FOURIER MULTIPLIERS ON ANISOTROPIC MIXED-NORM SPACES OF DISTRIBUTIONS

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Abstract. A new general Hörmander type condition involving anisotropies and mixed norms is introduced, and boundedness results for Fourier multipliers on anisotropic Besov and Triebel-Lizorkin spaces of distributions with mixed Lebesgue norms are obtained. As an application, the continuity of such operators is established on mixed Sobolev and Lebesgue spaces too. Some lifting properties and equivalent norms are obtained as well.

1. Introduction

The study of spaces of functions and distributions and operators on such spaces play an essential role in harmonic analysis. Several branches of both pure and applied mathematics make extensive use of such spaces, including in the study of partial differential equations, approximation theory, probability, statistics, and signal processing.

Some of the most general and applicable families of functions spaces in analysis are the Besov and Triebel-Lizorkin spaces. The two families are interesting in their own right, but their importance also stem from the fact that several of the classical function spaces such as Lebesgue, Hardy, BMO, Sobolev, and Hölder spaces can be recovered as special cases. The Besov and Triebel-Lizorkin spaces have been studied for many different reasons and in a variety of settings and circumstances. For further details we refer the reader to [7,9,12,20,22,25,26,37] and to the references found therein.

The purpose of this article is to study Fourier multipliers in the general setting of anisotropic Triebel-Lizorkin spaces based on mixed-norm Lebesgue spaces. Let us now elaborate further on this particular setting.

Anisotropic phenomena appear naturally in various fields of analysis, both pure and applied. A classical example found in [28] is the case of differential operators with anisotropic symbols. Such operators naturally introduce a need for anisotropic function classes containing functions compatible with the particular anisotropy. The notion of anisotropic Besov spaces goes back to Nikol’skiĭ [31] and the notion of anisotropic Triebel-Lizorkin spaces can be found in Triebel’s book [34, p. 269]. More recently, anisotropic Besov spaces in a more general setting have been studied by Bownik [7] and the anisotropic Triebel-Lizorkin spaces have been studied by Bownik and Ho [9].

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Mixed-norm Lebesgue spaces are useful as a framework for several problems arising in physics demanding different regularity in every direction (e.g., in time and in space). Mixed-norm Besov and Triebel-Lizorkin spaces have been studied during this decade by Johnsen and Sickel as one may see for example [20]. Here we will work on anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

Fourier multipliers form one of the fundamental and most important classes of operators in harmonic analysis. Their importance is emphasized by their close link to partial differential operators through the Fourier transform, and there has been a continuous interest in the study of boundedness properties of multipliers on $L^p$ and other spaces since the seminal work by Marcinkiewicz [29], Mihlin [30] and Hörmander [19]. Numerous variations and generalizations of the above works have been produced during the past years, see [1–3, 15, 27, 28, 36] and the references therein.

In this article we prove boundedness of a suitable class of Fourier multipliers on anisotropic mixed-norm Besov and Triebel-Lizorkin spaces with mixed Lebesgue norms. Moreover, we also introduce a new general Hörmander-type class of multipliers naturally adapted to mixed norms and to the general anisotropic setting.

Let us summarize the main contributions in this paper.

(α) We introduce a new and general condition, involving mixed norms and the anisotropy, for Hörmander multipliers, see [3–23].

(β) We prove the boundedness of Fourier multipliers on anisotropic mixed-norm Besov and Triebel-Lizorkin space, see Theorem 3.6.

(γ) The continuity of Fourier multipliers on mixed Lebesgue and Sobolev spaces will be obtained, under the new condition as well, see Corollaries [4, 14, 15, 16].

(δ) An equivalent norm characterization for the anisotropic mixed-norm Besov and Triebel-Lizorkin spaces is revisited, see Corollary 4.3.

Notation. We will denote by $\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ the Fourier transform of (suitably nice) $f$, where $x\cdot\xi := x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product on $\mathbb{R}^n$. The inverse Fourier transform is then given by $\mathcal{F}^{-1}f(x) := \hat{f}(-x)$. For $\vec{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ with $t_1, \ldots, t_n \neq 0$, we set $\frac{1}{\vec{t}} := (\frac{1}{t_1}, \ldots, \frac{1}{t_n})$. The sets of positive and non-negative integers will be denoted by $\mathbb{N}$ and $\mathbb{N}_0$ respectively. For $\gamma$ a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n_0$ we denote by $|\gamma| := \gamma_1 + \cdots + \gamma_n$ its length and we set $\partial^\gamma f := \partial_1^{\gamma_1}\cdots\partial_n^{\gamma_n} f$. By $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz class on $\mathbb{R}^n$ and by $\mathcal{S}'$ its dual; the tempered distributions. The $N \in \mathbb{N}_0$ times differential functions on $\mathbb{R}^n$ is denoted by $\mathcal{C}^N$. Finally, any positive constant will be denoted $c$, or as $c_\alpha$ if it depends on a significant parameter $\alpha$.

2. Preliminaries

2.1. Mixed norm Lebesgue spaces. Let $\vec{p} = (p_1, \ldots, p_n)$, with $0 < p_1, \ldots, p_n \leq \infty$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that $f \in L^{\vec{p}} = L^{\vec{p}}(\mathbb{R}^n)$ if

$$
\|f\|_{\vec{p}} := \left( \int_{\mathbb{R}} \cdots \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1}dx_1 \right)^{p_2/p_1} \cdots dx_n \right)^{1/p_n} \right) < \infty,
$$

with the standard modification when $p_k = \infty$, for some $1 \leq k \leq n$. The quasi-norm $\|\cdot\|_{\vec{p}}$ is a norm when $\min(p_1, \ldots, p_n) \geq 1$ and turns $L^{\vec{p}}$ into a Banach space. For further properties of $L^{\vec{p}}$ see for example [13, 16, 27].
Let \( R \subset \mathbb{R}^n \). We denote by \( \|f\|_{L^p(R)} := \|f\chi_R\|_p \), where \( \chi_R \) is the characteristic function of \( R \). For example when \( R := I_1 \times \cdots \times I_n \subset \mathbb{R}^n \) is a rectangle, we obtain
\[
\|f\|_{L^p(R)} = \left( \int_{I_n} \cdots \left( \int_{I_2} \left( \int_{I_1} |f(x_1, \ldots, x_n)|^{p_1} \ dx_1 \right)^{\frac{p_2}{p_2}} \ dx_2 \right)^{\frac{p_3}{p_3}} \cdots \ dx_n \right)^\frac{1}{n}.
\]

For \( \tilde{p} \in [1, \infty]^n \) we define the conjugate \( \tilde{p} := (p_1', \ldots, p_n') \in [1, \infty]^n \) by requiring that \( 1/p_k + 1/p_k' = 1 \) for every \( k = 1, \ldots, n \).

The mixed Hölder-inequality (see e.g. [3]) is the following estimate: For every \( \tilde{p} = [1, \infty]^n \), \( f \in L^{\tilde{p}} \) and \( g \in L^p \) we have
\[
\left( \int_{\mathbb{R}^n} f(x)g(x)dx \right)^{\frac{1}{\tilde{p}}} \leq \|f\|_{p} \|g\|_{\tilde{p}}.
\]

We will also need an adapted version of the Hausdorff-Young inequality. The mixed Hausdorff-Young’s Theorem [3] asserts that if \( \tilde{t} = (t_1, \ldots, t_n) \) with \( 1 \leq t_1 \leq t_{n-1} \leq \cdots \leq t_1 \leq 2 \), then for every \( f \in \mathcal{S} \),
\[
\|\hat{f}\|_{\tilde{t}} \leq \|f\|_{\tilde{t}}.
\]

### 2.2. Anisotropic geometry
Let \( b, x \in \mathbb{R}^n \) and \( \lambda > 0 \). We denote by \( \lambda^b x := (\lambda^{b_1} x_1, \ldots, \lambda^{b_n} x_n) \). We fix a vector \( \vec{a} \in [1, \infty)^n \), and introduce the anisotropic quasi-homogeneous norm \(| \cdot |_{\vec{a}}\) as follows: We set \(|0|_{\vec{a}} := 0\), and for \( x \neq 0 \) we let \(|x|_{\vec{a}} := \lambda_0\), where \( \lambda_0 \) is the unique positive number such that \( |\lambda^{-\vec{a}} x| = 1\). One observes immediately that
\[
|\lambda^{\vec{a}} x|_{\vec{a}} = \lambda |x|_{\vec{a}}, \text{ for every } x \in \mathbb{R}^n, \eta > 0.
\]
From this we notice that \(| \cdot |_{\vec{a}}\) is not a norm unless \( \vec{a} = (1, \ldots, 1) \), where it coincides with the Euclidean norm.

We have a link between the anisotropic and the Euclidean geometry (see [8][9]): There exist constants \( c_1, c_2 > 0 \) such that for every \( x \in \mathbb{R}^n \),
\[
c_1 (1 + |x|_{\vec{a}})^{a_m} \leq 1 + |x| \leq c_2 (1 + |x|_{\vec{a}})^{a_M}.
\]

where we denoted \( a_m := \min_{1 \leq j \leq n} a_j \), \( a_M := \max_{1 \leq j \leq n} a_j \).

Finally, we will need the so-called homogeneous dimension:
\[
\nu := |\vec{a}| = a_1 + \cdots + a_n.
\]

### 2.3. Anisotropic mixed-norm Triebel-Lizorkin and Besov spaces
In this section we define the anisotropic mixed-norm smoothness spaces needed for our analysis, and we discuss some corresponding Fefferman-Stein vector-valued maximal function estimates.

Let \( \varphi_0 \in \mathcal{S} \) (the class of Schwartz functions) be such that
\[
\text{supp} (\widehat{\varphi_0}) \subseteq 2^\vec{a}[-2, 2]^n =: R_0,
\]
\[
|\widehat{\varphi_0}(\xi)| \geq c > 0 \text{ if } \xi \in 2^\vec{a}[-5/3, 5/3]^n.
\]

and let \( \varphi \in \mathcal{S} \) be such that
\[
\text{supp} (\varphi) \subseteq [-2, 2]^n \setminus (-1/2, 1/2)^n =: \tilde{R}_1,
\]
\[
|\varphi(\xi)| \geq c > 0 \text{ if } \xi \in [-5/3, 5/3]^n \setminus (-3/5, 3/5)^n.
\]
Note that it is possible to choose \( \varphi, \varphi_0 \) satisfying the partition of unity condition
\[
\widehat{\varphi}_0(\xi) + \sum_{j=1}^{\infty} \hat{\varphi}(2^{-j} \xi) = 1 \quad \text{for every } \xi \in \mathbb{R}^n.
\]

We define the “rectangular version” of the annulus by
\[
R_j := 2^{-j} R_1, \quad j \geq 1 \quad \text{and} \quad R_j := \emptyset, \quad j < 0.
\]

Note that the punctured rectangle \( \widetilde{R}_1 \) can be expressed as the (almost) disjoint union of \( k_n := 2^{3n} - 2^n \) closed dyadic cubes \( \{Q_\mu\}_{\mu=1}^{k_n} \) of side-length \( 2^{-2} \). Then for every \( j \geq 1 \) we have that \( R_j = \bigcup_{\mu=1}^{k_n} 2^{2j} Q_\mu \). For every \( \bar{p} \in (0, \infty]^n \), we have a two sided estimate of the mixed-norm on \( R_j \) as
\[
\|f\|_{L^\bar{p}(R_j)} \asymp \sum_{\mu=1}^{k_n} \|f\|_{L^\bar{p}(2^{2j} Q_\mu)},
\]
where the constants in the equivalence depends only on \( \bar{p} \) and \( n \).

We denote by \( \varphi_j(x) := 2^{nj} \varphi(2^{j} x), \quad j \in \mathbb{N} \). We then have \( \widehat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j} \xi) \), so by (2.3)
\[
\text{supp } (\widehat{\varphi}_j) \subseteq 2^{2j} \text{supp } (\hat{\varphi}) \subseteq R_j \quad \text{for every } j \in \mathbb{N}.
\]

Let us now recall the definition of anisotropic mixed-norm Triebel-Lizorkin and Besov spaces (see for example [20]).

For \( s \in \mathbb{R} \), \( \bar{p} \in (0, \infty)^n \), \( q \in (0, \infty] \) and \( \bar{a} \in [1, \infty)^n \), the anisotropic mixed-norm Triebel-Lizorkin space \( F^s_{\bar{p}q}(\bar{a}) \) is defined, as the set of all \( f \in S' \) such that
\[
\|f\|_{F^s_{\bar{p}q}(\bar{a})} := \left( \left( \sum_{j=0}^{\infty} (2^j s |\varphi_j * f|)^q \right)^{1/q} \|_{\bar{p}} \right)^{1/q} < \infty,
\]
for \( s \in \mathbb{R} \), \( \bar{p} \in (0, \infty)^n \), \( q \in (0, \infty] \) and \( \bar{a} \in [1, \infty)^n \), the anisotropic mixed-norm Besov space \( B^s_{\bar{p}q}(\bar{a}) \) is defined, as the set of all \( f \in S' \) such that
\[
\|f\|_{B^s_{\bar{p}q}(\bar{a})} := \left( \sum_{j=0}^{\infty} (2^j s |\varphi_j * f|)^q \right)^{1/q} \|_{\bar{p}} < \infty,
\]
with the \( \ell_q \)-norm replaced by \( \text{sup}_{j} \) if \( q = \infty \) for both \( F^s_{\bar{p}q}(\bar{a}) \) and \( B^s_{\bar{p}q}(\bar{a}) \). Note that \( F^s_{\bar{p}q}(\bar{a}) = F^s_{\bar{p}q}(\bar{a}) \) for \( \bar{p} = (p, \ldots, p) \) (and the same holds true for the Besov spaces).

Further properties of \( F^s_{\bar{p}q}(\bar{a}) \) and \( B^s_{\bar{p}q}(\bar{a}) \) can be found in [17, 21, 32, 33].

**Remark 2.1.** One can easily verify that the definition of mixed-norm Triebel-Lizorkin and Besov spaces based on test functions with Fourier transforms having supports in a classical annulus, such as considered in e.g. [20], is equivalent to the above definitions.

### 2.4. Maximal operators.
Maximal operators will be an essential tool in the proof of our main result. Let \( 1 \leq k \leq n \). We define for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \),
\[
M_k f(x) := \sup_{I \in I^n_x} \frac{1}{|I|} \int_I |f(x_1, \ldots, y_k, \ldots, x_n)| \, dy_k,
\]
where \( I^n_x \) is the set of all intervals \( I \) in \( \mathbb{R}_{x_k} \) containing \( x_k \).
We will use extensively the following \textit{iterated maximal operator}: for \( f \in L^1_{	ext{loc}}(\mathbb{R}^n) \) we let
\begin{equation}
\mathcal{M}_\vec{r}f(x) := \left( M_n \left( \cdots M_2(M_1 |f|^{r_1})^{r_2 / r_1} \cdots \right)_{r_n / r_{n-1}}^{r_n} \right)^{1 / r_n}(x), \quad \vec{r} \in (0, \infty)^n, \quad x \in \mathbb{R}^n.
\end{equation}

We shall need a variation of \textit{Fefferman-Stein} vector-valued maximal inequality (see \cite{Fefferman1972, Fefferman1971}): If \( \vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n, \quad q \in (0, \infty] \) and \( \vec{r} = (r_1, \ldots, r_n) \in (0, \infty)^n \) with \( r_k < \min(p_1, \ldots, p_k, q) \) for every \( k = 1, \ldots, n \) then
\begin{equation}
\left\| \left( \sum_{j \geq 0} (\mathcal{M}_\vec{r}(f_j))^{q} \right)^{1/q} \right\|_{\vec{p}} \leq c \left\| \left( \sum_{j \geq 0} |f_j|^{q} \right)^{1/q} \right\|_{\vec{p}}.
\end{equation}

By \cite{Fefferman1971} Proposition 3.11], we obtain the following compound result: For every \( \vec{r} \in (0, \infty)^n \) there exists a constant \( c > 0 \), such that for every \( \vec{b} = (b_1, \ldots, b_n) \in (0, \infty)^n \) and \( f \) with \( \text{supp}(f) \subset [-b_1, b_1] \times \cdots \times [-b_n, b_n] \),
\begin{equation}
\sup_{z \in \mathbb{R}^n} \frac{|f(x-z)|}{(1 + |b_1 z_1|)^{1/r_1} \cdots (1 + |b_n z_n|)^{1/r_n}} \leq c \mathcal{M}_\vec{r}f(x), \quad x \in \mathbb{R}^n.
\end{equation}

3. Fourier multipliers

3.1. \textbf{Anisotropic Fourier multipliers}. One of the most classical problems in harmonic analysis is the boundedness of Fourier multipliers between suitable function (or distribution) smoothness spaces. A bounded function \( m = m(\xi) \) on \( \mathbb{R}^n \) is called a multiplier. The corresponding Fourier multiplier operator is given by
\begin{equation}
T_m f(x) := \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad \text{for every } x \in \mathbb{R}^n, \quad f \in \mathcal{S}.
\end{equation}

The question of boundedness of \( T_m \) is extremely well studied in the Euclidean setting as well as on manifolds, groups, symmetric spaces, and in many other settings. See for example \cite{Triebel1978, Triebel1983, Triebel1992, Triebel1983a, Triebel1983b} and the references therein.

Fourier multipliers on Triebel-Lizorkin spaces have been studied by Triebel in \cite{Triebel1978}. For anisotropic Besov and Triebel-Lizorkin spaces we refer to the articles \cite{Yang2015, Yang2016} of Bényi and Bownik.

Yang and Yuan in \cite{Yang2017} introduced some general scales; Triebel-Lizorkin-type spaces. For Fourier multipliers on such spaces, see D. Yang et al. \cite{Yang2018}.

Perhaps the most well-known multiplier conditions (with respect to an isotropic geometry) are the following \textit{Mihlin and Hörmander conditions}, which we will state as \( L^\infty \) and \( L^2 \) conditions for compatibility with the new condition that we are going to introduce below.

Let \( \alpha \in \mathbb{R}, \quad N \in \mathbb{N} \) and \( m \in \mathcal{C}^{N} \), we say that \( m \) satisfies the:

1. \( L^\infty \)-condition when
\begin{equation}
\sup_{|\gamma| \leq N} \sup_{\xi \in \mathbb{R}^n} \left| (1 + |\xi|^\alpha)^{-\alpha + \vec{a} \cdot \gamma} \partial^\gamma m(\xi) \right| \leq \infty.
\end{equation}

2. \( L^2 \)-condition when
\begin{equation}
\sup_{|\gamma| \leq N} \sup_{j \geq 0} \left\{ 2^{-j \alpha + \vec{a} \cdot \gamma} 2^{-j\nu/2} \| \partial^\gamma m \|_{L^2(R_j)} \right\} \leq \infty,
\end{equation}

where \( R_j \) are as in (2.6) and (2.11).
Remark 3.1. 1. The conditions (3.21) and (3.22) are the anisotropic analogues of the classical inhomogeneous ones, see [3][3][3][30], with the extra parameter $\alpha \in \mathbb{R}$, which allows us to interplay between different smoothness levels as in [3][3][30].

2. Under the isotropic versions of the above conditions, Antonić and Ivec in the recent paper [11], proved the boundedness of Fourier multipliers on mixed Lebesgue and Sobolev spaces.

3. It is not hard to see that $\|1\|_{L^2(R_j)} = c2^{\nu j/2}$, for every $j \in \mathbb{N}_0$, so the $L^2$-condition is sharper than the $L^\infty$-condition.

4. Multipliers on anisotropic homogeneous mixed-norm spaces are considered by the present authors in [13].

3.2. A new class of multipliers. Here we introduce a new class of multipliers replacing the conditions (3.21) and (3.22) above by a mixed-norm $L^\gamma$-condition of Hörmander type. Before of this we give the following definition

Definition 3.2. A vector $\vec{t} = (t_1, \ldots, t_n) \in [1, 2]^n$ with $1 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 2$ will be called admissible. We also denote by $t := \frac{1}{t_1} + \cdots + \frac{1}{t_n}$.

We proceed now to define multiplier classes with respect to both the anisotropy and the mixed-norms.

Definition 3.3. Let $\vec{a} \in [1, \infty)^n$. Given $\alpha \in \mathbb{R}$, an admissible $\vec{t}$ and $N \in \mathbb{N}$, we say that the multiplier $m \in C^N(\mathbb{R}^n)$ satisfies the $L^\vec{t}$-condition, or that it belongs to the class $A(\alpha, \vec{t}, N)$, if

$$A_{\alpha, \vec{t}, N}(m) := \sup_{|\gamma| \leq N} \sup_{j \geq 0} \left\{2^{-j\alpha}2^{\vec{a} \cdot \gamma}2^{-j\frac{\vec{a}}{2} \cdot \gamma} \|\partial^\gamma m\|_{L^\gamma(R_j)} \right\} < \infty,$$

where the $R_j$'s are as in (2.6) and (2.11).

We have the followings remarks pertaining to Definition 3.3

Remark 3.4. Note that:

1. When $\vec{t} = (2, \ldots, 2)$, the $L^\vec{t}$-condition (3.23) coincides with the Hörmander $L^2$-condition (3.22).

2. We observe that $\|1\|_{L^\vec{t}(R_j)} = c2^{j\frac{\vec{a}}{2}}$, for every $j \in \mathbb{N}_0$, and $\vec{t}$ admissible.

   Then from (2.4), we conclude that the $L^\vec{t}$-condition is sharper than the $L^\infty$-condition.

3. For every $\vec{t}, \vec{r} \in [1, 2]^n$, with $\vec{t} \leq \vec{r}$ (i.e., $t_j \leq r_j$, $j = 1, \ldots, n$) we have

$$2^{-j\nu} \|f\|_{L^\vec{t}(R_j)} \leq 2^{-j\frac{\vec{a}}{2} \cdot \vec{r}} \|f\|_{L^\vec{r}(R_j)} \leq 2^{-j\frac{\vec{a}}{2} \cdot \vec{r}} \|f\|_{L^\infty(R_j)} \leq 2^{-j\mu \nu/2} \|f\|_{L^\infty(R_j)}.$$

3.3. The main result. We now proceed to state our main result. However, we need first to fix some notation.

Definition 3.5. For every vector $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$ and every $q \in (0, \infty]$, we set $\mu_j := \min(p_1, \ldots, p_j, q)$, $j = 1, \ldots, n$ for the case of Triebel-Lizorkin and $\mu_j := \min(p_1, \ldots, p_j)$, $j = 1, \ldots, n$ for the case of Besov spaces. In every case we denote by $\mu := \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}$.

Our main multiplier result is the following.

Theorem 3.6. Let $\alpha, s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$, $\vec{a} \in [1, \infty)^n$, an admissible vector $\vec{t}$ and $N \in \mathbb{N}$ with $N > \mu + t$, where $t, \mu$ as in Definitions 3.2 and 3.5.
If \( m \) is a multiplier in the class \( \mathcal{A}(\alpha, \vec{t}, N) \), then the Fourier multiplier \( T_m \) is bounded from \( F_{\vec{p}_q}^{s, \alpha}(\vec{d}) \) to \( F_{\vec{p}_q}^{s, \alpha}(\vec{d}) \) and from \( B_{\vec{p}_q}^{s, \alpha}(\vec{d}) \) to \( B_{\vec{p}_q}^{s, \alpha}(\vec{d}) \).

**Proof.** We shall treat only the case of Triebel-Lizorkin spaces. The Besov space \( (2.13) \), we can verify that there exists \( M \in \mathbb{N} \) such that, for every \( j \geq 0 \)

\[
\hat{\varphi}_j = \sum_{k=j-M}^{j+M} \hat{\varphi}_k \hat{\varphi}_j,
\]

with the convention that \( \hat{\varphi}_k \equiv 0 \) if \( k < 0 \).

Let \( f \in F_{\vec{p}_q}^{s, \alpha}(\vec{d}) \). We have for every \( \xi \in \mathbb{R}^n \), using equation \((3.20)\),

\[
\hat{\varphi}_j(\xi) \overline{T_m f(\xi)} = \hat{\varphi}_j(\xi) m(\xi) \hat{f}(\xi) = \sum_{k=j-M}^{j+M} m(\xi) \hat{\varphi}_k(\xi) \hat{\varphi}_j(\xi) \hat{f}(\xi).
\]

We set

\[
m_{(j)}(\xi) := 2^{-ja} m(\xi) \sum_{k=j-M}^{j+M} \hat{\varphi}_k(\xi), \quad \xi \in \mathbb{R}^n,
\]

and

\[
g_{(j)}(\xi) := m_{(j)}(2^{ja} \xi), \quad \xi \in \mathbb{R}^n.
\]

In the light of the above, and by the inverse Fourier transform \( \mathcal{F}^{-1} \), \((3.20)\) implies

\[
(\varphi_j * T_m f)(x) = 2^{ja} (\mathcal{F}^{-1}(m_{(j)} * (\varphi_j * f))(x), \quad x \in \mathbb{R}^n.
\]

From the selection of \( N \), we can find \( \varepsilon > 0 \) such that \( N = \mu + t + 2n\varepsilon \). We set

\[
N_k := \frac{1}{\mu_k} + \frac{1}{t_k} + 2\varepsilon, \quad \text{for every } k = 1, \ldots, n
\]

and hence \( N_1 + \cdots + N_n = N \). We also introduce the vector \( \vec{r} := (r_1, \ldots, r_n) \) where \( r_k := (1/\mu_k + \varepsilon)^{-1} \) for every \( k = 1, \ldots, n \) and thus

\[
\frac{1}{r_k} = N_k - (\varepsilon + \frac{1}{t_k}), \quad \text{for every } k = 1, \ldots, n.
\]

By \((3.20)\), we obtain

\[
| (\varphi_j * T_m f)(x) | \leq 2^{ja} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(m_{(j)})(y)||\varphi_j * f(x-y)| dy
\]

\[
\leq 2^{ja} \sup_{z \in \mathbb{R}^n} \left( \prod_{k=1}^n (1 + |2^{ja_k} z_k|)^{1/r_k} \right) \times I, \quad x \in \mathbb{R}^n,
\]

where we have put

\[
I := \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(m_{(j)})(y)| \prod_{k=1}^n (1 + |2^{ja_k} y_k|)^{1/r_k} dy.
\]
Since $\text{supp}(\varphi_j \ast f) \subset 2^j \mathbb{R}_1 \subset 2^j [−2, 2]^n$, we use the maximal inequality \cite{2.1.1} to obtain that
\begin{equation}
\sup_{z \in \mathbb{R}^n} \frac{|\varphi_j \ast f(x - z)|}{\prod_{k=1}^n (1 + |2^{jz} z_k|)^{1/r_k}} \leq c \mathcal{M}(\varphi_j \ast f)(x), \quad x \in \mathbb{R}^n.
\end{equation}

**Estimation of $I$.**

By the definition of $g_{(j)}$, we have $F^{-1}(m_{(j)})(y) = 2^{jv} F^{-1}(g_{(j)})(2^{j\bar{y}}y)$, so
\begin{equation}
I = \int_{\mathbb{R}^n} 2^{jv} |F^{-1}(g_{(j)})(2^{j\bar{y}}y)| \prod_{k=1}^n (1 + |2^{jz} y_k|)^{1/r_k} dy
\end{equation}
and
\begin{equation}
I = \int_{\mathbb{R}^n} |F^{-1}(g_{(j)})(x)| \prod_{k=1}^n (1 + |x_k|)^{1/r_k} dx,
\end{equation}
where for the last equality we changed to the variable $x := 2^{j\bar{y}} y$.

We now apply the mixed Hölder-inequality \cite{2.1} for the admissible $\bar{t} \in [1, 2]^n$ and obtain
\begin{equation}
I \leq \left\| F^{-1}(g_{(j)})(x) \prod_{k=1}^n (1 + |x_k|)^{N_k} \right\|_{\bar{t}} \left\| \prod_{k=1}^n (1 + |x_k|)^{-(\epsilon + \frac{\epsilon}{r_k})} \right\|_{\bar{t}}.
\end{equation}

Now it is easy to observe that
\[ \left\| \prod_{k=1}^n (1 + |x_k|)^{-(\epsilon + \frac{\epsilon}{r_k})} \right\|_{\bar{t}} = \prod_{k=1}^n \left(1 + |x_k| \right)^{-\left(\epsilon + \frac{\epsilon}{r_k}\right)} \leq c. \]

On the other hand, since
\[ \prod_{k=1}^n (1 + |x_k|)^{N_k} \leq (1 + |x|)^N \leq c_N \sum_{|\gamma| \leq N} |x|^{\gamma}, \]
we conclude that
\begin{equation}
I \leq c \left\| F^{-1}(g_{(j)})(x) \sum_{|\gamma| \leq N} |x|^\gamma \right\|_{\bar{t}} \leq c \sum_{|\gamma| \leq N} \left\| F^{-1}(g_{(j)})(x) x^\gamma \right\|_{\bar{t}},
\end{equation}
\begin{equation}
= c \sum_{|\gamma| \leq N} I_{\gamma}.
\end{equation}

**Estimation of $I_{\gamma}$.**

We have for every multi-index $|\gamma| \leq N$,
\begin{equation}
I_{\gamma} = \left\| F^{-1}(g_{(j)})(x) x^\gamma \right\|_{\bar{t}} = \left\| F^{-1}(\partial^\gamma g_{(j)})(\cdot) \right\|_{\bar{t}}
= \left\| F(\partial^\gamma g_{(j)})(\cdot) \right\|_{\bar{t}} = \left\| F(\partial^\gamma g_{(j)})(\cdot) \right\|_{\bar{t}}.
\end{equation}

By assumption, $\bar{t}$ is admissible, so we may apply the mixed Hausdorff-Young inequality \cite{2.2} to obtain
\begin{equation}
I_{\gamma} = \left\| F(\partial^\gamma g_{(j)})(\cdot) \right\|_{\bar{t}} \leq \left\| \partial^\gamma g_{(j)}(\cdot) \right\|_{\bar{t}}.
\end{equation}
From the definitions of \(m(j)\) and \(g(j)\), it follows that

\[
(\partial^\gamma g(j))(x) = 2^{j\alpha}(\partial^\gamma m(j))(2^j x).
\]

Moreover, by a change of variables, we observe that

\[
\|\partial^\gamma m(j)(2^j x)\|_\ell = 2^{-j\alpha} \|\partial^\gamma m(j)\|_\ell.
\]

Moreover, by Leibniz’s product rule,

\[
|\partial^\gamma m(j)(x)| \leq 2^{-j\alpha} \sum_{k=j-M}^{j+M} |\partial^\gamma (m\varphi_k)(x)|
\]

\[
\leq 2^{-j\alpha} \sum_{k=j-M}^{j+M} \sum_{\beta \leq \gamma} (j) \left| (\partial^\beta m)(x) (\partial^{\gamma-\beta} \varphi_k)(x) \right|
\]

\[
\leq c2^{-j\alpha} \sum_{\beta \leq \gamma} |\partial^\beta m(x)|,
\]

since \( |(\partial^{\gamma-\beta} \varphi_k)(x)| = 2^{-k\alpha}(\gamma-\beta) |(\partial^{\gamma-\beta} \varphi)(2^{-k} x)| \leq c \) as \( \beta \leq \gamma \) (recall that \( \varphi_k \equiv 0 \), for \( k < 0 \)).

Combining (3.36) with the above estimates, eq. (2.12), and the fact that \( \text{supp } (m(j)) \subseteq \bigcup_{k=j-M}^{j+M} R_k \), we arrive at

\[
I_\gamma \leq c2^{j\alpha} 2^{-j\alpha} \sum_{\beta \leq \gamma} \sum_{k=j-M}^{j+M} \left| (\partial^{\gamma-\beta} \varphi_k)(x) \right| \left| (\partial^\gamma (m\varphi_k))(x) \right|
\]

\[
\leq 2^{-j\alpha} \sum_{k=j-M}^{j+M} \sum_{\beta \leq \gamma} \left| (\partial^{\gamma-\beta} \varphi_k)(x) \right| \left| (\partial^\gamma m)(x) \right|
\]

\[
\leq c2^{j\alpha} 2^{-j\alpha} \sum_{k=j-M}^{j+M} \left| (\partial^{\gamma-\beta} \varphi_k)(x) \right| \left| (\partial^\gamma m)(x) \right|
\]

\[
\leq c2^{j\alpha} \sum_{\beta \leq \gamma} \sum_{k=j-M}^{j+M} 2^{-k\alpha} 2^{-k\alpha} \left| (\partial^{\gamma-\beta} \varphi_k)(x) \right| \left| (\partial^\gamma m)(x) \right|
\]

By (3.35) and (3.37) we have

\[
I \leq c A_{\alpha, \gamma}(m) \leq c,
\]

since \( m \) belongs to the family \( \mathcal{A}(\alpha, \gamma, N) \). Combining the last with (3.30) and (3.32) we deduce that

\[
|(\varphi_j * T_m f)(x)| \leq c2^{j\alpha} M_\mathcal{F}(\varphi_j * f)(x).
\]
We now pass to the Triebel-Lizorkin norm, and we apply the mixed-Fefferman-Stein maximal inequality (2.18) to conclude that

\[ \|T_m f\|_{F^{\bar{p}}(\bar{a})} \leq c \left\| \left( \sum_{j=0}^{\infty} (2^j |\varphi_j * T_m f|)^q \right)^{1/q} \right\|_{\bar{p}} \leq c \|f\|_{F^{\bar{p}}(\bar{a})}, \]

(3.40)

which concludes the proof.

\[ \square \]

**Remark 3.7.** Let us consider the case \( \bar{t} = (2, \ldots, 2) \). As mentioned in Remark 3.4, our condition coincides with the (anisotropic) Hörmander condition with the extra parameter \( \alpha \in \mathbb{R} \) (as in [10,36]). The smoothness level that we require for the multiplier \( m \) is

\[ N = \left\lfloor \mu + \frac{n}{2} \right\rfloor + 1 \text{ for } \mu = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}, \text{ where } \mu_j = \min(p_1, \ldots, p_j, q). \]

This means that we ask for

\[ N \leq N_0 := \left[ \frac{n}{\min(p_1, \ldots, p_n, q)} + \frac{n}{2} \right] + 1 \]

derivatives on \( m \).

When \( \bar{p} = (p, \ldots, p) \), the index \( N_0 \) becomes the same as the one appearing by Yang et al. in [36].

Sharp multiplier results on Triebel-Lizorkin spaces have been proved by Triebel in [35]. The main tool used by Triebel in [35] is complex interpolation, and currently such tools are not available in the mixed-norm setting.

4. Special Cases

4.1. Fourier multipliers on anisotropic mixed-norm Sobolev spaces. One of the main motivation for studying Triebel-Lizorkin and Besov spaces is that for specific choices of parameters many well-know spaces of harmonic analysis can be recovered. Let us mention the special case mixed-norm Sobolev and generalized Sobolev spaces, see [27]. Such spaces provide a natural setting for the study of partial differential operators.

Let \( \bar{p} \in (1, \infty)^n \) and \( \bar{k} \in \mathbb{N}_0^n \), the mixed-norm Sobolev space \( W_{\bar{p}}^{\bar{k}} \) is defined, as the set of all \( f \in S' \) such that

\[ \|f\|_{W_{\bar{p}}^{\bar{k}}} := \|f\|_{\bar{p}} + \sum_{j=1}^{n} \left\| \frac{\partial^{k_j} f}{\partial x_j^{k_j}} \right\|_{\bar{p}} < \infty. \]

(4.41)

Note that \( W_{\bar{p}}^{\bar{0}} = L^{\bar{p}} \), for every \( \bar{p} \in (1, \infty)^n \).

Now let \( \bar{p} \in (1, \infty)^n \), \( s \in \mathbb{R} \) and \( \bar{a} \in [1, \infty)^n \), the anisotropic mixed-norm generalized Sobolev space \( H_{\bar{p}}^{s}(\bar{a}) \) is defined, as the set of all \( f \in S' \) such that

\[ \|f\|_{H_{\bar{p}}^{s}(\bar{a})} := \left\| F^{-1} \left( (1 + |\xi|^{s/2}) \hat{f} \right) \right\|_{\bar{p}} < \infty. \]

(4.42)
We have the identifications:

1. When \( \check{p} \in (1, \infty)^n \), \( s \in \mathbb{R} \) and \( \check{a} \in [1, \infty)^n \), then
   \[
   F^{s}_{\check{p}2}(\check{a}) = H^s_{\check{p}}(\check{a}),
   \]
   with equivalent norms.

2. When \( \check{p} \in (1, \infty)^n \), \( s \in \mathbb{R} \), \( \check{k} \in \mathbb{N}_0^n \) and \( \check{a} \in [1, \infty)^n \), satisfying \( k_j = s/a_j \), for all \( j = 1, \ldots, n \), then
   \[
   F^s_{\check{p}2}(\check{a}) = W^{\check{k}}_{\check{p}},
   \]
   with equivalent norms. Especially \( F^0_{\check{p}2}(\check{a}) = L^{\check{p}} \).

Before we present our Corollaries, we fix an admissible \( \check{t} \) and we keep \( q = 2 \).

Then for every \( \check{p} \), we have \( \mu_j = \min(p_1, \ldots, p_n, 2) \), \( j = 1, \ldots, n \). Finally recall that
\[
t = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}
\]
and \( \mu = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} \).

By all the above, Theorem 3.6 implies the following:

**Corollary 4.1.** Let \( \alpha, s \in \mathbb{R} \), \( \check{p} \in (1, \infty)^n \), \( N \in \mathbb{N} \) and \( \check{a} \in [1, \infty)^n \). If \( m \in A(\alpha, \check{t}, N) \), for \( N > \mu + t \), then the Fourier multiplier \( T_m \) is bounded from \( H^{\alpha, \check{t}, N}_{\check{p}}(\check{a}) \) to \( H^{\alpha,\check{a}}_{\check{p}}(\check{a}) \).

By the identification of mixed-norm Triebel-Lizorkin with Sobolev spaces, Theorem 3.6 also offers the following Corollary:

**Corollary 4.2.** Let \( \alpha, s \in \mathbb{R} \), \( \check{p} \in (1, \infty)^n \), \( \check{a} \in [1, \infty)^n \) be such that
\[
\check{k} := \left( \frac{s}{a_1}, \ldots, \frac{s}{a_n} \right), \quad \check{t} := \left( \frac{\alpha}{a_1}, \ldots, \frac{\alpha}{a_n} \right) \in \mathbb{N}_0^n.
\]

If \( m \in A(\alpha, \check{t}, N) \), for \( N > \mu + t \), then the Fourier multiplier \( T_m \) is bounded from \( W^{\alpha, \check{t}+\check{k}}_{\check{p}} \) to \( W^{\check{k}}_{\check{p}} \).

Let us restrict our attention to the isotropic case; \( \check{a} = (1, \ldots, 1) \). Then we recover the following recent results by Antonić and Ivec [1]:

**Corollary 4.3.** Let \( \alpha, s \in \mathbb{N}_0 \) and \( \check{p} \in (1, \infty)^n \). If \( m \in A(\alpha, \check{t}, N) \), for \( N > \mu + t \), then the Fourier multiplier \( T_m \) is bounded from \( W^{\alpha, \check{t}+\check{k}}_{\check{p}} \) to \( W^{\check{k}}_{\check{p}} \). Especially when \( \alpha = 0 \), then \( T_m \) is bounded on \( L^{\check{p}} \).

4.2. **Equivalent characterizations.** We conclude the paper by considering one explicit example of a bounded Fourier multiplier that can be used to obtain an equivalent norm of anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

Let \( \check{a} \in [1, \infty)^n \). We consider the following anisotropic bracket,
\[
\langle x \rangle_{\check{a}} := |(1, x)|_{(1, \check{a})}, \quad x \in \mathbb{R}^n.
\]

This quantity has been studied in detail in [8]. It is known that \( \langle \cdot \rangle_{\check{a}} \in C^\infty(\mathbb{R}^n) \) and that for \( \alpha \in \mathbb{R} \), \( \gamma \in \mathbb{N}_0^n \), there are constants \( c_{\gamma}, c'_\gamma > 0 \) such that for every \( \xi \in \mathbb{R}^n \)
\[
|\partial^\gamma \langle \xi \rangle_{\check{a}} | \leq c_{\gamma} \langle \xi \rangle_{\check{a}}^{\alpha - \check{a} \cdot \gamma} \leq c'_\gamma (1 + |\xi|_{\check{a}})^{\alpha - \check{a} \cdot \gamma}.
\]

So the multiplier \( m_\alpha(\xi) := \langle \xi \rangle_{\check{a}}^{\alpha} \), \( \alpha \in \mathbb{R} \), satisfies the Mihlin condition \( (3.21) \) for arbitrary \( N \in \mathbb{N} \). By Theorem 3.6, the multiplier \( T_{m_\alpha} \) is bounded from \( F^{s}_{\check{p}q}(\check{a}) \) to \( F^{s}_{\check{p}q}(\check{a}) \) and from \( B^{s,\check{a}}_{\check{p}q}(\check{a}) \) to \( B^{s}_{\check{p}q}(\check{a}) \). Moreover, we observe that \( T_{m_\alpha} \circ T_{m_{-\alpha}} \) is the identity on \( S' \) and thus we have the following characterization:

**Corollary 4.4.** Let \( \alpha, s \in \mathbb{R} \), \( \check{p} = (p_1, \ldots, p_n) \in (0, \infty)^n \), \( q \in (0, \infty) \) and \( \check{a} \in [1, \infty)^n \). Then \( \|T_{m_\alpha} f\|_{B^{s,\check{a}}_{\check{p}q}(\check{a})} \) and \( \|T_{m_\alpha} f\|_{F^{s,\check{a}}_{\check{p}q}(\check{a})} \) are equivalent quasi-norms on \( B^{s}_{\check{p}q}(\check{a}) \) and \( F^{s}_{\check{p}q}(\check{a}) \), respectively.
References

[1] N. Antonić and I. Ivec, On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces. J. Math. Anal. Appl. 433 (2016), no. 1, 176-199.

[2] R.J. Bagby, An extended inequality for the maximal function. Proc. Amer. Math. Soc. 48 (1975), 419-122.

[3] A.I. Benedek and R. Panzone, The spaces $L^p$ with mixed norm. Duke Math. J. 28 (1961), 301-324.

[4] A. Bényi and M. Bownik, Anisotropic classes of homogeneous pseudodifferential symbols. Studia Math. 200 (2019), no. 1, 4166.

[5] A. Bényi and M. Bownik, Anisotropic classes of inhomogeneous pseudodifferential symbols. Collect. Math. 64 (2013), no. 2, 155-173.

[6] L. Borup and M. Nielsen, On anisotropic Triebel-Lizorkin type spaces, with applications to the study of pseudo-differential operators. J. Funct. Spaces Appl. 6 (2008), no. 2, 107-154.

[7] M. Bownik, Atomic and molecular decompositions of anisotropic Besov spaces. Math. Z. 250 (2005), no. 3, 539-571.

[8] M. Bownik, Anisotropic Hardy spaces and wavelets. Mem. Amer. Math. Soc. 164 (2003), no. 781, vi+122.

[9] M. Bownik and K.P. Ho, Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. Trans. Amer. Math. Soc. 358 (2006), no. 4, 1469-1510.

[10] Y.-K. Cho and D. Kim, A Fourier multiplier theorem on the Besov-Lipschitz spaces. Korean Math. Soc. 16 (2008), 85-90.

[11] X.T. Duong, El M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. 196 (2002), no. 2, 443-485.

[12] G. Cleanthous, A. G. Georgiadis and M. Nielsen, Anisotropic Mixed-Norm Hardy Spaces, J. Geom. Anal. 27 (2017), no. 4, 2758-2787.

[13] G. Cleanthous, A. G. Georgiadis and M. Nielsen, Molecular decomposition of anisotropic homogeneous mixed-norm spaces with applications to the boundedness of operators, Appl. Comput. Harm. Anal. (2017). DOI: 10.1016/j.acha.2017.10.001.

[14] X.T. Duong and L. Yan, Spectral multipliers for Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. J. Math. Soc. Japan 63 (2011), no. 1, 295-319.

[15] D. La Fernandez, Vector-valued singular integral operators on $L^p$-spaces with mixed norms and applications. Pac. J. Math. 129 (1987), no. 2, 257-275.

[16] A. G. Georgiadis, $H^p$-bounds for spectral multipliers on Riemannian manifolds. Bull. Sci. Math. 134 (2010), 759-766.

[17] A. G. Georgiadis and M. Nielsen, Pseudodifferential operators on anisotropic mixed-norm Besov and Triebel-Lizorkin spaces. Math. Nachr. 289 (2016), no. 16, 2019-2036.

[18] T. P. Hytönen, Anisotropic Fourier multipliers and singular integrals for vector-valued functions. Ann. Mat. Pura Appl. (4) 186 (2007), no. 3, 455-468.

[19] L. Hörmander, Estimates for translation invariant operators in $L^p$ spaces. Acta Math. 104 (1960), 93-140.

[20] J. Johnsen and W. Sickel, On the trace problem for Lizorkin-Triebel spaces with mixed norms. Math. Nachr. 281 (2008), no. 5, 669-696.

[21] J. Johnsen, S. M. Hansen and W. Sickel, Anisotropic, mixed-norm Lizorkin-Triebel spaces and diffeomorphic maps. J. Funct. Spaces 2014, Art. ID 964794, 15 pp.

[22] G. Kerkyacharian and P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces. Trans. Amer. Math. Soc. 367 (2015), 121-189.

[23] H. Kumano-go, Pseudodifferential operators. MIT Press, Cambridge, Mass.-London, 1981. Translated from the Japanese by the author, Rémi Vaillancourt and Michihiro Nagase.

[24] I. Kyrezi and M. Marias, $H^p$-bounds for spectral multipliers on graphs. Trans. Amer. Math. Soc. 361 (2009), no. 2, 1053-1067.

[25] G. Kyriazis, P. Petrushev and X. Yuan, Decomposition of weighted Triebel-Lizorkin and Besov spaces on the ball. Proc. Lond. Math. Soc. (3) 97 (2008), no. 2, 477-513.
[26] L. Liu, D. Yang and W. Yuan, Besov-type and Triebel-Lizorkin-type spaces associated with heat kernels. Collect. Math. 6 (2016), no. 2, 247-310.

[27] P. I. Lizorkin, Multipliers of fourier integrals and bounds of convolutions in spaces with mixed norms. Applications, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 225255, Engl. transl. Math. USSR Izv. 4 (1970).

[28] N. Lohoué and M. Marias, Multipliers on locally symmetric spaces. J. Geom. Anal. 24 (2014), no. 2, 627-648.

[29] J. Marcinkiewicz, Sur les multipicateurs des sérées de Fourier. Studia Math. 8 (1939), 78-91.

[30] S. G. Mihlin, On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 701-703 (Russian).

[31] S. M. Nikol’skiĭ, Approximation of functions of several variables and imbedding theorems. Springer-Verlag, New York-Heidelberg, 1975.

[32] H.-J. Schmeisser, Maximal inequalities and Fourier multipliers for spaces with mixed quasinorms. Applications, Z. Anal. Anwendungen (1984), no. 3, 153-166.

[33] H.-J. Schmeisser and H. Triebel, Topics in Fourier Analysis and Function Spaces. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987. Published also by John Wiley, Chichester, 1987.

[34] H. Triebel, Theory of function spaces. Monographs in Math. Vol. 78, Birkhauser, Verlag, Basel, 1983.

[35] H. Triebel, Complex interpolation and Fourier multipliers for the spaces $B^s_{p,q}$ and $F^s_{p,q}$ of Besov-Hardy-Sobolev type: the case $0 < p \leq \infty$, $0 < q \leq \infty$. Math. Z. 176 (1981), no. 4, 495-510.

[36] D. Yang, W. Yuan and C. Zhao, Fourier multipliers on Triebel-Lizorkin-type spaces. J. Funct. Spaces Appl. 2012, Art. ID 431016, 37 pp.

[37] D. Yang and W. Yuan, New Besov-type spaces and Triebel-Lizorkin-type spaces including Q spaces. Math. Z. 265 (2010), no. 2, 451-480.

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