A NEW REALISATION OF THE $i$-QUANTUM GROUP $U^j(n)$

JIE DU AND YADI WU

Abstract. We follow the approach developed in [BLM90] and modified in [DF10] to investigate a new realisation for the $i$-quantum groups $U^j(n)$, building on the multiplication formulas discovered in [BKLW18, Lem. 3.2]. This allows us to present $U^j(n)$ via a basis and multiplication formulas by generators. We also establish a surjective algebra homomorphism from a Lusztig type form of $U^j(n)$ to integral $q$-Schur algebras of type $B$. Thus, base changes allow us to relate representations of the $i$-quantum hyperalgebras of $U^j(n)$ to representations of finite orthogonal groups of odd degree in non-defining characteristics. This generalises part of Dipper–James’ type $A$ theory to the type $B$ case.

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1. Introduction

Arising from representations of finite groups of Lie type, Iwahori–Hecke algebras play an important role in the study of unipotent principal blocks. In late 1980s, Dipper–James [DJ89] introduced $q$-Schur algebras to study the representations of finite general linear groups in non-defining characteristics. In an entirely different context, these algebras appeared earlier in the study of the Schur–Weyl duality for a quantum linear group or quantum $\mathfrak{gl}_n$. Thus, $q$-Schur algebras naturally link representations of quantum $\mathfrak{gl}_n$ with those of finite general linear groups. It is natural to expect that such a connection extends to finite orthogonal and symplectic groups. However, example calculations on characters showed that this is not the case. In fact, from the invariant theory point of view, it is the Brauer algebra (or the BMW-algebra in the quantum case), instead of a group (or Hecke) algebra, that is involved in the Schur–Weyl duality.

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Recently, in their study of canonical bases for quantum symmetric pairs, Bao and Wang [BW18] introduced certain co-ideal subalgebras $U^i$ and $U^i$ of quantum linear groups, whose corresponding quantum symmetric pairs in [Le03] are of type AIII. We follow [CLW] to call them $i$-quantum groups. Bao–Wang further proved that these $i$-quantum groups pair with the Hecke algebras of type $B$ in a Schur–Weyl duality, where two types of Hecke endomorphism algebras or $q$-Schur algebras play the bridging role. More interestingly, as revealed in [BKLW18], these $q$-Schur algebras arise naturally from finite orthogonal groups of odd degree or finite symplectic groups.

With an entirely different motivation, the structure and representations of general Hecke endomorphism algebras were investigated by B. Parshall, L. Scott, and the first author over twenty years ago. They made a stratification conjecture about their structure. A slightly modified version of this conjecture has recently been proved in [DPS] and applications to representations of finite groups of Lie type have been obtained. In order to extend these applications to $i$-quantum groups, we need to lift Bao-Wang’s duality to the integral level, and hence, to the roots-of-unity level. This is the aim of the current paper.

Building on the work of Bao et al [BKLW18], we will define an algebra homomorphism $\phi^i$ from the $i$-quantum groups $U^i(n)$ into the direct product of the corresponding $q$-Schur algebras $S^i(n,r)_{Q(\nu)}$ of type $B$. We mainly focus on the determination of the homomorphic image of $\phi^i$ in terms of a BLM basis $\{A(j)\}_{A^i}$ (cf. [BLM90]). This then allows us to investigate the Lusztig type forms and their associated hyperalgebras for the $i$-quantum groups through a certain monomial basis. We thus establish an algebra epimorphism from the integral $i$-quantum group to the integral $q$-Schur algebras of type $B$. Note that, with this epimorphism, the representation category of $i$-$q$-Schur algebras becomes a full subcategory of that of the $i$-quantum group (or $i$-quantum hyperalgebra). We further prove that the map $\phi^i$ is injective. This gives a new realisation of the $i$-quantum group $U^i(n)$ in terms of the basis $\{A(j)\}_{A^i}$ together with explicit multiplication formulas by generators.

Just like the original realisation by Beilinson–Lusztig–MacPherson for quantum $gl_n$, this work laid down a foundation for a further study of the above mentioned $i$-quantum hyperalgebra. On the other hand, it should be plausible that a similar realisation can be obtained for $i$-quantum groups $U^i(n)$ although the triangular relation between two bases is a bit subtle in this case. We will get these done in a forthcoming paper.

We organise the paper as follows. We first review in §2 the definition of $i$-quantum groups $U^i(n)$ and their realisation as coideal subalgebra of the quantum linear group $U(gl_{2r+1})$. In particular, the natural representation $\Omega$ of $U(gl_{2r+1})$ and its tensor product $\Omega^{\otimes r}$ restrict to become $U^i(n)$-modules. In §3, we introduce $q$-Schur algebra of type $B$ through finite orthogonal groups $O_{2r+1}$ as well as (Iwahori–)Hecke algebras of type $B_r$. We will also review the Schur–Weyl duality discovered by Bao–Wang [BW18]. Section 4 is quite parallel to [BLM90] §5.2–§5.4. Building on [BKLW18] Lem. 3.2, we derive certain multiplication formulas in $S^i(n,r)_{Q(\nu)}$. The structure constants in these formulas are independent of $r$. This allows to extend these formulas to get similar formulas in the direct product $S^i(n)_{Q(\nu)}$ of $S^i(n,r)_{Q(\nu)}$ (Theorem 4.4). Using a triangular relation in [BKLW18] Thm. 3.10, we prove in §5 that the subspace $A^i(n)$ spanned by all $A(j)$ is a subalgebra of $S^i(n)_{Q(\nu)}$ (Theorem 5.2). The above mentioned homomorphism $\phi^i$ has the image $A^i(n)$ (Theorem 6.1). Using $\phi^i$, we then lift Bao–Wang’s epimorphism to the integral level (Theorem 6.5). Finally, in the last section, we prove that $\phi^i$ is injective and thus, we establish the new realisation (Theorem 7.1).

Some notations. For a positive integer $a$, let

$$[1,a] = \{1, 2, \ldots, a\}, \quad [1,a] = \{1, 2, \ldots, a - 1\}.$$
Let $\mathbb{Z} = \mathbb{Z}[v, v^{-1}]$ be the integral Laurent polynomial ring. For $n > 0$, we set

$$[n] = \frac{v^{2n} - 1}{v^2 - 1}, \quad [n] = \frac{v^n - v^{-n}}{v - v^{-1}} \quad \text{and} \quad [n]! = [1][2] \ldots [n].$$

Set $[0] = [0] = 0$ and $[0]^2 = 1$. We also define, for $s, t \in \mathbb{Z}$ with $t > 0$,

$$\left[ \frac{s}{t} \right] = \prod_{i=1}^{t} \frac{v^{s-i+1} - v^{-(s-i+1)}}{v^i - v^{-i}}, \quad \left[ K; \frac{s}{t} \right] = \prod_{i=1}^{t} \frac{Kv^{s-i+1} - K^{-1}v^{-(s-i+1)}}{v^i - v^{-i}},$$

where $K$ is an element in a $\mathbb{Q}(v)$-algebra.

2. The $i$-quantum group $U^j(n)$: A first realisation

In the study of quantum symmetric pairs, Bao and Wang introduced the following quantum algebra $U^j(n)$, extracted from a quantum symmetric pair of type AIII in [Le03, §7]. Here we follow the definition from [BLW13, §4.3] and the $n$ indicates the rank of the $i$-quantum group.

**Definition 2.1.** The algebra $U^j(n)$ is defined to be the associative algebra over $\mathbb{Q}(v)$ generated by $e_i, f_i, d_a, d_a^{-1}, a \in [1, n], a \in [1, n+1]$ subject to the following relations: for $i, j \in [1, n], a, b \in [1, n+1],$

(iQG1) $d_a d_a^{-1} = d_a^{-1} d_a = 1, d_a d_b = d_b d_a$;

(iQG2) $d_a e_i d_a^{-1} = v^{\delta_{b,i} - \delta_{a,j} + 1} e_i, d_a f_j d_a^{-1} = v^{-\delta_{b,j} + \delta_{a,j} + 1} f_j$, if $a \leq n$;

$$d_{n+1} e_j d_{n+1}^{-1} = v^{-2\delta_{a,j} + \delta_{b,j} + 1} e_j, d_{n+1} f_j d_{n+1}^{-1} = v^{2\delta_{a,j} - 1} f_j;$$

(iQG3) $e_i f_j - f_j e_i = \delta_{i,j} \frac{d_{n+1}^{-1} - d_{n+1}}{v - v^{-1}}$, if $i \neq j$;

(iQG4) $e_i e_j = e_j e_i, f_i f_j = f_j f_i$, if $|i - j| > 1$;

(iQG5) $e_i^2 e_j + e_j e_i^2 = [2] e_i e_j e_i, f_i^2 f_j + f_j f_i^2 = [2] f_i f_j f_i$, if $|i - j| = 1$;

(iQG6) $f_a^2 e_n + e_n f_a^2 = [2] (f_a e_n e_n - (v d_n d_n^{-1} + v^{-1} d_n^{-1} d_{n+1}) f_n), e_n f_a^2 + f_n e_n^2 = [2] (e_n f_a e_n - e_n (v d_n d_n^{-1} + v^{-1} d_n^{-1} d_{n+1})).$

Note that the subalgebra generated by $e_i, f_i, d_a, d_a^{-1}, i \in [1, n], a \in [1, n]$ is isomorphic to the quantum linear group $U(\mathfrak{gl}_n)$ in the following sense.

**Definition 2.2.** The quantum linear group is a Hopf algebra $U(\mathfrak{gl}_N)$ over $\mathbb{Q}(v)$ with generators

$$E_a, F_a, K_j^{\pm 1}, a \in [1, N], j \in [1, N],$$

and relations:

(QG1) $K_i K_j = K_j K_i, \quad K_i K_j^{-1} = K_j^{-1} K_i = 1$;

(QG2) $E_a F_b = v^{\delta_{a,b} - \delta_{a,b} + 1} E_b K_j, \quad K_j F_b = v^{-\delta_{a,b} + \delta_{a,b} + 1} F_b K_j$;

(QG3) $[E_a, F_b] = \delta_{a,b} \frac{K_a - K_a^{-1}}{v - v^{-1}}$, where $K_a = K_a K_a^{-1};$

(QG4) $E_a E_b = E_b E_a, \quad F_a F_b = F_b F_a$, if $|a - b| > 1$;

(QG5) For $a, b \in [1, N]$ with $|a - b| = 1$,

$$E_a^2 E_b - (v + v^{-1}) E_a E_b E_a + E_b E_a^2 = 0,$$

$$F_a^2 F_b - (v + v^{-1}) F_a F_b F_a + F_b F_a^2 = 0.$$
The first realisation of $\mathbf{U}(n)$ is to embed it into the quantum group $\mathbf{U}(\mathfrak{gl}_{2n+1})$; see [BKLW18, Prop. 4.5]. \footnote{See also [BW18, Prop. 6.2] for the first version.}

**Lemma 2.3.** There is an injective $\mathbb{Q}(v)$-algebra homomorphism $\iota : \mathbf{U}(n) \to \mathbf{U}(\mathfrak{gl}_{2n+1})$ defined, for $i \in [1, n]$, by

\[
\begin{align*}
    d_i & \mapsto K_i^{-1} K_{2n+2-i}^{-1}, \\
    d_{n+1} & \mapsto v^{-1} K_{n+1}^{-2}, \\
    e_i & \mapsto F_i + \tilde{K}_i^{-1} E_{2n+1-i}, \\
    f_i & \mapsto E_i \tilde{K}_{2n+1-i} + F_{2n+1-i}.
\end{align*}
\]

Moreover, relative to the coalgebra structure, $\iota(\mathbf{U}(n))$ is a coideal of $\mathbf{U}(\mathfrak{gl}_{2n+1})$.

Let $\Omega = \Omega_{2n+1}$ be the natural representation of $\mathbf{U}(\mathfrak{gl}_{2n+1})$ with a $\mathbb{Q}(v)$-basis $\{\omega_1, \ldots, \omega_{2n+1}\}$ via the following actions:

\[
E_h \omega_i = \delta_{i,h+1} \omega_h, \quad F_h \omega_i = \delta_{i,h} \omega_{h+1}, \quad K_j \omega_i = v^{\delta_{i,j}} \omega_i.
\] (2.3.1)

Then $\Omega^r$ becomes a $\mathbf{U}(\mathfrak{gl}_{2n+1})$-module via the actions:

\[
\begin{align*}
    E_h \omega_i & = \Delta^{(r-1)}(E_h) \omega_i, \\
    F_h \omega_i & = \Delta^{(r-1)}(F_h) \omega_i, \\
    K_j \omega_i & = \Delta^{(r-1)}(K_j) \omega_i,
\end{align*}
\]

where $\omega_i = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$ for $i = (i_1, \ldots, i_r)$, and

\[
\Delta^{(r-1)}(K_j) = K_j \otimes \cdots \otimes K_j,
\]

\[
\Delta^{(r-1)}(E_h) = \sum_{j=1}^r 1 \otimes \cdots \otimes 1 \otimes E_h \otimes \tilde{K}_h \otimes \cdots \otimes \tilde{K}_{j-1}^h.
\]

\[
\Delta^{(r-1)}(F_h) = \sum_{j=1}^r \tilde{K}_h^{-1} \otimes \cdots \otimes \tilde{K}_{j-1}^{-1} \otimes F_h \otimes 1 \otimes \cdots \otimes 1.
\]

Thus, we obtain a $\mathbb{Q}(v)$-algebra homomorphism

\[
\rho_r : \mathbf{U}(\mathfrak{gl}_{2n+1}) \to \text{End}(\Omega^r).
\] (2.3.3)

It is well-known that the image $\text{im}(\rho_r)$ is the commutant subalgebra relative to a right action of the Hecke algebra of type $A$ (see Theorem 3.5 below). This is called a $q$-Schur algebra (of type $A$). We will soon see in next section that when we restrict $\rho_r$ to the subalgebra $\text{im}(\iota)$:

\[
\rho_r^\dagger := \rho_r \circ \iota : \mathbf{U}(n) \xrightarrow{\iota} \mathbf{U}(\mathfrak{gl}_{2n+1}) \xrightarrow{\rho_r} \text{End}(\Omega^r),
\]

the image $\text{im}(\rho_r^\dagger)$ is the commutant subalgebra relative to a right action of the Hecke algebra of type $B$. This is called the $q$-Schur algebra of type $B$; see [BW18, Thm. 6.27].

**Lemma 2.4.** The algebra $\mathbf{U}(n)$ has an involutive automorphism $\omega$ defined by

\[
\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(d_i) = d_i^{-1} \quad (1 \leq i \leq n), \quad \omega(d_{n+1}) = v^{-1} d_{n+1}^{-1}.
\]

\textit{Proof.} This can be directly checked by the relations above or modified from [BW18, Lem. 6.1](1); compare [DDPW08, Lem 6.5(1)]. \hfill \Box

\footnote{We corrected a typo in the correspondence $d_{n+1} \mapsto v K_{n+1}^{-2}$ there.}
3. The \( q \)-Schur algebra of type \( B \)

For any field \( k \), let \( \text{GL}_n(k) \) be the general linear group over \( k \) and consider the group isomorphism

\[
\vartheta : \text{GL}_n(k) \rightarrow \text{GL}_n(k), \ x \mapsto J^{-1}(x^t)^{-1}J,
\]

where \( J \) has entries \( J_{i,j} = 1 \) whenever \( i + j = n + 1 \) and 0 otherwise. The orthogonal group

\[
O_n(k) := \{ x \in \text{GL}_n(k) \mid J = x^tJx \} \quad (\text{where char}(k) \neq 2)
\]

is the fixed-point group of \( \vartheta \). Let \( G(q) := O_{2r+1}(k) \) for \( k = \mathbb{F}_q \), the finite field of \( q \) elements. Let

\[
\Lambda(n+1,r) = \{ \lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \in \mathbb{N}^{n+1} \mid \lambda_1 + \cdots + \lambda_{n+1} = r \}.
\]

Define the bijection

\[
\sim : \Lambda(n+1,r) \rightarrow \tilde{\Lambda}(n+1,r) \subseteq \Lambda(2n+1,2r+1), \quad \lambda \mapsto \tilde{\lambda} := (\lambda_1, \ldots, \lambda_n, 2\lambda_{n+1} + 1, \lambda_{n+1}, \ldots, \lambda_1),
\]

where \( \tilde{\Lambda}(n+1,r) \) is the image of \( \Lambda(n+1,r) \) of the map. For \( \lambda \in \Lambda(n+1,r) \), let \( P_{\lambda}(q) \) be the standard parabolic subalgebra of \( \text{GL}_{2r+1}(\mathbb{F}_q) \) associated with \( \tilde{\lambda} \), consisting of upper quasi-triangular matrices with blocks of sizes \( \tilde{\lambda}_i \) on the diagonal. Let

\[
P_{\lambda}(q) = P_{\tilde{\lambda}}(q) \cap G(q).
\]

Then \( G(q) \) acts on the set \( P_{\lambda}(q) \backslash G(q) \) of left cosets \( P_{\lambda}(q)g \) in \( G(q) \). For any commutative ring \( R \), this action induces a permutation representation over \( R \) which is isomorphic to the induced representations \( \text{Ind}_{P_{\lambda}(q)}^{G(q)}1_R \) of the Hecke algebra \( G(q) \) and define

\[
\mathcal{E}_{q,R}(n,r) = \text{End}_{RG(q)} \left( \bigoplus_{\lambda \in \Lambda(n+1,r)} \text{Ind}_{P_{\lambda}(q)}^{G(q)}1_R \right)^{\text{op}}.
\]

(3.0.2)

This is called the \( q \)-Schur algebra of type \( B \) (compare the type \( A \) case in [DJ89]).

This algebra has the following interpretation of Hecke endomorphism algebra. Let \( \mathcal{H}(B_r) \) be the Hecke algebra over \( \mathbb{Z} = \mathbb{Z}[v, v^{-1}] \) associated with the Coxeter system \((W,S)\) of type \( B_r \), where \( S = \{ s_1, \ldots, s_{r-1}, s_r \} \) has the Dynkin diagram:

\[
\begin{array}{c}
1 \quad 2 \quad 3 \quad \ldots \quad r-1 \quad r
\end{array}
\]

Then it is generated by \( T_i = T_{s_i} \) for \( 1 \leq i \leq r \) subject to the relations:

\[
T_i^2 = (v^2 - 1)T_i + v^2, \ \forall i; \ \ T_iT_j = T_jT_i, \ \ |i - j| > 2;
\]

\[
T_jT_{j+1}T_j = T_{j+1}T_jT_{j+1}, \ \ 1 \leq j < r - 1;
\]

\[
T_{r-1}T_rT_{r-1}T_r = T_rT_{r-1}T_rT_{r-1}.
\]

It has a basis \( \{ T_w \}_{w \in W} \). The subalgebra generated by \( T_1, \ldots, T_{r-1} \) is the Hecke algebra \( \mathcal{H}(S_r) \) associated with the symmetric group \( S_r \).

For \( \lambda \in \Lambda(n+1,r) \), let \( W_\lambda \) be the parabolic subgroup of \( W \) generated by

\[
S' \setminus \{ s_{\lambda_1 + \cdots + \lambda_i} \mid i \in [1,n] \},
\]

and let \( x_\lambda = \sum_{w \in W_\lambda} T_w \). The Hecke endomorphism \( \mathbb{Z} \)-algebra:

\[
S'/(n,r) = \text{End}_{\mathcal{H}(B_r)}(\mathcal{F}(n,r)), \quad \text{where } \mathcal{F}(n,r) = \bigoplus_{\lambda \in \Lambda(n+1,r)} x_\lambda \mathcal{H}(B_r)
\]

(3.0.3)

is called the (generic) \( q \)-Schur algebra of type \( B \).
In order to label the standard basis of $S^j(n, r)$ by matrices, we consider the graph automorphism of the symmetric group $\mathfrak{S}_{2r+1} = \mathfrak{S}_{\{1, 2r+1\}}$:

$$\sigma : \mathfrak{S}_{2r+1} \rightarrow \mathfrak{S}_{2r+1}, (i, j) \mapsto (2r + 2 - i, 2r + 2 - j)$$

for all $i, j \in \{1, 2r+1\}$.

If $\theta \in \mathfrak{S}_{2r+1}$ denotes the permutation sending $i$ to $2r + 2 - i$, then, for any $\pi \in \mathfrak{S}_{2r+1}$,

$$\sigma(\pi) = \theta \circ \pi \circ \theta$$

and $\sigma(i, j) = (\theta(i), \theta(j))$. Further, we may identify $W$ as the fixed-point subgroup $\mathfrak{S}_2^{\times}$ of $\sigma$ with

$$s_i = (i, i+1)(2r+2-i, 2r+2-i) \quad (1 \leq i < r), \quad s_r = (r, r+1)(r+1, r+2)(r, r+1).$$

For the parabolic subgroup $W_{\lambda}$, $\lambda \in \Lambda(n+1, r)$, if

$$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n, 2\lambda_{n+1} + 1, \lambda_{n+1}, \ldots, \lambda_1) \in \Lambda(2n+1, 2r+1).$$

as in (3.0.1), then $\sigma$ stabilises the Young subgroup $\mathfrak{S}_\lambda$ and $W_{\lambda} = \mathfrak{S}_\lambda^{\sigma}$ is the fixed-point subgroup.

**Lemma 3.1.** For $\lambda, \mu \in \Lambda(n+1, r)$ and $\tilde{d} \in \mathfrak{S}_{2r+1}$, suppose the double coset $\mathfrak{S}_\lambda \tilde{d} \mathfrak{S}_\mu$ defines a $(2n+1) \times (2n+1)$ matrix $A = (a_{i,j})$ over $\mathbb{N}$ whose entries sum to $2r + 1$. Then $\mathfrak{S}_\lambda \tilde{d} \mathfrak{S}_\mu$ is stabilised by $\sigma$ if and only if $a_{i,j} = a_{N+1-i, N+1-j}$ for all $i, j \in \{1, N\}$ ($N = 2n+1$).

**Proof.** Let $R_i^\lambda = \{ \lambda_1, \ldots, \lambda_i, 1, \lambda_i+1, \ldots, \lambda_{i+1} \}$. Then $a_{i,j} = |R_i^\lambda \cap \tilde{d} R_j^\mu|$ and $\theta(R_i^\lambda) = R_N^{\lambda_{N+1-i}}$. Note that $\sigma$ stabilises $\mathfrak{S}_\lambda$ and $\mathfrak{S}_\mu$. Thus, $\sigma$ stabilises $\mathfrak{S}_\lambda \tilde{d} \mathfrak{S}_\mu$ if and only if $|R_i^\lambda \cap \tilde{d} R_j^\mu| = |R_N^{\lambda_{N+1-i}} \cap \tilde{d} R_{N+1-j}^\mu| = a_{N+1-i, N+1-j}$ for all $i, j$. (See, e.g., [DDPW03, Lem. 4.14].) However, $|R_i^\lambda \cap \sigma(\tilde{d}) R_j^\mu| = |\theta(R_i^\lambda) \cap \tilde{d} R_j^\mu| = a_{i,j}$. \hfill $\square$

For $\lambda, \mu \in \Lambda(n+1, r)$, let $D_{\lambda, \mu}$ be the minimal length representatives of all double cosets $W_{\lambda} w W_{\mu}$.

For $N = 2n+1$, let

$$\Xi_{2n+1} = \{ A = (a_{i,j}) \in \text{Mat}_N(\mathbb{N}) \mid a_{i,j} = a_{N+1-i, N+1-j}, \forall i, j \in \{1, N\} \},$$

$$\Xi_{2n+1}^{\text{diag}} = \{ A - \text{diag}(a_{1,1}, a_{2,2}, \ldots, a_{N,N}) \mid A = (a_{i,j}) \in \Xi_{2n+1} \},$$

$$\Xi_{2n+1, 2r+1} = \{ A = (a_{i,j}) \in \Xi_{2n+1} \mid |A| := \sum_{i,j} a_{i,j} = 2r + 1 \}. \quad (3.1.1)$$

**Corollary 3.2.** There is a bijection

$$m : \{ (\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n+1, r), d \in D_{\lambda, \mu} \} \rightarrow \Xi_{2n+1, 2r+1}.$$ 

**Proof.** The matrix $m(\lambda, d, \mu)$ is the matrix associated with the double coset $\mathfrak{S}_\lambda \tilde{d} \mathfrak{S}_\mu$. The assertion follows from the lemma above. \hfill $\square$

Recall the basis $\{ e_A \mid A \in \Xi_{2n+1, 2r+1} \}$ for $E_{q,R}(n, r)$ introduced in [BKLW18, §2.3].

**Theorem 3.3.** By specialising $v^2$ to $q = q1_R \in R$, there is an algebra isomorphism

$$\xi : E_{q,R}(n, r) \rightarrow S^j(n, r) := S^j(n, r) \otimes_{\mathbb{Z}} R, \ e_A \mapsto \xi_A^d,$$

where $m(\lambda, d, \mu) = A$ and $\xi_A^d$ is the $\mathcal{H}(B_r)$-module homomorphism defined by

$$\xi_A^d(x_{\nu}) = \delta_{\mu, \nu} \sum_{w \in W_{\lambda} w W_{\mu}} T_w.$$

We will identify the two basis elements $e_A = \xi_A^d$ in the sequel.
Remark 3.4. The proof is omitted as it is standard, extending Iwahori’s original isomorphism $\text{End}_{R G(q)}(\text{ind}_{P'(q)}^{G(q)} 1_R) \cong \mathcal{H}(B_r)_R$. See [DDPW08, Thm. 13.15]. It should be noted that a more general result for all finite types can be found in [LW, Thm. 4.2].

We record the fact that, if $D = \text{diag}(\tilde{\lambda}) \in \Xi_{2n+1,2r+1}$ for some $\lambda \in \Lambda(n+1,r)$ is diagonal, then

$$e_D e_A = \delta_{\lambda, \text{ro}(A)} e_A, \quad e_A e_D = \delta_{\text{co}(A), \tilde{\lambda}} e_A$$

(3.4.1)

for all $A = (a_{i,j}) \in \Xi_{2n+1,2r+1}$, where

$$\text{ro}(A) := (\sum_j a_{1,j}, \sum_j a_{2,j}, \ldots, \sum_j a_{2n+1,j})$$

$$\text{co}(A) := (\sum_i a_{i,1}, \sum_i a_{i,2}, \ldots, \sum_i a_{i,2n+1})$$

We end this section with Bao-Wang’s Schur duality. We need to interpret the $\mathcal{H}(B_r)$-module $\mathcal{T}(n, r)$ in terms of the tensor space $\Omega^{\otimes r}$; see [DPS]. Let

$$I(2n+1, r) = \{i = (i_1, \ldots, i_r) \mid i_j \in [1, 2n+1], \forall j\}.$$  

(3.4.2)

Then $\omega_i := \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$, $i \in I(2n+1, r)$, form a basis for $\Omega^{\otimes r}$.

Recall the action of $\mathcal{H}(\mathfrak{g}_r)$ on $\Omega^{\otimes r}$ defined in [DDPW08, (14.6.4)]: for $1 \leq j < r$,

$$\omega_i T_j = \begin{cases} v \omega_{i s_j} & \text{if } i_j < i_{j+1}; \\ v^2 \omega_i & \text{if } i_j = i_{j+1}; \\ (v^2 - 1) \omega_i + v \omega_{i s_j} & \text{if } i_j > i_{j+1}. \end{cases}$$

(3.4.3)

We extend the action to $\mathcal{H}(B_r)$ by setting

$$\omega_i T_r = \begin{cases} v \omega_{i s_r} & \text{if } i_j < r + 1; \\ v^2 \omega_i & \text{if } i_j = r + 1; \\ (v^2 - 1) \omega_i + v \omega_{i s_r} & \text{if } i_j > r + 1. \end{cases}$$

(3.4.4)

Here $s_j = (j, j + 1)$ and $i s_j$ is the place permutation:

$$(i_1, \ldots, i_{r-1}, i_r)s_j = \begin{cases} (i_1, \ldots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \ldots, i_r), & \text{if } j < r; \\ (i_1, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_{r-1}, i_{r+2}), & \text{if } j = r, \end{cases}$$

where $i_{r+2} = N + 1 - i_{r-1}$.

As usual, we will call a surjective homomorphism an epimorphism. Let $\mathcal{H}(\mathfrak{g}_r) = \mathcal{H}(\mathfrak{g}_r)_{q(v)}$, etc.

Theorem 3.5. (1) The $U(\mathfrak{gl}_{2n+1})$ action on $\Omega^{\otimes r}$ commutes with the action of $\mathcal{H}(\mathfrak{g}_r)$ and the bimodule structure induces epimorphisms (cf. (3.2.3))

$$\rho_r : U(\mathfrak{gl}_{2n+1}) \longrightarrow \text{End}_{\mathcal{H}(\mathfrak{g}_r)}(\Omega^{\otimes r}), \quad \rho^\vee_r : \mathcal{H}(\mathfrak{g}_r) \longrightarrow \text{End}_{U(\mathfrak{gl}_{2n+1})}(\Omega^{\otimes r}).$$

(2) [DPS, Lem. 5.3.8] There is an $\mathcal{H}(B_r)$-module isomorphism $\mathcal{T}(n, r)_{q(v)} \cong \Omega^{\otimes r}$. Hence,

$$\text{End}_{\mathcal{H}(B_r)}(\Omega^{\otimes r}) \cong S^r(n, r)_{q(v)}.$$  

(3) [BW18, Thm. 6.27] The actions of $U^J(n)$ and $\mathcal{H}(B_r)$ commute and satisfy the double centraliser property as stated in (1). In particular, the algebra homomorphism $\rho_r$ restricts to an algebra epimorphism

$$\rho_r^J = \rho_r \circ i : U^J(n) \longrightarrow \text{End}_{\mathcal{H}(B_r)}(\Omega^{\otimes r}).$$

(3.5.1)
Remarks 3.6. (1) The \( \mathbb{Q}(v) \)-algebra \( S(2n + 1, r)_{\mathbb{Q}(v)} := \mathrm{End}_{\mathbb{Q}(v)}(\mathbb{Q}^{\otimes r}) \) is known as the \( q \)-Schur algebra \( (q = v^2) \). By identifying \( S^j(n, r)_{\mathbb{Q}(v)} \) with \( \mathrm{End}_{\mathbb{Q}(v)}(\mathbb{Q}^{\otimes r}) \) under the isomorphism in (2), we may regard \( S^j(n, r)_{\mathbb{Q}(v)} \) as a subalgebra of \( S(2n + 1, r)_{\mathbb{Q}(v)} \). Clearly, these relations hold at the integral level.

(2) The epimorphisms \( \rho^j \) induce an algebra homomorphism

\[
\rho^j : U^j(n) \rightarrow \prod_{r \geq 0} \mathrm{End}_{\mathbb{Q}(v)}(\mathbb{Q}^{\otimes r}), \ u \mapsto (\rho^j_k(u))_{\gamma \geq 0}.
\]

One of the aims of this paper is to determine a basis for the image \( \text{im}(\rho^j) \) and to show that \( \rho^j \) induces an isomorphism \( U^j(n) \cong \text{im}(\rho^j) \).

(3) Lai, Nakano and Xiang considered the representation theory of \( S^j(n, r)_k \) over a field \( k \). In particular they realized the aforementioned algebra as the dual of the \( r \)th homogeneous component of the quotient of the coordinate algebra of the quantum matrix space by a right ideal that is also a coideal. This shows there is a natural polynomial representation theory (see \[LNX\], Section 2.4).

Moreover, under a certain invertibility condition (i.e., the “semisimple bottom” condition in the sense of \[DR00\]), the structure and representations of these algebras including quasihereditariness and cellularity were investigated \[LNX\] Sections 5-6]. In turn, they obtained a concrete realization for the category \( \mathcal{O} \) of rational Cherednik algebras of type \( B \) together with the Knizhnik–Zamolodchikov functor in terms of the module category of \( S^j(n, r)_k \) and its corresponding Schur functor (see \[LNX\] Section 8).

4. Some multiplication formulas

We now derive some multiplication formulas and their associated stabilisation property in the \( q \)-Schur algebra of type \( B \). This work is built on the formulas in \[BKLW18\] Lem. 3.2.

For \( i, j \in [1, 2n + 1] \), let \( E_{i,j} \) be the standard matrix units in \( \text{Mat}_{2n+1}(\mathbb{N}) \). Let

\[
E_{i,j}^\theta = E_{i,j} + E_{2n+2-i,2n+2-j} = E_{2n+2-i,2n+2-j}.
\]

Note that \( E_{n+1,n+1}^\theta = 2E_{n+1,n+1} \). Let

\[
e_{i,j}^\theta = \begin{cases} 2 & \text{if } i = j = n + 1, \\ 1 & \text{otherwise}, \end{cases} \quad \text{and } e_i^\theta = \text{ro}(E_{i,i}^\theta).
\]

Then \( e_{i,j}^\theta \) is the \((i, j)\)-entry of \( E_{i,j}^\theta \) and \( e_i^\theta = e_i + e_{2n+2-i} \), where \( e_i = (0, \ldots, 0, 1, 0, \ldots 0) \in \mathbb{Z}^{2n+1} \) form the standard basis for \( \mathbb{Z}^{2n+1} \).

Recall the dimension \( d(A) \) of the orbit \( \mathcal{O}_A \) and the dimension \( r(A) \) of the image of \( \mathcal{O}_A \) under the first projection (see \[BKLW18\] (3.16)).

\[
d(A) - r(A) = \frac{1}{2} \left( \sum_{i \geq k, l < i} a_{ij} a_{kl} - \sum_{j < n+1 \leq i} a_{ij} \right).
\]

We normalise the \( \mathbb{Z} \)-basis \( \{ e_A \mid A \in \Xi_{2n+1,2r+1} \} \) for \( S^j(n, r) \) by setting

\[
[A] = v^{-d(A)+r(A)} e_A.
\]

The following multiplication formulas are special cases of \[BKLW18\] Thm. 3.7] by taking \( R = 1 \). We can also derive them directly by \[BKLW18\] Lem. 3.2. For notational clarity, we

\[\text{We thank Yiqiang Li for sending us this simplified version.}\]
extend the usual Kronecker delta $\delta_{i,j}$ to define

$$\delta_{i,j}^\le = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$$

Let, for $A \in \Xi_{2n+1,2r+1}$ and $h \in [1,n]$,

$$\beta_p(A,h) = \sum_{j \geq p} a_{h,j} - \sum_{j > p} a_{h+1,j} + \delta_{h,n} \delta_{p,n},$$

$$\beta'_p(A,h) = \sum_{j \leq p} a_{h+1,j} - \sum_{j < p} a_{h,j}. \quad (4.0.1)$$

Recall $\tilde{\lambda}$ in (3.0.1) for $\lambda \in \Lambda(n+1,r)$. We often use $\tilde{\lambda}$ to denote the diagonal matrix $\text{diag}(\tilde{\lambda})$ for simplicity. Thus, for any $A \in \Xi_{2n+1}^{\text{diag}}$, $A + \tilde{\lambda}$ really mean $A + \text{diag}(\tilde{\lambda})$.

**Lemma 4.1.** Suppose that $h \in [1,n]$, $\lambda \in \Lambda(n+1,r-1)$, and $A \in \Xi_{2n+1,2r+1}$. The following multiplication formulas hold in $S'(n,r)$:

(a) $[E^\theta_{h,h+1} + \tilde{\lambda}] \cdot [A] = \delta_{h+1,\text{ro}(A)} \sum_{p\in[1,2n+1]} v^\beta_p(A,h) [a_{h,p} + 1] [A + E^\theta_{h,p} - E^\theta_{h+1,p}].$

(b) $[E^\theta_{h+1,h} + \tilde{\lambda}] \cdot [A] = \delta_{h+1,\text{ro}(A)} \sum_{p\in[1,2n+1]} v^{\beta'_p(A,h)} [a_{h+1,p} + 1] [A - E^\theta_{h,p} + E^\theta_{h+1,p}].$

We now extend these formulas to a certain spanning set for $S'(n,r)$. Recall the notation $\Xi_{2n+1}^{\text{diag}}$ in (3.1.1).

For $A \in \Xi_{2n+1}^{\text{diag}}$ and $j = (j_1,j_2,\ldots,j_N) \in \mathbb{Z}^{2n+1}$, define

$$A(j,r) = \begin{cases} \sum_{\lambda \in \Lambda(n+1,r-\frac{|A|}{2})} v^{\tilde{\lambda}j}[A + \tilde{\lambda}], & \text{if } |A| \leq 2r, \\ 0, & \text{if } |A| > 2r, \end{cases} \quad (4.1.1)$$

where $\tilde{\lambda} \cdot j = \sum_{i=1}^{2n+1} \tilde{\lambda}_i j_i$ with $\tilde{\lambda}$ defined as in (3.0.1). Note that

$$\{ \tilde{\lambda} \mid \lambda \in \Lambda(n+1,r-\frac{1}{2}|A|) \} = \{ \mu \in \mathbb{N}^{2n+1} \mid A + \mu \in \Xi_{2n+1,2r+1} \}.$$

In particular, if $O$ denotes the zero matrix, $e_i \in \mathbb{Z}^{2n+1}$ as above, and $0 = (0,\ldots,0) \in \mathbb{Z}^{2n+1}$, we have $E^\theta_{h,h+1}(0,r) = \sum_{\lambda \in \Lambda(n+1,r-1)} [E^\theta_{h,h+1} + \tilde{\lambda}]$ and

$$O(e_i,r) = O(e_{2n+2-i},r) = \begin{cases} \sum_{\lambda \in \Lambda(n+1,r)} v^{\lambda j}(\tilde{\lambda}), & \text{if } 1 \leq i \leq n; \\ \sum_{\lambda \in \Lambda(n+1,r)} v^{2\lambda_{n+1}+1}(\tilde{\lambda}), & \text{if } i = n + 1. \end{cases} \quad (4.1.2)$$

For $1 \leq h \leq n$, we put $a_h = e_h - e_{h+1}$ and $a^-_h = -e_h - e_{h+1}$. The following multiplication formulas are the type $B$ counterpart of [BLM90] Lem. 5.3.

**Theorem 4.2.** Maintain the notations introduced above. For $N = 2n+1$, $j = (j_1,j_2,\ldots,j_N) \in \mathbb{Z}^N$, $h \in [1,n]$, and $A = (a_{i,j}) \in \Xi_{2n+1}^{\text{diag}}$, the following multiplication formulas hold in $S'(n,r)$ for all $r \geq \frac{|A|}{2}$:

1. $O(j,r) A(j',r) = v^{ro(A)} j A(j+j',r)$,
2. $A(j',r) O(j,r) = v^{co(A)} j A(j+j',r)$;
\( E_{h,h+1}(0,r) \cdot A(j,r) = \sum_{1 \leq p < h} v_{\beta_p(A,h)}[a_{h,p} + 1](A + E_{h+1,p}^\theta)(j + \alpha_h, r) \)

\[ \begin{align*}
&+ \frac{\varepsilon}{1 - v^{-2}} \left( (A - E_{h+1,h}^\theta)(j + \alpha_h, r) - (A - \tilde{E}_{h+1,h}^\theta)(j + \alpha_h, r) \right) \\
&+ \frac{\varepsilon}{1 - v^{-2}} \left( (A - E_{h+1,h}^\theta)(j - \alpha_h, r) - (A - \tilde{E}_{h+1,h}^\theta)(j - \alpha_h, r) \right) \\
&+ \sum_{h+1 < p \leq N} v_{\beta_p(A,h)}[a_{h,p} + 1](A + E_{h+1,p}^\theta)(j - \alpha_h, r),
\end{align*} \]

where \( \varepsilon = \delta_{1,a_{h+1,h}}^\leq \).

(3) \( E_{h+1,h}(0,r) \cdot A(j,r) = \sum_{1 \leq p < h} v_{\beta_p(A,h)}[a_{h,p} + 1](A - E_{h+1,p}^\theta)(j, r) \)

\[ \begin{align*}
&+ v_{\beta_h(A,h)}[a_{h,h} + 1](A + E_{h+1,h}^\theta)(j, r) \\
&+ \frac{\varepsilon'}{1 - v^{-2}} \left( (A - E_{h+1,h}^\theta)(j - \alpha_h, r) - (A - \tilde{E}_{h+1,h}^\theta)(j - \alpha_h, r) \right) \\
&+ \sum_{h+1 < p \leq N} v_{\beta_p(A,h)}[a_{h,p} + 1](A - E_{h+1,p}^\theta)(j - \alpha_h, r),
\end{align*} \]

where \( \varepsilon' = \delta_{1,a_{h+1,h}}^\leq \).

Proof. Recall the map in (3.0.1). For \( \lambda \in \Lambda(n+1,r), \mu \in \Lambda(n+1,r - \frac{|A|}{2}) \) with \( A + \tilde{\mu} \in \Xi_{2n+1,2r+1} \), (3.4.1) implies \( \tilde{\lambda}[A + \tilde{\mu}] = [A + \tilde{\mu}] \iff \tilde{\lambda} = \text{ro}(A) + \tilde{\mu} \). Thus,

\[ O(j,r)A(j',r) = \sum_{\lambda \in \Lambda(n+1,r)} \sum_{\mu \in \Lambda(n+1,r - \frac{|A|}{2})} v^{\lambda\lambda+j\lambda'\lambda'}[\tilde{\lambda}[A + \tilde{\mu}]
\]

\[ = \sum_{\mu \in \Lambda(n+1,r - \frac{|A|}{2})} v^{\text{ro}(A)\lambda} \sum_{\lambda \in \Lambda(n+1,r - \frac{|A|}{2})} v^{\tilde{\mu}\lambda\lambda'}[A + \tilde{\mu}]
\]

\[ = v^{\text{ro}(A)\lambda} A(j + j', r). \]

The proof for the second formula in (1) is similar.

We now prove (2). By definition, we have,

\[ E_{h,h+1}^\theta(0,r).A(j,r) = \sum_{\lambda \in \Lambda(n+1,r-1)} \sum_{\mu \in \Lambda(n+1,r - \frac{|A|}{2})} v^{\tilde{\mu}\lambda}[E_{h,h+1}^\theta + \tilde{\lambda}[A + \tilde{\mu}]
\]

\[ = \sum_{\mu \in \Lambda(n+1,r - \frac{|A|}{2})} v^{\tilde{\mu}\lambda}[E_{h,h+1}^\theta + \text{ro}(A) + \tilde{\mu} - \text{co}(E_{h,h+1}^\theta)][A + \tilde{\mu}]
\]

Let \( A + \tilde{\mu} = (a_{i,j}) \). Then \( a_{i,i}^{\mu} = a_{i,i} + \mu_i \) and \( a_{i,j}^{\mu} = a_{i,j} \) for \( i \neq j \). For the number \( \beta_p(A,h) \) in (4.0.1), we have

\[ \beta_p(A + \tilde{\mu}, h) = \begin{cases} \beta_p(A,h) + \tilde{\mu}_h - \tilde{\mu}_{h+1} & \text{if } p \leq h, \\
\beta_p(A,h) & \text{if } p \geq h + 1. \end{cases} \]
By Proposition 3, 11 and noting \( \text{co}(E_{h,h+1}^\theta) = e_{h+1} + e_{N-h} = e_{h+1}^\theta \), we obtain that

\[
E_{h,h+1}^\theta + \rho(A) + \mu - \text{co}(E_{h,h+1}^\theta) = [E_{h,h+1}^\theta + \rho(A) + \mu - e_{h+1}^\theta][A + \mu] \\
= \sum_{p \in [1,N]} \sum_{a_{h+1,p}^\mu \geq 2^{h+1}} v^\beta_{p}(A+\mu,h) [a_{h+1,p}^\mu] [A + \mu + E_{h,h+1}^\theta] \\
= \sum_{1 \leq p < h} v^\beta_{p}(A+\mu,h) [a_{h+1,p}^\mu + 1] [A + \mu + E_{h,h+1}^\theta] \\
+ \varepsilon \sum_{\mu} v^\beta_{\mu}(A+\mu,h) [a_{h+1,h}^\mu + 1] [A + \mu + E_{h,h+1}^\theta] \\
+ \sum_{\mu \geq h+1} v^\beta_{h+1}(A+\mu,h) [a_{h+1,h} + 1] [A + \mu + E_{h,h+1}^\theta] \\
+ \sum_{h+1 < p \leq N} v^\beta_{p}(A,h) [a_{h+1,p}^\mu + 1] [A + \mu + E_{h,h+1}^\theta],
\]

where \( \mu \) runs over \( \Lambda(n+1, r - |A|) \). The first and last summations give the required form in (2). It remains to compute the second and third summations. If \( a_{h+1,h} = a_{h+1,h}^\mu \geq 1 \), then \( \varepsilon = 1 \) and

\[
v^\beta_{h}(A+\mu,h) [a_{h+1,h}^\mu + 1] [A + \mu + E_{h,h+1}^\theta] \\
= v^\beta_{h}(A,h) [\mu_h + 1] [A - E_{h,h+1}^\theta + \mu + e_h^\theta] \\
= \frac{v^\beta_{h}(A,h) - j_h - j_{N+1-h} - 1}{1 - v^{-2}} v^{\tilde{\beta}_{h}}(A,h)_j + j_{N+1-h} + 1 \left( 1 - v^{-2} \tilde{\mu}(h+1) \right) [A - E_{h,h+1}^\theta + \tilde{\mu} + e_h^\theta]
\]

Since \( v^{\tilde{\lambda}}(j + e_{h+1}) - v^{\tilde{\lambda}}(j - e_{h+1}) = 0 \) whenever \( \lambda_h = 0 \) (so \( \tilde{\lambda}_h = \tilde{\lambda}_{N+1-h} = 0 \)), it follows that

\[
\sum_{\mu \in \Lambda(n+1, r - |A|)} \left( v^{\tilde{\mu}}(j + e_{h+1}) - v^{\tilde{\mu}}(j - e_{h+1}) \right) [A - E_{h,h+1}^\theta + \tilde{\mu} + e_h^\theta] \\
= \sum_{\lambda \in \Lambda(n+1, r - |A|) + 1} \left( v^{\tilde{\lambda}}(j + \alpha_h) - v^{\tilde{\lambda}}(j + \alpha_h) \right) [A - E_{h,h+1}^\theta + \tilde{\lambda}] \\
= (A - E_{h,h+1}^\theta)(j + \alpha_h, r) - (A - E_{h,h+1}^\theta)(j + \alpha_h, r).
\]
giving the second term in (2). Finally, for the third summation, we have
\[
\sum_{\mu \in \Lambda(n+1, r - \frac{|A|}{2}) \setminus \bar{\rho}} v^{\beta_{h+1}(A, h) + j} \prod_{a_{h, h+1}} (A + \bar{\mu} - E_{h, h+1}^\theta - E_{h+1, h+1}^\theta]
\]
\[
= v^{\beta_{h+1}(A, h) + j} \prod_{a_{h, h+1}} (A + E_{h, h+1}^\theta + \bar{\mu} - E_{h+1, h+1}^\theta]
\]
\[
= v^{\beta_{h+1}(A, h) + j} \prod_{a_{h, h+1}} (A + E_{h, h+1}^\theta)(j, r).
\]
Here, we have used an obvious bijection
\[
\left\{ \mu \in \Lambda(n + 1, r - \frac{|A|}{2}) \mid \mu_{h+1} \geq 1 \right\} \rightarrow \Lambda(n + 1, r - \frac{|A|}{2} - 1)
\]
\[
(\mu_1, \ldots, \mu_h, \mu_{h+1}, \mu_{h+2}, \ldots, \mu_{n+1}) \mapsto (\mu_1, \ldots, \mu_h, \mu_{h+1} - 1, \mu_{h+2}, \ldots, \mu_{n+1}).
\]
This proves (2). The proof of (3) is similar. 

Remark 4.3. If one compares these multiplication formulas with those given in [BLM90 Lem. 5.3] (or [DDPW08 Thm. 13.27]), they are very similar except the adjustments needed for the $h = n$ case. Of course, you may also see the difference arising from the symmetry of the matrices involved.

Let
\[
S^j(n)_{Q(v)} = \prod_{r \geq 0} S^j(n, r)_{Q(v)}.
\]

We will write the elements in $S^j(n)_{Q(v)}$ as formal infinite series. Define, for $A \in \mathbb{Z}^n_{2n+1}$ and $j \in \mathbb{Z}^{2n+1}$,
\[
A(j) := \sum_{r \geq 0} A(j, r) \in S^j(n)_{Q(v)}.
\]

For convenient use later, Theorem 4.2 is rewritten as follows.

Theorem 4.4. For $N = 2n + 1$, $j = (j_1, j_2, \ldots, j_N) \in \mathbb{Z}^N$, $h \in [1, n]$, and $A = (a_{ij}) \in \mathbb{Z}^n_{2n+1}$, the following multiplication formulas hold in $S^j(n)_{Q(v)}$:

1. $O(j)A(j') = v^{\nu(A)}j A(j + j')$, $A(j')O(j) = v^{\nu(A)}j A(j + j')$;
2. $E_{h, h+1}^\theta(0) \cdot A(j) = \sum_{1 \leq p < h} \sum_{a_{h, p+1} \geq 1} v^{\beta_p(A, h)} (a_{h, p} + 1)(A + E_{h, p}^\theta - E_{h+1, p}^\theta)(j + \alpha_h)
\]
\[
+ \delta_{a_{h, h+1}} v^{\beta_{h+1}(A, h) - j + j_{N+1} - h - 1} (1 - v^2) (A - E_{h+1, h}^\theta)(j + \alpha_h) - (A - E_{h+1, h}^\theta)(j + \alpha_h^\theta)
\]
\[
+ v^{\beta_{h+1}(A, h) + j} \prod_{a_{h, h+1}} (A + E_{h, h+1}^\theta)(j)
\]
\[
+ \sum_{h+1 < p \leq N} \sum_{a_{h, h+1} \geq 1} v^{\beta_p(A, h)} (a_{h, p} + 1)(A + E_{h, p}^\theta - E_{h+1, p}^\theta)(j_{h, h+1} + 1).
\]
Proof. This is clear since the coefficients in the multiplication formulas in Theorem 4.2 are independent of \( r \) for all \( r \geq \frac{1}{2} |A| \).

\[ \text{Corollary 4.5. For } h \in [1, n], \text{ we have in } S^1(n)_{Q(v)} \]
\[ (1) \ E_{h,h+1}^0(0)^m = [m]! (mE_{h,h+1}^0)(0); \]
\[ (2) \ E_{h,h+1}^0(0)^m = [m]! (mE_{h,h+1}^0)(0). \]

Proof. We only prove (1); the proof of (2) is similar. For \( A = mE_{h,h+1}^0 \), the only non-zero entry in row \( h + 1 \) is the diagonal entry and
\[ \beta_{h+1}(A, h) = \sum_{j \geq h+1} a_{h,j} - \sum_{j > h+1} a_{h+1,j} + \delta_{h,n} \delta_{h+1,n} = m. \]

Thus, by Theorem 4.4(2),
\[ E_{h,h+1}^0(0)^2 = v[1 + 1](2E_{h,h+1}^0)(0) = [2]^1 (2E_{h,h+1}^0)(0). \]

Now the general case follows from an induction. \( \square \)

5. The subalgebra \( \mathfrak{U}^q(n) \)

We now prove that the subspace of \( S^1(n)_{Q(v)} \):
\[ \mathfrak{U}^q(n) = \text{span}\{ A(j) \mid A \in \Xi^{0}_{2n+1}, j \in \mathbb{Z}^N \} \]
is indeed a subalgebra.

As in [BKLW18] or [BLM90, 5.3], we define a preorder \( \preceq \) on \( \Xi^{0}_{2n+1} \) as follows.
\[ A \preceq B \iff \sum_{r \leq i, s \geq j} a_{rs} \leq \sum_{r \leq i, s \geq j} b_{rs}, \text{ for all } 1 \leq i < j \leq 2n + 1. \]

Clearly, \( A \preceq B \iff \sum_{r \leq i, s \leq j} a_{rs} \leq \sum_{r \leq i, s \leq j} b_{rs}, \) for all \( i > j \). We write \( A < B \) if \( A \preceq B \) and \( B \not\preceq A \).

Let
\[ \mathcal{T}_{2n+1} = \{(i, h, j) \mid 1 \leq j \leq h < i \leq 2n + 1\}. \]
We order the set as in [BKLW18 Thm. 3.10]:
\[ (i, h, j) \preceq (i', h', j') \iff i < i' \text{ or } i = i', j < j' \text{ or } i = i', j = j', h > h'. \] (5.0.1)

This order modifies the order \( \leq_i \) defined in [DDPW08 (13.7.1)]. For example, the first few elements in \( (\mathcal{T}_{2n+1}, \leq) \) are
\[ (2, 1, 1), (3, 2, 1), (3, 1, 1), (3, 2, 2), (4, 3, 1), \ldots (4, 3, 2), (4, 2, 2), (4, 3, 3), \ldots, (N, N - 1, N - 1). \] (5.0.2)
For $A \in \Xi_{2n+1}^{0\text{diag}}$, let
$$m^A,0 := \prod_{(i,h,j) \in (2n+1)^3} (a_{i,j}E_{h+1}^\theta)(0), \quad (5.0.3)$$
where the product is taken with respect to the order $\leq$. Thus, by (5.0.2), the leading term of the product $m^A,0$ is $(a_{2,1}E_{2,1}^\theta)(0)$ and the ending term is $(a_{N,N-1}E_{N,N-1}^\theta)(0)$ ($N = 2n + 1$).

**Lemma 5.1.** For each $A \in \Xi_{2n+1}^{0\text{diag}}$, we have
$$m^A,0 = A(0) + \sum_{B \in \Xi_{2n+1}^{0\text{diag}} \backslash B < A} g_{A,B,j}B(j) = A(0) + (\text{lower terms}).$$

**Proof.** Repeatedly applying Theorem 4.4 yields
$$m^A,0 = \sum_{B \in \Xi_{2n+1}^{0\text{diag}} \backslash B < A} g_{A,B,j}B(j).$$
It suffices to prove that $g_{A,A,0} = 1$ and $B < A$ whenever $g_{A,B,j} \neq 0$. Consider the $r$-th component of $m^A,0$:
$$\pi_r(m^A,0) = \prod_{1 \leq j \leq h < i \leq N} (a_{i,j}E_{h+1}^\theta)(0, r)$$
$$= \prod_{1 \leq j \leq h < i \leq N} \sum_{\mu_{i,h,j} \in \Lambda(n+1,r-a_{i,j})} [a_{i,j}E_{h+1}^\theta + \tilde{\mu}_{i,h,j}]$$
$$= \sum_{\mu_{i,h,j} \in \Lambda(n+1,r-a_{i,j})} \prod_{1 \leq j \leq h < i \leq N} [a_{i,j}E_{h+1}^\theta + \tilde{\mu}_{i,h,j}].$$
If such a product $\prod_{1 \leq j \leq h < i} [a_{i,j}E_{h+1}^\theta + \tilde{\mu}_{i,h,j}] \neq 0$, by [BKLW18 Thm. 3.10], there exists $\lambda \in \Lambda(n+1, r - \frac{|A|}{2})$ such that
$$\prod_{1 \leq j \leq h < i} [a_{i,j}E_{h+1}^\theta + \tilde{\mu}_{i,h,j}] = [A + \tilde{\lambda}] + (\text{lower terms}).$$
Since $\text{ro}(a_{2,1}E_{2,1}^\theta + \tilde{\mu}_{2,1,1}) = \text{ro}(A + \tilde{\lambda})$ and $\text{co}(a_{N,N-1}E_{N,N-1}^\theta + \tilde{\mu}_{N,N-1,N-1}) = \text{co}(A + \tilde{\lambda})$, we have $\tilde{\lambda} = \tilde{\mu}_{2,1,1} + e_\theta^\lambda - \text{ro}(A)$. Thus, with the notation in [BKLW18 Thm. 3.10], we have $D_{2,1,1}^\lambda := \tilde{\mu}_{2,1,1}$ and $D_{N,N-1,N-1}^\lambda := \tilde{\mu}_{N,N-1,N-1}$, and all other $D_{i,h,j}^\lambda = \tilde{\mu}_{i,h,j}$ are completely determined by $A + \tilde{\lambda}$. Hence,
$$\pi_r(m^A,0) = \sum_{\lambda \in \Lambda(n+1, r - \frac{|A|}{2})} \prod_{1 \leq j \leq h < i \leq N} [a_{i,j}E_{h+1}^\theta + D_{i,h,j}^\lambda]$$
$$= \sum_{\lambda \in \Lambda(n+1, r - \frac{|A|}{2})} ([A + \tilde{\lambda}] + (\text{lower terms}))$$
$$= A(0, r) + \sum_{B \in \Xi_{2n+1}^{0\text{diag}} \backslash B < A} g_{A,B,j}B(j, r).$$
as desired.

Theorem 5.2. The vector space $\mathfrak{A}^i(n) = \text{span}\{A(j) \mid A \in \Xi_{2n+1,j}^{0,1,2r+1}, j \in \mathbb{Z}^{2n+1}\}$ of the algebra $S^i(n)_{Q(v)}$ is a subalgebra which is generated by

$$E_{h,h+1}^\theta(0), \quad E_{h+1,h}^\theta(0), \quad O(\pm e_i),$$

for all $1 \leq h \leq n$ and $1 \leq i \leq n + 1$, and is presented by the multiplication formulas in Theorem 4.4.

**Proof.** Let $\mathfrak{A}'$ be the subalgebra generated by

$$E_{h,h+1}^\theta(0), \quad E_{h+1,h}^\theta(0), \quad O(\pm e_i).$$

By Theorem 4.4 we have $\mathfrak{A}' \subseteq \mathfrak{A}^i(n)$. We now prove $\mathfrak{A}^i(n) \subseteq \mathfrak{A}'$. We prove all $A(j) \in \mathfrak{A}'$ by induction on $\|A\|$, where

$$\|A\| = \sum_{1 \leq i < j \leq N} \frac{(j-i)(j-i+1)}{2} (a_{ij} + a_{ji}).$$

If $\|A\| = 0$, then $A = O$, the zero matrix, and $A(j) = O(j) = \prod_{i=1}^N O(e_i)^{j_i} \in \mathfrak{A}'$. Assume now $\|A\| > 0$ and $B(j) \in \mathfrak{A}'$ for $\|B\| < \|A\|$. Firstly, by Corollary 5.3 we have

$$(a_{ij}E_{h+1,h}^\theta(0)) = \frac{E_{h+1,h}^\theta(0)^{a_{ij}}}{[a_{ij}]!} \in \mathfrak{A}'.
$$

Thus, for all $A \in \Xi_{2n+1,j}^{0,1,2r+1}, j \in \mathbb{Z}^N$, $m^{A,0}, O(j)m^{A,0} \in \mathfrak{A}'$. By Lemma 5.1

$$v^{-\text{ro}(A)}j O(j)m^{A,0} = A(j) + \sum_{B \in \Xi_{2n+1,j'}^{0,1,2r+1} j' \in \mathbb{Z}^N} g'_{A,B}j B(j').$$

Since $B \prec A$ implies that $\|B\| < \|A\|$, it follows from induction that all $B(j') \in \mathfrak{A}'$. Hence, $A(j) \in \mathfrak{A}'$. □

For $A \in \Xi_{2n+1,j}^{0,1,2r+1}, j \in \mathbb{Z}^N$, let $m^{A,j} = O(j)m^{A,0}$.

**Corollary 5.3.** The $Q(v)$-algebra $\mathfrak{A}^i(n)$ has bases

$$\mathfrak{B} = \{A(j) \mid A \in \Xi_{2n+1,j}^{0,1,2r+1}, j \in \mathbb{Z}^N\} \quad \text{and} \quad \mathfrak{M} = \{m^{A,j} \mid A \in \Xi_{2n+1,j}^{0,1,2r+1}, j \in \mathbb{Z}^N\}.$$

**Proof.** The assertion for $\mathfrak{B}$ is standard with an argument involving Vandermonde determinant. The assertion for $\mathfrak{M}$ follows from the assertion for $\mathfrak{B}$ and the triangular relation in Lemma 5.1. □

**Corollary 5.4.** The canonical projection from $S^i(n)_{Q(v)}$ onto $S^i(n,r)_{Q(v)}$ restricts to a $Q(v)$-algebra epimorphism

$$\pi_r : \mathfrak{A}^i(n) \to S^i(n,r)_{Q(v)}.$$

**Proof.** It suffice to prove that, for a fixed $A \in \Xi_{2n+1,j}^{0,1,2r+1}$,

$$\text{span}\{A(j,r) \mid j \in \mathbb{Z}^{2n+1}\} = \text{span}\{[A + \tilde{\lambda}] \mid \lambda \in \Lambda(n + 1, r - \frac{1}{2}\|A\|)\}.$$

This is clear from the definition of $A(j,r)$ in (4.1.1). □
6. Lifting Bao–Wang’s Schur duality to the integral level

In [BW13] Thm. 6.27, a \((U(n), \mathfrak{H}(B_\gamma))\)-duality via the tensor space \(\Omega_{2r}\) is established; see Theorem [4.5.3]. In this section, we will define an algebra epimorphism \(\phi^\gamma : U^\gamma(n) \to \mathfrak{H}(B_\gamma)\) via the subalgebra \(\mathfrak{U}(n)\) and prove that \(\phi^\gamma\) maps a Lusztig type form \(U^\gamma(n)_Z\) of \(U^\gamma(n)\) onto the integral \(q\)-Schur algebra \(\mathfrak{S}(n, r)\). We will compare \(\phi^\gamma\) with the epimorphism \(\rho^\gamma\) given in [3.5.1] in next section.

**Theorem 6.1.** There is a \(\mathbb{Q}(v)\)-algebra epimorphism

\[ \phi^\gamma : U^\gamma(n) \to \mathfrak{U}(n) \]

such that \(e_h \mapsto E_{h,h+1}^\theta(0), \ f_h \mapsto E_{h+1,h}^\theta(0), \ d^\pm_h \mapsto O(\pm e_h), \) and \(d^-_{n+1} \mapsto v^{-1}O(\pm e_{n+1})\) for all \(1 \leq h \leq n\).

**Proof.** We must prove that the relations (iQG1)–(iQG6) in Definition 2.1 are all satisfied for

\[ e_h = E_{h,h+1}^\theta(0), \ f_h = E_{h+1,h}^\theta(0), \ d_h = O(e_h), \ d_{n+1} = v^{-1}O(e_{n+1}). \]

Since relations (iQG1)–(iQG5) are more or less the defining relations for quantum \(\mathfrak{g}_n\), by Remark 4.3, the proof is almost the same as the proof of [DDPW08] Thm. 13.33. We now prove (iQG6). We only check the first relation here:

\[ f^2_n e_n + e_n f^2_n = [2](f_n e_n f_n - (vd^-_{n+1}d^-_{n+1}f_n)). \] (6.1.1)

First, compute \(f^2_n e_n = E_{n+1,n}^\theta(0)^2E_{n+1,n}^\theta(0).\) By Theorem 4.4(3), we have

\[ f_n e_n = E_{n+1,n}^\theta(0)E_{n+1,n}^\theta(0) = v^{\beta_{n+1}}(E_{n+1,n}^\theta(0)E_{n+1,n}^\theta(0) + E_{n+1,n}^\theta(0)) \]
\[ + \frac{v^{\beta_{n+1}}(E_{n+1,n}^\theta(0)E_{n+1,n}^\theta(0))}{1 - v^{-2}}(v^{-2}(E_{n+1,n}^\theta(0) - E_{n+1,n}^\theta(0))(-\alpha_n) - (E_{n+1,n}^\theta(0) - E_{n+1,n}^\theta(0))(\alpha_n)) \]
\[ = (E_{n+1,n}^\theta(0) + E_{n+1,n}^\theta(0))(0) + \frac{v^{-2}O(-\alpha) - O(\alpha)}{1 - v^{-2}}. \]

Further, we see that,

\[ E_{n+1,n}^\theta(0)(E_{n+1,n}^\theta(0) + E_{n+1,n}^\theta(0))(0) = v^{\beta_{n+1}}(E_{n+1,n}^\theta(0) + E_{n+1,n}^\theta(0))[2](E_{n+1,n}^\theta(0) + 2E_{n+1,n}^\theta(0)) \]
\[ + \frac{v^{\beta_{n+1}}(E_{n+1,n}^\theta(0) + E_{n+1,n}^\theta(0))}{1 - v^{-2}}(v^{-2}E_{n+1,n}^\theta(0)(-\alpha_n) - E_{n+1,n}^\theta(0)(\alpha_n)) \]
\[ = v[2](E_{n+1,n}^\theta(0) + 2E_{n+1,n}^\theta(0))(0) + \frac{v^{-1}E_{n+1,n}^\theta(0)(-\alpha_n) - \frac{v}{1 - v^{-2}}E_{n+1,n}^\theta(0)(\alpha_n)}{1 - v^{-2}}. \] (6.1.2)

and \(E_{n+1,n}^\theta(0)O(-\alpha_n) = v^{-1}E_{n+1,n}^\theta(-\alpha_n), \) \(E_{n+1,n}^\theta(0)O(\alpha_n) = v^{-1}E_{n+1,n}^\theta(\alpha_n).\) Thus, we have

\[ f^2_n e_n = E_{n+1,n}^\theta(0)^2E_{n+1,n}^\theta(0) \]
\[ = v[2](E_{n+1,n}^\theta(0) + 2E_{n+1,n}^\theta(0))(0) + \frac{v^{-1}}{1 - v^{-2}}E_{n+1,n}^\theta(-\alpha_n) - \frac{v}{1 - v^{-2}}E_{n+1,n}^\theta(\alpha_n). \]
Second, compute \( e_n f_n^2 = E_{n,n+1}^\theta(0) E_{n+1,n}^\theta(0)^2 \). Since \( E_{n+1,n}^\theta(0)^2 = [2]^1(2E_{n+1,n}^\theta)(0) \), by Corollary 4.3, it follows from Theorem 4.4(2) that

\[
e_n f_n^2 = \frac{[2]^{\alpha_n}}{1 - v^{-2}} \left[ (2E_{n+1,n}^\theta - E_{n,n+1}^\theta) (\alpha_n) - (2E_{n+1,n}^\theta - E_{n,n+1}^\theta) (\alpha_n^-) \right] + \frac{[2]^{\alpha_n}}{1 - v^{-2}} E_{n+1,n}^\theta(0) + \frac{[2]^{\alpha_n}}{1 - v^{-2}} E_{n,n+1}^\theta(0)
\]

Finally, compute the right hand side \( f_n e_n f_n - (vd_n d_{n+1}^{-1} + v^{-1}d_{n+1}^{-1} d_{n+1}) f_n \). Similar to \( e_n f_n^2 \),

\[
e_n f_n = \frac{E_{n,n+1}^\theta(0) E_{n+1,n}^\theta(0)}{1 - v^{-2}}
\]

Further, we have

\[
E_{n+1,n}^\theta(0) E_{n,n+2}^\theta(0) = \frac{E_{n+1,n}^\theta(0) E_{n,n+2}^\theta(0)}{1 - v^{-2}} \left[ (-\alpha_n) + (E_{n+1,n}^\theta + E_{n,n+2}^\theta)(0) \right]
\]

This together with (6.1.2) and Theorem 4.3(1) gives

\[
f_n e_n f_n = \frac{E_{n,n+1}^\theta(0) (E_{n,n+2}^\theta(0) - (\alpha_n)) + \frac{1}{1 - v^{-2}} (O(\alpha_n) - O(\alpha_n^-))}{2}
\]

Again, by Theorem 4.3(1),

\[
vd_n d_{n+1}^{-1} f_n + v^{-1}d_{n+1}^{-1} f_n = \frac{v^2 O(-e_n + e_{n+1}) E_{n+1,n}^\theta(0) + v^{-2} O(-e_n + e_{n+1}) E_{n+1,n}^\theta(0)}{2}
\]

Now (6.1.1) follows from coefficient equating of both sides:

| Term | Coefficient |
|------|-------------|
| \((E_{n,n+1}^\theta + 2E_{n+1,n}^\theta)(0)\) | \([v][2] + [2]^{v^{-2}} = [2][2]\) |
| \((E_{n+1,n}^\theta + E_{n,n+2}^\theta)(0)\) | \([2] = [2]\) |
| \(E_{n+1,n}^\theta(-\alpha_n)\) | \(\frac{v^{-1}}{1 - v^{-2}} + \frac{v^{-3}}{1 - v^{-2}} = [2](1 + \frac{v^{-2}}{1 - v^{-2}} - 1)\) |
| \(E_{n+1,n}^\theta(\alpha_n)\) | \(\frac{[2]^{v^{-3}}}{1 - v^{-2}} = [2]\left(\frac{1}{1 - v^{-2}} - 1\right)\) |
| \(E_{n+1,n}^\theta(\alpha_n^-)\) | \(\frac{v^{-1}}{1 - v^{-2}} + \frac{v^{-1}}{1 - v^{-2}} - \frac{[2]^{v^{-3}}}{1 - v^{-2}} = [2]\left(-\frac{1}{1 - v^{-2}} - \frac{v^{-2}}{1 - v^{-2}}\right)\) |

which can be checked easily.

Recall the canonical projection map \( \pi_r : 2U^\theta(n) \to S^\theta(n, r)_{Q(n)} \) in Corollary 5.4. This together with the above result gives the following.
Corollary 6.2. There is an algebra epimorphism

\( \phi_r^1 = \pi_r \circ \phi^1 : U^j(n) \to S^j(n, r)_{Q(r)}. \)

Remark 6.3. We remark that the algebra homomorphism \( \phi_r^1 \) has been established in \textbf{BKLW18} Prop. 3.1] (compare \textbf{BKLW18} Lem. 3.12)].\(^3\) However, the proof there is geometric via a geometric setting of \( S^j(n, r) \). See \textbf{LL} for an algebraic approach involving type \( B \) Hecke algebras of two parameters.

We now use a Lusztig type form \( U^j(n)_Z \) for \( U^j(n) \) and show that \( \phi_r^1 \) restricts to a \( Z \)-algebra epimorphism.

Define

\[
\mathfrak{k}_i := \begin{cases} 
O(e_i), & \text{if } 1 \leq i \leq n, \\
v^{-1}O(e_{n+1}), & \text{if } i = n + 1.
\end{cases}
\]

and define \( \mathfrak{k}_{i,r} = \pi_r(\mathfrak{k}_i) \). Recall the notations \( \left[ \frac{\mathfrak{k}_{i,r}; 0}{\lambda_i} \right] := \prod_{j=1}^{\lambda_i} \frac{k_{i,r} v_{j+1} - k_{i,r} v_j}{v_{j+1} - v_j} \) at the end of §1 and \( \tilde{\lambda} \) in \((3.0.1)\).

Lemma 6.4. For any \( \lambda \in \Lambda(n+1, r) \), we have in \( S^j(n, r) \)

\[
\prod_{i=1}^{n} \left[ \frac{\mathfrak{k}_{i,r}; 0}{\lambda_i} \right] \cdot \left[ \frac{\mathfrak{k}_{n+1,r}; 0}{\lambda_{n+1}} \right] v^2 = \left[ \tilde{\lambda} \right].
\]

Proof. We only outline the proof; missing details can be found in \textbf{DDPW08} p.572. For \( 1 \leq i \leq n + 1, 0 \leq j \leq r \), let

\[
\mathbb{D}_i(j) = \sum_{\lambda \in \Lambda(n+1, r)} \left[ \frac{\mathfrak{k}_{i,r}; 0}{\lambda_i} \right] \left[ \frac{\mathfrak{k}_{n+1,r}; 0}{\lambda_{n+1}} \right] v^2 \lambda_{n+1} + 1 \left[ \tilde{\lambda} \right] = \sum_{j=0}^{r} v^2 j \mathbb{D}_{n+1}(j)
\]

Then, by \((4.1.2)\), we have \( \sum_{j=0}^{r} v^j \mathbb{D}_i(j) = \mathfrak{k}_{i,r} \) for \( 1 \leq i \leq n \), and

\[
\mathfrak{k}_{n+1,r} = v^{-1} \sum_{\lambda \in \Lambda(n+1, r)} v^2 \lambda_{n+1} + 1 \left[ \tilde{\lambda} \right] = \sum_{j=0}^{r} v^2 j \mathbb{D}_{n+1}(j)
\]

Thus, for \( 1 \leq i \leq n \), \( \left[ \frac{\mathfrak{k}_{i,r}; 0}{\lambda_i} \right] = \sum_{j \geq \lambda_i} \left[ \frac{j}{\lambda_i} \right] \mathbb{D}_i(j) \), and

\[
\left[ \frac{\mathfrak{k}_{n+1,r}; 0}{\lambda_{n+1}} \right] v^2 = \prod_{s=1}^{\lambda_{n+1}} \frac{\mathbb{D}_{n+1}(s) - \mathbb{D}_{n+1}(s-1)}{v^2 s - v^{-2 s}} = \prod_{s=1}^{\lambda_{n+1}} \left( \sum_{j=0}^{r} \frac{v^2 j - v^{-2 j}}{v^2 s - v^{-2 s}} \mathbb{D}_{n+1}(j) \right)
\]

Hence,

\[
\prod_{i=1}^{n} \left[ \frac{\mathfrak{k}_{i,r}; 0}{\lambda_i} \right] \cdot \left[ \frac{\mathfrak{k}_{n+1,r}; 0}{\lambda_{n+1}} \right] v^2 = \sum_{j_1 \geq \lambda_1, \ldots, j_{n+1} \geq \lambda_{n+1}} \left( \prod_{i=1}^{n} \left[ \frac{j_i}{\lambda_i} \right] \cdot \left[ \frac{j_{n+1}}{\lambda_{n+1}} \right] v^2 \mathbb{D}_1(j_1) \ldots \mathbb{D}_{n+1}(j_{n+1}) = \left[ \tilde{\lambda} \right],
\]

as desired.\( \square \)

\(^3\)The notation in \textbf{BKLW18} Lem. 3.12] has been twisted by the involution \( \omega \) in Lemma 2.4.
Let $U^j(n)_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra generated by divided powers
\[
e_i^{(m)} := \frac{e_i^n}{[m]!}, \quad f_i^{(m)} := \frac{f_i^n}{[m]!}, \quad d_i, \quad \left[ \frac{d_i; 0}{t} \right], \quad d_{n+1}, \quad \left[ \frac{d_{n+1}; 0}{t} \right],
\]
for all $m, t \in \mathbb{N}$ and $1 \leq i \leq n$.

Recall the elements $m^{A,0}$ defined in (5.0.3). This is a product of $(a_{i,j}E_{h+1,h}^\theta(0))$ for $1 \leq j \leq h < n \leq 2n + 1$ in the order defined in (5.0.1). We now define $M^{A,0}$ by replacing the factor $(a_{i,j}E_{h+1,h}^\theta(0))$ for $h \leq n$ by $f_h(a_{i,j})$ and replacing $(a_{i,j}E_{h+1,h}^\theta(0))$ for $h > n$ by $e_{2n+1-h}$. Then $M^{A,0} \in U^j(n)_\mathbb{Z}$.

For any $A \in \Xi_{2n+1}^{\text{diag}}$ and $\lambda \in \mathbb{N}^{n+1}$, define elements in $\mathfrak{A}^j(n)_\mathbb{Z}$ and $U^j(n)_\mathbb{Z}$:
\[
m^{A,\lambda} = \left( \prod_{i=1}^n \left[ \frac{k_i; 0}{\lambda_i} \right] \cdot \left[ \frac{k_{n+1}; 0}{\lambda_{n+1}} \right] \right) m^{A,0}, \quad M^{A,\lambda} = \left( \prod_{i=1}^n \left[ \frac{d_i; 0}{\lambda_i} \right] \cdot \left[ \frac{d_{n+1}; 0}{\lambda_{n+1}} \right] \right) M^{A,0}.
\]

**Theorem 6.5.** The epimorphism $\phi^j_t$ in Corollary 6.2 induces by restriction a $\mathbb{Z}$-algebra epimorphism $\phi^j_t : U^j(n)_\mathbb{Z} \rightarrow S^j(n, r)$ such that
\[
e_i^{(m)} \mapsto (mE_{h+1,h}^\theta(0), r), \quad f_i^{(m)} \mapsto (mE_{h+1,h}^\theta(0), r), \quad d_i \mapsto k_i.r.
\]

**Proof.** Let $\mathfrak{A}^j(n)_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra generated by
\[
(mE_{h+1,h}^\theta(0), mE_{i+1,i}^\theta(0), k_i, \left[ \frac{k_i; 0}{t} \right], k_{n+1}, \left[ \frac{k_{n+1}; 0}{t} \right], v^2)
\]
for all $m, t \in \mathbb{N}$ and $1 \leq i \leq n$. Since the epimorphism $\phi_t^j$ in Theorem 6.1 sends the generators for $U^j(n)_\mathbb{Z}$ onto the generators of $\mathfrak{A}^j(n)_\mathbb{Z}$, it follows that restricting to $U^j(n)_\mathbb{Z}$ results in a $\mathbb{Z}$-algebra epimorphism $\phi_t^j : U^j(n)_\mathbb{Z} \rightarrow \mathfrak{A}^j(n)_\mathbb{Z}$. On the other hand, the canonical projection $\pi_r : \mathfrak{A}^j(n) \rightarrow S^j(n, r)_\mathbb{Q}(v)$ sends the generators of $\mathfrak{A}^j(n)_\mathbb{Z}$ to elements in $S^j(n, r)$. Thus, $\phi_t^j = \pi_r \circ \phi_t^j$ defines a $\mathbb{Z}$-algebra homomorphism
\[
\phi_t^j : U^j(n)_\mathbb{Z} \rightarrow S^j(n, r).
\]

It remains to prove that $\phi_t^j$ is surjective.

For $A \in \Xi_{2n+1}$, let $A'$ is obtained from $A$ by replacing the diagonal entries with zeros and let
\[
m^{(A)} := m^{A', \text{ro}(A)}.
\]
Then, by definition, $\phi_t^j(M^{(A)}) = m^{(A)}$. Now apply $\pi_r$ to $m^{(A)}$. For $\bar{\lambda} = \text{ro}(A)$, by Lemma 6.4 and [BKLW13, Thm. 3.10] (cf. the proof of Lemma 5.1), we have
\[
\pi_r(m^{(A)}) = \pi_r \left( \prod_{i=1}^n \left[ \frac{k_i; 0}{\lambda_i} \right] \cdot \left[ \frac{k_{n+1}; 0}{\lambda_{n+1}} \right] \right) \pi_r(m^{A',0}),
\]
\[
= [\text{ro}(A)] \prod_{(j,h,i) \in \{2n+1, \leq\}} (a_{i,j}E_{h+1,h}^\theta(0), r)
\]
\[
= [A] + \text{lower terms}.
\]

Hence, the set $\{ \pi_r(m^{(A)}) \mid A \in \Xi_{2n+1, 2r+1} \}$ spans $S^j(n, r)$. This proves the surjectivity of $\phi_t^j$.

**Remarks 6.6.** (1) Due to the integral nature, we may specialise $\mathbb{Z}$ to any commutative ring $k$ to get a $k$-algebra epimorphism
\[
\phi_t^{j,k} : U^j(n)_k \rightarrow S^j(n, r)_k.
\]
Thus, if $k$ is a field, the representation category of $S^i(n, r)_k$ is a full subcategory of that of the hyperalgebra $U^i(n)_k$ of $U^i(n)$. In this way we link representations of the $i$-quantum groups (or $i$-quantum hyperalgebras) $U^i(n)_k$ with those of the Hecke algebras of of type $B$.

(2) Consider now representations of finite orthogonal groups $G(q) = GL_n(F_q)$ in non-defining characteristics. Those involves in (3.0.2) are related to the unipotent principal block which can be determined through the $q$-Schur algebra defined in (3.0.2). Theorem 6.5 extends further this relation to $i$-quantum groups. It would be conceivable that much part of the classical Dipper–James theory can be generalised to finite orthogonal groups.

(3) We remark that a similar relation in terms of a certain $i$-quantum coordinate algebra is developed in [LNX]. See Remark 3.6(3).

(4) The set (see (6.4.1))

$$\{d^i_1 \cdots d^{i_{n+1}}_n M^A, \lambda | A \in \Xi_2^{n+1}, \lambda \in \mathbb{N}^{n+1}, \tau_i \in \{0,1\}\}$$

forms a $Q(v)$-basis for $U^i(n)$ (cf. [DDPW08] Lem. 6.47]. It is reasonable to believe that the set forms a $Z$-basis for the integral form $U^i(n)_Z$.

(5) The integral form of the modified $i$-quantum group $\hat{U}^i(n)$ and its $i$-canonical basis have already been studied in [BKLW18] and its appendix by Bao–Li–Wang.

7. A NEW REALISATION OF $U^i(n)$

The main purpose in this section is to prove that the $Q(v)$-algebra homomorphism $\phi^i$ in Theorem 6.1 is in fact an isomorphism. Thus, we obtain a new realisation for the $i$-quantum group $U^i(n)$ by explicitly presenting its regular representation in terms of a basis and multiplication formulas in Theorem 4.4, i.e., the matrix representations for generators.

We explain the idea of the proof as follows. Let $S(2n+1, r) = S(2n+1, r)_{Q(v)}$ and $S^i(n, r) = S^i(n, r)_{Q(v)}$ be the $q$-Schur algebras over $Q(v)$ of type $A$ and $B$, respectively. By Theorem 3.5(1), the algebra epimorphisms $\rho_r : U(gl_{2n+1}) \rightarrow S(2n+1, r)$ induce a $Q(v)$-algebra monomorphism (see, e.g. [DG14])

$$\rho = \prod_{r \geq 0} \rho_r : U(gl_{2n+1}) \rightarrow S(2n+1) := \prod_{r \geq 0} S(2n+1, r).$$

It is known from Theorem 3.5(2) that $S^i(n, r) \subset S(2n+1, r)$. Thus, by the embedding $i$ in Lemma 2.3, both $U^i(n)$ and $S^i(n)$ are subalgebras of $U(gl_{2n+1})$ (via $i$) and $S(2n+1)$, respectively. The idea of proving that $\phi^i$ is injective is to show that $\phi^i$ is the restriction $\rho^i$ of $\rho$ to the image of $i$. In other words, we need to prove that $\phi^i$ coincides with $\rho^i$ via the isomorphism given in Theorem 3.5(2). Thus, we must prove that the action of $U^i(n)$ on $\Omega^{\otimes r}$ via $\iota$ coincides with the action of $S^i(n, r)$ on $\Omega^{\otimes r}$ via $\phi^i$.

We need some preparation. There are two cases to consider.

If $n \geq r$, then the basis element $e_\emptyset := [\text{diag}(\emptyset)] \in S^i(n, r)$ is an idempotent, where

$$\emptyset = \{(1, \ldots, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots, 1)\} \in \mathbb{N}^{2n+1},$$

and $e_\emptyset S^i(n, r) e_\emptyset \cong H(B_r)$, $S^i(n, r) e_\emptyset \cong T(n, r)$ (see (3.0.3)). This gives an $S^i(n, r)$-$H(B_r)$-bimodule structure on $S^i(n, r) e_\emptyset$. On the other hand, the tensor space $\Omega^{\otimes r}$ is an $S^i(n, r)$-$H(B_r)$-bimodule via (3.4.3) and (3.4.4). Moreover, there is an $S^i(n, r)$-$H(B_r)$-bimodule isomorphism

$$\eta_r : \Omega^{\otimes r} \rightarrow S^i(n, r) e_\emptyset, \quad \omega_i \mapsto [A_i],$$
where \( i = (i_1, i_2, \ldots, i_r) \in I(2n + 1, r) \) and \( A_i = (a_{k,l}) \in \Xi_{2n+1,2r+1} \) defined for \( N = 2n + 1 \) by
\[
ak_{k,l} = \begin{cases} 
\delta_{k,i_l}, & \text{if } l \in [1, r]; \\
\delta_{k,n+1} & \text{if } l = n + 1, \\
a_{N+1-k,N+1-l} & \text{if } l \in [N + 1 - r, N] \\
0, & \text{for the remaining columns.}
\end{cases}
\] (7.0.1)

Note that \( \text{co}(A_i) = (1^r, 0^{n-r}, 1, 0^{n-r}, 1^r) = \emptyset \).

We remark that the isomorphism \( \eta_r \) is given in \[\text{BKLW18}\] (compare \[\text{BKLW18}\] (2.9),(2.10)) with \[\text{BKLW18}\] (4.11),(4.12), where \( \tilde{e}_i \) is the \([A_i]\) here and \( v_i \) is the \( \omega_i \) here.

If \( n < r \), then we may identify \( \Xi_{2n+1,2r+1} \) as a subset of \( \Xi_{2r+1,2r+1} \) via the following embedding:
\[
\Xi_{2n+1,2r+1} \longrightarrow \Xi_{2r+1,2r+1}, \quad A \mapsto A^\circ = \begin{pmatrix} X & 0 & 0 & Y \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & X' \\
\end{pmatrix},
\] (7.0.2)

where \( X, X', Y, Y' \) are \( n \times n \) matrices, \( - - \) and \( | | \) represent the \( n + 1 \)th row and column of \( A \), and the zeros in \( A^\circ \) represent zero matrices of appropriate sizes. Thus, if \( n < r \), we may regard \( S^j(n, r) \) as a centraliser subalgebra of \( S^j(r, r) \) via the induces embedding \([A] \mapsto [A^\circ]\).

The embedding \([A] \mapsto [A^\circ]\) induces an embedding
\[
\tilde{\Lambda}(n+1, r) = \{ \text{ro}(A) \mid A \in \Xi_{2n+1,2r+1} \} \longrightarrow \tilde{\Lambda}(r+1, r), \text{ro}(A) \mapsto \text{ro}(A^\circ) = \text{ro}(A^\circ).
\]

Let \( f = \sum_{\lambda \in \Lambda(n+1, r)} [\tilde{\Lambda}^\circ] \). Then there is an algebra isomorphism \( fS^j(r, r) f \cong S^j(n, r) \). This induces an \( S^j(n, r):H(B_r)\)-bimodule isomorphism
\[
\eta_r : \Omega^\otimes r \longrightarrow fS^j(r, r) e_\emptyset, \quad \omega_i \longmapsto [A_i].
\]

Thus, as stated in Theorem 3.15(2), \( \eta_r \) induces an \( \mathbb{Q}(v)\)-algebra isomorphism in both cases:
\[
\tilde{\eta}_r : \text{End}_{\mathbb{K}(B_r)}(\Omega^\otimes r) \longrightarrow S^j(n, r).
\]

Recall the automorphism \( \omega \) in Lemma 2.4.

**Theorem 7.1.** The \( \mathbb{Q}(v)\)-algebra epimorphism \( \phi_r^i \circ \omega : U^j(n) \longrightarrow S^j(n, r) \) factors through the epimorphism \( \rho_r^i = \rho_r \circ i : U^j(n) \longrightarrow \text{End}_{\mathbb{K}(B_r)}(\Omega^\otimes r) \), that is, \( \tilde{\eta}_r \circ \rho_r^i = \phi_r^i \circ \omega \). Hence, the \( \mathbb{Q}(v)\)-algebra epimorphism
\[
\phi^j = \prod_{r \geq 0} \phi_r^j : U^j(n) \longrightarrow \mathfrak{A}^j(n) \subset \prod_{r \geq 0} S^j(n, r)
\]
is an isomorphism.

**Proof.** The commutative relations \( \tilde{\eta}_r \circ \rho_r^i = \phi_r^i \circ \omega \) for all \( r \geq 0 \) implies that the following diagram commutes:
\[
\begin{array}{ccc}
U^j(n) & \xrightarrow{\rho^j} & \prod_{r \geq 0} \text{End}_{\mathbb{K}(B_r)}(\Omega^\otimes r) \\
\omega \downarrow & & \tilde{\eta} \downarrow \\
U^j(n) & \xrightarrow{\phi^j} & \mathfrak{A}^j(n) \subset \prod_{r \geq 0} S^j(n, r)
\end{array}
\]
where $\tilde{\eta} = \prod_{r \geq 0} \tilde{\eta}_r$. Since $\rho = \prod_{r \geq 0} \rho_r : U(\mathfrak{g}l_{2n+1}) \to \prod_{r \geq 0} \text{End}_{\mathfrak{h}^q}(\Omega^r)$ is a monomorphism, it follows that the $\rho^i$ is injective. The commutative diagram shows that $\phi^i$ is injective. Hence, $\phi^i$ is an isomorphism.

It remains to prove $\tilde{\eta}_r \circ \rho^i_r = \phi^i_r \circ \omega$. We simply compare the actions of $\iota(d_i), \iota(e_i), \iota(f_i)$ on $\omega_1 \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r}$, respectively, with the actions on $[A_i]$ of 

$$
\pi_r \phi^i_r (d_i) = O(-e_i, r), \quad \pi_r \phi^i_r (e_i) = E^g_{h+1, h}(0, r), \quad \pi_r \phi^i_r (f_i) = E^g_{h, h+1}(0, r),
$$

where $A_i$ is defined in (7.0.1). By the embedding in (7.0.2), it suffices to assume that $n \geq r$. Let $N = 2n + 1$ as usual.

For $1 \leq h \leq n$ and $i = (i_1, \ldots, i_r) \in I(N, r)$ as in (3.4.2), we have by (2.3.1),

$$
\iota(d_h)(\omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r}) = K^{-1}_h K^{-1}_N \omega_{i_1} \otimes \cdots \otimes K^{-1}_h K^{-1}_N \omega_{i_r},
$$

where $g_h = -|\{ k \mid 1 \leq k \leq r, i_k = h \}| - |\{ k \mid 1 \leq k \leq r, i_k = N + 1 - h \}|$.

On the other hand,

$$
O(-e_h, r)[A_i] = \sum_{\lambda \in \Lambda(A(n+1, r))} v^{-\lambda} \tilde{\lambda} \cdot [A_i] = v^{-\text{ro}(A_i)h}[\text{ro}(A_i)] [A_i] = v^{-\text{ro}(A_i)h} [A_i],
$$

where, by (7.0.1) and noting $h \neq n + 1$,

$$
\text{ro}(A_i)h = |\{ l \mid l \in [1, r], a_{h, l} = 1 = \delta_{h, i_l} \} \cup \{ l \mid l \in [N - r + 1, N], a_{h, l} = 1 \}| = |\{ l \mid l \in [1, r], i_l = h \} \cup \{ l \mid l \in [N - r + 1, N], i_{N+1-l} = N + 1 - h \}| = |\{ k \mid 1 \leq k \leq r, i_k = h \}| + |\{ k \mid 1 \leq k \leq r, i_k = N + 1 - h \}| = g_h.
$$

Now, for $h = n + 1$,

$$
\iota(d_{n+1})(\omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r}) = v^{-1} K^{-2}_{n+1, h_1} \omega_{i_1} \otimes \cdots \otimes K^{-2}_{n+1, h_r} \omega_{i_r} = v^{g_{n+1}} \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r}
$$

where $g_{n+1} = -1 - 2|\{ k \mid 1 \leq k \leq r, i_k = n + 1 \}|$. But

$$
O(-e_{n+1}, r)[A_i] = \sum_{\lambda \in \Lambda(A(n+1, r))} v^{-\tilde{\lambda}_{n+1}} \tilde{\lambda} \cdot [A_i] = v^{-\text{ro}(A_i)_{n+1}}[\text{ro}(A_i)] [A_i] = v^{-\text{ro}(A_i)_{n+1}} [A_i],
$$

where, by (7.0.1),

$$
\text{ro}(A_i)_{n+1} = 2|\{ k \mid 1 \leq k \leq r, a_{n+1, k} = \delta_{n+1, i_k} = 1 \}| + a_{n+1, n+1} = 2|\{ k \mid 1 \leq k \leq r, i_k = n + 1 \}| + 1 = -g_{n+1}.
$$

This proves $\tilde{\eta}_r \circ \rho^i_r (d_i) = \phi^i_r \circ \omega (d_i)$ for all $i = 1, 2, \ldots, n + 1$. 


We now use the short notation $\omega_i \omega_2 \cdots \omega_i$ for a tensor product $\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i$. By Lemma 2.3 [2.3.2], and noting $K_i = K_i K_i^{-1}$, we have

$$
\iota(\epsilon_h)(\omega_1 \omega_2 \cdots \omega_i) = (F_h + \tilde{K}_h^{-1} E_{N-h})(\omega_1 \omega_2 \cdots \omega_i)
+ \sum_{l=1}^r \tilde{K}_h^{-1} \omega_1 \cdots \tilde{K}_h^{l-1} \omega_i \cdot F_h \omega_i \cdot \omega_{l+1} \cdots \omega_i
+ \sum_{l=1}^r \tilde{K}_h^{-1} \omega_1 \cdots \tilde{K}_h^{-1} \omega_{l+1} \cdots \tilde{K}_h^{-1} \omega_i \cdot (E_{N-h} \omega_i) \cdot \tilde{K}_h^{-1} \tilde{K}_{N-h} \omega_{l+1} \cdots \tilde{K}_h^{-1} \tilde{K}_{N-h} \omega_i
= \sum_{i=1}^r (v^{f_1(l)}(\omega_1 \cdots \omega_{i-1} \omega_{h+1} \omega_{i+1} \cdots \omega_i).
$$

where $f_1(l) = |\{k \mid 1 \leq k < l, i_k = h + 1\}| - |\{k \mid 1 \leq k < l, i_k = h\}|$ and

$$
f_2(l) = -|\{k \mid l < k \leq r, i_k = N - h + 1\}| + |\{k \mid l < k \leq r, i_k = N - h\}|
+ |\{k \mid 1 \leq k \leq r, i_k = h + 1\}| - |\{k \mid 1 \leq k \leq r, i_k = h\}|.
$$

Thus, since $\delta_{N-h,h+1} = \delta_{h,n}$, $\delta_{N-h,h} = 0$, and the summands in the second sum survive only when $i_l = N + 1 - h$, we may assume $i_l \neq h, h + 1$ and

$$
f_1(l) + f_2(l) + \delta_{N-h,h+1} - \delta_{N-h,h}
= -|\{k \mid l < k \leq r, i_k = N - h + 1\}| + |\{k \mid l < k \leq r, i_k = N - h\}|
+ |\{k \mid 1 \leq k \leq r, i_k = h + 1\}| - |\{k \mid 1 \leq k \leq r, i_k = h\}| + \delta_{h,n}.
$$

On the other hand, for the same $i$ with $A_i = (a_{k,l})$ and $\bar{\mu} = \text{ro}(A_i) - (\epsilon_h + e_{N-h+1})$, since $a_{h,i} = 1$ forces $a_{h+1,i} = 0$ and $a_{k,n+1} = 0$ for all $n + 1 \neq k \in [1, N]$ by (7.0.1), it follows that

$$
E_{h+1,h}^\theta(0,r)[A_i] = \sum_{\lambda \in \Lambda^{(n+1-r,1)}} [E_{h+1,h}^\theta + \bar{\lambda}] \cdot [A_i] = [E_{h+1,h}^\theta + \bar{\mu}] [A_i]
= \sum_{l \in [1,N], a_{h,l} \geq 1} v^{\beta_l^{i_h}(A_i,h)} [A_i - E_{h,l}^\theta + E_{h+1,l}^\theta]
= \sum_{l \in [1,N], a_{h,l} \geq 1} v^{\beta_l^{i_h}(A_i,h)} [A_i - E_{h,l}^\theta + E_{h+1,l}^\theta]
+ \sum_{l \in [N-r+1,N], a_{h,l} = \delta_{N+1-h,l+1} = 1} v^{\beta_l^{i_h}(A_i,h)} [A_i - E_{h,l}^\theta + E_{h+1,l}^\theta].
$$
where $\beta_i^r(A_i, h) = \sum_{k \leq l} a_{h+1,k} - \sum_{k < l} a_{h,k}$. Since the bijection from $[N - r + 1, N]$ to $[1, r]$ sending $l$ to $l' = N + 1 - l$ permutes the summands in the second sum, it follows that

$$E_{h+1,h}^\theta(0,r),[A_i] = \sum_{l \in [1,r], \ i = h} \nu_{h}^\theta(A_i, h)[A_i - E_{h,l}^\theta + E_{h+1,l}^\theta]$$

$$+ \sum_{l \in [1,r], \ i = N+1-h} \nu_{N+1-l}^\theta(A_i, h)[A_i - E_{h,N+1-l}^\theta + E_{h+1,N+1-l}^\theta]$$

We now compare this with (7.1.1) via $\eta_r$. Since, for $i_t = h$,

$$\eta_r(\omega_{i_1} \cdots \omega_{i_r}) = [A_i - E_{h,l}^\theta + E_{h+1,l}^\theta]$$

and, for $i_t = N + 1 - h$,

$$\eta_r(\omega_{i_1} \cdots \omega_{i_r}) = [A_i - E_{N+1-h,l}^\theta + E_{N-h,l}^\theta] = [A_i - E_{h,N+1-l}^\theta + E_{h+1,N+1-l}^\theta],$$

it remains to prove that

$$\beta_i^r(A_i, h) = f_1(l), \quad \text{if } i_t = h;$$

$$\beta_{N+1-l}^r(A_i, h) = f_1(l) + f_2(l) + \delta_{N-h,h+1} - \delta_{N-h,h}, \quad \text{if } i_t = N + 1 - h. \quad (7.1.2)$$

The first equation is clear since

$$\beta_i^r(A_i, h) = \sum_{k \leq l} a_{h+1,k} - \sum_{k < l} a_{h,k}$$

$$= \left| \{ k \mid 1 \leq k \leq l, a_{h+1,k} = \delta_{h+1,i_k} = 1 \} \right| - \left| \{ k \mid 1 \leq k < l, a_{h,k} = \delta_{h,i_k} = 1 \} \right|$$

$$= \left| \{ k \mid 1 \leq k < l, i_k = h + 1 \} \right| - \left| \{ k \mid 1 \leq k < l, i_k = h \} \right| \quad (\text{as } i_t = h)$$

$$= f_1(l).$$

For the second, since $l \in [1, r] \iff N + 1 - l \in [N - r + 1, N]$, by (7.0.1),

$$\beta_{N+1-l}^r(A_i, h) = \sum_{1 \leq k < N+1-l} a_{h+1,k} - \sum_{1 \leq k < N+1-l} a_{h,k}$$

$$= \left| \{ k \mid 1 \leq k \leq r, a_{h+1,k} = 1 \} \right| + \delta_{h,n} + \left| \{ k \mid N - r + 1 \leq k \leq N + 1 - l, a_{h+1,k} = 1 \} \right|$$

$$- \left| \{ k \mid 1 \leq k \leq r, a_{h,k} = 1 \} \right| - \left| \{ k \mid N - r + 1 \leq k < N + 1 - l, a_{h,k} = 1 \} \right|$$

$$= \left| \{ k \mid 1 \leq k \leq r, i_k = h + 1 \} \right| + \delta_{h,n} + \left| \{ k \mid N - r + 1 \leq k \leq N + 1 - l, i_{N+1-k} = N - h \} \right|$$

$$- \left| \{ k \mid 1 \leq k \leq r, i_k = h \} \right| - \left| \{ k \mid N - r + 1 \leq k < N + 1 - l, i_{N+1-k} = N + 1 - h \} \right|$$

$$= \left| \{ k \mid 1 \leq k \leq r, i_k = h + 1 \} \right| + \delta_{h,n} + \left| \{ k \mid l \leq k \leq r, i_k = N - h \} \right|$$

$$- \left| \{ k \mid 1 \leq k \leq r, i_k = h \} \right| - \left| \{ k \mid l \leq k \leq r, i_k = N + 1 - h \} \right|$$

$$= f_1(l) + f_2(l) + \delta_{N-h,h+1} - \delta_{N-h,h},$$

proving (7.1.2). This proves $\tilde{\eta}_r \circ \rho_i^\phi(e_h) = \phi_i^\rho \circ \omega(e_h)$ for all $i = 1, 2, \ldots, n$. The proof of $\tilde{\eta}_r \circ \rho_i^\phi(f_h) = \phi_i^\rho \circ \omega(f_h)$ is similar (and symmetric). \qed

By abuse of notation, let $A(j) = (\phi^\rho)^{-1}A(j)$. Then the basis \{ $A(j)$ $\} \in \mathbb{Z}_{2n+1,2r+1} \in \mathbb{Z}^{2n+1}$ for $\mathfrak{U}^q(n)$ gives rise to a new basis for $\mathfrak{U}^q(n)$. We now have a new presentation for $\mathfrak{U}^q(n)$ (cf. Theorem 5.2).

**Corollary 7.2.** The $i$-quantum group $\mathfrak{U}^q(n)$ is a $\mathbb{Q}(v)$-algebra with basis

$$\{ A(j) \mid A \in \mathbb{Z}_{2n+1,2r+1} \in \mathbb{Z}^{2n+1} \},$$
which has generators
\[ E^0_{h,h+1}(0), \ E^0_{h+1,h}(0), \ O(\pm e_i), \text{ for all } 1 \leq h \leq n, 1 \leq i \leq n+1, \]
and relations (1) \( O(\pm e_i)A(j) = v^\pm\rho(A)i_A(j \pm e_i) \) together with (2) and (3) in Theorem 4.4.

In other words, the multiplication formulas (1)–(3) give rise to the matrix form of the regular representation of \( U^j(n) \).

**Remark 7.3.** We expect to investigate applications of this new realisation for \( U^j(n) \). For example, the existence of PBW type bases for \( U^j(n) \) seems not clear. It is very plausible to construct such a basis by using the divided powers of “root vectors” \( E^0_{i,j}(0) \) for all \( 1 \leq j < i \leq 2n + 1 \) and to establish a triangular relation with the integral monomial basis discussed in Remark 3.6(4).

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J. D., School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

Email address: j.du@unsw.edu.au

Y. W., Department of Mathematics, Tongji University, Shanghai, 200092, China

Email address: 1710062@tongji.edu.cn