On a special type of Ma-Minda function

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Abstract. We introduce a primitive class of analytic functions, by specializing in many well-known classes, classify Ma-Minda functions based on its conditions and their interesting geometrical aspects. Further, study a newly defined subclass of starlike functions involving a special type of Ma-Minda function introduced here for obtaining inclusion and radius results. We also establish some majorization, Bloch function norms, and other related problems for the same class.

§1 Introduction

Let \( A \) be the set of all normalized analytic functions \( f \) of the form \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) defined on the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). The subset \( S \) of \( A \) denotes the class of univalent normalized analytic functions. We say, \( f \) is subordinate to \( g \), denoted by \( f \prec g \) or \( f(z) \prec g(z) \), if there exists a Schwarz function \( \omega \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) such that \( f(z) = g(\omega(z)) \), for all \( z \in \mathbb{D} \), where \( f \) and \( g \) are analytic functions. Moreover, if \( g \) is univalent in \( \mathbb{D} \), \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(\mathbb{D}) \subseteq g(\mathbb{D}) \). Define \( g, h \in A \) as:

\[
g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad \text{and} \quad h(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{1}
\]

The convolution (Hadamard Product) of \( g \) and \( h \) is given by \( (g * h)(z) = z + \sum_{n=2}^{\infty} g_n h_n z^n \). We introduce now the following primitive class, which specializes in several well-known classes:

\[
A(g, h, \varphi) = \{ f \in A : (f * g)(z)/(f * h)(z) \prec \varphi(z), \varphi \text{ is analytic univalent in } \mathbb{D} \text{ and } \varphi(0) = 1 \}.
\]

In 1985, Padmanabhan and Parvatham [22], considered the class \( A(K_a * g_1, K_a * h_1, \varphi) \), where \( K_a(z) = z/(1 - z)^a \) (a Real), \( g_1(z) := z/(1 - z)^2 \) and \( h_1(z) := z/(1 - z) \) by imposing additional conditions on \( \varphi \), namely it is convex and \( \Re \varphi > 0 \). Later in the year 1989, Shanmugam [27] extended \( A(K_a * g_1, K_a * h_1, \varphi) \) to \( A(g * g_1, g * h_1, \varphi) \) by considering a more general \( g \) in place of \( K_a \).
of \( K_\alpha(z) \). In 1992, Ma and Minda [17] tweaked the conditions on \( \varphi \), which we shall denote by \( \phi \) to introduce their own subclasses of starlike and convex functions, namely

\[
\mathcal{S}^*(\phi) = \{ f \in \mathcal{S} : zf'(z)/f(z) < \phi(z) \} \quad \text{and} \quad \mathcal{C}(\phi) = \{ f \in \mathcal{S} : 1 + zf''(z)/f'(z) < \phi(z) \}.
\]

(2)

Let the Taylor series expansion of such \( \phi(z) \) be of the form:

\[
\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \quad (B_1 > 0).
\]

(3)

Note that \( \mathcal{S}^*(\phi) := \mathcal{A}(g_1, h_1, \phi) \) and \( \mathcal{C}(\phi) := \mathcal{A}(g_2, h_1, \phi) \) whenever \( f \) is univalent and \( g_2(z) := (z + z^2)/(1 - z)^3 \). We classify now the Ma-Minda function in the following definition on the basis of its conditions:

**Definition 1.1.** An analytic univalent function \( \phi \) with \( \phi'(0) > 0 \), satisfying:

A. \( \Re \phi(z) > 0 \quad (z \in \mathbb{D}) \)

B. \( \phi(\mathbb{D}) \) symmetric about the real axis and starlike with respect to \( \phi(0) = 1 \)

is called a Ma-Minda function, we denote the class of all such functions by \( \mathcal{M} \). If the condition above alone is relaxed, the resulting function, we call it a non-Ma-Minda of type-A, the class of all such functions is denoted by \( \mathcal{M}_\Lambda \).

Recently, the classes given in (2) were studied extensively for different choices of \( \phi \). Prominently, Aouf et al. [3], studied the class \( \mathcal{S}^*(q_c) \), where \( q_c = \sqrt{1 + cz} \ (0 < c \leq 1) \), Robertson [25] introduced the class of starlike functions of order alpha \( (0 \leq \alpha < 1) \), denoted by \( \mathcal{S}^*(\alpha) \) by opting \( \phi(z) \) to be \( (1 + (1 - 2\alpha)z)/(1 - z) \) and when \( \phi(z) = ((1 + z)/(1 - z))^n \), \( \mathcal{S}^*(\alpha) \) reduces to the class of strongly starlike functions of order \( \eta \), which can be represented in terms of argument as \( \mathcal{S}^*(\eta) := \{ f \in \mathcal{A} : |\arg zf'(z)/f(z)| < \eta \pi/2, \ (0 < \eta \leq 1) \} \). Let \( \mathcal{S}_P := \mathcal{S}^*(\phi_{PAR}(z)) \), introduced by Ronning [26], the class of parabolic starlike functions, where \( \phi_{PAR}(z) := 1 + (2/\pi^2)(\log((1 + \sqrt{z})/(1 - \sqrt{z}))^2), \quad \Re \sqrt{z} \geq 0 \). Consider the class \( k - \mathcal{S}^T \) \( (k \geq 0) \) of \( k- \) starlike functions, which was introduced by Kanas and Wiśniowska [11] as follows:

\[
k - \mathcal{S}^T = \{ f \in \mathcal{S} : \Re(zf'(z)/f(z)) > k|zf'(z)/f(z) - 1| \} \quad (z \in \mathbb{D}),
\]

for \( k = 1 \), the above class coincides with \( \mathcal{S}_P \), which is characterized by the expression given as \( \Re(zf'(z)/f(z)) > |zf'(z)/f(z) - 1| \) \( (z \in \mathbb{D}) \). Further, \( k - \mathcal{S}^T \) was generalized by adding a parameter \( \alpha \) and defined by a condition

\[
\Re(zf'(z)/f(z)) > k|zf'(z)/f(z) - 1| + \alpha \quad (z \in \mathbb{D}),
\]

denoted by \( \mathcal{S}^T(k, \alpha) \). Let us consider the domain \( \Omega_{k, \alpha} = \{ w \in \mathbb{C} : \Re w > |w - 1| + \alpha \} \), whose boundary represents an ellipse for \( k > 0 \), a parabola for \( k = 1 \) and a hyperbola for \( 0 < k < 1 \).

Let \( \mathcal{M}(\beta) \) and \( \mathcal{N}(\beta) \) be the subclasses of \( \mathcal{S} \) consisting of the functions \( f(z) \) which respectively, satisfy \( \Re(zf'(z)/f(z)) < \beta \) and \( \Re(1 + zf''(z)/f'(z)) < \beta \) \( (\beta > 1; z \in \mathbb{D}) \). Uralegaddi et al. [30] investigated the class \( \mathcal{N}(\beta) \) for \( 1 < \beta < 4/3 \).

The authors in [4,8,10,18] dealt with the radius, inclusion and differential subordination results for the classes involving \( \phi(z) \). Many authors have determined the coefficient bounds for the classes associated with \( \phi(z) \) (see [14,15,19,23,24]). In the past, authors considered non-Ma-Minda functions, for instance, Kargar et al. [13] and Uralegaddi et al. [30] considered functions in \( \mathcal{M}_\Lambda \) to define their classes, see also [12].

The Ma-Minda function \( \phi \) is considered as univalent and therefore \( \phi'(0) \neq 0 \). Since \( \phi(\mathbb{D}) \)
is symmetric about the real axis and if $\phi'(0)$ is any non-zero real number, then $\phi$ has real coefficients. Now to address the distortion theorem, Ma-Minda perhaps restricted $\phi'(0)$ to be positive instead of any non-zero real number. However, it has no influence in establishing the coefficient, radius, inclusion, subordination, and other results for the classes $C(\phi)$ and $S^*(\phi)$. This very fact, which is under gloom until now, has been brought to daylight in this paper by replacing the condition $\phi'(0) > 0$ with $\phi'(0) < 0$. Note that $\phi(z)$ and $\Phi(z) := \phi(-z)$ both map unit disk to the same image but different orientation. Thus $\Phi(z)$ differs from its Ma-Minda counterpart by mere a rotation and is therefore non-typically real, but still, image domain invariant and all the rest properties are intact. So $\Phi(z)$ can be considered as a special type of Ma-Minda function. We premise now the above notion in the following definition:

**Definition 1.2.** An analytic univalent function $\Phi$ defined on the unit disk $D$ is said to be a special type of Ma-Minda if $\Re \Phi(D) > 0$, $\Phi(D)$ is symmetric with respect to the real axis, starlike with respect to $\Phi(0) = 1$ and $\Phi'(0) < 0$. Further, it has a power series expansion of the form:

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n = 1 + C_1 z + C_2 z^2 + \cdots \quad (C_1 < 0).$$

The class of all such special type of Ma-Minda functions are denoted by $\mathcal{M}^\circ$.

Recently, Altinkaya et al. [2] considered a special type of Ma-Minda function $g(z) = \alpha(1-z)/(\alpha - z)$, $(\alpha > 1)$ to define and study their class $P(\alpha)$. Now the classes $S^*(\Phi)$ and $C(\Phi)$ can be defined on the similar lines of (2). We introduce here a special type of Ma-Minda function, given by

$$\psi(z) := 1 - \log(1+z) = 1 - z + z^2/2 - z^3/3 + \cdots, \quad (4)$$

which maps the unit disk onto a parabolic region, see Figure 1 for its boundary curve $\tau$. We list in Table 1, a few examples of $\phi \in \mathcal{M}$ and its counter part $\Phi \in \mathcal{M}^\circ$:

| $\phi(z)$                     | $\Phi(z)$                     |
|------------------------------|------------------------------|
| $\cos \sqrt{-z}$            | $\cos \sqrt{z}$             |
| $\sqrt{1+z}$                | $\sqrt{1-z}$                |
| $1 - \log(1-z)$             | $1 - \log(1+z)$             |

**Distortion and Growth Theorems:** Let us define the functions $d_{\phi_n}(z)$ and $t_{\phi_n}(z)$ exactly as $d_{\phi_n}(z)$ and $t_{\phi_n}(z)$, respectively in [17]. Thus the structural formula of $d_{\phi_n}$ and $t_{\phi_n}$ is given by:

$$d_{\phi_n}(z) = \exp \int_0^z \frac{\Phi(t^n) - 1}{t} dt \quad \text{and} \quad t_{\phi_n}(z) = z \exp \int_0^z \frac{\Phi(t^n) - 1}{t} dt. \quad (5)$$

Ma and Minda [17] proved the distortion and growth theorems for the classes $C(\phi)$ and $S^*(\phi)$ where $\phi \in \mathcal{M}$. These results do not hold for functions in $\mathcal{M}^\circ$ which can be easily verified when $\Phi = \psi$, given by (4). Thus functions in $C(\psi)$ violate distortion theorem, which shows that $\phi'(0) > 0$ is mandatory to obtain the distortion theorem for functions in $\mathcal{M}$. 
Remark 1.3. Let \( \phi \in \mathcal{M} \) and its counter part \( \Phi \in \mathcal{M}^2 \) then \( \Phi(\mathbb{D}) = \phi(\mathbb{D}) \), which implies \( C(\Phi) = C(\phi) \) and \( S^*(\Phi) = S^*(\phi) \). Therefore to obtain distortion and growth theorems for functions in \( C(\Phi) \) and \( S^*(\Phi) \), it is sufficient to replace \( \phi(z) \) by \( \Phi(-z) \), in the result [17, Corollary 1, p. 159].

Using the above Remark and the fact \( d_\phi(z) = d'_\phi(-z) \), we deduce the following result:

**Theorem 1.4.** Let \( |z_0| = r < 1 \).

1. Suppose \( f \in C(\Phi) \). Then
   - (i) Growth Theorem: \( d_\phi(r) \leq |f(z_0)| \leq d_\phi(-r) \).
   - (ii) Distortion Theorem: \( d'_\phi(r) \leq |f'(z_0)| \leq d'_\phi(-r) \).

2. Suppose \( f \in S^*(\Phi) \). Then
   - (i) Growth Theorem: \( t_\phi(r) \leq |f(z_0)| \leq -t_\phi(-r) \).
   - (ii) Distortion Theorem: \( t'_\phi(r) \leq |f'(z_0)| \leq t'_\phi(-r) \), when we additionally assume \( \min_{|z|=r} |\Phi(z)| = |\Phi(r)| \) and \( \max_{|z|=r} |\Phi(z)| = |\Phi(-r)| \).

Equality holds for some non zero \( z_0 \) if and only if \( f \) is a rotation of \( d_\phi \), and \( t_\phi \), respectively for (1) and (2).

We introduce now the following classes involving the special type of Ma-Minda function \( \psi \):

\[
S_i^* := \left\{ f \in S : \frac{zf'(z)}{f(z)} < 1 - \log(1 + z) \right\} \quad \text{and} \quad C_i := \left\{ f \in S : 1 + \frac{zf''(z)}{f'(z)} < 1 - \log(1 + z) \right\}.
\]

By the structural formula (5), we get a function \( f \in S_i^* \) if and only if there exists an analytic function \( q \), satisfying \( q(z) \prec \psi(z) \) such that

\[
f(z) = z \exp \left( \int_0^z \frac{q(t) - 1}{t} \, dt \right). \tag{6}
\]

Now, we give some examples of the functions in the class \( S_i^* \). For this, let us assume

\[
\psi_1(z) = 1 - \frac{z}{6}, \quad \psi_2(z) = \frac{4 - z}{4 + z}, \quad \psi_3(z) = 1 - z \sin \frac{z}{4} \quad \text{and} \quad \psi_4(z) = \frac{8 - 2z}{8 - z}.
\]

A geometrical observation leads to \( \psi_i(\mathbb{D}) \subset \psi(\mathbb{D}) \) (\( i = 1, 2, 3, 4 \)). Thus \( \psi_i(z) \prec \psi(z) \). Now, the functions \( f_i \)'s belonging to the class \( S_i^* \) corresponding to each of the functions \( \psi_i \)'s are determined by the structural formula (6) as follows:

\[
f_1(z) = z \exp \left( -\frac{z}{6} \right), \quad f_2(z) = \frac{16z}{(4 + z)^2}, \quad f_3(z) = z \exp \left( 4 \left( 1 + \cos \frac{z}{4} \right) \right) \quad \text{and} \quad f_4(z) = z - \frac{z^2}{8}.
\]

In particular, for \( q(z) = \psi(z) = 1 - \log(1 + z) \), the corresponding function obtained as follows:

\[
f_0(z) = z \exp \left( \int_0^z \frac{-\log(1 + t)}{t} \, dt \right) = z - z^2 + \frac{3}{4} z^3 - \frac{19}{36} z^4 + \frac{107}{288} z^5 + \cdots, \tag{7}
\]

acts as an extremal function in many cases for \( S_i^* \).

**Remark 1.5.** The distortion and growth theorems for \( C_i \) and \( S_i^* \) can be obtained from that of \( C(\Phi) \) and \( S^*(\Phi) \), given in Theorem 1.4.

Here, we establish inclusion results, radius problems, majorization results and estimation of the Bloch function norm for the functions in the class \( S_i^* \). In the coefficient bound section, we
consider the class:
\[ \mathcal{A}(g,h,\phi) =: \mathcal{M}_{g,h}(\phi) = \{ f \in \mathcal{A} : (f * g)(z) / (f * h)(z) \prec \phi(z), \phi \in \mathcal{M} \}, \]
where Taylor series expansion of \( g, h \) is given by (1) and \( g_n, h_n > 0 \) with \( g_n - h_n > 0 \).
This class is defined in [20] and authors have obtained Fekete-Szegö bound for the same. We determine the bounds of fourth coefficient \( |a_4| \), second Hankel determinant \( |a_2a_4 - a_2^2| \) and the quantity \( |a_2a_3 - a_4| \) for the functions in the class \( \mathcal{M}_{g,h}(\phi) \). The importance of this class lies in unification of various subclasses of \( \mathcal{S} \), discussed in detail in the coefficient section. Some of our results reduce to many earlier known results of Lee et al. [15], Mishra et al. [19] and Singh [29].

In view of (8), we also consider the class \( \mathcal{M}_{g,h}(\Phi) \) for \( \Phi \in \mathcal{M}^o \). Now, we introduce the class:
\[ \mathcal{M}_\alpha(\Phi) = \{ f \in \mathcal{A} : (z^{1/\alpha}f(z) + az^2f''(z))/(\alpha zf'(z) + (1 - \alpha)f(z)) \prec \Phi(z), \ 0 \leq \alpha \leq 1 \} . \]
Note that when \( g(z) = (z(1 + 2\alpha - 1z))/ (1 - z)^3 \) and \( h(z) = (z(1 + \alpha - 1z))/ (1 - z)^2 \), we have \( \mathcal{M}_{g,h}(\Phi) =: \mathcal{M}_\alpha(\Phi) \). Further, the power series expansion of \( g \) and \( h \), respectively yield
\[ g_2 = 2(1 + \alpha), \ g_3 = 3(1 + 2\alpha), \ g_4 = 4(1 + 3\alpha) \ldots \quad \text{and} \quad h_2 = 1 + \alpha, \ h_3 = 1 + 2\alpha, \ h_4 = 1 + 3\alpha \ldots \]

By setting \( \mathcal{M}_\alpha(\psi) =: \mathcal{S}_l(\alpha) \), then \( \mathcal{S}_l(0) = \mathcal{S}_l^* \) and \( \mathcal{S}_l(1) = \mathcal{C}_l \). We obtain the sharp bounds of initial coefficients such as \( a_2, \ a_3, \ a_4 \) and \( a_5 \), Fekete-Szegô functional, second Hankel determinant for functions in \( \mathcal{S}_l(\alpha) \). Further, using these sharp bounds, we estimate the third Hankel determinant bound for the functions in \( \mathcal{S}_l(\alpha) \). We need the following lemma to support our results.

**Lemma 1.6.** [9] Let \( p \in \mathcal{P} \) be of the form \( 1 + \sum_{n=1}^{\infty} p_n z^n \). Then
\[ 2p_2 = p_1^2 + x(4 - p_1^2), \]
\[ 4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y \]
for some \( x \) and \( y \) such that \( |x| \leq 1 \) and \( |y| \leq 1 \).

### §2 Radius problems

Besides majorization, this section chiefly focuses on estimating various radius constants associated with \( \mathcal{S}_l^* \). We begin with establishing the following bounds meant for \( \mathcal{S}_l^* \):

**Theorem 2.1.** Let \( f \in \mathcal{S}_l^* \). Then we have for \( |z| = r < 1 \),
\[ 1 - \log(1 + r) \leq \text{Re}(zf'(z)/f(z)) \leq 1 - \log(1 - r) \]
and
\[ |\text{Im}(zf'(z)/f(z))| \leq \tan^{-1} \left( \frac{r}{\sqrt{1 - r^2}} \right) . \]

**Proof.** Since \( f \in \mathcal{S}_l^* \), we have \( zf'(z)/f(z) \prec 1 - \log(1 + z) \). Thus, by the definition of subordination, we have
\[ zf'(z)/f(z) = 1 - \log |q(z)| - i \text{arg}(q(z)), \]
where \( q(z) = 1 + \omega(z) \), \( \omega \) is a Schwarz function satisfying \( \omega(0) = 0 \) and \( |\omega(z)| \leq |z| \). Let \( q(z) = u + iv \), then \( (u - 1)^2 + v^2 < 1 \). For \( |z| = r \), we have
\[ |q(z) - 1| \leq r \].

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Squaring both sides of the above equation yields

\[ T : (u - 1)^2 + v^2 \leq r^2. \] (14)

Clearly \( T \) represents the equation of the disk with center: \((1, 0)\) and radius \( r \), for which the point \((0, 0)\) lies outside the disk \( T \). From (13) and (14), we have \(|q(z)| \leq 1 + r\) and \(|q(z)| \geq 1 - r\). Since, \( \log x \) is an increasing function on \([1, \infty)\), we have \( \log(1 - r) \leq \log|q(z)| \leq \log(1 + r) \). Thus we get the desired result (10) by taking the real part in (12). If \( v = au \), representing the equation of the tangent to the boundary of disk \( T \), which passes through the origin \( O \), then the tangent and the boundary of the disk have a common point, hence from (14), we obtain

\[(1 + a^2)u^2 - 2u + 1 - r^2 = 0.\]

By the definition of tangent, we get

\[ 1 - (1 + a^2)(1 - r^2) = 0, \]

which yields \( a = \pm r/\sqrt{1 - r^2} \). Therefore we have

\[ \tan^{-1}\left(-r/\sqrt{1 - r^2}\right) \leq \arg q(z) \leq \tan^{-1}\left(r/\sqrt{1 - r^2}\right). \]

Thus we obtain the desired result (11). \( \Box \)

**Theorem 2.2.** Let \( f \in \mathcal{S}_1^\ast \). Then the followings hold:

(i) \( f \) is starlike of order \( \alpha \) in \(|z| < \exp(1 - \alpha) - 1\) whenever \( 1 - \log 2 \leq \alpha < 1 \).

(ii) \( f \in \mathcal{M}(\beta) \) in \(|z| < 1 - \exp(1 - \beta)\) whenever \( \beta > 1 \).

(iii) \( f \) is convex of order \( \alpha \) in \(|z| < \tilde{r}(\alpha) < 1\) whenever \( 0 < \alpha < 1 \), where \( \tilde{r}(\alpha) \) is the smallest positive root of the equation:

\[ (1 - r)(1 - \log(1 + r))(1 - \log(1 + r) - \alpha) - r = 0, \]

for the given value of \( \alpha \).

(iv) \( f \) is strongly starlike of order \( \gamma \) in \(|z| < r(\gamma)\) whenever \( 0 < \gamma \leq \gamma_0 \approx 0.514674 \), where

\[ r(\gamma) = \sqrt{2(1 - (1 + \tan^2(\tan(\gamma\pi/2)))^{-1/2})}. \] (16)

(v) \( f \) is \( k \)-starlike function in \(|z| < r(k)\) whenever \( k > 0 \), where \( r(k) \) is the smallest positive root of the equation

\[ 1 + r - e(1 - r)^k = 0, \]

for the given value of \( k \). In particular, for \( k = 1 \), \( f \) is parabolic starlike in \(|z| < \frac{e - 1}{e + 1}\). \( \Box \)

**Proof.** (i) Since \( f \in \mathcal{S}_1^\ast \), we obtain the following from Theorem 2.1:

\[ \Re(zf'(z)/f(z)) \geq 1 - \log(1 + r), \quad |z| = r < 1, \]

which yields the inequality \( \Re(zf'(z)/f(z)) > \alpha \), whenever \( 1 - \log 2 \leq \alpha < 1 \), which holds true in the open disk of radius \( \exp(1 - \alpha) - 1 \). For the function \( f_0 \), given in (7) and \( z_0 = \exp(1 - \alpha) - 1 \), we have \( \Re(z_0f_0'(z)/f_0(z)) = \alpha \). Hence this result is sharp.

(ii) From Theorem 2.1, we get

\[ \Re(zf'(z)/f(z)) \leq 1 - \log(1 - r), \quad |z| = r < 1, \]

which yields the following inequality \( \Re(zf'(z)/f(z)) < \beta \), for \( \beta > 1 \), which holds true in the open disk of radius \( 1 - \exp(1 - \beta) \). For the function \( f_0 \), given in (7) and \( z_0 = \exp(1 - \beta) - 1 \), we get \( \Re(z_0f_0'(z)/f_0(z)) = \beta \). Therefore the result is sharp.

(iii) Let \( f \in \mathcal{S}_1^\ast \). Now, \( f \in \mathcal{C}_1(\alpha) \), whenever \( \Re(1 + zf''(z)/f'(z)) > \alpha \). Since we have
\[ zf'(z)/f(z) = 1 - \log(1 + \omega(z)), \] where \( \omega \) is a Schwarz function, we obtain
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \Re (1 - \log(1 + \omega(z))) - \Re \left( \frac{z\omega'(z)}{(1 + \omega(z))(1 - \log(1 + \omega(z)))} \right). \tag{18}
\]
The function \( \omega \) satisfies the following inequality, given in [21]
\[ |\omega'(z)| \leq (1 - |\omega(z)|^2)/(1 - |z|^2). \tag{19}\]
Using the above result, (18) reduces to
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{(1-r)(1-\log(1+r))}{(1-r)(1-\log(1+r))} =: \nu(r).
\]
To complete the proof, it suffices to show
\[ \nu(r) - \alpha = \frac{(1-r)(1-\log(1+r))(1-\log(1+r) - \alpha) - r}{(1-r)(1-\log(1+r))} > 0,
\]
equivalently \( \nu(r,\alpha) = (1-r)(1-\log(1+r))(1-\log(1+r) - \alpha) - r > 0 \). Clearly, \( \nu(0,\alpha) > 0 \) and \( \nu(1,\alpha) < 0 \) for all \( \alpha \in [0,1] \). Thus, there must exist \( \tilde{r}(\alpha) \) such that \( \nu(r,\alpha) \geq 0 \) for all \( r \in [0,\tilde{r}(\alpha)] \), where \( \tilde{r}(\alpha) \) is the smallest positive root of the equation (15). Hence the result follows.

(iv) Since \( f \in S^*_1 \), we have \( |\arg zf'(z)/f(z)| \leq |\arg(1-\log(1+z))| \). Now, \( f \in SS^*(\gamma) \) whenever
\[ |\arg \left( 1 - \frac{1}{2} \log(|1+x|^2 + y^2) - i \arctan \frac{y}{1+x} \right) | \leq \gamma \pi/2,
\]
for \( |z| = \sqrt{x^2 + y^2} < 1 \). Consider the function \( f_0 \), given in (7) and let us assume
\[ z_0 = \frac{1}{\sqrt{1+A^2}} - 1 + i \frac{A}{\sqrt{1+A^2}},
\]
where \( A = \tan(\tan \gamma \pi/2) \). We have \( |z_0| = \sqrt{2(1-1/(\sqrt{1+A^2}) < 1 \), whenever \( \gamma \leq \gamma_0 \) and
\[
|\arg(1 - \log(1 + z_0))| = |\arg(z_0f'(z_0)/f_0(z_0))|
\] is given by (16) and therefore the result is sharp.

(v) Let \( f \in S^*_1 \). Now, \( f \in k - ST \) whenever \( \Re (1 - \log(1 + \omega(z))) > k|\log(1 + \omega(z))| \) (where \( z \in D \), for some Schwarz function \( \omega \). Let \( g(\omega(z)) := |\log(1 + \omega(z))| = |\log(1 + \omega(z)) + i \arg(1 + \omega(z))| \), where \( \omega(z) = \Re w, R \leq |z| = r < 1 \) and \( -\pi < t < \pi \). Now, consider the function
\[ g^2(R,t) = \left( \frac{1}{2} \log(1 + R^2 + 2R \cos t) \right)^2 + \left( \sqrt{R \sin t} \right)^2 \]
and let us assume
\[ h(R,t) := R \left( \frac{2(R + \cos t) \sqrt{R \sin t} - \log(1 + R^2 + 2R \cos t) \sin t}{1 + R^2 + 2R \cos t} \right).
\]
Then clearly, the function \( h(R,t) \geq 0 \) for \( t \in [0,\pi] \) and \( h(R,t) \leq 0 \) for \( t \in (-\pi,0] \). Thus
\[ \max_{-\pi < t < \pi} g(R,t) = \max \{ g(R,-\pi), g(R,\pi) \}, \]
which yields
\[ |\log(1 + \omega(z))| \leq |\log(1 - R)| \leq |\log(1 - r)|. \tag{20}\]
Using the inequality (20) and Theorem 2.1, the result follows by showing \(1 - \log(1 + r) \geq k|\log(1 - r)|\), whenever \(m(r, k) := 1 + r - e(1 - r)^k \leq 0\), which clearly holds for \(r \in [0, r(k)]\). Hence the result follows.

Our next result involves the concept of majorization. For any two analytic functions \(f\) and \(g\), we say \(f\) is majorized by \(g\) in \(\mathbb{D}\), denoted by \(f \ll g\), if there exists an analytic function \(\mu(z)\) in \(\mathbb{D}\), satisfying

\[|\mu(z)| \leq 1\] and \(f(z) = \mu(z)g(z)\).

The following majorization result involves the class \(S^*_1\):

**Theorem 2.3.** Let \(f \in A\). Suppose that \(f \ll g\) in \(\mathbb{D}\), where \(g \in S^*_1\). Then we have the inequality \(|f'(z)| \leq |g'(z)|\), for \(|z| \leq r\), where \(\tilde{r}\) is the smallest positive root of the following equation:

\[(1 - r^2)(1 - \log(1 + r)) - 2r = 0.\] (21)

**Proof.** Since \(g \in S^*_1\), we have \(zg'(z)/g(z) < 1 - \log(1 + z)\). Then there exists a Schwarz function \(\omega(z)\) such that

\[zg'(z)/g(z) = 1 - \log(1 + \omega(z)).\] (22)

Let \(\omega(z) = Re^{it}\), \(R \leq |z| < 1\) and \(-\pi < t < \pi\). Consider the function

\[h(R, t) := |1 - \log(1 + Re^{it})|^2 = \left(1 - \frac{1}{2} \log(1 + R^2 + 2R \cos t)\right)^2 + \left(\arctan \frac{R \sin t}{1 + R \cos t}\right)^2,
\]

which upon differentiation with respect to \(t\) yields

\[h_t(R, t) := \frac{R(2(R + \cos t) \arctan \frac{R \sin t}{1 + R \cos t} - (-2 + \log(1 + R^2 + 2R \cos t)) \sin t)}{1 + R^2 + 2R \cos t}.
\]

Then clearly, the function \(h_t(R, t) \geq 0\) for \(t \in [0, \pi]\) and \(h_t(R, t) \leq 0\) for \(t \in (-\pi, 0]\). Thus \(\min_{-\pi < t < \pi} h(R, t) = h(R, 0)\), which yields \(|1 - \log(1 + \omega(z))| \leq 1 - \log(1 + R) \geq 1 - \log(1 + r)\).

Further, condition (22) yields

\[
\frac{|g(z)|}{|g'(z)|} = |\frac{|z|}{1 - \log(1 + \omega(z))}| \leq \frac{r}{1 - \log(1 + r)}.\] (23)

By the definition of majorization, we get \(f(z) = \mu(z)g(z)\), which upon differentiation, gives

\[f'(z) = \mu(z)g'(z) + g(z)\mu'(z) = g'(z)(\mu(z) + \mu'(z)g(z)/g'(z)).\] (24)

The function \(\mu\) satisfies the inequality (19), thus using (19) for \(\mu\) and substituting the inequality (23) in (24), we get

\[|f'(z)| \leq K(r, \zeta)|g'(z)|,
\]

where \(K(r, \zeta) = \zeta + r(1 - \zeta^2)/((1 - r^2)(1 - \log(1 + r)))\) for \(|\mu(z)| = \zeta\) \((0 \leq \zeta \leq 1)\). To achieve the result, it suffices to show that

\[1 - K(r, \zeta) = \frac{(1 - \zeta)((1 - r^2)(1 - \log(1 + r)) - r(1 + \zeta))}{(1 - r^2)(1 - \log(1 + r))} \geq 0,
\]

equivalent to show \(\eta(r, \zeta) := (1 - r^2)(1 - \log(1 + r)) - r(1 + \zeta) \geq 0\), which clearly holds for \(r \in [0, \tilde{r}]\). Hence the proof is complete. \(\square\)
§3 Inclusion relations

In this section, we give inclusion relations between the classes \( S^*_l \) and various other subclasses of starlike functions, namely \( S^*(\alpha) \), \( SS^*(\gamma) \), \( ST(1, \alpha) \) and \( S^*(q_c) \).

**Theorem 3.1.** The class \( S^*_l \) satisfies the following results:

(i) \( S^*_l \subset S^*(\alpha) \subset S^* \) for \( 0 \leq \alpha \leq 1 - \log 2 \).

(ii) \( S^*_l \subset SS^*(\gamma) \subset S^* \) for \( 2 \tilde{f}(\theta_0)/\pi \leq \gamma \leq 1 \), where \( \theta_0 \) is the smallest positive root of the equation \(-2 + \log(2(1 + \cos \theta)) + \theta \tan \theta/2 = 0 \) and \( \tilde{f}(	heta) = \arg(1 - \log(1 + e^{i\theta})) \), \( \theta \in [0, \pi) \).

(iii) \( S^*_l \subset ST(1, \alpha) \) for \( \alpha \leq 1 - 2 \log 2 \).

(iv) \( S^*(q_c) \subset S^*_l \subset S^* \) for \( c \leq c_0 \), where \( c_0 = \log 2(2 - \log 2) \).

The above constants in each part are the best possible. The pictorial representation of the result is depicted in **Figure 1**.

![Figure 1](image-url)  
**Figure 1.** Boundary curves of best dominants and subordinate of \( \psi(z) = 1 - \log(1 + z) \).

**Proof.** (i) Since \( f \in S^*_l \), we have \( zf'(z)/f(z) \prec 1 - \log(1 + z) \). Then, for \( \theta \in [0, \pi] \), the following holds:

\[
1 - \log 2 = \min_{|z|=1} \Re(1 - \log(1 + z)) < \Re(zf'(z)/f(z)).
\]

Hence, the result follows.

(ii) Let \( f \in S^*_l \). Then, we have

\[
|\arg(zf'(z)/f(z))| \leq \max_{|z|=1} |\arg(1 - \log(1 + z))| = \max_{-\pi \leq \theta \leq \pi} |\arg(1 - \log(1 + e^{i\theta}))| = \max_{-\pi \leq \theta \leq \pi} |\tilde{f}(\theta)|.
\]

Due to the symmetry of the function \( \tilde{f}(\theta) \), we consider \( \theta \in [0, \pi] \) and \( \tilde{f}(\theta) = 0 \) yields \(-2 + \log(2(1 + \cos(\theta_0))) + \theta_0 \tan(\theta_0/2) = 0 \), where \( \theta_0 \approx 1.37502 \). A calculation shows that \( \tilde{f}''(\theta) < 0 \), which implies \( \max_{0 \leq \theta \leq \pi} \tilde{f}(\theta) = \tilde{f}(\theta_0) \approx 0.88329 \). Thus \( f \in SS^*(\gamma) \), for given \( \gamma \).

(iii) Let us consider the domain \( \Omega_{\lambda} := \{ w \in \mathbb{C} : \Re w > \arg(z - 1) + \alpha \} \), whose boundary represents a parabola, for \( w = x + iy \), given by: \( x = \frac{y^2}{2(1 - \alpha)} + \frac{1 + \alpha}{2} \), whose vertex is given by: \( ((1 + \alpha)/2, \theta_0) \).
In order to prove the result, it suffices to show
\[ h(\theta) := \text{Re}(1 - \log(1 + z)) - |\log(1 + z)| \]
\[ = 1 - \frac{1}{2} \log(2(1 + \cos \theta)) - \sqrt{\frac{1}{4} \log^2(2(1 + \cos \theta)) + \frac{\theta^2}{4}} > \alpha, \]
for \( z = e^{i\theta} \). A numerical computation shows that \( \min_{-\pi \leq \theta \leq \pi} h(\theta) = h(0) = 1 - 2 \log 2 \). Hence the result follows.

(iv) Since \( f \in S^\ast(q_c) \), we have \( zf'(z)/f(z) \prec \sqrt{1 + cz} \) and
\[ \sqrt{1 - \epsilon} = \min_{|z|=1} \sqrt{1 + cz} < \text{Re}(zf'(z)/f(z)) < \max_{|z|=1} \sqrt{1 + cz} = \sqrt{1 + \epsilon}. \]
Similar analysis can be carried out for the imaginary part bounds and therefore by using Theorem 2.1, we get the result.

\[ \Box \]

§4 Coefficient bounds

This section deals with various coefficient related bound estimates. Here we need the function \( H(q_1, q_2) \), given in [1, Lemma 3], to establish our results in what follows.

Remark 4.1. Murugusundaramoorthy et al. [20, Theorem 6.1] obtained the result of Fekete-Szegö functional bound for functions in the class \( M_{g,h}(\Phi) \). Since \( M_{g,h}(\Phi(z)) = M_{g,h}(\phi(z)) \), we state below the parallel result for functions in the class \( M_{g,h}(\Phi(z)) \) by simply replacing each \( B_i \) by \((-1)^i C_i \), the result is needed to prove our subsequent example.

Theorem 4.2. For \( f \in M_{g,h}(\Phi) \), we have the following sharp result
\[ |a_3 - ta_2^2| \leq \begin{cases} \frac{C_2}{g_3 - h_3} - \frac{tC_1^2}{(g_2 - h_2)^2} + \frac{(g_2 h_2 - h_3^2)C_1^2}{(g_3 - h_3)(g_2 - h_2)^2}, & t \leq \kappa_1; \\ -\frac{g_3 - h_3}{-(g_3 - h_3)^2} - \frac{tC_1^2}{(g_2 - h_2)^2} - \frac{(g_2 h_2 - h_3^2)C_1^2}{(g_3 - h_3)(g_2 - h_2)^2}, & t \geq \kappa_2; \end{cases} \]
where \( \kappa_1 = (g_2 - h_2)^2(C_2 + C_1) + h_2(g_2 - h_2)C_2^2/(g_3 - h_3)C_1^2 \) and \( \kappa_2 = (g_2 - h_2)^2(C_2 - C_1) + h_2(g_2 - h_2)C_2^2/(g_3 - h_3)C_1^2 \).

Remark 4.3. We notice that the bound of Fekete-Szegö stated in [20, Theorem 6.1], namely \( |a_3 - \mu a_2^2| \leq B_1/(g_3 - h_3) \) when \( \sigma_1 \leq \mu \leq \sigma_2 \), is incorrect and should be \( |a_3 - \mu a_2^2| \leq B_1/(g_3 - h_3) \), which is appropriately corrected in Theorem 4.2.

Evidently, the class \( M_{g,h}(\phi) \) unifies various subclasses of \( S \) for different choices of \( g \) and \( h \). A few of the same are enlisted below for ready reference:
\[ \begin{align*}
(f+g)(z) & = \frac{zf'(z)}{(f+g)(z)}, & g(z) = \frac{z+1}{(1+\alpha z)}; \\
(f+g)(z) & = \frac{zf'(z)}{(f+g)(z)}, & g(z) = \frac{z+(\alpha+1)\alpha}{(1+\alpha z)}, \\
(f+g)(z) & = \frac{zf'(z)}{(f+g)(z)}, & g(z) = \frac{z+(\alpha+1)\alpha}{(1+\alpha z)}, \\
(f+g)(z) & = \frac{zf'(z)}{(f+g)(z)}, & g(z) = \frac{z+(\alpha+1)\alpha}{(1+\alpha z)}, \\
(f+g)(z) & = \frac{zf'(z)}{(f+g)(z)}, & g(z) = \frac{z+(\alpha+1)\alpha}{(1+\alpha z)}. \\
\end{align*} \]

In the following example, we establish a Fekete-Szegö result for the class \( S_\alpha(\alpha) \):

Example 4.4. Let $f \in S_1(\alpha)$. Then

$$|a_3 - ta_2^2| \leq \begin{cases} \frac{3}{4(1+2\alpha)} - \frac{t}{(1+\alpha)^2}, & t \leq \frac{(1+\alpha)^2}{4(1+2\alpha)} =: \kappa_1; \\ \frac{1}{2(1+2\alpha)}, & \frac{(1+\alpha)^2}{4(1+2\alpha)} \leq t \leq \frac{5(1+\alpha)^2}{4(1+2\alpha)}; \\ \frac{3}{(1+\alpha)^2} - \frac{3}{4(1+2\alpha)}, & t \geq \frac{5(1+\alpha)^2}{4(1+2\alpha)} =: \kappa_2. \end{cases}$$

The result is sharp.

We omit the proof as it is a direct application of Theorem 4.2.

Lemma 4.5. Let $f \in S_1(\alpha)$. Then

(i) $|a_3 - a_2^2| \leq 1/(2(1+2\alpha))$,  
(ii) $|a_3| \leq 3/(4(1+2\alpha))$.

These inequalities are sharp.

The proof follows directly from Example 4.4.

Theorem 4.6. Let $f \in M_{g,h}(\phi)$ and either

$$(g_3 - h_3)^2 \leq L \quad \text{or} \quad L < (g_3 - h_3)^2 \leq 2L$$

holds, where $L = (g_2 - h_2)(g_4 - h_4)$. Then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{B^2}{(g_3 - h_3)^2}, & \text{when } X \text{ holds}; \\ \frac{|M|}{(g_2 - h_2)^4(g_3 - h_3)^2(g_4 - h_4)}, & \text{when } Y_1 \text{ or } Y_2 \text{ holds}; \\ \frac{B^2(|T| + B_1(g_3 - h_3)^2(g_2 - h_2) - 2B_1(g_2 - h_2)^2(g_4 - h_4))^2}{4S(g_3 - h_3)^2(g_4 - h_4)} + \frac{B^2}{(g_3 - h_3)^2}, & \text{when } Z \text{ holds}, \end{cases}$$

where,

$$X = |M| - B_1^2(g_2 - h_2)^4(g_4 - h_4) \leq 0 \quad \text{and} \quad (27)$$

$$Y_1 = |T| + B_1(g_3 - h_3)^2(g_2 - h_2) - 2B_1(g_2 - h_2)^2(g_4 - h_4) \geq 0 \quad \text{and} \quad 2|M| - B_1|T|(g_2 - h_2)^2 - B_1^2(g_3 - h_3)^2(g_2 - h_2)^3 \geq 0,$$

$$Y_2 = |M| - B_1^2(g_2 - h_2)^4(g_4 - h_4) \geq 0 \quad \text{and} \quad |T| + B_1(g_3 - h_3)^2(g_2 - h_2) - 2B_1(g_2 - h_2)^2(g_4 - h_4) \leq 0,$$

$$Z = |T| + B_1(g_3 - h_3)^2(g_2 - h_2) - 2B_1(g_2 - h_2)^2(g_4 - h_4) > 0 \quad \text{and} \quad 2|M| - B_1|T|(g_2 - h_2)^2 - B_1^2(g_3 - h_3)^2(g_2 - h_2)^3 \leq 0,$$

with $S = -(|M| - B_1|T|(g_2 - h_2)^2 - B_1^2(g_3 - h_3)^2(g_2 - h_2)^3 + B_1^2(g_2 - h_2)^4(g_4 - h_4)),$

$$M = B_1^2(-h_2^2(g_2 - h_2)^2(g_4 - h_4) + (g_3 - h_3)(g_2g_3h_2^2 - g_3h_2^3 + g_2^2h_3h_4 - 3g_2h_3^2h_4 + 2h_2^3h_3 + (g_3 - h_3)(g_2h_2^2 + h_2^3))) - B_2^2(g_2 - h_2)^4(g_4 - h_4) + B_1B_3(g_3 - h_3)^2.$$
The series expansion of the functions \( f, g \) and \( h \) yields

\[
(f \ast g)(z)/(f \ast h)(z) = 1 + a_2(g_2 - h_2)z + (a_3(g_3 - h_3) + a_2^2h_2(g_2 - g_2))z^2 + (a_4(g_4 - h_4) - a_2(a_2^2h_2^2(-g_2 + h_2)) + a_3(g_3h_2 + g_2h_3 - 2h_2h_3))z^3 + \cdots.
\]

(31)

Here, we define a function \( p \) in \( P \) as follows:

\[
p(z) = (1 + \omega(z))/(1 - \omega(z)) = 1 + p_1z + p_2z^2 + \cdots.
\]

(32)

Then, we have \( \omega(z) = (p(z) - 1)/(p(z) + 1) \). Clearly, \( \omega \) is a Schwarz function. Since \((f \ast g)(z)/(f \ast h)(z) < \phi(z)\), we get

\[
(f \ast g)(z)/(f \ast h)(z) = \phi(\omega(z)).
\]

(33)

Now, using (31), (3) and expression of \( \omega \) in terms of \( p \) in (33), we get

\[
a_2 = \frac{B_1p_1}{2(g_2 - h_2)}, \quad a_3 = \frac{B_2p_1^2(g_2 - h_2) - B_1(p_1^2 - 2p_2)(g_2 - h_2) + B_1^2p_1h_2}{4(g_2 - h_2)(g_3 - h_3)}
\]

(34)

and

\[
a_4 = \frac{1}{8(g_2 - h_2)(g_3 - h_3)(g_4 - h_4)}(p_1(-2B_2p_1^2 + 3B_3p_1^2 + 4B_2p_2)(g_2 - h_2)(g_3 - h_3)
\]

\[
+ B_1^2p_1h_2h_3 - B_1^2p_1(p_1^2 - 2p_2)(g_3h_2 + (g_2 - 2h_2)h_3) + B_1(p_1^2g_3 + (B_2 - 1)h_3)
\]

\[
+ (B_2 - 1)g_3 + h_3 - 2B_2h_3)) - 4p_1p_2(g_2 - h_2)(g_3 - h_3) + 4p_3(g_2 - h_2)(g_3 - h_3)).
\]

(35)

We assume \( p_1 = p \in [0, 2] \) and upon substituting the values of \( p_2 \) and \( p_3 \), given in Lemma 1.6, in the expression \( a_2a_4 - a_3^2 \), we get

\[
a_2a_4 - a_3^2 = \frac{1}{16(g_2 - h_2)^4(g_3 - h_3)^2(g_4 - h_4)}(p^4M - p^2x(4 - p^2)(g_2 - h_2)^2B_1T - (4 - p^2)^2x^2
\]

\[
+ B_1^2g_2 - h_2)^4(g_4 - h_4) - p^2x^2(4 - p^2)B_1^2(g_3 - h_3)^2(g_2 - h_2)^3
\]

\[
+ 2B_1p^4(4 - p^2)(g_3 - h_3)^2(g_2 - h_2)^3g(1 - |x|^2)),
\]

where \( M \) and \( T \) are given in (29) and (30), respectively. Applying triangular inequality in the above equation with the assumption that \( |x| = \rho \), we get

\[
|a_2a_4 - a_3^2| \leq \frac{1}{16g_2 - h_2)^4(g_3 - h_3)^2(g_4 - h_4)}(|M|p^4 + p^2(4 - p^2)p^2B_1^2(g_3 - h_3)^2(g_2 - h_2)^3
\]

\[
+ B_1T|p^2p^2(4 - p^2)(g_2 - h_2)^2 + 2B_1^2(g_3 - h_3)^2(g_2 - h_2)^3p(4 - p^2)(1 - \rho^2)
\]

\[
+ B_1^2(g_2 - h_2)^4(g_4 - h_4)(4 - p^2)^2\rho^2) =: G(p, \rho).
\]

The function \( G(p, \rho) \) is an increasing function of \( p \) in the closed interval \([0, 1]\), when either of the conditions in (26) hold. Thus \( \max_{0 \leq p \leq 1} G(p, \rho) = G(p, 1) =: F(p) \). On solving further, we get

\[
F(p) := \frac{1}{16g_2 - h_2)^4(g_3 - h_3)^2(g_4 - h_4)}(Ap^4 + Bp^2 + C).
\]

(36)
We recall that
\[
\max_{0 \leq t \leq 4} (A^2 + Bt + C) = \begin{cases} 
C, & B \leq 0, A \leq -B/4; \\
16A + 4B + C, & B \geq 0, A \geq -B/8 \text{ or } B \leq 0, A \geq -B/4; \\
(4AC - B^2)/(4A), & B > 0, A \leq -B/8. 
\end{cases}
\] (37)

From (36) and (37), we get the desired result. \(\square\)

**Remark 4.7.** In view of the first case of (25), Theorem 4.6 reduces to the result obtained by Lee et al. [15] which gives the sharp second Hankel determinant bound for functions in the class \(S^*(\phi)\).

**Remark 4.8.** We notice that the bound evaluated in [15, Theorem 1] for case (3), has an error and it should be:
\[
|a_2a_4 - a_3^2| \leq \frac{B_1^2}{12} \left( \frac{3|4B_1B_3 - B_1^2 - 3B_2^2| - 4B_1|B_2| - 4B_1^2 - |B_2|^2}{|4B_1B_3 - B_1^2 - 3B_2^2| - 2B_1|B_2| - B_1^2} \right).
\]

**Remark 4.9.** In view of the fifth case of (25) for \(\phi(z) = (1 + z)/(1 - z)\), Theorem 4.6 reduces to a result obtained in [29].

**Remark 4.10.** The second Hankel determinant bound for the functions in the class \(M_{g,h}(\Phi)\) can be obtained from Theorem 4.6 by replacing each \(B_i\) by \((-1)^iC_i\).

Since \(M_{\alpha}(\phi(z)) = M_{\alpha}(\phi(-z)) = M_{\alpha}(\Phi(z))\). Thus using Theorem 4.6 and Remark 4.10, we obtain the following result by taking \(\Phi(z) = 1 - \log(1 + z)\) in Remark 4.10.

**Lemma 4.11.** Let \(f \in S_0(\alpha)\). Then
\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
1, & 0 \leq \alpha \leq \frac{2 + \sqrt{11}}{11}; \\
\frac{4(1 + 2\alpha)^2}{31\alpha^4 + 136\alpha^3 - 14\alpha^2 - 24\alpha - 3}, & \frac{2 + \sqrt{11}}{11} \leq \alpha \leq 1. 
\end{cases}
\]

We recall that the function \(f \in S\) is in \(S_0^+(\phi)\) and \(C_\alpha(\phi)\) if it satisfies the subordination \((2zf'(z))/(f(z) - f'(-z)) \prec \phi(z), z \in \mathbb{D}\) and \((2zf'(z))/(f(z) - f'(-z))' \prec \phi(z), z \in \mathbb{D}\), respectively. The following couple of corollaries can be obtained from Theorem 4.6, in view of the fourth and fifth cases of (25), respectively.

**Corollary 4.12.** Let \(f \in S_0^+(\phi)\) and
\[
X: |M| - 64B_1^2 \leq 0 \text{ and } |T| - 24B_1 \leq 0.
\]
\[
Y: |M| - 2B_1|T| - 16B_1^2 \geq 0 \text{ and } |T| - 24B_1 \geq 0 \text{ or } |M| - 64B_1^2 \geq 0 \text{ and } |T| - 24B_1 \leq 0.
\]
\[
Z: |M| - 2B_1|T| - 16B_1^2 \leq 0 \text{ and } |T| - 24B_1 > 0,
\]
where \(M = 16B_1^2B_2 - 64B_1^2 + 32B_1B_3\) and \(T = 16B_2 - 4B_1^2\). Then we have
\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
B_1^2/4, & \text{when } X \text{ holds}; \\
|M|/256, & \text{when } Y \text{ holds}; \\
\frac{B_1^2}{4} - \frac{B_1^2(|T| - 24B_1)^2}{64(|M| - 4B_1|T| + 32B_1^2)} & \text{when } Z \text{ holds}.
\end{cases}
\]
Corollary 4.13. Let \( f \in C_s(\phi) \) and

\[
X : |M| - 4096B_1^2 \leq 0 \text{ and } |T| - 368B_1 \leq 0.
\]

\[
Y : |M| - 8B_1|T| - 1152B_1^2 \geq 0 \text{ and } |T| - 368B_1 \geq 0, \text{ or } |T| - 368B_1 \leq 0 \text{ and } |M| - 4096B_1^2 \geq 0.
\]

where \( M = 128(9B_1^2B_2 - 32B_2^2 + 18B_1B_3) \) and \( T = 8(28B_2 - 9B_1^2) \). Then we have

\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
    \frac{|M|}{147456}, & \text{when } X \text{ holds;} \\
    \frac{B_1^2}{36}, & \text{when } Y \text{ holds;} \\
    \frac{B_1^2(|T| - 368B_1)^2}{2304(|M| - 16B_1|T| + 1792B_1^2)}, & \text{when } Z \text{ holds.}
\end{cases}
\]

Remark 4.14. When \( \phi(z) = (1 + z)/(1 - z) \), the Corollaries 4.12 and 4.13 reduce to the results obtained in [19] for the classes \( S_1^* \) and \( C_s \) of starlike functions and convex functions with respect to symmetric points, respectively.

Remark 4.15. Note that the second Hankel determinant bound for the functions in the classes \( S_1^*(\Phi) \) and \( C_s(\Phi) \) can be obtained from the Corollaries 4.12 and 4.13, respectively by replacing each \( B_i \) by \((-1)^iC_i\).

Expressing the fourth coefficient \( a_4 \) for the function \( f \in M_{g,h}(\phi) \) in terms of the Schwarz function \( \omega(z) = 1 + \omega_1z + \omega_2z^2 + \cdots \), we obtain the bound of \( a_4 \) as follows:

\[
a_4 \leq H(q_1, q_2)B_1/(g_4 - h_4), \tag{38}
\]

where

\[
q_1 = (2B_2(g_2 - h_2)(g_3 - h_3) + B_1^2(g_2h_2 + g_3h_3 - 2h_2h_3))/(B_1(g_2 - h_2)(g_3 - h_3)) \tag{39}
\]

and

\[
q_2 = (B_3(g_2 - h_2)(g_3 - h_3) + B_1B_2(g_2h_3 + g_3h_2 - 2h_2h_3))/(B_1(g_2 - h_2)(g_3 - h_3)). \tag{40}
\]

Remark 4.16. In view of the first case of (25), for \( \phi(z) = \sqrt{1+z} \), the above result (38) reduces to the result obtained in [24].

Remark 4.17. Note that the bound for the fourth coefficient for the functions in the class \( M_{g,h}(\Phi) \) can be obtained from (38), (39) and (40) by replacing each \( B_i \) by \((-1)^iC_i\).

Lemma 4.18. Let \( f \in S_1(\alpha) \), then \( |a_4| \leq 19/(36(1 + 3\alpha)) \). The result is sharp.

We omit the proof as it follows directly from (38).

Expressing the expression \( a_2a_4 - a_3^2 \) for the function \( f \in M_{g,h}(\phi) \) in terms of the Schwarz function \( \omega(z) = 1 + \omega_1z + \omega_2z^2 + \cdots \), we obtain the bound as follows:

\[
|a_2a_4 - a_3^2| \leq H(q_1, q_2)B_1/(g_4 - h_4),
\]

where

\[
q_1 = \frac{2B_2(g_2 - h_2)^2(g_3 - h_3) + B_1^2(g_2 - h_2)(-g_4 + g_3h_2 + g_2h_3 - 2h_2h_3 + h_4)}{B_1(g_2 - h_2)^2(g_3 - h_3)}, \tag{41}
\]
\[ q_2 = \frac{1}{B_1(g_2 - h_2)^2(g_3 - h_3)} (B_3(g_2 - h_2)^2(g_3 - h_3) + B_1B_2(g_2 - h_2)(-g_4 + g_3h_2 + g_2h_3 - 2h_2h_3 + h_4)) + B_1^2h_2(-g_4 + g_3h_2 - h_2h_3 + h_4). \] (42)

As a consequence, we have the following result.

**Lemma 4.19.** Let \( f \in S_1(\alpha) \). Then \( |a_2a_3 - a_4| \leq 1/(3(1 + 3\alpha)) \). The result is sharp whenever \( (z'f(z) + \alpha z^2 f''(z))/(\alpha z f'(z) + (1 - \alpha)f(z)) = 1 - \log(1 + z^2) \).

**Theorem 4.20.** Let \( f \in S_1(\alpha) \). Then, we have
\[ |a_5| \leq \frac{107}{288(1 + 4\alpha)}. \]
The result is sharp.

**Proof.** The equations (9), (31), (32) and (33) with \( \phi(z) \) in place of \( \phi(z) \), yield \( a_5 \) in terms of \( p_1, p_2, p_3 \) and \( p_4 \) as follows
\[ |a_5| = \frac{1}{4(1 + 4\alpha)} \left| \begin{array}{c} 695 \frac{p_1^4}{1152} - 27 \frac{p_1^2p_2}{16} + \frac{1}{2}p_2^2 + \frac{13}{12}p_1p_3 - \frac{1}{2}p_4 \end{array} \right| \]
\[ = \frac{1}{8(1 + 4\alpha)} \left( |P| + \frac{1}{6}|p_1Q| + \frac{1}{24}|p_1^2R| \right), \]
where \( P = p_1^4 - 3p_1^2p_2 + 2p_1p_3 - p_4, Q = p_3 - 2p_1p_2 + p_3^2 \) and \( R = p_2 - (23/24)p_1^2 \). Since \( |P|, |Q| \leq 2 \) from [16] and \( |R| \leq 2 \) from [17], we get
\[ |a_5| \leq \frac{1}{8(1 + 4\alpha)} \left( 2 + \frac{2}{3} + \frac{1}{12} |p_1|^2 - \frac{1}{576} |p_1|^4 \right). \]

Let us assume \( G(p_1) := |p_1|^2/12 - |p_1|^4/576. \) Then, the formula given in (37) yields the bound when \( A = -1/576, B = 1/12 \) and \( C = 0 \). Letting \( p_1 = 1, p_2 = 2, p_3 = -2/3 \) and \( p_4 = -1/64 \) shows that the result is sharp.

Recall that \( |H_3(1)| \leq |a_3||a_2a_4 - a_2^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \). Using Lemmas 4.5, 4.11, 4.18, 4.19 and Theorem 4.20, we can estimate the bound for \( H_3(1) \) for the class \( S_1(\alpha) \), which is stated below in the following theorem:

**Theorem 4.21.** Let \( f \in S_1(\alpha) \). Then
\[ |H_3(1)| \leq g(\alpha), \]
where
\[ g(\alpha) = \frac{949 + 11388\alpha + 52493\alpha^2 + 114974\alpha^3 + 117180\alpha^4 + 42568\alpha^5}{1728(1 + 4\alpha)(1 + 3\alpha)^2(1 + 2\alpha)^4}, \]
when \( 0 \leq \alpha \leq (2 + \sqrt{15})/11 \) and
\[ g(\alpha) = \frac{1}{1728(1 + \alpha)(1 + 4\alpha)(1 + 3\alpha)^2(1 + 2\alpha)^4(61\alpha^2 - 20\alpha - 5)} \left( -5069 - 76035\alpha - 385994\alpha^2 - 619570\alpha^3 + 831511\alpha^4 + 3545777\alpha^5 + 3327024\alpha^6 + 1298324\alpha^7 \right), \]
when \( (2 + \sqrt{15})/11 \leq \alpha \leq 1 \).
Remark 4.22. Taking \( \alpha = 0 \) and 1, we get all the above bounds for the classes \( S^*_l \) and \( C_l \), respectively.

On the similar lines of the estimation of Third Hankel determinant for functions in \( SL^* \) in [5], we compute the same for \( f \in S^*_l \).

Theorem 4.23. Let \( f \in S^*_l \), then
\[
|H_3(1)| \leq \frac{1}{9}.
\]
The result is sharp.

Proof. The proof is on the similar lines of the proof of [5, Theorem 2.1], however the computation involves altogether new values. Let
\[
\tilde{f}(z) = z \exp \left( \int_0^z \frac{-\log(1 + t^3)}{t} \, dt \right) = z - \frac{z^4}{3} + \cdots,
\]
clearly which belongs to \( S^*_l \). The equality holds for the above defined function \( \tilde{f} \), as \( a_2 = a_3 = a_5 = 0 \) and \( a_4 = -1/3 \).

Let us define a function \( f_n \) in the class \( A \) as
\[
f_n(z) = z + a_{2,n} z^2 + a_{3,n} z^3 + \cdots = z + \sum_{m=2}^\infty a_{m,n} z^m.
\]
In view of the fact \( 1 - \log(1 + z^n) \prec 1 - \log(1 + z) \) for all \( n \geq 1 \), let us consider the subclass \( S^*_{l,n} \) of \( S^*_l \) consisting of the functions \( f_n \) satisfying
\[
z f_n'(z)/f_n(z) = 1 - \log(1 + z^n) \quad (n \geq 1),
\]
which upon simplification yields \( z f_n'(z) = f_n(z) (1 - \log(1 + z^n)) \). Further, we have
\[
\sum_{j=1}^\infty \left( \sum_{k=1}^\infty (-1)^k \frac{a_j}{k} z^{nk+j} \right) = \sum_{s=1}^\infty (s-1) a_s z^s.
\]
On comparing the coefficients of like power terms on either side of the above equation, we get a special pattern due to which we conjecture the following:

Conjecture 4.24. Let \( f_n \in S^*_{l,n} \). Then we have \( |a_{m,n}| \leq |a_{m,1}| \).

§5 Further results

We recall that the set \( B \) is the space of all Bloch functions. An analytic function \( f \) is said to be a Bloch function if it satisfies
\[
\kappa_B(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty. \tag{43}
\]
Also, \( B \) is a Banach space with the norm \( || \cdot ||_B \) defined by
\[
||f||_B = |f(0)| + \kappa_B(f), \quad f \in B. \tag{44}
\]
Now, we give below a result involving Bloch function norm for the functions in the class \( S^*_l \):

Theorem 5.1. The set \( S^*_l \subseteq B \). Further, if \( f \in S^*_l \), then \( ||f||_B \leq x \approx 1.27429 \).

Proof. If \( f \in S^*_l \), then \( z f'(z)/f(z) = 1 - \log(1 + \omega(z)) =: g(z) \). By the structural formula, given in (6), we get
\[
f(z) = z \exp \left( \int_0^z \frac{g(t) - 1}{t} \, dt \right).
\]
Upon differentiating \( f \) and further considering the modulus, we obtain
\[
|f'(z)| = |g(z)| \exp \left( \frac{\int_0^z g(t) - 1}{t} \, dt \right) \leq |1 - \log(1 + \omega(z))| \exp \left( \int_0^z \frac{\log(1 + \omega(t))}{|t|} \, dt \right) .
\]
(45)

Let \( t = re^{i\theta} \) and \( \omega(t) = Re^{i\theta_2} \), where \( R \leq r = |t| < 1 \) and \( -\pi < \theta_1, \theta_2 < \pi \). Now, by using the similar analysis carried out in the proof of part (v) of Theorem 2.2 and in Theorem 2.3, equation (45) reduces to
\[
|f'(z)| \leq (1 - \log(1 - R)) \exp \left( |\log(1 - R)| \int_{-\pi}^\pi e^{i\theta} \, d\theta \right) \leq 1 - \log(1 - r).
\]

Thus we have \( g(r) := (1 - |z|^2)|f'(z)| \leq (1 - r^2)(1 - \log(1 - r)) \), which upon differentiation, gives \( g'(r) = 1 - r + 2r \log(1 - r) \). Taking \( g'(r) = 0 \), yields \( r_0 \approx 0.453105 \). Now \( g''(r_0) < 0 \), yields \( \max_{0 < r < 1} g(r) = g(r_0) \approx 1.27429 < \infty \). Using (43), we obtain \( S^*_1 \subseteq B \). We can now, estimate the norm \( ||f||_B \) for the functions in the class \( S^*_1 \). Now, by using the definition of norm, given in (44), we have
\[
||f||_B \leq f(0) + 1.27429 .
\]

By the normalization of the function \( f \), the result follows now at once.

The following theorem gives the sufficient condition for the given function \( g \) to belong to the class \( S^*_1 \).

**Theorem 5.2.** Let \( m, n \geq 1 \) and \( 0 \leq \lambda \leq 1 \). Then, \( g(z) = z \exp(\alpha) \in S^*_1 \), where
\[
\alpha = \sum_{k=1}^\infty \frac{1}{k^2} \lambda \left( \frac{(-z)^{nk}}{n} - \frac{(-z)^{nk}}{m} \right) + \frac{(-z)^{nk}}{m} .
\]

**Proof.** For the given \( \alpha \), we have
\[
g(z) = z \exp \left( \frac{(\lambda - 1)}{m} z^m \left( 1 - \frac{z^m}{4} + \frac{z^{2m}}{9} - \cdots \right) - \frac{\lambda}{n} z^n \left( 1 - \frac{z^n}{4} + \frac{z^{2n}}{9} - \cdots \right) \right) .
\]

Then, we have
\[
\frac{zg'(z)}{g(z)} = 1 + (\lambda - 1)z^m \left( 1 - \frac{z^m}{2} + \frac{z^{2m}}{3} - \cdots \right) - \lambda z^n \left( 1 - \frac{z^n}{2} + \frac{z^{2n}}{3} - \cdots \right) = \lambda(1 - \log(1 + z^m)) + (1 - \lambda)(1 - \log(1 + z^n)).
\]

We observe that \( 1 - \log(1 + z^t) < 1 - \log(1 + z) =: \psi(z) \) for all \( t \geq 1 \) and the function \( \psi \) is convex in \( |z| < 1 \). Thus the result follows at once when \( 0 \leq \lambda \leq 1 \).

When \( m = n \), the above theorem yields the following result:

**Corollary 5.3.** Let \( n \geq 1 \) and \( g(z) = z \exp(\alpha) \), where
\[
\alpha = \frac{1}{n} \left( \sum_{k=1}^\infty \frac{(-z)^{nk}}{k^2} \right) .
\]

We have \( g \in S^*_1 \).

**Theorem 5.4.** The class \( S^*_1 \) is not a vector space.
Proof. For if, the class $S^*_l$ is a vector space, then the class preserves additive property, that is, whenever two functions belong to the class $S^*_l$, then their sum also belongs to the class $S^*_l$. Let $f_1$ and $f_2 \in S^*_l$. Then, using (6), we obtain

$$f_1(z) = z \exp \left( \int_0^z \frac{\log(1 + \omega_1(t))}{t} \, dt \right) \quad \text{and} \quad f_2(z) = z \exp \left( \int_0^z \frac{\log(1 + \omega_2(t))}{t} \, dt \right),$$

for some Schwarz functions $\omega_1$ and $\omega_2$. Thus, the sum of the functions, $f_1 + f_2$ to be in $S^*_l$, there should exist some Schwarz function $\omega(z)$ such that

$$\omega(z) = \frac{\exp(-z(A'(z) \exp A(z) + B'(z) \exp B(z))) - 1}{\exp A(z) + \exp B(z)},$$

where

$$A(z) = \int_0^z \frac{\log(1 + \omega_1(t))}{t} \, dt \quad \text{and} \quad B(z) = \int_0^z \frac{\log(1 + \omega_2(t))}{t} \, dt.$$

Then $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $A$ and $B$. Let $\omega_1(z) = z$ and $\omega_2(z) = z^2$. We observe that $\omega(z) \approx 1.03053$ at $z = -(\frac{1}{2} + i\frac{3}{2})$, which contradicts the existence of Schwarz function $\omega(z)$ with $|\omega(z)| < 1$. Hence the assertion follows.

The following theorem is an immediate consequence of the growth Theorem of $S^*(\Phi)$.

**Theorem 5.5.** Let $f \in S^*_l$. Then we have

$$|f(z)| \leq |z| \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \right) = |z|L \quad (z \in \mathbb{D}),$$

where $L \approx 0.822467$.

**Proof.** In view of Remark 1.5, we get

$$t_\Phi(r) \leq |f(z)| \leq -t_\Phi(-r),$$

For $|z| = r$, we have

$$\log \frac{|f(z)|}{z} \leq \int_0^r \frac{\log(1 + t)}{t} \, dt \leq \int_0^1 \frac{\log(1 + t)}{t} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$ 

The convergent nature of the series on the right side of the above equality yields the desired result at once.

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