Extracting work from mixing indistinguishable systems: A quantum Gibbs “paradox”

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The classical Gibbs paradox concerns the change in entropy upon mixing two gases. Whether or not an observer assigns an entropy increase to the process depends on their ability to distinguish the gases. A resolution is provided by realising that an “ignorant” observer, who cannot distinguish the gases with devices in their lab, has no way of extracting work by mixing them. Moving the thought experiment into the quantum realm, we reveal new and surprising behaviour. We show that the ignorant observer can in fact extract work from mixing different gases, even if the gases could not be directly distinguished. Moreover, in a macroscopic limit that classically recovers the ideal gas from statistical mechanics, there is a marked divergence in the quantum case: as much work can be extracted as if the gases had been fully distinguishable. Our analysis reveals that the ignorant observer assigns more microstates to the system than are found by naive state-counting in semiclassical statistical mechanics. This effect demonstrates the importance of carefully accounting for the level of knowledge of an observer, and its implications for genuinely quantum modifications to thermodynamics.

1. INTRODUCTION

Despite its phenomenological beginnings, thermodynamics has been inextricably linked throughout the past century with the abstract concept of information. Such connections have proven essential for solving apparent paradoxes in a variety of thought experiments, notably including Maxwell’s demon [1] and Loschmidt’s paradox [2]. This integration between classical thermodynamics and information is also one of the main motivating factors in the extension of the theory to the quantum realm, where information held by the observer plays a similarly fundamental role [3].

In this work, we study the transition from classical to quantum thermodynamics in the context of the Gibbs paradox [4–6]. This thought experiment considers two gases on either side of a box, separated by a partition and with equal volume and pressure on each side. If the gases are identical, then the box is already in thermal equilibrium, and nothing changes after removal of the partition. If the gases are distinct, then they mix and expand to fill the volume independently, approaching thermal equilibrium with a corresponding increase in entropy. The (supposed) paradox can be summarised as follows: what if the gases differ in some unobservable or negligible way – should we ascribe an entropy increase to the mixing process or not? This question sits uncomfortably with the view that thermodynamical entropy is an objective physical quantity.

Various resolutions have been described, from phenomenological thermodynamics to statistical mechanics perspectives, and continue to be analysed [6–8]. A crucial insight by Jaynes [9] assures our discomfort at the observer-dependent nature of the entropy change. For an informed observer, who sees the difference between the gases, the entropy increase has physical significance in terms of the work extractable through the mixing process – in principle, they can build a device that couples to the two gases separately (for example, through a semi-permeable membrane) and thus let each of them do work on an external weight independently. An ignorant observer, who has no access to the distinguishing degree of freedom, has no device in their laboratory that can exploit the difference between the gases, and so cannot extract work. For Jaynes, there is no paradox as long as one considers the abilities of the experimenter – a viewpoint central to the present work.

We study the Gibbs mixing process for quantum gases composed of identical bosons or fermions. This is motivated by the recognition that the laws of thermodynamics must be modified to account for quantum effects such as coherence [10], which can lead to enhanced performance of thermal machines [11–13]. The thermodynamical implications of identical quantum particles have received renewed interest for applications such as Szilard engines [14–15], thermodynamical cycles [16–17] and energy transfer from boson bunching [18]. Moreover, the particular quantum properties of identical particles, including entanglement, can be valuable resources in quantum information processing tasks [19–21].

We consider a toy model of an ideal gas with non-interacting quantum particles, distinguishing the two gases by a spin-like degree of freedom. We describe the mixing processes that can be performed by both informed and ignorant observers, taking into account their different levels of control, and calculate the corresponding entropy changes and thus work extractable by each. For the informed observer, we recover the same results as obtained by classical statistical mechanics arguments. However, for the ignorant observer, there is a marked divergence from the classical case. Counter-intuitively, the ignorant observer can typically extract more work from distinguishable gases – even though they appear indistinguishable – than from truly identical gases. In the continuum and large particle number limit which classically recovers the ideal gas, we find that this divergence is maximal: the ignorant observer can extract as much work from apparently indistinguishable gases as the informed observer.
Our analysis hinges on the symmetry properties of quantum states under permutations of particles. For the ignorant observer, these properties lead to non-trivial restrictions on the possible work extraction processes. Viewed another way, the microstates of the system described by the ignorant observer are highly non-classical entangled states. This implies a fundamentally different way of counting microstates, and therefore computing entropies, than is done classically or even in semi-classical treatments of quantum gases. Therefore we uncover a genuinely quantum thermodynamical effect in the Gibbs mixing scenario.

2. SET-UP

We consider a gas of \( N \) particles inside a box, such that each particle has a position degree of freedom, denoted \( x \), and a second degree of freedom which acts as a label to distinguish the gases. Since we only consider the case of two types of gases, this is a two-dimensional degree of freedom and we refer to this as the "spin" \( s \) (although it need not correspond to an actual angular momentum). Classically, the two spin labels are \( \uparrow, \downarrow \), and their quantum analogues are orthogonal states \( |\uparrow\rangle, |\downarrow\rangle \) [22].

Following the traditional presentation of the Gibbs paradox, the protocol starts with two independent gases on different sides of a box: \( n \) on the left and \( m = N - n \) on the right (see Fig. 1). We consider each side to initially be thermalised with an external heat bath \( B \) at temperature \( T \).

In our toy model, each side of the box consists of \( d/2 \) “cells” (\( d \) is even) representing different states that can be occupied by each particle. These states are degenerate in energy, such that the Hamiltonian of the particles vanishes. This might seem like an unrealistic assumption; however, this model contains the purely combinatorial (or “state-counting”) statistical effects, first analysed by Boltzmann [23], that are known to recover the entropy changes for a classical ideal gas [8, 24, 25]. One could instead think of this setting as approximating a non-zero Hamiltonian in the high-temperature limit, such that each cell is equally likely to be occupied in a thermal state.

Work extraction can be modelled in various ways in quantum thermodynamics. In the resource-theoretic approach based on thermal operations [26, 27], one keeps track of all resources by modelling the system (here, the particles), heat bath and work battery as interacting quantum systems. The work battery is an additional system with non-degenerate Hamiltonian whose energy changes are associated with work done by or on the system (generalising the classical idea of a weight being lifted and lowered).

We consider the extractable work to be given by the difference in free energy \( F \) between initial and final states, where \( F(\rho) = U(\rho) - k_B T S(\rho), U(\rho) = \text{tr}(\rho H) \) being the mean energy (zero in our case) and \( S(\rho) = -\text{tr}(\rho \ln \rho) \) the von Neumann entropy in natural units. The extractable work in a process that takes \( \rho \) to \( \rho' \) is then

\[
W = F(\rho) - F(\rho') = k_B T \left[ S(\rho') - S(\rho) \right].
\]  

Thus the extractable work is simply the change in von Neumann entropy. This is generally an over-simplification for small systems, in which work is a random variable with non-negligible fluctuations [28, 29]. However, Equation (1) will turn out to be sufficient for our purposes in the sense of mean extractable work. In addition, in Section 6.5 we do characterise fluctuations around the mean.

Our analysis compares the work extracted by two observers with different levels of knowledge: the informed observer, who can tell the difference between the two gases, and the ignorant observer, who cannot. The difference between these observers is that the former has access to the spin degree of freedom \( s \), whereas the latter does not.

It is important to point out that, for the informed observer, the spin acts as a "passive" degree of freedom, meaning that it can be measured but not actively changed. In other words, the two types of gases cannot be converted into each other. This assumption is always implicitly present in discussions of the Gibbs paradox – without it, the distinguishing degree of freedom would constitute another subsystem with its own entropy changes.

3. CLASSICAL CASE

In our setting, one defines the classical state space on which states are specified by counting how many particles exist with a certain position \( x \) and spin \( s \). This is the correct way of describing the state space of indistinguishable classical particles,
according to the informed observer \[30\]. The ignorant observer has a different state space given by a coarse-graining operation – the classical equivalent of “tracing out” the spin degree of freedom. In Appendix A we give a formal construction of the state spaces and prove that the ignorant observer can extract only as much work from two different gases as from a single gas. This recovers Jaynes’ original statement \[9\]. The result may be intuitively obvious, but it establishes that we fairly compare the classical and quantum cases by “playing the same game”.

The amount of extractable work in the classical case is easily found by state counting. Consider the gas initially on the left side – the number of ways of distributing \(n\) particles among \(d/2\) cells is \(\binom{n+d/2-1}{n}\). In the thermal state, each configuration occurs with equal probability. Therefore the initial entropy, also including the gas on the right, is \(\ln \binom{n+d/2-1}{n} + \ln \binom{m+d/2-1}{m}\). For distinguishable gases, each gas can deliver work independently, with an equal distribution over \(\binom{n+d/2-1}{n}\) configurations. For indistinguishable gases, the final thermal state is described as an equal distribution over all ways of putting \(N = n + m\) particles into \(d\) cells, of which there are \(\binom{N+d-1}{N}\). Hence the entropy change in each case is

\[
\Delta S = \ln \binom{n+d-1}{n} + \ln \binom{m+d-1}{m} - \ln \binom{n+d/2-1}{n} - \ln \binom{m+d/2-1}{m}
\]

(distinguishable),

\[
\Delta S = \ln \binom{N+d-1}{N} - \ln \binom{n+d/2-1}{n}
\]

(indistinguishable). (2)

Note that \(\Delta S \neq 0\) even in the indistinguishable case, which may seem at odds intuitively with the result for an ideal gas. However, one can check that \(\Delta S = O(\ln N)\) in the limit of large \(d\) (whereby the box becomes a continuum) and large \(N\). This is negligible compared with the ideal gas expression of \(N\ln 2\) for distinguishable gases \[31\].

It is worth noting that a classical analogue of fermions can be made by importing the Pauli exclusion principle, so that two or more particles can never occupy the same cell. This has the effect of replacing the binomial coefficients of the form \(\binom{N+d-1}{N}\) in \(2\) and \(3\) by \(\binom{N}{n}\).

4. QUANTUM CASE

4.1. Informed observer

Compared with the classical case, we must be more explicit about the role of the spin \(s\) as a “passive” degree of freedom for the informed observer. This observer is permitted to measure the spin of each particle in the fixed basis \(\{\uparrow, \downarrow\}\), and then act accordingly on the spatial part. For identical gases, the result is of course the same as for the ignorant observer, and the classical case \[7\]. For distinguishable gases, each gas behaves as an independent subsystem; thus, the entropy changes are the same as for classical distinguishable gases \[2\].

The remainder of this section is devoted to the ignorant observer, for which we find a departure from the classical case.

4.2. Hilbert space

The peculiarities of the quantum case stem from a careful look at the Hilbert space structure. The Hilbert space of a single particle is a product \(H_0 = H_x \otimes H_s\) of a part for the spatial degree of freedom \(x\) and a part for the spin \(s\). Since there are \(d\) cell modes and two spin states, these parts have dimensions \(\dim H_0 = d, \dim H_s = 2\). For \(N\) distinguishable particles, the state space would be \(H_0^{\otimes N}\). However, for bosons and fermions, which are quantum indistinguishable particles, states lie in the symmetric and antisymmetric subspaces, respectively (in first quantisation). The symmetry referred to here is the behaviour of the wavefunction under permutations of particles: for bosons, there is no change, whereas for fermions, each swap of a pair incurs a minus sign in the global phase. The physical Hilbert space of \(N\) particles can then be written as

\[
\mathcal{H}_N = P_{+(-)} \left( H_x^{\otimes N} \otimes H_s^{\otimes N} \right),
\]

where \(P_{+(-)}\) is the projector onto the (anti-)symmetric subspace.

Since each particle carries a position and spin state, a permutation \(\Pi\) of particles is applied simultaneously to these two parts: \(\Pi\) acts on the above Hilbert space in the form \(\Pi_x \otimes \Pi_s\). The requirement of an overall (anti-)symmetric wavefunction then results in an effective coupling of these two degrees of freedom via their symmetries. For a familiar example, consider two particles. The spin state space can be broken down into the symmetric “triplet” subspace spanned by \(\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}\) and \(\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}\), and the antisymmetric “singlet” subspace consisting of \(\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}\). For bosons, the overall symmetry requirement imposes that a triplet spin state be paired with a symmetric spatial wavefunction, and a singlet spin state with an antisymmetric spatial function. For fermions, opposite symmetries are paired.

With more than two particles, the description of symmetries is more complex, but the main idea of paired symmetries remains the same. Following \[32\], our main tool is Schur-Weyl duality \[33\], which decomposes

\[
\mathcal{H}_x^{\otimes N} = \bigotimes_{A} H_{x_A}^{\otimes} \otimes K_{x_A}^{\otimes},
\]

where \(\lambda\) runs over all Young diagrams of \(N\) boxes and no more than \(d\) rows \[34\]. In technical terms, \(H_{x_A}\) and \(K_{x_A}\) carry irreducible representations (irreps) of the unitary group \(U(d)\) and the permutation group \(S_N\) of \(N\) particles, respectively. More concretely, a non-interacting unitary operation on the positions of all the particles, \(u_x^{\otimes N}\), is represented in the decomposition \[5\] as an independent rotation within each of the \(H_{x_A}\) spaces. The term “irreducible” refers to the fact that each of these spaces may be fully explored by varying the unitary \(u_x\). Similarly, a
permutation of the spatial part of the wavefunction is represented by an action on each $K^s$ space. Thus each block labelled by $\lambda$ in the decomposition (5) has a specific type of permutation symmetry.

The same decomposition works for the spin part $\mathcal{H}^{\otimes N}$. However, since this degree of freedom is two-dimensional, each $\lambda$ is constrained to have no more than two rows. We can think of $s$ as describing a total angular momentum formed of $N$ spin-1/2 particles, and in fact $s$ can be replaced by a total angular momentum eigenvalue $J$ varying over the range $N/2, N/2 - 1, \ldots$.

After putting the spatial and spin decompositions together, projecting onto the overall (anti-)symmetric subspace causes the symmetries of the two parts to be linked. For bosons, the $\lambda$ label for $x$ and $s$ must be the same; for fermions, they are transposes of each other (i.e., related by interchanging rows and columns). This results in the form

$$\mathcal{H}_N = \bigoplus_{A} \mathcal{H}_x^A \otimes \mathcal{H}_s^A$$

for bosons,

$$\mathcal{H}_N = \bigoplus_{A} \mathcal{H}_x^{A^T} \otimes \mathcal{H}_s^{A}$$

for fermions. (6)

Instead of the label $\lambda$, from now on we use the angular number $J$ and generally write this decomposition as $\bigoplus_{J} \mathcal{H}^J \otimes \mathcal{H}^J_x$ — bearing in mind that $\mathcal{H}^J_x$ is different for bosons and fermions. In terms of the earlier $N = 2$ example, $J = 1$ corresponds to the spin triplet subspace, and $J = 0$ to the spin singlet.

Another way of describing the decomposition (6) is that it provides a convenient basis $|J, q\rangle_x |J, M\rangle_s |\phi_J\rangle_{xx}$, known as the Schur basis $[35]$. Here, $\{|J, q\rangle_x\}_q$ is a basis for $\mathcal{H}^J_t$, and $\{|J, M\rangle_s\}_M$ a basis for $\mathcal{H}^J_x$. $M = -J, -J + 1, \ldots, J$ can be interpreted as the total angular momentum quantum number along the $z$-axis. $|\phi_J\rangle_{xx} \in \mathcal{K}^J_x \otimes \mathcal{K}^J_s$ is a state shared between the $x$ and $s$ degrees of freedom.

4.3. Thermalisation for ignorant observer

Since the ignorant observer cannot interact with the spin degree of freedom, their effective state space is described by tracing out the factor $\mathcal{H}_s$ for each particle. In terms of the decomposition (6) and corresponding basis described above, this means that an initial density matrix $\rho$, after tracing out $s$, is of the form

$$\rho_x := \text{tr}_s \rho = \bigoplus_{J} p_J \rho^J_x \otimes \text{tr}_s |\phi_J\rangle_{xx},$$

(7)

where $\rho^J_x$ is a density matrix on $\mathcal{H}^J_x$, occurring with probability $p_J$. Note that there is no coherence between different values of $J$, and that the components $\rho^J_x$ are mutually perfectly distinguishable by a measurement of their $J$.

An additional constraint on the allowed operations is that they must preserve the bosonic or fermionic exchange symmetry. Any global unitary $U_{\text{BW}}$, coupling the spatial degree of freedom of the particles to the heat bath and work battery, must therefore commute with permutations on the spatial part: $[U_{\text{BW}}, \Pi_{\text{x}}] = 0$ for all $\Pi$. By Schur’s Lemma, such a unitary must decompose as $U = \bigoplus_{J} U^J \otimes I^J$, where $U^J$ operates on the $\mathcal{H}^J_x$ component, with an identity $I^J$ on $\mathcal{K}^J_s$. Hence each $J$ component is operated upon independently, the spin eigenvalue $J$ being conserved.

In summary, therefore, the ignorant observer may engineer any thermal operation extracting work separately from each $J$.

FIG. 2. Two diagrams representing the mixing of indistinguishable (bosonic) quantum gases from the perspective of the informed (left) and ignorant (right) observers. Initially, $n$ spin-$\uparrow$ particles are found on the left and $m$ spin-$\downarrow$ on the right. The particles are then allowed to mix while coupling to an external heat bath and work battery. The informed observer describes microstates via the number of particles in each cell, and their respective spins. The ignorant observer cannot tell the spins states, but describes microstates (schematically depicted here by different colours) as superpositions of cell configurations, determined by the decomposition (6).
component. We can think of their operations being conditioned on the spatial symmetry type, and although $J$ is observed to fluctuate randomly, a certain amount of work is extracted for each $J$ (see Section 6.6.5 for a more detailed analysis).

The question of optimal work extraction thus reduces to calculating the entropy of the initial state (7) and finding the maximum entropy final state. The fully thermalised final state seen by the ignorant observer is maximally mixed within each $J$ block:

$$\rho_s' = \sum_J p_J I^J_s \otimes \text{tr}_s |\phi_J\rangle\langle\phi_J|_{s'},$$

where $I^J_s$ is the identity on $\mathcal{H}^J_s$ and $d_J$ is the corresponding dimension.

The overall entropy change is the average over all $J$, found to be (with details in Appendix B):

$$\Delta S_{\text{igno}} = \sum_J p_J \Delta S^J_{\text{igno}},$$

$$= \sum_J p_J \ln d_J^B - \ln \left(\frac{n + d/2 - 1}{n}\right) - \ln \left(\frac{m + d/2 - 1}{m}\right)$$

for bosons, and

$$\Delta S_{\text{igno}} = \sum_J p_J \Delta S^J_{\text{igno}},$$

$$= \sum_J p_J \ln d_J^F - \ln \left(\frac{d/2}{n}\right) - \ln \left(\frac{d/2}{m}\right)$$

for fermions. Expressions for the dimensions $d^B_J, d^F_J$ are found in Appendix C.

$$d^B_J = \frac{(2J + 1) \left(\frac{N}{2} - J + d - 2\right)! \left(\frac{N}{2} + J + d - 1\right)!}{\left(\frac{N}{2} - J\right)! \left(\frac{N}{2} + J + d - 1\right)! (d - 1)(d - 2)!},$$

$$d^F_J = \frac{(2J + 1) \left(\frac{N}{2} + J + 1\right)! \left(\frac{N}{2} - J\right)!}{\left(\frac{N}{2} + J + 1\right)! \left(\frac{N}{2} - J\right)! (d - \frac{N}{2} + J + 1)! (d - \frac{N}{2} - J)!}.$$ (11)

The probabilities $p_J$ are found (see Appendix B) from the Clebsch-Gordan coefficients $C(j_1, m_1; j_2, m_2; J, M)$ describing the coupling of two spins with angular momentum quantum numbers $(j_1, m_1), (j_2, m_2)$ into overall quantum numbers $(J, M)$. Here, the two spins are the groups of particles on the left and right, respectively.

For identical gases, all particles have spins in the same direction, so the spin wavefunction is simply $|\uparrow\rangle \otimes^{N}$. This state lies fully in the subspace of maximal total spin eigenvalue, $J = M = N/2$ — which is also fully symmetric with respect to permutations. Thus the spin part factorises out (i.e., there is no correlation between spin and spatial degrees of freedom). It is then clear that dimension counting reduces to the classical logic of counting ways to distribute particles between cells. Indeed, the dimension of the subspace $\mathcal{H}^{N/2}_s$ is $d^B_{N/2} = \binom{N+d-1}{N}$ for bosons and $d^F_{N/2} = \binom{d}{N}$ for fermions. It follows that we recover the entropy as the classical case of indistinguishable particles [3].

For orthogonal spins, there are $n$ spin-$\uparrow$ and $m$ spin-$\downarrow$, leading to $M = (n - m)/2$ and a distribution over different values of $J$ according to

$$p_J = \frac{(2J + 1)!m!n!}{\left(\frac{N}{2} + J + 1\right)!\left(\frac{N}{2} - J\right)!}.$$ (12)

The resulting entropies and significant limits are discussed in Section 6.

### 5. EXAMPLE

We give an example with $n = m = 1$ to demonstrate the mechanism behind the state space decomposition. For two particles, there are only two values of $J$, corresponding to the familiar singlet and triplet subspaces:

$$\mathcal{H}_0^s = \text{span} \{ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \},$$

$$\mathcal{H}_1^s = \text{span} \{ |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \}.$$ (13)

Consider a spatial configuration where a spin-$\uparrow$ particle is on the left in cell $i$, and a spin-$\downarrow$ is on the right in cell $j$. For bosons, the properly symmetrised wavefunction is

$$|\psi_{ij}\rangle := \frac{1}{\sqrt{2}} \left( |i\downarrow j\uparrow\rangle_s |\uparrow\downarrow\rangle_s + |i\uparrow j\downarrow\rangle_s |\downarrow\uparrow\rangle_s \right)$$

$$= \frac{1}{\sqrt{2}} \left( |i\downarrow j\uparrow\rangle_s - |i\uparrow j\downarrow\rangle_s \right),$$

$$|\psi_{ij}\rangle = \frac{1}{\sqrt{2}} \left( |i\downarrow j\uparrow\rangle_s + |i\uparrow j\downarrow\rangle_s \right).$$ (14)

So $p_0 = 1/2$, and the spatial component of this state is conditionally pure for both $J$. The initial thermal state is a uniform mixture of all such $|\psi_{ij}\rangle$, with $(d/2)^2$ terms. Thus $S(\rho_0^s) = S(\rho_1^s) = 2(\ln d - \ln 2)$. For the final thermal state, we observe that

$$\mathcal{H}_0^s = \text{span} \{ |ij\rangle \}$$

$$\mathcal{H}_1^s = \text{span} \{ |ij\rangle + |ji\rangle \}$$

where $i, j$ now label cells either on the left or right. The corresponding dimensions are $d_0 = d(d - 1)/2$, $d_1 = d(d + 1)/2$. Within the $J = 0$ subspace, the entropy change is $\ln(d(d - 1)/2) - 2 \ln d + 2 \ln 2 = \ln(1 - 1/d) + \ln 2$, and for $J = 1$, it is $\ln(d(d + 1)/2) - 2 \ln d + 2 \ln 2 = \ln(1 + 1/d) + \ln 2$. Overall, therefore,

$$\Delta S_{\text{igno}} = \frac{1}{2} \ln \left(1 - \frac{1}{d}\right) + \frac{1}{2} \ln \left(1 + \frac{1}{d}\right) + \ln 2$$

$$= \frac{1}{2} \ln \left(1 - \frac{1}{d^2}\right) + \ln 2.$$ (16)

For the informed observer, we have $\Delta S_{\text{info}} = 2 \ln 2$. For identical gases, we find $\Delta S_{\text{den}} = \ln(1 + 1/d) + \ln 2$, strictly greater than $\Delta S_{\text{igno}}$, but the two become equal in the limit $d \to \infty$. 


Repeating the same calculation with fermions, the symmetric and antisymmetric states now pair up oppositely. Then \( \Delta S_{\text{info}} \) is the same as for bosons. However, we have \( \Delta S_{\text{idem}} = \ln(1 - 1/d) + \ln 2 < \Delta S_{\text{igno}} \). Unlike for bosons, two distinguishable fermions permit more extractable work by the ignorant observer than two identical fermions!

6. RESULTS AND LIMITS

In Figure 3 we plot both \( \Delta S_{\text{info}} \) and \( \Delta S_{\text{igno}} \) as a function of dimension for bosons and fermions. Below we analyse the special cases and limits which emerge from these expressions, a summary of which can be found in Table I.

6.1. Special cases

With bosons, there are two special cases in which it is easily proven that distinguishable gases are less useful than indistinguishable ones for the ignorant observer. The first case is the example above, with \( n = m = 1 \). In addition, for \( d = 2 \), we have \( d^B_j = 2J + 1 \) so the largest subspace is that with maximal \( J = N/2 \). The largest entropy change is then obtained when \( p_{N/2} = 1 \), which is satisfied precisely for indistinguishable gases.

For fermions, we see from Figure 3 that the greatest work – for both informed and ignorant observers – is obtained for small \( d \). An intuitive explanation of this is that the Pauli exclusion principle causes the initial state to be constrained and thus have low entropy. For example, with the minimal dimension \( d = 2n = 2m \), we have \( \Delta S_{\text{info}} = 2 \ln \left( \frac{2^m}{n^m} \right) \approx 4n \ln 2 \) to leading order when \( n \) is large. The ignorant observer can do almost as well: the state is entirely contained in the \( J = 0 \) subspace, with

\[
d^F_0 = \frac{(2n)!((2n + 1)!)}{(n!)^2(2n + 1)} = \frac{2n + 1}{(n + 1)^2} \approx 2n, \tag{17}
\]

giving \( \Delta S_{\text{igno}} \approx 4n \ln n \) for large \( n \). This is twice as much as for the classical ideal gas.

6.2. Low density limit

The most interesting conclusion is reached in the limit of large \( d \gg n^2 \), which we term the low density limit. For simplicity, we take \( n = m \). To lowest order in \( n^2/d \), we find

\[
\Delta S_{\text{igno}} \approx \Delta S_{\text{info}} - H(p) - \frac{n^2}{2d^2}, \tag{18}
\]

where \( H(p) = -\sum p_j \ln p_j \) is the Shannon entropy of the distribution \( p_j \). Thus, as \( d \to \infty \), the ignorant observer can extract as much work as the informed one, minus an amount \( H(p) \). This gap is evident from the graphs in Figure 3.

Now consider the limit \( d \gg n^2, n \gg 1 \), with both low density and large particle number. Classically, this limit recovers ideal gas behaviour – the large dimension limit can be thought of as letting the box become a continuum. In Appendix E, we show that \( H(p) \) (which depends only on \( n \), not \( d \)), behaves as

\[
H(p) \approx \frac{1}{2} \ln n + 0.595..., \tag{19}
\]

with a correction going to zero as \( n \to \infty \). Recall that the entropy change for the informed observer is approximately \( 2n \ln 2 \) in this limit. Therefore the deficit \( H(p) \), which is logarithmic, becomes negligible compared with the value \( 2n \ln 2 \). Thus the ignorant observer can extract essentially as much work as the informed observer: \( \Delta S_{\text{igno}} \approx \Delta S_{\text{info}} \approx 2n \ln 2 \). This result is remarkable because it shows an extreme departure from the classical case in the macroscopic limit.

6.3. Explaining the low density limit

An important feature of the low density limit is that the final entropy becomes as large as it could possibly be: \( p_{\uparrow} \), becomes maximally mixed over its whole state space. This is true for any \( N \), not just large numbers. We now give an explanation of this phenomenon, which proceeds by counting the number of mutually orthogonal states which can be accessed by the ignorant observer.

The important point about the low density limit is that we can almost always assume that no particles sit on top of each other – that is, almost all states are such that precisely \( N \) cells are occupied, each with a single particle. More formally, the number of ways of putting \( N \) bosonic particles into \( d \) cells is \( \binom{N + d - 1}{N} \approx \frac{d^N}{N!} \) when \( d \) is large, where the approximation means the ratio of the two sides is close to unity. Let us refer to each of these \( \binom{d}{n} \) choices of (singly) occupied cells as a cell configuration. For each cell configuration, there are \( \binom{N}{n} \) spin configurations, i.e., ways of distributing the \( n \) spin-up and \( n \) spin-down particles. In classical physics, the ignorant observer is unable to distinguish any of the spin configurations
FIG. 3. Series of plots showing $\Delta S_{\text{info}} - \Delta S_{\text{igno}}$ against the total cell number $d$ of the system. Figures in the top row are for bosonic systems of differing particle number $n$ and figures in the bottom row show the same for fermionic systems. Note that we have taken the initial number of particles on either side of the box to be equal, $n = m$ in all cases. For comparison, all four figures also display the classical changes in entropy for an informed/ignorant observer. The behaviour of the deficit between $\Delta S$ for an informed/ignorant observer of quantum particles agrees with the low density limit in equation (18) where we can see $\Delta S_{\text{info}}$ tending to the classical limit $2n \ln(2)$ with $\Delta S_{\text{igno}}$ trailing behind by a deficit of $n^2/d^2 + H(p)$. Additionally, by comparing the different plots, we can see the low-dimensional fermionic advantage where the change in entropy is even greater than the classical $2n \ln(2)$ value.

corresponding to a single cell configuration. In quantum mechanics, remarkably, there are precisely $\binom{N}{n}$ states which can be fully distinguished by the ignorant observer, each being a superposition of different spin configurations.

Let us choose a single cell configuration – without loss of generality, we choose cells $1, \ldots, N$ to be occupied. The state of a spin configuration is denoted as a permutation of

$$|\uparrow_1 \cdots \uparrow_n \downarrow_{n+1} \cdots \downarrow_N\rangle \in (\mathbb{C}^2)^{\otimes N},$$

(20)

where each cell is treated as a qubit with basis states $|\uparrow\rangle, |\downarrow\rangle$ according to which type of spin occupies it. (Note that the subsystems being labelled are here the occupied cells, not particles.)

Again using Schur-Weyl duality, the state space of $N$ qubits can be decomposed as

$$(\mathbb{C}^2)^{\otimes N} = \bigoplus_J \mathcal{H}^J \otimes \mathcal{K}^J.$$  

(21)

due to this decomposition, there is a natural basis $|J, M, p\rangle$, where SU(2) spin rotations $u^{\otimes N}_s$ act on the $M$ label (denoting the eigenvalue of the total $z$-direction spin), and permutations $\Pi$ of the $N$ cells act on the $p$ label.

How do we represent the effective state seen by the ignorant observer? In the representation used here, this corresponds to twirling over the spin states, i.e., performing a Haar measure average over all spin rotations $u^{\otimes N}_s$ [36]. In the basis $|J, M, P\rangle$, however, this is a straightforward matter of tracing out the $\mathcal{H}^J$ subspaces, since only these are acted on by the twirling operation. Thus the ignorant observer has access to states labelled as $|J, P\rangle$.

How much information has been lost by tracing out $\mathcal{H}^J$? In fact, none – the label $M = (n - m)/2$ is fixed. Therefore the experimenter can perfectly distinguish all the basis states $|J, P\rangle$ – and there are just as many of these as there are spin configurations, namely $\binom{N}{n}$. 

Due to this decomposition, there is a natural basis $|J, M, p\rangle$, where SU(2) spin rotations $u^{\otimes N}_s$ act on the $M$ label (denoting the eigenvalue of the total $z$-direction spin), and permutations $\Pi$ of the $N$ cells act on the $p$ label.
For example, take \( n = m = 1 \): the two spin configurations are \( |↑↓⟩, |↓↑⟩ \), and for some pair of occupied cells, the two distinguishable states are

\[
|J = 1, M = 0, p = 0⟩ = \frac{1}{\sqrt{2}} (|↑↓⟩ + |↓↑⟩),
\]

\[
|J = 0, M = 0, p = 0⟩ = \frac{1}{\sqrt{2}} (|↑↓⟩ - |↓↑⟩). \tag{22}
\]

Since these are respectively in the triplet and singlet subspaces, they remain orthogonal even after twirling. They can be distinguished by mixing the cells at a balanced beam splitter: it is easy to show that the symmetric state ends up with a superposition of both particles in cell 1 and both in cell 2, while the antisymmetric state ends up with one particle on each side. Therefore, after this beam splitter, the two states can be distinguished by counting the total particle number in each cell.

A slightly more complex example is with \( n = 2, m = 1 \). Then the distinguishable basis states for three occupied cells are

\[
|J = \frac{3}{2}, M = \frac{1}{2}, p = 0⟩ = \frac{1}{\sqrt{3}} (|↑↑↓⟩ + |↑↓↑⟩ + |↓↑↑⟩),
\]

\[
|J = \frac{1}{2}, M = \frac{1}{2}, p = 0⟩ = \frac{1}{\sqrt{2}} (|↓↑↑⟩ + |↑↓↑⟩), \tag{23}
\]

\[
|J = \frac{1}{2}, M = \frac{1}{2}, p = 1⟩ = \frac{1}{\sqrt{3}} (|↑↑↑⟩ - \frac{1}{\sqrt{6}} (|↑↓↑⟩ + |↓↑↑⟩)).
\]

Observe that the argument in this section does not depend in any way on the exchange statistics of the particles, explaining why we see the same for limit for bosons and fermions.

### 6.4. Quantumness of the protocol

The above discussion of the low density limit clarifies the fundamental reason why the quantum ignorant observer performs better than the classical one. The distinguishable states comprising the final thermalised state are superpositions of different spin configurations. We might describe a classical observer within the quantum setting as one who is limited to operations diagonal in the basis of cell configurations – that is, they are only able to count the number of particles occupying each cell. For such an observer, all the different superposition states are completely indistinguishable.

A crucial question is then: how difficult is it to engineer the quantum protocol for the ignorant observer? We can imagine that the heat bath and work battery might naturally couple to the system in the cell occupation basis (if this is the basis that emerges in the classical case). The required coupling is in the Schur basis \( |J, q⟩ \), which are generally highly entangled between cells. A sense of their complexity is given by the unitary that rotates the Schur basis to the computational basis, known as the Schur transform. Efficient algorithms to implement this transform have been found [37], with a quantum circuit whose size is polynomial in \( N, d, \ln(1/\epsilon) \), allowing for error \( \epsilon \) related to the quantum Fourier transform, an important subroutine in many quantum algorithms. Thus, while the Schur transform can be implemented efficiently, it appears that engineering the required work extraction protocol – in the absence of fortuitous symmetries in the physical systems being used – may be as complex as universal quantum computation.

### 6.5. Work fluctuations

The work extraction protocol we have presented is not deterministic: for each value of \( J \), a different amount of work is extracted with probability \( p_J \). This is typically expected of thermodynamics of small systems; however, in classical macroscopic thermodynamics, such fluctuations are negligible. We can ask whether the same is true of the work extracted by the ignorant observer in the quantum case, especially in the low density and large particle number limits.

One informative way of quantifying the fluctuations is via the variance of entropy change. Let us denote the entropy change for each \( J \) by \( \Delta S_{\text{igno}}(J) \). The mean is just \( \Delta S_{\text{igno}} = \sum_J p_J \Delta S_{\text{igno}}(J) \), and the variance is \( V(\Delta S_{\text{igno}}) = \sum_J p_J \Delta S_{\text{igno}}(J)^2 - \Delta S_{\text{igno}} \). This can be computed straightforwardly from our expressions for \( p_J, d_J \), and approximated in various limits.

Consider first a high density BEC-limit case with \( d = 2 \) and \( N = 2n \gg 1 \) bosons. We have \( d_B^n = 2J + 1 \), and using the techniques of Appendix E, \( p_J \approx \frac{\gamma_J}{2} e^{-J^2/2} \). Then \( \Delta S_{\text{igno}} = \sum_J p_J \ln(2J + 1) \approx 1 \frac{1}{2} \ln n + 2 - \frac{1}{2} \ln n + 0.405 \). Similarly, we compute \( V(\Delta S_{\text{igno}}) = \sum_J p_J [\ln(2J + 1)]^2 \approx \frac{\pi^2}{27} \approx 0.411 \). Therefore the mean work dominates its fluctuations (logarithmic versus a constant).

Next, consider the closest analogue for fermions: the case of minimal dimension \( d = 2n = 2m \). Recall that \( \Delta S_{\text{igno}} \approx \Delta S_{\text{info}} \approx 4n \ln 2 \) for large \( n \). Since \( p_0 = 1 \), work extraction is in fact completely deterministic in this case.

Finally, take the low density limit. As found before, for both bosons and fermions, \( \Delta S_{\text{igno}} \approx 2n \ln 2 - \text{linear in } n - \text{ and yet we still find a constant } V(\Delta S_{\text{igno}}) \approx \frac{\pi^2}{27} \).

In these macroscopic limits, therefore, work extraction is either fully deterministic or effectively deterministic in that the fluctuations are negligible compared with the mean.

### 7. DISCUSSION

In contrast to the classical setting of the Gibbs paradox, we have shown that quantum mechanics permits the extraction of work from apparently indistinguishable gases, without access to the degree of freedom that distinguishes them. It is notable that the lack of information about this “spin” degree of freedom does not in principle impede an experimenter at all in a suitable macroscopic limit with large particle number and low density – the thermodynamical value of the two gases is as great as if they had been fully distinguishable.

The underlying mechanism can be seen as a generalisation of the famous Hong-Ou-Mandel effect in quantum optics [32, 38, 39]. In this effect, polarisation may play the role of the spin.
Then a non-polarising beam splitter plus photon detectors are able to detect whether a pair of incoming photons are similarly polarised. The whole apparatus is polarisation-independent and thus accessible to the ignorant observer. Given this context, it is therefore not necessarily surprising that quantum Gibbs mixing can give different results to the classical case. However, the result of the low density limit is not readily apparent. This limit is reminiscent of the result in quantum reference frame theory [56] that the lack of a shared reference frame presents no obstacle to communication given sufficiently many transmitted copies [40].

Our work has implications for the fundamental limits of thermal machines whose working media are identical quantum particles. It is therefore important to apply our results to more concrete and practical heat engine proposals (such as Refs. [16, 18, 41]) to see how they can be modified to obtain the enhancements predicted here. The question of the maximal enhancement in the macroscopic limit is particularly compelling given the rapid progress in the manipulation of large quantum systems [42].

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Next, we describe making the particles indistinguishable by coarse graining. The top-level, coarse-grained state space \( \tilde{\Sigma} \) is the set of types of \( x \)-vectors. In other words, the ignorant observer only has the ability to say how many particles are at each position,

\[
\Sigma_N = \{ (x,s) | x \in [d]^N, s \in [S]^N \},
\]

\[
(A1)
\]

where \( |k| = \{ 1, 2, \ldots, k \} \). This can be expressed as a Cartesian product \( \Sigma_N = \Sigma_N^x \times \Sigma_N^s \) of the individual spaces for each degree of freedom. A probability distribution \( p(x,s) \) on this classical state space can then be represented as a vector \( \mathbf{p} \in \mathbb{R}^d \otimes \mathbb{R}^S \).

A general operation on a classical state space will be taken to be a stochastic map. With respect to the tensor product structure, an operation that is independent of the spin degree of freedom is then of the form

\[
\mathbf{T}_x \cdot \mathbf{x} = \mathbf{T}_x \pi(x) \quad \forall \pi \in \Sigma_N \quad \forall x, x'.
\]

(A2)

Our goal is to describe how this operation looks from the point of view of an ignorant observer, who has no access to the spin and regards the particles as indistinguishable.

The first step is simply to trace out the spin degree of freedom, which amounts to finding the marginal probability distribution on \( \Sigma^x_N \). Since \( p'(x) = \sum_s p(x,s) \), it is clear that

\[
p' = (T \otimes I) p \Rightarrow p'^x = T p^x,
\]

and so we can safely ignore all of the spin information.

Next, we describe making the particles indistinguishable by coarse graining. The top-level, coarse-grained state space \( \tilde{\Sigma}^x_N \) is the set of types of \( x \)-vectors. In other words, the ignorant observer only has the ability to say how many particles are at each position,
thus specifying a type $t = (t_1, \ldots, t_d)$, where $t_i$ is the number of particles at position $i$. Each $x$ has an associated type $t(x)$; we denote this relationship by $x \sim t$. The number of different $x$ vectors with a given type $i$ is the multinomial coefficient $\binom{N}{t}$. The coarse-grained probability distribution is obtained via $\tilde{p}^L = Mp^L$, where $M$ and its Moore-Penrose pseudo-inverse are given by

$$\tilde{M}_{t,x} = \delta(x \sim t), \quad \tilde{M}_{x,t}^{-1} = \binom{N}{t}^{-1} \delta(x \sim t). \quad \text{(A4)}$$

In general (without any assumptions on $T$), the stochastic dynamics of $p^L$ may not translate into stochastic dynamics of $\tilde{p}$, in which case information about traced out degrees of freedom leaks into the coarse-grained distribution. We show that the permutation symmetry $\binom{A}{2}$ does in fact lead to consistent dynamics for the ignorant observer.

The general necessary and sufficient condition for $\tilde{p}^L$ to evolve under some stochastic map $\tilde{T}$ is

$$PT = PTP, \quad \text{where} \quad P = M^{-1}M, \quad \text{(A5)}$$

in which case we have $\tilde{T} = MTM^{-1}$. Here, we find $P_x \cdot x = \binom{N}{t(x)}^{-1} \delta_{t(x),t(x)}$, so

$$(PT)_{x',x} = \sum_x \binom{N}{t(x')}^{-1} \delta_{t(x'),t(x')} T_{x',x} = \frac{1}{N!} \sum_{\pi \in S_N} T_{\pi(x'),x}, \quad \text{(A6)}$$

$$(PTP)_{x',x} = \sum_{x''} (PT)_{x'',x''} P_{x'',x}$$

$$= \frac{1}{N!} \sum_{x''} \sum_{\pi' \in S_N} T_{\pi'(x''),x''} \binom{N}{t(x')}^{-1} \delta_{t(x'),t(x'')} = \frac{1}{N!} \sum_{\pi' \in S_N} \frac{1}{(N!)^2} T_{\pi'(x''),\pi(x)}$$

$$= \frac{1}{N!} \sum_{\pi' \in S_N} \frac{1}{(N!)^2} T_{\pi^{-1}o\pi'(x''),x} = \frac{1}{N!} \sum_{\pi \in S_N} T_{\pi'(x'),x}. \quad \text{(A7)}$$

having used the property $\binom{A}{2}$ in the penultimate line. Thus the consistency condition $\binom{A}{5}$ holds, and one finds (after some algebra)

$$T_{t',t} = \sum_{x' \sim t'} T_{x',x} \quad \text{for any} \quad x \sim t. \quad \text{(A8)}$$

Finally, we address the question of whether the ignorant observer has full control over their state space $\Sigma_N$, i.e., whether $\tilde{T}$ has the freedom to be any stochastic matrix. It is not hard to see that this is indeed true. The stochastic matrix $T$ is constrained only by permutation symmetry $\binom{A}{2}$ – so it is determined by the minimal set of values $\{T_{x',x_0(t)} \mid t, t'\}$ (which are independent, apart from conservation of probability), where $x_0(t)$ is any fixed string of type $t$, say $x_0(t) = (1^n, 2^m, \ldots, d^d)$. Similarly, due to $\binom{A}{8}$, $\tilde{T}$ is determined by $\{\sum_{x' \sim t'} T_{x',x_0(t)} \mid t, t'\}$. It is thus clear that there is enough freedom in $T$ to be able to make $\tilde{T}$ any stochastic matrix.

Overall, we have shown that, when an ignorant observer – who sees the particles as indistinguishable and cannot access their spin – applies operations that are symmetric on the particles and do not touch their spin, it is possible to manipulate the coarse-grained classical state space with no restriction.

An example: Take the case of $N = d = 2$ (having already ignored the spin for simplicity). Then we can label the states in $\Sigma_N$ by \{LL, RR, LR, RL\} (referring to whether each particle is “left” or “right”). The most general $T$ with permutation symmetry is then

$$T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \\ c & f & j \end{pmatrix}, \quad a + b + 2c = d + e + 2f = g + h + i + j = 1. \quad \text{(A9)}$$
where each element is of course non-negative. The coarse-grained space \( \tilde{\Sigma}_X \) consists of \{(2, 0), (0, 2), (1, 1)\}, indicating the number of left and right particles. The corresponding coarse-graining matrix and its pseudo-inverse are

\[
M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},
\]

resulting in

\[
\tilde{T} = \begin{pmatrix} a & d & g \\ b & e & h \\ 2c & 2f & i + j \end{pmatrix},
\]

which evidently can be an arbitrary stochastic matrix.

**Appendix B: Details for quantum ignorant observer**

In this section, we provide additional details for the entropy change as seen by the ignorant observer.

Recall that Schur-Weyl duality [35, Chapter 5] provides the decomposition

\[
\mathcal{H}_x^\otimes N = \bigoplus_{\lambda} \mathcal{H}_x^\lambda \otimes \mathcal{K}_x^\lambda,
\]

where \( \lambda \) runs over all Young diagrams containing \( N \) boxes and no more than \( d \) rows. A Young diagram \( \lambda \) is a set of unlabelled boxes arranged in rows, with non-increasing row length from top to bottom. We can equivalently describe \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \), where \( \lambda_i \) is the number of boxes in row \( i \). For example, \( \begin{array}{ccc} & & \\
& & \\
* & & \\
\end{array} \) would be denoted \((3, 1)\) (where \( N = 4, d = 2 \)).

\( \mathcal{H}_x^\lambda \) and \( \mathcal{K}_x^\lambda \) carry irreps of \( U(d) \) and \( S_N \) respectively, corresponding to irreducible subspaces under the actions of single-particle unitary rotations \( u^\otimes N \otimes I^\otimes N \) and particle label permutations \( \Pi \otimes I^\otimes N \), each of which act only on the spatial part. The same decomposition works for the spin part \( \mathcal{H}_s^\otimes N \), although now the Young diagrams \( \lambda \) have maximally two rows. In fact, they correspond to the familiar SU(2) irreps with total angular momentum \( J \), via \( \lambda = (N/2 + J, N/2 - J) \).

After putting the spatial and spin decompositions together, projecting onto the overall (anti-)symmetric subspace causes the symmetries of the two components to be linked. For bosons, the overall symmetric subspace (itself a trivial irrep of \( S_N \)) occurs exactly once in \( \mathcal{K}_s^\lambda \otimes \mathcal{K}_s^{\lambda'} \) if and only if \( \lambda = \lambda' \), and otherwise does not [44, Section 7-13]. Thus we have

\[
\mathcal{H}_N = \bigoplus_{\lambda, \lambda'} \mathcal{H}_s^\lambda \otimes \mathcal{H}_s^{\lambda'} \otimes P_+ \left[ \mathcal{K}_s^\lambda \otimes \mathcal{K}_s^{\lambda'} \right]
\]

where \( P_+ \) is the projector onto the antisymmetric subspace, denoting the transpose of the Young diagram in which rows and columns are interchanged; thus,

\[
\mathcal{H}_N = \bigoplus_{\lambda} \mathcal{H}_s^{\lambda T} \otimes \mathcal{H}_s^\lambda \quad \text{(fermions)}.
\]

For fermions, the only difference is that the projector \( P_- \) onto the antisymmetric subspace enforces \( \lambda' = \lambda^T \), denoting the transpose of the Young diagram in which rows and columns are interchanged; thus,

\[
\mathcal{H}_N = \bigoplus_{\lambda} \mathcal{H}_s^{\lambda T} \otimes \mathcal{H}_s^\lambda \quad \text{(fermions)}.
\]

Due to the use of a two-dimensional spin, we employ the correspondence \( J \leftrightarrow \lambda = (N/2 + J, N/2 - J, 0, 0, \ldots) \) (with a total of \( d \) rows) to replace the label \( \lambda \) by \( J \).

Let us first consider the bosonic case. Thanks to the decomposition (B2), a state \( \rho \) (as seen by the informed observer) can be written in terms of the basis \( |J, q\rangle_x |J, M\rangle_x |\phi\rangle_{sx} \), where \( |J, q\rangle \in \mathcal{H}_s^J \), \( |J, M\rangle \in \mathcal{H}_s^M \), \( |\phi\rangle \in \mathcal{K}_s^\lambda \otimes \mathcal{K}_s^{\lambda'} \), as described in the main text. The ignorant observer sees the reduced state after tracing out the spin part, of the form

\[
\rho_x = \text{tr}_s \rho = \bigoplus_{J} p_J \rho_x^J \otimes \text{tr}_s |\phi\rangle \langle \phi|_{sx}.
\]

The entropy of this state is

\[
S(\rho_x) = H(p) + \sum_J p_J \left( S(\rho_x^J) + S(\text{tr} \, |\phi\rangle \langle \phi|_{sx}) \right).
\]
where $H(p) := - \sum_J p_J \ln p_J$ is the Shannon entropy of the probability distribution $p_J$.

As argued in the main text, the fully thermalised final state is of the form

$$
\rho_s^\prime = \bigoplus_J p_J \frac{I}{d_J} \otimes \text{tr}_s |\phi_J\rangle\langle\phi_J|_{xs},
$$

(B6)

with entropy

$$
S(\rho_s^\prime) = H(p) + \sum_J p_J \left[ \ln d_J + S(\text{tr}_s |\phi_J\rangle\langle\phi_J|_{xs}) \right].
$$

(B7)

An example of a channel that achieves the mapping from $\rho_s$ to $\rho_s^\prime$ – albeit without a coupling to a heat bath or work battery – is the so-called “twirling” operation. This is a probabilistic average over all single-particle unitary rotations $u^N_x$:

$$
\mathcal{T}_x(\rho) = \int \text{d}\mu(u_x) u^N_x \rho u^N_x, \tag{B8}
$$

where $\mu$ is the Haar measure over the group $U(d)$.

The entropy change for the ignorant observer is therefore

$$
\Delta S_{\text{ign}} = S(\rho_s^\prime) - S(\rho_s) = \sum_J p_J \left[ \ln d_J - S(\rho_s^J) \right].\tag{B9}
$$

(Note that the states $\phi_J$ do not enter into the entropy change.) Our goal is therefore to determine the probabilities $p_J$, dimensions $d_J$, and the entropy of the component states $\rho_s^J$.

The case of indistinguishable gases is dealt with in the main text: the state is fully in the subspace $J = N/2$, corresponding to the spatially symmetric subspace for bosons and spatially antisymmetric for fermions.

For gases of different spins, the state is such that all particles are in $|\uparrow\rangle$ and all on the right are in $|\downarrow\rangle$. Before getting to the thermal state, first consider a pure state in which $n_i$ particles are in each cell $i$ on the left, and $m_i$ in each cell $i$ on the right (such that $\sum_i n_i = \sum_i m_i = m$). This spatial configuration is denoted by the pair of vectors $(n, m)$. The properly symmetrised wavefunction is

$$
|\psi(n, m)\rangle = \mathcal{N}(n, m) \sum_{\pi \in S_N} \pi |n\rangle \otimes \pi |m\rangle,
$$

(B10)

$$
|n, m\rangle := |n_1^{m_1}\rangle |n_2^{m_2}\rangle \cdots |n_i^{m_i}\rangle \cdots ,
$$

where $\pi$ runs over permutations of the $N$ particles that lead to distinct terms, and $\mathcal{N}(n, m)$ is a normalisation factor (such that $\mathcal{N}(n, m)^{-2}$ is the number of distinct terms in the sum). We determine the $p_J$ via the expectation value of the projector $P_s^J$ onto the subspace $H_s^J$:

$$
\langle\psi(n, m)| P_s^J |\psi(n, m)\rangle = \mathcal{N}(n, m)^2 \sum_{\pi, \pi'} \langle\psi(n, m)| \pi' \pi |n, m\rangle \langle\uparrow m | \pi' P_s^J \pi |\uparrow m\rangle
$$

$$
= \mathcal{N}(n, m)^2 \sum_{\pi} \langle\uparrow m | \pi P_s^J \pi |\uparrow m\rangle,
$$

(B11)

where the second line holds because any pair of $\pi, \pi'$ giving rise to distinct terms in (B10) also have different actions on $|n, m\rangle$. Now we use Clebsch-Gordan coefficients to evaluate each term in this last sum. First note that we can express $|\uparrow m\rangle$ as a combined spin with $J_1 = M_1 = n/2$, and similarly $|\uparrow m\rangle$ as a spin with $J_2 = -M_2 = m/2$. The Clebsch-Gordan coefficient $C(n/2, n/2; m/2, m/2; J, J)$ is precisely the amplitude for this state in the $J$ subspace. This is unchanged by the inclusion of a permutation $\pi$, so (B11) simplifies to

$$
\langle\psi(n, m)| P_s^J |\psi(n, m)\rangle = \left| C(n, n; m, m; J, -m) \right|^2.
$$

(B12)

Now it remains to consider the correct initial state, which is a uniform probabilistic mixture of all $|\psi(n, m)\rangle$ with a fixed number of particles $n, m$ on the left and right, respectively. Since the Clebsch-Gordan coefficient is the same for all such configurations, we have

$$
p_J = \left| \frac{C(n, n; m, m; J, -m)}{(2J + 1)n!m!} \right|^2
$$

$$
= \frac{(N/2 + J + 1)! (N/2 - J)!}{(N/2 + J + 1)! (N/2 - J)!}.
$$

(B13)
Finally, we determine the entropy of each $\rho_s^J$ component. Using the basis $|J, q\rangle_s, |J, M\rangle_s, |\phi_J\rangle_{xs}$ provided by the Schur-Weyl decomposition, we have

$$|\psi(n, m)\rangle = \sum_J \sqrt{p_J} |\psi(n, m, J)\rangle_s |J, \frac{n-m}{2}\rangle_s |\phi_J\rangle_{xs}.$$  \hfill (B14)

Here, $|\psi(n, m, J)\rangle_s \in \mathcal{H}^d_s$ is some linear combination of the $|J, q\rangle_s$ – without needing to determine these states entirely, it will be sufficient to note that they are orthogonal for different configurations $(n, m)$. This follows from the fact that different $|\psi(n, m)\rangle$ are fully distinguishable just by measuring the occupation numbers of $d_i$. Below, which lists the values of $\tilde{\lambda}$

$$\text{tr}_s |\psi(n, m)\rangle = \bigoplus_J p_J |\psi(n, m, J)\rangle \otimes \text{tr}_s |\phi_J\rangle |\phi_J\rangle_{xs},$$  \hfill (B15)

From orthogonality of the $|\psi(n, m, J)\rangle$, it follows that

$$S(\rho_s^J) = \ln \left( \frac{n+d/2-1}{n} \right) + \ln \left( \frac{m+d/2-1}{m} \right).$$  \hfill (B16)

Inserted into (B9), this results in the claimed entropy changes [9, 10].

Appendix C: Dimension counting

From [33, Chapter 7], we have (labelling by $\lambda$ instead of $J$)

$$\dim \mathcal{H}^d_s = \prod_{1 \leq i < j \leq d} (\tilde{\lambda}_i - \tilde{\lambda}_j),$$

$$\prod_{m=1}^{d-\frac{d}{2}} m!,$$

$$\tilde{\lambda} := \lambda + (d-1, d-2, \ldots, 0).$$  \hfill (C1)

First take the bosonic case. Since the Young diagram for the SU(2) spin representation has no more than two rows, the same $\lambda$ labelling the spatial part has no more than two non-zero rows. Hence we have $\tilde{\lambda} = (\frac{N}{2} + J + d - 1, \frac{N}{2} - J + d - 2, d - 3, d - 4, \ldots, 0)$. Calculating the product in the numerator of (C1) is aided by the table below, which lists the values of $\tilde{\lambda}_i - \tilde{\lambda}_j$, where $i$ labels the row and $j > i$ labels the column:

|   | 2 | 3 | 4 | 5 | \ldots | $d-1$ | $d$ |
|---|---|---|---|---|---|---|---|
| 1 | $2J + 1$ | $\frac{N}{2} + J + 2$ | $\frac{N}{2} + J + 3$ | $\ldots$ | $\frac{N}{2} + J + d - 1$ | $\ldots$ | $\ldots$ |
| 2 | $\frac{N}{2} - J + 1$ | $\frac{N}{2} - J + 2$ | $\ldots$ | $\ldots$ | $\frac{N}{2} - J + d - 2$ | $\ldots$ | $\ldots$ |
| 3 | $1$ | $2$ | $\ldots$ | $\ldots$ | $d - 3$ | $\ldots$ | $\ldots$ |
| 4 | $1$ | $\ldots$ | $\ldots$ | $d - 4$ | $\ldots$ | $\ldots$ | $\ldots$ |
| \vdots | | | | | | | |
| $d-2$ | | | | | | | |
| $d-1$ | | | | | | | |

The product of the terms in the first row is

$$(2J + 1) \left( \frac{N}{2} + J + d - 1 \right)! \left( \frac{N}{2} + J + 1 \right)!,$$  \hfill (C3)

the second row gives

$$\left( \frac{N}{2} - J + d - 2 \right)! \left( \frac{N}{2} - J \right)!,$$  \hfill (C4)

and the remaining rows give

$$\prod_{m=1}^{d-3} m!.$$  \hfill (C5)
Putting these into \((C1)\) results in the expression for \(d^E_{N,J}\) in \((\Pi)\). For fermions, we instead use the transpose of the Young diagram, with
\[
\lambda^T = (2, \ldots, 2, 1, \ldots, 1).
\] (C6)

An important restriction on \(\lambda^T\) is that the number of rows can never be greater than the dimension, so \(N/2 + J \leq d\). We find
\[
\tilde{\lambda}^T = (d + 1, d, d - 1, \ldots, d - N/2 + J + 2, d - N/2 + J + 1, \ldots, d - N/2 - J - 1, d - N/2 - J).\] (C7)

As before, the differences \(\tilde{\lambda}^T_i - \tilde{\lambda}^T_j\) can be arranged as follows:

\[
\begin{array}{cccccccccccc}
2 & 3 & \ldots & \frac{N}{2} - J & \frac{N}{2} - J + 1 & \frac{N}{2} - J + 2 & \ldots & \frac{N}{2} + J & \frac{N}{2} + J + 1 & \frac{N}{2} + J + 2 & \ldots & d - 1 & d \\
1 & 2 & \ldots & \frac{N}{2} - J & \frac{N}{2} - J + 1 & \frac{N}{2} - J + 2 & \ldots & \frac{N}{2} + J & \frac{N}{2} + J + 1 & \frac{N}{2} + J + 2 & \ldots & d - 1 & d + 1 \\
2 & 1 & \ldots & \frac{N}{2} - J & \frac{N}{2} - J + 1 & \frac{N}{2} - J + 2 & \ldots & \frac{N}{2} + J & \frac{N}{2} + J + 1 & \frac{N}{2} + J + 2 & \ldots & d - 1 & d \\
& & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\frac{N}{2} - J - 1 & 1 & 3 & 4 & \ldots & 2J + 2 & 2J + 4 & 2J + 5 & \ldots & d - (\frac{N}{2} - J) + 2 & d - (\frac{N}{2} - J) + 3 \\
\frac{N}{2} - J & 2 & 3 & \ldots & 2J + 1 & 2J + 3 & 2J + 4 & \ldots & d - (\frac{N}{2} - J) + 1 & d - (\frac{N}{2} - J) + 2 \\
\frac{N}{2} - J + 1 & 1 & \ldots & 2J - 1 & 2J + 1 & 2J + 2 & \ldots & d - (\frac{N}{2} - J) - 1 & d - (\frac{N}{2} - J) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\frac{N}{2} + J - 1 & 1 & 3 & 4 & \ldots & d - (\frac{N}{2} + J) + 1 & d - (\frac{N}{2} + J) + 2 \\
\frac{N}{2} + J & 2 & 3 & \ldots & d - (\frac{N}{2} + J) & d - (\frac{N}{2} + J) + 1 \\
\frac{N}{2} + J + 1 & 1 & \ldots & d - (\frac{N}{2} + J) - 2 & d - (\frac{N}{2} + J) - 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\frac{N}{2} + J + 2 & 1 & 2 & \ldots & d - (\frac{N}{2} + J) & 1 & 1 \\
\end{array}
\] (C8)

Here, the blue and red lines indicate the division into the three main index groups. We want to calculate the product of all rows in the table. The bottom group of rows gives
\[
\prod_{m=1}^{d-(N/2-J)-1} m!.
\] (C9)

The next group up, being careful to discount the terms lost due to the jump at column \(j = N/2 + J\), gives
\[
\prod_{m=d-(N/2+J)+1}^{d-(N/2-J)} \frac{m!}{m-(d-(N/2+J))}.
\] (C10)

Finally, the top group of rows, noting the additional jump at \(j = N/2 - J + 1\), gives
\[
\prod_{m=d-(N/2-J)-2}^{d+1} \frac{m!}{m-(d-(N/2+J))} \frac{m!}{m-(d-(N/2-J)+1)}.
\] (C11)

Inserting into \((\Pi)\), we need to divide the product of the above three terms by \(\prod_{m=1}^{d+1} m!\). This factor cancels all the factorials present in the above three expressions, with the exception of the top two rows, and contributes two factorials occurring at \(m = d(N/2 + J), d - (N/2 - J) + 1\). Therefore we have
\[
d^E_{N,J} = \prod_{r=d-N/2-J+1}^{d-N/2-J} \frac{1}{r - d + N/2 + J} \prod_{m=d-N/2-J+2}^{d+1} \frac{1}{(m - d + N/2 + J)(m - d + N/2 - J - 1)} \frac{d!(d+1)!}{(2J+1)!} = \frac{1}{(2J+1)!} \cdot \frac{d!(d+1)!}{(N/2 + J + 1)!(N/2 - J)!}.
\] (C12)
Appendix D: Low density limit

1. Bosons

Here we prove equation (18) for bosons. For simplicity, we take \( n = m \). The result rests on the observation that, for sufficiently large \( d \), the ratio \( \frac{d B^2 / p J}{(n + d - 1)^2} \approx \left( \frac{n + d - 1}{n} \right)^2 \). We have

\[
\frac{d B^2 / p J}{(n + d - 1)^2} = \frac{(d - 1)!(d + n - 1)!(d + n - J - 2)!}{(d - 2)!(d + n - 1)!} \\
= (d - 1) \prod_{k=0}^{J-2} \frac{(d + n + k)}{\prod_{k=0}^{J-1} (d + n - J - 1 + k)} \\
= \left( 1 - \frac{1}{d} \right) \prod_{k=0}^{J-1} \frac{(1 + [n + k]/d)}{(1 + [n - J - 1 + k]/d)^{-1}} \\
\tag{D1}
\]

Letting \( x_k = [n + k]/d \), we have

\[
\prod_{k=0}^{J-1} (1 + [n + k]/d) = \sum_{k=0}^{J-1} x_k + \sum_{0 < k < l} x_k x_l + O(\epsilon^3) \\
= \sum_{k=0}^{J-1} x_k + \frac{1}{2} \left[ \left( \sum_{k=0}^{J-1} x_k \right)^2 - \sum_{k=0}^{J-1} x_k^2 \right] + O(\epsilon^3) \\
=: B_1 + B_2 + O(\epsilon^3), \tag{D2}
\]

where the first and second order terms are evaluated to be

\[
B_1 = \frac{J(2n + J - 1)}{2d}, \tag{D3}
\]

\[
B_2 = \frac{J(J - 1)(J[12n - 7] + 12n[n - 1] + 3J^2 + 2)}{24d^2}, \tag{D4}
\]

and \( \epsilon = n^2/d \). Similarly, letting \( y_k = [n - J - 1 + k]/d \),

\[
\prod_{k=0}^{J} (1 + [n - J - 1 + k]/d) = \sum_{k=0}^{J} y_k + \frac{1}{2} \left[ \left( \sum_{k=0}^{J} y_k \right)^2 - \sum_{k=0}^{J} y_k^2 \right] + O(\epsilon^3) \\
=: C_1 + C_2 + O(\epsilon^3), \tag{D5}
\]

with

\[
C_1 = \frac{(J + 1)(2n - J - 2)}{2d}, \tag{D6}
\]

\[
C_2 = \frac{J(J + 1)(12n^2 - 12n[J + 2] + 3J^2 + 11J + 10)}{24d^2}. \tag{D7}
\]

We then have

\[
\frac{d B^2 / p J}{(n + d - 1)^2} = \left( 1 - \frac{1}{d} \right) (1 + B_1 + B_2)(1 + C_1 + C_2)^{-1} + O(\epsilon^3) \\
= 1 + R_1 + R_2 + O(\epsilon^3), \tag{D8}
\]

\[
R_1 = B_1 - C_1 - \frac{1}{d} \\
= \frac{J(J + 1) - n}{d}, \tag{D9}
\]

\[
R_2 = B_2 - C_2 + \frac{B_1}{d} + \frac{C_1}{d} - B_1 C_1 \\
= \frac{2n^2 - 2n(2J[J + 1] + 1) + J(J + 1)(J^2 + J + 2)}{2d^2}. \tag{D10}
\]
We now use this to compute the deficit in the change of entropy, as compared with the entropy for the informed observer:

\[
\Delta S_{\text{igno}} - \Delta S_{\text{info}} = \sum_j p_J \ln \left( \frac{d^J}{p_J \left( \frac{n+J-1}{n} \right)^2} \right) + p_J \ln p_J
\]

\[
= \sum_j p_J \ln \left( 1 + R_1 + R_2 + O(\varepsilon^3) \right) - H(p)
\]

\[
= \sum_j p_J \left( R_1 + R_2 - \frac{R_1^2}{2} \right) + O(\varepsilon^3) - H(p),
\] (D11)

having used the expansion \( \ln(1 + x) = x - x^2/2 + \ldots \) for small \( x \).

In order to compute the first and second order terms in (D11) exactly, we need the following sums involving binomial coefficients:

\[
\sum_{j=0}^{n} \binom{2n+1}{n+j+1}(2j+1) = (2n+1)\binom{2n}{n},
\] (D12)

\[
\sum_{j=0}^{n} \binom{2n+1}{n+j+1}(2j+1)(J+1) = (2n)(2n+1)\binom{2n-1}{n-1}.
\] (D13)

These are both proved using the easily checked identity

\[
\frac{N-2k}{N} \binom{N}{k} = \binom{N-1}{k} - \binom{N-1}{k-1}.
\] (D14)

For (D12), we have (setting \( k = n - J, \ N = 2n + 1 \))

\[
\sum_{j=0}^{n} \binom{2n+1}{n+j+1}(2j+1) = \sum_{j=0}^{n} \binom{2n+1}{n-J}(2j+1) = \sum_{k=0}^{n} \binom{2n+1}{k}(2n+1-2k) = (2n+1)\binom{2n}{n}.
\] (D15)

Similarly, for (D13),

\[
\sum_{j=0}^{n} \binom{2n+1}{n+j+1}(2j+1)(J+1) = \sum_{k=0}^{n} \binom{2n+1}{k}(2n+1-2k)(n-k)(n-k+1) = \sum_{k=0}^{n} \left( \frac{2n}{k} \right) (n-k)(n-k+1) = (2n+1)\sum_{k=0}^{n-1} \left( \frac{2n}{k} \right) (n-k)(n-k+1) = (2n+1)\sum_{k=0}^{n-1} \left( \frac{2n}{k} \right) [(n-k+1) - (n-k)] = (2n+1)\sum_{k=0}^{n-1} \left( \frac{2n}{k} \right) (2n-2k),
\] (D16)
and by using (D14) with $N = 2n$,

$$\sum_{j=0}^{n} \binom{2n+1}{n+j+1}(2J+1)J(J+1) = (2n+1)(2n) \sum_{k=0}^{n-1} \binom{2n-1}{k} - \binom{2n-1}{k-1}$$

$$= (2n+1)(2n) \binom{2n-1}{n-1}.$$  \hfill (D17)

Recall that

$$p_j = \frac{1}{2J+1} \binom{2n+1}{n+j+1}(2J+1),$$

so the first order contribution is

$$\sum_{j=0}^{n} p_j R_1(J) = \sum_{j=0}^{n} \frac{p_j}{d} J(J+1) - n $$

$$= -\frac{n}{d} + \frac{(n!)^2}{d(2n+1)!} \binom{2n+1}{n+j+1}(2J+1)J(J+1) $$

$$= -\frac{n}{d} + \frac{(n!)^2}{d(2n+1)!} (2n+1)(2n) \binom{2n-1}{n-1} $$

$$= -\frac{n}{d} + \frac{(n!)^2(2n+1)(2n)(2n-1)!}{d(2n+1)!(n-1)!(n!)}. $$

$$= -\frac{n}{d} + \frac{n}{d} = 0.$$  \hfill (D19)

The second order is

$$\sum_{j=0}^{n} p_j \left[ R_2(J) - \frac{R_1(J)^2}{2} \right] = \sum_{j=0}^{n} p_j \frac{n(n-2) - 2(n-1)J(J+1)}{2d^2} $$

$$= \frac{n(n-2)}{2d^2} - \frac{2(n-1)}{2d^2} \sum_{j=0}^{n} p_j J(J+1) $$

$$= \frac{n(n-2)}{2d^2} - \frac{2(n-1)}{2d^2} J(J+1) $$

$$= -\frac{n^2}{2d^2}. $$

Therefore, substituting the above into (D11), we have

$$\Delta S_{\text{igno}} - \Delta S_{\text{info}} = -H(p) - \frac{n^2}{2d^2} + O\left(\frac{n^3}{d^3}\right). $$

\hfill (D20)

2. Fermions

The method is the same as for bosons. We expand $\frac{d^2 H_{\text{igno}}}{d^2 J}$ to second order. Letting $z_k = [k - n - J]/d$, we have

$$\prod_{k=1}^{J} \left(1 + \frac{[k - n - J]}{d}\right) = F_1 + F_2 + O(e^3),$$

\hfill (D22)

where

$$F_1 = \sum_{k=1}^{J} z_k = \frac{-J(2n + J - 1)}{2d},$$

\hfill (D23)

$$F_2 = \frac{1}{2} \left[ F_1^2 - \sum_{k=1}^{J} z_k^2 \right] = \frac{J(J-1)(2 + 3J^2 + 12n[n-1] + J[12n-7])}{24d^2}. $$

\hfill (D24)
Similarly, letting $w_k = [k-n+1]/d$,

$$\prod_{k=0}^{J}(1 + [k-n+1]/d) = G_1 + G_2 + O(\epsilon^3), \quad (D25)$$

where

$$G_1 = \sum_{k=0}^{J} w_k + \frac{(J+1)(J-2n+2)}{2d}, \quad (D26)$$

$$G_2 = \frac{1}{2} \left[ G_1^2 - \sum_{k=0}^{J} w_k^2 \right] = \frac{J(J+1)(10+11J+3J^2-12n[J+2]+12n^2)}{24d^2}. \quad (D27)$$

We then have

$$\frac{dF}{dp} \bigg|_{(d^3)^{1/2}} = \left( 1 + \frac{1}{d} \right) (1 + F_1 + F_2)(1 + G_1 + G_2)^{-1} + O(\epsilon^3)$$

$$= 1 + T_1 + T_2 + O(\epsilon^3), \quad (D28)$$

$$T_1 = F_1 - G_1 + \frac{1}{d} \left[ -J(J+1)+n \right], \quad (D29)$$

$$T_2 = F_2 - G_2 + \frac{F_1}{d} - \frac{G_1}{d} - F_1 G_1$$

$$= \frac{2n^2 - 2n(2J[J+1]+1) + J(J+1)(J^2+J+2)}{2d^2}. \quad (D30)$$

Note that compared with the boson case, $T_1 = -R_1$, $T_2 = R_2$, thus the first order vanishes and we again have

$$\Delta S_{\text{igno}} - \Delta S_{\text{info}} = -H(p) - \frac{n^2}{2d^2} + O \left( \frac{n^3}{d^3} \right). \quad (D31)$$

Appendix E: Entropy $H(p)$ for large particle number

Here, we evaluate the entropy $H(p)$ for large particle number. We take $n = m \gg 1$. Starting from (B13), we can rewrite

$$p_j = (2J+1) \frac{(n!)^2}{(2n+1)!} \binom{2n+1}{n+J+1}$$

$$= (2J+1) \frac{(n!)^2 2^{2n+1}}{(2n+1)!} b(n+J+1), \quad (E1)$$

where $b(n+J+1) = 2^{-(2n+1)} \binom{2n+1}{n+J+1}$ follows a binomial distribution with $N+1$ trials and a success probability of $1/2$.

Using Stirling’s approximation in the form $n! = \sqrt{2\pi n} e^{n+O(1/n)}$, we have

$$\frac{(n!)^2}{(2n+1)!} = \frac{n^{2n+1} e^{-2n+O(1/n)}}{\sqrt{2\pi(2n+1)2^{2n+3/2} e^{-2n-1+O(1/n)}}}$$

$$= (\sqrt{2\pi e}) \left( \frac{n}{2n+1} \right)^{2n+1} \frac{1}{(2n+1)^{1/2}} [1 + O(1/n)]$$

$$= \frac{\sqrt{2\pi e}}{2^{2n+1} \left( 1 + \frac{1}{2n} \right)^{2n+1} (2n+1)^{1/2}} [1 + O(1/n)]$$

$$= \frac{\sqrt{2\pi e}}{2^{2n+1} \left( e + O(1/n) \right)^{2n+1} (2n+1)^{1/2}} [1 + O(1/n)]$$

$$= \frac{1}{2^{2n+1}} \sqrt{\frac{2\pi}{2n+1}} [1 + O(1/n)]. \quad (E2)$$
Using a local version of the central limit theorem [47, Chapter VII, Theorem 1], we can approximate $b(n + J + 1)$ by a normal distribution with mean $(2n + 1)/2$ and variance $(2n + 1)/4$, obtaining

$$p_J = (2J + 1) \sqrt{\frac{2\pi}{2n + 1}} [1 + O(1/n)] \left[ e^{-\frac{(J + 1)^2}{8n + 1}} + o(n^{-1/2}) \right]$$

$$= (2J + 1) \left[ e^{-\frac{(J + 1)^2}{8n + 1}} + o(1/n) \right] [1 + O(1/n)]$$

$$= (2J + 1) e^{-\frac{(J + 1)^2}{8n + 1}} [1 + o(1)][1 + O(1/n)]$$

$$= (2J + 1) e^{-\frac{(J + 1)^2}{8n + 1}} [1 + o(1)], \quad \text{(E3)}$$

where $o(f)$ denotes an error term going to zero strictly faster than $f$. Then

$$\ln p_J = \ln(2J + 1) - \ln(n + 1/2) - \frac{(J + 1)^2}{n + 1/2} + o(1), \quad \text{(E4)}$$

so the entropy is approximated by

$$H(p) = - \sum_{J=0}^{n} p_J \left[ \ln(2J + 1) - \ln(n + 1/2) - \frac{(J + 1)^2}{n + 1/2} + o(1) \right]$$

$$= \ln(n + 1/2) + o(1) + [1 + o(1)] \sum_{J=0}^{n} (2J + 1) e^{-\frac{(J + 1)^2}{8n + 1}} \left[ - \ln(2J + 1) + \frac{(J + 1)^2}{n + 1/2} \right]. \quad \text{(E5)}$$

For large $n$, we expect that the sum can be approximated by an integral. To show this, we can use the simplest version of the Euler-Maclaurin formula:

$$\sum_{J=0}^{n} f(J) = \int_{0}^{n} f(x) \, dx + \int_{0}^{n} \left( x - \lfloor x \rfloor - \frac{1}{2} \right) f'(x) \, dx + \frac{f(0) + f(n)}{2}.$$ 

$$f(x) := (2x + 1) e^{-\frac{(x + 1)^2}{8n + 1}} \left[ - \ln(2x + 1) + \frac{(x + 1)^2}{n + 1/2} \right]. \quad \text{(E6)}$$

Firstly, we have

$$f(0) = e^{-\frac{1}{8n + 1}} \cdot \frac{1}{4(n + 1/2)} = O(n^{-2}),$$

$$f(n) = 2e^{-n} [- \ln(2n + 1) + (n + 1/2)] = O(ne^{-n}). \quad \text{(E7)}$$

Along these lines, it is not hard to see that shifting the initial point from $x = 0$ to $x = 1/2$ leads to an $o(1)$ error, so we change variables to $y = x + 1/2$ and let $g(y) := f(y - 1/2)$. Additionally, the upper limit can be extended to infinity with an error which can be verified to be $O(e^{-n} \text{poly}(n, \ln n))$. For the remainder integral, we let $k = (n + 1/2)^{-1}$ and use

$$g(y) = 2ke^{-ky^2} \left[ -y \ln(2y) + ky^3 \right],$$

$$g'(y) = 2ke^{-ky^2} \left[ 2ky^2 \ln(2y) - 2k^2y^4 - \ln(2y) - 1 + 3ky^2 \right]. \quad \text{(E8)}$$

Together with $|y - \lfloor y \rfloor - 1/2| \leq 1/2$, we have

$$\int_{0}^{\infty} \left( y - \lfloor y \rfloor - \frac{1}{2} \right) g'(y) \, dy \leq \int_{0}^{\infty} 2k^3y \ln(2y)e^{-ky^2} \, dy + \int_{0}^{\infty} 2k^3y^4e^{-ky^2} \, dy + \int_{0}^{\infty} k \ln(2y)e^{-ky^2} \, dy + \int_{0}^{\infty} ke^{-ky^2} \, dy + \int_{0}^{\infty} 3k^2y^2e^{-ky^2} \, dy \quad \text{(E9)}$$
in which the individual integrals can be evaluated with the highest order being $O\left(\frac{\ln n}{n}\right) = o(1)$.

Overall, therefore,

$$\sum_{J=0}^{n} f(J) = \int_{0}^{\infty} g(y) \, dy + o(1)$$

$$= \frac{1}{2} (\ln k + \gamma) - \ln 2 + 1 + o(1)$$

$$= -\frac{1}{2} \ln n + \frac{\gamma}{2} - \ln 2 + 1 + o(1), \quad (E10)$$

where $\gamma = 0.557 \ldots$ is the Euler-Mascheroni constant. Putting this into (E5),

$$H(p) = \frac{1}{2} \ln n + \frac{\gamma}{2} - \ln 2 + 1 + o(1)$$

$$= \frac{1}{2} \ln n + 0.595 \ldots + o(1). \quad (E11)$$