Delay time of waves performing Lévy flights in 1D random media

L. A. Razo-López1,*, A. A. Fernández-Marín1, J. A. Méndez-Bermúdez1, J. Sánchez-Dehesa2, and V. A. Gopar3,†
1Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla 72570, Mexico.
2Departamento de Ingeniería Electrónica, Universitat Politècnica de València, Camino de Vera s. n. (Edificio 7F), ES-46022, Valencia, Spain.
3Departamento de Física Teórica, Facultad de Ciencias, and BIFI, Universidad de Zaragoza, Pedro Cerbuna 12, ES-50009 Zaragoza, Spain.
(Dated: March 20, 2020)

The time that waves spend inside 1D random media with the possibility of performing Lévy flights is experimentally and theoretically studied. The dynamics of quantum and classical wave diffusion has been investigated in canonical disordered systems via the delay time. We show that a wide class of disorder–Lévy disorder–leads to strong random fluctuations of the delay time; nevertheless, the tail of the distribution and the average of the delay time are insensitive to Lévy flights. Our results reveal a universal character of wave propagation that goes beyond standard Brownian wave-diffusion.

A wave packet launched into a scattering region can penetrate that region and it may be reflected eventually. Thus, one might wonder how much time the wave packet has spent inside the media. This fundamental question was addressed by Wigner and Smith [1, 2]. It was shown that the delay time $\tau_R$ of a wave packet is related to the derivative of the reflection phase $\theta_R$ with respect to the frequency $\omega$: $\tau_R = d\theta_R/d\omega$.

The delay time has received attention in many disciplines since it reveals information on the scattering medium and, therefore, it has also been of interest from an application point of view; e.g., the delay time is a fundamental quantity in imaging of tissues in optical coherence tomography [3].

A major issue in wave transport is the presence of disorder, which gives a random character to this phenomenon. Moreover, if waves propagate coherently through 1D random media, complex interference effects emerge, such as the widely studied Anderson localization [14–16]: an exponential decay in space of classical and quantum waves.

Since disorder is ubiquitous in real systems, there has been a great interest in studying the effects of Anderson localization on dynamical quantities such as the delay time. Microwave experiments have been performed to analyze statistical properties of wave dynamics [6, 8], while several theoretical approaches have been developed to describe the delay-time statistics (see Ref. [3] for a review).

Remarkably, it has been demonstrated that some statistical properties of the delay time are invariant in the sense that they are independent of the details of the medium. For instance, the inverse square power decay of the distribution of $\tau_R$ has been predicted in semi-infinite 1D systems [10] and also studied in higher dimensions [11–13]. The average delay time is proportional to the mean length of trajectories [14], which was predicted, and recently observed experimentally, to be invariant with respect to details of the scattering region [15, 16].

Previous experimental and theoretical works on the delay time in 1D consider only disorder models that lead to Anderson localization, however, there is a wide class of disorder–Lévy disorder–that leads to delocalization or anomalous localization, in relation to the Anderson localization [17, 18]. Anomalous localization finds its origin in the nonzero probability that waves travel a long distance without being scattered; these events–Lévy flights–are scarce but have a large impact.

Here, we experimentally and theoretically study the delay time of reflected microwaves in a medium characterized by random spacings of scatterers following a Lévy type distribution. Experiments in waveguides with a standard disorder (i.e., with random scatterer separations following a non-heavy tailed distribution) are also performed to compare results with those of Lévy disorder. Additionally, numerical simulations are performed to overcome some practical limitations of experiments and to obtain further support of our model. We calculate the distribution of the delay time and, furthermore, our results allow us to conclude that universal features of the delay time in canonical disordered media go beyond standard Brownian models of wave diffusion, despite the fact that the presence of Lévy flights leads to stronger random fluctuations of the delay time.

Lévy statistics have been found in a broad range of contexts that go from foraging patterns of marine preda-
FIG. 2. Experimental delay-time distributions (histograms) for Lévy waveguides characterized by (a) $\alpha = 1/2$ and $(-\ln T) = 4.7$ at 9.9 GHz (red histogram) and (b) $\alpha = 3/4$ and $(-\ln T) = 12$ at 11.2 GHz (green histogram). The blue histogram in (b) corresponds to random waveguides with ordinary (Gaussian) disorder with $(-\ln T) = 12$ at 11.2 GHz. The histograms were constructed with (a) 4500 and (b) 1800 data. Insets show $p(\tau_R)$ in a logarithmic scale. Red, green [blue] solid curves show the theoretical predictions from Eq. (4) [Eq. (3)]. Experimental delay times for a typical realization of the disorder of waveguides with (c) $\alpha = 1/2$ and (d) $3/4$. Black dots represent the average of $\tau_R$ over frequency windows $\Delta \nu = 0.4$ GHz. The horizontal black dashed lines are the averages of $\tau_R$ over the complete frequency window (8-12 GHz).

Lévy random processes are characterized by probability distributions whose tail decay like a power-law, i.e., if $x$ is a random variable with probability density $p(x)$, then $p(x) \sim 1/x^{1+\alpha}$ for $x \gg 1$ [27]. Fluctuations of random variables that follow Lévy statistics are so large that the first and second moments diverge for $0 < \alpha < 1$.

Experimental results. Microwaves are launched into an aluminum waveguide containing 2.5 mm thick dielectric slabs whose separations follow a Lévy distribution (see Fig. 1). We work in a frequency range where a single transport channel is supported. Two different Lévy distributions characterized by their power-law decay have been chosen: $\alpha = 1/2$ and $3/4$. Additionally, a conventional disordered microwave waveguide with random spacing between slabs following a Gaussian distribution has been built. Using a network vector analyzer, we measure the $2 \times 2$ scattering matrix $S$:

$$S = \left( \begin{array}{cc} \sqrt{R} e^{i\theta_R} & \sqrt{T} e^{i\theta_T} \\ \sqrt{T} e^{i\theta_T} & \sqrt{R} e^{i\theta_R} \end{array} \right),$$

where $R$ and $T$ are the reflection and transmission coefficients, respectively. Measurements of $S$ are thus collected over different disorder realizations.

From the collected $S$-matrices, we obtain $\tau_R$ and its probability distribution function $p(\tau_R)$. Figures 2(a) and 2(b) show the distribution $p_\alpha(\tau_R)$ with $\alpha = 1/2$ (red color) and $3/4$ (green color), respectively. The insets show $p_\alpha(\tau_R)$ on a logarithmic scale for a better visualization of the tail. In Fig. 2(b), the delay-time distribution (blue histogram) for conventional disorder is also shown. Both distributions in Fig. 2(b) have the same average value $\langle \ln T \rangle$. We can observe, however, that the profile of both distributions (green and blue histograms) are different.

The distributions $p_\alpha(\tau_R)$ from our model, which we introduce below, are shown (solid lines) in Figs. 2(a) and 2(b). It is observed in Fig. 2(a) that for $\alpha = 1/2$, $p_\alpha(\tau_R)$ shows a small peak in the tail, which we attribute to scattering processes that reach the right boundary of the waveguide; in Lévy disordered samples those processes are favored since waves can travel long distances without being scattered. In contrast, for ordinary disordered systems, $p(\tau_R)$ decays monotonically and for $\tau_R \gg 1$, $p(\tau_R) \sim 1/\tau_R^2$ [10]. The distribution for $\alpha = 3/4$ in Fig. 2(b) also exhibits a peak but it is smoother and occurs at a larger value of $\tau_R$, outside of the time range shown in Fig. 2(b).

The trend of the experimental distributions (histograms) is well described by the model (solid lines), despite the fact that the statistics of $\tau_R$ is extracted from a limited amount of experimental data and the presence of a small tail for negative values of $\tau_R$ observed in Figs. 2(a) and 2(b). Negative delay times are not considered in our model and are thus a source for discrepancies between experimental and theoretical results. It has been proposed that those negative values are due to a strong distortion of the wave packet produced due to interference between incident and promptly reflected waves [28-31].

We now address an invariance property of the mean path of trajectories with respect to the details of the disordered medium. This invariance property is equivalent to the independence of the average delay time with energy since both quantities are proportional [14]. To illustrate this invariance, in Figs. 2(c) and 2(d), $\tau_R$ is plotted as a function of the linear frequency $\nu$ for typical samples with $\alpha = 1/2$ and $3/4$, respectively. We
see strong fluctuations of $\tau_R$, however, the average delay-time over frequency windows $\Delta \nu (=0.4 \text{GHz})$ are essentially independent of the frequency, as it is observed in both figures (dots). Moreover, the average over the whole frequency window (horizontal dashed line) has the same value ($3.4 \times 10^{-9} \text{s}$) for both cases: $\alpha = 1/2$ and $3/4$, and thus, it is independent of particularities of the medium. We will address later this point in more detail.

Model. For conventional disorder and within a random matrix approach to localization [32,33], the mean free path $\ell$ determines the statistical properties of the transport and can be obtained from the average $s \equiv \langle -\ln T \rangle = L/\ell$, which is proportional to the number of scatterers $n$ in the system, i.e., $\langle -\ln T \rangle = bn$ with $b$ a constant [34]. If the random spacing between scatterers follows a Lévy distribution, the number of scatterers in a system of length $L$ is subject to strong random fluctuations. Such fluctuations are described by the probability density $\Pi_L(n;\alpha)$ given by [17]: $\Pi_L(n;\alpha) = 2Lq_{\alpha,1} \left( L/(2n)^{1/\alpha} \right) / \alpha (2n)^{(1+1/\alpha)}$, where $q_{\alpha,1}(x)$ is the probability density function of the Lévy distribution with exponent $\alpha$ and scale parameter $c$. For $x \gg 1$, $q_{\alpha,1}(x) \sim c/x^{1+\alpha}$. Therefore, with the knowledge of $\Pi_L(n;\alpha)$ and the delay-time probability density for canonical disordered systems, $p_s(\tau_R)$, we write the probability density $p_\alpha(\tau_R)$ for Lévy disordered systems as

$$p_\alpha(\tau_R) = \int_0^\infty p_s(\tau_R) \Pi_L(n;\alpha) \, dn. \quad (2)$$

The probability density $p_s(\tau_R)$ in its full generality, however, remains an open problem. For semi-infinite disordered systems, assuming zero transmission, the limit ($L \rightarrow \infty$) delay-time distribution $p_\infty(\tau_R)$ is given by [10, 35–39]: $p_\infty(\tau_R) = \tau_R a/\tau_0^2 \exp (-\tau_R/\tau_0)$, where $\tau_0$ is the scattering time of the disorder. Real systems, however, are finite and finite-size effects may be of relevance. Our experiments, in particular, are performed in 2 m long waveguides and microwaves can be transmitted.

From Eq. (2), we need an expression for the time delay distribution as a function of the number of scatterers in a system of length $L$, i.e., in terms of $s$. We model $p_s(\tau_R)$ using a relationship between $\tau_R$ in the absence of absorption and the reflection $R$ in the presence of weak absorption, namely [39–41]: $R = 1 - \tau_R/\tau_0$, where $\tau_0$ is the absorption time and it is assumed that the transmission is negligible. Let us consider a system of finite length; in this case $\tau_R$ can exceed $\tau_0$ since, for instance, $\tau_R$ is infinite for transmitted waves, i.e., those waves do not come back. As a rough approximation, we assume that the previous relation between $R$ and $\tau_R$ is still valid for systems with small transmission. Thus, $\tau_R/\tau_0 = 1 - R \equiv (1 + \lambda)^{-1}$. The distribution of the variable $\lambda$ is given by the so-called Laguerre ensemble [42,43]: $p(\lambda) \propto \exp(-\tau_0 \lambda/\tau_0)$ with $\lambda > 0$. After the change of variable $\lambda \rightarrow \tau_R$ in $p(\lambda)$, we write the normalized distribution $p_s(\tau_R)$ as

$$p_s(\tau_R) = \int_0^\infty p_s(z,\alpha,\xi)(\tau_R) q_{\alpha,1}(z) \, dz, \quad (4)$$

where $p_s(z,\alpha,\xi)(\tau_R)$ is given in Eq. (3) with $s$ replaced by $s(z,\alpha,\xi)$. The theoretical distributions $p_s(\tau_R)$ in Fig. 2 (a) and 2 (b) (solid lines) have been calculated using Eq. (4), where the absorption time $\tau_0$ is used to fit the experimental results.

Numerical simulations are now performed for further

![Fig. 3. Numerical delay-time distributions with parameters $\alpha = 1/2$ (red histogram) and $\alpha = 3/4$ (green histogram) for random waveguides with conventional (Gaussian) disorder (blue histogram). In all cases $(-\ln T) = 10$. The histograms were constructed with $10^9$ disorder realizations. The solid curves are the theoretical predictions from Eqs. (3) and (4) for standard and Lévy disorder, respectively. The dashed line proportional to $1/\tau_R^2$ is plotted to guide the eye.](image-url)
disordered systems characterized by \((a,c)\) \(\alpha = 1/2\) and \((b,d)\) \(\alpha = 3/4\) with \((a,b)\) \(\xi = 10\) and \((c,d)\) \(\xi = 4\). Each histogram was obtained from \(5 \times 10^5\) disorder realizations. The solid curves are the theoretical predictions from Eq. (4). Insets show \(p_\alpha(\tau_R)\) in a logarithmic scale. The dashed lines follow the law \(1/\tau_R\).

FIG. 4. Numerical distributions \(p_\alpha(\tau_R)\) (histograms) for Lévy disordered systems characterized by \((a,c)\) \(\alpha = 1/2\) and \((b,d)\) \(\alpha = 3/4\) with \((a,b)\) \(\xi = 10\) and \((c,d)\) \(\xi = 4\). Each histogram was obtained from \(5 \times 10^5\) disorder realizations. The solid curves are the theoretical predictions from Eq. (4). Insets show \(p_\alpha(\tau_R)\) in a logarithmic scale. The dashed lines follow the law \(1/\tau_R\).

support of Eq. (4) and to reveal universal properties. With simulations we greatly increase the number of disorder realizations and absorption that reduces Lévy flights effects is not considered.

Figure 3 compares numerical and theoretical results from Eq. (4) for two parameters of the Lévy disorder: \(\alpha = 1/2\) and \(3/4\), but with the same value \(\xi = 10\). The delay-time distribution for conventional disorder with \(s = 10\) is also shown. The distribution profiles in Fig. 3 are different and show the impact of Lévy flights, however, they share some properties that are not evident because of the different time scale of each case.

In order to compare \(p_\alpha(\tau_R)\) for different system parameters, we divide \(\tau_R\) by \(2L/v_g\). In this manner, Fig. 4 shows \(p_\alpha(\tau_R)\) for \(\alpha = 1/2\) and \(3/4\), left and right panels, respectively, and \(\xi = 10\) and \(4\), upper and lower panels, respectively. A notorious difference with respect to the distribution for standard disordered systems (Fig. 4 blue line) is that a peak appears at \(\tau_R/(2L/v_g) = 1\), which is precisely the time that a wave would spend on traveling back and forth between the boundaries of the waveguide in absence of the disorder. Also, in the absence of absorption, we have identified \(\tau_0 = \langle \tau_R \rangle\). Thus, since \(\langle \tau_R \rangle\) is extracted from the numerical simulations, there is no free fitting parameters in the theoretical distributions in Fig. 4.

For short systems [Figs. 4(c) and 4(d)], we notice a deviation, mainly at the distribution tails, of the theoretical predictions (solid lines) with respect to the numerical results. Also, the numerical simulations start to deviate from the \(1/\tau_R^2\) decay [see insets in Figs. 4(c) and 4(d)], which is expected since the transmission is higher as the waveguide becomes shorter.

We now show that some properties of the delay time go beyond canonical disorder models. We have mentioned the inverse square decay of the delay time distributions for \(\tau_R \gg 1\) obtained in 1D semi-infinite Anderson localized systems; indeed, such power-law behaviour has been explained by resonance models in the localized regime \(10, 13, 14\). As we see in the insets of Figs. 4(a) and 4(b), for \(\alpha = 1/2\) and \(3/4\), respectively, both distributions decay as \(1/\tau_R^2\) (dashed lines). Actually, from our model, Eqs. (4) and (4), we find that \(p_\alpha(\tau_R) \sim 1/\tau_R^2\) for \(\tau_R \gg 1\).

In addition, the average \(\langle \tau_R \rangle\) is a linear function of the system length: \(\langle \tau_R \rangle = L/v_g\), as shown in Fig. 5(a) for \(\alpha = 1/2\) and \(3/4\) (inset), which is also observed in standard disordered systems \(10, 37\). Furthermore, an interesting invariance property of the mean length of random walk trajectories with respect to the details of the disorder \(15\) has been recently investigated in optical experiments \(16\). The invariance of the mean path length is equivalent to the independence of the average delay-time to the energy. We have already shown experimental evidence of this invariance in Figs. 2(c) and 2(d). Figure 5(b) provides further numerical evidence by showing the average \(\langle \tau_R \rangle\) of disordered systems of different lengths characterized by \(\alpha = 1/2\) and \(3/4\). It is observed that \(\langle \tau_R \rangle\) is constant with the linear frequency \(\nu\). Thus, these results give evidence that the invariance of the mean path length observed in experiments of light...
propagation. Additionally, the linear dependence of the average delay time with the system length in ordinary disordered media is not affected by the presence of Lévy flights. We remark that the statistics of the time delay do not depend on the details of the disorder: it only depends on the power $\alpha$ of the tail. All together, our results reveal a universal character of wave propagation that goes beyond standard Brownian models.

We point out that in Lévy disordered systems with $\alpha < 1$, the mean free path is meaningless since it diverges, in contrast to canonical disordered systems in which the mean free path settles the wave statistics. In the presence of Lévy flights two quantities fix the transport statistics: $\alpha$ that characterizes the Lévy flights and the logarithmic transmission average. These differences between anomalous and Anderson localizations are preserved for infinite systems.

A good agreement between experiments and numerical results with the theory is observed, however, we considered structures with a single transport channel or 1D systems. Although nowadays 1D wave transport is of relevance in some areas, it would be desirable to extend our study to higher dimensions. Finally, we stress that the ideas and results presented here are so general that can be applied from classical to quantum waves in disordered media.

A. A. F.-M. thanks the hospitality of the Laboratoire d’Acoustique de l’Université du Mans, France, where part of this work was done. J. A. M.-B. gratefully acknowledges to Departamento de Matemática Aplicada e Estatística, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo during which this work was completed. J.A.M.-B. was supported by FAPESP (Grant No. 2019/06931-2), Brazil. A. A. F.-M. thanks partial support by RFI Le Mans Acoustique and by the project HYPERMETA funded under the program toiles Montantes of the Region Pays de la Loire. V. A. G. acknowledges support by MCIU (Spain) under the Project number PGC2018-094684-B-C22.

* Present address: Université Côte d’Azur, CNRS, Institut de Physique de Nice, Parc Valrose. 06100 Nice, France.
† Email: gopar@unizar.es

---

[1] E. P. Wigner. Lower limit for the energy derivative of the scattering phase shift. Phys. Rev., 98:145–147, 1955.
[2] F. T. Smith. Lifetime matrix in collision theory. Phys. Rev., 119:2098–2098, 1960.
[3] A F Fercher, W Drexler, C K Hitzenberger, and T Lasser. Optical coherence tomography - principles and applications. Reports on Progress in Physics, 66(2):239–303, jan 2003.
[4] P. W. Anderson. Absence of diffusion in certain random lattices. Phys. Rev., 109:1492–1505, 1958.
[5] Ad Lagendijk, Bart van Tiggeleen, , and Diederik S. Wiersma. Fifty years of anderson localization. Physics Today, 62, 2009.
[6] A. Z. Genack, P. Sebbah, M. Stoytchev, and B. A. van Tiggeleen. Statistics of wave dynamics in random media. Phys. Rev. Lett., 82:715–718, 1999.
[7] P. Sebbah, O. Legrand, and A. Z. Genack. Fluctuations in photon local delay time and their relation to phase spectra in random media. Phys. Rev. E, 59:2406–2411, Feb 1999.
[8] A. A. Chabanov and A. Z. Genack. Statistics of dynamics of localized waves. Phys. Rev. Lett., 87:233903, Nov 2001.
[9] C. Texier. Wigner time delay and related concepts: Application to transport in coherent conductors. Physica E: Low-dimensional Systems and Nanostructures, 82:16–33, 2016.
[10] Christophe Texier and Alain Comtet. Universality of the wigner time delay distribution for one-dimensional random potentials. Phys. Rev. Lett., 82:4220–4223, May 1999.
[11] H. Schomerus, K. J. H. van Bemmel, and C. W. J Beenakker. Coherent backscattering effect on wave dynamics in a random medium. Europhysics Letters (EPL), 52(5):518–524, 2000.
[12] Fuming Xu and Jian Wang. Statistics of wigner delay time in anderson disordered systems. Phys. Rev. B, 84:024205, Jul 2011.
[13] A. Ossipov. Scattering approach to anderson localization. Phys. Rev. Lett., 121:076601, Aug 2018.
[14] R. Pierrat, P. Ambichl, S. Gigan, A. Haber, R. Carminati, and S. Rotter. Invariance property of wave scattering through disordered media. Proceedings of the National Academy of Sciences, 111(50):17765–17770, 2014.
[15] S Blanco and R Fournier. An invariance property of diffusive random walks. Europhysics Letters (EPL), 61(2):168–173, jan 2003.
[16] Romolo Savo, Romain Pierrat, Ulysse Najar, Rémi Carminati, Stefan Rotter, and Sylvain Gigan. Observation of mean path length invariance in light-scattering media. Science, 358(6364):765–768, 2017.
[17] F. Falceto and V. A. Gopar. Conductance through quantum wires with lévy-type disorder: Universal statistics in anomalous quantum transport. EPL (Europhysics Letters), 92(5):57014, 2010.
[18] Ilias Amanatidis, Ioannis Kletoiagis, Fernando Falceto, and Victor A. Gopar. Conductance of one-dimensional quantum wires with anomalous electron wave-function localization. Phys. Rev. B, 85:235450, Jun 2012.
[19] I. Kletoiagis, I. Amanatidis, and V. A. Gopar. Conductance through disordered graphene nanoribbons: Standard and anomalous electron localization. Phys. Rev. B, 88:205414, 2013.
[20] R. N. Mantegna and H. E. Stanley. Scaling behaviour in the dynamics of an economic index. Nature, 376(6535):46–49, 1995.
[21] A. M. Reynolds and N. T. Ouellette. Swarm dynamics may give rise to lévy flights. Scientific Reports, 6:30515 EP –, 2016.
[22] Rajesh Patel and Rasbindu V. Mehta. Lévy distribution of time delay in emission of resonantly trapped light in ferrodispersions. volume 6, page 069503, 2012.
[23] Jean-Philippe Bouclaud and Antoine Georges. Anoma-
lous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Physics Reports*, 195(4-5):127–293, Nov 1990.

[24] M. F. Shlesinger, J. Klafter, and G. Zumofen. Above, below and beyond brownian motion. *American Journal of Physics*, 67(12):1253–1259, 1999.

[25] E. Barkai, V. Fleurov, and J. Klafter. One-dimensional stochastic lévy-lorentz gas. *Phys. Rev. E*, 61:1164–1169, Feb 2000.

[26] P. Barthelemy, J. Bertolotti, and D. S. Wiersma. A lévy flight for light. *Nature*, 453:495 EP –, 05 2008.

[27] V. V. Uchaikin and V. M. Zolotarev. *Chance and Stability. Stable distributions and their applications*. Wiley, 1998.

[28] C. G. B. Garrett and D. E. McCumber. Propagation of a gaussian light pulse through an anomalous dispersion medium. *Phys. Rev. A*, 1:305–313, 1970.

[29] Dalitz Richard Henry and Moorhouse R. G. What is a resonance. *Proc. R. Soc. Lond. A*, 318, September 1970.

[30] S. Chu and S. Wong. Linear pulse propagation in an absorbing medium. *Phys. Rev. Lett.*, 48:738–741, 1982.

[31] M. Durand, S. M. Popoff, R. Carminati, and A. Goetschy. Optimizing light storage in scattering media with the dwell-time operator. *Phys. Rev. Lett.*, 123:243901, Dec 2019.

[32] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher. New method for a scaling theory of localization. *Phys. Rev. B*, 22:3519–3526, 1980.

[33] P. A. Mello and N. Kumar. *Quantum Transport in Mesoscopic Systems: Complexity and Statistical Fluctuation*. Oxford University Press, 2004.

[34] P. A. Mello. Central limit theorems on groups. *Journal of Mathematical Physics*, 27(12):2876–2891, 1986.

[35] A. M. Jayannavar, G. V. Vijayagovindan, and N. Kumar. Energy dispersive backscattering of electrons from surface resonances of a disordered medium and 1/f noise. *Zeitschrift für Physik B Condensed Matter*, 75(1):77–79, Mar 1989.

[36] J. Heinrichs. Invariant embedding treatment of phase randomisation and electrical noise at disordered surfaces. *Journal of Physics: Condensed Matter*, 2(6):1559–1568, feb 1990.

[37] A. Comtet and C. Texier. On the distribution of the wigner time delay in one-dimensional disordered systems. *Journal of Physics A: Mathematical and General*, 30(23):8017–8025, 1997.

[38] C. J. Bolton-Heaton, C. J. Lambert, Vladimir I. Fal’ko, V. Prigodin, and A. J. Epstein. Distribution of time constants for tunneling through a one-dimensional disordered chain. *Phys. Rev. B*, 60:10569–10572, Oct 1999.

[39] C. W. J. Beenakker. *Dynamics of localization in a waveguide*, pages 489–508. Springer Netherlands, 2001.

[40] Valerii I Klyatskin and Aleksandr I Saichev. Statistical and dynamic localization of plane waves in randomly layered media. *Soviet Physics Uspekhi*, 35(3):231–247, mar 1992.

[41] S. Anantha Ramakrishna and N. Kumar. Imaginary potential as a counter of delay time for wave reflection from a one-dimensional random potential. *Phys. Rev. B*, 61:3163–3165, Feb 2000.

[42] C. W. J. Beenakker, J. C. J. Paasschens, and P. W. Brouwer. Probability of reflection by a random laser. *Phys. Rev. Lett.*, 76:1368–1371, 1996.

[43] We have verified that the laguerre ensemble describes well our experimental results for the reflection of canonical disordered waveguides.

[44] Tsampikos Kottos. Statistics of resonances and delay times in random media: beyond random matrix theory. *Journal of Physics A: Mathematical and General*, 38(49):10761–10786, nov 2005.