Fast Distributed Coordination of Distributed Energy Resources Over Time-Varying Communication Networks

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Abstract—In this paper, we consider the problem of optimally coordinating the response of a group of distributed energy resources (DERs) to meet total electric power demand while minimizing the total generation cost and respecting the DER capacity limits. This problem can be cast as a convex optimization problem, where the global objective is to minimize a sum of convex functions corresponding to the costs of generating power from the DERs while satisfying linear inequality constraints corresponding to the DER capacity limits and a linear equality constraint corresponding to the total power generated by the DERs being equal to the total power demand. We develop distributed algorithms to solve the DER coordination problem over time-varying communication networks with either (i) bidirectional or (ii) unidirectional communication links. The algorithms proposed for directed communication graphs have the geometric convergence rate when communication out-degrees are unknown to agents. The algorithms can be seen as the distributed versions of a centralized primal-dual algorithm. We showcase the algorithms using the standard IEEE 39–bus test system, and compare their performance against that of existing ones.

I. INTRODUCTION

It is envisioned that present-day power grids mainly depend on centralized power generation stations that will transition towards more decentralized power generation mostly based on DERs. One of the obstacles in making this shift happen is to find effective control strategies for coordinating DERs. Due to high renewable intermittency in future power grids, DERs will need to more frequently adjust their set-points, which entails development of fast control strategies. Also, because of the communication overhead, it may not be feasible to use a centralized approach to coordinate a large number of DERs over a large geographic area. This necessitates development of distributed control strategies for DER coordination that scale well to power systems of large size.

In this work, we consider a group of DERs and electrical loads that are interconnected by an electric power network and interfaced with a communication network. Each DER is endowed with a power generation cost function, which is unknown to other DERs, and can only be operated within its capacity constraints. A computing device attached to each DER is able to communicate with the computing devices of other DERs located within its communication range. Then, the objective is to determine optimal power outputs of the DERs in a distributed manner to satisfy total electric power demand while minimizing the total generation cost and respecting the DER capacity limits. This DER coordination problem can be cast as a convex optimization problem (see, e.g., [1]–[7]), where the global objective is to minimize a sum of convex functions corresponding to the costs of generating power from the DERs while satisfying linear inequality constraints on the power produced by each DER and a linear equality constraint corresponding to the total generated power being equal to the total power consumed by electrical loads.

Since we aim to solve the DER coordination problem in a distributed manner, we also consider the problem of achieving resilient and fault-tolerant coordination of DERs, which is central to the distributed implementation. Such coordination requires a control design that is robust to communication delays and random data packet losses. In this paper, we focus on the challenges that arise due to the time-varying nature of the underlying communication network, and address the DER coordination problem via distributed algorithms that are capable of operating over time-varying communication graphs with either (i) bidirectional or (ii) unidirectional communication links. These algorithms also have geometric convergence rate, which is a desirable feature for ensuring fast performance. We believe that the proposed algorithms can be extended to solve more complex DER coordination problems with additional constraints, e.g., line flow constraints, voltage constraints, or reactive power balance constraints, as long as they are linear and have a separable structure, i.e., each constraint is local or involves only a pair of neighboring nodes.

A vast body of work has focused on solving the DER coordination problem in a distributed way (see, e.g., [1]–[11]). Earlier works focused on time-invariant communication networks (see, e.g., [1], [2], [8], [9]). In one of the earliest works, the authors of [1] proposed a distributed approach, in which agents’ local estimates are driven to the optimal incremental cost via the leader-follower consensus algorithm. The authors of [2] utilize the so-called ratio-consensus algorithm (see, e.g., [12], [13]) to distributively compute the solution to the dual formulation of the DER coordination problem. Later works focused on time-varying communication networks (see, e.g., [3]–[7]). In [7], the authors propose a robustified version of the so-called subgradient-push method (see, e.g., [14]) that...
operates over time-varying directed communication networks; the algorithm utilizes the so-called push-sum protocol (see, e.g., [15]–[17]) to converge to a consensual solution. In [3], the authors propose a distributed algorithm that uses a consensus term to converge to a common incremental cost, and a subgradient term to satisfy the total load demand; the algorithm is designed assuming that generation cost functions are quadratic. However, convergence of the algorithms proposed in [7] and [8] is not guaranteed to be geometrically fast and might be slow due to the fact that the algorithms use a diminishing stepsize. In [10] and [11], the authors propose distributed algorithms based on the dual-ascent method that have the geometric convergence rate but require the agents to know their communication out-degrees.

Our starting point in the design of the algorithms is a primal-dual algorithm (first order Lagrangian method), where the dual variable associated with the power balance constraint depends on the total power imbalance (supply-demand mismatch). We then develop distributed versions of this primal-dual algorithm by having DERs closely emulate the iterations of the primal-dual algorithm. To this end, each node with a DER maintains an estimate of the dual variable and updates it using a local gradient term to satisfy the total load demand; the algorithm is said to converge geometrically fast over time-varying directed communication networks. In this work, we present a distributed algorithm for solving (1) geometrically fast over time-varying communication networks.

B. Cyber Layer

Here, we introduce the cyber layer model for representing the communication network interconnecting the nodes with a DER. Let $G(0) = (\mathcal{V}, \mathcal{E}(0))$ denote the nominal communication graph, where $\mathcal{E}(0)$ is the set of all communication links. During any time interval $(t_k, t_{k+1})$, successful data transmissions among the DERs can be captured by graph $G(c)[k] = (\mathcal{V}, \mathcal{E}(c)[k])$, where $\mathcal{V}$ is the set of DERs, and $\mathcal{E}(c)[k] \subseteq \mathcal{E}(0)$ is the set of active communication links.

When $G(0)$ is undirected, $\{i, j\} \in \mathcal{E}(0)[k]$ if nodes $i$ and $j$ simultaneously exchange information with each other during time interval $(t_k, t_{k+1})$. Let $\mathcal{N}_i[k]$ denote the set of neighbors of node $i$ during time interval $(t_k, t_{k+1})$, i.e., $\mathcal{N}_i[k] := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}(0)[k]\}$.

When $G(0)$ is directed, $(i, j) \in \mathcal{E}(c)[k]$ if node $j$ receives information from node $i$ during time interval $(t_k, t_{k+1})$ but not necessarily vice versa. Let $\mathcal{N}_i^+[k]$ and $\mathcal{N}_i^-[k]$ denote the sets of out-neighbors and in-neighbors of node $i$, respectively, during time interval $(t_k, t_{k+1})$, i.e., $\mathcal{N}_i^+[k] := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}(c)[k]\}$ and $\mathcal{N}_i^-[k] := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}(c)[k]\}$. We define the node $i$ (communication) out-degree (including itself) to be $D_i^+[k] := |\mathcal{N}_i^+[k]| + 1$. Let $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}(0)\}$, and $d_i^+ := |\mathcal{N}_i| + 1$ denotes the nominal out-degree.

Regarding the communication model, we also make the following standard assumption (see, e.g., [14], [18]).

Assumption 2. When $G(0)$ is undirected (directed), there exists some positive integer $B$ such that the graph $\bigcup_{l=kB}^{(k+1)B-1} \mathcal{E}(c)[l]$ is connected (strongly connected) for $k = 0, 1, \ldots$. 

III. DER COORDINATION OVER TIME-VARYING UNDIRECTED GRAPHS

In this section, we present a distributed algorithm for solving the DER coordination problem (1) over time-varying undirected communication graphs.
A. Distributed Primal-Dual Algorithm

Our starting point to solve (1) is the following primal-dual algorithm:

\[
p[k + 1] = \left[ p[k] - s \nabla f(p[k]) + s \xi \lambda[k] \right]_{\mathcal{P}}, \tag{2a}
\]

\[
\lambda[k + 1] = \lambda[k] - s (1^T p[k] - \ell), \tag{2b}
\]

where \( [\cdot]^\mathcal{P} \) denotes the projection onto the interval \([p, \bar{p}]\), \( s \geq 0 \) is a constant stepsize, \( \mathbf{1} \) denotes the all-ones vector, and \( \xi \in (0, 1) \) is a constant parameter. \( \ell \) is the dual variable associated with the power balance constraint, \( 1^T p = \ell \). The algorithm (2) does not conform to the general communication model described in Section II-B because in order to execute it, the total power imbalance, \( 1^T p[k] - \ell \), is needed to update \( \lambda[k] \).

To design a distributed version of (2), each node \( n \) needs a local estimate of \( \lambda[k] \), denoted by \( \lambda_n[k] \). To update \( \lambda_n[k] \), it should also have an estimate of the total power imbalance, \( 1^T p[n] - \ell \). One such estimate can be constructed purely based on the local power imbalance, \( \tilde{\lambda}(p,n) \), where \( \tilde{\lambda} \) is some estimate of the total power imbalance, \( \lambda_n[k] \), and \( \tilde{\lambda} \) can be one, which leads us to the following distributed algorithm:

\[
p_n[k + 1] = p_n[k] - s \nabla f(p_n[k]) + s \xi \lambda_n[k], \tag{3a}
\]

\[
\lambda_n[k + 1] = (1 - \sum_j a_{ij}[k]) \lambda_n[k] + \sum_j a_{ij}[k] \lambda_{j}[k] - s \tilde{\lambda}(p_n[k] - \ell), \tag{3b}
\]

where \( s[k] \) is a stepsize, \( a_{ij}[k] = \bar{a}_{ij}[k] \geq \eta \) if \( \{i, j\} \in \mathcal{E}(c)[k] \), \( a_{ij}[k] = 0 \) if \( \{i, j\} \notin \mathcal{E}(c)[k] \), and the constant \( \eta > 0 \) is chosen so that \( 1 - \sum_j a_{ij}[k] \geq \eta \). Even if \( \tilde{\lambda} \) is an accurate estimate of \( \lambda_n[k] \), it is a very crude estimate of the total power imbalance, \( 1^T p[k] - \ell \), and results in slow convergence as will be demonstrated later via numerical simulations.

A better approach is to let each node estimate the total power imbalance by using its local power imbalance and the estimates of its neighbors. To elaborate on this further, we let \( y_i \) denote node \( i \)’s estimate of the total power imbalance. Then, once we update \( y_i \) is as follows:

\[
y_i[k + 1] = (1 - \sum_j a_{ij}[k]) y_i[k] + \sum_j a_{ij}[k] y_j[k] + \hat{n}(p_i[k + 1] - p_i[k]), \tag{4}
\]

where \( y_0[i] = \hat{n}(p_0[i] - \ell_i) \). In [4], node \( i \) first computes the average of its estimate and the estimates of its neighbors, and then, adds \( \hat{n}(p_i[k + 1] - p_i[k]) \) to ensure that the average of all total power imbalance estimates is always equal to \( \frac{1}{n} 1^T p[n] - \ell \), which is equal to the total power imbalance, \( 1^T p[k] - \ell \) if \( \hat{n} = n \). This second step allows local estimates to remain close to the total power imbalance. Below, we combine the primal and dual variables updates with the update of the total power imbalance estimate:

\[
p_n[k + 1] = p_n[k] - s \nabla f(p_n[k]) - s \xi \lambda_n[k], \tag{5a}
\]

\[
\lambda_n[k + 1] = (1 - \sum_j a_{ij}[k]) \lambda_n[k] + \sum_j a_{ij}[k] \lambda_{j}[k] - s \tilde{\lambda}(p_n[k] - \ell), \tag{5b}
\]

In (5b), node \( n \) computes the average of its estimate and the estimates of its neighbors, which yields a good estimate of \( \lambda[k] \). We note that (5) closely emulates the centralized primal-dual algorithm in (2) and, as we will show later, converges geometrically fast.

To examine (5) more closely, we rewrite the \( p_n \)-updates in (5a) in vector form as follows:

\[
p[k + 1] = \left[ p[k] - s \nabla f(p[k]) + s \xi \lambda[k] \right]_{\mathcal{P}}, \tag{6a}
\]

\[
\lambda[k + 1] = \lambda[k] - s (1^T p[k] - \ell), \tag{6b}
\]

where \( e[k] \equiv \lambda[k] - \frac{\lambda_n[k]}{n} 1^T \lambda[k], \lambda[k] = (\lambda_1[k], \ldots, \lambda_n[k])^T, \) and \( \lambda_n[k] = \frac{1}{n} 1^T \lambda[k] \). Here, \( e \) is the disturbance generated when a communication link becomes inactive. Without the disturbance \( e \), (6) has almost the same form as (2). As illustrated in Figure 1, the algorithm (6) can be viewed as a feedback interconnection of the nominal centralized algorithm (6), and the disturbance \( e \), where the convergence error is denoted by

\[
z[k] := \left[ p[k] - p^* \right],
\]

where \( (p^*, \lambda^*) \) is the equilibrium of (6) when \( e[k] \equiv 0, \forall k \). Finding the relationship between the feedback system gain and the step-size \( s \) allows to quantify the effect of the feedback system on the convergence error. We later show that the gain can be decreased by decreasing the step-size. Furthermore, if the gain is sufficiently small, then, the feedback loop does not amplify the energy of the convergence error as it passes through the system, and the error decays to zero, which follows from the small-gain theorem. To carry out the small-gain-theorem-based analysis, we adopt the appropriate metric for measuring the energy of the signals of interest:

\[
\|z\|_2^K := \max_{0 \leq k \leq K} a^{-k}\|z[k]\|_2,
\]

for some \( a \in (0, 1) \), where \( \|\cdot\|_2 \) is the Euclidean norm; this definition of energy was used previously in [18]. If the energy of the convergence error, \( \|z\|_2^K \), is bounded for all \( K > 0 \), then, \( a^{-K}\|z[k]\|_2 \) is always bounded, and \( z[k] \) converges to...
zero at a geometric rate $O(a^k)$.

In order to invoke the small-gain theorem, we must first show that the following relations between the energy of the convergence error and that of the disturbance hold:

**R1.** $\|z\|_2^a \leq \alpha_1 \|e\|_2^a + \beta_1$ for some positive $\alpha_1$ and $\beta_1$.

**R2.** $\|e\|_2^a \leq \alpha_2 \|z\|_2^a + \beta_2$ for some positive $\alpha_2$ and $\beta_2$.

The results R1 and R2 are equivalent to having that the systems $H_1$ and $H_2$ in Figure 1 are finite-gain stable. From R1 and R2, it can be determined that the feedback system gain is $\alpha_1 + \alpha_2$. Note that, for sufficiently small $s$, the gain $\alpha_1 + \alpha_2$ becomes strictly smaller than 1, and $\|z\|_2^a$ becomes bounded for all $K > 0$. Thus, $z[k]$ converges to zero at a geometric rate $O(a^k)$.

Next, we show that the relations R1 and R2 hold and present the convergence results for the algorithm (5).

**B. Convergence Analysis**

In the following result, we show that the system $H_1$ is finite-gain stable.

**Proposition 1.** Let Assumptions [1] and [2] hold. Then, under (6), we have that

$$\|z\|_2^a \leq \alpha_1 \|e\|_2^a + \beta_1,$$

(7)

for some positive $\alpha_1$ and $\beta_1$, $a \in (0,1)$, sufficiently small $s > 0$, and for any $\xi \in (0, \frac{n}{\eta})$.

**Proof.** Let

$$G[k] := p[k] - s \nabla f(p[k]) + \frac{s}{n} \hat{w} \lambda[k] + s \lambda e[k],$$

$$H[k] := \lambda[k] - s (1^T p[k] - \ell),$$

$$G(p^*, \lambda^*) := p^* - s \nabla f(p^*) + \frac{s}{n} \lambda^*.$$

It can be shown that

$$\left\| \left[ \begin{array}{c} p[k+1] - p^* \\ \lambda[k+1] - \lambda^* \\ \end{array} \right] \right\| \leq \left\| \left[ \begin{array}{c} G[k] \\ H[k] \end{array} \right] - \left[ \begin{array}{c} G(p^*, \lambda^*) \\ \lambda^* \end{array} \right] \right\|.$$  

(8)

It follows from the mean value theorem [19, Theorem 5.1] applied to each row that

$$\left[ \begin{array}{c} G[k] \\ H[k] \end{array} \right] - \left[ \begin{array}{c} G(p^*, \lambda^*) \\ \lambda^* \end{array} \right] = A[k] \left[ \begin{array}{c} p[k] - p^* \\ \lambda[k] - \lambda^* \end{array} \right] + s \xi \left[ \begin{array}{c} e[k] \\ 0 \end{array} \right],$$

(9)

where

$$A[k] := \left[ \begin{array}{cc} I - s \nabla^2 f(v[k]) & s \xi \frac{\hat{w}}{n} \\ -s1^T & 1 \end{array} \right],$$

(10)

and $v_{ij}[k]$ lies on the line segment connecting $p_i[k]$ and $p_j^*$. Define

$$B[k] := \left[ \begin{array}{cc} \nabla^2 f(v[k]) - \xi \frac{\hat{w}}{n} \\ 1 \end{array} \right]$$

(11)

so that $A[k] = I - sB[k]$. We show that all eigenvalues of $B[k]$ have a strictly positive real part. Suppose $\mu$ is an eigenvalue of $B[k]$ and $[v^H, w^H]^H$ is an eigenvector corresponding to $\mu$, where $x^H$ denotes the Hermitian transpose of $x$. Then,

$$\text{Re} \left( [v^H, w^H]^H B[k] \left[ \begin{array}{c} v \\ w \end{array} \right] \right) = \text{Re} \left( \mu [v^H, w^H] \left[ \begin{array}{c} v \\ w \end{array} \right] \right) = \text{Re}(\mu)(\|v\|_2^2 + \|w\|_2^2),$$

(12)

where $\| \cdot \|_2$ is the Euclidean norm. We also have that

$$\text{Re} \left( [v^H, w^H]^H B[k] \left[ \begin{array}{c} v \\ w \end{array} \right] \right) = \text{Re} \left( v^H \nabla^2 f(v[k]) v \right) - \frac{\xi}{n} v^H w + w^H 1^T v.$$  

(13)

Since $v[k] \in [p, \bar{p}]$, $\nabla^2 f(v[k])$ is positive definite. From the fact that

$$B[k] \left[ \begin{array}{c} v \\ w \end{array} \right] = \mu \left[ \begin{array}{c} v \\ w \end{array} \right],$$

we have that $1^T v = \mu w$, from which it follows that

$$w^H 1^T v = \mu \|w\|_2^2 \geq 0,$$

Therefore, $v^H w = w^H 1^T v = \mu \|w\|_2^2 \geq 0$, from which it follows that

$$\text{Re} \left( - \frac{\xi}{n} v^H 1^T v + w^H 1^T v \right) \geq 0,$$

and

$$\text{Re} \left( [v^H, w^H]^H B[k] \left[ \begin{array}{c} v \\ w \end{array} \right] \right) \geq \text{Re} \left( v^H \nabla^2 f(v[k]) v \right) > 0,$$

(14)

\forall v \neq 0. If $\text{Re}(\mu) = 0$, then, it follows from (14) that $v = 0$, and

$$B[k] \left[ \begin{array}{c} 0 \\ w \end{array} \right] = 0,$$

(15)

from which we conclude that $1^T w = 0$, and $w = 0$, which contradicts the fact that $[v^H, w^H]^H \neq 0$. Therefore, all eigenvalues of $B[k]$ have a strictly positive real part, and, for sufficiently small $s$, the spectral radius of $A[k]$ denoted by $\rho(A[k])$ is strictly less than 1.

In the following, we show that there exists an induced matrix norm $\| \cdot \|$ such that $\|A[k]\| < 1, \forall k$. Let

$$A[k] = A + s \hat{A}[k],$$

(16)

where

$$A = \left[ \begin{array}{cc} I - s \nabla^2 f(p) & s \xi \frac{\hat{w}}{n} \\ -s1^T & 1 \end{array} \right],$$

(17)

$$\hat{A}[k] = \left[ \begin{array}{cc} \nabla^2 f(p) - \nabla^2 f(v[k]) & 0 \\ 0 & 0 \end{array} \right].$$

(18)

By the Schur triangularization theorem (see, e.g., [20, Theorem 2.3.1]), there is a unitary matrix $U$ and an upper triangular matrix $S$ such that $A = U S U^H A U$. Let $D_t := \text{diag}(t, t^2, \ldots, t^n)$. In the following, we use the fact that if $\|A\|$ is a matrix norm, then, $\|S^{-1} A S \|$ is also a matrix norm, for any real $A$ and nonsingular $S$ (see, e.g., [20, Theorem 5.6.7]). We choose the following matrix norm:

$$\|A[k]\| := \left\| (U^H D_t^{-1})^{-1} A[k] U D_t^{-1} \right\|_1 = \|D_t U A[k] U^H D_t^{-1} \|_1$$

(19)

where $\|a_{ij}\|_1 := \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$. Choosing sufficiently
small $s$ and large $t$ will make $\|A[k]\| \leq \rho(A[k]) + \epsilon$ for any $\epsilon > 0$, and, in particular, $\|A[k]\| \leq \gamma, \forall k$ and some $\gamma < 1$. By [20, Theorem 5.6.26], there is an induced matrix norm $\|\cdot\|$ such that $\|A[k]\| \leq \|A\|$. Taking $\|\cdot\|$ on both sides of (9) and applying the triangle inequality gives
\[
\|G[k] - (G(p^*, \lambda^*) + \lambda^* \lambda^*) \| \leq \|A[k]\|\|G[p^*] - p^* + \lambda^* \lambda^* \| + \xi |e[k]| + \|e[k]\|, \tag{20}
\]
and applying (8) gives
\[
\|p[k] - p^*\| \leq \|A[k]\|\|p[k] - p^*\| + \xi |e[k]| + \|e[k]\| \leq \gamma \|p[k] - p^*\| + \xi \|e[k]\|. \tag{21}
\]
We multiply both sides of (21) by $a^{-(k+1)}$ to obtain
\[
a^{-(k+1)}\|z[k + 1]\| \leq a^{-k}\|z[k]\| + \frac{\gamma a^{-k}}{a^{-(k+1)}} + \frac{s\xi a^{-k}}{a^{-(k+1)}}|e[k]| \tag{22}.
\]
We take $\max_{0 \leq k \leq K} (\cdot)$ on both sides of (22) to obtain
\[
\max_{0 \leq k \leq K} a^{-(k+1)}\|z[k + 1]\| \leq \frac{\gamma}{a} \max_{0 \leq k \leq K} a^{-k}\|z[k]\| + \frac{s\xi}{a} \max_{0 \leq k \leq K} a^{-k}|e[k]| \tag{23},
\]
which can be written as
\[
\|z\|a,k+1 \leq \frac{\gamma}{a}\|z\|a,k + \frac{s\xi}{a}\|e\|a,K + \|z[0]\|, \tag{24}
\]
and, thus,
\[
\|z\|a,K \leq \frac{s\xi}{a - \gamma}\|e\|a,K + \frac{a}{a - \gamma}\|z[0]\|, \tag{25}
\]
where $\|z\|a,K := \max_{0 \leq k \leq K} a^{-k}\|x[k]\|$ for a sequence $\{x[k]\}_{k=0}^{K}$. Because $\|\cdot\|_2 \leq \alpha \|\cdot\|_a$ and $\|A\| \leq \beta \|\cdot\|$ for some $\alpha$ and $\beta$, we have that $\|z\|a,K \leq \|z\|_2 / \alpha$, $\|e\|a,K \leq \beta \|e\|_2$, and, hence,
\[
\|z\|_2 a,K \leq \frac{s\xi \beta}{a - \gamma}\|e\|_2 a + \frac{a}{a - \gamma}\|z[0]\|,
\]
which yields (22) for some positive $\alpha_1$ and $\beta_1$.

We omit the proof of the next result, where we show that the system $\mathcal{H}_2$ is finite-gain stable, since it is analogous to that of a similar result proposed for directed communication graphs in Section [IV].

**Proposition 2.** Let Assumptions [7] and [2] hold. Then, under the algorithm [5], we have that
\[
\|e\|_2 a,K \leq \alpha_1 + \beta_2, \tag{26}
\]
for some positive $\alpha_2$ and $\beta_2$, $a \in (0, 1)$, and sufficiently small $s > 0$.

Now, we show the convergence of the algorithm [5] by applying the small-gain theorem to the results in Propositions [1]–[5].

**Proposition 3.** Let Assumptions [7] and [2] hold. Then, under the algorithm [5],
\[
\|z\|a,K \leq \beta, \tag{27}
\]
for some $\beta > 0$, $a \in (0, 1)$, and sufficiently small $s > 0$. In particular, $(p_t[k], \lambda_t[k]), \forall i$, converges to $(p^*, \lambda^*)$ at a geometric rate $O(a^\gamma)$.

**Proof.** By using Propositions [1] and [2] it follows that
\[
\|z\|_2 a,K \leq \alpha_1 \|e\|_2 K + \beta_1 \leq \alpha_1 (s\|\|_2 K + \beta_2) + \beta_1, \tag{28}
\]
which after rearranging the terms results in
\[
\|z\|_2 a,K \leq \frac{\alpha_1 \beta_2 + \beta_1}{1 - s\alpha_1\beta_2} =: \beta. \tag{29}
\]
Hence, for sufficiently small $s$, $s\alpha_1\beta_2 < 1$, and $\beta$ is finite. \hfill \Box

### IV. DER Coordination Over Time-Varying Directed Graphs

In this section, we present a distributed algorithm for solving the DER coordination problem [4] over time-varying directed communication graphs.

#### A. Distributed Primal-Dual Algorithm

For directed graphs, we propose the following distributed primal-dual algorithm:

\[\lambda_i[k + 1] = \sum_{j \in \mathcal{N}_i^{-}[k] \cup \{i\}} \frac{\lambda_j[k] - sy_j[k]}{D_j\gamma[k]}, \tag{30a}\]
\[v_i[k + 1] = \sum_{j \in \mathcal{N}_i^{-}[k] \cup \{i\}} \frac{v_j[k]}{D_j\gamma[k]}, \tag{30b}\]
\[x_i[k + 1] = \frac{\lambda_i[k + 1]}{v_i[k + 1]}, \tag{30c}\]
\[p_i[k + 1] = \left[ p_i[k] - s\nabla f_i(p_i[k]) + s\xi x_i[k + 1] \right]_{p_i}, \tag{30d}\]
\[y_i[k + 1] = \sum_{j \in \mathcal{N}_i^{-}[k] \cup \{i\}} \frac{y_j[k + 1]}{D_j\gamma[k]} + \hat{n}(p_i[k + 1] - p_i[k]), \tag{30e}\]

where $\lambda_i[k]$ is the node $i$ estimate of the dual variable, $x_i[k], y_i[k]$ is the node $i$ estimate of the total power imbalance, $1/p[k] - \ell$, the algorithm [30] is initialized with $\lambda_i[0] = 0$, $v_i[0] = 1$, and $y_i[0] = \hat{n}(p_i[0] - \ell_i)$. In [30], node $i$ estimate of the dual variable $x_i[k], \lambda_i[k]$, is updated using the so-called Push-DiGing algorithm proposed in [18].

**Remark 1.** The algorithm [30] is similar to the algorithm in [17] in that they both use the Push-DiGing algorithm and the gradient tracking idea in [18] to update the dual variables, and both require agents to know their communication outgoing-degree $D_j\gamma[k]$’s at each iteration $k$. However, there are a few subtle differences that can be pointed out. The algorithm [30] is based on the first order Lagrangian method [21], while the algorithm in [17] is based on the dual-ascent method [22]. The
Proof. 0

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Proposition 4. Let Assumptions 1 and 2 hold. Then, under (31), we have that

\[ R. \quad \|z\|_{2}^{\alpha_{K},C} \leq \alpha_{1}\|e\|_{2}^{\alpha_{K},C} + \beta_{1}, \]  

for some positive \( \alpha_{1} \) and \( \beta_{1} \), \( \alpha \in (0,1) \), sufficiently small \( s > 0 \), and for any \( \xi \in (0, \frac{1}{2}) \), where

\[ z[k] := \left[ \frac{p[k] - p^{*}}{\lambda[k] - \lambda^{*}} \right], \]

and \( (p^{*}, \lambda^{*}) \) is the equilibrium of (31) when \( e[k] \equiv 0, \ \forall k \).

In the next result, we show that R2 holds, i.e., the system \( \mathcal{H}_{2} \) is finite-gain stable.

Proposition 5. Let Assumptions 1 and 2 hold. Then, under the algorithm (30), we have that

\[ R2. \quad \|e\|_{2}^{\alpha_{K},C} \leq s\alpha_{2}\|z\|_{2}^{\alpha_{K},C} + \beta_{2}, \]

for some positive \( \alpha_{2} \) and \( \beta_{2} \), \( \alpha \in (0,1) \), and sufficiently small \( s > 0 \).

Proof. Let \( V[k] := \text{diag}(v[k]) \), and \( C[k] \in \mathbb{R}^{n \times n} \) with \( C\alpha[k] := 1/(D^{\alpha}[k]) \) and

\[ C\alpha_{ij}[k] := \begin{cases} \frac{1}{D^{\alpha}[k]} & \text{if } (j, i) \in E^{(\alpha)}[k], \\ 0 & \text{else.} \end{cases} \]

We rewrite (30) as follows:

\[ x[k+1] = (V[k+1])^{-1}C[k](V[k]x[k] - sy[k]) \]

\[ h[k+1] = (V[k+1])^{-1}y[k+1] \]

\[ = (V[k+1])^{-1}(C[k]V[k]h[k] + \hat{n}(p[k+1] - p[k])) \]

\[ = \hat{R}[k](x[k] - sh[k]), \]  

(35a)

\[ h[k+1] = (V[k+1])^{-1}y[k+1] \]

\[ = (V[k+1])^{-1}(C[k]V[k]h[k] + \hat{n}(p[k+1] - p[k])) \]

\[ = \hat{R}[k](x[k] - sh[k]), \]  

(35b)

\[ p[k+1] = [p[k] - s\nabla f(p[k])] - s^{2}\frac{\hat{n}}{n}(1^{\top}p[k] - \ell) \]

\[ + s\xi\frac{\hat{n}}{n}\lambda[k] + s\xi e[k + 1] \]

\[ = (V[k+1])^{-1}(C[k]V[k]h[k] + \hat{n}(p[k+1] - p[k])), \]  

(35c)

where \( h[k] := (V[k])^{-1}y[k], \hat{R}[k] := (V[k+1])^{-1}C[k]V[k] \).

Let \( \delta[k+1] := \hat{n}(p[k+1] - p[k]), \) then by using the triangle inequality we obtain that

\[ \|\delta[k+1]\|_{2} \leq \hat{n}\|p[k+1] - p^{*} - p[k] + p^{*}\|_{2} \]

\[ \leq \hat{n}\|z[k+1]\|_{2} + \|z[k]\|_{2}. \]

We take \( \max_{0 \leq k \leq K}(\cdot) \) on both sides of (35) to obtain

\[ \max_{0 \leq k \leq K}\|\delta[k+1]\|_{2} \leq \hat{n}\left( \max_{0 \leq k \leq K}\|z[k+1]\|_{2} \right. \]

\[ + \left. \max_{0 \leq k \leq K}\|z[k]\|_{2} \right), \]

(36)

from which it follows that

\[ \|\delta\|_{2}^{\alpha_{K},C} \leq 2\hat{n}\|z\|_{2}^{\alpha_{K},C} + \|\delta[0]\|_{2}. \]

(37)

By noting that \( \lambda[k] = V[k]x[k] \), we have that

\[ \|e[k]\|_{2} \leq \|x[k] - 1\lambda[k]\|_{2} + \|\hat{x}[k]\|_{2} \]

\[ = \frac{1}{n}\|1^{\top}x[k] - 1^{\top}V[k]x[k]\|_{2} + \|\hat{x}[k]\|_{2} \]

\[ = \frac{1}{n}\|1^{\top}(1 - v[k])^{\top}x[k]\|_{2} + \|\hat{x}[k]\|_{2} \]

\[ = \frac{1}{n}\|1^{\top}(1 - v[k])^{\top}(I - \frac{1}{n}1^{\top})x[k]\|_{2} + \|\hat{x}[k]\|_{2} \]

\[ = \frac{1}{n}\|1^{\top}(1 - v[k])\|_{2}\|\hat{x}[k]\|_{2} + \|\hat{x}[k]\|_{2} \leq 2\|\hat{x}[k]\|_{2}, \]

(38)

where we used the fact that \( v[k]^{\top}1 = n, \) and \( 0 \leq v[k] \leq 1 \). For further analysis, we invoke the following results [18, Lemmas 15, 16]:

Lemma 1.

\[ \|\hat{h}\|_{2}^{\alpha_{K},C} \leq \gamma_{1}\|\delta\|_{2}^{\alpha_{K},C} + \gamma_{2}, \]

(40)

for some \( \gamma_{1} \) and \( \gamma_{2} \). [Precise values can be found in [18, Lemma 15].]

Lemma 2.

\[ \|\hat{x}\|_{2}^{\alpha_{K},C} \leq s\gamma_{3}\|\hat{h}\|_{2}^{\alpha_{K},C} + \gamma_{4} \]

(41)

for some \( \gamma_{3} \) and \( \gamma_{4} \). [Precise values can be found in [18, Lemma 16].]
By using (41), (40), and (38) in (39), we obtain
\[ \|e\|^2_{2,K} \leq 2\|\hat{e}\|^2_{2,K} \leq 2s\gamma_3\|\delta\|^2_{2,K} + 2\gamma_4 \]
\[ \leq 2s\gamma_3\|\delta\|^2_{2,K} + 2s\gamma_2\gamma_3 + 2\gamma_4 \]
\[ \leq 4s\gamma_3\gamma_2\|\delta\|^2_{2,K} + 2s\gamma_2\gamma_3 + 2\gamma_4 \]
\[ + 2s\gamma_1\gamma_3\|\delta(0)\|^2_2, \]
which yields (33) for some positive \( \alpha_1 \), and \( \beta_1 \).

Now, we show the convergence of the algorithm (30) by applying the small-gain theorem to the results in Propositions 4–5.

Proposition 6. Let Assumptions 1 and 2 hold. Then, under the algorithm (30),
\[ \|z\|^a_{2,K} \leq \beta, \]
for some \( \beta > 0, a \in (0,1) \), and sufficiently small \( s > 0 \). In particular, \( (p_i[k], \lambda_i[k]) \), \( \forall i \), converges to \( (p^*, \lambda^*) \) at a geometric rate \( O(a^k) \).

C. Robustified Distributed Primal-Dual Algorithm

To execute the primal-dual algorithm (30), each node needs to know its out-degree, which, in practice, may not be readily available. This limitation can be fixed by letting nodes exchange the so-called running sums and use them in the averaging step similar to the robust ratio-consensus algorithm in [13]. This idea was used in [7] to solve the same issue in [13].

1) Approach 1: We let node \( j \) broadcast the running sums \( \sum_{t=1}^{k} \frac{\lambda_i[j]}{d_j} \), \( \sum_{t=1}^{k} \frac{v_i[j]}{d_j} \), and \( \sum_{t=1}^{k} \frac{w_i[j]}{d_j} \), where we recall that \( d_j^+ := |N_i| + 1 \) is the nominal out-degree. For each in-neighbor \( j \in N_i^- \), node \( i \) stores a half of the received running sums

\[ \lambda_{ij}[k+1] = \begin{cases} 
\lambda_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{\lambda_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
\lambda_{ij}[k] & \text{otherwise},
\end{cases} \]
\[ v_{ij}[k+1] = \begin{cases} 
v_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{v_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
v_{ij}[k] & \text{otherwise},
\end{cases} \]
\[ y_{ij}[k+1] = \begin{cases} 
y_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{y_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
y_{ij}[k] & \text{otherwise},
\end{cases} \]
and uses the other half of the received running sums in the following updates:
\[ \lambda_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} \left( \lambda_{ij}[k+1] - \lambda_{ij}[k] - sy_{ij}[k+1] \right), \]
\[ v_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} v_{ij}[k+1] - v_{ij}[k], \]
\[ x_i[k+1] = \frac{\lambda_i[k+1]}{v_i[k+1]}, \]
\[ p_i[k+1] = \left[ p_i[k] - s\nabla f_i(p_i[k]) + s\xi x_i[k+1] \right]_{\mathbb{P}_i}, \]
\[ y_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} y_{ij}[k+1] - y_{ij}[k] + \hat{n}(p_i[k+1] - p_i[k]). \]

2) Approach 2: We let node \( j \) broadcast the running sums \( \sum_{t=1}^{k} \frac{\lambda_i[j]}{d_j} \), \( \sum_{t=1}^{k} \frac{v_i[j]}{d_j} \), and \( \sum_{t=1}^{k} \frac{w_i[j]}{d_j} \) (without adding the most recent estimate values), and the most recent estimate values \( \lambda_j[k], v_j[k], \) and \( y_j[k] \). For each in-neighbor \( j \in N_i^- \), node \( i \) stores one half of the received running sums

\[ \lambda_{ij}[k+1] = \begin{cases} 
\lambda_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{\lambda_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
\lambda_{ij}[k] & \text{otherwise},
\end{cases} \]
\[ v_{ij}[k+1] = \begin{cases} 
v_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{v_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
v_{ij}[k] & \text{otherwise},
\end{cases} \]
\[ y_{ij}[k+1] = \begin{cases} 
y_{ij}[k] + \frac{1}{2} \sum_{t=1}^{k} \frac{y_{ij}[t]}{d_j^+} & \text{if } j \in N_i^+, \\
y_{ij}[k] & \text{otherwise},
\end{cases} \]
and uses the other half of the received running sums and the estimate values as follows:
\[ \lambda_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} \left( \frac{\lambda_j[k] - sy_{ij}[k]}{d_j^+} + \lambda_{ij}[k+1] \right) \]
\[ - \lambda_{ij}[k] - sy_{ij}[k+1] + sy_{ij}[k] \]
\[ v_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} \left( \frac{v_j[k]}{d_j^+} + v_{ij}[k+1] - v_{ij}[k] \right), \]
\[ x_i[k+1] = \frac{\lambda_i[k+1]}{v_i[k+1]}, \]
\[ p_i[k+1] = \left[ p_i[k] - s\nabla f_i(p_i[k]) + s\xi x_i[k+1] \right]_{\mathbb{P}_i}, \]
\[ y_i[k+1] = \sum_{j \in N_i^- \cup \{i\}} \left( \frac{y_j[k]}{d_j^+} + y_{ij}[k+1] - y_{ij}[k] \right) + \hat{n}(p_i[k+1] - p_i[k]). \]

Below, we present the convergence results for the robustified algorithms (45) and (47).

Proposition 7. Let Assumptions 1 and 2 hold. Then, under
the algorithm \(45\),
\[
\|z\|_{2}^{\alpha,n,K} \leq \beta, \tag{48}
\]
for some \(\beta > 0\), \(\alpha \in (0,1)\), sufficiently small \(s > 0\), and any \(\xi \in (0, \frac{1}{n}]\). In particular, \(\pi_{i}^{\alpha} = \lambda_{i}^{\alpha}\), \(\forall i\), converges to \((p^{*}, \lambda^{*})\) at a geometric rate \(O(a^{\alpha})\).

Proof. The convergence analysis of the algorithm \(45\) is based on the idea of augmenting the network of nodes with a set of virtual nodes denoted by \(S = \{\mathcal{E}(0) + 1, \ldots, \mathcal{E}(0) + n\}\), where the virtual nodes correspond to the edges in \(\mathcal{E}(0)\) through a one-to-one map \(I\) such that \(I(i,j) = k\), for \(k \in S\) and \((i,j) \in \mathcal{E}(0)\). For \(i \in S\), let \(\mathcal{N}_{i}^{+}\) := \(\{j \in \mathcal{V} : I(j,l) = i, \text{ for some } l \in \mathcal{V}, (j,l) \in \mathcal{E}(0)[k]\}\), \(\mathcal{N}_{i}^{-}\) := \(\{j \in \mathcal{V} : I(l,j) = i, \text{ for some } l \in \mathcal{V}, (l,j) \in \mathcal{E}(0)[k]\}\), and \(D_{i}^{k} := |\mathcal{N}_{i}^{-}| + 1\). Note that \(j \in \mathcal{N}_{i}^{-}\) \(\forall k\), if \(I(l,j) = i\) for some \(l \in \mathcal{V}\). For \(i \in \mathcal{V}\), let \(\mathcal{N}_{i}^{+} := \{j \in S : I(l,i) = j, \text{ for some } l \in \mathcal{V}, (l,i) \in \mathcal{E}(0)[k]\}\). We let each virtual node \(i \in S\) maintain and update variables \(\lambda_{i}\) and \(v_{i}\) as follows:

\[
\lambda_{i}[k + 1] = \begin{cases} 
\lambda_{i}[k] + \frac{\lambda_{i}[k] - sy_{j}[k]}{2D_{i}^{+}[k]}, & j \in \mathcal{N}_{i}^{-}[k], \\
\frac{1}{2}\lambda_{i}[k] + \frac{\lambda_{i}[k] - sy_{j}[k]}{2D_{i}^{+}[k]}, & \text{otherwise},
\end{cases} \tag{49a}
\]

\[
v_{i}[k + 1] = \begin{cases} 
\frac{v_{i}[k] + sy_{j}[k]}{2D_{i}^{+}[k]}, & j \in \mathcal{N}_{i}^{-}[k], \\
\frac{1}{2}v_{i}[k] + \frac{v_{i}[k] + sy_{j}[k]}{2D_{i}^{+}[k]}, & \text{otherwise},
\end{cases} \tag{49b}
\]

\[
y_{i}[k + 1] = \begin{cases} 
\frac{y_{i}[k] + sy_{j}[k]}{2D_{i}^{+}[k]}, & j \in \mathcal{N}_{i}^{-}[k], \\
\frac{1}{2}y_{i}[k] + \frac{y_{i}[k] + sy_{j}[k]}{2D_{i}^{+}[k]}, & \text{otherwise},
\end{cases} \tag{49c}
\]

Note that node \(i \in \mathcal{V}\) updates in \(45\) and \(i \in S\) updates in \(49\) can be written in the following form:

\[
\lambda_{i}[k + 1] = \frac{\lambda_{i}[k] - sy_{j}[k]}{2D_{i}^{+}[k]} + \sum_{j \in \mathcal{N}_{i}^{-}[k]} \lambda_{j}[k] - sy_{j}[k] \\
+ \sum_{j \in \mathcal{N}_{i}^{-}[k]} \lambda_{j}[k] - sy_{j}[k], \tag{50a}
\]

\[
v_{i}[k + 1] = \frac{v_{i}[k] + sy_{j}[k]}{2D_{i}^{+}[k]} + \sum_{j \in \mathcal{N}_{i}^{-}[k]} \frac{v_{j}[k]}{2D_{j}^{+}[k]} \\
+ \sum_{j \in \mathcal{N}_{i}^{-}[k]} \frac{v_{j}[k]}{2D_{j}^{+}[k]}, \tag{50b}
\]

\[
x_{i}[k + 1] = \frac{\lambda_{i}[k + 1]}{v_{i}[k + 1]} \tag{50c}
\]

\[
p_{i}[k + 1] = \left[p_{i}[k] - s\nabla f_{i}(p_{i}[k]) + s\xi x_{i}[k + 1]\right]_{E}, \tag{50d}
\]

\[
y_{i}[k + 1] = \frac{y_{i}[k]}{2D_{i}^{+}[k]} + \sum_{j \in \mathcal{N}_{i}^{-}[k]} \frac{y_{j}[k]}{2D_{j}^{+}[k]} + \sum_{j \in \mathcal{N}_{i}^{-}[k]} \frac{y_{j}[k]}{D_{j}^{+}[k]}
+ \hat{n}(p_{i}[k + 1] - p_{i}[k]), \tag{50e}
\]

where, for \(i \in S\), \(p_{i} = p_{i}^{\alpha} = 0\), \(p_{i}[0] = 0\), \(f_{i}(x) := x^{2}\), \(\lambda_{i}[0] = 0\), \(v_{i}[0] = 1\), \(y_{i}[0] = 0\). We redefine \(C[k] \in \mathbb{R}^{n + |\mathcal{E}(0)|} \times (n + |\mathcal{E}(0)|)\) such that \(C_{ii}[k] := \frac{3}{D_{i}^{+}[k]}\) and

\[
C_{ij}[k] := \begin{cases} \frac{1}{2D_{i}^{+}[k]}, & j \in \mathcal{N}_{i}^{+}[k], \\
\frac{D_{j}^{+}[k]}{D_{i}^{+}[k]}, & j \in \mathcal{N}_{i}^{-}[k], \\
0, & \text{else.}
\end{cases} \tag{51}
\]

Note that \(C[k]\) is column stochastic, and its diagonal entries are always strictly positive, in particular, \(C_{ii}[k] \geq \frac{1}{n}, \forall k\). Then, \(50\) can be written in vector form:

\[
x[k + 1] = (V[k + 1]^{-1})C[k](V[k][x[k] - sy[k]]) \\
= \hat{R}[k][x[k] - sh[k]], \tag{52a}
\]

\[
h[k + 1] = (V[k + 1]^{-1})y[k + 1] \\
= (V[k + 1]^{-1})C[k]V[k]h[k] \\
+ \hat{n}(p[k + 1] - p[k]), \tag{52b}
\]

\[
p[k + 1] = \left[p[k] - s\nabla f(p[k]) - s^{2}\hat{n}\frac{1}{n}(1^{T}p[k] - \ell) \\
+ s\xi\frac{\hat{n}}{n}\|\mathcal{N}[k]1 + s\xi e[k + 1]\right]_{E}, \tag{52c}
\]

where \(h[k] := (V[k])^{-1}y[k], \hat{R}[k] := (V[k] + 1)^{-1}C[k]V[k]\). Since \(52\) has exactly the same form as \(35\), the rest of the proof is identical to that of Proposition \(6\). □

The proof of the next result is omitted since it is analogous to that of Proposition \(7\).

**Proposition 8.** Let Assumptions \(7\) and \(2\) hold. Then, under the algorithm \(47\),
\[
\|z\|_{2}^{\alpha,K} \leq \beta, \tag{53}
\]
for some \(\beta > 0\), \(\alpha \in (0,1)\), sufficiently small \(s > 0\), and any \(\xi \in (0, \frac{1}{n}]\). In particular, \(\pi_{i}^{\alpha} = \lambda_{i}^{\alpha}\), \(\forall i \in \mathcal{V}\), converges to \((p^{*}, \lambda^{*})\) at a geometric rate \(O(a^{\alpha})\).

**V. Simulations**

In this section, we present numerical results that illustrate the performance of the proposed algorithms using the IEEE 39-bus test system \(23\), the topology of which is given in Fig. \(2\). In this example, each bus has one electrical load and one DER, and \(1\) is solved to minimize the generation cost while the total power demand by the loads is supplied. We randomly pick the load demands and maximum and minimum generation capacity constraints of the DERs. For each \(i\), we choose \(f_{i}(p_{i}) = a_{i}p^{2}_{i}\), where \(a_{i} > 0\) is randomly selected. First, we consider the case when the communication graph is undirected, where any two nodes are connected by a bidirectional communication link if there is an electrical line between them. Then, we consider the case when the communication graph is directed, where any two nodes are connected by a unidirectional communication link if there is an electrical line between them, and some nodes are connected with two opposite unidirectional communication links. We assign the orientations of the communication links such that the nominal communication graph \(\mathcal{G}(0)\) is strongly connected.
We assume that out-degrees, \( D_i^+ [k] \), are unknown to DERs. In both cases, communication links fail with probability 0.2.

A. Undirected Communication Graphs

We first conduct numerical experiments with the distributed primal-dual algorithm (5), for convenience referred to as \( \text{PD}_1 \). We also compare its performance with that of the distributed algorithm (3), referred to as \( \text{PD}_2 \), in which we recall that every node uses a local power imbalance to update its local estimate of the dual variable. The proposed algorithm \( \text{PD}_1 \) uses a constant stepsize \( s = 0.015 \) and \( \xi = 0.06 \). In contrast, \( \text{PD}_2 \) needs to use a diminishing stepsize of the form \( s[k] = a / (k + b) \), where \( a > 0 \) and \( b > 0 \) are the parameters, in order to guarantee convergence. Both algorithms are initialized with \( p[0] = 0 \).

From Figure 3, which shows the convergence of the Euclidean norm of the error between local estimates and optimal solution, \( \| p[k] - p^* \|_2 \) for both algorithms, it can be seen that \( \text{PD}_1 \) outperforms \( \text{PD}_2 \) and has a geometric convergence speed.

B. Directed Communication Graphs

We compare the performance of the proposed resilient distributed algorithms (45) and (47), for convenience referred to as \( A_1 \) and \( A_2 \), against that of the distributed algorithms proposed in (10), (7), (3), referred to as \( A_3 \), \( A_4 \), and \( A_5 \), respectively.

The algorithms \( A_1 \), \( A_2 \) and \( A_3 \) use a constant stepsize \( s \). In contrast, \( A_4 \) and \( A_5 \) need to use a diminishing stepsize in order to guarantee convergence. However, if the stepsize is constant and sufficiently small, \( A_4 \) and \( A_5 \) can still achieve convergence within a small error. We tested the performance of \( A_4 \) using different diminishing stepsizes of the form \( s[k] = a / (k + b) \), where \( a > 0 \) and \( b > 0 \) are the parameters.

\( A_4 \) outperforms \( A_5 \), for all algorithms. To test \( A_5 \), we used \( (\alpha_k, \beta_k) = (0.003, 0.3) \) (see, e.g., [3]) and \( (\alpha_k, \beta_k) = (\frac{20}{k+1000}, 0.3) \), which in this numerical example worked better than the stepsizes used in the numerical simulations in [3].

Figure 4 shows the evolution of the Euclidean norm of the error between local estimates and the optimal solution, \( \| p[k] - p^* \|_2 \), for algorithms \( A_1 \)–\( A_5 \). Figure 4 shows the evolution of the Euclidean norm of the error between local estimates and the optimal solution, \( \| p[k] - p^* \|_2 \), for all algorithms. It can be seen that \( A_1 \) and \( A_2 \) outperform \( A_3 \), \( A_4 \) and \( A_5 \) and have the geometric convergence speed. \( A_3 \) fails to converge because out-degrees, \( D_i^+ [k] \), are unknown to DERs. Through numerical simulations, we observed that it is in general difficult to choose the right values for \( a \) and \( b \) in order for \( A_4 \) to operate well. In fact, if the ratio \( a/b \) is large, \( A_4 \) might exhibit an oscillatory behavior. But setting \( a/b \) to a small value results in a slow convergence.

We also observed that \( A_2 \) is typically faster than \( A_1 \) but requires each node to communicate more information, the running sums and the most recent estimate values. In contrast, \( A_1 \) requires each node to communicate only the running sums.
VI. CONCLUSION

We presented distributed algorithms for solving the DER coordination problem over time-varying communication graphs. The algorithms have geometric convergence rate. One important future direction is to extend the proposed algorithms to solve more complex possibly multi-period DER coordination problems with additional constraints, e.g., line flow constraints, voltage constraints, or reactive power balance constraints.

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