Stationary axisymmetric exteriors for perturbations of isolated bodies in general relativity, to second order

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Abstract

Perturbed stationary axisymmetric isolated bodies, e.g. stars, represented by a matter-filled interior and an asymptotically flat vacuum exterior joined at a surface where the Darmois matching conditions are satisfied, are considered. The initial state is assumed to be static. The perturbations of the matching conditions are derived and used as boundary conditions for the perturbed Ernst equations in the exterior region. The perturbations are calculated to second order. The boundary conditions are overdetermined: necessary and sufficient conditions for their compatibility are derived. The special case of perturbations of spherical bodies is given in detail.

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I. INTRODUCTION

One would like to have global solutions for rotating objects in general relativity consisting of a matter-filled interior region and a vacuum, asymptotically flat, exterior, with the aim of modelling planets, stars, star clusters, galactic nuclei or galaxies. The interior and exterior would be matched across a boundary, the surface of the object, Σ. Finding such global models is very difficult even for axially symmetric configurations in equilibrium. So far there are no explicit global models known other than those for spherical stars, which must be non-rotating, and Neugebauer and Meinel’s disc of dust [1], which has no interior (the matter source has zero thickness and is described by jumps in the metric derivatives).

There have been several recent studies of the structure of the underlying equations for the problem. The two regions can be treated independently subject to the matching at the boundary. For a given interior, this fixes “Cauchy data” giving both the metric and its derivative at the boundary, i.e. gives both Dirichlet and Neumann boundary conditions for the elliptic equations governing the vacuum exterior. A compatibility problem therefore arises. Uniqueness of the exterior solution has been shown in [2, 3] (independent of any non-circularity in the interior), and necessary conditions on the Cauchy data for the existence of the exterior also exist [4]. Correspondingly, the interior must satisfy these conditions in order to describe an isolated rotating compact object in equilibrium. Moreover, the conditions were found to be sufficient for static exteriors [5], and it has been conjectured the same holds for the stationary case, but this issue is still under investigation [4].

However, applying these conditions in specific situations has proved very difficult. For example, we had hoped to be able to use them to give a definitive treatment of the matching problem for the Wahlquist solution [6–8] but were unable to evaluate the required integrals in closed form. The understanding of rotating objects has advanced over the years through two major approaches, numerical relativity and perturbation theory. The latter has proved to be very useful and has been widely employed, leading for example to a well-developed theory of slowly rotating stars. Probably the key paper in the theory of slowly rotating stars is that of Hartle [9] (see also [10, 11]). He discussed the case of a rigidly rotating non-singular perfect fluid interior with reflection symmetry, with perturbations dependent only on the slow rotation imposed on a static background. The uniformity of rotation was justified by the argument that configurations minimizing the total mass-energy must rotate uniformly, and hence so must all stable configurations [9, 11]. He implicitly assumed spherical symmetry of the background (an assumption later shown to be true for many cases [12]), thus excluding convective motions in the interior, and the admissibility and $C^2$ character of the coordinates used, which is greater differentiability than required by the usual matchings [13].

With the aim of studying models for stars (not necessarily in equilibrium) using perturbation theory, there has been another direction of research. This has focused on the study of general perturbations around configurations constructed by the matching of static spherically symmetric spacetimes. The first works in this direction were the classical papers by Gerlach and Sengupta [14, 15], whose procedure was better justified by Martín-García and Gundlach [16] who considered gauge-independent quantities. The complete formulation of the relevant perturbed matching conditions to first order, and in general, was eventually given in independent papers by Battye and Carter [17] and Mukohyama [18]. Since these
works are not in principle aimed at isolated stars in equilibrium, there are no restrictions on the exterior due to asymptotic flatness or stationarity, and only first order perturbations are considered. The latter is important, since, as was pointed out in [9], we need to go to second order for isolated stars in equilibrium in order to obtain the effects of slow rotation on the shape of the star. The conditions to second order have been obtained only recently by one of us [19].

Yet another method used for the construction of models of slowly rotating stars has recently [20, 21] produced a particular model to second order in the approximation for a rigidly rotating and constant density perfect-fluid interior. The method is based on a two-parameter perturbation scheme, combining the post-Minkowskian and slow-rotation approximations, both taken up to the first non-linear level. The drawback of the model is inherent to the method, since there is no rotating exact Newtonian limit.

In this paper we re-examine the bases of the perturbed matching theory. An obvious question is, therefore, what new things can be said at this fundamental level? In our opinion, there are several issues that need to be clarified. Firstly, have all the hypotheses involved in the analysis been spelt out in full detail? Secondly, if not, what are they, and are they indispensable? This is especially relevant in the matching of spacetimes, which lies at the very heart of any approach dealing with finite objects with boundary, such as models of stars. Thirdly, most, if not all, of the analyses in the literature deal with a specific matter model in the interior, usually a rigidly rotating perfect fluid, and the results do depend strongly on this assumption. It is natural to ask how one can develop a theory which is as independent as possible of the interior matter model (or even fully independent). Achieving this is clearly of interest, since the resulting theory could be applied to many different situations, from differentially rotating perfect fluids to convective fluids, viscous fluids, or even totally different matter models like elastic bodies. The aim of this paper is to answer all these questions in detail.

One disadvantage of our approach, of course, is that without assuming a particular matter model we cannot study important questions like uniqueness or existence of interior solutions for given boundary data. However, even with specific interiors, the issue of existence of global models for self-gravitating isolated rotating objects is a very difficult one and extremely few results are known. The only general theorem in relativity of this type to date refers to stationary and axially symmetric rigidly rotating fluids with spatially compact sections. Existence of such rotating stars was proven by Uwe Heilig [22] for relativistic configurations close to Newtonian models and sufficiently small (but finite) angular velocity. In perturbation theory the situation is not much better, although the results by Hartle [9] indicate a possible path towards a rigorous existence theorem of rigidly rotating perturbations of a fluid around a static spherically symmetric configuration.

We believe that this paper is the first to give a comprehensive and consistent theory of rotating stationary and axisymmetric objects to first and second order in perturbation theory from first principles. We start from the Darmois form of the matching conditions, i.e. we do not a priori assume admissibility of particular coordinates, and specialize this form to the stationary axisymmetric case. We do not assume that the background about which we perturb is spherically symmetric (unlike previous work), though we shall give the specialization to this case, nor, as mentioned above, do we restrict the type of matter in
the interior region. The main restrictions are that: we consider only perturbations about static solutions; we assume that the exterior is an asymptotically flat vacuum and does not contain an ergoregion; and we assume that the axis is everywhere regular.

It would in principle be possible to undertake a parallel study of the more general case where the initial configuration is stationary rather than static, but we have not explored this at any length. Since necessary conditions for the existence of a matching of a given interior to an asymptotically flat exterior can be found in the fully non-linear regime for this case, they can also be found for the perturbations. However, proving sufficiency, even if possible, appears to be considerably more difficult.

In section II we give the general conditions for matching, specialized to the stationary axisymmetric case and expressed using Weyl coordinates in the exterior. In section III we introduce the perturbation scheme, in particular introducing some gauge choices. The perturbations in the exterior up to second order are considered in section IV: we give the perturbed Ernst equations together with the perturbed Cauchy data conditions resulting from the matching. Necessary and sufficient conditions on these data for a solution to the exterior equations are derived in section V (recall that sufficiency has not yet been shown for the corresponding full equations). These conditions are first expressed in terms of the equality of volume integrals over the exterior and surface integrals involving the Cauchy data and a principal function on the surface itself, the latter being defined as a solution to a partial differential equation (PDE). This form is then reduced to integrals on the boundary surface by first expressing the integrands as divergences of one-forms, which themselves are ultimately defined as solutions of ordinary differential equations (ODE). The final results are thus found in the form of integrals over the parameter which runs along the meridians of the boundary surface. Lastly we specialize the static background to the spherically symmetric case in section VI: here some of the functions can be found explicitly, since the exterior background must be the Schwarzschild solution. We note that we do not in this paper use the assumptions of Hartle’s work [9, 10]: we intend in a later paper to spell out fully the relationship of his approach and ours.

II. MATCHING WITH WEYL COORDINATES

In our problem the spacetime is composed of two regions with boundary, namely the interior \((\mathcal{V}^I, g^I)\) and the exterior \((\mathcal{V}^E, g^E)\) (where \(I\) and \(E\) stand for interior and exterior from now on). Each region, as well as its boundary, is assumed to be stationary and axially symmetric, and to have a regular axis of symmetry (see e.g [23]). We make no specific assumption on the energy-momentum tensor of \((\mathcal{V}^I, g^I)\): not even the so-called circularity condition [24] in the interior will be assumed (which for fluids without energy flux would be equivalent to the absence of convective motions [25, 26]). In other words we shall not assume that the orbits of the stationary axisymmetric isometry group in the interior region are orthogonally transitive (i.e. the metric might not admit coordinates adapted to the isometries in which the line-element becomes block diagonal). We do not, however, treat interiors containing non-gravitational fields which propagate to the exterior (such as electromagnetic fields, for instance). This means that the exterior region will be taken to be
vacuum. Moreover, all matter configurations will be taken to be isolated, and the spacetimes are assumed to be asymptotically flat.

Let us denote by $\Sigma^I$ and $\Sigma^E$ the respective boundaries of $(\mathcal{V}^I, g^I)$ and $(\mathcal{V}^E, g^E)$. For us to be able to identify them, these hypersurfaces must be diffeomorphic to one another. As usual, this is best described by taking an abstract three-dimensional manifold $\sigma$ and two embeddings $\chi^I$ and $\chi^E$, where $\chi^{I/E} : \sigma \to \mathcal{V}^{I/E}$, such that $\Sigma^{I/E} = \chi^{I/E}(\sigma)$. The point-to-point identification of the boundaries is $\chi^I \circ (\chi^E)^{-1}$ defined on $\Sigma^E$.

We assume also that in general the spacetimes admit no further local symmetries beyond stationarity and axial symmetry (though when perturbing we shall allow the background spacetime to be spherically symmetric). Then the axial Killing vector is uniquely fixed by demanding that its orbits are closed and the Killing coordinate has periodicity $2\pi$. Let us denote by $\vec{\eta}^I$ and $\vec{\eta}^E$ these unique axial Killing vectors in the interior and the exterior spacetimes, respectively. As $(\mathcal{V}^E, g^E)$ is asymptotically flat, it admits a unique Killing vector, denoted by $\vec{\xi}^E$, which is unit timelike at infinity. In the interior region there is no equivalent way of fixing the stationary Killing vector uniquely before the matching is performed. Two different situations might be considered, one in which the interior $(\mathcal{V}^I, g^I)$ is unknown and needs to be determined, and the other where it is explicitly known. In either case, we can choose $\vec{\xi}^I$ to be the unique Killing vector which matches continuously with $\vec{\xi}^E$ (recall that the boundaries are also assumed to be stationary and axially symmetric so that we have a “symmetry-preserving matching” [27]), i.e. we can propagate the exterior Killing vector into the interior, but in the second case this need not agree with the timelike Killing vector used in writing $g^I$, so we have to introduce two extra parameters in the interior metric in order to allow the most general matching. This is well understood and will not be discussed further here; see [2]. The isometry group being two-dimensional and the orbits of $\vec{\eta}^{I/E}$ closed, the Killing vectors $\vec{\xi}^{I/E}$ and $\vec{\eta}^{I/E}$ commute ([28], see also [29]). Thus, away from the symmetry axis, there exist local coordinates $\{T, \bar{\Phi}, \bar{x}^A\}$ $(A, B \cdots = 2, 3)$ in $(\mathcal{V}^I, g^I)$ such that $\vec{\xi}^I = \partial_T$ and $\vec{\eta}^I = \partial_{\Phi}$.

For the exterior metric, we do not allow ergoregions (as we aim to model stars rather than black holes) and the vacuum field equations then imply the local existence of Weyl coordinates $\{t, \phi, \rho, z\}$ which satisfy the conditions that (i) they are adapted to the isometries, i.e. $\vec{\xi}^E = \partial_t$ and $\vec{\eta}^E = \partial_{\phi}$, (ii) $\rho = 0$ defines the axis of symmetry and (iii) the metric $g^E$ takes the local form

$$
\text{d}s_E^2 = -e^{2U} (\text{d}t + A \text{d}\phi)^2 + e^{-2U} \left[ e^{2k} (\text{d}\rho^2 + \text{d}z^2) + \rho^2 \text{d}\phi^2 \right],
$$

where $U$, $A$ and $k$ are functions of $\rho$ and $z$ only. The scalar $\rho$ is intrinsically defined as the non-negative solution of $\rho^2 = -((\vec{\xi}^E, \vec{\xi}^E)_{g^E}(\vec{\eta}^E, \vec{\eta}^E)_{g^E} + (\vec{\xi}^E, \vec{\eta}^E)^2_{g^E}$ where $(,)_g$ denotes the scalar product with respect to a metric $g$. $z$ is also intrinsically defined as the scalar (unique up to a sign and a constant shift) such that $\text{d}z$ is orthogonal to $\text{d}\rho$ (with respect to the metric $g^E$) and $\text{d}z$ and $\text{d}\rho$ have the same norm. The coordinate freedom in (1) consists only of constant shifts of $t$, $\phi$ and $z$. For later use, let us recall that stationary and axially symmetric vacuum spacetimes admit, locally, a scalar function $\Omega$ (the twist potential) defined, up to an additive constant, by

$$
\rho \Omega, = -e^{4U} A, \quad \rho \Omega, = e^{4U} A, \quad \rho \Omega, = e^{4U} A.
$$
If, moreover, \((\mathcal{V}^E, g^E)\) is simply connected \(\Omega\) can be defined globally. In asymptotically flat spacetimes the additive constant is fixed so that \(\Omega \to 0\) at spatial infinity. The scalars \(U\) and \(\Omega\) are intrinsically defined by \(e^{2U} = -(\xi^E, \xi^E)\) and \(d\Omega = \ast (\xi^E \wedge d\xi^E)\), where \(\xi^E = g^E(\xi^E, \cdot)\) and \(\ast\) denotes the Hodge dual [39]. The Einstein vacuum field equations reduce to a complex second order PDE for \((U, \Omega)\) (a pair of real PDEs in the form used below), the Ernst equation(s), together with quadratures for the metric functions \(A\) and \(K\).

With this choice of coordinates in \((\mathcal{V}^E, g^E)\), and since the stationary and axial Killing vectors are tangent to the boundary \(\Sigma^I\), it follows that there exist local coordinates \(\{\tau, \varphi, \mu\}\) on the abstract manifold \(\sigma\) and functions \(T(\mu), \Phi(\mu)\) and \(\bar{x}_I(\mu)\) (see [3], and [2] for the orthogonally transitive case), such that the embedding \(\chi^I\) reads

\[
\chi^I : \{T = \tau + T(\mu), \Phi = \varphi + \Phi(\mu), \bar{x}_I = \bar{x}_I(\mu)\}
\]

and, moreover, \(d\chi^E(\partial_\tau) = \xi^E|_{\Sigma^I}, d\chi^E(\partial_\varphi) = \eta^I|_{\Sigma^I}\) and \(d\chi^E(\partial_\mu)\) is orthogonal to \(\xi^E\) and \(\eta^I\) on \(\Sigma^I\). It can easily be checked that \(\{\tau, \varphi, \mu\}\) is defined uniquely except for constant shifts of \(\tau\) and \(\varphi\) and redefinitions \(\mu(\mu')\). In terms of these coordinates on \(\sigma\), the exterior embedding \(\chi^E\) is forced by the matching conditions (more precisely, by the continuity of the first fundamental form) to take the following form

\[
\chi^E : \{t = \tau, \phi = \varphi, \rho = \rho(\mu), z = z(\mu)\}
\]

for some functions \(\rho(\mu)\) and \(z(\mu)\). Constant shifts in \(t\) and \(\phi\) are in principle allowed, but they can be set to zero without loss of generality by exploiting the freedom in the exterior coordinates. With the embeddings at hand, we can discuss the necessary and sufficient conditions for the interior and exterior spacetimes to be joinable (i.e. to produce a spacetime \((\mathcal{V}, g)\) with continuous metric and no surface layers in the Riemann tensor [13]). These are the well-known Darmois conditions [30] which require that the first and second fundamental forms inherited on \(\sigma\) from both sides agree. In our case, this set of matching conditions can be conveniently rewritten as follows (see [31, 32], [2] for the case where the interior is orthogonally transitive, and [3] for the generalization to an arbitrary interior):

(a) Conditions on the interior hypersurface, given by the Israel conditions [33]

\[
\alpha^I_{\alpha} n^I_{\beta} S^I_{\alpha \beta} |_{\Sigma^I} = 0, \quad n^I_{\alpha} e^I_\alpha S^I_{\alpha \beta} |_{\Sigma^I} = 0,
\]

where \(S^I_{\alpha \beta}\) is the Einstein tensor of \(g^I\), \(n^I\) is a normal vector to \(\Sigma^I\) in \((\mathcal{V}^I, g^I)\) and \(e^I_\alpha\) are any three independent vectors tangent to \(\Sigma^I\). In principle, any choice of normal vector \(n^I\) is suitable for writing (3). However, for the remaining matching conditions it is convenient to fix this vector so that it points to the interior of \((\mathcal{V}^I, g^I)\) and has norm

\[
(n^I, n^I)_{g^I} |_{\Sigma^I} = (e^I, e^I)_{g^I} |_{\Sigma^I},
\]

where \(e^I = d\chi^I(\partial_\mu)\). Note that redefining \(\mu\) on \(\sigma\) changes \(e^I\) and hence \(n^I\). Nevertheless, all expressions below are easily checked to be invariant under rescalings \(\mu(\mu')\).
Conditions (3) determine which (if any) hypersurfaces in a given interior metric are candidates to match to an empty exterior. If the matter content inside is, for instance, a perfect fluid with no convective motion, these conditions reduce to \( p = 0 \), where \( p \) is the pressure of the fluid. Generically (3) consists of four independent conditions, which means that only for special interior metrics will hypersurfaces matching with vacuum exist. Whenever the circularity condition is satisfied in the interior, two of (3) become trivial.

(b) Definition of the exterior matching hypersurface. The functions \( \rho(\mu) \) and \( z(\mu) \) determining the form of the exterior surface are uniquely fixed by

\[
\rho(\mu) = \alpha|\Sigma^I, \quad \dot{z}(\mu) = -\vec{n}^I(\alpha)|\Sigma^I,
\]

where \( \alpha^2 \equiv -\xi^I, \xi^I)g^I(\eta^I, \eta^I)g^I + (\xi^I, \eta^I)^2, \alpha \geq 0 \), and the dot denotes the derivative with respect to \( \mu \). The additive constant in \( z(\mu) \) is inessential given the shift freedom \( z \to z + \text{const} \).

(c) Boundary conditions for the exterior problem. The rest of the matching conditions provide the following data on the exterior metric functions \( U \) and \( A \) on \( \Sigma^E \)

\[
U|_{\Sigma^E} = V|_{\Sigma^I}, \quad \vec{n}^E(U)|_{\Sigma^E} = \vec{n}^I(V)|_{\Sigma^I}, \quad (6)
\]

\[
A|_{\Sigma^E} = B|_{\Sigma^I}, \quad \vec{n}^E(A)|_{\Sigma^E} = \vec{n}^I(B)|_{\Sigma^I}, \quad (7)
\]

where \( \vec{n}^E = -\dot{z} \partial_\mu + \dot{\rho} \partial_\rho |_{\Sigma^E}, e^{2V} = -(\xi^I, \vec{\xi}^I)g^I, \) and \( B = (\vec{\xi}^I, \vec{\eta}^I)g^I / (\xi^I, \xi^I)g^I \). In order to ensure that \( \vec{n}^E \) points to the exterior of \( (V^E, g^E) \) we choose \( \rho \) to be an increasing function of \( \mu \) at the south pole (i.e. at the intersection of \( \Sigma^E \) with the symmetry axis having minimum value of \( z \), which we assume exists). This choice constrains the allowed rescalings \( \mu(\mu') \) to be strictly increasing.

The conditions for \( A \) translate into boundary conditions for the twist potential as follows

\[
\dot{\Omega}|_{\Sigma^E} = \Omega, \rho \dot{\rho} + \Omega, z \dot{z}|_{\Sigma^E}
\]

\[
= -\frac{e^{AU}}{\rho} \vec{n}^E(A)|_{\Sigma^E}
\]

\[
= -\frac{e^{AV}}{\alpha} \vec{n}^I(B)|_{\Sigma^I}, \quad (8)
\]

\[
\vec{n}^E(\Omega)|_{\Sigma^E} = -\frac{e^{AU}}{\rho} \frac{d}{d\mu}(A|_{\Sigma^E})
\]

\[
= -\frac{e^{AV}}{\alpha} \frac{d}{d\mu}(B|_{\Sigma^I}).
\]

The right hand sides of (6), (7) and (8) are known once the interior is known. Thus the matching conditions fix the normal derivative of \( \Omega \) on \( \Sigma^E \) uniquely and \( \Omega \) on \( \Sigma^E \) up to an additive constant. This constant is in principle relevant as \( \Omega \) has been defined so that it vanishes at infinity. However, it can be proven \[2\] that there is at most one value of the additive constant for which the exterior vacuum field equations with the (overdetermined) boundary data (6) and (8) are compatible.
As usual in perturbation theory, we consider a one-parameter family \((\mathcal{V}_\epsilon, g_\epsilon)\) of 4-dimensional spacetimes, differentiable in \(\epsilon\), and think of perturbations in terms of derivatives of the metric with respect to \(\epsilon\), evaluated at \(\epsilon = 0\), which requires working on a single manifold (more precisely, one manifold for the interior region and one manifold for the exterior region). Quantities in the previous section, except for the coordinates, as we shall discuss below, will now bear a subscript \(\epsilon\).

Let us discuss the interior region (the exterior case will be similar). In order to deal with a single manifold, we need to identify in some way points of different spacetimes \(\mathcal{V}_\epsilon^I\) in the \(\epsilon\)-family. If the manifolds were without boundary and diffeomorphic to each other we could take any diffeomorphism, smooth in \(\epsilon\), between, say, \(\mathcal{V}_0^I\) and \(\mathcal{V}_\epsilon^I\) in order to identify them. It is clear that such an identification is not unique, and that there is no canonical choice, because we can perform a diffeomorphism \(\Xi_\epsilon\) of \(\mathcal{V}_\epsilon^I\) onto itself, before applying the diffeomorphism above. This freedom in performing the identification is the heart of the gauge freedom inherent to perturbation theory (see e.g. [34]).

Once an identification has been chosen, we have a single manifold \(\mathcal{V}_0^I\) and a collection of metrics \(g_\epsilon^I\) defined on it. In perturbation theory (up to \(n\)-th order), only the background metric \(g_0^I\) and the first \(n\) derivatives of \(g_\epsilon^I\) evaluated at \(\epsilon = 0\) are of interest. Thus we have a background spacetime \((\mathcal{V}_0^I, g_0^I)\) and \(n\) symmetric tensor fields, \(K_a^I (a = 1, \cdots n)\), defined on it (the perturbations). The freedom in the identification of spacetimes translates into the gauge freedom in perturbation theory and is defined by \(n\) vector fields on \(\mathcal{V}_0^I\). For instance, to first order, and denoting by \(K_1^I\) the first order perturbation tensor, the gauge freedom is \(K_1^I = K_1^I + \mathcal{L}_{\vec{s}_1} g_0\) where \(\vec{s}_1\) is the first order gauge vector and \(\mathcal{L}\) denotes the Lie derivative. For higher order perturbations, the gauge transformations are more complicated as they also involve all lower order terms.

These issues are all well-understood and would have deserved no inclusion here except that in our case the manifolds we are considering are with boundary. How to define the identification in this case has recently been discussed in [19]. We repeat the main idea here for completeness. For each \(\epsilon\), \(\mathcal{V}_\epsilon^I\) is a manifold with boundary \(\Sigma_\epsilon^I\). Thus identifying them via diffeomorphisms requires, strictly speaking, that boundaries are mapped into boundaries. However, if we view each manifold \(\mathcal{V}_\epsilon^I\) as a closed subset of a larger manifold without boundary, the condition that the boundaries are mapped to each other strongly restricts the gauge freedom (at least near the boundaries) and this may not be suitable for the problem at hand. It is more convenient to let the boundaries “move” freely in the identification. Perturbation tensors can still be defined everywhere on the background spacetime \(\mathcal{V}_0^I\) with boundary \(\Sigma_0^I\) as follows. For points away from the background boundary \(\Sigma_0^I\) the usual procedure obviously works. So we only need to worry about how to define perturbations on \(\Sigma_0^I\). At each point \(p \in \Sigma_0^I\) and for small enough positive \(\epsilon\), the corresponding identified point in \(\mathcal{V}_\epsilon^I\) will lie on \(\Sigma_\epsilon^I\) or move towards the interior of \(\mathcal{V}_\epsilon^I\) or move towards its exterior (within the larger manifold). In the first two cases we can define the perturbation tensors by defining the derivatives as one-sided limits with \(\epsilon \to 0\), \(\epsilon > 0\). In the third case, the point \(p\) moves towards the interior of \(\mathcal{V}_\epsilon^I\) for negative values of \(\epsilon\). So derivatives can again be defined if we take the one-sided limits with \(\epsilon < 0\). Note that this construction is independent of the larger
manifold with boundary, which can be dispensed with altogether. Hence we can, as before, define \( n \) symmetric perturbation tensors \( K^I_a \) on the background manifold with boundary \((\mathcal{V}_0^I, g_0^I)\) up to and including the boundary. Obviously, exactly the same considerations hold for the exterior region.

Following the discussion above, we can introduce coordinates \( \{T, \Phi, \bar{x}^A_I\} \) for each spacetime \((\mathcal{V}_I^I, g_I^I)\). In principle, the coordinates themselves should have an \( \epsilon \) label. However, we can make this unnecessary by propagating the coordinates from \((\mathcal{V}_0, g_0)\), using the identifications, so that points with the same coordinate values \( \{T, \Phi, \bar{x}^A_I\} \) in the interior regions of different spacetimes are identified. This choice reduces the gauge freedom available since we are only left with the changes of identification describable by coordinate changes of the type \( T \rightarrow T + T_1(\bar{x}^A_I, \epsilon), \Phi \rightarrow \Phi + \Phi_1(\bar{x}^A_I, \epsilon), \bar{x}^A_I \rightarrow \bar{x}^A_I(\bar{x}_I, \epsilon) \). This (partial) gauge fixing is in fact very useful in the sequel. In this gauge, let us denote by \( K_1^I \) and \( K_2^I \) the first and second perturbations of the metric tensor on the interior background \((\mathcal{V}_0^I, g_0^I)\), with boundary \( \Sigma_0^I \).

In the exterior regions, we analogously relate the spacetimes \((\mathcal{V}_I^E, g_I^E)\) so that points with the same Weyl coordinates \( \{t, \phi, \rho, z\} \) in different spacetimes are identified. Since the Weyl coordinates are unique except for constant shifts in \( t, \phi \) and \( z \), the freedom in performing the identification is reduced to \( t \rightarrow t + \beta_0(\epsilon), \phi \rightarrow \phi + \beta_1(\epsilon), z \rightarrow z + \beta_2(\epsilon) \). Thus the gauge freedom in the exterior region is reduced even more strongly than in the interior, which is also useful. The exterior background is therefore \((\mathcal{V}_0^E, g_0^E)\) with metric

\[
g_0 = -e^{2U_0}dt^2 + e^{-2U_0} \left[ d\rho^2 + dz^2 \right] + \rho^2 d\phi^2,
\]

and the perturbation tensors (up to second order) take the following form

\[
K_1^E = -2e^{2U_0}U_0' dt^2 - 2e^{2U_0}A_0' dt d\phi + 2e^{2U_0}e^{2k_0} (-U_0' + k_0') \left( d\rho^2 + dz^2 \right) - 2e^{-2U_0}U_0' \rho^2 d\phi^2,
\]

\[
K_2^E = -2e^{2U_0} \left( U_0'' + 2U_0'^2 \right) dt^2 - 2e^{2U_0} (A_0'' + 4A_0'U_0') dt d\phi - 2 \left[ e^{2U_0} A_0'' \right. \\
\left. + e^{-2U_0} \rho^2 \left( U_0'' - 2U_0'^2 \right) \right] d\phi^2 + 2e^{-2U_0}e^{2k_0} \left[ k_0'' - U_0'' + 2 (k_0' - U_0') \right] \left( d\rho^2 + dz^2 \right),
\]

which follow directly from (1) by taking \( \epsilon \) derivatives. Since a gauge choice is involved in the definition, we shall call them perturbation tensors in Weyl form or in Weyl gauge. In (9) and (10), \( U_0, k_0, U_0', A_0, k_0'', A_0'' \) are functions of \( \rho \) and \( z \) and are defined by \( U_0 = U_{\epsilon=0}, U_0' = \partial_{\epsilon}U_{\epsilon=0}, U_0'' = \partial_{\epsilon}^2U_{\epsilon=0} \), etc.

IV. THE SECOND ORDER PERTURBED EXTERIOR PROBLEM

Let us start by recalling the vacuum field equations for (1). As already mentioned, we only need to concentrate on the equations for \( U_{\epsilon} \) and \( \Omega_{\epsilon} \), since the remaining field equations for \( k_{\epsilon} \) and \( A_{\epsilon} \) reduce to quadratures.

Let us consider Euclidean space \((\mathbb{E}^3, \gamma)\) in cylindrical coordinates, so \( \gamma = d\rho^2 + dz^2 + \rho^2 d\phi^2 \). For each value of \( \epsilon \), let us consider an axially symmetric surface \( \Sigma_\epsilon \) defined by \( \chi_\epsilon = \{ \rho = \rho_\epsilon(\mu), z = z_\epsilon(\mu), \phi = \varphi \} \) with the same functions \( \rho_\epsilon(\mu) \) and \( z_\epsilon(\mu) \) as those defining \( \Sigma_\epsilon^E \) in \( \mathcal{V}_\epsilon^E \). Let us denote by \( D_\epsilon \subset \mathbb{E}^3 \) the exterior region of \( \Sigma_\epsilon \). The use of Weyl coordinates in \((\mathcal{V}_\epsilon^E, g_\epsilon^E)\)
allows us to consider \( \{ U_\epsilon, \Omega_\epsilon \} \) as fields on \( D_\epsilon \). The vacuum field equations are equivalent to the so-called Ernst equations on \( D_\epsilon \)

\[
\begin{align*}
\triangle_\gamma U_\epsilon + \frac{1}{2} e^{-4U_\epsilon} (d\Omega_\epsilon, d\Omega_\epsilon)_\gamma &= 0, \\
\triangle_\gamma \Omega_\epsilon - 4 (d\Omega_\epsilon, dU_\epsilon)_\gamma &= 0,
\end{align*}
\]

(11)

where \( \triangle_\gamma \) is the Laplacian of the metric \( g \). This set of equations is supplemented with the asymptotic values \( U_\epsilon \to 1, \Omega_\epsilon \to 0 \) at infinity plus the boundary data on \( \Sigma_\epsilon \) coming from the matching conditions. As already mentioned, these data determine both \( \{ U_\epsilon, \Omega_\epsilon \} \) on \( \Sigma_\epsilon \) and its normal derivatives (except for an additive constant in \( \Omega_\epsilon \)). Thus we are dealing with Cauchy data for the elliptic system of equations (11). This is an overdetermined problem and we should not expect solutions to exist for arbitrary data. That expresses the fact that given an arbitrary stationary and axially symmetric interior metric (even if it is perfect fluid, say), there will in general be no stationary and axially symmetric vacuum exterior solution matching with it and also asymptotically flat. Thus existence for the exterior problem is an important issue. This being true for all \( \epsilon \), the same will happen for the perturbed matching and field equations, as we discuss next. In the following subsections, we first derive the perturbed field equations (subsection IV A) and then the boundary conditions (subsection IV B). Section V is devoted to discussing under what conditions the Cauchy data of the perturbed field equations are compatible.

### A. The Ernst equations up to second order

Let us obtain the systems of equations satisfied by the different orders in \( \epsilon \) of \( U_\epsilon \) and \( \Omega_\epsilon \). Differentiating (11) with respect to \( \epsilon \) we get

\[
\begin{align*}
\triangle_\gamma U_\epsilon' + e^{-4U_\epsilon} (d\Omega_\epsilon, d\Omega_\epsilon')_\gamma - 2e^{-4U_\epsilon} U_\epsilon'' (d\Omega_\epsilon, d\Omega_\epsilon)_\gamma &= 0, \\
\triangle_\gamma \Omega_\epsilon' - 4 (d\Omega_\epsilon', dU_\epsilon)_\gamma - 4 (d\Omega_\epsilon, dU_\epsilon')_\gamma &= 0.
\end{align*}
\]

(12)

Differentiating once more we obtain

\[
\begin{align*}
\triangle_\gamma U_\epsilon'' + e^{-4U_\epsilon} \left[ -8U_\epsilon' (d\Omega_\epsilon, d\Omega_\epsilon')_\gamma + (d\Omega_\epsilon, d\Omega_\epsilon')_\gamma + (d\Omega_\epsilon', d\Omega_\epsilon')_\gamma \\
-2U'' (d\Omega_\epsilon, d\Omega_\epsilon)_\gamma + 8U'^2 (d\Omega_\epsilon, d\Omega_\epsilon)_\gamma \right] &= 0, \\
\triangle_\gamma \Omega_\epsilon'' - 8 (d\Omega_\epsilon', dU_\epsilon)_\gamma - 4 (d\Omega_\epsilon'', dU_\epsilon)_\gamma - 4 (d\Omega_\epsilon, dU_\epsilon'_\gamma) &= 0.
\end{align*}
\]

(13)

The systems for the zeroth, first and second order are now obtained by evaluating the systems (11), (12) and (13) at \( \epsilon = 0 \). As already mentioned, we are interested in studying perturbations of static objects. Thus the background exterior metric satisfies \( \Omega_0 = 0 \) and hence \( U_0 \) is a solution of the Laplace equation

\[
\triangle_\gamma U_0 = 0.
\]

(14)
In fact we are primarily interested in spherical backgrounds so that the exterior background metric is the Schwarzschild metric and $U_0$ is the corresponding Schwarzschild solution. Nevertheless, for the sake of generality we shall keep an arbitrary static and axially symmetric background until explicitly stated. The equations for the first-order perturbation of the static background $\{U'_0, \Omega'_0\}$, which will be called first-order linearized Ernst equations, read

$$
\begin{align*}
\Delta_\gamma U'_0 &= 0, \\
\Delta_\gamma \Omega'_0 - 4 (d\Omega'_0, dU'_0)_\gamma &= 0,
\end{align*}
$$

while for the second order perturbation $\{U''_0, \Omega''_0\}$ we get the second order perturbed Ernst equations

$$
\begin{align*}
\Delta_\gamma U''_0 + e^{-4U_0} (d\Omega'_0, dU'_0)_\gamma &= 0, \\
\Delta_\gamma \Omega''_0 - 4 (d\Omega''_0, dU'_0)_\gamma - 8 (d\Omega'_0, dU'_0)_\gamma &= 0.
\end{align*}
$$

B. Cauchy boundary data up to second order

In order to obtain the boundary data for the exterior problem we should, in a general setting, consider two background spacetimes $(V^I_0, g^I_0)$ and $(V^E_0, g^E_0)$ which match across the unperturbed boundaries $\Sigma^I_0$ and $\Sigma^E_0$ according to the standard matching conditions. Moreover, the first and second order perturbed metric tensors $K^I_a$ and $K^E_a$ ($a = 1, 2$) should satisfy suitable conditions on $\Sigma^I_0$ and $\Sigma^E_0$ coming from suitable first and second $\epsilon$-derivatives of the full matching conditions. These conditions may be called first and second order perturbed matching conditions. Their explicit form in full generality has been obtained by one of us [19]. These can be specialized to find the perturbed matching conditions of interest in this paper. However, for the sake of self-consistency we shall follow an alternative procedure which is well adapted to the stationary and axially symmetric problem we have at hand. Since, as we saw in Sect. III, the boundary conditions for all $\epsilon$ reorganize themselves into an elegant form and the exterior problem reduces to the Ernst equation for $(U_\epsilon, \Omega_\epsilon)$ alone, we need only concentrate on the perturbed boundary conditions for these objects. Note that we have introduced coordinates $\{\tau, \varphi, \mu\}$, with no $\epsilon$ label, on the abstract matching manifold $\sigma_\epsilon$. The reason is the same as before, i.e. we identify points in different $\sigma_\epsilon$ having the same coordinate values $\{\tau, \varphi, \mu\}$.

Given the interior family of metrics, the matching conditions give us (for every $\epsilon$) two functions $\rho_\epsilon(\mu)$ and $z_\epsilon(\mu)$. Thus we have a family of axially symmetric surfaces $\Sigma_\epsilon$ in $E^3$, all of them diffeomorphic to $\sigma_0$. To avoid repetition, we collect together here the assumptions and notation we shall use.

**Assumptions:** Let $(V^I_0, g^I_0)$ and $(V^E_0, g^E_0)$ be two static and axially symmetric spacetimes which can be joined across their static and axially symmetric boundaries $\Sigma^I_0, \Sigma^E_0$. Let $(V^E_0, g^E_0)$ be vacuum and asymptotically flat and choose, locally, Weyl coordinates (9). Let $K^I_{1/E}$ and $K^E_{2/E}$ be symmetric tensors on $V^I_{0/E}$, invariant under the static and axial isometries and such that $K^I_{1/2}$ take the Weyl form (10). Take any metric $g^I_\epsilon$ in $V^I_0$ such that $g^I_\epsilon = g^I_0 + \epsilon K^I_1 + \frac{1}{2} \epsilon^2 K^I_2 + O(\epsilon^3)$ and denote by $\xi^I_\epsilon$ and $\eta^I_\epsilon$ its stationary and axial Killing vectors. Let $(V^I_\epsilon, g^I_\epsilon)$ admit a hypersurface $\Sigma^I_\epsilon$ where it can be locally matched to a
vacuum exterior. Let the identification of $\Sigma^I/E$ with $\sigma_0$ be made through the common use of coordinates $\{\mu, \varphi\}$ in both manifolds. On $\sigma_0$ let functions $\rho_0(\mu), \rho'_0(\mu), z'_0(\mu), z''_0(\mu)$ be defined by $\rho_0(\mu) = \rho_0 + \epsilon\rho'_0 + \frac{1}{2}\epsilon^2\rho''_0 + O(\epsilon^3) = \alpha_0|\Sigma_0, \ z'(\mu) = z_0 + \epsilon z'_0 + \frac{1}{2}\epsilon^2z''_0 + O(\epsilon^3) = -n_I(\alpha_0)|\Sigma_0$, (where $\alpha_0$ and $\tilde{n}_I$ are as in (4) and (5), evaluated at $\Sigma_0$), and functions $V_\epsilon, \tilde{n}V_\epsilon, W_\epsilon, \tilde{n}W_\epsilon, V_0(\mu), V_0''(\mu), \tilde{n}V_0(\mu), \tilde{n}V_0''(\mu), W_0(\mu), W_0''(\mu), \tilde{n}W_0(\mu), \tilde{n}W_0''(\mu)$ by

$$
V_\epsilon = V_0 + \epsilon V'_0 + \frac{1}{2}\epsilon^2 V''_0 + O(\epsilon^3) = \frac{1}{2}\log \left( -\left( \tilde{\xi}'_\epsilon, \tilde{\xi}'_\epsilon \right)_{\bar{g}_\epsilon} \right) |_{\Sigma_\epsilon},
$$

$$
\tilde{n}V_\epsilon = \tilde{n}V_0 + \epsilon \tilde{n}V'_0 + \frac{1}{2}\epsilon^2 \tilde{n}V''_0 + O(\epsilon^3) = \frac{\tilde{n}_I(\tilde{\xi}'_\epsilon, \tilde{\xi}'_\epsilon)_{\bar{g}_\epsilon}}{2(\tilde{\xi}'_\epsilon, \tilde{\xi}'_\epsilon)_{\bar{g}_\epsilon}} |_{\Sigma_\epsilon},
$$

$$
W_\epsilon = \epsilon W'_0 + \frac{1}{2}\epsilon^2 W''_0 + O(\epsilon^3) = -\frac{(\tilde{\xi}'_\epsilon, \tilde{\xi}'_\epsilon)_{\bar{g}_\epsilon}^2}{\alpha_\epsilon}(\tilde{\xi}'_\epsilon, \tilde{n}_I)_{\bar{g}_\epsilon} |_{\Sigma_\epsilon},
$$

$$
\tilde{n}W_\epsilon = \epsilon \tilde{n}W'_0 + \frac{1}{2}\epsilon^2 \tilde{n}W''_0 + O(\epsilon^3) = -\frac{(\tilde{\xi}'_\epsilon, \tilde{\xi}'_\epsilon)_{\bar{g}_\epsilon}^2}{\alpha_\epsilon} \frac{d}{d\mu} (\tilde{\xi}_\epsilon, \tilde{n}_I)_{\bar{g}_\epsilon} |_{\Sigma_\epsilon}.
$$

Note that the staticity implies that the background values are $\Omega_0 = W_0(\mu) = 0$ and $\tilde{n}\Omega_0 = \tilde{n}W_0(\mu) = 0$. As shown in the previous section, the matching conditions imply that $V_\epsilon, \tilde{n}V_\epsilon, W_\epsilon$ and $\tilde{n}W_\epsilon$ are provided by the interior metrics, and give the boundary values $U_\epsilon|\Sigma_0 = V_\epsilon, \tilde{n}_\epsilon(U_\epsilon)|\Sigma_0 = \tilde{n}V_\epsilon, \Omega_\epsilon|\Sigma_0 = W_\epsilon$ and $\tilde{n}_\epsilon(\Omega_\epsilon)|\Sigma_0 = \tilde{n}W_\epsilon$ for the exterior problem. Note that the right hand sides of the four equations just given are functions of $\mu$ and $\epsilon$ alone, while the left hand sides are functions on $\mathbb{R}^3$ evaluated on a (moving) surface $\Sigma_\epsilon$. We can now take derivatives of these expressions with respect to $\epsilon$ (at constant $\mu$) in order to obtain the perturbed boundary conditions for the perturbation functions $U'_\epsilon, \Omega'_0, U''_\epsilon$ and $\Omega''_0$. It is clear that when taking derivatives on the right hand sides of (17) two types of terms appear, namely those coming from the explicit dependence on $\epsilon$ in the functions and those coming from the fact that the surfaces $\Sigma_\epsilon$ also depend on $\epsilon$. The latter will involve $\partial_\mu$ and $\partial_\nu$ (and higher) derivatives of lower order terms $U'_0, \Omega'_0, U''_0$ and $\Omega''_0$, all of them evaluated at the unperturbed surface $\Sigma_0$. Our aim is to express everything in terms of the twelve functions $V_0(\mu), V'_0(\mu), V''_0(\mu), \tilde{n}V_0(\mu), \tilde{n}V'_0(\mu), \tilde{n}V''_0(\mu), W_0(\mu), W'_0(\mu), W''_0(\mu), \tilde{n}W_0(\mu), \tilde{n}W'_0(\mu), \tilde{n}W''_0(\mu)$ of $\mu$, which correspond exactly to the boundary information coming from the interior through the background matching and the first and second order perturbed matching conditions.

Let us take a fixed point $p \in \sigma_0$ and consider the trajectory in $\mathbb{R}^3$ defined by $\chi_\epsilon(p)$ when $\epsilon$ varies. The first and second derivatives along this trajectory, evaluated at $\epsilon = 0$, give us two vectors on $p$ and hence two vector fields on $\Sigma_0$. They determine how the matching surface moves within $\mathbb{R}^3$ to second order in approximation theory. Denoting these vector fields by $\tilde{Z}_1$ and $\tilde{Z}_2$, we obviously have

$$
\tilde{Z}_1 = \rho'_0 \partial_\mu + z'_0 \partial_\nu|_{\Sigma_0}, \quad \tilde{Z}_2 = \rho''_0 \partial_\mu + z''_0 \partial_\nu|_{\Sigma_0},
$$

where, as before, $\rho'_0 \equiv \partial_\epsilon \rho_\epsilon|_{\epsilon=0}, \rho''_0 \equiv \partial_\epsilon \partial_\mu \rho_\epsilon|_{\epsilon=0},$ etc. Two further relevant vectors on $\Sigma_0$ are
the first and second perturbations of the normals \( \tilde{n}_\epsilon \). Explicitly

\[
\frac{\partial \tilde{n}_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = -z_0' \partial_\rho + \dot{\rho}_0 \partial_z |_{\Sigma_0},
\]

\[
\frac{\partial^2 \tilde{n}_\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} = -z_0'' \partial_\rho + \ddot{\rho}_0 \partial_z |_{\Sigma_0}.
\]

(19)

Since we want to rewrite everything in terms of intrinsic objects on the unperturbed surface \( \Sigma_0 \), we need to use a basis adapted to this surface. A convenient choice consists of the unperturbed tangent and normal vectors, namely \( \tilde{e} \equiv d\chi_0(\partial_\mu) = \dot{\rho}_0 \partial_\rho + z_0 \partial_z |_{\Sigma_0} \) and \( \tilde{n} = \tilde{n}_\epsilon |_{\epsilon=0} \) (for notational convenience we drop a subindex 0 both in \( \tilde{e} \) and in \( \tilde{n} \)). Writing down the vectors \( \{ \partial_\rho, \partial_z \} \) on \( \Sigma_0 \) in terms of \( \tilde{e} \) and \( \tilde{n} \) we get

\[
\partial_z|_{\Sigma_0} = \frac{1}{z_0^2 + \dot{\rho}_0^2} \left( z_0 \tilde{e} + \dot{\rho}_0 \tilde{n} \right),
\]

\[
\partial_\rho|_{\Sigma_0} = \frac{1}{z_0^2 + \dot{\rho}_0^2} \left( \dot{\rho}_0 \tilde{e} - z_0 \tilde{n} \right).
\]

(20)

We can now express \( \tilde{Z}_1, \tilde{Z}_2, \frac{\partial \tilde{n}_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0}, \frac{\partial^2 \tilde{n}_\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0}, \tilde{e}, \tilde{n} \), in terms of \( \{ \tilde{e}, \tilde{n} \} \). When doing this, six scalar fields on \( \Sigma_0 \) appear in a natural way. Denoting them by \( P_1, Q_1, P_2, Q_2, X_0 \) and \( X_1 \) we have

\[
\tilde{Z}_1 = P_1 \tilde{e} + Q_1 \tilde{n}, \quad \tilde{Z}_2 = P_2 \tilde{e} + Q_2 \tilde{n},
\]

\[
\frac{\partial \tilde{n}_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = \left( -\frac{dQ_1}{d\mu} + P_1 X_0 - Q_1 X_1 \right) \tilde{e} + \left( \frac{dP_1}{d\mu} - P_1 X_1 + Q_1 X_0 \right) \tilde{n},
\]

\[
\frac{\partial^2 \tilde{n}_\epsilon}{\partial \epsilon^2} \bigg|_{\epsilon=0} = \left( -\frac{dQ_2}{d\mu} + P_2 X_0 - Q_2 X_1 \right) \tilde{e} + \left( \frac{dP_2}{d\mu} + P_2 X_1 + Q_2 X_0 \right) \tilde{n},
\]

where

\[
P_1 = \frac{\dot{\rho}_0 \rho'_0 + z_0 z'_0}{\rho''_0 + z'_0^2}, \quad Q_1 = \frac{\dot{\rho}_0 z'_0 - z_0 \rho'_0}{\rho''_0 + z'_0^2}, \quad P_2 = \frac{\dot{\rho}_0 z''_0 + z_0 \rho''_0}{\rho''_0 + z'_0^2}, \quad Q_2 = \frac{\dot{\rho}_0 z''_0 - z_0 \rho''_0}{\rho''_0 + z'_0^2},
\]

\[
X_0 = \frac{\dot{\rho}_0 z'_0 - z_0 \rho'_0}{\rho''_0 + z'_0^2}, \quad X_1 = \frac{\dot{\rho}_0 \rho'_0 + z_0 z'_0}{\rho''_0 + z'_0^2}.
\]

(21)

Having defined these objects we can evaluate the first and second order perturbations of the matching conditions. The statement of the result is rather lengthy.

**Proposition IV.1** Under the Assumptions stated above, the metrics \( g^E \) and \( g^E = g^E_0 + \epsilon K^E_1 + \frac{1}{2} \epsilon^2 K^E_2 \) match perturbatively to second order on \( \Sigma_0^{I/E} \) if and only if the following
conditions are satisfied

\[ U_0|_{\Sigma_{\delta}} = V_0, \quad \bar{n}(U_0)|_{\Sigma_{\delta}} = \bar{n}V_0, \quad U_0'|_{\Sigma_{\delta}} = V_0' - P_1 \frac{dV_0}{d\mu} - Q_1 \bar{n}V_0, \]

\[ \bar{n}(U_0')|_{\Sigma_{\delta}} = \bar{n}V_0' + \frac{d}{d\mu} \left( Q_1 \frac{dV_0}{d\mu} \right) - d \left( P_1 (\bar{n}V_0) \right) d\mu + Q_1 \left( \frac{\dot{\rho}_0}{\rho_0} \frac{dV_0}{d\mu} - \frac{\dot{z}_0}{\rho_0} \bar{n}V_0 \right), \]

\[ U_0''|_{\Sigma_{\delta}} = V_0'' - 2P_1 \frac{dV_0'}{d\mu} - 2Q_1 \bar{n}V_0' + \frac{d}{d\mu} \left( (P_1^2 - Q_1^2) \frac{dV_0}{d\mu} \right) + \frac{d}{d\mu} \left( 2P_1Q_1\bar{n}V_0 \right) \]

\[ + \left( -P_2 + P_1X_1 + 2P_1Q_1X_0 - Q_1^2X_1 - Q_1^2 \frac{\dot{\rho}_0}{\rho_0} \right) \frac{dV_0}{d\mu} \]

\[ + \left( -Q_2 - P_2X_0 + 2P_1Q_1X_1 + Q_1^2X_0 + Q_1^2 \frac{\dot{z}_0}{\rho_0} \right) \bar{n}V_0, \]

\[ \bar{n}(U_0'')|_{\Sigma_{\delta}} = \bar{n}V_0'' + 2 \frac{d}{d\mu} \left( Q_1 \frac{dV_0'}{d\mu} \right) - 2 \frac{d}{d\mu} \left( P_1 \bar{n}V_0' \right) + 2Q_1 \left( \frac{\dot{\rho}_0}{\rho_0} \frac{dV_0}{d\mu} - \frac{\dot{z}_0}{\rho_0} \bar{n}V_0' \right) \]

\[ - \frac{d^2}{d\mu^2} \left( 2P_1Q_1 \frac{dV_0}{d\mu} \right) + \frac{d^2}{d\mu^2} \left( (P_1^2 - Q_1^2) \bar{n}V_0 \right) \]

\[ + \frac{d}{d\mu} \left\{ \left[ Q_2 + (P_1^2 - Q_1^2)X_0 - 2P_1Q_1 \frac{\dot{\rho}_0}{\rho_0} - 2P_1Q_1X_1 - Q_1^2 \frac{\dot{z}_0}{\rho_0} \right] \frac{dV_0}{d\mu} \right\} \]

\[ + \frac{d}{d\mu} \left\{ \left[ -P_2 + (P_1^2 - Q_1^2)X_1 + 2P_1Q_1 \frac{\dot{z}_0}{\rho_0} + 2P_1Q_1X_0 - Q_1^2 \frac{\dot{\rho}_0}{\rho_0} \right] \bar{n}V_0 \right\} \]

\[ + \left( Q_2 + (P_1^2 - Q_1^2)X_0 - 2P_1Q_1 \frac{\dot{\rho}_0}{\rho_0} \right) \left( \frac{\dot{\rho}_0}{\rho_0} \frac{dV_0}{d\mu} - \frac{\dot{z}_0}{\rho_0} \bar{n}V_0 \right) \]

\[ + \left( 2P_1Q_1X_0 - Q_1^2 \frac{\dot{\rho}_0}{\rho_0} \right) \left( \frac{\dot{\rho}_0}{\rho_0} \bar{n}V_0 + \frac{\dot{z}_0}{\rho_0} \frac{dV_0}{d\mu} \right) \), \]

\[ \Omega_0'|_{\Sigma_{\delta}} = W_0', \quad \bar{n}(\Omega_0')|_{\Sigma_{\delta}} = \bar{n}W_0', \quad \Omega_0''|_{\Sigma_{\delta}} = W_0'' - 2P_1 \frac{dW_0'}{d\mu} - 2Q_1 \bar{n}W_0'. \]

\[ \bar{n}(\Omega_0'')|_{\Sigma_{\delta}} = \bar{n}W_0'' + 2 \frac{d}{d\mu} \left( Q_1 \frac{dW_0'}{d\mu} \right) - 2 \frac{d}{d\mu} \left( P_1 \bar{n}W_0'' \right) \]

\[ + 2Q_1 \left[ \left( \frac{\dot{\rho}_0}{\rho_0} - 4 \frac{dV_0}{d\mu} \right) \frac{dW_0'}{d\mu} - \left( \frac{\dot{z}_0}{\rho_0} + 4 \bar{n}V_0 \right) \bar{n}W_0' \right]. \]

where \( P_1, P_2, Q_1, Q_2, X_1, X_2 \) are defined in (21).

The actual necessity and sufficiency arises from the corresponding properties of the general Darmois conditions. The main work in the proof is the direct but cumbersome calculation needed to arrive at the formulae: we leave this to the Appendix.

V. COMPATIBILITY CONDITIONS

In the previous section we found the overdetermined boundary data for \( U_0', \Omega_0', U_0'', \Omega_0'' \) in terms of the interior metric and perturbation tensors. Since the equations they satisfy are elliptic we need to determine under what conditions asymptotically flat solutions exist. Asymptotic flatness requires that, for any value of \( \epsilon \), \( \lim_{\rho^2+z^2 \to \infty} U_0 = 1 \) and \( \lim_{\rho^2+z^2 \to \infty} \Omega_0 = 0 \). This implies \( \lim_{\rho^2+z^2 \to \infty} U_0' = \lim_{\rho^2+z^2 \to \infty} \Omega_0' = \lim_{\rho^2+z^2 \to \infty} U_0'' = \lim_{\rho^2+z^2 \to \infty} \Omega_0'' = 0 \).
The problem of determining the Neumann data (i.e. the normal derivative of the function on the boundary \( \partial \Omega \) of a domain \( \Omega \)) in terms of the Dirichlet data (the value of the function on \( \partial \Omega \)), so that the Cauchy problem for an elliptic equation on \( \Omega \) is solvable, is called obtaining the Dirichlet-Neumann map. Many results are known on this problem, including deriving general properties of the map, determining the coefficients of the elliptic equation from the Dirichlet-Neumann map and finding explicit representations of this map for especially simple equations and domains. Good references on this topic are [35] and [36]. Our problem can then be phrased as saying that we want to find explicit restrictions on the boundary data so that they solve the Dirichlet-Neumann problem for the linearized Ernst equations on axially symmetric domains.

While the equation for \( U_0' \) is just the Laplace equation in Euclidean space, the equation for \( \Omega_0' \) has, in addition, lower order terms. In order to treat all cases at the same time it turns out to be convenient to define a conformally flat metric \( \tilde{\gamma} = e^{-8t/\alpha} \gamma \) on \( D_0 \), in terms of which the second equations in (15)-(16) can be rewritten as

\[
\Delta_{\tilde{\gamma}} \Omega_0' = 0, \\
\Delta_{\tilde{\gamma}} \Omega_0'' = 8 (d\Omega_0', dU_0')_{\tilde{\gamma}}.
\]

Thus all equations for \( U_0', U_0'', \Omega_0', \Omega_0'' \) can be collectively written as

\[
\Delta_{\tilde{\gamma}} u = j,
\]

where \( u = u(\rho, z) \) stands for \( U_0, U_0' \), etc..., and \( j = j(\rho, z) \) represents the inhomogeneous terms in the second order perturbation equations. The metric \( \tilde{\gamma} \) corresponds to either \( \gamma \), for the \( U \)-equations, or \( \tilde{\gamma} \), for the \( \Omega \)-equations. The domain \( (D_0, \tilde{\gamma}) \) is clearly non-compact because \( \tilde{\gamma} \) is an asymptotically flat metric. Thus the compatibility conditions for the boundary values of \( U_0', U_0'', \Omega_0', \Omega_0'' \) can be studied as particular cases of the compatibility conditions of the Cauchy problem for the general inhomogeneous Poisson equation (22) defined on a non-compact asymptotically flat region \( (D_0, \tilde{\gamma}) \) with a boundary \( \partial D_0 = \{ \rho = \rho_0(\mu), z = z_0(\mu), \phi = \varphi \} \). Furthermore, we shall assume that \( j \) tends to zero at infinity at least like \( 1/r^4 \) where \( r \equiv \sqrt{\rho^2 + z^2} \) (this requirement is fulfilled in the cases we are concerned with due to asymptotic flatness).

We start with some well-known facts from potential theory. A simple consequence of Gauss’ theorem is Green’s identity which, for any compact domain \( K \subset D_0 \) with \( C^1 \) boundary \( \partial K \) and any \( C^2 \) function \( \psi \) on \( K \) with \( C^1 \) extension to \( \overline{K} \cup \partial K \), reads

\[
\int_K (\psi \Delta_{\tilde{\gamma}} u - u \Delta_{\tilde{\gamma}} \psi) d\gamma = \int_{\partial K} \big[ \psi \tilde{n}_{\tilde{\gamma}}(u) - u \tilde{n}_{\tilde{\gamma}}(\psi) \big] dS_{\tilde{\gamma}},
\]

where \( \tilde{n}_{\tilde{\gamma}} \) is a unit (with respect to \( \tilde{\gamma} \)) normal vector pointing out from \( K \), \( d\gamma \) is the volume form of \( (D_0, \tilde{\gamma}) \) and \( dS_{\tilde{\gamma}} \) is the induced surface element of \( \partial K \). We intend to apply this identity to a function \( \psi \) that (i) solves the Laplace equation \( \Delta_{\tilde{\gamma}} \psi = 0 \) on \( D_0 \), (ii) admits a \( C^1 \) extension to \( \partial D_0 \) and (iii) decays at infinity in such a way that \( \psi \sqrt{\rho^2 + z^2} \) is a bounded function on \( D_0 \). A function \( \psi \) satisfying these three properties is called a regular \( \tilde{\gamma} \)-harmonic function on \( D_0 \) (if a function satisfies just (ii) and (iii) and is \( C^2 \) on \( D_0 \) we shall call it regular).
For such a function we can take $K = D_0$ in (23) because the integral over the boundary “at infinity” can be easily shown to vanish. Thus

$$\int_{\partial D_0} \left[ \psi \vec{n}_\gamma(u) - u \vec{n}_\gamma(\psi) \right] dS_\gamma = \int_{D_0} \psi j \eta_\gamma. \quad (23)$$

This expression relates the overdetermined boundary data on $\partial D_0$ to a volume integral of the inhomogeneous term $j$. Denoting the boundary data for $u$ by $u|_{\partial D_0} \equiv f_0$ and $\vec{n}_\gamma(u)|_{\partial D_0} \equiv f_1$, we have, explicitly,

$$\int_{\partial D_0} \left[ \psi f_1 - f_0 \vec{n}_\gamma(\psi) \right] dS_\gamma = \int_{D_0} \psi j \eta_\gamma, \quad (24)$$

which are clearly necessary conditions on the boundary data for existence of a regular solution $u$. It is natural to ask whether such conditions are also sufficient. More precisely, assume that a continuous function $j$ is given on $D_0$ such that $r^4 j$ is bounded at infinity. Give also two arbitrary continuous functions $f_0$ and $f_1$ on $\partial D_0$ which satisfy (24) for any choice of regular $\gamma$-harmonic function $\psi$. We want to check whether there always exists a function $u$ satisfying the Poisson equation $\Delta_\gamma u = j$, with $ru$ bounded at infinity and such that the boundary equations $u|_{\partial D_0} = f_0$, $\vec{n}_\gamma(u)|_{\partial D_0} = f_1$ are satisfied. The answer is yes as we prove next. Consider the Dirichlet problem $\Delta_\gamma u = j$ with $u|_{\partial D_0} = f_0$. Standard elliptic theory tells us that this problem always admits a unique solution $u$ which tends to zero at infinity. Let us define $\tilde{f}_1$ on $\partial D_0$ by $\tilde{f}_1 \equiv \vec{n}_\gamma(u)|_{\partial D_0}$. Since $u$ solves the Poisson equation, it follows from (23) that, for any regular $\gamma$-harmonic function $\psi$,

$$\int_{\partial D_0} \left[ \psi \tilde{f}_1 - f_0 \vec{n}_\gamma(\psi) \right] dS_\gamma = \int_{D_0} \psi j \eta_\gamma. \quad (25)$$

Our assumption is that $f_0$ and $f_1$ satisfy (24) for any such $\psi$. Subtracting (24) and (25) we get $\int_{\partial D_0} \psi (f_1 - \tilde{f}_1) dS_\gamma = 0$. However, since we can take any regular $\gamma$-harmonic function $\psi$, the fact that the Dirichlet problem for the Laplace equation always admits a solution allows us to choose $\psi|_{\partial D_0}$ to be any continuous function. This readily implies $f_1 = \tilde{f}_1$ and hence that the overdetermined boundary data admits a decaying solution, as claimed.

However, the compatibility condition (24) has a disadvantage, namely that it must be checked for an arbitrary decaying solution $\psi$ of the Laplace equation. This makes it useless in practical terms. Our aim is to reduce the number of solutions $\psi$ that must be checked in (24) in order to ensure compatibility of $f_0$ and $f_1$. Here is where axial symmetry plays an essential role. We restrict ourselves to axially symmetric functions $f_0$ and $f_1$ on an axially symmetric boundary $\partial D_0$ and we assume further that $\partial D_0$ is simply connected, so that it is diffeomorphic to a 2-sphere. Using coordinates $\mu$ and $\varphi$ on $\partial D_0$, let us denote by $\mu_{SD}$ and $\mu_N$ (with $\mu_S < \mu_N$) the only two values of $\mu$ at which $\partial D_0$ intersects the axis of symmetry. Let us also assume that they satisfy $z_0(\mu_S) < z_0(\mu_N)$, i.e. that $\mu$ increases from the “south” pole of the object (corresponding to $\mu_S$) to the “north” pole (corresponding to $\mu_N$). With this assumption, a direct calculation shows that $dS_\gamma \vec{n}_\gamma = \rho_0 d\mu d\varphi \vec{n}$, where $\vec{n}$ is our usual normal vector $\vec{n} = -z_\gamma \partial_\rho + \rho_0 \partial_\varphi|_{D_0}$. Similarly $dS_\gamma \vec{n}_\gamma = e^{-4V_0(z_0)} \rho_0 d\mu d\varphi \vec{n}$. For $\gamma = \gamma$, (23) becomes, after a trivial angular integration,

$$\int_{\mu_N}^{\mu_S} \left[ \psi \vec{n}(u) - u \vec{n}(\psi) \right] \rho_0|_{\Sigma_0} d\mu = \frac{1}{2\pi} \int_{D_0} \psi j \eta_\gamma. \quad (26)$$
For $\hat{\gamma} = \hat{\gamma}$, (23) becomes
\begin{equation}
\int_{\mu_S}^{\mu_N} \left[ \psi \bar{n}(u) - u \bar{n}(\psi) \right] \rho_0 e^{-4u_0} \big|_{\Sigma_0} \, d\mu = \frac{1}{2\pi} \int_{D_0} \psi j e^{-12u_0} \eta_N. \tag{27}
\end{equation}

We want to choose a reduced set of functions $\psi$ for which the argument used above to show consistency of the data still holds. The following Lemma is probably known although we could not find an explicit reference for it.

**Lemma V.1** Let $h : [\mu_S, \mu_N] \to \mathbb{R}$ be a continuous function satisfying
\begin{equation}
\int_{\mu_S}^{\mu_N} \frac{h(\mu) \, d\mu}{\sqrt{\rho_0^2(\mu) + (z_0(\mu) - y)^2}} = 0
\end{equation}
for any constant $y \in (z_S, z_N)$, where $z_S \equiv z_0(\mu_S)$ and $z_N \equiv z_0(\mu_N)$. Then $h \equiv 0$.

**Proof:** Let us define a function $\tilde{h}$ on $\partial D_0$ by extending $h$ in an axially symmetric way, i.e. $\tilde{h}(\mu, \varphi) \equiv h(\mu)$. For any point $q \in \mathbb{R}^3$ let us define the function
\begin{equation}
Y_h(q) = \int_{\partial D_0} \frac{\tilde{h}}{\text{dist}(\cdot, q)} \, dS_\gamma,
\end{equation}
where $\text{dist}(\cdot, q)$ denotes Euclidean distance between a point on $\partial D_0$ and the point $q$. This function, also called “single layer potential”, is well-defined throughout $\mathbb{R}^3$ (including the axis of symmetry and the surface $\partial D_0$ [37]), is axially symmetric and vanishes at infinity. Moreover, potential theory tells us that $Y_h$ is $C^0$ everywhere (including $\partial D_0$) and satisfies $\Delta \gamma Y_h = 0$ except on $\partial D_0$. The function $\tilde{h}$ is directly related to the jumps of the first normal derivative of $Y_h$ on $\partial D_0$ [37] ($Y_h$ can be physically interpreted as the potential created by a surface layer sitting on $\partial D_0$). Thus $h$ vanishes if and only if $Y_h$ is $C^1$ on $\mathbb{R}^3$.

The hypothesis of the Lemma tells us that $Y_h$ vanishes on the piece of the axis lying between the south and the north pole of $\partial D_0$. Regularity of the Laplace equation shows that $Y_h$ is analytic except at $\partial D_0$. If $Y_h$ is identically zero in some neighbourhood of the axis inside $\partial D_0$, then continuity at $\partial D_0$ and analyticity implies $Y_h \equiv 0$ everywhere and $h = 0$ would follow. Let us thus assume that there is a neighbourhood $\mathcal{U}$ of a point lying on the axis between the south and north poles in which $Y_h$ is not identically zero. Analyticity in Cartesian coordinates implies that $Y_h$ depends analytically on $\rho^2$ and $z$ in cylindrical coordinates. Not being identically zero, there must exist a minimum value $k \in \mathbb{N}$ and an analytic (not identically zero) function $g(z)$ such that $Y_h|_U = g(z)\rho^{2k} + O(\rho^{2k+2})$. The fact that $Y_h$ vanishes on the axis demands $k \geq 1$. Substitution into the Laplace equation $\partial_{\rho\rho} Y_h + \rho^{-1} \partial_{\rho} Y_h + \partial_z Y_h = 0$ yields $4k^2 g(z)\rho^{2k-2} + O(\rho^{2k}) = 0$ which is a contradiction. This completes the proof. $\square$

This Lemma already suggests which subclass of regular $\hat{\gamma}$-harmonic functions needs to considered for the compatibility of the boundary conditions. For the flat metric $\gamma$, a natural class of axially symmetric harmonic functions is
\begin{equation}
\psi_y(\rho, z) \equiv \frac{1}{\sqrt{\rho^2 + (z - y)^2}}, \quad y \in (z_S, z_N). \tag{28}
\end{equation}
This expression is singular only at $\rho = 0, z = y$, which lies on the axis of symmetry outside $D_0$. Thus $\psi_y$ is indeed a regular $\gamma$-harmonic function on $D_0$.

For the $\tilde{\gamma}$ metric (corresponding to the $\Omega$-equations), we need to find a suitable class of axially symmetric solutions of $\triangle u = 0$. In order to find them more easily let us consider the semiplane $\mathbb{K} = \mathbb{R}^+ \times \mathbb{R}$ defined as the subset $\{ \phi = \text{const.}, \rho \geq 0 \}$ of $\mathbb{E}^3$ in cylindrical coordinates. Working directly on $\mathbb{K}$ has the advantage that axial symmetry is incorporated into the calculations from the very beginning. $\mathbb{K}$ is endowed with a flat metric $d\rho^2 + dz^2$.

We choose the orientation so that $\star d\rho = -dz$ (and hence $\star dz = d\rho$). Obviously any axially symmetric function in $\mathbb{E}^3$ immediately defines a function on $\mathbb{K}$. Similarly a function on $\mathbb{K}$ defines an axially symmetric function on $\mathbb{E}^3$. We shall use the same symbol to denote both functions (the precise meaning should be clear from the context). The field equations (15)-(16) can be translated into equations on $\mathbb{K}$. It is straightforward to check that the $\gamma$-Laplace equation (i.e. the equation satisfied by $U_0$ or $U'_0$) becomes simply $d(\rho \star dU_0) = 0$ outside the axis of symmetry. For the $\tilde{\gamma}$-Laplace equation, we first note the following simple identity, valid for any pair of functions $f_1, f_2$ on $\mathbb{K}$,

$$df_1 \wedge \star df_2 = -\star df_1 \wedge df_2 = -(df_1, df_2)_\gamma d\rho \wedge dz.$$  
Thus the second equation in (15) can be rewritten, away from the axis of symmetry, as

$$d(\rho \star d\Omega'_0) - 4\rho d\Omega'_0 \wedge \star dU_0 = 0. \quad (29)$$

Our aim is to find suitable regular $\tilde{\gamma}$-harmonic functions, i.e. suitable solutions of this equation on the domain $\mathbb{K}^E$ corresponding to the exterior domain $D_0$. More precisely, the surface $\Sigma_0 \subset \mathbb{E}^3$ projects into a line $c_0$ in $\mathbb{K}$ defined parametrically as $\{ z = z_0(\mu), \rho = \rho_0(\mu) \}$. This line separates $\mathbb{K}$ into two regions, the exterior (denoted by $\mathbb{K}^E$) and the interior. The assumption we made on the topology of $\Sigma_0$, namely that it has vanishing genus, implies that $\mathbb{K}^E$ is simply connected and that $c_0$ intersects $\rho = 0$ at two values of $z$, namely $z_S$ and $z_N$. In order to determine suitable solutions of $\triangle \tilde{\gamma} u = 0$ the following lemma is useful.

**Lemma V.2** For any $y \in (z_S, z_N)$ and for any point $(\rho, z) \in \mathbb{K}^E$ define $\Upsilon_y(\rho, z) \in [0, 2\pi)$ by

$$\cos \Upsilon_y(\rho, z) \equiv \frac{z - y}{\sqrt{\rho^2 + (z - y)^2}},$$

$$\sin \Upsilon_y(\rho, z) \equiv \frac{\rho}{\sqrt{\rho^2 + (z - y)^2}}.$$

Then the PDE

$$dZ_y = \cos \Upsilon_y \, dU_0 + \sin \Upsilon_y \, \star dU_0, \quad (30)$$

with boundary condition $\lim_{\rho^2 + z^2 \to \infty} Z_y = 0$ admits a unique solution on $\mathbb{K}^E$.

**Proof:** The function $\Upsilon_y$ is regular everywhere in $\mathbb{K}$ except at $\{ z = \rho = 0 \}$ which lies outside $\mathbb{K}^E$. Similarly $U_0$ is regular everywhere on this simply connected domain. So, in order to prove existence we only need to show that $d(\cos \Upsilon_y \, dU_0 + \sin \Upsilon_y \, \star dU_0) = 0$ on $\mathbb{K}^E$.

A simple calculation which uses only the fact that $U_0$ is a flat-harmonic function on $D_0$ gives
Using the definition of $\Upsilon_y$ above it is immediate to show that the term in parenthesis on the right hand side vanishes. Thus existence of $Z_y$ follows. Furthermore, asymptotic flatness implies that the right hand side of (30) tends to zero like $1/r^2$. Thus $Z_y$ is bounded at infinity and we can impose boundary data there. Since the general solution of (30) depends on an arbitrary additive constant, the lemma follows. □

Note that both $\Upsilon_y$ and $U_0$ are analytic functions on $\mathbb{K}^E$. It immediately follows that $Z_y$ is also analytic on this domain. Let us now define the function

$$
\Psi_y(\rho, z) = \frac{e^{2U_0-2Z_y}}{\sqrt{\rho^2 + (z - y)^2}}, \quad y \in (z_S, z_N).
$$

This function solves $\Delta \tilde{\gamma} u = 0$, as we show next.

**Lemma V.3** The function $\Psi_y$ defined in (31) is a regular $\tilde{\gamma}$-harmonic function on $D_0$.

**Proof:** We need to show (i) $\Delta \tilde{\gamma} \Psi_y = 0$ on $D_0$, (ii) $\Psi_y$ admits a $C^1$ extension to $\partial D_0$ and (iii) $r \Psi_y$ is bounded at infinity. Property (iii) is immediate from the corresponding property of $Z_y$. Furthermore $U_0$ is $C^1$ on $\partial D_0 \cup D_0$, which implies, from the PDE (30), that $Z_y$ is also $C^1$ on this subset. In addition $\Upsilon_y$ is analytic except at the point $\{z = y, \rho = 0\}$; hence (ii) holds. In order to prove (i) it is sufficient to check the differential equation

$$
\mathbf{d} (\rho \ast \mathbf{d} \Psi_y) - 4 \rho \mathbf{d} \Psi_y \wedge \ast \mathbf{d} U_0 = 0,
$$

which is equivalent to $\Delta \tilde{\gamma} \Psi_y = 0$ except on the symmetry axis $\rho = 0$ ( $\Delta \tilde{\gamma} \Psi_y = 0$ on the axis follows from the fact that $\Psi_y$ is $C^2$ on $D_0$ – in fact analytic). Note that $\Psi_y e^{-2U_0+2Z_y}$ is just the flat-space harmonic function $1/r$ where $r$ is the distance from $\rho = 0$, $z = y$. Using this and $\mathbf{d} Z_y \wedge \ast \mathbf{d} U_0 = \mathbf{d} U_0 \wedge \ast \mathbf{d} U_0$, a simple calculation gives the result. □

We can now state the necessary and sufficient conditions that the boundary data must satisfy so that $\Delta \tilde{\gamma} u = j$ admits a decaying solution.

**Theorem V.1** Let $f_0$, $f_1$ be continuous axially symmetric functions on a $C^1$ simply connected, axially symmetric surface $\Sigma_0$ of $\mathbb{E}^3$. Let this surface be defined in cylindrical coordinates by $\{\rho = \rho_0(\mu), z = z_0(\mu), \phi = \phi\}$, where $\mu$ takes values in $[\mu_S, \mu_N]$ and $\mu_S < \mu_N$ are the only solutions of $\rho_0(\mu) = 0$. Call $z_S \equiv z(\mu_S)$ and $z_N \equiv z(\mu_N)$ and assume $z_S < z_N$ (i.e. that these values correspond to the “south” and “north” poles of the surface, respectively). Denote by $D_0$ the exterior region of this surface and let $j$ be any axially symmetric function on $D_0$ such that $r^4 j$ is bounded at infinity. Let $\gamma$ be the flat metric and $\tilde{\gamma} = e^{-8U_0} \gamma$, where $U_0$ is any regular flat-harmonic function on $D_0$. Then the Cauchy problem

$$
\Delta \tilde{\gamma} u = j, \quad u|_{\Sigma_0} = f_0, \quad \vec{n}(u)|_{\Sigma_0} = f_1,
$$

holds. In order to prove (i) it is sufficient to check the differential equation

$$
\mathbf{d} (\rho \ast \mathbf{d} \Psi_y) - 4 \rho \mathbf{d} \Psi_y \wedge \ast \mathbf{d} U_0 = 0,
$$

which is equivalent to $\Delta \tilde{\gamma} \Psi_y = 0$ except on the symmetry axis $\rho = 0$ ( $\Delta \tilde{\gamma} \Psi_y = 0$ on the axis follows from the fact that $\Psi_y$ is $C^2$ on $D_0$ – in fact analytic). Note that $\Psi_y e^{-2U_0+2Z_y}$ is just the flat-space harmonic function $1/r$ where $r$ is the distance from $\rho = 0$, $z = y$. Using this and $\mathbf{d} Z_y \wedge \ast \mathbf{d} U_0 = \mathbf{d} U_0 \wedge \ast \mathbf{d} U_0$, a simple calculation gives the result. □

We can now state the necessary and sufficient conditions that the boundary data must satisfy so that $\Delta \tilde{\gamma} u = j$ admits a decaying solution.

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$$
\Delta \tilde{\gamma} u = j, \quad u|_{\Sigma_0} = f_0, \quad \vec{n}(u)|_{\Sigma_0} = f_1,
$$

holds. In order to prove (i) it is sufficient to check the differential equation

$$
\mathbf{d} (\rho \ast \mathbf{d} \Psi_y) - 4 \rho \mathbf{d} \Psi_y \wedge \ast \mathbf{d} U_0 = 0,
$$

which is equivalent to $\Delta \tilde{\gamma} \Psi_y = 0$ except on the symmetry axis $\rho = 0$ ( $\Delta \tilde{\gamma} \Psi_y = 0$ on the axis follows from the fact that $\Psi_y$ is $C^2$ on $D_0$ – in fact analytic). Note that $\Psi_y e^{-2U_0+2Z_y}$ is just the flat-space harmonic function $1/r$ where $r$ is the distance from $\rho = 0$, $z = y$. Using this and $\mathbf{d} Z_y \wedge \ast \mathbf{d} U_0 = \mathbf{d} U_0 \wedge \ast \mathbf{d} U_0$, a simple calculation gives the result. □

We can now state the necessary and sufficient conditions that the boundary data must satisfy so that $\Delta \tilde{\gamma} u = j$ admits a decaying solution.
where $\vec{n}|\Sigma_0 = -\hat{z}_0 \partial_\rho + \rho_0 \partial_z$, admits a regular solution if and only if the compatibility conditions

$$\hat{\gamma} = \gamma : \int_{\mu S}^{\mu N} \left[ \psi_y f_1 - f_0 \vec{n}(\psi_y) \right] \rho_0|\Sigma_0 \, d\mu = \frac{1}{2\pi} \int_{D_0} \psi_y j \eta_\gamma, \quad \forall y \in (z_S, z_N) \quad (32)$$

$$\hat{\gamma} = \gamma : \int_{\mu S}^{\mu N} \left[ \Psi_y f_1 - f_0 \vec{n}(\Psi_y) \right] \rho_0 e^{-4U_0}|\Sigma_0 \, d\mu = \frac{1}{2\pi} \int_{D_0} \Psi_y j e^{-12U_0} \eta_\gamma, \quad \forall y \in (z_S, z_N) \quad (33)$$

are satisfied, where $\psi_y$ and $\Psi_y$ are given in (28) and (31) respectively.

Proof: We give the proof for $\hat{\gamma} = \gamma$; the case $\hat{\gamma} = \gamma$ follows by setting $U_0 = 0$ everywhere. Necessity follows directly from Green’s identity (23). In order to prove sufficiency, let $u$ be the unique regular solution of the Dirichlet problem $\triangle u = j$, $u|\Sigma_0 = f_0$ on $D_0$ (which is known to exist). Define $\vec{n}(u)|\Sigma_0 = \vec{f}_1$. Green’s identity and the fact that $\Psi_y$ is a regular $\gamma$-harmonic function implies

$$\int_{\mu S}^{\mu N} \left[ \Psi_y \vec{f}_1 - f_0 \vec{n}(\Psi_y) \right] \rho_0 e^{-4U_0}|\Sigma_0 \, d\mu = \frac{1}{2\pi} \int_{D_0} \Psi_y j e^{-12U_0} \eta_\gamma.$$

Subtracting (33) we get $\int_{\mu S}^{\mu N} \Psi_y (f_1 - \vec{f}_1) \rho_0 e^{-4U_0}|\Sigma_0 \, d\mu = 0$. We now apply Lemma V.1 with the function $h \equiv e^{-2U_0 - 2z_y}|\Sigma_0 \rho_0(f_1 - \vec{f}_1)$. It follows that $h = 0$ and hence $\vec{f}_1 = f_1$. Thus the Cauchy problem is solvable and the theorem follows. □

For the first order perturbations, the inhomogeneous term $j$ vanishes and this theorem provides necessary and sufficient conditions involving the boundary data only (and hence conditions on the interior perturbations via the perturbed matching conditions described in the previous section).

For the second order perturbations things are not so easy because the equations are inhomogeneous ($j \neq 0$). In principle, one would need to integrate the first order functions $U_0', \Omega_0'$ in order to compute $j$ and hence $\int_{D_0} \psi_y j \eta_\gamma$ (and the corresponding expression for $\hat{\gamma}$). In some practical situations, an alternative procedure would be finding a particular solution $u_\rho$ of

$$\triangle u = j.$$

so that the homogeneous compatibility conditions (i.e. with $j = 0$) can be applied to the function $u_h = u - u_\rho$, which solves the homogeneous equation $\triangle u_h = 0$. More specifically we should need to check whether the Cauchy data $\{f_0 - u_\rho|\Sigma_0, f_1 - \vec{n}(u_\rho)|\Sigma_0\}$ (where $f_0$ and $f_1$ are the Cauchy data for $u$) satisfy the homogeneous compatibility conditions in theorem V.1. Obviously this approach relies heavily on knowing a particular solution of the inhomogeneous equation.

It is clear that, generically, we shall not be able to integrate the first order equations explicitly or find a particular solution of the inhomogeneous equation. We should still like to be able to treat the problem in a satisfactory way. The key idea which enables us to do so is to rewrite the volume integrals on the right-hand sides of (32) or (33) as surface integrals, and hence rewrite everything in terms of boundary data (or integrals thereof). Suppose that we were able to find an axially symmetric vector $\vec{T} = T^\rho \partial_\rho + T^z \partial_z$ on $D_0$ such that (with $\nabla_a$ denoting covariant derivative with respect to $\gamma$) $\nabla_a T^a = \psi_y j$ (for $\gamma$) or $\nabla_a T^a = \Psi_y j e^{-12U_0}$ (for $\hat{\gamma}$). Then we could use Gauss’ identity to transform the volume integral into a surface integral on $\Sigma_0$. We now show that this is indeed possible. As before, it is useful to work on
the two-dimensional space $K^E$. Defining the one-form $\mathbf{T} \equiv T^\rho d\rho + T^z dz$ we can translate the equation above for $\bar{T}$ into an equation for $\mathbf{T}$ on $K^E$ by means of the general identity $d(\rho \star \mathbf{T}) = -\rho \nabla_a T^a d\rho \wedge dz$.

We start with the equation for $U_0''$ in (16). We obviously have $j = -e^{-4U_0}(d\Omega_0', d\Omega_0')$, so that the equation for $\mathbf{T}$ reads, using the explicit form (28) for $\psi_y$,

$$d(\rho \star \mathbf{T}) + \sin \Upsilon_y e^{-4U_0} d\Omega_0' \wedge \star d\Omega_0' = 0.$$  \hfill (34)

Our aim is to find a solution of this equation. In order to do this, the following Lemma turns out to be useful.

**Lemma V.4** Let $\Omega_0'$ be a solution of the second equation in (15) and $Z_y$ be defined as in (30). Then the equations

$$dS_1 = e^{-2U_0 + 2Z_y} [-(1 + \cos \Upsilon_y) d\Omega_0' - \sin \Upsilon_y \star d\Omega_0'],$$
$$dS_2 = e^{-2U_0 - 2Z_y} [(1 - \cos \Upsilon_y) d\Omega_0' - \sin \Upsilon_y \star d\Omega_0'], \hfill (35)$$

admit unique solutions which tend to zero at infinity.

**Proof:** The solutions, if they exist, are unique except for additive constants. Asymptotic flatness of $\Omega_0'$ implies that $S_1$ and $S_2$ each tend to a constant at infinity; these additive constants can be chosen so that $S_1$ and $S_2$ vanish at infinity. Thus the only non-trivial part of the proof is to show existence of the solutions. Simple-connectedness of $K^E$ implies that we only need to check $\omega \equiv d(e^{-2U_0 + 2Z_y} [- (\delta + \cos \Upsilon_y) d\Omega_0' - \sin \Upsilon_y \star d\Omega_0]) = 0$, where $\delta = \pm 1$. A straightforward, if somewhat long, calculation using the equations for $U_0$, $\Omega_0'$ and $Z_y$ gives

$$\omega = e^{2Z_y - 2U_0} \sin \Upsilon_y \left[ d\Upsilon_y + \frac{\star d\Upsilon_y}{\tan \Upsilon_y} + \frac{dz}{\rho} \right] \wedge d\Omega_0' = 0,$$

where again the explicit expression for $\Upsilon_y$ is used in the last equality. \hfill $\Box$

The equations for $S_1$ and $S_2$ imply $dS_1 \wedge dS_2 = 2 \sin \Upsilon_y e^{-4U_0} d\Omega_0' \wedge \star d\Omega_0'$, which is the crucial fact allowing us to solve (34). Indeed, defining

$$T_1 \equiv \frac{1}{2 \rho} S_1 \star dS_2, \hfill (36)$$

it is immediate to check that (34) is satisfied for $\mathbf{T} = T_1$. This expression is apparently singular at $\rho = 0$. However, on the axis of symmetry we have $\cos \Upsilon_y = +1$ or $\cos \Upsilon_y = -1$ depending on whether we are above the north pole or below the south pole, respectively. Consequently, we have $S_1 = 0$ on the subset of the symmetry axis below the south pole (from the equation it satisfies and the fact that $S_1$ vanishes at infinity) and $dS_2$ vanishes on the axis above the north pole. Combining this with analyticity of $\Omega_0'$ everywhere on $D_0$ (including the axis), regularity of $T_1$ follows.

Considering the equation for $\Omega''_0$, in this case we have $j = 8 (d\Omega_0', dU'_0)$. Using the explicit expression (31) for $\Psi_y$, the equation we need to satisfy is

$$\nabla_a T^a = \frac{8}{\sqrt{\rho^2 + (z - y)^2}} e^{-2U_0 - 2Z_y} (d\Omega_0', dU'_0),$$
which, in terms of exterior forms, reads
\[
d (\rho \star T) = 8 \sin \Upsilon_y e^{-2 \Upsilon_0 - 2 Z_y} d\Omega_0 \land * dU_0'.
\] (37)

We now use the fact that \(U_0'\) is a flat-harmonic function. Thus (compare Lemma V.2) we can uniquely define a function \(Z_y'\) on \(\mathbb{K}^E\) by the equation
\[
d Z_y' = \cos \Upsilon_y dU_0' + \sin \Upsilon_y * dU_0', \quad \lim_{\rho^2 + z^2 \to \infty} Z_y' = 0.
\]

It is then straightforward to check that
\[
T_2 \equiv -4 \frac{\rho}{S_2} \star d \left( Z_y' + U_0' \right)
\] (38)
satisfies (37). Again, regularity at the axis follows because \(S_2\) vanishes on the axis above the north pole and \(d(Z_y' + U_0')\) is zero on the axis below the south pole.

We end this section by summarizing its main results in the form of the following Theorem.

\textbf{Theorem V.2} Let the assumptions and notation of theorem V.1 hold. Then
(i) the Cauchy boundary value problem
\[
\Delta_y U_0' = 0, \quad U_0'|_{\Sigma_0} = f_0, \quad \bar{n} (U_0') |_{\Sigma_0} = f_1,
\]
admits a regular solution on \(D_0\) if and only if
\[
\int_{\mu}^\mu \left[ \psi_y f_1 - f_0 \bar{n}(\psi_y) \right] \rho_0 |_{\Sigma_0} d\mu = 0, \quad \forall y \in (z_S, z_N),
\]
(ii) the Cauchy boundary value problem
\[
\Delta_y \Omega_0' - 4 (d\Omega_0', dU_0)_{\gamma} = 0, \quad \Omega_0'|_{\Sigma_0} = f_0, \quad \bar{n} (\Omega_0') |_{\Sigma_0} = f_1,
\]
admits a regular solution on \(D_0\) if and only if
\[
\int_{\mu}^\mu \left[ \Psi_y f_1 - f_0 \bar{n}(\Psi_y) \right] \rho_0 e^{-4 U_0'} |_{\Sigma_0} d\mu = 0, \quad \forall y \in (z_S, z_N),
\]
(iii) the Cauchy boundary value problem
\[
\Delta_y U_0'' + e^{-4 U_0'} (d\Omega_0', dU_0')_{\gamma} = 0, \quad U_0'|_{\Sigma_0} = f_0, \quad \bar{n} (U_0'') |_{\Sigma_0} = f_1,
\]
admits a regular solution on \(D_0\) if and only if
\[
\int_{\mu}^\mu \left[ \psi_y f_1 - f_0 \bar{n}(\psi_y) - \bar{n} (T_1) \right] \rho_0 |_{\Sigma_0} d\mu = 0, \quad \forall y \in (z_S, z_N),
\]
and (iv) the Cauchy boundary value problem
\[
\Delta_y \Omega_0'' - 8 (d\Omega_0', dU_0')_{\gamma} - 4 (d\Omega_0', dU_0)_{\gamma} = 0, \quad \Omega_0'|_{\Sigma_0} = f_0, \quad \bar{n} (\Omega_0'') |_{\Sigma_0} = f_1,
\]
admits a regular solution on \(D_0\) if and only if
\[
\int_{\mu}^\mu \left[ (\Psi_y f_1 - f_0 \bar{n}(\Psi_y)) e^{-4 U_0'} - \bar{n} (T_2) \right] \rho_0 |_{\Sigma_0} d\mu = 0, \quad \forall y \in (z_S, z_N),
\]
where \(\psi_y, \Psi_y, T_1\) and \(T_2\) are given in (28), (31), (36) and (38) respectively.
Remark. Since writing down $T_1$ and $T_2$ requires the integration of $dS_2$ it may seem at first sight that one has not really gained anything with respect to the volume integral in theorem V.1. The difference however is substantial because we only need to know $S_2$ on the boundary $\Sigma_0$. So by projecting equation (35) into the boundary we get an ODE for $S_2|_{\Sigma_0}$ which can in principle be solved by quadratures. Hence the problem reduces to performing integrals, which is of course much easier than solving PDEs. Thus the compatibility conditions are truly written in terms of the boundary data alone, as we wished.

VI. PERTURBATIONS AROUND SPHERICALLY SYMMETRIC STATIC BACKGROUND CONFIGURATIONS

Everything we have discussed so far holds for any stationary and axially symmetric perturbation of a static and axially symmetric background. Of course in many cases of physical interest, and especially for perfect fluids, one expects that equilibrium (non-rotating) configurations of isolated bodies are spherically symmetric. In the case of perfect fluids this has been rigorously proven for a large class of equations of state [12]. It is therefore of interest to specialize the previous results to the case when the background spacetime is in fact spherically symmetric. This implies in particular that the exterior background metric, being vacuum, corresponds to the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $m$ is the total mass of the spacetime. The static Killing vector is given by $\tilde{\xi}_0^E = \partial_t$, and we fix the axial Killing vector to be $\tilde{\eta}_0^E = \partial_\phi$ in these coordinates. (The non-uniqueness in the choice of axial symmetry in a spherically symmetric situation is precisely what allows us to orient the coordinate system so that this vector is the limit of the axial symmetries of the perturbed spaces.) From the definition of $U_0$ we get

$$U_0 = \frac{1}{2} \ln \left(1 - \frac{2m}{r}\right). \quad (39)$$

The intrinsic definition of the Weyl coordinate $\rho$, $\rho^2 = - (\tilde{\xi}_0^E, \tilde{\xi}_0^E)_{\tilde{g}_0^E} (\tilde{\eta}_0^E, \tilde{\eta}_0^E)_{\tilde{g}_0^E} + (\tilde{\xi}_0^E, \tilde{\eta}_0^E)^2_{\tilde{g}_0^E}$,
gives

$$\rho = r \sin \theta \sqrt{1 - \frac{2m}{r}}. \quad (40)$$

In order to determine $z$ we use $(dz, d\rho)_{\tilde{g}_0^E} = 0$, $(dz, dz)_{\tilde{g}_0^E} = (d\rho, d\rho)_{\tilde{g}_0^E}$, which can be solved to give $z = z_0 \pm (r - m) \cos \theta$, where $z_0$ is a constant. With these coordinate changes, $\{r, \theta, \phi\}$ can be regarded as coordinates on $(\mathbb{E}^3, \gamma)$. Choosing $\{dr, d\theta\}$ to be positively oriented on a plane $\phi = \text{const.}$, i.e. that $*dr \propto d\theta$ with a positive proportionality factor, and choosing (as we have above) that $\{dz, d\rho\}$ is also positively oriented, the + sign above is selected. We can further choose $z_0 = 0$ without loss of generality, so we have

$$z = (r - m) \cos \theta. \quad (41)$$
Spherical symmetry of the whole background spacetime requires that the surface of the body $\Sigma^E_0$ is defined by $r = r_0(> 2m)$. The embedding for this surface can be chosen to be $\chi^E_0: \{\tau, \varphi, \mu\} \rightarrow \{t = \tau, \phi = \varphi, r = r_0, \theta = \pi - \mu\}$, with $\varphi \in [0, 2\pi)$ and $\mu \in [0, \pi]$. This choice for $\mu$ is motivated by the fact that it increases from the south to the north pole of the body, as required. In fact, we have $\mu_S = 0$ and $\mu_N = \pi$. The range for the constant $y$ introduced in the previous section is then given by $-r_0 + m < y < r_0 - m$. With this embedding we clearly have

$$\rho_0(\mu) = r_0 \sin \mu \sqrt{1 - \frac{2m}{r_0}}, \quad z_0(\mu) = -(r_0 - m) \cos \mu$$

The vector $\vec{e}_0 = d\chi^E_0(\partial_{\mu})$ has norm $r_0^2$, which implies that $\vec{n}_0$ (which is defined to have the same norm) reads

$$\vec{n}_0 = -r_0 \sqrt{1 - \frac{2m}{r_0}} \partial_r \bigg|_{\Sigma_0}. \quad (43)$$

We can now write down the first and second order perturbed Ernst equations in Schwarzschild coordinates. Performing the coordinate transformation (40), (41) the flat metric $\gamma$ in $E^3$ becomes

$$\gamma = \frac{(r - m)^2 - m^2 \cos^2 \theta}{r^2} \left( \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right) + r (r - 2m) \sin^2 \theta d\phi^2,$$

which implies the following form for the first order Ernst equations (15)

$$r (r - 2m) U''_{0,rr} + 2(r - m) U'_{0,r} + U'_{0,\theta \theta} + \frac{\cos \theta}{\sin \theta} U'_{0,\theta} = 0,$$

$$r (r - 2m) \Omega''_{0,rr} + 2(r - 3m) \Omega'_{0,r} + \Omega'_{0,\theta \theta} + \frac{\cos \theta}{\sin \theta} \Omega'_{0,\theta} = 0. \quad (44)$$

The second order equations (16) become

$$r (r - 2m) U''_{0,rr} + 2(r - m) U''_{0,r} + U''_{0,\theta \theta} + \frac{\cos \theta}{\sin \theta} U''_{0,\theta} + \frac{r^3 \Omega'_{0,rr}^2}{r - 2m} + \frac{r^2 \Omega'_{0,\theta}^2}{(r - 2m)^2} = 0,$$

$$r (r - 2m) (\Omega''_{0,rr} - 8 \Omega'_{0,r} U'_{0,\theta}) + 2(r - 3m) \Omega''_{0,r} + \Omega''_{0,\theta \theta} + \frac{\cos \theta}{\sin \theta} \Omega''_{0,\theta} - 8 \Omega'_{0,\theta} U'_{0,\theta} = 0.$$

Let us analyze the equation for $U'_0$. Expanding in axisymmetric spherical harmonics

$$U'_0 = \sum_{l=0}^{\infty} u^{(1)}_l(r) P_l(\cos \theta),$$

we get a collection of ordinary differential equations which, after performing the change of variables $r = m(1 + x)$, read

$$(1 - x^2) u^{(1)}_{l,xx} - 2x u^{(1)}_{l,x} + l(l + 1) u^{(1)}_l = 0.$$
These equations can be solved as \( u_1^{(1)} = a_l P_l(x) + b_l P_l(x) \int_{r}^{x} \frac{dy}{(1-y^2)} \), where \( P_l(x) \) is the \( l \)-th Legendre polynomial and \( a_l \) and \( b_l \) are constants (note that the integral in the second summand converges). Asymptotic flatness demands that \( U'_0 \) tends to zero when \( x \to \infty \). Thus \( a_l = 0 \) for all \( l \) and the most general solution for \( U'_0 \) is

\[
U'_0 = \sum_{l=0}^{\infty} b_l P_l(\cos \theta) P_l \left( \frac{r}{m} - 1 \right) \int_{+\infty}^{\frac{r}{m} - 1} dy \frac{(1-y^2)}{(1-y^2)^2(y)},
\]

This expression also gives the general solution of the homogeneous part of \( U''_0 \). For \( \Omega'_0 \), the expansion in spherical harmonics \( \Omega'_0 = \sum_{l=0}^{\infty} w_l^{(1)}(r) P_l(\cos \theta) \) transforms (44) into

\[
(1-x^2) w_{l,xx}^{(1)} - 2(x-2) w_{l,x}^{(1)} + l(l+1) w_l^{(1)} = 0
\]

where again the \( x \) variable has been used. The general solution is now \( w_l^{(1)} = c_l P_l^{(-2,2)}(x) + d_l P_l^{(-2,2)}(x) \int_{+\infty}^{\frac{r}{m} - 1} dy \) \( \frac{(y-1)dy}{(1+y)^2 P_l^{(-2,2)}(y)} \), where \( P_l^{(-2,2)}(x) \) is the Jacobi polynomial and \( c_l \) and \( d_l \) are constants (we are using the notation of [38]). Imposing the condition that the solution tends to zero at infinity we conclude that \( c_l = 0 \) and we can write the general solution as

\[
\Omega'_0 = \sum_{l=0}^{\infty} d_l P_l(\cos \theta) P_l^{(-2,2)} \left( \frac{r}{m} - 1 \right) \int_{+\infty}^{\frac{r}{m} - 1} dy \frac{(y-1)dy}{(1+y)^2 P_l^{(-2,2)}(y)}.
\]

Having obtained these solutions, we consider next the boundary data that these functions must satisfy. In Proposition IV.1 we obtained expressions that involve only tangential derivatives of several scalar objects. We need to evaluate them. First of all we notice that spherical symmetry implies that

\[
\bar{c}(U_0|_{\Sigma_0}) = \bar{c}(\bar{n}(U_0)|_{\Sigma_0}) = 0,
\]

which substantially simplifies the boundary conditions. The explicit forms of \( \bar{n}(U_0)|_{\Sigma_0} \), \( X_0 \) and \( X_1 \) are directly obtained from their definitions as

\[
\frac{\bar{n}(U_0)|_{\Sigma_0}}{-1} = \frac{-1}{\sqrt{x_0^2 - 1}}, \quad x_0 = \frac{x_0^2 - \cos^2 \mu}{x_0^2 - \cos^2 \mu}, \quad X_0 = -\frac{x_0^2 - \cos^2 \mu}{x_0^2 - \cos^2 \mu}, \quad X_1 = \frac{\cos \mu \sin \mu}{x_0^2 - \cos^2 \mu},
\]

where \( x_0 \) is the value of \( x \) on the surface, i.e. \( x_0 = r_0/m - 1 \). Since the background spacetime satisfies the matching conditions, we obviously must have \( V_0 = 1/2 \log((x_0 - 1)/(x_0 + 1)) \) and \( \bar{n}V_0 = -1/\sqrt{x_0^2 - 1} \). The functions \( P_1, Q_1, P_2 \) and \( Q_2 \) are still arbitrary because they determine how the unperturbed surface is deformed to first and second order. The same is true regarding the functions \( V'_0, \bar{n}V'_0, V''_0, \bar{n}V''_0 \) for the \( U \)-boundary conditions and \( W'_0, \bar{n}W'_0, W''_0, \bar{n}W''_0 \) for the \( \Omega \)-boundary conditions, which depend on the interior perturbations. Using all these expressions, the boundary conditions described in Proposition IV.1 simplify, for spherical backgrounds, to
admit asymptotically flat solutions. In particular we need to find the functions (40) and (41) we get the explicit form

\[ Z_0' |_{\Sigma_0} = V_0' + Q_1 \frac{1}{\sqrt{x_0^2 - 1}}, \quad \tilde{n}(U_0') |_{\Sigma_0} = \tilde{n}V_0' + \left( \frac{dP_1}{d\mu} + Q_1 \frac{x_0}{\sqrt{x_0^2 - 1}} \right) \frac{1}{\sqrt{x_0^2 - 1}}. \]

\[ U_0'' |_{\Sigma_0} = V_0'' - 2P_1 \frac{dV_0'}{d\mu} - 2Q_1 \tilde{n}V_0' - \left( 2 \frac{d(P_1Q_1)}{d\mu} - Q_2 + Q_1^2 \frac{x_0}{\sqrt{x_0^2 - 1}} \right) \frac{1}{\sqrt{x_0^2 - 1}} \]

\[ + \left( P_1^2 - Q_1^2 \right) x_0 \frac{x_0}{x_0^2 - \cos^2 \mu} \right) \frac{1}{\sqrt{x_0^2 - 1}}. \]

\[ \tilde{n}(U_0'') |_{\Sigma_0} = \tilde{n}V_0'' + 2 \frac{d}{d\mu} \left( Q_1 \frac{dV_0'}{d\mu} \right) - 2 \frac{d(P_1 \tilde{n}V_0')}{d\mu} + 2Q_1 \left( \frac{\cos \mu \frac{dV_0'}{d\mu} \sin \mu}{\sin \mu d\mu} - \frac{x_0}{\sqrt{x_0^2 - 1}} \tilde{n}V_0' \right) \]

\[ - \left[ \frac{d^2}{d\mu^2} \left( P_1^2 - Q_1^2 \right) + \frac{d}{d\mu} \left( -P_2 + \left( P_1^2 - Q_1^2 \right) \frac{x_0}{x_0^2 - \cos^2 \mu} \right) \right] \frac{1}{\sqrt{x_0^2 - 1}} \]

\[ + \left( Q_2 - \left( P_1^2 - Q_1^2 \right) x_0 \frac{x_0}{x_0^2 - \cos^2 \mu} - 2P_1Q_1 \frac{x_0}{\sqrt{x_0^2 - 1}} \right) \frac{x_0}{x_0^2 - 1} \]

\[ \Omega_0' |_{\Sigma_0} = W_0', \quad \tilde{n}(\Omega_0') |_{\Sigma_0} = \tilde{n}W_0' , \quad \Omega_0'' |_{\Sigma_0} = W_0'' - 2P_1 \frac{dW_0'}{d\mu} - 2Q_1 \tilde{n}W_0'. \]

\[ \tilde{n}(\Omega_0'') |_{\Sigma_0} = \tilde{n}W_0'' + 2 \frac{d}{d\mu} \left( Q_1 \frac{dW_0'}{d\mu} \right) - 2 \frac{d(P_1 \tilde{n}W_0')}{d\mu} + 2Q_1 \left[ \frac{\cos \mu \frac{dW_0'}{d\mu} \sin \mu}{\sin \mu d\mu} - \frac{x_0 - 4}{\sqrt{x_0^2 - 1}} \tilde{n}W_0' \right]. \]

Our next aim is to find what compatibility conditions these data must satisfy in order to admit asymptotically flat solutions. In particular we need to find the functions \( \psi_y \) and \( \Psi_y \) defined in the previous section. Recalling the definition \( \psi_y = 1/\sqrt{\rho^2 + (z - y)^2} \) and using (40) and (41) we get the explicit form

\[ \psi_y = \frac{1}{\sqrt{m^2 x^2 + y^2 - 2mxy \cos \theta - m^2 \sin^2 \theta}}, \]

where \( y \) is a constant satisfying \( |y| \leq mx_0 \). For \( \Psi_y \) we first need to integrate the partial differential equation (30) for \( Z_y \). Contracting this equation with \( \partial \theta \) and using the fact that \( U_0 \) depends only on \( r \) we get

\[ \frac{\partial Z_y}{\partial \theta} = \frac{m \sin \theta}{\sqrt{m^2 x^2 + y^2 - 2mxy \cos \theta - m^2 \sin^2 \theta}}, \]

which, together with the boundary condition at infinity, gives

\[ e^{-Z_y} = \frac{Z_0(x)}{(y - m) \sqrt{x^2 - 1}} \left[ yx - m \cos \theta - \sqrt{m^2 x^2 + y^2 - 2mxy \cos \theta - m^2 \sin^2 \theta} \right] \]
where \( Z_0 \) is an arbitrary function of \( x \) obeying \( Z_0 \to 1 \) as \( x \to \infty \). (Using L'Hôpital's rule one can easily check that this form for \( e^{-Z_0} \) has the correct limit at the special value \( y = m \).) Then contracting (30) with \( \partial_r \) we find that \( Z_0 \) is constant. Thus

\[
e^{-Z_0} = \frac{yx - m \cos \theta - \sqrt{m^2 x^2 + y^2 - 2m x y \cos \theta - m^2 \sin^2 \theta}}{(y - m) \sqrt{x^2 - 1}}.
\]

We can finally write down the explicit expression for \( \Psi_y \), which reads

\[
\Psi_y = \frac{(yx - m \cos \theta - \sqrt{m^2 x^2 + y^2 - 2m x y \cos \theta - m^2 \sin^2 \theta})^2}{(y - m)^2 (x + 1)^2 \sqrt{m^2 x^2 + y^2 - 2m x y \cos \theta - m^2 \sin^2 \theta}}.
\]

Having obtained \( \psi_y \) and \( \Psi_y \) explicitly, the compatibility conditions that the boundary data for \( U_0', \Omega_0', U_0'' \) and \( \Omega_0'' \) must satisfy in the spherically symmetric case are direct consequences of theorem V.2 in the previous section.

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**APPENDIX: THE PERTURBED BOUNDARY CONDITIONS**

In section IV we introduced four functions \( V_\epsilon, \vec{n} V_\epsilon, W_\epsilon \) and \( \vec{n} W_\epsilon \) such that the full matching conditions for the Ernst potential are \( U_\epsilon|_{\Sigma_\epsilon} = V_\epsilon, \vec{n}_\epsilon(U_\epsilon)|_{\Sigma_\epsilon} = \vec{n} V_\epsilon, \Omega_\epsilon|_{\Sigma_\epsilon} = W_\epsilon, \vec{n}_\epsilon(\Omega_\epsilon)|_{\Sigma_\epsilon} = \vec{n} W_\epsilon \). We want to take first and second derivatives with respect to \( \epsilon \) (at fixed \( \mu \)) in order to obtain the perturbed matching conditions in Proposition IV.1. Since the calculations for \( U_\epsilon \) and \( \Omega_\epsilon \) are very similar, let us introduce a function \( F_\epsilon(\mu, \epsilon) \) which satisfies

\[
F_\epsilon|_{\Sigma_\epsilon} = H_\epsilon, \quad \vec{n}_\epsilon(F_\epsilon)|_{\Sigma_\epsilon} = J_\epsilon,
\]

for some functions \( H_\epsilon(\mu, \epsilon), J_\epsilon(\mu, \epsilon) \). We shall obtain the perturbed expressions for this generic function and then specialize to \( U_\epsilon \) and to \( \Omega_\epsilon \). To do so we need to differentiate (A.1) with respect to \( \epsilon \). The right hand sides are defined as functions of \( (\mu, \epsilon) \) so that for
them \(d/d\epsilon \equiv \partial/\partial \epsilon\), but on the left hand sides we have functions such as \(F(\rho, z, \epsilon)|_{\Sigma_\epsilon} = F(\rho(\mu, \epsilon), z(\mu, \epsilon), \epsilon)\). For the latter we have to take a convective derivative with velocity \(\bar{Z}_{1,\epsilon}\), defined by the obvious generalization of (18) when we evaluate at \(\Sigma_\epsilon\) instead of \(\Sigma_0\). \(\bar{Z}_{2,\epsilon}\) is defined similarly.

Evaluating (A.1) at \(\epsilon = 0\), we immediately get the unperturbed boundary data \(F_0|_{\Sigma_0} = H_0\), \(\bar{n}(F_0)|_{\Sigma_0} = J_0\), and taking the first convective derivative of \(F_\epsilon\) gives (on simple rearrangement)

\[
F_\epsilon' = H_0' - P_1 \partial_\mu(F_\epsilon|_{\Sigma_\epsilon}) - Q_1 \bar{n}(F_0)|_{\Sigma_0}.
\]

To obtain the first derivative of \(\bar{n}_\epsilon(F_\epsilon) = n_\epsilon^i \partial_i F_\epsilon\) we calculate

\[
\frac{d}{d\epsilon}(\bar{n}_\epsilon(F_\epsilon))|_{\Sigma_\epsilon} = \frac{\partial n^i_\epsilon}{\partial \epsilon} \partial_\epsilon F_\epsilon + n^i_\epsilon \partial_\epsilon \frac{dF_\epsilon}{d\epsilon}|_{\Sigma_\epsilon}, \quad \frac{\partial n^i_\epsilon}{\partial \epsilon} \partial_\epsilon F_\epsilon + Z^i_{1,\epsilon} n^j_\epsilon \partial_j F_\epsilon + n^i_\epsilon \partial_\epsilon F_\epsilon|_{\Sigma_\epsilon}.
\]

Substituting for \(\bar{Z}_1\) and \(\bar{n}_\epsilon\) from (18) and (19), using the identity

\[
e^i n^j \partial_i \partial_j F_0|_{\Sigma_0} = \partial_\mu(\bar{n}(F_0)|_{\Sigma_0}) - (X_0 \partial_\mu(F_0|_{\Sigma_0}) + X_1 \bar{n}(F_0)|_{\Sigma_0}), \quad (A.2)
\]

which like subsequent similar identities follows from integration by parts using \(\partial_\mu(\bar{c}) = X_1 \bar{c} - X_0 \bar{n}\) and \(\partial_\mu(\bar{n}) = X_0 \bar{c} + X_1 \bar{n}\), and rearranging, we obtain, at \(\epsilon = 0\),

\[
\bar{n}(F_0')|_{\Sigma_0} = J_0' - \partial_\mu(\bar{n}(F_0)|_{\Sigma_0}) - Q_1 \left( n^i n^j \partial_i \partial_j F_0|_{\Sigma_0} \right) + \left( \frac{dQ_1}{d\mu} + Q_1 X_1 \right) \partial_\mu(F_0|_{\Sigma_0}) - \left( \frac{dP_1}{d\mu} + Q_1 X_0 \right) \bar{n}(F_0)|_{\Sigma_0}.
\]

The second derivative of \(F_\epsilon\) follows by applying the convective derivative twice. Thus we obtain

\[
H''_\epsilon = (F'_\epsilon + Z^i_{1,\epsilon} \partial_i F_\epsilon)' + Z^i_{1,\epsilon} \partial_i (F'_\epsilon + Z^j_{1,\epsilon} \partial_j F_\epsilon)
\]

\[
= F''_\epsilon + 2Z^i_{1,\epsilon} \partial_i F'_\epsilon + Z^j_{2,\epsilon} \partial_j F_\epsilon + Z^j_{1,\epsilon} Z^i_{1,\epsilon} \partial_i \partial_j F_\epsilon.
\]

Substituting the values of \(\bar{Z}_1\) and \(\bar{Z}_2\) from (18), using the identities (A.2) and

\[
e^i e^j \partial_i \partial_j F_0|_{\Sigma_0} = \partial_\mu(F_0|_{\Sigma_0}) - (X_1 \partial_\mu(F_0|_{\Sigma_0}) - X_0 \bar{n}(F_0)|_{\Sigma_0}), \quad (A.4)
\]

and rearranging we obtain

\[
F''_0|_{\Sigma_0} = H''_0 - 2P_1 \partial_\mu(F_0')|_{\Sigma_0} - 2Q_1 \bar{n}(F_0')|_{\Sigma_0} - Q^2_1 \left( n^i n^j \partial_i \partial_j F_0|_{\Sigma_0} \right) - 2P_1 Q_1 \bar{n}(F_0)|_{\Sigma_0} - P^2_1 \partial_\mu(F_0|_{\Sigma_0})
\]

\[- (Q_2 + P^2_1 X_0 - 2P_1 Q_1 X_1) \bar{n}(F_0)|_{\Sigma_0} - (P_2 - P^2_1 X_1 - 2P_1 Q_1 X_0) \partial_\mu(F_0|_{\Sigma_0}).
\]
The last of these expressions that we need is the second derivative of $\vec{n}(F,\epsilon)$. This we can obtain by calculating

$$[\vec{n}(F,\epsilon)]''_{\Sigma} = \frac{\partial^2 n^j}{\partial \epsilon^2} \partial_\epsilon F_\epsilon + 2 \frac{\partial n^i}{\partial \epsilon} (\partial_\epsilon F'_\epsilon + Z^j_{\epsilon,\epsilon} \partial_\epsilon \partial_j F_\epsilon) + n^i (Z^j_{\epsilon,\epsilon} \partial_\epsilon \partial_j F_\epsilon + Z^j_{\epsilon,\epsilon} Z^k_{\epsilon,\epsilon} \partial_\epsilon \partial_j \partial_k F_\epsilon + 2 Z^j_{\epsilon,\epsilon} \partial_\epsilon \partial_j F'_\epsilon + \partial_\epsilon F''_\epsilon) \bigg|_{\Sigma}$$

The terms in $F'_\epsilon$ and $F''_\epsilon$ on the right at $\epsilon = 0$ are readily evaluated on substituting for $\frac{\partial \vec{n}}{\partial \epsilon}$ and $\vec{Z}_1$ from (19) and (18), and again using the identities (A.2) and (A.4). They lead to $\vec{n}(F''_\epsilon)|_{\Sigma_0}$ and

$$2 Q_1 \left( n^i n^j \partial_i \partial_j F'_0|_{\Sigma_0} \right) + 2 \partial_\mu (P_1 \vec{n}(F'_0)|_{\Sigma_0}) - \left( 2 \frac{dQ_1}{d\mu} + 2 Q_1 X_1 \right) \partial_\mu (F'_0)|_{\Sigma_0} + 2 Q_1 X_0 \vec{n}(F'_0)|_{\Sigma_0}$$

respectively. To evaluate $n^i Z^j_1 Z^k_1 \partial_i \partial_j \partial_k F_0|_{\Sigma_0}$ in the form we require, we will need the identities

$$e^i e^j n^k \partial_i \partial_j \partial_k F_0|_{\Sigma_0} = \partial_{\mu\nu} (\vec{n}(F_0)|_{\Sigma_0}) - X_0 \partial_{\mu}(F_0|_{\Sigma_0}) - X_1 \partial_{\mu}(\vec{n}(F_0)|_{\Sigma_0}) - \dot{X}_0 \partial_{\mu}(F_0|_{\Sigma_0}) \quad (A.5)$$

$$- \dot{X}_1 \vec{n}(F_0)|_{\Sigma_0} - X_0 e^i e^j \partial_i \partial_j F_0|_{\Sigma_0} + 2 X_1 e^i n^j \partial_i \partial_j F_0|_{\Sigma_0} + X_0 n^i n^j \partial_i \partial_j F_0|_{\Sigma_0},$$

$$e^i n^j n^k \partial_i \partial_j \partial_k F_0|_{\Sigma_0} = \partial_\mu (n^i n^j \partial_i \partial_j F_0|_{\Sigma_0}) - 2 X_0 \partial_\mu (\vec{n}(F_0)|_{\Sigma_0}) - 2 X_1 (n^i n^j \partial_i \partial_j F_0|_{\Sigma_0}) \quad (A.6)$$

$$+ 2 X_0 (X_0 \partial_\mu (F_0|_{\Sigma_0}) + X_1 \vec{n}(F_0)|_{\Sigma_0}).$$

Using these, the previous identities (A.2) and (A.4), and the values of $\vec{Z}_1$, $\vec{Z}_2$, $\frac{\partial \vec{n}}{\partial \epsilon}$ and $\frac{\partial^2 \vec{n}}{\partial \epsilon^2}$ from (18) and (19), we obtain

$$\vec{n}(F''_0)|_{\Sigma_0} = J''_0 - 2 Q_1 \left( n^i n^j \partial_i \partial_j F'_0|_{\Sigma_0} \right) - 2 \partial_\mu (P_1 \vec{n}(F'_0)|_{\Sigma_0})$$

$$+ \left( 2 \frac{dQ_1}{d\mu} + 2 Q_1 X_1 \right) \partial_\mu (F'_0)|_{\Sigma_0} - 2 Q_1 X_0 \vec{n}(F'_0)|_{\Sigma_0}$$

$$- Q_1^2 \left( n^i n^j n^k \partial_i \partial_j \partial_k F_0|_{\Sigma_0} \right) - 2 P_1 Q_1 \partial_\mu \left( n^i n^j \partial_i \partial_j F_0|_{\Sigma_0} \right) - P_1^2 \partial_{\mu\nu} (\vec{n}(F_0)|_{\Sigma_0})$$

$$- \left( Q_2 + 2 Q_1^2 X_0 + 2 Q_1 \frac{dP_1}{d\mu} + P_1^2 X_0 - 2 P_1 Q_1 X_1 \right) (n^i n^j \partial_i \partial_j F_0|_{\Sigma_0}) \quad (A.7)$$

$$+ \left( - P_2 + \frac{d(Q_1^2 - P_2^2)}{d\mu} + P_1^2 X_1 + 2 Q_1 X_1 \right) \partial_\mu (\vec{n}(F_0)|_{\Sigma_0})$$

$$+ \left( 2 P_1 \frac{dQ_1}{d\mu} + 2 P_1 Q_1 X_1 \right) \partial_{\mu\nu}(F_0|_{\Sigma_0})$$

$$+ \left[ - \frac{dP_2}{d\mu} + 2 \frac{d(P_1^2 X_1)}{d\mu} - X_1 \frac{d(Q_1^2)}{d\mu} + 2 P_1 X_0 \frac{dQ_1}{d\mu} \right]$$

$$- Q_2 X_0 - P_1^2 X_0^2 + 2 P_1 Q_1 X_0 X_1 - 2 Q_1^2 X_0 X_1 \vec{n}(F_0)|_{\Sigma_0}$$

$$+ \left[ \frac{dQ_2}{d\mu} + \frac{d(P_1^2 X_0)}{d\mu} - X_0 \frac{d(Q_1^2)}{d\mu} - 2 P_1 X_1 \frac{dQ_1}{d\mu} \right]$$

$$+ Q_2 X_1 + P_2^2 X_0 X_1 - 2 P_1 Q_1 X_1^2 - 2 Q_1^2 X_0 X_1 \partial_\mu (F_0|_{\Sigma_0}).$$

These expressions are in fact identities which hold for any axially symmetric function, in particular for $U_\epsilon$ and $\Omega_\epsilon$, but they are not in a form given by the boundary data, since they
involve \((n^i n^j \partial_i \partial_j F_0 |\Sigma_0)\) and \((n^i n^j n^k \partial_i \partial_j \partial_k F_0 |\Sigma_0)\). However, because the functions \(U_0, U'_0, \Omega'_0\) satisfy the elliptic equations (14) and (15) which relate second order tangential to second order normal derivatives we can eliminate the second and third order transverse derivatives. Since \(\gamma^{ij}|\Sigma_0 = \frac{1}{\Sigma_0} (n^i n^j + e^i e^j) + \frac{1}{\rho_0} (\partial_\rho)^i (\partial_\rho)^j |\Sigma_0, S_0 (\partial_{\rho\rho} + \partial_{zz}) F = (n^i n^j + e^i e^j) \partial_i \partial_j F\) and we immediately find the identity

\[
(n^i n^j \partial_i \partial_j F_0 |\Sigma_0) \equiv S_0 (\Delta, F_0 |\Sigma_0) - \partial_{\mu \mu} (F_0 |\Sigma_0) + X_1 \partial_{\mu} (F_0 |\Sigma_0) - X_0 \bar{n} (F_0 |\Sigma_0) - \frac{\delta_0}{\rho_0} \partial_{\mu} (F_0 |\Sigma_0) + \frac{\dot{\delta}_0}{\rho_0} \bar{n} (F_0 |\Sigma_0).
\]

Applying this identity to \(U_0, U'_0\) and \(\Omega'_0\) and using the field equations they satisfy, we obtain the following expressions for the transverse-transverse derivatives on the boundary

\[
(n^i n^j \partial_i \partial_j U_0 |\Sigma_0) = - \partial_{\mu \mu} (U_0 |\Sigma_0) + \left( X_1 - \frac{\dot{\rho}_0}{\rho_0} \right) \partial_{\mu} (U_0 |\Sigma_0) - \left( X_0 - \frac{\dot{\delta}_0}{\rho_0} \right) \bar{n} (U_0 |\Sigma_0), \quad (A.9)
\]

\[
(n^i n^j \partial_i \partial_j U'_0 |\Sigma_0) = - \partial_{\mu \mu} (U'_0 |\Sigma_0) + \left( X_1 - \frac{\dot{\rho}_0}{\rho_0} \right) \partial_{\mu} (U'_0 |\Sigma_0) - \left( X_0 - \frac{\dot{\delta}_0}{\rho_0} \right) \bar{n} (U'_0 |\Sigma_0), \quad (A.10)
\]

\[
(n^i n^j \partial_i \partial_j \Omega'_0 |\Sigma_0) = 4 \bar{n} (\Omega'_0 |\Sigma_0) \bar{n} (U'_0 |\Sigma_0) + 4 \partial_{\mu} (\Omega'_0 |\Sigma_0) \partial_{\mu} (U_0 |\Sigma_0) - \partial_{\mu \mu} (\Omega'_0 |\Sigma_0) + \left( X_1 - \frac{\dot{\rho}_0}{\rho_0} \right) \partial_{\mu} (\Omega'_0 |\Sigma_0) - \left( X_0 - \frac{\dot{\delta}_0}{\rho_0} \right) \bar{n} (\Omega'_0 |\Sigma_0).
\]

It only remains to evaluate \((n^i n^j n^k \partial_i \partial_j \partial_k U_0 |\Sigma_0)\) in terms of known boundary data. In order to do that we need a similar identity but now involving third derivatives. A straightforward calculation gives

\[
(n^i n^j n^k \partial_i \partial_j \partial_k F_0 |\Sigma_0) \equiv S_0 \bar{n} (\Delta, F_0 |\Sigma_0) - \partial_{\mu \mu} (\bar{n} (F_0 |\Sigma_0)) + 2 X_0 \partial_{\mu \mu} (F_0 |\Sigma_0)
\]

\[
+ \left( 3 X_1 - \frac{\dot{\rho}_0}{\rho_0} \right) \partial_{\mu} (\bar{n} (F_0 |\Sigma_0)) + \left( - X_0 + \frac{\dot{\delta}_0}{\rho_0} \right) (n^i n^j \partial_i \partial_j F_0 |\Sigma_0)
\]

\[
+ \left( \dot{X}_0 - 3 X_0 X_1 + \frac{\dot{\rho}_0}{\rho_0} X_0 - \frac{\dot{\rho}_0 \dot{\delta}_0}{\rho_0^2} \right) \partial_{\mu} (F_0 |\Sigma_0)
\]

\[
+ \left( \dot{X}_1 - 2 X_1^2 + X_0^2 + \frac{\dot{\rho}_0}{\rho_0} X_1 + \frac{\dot{\delta}_0^2}{\rho_0^2} \right) \bar{n} (F_0 |\Sigma_0),
\]

which, again, holds for any axially symmetric function in \(E^3\). Applying it to \(U_0\), and using \(\Delta, U_0 = 0\) and (A.9), we obtain

\[
(n^i n^j n^k \partial_i \partial_j \partial_k U_0 |\Sigma_0) \equiv - \partial_{\mu \mu} (\bar{n} (U_0 |\Sigma_0)) + \partial_{\mu \mu} (U_0 |\Sigma_0) \left( 3 X_0 - \frac{\dot{\delta}_0}{\rho_0} \right) +
\]

\[
+ \partial_{\mu} (\bar{n} (U_0 |\Sigma_0)) \left( 3 X_1 - \frac{\dot{\rho}_0}{\rho_0} \right) +
\]

\[
+ \partial_{\mu} (U_0 |\Sigma_0) \left( \dot{X}_0 - 4 X_0 X_1 + 2 \frac{\dot{\rho}_0}{\rho_0} X_0 + \frac{\dot{\delta}_0}{\rho_0} X_1 - \frac{2 \dot{\rho}_0 \dot{\delta}_0}{\rho_0^2} \right)
\]

\[
+ \bar{n} (U_0 |\Sigma_0) \left( \dot{X}_1 - 2 X_1^2 + 2 X_0^2 + \frac{\dot{\rho}_0}{\rho_0} X_1 - \frac{2 \dot{\delta}_0}{\rho_0} X_0 + \frac{2 \dot{\delta}_0^2}{\rho_0^2} \right), \quad (A.12)
\]
Substituting the transverse-transverse derivatives (A.9), (A.10), and (A.12) into (A.3), (A.5) and (A.7) with the substitutions $F \rightarrow U$, $H \rightarrow V$ and $J \rightarrow \vec{n}V$ we find the Cauchy $U$-boundary data in Proposition IV.1. Note that in this evaluation we need to use the formulae for $U'_0|_\Sigma_0$ and $\vec{n}(U'_0)|_\Sigma_0$ in those for $U''_0|_\Sigma_0$ and $\vec{n}(U''_0)|_\Sigma_0$. Finally, the substitution $F \rightarrow \Omega$, $H \rightarrow W$ and $J \rightarrow \vec{n}W$ and use of (A.11) and the fact that $\Omega_0 = 0$ gives us the $\Omega$-boundary data. This completes the proof of the Proposition.

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[39] We shall be using Hodge duals with respect to several metrics in this paper. The metric being used in each case should be clear from the context.