On the basis of Braaten and Segel’s representation of the electro-

magnetic dispersion relations in a QED plasma we check the numerical

accuracy of several published analytic approximations to the plasma

neutrino emission rates. As we find none of them satisfactory we de-

rive a new analytic approximation which is accurate to within 4%

where the plasma process dominates. The correct emission rates in

the parameter regime relevant for the red giant branch in globular

clusters are larger by about $10 - 20\%$ than those of previous stellar

evolution calculations. Therefore, the core mass of red giants at the

He flash is larger by about $0.005M_\odot$ or 1% than previously thought.

Our bounds on neutrino magnetic dipole moments remain virtually

unchanged.

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1 Introduction

In stars with densities below nuclear, neutrinos are emitted by the plasma process $\gamma \rightarrow \nu \nu$, the photo process $\gamma e^- \rightarrow e^- \nu \nu$, the pair process $e^- e^+ \rightarrow \nu \nu$, and by bremsstrahlung $e^- (Z e) \rightarrow (Z e)e^- \nu \nu$. In Fig. 1 we show the regions of density and temperature where each of these processes dominates over the others. The plasma process is particularly interesting because it does not occur in vacuum and yet dominates the stellar emission rates in a large range of temperatures and densities. Notably white dwarfs and the degenerate cores of low-mass red giants fall in this parameter region. Evolutionary calculations for these stars can be tested with great statistical significance. For example, globular cluster observations allow one to determine the core mass at the helium flash to within about $0.012 \, M_\odot$ or 2.5% at a 1σ statistical confidence level (Raffelt 1990, Raffelt and Weiss 1992). Therefore, it is important to identify possible systematic effects that would enter at this or at a larger level. If the standard neutrino loss rates are multiplied with a factor $F_\nu$ where $F_\nu = 1$ refers to the standard case, the core mass at the helium flash varies approximately as $\delta M_c = 0.020 \, M_\odot \delta F_\nu$. This strong sensitivity to the neutrino luminosity has been used to constrain possible non-standard contributions such as the plasmon decay by virtue of neutrino dipole moments (Raffelt 1990, Raffelt and Weiss 1992).

In practical stellar evolution calculations the neutrino loss rates are implemented in the form of simple analytic approximations. Widely used versions are those of Beaudet, Petrosian, and Salpeter (1967), Munakata, Kohyama, and Itoh (1985), Schinder et al. (1987), and Itoh et al. (1989). With regard to the plasma process, the results of Beaudet et al., Munakata et al., and Schinder et al. agree with each other to better than 1% if the same neutrino coupling strength to electrons is used, while Itoh et al. made an attempt to improve the accuracy of the rates at low temperatures.

It turns out that all of these rates are relatively poor approximations for $T \lesssim 10^8 \, K$ which is relevant for low-mass stars. They were optimized for higher temperatures and correspondingly higher densities in the diagonal band of Fig. 1 where the plasma process dominates. At higher densities, however, the approximate photon dispersion relation that had been used in all of these works is a rather poor approximation (Braaten 1991) with the result that the above plasma emission rates are bad approximations everywhere. In response, Itoh et al. (1992) have derived a new analytic formula which is claimed to fit the exact results to better than 5% in the region where the plasma process dominates. Independently, Blinnikov and Dunina-Barkovskaya (1992) have published rates which were optimized for low-mass stars.
Since then Braaten and Segel (1993) have devised an approach to the photon dispersion relation which, for given $T$ and $\rho$, reduces the calculation of the plasma emission rate essentially to the numerical evaluation of a few one-dimensional integrals. Their method for the first time offers a simple and practical approach to check the accuracy of all of the above emission rates as well as the approximations that were used to study non-standard neutrino emission (Raffelt 1990, Raffelt and Weiss 1992, Blinnikov and Dunina-Barkovskaya 1992, Castellani and Degl’Innocenti 1992). On the basis of this method we found none of the above approximations satisfactory and thus have derived a new analytic approximation to the standard rate for the plasma process which is accurate to better than 4%. For the non-standard emission rates we found that a simple scaling of the standard rates as in Raffelt and Weiss (1992) introduces only a small error (less than 5%) for low-mass stars so that bounds on neutrino dipole moments are mostly affected by the accuracy of the standard emission rates.

We begin in Sect. 2 by adapting the results of Braaten and Segel (1993) to our calculation of standard and non-standard emission rates by the plasma process. Besides the standard neutrino interactions we include the possibility of the coupling of right-handed neutrinos to electrons (Fukugita and Yanagida 1990), neutrino dipole moments, and neutrino electric “millicharges”. In Sect. 3 we perform a numerical test of all of the above analytic approximations, and we derive and test a new version. We also check the accuracy of a simple scaling of the standard rates to obtain the non-standard ones. In Sect. 4 we discuss the accuracy of previous calculations of the core-mass at the helium flash as well as the accuracy of previous bounds on neutrino properties, and we derive bounds on the Fukugita and Yanagida model as well as on neutrino millicharges.

2 Exact Emission Rates

2.1 Electromagnetic Excitations in a Medium

The behavior of electromagnetic excitations in a linear medium is best understood on the basis of the wave equation for the vector potential $A$ in Feynman gauge, $[K^2 - \Pi] A = j_{\text{ext}}$, where $j_{\text{ext}}$ is an external electromagnetic current and $\Pi$ is the polarization tensor. Moreover, $K^2 = \omega^2 - k^2$ where $\omega$ is the frequency while $k = |k|$ the modulus of the momentum coordinate. In an isotropic medium which also remains invariant under a parity transformation, the polarization tensor can be expressed as (Nieves and Pal 1989a,b) $\Pi = \pi_T Q_T + \pi_L Q_L$ where $Q_T$ is a $K$-dependent projector on the subspace.
transverse to $k$ while $Q_L$ projects on the longitudinal subspace, and in addition $Q_{T,L}K = 0$ so that $\Pi K = 0$ as required by gauge invariance. Explicit expressions for $Q_{T,L}$ in terms of $K$ were given, for example, by Weldon (1982). He also showed that in the rest-frame of the medium $\pi_L = -\Pi_{00} K^2/k^2$ and $\pi_T = (\Pi^\mu_{\mu} - \pi_L)/2$.

If $\Pi$ is expressed in this way, it is clear that the homogeneous wave equation $(K^2 - \Pi)A = 0$ has non-trivial solutions only for

$$\omega^2 - k^2 - \pi_s(\omega, k) = 0, \quad s = T \text{ or } L,$$

a relationship which implicitly defines the dispersion relations for transverse and longitudinal propagating modes ("photons" and "plasmons" or "transverse and longitudinal plasmons").

In order to calculate the decay rate into neutrinos of such modes we need to normalize properly the amplitude of the quantized excitations. To this end we consider a mode $k$, polarization $s = T$ or $L$, with the corresponding frequency $\omega_{s,k}$ so that the dispersion relation Eq. (1) is obeyed. Then we expand $\pi_s(\omega, k) = \pi_s(\omega_{s,k}, k) + \partial_\omega^2 \pi_s(\omega_{s,k}, k)(\omega^2 - \omega_{s,k}^2)$ so that the equation of motion for the amplitude $A_{s,k}$ is that of a harmonic oscillator. Equivalently, we write the photon propagator near its poles in the form $Z_s/(\omega^2 - \omega_{s,k}^2)$. Either way, one finds that the squared matrix element of a process with an external electromagnetic excitation of momentum $k$ and polarization $s$ carries a factor

$$Z_s = [1 - \partial_\omega^2 \pi_s(\omega_{s,k}, k)]^{-1}.$$  

In a fully ionized plasma the polarization tensor is simply given in terms of the forward scattering amplitudes of photons on the electrons and positrons. To lowest order in $\alpha = e^2/4\pi$ it is found to be

$$\Pi_{\mu\nu} = -8\pi\alpha \int \frac{d^3p}{(2\pi)^3} \frac{f(E_p)}{E_p} \frac{P\cdot K (K_\mu P_\nu + K_\nu P_\mu) - K^2 P_\mu P_\nu - (P\cdot K)^2 g_{\mu\nu}}{(P\cdot K)^2 - K^4/4}$$

where $P = (E_p, p)$ is an $e^-$ or $e^+$ four-momentum. The sum of the phase-space occupation numbers for these particles is

$$f(E) = \frac{1}{e^{(E-\mu)/T} + 1} + \frac{1}{e^{(E+\mu)/T} + 1}$$

with $\mu$ the electron chemical potential and $T$ the temperature. On the basis of Eq. (3) it is, in principle, straightforward to derive $\pi_s$ and $Z_s$ for a given $\mu$ and $T$. 

4
2.2 Analytic Representation for $\Pi$ and $Z$

Braaten and Segel (1993) have shown that ignoring the $K^4/4$ term in the denominator of Eq. (3) introduces an error which appears only on the $\alpha^2$ level so that to $O(\alpha)$ it can be ignored. In fact, ignoring this term provides a better approximation to the $O(\alpha)$ result than keeping it because $\pi_T$ then remains real for all conditions as it must because the imaginary part from the $\gamma T \rightarrow e^+e^-$ process which otherwise appears at sufficiently large plasma frequencies is unphysical (Braaten 1991).

Once the $K^4/4$ term has been dropped, the angular part of the integral in Eq. (3) can be done analytically, leaving one with a one-dimensional integral over electron momenta which can be done analytically in the classical, degenerate, and relativistic limit. In these cases Braaten and Segel found

\[
\pi_T(\omega, k) = \omega_P^2 \left[ 1 + \frac{1}{2} G(v_\ast^2 k^2/\omega^2) \right],
\]
\[
\pi_L(\omega, k) = \omega_P^2 \left[ 1 - G(v_\ast^2 k^2/\omega^2) \right] + v_\ast^2 k^2 - k^2,
\]
(5)

where $v_\ast = \omega_1/\omega_P$ has the interpretation of a typical velocity of the electrons in the medium. The plasma frequency $\omega_P$ and the frequency $\omega_1$ are defined by

\[
\omega_P^2 \equiv \frac{4\alpha}{\pi} \int_0^\infty dp \left( v - \frac{1}{3} v^3 \right) p f(E_p),
\]
\[
\omega_1^2 \equiv \frac{4\alpha}{\pi} \int_0^\infty dp \left( \frac{5}{3} v^3 - v^5 \right) p f(E_p),
\]
(6)

where $v = p/E_p$ is the electron or positron velocity. In the classical limit one has $v_\ast = (5T/m_e)^{1/2}$, in the degenerate limit $v_\ast = v_F$ (velocity at the Fermi surface), and in the relativistic limit $v_\ast = 1$. The function $G$ is defined by

\[
G(x) \equiv \frac{3}{x} \left[ 1 - \frac{2x}{3} - \frac{1 - x}{2\sqrt{x}} \log \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right] = 6 \sum_{n=1}^\infty \frac{x^n}{(2n + 1)(2n + 3)}.
\]
(7)

We have plotted $G(x)$ in Fig. 2; note that $G(0) = 0$, $G(1) = 1$, and $G'(1) = \infty$.

Braaten and Segel then claim (and we have checked) that these results apply approximately for all conditions. The deviations between $\pi_s(\omega, k)$ given by Eq. (3) and by the proper integral over the $e^+e^-$ phase space are always

\textsuperscript{1}Braaten and Segel used Coulomb rather than Feynman gauge so that we had to translate their longitudinal expression accordingly. We preferred to follow Weldon (1982) in the choice of gauge because the dispersion relations and plasmon decay rates then have the same form for transverse and longitudinal excitations.
so small that Eq. (5) can be considered exact to $O(\alpha)$. As a higher precision would require an $O(\alpha^2)$ QED calculation, nothing is gained by evaluating the full integrals. Therefore, the above results are as exact as an $O(\alpha)$ result can be.

In order to calculate the plasmon decay rates we also need the amplitude normalization factors. Braaten and Segel (1993) found the analytic representations

\begin{align*}
Z_{T,k} &= \frac{2\omega_T^2(\omega_T^2 - v_s^2 k^2)}{\omega_T^2(3\omega_P^2 - 2\pi_T,k) + (\omega_T^2 + k^2)(\omega_T^2 - v_s^2 k^2)}, \\
Z_{L,k} &= \frac{2\omega_L^2(\omega_L^2 - v_s^2 k^2)}{[3\omega_P^2 - (\omega_L^2 - v_s^2 k^2)]\pi_L,k},
\end{align*}

(8)

In these expressions $\pi_{s,k} \equiv \pi_s(\omega_{s,k},k)$ is the “on shell” value of $\pi_s$ for excitations with momentum $k$, i.e. $\omega_{s,k}^2 - k^2 = \pi_{s,k}$.

The decay $\gamma_s \to \pi\nu$ is only possible if $\gamma_s$ has a time-like four-momentum, $\omega_{s,k} > k$. For longitudinal excitations this condition is only fulfilled for $k < k_{\text{max}}$ where

$$
k_{\text{max}} = \omega_P \left[ \frac{3}{v_s^2} \left( \frac{1}{2v_s} \log \frac{1 + v_s}{1 - v_s} - 1 \right) \right]^{1/2}
$$

(9)
to the same level of approximation.

In the Braaten and Segel representation the on-shell values $\pi_{s,k}$ and thus the dispersion relations depend only on the medium parameters $\omega_P$ and $v_s$. However, media are more readily characterized by their density and temperature. Therefore, in Fig. 3 we show contours of constant $v_s$ and $\gamma \equiv \omega_P/T$ in the region of $\rho$ and $T$ relevant for the plasma process in stars. In the shaded area the plasma process contributes more than 10% to the total neutrino luminosity. Evidently, it is important only for $0.3 \lesssim \gamma \lesssim 30$. Moreover, the $v_s$ contours are almost perfect vertical straight lines in this regime, i.e. the medium is degenerate.

The polarization functions $\pi_{s,k}/\omega_P^2$ and the amplitude factors $Z_{s,k}$ are functions only of $v_s$ and of $k/\omega_P$. In Fig. 4 we show contours for these functions in the $v_s$-$k$-plane for transverse excitations. We have $1 < \pi_{T,k}/\omega_P^2 < 3/2$ and $Z_{T,k} < 1$. The deviation of $Z_{T,k}$ from unity is always small. In Fig. 5 we show similar contours for the longitudinal case. We have $\pi_{L,k}/\omega_P^2 < 1$; the occurrence of the plasma process in addition requires $0 < \pi_{L,k}$. The contour $\pi_{L,k} = 0$ is identical with the function $k_{\text{max}}(v_s)$ given in Eq. (9). The amplitude function $Z_{L,k}$ diverges when $k \to k_{\text{max}}$. Therefore, we show in Fig. 5 (lower panel) contours for $Z_{L,k}^* \equiv Z_{L,k} \pi_{L,k}/\omega_{L,k}^2$ instead.\footnote{Our $Z_{L,k}^*$ is what Braaten and Segel (1993) call $Z_L(k)$.}
2.3 Plasmon Decay Rates

In order to calculate the neutrino emission rates for the standard and several non-standard interaction models it remains to calculate the decay rates for the process $\gamma_s \rightarrow \nu \nu$. In the standard model one finds (Adams, Ruderman and Woo 1963, Zaidi 1965, Dicus 1972)

$$\Gamma_{s,k} = C_{V}^2 \frac{G_F^2}{48\pi^2\alpha} \frac{Z_{s,k} \pi_{s,k}^3}{\omega_{s,k}}. \quad (10)$$

The effective vector coupling constant is

$$C_V^2 \equiv \sum_{i=1}^{3} C_{V,i}^2 = \left( \frac{1}{2} + 2\sin^2\Theta_W \right)^2 + 2\left( \frac{1}{2} - 2\sin^2\Theta_W \right)^2, \quad (11)$$

where $C_{V,i}$ is the effective neutral-current vector coupling constant of neutrino flavor $i$ to the electrons. With a weak mixing angle of $\sin^2\Theta = 0.2325 \pm 0.0008$ this is $C_V^2 = (0.9312 \pm 0.0031) + 2(0.0012 \pm 0.0001) = 0.9325 \pm 0.0033$ where the first term is from $\gamma_s \rightarrow \bar{\nu}_e\nu_e$ while the second term is from $\nu_\mu\nu_\mu$ and $\bar{\nu}_\tau\nu_\tau$. Thus, even if these latter flavors were too heavy to be produced by plasmon decay the value of $C_V^2$ would change by less than its uncertainty. The contribution of the axial neutral current is always negligible.

As a first non-standard coupling we consider a model by Fukugita and Yanagida (1990) which was devised to obtain a large neutrino magnetic dipole moment and a strong effective coupling of right-handed $\nu_e$’s to electrons. The relevant part of the Lagrangian is

$$L_{\nu R e L} = g \phi \bar{\nu}_{\nu R} \psi_{e L} + h.c. \quad (12)$$

where $\phi$ is a scalar field of mass $M$ and $g$ is a dimensionless coupling constant. For low energies this interaction produces an effective neutral-current coupling of r.h. neutrinos to electrons. The corresponding effective vector coupling constant is

$$C_{V,R}^2 = \frac{g^4}{32M^4G_F^2}. \quad (13)$$

In this model the rate for $\gamma_s \rightarrow \bar{\nu}_e R\nu_{e,R}$ is found by inserting $C_{V,R}^2$ instead of $C_V^2$ into Eq. (10).

Next, we consider direct couplings of neutrinos with the electromagnetic field. The least exotic possibility is that of neutrino dipole moments with an effective Lagrangian

$$L_{\mu} = \frac{1}{2} \sum_{i,j=1}^{3} \left( \mu_{ij} \bar{\psi}_i \sigma_{\mu\nu} \psi_j + \epsilon_{ij} \bar{\psi}_i \sigma_{\mu\nu} \gamma_5 \psi_j \right) F_{\mu\nu}, \quad (14)$$
where \( \mu_{ij} \) and \( \epsilon_{ij} \) are matrices of magnetic and electric dipole and transition moments, \( F \) is the electromagnetic field tensor, and the sum is over neutrino flavors. We define an effective dipole moment by

\[
\mu^2_\nu \equiv \sum_{i,j=1}^{3} \left( |\mu_{ij}|^2 + |\epsilon_{ij}|^2 \right)
\]

with the restriction that the sum should only run over those flavors which are light enough to be produced by the plasma process: \( m_i + m_j \lesssim \omega_P \). Then we find for the plasmon decay rate

\[
\Gamma_{s,k} = \frac{\mu^2_\nu}{24\pi} \frac{Z_{s,k} \pi_{s,k}^2}{\omega_{s,k}}.
\]

This result is in agreement with Sutherland et al. (1976) except for their \( Z_{L,k} \).

Finally, we assume that neutrino flavor \( i \) carries a “millicharge” \( q_i e \). In this case we find

\[
\Gamma_{s,k} = \frac{q^2_\nu \alpha}{3} \frac{Z_{s,k} \pi_{s,k}}{\omega_{s,k}},
\]

where \( \alpha = e^2/4\pi \) is the (electron) fine structure constant and \( q^2_\nu \equiv \sum_{i=1}^{3} q^2_i \).

### 2.4 Neutrino Emission Rates

If transverse and longitudinal electromagnetic excitations of momentum \( k \) can decay according to \( \gamma_s \rightarrow \bar{\nu} \nu \) with a rate \( \Gamma_{s,k} \) then the energy-loss rate per unit volume of a medium at temperature \( T \) is \( Q = Q_T + Q_L \) where

\[
Q_T = 2 \int_0^\infty \frac{dk}{2\pi^2} \frac{k^2}{2\pi^2} \frac{\Gamma_{T,k} \omega_{T,k}}{e^{\omega_{T,k}/T} - 1},
\]

\[
Q_L = \int_{k_{\text{max}}} k^2 \frac{dk}{2\pi^2} \frac{\Gamma_{L,k} \omega_{L,k}}{e^{\omega_{L,k}/T} - 1}.
\]

Inserting the results of the previous section into this equation we find for the various interaction models

\[
Q_V = C^2_V \frac{G_F^2}{96\pi^4 \alpha} T^3 \omega_P^6 Q_3,
\]

\[
Q_\mu = \mu^2_\nu \frac{48\pi^3}{4} T^3 \omega_P^4 Q_2,
\]

\[
Q_q = \frac{q^2_\nu \alpha}{6\pi^2} T^3 \omega_P^2 Q_1.
\]
The dimensionless emission rates are \( Q_n \equiv (Q^T_n + Q^L_n) \) where
\[
Q^T_n \equiv 2 \int_0^\infty \frac{dk}{T^3} k^2 Z_{T,k} \left( \frac{\pi_{T,k}}{\omega_P^2} \right)^n \frac{1}{e^{\omega_{T,k}/T} - 1},
\]
\[
Q^L_n \equiv \int_{k_{\text{max}}}^{k_{\text{max}}} \frac{dk}{T^3} k^2 Z_{L,k} \left( \frac{\pi_{L,k}}{\omega_P^2} \right)^n \frac{1}{e^{\omega_{L,k}/T} - 1}.
\] (20)

In the Braaten and Segel approximation the \( Q^a_n \) are only functions of \( v^* \) and \( \gamma = \omega_P/T \).

In Fig. 6 we show contours in the \( v^*-\gamma \)-plane of \( Q^L_3/Q^T_3 \), i.e. the ratio between longitudinal and transverse emissivity for the standard-model interactions. Corresponding contours in the \( \rho-T \)-plane are shown as dashed lines in Fig. 1. Evidently, the longitudinal process is of importance only in a relatively narrow region near \( \gamma = 10 \).

In a practical calculation of anomalous neutrino losses one will scale the standard luminosity appropriately. The relevant ratios are
\[
\frac{Q_\mu}{Q_V} = \frac{\mu_\nu^2 2\pi\alpha}{C^2 V G^2 F^4 \omega_P^4} \frac{Q_2}{Q_3} = 0.318 \mu_{12}^2 \left( \frac{10 \text{ keV}}{\omega_P} \right)^2 \frac{Q_2}{Q_3},
\]
(21)
where \( \mu_{12} \equiv \mu_\nu/10^{-12}(e/2m_e) \) and
\[
\frac{Q_q}{Q_V} = \frac{q_{\nu}^2 (4\pi\alpha)^2}{C^2 V G^2 F^4 \omega_P^4} \frac{Q_1}{Q_3} = 0.664 q_{14}^2 \left( \frac{10 \text{ keV}}{\omega_P} \right)^4 \frac{Q_1}{Q_3},
\]
(22)
where \( q_{14} \equiv q_\nu/10^{-14} \). In Fig. 7 we show contours of \( Q_1/Q_3 \) and \( Q_2/Q_3 \).

### 3 Numerical Emission Rates

Even though the methods described in Sect. 2 allow for a relatively simple calculation of the plasma neutrino emission rates, one still needs to evaluate several numerical integrals in order to determine the energy loss rate for given values of density and temperature so that this procedure is not suitable to be coupled directly with a stellar evolution code. However, we can use these results to test the accuracy of widely used analytic approximation formulae.

It turns out that for the plasma process the analytic approximations of Beaudet, Petrosian, and Salpeter (1967), Munakata, Kohyama, and Itoh (1985), and Schinder et al. (1987) all agree with each other to an astonishing accuracy if the same value for \( C^2 V \) is used. In Fig. 8a we compare the numerical rates of Beaudet et al. (1967) with the exact results obtained by the Braaten and Segel method. We show contours for the relative deviation...
in percent, \( Q_{\text{analytic}}^{\text{tot}} / Q_{\text{exact}}^{\text{tot}} - 1 \). We stress that what is plotted is the error of the total emission rate; the plasma process alone may be more inaccurate than shown in regions where it does not dominate. In this and the following figures we have used the rates of Itoh et al. (1989) for the other emission processes. Thus, \( Q_{\text{analytic}}^{\text{tot}} \) and \( Q_{\text{exact}}^{\text{tot}} \) differ only in the treatment of the plasma process. From Fig. 8a it is evident that the numerical rates are rather poor approximations almost everywhere. Nevertheless, it is these rates that have been used in virtually all stellar evolution calculations.

Recently Itoh et al. (1992) have published numerical rates for the plasma process which are claimed to be more accurate than 5% wherever the plasma process dominates. In Fig. 8b we put this claim to a test and find that there remain substantial regions with much larger errors. We have checked that at least some of these deviations also occur between their tabulated emission rates and their fitting formula.

Blinnikov and Dunina-Barkovskaya (1992) have also derived a new analytic approximation which is optimized in the region of small temperatures and densities relevant for low-mass stars. We compare their rates with the exact ones in Fig. 8c. Indeed, these rates are rather good fits for \( T \lesssim 10^8 \) and \( \rho \lesssim 10^6 \), but for higher \( T \) or \( \rho \) they are unrelated to the exact results. These approximation formulae involve multiplicative factors which depend on \( T \) alone and which approach 1 for \( T \to 0 \). If we set these factors to 1 for all \( T \) the errors of the resulting simplified emission rates are shown in Fig. 8d. The fit is not worse in the low-\( T \) and low-\( \rho \) region, and much better otherwise!

As we find none of the published fitting formulae satisfactory we have derived a new one. To this end we have started with the \( T = 0 \) version of Blinnikov and Dunina-Barkovskaya (1992) and then “flattened” the errors with an extra correction factor \( f_{xy} \). As a result we have come up with \( Q_{\text{plas}} = C V Q_{\text{approx}} \) (in erg cm\(^{-3}\) s\(^{-1}\)) where

\[
Q_{\text{approx}} = 3.00 \times 10^{21} \lambda^9 \gamma^6 e^{-\gamma} (f_T + f_L) f_{xy},
\]

(23)

where \( \lambda = T/m_e \) and \( \gamma = \omega_0/T \) where \( \omega_0 \) is the zero-temperature plasma frequency, \( \omega_0^2 = 4\pi\alpha N_e/E_F \). Numerically,

\[
\lambda = 1.686 \times 10^{-10} T
\]

\[
\gamma^2 = \frac{1.1095 \times 10^{11} \rho/\mu_e}{T^2 [1 + (1.019 \times 10^{-6} \rho/\mu_e)^{2/3} ]^{1/2}},
\]

(24)

with \( T \) in K, \( \rho \) in g/cm\(^3\), and \( \mu_e \) the number of baryons per electron. Moreover, we have

\[
f_T = 2.4 + 0.6 \gamma^{1/2} + 0.51 \gamma + 1.25 \gamma^{3/2},
\]
\[ f_L = \frac{8.6 \gamma^2 + 1.35 \gamma^{7/2}}{225 - 17 \gamma + \gamma^2}. \]  

(25)

The coefficients here are slightly different from those used by Blinnikov and Dunina-Barkovskaya (1992). Finally, we define

\[
\begin{align*}
\frac{x}{\gamma} &= \frac{1}{6} \left[ +17.5 + \log_{10}(2\rho/\mu_e) - 3\log_{10}(T) \right], \\
\frac{y}{\gamma} &= \frac{1}{6} \left[ -24.5 + \log_{10}(2\rho/\mu_e) + 3\log_{10}(T) \right],
\end{align*}
\]

(26)

where \(x\) is a coordinate transverse to the diagonal band in Fig. 1 where the plasma process is important, and \(y\) is along this band. If \(|x| > 0.7\) or \(y < 0\) we use \(f_{xy} = 1\) and otherwise

\[
f_{xy} = 1.05 + \left[ 0.39 - 1.25 x - 0.35 \sin(4.5 x) - 0.3 \exp\left\{-(4.5 x + 0.9)^2\right\} \right] \times \exp\left\{ - \left[ \frac{\min(0, y - 1.6 + 1.25 x)}{0.57 - 0.25 x} \right]^2 \right\}.
\]

(27)

We show the errors of our fitting formula in Fig. 8e; it is better than 5\% everywhere and better than 4\% almost everywhere.

4 Discussion and Summary

In low-mass stars helium ignites in the cores of red giant stars under degenerate conditions. Helium burning depends extremely sensitively on temperature and density so that even relatively minor changes in the neutrino cooling rates of the core change the ignition point and thus the core mass \(M_{\text{tip}}\) at the tip of the red giant branch, which in turn affects the luminosity during the subsequent horizontal branch evolution. All previous calculations of the core mass at helium flash seem to have used the Beaudet et al. (1967), the Munakata et al. (1985), or the Schinder et al. (1987) rates, all of which are equivalent with regard to the plasma process and thus underestimate the emission rate as shown in Fig. 8a.

In order to illustrate the relevant range of parameters we show in Fig. 9 the evolution of the central density and temperature of a 0.80 \(M_\odot\) star where the tail ends of the arrows mark the conditions when the surface brightness is at the indicated magnitudes. In Fig. 10 we show the red giant part of this track overlaid with the errors of the Munakata et al. (1985) rates that were used in our previous calculations (Raffelt and Weiss 1992). The error contours are virtually the same as those shown in Fig. 8a. Near the helium flash the average neutrino emission rate was underestimated by around 15\%.
Raffelt and Weiss (1992) found that the core mass at the helium flash varies approximately as $\delta M_c = 0.020 \delta F_\nu$ if the standard neutrino loss rates are multiplied with a factor $F_\nu$, a result which agrees with the previous calculations of Sweigart and Gross (1978). Therefore, the core mass at the helium flash will be increased by about $0.004 M_\odot$. In order to confirm this estimate we have re-calculated run 11 of Raffelt and Weiss (1992) with the analytic emission rates derived in this paper. For $M = 0.80 M_\odot$, $Z = 10^{-4}$, and $Y_0 = 0.22$ we previously found $M_{\text{tip}} = 0.497 M_\odot$ for the core mass at helium ignition (at the tip of the red giant branch) while we now find $M_{\text{tip}} = 0.503 M_\odot$. Therefore, we find an increase of $\delta M_{\text{tip}} = 0.006 M_\odot$, in reasonable agreement with our simple estimate. Thus, using the correct neutrino emission rates changes the core mass at the helium flash by a small but noticeable amount.

In order to constrain neutrino dipole moments Raffelt and Weiss (1992) as well as Blinnikov and Dunina-Barkovskaya (1992) and Castellani and Degl’Innocenti (1992) have included the increased plasma losses in evolutionary calculations by scaling the standard rates with a certain factor. Instead of the exact ratio given by Eq. (21) they used $Q_2/Q_3 = 1$ and $\omega_0^2 = 4 \pi \alpha N_e/E_F$ which is the zero-temperature value for the plasma frequency. In Fig. 11 we show the evolutionary track of Fig. 9 in the plane of $\gamma = \omega_P/T$ and $v_*$. Evidently on the RGB the interior of the core has an almost constant value $\gamma \approx 3$. A comparison with the upper panel of Fig. 7 reveals that by using $Q_2/Q_3 = 1$ Raffelt and Weiss (1992) have underestimated the dipole-induced emission rate by about 5%. An additional small error occurs by using $\omega_0$ instead of $\omega_P$. In Fig. 12 (upper panel) we show the error of the dipole-induced emission rate if it had been scaled with the correct standard rate. These errors are so small that this scaling procedure remains well justified.

In Fig. 12 (lower panel) we show the compound error of the dipole-induced emission rate when coupled with the Munakata et al. (1985) emission rates. Near helium ignition the emission rate was underestimated by $15 - 20\%$. Because the core-mass and brightness at the helium flash vary approximately linearly with $\mu_\nu$ in the range of interest, for a given value of $\mu_\nu$ these quantities are changed by about 10% more than given in Raffelt and Weiss (1992). This is a negligible change with regard to bounds on $\mu_\nu$.

The bounds on neutrino dipole moments discussed in Raffelt (1990) and Raffelt and Weiss (1992) crudely amount to the constraint that the total neutrino luminosity must not exceed about twice its standard value in order to maintain the beautiful agreement between the observed and calculated properties of globular cluster stars. We have already discussed that on the RGB we may set $Q_2/Q_3 = 1$ in Eq. (21) without introducing a large error,
and similarly we may set \( Q_1/Q_3 = 1 \) in Eq. (22), although the error here is somewhat larger. In any case, these approximations lead to an underestimation of the non-standard emission rates and thus to conservative bounds. Moreover, near the helium flash the density is about \( 10^6 \text{ g/cm}^3 \) (see Fig. 10) so that \( \omega_p^2 = 17.8 \text{ keV} \) in the center of the star. Therefore, in the center of the star we have for the total energy-loss rate from Eqs. (21) and (22)

\[
Q_{\text{tot}}/Q_V = 1 + 1.07 C_{V,R}^2 + 0.100 \mu_{12}^2 + 0.066 q_{14}^2 .
\] (28)

Away from the center the density and thus \( \omega_p \) is lower so that the coefficients of \( \mu_{12}^2 \) and of \( q_{14}^2 \) would be larger if averaged properly over the core. Hence this expression, again, is a conservative estimate of the non-standard neutrino losses. With \( Q_{\text{tot}}/Q_V < 2 \) as a formal criterion we find the bounds

\[
C_{V,R} < 0.9 ,
\mu_\nu < 3\times10^{-12} (e/2m_e) ,
q_\nu < 4\times10^{-14} .
\] (29)

In the model of Fukugita and Yanagida (1990) one has both a dipole moment and r.h. interactions for the \( \nu_e \)'s so that in that model the constraints on the dipole moment are more restrictive. The bound on \( \mu_\nu \) alone is virtually the same as that from the more detailed analysis of Raffelt (1990) and Raffelt and Weiss (1992). The bound on \( q_\nu \) is slightly more restrictive than that found by Bernstein, Ruderman, and Feinberg (1963).

In summary, we have discussed in detail the neutrino emission rates from the plasma process due to standard and non-standard interactions. By means of Braaten and Segel's (1993) representation of the QED dispersion relations we have tested the accuracy of widely used analytic approximation formulae, none of which are found satisfactory. For the first time we have derived an approximation formula which is accurate on the 4% level wherever the plasma process dominates. The correct emission rate leads to a slightly increased core mass at the helium flash in low-mass stars, and to a slightly increased sensitivity to non-standard neutrino losses. While these changes are noticable they are so small that previous bounds on neutrino dipole moments remain virtually unchanged.
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Figure Captions

Figure 1
Regions of density and temperature where the different neutrino emission processes contribute more than 90% of the total. $\mu_e$ is the mean number of baryons per electron. The bremsstrahlung contribution is for helium. The dashed lines are contours for the indicated values of $Q_L/Q_T$, i.e. the contribution of the longitudinal relative to the transverse plasma process.

Figure 2
Function $G(x)$ as defined in Eq. (7).

Figure 3
Contours of $\gamma = \omega_P/T$ and $v_*$ as defined in Eq. (6).

Figure 4
Contours for $\pi_{T,k} = \omega_{T,k}^2 - k^2$ in units of $\omega_P^2$ and for $Z_{T,k}$.

Figure 5
Contours for $\pi_{L,k} = \omega_{L,k}^2 - k^2$ in units of $\omega_P^2$ and for $Z_{L,k}^* = Z_{L,k} \pi_{L,k} / \omega_{L,k}^2$. The $\pi_{L,k} = 0$ contour corresponds to $k_{\text{max}}(v_*)$ of Eq. (9). In the cross-hatched region longitudinal plasmons have a space-like four momentum and thus cannot decay.

Figure 6
Contours for $Q_{L}^3/Q_{T}^3$ as defined in Eq. (19). This ratio is identical to the ratio of the longitudinal and transverse emission rates in the standard model.

Figure 7
Contours for $Q_2/Q_3$ and $Q_1/Q_3$ as defined in Eq. (20).
Figure 8
(a)–(e) Comparison of the analytic plasma emission rates of the indicated authors with the exact rates. Except for (e) the contours are at multiples of ±10%. The errors in (a) of the Beaudet et al. (1967) rates are the same for Munakata et al. (1985) and Schinder et al. (1987).

Figure 9
Evolution of the central density and temperature of a 0.80 $M_\odot$ star up the red giant branch (RGB), and then from the horizontal branch (HB) up the asymptotic giant branch (AGB). The rear-ends of the arrows are at the indicated values for the surface brightness.

Figure 10
Error in % of the standard neutrino emission rates used in the evolutionary calculations of Raffelt and Weiss (1992). The evolutionary track is that of Fig. 9.

Figure 11
Evolutionary track of Fig. 9 in the $\gamma$-$v_e$-plane.

Figure 12
Error in % of the dipole moment emission rates used in the calculations of Raffelt and Weiss (1992). Upper panel: Error relative to the standard rates. Lower panel: Compound error if combined with the analytic standard rates of Munakata et al. (1985).