A numerical calculation method for frequency-shift of a plasma wave using wave digital filters

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Abstract. It is difficult to measure any quantity in plasmas, since the physical quantity is usually measured only under very severe experimental conditions. For instance, it is often observed that the frequency of the wave changes during propagation and also different waves from the incident one are newly stimulated. Therefore, proposed is the most suitable method for computer experiment using technique of signal processing.

In plasmas, various phenomena are described by a fluid theory. The fluid theory consists of a system of Maxwell's equations and that of Euler's equations. Applying the technique of wave digital filters (WDF) to these equation systems, the partial differential equations are integrated directly. Considering all physical quantities as currents, a system of partial differential equation systems is transformed into an equivalent Kirchhoff circuit. Wave-flow diagrams corresponding to their Kirchhoff circuits are obtained by the ordinary WDF procedures, which are algorithms themselves for numerical calculations. Therefore, we may safely say that these algorithms construct a computer experimental apparatus, which simulates many plasma experiments concerning propagation properties of plasma waves. Here, we simulate frequency-shift phenomena of some plasma wave by means of the function with respect to spectrum analysis of this apparatus.

1. Introduction

Some normal form of the ordinary differential equation of the first order can be solved by only integration as inverse manipulation of differentiation. Even if the problem cannot be solved analytically, the problem is solved by integrating numerically with the aid of trapezoidal rule and so on. This technique cannot be directly applied to partial differential equations from the mathematical point of view.

However, from the viewpoint of signal processing, it is realized to get the numerical solution for the partial differential equations by considering multidimensional operator of integration as an inverse operator, because mathematical manipulations such as differentiation and integration and so on are considered to be operators.

Some system of partial differential equations describing physical system can be transformed into an electric circuit. According to the Maxwell's analogy proposed in 1873, there has a merit that it is easy
for us to treat the physical systems in computer if the physical systems are transformed into the
circuits, since the mathematical models for the circuits corresponding to the various physical systems
become the same ones when we treat them in computer.

From this point of view, a new method for numerically solving partial differential equations that
describe physical systems has been proposed so far [1-3]. The fundamental principle of this method is
as follows: In plasmas, various phenomena are described by a fluid theory. In the fluid theory, a
system of plasma equations consists of a system of Maxwell's equations and that of Euler's equations.
Euler's equation system is nonlinear, while Maxwell's one is linear. However, this method can treat
their equations consistently whether they are linear or not. An outline of this method is as follows: At
first, a system of partial differential equations describing some physical system is transformed into
equivalent Kirchhoff circuits as electric circuits. And these circuits are transformed into wave digital
filter (WDF) structure by digitizing based on wave digital filter theory. Obtained WDF structure leads
to a numerical algorithm which is reduced to a solution in time-domain. Since the sampling of all
variables is done along the time axis in the field of digital signal processing by nature, this method is
considered to be a kind of FDTD method. Therefore, it is considered that this method is justified and
also suitable for the analysis of time-dependent or transient phenomena [4,5]. In addition, the stability
of numerical calculation is guaranteed, because the numerical calculation is carried out after
constructing the digital filter structure.

Furthermore, it is found that this numerical method is very suitable for some computer experiments.
Here, using this virtual plasma experimental apparatus, many plasma experiments concerning
propagation properties of plasma waves are carried out and this method is furthermore applied to
plasma scattering phenomena.

2. Basic plasma equations
First, Maxwell's equations are shown as

\[ \nabla \times H = j + \frac{\partial (\varepsilon_0 E)}{\partial t}, \]  (1)

\[ \nabla \times E = -\frac{\partial (\mu_0 H)}{\partial t}, \]  (2)

where \( j \) is given by

\[ j = -n_e q_e v_e, \]  (3)

and \( n_e, v_e \) and \( -q_e \) stand for density, velocity and charge of electron, respectively.

Second, as for a system of Euler's equations, ion is considered to be static as background for high
frequency phenomena like around 0.1 GHz, because the mass of ion is much heavier than that of
electron. From this, we have only to consider the electron fluid alone for Euler's equation, continuity
equation and equation of state, which are shown as

\[ m_e n_e \left[ \frac{\partial v_e}{\partial t} + (v_e \cdot \nabla)v_e \right] + \nabla p_e = f, \]  (4)

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0, \]  (5)

\[ p_e = C (m_e n_e)^{\gamma_e}, \]  (6)

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where \( m_e, p_e \) and \( \kappa_e \) stand for mass, pressure and the ratio of specific heats of electron fluid, respectively. \( f \) is an external force against electron particle, which expresses the electric force and the Lorentz force shown as

\[
f = - n_e q_e \left[ E + v_e \times \mu_0 (H_0 + H) \right],
\]

where \( H_0 \) denotes a static magnetic field.

### 3. Equivalent Kirchhoff circuits

At first, usual coordinates \((x, y, z)\) are replaced with \((t_1, t_2, t_3)\) and furthermore \( t_4 \) is added in place of time, \( t \), where the relation between \( t_4 \) and \( t \) is given by

\[
v_4 = \frac{dt_4}{dt}.
\]

Treating the problem in four-dimensional manifold [6, 7], physical variables are functions of four-dimensional vector, \( t = (t_1, t_2, t_3, t_4)^T \). When partial derivative with respect to \( t_i \) is denoted as \( D_i \), the operator vector \( D \) is defined as

\[
D = (D_1, D_2, D_3, D_4)^T.
\]

And furthermore introducing offset sampling [8], \( t \) and \( D \) are transformed into

\[
t' = (t'_1, t'_2, t'_3, t'_4)^T
\]

\[
D' = (D'_1, D'_2, D'_3, D'_4)^T,
\]

respectively.

Here, problems are analyzed from a viewpoint of multidimensional signal processing [9,10]. Therefore, when we transform plasma equations into equivalent electric circuits, the supplementary operator vector as

\[
D'' = (D''_1, D''_2, D''_3, D''_4, D''_5, D''_6, D''_7),
\]

is also introduced, where 4-dimensional space is embedded into 7-dimensional one [3].

From inverse point of view, embedding is considered to be an extension into higher dimensional space. It is said that lower dimensional problems can be often solved by transforming them into higher dimensional problems, which is safely said to be called as "Oka's lifting principle" [11]. We have transformed a system of Maxwell's equations as follows:

\[
D'_7 (e^\omega \hat{E}_1) + (D'_3 - D'_6) \hat{H}_2 + (D'_5 - D'_2) \hat{H}_3 + \hat{j}_1 = 0,
\]

\[
D'_7 (e^\omega \hat{E}_2) + (D'_1 - D'_4) \hat{H}_1 + (D'_6 - D'_3) \hat{H}_2 + \hat{j}_2 = 0,
\]

\[
D'_7 (e^\omega \hat{E}_3) + (D'_2 - D'_5) \hat{H}_2 + (D'_4 - D'_1) \hat{H}_3 + \hat{j}_3 = 0,
\]

\[
D'_7 (\mu^\omega \hat{H}_1) + (D'_6 - D'_3) \hat{E}_2 + (D'_2 - D'_5) \hat{E}_3 = 0,
\]

\[
D'_7 (\mu^\omega \hat{H}_2) + (D'_4 - D'_1) \hat{E}_3 + (D'_3 - D'_6) \hat{E}_1 = 0,
\]
where hatted physical quantities stand for normalized ones.

On the other hand, a system of Euler's equations are also transformed as follows [12]:

\[
\sum_{k=1}^{6} \eta_{0k} D_{k}^2 (\eta_{hk} \hat{p}_k) + \sum_{k=1}^{3} [D_{k}^2 (\hat{v}_d + \hat{v}_d) + D_{k+3} (\hat{v}_d - \hat{v}_d)] = 0,
\]

\[
\sum_{k=1}^{6} \eta_{dk} D_{k}^2 (\eta_{hk} \hat{v}_d) + D_{l}^2 (\hat{v}_d + \hat{p}_e) + D_{l+3} (\hat{v}_d - \hat{p}_e) = \hat{f}_d \quad (l = 1, 2, 3).
\]

Considering all physical quantities as currents and also replacing the differential operators with inductances, a system of partial differential equations is transformed into what is called an equivalent Kirchhoff circuit, which is named by considering that each equation of the system mentioned above represents the Kirchhoff's second law. Kirchhoff circuit of Maxwell's equations is shown in figure 1. On the other hand, Kirchhoff circuit of Euler's equations is shown in figure 2. Resistances are not used in these circuits because of the losslessness of the physical system. A pair of zigzag lines facing each other stands for an ideal (lossless) transformer traditionally [13].
5. Computer experiments and results

As an example, analyzed are propagation properties of some plasma wave in addition to scattering phenomena from a cylindrical conductor as an obstacle immersed in a plasma which is magnetized perpendicularly to the paper plane in figure 6 [14,15]. When a plane electromagnetic wave with stimulated longitudinal waves is incident from left-hand side on a cylindrical conductor whose radius \( a \) is given by \( ka = 1 \), calculations are done in the area \( 3\lambda \times 3\lambda \), where \( \lambda \) is wavelength of the incident electromagnetic wave at initial time.

Figure 3. Wave-flow diagram for Maxwell's Euler's equations.

Figure 4. Wave-flow diagram for equations.

Figure 5. Wave-flow diagram for connecting network.

Figure 6. Setup for numerical calculations.

Boundary conditions in this case are as follows: For Maxwell's equations, tangential component of the electric field on the boundary surface is null because of continuity condition of tangential
components of electric fields between both media. And for Euler's equations, normal components of both velocity and external force are null, because the cylindrical conductor is considered merely as a rigid body. Furthermore, by similar reason, null is the first derivative of both the tangential component of velocity and the pressure with respect to normal direction to the boundary surface.

Figure 7. Observed waveform at the point P. Figure 8. Frequency spectrum of the observed waveform.

It is often observed that the frequency of the wave changes during propagation and also different waves from the incident one are newly stimulated. To inspect these effects, the electric field is observed at a point P behind the conductor cylinder in the plasma area of the plasma experimental apparatus as shown in figure 6. Electric field observed at the point P is recorded automatically and is drawn as in figure 7. After all, frequency spectrum as shown in figure 8, which corresponds to the waveform in figure 7, is also obtained. Such a function like this is contained in this virtual plasma experimental apparatus.

Figure 9. Observed waveform for \(\omega_c / \omega = 2.5\). Figure 10. Observed waveform for \(\omega_c / \omega = 4.0\).
Since figures 7 and 8 show waveform and spectrum, respectively when the frequency of the incident wave corresponds to the hybrid resonant one, the calculated results are very interesting, where electron density, static magnetic field and wavelength of incident wave are $1.0 \times 10^{15}$ $[\text{m}^{-3}]$, $\omega_c/\omega = 0.321$ and 1.0 [m], respectively.

As a result, in the frequency spectrum of figure 8, it is clearly found that the third peak is appeared whose frequency is higher than that of the second peak which corresponds to that of the incident wave. In addition, the frequency of the first peak is lower than that of any of other two peaks. That is why the frequency of the first peak corresponds to that of the envelope of the observed waveform.

**Figure 11.** Observed waveform for $\omega_c/\omega = 7.0$.

**Figure 12.** Corresponding frequency spectrum to the waveform for $\omega_c/\omega = 2.5$.

**Figure 13.** Corresponding frequency spectrum to waveform for $\omega_c/\omega = 4.0$. 
Figure 14. Corresponding frequency spectrum to the waveform for $\frac{\omega_c}{\omega} = 7.0$.

And furthermore, figures 9 ~ 11 show the observed waveforms corresponding to the static magnetic fields $\omega_c / \omega = 2.5, 4.0$ and 7.0, respectively, where the electron density is $5.574 \times 10^{14}$ [m$^{-3}$]. Here, in figures 9 ~ 11, only longitudinal components of the electric field are drawn, because separated effects from the incident transverse electromagnetic wave are considered to be emphasized.

On the other hand, figures 12 ~ 14 show the frequency spectra corresponding to the waveforms for figures 9 ~ 11, respectively.

Figure 15. Peak frequency–shift of the spectrum against static magnetic field $\frac{\omega_c}{\omega}$.

Watching these figures, it is easily found that ripples on the swell become smaller according as applied magnetic field becomes stronger. This corresponds to the fact that the peak of the frequency spectrum around 0.3GHz as the incident wave becomes smaller similarly.
From these figures, it is clearly found that the stronger the static magnetic field becomes, the lower
the frequency of the main peak becomes. Figure 15 shows that the frequency of the main peak in the
spectrum of the waveform of the propagating plasma wave shifts lower against increasing magnetic
field from $\omega_0 / \omega = 0$ to $\omega_1 / \omega = 10$.

Since this apparatus is a virtual experimental one, it is natural that this method is suitable for the
time-dependent phenomena. However, even in steady state, this method is applicable if the time is
sufficiently elapsed. The scattering pattern is also actually obtained by this manner.

Figure 16. A scattering pattern of the longitudinal plasma wave.

Figure 16 shows an example of scattering pattern as a steady state of the longitudinal plasma wave
against the conductor cylinder when electron density, static magnetic field and wavelength of incident
wave are $1.0 \times 10^{15}$ [m$^{-3}$], $\omega_0 / \omega = 2.0$ and 1.0 [m], respectively.

6. Conclusion

In calculating the frequency-shift phenomena of a plasma wave during propagation, used is the so-
called wave-digital method which utilizes the principle of the multidimensional wave digital filters.

Considering all physical quantities as currents, transforming partial differential equations into
equivalent electric circuits by using Kirchhoff's second law and moreover digitizing them, various
parameters and derived phenomena in plasmas are calculated with time. In other words, this method is
a kind of FDTD method, because all variables are sampled along the time axis as usual procedure in
digital signal processing, and spatial and physical variables are calculated every sampling time.
Therefore, the validity of this method is guaranteed.

Furthermore, other many advantages are added by transforming a system of differential equations
into a circuit network system, because differential equations themselves in the field of mathematics
can be grasped as a physical system based on the energy principle. Therefore, singular types of partial
differential equations, as is often shown in the field itself of mathematics, are naturally excluded.
There are some instances: one is that, by introducing the power wave, this method can treat the partial
differential equations consistently whether they are linear or not, another is that inconveniences
inherent in mathematical or numerical manipulations are avoided by introducing circuit concepts like
stability of the circuit system as an example, and so forth.
From this, applying the wave-digital method to the plasma equation system consisting of a system of Maxwell's equations and that of Euler's equations, a plasma simulator is actualized. As plasma is the dispersive medium, it is often observed that the frequency of the wave changes during propagation and also different waves from the incident one are newly stimulated.

Observing the electric field at an arbitrary point in the plasma area of the virtual experimental apparatus, we can inspect them.

Accordingly, we may safely say that this virtual plasma experimental apparatus is called wave-digital plasma simulator. It is considered that this wave-digital plasma simulator is promising for the virtual experiments such as nonlinear interaction among various plasma waves.

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