INVERSE PROBLEMS FOR THE HEAT EQUATION
WITH MEMORY

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ABSTRACT. We study inverse boundary problems for one dimensional linear integro-differential equation of the Gurtin–Pipkin type with the Dirichlet-to-Neumann map as the inverse data. Under natural conditions on the kernel of the integral operator, we give the explicit formula for the solution of the problem with the observation on the semiaxis $t > 0$. For the observation on finite time interval, we prove the uniqueness result, which is similar to the local Borg–Marchenko theorem for the Schrödinger equation.

1. Introduction and the main results.

1.1. Spectral and dynamical inverse problems. Inverse spectral theory for the Schrödinger equation on the semiaxis and a finite interval was created starting with the classical works by Borg, Levinson, Gelfand, Levitan, Krein, Marchenko (see, e.g. [13] and references therein). Then these investigations were complimented by dynamical inverse problems and methods (see, e.g. [5]). About interesting connections between inverse spectral and dynamical problems see [2]. In the present paper we study inverse problem for a nonstandard partial differential equation — heat equation with memory and we use spectral and dynamical methods.

1.2. Gurtin–Pipkin type equations. It is known that the classical heat equation has a non-physical property, namely the infinite speed of propagation of singularities. Based on the modified Fourier law, Gurtin and Pipkin [7] introduced a

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model of heat transfer with a finite propagation speed. In the present paper we consider a form of this model described by the linear integro-differential equation

\begin{equation}
\theta_t(x, t) = \int_0^t k(t-s) \theta_{xx}(x, s) ds, \quad t > 0, \quad x \in (0, L), \quad L \leq \infty,
\end{equation}

the initial condition

\begin{equation}
\theta(\cdot, 0) = 0
\end{equation}

and the boundary conditions

\begin{equation}
\theta(0, t) = f(t), \quad \theta(L, t) = 0.
\end{equation}

The latter condition will be omitted in the case \( L = \infty \). We will write \( \theta^f \) if we need to specify dependence on the boundary function. Conditions on the kernel \( k \) will be discussed in Section 1.2.

Another form of the Gurtin–Pipkin model (an isotropic viscoelastic model) is described by the integro-differential equation of the second order in time:

\begin{equation}
\ddot{u}(x, t) = a \dddot{u}(x, t) + \int_0^t k(t-s) u_{xx}(x, s) ds, \quad a > 0, \quad x \in (0, L), \quad t > 0.
\end{equation}

The following form of the heat equation with memory,

\begin{equation}
\dot{u}(x, t) = u_{xx} + \int_0^t k(t-s) u_{xx}(x, s) ds, \quad x \in (0, L), \quad t > 0,
\end{equation}

can also be found in the literature.

We will consider the equation (1) because in this form the integral term plays the most important role. In some interpretations of the equation this means the absence of the latent heat. Note that the differentiation of (1) with respect to \( t \) leads to the equation of the form (4). The system described by (5) has the infinite speed of propagation of singularities and it is closer, in this sense, to the heat equation than (1).

Equation (1) can be treated as a perturbed wave equation. In the case \( k(t) = \text{const} = a^2 \), equation (1) is in fact an integrated wave equation. Indeed, differentiating (1) we obtain \( u_{tt} = a^2 u_{xx} \). If \( k(t) = e^{-bt} \), then differentiation gives a damped wave equation \( u_{tt} = u_{xx} - bu_t \). In the singular case \( k(t) = \delta(t) \) the equation (1) becomes the heat equation.

1.3. Well-posedness of initial boundary value problems. Well-posedness of initial boundary value problems for the Gurtin–Pipkin type equations was studied by many authors. In particular, regularity of the solutions in the Sobolev spaces was a topic of a series of papers of V.Vlasov and his coauthors, see, e.g.,[19] and also [10]. For the case of a finite interval regularity results for several forms of Gurtin–Pipkin type equations can be found in the book [17]. Note that in these papers the kernel \( k \) is assumed to be continuous and positive at the origin.

For the case of a finite space interval the solution can be obtained using the Fourier approach (see the example in section 3.2). The solution can also be constructed by the Laplace transform, see section 2.
Now we state the assumption on the kernel that we use in the present paper. In what follows we suppose that the Laplace transform of $k$ satisfies for large $|z|$ the condition
\begin{equation}
K(z) = \mathcal{L}(k(t)) = \frac{a^2}{z} + O\left(\frac{1}{z^2}\right), \quad a > 0, \ \Re z > 0. 
\end{equation}
We note that if $k'$ is bounded, then integration by parts implies (6) with $k(0) = a^2 > 0$.

**Remark 1.** If $k(0) = 0$ or $k$ has a singularity, the solution may not exist even in a weak sense. In the example at the end of the paper we show that in the case $k(t) = t^2$ the equation (1) has no solution in Sobolev spaces.

1.4. **The statement of the inverse problem.** Let $T \leq \infty$, $f \in L^2(0, T)$ and $\theta^f$ be a solution to the initial boundary value problem (1) – (3). We introduce the response operator $r^T : L^2(0, T) \rightarrow L^2_{\text{loc}}(0, T)$ with the domain $\{f \in H^1(0, T), f(0) = 0\}$, acting by the rule
\begin{equation}
(r^T f)(t) = \theta^f_x(0, t), \quad t \in (0, T). 
\end{equation}
Our inverse problem is formulated as follows: given the response operator $r^T$, to recover the kernel $k$ on the maximal possible interval $[0, T_0]$. (We will demonstrate that this interval is exactly $[0, T]$.)

Such a statement of inverse problem is standard for the hyperbolic type equations, see, e.g. [1]. However, the known methods do not work in our situation. For example, the boundary control method successfully used in [1] for recovering a matrix potential $Q(x)$ is not applicable to the problem where unknown coefficient depends on time.

1.5. **A brief survey of known results on inverse problems for equations with memory.** There are many papers concerning inverse problems for partial differential equations with memory. However, almost all of them deal with unknown sources or (in multidimensional cases) with unknown spatial part of the kernel $k(x, t)$ if it has a form $k(x, t) = h(t)p(x)$. Here we mention several papers devoted to inverse problems similar to ours.

In [6] the equation of the second order with memory in $\mathbb{R}^3$ was studied:
\begin{equation}
\begin{aligned}
&u_{tt} = \Delta u - k \ast u, \quad \text{in } \mathbb{R}^3, \quad u|_{t=0} = 0, \quad u|_{t=0} = \delta(x - x_0).
\end{aligned}
\end{equation}
The inverse data is the value of the scattering wave at $x_0$ for $0 < t < T$. The authors demonstrate the uniquely stable identification of $k$ on the time interval $[0, T - |x_0|]$.

In [11] the inverse problem for the system describing by the equation
\begin{equation}
\beta u_t = u_{xx} - m \ast u_{xx}, \quad x \in (0, 1),
\end{equation}
was studied. The properties of this system is close to a parabolic type equation. The authors prove that by, for instance, given $u(x_0, t)$, $t > 0$, it is possible to recover the kernel $m$.

In [3, 15, 16] the inverse problem for the system (1), (2) was studied on a finite spatial interval $(0, L)$ with the help of the Fourier method. A linear algorithm reconstructing the kernel $k$ from two boundary observation was developed. One of the observations corresponds to a nonzero initial condition.
1.6. **Local uniqueness property.** In the present paper we study inverse problem for the system (1) – (3) on a finite spatial interval and on the semi-axis with the help of the Laplace transform. In the case of the infinite time of observation we obtain explicit formulas that allow recovering the Laplace transform of the kernel $k$.

In the case of a finite time of observation we prove the uniqueness of the solution to the inverse problem and obtain the local uniqueness result similar to the local Borg–Marchenko theorem for the Schrödinger equation [18, 4]. Our approach is based on the Laplace transform and uses some basic facts of the Hardy space theory.

We recall now the local Borg–Marchenko theorem. The following uniqueness result was proved in [18] (see also a very short proof in [4]). If two Weyl-Titchmarsh m-functions, $m_j(z), j = 1, 2$, for two Schrödinger equations

$$\frac{d^2 \psi}{dx^2} - q_j(x) \psi = z \psi, \quad x > 0,$$

with some regular condition at $x = 0$ are exponentially close, that is,

$$|m_1(z) - m_2(z)| \leq C e^{-2\sqrt{z}a}, \Im\sqrt{z} > 0,$$

then $q_1 = q_2$ on $[0, a]$. This result may be considered as a local version of the celebrated Borg–Marchenko uniqueness theorem [14]. Here and below we choose the principal branch of the square root, $\sqrt{z} > 0$ for $z > 0$.

The dynamical version of the local Borg–Marchenko theorem, expressed through the response operator, was discussed in [2]. In the present paper we prove a similar local uniqueness result for the kernel of the heat equation with memory.

2. **The stationary inverse problem, $T = \infty$.** The inverse data is $r^\infty$, the observation of $\theta_x(0, t)$ for all $t > 0$.

We suppose that the solution to the IBVP (1), (2) does not grow too fast in order to be able to apply the Laplace transform. This assumption can be justified using the integral representation of the solution presented in [17] (section 1.2, formula (1.10)) and the condition (6).

We apply the Laplace transform to (1), (2), denote the images by capitals and obtain the family of ODEs depending on $z$ as a parameter:

$$z \Theta(x, z) = K(z) \Theta_{xx}(x, z), \quad x \in (0, L), \quad \Theta(0, z) = F(z).$$

For every $z$ this differential (in $x$) equation has constant coefficients. We set $\omega(z) = \sqrt{z/K(z)}$ and consider separately the cases of a finite and the infinite interval $(0, L)$.

2.1. **The case $L = \infty$.**

**Theorem 2.1.** Let $k$ satisfy (6) and $L = \infty$. Then the kernel $k$ can be uniquely recovered from $r^\infty$.

**Proof.** The solution to (8) which is bounded in the right half $z$-plane is

$$\Theta(x, z) = F(z) e^{-\omega(z)x}.$$

Then the Laplace transform of the response $(r^\infty f)(t) = \theta^f_x(0, t)$ is

$$R(z) = \Theta_x(0, z) = -F(z) \omega(z) = -F(z) \sqrt{z/K(z)}.$$

Evidently, we can find $K(z)$ via the data $R$ and the given $F$. $\square$
2.2. The case of a finite interval, $L < \infty$.

**Theorem 2.2.** Let $k$ satisfy (6) and $L < \infty$. Then the kernel $k$ can be uniquely recovered from $r^\infty$.

**Proof.** In this case the solution $\Theta$ satisfies the equation (8) and zero boundary condition at $x = L$. Thus, we have the problem

$$\Theta_{xx}(x, z) = \omega^2 \Theta(x, z), \quad \Theta(0, z) = F(z), \quad \Theta(L, z) = 0.$$ 

First, taking into account only the boundary condition at $x = 0$, we obtain

$$\Theta(x, z) = F(z) \cosh[\omega(z)x] + \Phi(z) \sinh[\omega(z)x].$$

The boundary condition at $x = L$ implies

$$\Theta(L, z) = F(z) \cosh[\omega(z)L] + \Phi(z) \sinh[\omega(z)L] = 0.$$ 

Then for $\omega(z)L \neq \pi n, n \in \mathbb{Z},$

$$\Phi(z) = -\frac{F(z) \cos[\omega(z)L]}{\sin[\omega(z)L]}.$$ 

From the other hand,

$$R(z) = \Theta_x(0, z) = \omega(z)\Phi(z) = -\omega(z) \frac{F(z) \cos[\omega(z)L]}{\sin[\omega(z)L]}.$$ 

It is possible now to recover $\omega(z)$, then $K(z)$ and $k(t)$.

3. Non-stationary inverse problem, $T < \infty$.

3.1. The local uniqueness. First, we consider the case $L = \infty$. Our inverse data is the response operator $r^T$ acting in $L^2(0,T)$ defined on the functions $f \in H^1(0,T), f(0) = 0$. It is convenient to extend the functions $f$ from the domain of $r^T$ and the corresponding responses, $(r^T f)(t)$, by zero for $t \geq T$.

We will use in the Hardy space $H^2(\mathbb{R}z > 0)$ in the right half plane. The inner product in this space is defined by the formula

$$\int_{-\infty}^{\infty} f(iy) \overline{g(iy)} \, dy,$$

and by the Paley-Wiener theorem we have:

$$H^2(\mathbb{R}z > 0) = \mathcal{L}L^2(0,\infty), \quad e^{-zT}H^2(\mathbb{R}z > 0) = \mathcal{L}L^2(T,\infty),$$

where $\mathcal{L}$ denotes the Laplace transform. We note that the Laplace image of the domain of $r^T$ is the set of $F$ such that $F$ and $zF$ are in the Hardy space.

It is important that, by the condition (6), the function $K$ can have only a finite number of zeros in the right half plane. For simplicity we assume now that $K(z)$ has no zeros there: $K(z) \neq 0, \Re z > 0$. This takes place, in particular, if $k$ is a non-increasing function. In general case, we may consider the Hardy space in the half plane $\Re z > \text{const} > 0$, where $K(z)$ has no zeros.

**Theorem 3.1.** Two kernels $k_1(t)$ and $k_2(t)$ coincide on the interval $0 \leq t \leq T$, if and only if the response operators $r^T_1$ and $r^T_2$ coincide.
Proof. The part ‘only if’. We have

\[ K_1(z) - K_2(z) = \int_T^\infty e^{-zt}(k_1(t) - k_2(t))dt. \]

This means that

(10) \[ e^{zT}(K_1(z) - K_2(z)) \in H^2(\Re z > 0). \]

We take a function \( f \) from the domain of the response operator and set \( r_j(t) = (r_j^T f)(t), R_j(z) = (L r_j)(z), j = 1,2 \). From the explicit expression (9) for the Laplace transform of the responses we get

\[ R_1(z) - R_2(z) = \frac{\sqrt{z}}{\sqrt{K_1 K_2}} F(z) \left( \sqrt{K_1} - \sqrt{K_2} \right) \]

or

(11) \[ R_1(z) - R_2(z) = \Phi(z) F(z) \left[ z(K_1 - K_2) \right] \]

with

\[ \Phi(z) = \frac{1}{\sqrt{K_1 K_2} (\sqrt{K_1} + \sqrt{K_2}) \sqrt{z}}, \]

By the main assumption (6) we obtain the asymptotics

\[ \frac{1}{\sqrt{K_1 K_2} (\sqrt{K_2} + \sqrt{K_1}) \sqrt{z}} = cz \left( 1 + O\left( \frac{1}{z} \right) \right), \]

therefore, the function \( \Phi F \) is in the Hardy space. Because the bounded function \( z(K_2 - K_1) \) has the inner factor \( e^{-zt} \) by (10), we conclude that

\[ e^{zT}(R_1 - R_2) \in H^2(\Re z > 0), \]

what is equivalent to \( (r_1^T f)(t) = (r_2^T f)(t), 0 \leq t \leq T. \)

Part ‘if’. Using the Paley–Wiener theorem, the equality

\[ r_1^T = r_2^T, \]

can be written as

\[ e^{zT}(R_1 - R_2) \in H^2(\Re z > 0). \]

From (11) we see that

\[ e^{zT}(K_1 - K_2) \in H^2(\Re z > 0). \]

The Paley–Wiener theorem implies now that \( k_1(t) = k_2(t) \) for \( 0 \leq t \leq T. \)

Remark 2. If we expand the response operator to distribution we may obtain the expression of the response operator via the response \( r_3^\infty \) to the Dirac delta function. Indeed, by (9)

\[ \mathcal{L}(r^\infty f) = -F \sqrt{z/K}, \]

that implies

\[ \mathcal{L}(r^\infty \delta) = -\sqrt{z/K}. \]

So, we may conclude that

\[ \mathcal{L}(r^\infty f) = \mathcal{L}(r^\infty \delta) \mathcal{L}(f) \quad \text{and} \quad r^\infty f = (r^\infty \delta) * f. \]
It is clear that the operator \( r^T \) does not determine the kernel \( k(t) \) for all \( t > 0 \), but it is possible to derive an explicit integral relation between the Laplace transforms of \( r^T f \) and \( k \). We take \( f \) from the domain of \( r^\infty \) and set \( r(t) = (r^\infty f)(t) \), \( R(z) = (Lr)(z) \). Evidently, we can not find the whole \( R \) from the inverse data, but we know its projection \( R_T \) onto the subspace
\[
H^2(\mathbb{R}z > 0) \ominus e^{-z}H^2(\mathbb{R}z > 0).
\]

Now we write \( R_T(z) \) in the terms of \( R(z) \):
\[
R_T(z) = \mathcal{L}[\hat{\chi}_{[0,T]}r(t)](z) = \int_0^T e^{-tz} \hat{\chi}_{[0,T]}r(t)dt
\]
and
\[
= \int_0^T dt e^{-tz} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{pt} R(p).
\]

Using the substitution \( p = iy \) and changing the order of integration we obtain
\[
R_T(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy R(iy) \int_0^T dt e^{t(iy-z)}
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{T(iy-z)} - \frac{1}{iy} R(iy) dy.
\]

Using now formula (9) we obtain the relation between \( R_T \) and \( K \):
\[
R_T(z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{T(iy-z)} - 1}{iy} F(iy) \sqrt{\frac{iy}{K(iy)}} dy.
\]

Thus the problem is to understand what information about \( k \) can be extracted from the with known \( R_T(z) \).

Let now \( L < \infty \) and \( x \in (0,L) \).

**Theorem 3.2.** *Let \( T \leq L/a \). Then \( k_1(t) = k_2(t) \), \( 0 \leq t \leq T \), if and only if \( r^T_1 = r^T_2 \).*

**Proof.** Since the system (1), (2) has a finite speed of the wave propagation equal to \( a \), the solution of the equation (1) with \( L < \infty \) coincides with the solution for \( L = \infty \) for times \( t \leq L/a \). Theorem 4 follows now from Theorem 3. \( \Box \)

**3.2. Example.** Let us give an example of a ‘non Sobolev’ solutions of (1), (2) in the case where \( k(0) = 0 \) (and (6) is not true) Take a smooth kernel vanishing at the origin, say, \( k(t) = t \). Its Laplace image is \( K(z) = 1/z^2 \).

We will find the solution as a series in sine functions (the eigenfunction of the operator \( d^2/dx^2 \) with the Dirichlet boundary conditions)
\[
\theta(x,t) = \sum_{n=1}^{\infty} \theta_n(t) \sin nx.
\]
Let the initial data be
\[
\theta(0,x) = \sum_{n=1}^{\infty} \xi_n \sin nx.
\]

For the Laplace image of \( u_n \) we have [8]
\[
\Theta_n(z) = \frac{\xi_n}{z + n^2 K(z)} = \frac{z^2 \xi_n}{z^3 + n^2}.
\]
and the pre-image $\theta_n(t)$ of $\Theta_n(z)$ is
\[
\mathcal{L}^{-1} \left( \frac{z^3 \xi_n}{z^4 + n^2} \right) = \frac{\xi_n}{3} \left( e^{-n^{2/3} t} + e^{\exp(-\pi i/3)n^{2/3} t} + e^{\exp(\pi i/3)n^{2/3} t} \right).
\]

Hence $n^{-P}\theta_n(t)$ is in $\ell^2$ for fixed $t$ only if the coefficients decrease exponentially as $n^{P}\exp(-n^{2/3} t/2)$.

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