Wilf Equivalence for the Charge Statistic

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Abstract

Savage and Sagan have recently defined a notion of $st$-Wilf equivalence for any permutation statistic $st$ and any two sets of permutations $\Pi$ and $\Pi'$. In this paper we give a thorough investigation of $st$-Wilf equivalence for the charge statistic on permutations and use a bijection between the charge statistic and the major index to prove a conjecture of Dokos, Dwyer, Johnson, Sagan and Selsor regarding powers of 2 and the major index.

1 Background

Let $S_n$ be the symmetric group of all permutations of the set $[n] = \{1, 2, \ldots, n\}$ and suppose $\pi = a_1a_2\cdots a_n$ and $\sigma = b_1b_2\cdots b_n$ are two permutations in $S_n$. We say that $\pi$ is order isomorphic to $\sigma$ if $a_i < a_j$ if and only if $b_i < b_j$. For any $\pi \in S_n$ and $\sigma \in S_k$ for $k \leq n$, we say that $\pi$ contains a copy of $\sigma$ if $\pi$ has a subsequence that is order isomorphic to $\sigma$. If $\pi$ contains no subsequence order isomorphic to $\sigma$ then we say that $\pi$ avoids $\sigma$.

Now let $\Pi$ be a subset of permutations in $S_n$ and define $Av_n(\Pi)$ as the set of permutations in $S_n$ which avoid every permutation in $\Pi$. Two sets of permutations $\Pi$ and $\Pi'$ are said to be Wilf equivalent if $|Av_n(\Pi)| = |Av_n(\Pi')|$. If $\Pi$ and $\Pi'$ are Wilf equivalent, we write $\Pi \equiv \Pi'$.

Savage and Sagan [4] defined a $q$-analogue of Wilf equivalence by considering any permutation statistic $st$ from $S_n \to N$, where $N$ is the set of nonnegative integers, and letting

$$F_n^{st}(\Pi; q) = \sum_{\sigma \in Av_n(\Pi)} q^{st(\sigma)}.$$
They defined $\Pi$ and $\Pi'$ to be \textit{st-Wilf equivalent} if $F_n^{st}(\Pi; q) = F_n^{st}(\Pi'; q)$ for all $n \geq 0$. In this case, we write $\Pi \equiv^{st} \Pi'$. We will use $[\Pi]_n^{st}$ to denote the st-Wilf equivalence class of $\Pi$. If we set $q = 1$ in the generating function above we have $F_n^{st}(\Pi; 1) = |Av_n(\Pi)|$, thus st-Wilf equivalence implies Wilf equivalence.

In [1], Dokos, Dwyer, Johnson, Sagan and Selsor give a thorough investigation of st-Wilf equivalence for both the major index and the inversion statistic. Our goal in this paper is to give a similarly thorough investigation for another well known Mahonian statistic, the charge statistic. In Section 2, we give the necessary definitions for the material covered and in Section 3 we discuss charge Wilf equivalence for subsets $\Pi \in S_3$. In Section 4 we state and prove a conjecture of Dokos, Dwyer, Johnson, Sagan and Selsor by showing it is equivalent to a similar statement for the charge statistic. Our proof of this result uses some basic facts about standard Young tableaux and the Robinson-Schensted correspondence. We close the paper with some directions for further research.

2 Definitions

Throughout this paper we will utilize some basic operations on permutations, namely the \textit{inverse}, the \textit{reverse} and the \textit{complement}. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, the inverse is the standard inverse operation on permutations, the reverse is

$$\pi^r = \pi_n \cdots \pi_2 \pi_1$$

and the complement is

$$\pi^c = n + 1 - \pi_1 \ n + 1 - \pi_2 \ \cdots \ n + 1 - \pi_n.$$ 

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, define the \textit{descent set} of $\pi$ to be $\text{Des}(\pi) = \{i | \pi_i > \pi_{i+1}\}$. The \textit{major index} of a permutation, first defined by MacMahon [3], is then defined as

$$maj(\pi) = \sum_{i \in \text{Des}(\pi)} i.$$

For example, for $\pi = 3 \ 2 \ 8 \ 5 \ 7 \ 4 \ 6 \ 1 \ 9$, $\text{Des}(\pi) = \{1, 3, 5, 7\}$ and $maj(\pi) = 1 + 3 + 5 + 7 = 16$. 

2
Let $\pi$ be a permutation in $S_n$. For any $i$ in the permutation, define the charge value of $i$, $\text{chv}(i)$, recursively as follows:

\begin{align*}
\text{chv}(1) &= 0 \\
\text{chv}(i) &= 0 \text{ if } i \text{ is to the right of } i - 1 \text{ in } \pi \\
\text{chv}(i) &= n + 1 - i \text{ if } i \text{ is to the left of } i - 1 \text{ in } \pi
\end{align*}

Now for $\pi \in S_n$, define the charge of $\pi$, $\text{ch}(\pi)$, to be

$$
\text{ch}(\pi) = \sum_{i=1}^{n} \text{chv}(i).
$$

In the following example for $\pi = 3 \ 2 \ 8 \ 5 \ 7 \ 4 \ 6 \ 1 \ 9$, the charge values of each element are given below the permutation:

$$
\begin{array}{cccccccccccc}
\pi &=& 3 & 2 & 8 & 5 & 7 & 4 & 6 & 1 & 9 \\
\text{ch}(\pi) &=& 7 & 8 & 2 & 5 & 3 & 0 & 0 & 0 & 0
\end{array}
$$

end $\text{ch}(\pi) = 7 + 8 + 2 + 5 + 3 = 25$. The definition of the charge statistic was first given by Lascoux and Schützenberger [2].

Define

$$
\text{Ch}_n(\Pi; q) = F_n^{\text{ch}}(\Pi; q) = \sum_{\sigma \in A_{\text{Av}}(\Pi)} q^{\text{ch}(\sigma)}.
$$

### 3 Equivalence for permutations in $S_3$

In this section, we will consider the polynomials $\text{Ch}_n(\Pi; q)$ where $\Pi \subseteq S_3$. To begin, fix $n \geq 0$ and let $\pi \in S_n$. Define $f(\pi) = ((\pi^r)^c)^{-1}$. It is well known that each of the operations of reverse, complement and inverse are bijections on $S_n$ so $f$ is a bijection from $S_n$ to $S_n$.

**Lemma 1.** Fix $n \geq 0$ and let $\pi \in S_n$. Then $\text{maj}(\pi) = \text{ch}(f(\pi))$.

**Proof.** We will show that if there is no descent in position $i$ in $\pi$ then $n+1-i$ has a charge value of 0 in $f(\pi)$ and if there is a descent in position $i$ in $\pi$ then the charge value of $n+1-i$ in $f(\pi)$ is $(n+1) - (n+1-i) = i$.

If there is no descent in position $i$ in $\pi$, then $\pi_i < \pi_{i+1}$. Let $\pi_i = j$ and $\pi_{i+1} = k$, so $j < k$. When we apply the reverse operation to $\pi$, we obtain
\(\pi_{n-i} = k\) and \(\pi_{n+1-i} = j\). Then \((\pi_{n-i})^c = n+1-k\) and \((\pi_{n-i+1})^c = n+1-j\) and since \(j < k\), \(n+1-k < n+1-j\). When we apply the inverse operation we have \(\pi_{n+1-k} = n-i\) and \(\pi_{n+1-j} = n+1-i\), thus \(n+1-i\) is to the right of \(n-i\) in \(f(\pi)\) so \(n+1-i\) has a charge value of 0.

If there is a descent in position \(i\) in \(\pi\) then \(\pi_i > \pi_{i+1}\). Let \(\pi_i = j\) and \(\pi_{i+1} = k\), so \(j > k\). When we apply the reverse operation, we obtain \(\pi_{n-i} = k\) and \(\pi_{n+1-i} = j\). Then \((\pi_{n-i})^c = n+1-k\) and \((\pi_{n+1-i})^c = n+1-j\) and since \(j > k\), \(n+1-k > n+1-j\). When we apply the inverse operation, we have \(\pi_{n+1-k} = n-i\) and \(\pi_{n+1-j} = n+1-i\), thus \(n+1-i\) is to the left of \(n-i\) in \(f(\pi)\) so \(n+1-i\) has a charge value of \((n+1) = (n+1-i) = i\).

Lemma 2. Fix \(n \geq 0\). Then

\[
\begin{align*}
  f : Av_n(123) &\to Av_n(123) \\
  f : Av_n(132) &\to Av_n(213) \\
  f : Av_n(213) &\to Av_n(132) \\
  f : Av_n(231) &\to Av_n(231) \\
  f : Av_n(312) &\to Av_n(312) \\
  f : Av_n(321) &\to Av_n(321).
\end{align*}
\]

Proof. We will instead prove that for a fixed \(n \geq 0\),

\[
\begin{align*}
  f : Av_n(123)^C &\to Av_n(123)^C \\
  f : Av_n(132)^C &\to Av_n(213)^C \\
  f : Av_n(213)^C &\to Av_n(132)^C \\
  f : Av_n(231)^C &\to Av_n(231)^C \\
  f : Av_n(312)^C &\to Av_n(312)^C \\
  f : Av_n(321)^C &\to Av_n(321)^C.
\end{align*}
\]

Suppose \(\pi \in S_n\) contains a 123-pattern. Then there exists an \(i < j < k\) such that \(\pi_i = a\), \(\pi_j = b\) and \(\pi_k = c\) with \(a < b < c\). Then in \(\pi^r\), \(\pi_{n+1-k} = c\), \(\pi_{n+1-j} = b\) and \(\pi_{n+1-i} = a\). In \((\pi^r)^c\) we have \(\pi_{n+1-k} = n+1-c\), \(\pi_{n+1-j} = n+1-b\) and \(\pi_{n+1-i} = n+1-a\) where \(n+1-c < n+1-b < n+1-a\). Finally, in \(((\pi^r)^c)^{-1}\) we have \(\pi_{n+1-c} = n+1-k\), \(\pi_{n+1-b} = n+1-j\) and \(\pi_{n+1-a} = n+1-i\). Since \(i < j < k\), \(n+1-k < n+1-j < n+1-k\) thus \(\pi_{n+1-c}\), \(\pi_{n+1-b}\) and \(\pi_{n+1-a}\) form a \((123)\)-pattern in \(f(\pi)\). The proofs of the other bijections are similar and are left to the reader.

\[\square\]
We now prove the following result:

**Theorem 1.** We have

\[
\begin{align*}
[123]_{ch} &= \{123\} \\
[321]_{ch} &= \{321\} \\
[312]_{ch} &= \{312, 132\} = [132]_{ch} \\
[213]_{ch} &= \{213, 231\} = [231]_{ch}.
\end{align*}
\]

**Proof.** Dokos, Dwyer, Johnson, Sagan and Selsor [1] proved

\[
\begin{align*}
[123]_{maj} &= \{123\} \\
[321]_{maj} &= \{321\} \\
[132]_{maj} &= \{132, 231\} = [231]_{maj} \\
[213]_{maj} &= \{213, 312\} = [312]_{maj}.
\end{align*}
\]

Recall that

\[
[123]_{maj} = \{\pi \in S_3| F_{n}^{maj}(123; q) = F_{n}^{maj}(\pi; q)\}
\]

or

\[
[123]_{maj} = \{\pi \in S_3| \sum_{\sigma \in Av_n(123)} q^{maj}(\sigma) = \sum_{\sigma \in Av_n(\pi)} q^{maj}(\sigma)\}.
\]

Since the function \( f \) defined above takes \( Av_n(123) \) to \( Av_n(123) \) and takes the major index to the charge statistic, we can apply the function \( f \) to the equation above to obtain \( [123]_{ch} = \{123\} \). Similarly, by Lemma 2 we have that \( f \) takes \( Av_n(321) \) to \( Av_n(321) \), thus \( [321]_{ch} = \{321\} \). Since \( f \) takes \( Av_n(213) \) to \( Av_n(132) \) and \( Av_n(312) \) to \( Av_n(312) \) we have \( [132]_{ch} = [312]_{ch} = \{312, 132\} \). Finally, since \( f \) takes \( Av_n(231) \) to \( Av_n(231) \) and \( Av_n(132) \) to \( Av_n(213) \) we have \( [213]_{ch} = [231]_{ch} = \{213, 231\} \).

\( \square \)

Utilizing the same function \( f \) and results of [1], we can obtain the following results for larger subsets of \( S_3 \).

**Theorem 2.** For \( \Pi \in S_3 \) with \( |\Pi| = 2 \) and \( \Pi \neq \{123, 321\} \), we have

\[
[132, 213]_{ch} = \{\{132, 213\}, \{213, 312\}, \{132, 231\}, \{231, 312\}\}.
\]

All other ch-Wilf equivalence classes under the given conditions contain a single pair.
Proof. Dokos, Dwyer, Johnson, Sagan and Selsor prove the following theorem (Theorem 5.1) in [1]:

\begin{theorem}
We have
\[
[132, 213]_{maj} = \{\{132, 213\}, \{132, 312\}, \{213, 231\}, \{231, 312\}\}.
\]
All other maj-Wilf equivalence classes for \( \Pi \in S_3 \) with \( |\Pi| = 2 \) and \( \Pi \neq \{123, 321\} \) contain a single pair.

Applying our function \( f \) to this result gives our theorem for the charge statistic.
\end{theorem}

Dokos, Dwyer, Johnson, Sagan and Selsor go on to classify the maj-Wilf equivalence classes for all subsets of \( S_3 \) and one can translate these results into equivalent statements for the charge statistic utilizing the function \( f \) if desired.

4 A Conjecture of Dokos, Dwyer, Johnson, Sagan and Selsor

In their paper defining \( st \)-Wilf equivalence, Dokos, Dwyer, Johnson, Sagan and Selsor [1] state the following conjecture (Conjecture 3.6):

\begin{conjecture}
For all \( k \geq 0 \) we have
\[
< q^i > M_{2k-1}(321; q) = \begin{cases} 
1 & \text{if } i = 0, \\
n \text{an even number} & \text{if } i \geq 1.
\end{cases}
\]
\end{conjecture}

We will prove the analogous statement for the charge statistic in this section.

Before proving this theorem, we need some preliminary results and definitions. It is a well-known result of the Robinson-Schensted correspondence that if two permutations have the same \( P \) tableau then the charge statistic on those two permutations is the same. In addition, one can easily show that a permutation is 321-avoiding if and only if the \( P \) tableau contains at most 2 rows. There is only one tableau with 1 row, namely \( P = 1 2 3 \cdots n \) and this tableau corresponds with the permutation \( \pi = 1 2 3 \cdots n \) which has a charge value of zero.
We will now inductively define a bijection $\phi_{2^{k-1}}$ from the set of 2-row tableau of size $2^k - 1$ to the set of 2-row tableau of size $2^k - 1$. If $k = 1$, there are no 2-row tableau of size 1 so we begin with $k = 2$. There are two tableau of size $2^2 - 1 = 3$ and we define $\phi_3$ as:

\[
\phi_3 \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}
\]

Now suppose that $\phi_{2^{k-1}-1}$ is a bijection from the set of 2-row tableau of size $2^{k-1} - 1$ to the set of 2-row tableau of size $2^{k-1} - 1$. Define $\phi_{2^{k}-1}$ as follows:

Let $T$ be a 2-row tableau of size $2^k - 1$ and let $S$ be the portion of $T$ containing the numbers 1, 2, \ldots, $2^{k-1} - 1$.

**Case 1:** If $S$ is a 2-row tableau then define $\phi_{2^{k-1}}(T)$ as the tableau $T$ with $S$ replaced by $\phi_{2^{k-1}-1}(S)$. Since $\phi_{2^{k-1}-1}$ is a bijection then $\phi_{2^{k}-1}$ is a bijection on this set of tableau.

For example,

**Example 1.** Let $k = 4$ and

$$T = \begin{array}{cccccccc}
1 & 2 & 4 & 5 & 7 & 8 & 9 & 13 & 15 \\
3 & 6 & 10 & 11 & 12 & 14 \\
\end{array}$$

Then

$$S = \begin{array}{cccc}
1 & 2 & 4 & 5 & 7 \\
3 & 6 \\
\end{array},$$

and if

$$\phi(S) = \begin{array}{cccc}
1 & 3 & 4 & 5 & 7 \\
2 & 6 \\
\end{array}$$

then

$$\phi_{2^4-1}(T) = \begin{array}{cccccccc}
1 & 3 & 4 & 5 & 7 & 8 & 9 & 13 & 15 \\
2 & 6 & 10 & 11 & 12 & 14 \\
\end{array}.$$
Form $\phi_{2k-1}(T)$ by swapping the positions of $2k$, $2k + 1$, $\ldots$, $2k + l$ and $2k + l + 1$, $2k + l + 2$, $\ldots$, $2k + 2l$ in $T$. Clearly this is a bijection on this set of tableaux. For example,

**Example 2.** Let $k = 4$ and let

$$T = \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 14 \\
10 & 12 & 13 & 15
\end{array}.$$  

Then

$$\phi_{2^4-1}(T) = \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 14 \\
11 & 12 & 13 & 15
\end{array}.$$  

Now we can move on to some necessary preliminary results.

**Lemma 3.** For all $k \geq 1$, there are an odd number of 321-avoiding permutations in $S_{2k-1}$.

**Proof.** Every 321-avoiding permutation corresponds to a pair of tableau with 1 or 2 rows. There is only one pair of 1-row tableau, namely $(P, P)$ where $P = 1 \ 2 \ 3 \ \cdots \ 2k - 1$. The number of pairs of 2-row tableau is $m^2$ where $m$ is the number of 2-row tableau of size $2k - 1$. Since $\phi_{2k-1}$ is a bijection on the set of 2-row tableau of size $2k - 1$ with $\phi_{2k-1}(T) \neq T$ for any $T$ (by definition) and $\phi_{2k-1} (\phi_{2k-1}(T)) = T$, then $m$ is even and thus $m^2$ is even.

**Corollary 1.** There are an even number of 2-row tableau of size $2^k - 1$.

**Theorem 4.** For all $k \geq 0$ we have

$$< q^i > Ch_{2^k-1} (321; q) = \begin{cases} 
1 & \text{if } i = 0, \\
\text{an even number} & \text{if } i \geq 1. 
\end{cases}$$

**Proof.** Since any inversion in a permutation $\pi$ introduces a charge value, the only permutation in $S_{2k-1}$ with a charge value of zero is $\pi = 1 \ 2 \ 3 \ \cdots \ n$, which corresponds to the only 1-row tableau of size $n$. Thus all other permutations in $Av(321)$ correspond to pairs of 2-row tableau under the Robinson-Schensted correspondence. By Corollary, there are $m$ 2-row tableau of size $2^k - 1$ and $m$ is even.

Choose an ordered pair of distinct 2-row tableaux of size $2^k - 1$, say $A$ and $B$. This can be done in $m(m-1)$ ways so there are an even number of ordered
pairs \((A, B)\) where \(A \neq B\). Then since any two permutations which give rise to the same \(P\) tableau under the Robinson-Schensted correspondence have the same charge, the permutations corresponding to \((A, A)\) and \((A, B)\) have the same charge.

Thus the set of pairs of 2-row tableau of size \(2^k - 1\) can be partitioned into pairs of pairs that give rise to a pair of permutations with the same charge. This gives our result.

We now obtain the conjecture of Dokos, Dwyer, Johnson, Sagan and Selsor as a Corollary.

**Corollary 2.** For all \(k \geq 0\) we have

\[
< q^i > M_{2^k - 1}(321; q) = \begin{cases} 
1 & \text{if } i = 0, \\
\text{an even number} & \text{if } i \geq 1.
\end{cases}
\]

*Proof.* The function \(f\) defined in Section 3 is a bijection from \(Av(321)\) to \(Av(321)\) that takes the major index to the charge statistic, thus applying \(f\) to this set gives the result.

**References**

[1] Dokos, Dwyer, Johnson, Sagan and Selsor, Permutation patterns and statistics. Preprint [arXiv:1109.4976](http://arxiv.org/abs/1109.4976).

[2] Lascoux, A. and Schutzenberger, M.P., Sur une conjecture do H.O. Foulkes, *C.R. Acad. Sc. Paris*, 286A (1978) 323-324.

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[4] Bruce E. Sagan and Carla D. Savage, Mahonian pairs. Preprint [arXiv:1101.4332](http://arxiv.org/abs/1101.4332).