The Dimensional-Reduction Anomaly

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Abstract

In a wide class of $D$-dimensional spacetimes which are direct or semi-direct sums of a $(D-n)$-dimensional space and an $n$-dimensional homogeneous “internal” space, a field can be decomposed into modes. As a result of this mode decomposition, the main objects which characterize the free quantum field, such as Green functions and heat kernels, can effectively be reduced to objects in a $(D-n)$-dimensional spacetime with an external dilaton field. We study the problem of the dimensional reduction of the effective action for such spacetimes. While before renormalization the original $D$-dimensional effective action can be presented as a “sum over modes” of $(D-n)$-dimensional effective actions, this property is violated after renormalization. We calculate the corresponding anomalous terms explicitly, illustrating the effect with some simple examples.

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1 Introduction

Simplifications connected with an assumption of symmetry play an important role in the study of physical effects in a curved spacetime. In this paper we consider quantum fields propagating in a $D$-dimensional spacetime which is a semi-direct sum of a $(D-n)$-dimensional space and an $n$-dimensional homogeneous “internal” space. Such a field can be decomposed into modes. As a result of this mode decomposition, the main objects which characterize the free quantum field, such as Green functions and heat kernels, can effectively be reduced to objects in a $(D-n)$-dimensional spacetime with an external dilaton field. Our aim is to study the problem of the dimensional reduction of the effective action for such spacetimes. We shall demonstrate that while before renormalization the original $D$-dimensional effective action can be presented as a “sum over modes” of $(D-n)$-dimensional effective actions, this property is generically violated after renormalization. We call this effect the dimensional-reduction anomaly.

First of all, there is an evident general reason why $D$-dimensional renormalization is not equivalent to renormalization of the $(D-n)$-dimensional effective theory. Namely, the number of divergent terms of the Schwinger-DeWitt series which is used to renormalize a given object depends on the number of dimensions. What is much more interesting is that this is not the only reason for the presence of the dimensional-reduction anomaly. The aim of this paper is to discuss this problem and to derive explicit expressions for the dimensional-reduction anomaly in a four-dimensional spacetime with special choices of one- and two-dimensional homogeneous internal spaces.

In some aspects the problem we study is related to the study of the momentum-space representation of the ultraviolet divergences discussed by Bunch and Parker [1]. Nevertheless, there exists a very important difference. Namely, Bunch and Parker used the Fourier transform with respect to all $D$ dimensions ($D=4$ in their paper) in a spacetime without symmetries. We are making mode decompositions with respect to $n$ internal dimensions only. Moreover, due to the symmetry we effectively rewrite the original $D$-dimensional theory in terms of a set of $(D-n)$-dimensional effective theories with a dilaton field. This representation, which was absent in the paper by Bunch and Parker, allows us to make the comparison of renormalization in $D$- and $(D-n)$-dimensional theories.

The dimensional-reduction anomaly discussed in this paper might have interesting applications. One of them is connected with black-hole physics. Recently there has been much interest in the study of the Hawking radiation in two-dimensional dilaton gravity models of a black hole. This study was initiated by the paper [2]. In this and other papers on the subject (see e.g. [3]–[6]) it is either explicitly or implicitly assumed that two-dimensional calculations (at least for the special choice of the dilaton field corresponding to the spherical reduction of the four-dimensional Schwarzschild spacetime) correctly reproduce the $s$-mode contribution to the stress-energy tensor of the four-dimensional theory. Generally speaking, in the presence of the dimensional-reduction anomaly this is not true (see also the discussion in [7]).

It is interesting to note that in a numerical study of the vacuum polarization in black holes where the mode decomposition was used, it has been demonstrated that one always needs to add extra contributions to the terms of the series in order to ensure convergence of the series (see e.g. [8, 9, 10]). This fact, as we shall see, is a direct manifestation of the dimensional-reduction anomaly.

The paper is organized as follows. Section 2 contains a general discussion of the
dimensional reduction of a free quantum field theory in a gravitational background. In Section 3 we discuss simple examples of the dimensional reduction of four-dimensional flat spacetime and illustrate the effect of the dimensional-reduction anomaly. In Sections 4 and 5 we derive the dimensional-reduction anomaly for $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and $\langle \hat{T}^{\mu \nu} \rangle_{\text{ren}}$ in a four-dimensional spacetime for $(3 + 1)$ and $(2 + 2)$ reductions. We discuss the obtained results in the Conclusion. In our work we use dimensionless units where $G = c = \hbar = 1$, and the sign conventions of [12] for the definition of the curvature.

2 Dimensional Reduction of the Heat Kernel and the Effective Action

Consider a $D$-dimensional spacetime with a metric of the form

$$ds^2 = g_{\mu \nu} dX^\mu dX^\nu = dh^2 + e^{-(4\phi/n)} d\Omega^2, \quad X^\mu = (x^a, y^i),$$

where $d\Omega^2$ is the metric of an $n$-dimensional homogeneous space. In other words, the metric $ds^2$ is a semi-direct sum of $dh^2$ and $d\Omega^2$. We call the $n$-dimensional space with metric $d\Omega^2$ the internal space, and the scalar field $\phi(x^c)$ on the $(D - n)$-dimensional manifold with metric $dh^2$ the dilaton field. Let us emphasize that the normalization of the dilaton field $\phi$ is a question of convenience. We fix this normalization by requiring that $\sqrt{g}$ for the metric (2.1) be proportional to $\exp(-2\phi)$ for any number of internal dimensions $n$. Well-known examples of metrics of the form (2.1) are those of spherical spacetimes, and metrics connected with a dimensional reduction in Kaluza-Klein theories.

Let $\hat{\Phi}$ be a free scalar quantum field propagating in the spacetime (2.1) and obeying the equation

$$F \hat{\Phi}(X) = 0,$$

with field operator

$$F = \Box - V - m^2.$$

Note that we explicitly separate the mass term $m^2$ from the potential $V$. The latter may contain an interaction with the curvature, $\xi R$, for a non-minimally coupled field, but is not fixed at the moment. We only assume that when calculated on the background (2.1) the potential $V$ is independent of the $y^a$ coordinates.

Using the line element (2.1), the operator $\Box$ becomes

$$\Box = \Delta_h - 2\nabla \phi \cdot \nabla + e^{(4\phi/n)} \Delta_\Omega,$$

where $\Delta_h, \Delta_\Omega$ are the d’Alembertians corresponding to the metrics $h_{ab}, \Omega_{ij}$ respectively, and $\nabla$ is understood to denote the covariant derivative with respect to the metric $h_{ab}$.

Considerable simplification of the problem in spacetime (2.1) is connected with the fact that for a wide class of homogeneous metrics the eigenvalue problem

$$\Delta_\Omega Y(y) = -\lambda Y(y),$$

is well-studied. We denote by $Y_{\lambda W}$ harmonics, that is eigenfunctions of (2.6), and use a (collective) index $W$ to distinguish between different solutions of (2.6) for the same $\lambda$. We
assume standard normalization and orthogonality conditions,
\[
\int dy \Omega^{1/2} Y_{\lambda W}(y) \bar{Y}_{\lambda W'}(y) = \delta_{\lambda \lambda'} \delta_{WW'}, \quad (2.7)
\]
\[
\sum_{\lambda W} Y_{\lambda W}(y) \bar{Y}_{\lambda W}(y') = \delta(y, y') \equiv \Omega^{-1/2} \delta^n(y - y'). \quad (2.8)
\]
We denote by \(\mathcal{N}(\lambda)\) the degeneracy of the eigenvalue \(\lambda\), that is \(\mathcal{N}(\lambda) = \sum_{W,W'} \delta_{WW'}\).

The field \(\hat{\Phi}\) can be decomposed into modes
\[
e^\phi \varphi_{\lambda p}(x) Y_{\lambda W}(y), \quad (2.9)
\]
where the functions \(\varphi_{\lambda p}\) obey the equation
\[
\mathcal{F}_\lambda \varphi_{\lambda p}(x) = 0, \quad (2.10)
\]
\[
\mathcal{F}_\lambda = \Delta_h - V_\lambda[\phi] - m^2. \quad (2.11)
\]
The index \(p\) is an additional quantum number which enumerates solutions for a given \(\lambda\) and \(V_\lambda[\phi]\) is
\[
V_\lambda[\phi] = \lambda \epsilon^{4\phi/n} + (\nabla \phi)^2 - \Delta_h \phi + V. \quad (2.12)
\]
In other words, by expanding the field in modes we effectively reduce the original \(D\)-dimensional problem to a similar problem in \((D - n)\)-dimensional space with an effective potential \(V_\lambda\) depending on the “dilaton” field \(\phi\).

It is not difficult to show that at least formally the effective action for the quantum field \((2.3)\) allows a similar dimensional reduction. In order to demonstrate this, consider a heat kernel \(K(X, X'|s)\) for the problem \((2.3)\), which is the solution of the equation
\[
\left( \frac{\partial}{\partial s} - F \right) K(X, X'|s) = 0, \quad (2.13)
\]
\[
K(X, X'|s = 0) = \delta^D(X, X') = e^{2\phi(x)} \frac{\delta^{(D-n)}(x - x')}{\sqrt{h}} \frac{\delta^n(y - y')}{\sqrt{\Omega}}. \quad (2.14)
\]
The effective action is defined as
\[
W = -\frac{1}{2} \int_0^\infty ds \frac{ds}{s} tr_X K(X, X'|s). \quad (2.15)
\]
Here and later the trace operation is understood as
\[
tr_X A(X, X') = \int dX \sqrt{g} A(X, X) = \int dx \sqrt{h} \int dy \sqrt{\Omega} e^{-2\phi(x)} A(x, y; x, y). \quad (2.16)
\]
By decomposing \(K(X, X'|s)\) into harmonics one can write
\[
K(X, X'|s) = e^{\phi(x) + \phi(x')} \sum_{\lambda W} K(x, x'; \lambda|s) Y_{\lambda W}(y) \bar{Y}_{\lambda W}(y'). \quad (2.17)
\]
\[\text{We write summation over indices assuming that the spectrum is discrete. For a continuous spectrum one must replace summation by integration over the spectrum. In what follows we shall assume that this rule is automatically applied.}\]
Using relations (2.5)–(2.8), it is easy to verify that the reduced heat kernel \( K(x, x'; \lambda|s) \) obeys the relations

\[
\left( \frac{\partial}{\partial s} - F_\lambda \right) K(x, x'; \lambda|s) = 0, \tag{2.18}
\]

\[
K(x, x'; \lambda|s = 0) = \delta^{(D-n)}(x, x') = \frac{\delta^{(D-n)}(x - x')}{\sqrt{h}}. \tag{2.19}
\]

Here the operator \( F_\lambda \) is given by (2.10)–(2.11).

Substituting representation (2.17) into the effective action (2.15), integrating over the \( y \)-variables, and using (2.7), one gets

\[
W = \sum_\lambda N(\lambda) W_\lambda, \tag{2.20}
\]

where

\[
W_\lambda = -\frac{1}{2} \int_0^\infty ds \text{tr}_x K(x, x'; \lambda|s). \tag{2.21}
\]

Here

\[
\text{tr}_x K(x, x'; \lambda|s) = \int dx \sqrt{h} K(x, x; \lambda|s), \tag{2.22}
\]

and \( N(\lambda) \) is the degeneracy factor of the eigenvalue \( \lambda \). Relations (2.20)–(2.21) can be interpreted as the mode decomposition of the effective action.

It should be emphasized that the above relations for the effective action are strictly formal because of the presence of ultraviolet divergences. In order to obtain the renormalized value of the effective action in \( D \)-dimensional spacetime one must subtract from \( \text{tr}_x K(X, X'|s) \) the first \( N_D \) terms of the Schwinger-DeWitt expansion of the heat kernel, where

\[
N_D = \begin{cases} 
\frac{D}{2} + 1 & \text{for } D \text{ even}, \\
\frac{D+1}{2} & \text{for } D \text{ odd}.
\end{cases} \tag{2.23}
\]

Our main observation is that this procedure destroys the formal representation (2.20), so that after renormalization one gets

\[
W^\text{ren} = \sum_\lambda N(\lambda) [W_\lambda^\text{ren} + \Delta W_\lambda]. \tag{2.24}
\]

In this expression \( W_\lambda^\text{ren} \) is understood as the renormalized effective action of the \( (D-n) \)-dimensional theory with a dilaton field, (2.10), where the renormalization is performed by subtracting the first \( N_{D-n} \) terms of the Schwinger-DeWitt expansion for the operator \( F_\lambda \). We call the additional contribution \( \Delta W_\lambda \) the dimensional-reduction anomaly. A representation similar to (2.24) is also valid for \( \langle \hat{\Phi}^2 \rangle^\text{ren} \),

\[
\langle \hat{\Phi}^2 \rangle^\text{ren} = e^{2\phi} \sum_\lambda N(\lambda) \left[ \langle \hat{\Phi}^2 \rangle^\text{ren}_\lambda + \Delta \langle \hat{\Phi}^2 \rangle_\lambda \right]. \tag{2.25}
\]

One might observe that there exists a relationship between the dimensional-reduction anomaly and the so-called multiplicative anomaly \([13, 14]\). Formally, we can write

\[
F = \prod_{\lambda, W} F_\lambda, \tag{2.26}
\]
\[- \frac{1}{2} \log \det F = - \frac{1}{2} \sum_{\lambda} N(\lambda) \log \det F_\lambda. \tag{2.27}\]

The latter relation is nothing but (2.20). The violation of the formal relation (2.27) for products of operators after renormalization is known as the multiplicative anomaly.

The aim of this paper is to discuss special examples of the dimensional-reduction anomaly. In what follows, we restrict ourselves to the physically interesting case where the number of spacetime dimensions is 4, and the number of dimensions of the “internal” homogeneous space is 1 or 2. We also restrict ourselves to manifolds of Euclidean signature. In each case the anomaly is found as the difference between the renormalization terms for the \((D - n)\)-dimensional theory and the mode-decomposed renormalization terms from \(D\) dimensions.

3 Flat Space Examples of the Dimensional-Reduction Anomaly

The dimensional-reduction anomaly can occur in even the simple case of mode decomposition in a flat spacetime. In order to demonstrate this, let us consider a free massive scalar field \(\hat{\Phi}\) obeying the equation

\[F \hat{\Phi} = (\Box - m^2)\hat{\Phi} = 0 \tag{3.1}\]

in four-dimensional flat Euclidean space. The Euclidean Green function for this equation is

\[G^0(X, X') = \frac{m}{4\pi^2 \sqrt{2\sigma}} K_1(m\sqrt{2\sigma}), \tag{3.2}\]

where \(2\sigma\) is the square of the geodesic distance from \(X\) to \(X'\), and \(K_1\) is a modified Bessel function. If there exists a boundary \(\Sigma\) surrounding the region \(\mathcal{M}\) under consideration and the field obeys a non-trivial boundary condition at \(\Sigma\), or equation (3.1) includes a non-vanishing potential \(V\) which vanishes in \(\mathcal{M}\), then the Green function would differ from \(G^0\). We denote this Green function by \(G(X, X')\). The renormalized Green function in this case is defined as\(^2\)

\[G^{\text{ren}}(X, X') = G(X, X') - G^0(X, X'). \tag{3.3}\]

We also have

\[\langle \hat{\Phi}^2(X) \rangle^{\text{ren}} = \lim_{X' \to X} G^{\text{ren}}(X, X'). \tag{3.4}\]

It is evident that \(\langle \hat{\Phi}^2(X) \rangle^{\text{ren}} = 0\) in the absence of the external field \(V\) and boundaries.

In this section we derive the dimensional-reduction anomaly in \(\langle \hat{\Phi}^2(X) \rangle^{\text{ren}}\) for two examples of mode decompositions in flat space. In each case, the anomaly is calculated as the difference between the subtraction terms used to renormalize the Green function \(G\) of the dimensionally-reduced theory, and the \(G^0\) used to renormalize the four-dimensional Green function \(G\). These calculations are easily repeated for the heat kernel, allowing one to obtain the anomaly in the effective action in a similar manner.

\(^2\)In a generic four-dimensional curved background, one has to subtract the first two terms in the Schwinger-DeWitt expansion for the heat kernel of the operator \(F\).
3.1 Spherical reduction

For the first example we consider the case when the external field and/or boundary is spherically symmetric and perform decomposition into spherical harmonics. The metric in spherical coordinates \( X^\mu = (t, r, \theta, \varphi) \) is given by

\[
\begin{align*}
    ds^2 &= dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\
    &\quad \text{(3.5)}
\end{align*}
\]

and for the square of the geodesic distance from \( X \) to \( X' \) we have

\[
\begin{align*}
    2\sigma &= (\Delta t)^2 + (\Delta r)^2 + 2rr' (1 - \cos \lambda), \\
    &\quad \text{(3.6)}
\end{align*}
\]

where \( \Delta t = t - t', \Delta r = r - r', \) and

\[
\cos \lambda = \cos \theta \cos \theta' + \sin \theta \sin \theta' \sin (\varphi - \varphi').
\]

The Green function \( G^0 \) can be decomposed into spherical harmonics \( Y_{\ell m}(\theta, \varphi) \),

\[
G^0(X, X') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi) Y_{\ell m}(\theta', \varphi') \frac{G^0_\ell(x, x')}{rr'},
\]

where \( x = (t, r) \) and \( x' = (t', r') \). The two-dimensional Green function is a solution of the equation

\[
\mathcal{F}_\ell G^0_\ell(x, x') \equiv (2\Box - m^2 - V_\ell)G^0_\ell(x, x') = -\delta(t - t') \delta(r - r'),
\]

where \( 2\Box = \partial_t^2 + \partial_r^2 \), and

\[
V_\ell(r) = \frac{\ell(\ell + 1)}{r^2}.
\]

It is possible to write representations similar to (3.8) for the Green functions \( G \) and \( G^{\text{ren}} \), and using these representations to write the renormalized value of \( \langle \hat{\Phi}^2 \rangle \) in the form

\[
\langle \hat{\Phi}^2(x) \rangle^F_{\text{ren}} = \frac{1}{4\pi r^2} \lim_{x' \to x} \sum_{\ell=0}^{\infty} (2\ell + 1) G^{\text{ren}}_\ell(x, x'),
\]

where

\[
G^{\text{ren}}_\ell(x, x') = G_\ell(x, x') - G^0_\ell(x, x').
\]

The Green functions \( G^0_\ell \) can be obtained either by solving equation (3.9) or by decomposing the known function \( G^0 \) into spherical harmonics. Decomposing \( G^0 \) we get

\[
\frac{G^0_\ell(x, x')}{rr'} = \frac{1}{2\pi} \int_{-1}^{1} dz \, P_\ell(z) \frac{mK_1(m\sqrt{(\Delta t)^2 + (\Delta r)^2 + 2rr'(1 - z))}}{\sqrt{(\Delta t)^2 + (\Delta r)^2 + 2rr'(1 - z)}},
\]

with \( z = \cos \lambda \). Using the integral representation for \( K_1 \) (see Appendix C.1) this becomes

\[
\frac{G^0_\ell(x, x')}{rr'} = \frac{1}{8\pi} \int_{0}^{\infty} ds \frac{1}{s^2} \exp \left( -m^2 s - \frac{(\Delta t)^2 + (\Delta r)^2 + 2rr'}{4s} \right) \int_{-1}^{1} dz \, P_\ell(z) \exp \left( \frac{2rr'}{4s} z \right).
\]

The integral over \( z \) can be taken (see e.g. [15], vol.2, eq.2.17.5.2):

\[
\int_{-1}^{1} dz \, P_\ell(z) \exp (-pz) = (-1)^\ell \sqrt{\frac{2\pi}{p}} I_{\ell + 1/2}(p),
\]

\[
\int_{-1}^{1} dz \, P_\ell(z) \exp (-pz) = (-1)^\ell \sqrt{\frac{2\pi}{p}} I_{\ell + 1/2}(p),
\]
where \( I_{\ell+1/2} \) is a modified Bessel function. Since for \( \ell \geq 1 \) the expression in the right-hand side of (3.13) vanishes at \( p = 0 \), one can easily show that the functions \( G^0_\ell(x, x') \) vanish for all \( \ell \) when either \( r = 0 \) or \( r' = 0 \). This property will be of some importance for the anomaly.

Using the following representation for the function \( I_{\ell+1/2} \)

\[
I_{\ell+1/2}(p) = \frac{1}{\sqrt{2\pi p}} \sum_{k=0}^{\ell} \frac{(\ell + k)!}{k! (\ell - k)!} \frac{1}{(2p)^k} \left[ (-1)^k p^k - (-1)^\ell e^{-p} \right],
\]

(see, for example, 8.467 of [16]), we obtain for the reduced Green functions

\[
G^0_\ell(x, x') = \frac{1}{2\pi} \sum_{k=0}^{\ell} \frac{(\ell + k)!}{k! (\ell - k)!} \left[ (-1)^k \frac{(\Delta t)^2 + (\Delta r)^2)^{k/2}}{(2mr' r)^k} K_k(m \sqrt{\Delta t^2 + (\Delta r)^2}) - (-1)^\ell \frac{(\Delta t)^2 + (\Delta r)^2 + 4rr')^{k/2}}{(2mr' r)^k} K_k(m \sqrt{\Delta t^2 + (\Delta r)^2 + 4rr'}) \right].
\]

(3.17)

Until now we were working with the series representation for the four-dimensional Green function \( G^0_\ell \). On the other hand, one can start with the two-dimensional theory defined by (3.9). In order to calculate the two-dimensional quantity \( \langle \hat{\Phi}^2(X) \rangle^\text{ren} \) for the two-dimensional operator \( \mathcal{F}_\ell \), one subtracts from the Green function \( G_\ell(x, x') \) the free-field Green function\(^3\), and takes the coincidence limit:

\[
\langle \hat{\Phi}^2(x) \rangle^\text{ren} = \lim_{x' \rightarrow x} \left[ G_\ell(x, x') - G^\text{div}_\ell(x, x') \right],
\]

(3.18)

where the two-dimensional free-field Green function is

\[
G^\text{div}_\ell(x, x') = \frac{1}{2\pi} K_0(m \sqrt{\Delta t^2 + (\Delta r)^2}).
\]

(3.19)

Note that \( G^\text{div}_\ell \) is identical to the first term in the first sum for \( G^0_\ell \) in (3.17).

By comparing (3.11) with (3.12) we get

\[
\langle \hat{\Phi}^2(x) \rangle^\text{F} = \frac{1}{4\pi^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \langle \hat{\Phi}^2(x) \rangle^\text{ren}_\ell + \Delta \langle \hat{\Phi}^2(r) \rangle_{\ell} \right],
\]

(3.20)

where the anomalous term \( \Delta \langle \hat{\Phi}^2(r) \rangle_{\ell} \) is given by

\[
\Delta \langle \hat{\Phi}^2(r) \rangle_{\ell} = \lim_{x' \rightarrow x} \left[ G^{\text{div}}_\ell(x, x') - G^0_\ell(x, x') \right]
\]

\[
= \frac{1}{4\pi} \sum_{k=1}^{\ell} \frac{(\ell + k)!}{(\ell - k)! k!} K_k(2mr) \frac{1}{(mr)^k} \left[ \frac{1}{(mr)^2} - K_0(2mr) - \frac{2}{(mr)} K_1(2mr) \right]
\]

(3.21)

(3.22)

For example,

\[
\Delta \langle \hat{\Phi}^2(r) \rangle_{\ell=0} = \frac{1}{2\pi} K_0(2mr)
\]

(3.23)

\[
\Delta \langle \hat{\Phi}^2(r) \rangle_{\ell=1} = \frac{1}{2\pi} \left[ \frac{1}{(mr)^2} - K_0(2mr) - \frac{2}{(mr)} K_1(2mr) \right]
\]

\[
\Delta \langle \hat{\Phi}^2(r) \rangle_{\ell=2} = \frac{1}{2\pi} \left[ \frac{3}{(mr)^2} - \frac{6}{(mr)^4} + 6 \frac{6}{(mr)} K_0(2mr) + \frac{6}{(mr)} K_1(2mr) + \frac{12}{(mr)^2} K_2(2mr) \right].
\]

\(^3\) In two dimensions, renormalization consists of subtracting only the first term of the Schwinger-DeWitt expansion of the heat kernel for the operator \( \mathcal{F}_\ell \). In flat spacetime this is equivalent to the subtraction of the free-field Green function, even in the presence of the dilaton field and a nontrivial potential \( V \).
Relation (3.20) explicitly demonstrates that \( \langle \hat{\Phi}^2(x) \rangle_{\text{ren}} \) can be expressed as a sum over modes of \( \langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^\ell \) from the corresponding two-dimensional theory. However, to obtain the correct result we see that each term in the decomposition must be modified by adding the state-independent quantity \( \Delta \langle \hat{\Phi}^2(r) \rangle^\ell \) of (3.21). Failure to account for these extras terms would result, for example, in a nonzero value for \( \langle \hat{\Phi}^2(x) \rangle_{\text{ren}} \) in the Minkowski state.

The violation of the expected mode decomposition for a physical observable in the process of dimensional reduction due to the renormalization procedure is called the dimensional-reduction anomaly. In this particular example the dimensional-reduction anomaly is given explicitly by (3.22). There are several reasons why this anomaly arises. First, the number of divergent terms which are to be subtracted in the renormalization procedure is different in 4 and 2 dimensions (see the footnotes on pages 6 and 8). However, it is easily shown that even if we make an additional subtraction of the next term of the Schwinger-DeWitt expansion in two dimensions, for \( \ell \geq 2 \) there remains a non-vanishing “local” part of the dimensional-reduction anomaly which is given by the first sum in (3.22), with \( k \geq 2 \). Besides this local contribution, in our particular case there is a “non-local” part to the anomaly given by the second sum. It is present because the two-dimensional \((t, r)\) space is a half-plane and the two-dimensional field vanishes at the boundary \( r = 0 \). While the mode-decomposed subtraction term \( G^0_\ell \) from four dimensions obeys this boundary condition, the two-dimensional subtraction term \( G^\text{div}_\ell \) does not.

### 3.2 Rindler space

As a second example of the dimensional-reduction anomaly in flat spacetime we consider the decomposition of a scalar field into Rindler time modes. The metric for (Euclidean) Rindler space is given by

\[
ds^2 = z^2 dt^2 + dz^2 + dx^2 + dy^2,
\]

where \( t \in [0, 2\pi) \), \( z \in [0, \infty) \), \( x, y \in (-\infty, \infty) \). This line element may be obtained from the flat-space line element

\[
ds^2 = dT^2 + dX^2 + dY^2 + dZ^2
\]

by the coordinate transformation

\[
T = z \sin(t), \quad X = x, \\
Z = z \cos(t), \quad Y = y,
\]

and is clearly just flat space in polar coordinates. Hence, the free-field Green function is given by (3.2) with

\[
2\sigma = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + 2zz'[1 - \cos(\Delta t)]
\]

This Green function may be decomposed in terms of the Rindler “time” modes \( \cos(k\Delta t) \) as an ordinary Fourier cosine series,

\[
G^0(X, X') = \frac{1}{2\pi} \frac{G_0^0(x, x')}{\sqrt{zz'}} + \sum_{k=1}^{\infty} \frac{\cos(k\Delta t)}{\pi} \frac{G_k^0(x, x')}{\sqrt{zz'}},
\]

\[
\frac{G_k^0(x, x')}{\sqrt{zz'}} = \int_0^{2\pi} d(\Delta t) \cos(k\Delta t) G^0(X, X') ; \ k \geq 0,
\]
where \( \mathbf{x} = (x, y, z) \). The three-dimensional Green function \( G^0_k(\mathbf{x}, \mathbf{x}') \) is a solution of the equation

\[
  \mathcal{F}_k G^0_k(\mathbf{x}, \mathbf{x}') \equiv (\Box - m^2 - V_k) G^0_k(\mathbf{x}, \mathbf{x}') = -\delta(x-x') \delta(y-y') \delta(z-z') , \tag{3.30}
\]

where \( \Box = \partial^2_x + \partial^2_y + \partial^2_z \), and

\[
  V_k(z) = \frac{4k^2 - 1}{4z^2} . \tag{3.31}
\]

As before, we may obtain an explicit expression for \( G^0_k(\mathbf{x}, \mathbf{x}') \) by decomposing the four-dimensional Green function, as in \([3.23]\). Using the integral representation for \( K_1 \) from Appendix C.3, combined with the integrals

\[
  \int_0^{2\pi} dx \cos(kx) \exp(p \cos(x)) = 2\pi I_k(|p|) \tag{3.32}
\]

and

\[
  \int_0^\infty dx \frac{1}{x} \exp(-ax - \frac{b}{x}) I_k(cx) = 2I_k(\sqrt{b(a+c) - \sqrt{b(a-c)}})K_k(\sqrt{b(a+c) + \sqrt{b(a-c)}}) \tag{3.33}
\]

(see, for example, 3.937.2 of \([16]\) and 2.15.6.4 of \([15]\) respectively), one can show that

\[
  \frac{G^0_k(\mathbf{x}, \mathbf{x}')}{\sqrt{z^2 + z'^2}} = -\frac{m^2}{4\pi} \left[ \left( I_{k+1}(\alpha_-) + \frac{k}{\alpha_-} I_k(\alpha_-) \right) K_k(\alpha_+) \left( \frac{1}{md_+} - \frac{1}{md_-} \right) - I_k(\alpha_-) \left( K_{k+1}(\alpha_-) + \frac{k}{\alpha_+} K_k(\alpha_-) \right) \left( \frac{1}{md_+} + \frac{1}{md_-} \right) \right] , \tag{3.34}
\]

where we have defined

\[
  \alpha_\pm = \frac{m}{2}(d_\pm \pm d_-) , \quad d_\pm = \sqrt{(x-x')^2 + (y-y')^2 + (z \pm z')^2} . \tag{3.35}
\]

The physical observable of interest, \( \langle \hat{\Phi}^2 \rangle^F_{\text{ren}} \), can be calculated from the Green function \( G(X, X') \) in the presence of a \( t \)-independent boundary or potential using \((3.3, 3.4)\). Decomposing \( G \) in the same manner as \( G^0 \) then allows us to write a decomposed form for \( \langle \hat{\Phi}^2 \rangle^F_{\text{ren}} \) in analogy to \((3.11, 3.12)\):

\[
  \langle \hat{\Phi}^2(\mathbf{x}) \rangle^F_{\text{ren}} = \frac{1}{\pi z} \lim_{\mathbf{x}' \to \mathbf{x}} \left\{ \frac{1}{2} [G_0(\mathbf{x}, \mathbf{x}') - G^0_0(\mathbf{x}, \mathbf{x}')] + \sum_{k=1}^\infty [G_k(\mathbf{x}, \mathbf{x}') - G^0_k(\mathbf{x}, \mathbf{x}')] \right\} . \tag{3.36}
\]

On the other hand, the renormalized value of \( \langle \hat{\Phi}^2 \rangle \) for the three-dimensional operator \( \mathcal{F}_k \) in \((3.30)\) is obtained by subtracting from the full three-dimensional Green function \( G_k(\mathbf{x}, \mathbf{x}') \) not \( G^0_k(\mathbf{x}, \mathbf{x}') \), but rather the first term in the Schwinger-DeWitt expansion for \( \mathcal{F}_k \), denoted \( G^\text{div}_k(\mathbf{x}, \mathbf{x}') \). Specifically, for each \( k \) we have

\[
  \langle \hat{\Phi}^2(\mathbf{x}) \rangle^F_{\text{ren}} = \lim_{\mathbf{x}' \to \mathbf{x}} [G_k(\mathbf{x}, \mathbf{x}') - G^\text{div}_k(\mathbf{x}, \mathbf{x}')] , \tag{3.37}
\]

where

\[
  G^\text{div}_k(\mathbf{x}, \mathbf{x}') = \left( \frac{m}{4\pi^2\sqrt{2\sigma}} \right)^{\frac{1}{2}} K_{\frac{3}{2}}(m\sqrt{2\sigma}) . \tag{3.38}
\]

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Comparing (3.36), (3.37), we find
\[
\langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^F = \frac{1}{\pi z} \left\{ \frac{1}{2} \langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^F + \Delta \langle \hat{\Phi}^2(z) \rangle_0 + \sum_{k=1}^{\infty} \left[ \langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^F + \Delta \langle \hat{\Phi}^2(z) \rangle_k \right] \right\},
\]
(3.39)
where for each \( k \) the anomaly is
\[
\Delta \langle \hat{\Phi}^2(z) \rangle_k = \lim_{x' \to x} \left[ G_{\text{div}}^F(x, x') - G_0^F(x, x') \right]
= \frac{m}{4\pi} \left[ -1 + mz \left( I_{k+1}(mz)K_{k+1}(mz) + I_k(mz)K_k(mz) \right) \right.
\left. + k \left( I_k(mz)K_{k+1}(mz) - I_{k+1}(mz)K_k(mz) \right) \right].
\]
(3.40)

Once again, the anomaly is seen as the difference in the subtraction terms used to renormalize the higher- and lower-dimensional theories. It is easily shown that (3.41) falls off as \( O(z^{-1}) \) for large \( mz \), and diverges as \( O(z^{-1}) \) when \( mz \to 0 \) for \( k \neq 0 \). For \( k = 0 \) the anomaly is finite as \( mz \to 0 \).

These results are qualitatively very similar to those from the spherical decomposition of flat spacetime in the previous section. This should not be surprising, considering that the Euclidean Rindler space (3.24) is simply flat space in polar coordinates. As a result, the calculations of this section amount to a one-dimensional “spherical” decomposition of flat spacetime.

4 The Dimensional-Reduction Anomaly for \( \langle \hat{\Phi}^2 \rangle_{\text{ren}} \)

In the previous section we examined the dimensional-reduction anomaly for mode decompositions in flat space. In the next two sections we extend these calculations to more general, curved spaces. While our chief aim is the study of the dimensional-reduction anomaly in the effective action, in order to illustrate more clearly the effect of curvature on local reduction anomalies we begin with the simpler case of the anomaly for \( \langle \hat{\Phi}^2 \rangle_{\text{ren}} \).

At this point some conventions on notation are in order. Henceforth the scalar field operator in four dimensions is denoted by \( F \), and its Green function is \( G^F \). The calligraphic symbols \( F_\omega \) and \( G^F_\omega \) represent the corresponding quantities in the dimensionally-reduced theory. In both cases the subtraction terms for renormalization are identified by the subscript \( ()_{\text{div}} \) (rather than \( ()^0 \) as in the previous section). Also, the mode decomposition of \( G^F_{\text{div}} \) will be denoted by \( G^F_{\text{div}}(\omega) \) to distinguish it more clearly from \( G^F_{\text{div}} \).

4.1 (1+3)-reduction

We begin with the case \( n = 1 \) and write the metric (2.1) in the form \((a, b = 1, 2, 3) \)
\[
ds^2 = g_{\mu\nu} \, dX^\mu \, dX^\nu = e^{-4\phi(x)} \, dt^2 + h_{ab}(x) \, dx^a \, dx^b.
\]
(4.1)

We assume that \( t \in (-\infty, \infty) \), corresponding to a zero-temperature state. The scalar field operator is taken to be \( F = \Box - V - m^2 \), where the potential \( V \) is independent of Euclidean “time” \( t \). It is easy to see that the operator \( \Delta_\Omega \) for the metric (4.1) is \( \partial^2 / \partial t^2 \). Hence, the mode decomposition in terms of its eigenvalues is simply the standard Fourier transform with
\[
Y(t) = \exp(\pm i\omega t).
\]
(4.2)
The bare $\langle \hat{\Phi}^2 \rangle$ is obtained from the coincidence limit of the Green function,

$$\langle \hat{\Phi}^2(X) \rangle^F = \lim_{X' \to X} G^F(X, X').$$  \hspace{1cm} (4.3)

For a general four-dimensional space, the divergences of the Green function in the coincidence limit come from the first two terms of the Schwinger-DeWitt expansion of the heat kernel,

$$G^F_{\text{div}}(t, x; t', x') = \int_0^\infty ds \frac{1}{(4\pi s)^2} \exp \left\{ -m^2 s - \frac{2\sigma + \epsilon^2}{4s} \right\} \left[ \Re_0^{\Box-V} + s\Re_1^{\Box-V} \right].$$  \hspace{1cm} (4.4)

Here $\sigma = \sigma(X, X')$ is one-half of the square of the geodesic distance between points $X = (t, x^a)$ and $X' = (t', x'^a)$, and

$$\Re_n^{\Box-V} = \Delta^{\frac{1}{2}}(X, X') a_n^{\Box-V}(X, X'),$$  \hspace{1cm} (4.5)

where $\Delta(X, X')$ is the Van Vleck determinant,

$$\Delta(X, X') = \frac{1}{\sqrt{g(X)} \sqrt{g(X')}} \det \left[ \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X'^\nu} \sigma(X, X') \right],$$  \hspace{1cm} (4.6)

and the first few Schwinger-DeWitt coefficients in the coincidence limit $X' \to X$ are

$$a_0^{\Box-V} = 1,$$
$$a_1^{\Box-V} = \frac{1}{6} 4R - V,$$
$$a_2^{\Box-V} = \frac{1}{180} \left[ 4R_{\alpha \beta \gamma \delta} 4R^{\alpha \beta \gamma \delta} - 4R_{\alpha \beta} 4R^{\alpha \beta} \right] + \frac{1}{2} \left( \frac{1}{6} 4R - V \right)^2 + \frac{1}{30} \Box^4 R - \frac{1}{6} \Box V.$$  \hspace{1cm} (4.7)

Expansions of $\sigma$ and the $\Re_n^{\Box-V}$ for the metric $(4.1)$ with $x = x'$ and $e^{-2\phi(t - t')}$ small are given in Appendix A.

Clearly, the Green function and Schwinger-DeWitt coefficients depend on the form of the field equation. We make this dependence explicit by providing such quantities with a corresponding superscript. The superscript for the $a_n$ does not include the mass term from the operator $F$, as it is accounted for separately in $(4.4)$. We also include a cut-off parameter $\epsilon$ in the exponent of the integrand in $(4.4)$ which ensures convergence of the integral for small $s$ with $\sigma$ vanishing.

Our purpose is to compare the divergences of four- and three-dimensional theories related by a Fourier time transform. Unfortunately, it is not possible to evaluate the Fourier transform of $(4.4)$ exactly for general $h_{ab}$. Our response is to make point splitting in the $t$-direction, expanding all $t$-dependent quantities in powers of the curvature, and truncating all expressions at first order in the curvature (two derivatives of the dilaton or metric). Denoting $\tau \equiv e^{-2\phi(t - t')}$ and putting $x = x'$ we have up to first order in the curvature

$$2\sigma(t, x; t', x) = \tau^2 - \frac{1}{3} (\nabla \phi)^2 \tau^4,$$
$$\Re_0^{\Box-V} = 1 + \frac{1}{6} \Box \phi \tau^2,$$
$$\Re_1^{\Box-V} = \frac{1}{6} R - V + \frac{2}{3} \Box \phi.$$  \hspace{1cm} (4.8-4.10)
See Appendix \[A\] for details. Here $R$ is understood to be the scalar curvature for the three-metric $h$, and is related to the four-dimensional curvature $^4R$ via
\[
^4R = R + 4 \Box \phi . \tag{4.11}
\]
Substituting (4.8) into (4.4), expanding the exponent and keeping in the exponent only terms which are quadratic in $\tau$, we get
\[
G^{F}_{\text{div}}(t, x; t', x) = \int_{0}^{\infty} \frac{ds}{(4\pi s)^{2}} \exp \left\{ -m^{2}s - \frac{\tau^{2} + \epsilon^{2}}{4s} \right\} \left[ \left( 1 + \frac{(\nabla \phi)^{2}}{12s} \right) \gamma^{0}_{0} - V + \frac{m^{2}}{s} \gamma^{0}_{0} - V \right] . \tag{4.12}
\]
The integral over the parameter $s$ can be taken with the following result (see Appendix \[C.1\]):
\[
G^{F}_{\text{div}}(t, x; t', x) = \frac{1}{8\pi^{2}} \left[ \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) K_{0}(z) + m^{2} \left( 1 + \frac{\Box \phi}{6} \right) K_{1}(z) \right.
\]
\[
+ m^{4} \left( \frac{\Box \phi}{12} \right) K_{2}(z) \left\{ \gamma + \frac{1}{2} \ln \left( \frac{m^{2}r^{2}}{4} \right) \right\} \tag{4.13}
\]
Here $z \equiv m\sqrt{\tau^{2} + \epsilon^{2}}$, \[K_{\nu}(z) = \left( \frac{2}{z} \right)^{\nu} K_{\nu}(z) , \tag{4.14}\]
and the $K_{\nu}$ are modified Bessel functions. Putting $\epsilon = 0$ and expanding $G^{F}_{\text{div}}$ in a Laurent series in $\tau$ we get
\[
G^{F}_{\text{div}}(t, x; t', x) = \frac{1}{4\pi^{2} \tau^{2}} + \frac{1}{8\pi^{2}} \left\{ m^{2} - \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) \right\} \left\{ \gamma + \frac{1}{2} \ln \left( \frac{m^{2}r^{2}}{4} \right) \right\}
\]
\[
- \frac{m^{4}}{16\pi^{2}} + \left( \frac{\Box \phi}{12\pi^{2}} + \frac{\Box \phi}{24\pi^{2}} + \ldots \right) . \tag{4.15}
\]
Here $\gamma$ is the Euler constant, and the dots denote terms of higher order in $\tau$. The terms displayed are just the usual DeWitt-Schwinger expansion for the divergent parts of $G^{F}$ for the metric (4.1).

The renormalized value of $\langle \hat{\phi}^{2} \rangle^{F}$ can be written in the form
\[
\langle \hat{\phi}^{2}(t, x) \rangle^{F}_{\text{ren}} = \lim_{\epsilon \rightarrow 0} \lim_{t' \rightarrow 0} \left[ G^{F}(t, x; t', x) - G^{F}_{\text{div}}(t, x; t', x) \right] . \tag{4.16}
\]
Taking the limit $\epsilon \rightarrow 0$ in this expression is a trivial operation since the difference in the square brackets is already a finite quantity.

Let us now analyze what happens when we mode decompose $G^{F}_{\text{div}}$ and compare to the corresponding divergent terms from the three-dimensional theory. The Fourier time-transform pair is defined as
\[
G^{F}_{\text{div}}(x; x'|\omega) = e^{-i\omega(t-t')} \int_{-\infty}^{\infty} d(t-t') e^{i\omega(t-t')} G^{F}_{\text{div}}(t, x; t', x') , \tag{4.17}
\]
\[
G^{F}_{\text{div}}(t, x; t', x') = e^{i\omega(t-t')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G^{F}_{\text{div}}(x; x'|\omega) . \tag{4.18}
\]
Since $G^{F}_{\text{div}}(t, x; t', x')$ depends only on the difference $t-t'$, the function $G^{F}_{\text{div}}(x; x'|\omega)$ does not depend on $t$ and $t'$. Calculating the integral in (4.17) using (4.13) and (C.5) we obtain
\[
G^{F}_{\text{div}}(x; x'|\omega) = \frac{1}{4\pi} \left\{ \frac{1}{\epsilon} - \mu + \frac{1}{2\mu} \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) + \frac{m^{2}\Box \phi}{6\mu^{3}} + \frac{m^{4}(\Box \phi)^{2}}{2\mu^{5}} + O(\epsilon) \right\} . \tag{4.19}
\]
where $\mu \equiv \sqrt{m^2 + e^{2\phi}c^2}$. Meanwhile, the operator $\mathcal{F}_\omega$ which determines the reduced equation of motion (2.11) is
\[ \mathcal{F}_\omega = \Delta_h - V_\omega[\phi] - m^2, \] (4.20)
where
\[ V_\omega[\phi] = \omega^2 e^{4\phi} + (\nabla \phi)^2 - \Delta_h \phi + V. \] (4.21)
The divergent part of the Green function for the operator $\mathcal{F}_\omega$ in three dimensions is generated by the first term in the Schwinger-DeWitt expansion of the heat kernel and is
\[ G_{\text{div}}^{\mathcal{F}_\omega}(x; x) = \int_0^\infty ds \frac{1}{(4\pi s)^{\frac{5}{2}}} \exp \left\{ -m^2 s - \frac{e^2}{4s} \right\} = \frac{1}{4\pi} \left[ \frac{1}{\epsilon} - m + O(\epsilon) \right]. \] (4.22)
Hence, if we start with a three-dimensional theory with the field equation
\[ \mathcal{F}_\omega \hat{\Phi}(x) = 0, \] (4.23)
we will obtain for the renormalized value of $\langle \hat{\Phi}^2 \rangle_{\mathcal{F}_\omega}$ the representation
\[ \langle \hat{\Phi}^2(x) \rangle_{\mathcal{F}_{\omega}}^{\text{ren}} = \lim_{\epsilon \to 0} \left[ G^{\mathcal{F}_\omega}(x; x) - G^{\mathcal{F}_\omega}_{\text{div}}(x; x) \right]. \] (4.24)
By comparing (4.16) with (4.24) we can get the following relation between $\langle \hat{\Phi}^2 \rangle$ in the four- and three-dimensional theories:
\[ \langle \hat{\Phi}^2 \rangle_{\mathcal{F}_{\omega}}^{\text{ren}} = e^{2\phi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \langle \hat{\Phi}^2 \rangle_{\mathcal{F}_\omega}^{\text{ren}} + \Delta \langle \hat{\Phi}^2 \rangle_{\omega} \right]; \] (4.25)
where
\[ \Delta \langle \hat{\Phi}^2 \rangle_{\omega} = \lim_{\epsilon \to 0} \left[ G^{\mathcal{F}_\omega}_{\text{div}}(x; x) - G^{\mathcal{F}_\omega}(x; x|\omega) \right] \] (4.26)
\[ = \frac{1}{4\pi} \left[ \mu - m - \frac{1}{2\mu} \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) - \frac{m^2 \Box \phi}{6\mu^3} - \frac{m^4 (\nabla \phi)^2}{2\mu^5} \right]. \] (4.27)
(compare to (2.23)). Since $\Delta \langle \hat{\Phi}^2 \rangle_{\omega}$ does not vanish we have another example of the dimensional-reduction anomaly.

The anomalous term $\Delta \langle \hat{\Phi}^2 \rangle_{\omega}$ is finite and can be written in the form
\[ \Delta \langle \hat{\Phi}^2 \rangle_{\omega} = \Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\xi + \Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\flat, \] (4.28)
where
\[ \Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\xi = \frac{1}{4\pi} \left[ \omega + \frac{m^2}{2\omega} - m - \frac{1}{2\omega} \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) \right], \] (4.29)
\[ \Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\flat = \frac{1}{4\pi} \left[ \mu - \omega - \frac{m^2}{2\omega} - \frac{1}{2} \left( \frac{1}{6} R - V + \frac{2}{3} \Box \phi \right) \left( \frac{1}{\mu} - \frac{1}{\omega} \right) - \frac{m^2 \Box \phi}{6\mu^3} - \frac{m^4 (\nabla \phi)^2}{2\mu^5} \right]. \] (4.30)
Here and later we use the notation $\omega = e^{2\phi}$. The quantity $\Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\xi$ is that part of the anomaly which dominates at high frequency $\omega$; it consists of all terms of $O(\omega^{-1})$ and higher in the large-$\omega$ expansion of $\Delta \langle \hat{\Phi}^2 \rangle_{\omega}$. These are the terms which diverge in the integration over $\omega$, and hence which lead to the divergences in the four-dimensional Green function as $t - t' \to 0$. The part of the anomaly that remains when $\Delta \langle \hat{\Phi}^2 \rangle_{\omega}^\xi$ is subtracted...
off is denoted by $\Delta(\hat{\Phi}^2)^\flat_\omega$. It is of $O(\omega^{-1})$ for high frequencies, so the Fourier transform of $\Delta(\hat{\Phi}^2)^\flat_\omega$ is finite as $t - t' \to 0$.

It should be emphasized that, generally speaking, the procedure of determining the Fourier transform for the point-split divergent part of the Green function is not unique. The short distance Schwinger-DeWitt expansion guarantees us only the correct reproduction of terms which are singular in the coincidence limit $t - t' \to 0$. For this reason, only the high-frequency part $(\hat{\Phi}^2)^\sharp_\omega$ which is responsible for the short distance behavior of $G^F_{\text{div}}(t, x; t', x)$ in four dimensions is a universal function. A particular form of the low frequency part of the anomalous term, $(\hat{\Phi}^2)^\♭_\omega$, may depend on the concrete form of the continuation of $G^F_{\text{div}}(t, x; t', x)$ for large separation of points $t$ and $t'$. Our choice (4.13), though not unique, possesses a number of pleasant properties. First of all, in a flat space-time it is identical to the Green function of the scalar field for both the massive and massless cases. Moreover, the reduction anomaly obtained by using this prescription is directly related with a so-called analytic approximation for $(\hat{\Phi}^2)^F_{\text{ren}}$ derived in [9]. In order to demonstrate this let us take the inverse Fourier transform of $(\hat{\Phi}^2)^\♭_\omega$,

$$
(\hat{\Phi}^2)^F_{\text{approx}} = \int_0^\infty \frac{d\omega}{\pi} \Delta(\hat{\Phi}^2)^\flat_\omega.
$$

Performing the integration (see Appendix C.2), we obtain

$$
(\hat{\Phi}^2)^F_{\text{approx}} = \frac{m^2}{16\pi^2} + \frac{1}{16\pi^2} \left( \frac{1}{6} 4R - V - m^2 \right) \ln \frac{m^2 e^{-4\phi}}{4\eta^2} - \frac{\Box \phi}{24\pi^2} - \frac{(\nabla \phi)^2}{12\pi^2}.
$$

The parameter $\eta$ is a low-frequency cut-off which is required to make the integral convergent. It corresponds to a well-known ambiguity in the renormalization prescription. This ambiguity is absent for a conformally-invariant theory, when $m = 0$ and the parameter of the non-minimal coupling $\xi$ takes its conformal value $\xi = 1/6$.

For the special case of a static spherically symmetric spacetime this reproduces exactly the analytic approximation of Anderson [4]. Relation (4.32) also reproduces the zero-temperature Killing approximation [11] for a massless conformally-coupled field in a static spacetime. Relation (4.32) can be considered in fact as an extension of these results to the general case when the spacetime is static, but not necessary spherically symmetric, and the field equation includes an arbitrary mass and potential $V$. We shall discuss the origin of this approximation, its finite temperature generalization, and its properties elsewhere [17].

4.2 (2+2)-reduction

Let us discuss now the dimensional-reduction anomaly for $(\hat{\Phi}^2)^{\text{ren}}$ for the case when the metric of the internal space is flat and two-dimensional; that is, the spacetime metric is of the form $(A, B = 2, 3)$

$$
ds^2 = e^{-2\phi(x)}(dt_0^2 + dt_1^2) + h_{AB}(x)dx^A dx^B.
$$

First of all, it should be noticed that this metric is a special case of the metric (1.11) when the dilaton field does not depend on one of the coordinates $x^a$. Equation (1.33) can be

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4The symbol $\sharp$ (sharp) is borrowed from musical notation to denote the high-frequency part of the anomaly; the remainder is labelled with $\flat$ (flat).
obtained from (4.1) by rescaling the dilaton field $\phi \to \phi/2$ and putting
\[ h_{ab}dx^a dx^b = e^{-2\phi(x)} dt_1^2 + h_{AB}(x)dx^A dx^B. \] (4.34)

For the divergent part of the four-dimensional Green function expanded to first order in the curvature we have an expression similar to (4.13),
\[ G^F_{\text{div}}(t, x; t', x) = \frac{1}{8\pi^2} \left[ m^2K_1(z) + \frac{(\nabla \phi)^2}{48} m^4\tau^4K_2(z) + \frac{\Box \phi}{12} m^2\tau^2K_1(z) \right. \]
\[ \left. + \left( \frac{1}{6}R - V + \frac{2}{3} \Box \phi + \frac{1}{3}(\nabla \phi)^2 \right) K_0(z) \right]. \] (4.35)

See Appendix B. Here $R$ refers to the curvature of the two-dimensional space with metric $h_{AB}$, and we define
\[ z = m\sqrt{\tau^2 + \epsilon^2}, \quad \tau^2 = e^{-2\phi}t^2 \equiv e^{-2\phi}(t_0^2 + t_1^2). \] (4.36)

To obtain the mode decomposition of the divergent part of the Green function we make a Fourier transform similar to (4.17)
\[ G^F_{\text{div}}(x; x'|p) = e^{-(\phi + \phi')} \int_{-\infty}^{\infty} dt \cdot e^{ip(t-t')}G^F_{\text{div}}(t, x; t', x'), \] (4.37)
\[ G^F_{\text{div}}(t, x; t', x') = e^{(\phi + \phi')} \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} e^{-ip(t-t')}G^F_{\text{div}}(x; x'|p). \] (4.38)

Here we use the vector notations $p = (p_0, p_1)$ and $pt = p_0t_0 + p_1t_1$. We also denote $p^2 = p^2$. Since the function $G^F_{\text{div}}$ depends only on the difference $t - t'$, its Fourier transform depends only on $p$. Calculating the integrals we obtain
\[ G^F_{\text{div}}(x; x'|p) = \frac{1}{2\pi} \left[ -\left\{ \gamma + \frac{1}{2} \ln \left( \frac{\mu^2 \epsilon^2}{4} \right) \right\} + \frac{1}{\mu^2} \left( \frac{1}{6}R + \frac{2}{3} \Delta \phi - (\nabla \phi)^2 - V \right) \right. \]
\[ \left. + \frac{m^2}{6\mu^4} \Box \phi + \frac{m^4}{3\mu^6}(\nabla \phi)^2 \right] + O(\epsilon), \] (4.39)
where
\[ \mu = \sqrt{m^2 + \rho^2}, \quad \rho = e^\phi p. \] (4.40)

The adopted mode expansion into plane waves $\exp(\mathbf{ip}\mathbf{x})$ reduces the initial problem for the four-dimensional operator $F = \Box - V - m^2$ to two-dimensional problems with a dilaton-dependent potential. The corresponding wave operator $\mathcal{F}_p$ is
\[ \mathcal{F}_p = \Delta_h - V_p[\phi] - m^2, \] (4.41)
where
\[ V_p[\phi] = \rho^2 + (\nabla \phi)^2 - \Delta_h \phi + V. \] (4.42)

The divergent part of the two-dimensional Green function for the operator $\mathcal{F}_p$ can be obtained by using the Schwinger-DeWitt expansion for this operator. In two dimensions this divergent part is generated by only the first term in the Schwinger-DeWitt expansion of the heat kernel and is
\[ G^\mathcal{F}_p_{\text{div}}(x; x) = \int_0^\infty ds \frac{1}{4\pi s} \exp \left\{ -m^2s - \frac{\epsilon^2}{4s} \right\} = -\frac{1}{2\pi} \left\{ \gamma + \frac{1}{2} \ln \left( \frac{m^2 \epsilon^2}{4} \right) \right\} + O(\epsilon). \] (4.43)
We define the four- and two-dimensional renormalized values \(\langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^{F}\) and \(\langle \hat{\Phi}^2(x) \rangle_{\text{ren}}^{F_p}\) by expressions similar to (4.16) and (4.24) respectively. By comparing these definitions, and using relations (4.39) and (4.43) we obtain the representation

\[
\langle \hat{\Phi}^2 \rangle_{\text{ren}}^{F} = e^{2\phi} \int \frac{dP}{(2\pi)^2} \left[ \langle \hat{\Phi}^2 \rangle_{\text{ren}}^{F_p} + \Delta \langle \hat{\Phi}^2 \rangle_{\text{p}} \right],
\]

where the dimensional-reduction anomaly \(\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}}\) is

\[
\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}} = \frac{1}{4\pi} \left[ \ln \left( \frac{\mu^2}{m^2} \right) - \frac{1}{\mu^2} \left( \frac{1}{6} R + \frac{2}{3} \Delta \phi - (\nabla \phi)^2 - V \right) - \frac{m^2}{3\mu^4} \left( \Delta \phi - 2(\nabla \phi)^2 \right) - \frac{2m^4}{3\mu^6} (\nabla \phi)^2 \right] \quad (4.45)
\]

\[
= \frac{1}{4\pi} \left[ \ln \left( \frac{\mu^2}{m^2} \right) - \frac{1}{\mu^2} \left( \frac{1}{6} R - V \right) - \frac{m^2}{3\mu^4} \Delta \phi - \frac{2m^4}{3\mu^6} (\nabla \phi)^2 \right] . \quad (4.46)
\]

For convenience, we give here two different forms of the representation for \(\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}}\). In the first equality, (4.45), all the quantities such as the curvature, covariant derivatives, and so on are two-dimensional objects calculated for two-metric \(h_{AB}\). In the second equality, (4.46), the same expression is written in terms of four-dimensional objects defined for the four-metric (4.33).

The part of \(\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}}\) which dominates at large “momentum” \(p\) and which is responsible for the divergences of the four-dimensional Green function in the coincidence limit \(t - t' \to 0\) is

\[
\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}} = \frac{1}{4\pi} \left[ \ln \frac{\mu^2}{m^2} - \frac{1}{\mu^2} \left( \frac{1}{6} R - V - m^2 \right) \right]. \quad (4.47)
\]

Defining the sub-leading part \(\Delta \langle \hat{\Phi}^2 \rangle_{\text{p}}^{b}\) of the anomaly and \(\langle \hat{\Phi}^2 \rangle_{\text{approx}}^{F}\) by relations similar to (4.28) and (4.31), respectively, we obtain an expression for \(\langle \hat{\Phi}^2 \rangle_{\text{approx}}^{F}\) which is identical to (4.32). One can expect this result, since the \((2+2)\) reduction may be considered as a special case of the \((3+1)\) reduction.

5 The Dimensional-Reduction Anomaly for the Effective Action

5.1 \((1+3)\)-reduction

For the static spacetime (4.1), the calculation of the anomaly in the effective action \(W^{F}\)

\[
W^{F}[g] = \int dX \sqrt{g} L^{F} \quad (5.1)
\]

proceeds analogously to the calculation of the anomaly in \(\langle \hat{\Phi}^2 \rangle\). To analyse the divergent part of the effective action we introduce first a point-split version of \(L^{F}\). Using the Schwinger-DeWitt expansion for the heat kernel, we have for the divergent part of \(L^{F}\)

\[
L_{\text{div}}^{F}(t, x; t', x) = -\frac{1}{2} \int_{0}^{\infty} ds \frac{1}{(4\pi s)^{2}} \exp \left\{ -m^2 s - \frac{2\sigma + \epsilon^2}{4s} \right\} \left[ \mathfrak{R}_{0}^{\square - V} + s\mathfrak{R}_{1}^{\square - V} + s^2\mathfrak{R}_{2}^{\square - V} \right]. \quad (5.2)
\]
As earlier, the points are split in the \( t \) direction. Since the internal space is homogeneous, the point-split Lagrangian depends on \( t - t' \).

Faced with the same problem as before, we expand \( \sigma \) and the \( \Re_n^{\Box-V} \) in terms of \( \tau \equiv e^{-2\phi}(t - t') \) for \( x = x' \), this time truncating at second order in the curvature (four derivatives of the metric or dilaton). Writing

\[
2\sigma(t, x; t', x) = \tau^2 + u\tau^4 + v\tau^6 + \cdots ,
\]

\[
\Re_n^{\Box-V}(t, x; t', x) = \Re_n^{\Box-V}(0) + \Re_n^{\Box-V}(2) + \Re_n^{\Box-V}(4) + \cdots , \tag{5.3}
\]

we have \( \Re_0^{\Box-V}(0) = 1 \), while \( u, v \), and the other \( \Re_n^{\Box-V} \) may be found in Appendix A. Inserting and truncating at \( O(R^2) \) gives

\[
L_{\text{div}}^F(t, x; t', x) = -\frac{1}{(4\pi)^2} \left[ m^4 \mathcal{K}_2(z) - \frac{1}{4}(u\tau^4 + v\tau^6)m^6 \mathcal{K}_3(z) + \frac{1}{32} u^2 \tau^8 m^8 \mathcal{K}_4(z) \right. \\
+ \Re_{0(2)}^{\Box-V} \tau^2 m^4 \left( 2(z) - \frac{1}{4} u\tau^4 m^2 \mathcal{K}_3(z) \right) + \Re_{0(4)}^{\Box-V} \tau^4 m^4 \mathcal{K}_2(z) \\
+ \Re_{1(0)}^{\Box-V} m^2 \left( 1(z) - \frac{1}{4} u\tau^4 m^2 \mathcal{K}_2(z) \right) + \Re_{1(2)}^{\Box-V} \tau^2 m^2 \mathcal{K}_1(z) \\
+ \Re_{2(0)}^{\Box-V} \mathcal{K}_0(z) \right] . \tag{5.4}
\]

The function \( \mathcal{K}_\nu(z) \) is defined by \( (4.14) \).

We define the Fourier transform of \( L_{\text{div}}^F \) as in \( (4.17) \). Evaluating the transform as before yields

\[
L_{\text{div}}^F(x|\omega) = -\frac{1}{4\pi} \left\{ \frac{1}{\epsilon^3} + \frac{1}{2\epsilon} \left[ -3u + 2\Re_{0(2)}^{\Box-V} + \Re_{1(0)}^{\Box-V} - \mu^2 \right] \\
+ \frac{1}{3} \mu^3 - \frac{1}{2} \mu \left( -3u + 2\Re_{0(2)}^{\Box-V} + \Re_{1(0)}^{\Box-V} \right) \\
+ \frac{5u\omega^2}{2\mu} + \frac{m^2 u\omega^2}{2\mu} - \frac{15m^6 v}{2\mu^5} + \frac{105m^8 u^2}{8\mu^9} \\
+ \Re_{0(2)}^{\Box-V} \left[ -\frac{\omega^2}{\mu} - \frac{15m^6 u}{2\mu^7} \right] + \Re_{0(4)}^{\Box-V} \left[ \frac{3m^4}{\mu^5} \right] \\
+ \Re_{1(0)}^{\Box-V} \left[ -\frac{3m^4 u}{4\mu^5} \right] + \Re_{1(2)}^{\Box-V} \left[ \frac{m^2}{2\mu^3} \right] + \Re_{2(0)}^{\Box-V} \left[ \frac{1}{4\mu} \right] + O(\epsilon) \right\} . \tag{5.5}
\]

Meanwhile, for the three-dimensional theory with the field operator \( \mathcal{F}_\omega \) given by \( (4.20) \) we have

\[
L_{\text{div}}^{\mathcal{F}_\omega}(x) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{\frac{3}{2}}} \exp \left\{ -m^2 s - \frac{\epsilon^2}{4s} \right\} \left[ \Re_0^{\Delta_h-V_\omega} + s\Re_1^{\Delta_h-V_\omega} \right] \\
= -\frac{1}{4\pi} \left\{ \frac{1}{\epsilon^3} + \frac{1}{2\epsilon} \left( \frac{1}{6} R - V_\omega - m^2 \right) + \frac{m^3}{3} - \frac{m}{2} \left( \frac{1}{6} R - V_\omega \right) \right\} + O(\epsilon) . \tag{5.6}
\]

Here \( V_\omega \) is the effective potential of the three-dimensional system, and is given by \( (4.21) \).

The renormalized effective actions in four and three dimensions are obtained by subtracting from the exact effective action its divergent part, as given by \( (5.4) \) and \( (5.6) \).
respectively. By comparing these divergent parts using (4.17, 4.18) and (5.5) we can write

\[ L_{\text{approx}}^F(x) = e^{2\phi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ L_{\text{approx}}^F(x) + \Delta L_\omega(x) \right], \quad (5.7) \]

where \( \Delta L_\omega \) is the term representing the dimensional-reduction anomaly of the renormalized effective action,

\[
\Delta L_\omega = \frac{1}{4\pi} \left\{ \frac{1}{3} (\mu^3 - m^3) - \frac{1}{2} (\mu - m) \left( \frac{1}{6} R - V + (\nabla \phi)^2 + \Delta_h \phi \right) - \frac{1}{2} m\bar{\omega}^2 \\
+ \mu \left( \frac{5}{2} u - 3\mathcal{R}_{0(2)}^{\square-V} \right) + \frac{1}{\mu} \left( -2m^2u + m^2\mathcal{R}_{0(2)}^{\square-V} + \frac{1}{4} \mathcal{R}_{2(0)}^{\square-V} \right) \\
+ \frac{m^2}{\mu^3} \left( -\frac{m^2}{2} u + \frac{1}{2} \mathcal{R}_{1(2)}^{\square-V} \right) + \frac{m^4}{\mu^5} \left( 3\mathcal{R}_{2(0)}^{\square-V} - \frac{3}{4} 3u\mathcal{R}_{1(0)}^{\square-V} \right) \\
+ \frac{m^6}{\mu^7} \left( -\frac{15}{2} u - \frac{15}{2} u\mathcal{R}_{0(2)}^{\square-V} \right) + \frac{m^8}{\mu^9} \left( \frac{105}{8} u^2 \right) \right\}. \quad (5.8)
\]

As earlier, we write

\[
\Delta L_\omega = \Delta L_\omega^\sharp + \Delta L_\omega^\flat, \quad (5.9)
\]

where \( \Delta L_\omega^\flat \) is the part of the anomaly which dominates at high frequencies (\( \omega \to \infty \)),

\[
\Delta L_\omega^\flat = \frac{1}{4\pi} \left\{ \frac{1}{3} \omega^3 - \frac{1}{2} m\bar{\omega}^2 + \bar{\omega} \left[ \frac{m^2}{2} - \frac{1}{2} \left( \frac{1}{6} R - V - (\nabla \phi)^2 + \Delta_h \phi \right) + \frac{5u}{2} - 3\mathcal{R}_{0(2)}^{\square-V} \right] \\
+ \left[ \frac{m}{2} \left( \frac{1}{6} R - V - (\nabla \phi)^2 + \Delta_h \phi \right) - \frac{m^3}{3} \right] \\
+ \frac{1}{\omega} \left[ \frac{3m^4}{8} - \frac{m^2}{4} \left( \frac{1}{6} R - V - (\nabla \phi)^2 + \Delta_h \phi + 3u - 2\mathcal{R}_{0(2)}^{\square-V} \right) + \frac{1}{4} \mathcal{R}_{2(0)}^{\square-V} \right] \right\}. \quad (5.10)
\]

By subtracting this large-\( \omega \) limit from the anomaly (5.8) and making the inverse Fourier transform, we can construct an approximate effective Lagrangian for the four-dimensional theory:

\[
L_{\text{approx}}^F = \int_0^\infty \frac{d\omega}{\pi} \Delta L_\omega^\flat. \quad (5.11)
\]

Performing the \( \omega \)-integration (see Appendix C.2) gives

\[
L_{\text{approx}}^F = \frac{3m^4}{128\pi^2} + \frac{m^2}{32\pi^2} \left[ -\frac{1}{6} R + V + (\nabla \phi)^2 - \Delta_h \phi + u - 2\mathcal{R}_{0(2)}^{\square-V} \right] \\
+ \frac{1}{8\pi^2} \left[ \mathcal{R}_{1(2)}^{\square-V} + 4\mathcal{R}_{0(4)}^{\square-V} - u\mathcal{R}_{1(0)}^{\square-V} - 8u\mathcal{R}_{0(2)}^{\square-V} - 8v + 12u^2 \right] \\
+ \frac{1}{32\pi^2} \left[ -\frac{m^4}{2} + m^2 \left( \frac{1}{6} R - V - (\nabla \phi)^2 + \Delta_h \phi + 3u - 2\mathcal{R}_{0(2)}^{\square-V} \right) \\
- \mathcal{R}_{2(0)}^{\square-V} \right] \ln \left( \frac{m^2 e^{-4\phi}}{4\eta^2} \right). \quad (5.12)
\]

The effective action corresponding to this Lagrangian may be simplified considerably using integration by parts. Substituting for the \( u, v \), and \( \mathcal{R}_{n(k)}^{\square-V} \) from Appendix A and
neglecting surface terms, one can show that the effective action for the important case
\( V = \xi^4 R \) may be written as

\[
W^F_{\text{approx}} = \int d^4 x \sqrt{g} \left\{ -\frac{1}{64 \pi^2} \ln \left( \frac{m^2 e^{-4\phi}}{4\eta^2} \right) \left[ m^4 + \frac{1}{90} \left( 4R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + \Box^4 R \right) \right] + \frac{3m^4}{128\pi^2} - \frac{m^2 (\nabla \phi)^2}{24\pi^2} + \frac{1}{360\pi^2} \left[ 4R_{\alpha\beta\gamma\delta} \phi^{\alpha\beta\gamma\delta} - \frac{3}{2} (\Box \phi)^2 - 4\Box \phi (\nabla \phi)^2 - 4(\nabla \phi)^4 \right] \right\}
\]

\[
+ \left( \frac{\xi - \frac{1}{6}}{64 \pi^2} \right)^2 \int d^4 x \sqrt{g} \left\{ m^2 4R - \frac{4}{3} 4R (\nabla \phi)^2 + \ln \left( \frac{m^2 e^{-4\phi}}{4\eta^2} \right) \left[ -m^2 4R + \frac{1}{6} \Box^4 R \right] \right\}
\]

Note that (5.13) has been written in terms of four-dimensional quantities. It may be shown [17] that the stress-energy tensor resulting from the variation of (5.13) in the special case of a static, spherically symmetric spacetime coincides with the analytic approximation of Anderson et al [10] for the zero-temperature case. Furthermore, in the massless, conformally-coupled limit (5.13) coincides with the zero-temperature Killing approximation [11].

### 5.2 (2+2)-reduction

As we already mentioned, the (2+2)-reduction is a special case of the “static”-spacetime reduction. The calculation of the dimensional-reduction anomaly is very similar to the calculations of the previous subsection – straightforward but quite involved. We do not reproduce the details of these calculations here but simply give the final results.

The mode decomposition of the renormalized effective Lagrangian for the operator
\( F = \Box - V - m^2 \) has the form

\[
L^F_{\text{ren}} = e^{2\phi} \int \frac{dP}{(2\pi)^2} \left[ L^F_{\text{ren}} + \Delta L_P \right],
\]

where the dimensional-reduction anomaly \( \Delta L_P \) is

\[
\Delta L_P = \frac{1}{8\pi} \left\{ -\bar{p}^2 + \left( \frac{1}{6} R - V + \Delta \phi - (\nabla \phi)^2 - m^2 - \bar{p}^2 \right) \ln \left( \frac{m^2}{\mu^2} \right) \right. \\
+ \left( 12u - 4\Re^{\Box-V}_{0(2)} \right) + \frac{1}{\mu^2} \left( \Re^{\Box-V}_{2(0)} - 8m^2 u + 4m^2 \Re^{\Box-V}_{0(2)} \right) \\
+ \frac{m^2}{\mu^4} \left( -4m^2 u + 4\Re^{\Box-V}_{1(2)} \right) + \frac{m^4}{\mu^6} \left( -8u \Re^{\Box-V}_{1(0)} + 32 \Re^{\Box-V}_{0(4)} \right) \\
+ \frac{m^6}{\mu^8} \left( -96v - 96u \Re^{\Box-V}_{0(2)} \right) + \frac{m^8}{\mu^{10}} \left( 192u^2 \right) \right\}.
\]

Recall that \( \bar{p} \) and \( \mu \) are given by relation (4.40). By subtracting the high-frequency part and taking the inverse Fourier transform we obtain a result for \( W^F_{\text{approx}} \) which identically coincides with (5.13).
6 Conclusion

In the presence of a continuous spacetime symmetry, when the field equation can be solved by decomposition of the field into harmonics, one can easily obtain similar mode-decomposed expressions for the field fluctuations and the effective action. Each of the terms in this decomposition coincides with the corresponding object (the fluctuations or the effective action) of the lower-dimensional theory obtained by the reduction. We have demonstrated that because of the ultraviolet divergences these decompositions have only formal meaning, as the renormalization violates the exact form of such representations. As a result, the expression for the renormalized expectation value of the object in the “physical” spacetime can be obtained by summing the contributions of corresponding lower-dimensional quantities only if additional anomalous terms are added to each of the modes. We call this effect the dimensional-reduction anomaly.

In the general case, there can be different origins of the dimensional-reduction anomaly. In particular, such an anomaly may arise when the lower-dimensional manifold which arises as the result of the dimensional reduction has a non-trivial topology or has boundaries. We demonstrated how this kind of situation arises in the simple examples of the reduction of four-dimensional flat spacetime in spherical modes (Section 3.1) and in Rindler time modes (Section 3.2). In addition to these “global” contributions to the dimensional-reduction anomaly, there also exist “local” contributions. The corresponding anomalous terms are local invariants constructed from the curvature, the dilaton field, and their covariant derivatives. Our main objective was the study of such “local” dimensional-reduction anomalies. We derived the expressions for such anomalies for (3 + 1)- and (2 + 2)- spacetime reductions with a flat internal space. The remarkable fact that was discovered in this analysis is that the calculated dimensional-reduction anomaly is closely related to the analytical approximation in static spacetimes developed in [9, 10]. We shall investigate this intriguing relationship further in a future publication [17].

The dimensional-reduction anomaly discussed in the present paper might have many interesting applications. For example, it may be important for calculations of the contribution of individual modes with a fixed angular momentum to the stress-energy tensor and the flux of Hawking radiation in the spacetime of a black hole. In order to obtain the contribution of each mode in the presence of the dimensional-reduction anomaly, the results of two-dimensional calculations must be modified by terms produced by the anomaly. The calculated dimensional-reduction anomaly for the effective action, or the corresponding result for the spherical reduction, allows one to find directly the difference between the contribution of a given mode to the stress-energy tensor, and the stress-energy tensor of the corresponding lower-dimensional reduced theory. This may also be important for the discussion of self-consistent solutions which incorporate the back-reaction of the quantum fields. We hope to return to the discussion of this and other related topics in subsequent publications.

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A Small-Distance Expansions for the (1+3) Reduction of Static Space

The line element for a static space may be written in the form

$$ds^2 = e^{-4\phi(x)} dt^2 + h_{ab}(x) dx^a dx^b.$$  \hfill (A.1)

For this static metric, $h_{ab}$ is an induced 3-metric, and $n^a = e^{2\phi(x)} \delta^a_t$ is a unit vector normal to the surface $t = \text{const}$. The extrinsic curvature $K_{ab}$ on $t = \text{const}$ surfaces vanishes. The nonvanishing Christoffel symbols are

$$4\Gamma^a_{bc}[g] = \Gamma^a_{bc}[h], \quad 4\Gamma^a_{00} = 2e^{-4\phi} \phi^a, \quad 4\Gamma^0_{0a} = -2\phi_{,a}.$$  \hfill (A.2)

Because we will be using some quantities defined in terms of the full four-dimensional metric $g$ and others in terms of the three-dimensional metric $h$, some conventions on notation are in order. Henceforth four-dimensional curvatures and covariant derivatives will be denoted by $^4R_{\alpha\beta\gamma\delta}$ and $(\cdot)_a$ respectively, while $\Box$ is understood to represent the d’Alembertian with respect to $g$. All other curvatures and covariant derivatives are understood to be calculated using the three-metric $h$. In particular, $\nabla$ and $(\cdot)_a$ are three-dimensional covariant derivatives, and $\Delta = \Delta_h$ is the three-dimensional d’Alembertian. For the dilaton $\phi$ we shall understand $\phi_a$, $\phi_{ab}$, etc. to denote multiple three-dimensional covariant derivatives of $\phi$.

With these conventions we have, for example,

$$\Box \phi = \Delta \phi - 2(\nabla \phi)^2.$$  \hfill (A.3)

It is convenient to define the following three-dimensional tensor which occurs naturally in the $3 + 1$ reduction:

$$T_{ab} = 2 [\phi_{ab} - 2\phi_a \phi_b], \quad T = T_a^a = 2 \left[\Delta \phi - 2(\nabla \phi)^2\right] = 2\Box \phi.$$  \hfill (A.4)

(A.5)

In terms of $T_{ab}$ the only nonvanishing components of the four-dimensional curvatures are

$$^4R_{abcd} = R_{abcd}, \quad ^4R_{0\alpha\beta\delta} = e^{-4\phi} T_{ab},$$

$$^4R_{ab} = R_{ab} + T_{ab}, \quad ^4R_{00} = e^{-4\phi} T, \quad ^4R = R + 2T.$$  \hfill (A.6)

(A.7)

We shall also need the following expressions for $^4R_{\alpha\beta\gamma\delta}, ^4R_{\alpha\beta\gamma}, ^4R_{\alpha\beta}$, and $^4R_{\alpha\beta\gamma}$:

$$^4R_{ab,\cdot c} = (R_{ab} + T_{ab})_{,c},$$

$$^4R_{ab,00} = -2e^{-4\phi} \left[ (R_{ab} + T_{ab}) \phi^b - T \phi_a \right],$$

$$^4R_{00,\cdot c} = e^{-4\phi} T_{,c},$$

$$^4R_{ab,\cdot cd} = (R_{ab} + T_{ab})_{,cd},$$

$$^4R_{ab,00} = -2e^{-4\phi} \left[ (R_{ab} + T_{ab}) \phi^c - 4\phi_a \phi_b T + 2\phi_a (R_{bc} + T_{bc}) \phi^c + 2\phi_b (R_{ac} + T_{ac}) \phi^c \right],$$

$$^4R_{ab,0c} = -2e^{-4\phi} \left[ (R_{ab} + T_{ab}) \phi^b + 2\phi_b (R_{ab} + T_{ab}) \phi_c - 2\phi_a \phi_c T - \phi_a T_{,c} \right],$$

$$^4R_{ab,0\cdot c} = -2e^{-4\phi} \left[ (R_{ab} + T_{ab}) \phi^b + 2\phi_b (R_{ab} + T_{ab}) \phi_c - 2\phi_a \phi_c T - \phi_a T_{,c} \right],$$

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\[ 4R_{ab;cd} = -2e^{-4\phi} \left[(R_{ab} + T_{ab}) \phi^b - T\phi_a\right]_{|d}, \]
\[ 4R_{00;cd} = e^{-4\phi} T_{0cd}, \]
\[ 4R_{00;00} = 2e^{-8\phi} \left[4(R_{ab} + T_{ab}) \phi^a \phi^b - 4T \phi^a \phi_a - T_{|a} \phi^a\right], \] (A.9)
\[ 4R_{a} = (R + 2T)_{|a}, \]
\[ 4R_{ab} = (R + 2T)_{|ab}, \]
\[ 4R_{00} = -2e^{-4\phi} (R + 2T)_{|a} \phi^a. \] (A.10)

For the mode decomposition (Fourier time transform) we need to know the behaviour of the two-point functions \( \sigma \) and \( \Delta^{1/2}a_n^{-V} \) for points \( x^\alpha = (t, x) \) and \( x'^\alpha = (t', x) \) where \( \tau \equiv e^{-2\phi}(t - t') \) is small. For \( \sigma \) one can easily show that

\[
2\sigma(t, x; t', x) = \tau^2 - \frac{1}{3} \phi_a \phi_a \tau^4 + \frac{1}{45} \left[8(\phi^a \phi_a)^2 - 3\phi^a \phi^b \phi_{ab}\right] \tau^6 + \cdots , \\
\sigma_a(t, x; t', x) = -\phi_a \tau^2 + \frac{1}{6} \left[\phi^b \phi_b \phi_a - \phi^b \phi_{ba}\right] \tau^4 + \cdots , \] (A.11)
\[
\sigma_t(t, x; t', x) = e^{-2\phi} \left[1 - \frac{2}{3} \phi^a \phi_a \tau^2 + \frac{1}{15} \left[8(\phi^a \phi_a)^2 - 3\phi^a \phi^b \phi_{ab}\right] \tau^4 + \cdots \right] .
\]

Combining the above expressions with the results of [18, 19] for small-\( \sigma^a \) expansions of the \( \Delta^{1/2}a_n^{-V} \), it is easily shown that for the operator \( \Box - V \), where the potential function \( V \) is independent of \( t \), the first three \( \Delta^{1/2}a_n^{-V} \) are

\[ \Delta^{1/2}a_0^{-V} = 1 + \frac{1}{12} T \tau^2 + \frac{1}{360} \left[6(R_{ab} + T_{ab}) \phi^a \phi^b - 16T \phi^a \phi_a + 6T_{|a} \phi^a \right. \\
+ \left. \frac{5}{4}T^2 + T^{ab}T_{ab}\right] \tau^4 + O(\tau^6), \]
\[ \Delta^{1/2}a_1^{-V} = \left(\frac{1}{6}R - V + \frac{1}{3}T\right) + \left\{\frac{1}{2} \left(\frac{1}{6}R - V + \frac{1}{3}T\right)_{|a} \phi^a + \left(\frac{1}{3}V_{|a} \phi^a - \frac{1}{12}VT\right)_{|a}\right\} \tau^2 \\
+ \frac{1}{360} \left[3 \Delta T + 6T^2 + 6T^{ab}T_{ab} + 5RT + 2R^{ab}T_{ab} + 24(R_{ab} + T_{ab}) \phi^a \phi^b \right. \\
- 24T \phi^a \phi_a - 18R_{|a} \phi^a - 42T_{|a} \phi^a\right\} \tau^2 + O(\tau^4), \] (A.12)
\[ \Delta^{1/2}a_2^{-V} = \frac{1}{2} \left(\frac{1}{6}R - V + \frac{1}{3}T\right)^2 + \frac{1}{30} \left[\Delta(R + 2T) - 2(R + 2T)_{|a} \phi^a\right] - \frac{1}{6} \left[\Delta V - 2V_{|a} \phi^a\right] \\
+ \frac{1}{180} \left[R_{abcd}R^{abcd} + 4T^{ab}T_{ab} - \left(R^{ab} + T^{ab}\right)(R_{ab} + T_{ab}) - T^2\right] + O(\tau^2). \]

**B Small-Distance Expansions for the (2+2) Reduction**

Consider a spacetime with the line element (4.33). In this case we will be using some quantities defined in terms of the full four-dimensional metric \( g_{\mu\nu} \), and others in terms of \( \phi^a \) and \( \phi_{ab} \) as in (A.9). The Van Vleck determinant \( \Delta [4.7] \) is not to be confused with our notation for the three-dimensional covariant derivative.
the two-dimensional metric $h_{AB}$ ($A,B = 2,3$). In analogy to the $(3 + 1)$-splitting case, four-dimensional curvatures and covariant derivatives will be denoted by $^4R\cdots$ and $(\cdots)_A$ respectively, while $\Box$ is understood to represent the d’Alembertian with respect to $g_{\mu\nu}$.

All other curvatures and covariant derivatives are understood to be calculated using the two-metric $h_{AB}$. In particular, $\nabla$ and $(\cdots)_A$ are two-dimensional covariant derivatives, and $\Delta = \Delta_h$ is the two-dimensional d’Alembertian. For the dilaton $\phi$ we shall understand $\phi_A$, $\phi_{AB}$, etc. to denote multiple two-dimensional covariant derivatives of $\phi$. For example, with these conventions, the four-dimensional d’Alembertian of a $y$-independent scalar $S$ decomposes to

$$\Box S = \Delta S - 2\nabla\phi \cdot \nabla S .$$  \hspace{1cm} (B.1)

In particular,

$$\Box \phi = \Delta \phi - 2(\nabla\phi)^2 = -\frac{1}{2}e^{2\phi}\Delta e^{-2\phi} .$$  \hspace{1cm} (B.2)

For the given line element, the nonvanishing Christoffel symbols are ($i,j = 0,1$)

$$\Gamma^A_{BC} = g^{AD}\Gamma_{DBC} = \frac{1}{2}h^{AD}(h_{DB,C} + h_{CD,B} - h_{BC,D}) ,$$

$$\Gamma^i_{ij} = \phi^A e^{-2\phi}\eta_{ij} = \phi^A g_{ij} ,$$

$$\Gamma^i_{jA} = -\phi_A e^{\phi}g^i_j ,$$

$$\Gamma_{Aij} = -\Gamma_{ijA} = \phi_A e^{-2\phi}\eta_{ij} = \phi_A g_{ij} .$$  \hspace{1cm} (B.3)

Meanwhile, the only nonvanishing components of the four-dimensional curvatures are

$$^4R_{ABCD} = \frac{1}{2}R(g_{AC}g_{BD} - g_{AD}g_{BC}) ,$$

$$^4R_{AmBn} = g_{mn}[\phi_{AB} - \phi_A\phi_B] ,$$

$$^4R_{ijkm} = -\phi_A \phi^A(g_{ik}g_{jm} - g_{im}g_{jk}) .$$  \hspace{1cm} (B.4)

$$^4R_{AB} = \frac{1}{2}Rg_{AB} + 2[\phi_{AB} - \phi_A\phi_B] ,$$

$$^4R_{mn} = g_{mn}[\phi^A_A - 2\phi^A\phi_A] ,$$

$$^4R = R + 4\phi^A_A - 6\phi^A\phi_A .$$  \hspace{1cm} (B.5)

For comparison, note that

$$^2R_{ABCD} = \frac{1}{2}R(g_{AC}g_{BD} - g_{AD}g_{BC}) ,$$

$$^2R_{AB} = \frac{1}{2}Rg_{AB} , \quad ^2R = R .$$  \hspace{1cm} (B.7)

We shall also need the following components of $^4R_{\alpha\beta;\gamma}$, $^4R_{\alpha\beta;\gamma\delta}$, $^4R_{\alpha;\gamma}$, and $^4R_{\alpha\beta}$:

$$^4R_{AB;C} = \frac{1}{2}g_{AB}R_C + 2[\phi_{AB} - \phi_A\phi_B]g_C ,$$

$$^4R_{Am;n} = g_{mn}[\frac{1}{2}R\phi_A + \phi_A\Delta\phi - 2\phi^B\phi_{BA}] ,$$

$$^4R_{mn;A} = g_{mn}[\Delta\phi - 2(\nabla\phi)^2]g_A .$$  \hspace{1cm} (B.8)
\[
4R_{mn;AB} = g_{mn} [\Delta \phi - 2(\nabla \phi)^2]_{AB},
\]
\[
4R_{ij;km} = (g_{ik}g_{jm} + g_{im}g_{jk}) \left[ \frac{1}{2} R(\nabla \phi)^2 - (\nabla \phi)^2 \Delta \phi + 2\phi^A \phi^B \phi_{AB} \right] - (g_{ij}g_{km}) [\Delta \phi - 2(\nabla \phi)^2]_{A} \phi^A ,
\] (B.9)
\[
4R_{;A} = \left[ R + 4\Delta \phi - 6(\nabla \phi)^2 \right]_{;A},
\] (B.10)
\[
4R_{AB} = \left[ R + 4\Delta \phi - 6(\nabla \phi)^2 \right]_{AB},
\]
\[
4R_{mn} = -g_{mn} [R + 4\Delta \phi - 6(\nabla \phi)^2]_{A} \phi^A .
\] (B.11)

Note again that operators and curvatures are with respect to the two-dimensional metric \( h_{AB} \) unless explicitly labelled otherwise.

We now write out the expansions of \( \sigma(x, x; t - t') \) and the \( \Delta^{1/2} a_n(x, x; t - t') \) for small separations. Defining \( \tau^i \equiv e^{-\phi}(t^i - t'^i) \), we have
\[
2\sigma(x, x; \tau) = \tau^2 - \frac{1}{12} (\nabla \phi)^2 \tau^4 + \frac{1}{360} \left[ 4(\nabla \phi)^4 - 3\phi^A \phi^B \phi_{AB} \right] \tau^6 + \ldots ,
\]
\[
\sigma^i(x, x; \tau) = e^\phi \tau^i \left[ 1 - \frac{1}{6} \phi^A \phi^A \tau^2 + \frac{1}{120} \left[ 4(\phi^A \phi^A)^2 - 3\phi^A \phi^B \phi_{AB} \right] \tau^4 + \ldots \right] ,
\]
\[
\sigma^a(x, x; \tau) = -\frac{1}{2} \phi^A \tau^2 - \frac{1}{24} \phi_B \left[ \phi^{BA} - 2\phi^B \phi^A \right] \tau^4 + \frac{1}{720} \left[ -12(\phi^B \phi_B)^2 \phi^A + 8\phi^B \phi_B \phi_C \phi^C \phi_{BC} + 9\phi^A \phi^B \phi^C \phi_{BC} - 3\phi^B \phi_{BC} \phi^C \phi_{BC} - 3\phi^B \phi_{BC} \phi^C \phi_{BC} \right] \tau^6 + \ldots .
\] (B.12)

Combining these expressions with the results of [13, 19], it is easily shown that for the operator \( \Box - V \), where the potential function \( V \) is independent of \( t' \), the first three \( \Delta^{1/2} a_n^{\Box - V} \) are
\[
\Delta^{1/2} a_0^{\Box - V} = 1 + \frac{1}{12} \left[ \Delta \phi - 2(\nabla \phi)^2 \right] \tau^2 + \frac{1}{1440} \left[ 3R(\nabla \phi)^2 + 48(\nabla \phi)^4 - 36(\nabla \phi)^2 \Delta \phi - 44\phi^A \phi^B \phi_{AB} + 5(\Delta \phi)^2 + 4\phi^A \phi^B \phi_{AB} + 12\phi^A (\Delta \phi)_A \right] \tau^4 + O(\tau^6) ,
\]
\[
\Delta^{1/2} a_1^{\Box - V} = \left[ \frac{1}{6} R - V + \frac{2}{3} \phi^A - \phi^A \phi_A \right]
\]
\[
+ \frac{1}{2} \left[ -\frac{1}{6} V_A \phi^A - \frac{1}{6} V [\Delta \phi - 2(\nabla \phi)^2] + \frac{1}{30} R \phi^A + R \left[ \frac{1}{30} \Delta \phi - \frac{2}{45} (\nabla \phi)^2 \right] \right]
\]
\[
+ \frac{1}{180} \left[ 60(\nabla \phi)^4 - 62(\nabla \phi)^2 \Delta \phi - 52\phi^A \phi^B \phi_{AB} + 16(\Delta \phi)^2 - 4\phi^A \phi_{AB} + 18\phi^A (\Delta \phi)_A - 12\phi^A (\Delta \phi)_A + 3\Delta^2 \phi \right] \tau^2 + O(\tau^4) ,
\] (B.13)
\[
\Delta^{1/2} a_2^{\Box - V} = \frac{1}{2} \left[ \frac{1}{6} R - V + \frac{2}{3} \phi^A - \phi^A \phi_A \right]^2 - \frac{1}{6} \Delta V + \frac{1}{3} V_A \phi^A + \frac{1}{30} \Delta R - \frac{1}{15} R_A \phi^A
\]
\[
+ \frac{1}{180} \left[ \frac{1}{2} R^2 - 2R [\Delta \phi - (\nabla \phi)^2] + 8(\nabla \phi)^2 \Delta \phi + 136\phi^A \phi^B \phi_{AB} - 2(\Delta \phi)^2 \right]
\]
\[ -68\phi^{AB}\phi_{AB} - 72\phi^A\Delta(\phi_A) - 48\phi^A(\Delta\phi)_A + 24\Delta^2\phi \right] + O(\tau^2). \]

C Useful Formulae

C.1 The Modified Bessel Function \( K_\nu \)

The modified Bessel functions \( K_\nu(z) \) may be defined via the integral

\[ \int_0^\infty dx x^{-1-\nu} \exp \left\{ -x - \frac{z^2}{4x} \right\} = 2 \left( \frac{2}{z} \right)^\nu K_\nu(z) \] \quad (C.1)

It may be shown that \( K_{-\nu}(z) = K_\nu(z) \). Furthermore, for \( \nu > 0 \) the \( K_\nu(z) \) obey the differential relation

\[ \left( -1 \frac{d}{dz} \right)^n z^\nu K_\nu(z) = z^{\nu-n} K_{\nu-n}(z) \] \quad (C.2)

In particular, for \( z = m\sqrt{\tau^2 + \epsilon^2} \) one can easily show that

\[ \frac{1}{z^n} K_n(z) = \left( -\frac{1}{m^2\epsilon} \frac{d}{d\epsilon} \right)^n K_0(z) \] \quad (C.3)

Combining (C.3) with integral (6.677) of \[16\],

\[ \int_{-\infty}^{\infty} d\tau \cos(\bar{\omega}\tau) K_0(m\sqrt{\epsilon^2 + \tau^2}) = \frac{\pi}{\sqrt{m^2 + \bar{\omega}^2}} \exp(-\epsilon\sqrt{m^2 + \bar{\omega}^2}), \] \quad (C.4)

allows us to evaluate the (1+3)-splitting Fourier transforms of Sections 4.1, 5.1 as follows:

\[ \int_{-\infty}^{\infty} d\tau \cos(\bar{\omega}\tau) \tau^{2k} \frac{1}{(m\sqrt{\tau^2 + \epsilon^2})^n} K_n(m\sqrt{\epsilon^2 + \tau^2}) = \]

\[ = (-1)^k \left( \frac{d^{(2k)}}{d\bar{\omega}^{(2k)}} \right) \left( -\frac{1}{m^2\epsilon} \frac{d}{d\epsilon} \right)^n \frac{\pi}{\sqrt{m^2 + \bar{\omega}^2}} \exp(-\epsilon\sqrt{m^2 + \bar{\omega}^2}). \] \quad (C.5)

For convenience, we have used the notation \( \bar{\omega} = e^{2\phi} \) introduced in Section 4.1.

C.2 Integrals of \( \mu^n \) for the (1+3) Reduction

For \( \mu = \sqrt{m^2 + x^2} \) it is easily shown that for large \( \bar{\omega} \)

\[ \int_0^\infty dx \mu^3 = \frac{1}{4} \bar{\omega}^4 + \frac{3}{4} m^2 \bar{\omega}^2 + \frac{9}{32} m^4 + \frac{3}{8} m^4 \ln \frac{2\bar{\omega}}{m}, \]

\[ \int_0^\infty dx \mu = \frac{1}{2} \bar{\omega}^2 + \frac{1}{4} m^2 + \frac{1}{2} m^2 \ln \frac{2\bar{\omega}}{m}, \] \quad (C.6)

\[ \int_0^\infty dx \frac{1}{\mu} = \ln \frac{2\bar{\omega}}{m}. \]

In addition, for \( n \geq 1 \),

\[ \int_0^\infty dx \mu^{-(2n+1)} = \frac{1}{m^{2n}} \frac{2^{n-1} (n-1)!}{(2n-1)!!}. \] \quad (C.7)

See, for example, (2.271) of \[10\]. These results are sufficient to perform the sum over modes in \( \{4.31, 5.11\} \).
C.3 Formulae for the \((2+2)\) Reduction

The Fourier transforms of Sections 4.2, 5.2 were computed before performing the \(s\)-integration by expanding all \(z\)-dependent quantities for small curvatures and using

\[
\int_{-\infty}^{\infty} d^2z \exp \left\{ -\frac{1}{4s} z^2 + i\bar{p}z \right\} z^{2n} = (4\pi s) e^{-\bar{p}^2 s} (4s)^n I_n \, ,
\]

where \(\bar{p} = e^{\phi}p\) and

\[
\begin{align*}
I_0 & = 1 \\
I_1 & = 1 - \bar{p}^2 s \\
I_2 & = 2 - 4\bar{p}^2 s + \bar{p}^4 s^2 \\
I_3 & = 6 - 18\bar{p}^2 s + 9\bar{p}^4 s^2 - \bar{p}^6 s^3 \\
I_4 & = 24 - 96\bar{p}^2 s + 72\bar{p}^4 s^2 - 16\bar{p}^6 s^3 + \bar{p}^8 s^4 \, .
\end{align*}
\]

For the summation over modes referred to in Sections 4.2, 5.2 one may use the integrals

\[
\begin{align*}
\int_0^\bar{p} dx x \ln \frac{\mu^2}{m^2} & = \frac{1}{2} (\bar{p}^2 + m^2) \ln \frac{\bar{p}^2 + m^2}{m^2} - \frac{1}{2} \bar{p}^2 , \\
\int_0^\bar{p} dx x \mu^{-2} & = \frac{1}{2} \ln \frac{\bar{p}^2 + m^2}{m^2} ,
\end{align*}
\]

and for \(n > 1\),

\[
\int_0^{\infty} dx x \mu^{-2n} = \frac{1}{2(n-1)m^{2(n-1)}}
\]

where \(\mu = \sqrt{m^2 + x^2}\).

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