Uniformly frustrated Josephson junction array in trapped Bose-Einstein condensates

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Abstract. We investigate the ground state structure of two-dimensional Josephson junction arrays consisting of Bose-Einstein condensates trapped by both a rotating harmonic potential and a corotating deep optical lattice with square symmetry. Monte Carlo simulations of the uniformly frustrated XY model in a wide range of the frustration parameter reveal that the stable configuration of vortices for a simple fractional number is a staircase structure, consisting of diagonal chains of vortices with periodic lattice of unit cells.

1. Introduction
Ultracold Bose gases in an optical lattice (OL) provide an ideal testing ground for the study of many-body effects associated with the model Hamiltonian in conventional condensed matter systems [1]. Bose-Einstein condensates (BECs) loaded in a deep OL form a bosonic Josephson junction array (JJA) [2, 3], where many condensates are localized at potential minima and there is small overlap of the wave function between the nearest neighbor sites. Recently, a two-dimensional (2D) JJA of BECs was studied, demonstrating thermally-activated vortex generation associated with the Berezinskii, Kosterlitz and Thouless mechanism [4, 5].

In this work, we study the rotation effect on the 2D bosonic JJA. Rotation of a 2D shallow OL can be achieved experimentally [6] and studied theoretically [7, 8, 9]. If the amplitude of the OL is strong, the situation is analogous to the superconducting JJA in a uniform magnetic field, which can be modeled by the uniformly frustrated XY model (UFXYM) [10, 11, 12, 13]

\[ H = -J \sum_{\langle jj' \rangle} \cos(\theta_j - \theta_{j'} + A_{jj'}). \]  

Here, \( J > 0 \) denotes the coupling constant, \( \theta_j \) the phase of the superconducting node at a site \( j \), and \( \langle jj' \rangle \) near neighbors. The vector potential \( A_{jj'} \) satisfies the constraint \( \sum A_{jj'} = 2\pi f \) with the frustration parameter \( f \), where the summation is taken over the perimeter of a plaquette. In the case of the superconductor, \( f \) is the magnetic flux piecing the plaquette in units of the flux quantum. The competition of two length scales — the mean separation of magnetic fluxes (vortices) induced by the frustration and the period of underlying lattice — yields a rich variety of the ground state structures and phase transitions [10, 11, 12, 13].

The 2D bosonic JJA can be also mapped to the UFXYM [14]. Since actual BECs are confined by both an OL and a harmonic potential, the coupling constant \( J \) becomes spatially inhomogeneous. We study the ground state structure of the 2D rotating bosonic JJA in a
Then, bosons can be described by the Bose-Hubbard model [15]. The lowest and first excited band, the particles are confined to the lowest Wannier orbitals. When the energy due to interaction and rotation is small compared to the energy gap between the lowest and first excited band, the particles are confined to the lowest Wannier orbitals. Then, bosons can be described by the Bose-Hubbard model [15].

\[ \hat{H} = -\sum_{\langle j,j' \rangle} \frac{t_{jj'}}{2}(\hat{a}_j \hat{a}_{j'} e^{-iA_{j,j'}} + \text{h.c.}) + \sum_j E_j \hat{N}_j + \sum_j \frac{U_j}{2} \hat{N}_j(\hat{N}_j - 1), \]

(2)

where \( \hat{a}_j \) (\( \hat{N}_j = \hat{a}_j \hat{a}_j^\dagger \)) is the boson annihilation (number) operator at the \( j \)th site, and the minimum points of the 2D OL is \( jd = (j_x,j_y)d \) with the integers \( j_x, j_y \) and the lattice constant \( d \). The parameters \( t_{jj'}, E_j, U_j \) represent the hopping matrix element, the energy offset of each lattice site, and the on-site energy, respectively, being calculated from the Wannier functions \( w_j(r) \) localized at the \( j \)th site. The effect of rotation is described by the vector potential \( A_{j,j'} = (m/\hbar) \int_{r_j}^{r_{j'}} A(r') \cdot dr' \) with \( A = \Omega \times r \) and the constraint \( \sum A_{j,j'} = 2\pi f \), where the sum is taken around any unit cell of the 2D array. The constant \( f \) is the frustration parameter, being given by the average number of vortices per unit cell:

\[ f = \frac{m}{2\pi \hbar} \int A \cdot dr = \frac{1}{\kappa} \int_{\text{plaq}} \left[ \nabla \times A \right] \cdot ndS = \frac{2\Omega d^2}{\kappa} \]

(3)

with the quantum circulation \( \kappa = \hbar/m \). The total particle number is fixed as \( N = \sum_j N_j \).

If the number of atoms per site is large (\( N_j \gg 1 \)), the operator can be expressed in terms of its amplitude and phase, the amplitude being subsequently approximated by the \( c \)-number as \( \hat{a}_j \simeq \sqrt{N_j} e^{i\hat{\theta}_j} \). Then, Eq. (2) reduces to the UFXYM

\[ \hat{H} = -\sum_{\langle j,j' \rangle} J_{j,j'} \cos(\theta_j - \theta_{j'} + A_{j,j'}) \]

(4)

where we have used the phase representation \( \hat{N}_j = N_j - i\partial/\partial \theta_j, \hat{\theta}_j = \theta_j \) and the notation \( J_{j,j'} = t_{jj'} \sqrt{N_jN_{j'}} \). This reduction is valid when the Josephson regime \( J_{j,j'}/N_j^2 \ll U_j \) and \( J_{j,j'} \ll U_j \) is satisfied [5]. To obtain Eq. (4), we have made a couple of approximations and assumptions: (a) We have omitted the energy of the equilibrium state to write down the form as Eq. (4). (b) The equilibrium density is determined so as to satisfy \( E_j + U_j N_j = 0 \), which is the minimizing condition of the energy stated in (a). (c) The Wannier functions can be decomposed as \( w_j(r) = u_0(x-j_xd, y-j_yd) v_j(z) \) with the site-independent transverse part \( u_0(x,y) \) and the site-dependent longitudinal part \( v_j(z) \) [16]. Since the atoms are tightly confined by 2D OL, the contribution arising from the two-body interactions is negligible to evaluate \( u_0(x,y) \). The longitudinal part \( v_j(z) \) is approximated by the Thomas-Fermi form with an inverted parabolic profile. Using the normalization condition \( \int w_j(r)^2dr = 1 \) and \( \sum_j N_j = N \), we can obtain

\[ N_j = \frac{5N}{2\pi J_{j,\max}^2} \left( 1 - \frac{j_x^2 + j_y^2}{J_{j,\max}^2} \right)^{3/2}. \]

(5)
Here, \( N_j \) = 0 for \( |j| > j_{\max} \) because of the harmonic confinement. Using these \( N_j \) and \( w_j(r) \), we confirm that the conditions \( J_{j,j'}/N_j^2 \ll U_j \) and \( J_{j,j'} \gg U_j \) are certainly satisfied. More detailed derivation will be reported elsewhere.

3. Simulation results

We perform Monte Carlo simulations of the Hamiltonian Eq. (4) to study the ground state properties of this system as a function of the frustration parameter \( f \). We use the Metropolis algorithm and choose the symmetric gauge for the vector potential \( A_{j,j'} \). The temperature is gradually decreased from high temperatures to zero according to the stimulated annealing. Since there are many metastable state caused by the frustration, we change the annealing rates in the several hundred simulations, taking the steady solution with the lowest energy as the ground state.

It is known that the UFXYM of Eq. (1) exhibits rich ground state structures depending on the parameter \( f \) \([10, 11, 12]\). For rational \( f = p/q \), the ground state is periodic on the \( q \times q \) cell in most cases. The striking difference of Eq. (1) and Eq. (4) of the bosonic JJA is the inhomogeneous coupling \( J_{j,j'} \propto \sqrt{N_j N_j'} \). Also, it should be noted that the range of \( f \) is restricted by the harmonic potential because rotation frequency \( \Omega \) cannot exceed \( \omega_\perp \), that is, \( f < \omega_\perp /\pi a_\perp^2 = 0.78 \) in our case.

Figure 1 represents the total energy and the typical vortex patterns of the ground state as a function of \( f \). The energy curve has a non-monotonic behavior characterized by some minima at the simple rational values. These features are reflected in the bottom edge of Hofstadter butterfly spectrum \([12]\). Figures 1(a)-(l) represent the ground state for some values of \( f \) at the simple rational values. These features are reflected in the bottom edge of Hofstadter function of lattice with a unit cell of vortex configurations at these minima possess simple periodic structures, forming a Bravais lattice.

Between these energy minima, we obtain characteristic intermediate structures consisting of the domains of simple periodic Bravais lattice. Since the ground state has typically \( q \times q \) periodic unit cell, it is difficult to obtain the periodic structure for large \( q \) in the finite-size system. The periodicity is easily broken near the condensate edge due to the weak couplings \([17]\), the structural change being of crossover. This is contrast to the homogeneous model Eq. (1) where the vortex patterns and accompanying domain walls form diagonal lines for a square lattice. This broken periodicity does not become noticeable as \( f \) increases, because the system size expands due to the centrifugal effect and approaches the homogeneous limit. For \( 1/3 < f < 1/2 \) the results reproduce the results obtained by the Coulomb gas model \([13]\). They consist of diagonal domains of the \( f = 1/2 \) checkerboard configuration, separated by domain walls (or domains) of the \( f = 1/3 \) structure. For \( f > 0.425 \), the ground-state structures are the \( f = 1/2 \) checkerboard pattern with a low concentration of missing vortices. The energy is reflection symmetric about \( f = 1/2 \), and the periodic structure for \( f > 1/2 \) is equivalent to those of \( 1 - f \), but the condensate size is expanded and vortices are replaced by “vacancies”; an example is shown in Fig. 1 (l).

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Figure 1. Ground state energy and vortex lattice structures in a bosonic JJA under rotation. Top panel shows the total energy (normalized by $J_{(0,0)/(1,0)}$ for $f = 0$) as a function of $f$. The bottom panels from (a) to (l) represent the discretized condensate density $N_{j,j'}$ (black-white contour plot) and the positions of vortices marked by red or gray circles. Each square in the density corresponds to the site (minima of the OL) and vortices are located at the corners of the squares (maxima of the OL). The positions of vortices are calculated by the current circulation $P \sin(\theta_j - \theta_{j'} + A_{j,j'})$ with the plaquette sum. Following the typical experimental conditions such as $^{87}$Rb atoms used in the JILA experiments [4, 6], the parameter values are $N = 6 \times 10^5$, a s-wave scattering length $a = 5.29$ nm, trapping frequencies $\omega_\perp = 11.5 \times 2\pi$ Hz, $\omega_z = 50 \times 2\pi$ Hz, an amplitude of OL $V_0 = 65\hbar \omega_\perp$, and a length of one side of the square $d = 5 \mu$m.

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