Effects of Gravity and Finite Temperature on the Decay of the False Vacuum

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I have calculated the exponential suppression factor in the decay rate of the false vacuum (per unit volume) for a real scalar field at finite temperature, in the presence of gravity, in the thin-wall approximation. Temperatures are assumed to be much greater than the inverse of the nucleation radius. The value of a local minimum of the scalar potential is arbitrary. Thus, both true and false vacua may have arbitrary cosmological constants.

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I. INTRODUCTION

Vacuum decay occurs when the potential for some scalar field has a local minimum which is not the global minimum. If the field starts out in the local minimum, the resulting “false vacuum” can remain metastable for some time, until it decays by tunneling. The decay of the false vacuum has been studied by many authors. For example, the exponential decay factor was first found by [1] then later more rigorously by [2]. The prefactor which contains some of the basic quantum mechanical modifications to the decay probability was found for a single scalar field in [2] and for spontaneous symmetry breaking and internal degrees of freedom in [3] and [4] respectively.

This effect is modified in the presence of gravity [6], [7] or at finite temperature [8]. As demonstrated in Ref. [7], gravity enhances the decay process when the average of the two minima is positive and diminishes it if both minima are negative. If the false minimum is positive, but the average is negative, the behavior is slightly more complex: the vacuum decay rate decreases if the gravitational effects are small and increases if the gravitational effects become large.

Reference [3], gives an excellent explanation of how the decay rate changes at finite temperatures. First, the potential is changed, second the action integral is taken over a periodic time interval of period $\frac{1}{\beta}$. Thus, if the bubble radius is much larger than this period, we must change from a bubble that is a 4-dimensional sphere to one that is a 4-dimensional cylinder and whose cross section is a 3-dimensional sphere. The transition between these two limits is hinted at but not fully explored in the reference. I am not aware of any work which explores this more thoroughly, nor will I attempt to here.

In this paper I will combine these procedures to get a description of the decay rate of the false vacuum in the presence of gravity and at a finite temperature. Some work in this direction has been done before [9]. However, they considered only the change in the potential caused by the finite temperatures, and did not consider the change in the action integral.

Other exotic decay possibilities have also been explored. [10] explores the possibility that in an expanding universe, the true vacuum may tunnel back to a false vacuum. [11] and [12] explore the possibility of tunneling decays in the absence of minima and barriers in the potential in flat and curved space respectively. [13] shows how gravitational stabilization could explain the smallness of the cosmological constant. These ideas in conjunction with the results of this paper may provide interesting avenues for further study.

II. CALCULATION OF BOUNCE AT FINITE TEMPERATURE

Let us consider a scalar field with a potential that has at least two non-degenerate local minima at a given temperature. We will call $\phi$ in the greater (false) vacuum state $\phi_+$ and the potential at this point $U_+ \equiv U(\phi_+, T)$. Similarly, $\phi$ at the lesser (possibly true) vacuum state is $\phi_-$ and the potential there is $U_- \equiv U(\phi_-, T)$.

Vacuum decay will begin when a bubble of the lesser state nucleates in the greater state. The nucleation rate per unit volume is $A \exp[-B/\hbar](1 + O(\hbar))$. $B$ is the action of the bounce. (Or more accurately the difference between the action of $\phi_+$ and the action of the bounce.) The bounce, $\phi_b$ is a choice of $\phi$ which

1. solves the euclidean equations of motion,
2. transverses the barrier between the two vacua, and
3. minimizes the action for these two criteria.

In the absence of gravity or finite temperatures, the bounce is spherically symmetric in four euclidean spacetime dimensions and is described by a bubble of lesser (true) vacuum around $\rho = 0$ (where $\rho$ is the radial variable), a thin shell at $\bar{\rho}$ where the value of $\phi$ changes from $\phi_-$ to $\phi_+$ and then the outside of the bubble is simply $\phi_+$. Following Linde [8], we notice that because we are considering a system in thermal equilibrium, the action integral must be periodic in imaginary time with a period $\beta = 1/T$. If the radius of the nucleating bubble is much smaller than $\beta$ then this change is irrelevant as the appropriate periodic system is just a series of such bubbles a distance $\beta$ apart. However, if the temperature is much
higher so that β is very small compared with the radius of the nucleating bubble, then the bounce corresponds to a cylinder. That is to say a three sphere in the spatial directions and constant in time. The integration over time is then simply multiplication by β = 1/T.

In the absence of gravity it can be proved that the solution to the bounce is spherically symmetric at zero temperature in 4 Euclidean dimensions, and is a 4-cylinder (spherically symmetric in three orthogonal dimensions) at high temperature. In the presence of gravity this is only an assumption, so these results are only a lower bound on the nucleation rate. However, the assumption seems reasonable and no counter-example is known.

In evaluating the effects of gravity I will follow [6], and begin with the action for a scalar field including gravity

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi, T) - (16\pi G)^{-1} R \right], \]

and a metric of the form

\[ (ds)^2 = (d\tau)^2 + (d\xi)^2 + \rho(\xi)^2 (d\Omega)^2. \]

Computing the Euclidean equations of motion is now straightforward. The results follow. By varying with respect to \( \phi \) we find the equation

\[ \phi'' + 2\rho' \rho \phi' = \frac{dU}{d\phi}. \]

Varying with respect to the metric yields the Einstein Equation

\[ G_{\mu\nu} = \kappa T_{\mu\nu}, \]

the \( \xi \) component of which becomes

\[ (1 + \kappa \rho^2(\frac{1}{2} \phi'^2 - U)). \]

Finally the action can be rewritten as

\[ S_E = \frac{4\pi}{T} \int d\xi \left[ \rho^2 \left( \frac{1}{2} \phi'^2 + U \right) + \frac{1}{\kappa} (2\rho \phi'' + \rho'^2 - 1) \right]. \]

At this point Coleman [6] finds \( \phi \) and \( \rho \) in terms of \( \xi \). This calculation is important for determining the validity of the thin-wall approximation. I will not repeat this calculation as it is unnecessary to understand the derivation of the bounce action. The calculation is the same as in the reference.

We can now simplify the action [6] by using integration by parts to eliminate the second derivative term, and equation [5] to eliminate \( \rho' \) which yields

\[ S_E = \frac{8\pi}{T} \int d\xi \left( \rho^2 U - \frac{1}{\kappa} \right) + \text{surface terms}. \]

The bounce action is then

\[ B \equiv S_E(\phi_b) - S_E(\phi_+). \]

This can be divided into three parts. Outside the wall, the wall itself, and inside the wall.

Because the bounce solution and greater vacuum are identical outside of the wall, there is no contribution to the bounce from outside the wall. (This is also why surface terms can be safely neglected.)

\[ B_{\text{outside}} = 0 \]

At the wall it is useful to define \( U_0(\phi, T) \) such that

\[ U_0(\phi, T) = U(\phi, T) + O(U_+ - U_-), \]

\[ U_0(\phi_+, T) = U_0(\phi_-, T), \]

and \( dU_0/d\phi = 0 \) at both \( \phi_+ \) and \( \phi_- \). We can now approximate \( \rho \) as \( \bar{\rho} \) and \( U(\phi, T) \) as \( U_0(\phi, T) \) to get

\[ B_{\text{wall}} = \frac{8\pi}{T} \rho^2 \int d\xi \left[ U_0(\phi, T) - U_0(\phi_+ , T) \right] = \frac{4\pi}{T} \rho^2 S_1. \]

Inside the wall, \( \phi \) is constant so that from equation [6] we get

\[ d\xi = d\rho \left( 1 - \kappa \rho^2 U \right)^{-1/2}, \]

so that

\[ B_{\text{in}} = -\frac{8\pi}{T} \int_0^{\bar{\rho}} d\rho \left( \sqrt{1 - \kappa \rho^2 U} \right) \bar{\rho} \left( -\phi_- + \phi_+ \right). \]

I should note that when \( U < 0 \) the \( \sqrt{1 - \kappa \rho^2 U} \) must be changed to an inverse hyperbolic \( \sinh \). (This can also be accomplished by simply dropping the absolute value signs.) Otherwise the equation is unchanged.

### III. SIMPLE CASES

While I will, in the next section, explore the general case at finite temperature (as Parke [6] did for the zero temperature case), it is illustrative to begin by considering the relatively simple cases given in [6]. It is useful to understand the simpler solutions first as a reference and check for the more general solution. It is also worth noting that in finding the extrema of the bounce solutions it is much easier to work from the integral form of equation [12].

#### A. Null True Vacuum

We begin with the case where the true vacuum is 0 (null) and the false vacuum is small and positive.

\[ U(\phi_+, T) = \epsilon, \quad U(\phi_-, T) = 0. \]
In this case we find
\[ \bar{\rho} = \frac{2S_1}{\epsilon + \kappa S_1^2} = \frac{\bar{\rho}_0}{1 + (\frac{\kappa}{4\pi})^2} \quad (14) \]

Where \( \bar{\rho}_0 = \frac{2S_1}{\epsilon} \), in agreement with Linde's work, and \( \Lambda = (\kappa\epsilon)^{-1/2} \).

The bounce action can not be put in quite as simple a form as in Coleman's paper, but it can be written as
\[ B = \frac{4\pi}{T} \Lambda^3 \epsilon \left[ \arcsin \left( \frac{2\alpha}{1 + \alpha^2} \right) - \left( \frac{2\alpha}{1 + \alpha^2} \right) \right] \quad (15) \]
\[ = \frac{4\pi}{T} \Lambda^3 \epsilon \left[ \arccos \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right) - \left( \frac{2\alpha}{1 + \alpha^2} \right) \right] \quad (16) \]

Where \( \alpha = \frac{\epsilon}{4\pi} \). We can easily verify that this gives Linde's result as \( \Lambda \to \infty \). The need for the final change to an inverse cosine is apparent when you realize that the first form traces both the domain and range of the inverse sine twice yielding incorrect results. The second form traces the entire domain and range of the inverse cosine once giving correct answers.

**B. Null False Vacuum**

The opposite case where the false vacuum is 0 (null) and the true vacuum is small and negative can also be worked out rather simply.
\[ U(\phi_+) = 0, \quad U(\phi_-) = -\epsilon. \quad (17) \]
We find
\[ \bar{\rho} = \frac{2S_1}{\epsilon - \kappa S_1^2} = \frac{\bar{\rho}_0}{1 - (\frac{\kappa}{4\pi})^2}, \quad (18) \]
and the bounce is
\[ B = \frac{4\pi}{T} \Lambda^3 \epsilon \left[ \frac{2\alpha}{1 - \alpha^2} - \sinh^{-1} \left( \frac{2\alpha}{1 - \alpha^2} \right) \right] \quad (19) \]
\[ = \frac{4\pi}{T} \Lambda^3 \epsilon \left[ \frac{2\alpha}{1 - \alpha^2} - \cosh^{-1} \left( \frac{1 + \alpha^2}{1 - \alpha^2} \right) \right]. \]

As in the zero temperature case, the stabilizing effect is still present. That is to say that if \((\frac{\kappa}{4\pi})^2 > 1\) then the new vacuum state is not large enough to hold the bubble and a decay can never nucleate.

**IV. THE GENERAL CASE**

In general neither the greater (false) vacuum \( U_+ \) nor the lesser (true) vacuum \( U_- \) is 0. When we allow for this the equations get far more complicated though still reasonably tractable. We find that
\[ \bar{\rho}^2 = \frac{4S_1^2}{\kappa^2 S_1^4 + 2S_1^2 \kappa(U_+ + U_-) + (U_- - U_+)^2} \]
\[ = \bar{\rho}_0^2 \frac{1 + 2\left(\frac{\kappa}{4\pi}\right)^2 + \left(\frac{\kappa}{4\pi}\right)^4}{1 + 2\left(\frac{\kappa}{4\pi}\right)^2 + \left(\frac{\kappa}{4\pi}\right)^4}. \quad (20) \]

As in the previous section, \( \bar{\rho}_0 \) is the critical radius in the absence of gravity
\[ \bar{\rho}_0 = \frac{2S_1}{U_+ - U_-}. \quad (21) \]
Also
\[ \lambda^2 = [\kappa(U_+ + U_-)]^{-1}, \quad (22) \]
and
\[ \alpha^2 = [\kappa(U_+ - U_-)]^{-1}. \quad (23) \]

Now when we add the parts of the bounce from the wall and the interior of the bubble, we find that the simple square-root terms cancel leaving only the arcsin terms so that
\[ B = \frac{4\pi}{\kappa T} \left[ \arcsin \frac{\sqrt{U_+ \bar{\rho}}}{U_+} - \arcsin \frac{\sqrt{U_- \bar{\rho}}}{U_-} \right] \quad (24) \]

To check this we take the limit in which \( \kappa \to 0 \) (small gravity) and find
\[ B_0 = \frac{2\pi}{3T} \rho_0^3 \frac{16\pi S_1^3}{3T \epsilon^2} \quad (25) \]
in agreement with 8.

As in 7 it is useful to separate out the zero gravity portion of the bounce and write
\[ B = B_0 r[(\bar{\rho}_0/2\Lambda)^2, \Lambda^2/\lambda^2] \quad (26) \]

There are many ways to write \( r(x, y) \). These are the two which I find most clear and illuminating.
\[ r(x, y) = \frac{3}{2x^{3/2}} \int_0^x \frac{z^{1/2}}{1 + 2yz + z^2} dz \quad (27) \]
\[ = \frac{3}{2\sqrt{2x^{3/2}}} \left[ \frac{1}{\sqrt{y + 1}} \arccos \frac{1 - x}{\sqrt{1 + 2yx + x^2}} + \frac{1 + x}{\sqrt{1 + 2yx + x^2}} \right] \]

Both forms are relatively simple, though for \( y < 1 \) equation 27 is not explicitly real. (That is to say two factors of \( i \) arise and cancel.) The integral form 26 is always explicitly real, and deals with the limits as \( y \to \pm 1 \) and \( x \to 0 \) more easily. However, actually evaluating the integral is very difficult to do in closed form. So both forms are useful. Also, if \( y = \pm 1 \) and \( x = \alpha^2 \) both equations 26 and 27 reduce to forms consistent with equations 16 and 18.

Figure 4 shows \( B/B_0 \) (\( r \)) as a function of \( \alpha = \frac{\sqrt{\kappa}}{4\pi} \) for various values of \( \Lambda^2/\lambda^2 \). It is clear from the figure that all the same basic features exist as in the zero temperature case.
FIG. 1: The ratio \(B/B_0(= r)\) as a function of \((\tilde{\rho}_0/2\Lambda)\) for various values of \(\Lambda^2/\lambda^2\). The number next to each line gives the value of \(\Lambda^2/\lambda^2\) for that specific line.

1. For \(\Lambda^2/\lambda^2 > 0\) (which implies \(U_+ + U_- > 0\)), gravity lessens the bounce action and thus increases the rate of decay. It is interesting to consider two limits of this case.
   
   (a) If \(\Lambda^2/\lambda^2 \ll (\tilde{\rho}_0/2\Lambda)^2 \ll \Lambda^2/\lambda^2\), then
   
   \[
   r[(\tilde{\rho}_0/2\Lambda)^2, \Lambda^2/\lambda^2] = \frac{3}{2} \left\{ \frac{2}{(\tilde{\rho}_0/2\Lambda)^2 \Lambda^2/2\lambda^2} \right\} = \frac{6\lambda^2}{\tilde{\rho}_0^2} \quad (28)
   \]

   (b) \((\tilde{\rho}_0/2\Lambda)^2 \gg 1\) and \((\tilde{\rho}_0/2\Lambda)^2 \gg \Lambda^2/\lambda^2\) then
   
   \[
   r[(\tilde{\rho}_0/2\Lambda)^2, \Lambda^2/\lambda^2] = \frac{3\pi}{2 \sqrt{2(\Lambda^2/\lambda^2 + 1)(\tilde{\rho}_0/2\Lambda)^3}} = \frac{3\pi}{4} \frac{U_+ - U_-}{U_+/(\tilde{\rho}_0/2\Lambda)^3}. \quad (29)
   \]

   (The last equation still applies in the next situation.) So we see that the system will change from quadratic to cubic falloff around
   
   \[
   (\tilde{\rho}_0/2\Lambda) \cdot \lambda/\Lambda = 1. \quad (30)
   \]

2. For \(-1 < \Lambda^2/\lambda^2 < 0\) (which implies \(U_+ > 0\), \(U_+ + U_- < 0\)) gravity increases the bounce action for small values of \((\tilde{\rho}_0/2\Lambda))\), but the bounce action reaches a maximum and for large values of \((\tilde{\rho}_0/2\Lambda))\) the action falls off as it does for positive \(\Lambda^2/\lambda^2\). The maximum action will occur when the value of \(\tilde{\rho}_0/2\Lambda\) is slightly larger than the value which minimizes \(1 + 2\Lambda^2/\lambda^2(\tilde{\rho}_0/(2\Lambda))^2 + (\tilde{\rho}_0/(2\Lambda))^4\).

3. For \(\Lambda^2/\lambda^2 \leq -1\) (which implies \(U_+, U_- < 0\)) gravity increases the bounce action which goes to infinity (completely stabilizing the vacuum) when

\[1 + 2\Lambda^2/\lambda^2(\tilde{\rho}_0/(2\Lambda))^2 + (\tilde{\rho}_0/(2\Lambda))^4 = 0.\]

In other words, the greater vacuum is stable if

\[\frac{(\tilde{\rho}_0/2\Lambda)^2}{\Lambda^2/\lambda^2} \geq -\Lambda^2/\lambda^2 - \sqrt{\Lambda^4/\lambda^4 - 1}, \quad (31)\]

which is the same as when

\[S_1\sqrt{8\pi G} \geq -\sqrt{-U_-} - \sqrt{-U_+}. \quad (32)\]

V. RANGE OF VALIDITY

These results are valid under the same conditions as in \(R\). That is \(B \gg 1\) so that the semi-classical treatment is valid, and all length scales \(\tilde{\rho}_0, |\lambda|, \text{ and } \Lambda\) must be much greater than the thickness of the wall.

The most restrictive of these is likely to be \(|\lambda|\) because while we set the difference between \(U_+\) and \(U_-\) to be small in all cases, we have never yet required that these potentials themselves be small. (In the absence of gravity the value of the potentials themselves is irrelevant. Only the difference mattered.)

The thickness of the wall is determined by an equation of the approximate form

\[\Delta \xi = \int_{(\phi_+ - \phi_-)}^{(\phi_+ + \phi_-)} d\phi \left\{ \frac{2}{U_0(\phi, T - U_0(\phi_\pm, T))} \right\}^{-\frac{1}{2}} \sim \frac{\Delta \phi}{(U_{\text{max}} - U_{\text{min}})^{1/2}} \quad (33)\]

\[\Delta \phi \lesssim U_{\text{max}} - U_{\text{min}} \gg (\Delta \phi)^2 \kappa. \quad (34)\]

As long as gravity is weak and the difference between \(\phi_+\) and \(\phi_-\) is not too great, this should easily be satisfied by a moderately high barrier. As gravity or the difference between the values of \(\phi\) increases the barrier must be significantly greater than the actual values of the two minima in the potential.

We should also note that to work in the finite temperature case we assumed \(T \gg \frac{1}{\Sigma}\). In this case the bounce action will always be significantly less than in the zero temperature case by a factor of about

\[\frac{B(T)/B(0)}{\frac{1}{\rho T}} \sim \frac{\epsilon}{S_1 T}. \quad (35)\]

(Because the potential also changes at finite temperature, there is no reason to improve this estimate.)

VI. CONCLUSION

We have seen in this paper that many of the basic features of vacuum decay in the presence of gravity are
unchanged at finite temperature. Gravity still enhances the decay of positive energy vacua and suppresses the decay of negative energy vacua, and there is still a middle region of decay from a positive energy vacuum to a negative energy vacuum (whose energy has a greater magnitude) which is suppressed when gravitational effects are small and enhanced as they increase. Also we see that thermal effects increase the rate of decay just as they did in flat space. There are many interesting avenues for further study including the possibility that the true vacuum may actually "decay" to the false vacuum, the effects of such tunneling in very flat potentials, and potential implications for cosmology.

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