Rough differential equations driven by signals in Besov spaces

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Abstract

Rough differential equations are solved for signals in general Besov spaces unifying in particular the known results in Hölder and $p$-variation topology. To this end the paracontrolled distribution approach, which has been introduced by Gubinelli, Imkeller and Perkowski [24] to analyze singular stochastic PDEs, is extended from Hölder to Besov spaces. As an application we solve stochastic differential equations driven by random functions in Besov spaces and Gaussian processes in a pathwise sense.

Key words: Besov regularity, Itô map, Paradifferential calculus, Rough differential equation, Geometric Besov rough path, Stochastic differential equation, Young integration.

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1 Introduction

Differential equations belong to the most fundamental objects in numerous areas of mathematics gaining extra interest from their various fields of applications. A very important sub-class of classical ordinary differential equations (ODEs) are controlled ODEs, whose dynamics are given by

$$du(t) = F(u(t))\xi(t), \quad u(0) = u_0, \quad t \in \mathbb{R},$$

where $u_0 \in \mathbb{R}^m$ is the initial condition, $u: \mathbb{R} \to \mathbb{R}^m$ is a continuous function, $d$ denotes the differential operator and $F: \mathbb{R}^m \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a family of vector fields on $\mathbb{R}^m$. In such a dynamic $\xi: \mathbb{R} \to \mathbb{R}^n$ typically models the input signal and $u$ the output.

If the signal $\xi$ is very irregular, for instance if $\xi$ has the regularity of white noise, equation (1) is called rough differential equation (RDE). Starting with the seminal paper by Lyons [37], the theory of rough paths has been developed to solve and analyze rough differential equations over the last two decades. A significant insight due to Lyons [37] was that the driving signal $\xi$ must be enhanced to a "rough path" in some sense, in order to solve the RDE (1) and to restore the continuity of the Itô map defined by $\xi \mapsto u$ in a $p$-variation topology, cf. [36, 38, 19]. In particular, the rough path framework allows for treating important examples as stochastic differential equations in a non-probabilistic setting. Parallel to the $p$-variation results, rough differential equations have been analyzed in the Hölder topology with similar tools, cf. [20, 14].

One core goal of this article is to unify the approach via the $p$-variation and the one via the Hölder topology in a common framework. To this end, we deal with rough differential equations on the very large and flexible class of Besov spaces $B^\alpha_{p,q}$, noting that, loosely speaking, the space of $\alpha$-Hölder regular functions is given by the Besov space $B^\alpha_{\infty,\infty}$ and that the $p$-variation scale corresponds to $B^{1/p}_{p,\infty}$ (see [4]). The results by Zähle [46, 47, 48], who set up integration for functions in Sobolev–Slobodeckij spaces via fractional calculus, are covered by our results as well. In fact, Besov spaces unify numerous function spaces, including also Sobolev spaces and Bessel-potential

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spaces, for a comprehensive monograph we refer to Triebel [43]. Furthermore, different types of Besov spaces and Besov embeddings already appear naturally in various applications of rough path theory. Let us mention, for instance, their use to derive large deviation principles [32, 31], a non-Markovian Hörmander theory for RDEs [18] and certain embedding results in the context rough path [23, 17].

Due to this generality, studying solutions to the RDE (1) on Besov spaces is a highly interesting, but challenging problem. In a first step, provided the driving signal $\xi$ is in $B^{\alpha-1}_{p,q}$ for $\alpha > 1/2$, $p \geq 2$, $q \geq 1$, the existence and uniqueness of a solution $u$ to the RDE (1) is proven, see Theorem 3.2, and further it is shown that the corresponding Itô map is locally Lipschitz continuous with respect to the Besov topology, see Theorem 5.8. In particular, with these results we recover the classical Young integration [45] on Besov spaces.

In order to handle a more irregular driving signal $\xi$ in $B^{\alpha-1}_{p,q}$ for $\alpha > 1/3$, $p \geq 3$, $q \geq 1$, the path itself has to be enhanced with an additional information, say $\pi(\vartheta, \xi)$, which always exists for a smooth path $\xi$ and corresponds to the first iterated integral in rough path theory. In the spirit of the usual notion of geometric rough path, this leads naturally to the new definition of the space of geometric Besov rough paths $B^\alpha_{p,q}$, cf. Definition 5.1. Starting with a smooth path $\xi$, it is shown that the Itô map associated to the RDE (1) extends continuously to the space of geometric Besov rough path, cf. Theorem 5.10. As a consequence there exists a unique pathwise solution to the RDE (1) driven by a geometric Besov rough path. Note that due to $\alpha > 1/p$ our results are restricted to continuous solutions, which seems to appear rather naturally, see Remark 5.12 for a discussion. Especially, for signals which are not self-similar like Brownian motion but whose regularity is determined by rare singularities, we can profit from measuring regularity in general Besov norms.

The immediate and highly non-trivial problem appearing in equation (1) is that the product $F(u)\xi$ is not well-defined for very irregular signals. While classical approaches as rough path theory formally integrate equation (1) and then give the appearing integral a meaning, the first step of our analysis is to give a direct meaning to the product in (1). Our analysis relies on the notion of paracontrolled distributions, very recently introduced by Gubinelli et al. [24] on the Hölder spaces $B^\alpha_{\infty, \infty}$. Their key insight is that by applying Bony’s decomposition to $F(u)\xi$ the appearing resonant term can be reduced to the resonant term $\pi(\vartheta, \xi)$ of $\xi$ and its antiderivative $\vartheta$, using a controlled ansatz to the solution $u$. The resonant term $\pi(\vartheta, \xi)$ turns out to be the necessary additional information to show the existence of a pathwise solution and corresponds to the first iterated integral in rough path theory as already mentioned above.

Generalizing the approach from [24] to Besov spaces poses severe additional problems, which are solved by using the Besov space characterizations via Littlewood-Paley blocks as well as the one via the modulus of continuity. Besov spaces are a Banach algebra if and only if $p = q = \infty$ such that in general our results can only rely on pointwise multiplier theorems, Bony’s decomposition and Besov embeddings. We thus need to generalize certain results in [2] and [24], including the commutator lemma, see Lemma 4.4. A second difficulty is that $u \in B^\alpha_{p,q}$ imposes an $L^p$-integrability condition on $u$. To overcome this problem, we localize the signal and consider a weighted Itô(-Lyons) map, both done in a way that does not change the dynamic of the RDE on a compact interval around the origin.

The paracontrolled distribution approach [24] offers an extension of rough path theory to a multiparameter setting as also done by the innovative theory of regularity structures developed by Hairer [25]. While Hairer’s theory presumably has a much wider range of applicability, both successfully give a meaning to many stochastic partial differential equations (PDEs) like the KPZ equation [26, 27] and the dynamical $\Phi^4_3$ equation [24, 28] just to name two. Even if the approach of Gubinelli et al. [24] may not be a systematic theory as regularity structures, it comprises some advantages. The approach works with already well-studied tools like Bony’s paraproduct and Littlewood-Paley theory, which leads to globally defined objects rather than the locally operating “jets” appearing in the theory of regularity structures. Since for stochastic PDEs the question about the “most suitable” function spaces seems not to be settled yet, it might be quite promising on its own to extend [24] to a more general foundation as we do by working with general Besov spaces. For instance, let us refer to the very recent work of Hairer and Labbé [25], where the
theory of regularity structures is adapted to a setting of weighted Besov spaces.

In probability theory the prototypical example of the differential equation (1) is a stochastic differential equation driven by a fractional Brownian motion \( B^H \) with Hurst index \( H > 0 \). It is well-known that the Besov regularity of such a fractional Brownian motion is \( B^{H,p}_\infty \) for \( p \in [1, \infty) \) and thus the results of the present paper are applicable. For our Besov setting, an even more interesting example coming from stochastic analysis, recalling for example the Karhunen-Loève theorem, are Gaussian processes and stochastic processes given by a basis expansion with random coefficients, see e.g. Friz et al. [21]. The Besov regularity of such random functions can be determined sharply and they are well-studied for instance when investigating the regularity of solutions for certain stochastic PDEs [11] or in non-parametric Bayesian statistics [1, 4]. In order to make our results about RDEs accessible for these examples, we prove all the required sample path properties in Section \( \mathcal{A} \), especially the existence of the resonant term is provided.

This work is organized as follows. Section \( \mathcal{B} \) introduces the functional analytic framework and gives some preliminary results. In Section \( \mathcal{C} \) we recover Young integration on Besov spaces and deal with differential equations driven by paths with regularity \( \alpha > 1/2 \). The analytic foundation of the paracontrolled distribution approach on general Besov spaces is presented in Section \( \mathcal{D} \). The application of the paracontrolled ansatz to rough differential equations is developed in Section \( \mathcal{E} \) and in Section \( \mathcal{F} \) it is used to solve certain stochastic differential equations. In Appendix \( \mathcal{A} \) some known results about Besov spaces are recalled and the proof for the local Lipschitz continuity of the Itô map is given.

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2 Functional analytic preliminaries

For our analysis we need to recall the definition of Besov spaces, some elements of the Littlewood-Paley theory and Bony’s paraproduct. For the properties of Besov spaces we refer to Triebel [43]. The calculus of Bony’s paraproduct is comprehensively studied by Bahouri et al. [2], from which we also borrow most of our notation.

For the sake of clarification let us mention that \( L^p(\mathbb{R}^d, \mathbb{R}^m) \) denotes the space of Lebesgue \( p \)-integrable functions for \( p \in (0, \infty) \) and \( L^\infty(\mathbb{R}^d, \mathbb{R}^m) \) denotes the space of bounded functions with the (quasi-)norms \( \| \cdot \|_{L^p} \), \( p \in (0, \infty) \). The space of \( \alpha \)-Hölder continuous functions \( f: \mathbb{R}^d \to \mathbb{R}^m \) is denoted by \( C^\alpha \) equipped with the Hölder norm

\[
\| f \|_\alpha := \sum_{|k| < \lfloor \alpha \rfloor} \| f^{(k)} \|_{L^\infty} + \sum_{|k| = \lfloor \alpha \rfloor} \sup_{x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x - y|^{|\alpha| - \lfloor \alpha \rfloor}},
\]

where \( k \) denotes multi-indices with usual conventions and where \( \lfloor \alpha \rfloor \) denotes the integer part of \( \alpha > 0 \). For operator valued functions \( F: \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) we write \( F \in C^n \), \( n \in \mathbb{N} \), if \( F \) is bounded, continuous and \( n \)-times differentiable with bounded and continuous derivatives, and we use the abbreviation \( C := C^0 \). The first and second derivative are denoted by \( F' \) and \( F'' \), respectively, and higher derivatives by \( F^{(n)} \). On the space \( C^n \) we introduce the norm

\[
\| F \|_\infty := \sup_{x \in \mathbb{R}^m} \| F(x) \| \quad \text{and} \quad \| F \|_{C^n} := \| F \|_\infty + \sum_{j=1}^n \| F^{(n)} \|_\infty,
\]

for \( n \geq 1 \), where \( \| \cdot \| \) denotes the corresponding operator norms.
The presumably most fundamental way to define Besov spaces is given via the modulus of continuity of a function \( f \in L^p(\mathbb{R}^d, \mathbb{R}^m) \)
\[
\omega_p(f, \delta) := \sup_{0 < |h| < \delta} \| f(\cdot) - f(\cdot - h) \|_{L^p} \quad \text{for} \quad p, \delta > 0.
\]

For \( p, q \in [1, \infty] \) and \( \alpha \in (0, 1) \) Besov spaces are defined as
\[
B^{\alpha}_{p,q}(\mathbb{R}^d) := B^{\alpha}_{p,q}(\mathbb{R}^d, \mathbb{R}^m) := \left\{ f \in L^p(\mathbb{R}^d, \mathbb{R}^m) : \| f \|_{\omega_{\alpha,p,q}} < \infty \right\}
\]
with
\[
\| f \|_{\omega_{\alpha,p,q}} := \| f \|_{L^p} + \left( \int_{\mathbb{R}^d} |h|^{-\alpha q} \omega_p(f, |h|)^{q} \frac{dh}{|h|^d} \right)^{1/q}
\]
and the usual modification if \( q = \infty \). If \( d = 1 \) (and no confusion arises from the dimension \( m \)) we subsequently abbreviate \( L^p := L^p(\mathbb{R}, \mathbb{R}^m) \) and \( B^{\alpha}_{p,q} := B^{\alpha}_{p,q}(\mathbb{R}, \mathbb{R}^m) \). In \( B^{\alpha}_{p,q}(\mathbb{R}^d) \) the regularity \( \alpha \) is measured in the \( L^p \)-norm while \( q \) is basically a fine tuning parameter in view of the embedding \( B^{\alpha}_{p,q}(\mathbb{R}) \subseteq B^{\beta}_{p,q}(\mathbb{R}^d) \) for \( \beta < \alpha \) and any \( q_1, q_2 \geq 1 \). The classical Hölder spaces and Sobolev spaces are recovered as the special cases \( B^{\alpha}_{\infty,\infty}(\mathbb{R}^d) \) (for non-integer \( \alpha \)) and \( B^{2}_{2,2}(\mathbb{R}^d) \), respectively. Alternatively, Besov spaces can be characterized in terms of a Littlewood-Paley decomposition. Since our analysis mainly relies on this latter characterization, we describe it subsequently.

We write \( \mathcal{S}(\mathbb{R}^d) := \mathcal{S}(\mathbb{R}^d, \mathbb{R}^m) \) for the space of Schwartz functions on \( \mathbb{R}^d \) and denote its dual by \( \mathcal{S}'(\mathbb{R}^d) \), which is the space of tempered distributions. For a function \( f \in L^1 \) the Fourier transform is defined by
\[
\mathcal{F}f(z) := \int_{\mathbb{R}^d} e^{-i(z \cdot x)} f(x) \, dx
\]
and so the inverse Fourier transform is given by \( \mathcal{F}^{-1}f(z) := (2\pi)^{-d} \mathcal{F}f(-z) \). If \( f \in \mathcal{S}'(\mathbb{R}^d) \), then the usual generalization of the Fourier transform is considered. The Littlewood-Paley theory is based on localization in the frequency domain. Let \( \chi \) and \( \rho \) be non-negative infinitely differentiable radial functions on \( \mathbb{R}^d \) such that

(i) there is a ball \( B \subseteq \mathbb{R}^d \) and an annulus \( A \subseteq \mathbb{R}^d \) satisfying \( \text{supp} \, \chi \subseteq B \) and \( \text{supp} \, \rho \subseteq A \),

(ii) \( \chi(z) + \sum_{j \geq 0} \rho(2^{-j} z) = 1 \) for all \( z \in \mathbb{R}^d \),

(iii) \( \text{supp}(\chi) \cap \text{supp}(\rho(2^{-j} \cdot)) = 0 \) for \( j \geq 1 \) and \( \text{supp}(\rho(2^{-1} \cdot)) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset \) for \( |i - j| > 1 \).

We say a pair \((\chi, \rho)\) with these properties is a dyadic partition of unity and we throughout use the notation
\[
\rho_{-1} := \chi \quad \text{and} \quad \rho_j := \rho(2^{-j} \cdot) \quad \text{for} \quad j \geq 0.
\]

For the existence of such a partition we refer to [2, Prop. 2.10]. Taking a dyadic partition of unity \((\chi, \rho)\), the Littlewood-Paley blocks are defined as
\[
\Delta_{-1} f := \mathcal{F}^{-1}(\rho_{-1} \mathcal{F}f) \quad \text{and} \quad \Delta_j f := \mathcal{F}^{-1}(\rho_j \mathcal{F}f) \quad \text{for} \quad j \geq 0.
\]

Note that \( \Delta_j f \) is a smooth function for every \( j \geq -1 \) and for every \( f \in \mathcal{S}'(\mathbb{R}^d) \) we have
\[
f = \sum_{j \geq -1} \Delta_j f := \lim_{j \to \infty} S_j f \quad \text{with} \quad S_j f := \sum_{i \leq j - 1} \Delta_i f.
\]

For \( \alpha \in \mathbb{R} \) and \( p, q \in (0, \infty) \) the Besov space can be characterized in full generality as
\[
B^{\alpha}_{p,q}(\mathbb{R}^d, \mathbb{R}^m) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m) : \| f \|_{\omega_{\alpha,p,q}} < \infty \right\}
\]
with
\[
\| f \|_{\omega_{\alpha,p,q}} := \left\| (2^{jn} \| \Delta_j f \|_{L^p})_{j \geq -1} \right\|_{\ell^q}.
\]

According to [43, Thm. 2.5.12], the norms \( \| \cdot \|_{\omega_{\alpha,p,q}} \) and \( \| \cdot \|_{\alpha,p,q} \) are equivalent for \( p, q \in (0, \infty) \) and \( \alpha \in \left( \frac{d}{\min(3, d + 1)} - d, 1 \right) \). \( B^{\alpha}_{p,q}(\mathbb{R}^d) \) is a quasi-Banach space and if \( p, q \geq 1 \), it is Banach space,
Although the (quasi-)norm \(\|\cdot\|_{\alpha,p,q}\) depends on the dyadic partition \((\chi, \rho)\), different dyadic partitions of unity lead to equivalent norms.

We will frequently use the notation \(A_\vartheta \lesssim B_\vartheta\), for a generic parameter \(\vartheta\), meaning that \(A_\vartheta \lesssim C B_\vartheta\) for some constant \(C > 0\) independent of \(\vartheta\). We write \(A_\vartheta \sim B_\vartheta\) if \(A_\vartheta \lesssim B_\vartheta\) and \(B_\vartheta \lesssim A_\vartheta\). For integers \(j_\vartheta, k_\vartheta \in \mathbb{Z}\) we write \(j_\vartheta \lesssim k_\vartheta\) if there is some \(N \in \mathbb{N}\) such that \(j_\vartheta \leq k_\vartheta + N\), and \(j_\vartheta \sim k_\vartheta\) if \(j_\vartheta \lesssim k_\vartheta\) and \(k_\vartheta \lesssim j_\vartheta\).

In view of the RDEs \((\mathbf{R})\) we need to study the product of two distributions. The standard estimate, cf. Triebel \cite{triest} (24) on p. 143,

\[
\|fg\|_{\alpha,p,q} \lesssim \|f\|_{\alpha,\infty,q}\|g\|_{\alpha,p,q}
\]

applies only for \(\alpha > 0\) and \(p,q \geq 1\). However, in the context of RDEs the regularity \(\alpha\) of the involved product will typically be negative. Given \(f \in B^{\alpha}_{p_1,q_1}(\mathbb{R}^d)\) and \(g \in B^{\beta}_{p_2,q_2}(\mathbb{R}^d)\), at least formally we can decompose the product \(fg\) in terms of Littlewood-Paley blocks as

\[
fg = \sum_{j \geq -1} \sum_{k \geq -1} \Delta_j f \Delta_k g = T_j g + T_k f + \pi(f,g),
\]

where

\[
T_j g := \sum_{|j-i| \leq 1} \Delta_{j-1} f \Delta_j g, \quad \text{and} \quad \pi(f,g) := \sum_{|i-j| > 1} \Delta_i f \Delta_j g.
\]

We call \(\pi(f,g)\) the resonant term. This decomposition was introduced by Bony \cite{bony} and it comes with the following estimates:

**Lemma 2.1.** Let \(\alpha, \beta \in \mathbb{R}\) and \(p_1, p_2, q_1, q_2 \in [1, \infty]\) and suppose that

\[
\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} \leq 1.
\]

(i) For any \(f \in L^{p_1}(\mathbb{R}^d)\) and \(g \in B^{\beta}_{p_2,q_2}(\mathbb{R}^d)\) we have

\[
\|T_j g\|_{\beta,p,q} \lesssim \|f\|_{L^{p_1}}\|g\|_{\beta,p_2,q_2}.
\]

(ii) If \(\alpha < 0\), then for any \((f,g) \in B^{\alpha}_{p_1,q_1}(\mathbb{R}^d) \times B^{\beta}_{p_2,q_2}(\mathbb{R}^d)\) we have

\[
\|T_j g\|_{\alpha+\beta,p,q} \lesssim \|f\|_{\alpha,p_1,q_1}\|g\|_{\beta,p_2,q_2}.
\]

(iii) If \(\alpha + \beta > 0\), then for any \((f,g) \in B^{\alpha}_{p_1,q_1}(\mathbb{R}^d) \times B^{\beta}_{p_2,q_2}(\mathbb{R}^d)\) we have

\[
\|\pi(f,g)\|_{\alpha+\beta,p,q} \lesssim \|f\|_{\alpha,p_1,q_1}\|g\|_{\beta,p_2,q_2}.
\]

**Proof.** The last claim is Theorem 2.85 in \cite{triest}. For the first claim and the second one we slightly generalize their Theorem 2.82. Since \(\rho_j\) is supported on \(2^j\) times an annulus and the Fourier transform of \(S_{k-1} f \Delta_k g\) is supported on \(2^k\) times another annulus, it holds \(\Delta_j T_j g = \Delta_j \sum_{j-k} S_k f \Delta_k g\). Using that \(\Delta_j\) is a convolution with \(\mathcal{F}^{-1}\rho_j = 2^{jd}\mathcal{F}^{-1}\rho(2^j \cdot)\), \(j \geq 0\), Young’s inequality yields for any function \(h \in L^p(\mathbb{R}^d)\) that \(\|\Delta_j h\|_{L^p} \lesssim \|\mathcal{F}^{-1}\rho\|_{L^1}\|h\|_{L^p}\). Together with Hölder’s inequality we obtain for any \(j \geq -1\)

\[
\left\|\Delta_j \left( \sum_{k \geq -1} S_k f \Delta_k g\right) \right\|_{L^p} \lesssim \sum_{k \geq j} \|S_{k-1} f \Delta_k g\|_{L^p} \lesssim \sum_{k \geq j} \|S_{k-1} f\|_{L^{p_1}} \|\Delta_k g\|_{L^{p_2}}.
\]

Since \(\lim_{k \to \infty} \|S_{k-1} f\|_{L^{p_1}} = \|f\|_{L^{p_1}}\), assertion (i) follows from

\[
\left\|T_j g\|_{\beta,p,q} \lesssim \|2^{jd} \sum_{j-k} S_k f\|_{L^{p_1}} \|\Delta_k g\|_{L^{p_2}}\right\|_{L^q} 
\]

\[
\lesssim \|f\|_{L^{p_1}} \|2^{jd} \Delta_j g\|_{L^{p_2}} = \|f\|_{L^{p_1}} \|g\|_{\beta,p_2,q_2}.
\]

\[
5
\]
For (ii) another application of Hölder’s inequality yields
\[
\|Tfg\|_{\alpha + \beta, p, q} \lesssim \left(2^{(\alpha + \beta)} \sum_{j \neq k} \|S_{j,k}f\|_{p, r_1} \|\Delta_k g\|_{L^{p_2}}\right)_{t_0}
\]
\[
\lesssim \left(2^{\alpha}\|S_{j}f\|_{L^{p_1}}\right)_{t_0} \left(2^{\beta}\|\Delta_k g\|_{L^{p_2}}\right)_{t_0} \lesssim \left(2^{\alpha}\|S_{j}f\|_{L^{p_1}}\|g\|_{\beta, p_2, q_2}\right).
\]
Finally, we apply Lemma A.3 to conclude that \((2^{\alpha}\|S_{j}f\|_{L^{p_1}})_{t_0}\) and that
\[
\|(2^{\alpha}\|S_{j}f\|_{L^{p_1}})_{t_0}\|_{L^{p_1}} \lesssim \|f\|_{\alpha, p_1, q_1}.
\]

We finish this section with two elementary lemmas, which seem to be non-standard (cf. Lemma A.4 and A.10 in [24] for the Hölder case). To control the norm of an antiderivative with respect to the function itself will play an important role, naturally restricted to the case \(d = 1\). The following lemma provides the counterpart to the well-known estimate \(\|F'\|_{\alpha - 1, p, q} \lesssim \|F\|_{\alpha, p, q}\) for any \(F \in B^\alpha_{p, q}\), cf. Triebel [13, Thm. 2.3.8]. For \(p < \infty\) the antiderivative will in general have no finite \(L^p\)-norm such that we have to apply a weighting function to ensure integrability.

**Lemma 2.2.** Let \(p \in (1, \infty)\) and \(\alpha \in (1/p, 1)\). For every \(f \in B^\alpha_{p, q}(\mathbb{R})\) there exists a unique function \(F: \mathbb{R} \rightarrow \mathbb{R}^m\) such that \(F' = f\) and \(F(0) = 0\). Moreover, it holds for any fixed \(\psi \in C^1\) satisfying \(C_\psi := \|\psi\|_{C^1} + \sum_{j,k \in \{0,1\}} \|\psi^{(j,k)}(t)\|_{L^p} < \infty\) that
\[
\|\psi F\|_{\alpha, p, q} \lesssim C_\psi \|f\|_{\alpha - 1, p, q}.
\]
In particular, for any smooth \(\psi\) with \(\text{supp} \psi \subseteq [-T, T]\) for some \(T > 0\) one has
\[
\|\psi F\|_{\alpha, p, q} \lesssim (1 \lor T^2) \|\psi\|_{C^1} \|f\|_{\alpha - 1, p, q}.
\]

**Proof.** Since differentiating in spatial domain corresponds to multiplication in Fourier domain, we set
\[
G(t) := \sum_{j \geq 0} F^{-1}\left[\frac{1}{iu} \rho_j(u) F(f(u))\right](t) \quad \text{and} \quad H(t) := \int_0^t \Delta_{-1} f(s) \text{d}s, \quad t \in \mathbb{R}.
\]
Provided
\[
\|\psi G\|_{\alpha, p, q} \lesssim \|\psi\|_{C^1} \|G\|_{\alpha, p, q} \lesssim \|\psi\|_{C^1} \|f\|_{\alpha - 1, p, q}, \quad \|\psi H\|_{\alpha, p, q} \lesssim C_\psi \|f\|_{\alpha - 1, p, q} \tag{5}
\]
and noting that \(B^\alpha_{p, q} \subseteq C(\mathbb{R})\) for \(\alpha > 1/p\), the function \(F := G + H - G(0)\) satisfies \(F' = f\) and the asserted norm estimate. Uniqueness follows because any distribution with zero derivative is constant.

It remains to verify (5). Concerning \(G\), we obtain for each Littlewood-Paley block, using \(\text{supp}(\rho_j) \cap \text{supp}(\rho_k) = \emptyset\) for all \(j, k \geq -1\) with \(|j - k| > 1\),
\[
\Delta_k G = \sum_{j = (k-1) \lor 0}^{k+1} F^{-1}\left[\frac{1}{iu} \rho_j(u) \rho_j(u) F(f(u))\right] = \left(\sum_{j = (k-1) \lor 0}^{k+1} F^{-1}\left[\frac{1}{iu} \rho_j(u)\right]\right) \Delta_k f.
\]
Using twice a substitution, we have for \(j \geq 0\)
\[
\|F^{-1}[\rho_j(u)]\|_{L^1} = \left\|F^{-1}[\frac{\rho(u)}{iu}]\right\|_{L^1} = 2^{-j} \left\|F^{-1}[\frac{\rho(u)}{iu}]\right\|_{L^1}.
\]
Hence, Young’s inequality yields
\[
\|G\|_{\alpha, p, q} = \left\|\left(2^{\alpha} \|\Delta_k G\|_{L^p}\right)_{k} \right\|_{L^{\alpha, p, q}} \lesssim \left\|\left(2^{(\alpha - 1)k} \|F[\rho(u)/(iu)]\|_{L^1}\|\Delta_k f\|_{L^p}\right)_{k} \right\|_{L^{\alpha, p, q}} \lesssim \|f\|_{\alpha - 1, p, q}.
\]
To show the second part of (5), we use \(\|\psi H\|_{\alpha, p, q} \lesssim \|\psi H\|_{1, p, \infty} \lesssim \|\psi H\|_{L^p} + \|H\|_{L^p}\) due to \(\alpha < 1\). Hölder’s inequality yields for \(\tilde{p} := \frac{p}{p-1}\) with the usual modification for \(p = \infty\) that
\[
\|\psi H\|_{L^p} \lesssim \|\Delta_{-1} f\|_{L^{\rho}} \|\psi(t)\|_{L^{1/\tilde{p}}} \lesssim \|(1 \lor t)\|_{L^p} \|f\|_{\alpha - 1, p, q}\]
and similarly
\[ \|\psi H\|_{L^p} \leq \|\psi H\|_{L^p} + \|\psi \Delta_{-1} f\|_{L^p} \lesssim \|\Delta_{-1} f\|_{L^p} (\|\psi'(t)\|_{L^1} + \|\psi\|_{\infty}) \lesssim (\|\psi\|_{\infty} + \|(1+|t|)\psi'(t)\|_{L^p})\|f\|_{\alpha-1,p,q}. \]

For later reference we finally investigate the scaling operator \( \Lambda_\lambda \), given by \( \Lambda_\lambda f(\cdot) := f(\lambda \cdot) \) for any \( \lambda > 0 \) and any function \( f \), on Besov spaces.

**Lemma 2.3.** For \( \alpha \neq 0, p, q \geq 1 \) and all \( f \in B^\alpha_{p,q}(\mathbb{R}^d) \) we have
\[ \|\Lambda_\lambda f\|_{\alpha,p,q} \lesssim (1 + |\lambda| \log |\lambda|) \lambda^{-d/p} \|f\|_{\alpha,p,q}. \]

**Proof.** Using \( \Lambda_\lambda (Ff) = \kappa^{-d} F[\Lambda_\lambda^{-1} f] \) for \( \kappa > 0, f \in B^\alpha_{p,q}(\mathbb{R}^d) \), we first deduce
\[
\Delta_j(\Lambda_\lambda f) = \lambda^{-d} F^{-1}[\rho_j \Lambda_\lambda^{-1} (Ff)] = F^{-1}[\rho_j (\lambda \cdot) Ff(\lambda \cdot) \cup A]
\]
and similarly
\[ \lambda > 0 \]
for all \( \lambda > 0 \). For \( j \geq 0 \) the Fourier transform of \( \Lambda_\lambda (\Delta_j f) \) is consequently supported in \( \lambda 2^j A \), where \( A \) is the annulus containing the support of \( \rho \), and we have \( \Delta_k (\Lambda_\lambda \Delta_j f) \neq 0 \) only if \( 2^k \sim \lambda 2^j \).

Together with \( \|\Delta_k f\|_{L^p} \leq \|F^{-1} \rho_k\|_{L^1} \|f\|_{L^p} \lesssim \|f\|_{L^p} \) by Young’s inequality we obtain
\[ \|\Delta_k \Lambda_\lambda f\|_{L^p} \leq \sum_{j: 2^j \sim \lambda 2^j} \|\Delta_k \Delta_j f\|_{L^p} \lesssim \lambda^{-d/p} \sum_{j: 2^j \sim \lambda 2^j} \|\Delta_j f\|_{L^p} \quad \text{for } k \geq 0. \]

Applying again Young’s inequality to the sequences \( a := \left( \|\mathbb{I}_{[-|\log \lambda|,|\log \lambda|]}(k)\right)_k \) and \( (2^\alpha |\Delta_j f|_{L^p})_j \), we infer
\[
\left( \sum_{j: 2^j \sim \lambda 2^j} \lambda^\alpha 2^{ja} \|\Delta_j f\|_{L^p} \right)_{k \geq 0} \|e_t\|^q \lesssim \lambda^{-d/p} \lambda^\alpha \sum_{j: 2^j \sim \lambda 2^j} \|\Delta_j f\|_{L^p} |\log \lambda|^{a - d/p} \|f\|_{\alpha,p,q}.
\]

Finally, we obtain analogously for \( k = -1 \) that
\[ \|\Delta_{-1} \Delta_\lambda f\|_{L^p} \lesssim \lambda^{-d/p} \sum_{j: \lambda 2^j \leq 1} \|\Delta_j f\|_{L^p} \lesssim \lambda^{-d/p} \|f\|_{\alpha,p,q} \sum_{j: \lambda 2^j \leq 1} 2^{-aj} \lesssim (1 + \lambda^\alpha) \lambda^{-d/p} \|f\|_{\alpha,p,q}. \]

### 3 Young integration revisited

In the present section we start to consider the differential equation (1), which was given by
\[ du(t) = F(u(t))\xi(t), \quad u(0) = u_0, \quad t \in \mathbb{R}, \]
where \( u_0 \in \mathbb{R}^m, u: \mathbb{R} \to \mathbb{R}^m \) is a continuous function and \( F: \mathbb{R}^m \to L(\mathbb{R}^n, \mathbb{R}^m) \). Assuming our driving signal \( \xi: \mathbb{R} \to \mathbb{R}^n \) is smooth enough, the differential equation (1) is well-defined and can be equivalently written in its integral form
\[ u(t) = u_0 + \int_0^t F(u(s))\xi(s) \, ds, \quad t \in [0, \infty), \]
and analogously for \( t \in (-\infty, 0) \). According to Young [43], the involved integral can be defined as limit of Riemann sums as long as the driving signal \( \xi \) is the derivative of a path \( \vartheta \) which is of finite \( p \)-variation for \( p < 2 \). Then, equation (1) admits a unique solution on every bounded interval \([-T, T] \subseteq \mathbb{R} \) if \( F \in C^2 \) (see modern books as [38, Theorem 1.28] or [33, Theorem 1]). This result was first proven by Lyons [35] using a Picard iteration. The case of a \( 1/p \)-Hölder continuous driving path \( \vartheta \) was treated by Ruzmaikina [42]. Since then it is still of great interest
to find new approaches to \(6\): Gubinelli \cite{22} has introduced the notion of controlled paths, Davie \cite{15} has shown the convergence of an Euler scheme, Hu and Nualart \cite{30} have used techniques from fractional calculus and Lejay \cite{33} has developed a simple approach similar to \cite{32}.

In this section we recover the analogous results on Besov spaces with a special focus on the situation when \(F\) is a linear functional. For a discussion of the importance of linear RDEs we refer to Coutin and Lejay \cite{13} and references therein.

We first note that the function \(F(u)\) inherits its regularity from the regularity of \(u\). More precisely, \cite[Thm. 2.87]{2} shows for \(u \in B^{\alpha}_{p,q}\) satisfying \(\|u\|_{\infty} < \infty\) and a family of sufficient regular vector fields \(F\) with \(F(0) = 0\) (or \(p = \infty\)) that

\[
\|F(u)\|_{\alpha,p,q} \lesssim \left( \sum_{k=1}^{\lceil \alpha \rceil} \sup_{|x| \leq \|u\|_{\infty}} \|F^{(k)}(x)\| \right) \|u\|_{\alpha,p,q} \lesssim \|F\|_{C^{\lceil \alpha \rceil}} \|u\|_{\alpha,p,q},
\]

(7)

denoting the smallest integer larger or equal than \(\alpha > 0\) by \([\alpha]\) and provided the norms on the right-hand side are finite. If the product \(F(u)\xi\) is regular enough, we can understand the differential equation \((\text{8})\) in its integral form \((\text{11})\) where the integral is given by the antiderivative of the product, i.e.

\[
d\left( \int_{0}^{t} F(u(s))\xi(s) \, ds \right) = F(u(t))\xi(t) \quad \text{and} \quad \int_{0}^{t} F(u(s))\xi(s) \, ds = 0.
\]

In view of Lemma \(2.2\) the solution \(u\) of \((\text{1})\) cannot be expected to be contained in \(B^{\alpha}_{p,q}\). Therefore, we consider instead a localized version of the differential equation. Alternatively, the solution of the RDE \((\text{11})\) could be studied in homogenous or weighted Besov spaces, which can only lead to very similar results. In order to provide our results in the most commonly used notion of Besov spaces, we focus on localized equations. We impose the following standing assumption:

**Assumption 3.1.** Let \(\varphi: \mathbb{R} \rightarrow \mathbb{R}_{+}\) be fixed smooth function with support \([-2,2]\) and equal to 1 on \([-1,1]\). Denote \(\varphi_T(t) := \varphi(t/T)\) for \(T > 0\).

**Theorem 3.2.** Let \(T > 0\), \(\alpha \in (1/2,1]\) and assume that \(\xi \in B^{\alpha-1}_{p,q}\) for \(p \in [2,\infty]\) and \(q \in [1,\infty]\). If \(F: \mathbb{R}^{n} \rightarrow L(\mathbb{R}^{n},\mathbb{R}^{m})\) is a linear mapping, then for every \(u_{0} \in \mathbb{R}^{d}\) there exists a unique global solution \(u \in B^{\alpha}_{p,q}\) to the Cauchy problem

\[
u(t) = \varphi_{T}(t)u_{0} + \varphi_{T}(t) \int_{0}^{t} F(u(s))\xi(s) \, ds, \quad t \in \mathbb{R},
\]

(8)

with the usual convention for \(t < 0\). This result extends to nonlinear \(F \in C^{2}\) if \(p = \infty\).

**Proof.** Step 1: First we establish a contraction principle under the assumption that \(\|F\|_{C^{2}}\) is sufficiently small. Without loss of generality we may assume \(u_{0} = 0\). Following a fixed point argumentation, we consider the solution map

\[
\Phi: B^{\alpha}_{p,q} \rightarrow B^{\alpha}_{p,q}, \quad u \mapsto \tilde{u} := \varphi_{T} \int_{0}^{t} F(u(s))\xi(s) \, ds, \quad t \in \mathbb{R}.
\]

In order to verify that \(\Phi\) is indeed well-defined, we use Lemma \(2.2\) to observe

\[
\|\varphi_{T}F\|_{\alpha,p,q} \lesssim (1 + T^{2})(1 + T^{-1})\|\varphi\|_{C^{1}}\|f\|_{\alpha-1,p,q} \lesssim C_{T,\varphi}\|f\|_{\alpha-1,p,q},
\]

where \(C_{T,\varphi} := (T^{-1} + T^{2})\|\varphi\|_{C^{1}},\) for any given \(f \in B^{\alpha-1}_{p,q}\) with \(dF = f\) and \(F(0) = 0\). We thus have

\[
\|\Phi(u)\|_{\alpha,p,q} = \left\|\varphi_{T} \left( \int_{0}^{t} F(u(s))\xi(s) \, ds \right) \right\|_{\alpha,p,q} \lesssim C_{T,\varphi}\|F(u)\|_{\alpha-1,p,q}.
\]
Applying Bony’s decomposition, the Besov embedding $B^{2\alpha-1}_{p/2,q} \subseteq B^{\alpha-1}_{p,q}$ (cf. [13, Thm. 2.7.1]) for $p > 1/\alpha$ and Lemma 2.2, we obtain

$$\|\Phi(u)\|_{\alpha,p,q} \lesssim C_{T,\varphi} \left( \|T_{F(u)}\|_{\alpha-1,p,q} + \|\pi(F(u),\xi)\|_{2\alpha-1,p/2,q/2} + \|T_{\xi}(F(u))\|_{\alpha-1,p,q} \right)$$

$$\lesssim C_{T,\varphi} \left( \|F(u)\|_{\infty} \|\xi\|_{\alpha-1,p,q} + \|F(u)\|_{\alpha,p,2q} \|\xi\|_{\alpha-1,p,2q} + \|\|\|_{\alpha-1,p,q}\|F(u)\|_{[0,\infty,\infty]} \right).$$

Using the embeddings $B^{\alpha}_{p,q} \subseteq L^\infty$ and $B^{\alpha}_{p,q} \subseteq B^{0}_{1,\infty}$ for $\alpha > 1/p$ and (7), we deduce that

$$\|\Phi(u)\|_{\alpha,p,q} \lesssim C_{T,\varphi} \|F\|_{\infty} \|\xi\|_{\alpha-1,p,q} \|u\|_{\alpha,p,q}. \quad (9)$$

To apply Banach’s fixed point theorem, it remains to show that $\Phi$ is a contraction. For $u, \tilde{u} \in B^{\alpha}_{p,q}$, Lemma 2.2 again yields

$$\|\Phi(u) - \Phi(\tilde{u})\|_{\alpha,p,q} \lesssim C_{T,\varphi} \left( \|F'(u) - F'(\tilde{u})\|_{\alpha-1,p,q} \right)$$

$$\lesssim C_{T,\varphi} \int_0^1 \|F'(u + t(\tilde{u} - u))(\tilde{u} - u)\|_{\alpha-1,p,q} \, dt.$$ 

Denoting by $v_t := F'(u + t(\tilde{u} - u))(\tilde{u} - u)$, we conclude as above

$$\|\Phi(u) - \Phi(\tilde{u})\|_{\alpha,p,q} \lesssim C_{T,\varphi} \int_0^1 \left( \|F'(u + t(\tilde{u} - u))(\tilde{u} - u)\|_{\alpha-1,p,q} \right) \, dt.$$

By the standard estimate (8), we obtain

$$\|\Phi(u) - \Phi(\tilde{u})\|_{\alpha,p,q} \lesssim C_{T,\varphi} \left( \int_0^1 \|F'(u + t(\tilde{u} - u))\|_{\alpha-1,p,q} \right) \|\xi\|_{\alpha-1,p,q} \|u - \tilde{u}\|_{\alpha,p,q}. \quad (10)$$

Hence, if $F$ is linear and $\|F'\|_{\infty}$ is small enough, $\Phi$ is a contraction. Provided $p = \infty$ and $F \in C^2$, it suffices if $\|F'\|_{C^1}$ is sufficiently small:

$$\|\Phi(u) - \Phi(\tilde{u})\|_{\alpha,p,q} \lesssim C_{T,\varphi} \|F'\|_{C^1} \left( \|u\|_{\alpha-1,p,q} + \|\tilde{u}\|_{\alpha-1,p,q} \right) \|\xi\|_{\alpha-1,p,q} \|u - \tilde{u}\|_{\alpha-1,p,q}. \quad (11)$$

**Step 2:** In order to ensure that $\|F'\|_{C^1}$ is small enough, we scale $\xi$ as follows: For some fixed $\varepsilon \in (0, 1/p)$ and for some $\lambda \in (0, 1)$ to be chosen later we set

$$\xi^\lambda := \lambda^{1-\alpha+1/p+\varepsilon} \Lambda_\lambda \xi,$$

where we recall the scaling operator $\Lambda_\lambda f = f(\lambda)$ for $f \in S'$. Lemma 2.2 yields

$$\|\xi^\lambda\|_{\alpha-1,p,q} = \lambda^{1-\alpha+1/p+\varepsilon} \|\Lambda_\lambda \xi\|_{\alpha-1,p,q} \lesssim (\lambda^\varepsilon \log \lambda + \lambda^{1-\alpha+\varepsilon}) \|\xi\|_{\alpha-1,p,q} \lesssim \|\xi\|_{\alpha-1,p,q}.$$

For $\lambda > 0$ sufficiently small Step 1 provides a unique global solution $u^\lambda \in B^{\alpha}_{p,q}$ to the (localized) differential equation

$$u^\lambda(t) = \varphi_T(t) u_0 + \varphi_T(t) \int_0^t \lambda^{\alpha-1/\varepsilon-\varepsilon} F(u^\lambda(s)) \xi^\lambda(s) \, ds,$$

for all $u_0 \in \mathbb{R}$. Setting now $u := \Lambda_{\lambda^{-1}} u^\lambda$, we have constructed a unique solution to

$$u(t) = \Lambda_{\lambda^{-1}} u^\lambda(t) = \varphi_{\lambda T}(t) u_0 + \varphi_{\lambda T}(t) \int_0^t F(u(s)) \xi(s) \, ds,$$

which coincides with (8) on $[-\lambda T, \lambda T]$.

**Step 3:** Since the choice of $\lambda$ does not depend on $u_0$, we can iteratively apply Step 2 on intervals of length $2\lambda T$ to construct a unique global solution $u \in B^{\alpha}_{p,q}$ to equation (8). \qedsymbol
In this simple setting it turns out that the Itô map $S$ defined by

$$S : \mathbb{R}^d \times B_{p,q}^{\alpha-1} \to B_{p,q}^{\alpha} \quad \text{via} \quad (u_0, \xi) \mapsto u,$$

where $u$ denotes the solution of the (localized) Cauchy problem \eqref{localCauchy}, is a locally Lipschitz continuous map with respect to the Besov norm.

**Theorem 3.3.** Let $\alpha \in (1/2, 1]$, $q \in [1, \infty]$ and $F: \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. If either $F$ is a linear mapping and $p \in [2, \infty]$ or $F \in C^2$ and $p = \infty$, then the Itô map $S$ given by \eqref{ItôMap} is locally Lipschitz continuous.

**Proof.** Let $u_0^i \in \mathbb{R}^d$, $\xi^i \in B_{p,q}^{\alpha-1}$ be such that $\|\xi^i\|_{\alpha-1,p,q} \leq R$ and $|u_0^i| \leq R$ for some $R > 0$ and denote by $u^i$ the unique solution to corresponding Cauchy problems \eqref{localCauchy} for $i = 1, 2$, which exists thanks to Theorem 3.2. In order to avoid repetition, we just consider a linear mapping $F$. The non-linear case works analogously.

**Step 1:** Suppose that $\|F'\|_\infty$ is sufficiently small. Recalling $C_{T,\varphi} = (T^{-1} \vee T^2)\|\varphi\|_{C^1}$, we deduce similarly to \eqref{est1} that

$$\|u^i\|_{\alpha,p,q} \lesssim \|\varphi_T\|_{\alpha,p,q}|u_0^i| + C_{T,\varphi}\|F'\|_\infty \|\xi^i\|_{\alpha-1,p,q} \|u^i\|_{\alpha,p,q},$$

which, provided $\|F'\|_\infty$ is small enough, depending only on $R$, $\varphi$ and $T$, leads to

$$\|u^i\|_{1,p,q} \lesssim \|\varphi_T\|_{\alpha,p,q} R, \quad \text{for} \quad i = 1, 2.$$

For the difference $u^1 - u^2$ we have

$$\|u^1 - u^2\|_{\alpha,p,q} \lesssim \|\varphi_T(u_0^1 - u_0^2)\|_{\alpha,p,q} + \|\varphi_T\|_{\alpha,p,q} \\left( \int_0^T F(u(s))\xi(s) \, ds - \int_0^T F(u(s))\xi^2(s) \, ds \right)_{\alpha,p,q}$$

$$\lesssim \|\varphi_T\|_{\alpha,p,q}\|u_0^1 - u_0^2\|_{\alpha,p,q} + \|\varphi_T\|_{\alpha,p,q} \\left( \int_0^T \|F(u^1(s)) - F(u^2(s))\| \xi^2(s) \, ds \right)_{\alpha,p,q}$$

$$+ C_{T,\varphi}\|F(u^2)(\xi^1 - \xi^2)\|_{\alpha-1,p,q}.$$

The second term can be estimated as in \eqref{est2} and for the last one Bony’s decomposition, Lemma 2.1, and \eqref{est2} yield

$$\|F(u^2)(\xi^1 - \xi^2)\|_{\alpha-1,p,q} \lesssim \|F(u^2)\|_{\alpha,p,q}\|\xi^1 - \xi^2\|_{\alpha-1,p,q} \lesssim \|F'\|_\infty \|u^1\|_{\alpha,p,q}\|\xi^1 - \xi^2\|_{\alpha-1,p,q}.$$

Therefore, we can combine the above estimates to

$$\|u^1 - u^2\|_{\alpha,p,q} \lesssim C_{T,\varphi}\left( |u_0^1 - u_0^2| + \|\varphi_T\|_{\alpha,p,q}\|F'\|_\infty R \|\xi^1 - \xi^2\|_{\alpha-1,p,q} \right.$$

$$\left. + \left( \int_0^T \|F'(u^1 + t(u^2 - u^1))\|_{\alpha-1,\infty,q} \, dt \right)_{\alpha-1,\infty,q} \right) \|u^1 - u^2\|_{\alpha,p,q}.$$

If $F$ is linear with sufficiently small $\|F\|_{C^1}$, we obtain the desired estimate by rearranging:

$$\|u^1 - u^2\|_{\alpha,p,q} \lesssim C_{T,\varphi}\left( |u_0^1 - u_0^2| + \|\varphi_T\|_{\alpha,p,q}\|F\|_{C^1} R \|\xi^1 - \xi^2\|_{\alpha-1,p,q} \right).$$

**Step 2:** The assumption on $\|F'\|_\infty$ can be translated to an assumption on $T$ using the same scaling argument as in Step 2 in the proof of Theorem 3.2. More precisely, we define $\xi_{\lambda,1}$ and $\xi_{\lambda,2}$ for $\lambda > 0$ as in \eqref{scaling} and note $\|\xi_{\lambda,i}\|_{\alpha,p,q} \lesssim R$ for $i = 1, 2$. Therefore, for sufficiently small $\lambda$ there exists a unique solution $u^\lambda$ to \eqref{scaling} for $i = 1, 2$. Setting again $u^i := \Lambda_{\lambda-1} u^\lambda$ and applying twice Lemma 2.3 together with Step 1 gives

$$\|u^1 - u^2\|_{\alpha,p,q} \lesssim (1 + \lambda^{-\alpha} \log \lambda^{-1})^{1/\beta} \|u^\lambda - u^\lambda\|_{\alpha,p,q}$$

$$\lesssim C_{T,\varphi}(1 + \lambda^{-\alpha} \log \lambda^{-1})^{1/\beta} \left( |u_0^1 - u_0^2| + \|\varphi_T\|_{\alpha,p,q}\|F'\|_\infty R \|\xi^1 - \xi^2\|_{\alpha-1,p,q} \right)$$

$$\lesssim C_{T,\varphi}(1 + \lambda^{-\alpha} \log \lambda^{-1})^{1/\beta} \left( |u_0^1 - u_0^2| + \|\varphi_T\|_{\alpha,p,q}\|F'\|_\infty R \|\xi^1 - \xi^2\|_{\alpha-1,p,q} \right).$$
Let $\mu_0 \in \mathcal{Z}$ be such that $\mu_0(t_0 + \varepsilon) = 1$, $\varepsilon \in [-\frac{1}{2} \lambda T, \frac{1}{2} \lambda T]$, for anchor points $t_j \in \mathbb{R}$ with $t_0 = 0$ and $|t_j - t_{j-1}| \leq \lambda T/2$ and fulfilling

$|\text{supp } \mu_j| := \text{sup}\{|x-y| : x,y \in \text{supp } \mu_j\} \leq \lambda T$ \text{ and } $\sum_{j \in \mathbb{Z}} \mu_j(x) = 1$ \text{ for all } $x \in \mathbb{R}$.

Since the $u^i$ for $i = 1,2$ have compact support, there is some $N \in \mathbb{N}$ such that one has, using (2).

$$
\|u^1 - u^2\|_{\alpha,p,q} \leq \sum_{j=-N}^{N} \|\mu_j(u^1 - u^2)\|_{\alpha,p,q} \leq \sum_{j=-N}^{N} \|\mu_j\|_{C^1} \|u^1_j - u^2_j\|_{\alpha,p,q},
$$

where $u^1_j$ is the unique solution to

$$
u^1_j(t) = \varphi \lambda T(t-t_j)u^1_j + \varphi \lambda T(t-t_j) \int_{t_j}^{t} F(u^1_j(s))\xi(s) ds,
$$

with initial condition $u^1_j := u^1(t_j)$ for $i = 1,2$. Noting that $|u^1_j - u^2_j| \leq \|u^1_j - u^2_j\|_{\alpha,p,q}$ for $j \geq 0$ and similarly for negative $j$, Step 2 yields

$$
\|u^1 - u^2\|_{\alpha,p,q} \leq C_{T,\varphi} \left(\|u^1_0 - u^2_0\| + \|\varphi\|_{C^1} \|F\|_\infty R \|\xi^1 - \xi^2\|_{\alpha-1,p,q}\right).
$$

To extend these results to nonlinear functions $F$ for $p < \infty$ and to less regular driving signals $\xi$, more precisely $\xi \in B^{r-1}_{p,q}$ for $\alpha \in (1/3,1/2)$, is the aim of the following two sections.

4 Linearization and commutator estimate

In order to deal with more irregular driving signals $\xi \in B^{r-1}_{p,q}$, we shall apply Bony’s decomposition to rigorously define the product $F(u)\xi$, which appears in the RDE (6). Let us first formally decompose $F(u)\xi$ and analyze the Besov regularity of the different terms as follows

$$
F(u)\xi = F(u)\xi + \pi(F(u),\xi) + \tilde{T}_F(F(u),\xi). \tag{15}
$$

The first term $T_F(u)\xi$ is in $B^{r-1}_{p,q}$ due to Lemma 19 and the boundedness of $F$. The regularity of the third term $T_\xi F(u) \in B^{2r-1}_{p,2/q}$ for $\alpha < 1$ can also be deduced from Lemma 19 since naturally the solution $u$ has regularity $B^{3\alpha-1}_{p,3/2}$ and thus $F(u) \in B^{2\alpha}_{p,q}$ by 7. The regularity estimate of the resonant term can be applied only if $2\alpha - 1 > 0$. This is the main reason, why it was possible for $\alpha \in (1/2,1]$ to show the existence of a solution to the (localized) RDE 1 in Section 3 without taking any additional information about $\xi$ into account. However, this high Besov regularity assumption on $\xi$ is violated in most of the basic examples from stochastic analysis as for instance for stochastic differential equations driven by Brownian motion or martingales. The aim of this section is to reduce the resonant term $\pi(F(u),\xi)$ to $\pi(u,\xi)$:

**Proposition 4.1.** Let $\alpha \in (\frac{1}{3},\frac{1}{2})$, $p \in [3,\infty]$ and $F \in C^{2+\gamma}(\mathbb{R})$ for some $\gamma \in (0,1]$ satisfying $F(0) = 0$. Then there is a map $\Pi_F : B^{3\alpha-1}_{p,3/2}(\mathbb{R}) \times B^{2\alpha}_{p,2/q}(\mathbb{R}) \rightarrow B^{2\alpha-1}_{p,2/q}(\mathbb{R})$ such that for any $u \in B^{\alpha}_{p,\infty}(\mathbb{R})$ and $\xi \in B^{\alpha}_{p,\infty}(\mathbb{R})$ we have

$$
\pi(F(u),\xi) = F'(u)\pi(u,\xi) + \Pi_F(u,\xi) \tag{16}
$$
with
\[ \|\Pi_F(u, \xi)\|_{\alpha-1, p/3, \infty} \lesssim \|F\|_{C^{\alpha}} \|u\|_{\alpha, p, \infty}^2 \|\xi\|_{\alpha-1, p, \infty}. \] (17)

Moreover, \( \Pi_F \) is locally Hölder continuous satisfying for any \( u^1, u^2 \in B^\alpha_{p,q}(\mathbb{R}) \) and \( \xi^1, \xi^2 \in B^\alpha_{p,q}(\mathbb{R}) \)
\[ \|\Pi_F(u^1, \xi^1) - \Pi_F(u^2, \xi^2)\|_{\alpha-1, p/3, \infty} \lesssim \|F\|_{C^{\alpha}} C(u^1, u^2, \xi^1, \xi^2) \left( \|u^1 - u^2\|_{\infty} + \|u^1 - u^2\|_{\alpha, p, \infty} + \|\xi^1 - \xi^2\|_{\alpha-1, p, \infty} \right) \]
where \( C(u^1, u^2, \xi^1, \xi^2) := \|u^1\|_{\alpha, p, \infty}^2 \wedge \|u^2\|_{\alpha, p, \infty}^2 + \left( \|u^1\|_{\alpha, p, \infty} + \|u^2\|_{\alpha, p, \infty} \right) \left( 1 + \|\xi^1\|_{\alpha-1, p, \infty} \wedge \|\xi^2\|_{\alpha-1, p, \infty} \right). \)

As we will see in the next section, it suffices to consider only \( q = \infty \) in Proposition 11.1. Taking into account the embedding \( B^\alpha_{p,q} \subset B^\alpha_{p,\infty} \) for any \( q \in [1, \infty] \), this case corresponds to the weakest Besov norm for fixed \( \alpha \) and \( p \).

In order to prove this proposition, we need the subsequent lemmas. As the first step, we show the following paralinearization result, which is a slight generalization of Theorem 2.92 in [2]. Our proof is inspired by [24, Lem. 2.6] and relies on the characterization of Besov spaces via the modulus of continuity. We obtain that the composition \( F(u) \) can be written as a paraproduct of \( F'(u) \) and \( u \) up to some more regular remainder.

**Lemma 4.2.** Let \( 0 < \beta \leq \alpha < 1 \) and \( F \in C^{1+\beta/\alpha}(\mathbb{R}^m) \). Let \( p \geq \beta/\alpha + 1 \) and define \( p' := \alpha p/\alpha + 1 \). Then for any \( g \in B^\alpha_{p,q}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) there is some \( R_F(g) \in B^\alpha_{p',\infty}(\mathbb{R}^d) \) satisfying
\[ F(g) - F(0) = T^\alpha_F(g) + R_F(g) \] and
\[ \|R_F(g)\|_{\alpha+\beta, \infty'} \lesssim \|F\|_{C^{1+\beta/\alpha}} \|g\|_{\alpha, \infty}. \]
Moreover, if \( F \in C^{2+\gamma} \) for some \( \gamma \in (0, 1] \) and if \( p > 2 \vee 1/\alpha \) then the map
\[ R_F : B^\alpha_{p,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \to B^{2\alpha}_{p,\infty}(\mathbb{R}^d) \]
is locally Hölder continuous with
\[ \|R_F(g) - R_F(h)\|_{\alpha, p/2, \infty} \lesssim \|F\|_{C^{2+\gamma}} \left( \|g\|_{\alpha, p, \infty}^2 + \|h\|_{\alpha, p, \infty} + \|g\|_{\alpha, p, \infty} + \|h\|_{\alpha, p, \infty} \right) \left( \|g - h\|_{\infty} + \|g - h\|_{\alpha, p, \infty} \right). \]

**Proof.** The remainder \( R_F(g) \) is given by
\[ R_F(g) = F(g) - F(0) - T^\alpha_F(g) = \sum_{j \geq 1} F_j \quad \text{with} \quad F_j := \Delta_j(F(g) - F(0)) - S_{j-1}(F'(g)) \Delta_j g. \]

For \( j \leq 0 \) Young’s inequality and the Lipschitz continuity of \( F \) yield
\[ \|F_j\|_{L^p} = \|\Delta_j(F(g) - F(0))\|_{L^p} \leq \|F^{-1}\rho_j\|_{L^p} \|F(g) - F(0)\|_{L^p} \lesssim \|F\|_{C^{1+\gamma}} \|g\|_{L^p} \]
and we have \( \|F_j\|_{L^p} \lesssim \|F_j\|_{L^p} \lesssim \|F_j\|_{L^{p/\beta}} \lesssim \|F_j\|_{L^{1-\beta/\alpha}} \|F_j\|_{L^{1+\beta/\alpha}} \).

For \( j > 0 \) we have \( \Delta_j F(0) = 0 \) and the Fourier transform of \( F_j \) is supported in \( 2^j \) times some annulus. Defining the kernel functions \( K_j := F^{-1}\rho_j \) and \( K_{<j-1} := \sum_{k<j-1} K_k \) and using that \( \int K_j(x) \, dx = \rho_j(0) = 0 \), the blocks \( F_j \) can be written as convolution
\[ F_j(x) = \int_{\mathbb{R}^d} K_j(x-y) K_{<j-1}(x-z) \left( F(g(y)) - F'(g(z)) g(y) \right) dy \, dz \]
\[ = \int_{\mathbb{R}^d} K_j(x-y) K_{<j-1}(x-z) \left( F(g(y)) - F(g(z)) - F'(g(z))(g(y) - g(z)) \right) dy \, dz \]
\[ = \int_{\mathbb{R}^d} K_j(x-y) K_{<j-1}(x-z) \left( (F'(g(z) + \xi_{yz}(g(y) - g(z))) - F'(g(z))(g(y) - g(z)) \right) dy \, dz, \] (18)
where we used in the last equality the mean value theorem for intermediate points $\xi_{yz} \in [0,1]$. By the Hölder continuity of $F'$ the above display can be estimated by

$$|F_j(x)| \leq \|F\|_{C^{1+\beta/\alpha}} \int_{\mathbb{R}^2} |K_j(x - y)K_{<j-1}(x - z)||\xi_{yz}^{\beta/\alpha}||g(y) - g(z)||^{\beta/\alpha + 1} dy dz$$

$$\leq \|F\|_{C^{1+\beta/\alpha}} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(z)||g(x - y) - g(x - z)||^{\beta/\alpha + 1} dy dz.$$  

Now we can estimate the $L^{p'}$-norm of the integral by the integral of the $L^{p'}$-norm, which yields

$$\|F_j\|_{L^{p'}} \leq \|F\|_{C^{1+\beta/\alpha}} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(y - h)| \omega_p(|h|) \|g\|_{L^{1+\beta/\alpha}} dy dh$$

$$\leq \|F\|_{C^{1+\beta/\alpha}} \|g\|_{L^{1+\beta/\alpha}} \left( \int \|h||^{\alpha + \beta + d/q} \int |K_j(y)K_{<j-1}(y - h)| dy \right)^{1/q'}$$

(19)

(with $d/q := 0$ for $q = \infty$ and the usual modification for $q' = \infty$). Abbreviating $\delta := \alpha + \beta + d/q$, the last integral can be written as

$$\|h||^{\delta} \left( |K_j| \ast |K_{<j-1}(\cdot))|\right)(h) \right\|_{L^{\delta'}}$$

$$\leq \|h||^{\delta} \left( |K_j| \ast |K_{<j-1}(\cdot))|\right)(h) \right\|_{L^{\delta'}}$$

$$\leq \|h||^{\delta} \left( |K_j| \ast |K_{<j-1}(\cdot))|\right)(h) \right\|_{L^{\delta'}}$$

where we apply Young’s inequality in the last estimate. Due to $K_j = \mathcal{F}^{-1} \rho_j = (2\pi)^{-d/2} j^d \mathcal{F} \rho(2^j \cdot)$, we see easily that $\|h||^{\delta} \left( |K_j| \ast |K_{<j-1}(\cdot))|\right)(h) \right\|_{L^{\delta'}} \leq 2^{-j(\alpha + \beta)}$ and $\|K_j\|_{L^1} \leq 1$. To bound similarly the norms of $K_{<j-1}$ note that $\mathcal{F}K_{<j-1}$ is uniformly bounded and supported on a ball with radius of order $2^j$. We conclude

$$\|F_j\|_{L^{p'}} \leq 2^{-j(\alpha + \beta)} \|F\|_{C^{1+\beta/\alpha}} \|g\|_{L^{1+\beta/\alpha}}^{1+\beta/\alpha}$$

The claimed bound $\|R_F(g)\|_{\alpha + \beta, p, \infty}$ thus follows from Lemma A and choosing $q = \infty$.

To show the Hölder continuity, we will apply similar arguments. For convenience we define $\Delta f(y, z) := f(y) - f(z)$ for any function $f$. Using the additional regularity of $F$, we obtain from (18) that

$$F_j(x) = \int_{\mathbb{R}^2} K_j(x - y)K_{<j-1}(x - z) \int_0^1 \left( F'(g(z) + s\Delta g(y, z)) - F'(g(z)) \right) \Delta g(y, z) ds dy dz$$

$$= \int_{\mathbb{R}^2} K_j(x - y)K_{<j-1}(x - z) \int_0^1 \int_0^1 sF''(g(z)) + r s \Delta g(y, z) \Delta g(y, z)^2 dr ds dy dz.$$

Hence, we can write

$$R_F(g) = R_F(h) = \sum_{j \geq -1} G_j$$
with
\[ G_j(x) = \int_{\mathbb{R}^2} \int_0^1 \int_0^1 K_j(x-y)K_{<j-1}(x-z)s \left( F''(g(z) + rs\Delta g(y,z))\Delta g(y,z)^2 \right) \, dr \, ds \, dy \, dz \]
\[ = \int_{\mathbb{R}^2} \int_0^1 \int_0^1 K_j(x-y)K_{<j-1}(x-z)s \left( \left( F''(g(z) + rs\Delta g(y,z)) - F''(h(z) + rs\Delta h(y,z)) \right) \right) \times \Delta g(y,z)^2 + F''(h(z) + rs\Delta h(y,z)) \left( \Delta g(y,z)^2 - \Delta h(y,z)^2 \right) \, dr \, ds \, dy \, dz. \]

The Hölder continuity of \( F'' \) yields
\[ |G_j(x)| \leq \| F \|_{C^{2,\gamma}} \int_{\mathbb{R}^2} \int_0^1 \int_0^1 |K_j(x-y)K_{<j-1}(x-z)| \left( |(g-h)(z) + rs\Delta(g-h)(y,z)| \right)^\gamma \times \left( |\Delta g(y,z)|^2 + |\Delta (g-h)(y,z)| \right) \, dr \, ds \, dy \, dz \]
\[ \leq \| F \|_{C^{2,\gamma}} \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \left( |g-h|_\infty |\Delta g(x,y,x-z)|^2 \right. \left| \Delta h(x,y,x-z) \right| |L^p| + \left| \Delta h(x,y,x-z) \right| |L^p| \right) \, dy \, dz \]
\[ \leq \| F \|_{C^{2,\gamma}} \int_{\mathbb{R}^2} \left( |K_j| \right) \right| K_{<j-1}(\cdot) \right) \left| \right| \left( |g-h|_\infty |\Delta g(x,y,x-z)|^2 \right. \left| \Delta h(x,y,x-z) \right| |L^p| + \left| \Delta h(x,y,x-z) \right| |L^p| \right) \, dy \, dz \]
\[ \leq \| F \|_{C^{2,\gamma}} \left( |g-h|_\infty \|g\|_a,p,2q + |g-h|_a,p,2q \right) \left( |g|_a,p,2q + |h|_a,p,2q \right) 2^{-j2\alpha}. \]

The claimed bound follows again from Lemma A.1 and the symmetry in \( g \) and \( h \).

In the situation of Proposition 4.1, we conclude
\[ F(u) = T_{F(u)}u + R_F(u) \quad \text{with} \quad \| R_F(u) \|_{2a,p/2,\infty} \lesssim \| u \|_{2a,p,\infty}^2. \]

Due to this linearization it remains to study \( \pi(T_{F(u)}u, \xi) \). For Hölder continuous functions Gubinelli et al. [2], Lem. 2.4] have shown that the terms \( \pi(T_{F(u)}u, \xi) \) and \( F'(u)\pi(u, \xi) \) only differ by a smoother remainder. To find an estimate of the regularity for the commutator
\[ \Gamma(f, g, h) := \pi(T_fg, h) - f \pi(g, h) \]
(20)
in general Besov norms, we first prove the following auxiliary lemma, cf. [2, Lem. 2.97].

**Lemma 4.3.** Let \( p, p_1, p_2 \geq 1 \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \). Then for \( \alpha \in (0, 1) \) and for any \( f \in B^a_{p,1,\infty}(\mathbb{R}^d) \) and \( g \in L^{p_2}(\mathbb{R}^d) \) the operator \( \Delta_j, f \) satisfies
\[ \| [\Delta_j, f]g \|_{L^p} \lesssim 2^{-j\alpha} \| f \|_{a,p_1,\infty} \| g \|_{L^{p_2}}. \]

**Proof.** Since \( \Delta_j f = (F^{-1} \rho_j) * f \), we observe
\[ [\Delta_j, f]g(x) = F^{-1} \rho_j * (fg)(x) - f(F^{-1} \rho_j * g)(x) \]
\[ = \int_{\mathbb{R}} F^{-1} \rho_j(y)(f(x-y) - f(x))g(x-y) \, dy, \quad x \in \mathbb{R}^d. \]
Minkowski’s and Hölder’s inequalities yield
\[
\|\langle \Delta_j, f \rangle \|_{L^p} \leq \int_{\mathbb{R}} \|F^{-1} \rho_j(y) \langle f(\cdot - y) - f \rangle \|_{L^p} \ dy
\]
\[
\leq \|g\|_{L^{p_2}} \int_{\mathbb{R}} \|F^{-1} \rho_j(y)\| \|f(\cdot - y) - f\|_{L^{p_1}} \ dy.
\]
With the modulus of continuity \(\varphi\) and the corresponding Besov norm, we obtain
\[
\|\langle \Delta_j, f \rangle \|_{L^p} \leq \|g\|_{L^{p_2}} \int_{\mathbb{R}} \|F^{-1} \rho_j(y)\varphi_{\rho_1}(f, |y|)\| \ dy
\]
\[
\leq \|g\|_{L^{p_2}} \sup_{y \in \mathbb{R}^d} \{ |y|^{-\alpha} \varphi_{\rho_1}(f, |y|) \} \int_{\mathbb{R}} |y|^{\alpha} \|F^{-1} \rho_j(y)\| \ dy
\]
\[
\approx \|f\|_{a_0, p_1, \infty} \|g\|_{L^{p_2}} \| |y|^{\alpha} \|F^{-1} \rho_j(y)\|_{L^1}.
\]
For \(j = -1\) the previous \(L^1\)-norm is finite because \(\chi\) is smooth and compactly supported. For \(j \geq 0\) we additionally note that \(F^{-1} \rho_j = 2^{jd} F(2^j \cdot)\) implies
\[
\| |y|^{\alpha} \|F^{-1} \rho_j(y)\|_{L^1} = 2^{-j\alpha} \| |y|^{\alpha} \|F^{-1} \rho(y)\|_{L^1} \lesssim 2^{-j\alpha}.
\]
\[\blacklozenge\]

**Lemma 4.4.** Let \(\alpha \in (0, 1)\), \(\beta, \gamma \in \mathbb{R}\) such that \(\alpha + \beta + \gamma > 0\) and \(\beta + \gamma < 0\). Moreover, let \(p_1, p_2, p_3 \geq 1\) satisfy \(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p} \leq 1\) and let \(q \geq 1\). Then for \(f, g, h \in S(\mathbb{R}^d)\) the commutator operator from \((20)\) satisfies
\[
\|\Gamma(f, g, h)\|_{\alpha + \beta + \gamma, p, q} \lesssim \|f\|_{a_0, p_1, q} \|g\|_{b_0, p_2, q} \|h\|_{\gamma, p_3, q}.
\]
Therefore, \(\Gamma\) can be uniquely extended to a bounded trilinear operator \(\Gamma : B^\alpha_{p_1, q}(\mathbb{R}^d) \times B^\beta_{p_2, q}(\mathbb{R}^d) \times B^\gamma_{p_3, q}(\mathbb{R}^d) \to B^{\alpha + \beta + \gamma}_{p, q}(\mathbb{R}^d)\).

**Proof.** Let \(f, g, h \in S(\mathbb{R}^d)\). Using \(T_f g = \sum_{k \geq -1} \sum_{i \geq k+2} \Delta_k f \Delta_i g = \sum_{k \geq -1} \Delta_k f (g - S_{k+2} g)\), we decompose
\[
\Gamma(f, g, h) = \pi(T_f g, h) - \pi(f, g, h)
\]
\[
= \sum_{j \geq -1} \sum_{i \geq j} (\Delta_i(T_f g) \Delta_j h - f \Delta_i \Delta_j h)
\]
\[
= \sum_{j \geq -1} \sum_{i \geq j} \left( \Delta_i \left( (\Delta_k f) (g - S_{k+2} g) \right) - \Delta_k f \Delta_i g \right) \Delta_j h
\]
\[
= - \sum_{k \geq -1} \sum_{j \geq -1} \sum_{i \geq j} \Delta_k f \Delta_i (S_{k+2} g) \Delta_j h + \sum_{j \geq -1} \sum_{k \geq -1} \sum_{i \geq j} \left( \Delta_i \Delta_k f (g - S_{k+2} g) \right) \Delta_j h.
\]
\[
= \text{a}_k + \text{b}_j
\]
(21)

We will separately estimate both sums in the following.
For \(k \geq -1\) we have \(\Delta_i(S_{k+2} g) = 0\) for \(i > k + 2\) due to property [iii] of the dyadic partition of unity. Consequently,
\[
a_k = \sum_{i = -1}^{k+2} \sum_{j \geq -1} \sum_{i \geq j} \Delta_k f \Delta_i(S_{k+2} g) \Delta_j h
\]
and its Fourier transform satisfies \(\text{supp} \hat{F} a_k \subseteq 2^k \mathcal{B}\) for some ball \(\mathcal{B}\). Hölder’s inequality yields
\[
\|a_k\|_{L^p} \leq \|\Delta_k f\|_{L^{p_1}} \sum_{i = -1}^{k+2} \sum_{j \geq -1} \sum_{i \geq j} \|\Delta_i(S_{k+2} g)\|_{L^{p_2}} \|\Delta_j h\|_{L^{p_3}}.
\]
Owing to $\Delta_i(S_{k+2}g) = \Delta_ig$ for $i \leq k$ and $\|\Delta_i\Delta_kg\|_{L^2} \leq \|F^{-1}p_i\|_{L^1}\|\Delta_kg\|_{L^2} \lesssim \|\Delta_kg\|_{L^2}$ by Young’s inequality, we have

$$
\|a_k\|_{L^p} \lesssim \|\Delta_kf\|_{L^{p_1}} \sum_{i=0}^{k+2} \sum_{j \geq i+1} \|\Delta_jg\|_{L^2} \|\Delta_jh\|_{L^3}
\lesssim \|\Delta_kf\|_{L^{p_1}} \|g\|_{\beta,p_2,\infty} \|h\|_{\gamma,p_3,\infty} \sum_{i=0}^{k+2} 2^{-i(\beta+\gamma)} \lesssim 2^{-k(\beta+\gamma)} \|\Delta_kf\|_{L^{p_1}} \|g\|_{\beta,p_2,\infty} \|h\|_{\gamma,p_3,\infty},
$$

using $\beta + \gamma < 0$ in the last estimate. Since $2^{k\alpha}\|\Delta_kf\|_{L^{p_1}} \in \ell^q$, Lemma 4.2 yields

$$
\left\| \sum_{k \geq 1} a_k \right\|_{\alpha+\beta+\gamma,\infty} \lesssim \|f\|_{\alpha,p_1,q} \|g\|_{\beta,p_2,\infty} \|h\|_{\gamma,p_3,\infty}.
$$

Now, let us consider the second sum in 21. Note that

$$
b_j = \sum_{i; |i-j| \leq 1} \sum_{k \geq 1} \sum_{k \geq 1-i} \sum_{k < 1+i} (|\Delta_i, \Delta_kf|) \Delta_j h = \sum_{i; |i-j| \leq 1} \sum_{k \geq 1} \sum_{k < 1+i} (|\Delta_i, S_{i-1}f|) \Delta_j h.
$$

Since the support of the Fourier transform of $S_{i-1}f\Delta_i g$ is of the form $2^i A$ for some annulus $A$, we have that

$$
|\Delta_i, S_{i-1}f| \Delta_i g = \Delta_i(S_{i-1}f \Delta_i g) - (S_{i-1}f)(\Delta_i g)
$$

vanishes if $|i-l| \geq N$ for some $N \in \mathbb{N}$. Therefore, $b_j = \sum_{i; |i-j| \leq 1} \sum_{k \geq 1} (|\Delta_i, S_{i-1}f|) \Delta_j h$ has a Fourier transform supported on $2^i A$ for some annulus. Using Hölder’s inequality and Lemma 13, we estimate

$$
\|b_j\|_{L^p} \lesssim \sum_{i; |i-j| \leq 1} \sum_{l \geq 1} 2^{-i\alpha} \|S_{k-1}f\|_{\alpha,p_1,\infty} \|\Delta_l g\|_{L^2} \|\Delta_j h\|_{L^3}
\lesssim 2^{-j(\alpha+\beta+\gamma)} \|f\|_{\alpha,p_1,\infty} \left(2^{j\beta} \sum_{l \geq 1} \|\Delta_l g\|_{L^2}\right) 2^{j\gamma} \|\Delta_j h\|_{L^3}.
$$

For any $q_2, q_3 \geq q$ satisfying $\frac{1}{q} = \frac{1}{q_2} + \frac{1}{q_3}$ Hölder’s inequality and Lemma 4.2 yield then

$$
\left\| \sum_{j \geq 1} b_j \right\|_{\alpha+\beta+\gamma,\infty} \lesssim \|f\|_{\alpha,p_1,\infty} \|g\|_{\beta,p_2,\infty} \|h\|_{\gamma,p_3,\infty}.
$$

To obtain the claimed norm bound, recall that $B^\alpha_{p,q}(\mathbb{R}^d)$ continuously embeds into $B^\alpha_{p,q}(\mathbb{R}^d)$ for any $q \leq q'$.

For $p, q < \infty$ the Schwartz space $S(\mathbb{R}^d)$ is dense $B^\alpha_{p,q}(\mathbb{R}^d)$ for any $\alpha \in \mathbb{R}$ such that there is a unique extension of $C$ on $B^\alpha_{p,q}(\mathbb{R}^d) \times B^\beta_{p,q}(\mathbb{R}^d) \to B^\alpha_{p,q}(\mathbb{R}^d)$. For $p = \infty$ or $q = \infty$ a similar argument as in [24, Lem. 2.4] applies.

Combining the previous results, we obtain the following corollary, cf. [24, Lem. 2.7], which immediately implies Proposition 1.1 due to the embedding $B^\alpha_{p,q} \subseteq L^\infty$ for $\alpha > 1/p$ and $d = 1$.

**Corollary 4.5.** Let $p_1, p_2 \in [1, \infty]$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leq 1$. Let $\alpha \in (0,1)$ and $\beta < 0$ such that $2\alpha + \beta > 0$ and $\alpha + \beta < 0$. Further, suppose $F \in C^{\gamma+1}(\mathbb{R}^m)$ for some $\gamma \in (0,1]$ satisfying $F(0) = 0$. Then there exists a map $\Pi_F: B^\alpha_{p_1,\infty}(\mathbb{R}^d) \times B^\beta_{p_2,\infty}(\mathbb{R}^d) \to B^\alpha_{p,q}(\mathbb{R}^d)$ such that

$$
\pi(F(f),g) = F'(f)\pi(f,g) + \Pi_F(f,g)
$$

and

$$
\|\Pi_F(f,g)\|_{2\alpha+\beta,\infty} \lesssim \|F\|_{C^{\gamma+1}} \|f\|_{\alpha,p_1,\infty} \|g\|_{\beta,p_2,\infty}.
$$
For $f_1, f_2 \in B_{p_1,\infty}^\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $g_1, g_2 \in B_{p_2,\infty}^\beta(\mathbb{R}^d)$ we have furthermore
\[
\|\Pi_F(f_1, g_1) - \Pi_F(f_2, g_2)\|_{2\alpha + \beta, p, \infty} \lesssim \|\Pi\|_{C^{2+}} \left( \|f_1\|^2_{\alpha, p, \infty} \wedge \|f_2\|^2_{\alpha, p, \infty} + \left( \|f_1\|_{\alpha, p, \infty} + \|f_2\|_{\alpha, p, \infty} \right) \times \left( 1 + \|g_1\|_{\beta, p, \infty} \wedge \|g_2\|_{\beta, p, \infty} \right) \right)
\times \left( \|f_1 - f_2\|_{\infty} + \|f_2\|_{\alpha, p, \infty} + \|g_1 - g_2\|_{\beta, p, \infty} \right).
\]

**Proof.** Setting $\Pi_F(f, g) := \Gamma(F'(f), f, g) + \pi(R_F(f), g)$, we can write
\[
\pi(F(f, g)) = F'(f)\pi(f, g) + \Gamma(F'(f), f, g) + \pi(R_F(f), g) = F'(f)\pi(f, g) + \Pi_F(f, g).
\]

Lemmas 4.1 and 4.2 yield
\[
\|\Pi_F(f, g)\|_{2\alpha + \beta, p, \infty} \leq \|\Gamma(F'(f), f, g)\|_{2\alpha + \beta, p, \infty} + \|\pi(R_F(f), g)\|_{2\alpha + \beta, p, \infty}
\leq \|F'(f)\|_{\alpha, p, \infty} \|f\|_{\alpha, p, \infty} \|g\|_{\beta, p, \infty} + \|R_F(f)\|_{2\alpha, p_1, 2, \infty} \|g\|_{\beta, p_2, \infty}
\leq (\|F'(f)\|_{\alpha, p, \infty} + \|F\|_{L^2} \|f\|_{\alpha, p, \infty}) \|f\|_{\alpha, p, \infty} \|g\|_{\beta, p_2, \infty},
\]
where we again used Besov embeddings. Finally, we apply (7).

The bound of $\|\Pi_F(f_1, g_1) - \Pi_F(f_2, g_2)\|_{2\alpha + \beta, p, \infty}$ follows from analogous estimates, using the argument-wise linearity of $\Gamma$ and $\pi$, the Hölder continuity of $R_F$ from Lemma 4.2 and
\[
\|F'(f_1) - F'(f_2)\|_{\alpha, p, q} = \left\| \int_0^1 F''(f_1 + s(f_2 - f_1))(f_1 - f_2) \right\|_{\alpha, p, q}
\leq \int_0^1 \|F''(f_1 + s(f_2 - f_1))(f_1 - f_2)\|_{\alpha, p, q} ds
\leq \|F''\|_{L^\infty} \left| \int_0^1 f_1 - f_2 \right|_{\alpha, p, q} (22)
\]
for any $q \in [1, \infty]$.

5 The paracontrolled ansatz

Assuming that the driving signal $\xi$ satisfies $\xi \in B_{p,q}^\alpha$ for $\alpha > 1/3$, we come back to the RDE (1). Recall that it was given by
\[
du(t) = F(u(t))\xi(t), \quad u(0) = u_0, \quad t \in \mathbb{R},
\]
where $u_0 \in \mathbb{R}^m, u : \mathbb{R} \to \mathbb{R}^m$ is a continuous function and $F : \mathbb{R}^m \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a family of vector fields on $\mathbb{R}^m$. In Section 5 we have already considered the case $\alpha > 1/2$. The classical way to continuously extend Young’s approach to more irregular driving signals is Lyons’ rough path theory, which additionally to the signal $\xi$ needs to handle the corresponding “iterated integral”.

As an alternative, we use in the present section a new paracontrolled ansatz similar to Gubinelli et al. [24]. We postulate that the solution $u$ of the RDE (1) is of the form
\[
u = T_u \vartheta + u^\sharp
\]
with $\vartheta, u^\vartheta \in B_{p,q}^\alpha$ and a remainder $u^\# \in B_{2/p,2,q}^{2\alpha}$. Decomposing $F(u)\xi$ in terms of Littlewood-Paley blocks and linearizing $F$ by Proposition 4.3, we have
\[
F(u)\xi = T_{F(u)}\xi + \pi(F(u), \xi) + T_\xi(F(u)) = T_{F(u)}\xi + F'(u)\pi(u, \xi) + \Pi_F(u, \xi) + T_\xi(F(u)).
\]

The presumed controlled structure yields that understanding the (problematic) term $\pi(u, \xi)$ reduces further to the analysis of $\pi(\vartheta, \xi)$ owing to the commutator from (20):
\[
\pi(u, \xi) = \pi(T_u \vartheta, \xi) + \pi(u^\#, \xi) = u^\vartheta \pi(\vartheta, \xi) + \Gamma(u^\vartheta, \vartheta, \xi) + \pi(u^\#, \xi).
\]

\[
\in B_{p/3, q}^{2\alpha - 1}, \quad \in B_{p/3, q}^{2\alpha - 1}
\]

\[\[
\]
Plugging the paracontrolled ansatz into the RDE \( \mathbb{1} \), the Leibniz rule and the above observation yield

\[
T_u \vartheta \mathrm{d}\vartheta + T_{u^\vartheta} \vartheta + \mathrm{d}u = T_F(u) \vartheta + F'(u) \vartheta \mathrm{d}\vartheta + \Pi_F(u) \vartheta + T_\xi(F(u)).
\]

Comparing the least regular terms on the left-hand and on the right-hand side, we choose \( \vartheta \) as the solution to \( \vartheta \mathrm{d}\vartheta = \vartheta \) with \( \vartheta(0) = 0 \) and \( u^\vartheta = F(u) \).

As already noted in Section 3, we cannot expect \( \vartheta \) to be contained in any Besov space (cf. Lemma \( \mathbb{22} \)). This requirement would especially be violated in most interesting examples from probability theory, for instance, \( \vartheta \) being Brownian motion or a martingale. In order to circumvent this issue, we use again the localizing function \( \varphi \) from Assumption \( \mathbb{7.4} \). Still relying on \( \vartheta \mathrm{d}\vartheta = \vartheta \) and \( \vartheta(0) = 0 \), we introduce the local version of the signal

\[
\vartheta_\varphi := \varphi \vartheta \quad \text{and} \quad \xi_\varphi := \vartheta_\varphi \varphi + \varphi_\vartheta \vartheta.
\]

The corresponding localized RDE is then given by

\[
du = F(u) \xi_\varphi, \quad u(0) = u_0.
\] (23)

This differential equation coincides with the original one on the interval \( [-T, T] \) due to \( \varphi(t) = 1 \) and \( \varphi'(t) = 0 \) for \( |t| \leq T \).

Summarizing briefly the above discussion, we need two additional pieces of information about very irregular signals. Namely, \( \xi_\varphi \) has to be the derivative of a path \( \vartheta_\varphi \) with compact support and the resonant term \( \pi(\vartheta_\varphi, \xi_\varphi) \) has to be well-defined. This precisely corresponds to the classical rough path theory, where a path \( \vartheta \) defined on some compact interval is enhanced with the information of the iterated integral \( \int \vartheta \mathrm{d}\vartheta \).

Analogously to the notion of geometric rough path (cf. for example Section 2.2. in \( \mathbb{16} \)), we introduce now the notion of geometric Besov rough path:

**Definition 5.1.** Let \( T > 0 \) and let \( C^\infty_p \) be the space of smooth functions \( \vartheta_\varphi : \mathbb{R} \to \mathbb{R}^n \) with support \( \vartheta_\varphi \subseteq [-2T, 2T] \) and \( \vartheta_\varphi(0) = 0 \). The closure of the set \( \{ (\vartheta_\varphi, \pi(\vartheta_\varphi, \mathrm{d}\vartheta_\varphi)) : \vartheta_\varphi \in C^\infty_p \} \subseteq B^0_{p,q} \times B^2_{p,2/2,q-1} \) with respect to the norm \( \| \cdot \|_{\alpha_p,q} + \| \cdot \|_{2\alpha_p-1,2/2,q} \) is denoted by \( B^{0,\alpha}_{p,q} \) and \((\vartheta_\varphi, \eta_\varphi) \in B^{0,\alpha}_{p,q} \) is called geometric Besov rough path.

Even with the driving signal \( (\vartheta, \eta) \in B^{0,\alpha}_{p,q} \) we unfortunately cannot expect in general that the solution \( u \) to the Cauchy problem \( \mathbb{23} \) with \( \xi_\varphi = \vartheta_\varphi \mathrm{d}\vartheta_\varphi \) lies in any Besov spaces \( B^{\alpha}_{p,q} \) for finite \( p \) and \( q \). On the other hand, Besov spaces on the compact domain \( [-T, T] \) seem not convenient for the paraproduct approach since Littlewood-Paley theory and Bony’s paraproduct are from their very nature constructed on the whole real line. It appears to be natural to instead consider a weighted version of the Itô-Lyons \( \tilde{S} \) map given by

\[
\tilde{S} : \mathbb{R}^n \times B^{0,\alpha}_{p,q} \to B^{\alpha}_{p,q} \quad \text{via} \quad (u_0, \vartheta_\varphi, \pi(\vartheta_\varphi, \mathrm{d}\vartheta_\varphi)) \mapsto \psi u,
\] (24)

where \( u \) solves \( \mathbb{23} \) with \( \xi_\varphi = \vartheta_\varphi \mathrm{d}\vartheta_\varphi \) and \( \psi : \mathbb{R} \to (0, \infty) \) is a regular weight function being constant one on \( [-2T, 2T] \). Consequently, provided \( \vartheta_\varphi \in C^\infty_p \) with \( \xi_\varphi = \vartheta_\varphi \mathrm{d}\vartheta_\varphi \) the weighted solution \( \tilde{u} := \psi u \) possesses the dynamic

\[
d\tilde{u} = \psi \mathrm{d}u + \psi' u = F(\tilde{u}) \xi_\varphi + \frac{\psi'}{\psi} \tilde{u}, \quad \tilde{u}(0) = u_0.
\] (25)

Let us emphasize that also this weighted differential equation still coincides with the original RDE \( \mathbb{1} \) restricted to the interval \( [-T, T] \). While the very recently developed semigroup approach to paracontrolled calculus by Bailleul and Bernicot \( \mathbb{3} \) might allow for working without the weight \( \psi \), this would lead to non-standard Littlewood-Paley blocks and Besov spaces.

The aim is now to continuously extend the weighted Itô-Lyons map \( \tilde{S} \) from smooth functions with support in \([ -2T, 2T ] \) to the geometric Besov rough paths or more precisely from the domain \( \mathbb{R}^d \times \{ (\vartheta_\varphi, \pi(\vartheta_\varphi, \mathrm{d}\vartheta_\varphi)) : \vartheta_\varphi \in C^\infty_p \} \) to \( \mathbb{R}^d \times B^{0,\alpha}_{p,q} \). For this purpose we specify our assumptions on the weight function \( \psi \) as follows:
**Assumption 5.2.** For any $\mathcal{T} > 0$ let $\psi = \psi_{\mathcal{T}} \in B_{p,q}^\alpha \cap C^1$ be a strictly positive function which is equal to one on $[-2\mathcal{T}, 2\mathcal{T}]$ and suppose that there exist two constants $C_\psi, c_\psi > 0$ such that $\|\psi/\psi\|_\infty \lesssim C_\psi$ and $\max\{\psi(2\mathcal{T} + 1), \psi(-2\mathcal{T} - 1)\} > c_\psi$.

The conditions on $\psi$ are quite weak and allow for a large variety of weight functions as illustrated by the following examples.

**Example 5.3.** Let $\alpha \in (0, 1)$, $\mathcal{T} > 0$ and $\kappa \in (0, 1)$.

(i) The function

$$\psi_{\mathcal{T}}(t) := \begin{cases} 1, & |t| \leq 2\mathcal{T}, \\ \exp\left(-\frac{\kappa (|t| - 2\mathcal{T})^2}{1 + |t| - 2\mathcal{T}}\right), & |t| > 2\mathcal{T}, \end{cases}$$

satisfies Assumption 5.2 for $C_\psi = \kappa$ and $c_\psi = \epsilon^{-1/2}$.

(ii) The function

$$\tilde{\psi}_{\mathcal{T}}(t) := \begin{cases} 1, & |t| \leq 2\mathcal{T}, \\ (1 + \kappa(|t| - 2\mathcal{T})^2)^{-1/2}, & |t| > 2\mathcal{T}, \end{cases}$$

satisfies Assumption 5.2 for $C_\psi = \sqrt{\kappa}$ and $c_\psi = 1/4$.

For later reference let us remark a property which makes weight functions fulfilling Assumption 5.2 so suitable in our context.

**Remark 5.4.** For any two weight functions $\psi$ and $\tilde{\psi}$ satisfying Assumption 5.2, the resulting weighted Besov norms of the solution $u$ are equivalent. More precisely, it is elementary to show

$$\|\psi u\|_{\alpha,p,q} \lesssim (1 + c_\psi^{-1}\|\tilde{\psi} - \psi\|_{\alpha,p,q})\|\tilde{\psi} u\|_{\alpha,p,q}$$

for any $u \in B_{p,q}^\alpha$ which is constant on $(-\infty, -2\mathcal{T}]$ and on $[2\mathcal{T}, \infty)$.

In order to analyze the weighted RDE (25), we modify our ansatz to $\tilde{u} = T_{F(\tilde{u})}\vartheta_{\mathcal{T}} + u^\#$, where $u^\# \in B_{p/2,q}^{2\alpha}, \vartheta_{\mathcal{T}} \in C_\mathcal{T}^\infty$.

Roughly speaking, in the terminology of 24 the pair $(\tilde{u}, F(\tilde{u})) \in (B_{p,q}^\alpha)^2$ is said to be paracontrolled by $\vartheta_{\mathcal{T}} \in B_{p,q}^\alpha$. The dynamic of $u^\#$ is characterized in the next lemma.

**Lemma 5.5.** Let $u_0 \in \mathbb{R}^n$, let $\vartheta_{\mathcal{T}} \in C_\mathcal{T}^\infty$ with derivative $\vartheta_{\mathcal{T}} = d\vartheta_{\mathcal{T}}$ and suppose that $\psi$ satisfies Assumption 5.2. Then the following conditions are equivalent:

(i) $u$ is the solution to the ODE (25),

(ii) $u$ can be written as $u = \psi^{-1}\tilde{u}$ where $\tilde{u}$ solves the ODE (24),

(iii) $\tilde{u}$ can be written as $\tilde{u} = T_{F(\tilde{u})}\vartheta_{\mathcal{T}} + u^\#$ where $u^\#$ solves

$$du^\# = F(\tilde{u})\vartheta_{\mathcal{T}} - d(T_{F(\tilde{u})}\vartheta_{\mathcal{T}}) + \frac{\psi'}{\psi^{1/2}}\tilde{u}, \quad u^\#(0) = u_0 - T_{F(\tilde{u})}\vartheta_{\mathcal{T}}(0). \tag{26}$$

**Proof.** For the equivalence between (i) and (ii) note that $u = \psi^{-1}\tilde{u}$ is well-defined by Assumption 5.2 and that we have by the Leibniz rule

$$du = d(\psi^{-1}\tilde{u}) = \psi^{-1}d\tilde{u} - \frac{\psi'}{\psi^{1/2}}\tilde{u} = F(u)\vartheta_{\mathcal{T}}, \quad u(0) = \psi^{-1}(0)\tilde{u}(0) = u_0.$$

The equivalence between (ii) and (iii) follows by combining $\tilde{u} = T_{F(\tilde{u})}\vartheta_{\mathcal{T}} + u^\#$ and (25), which yields

$$du^\# = d\tilde{u} - d(T_{F(\tilde{u})}\vartheta_{\mathcal{T}}) = F(\tilde{u})\vartheta_{\mathcal{T}} - d(T_{F(\tilde{u})}\vartheta_{\mathcal{T}}) + \frac{\psi'}{\psi^{1/2}}\tilde{u}$$

and due to $\tilde{u}(0) = u(0) = u_0$ the initial condition satisfies $u^\#(0) = u_0 - T_{F(\tilde{u})}\vartheta_{\mathcal{T}}(0)$. □
As we have seen in the discussion at the beginning of the present section, we want to reduce the resonant term \( \pi(F(\tilde{u}), \xi_T) \) to the resonant term \( \pi(\vartheta_T, \xi_T) \). Indeed, this is possible as proven in the following proposition. The specific form of \( u \) allows to improve the quadratic estimate \( ^{17} \) in Proposition 4.1 to a linear one. Its proof is inspired by Lemma 5.2 by Gubinelli et al. \( ^{24} \).

**Proposition 5.6.** Let \( \alpha \in (\frac{1}{T}, \frac{1}{2}) \), \( p \geq 3 \), \( q \geq 1 \), and \( F \in C^2 \) with \( F(0) = 0 \). If \( \vartheta_T \in C^\infty \) with derivative \( \xi_T = d\vartheta_T \), then for \( \tilde{u} = T_{F(\tilde{u})} \vartheta_T + u^\# \) with \( \tilde{u} \in B_{p,q}^\alpha \) and \( u^\# \in B_{p,q}^{2\alpha} \) one has

\[
\| \pi(F(\tilde{u}), \xi_T) \|_{2\alpha-1,p/2,q} \lesssim \left( \| F \|_{C^2} \vee \| F \|_{C^1}^2 \right) \left( \| u \|_{\alpha,p,q} + \| u^\# \|_{2\alpha,p/2,q} \right)
\times \left( \| \vartheta_T \|_{\alpha,p,q} + \| \vartheta_T \|_{\alpha,p,q}^2 + \| \pi(\vartheta_T, \xi_T) \|_{2\alpha-1,p/2,q} \right).
\]

**Proof.** Step 1: To avoid the quadratic estimate, we first need a modified version of Lemma 4.2. We will borrow some notation from the proof of this former lemma. For brevity we define \( v_u := T_{F(\tilde{u})} \vartheta_T \) and recall \( \tilde{u} := \psi/u \) such that \( \tilde{u} = v_u + u^\# \). We write

\[
F(\tilde{u}) - F(0) = T_{F(\tilde{u})} \vartheta_T + R_F(\tilde{u})
\]

with

\[
R_F(\tilde{u}) = \sum_{j \geq 1} F_j \quad \text{with} \quad F_j := \Delta_j(F(\tilde{u}) - F(0)) - S_{-1}(F'(\tilde{u})) \Delta_j \tilde{u}.
\]

For \( j \leq 0 \), we saw in Lemma 4.2 that \( \| F_j \|_{L^{p/2}} \lesssim \| F \|_{C^1} \| \tilde{u} \|_{L^{p/2}} \) which yields

\[
\| F_j \|_{L^{p/2}} \lesssim \| F \|_{C^1}(\| v_u \|_{L^{p/2}} + \| u^\# \|_{L^{p/2}})
\]

\[
\lesssim \| F \|_{C^1}(\| T_{F(\tilde{u})} \vartheta_T \|_{L^{p/2}} + \| u^\# \|_{L^{p/2}}).
\]

For \( j > 0 \), we deduce from 18 and our ansatz that

\[
|F_j| = \left| \int_{\mathbb{R}^2} K_j(x-y)K_{<j-1}(x-z) \left( (F'(\tilde{u}(y)) + \xi_{g_{jz}(\tilde{u}(y) - \tilde{u}(z))) - F'(\tilde{u}(y))) \right)
\times (v_u(y) - v_u(x) + u^\#(y) - u^\#(x)) \right| dy dz,
\]

\[
\lesssim \| F \|_{C^2} \int_{\mathbb{R}^2} |K_j(x-y)K_{<j-1}(x-z)||\tilde{u}(y) - \tilde{u}(z)||v_u(y) - v_u(x)|| dy dz
\]

\[
+ 2\| F \|_{C^1} \int_{\mathbb{R}^2} |K_j(x-y)K_{<j-1}(x-z)||u^\#(y) - u^\#(z)|| v_u(y) - v_u(x) || dy dz.
\]

Proceeding as in proof of Lemma 4.2 and applying Hölder’s inequality, we obtain for \( q^* = q/(q-1) \)

\[
\| F_j \|_{L^{p/2}} \lesssim \| F \|_{C^2} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(z)||\tilde{u}(x) - (y - z) - \tilde{u}(z)||_{L^p}
\times \| (v_u(x) - (y - z)) - v_u(x) \|_{L^p} dy dz
\]

\[
+ 2\| F \|_{C^1} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(z)||u^\#(x) - (y - z) - u^\#(z)||_{L^{p/2}} dy dz
\]

\[
\lesssim \| F \|_{C^2} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(y - h)|\omega_p(|v_u|, |h|)\omega_p(v_u, |h|) dy dh.
\]

\[
+ 2\| F \|_{C^1} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(y - h)|\omega_p(u^\#, |h|) dy dh
\]

\[
\lesssim \| F \|_{C^2} \left( \| h \|^{2\alpha+1/q} \left( |K_j| \ast |K_{<j-1}(-\cdot)| \right)(h) \right)
\times \left( \| h \|^{-\alpha}\omega_p(|v_u|, |h|) \right)_{\infty}
\times \left( \| h \|^{-\alpha-1/q}\omega_p(|v_u|, |h|) \right)_{L^q}
\]

\[
+ 2\| F \|_{C^1} \int_{\mathbb{R}^2} |K_j(y)K_{<j-1}(y - h)|\omega_{p/2}(u^\#, |h|) dy dh
\]

\[
\lesssim \| F \|_{C^2} \left( \| v_u \|_{\alpha,p,q} + \| u^\# \|_{2\alpha,p/2,q} \right).
\]

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Due to Lemma 2.1 one further has
\[ \|v_u\|_{\alpha,p,\infty} = \|T_{F(u)}\vartheta_T\|_{\alpha,p,\infty} \lesssim \|T_{F(u)}\vartheta_T\|_{\alpha,p,q} \lesssim \|F\|_\infty \|\vartheta_T\|_{\alpha,p,q} \]
and thus Lemma 3.1 gives
\[ \|R_F(\tilde{u})\|_{2\alpha,p,2/\infty} \lesssim \|F\|_{C^2}(1 + \|F\|_\infty \|\vartheta_T\|_{\alpha,p,q})(\|\tilde{u}\|_{\alpha,p,q} + \|u^\#\|_{2\alpha,p,2/\infty}). \]  
(27)

Step 2: Plugging in the ansatz once again and keeping the definition of our commutator (20) in mind, we decompose
\[
\pi(F(\tilde{u}),\xi_T) = \pi(T_{F(\tilde{u})}\tilde{u},\xi_T) + \pi(R_F(\tilde{u}),\xi_T) \\
= \pi(T_{F(\tilde{u})}T_{F(\tilde{u})}\vartheta_T,\xi_T) + \pi(T_{F(\tilde{u})}u^\#,\xi_T) + \pi(R_F(\tilde{u}),\xi_T) \\
= F'(\tilde{u})\pi(T_{F(\tilde{u})}\vartheta_T,\xi_T) + \Gamma(F(\tilde{u}),T_{F(\tilde{u})}\vartheta_T,\xi_T) + \pi(T_{F(\tilde{u})}u^\#,\xi_T) + \pi(R_F(\tilde{u}),\xi_T) \\
= F'(\tilde{u})F(\tilde{u})\pi(\vartheta_T,\xi_T) + F'(\tilde{u})\Gamma(F(\tilde{u}),\vartheta_T,\xi_T) + \pi(T_{F(\tilde{u})}u^\#,\xi_T) + \pi(R_F(\tilde{u}),\xi_T).
\]  
(28)

Therefore, we can bound \(\|\pi(F(\tilde{u}),\xi_T)\|_{2\alpha-1,p,2/\infty}\) by estimating these five terms separately. We will apply the following bound which holds owing to the Besov embedding \(B^{2\alpha-1}_{p/3,q/2} \subseteq B^{2\alpha-1}_{p/2,q/2}\) due to \(\alpha > 1/p\) and which uses Bony’s estimates and \(2\alpha - 1 < 0\) for \(f \in L^\infty \cup B^{\alpha}_{p,\infty}\) and \(g \in B^{2\alpha-1}_{p/2,q/2}\) it holds
\[
\|fg\|_{2\alpha-1,p,2/\infty/q/2} \lesssim \|fg\|_{2\alpha-1,p,2/\infty/q/2} + \|(f,g)\|_{3\alpha-1,p/3,q/2} + \|T_\vartheta f\|_{2\alpha-1,p,2/\infty/q/2} \\
\lesssim \|f\|_\infty \|g\|_{2\alpha-1,p,2/\infty/q/2} + \left(\|f\|_{L^\infty} \|g\|_{3\alpha-1,p/3,q/2} \wedge \|f\|_{\alpha,p,\infty} \|g\|_{2\alpha-1,p,2/\infty/q/2}\right) \\
+ \|g\|_{2\alpha-1,p,2/\infty/q/2} \|f\|_{L^\infty} \\
\lesssim \left(\|f\|_\infty \|g\|_{3\alpha-1,p/3,q/2} \wedge \|f\|_{\alpha,p,\infty} \|g\|_{2\alpha-1,p,2/\infty/q/2}\right). \]  
(29)

Furthermore, note for the following estimates that \(\|\xi_T\|_{\alpha-1,p,q} \lesssim \|\vartheta_T\|_{\alpha,p,q}\) thanks to the lifting property of Besov spaces, cf. [13, Thm. 2.3.8].

Applying (20) and (7) to \(F := F'F\), we obtain for the first summand
\[
\|F'(\tilde{u})F(\tilde{u})\pi(\vartheta_T,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2} \lesssim \tilde{F}(\tilde{u})\|\vartheta_T\|_{\alpha,p,\infty} \|\vartheta_T\|_{\alpha,p,\infty} \|\pi(\vartheta_T,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2} \\
\lesssim \|F\|_{C_1} \|F\|_{C_2} \|\tilde{u}\|_{\alpha,p,q} \|\pi(\vartheta_T,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2}.
\]

For the second term the above estimate (29) and Lemma 3.4 yield
\[
\|F'(\tilde{u})\Gamma(F(\tilde{u}),\vartheta_T,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2} \lesssim \|F'\|_\infty \|\Gamma(F(\tilde{u}),\vartheta_T,\xi_T)\|_{3\alpha-1,p/3,q} \\
\lesssim \|F'\|_\infty \|F(\tilde{u})\|_{\alpha,p,q} \|\vartheta_T\|_{\alpha,p,q} \|\xi_T\|_{\alpha-1,p,q} \\
\lesssim \|F\|_{C_1} \|\tilde{u}\|_{\alpha,p,q} \|\vartheta_T\|_{\alpha,p,q}^2,
\]
where (7) is used in the last line. Lemmas 2.1 and 3.4 again together with (7) gives for the third term
\[
\|\Gamma(F'(\tilde{u}),T_{F(\tilde{u})}\vartheta_T,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2} \lesssim \|F'(\tilde{u})\|_{\alpha,p,q} \|T_{F(\tilde{u})}\vartheta_T\|_{\alpha,p,q} \|\xi_T\|_{\alpha-1,p,q} \\
\lesssim \|F\|_{C_1} \|\tilde{u}\|_{\alpha,p,q} \|\vartheta_T\|_{\alpha,p,q}^2.
\]

The second last term in (28) can be estimated by
\[
\|\pi(T_{F'(\tilde{u})}u^\#,\xi_T)\|_{2\alpha-1,p,2/\infty/q/2} \lesssim \|T_{F'(\tilde{u})}u^\#\|_{2\alpha,p,2/\infty/q/2} \|\xi_T\|_{\alpha-1,p,q} \\
\lesssim \|F'\|_\infty \|u^\#\|_{2\alpha,p,2/\infty/q/2} \|\vartheta_T\|_{\alpha,p,q}.
\]
where a Besov embedding, Lemma 2.1 and (7) are used. Finally, for the last term, note that there is some \( \varepsilon \in (0, \alpha - 1/2) \) such that \( 3\alpha - 1 - \varepsilon > 0 \). Applying Lemma 2.1 Step 1 and Besov embeddings, we get

\[
\|\pi(R_F(\tilde{u}), \xi_T)\|_{2\alpha-1,p/2,q} \lesssim \|\pi(R_F(\tilde{u}), \xi_T)\|_{3\alpha-1-\varepsilon,p/3,q} \\
\lesssim \|R_F(\tilde{u})\|_{2\alpha-1,p/2,q} \|\xi_T\|_{\alpha-1,p,q} \\
\lesssim \|F\|_{C^2(1 + \|\xi_T\|_{\alpha,p,q})} (\|\tilde{u}\|_{\alpha,p,q} + \|\nabla \tilde{u}\|_{2\alpha,p/2,q}) \|\xi_T\|_{\alpha,p,q}.
\]

These five estimates combined lead to the asserted bound.

\( \square \)

**Remark 5.7.** The requirement \( F(0) = 0 \) seems to be a purely technical assumption. In view of Lemma 4.2, we can decompose in general \( \pi(F(\tilde{u}), \xi_T) = \pi(F(F(\tilde{u}) - F(0), \xi_T) + \pi(F(0), \xi_T) \). If \( p = \infty \) the additional term can be easily estimated. If \( p < \infty \), it seems more reasonable to decompose \( F(\tilde{u})\xi_T = (F(\tilde{u}) - F(0))\xi_T + F(0)\xi_T \) at the beginning. Hence, we decided to assume the condition \( F(0) = 0 \). Otherwise, all estimates would become even more involved by keeping track of the additional term due to \( F(0) \neq 0 \) without needing conceptional new ideas.

Having established a linear upper bound for the resonant term \( \pi(F(\tilde{u}), \xi_T) \), we deduce the boundedness of the solution to the localized RDE (23) in the weighted Besov norm.

**Corollary 5.8.** Let \( \alpha \in (1/3, 1/2) \), \( p \geq 3 \), \( q \geq 1 \) and \( F \in C^2 \) with \( F(0) = 0 \). Let \( \vartheta_T \in C_\infty^\infty \) with derivative \( \xi_T = d\vartheta_T \). If the bound

\[
\|F\|_{C^2} \vee \|F\|_{C^2}^2 < c(T^3 \vee 1) \left( \|\vartheta_T\|_{\alpha-1,p,q} + \|\vartheta_T\|_{\alpha,p,q}^2 + \|\pi(\vartheta_T, \xi_T)\|_{2\alpha-1,p/2,q} \right)^{-1}
\]

holds for a universal constant \( c > 0 \), independent of \( \vartheta, F, u_0 \) and if \( \psi \) satisfies Assumption 5.6 for some sufficiently small \( C_\psi \), then the solution \( u \) to (23) satisfies

\[
\|\psi u\|_{\alpha,p,q} \lesssim (T^2 \vee 1)(\|u(0)\| + (\|F\|_{C^2} \vee \|F\|_{C^2}^2)(\|\vartheta_T\|_{\alpha,p,q} + 1) \\
\times (\|\vartheta_T\|_{\alpha,p,q} + \|\vartheta_T\|_{\alpha,p,q}^2 + \|\pi(\vartheta_T, \xi_T)\|_{2\alpha-1,p/2,q})).
\]

**Proof.** We recall the characterization of \( \tilde{u} = \psi u \) from Lemma 5.5. In order to obtain the desired estimate of the norm, we apply Bony’s decomposition and calculate

\[
\begin{align*}
\text{du}^\# &= F(\tilde{u})\xi_T - d(T_F(\tilde{u})\vartheta_T) + \frac{\psi'}{\psi}\tilde{u} \\
&= T_F(\tilde{u})\xi_T + \pi(F(\tilde{u}), \xi_T) + T_{\xi_T}(F(\tilde{u})) - d(T_F(\tilde{u})\vartheta_T) + \frac{\psi'}{\psi}\tilde{u} \\
&= \pi(F(\tilde{u}), \xi_T) + T_{\xi_T}(F(\tilde{u})) - T_{d(F(\tilde{u}))}\vartheta_T + \frac{\psi'}{\psi}\tilde{u}. \quad (30)
\end{align*}
\]

We bound the \( B^{2\alpha-1}_{p/2,q} \)-norm of these four terms separately. The first term is bounded by Proposition 5.6. To estimate the second term in (30), Lemma 2.1 (7) and a Besov embedding yield

\[
\|T_{\xi_T}(F(\tilde{u}))\|_{2\alpha-1,p/2,q} \lesssim \|F\|_{C^1} \|\xi_T\|_{\alpha-1,p,2q} \|\tilde{u}\|_{\alpha,p,2q} \\
\lesssim \|F\|_{C^1} \|\vartheta_T\|_{\alpha,p,q} \|\tilde{u}\|_{\alpha,p,q}.
\]

The third term in (30) can be estimated with the lifting property of Besov spaces, Lemma 2.1 (7) and a Besov embedding

\[
\|T_{d(F(\tilde{u}))}\vartheta_T\|_{2\alpha-1,p/2,q} \lesssim \|d(F(\tilde{u}))\|_{\alpha-1,p,2q} \|\vartheta_T\|_{\alpha,p,2q} \\
\lesssim \|F\|_{\alpha,p,2q} \|\vartheta_T\|_{\alpha,p,2q} \|\tilde{u}\|_{\alpha,p,2q} \lesssim \|F\|_{C^1} \|\tilde{u}\|_{\alpha,p,q} \|\vartheta_T\|_{\alpha,p,q}.
\]

For the last term in (30), we note the norm equivalence \( \|\psi u\|_{L^{p/2}} \sim \|\tilde{u}\|_{L^{p/2}} \) with for \( u \) being constant outside of \([-2T, 2T]\), where we set \( \psi := \psi_2 \) for another weight function \( \psi_2 \) satisfying
Hence, combining the last two inequalities leads to
\[ \|\tilde{u}\|_{L^{p/2}} \lesssim \|\psi_2\tilde{u}\|_{L^{p/2}} \lesssim \|\psi_2\|_{L^p} \|\tilde{u}\|_{L^p} \] 
by Hölder’s inequality. Since \( 2\alpha - 1 < 0 \), a Besov embedding yields
\[ \|\tilde{u}\|_{L^{p/2}} \lesssim \|\tilde{u}\|_{2\alpha-1,p/2,q} \lesssim \|\tilde{u}\|_{L^p} \lesssim (\mathcal{T} \lor 1) \|\tilde{u}\|_{L^p}. \]
Combining all the above estimates, we obtain
\[ \|d\tilde{u}\|_{2\alpha-1,p/2,q} \lesssim C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|d\tilde{u}\|_{2\alpha-1,p/2,q}) + (\mathcal{T} \lor 1) \|\tilde{u}\|_{\alpha,p,q} \]
with
\[ C_{\xi, \vartheta} := \|\vartheta\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q} + \|\pi(\tilde{\vartheta}, \xi T)\|_{2\alpha-1,p/2,q}. \]
Applying again the lifting property of Besov spaces \([43, \text{Thm. 2.3.8}]\) together with the definition of \( d\tilde{u} \), \( \tilde{u}\|_{L^p} \lesssim (\mathcal{T} \lor 1) \|\tilde{u}\|_{L^p} \) and the compact support of \( \tilde{\vartheta} \), we have
\[ \|d\tilde{u}\|_{2\alpha-1,p/2,q} \lesssim \|\tilde{u}\|_{L^p} + \|d\tilde{u}\|_{2\alpha-1,p/2,q} \]
\[ \lesssim (\mathcal{T} \lor 1)(\|F\|_{\alpha,p,q} + \|\tilde{u}\|_{\alpha,p,q} + \|\vartheta\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q}). \]
Hence, combining the last two inequalities leads to
\[ \|d\tilde{u}\|_{2\alpha-1,p/2,q} \lesssim C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|d\tilde{u}\|_{2\alpha-1,p/2,q}) \]
\[ + (\mathcal{T} \lor 1)C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|\vartheta\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q}) \]
If \( C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2) \) is sufficiently small, we thus obtain
\[ \|d\tilde{u}\|_{2\alpha-1,p/2,q} \]
\[ \lesssim (\mathcal{T} \lor 1)C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|F\|_{\alpha,p,q} + \|\vartheta\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q} + 1). \]
In combination with the ansatz and the bounds from above, Lemma \([24]\) reveals
\[ \|\tilde{u}\|_{\alpha-1,p,q} \lesssim \|d(T\tilde{F}(\tilde{\vartheta})\vartheta\tilde{T})\|_{\alpha-1,p,q} + \|d\tilde{u}\|_{\alpha-1,p,q} \]
\[ \lesssim \|T\tilde{F}(\tilde{\vartheta})\vartheta\tilde{T}\|_{\alpha-1,p,q} + \|T\tilde{F}(\tilde{\vartheta})\xi T\|_{\alpha-1,p,q} + \|d\tilde{u}\|_{\alpha-1,p,q} \]
\[ \lesssim (\mathcal{T} \lor 1)C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q} + 1). \]
Due to Remark \([5.3]\) applied to \( \tilde{\vartheta} = \psi_2 \), we can apply Lemma \([22]\) to obtain
\[ \|\tilde{u}\|_{\alpha,p,q} \lesssim \|\psi_2\tilde{u}\|_{\alpha,p,q} \lesssim (\mathcal{T} \lor 1)(\|u(0)\| + \|d\tilde{u}\|_{\alpha-1,p,q}) \]
\[ \lesssim (\mathcal{T} \lor 1)C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{u}\|_{\alpha,p,q} + \|\tilde{\vartheta}\|_{\alpha,p,q} + 1) \]
\[ + (\mathcal{T} \lor 1)(\|u(0)\| + C_{\xi, \vartheta}(\|F\|_{C^2} \lor \|F\|_{C^2}^2)(\|\tilde{\vartheta}\|_{\alpha,p,q} + 1)). \]
For \( D \) smaller than some universal constant we conclude the assertion. \( \square \)

For any \( F \in C^3 \) and \( \|F\|_{C^3} \) small enough, the following lemma reveals that the weighted Itô-Lyons map \( \tilde{S} \) as introduced in \([24]\) is locally Lipschitz continuous with respect to the Besov norms on \( \mathbb{R}^d \times B_{p,q}^{\alpha-1} \times B_{p,q}^{\alpha-2} \) and thus it can be uniquely extended in a continuous way.
Lemma 5.9. Let $\alpha \in (1/3, 1/2)$, $p \geq 3$, $q \geq 1$ and let $F \in C^3$ with $F(0) = 0$. Assume $\psi$ is a weight function satisfying Assumption 6.2 and let $\vartheta \in C_0^\infty$ with derivative $\xi_T = d\vartheta_T$. Then there exists a polynomial on $\mathbb{R}^3$ such that, provided the bound

$$
\|F\|_{C^3} + \|F\|^2_{C^1} \leq P(T \vee 1, \|\vartheta_T\|_{\alpha,p,q}, \|\pi(\xi_T, \vartheta_T)\|_{2\alpha-1,p,q})^{-1},
$$

holds and $C_\psi$ is sufficiently small, there exists for every $w_0 \in \mathbb{R}^d$ a unique global solution $w \in S'$ with $\psi w \in B^p_{p,q}$ to the Cauchy problem (23). Furthermore, for fixed $T$, $\psi$ and the weighted Itô-Lyons map $\hat{S}$ is local Lipschitz continuous on $\mathbb{R}^d \times C_\psi^\infty$ around $(w_0, \vartheta_T, \pi(\vartheta_T, \xi_T))$.

The local Lipschitz continuity is the key ingredient to extend the weighted Itô-Lyons map from smooth paths to irregular ones. The proof works similarly to the proofs of Proposition 5.6 and Corollary 5.8 with an additional application of the Lipschitz result in Proposition 4.1. Due to the necessary, but quite lengthy estimations, we postpone the proof to Appendix A.2 with the hope to increase the readability of the paper.

Finally, we can state our main result: There exist a continuous extension of the weighted Itô-Lyons map $\hat{S}$ from $\mathbb{R}^d \times C_\psi^\infty$ to the domain $\mathbb{R}^d \times B^0_{p,q}$. Similarly to Theorem 5.10 we use a dilation argument together with a localization procedure to circumvent the assumption that $|F|_{C^3}$ has to be small. Allowing for general Besov spaces, this theorem generalizes Lyons’ celebrated Universal Limit Theorem 36, Thm. 6.2.2] and in particular [24, Thm. 3.3].

Theorem 5.10. Let $T > 0$, $\alpha \in (1/3, 1/2)$, $p \geq 3$, $q \geq 1$ and $F \in C^3$ with $F(0) = 0$. If the weight function $\psi$ satisfies Assumption 6.2 with $C_\psi$ sufficiently small, then the weighted Itô-Lyons map $\hat{S}$ as introduced in (24) can be continuously extended from $\mathbb{R}^d \times C_\psi^\infty$ to the domain $\mathbb{R}^d \times B^0_{p,q}$. In particular, there exists a unique solution to (24) for any geometric Besov rough path $(\vartheta_T, \pi(\vartheta_T, d\vartheta_T)) \in B^0_{p,q}$.

An elementary formulation of Theorem 5.10 is presented in the next lemma. The proof of Theorem 5.10 is then an immediate consequence.

Lemma 5.11. Assume the weight function $\psi$ satisfies Assumption 6.2 with $C_\psi$ sufficiently small. Let $T > 0$, $\alpha \in (1/3, 1/2)$, $p \geq 3$, $q \geq 1$ and $F \in C^3$ with $F(0) = 0$. Let further $w_0 \in \mathbb{R}^m$ be an initial condition and $(\vartheta_T, \eta_T) \in B^0_{p,q}$ be a geometric Besov rough path. Let $(\vartheta_T^n) \subseteq C_\psi^\infty$ be a sequence of initial conditions such that $(\vartheta_T^n, \eta_T^n) \subseteq B^0_{p,q}$ and $(w_0^n) \subseteq \mathbb{R}^m$ be a sequence of initial conditions such that $(w_0^n, \vartheta_T^n, \pi(\vartheta_T^n, \xi_T^n))$ converges to $(w_0, \vartheta_T, \eta_T)$ in $\mathbb{R}^m \times B_{p,q}^{3-1} \times B_{p,q}^{2-1}$. Denote by $u^n$ the unique solution to the Cauchy problem (23) with $w_0^n$ and $\vartheta_T^n$ for all $n \in \mathbb{N}$. Then there exists $u \in S'$ such that $\psi u \in B^0_{p,q}$ and $\psi u^n \to \psi u$ in $B^0_{p,q}$. The limit $u$ depends only on $(w_0, \vartheta_T, \eta_T)$ and not on the approximating family $(u^n, \vartheta_T^n, \pi(\vartheta_T^n, \xi_T^n))$.

Proof. In order to apply Lemma 5.11 we first need to ensure that $\|F\|_{C^3}$ is small enough. Thus, as similarly done in Step 2 of the proof of Theorem 5.10 we scale $\vartheta^n_T$: For some fixed $\varepsilon \in (0, \alpha - 1/p)$ and for $\lambda \in (0, 1)$ we set

$$
\vartheta^n_T := \lambda^{1-\alpha+1/p+\varepsilon} \Lambda \vartheta^n_T \quad \text{and} \quad \xi^n_T := \lambda^{1-\alpha+1/p+\varepsilon} \Lambda \xi^n_T,
$$

where we recall the scaling operator $\Lambda_f = f(\lambda)$ for $f \in S'$. Given this scaling, still $\xi^n_T = d\vartheta^n_T$ holds true and the corresponding norms of $\xi^n_T$ and $\vartheta^n_T$ can be controlled by the Lemmas 6.2 and 24, i.e.

$$
\|\xi^n_T\|_{\alpha-1,p,q} \leq \|\xi_T\|_{\alpha-1,p,q} \quad \text{and} \quad \|\vartheta^n_T\|_{\alpha,p,q} \leq (1 \vee T^2) \|\xi^n_T\|_{\alpha-1,p,q} \leq (1 \vee T^2) \|\xi_T\|_{\alpha-1,p,q}.
$$

Moreover, again using Lemma 24 we can estimate

$$
\|\pi(\vartheta^n_T, \xi^n_T)\|_{2\alpha-1,p/2,q} = \lambda^{1-2\alpha+2/p+2\varepsilon} \|\pi(\Lambda \vartheta^n_T, \Lambda \xi^n_T)\|_{2\alpha-1,p/2,q} \leq (\lambda^{2\varepsilon} |\log \lambda| + \lambda^{1-2\alpha+2\varepsilon}) \|\pi(\vartheta^n_T, \xi^n_T)\|_{2\alpha-1,p/2,q}.
$$

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Let us take once more the localization function \( \varphi \) from Assumption 3.1 and noticing that \( \varphi_{2T} \vartheta_T^n = \vartheta_T^n \) for all \( n \in \mathbb{N} \). Therefore, Lemma 5.9 provides for \( \lambda > 0 \) sufficiently small a unique global solution \( u^{n,\lambda} \in B^\alpha_{p,q} \)

\[
d u^{n,\lambda} = \lambda^{\alpha-1/p-\varepsilon} F(u^{n,\lambda})d(\varphi_{2T}\vartheta_T^n), \quad u^{n,\lambda}(0) = u_0^n.
\]

Setting now \( u^n := \Lambda_{\lambda-1}u^{n,\lambda} \), we have constructed a unique global solution to

\[
d u^n = F(u^n)d(\varphi_{2T}\vartheta_T^n), \quad u(0) = u_0^n.
\]

Since \( (u_0^n, \vartheta_T^{n,\lambda}, \pi(\vartheta_T^{n,\lambda})) \) converges to \( (u_0, \vartheta_T^n, \pi(\vartheta_T^n, \xi_T^n)) \) in \( \mathbb{R}^d \times B^\alpha_{p,2^{-k}q} \times B^\alpha_{p,2^{-k}q} \), the continuity of the Itô-Lyons map established in Lemma 5.2 implies that \( u^{n,\lambda} \) converges to some \( u^\lambda \) in \( B^\alpha_{p,q} \) weighted by \( \psi \). Therefore, the solution \( u^n \) converges to \( u := \Lambda_{\lambda-1}u^\lambda \) in \( B^\alpha_{p,q} \) weighted by \( \psi \), due to Lemma 2.3 and 2.8 in which we can see analogously to Step 2 of the proof of Theorem 3.3 We note that \( u \|_{[-\lambda T,\lambda T]} \) does not depend on \( \varphi_{\lambda T} \).

Following the same argumentation as in Step 3 of the proof of Theorem 3.3 we can iterate this construction of \( u^n \) and \( u \) on intervals of the length \( 2\lambda T \). In this way we end up with a continuous function \( u \) such that \( \psi u \in B^\alpha_{p,q} \) and \( \psi u^n \) converges to \( \psi u \) in \( B^\alpha_{p,q} \). Note that \( u \) depends only on \( (u_0, \vartheta_T, \pi(\vartheta_T, \xi_T)) \) but neither on approximating family \( (u_0^n, \vartheta_T^n, \pi(\vartheta_T^n, \xi_T^n)) \) nor on \( \varphi_{\lambda T} \).

While general Besov spaces contain functions with jumps, the paracontrolled distribution approach to rough differential equations as explored in the present section only studies continuous functions. Therefore, we think a discussion is in order why the paracontrolled distribution approach seems to be naturally restricted to continuous functions.

**Remark 5.12.** The results in Section 4 apply only to Besov spaces \( B^\alpha_{p,q} \) for \( p \geq 1 \). According to (17), our estimates result in a bound of the \( B^{\alpha-1}_{p/3,q} \)-norm. Consequently, we require \( p \geq 3 \) and \( \alpha > 1/p \) in order to have positive regularity. In particular, our main theorem applies only to the case \( \alpha > 1/p \) which implies that \( B^\alpha_{p,q} \) embeds into the space of continuous functions.

If we want to extend our results to discontinuous functions, corresponding to \( \alpha < 1/p \), then we could hope that it helps to verify the previous results for \( p < 1 \). Let us sketch some details on this idea, where we have to deal with the quasi-Banach space \( B^\alpha_{p,q} \) for \( p < 1 \). In that case the triangle inequality only holds true up to a multiplicative constant

\[
\|f + g\|_{\alpha,p,q} \leq 2^{1/p-1}(\|f\|_{\alpha,p,q} + \|g\|_{\alpha,p,q}) \quad \text{for } f, g \in B^\alpha_{p,q},
\]

Following the lines of the proof of Lemma 2.84 (or Lemma 2.49 respectively) in Bahouri et al. 2, we obtain in the case \( p \in (0,1), q > 1, \alpha > 1/p - 1 \), for \( u := \sum_j u_j \) with \( \text{supp } u_j \subseteq 2^j B \) for some ball \( B \) that

\[
\|u\|_{\alpha-(1/p-1),p,q} \lesssim \left( 2^{j\alpha}\|u_j\|_{L^p} \right)_j \|\xi_j\|_{\ell^p},
\]

provided the right-hand side is finite. For the commutator lemma in the case \( p \in (0,1) \) we thus cannot hope for more than the following: Replacing the assumption \( p \geq 1 \) with \( \alpha + \beta + \gamma > \left( \frac{1}{p} - 1 \right)/0 \) in the situation of Lemma 4.4 we conjecture

\[
\|\Pi(f,g,h)\|_{\alpha + \beta + \gamma - \left( \frac{1}{p} - 1 \right)/0,p,q} \lesssim \|f\|_{\alpha,p,q}\|g\|_{\beta,p,q}\|h\|_{\gamma,p,q}.
\]

Applying this bound to (17), we obtain for \( p \in (0,1) \)

\[
\|\Pi F(u,\xi)\|_{\alpha-1-(3/p-1),p/3,q} < \left( \|F'(u)\|_{\alpha,p,q} + \|u\|_{\alpha,p,q}\|\xi\|_{\alpha-1,p,q} \right).
\]

However, \( 3\alpha-1-(3/p-1) > 0 \) is equivalent to \( \alpha > 1/p \), which is the same condition as we had before, excluding discontinuous functions.

Alternatively, a higher order expansion in the linearization Lemma 4.2 could be studied (corresponding to more additional information). If such a second order expansion would succeed, we may have the condition \( 4\alpha - 1 > 0 \), but with the price of imposing \( p/4 \geq 1 \). Consequently, we would again obtain \( \alpha > 1/p \).

In conclusion, it appears natural that this approach is restricted to continuous functions.
6 Stochastic differential equations

The purely analytic results from the previous sections for rough differential equations allow for treating a large class of stochastic differential equations (SDEs) in a pathwise way. While we assumed so far that the driving signal $\xi$ of the RDE (11) is given by a deterministic function with a certain Besov regularity, we suppose from now on that $\xi$ is the distributional derivative of some continuous stochastic process $X$. Provided all involved stochastic objects live on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and setting $\xi := dX$, the RDE (14) becomes an SDE with the dynamic

$$du(t) = F(u(t))dX_t, \quad u(0) = u_0, \quad t \in [0, 1], \quad (33)$$

where $u_0$ is a random variable in $\mathbb{R}^m$ and $X$ is some $d$-dimensional stochastic process for simplicity on the interval $[0, 1]$.

Instead of relying on classical stochastic integration in order to give the SDE (33) a meaning, we shall demonstrate here that the results of Section 3 and 5 are feasible for a wide class of SDEs. For this propose the present section is devoted to show the required sample path properties of a couple of stochastic processes. This allows for solving SDEs which are beyond the scope of classical probability theory as well as for recovering well-known examples. Let us emphasize that we present here only a few exemplary stochastic processes to illustrate our results and do not aim for the most general class of stochastic processes.

Gaussian processes

A well-known but very common example for a stochastic driving signal $X$ is the fractional Brownian motion, cf. [12, 33]. A $d$-dimensional fractional Brownian motion $B^H = (B^H_1, \ldots, B^H_d)$ with Hurst index $H \in (0, 1)$ is a Gaussian process with zero mean, independent components, and covariance function given by

$$\mathbb{E}[B^H_i(t_1)B^H_i(t_2)] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \in [0, 1],$$

for $i = 1, \ldots, d$. The Besov regularity of (fractional) Brownian motion is already known for a long time due to Roynette [41] and Ciesielski et al. [10]: it holds ($H = \frac{3}{4}$) in the case $H > \frac{1}{2}$, the following lemma shows how to recover the results for SDEs with our machinery. It in particular covers the fractional Brownian motion.

**Lemma 6.1** ([24, Cor. 3.10]). Let $X$ be a centered $d$-dimensional Gaussian process with independent components whose covariance function fulfills for some $H \in (1/4, 1)$ the Coutin-Qian condition

$$\mathbb{E}|X_t - X_s|^2 \lesssim |t-s|^{2H} \quad \text{and} \quad \mathbb{E}|(X_{s+t} - X_s)(X_{s+t} - X_t)| \lesssim |t-s|^{2H-2r^2}, \quad (34)$$

for all $s, t \in \mathbb{R}$ and all $r \in [0, |t-s|]$. For every $\alpha < H$ and any smooth function $\varphi$ with compact support we have $\varphi X \in B^2_{\alpha, \infty}$. Moreover, there exists an $\eta \in B^{2\alpha-1}_{\alpha, \infty}$ such that for every $\delta > 0$ and every $\psi \in \mathcal{S}$ with $\int \psi(t) dt = 1$ it holds

$$\lim_{n \to \infty} \mathbb{P}(||\psi^{n} * (\varphi X) - (\varphi X)||_{\alpha, \infty} + \|\pi(\psi^{n} * (\varphi X), d(\psi^{n} * (\varphi X) - \eta)||_{2\alpha-1, \infty, \infty} > \delta) = 0,$$

where we denote $\psi^{n} := n\psi(n \cdot)$.
In other words, every $d$-dimensional Gaussian process $X$ satisfying the Coutin-Qian condition \( \mathbb{H} \) for some $H \in (1/3, 1/2)$ can be enhanced to a geometric Besov rough path and especially Theorem 6.10 can be applied to solve the SDE \( \mathbb{H} \), cf. Coutin and Qian [14] or Friz and Victoir [18].

**Stochastic processes via Schauder expansions**

Instead of approximating stochastic processes by processes with smooth sample paths, in probability theory it is often more convenient to construct a process via an expansion with respect to a basis of $L^2$. The presumably most famous construction of this type is the Karhunen-Loève expansion of Gaussian processes.

A classical construction of a Brownian motion on the interval $[0, 1]$ is the Lévy-Ciesielski construction based on Schauder functions. More generally, Schauder functions are a very frequently applied tool in stochastic analysis. Notably, they are used to investigate the Besov regularity of stochastic processes, cf. for example Ciesielski et al. [10] and Rosenbaum [40], and very recently Gubinelli et al. [23] constructed directly the rough path integral in terms of Schauder expansions.

The Schauder functions can be defined as the antiderivatives of the Haar functions. More explicitly they are given by

$$G_{j,k}(t) := 2^{-j/2} \psi(2^j t - (k - 1)) \quad \text{with} \quad \psi(t) := t \mathbb{1}_{[0,1/2]}(t) - (t - 1/2) \mathbb{1}_{(1/2,1]}(t), \quad t \in \mathbb{R},$$

for $j \in \mathbb{N}$ and $1 \leq k \leq 2^j$, and $G_{0,0}(0) := 1$. The Haar functions form a basis of $L^2([0, 1], \mathbb{R})$ and it is obvious that $G_{n,k} \in B^\beta_{p,q}$ for $0 < \beta < 1$ and $p,q \in [1, \infty]$ with $\beta > 1/p$, cf. [40, Prop. 9]. The next lemma explains why an approximation of stochastic processes in terms of Schauder expansions can also be used to show that a process can be enhanced to a geometric Besov rough path.

**Lemma 6.2.** Let $\alpha \in (1/3, 1/2)$, $\beta \in (1/2, 1]$, $p \geq 2$ and $q \geq 1$. Suppose $(f^n) \subseteq B^\beta_{p,q}$ is a sequence of functions such that $\supp f^n \subseteq [0, 1]$ for all $n \in \mathbb{N}$. If $(f^n, \pi(f^n, df^n))$ converges in $B^\alpha_{p,q} \times B^{2\alpha-1}_{p,q}$ to some $(f, \pi(f, df)) \in B^\alpha_{p,q} \times B^{2\alpha-1}_{p,q}$, then $(f, \pi(f, df)) \in B^{0,0}_{p,q}$.

**Proof.** Let us recall that $C^\infty_1$ is dense in $\{g \in B_{p,q}^\beta : \supp g \subseteq [0, 1]\}$. Hence, for every $n \in \mathbb{N}$ there exists a sequence of smooth functions $(f^n, m)_m \subseteq C^\infty_1$ such that $(f^n, df^n)$ converges to $(f^n, df^n)$ in $B_{p,q}^\beta \times B^{2\beta-1}_{p,q}$ as $m$ goes to infinity, where the convergence of the second component follows from the first one using the lifting property of Besov spaces. Since $\beta > 1/2$, we also have by Lemma 2.1 that $f^n, m, df^n, m$ converges to $f^n, df^n$ as $m$ goes to infinity. Therefore, taking a diagonal sequence there exists a sequence of smooth functions $(f^n, m)_n \subseteq C^\infty_1$ such that $(f, \pi(f, df)) = \lim_{n \to \infty} \pi(f^n, m, df^n)$ where the limit is taken in $B_{p,q}^\alpha \times B^{2\alpha-1}_{p,q}$.

Based on Lemma 6.2 it is now an immediate consequence of Theorem 6.5 and 6.6. in [23] that suitable hypercontractive processes and continuous martingales can be lifted to geometric Besov rough paths since the Lévy area term in [23] corresponds to our resonant term. Especially, all examples from probability theory in [23] are feasible with our results as well.

**Random functions via wavelet expansions: a prototypical example**

Random Fourier series have been enhanced to rough paths by Friz et al. [21]. Due to the localization of the trigonometric basis in Fourier domain, it is quite convenient to use in their examples also the paracontrolled approach. Working with Fourier series requires to localize the signal. Motivated by the previous construction, we shall instead consider stochastic processes which can be constructed as series expansion with random coefficients and with respect to a wavelet basis. There are several applications of such models, for instance, in non-parametric Bayesian statistics to construct priors on function spaces. One advantage is that the sample path regularity of such processes can be determined precisely, cf. Abramovich et al. [1], Cioica et al. [11] and Bochkina [4]. Note that
very similar calculations apply also to Fourier series, requiring some extra technical effort for the localization function.

Wavelets can be taken to be localized in the time domain as well as in the Fourier domain. The latter property is quite convenient when working with Littlewood-Paley theory as we demonstrate in the following. Let \( \{ \psi_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z} \} \) be an orthonormal wavelet basis of \( L^2(\mathbb{R}) \), where \( \psi_{j,k}(t) := 2^{j/2} \hat{\psi}(2^j t - k) \) for \( j \geq 1 \), \( k \in \mathbb{Z} \), \( t \in \mathbb{R} \), and \( \hat{\psi} \in L^2(\mathbb{R}) \). Then, any function \( f \in L^2(\mathbb{R}) \) can be written as

\[
f(t) := \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t), \quad t \in \mathbb{R}, \quad \text{with} \quad \langle f, \psi_{j,k} \rangle := \int_{\mathbb{R}} f(s) \psi_{j,k}(s) \, ds.
\]

Replacing the deterministic wavelet coefficients with real valued random variables \( (Z_{j,k})_{j,k} \), we now study stochastic processes of the type

\[
X_t := \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} 2^j Z_{j,k} \psi_{j,k}(t), \quad t \in \mathbb{R}. \tag{35}
\]

Without loss of generality, we truncated the series expansion in \( k \) since we always have to localize the signal in order to apply our results concerning RDEs, see the equations (8) and (23). Let us impose the following weak assumptions on \( (Z_{j,k})_{j,k} \):

**Assumption 6.3.** Let \( \{ \psi_{j,k} : j \in \mathbb{N}, k \in \mathbb{Z} \} \) be an orthonormal and band limited wavelet basis of \( L^2(\mathbb{R}) \) and suppose \( Z_{j,k} = A_{j,k} B_{j,k} \) for all \( j \geq 0 \) and \( k = -2^j, \ldots, 2^j \) where

- \( (A_{j,k})_{j,k} \) are random variables satisfying \( \mathbb{E}[A_{j,k}^p]^{1/p} \leq 2^{-j s} \) for some \( s > 0 \) and \( p \in \{2, 4\} \),
- \( \mathbb{E}[A_{j,k}] = 0 \) for all \( j, k \) and \( \mathbb{E}[A_{j,k} A_{m,n}] = 0 \) for \( j \neq m \) or \( k \neq n \),
- \( (B_{j,k})_{j,k} \) are Bernoulli random variables with \( \mathbb{P}(B_{j,k} = 1) = 2^{-j r} \) for some \( r \in [0, 1) \),
- \( \mathbb{E}[A_{j,k} B_{j,k} A_{m,n} B_{m,n}] = \mathbb{E}[A_{j,k} A_{m,n}] \mathbb{E}[B_{j,k} B_{m,n}] \) for all \( j, k, m, n \).

The assumption allows for a quite flexible class of stochastic processes although it is chosen in a way to keep the required analysis simple. Having in mind the construction of Brownian motion via Schauder functions, as mentioned before, the process \( X \) behaves like a Wiener process if \( (Z_{j,k})_{j,k} \) are i.i.d. standard normal distributed random variables with \( s = 1 \). In particular, the self-similar behavior of Brownian motion is then achieved because all wavelet coefficients at a level \( j \) are of the same order of magnitude (especially \( r = 0 \)). If instead \( r \in (0, 1) \), we expect only a number of \( 2 \cdot 2^{(1-r) s} \) non-zero wavelet coefficients at each level \( j \) and we consequently gain from measuring the regularity of \( X \) in a \( B^p_{p,q} \)-norm for some finite \( p \).

In order to profit from \( (Z_{j,k})_{j,k} \) being uncorrelated we choose an even number \( p \). Together with the requirement \( p \geq 3 \) in our uniqueness and existence theorem for RDEs (Theorem 5.10), we thus take \( p = 4 \). Keeping in mind that the Littlewood-Paley theory relies on decomposing functions into blocks with compact support in the Fourier domain, we postulate to take band limited wavelets, e.g. Meyer wavelets. Note that \( X \) then is not compactly supported, but exponentially concentrated on a fixed interval for an appropriate choice of \( \psi \). We obtain the following sample path regularity of \( X \):

**Lemma 6.4.** If \( X \) is defined as in (35) and satisfies Assumption 6.3 then \( X \in B^p_{p,1} \) almost surely for any \( \alpha < s + p - \frac{1}{2} \) and for \( p \in \{2, 4\} \).

**Proof.** Applying formally the Littlewood-Paley decomposition, one has \( X = \sum_{j \geq -1} \Delta_j X \) and for the sake of brevity we introduce the multi-indices \( \lambda = (j, k) \) with \( |\lambda| := j \). Noting that by the assumption on the wavelet basis \( \text{supp} \mathcal{F} \psi_\lambda \subseteq 2^{|\lambda|} \mathcal{A} \) for some annulus \( \mathcal{A} \) independent of \( \lambda \), we obtain \( \Delta_j \psi_\lambda = 0 \) if \( |j - |\lambda|| \) is larger than some fixed integer. Therefore, the Littlewood-Paley blocks are well-defined and given by

\[
\Delta_j X = \sum_{\lambda : |\lambda| = j} Z_\lambda \Delta_j \psi_\lambda \quad \text{for} \quad j \geq -1.
\]

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Further, let us remark that $X$ as given in (35) exists in $B^p_{p,1}$ if $\sum_j \Delta_j X$ exists as limit in $B^p_{p,1}$.

In order to show the claimed Besov regularity, we have to verify
\[ \|\Delta_j X\|_{L^p} \lesssim 2^{-j(s+r/2-1/2)} \quad \text{for} \quad j \geq -1, \quad p \in \{2, 4\}. \]

Let us focus on $p = 2$. The case $p = 4$ can be proved similarly relying on the estimates for the forth moments of $(Z_{\lambda})$, see also Lemma 6.6 below. For $j \geq -1$ we have
\[
\mathbb{E}[\|\Delta_j X\|_{L^2}^2] = \int_{\mathbb{R}} \mathbb{E}\left[\left(\sum_{\lambda} Z_{\lambda} \Delta_j \psi_{j,k}(t) \right)^2 \right] dt
\]
\[ = \sum_{\lambda, \lambda'} \mathbb{E}[Z_{\lambda} Z_{\lambda'}] \int \Delta_j \psi_{\lambda}(t) \Delta_j \psi_{\lambda'}(t) dt \lesssim \sum_{\lambda} 2^{-2(s+r)\|\lambda\|} \int (\Delta_j \psi_{\lambda})^2(t) dt, \]
where the last equality follows from $(Z_{\lambda})$ being mutually uncorrelated. Hence, we further estimate
\[
\mathbb{E}[\|\Delta_j X\|_{L^2}^2] \lesssim \sum_{\|\lambda\| \sim j} 2^{-2(s+r)\|\lambda\|} \|\Delta_j \psi_{\lambda}\|^2_{L^2}
\]
\[ \lesssim \sum_{j' \sim j} 2^{-(s+r)j'} \sum_{k=-2^{j'}} \|\psi_{j',k}\|^2_{L^2} = 2 \sum_{j' \sim j} 2^{-2j'(s+r/2-1/2)}. \]

By the Littlewood-Paley characterization of the Besov norm we conclude
\[
\mathbb{E}[\|X\|_{\alpha,p,1}] = \sum_{j \geq -1} 2^{j\alpha} \mathbb{E}[\|\Delta_j X\|_{L^p}] \lesssim \sum_{j \geq -1} 2^{(\alpha-s-r/2+1/2)},
\]
which is finite whenever $\alpha < s + r/2 - 1/2$.

**Remark 6.5.** With analogous estimates as in Lemma [6.3] it is easy to show that $X \in B^p_{p,1}$ a.s. for any $\alpha < s + \frac{r}{p} - \frac{1}{2}$ for any even $p \geq 2$ provided $\mathbb{E}[A_j^p]^{1/p} \lesssim 2^{-s}$ still holds for these higher powers.

The derivative of $X$ is naturally given by $dX_t = \sum_{j,k} Z_{j,k} \psi_{j,k}(t)$ for $t \in \mathbb{R}$. The crucial point is now, that we can indeed verify that the resonant term $\pi(X, dX)$ is in $B^{2s-1}_{2,1}$ almost surely due to the probabilistic nature of $X$. The following lemma highlights how the stochastic setting nicely complements the analytical foundation.

**Lemma 6.6.** Suppose $X$ is given by (35) and satisfies Assumption 6.3 then
\[ X \in B^\alpha_{4,1} \quad \text{and} \quad \pi(X, dX) \in B^{2\alpha-1}_{2,1}, \]
almost surely for any $\alpha < s + \frac{r}{p} - \frac{1}{2}$.

**Proof.** We start as in the classical proof of Bony’s estimate (Lemma [4.7] (iii), cf. [2] Thm. 2.85), and decompose
\[
\pi(X, dX) = \sum_{j \geq -1} R_j \quad \text{with} \quad R_j := \sum_{|\nu| \leq 1} (\Delta_j - \nu) X (\Delta_j dX).
\]

By the properties of the Littlewood-Paley blocks the Fourier transform of $R_j$ is supported in $2^j$ times some fixed ball. Consequently, $\Delta_{j'} R_j = 0$ if $|j' - j| \leq 1$ and thus
\[
\|\Delta_{j'} \pi(X, dX)\|_{L^2} = \left\| \sum_{j \geq j'} \Delta_{j'} R_j \right\|_{L^2} \lesssim \sum_{j \geq j'} \sum_{|\nu| \leq 1} \|\Delta_j - \nu X (\Delta_j dX)\|_{L^2}.
\]
Now we proceed similarly to Lemma 6.4 (using again the multi-indices $\lambda = (j,k)$):

\[
\begin{align*}
\mathbb{E}[\|\Delta_{j-\nu}X(\Delta_j dX)\|_{L^2}^2] &= \int_{\mathbb{R}} \mathbb{E}\left[ \left( \sum_{\lambda_1, \lambda_2} Z_{\lambda_1} Z_{\lambda_2}(\Delta_{j-\nu}\psi_{\lambda_1})(\Delta_j \psi'_{\lambda_2}) \right)^2 \right] dt \\
&= \sum_{\lambda_1 \neq \lambda_2; |\lambda| \sim j} \mathbb{E}[Z_{\lambda_1} Z_{\lambda_2} Z_{\lambda_3} Z_{\lambda_4}] \int_{\mathbb{R}} \langle \Delta_{j-\nu}\psi_{\lambda_1}\rangle \langle \Delta_j \psi'_{\lambda_2} \rangle \langle \Delta_{j-\nu}\psi_{\lambda_3} \rangle \langle \Delta_j \psi'_{\lambda_4} \rangle dt \\
&\leq \sum_{\lambda_1 \neq \lambda_2; |\lambda| \sim j} \mathbb{E}[Z_{\lambda_1}^2 Z_{\lambda_2}^2] \int_{\mathbb{R}} \langle \Delta_{j-\nu}\psi_{\lambda_1} \rangle^2 \langle \Delta_j \psi'_{\lambda_2} \rangle^2 + \langle \Delta_{j-\nu}\psi_{\lambda_1} \rangle \langle \Delta_j \psi'_{\lambda_2} \rangle \langle \Delta_{j-\nu}\psi_{\lambda_3} \rangle \langle \Delta_j \psi'_{\lambda_4} \rangle dt \\
&\quad + \sum_{\lambda_1 \neq \lambda_2; |\lambda| \sim j} \mathbb{E}[Z_{\lambda_1}^2] \int_{\mathbb{R}} \langle \Delta_{j-\nu}\psi_{\lambda_1} \rangle^2 \langle \Delta_j \psi'_{\lambda_2} \rangle^2 dt \\
&\leq \sum_{\lambda_1 \neq \lambda_2; |\lambda| \sim j} 2^{-2(4\nu s + 2r)j} \|\psi_{\lambda_1}\|_{L^4} \|\psi'_{\lambda_2}\|_{L^4} \left( \|\psi_{\lambda_1}\|_{L^4} \|\psi'_{\lambda_2}\|_{L^4} + \|\psi_{\lambda_2}\|_{L^4} \|\psi'_{\lambda_1}\|_{L^4} + \|\psi_{\lambda_2}\|_{L^4} \|\psi'_{\lambda_2}\|_{L^4} \right) \\
&\quad + \sum_{\lambda_1 \neq \lambda_2; |\lambda| \sim j} 2^{-2(4\nu s + r)j} \|\psi_{\lambda_1}\|_{L^4} \|\psi'_{\lambda_2}\|_{L^4}^2.
\end{align*}
\]

Plugging $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)$, we obtain

\[
\mathbb{E}[\|\Delta_{j-\nu}X(\Delta_j dX)\|_{L^2}^2] \lesssim 2^{-j(2s + r/2 - 2)}.
\]

The assertion follows from Lemma 6.2 by the compact support of $FR_j$ for $j \geq -1$.

Combining the two previous lemmas, we conclude that stochastic models of the form \eqref{eq:stochastic_model} are prototypical examples of geometric Besov rough paths, which were introduced in Definition 5.1 and thus Theorem 5.10 can be applied to the corresponding stochastic differential equations.

**Proposition 6.7.** Let $\phi$ satisfy Assumption 5.1 and $X = (X^1, \ldots, X^n)$ be an $n$-dimensional stochastic process. Suppose each component $X^d$, $d = 1, \ldots, n$, is of the form \eqref{eq:stochastic_model}, fulfills Assumption 5.3 for $s > \frac{r}{4}$ and the corresponding coefficients $(Z^d_{\lambda})$ and $(Z^m_{\lambda})$ are independent for $d \neq m$ and all $j, k$. Then, the localized process $\phi X$ can be enhanced to a geometric Besov rough path, that is $\phi X \in B_{s,1}^{\alpha,\alpha}$ almost surely for $\alpha \in (\frac{r}{4}, s + \frac{r}{4} - \frac{3}{2})$.

**Proof.** The regularity for each component $X^d$, $d = 1, \ldots, n$, is determined by Lemma 6.4 and thus $X \in B_{s,1}^{\alpha,\alpha}$ for $\alpha \in (\frac{r}{4}, s + \frac{r}{4} - \frac{3}{2})$. Furthermore, a smooth approximation is given by the projection of $X$ onto the first $J \geq 1$ Littlewood-Paley blocks as used in the proof of Lemma 6.4 or similarly by projecting on the first $J \geq 1$ wavelet resolution levels.

The resonant terms $\pi(X^d, dX^d)$, $d = 1, \ldots, n$, are constructed in Lemma 6.6 again by a smooth approximation in terms of Littlewood-Paley blocks. Due to the independence of the corresponding coefficients $(Z^d_{\lambda})$ and $(Z^m_{\lambda})$ for $d \neq m$, an analogous calculation shows that the resonant terms $\pi(X^d, dX^m)$ for $d \neq m$ exists as limit of the same approximation in terms of Littlewood-Paley blocks, too.

It remains to deduce the above results for the localized process $\phi X$ as well. The regularity and approximation of $\phi X$ is implied by Lemma 2.2. For the resonant term $\pi(\phi X, d(\phi X))$ we observe that

\[
\pi(\phi X, d(\phi X)) = \pi(\phi X, \phi' X) + \pi(\phi X, dX),
\]

where the first term turns out to be no issue thanks to Lemma 2.1. For the second one we apply Bony’s decomposition to $\phi X$ and our commutator lemma (Lemma 6.4) to get

\[
\pi(\phi X, \phi dX) = \phi \pi(X, \phi dX) + \phi \Gamma(\phi, X, \phi dX) + \pi(\phi, X, \phi dX) + X \pi(\phi, \phi dX) + \Gamma(X, \phi, \phi dX).
\]
Due to the regularity of \( \varphi \) and \( X \) it remains to only handle the first term. By another analogous application of the commutator lemma, we finally see that the approximation of the resonant term of the localized process can be deduced from the above approximation of the non-localized process and therefore \( \varphi X \in B^{\alpha,0}_{4,1} \).

\[ \square \]

A Appendix

A.1 Nonhomogeneous Besov spaces

In this part of the appendix we collect for the reader’s convenience some results which allow to estimate the Besov norm of a function. For a general introduction to Littlewood-Paley theory and Besov spaces we recommend Triebel [43] as well as Bahouri et al. [2].

Lemma A.1. \[2\] Lem. 2.69] Let \( A \subseteq \mathbb{R}^d \) be an annulus, \( \alpha \in \mathbb{R} \) and \( p, q \in [1, \infty] \). Suppose that \( (f_j) \) is a sequence of smooth functions such that

\[ \text{supp} \mathcal{F} f_j \subseteq 2^j A \quad \text{and} \quad \| (2^{\alpha j} \| f_j \|_{L^p})_j \|_{\ell^q} < \infty. \]

Then \( f := \sum_j f_j \) satisfies

\[ f \in B^\alpha_{p,q}(\mathbb{R}^d) \quad \text{and} \quad \| f \|_{\alpha, p,q} \lesssim \| (2^{\alpha j} \| f_j \|_{L^p})_j \|_{\ell^q}. \]

Lemma A.2. \[2\] Lem. 2.84] Let \( B \subseteq \mathbb{R}^d \) be a ball, \( \alpha \in \mathbb{R} \) and \( p, q \in [1, \infty] \). Suppose that \( (f_j) \) is a sequence of smooth functions such that

\[ \text{supp} \mathcal{F} f_j \subseteq 2^j B \quad \text{and} \quad \| (2^{\alpha j} \| f_j \|_{L^p})_j \|_{\ell^q} < \infty. \]

Then \( f := \sum_j f_j \) satisfies

\[ f \in B^\alpha_{p,q}(\mathbb{R}^d) \quad \text{and} \quad \| f \|_{\alpha, p,q} \lesssim \| (2^{\alpha j} \| f_j \|_{L^p})_j \|_{\ell^q}. \]

Lemma A.3. \[2\] Prop. 2.79] Let \( p, q \in [1, \infty] \), \( \alpha < 0 \) and \( f \) be a tempered distribution. Then, \( f \in B^\alpha_{p,q}(\mathbb{R}^d) \) if and only if

\[ (2^{\alpha j} \| S_j f \|_{L^p})_j \in \ell^q, \]

where we recall \( S_j f := \sum_{k=-1}^{j-1} \Delta_k f \). Furthermore, there exists a constant \( C > 0 \) such that

\[ C^{-|\alpha|+1} \| f \|_{\alpha, p,q} \leq \| (2^{\alpha j} \| S_j f \|_{L^p})_j \|_{\ell^q} \leq C \left( 1 + \frac{1}{|\alpha|} \right) \| f \|_{\alpha, p,q}. \]

A.2 Proof of Lemma [5.9] Lipschitz continuity

This subsection is devoted to the proof of Lemma [5.9]. For \( j = 1, 2 \) let \( u_j^j \in \mathbb{R}^d \) and \( \varphi_j \in C_0^\infty \) with derivative \( \xi_j = d \varphi_j \). Denote by \( u^j \), \( j = 1, 2 \), the solutions to corresponding Cauchy problems and \( \tilde{u} = \psi u \) for a weight function \( \psi \) satisfying Assumption [5.2]. Then Lemma [5.9] is proven if we can show that

\[ \| \tilde{u} - \tilde{u} \|_{\alpha, p,q} \leq C (\| \varphi_1 - \varphi_2 \|_{\alpha, p,q} + \| \pi (\varphi_1, \xi_1^j) - \pi (\varphi_2, \xi_2^j) \|_{L^{n-1,p/2}}), \]

for a constant \( C \) which does not depend on \( \tilde{u} \). Roughly speaking, the verification of this bound follows the pattern of the proofs of Proposition [5.1] and Corollary [5.8]. However, since Lemma [5.9] is essential for one of our main results, we shall present it here in full length.

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Taking another weight function $\psi_2$ fulfilling Assumption 5.2 and keeping Remark 5.3 in mind, we obtain

\[
\|\tilde{u}^1 - \tilde{u}^2\|_{\alpha,p,q} \lesssim \|\psi_2(\tilde{u}^1 - \tilde{u}^2)\|_{\alpha,p,q} \\
\lesssim (T^2 \vee 1)(|u^1(0) - u^2(0)| + |d(\tilde{u}^1 - \tilde{u}^2)|)_{\alpha-1,p,q} \\
\lesssim (T^2 \vee 1)(|u^1(0) - u^2(0)| + |d(T_F(\tilde{u}^1)\theta_T^1 - T_F(\tilde{u}^2)\theta_T^2)|)_{\alpha-1,p,q} + \|d(u^{#1} - u^{#2})\|_{\alpha-1,p,q}
\]

(36)

where Lemma 2.2 is used in the second line and the paracontrolled ansatz $\tilde{u}^j = T_F(\tilde{u}^j)\theta_T^j + u^{#j}$ in the third one. Let us continue by further estimating the term $d(T_F(\tilde{u}^1)\theta_T^1 - T_F(\tilde{u}^2)\theta_T^2)$. Applying the Leibniz rule and the triangle inequality leads to

\[
\|d(T_F(\tilde{u}^1)\theta_T^1 - T_F(\tilde{u}^2)\theta_T^2)\|_{\alpha-1,p,q} \\
\lesssim \|d(\tilde{u}^1)\|_{\alpha-1,p,q}\|\theta_T^1 - \theta_T^2\|_{0,\infty,\infty} + \|d(\tilde{u}^2)\|_{\alpha-1,p,q}\|\theta_T^2\|_{0,\infty,\infty} \\
+ \|F\|_{\infty}\|\xi_T^1 - \xi_T^2\|_{\alpha-1,p,q} + \|F\|_{\infty}\|\xi_T^2 - F(\tilde{u}^2)\|_{0,\infty}\|\xi_T^2\|_{\alpha-1,p,q} \\
\lesssim \|F(\tilde{u}^1)\|_{\alpha,p,q}\|\theta_T^1 - \theta_T^2\|_{\alpha-1,p,q} + \|F(\tilde{u}^2)\|_{\alpha,p,q}\|\theta_T^2\|_{\alpha-1,p,q} + \|F\|_{\infty}\|\xi_T^1 - \xi_T^2\|_{\alpha-1,p,q} \\
+ \|F\|_{\infty}\|\xi_T^2\|_{\alpha-1,p,q}\|\tilde{u}^1 - \tilde{u}^2\|_{\alpha-1,p,q} \\
\lesssim \|F\|_{C^1}\|\tilde{u}^1\|_{\alpha,p,q}\|\theta_T^1 - \theta_T^2\|_{\alpha-1,p,q} + \|F\|_{\infty}\|\theta_T^1\|_{\alpha,p,q}\|\tilde{u}^1 - \tilde{u}^2\|_{\alpha-1,p,q} + \|F\|_{\infty}\|\xi_T^1 - \xi_T^2\|_{\alpha-1,p,q} \\
+ \|F\|_{\infty}\|\xi_T^2\|_{\alpha-1,p,q}\|\tilde{u}^1 - \tilde{u}^2\|_{\alpha-1,p,q}
\]

(37)

It remains to consider the difference of derivatives $d\tilde{u}^{#j}$, which can be decomposed (cf. 39) into

\[
d\tilde{u}^{#j} = \pi(F(\tilde{u}^j), \xi_T^j) + T_{\xi_T^j}(F(\tilde{u}^j)) - T_{dF(\tilde{u}^j)}\theta_T^j + \frac{\psi'}{\psi}\tilde{u}^j \quad \text{for } j = 1, 2.
\]

Applying Proposition 5.4, we can rewrite the resonant term, differently than in the proof of Proposition 5.6 as

\[
\pi(F(\tilde{u}^j), \xi_T^j) = F'(\tilde{u}^j)\pi(\tilde{u}^j, \xi_T^j) + \Pi_F(\tilde{u}^j, \xi_T^j)
\]

and, taking the ansatz $\tilde{u}^j = T_F(\tilde{u}^j)\theta_T^j + u^{#j}$ into account and applying the commutator Lemma 4.5, we have

\[
\pi(\tilde{u}^j, \xi_T^j) = \pi(T_F(\tilde{u}^j)\theta_T^j, \xi_T^j) + \pi(u^{#j}, \xi_T^j) = F'(\tilde{u}^j)\pi(\theta_T^j, \xi_T^j) + \Gamma(F(\tilde{u}^j), \theta_T^j, \xi_T^j) + \pi(u^{#j}, \xi_T^j).
\]

Therefore, we decompose $d\tilde{u}^{#j}$ into the following seven terms

\[
d\tilde{u}^{#j} = F'(\tilde{u}^j)F(\tilde{u}^j)\pi(\theta_T^j, \xi_T^j) + F'(\tilde{u}^j)\Gamma(F(\tilde{u}^j), \theta_T^j, \xi_T^j) + F'(\tilde{u}^j)\pi(u^{#j}, \xi_T^j) + \Pi_F(\tilde{u}^j, \xi_T^j) \\
+ T_{\xi_T^j}(F(\tilde{u}^j)) - T_{dF(\tilde{u}^j)}\theta_T^j + \frac{\psi'}{\psi}\tilde{u}^j =: D_1 + \cdots + D_7.
\]
Let us tackle the differences of these seven terms: The first term is estimated as follows

\[ \|D_1^1 - D_2^1\|_{2a-1,p/2,q} = \|F'(\tilde{u}^1)\pi(\tilde{\varphi}_T, \xi_T^1) - F'(\tilde{u}^2)\pi(\tilde{\varphi}_T, \xi_T^2)\|_{2a-1,p/2,q} \]

\[ \lesssim \|F'(\tilde{u}^1)\|_{a,p,q} \|\tilde{u}^1\|_{2a-1,p/2,q} \]

\[ + \|F'(\tilde{u}^2)\|_{a,p,q} \|\tilde{u}^2\|_{2a-1,p/2,q} \]

\[ \lesssim \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{2a-1,p/2,q} \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^2)\|_{2a-1,p/2,q} \]

\[ \lesssim \|F\|_{2r}^2 \left( \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{2a-1,p/2,q} + \|\tilde{u}^2\|_{a,p,q} \right) \]

\[ \lesssim \|\tilde{u}^1\|_{a,p,q} \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{2a-1,p/2,q} \]

where we refer to (3), (7), (22) and (29) for explanations to the above estimates. Applying (29), Lemma 4.4 and Besov embeddings, we see for the next term that

\[ \|D_2^1 - D_2^2\|_{2a-1,p/2,q} = \|F'(\tilde{u}^1)\|_{a,p,q} \]

\[ \lesssim \|F\|_{\infty} \left( \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{2a-1,p/3,3} \right) \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^2)\|_{2a-1,p/3,3} \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{\infty} \|\tilde{u}^1\|_{a,p,q} \|	ilde{u}^2\|_{a,p,q} \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^2)\|_{\infty} \|\tilde{u}^1\|_{a,p,q} \|	ilde{u}^2\|_{a,p,q} \]

For the third term, again due to (29) as well as Lemma 4.4 and Besov embeddings, we obtain

\[ \|D_2^1 - D_2^3\|_{2a-1,p/2,q} = \|F'(\tilde{u}^1)\|_{a,p,q} \]

\[ \lesssim \|F\|_{\infty} \left( \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{2a-1,p/3,3} \right) \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^2)\|_{2a-1,p/3,3} \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^1)\|_{\infty} \|\tilde{u}^1\|_{a,p,q} \|	ilde{u}^2\|_{a,p,q} \]

\[ + \|\pi(\tilde{\varphi}_T, \xi_T^2)\|_{\infty} \|\tilde{u}^1\|_{a,p,q} \|	ilde{u}^2\|_{a,p,q} \]

Proposition 4.1 and the embedding \(B_{p/3,3}^{3a-1} \subseteq B_{p/2,2}^{2a-1}\) yield for the fourth term

\[ \|D_2^1 - D_2^4\|_{2a-1,p/2,q} = \|\Pi_F(\tilde{u}^1, \xi_T^1) - \Pi_F(\tilde{u}^2, \xi_T^2)\|_{2a-1,p/2,q} \]

\[ \lesssim \|\Pi_F(\tilde{u}^1, \xi_T^1)\|_{\infty} \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} \]

\[ + \|\Pi_F(\tilde{u}^2, \xi_T^2)\|_{\infty} \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} \]

where the constant \(C(\tilde{u}^1, \tilde{u}^2, \xi_T^1, \xi_T^2)\) is given in Proposition 4.1. The fifth term can be bounded by

\[ \|D_2^1 - D_2^5\|_{2a-1,p/2,q} = \|T_{\xi_T^1} F(\tilde{u}^1) - T_{\xi_T^2} F(\tilde{u}^2)\|_{2a-1,p/2,q} \]

\[ \lesssim \|T_{\xi_T^1} F(\tilde{u}^1)\|_{2a-1,p/2,q} + \|T_{\xi_T^2} F(\tilde{u}^2)\|_{2a-1,p/2,q} \]

\[ \lesssim \|F\|_{\infty} \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} + \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} \]

\[ \lesssim \|F\|_{\infty} \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} + \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} \]

\[ \lesssim \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} + \|\tilde{u}^1 - \tilde{u}^2\|_{a,p,q} \]
because of Lemma 2.1 and (22). For the sixth term, the lifting property [43, Thm. 2.3.8], an analog to (22) and (7) yield
\[
\|D_0^1 - D_0^2\|_{2\alpha-1,p/2,q} = \|T_{dF(\tilde{u})}\theta_T^1 - T_{dF(\tilde{u})}\theta_T^2\|_{2\alpha-1,p/2,q}
\lesssim \|T_{dF(\tilde{u})}-dF(\tilde{u})\theta_T^1\|_{2\alpha-1,p/2,q} + \|T_{dF(\tilde{u})}(\theta_T^1 - \theta_T^2)\|_{2\alpha-1,p/2,q}
\lesssim \|dF(\tilde{u})\|_{2\alpha-1,p,q} \|\theta_T^1\|_{\alpha,p,q} + \|dF(\tilde{u})\|_{2\alpha-1,p,q} \|\theta_T^2 - \theta_T^1\|_{\alpha,p,q}
\lesssim \|F(\tilde{u}) - F(\tilde{u})\|_{\alpha,p,q} \|\theta_T^1\|_{\alpha,p,q} + \|F(\tilde{u})\|_{\alpha,p,q} \|\theta_T^1 - \theta_T^2\|_{\alpha,p,q}
\lesssim \|F\|_{\infty} \|\theta_T^1\|_{\alpha,p,q} \|\tilde{u} - \tilde{u}\|_{\alpha,p,q} + \|F\|_{C^1} \|\tilde{u}\|_{\alpha,p,q} \|\theta_T^1 - \theta_T^2\|_{\alpha,p,q}.
\]
Since $2\alpha - 1 < 0$, the last difference $D_1^1 - D_1^2$ can be easily estimated by
\[
\|\tilde{u}(\tilde{1} - \tilde{2})\|_{2\alpha-1,p/2,q} \lesssim \|\tilde{u}(\tilde{1} - \tilde{2})\|_{L_p^2} \lesssim \|\tilde{u}\|_{\infty} \|\tilde{1} - \tilde{2}\|_{L_p^2} \lesssim (T \vee 1) \left|\frac{\tilde{u}'}{\tilde{u}}\right|_{\infty} \|\tilde{1} - \tilde{2}\|_{\alpha,p,q}.
\]
Defining the constants
\[
\tilde{C}_{\tilde{u},\tilde{u}} := 1 + \sum_{i=1,2} \left( \|\tilde{u}^j\|_{\alpha,p,q} + \|\tilde{u}^j\|_{\alpha,p,q} + \|u^{\#j}\|_{2\alpha-1,p/2,q} \right),
\]
\[
C_{\xi^j,\theta^j} := \|\theta_T^j\|_{\alpha,p,q} + \|\theta_T^j\|_{\alpha,p,q} + \|\pi(\theta_T^j, \xi_T^j)\|_{2\alpha-1,p/2,q}, \quad j = 1, 2,
\]
we altogether obtain
\[
\|d u^{\#,1} - d u^{\#,2}\|_{2\alpha-1,p/2,q} \lesssim \tilde{C}_{\tilde{u},\tilde{u}} \left( \|F\|_{C^1} + \|F\|_{C^2} \right) \times \left( \|\tilde{u} - \tilde{u}\|_{\alpha,p,q} + \|u^{\#1} - u^{\#2}\|_{2\alpha-1,p/2,q} \right)
+ \|\tilde{u} - \tilde{u}\|_{\alpha,p,q} + \|u^{\#1} - u^{\#2}\|_{2\alpha-1,p/2,q}
+ \|\pi(\theta_T^1, \xi_T^1) - \pi(\theta_T^2, \xi_T^2)\|_{2\alpha-1,p/2,q}
+ (T \vee 1) \left|\frac{\tilde{u}'}{\tilde{u}}\right|_{\infty} \|\tilde{u} - \tilde{u}\|_{\alpha,p,q}.
\]
The factor $\tilde{C}_{\tilde{u},\tilde{u}}$ is (locally) bounded since $\|\tilde{u}\|_{\alpha,p,q}$ and $\|\tilde{u}\|_{2\alpha-1,p/2,q}$ can be bounded by Corollary 5.8 and $u^{\#j}_{2\alpha-1,p/2,q}$ for $j = 1, 2$, can be bounded analogously to $5.11$ and $5.22$ by
\[
\|u^{\#j}\|_{2\alpha-1,p/2,q} \lesssim (T \vee 1) \left( \|F\|_{\infty} \|\theta_T^j\|_{L_p} + \|\tilde{u}\|_{L_p^2} \right)
+ C_{\xi^j,\theta^j} (\|F\|_{C^2} \vee \|F\|_{C^2}^2) \left( \|\tilde{u}\|_{\alpha,p,q} + \|F\|_{\infty} \|\theta_T^j\|_{\alpha,p,q} \right)
\lesssim (T \vee 1) \left( 1 + \|F\|_{C^2} \vee \|F\|_{C^2}^3 \right) (1 + \|\theta_T^j\|_{\alpha,p,q} + \|\tilde{u}\|_{\alpha,p,q}^\prime). \quad j = 1, 2.
\]
Relying on the lifting property of Besov spaces together with the definition of $u^{\#}$, $\|\tilde{u} - \tilde{u}\|_{L_p^2} \lesssim (T \vee 1) \|\tilde{u} - \tilde{u}\|_{L_p^2}$ and the compact support of $\theta_T^j$, we have
\[
\|u^{\#1} - u^{\#2}\|_{2\alpha-1,p/2,q} \lesssim (T \vee 1) \|F(\tilde{u}) - F(\tilde{u})\|_{\infty} \|\theta_T^1 - \theta_T^2\|_{L_p} + \|F\|_{\infty} \|\theta_T^1 - \theta_T^2\|_{L_p} + \|\tilde{u} - \tilde{u}\|_{L_p^2} \lesssim (T \vee 1) \left( \|F\|_{C^2} \vee \|F\|_{C^2}^2 \right) \left( \|\tilde{u}\|_{\alpha,p,q} + \|F\|_{\infty} \|\theta_T^j\|_{\alpha,p,q} \right)
+ (T \vee 1) \left( \|F\|_{C^2} \vee \|F\|_{C^2}^3 \right) \left( 1 + \|\theta_T^j\|_{\alpha,p,q} + \|\tilde{u}\|_{\alpha,p,q}^\prime \right) + \|u^{\#1} - u^{\#2}\|_{2\alpha-1,p/2,q}.
\]
Therefore, if \( \|F\|_{C^3} + \|F\|_{C^2} \) is sufficiently small, depending on \( \dot{C}_{\xi,0}, \dot{C}_{\check{u},u} \) and \( T \), then

\[
\begin{align*}
\|\dot{u}^{1,2} - \check{u}^{1,2}\|_{2a-1,p/2,q} & \lesssim (1 + \|\dot{\vartheta}_T\|_{a,p,q})\dot{C}_{\xi,\vartheta}\dot{C}_{\check{u},u} (T \lor 1)(\|F\|_{C^3} + \|F\|_{C^2}) \\
& \times (\|\check{u}^{1,2}\|_{a,p,q} + \|\dot{\vartheta}_T\|_{a-1,p,q} + \|\dot{\vartheta}_T - \vartheta_T\|_{a,p,q}) \\
& + \|\pi(\vartheta_T^{1,2},\xi_T^{1,2}) - \pi(\check{u}_T^{1,2},\check{u}_T^{1,2})\|_{2a-1,p/2,q}) + (T \lor 1)\|\check{u}^{1,2}_T\|_\infty \|\check{u}^{1,2} - \check{u}^{1,2}\|_{a,p,q}.
\end{align*}
\]

Plugging this estimate and (37) into (36), we obtain

\[
\begin{align*}
\|\check{u}^{1,2} - \check{u}^{1,2}\|_{a,p,q} & \lesssim (T^2 \lor 1)|u^{1,2}(0) - u^{2,2}(0)| + (1 + \|\vartheta_T^{1,2}\|_{a,p,q})C_{\check{u},\vartheta}\check{C}_{\check{u},u} (T \lor 1)(\|F\|_{C^3} + \|F\|_{C^2}) \\
& \times (\|\check{u}^{1,2}-\check{u}^{1,2}\|_{a,p,q} + \|\vartheta_T^{1,2}\|_{a-1,p,q} + \|\vartheta_T^{1,2} - \vartheta_T^{1,2}\|_{a,p,q}) \\
& + \|\pi(\vartheta_T^{1,2},\xi_T^{1,2}) - \pi(\check{u}_T^{1,2},\check{u}_T^{1,2})\|_{2a-1,p/2,q}) + (T^2 \lor 1)\|\check{u}^{1,2}_T\|_\infty \|\check{u}^{1,2} - \check{u}^{1,2}\|_{a,p,q}.
\end{align*}
\]

For a possibly smaller \( \|F\|_{C^3} + \|F\|_{C^2} \) and a sufficiently small \( \|\check{u}^{1,2}_T\|_\infty \), we conclude

\[
\begin{align*}
\|\check{u}^{1,2} - \check{u}^{1,2}\|_{a,p,q} & \lesssim (T^2 \lor 1)|u^{1,2}(0) - u^{2,2}(0)| + (1 + \|\vartheta_T^{1,2}\|_{a,p,q})C_{\check{u},\vartheta}\check{C}_{\check{u},u} (T \lor 1)(\|F\|_{C^3} + \|F\|_{C^2}) \\
& \times (\|\vartheta_T^{1,2} - \vartheta_T^{1,2}\|_{a,p,q} + \|\pi(\vartheta_T^{1,2},\xi_T^{1,2}) - \pi(\check{u}_T^{1,2},\check{u}_T^{1,2})\|_{2a-1,p/2,q}).
\end{align*}
\]

Finally, note again that \( \dot{C}_{\check{u},u} \) is (locally) bounded by Corollary 3.8.

\[\square\]

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