Reversible dynamics with closed time-like curves and freedom of choice

Germain Tobar$^1$ and Fabio Costa$^2$

$^1$School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia
$^2$Centre for Engineered Quantum Systems, School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia

The theory of general relativity predicts the existence of closed time-like curves (CTCs), which theoretically would allow an observer to travel back in time and interact with their past self. This raises the question of whether this could create a grandfather paradox, in which the observer interacts in such a way to prevent their own time travel. Previous research has proposed a framework for deterministic, reversible, dynamics in the presence of CTCs, where observers in distinct regions of spacetime can perform arbitrary local operations with no contradiction arising. However, only scenarios with up to three regions have been fully characterised, revealing only one type of process where the observers can verify to both be in the past and future of each other. Here we extend this characterisation to an arbitrary number of regions and find that there exist several inequivalent processes that can only arise in the presence of CTCs. This supports the view that complex dynamics is possible in the presence of CTCs, compatible with free choice of local operations and free of inconsistencies.

I. INTRODUCTION

The dominant paradigm in physics relies on the idea that systems evolve through time according to dynamical laws, with the state at a given time determining the entire history of the system.

General relativity challenges this view. The Einstein equations, describing the relationship between spacetime geometry and mass-energy [1], have counterintuitive solutions containing closed time like curves (CTCs) [2–9]. An event on such a curve would be both in the future and in the past of itself, preventing an ordinary formulation of dynamics according to an “initial condition” problem. The question then arises whether some more general type of dynamics is possible.

Although it is an open question whether CTCs are possible in our universe [10–14], considering dynamics beyond the ordinary temporal view is relevant to other research areas as well. In a theory that combines quantum physics with general relativity, it is expected that spacetime loses its classical properties [15, 16], possibly leading to indefinite causal structures [17–19]. In a quite different direction, it has been suggested that quantum physics could be reduced to some kind of “retrocausal” classical dynamics [20–30].

The main problem arising when abandoning ordinary causality is the so called “grand father paradox” [31]: a time traveller could kill her own grandfather and thus prevent her own birth, leading to a logical inconsistency. A popular approach holds that the grandfather paradox makes CTCs incompatible with classical physics, while appropriate modifications to quantum physics could restore consistency [32–46]. A common feature of the proposals within this approach is that they postulate a radical departure from ordinary physics even in regions of space-time devoid of CTCs, or in scenarios where the time travelling system does not actually interact with anything in the past [47, 48].

A different approach is the so called “process matrix formalism”, which takes as a starting point the local validity of the ordinary laws of physics and asks what type of global processes are compatible with this assumption [49–64]. This framework enforces that all operations that would normally be possible in ordinary spacetime should still be available in local regions. First considered in the quantum context, this approach has been applied to classical physics too, with the remarkable discovery of classical processes that are incompatible with any causal order between events [65–67].

In Ref. [68], a classical, deterministic version of the formalism was proposed as a possible model for CTCs. In this model, one considers a set of regions that do not contain any, but might be traversed by, CTCs. Agents in the regions receive a classical state from the past boundary, and send the system through the future boundary. Dynamics outside the regions determines the state each agent will observe in the past of the respective region, as a function of the states prepared by other agents. A simple characterisation was found for all processes involving up to three regions; furthermore, it was found that, for three regions, all non causally ordered processes are essentially equivalent.

In this work, we extend the characterisation of deterministic processes to an arbitrary number of regions. We provide some simple interpretation of the characterisation: when fixing the state on the future of all but two regions, the remaining two must be causally ordered, with only one directional signalling possible. We show, by explicit examples, that there are inequivalent, non causally ordered four-partite processes, which cannot be reduced to tripartite ones. Our results show that CTCs are not only compatible with determinism and with the local “free choice” of operations, but also with a rich and diverse range of scenarios and dynamical processes.
II. DETERMINISTIC PROCESSES

This section aims to revise and summarise the approach of Ref. [68] and the results that are relevant to the full characterisation of arbitrary deterministic, classical processes.

In ordinary dynamics, a process is a function that maps the state of a system at a given time to the state at a future time. Operationally, we can think of the state in the past as a 'preparation' and the one in the future as the outcome of a 'measurement'. To generalise this picture, we consider \( N \) spacetime regions, in which agents can perform arbitrary operations. In particular, each agent will observe a state coming from the past of the region and prepare a state to send out through the future. The key assumption is that the actions of the agents in the regions are independent from the relevant dynamics governing the exterior of the regions. In other words, agents retain their "freedom of choice" to perform arbitrary operations. In this approach, a process should determine the outcomes of measurements performed by an agent, as a function of the operations performed by the others.

To simplify the analysis, we assume that each region is connected and it has only space-like boundaries (one future and one past). This ensures that, in a CTC-free spacetime, each region is either in the future, in the past, or space-like to any other, so a violation of causal order between the regions can be attributed to a lack of causal order in the background spacetime. Furthermore, we shall assume that all the time-like curves contained in a region can be extended to curves that cross each boundary once (in particular, the regions contain no CTCs). This ensures that each region is locally indistinguishable from a region in ordinary spacetime and enables a simple characterisation of local operations as functions from past to future boundary.

A. The Process Function

In order to develop the formalism for classical, deterministic dynamics of local regions in the presence of CTCs, we will assign the boundaries of these local regions classical state spaces. The state spaces \( A_i \) and \( X_i \) denote the past and future boundaries respectively of a local region \( i \). Individual states will be denoted as \( a_i \in A_i \), \( x_i \in X_i \). A classical, deterministic operation in the local region will be denoted by the function \( f_i : A_i \rightarrow X_i \) (Fig. 1). The function \( f_i \) describes the transition from the input space to the output space for the local region \( i \). We denote \( D_i := \{ f_i : A_i \rightarrow X_i \} \) to be the set of all possible operations in region \( i \). In order to refer to a collection of objects for all regions, we drop the index. For example, the set of all possible inputs for \( N \) distinct local regions will be denoted \( A \equiv A_1 \times \ldots \times A_N \).

We will use the notation \( A_i^\prime = A_1 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_N \), \( a_i^\prime = \{ a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N \} \), etc., to denote collections with the component \( i \) removed. Appropriate ordering will be understood when joining variables, for example in expressions as \( a = a_i \cup a_i^\prime \), \( f(a) = f(a_i, a_i^\prime) \), and so on.

One of the requirements of a deterministic framework for local regions in the presence of CTCs, is that the framework must be able to predict the state on the past boundary of each local region. In the presence of CTCs, the past state of each local region can depend on all local operations (In a CTC free space-time, the past-boundary state of a region would only depend on operations in its past). The dependence on local operations can be described with a function \( \omega \equiv \{ \omega_1, ..., \omega_N \} : D \rightarrow A \) which determines the past state of each local region as a function of all local operations \( [68] \). The function \( \omega \) will be henceforth labelled as a process.

The function \( \omega \) will remain general, only being restricted by a weak form of locality. Locality requires that, once the state at the boundary of a region is fixed, the details of what happens inside the region should not be relevant to the exterior dynamics. The local field equations typically used in physics all satisfy this requirement. In order to formalise this requirement for locality, for every process \( \omega \) there must exist an additional function \( w : X \rightarrow A \) such that

\[
\omega (f) = w (f (\omega (f))) \quad \forall f \in D . \quad (1)
\]

We will henceforth refer to a function that satisfies the above consistency condition as a process function (Fig. 2). It has been shown in Ref. [68], that a necessary and sufficient condition for a process function is that \( w \circ f \) has a unique fixed point for every local operation \( f \):

\[
\forall f \exists ! a \text{ such that } w \circ f (a) = a . \quad (2)
\]

In the following, we will work with process functions, rather processes, and use the fixed point condition (2) as the defining property.

An important property of process functions is that an observer in a localised region cannot use it to send information back to herself. Intuitively, this prevents paradoxes, such as an agent attempting to warn her past self

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1 Ref. [68] defined deterministic processes through a different but equivalent self-consistency condition.
to avoid a particular event, thus removing the motivation for her to warn her past self. Formally, this requires that the input of each local region is independent of the output of the same region. Indeed, It was proven in Ref. [68] that each component of a process function \( w \) is independent of the output of the same region:

\[ w_i(x) = w_i(x_{\backslash i}), \quad (3) \]

where, \( x_{\backslash i} \) is the set of outputs of all regions except the \( i \)th region. Note that Eq. (3) implies that a process function can be described in terms of a set of functions \( w_1 : X_1 \to A_1, \ldots, w_n : X_n \to A_n \).

**Figure 2.** A process function \( w \) describes the interaction of distinct localised space-time regions with CTCs. The process function \( w \) formalises how in the presence of CTCs, the past state of each local region depends on the outputs of all the other regions.

**B. Reduced Processes**

Before we go into detail about the characterisation of process functions, we must firstly make some important definitions and consider some important properties of process functions.

**Definition 1.** Consider a function \( w : X \to A \), such that, for each region \( i = 1, \ldots, N \), \( w_i(x) = w_i(x_{\backslash i}) \). For a particular local operation \( f_i : A_i \to X_i \), we define the reduced function \( w^{f_i} : X_{\backslash i} \to A_{\backslash i} \) on the remaining regions through the composition of \( w \) with \( f_i \):

\[ w^{f_i}(x_{\backslash i}) := w_i(x_{\backslash i}, f_i(w_i(x_{\backslash i}))), \quad i \neq j. \quad (4) \]

This definition is important for formalising the intuition that if we fix the operation for a particular local region, there should still exist a process for the remaining regions. The definition of the reduced function plays an important role in the investigation of the properties of multipartite process functions.

**Lemma 1** (Lemma 3 in Ref. [68]). Given a function \( w : X \to A \), such that, for each region \( i = 1, \ldots, N \), \( w_i(x) = w_i(x_{\backslash i}) \), we have

(i) If \( w \) is a process function, then \( w^{f_i} \) is also a process function for every region \( i \) and operation \( f_i \).

(ii) If there exists a region \( i \) such that, for every local operation \( f_i \), \( w^{f_i} \) is a process function, then \( w \) is also a process function.

Following the result of this lemma, we can conclude that \( w \) is a process function if and only if the corresponding reduced function \( w^{f_i} \) is also a process function.

Definition 1 can be modified to apply to a process in which we fix a particular region’s output instead of fixing a particular local operation.

**Definition 2.** Consider a function \( w : X \to A \), such that, for each region \( i = 1, \ldots, N \), \( w_i(x) = w_i(x_{\backslash i}) \). For a particular region’s output \( x_i \in X_i \), we define the output reduced function \( w^{x_i} : X_{\backslash i} \to A_{\backslash i} \) on the remaining regions to denote the function in which we have fixed the output of the \( i \)th region:

\[ w^{x_i}(x_{\backslash i}) := \{w_1(x_{\backslash 1}), \ldots, w_{i-1}(x_{\backslash i-1}), w_{i+1}(x_{\backslash i+1}), \ldots, w_n(x_{\backslash n})\}. \quad (5) \]

We can apply definition 2 to denote \( w^{x_i(\backslash i)} \) as the output reduced function in which we fixed the outputs of all regions except regions \( i \) and \( j \).

While definition 1 and 2 are similar, the distinction between the reduced function and the output reduced function is important for the characterisation of multipartite process functions.

**C. Signalling**

In order to understand how different parties in distinct regions of spacetime signal to each other we must define what it means for one observer to signal to another observer.

Equation (3) describes how an observer in a local region can not signal to their own past. This is consistent with the following definition of no-signalling.

**Definition 3.** Given a process function \( w : X \to A \), we say that region \( j \) cannot signal to region \( i \) if

\[ w_i(x_j, x_{\backslash j}) = w_i(x'_j, x_{\backslash j}) \quad \forall x \in X, x'_j \in X_j, \quad (6) \]

which we can abbreviate as \( w_i(x) = w_i(x_{\backslash j}) \).

We define signalling as the negation of definition 3. Signalling is useful to establish whether a process is compatible with a given causal structure, as a region can only signal to regions in its future. This means that, in a spacetime without CTCs, signalling between regions
defines a relation of partial order. However, the presence of CTCs does not automatically allow arbitrary signalling, as the consistency condition (2) imposes strong constraints on the process function.

As we will show below, it is convenient to characterise process functions in terms of a more refined notion of signalling. In general, the possibility to signal from a region to another can depend on the outputs of all other regions. It is useful to capture this as follows:

**Definition 4.** Given two regions $i$ and $j$, a process function $w : \mathcal{X} \rightarrow \mathcal{A}$, and an output state $\tilde{x}_{(i,j)} \in \mathcal{X}_{(i,j)}$, we say that $j$ cannot signal to $i$ conditioned on $\tilde{x}_{(i,j)}$ if

$$w_i^\tilde{x}_{(i,j)}(x_j) = w_i^{\tilde{x}_{(i,j)}}(x_j') \quad \forall x_j, x_j' \in \mathcal{X}. \quad (7)$$

For example, for certain process functions $w$, there can exist $\tilde{x}_{(i,j)} \in \mathcal{X}_{(i,j)}$ such that each of the components $w_i^{\tilde{x}_{(i,j)}}$ and $w_j^{\tilde{x}_{(i,j)}}$ are a constant. In these cases neither region can signal to the other. However, there may also exist another choice of outputs $x_j' \in \mathcal{X}_{(i,j)}$ such that signalling occurs between regions $i$ and $j$. Note that signalling can also depend on the “sender’s” input state, although this fact would not play an important role in what follows.

We now have a framework which describes general deterministic dynamics in the presence of CTCs. This framework is characterized by the process function $w$ which maps the output states of each local region to the past boundary of each local region. Condition (2) allows freedom of choice for the operations performed by the observer in each region. Condition (3) guarantees that there is no paradox resulting from the operations performed in the presence of CTCs. In order to further understand communication between observers in the presence of CTCs, we must develop a characterisation of the process function $w$ which describes how these observers can communicate.

### D. Characterization of Process Functions

The simplest and most intuitive process functions are causally ordered ones. For example, consider three observers in three distinct regions which we label regions 1, 2, and 3 respectively. If there exists causal order between these regions such that $1 \prec 2 \prec 3$, then the process function is given by $w_1(x) = a$ (constant), $w_2(x) = w_2(x_1)$, and $w_3(x) = w_3(x_1, x_2)$. For such causally ordered process functions condition (2) is satisfied. However, causally ordered process functions are compatible without the presence of CTCs. We are interested in whether non-trivial process functions exist in the presence of CTCs. In order to answer whether non-trivial process functions exist in the presence of CTCs, we must develop a characterization of process functions for an arbitrary number of regions. In other words, we want to find a way to tell whether a generic function $w : \mathcal{X} \rightarrow \mathcal{A}$ satisfies the fixed point condition (2).

Ref. [68] characterised process functions with up to three regions. For a single region, condition (3) requires that the process function has to be a constant: $w(x) = a \forall x$. Bipartite process functions are characterized by three conditions:

(i) $w_1(x_1, x_2) = w_1(x_2)$,

(ii) $w_2(x_1, x_2) = w_2(x_1)$,

(iii) at least one of $w_1(x_2)$ or $w_2(x_1)$ is constant.

It is clear that (i) and (ii) follow from condition (3), while (iii) follows from condition (2). As a result, bipartite process functions only allow one-way signalling.

In order to characterise tripartite process functions, we must consider three distinct regions which we label 1, 2, and 3. This process function has three components $a_1 = w_1(x_2, x_3)$, $a_2 = w_2(x_1, x_3)$, and $a_3 = w_3(x_1, x_2)$. Ref. [68] proves the following characterisation of tripartite process functions, where the output variable of one region ‘switches’ the direction of signalling between the other two regions.

**Theorem 1** (Tripartite process functions, Theorem 3 in Ref. [68]). Three functions $w_1 : \mathcal{X}_2 \times \mathcal{X}_3 \rightarrow \mathcal{A}_1$, $w_2 : \mathcal{X}_1 \times \mathcal{X}_3 \rightarrow \mathcal{A}_2$, $w_3 : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{A}_3$ define a process function if and only if each of the output reduced functions

$$w^{x_3}(x_1, x_2) := \{w_1(x_2, x_3), w_2(x_1, x_3)\}, \quad (8)$$

$$w^{x_2}(x_1, x_3) := \{w_2(x_1, x_3), w_3(x_1, x_2)\}, \quad (9)$$

$$w^{x_1}(x_2, x_3) := \{w_1(x_2, x_3), w_3(x_1, x_2)\} \quad (10)$$

is a bipartite process function for every $x_3 \in \mathcal{X}_3$, $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$ respectively.

The properties defined in theorem 1 describe that, for every fixed output of one of the regions, at most one-way signalling is possible between the other two regions. Our goal is to prove a similar characterisation of multipartite process functions in terms of conditional signalling.

### III. CHARACTERIZATION OF MULTIPARTITE PROCESS FUNCTIONS

We are now ready to prove our core result: a characterisation of arbitrary multipartite process functions that generalises Theorem 1. There are in fact two distinct (but equivalent) ways to generalise Theorem 1: given an $N$-partite process function, one can check if all $N-1$-partite functions, obtained by fixing one output, are valid process functions. Alternatively, one can fix all but two outputs, and check if the remaining two regions are at most one-way signalling. Let us start with the first generalisation.

**Theorem 2** (N-partite process function). $P[N]$ functions $w_1 : \mathcal{X}_1 \rightarrow \mathcal{A}_1$, $w_2 : \mathcal{X}_2 \rightarrow \mathcal{A}_2$, ..., $w_N : \mathcal{X}_N \rightarrow \mathcal{A}_N$, define a process function $w$ if and only
Proof. Consider N space time regions, with the ith region’s input states denoted as $a_i \in A_i$, and its output states denoted as $x_i \in X_i$. We know that if w is a process function, then $w^{x_i}$ must also be a valid process function as proven in point (i) of Lemma 1. This proves one direction of the theorem.

In order to complete the proof, we need to prove the converse as well. If $w^{x_i}$ is a valid N-1 partite process function for $i \in \{1, 2, 3, ..., N\}$, then w is a valid N-partite process function. The proof will proceed by induction. Firstly, we will prove P[3], and then the implication P[N-1] ⇒ P[N]. P[3] is proven simply by applying theorem 1. Next, we assume the induction hypothesis is true for the P[N-1] case: for all $i \in \{1, 2, 3, ..., N\}$, $w^{x_i}$ is a valid N-2 partite process function for all $i \in \{1, 2, 3, ..., N\}$. We know that if $w$ is an N-partite process function, then $w^{x_i}$ must also be a valid process function by applying either point (i) of Lemma 1, or theorem 2 directly. We use the same logic to prove that for all $i, k \in \{1, 2, 3, ..., N\}$, $i \neq k$, $(w^{x_i})^{x_k}$ is a valid N-2 partite process function. Repeating the argument until we have fixed the output of all regions except two proves that if w is an N-partite process function, then $w^{x_i}(x_j)$ is a valid bipartite process function, proving one direction of the corollary.

In order to prove the converse, we begin by noting that we can write an arbitrary bipartite function as an output reduced tripartite process function: $w^{x_i}(x_j) = (w^{x_i(x_j)})^{x_k}$ for $i \neq j \neq l$. If, for all distinct $i, j, l \in \{1, 2, 3, ..., N\}$, $w^{x_i(x_j)}$ is a valid bipartite process function, then by Theorem 2, for all distinct $i, j, l \in \{1, 2, 3, ..., N\}$, $w^{x_i(x_j)}$ must be a valid tripartite process function. We can repeat this argument in order to conclude that for all distinct $i, j, k, l \in \{1, 2, 3, ..., N\}$, $w^{x_i(x_j)}$ must be a valid quadripartite process function. We can keep applying the same argument until we conclude that w is a valid N-partite process function, thus proving the reverse direction of the corollary and hence concluding the proof.

Corollary 1 explicitly demonstrates the condition that fixing the output of all regions except two arbitrarily picked l, j, i ∈ {1, 2, ..., N} determines the direction of signalling between regions l and j, and hence the remaining output reduced process function is a bipartite process function:

$$w^{x_i(x_j)}(x_l, x_j) : = \{ (w_l(x_j, x_{(l,j)})), w_j(x_l, x_{(l,j)}) \},$$

where at least one of the two component functions $w_l^{x_i(x_j)}$, $w_j^{x_i(x_j)}$ is a constant. The validity of a multipartite process function can be checked by ensuring that, for all $l, j \in \{1, 2, 3, ..., N\}$, $w^{x_i(x_j)}$ is a valid bipartite process function.

Theorem 2 and subsequently Corollary 1 demonstrates that multipartite process functions can at most be conditionally one-way signalling between any pair of regions. In other words, fixing the output of all regions except two, allows at most one-way signalling between the two remaining regions.
IV. EXAMPLES

The above characterisation of process functions allows us to consider specific examples that cannot occur in an ordinary, causally ordered spacetime. An example of such a process function in three spacetime regions was first presented in Ref. [67]. This tripartite process function can easily be extended to a quadripartite process function through the addition of a fourth party either in the past or future of the other three parties. However, the existence of the fourth party in the process function does not require the presence of CTCs. In the case where the fourth party is in the future of the other three parties, this simply corresponds to a fourth region where causal order exists from the other three parties to the fourth party. As a result, there is significant motivation to find quadripartite process functions incompatible with causal order between any subsets of parties (this is analogous to the “genuinely multipartite non-causal correlations” studied for quantum processes [60]).

Here, we present examples of such quadripartite process functions. Consider four parties in local regions of space time in the presence of CTCs. We define input variables $a_1$, $a_2$, $a_3$, $a_4 \in \{0, 1\}$. We define output variables $x_1$, $x_2$, $x_3$, $x_4 \in \{0, 1\}$. We define the binary addition operator $a \oplus b$: $a, b \in \{0, 1\} \rightarrow \{0, 1\}$ as

$$a \oplus b = \begin{cases} 0, & a = b \\ 1, & a \neq b \end{cases}$$

Using the above notation, we define the quadripartite process function $w$: $(x_1, x_2, x_3, x_4) \rightarrow (a_1, a_2, a_3, a_4)$ as

$$a_1 = x_4(x_2 \oplus 1)(x_3 \oplus 1)$$
$$a_2 = x_1(x_4 \oplus 1)(x_3 \oplus 1)$$
$$a_3 = x_2(x_1 \oplus 1)(x_4 \oplus 1)$$
$$a_4 = x_3(x_2 \oplus 1)(x_1 \oplus 1).$$

Applying either Lemma 2 or Theorem 2, one can check that this is a valid process function. Equation (12) defines a process function in which the input of each region depends non-trivially on the output of the other three regions. In this process, the output of two regions sets the direction of signalling between the other two. For example, Table I displays the resulting inputs of regions 3 and 4 for all the possible combinations of the outputs of regions 1 and 2.

| Output of region 1 ($x_1$) | Output of region 2 ($x_2$) | Input of region 3 ($a_3$) | Input of region 4 ($a_4$) | Direction of signalling |
|---------------------------|---------------------------|---------------------------|---------------------------|------------------------|
| 0                         | 0                         | 0                         | $x_3$                     | 3 signals to 4         |
| 0                         | 1                         | $x_4 \oplus 1$            | 0                         | 4 signals to 3         |
| 1                         | 0                         | 0                         | 0                         | No signalling          |
| 1                         | 1                         | 0                         | 0                         | No signalling          |

Table I. Inputs of region 3 and region 4 (denoted $a_3$ and $a_4$ respectively), for every possible combinations of the outputs of region 1 and region 2 (denoted $x_1$ and $x_2$ respectively). The displayed signalling structure is true regardless of which two regions we choose to fix the outputs, due to the symmetry between different components of equation (12).

Here it is clear that, depending on the choice of outputs for an observer in region 1 and another observer in region 2, the communication between regions 3 and 4 can either be non-existent (neither region can signal to each other), or at most one way signalling. As a result, Equation 12 is characterised by conditional signalling. For example, if the outputs of regions 1 and 2 are chosen to be $x_1 = 1$, $x_2 = 0$ respectively, then neither one of the observers in regions 3 and 4 can signal to each other. However, if the output of region 1 and region 2 are chosen to be $x_1 = 0$ and $x_2 = 0$ respectively, then an observer in region 3 can signal to an observer in region 4. Crucially, there exists combinations of outputs such that each observer can signal to an observer in another region.

It was found in Ref. [67] that, up to relabelling of parties or of inputs/outputs, there exists only one unique non trivial tripartite process function with binary inputs and outputs that is compatible with the presence of CTCs and incompatible with any causal order. In other words, all non-trivial tripartite functions compatible with the presence of CTCs are equivalent after a relabelling of parties or of states. However, this is not true for quadripartite process functions. There exists many quadripartite process functions that are not related to one another by relabelling of party or of states. An ex-

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2 This is not strictly true beyond binary state spaces (for example, the continuous variables example presented in Ref. [68] cannot be relabelled to be a three bits function). However, Theorem 1 implies that all non-causally-ordered tripartite process functions have the same causal structure, in the sense that the output space of each party can be divided in two subsets, with states within each subset corresponding to a fixed direction of signalling between the two other parties.
ample of a non-trivial quadripartite process function not related to the one given in equation (12) up to relabelling is

\[
a_1 = x_2(x_3 \oplus x_4) \\
a_2 = x_3(x_4(x_1 \oplus 1) \oplus 1) \\
a_3 = x_4(x_1 \oplus 1)(x_2 \oplus 1) \\
a_4 = x_1(x_2 \oplus 1)(x_3 \oplus 1).
\] (13)

It is easy to see that the signalling structure between the four components of equation (13) is different from the signalling structure between the four components of equation (12). For example, if we fix the outputs of regions 2 and 3 to be either \(x_2 = 1\), \(x_3 = 0\) or \(x_2 = 1\), \(x_3 = 1\), then an observer in region 4 can signal to an observer in region 1. As a result, equation (13) has produced a scenario in which there are two distinct choices for outputs for two components of the process function which result in the same signalling direction (region 4 to region 1). This scenario does not exist for any choices for the outputs of two distinct components of the process function described by equation (12).

We have shown that there exists distinct non-trivial quadripartite process functions which are compatible with the presence of CTCs and incompatible with any causal order. A numerical search for other quadripartite process functions satisfying theorem 2 revealed a large number of non-equivalent quadripartite process functions. In this paper we have presented two examples of such process functions. In comparison to tripartite process functions, quadripartite process functions allow a greater variety in the ways different regions can communicate without causal order in the presence of CTCs.

As demonstrated in Ref. [68], every process function can be extended to a reversible one, which in turn means they can be realised by reversible physical processes, such as bouncing billiard balls [69]. This suggests that the abstract examples we presented can indeed arise from solutions of dynamical equations in an appropriate geometry. (Binary variables would correspond to subsets of the state space of the dynamical model.)

V. CONCLUSIONS

We have developed a characterisation of deterministic processes in the presence of CTCs for an arbitrary number of localised regions. Our proofs have demonstrated that non-trivial time travel between multiple regions is consistent with the absence of a logical paradox as long as once the outputs of all but two regions are fixed, at most one-way signalling is possible.

The most significant result of our work is our discovery of distinct non-trivial quadripartite process functions which are compatible with the presence of CTCs. This demonstrates that when multiple local regions communicate with each other in the presence of CTCs, there is a broad range of communication scenarios which still allow freedom of choice for observers in each region without the development of a logical inconsistency such as a grand father paradox. The range of distinct communication scenarios which are consistent with the presence of CTCs proves that the way CTCs allow multiple observers in distinct regions to communicate is not incredibly restricted by a conflict with locality, free will and logical inconsistencies. As a result, we have demonstrated that there is a range of scenarios in which multiple observers can communicate without causal order in a classical framework.

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