A NOTE ON A WEAKLY COUPLED SYSTEM OF STRUCTURALLY DAMPED WAVES

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Abstract. In this note, we find the critical exponent for a system of weakly coupled structurally damped waves.

1. Introduction. Recently, K. Nishihara and Y. Wakasugi [17] studied the Cauchy problem of the weakly coupled system of damped wave equations

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v|^p, & t \geq 0, & x \in \mathbb{R}^n, \\
    v_{tt} - \Delta v + v_t &= |u|^q, & t \geq 0, & x \in \mathbb{R}^n, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{align*}
\]

(1)

where \( p, q > 1 \). They proved that global existence of small data solutions to (1) holds, in any space dimension \( n \geq 1 \), if

\[
    \alpha := \frac{\max\{p, q\} + 1}{pq - 1} < \frac{n}{2}.
\]

(2)

Condition (2) is equivalent to

\[
    \max\{p, q\} \left( \min\{p, q\} + 1 - p_F(n) \right) > p_F(n),
\]

(3)

where \( p_F(n) = 1 + 2/n \) is the Fujita critical exponent for the damped wave equation [18]. They also estimated the lifespan of the solution in the case of blow-up in finite time, which occurs for \( \alpha > n/2 \).

The result in [17] extended the previous one obtained in space dimension \( n \leq 3 \) by K. Nishihara [16]. The bound on the space dimension does not appear in the weakly coupled system of heat equations, and it is related to the regularity problems which appear in the estimates available for the linear damped wave equations

\[
\begin{align*}
    u_{tt} - \Delta u + 2au_t &= 0, & t \geq 0, & x \in \mathbb{R}^n, \\
    (u, u_t)(0, x) &= (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{align*}
\]

(4)

where \( a > 0 \). Using weighted energy estimates, as in [12, 18], this limitation has been overcome in [17]. We address the interested reader to the reference therein, for a more detailed overview about systems of damped waves.

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The main purpose of this note is to derive an analogous result for the system of structurally damped waves

\[
\begin{cases}
u_{tt} - \Delta v + 2b(-\Delta)^{\frac{1}{2}} v_t = |v|^q, \\
u_{tt} - \Delta u + 2a(-\Delta)^{\frac{1}{2}} u_t = |v|^p,
\end{cases}
\quad t \geq 0, \ x \in \mathbb{R}^n, \tag{5}
\]

where \(a, b > 0\). In order to do that, we take advantage of the better linear decay estimates available for linear wave equations with structural damping \((-\Delta)^{\frac{1}{2}} u_t\) (see, for instance, \([1, 15]\)):

\[
\begin{cases}
u_{tt} - \Delta v + 2b(-\Delta)^{\frac{1}{2}} v_t = 0, \\
u_{tt} - \Delta u + 2a(-\Delta)^{\frac{1}{2}} u_t = 0,
\end{cases}
\quad t \geq 0, \ x \in \mathbb{R}^n, \tag{6}
\]

In particular, for any \(n \geq 2\), we show that global existence of small data solutions to (5) holds if

\[
\max\{p, q\} \{\min\{p, q\} + 1 - p_K(n)\} > p_K(n), \tag{7}
\]

where \(p_K(n) = 1 + 2/(n - 1)\) is the critical exponent for the wave equation with structural damping \((-\Delta)^{\frac{1}{2}} u_t\) and power nonlinearity (see \([7, 9, 10]\)). We remark that \(p_K(n)\) is also the critical exponent obtained, by comparison method, by Kato for wave equations with variable coefficients and compactly supported data \([13]\). Condition (7) is equivalent to

\[
\alpha = \max\{p, q\} + 1 - \frac{n - 1}{2} < 0.
\]

**Theorem 1.** Let \(n \geq 2\) and \(p, q > p_K(n) - 1\), be such that (7) holds. Then, there exists \(\varepsilon > 0\) such that, for any \((u_0, v_0) \in L^1 \cap H^1 \cap L^\infty\) and \((u_1, v_1) \in L^1 \cap L^n\), with

\[
\mathcal{A} := \| (u_0, v_0) \|_{L^1 \cap H^1 \cap L^\infty} + \| (u_1, v_1) \|_{L^1 \cap L^n} \leq \varepsilon,
\]

there exists a solution

\[
(u, v) \in (C([0, \infty), L^1 \cap H^1 \cap L^\infty) \cap C^1([0, \infty), L^2))^2
\]

to (5). Moreover,

\[
\|u(t, \cdot)\|_{L^r} \leq C (1 + t)^{1-n(1-\frac{1}{r})+\gamma(p)} \mathcal{A}, \tag{9}
\]

\[
\|v(t, \cdot)\|_{L^r} \leq C (1 + t)^{1-n(1-\frac{1}{r})+\gamma(q)} \mathcal{A}, \tag{10}
\]

for any \(r \in [1, \infty]\), and

\[
\|(u_t, \nabla u)(t, \cdot)\|_{L^2} \leq C (1 + t)^{-\frac{3}{2}+\gamma(p)} \mathcal{A}, \tag{11}
\]

\[
\|(u_t, \nabla v)(t, \cdot)\|_{L^2} \leq C (1 + t)^{-\frac{3}{2}+\gamma(q)} \mathcal{A}, \tag{12}
\]

where

\[
\gamma(p) = (n - 1) (p_K(n) - p)_+, \tag{13}
\]

\[
\gamma(q) = (n - 1) (p_K(n) - q)_+, \tag{14}
\]

provided that \(p, q \neq p_K(n)\). Here \(\gamma(p)\) and \(\gamma(q)\) represent the possible loss of decay with respect to the corresponding linear estimates for \(u\) and \(v\) (see (22)-(24)). If \(p = p_K(n)\) or, respectively, \(q = p_K(n)\), then the loss of decay \((1 + t)^{\gamma(p)}\) or, respectively, \((1 + t)^{\gamma(q)}\), is replaced by \(\log(e + t)\).

In (13), (14) and through this paper, we denote \((a)_+ := \max\{a, 0\}\).

**Remark 1.** We notice that (7) implies that at least one among \(\gamma(p)\) and \(\gamma(q)\) is zero.
Remark 2. We may weaken the assumptions on the space of the data in different ways, if we are not interested for the solution to be in $L^1$, or in $L^\infty$, or if we are not interested in an estimate for the energy (see the proof of Theorem 1). However, a reasonable assumption for the regularity of the solution to guarantee the same critical exponent, is at least $(u,v) \in \mathcal{C}([0,\infty), L^p) \times \mathcal{C}([0,\infty), L^q)$. In fact, in this case, linear $L^1 - L^q$ and $L^1 - L^p$ estimates may be successfully applied to deal with the nonlinear term, via Duhamel’s principle.

The critical exponent in (7) is sharp. Indeed, we have the following.

**Theorem 2.** Let $n \geq 1$ and $p, q > 0$ with $pq > 1$. We consider the Cauchy problem

\[
\begin{align*}
\begin{cases}
  u_{tt} - \Delta u + 2(-\Delta)^{\frac{1}{2}} u_t = v^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
v_{tt} - \Delta v + 2(-\Delta)^{\frac{1}{2}} v_t = u^q, & t \geq 0, \quad x \in \mathbb{R}^n, \\
(u, u_t, v, v_t)(0, x) = (0, u_1, 0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

(15)

Let us assume that the data $u_1, v_1 \in L^q_{\text{loc}}$ are non-negative. If $n = 1$, or $n \geq 2$ and

\[
\max\{p, q\} (\min\{p, q\} + 1 - p_K(n)) < p_K(n),
\]

(16)

or, equivalently, $\alpha > (n-1)/2$, then there exists no global nontrivial $L^q_{\text{loc}} \times L^p_{\text{loc}}$ solution to (15).

We remark that $\min\{p, q\}$ may be smaller than 1 in Theorem 2.

We leave open the problem to study whether a nonexistence result holds in the limit case $\alpha = (n-1)/2$.

It is clear that one may also study the global existence of small data solutions for a system where a wave equation with classical damping is coupled with a structurally damped wave,

\[
\begin{align*}
\begin{cases}
  u_{tt} - \Delta u + 2 au_t = |v|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
v_{tt} - \Delta v + 2 b(-\Delta)^{\frac{1}{2}} v_t = |u|^q, & t \geq 0, \quad x \in \mathbb{R}^n, \\
(u, u_t, v, v_t)(0, x) = (0, u_0, 0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

(17)

where $a, b > 0$. In this note, we limit the study of (17) to space dimension $n = 2$ and we assume $q > p_F(2) = 2$. The restriction $q > 2$ allow us to avoid the use of both weighted energy estimates and $L^1 - L^\infty$ estimates for the wave equation with classical damping, since standard energy estimates on the $L^2$ basis are sufficient. We recall that $p_F(2) = 3$.

**Proposition 1.** Let $n = 2$ and $p, q > 2$ be such that

\[
q(p - 2) > 2.
\]

(18)

Then, there exists $\varepsilon > 0$ such that, for any $(u_0, v_0) \in L^1 \cap H^1$ and $(u_1, v_1) \in L^1 \cap L^2$, with

\[
\mathcal{A} := \|(u_0, v_0)\|_{L^1 \cap H^1} + \|(u_1, v_1)\|_{L^1 \cap L^2} \leq \varepsilon,
\]

(19)

there exists a solution

\[
(u, v) \in (\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2))^2
\]

to (17). Moreover,

\[
\|\partial_t^j \nabla^k u(t, \cdot)\|_{L^2} \leq C (1 + t)^{-\frac{j}{2} - \frac{k}{2} - \gamma(p)} \mathcal{A}, \quad \text{for } j + k = 0, 1,
\]

(20)

\[
\|\partial_t^j \nabla^k v(t, \cdot)\|_{L^2} \leq C (1 + t)^{-k-j} \mathcal{A}, \quad \text{for } j + k = 0, 1,
\]

(21)

where $\gamma(p) = (3-p)_+$, as in (13), represents the possible loss of decay with respect to the corresponding linear estimates for $u$. If $p = 3$ then $(1 + t)^\gamma(p)$ is replaced by $\log(1 + t)$ in (20).
Remark 3. It is clearly possible to extend Proposition 1 to space dimension \( n \geq 3 \), but some technical difficulties appear in the application of the linear estimates, due to the coupling of waves with different damping terms. In particular, the loss of decay should be carefully managed; as far as the employed estimates allow to keep this latter equal to \((1 + t)^{\gamma(p)}\), one may expect the critical exponent to be given by
\[
q(p + 1 - p_K(n)) > p_F(n),
\]
for \( n \geq 3 \) and \( q > p_F(n) \).

2. **Proof of Theorem 1.** In order to prove Theorem 1, by Duhamel’s principle, we may reduce to consider the linear equation in (6). We recall the linear estimates (see, for instance, \([6, 15]\)):
\[
\|u(t, \cdot)\|_{L^r} \leq C (1 + t)^{-n(1 - \frac{1}{n})} \|u_0\|_{L^r \cap L^\infty} + C (1 + t)^{1 - n(1 - \frac{1}{n})} \|u_1\|_{L^1 \cap L^n},
\]
for any \( r \in [1, \infty] \), and
\[
\|u(t, \cdot)\|_{L^\infty} \leq C \|u_0\|_{L^\infty} + C \|u_1\|_{L^n},
\]
\[
\|u_t, \nabla u(t, \cdot)\|_{L^2} \leq C (1 + t)^{-\frac{\gamma}{2}} (\|u_0\|_{H^1} + \|u_1\|_{L^1 \cap L^2}),
\]
\[
\|u_t, \nabla u(t, \cdot)\|_{L^2} \leq C (\|u_0\|_{H^1} + \|u_1\|_{L^2}).
\]
In the following, we denote by \( E_0(t, x) \) and \( E_1(t, x) \) the fundamental solutions to the linear equation, corresponding to the two initial data, namely,
\[
u(t, x) = E_0(t, x) *_{(x)} u_0(x) + E_1(t, x) *_{(x)} u_1(x)
\]
is the solution to (6).

**Proof of Theorem 1.** Let us define the space
\[
X := (C([0, \infty), L^1 \cap H^1 \cap L^\infty) \cap C^1([0, \infty), L^2))^2,
\]
with norm:
\[
\|(u, v)|_X := \sup_{t \geq 0} \left( (1 + t)^{-\gamma(p)} M(u) + (1 + t)^{-\gamma(q)} M(v) \right),
\]
where \( \gamma(p), \gamma(q) \) are as in (13), (14), and
\[
M(u) = (1 + t)^{-1} \|u(t, \cdot)\|_{L^1} + (1 + t)^{n-1} \|u(t, \cdot)\|_{L^\infty} + (1 + t)^{\frac{\gamma}{2}} \|\nabla u(t, \cdot)\|_{L^2},
\]
and the same for \( v \). If \( p = p_K(n) \) or, respectively, \( q = p_K(n) \), then we replace \((1 + t)^{-\gamma(p)}\) or, respectively, \((1 + t)^{-\gamma(q)}\), by \((\log(e + t))^{-1}\).

Now let us consider the integral operator \( N : X \to X \), defined by \( N[u, v] = A + (Fv, Gu) \), where
\[
A(t, x) = E_0(t, x) *_{(x)} (u_0, v_0)(x) + E_1(t, x) *_{(x)} (u_1, v_1)(x),
\]
\[
Fv(t, x) = \int_0^t E_1(t - s, x) *_{(x)} |v(s, x)|^p ds,
\]
\[
Gu(t, x) = \int_0^t E_1(t - s, x) *_{(x)} |u(s, x)|^q ds.
\]
Recalling that \( p, q > 1 \), global existence of small data solutions to (5) follows by standard contraction argument (see, for instance, \([10]\)), with the desired estimates (9), (10), (11), (12), as in Theorem 1, if we prove the a priori estimates
\[
\|N[u, v]\|_X \leq C_1 A + C_2 \|(u, v)\|_X + C_3 \|(u, v)\|_X^p,
\]
\[
\|N[u, v] - N[\tilde{u}, \tilde{v}]\|_X \leq C \|(u, v) - (\tilde{u}, \tilde{v})\|_X \left( \|(u, v)\|_X^{p-1} + \|(\tilde{u}, \tilde{v})\|_X^{q-1} \right).
\]
For the sake of brevity, we will prove only (27), being (28) analogous.

For any \( (u, v) \in X \), there exists \( C > 0 \) such that, for any \( s \geq 0 \), it holds

\[
\|u(s, \cdot)\|_{L^r} \leq C (1 + s)^{1 - n(1 - \frac{1}{p}) + \gamma(p)} \|u, v\|_X, \tag{29}
\]
\[
\|v(s, \cdot)\|_{L^r} \leq C (1 + s)^{1 - n(1 - \frac{1}{p}) + \gamma(q)} \|u, v\|_X, \tag{30}
\]

for any \( r \in [1, \infty] \), and

\[
\|(u_t, \nabla u)(s, \cdot)\|_{L^2} \leq C (1 + s)^{-\frac{2}{p} + \gamma(p)} \|u, v\|_X, \tag{31}
\]
\[
\|(v_t, \nabla v)(s, \cdot)\|_{L^2} \leq C (1 + s)^{-\frac{2}{q} + \gamma(q)} \|u, v\|_X, \tag{32}
\]

By the linear estimates (22), (24), recalling that \( \gamma(p), \gamma(q) \geq 0 \), it immediately follows that \( \|A\|_X \leq C_1 A \). On the other hand, by virtue of (22), (29), (30), we derive

\[
\|Fv(t, \cdot)\|_{L^1} \lesssim \int_0^t (1 + t - s)^{\frac{p}{2}} \|v(s, \cdot)\|_{L^p}^p ds \lesssim \|(u, v)\|_X^p \int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/p) + \gamma(p))p} ds,
\]
\[
\|Gu(t, \cdot)\|_{L^1} \lesssim \int_0^t (1 + t - s)^{\frac{q}{2}} \|u(s, \cdot)\|_{L^q}^q ds \lesssim \|(u, v)\|_X^q \int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/q) + \gamma(p))q} ds.
\]

Now let us distinguish two cases. If both \( p, q > p_K(n) \), then \( \gamma(p) = \gamma(q) = 0 \) and

\[
\int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/p))p} ds \lesssim 1 + t,
\]
\[
\int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/q))q} ds \lesssim 1 + t;
\]

hence, we trivially get

\[
\sup_{t \geq 0} (1 + t)^{-1} \left((1 + t)^{-\gamma(p)} \|Fv(t, \cdot)\|_{L^1} + (1 + t)^{-\gamma(q)} \|Gu(t, \cdot)\|_{L^1}\right) < \infty. \tag{33}
\]

Otherwise, let \( \min\{p, q\} \leq p_K(n) \), say \( p \leq p_K(n) \). Then, in particular, \( q > p_K(n) \) and \( \gamma(q) = 0 \). On the one hand, we obtain:

\[
\int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/p))p} ds \lesssim \begin{cases} (1 + t)^{1 + \gamma(p)} & \text{if } p < p_K(n), \\ (1 + t) \log(e + t) & \text{if } p = p_K(n). \end{cases}
\]

On the other hand, condition (7) is equivalent to

\[
(1 - n(1 - 1/q) + \gamma(p))q < -1,
\]

therefore

\[
\int_0^t (1 + t - s)(1 + s)^{(1 - n(1 - 1/q) + \gamma(p))q} ds \lesssim 1 + t,
\]

and the same if we replace \( (1 + s)^{\gamma(p)q} \) by \( (\log(e + s))^n \) when \( p = p_K(n) \). Summarizing, we obtained again (33). For the estimates of the integrals used above, see, for instance, Lemma 3.3 in [1].

We have to pay more attention to estimate the \( L^\infty \) norms of \( Fv \) and \( Gu \). We split the interval \([0, t]\) into \([0, t/2]\) and \([t/2, t]\), then we use (22) in the first one and (23) in the second
one. Therefore,
\[
\|Fv(t, \cdot)\|_{L^\infty} \lesssim \int_0^{t/2} (1 + t - s)^{-(\ell-1)} \|v(s, \cdot)\|_{L^p}^p \, ds + \int_{t/2}^t \|v(s, \cdot)\|_{L^p}^p \, ds
\]
\[
\lesssim \|(u, v)\|_X^p \int_0^{t/2} (1 + t - s)^{-(\ell-1)} (1 + s)^{(1-n(1/p) + \gamma(q))p} \, ds
\]
\[
+ \|(u, v)\|_X^p \int_{t/2}^t (1 + s)^{(1-n(1/np) + \gamma(q))p} \, ds.
\]
Due to
\[
1 + t - s \approx 1 + t \quad \text{for} \ s \in [0, t/2], \quad \text{and} \quad 1 + s \approx 1 + t \quad \text{for} \ s \in [t/2, t],
\]
we reduce to estimate
\[
(1 + t)^{-(\ell-1)} \int_0^{t/2} (1 + s)^{(1-n(1/np) + \gamma(q))p} \, ds, \quad \text{and} \quad (1 + t)^{2+(1-n+\gamma(q))p},
\]
and analogously for \(\|Gu(t, \cdot)\|_{L^\infty}\). We remark that the second term may be controlled by the first one. As we did for (33), we may distinguish two cases, and derive the desired estimate
\[
\sup_{t \geq 0} (1 + t)^{n-1} \left((1 + t)^{-\gamma(p)} \|Fv(t, \cdot)\|_{L^\infty} + (1 + t)^{-\gamma(q)} \|Gu(t, \cdot)\|_{L^\infty}\right) < \infty. \tag{34}
\]
We proceed similarly for \(\|\partial_t \nabla Fv\|_{L^2}\) and \(\|\partial_t \nabla Gu\|_{L^2}\), using (24) in \([0, t/2]\) and (25) in \([t/2, t]\). Now (31), (32) come into play. This concludes the proof of (27).

3. Proof of Theorem 2. To prove Theorem 2, we may use the test function method (first used for the damped wave equation with power nonlinearity in [19]), following the approach in [7, 10].

An essential role in the proof of Theorem 2 is played by the fact that any local or global solution to (15) with non-negative initial data \(u_1, v_1\), is non-negative. This property is used to extend the test function method to the system in (15), which contains the nonlocal term \((-\Delta)^{1/2} u_2\). To get this property, we set \(a = b = 1\) and \(u_0 = v_0 = 0\) in (5) (see [10]). The requirement of non-negativity of the solution does not appear for (1).

Proof of Theorem 2. We assume by contradiction that \((u, v) \in L^q_{loc} \times L^p_{loc}\) is a global solution to (15). Therefore, for any test function \(\psi \in C_c^\infty([0, \infty) \times \mathbb{R}^n)\) it holds
\[
\int_0^{\infty} \int_{\mathbb{R}^n} u \psi \, dx \, dt = \int_0^{\infty} \int_{\mathbb{R}^n} v^p \psi \, dx \, dt + \int_{\mathbb{R}^n} u_1(x) \psi(0, x) \, dx, \tag{35}
\]
\[
\int_0^{\infty} \int_{\mathbb{R}^n} v \psi \, dx \, dt = \int_0^{\infty} \int_{\mathbb{R}^n} u^q \psi \, dx \, dt + \int_{\mathbb{R}^n} v_1(x) \psi(0, x) \, dx.
\]
where
\[
L\psi := \psi_{tt} - \Delta \psi - 2(-\Delta)^{1/2} \psi_t.
\]
Let \(\phi \in C_c^\infty([0, \infty))\) be a nontrivial, nonincreasing function, compactly supported in \([0, 1]\), and let \(\ell > \max\{p', q'\}\). For any \(R > 1\), we set \(\psi(t, x) = \phi(t/R)^\ell \phi(|x|/R)^\ell\) in (35). Recalling that \(\phi, -\phi', u_1, v_1\) are nonnegative and that (see [3])
\[
(-\Delta)^\phi(|x|/R)^\ell \leq \ell \phi(|x|/R)^{\ell-1}(-\Delta)^\phi(|x|/R)
\]
for any $\theta \in (0,1]$ and $\ell > 1$, we may derive

\[
I_R := \int_0^\infty \int_{\mathbb{R}^n} v^\ell \psi \, dx \, dt \leq R^{-2} \ell \int_0^\infty \int_{\mathbb{R}^n} u^{\frac{\ell}{p+1}} h(t/R, |x|/R) \, dx \, dt,
\]

\[
\leq R^{-2} \ell \int_0^\infty \int_{\mathbb{R}^n} u^{\frac{\ell}{p+1}} |h(t/R, |x|/R)| \, dx \, dt,
\]

\[
J_R := \int_0^\infty \int_{\mathbb{R}^n} u^{\ell} \psi \, dx \, dt \leq R^{-2} \ell \int_0^\infty \int_{\mathbb{R}^n} v^{\frac{\ell}{q+1}} h(t/R, |x|/R) \, dx \, dt,
\]

\[
\leq R^{-2} \ell \int_0^\infty \int_{\mathbb{R}^n} v^{\frac{\ell}{q+1}} |h(t/R, |x|/R)| \, dx \, dt
\]

$h(t, |x|) = (\phi''(t) + (\ell - 1)(\phi')^2(t)/\phi(t))\phi(|x|)$

$- \phi(t)\Delta \phi(|x|) - 2 \ell \phi'(t)(-\Delta)^{\frac{1}{2}} \psi(|x|)$

We remark that, by virtue of the nonnegativity of $I_R$ and $J_R$, it follows that the functions $u^{\frac{\ell}{p+1}} h(t/R, |x|/R)$ and $v^{\frac{\ell}{q+1}} h(t/R, |x|/R)$ have nonnegative integrals over $[0, \infty) \times \mathbb{R}^n$, so that we can apply the triangle inequality for integrals.

We notice that $\phi'(t, x) h(t/R, |x|/R)$ is a bounded function with compact support in $[0, R] \times B_R$, for any $\epsilon > 0$, since $h$ is bounded.

Setting $\epsilon_p = (\ell - 1)/\ell - \ell/p$ and $\epsilon_q = (\ell - 1)/\ell - \ell/q$, by Hölder’s inequality, we obtain:

\[
I_R \lesssim R^{-2} \frac{1}{R} \left( \int_0^R \int_{B_R} (\psi^\ell |h(t/R, |x|/R)|)^{\frac{1}{\ell'}} \, dx \, dt \right)^{\frac{1}{\ell'}} \lesssim R^{-2+\frac{n+1}{p'}} J_R^{\frac{1}{\ell'}} , \tag{36a}
\]

\[
J_R \lesssim R^{-2} \frac{1}{R} \left( \int_0^R \int_{B_R} (\psi^\ell |h(t/R, |x|/R)|)^{\frac{1}{\ell'}} \, dx \, dt \right)^{\frac{1}{\ell'}} \lesssim R^{-2+\frac{n+1}{q'}} I_R^{\frac{1}{\ell'}} . \tag{36b}
\]

Let $q \geq p$. Combining (36a) and (36b), we obtain

\[
J_R \lesssim R^{-2+\frac{n+1}{p'}} I_R^{\frac{1}{\ell'}} \lesssim R^{-2+\frac{n+1}{q'}} I_R^{\frac{1}{\ell'}} .
\]

The power of $R$ is negative if, and only if, condition (16) holds. Therefore, in this case, being $pq > 1$, by

\[
J_R^{\frac{1}{pq}} \lesssim R^{-2+\frac{n+1}{pq}} I_R^{\frac{1}{\ell'}} \lesssim R^{-2+\frac{n+1}{pq} + \frac{1}{\ell'}(-2+\frac{n+1}{\ell'})} J_R^{\frac{1}{pq}},
\]

it follows that $u \equiv 0$ by Beppo- Levi convergence theorem. Therefore, $v \equiv 0$ as well. If $p > q$, we proceed as before, but estimating $I_R$ instead of $J_R$. \hfill \square

4. Proof of Proposition 1. To prove Proposition 1 we follow the the proof of Theorem 1, but we now combine the linear estimates for the solution to (6), given for $n = 2$ by (see [2, 10, 11]):

\[
\|\partial_x \nabla^k v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{k}{2}} (\|v_0\|_{L^1 \cap H^{j+k}} + \|v_1\|_{L^1 \cap L^2}), \tag{37}
\]

\[
\|\partial_x \nabla^k u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{1-\frac{k}{2}} (\|v_0\|_{H^{j+k}} + \|v_1\|_{L^2}), \tag{38}
\]

with the linear estimates for the solution to (4), given for $n = 2$ by (see [14]):

\[
\|\partial_t \nabla^k u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{k}{2}} (\|u_0\|_{L^1 \cap H^{j+k}} + \|u_1\|_{L^1 \cap L^2}), \tag{39}
\]

\[
\|\partial_t \nabla^k u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{k}{2}} (\|u_0\|_{H^{j+k}} + \|u_1\|_{L^2}) . \tag{40}
\]

In the following, we denote by $E_0(t, x)$ and $E_1(t, x)$ the fundamental solutions to (4), that is, its solution may be written as

\[
u = E_0(t, x) * (x) u_0(x) + E_1(t, x) * (x) u_1(x).
\]
Proof of Proposition 1. We define the space
\[ X := \left( C([0, \infty), H^1) \cap C^1([0, \infty), L^2) \right)^2, \]
with norm:
\[ \|(u, v)\|_X := \sup_{t \geq 0} \left( (1 + t)^{-\gamma(p)} M(u) + M(v) \right), \tag{41} \]
where
\[ M(u) = (1 + t)^{\frac{2}{5}} \|u(t, \cdot)\|_{L^2} + (1 + t)\|\nabla u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{2}{5}} \|u_t(t, \cdot)\|_{L^2}, \tag{42} \]
\[ M(v) = \|v(t, \cdot)\|_{L^2} + (1 + t)\|(v_t, \nabla v)(t, \cdot)\|_{L^2}. \tag{43} \]
By Gagliardo-Nirenberg inequality, for any \((u, v) \in X\) it holds
\[ \|u(s, \cdot)\|_{L^q} \lesssim (1 + s)^{-(1 - \frac{j}{q}) + \gamma(p)} \|(u, v)\|_X, \tag{44} \]
\[ \|v(s, \cdot)\|_{L^q} \lesssim (1 + s)^{-2(\frac{j}{q} - \frac{j}{2})} \|(u, v)\|_X, \tag{45} \]
for any \(r \in [2, \infty)\).

The integral operator \(N : X \to X\), is now defined by \(N[u, v] = \tilde{A} + (\tilde{F}v, Gu)\), where \(G\) is as in the proof of Theorem 1, and
\[ \tilde{A}(t, x) = \left( \tilde{E}_0 *_{(x)} u_0 + \tilde{E}_1 *_{(x)} u_1, E_0 *_{(x)} v_0 + E_1 *_{(x)} v_1 \right) \]
\[ \tilde{F}v(t, x) = \int_0^t \tilde{E}_1(t - s, x) *_{(x)} |v(s, x)|^p ds. \]
The remaining part of the proof follows is similar to the proof of Theorem 1. By virtue of (39), (40) and (45), we derive
\[ \left\| \partial^j_t \nabla^k \tilde{F}v(t, \cdot) \right\|_{L^2} \lesssim \int_{t/2}^{t/2} (1 + t - s)^{-\frac{j}{2} - j - \frac{k}{2}} \|v(s, \cdot)\|_{L^p \cap L^{2p}}^p ds \]
\[ + \int_{t/2}^{t} (1 + t - s)^{-j - \frac{k}{2}} \|v(s, \cdot)\|_{L^p}^p ds \]
\[ \lesssim \|(u, v)\|_X^p (1 + t)^{-\frac{j}{2} - j - \frac{k}{2}} \int_{t/2}^{t/2} (1 + s)^{-2(\frac{j}{2} - \frac{j}{q})} ds, \]
\[ + \|(u, v)\|_X^p (1 + t)^{-2(\frac{j}{2} - \frac{j}{q})} \int_{t/2}^{t} (1 + t - s)^{-j - \frac{k}{2}} ds, \]
for \(j + k = 0, 1\). Again, the second term may be controlled by the first one, and we obtain the desired estimate
\[ \left\| \partial^j_t \nabla^k \tilde{F}v(t, \cdot) \right\|_{L^2} \lesssim (1 + t)^{-\frac{j}{2} - j - \frac{k}{2} + \gamma(p)} \|(u, v)\|_X^p. \]

On the other hand, by virtue of (37), (38), and (44), we get
\[ \left\| \partial^j_t \nabla^k Gu(t, \cdot) \right\|_{L^2} \lesssim \int_0^t (1 + t - s)^{-j - k} \|u(s, \cdot)\|_{L^q \cap L^{2q}}^q ds \]
\[ \lesssim \|(u, v)\|_X^q \int_0^t (1 + t - s)^{-j - k} (1 + s)^{-(1 - 1/q) + \gamma(p)q} ds. \]
The desired estimate, i.e.
\[ \left\| \partial^j_t \nabla^k Gu(t, \cdot) \right\|_{L^2} \lesssim (1 + t)^{-j - k} \|(u, v)\|_X^q, \]
follows if
\[ (-1 + 1/q + \gamma(p))q < -1, \]
which is equivalent to (18), due to the assumption \(q > 2\). This concludes the proof of Proposition 1. \(\square\)
5. Some open problems. It is possible to study the general case of two wave equations with structural damping $(-\Delta)^\delta u_t$, with $\delta \in [0, 1]$, namely,

$$
\begin{align*}
   u_{tt} - \Delta u + 2a(-\Delta)^\delta u = |v|^p, & \quad t \geq 0, x \in \mathbb{R}^n, \\
   v_{tt} - \Delta v + 2b(-\Delta)^\sigma v_t = |u|^q, & \quad t \geq 0, x \in \mathbb{R}^n, \\
   (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & \quad x \in \mathbb{R}^n,
\end{align*}
$$

(46)

where $\delta, \sigma \in [0, 1]$ and $a, b > 0$. However, some difficulties arise. On the one hand, linear $L^1 - L^p$ estimates for

$$
   u_{tt} - \Delta u + 2a(-\Delta)^\delta u = f(t, x),
$$

(47)

appear to be not good enough for any exponent $p > 1$ in any space dimension $n \geq 1$, when $\delta \neq 1/2$. On the other hand, the use of weighted energy estimates introduced in [12, 18] for the wave equation with classical damping cannot be directly extended if $\delta \neq 0$.

In [8, 9], $L^1 - L^p$ optimal estimates for (47) with $\delta \in (0, 1/2)$, have been derived for $n < \bar{n}(p) \in [4, \infty)$, where $\bar{n}(p) \to \infty$ for any $p \geq 1$, as $\delta \to 1/2$. The application of these estimates may lead to a global existence result for (46), possibly in low dimension and/or with additional restrictions of the range for $p, q$.

Another generalization consists into replace the power nonlinearities $|v|^p$ and $|u|^q$ in (46) with other nonlinearities, for instance, with powers of derivatives of $u$ and $v$, as in [4]. Also, one may replace one or both power nonlinearities with nonlinear memory terms obtained by their Riemann-Liouville fractional integral, i.e.

$$
   \int_0^t (t-s)^{-\alpha} |u(s, x)|^p \, ds,
$$

with $\alpha \in (0, 1)$. For damped wave equations with a nonlinear memory term it has been shown [5, 6] that this latter brings, in general, a loss of decay in the estimates with respect to the linear problem. This effect would then be taken into account together with the loss of decay described by (13), (14) in Theorem 1.

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