A universal formula for the field enhancement factor

Debabrata Biswas
Bhabha Atomic Research Centre, Mumbai 400 085, INDIA
Homi Bhabha National Institute, Mumbai 400 094

The field enhancement factor (FEF) is an important quantity in field emission calculations since the tunneling electron current depends very sensitively on its magnitude. The exact dependence of FEF on the emitter height $h$, the radius of curvature at the apex $R_a$, as well as the shape of the emitter base is still largely unknown. In this work, a universal formula for the field enhancement factor is derived. It depends on the ratio $h/R_a$ and has the form

$$\gamma_a = \frac{2h}{\log(4h/R_a)} - \alpha$$

as $\gamma_a \approx a(b + h/R_a)\gamma$ with $0 < \sigma < 1$. These expressions for $\gamma_a$ seem to be very different from the ellipsoid and hyperboloid results and can at best be local approximations of a more general expression. A formula, applicable to a wide class of emitters has thus been elusive and it is our aim here to provide one that is universally applicable.

We shall approach the problem analytically using a general nonlinear line charge distribution and provide a formula that is universally valid and reduces to Eq. 1 for the ellipsoid. However, it depends on properties of the charge distribution that are a priori unknown. We therefore address the problem numerically and establish that Eq. 1 is still valid and $\alpha$ is an indicator of the absence of shielding by the emitter base. Thus, for the hyperboloid, $\alpha = 0$ while for an emitter top placed on a cylindrical post, $\alpha \simeq 2.6$, indicative of the absence of shielding.

THE LINE CHARGE MODEL

The problem of an emitter of height $h$ placed on grounded metallic plane and aligned along an external electrostatic field $-E_0\hat{z}$, can be modelled by a vertical line charge distribution and its image. Denoting the line charge density by $\Lambda(s)$, the potential at any point $(\rho, z)$ can be expressed as

$$V(\rho, z) = \frac{1}{4\pi\varepsilon_0} \left[ \int_0^L \frac{\Lambda(s)}{[\rho^2 + (z - s)^2]^{1/2}} ds - \int_0^L \frac{\Lambda(s)}{[\rho^2 + (z + s)^2]^{1/2}} ds \right] + E_0 z$$

where $L$ is the extent of the line charge distribution and $E_0$ is the magnitude of the electric field. The zero-potential contour then corresponds to the surface of the emitter and quantities such as the enhancement factor and the principle radii of curvature can be calculated numerically. The line charge distribution can be thought of...
as a projection of the surface charge of the emitter along the axis.

Amongst the unknowns in Eq. \[2\] are the parameters defining the line charge distribution and its extent \(L\). These can in principle be calculated by imposing the requirement that the potential should vanish along the surface of the emitter. A shortcoming of the line charge model is the absence of image charges due to the anode in the formulation of the problem. The results are therefore valid when the anode is sufficiently far away from the emitter. In the rest of the manuscript, we shall consider axially symmetric emitters and assume \(\Lambda(s)\) to be a nonlinear function of \(s\) unless stated otherwise.

**THE APEX FIELD ENHANCEMENT FACTOR**

We are interested in the field enhancement factor, \(\gamma_a\) at the emitter apex. For axially symmetric emitters aligned along \(\hat{z}\), this is defined as \(\gamma_a = -\frac{1}{E_0} \frac{\partial V}{\partial z}\big|_{\rho=0,z=h}\). Note that for parabolic emitter tips, it has recently been established [18] that \(\gamma = \gamma_a \cos \theta\) where \(\gamma\) is the enhancement factor at a point \((\rho, z)\) on the emitter surface while \(\cos \theta = (z/h)/[(\rho/R_a)^2 + (z/h)^2]^{1/2}\). Thus, the local field around the emitter apex can be determined if the apex field enhancement factor is known.

Our starting point for the FEF is Eq. \[2\]. At the apex,

\[
\frac{\partial V}{\partial z}\big|_{\rho=0,z=h} = -\frac{1}{4\pi\epsilon_0} \left[ \int_0^L \frac{\Lambda(s)}{(h-s)^2} ds - \int_0^L \frac{\Lambda(s)}{(h+s)^2} ds \right] + E_0
\]

so that on writing \(\Lambda(s) = sf(s)\), we have

\[
\frac{\partial V}{\partial z}\big|_{\rho=0,z=h} = -\frac{1}{4\pi\epsilon_0} \left[ \int_0^L ds \left\{ -\frac{f(s)}{h-s} + \frac{hf(s)}{(h-s)^2} - \frac{f(s)}{h+s} + \frac{hf(s)}{(h+s)^2} \right\} \right] + E_0. \tag{4}
\]

Using partial integrations,

\[
\frac{\partial V}{\partial z}\big|_{\rho=0,z=h} = -\frac{1}{4\pi\epsilon_0} \left[ f(L) \ln \left( \frac{h+L}{h-L} \right) (1-C_1) \right. \left. - f(L) \frac{2hL}{h^2-L^2} (1-C_0) \right] + E_0 \tag{5}
\]

where

\[
C_0 = \int_0^L f'(s) \frac{s/(h^2-s^2)}{f(L) L/(h^2-L^2)} ds \tag{6}
\]

\[
C_1 = \int_0^L f'(s) \ln \left( \frac{h+s}{h-L} \right) \frac{L}{f(L) L/(h^2-L^2)} ds. \tag{7}
\]

Note that \(L\) is the height of the line charge distribution and must extend almost till the apex height \(h\). Moreover, since the charge distribution is well behaved and can be expressed as a polynomial function of degree \(n\) (for cases of interest here, \(n \leq 5\)), it obeys Bernstein’s inequality [19]

\[
|f'(x)| \leq \frac{n}{(1-x^2)^{1/2}} \|f\| \tag{8}
\]

where \(x \in [-1,1]\) and \(\|f\|\) denotes the maximum value of \(f\) in this interval. With \(x = s/h\) and applying the inequality, it can be shown that \(C_0 \sim (h^2-L^2)^{1/2}\) is vanishingly small for sharp-tipped emitters. In contrast, \(C_1\) cannot be neglected due to the logarithmic dependence.

**Linear line charge density - the ellipsoid**

For a linear line charge as in case of an ellipsoidal emitter, \(f(s) = 0\) so that \(C_0 = 0 = C_1\). Denoting \(f(s) = \lambda\), a constant, the enhancement factor can be expressed as

\[
\gamma_a = \frac{|\lambda|}{4\pi\epsilon_0 E_0} \left[ \frac{2hL}{h^2-L^2} - \ln \left( \frac{h+L}{h-L} \right) \right] - 1 \tag{9}
\]

\[
\approx \frac{|\lambda|}{4\pi\epsilon_0 E_0} \left[ \frac{2h^2}{h^2-L^2} - \ln \left( \frac{4h^2}{h^2-L^2} \right) \right] - 1 \tag{10}
\]

where the last line assumes that \(h \simeq L\) as the line charge must extend almost till the apex height \(h\). Further, if \(h^2/(h^2-L^2)\) is large, the logarithmic part can be neglected so that

\[
\gamma_a = \frac{|\lambda|}{4\pi\epsilon_0 E_0} \left[ \frac{2h^2}{h^2-L^2} \right] \tag{11}
\]

The parameter \(\lambda\) can be determined by demanding that the potential in Eq. \[2\] vanishes at the apex for \(\Lambda(s) = \lambda s\). Thus,

\[
\lambda = -\frac{4\pi\epsilon_0 E_0 h}{h \ln \left( \frac{h+L}{h-L} \right) - 2L} \tag{12}
\]

using which, the field enhancement factor under the approximation \(L \simeq h\) can be expressed as

\[
\gamma_a = \frac{2h/R_0}{\ln(4h/R_0) - 2} \tag{13}
\]

where \(R_0 = (h^2-L^2)/h\). We show in the appendix that \(R_0 \simeq R_a\), the apex radius curvature for general sharp emitters. We are thus able to recover the field enhancement factor for an ellipsoid at least when \(h/R_a\) is large.
The nonlinear density

The general case corresponding to nonlinear $Λ(s)$ holds greater interest. Since $h \simeq L$, under approximations mentioned earlier, the enhancement factor reduces to

$$\gamma_a = \frac{|f(L)|}{4\pi\varepsilon_0 \varepsilon_a} \frac{2h^2}{h^2 - L^2}$$

(14)

where $f(L)$ can be determined following the procedure for $λ$.

As in the linear case, $Λ(L)$ can be determined by demanding that the potential vanishes at the emitter apex, $(\rho = 0, z = h)$. Thus,

$$4\pi\varepsilon_0 \varepsilon_a h = \int_0^L ds \left[ \frac{s f(s)}{h + s} - \frac{s f(s)}{h - s} \right]$$

(15)

$$= \int_0^L ds \left[ f(s) - \frac{h f(s)}{h + s} + f(s) - \frac{h f(s)}{h - s} \right].$$

(16)

On integrating by parts,

$$4\pi\varepsilon_0 \varepsilon_a h = -\left[ f(L) h \ln \left( \frac{h + L}{h - L} \right) \left( 1 - C_1 \right) - f(L) 2L \left( 1 - C_2 \right) \right]$$

(17)

where $C_2 = \int_0^L \frac{f'(s)}{f(s)} L^2 ds$. Thus,

$$f(L) = -\frac{4\pi\varepsilon_0 \varepsilon_a h}{h \ln \left( \frac{h + L}{h - L} \right) \left( 1 - C_1 \right) - 2L \left( 1 - C_2 \right)}$$

(18)

and hence with $L \simeq h$,

$$\gamma_a \simeq \left[ \frac{2h/R_0}{\alpha_1 \ln \left( 4h/R_0 \right) - \alpha_2} \right]$$

(19)

where $\alpha_1 = 1 - C_1$, $\alpha_2 = 2(1 - C_2)$ and $R_0 = (h^2 - L^2)/h \simeq R_a$. Eq. [19] is thus an expression for the field enhancement factor that is generally applicable to all shapes. The correction terms $C_1$ and $C_2$ are however a priori unknown since they depend on details of the charge distribution. We have tested Eq. [19] numerically for several emitter shapes. For example, in case of a conical base with a parabolic top with $h/R_0 = 35009$, $C_1 = -0.296$ and $C_2 = -1.116$. The numerically measured value of $\gamma_a$ is 6298 while Eq. [19] predicts 6294.

Eq. [19] is valid for a single emitter rather than a family of emitters having a similar base (such as conical). For a set of 10 emitters with a conical base and $h/R_a$ varying between 5000 to 200000, the best fitted value of $C_1 = -0.11657$ and $C_2 = -0.21655$. The average error in prediction of $\gamma_a$ was found to be 2.02%. As an alternate formula for field enhancement, Eq. [1] was fitted on the same data set of conical emitters with respect to the parameter $\alpha$. The best fit was for $\alpha = 0.88937$ and the average error in predicting $\gamma_a$ was 1.94%.

FIG. 1. The field enhancement factor for parabolic emitter tops with a conical base (squares) with the best fit using Eq. [1] (continuous line).

FIG. 2. The field enhancement factor for parabolic emitter tops with a cylindrical base (squares) with the best fit using Eq. [1] (continuous line).

Similarly for a family of 10 parabolic emitter tops on a cylindrical base, the fitted values using Eq. [19] are $C_1 = 0.07611$ and $C_2 = 0.062489$. On using Eq. [1] the fitted value of the parameter is $\alpha = 2.5794$. The average error in prediction is 0.73% for Eq. [19] and 0.76% for Eq. [1].

SUMMARY AND CONCLUSIONS

We have derived a formula for the field enhancement factor based on the line charge model. It reduces to the result for ellipsoid that can be obtained by solving Laplace’s equation in prolate spheroidal co-ordinates. In general, it depends on parameters that can be evaluated
only when the line charge distribution is known numerically. Interestingly, a set of optimally chosen parameters represents an entire family of emitters and can be used with acceptable errors.

An alternate formula (Eq. 1) that generalizes the known result for ellipsoid and hyperboloid was also used to study a set of emitters with conical and cylindrical bases respectively. The parameter \( \alpha \) is known to be 0 for the hyperboloid where shielding of field lines by the base is large. The conical base was found to have \( \alpha \approx 0.89 \). The ellipsoid is known to have \( \alpha = 2 \) while it was found to be 2.58 for a cylindrical base. Thus, \( \alpha \) appears to be an indicator of shielding by the emitter base with a larger value corresponding to lower shielding. We expect Eq. 1 to be of relevance even in emitter arrays/clusters where electrostatic shielding is significant when the mean spacing is smaller than the emitter height.

**APPENDIX: THE APEX RADIUS OF CURVATURE**

The apex radius of curvature \( R_a \) can be expressed as \( R_a = (\partial V/\partial z)/((\partial^2 V)/(\partial \rho^2)) \) evaluated at the apex. An expression for \( \partial^2 V/\partial z \) at the apex has already been derived and the form in Eq. 5 will be used. We shall now arrive at a form for \( \partial^2 V/\partial \rho^2 \). On differentiating Eq. 2 twice and evaluating at the apex,

\[
\frac{\partial^2 V}{\partial \rho^2} = -\frac{1}{4\pi \epsilon_0} \left[ \int_0^L \frac{sf(s)}{(h-s)^3} \, ds - \int_0^L \frac{sf(s)}{(h+s)^3} \, ds \right].
\]

(20)

This can be expressed as

\[
\frac{\partial^2 V}{\partial \rho^2} = -\frac{1}{4\pi \epsilon_0} \left[ \int_0^L ds \left\{ -\frac{f(s)}{(h-s)^2} - \frac{f(s)}{(h+s)^2} + \frac{hf(s)}{(h-s)^3} + \frac{hf(s)}{(h+s)^3} \right\} \right]
\]

(21)

which on integration by parts yields

\[
\frac{\partial^2 V}{\partial \rho^2} = \frac{1}{4\pi \epsilon_0} \left[ f(L) \frac{2L}{h^2 - L^2} (1 - C_0) - \frac{2h^2 Lf(L)}{(h^2 - L^2)^2} (1 - C_3) \right]
\]

(22)

where

\[
C_3 = \int_0^L \frac{f'(s)}{f(L) (h^2 - s^2)^2} \, ds.
\]

(23)

Using Bernstein’s inequality again, \( C_3 \sim (h^2 - L^2)^{3/2} \) which is vanishingly small for sharp emitters. Further, the second term in Eq. 22 is dominant. Thus,

\[
\frac{\partial^2 V}{\partial \rho^2} \simeq -\frac{1}{4\pi \epsilon_0} \left[ \frac{2h^3 f(L)}{(h^2 - L^2)^2} \right]
\]

(24)

and

\[
\frac{\partial V}{\partial z} \simeq -\frac{1}{4\pi \epsilon_0} \frac{2f(L)h^2}{h^2 - L^2}
\]

(25)

so that \( R_a \simeq \frac{h^2 - L^2}{h} \). Thus \( R_o \simeq R_a \).

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