COMPACT MULTIPLICATION OPERATORS ON SEMICROSSED PRODUCTS

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Abstract. We characterize the compact multiplication operators on a semicrossed product \( C_0(X) \times_\phi \mathbb{Z}_+ \) in terms of the corresponding dynamical system. We also characterize the compact elements of this algebra and determine the ideal they generate.

1. Introduction

Let \( A \) be a Banach algebra and \( a, b \in A \). The map \( M_{a,b} : A \to A \) given by \( M_{a,b}(x) = axb \) is called a multiplication operator. Properties of compact multiplication operators have been investigated since 1964 when Vala published his work “On compact sets of compact operators” [15]. Let \( X \) be a normed space and \( B(X) \) the algebra of all bounded linear maps from \( X \) into \( X \). Vala proved that a non-zero multiplication operator \( M_{a,b} : B(X) \to B(X) \) is compact if and only if both the operators \( a, b \in B(X) \) are compact. Also, in [16] Vala defines an element \( a \) of a normed algebra to be compact if the mapping \( x \mapsto axa \) is compact. This concept enabled the study of compactness properties of elements of abstract normed algebras. Ylinen in [17] studied compact elements for abstract C*-algebras and showed that \( a \) is a compact element of a C*-algebra \( A \) if and only if there exists an isometric \( \ast \)-representation \( \pi \) of \( A \) on a Hilbert space \( H \) such that the operator \( \pi(a) \) is compact.

Compactness questions have also been considered in the more general framework of elementary operators. A map \( \Phi : A \to A \), where \( A \) is a Banach algebra, is called elementary if \( \Phi = \sum_{i=1}^{m} M_{a_i,b_i} \) for some \( a_i, b_i \in A, i = 1, \ldots, m \). Fong and Sourour showed that an elementary operator \( \Phi : B(H) \to B(H) \), where \( B(H) \) is the algebra of bounded linear operators on a Hilbert space \( H \), is compact if and only if there exist compact operators \( c_i, d_i \in B(H), i = 1, \ldots, m \) such that \( \Phi = \sum_{i=1}^{m} M_{c_i,d_i} \). This result was expanded by Mathieu on prime C*-algebras [19] and later on general C*-algebras by Timoney [14].

Akemann and Wright [1] characterized the weakly compact multiplication operators on \( B(H) \), where \( H \) is a Hilbert space. Saksman and Tylli [12, 13] and Johnson and Schechtman [6] studied weak compactness of multiplication operators in a Banach space setting.

Moreover, strictly singular multiplication operators are studied by Lindström, Saksman and Tylli [8] and Mathieu and Tradacete [10].

Compactness properties of multiplication operators on nest algebras, a class of non selfadjoint operator algebras, are studied by Andreolas and Anoussis in [2].
particular they characterized the compact multiplication operators, the compact elements and the ideal generated by the compact elements.

In the present paper we study multiplication operators on a semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ where $X$ is a locally compact metrizable space, and $\phi : X \to X$ a homeomorphism. We characterize the compact multiplication operators in terms of the corresponding dynamical system. As a consequence, we obtain a characterization of the compact elements of the semicrossed product. We also characterize the ideal generated by the compact elements.

We would like to note that the equicontinuity condition appearing in the characterization of the compact multiplication operators on the semicrossed product, follows from the other conditions if $X$ is discrete or has no isolated points. However, in the general case this does not hold and thus the proof is considerably more elaborated.

2. COMPACT MULTIPLICATION OPERATORS ON SEMICROSSED PRODUCTS

Throughout this paper, $X$ will be a locally compact metrisable space and $\phi : X \to X$ a homeomorphism. The pair $(X, \phi)$ is called a dynamical system. An action of $\mathbb{Z}_+$ on $C_0(X)$ by isometric $\ast$-automorphisms $\alpha_n$, $n \in \mathbb{Z}_+$ is obtained by defining $\alpha_n(f) = f \circ \phi^n$. We write the elements of the Banach space $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $A = \sum_{n \in \mathbb{Z}_+} U^n f_n$ with the norm given by $\|A\|_1 = \sum \|f_n\|_{C_0(X)}$.

The multiplication on $\ell^1(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$U^n f U^m g = U^{n+m} (\alpha^n(f) g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\mathbb{Z}_+, C_0(X))$ is a Banach algebra.

The Banach algebra $\ell^1(\mathbb{Z}_+, C_0(X))$ can be faithfully represented as a (concrete) operator algebra on a Hilbert space. This is achieved by assuming a faithful action of $C_0(X)$ on a Hilbert space $\mathcal{H}_0$. Then, we can define a faithful contractive representation $\pi$ of $\ell^1(\mathbb{Z}_+, C_0(X))$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(\mathbb{Z}_+)$ by defining $\pi(U^n f)$ as

$$\pi(U^n f)(\xi \otimes e_k) = \alpha^k(f) \xi \otimes e_{k+n}.$$  

The semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ is the closure of the image of $\ell^1(\mathbb{Z}_+, C_0(X))$ in $\mathcal{B}(\mathcal{H})$ in the representation just defined, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. We will denote the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ by $\mathcal{A}$ and an element $\pi(U^n f)$ of $\mathcal{A}$ by $U^n f$ to simplify the notation. The closed unit ball of $\mathcal{A}$ will be denoted by $\mathcal{A}_1$. We refer to [3] and [4], for more information about the semicrossed product.

For $A = \sum_{n \in \mathbb{Z}_+} U^n f_n \in \ell^1(\mathbb{Z}_+, C_0(X))$ we call $f_n \equiv E_n(A)$ the $n$th Fourier coefficient of $A$. The maps $E_n : \ell^1(\mathbb{N}_+, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $\mathcal{A}$, and therefore they extend to contractions $E_m : \mathcal{A} \to C_0(X)$. Let $A \in \mathcal{A}$. If the set $\{m \in \mathbb{Z}_+ : E_m(A) \neq 0\}$ is finite, then $A$ is called a polynomial. If there exists a unique $m \in \mathbb{Z}_+$, such that $E_m(A) \neq 0$, then $A$ is called monomial.

Let $(X, \phi)$ be a dynamical system. Then, a point $x \in X$ is called recurrent if there exists a strictly increasing sequence $(m_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, such that $\lim_{k \to \infty} \phi^{m_k}(x) = x$. The set of the recurrent points of $(X, \phi)$ will be denoted by $X_r$. We will denote by $X_i$ the set of the isolated points of $X$, by $X_a$ the set of the accumulation points of $X$ and we set $X_{a,i} = \{x \in X_a : \exists(x_j) \subseteq X_i, \lim_{j \to \infty} x_j = x\}$. If $f \in C_0(X)$, we set $D(f) = \{x \in X : |f(x)| > 0\}$. 

Lemma 2.1. Let $M_{A,B} : A \to A$ be a compact multiplication operator, where $A,B \in A_1$, and $E_m(A) = f_m$, $E_n(B) = g_n$, for all $m \in \mathbb{Z}_+$. Then, $(f_m \circ \phi^n g_n)(X_a) = \{0\}$, for all $m,n,l \in \mathbb{Z}_+$.

Proof. We suppose that there exist $m, n, l \in \mathbb{Z}_+$ such that $(f_m \circ \phi^{n+l} g_n) (X_a) \neq \{0\}$. We define the following indices.

\begin{align*}
  n_0 &= \min \{ n \in \mathbb{Z}_+ | \exists m, l \in \mathbb{Z}_+: (f_m \circ \phi^{n+l} g_n) (X_a) \neq \{0\} \}, \\
  m_0 &= \min \{ m \in \mathbb{Z}_+ | \exists l \in \mathbb{Z}_+: (f_m \circ \phi^{n_0+l} g_n) (X_a) \neq \{0\} \}, \\
  l_0 &= \min \{ l \in \mathbb{Z}_+ | (f_m \circ \phi^{n_0+l} g_n) (X_a) \neq \{0\} \}.
\end{align*}

Let $x_0 \in X_a$, such that $(f_m \circ \phi^{n_0+l_0} g_n) (x_0) \neq 0$. Then, there exist an $\epsilon > 0$ and an open neighborhood $U_0$ of $x_0$ such that $|(f_m \circ \phi^{n_0+l_0} g_n) (x)| > 2\epsilon$, for all $x \in U_0$. Now, we consider the quantity

\[ \bar{g} = \inf_{x \in U_0} \{|g_n(x)|\} > 0. \]

We consider the function $J : X \to \mathbb{C}$, defined as follows.

\[ J(x) = \begin{cases} 0 & \text{if } m_0 + n_0 = 0 \\ \sum_{n=0, n \neq n_0}^{m_0+n_0} |(f_{m_0+n_0-n} \circ \phi^n g_n g_{n_0})(x)| & \text{if } m_0 + n_0 > 0. \end{cases} \]

If $m_0 + n_0 > 0$, we claim that $J(x_0) = 0$. Indeed, if $n < n_0$, then it follows from the definition of $n_0$ that $(f_{m_0+n_0-n} \circ \phi^n g_n)(X_a) = \{0\}$. Otherwise, if $n > n_0$ and $n \leq m_0 + n_0$, it follows from the definition of $m_0$ that $(f_{m_0+n_0-n} \circ \phi^n g_n)(X_a) = \{0\}$, since $m_0 + n_0 - n < m_0$. Therefore, there exists an open neighborhood $V_0$ of $x_0$, such that $J(x) < \epsilon \bar{g}$, for all $x \in V_0$, by the continuity of $J$.

If $m_0 + n_0 > 0$, we set $W_0 = U_0 \cap V_0$, otherwise, we set $W_0 = U_0$. Since $x_0 \in W_0 \cap X_a$, there exist a sequence of points $\{x_i\}_{i \in \mathbb{N}} \subseteq W_0$, a sequence of open subsets $\{W_i\}_{i \in \mathbb{N}} \subset W_0$ with $x_i \in W_i$ and $W_i \cap W_j = \emptyset$, for $i \neq j$ and a sequence of norm one functions $\{h_i\}_{i \in \mathbb{N}} \subseteq C_0(X)$ with $D(h_i) \subseteq W_i$ and $h_i(x_i) = 1$, for all $i \in \mathbb{N}$.

To complete the proof, we consider the sequence $\{U^i h_i \circ \phi^{-n_0}\}_{i \in \mathbb{N}}$ and we will prove that the sequence $\{M_{A,B}(U^i h_i \circ \phi^{-n_0})\}_{i \in \mathbb{N}}$ has no convergent subsequence. We estimate the quantity $\|M_{A,B}(U^i h_u \circ \phi^{-n_0}) - M_{A,B}(U^i h_v \circ \phi^{-n_0})\|_A$, for $u, v \in \mathbb{N}, u \neq v$. 
We note that \(|\mathbb{1}_n| \leq 1\), since \(D(h_u) \cap D(h_v) = \emptyset\). Therefore, we obtain

\[
\|M_{A,B} (U^{l_0} h_u \circ \phi^{-n_0}) - M_{A,B} (U^{l_0} h_v \circ \phi^{-n_0})\|_A \geq
\]

\[
\sum_{n=0}^{m_0+n_0} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\geq
\sum_{n=0}^{m_0+n_0} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\geq
\sum_{n=0}^{m_0+n_0} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\}

\[-\sum_{n=0}^{m_0+n_0} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\}

\[
\| (f_{m_0} \circ \phi^{n_0+l_0} g_n^2) (x_u) \] \[
- \sum_{n=0,n \neq n_0}^{} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\}

\[
\| (f_{m_0} \circ \phi^{n_0+l_0} g_n^2) (x_u) \] \[
- \sum_{n=0,n \neq n_0}^{} \int_{f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n (h_u \circ \phi^{-n_0} - h_v \circ \phi^{-n_0})}\}

which concludes the proof. \(\square\)

**Lemma 2.2.** Let \(M_{A,B} : A \to A\) be a compact multiplication operator, where \(A, B \in A_1\) and \(E_m(A) = f_m, E_m(B) = g_m\), for all \(m \in \mathbb{Z}^+\). Then, \(\lim_{l \to \infty} (f_m \circ \phi^{n+l_0} g_n) (x) = 0\), for all \(m, n \in \mathbb{Z}^+\) and \(x \in X_i\).

**Proof.** We suppose that there exist \(n, m \in \mathbb{Z}^+\) and \(x \in X_i\), such that \(\lim_{l \to \infty} (f_m \circ \phi^{n+l_0} g_n) (x) \neq 0\). We define the following indices.

\[
\begin{align*}
n_0 &= \min \left\{ n \in \mathbb{Z}^+ | \exists m \in \mathbb{Z}^+, x \in X_i : \lim_{l \to \infty} (f_m \circ \phi^{n+l_0} g_n) (x) \neq 0 \right\}, \\
m_0 &= \min \left\{ m \in \mathbb{Z}^+ | \exists x \in X_i : \lim_{l \to \infty} (f_m \circ \phi^{n_0+l_0} g_n) (x) \neq 0 \right\}.
\end{align*}
\]

Then, there exist an element \(x_0 \in X_i\), an \(\epsilon > 0\) and a strictly increasing sequence \(\{l_1 \in \mathbb{N} \subset \mathbb{Z}^+\}\) such that \(\int |(f_{m_0} \circ \phi^{n_0+l_0} g_n) (x_0)| \geq 2\epsilon\). We consider the sequence \(\{J(l)\}_{l \in \mathbb{Z}^+}\),

\[
J(l) = \begin{cases} 
0, & \text{if } m_0 + n_0 = 0 \\
\sum_{n=0,n \neq n_0}^{m_0+n_0} \int |(f_{m_0+n_0-n} \circ \phi^{n+l_0} g_n g_n) (x_0)|, & \text{if } m_0 + n_0 > 0.
\end{cases}
\]
If \( m_0 + n_0 > 0 \), we claim that there exists an \( L \in \mathbb{N} \), such that:

\[
J(l) = \sum_{n=0}^{n_0+m_0} |(f_{m_0+n_0-n} \circ \phi^{n+l}g_n g_{n_0})(x_0)| < \epsilon g_{n_0}(x_0), \quad \forall l > L.
\]

Indeed, if \( n < n_0 \), it follows from the definition of \( n_0 \), that \( \lim_{l \to \infty} (f_{m_0+n_0-n} \circ \phi^{n+l}g_n)(x_0) = 0 \). On the other hand, if \( n > n_0 \) and \( n \leq m_0 + n_0 \), it follows from the definition of \( m_0 \), that \( \lim_{l \to \infty} (f_{m_0+n_0-n} \circ \phi^{n+l}g_n)(x_0) = 0 \), since \( m_0 + n_0 - n < m_0 \). We choose a subsequence \( \{l_k\}_{k \in \mathbb{N}} \subseteq \{l_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}_+ \), such that \( l_{k+1} - l_k > m_0 + n_0 + 1 \), for all \( k \in \mathbb{N} \). We consider the sequence \( \{U^{l_k} \chi\}_{k \in \mathbb{N}} \), where \( \chi \) is the characteristic function of the singleton \( \{\phi^{n_0}(x_0)\} \). To complete the proof we will prove that the sequence \( \{M_{A,B}(U^{l_k} \chi)\}_{k \in \mathbb{N}} \) has no convergent subsequence.

We estimate the quantity \( \|M_{A,B}(U^{l_k} \chi) - M_{A,B}(U^{l_u} \chi)\|_{A'} \), for \( u, v \in \mathbb{N}, u < v \).

\[
\|M_{A,B}(U^{l_u} \chi) - M_{A,B}(U^{l_v} \chi)\|_{A} \geq \|E_{m_0+n_0+l_u}(M_{A,B}(U^{l_u} \chi) - M_{A,B}(U^{l_v} \chi))\|_{C_0(X)} = \|E_{m_0+n_0+l_u}(M_{A,B}(U^{l_v} \chi))\|_{C_0(X)},
\]

since by the assumption \( l_u - l_v > m_0 + n_0 + 1 \), the \((m_0 + n_0 + l_u)\)th Fourier coefficient of \( M_{A,B}(U^{l_v} \chi) \) is 0. We thus obtain

\[
\|M_{A,B}(U^{l_u} \chi) - M_{A,B}(U^{l_v} \chi)\|_{A} \geq \sum_{n=0}^{m_0+n_0} (f_{m_0+n_0-n} \circ \phi^{n+l_u} \chi \circ \phi^n g_n)(x_0) \geq \sum_{n=0}^{m_0+n_0} (f_{m_0+n_0-n} \circ \phi^{n+l_u} \chi \circ \phi^n g_{n_0})(x_0) \geq \sum_{n=0}^{m_0+n_0} (f_{m_0+n_0-n} \circ \phi^{n+l_u} \chi \circ \phi^n g_{n_0})(x_0) \geq \epsilon g_{n_0}(x_0),
\]

which concludes the proof. \( \square \)

**Lemma 2.3.** Let \( M_{A,B} : A \to A \) be a compact multiplication operator, where \( A, B \in A_1 \) and \( E_m(A) = f_m, \ E_m(B) = g_m \), for all \( m \in \mathbb{Z}_+ \). Then, \( (f_m \circ \phi^{n+l}g_n)(x) = 0 \), for all \( m, n, l \in \mathbb{Z}_+ \) and \( x \in X_r \).

**Proof.** If \( x \in X_k \cap X_r \), there exists a \( k_0 \in \mathbb{Z}_+ \) such that \( \phi^{k_0}(x) = x \). It follows from Lemma 2.2 that \( \lim_{l \to \infty} (f_m \circ \phi^{n+l+k_0}g_n)(x) = 0 \), for \( m, n, l \in \mathbb{Z}_+ \). We note that \( (f_m \circ \phi^{n+l+k_0}g_n)(x) = (f_m \circ \phi^{n+l}g_n)(x) \), for all \( i \in \mathbb{Z}_+ \), and hence \( (f_m \circ \phi^{n+l}g_n)(x) = 0 \). If \( x \in X_k \cap X_r \), the assertion follows from Lemma 2.1. \( \square \)
Lemma 2.4. Let $M_{A,B} : A \to A$ be a compact multiplication operator, where $A, B \in A_1$ and $E_m(A) = f_m$, $E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then, the sequence $\{f_m \circ \phi^{n+l}g_n\}_{l \in \mathbb{Z}_+}$ is pointwise equicontinuous, for all $m, n \in \mathbb{Z}_+$.

Proof. It follows from Lemma 2.1 that it is sufficient to prove that $\{f_m \circ \phi^{n+l}g_n\}_{l \in \mathbb{Z}_+}$ is pointwise equicontinuous on $X_{A,B}$. We suppose that there exist some $n_0, m_0 \in \mathbb{Z}_+$ and a point $x_0 \in X$, such that the sequence $\{f_{m_0} \circ \phi^{n_0+l}g_{n_0}\}_{l \in \mathbb{Z}_+}$ is not equicontinuous at $x_0$. We note that $(f_{m_0} \circ \phi^{n_0+l}g_{n_0})(x_0) = 0$, for all $l \in \mathbb{Z}_+$, by Lemma 2.1. Therefore, there exist an $\epsilon > 0$, a strictly increasing sequence $\{l_i\}_{i \in \mathbb{N}} \subset \mathbb{Z}_+$ and a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq X$, such that $\lim_{i \to \infty} x_i = x_0$ and $|\langle f_{m_0} \circ \phi^{n_0+l_i}g_{n_0}(x_i) \rangle| > \epsilon$, for all $i \in \mathbb{N}$. We note that the inclusion $\{x_i\}_{i \in \mathbb{N}} \subseteq X$ holds by Lemma 2.1. Furthermore, we may assume that $l_{i+1} - l_i > m_0 + n_0 + 1$, for all $i \in \mathbb{N}$. It follows from Lemma 2.3 that the elements $\{x_i\}_{i \in \mathbb{N}}$ are not periodic. Therefore, if $m_0 + n_0 > 0$, we have that $\phi^{m_0}(x_i) \neq \phi^n(x_i)$, for all $n \in \{0, \ldots, m_0 + n_0\} \setminus \{n_0\}$ and $i \in \mathbb{N}$. We consider the sequence $\{M_{A,B}(U^{l_i} \chi_i)\}_{i \in \mathbb{N}}$, where $\chi_i$ is the characteristic function of the set $\{\phi^{m_0}(x_i)\}$. Then, for $u < v$ we obtain,

$$\|M_{A,B}(U^{l_u} \chi_u) - M_{A,B}(U^{l_v} \chi_v)\|_A \geq \|E_{m_0+n_0+l_u} (M_{A,B}(U^{l_u} \chi_u) - M_{A,B}(U^{l_v} \chi_v))\|_{C_0(X)}$$

$$= \|E_{m_0+n_0+l_u} (M_{A,B}(U^{l_u} \chi_u))\|_{C_0(X)} \geq \left| \sum_{n=0}^{m_0+n_0} (f_{m_0+n_0-n} \circ \phi^{n+l_u} \chi_u \circ \phi^n g_n)(x_u) \right| > \epsilon,$$

since $x_u$ is not periodic. Therefore, the sequence $\{M_{A,B}(U^{l_i} \chi_i)\}_{k \in \mathbb{N}}$ has no Cauchy subsequence. \hfill $\square$

Proposition 2.5. Let $m, n \in \mathbb{Z}_+$ and $A = U^mf$, $B = U^ng \in A_1$. Then, the multiplication operator $M_{A,B} : A \to A$ is compact if and only if the following assertions are valid.

1. $(f \circ \phi^{n+l}g)(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$,
2. $\lim_{l \to \infty}(f \circ \phi^{n+l}g)(x) = 0$, for all $x \in X$,
3. The sequence $\{f \circ \phi^{n+l}g\}_{l \in \mathbb{Z}_+}$ is pointwise equicontinuous.

Proof. The forward direction is immediate by Lemmas 2.1, 2.2, and 2.4. We will show the opposite direction.

We will divide the proof in three steps:

1st step
In this step we construct an approximation of $M_{A,B}$ by multiplication operators $M_{A_k,B_k}$ with the property that the Fourier coefficients of $A_k, B_k$ are compactly supported.

We define the following sets, for $h \in C_0(X)$ and $k \in \mathbb{N}$.

$$D_k(h) = \left\{ x \in X : |h(x)| \geq \frac{1}{k} \right\}, \quad U_k(h) = \left\{ x \in X : |h(x)| > \frac{2}{3k} \right\}.$$  

It is obvious that $D_k(h) \subseteq U_k(h) \subseteq D_{2k}(h)$ and that the set $\overline{U_k(h)}$ is compact. If $\|f\| = 0$, the proof is trivial. Otherwise, we choose a natural number $k$, such
that $\frac{1}{k_0} < \min\{\|f\|, \|g\|\}$. By Urysohn’s lemma, there are norm one functions $v_{f_k}$ and $v_{g_k}$ in $C_0(X)$, for $k > k_0$, such that

$$v_{f_k}(x) = \begin{cases} 1, & x \in D_k(f) \\ 0, & x \in X \setminus U_k(f) \end{cases} \quad \text{and} \quad v_{g_k}(x) = \begin{cases} 1, & x \in D_k(g) \\ 0, & x \in X \setminus U_k(g) \end{cases}.$$ 

We define the functions $f_k = v_{f_k} \circ f$, $g_k = v_{g_k} \circ g$. It is immediate that $\|f - f_k\| \leq \frac{2}{k}$ and $\|g - g_k\| \leq \frac{2}{k}$. It follows that, if $A_k = U^m f_k$ and $B_k = U^n g_k$, we have that $\|A - A_k\| < \frac{2}{k}$ and $\|B - B_k\| < \frac{2}{k}$. Then, we can see that

$$\sup_{T \in A_n} \|M_{A,B}(T) - M_{A_k,B_k}(T)\| < \frac{4}{k}.$$ 

Hence, to prove that $M_{A,B}$ is compact, it suffices to show that there exists a natural number $k_0$ such that $M_{A_k,B_k}$ is compact, for all $k > k_0$.

**2nd step**

1st case

Firstly, we assume that $\overline{U_k(g)} \cap X_{a,i} = \emptyset$, for some $k' \in \mathbb{Z}_+$. It follows that $\overline{U_k(g)} \cap X_{a,i} = \emptyset$, for all $k > k'$. Let $k_0 = \max\{k, k'\}$ and $k > k_0$. If $\hat{x} \in \overline{U_k(g)} \cap X_{a,i}$, there exists an open neighbourhood $V_{\hat{x}}$ of $\hat{x}$, such that $|g(x)| > \frac{1}{2k}$, for all $x \in V_{\hat{x}}$. Furthermore, $\hat{x}$ is an accumulation point which in turn means that $(f \circ \phi^{n+l})(\hat{x}) = 0$, for all $l \in \mathbb{Z}_+$. Moreover, we recall that the family $\{f \circ \phi^{n+l}\}_{l \in \mathbb{Z}_+}$ is equicontinuous at $\hat{x}$. Therefore, there exists an open neighbourhood $V'_{\hat{x}}$ of $\hat{x}$, such that $|(f \circ \phi^{n+l})(x)| < \frac{1}{4k^2}$, for all $x \in V'_{\hat{x}}$ and $l \in \mathbb{Z}_+$. We set $W_{\hat{x}} = V_{\hat{x}} \cap V'_{\hat{x}}$. It follows that

$$\frac{1}{4k^2} > |(f \circ \phi^{n+l})(x)| = |f \circ \phi^{n+l}(x)||g(x)| \geq |f \circ \phi^{n+l}(x)| \cdot \frac{1}{2k},$$

and therefore, $|f \circ \phi^{n+l}(x)| \leq \frac{1}{2k}$, for all $x \in W_{\hat{x}}$ and $l \in \mathbb{Z}_+$. We denote by $\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}$ the set $\cup\{W_l \mid \hat{x} \in \overline{U_k(g)} \cap X_{a,i}\}$. It follows that $\phi^{n+l}(\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) \subseteq X \setminus \overline{U_k(f)}$, for all $l \in \mathbb{Z}_+$ and hence $(f_k \circ \phi^{n+l} g_k)(\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) = \{0\}$. Moreover, the set $\overline{U_k(g)} \setminus (\cup_{l \in \mathbb{Z}_+} W_{\hat{x}})$ is compact and $\overline{U_k(g)} \setminus (\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) \cap X_{a,i} = \emptyset$, which in turn implies that the set $\overline{U_k(g)} \setminus (\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) \cap X_i$ is finite. We denote by $I_k$ the set $\overline{U_k(g)} \setminus (\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) \cap X_i$ and by $\chi_{I_k}$ the characteristic function of $I_k$. We set $\tilde{g}_k = g_k \chi_{I_k}$. We note that $(f \circ \phi^{n+l})(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$ and hence $(f_k \circ \phi^{n+l} g_k)(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$. Furthermore, we have proved that $(f_k \circ \phi^{n+l} g_k)(\cup_{l \in \mathbb{Z}_+} W_{\hat{x}}) = \{0\}$. Since the function $g_k$ is supported in $U_k(g)$, we conclude that

$$f_k \circ \phi^{n+l} g_k = f_k \circ \phi^{n+l} \tilde{g}_k,$$

for all $l \in \mathbb{Z}_+$.

2nd case

We assume now that $\overline{U_k(g)} \cap X_{a,i} = \emptyset$, $\forall k \in \mathbb{N}$. The set $\overline{U_k(g)} \cap X_i$ is finite, since the set $\overline{U_k(g)}$ is compact and $\overline{U_k(g)} \cap X_{a,i} = \emptyset$. We denote by the same letter as in the 1st case, $I_k$ the set $\overline{U_k(g)} \cap X_i$ and by $\chi_{I_k}$ the characteristic function of $I_k$. We set $\tilde{g}_k = g_k \chi_{I_k}$. We note that $(f \circ \phi^{n+l})(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$ and
hence \((f_k \circ \phi^{n+l}g_k)(X_a) = \{0\}\), for all \(l \in \mathbb{Z}_+\). Since \(g_k\) is supported in \(U_k(y)\), we conclude that
\[
f_k \circ \phi^{n+l}g_k = f_k \circ \phi^{n+l}\tilde{g}_k,
\]
for all \(l \in \mathbb{Z}_+\). It follows that \(M_{A_k,B_k} = M_{A,k,B_k}\), where \(\tilde{B}_k = U^n\tilde{g}_k\).

3rd step

It follows that \(\lim_{l \to \infty} (f_k \circ \phi^{n+l}g_k)(y) = 0\), for all \(y \in I_k\), by assumption. We observe that \(\lim_{l \to \infty} (f_k \circ \phi^{n+l})(y) = 0\), since \(g_k(y) \neq 0\), which follows from the inclusion \(I_k \subseteq U_k(g_k)\). Therefore, there exists an \(L_0 \in \mathbb{N}\), such that \(\phi^{n+l} (y) \in (X \setminus U_k(f))\), for all \(l \geq L_0\) and for all \(y \in I_k\), since the set \(I_k\) is finite. Since \(f_k\) is supported in \(U_k(f)\), we obtain that \(f_k \circ \phi^{n+l}\tilde{g}_k = 0\), for all \(l \geq L_0\). It follows that
\[
M_{A_k,B_k}(U^lh) = 0,
\]
for all \(l \geq L_0\) and \(h \in C_0(X)\). Since \(\tilde{g}_k\) has finite support, the operator \(M_{A_k,\tilde{B}_k}\), is a finite rank operator and hence compact.

Let \(A\) be an element of the semicrossed product \(\mathcal{A}\). We consider the sequence 
\[
\{U^n f_n\}_{n \in \mathbb{Z}_+} \subseteq \mathcal{A},
\]
where \(f_n = E_n(A)\), for \(n \in \mathbb{Z}_+\). We note that the series \(\sum_{n \in \mathbb{Z}_+} U^n f_n\) does not converge to \(A\) in general. The \(k\)th arithmetic mean of \(A\) is defined to be the element \(A_k = \frac{1}{k+1} \sum_{i=0}^k S_i\), where \(S_i = \sum_{n=0}^i U^n f_n\). Then, the sequence \(\{A_k\}_{k \in \mathbb{Z}_+}\) is norm convergent to \(A\) [11, p. 524].

**Theorem 2.6.** Let \(A, B \in \mathcal{A}\) and \(E_m(A) = f_m \in C_0(X)\), \(E_m(B) = g_m \in C_0(X)\), for all \(m \in \mathbb{Z}_+\). The following statements are equivalent.

1. The multiplication operator \(M_{A,B} : \mathcal{A} \to \mathcal{A}\) is compact.
2. The following assertions are valid, for all \(m, n \in \mathbb{Z}_+\).
   a. \(f_m \circ \phi^{n+l}g_n)(X_a) = \{0\}\), for all \(l \in \mathbb{Z}_+\).
   b. \(\lim_{l \to \infty} (f_m \circ \phi^{n+l}g_n)(x) = 0\), for all \(x \in X_1\).
   c. The sequence \(\{f_m \circ \phi^{n+l}g_n\}_{l \in \mathbb{Z}_+}\) is pointwise equicontinuous.

**Proof.** It is sufficient to prove the theorem for \(A, B \in \mathcal{A}_1\).

The condition (1) implies the condition (2) by Lemmas 2.1, 2.2 and 2.4. We will show the opposite direction.

If \(A = \sum_{m=0}^p U^m f_m\) and \(B = \sum_{n=0}^q U^n g_n\), for \(p, q \in \mathbb{Z}_+\), we have
\[
M_{A,B} = \sum_{m=0}^p \sum_{n=0}^q M_{U^m f_m, U^n g_n}
\]
and the assertion follows from Proposition 2.3.

If \(A, B \in \mathcal{A}\) and \(k \in \mathbb{Z}_+\), we denote by \(A_k\) and \(B_k\) the \(k\)th arithmetic mean of \(A\) and \(B\) respectively. Since the Fourier coefficients of \(A\) and \(B\) satisfy the condition (2), the Fourier coefficients of \(A_k\) and \(B_k\) satisfy the condition (2) as well. Thus, the operator \(M_{A_k,B_k}\) is compact, for all \(k \in \mathbb{Z}_+\). The operator \(M_{A,B}\) is the norm limit of the sequence \(\{M_{A_k,B_k}\}_{k \in \mathbb{Z}_+}\) and hence it is compact.

As a corollary of the above theorem, we obtain the following characterization of the compact elements of the algebra \(\mathcal{A}\).
Corollary 2.7. Let $A \in A$ and $E_m(A) = f_m \in C_0(X)$, for all $m \in \mathbb{Z}_+$. Then, $A$ is compact element of $A$, if and only if the following conditions are satisfied, for all $m, n \in \mathbb{Z}_+$.

1. $(f_m \circ \phi^{n+l}f_n)(X_\alpha) = \{0\}$, for all $l \in \mathbb{Z}_+$.
2. $(f_m(X_r) = \{0\}$.
3. The sequence $(f_m \circ \phi^{n+l}f_n)_{l \in \mathbb{Z}_+}$ is pointwise equicontinuous.

Proof. It is sufficient to prove the corollary for $A \in A_1$.

Firstly, we will show the forward direction. The conditions (1) and (3) are satisfied by Theorem 2.3. Let $x$ be a recurrent point. Then, there exists a strictly increasing sequence $\{l_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}_+$, such that $\lim_{i \to \infty} \phi^{n+l_i}(x) = x$, which implies that $\lim_{i \to \infty} (f_m \circ \phi^{n+l_i}f_n)(x) = f_m^2(x)$. It follows from Lemma 2.3 that $f_m(x) = 0$.

Now, we will show the opposite direction. In view of Theorem 2.4, it suffices to prove that $\lim_{i \to \infty} (f_m \circ \phi^{n+l_i}f_n)(x) = 0$, for all $m, n \in \mathbb{Z}_+$ and $x \in X$. Let $m_0, n_0 \in \mathbb{Z}_+$ and $x_0 \in X$, such that $\lim_{n \to \infty} (f_m \circ \phi^{n+l_i}f_n)(x_0) \neq 0$. Then, there exists a natural number $k_0$ and a strictly increasing sequence $\{l_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}_+$, such that $|(f_m \circ \phi^{n+l_i}f_n)(x_0)| \geq \frac{1}{k_0}$, for all $i \in \mathbb{N}$. Hence, $|(f_m \circ \phi^{n+l_i}f_n)(x)| \geq \frac{1}{k_0}$, for all $i \in \mathbb{N}$, which implies that $\phi^{n+l_i}(x_0) \in D_{k_0}(f_m) = \{x \in X : f_m(x) \geq \frac{1}{k_0}\}$, for all $i \in \mathbb{N}$. By the condition (2), we obtain that $x_0 \in X_r \setminus X$ and hence, $\phi^{n+l_i}(x_0) \neq \phi^{n+l_j}(x_0)$, for $i \neq j$. Moreover, the set $D_{k_0}(f_m)$ is compact and hence, there exists a point $\tilde{x} \in X_{r,i} \cap D_{k_0}(f_m)$ and a subsequence $\{l_{i_j}\}_{j \in \mathbb{N}}$, such that $\lim_{j \to \infty} \phi^{n+l_{i_j}}(x_0) = \tilde{x}$. Let $W_{\delta}$ be an open neighbourhood of $\tilde{x}$, such that $\{f_m(x)\} \geq \frac{1}{k_0}$, for all $x \in W_{\delta}$. By the condition (1), we have that $(f_m \circ \phi^{n+l_{i_j}}f_n)(\tilde{x}) = 0$, for all $l \in \mathbb{N}$ and by (c) we have that the sequence $(f_m \circ \phi^{n+l_i}f_n)_{l \in \mathbb{Z}_+}$ is equicontinuous at $\tilde{x}$. Therefore, there is an open neighbourhood $W_{\tilde{x}}'$ of $\tilde{x}$, such that $|(f_m \circ \phi^{n+l_i}f_n)(x)| < \frac{1}{k_0}$, for all $l \in \mathbb{N}$ and $x \in W_{\tilde{x}}'$. Let $W = W_{\tilde{x}} \cap W_{\tilde{x}}'$ and $j_1, j_2 \in \mathbb{N}$ be such that $\phi^{n+l_{i_j_1}}(x_0), \phi^{n+l_{i_j_2}}(x_0) \in W$ and $l_{i_2} > l_{i_2} + n_0$. Then, $\frac{1}{5k_0} \geq |(f_m \circ \phi^{l_{i_2-j_2}}f_n)(\phi^{n+l_{i_j_1}}(x_0))| = |f_m(\phi^{n+l_{i_2}}(x_0))||f_n(\phi^{n+l_{i_j_1}}(x_0))| \geq \frac{1}{4k_0^2}$, which is a contradiction. \hfill \Box

Let us see now how Theorem 2.6 applies to two special cases. If $X$ is a discrete space or it has no isolated points, we obtain the following characterizations.

Corollary 2.8. Let $X$ be a discrete space, $A, B \in A$ and $E_m(A) = f_m, E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then, the following are equivalent.

1. The multiplication operator $M_{A,B} : A \to A$ is compact.
2. $(f_m \circ \phi^{n+l_i}g_n)(X_r) = \{0\}$, for all $m, n, l \in \mathbb{Z}_+$.
3. $\lim_{l \to \infty} (f_m \circ \phi^{n+l_i}g_n)(x) = 0$, for all $x \in X$.

Proof. We will show the implication (2) $\Rightarrow$ (3). Assume that there exist $m, n \in \mathbb{Z}_+$ and $x \in X \setminus X_r$, such that $\lim_{l \to \infty} (f_m \circ \phi^{n+l_i}g_n)(x) \neq 0$. Then, there exists an $\epsilon > 0$ and a strictly increasing sequence $\{l_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}_+$, such that $|(f_m \circ \phi^{n+l_i}g_n)(x)| \geq \epsilon$, for all $i \in \mathbb{N}$. Moreover, $x \notin X_r$, and therefore $\phi^{n+l_i}(x) \neq \phi^{n+l_j}(x)$, for $i \neq j$. This is a contradiction, since $f_m \in C_0(X)$. 

The implication (3) $\Rightarrow$ (1) follows from Theorem 2.6 and the implication (1) $\Rightarrow$ (2) follows from Lemma 2.3.

**Corollary 2.9.** Let $X$ be a space without isolated points, $A, B \in \mathcal{A}$ and $E_m(A) = f_m, E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then, the following are equivalent.

1. The multiplication operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ is compact.
2. $(f_m \circ \phi^{n+l}g_n)(X) = \{0\}$ for all $l, m, n \in \mathbb{N}$.
3. $M_{A,B} = 0$.

**Proof.** It follows immediately from Theorem 2.6.

The next example shows that the equicontinuity condition cannot be omitted in general.

**Example 2.10.** We consider the dynamical system $(X, \phi)$ where

$$X = \{0\} \cup \{x_n\}_{n \in \mathbb{Z}} \cup \{2\}, \quad x_n = \begin{cases} \frac{1}{m+1}, & n < 0 \\ \frac{1}{n+1}, & n \geq 0 \end{cases}$$

and $\phi$ is the homeomorphism

$$\phi(0) = 0, \quad \phi(x_n) = x_{n-1}, \quad \phi(2) = 2.$$

We define the monomials $A = U^1f$ and $B = U^1g$ of the semicrossed product $\mathcal{A}$ by the following formulae.

$$f(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{else} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x \leq 1 \end{cases}.$$

We observe that, $(f \circ \phi^{l+1}g)(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$ and $\lim_{l \to \infty}(f \circ \phi^{l+1}g)(x) = 0$, for all $x \in X_i$. However, the sequence $\{f \circ \phi^{l+1}g\}_{l \in \mathbb{Z}_+}$ is not equicontinuous at $x = 2$ and the multiplication operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ is not compact.

In the following theorem, we characterize the ideal generated by the set of compact elements of the semicrossed product $\mathcal{A}$. Recall that $Y \subseteq X$ is called wandering if the sets $\phi^{-1}(Y), \phi^{-2}(Y), \ldots$ are pairwise disjoint. If $\phi$ is a homeomorphism, this condition is equivalent to the condition that $\phi^m(Y) \cap \phi^n(Y) = \emptyset$ for all $m, n \in \mathbb{Z}_+, m \neq n$. A point $x \in X$ is called wandering if it possesses an open wandering neighborhood. Otherwise it is called non wandering. If $x$ is a non wandering point and $W$ is an open neighbourhood of $x$, then there exists $m \in \mathbb{N}$, such that $W \cap \phi^m(W) \neq \emptyset$. Note that we may assume that $m$ is arbitrarily large [2, p. 129]. We will denote by $X_w$ the set of wandering points of $X$. It is clear that $X_w$ is the union of all open wandering subsets of $X$.

**Theorem 2.11.** The ideal generated by the compact elements of $\mathcal{A}$ is the set

$$\{A \in \mathcal{A} \mid E_n(A)(X \setminus X_w) = \{0\}, \forall n \in \mathbb{Z}_+ \text{ and } E_0(A)(X_a) = \{0\}\}.$$

**Proof.** Let $A$ be a compact element of $\mathcal{A}$. We will show that $E_n(A)(X \setminus X_w) = \{0\}$, for all $n \in \mathbb{Z}_+$ and $E_0(A)(X_a) = \{0\}$. Let us denote by $f_n$ the $n$th Fourier coefficient, $E_n(A)$, of $A$, for all $n \in \mathbb{Z}_+$. We suppose that there exists an $n_0 \in \mathbb{Z}_+$ and a point $x_0 \in X \setminus X_w$, such that $f_{n_0}(x_0) \neq 0$. Without loss of generality, we assume that $|f_{n_0}(x_0)| = 1$. We note that by Corollary 2.7, $x_0$ cannot be isolated, since a non wandering isolated point is periodic and hence recurrent. Thus, $x_0$ is an accumulation point, and we have $(f_{n_0} \circ \phi^{n_0+l}f_{n_0})(x_0) = 0$, for all $l \in \mathbb{Z}_+$. Let $0 < \epsilon < 1/4$. The sequence $\{f_{n_0} \circ \phi^{n_0+l}f_{n_0}\}_{l \in \mathbb{Z}}$ is equicontinuous at $x_0$ and therefore,
there exists an open neighborhood $U$ of $x_0$, such that $|(f_{n_0} \circ \phi^{n_0+l}f_{n_0})(x)| < \epsilon$, for all $x \in U$ and $l \in \mathbb{Z}_+$. There exists an open neighborhood $V$ of $x_0$, such that $|f_{n_0}(x)| > 1 - \epsilon$, for all $x \in V$. Let $W = U \cap V$. Since $x_0$ is a non wandering point, there exists an $l_0 \in \mathbb{Z}_+$ such that $l_0 \geq n_0$ and $W \cap \phi^{l_0}(W) \neq \emptyset$. Let $y \in W$, such that $\phi^{l_0}(y) \in W$. Then,

$$
\epsilon > |(f_n \circ \phi^{l_0}f_n)(y)| = |f_n(\phi^{l_0}(y))||f_n(y)| > (1 - \epsilon)^2
$$

which is a contradiction. Furthermore, it is evident from Corollary 2.7 that $f_0(X_0) = \{0\}$. We observe that the conditions $f_n(X \setminus X_0) = \{0\}$, for all $n \in \mathbb{Z}_+$ and $f_0(X_0) = \{0\}$, we have already proved for a compact element $A$, are satisfied by the elements of the ideal generated by the compact elements as well.

To complete the proof, we will prove that if $A \in \mathcal{A}$ satisfies $E_n(A)(X \setminus X_0) = \{0\}$, for all $n \in \mathbb{Z}_+$ and $E_0(A)(X_0) = \{0\}$, then $A$ belongs to the ideal generated by the compact elements. We denote by $f_n$ the $n$th Fourier coefficient $E_n(A)$ of $A$, for all $n \in \mathbb{Z}_+$. It is sufficient to show that $U^n f_n$ belongs to the ideal generated by the compact elements, $\forall n \in \mathbb{Z}_+$.

First, we consider the case $n > 0$. Let $C$ be the set $\{h \in C_0(X) : D(h) = h^{-1}(C \setminus \{0\}) \text{ is wandering} \}$. Let $h \in C$. Since $D(h)$ is wandering and $n > 0$, the functions $(h \circ \phi^{n+l}h)$ are identically 0, for every $l \in \mathbb{Z}_+$. Hence, $U^n h$ satisfies the conditions (a) and (c) of Corollary 2.7. Since $X_r$ is contained in $X \setminus X_0$, condition (c) is also satisfied and $U^n h$ is a compact element of $A$. The norm closed algebra generated by $C$, is the ideal $\{f \in C_0(X) : f(x) = 0, \forall x \in X \setminus X_0 \}$ of $C_0(X)$. In particular, $f_n$ belongs to this algebra. We conclude that $U^n f_n$ belongs to the ideal generated by the compact elements.

We consider now the case $n = 0$. The proof is the similar to the proof in the case $n > 0$, considering the set $\mathcal{F} = \{h \in C_0(X) : h(X_0) = \{0\}, D(h) = h^{-1}(C \setminus \{0\}) \text{ is wandering} \}$ instead of $C$.

\[ \square \]

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