Spike statistics

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Abstract In this paper we explore stochastical and statistical properties of so-called recurring spike induced Kasner sequences. Such sequences arise in recurring spike formation, which is needed together with the more familiar BKL scenario to yield a complete description of generic spacelike singularities. In particular we derive a probability distribution for recurring spike induced Kasner sequences, complementing similar available BKL results, which makes comparisons possible. As examples of applications, we derive results for so-called large and small curvature phases and the Hubble-normalized Weyl scalar.

Keywords Singularities · BKL · Recurring spikes · Stochasticty

1 Introduction

It is no understatement to say that the work of Belinskii, Khalatnikov and Lifshitz (BKL) [1–3] has set the stage for much of subsequent investigations about the detailed nature of generic spacelike singularities. The starting point for their analysis is their ‘locality’ conjecture, which states that asymptotically toward a generic spacelike singularity in inhomogeneous cosmology the dynamics is local, in the sense that each spatial point is assumed to evolve toward the singularity individually and independently of its neighbors as a spatially homogeneous model [1,3]. A second important conjecture of BKL is that some sources, like perfect fluids with sufficiently soft equations of state
such as dust or radiation, lead to models that generically are asymptotically ‘vacuum dominated’, i.e., asymptotically toward a generic spacelike singularity the spacetime geometry is not influenced by the matter content, even though, e.g., the energy density blows up [1–3]. This leads to a picture of temporally oscillating Kasner states (i.e., vacuum Bianchi type I solutions, which have flat spatial curvature) where two subsequent Kasner states are determined by the vacuum Bianchi type II solutions, so-called single curvature transitions in the nomenclature of [4].1

Several recent papers [4–10] have provided further evidence for the BKL scenario, but it has also been found that there are timelines whose evolution is different, exhibiting ‘non-local’ recurring spike formation. Remarkably, BKL and recurring spike behavior are intimately linked in a hierarchical manner: by hierarchies of subsets and by a solution generating algorithm that links all building blocks for understanding asymptotics at generic spacelike singularities to the bottom of the hierarchy, which consists of the Kasner solutions.

In a recent paper [9] it was shown that recurring spike formation could be analytically described by means of combining the exact inhomogeneous solutions found by Lim [7] into concatenation blocks that connect families of timelines affected by spike formation at common initial and final Kasner states. It was also shown that the concatenation blocks come in two versions, so-called high velocity transitions, \( T_{Hi} \), and joint low/high velocity spike transitions, \( T_{Jo} \), where the word transition, as in the BKL case, signifies a change from one Kasner state to another. Hence both BKL dynamics and spike dynamics can be described in terms of temporally oscillating Kasner states, and these oscillations can be described by discrete maps.

The outline of the paper is as follows. In the next section we describe the BKL and spike maps, and the maps they in turn give rise to, and some of these maps properties; due to the central feature that all these maps describe changes in Kasner states, we refer to them collectively as Kasner maps. Furthermore, we derive a distribution function that allows one to statistically analyze properties associated with spike induced Kasner sequences, which complements previous corresponding BKL results. In Sect. 3 we explore some statistical features the maps give rise to. In particular we prove that statistically so-called small curvature phases dominate over large curvature phases, and we also give stochastical relations for the ‘Hubble-normalized’ Weyl scalar, for generic BKL as well as spike induced Kasner sequences. We conclude the paper with a few remarks about future possible research projects concerning explorations of chaotic features of recurring spike formation, which we leave for the interested reader.

2 Kasner maps

A Kasner state can be described in terms of the gauge invariant Kasner parameter \( u \), see e.g. [2,4,10,11].2 Next we describe the two types of discrete maps which determine the

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1 The term transition denotes a change between two different Kasner states; ‘single curvature’ refers to a single degree of freedom that describes the spatial curvature, which is excited during the transition.

2 The 1-parameter family of Kasner solutions is often given in terms of the line element \( ds^2 = -dt^2 + \sum_{i=1}^{3} t^2 p_i dx_i^2 + t^2 p_1 dy_1^2 + t^2 p_2 dz_2^2 \), where \( p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \). The Kasner parameter \( u \) can be defined via \( p_1 p_2 p_3 = -u^2 (1 + u)^2 / (1 + u + u^2)^3 \).
change in $u$ due to the transitions, beginning with the map that arises from the Bianchi type II solutions (i.e., single curvature transitions) in the BKL scenario. Throughout, the time direction is chosen to be toward the initial singularity.

2.1 The BKL map

The single curvature transitions induce the following BKL map $[2,11]$:

$$u^f = \begin{cases} u^i - 1 & \text{if } u^i \in [2, \infty), \\ (u^i - 1)^{-1} & \text{if } u^i \in [1, 2]. \end{cases}$$

(1)

This map is often referred to as the Kasner map, but since we here will use the term Kasner map as an umbrella term for maps that induce changes in a Kasner state, we instead refer to this particular Kasner map as the BKL map.

2.2 The BKL era map

Iterations of the BKL map generate, from every initial value $u_0 \in [1, \infty)$, a (finite or infinite) BKL sequence of Kasner epochs $(u_l)_{l=0,1,2,\ldots}$. We say that the Kasner epochs $u_l$ and $u_{l+1}$ belong to the same (BKL) era if $u_{l+1} - u_l = 1$, thereby leading to a partition of a BKL sequence of Kasner epochs into eras $[2,11]$, e.g.,

$$\cdots \rightarrow 1.14 \rightarrow 7.29 \rightarrow 6.29 \rightarrow 5.29 \rightarrow 4.29 \rightarrow 3.29 \rightarrow 2.29 \rightarrow 1.29 \rightarrow 3.45 \rightarrow 2.45 \rightarrow 1.45 \rightarrow 2.24 \rightarrow 1.24 \rightarrow \cdots$$

Let us denote the initial (= maximal) value of the Kasner parameter $u$ in era number $s$ by $u_s$, where $s = 0, 1, 2, \ldots$. Following $[2,11]$, we decompose $u_s$ into its integer part $k_s = \lfloor u_s \rfloor$ and its fractional part $\{u_s\}$, i.e.,

$$u_s = k_s + \{u_s\}. \quad (2)$$

The number $k_s$ represents the (discrete) length of era $s$, which is simply the number of Kasner epochs it contains. The final (= minimal) value of the Kasner parameter in era $s$ is given by $1 + \{u_s\}$, which implies that era number $(s + 1)$ begins with

$$u_{s+1} = \frac{1}{\{u_s\}}. \quad (3)$$

The map $u_s \mapsto u_{s+1}$ is (a variant of) the BKL era map, which we usually, for brevity, refer to as just the era map; starting from $u_0 = u_0$ it recursively determines $u_s$, $s = 0, 1, 2, \ldots$, and thereby the complete BKL Kasner sequence $(u_l)_{l=0,1,\ldots}$ of epochs.

The era map admits a straightforward interpretation in terms of continued fractions. Consider the continued fraction representation of the initial value, i.e.,
\[
\begin{align*}
\mathbf{u}_0 &= k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots}} = [k_0; k_1, k_2, k_3, \ldots].
\end{align*}
\]  

The fractional part of \( \mathbf{u}_0 \) is \([0; k_1, k_2, k_3, \ldots]\); since \( \mathbf{u}_1 \) is the reciprocal of \( \{\mathbf{u}_0\} \), it follows that

\[
\mathbf{u}_1 = [k_1; k_2, k_3, k_4, \ldots].
\]  

Therefore, the era map is simply a shift to the left in the continued fraction expansion,

\[
\mathbf{u}_s = [k_s; k_{s+1}, k_{s+2}, \ldots] \mapsto \mathbf{u}_{s+1} = [k_{s+1}; k_{s+2}, k_{s+3}, \ldots].
\]  

In terms of continued fractions, the Kasner sequence \((\mathbf{u}_t)_{t \in \mathbb{R}}\) generated by \( \mathbf{u}_0 = [k_0; k_1, k_2, \ldots] \) is given by

\[
\begin{align*}
\mathbf{u}_0 &= \mathbf{u}_0 = [k_0; k_1, k_2, \ldots] \mapsto [k_0 - 1; k_1, k_2, \ldots] \mapsto [k_0 - 2; k_1, k_2, \ldots] \mapsto \cdots \mapsto [1; k_1, k_2, \ldots] \\
\mathbf{u}_1 &= [k_1; k_2, k_3, \ldots] \mapsto [k_1 - 1; k_2, k_3, \ldots] \mapsto [k_1 - 2; k_2, k_3, \ldots] \mapsto \cdots \mapsto [1; k_2, k_3, \ldots] \\
\mathbf{u}_2 &= [k_2; k_3, k_4, \ldots] \mapsto [k_2 - 1; k_3, k_4, \ldots] \mapsto [k_2 - 2; k_3, k_4, \ldots] \mapsto \cdots.
\end{align*}
\]  

Since \( \mathbf{u}_s \in [k_s, k_s + 1) \), the number \( k_s \) represents the (discrete) length of era \( s \), which is simply the number of Kasner epochs it contains. Therefore, passing on to the stochastical interpretation of (generic) Kasner sequences of epochs, we find that the probability that a randomly chosen era \( s \) of a Kasner sequence \((\mathbf{u}_t)_{t \in \mathbb{R}}\) has length \( m \in \mathbb{N} \) corresponds to the probability that \( k_s = m \), or, equivalently, to the probability that \( \mathbf{u}_s \in [m, m + 1) \). Since the sequence \((k_0, k_1, k_2, \ldots)\) arises as the continued fraction expansion of \( \mathbf{u}_0 \) this probability in turn corresponds to the probability that a randomly chosen partial quotient in the continued fraction expansion is equal to \( m \). Hence we resort to Khinchin’s law [12], which states that the partial quotients of the continued fraction representation of a generic real number are distributed like a random variable whose probability distribution is given by

\[
K(m) = \log_2 \left( \frac{m + 1}{m + 2} \right) - \log_2 \left( \frac{m}{m + 1} \right),
\]  

which leads to

\[
\text{Probability(length of era} = n) = L(n) = K(n).
\]  

It follows that 42\% of the BKL eras of a generic BKL Kasner sequence of epochs contain merely one Kasner epoch; 17\% of the eras contain 2 Kasner epochs, and, e.g., \( 1.4 \times 10^{-12} \)\% of the eras consist of 100 Kasner epochs.
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2.3 The spike map

Let us now consider the Kasner map that is induced by the spike transitions $T_{Hi}$ and $T_{Jo}$. It is shown in [9] that, remarkably, the two types of transitions give rise to the same Kasner map which we refer to as the spike map. This map is obtained by applying the BKL map twice, see [9], which results in the following relations:

$$u_+ = \begin{cases} u_- - 2 & u_- \in [3, \infty), \\ (u_- - 2)^{-1} & u_- \in [2, 3), \\ ((u_- - 1)^{-1} - 1)^{-1} & u_- \in [3/2, 2), \\ (u_- - 1)^{-1} - 1 & u_- \in [1, 3/2]. \end{cases}$$  \hspace{1cm} (9)

2.4 The spike era map

Asymptotically we expect that spike dynamics is reduced to dynamics along timelines associated with a spatial ‘spike’ surface. Furthermore, this gives rise to recurring spikes described by sequences of $T_{Hi}$ and $T_{Jo}$, which yield iterations of the spike map [9]. Iterations of the spike map (9) generate, from every initial value $u_0 \in [1, \infty)$, a (finite or infinite) recurring spike-generated sequence of Kasner epochs $(u_l)_{l=0,1,2,...}$. We make a partition of $(u_l)_{l=0,1,2,...}$ into recurring spike-induced eras, which we denote as spike eras, or, for brevity, eras: $u_l$ and $u_{l+1}$ belong to the same era if $u_{l+1} = u_l - 2$. If $u_{l+1}$ arises from $u_l$ by one of the other three laws of (9), we speak of a change of era.

\[ \cdots \rightarrow 1.14 \rightarrow 7.29 \rightarrow 5.29 \rightarrow 3.29 \rightarrow 1.29 \rightarrow 2.45 \rightarrow 2.24 \rightarrow 4.16 \rightarrow 2.16 \rightarrow 6.14 \rightarrow 4.14 \rightarrow 2.14 \rightarrow \cdots \]

We denote the initial (= maximal) value of the Kasner parameter $u$ in era number $s$ (where $s = 0, 1, 2, \ldots$) by $u_s$. The spike map induces an era map $u_s \mapsto u_{s+1}$, which recursively determines $(u_s)_{s \in \mathbb{N}}$ from $u_0 = u_0$, and thereby the complete spike induced sequence $(u_l)_{l=0,1,2,...}$ of Kasner epochs.

The length of an era $s$ is determined by the value of $u_s$: If $u_s \in [m, m+1)$ for some $m \in \mathbb{N}$, then the length of the era is $m/2$, if $m$ is even, and $(m + 1)/2$, if $m$ is odd. In the stochastical framework, in analogy with (7), let $\bar{K}(m)$ denote the probability that a randomly chosen element of an era sequence $(u_s)_{s \in \mathbb{N}}$ lies in the interval $[m, m+1)$.

**Theorem 2.1** Let $(u_s)_{s \in \mathbb{N}}$ be a generic spike-induced sequence of eras; then the probability that a randomly chosen element of $(u_s)_{s \in \mathbb{N}}$ lies in the interval $[m, m+1)$ is

$$\bar{K}(m) = \log_3 \left( \frac{m+2}{m+3} \right) - \log_3 \left( \frac{m}{m+1} \right).$$  \hspace{1cm} (10)
Proof We perform the proof in three steps.

Step 1: Representation of the $\tau \sigma \pi$ map $u_1 \mapsto u_{s+1}$ by means of continued fractions. The spike map (9) yields the following representations in terms of continued fraction expansions of $u_\pm$ for the four different cases of (9):

$$ u_- = [2 + k; k_1, k_2, k_3, \ldots] \mapsto u_+ = [k; k_1, k_2, k_3, \ldots], \quad (11a) $$
$$ u_- = [2; k_1, k_2, k_3, k_4, \ldots] \mapsto u_+ = [k_1; k_2, k_3, \ldots], \quad (11b) $$
$$ u_- = [1; 1, k_2, k_3, k_4, \ldots] \mapsto u_+ = [k_2; k_3, k_4, \ldots], \quad (11c) $$
$$ u_- = [1; 1 + k, k_2, k_3, \ldots] \mapsto u_+ = [k; k_2, k_3, k_4, \ldots], \quad (11d) $$

where $k$ and $k_i$, $i = 1, 2, \ldots$, are positive natural numbers. Recall that (11b)–(11d) describe a change of $\tau \sigma \pi$. The spike map (11) leads to a representation of the $\tau \sigma \pi$ map $u_s \mapsto u_{s+1}$: If

$$ u_s = [\kappa_s; \kappa_{s+1}, \kappa_{s+2}, \ldots], $$

then

$$ u_{s+1} = \begin{cases} [\kappa_{s+1} - (\kappa_s \mod 2); \kappa_{s+2}, \kappa_{s+3}, \ldots] & \text{if } \kappa_{s+1} - (\kappa_s \mod 2) \neq 0, \\ [\kappa_{s+2}; \kappa_{s+3}, \kappa_{s+4}, \ldots] & \text{if } \kappa_{s+1} - (\kappa_s \mod 2) = 0, \end{cases} \quad (12) $$

when we use the standard notation $(\kappa \mod 2)$ for natural numbers (where we recall that $(\kappa \mod 2) = 0$ if $\kappa$ is even; $(\kappa \mod 2) = 1$ if $\kappa$ is odd).

Step 2: Iterate (12) and represent the elements $u_s, s = 0, 1, 2, \ldots$, of the $\tau \sigma \pi$ sequence in terms of the partial quotients of the continued fraction representation of $u_0$. Let $u_0$ be given by $u_0 = [k_0; k_1, k_2, \ldots]$. We begin by constructing from $(k_0, k_1, k_2, \ldots)$ an auxiliary sequence $(\kappa'_0, \kappa'_1, \kappa'_2, \ldots)$ according to $\kappa'_0 = k_0$ and

$$ \kappa'_{s+1} = k_{s+1} - (\kappa'_s \mod 2) = k_{s+1} - \left( \left[ \sum_{i=0}^{s} k_i \right] \mod 2 \right). \quad (13) $$

In the generic case, the number 0 will appear (infinitely often) in $(\kappa'_0, \kappa'_1, \kappa'_2, \ldots)$. We remove these entries and denote the arising (sub)sequence by $(k'_0, k'_1, k'_2, \ldots)$. A typical example is

$$ (k_0, k_1, k_2, \ldots) = (4, 3, 6, 5, 1, 2, 1, 7, 2, 7, 1, 1, 3, \ldots), $$
$$ (\kappa'_0, \kappa'_1, \kappa'_2, \ldots) = (4, 3, 5, 4, 1, 1, 0, 7, 1, 6, 1, 0, 3, \ldots), $$
$$ (k'_0, k'_1, k'_2, \ldots) = (4, 3, 5, 4, 1, 1, 7, 1, 6, 1, 3, \ldots). $$

Accordingly, we have

$$ k'_s = \kappa'_{s+z_s} = k_{s+z_s} - \left( \left[ \sum_{i=0}^{s+z_s-1} k_i \right] \mod 2 \right), \quad (14) $$
where \( z_s \) is the number of slots (i.e., zeros) that have been removed up to index number \( s \) of the sequence \((k'_0, k'_1, k'_2, \ldots)\).

Comparing (12) and (13) we find that the \( \bar{\varphi} \) sequence reads

\[
\varphi_0 = [k'_0; k_1, k_2, \ldots], \quad \varphi_1 = [k'_1; k_2, k_3, \ldots], \quad \varphi_2 = [k'_2; k_3, k_4, \ldots], \quad \ldots
\]

up the smallest index \( s \) for which \( k_{s+1} - (k'_s \mod 2) = 0 \); at that point, the second case of (12) becomes relevant, which leads to an omission of \( k'_{s+1} \) (= 0). The appropriate sequence that enters our representation of \((\varphi_s)_{s \in \mathbb{N}}\) is thus the constructed (sub)sequence \((k'_0, k'_1, k'_2, \ldots)\): Making use of \((k'_0, k'_1, k'_2, \ldots)\) we are able to give the element \( \varphi_s \) of the \( \bar{\varphi} \) sequence as

\[
\varphi_s = [k'_{s+z_s}; k_{s+z_s+1}, k_{s+z_s+2}, \ldots] = [k'_s; k_{s+z_s+1}, k_{s+z_s+2}, \ldots],
\]

where \( z_s \) is the number of omissions up to index number \( s \). It follows that the probability \( \bar{K}(m) \) that a randomly chosen element of \((\varphi_s)_{s \in \mathbb{N}}\) lies in the interval \([m, m + 1)\) is identical to the probability that a randomly chosen element of \((k'_0, k'_1, k'_2, \ldots)\) is equal to \( m \).

Step 3: Here we use the stochastical properties of sequences \((k'_0, k'_1, k'_2, \ldots)\) that arise from (generic) sequences \((k_0, k_1, k_2, \ldots)\) to derive the desired probability distribution. As a preliminary, we consider sums of the type \( \sum_{i=0}^{j} k_i \), where \((k_0, k_1, k_2, \ldots)\) are randomly distributed natural numbers, e.g., according to Khinchin’s law (7). Let \( 0 < p < 1 \) be the probability for an element of \((k_0, k_1, k_2, \ldots)\) to be odd (which is \( \sum_{m=1, m \text{ odd}} \bar{K}(m) \) in the case of Khinchin’s law). Note that \( p = 1/2 \) is not needed; in fact \( p \approx 0.6515 \) in the case of of Khinchin’s law. The sum \( \sum_{i=0}^{j} k_i \) is even or odd, if the number of odd summands is even or odd, respectively. Hence, making use of the binomial distribution, we find that the probability for \( \sum_{i=0}^{j} k_i \) to be even or odd is

\[
\sum_{i=0, i \text{ even}}^{j} \binom{j}{i} p^i (1 - p)^{j-i}, \quad \sum_{i=0, i \text{ odd}}^{j} \binom{j}{i} p^i (1 - p)^{j-i},
\]

respectively. The two probabilities are not exactly 1/2, but they converge to 1/2 rapidly as \( j \) grows, and hence the two probabilities are equal to 1/2 in the asymptotic limit of very long sequences.\(^3\)

Finally, let \( \varphi_0 = [k_0; k_1, k_2, \ldots] \) be a generic real number. Consider the truncated sequence \((k_0, k_1, \ldots, k_n)\), where \( n \gg 1 \), and denote the probability distribution of its elements by \( K_n \). In the limit \( n \to \infty \), \( K_n \) converges to Khinchin’s distribution \( K \), cf. (7). In the auxiliary sequence \((k'_0, k'_1, \ldots, k'_n)\) constructed from \((k_0, k_1, \ldots, k_n)\), the number zero appears in approximately 21% of the slots. This is because of (13): \( k'_s \) is 0 if and only if \( k_s \) equals 1 and \( \sum_{i=1}^{s-1} k_i \) is odd. The probability of the former event is \( K_n(1) \approx K(1) \), cf. (7), which is approximately 42% (because \( n \gg 1 \)); the probability

\(^3\) For \( j = 10 \), the error is already of the order of \( 10^{-6} \) in the case of Khinchin’s law; for a mere 100 elements, the error is already less than \( 10^{-50} \).
of the latter event is 1/2 in the limit $n \to \infty$. Since the two events are independent, we obtain a probability of $K_n(1)/2 \approx K(1)/2 \approx 21\%$ that the number 0 appears in a randomly chosen slot of $(\kappa_0', \kappa_1', \ldots, \kappa_n')$. Since the (sub)sequence $(k_0', k_1', \ldots, k_n')$ is constructed from $(\kappa_0', \kappa_1', \ldots, \kappa_n')$ by removing each appearance of the number 0, $(k_0', k_1', \ldots, k_n')$ is shorter than $(\kappa_0', \kappa_1', \ldots, \kappa_n')$ by 21%, i.e., $\bar{n} \approx 0.79n$. We are thus able to compute the probability $\bar{K}_n(m)$ that a randomly chosen element $k_s'$ of the sequence $(k_0', k_1', \ldots, k_n')$ is equal to $m$. There are two ways that the event $k_s' = m$ can occur. Either the corresponding element $k_{s+z_s}$, from which $k_s'$ is generated, see (14), is equal to $m$ and $\sum_{i=0}^{s+z_s-1} k_i$ is even, or it is equal to $m + 1$ and $\sum_{i=0}^{s+z_s-1} k_i$ is odd. Accordingly,

$$\bar{K}_n(m) = \frac{\frac{1}{2} K_n(m) + \frac{1}{2} K_n(m + 1)}{1 - \frac{1}{2} K_n(1)}.$$  \hspace{1cm} (16)

The limit $\bar{n} \to \infty$ corresponds to $n \to \infty$; since $K_n(m) \to \bar{K}(m)$ in this limit, (7) implies

$$\bar{K}(m) = \frac{\frac{1}{2} K(m) + \frac{1}{2} K(m + 1)}{1 - \frac{1}{2} K(1)} = \log_3 \left( \frac{m + 2}{m + 3} \right) - \log_3 \left( \frac{m}{m + 1} \right),$$ \hspace{1cm} (17)

which concludes the proof.

\textbf{Corollary 2.2} Let $(u_s)_{s \in \mathbb{N}}$ be a generic spike induced sequence of Kasner; then the probability that a randomly chosen Kasner in this sequence possesses length $n$ is given by

$$\text{Probability (length of Kasner } = n) =: \bar{L}(n) = \log_3 \left( \frac{2n + 1}{2n + 3} \right) - \log_3 \left( \frac{2n - 1}{2n + 1} \right).$$ \hspace{1cm} (18)

\textbf{Proof} This is a direct consequence of Theorem 2.1: The length of Kasner $s$, where $u_s = [x_s; x_{s+1}, x_{s+2}, \ldots]$, is $x_s/2$ if $x_s$ is even and $(x_s + 1)/2$ if $x_s$ is odd. Therefore, for Kasner $s$ to be of length $n$, $x_s$ must be either $2n$ or $2n - 1$. From (15) and (10) we see that the probability that one of these events occur is $\bar{K}(2n) + \bar{K}(2n - 1)$, which yields (18).

\section*{3 Statistics}

\subsection*{3.1 Comparison between BKL eras and spike eras}

Let us compare some consequences of the probability distribution (7), which determines the probabilities for prescribed lengths of BKL eras in BKL sequences of Kasner epochs according to Eq. (8), and the probability distribution (10), which determines the probabilities for prescribed lengths of Kasner in spike-induced sequences of Kasner epochs according to Eq. (18), see Tables 1 and 2. As seen in Table 2, Kasner have the tendency of being shorter than BKL eras. The probability that an Kasner contains merely
Table 1  Probabilities (in %) that a randomly chosen element of a BKL/spike-generated Kasner sequence of Kasner epochs \((u_l)_{l \in \mathbb{N}}\), is in the interval \([m, m + 1)\), \(m = 1, 2, 3, \ldots\)

| Sequence | 1     | 2     | 3     | 4     | 5 | 10    | 50    | 100    | 500    |
|----------|-------|-------|-------|-------|---|-------|-------|-------|-------|
| era      | 41.50 | 16.99 | 9.31  | 5.89  | 4.06| 1.20  | \(5.5 \times 10^{-2}\) | \(1.4 \times 10^{-2}\) | \(5.7 \times 10^{-4}\) |
| \(\overline{\text{era}}\) | 36.91 | 16.60 | 9.60  | 6.28  | 4.44| 1.39  | \(6.9 \times 10^{-2}\) | \(1.8 \times 10^{-2}\) | \(7.2 \times 10^{-4}\) |

These are the probability distributions \(K(m)\) and \(\overline{K}(m)\), see (7) and (10).

Table 2  Probabilities (in %) that a randomly chosen era/\(\overline{\text{era}}\) of a BKL/spike generated sequence of Kasner epochs, is of a prescribed length

| Length | 1 | 2 | 3 | 4 | 5 | 10 | 50 | 100 | 500 |
|--------|---|---|---|---|---|----|----|-----|-----|
| era    | 41.50 | 16.99 | 9.31 | 5.89 | 4.06 | 1.20 | \(5.5 \times 10^{-2}\) | \(1.4 \times 10^{-2}\) | \(5.7 \times 10^{-4}\) |
| \(\overline{\text{era}}\) | 53.50 | 15.87 | 7.75 | 4.61 | 3.06 | 0.83 | \(3.5 \times 10^{-2}\) | \(0.9 \times 10^{-2}\) | \(3.6 \times 10^{-4}\) |

These are the probability distributions \(L(m) = K(m)\) and \(\overline{L}(m)\), see (8) and (18).

one epoch is larger than 50 %; the probability that an \(\overline{\text{era}}\) consists of \(n > 1\) epochs is smaller than that of a BKL era. Asymptotically, for \(n \gg 1\), we have

\[
\text{Probability (length of era } = n) = (\log 2)^{-1} n^{-2} \left(1 - 2n^{-1} + O(n^{-2})\right),
\]

\[
\text{Probability (length of } \overline{\text{era}} = n) = (\log 3)^{-1} n^{-2} \left(1 - n^{-1} + O(n^{-2})\right),
\]

and hence the two probabilities are asymptotically proportional with a proportionality factor \(\log 2 / \log 3\).

3.2 Dominance of small curvature phases

Following [4] we first introduce \(\Upsilon > 3\), where we are foremost interested in the case \(\Upsilon \gg 1\). Then we define a small curvature phase of a BKL or spike induced sequence of Kasner epochs \((u_l)_{l \in \mathbb{N}}\) as a connected and inextendible piece \(\mathcal{L} \subset \mathbb{N}\) such that \(u_l > \Upsilon \forall l \in \mathcal{L}\). During a small curvature phase \(u_l\) is thus monotonically decreasing from a maximal value by the BKL map to a minimal value in the interval \((\Upsilon, \Upsilon + 1]\), while the spike map yields a monotonic decrease from a maximum value to a minimal value in the interval \((\Upsilon, \Upsilon + 2]\). The complement of the concept of a small curvature phase is a large curvature phase, which is defined as an (inextendible) piece of the sequence of Kasner epochs such that \(u_l \leq \Upsilon\) for all \(l\).

During small/large curvature phases, the ‘Hubble-normalized’ curvature is comparatively small/large, see Eq. (25) below. While a small curvature phase can be viewed as an era/\(\overline{\text{era}}\) that is terminated prematurely at \(\Upsilon\), a large curvature phase typically consists of many eras/\(\overline{\text{eras}}\); clearly, small and large curvature phases occur alternately. In the following BKL example, where the choice \(\Upsilon = 4\) has been made, the large curvature phase contains two and a half eras.
Combining the probabilistic viewpoint with the concept of small/large curvature phases lead to a fundamental result in the description of the BKL and spike induced Kasner sequences. For generic Kasner sequences \((u_l)_{l \in \mathbb{N}}\), small curvature phases dominate over large curvature phases in the following sense:

**Theorem 3.1** Let \((u_l)_{l \in \mathbb{N}}\) be a generic BKL or spike-induced Kasner sequence and let \(\Upsilon\) be arbitrarily large. Then for a randomly chosen epoch \(u\) the probability for the event \(u > \Upsilon\) is one and the probability for the event \(u \leq \Upsilon\) is zero.

**Proof** Let us first consider the BKL case and let us for simplicity choose \(\Upsilon \in \mathbb{N}\). We let \(u_0 = [k_0; k_1, k_2, \ldots]\) be a well approximable irrational number and consider the BKL Kasner sequence of epochs \((u_l)_{l \in \mathbb{N}}\) and the associated era sequence \((u_s)_{s \in \mathbb{N}}\) (where we recall \(u_s = [k_s; k_{s+1}, \ldots]\)). Consider the truncated sequence of eras \((u_s)_{s \leq n}\) and denote the associated probability distribution by \(K_n\) (which converges to Khinchin’s law \(K\) in the limit \(n \to \infty\)). Let \(u\) be a randomly chosen element from the associated sequence of epochs \((u_l)_{l \leq k_0 + \cdots + k_n}\). We denote by \(E_m\) the collection of epoch of eras of length \(m\) and by \(\tilde{P}_n(u \in E_m)\) the probability that the epoch \(u\) belongs to an era of length \(m\); it is given by

\[
\tilde{P}_n(u \in E_m) = \left( \sum_{m'} m' K_n(m') \right)^{-1} m K_n(m). \tag{19}
\]

The probability for \(u \leq \Upsilon\), which we denote by \(P_n(u \leq \Upsilon)\), is thus obtained from

\[
P_n(u \leq \Upsilon) = \sum_{m=1}^{\Upsilon-1} \tilde{P}_n(u \in E_m) + \sum_{m=\Upsilon}^{\infty} \tilde{P}_n(u \in E_m) \frac{\Upsilon-1}{m} = \left( \sum_{m'} m' K_n(m') \right)^{-1} \left[ \sum_{m=1}^{\Upsilon-1} m K_n(m) + (\Upsilon - 1) \sum_{m=\Upsilon}^{\infty} K_n(m) \right].
\]

For a generic number \(u_0\), in the asymptotic limit \(n \to \infty\), the probability \(K_n(m)\) converges to \(K(m)\) given by Khinchin’s law (7). Since

\[
K(m) = \frac{1}{\log 2} \left[ m^{-2} - 2m^{-3} + O\left(m^{-4}\right) \right] \quad (m \to \infty), \tag{20}
\]

the sum \(\sum_{m=\Upsilon}^{\infty} K_n(m)\) approaches the convergent series \(\sum_{m=\Upsilon}^{\infty} K(m)\) in the limit \(n \to \infty\); however, the sum \(\sum_{m} m K_n(m)\) diverges as \(n \to \infty\), so that

\[
P(u \leq \Upsilon) = \lim_{n \to \infty} P_n(u \leq \Upsilon) = 0, \tag{21}
\]

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irrespective of the value of $\Upsilon$. We conclude that the probability that an epoch $u$, which is randomly chosen from the Kasner sequence $(u_l)_{l \in \mathbb{N}}$, belongs to a large curvature phase ($u \leq \Upsilon$) is zero; with probability 1 a randomly chosen element belongs to a small curvature phase ($u > \Upsilon$).

It remains to consider the recurring spike case, where we choose an arbitrarily large $\Upsilon$ such that $\Upsilon = 2\nu + 1$ with $\nu \in \mathbb{N}$ for simplicity. Let $\bar{L}_n(m)$ denote the probability for an era of the truncated sequence to be of length $m$; $\bar{L}_n(m)$ converges to $\bar{L}(m)$ as $n \to \infty$, cf. (18). Then

$$\hat{P}_n(u \in \mathcal{E}_m) = \left( \sum_{m'} m' \bar{L}_n(m') \right)^{-1} m \bar{L}_n(m),$$

and the probability for $u \leq \Upsilon$ reads

$$P_n(u \leq \Upsilon) = \sum_{m=1}^{\nu} \hat{P}_n(u \in \mathcal{E}_m) + \sum_{m=\nu+1}^{\infty} \hat{P}_n(u \in \mathcal{E}_m) \frac{\nu}{m} \sum_{m', \bar{L}_n(m')} \bar{L}_n(m) \bar{L}_n(m').$$

In the case of a generic sequence, $\bar{L}_n(m)$ converges to $\bar{L}(m)$ as $n \to \infty$, where

$$\bar{L}(m) = \frac{1}{\log 3} \left[ m^{-2} - m^{-3} + O(m^{-4}) \right] \quad (m \to \infty).$$

Hence the sum $\sum_{m=\nu+1}^{\infty} \bar{L}_n(m)$ converges as $n \to \infty$. However, the sum $\sum_{m} m \bar{L}_n(m)$ diverges as $n \to \infty$ and $P(u \leq \Upsilon) = \lim_{n \to \infty} P_n(u \leq \Upsilon) = 0$. \hfill \Box

The underlying reason for the dominance of small curvature phases is hence the failure of the probability distributions (7), (10) [and (8), (18)], to generate finite expectation values, since

$$\sum_{m=1}^{\infty} m \bar{K}(m) = \sum_{m=1}^{\infty} m \bar{L}(m) = \infty, \quad \sum_{m=1}^{\infty} m \bar{K}(m) = \infty = \sum_{m=1}^{\infty} m \bar{L}(m),$$

which is due to the infinite tail of the distributions. Accordingly, the average length of an era is ill-defined.

3.3 Weyl stochastics

The magnetic part of the Weyl tensor for any Kasner state is zero, and hence the Hubble-normalized Weyl scalar is given by $\mathcal{W}_K^2 = \frac{1}{4} H^{-4} E_{a \beta} E^{a \beta}$, where $E_{a \beta}$ is the electric Weyl tensor and $H$ is the Hubble variable, which is related to the expansion $\theta$.
of the timelike reference congruence according to $H = \theta/3$, see [4,10]. Expressed in terms of the Kasner parameter $u$, $W^2_K$ is given by

$$W^2_K = -81 p_1 p_2 p_3 = \frac{81 u^2(1 + u)^2}{(1 + u + u^2)^3}. \quad (25)$$

The behavior of $W^2_K$ over generic sequences $(u_l)_{l \in \mathbb{N}}$ of Kasner epochs is statistically described by the following Theorem:

**Theorem 3.2** Let $u_1 = u_1 = [k_1; k_2, k_3, \ldots]$ be a generic real number, $(u_l)_{l \in \mathbb{N}}$ the associated BKL or spike induced sequence of epochs, and $(u_s)_{s \in \mathbb{N}}$ the associated BKL or spike induced sequence of eras. Consider the truncated sequence $(u_1, \ldots, u_n)$ of epochs/(eras). Then the average $\langle W^2_K \rangle_n$ of $W^2_K$ over all epochs of the first $n$ BKL and spike-induced eras satisfies

$$\langle W^2_K \rangle_n = 19.580317157(8) \frac{1 + o(1)}{\log_2 n} \frac{1}{1 + \chi(n)}, \quad (26)$$

where $\chi$ depends on if we consider an era or and era, but in both cases it is a function satisfying $\lim \inf_{n \to \infty} \chi(n) = 0$ and $\lim \sup_{n \to \infty} \chi(n) = \infty$.

**Remark** Although the results are similarly the same for BKL and spike-induced sequences of Kasner eras, note that the concepts of eras differ for the two cases.

**Remark** The function $\chi$ does not converge to zero exactly, but 'almost'. Specifically, $\chi(n)$ converges to zero as $t \in n \to \infty$, where $t$ is a subset of $\mathbb{N}$ minus a subset of logarithmic density zero. The average $\langle W^2_K \rangle_n$ (slowly) converges to zero as $n \to \infty$. Intuitively, this is due to the dominance of small curvature phases over large curvature phases.

**Proof** We first prove Theorem 3.2 for the BKL case and then for the recurring spike case. In the BKL case we have:

**Lemma 3.3** Let $u_1 = u_1 = [k_1; k_2, k_3, \ldots]$ be a generic real number, $(u_l)_{l \in \mathbb{N}}$ the associated BKL sequence of epochs, and $(u_s)_{s \in \mathbb{N}}$ the associated BKL sequence of eras. Consider the truncated sequence $(u_1, \ldots, u_n)$ of eras. Then the average $\langle W^2_K \rangle_n$ of $W^2_K$ over all epochs of the first $n$ eras satisfies

$$\langle W^2_K \rangle_n = 19.580317157(8) \frac{1 + o(1)}{\log_2 n} \frac{1}{1 + \chi(n)}, \quad (27)$$

where $\chi$ is a function satisfying $\lim \inf_{n \to \infty} \chi(n) = 0$ and $\lim \sup_{n \to \infty} \chi(n) = \infty$.

Let $n \gg 1$. The average of $W^2_K$ over all epochs ($= k_1 + \cdots + k_n$) of the first $n$ eras is

$$\langle W^2_K \rangle_n = \left( \sum_{i=1}^n k_i \right)^{-1} \sum_{i=1}^n W^2_K(u_i) = \left( \sum_{i=1}^n k_i \right)^{-1} \sum_{s=1}^n \sum_{j=1}^n W^2_K([u_s] + j).$$
where \( \{u_s\} = [0; k_{s+1}, \ldots] = u_{s+1}^{-1} \) is the fractional part of \( u_s \); recall that the integer part \([u_s]\) is equal to \( k_s \). Since \( n \gg 1 \) we may consider \( \{u_s\} =: \kappa \) as a random variable on \((0, 1)\); its probability distribution is \( w(\kappa) = (\log 2)^{-1}(1 + \kappa)^{-1} \), see [4] with \( \kappa = u^{-1} \) and \( w(\kappa)d\kappa = -p(u)du \). We are thus able to replace \( \sum_{j=1}^{k_s} W^2_k(\{u_s\} + j) \) by its weighted average \( \overline{W^2_k(k_s)} \) which is given by

\[
\overline{W^2_k(m)} := \frac{1}{m} \int_0^1 w(\kappa) \sum_{j=1}^m W^2_k(j + \kappa) d\kappa = \frac{1}{m} \sum_{j=1}^m w(\kappa) W^2_k(j + \kappa) d\kappa.
\]

Accordingly we obtain

\[
\langle W^2_k \rangle_n = \left( \sum_{i=1}^n k_i \right)^{-1} \sum_{s=1}^n \overline{W^2_k(k_s)} = \frac{\sum_m K_n(m) \overline{W^2_k(m)}}{1/n \sum_{i=1}^n k_i}, \quad (28)
\]

where \( K_n \) denotes the probability distribution associated with \((k_1, \ldots, k_n)\), which converges to \( K \) as \( n \to \infty \).

To compute the numerator of (28) we first note that

\[
\int_0^1 w(\kappa) W^2_k(j + \kappa) d\kappa = \int_0^1 w(\kappa) W^2_k(j + \bar{\kappa}) d\kappa \\
+ \int_0^1 w(\kappa) [W^2_k(j + \kappa) - W^2_k(j + \bar{\kappa})] d\kappa
\]

\[
= W^2_k(j + \bar{\kappa}) - 162 j^{-3} \int_0^1 w(\kappa)(\kappa - \bar{\kappa}) d\kappa + O(j^{-4})
\]

\[
= \int_{j+\bar{\kappa}-1/2}^{j+\bar{\kappa}+1/2} W^2_k(\kappa) d\kappa + O(j^{-4}) \quad (29)
\]

as \( j \to \infty \), when we use \( \bar{\kappa} = \int_0^1 \kappa w(\kappa) d\kappa = (\log 2)^{-1} - 1 \). Accordingly, for \( m > M \gg 1 \) we find

\[
\overline{W^2_k(m)} - \overline{W^2_k(M)} = \sum_{j=M+1}^m \int_0^1 w(\kappa) W^2_k(j + \kappa) d\kappa \\
= \int_{M+\bar{\kappa}+1/2}^{M+\bar{\kappa}+1/2} W^2_k(\kappa) d\kappa + O(M^{-3} - m^{-3})
\]

\[
= 81(M^{-1} - m^{-1}) - \frac{81}{\log 2} (M^{-2} - m^{-2}) + O(M^{-3} - m^{-3}).
\]
Since $\sum_m K_n(m)\overline{W}_K^2(m)$ converges to $\sum_m K(m)\overline{W}_K^2(m)$ as $n \to \infty$, we consider the latter series. Choosing $\mathbb{N} \ni M \gg 1$ we obtain

$$
\sum_{m=1}^{\infty} K(m)\overline{W}_K^2(m) = \sum_{m=1}^{M} K(m)\overline{W}_K^2(m) + \sum_{m=M+1}^{\infty} K(m) \left(\overline{W}_K^2(M) + \frac{81}{M}\right) - \frac{81}{m} + O\left(\frac{1}{M^2}\right)
$$

$$
= \sum_{m=1}^{M} K(m)\overline{W}_K^2(m) - \left(\overline{W}_K^2(M) + \frac{81}{M}\right) \log_2 \frac{M+1}{M+2} - \frac{81}{2 \log 2 M^2} + O\left(\frac{1}{M^3}\right).
$$

A numerical calculation then yields the result 19.580317157(8).

It remains to analyze the denominator of (28), which is the mean length of the first $n$ eras, or, in other words, the arithmetic mean of the first $n$ partial quotients of $[k_1; k_2, \ldots]$. From the theory of continued fractions we have the result that

$$
\sum_{i=1}^{n} k_i = n \log_2 n \left(1 + \chi(n)\right),
$$

where $\chi$ is as described in the lemma and the remark; see, e.g., [12,13]. By combining (30) with the result of the previous calculation the lemma is proved.

Let us now turn to the recurring spike case. By considering the union of all Kasner sequences and noting that the spike Kasner map is just the square of the BKL map it is perhaps intuitive that we will obtain a similar result as in the BKL case, but let us give a formal proof nonetheless.

**Lemma 3.4** Let $u_1 = u_1 = [k_1; k_2, k_3, \ldots]$ be a generic real number, $(u_i)_{i \in \mathbb{N}}$ the associated spike induced sequence of epochs, and $(u_i)_{i \in \mathbb{N}}$ the associated spike induced sequence of eras. Consider the truncated sequence $(u_1, \ldots, u_n)$ of $\overline{\text{eras}}$. Then the average $\langle W_K^2 \rangle_n$ of $W_K^2$ over all epochs of the first $n$ $\overline{\text{eras}}$ satisfies

$$
\langle W_K^2 \rangle_n = 19.580317157(8) \frac{1 + o(1)}{\log_2 n} \frac{1}{1 + \tilde{\chi}(n)},
$$

where $\tilde{\chi}$ is a function with the same properties as $\chi$ of Lemma 3.3.

**Remark** The statement of this lemma is formally identical to that of Lemma 3.3, which motivates the formulation of Theorem 3.2. Note, however, that the concept of an era of a spike induced sequence of Kasner epochs differs from that of a BKL sequence.
For the proof of the above lemma we need (13)–(15); in particular we recall that
\[ u_s = \left[ k'_s, k_s + z_s + 1, \ldots \right] \]. Let us denote the length of \( \text{eras} \) by \( l_s \), where \( l_s = l(k'_s) \) with
\[
 l(k') = \begin{cases} 
 \frac{k'}{2}, & \text{if } k' \text{ even}, \\
 \frac{k'+1}{2}, & \text{if } k' \text{ odd}. 
\end{cases}
\] (32)
The average of \( \mathcal{W}_K^2 \) over all epochs \( (= l_1 + \cdots + l_n) \) of the first \( n \) \( \text{eras} \) is
\[
\langle \mathcal{W}_K^2 \rangle_n = \left( \sum_{i=1}^{n} l_i \right)^{-1} \sum_{l=1}^{l_1+\cdots+l_n} \mathcal{W}_K^2(u_l) = \left( \sum_{i=1}^{n} l_i \right)^{-1} \sum_{s=1}^{l_s} \sum_{j=1}^{l_s} \mathcal{W}_K^2(2j - (k'_s \mod 2) + \{u_s\}) .
\] In complete analogy with the proof of Lemma 3.3 we are able to replace the inner sum by by the weighted average \( \overline{\mathcal{W}_K^2(k'_s)} \), where
\[
\overline{\mathcal{W}_K^2(k')} \ := \int_0^1 w(x) \sum_{j=1}^{l(k')} \mathcal{W}_K^2(2j - (k' \mod 2) + x) dx
\]
\[
= \sum_{j=1}^{l(k')} \int_0^1 w(x) \mathcal{W}_K^2(2j - (k' \mod 2) + x) dx .
\]
Accordingly,
\[
\langle \mathcal{W}_K^2 \rangle_n = \left( \sum_{i=1}^{n} l_i \right)^{-1} \sum_{s=1}^{l_s} \overline{\mathcal{W}_K^2(k'_s)} = \frac{\sum_{k'} \bar{K}_n(k') \overline{\mathcal{W}_K^2(k')} }{\frac{1}{n} \sum_{i=1}^{n} l_i} ,
\] (33)
where \( \bar{K}_n \) denotes the probability distribution associated with \( (k_1, \ldots, k_n) \), which converges to \( \bar{K} \) as \( n \to \infty \).
To compute the numerator of (33) we use (29), where \( j \) is replaced by \( 2j - k' \mod 2 \). In addition we make use of the relation
\[
\int_{2j-a-1/2}^{2j+a+1/2} \mathcal{W}_K^2(x) dx = \frac{1}{4} \int_{2j-a-3/2}^{2j+a+3/2} \mathcal{W}_K^2(x) dx + \frac{1}{2} \int_{2j-a-1/2}^{2j+a+1/2} \mathcal{W}_K^2(x) dx
\]
\[
+ \frac{1}{4} \int_{2j+a+1/2}^{2j+a+3/2} \mathcal{W}_K^2(x) dx + O(j^{-4}) ,
\]
which holds for all $a$ (e.g., $\tilde{x} - k' \mod 2$). Let $k' > K' >> 1$ with $k' \mod 2 = K' \mod 2$. We find

$$\bar{W}_K^2(k') - \bar{W}_K^2(K') = \sum_{j=l(k')-1}^{l(k')} \int_0^1 w(x) \bar{W}_K^2(2j - (k' \mod 2) + x) dx$$

$$= \frac{1}{4} \int_{K'+\tilde{x}+3/2}^{K'+\tilde{x}+1/2} \bar{W}_K^2(x) dx + \frac{1}{2} \int_{K'+\tilde{x}+1/2}^{K'+\tilde{x}+3/2} \bar{W}_K^2(x) dx$$

$$+ \frac{1}{4} \int_{K'+\tilde{x}+1/2}^{K'+\tilde{x}+3/2} \bar{W}_K^2(x) dx + O((K')^{-3})$$

$$= \frac{81}{2} ((K')^{-1} - (k')^{-1}) - \frac{81}{4 \log 2} (2 + \log 2) ((K')^{-2} - (k')^{-2}) + O((K')^{-3})$$

Since $\sum_{k'} \tilde{K}_n(k') \bar{W}_K^2(k')$ converges to $\sum_{k'} \tilde{K}(k') \bar{W}_K^2(k')$ as $n \to \infty$, we consider the latter series. Choosing $\mathbb{N} \ni K' >> 1$ we obtain

$$\sum_{k'=K'+1}^{\infty} \tilde{K}(k') \bar{W}_K^2(k') = \sum_{k'=K'+1}^{\infty} \tilde{K}(k') \left( \bar{W}_K^2(K') - 1 \right) + \frac{81}{2} \left( \frac{1}{K'} - \frac{1}{k'} \right)$$

$$+ O\left( \frac{1}{K'^2} \right) + \sum_{k'=K'+2}^{\infty} \tilde{K}(k') \left( \bar{W}_K^2(K') \right)$$

$$+ \frac{81}{2} \left( \frac{1}{K'} - \frac{1}{k'} \right) + O\left( \frac{1}{K'^2} \right),$$

and thus

$$\sum_{k'=1}^{\infty} \tilde{K}(k') \bar{W}_K^2(k') = \sum_{k'=1}^{K'} \tilde{K}(k') \bar{W}_K^2(k') - \bar{W}_K^2(K' - 1) \log_3 \frac{K' + 1}{K' + 2}$$

$$- \bar{W}_K^2(K') \log_3 \frac{K' + 2}{K' + 3} - \frac{81}{2} \frac{1}{K'} \log_3 \frac{K' + 1}{K' + 3}$$

$$- \frac{81}{2 \log 3} \frac{1}{K'^2} + O\left( \frac{1}{K'^3} \right).$$

A numerical calculation then yields the result $12.35380467921(1)$.

It remains to analyze the denominator of (28), which is the mean length of the first $n$ Æras. Using (32) we have

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} l(k'_i) = \frac{1}{2} \sum_{i=1}^{n} k'_i + \frac{1}{2} n \log_3 2,$$
where $\log_3 2$ is the probability for a randomly chosen element of $(k_1', k_2', \ldots)$ to be odd. From the construction of the auxiliary sequence $(k_1', k_2', \ldots, k_n')$, see (13)–(15), we recall that $(k_1', k_2', \ldots, k_n')$ is shorter than $(k_1', k_2', \ldots, k_n')$ by the fraction $\frac{1}{2} K(1) = 1 - (2 \log_3 2)^{-1}$; accordingly, we have $\hat{n} = 2(\log_3 2) n$. Furthermore, we obtain

$$
\sum_{i=1}^{n} l_i = \frac{1}{2} \sum_{i=1}^{\hat{n}} k_i' + \frac{1}{2} n \log_3 2 = \frac{1}{2} \sum_{i=1}^{\hat{n}} (k_i - \frac{1}{2}) + \frac{1}{2} n \log_3 2,
$$

because, on the average, the elements of $(k_1', \ldots, k_n')$ are smaller than the elements of $(k_1, \ldots, k_{\hat{n}})$ by one half; we again refer to the proof of Lemma 2.1. Therefore,

$$
\sum_{i=1}^{n} l_i = \frac{1}{2} \sum_{i=1}^{\hat{n}} k_i - \frac{1}{4} \hat{n} + \frac{1}{2} n \log_3 2 = \frac{1}{2} \sum_{i=1}^{\hat{n}} k_i = \frac{1}{2} \hat{n} \log_2 \hat{n} \left(1 + \chi(\hat{n})\right)
$$

$$
= n \log_3 2 \left(1 - \log_2 \log_2 3 + \log_2 n\right) \left(1 + \chi(n)\right) = n \log_3 n \left(1 + \tilde{\chi}(n)\right)
$$

(34)

Noting that $12.35380467921(1) \log_3 (2(\log_2) \frac{1}{2})^{-1} = 19.580317157(8)$, the lemma is proved. This concludes the proof of Theorem 3.2. \qed

**Remark** The results of Lemmas 3.3 and 3.4 differ slightly when expressed in terms of a different parameter; instead of $n$ we use the number $N$ of epochs up to (and including) era/era $n$. Let us restrict ourselves to the main point. In the context of Lemma 3.3 we have $N = \sum_{i=1}^{n} k_i = n \log_2 n$, cf. (30). Therefore, $\log_2 n = (\log 2)^{-1} W(N \log 2)$, where $W$ is the Lambert $W$ function (product logarithm). For $N \gg 1$ we thus have

$$
\log_2 n = (\log 2)^{-1} \left( \log(N \log 2) - \log(N \log 2) + o(1) \right)
$$

$$
= (\log 2)^{-1} \left( \log N - \log \log N + \log \log 2 + o(1) \right).
$$

Analogously, in the context of Lemma 3.4 we have $N = \sum_{i=1}^{n} l_i = n \log_3 n$, cf. (34). Therefore, $\log_2 n = (\log 2)^{-1} W(N \log 3)$ and

$$
\log_2 n = (\log 2)^{-1} \left( \log N - \log \log N + \log \log 3 + o(1) \right).
$$

### 4 Concluding remarks

The BKL and the BKL era maps have attracted a lot of attention in the literature, perhaps due to the tantalizing chaotic features that they indicate, see [11] and [14–30], and references therein. The present paper opens up similar exploration possibilities for recurring spike formation—we have only given a few examples of what can be done. As in the BKL case, one can derive probabilities for other quantities, but one can also use the explicit solutions that describe spike transitions to attempt to extend the associated maps in perhaps similar ways as have been done for the BKL case.
Another possibility is to focus on the state space picture in [9, 10], where [9] give some examples of so-called concatenated chains, where axes permutations are not quoted out (for a discussion, see [10]). In particular, one can explore periodic spike chains, which are the recurring spike analogues of the so-called heteroclinic cycles in the BKL case, which recently have attracted attention [31,32].

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