Abstract. Let \( k, n \) and \( r \) be integers with \( 0 < k < n \) and \( 1 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor \). We determine the facets of the \( r \)-stable \( n, k \)-hypersimplex \( \Delta_{n,k}^{\text{stab}(r)} \). As a result, it turns out that \( \Delta_{n,k}^{\text{stab}(r)} \) has exactly \( 2n \) facets for every \( r < \left\lfloor \frac{n}{k} \right\rfloor \). Furthermore, we study when \( \Delta_{n,k}^{\text{stab}(r)} \) is Gorenstein.

1. Introduction

The \( r \)-stable \( n, k \)-hypersimplex was introduced in [2]. First we recall the definition of the \( r \)-stable \( n, k \)-hypersimplex. Let \([n] = \{1, 2, \ldots, n\}\). The characteristic vector of a subset \( I \) of \([n]\) is the \((0, 1)\)-vector \( \epsilon_I = (\epsilon_1, \ldots, \epsilon_n) \) for which \( \epsilon_i = 1 \) for \( i \in I \) and \( \epsilon_i = 0 \) for \( i \notin I \). Fix integers \( k \) with \( 0 < k < n \) and let \( r \) be an integer with \( 1 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor \). We label the vertices of a regular \( n \)-gon embedded in \( \mathbb{R}^2 \) in a clockwise fashion from 1 to \( n \). A subset \( S \subset [n] \) is called \( r \)-stable if, for each pair \( i, j \in S \), the path of shortest length from \( i \) to \( j \) about the \( n \)-gon uses at least \( r \) edges. The \( r \)-stable \( n, k \)-hypersimplex, denoted by \( \Delta_{n,k}^{\text{stab}(r)} \), is the convex polytope in \( \mathbb{R}^n \) which is the convex hull of the characteristic vectors of all \( r \)-stable \( k \)-subsets of \([n]\). In particular the 1-stable \( n, k \)-hypersimplex \( \Delta_{n,k}^{\text{stab}(1)} \) is just the \( n, k \)-hypersimplex \( \Delta_{n,k} \) ([9, p.75]). These convex polytopes form the nested chain

\[ \Delta_{n,k} \supset \Delta_{n,k}^{\text{stab}(2)} \supset \Delta_{n,k}^{\text{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\text{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor\right)} \, . \]

A regular unimodular triangulation of \( \Delta_{n,k} \) which can be restricted to a regular unimodular triangulation of each polytope \( \Delta_{n,k}^{\text{stab}(r)} \) in the above chain is studied in [2].

In the present paper, we utilize the regular unimodular triangulation to compute the facets of \( \Delta_{n,k}^{\text{stab}(r)} \) and then study when \( \Delta_{n,k}^{\text{stab}(r)} \) is Gorenstein. In section 2 we recall the details of the regular unimodular triangulation of \( \Delta_{n,k}^{\text{stab}(r)} \). In section 3 we compute the facets of \( \Delta_{n,k}^{\text{stab}(r)} \) for \( r < \left\lfloor \frac{n}{k} \right\rfloor \) (Theorem 3.1). As a result, it turns out that \( \Delta_{n,k}^{\text{stab}(r)} \) has exactly \( 2n \) facets for every \( r < \left\lfloor \frac{n}{k} \right\rfloor \) (Corollary 3.25). Finally in section 4 we classify \( 1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor \) for which \( \Delta_{n,k}^{\text{stab}(r)} \) is Gorenstein (Theorem 4.8). We conclude that the Ehrhart \( \delta \)-vector of \( \Delta_{n,k}^{\text{stab}(r)} \) is unimodal if it is Gorenstein (Corollary 4.10).

2. The Regular Unimodular Triangulation of \( \Delta_{n,k}^{\text{stab}(r)} \)

In [7], Lam and Postnikov compare four different triangulations of the hypersimplex and show that they are identical. One construction of this triangulation, known as the circuit triangulation, is introduced in [7]. In [2], it is shown that the circuit triangulation restricts to a triangulation of each \( r \)-stable \( n, k \)-hypersimplex. For the purposes of this paper it will be helpful to recall the details of this construction.

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circuits in Theorem 2.2.

Theorem 2.1. [7, Lam and Postnikov] The collection of simplices \( \sigma(\omega) \) given by the minimal circuits in \( G_{n,k} \) are the maximal simplices of a triangulation of the hypersimplex \( \Delta_{n,k} \). We call this triangulation the circuit triangulation.

Denote the circuit triangulation of \( \Delta_{n,k} \) by \( \nabla_{n,k} \), and let \( \text{max} \ \nabla_{n,k} \) denote the set of maximal simplices of \( \nabla_{n,k} \). To simplify notation we will write \( \omega \) to denote the simplex \( \sigma(\omega) \in \text{max} \ \nabla_{n,k} \).

Theorem 2.3. [2] Braun and Solus] The triangulation \( \nabla_{n,k} \) induces a triangulation of the \( r \)-stable hypersimplex \( \Delta_{n,k}^{\text{stab}(r)} \). That is,

\[
\nabla_{n,k} \cap \Delta_{n,k}^{\text{stab}(r)}
\]

is a triangulation of \( \Delta_{n,k}^{\text{stab}(r)} \).

Figure 1. A minimal circuit in the graph \( G_{6,3} \).

(0, 1, 0, 0, 1, 1) \( \xrightarrow{2} \) (0, 0, 1, 0, 1, 1) \( \xrightarrow{6} \) (1, 0, 1, 0, 1, 0)

(0, 1, 0, 1, 0, 1) \( \xleftarrow{3} \) (0, 1, 1, 0, 0, 1) \( \xleftarrow{5} \) (0, 1, 0, 1, 0, 0)

Fix \( 0 < k < n \), and let \( G_{n,k} \) be the labeled, directed graph with the following vertices and edges. The vertices of \( G_{n,k} \) are all the vectors \( \epsilon_I \in \mathbb{R}^n \) where \( I \) is a \( k \)-subset of \([n]\). We think of the indices of a vertex of \( G_{n,k} \) modulo \( n \). Now suppose that \( \epsilon \) and \( \epsilon' \) are two vertices of \( G_{n,k} \) such that for some \( i \in [n] \) \( \epsilon_i, \epsilon_{i+1} = (1, 0) \) and \( \epsilon' \) is obtained from \( \epsilon \) by switching the order of \( \epsilon_i \) and \( \epsilon_{i+1} \). Then the directed and labeled edge \( \epsilon \rightarrow \epsilon' \) is an edge of \( G_{n,k} \). Hence, an edge of \( G_{n,k} \) corresponds to a move of a single 1 in a vertex \( \epsilon \) one spot to the right, and such a move can be done if and only if the next spot is occupied by a 0.

We are interested in the circuits of minimal length in the graph \( G_{n,k} \). Such a circuit is called a minimal circuit. Suppose that \( \epsilon \) is a vertex in a minimal circuit of \( G_{n,k} \). Then the minimal circuit can be thought of as a sequence of edges in \( G_{n,k} \) that moves each 1 in \( \epsilon \) into the position of the 1 directly to its right (modulo \( n \)). It follows that a minimal circuit in \( G_{n,k} \) has length \( n \).

An example of a minimal circuit in \( G_{6,3} \) is provided in Figure 1. Notice that for a fixed initial vertex of the minimal circuit the labels of the edges form a permutation \( \omega = \omega_1 \omega_2 \cdots \omega_n \in S_n \), the symmetric group on \( n \) elements. Moreover, the permutations corresponding to two different choices of initial vertex will always be equivalent modulo cyclic shifts \( \omega_1 \cdots \omega_n \sim \omega_{n} \omega_1 \cdots \omega_{n-1} \). By convention, we associate a minimal circuit in \( G_{n,k} \) with the permutation in \( S_n \) corresponding to the lexicographically maximal choice of starting vertex. This corresponds to picking the permutation consisting of the labels of the edges of the circuit for which \( \omega_n = n \).

Theorem 2.1. [7] Lam and Postnikov] A minimal circuit in the graph \( G_{n,k} \) is uniquely determined by the permutation \( \omega \) modulo cyclic shifts. A permutation \( \omega \in S_n \) such that \( \omega_n = n \) corresponds to a minimal circuit in the graph \( G_{n,k} \) if and only if the inverse permutation \( \omega^{-1} \) has exactly \( k - 1 \) descents.

We let \( (\omega) \) denote the minimal circuit in \( G_{n,k} \) corresponding to the permutation \( \omega \in S_n \) with \( \omega_n = n \). Let \( v(\omega) \) denote the set of vertices of \( (\omega) \), and let \( \sigma(\omega) \) denote the convex hull of \( v(\omega) \). Notice that \( \sigma(\omega) \) will always be an \((n-1)\)-simplex.
Let \( \nabla^{r}_{n,k} \) denote the triangulation of \( \Delta^{\text{stab}(r)}_{n,k} \) induced by \( \nabla_{n,k} \), and let \( \max \nabla^{r}_{n,k} \) denote the set of maximal simplices of \( \nabla^{r}_{n,k} \). Notice that we have the nesting of triangulations

\[
\nabla_{n,k} \supset \nabla^{2}_{n,k} \supset \nabla^{3}_{n,k} \supset \cdots \supset \nabla^{\left\lfloor \frac{n}{k} \right\rfloor}_{n,k}.
\]

In the coming section we will utilize this nesting of triangulations to compute the facets of \( \Delta^{\text{stab}(r)}_{n,k} \).

3. The Facets of \( \Delta^{\text{stab}(r)}_{n,k} \)

It is a well-known fact that the facets of the \( n, k \)-hypersimplex \( \Delta_{n,k} \subset \mathbb{R}^{n-1} \) are given by the supporting hyperplanes

\[
x_{\ell} = 0 \quad \text{for } \ell \in [n-1],
\]

\[
x_{\ell} = 1 \quad \text{for } \ell \in [n-1],
\]

\[
\sum_{i=1}^{n-1} x_{i} = k, \quad \text{and}
\]

\[
\sum_{i=1}^{n-1} x_{i} = k - 1.
\]

Let \( H \) denote the hyperplane in \( \mathbb{R}^{n} \)

\[
x_{1} + x_{2} + \cdots + x_{n} = k,
\]

and consider the affine isomorphism

\[
\varphi : \mathbb{R}^{n-1} \rightarrow H;
\]

\[
\varphi : (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}) \mapsto (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}, k - \left( \sum_{i=1}^{n-1} \alpha_{i} \right)).
\]

Notice that \( \varphi(\mathbb{Z}^{n-1}) = H \cap \mathbb{Z}^{n} \). Under the affine isomorphism \( \varphi \) we see that the supporting hyperplanes of \( \Delta_{n,k} \subset \mathbb{R}^{n} \) are given by intersecting the hyperplane \( H \) with the hyperplanes

\[
x_{\ell} = 0 \quad \text{for } \ell \in [n], \text{ and}
\]

\[
x_{\ell} = 1 \quad \text{for } \ell \in [n].
\]

In the following we compute the facets of the \( r \)-stable \( n, k \)-hypersimplex \( \Delta^{\text{stab}(r)}_{n,k} \). To do so, we compute the \((n-1)\)-dimensional hyperplanes that, when intersected with \( H \), give the supporting hyperplanes of \( \Delta_{n,k} \subset \mathbb{R}^{n} \). Our main result is the following theorem.

**Theorem 3.1.** Suppose \( r < \left\lfloor \frac{n}{k} \right\rfloor \). Then the facets of \( \Delta^{\text{stab}(r)}_{n,k} \) are given by intersecting the hyperplane \( H \) with the hyperplanes

\[
x_{\ell} = 0 \quad \text{for } \ell \in [n], \text{ and}
\]

\[
\sum_{i=\ell}^{\ell+r-1} x_{i} = 1 \quad \text{for } \ell \in [n].
\]

Notice that for a fixed \( n \) and \( k \) the inequality \( r < \left\lfloor \frac{n}{k} \right\rfloor \) includes all but the smallest polytope in the nested chain

\[
\Delta_{n,k} \supset \Delta^{\text{stab}(2)}_{n,k} \supset \Delta^{\text{stab}(3)}_{n,k} \supset \cdots \supset \Delta^{\text{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor \right)}_{n,k}.
\]
Fig. 2. The minimal circuit of the simplex \( \omega = 374185296(10) \in \text{max}\, \nabla_{10,3} \).

In fact, we will also compute the facets of the smallest polytope in this chain for \( n \equiv 1 \mod k \).

Notice also that, in some sense, the facets of the \( r \)-stable \( n,k \)-hypersimplex are a generalization of the facets of the \( n,k \)-hypersimplex. In particular, when \( r = 1 \) we have that

\[
\Delta_{n,k}^{\text{stab}(r)} = \Delta_{n,k}, \quad \text{and} \quad \sum_{i=\ell}^{\ell+r-1} x_i = x_i \quad \text{for every } \ell \in [n].
\]

An immediate consequence of Theorem 3.1 is that for a fixed \( n \) and \( k \) all but (possibly) the smallest polytope in the nested chain

\[
\Delta_{n,k} \supset \Delta_{n,k}^{\text{stab}(2)} \supset \Delta_{n,k}^{\text{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{r} \rfloor)}
\]

has \( 2n \) facets. This is an interesting geometric property since the number of vertices for each polytope in the above chain forms a strictly decreasing sequence.

**Computing Facets with a Nesting of Triangulations.** As we noted in section 2, we will use the nested triangulations \( \nabla_{r,n,k} \) to prove Theorem 3.1. To do so, we will use the following theorem of Lam and Postnikov [7].

**Theorem 3.2.** [7] Lam and Postnikov] Two simplices \( u \) and \( \omega \) are adjacent simplices (i.e. they share a facet) of \( \text{max}\, \nabla_{n,k} \) if and only if there exists some \( i \in [n] \) such that \( u_i - u_{i+1} \neq \pm 1 \mod n \) and \( \omega \) is obtained from \( u \) by switching \( u_i \) with \( u_{i+1} \).

We should note that since our goal is to study \((n-1)\)-dimensional polytopes embedded in the hyperplane \( H \subset \mathbb{R}^n \), we will refer to the \((n-2)\)-dimensional flats corresponding to supporting hyperplanes of these polytopes under the embedding \( \varphi \) simply as the supporting hyperplanes of the polytope. We do this because the simplices of the circuit triangulation are best described in \( \mathbb{R}^n \), and we will use these simplices to compute the facets of \( \Delta_{n,k}^{\text{stab}(r)} \).

**Remark 3.3.** If \( u \) and \( \omega \) are adjacent simplices in \( \text{max}\, \nabla_{n,k} \) then the supporting hyperplane, say \( H_{u,\omega} \), of the shared facet is spanned by the common vertices of \( u \) and \( \omega \). Moreover, since \( \nabla_{n,k} \) is a triangulation of \( \Delta_{n,k} \) then \( u \) and \( \omega \) must lie in opposite closed halfspaces defined by \( H_{u,\omega} \).

**Example 3.4.** Here is an example of two adjacent maximal simplices in the triangulation \( \text{max}\, \nabla_{10,3} \) of \( \Delta_{10,3} \). Consider the simplices \( \omega \) and \( u \) in \( \text{max}\, \nabla_{10,3} \) given by the circuits in Figures 2 and 3 respectively.
By Theorem 3.2, \( \omega \) and \( u \) are adjacent simplices since
\[
u_9 - u_{10} \mod 10 = 3 - 10 \mod 10 = -7 \mod 10 \neq \pm 1 \mod 10,
\]
and, modulo cyclic shifts, we have that
\[
74185296(10)3 \sim 374185296(10) = \omega.
\]
The two simplices \( \omega \) and \( u \) differ by a single vertex (the top left-most vertex in the circuits depicted in Figures 2 and 3), and the remaining vertices lie in the \((n - 2)\)-flat \( H_{u, \omega} = H \cap K \) where
\[
K = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{3} x_i = 1 \right\}.
\]
Moreover, \( \omega \) is contained in the closed halfspace
\[
K^+ = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{3} x_i \geq 1 \right\},
\]
and \( u \) is contained in the closed halfspace
\[
K^- = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{3} x_i \leq 1 \right\}.
\]
Hence, the adjacent simplices \( u \) and \( \omega \) share the supporting hyperplane \( H_{u, \omega} \), which is spanned by the common vertices of \( u \) and \( \omega \), and they lie in opposite closed halfspaces defined by this supporting hyperplane.

With Theorem 3.2 in hand, we are now ready to compute the facets of \( \Delta_{n,k}^{\text{stab}(r)} \). For now, we assume \( \Delta_{n,k}^{\text{stab}(r)} \) is \((n - 1)\)-dimensional. We will soon see that this is almost always true. In the following, for a convex polytope \( P \subset \mathbb{R}^N \) of dimension \( d \), we will let \( \partial P \) denote the boundary of \( P \), and \( \text{relint}(P) \) denote the relative interior of \( P \).

Once more, we remark that we only wish to study \((n - 1)\)-dimensional polytopes embedded in the hyperplane \( H \subset \mathbb{R}^n \). For this reason, we refer to the \((n - 2)\)-dimensional flats in \( \mathbb{R}^n \) that correspond to the supporting hyperplanes of these polytopes under the affine isomorphism \( \varphi \) simply as the supporting hyperplanes of the polytope. This will be useful because of the nice description of the simplices in the circuit triangulation in \( \mathbb{R}^n \).
Proposition 3.5. Suppose $\Delta_{n,k}^{\text{stab}(r)}$ is $(n - 1)$-dimensional with $r > 1$. Then the supporting hyperplanes of $\Delta_{n,k}^{\text{stab}(r)}$ are either supporting hyperplanes of $\Delta_{n,k}^{\text{stab}(r-1)}$ or they intersect the relative interior of $\Delta_{n,k}^{\text{stab}(r-1)}$.

Proof. Recall that $\Delta_{n,k}^{\text{stab}(r)} \subset \Delta_{n,k}^{\text{stab}(r-1)}$, and suppose that $F$ is a facet of $\Delta_{n,k}^{\text{stab}(r)}$. Let $H_F$ denote the supporting hyperplane of $F$. Then each $\alpha \in F$ is a point in $\Delta_{n,k}^{\text{stab}(r-1)}$. Hence, $\alpha \in \partial\Delta_{n,k}^{\text{stab}(r-1)}$ or $\alpha \in \text{relint} \Delta_{n,k}^{\text{stab}(r-1)}$. If there is some $\alpha \in F$ such that $\alpha \in \text{relint} \Delta_{n,k}^{\text{stab}(r-1)}$ then $H_F$ intersects $\text{relint} \Delta_{n,k}^{\text{stab}(r-1)}$. If there is no such $\alpha \in F$ then $F \subset \partial\Delta_{n,k}^{\text{stab}(r-1)}$, and is therefore contained in a facet of $\Delta_{n,k}^{\text{stab}(r-1)}$. Thus, $H_F$ is a supporting hyperplane of $\Delta_{n,k}^{\text{stab}(r-1)}$. \qed

We can think of the facets of $\Delta_{n,k}^{\text{stab}(r)}$ that share supporting hyperplanes with facets of $\Delta_{n,k}^{\text{stab}(r-1)}$ as having preserved (supporting) hyperplanes, and those facets that intersect the relative interior of $\Delta_{n,k}^{\text{stab}(r-1)}$ as having new (supporting) hyperplanes. Hence, to determine the facets of $\Delta_{n,k}^{\text{stab}(r)}$ it suffices to determine the set of preserved hyperplanes and the set of new hyperplanes. In the following, we let

$$H_\ell := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_\ell = 0\} \cap H,$$

and

$$H_{\ell,r} := \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=\ell}^{\ell+r-1} x_i = 1\right\} \cap H,$$

for $\ell \in [n]$ and $1 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor$.

The preserved supporting hyperplanes. We first determine the set of preserved hyperplanes of $\Delta_{n,k}^{\text{stab}(r)}$.

Proposition 3.6. Let $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$. The supporting hyperplanes $H_\ell$ for $\ell \in [n]$ of $\Delta_{n,k}$ are also supporting hyperplanes of $\Delta_{n,k}^{\text{stab}(r)}$. That is, the hyperplane $H_\ell$ is preserved for every $\ell \in [n]$ and $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. First notice that $\Delta_{n,k}^{\text{stab}(r)}$ clearly lies in the closed halfspace $x_i \geq 0$. This is because $\Delta_{n,k}$ lies in this closed halfspace. Hence, to prove the claim it suffices to identify a simplex $\omega \in \max \nabla^{r}_{n,k}$ such that $H_\ell$ is a supporting hyperplane of $\omega$.

Fix $i \in [n]$. When $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$ we may construct a minimal circuit in the graph $G_{n,k}$ that corresponds to a simplex in $\max \nabla^{r}_{n,k}$ with supporting hyperplane $H_\ell$. To do this, we first construct an initial vertex, say $e^r_\ell$, for this circuit in the following way:

1. Label $k$ 1’s as $1_1, 1_2, \ldots, 1_k$.
2. Place $1_1$ in entry $\ell - 1 - r$.
3. Place $1_2$ in entry $\ell - 1$.
4. For $t \geq 3$ place $1_t$ in entry $\ell - 1 + (t - 1)r$.
5. Make all other entries 0.

For $\ell = 2$, this produces the following vertex:

$$e^r_2 = (1_2, [ (2r - 1) \ 0 \ 's ], 1_3, [ (r - 1) \ 0 \ 's ], 1_4, [ (r - 1) \ 0 \ 's ], 1_5, \ldots, 1_k, [ \geq (r - 1) \ 0 \ 's ], 1_1, [ (r - 1) \ 0 \ 's ]).$$

Notice that for the vertex $e^r_\ell$ to be $r$-stable we need at least $(r - 1) 0$’s between each pair of consecutive 1’s. This is guaranteed by our construction for all pairs of 1’s except for $1_k$ and $1_1$. However, the total number of entries accounted for by the $k$ 1’s and the 0’s between the pairs $\{1_t, 1_{t+1}\}$ for $t \leq k - 1$ is

$$(k - 2)r + 2r + 1.$$
Here, we count \( r \) entries for each 1, with \( t \neq 2, k \), since this counts each such 1 and the \( r - 1 \) zeros following it. Then we count \( 2r \) entries for 1 and the \( 2r - 1 \) zeros that follow it. Finally we add 1 for the entry filled by \( 1_k \). Since we need at least \( r - 1 \) entries between \( 1_k \) and 1 then we want at least \( r - 1 \) entries between these two 1’s. In other words, we want

\[
(r - 1) + (k - 2)r + 2r + 1 \leq n.
\]

Now notice that since \( r < \left\lfloor \frac{n}{k} \right\rfloor \) then \( r + 1 \leq \left\lfloor \frac{n}{k} \right\rfloor \). Moreover, \( \left\lfloor \frac{n}{k} \right\rfloor = \frac{n - \alpha}{k} \) for some \( \alpha \in \{0, 1, \ldots, k-1\} \).

Hence,

\[
r + 1 \leq \frac{n - \alpha}{k},
\]

and so

\[
k(r + 1) + \alpha \leq n.
\]

Hence, \( k(r + 1) \leq n \). Finally, notice that

\[
(r - 1) + (k - 2)r + 2r + 1 = (k + 1)r \leq n.
\]

Thus, there are at least \( (r - 1) \) zeros between 1 and 1, and we conclude that \( \epsilon^\ell \) is \( r \)-stable.

Now we construct a minimal circuit in \( G_{n,k} \), which we will denote \( (\omega^\ell) \), starting with initial vertex \( \omega^\ell \). To do so, make the following sequence of moves.

(1) Move 12 \( r \) times.
(2) Move 11. Then move 1k. Then move 1k−1. Then move 1k−2. \ldots Then move 12.
(3) Repeat step (2) \( r - 1 \) more times.
(4) Move 1k until it rests in entry \( \ell - 1 \).

It is clear that this produces a minimal circuit in \( G_{n,k} \) since each 1t has moved precisely enough times to replace 1t+1. Moreover, every vertex in the circuit is \( r \)-stable. To see this, notice that 12 first makes \( r \) moves of its necessary 2r moves. Since 12 must make a total of 2r moves to replace 13 then the pair \{12, 13\} is still \( r \)-stable. Moreover, there are now 2r - 1 0’s between 11 and 12. So 11 may move once and not violate \( r \)-stability. Now there are \( r \) 0’s between 1k and 11. So 1k may move once without violating \( r \)-stability. Now there are \( r \) 0’s between 1k−1 and 1k. So 1k−1 may move once without violating \( r \)-stability, and so on. Hence, steps (2) and (3) preserve \( r \)-stability in the circuit \( (\omega^\ell) \). Finally, \( r \)-stability is also preserved in step (4) since all 1’s for \( t \neq k \) have already assumed the position of 1t+1. In particular, 11 is in the entry that was originally occupied by 12, and this entry is \((r - 1)\) entries away from entry \( \ell - 1 \). Hence, the minimal circuit \( (\omega^\ell) \) uses only \( r \)-stable vertices.

It follows that \( \omega^\ell \in \max \nabla_{n,k}^r \). Moreover, since \( r > 1 \), the simplex \( \omega^\ell \) has only one vertex satisfying \( x_\ell = 1 \), and this is the vertex following \( \epsilon^\ell \) in the circuit \( (\omega^\ell) \). Hence, all other vertices of \( \omega^\ell \) satisfy \( x_\ell = 0 \). So \( H_\ell \) is a supporting hyperplane of \( \omega^\ell \). Since \( \Delta_{n,k}^{stab(r)} \) lies in the closed halfspace \( x_\ell \geq 0 \) we conclude that \( H_\ell \) is a supporting hyperplane of \( \Delta_{n,k}^{stab(r)} \) for \( r < \left\lfloor \frac{n}{k} \right\rfloor \).

\[\square\]

**Example 3.7.** Here is an example of the circuit constructed in the proof of Proposition 3.6. Let \( n = 10, k = 3, r = 2, \) and \( \ell = 4 \). Then the vertex \( \epsilon^\ell \) is

\[
\epsilon^\ell = (1, 0, 1, 2, 0, 0, 0, 0, 1, 3, 0, 0, 0).
\]

The minimal circuit \( (\omega^\ell) \) produced by the algorithm in the proof of Proposition 3.6 is depicted in Figure 4. Notice \( \omega^\ell = 341752869(10) \in \max \nabla_{10,3}^r \), and the only vertex in \( (\omega^\ell) \) that satisfies \( x_\ell = 1 \) is the vertex immediately following \( \epsilon^\ell \).

The following theorem on the dimension of \( \Delta_{n,k}^{stab(r)} \) follows from the proof of Proposition 3.6.

**Theorem 3.8.** The polytope \( \Delta_{n,k}^{stab(r)} \) is \((n - 1)\)-dimensional for all \( r < \left\lfloor \frac{n}{k} \right\rfloor \).
Proposition 3.9. Suppose \( r > 1 \) and \( \Delta_{n,k}^{\text{stab}(r)} \) is \((n-1)\)-dimensional. Then the hyperplane \( H_{\ell,r-1} \) is not a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \). That is, \( H_{\ell,r-1} \) is not a preserved hyperplane for any \((n-1)\)-dimensional \( r \)-stable hypersimplices.

Proof. Suppose for the sake of contradiction that \( H_{\ell,r-1} \) is a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \). Since \( \Delta_{n,k}^{\text{stab}(r)} \) is \((n-1)\)-dimensional then there exists an \((n-1)\)-simplex \( \omega \in \max \nabla_{n,k}^r \) such that \( H_{\ell,r-1} \) is a supporting hyperplane of \( \omega \). In other words, every vertex in \( \omega \) satisfies

\[
\sum_{i=\ell}^{\ell+r-2} x_i = 1
\]

except for exactly one vertex, say \( \epsilon^* \). Since all vertices in \( \omega \) are \((0,1)\)-vectors, this means all vertices other than \( \epsilon^* \) have exactly one entry in the set of entries \( \{\ell, \ell+1, \ell+2, \ldots, \ell+r-2\} \) being 1 and all other entries in this set are 0. Similarly, the vertex \( \epsilon^* \) has a 0 in all entries in the set \( \{\ell, \ell+1, \ell+2, \ldots, \ell+r-2\} \). Since \( \omega \) is a minimal circuit this means that the move preceding the vertex \( \epsilon^* \) in \( \omega \) results in the only 1 in the set of entries \( \{\ell, \ell+1, \ell+2, \ldots, \ell+r-2\} \) leaving this set of entries. Similarly, the move following the vertex \( \epsilon^* \) in \( \omega \) results in a single 1 moving into the set of entries \( \{\ell, \ell+1, \ell+2, \ldots, \ell+r-2\} \). Suppose that

\[
\epsilon^* = (\ldots, \epsilon^*_{\ell-1}, \epsilon^*_\ell, \epsilon^*_\ell, \ldots, \epsilon^*_{\ell+r-2}, \epsilon^*_{\ell+r-1}, \ldots) = (\ldots, 1, 0, 0, \ldots, 0, 1, \ldots) .
\]

Then this situation looks like

\[
\begin{array}{cccccccc}
\epsilon^* &=& (1,0,1,0,0,0,1,0,0,0) & \longrightarrow & (1,0,1,0,0,1,0,0,0) & \longrightarrow & (1,0,0,0,1,0,0,0) & \longrightarrow & (1,0,0,0,0,1,0,0,0) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& (0,0,1,0,0,0,1,0,0,1) & \longrightarrow & (0,0,1,0,0,1,0,0,0) & \longrightarrow & (0,1,0,0,1,0,0,0) & \longrightarrow & (0,1,0,0,0,1,0,0,0) \\
& (0,0,1,0,0,0,1,0,1,0) & \longrightarrow & (0,0,1,0,0,1,0,1,0) & \longrightarrow & (0,1,0,0,1,0,1,0) & \longrightarrow & (0,1,0,0,0,1,1,0) \\
& (0,0,1,0,0,1,0,0,1,0) & \longrightarrow & (0,0,1,0,0,1,0,0,1,0) & \longrightarrow & (0,1,0,0,1,0,0,1,0) & \longrightarrow & (0,1,0,0,0,1,0,1,0) \\
\end{array}
\]

Figure 4. The minimal circuit of the simplex \( \omega^\ell = 341752869(10) \in \max \nabla_{10,3}^2 \).
Hence, neither the vertex preceding or following the vertex \( e^* \) is \( r \)-stable. For example, in the vertex following \( e^* \) there is a 1 in entries \( \ell \) and \( \ell + r - 1 \). This contradicts the fact that \( \omega \in \max \nabla_{n,k}^r \). Hence, \( H_{\ell,r-1} \) is not a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \).

To see why Proposition 3.9 will be useful suppose that Theorem 3.1 holds for \( \Delta_{n,k}^{\text{stab}(r-1)} \) for some \( 1 < r < \left\lfloor \frac{n}{2} \right\rfloor \). Then Propositions 3.6 and 3.9 tell us that the collection of preserved hyperplanes for \( \Delta_{n,k}^{\text{stab}(r)} \) is \( \{ H_{\ell} : \ell \in [n] \} \). In this way, we will be able to prove Theorem 3.1 by induction on \( r \).

The new supporting hyperplanes. We begin with a helpful definition.

**Definition 3.10.** Suppose \( u \) and \( \omega \) are a pair of simplices in \( \max \nabla_{n,k}^r \) satisfying

- \( u \in \max \nabla_{n,k}^r \),
- \( \omega \in \max \nabla_{n,k}^r \setminus \max \nabla_{n,k}^{r-1} \), and
- \( \omega \) uses exactly one vertex that is not \( r \)-stable, and this is the only vertex by which \( u \) and \( \omega \) differ.

We then say that the ordered pair of simplices \((u, \omega)\) is an \( r \)-supporting pair of the hyperplane \( H_{u,\omega} \), where \( H_{u,\omega} \) is the hyperplane spanned by the common vertices of \( u \) and \( \omega \).

**Example 3.11.** Let \( n = 10, k = 3 \), and \( r = 3 \). Recall the simplices \( \omega, u \in \max \nabla_{10,3}^3 \) from Example 3.4. Notice that

\[
u \in \max \nabla_{10,3}^3,
\]

and

\[
\omega \in \max \nabla_{10,3}^2 \setminus \max \nabla_{10,3}^3.
\]

Also notice that the only vertex in \( \omega \) that is not \( r \)-stable is

\[
(1,0,1,0,0,0,1,0,0,0),
\]

and this is the only vertex by which \( \omega \) and \( u \) differ. Hence, \((u, \omega)\) is an \( r \)-supporting pair of the hyperplane \( H_{u,\omega} \).

**Proposition 3.12.** Suppose \( r > 1 \) and \( \Delta_{n,k}^{\text{stab}(r)} \) is \((n-1)\)-dimensional. Suppose also that \( H_F \) is a supporting hyperplane of a facet \( F \) of \( \Delta_{n,k}^{\text{stab}(r)} \) such that \( H_F \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \neq \emptyset \). Then \( H_F \) is the supporting hyperplane of a pair of adjacent simplices \( u \) and \( \omega \) that form an \( r \)-supporting pair for the hyperplane \( H_{u,\omega} \), and \( H_F = H_{u,\omega} \).

**Proof.** Since \( H_F \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \neq \emptyset \) and \( \Delta_{n,k}^{\text{stab}(r)} \subset \Delta_{n,k}^{\text{stab}(r-1)} \) then \( F \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \neq \emptyset \). That is, there exists some \( \alpha \in F \) such that \( \alpha \in \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \). Now recall that \( \nabla_{n,k}^{r-1} \) is a triangulation of \( \Delta_{n,k}^{\text{stab}(r-1)} \) that restricts to a triangulation of \( \nabla_{n,k}^r \) of \( \Delta_{n,k}^{\text{stab}(r)} \). It follows that \( \nabla_{n,k}^{r-1} \setminus \nabla_{n,k}^r \) gives identical triangulations of \( \partial \Delta_{n,k}^{\text{stab}(r)} \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \). Since \( \Delta_{n,k}^{\text{stab}(r)} \) is \((n-1)\)-dimensional we may assume, without loss of generality, that \( \alpha \) lies in the relative interior of an \((n-2)\)-dimensional simplex in the triangulation of \( \partial \Delta_{n,k}^{\text{stab}(r)} \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \) induced by \( \nabla_{n,k}^r \) and \( \nabla_{n,k}^{r-1} \setminus \nabla_{n,k}^r \). Therefore, there exists some \( u \in \max \nabla_{n,k}^3 \) such that \( H_F \) is a supporting hyperplane of \( u \) and \( \alpha \in u \cap H_F \), and there exists some \( \omega \in \max \nabla_{n,k}^2 \setminus \max \nabla_{n,k}^r \) such that \( \alpha \in \omega \cap H_F \). Since
\[ \nabla_{n,k}^{r-1} \] is a triangulation of \( \Delta_{n,k}^{\text{stab}(r-1)} \) it follows that \( u \cap H_F = \omega \cap H_F \). Hence, \( u \) and \( \omega \) are adjacent simplices that share the supporting hyperplane \( H_F \), and they form an \( r \)-supporting pair \((u, \omega)\) with \( H_{u, \omega} = H_F \).

Suppose \((u, \omega)\) is an \( r \)-supporting pair. It will be helpful to understand the vertex of \( \omega \) that is not \( r \)-stable. To do so, we will use the following definition.

**Definition 3.13.** Let \( \epsilon \in \mathbb{R}^n \) be a vertex of \( \Delta_{n,k} \). A pair of 1’s in \( \epsilon \) is an \( \epsilon \)-pair if there are at least \( r - 1 \) 0’s separating the two 1’s.

**Example 3.14.** Consider the vertex \((1,0,1,0,0,0,1,0,0,0,0)\) of the simplex \( \omega \) depicted in Figure [2]. This circuit has exactly one 2-stable but not 3-stable pair of 1’s, and the other two pairs of 1’s are 3-stable. The 2-stable but not 3-stable pair is \((1,3)\) and the 3-stable pairs are \((3,7)\) and \((7,1)\).

Suppose a hyperplane \( K \) has the \( r \)-supporting pair \((u, \omega)\). We would now like to understand the possible \((r - 1)\)-stable but not \( r \)-stable vertices that can be used by the simplex \( \omega \). We call this vertex the key vertex of the \( r \)-supporting pair \((u, \omega)\).

**Proposition 3.15.** Suppose \((u, \omega)\) is an \( r \)-supporting pair, and let \( \epsilon \) be the key vertex of this pair. Then \( \epsilon \) has precisely one \((r - 1)\)-stable but not \( r \)-stable pair, say \((i, j)\), this pair is followed by an \((r + 1)\)-stable pair \((j, t)\), and all other pairs of 1’s in \( \epsilon \) are \( r \)-stable.

**Proof.** Consider the minimal circuit \((\omega)\) in the graph \( G_{n,k} \) associated with the simplex \( \omega \). Think of the key vertex \( \epsilon \) as the initial vertex of this circuit, and recall that each edge of the circuit corresponds to a move of exactly one 1 to the right by exactly one entry. Hence, in the circuit \((\omega)\) the vertex following \( \epsilon \) differs from \( \epsilon \) by a single right move of a single 1. Since \( \epsilon \) is the only vertex in \((\omega)\) that is \((r - 1)\)-stable but not \( r \)-stable then the move of this single 1 to the right by one entry must eliminate all pairs that are \((r - 1)\)-stable but not \( r \)-stable. Moreover, this move cannot introduce any new \((r - 1)\)-stable but not \( r \)-stable pairs. Since a single 1 can be in at most two pairs, and this 1 must move exactly one entry to the right, then this 1 must be in entry \( j \) in the pairs \((i, j)\) and \((j, t)\) where \((i, j)\) is \((r - 1)\)-stable but not \( r \)-stable, and \((j, t)\) is \((r + 1)\)-stable. Moreover, since the move of the 1 in entry \( j \) can only change the stability of the pairs \((i, j)\) and \((j, t)\) then it must be that all other pairs are \( r \)-stable.

**Example 3.16.** Consider the key vertex of the \( r \)-supporting pair \((u, \omega)\) from Example 3.11. This is the vertex \((1,0,1,0,0,0,1,0,0,0,0)\). We saw in Example 3.11 that this vertex has one 2-stable but not 3-stable pair of 1’s, \((1,3)\), and the other two pairs of 1’s are 3-stable. Notice now that the pair of 1’s \((3,7)\) is in fact 4-stable. Hence, the key vertex of the \( r \)-supporting pair \((u, \omega)\) indeed satisfies Proposition 3.15. Moreover, it is easy to see from the circuit \((\omega)\) depicted in Figure [2] that we must move the 1 in entry 3 of the key vertex to ensure that no other vertices in the circuit contain an \((r - 1)\)-stable but not \( r \)-stable pair.

Proposition 3.15 says that the key vertex of an \( r \)-supporting pair \((u, \omega)\) for \( H_{u, \omega} \) contains an \((r - 1)\)-stable but not \( r \)-stable pair followed by an \((r + 1)\)-stable pair, and all other pairs in the vertex are \( r \)-stable. We can construct all such possible key vertices by picking the \((r - 1)\)-stable but not \( r \)-stable pair, \((\ell, \ell + r - 1)\), then picking the next pair \((\ell + r - 1, j)\) to be \((r + 1)\)-stable, and then redistributing the remaining 1’s across the remaining entries of the vertex in an \( r \)-stable fashion. With this in mind, consider the following proposition.

**Proposition 3.17.** Suppose \( \omega \) and \( \omega' \) are simplices in \( \max \nabla_{n,k}^{r-1} \setminus \max \nabla_{n,k}^{r-1} \) that are in \( r \)-supporting pairs \((u, \omega)\) and \((u', \omega')\), respectively. Suppose also that the key vertices of \( \omega \) and \( \omega' \) have the same \((r - 1)\)-stable but not \( r \)-stable pair of 1’s. Then

\[
H_{u, \omega} = H_{u, \omega'} = H_{u', \omega'}.
\]
Hence, let $\omega$ denote the key vertices of the pairs $(u, \omega)$ and $(u', \omega')$, respectively. Then all vertices in $\omega$ other than $\epsilon$ span the hyperplane $H_{u, \omega}$, and all vertices in $\omega'$ other than $\epsilon'$ span the hyperplane $H_{u', \omega'}$. Since $\epsilon$ and $\epsilon'$ have the same $(r-1)$-stable but not $r$-stable pairs, say $(\ell, \ell' + r - 1)$, and $(\omega)$ and $(\omega')$ are minimal circuits with all other vertices being $r$-stable, then all other vertices in $\omega$ and $\omega'$ satisfy

$$
\sum_{i=\ell}^{\ell + r - 1} x_i = 1.
$$

Hence,

$$H_{u, \omega} = H_{\ell, r} = H_{u', \omega'}.
$$

**Proof.** Let $\epsilon$ and $\epsilon'$ denote the key vertices of the pairs $(u, \omega)$ and $(u', \omega')$, respectively. Then all vertices in $\omega$ other than $\epsilon$ span the hyperplane $H_{u, \omega}$, and all vertices in $\omega'$ other than $\epsilon'$ span the hyperplane $H_{u', \omega'}$. Since $\epsilon$ and $\epsilon'$ have the same $(r-1)$-stable but not $r$-stable pair, say $(\ell, \ell' + r - 1)$, and $(\omega)$ and $(\omega')$ are minimal circuits with all other vertices being $r$-stable, then all other vertices in $\omega$ and $\omega'$ satisfy

$$
\sum_{i=\ell}^{\ell + r - 1} x_i = 1.
$$

Hence,

$$H_{u, \omega} = H_{\ell, r} = H_{u', \omega'}.
$$

**Example 3.18.** Let $n = 9$, $k = 3$, and $r = 2$. Consider the simplices $\omega, \omega' \in \max \nabla_{9,3} \setminus \max \nabla_{9,3}^2$ with minimal circuits depicted in Figures 5 and 6, respectively.

These simplices are in $r$-supporting pairs $(u, \omega)$ and $(u', \omega')$ where

$$u = 531647829 \in \max \nabla_{9,3}^2,$$

and

$$u' = 316475829 \in \max \nabla_{9,3}^2.$$
The \( r \)-supporting pairs \((u, \omega)\) and \((u', \omega')\) have respective key vertices \((1, 1, 0, 0, 0, 1, 0, 0, 0, 0)\) and \((1, 1, 0, 0, 0, 1, 0, 0, 0, 0)\), and these vertices have the same \((r - 1)\)-stable but not \(r\)-stable pair, namely \((1, 2)\). Hence, \(\omega\) and \(\omega'\) are simplices in \(\nabla_{9,3} \setminus \max \nabla_{9,3}^2\) that are in \(r\)-supporting pairs for which the key vertices share the same \((r - 1)\)-stable but not \(r\)-stable pair. It is then easy to see that the vertices of the circuits \((\omega)\) and \((\omega')\) that are not the key vertices satisfy \(x_1 + x_2 = 1\). Hence,

\[
H_{u,\omega} = H_{1,2} = H_{u',\omega'}.
\]

**Remark 3.19.** Notice that Proposition [3.15] is not an “if and only if” statement. It is not necessary that each vertex that has an \((r - 1)\)-stable but not \(r\)-stable pair followed by an \((r + 1)\)-stable pair, and has all other pairs being \(r\)-stable needs to be the key vertex of an \(r\)-supporting pair of a supporting hyperplane of \(\Delta_{n,k}^{\text{stab}(r)}\). However, Proposition [3.17] says that if two such vertices happen to agree on their \((r - 1)\)-stable but not \(r\)-stable pair, and both happen to be vertices of simplices in \(r\)-supporting pairs then they are in \(r\)-supporting pairs of the same hyperplane and that hyperplane is \(H_{\ell,r}\) for some \(\ell \in [n]\).

**Example 3.20.** Let \(n = 10\), \(k = 3\), and \(r = 3\). In this case, the possible key vertices for \(r\)-supporting pairs come in two types.

| Type 1                              | Type 2                              |
|-------------------------------------|-------------------------------------|
| \(1, 0, 1, 0, 0, 0, 1, 0, 0, 0\)    | \(1, 0, 1, 0, 0, 0, 1, 0, 0, 0\)    |
| \((0, 1, 0, 1, 0, 0, 0, 1, 0)\)      | \((0, 1, 0, 1, 0, 0, 0, 1, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 1, 0, 0, 0, 1, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 0, 1, 0, 0, 1, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 1, 0, 0, 1, 0, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 1, 0, 0, 1, 0, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 1, 0, 0, 1, 0, 0)\)      |
| \((0, 0, 0, 0, 1, 0, 0, 0, 1)\)      | \((0, 0, 0, 1, 0, 0, 1, 0, 0)\)      |

The Type 1 vertices are all equivalent modulo cyclic shifts, as are the Type 2 vertices. Notice all of these vertices consist of an \((r - 1)\)-stable but not \(r\)-stable pair, followed by an \((r + 1)\)-stable pair, and have all other pairs being \(r\)-stable. The \((r - 1)\)-stable but not \(r\)-stable pair has its first 1 highlighted in red in the above table. Notice now that only the Type 1 vertices are used as key vertices in \(r\)-supporting pairs. To see this, recall that the first vertex in the Type 1 column is the key vertex in the \(r\)-supporting pair \((u, \omega)\) described in Example [3.11]. Similar to this example, every other Type 1 vertex is the key vertex in an \(r\)-supporting pair that uses the simplex \(u\). This is easily verified by switching the order of the moves around each vertex in the minimal circuit \((u)\), one at a time. On the other hand, notice that \(\left\lceil \frac{10}{3} \right\rceil = 3\), and \(10 \equiv 1 \mod 3\). Hence, by [2] Lemma 2.7, \(\Delta_{10,3}^{\text{stab}(3)}\) is a unimodular 9-simplex. Therefore, \(\{u\} = \max \nabla_{10,3}^2\), and each Type 1 vertex corresponds to a simplex in \(\max \nabla_{10,3}^2 \setminus \max \nabla_{10,3}^3\) that shares exactly one face with \(u\). Hence, no Type 2 vertex can be in a simplex adjacent to \(u\), and therefore these vertices cannot be in \(r\)-supporting pairs.

**Proposition 3.21.** Suppose \(r > 1\) and \(\Delta_{n,k}^{\text{stab}(r)}\) is \((n - 1)\)-dimensional. Suppose \(H_F\) is a supporting hyperplane of a facet \(F\) of \(\Delta_{n,k}^{\text{stab}(r)}\), such that \(H_F \cap \text{relint} \Delta_{n,k}^{\text{stab}(r-1)} \neq \emptyset\). Then

\[
H_F = H_{\ell,F}
\]

for some \(\ell \in [n]\).
Proposition 3.22. Suppose \( \ell \) for all \( H \) are supported by hyperplanes \( H \) \( r \)-stable but not \( \omega \). Proposition 3.15

Proof. By Proposition 3.12 the hyperplane \( H_F \) is given by some \( r \)-supporting pair \( (u, \omega) \). By Proposition 3.15 \( \omega \) has a unique vertex that is \( (r - 1) \)-stable but not \( r \)-stable with a unique \( (r - 1) \)-stable but not \( r \)-stable pair, say \( (\ell, \ell + r - 1) \), for some \( \ell \in [n] \). By Proposition 3.17 it follows that \( H_F = H_{\ell,r} \).

So far, we have shown that the facets of \( \Delta_{n,k}^{\text{stab}(r)} \) that intersect the relative interior of \( \Delta_{n,k}^{\text{stab}(r-1)} \) are supported by hyperplanes \( H_{\ell,r} \). We now show that \( H_{\ell,r} \) is a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \) for all \( \ell \in [n] \).

Proposition 3.22. Suppose \( r < \left\lfloor \frac{n}{k} \right\rfloor \) or \( n = kr + 1 \). The hyperplane \( H_{\ell,r} \) is a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \) for all \( \ell \in [n] \).

Proof. First we note that the result is clearly true for \( r = 1 \). So in the following we assume \( r > 1 \). Now notice that \( \Delta_{n,k}^{\text{stab}(r)} \) is in the closed halfspace given by \( \sum_{i=1}^{\ell+r-1} x_i \leq 1 \). This is because all vertices of \( \Delta_{n,k}^{\text{stab}(r)} \) are \( r \)-stable. Hence, it suffices to show that \( H_{\ell,r} \) supports an \( (n - 1) \)-simplex that uses only \( r \)-stable vertices and has \( H_{\ell,r} \) as a supporting hyperplane. To do so, we first construct an initial vertex, say \( \epsilon^\ell \), for this minimal circuit. Construct the vertex \( \epsilon^\ell \) as follows:

1. Label \( k \) 1’s as \( 1_1, 1_2, \ldots, 1_k \).
2. For \( t \in [k] \), place \( 1_t \) in entry \( \ell - 1 + (t - 1)r \).
3. Make all other entries 0.

It follows that \( \epsilon^\ell \in \mathbb{R}^n \) is \( r \)-stable. To see this, first notice that we have constructed \( \epsilon^\ell \) such that \( 1_t \) and \( 1_{t+1} \) are separated by \( r - 1 \) 0’s for all \( t \leq k - 1 \). Then notice that the \( k \) 1’s and the 0’s between \( 1_t \) and \( 1_{t+1} \) for all \( t \leq k - 1 \) account for

\[ k + (k - 1)(r - 1) = kr - (r - 1) = kr - r + 1 \]

entries of the vertex. Since \( n \geq kr + 1 \) then there are at least \( r \) entries (all filled with 0’s) between \( 1_k \) and \( 1_1 \). Hence, \( \epsilon^\ell \) is \( r \)-stable.

Moreover, since there are at least \( r \) 0’s between \( 1_k \) and \( 1_1 \) we can construct an \( r \)-stable circuit, which we denote \( (\omega^\ell) \), in the following fashion:

1. Move \( 1_k \). Then move \( 1_{k-1} \). Then move \( 1_{k-2} \). . . Then move \( 1_1 \).
2. Repeat step (1) \( r - 1 \) more times.
3. Move \( 1_k \) to entry \( \ell \).

Then \( \omega^\ell \in \max \nabla_{n,k}^r \) and \( H_{\ell,r} \) supports \( \omega^\ell \) since every vertex of \( (\omega^\ell) \) lies in \( H_{\ell,r} \) except for the vertex preceding the first move of \( 1_1 \) in the circuit \( (\omega^\ell) \). Hence, \( H_{\ell,r} \) is a supporting hyperplane of \( \Delta_{n,k}^{\text{stab}(r)} \).

Example 3.23. Here is an example of the circuit constructed in the proof of Proposition 3.22. Let \( n = 9, k = 3, r = 2, \) and \( \ell = 2 \). Then

\[
\epsilon^\ell = (1_1, 0, 1_2, 0, 1_3, 0, 0, 0, 0, 0) .
\]

The circuit \( (\omega^\ell) \) is depicted in Figure 4.

From this, it is easy to see that \( \omega^\ell \in \max \nabla_{3,3}^2 \). Moreover, thinking of \( \epsilon^\ell \) as the initial vertex of the circuit, only the third vertex in the circuit does not satisfy \( x_2 + x_3 = 1 \), and this is precisely the vertex preceding the first move of \( 1_1 \).
Propositions 3.21 and 3.22 the facets of $\Delta_{n,k}^{r}$ are supported by the hyperplanes $H_{\ell,r}$ for $\ell \in [n]$. We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** First recall that Theorem 3.1 is known to be true for $r = 1$. Now let $1 < r < \lfloor \frac{n}{2} \rfloor$. By Theorem 3.8 we know that $\Delta_{n,k}^{\text{stab}(r)}$ is $(n-1)$-dimensional. We proceed by induction on $r$. First let $r = 2$. By Proposition 3.26 we know that $H_{\ell}$ is a supporting hyperplane of $\Delta_{n,k}^{\text{stab}(2)}$ for all $\ell \in [n]$. By Proposition 3.29 we know that for every $\ell \in [n]$ the hyperplane $H_{\ell,1}$ is not a supporting hyperplane of $\Delta_{n,k}^{\text{stab}(2)}$. Hence, the collection of preserved supporting hyperplanes for $\Delta_{n,k}^{\text{stab}(2)}$ is given by $\{H_{\ell} : \ell \in [n]\}$, and all other supporting hyperplanes of $\Delta_{n,k}^{\text{stab}(2)}$ must intersect the relative interior of $\Delta_{n,k}$. Therefore, by Propositions 3.21 and 3.22 the remaining facets of $\Delta_{n,k}^{\text{stab}(2)}$ are supported by $H_{\ell,2}$ for $\ell \in [n]$. Now pick $2 < r < \lfloor \frac{n}{2} \rfloor$. By the inductive hypothesis, the supporting hyperplanes for $\Delta_{n,k}^{\text{stab}(r-1)}$ are $H_{\ell}$ and $H_{\ell,r-1}$ for $\ell \in [n]$. Now apply the same argument as for the base case of $r = 2$, and this completes the proof of Theorem 3.1.

The technique used in the proof of Theorem 3.1 also gives the following result on the facets of the unimodular $(n-1)$-simplex $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}$.

**Scholium 3.24.** Suppose $n \equiv 1 \mod k$. The facets of $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}$ have supporting hyperplanes $H_{\ell,r}$ for $\ell \in [n]$.

**Proof.** Recall [2] Lemma 2.7 implies that $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}$ is a unimodular $(n-1)$-simplex whenever $n \equiv 1 \mod k$. Notice also that Proposition 3.21 only requires $\Delta_{n,k}^{\text{stab}(r)}$ to be $(n-1)$-dimensional and $r > 1$. (It does not require that $r < \lfloor \frac{n}{k} \rfloor$.) Also, Proposition 3.22 holds for $n = kr + 1$. Moreover, since $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}$ is an $(n-1)$-simplex then it has precisely $n$ facets. Therefore, by Propositions 3.21 and 3.22 the facets of $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}$ must be $H_{\ell,r}$ for $\ell \in [n]$ whenever $\lfloor \frac{n}{k} \rfloor > 1$. In the case that $r = \lfloor \frac{n}{k} \rfloor = 1$, then $\Delta_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)} = \Delta_{k+1,k}$ which clearly has supporting hyperplanes $H_{\ell,1}$. □
Corollary 3.25. Fix $0 < k < n$. All but (possibly) the smallest polytope in the nested chain 

$$\Delta_{n,k} \supset \Delta_{n,k}^{\text{stab}(2)} \supset \Delta_{n,k}^{\text{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\text{stab} \left( \left\lfloor \frac{n}{r} \right\rfloor \right)}$$

has $2n$ facets.

This is an interesting geometric property since the number of vertices of these polytopes form a strictly decreasing sequence. In the special case when $n \equiv 1 \mod k$, by Scholium 3.24 we completely understand the facets of all polytopes in this chain, and the “possibly” is not necessary.

Remark 3.26. Fix $1 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor$. For all $\ell \in [n]$ let 

$$H_{\ell}^{(+)} := \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_\ell \geq 0 \} \cap H,$$

and 

$$H_{\ell,r}^{(-)} := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=\ell}^{\ell+r-1} x_i \leq 1 \right\} \cap H.$$

Then for $r < \left\lfloor \frac{n}{k} \right\rfloor$, and $r = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv 1 \mod k$, we have that 

$$\Delta_{n,k}^{\text{stab}(r)} = \left( \bigcap_{\ell=1}^{n} H_{\ell}^{(+)} \right) \cap \left( \bigcap_{\ell=1}^{n} H_{\ell,r}^{(-)} \right).$$

Remark 3.27. In this section we computed the facets of $\Delta_{n,k}^{\text{stab}(r)}$, a subpolytope of $\Delta_{n,k}^{\text{stab}(r-1)}$. Since $\Delta_{n,k}^{\text{stab}(r)} \subset \Delta_{n,k}^{\text{stab}(r-1)}$ each facet of $\Delta_{n,k}^{\text{stab}(r)}$ either lies in a facet of $\Delta_{n,k}^{\text{stab}(r-1)}$ or intersects the relative interior of $\Delta_{n,k}^{\text{stab}(r-1)}$. We first determined the set of facets of $\Delta_{n,k}^{\text{stab}(r)}$ that lie in the facets of $\Delta_{n,k}^{\text{stab}(r-1)}$. Then, using the geometry of the triangulation $\nabla_{n,k}^{r-1}$ and the fact that it restricts to a triangulation of $\Delta_{n,k}^{\text{stab}(r)}$, we determined the set of facets of $\Delta_{n,k}^{\text{stab}(r)}$ that intersect the relative interior of $\Delta_{n,k}^{\text{stab}(r-1)}$. In this way, we computed the facets of $\Delta_{n,k}^{\text{stab}(r)}$ via a nesting of triangulations of polytopes.

We remark that this technique may be interesting to apply to any pair of nested $d$-dimensional polytopes $Q \subset P$ for which there is triangulation of $P$ that restricts to a triangulation of $Q$.

4. The Gorenstein $r$-stable Hypersimplices

One application for the equations of the facets of a rational convex polytope is to determine whether or not the polytope is Gorenstein [5]. First we recall the definition of a Gorenstein polytope. Let $P \subset \mathbb{R}^N$ be a rational convex polytope of dimension $d$, and for an integer $q \geq 1$ let $qP := \{ qa : \alpha \in P \}$. Let $x_1, x_2, \ldots, x_N$, and $z$ be indeterminates over some field $K$. Given an integer $q \geq 1$, let $A(P)_q$ denote the vector space over $K$ spanned by the monomials $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_N^{\alpha_N} z^q$ for $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in qP \cap \mathbb{Z}^N$. Since $P$ is convex we have that $A(P)_p A(P)_q \subset A(P)_{p+q}$ for all $p$ and $q$. It then follows that the graded algebra

$$A(P) := \bigoplus_{q=0}^{\infty} A(P)_q$$

is finitely generated over $K = A(P)_0$. We call $A(P)$ the Ehrhart Ring of $P$, and we say that $P$ is Gorenstein if $A(P)$ is Gorenstein.

We now recall the combinatorial criterion given in [4] for an integral convex polytope $P$ to be Gorenstein. Let $\partial P$ denote the boundary of $P$ and let relint($P$) = $P - \partial P$. We say that $P$ is of
standard type if \( d = N \) and the origin in \( \mathbb{R}^d \) is contained in \( \text{relint}(P) \). When \( P \subset \mathbb{R}^d \) is of standard type we define its polar set

\[
P^* = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d : \sum_{i=1}^d \alpha_i \beta_i \leq 1 \text{ for every } (\beta_1, \beta_2, \ldots, \beta_d) \in P \right\}.
\]

The polar set \( P^* \) is again a convex polytope of standard type, and \( (P^*)^* = P \). We call \( P^* \) the dual polytope of \( P \).

**Remark 4.1.** An elementary fact in the theory of convex polytopes is the existence of an inclusion-reversing bijection between the faces of \( P \) and those of \( P^* \). This bijection is given as follows. Suppose \( (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d \), and \( K \) is the hyperplane in \( \mathbb{R}^d \) defined by the equation \( \sum_{i=1}^d \alpha_i x_i = 1 \). Then \( (\alpha_1, \alpha_2, \ldots, \alpha_d) \) is a vertex of \( P^* \) if and only if \( K \cap P \) is a facet of \( P \).

Notice that Remark 4.1 implies that the dual polytope of a rational polytope is always rational. However, it need not be that the dual of an integral polytope is always integral. If \( P \) is an integral polytope with integral dual we say that \( P \) is reflexive. This idea plays a key role in the following combinatorial characterization of Gorenstein polytopes.

**Theorem 4.2.** [De Negri and Hibi] Let \( P \subset \mathbb{R}^d \) be an integral polytope of dimension \( d \), and let \( q \) denote the smallest positive integer for which

\[
q(\text{relint}(P)) \cap \mathbb{Z}^d \neq \emptyset.
\]

Fix an integer point \( \alpha \in q(\text{relint}(P)) \cap \mathbb{Z}^d \), and let \( Q \) denote the integral polytope \( qP - \alpha \subset \mathbb{R}^d \). Then the polytope \( P \) is Gorenstein if and only if the polytope \( Q \) is reflexive.

Since the facets of \( \Delta_{n,k}^{\text{stab}(r)} \) for \( r < \left\lfloor \frac{n}{k} \right\rfloor \) are given in Theorem 3.1, we now wish to apply Remark 4.1 and Theorem 4.2 to determine exactly which of these \( r \)-stable hypersimplices are Gorenstein. We remark that we need not consider the case of \( \Delta_{n,k}^{\text{stab}(\left\lfloor \frac{n}{k} \right\rfloor)} \) when \( n \equiv 1 \mod k \) since this polytope is a unimodular \((n-1)\)-simplex and is therefore trivially Gorenstein. Hence, we only consider the case when \( r < \left\lfloor \frac{n}{k} \right\rfloor \). Recall that we computed the facets \( \Delta_{n,k}^{\text{stab}(r)} \subset \mathbb{R}^n \) as an \((n-1)\)-dimensional polytope embedded in the hyperplane \( H \) under the affine isomorphism \( \varphi \) defined in section 3. Therefore, we must first apply the inverse isomorphism \( \varphi^{-1} \) to ensure that \( \Delta_{n,k}^{\text{stab}(r)} \) is full-dimensional. Also recall that \( \varphi (\mathbb{Z}^{n-1}) = H \cap \mathbb{Z}^n \). Hence, we have the isomorphism of Ehrhart Rings as graded algebras

\[
A \left( \varphi^{-1} \left( \Delta_{n,k}^{\text{stab}(r)} \right) \right) \cong A \left( \Delta_{n,k}^{\text{stab}(r)} \right).
\]

Let \( P_{n,k}^{\text{stab}(r)} := \varphi^{-1} \left( \Delta_{n,k}^{\text{stab}(r)} \right) \).

**The facets and closed halfspaces for \( P_{n,k}^{\text{stab}(r)} \).** Recall from Remark 3.26 that

\[
\Delta_{n,k}^{\text{stab}(r)} = \left( \bigcap_{\ell=1}^n H_{\ell}^{(+)} \right) \cap \left( \bigcap_{\ell=1}^n H_{\ell,r}^{(-)} \right).
\]

We now give a description of the facets, and their associated closed halfspaces, for \( P_{n,k}^{\text{stab}(r)} \) in terms of those defining \( \Delta_{n,k}^{\text{stab}(r)} \). In the following, it will be convenient to let \( T = \{ \ell, \ell+1, \ell+2, \ldots, \ell+r-1 \} \) for \( \ell \in [n] \). We also let \( T^c \) denote the complement of \( T \) in \([n]\). Notice that for a fixed \( r < \left\lfloor \frac{n}{k} \right\rfloor \) and \( \ell \in [n] \), the set \( T \) is precisely the set of summands in the defining equation of the hyperplane \( H_{\ell,r} \).

We first consider the facets of \( \Delta_{n,k}^{\text{stab}(r)} \) with supporting hyperplanes \( H_{\ell,r} \). For these supporting hyperplanes we have two cases.
(1) First suppose that $n \notin T$. Then applying the affine isomorphism $\varphi^{-1}$ to $H_{\ell,r}$ gives

$$K_{\ell,r} := \varphi^{-1}(H_{\ell,r}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T} x_i = 1 \right\}.$$  

Similarly,

$$K_{\ell,r}^{(-)} := \varphi^{-1}(H_{\ell,r}^{(-)}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T} x_i \leq 1 \right\}.$$  

(2) Next suppose $n \in T$. Then making the substitution $x_n = k - \left( \sum_{i=1}^{n-1} x_i \right)$ gives

$$\sum_{i \in T \setminus \{n\}} x_i + k - \left( \sum_{i=1}^{n-1} x_i \right) = 1,$$

$$\sum_{i \in T^c} x_i = k - 1.$$  

Hence, for $n \in T$ we have

$$\bar{K}_{\ell,r} := \varphi^{-1}(H_{\ell,r}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T^c} x_i = k - 1 \right\}.$$  

Similarly,

$$\bar{K}_{\ell,r}^{(+)} := \varphi^{-1}(H_{\ell,r}^{(+)}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T^c} x_i \geq k - 1 \right\}.$$  

Next consider the hyperplanes and halfspaces $H_{\ell}$ and $H_{\ell}^{(+)}$ for $\ell \in [n]$. We again have two cases to consider.

(1) First suppose that $\ell \neq n$. Then

$$K_{\ell} := \varphi^{-1}(H_{\ell}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : x_i = 0 \right\},$$

and

$$K_{\ell}^{(+)} := \varphi^{-1}(H_{\ell}^{(+)}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : x_i \geq 0 \right\}.$$  

(2) Next suppose that $\ell = n$. Then since $x_n = k - \left( \sum_{i=1}^{n-1} x_i \right)$ we have that

$$0 = x_n = k - \left( \sum_{i=1}^{n-1} x_i \right),$$

or equivalently,

$$\sum_{i=1}^{n-1} x_i = k.$$  

Hence,

$$K_n := \varphi^{-1}(H_n) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i = k \right\},$$

and

$$K_n^{(-)} := \varphi^{-1}(H_n^{(+)}) = \left\{ (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i \leq k \right\}.$$  

Therefore, the facets of $P_{n,k}^{\text{stab}(r)}$ are
• $K_i \cap P_{n,k}^{\text{stab}(r)}$, for $i \in [n - 1]$,
• $K_n \cap P_{n,k}^{\text{stab}(r)}$,
• $K_{t,r} \cap P_{n,k}^{\text{stab}(r)}$, for $n \notin T$, and
• $\tilde{K}_{t,r} \cap P_{n,k}^{\text{stab}(r)}$, for $n \in T$.

Moreover, we may write $P_{n,k}^{\text{stab}(r)}$ as the intersection of closed halfspaces in $\mathbb{R}^{n-1}$

$$P_{n,k}^{\text{stab}(r)} = \left( \bigcap_{n \notin T} K_{t,r}^{(-)} \right) \cap \left( \bigcap_{n \in T} \tilde{K}_{t,r}^{(+)} \right) \cap \left( \bigcap_{i=1}^{n-1} K_i^{(+)} \right) \cap K_n^{(-)}.$$

The codegree of $P_{n,k}^{\text{stab}(r)}$. Given the above description of $P_{n,k}^{\text{stab}(r)}$, we would now like to determine the smallest positive integer $q$ for which $qP_{n,k}^{\text{stab}(r)}$ contains a lattice point in its relative interior. To do so, recall that for a lattice polytope $P$ of dimension $d$ we can define the (Ehrhart) $\delta$-polynomial of $P$. If we write this polynomial as

$$\delta_P(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \cdots + \delta_d z^d$$

then we call the coefficient vector $\delta(P) = (\delta_0, \delta_1, \delta_2, \ldots, \delta_d)$ the $\delta$-vector of $P$. We let $s$ denote the degree of $\delta_P(z)$, and we call $q = (d + 1) - s$ the codegree of $P$. It is a consequence of Ehrhart Reciprocity that $q$ is the smallest positive integer such that $qP$ contains a lattice point in its relative interior. Hence, we would like to compute the codegree of $P_{n,k}^{\text{stab}(r)}$. To do so requires that we first prove two Lemmas. In the following let $q = [\frac{n}{k}]$. Our first goal is to show that there is at least one integer point in relint $\left( qP_{n,k}^{\text{stab}(r)} \right)$ for $r < [\frac{n}{k}]$. We then show that $q$ is the smallest positive integer for which this is true. Recall that $q = \frac{n + \alpha}{k}$ for some $\alpha \in \{0, 1, \ldots, k - 1\}$. Also recall that for a fixed $0 < k < n$ we have the nesting of polytopes

$$P_{n,k}^{\text{stab}(2)} \supset P_{n,k}^{\text{stab}(3)} \supset \cdots \supset P_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor - 1)} \supset P_{n,k}^{\text{stab}(\lfloor \frac{n}{k} \rfloor)}.$$

Hence, if we identify an integer point inside relint $\left( qP_{n,k}^{\text{stab}(r)} \right)$ for every $r < [\frac{n}{k}]$, then this same integer point lives inside relint $\left( qP_{n,k}^{\text{stab}(r)} \right)$ for every $r < [\frac{n}{k}]$. With these facts in mind consider the following lemma.

Lemma 4.3. Fix $0 < k < n$ and suppose that $q = [\frac{n}{k}] = \frac{n + \alpha}{k}$ where $\alpha \in \{0, 1\}$. Then the integer point $(1, 1, \ldots, 1) \in \mathbb{R}^{n-1}$ lies inside relint $\left( qP_{n,k}^{\text{stab}(r)} \right)$ for every $r < [\frac{n}{k}]$.

Proof. It suffices to show that $(x_1, x_2, \ldots, x_{n-1}) = (1, 1, \ldots, 1)$ satisfies the set of inequalities

(i) $x_i > 0$, for $i \in [n - 1]$,
(ii) $\sum_{i=1}^{n-1} x_i < kq$,
(iii) $\sum_{i \in T} x_i < q$, for $n \notin T$, and
(iv) $\sum_{i \in T^c} x_i > (k - 1)q$, for $n \in T$.

We do this in two cases. First suppose that $\alpha = 0$. Then $k$ divides $n$ and $q = \frac{n}{k}$. Clearly, (i) is satisfied. To see that (ii) is also satisfied simply notice that

$$\frac{n - 1}{k} < \frac{n}{k}, \quad n - 1 < kq.$$

To see that (iii) is satisfied recall that $\#T = r$ and $r < [\frac{n}{k}] = q$. Finally, to see that (iv) is satisfied notice that $\#T^c = n - r$. So we must show that $n - r > (k - 1)q$. To see this, consider the following
equivalent inequalities.

\[ r < \frac{n}{k}, \]
\[ -r > -\frac{n}{k}, \]
\[ n - r > n - \frac{n}{k}, \]
\[ n - r > (k - 1)\frac{n}{k}. \]

Hence, \((1, 1, \ldots, 1) \in \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right) \) for \( r < \left\lfloor \frac{n}{k} \right\rfloor \) when \( \alpha = 0. \)

Now consider the case where \( \alpha = 1. \) Notice it suffices to consider the case when \( r = \left\lfloor \frac{n}{k} \right\rfloor - 1 \) since if \((1, 1, \ldots, 1) \in \text{relint} \left( qP_{n,k}^{\text{stab}(\left\lfloor \frac{n}{k} \right\rfloor - 1)} \right) \) then \((1, 1, \ldots, 1) \in \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right) \) for every \( r < \left\lfloor \frac{n}{k} \right\rfloor. \)

Inequalities (i), (ii), and (iii) are all satisfied in the same fashion as the case when \( \alpha = 0. \) So we need only check that (iv) is also satisfied. Again we show that \( n - r > (k - 1)q. \) Notice since \( \alpha = 1 \) then \( k \) does not divide \( n, \) and so \( \left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n}{k} \right\rfloor + 1. \) Hence, \( q = r + 2. \) It then follows that

\[ n + 2 > n + \alpha, \]
\[ \frac{n + 2}{k} > \frac{n + \alpha}{k}, \]
\[ \frac{n + 2}{k} > q, \]
\[ n + 2 > kq, \]
\[ n > kq - 2, \]
\[ n > k(r + 2) - 2, \]
\[ n > kr + 2k - 2 + r - r, \]
\[ n > (k - 1)(r + 2) + r, \]
\[ n - r > (k - 1)q. \]

Hence, (iv) is also satisfied when \( \alpha = 1. \) Thus, whenever \( \alpha \in \{0, 1\}, \) the lattice point \((1, 1, \ldots, 1) \in \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right) \) for every \( r < \left\lfloor \frac{n}{k} \right\rfloor. \) \( \square \)

Next we would like to identify an integer point in the relative interior of \( qP_{n,k}^{\text{stab}(r)} \) for \( r < \left\lfloor \frac{n}{k} \right\rfloor \) when \( \alpha \geq 2. \) In this case the point \((1, 1, \ldots, 1) \) does not always work, so we must identify another point. In the following, we identify such a point for \( r = \left\lfloor \frac{n}{k} \right\rfloor - 1, \) and it then follows that this point lies in \( \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right) \) for all other \( r < \left\lfloor \frac{n}{k} \right\rfloor. \) To do so, we construct the desired point using the notions of \( r \)-stability. Fix \( 0 < k < n \) such that \( q = \frac{n + \alpha}{k} \) for \( \alpha \geq 2, \) and let \( r = \left\lfloor \frac{n}{k} \right\rfloor - 1. \) This also fixes the value \( \alpha \in \{2, 3, \ldots, k - 1\}. \) Since \( r = \left\lfloor \frac{n}{k} \right\rfloor - 1 \) we may construct an \( r \)-stable vertex with \( k - 1 \)’s in entries \( n - r, n - 2r, n - 3r, \ldots, n - (k - 1)r. \)

Notice that this vertex is \( r \)-stable since the inequality

\[ (k - 1) + (k - 1)(r - 1) \leq n - r \]

holds. This equality implies that there are at least \( r \) 0’s between the \( n^{th} \) entry of the vertex and the \( n - (k - 1)r^{th} \) entry (read from right-to-left modulo \( n \)). In particular, this implies that the \( n^{th} \) entry (and the 1st entry) is occupied by a 0. To construct the desired vertex replace the 1’s in spots \( n - (\alpha + 1)r, n - (\alpha + 2)r, \ldots, n - (k - 1)r \)
with 0’s. Now add 1 to each entry of this lattice point. If the resulting point is \((x_1, x_2, \ldots, x_n)\) then replace the entry \(x_n = 1\) with the value

\[
kq - \left( \sum_{i=1}^{n-1} x_i \right).
\]

Call the resulting vertex \(\epsilon^\alpha\). Next consider the hyperplane

\[
H_q := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = kq \right\},
\]

and the affine isomorphism

\[
\widetilde{\varphi} : \mathbb{R}^{n-1} \rightarrow H_q,
\]

\[
\widetilde{\varphi} : (x_1, x_2, \ldots, x_{n-1}) \mapsto \left( x_1, x_2, \ldots, x_{n-1}, kq - \left( \sum_{i=1}^{n-1} x_i \right) \right).
\]

Notice that by our construction of \(\epsilon^\alpha\), the point \(\widetilde{\varphi}^{-1}(\epsilon^\alpha)\) is simply \(\epsilon^\alpha\) with the last coordinate deleted.

**Lemma 4.4.** Fix \(0 < k < n\) such that \(q = \left\lceil \frac{n}{k} \right\rceil = \frac{n + \alpha}{k}\) for \(2 \leq \alpha < k - 1\). Then for every \(r < \left\lfloor \frac{n}{k} \right\rfloor\) the lattice point

\[
\widetilde{\varphi}^{-1}(\epsilon^\alpha) \in \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right).
\]

**Proof.** It suffices to show that \((x_1, x_2, \ldots, x_{n-1}) = \widetilde{\varphi}^{-1}(\epsilon^\alpha)\) satisfies the set of inequalities

(i) \(x_i > 0\), for \(i \in [n-1]\),
(ii) \(\sum_{i=1}^{n-1} x_i < kq\),
(iii) \(\sum_{i \in T} x_i < q\), for \(n \notin T\), and
(iv) \(\sum_{i \in T^c} x_i > (k-1)q\), for \(n \in T\),

for \(r = \left\lceil \frac{n}{k} \right\rceil - 1\).

It is clear that (i) is satisfied. To see that (ii) is satisfied notice that

\[
\sum_{i=1}^{n-1} x_i = n - 1 + \alpha.
\]

Here, we count \(n - 1\), one for each coordinate of the point, and then we add \(\alpha\) since exactly \(\alpha\) coordinates are occupied by 2’s and all other coordinates are occupied by 1’s. It then follows that

\[
\sum_{i=1}^{n-1} x_i = n - 1 + \alpha < n + \alpha = k \left( \frac{n + \alpha}{k} \right) = kq.
\]

Hence, (ii) is satisfied. To see that (iii) is satisfied first notice that for \(T\) with \(n \notin T\)

\[
\sum_{i \in T} x_i = \begin{cases} r & \text{if } T \text{ contains no entry with value 2}, \\ r + 1 & \text{otherwise}. \end{cases}
\]

This is because we have chosen the 2’s to be separated by at least \(r - 1\) 0’s. Thus, since \(k\) does not divide \(n\) we have that

\[
\sum_{i \in T} x_i \leq r + 1 = \left\lceil \frac{n}{k} \right\rceil < q.
\]
Finally, to see the (iv) is satisfied first notice that for $T$ with $n \in T$

$$\sum_{i \in T} x_i = \begin{cases} n - r + \alpha - 1 & \text{if } T \text{ contains an entry with value } 2, \\ n - r + \alpha & \text{otherwise.} \end{cases}$$

Hence, we must show that $n - r + \alpha - 1 > (k - 1)q$. To see this, simply notice that

$$-\frac{n + \alpha}{k} + 1 > -\frac{n + \alpha}{k},$$

$$\alpha - \frac{n + \alpha}{k} + 1 > \alpha - \frac{n + \alpha}{k},$$

$$\alpha - \left(\frac{n + \alpha}{k} - 2\right) - 1 > \alpha - \frac{n + \alpha}{k},$$

$$\alpha - r - 1 > \alpha - \frac{n + \alpha}{k},$$

$$n - r + \alpha - 1 > n + \alpha - \frac{n + \alpha}{k},$$

$$n - r + \alpha - 1 > (k - 1)\left(\frac{n + \alpha}{k}\right),$$

$$n - r + \alpha - 1 > (k - 1)q.$$

Hence, $\epsilon^\alpha \in \text{relint} \left(q\mathcal{P}_{n,k}^{\text{stab}}\right)$ for every $r < \left\lfloor \frac{n}{k} \right\rfloor$. □

By Lemmas 4.3 and 4.4 we see that $q\mathcal{P}_{n,k}^{\text{stab}}$ contains a lattice point in its relative interior for $r < \left\lfloor \frac{n}{k} \right\rfloor$. We now show that $q = \left\lceil \frac{n}{k} \right\rceil$ is indeed the codegree of these polytopes.

**Theorem 4.5.** Let $r < \left\lfloor \frac{n}{k} \right\rfloor$. The codegree of $\mathcal{P}_{n,k}^{\text{stab}}$ is $q = \left\lceil \frac{n}{k} \right\rceil$.

**Proof.** First recall that $\mathcal{P}_{n,k}^{\text{stab}}$ is a subpolytope of $\mathcal{D}_{n,k}$. Hence, by a theorem of Stanley [8], it follows that $\delta\left(\mathcal{P}_{n,k}^{\text{stab}}\right) \leq \delta(\mathcal{D}_{n,k})$. Therefore, the codegree of $\mathcal{P}_{n,k}^{\text{stab}}$ is no smaller than the codegree of $\mathcal{D}_{n,k}$. In [6, Corollary 2.6], Katzman determines that the codegree of $\mathcal{D}_{n,k}$ is $q = \left\lceil \frac{n}{k} \right\rceil$. Since Lemmas 4.3 and 4.4 imply that $q\mathcal{P}_{n,k}^{\text{stab}}$ contains a lattice point inside its relative interior we conclude that the codegree of $\mathcal{P}_{n,k}^{\text{stab}}$ is $q = \left\lceil \frac{n}{k} \right\rceil$. □

As a corollary to Theorem 4.5 we note the following Ehrhart theoretical result for $r$-stable $n,k$-hypersimplices.

**Corollary 4.6.** Let $r < \left\lfloor \frac{n}{k} \right\rfloor$. Then the degree of the $\delta$-polynomial of $\mathcal{D}_{n,k}^{\text{stab}}$ is $n - \left\lceil \frac{n}{k} \right\rceil$.

**Proof.** Theorem 4.5 shows that the codegree of $\mathcal{D}_{n,k}^{\text{stab}}$ is $\left\lceil \frac{n}{k} \right\rceil$. Since $r < \left\lfloor \frac{n}{k} \right\rfloor$ then $\mathcal{D}_{n,k}^{\text{stab}}$ has dimension $n - 1$, and it follows that the degree of $\delta_{\mathcal{D}_{n,k}^{\text{stab}}}(z)$ is

$$s = (n - 1) + 1 - \left\lfloor \frac{n}{k} \right\rfloor = n - \left\lfloor \frac{n}{k} \right\rfloor.$$

□

Recall that if an integral polytope $P$ of dimension $d$ with codegree $q$ is Gorenstein then

$$\# \left(\text{relint} \left(qP\right) \cap \mathbb{Z}^d\right) = 1.$$

With this fact in hand, we have the following corollary.
Corollary 4.7. Suppose $0 < k < n$ and $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$, where $2 \leq \alpha \leq k - 1$. Then for every $r < \left\lfloor \frac{n}{k} \right\rfloor$

$$\# \left( \text{relint} \left( qP_{n,k}^{\text{stab}(r)} \right) \cap \mathbb{Z}^d \right) > 1.$$  

In particular, $\Delta_{n,k}^{\text{stab}(r)}$ is not Gorenstein.

Proof. First recall the method by which we constructed the lattice point $\epsilon^\alpha$. Notice that, since the 1st entry of $\epsilon^\alpha$ is 0, this method works just as well if we first construct an $r$-stable vertex with $k - 1$ 1’s in entries  

$$(n + 1) - r, (n + 1) - 2r, (n + 1) - 3r, \ldots, (n + 1) - (k - 1)r,$$

as opposed to the entries  

$$n - r, n - 2r, n - 3r, \ldots, n - (k - 1)r.$$  

Hence, if we produce a second vertex using this placement of the $k - 1$ 1’s, say $\zeta^\alpha$, then $\tilde{\varphi}(\zeta^\alpha)^{-1}$ also lies in the relative interior of $qP_{n,k}^{\text{stab}(r)}$. The details of the proof are analogous to the case of $\epsilon^\alpha$. $\square$

Translation of $\ell P_{n,k}^{\text{stab}(r)}$ to the origin. Notice that Corollary 4.7 and Lemma 4.3 together imply that the only possible Gorenstein $r$-stable hypersimplices are those for which  

$$(1,1,\ldots,1) \in \text{relint} \left( P_{n,k}^{\text{stab}(r)} \right).$$  

For these polytopes, namely those described in Lemma 4.3, we now consider the translated integral polytope

$$Q := qP_{n,k}^{\text{stab}(r)} - (1,1,\ldots,1).$$  

Notice, by our description of the facets of $P_{n,k}^{\text{stab}(r)}$, we have that the facets of $Q$ are defined by the hyperplanes

(a) $x_i = -1$, for $i \in [n-1],$

(b) $\sum_{i=1}^{n-1} x_i = kq - (n - 1),$

(c) $\sum_{i \in T} x_i = q - r$, for $n \notin T$, and

(d) $\sum_{i \in T^c} x_i = (k - 1)q - (n - r)$, for $n \in T$.

Recall that by Theorem 4.2, the polytope $\Delta_{n,k}^{\text{stab}(r)}$ will be Gorenstein if and only if the the polytope $Q$ is reflexive. In other words, we must apply Remark 4.1 to the facets of $Q$ to determine when all vertices of $Q^*$ will be integral. This is the content of the following theorem.

Theorem 4.8. Let $r < \left\lfloor \frac{n}{k} \right\rfloor$. Then $\Delta_{n,k}^{\text{stab}(r)}$ is Gorenstein if and only if $n = kr + k$.

Proof. First recall that by Corollary 4.7, we need only consider those polytopes described in Lemma 4.3. For these polytopes we must determine when the all vertices of $Q^*$ are integral. We do so by means of the inclusion-reversing bijection described in Remark 4.1. First consider the hyperplanes of type (a) in the above list. These are

$$x_i = -1$$

for $i \in [n - 1]$. Equivalently, we may write such a hyperplane as

$$\sum_{j=1}^{n-1} \beta_j x_j = 1,$$

where

$$\beta_j := \begin{cases} 0 & \text{if } j \neq i, \\ -1 & \text{if } j = i. \end{cases}$$

Hence, the corresponding vertex in $Q^*$ is integral.
Next consider the hyperplane given in (b):

$$\sum_{i=1}^{n-1} x_i = kq - (n - 1).$$

Recall that $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ for some $\alpha \in \{0, 1\}$. Hence, this hyperplane is equivalently expressed as

$$\sum_{i=1}^{n-1} x_i = k \left( \frac{n+\alpha}{k} \right) - (n - 1),$$

$$\sum_{i=1}^{n-1} x_i = \alpha + 1,$$

$$\sum_{i=1}^{n-1} \frac{1}{\alpha + 1} x_i = 1.$$

Therefore, for the corresponding vertex in $Q^*$ to be integral it must be that $\alpha = 0$.

It remains to consider the hyperplanes corresponding to the sets of indices $T$. First consider those of type (c) in the above list. We may equivalently write these hyperplanes as

$$\sum_{i \in T} x_i = 1,$$

Hence, for the corresponding vertex in $Q^*$ to be integral we need $q - r = 1$. Since $\alpha = 0$ we also have that $q = \frac{n}{k}$ where $k$ divides $n$. Thus, we have that $r = \frac{n}{k} - 1$, or equivalently, $n = kr + k$.

Finally, consider the hyperplanes of type (d) in the above list. When $n = kr + k$, we have that $q = r + 1$, and so

$$\sum_{i \in T^c} x_i = (k - 1)q - (n - r),$$

$$= (k - 1)(r + 1) - ((kr + k) - r),$$

$$= (k - 1)(r + 1) - ((k - 1)r + k),$$

$$= (k - 1)r + (k - 1) - (k - 1)r - k,$$

$$= -1.$$

Hence, the corresponding vertex of $Q^*$ is integral, and we conclude that, for $r < \left\lfloor \frac{n}{k} \right\rfloor$, the polytope $\Delta_{n,k}^{\text{stab}(r)}$ is Gorenstein if and only if $n = kr + k$.

\[ \square \]

**Remark 4.9.** Recall that for $n \equiv 1 \mod k$ the polytope $\Delta_{n,k}^{\text{stab}\left(\frac{n}{k}\right)}$ is a unimodular $(n-1)$-simplex \cite{2}. Hence, it is trivially Gorenstein. Therefore, for every $r \geq 1$ we have two Gorenstein $r$-stable $n,k$-hypersimplices, one of which is a $(n - 1)$-simplex, and they are given by $n = kr + k$ when $0 < k < n$, and $r = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv 1 \mod k$.

Thinking of this result in regards to the nested chain of polytopes

$$\Delta_{n,k} \supset \Delta_{n,k}^{\text{stab}(2)} \supset \Delta_{n,k}^{\text{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\text{stab}\left(\frac{n}{k}\right)}$$

it follows from Theorem 3.8 that the smallest polytope in this chain is Gorenstein for $n \equiv 1 \mod k$, and the second smallest polytope is Gorenstein when $n$ divides $k$. In both cases, no larger polytope in the chain is Gorenstein.

We also have the following corollary to Theorem 4.8.
Corollary 4.10. Let $r \geq 1$. The $r$-stable, $n,k$-hypersimplices $\Delta_{n,k}^{stab(r)}$ for $n = kr + k$ when $0 < k < n$, and $r = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv 1 \mod k$ have unimodal $\delta$-vectors.

Proof. By [2] Corollary 2.6 there exists a regular unimodular triangulation of $\Delta_{n,k}^{stab(r)}$. By Theorem 4.8 the polytope $\Delta_{n,k}^{stab(r)}$ is Gorenstein for $n = kr + k$ when $0 < k < n$, and $r = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv 1 \mod k$. By [3] Theorem 1 we conclude that the $\delta$-vector of $\Delta_{n,k}^{stab(r)}$ is unimodal for $n = kr + k$ when $0 < k < n$, and $r = \left\lfloor \frac{n}{k} \right\rfloor$ when $n \equiv 1 \mod k$. □

Theorem 4.8 is interesting since it demonstrates that the Gorenstein property is quite rare amongst the class of $r$-stable hypersimplices, and also because it extends the collection of $r$-stable hypersimplices known to have unimodal $\delta$-vectors given [2].

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