Maximal cocliques of a strongly regular graph with parameters (2048,276,44,36)

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1 Abstract

This article considers a strongly regular graph with parameters (2048,276,44,36) that is related to the extended binary Golay code. That graph is described in [3], [1] and [2].

The source package of this article contains a data file that encodes a sequence of maximal independence sets of that graph, covering all sizes from 20 to 67 and the size 72, and a Pascal program to check this assertion and to optionally generate a text file (to be read by the computer algebra system GAP) that contains the adjacency lists of that graph and the list of the independence sets.

2 Introduction

2.1 Strongly regular graphs

A simple loopless finite undirected graph is called strongly regular with parameter set \((v, k, \lambda, \mu)\), or shortly a \(\text{srg}(v, k, \lambda, \mu)\), iff it has exactly \(v\) vertices, each of them has exactly \(k\) neighbours, and the number of common neighbours of any two different vertices is \(\lambda\) if they are neighbours and \(\mu\) otherwise.

2.2 The considered \(\text{srg}(2048, 276, 44, 36)\)

Seemingly, the first published description of the graph is in [3], identified by S.24, in the “Notes added in proof”, there attributed to J. H. Conway and M. S. Smith. But herein, the following construction given in [1] will be used: “Take the \(2^{11} = 2048\) cosets of even weight of the extended binary Golay code as vertices, and join two cosets when they have representations differing by a vector of weight two.”

Let \(X\) the 24-dimensional vector space over the finite field \(GF(2)\), \(M\) the set of even-weight elements of \(X\), \(C\) the extended binary Golay code, and \(R\) the set of those elements of \(M\) that have the weight 0, have the weight 2, or have the weight 4 and the value 1 at the last position. Let \(w\) denote the weight function, \(m\) denote the function that returns the result of component-wise multiplication of two vectors from \(X\), and \(0\) denote the zero vector of \(X\).

If \(x \in X\) then \(x + x = 0\). If \(x, y \in X\) then \(x + y = x - y\).

\(C\) is known to be a 12-dimensional linear subspace of \(M\), consisting of 1, 759, 2576, 759 and 1, resp., vectors of weight 0, 8, 12, 16 and 24, resp.

\(|X| = 2^{24}, |M| = 2^{23}, |C| = 2^{12} = 4096, |R| = \binom{24}{0} + \binom{24}{2} + \binom{23}{3} = 1 + 276 + 1771 = 2048 = 2^{11}\).

Let \(x, y \in M\). \(w(x - y) = w(x + y)\) equals \(w(x) + w(y) - 2 \times w(m(x, y))\) and is therefore always even. Thus, \(x + y \in M\).
Let \( x, y \in R \wedge x \neq y \wedge z = x + y = x - y \). Because \( x \neq y \), \( z \neq 0 \). On the other hand, the weight of \( z \) is at most 6. This is obvious if the weight of \( x \) or of \( y \) is 2 or 0. In the remaining case, both vectors have weight 4 but also have the value 1 at the last position and so \( z \) can have the value 1 in at most \( 3 + 3 = 6 \) coordinates.

Thus, \( z \) is not in \( C \), \( x \) and \( y \) are not in the same coset. Because of the cardinalities of \( M, C \) and \( R \), each coset is represented by a unique element of \( R \).

To decide whether the two cosets represented by \( x \) and \( y \), resp., are to join, as the above construction says, we have to check whether there is a \( c \in C \) such that the \( w((x - y) + c) = 2 \).

Again, let \( z = x + y = x - y \). As just explained, \( w(z) \leq 6 \). If \( w(c) > 10 \) then \( w(z + c) > 10 - 6 = 4 \). Thus, \( w(c) = 8 \vee c = 0 \) is a derived precondition.

Case \( w(z) = 0 \):
If \( c = 0 \) then \( w(z + c) = w(z) = 2 \). The two cosets are not to join.
Case \( w(z) = 2 \):
If \( c = 0 \) then \( w(z + c) = w(z) = 2 \). The two cosets are to join.
Case \( w(z) = 4 \):
If \( c = 0 \) then \( w(z + c) = w(z) = 4 > 2 \). If \( w(c) = 8 \) then \( w(z + c) \geq 8 - 4 = 4 > 2 \). The two cosets are not to join.
Case \( w(z) = 6 \):
If \( c = 0 \) then \( w(z + c) = w(z) = 6 > 2 \). It remains to scan \( \{c \in C : w(c) = 8\} \).
Because \( w(c) = w(z) + 2 \), \( w(z + c) \geq 2 \).
Clearly, if \( w(z + c) = 2 \) then the two cosets are to join.
If \( w(z + c) = 2 \) then all 6 non-zero positions of \( z \) are non-zero positions of \( c \).
Assume that there is a \( d \in C \) such that \( w(d) = 8 \) and \( w(z + d) = 4 \). Then 5 of the non-zero positions of \( d \) are non-zero positions of \( z \) and thus non-zero positions of \( c \). This implies \( w(c + d) \leq 6 \) and finally \( c = d \). But in that case \( w(z + d) = w(z + c) = 2 \).
So, if \( w(z + c) = 4 \) then the two cosets are not to join.
If \( w(z + c) > 4 \), we cannot immediately decide whether the cosets are to join, continue the search if not all \( c \in C \) with weight 8 were checked.

### 2.3 Cliques and cocliques

The parameters of the considered graph imply that the independence number (size of the largest clique) cannot exceed 85: The smallest eigenvalue of the \( \{0, 1\} \)-adjacency matrix of the graph is \(-12 \). Thus, the Delsarte bound for the maximum clique size is \( 2048/(1 + 276/12) = 2048/24 = 85 + 1/3 \).

The current survey \[2\] contains in particular many propositions on maximal cliques and cocliques of individual strongly regular graphs. In the case of the graph considered herein, the subsection on cliques is rather precise but the subsection on cocliques just stated that the independence number is in the range from 50 to 84, giving neither a proof nor a dedicated reference.

By an (incomplete) extensive search I have found maximal (i.e., not extensible) cocliques of any size from 20 to 67 and of size 72.

The source package of this article includes the files SRG2048C2.PAS and SRG2048C.DAT .

The Pascal source file SRG2048C2.PAS has been (and can be) used to check that the vertex subsets encoded in SRG2048C.DAT represent one maximal coclique of each size from 20 to 67 and two maximal cocliques of size 72, and that the subgraphs induced by the complements of the two vertex subsets of size 72 are not isomorphic (to each other).
3 Computations

SRG2048C2.PAS is a console program for compilers compatible with Borland’s Turbo Pascal 4.0. The binary vectors of length 24 are encoded as (binary) integer values in the range from 0 to $2^{24} - 1$ (type name t.bin24). The addition and subtraction of those vectors is performed by applying the binary operator xor to the encoding integers.

The generator matrix for the Golay code given by the constant golay_bases_str as an array of 12 character strings of length 24 has been taken from the respective chapter of [2].

In the initial stage, the variable golay8 code is filled with the 759 code words of weight 8 (routine init_golay8_code) and the variable rep is filled with the representations of the 2048 cosets (routine init_rep).

The function routine adja calculates whether the two cosets whose representations are given as parameters are adjacent. For this purpose, it calls the function routine minw. The task of minw is to find a weight 8 code word such that the weight of the sum of that code word and the vector encoded by the function parameter value is at most 4 or as small as possible. If minw returns a value that is not 0, 2 or 4, then adja shows that value together with the message “invalid distance” and stops the execution. Probably, that can not happen, but I do not have a proof.

The routine read_and_check_cocliques expects that the binary file SRG2048C.DAT is in the current working directory and that it contains a sequence of vector sets in the form of strings where each string starts with a byte giving the size of the then following sequence of unsigned three-byte integer values (that encode the vectors). If the size is not in the range from 2 to 85, or if one of the vectors is not a proper coset representation, a message is displayed and the execution stops.

For each vector sequence found in the file, its size is displayed and it is checked to establish a maximal coclique of that size.

The size of the largest cocliques that I have found is 72. In order to show that there are (at least) two structurally different cocliques of that probably maximal size, the file contains two vertex sets of size 72 and the program calls the routine check_external_relation for each coclique of size at least 72 found in the file.

The first part of that routine counts the vertices out of the considered set having certain numbers of neighbours in that set and shows the resulting statistic. Because the result was the same (namely 8:480 10:960 12:536) for all cocliques of size 72 that I’ve investigated, a more detailed analysis has been added: The number of two-sets of vertices in the considered set having no common neighbour that is adjacent to exactly 8 vertices in the considered set is calculated and displayed. The results for the two vertex sets of size 72 contained in SRG2048C.DAT were 166 and 276, resp.

3.1 Optional data export for GAP

SRG2048C2.PAS has been build from its predecessor SRG2048C.PAS (published with version 1 of this article) by adding optional instructions to generate a text file SRG2048C.g (to be read by the computer algebra system GAP; about 2.6 MB). These instructions will be compiled into the executable file only if the compiler symbol GAP is defined. The content of SRG2048C.g would look like

\[
A := \ldots \ ; \\
MIS := \ldots \ ; \\
\text{LoadPackage("grape")} ; \\
\text{Gra := Graph(Group(()),[1..2048],OnPoints,} \\
\text{function(x,y) return (x in A[y]) ; end, true)} ;
\]

where the two five-dot sequences respectively stand for the list of adjacency lists of the vertices of the considered graph and the list of the maximal independence sets represented by lists of vertex numbers.
3.2 Compilation and execution

The command line instances (TPC, DCC32 and fpc, resp.) of the compilers have been used. In the case of fpc, the command line option -Mtp has been given (for compatibility with Turbo Pascal).

3.2.1 Without the optional data export

Actual compilations and executions have been done on two different computer systems:

**System 1:**
- 1 GHz Intel Pentium(R) III, running MS Windows 98SE.
- Compilers and execution times:
  - Turbo Pascal 5.5 : 2:12.81 min
  - Turbo Pascal 7.01 : 1:43.12 min
  - Delphi 4.0 build 5.37 : 5.00 s
  - Free Pascal 2.4.4 for i386 : 6.75 s

**System 2:**
- 2.8 GHz Intel Pentium(R) Dual-Core E5500, running (Linux distribution) Lubuntu 20.04 (64 bit).
- Compiler and execution time: Free Pascal 3.0.4 for x86_64 : 2.43 s

3.2.2 With the optional data export (compiler symbol GAP defined)

The command line option -dGAP should work for all mentioned compilers.

Actual compilations and executions have been done on system 2 as described above.

Compiler and execution time: Free Pascal 3.0.4 for x86_64 : 10.52 s

4 Remark

In August 2021, a few weeks after the appearance of version 1 of this article, Ivan Mogilnykh wrote to me that some time before he had together with his colleague Denis Krotov found two maximal independence sets of size 72 that are structurally different (from each other). He talked about (but did not publish) the first of those two sets at the Mal’tsev meeting 2020 (abstract in [4]). According to the eMails from I. Mogilnykh and D. Krotov that I received today, the calculation of the invariant used in my program to structurally distinguish the two sets of size 72 (giving 166 and 276, resp.) done for the sets found by them resulted in 336 and 166, resp. Thus, at least the first of their sets is structurally different from the two sets that I found.

References

[1] A. E. Brouwer, J. H. van Lint, *Strongly Regular Graphs and Partial Geometries*, Enumeration and design, pp. 85-122, Academic Press Canada, 1984

[2] A. E. Brouwer, H. van Maldeghem, *Strongly regular graphs*, Cambridge University Press, 2022

[3] X. L. Hubaut, *Strongly regular graphs*, Discrete Math., Vol. 13 (1975), 357-381.

[4] I. Y. Mogilnykh, *On codes with minimum distance 3 in the coset graph of the binary golay code*, Collection of abstracts of the Mal’tsev meeting 2020 (Novosibirsk State University), [http://math.nsc.ru/conference/malmeet/20/maltsiev20.pdf](http://math.nsc.ru/conference/malmeet/20/maltsiev20.pdf), page 52.

[5] Turbo Pascal versions 1.0, 3.02, and 5.5 (binaries only) [http://edn.embarcadero.com/museum/antiquesoftware](http://edn.embarcadero.com/museum/antiquesoftware)

For downloading one has to register or sign-in.
[6] *Free Pascal* (Open Source freeware)

Sources, documentation, and binaries for several systems

http://www.freepascal.org

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