EXTERIOR EXTENSION PROBLEMS FOR STRONGLY
ELLIPTIC OPERATORS: SOLVABILITY AND APPROXIMATION
USING FUNDAMENTAL SOLUTIONS

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Abstract. In this work we study three exterior extension problems for strongly
elliptic partial equations: the Cauchy problem (in a special statement), the
"analytical" continuation problem and the so called "inner" Dirichlet prob-
lem in the scale of the Sobolev spaces over a domain with relatively smooth
boundaries. We consider the existence of solutions to these problems, the dense
solvability and conditional well-posedness of these problems for a wide class of
strongly elliptic systems. We also consider the approximation of solutions to
these problems by a single layer potential and by a linear combination of "dis-
crete" fundamental solutions in relation to a narrower class of strongly elliptic
operators of the second order. The obtained results justify the applicability of
the indirect method of boundary integral equations and for numerical solving
the exterior extension problems.

INTRODUCTION

In this paper we consider three interconnected problems for elliptic partial dif-
fferential equations: a special case of the Cauchy problem with a doubly connected
boundary of the solution domain, the "analytical" continuation problem and a
problem that can be named the "inner" Dirichlet problem. These problems can be
classified as external extension problems.

Namely, let \( \Omega_0 \) and \( \Omega_1 \) be bounded domains in \( \mathbb{R}^n, n \geq 2 \), with sufficiently
smooth boundaries \( \partial \Omega_0 \) and \( \partial \Omega_1 \), such that \( \overline{\Omega_0} \subset \Omega_1 \) and let \( L \) be an elliptic linear
partial differential operator of order \( 2m, m \in \mathbb{N} \), with real analytic coefficients in a
neighbourhood of \( \overline{\Omega_1} \). We also fix a boundary Dirichlet system \( \{ B_j \}_{j=0}^{2m-1} \) of order
\( (2m-1) \) on \( \partial \Omega_0 \), see Definition 1.1 below; of course the standard system \( \{ \frac{\partial^{j}}{\partial \nu^{j}} \}_{j=0}^{2m-1} \)
of the normal derivatives with respect to \( \partial \Omega_0 \) belongs to this class. Let us roughly
formulate the problems, leaving the specification of function spaces for the next
section.

The first one is the Cauchy problem for \( L \).

Problem 0.1. Given data \( u_0, \ldots, u_{2m-1} \) on \( \partial \Omega_0 \), find a solution \( u \) to the homoge-
neous equation \( Lu = 0 \in \Omega_1 \setminus \overline{\Omega_0} \) satisfying in a suitable sense \( B_j u = u_j \) on \( \partial \Omega_0 \),
for all \( 0 \leq j \leq 2m-1 \).

The second one is the problem of the "analytic" continuation from a bounded
domain to a large one.

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problem, the single layer, the boundary elements method, the fundamental solution method.
Problem 0.2. Given data \( V \) in \( \Omega_0 \) satisfying the equation \( LV = 0 \) in \( \Omega_0 \) find a solution \( U \) to the homogeneous equation \( LU = 0 \) in \( \Omega_1 \) such that \( U = V \) in \( \Omega_0 \).

The third one is the so-called "inner Dirichlet problem"

Problem 0.3. Given data \( v_0, \ldots, v_{m-1} \) on \( \partial \Omega_0 \) find a solution \( U \) to the homogeneous equation \( LU = 0 \) in \( \Omega_1 \) such that \( B_j U = v_j \) on \( \partial \Omega_0 \) for all \( 0 \leq j \leq m - 1 \).

A typical example of Problem 0.1 is the Cauchy problem for the Laplace equation that were initially considered by Jacques Hadamard [27] as a famous example of an ill-posed problem. Since then the Cauchy problem for elliptic equations has been actively studying in various aspects, see, for instance, works by S.N. Mergeljan [50], E.M. Landis [38], M.M. Lavrent’ev [10], R. Lattés and J.-L. Lions [39], L.E. Payne [55], V.A. Kozlov, V.G. Maz’ya and A.V. Fomin [35], G. Alessandrini with co-authors [4] and others. It appears that the regularization methods (see, for instance, [82], [41]) are most effective for studying the problem. The book [81] gives a rather full description of solvability conditions for the homogeneous elliptic equations in the Sobolev spaces. Actually, the Cauchy problem for elliptic equations and systems of equations arises in many fields of science and technology, including geophysics, elasticity theory (see, for instance, [42]) and medical diagnostics (in particular, non-invasive electrocardiographic imaging [57]).

Problem 0.2 initially arose as the analytical continuation problem in the theory of functions of a complex variable and it was subsequently generalized to harmonic functions and functions governed by a wide class of elliptic equations in \( \mathbb{R}^n \). The long-time history of the study of various aspects of this problem includes, in particular, works of T. Carleman [14], N. Aronszajn [7], F. E. Browder [12], B.M. Weinstock [86]. A deep connection between Problems 0.1 and 0.2 was exploited in the paper [64], see also book by N. Tarkhanov [81]. The issues related to theoretical justifications of numerical solving of this problem have not been sufficiently studied. We can mention only one recent paper on quantifying the ill-conditioning of the classical analytic continuation problem, see L.N. Trefethen [83].

Problem 0.3 has an independent applied significance, since it arises in various fields of science and technology. In particular, is related to non-contact cardiac electrical mapping with the basket intracardiac catheter [26].

Moreover, Problems 0.1 and 0.2 can be reduced to Problem 0.3. This observation opens a way to construct numerical algorithms to obtain the Problem 0.1 solution and 0.2 via solving Problem 0.3.

Besides that, Problem 0.3 underlies the following special "extension" approach to numerical solving classical boundary value problems for elliptic equations. In this paper, we will take the Dirichlet problem for the second order strongly elliptic operators as a model example of such a classical problem. The Dirichlet problem reads as follows:

Problem 0.4. Let \( m = 1 \). Given data \( w_0 \) on \( \partial \Omega_0 \), find a solution \( w \) to the \( Lw = 0 \) in \( \Omega_0 \) satisfying in a suitable sense \( \partial \Omega_0 w = w_0 \) on \( \partial \Omega_0 \).

The "extension" approach for solving Problem 0.4 consists in setting a "virtual" arbitrary boundary \( \partial \Omega_1 \) so that domain \( \Omega_1 \) bounded to it includes the closure of domain \( \partial \Omega_0 \); \( \Omega_0 \subset \Omega_1 \). A solution \( w \) to Problem 0.4 in \( \Omega_0 \) is approximated by restricting to \( \Omega_0 \) the Problem 0.3 solution for the same elliptic operator \( L \) and the same boundary datum \( f_0 = w_0 \) on \( \partial \Omega_0 \) (i.e. \( Lu = 0 \) in \( \Omega_1 \), \( u = w_0 \) on \( \partial \Omega_0 \)).
Actually, the "extension" strategy is used relatively widely for numerical solving elliptic boundary value problems. For example, in relation to the finite element method the "extension" approach is known as the fictitious domain method [25], [8], [2]. The same approach is also used in the method of fundamental solutions, which will be discussed below.

The "extension" approach has some advantages. In particular, it allows a certain freedom of choice of geometry and smoothness of the prescribed embracing boundary \( \partial \Omega_1 \). However, the well-posedness of boundary value Problem 0.4 does not imply the well-posedness for Problem 0.3. Therefore, the applicability of the "extension" method needs a special justification which is provided in the presented paper.

The main objective of this paper is to consider a way to solve Problem 0.3 using the concept of fundamental solutions. We will consider this issue in several aspects: a) as an actual method for solving Problem 0.3 itself; b) as a method for solving Problem 0.1 and Problem 0.2 which are reduced to Problem 0.3; c) as a method for solving Problem 0.4 via the "extension" approach.

We will focus on two methods of solving problem 0.3. The first method is based on the representation of solution \( u \) to Problem 0.3 in \( \Omega_1 \) by a potential of the single layer defined on boundary \( \partial \Omega_1 \):

\[
(0.1) \quad u(x) = \int_{\partial \Omega_1} \Phi(x, y)v(y)ds_y, \quad x \in \Omega_1, \, y \in \partial \Omega_1,
\]

where \( \Phi(x, y) \) is the fundamental solution for operator \( L \) and \( v \) is the single layer density.

The proposed method can be considered in the context of the classical boundary integral equation method, which is based on the representation of the boundary value problems solutions by the single layer or (and) double layer potential. Its numerical realization is known as the boundary element method (BEM) (see, for instance, [61] or elsewhere).

The single layer version of BEM is the simplest for numerical implementation, however, it requires some caution. In certain cases, despite the fact that a boundary value problem has a unique solution, the integral equation for the single layer density may be unsolvable (some examples are given in section 3 of this paper). Moreover, the solvability of the equation depends on the specific type of fundamental solution.

The study of the solvability of the boundary integral equations has a long history, including influential works of S.G. Mikhlin [51], A.P. Calderón and A. Zygmund (see, for instance, [76]), M. Costabel [18], M. Costabel and W. L. Wendland [19].

Note that the integral equation method was mainly studied in relation to classical boundary value problems, such as the Dirichlet and Neumann problems. Its application to Problem 0.3 needs to be considered separately.

The second method for solving Problem 0.3 taken into our consideration is known as the method of fundamental solutions (MFS). It consists in approximating the solution of an elliptic equation in a bounded domain by a linear combination of fundamental solutions, whose singularities belongs to a discreet set of the isolated points of the prescribed boundary embracing the domain. More precisely, the solution \( u \) to Problem 0.3 in \( \Omega_1 \) is approximated as follows:
\[ u \approx u_N(x) = \sum_{j=1}^{N} \Phi(x, y_j) v_j, \quad x \in \Omega, \quad y_j \in \{y_j\}_{j=1}^{N} \subset \partial \Theta_{ex} \]

where \( v_j \) are real weight vector coefficients, \( \partial \Theta_{ex} \) is the "embracing" boundary, i.e., it is a boundary of a domain \( \Theta_{ex} \) such that \( \overline{\Omega}_1 \subset \Theta_{ex} \).

MFS can be viewed as a discrete version of the single layer potential method [21]. MFS has a number of attractive properties (see the discussion of its advantages in [70]). For instance, it does not require numerical integration. Therefore, MFS is widely used for numerical solving of classical boundary value problems for partial differential equations arising in various fields (see [15] and the literature cited there).

To justify the applicability of MFS to solving problems for an elliptic equation in a given domain, it is necessary to show the existence of real coefficients \( \{v_j\}_{j=1}^{N}, j, N \in \mathbb{N} \) so that finite or infinite sums of type (0.2) are dense in a space of solutions to the equation in the solutions domain. This fact has been shown for some strong elliptic equations including the Laplace, Helmholtz, biharmonic and polyharmonic equation (using various functional spaces) subject to some smoothness requirements of the boundary geometry and boundary condition (see [10], [71], [70], [72] and the literature cited in [15]). The question of further generalization of these results remains open.

Some other results adjoin this topic. The completeness of the set of the discrete fundamental solutions with a different localization of their singularities, namely: \( \Phi(x_i, y), x_i \in \partial \Omega_0, y \in \Omega_2 \setminus \overline{\Omega}_1 \cup \overline{\Omega}_1 \subset \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \) were studied by A.B. Bakushinsky [9], M.A. Alexidze [5], cf. also the pioneer result [60] by C. Runge for holomorphic functions where the Cauchy kernel was used, or [80, Theorem 4.2.1] for elliptic differential operators admitting left fundamental solutions. It was also shown that the linear combination of fundamental solutions with singularities in an arbitrary open set \( U \) outside the closure of a connected domain \( \Omega \) are dense (in the sense of uniform norm) in the space \( X = \{u \in C^m(\Omega) : Lu = 0 \text{ in } \Omega \} \cap C(\overline{\Omega}) \), see [12]. This result has been significantly expanded in works [50], [51], [72]. However, when singularities of the fundamental solution belong only boundary \( \partial \Omega \) of domain \( \Omega \), this statement not always valid [71].

In addition, we note that the single layer integral equation method (see [41], [45], [11]) and MFS (see [85] and [88]) have been used for numerical solving the Cauchy problem for the Laplace equation in a domain with the doubly-connected boundary (similar to Problem 0.1). In these works the single layer density was given on both boundaries \( \partial \Omega_0 \) and \( \partial \Omega_1 \) and the fundamental solutions singularities were given on the two closed surfaces \( \partial \Theta_{in} \) and \( \partial \Theta_{ex} \) such that \( \overline{\Omega}_1 \subset \Omega_0, \overline{\Omega}_1 \subset \Theta_{ex} \). In this paper, we do not consider the justification of these approaches focusing on the proposed method of reduction of the Cauchy problem to Problem 0.3 which assumes the single layer density or the fundamental solution singularities to be given only on one surface.

The paper is organized as follows. Section 1 is devoted to mathematical preliminaries such as the notations and the used function spaces. In section 2, we present some results on solvability and conditional well-posedness of extension Problems 0.1, 0.2, 0.3 for strongly elliptic systems in the form of the generalized Laplacian in bounded domains in \( \mathbb{R}^n \), \( n \geq 2 \), with relatively smooth boundaries, considering
their solutions in Sobolev-Slobodetskii spaces. In section 3 we investigate the representation and approximation of solutions to the strongly elliptic systems of the second order by the single layer potential and a weighted sum of the fundamental solutions. In section 4 we provide some results justifying the applicability of the single layer and MFS methods for approximation of solutions to Problems 1.1–1.4 for the strongly elliptic systems of the second order.

1. Mathematical preliminaries

Let \( \theta \) be a measurable set in \( \mathbb{R}^n \), \( n \geq 2 \). Denote by \( L^2(\theta) \) a Lebesgue space of functions on \( \theta \) with the standard inner product \( (u, v)_{L^2(\theta)} \). If \( D \) is a domain in \( \mathbb{R}^n \) with a Lipschitz boundary \( \partial D \), then for \( s \in \mathbb{N} \) we denote by \( H^s(D) \) the standard Sobolev space with the standard inner product \( (u, v)_{H^s(D)} \), see, for instance, [1]. It is well-known that this scale extends for all real \( s > 0 \). More precisely, given any non-integer \( s > 0 \), we use the so-called Sobolev-Slobodetskii space \( H^s(D) \), see [2]. We denote also by \( H^0_0(D) \) the closure of the subspace \( C_0^\infty(D) \) in \( H^s(D) \), where \( C_0^\infty(D) \) is the linear space of functions with compact supports in \( D \). Then the scale of Sobolev spaces can be extended for negative smoothness indexes, too. Namely, for \( s > 0 \), the space \( H^{-s}(D) \) can be identified with the dual of \( H^s_0(D) \) with respect to the pairing induced by \( (\cdot, \cdot)_{L^2(D)} \), see, for instance, [1].

If the boundary \( \partial D \) of the domain \( D \) is sufficiently smooth, then, using the standard volume form \( d\sigma \) on the hypersurface \( \partial D \) induced from \( \mathbb{R}^n \), we may consider the Sobolev-Slobodeckij spaces \( H^s(\partial D) \) on \( \partial D \), see, for instance, [1].

Recall that a linear (matrix) differential operator

\[
A(x, \partial) = \sum_{|\alpha| \leq m} A_\alpha(x)(x)\partial^{\alpha}
\]

of order \( m \) and with \((l \times k)\)-matrices \( A_\alpha(x) \) having entries from \( C_0^\infty(X) \) on an open set \( X \subset \mathbb{R}^n \), is called an operator with injective (principal) symbol on \( X \) if \( l \geq k \) and for its principal symbol

\[
\sigma(A)(x, \zeta) = \sum_{|\alpha| = m} A_\alpha(x)\zeta^\alpha
\]

we have \( \text{rang} (\sigma(A)(x, \zeta)) = k \) for any \( x \in X, \zeta \in \mathbb{R}^n \setminus \{0\} \). An operator \( A \) is called (Petrovsky) elliptic, if \( l = k \) and its symbol is injective.

An operator \( L(x, \partial) \) is called strongly elliptic if it is elliptic, its order \( 2m \) is even and there is a positive constant \( c_0 \) such that

\[
(1.1) \quad (-1)^m \Re (w^*\sigma(L)(x, \zeta) w) \geq c_0 |\zeta|^{2m}|w|^2
\]

for any \( x \in X, \zeta \in \mathbb{R}^n, w \in \mathbb{C}^k \)

where \( w^* = \overline{w}^T \) and \( w^T \) is the transposed vector for \( w \in \mathbb{C}^k \).

In fact, (1.1) yields that the so-called Garding inequality:

\[
(1.2) \quad \|u\|_{[H^m(X)]^k}^2 \leq c_1 \Re ((Lu, u)_{L^2(X)^k}) + c_2 \|u\|_{L^2(X)^k}^2
\]

for all \( u \in [H^m_0(X)]^k \) with some positive constants \( c_1, c_2 \) independent on \( u \).

Actually, if the principal symbol of \( A \) is injective then the operator \( A^* A \) is strongly elliptic of order \( 2m \) where

\[
A^*(x, \partial) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} (A_\alpha^*(x))(x)
\]
is the formal adjoint for $A$ with the adjoint matrices $A^*_a(x)$. The typical operator of such type is the Laplacian $-\Delta = \nabla^* \nabla = -\nabla \cdot \nabla$. For the generalized Laplacians $A^* A$ Gårding inequality (1.2) is equivalent to the following:

\[
(1.3) \quad \|u\|_{[H^m(X)]^k}^2 \leq c_1 \|Au\|_{[L^2(X)]^k}^2 + c_2 \|u\|_{[L^2(X)]^k}^2 \quad \text{for all } u \in [H^m(X)]^k
\]

with some positive constants $c_1, c_2$ independent on $u$.

Next, for a domain $D \subset X$ denote by $S_A(D)$ the space of solutions to the equation $Au = 0$ in $D$ in the sense of distributions. If the principal symbol of $A$ is injective then elements of $S_A(D)$ are actually $C^\infty$-smooth (vector-)functions in $D$.

We say that a differential operator $Q$ satisfies the so called Uniqueness condition in the small on $X$ or the Unique Continuation Property if

(US) for any domain $D \subset X$ if a solution $u \in S_Q(D)$ vanishes on an open subset $G \subset D$ then it is identically zero in $D$.

In the sequel we assume that the both operators $L$ and $L^*$ under the consideration possess the Unique Continuation Property (US) (in particular, we assume that the generalized Laplacian $A^* A$ possesses this property). This assumption provides that the operator admits a bilateral fundamental solution, say $\Phi$, for the generalized Laplacian $A^* A$ if the operator $A$ has injective principal symbol and satisfies (US) on $X$. Actually, in this case the Green function of the Dirichlet problem for the Laplacian $A^* A$ in $D$ is a bilateral fundamental solution for it.

More precisely, we begin with a definition.

**Definition 1.1.** A set of linear differential operators $\{B_0, B_1, \ldots, B_{m-1}\}$ is called a $(k \times k)$-Dirichlet system of order $(m-1)$ on $\partial D$ if: 1) the operators are defined in a neighbourhood of $\partial D$; 2) the order of the differential operator $B_j$ equals to $j$; 3) the map $\sigma(B_j)(x, u(x)) : \mathbb{C}^k \to \mathbb{C}^k$ is bijective for each $x \in \partial D$, where $\nu(x)$ will denote the outward normal vector to the hypersurface $\partial D$ at the point $x \in \partial D$.

A typical $(k \times k)$-Dirichlet system of order $(m-1)$ on $\partial D$ consists of

\[
\{I_k, I_k \frac{\partial}{\partial \nu}, \ldots, I_k \frac{\partial^{m-1}}{\partial \nu^{m-1}}\}
\]

where $I_k$ is the unit $(k \times k)$-matrix and $\frac{\partial^j}{\partial \nu^j}$ is $j$-th normal derivative with respect to $\partial D$. According to the Trace Theorem, see for instance [44] Ch. 1, § 8] and [52], if $\partial D \in C^s$, $s \geq m \geq 1$ then each operator $B_j$ induces a bounded linear operator

\[
(1.4) \quad B_j : H^s(D) \to H^{s-j/2}(\partial D).
\]

The main advantage of the use of the Dirichlet system is the following lemma.

**Lemma 1.2.** Let $\partial D \in C^s$, $s \geq m$ and $B = \{B_0, B_1, \ldots, B_{m-1}\}$ be a Dirichlet system of order $(m-1)$ on $\partial D$. Then for each set $\bigoplus_{j=0}^{m-1} u_j \in \bigoplus_{j=0}^{m-1} [H^{s-j-1/2}(\partial D)]^k$
there is a function \( u \in [H^s(D)]^k \) such that
\[
\oplus_{j=0}^{m-1} B_j u = \oplus_{j=0}^{m-1} u_j \text{ on } \partial D.
\]

**Proof.** See, for instance, [80, Lemma 8.3.3]. \( \square \)

Next we need the following useful (first) Green formula.

**Lemma 1.3.** Let \( m \in \mathbb{N}, \partial D \in C^m, \) \( A \) be a differential operator with injective symbol of order \( m \in \mathbb{N} \) in a neighbourhood of \( \bar{D} \) and \( B = \{ B_0, B_1, \ldots, B_{m-1} \} \) be a Dirichlet system of order \((m-1)\) on \( \partial D \). Then there is a Dirichlet system \( \tilde{B}^A = \{ \tilde{B}_0^A, \tilde{B}_1^A, \ldots, \tilde{B}_{m-1}^A \} \) on \( \partial D \) such that for all \( v \in [H^m(D)]^k, \ u \in [H^m(D)]^k \) we have
\[
\int_{\partial D} \left( \sum_{j=0}^{m-1} (\tilde{B}_j^A - v) \right)^* B_j u \, d\sigma = \int_D \left( v^* Au - (A^* v) u \right) \, dx.
\]

**Proof.** See, for instance, [59, Lemma 5.1.1]. \( \square \)

Now we recall the Existence and Uniqueness Theorem for the Dirichlet Problem related to strongly elliptic operators.

**Problem 1.4.** Given pair \( g \in [H^{s-2m}(D)]^k \) and \( \oplus_{j=0}^{m-1} w_j \in \oplus_{j=0}^{m-1} [H^{s-j/2}(\partial D)]^k \) find, if possible, a function \( w \in [H^s(D)]^k \) such that
\[
Lw = g \quad \text{in } D,
\]
\[
\oplus_{j=0}^{m-1} B_j w = \oplus_{j=0}^{m-1} w_j \quad \text{on } \partial D.
\]

The problem can be treated in the framework of the operator theory in Banach spaces, regarding \( L, \oplus_{j=0}^{m-1} B_j \) as an operator equation with the linear bounded operator
\[
(L, \oplus_{j=0}^{m-1} B_j) : [H^s(D)]^k \to [H^{s-2m}(D)]^k \times \oplus_{j=0}^{m-1} [H^{s-j/2}(\partial D)]^k, \ s \geq m.
\]

Recall that a problem related to operator equation \( Ru = f \) with a linear bounded operator \( R : X_1 \to X_2 \) in Banach spaces \( X_1, X_2 \) has the Fredholm property, if the kernel \( \ker(R) \) of the operator \( R \) and the co-kernel \( \text{coker}(R) \) (i.e. the kernel \( \ker(R^*) \) of its adjoint operator \( R^* : X_2^* \to X_1^* \)) are finite-dimensional vector spaces and the range of the operator \( R \) is closed in \( X_2 \).

**Theorem 1.5.** Let \( L \) be a strongly elliptic differential operator of order \( 2m, \ m \geq 1, \) with smooth coefficients in a neighbourhood \( X \) of \( \bar{D} \), \( \partial D \in C^s, \ s \geq m \) and \( B = \{ B_0, B_1, \ldots, B_{m-1} \} \) be a Dirichlet system of order \((m-1)\) on \( \partial D \). Then Problem \( L, \oplus_{j=0}^{m-1} B_j \) has the Fredholm property on the scale of the Sobolev spaces over \( D \). Namely, the dimensions of the spaces \([H^s(D) \cap H^m_0(D)]^k \cap S_L(D)\) and \([H^s(D) \cap H^m_0(D)]^k \cap S_{L^*}(D)\) are finite and there is a bounded linear operator
\[
G_D : [H^{s-2m}(D)]^k \to [H^s(D)]^k \cap [H^m_0(D)]^k,
\]
\[
\Pi_D^{(1)} : [H^s(D)]^k \cap [H^m_0(D)]^k \to [H^s(D)]^k \cap [H^m_0(D)]^k \cap S_L(D),
\]
\[
\Pi_D^{(2)} : [H^{s-2m}(D)]^k \to [H^s(D)]^k \cap [H^m_0(D)]^k \cap S_{L^*}(D),
\]
such that
\[
G_D L = I - \Pi_D^{(1)} \quad \text{on } [H^s(D) \cap H^m_0(D)]^k, \quad LG_D = I - \Pi_D^{(2)} \quad \text{on } [H^{s-2m}(D)]^k.
\]
Remark 1.8. Note that for the second order Laplacians $L = A^*A$ (i.e. for the case $m = 1$), the results of Lemmata 1.2, 1.3, Corollaries 1.6, 1.7 and Theorem 1.5 can be extended domains with Lipschitz boundaries using the classical method.
of non-negative Hermitian forms, see [54] or [66], [67] for the case of even more general problem with domains having non-smooth boundaries and with non-smooth boundary operators in (weighted) Sobolev spaces.

2. Exterior extension problems for strongly elliptic systems

We begin the section with the discussion of the ill-posed Cauchy problem for a $(k \times k)$-elliptic system $L$ of order $2m$ with $m \geq 1$ on $X$ in a very particular situation. Namely, let $\Omega_0$ and $\Omega_1$ be two bounded domains in $X$ such that $\overline{\Omega_0} \subset \Omega_1$.

**Problem 2.1.** Let $s \in \mathbb{Z}_+$ and $\partial \Omega_0 \in C^s$ if $s \geq m$ or $\partial \Omega_0 \in C^\infty$ if $s < m$. Let also $B = \{B_0, B_1, \ldots, B_{2m-1}\}$ be a Dirichlet system of order $(2m-1)$ on $\partial \Omega_0$. Given functions $u_j \in [H^{s-j-1/2}(\partial \Omega_0)]^k$, $0 \leq j \leq 2m-1$, find, if possible, a vector function $u \in [H^s(\Omega_1 \setminus \overline{\Omega_0})]^k$ such that

$$
\begin{aligned}
&Lu = 0 \\
&\oplus_{j=0}^{2m-1} B_j u = \oplus_{j=0}^{2m-1} u_j \\
&\text{on } \partial \Omega_0.
\end{aligned}
$$

As the Cauchy problem is generally ill-posed, the description of its solvability conditions is rather complicated. It appears that the regularization methods (see, for instance, [82]) are most effective for studying the problem. However, there are many different ways to realize the regularization, see, for instance, [40], [49], [35] for the Cauchy problem related to the second order elliptic equations. We follow idea of the book [81], that gives a rather full description of solvability conditions for the homogeneous elliptic equations. The even order of the system $L$ is unessential for Problem 2.1, but is essential to other extension problems consired in this section.

As we mentioned above, if we assume that both operators $L$ and $L^*$ possess the Unique continuation property (US) then $L$ admits a bilateral (left and right) fundamental solution $\Phi(x,y)$, see, for instance, [79, §2.3]. In particular, the following (representation) Green formula holds: for each $u \in [H^s(\Omega_1 \setminus \overline{\Omega_0})]^k$ we have

$$
\chi_{\Omega_1 \setminus \overline{\Omega_0}} u = T_{\partial \Omega_1}^{(B)}(Bu) - T_{\partial \Omega_0}^{(B)}(Bu),
$$

where $\chi_D$ is the characteristic function of the (bounded) domain $D$ in $\mathbb{R}^n$, and

$$
T_S^{(B)}(\oplus_{j=0}^{2m-1} u_j) = \int_S \left( \sum_{j=0}^{2m-1} (\tilde{B}_j^*(y)\Phi^*(x,y))^* u_j(y) \right) d\sigma(y),
$$

with a hypersurface $S$, $x \not\in S$, and the dual Dirichlet system $\tilde{B}^L$ for $B$ with respect to the (first) Green formula for the operator $L$.

Let us formulate a solvability criterion for Problem 2.1 under reasonable assumptions on $S$. We denote by $(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+$ the restriction of the potential $T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j)$ onto $\Omega_0$.

Similarly, the restriction of the potential $T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j)$ onto $\Omega_1 \setminus \overline{\Omega_0}$ will be denoted by $(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^-$.

Obviously, $LT_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j) = 0$ in $\Omega_1 \setminus \partial \Omega_0$, as a parameter dependent integrals, i.e. $(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+ \in S_L(\Omega_0)$ and, similarly, $(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^- \in S_L(\Omega_1 \setminus \overline{\Omega_0})$. 


Theorem 2.2. Let $L$ be an elliptic operator such that both $L$ and $L^*$ satisfy (US) on $X$ and $\Omega_0$ be a bounded domain in $X$. Let also $s \in \mathbb{Z}_+$ and $\partial \Omega_0 \in C^{\text{max}(s,2)}$ if $s \geq m$ or $\partial \Omega_0 \in C^\infty$ if $s < m$. If $\Omega_1 \setminus \Omega_0$ has no compact components in $\Omega_1$ then Problem (2.1) is densely solvable and it has no more than one solution. It is solvable if and only if there is a function $F \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1)$ and such that

$$F = (T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+ \text{ in } \Omega_0.$$  

Besides, the solution $u$, if exists, is given by the following formula:

$$u = (T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^- - F \text{ in } \Omega_1 \setminus \overline{\Omega_0}.$$  

Proof. All the statement, except the dense solvability follow from [51] Theorems 2.8 and 5.2. The dense solvability of the Cauchy problem was established in spaces of different types, see, for instance, [49, 50] for the Laplace operator in spaces of $C^1$-smooth functions or [64, Lemma 3.2] for general elliptic systems in the so-called Hardy spaces. In order to give some arguments about the dense solvability in this particular situation we may use Approximation Theorems for solutions to elliptic systems, see, for instance, [81, Ch. 5-8] and Theorems on the jump behaviour of the potential $T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j)$, see [73] Theorem 3.3.9 and [64, Lemma 2.7].

Indeed, the continuity of the potentials

$$\begin{align*}
(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+ : \oplus_{j=0}^{2m-1} [H^{s-j-1/2}(\partial \Omega_0)]^k \rightarrow [H^s(\Omega_0)]^k, \\
(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^- : \oplus_{j=0}^{m-1} [H^{s-j-1/2}(\partial \Omega_0)]^k \rightarrow [H^s(\Omega_1 \setminus \overline{\Omega_0})]^k, 
\end{align*}$$

follows from Theorems on the boundedness of the boundary potential operators related the pseudo differential operators satisfying the so-called transmission property, see, for instance, [58, §2.3.2.5] or [81, §2.4].

In particular, the continuity of the potentials mean that

$$\begin{align*}
(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+ \in S_L(\Omega_0) \cap [H^s(\Omega_0)]^k, \\
(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^- \in S_L(\Omega_1 \setminus \overline{\Omega_0}) \cap [H^s(\Omega_1 \setminus \overline{\Omega_0})]^k. 
\end{align*}$$

Next, according to Jump Theorems, for $0 \leq j \leq 2m - 1$ we have

$$B_j(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^- - B_j(T_{\partial \Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+ = u_j \text{ on } \partial \Omega_0,$$

where the type of boundary values depends on the order of the differential operators. Of course, if $s \geq 2m$ then all the boundary values under the consideration can be treated as the standard traces on $\partial \Omega$. Otherwise, if the orders of operators $B_j$ are greater than $s$ then we should pass to the so-called weak boundary values, see, for instance, [73] for harmonic functions or [81, §§9.3, 9.4] or [64, Definition 2.2] for general elliptic systems satisfying the Unique Continuation Property (US) and then the boundary should be sufficiently regular (for example, $C^\infty$-smooth for $0 \leq s < m$, but it always can be a high finite smoothness). In any case, under the assumptions above the mappings

$$\begin{align*}
B_j : [H^s(\Omega_0)]^k \cap S_L(\Omega_0) \rightarrow [H^{s-j-1/2}(\partial \Omega_0)]^k, \\
B_j : [H^s(\Omega_1 \setminus \overline{\Omega_0})]^k \cap S_L(\Omega_1 \setminus \overline{\Omega_0}) \rightarrow [H^{s-j-1/2}(\partial \Omega_0)]^k,
\end{align*}$$

are continuous.
Next, for a domain \( D \) in \( X \) we denote by \( S_L(D) \) the union \( \cup_{U \supseteq \mathcal{P}} S_L(U) \) of solutions on all open sets in \( X \) containing \( D \).

According to [81] Theorem 8.2.2 for \( s < 2m \) and [81] Theorems 8.1.2 and 8.1.3 for \( s \geq 2m \), the space \( S_L(\overline{\Omega}_1) \) is everywhere dense in \( S_L(\Omega_1 \setminus \Omega_0) \) and, similarly, the space \( S_L(\overline{\Omega}_1 \setminus \Omega_0) \) is everywhere dense in the space \( S_L(\Omega_1 \setminus \overline{\Omega}_0) \). In particular, the potential \((T_{\partial\Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^-\) can be approximated in the space \([H^s(\Omega_1 \setminus \overline{\Omega}_0)]^k\) by elements from \( S_L(\overline{\Omega}_1 \setminus \Omega_0) \).

The Runge type Theorems for solutions to elliptic systems, see, for instance, [50] for harmonic functions, [81] Theorems 5.1.11, 5.1.13, yields that the space \( S_L(\Omega_1) \) is everywhere dense in the space \( S_L(\overline{\Omega}_1) \), since \( \Omega_1 \setminus \Omega_0 \) has no compact components in \( \Omega_1 \). Hence, relations (2.4)–(2.6) imply that the potential \((T_{\partial\Omega_0}^{(B)}(\oplus_{j=0}^{2m-1} u_j))^+\) can be approximated in the space \([H^s(\Omega_0)]^k\) by elements from \( S_L(\overline{\Omega}_1) \).

Finally, the jump formulas (2.7) combined with the continuity relations (2.8)–(2.9) yield the possibility to approximate the Cauchy data \( \oplus_{j=0}^{2m-1} B_j u \) with \( u \) from the space \( S_L(\Omega_1 \setminus \Omega_0) \), which was to be proved. \( \square \)

Thus, Theorem 2.2 reduces Problem 2.1 to the following problem related to "analytic continuation" from an open subset of \( X \) to a larger one.

**Problem 2.3.** Given \( V \in [H^s(\Omega_0)]^k \cap S_L(\Omega_0) \), find, if possible, \( U \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1) \) such that \( U = V \in \Omega_0 \).

**Corollary 2.4.** Let \( L \) be an elliptic operator such that both \( L \) and \( L^* \) satisfy (US) on \( X \) and \( \Omega_0 \) be a bounded domain. Let also \( s \in \mathbb{Z}_+ \) and \( \partial\Omega_0 \in C^{\max(s,2)} \) if \( s \geq m \) or \( \partial\Omega_0 \in C^m \) if \( s < m \). If \( \Omega_1 \setminus \Omega_0 \) has no compact components in \( \Omega_1 \) then Problem 2.3 is densely solvable and it has no more than one solution. It is solvable if and only if Problem 2.1 is solvable for the data \( \oplus_{j=0}^{2m-1} u_j = \oplus_{j=0}^{2m-1} B_j V \) on \( \partial\Omega_0 \).

**Proof.** The uniqueness of the problem is provided by the Unique Continuation Property (US) on \( X \). The dense solvability follows from the same arguments as in the proof of Theorem 2.2 first, [81] Theorem 8.2.2 for \( s < 2m \) and [81] Theorems 8.1.2 and 8.1.3 for \( s \geq 2m \), implies that the space \( S_L(\overline{\Omega}_0) \) is everywhere dense in \( S_L(\Omega_0) \) and then the Runge type Theorems for solutions to elliptic systems, see, for instance, [81] for harmonic functions, [81] Theorems 5.1.11, 5.1.13, provides that the space \( S_L(\overline{\Omega}_1) \) is everywhere dense in the space \( S_L(\overline{\Omega}_0) \), since \( \Omega_1 \setminus \Omega_0 \) has no compact components in \( \Omega_1 \).

If \( U \) is a solution to Problem 2.3 then \( U \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1) \), and the Cauchy data \( \oplus_{j=0}^{2m-1} B_j U = \oplus_{j=0}^{2m-1} B_j V \), are well-defined on \( \partial\Omega_0 \). Obviously the restriction \( u = U|_{\partial\Omega_0} \) is the unique solution to Problem 2.1.

If \( u \in [H^s(\Omega_1 \setminus \overline{\Omega}_0)]^k \) is the solution to Problem 2.1 with the Cauchy data \( \oplus_{j=0}^{m-1} u_j = \oplus_{j=1}^{m} B_j V \) on \( \partial\Omega_0 \) then the function

\[
U = \begin{cases} 
V & \text{in } \Omega_0, \\
u & \text{in } \Omega_1 \setminus \overline{\Omega}_0
\end{cases}
\]

belongs to \([H^s(\Omega_0)]^k \cap S_L(\Omega_0) \) and \([H^s(\Omega_1 \setminus \overline{\Omega}_0)]^k \cap S_L(\Omega_1 \setminus \overline{\Omega}_0) \) and satisfies

\[
\oplus_{j=0}^{2m-1} B_j U^- = \oplus_{j=0}^{2m-1} B_j u = \oplus_{j=0}^{m-1} B_j V = \oplus_{j=0}^{m-1} B_j U^+ \text{ on } \partial\Omega_0,
\]

**Solution.**
on \( \partial \Omega_0 \), too. Then, by the Green formula for solutions to elliptic systems [2.1], see [64], Theorem 2.4,

\[
(2.11) \quad \chi_{\Omega_1}(u) u = \mathcal{J}_{\partial (\Omega_1 \setminus \Omega_0)}^{(B)} (\oplus_{j=0}^{2m-1} B_j u) \quad \text{in} \quad \Omega_1 \setminus \partial \Omega_0,
\]

\[
(2.12) \quad \chi_{\Omega_0} V = \mathcal{J}_{\partial \Omega_0}^{(B)} (\oplus_{j=0}^{2m-1} B_j V)
\]

and then, adding (2.12) to (2.11) and taking into account relations (2.10) and the orientation of the boundaries of the domains \( \Omega_0 \) and \( \Omega_1 \setminus \Omega_0 \) we see that

\[
(2.13) \quad U = \mathcal{J}_{\partial \Omega_1}^{(B)} (\oplus_{j=0}^{2m-1} B_j u) \quad \text{in} \quad \Omega_1 \setminus \partial \Omega_0.
\]

The right hand side of (2.13) belongs to \( S_L(\Omega_1) \) as a parameter depending integral and hence \( U \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1) \) is the unique solution to Problem 2.3. \( \Box \)

Let us consider one more "extension problem" that can be called an "inner" Dirichlet problem for a strongly elliptic operator \( L \).

**Problem 2.5.** Given Dirichlet data \( \oplus_{j=0}^{m-1} v_j \in [H^{s-j-1/2}(\partial \Omega_0)]^k \) if possible, \( v \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1) \) such that \( \oplus_{j=0}^{m-1} B_j v = \oplus_{j=0}^{m-1} v_j \) on \( \partial \Omega_0 \).

Though it depends on the Dirichlet data, this problem is closely related to Problems 2.1 and 2.3.

**Corollary 2.6.** Let \( L \) be a strongly elliptic operator such that both \( L \) and \( L^* \) satisfy (US) on \( X \) and \( \Omega_0 \) be a bounded domain. Let also \( s \in \mathbb{Z}_+ \) and \( \partial \Omega_0 \in C^{\max(s,2)} \) if \( s \leq m \) or \( \partial \Omega_0 \in C^\infty \) if \( s < m \). If (1.9) holds and \( \Omega_1 \setminus \Omega_0 \) has no compact components in \( \Omega_1 \) then Problem 2.3 is densely solvable and it has no more than one solution. Moreover, the following statements are equivalent:

- Problem 2.3 is solvable with the data \( \oplus_{j=0}^{m-1} v_j \in [H^{s-j-1/2}(\partial \Omega_0)]^k \);
- Problem 2.3 is solvable with \( V = \mathcal{P}_{\Omega_0}(\oplus_{j=0}^{m-1} v_j) \);
- Problem 2.3 is solvable for the Cauchy data \( \oplus_{j=0}^{2m-1} u_j \) where \( \oplus_{j=0}^{m-1} u_j = \oplus_{j=0}^{m-1} v_j \) and \( \oplus_{j=m}^{2m-1} u_j = \oplus_{j=m}^{2m-1} B_j \mathcal{P}_{\Omega_0}(\oplus_{j=0}^{m-1} v_j) \) on \( \partial \Omega_0 \).

**Proof.** Indeed, according to Corollary 1.6, the Poisson integral \( \mathcal{P}_{\Omega_0} \) induces an isomorphism of Banach spaces

\[
\mathcal{P}_{\Omega_0} : [H^{s-j-1/2}(\partial \Omega_0)]^k \to [H^s(\Omega_0)]^k \cap S_L(\Omega_0).
\]

For this reason any solution \( U \) to Problem 2.3 coincides with \( \mathcal{P}_{\Omega_0}(\oplus_{j=0}^{m-1} v_j) \) in \( \Omega_0 \) and this proves the equivalence between the first two statements of the corollary. In particular, the Uniqueness Theorem for Problem 2.3 and its dense solvability immediately follow from Corollary 2.4.

Moreover, as the operator

\[
\oplus_{j=m}^{2m-1} B_j \mathcal{P}_{\Omega_0} : [H^{s-j-1/2}(\partial \Omega_0)]^k \to [H^{s-j-1/2}(\partial \Omega_0)]^k
\]

is well-defined (of course, as we mentioned above, it should be understood in the sense of weak boundary values for \( s < 2m \)), because of the continuity of mapping (2.8), (2.9); it represents the so-called Dirichlet-to-Neumann operator in a sense. Combined with Theorem 2.2, this proves the equivalence of the second and the third statements of the corollary. \( \Box \)
Thus, Corollary 2.6 hints us that Problem 2.3 is of key importance for Problems 2.1 and 2.4. Its solvability conditions and formulas for approximate and exact solutions may be indicated in terms of the so-called bases with the double orthogonality property, see [63] and [81] Ch. 12. Hence the Cauchy Problem 2.1 and interior Dirichlet Problem 2.5 can be investigated with the use of these bases; of course, one may elaborate some iteration algorithms, see, for instance, [35] or invoke integral representations method, see [13, 40] or elsewhere.

We finish the section with another important issue, namely, the conditional well-posedness (see, for instance, [82]) of the problems considered above. With this purpose, given positive constant $\gamma$, denote by $[H^s_{L,\gamma}(\Omega_1)]^k$ the set of vectors $U \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1)$ such that

$$||U||_{[H^s(\Omega_1)]^k} \leq \gamma.$$  

We consider it as a metric space with the metric induced by the norm $|||.||_{[H^s(\Omega_1)]^k}$.

**Corollary 2.7.** Let $L$ be an elliptic operator such that both $L$ and $L^*$ satisfy (US) on $X$ and $\Omega_0$ be a bounded domain. Let also $s \in \mathbb{Z}_+$ and $\partial \Omega_0 \in C^{\max(s,2)}$ if $s \geq m$ or $\partial \Omega_0 \in C^\infty$ if $s < m$. If $\Omega_1 \setminus \Omega_0$ has no compact components in $\Omega_1$ then Problems 2.1, 2.3 are conditionally well-posed in the following senses:

1. if a sequence $\{U_i\} \subset [H^s_{L,\gamma}(\Omega_1)]^k$ converges to zero in $[H^s(\Omega_0)]^k$ then $\{U_i\}$ converges to zero in the local space $[H^s_{L,\gamma}(\Omega_1)]^k$ and weakly converges to zero in $[H^s(\Omega_1)]^k$;

2. if for a sequence $\{u_i\} \subset [H^s_{L,\gamma}(\Omega_1 \setminus \overline{\Omega_0})]^k$ the sequences $\{B_ju_i\}$ converge to zero in $[H^{s-j-1/2}(\partial \Omega_0)]^k$ for each $j$, $0 \leq j \leq 2m - 1$, then $\{u_i\}$ converges to zero in the space $[H^s_{L,\gamma}(\Omega_1 \setminus \Omega_0)]^k$.

If, moreover, $L$ is a strongly elliptic operator satisfying (1.9) then Problem 2.2 is conditionally well-posed in the following sense:

3. if for a sequence $\{U_i\} \subset [H^s_{L,\gamma}(\Omega_1)]^k$ the sequences $\{B_jU_i\}$ converge to zero in $[H^{s-j-1/2}(\partial \Omega_0)]^k$ for all $j$, $0 \leq j \leq m - 1$, then $\{U_i\}$ converges to zero in the space $[H^s_{L,\gamma}(\Omega_1)]^k$.

**Proof.** We begin with Problem 2.3. Actually, the topology of the space $[H^s_{L,\gamma}(\Omega_1)]^k$ is "metrizable". Namely, one may fix an exhaustion $\{D_\nu\}$ of the domain $\Omega_1$ by relatively compact domains $D_\nu \subset \Omega_1$, $\cup_\nu \overline{D_\nu} = \Omega_1$ and define a metric as follows:

$$\rho_{\Omega_1}(U, V) = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \frac{||U - V||_{[H^s(D_\nu)]^k}}{1 + ||U - V||_{[H^s(D_\nu)]^k}}.$$  

If a sequence $\{U_i\}$ belongs to $H^s_{L,\gamma}(\Omega_1)$ then it is bounded in this space. Then by the weak compactness principle one may extract a subsequence $\{U'_i\}$ weakly convergent to a vector $U_0 \in [H^s(\Omega_1)]^k$. Now Stieltjes-Vitali Theorem for solutions to elliptic systems implies that the subsequence $\{U'_i\}$ converges to $U_0$, in fact, in the space $C^{\infty}_{loc}(\Omega_1)$; in particular, this means $U_0 \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1)$.

On the other hand, according to the hypothesis of the theorem it converges to zero in the space $[H^s(\Omega_0)]^k$. Hence $U_0 = 0$ in $\Omega_0$ and, by the Unique Continuation Property, $U_0 = 0$ in $\Omega_1$.

Now, if the sequence $\{U_i\}$ does not converge to zero in $[H^s_{L,\gamma}(\Omega_1)]^k$ then there is a positive number $\varepsilon_0$ and a subsequence $\{U_i\}$ such that

$$\rho_{\Omega_1}(U_i, 0) \geq \varepsilon_0.$$  

(2.14)
Applying to the sequence \( \{\tilde{U}_i\} \) the arguments as above we may extract a subsequence \( \{\tilde{U}_i\} \) weakly convergent to zero in \([H^s(\Omega_1)]^k\) and convergent to zero in the space \( C^\infty_{\text{loc}}(\Omega) \), obtaining a contradiction with (2.14).

Similarly, if the sequence \( \{U_i\} \) does not converge to zero weakly in \([H^s(\Omega_1)]^k\) then it has a subsequence weakly converging to a non-zero element \( U_0 \in [H^s(\Omega_1)]^k \cap \cap S_L(\Omega_1) \). Repeating the arguments above we see that \( U_0 \equiv 0 \) in \( \Omega_1 \), i.e. we arrive at the contradiction.

We continue with Problem 2.21.

If a sequence \( \{u_i\} \) belongs to \([H^2_0,(\Omega_1 \setminus \overline{\Omega}_0)]^k\) then it is bounded in this space. Each element \( u_i \) of this sequence is the unique solution to the Cauchy problem 2.3 with the data \( \oplus_{j=0}^{2m-1} u_{j,i} \) with \( u_{j,i} = B_j u_i \) for \( 0 \leq j \leq 2m - 1 \). Then Theorem 2.2 implies that for the potential \( (\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^+ \) there is the unique extension \( \mathcal{F}_i \in H^s(\Omega_1) \cap S_L(\Omega_1) \) in the domain \( \Omega_1 \). Moreover, formula (2.2) means that

\[
\mathcal{F}_i = (\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^- - \chi_{\Omega_1 \setminus \overline{\Omega}_0} u_i.
\]

If the sequences \( \{B_j u_i\} \) converge to zero in \([H^{s-j-1/2}(\partial\Omega_0)]^k\) for each \( j, 0 \leq j \leq 2m - 1 \), then the continuity of the potentials, see (2.3), (2.4), implies

\[
(\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^+ \to 0 \text{ in } H^s(\Omega_0),
\]

\[
(\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^- \to 0 \text{ in } H^s(\Omega_1 \setminus \overline{\Omega}_0).
\]

In particular, \( \|\mathcal{F}_i\|_{H^s(\Omega_0)} \) converges to zero.

On the other hand,

\[
\|\mathcal{F}_i\|_{H^s(\Omega_1)}^2 = \|\mathcal{F}_i\|_{H^s(\Omega_0)}^2 + \|\mathcal{F}_i\|_{H^s(\Omega_1 \setminus \overline{\Omega}_0)}^2 \leq \|u_i\|_{H^s(\Omega_1 \setminus \overline{\Omega}_0)}^2 + 2\|\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^-\|_{H^s(\Omega_1 \setminus \overline{\Omega}_0)}^2 + 2\|\mathcal{T}_{\partial(\Omega)}(\oplus_{j=0}^{2m-1} u_{j,i}))^+\|_{H^s(\Omega_0)}^2,
\]

and then, by (2.15), (2.16), (2.17) and the hypothesis of the corollary, the sequence \( \{\mathcal{F}_i\} \) belongs to the space \( H^2_0,\tilde{\gamma}(\Omega_1) \) with some \( \gamma \geq \tilde{\gamma} > 0 \).

At this point, the already proved part (1) of this corollary yields that the sequence \( \{\mathcal{F}_i\} \) converges to zero in the local space \([H^s_{\text{loc}}(\Omega_1)]^k\).

Hence, by formula (2.22) and (2.24), we conclude that the sequence \( \{u_i\} \) converges to zero in the space \([H^s_{\text{loc}}(\Omega_1 \setminus \Omega_0)]^k\).

Finally, for Problem 2.26 we argue as follows. Each element \( U_i \) is the unique solution to Problem 2.3 with the Dirichlet data \( \{\oplus_{j=0}^{2m-1} u_{j,i} = \oplus_{j=0}^{m-1} B_j U_i\} \). Moreover, by Corollary 1.9 we have \( U_i = \mathcal{P}_{\Omega_0}(\oplus_{j=0}^{m-1} u_{j,i}) \). As the Dirichlet problem 1.4 for the operator \( L \) is well-posed under the hypothesis of this corollary, we see that \( \mathcal{P}_{\Omega_0}(\oplus_{j=0}^{m-1} u_{j,i}) \) converges to zero in \([H^s(\Omega_0)]^k\) if the sequences \( \{B_j U_i\} \) converge to zero in \([H^{s-j-1/2}(\partial\Omega_0)]^k\) for each \( j, 0 \leq j \leq m - 1 \). Now, the already proved part (1) of this corollary yields that the sequence \( \{U_i\} \) converges to zero in the local space \([H^s_{\text{loc}}(\Omega_1)]^k\). \( \square \)

3. ON REPRESENTATION OF SOLUTIONS BY THE SINGLE LAYER POTENTIAL

Now we would like to discuss the possibility to representation of solutions of second order strongly elliptic systems by a single layer potential. For harmonic functions of different classes the problem is known since 23 (see also 75, 90 and elsewhere). The matter is closely related to the theory of multidimensional singular integral equation, see, for instance, 54, 81.
We restrict ourselves with the second order strongly elliptic operator with smooth coefficients. Thus, without loss of generality we may assume that the operator $L$ is written in the divergence form:

$$Lu = -\sum_{i,j=1}^{n} \partial_j (L_{i,j} \partial_i u) + \sum_{i=1}^{n} L_i \partial_i u + L_0 u$$

with $(k \times k)$-matrices of smooth functions $L_{i,j}(x)$, $L_i(x)$, $L_0(x)$ on $\overline{X} \subset \mathbb{R}^n$. Of course, any generalized Laplacian $A^*A$ is automatically of the divergence form.

Fix a relatively compact domain $\Omega$ in $X$ with the Lipschitz boundary. Following [58], we additionally assume that $L$ satisfy following Korn type inequality that is much stronger than Gårding inequality (1.2) in $\Omega$:

$$(3.1) \quad \|u\|_{[H^1(\Omega)]^k}^2 - c_2 \|u\|_{[L^2(\Omega)]^k}^2 \leq \sum_{i,j=1}^{n} (L_{i,j} \partial_i u, \partial_j u)_{[L^2(\Omega)]^k} + \sum_{i=1}^{n} (L_i \partial_i u + L_0 u, u)_{[L^2(\Omega)]^k}$$

for all $u \in [H^1(\Omega)]^k$ with some positive constants $c_1$, $c_2$ independent on $u$. It is always fulfilled for uniformly strongly elliptic scalar operators with real entries. For a system $L$ or even for a scalar linear operator with complex-valued coefficients (3.1) can be unaffordable. Indeed, for the generalized second order Laplacians $A^*A$ this is equivalent to the following:

$$(3.2) \quad \|u\|_{[H^1(\Omega)]^k}^2 \leq c_1 \|Au\|_{[L^2(\Omega)]^k}^2 + c_2 \|u\|_{[L^2(\Omega)]^k}^2 \quad \text{for all} \quad u \in [H^1(\Omega)]^k$$

with some positive constants $c_1$, $c_2$ independent on $u$. Then the presence or absence of (3.2) depends on the operator $A$ involved in the factorisation. For instance, if $\Delta$ is the usual Laplace operator in $\mathbb{R}^2$ then $(-\Delta) = \nabla^* \nabla$ and (3.2) holds true. On the other hand, we also have $(-\Delta) = 4\partial_1 \partial_1$ where $\partial = (1/2)(\partial_x + i\partial_y)$ is the Cauchy-Riemann operator. In this case, (3.2) fails on holomorphic functions on $\Omega$ from the Sobolev class $H^1(\Omega)$.

The Lamé system from the elasticity theory $L = -\mu \Delta - (\mu + \lambda) \nabla \text{div}$ (with positive constants $\mu$ and $\lambda$) is known to be strongly elliptic (and even a generalized Laplacian) and satisfy (3.1), see the original paper by Korn [37] or classical book [22] by G. Fichera or elsewhere. However, similarly to the Laplace operator on complex-valued functions, this system admit a factorization that does not fit for the Korn inequality, see [60].

Actually, inequality (3.1) provides that the following Neumann Problem has the Fredholm property on the scale of the Sobolev spaces for the domain $\Omega \Subset X$, see, for instance, [68].

**Problem 3.1.** Given $k$-vector function $u_1 \in [H^{s-3/2}(\partial\Omega)]^k$ find, if possible a $k$-vector function $u \in [H^s(\Omega)]^k$ such that

$$\begin{cases}
Lu = 0 & \text{in } \Omega, \\
\partial_{L,v}u = u_1 & \text{on } \partial\Omega,
\end{cases}$$

where $\partial_{L,v} = \sum_{i,j=1}^{n} \nu_j L_{i,j} \partial_i$ is the "co-normal derivative" with respect to $\partial\Omega$ related to $L$ with the exterior unit normal vector $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ to the surface $\partial\Omega$ at the point $x$. 

In the case of the usual Laplace operator in $\mathbb{R}^2$ and $A = \nabla$ Problem (3.1) is the classical Neumann problem where $B_1$ coincides with the normal derivative while the operator $A = 2\bar{\partial}$ corresponds to the $\bar{\partial}$-Neumann problem with $\sum_{i,j=1}^n \nu_i \nu_j \partial_i = (\nu_1 + i\nu_2)\bar{\partial}$. The $\bar{\partial}$-Neumann problem is not Fredholm on the scale of the Sobolev spaces because the space of solutions of its homogeneous version is precisely infinite-dimensional space of holomorphic functions of the corresponding Sobolev class over $D$. The normal solvability of such non-Fredholm problems can be established by the method of sub-elliptic estimates, see for instance, [36], [56], [66]; however it may affect the behaviour of boundary integrals in the Green formula (2.1).

In our particular case, for the Dirichlet pairs

\begin{equation}
\tag{3.3}
B = (B_0 = I_k, B_1 = \sum_{j=1}^n \nu_j L_j u - \partial_{L_j \nu} \partial), \quad \bar{B} = (\bar{B}_0 = I_k, \bar{B}_1 = \partial_{L \nu} \nu)
\end{equation}

the first Green formula (1.4) reads as follows:

\[
\int_{\partial \Omega} ((\bar{B}_1 v)^* u - v^* B_1 v) d\sigma = \int_{\Omega} (v^* Lu - (L^* v)^* u) dx,
\]

where

\[
L^* v = -\sum_{i,j=1}^n \partial_i (L_{i,j}^* \partial_j v) - \sum_{i=1}^n \partial_i (L_i^* v) + L_0^* v.
\]

For the generalized Laplacians $A^* A$ the boundary operators (3.3) are the following:

\begin{equation}
\tag{3.4}
B_0 = \bar{B}_0 = I_k, \quad B_1 = \bar{B}_1 = \sum_{j=1}^n A_j^* \nu_j A.
\end{equation}

As before, we assume that both the operators $L$ and $L^*$ possess the Unique continuation property (US) on $X$. Then the potential $\mathcal{T}_S(u_0, u_1)$ corresponding to $L$ and the Dirichlet pairs (3.3) can be present as the sum

\begin{equation}
\tag{3.5}
\mathcal{T}_S^{(B)}(u_0, u_1)(x) = \mathcal{V}_S^{(B)}(u_1) + \mathcal{W}_S^{(B)}(u_0)
\end{equation}

where

\begin{equation}
\tag{3.6}
\mathcal{V}_S^{(B)}(u_1) = \int_S \Phi(x, y) u_1(y) d\sigma(y)
\end{equation}

is the single layer potential and

\begin{equation}
\tag{3.7}
\mathcal{W}_S^{(B)}(u_0) = -\int_S (\partial_{L^* \nu} \Phi^*(x, y))^* u_0(y) d\sigma(y)
\end{equation}

is the double layer potential. Note that in some books the classical double layer potential related to the Laplace operator is defined without the sign “-”; this affects on the sign in formula (3.3), see, for instance, [11].

We will be concentrated on domains with Lipschitz boundaries as the most important case for applications.

Let $\Omega$ be a relatively compact domains in $X$ with Lipschitz boundary. According to [13] Theorem 1] the operators

\begin{equation}
\tag{3.8}
\mathcal{V}_{\partial \Omega}^{(B)} : [H^{s-3/2}(\partial \Omega)]^k \to [H^s(\Omega)]^k
\end{equation}

\begin{equation}
\tag{3.9}
\mathcal{W}_{\partial \Omega}^{(B)} : [H^{s-1/2}(\partial \Omega)]^k \to [H^s(\Omega)]^k
\end{equation}

\begin{equation}
\tag{3.10}
B_0 \mathcal{V}_{\partial \Omega}^{(B)} : [H^{s-3/2}(\partial \Omega)]^k \to [H^{s-1/2}(\partial \Omega)]^k
\end{equation}
Let Theorem 3.2.

Moreover, taking in account Remark 1.8 we conclude that for any $s \in (\frac{1}{3}, \frac{2}{3})$ and $u \in [H^s(\Omega)]^k \cap S_L(\Omega)$ the (representation) Green formula

$$\chi_{\Omega}u = W_{\partial \Omega}^{(B)}B_0u + V_{\partial \Omega}^{(B)}B_1u$$

is still valid for the Lipschitz domain $\Omega$ under the assumptions above because the following boundary operator can be treated as continuous

$$B_1 : [H^s(\Omega)]^k \cap S_L(\Omega) \to [H^{s-3/2}(\partial \Omega)]^k,$$

see [18, Lemma 3.7].

This allows to extend Theorem 2.2 to the case of Lipschitz boundaries for $m = 1$.

**Theorem 3.2.** Let $L$ be a second order strongly elliptic operator such that both $L$ and $L^*$ satisfy (US) on $X$ and satisfy (1.9), (3.1). Let also $s \in (\frac{1}{2}, \frac{2}{3})$ and let $\Omega_1 \setminus \Omega_0$ be a bounded Lipschitz domain in $X$. If $\Omega_1 \setminus \Omega_0$ has no compact components in $\Omega_1$ then Problem (2.7) is densely solvable and it has no more than one solution. It is solvable if and only if there is a function $u \in [H^s(\Omega)]^k \cap S_L(\Omega)$ and such that

$$F = (W_{\partial \Omega_0}^{(B)}(u_0))^+ + (V_{\partial \Omega_0}^{(B)}(u_1))^+ \text{ in } \Omega_0.$$

Besides, the solution $u$, if exists, is given by the following formula:

$$u = (W_{\partial \Omega_0}^{(B)}(u_0))^+ + (V_{\partial \Omega_0}^{(B)}(u_1))^+ \text{ in } \Omega_1 \setminus \Omega_0.$$

**Proof.** The proof actually follows the same way as the proofs of Theorem 2.2. If $u$ is a solution to Problem (2.7) and $u \in [H^s(\Omega)]^k \cap S_L(\Omega_1)$ then

$$F = (W_{\partial \Omega_0}^{(B)}(u_0))^+ + (V_{\partial \Omega_0}^{(B)}(u_1))^+ - \chi_{\Omega_1 \setminus \Omega_0}u = -(W_{\partial \Omega}^{(B)}(B_0u_0)) - (V_{\partial \Omega}^{(B)}(B_1u))$$

that is obviously belongs to $[H^s(\Omega)]^k \cap S_L(\Omega_1)$.

Back, if there is a function $F \in [H^s(\Omega)]^k \cap S_L(\Omega_1)$ we may invoke the jump formula (2.7) for potentials. In this particular case they are still valid for Lipschitz surfaces and they have the following form:

$$B_1(V_{\partial \Omega_0}^{(B)}(u_1))^+ - B_1(V_{\partial \Omega_0}^{(B)}(u_1))^+ = u_1 \text{ on } \partial \Omega_1,$$

$$B_0(V_{\partial \Omega_0}^{(B)}(u_1))^+ - B_0(V_{\partial \Omega_0}^{(B)}(u_1))^+ = 0 \text{ on } \partial \Omega_0,$$

$$B_0(W_{\partial \Omega_0}^{(B)}(u_0))^+ - B_0(W_{\partial \Omega_0}^{(B)}(u_0))^+ = u_0 \text{ on } \partial \Omega_0,$$

$$B_1(W_{\partial \Omega_0}^{(B)}(u_0))^+ - B_1(W_{\partial \Omega_0}^{(B)}(u_0))^+ = 0 \text{ on } \partial \Omega_0$$

see [18, Lemma 4.1]. According to them the function $u$ defined by (3.14) is the solution to Problem (2.7).}

Now we may extend Corollary 2.3 to the Lipschitz domains for $m = 1$.

**Corollary 3.3.** Let $L$ be a second order strongly elliptic operator such that both $L$ and $L^*$ satisfy (US) on $X$ and satisfy (1.9), (3.1). Let also $s \in (\frac{1}{2}, \frac{2}{3})$ and let $\Omega_1 \setminus \Omega_0$ be bounded Lipschitz domains in $X$. If $\Omega_1 \setminus \Omega_0$ has no compact components in $\Omega_1$ then Problem (2.7) is densely solvable and it has no more than one solution. It is solvable if and only if Problem (2.7) is solvable for the data $u_0 = B_0v$ and $u_1 = B_1v$ on $\partial \Omega_0$. \qed
Proof. Actually, all the arguments are the same as in the proof of Corollary 2.4 we need to correct then in the part related to the dense solvability. But according to [SI Exercise 1.4.10] the bounded Lipschitz domain Ω0 has the so-called strong cone property and then \([H^s(Ω_1)]^k ∩ S_L(Ω_1)\) is dense in \([H^s(Ω_0)]^k ∩ S_L(Ω_0)\) because of [SI Corollary 8.4.2].

Thus we arrive at the end of our discussion of Problems 2.1, 2.3 and 2.5 for \(m = 1\) and Lipschitz boundaries.

**Corollary 3.4.** Let \(L\) be a second order strongly elliptic operator such that both \(L\) and \(L^*\) satisfy (US) on \(X\) and satisfy (1.9), (3.1). Let also \(s ∈ \left(\frac{1}{2}, \frac{3}{2}\right)\) and let \(Ω_j\) be bounded Lipschitz domains in \(X\). If \(Ω_1 \setminus Ω_0\) has no compact components in \(Ω_1\) then Problem 2.5 is densely solvable and it has no more than one solution. Moreover, the following statements are equivalent:

- Problem 2.5 is solvable with the data \(v_0 \in [H^{s-1/2}(∂Ω_0)]^k\);
- Problem 2.5 is solvable with \(V = P_{Ω_0}(v_0)\);
- Problem 2.1 is solvable for the Cauchy data \(u_0, u_1\) where \(u_0 = v_0\) and \(u_1 = B_1P_{Ω_0}(v_0)\) on \(∂Ω_0\).

**Corollary 3.5.** Let \(L\) be a second order strongly elliptic operator such that both \(L\) and \(L^*\) satisfy (US) on \(X\) and satisfy (1.9), (3.1). Let also \(s ∈ \left(\frac{1}{2}, \frac{3}{2}\right)\) and let \(Ω_j\) be bounded Lipschitz domains in \(X\). If \(Ω_1 \setminus Ω_0\) has no compact components in \(Ω_1\) then Problems 2.1, 2.3 are conditionally well-posed in the following senses:

- (1) if a sequence \(\{U_i\} ⊂ [H^s_{L,γ}(Ω)]^k\) converges to zero in \([H^s(Ω)]^k\) then \(\{U_i\}\) converges to zero in the local space \([H^s_{loc}(Ω)]^k\) weakly converges to zero in \([H^s(Ω)]^k\);
- (2) if for a sequence \(\{u_i\} ⊂ [H^s_{L,γ}(Ω_1 \setminus Ω_0)]^k\) the sequences \(\{B_ju_i\}\) converge to zero in \([H^{s-j-1/2}(∂Ω_0)]^k\) for each \(j, 0 ≤ j ≤ 2m - 1\), then \(\{u_i\}\) converges to zero in the space \([H^s_{loc}(Ω_1 \setminus Ω_0)]^k\).

If, moreover, \(L\) satisfies (1.9) then Problem 2.3 is conditionally well-posed in the following sense:

- (3) if for a sequence \(\{U_i\} ⊂ [H^s_{L,γ}(Ω)]^k\) the sequences \(\{B_jU_i\}\) converge to zero in \([H^{s-j-1/2}(∂Ω_0)]^k\) for all \(j, 0 ≤ j ≤ m - 1\), then \(\{U_i\}\) converges to zero in the space \([H^s_{loc}(Ω)]^k\).

Let us formulate the main result of this section related to the problem of representation of solutions to \(L\) as a single layer potential. The following expectable and rather known statement follows almost immediately from [18 Theorem 2 and 3].

**Theorem 3.6.** Let \(Ω\) be a relatively compact domains in \(X\) with Lipschitz boundary and \(L\) be a second order strongly elliptic operator satisfying (1.9), (3.1) and

\[
[H^0_{L}(Ω)]^k ∩ S_L(Ω) = [H^0_{L}(Ω)]^k ∩ S_L(Ω) = 0.
\]

If operator \((3.10)\) is injective then for any \(u ∈ [H^s(Ω)]^k ∩ S_L(Ω), s ∈ [1, \frac{3}{2})\), there is a unique function \(u_1 ∈ [H^{s-3/2}(∂Ω)]^k\) such that

\[
u = \frac{∂}{∂Ω}(u_1) \text{ in } Ω
\]

related to \(L\) and Dirichlet pairs \((3.3)\). Moreover, if \(∂Ω\) is a surface of class \(C^∞\), statements is still true for all \(s ∈ N\).
Proof: We begin the discussion for domains Lipschitz boundaries. First, we note that the operator \( (3.10) \) is strongly elliptic in the following sense (18 Theorem 2): there is a compact operator \( K : [H^{-1/2}(\partial\Omega)]^k \to [H^{1/2}(\partial\Omega)]^k \) and a positive constant \( C \) such that
\[
(3.18) \quad \Re \langle (B_0 V_{\partial\Omega}^{(B)} + K) w, w \rangle \geq C \| w \|^2_{[H^{-1/2}(\partial\Omega)]^k} \text{ for all } w \in [H^{-1/2}(\partial\Omega)]^k
\]
where the brackets \( \langle \cdot, \cdot \rangle \) denote the natural duality pairing between the Sobolev space \([H^s(\partial\Omega)]^k\) and its dual.

As it is well-known, estimate (3.18) is equivalent to the fact that the operator \( B_0 V_{\partial\Omega}^{(B)} \) admits a parametrix on the space \([H^{-1/2}(\partial\Omega)]^k\), i.e. there are compact operators \( K^{(i)} : [H^{-1/2}(\partial\Omega)]^k \rightarrow [H^{-1/2}(\partial\Omega)]^k\), \( K^{(r)} : [H^{1/2}(\partial\Omega)]^k \rightarrow [H^{1/2}(\partial\Omega)]^k\) and a bounded operator \( Q : [H^{1/2}(\partial\Omega)]^k \rightarrow [H^{-1/2}(\partial\Omega)]^k\) such that
\[
Q(B_0 V_{\partial\Omega}^{(B)}) + K^{(i)} = I \text{ on } [H^{-1/2}(\partial\Omega)]^k,
\]
\[
(B_0 V_{\partial\Omega}^{(B)})Q + K^{(r)} = I \text{ on } [H^{1/2}(\partial\Omega)]^k.
\]

Second, the regularity results (18 Theorem 3) allow us to define a parametrix for the operator \( B_0 V_{\partial\Omega}^{(B)} \) on the scale \([H^{s-3/2}(\partial\Omega)]^k \rightarrow [H^s(\Omega)]^k, s \in (\frac{1}{2}, \frac{3}{2})\), i.e. there are compact operators \( K_s^{(i)} : [H^{s-3/2}(\partial\Omega)]^k \rightarrow [H^{s-3/2}(\partial\Omega)]^k\), \( K_s^{(r)} : [H^{s-1/2}(\partial\Omega)]^k \rightarrow [H^{s-1/2}(\partial\Omega)]^k\) and a bounded operator \( Q : [H^{s-1/2}(\partial\Omega)]^k \rightarrow [H^{s-3/2}(\partial\Omega)]^k\) such that
\[
Q_s(B_0 V_{\partial\Omega}^{(B)}) + K_s^{(i)} = I \text{ on } [H^{s-3/2}(\partial\Omega)]^k,
\]
\[
(B_0 V_{\partial\Omega}^{(B)})Q_s + K_s^{(r)} = I \text{ on } [H^{s-1/2}(\partial\Omega)]^k.
\]

Next, formula (3.16) and Corollary 1.6 imply that a function \( u \in S_L(\Omega) \cap [H^s(\Omega)]^k \) satisfies (3.17) with a function \( u_1 \in [H^{s-3/2}(\partial\Omega)]^k \) if and only if
\[
(3.21) \quad B_0 u = B_0 V_{\partial\Omega}^{(B)}(u_1) \text{ on } \partial\Omega.
\]

According to (3.19), (3.20), on the space \([H^{s-3/2}(\partial\Omega)]^k\) we have
\[
B_0 V_{\partial\Omega}^{(B)} = B_0 V_{\partial\Omega}^{(B)} \left( Q_s(B_0 V_{\partial\Omega}^{(B)}) + K_s^{(i)} \right) = I + B_0 V_{\partial\Omega}^{(B)} K_s^{(i)} + K_s^{(r)} B_0 V_{\partial\Omega}^{(B)}.
\]

Thus, since compositions of compact and bounded operators are compact, we see that equation (3.21) reduces to a Fredholm operator equation.

Hence, it follows from Fredholm alternative that the operator \( (3.10) \) is continuously invertible if and only if it is injective for \( s \in (1, \frac{3}{2}) \). Thus, as \( (3.10) \) is injective, we conclude that equation (3.21) is always uniquely solvable in \([H^{s-3/2}(\partial\Omega)]^k\) with \( s \in [1, \frac{3}{2}) \).

Finally, if \( \partial\Omega \) is a surface of class \( C^\infty \) then the trace operators \( (1.4) \) are continuous for all \( s > 1/2 \). The potential operator \( (3.5) \) is bounded this \( s > 1/2 \) too, because of theorems related to pseudo differential operators satisfying the so-called transmission property, see, for instance, [58 §2.3.2.5] or [81 §2.4].

Hence operator \( (3.10) \) is continuous and the statement follows from the standard scheme of improvement of the regularity for solutions to elliptic pseudo-differential equations, that was to be proved. Actually, the statement of the theorem follows from more general results on strongly elliptic boundary operators by M. Costabel and W. L. Wendland [19 Theorem 3.7].
Let us indicate important cases where the operator \(3.10\) is injective.

**Theorem 3.7.** Let \(\Omega \Subset D\) be a relatively compact domains in \(X\) with Lipschitz boundary and \(L\) be a second order strongly elliptic operator satisfying \(1.9\), \(3.1\), \(3.4\) and
\[
(3.22) \quad [H^1_0(D \setminus \overline{\Omega})]^k \cap S_L(D \setminus \overline{\Omega}) = [H^1_0(D \setminus \overline{\Omega})]^k \cap S_{L^*}(D \setminus \overline{\Omega}) = 0.
\]
If \(\Phi = G_D\) is the Green function of the Dirichlet Problem \(1.4\) for \(L\) in \(D\) then for any \(u \in [H^s(\Omega)]^k \cap S_L(\Omega), \ s \in [1, \frac{3}{2})\), there is a unique function \(u_1 \in [H^{s-3/2}(\partial \Omega)]^k\) such that \(3.17\) is fulfilled. Moreover, if \(\partial \Omega\) is a surface of class \(C^\infty\), then the statements is still true for this \(s \in \mathbb{N}\).

**Proof.** Let us establish that operator \(3.10\) is injective under the hypothesis of this corollary.

Let a function \(v \in [H^{s-3/2}(\partial \Omega)]^k\) belongs to the kernel of the operator \(3.10\).

In order to prove that \(v\) equals to zero we invoke jump formulas \(3.15\).

As \(L \nu_{\partial \Omega}^A(v) = 0\) in \(D \setminus \partial \Omega\) and \(B^0 \nu_{\partial \Omega}^A(v) \equiv 0\) on \(\partial \Omega\), we see that \((\nu_{\partial \Omega}^A(v))|_{\Omega} \equiv 0\) in \(\Omega\) because \(3.10\) and Corollary \(1.6\). Similarly, since \(\Phi\) is the Green function of the Dirichlet Problem \(1.4\) for \(L\) in \(D\) we see that \((\nu_{\partial \Omega}^A(v))|_{\partial \Omega}^D\) is the solution to the homogeneous Dirichlet problem for \(L\) in \(D \setminus \overline{\Omega}\). Then \(3.22\) and Corollary \(1.6\) yield \((\nu_{\partial \Omega}^A(v))|_{\partial \Omega}^D \equiv 0\) in \(D \setminus \overline{\Omega}\). Now jump formula \(3.16\) (applied to \(\partial \Omega\) instead of \(\partial \Omega_0\)) implies that \(v = 0\) on \(\partial \Omega\), i.e. the operator \(3.10\) is injective. Finally, the Fredholm alternative mean that equation \(3.21\) is always uniquely solvable in \([H^{s-3/2}(\partial \Omega)]^k\) with \(s \in [1, \frac{3}{2})\), that was to be proved.

For the generalized Laplacians the statement of Theorem \(3.7\) can be formulated much shorter.

**Corollary 3.8.** Let \(\Omega \Subset D\) be a relatively compact domain in \(X\) with Lipschitz boundary and \(A\) be an operator with injective principal symbol satisfying \((US)\) on \(X\). If \(L = A^*A\) satisfies \(3.2\) and \(\Phi = G_D\) is the Green function of the Dirichlet Problem \(1.4\) for \(A^*A\) in \(D\) then for any \(u \in [H^s(\Omega)]^k \cap S_{A^*A}(\Omega), \ s \in [1, \frac{3}{2})\), there is a function \(u_1 \in [H^{s-3/2}(\partial \Omega)]^k\) such that \(3.17\) holds true. Moreover, if \(\partial \Omega\) is a surface of class \(C^\infty\), then the statements is still true for this \(s \in \mathbb{N}\).

**Proof.** If \(L = A^*A\) with a first order operator \(A\) having injective principal symbol and satisfying \((US)\) on \(X\) then \(1.9\), \(3.10\) and \(3.16\) hold true, see Corollary \(1.7\). As \(3.22\) is equivalent to \(3.1\) for \(L = A^*A\), we conclude that the statement of the corollary follows from Theorem \(3.7\). \(\square\)

As it is well known, in general, the solvability of the corresponding boundary singular integral equations on \(\partial \Omega\) is closely related to solvability of some ”interior” and ”exterior” boundary problems for the elliptic operator, see, for instance, \(8.2\) for classical solutions or \(20\), \(19\) for the Sobolev type spaces. In our particular case it is an ”exterior” Dirichlet problem, as Theorem \(3.7\) and Corollary \(3.8\) confirm. However an ”exterior” Dirichlet problem for general strongly elliptic operator \(L\) on \(\mathbb{R}^n \setminus \Omega\) can be rather complicated even in the case of scalar operators, see, for instance, \(71\).

At the end of this section, let us consider the classical case of second order elliptic operators with constant coefficients in \(\mathbb{R}^n\). In this case \(L\) admits a bilateral fundamental solution of the convolution type, say \(\Phi(x - y), \ x, y \in \mathbb{R}^n, \ x \neq y\, \text{and} \)
hence it is natural to consider potentials constructed with the use of this kernel. We will consider a rather particular situation. where \( n \geq 2 \) and \( A = \sum_{j=1}^{n} A_j \partial_j \) is a homogeneous first order operator with constant coefficients in \( \mathbb{R}^n \) having injective principal symbol. Then its Laplacian \( A^* A \) admits the fundamental solution of the convolution type in the form

\[
\Phi(x) = a \left( \frac{x}{|x|} \right) |x|^{2-n} + b(x) \ln |x|
\]

where \( a(\zeta) \) is a \((k \times k)\)-matrix of real analytic functions in a neighbourhood of the sphere \( \{|\zeta| = 1\} \) and \( b(\zeta) \) is a \((k \times k)\)-matrix of polynomials \( a_{p,q}(\zeta) \) of order \((2-n)\), see, for instance, [80, §Ra]. This means that \( b \equiv 0 \) and

\[
\Phi(x) = a \left( \frac{x}{|x|} \right) |x|^{2-n} \quad \text{for} \quad n \geq 3,
\]

and \( b \) is a matrix with constant entries for \( n = 2 \).

**Corollary 3.9.** Let \( \Omega \) be a relatively compact domain in \( X \) with Lipschitz boundary in \( \mathbb{R}^n \), \( n \geq 3 \) and \( A \) be a first order homogeneous operator with constant coefficients having injective principal symbol and satisfying (3.1). If \( \Phi \) is given by (3.24) then for any \( u \in [H^s(\Omega)]^k \cap S_L(\Omega), \quad s \in [1, \frac{3}{2}] \), there is a unique function \( u_1 \in [H^{s-\frac{3}{2}}(\partial\Omega)]^k \) such that (3.17) holds true. Moreover, if \( \partial\Omega \) is a surface of class \( C^\infty \), then the statements is still true for all \( s \in \mathbb{N} \).

**Proof.** Let \( A \) is a first order homogeneous operator with constant coefficients having injective principal symbol satisfying (3.1). Then \( A \) satisfies (US) condition in \( \mathbb{R}^n \).

Thus, according to Theorem 3.6, the remaining fact to establish is the injectivity of the corresponding operator (3.10).

Let a function \( v \in [H^{s-\frac{3}{2}}(\partial\Omega)]^k \) belongs to the kernel of the operator (3.10). Again, the Uniqueness of the Dirichlet problem for \( A^* A \) in \( \Omega \) means that \( \mathcal{V}_{\partial\Omega}^{(B)}(v) \) equals identically to zero in \( \Omega \). However, for \( s \geq 1 \), using integration by parts, we easily obtain

\[
0 = \langle A^* A(\mathcal{V}_{\partial\Omega}^{(B)}(v)), \mathcal{V}_{\partial\Omega}^{(B)}(v) \rangle_{\mathbb{R}^n \setminus \overline{\Omega}} = \lim_{R \to +\infty} \left( \| \mathcal{V}_{\partial\Omega}^{(B)}(v) \|_{L^2(B(0,R) \setminus \overline{\Omega})}^2 \right) - \int_{|x| = R} \sum_{j=1}^{n} \left( \mathcal{V}_{\partial\Omega}^{(B)}(v)(x) \right)^* \frac{x_j}{|x|} A_j^* A \mathcal{V}_{\partial\Omega}^{(B)}(v)(x) d\sigma(x).
\]

Clearly, as \( \Omega \) is bounded, then using the particular type (3.24) of the kernel \( \Phi(x-y) \), we see that there is a number \( R_1 > 0 \) such that

\[
|\mathcal{V}_{\partial\Omega}^{(B)}(v)(x)| \leq \| v \|_{[H^{s-\frac{3}{2}}(\partial\Omega)]^k} |x|^{2-n} \max_{p,q} \| a_{p,q} \left( \frac{x-y}{|x-y|} \right) |x-y|^{2-n} \|_{[H^2_{\partial\Omega^2}(\partial\Omega)]^k} \leq \frac{1}{C_{\Omega}^{(1)}} \| v \|_{[H^{s-\frac{3}{2}}(\partial\Omega)]^k} |x|^{2-n}
\]

with a positive constant \( C_{\Omega}^{(1)} \) independent on \( x \) with \( |x| \geq R_1 \) and \( v \).

On the other hand,

\[
\partial_j \Phi(x-y) = (\partial_j a) \left( \frac{x-y}{|x-y|} \right) \frac{\partial}{\partial y_j} \left( \frac{x-y}{|x-y|} \right) |x-y|^{2-n} + a \left( \frac{x-y}{|x-y|} \right) \partial_j |x-y|^{2-n},
\]

\[
|\partial_j \Phi(x-y)| \leq C |x|^{1-n} \left| \frac{x-y}{|x|} \right|^{1-n}
\]
with a constant $C$ independent on $x$ and $y$. Then, using the homogeneity of the operator $A$, we see that there is a number $R_2 > 0$ such that

$$|AV_{\partial\Omega}^{(B)}(v)(x)| \leq C^{(2)}_{\Omega} |x|^{1-n}$$

with a positive constant $C^{(2)}_{\Omega}$ independent on $x$ with $|x| \geq R_2$ and $v$.

Now estimates (3.25), (3.26) imply for

$$\left| \int_{|x|=R} \sum_{j=1}^{n} (V_{\partial\Omega}^{(B)}(v)(x))^* \frac{x_j}{|x|} A_j^* A_{\partial\Omega}^{(B)}(v)(x) d\sigma(x) \right| \leq R^{2-n}$$

for $R \geq \text{max}(R_1, R_2)$ and then

$$0 = \lim_{R \to +\infty} \|A_{\partial\Omega}^{(B)}(v)\|_{L^2(B(0,R) \setminus \Omega)}^2 = \|A_{\partial\Omega}^{(B)}(v)\|_{L^2(\mathbb{R}^n \setminus \Omega)}^2,$$

i.e. $A_{\partial\Omega}^{(B)}(v) = 0$ in $\mathbb{R}^n \setminus \Omega$.

Then jump formula (3.15) with the boundary operators (3.4) yields

$$v = B_1 (V_{\partial\Omega}^{(B)}(v)^- - B_1 (V_{\partial\Omega}^{(B)}(v)^+) = 0 \text{ on } \partial\Omega,$$

i.e. operator (3.10) is injective, that was to be proved.

\[\square\]

**Example 3.10.** Let $n \geq 2$. The typical example of a homogeneous first order operator $A$ with constant coefficients in $\mathbb{R}^n$ having injective principal symbol is the gradient operator $A = \nabla$. Of course, its Laplacian $A^* A = -\Delta$ satisfies (3.2). Its fundamental solution of type (3.23) is given by

$$\Phi(x) = \varphi_n(x) = \begin{cases} \frac{|x|^{2-n}}{(n-2)\sigma_n}, & n \geq 3, \\
\frac{1}{2\pi} \ln|x|, & n = 2, \end{cases}$$

where $\sigma_n$ is the square of the unit sphere in $\mathbb{R}^n$. Thus, we see that $\varphi_2$ does not have a sufficient decay at the infinity in order to prove Corollary 3.9 for $n = 2$ by the present arguments (estimate (3.25) is not fulfilled!). However, for $n = 2$ injectivity of the corresponding potential with logarithmic kernel may be established by more sophisticated methods under specific assumptions on the geometry of the curve $\partial\Omega$, see, for instance, [32], [39].

**Example 3.11.** Let $n \geq 2$. The typical example of a homogeneous second order strongly elliptic matrix operator with constant coefficients in $\mathbb{R}^n$ is the Lamé $(n \times n)$-system from the elasticity theory $\mathcal{L} = -\mu \Delta - (\mu + \lambda) \nabla \text{div}$ with the Lamé constants $\mu$ and $\lambda$ such that $\mu > 0$, $2\mu + \lambda > 0$. It is known to satisfy (3.1), see the original paper by Korn [31] or classical book [22]. Actually, it can be factorized as $\mathcal{L} = A^* A$ a first order homogeneous $(k \times n)$-matrix operator $A = \sum_{j=1}^{m} A_j \partial_j$. Of course, there are many such operators $A$. To introduce three of them we denote by $M_1 \otimes M_2$ the Kronecker product of matrices $M_1$ and $M_2$, by $\text{rot}_m$ we denote $\left(\frac{m^2-m}{2} \times m\right)$-matrix operator with the lines $\hat{e}_i \frac{\partial}{\partial x_j} - \hat{e}_j \frac{\partial}{\partial x_i}$, $1 \leq i < j \leq m$, representing the vorticity (or the standard rotation operator for $m = 2$, $m = 3$), and by $\mathbb{D}_m$ we denote $\left(\frac{(m^2+m)}{2} \times m\right)$-matrix operator with the lines $\sqrt{2} \hat{e}_i \frac{\partial}{\partial x_j} + \hat{e}_j \frac{\partial}{\partial x_i}$, $1 \leq i < j \leq m$, representing the deformation (the strain). The
we set:

\[ A^{(1)} = \left( \sqrt{\mu} D_m \right), A^{(2)} = \left( \sqrt{\mu} \nabla_m \otimes I_m - \frac{\lambda}{\sqrt{\mu + \lambda}} \text{div}_m \right), A^{(3)} = \left( \sqrt{\mu} \text{rot}_m \right), \]

here \( \lambda \geq 0 \), \( k_1 = (m^2 + m)/2 + 1 \) for the first operator, \( (\mu + \lambda) \geq 0 \), \( k_2 = m^2 + 1 \) for the second operator, and \( 2\mu + \lambda > 0 \), \( k_3 = (m^2 - m)/2 + 1 \) for the third operator.

Factorization as \( (A^{(1)})^* A^{(1)} \) or \( (A^{(2)})^* A^{(2)} \), it satisfies (3.2); see, for instance, [50, Examples 3 and 5] or [22]. Factorization \( (A^{(3)})^* A^{(3)} \) does not admit (3.2) because the Neumann problem [51] is not coercive in this case, [50, Example 4].

Its fundamental solution of type (3.23) is given by \( \Phi_n(x) = \left( \Phi_{ij}^{(n)}(x) \right)_{i,j=1,2, \ldots, n} \) with components

\[
\Phi_{ij}^{(n)}(x) = \frac{1}{2\mu(\lambda + 2\mu)} \left( \delta_{ij} (\lambda + 3\mu)g(x) - (\lambda + \mu) x \frac{\partial}{\partial x_i} \varphi_n(x) \right) \quad (i, j = 1, 2, \ldots, n),
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \varphi_n(x) \) is the standard fundamental solution to the Laplace operator in \( \mathbb{R}^n \). Thus, we see that \( \Phi_2 \) does not have a sufficient decay at the infinity in order to prove Corollary 3.9 for \( n = 2 \) by the present arguments (estimate (3.25) is not fulfilled!)

**Example 3.12.** Let \( a \) be a complex non-zero number and

\[ A = \left( \begin{array}{c} \nabla \\ a \end{array} \right); \]

this operator is not homogeneous. Then \( A^* A = |a|^2 - \Delta \) is the Helmholtz operator in \( \mathbb{R}^3 \) admitting the fundamental solutions

\[ \Phi_-(x) = \frac{e^{-|a||x|}}{4\pi|x|}, \quad \Phi_+(x) = \frac{e^{+|a||x|}}{4\pi|x|} \]

and (3.2) is obviously fulfilled. Clearly, we may repeat the arguments from the proof of Corollary 3.9 and prove that the representation via single layer potential is true for solutions to the Helmholtz equation if we choose \( \Phi_-(x) \) for the potential. However estimates (3.25), (3.26) are not true for \( \Phi_+(x) \) and the arguments fail for the corresponding potential.

Note that the Helmholtz operator in \( \mathbb{R}^3 \) may also have the form \((-|a|^2 - \Delta)\). However, in this case its standard fundamental solutions are

\[ \Phi_-(x) = \frac{e^{-|a||x|}}{4\pi|x|}, \quad \Phi_+(x) = \frac{e^{+|a||x|}}{4\pi|x|}; \]

and again estimates (3.25), (3.26) are not true for the potentials corresponding to both \( \Phi_-(x) \) and \( \Phi_+(x) \). Hence the arguments in the proof of Corollary 3.9 fail.

Finally, let us show how the theorems on the representation by the single layer potential and approximation theorems for solutions to elliptic systems help to clarify the question on the so-called *discrete approximation*.

**Corollary 3.13.** Let \( s, s' \in \mathbb{N} \) and let \( \Omega \) be a relatively compact domain in \( X \) with \( C^\infty \)-smooth boundary. Let also \( L \) be a second order strongly elliptic operator satisfying (1.9), (3.1), (3.10), and the injectivity condition for operator (3.11). If \( \Omega' \) is a relatively compact domain in \( \Omega \) with Lipschitz boundary such that set \( \Omega \setminus \Omega' \) has no compact components then for any the set of all single layer potentials of
type (3.17) with densities from the space $[H^{s-3/2}(\partial\Omega)]^k$ are dense in the space $[H^s(\Omega')]^k \cap S_L(\Omega')$.

Proof. Approximation Theorems for solutions to elliptic systems, see, for instance, [51 Theorems 5.1.11, 5.1.13, 8.2.2] imply that the space $[H^s(\Omega')]^k \cap S_L(\Omega')$ is dense in $[H^{s'}(\Omega')]^k \cap S_L(\Omega')$. Hence the statement follows from Theorem 3.6. □

The following corollary is just a specification of similar statements for various solutions to elliptic systems, see the pioneer result [60] by C. Runge for holomorphic operators admitting left fundamental solutions.

Corollary 3.14. Let $s' \in \mathbb{N}$ and let $\Omega$ be a relatively compact domain in $X$ with $C^\infty$-smooth boundary. Let also $L$ be a second order strongly elliptic operator satisfying (1.9), (3.1), (3.16) and the injectivity condition for operator (3.10). If $\{z_j\}_{j \in \mathbb{N}}$ is an everywhere dense set on $\partial\Omega$ and $\Omega'$ is a relatively compact domain in $\Omega$ with Lipschitz boundary such that set $\Omega \setminus \Omega'$ has no compact components then for any $u \in [H^{s'}(\Omega')]^k \cap S_L(\Omega')$ and any $\varepsilon > 0$ there are numbers $M(u, \varepsilon) \in \mathbb{N}$ and $k$-vectors $\{c_j(u, \varepsilon)\}_{j=1}^M$ such that

$$\left\|u(x) - \sum_{j=1}^M \Phi(x, z_j) c_j(u, \varepsilon)\right\|_{[H^{s'}(\Omega')]^k} < \varepsilon.$$ 

Proof. Fix a positive number $\varepsilon$. First, take domain $\Omega''$ such that $\Omega'' \Subset \Omega' \Subset \Omega$. Then, by the a priori estimates for elliptic systems, see, for instance, [24], we know that there is a number $c(s', \partial\Omega', \partial\Omega'') > 0$ dependent on the distance between $\partial\Omega'$ and $\partial\Omega''$ such that

$$\|u\|_{H^{s'}(\Omega')} \leq c(s', \partial\Omega', \partial\Omega'') \|u\|_{C((\Omega'))^k}.$$  

for all $u \in S_L(\Omega'') \cap [C(\Omega'')]^k$.

Second, using Corollary 3.13 for $s > \frac{n+2}{2}$, we pick a density a density $v_\varepsilon \in [H^{s-3/2}(\partial\Omega)]^k$ such that

$$\left\|u - \nabla_{\partial\Omega}^B(v_\varepsilon)\right\|_{[H^{s'}(\Omega')]^k} < \varepsilon / 2.$$  

By the Sobolev embedding theorems, see, for instance, [11 Ch. 4, Theorem 4.12]), the density $v_\varepsilon$ is actually continuous on $\partial\Omega$. In particular, the potential $\nabla_{\partial\Omega}^B(v_\varepsilon)$ is a Riemann integral depending on the parameter $x \in \Omega''$ as $\Omega'' \Subset \Omega$.

Note that the set $K = \overline{\Omega''} \times \partial\Omega$ is a compact in $\mathbb{R}^n \times \mathbb{R}^n$. By Cantor’s theorem, any continuous function on $K$ is uniformly continuous and hence for any $w \in C(K)$ there is a number $\delta_\varepsilon$ such that

$$|w(x', y') - w(x'', y'')| < \frac{\varepsilon}{2c(s', \partial\Omega', \partial\Omega'') \sigma(\partial\Omega)}$$  

if $x', x'' \in \overline{\Omega''}$, $y', y'' \in \partial\Omega$, and $|x' - x''|^2 + |y' - y''|^2$, where $\sigma(\partial\Omega)$ is the square $((n - 1)$-Jordan measure) of the hypersurface $\partial\Omega$.

As the function $\Phi^*(x, y)v_\varepsilon(y)$ is continuous on $K$, it is Riemann integrable over $\partial\Omega$. Then for any partition $P = \{P_i\}_{i=1}^N$ of $\partial\Omega$ by measurable sets $P_i \subset \partial\Omega$ we
have for all \( x \in \Omega' \):

\[
\mathcal{V}_{\partial \Omega}^{(B)}(v_\epsilon)(x) = \sum_{i=1}^{N} \mathcal{V}_{P_i}^{(B)}(v_\epsilon)(x)
\]

Then, by the integral mean value theorem, there are points \( y_{i,x} \in P_i \) such that

\[
(3.31) \quad \mathcal{V}_{\partial \Omega}^{(B)}(v_\epsilon)(x) = \sum_{i=1}^{N} \Phi(x, y_{i,x}) v_\epsilon(y_{i,x})
\]

Next, we choose \( N = N(\epsilon) \) and \( \{ P_i = P_i(\epsilon) \}_{i=1}^{N} \) to be relatively open sets on \( \partial \Omega \) with piece-wise smooth boundaries and such that the diameter of each set is less than the number \( \delta_\epsilon \) related to \( (3.30) \). Since the set of points \( \{ z_j \} \) is every where dense in \( \partial \Omega \) we conclude that each \( P_i \) contains at least one point \( z_{i}(\epsilon) \in \{ z_j \} \).

Hence \( (3.30) \) yields for each \( x \in \Omega' \):

\[
(3.32) \quad \left| \sum_{j=1}^{N(\epsilon)} \left( \Phi(x, y_{i,x}) v_\epsilon(y_{i,x}) - \Phi(x, z_{i}(\epsilon)) v_\epsilon(z_{i}(\epsilon)) \right) \sigma(P_i(\epsilon)) \right| \leq \varepsilon \sum_{j=1}^{N(\epsilon)} \varepsilon \sigma(P_i(\epsilon)) = \frac{\varepsilon}{2(\varepsilon', \partial \Omega', \partial \Omega'')} \varepsilon.
\]

Finally, combining \( (3.28) \), \( (3.29) \), \( (3.31) \), \( (3.32) \) we conclude that the statement of the corollary holds true with \( M(\epsilon) = \max_{1 \leq i \leq N(\epsilon)} I(\epsilon) \) and the vectors

\[
e_j(u, \epsilon) = \begin{cases} \sigma(P_i(\epsilon)) v_\epsilon(z_{i}(\epsilon)) & \text{if } j = i(\epsilon), \\ 0 & \text{if } j \neq i(\epsilon). \end{cases}
\]

This finishes the proof. \( \square \)

Note that the described scheme allows us to use the set of points \( \{ z_j \}_{j=1}^{M(\epsilon)} \) after refining the partition \( \{ P_j \}_{j=1}^{N(\epsilon)} \) for a new positive number \( \epsilon' < \epsilon \), adding new points from the set \( \{ z_j \}_{j=1}^{\infty} \) related to the new elements of the refined partition that do not contain the elements of the set \( \{ z_j \}_{j=1}^{M(\epsilon)} \).

4. Some Methods for Solving the Exterior Extension Problems for Strongly Elliptic Operators of the Second Order

In this section we consider some methods for constructing solutions to Problems 0.1, 0.2, 0.3 (or, more precisely, Problems 2.1, 2.3, 2.5) respectively which are based on the use of fundamental solutions of the corresponding elliptic equations. More specifically, we will focus on the so-called indirect method of boundary integral equations in terms of the single layer and on the method of fundamental solutions. We also considered the “extension approach” for approximation of the solution to the Dirichlet Problem 0.4 (more precisely, Problem 1.4 for \( m = 1 \)).

In this section we assume that \( m = 1 \), \( L \) is a second order strongly elliptic operator with smooth coefficients such that \( L \) and \( L^* \) satisfy (US) property in \( X \) and requirements \( (3.1) \), \( (3.10) \) and the injectivity condition for the single layer operator \( (3.10) \) in the respective domains. We assume also that the hypotheses of Corollary 1.6 and Corollary 2.7 or Corollary 3.5 are fulfilled.

As in the previous sections, we assume that \( \Omega_0 \) is a relatively compact domain in \( X \) and \( \Omega_1 \) is a bounded domain in \( X \) such that \( \Omega_0 \subset \Omega_1 \) and \( \Omega_1 \setminus \Omega_0 \).
has no compact components; solutions to the boundary value problems belong to $[H^s(\Omega_1)]^k \cap S_L(\Omega_1)$. In this section we will use the following requirements to the domain boundaries:

1) the boundaries $\partial \Omega_0$ and $\partial \Omega_1$ belongs to $C^s$-class of smoothness if $s \in \mathbb{N}$, $s \geq 2$

2) both boundaries $\partial \Omega_0$ and $\partial \Omega_1$ are Lipschitz ones if $s \in [1, 3/2)$.

We denote the restrictions of $B_0u$ and $B_1u$ onto $\partial \Omega_i$ as $B_{0,\partial \Omega_i}$ and $B_{1,\partial \Omega_i}$ respectively. We also denote $B_{0,\partial \Omega_0}u$ as $u_0$ and $B_{1,\partial \Omega_1}u$ as $u_1$.

We begin our consideration with the direct boundary integral equations method

The indirect boundary integral equation method in terms of the single layer consists in computing the solution to Problem 2.5 as a potential of the single layer given on $\partial \Omega_1$:

$$u(x) = V_{\partial \Omega_1} v(z), x \in \Omega_1, z \in \partial \Omega_1,$$

where $v$ is a density of the single layer. The unknown single layer density $v$ can be found as a solution to the operator equation:

$$V_{\partial \Omega_1} v(z) = f(y), z \in \partial \Omega_1, y \in \partial \Omega_0,$$

where $f = B_0u$ on $\partial \Omega_0$ is the boundary datum of Problem 2.5.

**Corollary 4.1.** If a solution $u \in [H^s(\Omega_1)]^k \cap S_L(\Omega_1)$ to Problem 2.5 exists, then the solution $v \in [H^{s-3/2}(\partial \Omega_1)]^k$ to operator equation (4.1) exists and it is unique. Equation (4.1) is densely solvable. Any solution $v \in [H^s(\partial \Omega_1)]^k \cap S_L(\Omega_1)$ to Problem 2.5 can be presented in form (4.1) where $v \in [H^{s-3/2}(\partial \Omega_1)]^k$ is the solution to equation (4.1).

**Proof.** The statements follow immediately from Corollary 3.5 and Theorem 3.6. $\square$

**Corollary 4.2.** Let $\{v^{(i)}\} \subset [H^{s-\frac{3}{2}}(\partial \Omega_1)]^k$ be a bounded sequence.

If the sequence $f^{(i)}$ of the boundary datum of Problem 2.5 strongly convergence to zero in $[H^{s-1/2}(\partial \Omega_0)]^k$ then:

- the sequence $\{v^{(i)}\}$ given by formula (4.1) with $v = v^{(i)}$ strongly converges to zero in $[H^s(\Omega_1)]^k \cap S_L(\Omega_0)$, weakly converges to zero in $[H^s(\Omega_1)]^k \cap S_L(\Omega_1)$ and it converges to zero in the local space $[H_{loc}^s(\Omega_1)]^k$;

- the sequence $B_{0,\partial \Omega_0} v$ weakly converges to zero in $[H^{s-1/2}(\partial \Omega_1)]^k$ and the sequence $\{u^{(i)}\}$ is weakly converges to zero in $[H^{s-3/2}(\partial \Omega_1)]^k$.

**Proof.** First, note that $B_0 V_{\partial \Omega_1} v^{(i)} = B_0 u^{(i)} = f^{(i)}$ and according to Theorem 3.6 each $u^{(i)}$ can be presented by formula (4.1). As the sequence $\{v^{(i)}\} \subset [H^{s-\frac{3}{2}}(\partial \Omega_1)]^k$ is bounded, the sequence $\{u^{(i)}\}$ is bounded in the space $[H^s(\Omega_1)]^k$ because of the boundedness of the single layer potential $V_{\partial \Omega_1}^{(B)} : [H^{s-3/2}(\partial \Omega_1)]^k \to [H^s(\Omega_1)]^k$, see $\S 3.2.5$ or $\S 2.4$ for $s \in \mathbb{N}$ or $\S 1.3$ Theorem 1 for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, i.e. $\{u^{(i)}\} \subset [H_{loc}^s(\Omega_1)]^k$ with some non-negative number $\gamma$.

Then Corollary 3.5 states that if $\{B_0 u^{(i)}\}$ convergence to zero in $[H^{s-1/2}(\partial \Omega_0)]^k$ then the Problem 2.5 solution sequence $\{u^{(i)}\}$ strongly converges to zero in the
space \([H^s(\Omega_0) \cap S_L(\Omega_0)]^k\), weakly converges to zero in \([H^s(\Omega_1)]^k \cap S_L(\Omega_1)\) and it converges to zero in the local space \(H^s_{\text{loc}}(\Omega_1)]^k\).

Second, the weak convergence of \(\{u(i)\}\) in \([H^s(\Omega_1)]^k \cap S_L(\Omega_1)\) to zero implies the weak convergence of \(B_0u(i)\) in \([H^{s-1/2}(\partial\Omega_1)]^k\) due to the continuity of the trace operator \(B_0\). As it was shown in the proof of Theorem 3.6, the operator
\[
\mathcal{V}_{\partial\Omega_1} : [H^{s-3/2}(\partial\Omega_1)]^k \to [H^{s-1/2}(\partial\Omega_1)]^k
\]
is continuously invertible under our assumptions. This fact provides a weak convergence of the sequence \(\{v(i)\}\) in \([H^{s-3/2}(\partial\Omega_1)]^k\).

The method of fundamental solutions (MFS) for solving Problem 2.5 consists in setting an additional larger bounded domain \(\Omega_2 \subset X\) such that \(\Omega_1 \subset \Omega_2\) and representation of the solution to Problem 2.5 in \(\Omega_1\) by a weighted sum of fundamental solutions whose singularities are located on the boundary \(\partial\Omega_2\) of domain \(\Omega_2\):

\[
(4.3) \quad u(x) = \sum_{j=1}^N \Phi(x, z_j)c_j,
\]
where \(x \in \Omega_1\), \(\{z_j\}_{j=1}^N\) is a set of isolated points of \(\partial\Omega_2\) and \(c_j\) are some vectors from \(\mathbb{R}^k\).

We consider the case when \(\Omega_2\) has a \(C^\infty\)-smooth boundary \(\partial\Omega_2\) and \(\Omega_2 \setminus \Omega_1\) has no compact components. Under these assumptions the possibility of approximation of the solution to Problem 2.5 by this method and the conditional stability of this approximation is justified by the following results.

**Corollary 4.3.** If \(\{z_j\}_{j \in \mathbb{N}}\) is an everywhere dense set of points on \(\partial\Omega_2\) then for any boundary datum \(f \in [H^{s-1/2}(\partial\Omega_0)]^k\) of Problem 2.5 and for any solutions \([H^s(\Omega_1)]^k \cap S_L(\Omega_1)\) to Problem 2.5 (if it exists) and any \(\varepsilon > 0\) there are numbers \(M(u, \varepsilon)\) and the weight coefficients vector \(\{c_j(u, \varepsilon)\}_{j=1}^{M(u, \varepsilon)}\) such that

\[
\left\| u - \sum_{j=1}^M \Phi(x, z_j)c_j \right\|_{[H^s(\Omega_1)]^k} < \varepsilon, \quad \left\| f - \sum_{j=1}^M \Phi(x, z_j)c_j \right\|_{[H^{s-1/2}(\partial\Omega_1)]^k} < \varepsilon.
\]

**Proof.** The statements follow immediately from Corollary 3.14 and the continuity of the trace operator \(B_0\).

**Corollary 4.4.** Let \(\{z_j\}_{j \in \mathbb{N}} \subset \partial\Omega_2\). If a sequence \(\{u(i)\} = \left\{ \sum_{j=1}^\infty \Phi(x, z_j)c_j(i) \right\}_{x \in \partial\Omega_1}\) is bounded in \([H^s(\Omega_1)]^k\) and the sequence of series \(\{u_0(i) = B_0u(i)\}_{x \in \partial\Omega_0}\) strongly converges to zero in \([H^{s-1/2}(\partial\Omega_0)]^k\) then the sequence of series \(\{u(i)\}\) strongly converges to zero in \([H^s(\Omega_0)]^k \cap S_L(\Omega_0)\), weakly converges to zero in the space \([H^s(\Omega_1)]^k\), it converges to zero also in the local space \([H^s_{\text{loc}}(\Omega_1)]^k\) and the sequence of series \(\{B_0u(i)\}_{x \in \Omega_1}\), weakly converges to zero in \([H^{s-1/2}(\partial\Omega_1)]^k\).

**Proof.** The statements follow from Corollary 3.15 and Corollary 3.14 and the continuity of the trace operator \(B_0\).
Next, we will discuss the "extension" method for approximate solving the interior Dirichlet Problem 0.4 or, more precisely, Problem 1.4 for the operator \( L \) in \( \Omega_0 \) with the Dirichlet condition \( u_{|\partial\Omega_0} \).

The method consists in introducing a virtual embracing boundary \( \partial\Omega_v \) such that \( \Omega_0 \subset \Omega_v \) and solving Problem 2.5 in \( \Omega_v \) with the same condition \( u_{|\partial\Omega_0} \) on \( \partial\Omega_0 \). A restriction of the solution to Problem 2.5 on \( \Omega_0 \) is considered as an approximation of the Dirichlet problem solution \( u \) in \( \Omega_0 \).

To solve Problem 2.5 taking into account the approximation of the Problem 1.4 solution, one can use the integral equation method discussed above, as well as the method of fundamental solutions. For this purpose, the virtual surface \( \partial\Omega_v \) can be considered as \( \partial\Omega_1 \) for the direct and indirect methods of integral equations and as \( \partial\Omega_2 \) for the MSF. In the last case there is no need to explicitly set the intermediate boundary \( \partial\Omega_1 \).

Corollaries 4.1-4.4 justify the applicability these methods for solving Problem 1.4 in \( [H^s(\Omega_0)]^k \cap S_2(\Omega_0) \) subject to the appropriate assumptions about the smoothness of the boundary \( \partial\Omega_0 \) and the prescribed boundary \( \partial\Omega_v \). Namely, for the integral equation methods 1) the boundaries \( \partial\Omega_0 \) and \( \partial\Omega_v \) belong to \( C^s \)-class of smoothness if \( s \in \mathbb{N}, s \geq 2 \) or 2) both boundaries \( \partial\Omega_0 \) and \( \partial\Omega_v \) are Lipschitz ones if \( s \in [1, 3/2) \). For the MFS \( \partial\Omega_0 \) is assumed to be of the same class of smoothness and \( \partial\Omega_v \) has \( C^\infty \) smoothness.

In particular, Corollaries 4.1-4.4 guarantee an arbitrarily accurate approximation of the solution of Problem 1.4 and the stability of its solution to small perturbations of the boundary condition in the respective norms.

The same technique of reduction to the problem 2.5 can also be used for solving Problem 2.3 of "analytical" continuation from the domain \( \partial\Omega_0 \) to the large domain \( \partial\Omega_1 \).

Actually, knowing the datum \( V \in [H^s(\Omega_0)]^k \cap S_2(\Omega_0) \) implies knowing its trace \( B_0u = [H^{s-1/2}(\partial\Omega_0)]^k \) on \( \partial\Omega_0 \). Therefore, as we showed above solving Problem 2.3 can be reduced to solving Problem 2.5 with the boundary datum \( B_0 V \) on \( \partial\Omega_0 \) using the integral equation methods or MFS. All the statements of Corollaries 4.1-4.4 for applying these methods to solve Problem 2.5 are valid under the above assumption.

Finally, we consider some ways to compute a solution \( u \in [H^s(\Omega_1 \setminus \Omega_0)]^k \cap S_2(\Omega_1 \setminus \Omega_0) \) to the Cauchy Problem 2.4 with a pair of the Cauchy data \( \{u_{00}, u_{10}\}, u_{00} \in [H^{s-1/2}(\partial\Omega_0)]^k, u_{10} \in [H^{s-3/2}(\partial\Omega_0)]^k \).

These methods are based on the application of Theorem 3.14. It includes the following stages:

\[
\begin{align*}
&\text{a) computing the analytical continuation of } F = (W_{0\partial\Omega}(u_{00}))^+ + (V_{0\partial\Omega}(u_{10}))^+ \text{ from domain } \Omega_0 \text{ to domain } \Omega_1, \\
&\text{b) computing the function } (W_{0\partial\Omega}(u_{00}))^- + (V_{0\partial\Omega}(u_{10}))^- \text{ in } \Omega_1 \setminus \Omega_0; \\
&\text{c) computation of the Problem 2.4 solution } u \text{ in } \Omega_1 \setminus \Omega_0 \text{ by formula (3.14).}
\end{align*}
\]

The idea of the proposed methods is to replace the analytical continuation with the solution to the Problem 2.5 with an appropriate boundary datum.

The boundary datum of Problem 2.5 can be set on \( \partial\Omega_0 \). We denote it as \( f_{\partial\Omega_0} \).

The boundary datum can be defined as:

\[
f_{\partial\Omega_0}(x) = \lim_{y \to x} F(y), x \in \partial\Omega_0, y \in \Omega_0.
\]

According the definition of function \( F \), almost everywhere on \( \partial\Omega_0 \)

\[
(4.4) \quad f_{\partial\Omega_0}(x) = -\frac{1}{2}u_{00} + \text{v.p.}(W_{0\partial\Omega}(u_{00}))^+ + B_{0,\partial\Omega}(V_{0\partial\Omega}(u_{10}))^+.
\]
where v.p.$W_{\partial\Omega_0}(u_{00})$ denotes the principal value of the double layer potential $(W^{(B)}_{\partial\Omega_0}(u_{00}))(x)$ (see (4.7)) at the point $x \in \partial\Omega_0$.

The application of the formula (4.4) requires the calculation of singular integrals. To avoid this, we can set a "virtual" surface $\partial\Omega_{\text{int}}$, which bounds a relatively compact domain $\Omega_{\text{int}}$ such that $\Omega_{\text{int}} \subset \Omega_0$. We assume that $\partial\Omega_{\text{int}}$ belongs to $C^s$-class of smoothness if $s \in \mathbb{N}$, $s \geq 2$ or $\partial\Omega_{\text{int}}$ is the Lipschitz surface if $s \in [1, 3/2)$.

Instead of the analytical continuation of $\mathcal{F}$ from $\Omega_0$ to $\Omega_1$ we can perform the analytical continuation of $\mathcal{F}$ from $\Omega_{\text{int}}$ to $\Omega_1$. Obviously, if the analytical continuation of $\mathcal{F}$ from $\Omega_0$ to $\Omega_1$ exists, the analytical continuation of $\mathcal{F}$ from $\Omega_{\text{int}}$ to $\Omega_1$ also exists; according the Unicite Continuation property, it coiincides with the analytical continuation of $\mathcal{F}$ from $\Omega_0$ to $\Omega_1$.

To perform that analytical continuation we can solve Problem 2.5 in the following statement: to find a function $\mathcal{F}$ in $\Omega_1$ such that $L\mathcal{F} = 0$ in $\Omega_1$ s.t. $B_{0, \partial\Omega_{\text{int}}} \mathcal{F} = f_{\partial\Omega_{\text{int}}}$. The boundary datum $f_{\partial\Omega_{\text{int}}}$ is computed as:

$$f_{\partial\Omega_{\text{int}}}(x) = B_{0, \partial\Omega_{\text{int}}} (W_{\partial\Omega_0}(u_{00}))^+ + B_{0, \partial\Omega_{\text{int}}} (V_{\partial\Omega_0}(u_{01}))^+$$

The reduction of the Cauchy problem to Problem 2.5 can also be performed in a slightly different form (cf. with [33]).

According formula (5.11), the trace $\phi = B_{0, \partial\Omega_1} u$ on $\partial\Omega_1$ of the Cauchy problem solution $u$ is equal to:

$$\phi = B_{0, \partial\Omega_1} (W_{\partial\Omega_0}(u_{00}))^- + B_{0, \partial\Omega_1} (V_{\partial\Omega_0}(u_{01}))^- - B_{0, \partial\Omega_1} \mathcal{F}.$$

Let us consider the Dirichlet problem: $Lu = 0$, $u \in [H^s(\Omega_1)]^k$, $B_{0, \partial\Omega_1} u = \phi$, $\phi \in [H^{s-1/2}(\partial\Omega_1)]^k$, and introduce an operator $D : [H^{s-1/2}(\partial\Omega_1)]^k \to [H^{s-1/2}(\partial\Omega_0)]^k$ which maps $\phi$ on $\partial\Omega_1$ to $\hat{f} = B_{0, \partial\Omega_0} u$ on $\partial\Omega_0$.

Taking into account (4.4), we can see that

$$\hat{f} = D\phi = D(B_{0, \partial\Omega_1} (W_{\partial\Omega_0}(u_{00}))^- + B_{0, \partial\Omega_1} (V_{\partial\Omega_0}(u_{01}))^-) - f_{\partial\Omega_0}. \tag{4.6}$$

Thus, the trace on $\partial\Omega_1$ of the solution to Problem 2.5 with boundary data $\hat{f}$ coincides with the trace on $\partial\Omega_1$ of the solution to the Cauchy problem. Therefore, the trace on $\partial\Omega_1$ of the solution to the Cauchy problem can be found by solving Problem 2.5 with that boundary datum $\hat{f}$.

Operator $D$ can be presented, for instance, as: $D\phi = B_{0, \partial\Omega_0} V_{\partial\Omega_1} (B_{0, \partial\Omega_1} V_{\partial\Omega_0})^{-1} \phi$. Note, that according to Corollary 1.1 and Theorem 5.6 operator $B_{0, \partial\Omega_1} V_{\partial\Omega_0} : [H^{s-3/2}(\partial\Omega_1)]^k \to [H^{s-1/2}(\partial\Omega_0)]^k$ is continuously invertible.

To justify the applicability of the reduction of the Cauchy problem to Problem 2.5 we note that transform $(W_{\partial\Omega_0}(u_{00}))^-, (V_{\partial\Omega_0}(u_{01}))^-$ from the space $\{[H^{s-1/2}(\partial\Omega_0)]^k, [H^{s-3/2}(\partial\Omega_0)]^k\}$ to $[H^s(\Omega_1 \setminus \Omega_0)]^k$ is continuous. Transforms (4.4), (4.9) and (4.13) also continuously map the Cauchy data $\{u_{00}, u_{01}\}$ from the space $\{[H^{s-1/2}(\partial\Omega_0)]^k, [H^{s-3/2}(\partial\Omega_0)]^k\}$ to $f_{\partial\Omega_0} \in [H^{s-1/2}(\partial\Omega_0)]^k$ and $f_{\partial\Omega_{\text{int}}} \in [H^{s-1/2}(\partial\Omega_{\text{int}})]^k$ respectively.

To solve Problem 2.5 (or approximate its solution) one can use the integral equations method of MFC. All the statements of Corollaries 4.1-4.4 hold for the boundary data of Problem 2.5 which are obtained by (4.4), (4.9) and (4.10) from the Cauchy data $\{u_{00} \in [H^{s-1/2}(\partial\Omega_0)]^k, u_{01} \in [H^{s-3/2}(\partial\Omega_0)]^k\}$.

In conclusion, we would like to make a few short remarks about the numerical implementation of the methods considered above.
The indirect boundary integral equation method for solving Problem 2.5 requires numerical solving operator equation (4.2) and computing the problem solution in $\Omega_1$ via formula (4.1) by numerical integration.

The conventional way of numerical solving linear operator equations consists of approximating them by systems of linear algebraic equations. The boundary element method (BEM) (in the collocation or Galerkin version) can be employed for this propose. Note that the results obtained in this paper for the second order elliptic operators include the case of a domain with Lipschitz boundaries. This justifies the use of the most popular version of BEM, which assumes a polygonal approximation (triangulation) of the boundaries. For the details of BEM implementation, see, for instance [61].

A matrix of sufficiently large dimension obtained by BEM approximation of operator of equation (4.1) is expected to be ill-conditioned. Therefore, to solve a system of linear algebraic equations approximating operator equation (4.1), it is necessary to apply regularization algorithms. Regularization algorithms of the Tikhonov type (which provide the boundedness of the respective norms of the solutions in finite-dimensional spaces) can be used for this purpose. Corollary 4.2 justifies the applicability of this regularization approach.

To assemble the BEM matrix and calculate the problem solutions in $\Omega_1$ in a neighbourhood of $\partial \Omega_1$ by (4.1), they have to compute weak singular and near weak singular integrals. MFS is free from this encumbrance. When using MFS to solve Problem 2.5, the unknown coefficients $c_i$ of the expansion (4.3) of the problem solution by the system of fundamental solutions can be found by setting of collocation points $\{z_j\}_{j=1}^N \subset \partial \Omega_2$ and $\{x_i\}_{i=1}^N \subset \partial \Omega_0$ and solving the following system of linear algebraic equations:

\[
\sum_{j=1}^N \Phi(x_i, z_j)c_j = u_{00}(x_i)
\]

where $u_{00}$ is the boundary datum of Problem 2.5.

Some rules for choosing the collocation points that ensure the uniqueness of solution to equation (4.7) can be found in [15].

In general, the matrix of equation (4.7) is typically ill-conditioned and application of regularization methods for solving (4.7) is required. Corollary 4.3 justifies the applicability of the Tikhonov-type regularization.

The "extension" approach for approximation of the Problem 1.4 solution via numerical solving Problem 2.5 by BEM or MFS leads to the ill-conditioned matrices. Despite the fact that solutions to the corresponding linear algebraic equation systems are very sensitive to errors of the conditions, the final solutions to Problem 1.4 in $\Omega_0$ is stable. The statements of Corollary 4.2 and Corollary 4.4 explain this fact. However, too large values of the linear algebraic systems solutions can cause technical problems in the computer calculations. In this situation the double-precision arithmetic or using some regularization methods (see, for instance, [6]) can be necessary.

Note that the conditionality of that BEM or MFS matrixes as well as the convergence rate of the approximation of the solution to Problem 1.4 depends on the geometry of domain $\Omega_0$ and the embracing boundary $\partial \Omega_2$. These regularities are
still insufficiently investigated. Some observations on these issues (related to MFS) have been summarised in [15].

The numerical solving of the Cauchy problem with the proposed methods includes a preliminary computation of the boundary datum for Problem 2.5. The computations by formula (4.6) require inversion of operator $B_0, \p_{\p_1} V_{\p_{\p_1}}$. The operator inversion can be computed by BEM. Corollary 1.6 and Theorem 3.6 allow us to conclude that the BEM matrix which approximates this operator is well-conditioned for the common versions of BEM (see, for instance 61 for more details).

The implementation of formulas (4.4) and (4.6) require computation of singular and weak singular integrals. Moreover, obtaining the solution component $(W_{\p_0}(u_{00}))^- + (V_{\p_0}(u_{10}))^-$ in $\Omega_1 \setminus \Omega_0$ in a neighbourhood of $\p \Omega_0$ requires computing near singular and near weak singular integrals. The technique of numerical computation of the singular integrals can be found, for instance, in [69], [29] and literature cited there.

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