi Y_2,$$

$$q^{-2}Y_0Y_{-2} = q^2Y_{-2}Y_0 + \varpi\left[\frac{4}{6}\right]^3\xi Y_{-2},$$

$$q^{-3}Y_2Y_{-1} = q^3Y_{-1}Y_2 + \varpi\left[\frac{3}{6}\right]^4\xi Y_1,$$

$$q^{-3}Y_1Y_{-2} = q^3Y_{-2}Y_1 + \varpi\left[\frac{3}{6}\right]^4\xi Y_{-1},$$

$$q^{-1}Y_1Y_0 = qY_0Y_1 + \varpi\left[\frac{3}{6}\right]^6\xi Y_1,$$

$$q^{-1}Y_0Y_{-1} = qY_{-1}Y_0 + \varpi\left[\frac{3}{6}\right]^6\xi Y_{-1},$$

$$q^{-1}\mu_{11}Y_2Y_{-2} = q\mu_{12}Y_{-2}Y_2 - F_1Y_0^2 + F_2Y_0,$$

$$q^{-1/2}Y_1Y_{-1} = -q^{1/2}Y_{-1}Y_1 + \omega Y_0^2 + \varpi + q^{-1}\xi Y_0,$$

$$Y^2_1 = \frac{1}{\mu_{31}}(Y_2Y_0 - Y_0Y_2), \quad Y^2_{-1} = \frac{1}{\mu_{31}}(Y_0Y_{-2} - Y_{-2}Y_0),$$

16
where

\[ F_1 = \varpi (q^2 + q + 1 + q^{-1} + q^{-2}) \frac{[2]^2}{[4]}, \]
\[ F_2 = \varpi (q^2 + 2q + 2q^{-1} + q^{-2}) \frac{[3]! [2]}{[6] [4]} \xi. \quad (6.11) \]

Classical commutation properties immediately follow from (6.10) in the limit \( q \to 1 \). Three said constraints are given by

\[ Y_2 Y_{-1} = q^2 \frac{[3]}{[4]} \left( \frac{[3]!}{[4]} \right)^{1/2} Y_1 Y_0 + q^{1/2} \frac{[2]}{[6]} \left( \frac{[3]!}{[4]} \right)^{1/2} \xi Y_1, \]
\[ Y_1 Y_{-2} = q^2 \frac{[3]}{[4]} \left( \frac{[3]!}{[4]} \right)^{1/2} Y_0 Y_{-1} + q^{1/2} \frac{[2]}{[6]} \left( \frac{[3]!}{[4]} \right)^{1/2} \xi Y_{-1}, \quad (6.12) \]
\[ Y_0^2 = q^{-1} \frac{[4]}{[2]} Y_2 Y_{-2} - q^{-1/2} (q + q^{-1}) \mu_{12} Y_1 Y_{-1} - q^{-3/2} \frac{[3]!}{[6]} \xi Y_0. \]

The classical limit of these constraints are not required in the commutative case. However, we need the constraints to satisfy the two important requirements given in Sec.V.B. Verification of these conditions is straightforward but requires dull lengthy computation. Since the two conditions are satisfied, we define a one-parameter family of \( \mathcal{A} \)-covariant quantum superspheres by the radius relation (6.3), the commutation relations (6.10) and the constraints (6.12). We denote this quantum supersphere by \( S^0_{q, \xi} \). The superscript 0 indicates the parity \( \lambda = 0 \) and \( \xi \) is a free parameter which does not have classical counterpart. The origin of the parameter and the fact that the quantum superspheres can not have more parameters clearly follows from the formulation in Sec. V.A.

**B. Properties of \( S^0_{q, \xi} \)**

In this subsection, three properties of the quantum supersphere \( S^0_{q, \xi} \) are investigated. First, we consider a realization of \( S^0_{q, \xi} \) in terms of elements of \( \mathcal{A} \). Then it turns out that this realization admits an infinitesimal characterization of \( S^0_{q, \xi} \). A representation of \( S^0_{q, \xi} \) in terms of a \( \mathcal{U} \)-covariant oscillator is also given.

Analogous to the example of Podleś \( q \)-sphere, embedding of \( S^0_{q, \xi} \) in \( \mathcal{A} \) may be done by realizing its generators in terms of entries of \( T^2(0) \) matrix. Denoting \( T^2(0) \) by \( T \), it is straightforward to verify that the embedding is given by

\[ Y_2 = g_1 T_{2,2} + g_2 T_{0,2} + g_3 T_{-2,2}, \]
\[ Y_1 = g_1 T_{2,1} + g_2 T_{0,1} + g_3 T_{-2,1}, \]
\[ Y_0 = g_1 T_{2,0} + g_2 T_{0,0} + g_3 T_{-2,0}, \]
\[ Y_{-1} = g_1 T_{2,-1} + g_2 T_{0,-1} + g_3 T_{-2,-1}, \]
\[ Y_{-2} = g_1 T_{2,-2} + g_2 T_{0,-2} + g_3 T_{-2,-2}, \quad (6.13) \]

where the coefficients \( g_1, g_2 \) and \( g_3 \) need to satisfy the constraint

\[ g_1 g_3 = \frac{[3]!}{[4]} g_2. \quad (6.14) \]
In this embedding, the radius \( r \) and the parameter \( \xi \) are given as functions of \( g_2 \):

\[
r = ([2]g_2)^2, \quad \xi = \frac{[6]}{[3]}g_2.
\]  

(6.15)

This embedding allows us to treat \( S^0_{q,\xi} \) as a subalgebra of \( \mathcal{A} \). This fact suggests that \( S^0_{q,\xi} \) has an infinitesimal characterization \( \text{a la} \) Koornwinder \(^{13} \). To demonstrate this, we extend the left and right actions of \( U_q[su(2)] \) on \( SU_q(2) \) defined in Ref. 13 to supersymmetric case. In the followings we assume \( u, v \in \mathcal{U}; a, b \in \mathcal{A} \), and use Sweedler’s notation to denote coproducts: \( e.g. \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \). With a slight change of notation from Ref. 13, we now define elements \( u \odot a \) and \( a \odot u \) of \( \mathcal{A} \) by

\[
u \odot a = (\text{id} \otimes u)(\Delta(a)) = \sum (-1)^{\hat{u}\hat{a}}(1) a_{(1)} \langle u, a_{(2)} \rangle, \quad (6.16)
\]

\[
a \odot u = (-1)^{\hat{a}\hat{u}}(u \otimes \text{id})(\Delta(a)) = \sum (-1)^{\hat{a}\hat{u}} \langle u, (a_{(1)}) a_{(2)} \rangle. \quad (6.17)
\]

The coassociativity of \( \mathcal{A} \) then leads to

\[
(uv) \odot a = u \odot (v \odot a), \quad a \odot (uv) = (a \odot u) \odot v. \quad (6.18)
\]

Moreover, we also have

\[
u \odot (ab) = \sum (-1)^{\hat{u}\hat{a}}(u_{(1)} \odot a)(u_{(2)} \odot b), \quad (6.19)
\]

\[
(ab) \odot u = \sum (-1)^{\hat{a}\hat{b}}(a \odot u_{(1)})(b \odot u_{(2)}).
\]

Thus, \( u \odot a \) and \( a \odot u \) define left and right actions of \( u \) on \( a \), respectively. The actions of generators of \( \mathcal{U} \) on \( T^\ell \)-matrix of \( \mathcal{A} \) are calculated by using (2.11). Explicitly, the left actions are given by

\[
K^{\pm 1} \odot T^{\ell}_{m_1 m_2}(\lambda) = q^{m_2/2} T^{\ell}_{m_1 m_2}(\lambda),
\]

\[
v_+ \odot T^{\ell}_{m_1 m_2}(\lambda) = (-1)^{\ell + m_1 + \lambda} \sqrt{[\ell - m_2][\ell + m_2 + 1]} g T^{\ell}_{m_1 m_2 + 1}(\lambda), \quad (6.20)
\]

\[
v_- \odot T^{\ell}_{m_1 m_2}(\lambda) = (-1)^{m_1 + m_2 + \lambda + 1} \sqrt{[\ell + m_2][\ell - m_2 + 1]} g T^{\ell}_{m_1 m_2 - 1}(\lambda),
\]

while the right actions read

\[
T^{\ell}_{m_1 m_2}(\lambda) \odot K^{\pm 1} = q^{m_2/2} T^{\ell}_{m_1 m_2}(\lambda),
\]

\[
T^{\ell}_{m_1 m_2}(\lambda) \odot v_+ = (-1)^{\ell + m_2 + \lambda} \sqrt{[\ell - m_1 + 1][\ell + m_1]} g T^{\ell}_{m_1 - m_2 - 1}(\lambda), \quad (6.21)
\]

\[
T^{\ell}_{m_1 m_2}(\lambda) \odot v_- = (-1)^{m_1 + m_2 + \lambda} \sqrt{[\ell + m_1 + 1][\ell - m_1 - 1]} g T^{\ell}_{m_1 + m_2 + 1}(\lambda).
\]

An element \( u \in \mathcal{U} \) possessing a coproduct structure \( \Delta(u) = g \otimes u + u \otimes g^{-1} \) with \( g \in \mathcal{U} \) being a group-like element, is known as \textit{twisted primitive} with respect to \( g \). For a twisted primitive element \( u \), it is straightforward to verify that

\[
u \odot a = 0 \quad \text{and} \quad u \odot b = 0 \implies u \odot (ab) = 0, \quad (6.22)
\]

\[
a \odot u = 0 \quad \text{and} \quad b \odot u = 0 \implies (ab) \odot u = 0. \quad (6.23)
\]
Thus a set of elements of $\mathcal{A}$ annihilated by a twisted primitive element $u$ form a subalgebra of $\mathcal{A}$. Indeed, the quantum supersphere $S^0_{q,\xi}$ embedded into $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ that is annihilated by the twisted primitive element $\mathcal{P}_R$

$$\mathcal{P}_R = -\sqrt{g_3}v_+ + \sqrt{g_1}v_-,$$

$$Y_k \odot \mathcal{P}_R = 0, \quad k = \pm 2, \pm 1, 0.$$  \hfill (6.24)

The algebra $\mathcal{U}$ has three twisted primitive elements: $K - K^{-1}, v_+$ and $v_-$. However, $\mathcal{P}_R$ consists of only odd twisted primitive elements. This is a difference from the $q$-sphere for $SU_q(2)$. In that example, all the twisted primitive elements contribute to the annihilation operator of $q$-sphere.

We now turn to an oscillator realization of $S^0_{q,\xi}$. In Ref. 20, a $\mathcal{U}$-covariant oscillator algebra is introduced. This oscillator algebra is generated by a pair of even creation/annihilation operators ($\bar{a}, a$), and an odd operator $c$ obeying the relations

$$\bar{a}c = qca, \quad ac = q^{-1}ca, \quad a\bar{a} - q^{-2}\bar{a}a = 1,$$

$$c^2 = q^{-1}[2]\bar{a}a + \varpi^{-1}. $$ \hfill (6.26)

These relations are determined by two steps. First, the action of $\mathcal{U}$ on the oscillator is defined via the coproduct of $\mathcal{U}$. Then the commutation properties of the oscillator is fixed by demanding that the triplet ($\bar{a}, c, -a$) transforms under the $\ell = 1$ representation of $\mathcal{U}$. This suggests that the $\mathcal{U}$-covariant oscillator has a close kinship to the quantum space discussed in Sec.V.B. Indeed, the $\mathcal{U}$-covariant oscillator is isomorphic to the quantum plane for a special value of $r$:

$$z_1 = \bar{a}, \quad z_0 = c, \quad z_{-1} = -a, \quad r = -q\varpi^{-1}. $$ \hfill (6.27)

Employing the $\mathcal{U}$-covariant oscillator, it is possible to realize $S^0_{q,\xi}$:

$$Y_2 = \bar{a}^2, \quad Y_1 = q^{-1/2}\left[\begin{array}{c}4 \\ 2 \end{array}\right]^{1/2}\bar{a}c,$$

$$Y_0 = -\sqrt{\frac{4!}{q[2]}}\bar{a}a - q^{-1/2}\frac{\varpi}{\varpi}\left[\begin{array}{c}4 \\ 3! \end{array}\right]^{1/2},$$

$$Y_{-1} = -q^{-1/2}\left[\begin{array}{c}4 \\ 2 \end{array}\right]^{1/2}ca, \quad Y_{-2} = a^2. $$ \hfill (6.28)

In this realization, the radius $r$ and the parameter $\xi$ of $S^0_{q,\xi}$ assume the following values

$$r = q^2\frac{[4]}{[2]}\frac{[4]}{[3]!} \varpi^{-1/2}, \quad \xi = \frac{[6]}{[3]!}r^{1/2}. $$ \hfill (6.29)

An advantage of this realization is that we may represent $S^0_{q,\xi}$ with matrices, since matrix representation of $\mathcal{U}$-covariant oscillator via that of Biedenharn-Macfarlane $q$-boson algebra exists$^{20}$. 

19
VII. CONCLUDING REMARKS

We have developed the general prescription for constructing $\mathcal{A}$-covariant algebras. By the method, four $\mathcal{A}$-covariant algebras have been obtained, namely, three quantum superspaces and a one-parameter family of quantum superspheres. The special cases of the quantum superspaces correspond to the Manin’s quantum superplane and $\mathcal{U}$-covariant oscillator algebra. The quantum superspheres are realized by $\mathcal{A}$ so that it can be regarded as a subalgebra of $\mathcal{A}$. This subalgebra is characterized by the fact that it is annihilated by the right action of a particular combination of the twisted primitive elements of $\mathcal{U}$. These are the similarities to the $q$-spheres for $SU_q^0(2)$. It has also been shown that the quantum superspheres have $\mathcal{U}$-covariant oscillator realization that allows us to have matrix representations of the quantum supersphere.

We believe that the results of this paper are useful for making progress in constructing supersymmetric versions of noncommutative geometry. For instance, we may consider differential calculi on the quantum supersphere, then compute its curvature, metric and so on based on the framework of Ref. 5. In connection with the classification of differential calculi, it is interesting to determine the dual coalgebra of $S^0_{q,\xi}$. The corresponding computation for $SU_q(2)$ $q$-sphere was made recently and applied to the classification of differential calculi11. Furthermore, the general method in Sec. V.A is applied to construct higher dimensional quantum superspaces by taking higher values of $\ell$.

It is worth pointing out that, because of the similarity of the representation theory of $\mathcal{U}$ to that of $U_q(su(2))$, the method developed in Sec. V.A is valid for $SU_q(2)$. This procedure allows us to treat quantum plane, deformed oscillator and Podleś $q$-sphere in a unified way. The reason for $q$-spheres being a one-parameter family becomes clear in this framework. It is also possible to construct hitherto unknown higher dimensional $SU_q(2)$-covariant quantum spaces. We will present these results elsewhere.

ACKNOWLEDGEMENTS

The work of N.A. is partially supported by the grants-in-aid from JSPS, Japan (Contract No. 15540132). The work of the other author (R.C.) is partially supported by the grant DAE/2001/37/12/BRNS, Government of India.

Appendix A CGC OF $\mathcal{U}$

In this appendix, the general expression of CGC in our conventions is derived. Let us write the highest weight vector as

$$e_{\ell}^\xi(\ell_1, \ell_2, \Lambda) = \sum_{m_1, m_2} A_{m_1, m_2} e_{m_1}^{\ell_1} (\lambda) \otimes e_{m_2}^{\ell_2} (\lambda),$$

(A.1)

where, for simplicity of notation, the CGC is denoted by $A_{m_1, m_2}$. The highest weight condition $\Delta(u_+) e_{\ell}^\xi(\ell_1, \ell_2, \Lambda) = 0$ gives the recurrence relation for $A_{m_1, m_2}$

$$\sqrt{[\ell_1 - m_1]}[\ell_1 + m_1 + 1]q^{-m_2/2}A_{m_1, m_2}$$

$$- (-1)^{\ell_1 - m_1 + \lambda} q^{(m_1 + 1)/2} \sqrt{[\ell_2 + m_2][\ell_2 - m_2 + 1]}A_{m_1 + 1, m_2 - 1} = 0.$$  

(A.2)
Thus the norm of the highest weight vector reads

\[ A_{m_1m_2} = (-1)^{\ell_1 - m_1} (\ell_1 - m_1 + 1)/(\ell_1 - m_1 + 2) q^{(\ell_1 - m_1)/2} \times \left( \frac{[\ell_1 + \ell_2 - \ell]! [\ell_1 + m_1]! [\ell_2 + m_2]!}{[\ell_2 - \ell_1 + \ell]! [2\ell_1]! [\ell_1 - m_1]! [\ell_2 - m_2]!} \right)^{1/2} A_{\ell_1 \ell - \ell_1}. \] (A.3)

The normalization of the highest weight vector (A.1) determines \( A_{\ell_1 \ell - \ell_1} \) as follows:

\[ (e_\ell^\epsilon(\ell_1, \ell_2, \Lambda), e_\ell^\epsilon(\ell_1, \ell_2, \Lambda)) = \sum_{m_1, m_2} (-1)^{\ell_1 - m_1}(\ell_2 - m_2 + \lambda) A_{m_1m_2}^2 \]

\[ = (-1)^{\ell_1 + \lambda}(\ell_2 + \ell + \lambda) \frac{[\ell_1 + \ell_2 - \ell]!}{[\ell_2 - \ell_1 + \ell]! [2\ell_1]} A_{\ell_1 \ell - \ell_1}^2 \times \sum_{m_1} (-1)^{m_1}(\ell_1 + \ell_2 + \ell + 1) q^{(\ell_1 - m_1)/(\ell_1 - m_1)} \frac{[\ell_1 + m_1]! [\ell_2 + \ell - m_1]!}{[\ell_1 - m_1]! [\ell_2 - \ell + m]!} \]

The summation over \( m_1 \) is computed by using the formula (B.5). Setting

\[ a = \ell_1 - m_1, \quad k = \ell_1 + \ell_2 - \ell, \quad -n = \ell_1 - \ell_2 + \ell + 1, \quad -r = -\ell_1 + \ell_2 + \ell + 1 \]

in (B.5), while noticing that all these quantities are positive integers, we obtain

\[ \sum_{m_1} (-1)^{\ell_1 - m_1}(\ell_1 + \ell_2 + \ell + 1) q^{(\ell_1 - m_1)/(\ell_1 - m_1)} \frac{[\ell_1 + m_1]! [\ell_2 + \ell - m_1]!}{[\ell_1 - m_1]! [\ell_2 - \ell + m]!} \]

\[ = q^{-(\ell_1 + \ell_2 + \ell + 1)/(\ell_1 - m_1)/2} [\ell_1 + \ell_2 + \ell + 1]! [\ell_1 - \ell_2 + \ell]! [-\ell_1 + \ell_2 + \ell]! \]

\[ \frac{[\ell_1 + \ell_2 - \ell]! [2\ell_1]!}{[\ell_1 + \ell_2 - \ell]! [2\ell_1]!} \]

Thus the norm of the highest weight vector reads

\[ ||e_\ell^\epsilon(\ell_1, \ell_2, \Lambda)||^2 \]

\[ = (-1)^{\lambda(\ell_1 + \ell_2 + \ell + 1)/4} q^{-(\ell_1 + \ell_2 + \ell + 1)/(\ell_1 - m_1)/2} A_{\ell_1 \ell - \ell_1}^2 \frac{[\ell_1 + \ell_2 + \ell + 1]! [\ell_1 - \ell_2 + \ell]!}{[2\ell_1]! [2\ell_1]!} \]

This leads to

\[ A_{\ell_1 \ell - \ell_1} = q^{-(\ell_1 + \ell_2 + \ell + 1)/(\ell_1 - m_1)/4} \left( \frac{[2\ell_1]! [2\ell_1]!}{[\ell_1 + \ell_2 + \ell + 1]! [\ell_1 - \ell_2 + \ell]!} \right)^{1/2}, \] (A.4)

where the phase is chosen such that the expression coincides the result in Ref. 17.

To obtain the other vectors in this irreducible representation we need the following results, which may be verified by induction:

\[ v_k^m e_\ell^\epsilon(\lambda) = (-1)^{(\ell - m)k + k(k+1)/2} \left( \frac{[\ell + m]! [\ell - m + k]!}{[\ell - m]! [\ell + m - k]!} q^k \right)^{1/2} e_{m-k}^\epsilon(\lambda), \] (A.5)

\[ \Delta(v_n^m) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^{k(n-k)} v_n^{n-k} K^k \otimes v_k^K n^k. \] (A.6)
Using (A.5) we obtain
\[ e_m^\ell (\ell_1, \ell_2, \Lambda) = (-1)^{\ell-m}(\ell-m+1)/2 \left( \frac{[\ell + m]!}{[2\ell]! [\ell - m]! q^{\ell-m}} \right)^{1/2} \Delta(v_{\ell-m}^\ell) e_m^\ell (\ell_1, \ell_2, \Lambda). \] (A.7)

The right hand side is computed by using (A.6) and (A.1). After some algebra, we derive
\[
\Delta(v_{\ell-m}^\ell) e_m^{\ell_1}(\lambda) \otimes e_m^{\ell_2}(\lambda) = \sum_k \left[ \begin{array}{c} \ell - m \\ k \end{array} \right] (-1)^{k(\ell_2-m+\lambda)+(\ell_1-m_1)(\ell-m)+(\ell-m)(\ell-m+1)/2} q^{km_1/2+(-\ell+m+k)m_2/2} \\
\times \left( \frac{[\ell_1 + m_1]! [\ell_2 + m_2]! [\ell_1 + \ell - m_1 - m - k]! [\ell_2 - m_2 + k]!}{[\ell_1 - m_1]! [\ell_2 - m_2]! [\ell_1 - \ell + m_1 + m + k]! [\ell_2 + m_2 - k]! q^{\ell-m}} \right)^{1/2} \\
\times e_m^{\ell_1}(\lambda) \otimes e_m^{\ell_2}(\lambda). \] (A.8)

Equations (A.3) and (A.4) allow us to derive
\[
e_m^\ell (\ell_1, \ell_2, \Lambda) = \sum_{m_1, m_2} (-1)^{\ell_1-\ell+m_2}(\ell-m+\lambda)+(\ell_1-\ell+m_2)(\ell_1-\ell+m_2+1)/2 \\
\times q^{m_2(m_2+1)/2+(\ell_1-\ell_2)(\ell_1+\ell_2+1)/4-\ell(\ell+1)/4} \\
\times \left( \frac{[2\ell+1]}{[\ell_1 + \ell + 2 - \ell]! [\ell + m]! [\ell - m]! [\ell_1 - m_1]! [\ell_2 - m_2]!}{[\ell_1 + \ell_2 + \ell + 1]! [\ell_1 - \ell_2 + \ell]! [-\ell_1 + \ell_2 + \ell]! [\ell_1 + m_1]! [\ell_2 + m_2]!} \right)^{1/2} \\
\times \sum_k (-1)^{k(k-1)/2+k(\ell_1+\ell_2-m)} q^{k(\ell+1)/2} \\
\times \left[ \begin{array}{c} \ell_1 - \ell - m_2 - k \\ k \end{array} \right] [\ell_2 + m_2 + k]! [\ell_1 - \ell + m_2 + k]! [\ell_2 - m_2 - k]! e_m^{\ell_1}(\lambda) \otimes e_m^{\ell_2}(\lambda). \] (A.9)

**Appendix B  SUMMATION FORMULA FOR KULISH SYMBOL**

A summation formula for Kulish symbol which is used in the previous section is derived in this appendix. Corresponding binomial coefficient is usually defined for \( y \geq x \geq 0 \) as
\[
\left[ \begin{array}{c} y \\ x \end{array} \right] = \frac{[y]!}{[y-x]! [x]!} = \frac{[y][y-1] \cdots [y-x+1]}{[x]!}. \] (B.1)

We may extend this to \( y < 0 \) by using the property
\[ [-n] = (-1)^{n+1}[n], \quad n \geq 0, \] (B.2)
and define the binomial coefficient for negative \( y \) by the right most formula of (B.1):
\[
\left[ \begin{array}{c} y \\ x \end{array} \right] = \frac{[-(-y)][-(-y+1)] \cdots [-(-y+x-1)]}{[x]!} = (-1)^{xy+x(x+1)/2} \frac{[-y][-y+2] \cdots [-y+x-1]}{[x]!}. \]
Thus for \( y < 0 \) and \( x \geq 0 \), the binomial coefficient is given by

\[
\binom{y}{x} = (-1)^{xy+x(x+1)/2} \binom{x-y-1}{x}.
\] (B.3)

The following formula, proved by induction, is found in Ref. 17:

\[
\binom{n+r}{k} = \sum_{a} \binom{n}{k-a} \binom{r}{a} (-1)^{(k-a)(r-a)} q^{(r-a)(n+r)/2-r(n-k+r)/2},
\] (B.4)

where \( a \) runs over any positive integers such that the arguments of \([x]\) in the binomial coefficients are non-negative. We assume \( n, r < 0 \), and apply (B.3) to this formula

\[
\text{l.h.s.} = (-1)^{k(n+r)+k(k+1)/2} \binom{k-n-r-1}{k},
\]
\[
\text{r.h.s.} = (-1)^{k(n+r)+k(k+1)/2} q^{rk/2} \sum_{a} (-1)^{an} q^{-a(n+r)/2} \binom{k-a-n-1}{k-a} \binom{a-r-1}{a}.
\]

From this, the following summation formula of Kulish symbols is obtained

\[
\sum_{a} (-1)^{an} q^{-a(n+r)/2} \frac{(k-a-n-1)! [a-r-1]!}{[k-a]! [a]!} = q^{-rk/2} \frac{(k-n-r-1)! [-n-1]! [-r-1]!}{[k]! [-n-r-1]!}.
\] (B.5)
Appendix C  CGC FOR $1 \otimes 1 = 2 \oplus 1 \oplus 0$

The following tables contain the values of $C_{m_1}^{1 \ell} m_{m_1-m}^{1 \ell}$ for a given $\ell$. The columns provide the values of $m_1$, while the rows indicate $m$. The rightmost column of Table 1, titled as "OF", indicates the overall factors that are common to all entries in the row. In Table 2 and Table 3, the overall factors (OF) are common for all entries of the tables.

|   | 1   | 0   | −1   | OF             |
|---|-----|-----|------|----------------|
| 2 | 1   | 0   | 0    | 1              |
| 1 | $(−1)^\lambda q^{1/2}$ | $q^{-1/2}$ | 0    | $(\binom{2}{4})^{1/2}$ |
| 0 | $q$  | $(−1)^\lambda \binom{2}{4}$ | $q^{-1}$ | $\frac{2}{\sqrt{4!}}$ |
| −1| 0   | $q^{1/2}$ | $(−1)^\lambda q^{-1/2}$ | $(\binom{2}{4})^{1/2}$ |
| −2| 0   | 0   | 1    | 1              |

Table 1: $\ell = 2$

|   | 1   | 0   | −1   |
|---|-----|-----|------|
| 1 | $q^{-1/2}$ | $(−1)^\lambda+1 q^{1/2}$ | 0    |
| 0 | $(−1)^\lambda+1$ | $q^{1/2} + q^{-1/2}$ | $(−1)^\lambda$ |
| −1| 0   | $(−1)^\lambda+1 q^{-1/2}$ | $q^{1/2}$ |

Table 2: $\ell = 1$, OF = $\left(\binom{2}{4}\right)^{1/2}$.

|   | 1   | 0   | −1   |
|---|-----|-----|------|
| 1 | $q^{-1/2}$ | $−(−1)^\lambda+1$ | $−q^{1/2}$ |
| 0 | $(−1)^\lambda+1$ | $−1$ | $q^{1/2}$ |

Table 3: $\ell = 0$, OF = $\frac{1}{\sqrt{3}}$.

Appendix D  CGC FOR $2 \otimes 2 = 4 \oplus 3 \oplus 2 \oplus 1 \oplus 0$

The following tables contain the values of $C_{m_1}^{2 \ell} m_{m_1-m}^{2 \ell}$ for a given $\ell$. The columns provide the values of $m_1$, while the rows indicate $m$. The rightmost column, titled as "OF", indicates the overall factors common to all entries in the row.
|   | 2   | 1   | 0   | −1  | −2  | OF |
|---|-----|-----|-----|-----|-----|----|
| 4 | 1   | 0   | 0   | 0   | 0   | 1  |
| 3 | $(−1)^λq$ | $q^{-1}$ | 0   | 0   | 0   | $\frac{[4]}{[8]}^{1/2}$ |
| 2 | $q^2$ | $(−1)^λ \left( \frac{[2][4]}{[3]} \right)^{1/2}$ | $q^{-2}$ | 0   | 0   | $\frac{[3][4]}{[7][8]}^{1/2}$ |
| 1 | $(−1)^λ q^3 \sqrt{[4]}!$ | $q[4][3]$ | $(−1)^λ q^{-1}[4][3]$ | $q^{-3} \sqrt{[4]}!$ | 0   | $\frac{[5][8]}{[8]}^{1/2}$ |
| 0 | $q^4$ | $(−1)^λ q^2[4]$ | $[4][3]$ | $(−1)^λ q^{-2}[4]$ | $q^{-4}$ | $\sqrt{[8]}!$ |
| −1 | 0   | $q^3 \sqrt{[4]}!$ | $(−1)^λ q[4][3]$ | $q^{-1}[4][3]$ | $(−1)^λ q^{-3} \sqrt{[4]}!$ | $\frac{[5][8]}{[8]}^{1/2}$ |
| −2 | 0   | 0   | 0   | $q^2$ | $(−1)^λ \left( \frac{[2][4]}{[3]} \right)^{1/2}$ | $q^{-2}$ | $\frac{[3][4]}{[7][8]}^{1/2}$ |
| −3 | 0   | 0   | 0   | 0   | $q$ | $(−1)^λ q^{-1}$ | $\frac{[4]}{[8]}^{1/2}$ |
| −4 | 0   | 0   | 0   | 0   | 1   | 1  |

Table 4: $\ell = 4$
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\ell & 2 & 1 & 0 & -1 & -2 & \text{OF} \\
\hline
3 & q^{-1} & (-1)^{\lambda+1}q & 0 & 0 & 0 & \left(\frac{[4]}{8}\right)^{1/2} \\
\hline
2 & (-1)^{\lambda+1} & \mu_{31} & (-1)^{\lambda} & 0 & 0 & \left(\frac{[4]!}{[6][8]}\right)^{1/2} \\
\hline
1 & q & (-1)^{\lambda+1}\mu_{32} & \mu_{33} & (-1)^{\lambda+1}q^{-1} & 0 & \left(\frac{[7][3][8]}{4!}\right)^{1/2} \\
\hline
0 & (-1)^{\lambda+1}q^2 & q^{1/2}([3] + q^2) & (-1)^{\lambda}[3]\omega & q^{-1/2}([3] + q^{-2}) & (-1)^{\lambda}q^{-2} & \left(\frac{[7][3][8]}{4!}\right)^{1/2} \\
\hline
-1 & 0 & (-1)^{\lambda+1}q & \mu_{32} & (-1)^{\lambda+1}\mu_{33} & q^{-1} & \left(\frac{[7][3][8]}{4!}\right)^{1/2} \\
\hline
-2 & 0 & 0 & (-1)^{\lambda+1} & \mu_{31} & (-1)^{\lambda} & \left(\frac{[6][8]}{4!}\right)^{1/2} \\
\hline
-3 & 0 & 0 & 0 & (-1)^{\lambda+1}q^{-1} & q & \left(\frac{[4]}{8}\right)^{1/2} \\
\hline
\end{array}
\]

\[\mu_{31} = \sqrt{\frac{4!}{[2]^2}}\omega, \quad \mu_{32} = \left[\frac{4}{[3]!}\right]^{1/2} (q^2 + q^{-1/2}[2]), \quad \mu_{33} = \left[\frac{4}{[3]!}\right]^{1/2} (q^{-2} - q^{1/2}[2]).\]

Table 5: \( \ell = 3 \)
\[ \mu_{21} = \frac{2}{4}(q^{-2} + q^{3/2}[2]), \quad \mu_{22} = \frac{2}{4}(q^2 - q^{-3/2}[2]), \quad \mu_{23} = \frac{2}{4}\left(\frac{8}{4} + 2\frac{4}{2} + 1\right). \]

Table 6: \( \ell = 2 \)
\[
\begin{align*}
\text{Table 7: } \ell = 1 & & \\
\hline
& 2 & 1 & 0 & -1 & -2 & \text{OF} \\
\hline
1 & q^{-3/2} & (-1)^{\lambda+1}q^{-1/2} & (-1)^{\lambda+1/2} & (-1)^{\lambda}q^{3/2} & 0 & \left(\frac{[3]!}{[4]}\right)^{1/2} \\
0 & (-1)^{\lambda+1}q^{-1/2} & q^{-1}\mu_{11} & (-1)^{\lambda+1}\left(\frac{[3]!}{[4]}\right) & -q\mu_{12} & (-1)^{\lambda+1}q^{1/2} & \left(\frac{[3][4]}{[5][6]}\right)^{1/2} \\
-1 & 0 & (-1)^{\lambda+1}q^{-3/2} & q^{-1/2}\left(\frac{[3]!}{[4]}\right) & (-1)^{\lambda}q^{1/2}\left(\frac{[3]!}{[4]}\right) & -q^{3/2} & \left(\frac{[3]!}{[5][6]}\right)^{1/2} \\
\hline
\end{align*}
\]

\[
\mu_{11} = q + q^{-1}\frac{[2]}{[4]}, \quad \mu_{12} = q^{-1} + q\frac{[2]}{[4]}.
\]

Table 8: \(\ell = 0\)
References

1 see for example, R. J. Szabo, Phys. Rep. 378, 207 (2003).

2 Yu. I. Manin, Comm. Math. Phys. 123, 163 (1989).

3 T. Kobayashi and T. Uematsu, Z. Phys. C:Part. Fields 56, 193 (1992).

4 S. K. Soni, J. Phys. A:Math. Gen. 24, 619 (1991).

5 N. Aizawa and R. Chakrabarti, J. Math. Phys. 45, 1623 (2004).

6 H. Saleur and B. Wehefritz-Kaufmann, Nucl. Phys. B628, 407 (2002); ibid. B663, 443 (2003).

7 P. Podleś, Lett. Math. Phys. 14, 193 (1987).

8 P. Podleś, Lett. Math. Phys. 18, 107 (1989).

9 P. Podleś, Comm. Math. Phys. 150, 167 (1992).

10 J. Apel and K. Schműdgen, Lett. Math. Phys. 32, 25 (1994).

11 I. Heckenberger and S. Kolb, J. Algebra 263, 193 (2003).

12 M. Noumi and K. Mimachi, Comm. Math. Phys. 128, 521 (1990).

13 M.S. Dijkhuizen and T. Koornwinder, Geom. Dedicata 52, 291 (1994).

14 L.L. Vaksman and Ya. S. Soibelman, Leningrad Math. J. 2, 1023 (1991).

15 R. Chakrabarti and J. Segar, Mod. Phys. Lett. A16, 1731 (2001).

16 P.P. Kulish and N.Yu. Reshetikhin, Lett. Math. Phys. 18, 143 (1989).

17 P. Minnaert and M. Mozrzymas, J. Math. Phys. 35, 3132 (1994); J. Phys. A:Math. Gen. 28, 669 (1995).

18 N.Yu. Reshetikhin, L.A. Takhtajan, and L.D. Faddeev, Leningrad Math. J. 1, 193, (1990).

19 A. Klimyk and K. Schműdgen, Quantum Groups and Their Representations (Springer,1997), Sec. 1.3.5.

20 F. Thuillier and J.-C. Wallet, Phys. Lett. B323, 153 (1994).