GLOBAL WELL-POSEDNESS AND LARGE TIME ASYMPTOTIC BEHAVIOR OF STRONG SOLUTIONS TO THE CAUCHY PROBLEM OF THE 2-D MHD EQUATION

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Abstract. This paper is concerned with the Cauchy problem of the two-dimensional MHD system with magnetic diffusion. It was proved that the MHD equations have a unique global strong solution around the equilibrium state \((0, e_1)\). Furthermore, the \(L^2\) decay rate of the velocity and magnetic field is obtained.

Keywords: MHD equations; Global well-posedness; decay estimate

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1. Introduction

In this paper, we consider the following two-dimensional magneto-hydrodynamical system:

\[
\begin{aligned}
&\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = B \cdot \nabla B, \\
&\partial_t B - \Delta B + u \cdot \nabla B = B \cdot \nabla u, \\
&\text{div} \ u = \text{div} \ B = 0, \\
&u(0, x) = u_0(x), \quad B(0, x) = B_0(x).
\end{aligned}
\]

Here \(t \geq 0, x \in \mathbb{R}^2\), \(u = (u_1(t, x), u_2(t, x))\) and \(B = (B_1(t, x), B_2(t, x))\) are vector fields representing the velocity and the magnetic field, respectively; the scalar function \(p = p(t, x)\) denotes the usual pressure. Recall that \(\Delta := (\partial_1^2 + \partial_2^2), \nabla := (\partial_1, \partial_2)\). This MHD system (1.1) with zero diffusivity in the equation for magnetic field can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. We refer to [2] for some detailed discussions on the relevant physical background of this system.

Recently, there are many works developed to the study of the MHD system. When the magnetic diffusion is included, G. Duvaut and J-L. Lions [6] established the local existence and uniqueness of a solution in the Sobolev space \(H^s(\mathbb{R}^d), s > d\), and proved global existence of the solution for small initial data. Moreover, M. Sermange and R. Temam [17] examined some properties of these solutions. In particular, the 2-D local strong solution has been proved to be global and unique. There are also important progress in the case without magnetic diffusion. For instance, Cao, Regmi, Wu and Yuan [3, 4, 5] where the authors studied the global regularity of the 2-D MHD equations with partial dissipation and additional magnetic diffusion for any data in \(H^s(\mathbb{R}^2), s \geq 2\).

For the 2-D system (1.1) around the equilibrium state \((0, e_1)\). Lin, Xu and Zhang [10] proved the global well-posedness which is close to some non-trivial steady state. In [13], the authors further established the global existence and time decay rate of smooth solutions for general perturbations, which confirms the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity. In a recent remarkable paper [14], Ren, Xiang and Zhang proved the unique global strong solution for both the non-slip boundary condition and Navier slip boundary condition on the velocity. For the 3-D system, [15] proved the global existence and decay estimate of smooth solution, and a similar
global existence result as [10] has been established by Xu and Zhang [16]. In [9], Lei proved the global regularity of axially symmetric solutions to the MHD system.

Motivated by [10] [13] [16], we will investigate small perturbations of the system (1.1) around the equilibrium state \((0, e_1)\). Thus, we can set \(b = B - e_1\) and reformulate our first problem as follows:

\[
\begin{align*}
\frac{\partial_t u - \Delta u - \partial_1 b + \nabla p + u \cdot \nabla u = b \cdot \nabla b,}{}
\frac{\partial_t b - \Delta b - \partial_1 u + u \cdot \nabla b = b \cdot \nabla u,}{}
\text{div } u = \text{div } b = 0,
\end{align*}
\]

(1.2)

Our main result is stated as follows.

**Theorem 1.1.** Let \(s > 2\), and \((u_0, b_0) \in H^s(\mathbb{R}^2)\) with \(\text{div } u_0 = \text{div } b_0 = 0\). Then the MHD system (1.2) has a unique global solution \((u, b)\) on \([0, +\infty)\) such that

\[
(u, b) \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^2)).
\]

Moreover, motivated by [10] the \(L^2\) decay rate of the velocity and magnetic field is obtained.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if in addition \((u_0, b_0) \in \dot{H}^{-\varepsilon}\) for some \(\varepsilon \in (0, 1)\). Then the solution \((u, b)\) obtained in Theorem 1.1 has the following decay estimates

\[
\|(u, b)\|_{L^2} \leq C(e + t)^{-\kappa},
\]

with \(\kappa = \min(\varepsilon, \frac{1}{4})\).

The rest of this paper is organized as follows. In section 2, we collect some elementary facts and inequalities which will be used later. Section 3 is devoted to the local well-posedness of the MHD system (1.2). In section 4, we will give the proof of Theorem 1.1. Finally, we will show the proof of Theorem 1.2 in section 5.

Let us complete this section with the notations we are going to use in this context.

**Notations:** Let \(A, B\) be two operators, we denote \([A, B] = AB - BA\), the commutator between \(A\) and \(B\). By \(a \lesssim b\), we mean that there is a uniform constant \(C\), which may be different on different lines, such that \(a \leq Cb\). We shall denote by \((a, b)\) or \((a, b)_{L^2}\) the \(L^2(\mathbb{R}^2)\) inner product of \(a\) and \(b\), and \(\int a \cdot b = \int_{\mathbb{R}^2} a(x)b(x)\). We always denote the Fourier transform of a function \(u\) by \(\hat{u}\) or \(\mathcal{F}(u)\). For \(s \in \mathbb{R}\), we denote the pseudo-differential operator \(\Lambda^s := (1 - \Delta)^{\frac{s}{2}}\) with the Fourier symbol \((1 + |\xi|^2)^{\frac{s}{2}}\).

For \(X\) a Banach space and \(I\) an interval of \(\mathbb{R}\), we denote by \(C(I; X)\) the set of continuous functions on \(I\) with values in \(X\), and by \(C_b(I; X)\) the subset of bounded functions of \(C(I; X)\). For \(q \in [1, +\infty]\), the notation \(L^q(I; X)\) stands for the set of measurable functions on \(I\) with values in \(X\), such that \(t \mapsto \|f(t)\|_X\) belongs to \(L^q(I)\). For a vector \(v = (v_1, v_2) \in X\), we mean that all the components \(v_i\) \((i = 1, 2)\) of \(v\) belong to the space \(X\).

2. PRELIMINARIES

In this section, we will give some elementary facts and useful lemmas which will be used in the next section.

Let us first recall some basic facts about the regularizing operator called a mollifier, see [11] for more details. Given any radial function

\[
\rho(|x|) \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \int_{\mathbb{R}^N} \rho \, dx = 1.
\]

define the mollification \(J_\varepsilon u\) of \(u \in L^p(\mathbb{R}^N), 1 \leq p \leq \infty\), by

\[
(J_\varepsilon u)(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x - y}{\varepsilon}\right) u(y) \, dy, \quad \varepsilon > 0
\]

(2.1)
Mollifiers have several well-known properties: (i). \( J_c u \) is a \( C^\infty \) function; (ii). for all \( u \in \mathcal{C}^0(\mathbb{R}^N) \), \( J_c u \to u \) uniformly on any compact set \( \Omega \) in \( \mathbb{R}^N \) and \( \| J_c u \|_{L^\infty} \leq \| u \|_{L^\infty} \); (iii). mollifiers commute with distribution derivatives, \( D^\alpha J_c u = J_c D^\alpha u \); (iv). for all \( u \in H^m(\mathbb{R}^N) \), \( J_c u \) converges to \( u \) in \( H^m \) and the rate of convergence in the \( H^{m-1} \) norm is linear in \( \varepsilon \): \( \lim_{\varepsilon \to 0} \| J_c u - u \|_{H^{m-1}} = 0 \), \( \| J_c u - u \|_{H^{m-1}} \leq C \| u \|_{H^m} \); (v). for all \( u \in H^m(\mathbb{R}^N), k \in \mathbb{Z}^+ \cup 0 \), and \( \varepsilon > 0 \), \( \| J_c u \|_{H^{m+k}} \leq \frac{C(m,k)}{\varepsilon^{k+1}} \| u \|_{H^m} \), \( \| J_c u \|_{L^\infty} \leq \frac{C(k)}{\varepsilon} \| u \|_{L^2} \).

Next we recall the Leray projection operator \( P \).

**Lemma 2.1.** ([11]) The Leray projection \( P \) is defined by \( P(u) := u - \nabla \Delta^{-1} (\nabla \cdot u) \). Then

(i)\( P \) commutes with the distribution derivatives, \( P D^\alpha u = D^\alpha P u, \forall u \in H^m, \|\alpha\| \leq m \).

(ii)\( P \) commutes with mollifiers \( J_c \), \( P(J_c u) = J_c (P u) \), \( \forall \varepsilon > 0 \).

(iii)\( P \) is symmetric, \( (P u, v)_{H^s} = (u, P v)_{H^s} \forall \sigma \in \mathbb{R} \).

**Lemma 2.2** (Aubin-Lions’s lemma, [18]). Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow E \). Then the following embeddings are compact:

(i)\( \{ \varphi : \varphi \in L^q([0,T];X), \frac{\partial \varphi}{\partial t} \in L^1([0,T];Y) \} \hookrightarrow L^q([0,T];E) \) if \( 1 \leq q \leq \infty \);

(ii)\( \{ \varphi : \varphi \in L^\infty([0,T];X), \frac{\partial \varphi}{\partial t} \in L^r([0,T];Y) \} \hookrightarrow C([0,T];E) \) if \( 1 \leq r \leq \infty \).

**Lemma 2.3** (Calculus inequalities, [8]). Let \( s > 0 \). Then the following two estimates are true:

(i)\( \| uv \|_{H^s} \leq C \{ \| u \|_{L^\infty} \| v \|_{H^s} + \| u \|_{H^s} \| v \|_{L^\infty} \} \);

(ii)\( \| uv \|_{H^s} \leq C \| u \|_{H^s} \| v \|_{H^s} \) for all \( s > \frac{N}{2} \);

(iii)\( \| [\Delta^s, u] v \|_{L^2(\mathbb{R})} \leq C(\| u \|_{H^s} \| v \|_{L^\infty(\mathbb{R})} + \| \nabla u \|_{L^\infty(\mathbb{R})} \| v \|_{H^{s-1}(\mathbb{R})}) \);

where all the constants \( C \)’s are independent of \( u \) and \( v \).

**Lemma 2.4** (Logarithmic Sobolev interpolation inequality, [1]). For any \( f \in H^1(\mathbb{R}^2) \) with \( \nabla f \in L^p(\mathbb{R}^2) \) for some \( 2 < p < \infty \), there holds

\[
(2.2) \quad \sum_{i,j=1}^{2} \| \Delta^{-1} \partial_i \partial_j f \|_{L^\infty} \leq C(\| f \|_{L^2} + \| f \|_{L^\infty} \log(e + \| \nabla f \|_{L^p})).
\]

**Lemma 2.5** (Gagliardo-Nirenberg inequality, [12]). For \( q \in [2, \infty) \), \( r \in (2, \infty) \) and \( s \in (1, \infty) \), there exists some generic constant \( C > 0 \) which may depend on \( q, r, \) and \( s \) such that for \( f \in H^1(\mathbb{R}^2) \) and \( g \in L^s(\mathbb{R}^2) \cap W^{1,r}(\mathbb{R}^2) \), there hold

\[
\| f \|_{L^q(\mathbb{R}^2)}^q \leq C \| f \|_{L^2(\mathbb{R}^2)} \| \nabla f \|_{L^2(\mathbb{R}^2)}^{q-2},
\]

\[
\| g \|_{C(\mathbb{R}^2)} \leq C \| g \|_{L^s(\mathbb{R}^2)} \| \nabla g \|_{L^r(\mathbb{R}^2)}^{2r/(2r+s(r-2))} \| \nabla g \|_{L^r(\mathbb{R}^2)}^{2r/(2r+s(r-2))}.
\]

### 3. Local Well-Posedness

This section is devoted to the proof of the local well-posedness of the system (1.2).

**Theorem 3.1.** Let \( s > 2 \), and \( (u_0, b_0) \in H^s(\mathbb{R}^2) \) with \( \text{div} u_0 = \text{div} b_0 = 0 \). Then there exist \( T > 0 \) and a unique solution \( (u, b) \) on \([0,T]\) of the MHD system (1.2) such that

\( (u, b) \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^2)) \).

**Proof of Theorem 3.1** In order to prove the existence of solutions for the system (1.2), we firstly solve an approximate problem, next we perform the uniform estimates for the approximate solutions, and then by a compactness argument, we obtain the existence part of...
Theorem 3.1 With the existence of the solution to (1.2) in hand, we apply Gronwall’s inequality to prove the uniqueness part of Theorem 3.1 The approach of proof is then divided the following four steps.

Step 1: Construction of smooth approximate solution

We introduce the following approximate system of (1.2):

\[
\begin{align*}
\frac{\partial u^\varepsilon}{\partial t} &= \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon b^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon b^\varepsilon) - \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon u^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\partial_1 b^\varepsilon)] + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\Delta u^\varepsilon)], \\
\frac{\partial b^\varepsilon}{\partial t} &= \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon b^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) - \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon u^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\partial_1 u^\varepsilon)] + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\Delta b^\varepsilon)], \\
\tilde{u}_0^\varepsilon(x) &= u_0^\varepsilon(x), \quad b_0^\varepsilon(x) = b_0^\varepsilon(x),
\end{align*}
\]

where \(\mathcal{J}_\varepsilon\) denotes mollifier operator and \(\mathbb{P}\) denotes Leray’s projection operator. The regularized equation (3.1) reduces to an ordinary differential system:

\[
\begin{align*}
\frac{d}{dt} (u^\varepsilon, b^\varepsilon) = \\
\begin{pmatrix}
\mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\partial_1 b^\varepsilon)] + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\Delta u^\varepsilon)] \\
\mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\partial_1 u^\varepsilon)] + \mathbb{P}_{\varepsilon}[\mathcal{J}_\varepsilon (\Delta b^\varepsilon)]
\end{pmatrix}, \\
u^\varepsilon(0, x) = u_0^\varepsilon(x), \quad b^\varepsilon(0, x) = b_0^\varepsilon(x).
\end{align*}
\]

The classical Picard Theorem ensures that the (3.1) has a unique smooth solution \((u^\varepsilon, b^\varepsilon) \in C([0, T_\varepsilon); H^s(\mathbb{R}^d))\) for some \(T_\varepsilon > 0\).

Step 2: Uniform estimates to the approximate solutions

Applying the operator \(\Lambda^s\) to the system (3.1) and then taking the \(L^2\) inner product, we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\Lambda^s u^\varepsilon\|^2_{L^2} + \|\Lambda^s b^\varepsilon\|^2_{L^2}) &= \int \Lambda^s \mathcal{J}_\varepsilon [\mathcal{J}_\varepsilon b^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon b^\varepsilon) \cdot \Lambda^s u^\varepsilon dx \\
&\quad - \int \Lambda^s \mathcal{J}_\varepsilon [\mathcal{J}_\varepsilon u^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \Lambda^s u^\varepsilon dx + \int \Lambda^s \mathcal{J}_\varepsilon (\Delta u^\varepsilon) \cdot \Lambda^s u^\varepsilon dx \\
&\quad + \int \Lambda^s \mathcal{J}_\varepsilon [\mathcal{J}_\varepsilon u^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \Lambda^s b^\varepsilon dx - \int \Lambda^s \mathcal{J}_\varepsilon [\mathcal{J}_\varepsilon u^\varepsilon] \cdot \nabla (\mathcal{J}_\varepsilon b^\varepsilon) \cdot \Lambda^s b^\varepsilon dx \\
&\quad + \int \Lambda^s \mathcal{J}_\varepsilon (\Delta b^\varepsilon) \cdot \Lambda^s b^\varepsilon dx \\
&= \sum_{n=1}^{6} I_n.
\end{align*}
\]

Thanks to \(\text{div} u = \text{div} b = 0\), we may get from a standard commutator’s process that

\[
|I_1| \leq \frac{1}{2} \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty} \|\Lambda^s \mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \|\Lambda^s \mathcal{J}_\varepsilon b^\varepsilon\|_{L^2} \\
+ C \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty} \|\Lambda^{s-1} \nabla (\mathcal{J}_\varepsilon b^\varepsilon)\|_{L^2} \|\mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty} \|\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \\
\leq C \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty} \|\Lambda^s \mathcal{J}_\varepsilon b^\varepsilon\|_{L^2} \|\Lambda^s \mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \leq C \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty} (\|\mathcal{J}_\varepsilon b^\varepsilon\|_{H^s}^2 + \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^s}^2).
\]

Similarly, we have

\[
|I_2| \leq C \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^s}^2,
\]

\[
|I_4| \leq \frac{1}{2} \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} \|\Lambda^s \mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \|\Lambda^s \mathcal{J}_\varepsilon b^\varepsilon\|_{L^2} \\
+ C \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} \|\Lambda^{s-1} \nabla (\mathcal{J}_\varepsilon u^\varepsilon)\|_{L^2} \|\nabla (\mathcal{J}_\varepsilon b^\varepsilon)\|_{L^\infty} \|\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \|\mathcal{J}_\varepsilon b^\varepsilon\|_{H^s} \\
\leq C (\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} + \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty})(\|\mathcal{J}_\varepsilon b^\varepsilon\|_{H^s}^2 + \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^s}^2),
\]
and
\begin{equation}
|I_5| \leq C(\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} + \|\nabla \mathcal{J}_\varepsilon b^\varepsilon\|_{L^\infty})(\|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{H^s}^2 + \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^s}^2).
\end{equation}

Thanks to integrating by parts, we obtain
\begin{equation}
|I_3| = \int \Lambda^s \mathcal{J}_\varepsilon(\Delta u^\varepsilon) \cdot \Lambda^s u^\varepsilon \, dx = -\int |\Lambda^s \mathcal{J}_\varepsilon(\nabla u^\varepsilon)|^2 \, dx = -\|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{H^s}^2.
\end{equation}

Similarly, we have
\begin{equation}
|I_6| = \int \Lambda^s \mathcal{J}_\varepsilon(\Delta b^\varepsilon) \cdot \Lambda^s b^\varepsilon \, dx = -\int |\Lambda^s \mathcal{J}_\varepsilon(\nabla b^\varepsilon)|^2 \, dx = -\|\mathcal{J}_\varepsilon \nabla b^\varepsilon\|_{H^s}^2.
\end{equation}

Substituting (3.3), (3.8) into (3.2) leads to
\begin{equation}
\frac{d}{dt} \|\langle u^\varepsilon, b^\varepsilon\rangle\|_{H^s}^2 + \|\mathcal{J}_\varepsilon \nabla u^\varepsilon, \mathcal{J}_\varepsilon \nabla b^\varepsilon\|_{L^\infty}^2 \|\langle u^\varepsilon, b^\varepsilon\rangle\|_{H^s}^2 \\
\leq C \|\mathcal{J}_\varepsilon \nabla u^\varepsilon, \mathcal{J}_\varepsilon \nabla b^\varepsilon\|_{L^\infty}^2 \|\langle u^\varepsilon, b^\varepsilon\rangle\|_{H^s}^2 \leq C \|\langle u_0, b_0\rangle\|_{H^s},
\end{equation}

Therefore, by the bootstrap argument, we may get that, there is a positive time $T = T(u_0, \theta_0) \leq T_\varepsilon$ independent of $\varepsilon$ such that for all $\varepsilon$,
\begin{equation}
\sup_{0 \leq t \leq T} \|\langle u^\varepsilon, b^\varepsilon\rangle\|_{H^s} \leq C \|\langle u_0, b_0\rangle\|_{H^s},
\end{equation}

which along with (3.9) implies that
\begin{equation}
\{\langle u^\varepsilon, b^\varepsilon\rangle\}_{\varepsilon > 0} \quad \text{is uniformly bounded in} \quad C([0, T]; H^s(\mathbb{R}^2)),
\end{equation}

and
\begin{equation}
\{\nabla u^\varepsilon, \nabla b^\varepsilon\}_{\varepsilon > 0} \quad \text{is uniformly bounded in} \quad L^2([0, T]; H^s(\mathbb{R}^2)).
\end{equation}

Furthermore, there holds
\begin{equation}
\{\langle \frac{du^\varepsilon}{dt}, \frac{db^\varepsilon}{dt}\rangle\}_{\varepsilon > 0} \quad \text{is uniformly bounded in} \quad C([0, T]; H^{s-2}(\mathbb{R}^2)).
\end{equation}

**Step 3: Convergence**

With (3.10), (3.11), (3.12), and (3.13), The Aubin-Lions’s compactness lemma ensures that there exist a subsequence of $\{\langle u^\varepsilon, b^\varepsilon\rangle\}_{\varepsilon > 0}$ (still denoted by $\{\langle u^\varepsilon, b^\varepsilon\rangle\}_{\varepsilon > 0}$) converges to some limit $(u, b)$ on $[0, T]$, which solves (1.2). Moreover, there hold
\begin{equation}
(u, b) \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2([0, T]; H^{s+1}(\mathbb{R}^2)).
\end{equation}

**Step 4: Uniqueness of the solution**

Let $(u^1, b^1)$ and $(u^2, b^2)$ be two solutions of (1.2) with the same initial data and satisfy (3.14). We denote $u^{1,2} := u^1 - u^2, \ b^{1,2} := b^1 - b^2, \ p^{1,2} := p^1 - p^2$. Then $(u^{1,2}, p^{1,2})$ satisfies
\begin{equation}
\begin{cases}
\partial_t u^{1,2} - \Delta u^{1,2} - \partial_t b^{1,2} + u^1 \cdot \nabla u^{1,2} + u^{1,2} \cdot \nabla u^1 + \nabla p^{1,2} = b^1 \cdot \nabla b^{1,2} + b^{1,2} \cdot \nabla b^2, \\
\partial_t b^{1,2} - \Delta b^{1,2} - \partial_t u^{1,2} + u^1 \cdot \nabla b^{1,2} + u^{1,2} \cdot \nabla b^1 + \nabla b^2 = b^1 \cdot \nabla u^{1,2} + b^{1,2} \cdot \nabla u^2, \\
\text{div} \ u^{1,2} = \text{div} \ b^{1,2} = 0,
\end{cases}
\end{equation}

with $(u^{1,2}, b^{1,2})|_{t=0} = (0, 0)$.

Taking $L^2(\mathbb{R}^2)$ energy estimate, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} (\|u^{1,2}\|_{L^2}^2 + \|b^{1,2}\|_{L^2}^2) + (u^1 \cdot \nabla u^{1,2}, u^{1,2}) + (u^{1,2} \cdot \nabla u^2, u^{1,2}) \\
+ (u^1 \cdot \nabla b^{1,2}, b^{1,2}) + (u^{1,2} \cdot \nabla b^2, b^{1,2}) \\
= (b^1 \cdot \nabla b^{1,2}, u^{1,2}) + (b^{1,2} \cdot \nabla b^2, u^{1,2}) + (b^1 \cdot \nabla u^{1,2}, b^{1,2}) + (b^{1,2} \cdot \nabla u^2, b^{1,2}) \\
+ (\Delta u^{1,2}, u^{1,2}) + (\Delta b^{1,2}, b^{1,2}).
\end{equation}
Thanks to integration by parts and the divergence theorem, we get
\[(u^1 \cdot \nabla u^{1,2}, u^{1,2}) = \int (u^1 \cdot \nabla) u^{1,2} \cdot u^{1,2} \, dx = \int u^1 \cdot \frac{1}{2} \nabla|u^{1,2}|^2 \, dx\]
\[(3.16) = - \int \frac{1}{2} |u^{1,2}|^2 \text{div} \, u^1 \, dx = 0.\]

Similarly, we have
\[(3.17) (u^1 \cdot \nabla b^{1,2}, b^{1,2}) = 0.\]

On the other hand, we get from Hölder’s inequality and Sobolev embedding theorem that
\[(3.18) |(u^{1,2} \cdot \nabla u^2, u^{1,2})| \leq \|u^{1,2}\|_{L^2} \|\nabla u^2\|_{L^\infty} \leq C \|u^2\|_{H^s(\mathbb{R}^2)} \|u^{1,2}\|^2_{L^2},\]
and we have
\[(3.19) |(u^{1,2} \cdot \nabla b^2, b^{1,2})| \leq \|u^{1,2}\|_{L^2} \|b^{1,2}\|_{L^2} \|\nabla b^2\|_{L^\infty} \leq C \|b^2\|_{H^s(\mathbb{R}^2)} \|u^{1,2}\|^2_{L^2} + \|b^{1,2}\|^2_{L^2}.\]

Similarly, we have
\[(3.20) |(b^{1,2} \cdot \nabla b^2, u^{1,2})| \leq \|b^{1,2}\|_{H^s(\mathbb{R}^2)} \|u^{1,2}\|^2_{L^2} + \|b^{1,2}\|^2_{L^2},\]
\[(3.21) |(b^{1,2} \cdot \nabla u^2, b^{1,2})| \leq \|b^2\|_{H^s(\mathbb{R}^2)} \|b^{1,2}\|^2_{L^2},\]
\[(3.22) |(b^{1,2} \cdot \nabla b^2, u^{1,2})| \leq \|\nabla b^{1,2}\|_{L^2} \|b^{1,2}\|_{L^2} \|\nabla b^2\|_{L^\infty} \leq C \|b^{1,2}\|_{H^s(\mathbb{R}^2)} \|b^{1,2}\|^2_{L^2} + C |u^{1,2}|^2_{L^2},\]
and
\[(3.23) |(b^{1,2} \cdot \nabla u^{1,2}, b^{1,2})| \leq C \|b^1\|_{H^s(\mathbb{R}^2)} \|\nabla u^{1,2}\|^2_{L^2} + C |b^{1,2}|^2_{L^2}.\]

Using the integration of parts, we get
\[(3.24) |(\Delta u^{1,2}, u^{1,2})| = \int \Delta u^{1,2} \cdot u^{1,2} \, dx = - \|\nabla u^{1,2}\|^2_{L^2},\]
and
\[(3.25) |(\Delta b^{1,2}, b^{1,2})| = \int \Delta b^{1,2} \cdot b^{1,2} \, dx = - \|\nabla b^{1,2}\|^2_{L^2}.\]

Thus, Inserting (3.16)-(3.25) into (3.15) leads to
\[\frac{d}{dt} \left(\|u^{1,2}\|^2_{L^2} + \|b^{1,2}\|^2_{L^2} + \|\nabla u^{1,2}\|^2_{L^2} + \|\nabla b^{1,2}\|^2_{L^2}\right) \leq C \left(\|u^{1,2}\|^2_{L^2} + \|b^{1,2}\|^2_{L^2}\right),\]
where we took \(\eta\) sufficiently small.

Therefore, it follows from Gronwall’s inequality that \(u^{1,2}(t) = 0\) and \(b^{1,2}(t) = 0\) for all \(t \in [0, T]\).

The proof of Theorem 3.1 is completed. \(\square\)

4. Proof of Theorem 1.1

In the section, we prove the global well-posedness of the MHD system (1.2).

Proof of Theorem 1.1. First of all, Theorem 3.1 provides us a local strong solution \((u, b) \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2([0, T]; H^{s+1}(\mathbb{R}^2)).\) Let \(T^*\) be the maximal existence time of the solution. It suffices to prove \(T^* = +\infty\), we will argue contradiction argument. Hence, assume \(T^* \leq +\infty\) in what follows.

Taking \(L^2\) energy estimate to the system (1.2), it is easy to find that
\[(4.1) \| (u, b) \|_{L^\infty(0, T^*; L^2)} + \| (\nabla u, \nabla b) \|_{L^2(0, T^*; L^2)} = \| (u_0, b_0) \|_{L^2},\]
which implies that \((u, b)\) is bounded in \(L^4(0, T^*; L^4(\mathbb{R}^2))\) by Gagliardo-Nirenberg inequality.
Next, multiplying the first equation in (1.2) by $\Delta u$ and the second equation in (1.2) by $\Delta b$, adding the results and integrating by parts, we obtain

\[
\frac{d}{dt}(\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + 2(\|\Delta u\|^2_{L^2} + \|\Delta u\|^2_{L^2}) = (- u \cdot \nabla u, \Delta u) + (- u \cdot \nabla b, \Delta b) \leq \|u \cdot \nabla u\|_{L^2}\|\Delta u\|_{L^2} + \|u \cdot \nabla b\|_{L^2}\|\Delta b\|_{L^2}.
\]

By Gagliardo-Nirenberg’s inequality, we have

\[
\|u \cdot \nabla u\|_{L^2} \leq C\|u\|_{L^4}\|\nabla u\|_{L^4} \leq C\|u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}},
\]

and

\[
\|u \cdot \nabla b\|_{L^2} \leq C\|u\|_{L^4}\|\nabla b\|_{L^4} \leq C\|u\|_{L^2}^{\frac{3}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}}.
\]

Then we get by Young’s inequality that

\[
\frac{d}{dt}(\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + 2(\|\Delta u\|^2_{L^2} + \|\Delta u\|^2_{L^2}) \leq C(\|u\|^2_{L^2}\|\nabla u\|^4_{L^2} + \|u\|^2_{L^2}\|\nabla u\|^2_{L^2}\|\nabla b\|^2_{L^2}) \leq C\|\nabla u\|^2_{L^2}(\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}),
\]

which along with Gronwall’s inequality ensures that $(u, b)$ is bounded in $L^2(0, T^*; H^2(\mathbb{R}^2))$.

On the other hand, applying the operator $\Lambda^s$ to the system (1.2), we obtain:

\[
\begin{aligned}
\partial_t \Lambda^s u - \Delta \Lambda^s u - \partial_1 \Lambda^s b + \nabla \Lambda^s p + \Lambda^s(u \cdot \nabla u) &= \Lambda^s(b \cdot \nabla b), \\
\partial_1 \Lambda^s b - \Delta \Lambda^s b - \partial_1 \Lambda^s u + \Lambda^s(u \cdot \nabla b) &= \Lambda^s(b \cdot \nabla u), \\
\text{div } u = \text{div } b &= 0, \\
u(0, x) = u_0(x), \quad b(0, x) = b_0(x).
\end{aligned}
\]

Taking the $L^2$ inner product, we may obtain

\[
\frac{1}{2} \frac{d}{dt}(\|\Lambda^s u\|^2_{L^2} + \|\Lambda^s b\|^2_{L^2}) = \int \Lambda^s(b \cdot \nabla b) \cdot \Lambda^s u dx - \int \Lambda^s(u \cdot \nabla u) \cdot \Lambda^s u dx + \int \Lambda^s \Delta u \cdot \Lambda^s u dx + \int \Lambda^s \Delta b \cdot \Lambda^s b dx
\]

\[
\begin{aligned}
&+ \int \Lambda^s(b \cdot \nabla u) \cdot \Lambda^s b dx - \int \Lambda^s(u \cdot \nabla b) \cdot \Lambda^s b dx + \int \Lambda^s \Delta b \cdot \Lambda^s b dx \\
&:= \sum_{n=1}^{6} I_n.
\end{aligned}
\]

Thanks to div $u = \text{div } b = 0$, we may get from a standard commutator’s process that

\[
I_1 = \int (b \cdot \nabla \Lambda^s b) \cdot \Lambda^s u dx + \int \left(\Lambda^s(b \cdot \nabla b) - b \cdot \nabla(\Lambda^s b)\right) \cdot \Lambda^s u dx,
\]

which follows from Lemma 2.3 and Hölder’s inequality that

\[
|I_1| \leq \frac{1}{2} \|\nabla b\|_{L^\infty} \|\Lambda^s u\|_{L^2} \|\Lambda^s b\|_{L^2}
\]

\[
+ C\{\|\nabla b\|_{L^\infty} \|\Lambda^{-1} \nabla b\|_{L^2} + \|\Lambda^s b\|_{L^2} \|\nabla b\|_{L^\infty}\} \|\Lambda^s u\|_{L^2}
\]

\[
\leq C\|\nabla b\|_{L^\infty} \|\Lambda^s b\|_{L^2} \|\Lambda^s u\|_{L^2} \leq C\|\nabla b\|_{L^\infty}(\|b\|_{H^s}^2 + \|u\|_{H^s}^2).
\]
Similarly, we have
\begin{equation}
|I_2| \leq C \| \nabla u \|_{L^\infty} \| u \|_{H^s}^2,
\end{equation}
\begin{equation}
|I_4| \leq \frac{1}{2} \| \nabla b \|_{L^\infty} \| \Lambda^s u \|_{L^2} \| \Lambda^s b \|_{L^2}
+ C \{ \| \nabla u \|_{L^\infty} \| \Lambda^{s-1} \nabla b \|_{L^2} + \| \Lambda^s u \|_{L^2} \| \nabla b \|_{L^\infty} \} \| \Lambda^s b \|_{L^2}
\leq C (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) (\| b \|_{H^s}^2 + \| u \|_{H^s}^2),
\end{equation}
and
\begin{equation}
|I_5| \leq C (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) (\| b \|_{H^s}^2 + \| u \|_{H^s}^2).
\end{equation}
Thanks to integrating by parts, we obtain
\begin{equation}
|I_3| = \int \Lambda^s \Delta u \cdot \Lambda^s u dx = - \int |\Lambda^s \nabla u|^2 dx = -\| \nabla u \|_{H^s}^2.
\end{equation}
Similarly, we have
\begin{equation}
|I_6| = \int \Lambda^s \Delta b \cdot \Lambda^s b dx = - \int |\Lambda^s \nabla b|^2 dx = -\| \nabla b \|_{H^s}^2.
\end{equation}
Inserting (4.8)-(4.13) into (4.7), we obtain
\begin{equation}
\frac{d}{dt} (\| u \|_{H^s}^2 + \| b \|_{H^s}^2) + \| \nabla u \|_{H^s}^2 + \| \nabla b \|_{H^s}^2 \leq C (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) (\| u \|_{H^s}^2 + \| b \|_{H^s}^2).
\end{equation}
Hence, we obtain from Gronwall’s inequality
\begin{equation}
\| u(t) \|_{H^s}^2 + \| b(t) \|_{H^s}^2 \leq C e^{\int_0^t (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) d\tau} (\| u \|_{H^s}^2 + \| b \|_{H^s}^2).
\end{equation}
Now by the above arguments and Lemma 2.4, we know that
\begin{equation}
\int_0^t (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) d\tau
\leq C \left( \int_0^t (1 + \log(e + \| u(\tau) \|_{H^s}^2 + \| b(\tau) \|_{H^s}^2) d\tau \right),
\end{equation}
for any $t \in [0, T^*)$.
Thus, we have
\begin{equation}
\| u(t) \|_{H^s}^2 + \| b(t) \|_{H^s}^2
\leq (\| u_0 \|_{H^s}^2 + \| b_0 \|_{H^s}^2) e^{C \left( \int_0^t (1 + \log(e + \| u(\tau) \|_{H^s}^2 + \| b(\tau) \|_{H^s}^2) d\tau \right),
\end{equation}
which implies that
\begin{equation}
\log(e + \| u(t) \|_{H^s}^2 + \| b(t) \|_{H^s}^2) \leq C(T^*, u_0, b_0)
+ C \left( \int_0^t \log(e + \| u(\tau) \|_{H^s}^2 + \| b(\tau) \|_{H^s}^2) d\tau \right).
\end{equation}
Since $(u, b)$ is bounded in $L^2(0, T^*; H^2(\mathbb{R}^2))$, Gronwall’s inequality ensures that $(u, b)$ is also bounded in $L^\infty(0, T^*; H^s(\mathbb{R}^2))$. Thus, the solution can be extended after $t = T^*$, Which contradicts with the definition of $T^*$. Hence $T^* = +\infty$. $\square$
5. Proof of Theorem 1.2

The main goal of this subsection is to prove (1.3). Motivated by [19].

Proof of Theorem 1.2. Firstly, thanks to (1.4), one has

\begin{equation}
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\end{equation}

Applying Schonbek’s strategy, by splitting the phase-space $\mathbb{R}^2$ into two time-dependent parts, we get

\[ \|(\nabla u, \nabla b)\|_{L^2}^2 = \int_{S(t)} |\xi|^2 (|\widehat{u}(t, \xi)|^2 + |\widehat{b}(t, \xi)|^2) d\xi + \int_{S(t)^c} |\xi|^2 (|\widehat{u}(t, \xi)|^2 + |\widehat{b}(t, \xi)|^2) d\xi, \]

where $S(t) := \{\xi \in \mathbb{R}^2 : |\xi| \leq C_1^{1/2} g(t)\}$ and $g(t)$ satisfies $g(t) \leq C(1 + t)^{-1/2}$, which will be chosen later on. Then we obtain

\begin{equation}
\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + g^2(t)(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2)
\leq g^2(t) \int_{S(t)} (|\widehat{u}(t, \xi)|^2 + |\widehat{b}(t, \xi)|^2) d\xi.
\end{equation}

To deal with the low frequency part of $u$ and $b$ on the right-hand side of (5.2), we rewrite the equations $u$ and $b$ in (1.2) as

\begin{equation}
\begin{cases}
\partial_t u - \Delta u + \nabla p = b \cdot \nabla b - u \cdot \nabla u + \partial_1 b, \\
\partial_t b - \Delta b = b \cdot \nabla u - u \cdot \nabla b + \partial_1 u, \\
\text{div } u = \text{div } b = 0, \\
u(0, x) = u_0(x), \quad b(0, x) = b_0(x).
\end{cases}
\end{equation}

The system (5.3) is equivalent to the system

\begin{equation}
\begin{cases}
\partial_t u - \Delta u = \mathbb{P}(\text{div } (b \otimes b) + \text{div } (-u \otimes u) + \partial_1 b), \\
\partial_t b - \Delta b = \text{div } (b \otimes u) + \text{div } (-u \otimes b) + \partial_1 u, \\
\text{div } u = \text{div } b = 0, \\
u(0, x) = u_0(x), \quad b(0, x) = b_0(x),
\end{cases}
\end{equation}

where $\mathbb{P}$ is the Leray projection operator.

First, Consider the first equation in (5.3), by using Duhamel’s principle, we get

\[ u = e^{-t\Delta} u_0 + \int_0^t e^{(t-t')\Delta} \mathbb{P}(\text{div } (b \otimes b) + \text{div } (-u \otimes u) + \partial_1 b) dt'. \]

Taking Fourier transform with respect to $x$ variables gives rise to

\[ |\widehat{u}(t, \xi)| \lesssim e^{-t|\xi|^2} |\widehat{u}_0(\xi)| + \int_0^t e^{-(t-t')|\xi|^2} |\xi||\mathcal{F}_x(b \otimes b) + |\mathcal{F}_x(u \otimes u)| + |\mathcal{F}_x(\partial_1 b)| dt'. \]

so that

\begin{equation}
\int_{S(t)} |\widehat{u}(t, \xi)|^2 d\xi \lesssim \int_{S(t)} e^{-2t|\xi|^2} |\widehat{u}_0(\xi)|^2 d\xi
+ g^4(t) \int_0^t \|\mathcal{F}_x(b \otimes b)\|_{L^\infty} + \|\mathcal{F}_x(u \otimes u)\|_{L^\infty} dt' \]
+ g^4(t) \int_0^t \|\mathcal{F}_x(\partial_1 b)\|_{L^\infty} dt'.
\end{equation}
Applying (5.1) gives
\[(5.6)\] \[
\left( \int_0^t \| F_x(b \otimes b) \|_{L^\infty dt'} \right)^2 \leq \left( \int_0^t \| b \otimes b \|_{L^1 dt'} \right)^2 = \left( \int_0^t \| b(t') \|_{L^2}^2 dt' \right)^2,
\]
\[(5.7)\] \[
\left( \int_0^t \| F_x(u \otimes u) \|_{L^\infty dt'} \right)^2 \leq \left( \int_0^t \| u \otimes u \|_{L^1 dt'} \right)^2 = \left( \int_0^t \| u(t') \|_{L^2}^2 dt' \right)^2,
\]
and by using Hölder’s inequality, we have
\[(5.8)\] \[
\left( \int_0^t \| F_x(\partial_1 b) \|_{L^\infty dt'} \right)^2 \leq \left( \int_0^t \| \nabla b \|_{L^1 dt'} \right)^2 \leq C.
\]
Similarly, we consider the second equation in (5.4), by using Duhamel’s principle and taking Fourier transform with respect to \( x \) variables gives rise to
\[
|\widehat{b}(t, \xi)| \lesssim e^{-t|\xi|^2} |\widehat{b}_0(\xi)|
\]
\[
+ \int_0^t e^{-(t-t')|\xi|^2} |\xi| (|F_x(b \otimes u)| + |F_x(u \otimes b)| + |F_x(\partial_1 u)|) dt',
\]
so that
\[(5.9)\] \[
\int_{S(t)} \widehat{b}(t, \xi)^2 d\xi \lesssim \int_{S(t)} e^{-2t|\xi|^2} |\widehat{b}_0(\xi)|^2 d\xi
\]
\[
+ g^4(t)[\int_0^t (\| F_x(b \otimes u) \|_{L^\infty} + \| F_x(u \otimes b) \|_{L^\infty} dt')^2
\]
\[
+ g^2(t)[\int_0^t \| F_x(\partial_1 u) \|_{L^\infty} dt']^2.
\]
Applying (5.1) gives
\[(5.10)\] \[
\left( \int_0^t \| F_x(b \otimes u) \|_{L^\infty} + \| F_x(u \otimes b) \|_{L^\infty} dt' \right)^2 \leq \left( \int_0^t \| u \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| b \|^2_{L^2} dt' \right)^2
\]
\[
\lesssim \left( \int_0^t (\| u \|^2_{L^2} + \| b \|^2_{L^2} dt') \right)^2,
\]
and by using Hölder’s inequality, we have
\[(5.11)\] \[
\left( \int_0^t \| F_x(\partial_1 u) \|_{L^\infty} dt' \right)^2 \leq \left( \int_0^t \| \nabla u \|_{L^1} dt' \right)^2 \leq C.
\]
Noticing the fact \((u_0, b_0) \in \dot{H}^{-\varepsilon}\), one has
\[(5.12)\] \[
\int_{S(t)} e^{-2t|\xi|^2} |\widehat{u}_0(\xi)|^2 d\xi = \int_{S(t)} \frac{e^{-2t|\xi|^2}}{(1 + |\xi|^2)^{-\frac{\varepsilon}{2}}} (1 + |\xi|^2)^{-\frac{\varepsilon}{2}} |\widehat{u}_0(\xi)|^2 d\xi \lesssim (1 + t)^{-2\varepsilon},
\]
similarly, we have
\[(5.13)\] \[
\int_{S(t)} e^{-2t|\xi|^2} |\widehat{b}_0(\xi)|^2 d\xi \lesssim (1 + t)^{-2\varepsilon}.
\]
Then as \( g(t) \lesssim (1 + t)^{-\frac{1}{2}} \), we deduce from (5.5)-(5.13) that
\[(5.14)\] \[
\int_{S(t)} (|\widehat{u}(t, \xi)|^2 + |\widehat{b}(t, \xi)|^2) d\xi \lesssim (1 + t)^{-2\kappa} + g^4(t)[\int_0^t (\| u(t') \|^2_{L^2} + \| b(t') \|^2_{L^2}) dt']^2,
\]
with \( \kappa := \min\{\frac{1}{2}, \varepsilon\} \).
Substituting (5.14) to (5.2) results in
\[
\frac{d}{dt} \|(u, b)(t)\|_{L^2}^2 + g^2(t) \|(u, b)(t)\|_{L^2}^2 
\lesssim g^2(t)(1 + t)^{-2\gamma} + g^6(t) \left( \int_0^t \|(u, b(t)\|_{L^2}^2 dt' \right)^2.
\]
(5.15)

Therefore, we get
\[
e^{\int_0^t g^2(t')dt'} \|(u, b)(t)\|_{L^2}^2 \lesssim \|(u_0, b_0)\|_{L^2}^2
\]
\[
+ \int_0^t e^{\int_0^{t'} g^2(\tau)d\tau}(g^2(t')(1 + t')^{-2\gamma} + g^6(t')(\int_0^{t'} \|(u, b(\tau)\|_{L^2}^2 d\tau)^2) dt'.
\]
(5.16)

Now taking \(g^2(t) = \frac{3}{(e + t)\ln(e + t)}\) in (5.16), we deduce from (5.1) that
\[
\|(u, b)(t)\|_{L^2}^2 \ln^3(e + t)
\]
\[
\lesssim 1 + \int_0^t \frac{\ln^2(e + t')}{(e + t')^{1+2\gamma}} + \frac{1}{(e + t')^3} (\int_0^{t'} \|(u, b(\tau)\|_{L^2}^2 d\tau)^2) dt'
\]
\[
\lesssim 1 + \int_0^t \frac{1}{(e + t')^3} dt' \lesssim \ln(e + t),
\]
which implies
\[
\|(u, b)(t)\|_{L^2} \lesssim \ln^{-1}(e + t),
\]
(5.17)

and
\[
\int_0^t \|(u, b(t'))\|_{L^2}^2 dt' \lesssim (e + t) \ln^{-2}(e + t).
\]
(5.18)

Thanks to (5.16), (5.18) and (5.19), we may get by all standard iteration argument as [19] for the classical 2-D Navier-Stokes system (see also [7]) that
\[
\|(u, b)(t)\|_{L^2} \lesssim (e + t)^{-\gamma}.
\]
(5.19)

The proof of Theorem 1.2 is completed.
\[
\square
\]

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