ON A PROPERTY OF THE INEQUALITY CURVE $\lambda(p)$

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Abstract

The Zenga (1984) inequality curve $\lambda(p)$ is constant in $p$ for Type I Pareto distributions. We show that this property holds exactly only for the Pareto distribution and, asymptotically, for distributions with power tail with index $\alpha$, with $\alpha > 1$. Exploiting these properties one can develop powerful tools to analyze and estimate the tail of a distribution.

Keywords: Tail index, inequality curve, non-parametric estimation

1 Introduction

Let $X$ be a positive random variable with finite mean $\mu$, distribution function $F$, and probability density $f$. The inequality curve $\lambda(p)$ \cite{15} is defined as:

$$\lambda(p) = 1 - \frac{\log(1 - Q(F^{-1}(p)))}{\log(1 - p)}, \quad 0 < p < 1,$$

where $F^{-1}(p) = \inf\{x: F(x) \geq p\}$ is the generalized inverse of $F$ and $Q(x) = \int_0^x tf(t)dt/\mu$ is the first incomplete moment. $Q$ can be defined as a function of $p$ via the Lorenz curve

$$L(p) = Q(F^{-1}(p)) = \frac{1}{\mu} \int_0^p F^{-1}(t)dt. \quad (2)$$

The curve $\lambda(p)$ has the property of being constant for Type I Pareto distributions and, as it will be shown, this property holds asymptotically for distributions $F$ satisfying

$$\bar{F}(x) = x^{-\alpha}L(x), \quad (3)$$

where $\bar{F} = 1 - F$, and $L(x)$ is a slowly varying function, that is $L(tx)/L(x) \rightarrow 1$ as $x \rightarrow \infty$, for any $t > 0$. We will say that $\bar{F}$ is regularly varying (RV) at infinity with index $-\alpha$, denoted as $\bar{F} \in RV_{-\alpha}$. The parameter $\alpha > 0$ is usually referred to as tail index; alternatively, in the extreme value (EV) literature it is typical to refer to the EV index $\gamma > 0$ with $\alpha = 1/\gamma$ (see e.g. \cite{12}).

These properties can be exploited in order to develop estimator an estimator of the tail index as well as a goodness of fit test for the Pareto distribution.
Probably the most well-known estimator of the tail index is the Hill \cite{6} estimator, which exploits the $k$ upper order statistics. The Hill estimator may suffer from high bias and is heavily dependent on the choice of $k$ (see e.g. \cite{1}). It has been thoroughly studied and several generalization have appeared in the literature. For recent review of estimation procedures for the tail index of a distribution see \cite{2}.

The approach to estimation proposed here, directly connected to the inequality curve $\lambda(p)$ has a nice graphical interpretation and could be used to develop graphical tools for tail analysis. Another graph-based method is to be found in \cite{9}, which exploits properties of the QQ-plot; while a recent approach based on the asymptotic properties of the partition function, a moment statistic generally employed in the analysis of multi-fractality, has been introduced by \cite{4}; see also \cite{8} which analyzes the real part of the characteristic function at the origin. For other related works see \cite{10}, \cite{11}, \cite{13}.

2 Properties of $\lambda(p)$

For a Type I Pareto distribution \cite[573 ff.]{7} with

$$F(x) = 1 - (x/x_0)^{-\alpha}, \quad x \geq x_0$$

it holds that $\lambda(p) = 1/\alpha$, i.e. $\lambda(p)$ is constant in $p$. This is actually an if-and-only-if result, as we formalize in the following lemma:

\textbf{Lemma 1.} The curve $\lambda(p)$ defined in (1) is constant in $p$ if, and only if, $F$ satisfies (4).

\textbf{Proof.} It is trivially verified that if $F$ satisfies (4) then $\lambda(p) = 1/\alpha$. Suppose now that $\lambda(p) = k$, $p \in (0, 1)$, where $k$ is some constant. Then it must hold that $1 - H(p) = (1 - p)^k$ or equivalently, after some algebraic manipulation,

$$\int_0^p F^{-1}(u)du = \mu[1 - (1 - p)^k]$$

Taking derivatives on both sides we have that

$$\frac{d}{dp} \int_0^p F^{-1}(u)du = \frac{d}{dp}\mu[1 - (1 - p)^k],$$

which gets

$$F^{-1}(p) = \mu k(1 - p)^{k-1}$$

from which, setting $x_p = F^{-1}(p)$, which implies $p = F(x_p)$, it follows that, after some further elementary manipulations,

$$\left(\frac{x_p}{\mu k}\right)^{1/(k-1)} = 1 - F(x_p).$$

Setting $1/(k - 1) = -\alpha$, properly normalized, the above $F$ follows \cite{4}, \hfill \Box
See [15] for a detailed analysis and calculations of \( \lambda(p) \) for other probability distributions. The following result can also be stated, asymptotically for the case where \( \bar{F} \) satisfies (3) as it is stated in the next lemma.

For this purpose write

\[
\lambda(x) = 1 - \frac{\log(1 - Q(x))}{\log(1 - F(x))}
\]  

(7)

**Lemma 2.** If \( \bar{F} \) satisfies (3), then \( \lim_{x \to \infty} \lambda(x) = 1/\alpha \).

**Proof.** Assume (3), since \( \bar{F}(x) = \int_{x}^{\infty} f(t) dt \); by Karamata’s theorem it follows that the density \( f(x) = L(x)x^{-(\alpha+1)} \) as \( x \to \infty \); again, by Karamata’s theorem:

\[
\mu(1 - Q(x)) = \int_{x}^{\infty} L(t)t^{1-(\alpha+1)} dt = L(x)x^{-\alpha+1}, \quad x \to \infty.
\]

Then, as \( x \to \infty \),

\[
\lambda(x) = 1 - \frac{\log(\mu^{-1}L(x)x^{1-\alpha})}{\log(L(x)x^{-\alpha})}
\]

\[
= 1 - \frac{\log(L(x)x^{-\alpha})}{\log(L(x)x^{-\alpha})} + \frac{1}{\alpha} \log(L(x)x) + \frac{1}{\alpha} \log(\mu)
\]

\[
= \frac{1}{\alpha} + O \left( \frac{1}{\log L(x)x} \right).
\]

\[
(8)
\]

A tail property of Pareto type I distribution is worth of being noted. Let \( X \) be a random variable distributed according to (4) – that is, \( X \sim \text{Pareto}(\alpha, x_0) \) –, the following property holds for any \( x_1 > x_2 > x_0 \):

\[
P[X > x_1 | X > x_2] = \left( \frac{x_1}{x_2} \right)^{-\alpha},
\]

hence, the truncated random variable \( (X | X > x_2) \) is distributed as \( \text{Pareto}(\alpha, x_2) \).

The implications of this property are twofold. Firstly, the truncated random variable is still distributed according to (4), thus Lemma 1 still applies. Secondly, the tail index \( \alpha \) is the same both for original and for truncated random variable, thus function \( \lambda(p) \) can be used for the estimation of \( \alpha \) regardless of the truncation threshold \( x_2 \).

The same property we have just outlined holds asymptotically for distribution functions satisfying (3).

Figure 1 reports the empirical curve \( \hat{\lambda}(p) \) as a function of \( p \) for a Pareto distribution defined by (4) with \( \alpha = 2 \) and \( x_0 = 1 \), denoted with \( \text{Pareto}(2, 1) \) and a Fréchet distribution with \( F(x) = \exp(-x^{-\alpha}) \) for \( x \geq 0 \) and \( \alpha = 2 \), denoted by Fréchet(2) at different truncation thresholds. Note the remarkably regular behavior or the curves
and the closeness to the theoretical form for the Fréchet case already for low levels of truncation.

Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of the sample, $\mathbb{I}_A$ the indicator function of the event $A$. To estimate $\lambda(p)$, define the preliminary estimates

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{(X_i \leq x)} \quad Q_n(x) = \frac{\sum_{i=1}^{n} X_i \mathbb{I}_{(X_i \leq x)}}{\sum_{i=1}^{n} X_i} \quad (9)$$

Under the Glivenko-Cantelli theorem (see e.g. [13]) it holds that $F_n(x) \to F(x)$ almost surely and uniformly in $0 < x < \infty$; under the assumption that $E(X) < \infty$, it holds that $Q_n(x) \to Q(x)$ almost surely and uniformly in $0 < x < \infty$. $F_n$ and $Q_n$ are both step functions with jumps at $X_{(1)}, \ldots, X_{(n)}$. The jumps of $F_n$ are of size $1/n$ while the jumps of $Q_n$ are of size $X_{(j)}/T$ where $T = \sum_{i=1}^{n} X_{(i)}$. Define the empirical counterpart of $L$ as follows:

$$L_n(p) = Q_n(F_n^{-1}(p)) = \frac{\sum_{j=1}^{i} X_{(j)}}{T}, \quad \frac{i}{n} \leq p < \frac{i+1}{n}, \quad i = 1, 2, \ldots, n - 1, \quad (10)$$

where $F_n^{-1}(p) = \inf\{x : F_n(x) \geq p\}$. To estimate $\alpha$ define

$$\hat{\lambda}_i = 1 - \frac{\log(1 - L_n(p_i))}{\log(1 - p_i)}, \quad p_i = \frac{i}{n}, \quad i = 1, 2, \ldots, n - \lfloor \sqrt{n} \rfloor. \quad (11)$$

and let $\hat{\alpha} = 1/\hat{\lambda}$ where $\hat{\lambda}$ is the mean of the $\hat{\lambda}_i$'s. The choice of $i = 1, \ldots, n - \lfloor \sqrt{n} \rfloor$ guarantees that $\hat{\lambda}_i$ is consistent for $\lambda_i$ for each $p_i = i/n$ as $n \to \infty$. 

![Figure 1: Plot of $\hat{\lambda}(p)$ and $p$ for Pareto(2, 1) (solid line) and Fréchet(2) (dashed line) at various levels of truncation. Sample size $n = 500$. Horizontal line at $1/\alpha = 0.5$.](image)
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