Details of Second-Order Partial Derivatives of Rigid-Body Inverse Dynamics

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Abstract

This document provides full details of second-order partial derivatives of rigid-body inverse dynamics. Several properties and identities using an extension of Spatial Vector Algebra for tensorial use are listed, along with their detailed derivations. Using those, the expressions for second-order derivatives are derived step-by-step in detail. The expressions build upon previous work by the authors on first-order partial derivatives of inverse dynamics.

I. NOTATIONS, SYMBOLS AND ACRONYMS

Cartesian 3D vectors are denoted by lower-case letters with an overhead bar ($\bar{v}$). Spatial vectors are denoted by lower-case bold letters ($\mathbf{a}$). Matrices are denoted by upper-case bold letters ($\mathbf{M}$), while third-order tensors are denote by upper-case calligraphic letters ($\mathcal{A}$).

Acronyms:
1) RHS : Right-hand-side
2) w.r.t : with respect to
3) FO : first-order
4) SO : second-order
5) SVA : Spatial Vector Algebra

Section II is an introduction on Spatial Vector Algebra (SVA) \(^1\), also covered in Ref. [3, App A]. Section III provides an extension of SVA for spatial motion matrices, and spatial cross-product operators associated with them. Then, the properties of spatial matrices (numbered K1-K16) are listed and derived in detail. Then, Sections IV and V provide the derivations of SO partial derivatives of ID w.r.t $q$ and $\dot{q}$ respectively. Section VI provides the SO cross-derivatives w.r.t $q$ and $\dot{q}$, while Section VII gives the FO partial derivatives of $M(q)$ w.r.t $q$. At the end, Section VIII provides the matrix and vector form of expressions used in the Algorithm developed.

II. SPATIAL VECTOR ALGEBRA IDENTITIES AND PROPERTIES

A. Spatial Vector Algebra

A body $k$ with spatial velocity $^k\mathbf{v}_k \in M^6$ in the body frame is decomposed in its angular and linear components as:

$$^k\mathbf{v}_k = \begin{bmatrix} ^k\omega_k \\ ^k\mathbf{v}_k \end{bmatrix}$$

where $^k\omega_k \in \mathbb{R}^3$ is the angular velocity of the body in a coordinate frame fixed to the body, while $^k\mathbf{v}_k \in \mathbb{R}^3$ is the linear velocity of the body-fixed point at the origin of the body frame. Spatial vectors can also be expressed in the ground frame. For example, the spatial velocity of the body $k$ in the ground frame is denoted as $^0\mathbf{v}_k$. In this case, the linear velocity is associated with the body-fixed point on body $k$ that is coincident with the origin of

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the ground frame. The net spatial force $0f_k \in F^6$ defined in Eq. 2 on the body can be calculated from the spatial equation of motion (Eq. 3):

$$0f_k = \begin{bmatrix} 0 \bar{n}_k \\ 0 \bar{f}_k \end{bmatrix}$$ (2)

$$0f_k = 0I_k^0 a_k + 0v_k \times^* 0I_k^0 v_k$$ (3)

where $0\bar{n}_k \in \mathbb{R}^3$ is the net moment on the body about the origin of the ground frame, $0\bar{f}_k \in \mathbb{R}^3$ is the net linear force on body, $0I_k$ is the spatial inertia of the body $k$ that maps motion vectors to force vectors, and $0a_k \in M^6$ is the spatial acceleration of the body. The transformation matrix $iX_j$ is used to transform vectors in frame $j$ to frame $i$ is defined as:

$$iX_j = \begin{bmatrix} iR_j \\ -iR_j(\bar{p}_{i/j} \times) \end{bmatrix}$$ (4)

where $iR_j \in \mathbb{R}^{3\times3}$ is the rotation matrix from frame $j$ to frame $i$, $\bar{p}_{i/j} \in \mathbb{R}^3$ is the Cartesian vector from origin of frame $j$ to $i$, and $0$ is the $3 \times 3$ zero matrix. $\bar{p} \times$ is the 3D vector cross product on the elements of $\bar{p}$, defined as:

$$\bar{p} \times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$ (5)

The spatial transformation matrix $0X_k$ can be used to obtain spatial velocity vector $0v_k$ from the vector $k v_k$ as:

$$0v_k = 0X_k^k v_k$$ (6)

A spatial cross-product operator between two motion vectors $(v, u)$, written as $(v \times u)$, is given by (Eq. 7). This operation can be understood as providing the time rate of change of $u$, when $u$ is moving with a spatial velocity $v$. A spatial cross-product between a motion and a force vector is written as $(v \times^*)f$, and defined in Eq. 8:

$$v \times = \begin{bmatrix} \bar{\omega} \times & 0 \\ \bar{v} \times & \bar{\omega} \times \end{bmatrix}$$ (7)

$$v \times^* = \begin{bmatrix} \bar{\omega} \times & \bar{v} \times \\ 0 & \bar{\omega} \times \end{bmatrix}$$ (8)

An operator $\bar{\times}^*$ (Eq. [9]) is defined by swapping the order of the cross product, such that $(f \bar{\times}^*)v = (v \times^*)f$ [4].

$$f \bar{\times}^* = \begin{bmatrix} -\bar{n} \times & -\bar{f} \times \\ -\bar{f} \times & 0 \end{bmatrix}$$ (9)

Hence, the three spatial vector cross-products can be considered as binary operators using the above definitions to map between motion and force vector spaces as [11]:

$$\times : M^6 \times M^6 \rightarrow M^6$$

$$\times^* : M^6 \times F^6 \rightarrow F^6$$

$$\bar{\times}^* : F^6 \times M^6 \rightarrow F^6$$ (10)

### B. Kinematics

The spatial velocity of a body $i$ in a connectivity tree in given as:

$$v_i = \sum_{l \preceq i} v_{J_l}$$ (11)

where $v_{J_l}$ is the spatial joint velocity, given as $v_{J_l} = S_i \dot{q}_l$. The quantity $S_i$ is the joint motion sub-space matrix for a multi-DoF joint $l$ that precedes joint $i$ (i.e., $l \preceq i$) in the kinematic tree. The kinematic quantities $a_i$, $\gamma_i$, and $\xi_i$ are defined in Ref. [2] Sec III:

$$a_i = \sum_{l \preceq i} (S_i \dot{q}_l + v_l \times S_i \dot{q}_l) + a_0$$

$$\gamma_i = \sum_{l \preceq i} S_i \ddot{q}_l$$

$$\xi_i = \sum_{l \preceq i} v_l \times S_i \dot{q}_l$$ (12)
Two more spatial quantities $\hat{\Psi}_j$, and $\tilde{\Psi}_i$ defined in Ref. [2, Sec IV] are:

$$\hat{\Psi}_j = \dot{v}_{\lambda(j)} \times S_j$$
$$\tilde{\Psi}_i = a_{\lambda(j)} \times S_j + v_{\lambda(j)} \times \tilde{\Psi}_j$$  \hspace{1cm} (13)

A body-level Coriolis matrix $B_k$ is defined [2] as:

$$B_k = \frac{1}{2}((v_k \times) I_k - I_k (v_k \times) + (I_k v_k) \tilde{S})$$  \hspace{1cm} (14)

If in Eq. [14] instead of $v_k$, some other spatial quantity $m$ is used, then the body-Coriolis matrix is denoted with a square bracket as $B_k[m]$, and defined as:

$$B_k[m] = \frac{1}{2}((m \times) I_k - I_k (m \times) + (I_k m) \tilde{S})$$  \hspace{1cm} (15)

Composite term for $B_k[m]$ is $B^C_k[m] = \sum_{j \geq k} B_i[m]$.

C. Dynamics

The generalized force vector on a joint $i$ can be calculated as:

$$\tau_i = S_i^T f^C_i$$  \hspace{1cm} (16)

where, $f^C_i = \sum_{k \geq i} f_k$ is the net spatial force transmitted across joint $i$. The first-order partial derivatives of $\tau_i$ with respect to joint configuration $(q)$ and velocity $(\dot{q})$ for the case $j \leq i$ were derived in Ref. [2, Sec. IV] as:

$$\frac{\partial \tau_i}{\partial q_j} = S_j^T \left[ 2B_i^C \hat{\Psi}_j + S_i^T I_i^C \tilde{\Psi}_j \right]$$  \hspace{1cm} (17)

$$\frac{\partial \tau_i}{\partial \dot{q}_j} = S_j^T \left[ 2B_i^C \hat{\Psi}_j + I_i^C (\dot{\Psi}_j + \dot{S}_j) \right] (j \neq i)$$  \hspace{1cm} (18)

$$\frac{\partial \tau_i}{\partial q_j} = S_j^T \left[ 2B_i^C S_j + I_i^C (\dot{\Psi}_j + \dot{S}_j) \right] (j \neq i)$$  \hspace{1cm} (19)

$$\frac{\partial \tau_i}{\partial \dot{q}_j} = S_j^T \left[ 2B_i^C S_j + I_i^C (\dot{\Psi}_j + \dot{S}_j) \right] (j \neq i)$$  \hspace{1cm} (20)

where, the quantities $I_i^C = \sum_{k \geq i} I_k$, $B^C_i = \sum_{k \geq i} B_k$ are the quantities defined for an entire sub-tree.

D. Properties of Spatial Vectors

Assuming $u, v, m \in M^6$, and $f \in F^6$, many spatial vector properties [11] are utilized herein:

P1. $u \times v = -v \times u$

P2. $(v \times m) \times = (v \times (m \times)) - (m \times (v \times))$

P3. $(v \times m)^* = (v^* \times (m^*)) - (m^* \times (v^*))$

P4. $(v^* \times f) \tilde{S}^* = (v^* \times (f \tilde{S}^*)) - (f \tilde{S}^* \times (v^*))$

P5. $(u \times v)^T f = -v^T (u \times f)$

P6. $(u \times f)^T \dot{v} = -\dot{f}^T (u \times v)$

P7. $u^T (v^* \times f) = f^T (u \times v)$

P8. $(u \times v)^T \dot{u} = -\dot{v}^T (u \times u)$

P9. $(u \times f)^T \dot{u} = -\dot{f}^T u \times$

P10. $u \times v \times m = u \times (v \times m)$

E. Identities by Perturbing All DoFs of a multi-DoF Joint

Ref. [2] provides the following identities with their derivations in Ref. [3]. These give partial derivatives of some kinematic and dynamic identities by perturbing the full configuration $(q)$ or velocity vector $(\dot{q})$ of a multi-DoF joint.

J1. $\frac{\partial s}{\partial q_j} = \begin{cases} s_{j,p} \times S_i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
Another rotation is defined as the combination of \(B\) allows taking a cross-product-like operation \(V\) and defined by carrying out the usual spatial cross-product operator \(\mathcal{Y}\) tensor, denoted as \(V\) provides cross products.

For any tensor \(A\), the motion space \(M\) is extended to a space of spatial-motion matrices \(M^{6 \times n}\), where each column of such a matrix is a usual spatial-motion vector. For any \(U \in M^{6 \times n}\) a new spatial cross-product operator \(U \tilde{\times}\) is considered, and defined by carrying out the usual spatial cross-product operator \(\times\) on each column of \(U\). The result is a third-order tensor in \(\mathbb{R}^{6 \times 6 \times n}\), where each \(6 \times 6\) matrix in the 1-2 dimension is a result of the spatial cross-product operator on a column of \(U\).

Given two spatial motion matrices, \(V \in M^{6 \times n_v}\) and \(U \in M^{6 \times n_u}\), we can now define a cross-product operation between them as \((V \tilde{\times})U \in M^{6 \times n_u \times n_v}\) via a tensor-matrix product. Such an operation, denoted as \(Z = AB\) is defined as:

\[
Z_{i,j,k} = \sum_{\ell} A_{i,\ell,k} B_{\ell,j}
\]  

(21)

defined as:

\[
\Psi_j = \frac{\partial}{\partial q_j} \mathcal{Y}_{i,j,\ell} = \sum_{\ell} B_{i,\ell,k} A_{\ell,j,k}
\]

(22)

III. TENSORIAL IDENTITIES AND PROPERTIES

Two types of tensor rotations are defined for this paper:

1) \(A^\top\): Transpose along the 1-2 dimension. This operation can also be understood as the usual matrix transpose of each matrix moving along the third dimension in the tensor. If \(A^\top = B\), then \(A_{i,j,k} = B_{j,i,k}\).

2) \(A^{R}\): Rotation of elements along the 2-3 dimension. If \(A^{R} = B\), then \(A_{i,j,k} = B_{i,k,j}\).

Another rotation is defined as the combination of \((\tilde{R})\) followed by \((\mathcal{Y})\). For example, if \(A^{\tilde{R},\mathcal{Y}} = B\), then \(A_{i,j,k} = B_{k,i,j}\).
A. Properties of Spatial Matrices

Some additional properties using the operators defined above are as follows. These properties are a natural extension of the spatial vector properties from Sec. I.4. Assuming $v \in \mathbb{M}^6, f \in \mathbb{F}^6, U \in \mathbb{M}^{6 \times n}, F \in \mathbb{F}^{6 \times n}, V \in \mathbb{M}^{6 \times l}, B \in \mathbb{R}^{n_1 \times n_2}, Y \in \mathbb{R}^{n_2 \times n_3 \times n_4}$.

- $U \hat{\times} = -(U \hat{\times})^\top$
- $-V^T (U \hat{\times}^*) = (U \hat{\times}V)^\top$
- $-V^T (U \hat{\times}^*) F = (U \hat{\times} V)^\top F$
- $(U \hat{\times} v) = -v \times U$
- $U \hat{\times} F = (F \hat{\times} U)^\top$
- $F \hat{\times} U = (U \hat{\times} F)^\top$
- $(\lambda U) \hat{\times} = \lambda (U \hat{\times})$
- $U \hat{\times} V = -(V \hat{\times} U)^\top$
- $(v \times U) \hat{\times} = v \times U \hat{\times} - U \hat{\times} v \times$
- $(v \times U) \hat{\times}^* = v \times U \hat{\times}^* - U \hat{\times}^* v \times$
- $(U \hat{\times} v) \hat{\times}^* = U \hat{\times} v \hat{\times}^* - v \hat{\times}^* U \hat{\times}$
- $(U \hat{\times}^* F)^\top = -F^T (U \hat{\times})$
- $V^T (U \hat{\times}^* F) = (V \hat{\times} U)^\top F = (F^T (V \hat{\times} U)^\top)^\top$
- $v \hat{\times}^* F = F \hat{\times}^* v$
- $f \hat{\times} U = U \hat{\times}^* f$
- $V^T (U \hat{\times}^* F)^\top = ((V \hat{\times} U)^\top)^\top F$
- $V^T (U \hat{\times}^* F)^\top = [-U^T (V \hat{\times}^* F)^\top]^\top$
- $V^T (U \hat{\times}^* F)^\top = (V^T (U \hat{\times}^* F))^\top$
- $(B Y)^\top = Y^\top B^\top$

Using the properties and spatial matrix operators defined above, some more identities are derived as follows:

- $\frac{\partial S_i}{\partial q_j} = \begin{cases} S_j \hat{\times} S_i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial S_i}{\partial q_j} = \begin{cases} \Psi_j \hat{\times} S_i + S_j \hat{\times} \dot{S}_i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial (v_j \times S_j)}{\partial q_i} = \begin{cases} S_j \hat{\times} (v_j \times S_j), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial \Psi_j}{\partial q_i} = \begin{cases} \dot{\Psi}_j \times S_i + S_j \hat{\times} S_i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial I_i}{\partial q_j} = \begin{cases} S_j \hat{\times}^* I_i - I_i (S_j \hat{\times}), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial I^C_i}{\partial q_j} = \begin{cases} S_j \hat{\times}^* I^C_i - I^C_i (S_j \hat{\times}), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial a_i}{\partial q_j} = \begin{cases} \dot{\Psi}_j - v_i \times \dot{\Psi}_j - a_i \times S_j, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial (I_i a_i)}{\partial q_j} = \begin{cases} S_j \hat{\times}^* (I_i a_i) + I_i (v_i \times \dot{\Psi}_j), & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
- $\frac{\partial \psi_j}{\partial q_i} = \begin{cases} \dot{\Psi}_j \times S_i + 2 \Psi_j \hat{\times} \dot{\Psi}_i + S_j \hat{\times} \dot{\Psi}_i, & \text{if } j \leq i \\ 0, & \text{otherwise} \end{cases}$
The identities above are derived in detail as follows.

K1 From identity J1, the directional derivative of joint motion subspace matrix \( S_i \) along the \( p^{th} \) free-mode of joint \( j \) is:

\[
\frac{\partial S_i}{\partial q_{j,p}} = s_{j,p} \times S_i \tag{23}
\]

Collectively for each dimension \( p \) for joint \( j \):

\[
\frac{\partial S_i}{\partial q_j} = S_j \times S_i \tag{24}
\]

K2 For the partial derivative of \( \dot{S}_i \) w.r.t \( q_j \), we use the definition of \( \dot{S}_i \) and the product rule for \( j \leq i \) as:

\[
\frac{\partial \dot{S}_i}{\partial q_j} = \frac{\partial v_i}{\partial q_j} \times S_i + v_i \times \frac{\partial S_i}{\partial q_j} \tag{25}
\]

The directional partial derivative of the spatial velocity of a body \( i \), \( v_i \) with respect to \( q_{j,p} \), where \( j \leq i \) is given as:

\[
\frac{\partial v_i}{\partial q_{j,p}} = \sum_{j \leq l \leq i} \frac{\partial S_l}{\partial q_{j,p}} q_l \tag{26}
\]

Using J1:

\[
\frac{\partial v_i}{\partial q_{j,p}} = \sum_{j \leq l \leq i} s_{j,p} \times S_i \dot{q}_l \tag{27}
\]

Using P1:

\[
\frac{\partial v_i}{\partial q_{j,p}} = -\sum_{j \leq l \leq i} S_i \dot{q}_l \times s_{j,p} \tag{28}
\]

Eq \( 28 \) is written for all DoFs collectively as:

\[
\frac{\partial v_i}{\partial q_j} = -\sum_{j \leq l \leq i} S_l \dot{q}_l \times S_j \tag{29}
\]

Using the definition of \( v_i \):
\[
\frac{\partial v_i}{\partial q_j} = (v_{\lambda(j)} - v_i) \times S_j
\]  
(30)

Using Eq. 30 and K1

\[
\frac{\partial \dot{S}_i}{\partial q_j} = ((v_{\lambda(j)} - v_i) \times S_j) \times S_i + v_i \times (S_j \times S_i)
\]  
(31)

Expanding terms, and using the definition of \(\dot{\Psi}_j\) (Eq. 13),

\[
\frac{\partial \dot{S}_i}{\partial q_j} = (\dot{\Psi}_j - v_i \times S_j) \times S_i + v_i \times (S_j \times S_i)
\]  
(32)

Using M9 and property P10:

\[
\frac{\partial \dot{S}_i}{\partial q_j} = \dot{\Psi}_j \times S_j + S_j \times \dot{S}_i
\]  
(33)

Cancelling terms and using the definition of \(\dot{S}_i\):

\[
\frac{\partial \dot{S}_i}{\partial q_j} = \dot{\Psi}_j \times S_i + S_j \times \dot{S}_i
\]  
(34)

K3 Using the product rule:

\[
\frac{\partial (v_{J_i} \times S_i)}{\partial q_j} = \left( \frac{\partial v_{J_i}}{\partial q_j} \right) \times S_i + v_{J_i} \times \frac{\partial S_i}{\partial q_j}
\]  
(35)

The partial derivative of the joint velocity \(v_{J_i}\) with respect to \(q_j\) for \(j \leq i\) is now calculated. The directional derivative of \(v_{J_i}\) along the \(p^{th}\) free-mode of joint \(j\) is:

\[
\frac{\partial v_{J_i}}{\partial q_{j,p}} = \frac{\partial (S_i \dot{q}_i)}{\partial q_{j,p}}
\]  
(36)

Using J1:

\[
\frac{\partial v_{J_i}}{\partial q_{j,p}} = s_{j,p} \times S_i \dot{q}_i
\]  
(37)

Using property P1 (Sec. II.4):

\[
\frac{\partial v_{J_i}}{\partial q_{j,p}} = -v_{J_i} \times s_{j,p}
\]  
(38)

Collectively, for all DoFs of joint \(j\):

\[
\frac{\partial v_{J_i}}{\partial q_j} = -v_{J_i} \times S_j
\]  
(39)

Using Eq. 39 and K1 in Eq. 35

\[
\frac{\partial (v_{J_i} \times S_i)}{\partial q_j} = -(v_{J_i} \times S_j) \times S_i + v_{J_i} \times (S_j \times S_i)
\]  
(40)

Using property M9

\[
\frac{\partial (v_{J_i} \times S_i)}{\partial q_j} = -v_{J_i} \times S_j \times S_i + S_j \times v_{J_i} \times S_i + v_{J_i} \times (S_j \times S_i)
\]  
(41)

Cancelling terms:

\[
\frac{\partial (v_{J_i} \times S_i)}{\partial q_j} = S_j \times (v_{J_i} \times S_i)
\]  
(42)

K4 Using the definition of \(\dot{\Psi}_i\) (Eq. 13):

\[
\frac{\partial \dot{\Psi}_i}{\partial q_j} = \frac{\partial (\dot{S}_i - v_{J_i} \times S_i)}{\partial q_j}
\]  
(43)
Using identities $K2$ and $K3$ for $j \preceq i$:

$$\frac{\partial \Psi_i}{\partial q_j} = \dot{\Psi}_j \times S_i + S_j \times \dot{S}_i - S_j \times (v_{j_i} \times S_i)$$  \hfill (44)

Combining the second and third terms on the RHS, and using the definition of $\dot{\Psi}_i$ (Eq. 13):

$$\frac{\partial \Psi_i}{\partial q_j} = \dot{\Psi}_j \times S_i + S_j \times \dot{\Psi}_i$$  \hfill (45)

The directional partial derivative of $I_i$ w.r.t to the $p^{th}$ free-mode of a joint $j$ is given by (1):

$$\frac{\partial I_i}{\partial q_{j,p}} = s_{j,p} \times I_i - I_i (s_{j,p} \times)$$  \hfill (46)

Collectively, for all free modes $p$ of the joint $j$, the partial derivative of $I_i$ wrt $q_j$ can be written as:

$$\frac{\partial I_i}{\partial q_j} = S_j \times I_i - I_i (S_j \times)$$  \hfill (47)

K6 Consider first the case when $i \succeq j$. Using K5, we have:

$$\frac{\partial I_i}{\partial q_j} = S_j \times I_i - I_i (S_j \times)$$  \hfill (48)

Taking the partial derivative of $I^C_i$:

$$\frac{\partial I^C_i}{\partial q_j} = \frac{\partial}{\partial q_j} \sum_{k \succeq i} (S_j \times I_k - I_k (S_j \times))$$  \hfill (49)

Summing over $k$:

$$\frac{\partial I^C_i}{\partial q_j} = S_j \times I^C_i - I^C_i (S_j \times)$$  \hfill (50)

For the case $j \succ i$:

$$\frac{\partial I^C_i}{\partial q_j} = \sum_{k \succeq j} (S_j \times I_k - I_k (S_j \times))$$  \hfill (51)

Summing over $k$:

$$\frac{\partial I^C_i}{\partial q_j} = S_j \times I^C_j - I^C_j (S_j \times)$$  \hfill (52)

K7 Using the definition of $\alpha_i = \gamma_i + \xi_i + a_{gi}$ (See Ref. 1), and taking the partial derivative wrt $q_j$:

$$\frac{\partial \alpha_i}{\partial q_j} = \frac{\partial \gamma_i}{\partial q_j} + \frac{\partial \xi_i}{\partial q_j} + \frac{\partial a_{gi}}{\partial q_j}$$  \hfill (53)

The quantity $a_{gi}$ is constant, it results into a zero partial derivative. Hence,

$$\frac{\partial \alpha_i}{\partial q_j} = \frac{\partial \gamma_i}{\partial q_j} + \frac{\partial \xi_i}{\partial q_j}$$  \hfill (54)

Using J6, and J7, for $j \preceq i$:

$$\frac{\partial \alpha_i}{\partial q_j} = (v_{\lambda(j)} - v_i) \times \dot{\Psi}_j + (\xi_{\lambda(j)} - \xi_i) \times S_j + (\gamma_{\lambda(j)} - \gamma_i) \times S_j$$  \hfill (55)

Combining second and the third terms:
\[ \frac{\partial a_i}{\partial q_j} = (v \xi_i - v_{\xi_i}) \times \Psi_j + (\xi_{\lambda(j)} + \gamma_{\lambda(j)}) - (\xi_i + \gamma_i) \times S_j \] (58)

Since \( a_{g_i} = a_{g_{\lambda(j)}} = a_g \), adding \( a_{g_i} \) and subtracting \( a_{g_i} \):

\[ \frac{\partial a_i}{\partial q_j} = (v \lambda(j) - v_i) \times \Psi_j + (\xi \lambda(j) + \gamma \lambda(j) + a_{g_{\lambda(j)}} - (\xi_i + \gamma_i + a_{g_i})) \times S_j \] (59)

Using the definition of \( a_{\lambda(j)} \) and \( a_i \):

\[ \frac{\partial a_i}{\partial q_j} = (v \lambda(j) - v_i) \times \Psi_j + (a_{\lambda(j)} - a_i) \times S_j \] (60)

Using the definition of \( \Psi_j \) (Eq. [13]), and simplifying:

\[ \frac{\partial a_i}{\partial q_j} = \Psi_j - v_i \times \Psi_j - a_i \times S_j \] (61)

Using the product rule of differentiation to take the partial derivative of \( I_i a_i \) wrt \( q_j \) as:

\[ \frac{\partial (I_i a_i)}{\partial q_j} = \left( \frac{\partial I_i}{\partial q_j} \right) a_i + I_i \left( \frac{\partial a_i}{\partial q_j} \right) \] (62)

Using [K5] and [K7] as:

\[ \frac{\partial (I_i a_i)}{\partial q_j} = (S_j \times I_i - I_i (S_j \times)) a_i + I_i (\Psi_j - v_i \times \Psi_j - a_i \times S_j) \] (63)

Expanding:

\[ \frac{\partial (I_i a_i)}{\partial q_j} = S_j \times I_i a_i - I_i (S_j \times) a_i + I_i \Psi_j - I_i v_i \times \Psi_j - I_i a_i \times S_j \] (64)

Using the property [M4] in the second term on RHS:

\[ \frac{\partial (I_i a_i)}{\partial q_j} = S_j \times I_i a_i + I_i (a_i \times S_j) + I_i \Psi_j - I_i v_i \times \Psi_j - I_i a_i \times S_j \] (65)

Cancelling terms:

\[ \frac{\partial (I_i a_i)}{\partial q_j} = S_j \times (I_i a_i) + I_i \Psi_j - I_i (v_i \times \Psi_j) \] (66)

Taking the partial derivative of \( \Psi_i \) (Eq. [13]) and using the product rule results in:

\[ \frac{\partial \Psi_i}{\partial q_j} = \left( \frac{\partial a_{\lambda(i)}}{\partial q_j} \right) \times S_i + a_{\lambda(i)} \times \left( \frac{\partial S_i}{\partial q_j} \right) + \left( \frac{\partial v_{\lambda(i)}}{\partial q_j} \right) \times \Psi_i + v_{\lambda(i)} \times \left( \frac{\partial \Psi_i}{\partial q_j} \right) \] (67)

Using [K7] for the first term, [K1] for the second term, Eq. [30] for the third term, and [K4] for the last term results in:

\[ \frac{\partial \Psi_i}{\partial q_j} = (\Psi_j - v_{\lambda(i)} \times \Psi_j - a_{\lambda(i)} \times S_j) \times S_i + a_{\lambda(i)} \times (S_j \times S_i) + \left( v_{\lambda(i)} \times S_i \right) \times \Psi_i + v_{\lambda(i)} \times (\Psi_j \times S_i + S_j \times \Psi_i) \] (68)

Expanding terms using the property [M9] repeatedly, and using the definition of \( \Psi_j \) (Eq. [13]):

\[ \frac{\partial \Psi_j}{\partial q_j} = \Psi_j \times S_i + v_{\lambda(i)} \times S_i + \Psi_j \times v_{\lambda(i)} \times S_i - a_{\lambda(i)} \times S_j \times S_i + S_j \times a_{\lambda(i)} \times S_i \]
\[ + a_{\lambda(i)} \times S_j \times S_i + v_{\lambda(i)} \times S_j \times \Psi_i + S_j \times v_{\lambda(i)} \times \Psi_i \]
\[ + v_{\lambda(i)} \times \Psi_j \times S_i + v_{\lambda(i)} \times S_j \times \Psi_i \] (69)

Cancelling terms:
\[
\frac{\partial \dot{\Psi}_i}{\partial q_j} = \dot{\Psi}_j \times S_i + \dot{\Psi}_j \times v_{\lambda(i)} \times S_i + S_j \times a_{\lambda(i)} \times S_i
\]

\[
+ \dot{\Psi}_j \times \dot{\Psi}_i + S_j \times v_{\lambda(i)} \times \dot{\Psi}_i
\]

(70)

Using the definition of \( \dot{\Psi}_i \) (Eq. 13) and collecting terms:

\[
\frac{\partial \dot{\Psi}_i}{\partial q_j} = \dot{\Psi}_j \times S_i + 2\dot{\dot{\Psi}}_j \times \dot{\Psi}_i + S_j \times \dot{a}_{\lambda(i)} \times S_i
\]

(71)

Using the definition of \( \dot{\Psi}_i \) (Eq. 13) results in:

\[
\frac{\partial \ddot{\Psi}_i}{\partial q_j} = \dot{\Psi}_j \times S_i + 2\dot{\dot{\Psi}}_j \times \dot{\Psi}_i + S_j \times \dot{\dot{\Psi}}_i
\]

(72)

**K10** We consider first the case when \( i \geq j \). Using the definition of \( B_i \) (see Ref. 2), we know: where \( B_i^C = \sum_{k \geq i} B_k \).

For the case, \( j \leq i \), taking partial derivative wrt \( q_j \)

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \frac{\partial (v_k \times^s) I_k}{\partial q_j} - \frac{\partial I_k (v_k \times^s)}{\partial q_j} + \frac{\partial (I_k v_k) \bar{v}^s}{\partial q_j} \right)
\]

(73)

Using the product rule of differentiation:

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \frac{\partial v_k}{\partial q_j} \times I_k + v_k \times^s \frac{\partial I_k}{\partial q_j} - \frac{\partial I_k v_k}{\partial q_j} \times -I_k \left( \frac{\partial v_k}{\partial q_j} \right) \times + \left( \frac{\partial I_k v_k}{\partial q_j} + I_k \frac{\partial v_k}{\partial q_j} \right) \bar{v}^s \right)
\]

(74)

Using Eq. 30 and K5

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \left( (v_{\lambda(j)} - v_k) \times S_j \right) \times I_k + v_k \times^s \left( S_j \times I_k - I_k (S_j \times \bar{v}) \right) \right)
\]

\[
- \left( (S_j \times I_k - I_k (S_j \times \bar{v})) v_k \times -I_k ((v_{\lambda(j)} - v_k) \times S_j) \bar{v} + \right)
\]

\[
(S_j \times I_k v_k + I_k ((v_{\lambda(j)} - v_k) \times S_j)) \bar{v}^s \right)
\]

(75)

Using the definition of \( \dot{\Psi}_j \) (Eq. 13), and expanding terms:

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \left( (\dot{\Psi}_j - v_k \times S_j) \times I_k + v_k \times^s \left( S_j \times I_k - v_k \times^s I_k (S_j \times \bar{v}) \right) \right)
\]

\[
- \left( S_j \times I_k v_k \times + I_k (S_j \times \bar{v}) v_k \times -I_k (\dot{\Psi}_j - v_k \times S_j) \bar{v} + \right)
\]

\[
(S_j \times I_k v_k - I_k (S_j \times \bar{v}) v_k + I_k (\dot{\Psi}_j - I_k v_k) \times S_j) \bar{v}^s \right)
\]

(76)

using properties M9 and M10 to expand terms:

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \left( (\dot{\Psi}_j \times I_k - v_k \times S_j \times I_k + S_j \times v_k \times^s I_k - v_k \times^s I_k (S_j \times \bar{v}) \right)
\]

\[
- \left( S_j \times I_k v_k \times + I_k (S_j \times \bar{v}) v_k \times -I_k (\dot{\Psi}_j - v_k \times S_j) \bar{v} + \right)
\]

\[
(S_j \times S_j v_k - I_k (S_j \times \bar{v}) v_k + I_k (\dot{\Psi}_j - I_k v_k) \times S_j) \bar{v}^s \right)
\]

(77)

Cancelling terms, and using property M4, M11

\[
\frac{\partial B_i^C}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \dot{\Psi}_j \times I_k + S_j \times v_k \times^s I_k - v_k \times^s I_k (S_j \times \bar{v}) \right)
\]

\[
- \left( S_j \times I_k v_k \times -I_k \dot{\Psi}_j \bar{v} + I_k v_k \times S_j \bar{v} + \right)
\]

\[
S_j \times (I_k v_k) \bar{v}^s - \left( I_k v_k \right) \bar{v}^s \bar{v} + \left( I_k \dot{\Psi}_j \right) \bar{v}^s \right)
\]

(78)
Collecting the $\dot{\Psi}_j$ terms:

$$\frac{\partial B^C_i}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \dot{\Psi}_j \times (\dot{I}_k \times \dot{I}_k) - (\dot{I}_k \times \dot{I}_k) \dot{\Psi}_j \right)$$

$$+ S_j \times (\dot{v}_k \times \dot{v}_k) - (\dot{v}_k \times \dot{v}_k) S_j$$

Using the definition of $B^C_i$:

$$\frac{\partial B^C_i}{\partial q_j} = B^C_i [\dot{\Psi}_j] + S_j \times B^C_i S_j$$

(79)

where $B^C_i [\dot{\Psi}_j] = \sum_{k \geq i} B_k [\dot{\Psi}_j]$, with $B_k [\dot{\Psi}_j]$:

$$B_k [\dot{\Psi}_j] = \frac{1}{2} \left( (\dot{\Psi}_j \times \dot{I}_k) - (\dot{I}_k \times \dot{\Psi}_j) \right)$$

(80)

(81)

For the case, $j > i$, taking partial derivative wrt $q_j$

$$\frac{\partial B^C_i}{\partial q_j} = \frac{1}{2} \sum_{k \geq i} \left( \frac{\partial (\dot{v}_k \times \dot{v}_k)}{\partial q_j} - \frac{\partial (\dot{I}_k \times \dot{v}_k)}{\partial q_j} + \frac{\partial (\dot{I}_k \times \dot{I}_k)}{\partial q_j} \right)$$

(82)

Similar steps for this case results in:

$$\frac{\partial B^C_i}{\partial q_j} = B^C_i [\dot{\Psi}_j] + S_j \times B^C_i S_j$$

(83)

Using the definition of $f_i$ (Eq. 3) to take the partial derivative as:

$$\frac{\partial f_i}{\partial q_j} = \left( \frac{\partial I_i a_i}{\partial q_j} \right) + \left( \frac{\partial v_i}{\partial q_j} \right) \times I_i v_i + v_i \times \left( \frac{\partial I_i v_i}{\partial q_j} \right)$$

(84)

Using J5, Eq. 50 and K8

$$\frac{\partial f_i}{\partial q_j} = S_j \times \left( I_i a_i + I_i \dot{\Psi}_j - I_i (v_i \times \dot{\Psi}_j) \right) + (v_i - \dot{v}_i) S_j \times I_i v_i +$$

$$v_i \times \left( (I_i v_i) \times S_j + v_i \times I_i \dot{\Psi}_j \right)$$

(85)

Using the definition of $\dot{\Psi}_j$ (Eq. 13), and expanding terms:

$$\frac{\partial f_i}{\partial q_j} = S_j \times \left( I_i a_i + I_i \dot{\Psi}_j - I_i (v_i \times \dot{\Psi}_j) \right) + (\dot{\Psi}_j - v_i \times S_j) \times I_i v_i +$$

$$v_i \times \left( (I_i v_i) \times S_j + v_i \times I_i \dot{\Psi}_j \right)$$

(86)

Using the property M10 to expand terms:

$$\frac{\partial f_i}{\partial q_j} = S_j \times \left( I_i a_i + I_i \dot{\Psi}_j - I_i (v_i \times \dot{\Psi}_j) \right) + \dot{\Psi}_j \times I_i v_i - v_i \times S_j \times I_i v_i +$$

$$v_i \times \left( (I_i v_i) \times S_j + v_i \times I_i \dot{\Psi}_j \right)$$

(87)

Using property M13 so that $(I_i v_i) \times S_j = S_j \times (I_i v_i)$ to get:

$$\frac{\partial f_i}{\partial q_j} = S_j \times \left( I_i a_i + I_i \dot{\Psi}_j - I_i (v_i \times \dot{\Psi}_j) \right) + \dot{\Psi}_j \times I_i v_i - v_i \times S_j \times I_i v_i + S_j \times v_i \times I_i v_i +$$

$$v_i \times \left( S_j \times (I_i v_i) + v_i \times I_i \dot{\Psi}_j \right)$$

(88)

Cancelling terms:

$$\frac{\partial f_i}{\partial q_j} = S_j \times \left( I_i a_i + I_i \dot{\Psi}_j - I_i (v_i \times \dot{\Psi}_j) \right) + \dot{\Psi}_j \times I_i v_i + S_j \times v_i \times I_i v_i + v_i \times I_i \dot{\Psi}_j$$

(89)

Using M15 again for the fourth term on RHS, and collecting terms:
\[ \frac{\partial f_i}{\partial q_j} = S_j \dot{x}^* (I_i a_i + v_i \times I_i v_i) + I_i \ddot{q} + (I_i v_i) \dot{\vec{\psi}}_j + (I_i v_i) \ddot{v} + I_i \dot{v}_i \]  

(90)

Using the definition of \( f_i \):
\[ \frac{\partial f_i}{\partial q_j} = S_j \dot{x}^* f_i + I_i \ddot{q} + \left( -\frac{1}{2} I_i (v_i \times) + \frac{1}{2} (I_i v_i) \dot{\vec{\psi}}_j + \frac{1}{2} v_i \times I_i \right) \dot{\vec{\psi}}_j \]  

(91)

Using the definition of \( B_i \) to get:
\[ \frac{\partial f_i}{\partial q_j} = I_i \ddot{q} + S_j \dot{x}^* f_i + 2B_i \dot{\vec{\psi}}_j \]  

(92)

**K12** This identity follows directly from **K11** using the definition of \( f^C_i = \sum_{k \geq i} f_k \) and taking the partial derivative for the case \( j \leq i \):
\[ \frac{\partial f^C_i}{\partial q_j} = \sum_{k \geq i} \left( \frac{\partial f_k}{\partial q_j} \right) \]  

(93)

Using **K11**
\[ \frac{\partial f^C_i}{\partial q_j} = \sum_{k \geq i} (I_k \ddot{q} + S_j \dot{x}^* f_k + 2B_k \dot{\vec{\psi}}_j) \]  

(94)

Summing over the index \( k \):
\[ \frac{\partial f^C_i}{\partial q_j} = I_i \ddot{q} + S_j \dot{x}^* f_i + 2B_i \dot{\vec{\psi}}_j \]  

(95)

For the case \( j > i \), we follow similar steps to get:
\[ \frac{\partial f^C_i}{\partial q_j} = I_j \ddot{q} + S_j \dot{x}^* f_j + 2B_j \dot{\vec{\psi}}_j \]  

(96)

**K13** Using **K1** to take the partial derivative of \( S_i^\top \):
\[ \frac{\partial S_i^\top}{\partial q_j} = (S_j \dot{x}^* S_i)^\top \]  

(97)

Using the property **M1** to get:
\[ \frac{\partial S_i^\top}{\partial q_j} = -S_i^\top S_j \dot{x}^* \]  

(98)

**K14** Using the definition of \( \dot{S}_i = v_i \times S_i \), and taking the partial derivative w.r.t \( q_j \) as:
\[ \frac{\partial \dot{S}_i}{\partial q_j} = \frac{\partial (v_i \times S_i)}{\partial q_j} \]  

(99)

Using identity J8 results for \( j \leq i \) in:
\[ \frac{\partial \dot{S}_i}{\partial q_j} = S_j \dot{x}^* S_i \]  

(100)

**K15** Using the definition of \( \dot{\vec{\psi}}_i = (v_i - v_{j_i}) \times S_i \):
\[ \frac{\partial \dot{\vec{\psi}}_i}{\partial q_j} = \left( \frac{\partial v_i}{\partial q_j} - \frac{\partial v_{j_i}}{\partial q_j} \right) \times S_i \]  

(101)

Using J8 for the first term for \( j \leq i \), and the definition of \( v_{j_i} = S_i \dot{q}_i \) for the second term:
\[ \frac{\partial \dot{\vec{\psi}}_i}{\partial q_j} = \left( \frac{\partial v_i}{\partial q_j} - \frac{\partial v_{j_i}}{\partial q_j} \right) \times S_i \]  

(102)
For \( j = i \), the expression results in 0 since
\[
\frac{\partial v_j}{\partial q_j} = \frac{\partial v_j}{\partial q_j} = S_j
\]

But for the case \( j < i \), the expression is:
\[
\frac{\partial \psi_i}{\partial q_j} = S_j \times S_i
\]  
(103)

Using the definition of \( B^C_i \) (Eq. 14) and taking the partial derivative w.r.t \( \dot{q}_j \) for the case \( j \leq i \):
\[
\frac{\partial B^C_i}{\partial q_j} = \frac{1}{2} \sum_{l \succeq i} \left( \frac{\partial (v_l \times^*)}{\partial q_j} I_l - \frac{\partial I_l (v_l \times)}{\partial q_j} \right. + \left. \frac{\partial (I_l v_l) \times^*}{\partial q_k} \right)
\]  
(104)

Using J8:
\[
\frac{\partial B^C_i}{\partial q_j} = \frac{1}{2} \sum_{l \succeq i} \left( S_j \times^* I_l - I_l (S_j \times^*) + (I_l S_j) \times^* \right)
\]  
(105)

Summing over the index \( l \) results in:
\[
\frac{\partial B^C_i}{\partial q_j} = B^C_i[S_j]
\]  
(106)

where \( B^C_i[S_j] = \sum_{l \succeq i} B_i[S_j] \), and \( B_i[S_j] \) is given as:
\[
B_i[S_j] = \frac{1}{2} \left( (S_j \times^*) I_l - I_l (S_j \times) + (I_l S_j) \times^* \right)
\]  
(107)

For the case with \( j > i \), we follow the similar procedure to get:
\[
\frac{\partial B^C_i}{\partial q_j} = B^C_j[S_j]
\]  
(108)
IV. SECOND ORDER PARTIAL DERIVATIVES OF ID WRT $q$

We take the partial derivative of Eq. [17] and Eq. [18] w.r.t. $q_k$ where the index $k$ results in three cases $k \leq j \leq i$ (A), $j < k < i$ (B) and $j > i > k$ (C). We individually solve for the three cases as follows:

A. $k \leq j \leq i$

1) 1A

Taking the partial derivative of Eq. [17] w.r.t. $q_k$:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \frac{\partial (S_i^T [2B_i^C] \dot{\Psi}_j)}{\partial q_k} + \frac{\partial (S_i^T I_i^C \dot{\Psi}_j)}{\partial q_k}
$$

(109)

Considering the first term in Eq. [109] and using the product rule of differentiation:

$$
\frac{\partial (S_i^T [2B_i^C] \dot{\Psi}_j)}{\partial q_k} = 2 \left( \frac{\partial S_i^T}{\partial q_k} \right) B_i^C \dot{\Psi}_j + 2 S_i^T \left( \frac{\partial B_i^C}{\partial q_k} \dot{\Psi}_j + B_i^C \frac{\partial \dot{\Psi}_j}{\partial q_k} \right)
$$

(110)

Using [K4] [K10] and [K13]

$$
\frac{\partial (S_i^T [2B_i^C] \dot{\Psi}_j)}{\partial q_k} = 2 \left( -S_i^T S_k \times^* B_i^C \dot{\Psi}_j + 2 S_i^T (B_i^C [\dot{\Psi}_k] + S_k \times^* B_i^C - B_i^C (S_k \times)) \dot{\Psi}_j + 2 S_i^T B_i^C (\dot{\Psi}_k \times S_j + S_k \times \dot{\Psi}_j) \right)
$$

(111)

Expanding terms:

$$
\frac{\partial (S_i^T [2B_i^C] \dot{\Psi}_j)}{\partial q_k} = -2 S_i^T S_k \times^* B_i^C \dot{\Psi}_j + 2 S_i^T (B_i^C [\dot{\Psi}_k] - B_i^C (S_k \times)) \dot{\Psi}_j + 2 S_i^T S_k \times^* B_i^C \dot{\Psi}_j + 2 S_i^T B_i^C (\dot{\Psi}_k \times S_j + S_k \times \dot{\Psi}_j)
$$

(112)

Cancelling terms lead to:

$$
\frac{\partial (S_i^T [2B_i^C] \dot{\Psi}_j)}{\partial q_k} = 2 S_i^T (B_i^C [\dot{\Psi}_k] \dot{\Psi}_j + B_i^C \dot{\Psi}_k \times S_j)
$$

(113)

Now, considering the second term in Eq. [109] and using the product rule:

$$
\frac{\partial (S_i^T I_i^C \dot{\Psi}_j)}{\partial q_k} = \left( \frac{\partial S_i^T}{\partial q_k} \right) I_i^C \dot{\Psi}_j + S_i^T \left( \frac{\partial I_i^C}{\partial q_k} \dot{\Psi}_j + I_i^C \frac{\partial \dot{\Psi}_j}{\partial q_k} \right)
$$

(114)

Using the identities [K6] [K9] and [K13]

$$
\frac{\partial (S_i^T I_i^C \dot{\Psi}_j)}{\partial q_k} = (-S_i^T S_k \times^*) I_i^C \dot{\Psi}_j + S_i^T (S_k \times^* I_i^C - I_i^C (S_k \times)) \dot{\Psi}_j + S_i^T I_i^C (\dot{\Psi}_k \times S_j + \dot{\Psi}_j \times S_k + S_k \times \dot{\Psi}_j)
$$

(115)

Cancelling terms lead to:

$$
\frac{\partial (S_i^T I_i^C \dot{\Psi}_j)}{\partial q_k} = S_i^T I_i^C \dot{\Psi}_k \times S_j + 2 S_i^T I_i^C \dot{\Psi}_k \times S_j
$$

(116)

Adding Eq. [113] and [116]

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = 2 S_i^T (B_i^C [\dot{\Psi}_k] \dot{\Psi}_j + B_i^C (S_j \times \dot{\Psi}_k)) \dot{\Psi}_j + 2 S_i^T I_i^C \dot{\Psi}_k \times S_j + S_i^T I_i^C \dot{\Psi}_k \times \dot{\Psi}_j
$$

(117)

Re-arranging common terms and using the property [M8]

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = 2 S_i^T B_i^C [\dot{\Psi}_k] \dot{\Psi}_j - 2 S_i^T B_i^C (S_j \times \dot{\Psi}_k) \dot{\Psi}_j + 2 S_i^T I_i^C \dot{\Psi}_k \times \dot{\Psi}_j - S_i^T I_i^C (S_j \times \dot{\Psi}_k) \dot{\Psi}_j
$$

(118)

The first term with $B_i^C [\dot{\Psi}_k]$ in Eq. [118] is mixed since it comprises of quantities with two indices $i$ and $k$. Using the definition of $B_i^C [\dot{\Psi}_k]$ in Eq. [119]
\[ B_i^C[\Psi_k] = \frac{1}{2} \left( (\Psi_k \times ^* I_i^C \Psi_k) + (I_i^C \Psi_k) \times ^* \right) \] (119)

\[ \frac{\partial^2 \tau_k}{\partial q_j \partial q_k} = S_i^T (\Psi_k \times ^* I_i^C \Psi_k + (I_i^C \Psi_k) \times ^*) \Psi_j - 2S_i^T B_i^C (S_j \times \Psi_k) \tilde{R} - S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (120)

Simplifying terms:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T (\Psi_k \times ^* I_i^C \Psi_k + (I_i^C \Psi_k) \times ^*) \Psi_j - 2S_i^T B_i^C (S_j \times \Psi_k) \tilde{R} - S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (121)

Using property $[\text{M13}]$ for the first term inside the parenthesis, $[\text{M19}]$ for the second term inside the parenthesis, $[\text{M6}]$ for the third term inside the parenthesis, and noting that $I_i^C$ is symmetric:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = (S_i \times \Psi_k) \tilde{R}^\top + S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (122)

Using $[\text{M19}]$ for the first term, $[\text{M8}]$ for the second term, and $[\text{M17}]$ for the third term:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = (S_i \times \Psi_k) \tilde{R}^\top + S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (123)

Using $[\text{M13}]$ again for the second term,

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T B_i^C (S_j \times \tilde{\Psi}_k) \tilde{R} - S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (124)

Finally, using $[\text{M18}]$ for the first term, and $[\text{M5}]$ for the second term:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T B_i^C (S_j \times \tilde{\Psi}_k) \tilde{R} - S_i^T (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (125)

Using the definition of $B_i^C[S_j]$ (Eq. 81) to combine the first three terms:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ \Psi_j^\top \left[ 2B_i^C[S_j] \tilde{\Psi}_k \right] \right]^\top \tilde{R}^\top + S_i^\top B_i^C (S_j \times \tilde{\Psi}_k) \tilde{R} - S_i^\top (I_i^C S_j \times \tilde{\Psi}_k) \tilde{R} \] (126)

Using $[\text{M19}]$ for the second and $[\text{M18}] [\text{M19}]$ for the third term:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ \Psi_j^\top \left[ 2B_i^C[S_j] \tilde{\Psi}_k \right] \right]^\top \tilde{R}^\top + 2 \left[ (S_j \times \tilde{\Psi}_k) \tilde{R}^\top, \tilde{\Psi}_k^\top, I_i^C S_j \right] \] (127)

Using $[\text{M13}]$ for the second and the third terms:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ \Psi_j^\top \left[ 2B_i^C[S_j] \tilde{\Psi}_k \right] \right]^\top \tilde{R}^\top - 2 \left[ (S_j \times \tilde{\Psi}_k)^\top, (I_i^C S_j \times \tilde{\Psi}_k) \right] \] (128)

Finally, using $[\text{M5}]$ for the second and third terms results in:

\[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ \Psi_j^\top \left[ 2B_i^C[S_j] \tilde{\Psi}_k \right] \right]^\top + S_j^\top \left( (I_i^C S_j \times \tilde{\Psi}_k) \right) \tilde{R} \] (129)
2) 2A

By exploiting Hessian block symmetry, the term $\frac{\partial^2 \tau_i}{\partial q_i \partial q_j}$ is calculated using $\frac{\partial^2 \tau_i}{\partial q_i \partial q_k}$ (Eq. 129). Due to the symmetry along the 2-3 dimensions, it results into a 3D rotation:

$$\frac{\partial^2 \tau_i}{\partial q_i \partial q_j} = \left[ \frac{\partial^2 \tau_i}{\partial q_i \partial q_k} \right] \hat{r}$$ (130)

Since the previous case (Eq. 129) covers the condition $k = j$, this case can be solved for a stricter condition $k < j < i$.

B. $j < k < i$

1) 1B

For this case, since $j < i$, we take the partial derivative of Eq. 131 wrt $q_k$ and use the product rule as:

$$\frac{\partial^2 \tau_i}{\partial q_i \partial q_k} = S_j^T \left( \frac{\partial B^C}{\partial q_k} \tilde{\Psi}_i + 2B^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} + \frac{\partial I^C}{\partial q_k} \tilde{\Psi}_i + I^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} + \left( \frac{\partial f_i^C}{\partial q_k} \right) \tilde{x}^* S_i + (f_i^C)^* \tilde{x}^* \frac{\partial S_i}{\partial q_k} \right)$$ (131)

Considering the first two terms in Eq. 131 and using the identities [K4] and [K10]

$$S_j^T \left( \frac{\partial B^C}{\partial q_k} \tilde{\Psi}_i + 2B^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} \right) = 2S_j^T \left( B_i^C \tilde{\Psi}_k + S_k \tilde{x}^* B_i^C - B_i^C(S_k \tilde{x}) \right) \tilde{\Psi}_i + 2S_j^T B_i^C(\tilde{\Psi}_k \tilde{x} S_i + S_k \tilde{x} \tilde{\Psi}_i)$$ (132)

Cancelling terms result in:

$$S_j^T \left( \frac{\partial B^C}{\partial q_k} \tilde{\Psi}_i + 2B^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} \right) = 2S_j^T \left( (B_i^C \tilde{\Psi}_k + S_k \tilde{x}^* B_i^C) \tilde{\Psi}_i + B_i^C \tilde{\Psi}_k \tilde{x} S_i \right)$$ (133)

Considering the next two terms in Eq. 131 and using the identities [K6] and [K9] as:

$$S_j^T \left( \frac{\partial I^C}{\partial q_k} \tilde{\Psi}_i + I^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} \right) = S_j^T \left( (S_k \tilde{x}^* I_i^C - I_i^C(S_k \tilde{x})) \tilde{\Psi}_i + I_i^C(\tilde{\Psi}_k \tilde{x} S_i + 2\tilde{\Psi}_k \tilde{x} \tilde{\Psi}_i) \right)$$ (134)

Cancelling terms result in:

$$S_j^T \left( \frac{\partial I^C}{\partial q_k} \tilde{\Psi}_i + I^C_i \frac{\partial \tilde{\Psi}_i}{\partial q_k} \right) = S_j^T \left( S_k \tilde{x}^* I_i^C \tilde{\Psi}_i + I_i^C(\tilde{\Psi}_k \tilde{x} S_i + 2\tilde{\Psi}_k \tilde{x} \tilde{\Psi}_i) \right)$$ (135)

Finally, considering the last two terms in Eq. 131 and using the identities [K1] and [K12] as:

$$S_j^T \left( \frac{\partial f_i^C}{\partial q_k} \tilde{x}^* S_i + (f_i^C)^* \tilde{x}^* \frac{\partial S_i}{\partial q_k} \right) = S_j^T \left( (I_i^C \tilde{\Psi}_k + S_k \tilde{x}^* f_i^C + 2B_i^C \tilde{\Psi}_k) \tilde{x} S_i + (f_i^C)^* \tilde{x} S_k \tilde{x} S_i \right)$$ (136)

Using the property [M11] and expanding terms we get:

$$S_j^T \left( \frac{\partial f_i^C}{\partial q_k} \tilde{x}^* S_i + (f_i^C)^* \tilde{x}^* \frac{\partial S_i}{\partial q_k} \right) = S_j^T \left( (I_i^C \tilde{\Psi}_k + 2B_i^C \tilde{\Psi}_k) \tilde{x} S_i + S_k \tilde{x}^* f_i^C \tilde{x} S_i \right)$$ (137)

Cancelling terms lead to:

$$S_j^T \left( \frac{\partial f_i^C}{\partial q_k} \tilde{x}^* S_i + (f_i^C)^* \tilde{x}^* \frac{\partial S_i}{\partial q_k} \right) = S_j^T \left( (I_i^C \tilde{\Psi}_k + 2B_i^C \tilde{\Psi}_k) \tilde{x} S_i + S_k \tilde{x}^* f_i^C \tilde{x} S_i \right)$$ (138)

Adding the terms in Eq. 132 135 and 138 to get:

$$\frac{\partial^2 \tau_i}{\partial q_i \partial q_k} = S_j^T \left( 2(B_i^C \tilde{\Psi}_k + S_k \tilde{x}^* B_i^C) \tilde{\Psi}_i + 2B_i^C \tilde{\Psi}_k \tilde{x} S_i + S_k \tilde{x}^* I_i^C \tilde{\Psi}_i + I_i^C(\tilde{\Psi}_k \tilde{x} S_i + 2\tilde{\Psi}_k \tilde{x} \tilde{\Psi}_i) \right)$$ (139)
Re-arranging terms:
\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 \left( B_i^C [\dot{\psi}_k] + S_k \times B_i^C \right) \dot{\psi}_i + 2 B_i^C (\dot{\psi}_k \times S_i) + 2 (B_i^C \dot{\psi}_k) \times S_i \\
+ S_k \times (I_i^C \dot{\psi}_i) + I_i^C (2 \dot{\psi}_k \times \dot{\psi}_j) + (I_i^C \dot{\psi}_k) \times S_i + I_i^C \dot{\psi}_k \times S_i + S_k \times f_i^C \times S_i \right)
\]
(140)

Using the property [M6] and [M8] and re-arranging terms:
\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 \left( B_i^C [\dot{\psi}_k] + S_k \times B_i^C \right) \dot{\psi}_i - 2 B_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i + 2 (S_i \times B_i^C \dot{\psi}_k) \times \dot{\psi}_i \\
+ S_k \times (I_i^C \dot{\psi}_i) + I_i^C (2 \dot{\psi}_k \times \dot{\psi}_j) + (S_i \times I_i^C \dot{\psi}_k) \times S_i - I_i^C (S_i \times \dot{\psi}_k) \times S_i + S_k \times f_i^C \times S_i \right)
\]
(141)

Collecting terms with commonality and using the property [M8]:
\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 B_i^C [\dot{\psi}_k] \dot{\psi}_i + S_k \times (2 B_i^C \dot{\psi}_i + I_i^C \dot{\psi}_i + f_i^C \times S_i) \\
+ 2 (S_i \times B_i^C \dot{\psi}_k) \times \dot{\psi}_i - 2 B_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i - 2 I_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i - I_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i \right)
\]
(142)

Using property [M18]:
\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 B_i^C [\dot{\psi}_k] \dot{\psi}_i + S_k \times (2 B_i^C \dot{\psi}_i + I_i^C \dot{\psi}_i + f_i^C \times S_i) + 2 (S_i \times B_i^C \dot{\psi}_k) \times \dot{\psi}_i \\
- 2 B_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i - 2 I_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i - I_i^C (S_i \times \dot{\psi}_k) \times \dot{\psi}_i \right)
\]
(143)

We switch the index \(k\) and \(j\) to get the case \(k < j \leq i\), and the expression changes to:
\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_k^T \left( 2 B_i^C [\dot{\psi}_j] \dot{\psi}_i + S_j \times (2 B_i^C \dot{\psi}_j + I_i^C \dot{\psi}_j + f_i^C \times S_i) + 2 (S_i \times B_i^C \dot{\psi}_j) \times \dot{\psi}_i \\
- 2 B_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i - 2 I_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i - I_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i \right)
\]
(144)

The first term is mixed, and has two indices \(i\) and \(j\). Using the definition of \(B_i^C [\dot{\psi}_j]\) (Eq. 119) to expand the first term as:
\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_k^T \left( (I_i^C \dot{\psi}_j) \times \dot{\psi}_i - I_i^C (\dot{\psi}_j \times \dot{\psi}_i) + (\dot{\psi}_i \times (I_i^C \dot{\psi}_j)) \times \dot{\psi}_i + S_j \times (2 B_i^C \dot{\psi}_j + I_i^C \dot{\psi}_j + f_i^C \times S_i) \\
+ 2 (S_i \times B_i^C \dot{\psi}_j) \times \dot{\psi}_i - 2 B_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i - 2 I_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i + \right)
(145)

Using the properties [M5] [M6] [M8] and [M17] as:
\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_k^T \left( (I_i^C \dot{\psi}_j) \times \dot{\psi}_i - I_i^C (\dot{\psi}_j \times \dot{\psi}_i) + (\dot{\psi}_i \times (I_i^C \dot{\psi}_j)) \times \dot{\psi}_i + S_j \times (2 B_i^C \dot{\psi}_j + I_i^C \dot{\psi}_j + \\
+ f_i^C \times S_i) + 2 (S_i \times B_i^C \dot{\psi}_j) \times \dot{\psi}_i - 2 B_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i - 2 I_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i + \right)
(146)

Simplifying:
\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_k^T \left( (I_i^C \dot{\psi}_j) \times \dot{\psi}_i - I_i^C (\dot{\psi}_j \times \dot{\psi}_i) + (\dot{\psi}_i \times (I_i^C \dot{\psi}_j)) \times \dot{\psi}_i + S_j \times (2 B_i^C \dot{\psi}_j + I_i^C \dot{\psi}_j + \\
+ f_i^C \times S_i) + 2 (S_i \times B_i^C \dot{\psi}_j) \times \dot{\psi}_i - 2 B_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i + S_i \times I_i^C \dot{\psi}_j \times \dot{\psi}_i - (I_i^C (S_i \times \dot{\psi}_j) \times \dot{\psi}_i) + \right)
(147)

Using the definition of \(B_i^C [\dot{\psi}_j]\) (Eq. 119), and taking the tensor rotation out:
\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_k^T \left( [2 (B_i^C [\dot{\psi}_j] + S_i \times B_i^C - B_i^C (S_i \times \dot{\psi}_j) \dot{\psi}_j + (S_i \times I_i^C - I_i^C S_i \times \dot{\psi}_j) \dot{\psi}_j] \times \dot{\psi}_i + \right)
(148)
S_j \times (2 B_i^C \dot{\psi}_j + I_i^C \dot{\psi}_j + f_i^C \times S_i), (k < j \leq i)
2) \(2B\)

The expression for \(\frac{\partial^2 \tau_k}{\partial q_j \partial q_i}\) for the condition \((k \prec j \preceq i)\) can be calculated using \(\frac{\partial^2 \tau_k}{\partial q_i \partial q_j}\) (Eq. 148). A 3D rotation occurs as a result of the symmetry along the 2-3 dimension. Since the previous case (Eq. 148) covers the condition \(j = i\), this case is solved for a stricter condition \((k \prec j \prec i)\).

\[
\frac{\partial^2 \tau_k}{\partial q_j \partial q_i} = \left[ \frac{\partial^2 \tau_k}{\partial q_i \partial q_j} \right]_{\hat{R}} , (k \prec j \prec i) \tag{149}
\]

C. \(j \preceq i \prec k\)

1) \(1C\)

For this case we take the partial derivatives of Eq. 17 and apply the product rule of differentiation to get:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = 2S_i^\top \left( \frac{\partial B_i^C}{\partial q_k} \dot{\Psi}_j \right) + S_i^\top \left( \frac{\partial I_i^C}{\partial q_k} \ddot{\Psi}_j \right) \tag{150}
\]

Applying identities \(K6\) and \(K10\) to get:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = 2S_i^\top \left( B_k^C [\dot{\Psi}_j] + S_k \times B_k^C - B_k^C (S_k \times) \right) \dot{\Psi}_j + S_i^\top \left( S_k \times I_k^C - I_k^C (S_k \times) \right) \dot{\Psi}_j \tag{151}
\]

First switching the indices \(k\) and \(j\), followed by \(j\) and \(i\) to get the case \(k \preceq j \prec i\), Eq. 151 now converts to:

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = 2S_j^\top \left( B_i^C [\dot{\Psi}_k] + S_j \times B_i^C - B_i^C (S_i \times) \right) \dot{\Psi}_j + S_j^\top \left( S_i \times I_i^C - I_i^C (S_i \times) \right) \dot{\Psi}_k , (k \preceq j \prec i) \tag{152}
\]

2) \(2C\)

To exploit the block Hessian symmetry, \(\frac{\partial^2 \tau_k}{\partial q_j \partial q_i}\) is calculated using \(\frac{\partial^2 \tau_k}{\partial q_i \partial q_j}\) (Eq. 152). A 3D rotation along the 2-3 dimension occurs as a result of it.

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = \left[ \frac{\partial^2 \tau_j}{\partial q_k \partial q_i} \right]_{\hat{R}} , (k \preceq j \prec i) \tag{153}
\]
V. SECOND ORDER PARTIAL DERIVATIVES OF ID WRT $\dot{q}$

Now we take the partial derivatives of Eq. [19] and Eq. [20] wrt $\dot{q}_k$ to get the second-order partial derivative of $\tau_i$ wrt $\dot{q}_k$. We consider 5 cases as follows:

A. $k < j \leq i$

1) $1A$

We take the partial derivative of Eq. [19] wrt $\dot{q}_k$ using the product rule of differentiation as:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T \left[ 2 \frac{\partial B_i^C}{\partial q_k} S_j + I_i^C \left( \frac{\partial S_i}{\partial q_k} + \frac{\partial \dot{S}_i}{\partial q_k} \right) \right] \quad (154)$$

Using the identities [K14][K16] we get:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T \left[ 2B_i^C[S_k]S_j + I_i^C (S_k \times S_j + S_k \times S_j) \right] \quad (155)$$

Upon simplifying:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = 2S_i^T \left[ B_i^C[S_k]S_j + I_i^C (S_k \times S_j) \right] \quad (156)$$

Using the definition of $B_i^C[S_k]$ (Eq. [157]) to expand the first term as:

$$B_i^C[S_j] = \frac{1}{2} ((S_j \times \ldots) I_i^C - I_i^C (S_j \times \ldots) + (I_i^C S_j) \times \ldots) \quad (157)$$

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T \left[ S_k \times I_i^C S_j + I_i^C (S_k \times S_j) + 2I_i^C (S_k \times S_j) \right] \quad (158)$$

Simplifying terms:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^T \left[ S_k \times I_i^C S_j + I_i^C (S_k \times S_j) + (I_i^C S_k) \times S_j \right] \quad (159)$$

Using $M13$ for the first term, $M19$ for the second, and $M6$ for the third term results in

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^T I_i^C S_i \times S_k \right] \tilde{R} + \left[ (S_k \times S_j) \times I_i^C S_i \right] \tilde{R} + S_i^T (S_k \times I_i^C S_k) \tilde{R} \quad (160)$$

Using $M18$ for the first term, using $M8$ for the second term, and $M17$ for the third term, we get:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^T (I_i^C S_i \times S_k) \right] \tilde{R} - \left[ (S_k \times S_j) \times I_i^C S_i \right] \tilde{R} - \left[ S_j^T (S_k \times I_i^C S_k) \right] \tilde{R} \quad (161)$$

Using $M13$ for the second term:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^T (I_i^C S_i \times S_k) \right] \tilde{R} - \left[ S_j^T (S_k \times I_i^C S_i) \right] \tilde{R} - \left[ S_j^T (S_k \times I_i^C S_k) \right] \tilde{R} \quad (162)$$

Using $M8$ for the second term:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^T (I_i^C S_i \times S_k) \right] \tilde{R} - \left[ S_j^T (I_i^C S_i \times S_k) \right] \tilde{R} - \left[ S_j^T (S_k \times I_i^C S_k) \right] \tilde{R} \quad (163)$$

Using the definition of $B_i^C[S_j]$ (Eq. [157]), we get:

$$\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ S_j^T (2B_i^C[S_k]S_k) \right] \tilde{R}, (k < j \leq i) \quad (164)$$

2) $2A$

Exploiting the block Hessian symmetry, $\frac{\partial^2 \tau_i}{\partial q_k \partial q_j}$ is calculated using $\frac{\partial^2 \tau_i}{\partial q_j \partial q_k}$ (Eq. [164]). Due to the symmetry between the 2-3 dimensions, a 3D rotation occurs as:

$$\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = \left[ \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} \right] \tilde{R}, (k < j \leq i) \quad (165)$$
B. \( k = j \leq i \)

This case is similar to 1A except for the term \( \frac{\partial^2 \tau_i}{\partial q_j \partial q_k} \), which results into 0, based on the identity [K15]. Hence the expression for this case results in:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^\top \left[ 2B_i^C[S_k]S_j + I_i^C(S_k\times S_j) \right] \tag{166}
\]

Using the definition of \( B_i^C[S_k] \) (Eq. 157) to expand the first term as:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^\top [S_k\times I_i^C - I_i^C(S_k\times) + (I_i^C S_k)\times* + I_i^C(S_k\times)] S_j \tag{167}
\]

Cancelling terms:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^\top [S_k\times* I_i^C + (I_i^C S_k)\times*] S_j \tag{168}
\]

Using [M13] for the first term and [M6] for the second term:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^\top I_i^C(S_i\times S_k)^\mathbf{R} \right]^\top + S_i^\top (S_j \times* I_i^C S_k)^\mathbf{R} \tag{169}
\]

Now using [M18] for the first term and [M17] for the second term results in:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \left[ S_j^\top (I_i^C S_i\times S_k)^\mathbf{R} \right]^\top - \left[ S_j^\top (S_i \times* I_i^C S_k)^\mathbf{R} \right]^\top \tag{170}
\]

Final expression can be written as:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - \left[ S_j^\top ((S_i \times* I_i^C - I_i^C S_i\times) S_k) S_j \right] \tag{171}
\]

C. \( j < k < i \)

1) 1C

For this case, since \( j < i \), we take the partial derivative of Eq. 20 w.r.t. \( q_k \), noting that \( \frac{\partial S_i}{\partial q_k} = \frac{\partial I_i^C}{\partial q_k} = 0 \) as:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^\top \left[ 2\frac{\partial B_i^C}{\partial q_k} S_i + I_i^C \left( \frac{\partial \Psi_i}{\partial q_k} + \frac{\partial \dot{S}_i}{\partial q_k} \right) \right] \tag{172}
\]

Using identities [K13], [K16] results in:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^\top \left[ 2B_i^C[S_k]S_i + I_i^C(S_k\times S_i + S_k\times S_i) \right] \tag{173}
\]

Upon simplifying:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = 2S_j^\top \left[ B_i^C[S_k]S_i + I_i^C(S_k\times S_i) \right] \tag{174}
\]

Switching the indices \( k \) and \( j \) to get the case \( k < j < i \) for the term \( \frac{\partial^2 \tau_k}{\partial q_i \partial q_j} \), as:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = 2S_i^\top \left[ B_i^C[S_j]S_i + I_i^C(S_j\times S_i) \right] \tag{175}
\]

Using the definition of \( B_i^C[S_j] \) (Eq. 157), and expanding the first term as:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^\top \left[ S_j\times* I_i^C - I_i^C(S_j\times) + (I_i^C S_j)\times* + 2I_i^C(S_j\times) \right] S_i \tag{176}
\]

Simplifying:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^\top \left[ S_j\times* I_i^C S_i + I_i^C(S_j\times)S_i + (I_i^C S_j)\times* S_i \right] \tag{177}
\]
Using the properties $M5$, $M8$ and $M6$ respectively for the first, second and the third term as:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left( (S_i \times \dot{I}_i^C S_j)^\dddot{R} - \dot{I}_i^C (S_i \times S_j)^\dddot{R} + ((I_i^C S_i)^{\times^*} S_j)^\dddot{R} \right)$$

(178)

Simplifying, and using the definition of $B_i^C[S_i]$ (Eq. 157):

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ 2B_i^C[S_i]S_j + I_i^C (S_j \times S_i) \right], (k \prec j \prec i)$$

(179)

2) 2C

To get the term $\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j}$ for the condition $k \prec j \prec i$, we use the block Hessian symmetry for the term $\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j}$ (Eq. 179). A 3D rotation occurs due to symmetry between the 2-3 dimension.

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_j \partial \dot{q}_i} = \left[ \frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} \right]^\dddot{R}, (k \prec j \prec i)$$

(180)

D. $j \prec k = i$

This case is similar to 1C except the expression for $\frac{\partial \Psi_j}{\partial \dot{q}_k}$, which is 0 using the identity $K15$. Switching the indices $j$ and $k$ results into the case $k \prec j = i$, and the expression:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ 2B_i^C[S_j]S_i + I_i^C (S_j \times S_i) \right]$$

(181)

Using the definition of $B_i^C[S_j]$, expanding the first term as:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ S_j \times^* I_i^C - I_i^C (S_j \times) + (I_i^C S_j)^{\times^*} + I_i^C (S_j \times) \right] S_i$$

(182)

Cancelling terms:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ S_j \times^* I_i^C S_i + (I_i^C S_j)^{\times^*} S_i \right]$$

(183)

Using the properties $M5$ and $M6$ as:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ ((I_i^C S_i)^{\times^*} S_j)^\dddot{R} + (S_i \times^* (I_i^C S_j)) \right]$$

(184)

The final equation is:

$$\frac{\partial^2 \tau_k}{\partial \dot{q}_i \partial \dot{q}_j} = S_k^\top \left[ ((I_i^C S_i)^{\times^*} + S_i \times^* I_i^C) S_j \right]^\dddot{R}, (k \prec j = i)$$

(185)

E. $j \preceq i < k$

1) $1E$

Taking the partial derivative of $\frac{\partial \tau_i}{\partial \dot{q}_i}$ (Eq. 19) wrt $\dot{q}_k$ for this case results in:

$$\frac{\partial^2 \tau_i}{\partial \dot{q}_j \partial \dot{q}_k} = S_i^\top \left[ 2\frac{\partial B_i^C}{\partial \dot{q}_k} S_j + I_i^C \left( \frac{\partial \Psi_j}{\partial \dot{q}_k} + \frac{\partial S_j}{\partial \dot{q}_k} \right) \right]$$

(186)

Using the identities $K14$ and $K16$ results in:

$$\frac{\partial^2 \tau_i}{\partial \dot{q}_j \partial \dot{q}_k} = S_i^\top \left[ 2B_i^C[S_k] S_j \right]$$

(187)

Switching the indices $j$ and $k$, and then $i$ and $j$ to get the case $k \preceq j \preceq i$ for $\frac{\partial^2 \tau_i}{\partial \dot{q}_j \partial \dot{q}_k}$ as:

$$\frac{\partial^2 \tau_j}{\partial \dot{q}_k \partial \dot{q}_i} = S_j^\top \left[ 2B_i^C[S_i] S_k \right], (k \preceq j \preceq i)$$

(188)
2) 2E

Exploiting the block Hessian symmetry, we get the term $\frac{\partial^2 \tau_j}{\partial q_i \partial q_k}$ for the case $k \preceq j < i$. A 3D rotation occurs as a result of the symmetry between 2-3 dimensions.

$$\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = \left[ \frac{\partial^2 \tau_j}{\partial q_k \partial q_i} \right] \tilde{R}, \quad (k \preceq j < i)$$

(189)
VI. CROSS SECOND-ORDER PARTIAL DERIVATIVES OF ID W.R.T. \( \dot{q} \) AND \( q \)

In this section, the second-order cross partial derivatives of ID w.r.t \( q \) and \( \dot{q} \) are derived. For simplifying the algebra, first-order partial derivative of \( \frac{\partial \mathbf{r}}{\partial q} \) (Eq. \[19\]) w.r.t \( q \) are taken. Hence, 3 cases each for \( \frac{\partial \mathbf{r}}{\partial q} \) and \( \frac{\partial \mathbf{r}}{\partial \dot{q}} \) w.r.t \( q_k \) are considered pertaining to \( k \leq j \leq i, j < k \leq i, \) and \( j \leq i < k \) as follows:

A. \( k \leq j \leq i, \) IA, 2A

1) IA

Since for this case \( j \leq i, \) we first take the derivative of \( \frac{\partial \mathbf{r}}{\partial q} \) (Eq. \[19\]) w.r.t \( q_k \) as:

\[
\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} = \frac{\partial S_i^T}{\partial q_k} (2B^C_i \mathbf{S}_j + I^C_i (\dot{\Psi}_j + \dot{S}_j)) + \frac{\partial S_i^T}{\partial q_k} \left( 2 \frac{\partial B^C_i}{\partial q_k} \mathbf{S}_j + 2B^C_i \frac{\partial S_i}{\partial q_k} + \frac{\partial I^C_i}{\partial q_k} (\dot{\Psi}_j + \dot{S}_j) + I^C_i \left( \frac{\partial \dot{\Psi}_j}{\partial q_k} + \frac{\partial \dot{S}_j}{\partial q_k} \right) \right)
\]

Considering only the first term \( \frac{\partial S_i^T}{\partial q_k} (2B^C_i \mathbf{S}_j + I^C_i (\dot{\Psi}_j + \dot{S}_j)) \) in Eq. \[190\] and using the identity \[K13\]

\[
\frac{\partial S_i^T}{\partial q_k} (2B^C_i \mathbf{S}_j + I^C_i (\dot{\Psi}_j + \dot{S}_j)) = -S_i^T S_k \times^* (2B^C_i \mathbf{S}_j + I^C_i (\dot{\Psi}_j + \dot{S}_j))
\]

(191)

Considering the next two terms in Eq. \[190\] and using the identities \[K1\] and \[K10\] as:

\[
S_i^T \left( 2 \frac{\partial B^C_i}{\partial q_k} \mathbf{S}_j + 2B^C_i \frac{\partial S_i}{\partial q_k} \right) = S_i^T \left( 2 \left[ B^C_i \left[ \dot{\Psi}_k \right] + S_k \times^* B^C_i - B^C_i (S_k \times) \right] \mathbf{S}_j + 2B^C_i (S_k \times \mathbf{S}_j) \right)
\]

(192)

Cancelling terms lead to:

\[
S_i^T \left( 2 \frac{\partial B^C_i}{\partial q_k} \mathbf{S}_j + 2B^C_i \frac{\partial S_i}{\partial q_k} \right) = 2S_i^T \left( B^C_i \left[ \dot{\Psi}_k \right] + S_k \times^* B^C_i \right) \mathbf{S}_j
\]

(193)

Now considering the next term in Eq. \[190\] and using the identity \[K6\] as:

\[
S_i^T \frac{\partial I^C_i}{\partial q_k} (\dot{\Psi}_j + \dot{S}_j) = S_i^T (S_k \times^* I^C_i - I^C_i (S_k \times)) (\dot{\Psi}_j + \dot{S}_j)
\]

(194)

Expanding:

\[
S_i^T \frac{\partial I^C_i}{\partial q_k} (\dot{\Psi}_j + \dot{S}_j) = S_i^T S_k \times^* I^C_i \dot{\Psi}_j - S_i^T I^C_i (S_k \times) \dot{\Psi}_j + S_i^T S_k \times^* I^C_i \dot{S}_j - S_i^T I^C_i (S_k \times) \dot{S}_j
\]

(195)

Now considering the last term in Eq. \[190\] and using the identity \[K2\] and \[K4\] as:

\[
S_i^T I^C_i \left( \frac{\partial \dot{\Psi}_j}{\partial q_k} + \frac{\partial \dot{S}_j}{\partial q_k} \right) = S_i^T I^C_i (\dot{\Psi}_k \times \mathbf{S}_j + S_k \times \dot{\Psi}_j) + S_i^T I^C_i (\dot{\Psi}_k \times \dot{S}_j + S_k \times \dot{S}_j)
\]

(196)

Adding the terms in Eq. \[191\] \[193\] \[195\] \[196\] results in:

\[
\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} = -S_i^T S_k \times^* (2B^C_i \mathbf{S}_j + I^C_i (\dot{\Psi}_j + \dot{S}_j)) + 2S_i^T \left( B^C_i \left[ \dot{\Psi}_k \right] + S_k \times^* B^C_i \right) \mathbf{S}_j + S_i^T S_k \times^* I^C_i \dot{\Psi}_j - S_i^T I^C_i (S_k \times) \dot{\Psi}_j + S_i^T S_k \times^* I^C_i \dot{S}_j - S_i^T I^C_i (S_k \times) \dot{S}_j + S_i^T I^C_i (\dot{\Psi}_k \times \mathbf{S}_j + S_k \times \dot{\Psi}_j) + S_i^T I^C_i (\dot{\Psi}_k \times \dot{S}_j + S_k \times \dot{S}_j)
\]

(197)

Cancelling terms lead to:

\[
\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} = 2S_i^T \left( B^C_i \left[ \dot{\Psi}_k \right] \mathbf{S}_j + I^C_i (\dot{\Psi}_k \times) \right)
\]

(198)

Using the definition of \( B^C_i \left[ \dot{\Psi}_k \right] \) (Eq. \[119\] to expand the first term as:

\[
\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} = S_i^T ( \dot{\Psi}_k \times^* I^C_i - I^C_i (\dot{\Psi}_k \times) + (I^C_i \dot{\Psi}_k \times^* + 2I^C_i (\dot{\Psi}_k \times)) \mathbf{S}_j
\]

(199)

Simplifying:
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_j^T (\tilde{\Psi}_k \tilde{x}^* I_i^C + I_i^C (\tilde{\Psi}_k \tilde{x}) + (I_i^C \tilde{\Psi}_k) \tilde{x}^*) S_j
\] (200)

Using [M13] for the first term, [M19] for the second term, [M6] for the third term, we get:
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = (S_j \tilde{x} \tilde{\Psi}_k) \tilde{R}_i \tilde{S}_j + [(\tilde{\Psi}_k \tilde{x} S_j)^\top I_i^C S_j] \tilde{R}^\top + S_j^T (S_j \tilde{x}^* I_i^C \tilde{\Psi}_k) \tilde{R}
\] (201)

Using [M19] for the first term, [M8] for the second term, and [M17] for the third term,
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = [(I_i^C S_j)^\top (S_j \tilde{x} \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top - [(S_j \tilde{x} \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top - [S_j^T (S_j \tilde{x}^* I_i^C \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top
\] (202)

Simplifying the first term, using [M13] for the first term leads to
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = [S_j^T (I_i^C S_j \tilde{x} \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top - [S_j^T (\tilde{\Psi}_k \tilde{x}^* I_i^C S_i)] \tilde{R}^\top - [S_j^T (S_j \tilde{x}^* I_i^C \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top
\] (203)

Using [M18] for the first term, and [M5] for the second term:
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = [S_j^T (I_i^C \tilde{S}_j \tilde{x} \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top - [S_j^T ((I_i^C S_i) \tilde{x} \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top - [S_j^T (S_i \tilde{x}^* I_i^C \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top
\] (204)

Using the definition of $B_i^C[S_i]$ (Eq. 157), we get:
\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = - [S_j^T (2B_i^C[S_i] \tilde{\Psi}_k) \tilde{R}] \tilde{R}^\top, (k \leq j < i)
\] (205)

2) $2A (j \neq i)$

For this case, we take the partial derivative of $\frac{\partial \tau_i}{\partial q_k}$ (Eq. 20) w.r.t $q_k$. Due to $j \neq i$, the condition becomes $k \leq j < i$.
\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = \frac{\partial S_j^T}{\partial q_k} \left( 2B_i^C S_i + I_i^C (\tilde{\Psi}_i + \tilde{S}_i) \right) + S_j^T \left( \frac{\partial B_i^C}{\partial q_k} S_i + 2B_i^C \frac{\partial S_i}{\partial q_k} + \frac{\partial I_i^C}{\partial q_k} (\tilde{\Psi}_i + \tilde{S}_i) + I_i^C \left( \frac{\partial \tilde{\Psi}_i}{\partial q_k} + \frac{\partial \tilde{S}_i}{\partial q_k} \right) \right)
\] (206)

Considering only the first term in Eq. 206 and using [K13]
\[
\frac{\partial S_j^T}{\partial q_k} \left( 2B_i^C S_i + I_i^C (\tilde{\Psi}_i + \tilde{S}_i) \right) = -S_j^T S_k \tilde{x}^* \left( 2B_i^C S_i + I_i^C (\tilde{\Psi}_i + \tilde{S}_i) \right)
\] (207)

Now, considering the next two terms in Eq. 206 and using [K1] and [K10] as:
\[
S_j^T \left( 2 \frac{\partial B_i^C}{\partial q_k} S_i + 2B_i^C \frac{\partial S_i}{\partial q_k} \right) = S_j^T \left( 2 \left( B_i^C \left[ \tilde{\Psi}_k \right] + S_k \tilde{x}^* B_i^C - B_i^C (S_k \tilde{x}) \right) S_i + 2B_i^C (S_k \tilde{x} S_i) \right)
\] (208)

Cancelling terms lead to:
\[
S_j^T \left( 2 \frac{\partial B_i^C}{\partial q_k} S_i + 2B_i^C \frac{\partial S_i}{\partial q_k} \right) = 2S_j^T \left( B_i^C \left[ \tilde{\Psi}_k \right] + S_k \tilde{x}^* B_i^C \right) S_i
\] (209)

Considering the next term in Eq. 206 and using the identity [K6] as:
\[
S_j^T \frac{\partial I_i^C}{\partial q_k} (\tilde{\Psi}_i + \tilde{S}_i) = S_j^T (S_k \tilde{x}^* I_i^C - I_i^C (S_k \tilde{x})) (\tilde{\Psi}_i + \tilde{S}_i)
\] (210)

Expanding the terms results into
\[
S_j^T \frac{\partial I_i^C}{\partial q_k} (\tilde{\Psi}_i + \tilde{S}_i) = S_j^T S_k \tilde{x}^* I_i^C \tilde{S}_i - S_j^T I_i^C (S_k \tilde{x}) \tilde{\Psi}_i + S_j^T S_k \tilde{x}^* I_i^C \tilde{S}_i - S_j^T I_i^C (S_k \tilde{x}) \tilde{S}_i
\] (211)

Now considering the last term in Eq. 206 and using the identity [K2] and [K4] as:
\[
S_j^T I_i^C \left( \frac{\partial \tilde{\Psi}_i}{\partial q_k} + \frac{\partial \tilde{S}_i}{\partial q_k} \right) = S_j^T I_i^C (\tilde{\Psi}_k \tilde{x} S_i + S_k \tilde{x} \tilde{\Psi}_i) + S_j^T I_i^C (\tilde{\Psi}_k \tilde{x} S_i + S_k \tilde{x} \tilde{S}_i)
\] (212)
Now adding the terms in Eq. [207][209][211][212] results in:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = -S_j S_k \times (2B_i^C S_i + I_i^C (\Psi_i + \dot{S}_i)) + 2S_j (B_i^C [\dot{\Psi}_k] + S_k \times B_i^C) S_i + S_j S_k \times I_i^C (\Psi_k \times S_k \times \dot{S}_i - \dot{S}_i I_i^C (S_k \times \dot{S}_i) + S_j I_i^C (\dot{\Psi}_k \times S_i + S_j \times \dot{S}_i) + S_j I_i^C (\dot{\Psi}_k \times S_i + S_k \times \dot{S}_i) + S_j J_i^C (\Psi_k \times S_i + S_k \times S_i)
$$

Cancelling terms leads to:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = 2S_j (B_i^C [\dot{\Psi}_k] S_i + I_i^C (\dot{\Psi}_k \times S_i))
$$

Expanding the first term upon using the definition of $B_i^C [\dot{\Psi}_k]$ (Eq. [119]) as:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j (\dot{\Psi}_k \times \dot{I}_i^C - I_i^C \dot{\Psi}_k \times \dot{S}_i + (I_i^C \dot{\Psi}_k \times S_i + (I_i^C \dot{\Psi}_k \times S_i)) S_i
$$

Simplifying:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j (\dot{\Psi}_k \times \dot{I}_i^C - I_i^C \dot{\Psi}_k \times \dot{S}_i + (I_i^C \dot{\Psi}_k \times S_i + (I_i^C \dot{\Psi}_k \times S_i)) S_i
$$

Using the properties [M5][M8] and [M6] for the first, second and the third term respectively:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j (((I_i^C \dot{S}_i) \dot{\Psi}_k \times \dot{\Psi}_k) \dot{S}_i - (I_i^C \dot{S}_i \times \dot{\Psi}_k) \dot{S}_i) + (S_i \times I_i^C \dot{\Psi}_k) \dot{S}_i)
$$

Using the definition of $B_i^C [S_i]$ (Eq. [157]), the final expression is:

$$
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j [2B_i^C [\dot{\Psi}_k] S_i] R, (k \preceq j \preceq i)
$$

B. $j \prec k \leq i$, 1B, 2B

1) 1B

For this case, the partial derivative of Eq. [19] w.r.t $q_k$ for the case $j \prec k \preceq i$ is taken as:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = \frac{\partial S_j}{\partial q_k} \left(2B_i^C S_j + I_i^C (\dot{\Psi}_j + \dot{S}_j)\right) + S_j \left(2 \frac{\partial B_i^C}{\partial q_k} S_j + \frac{\partial I_i^C}{\partial q_k} (\dot{\Psi}_j + \dot{S}_j)\right)
$$

Using the identities [K6][K10] and [K13] as:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = -S_j S_k \times (2B_i^C S_j + I_i^C (\dot{\Psi}_j + \dot{S}_j)) + S_j \left(2 (B_i^C [\dot{\Psi}_k] + S_k \times B_i^C - B_i^C (S_k \times \dot{S}_j)) S_j + S_j \times I_i^C (S_k \times \dot{S}_j)\right)
$$

Cancelling terms:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_j \left(2 (B_i^C [\dot{\Psi}_k] - B_i^C (S_k \times \dot{S}_j)) S_j + S_j \times I_i^C (S_k \times \dot{S}_j)\right)
$$

Simplifying:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_j \left(2 (B_i^C [\dot{\Psi}_k] S_j - 2B_i^C (S_k \times S_j - I_i^C (S_k \times \dot{S}_j)) (\dot{\Psi}_j + \dot{S}_j)\right)
$$

Switching the indices $j$ and $k$ to get the case $k \preceq j \preceq i$, results in:

$$
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_j \left(2 (B_i^C [\dot{\Psi}_j] S_k - 2B_i^C (S_j \times S_k - I_i^C (S_j \times \dot{S}_k)) (\dot{\Psi}_k + \dot{S}_k)\right)
$$

Using the definition of $B_i^C [\dot{\Psi}_j]$ (Eq. [119]) (Eq. [157]) to expand the first term as:
\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = S_k^T (\hat{\Psi}_j \hat{\Psi}_j^\top S_k - I_i^C \hat{\Psi}_j) S_k + (I^C_i \hat{\Psi}_j)^* S_k - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(224)

Using [M8] in the first term, [M8] for the second term, [M6] for the third term,

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = [S^T_k (I^C_i (S_j \hat{\Psi}_j)) \hat{R}]^\top + S^T_i (S_k \hat{\Psi}_j)^\top \hat{R} + S^T_i [S_k \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)] \hat{R} - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(225)

Using [M18] in the first term as, [M19] for the second term, and using [M17] for the third term,

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = [S^T_k (I^C_i (S_j \hat{\Psi}_j)) \hat{R}]^\top + [(S_k \hat{\Psi}_j)^\top \hat{R} \hat{I}^C_i S_j] \hat{R} - [S^T_k (S_j \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)) \hat{R}]^\top - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(226)

Using [M13] for the second term,

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = [S^T_k (I^C_i (S_j \hat{\Psi}_j)) \hat{R}]^\top + [S^T_k (S_j \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)) \hat{R}]^\top - [S^T_k (S_j \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)) \hat{R}]^\top - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(227)

Using [M5] for the second term,

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = [S^T_k (I^C_i (S_j \hat{\Psi}_j)) \hat{R}]^\top + [S^T_k (S_j \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)) \hat{R}]^\top - [S^T_k (S_j \hat{\Psi}_j^* (I^C_i \hat{\Psi}_j)) \hat{R}]^\top - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(228)

Using the definition of $B^C_i [S_j]$ (Eq. [157]), we get:

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = - [S^T_k (2B^C_i [S_j] \hat{\Psi}_j) \hat{R}]^\top + 2 [S^T_k ((I^C_i S_j) \hat{\Psi}_j \hat{R})^\top - 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k)
\]

(229)

Using [M8] for the third and fourth term,

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = - [S^T_k (2B^C_i [S_j] \hat{\Psi}_j) \hat{R}]^\top + 2 [S^T_k ((I^C_i S_j) \hat{\Psi}_j \hat{R})^\top + 2 S_i^T B^C_i (S_j \hat{\Psi}_j) S_k - S_i^T I^C_i (S_j \hat{\Psi}_j)(\hat{\Psi}_k + \hat{S}_k) \hat{S}_j]
\]

(230)

Using [M19] for the third and fourth terms:

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = - [S^T_k (2B^C_i [S_j] \hat{\Psi}_j) \hat{R}]^\top + 2 [S^T_k ((I^C_i S_j) \hat{\Psi}_j \hat{R})^\top + [(\hat{\Psi}_k + \hat{S}_k)^\top \hat{S}_j S_j + (\hat{\Psi}_k + \hat{S}_k)^\top (\hat{\Psi}_k + \hat{S}_k) S_j S_j] \hat{R}
\]

(231)

Using [M13] for the third and fourth terms:

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = - [S^T_k (2B^C_i [S_j] \hat{\Psi}_j) \hat{R}]^\top + 2 [S^T_k ((I^C_i S_j) \hat{\Psi}_j \hat{R})^\top + [(\hat{\Psi}_k + \hat{S}_k)^\top \hat{S}_j S_j + (\hat{\Psi}_k + \hat{S}_k)^\top S_j \hat{\Psi}_j \hat{R})^\top \hat{I}^C_i S_j \hat{S}_j \hat{R}]^\top
\]

(232)

Using [M8] for the third and fourth terms to finally get:

\[
\frac{\partial^2 \tau_i}{\partial q_k \partial q_j} = - [S^T_k (2B^C_i [S_j] \hat{\Psi}_j) \hat{R}]^\top + 2 [S^T_k ((I^C_i S_j) \hat{\Psi}_j \hat{R})^\top + [(\hat{\Psi}_k + \hat{S}_k)^\top ((\hat{\Psi}_k + \hat{S}_k) \hat{\Psi}_j \hat{R})^\top \hat{I}^C_i S_j \hat{S}_j \hat{R}]^\top, (k \neq j \leq i)
\]

Simplifying
\[
\frac{\partial^2 \tau_i}{\partial q_i \partial q_j} = \left[ S_k^T (-2B^C [S_i] \dot{\Psi}_j + (2B_i^C^T S_i) \tilde{x}^* S_j + 2(I_i^C S_i) \tilde{x}^* \dot{\Psi}_j) \right] \hat{R} + (\dot{\Psi}_k + \dot{S}_k)^T ((I_i^C S_i) \tilde{x}^* S_j) \hat{R} \right], (k \neq j \leq i)
\]

(234)

2) 2B (j \neq i)
In this case, we take the partial of Eq. [20] w.r.t \( q_k \). This case condition \( j \neq k \leq i \) satisfies the assumption \( j \neq i \) for using Eq. [20]

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 \frac{\partial B^C}{\partial q_k} S_i + 2B^C_i \frac{\partial S_k}{\partial q_k} + \frac{\partial I_i^C}{\partial q_k} (\dot{\Psi}_i + \dot{S}_i) + I_i^C (\frac{\partial \dot{\Psi}_i}{\partial q_k} + \frac{\partial \dot{S}_i}{\partial q_k}) \right)
\]

(235)

Using the identities [K1] [K2] [K4] [K6] and [K10] results in:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 (B_i^C [\dot{\Psi}_k] + S_k \tilde{x}^* B_i^C S_i) + S_j^T \tilde{x}^* I_i^C \dot{\Psi}_i + S_j^T S_k \tilde{x}^* \dot{S}_i \right) + S_j^T I_i^C (\dot{\Psi}_k \tilde{x}^* S_i + S_k \tilde{x}^* \dot{\Psi}_i) + S_j^T I_i^C (\dot{S}_k \tilde{x}^* S_i + S_k \tilde{x}^* \dot{\Psi}_i)
\]

(236)

Cancelling terms lead to:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^T \left( 2 (B_i^C [\dot{\Psi}_k] + S_k \tilde{x}^* B_i^C S_i) + S_j^T S_k \tilde{x}^* I_i^C \dot{\Psi}_i + S_j^T S_k \tilde{x}^* I_i^C \dot{S}_i \right)
\]

(237)

Collecting terms:

\[
\frac{\partial^2 \tau_i}{\partial q_i \partial q_j} = 2S_j^T \left( B_i^C [\dot{\Psi}_k] S_i + S_k \tilde{x}^* B_i^C S_i + I_i^C (\dot{\Psi}_j \tilde{x}^* S_i) \right) + S_j^T S_k \tilde{x}^* I_i^C (\dot{\Psi}_i + \dot{S}_i)
\]

(238)

Switching indices \( j \) and \( k \) to get the case \( k \neq j \leq i \):

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = 2S_i^T \left( B_i^C [\dot{\Psi}_k] S_j + S_j \tilde{x}^* B_i^C S_i + I_i^C (\dot{\Psi}_j \tilde{x}^* S_i) \right) + S_i^T S_j \tilde{x}^* I_i^C (\dot{\Psi}_i + \dot{S}_i)
\]

(239)

Using the definition of \( B_i^C [\dot{\Psi}_j] \) (Eq. [119]) to expand the first term as:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^T \left( \dot{\Psi}_j \tilde{x}^* I_i^C S_i + I_i^C (\dot{\Psi}_j \tilde{x}^* S_i) \right) + S_i^T S_j \tilde{x}^* (2B_i^C S_i + I_i^C (\dot{\Psi}_i + \dot{S}_i))
\]

(240)

Simplifying terms:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^T \left( \dot{\Psi}_j \tilde{x}^* I_i^C S_i + I_i^C (\dot{\Psi}_j \tilde{x}^* S_i) + (I_i^C \dot{\Psi}_j) \tilde{x}^* S_i \right) + S_i^T S_j \tilde{x}^* (2B_i^C S_i + I_i^C (\dot{\Psi}_i + \dot{S}_i))
\]

(241)

Using the [M5] for the first term, [M8] and [M17] for the second term, and [M6] for the third term as:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^T \left( (I_i^C S_i) \tilde{x}^* \dot{\Psi}_j \right) \hat{R} + (S_i \tilde{x}^* (I_i^C \dot{\Psi}_j)) \hat{R} + S_i^T S_j \tilde{x}^* (2B_i^C S_i + I_i^C (\dot{\Psi}_i + \dot{S}_i))
\]

(242)

Using the definition of \( B_i^C [S_i] \) (Eq. [117]), and the property [M5]

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^T \left[ 2B_i^C [S_i] \dot{\Psi}_j \right] \hat{R} + S_i^T \left[ (2B_i^C S_i + I_i^C (\dot{\Psi}_i + \dot{S}_i)) \tilde{x}^* S_j \right] \hat{R}
\]

(243)

Simplifying:

\[
\frac{\partial^2 \tau_k}{\partial q_i \partial q_j} = S_i^T \left[ 2B_i^C [S_i] \dot{\Psi}_j + (2B_i^C S_i + I_i^C (\dot{\Psi}_i + \dot{S}_i)) \tilde{x}^* S_j \right] \hat{R}, (k \neq j \leq i)
\]

(244)
C. \( j \preceq i \prec k \), 1C, 2C

1) 1C

For this case, the partial derivative of \( \frac{\partial \tau_i}{\partial q_j} \) (Eq. 19) w.r.t \( q_k \) is taken as:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^\top \left( 2 \frac{\partial B^C_k}{\partial q_k} S_j + \frac{\partial I^C_k}{\partial q_k} (\dot{\Psi}_j + \dot{S}_j) \right)
\]  

(245)

Using the identities [K6] and [K10] for the first and the second term respectively:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = S_i^\top \left( 2 \left( B^C_k [\dot{\Psi}_k] + S_k \times B^C_k - B^C_k (S_k \times) \right) S_j + (S_k \times I^C_k - I^C_k (S_k \times)) (\dot{\Psi}_j + \dot{S}_j) \right)
\]

(246)

First, switching the indices \( j \) and \( k \), and then \( i \) and \( j \) to get \( k \preceq j \prec i \) to get:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^\top \left( 2 \left( B^C_k [\dot{\Psi}_k] + S_k \times B^C_k - B^C_k (S_k \times) \right) S_i + (S_k \times I^C_k - I^C_k (S_k \times)) (\dot{\Psi}_j + \dot{S}_j) \right), (k \preceq j \prec i)
\]  

(247)

2) 2C (\( j \neq i \))

In this case, we take the partial derivative of \( \frac{\partial \tau_i}{\partial q_j} \) (Eq. 20) w.r.t \( q_k \). Assumption for using Eq. 20 results in the condition \( j \neq i \), which results in this case condition as \( j \prec i \prec k \).

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^\top \left( 2 \frac{\partial B^C_i}{\partial q_k} S_i + \frac{\partial I^C_i}{\partial q_k} (\dot{\Psi}_i + \dot{S}_i) \right)
\]

(248)

Using identities [K6] and [K10] for the first and the second term as:

\[
\frac{\partial^2 \tau_j}{\partial q_i \partial q_k} = S_j^\top \left( 2 \left( B^C_i [\dot{\Psi}_k] + S_k \times B^C_k - B^C_k (S_k \times) \right) S_i + (S_k \times I^C_k - I^C_k (S_k \times)) (\dot{\Psi}_j + \dot{S}_j) \right)
\]

(249)

Like the previous case (1C), first switching the indices \( j \) and \( k \), and then \( i \) and \( j \) to get the case \( k \prec j \prec i \) results in:

\[
\frac{\partial^2 \tau_k}{\partial q_j \partial q_i} = S_k^\top \left( 2 \left( B^C_i [\dot{\Psi}_j] + S_j \times B^C_j - B^C_j (S_j \times) \right) S_k + (S_j \times I^C_j - I^C_j (S_j \times)) (\dot{\Psi}_k + \dot{S}_k) \right), (k \prec j \prec i)
\]

(250)
VII. First Order Partial Derivative of $M(q)$ w.r.t $q$

Here, we present the cross second-order partial derivatives of ID w.r.t $\bar{q}$ and $q$ that results in $\frac{\partial M}{\partial q}$. The lower-triangle of $M(q)$ for the case $j \leq i$ is given as:

$$M_{ji} = S_j^\top I_i^C S_i$$

(251)

We take the partial derivative of $M_{ji}$ w.r.t $q_k$ by considering 3 cases pertaining to $k \leq j \leq i$, $j \prec k \leq i$, and $j \leq i \prec k$ as follows:

A. $k \leq j \leq i$

1) $1A$

For this case, we use the product chain rule to get:

$$\frac{\partial M_{ji}}{\partial q_k} = \frac{\partial S_j^\top}{\partial q_k} I_i^C S_i + S_j^\top \left( \frac{\partial I_i^C}{\partial q_k} S_i + I_i^C \frac{\partial S_i}{\partial q_k} \right)$$

(252)

Using identities $\text{K1}$, $\text{K6}$ and $\text{K13}$ we get:

$$\frac{\partial M_{ji}}{\partial q_k} = -S_j^\top \tilde{S}_k \times I_i^C S_i + S_j^\top (S_k \times I_i^C - I_i^C (S_k \times)) S_i + S_j^\top I_i^C S_k \times S_i$$

(253)

Cancelling terms lead to:

$$\frac{\partial M_{ji}}{\partial q_k} = 0, (k \leq j \leq i)$$

(254)

2) $2A$

Due to symmetry in $M(q)$, we get:

$$\frac{\partial M_{ij}}{\partial q_k} = 0, (k \leq j \leq i)$$

(255)

B. $j \prec k \leq i$

1) $1B$

For this case, we use the product rule of differentiation:

$$\frac{\partial M_{ji}}{\partial q_k} = S_j^\top \left( \frac{\partial I_i^C}{\partial q_k} S_i + I_i^C \frac{\partial S_i}{\partial q_k} \right)$$

(256)

Now using with identities $\text{K6}$ and $\text{K1}$ as:

$$\frac{\partial M_{ji}}{\partial q_k} = S_j^\top (S_k \times I_i^C - I_i^C (S_k \times)) S_i + S_j^\top I_i^C S_k \times S_i$$

(257)

Cancelling terms lead to:

$$\frac{\partial M_{ji}}{\partial q_k} = S_j^\top (S_k \times I_i^C) S_i$$

(258)

We switch the indices $k$ and $j$ to get the form $k \prec j \leq i$:

$$\frac{\partial M_{kj}}{\partial q_j} = S_k^\top (S_j \times I_i^C) S_i$$

(259)

using the property $\text{M5}$:

$$\frac{\partial M_{kj}}{\partial q_j} = S_k^\top ((I_i^C S_i) \times S_j) \bar{R}, (k \prec j \leq i)$$

(260)

2) $2B$

Using the symmetry for $M(q)$, we get the expression for $\frac{\partial M_{ik}}{\partial q_j}$ as:

$$\frac{\partial M_{ik}}{\partial q_j} = \left[ \frac{\partial M_{ki}}{\partial q_j} \right]^\top, (k \prec j \leq i)$$

(261)
C. $j \preceq i < k$

1) 1C
For this case we get:
\[
\frac{\partial M_{ji}}{\partial q_k} = S_j^\top \left( \frac{\partial I^C_i}{\partial q_k} \right) S_i
\]  
(262)
using identity [K6] we get:
\[
\frac{\partial M_{ji}}{\partial q_k} = S_j^\top (S_k \hat{\times} I_k^C - I_k^C (S_k \hat{\times} ))S_i
\]  
(263)
Switching first the indices $k$ and $j$, and the indices $j$ and $i$ to get the case $k \preceq j < i$ results in the term:
\[
\frac{\partial M_{kj}}{\partial q_i} = S_k^\top (S_i \hat{\times} I_i^C - I_i^C (S_i \hat{\times} ))S_j , (k \preceq j < i)
\]  
(264)
2) 2C
Using the symmetry for $M(q)$, we get the expression for $\frac{\partial M_{jk}}{\partial q_i}$ as:
\[
\frac{\partial M_{jk}}{\partial q_i} = \left[ \frac{\partial M_{kj}}{\partial q_i} \right]^\top , (k \preceq j < i)
\]  
(265)
VIII. Efficient Implementation

For implementation simplicity, the expressions for SO partials are reduced from tensor to matrix and vector form. Each of the expressions listed in Sec. IV, V, and VI are written in a form to avoid tensor operations. For this purpose, they are evaluated only for one DoF \( (p \tilde{\psi}_{k,r}) \) at a time. The spatial matrix operators \( \tilde{\psi}_k \) and \( \tilde{\psi}_k^* \) automatically reduce to spatial vector operators \( \tilde{\psi}_k \) and \( \tilde{\psi}_k^* \) respectively.

1. SO Partial w.r.t \( q \)

1A) Considering Eq. 129 and evaluating it for only the \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \) results into:

\[
\frac{\partial^2 \tau_i}{\partial q_j \partial q_k} = -\left[ \psi_{j,t}(2B^C_i[s_{i,p}]\tilde{\psi}_{k,r})^T + 2s_{j,t}(B^C_{i} s_{i,p})\tilde{\psi}_{k,r}^T \right] + s_{j,t}(I^C_i s_{i,p} \tilde{\psi}_{k,r})^T, \quad (k \geq j \geq i)
\]

(266)

where \( s_{i,p} \) is a column vector of \( S_i \), \( s_{j,t} \) is the \( t^\text{th} \) column vector of \( S_j \), and \( \tilde{\psi}_{k,r} \) is the \( r^\text{th} \) column vector of \( \tilde{\psi}_k \). The tensor \( B^C_i[S_i] \) (Eq. 157) takes the vector \( s_{i,p} \) as an argument and reduces to a matrix. Since the terms inside the 2-3 3D rotation \( (\tilde{\psi}) \) are vectors, the 3D 2-3 rotation is no longer needed and can be safely removed. The 3D 1-2 rotation \( (\tilde{\psi}) \) falls out, since it operates on a scalar, resulting in:

\[
\frac{\partial^2 \tau_{i,p}}{\partial q_{j,t} \partial q_{k,r}} = -\tau_{j,t}^T 2B^C_i[s_{i,p}]\tilde{\psi}_{k,r} - 2s_{j,t}(B^C_i s_{i,p})\tilde{\psi}_{k,r} + s_{j,t}(I^C_i s_{i,p} \tilde{\psi}_{k,r})
\]

(267)

2A) Considering Eq. 130 for only the \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \) results in:

\[
\frac{\partial^2 \tau_{i,p}}{\partial q_{j,r} \partial q_{k,r}} = \left[ \frac{\partial^2 \tau_{i,p}}{\partial q_{j,r} \partial q_{k,r}} \right]^T
\]

(268)

Since the quantity \( \frac{\partial^2 \tau_{i,p}}{\partial q_{j,t} \partial q_{k,r}} \) is a scalar, the 3D 2-3 rotation \( (\tilde{\psi}) \) falls out:

\[
\frac{\partial^2 \tau_{i,p}}{\partial q_{j,r} \partial q_{k,r}} = \frac{\partial^2 \tau_{i,p}}{\partial q_{j,t} \partial q_{k,r}}
\]

(269)

1B) Eq. 148 is evaluated for \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \). The 2-3 3D rotation \( (\tilde{\psi}) \) on the tensorial terms can be removed due to the resulting expression being a scalar:

\[
\frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} = s_{k,r}^T \left( 2(B^C_i[\tilde{\psi}_{i,p}] + s_{i,p} \times B^C_i \times s_{i,p}) \tilde{\psi}_{j,t} + (s_{i,p} \times I^C_i s_{i,p} \times I^C_i) \tilde{\psi}_{j,t}^T \right)
\]

(270)

2B) Eq. 149 is considered only for \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \) as:

\[
\frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} = \left[ \frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} \right]^T
\]

(271)

In this case, since the quantity \( \frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} \) is a scalar, the 3D 2-3 rotation has no affect on it.

\[
\frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} = \left[ \frac{\partial^2 \tau_{k,r}}{\partial q_{j,t} \partial q_{i,p}} \right]^T
\]

(272)

1C) Considering Eq. 152 for the \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \) results into:

\[
\frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}} = 2s_{j,t}(B^C_i[\tilde{\psi}_{i,p}] + s_{i,p} \times B^C_i \times s_{i,p}) \tilde{\psi}_{k,r} + s_{j,t}(s_{i,p} \times I^C_i s_{i,p}) \tilde{\psi}_{k,r}
\]

(273)

2C) For this case, Eq. 153 is considered for \( p^\text{th} \) DoF of joint \( i \), \( t^\text{th} \) DoF of joint \( j \), and \( r^\text{th} \) DoF of joint \( k \) as:

\[
\frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}} = \left[ \frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}} \right]^T
\]

(274)
Since the quantity \( \frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}} \) is a scalar, the 3D 2-3 rotation doesn’t affect it, resulting into:

\[
\frac{\partial^2 \tau_{j,t}}{\partial q_{i,p} \partial q_{k,r}} = \frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}}
\]  

(275)

2. SO Partials w.r.t \( \dot{q} \)

All of the cases in Sec. VI are considered for only the \( p \text{th} \) DoF of joint \( i \), \( t \text{th} \) DoF of joint \( j \), and \( r \text{th} \) DoF of joint \( k \). The 3D 2-3 tensor rotation (\( \bar{R} \)) is dropped in the expressions due to reduction to scalar form. The 3D 1-2 tensor rotation (\( \bar{R}^{\top} \)) is also dropped due to reduction to the scalar form.

1A) Eq. 164 for \( p \text{th} \) DoF of joint \( i \), \( t \text{th} \) DoF of joint \( j \), and \( r \text{th} \) DoF of joint \( k \) becomes:

\[
\frac{\partial^2 \tau_{i,p}}{\partial \dot{q}_{j,t} \partial \dot{q}_{k,r}} = -s_{j,t}^{\top} \left[ 2B_i^{C}[s_{i,p}][s_{k,r}] \right], (k < j \leq i)
\]  

(276)

2A) Eq. 165 for \( p \text{th} \) DoF of joint \( i \), \( t \text{th} \) DoF of joint \( j \), and \( r \text{th} \) DoF of joint \( k \) becomes:

\[
\frac{\partial^2 \tau_{i,p}}{\partial \dot{q}_{j,t} \partial \dot{q}_{k,r}} = \frac{\partial^2 \tau_{i,p}}{\partial \dot{q}_{k,r} \partial \dot{q}_{j,t}}
\]  

(277)

B) Eq. 171 becomes:

\[
\frac{\partial^2 \tau_{i,p}}{\partial \dot{q}_{j,t} \partial \dot{q}_{k,r}} = -s_{j,t}^{\top} \left[ (s_{i,p} \times I^C_i)^{\times} - (s_{i,p} \times I^C_i) \right] s_{k,r}
\]  

(278)

1C) Eq. 179 becomes:

\[
\frac{\partial^2 \tau_{k,r}}{\partial \dot{q}_{i,p} \partial \dot{q}_{j,t}} = s_{k,r}^{\top} \left[ 2B_i^{C}[s_{i,p}][s_{j,t}] \right]
\]  

(279)

2C) Eq. 180 becomes:

\[
\frac{\partial^2 \tau_{k,r}}{\partial \dot{q}_{i,p} \partial \dot{q}_{j,t}} = \frac{\partial^2 \tau_{k,r}}{\partial \dot{q}_{i,p} \partial \dot{q}_{j,t}}
\]  

(280)

D) Eq. 185 becomes:

\[
\frac{\partial^2 \tau_{k,r}}{\partial \dot{q}_{i,p} \partial \dot{q}_{j,t}} = s_{k,r}^{\top} \left[ (s_{i,p} \times I^C_i)^{\times} + (I^C_i s_{i,p})^{\times} \right] s_{j,t}
\]  

(281)

1E) Eq. 188 becomes:

\[
\frac{\partial^2 \tau_{j,t}}{\partial \dot{q}_{k,r} \partial \dot{q}_{i,p}} = s_{j,t}^{\top} \left[ 2B_i^{C}[s_{i,p}][s_{k,r}] \right]
\]  

(282)

2E) Eq. 189 becomes:

\[
\frac{\partial^2 \tau_{j,t}}{\partial \dot{q}_{k,r} \partial \dot{q}_{i,p}} = \frac{\partial^2 \tau_{j,t}}{\partial \dot{q}_{k,r} \partial \dot{q}_{i,p}}
\]  

(283)

Here, the 2-3 rotation is dropped since the term \( \frac{\partial^2 \tau_{j,t}}{\partial q_{k,r} \partial q_{i,p}} \) is a scalar, and the 2-3 rotation has no effect on it.

3. Cross SO Partials w.r.t \( q \) and \( \dot{q} \):

Similar to previous case, expressions listed in Sec. VI are converted to the matrix form by considering only the \( p \text{th} \) DoF of joint \( i \), \( t \text{th} \) DoF of joint \( j \), and \( r \text{th} \) DoF of joint \( k \). This reduction results in dropping out of the tensor 2-3 rotation in all the cases.

1A) Eq. 205 becomes:
\[
\frac{\partial^2 \tau_{i,p}}{\partial q_{j,t} \partial q_{k,r}} = -s_{j,t}^\top (2B_i^C [s_{i,p}] \dot{\psi}_{k,r}) \tag{284}
\]

2A) Eq. 218 becomes:

\[
\frac{\partial^2 \tau_{i,t}}{\partial q_{i,p} \partial q_{k,r}} = s_{j,t} 2B_i^C [s_{i,p}] \dot{\psi}_{k,r} \tag{285}
\]

1B) Eq. 234 becomes:

\[
\frac{\partial^2 \tau_{i,t}}{\partial q_{i,p} \partial q_{k,r}} = s_{j,t} 2B_i^C [s_{i,p}] \dot{\psi}_{k,r} \tag{285}
\]

1C) Eq. 247 becomes:

\[
\frac{\partial^2 \tau_{i,t}}{\partial q_{i,p} \partial q_{j,t}} = s_{j,t} 2B_i^C [s_{i,p}] \dot{\psi}_{i,p} \tag{288}
\]

2B) Eq. 244 becomes:

\[
\frac{\partial^2 \tau_{i,t}}{\partial q_{i,p} \partial q_{j,t}} = s_{j,t} 2B_i^C [s_{i,p}] \dot{\psi}_{i,p} + (\dot{\psi}_{i,p} + \dot{s}_{i,p}) \tag{289}
\]

2C) Eq. 250 becomes:

\[
\frac{\partial^2 \tau_{i,t}}{\partial q_{i,p} \partial q_{j,t}} = s_{j,t} 2B_i^C [s_{i,p}] \dot{\psi}_{i,p} + 2B_i^C \dot{s}_{i,p} + (\dot{\psi}_{i,p} + \dot{s}_{i,p}) \tag{290}
\]

4. FO Partial of \( M(q) \) w.r.t. \( q \): All the expressions for FO partial of \( M(q) \) are converted to scalar forms by considering them only for the \( p^{th} \) DoF of joint \( i \), \( t^{th} \) DoF of joint \( j \), and \( r^{th} \) DoF of joint \( k \).

1A) Eq. 254 becomes

\[
\frac{\partial M_{(j,t)(i,p)}}{\partial q_{k,r}} = 0 \tag{291}
\]

2A) Eq. 255 becomes

\[
\frac{\partial M_{(i,p)(j,t)}}{\partial q_{k,r}} = 0 \tag{292}
\]

1B) Eq. 260 becomes:

\[
\frac{\partial M_{(k,r)(i,p)}}{\partial q_{j,t}} = s_{j,t} (I_i^C s_{i,p}) \tag{293}
\]

2B) Eq. 261 becomes

\[
\frac{\partial M_{(i,p)(k,r)}}{\partial q_{j,t}} = \frac{\partial M_{(k,r)(i,p)}}{\partial q_{j,t}} \tag{294}
\]

The tensor 1-2 rotation falls out in this case, since the quantity \( \frac{\partial M_{(k,r)(i,p)}}{\partial q_{j,t}} \) is a scalar, and 1-2 tensor rotation has no effect on it.

1C) Eq. 264 becomes:

\[
\frac{\partial M_{(k,r)(j,t)}}{\partial q_{i,p}} = s_{j,t} (s_{i,p} I_i^C - I_i^C s_{i,p}) \tag{295}
\]

2C) Eq. 265 becomes:
\[
\frac{\partial M_{(j,t)(k,r)}}{\partial q_{i,p}} = \frac{\partial M_{(k,r)(j,t)}}{\partial q_{i,p}}
\]

(296)

In this case, since \(\frac{\partial M_{(k,r)(j,t)}}{\partial q_{i,p}}\) is a scalar, the 1-2 tensor rotation falls off.

IX. SUMMARY

The details of SO partial derivatives of inverse dynamics are provided in these notes. An efficient implementation is described and is used to develop an associated algorithm, that can be found in the paper \([5]\) associated with these notes.

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