A BLURRED VIEW OF VAN DER WAERDEN TYPE THEOREMS

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Dedicated to the memory of Ronald Graham

Abstract. Let $AP_k = \{a, a+d, \ldots, a+(k-1)d\}$ be an arithmetic progression. For $\varepsilon > 0$ we call a set $AP_k(\varepsilon) = \{x_0, \ldots, x_{k-1}\}$ an $\varepsilon$-approximate arithmetic progression if for some $a$ and $d$, $|x_i - (a+id)| < \varepsilon d$ holds for all $i \in \{0,1,\ldots,k-1\}$. Complementing earlier results of Dumitrescu [4], in this paper we study numerical aspects of Van der Waerden, Szemerédi and Furstenberg–Katznelson like results in which arithmetic progressions and their higher dimensional extensions are replaced by their $\varepsilon$-approximation.

1. Introduction

For a natural number $N$ we set $[N] = \{1,2,\ldots,N\}$. Assume that $[N]$ is colored by $r$ colors. We denote by

$$N \rightarrow (AP_k)_r$$

the fact that any such $r$-coloring yields a monochromatic arithmetic progression $AP_k$ of length $k$. With this notation the well known Van der Waerden’s theorem can be stated as follows.

Theorem 1.1. For every positive integers $r$ and $k$, there exists a positive integer $N$ such that $N \rightarrow (AP_k)_r$.

The minimum $N$ with the property of Theorem 1.1 is called the Van der Waerden number of $r,k$ and is denoted by $W(k,r)$. In other words, $W(k,r)$ is the minimum integer $N$ such that any $r$-coloring of $[N]$ contains a monochromatic arithmetic progression of length $k$. Much effort was put to determine lower and upper bounds for $W(k,r)$, but the problem remains widely open. As an illustration, the best known bounds for $W(k,2)$ are

$$\frac{2^k}{k^{o(1)}} \leq W(k,2) \leq 2^{2^{2k+9}},$$

where $o(1) \rightarrow 0$ as $k \rightarrow \infty$. The lower bound is due to Szabo [25] while the upper bound is a celebrated result of Gowers on Szemerédi’s theorem [10]. It is good to remark that when $k$ is a prime the lower bound can be improved to $W(k+1,2) \geq k2^k$ by a construction of Berlekamp [2].

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Ron Graham was keenly interested in the research leading to improvements of the upper bound of $W(k, 2)$ and motivated it by monetary prizes. Currently open is his $1000 award for the proof that $W(k, 2) < 2^{k^2}$ (see [14]). During his career he also contributed to related problems in the area (see [3,12,13]). For instance, together with Erdős [6], Graham proved a canonical version of Van der Waerden: Every coloring of $\mathbb{N}$, not necessarily with finitely many colors, contains either an monochromatic arithmetic progression or a rainbow arithmetic progression, i.e., a progression with every element of distinct color.

Inspired by the works of [4] and [16], we are interested in the related problem where we replace an arithmetic progression by an perturbation of it.

**Definition 1.2.** Given $\varepsilon > 0$, a set $X = \{x_0, \ldots, x_{k-1}\} \subseteq [N]$ is an $\varepsilon$-approximate $AP_k(\varepsilon)$ of an arithmetic progression of length $k$ if there exists $a \in \mathbb{R}$ and $d > 0$ such that $|x_i - (a + id)| < \varepsilon d$.

In other words, an $AP_k(\varepsilon)$ is just a transversal of $\bigcup_{i=0}^{k-1} B(a + id; \varepsilon d)$, where $B(a + id; \varepsilon d)$ is the open ball centered at $a + id$ of radius $\varepsilon d$. Depending on the choice of $\varepsilon$, an $AP_k(\varepsilon)$ can be different from an $AP_k$. For example, if $\varepsilon = 1/3$, then $a = 0.8$ and $d = 2.4$ testifies that $\{1, 3, 6\}$ is an $\varepsilon$-approximate arithmetic progression of length 3, but it is not an arithmetic progression itself.

For integers $r$, $k$, and $\varepsilon > 0$, let

$$W_{\varepsilon}(k, r) = \min\{N : N \rightarrow (AP_k(\varepsilon))_r\}.$$  

That is, $W_{\varepsilon}(k, r)$ is the smallest $N$ with the property that any coloring of $[N]$ by $r$ colors yields a monochromatic $AP_k(\varepsilon)$. Our first result shows that one can obtain sharper bounds to the Van der Waerden problem by replacing $AP_k$ to $AP_k(\varepsilon)$.

**Theorem 1.3.** Let $r \geq 1$. There exists a positive constant $\varepsilon_0$ and a real number $c_r$ depending on $r$ such that the following holds. If $0 < \varepsilon \leq \varepsilon_0$ and $k \geq 2^r r! \varepsilon^{-1} \log^r (1/5\varepsilon)$, then

$$0 \leq \frac{k^r}{\varepsilon^{-1} \log(1/\varepsilon)(r!)^{-1}} \leq W_{\varepsilon}(k, r) \leq \frac{2k^r}{\varepsilon^{r-1}}.$$  

Similar as in the previous discussion we will write $N \rightarrow_{\alpha} AP_k$ (or $N \rightarrow_{\alpha} AP_k(\varepsilon)$) to denote that any subset $S \subseteq [N]$ with $|S| \geq \alpha N$ necessarily contains an arithmetic progression $AP_k$ (or $AP_k(\varepsilon)$, respectively). Answering a question of Erdős and Turán [7], Szemerédi proved the following celebrated result:

**Theorem 1.4.** For any $\alpha > 0$ and a positive integer $k$, there exists an integer $N_0$ such that for every $N \geq N_0$ the relation $N \rightarrow_{\alpha} AP_k$ holds.

Basically Szemerédi theorem states that any positive proportion of $\mathbb{N}$ contains an arithmetic progression of length $k$. Not much later Furstenberg [9] gave an alternative proof of Theorem 1.4 using Ergodic theory. Extending [9], Furstenberg and Katznelson [8] were able to prove a multidimensional version of Szemerédi’s theorem:
An $m$-dimensional cube $C(m, k)$ is a set of $k^m$ points in $m$-dimensional Euclidean lattice $\mathbb{Z}^m$ such that

$$C(m, k) = \{ \bar{a} + d\bar{v} : \bar{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m \text{ and } \bar{v} = (v_1, \ldots, v_m) \in \{0, 1, \ldots, k - 1\}^m \}.$$ 

That is, $C(m, k)$ is a homothetic translation of $[k]^m$. As in the one dimensional case, for $\alpha > 0$ and integers $m$, $k$ and $N$ we will write $[N]^m \to_\alpha C(m, k)$ to mean that any subset $S \subseteq [N]^m$ with $|S| \geq \alpha N^m$ contains a cube $C(m, k)$. The following is the multidimensional version of Theorem 1.4 proved in [8].

**Theorem 1.5.** For any $\alpha > 0$ and positive integers $k$ and $m$, there exists an integer $N_0$ such that for every $N \geq N_0$ the relation $[N]^m \to_\alpha C(m, k)$ holds.

Define $f(N, m, k)$ as the maximum size of a subset $A \subseteq [N]^m$ without a cube $C(m, k)$. Note that $f(N, 1, k)$ corresponds to the maximal size of a subset $A \subseteq [N]$ without an arithmetic progression $\text{AP}_k$. Theorems 1.4 and 1.5 give us that $f(N, m, k) = o(N^m)$. Determining bounds for $f(N, m, k)$ is a long standing problem in additive combinatorics. For $m = 1$ the best current bounds are

$$N \exp\left( -c_k (\log N)^{1/\log 2k} \right) \leq f(N, 1, k) \leq \frac{N}{(\log \log N)^{2^{-2k+9}}}$$

where $c_k$ is a positive constant depending only on $k$. The upper bound is due to Gowers [10], while the lower bound with best constant $c_k$ is due to O’Bryant [21].

For larger $m$ it is worth mentioning that Furstenberg–Katznelson proof of Theorem 1.5 uses Ergodic theory and gives us no quantitative bounds on $f(N, m, K)$. Purely combinatorial proofs were given later based on the hypergraph regularity lemma in [11] and [20, 23]. Those proofs give quantitative bounds which are incomparably weaker than the one for $m = 1$. For instance, in [19] Moshkovitz and Shapira proved that the hypergraph regularity lemma gives a bound of the order of the $k$-th Ackermann function.

Now we consider $\varepsilon$-approximate versions of Theorems 1.4 and 1.5.

**Definition 1.6.** Given $\varepsilon > 0$, a set $X = \{ x_\bar{v} : \bar{v} \in \{0, 1, \ldots, k - 1\}^m \} \subseteq [N]^m$ is an $\varepsilon$-approximate cube $C_\varepsilon(m, k)$ if there exists $\bar{a} \in \mathbb{R}^m$ and $d > 0$ such that $||x_\bar{v} - (\bar{a} + d\bar{v})|| < \varepsilon d$.

For integers $N$, $m$, $k$ and $\varepsilon > 0$, let $f_\varepsilon(N, m, k)$ be the maximal size of a subset $A \subseteq [N]^m$ without an $C_\varepsilon(m, k)$. Dimitrescu showed an upper bound for $f_\varepsilon(N, m, k)$ in [4]. We complement his result by also providing a lower bound to the problem.

**Theorem 1.7.** Let $m \geq 1$ and $k \geq 3$ be integers and $0 < \varepsilon < 1/125$. Then there exists an integer $N_0 := N_0(k, \varepsilon)$ and positive constants $c_1$ and $c_2$ depending only on $k$ and $m$ such that

$$N^{m-c_1(\log (1/\varepsilon))^{1/\ell}} \leq f_\varepsilon(N, m, k) \leq N^{m-c_2(\log (1/\varepsilon))^{-1}},$$

for $N \geq N_0$ and $\ell = \lceil \log_2 k \rceil$. 

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The paper is organized as follows. In Section 2, we present a proof of Theorem 1.3. The upper bound is an iterated blow-up construction, while the lower bound is given by an ad-hoc inductive coloring. We prove Theorem 1.7 in Section 3. The lower bound uses the current lower bounds for $f(N, 1, k)$, while the upper bound is given by an iterated blow-up construction combined with an averaging argument.

2. Proof of Theorem 1.3

2.1. Upper bound. We start with the upper bound. Given $r \geq 1$ colors, we consider the following $r$-iterated blow-up of an $AP_k$ given by the set of integers

$$B_r = \{b_0 + tb_1 + \ldots + t^{r-1}b_{r-1} : (b_0, \ldots, b_{r-1}) \in \{0, 1, \ldots, k-1\}^r, t = \lceil k/\varepsilon \rceil \}$$

Note that $B_r$ is a set of size $|B_r| = k^r$ and $\text{diam}(B_r) \leq (k-1)(1 + t + \ldots + t^{r-1}) < 2(k-1)t^{r-1}$. It turns out that any $r$-coloring of $B_r$ contains a monochromatic $AP_k(\varepsilon)$. In particular, this implies that $W_\varepsilon(k, r) \leq \text{diam}(B_r) + 1 \leq 2k^r/\varepsilon^{r-1}$.

Proposition 2.1. Any $r$-coloring of $B_r$ has a monochromatic $AP_k(\varepsilon)$.

Proof. We prove the proposition by induction on the number of colors $r$. For $r = 1$, one can see that $B_1 = [k]$, which is an arithmetic progression of length $k$ and in particular a $AP_k(\varepsilon)$. Now suppose that any $(r-1)$-coloring of $B_{r-1}$ contains a monochromatic $AP_k(\varepsilon)$. Consider an $r$-coloring of $B_r$. Note that we can partition $B_r = \bigcup_{i=0}^{k-1} B_{r,i}$ where

$$B_{r,i} = \{b_0 + \ldots + t^{r-2}b_{r-2} + it^{r-1} : (b_0, \ldots, b_{r-2}) \in \{0, 1, \ldots, k-1\}^{r-1}, t = \lceil k/\varepsilon \rceil \}$$

That is, for every $0 \leq i \leq k-1$, the set $B_{r,i}$ is a translation of $B_{r-1}$ by $it^{r-1}$.

Consider a transversal $X = \{x_0, \ldots, x_{k-1}\}$ of $B_r = \bigcup_{i=0}^{k-1} B_{r,i}$ with $x_i \in B_{r,i}$ for every $0 \leq i \leq k-1$. Let $a = \text{diam}(B_{r-1})/2$ and $d = t^{r-1}$. Since $x_i \in B_{r,i}$ implies that $it^{r-1} \leq x_i \leq it^{r-1} + \text{diam}(B_{r-1})$, we obtain that

$$|x_i - (a + id)| \leq \frac{\text{diam}(B_{r-1})}{2} \leq \frac{k^{r-1}}{\varepsilon^{r-2}} \leq \varepsilon d$$

and $X$ is an $\varepsilon$-approximate $AP_k(\varepsilon)$. Therefore, if some color $c$ is present in each of the sets $B_{r,i}$ for $0 \leq i \leq r-1$, we could select $X$ to be a monochromatic $AP_k(\varepsilon)$. Consequently we may assume that there is no monochromatic transversal in $B_r$, which means that there exists an index $i$ such that $B_{r,i}$ is colored with at most $(r-1)$ colors. Since $B_{r,i}$ is just a translation of $B_{r-1}$, by induction hypothesis we conclude that there exists a monochromatic $AP_k(\varepsilon)$ inside $B_{r,i}$. \hfill \square

2.2. Lower bound. In order to construct a large set avoiding $\varepsilon$-approximate $AP_k(\varepsilon)$ we need some preliminary results. Given a real number $D > 0$, we define an $(r-1, 1; D)$-alternate labeling of $\mathbb{R}$ to be an labeling $\chi : \mathbb{R} \to \{-1, +1\}$ such that
\[
\chi(x) = \begin{cases} 
+1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left( irD + mD, \left( i + \frac{r-1}{r} \right) rD + mD \right], \\
-1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left( \left( i + \frac{r-1}{r} \right) rD + mD, (i + 1)rD + mD \right]. 
\end{cases}
\]

for some \( m \in \mathbb{Z} \). That is, \( \chi \) is a periodic labeling of \( \mathbb{R} \) with period \( rD \), where we partition \( \mathbb{R} \) into disjoint intervals of length \( D \) and label them alternating between \( r-1 \) consecutive intervals of label \(+1\) and one of label \(-1\). The restriction of an \((r-1, 1; D)\)-alternate labeling to \( \mathbb{Z} \) will be of great importance for us. The following lemma roughly characterizes the common difference of any large monochromatic approximate arithmetic progression in such a labeling.

**Lemma 2.2.** Let \( D, \delta > 0 \), \( m \) be a positive integer with \( \delta \leq \frac{1}{2r(r+1)} \) and \( \chi : \mathbb{R} \to \{-1, +1\} \) be an \((r-1, 1; D)\)-alternate labeling of \( \mathbb{R} \). If there exist \( a, d \in \mathbb{R} \) and an integer \( \ell \) such that

\[
d \notin \bigcup_{i \in \mathbb{Z}} \left( \left( \frac{i}{q} - \delta \right) rD, \left( \frac{i}{q} + \delta \right) rD \right],
\]

and that \( B = \bigcup_{i=0}^{\ell-1} B(a + id, \delta rD) \) has a monochromatic transversal of label \(+1\), then \( \ell \leq 3r/\delta \).

**Proof.** We may assume without loss of generality that \( \chi \) is the following labeling of \( \mathbb{R} \):

\[
\chi(x) = \begin{cases} 
+1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left( irD, \left( i + \frac{r-1}{r} \right) rD \right], \\
-1, & \text{if } x \in \bigcup_{i \in \mathbb{Z}} \left( \left( i + \frac{r-1}{r} \right) rD, (i + 1)rD \right]. 
\end{cases}
\]

That is, we may assume that \( m = 0 \) in the definition of an alternate labeling. Also, during the proof we shall write \( \overline{x} \) to be the representative of \( x \) modulo \( rD \) in the interval \( (0, rD] \), i.e., the number \( 0 < \overline{x} \leq rD \) such that \( x - \overline{x} = brD \) for some integer \( b \in \mathbb{Z} \).

We start by claiming that there exists \( 1 \leq s \leq r \) such that

\[
\overline{sd} \in \left[ \delta rD, \frac{rD}{r+1} \right] \cup \left[ \left( 1 - \frac{1}{r+1} \right) rD, (1 - \delta)rD \right]. \tag{1}
\]

First note by our hypothesis that

\[
d \notin \left( \left( \frac{i}{q} - \delta \right) rD, \left( \frac{i}{q} + \delta \right) rD \right)
\]

for every \( i \in \mathbb{Z} \) and \( 1 \leq q \leq r \). Therefore,

\[
qd \notin \left( \left( i - \delta \right) rD, \left( i + \delta \right) rD \right) \subseteq \left( \left( i - q\delta \right) rD, \left( i + q\delta \right) rD \right) \tag{2}
\]

for every \( i \in \mathbb{Z} \) and \( 1 \leq q \leq r \).

Now consider the partition \( (0, rD] = \bigcup_{j=0}^{\ell(r+1)-1} \left[ \frac{irD}{r+1}, \frac{(i+1)rD}{r+1} \right] \). If there exists \( 1 \leq s \leq r \) such that \( \overline{sd} \) is in the two outer intervals above, i.e., in either \( \left( 0, \frac{rD}{r+1} \right] \) or \( \left( \left( 1 - \frac{1}{r+1} \right) rD, rD \right] \), then by (2) we obtain that \( s \) satisfies (1). Otherwise, assume that there is no \( 1 \leq s \leq r \) with \( \overline{sd} \) in the two outer intervals. Then by the pigeonhole principle there exist \( 1 \leq p < q \leq r \) and an index \( j \) such that \( \overline{pd}, \overline{qd} \in \left[ \frac{irD}{r+1}, \frac{(i+1)rD}{r+1} \right] \). Consequently, we have that \( \overline{qd} - \overline{pd} \in \left( -\frac{rD}{r+1}, \frac{rD}{r+1} \right) \).
By letting \( s = q - p \) we obtain that
\[
\overline{sd} \in \left( 0, \frac{rD}{r+1} \right] \cup \left( \left( 1 - \frac{1}{r+1} \right) rD, rD \right],
\]
for \( 1 \leq s \leq r \), which is a contradiction. Therefore, condition (1) is always satisfied for some \( s \).

Let \( 1 \leq s \leq r \) be the number satisfying (1) and consider the subset
\[
B' = \bigcup_{i=0}^{\ell'} B(a + isd, \delta rD) \subseteq B,
\]
where \( \ell' = \lfloor (\ell - 1)/s \rfloor \). That is, if we see \( B \) as the arithmetic progression of intervals of length \( \delta rD \), size \( \ell \) and common difference \( d \), then \( B' \) is a subarithmetic progression of \( B \) with common difference \( sd \). Since \( B \) has a monochromatic transversal labeled +1, then \( B' \) also has a monochromatic transversal labeled +1. Hence, because \( \bigcup_{i \in \mathbb{Z}} \left( irD, (i + \frac{r-1}{r}) rD \right) \) are the elements of label +1 in our \((r - 1, 1; D)\)-alternate labeling, we have that
\[
\{a, a + sd, \ldots, a + \ell' sd\} \subseteq \bigcup_{i \in \mathbb{Z}} \left( (i - \delta)rD, \left( i + \frac{r-1}{r} + \delta \right) rD \right).
\]

Suppose that \( \overline{sd} \in \left[ \delta rD, \frac{1}{r+1} rD \right] \). Since the coloring \( \chi \) is periodic modulo \( rD \), we may assume without loss of generality that \( sd \in \left[ \delta rD, \frac{1}{r+1} rD \right] \). We claim that there exists an integer \( p \) such that \( \{a, a + sd, \ldots, a + \ell' sd\} \subseteq \left( (p - \delta)rD, (p + \frac{r-1}{r} + \delta) rD \right) \). Suppose that this is not the case. Because \( sd > 0 \) there exist integers \( p < q \) and \( 0 \leq i \leq \ell' - 1 \) such that \( a + isd \in \left( (p - \delta)rD, (p + \frac{r-1}{r} + \delta) rD \right) \) and \( a + (i + 1)sd \in \left( (q - \delta)rD, (q + \frac{r-1}{r} + \delta) rD \right) \). A computation shows that
\[
sd = a + (i + 1)sd - (a + isd) > (q - \delta)rD - \left( p + \frac{r-1}{r} + \delta \right) rD \geq (1 - 2\delta r)D \geq \frac{rD}{r+1}
\]
for \( \delta \leq \frac{1}{2r(r+1)} \), which contradicts our assumption on \( sd \).

Hence, there exists \( p \) such that \( a, a + \ell' sd \in \left( (p - \delta)rD, (p + \frac{r-1}{r} + \delta) rD \right) \), which implies that
\[
\ell' sd = (a + \ell' sd) - a \leq \left( p + \frac{r-1}{r} + \delta \right) rD - (p - \delta)rD = (r - 1)D + 2\delta rD.
\]
Since \( sd \geq \delta rD \), we obtain that
\[
\ell' sd \geq \left( \frac{\ell - 1}{s} \right) \delta rD \geq \frac{\ell \delta rD}{2s} \geq \frac{\delta \ell D}{2}
\]
for \( \ell > r \geq s \). The last two computations combined with the fact that \( \delta \leq \frac{1}{2r(r+1)} \leq \frac{1}{4} \) gives us that
\[
\ell \leq \frac{2(r - 1)D + 4\delta rD}{\delta D} \leq \frac{2(r - 1)}{\delta} + 4r \leq \left( \frac{2}{\delta} + 4 \right) r \leq \frac{3r}{\delta}
\]
Now assume that \( \overline{sd} \in \left[ (1 - \frac{1}{r+1}) rD, (1 - \delta)rD \right] \). By the periodicity of \( \chi \), we may assume without loss of generality that \( sd \in \left[ -\frac{rD}{r+1}, -\delta rD \right] \). By rewriting \( \{a, a + sd, \ldots, a + \ell' sd\} \)
as \( \{a', a' + sd', \ldots, a' + \ell_sd'\} \) with \( a' = a + \ell'sd \) and \( d' = -d \), we are back to the previous case and again \( \ell \leq 3r/\delta \).

Although it is convenient to prove Lemma 2.2 using an alternate labeling of \( \mathbb{R} \), the lower bound construction will use alternate labelings of set of integers. With this in mind, we give the following companion definition.

Given positive integers \( D, r \) and \( t \), an \((r - 1,1; D)\)-alternate labeling of the set \([rtD]\) is a labeling \( \chi' : [rtD] \to \{-1,+1\} \) such that \( \chi'(x) = \chi(x) \), where \( \chi \) is an \((r - 1,1; D)\)-alternate labeling of \( \mathbb{R} \). In other words, an alternate labeling of a set of integers is just the restriction of an alternate labeling of \( \mathbb{R} \) to the set. Note that by this definition, there exists \( r \) distinct \((r - 1,1; D)\)-alternate labelings of \([rtD]\). A \( D\)-block of \([rtD]\) is a block of \( D \) consecutive integers of the form \([iD + 1, (i + 1)D]\). One can note that the \( D\)-blocks form a partition of \([rtD]\) and each \( D\)-block is monochromatic in an \((r - 1,1; D)\)-alternate labeling of \([rtD]\).

Finally, note that given an alternate labeling \( \chi' \) of a set \([rtD]\) we can extend back to an alternate labeling of \((0, rtD)\) by labeling the entire interval \((iD, (i + 1)D)\) with the same label as the \( D\)-block of integers \([iD + 1, (i + 1)D]\). Since the labeling is periodic, it is now easy to extend back to a labeling \( \chi \) of \( \mathbb{R} \).

The next result is a consequence of the proof of Lemma 2.2.

**Proposition 2.3.** Let \( D, r, t \) and \( \ell \) be positive integers with \( \ell \geq t(r + 1) + 2 \) and \( 0 < \varepsilon < 1/2r \) be a real number. If \([rtD]\) is colored by an \((r - 1,1; D)\)-alternate labeling and \( X \subseteq [rtD] \) is a monochromatic \( \mathbb{A}_{\ell}(\varepsilon) \) of label +1, then there exists \( 0 \leq i \leq rt - 1 \) such that the \( D\)-block \([iD + 1, (i + 1)D]\) satisfies \( |X \cap [iD + 1, (i + 1)D]| \geq \ell/(r - 1) \).

**Proof.** Write \( X = \{x_0, \ldots, x_{\ell-1}\} \). Since \( X \) is an \( \mathbb{A}_{\ell}(\varepsilon) \), there exists \( a \in \mathbb{R}, d > 0 \) such that \( |x_i - (a + id)| < \varepsilon d \). Therefore, a computation shows that

\[
rtD > |x_{\ell-1} - x_0| \geq a + (\ell - 1)d - a - 2\varepsilon d = (\ell - 1 - 2\varepsilon)d,
\]

which implies that

\[
d \leq \frac{rtD}{\ell - 2} \leq \frac{rd}{r + 1} \tag{3}
\]

for \( \ell \geq t(r + 1) + 2 \).

Similarly as in the proof of Lemma 2.2, we will show that all the elements of \( X \) are inside an interval of \((r - 1)\) consecutive \( D\)-blocks of label +1.

Suppose that this was not the case. Since non-consecutive \( D\)-blocks of label +1 are at a distance of at least \( D \) elements, then there exists \( x_i \) and \( x_{i+1} \) such that \( |x_{i+1} - x_i| \geq D \). However, in view of \( \varepsilon < 1/2r \) and (3), we obtain

\[
|x_{i+1} - x_i| \leq |x_{i+1} - (a + (i + 1)d)| + |a + (i + 1)d - (a + id)| + |x_i - (a + id)| \leq (1 + 2\varepsilon)d < D,
\]
which is a contradiction. The result now follows by an application of the pigeonhole principle.

Note that Proposition 2.3 already gives us a lower bound for the case $r = 2$. Indeed, we will prove that an $(1, 1; k - 1)$-alternate labeling of $\left\lceil \frac{2(k-1)(k-2)}{3} \right\rceil$ does not contain a monochromatic $\text{AP}_k(\varepsilon)$ for $\varepsilon < 1/4$ and sufficiently large $k$.

Suppose that this is not the case. Since an $(1, 1; k - 1)$-alternate labeling is symmetric, we may assume that there is a monochromatic $\text{AP}_k(\varepsilon)$ of label +1. Applying Proposition 2.3 with $r = 2$, $t = (k - 2)/3$, $D = k - 1$ and $\ell = k$ gives us that there exists a $(k - 1)$-block of the form $[i(k - 1) + 1, (i + 1)(k - 1)]$ such that $|X \cap [i(k - 1) + 1, (i + 1)(k - 1)]| \geq k$, which contradicts the size of the block.

Unfortunately, the argument above does not give a lower bound depending on $\varepsilon$. To achieve such a bound we will need to refine the previous construction, but first we need one more preliminary result.

The second Chebyshev function $\psi(x)$ is defined to be the logarithm of the least common multiple of all positive integers less or equal than $x$. The following bound on $\psi(x)$ will be useful for us.

**Theorem 2.4 ([24], Theorem 7).** If $x \geq 10^8$, then $|\psi(x) - x| < cx/\log x$ for some positive constant $c$.

In particular, Theorem 2.4 asserts that for sufficiently large $n$ we have

$$\text{lcm}(1, \ldots, n) = e^{n+O(n/\log n)}. \quad (4)$$

We are now ready to prove the lower bound of Theorem 1.3.

**Theorem 2.5.** Let $r \geq 1$. There exists a positive constant $\varepsilon_0$ and a real number $c_r$ depending on $r$ such that the following holds. If $0 < \varepsilon \leq \varepsilon_0$ and $k \geq 2^r r! \varepsilon^{-1} \log^r (1/5\varepsilon)$ is a integer, then there exist an integer $N := N(\varepsilon, k, r)$ satisfying

$$N \geq c_r \frac{k^r}{\varepsilon^{r-1} \log(1/\varepsilon)(r+1)^{1-1}};$$

so that $[N]$ admits an $r$-coloring without monochromatic $\text{AP}_k(\varepsilon)$.

**Proof.** The proof is by induction on the number of colors $r$. For $r = 1$, the result clearly holds for $N(\varepsilon, k, 1) = k - 1$ since there is no $\text{AP}_k(\varepsilon)$, or even $\text{AP}_k$, on $(k - 1)$ terms. Now suppose that for any $\varepsilon$ and $k$ such that $0 < \varepsilon \leq \varepsilon_0$ and $k \geq 2^r r! \varepsilon^{-1} \log^r (1/5\varepsilon)$, there exists $N(\varepsilon, k, r - 1)$ and a $(r - 1)$-coloring of $[N(\varepsilon, k, r - 1)]$ satisfying the conclusion

\footnote{Strictly speaking we should use the set $[2 \{k-2\} (k-1)]$, since $\frac{k-2}{3}$ is not necessarily an integer. However, during our exposition we will not bother with this type of detail since it has no significant effect on arguments or results}
of the statement. We want to find an integer \( N_1 \) so that \([N_1]\) has a \( r\)-coloring without monochromatic \( \text{AP}_k(\varepsilon) \).

To do that we start with some choice of variables. Let
\[
N_0 = \left( \varepsilon, \frac{k}{rs}, r-1 \right), \quad s = \frac{1}{0.9} \log(1/5\varepsilon), \quad w = \frac{e^{0.9s}}{s(r-1)!}, \quad t = \frac{k}{2rs}, \quad D_j = \frac{s-j+1}{s} N_0
\]
be integers for \( 1 \leq j \leq s/2 \). Note that although \( s, w, t \) and \( \{D_j\}_{1 \leq j \leq s/2} \) might not be integers, we prefer to write in this way, since it simplifies the exposition and has no significant effect on the arguments. Moreover, the integer \( N_0 \) always exists since by hypothesis
\[
\frac{k}{rs} \geq 2^r r! \varepsilon^{-1} \log^r (1/5\varepsilon) \geq 2^{r-1} (r-1)! \varepsilon^{-1} \log^{r-1} (1/5\varepsilon).
\]

Let \( N_1 = rwt(D_1 + \ldots + D_{s/2}) \). We are going to define a coloring \( \varphi : [N_1] \to [r] \) not admitting monochromatic \( \text{AP}_k(\varepsilon) \). To this end we partition \([N_1]\) into consecutive intervals following the four steps below:

- First we partition \([N_1]\) into \([N_1] = Y_1 \cup \ldots \cup Y_w\), where \( Y_i \) are consecutive intervals and \( |Y_i| = rt(D_1 + \ldots + D_{s/2}) \) for every \( i = 1, \ldots, w \).
- Each \( Y_i \) is partitioned into \( Y_i = Y_{i,1} \cup \ldots \cup Y_{i,s/2} \), where \( Y_{i,j} \)'s are consecutive intervals and \( |Y_{i,j}| = rtD_j \) for every \( j = 1, \ldots, s/2 \).
- Each \( Y_{i,j} \) is partitioned into \( Y_{i,j} = Z_{i,j}^1 \cup \ldots \cup Z_{i,j}^{i,j} \), where \( Z_{i,j}^{i,j} \)'s are consecutive intervals and \( |Z_{i,j}^{i,j}| = rtD_j \) for every \( u = 1, \ldots, t \).
- Each \( Z_{i,j}^{i,j} \) is partitioned into \( Z_{i,j}^{i,j} = Z_{i,j,1}^{i,j} \cup \ldots \cup Z_{i,j,v}^{i,j} \), where \( Z_{i,j,v}^{i,j} \)'s are consecutive intervals and \( |Z_{i,j,v}^{i,j}| = D_j \) for every \( v = 1, \ldots, r \).

More explicitly, we define
\[
\alpha_i = (i-1)rt(D_1 + \ldots + D_{s/2}), \quad i \in [w]
\]
\[
\beta_{i,1} = \alpha_i, \quad i \in [w]
\]
\[
\beta_{i,j} = rt(D_1 + \ldots + D_{j-1}) + \alpha_i, \quad (i, j) \in [w] \times [2, s/2]
\]
\[
\gamma_{i,j,u} = (u-1)rtD_j + \beta_{i,j}, \quad (i, j, u) \in [w] \times [s/2] \times [t]
\]
\[
\sigma_{i,j,u,v} = (v-1)D_j + \gamma_{i,j,u}, \quad (i, j, u, w) \in [w] \times [s/2] \times [t] \times [r]
\]

Therefore, our intervals can be written as
\[
Y_i = [\alpha_i + 1, \alpha_i + rt(D_1 + \ldots + D_{s/2})], \quad i \in [w]
\]
\[
Y_{i,j} = [\beta_{i,j} + 1, \beta_{i,j} + rtD_j], \quad (i, j) \in [w] \times [s/2]
\]
\[
Z_{i,j}^{i,j} = [\gamma_{i,j,u} + 1, \gamma_{i,j,u} + rtD_j], \quad (i, j, u) \in [w] \times [s/2] \times [t]
\]
\[
Z_{i,j,v}^{i,j} = [\sigma_{i,j,u,v} + 1, \sigma_{i,j,u,v} + D_j], \quad (i, j, u, v) \in [w] \times [s/2] \times [t] \times [r]
\]

Finally, we describe the coloring \( \varphi : [N_1] \to [r] \) on the intervals \( Z_{i,j,v}^{i,j} \). By induction hypothesis, given any set \( C \) of \( r - 1 \) colors there exists a coloring \( \varphi_C : [N_0] \to C \) with
no monochromatic $A_{P_{k/f/r}}(\varepsilon)$. Fix $Z_{u,v}^{i,j}$ with $(i,j,u,v) \in [w] \times [s/2] \times [t] \times [r]$. We color $Z_{u,v}^{i,j}$ by the same coloring as the first $D_j$ elements of $[N_0]$ when $[N_0]$ is colored by $\varphi_{[r]-\{v\}}$. That is, the coloring $\varphi$ restricted to $Z_{u,v}^{i,j}$ only uses $r-1$ colors and does not contain a monochromatic $A_{P_{k/f/r}}(\varepsilon)$.

To prove that the coloring $\varphi$ is free of $A_P(k,\varepsilon)$ we are going to show that there is no $a \in \mathbb{R}$ and $d > 0$ such that $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$ has a monochromatic transversal in $[N_1]$. Suppose the opposite and assume that there exists $a$ and $d$ such that $\bigcup_{i=0}^{k-1} B(a + id, \varepsilon d)$ has a monochromatic transversal $X = \{x_0, \ldots, x_{k-1}\} \subseteq [N_1]$ of color $c \in [r]$. Since all the balls have radius $\varepsilon d$, we obtain that $\{a, a + d, \ldots, a + (k-1)d\} \subseteq (1 - \varepsilon d, N_1 + \varepsilon d)$, which gives that $(k-1)d \leq (N_1 - 1) + 2\varepsilon d$. By (5) and by the fact that $\varepsilon \leq \varepsilon_0$ we have that

$$d \leq \frac{N_1 - 1}{k - 1 - 2\varepsilon} \leq \frac{2N_1}{k} = \frac{2rtw(D_1 + \ldots + D_{s/2})}{k} = \frac{wN_0}{s^2} \left(s + \ldots + \left(\frac{s}{2} + 1\right)\right) \leq \frac{wN_0}{2}, \quad (6)$$

for sufficiently small $\varepsilon_0$.

For a fixed $Y_{i,j} = \bigcup_{u=1}^{r} \bigcup_{v=1}^{z} Z_{u,v}^{i,j}$ we define an auxiliary labeling $\chi_{i,j} : Y_{i,j} \to \{-1, +1\}$ of $Y_{i,j}$ such that every $D_j$-block $Z_{u,v}^{i,j}$ is monochromatic and

$$\chi_{i,j}(Z_{u,v}^{i,j}) = \begin{cases} 
+1, & \text{if } v \neq c, \\
-1, & \text{if } v = c.
\end{cases}$$

In other words, every element of a $D_j$-block $Z_{u,v}^{i,j}$ is of label $-1$ if the coloring $\varphi$ restricted to $Z_{u,v}^{i,j}$ has the same coloring of the first $D_j$ elements of $\varphi_C : [N_0] \to C$, where $C = [r] - \{c\}$, i.e., the set of colors missing the color $c$. Otherwise, we label all the elements in $Z_{u,v}^{i,j}$ by $+1$. It is not difficult to check that $\chi_{i,j}$ is an $(r-1,1; D_j)$-alternate labeling of $Y_{i,j}$. Moreover, since $X$ is monochromatic of color $c$ and $Z_{u,v}^{i,j}$ is colored by $\varphi_{[r]-\{c\}}$, we obtain that $X \cap Z_{u,v}^{i,j} = \emptyset$. This implies that every element of $X \cap Y_{i,j}$ is labeled $+1$. Finally, in order to apply Lemma 2.2, we extend the labeling $\chi_{i,j}$ to the set of real numbers $(\beta_{i,j}/\beta_{i,j} + rtD_j)$ by labeling the entire interval $(\sigma_{i,j,u,v} + \sigma_{i,j,u,v} + D_j)$ by color $\chi_{i,j}(Z_{u,v}^{i,j})$ for every $u,v \in [t] \times [r]$.

The main idea of the proof is based on the fact that for $d$ not too small, there exists an index $j_0$ such that $d$ is far from certain fractions involving $D_{j_0}$. We will then imply by Lemma 2.2 that the number of elements of $X$ in $Y_{i,j_0}$ is “small”. It turns out that this fact is enough to restrict the entire location of $X$ to just a few $Y_{i,j}$’s. Then by the pigeonhole principle and Proposition 2.3 we can show that there exists a $D_j$-block $Z_{u,v}^{i,j}$ with large intersection with $X$, which contradicts the inductive coloring of $Z_{u,v}^{i,j}$.

The next proposition elaborates more on the existence of such a $j_0$.

**Proposition 2.6.** If $d > \frac{N_0}{s(r-1)!}$, then there exists index $1 \leq j_0 \leq s/2$ such that

$$d - \frac{mD_{j_0}}{(r-1)!} \geq \frac{N_0}{2s(r-1)!}.$$
for every $m \in \mathbb{Z}$. 

**Proof.** Let $M_0 = \frac{N_0}{s(r-1)!}$. Note that by (5) we can write

$$\frac{D_j}{(r-1)!} = (s-j+1)\frac{N_0}{s(r-1)!} = (s-j+1)M_0,$$

for every $1 \leq j \leq s/2$. Therefore, every number of the form $\frac{mD_j}{(r-1)!}$ for $m \in \mathbb{Z}^+$ and $1 \leq j \leq s/2$ is a multiple of $M_0$. Moreover, the least non-zero common term among the sequences $\{\frac{mD_j}{(r-1)!}\}_{m \in \mathbb{Z}^+}$ for $1 \leq j \leq s/2$, i.e.,

$$\min \bigcap_{1 \leq j \leq s/2} \left\{ \frac{mD_j}{(r-1)!} : m \in \mathbb{Z}^+ \right\} = \min \bigcap_{1 \leq j \leq s/2} \{m(s-j+1)M_0 : m \in \mathbb{Z}^+\}$$

is equal to $LM_0$, where $L = \text{lcm}(s/2+1, \ldots, s)$.

Since every number in $\{1, \ldots, s/2\}$ has a nontrivial multiple inside $\{s/2+1, \ldots, s\}$ we obtain by (4) that

$$L = \text{lcm}(s/2+1, \ldots, s) = \text{lcm}(1, \ldots, s) = e^{|\log(s/\log s)|} \geq e^{0.9s},$$

for $s = \frac{1}{0.9} \log(1/5\varepsilon) \geq \frac{1}{0.9} \log(1/5\varepsilon_0)$ and $\varepsilon_0$ sufficiently small. Hence, by (5) and (6) we have

$$d \leq \frac{wN_0}{2} = \frac{N_0 e^{0.9s}}{2s(r-1)!} \leq \frac{LN_0}{2s(r-1)!} = \frac{L}{2} M_0.$$

Let $pM_0$ be the multiple of $M_0$ closest to $d$. Since $d > M_0$, we clearly have that $p \neq 0$. By definition,

$$pM_0 = \frac{pN_0}{s(r-1)!} \leq d + \left| d + \frac{pN_0}{s(r-1)!} - d \right| \leq d + \frac{M_0}{2} < LM_0.$$

Therefore, by the minimality of $LM_0$, there exists an index $1 \leq j_0 \leq s/2$ such that $pM_0$ is not a multiple of $\frac{D_{j_0}}{(r-1)!} = (s-j_0+1)M_0$. Since, by the definition of $p$, all the other numbers of the form $mM_0$ have distance at least $\frac{M_0}{2} = \frac{N_0}{2s(r-1)!}$ to $d$, Proposition 2.6 follows. 

We now prove that there exists a set $Y_{i,j}$ with a large proportion of elements of $X$.

**Proposition 2.7.** There exist indices $(i_1, j_1) \in [w] \times [s/2]$ such that $|X \cap Y_{i_1,j_1}| \geq k/s$.

**Proof.** Let $I \subseteq [w] \times [s/2]$ be set of pair of indices defined by

$$I = \{(i,j) \in [w] \times [s/2] : X \cap Y_{i,j} \neq \emptyset \},$$

and let $\mathcal{Y} = \bigcup_{(i,j) \in I} Y_{i,j}$. By (5) and (6) we obtain that the difference between two consecutive terms of $X$ is bounded by

$$|x_{h+1} - x_h| \leq (1+2\varepsilon)d \leq (1+2\varepsilon)\frac{e^{0.9s}N_0}{2s(r-1)!} < \frac{kN_0}{4s} \leq \frac{k(s-j+1)N_0}{2s^2} = rtD_j = |Y_{i,j}|,$$
for \( k \geq 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon) \geq \varepsilon^{-1}/(r-1)! \). That is, the difference between two consecutive terms of \( X \) is smaller than the size of an interval \( Y_{i,j} \) for \((i,j) \in [w] \times [s/2] \). This implies that all intervals in \( \mathcal{Y} \) must be consecutive. Recall that by construction two intervals \( Y_{i,j} \) and \( Y_{i',j'} \) are consecutive if \((i,j)\) and \((i',j')\) are consecutive in the lexicographical ordering of \([w] \times [s/2] \).

If \(|I| \leq 2\), then by the pigeonhole principle there exist indices \((i_1,j_1)\) such that \(|X \cap Y_{i_1,j_1}| \geq k/2 \geq k/s\) for \( \varepsilon_0 \) sufficiently small. Thus we may assume that \(|I| > 3\). This implies that there exists at least one pair of indices \((i',j')\) such that \( Y_{i',j'} \) is neither the first or last interval of \( \mathcal{Y} \).

Let \( X \cap Y_{i',j'} = \{x_h, \ldots, x_{h+b-1}\} \), where \( b = |X \cap Y_{i',j'}| \). Since \( Y_{i',j'} \) is not one of intervals in the extreme of \( \mathcal{Y} \), we obtain that \( 2 \leq h \leq h+b-1 \leq k-1 \) and in particular there exists points \( x_{h-1} \) and \( x_{h+b} \) outside of \( Y_{i',j'} \). Then a simple computation gives us that

\[
|Y_{i',j'}| \leq |x_{h+b} - x_{h-1}| \leq (b + 1 + 2\varepsilon)d < 2bd
\]

and consequently

\[
|X \cap Y_{i',j'}| = b > \frac{|Y_{i',j'}|}{2d} \tag{7}
\]

for any \( Y_{i',j'} \) not on the extremes of \( \mathcal{Y} \).

We split the proof into two cases depending on the size of \( d \). If \( d \leq \frac{N_0}{s(r-1)!} \), then (5) and (7) give that

\[
|X \cap Y_{i',j'}| > \frac{|Y_{i',j'}|}{2d} = \frac{rtD_{j'}}{2d} \geq \frac{k(s-j'+1)(r-1)!}{4s} \geq \frac{k(r-1)!}{8} \geq \frac{k}{s}
\]

for every \( Y_{i',j'} \) not on the extremes and sufficiently large \( s \). Taking \((i_1,j_1)\) as one such \((i',j')\) gives the desired result.

Now suppose that \( d > \frac{N_0}{s(r-1)!} \). Let \( j_0 \) be the index provided by Proposition 2.6. In particular, it holds that

\[
\left|d - \frac{mrD_{j_0}}{q}\right| \geq \frac{N_0}{2s(r-1)!} \tag{8}
\]

for every \( m \in \mathbb{Z} \) and \( 1 \leq q \leq r \). Suppose that \( X \cap Y_{i,j_0} \neq \emptyset \) for some \( 1 \leq i \leq w \). Our goal is to apply Lemma 2.2 with \( D = D_{j_0}, \delta = 1/4sr! \) to the interval \((\min(Y_{i,j_0}) - 1, \max(Y_{i,j_0})] = (\beta_{i,j_0}, \beta_{i,j_0} + rtD_j] \) labeled with our extension of \( \chi_{i,j_0} \). In order to verify the assumptions of the lemma note that

\[
\frac{N_0}{2s(r-1)!} = \frac{D_{j_0}}{2(s-j_0 + 1)(r-1)!} \geq \frac{D_{j_0}}{2s(r-1)!} > \delta rD_{j_0}
\]

and therefore by (8) we have

\[
d \notin \bigcup_{m \in \mathbb{Z}} \bigcup_{1 \leq q \leq r} \left( \left( \frac{m}{q} - \delta \right) rD_{j_0}, \left( \frac{m}{q} + \delta \right) rD_{j_0} \right).
\]
Consequently, the conclusion of the lemma gives to us that any arithmetic progression of intervals of radius \( \delta r D_{j_0} \) with common difference \( d \) and a monochromatic transversal of label +1 inside the interval \( (\min(Y_{i,j_0}) - 1, \max(Y_{i,j_0})] \) has length bounded by \( 3r/\delta \). This is true in particular for \( \bigcup_{i=0}^{k-1} B(a + id, \varepsilon d) \), since by (5) and (6) we have
\[
\varepsilon d \leq \frac{\varepsilon w N_0}{2} = \frac{N_0}{10s(r - 1)!} = \frac{D_{j_0}}{10(s - j_0 + 1)(r - 1)!} \leq \frac{D_{j_0}}{5s(r - 1)!} < \delta r D_{j_0}.
\]
Hence, because \( X \) is transversal of label +1 of \( \bigcup_{i=0}^{k-1} B(a + id, \varepsilon d) \), the conclusion of Lemma 2.2 gives for \( k \geq 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon) > \frac{32}{3} r^2 \varepsilon^{-1} \log(1/5\varepsilon) \) that
\[
|X \cap Y_{i,j_0}| \leq \frac{3r}{\delta} = 12sr!r = \frac{40}{3} r! r \log(1/5\varepsilon) < \frac{5}{4} \varepsilon (r - 1)! k. \tag{9}
\]
However, by (5), (6) and (7) we have
\[
|X \cap Y_{i',j'}| > \frac{|Y_{i',j'}|}{2d} = \frac{rt D_{j'}}{2d} \geq \frac{1}{w N_0} \cdot \frac{k(s - j' + 1)N_0}{2s^2} \geq \frac{k}{4w s} \geq \frac{5}{4} \varepsilon (r - 1)! k \tag{10}
\]
for any \( Y_{i',j'} \) in the middle of \( Y \). Comparing (9) and (10) yields that \( |X \cap Y_{i,j_0}| < |X \cap Y_{i',j'}| \) for any interval \( Y_{i',j'} \) in the middle of \( Y \). Thus \( Y_{i,j_0} \) cannot be a middle interval and we obtain that if \( (i, j_0) \in I \), then \( Y_{i,j_0} \) is either the first or last interval of \( Y \). Therefore, we can have at most two occurrences of \( j_0 \) in \( I \) and consequently the entire location of \( I \) is contained between those two occurrences, i.e., \( I \subseteq \{ (i, j_0), (i, j_0 + 1), \ldots, (i + 1, j_0 - 1), (i + 1, j_0) \} \) for some \( 1 \leq i \leq w - 1 \). Hence, the set \( I \) has at most \( s/2 + 1 \) elements and by the pigeonhole principle there exists a pair of indices \( (i_1, j_1) \in I \) such that \( |X \cap Y_{i_1,j_1}| \geq k/(s/2 + 1) \geq k/s \). \( \square \)

Let \( (i_1, j_1) \) be the indices given by Proposition 2.7. Next we apply Proposition 2.3 to the set \( Y_{i_1,j_1} \) labeled by \( \chi_{i_1,j_1} \) with \( D = D_{j_1}, \ell = k/s \) and \( \varepsilon \)-approximate arithmetic progression \( X \cap Y_{i_1,j_1} \). Note that by (5) the hypothesis concerning \( r, t \) and \( \ell \) in the statement holds since
\[
t(r + 1) + 2 = \frac{r + 1}{2r s} + 2 < \frac{k}{s} = \ell
\]
for \( r \geq 2 \) and \( k \geq 2^r r! \varepsilon^{-1} \log^r(1/5\varepsilon) \geq 80 \log(1/5\varepsilon)/9 \). Also a \( D_{j_1} \)-block of \( Y_{i_1,j_1} \) is an interval of the form \( Z_{u,v}^{i_1,j_1} \). Hence, by the conclusion of the proposition, there exists \( Z_{u,v}^{i_1,j_1} \) such that \( |X \cap Z_{u,v}^{i_1,j_1}| \geq \ell/(r - 1) > k/rs \). Since each set \( Z_{u,v}^{i_1,j_1} \) was \( (r - 1) \)-colored inductively not to contain an \( AP_{k/rs}(\varepsilon) \), we reach a contradiction. Thus there is no monochromatic \( AP_{k}(\varepsilon) \) in \( \lfloor N_1 \rfloor \). In view of (5) we have
\[
N_1 = rwt(D_1 + \ldots + D_{s/2}) = \frac{k e^{0.98 N_0}}{2 s^3 (r - 1)!} \left( s + \ldots + \left( \frac{s}{2} + 1 \right) \right)

\geq \frac{k N_0}{40 \varepsilon s (r - 1)!} \geq \frac{k N_0}{50 (r - 1)! \varepsilon \log(1/\varepsilon)}.
\]
Consequently, in view of \( s = O(\log(1/\varepsilon)) \) we obtain by induction that

\[
N_1 \geq \frac{k}{50(r-1)! \varepsilon \log(1/\varepsilon)} \cdot \frac{c_r \left( \frac{k}{r\varepsilon} \right)^{r-1}}{\varepsilon^{r-2} \log(1/\varepsilon)^{\left(\frac{r}{2}\right)-1}} \geq c_r \frac{k^r}{\varepsilon^{r-1} \log(1/\varepsilon)^{\left(\frac{r+1}{2}\right)-1}}.
\]

\[\square\]

3. Proof of Theorem 1.7

3.1. Lower bound. For positive integers \( k \) and \( N \), recall that \( f(N, 1, k) \), sometimes denoted by \( r_k(N) \), is defined to be the size of the largest set \( A \subseteq [N] \) without an arithmetic progression of length \( k \). A classical result of Behrend \([1]\) shows that,

\[ f(N, 1, 3) > N \exp(-c_1 \sqrt{\log N}), \]

for a positive constant \( c_1 \) (see \([5, 15]\) for slightly improvements). In \([22]\) (See also \([18]\)) the argument was generalized to yield that

\[ f(N, 1, k) > N \exp\left(-c_1 (\log N)^{1/r}\right), \tag{11} \]

where \( r = \lfloor \log_2 k \rfloor \) and \( k \geq 3 \) and \( c_1 \) is a constant depending only on \( k \). We will use the last result as a building block for our construction.

Before we turn our attention to the lower bound construction, we will state a preliminary result about \( \varepsilon \)-approximate arithmetic progressions. Given a set of \( k \) integers, one can identify them as an \( \text{AP}_k \) by the common difference between the elements. Unfortunately, the same is not true for an \( \text{AP}_k(\varepsilon) \). On the positive side, the next result shows that if a set of \( k \) elements is an \( \text{AP}_k(\varepsilon) \), then the differences of consecutive terms are almost equal.

\textbf{Proposition 3.1.} \textit{Given }\( 0 < \varepsilon < 1/10 \), let \( X = \{x_0, \ldots, x_{k-1}\} \) be an \( \text{AP}_k(\varepsilon) \). Then for every pair of indices \( 0 \leq i, j \leq k-2 \) the following holds

\[
\left| \frac{x_{j+1} - x_j}{x_{i+1} - x_i} - 1 \right| < 5\varepsilon.
\]

\textit{Proof.} Since \( X \) is an \( \text{AP}_k(\varepsilon) \), there exist \( a \) and \( d \) such that \( |x_i - (a + id)| < \varepsilon d \) for \( 0 \leq i \leq k-1 \). Therefore, a simple computation shows that

\[
1 - 5\varepsilon < \frac{1 - 2\varepsilon}{1 + 2\varepsilon} d < \frac{|x_{j+1} - x_j|}{|x_{i+1} - x_i|} < \frac{1 + 2\varepsilon}{1 - 2\varepsilon} d < 1 + 5\varepsilon
\]

for \( 0 < \varepsilon < 1/10 \) and \( 0 \leq i, j \leq k - 2 \). \( \square \)

We now prove the lower bound of Theorem 1.3 for one dimension.

\textbf{Lemma 3.2.} \textit{Let }\( k \geq 3 \) and \( 0 < \varepsilon \leq 1/125 \). Then there exists a positive constant \( c_1 \) depending only on \( k \) and an integer \( N_0 := N_0(k, \varepsilon) \) such that the following holds. If \( N \geq N_0 \), then there
exists a set $A \subseteq [N]$ without $AP_k(\varepsilon)$ such that

$$|A| \geq N^{1-c_1(\log(1/\varepsilon))^{1/\ell}}$$

for $\ell = \lceil \log_2 k \rceil$.

Proof. For integers $a, b$, let $S_k([a, b])$ be the largest subset in the interval $[a, b]$ without any arithmetic progression $AP_k$ of length $k$. By a simple translation, one can note that $S_k([a, b])$ has the same size as $S_k([b - a + 1])$ and by (11) we have

$$|S_k([a, b])| = f(b - a + 1, 1, k) \geq (b - a + 1) \exp \left( -c(\log(b - a + 1))^{1/\ell} \right), \quad (12)$$

for a positive constant $c$ and $\ell = \lceil \log_2 k \rceil$.

Let $q = \frac{1}{2\varepsilon} \geq 5$ be an integer and $h$ be largest exponent such that $q^h \leq N < q^{h+1}$. For such a choice of $q$ and $h$, we construct the set

$$A = \left\{ s \in [N] : a = s_0 + s_1 q + \ldots + s_{h-1} q^{h-1} \right\},$$

where $s_{h-1} \in S_k([0, q - 1])$ and $s_i \in S_k([2q/5, 3q/5])$ for $0 \leq i \leq h - 2$. Our goal is to show that $A$ satisfies the conclusion of Lemma 3.2.

First note by (12) that

$$|A| = |S_k([0, q - 1])| \cdot |S_k([2q/5, 3q/5])|^{h-1} \geq \frac{q}{\exp(c(\log q)^{1/\ell})} \cdot \left( \frac{q}{5 \exp(c(\log q/5)^{1/\ell})} \right)^{h-1} \geq \frac{q^h}{\exp(c(\log q)^{1/\ell})(5 \exp(c(\log q/5)^{1/\ell}))^{h-1}} \geq \frac{N}{5^{h-1} q \exp(c(\log q)^{1/\ell})^h} \geq \frac{N}{q \exp(c'h(\log q)^{1/\ell})},$$

and in view of $h \leq \frac{\log N}{\log q}$ and our choice of $q$ we obtain that

$$|A| \geq \frac{20\varepsilon N}{\exp(c' \log N(\log q)^{1/\ell-1})} \geq N^{1-c_1 \log(1/\varepsilon)^{1/\ell-1}}$$

for sufficiently large $N$ and appropriate constant $c_1$ depending only on $k$. Therefore the set $A$ has the desired size. It remains to prove that $A$ is $AP_k(\varepsilon)$-free.

Suppose that there exists an $\varepsilon$-approximate arithmetic progression $X = \{x_0, \ldots, x_{k-1}\}$ in $A$. For each $0 \leq i \leq k - 1$, write $x_i = \sum_{j=0}^{h-1} x_{i,j} q^j$. Since all $x_i$'s are distinct, there exists a maximal index $j_0$ such that the elements of $X_{j_0} = \{x_{i,j_0} : 0 \leq i \leq k - 1\}$ are not all equal. By construction of $A$ the set $X_{j_0}$ fails to be an $AP_k$. Therefore there exists two indices $0 \leq i_1, i_2 \leq k - 2$ such that

$$|x_{i_1+1,j_0} - x_{i_1,j_0}| \neq |x_{i_2+1,j_0} - x_{i_2,j_0}|. \quad (13)$$
For $0 \leq i \leq k - 1$, note that
\[
|x_{i+1} - x_i| = \left| \sum_{j=0}^{j_0-1} (x_{i+1,j} - x_{i,j})q^j \right| = \left| \sum_{j=0}^{j_0} (x_{i+1,j} - x_{i,j})q^j \right|
\]
by the maximality of $j_0$. Thus by the triangle inequality we obtain that
\[
\left| x_{i+1} - x_i \right| - \left| x_{i+1,j_0} - x_{i,j_0} \right| q^{j_0} \leq \sum_{j=0}^{j_0-1} \left| x_{i+1,j} - x_{i,j} \right| q^j.
\tag{14}
\]
Moreover, recalling that $x_{i,j} \in [2q/5, 3q/5]$ for $0 \leq j \leq h - 2$ we infer that
\[
\sum_{j=0}^{j_0-1} \left| x_{i+1,j} - x_{i,j} \right| q^j \leq \sum_{j=0}^{j_0-1} \frac{q^{j+1}}{5} \leq \frac{2q^{j_0}}{5}
\]
for $q \geq 2$. The last inequality combined with (14) gives us that
\[
\left| x_{i+1} - x_i \right| - \left| x_{i+1,j_0} - x_{i,j_0} \right| q^{j_0} \leq \frac{2}{5} q^{j_0},
\tag{15}
\]
for $0 \leq i \leq k - 2$. Hence by (13) we have that
\[
\left| x_{i+1} - x_i \right| - \left| x_{i+1,j_0} - x_{i,j_0} \right| q^{j_0} \geq \left| x_{i+1,j_0} - x_{i,j_0} \right| q^{j_0} - \frac{4}{5} q^{j_0}
\]
\[
\geq q^{j_0} - \frac{4}{5} q^{j_0} = q^{j_0} \tag{16}
\]
On the other hand, Proposition 3.1 for $i_1$ and $i_2$ together with (15) gives us that
\[
\left| x_{i_2+1} - x_{i_2} \right| - \left| x_{i_1+1} - x_{i_1} \right| < 5 \varepsilon \left| x_{i_1+1} - x_{i_1} \right| < 5 \varepsilon \left( \left| x_{i_1+1,j_0} - x_{i_1,j_0} \right| + \frac{2}{5} \right) q^{j_0}.
\]
Since $x_{i,j_0} \in [0, q - 1]$ for every $0 \leq i \leq k - 1$ and $\varepsilon q = 1/25$ we have
\[
\left| x_{i_2+1} - x_{i_2} \right| - \left| x_{i_1+1} - x_{i_1} \right| < 5 \varepsilon q^{j_0+1} = \frac{q^{j_0}}{5},
\]
which contradicts (16). \qed

For higher dimensions the result follow as a corollary of Lemma 3.2. Recall by Definition 1.6 that an $\varepsilon$-approximate cube $C_\varepsilon(m, k)$ is just an multidimensional version of an $\text{AP}_k(\varepsilon)$

**Corollary 3.3.** Let $k \geq 3$ and $m \geq 1$ be integers and $0 < \varepsilon \leq 1/125$. Then there exists an integer $N_0 := N_0(k, \varepsilon)$ and a positive constant $c_1$ depending on $k$ such that the following holds. If $N \geq N_0$, then there exists a set $S \subseteq [N]^m$ without $C_\varepsilon(m, k)$ such that
\[
|S| \geq N^{m-c(\log(1/\varepsilon))^{\ell-1}}
\]
for $\ell = \lceil \log_2 (k - 1) \rceil$.

**Proof.** Let $N_0$ be the integer given by Lemma 3.2 and let $A \subseteq [N]$ be the set such that $A$ has no $\text{AP}_k(\varepsilon)$ for $N \geq N_0$. Set $S = A \times [N]^{m-1}$, i.e., $S = \{(s_1, \ldots, s_m) : s_i \in A, s_2, \ldots, s_m \in [N] \}$. 

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Note that $S$ has the desired size since

$$|S| = N^{m-1}|A| \geq N^{m-c(\log(1/\varepsilon))^{1-\varepsilon}}.$$ 

We claim that $S$ is free of $C_{\varepsilon}(m,k)$. 

Suppose that the claim is not true and let $X = \{x_{\bar{v}} : \bar{v} \in \{0, \ldots, k-1\}^m\}$ be an $C_{\varepsilon}(m,k)$ in $S$. By definition there exists $\bar{a} \in \mathbb{R}^m$ and $d > 0$ such that $||x_{\bar{v}} - (\bar{a} + d\bar{v})|| < \varepsilon d$ for every $\bar{v} \in \{0, \ldots, k-1\}^m$. In particular, when applied to $\{te_1 = (t,0, \ldots, 0) : 0 \leq t \leq k-1\}$ the observation gives us that

$$|x_{te_1} - (a_1 + td)| \leq \left( (x_{te_1,1} - (a_1 + dt))^2 + \sum_{i=2}^{m} (x_{te_1,i} - a_i)^2 \right)^{1/2} = ||x_{te_1} - (\bar{a} + dte_1)|| < \varepsilon d$$

Therefore, the set $\{x_{te_1}\}_{0 \leq t \leq k-1} \subseteq A$ is an $A_{m}(\varepsilon)$, which contradicts our choice of $A$. \qed

3.2. Upper bound. As in the upper bound of $W_{\varepsilon}(k,r)$, our proof of the upper bound of $f_{\varepsilon}(N,m,k)$ will use an iterative blow-up construction. It is worth to point out that a similar proof was obtained independently by Dumitrescu in [4]. While both proofs use a blow-up construction, the author of [4] finishes the proof with a packing argument. Here we will follow the approach of [16, 17], which uses an iterative blow-up construction combined with an average argument to estimate the largest subset of a grid without a class of configurations of a given size. This approach allows us to slightly improve the constants in the result.

The proof is split into two auxiliary lemmas.

Lemma 3.4. Given positive real numbers $\alpha, \varepsilon > 0$ and integers $m \geq 1$ and $k \geq 3$, there exists $N_0 := N_0(\alpha, \varepsilon, m, k) \leq (k\sqrt{m/\varepsilon})^{2k^m\log(1/\alpha)}$ and a subset $A \subseteq [N]^m$ with the property that any $X \subseteq A$, $|X| \geq \alpha|A|$ contains a $C_{\varepsilon}(m,k)$.

Proof. For $m$ and $k$, let $\Delta$ be the standard cube $C(m,k)$ of dimension $m$ over $\{0, \ldots, k-1\}$, i.e., $\Delta$ is the set of all $m$-tuples $\bar{v} = \{v_1, \ldots, v_m\} \in \{0, \ldots, k-1\}^m$. Viewing $\Delta$ as an $m$-dimensional lattice in the Euclidean space, we note that $\text{diam}(\Delta) = (k-1)\sqrt{m}$, while the minimum distance between two vertices in $\Delta$ is one.

Similarly as in the proof of the upper bound of Theorem 1.3, we consider an iterated blow-up of the cube. For integers $r$ and $t = k\sqrt{m/\varepsilon}$, let $A_r$ be the following $r$-iterated blow-up of a cube

$$A_r = \left\{ \bar{v}_0 + t\bar{v}_1 + \ldots + t^{r-1}\bar{v}_{r-1} : \bar{v}_0, \ldots, \bar{v}_{r-1} \in \Delta, \ t = \frac{k\sqrt{m}}{\varepsilon} \right\}.$$ 

Alternatively, we can view $A_r$ as the product $\prod_{i=1}^{m} B_r^{(i)}$ of $m$ identical copies of

$$B_r = \left\{ b_0 + tb_1 + \ldots + t^{r-1}b_{r-1} : (b_0, \ldots, b_{r-1}) \in \{0, 1 \ldots, k-1\}^r, \ t = \frac{k\sqrt{m}}{\varepsilon} \right\},$$
an $r$-iterated blow-up of the standard $A_{P_k}$. Note by the construction that $|A_r| = k^m$. The next proposition shows that fixed $\alpha > 0$, for a sufficiently large $r$ any $\alpha$-proportion of $A_r$ will contain a $C_{\varepsilon}(m, k)$.

**Proposition 3.5.** Let $0 < \alpha < 1$ be a real number and $r$ a positive integer such that $\alpha > (k^{m-1} / k^m)^r$. Then every $X \subseteq A_r$ with $|X| \geq \alpha |A_r|$ contains a $C_{\varepsilon}(m, k)$.

**Proof.** The proof is by induction on $r$. If $r = 1$, then $A_1 = \Delta$ and $\alpha > (k^{m-1} / k^m)$. Let $X \subseteq A_1$ with $|X| \geq \alpha |A_1|$. Thus

$$|X| \geq \alpha |A_1| > \frac{k^m - 1}{k^m} \cdot k^m = k^m - 1,$$

which implies that $X = \Delta$. So $X$ contains a cube $C(k, m)$ and in particular an $\varepsilon$-approximate cube.

Now suppose that the proposition is true for $r - 1$ and we want to prove it for $r$. First, we partition $A_r$ into $\bigcup_{\bar{u} \in \Delta} A_{r, \bar{u}}$, where

$$A_{r, \bar{u}} = \left\{ \bar{v}_0 + t\bar{v}_1 + \ldots + t^{r-2}\bar{v}_{r-2} + t^{r-1}\bar{u} : \bar{v}_0, \ldots, \bar{v}_{r-2} \in \Delta, t = \frac{k\sqrt{m}}{\varepsilon} \right\}.$$

Note that by definition $A_{r, \bar{u}}$ is a translation of $A_{r-1}$ by $t^{r-1}\bar{u}$. In particular, this implies that $|A_{r, \bar{u}}| = k^{(r-1)m}$. Let $X \subseteq A_r$ with $|X| \geq \alpha |A_r|$ be given. We will distinguish two cases:

**Case 1:** $X \cap A_{r, \bar{u}} \neq \emptyset$ for all $\bar{u} \in \Delta$.

For each $\bar{u} \in \Delta$ choose an arbitrary vector $w(\bar{u}) \in X \cap A_{r, \bar{u}}$. We will observe that $\{w(\bar{u})\}_{\bar{u} \in \Delta}$ forms a $C_{\varepsilon}(m, k)$. To testify that, set $\bar{a} = (0, \ldots, 0)$ and $d = t^{r-1}$. Write $w(\bar{u}) = \sum_{i=0}^{r-2} t^i \bar{w}_i + t^{r-1}\bar{u}$ with $\bar{w}_i \in \Delta$. Thus, a computation shows that

$$\|w(\bar{u}) - (\bar{a} + d\bar{u})\| = \|w(\bar{u}) - t^{r-1}\bar{u}\| = \left\| \sum_{i=0}^{r-2} t^i \bar{w}_i \right\| \leq \sum_{i=0}^{r-2} t^i \|\bar{w}_i\|$$

for $\bar{w}_0, \ldots, \bar{w}_{r-2} \in \Delta$. Since $\text{diam}(\Delta) = (k - 1)\sqrt{m}$, it follows that

$$\|w(\bar{u}) - (\bar{a} + d\bar{u})\| \leq (k - 1)\sqrt{m} \left( \sum_{i=0}^{r-2} t^i \right) \leq kt^{r-2}\sqrt{m} < \varepsilon t^{r-1} = \varepsilon d,$$

by our choice of $t$. Since $\{w(\bar{u})\}_{\bar{u} \in \Delta} \subseteq X$, we conclude that $X$ contains an $C_{\varepsilon}(m, k)$.

**Case 2:** There exists $\bar{u}_0 \in \Delta$ with $X \cap A_{r, \bar{u}_0} = \emptyset$.

Since $|X| \geq \alpha |A_r|$ and $|\Delta| = k^m$, by an average argument there exists $\bar{u}_1 \in \Delta$ such that

$$|X \cap A_{r, \bar{u}_1}| \geq \alpha |A_r| = \frac{\alpha k^m |A_{r-1}|}{k^m - 1}.$$

Set $X' = X \cap A_{r, \bar{u}_1}$ and $\alpha' = \frac{\alpha k^m}{k^m - 1}$. Note that

$$\alpha' = \frac{\alpha k^m}{k^m - 1} > \left( \frac{k^m - 1}{k^m} \right)^r \cdot \frac{k^m}{k^m - 1} = \left( \frac{k^m - 1}{k^m} \right)^{r-1}.$$
Therefore, viewing $A_{r,u}$ as a copy of $A_{r-1}$ by the induction assumption we obtain that $X' \subseteq X$ contains an $C_\varepsilon(m,k)$. 

Let $r$ be the smallest integer such that $\left(\frac{k^{m-1}}{k^m}\right)^r < \alpha$ and set $A = A_r$. A computation shows that

$$r = \left\lceil \frac{\log(1/\alpha)}{\log \frac{k^m}{k^{m-1}}} \right\rceil < 2k^m \log(1/\alpha).$$

Therefore by Proposition 3.5 we have that any set $X \subseteq A$ with $|X| \geq \alpha |A|$ contains an $C_\varepsilon(m,k)$. Finally, by the construction of $A$ we have that $A \subseteq [N_0]^m$ for

$$N_0 \leq \text{diam}(B_r) + 1 = (k-1)(1 + t + \ldots + t^{r-1}) + 1 \leq kt^{r-1} \leq \left(\frac{k \sqrt{m}}{\varepsilon}\right)^{2k^m \log(1/\alpha)}.$$ 

Lemma 3.4 gives us a set $A \subseteq [N]^m$ such that any $\alpha$-proportion contains a $C_\varepsilon(m,k)$. However, this is still not good enough, since to obtain an upper bound we need a similar result for $[N]^m$. The next lemma shows by an average argument that the property of $A$ can be extended to $[N]^m$ by losing a factor of a power of two in the proportion $\alpha$.

**Lemma 3.6.** Let $A \subseteq [N]^m$ be a configuration in the grid. For any $X \subseteq [N]^m$ with $|X| \geq \alpha N^m$, there exists a translation $A'$ of $A$ such that $|X \cap A'| \geq \frac{\alpha}{2m}|A'|$.

**Proof.** Consider a random translation $A' = A + \bar{u}$, where $\bar{u} = (u_1, \ldots, u_m)$ is an integer vector chosen uniformly inside $[-N+1, N]^m$. For every vector $\bar{x} \in X$, there exists exactly $|A|$ elements $\bar{v} \in [-N+1, N]^m$ such that $\bar{x} - \bar{v} \in A$. This means that $P(\bar{x} \in A') = P(\bar{x} - \bar{v} \in A) = \frac{|A|}{(2N)^m}$. Therefore

$$E(|X \cap A'|) = \sum_{\bar{x} \in X} P(\bar{x} \in A') = \frac{|X||A|}{(2N)^m} \geq \frac{\alpha}{2m}|A| = \frac{\alpha}{2m}|A'|$$

Consequently, by the first moment method, there is $\bar{u}$ and $A'$ satisfying our conclusion. 

We finish the section putting everything together.

**Proposition 3.7.** Let $N$, $m$ and $k$ be integers and $\varepsilon > 0$. Then there exists a positive constant $c_2$ depending only on $k$ and $m$ such that the following holds. If $S \subseteq [N]^m$ is such that

$$|S| > N^{m-c_2(\log(1/\varepsilon))^{-1}},$$

then $S$ contains an $C_\varepsilon(m,k)$.

**Proof.** Set $\alpha_0 = 2^m N^{c'(\log(1/\varepsilon))^{-1}}$ where $c' = (4k^m \log(k \sqrt{m}))^{-1}$. Let $N_0 = N_0(\alpha_0/2^m, \varepsilon, m, k)$ be the integer obtained by Lemma 3.4 and $A \subseteq [N_0]$ be the set such that any $X \subseteq A$
with \(|X| \geq \frac{2m}{\alpha_0}|A|\) contains an \(C_\varepsilon(m,k)\). Note that
\[
N_0 \leq \left(\frac{k\sqrt{m}}{\varepsilon}\right)^{2k^m \log(2^m/\alpha_0)} = \exp\left(\frac{2c'k^m \log N \log(k\sqrt{m}/\varepsilon)}{\log(1/\varepsilon)}\right)
\leq \exp\left(4c'k^m \log N \log(k\sqrt{m})\right) = N,
\]
which implies that \(A \subseteq [N]\).

Let \(S \subseteq [N]\) with \(|S| \geq \alpha_0 N^m\). Then by Lemma 3.6, there exists a translation \(A'\) of \(A\) such that \(|S \cap A'| \geq \frac{\alpha_0}{2m}|A'|\). Hence, by Lemma 3.4, the set \(S\) contains a \(C_\varepsilon(m,k)\). The result now follows since
\[
|S| \geq \alpha_0 N^m = 2^m N^{m-c'(\log(1/\varepsilon))^{-1}} > N^{m-c_2(\log(1/\varepsilon))^{-1}}
\]
for appropriate \(c_2\).

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