I. INTRODUCTION

Plasmon spectrum and polarization operator of 1, 2, and 3 dimensional electron gas are calculated by $T = 0$ Green function technique. It is shown that this field theory method gives probably the simplest pedagogical derivation of the statistical problem for the response function. The explanation is complimentary to the standard courses on condensed matter and plasma physics of the level of IX volume of Landau-Lifshitz encyclopedia on theoretical physics.

II. MODEL

The dielectric formalism gives us the equation for the dielectric polarizability

$$\varepsilon_r(\Omega, \mathbf{q}) = 1 - V(\Omega, \mathbf{q})\Pi(\Omega, \mathbf{q}).$$

In that expression $V(\Omega, \mathbf{q})$ means photon propagator, which in non-relativistic approximation

$$v_{gr} = \frac{\partial \omega}{\partial k} \ll c$$
is the Fourier transformation of the Coulomb potential
\[ V^{(2D)}(\Omega, q) \approx \frac{2\pi e^2}{q}, \quad V^{(3D)}(\Omega, q) \approx \frac{4\pi e^2}{q^2}. \] 

The polarization operator \( \Pi \) is the functional derivative of the electron density with respect to small perturbations of the electric potential. The diagram rules described in Ref. [1], section 13 give for the simple electron loop
\[ \Pi(\Omega, q) = (-i)(-1) \int_P iG_{\alpha\beta}(P)G_{\beta\alpha}(P + Q) \frac{d^{D+1}P}{(2\pi)^{D+1}}, \] 
where \( G_{\alpha\beta}(P) \) are the Green functions of degenerated electron gas
\[ G_{\alpha\beta}(P) = \delta_{\alpha\beta}G(P) = \frac{\delta_{\alpha\beta}}{\omega - \eta_p(1 - i0)}. \] 

We follow standard notations: \( \eta_p = \varepsilon_p - \mu \), \( \mu \) is the Fermi energy, \( \varepsilon_p = p^2/2m \) is the electron kinetic energy and \( P = (\omega, p) \), \( Q = (\Omega, q) \). The dispersion of longitudinal plasma waves \( \Omega(q) \) is the solution of
\[ \varepsilon_l(\Omega, q) = 0. \] 

The first step is calculating the polarization operator. Using
\[ G(P)G(P + Q) = \frac{1}{\omega - \eta_p(1 - i0)} \frac{1}{\omega + \Omega - \eta_{p+q}(1 - i0)} = \frac{1}{\eta_p - \eta_{p+q} + \Omega \left[ \frac{1}{\omega - \eta_p(1 - i0)} - \frac{1}{\omega + \Omega - \eta_{p+q}(1 - i0)} \right]^t}, \] 
the \( \omega \) integration gives the electron filling factors
\[ n_p = \frac{1}{2\pi i} \int \frac{d\omega}{\omega - \eta_p(1 - i0)} = \theta(p_F - p). \] 

In such a way we obtain
\[ \Pi = -2i \int \frac{d^Dp}{(2\pi)^D} \int \frac{d\omega}{2\pi} G(P)G(P + Q) = \]
\[ = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{n_p - n_{p+q}}{\eta_p - \eta_{p+q} + \Omega + i0} = I_1 + I_2, \] 
where the evanescent imaginary part of the frequency corresponds to the Landau rule for adiabatic appearance of the perturbation, see Ref. [2] secs. 29, 40, Eqs. (29.6), (40.15). Here for brevity we use the notations
\[ I_1(\Omega, q) = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{n_p}{\eta_p - \eta_{p+q} + \Omega}, \quad I_2(\Omega, q) = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{n_p}{\frac{p^2}{2m} - \frac{(p+q)^2}{2m} + \Omega}. \]
$$I_2(\Omega, q) = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{-n_{p+q}}{\eta_p - \eta_{p+q} + \Omega} = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{-n_{p+q}}{\frac{p^2}{2m} - \frac{(p+q)^2}{2m} + \Omega} =$$

$$= 2 \int \frac{d^Dp}{(2\pi)^D} \frac{n_{p+q}}{\frac{p^2}{2m} - \frac{(p+q)^2}{2m} - \Omega}. \tag{10}$$

We change the variables $p' = p + q$. Then $dp = dp'$, $n_{p+q} = n_{p'}$ and

$$I_2(\Omega, q) = 2 \int \frac{d^Dp}{(2\pi)^D} \frac{n_p}{\frac{p^2}{2m} - \frac{(p-q)^2}{2m} - \Omega} = I_1(-\Omega, -q). \tag{11}$$

The general formula (8) for the response function of the electron gas is given in many textbooks on statistical and solid state physics.

In the next section we will analyze the simple case of $D = 1$ which is discussed in [3].

### III. 1-DIMENSIONAL CASE

For the case of $D = 1$ we assume $q \geq 0$, and

$$I_1 = 2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{n_p}{\frac{p^2}{2m} - \frac{(p+q)^2}{2m} + \Omega} = \frac{2m}{\pi} \int_{-\infty}^{\infty} \frac{\theta(p_F - |p|)dp}{2m \Omega - q^2 - 2pq} =$$

$$= \frac{2m}{\pi} \int_{-p_F}^{p_F} \frac{dp}{2m \Omega - q^2 - 2pq} = -\frac{m}{\pi q} \int_{-p_F}^{p_F} \frac{dp}{p + \left(\frac{q^2}{2} - \frac{m\Omega}{q}\right)} - i0. \tag{12}$$

Then

$$I_1 = \frac{m}{\pi q} \ln \left|\frac{2m \Omega - q^2 + 2p_F q}{2m \Omega - q^2 - 2p_F q}\right| - \frac{im}{q} \theta \left(1 - \left|\frac{q}{2p_F} - \frac{m\Omega}{p_F q}\right|\right). \tag{13}$$

The velocity on Fermi level is $v_F = p_F/m$, and

$$I_1 = \frac{m}{\pi q} \ln \left|\frac{\Omega - \frac{q^2}{2m} + v_F q}{\Omega - \frac{q^2}{2m} - v_F q}\right| - \frac{im}{q} \theta \left(1 - \left|\frac{q}{2p_F} - \frac{m\Omega}{p_F q}\right|\right), \tag{14}$$

$$I_2 = \frac{m}{\pi q} \ln \left|\frac{\Omega + \frac{q^2}{2m} - v_F q}{\Omega + \frac{q^2}{2m} + v_F q}\right| + \frac{im}{q} \theta \left(1 - \left|\frac{q}{2p_F} + \frac{m\Omega}{p_F q}\right|\right). \tag{15}$$

Owing to the symmetry of the polarization operator

$$\Pi(Q) = \Pi(-Q),$$
we obtain

\[
\Pi = \frac{m}{\pi q} \ln \left| \frac{(\Omega - \frac{q^2}{2m} + v_F q)(\Omega + \frac{q^2}{2m} - v_F q)}{(\Omega - \frac{q^2}{2m} - v_F q)(\Omega + \frac{q^2}{2m} + v_F q)} \right| - \frac{im}{q} \left[ \theta \left( 1 - \left| \frac{q}{2p_F} - \frac{m\Omega}{p_F q} \right| \right) - \theta \left( 1 - \left| \frac{q}{2p_F} + \frac{m\Omega}{p_F q} \right| \right) \right] =
\]

\[
= \frac{m}{\pi q} \ln \left| 1 - \left( \frac{\frac{q^2}{2m} - v_F q}{\Omega} \right)^2 \right| - \frac{im}{q} \left[ \theta \left( 1 - \left| \frac{q}{2p_F} - \frac{m\Omega}{p_F q} \right| \right) - \theta \left( 1 - \left| \frac{q}{2p_F} + \frac{m\Omega}{p_F q} \right| \right) \right].
\]

(16)

In hydrodynamic approximation (long waves and phase velocity much larger than Fermi velocity)

\[
\frac{q^2}{2m} \ll \Omega, \quad q \to 0, \quad v_F \ll \frac{\Omega}{q} = v_\varphi,
\]

we use

\[(1 + \epsilon)^n \approx 1 + n\epsilon \quad \text{for} \quad \epsilon \ll 1.\]

Then

\[
\Pi \approx \frac{m}{\pi q} \ln \left| 1 - \left( \frac{\frac{q^2}{2m} - v_F q}{\Omega} \right)^2 \right| + \left| 1 + \left( \frac{\frac{q^2}{2m} + v_F q}{\Omega} \right)^2 \right| \approx
\]

\[
\approx \frac{m}{\pi q} \ln \left| 1 - \left( \frac{\frac{q^2}{2m} - v_F q}{\Omega} \right)^2 \right| + \left( \frac{\frac{q^2}{2m} + v_F q}{\Omega} \right)^2 \approx
\]

\[
= \frac{m}{\pi q} \ln \left| 1 + \frac{2v_F q^3}{m\Omega^2} \right|.
\]

(18)

Using

\[
\ln(1 + \epsilon) \approx \epsilon \quad \text{for} \quad \epsilon \ll 1,
\]

we get

\[
\Pi \approx \frac{m}{\pi q} \frac{2v_F q^3}{m\Omega^2} =
\]

\[
= \frac{2p_F q^2}{\pi m\Omega^2}.
\]

(19)

Let us introduce D-dimensional density of states

\[
n^{(D)} = 2 \frac{V_F}{(2\pi)^D}, \quad V_F(p) = \frac{\pi^{D/2}}{(D/2)!} p^D, \quad V_F(p_F) = 2p_F \quad \text{for} \quad D = 1,
\]

(20)

where \(V_F\) is the volume of the Fermi sphere. In our case

\[
n^{(1D)} = 2 \frac{2p_F}{2\pi} = \frac{2p_F}{\pi}.
\]

(21)
Then
\[ \Pi = \frac{q^2 n^{(1D)}}{m\Omega^2}. \]  
(22)

Calculating the Fourier transformation, it can be shown [3, 4, 5] that the photon propagator in long wavelength approximation \((q \to 0)\) obeys
\[ V^{(1D)}(\Omega, q) \approx 2e^2 \ln \left( \frac{1}{qa} \right), \]  
(23)
where \(a \geq 0, \ a \to 0\) is a regularization parameter. More precisely
\[ V^{(1D)}(\Omega, q) = e^2 \int_{-\infty}^{\infty} \frac{e^{ix} dx}{|x|}. \]  
(24)

This integral is not convergent – we have to regularize it introducing a small diameter of the wire.

We give two schemes:

1) \[ V^{(1D)}(\Omega, q) = e^2 \int_{-\infty}^{\infty} \frac{e^{ix} dx}{\sqrt{x^2 + a^2}} = -\frac{\pi e^2}{2} \left( N_0(iaq) + N_0(-iaq) \right) \approx 2e^2 \ln \left( \frac{2}{\gamma qa} \right), \]  
(25)

2) \[ V^{(1D)}(\Omega, q) = 2e^2 \int_{a}^{\infty} \frac{\cos(x) dx}{x} = -2e^2 \text{Ci}(qa) \approx -2e^2(\gamma + \ln(qa)) = 2e^2 \ln \left( \frac{1}{e^\gamma qa} \right). \]  
(26)

Here \(\gamma\) is Euler’s constant. As it is written above, both schemes yield
\[ V \approx 2e^2 \ln \left( \frac{1}{qa} \right). \]

In order to find the dispersion we have to solve the equation (27)
\[ V^{(1D)}(\Omega, q) = \frac{2e^2 q^2 n^{(1D)}}{m\Omega^2} \ln \left( \frac{1}{qa} \right) = 1. \]  
(27)

As a result, for dispersion of plasma waves we obtain
\[ \Omega^2(q) = \frac{2e^2 n^{(1D)}}{m} \ln \left( \frac{1}{qa} \right) q^2, \]  
(28)
in agreement with the derivation in [3].

In the next section we will discuss the case of \(D = 2\) which is for first time was discussed in [6].

IV. 2-DIMENSIONAL CASE

For the case of \(D = 2\) we introduce polar variables
\[ p = p(\cos \theta, \sin \theta), \quad q = q(1, 0), \]
\[ d^2 p = pdp\theta, \quad p \cdot q = pq \cos \theta, \]
and

$$I_1 = 2 \int \frac{d^2 p}{(2\pi)^2} \frac{n_p}{\frac{p^2}{2m} - \frac{(p+q)^2}{2m} + \Omega} = \frac{4m}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \frac{p \theta (p_F - p) dp d\theta}{2m\Omega - q^2 - 2p q \cos \theta}. \quad (29)$$

If we use

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a - b \cos \theta} = \frac{\text{sign}(a)}{\sqrt{a^2 - b^2}}, \quad (30)$$

then

$$I_1 = \frac{2m}{\pi} \text{sign}(2m\Omega - q^2) \int_0^{p_F} \frac{p dp}{\sqrt{(2m\Omega - q^2)^2 - 4p^2q^2}} = \frac{m \text{sign}(2m\Omega - q^2)}{2qa^2} \left[ |2m\Omega - q^2| - \sqrt{(2m\Omega - q^2)^2 - 4p^2q^2} \right]. \quad (31)$$

The velocity on Fermi level is $v_F = p_F/m$, and

$$I_1(\Omega, q) = \frac{m^2}{\pi a^2} \left[ \Omega - \frac{q^2}{2m} - \text{sign}(\Omega - \frac{q^2}{2m}) \sqrt{\left( \Omega - \frac{q^2}{2m} \right)^2 - q^2v_F^2} \right], \quad (32)$$

$$I_2(\Omega, q) = \frac{m^2}{\pi a^2} \left[ -\Omega - \frac{q^2}{2m} + \text{sign}(\Omega + \frac{q^2}{2m}) \sqrt{\left( \Omega + \frac{q^2}{2m} \right)^2 - q^2v_F^2} \right]. \quad (33)$$

Owing to the symmetry of the polarization operator

$$\Pi(Q) = \Pi(-Q),$$

we obtain for big enough frequency

$$\Pi = \frac{m^2}{\pi a^2} \left[ -\frac{q^2}{m} + \sqrt{\left( \frac{q^2}{2m} \right)^2 - q^2v_F^2} - \sqrt{\left( \Omega - \frac{q^2}{2m} \right)^2 - q^2v_F^2} \right]. \quad (34)$$

Working in hydrodynamic approximation (17), we introduce the notation

$$\epsilon = \frac{\Omega q^2/m}{\Omega^2 + \left( \frac{q^2}{2m} \right)^2 - q^2v_F^2} \ll 1.$$ 

In these terms we have for the polarization operator

$$\Pi = \frac{m^2}{\pi a^2} \left[ -\frac{q^2}{m} + \sqrt{\Omega^2 + \left( \frac{q^2}{2m} \right)^2 - q^2v_F^2} \left( \sqrt{1 + \epsilon} - \sqrt{1 - \epsilon} \right) \right]. \quad (35)$$

Using

$$\sqrt{1 + \epsilon} - \sqrt{1 - \epsilon} \approx \epsilon,$$
we arrive at
\[
\Pi = \frac{m^2}{\pi q^2} \left[ -\frac{q^2}{m} + \frac{\Omega q^2 / m}{\sqrt{\Omega^2 + (\frac{q^2}{2m})^2 - q^2 v_F^2}} \right] = \frac{m}{\pi} \left[ \frac{\Omega}{\sqrt{\Omega^2 - q^2 v_F^2}} - 1 \right]
\]
(36)
For the 2D case the electron density
\[
n^{(2D)} = \frac{2\pi p_F^2}{(2\pi)^2} = \frac{p_F^2}{2\pi}.
\]
(37)
Then
\[
\Pi = \frac{q^2 n^{(2D)}}{m \Omega^2}.
\]
(38)
Again for the dispersion we solve the equation (5)
\[
V \Pi = \frac{2\pi e^2 q n^{(2D)}}{m \Omega^2} = 1.
\]
(39)
As a result for dispersion of 2D plasma waves, we get
\[
\Omega^2(q) = \frac{2\pi e^2 n^{(2D)}}{m} q.
\]
(40)
Another important case is the static one (\(\Omega = 0\)) for 2D electron gas where for \(q \leq 2p_F\) we have \(\text{Re}(\Pi) = -\frac{m}{\pi}\) [6].

In the next section we will discuss the \(D = 3\) case which is described in the monograph by Abrikosov, Gor’kov and Dzyaloshinsky [7].

V. 3-DIMENSIONAL CASE

Analogously for \(D = 3\) we introduce spherical coordinates
\[
\mathbf{p} = p(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \mathbf{q} = q(0, 0, 1), \quad d^3p = p^2 \sin \theta \; dp \; d\theta \; d\varphi, \quad \mathbf{p} \cdot \mathbf{q} = pq \cos \theta.
\]
Then (we are not interested in the imaginary part)

\[ I_1 = 2 \int \frac{d^3p}{(2\pi)^3} \left( \frac{p^2}{2m} - \frac{q^2}{2m} + \Omega \right) \]
\[ = \frac{4m}{(2\pi)^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi p^2 (p_\Omega - \Omega) \sin \theta \, dp \, d\theta \, dp' \]
\[ = \frac{m}{2\pi^2 q} \int_0^{p_F} p \ln \left| \frac{2m\Omega - q^2 + 2p q}{2m\Omega - q^2 - 2p q} \right| \, dp. \]  

(41)

To simplify the notations we introduce \( a = 2m\Omega - q^2 \), \( b = 2q \), so

\[ \int_0^{p_F} p \ln \left| \frac{a + bp}{a - bp} \right| \, dp = \frac{1}{2} p_F^2 \ln \left| \frac{a + bp_F}{a - bp_F} \right| - ab \int_0^{p_F} \frac{p^2 \, dp}{a^2 - b^2 p^2} \]
\[ = \left( \frac{1}{2} p_F^2 - \frac{a^2}{b^2} \right) \ln \left| \frac{a + bp_F}{a - bp_F} \right| - \frac{ab p_F}{b} \]  

(42)

and

\[ I_1 = \frac{m}{4\pi^2 q} \left( p_F^2 - \frac{(2m\Omega - q^2)^2}{4q^2} \right) \ln \left| \frac{2m\Omega - q^2 + 2p_F q}{2m\Omega - q^2 - 2p_F q} \right| \]
\[ + (2m\Omega - q^2) \frac{mp_F}{4\pi^2 q^2}. \]

\[ I_2 = -\frac{m}{4\pi^2 q} \left( p_F^2 - \frac{(2m\Omega + q^2)^2}{4q^2} \right) \ln \left| \frac{2m\Omega + q^2 + 2p_F q}{2m\Omega + q^2 - 2p_F q} \right| \]
\[ - (2m\Omega + q^2) \frac{mp_F}{4\pi^2 q^2}. \]  

(43)

(44)

Finally we obtain

\[ \Pi = \frac{m}{4\pi^2 q} \left( p_F^2 - \frac{(2m\Omega - q^2)^2}{4q^2} \right) \ln \left| \frac{2m\Omega - q^2 + 2p_F q}{2m\Omega - q^2 - 2p_F q} \right| \]
\[ - \frac{m}{4\pi^2 q} \left( p_F^2 - \frac{(2m\Omega + q^2)^2}{4q^2} \right) \ln \left| \frac{2m\Omega + q^2 + 2p_F q}{2m\Omega + q^2 - 2p_F q} \right| - \frac{mp_F}{2\pi^2}. \]  

(45)

The Fourier transformation of the static polarization operator gives the well known from the physics of magnetism RKKY interaction. Technical details are given in Ref. 8.

Introducing the notations

\[ g(\Omega) = \frac{m(\Omega^2 - q^2 p_F^2)}{2q^2 V_F} \ln \frac{\Omega + q p_F}{\Omega - q p_F}, \quad \Omega_\pm = \Omega \pm q^2 / 2m, \]  

(46)

\[ \Omega_c = \sqrt{4\pi n(3D)c^2 / m}, \quad n^{(3D)} = 2\pi^2 p_F^3 / 3\pi^2, \]  

(47)

the longitudinal polarizability takes the form

\[ \epsilon_l(\Omega, q) - 1 = \frac{3\Omega^2}{2q^2 V_F^2} \left( 1 - g(\Omega_+) + g(\Omega_-) \right). \]  

(48)
As we did before, we use the hydrodynamical approximation (17), and

\[
\ln \left| \frac{1 + x}{1 - x} \right| \xrightarrow{x \to 0} 2x + \frac{2}{3}x^3, \quad x = \frac{2p_F q}{2m\Omega \pm q^2}.
\]

That is why equation (45) transforms into

\[
\Pi = \frac{m}{4\pi^2q} \left( p_F^2 - \frac{(2m\Omega - q^2)^2}{4q^2} \right) \left( \frac{4p_F q}{2m\Omega - q^2} + \frac{16p_F^3q^3}{3(2m\Omega - q^2)^3} \right) - \frac{mp_F}{4\pi^2q} \left( p_F^2 - \frac{(2m\Omega + q^2)^2}{4q^2} \right) \left( \frac{4p_F q}{2m\Omega + q^2} + \frac{16p_F^3q^3}{3(2m\Omega + q^2)^3} \right) - \frac{mp_F}{2\pi^2}
\]

\[
= \left( 1 - \frac{1}{3} \right) \frac{mp_F^3}{\pi^2} \left( \frac{1}{2m\Omega - q^2} - \frac{1}{2m\Omega + q^2} \right) + \frac{4mp_F^3q^2}{3\pi^2} \left( \frac{1}{(2m\Omega - q^2)^3} - \frac{1}{(2m\Omega + q^2)^3} \right) = \frac{p_F^3q^2}{3\pi^2m\Omega^2}.
\]

Taking into account the density of states we can write the polarization operator in another form

\[
\Pi = \frac{q^2n^{(3D)}}{m\Omega^2}.
\]

This hydrodynamic asymptotic is valid for arbitrary dimensions.

Turning back in (5)

\[
V\Pi = \frac{4\pi e^2 q^2 n^{(3D)}}{q^2} = 1,
\]

we finally obtain the well-known result for bulk plasma oscillations

\[
\Omega^2(q) = \frac{4\pi e^2 n^{(3D)}}{m} = \text{const.}
\]

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