A Software Package for Queueing Networks and Markov Chains analysis

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Queueing networks and Markov chains are widely used for conducting performance and reliability studies. In this paper we describe the queueing package, a free software package for queueing networks and Markov chain analysis for GNU Octave. The queueing package provides implementations of numerical algorithms for computing transient and steady-state performance measures of discrete and continuous Markov chains, and for steady-state analysis of single-station queueing systems and queueing networks. We illustrate the design principles of the queueing package, describe its most salient features and provide some usage examples.

CCS Concepts: • Mathematics of computing → Queueing theory; Markov processes; Mathematical software.

Additional Key Words and Phrases: Queueing Networks, Markov Chains, Mean Value Analysis

1 INTRODUCTION

Queueing Networks (QNs) and Markov chains are powerful modeling notations that are commonly used for capacity planning, bottleneck analysis and performance evaluation of systems [7]. Analyzing QNs and Markov chains involves the computation of metric such as the system throughput of a QN, or the stationary state occupancy probabilities of a Markov chain. Symbolic, numerical, and simulation-based techniques have been developed to compute these metrics.

In this paper we describe the queueing package for GNU Octave, a free environment for numerical computing [16]. The queueing package provides implementations of numerical algorithms for (i) transient and stationary analysis of discrete and continuous Markov chains; (ii) stationary analysis of single-station queueing systems; (iii) stationary analysis of some classes of product-form Queueing Network.

Although QNs and Markov chains are well studied topics, relatively few computer implementations of solution algorithms are available and actively supported [12]. Table 1 lists some software tools that are relevant for this paper.

JMT [6] is a Java package for workload characterization, bottleneck analysis, and QN modeling. JMT has a GUI that simplifies the definition and analysis of QN models, although it can also be used from the command line. JMT uses a simulation engine as its main solution technique, so it can support extended features (non-Markovian queues, fork/join systems, passive resources, and others) that are difficult if not impossible to handle numerically.

The LINE solver [11] is a free MATLAB toolbox for analyzing extended and layered queueing networks [17]. Extended QNs support features, such as simultaneous resource possession, fork/join systems, finite capacity regions and others, allowing more accurate models to be defined. This comes with the drawback that extended QNs are more difficult to analyze numerically. LINE can delegate the solution of these models to external solvers such as JMT.

PDQ [19] is an implementation of the Mean Value Analysis (MVA) algorithm for closed, single-class networks. PDQ provides bindings for different programming languages: at the time of writing, C, Perl, Python and R are supported.

The queueing package for R [9] (that, despite the name, is unrelated to the software described in this paper) is a free package for analyzing product-form QNs written in the R language [29]. It supports product-form open and closed, single and multiclass networks.
SHARPE (Symbolic Hierarchical Automated Reliability and Performance Evaluator) [35] is a hierarchical modeling tool that supports any combination of different types of performance and reliability models (product-form queueing networks, Petri nets, Markov chains, fault trees). SHARPE has both a command-line and a graphical interface, and has been under development since the early 80s. It is the only tool of those reviewed that has a non-free license.

The \texttt{queueing} package presented in this paper is somewhat orthogonal to above tools, in the sense that it has been developed around specific design goals which are only partially considered by other packages; of course, this implies that it has some limitations which might be addressed by other tools.

One of the design goals of the \texttt{queueing} package is to provide reference implementations of some fundamental "textbook" algorithms for QN and Markov chain analysis, like other research communities are doing since a long time (e.g., linear algebra algorithms). To this aim, efficiency has sometimes been sacrificed in favor of code readability. The availability of reference implementations is useful also for teaching purposes: students can immediately put the textbook algorithms at work to solve practical problems, encouraging "learning by doing". The author is aware of several Universities that are using the \texttt{queueing} package to teach performance modeling classes.

The GNU Octave language, being a large subset of the MATLAB language, is well suited for implementing numerical algorithms that operate on arrays and matrices in a concise and understandable way. Moreover, it allows complex performance studies can be done quickly, since models involving repetitive or embedded structure can be defined programmatically. Parametric model evaluation or ad-hoc analyses are also possible. The \texttt{queueing} package has been contributed to the Octave-forge public repository (https://octave.sourceforge.io/). This means that \texttt{queueing} can be easily installed from the Octave prompt using the standard command \texttt{pkg install}.

Any design decision inevitably carries some drawbacks. The GNU Octave environment allows a great degree of flexibility, but imposes a steep learning curve that might deter the occasional user. The focus on well-known classic algorithms neglects more recent results or less frequently used techniques. Yet, both issues can be addressed. A more comfortable interface, e.g., a GUI, can be built either as an independent application, or by leveraging existing tools (e.g., JMT) and then delegating the computations to the \texttt{queueing} package. More algorithms can be implemented and contributed for inclusion in the \texttt{queueing} package, that is free software and as such can be extended by anyone. A few contributors already did so.

This paper is structured as follows. In Section 2 we illustrate the design principles behind the \texttt{queueing} package. The next sections are devoted to illustrate the functions for analyzing Markov chains (Section 3), single-station queueing systems (Section 4) and queueing networks.
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(Section 5). The presentation focuses on the features provided, rather than the algorithmic details; comprehensive bibliographic references are provided for the interested reader. Although this paper is not intended to be a substitute of the package user’s manual, a few examples will be shown to better illustrate its use. Finally, concluding remarks are given in Section 6.

2 DESIGN PRINCIPLES

The queueing package is a collection of functions for computing transient and steady-state performance measures of queueing networks and Markov chains. It has been under development over the last decade to support the author’s research and teaching activity in the area of performance modeling of systems. The queueing package consists of a set of m-files written in the GNU Octave [16] dialect of the MATLAB programming language; therefore, queueing does not require any special installation procedure, nor does it require a compiler to generate executable code.

The decision of targeting GNU Octave was made at the beginning of the development effort. GNU Octave started its existence as a free MATLAB clone, but ended up providing extensions and additional features, some of which have been exploited by queueing (see below). More importantly, GNU Octave is free software and runs on all major operating systems, so it does not represent an entry barrier for potential users.

GNU Octave supports most of the standard MATLAB syntax, plus some extensions. For example, `!` can be used as the logical `not` operator; structured blocks such as the `if` and `for` constructs can be terminated with the `endif` and `endfor` keywords, respectively, to improve readability. The Texinfo markup notation [14] can be used for the documentation text embedded in function files. This feature has been extensively used: the documentation of each function in the queueing package can be displayed using the `help` command during interactive sessions. The user’s manual, in PDF and HTML formats, is built from the Texinfo documentation extracted from the source files. This guarantees that the user’s manual is always consistent with the help text.

Naming conventions. Most of the functions in the queueing package obey a common naming convention. Function names are the concatenation of several parts, beginning with a prefix that indicates the class of problems the function addresses:

- `ctmc-` Functions dealing with continuous-time Markov chains
- `dtmc-` Functions dealing with discrete-time Markov chains
- `qs-` Functions dealing with single-station queueing systems
- `qn-` Functions dealing with queueing networks

Functions that handle Markov chains (Section 3) start with either the `ctmc` or `dtmc` prefix, that may be followed by a string that hints at what the function does:

- `-bd` Birth-Death process
- `-mtta` Mean Time to Absorption
- `-fpt` First Passage Times
- `-exps` Expected Sojourn Times
- `-taexps` Time-Averaged Expected Sojourn Times

Therefore, function `ctmcbd` returns the infinitesimal generator matrix for a continuous birth-death process, while `dtmcbd` returns the transition probability matrix for a discrete birth-death process. Functions `ctmc` and `dtmc` (without any suffix) compute steady-state and transient state occupancy probabilities for Continuous Time Markov Chains (CTMCs) and Discrete Time Markov Chains (DTMCs), respectively.

Functions whose name starts with `qs-` deal with single station queueing systems (Section 4). The suffix describes the type of system, e.g., `qsmm1` for $M/M/1$, `qnmnm` for $M/M/m$ and so on.
Finally, functions whose name starts with qn- deal with queueing networks (Section 5). The character that follows indicates the type of network (o = open network, c = closed network), and whether there is a single (s) or multiple (m) customer classes.

- os- Open, single-class network
- om- Open, multiclass network
- cs- Closed, single-class network
- cm- Closed, multiclass network
- mix- Mixed network with open and closed classes of customers

The last part of the function name indicates what the function computes:

- aba Asymptotic Bounds
- bsb Balanced System Bounds
- gb Geometric Bounds
- pb PB Bounds
- cb Composite Bounds
- mva Mean Value Analysis (MVA)
- cmva Conditional MVA
- mvald MVA with load-dependent servers
- mvabs Approximate MVA using Bard and Schweitzer’s approximation
- mvablo Approximate MVA for blocking queueing networks
- conv Convolution algorithm
- convld Convolution algorithm with load-dependent servers

Validation. One important issue of numerical software is to make sure that the computed results are correct. Almost all functions in the queueing package include unit tests embedded as specially-formatted comments inside the source code. The unit tests are used to check the results against reference values from the literature. When reference results are not available, cross-validation with the output of different functions on the same model (if available), or with the output of other packages have been used. For example, a closed product-form network can be analyzed by MVA or using the convolution algorithm; therefore it is possible to apply the functions qncsmva() and qncsconv() on the same model and check whether their results agree up to known numerical problems [10]. Results have also been compared with those produced by different tools. This was helpful to investigate an issue with the qncmmva() function, whose result on the model described in [33, Figure 7, p. 9] did not agree with the one reported in that paper. The model was analyzed with Java Modeling Tools (JMT) that confirmed the values computed by the queueing package.

3 MARKOV CHAINS

A stochastic process is a set of random variables \( \{X(t), \ t \in T\} \) where each \( X(t) \) is indexed by a time parameter \( t \in T \). The state space is the set of all possible values of \( X(t) \). A time-homogeneous Markov chain is a stochastic process over the discrete state space \( \{1, \ldots, N\} \) for some given \( N \). In a DTMC, the time parameter \( t \) assumes the discrete values in \( T = \{0, 1, \ldots\} \), while in a CTMC the time parameter assumes values in \( T = [0, +\infty) \).

In a time-homogeneous DTMC the conditional probability \( p_{i,j} = \Pr\{X(n+1) = j \mid X(n) = i\} \) that the system is in state \( j \) at time \( n+1 \), given that the system was in state \( i \) at time \( n \), is independent from \( n \), so that we can define a DTMC as a stochastic matrix \( P \in \mathbb{R}^{N \times N} \), where \( p_{i,j} = \Pr\{X(n+1) = j \mid X(n) = i\} \) is the transition probability from state \( i \) to state \( j, i \neq j \).

Similarly, in a time-homogeneous CTMC the conditional probability \( p_{i,j}(u,v) = \Pr\{X(v) = j \mid X(u) = i\} \), \( v \geq u \), only depends on the time difference \( t = v - u \), and not on the specific values
of \(u\) and \(v\), so that we have \(p_{ij}(t) = \Pr\{X(u + t) = j \mid X(u) = i\} = \Pr\{X(t) = j \mid X(0) = i\}\) for each \(t \geq 0\). The evolution of a CTMC is defined by a generator matrix \(Q \in \mathbb{R}^{N \times N}\) where \(q_{ij}\) is the transition rate from state \(i\) to state \(j \neq i\). The diagonal elements \(q_{ii}\) are defined in such a way that the sum of each row is zero, i.e., \(1Q = 0\) (1 and 0 denote suitably sized row vectors of 1 and 0, respectively).

Let \(\pi_i(0)\) be the probability that the system is in state \(i\) at time 0. It can be shown that the state occupancy probabilities \(\pi(t) = (\pi_1(t), \ldots, \pi_N(t))\) at time \(t\) can be computed as:

\[
\pi(n) = \pi(0)P^n \quad \text{(DTMC)} \quad \pi(t) = \pi(0)e^{Qt} \quad \text{(CTMC)}
\]

Under certain conditions \([7]\) a Markov chain has a unique stationary distribution \(\pi\) that is independent from the initial state. The stationary distribution can be computed by solving the linear systems:

\[
\begin{align*}
\pi P &= \pi \\
\pi 1^\top &= 1 \\
\pi Q &= 0 \\
\pi 1^\top &= 1
\end{align*}
\]

where \(1^\top\) is a column vector of 1.

Functions \(\text{dtmc()}\) and \(\text{ctmc()}\) compute the transient or stationary state occupancy probabilities of a DTMC and CTMC, respectively, using a direct implementations of equations (1) and (2), respectively. For example, the expression \(p_n = \text{dtmc}(P, n, p0)\) computes the state occupancy probability vector \(p_n\) after \(n\) steps of a DTMC with stochastic matrix \(P\) and initial state probabilities \(p0\). If invoked with a single parameter as in \(p = \text{dtmc}(P)\), the function computes the stationary state distribution vector \(p\) or \(\text{ctmc()}\) can be used in a similar way to analyze CTMCs.

The queueing package provides other functions that compute metrics used in reliability and performability studies. The Mean Time To Absorption (MTTA) of a DTMC is defined as the average number of transitions required to reach an absorbing state, given the initial occupancy probability vector \(\pi(0)\) (a state is absorbing if it has no outgoing transitions). The MTTA can be computed from the fundamental matrix \(N = (I - P_r)^{-1}\), where \(P_r\) is the restriction of the transition matrix \(P\) to transient states only, and \(I\) is a suitably sized square identity matrix. Given initial state occupancy probabilities \(\pi(0)\), the mean number of steps before entering any absorbing state is:

\[
\text{MTTA} = \pi(0)(1N)^\top
\]

Other metrics of interest include the first passage time \(M_{i,j}\), defined as the average number of transitions before state \(j\) is visited for the first time, starting from state \(i\). Finally, the mean sojourn time \(L_i(n)\) is the expected number of visits to state \(i\) during the first \(n\) transitions, for given initial state occupancy probabilities; the ratio \(L_i(n)/n\) is called time-averaged mean sojourn time. All these concepts can be easily defined for continuous-time Markov chains as well.

Birth-death processes are a subclass of Markov chains that are at the basis, among other things, of the analysis of single-station queueing systems (see Section 4). In a \((N+1)\)-states birth-death
| Type          | Description                                      |
|--------------|--------------------------------------------------|
| Continuous   | Discrete                                         |
| ctmc()       | dtmc()                                           |
| ctmccbd()    | dtmccbd()                                        |
| ctmcexps()   | dtmcexps()                                       |
| ctmctaexps() | dtmctaexps()                                     |
| ctmcfpt()    | dtmcfpt()                                        |
| ctmcmmtta()  | dtmcmmtta()                                      |

Stationary/Transient state occupancy probabilities
Birth-Death process
Mean Sojourn Times
Time-Averaged Mean Sojourn Times
First Passage Times
Mean Time to Absorption

Table 2. Functions for Markov chains analysis

process, the transition probability (resp. rate) from state $i$ to $(i + 1)$ is $b_i$, and the transition probability (resp. rate) from state $(i + 1)$ to $i$ is $d_i$, $i = 1, \ldots, N$ (Figure 1). Function \( P = \text{dtmcbd}(b, d) \) returns a stochastic matrix \( P \) for a birth-death process with birth rates \( b = (b_1, \ldots, b_N) \) and death rates \( d = (d_1, \ldots, d_N) \). Function \( Q = \text{ctmc}(b, d) \) does the same for the continuous case, with the obvious difference that \( b \) and \( d \) are birth and death rates instead of probabilities.

Table 2 lists the functions that compute the performance metrics described above for.

**Example**

Let us consider the reliability model of a multiprocessor system shown in Figure 2 and originally described in [20]. The system consists of two processors, each subject to failures with Mean Time To Failure (MTTF) \( 1/y \). States labeled \( n \in \{0, 1, 2\} \) denote that there are \( n \) working processors. If one processor fails, it can be recovered (state RC) with probability \( c \); recovery takes time \( 1/\beta \). When the system can not be recovered, a reboot is required (state RB) that brings down the entire system for time \( 1/\alpha > 1/\beta \). The mean time to repair a failed processor is \( 1/\delta \). The system is operational if there is at least one working processor.

The model above can be represented as a CTMC with five states \( \{2, RC, RB, 1, 0\} \). The following fragment of GNU Octave code defines the stochastic matrix \( Q \) of the CTMC in Figure 2(a), and the uses the function \( \text{ctmc()} \) to compute the steady state occupancy probability vector \( p \) (parameter values are taken from [20]):

```octave
mm = 60; hh = 60*mm; dd = 24*hh; yy = 365*dd;
a = 1/(10*mm); # 1/a = duration of reboot (10 min)
b = 1/30; # 1/b = reconfiguration time (30 sec)
g = 1/(5000*hh); # 1/g = processor MTTF (5000 h)
```

![Fig. 2. Reliability Model for a dual-processor system (from [20])](image)
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\[ d = \frac{1}{4 \times \text{hh}}; \quad \# \ 1/d = \text{processor MTTR (4 h)} \]
\[ c = 0.9; \quad \# \ \text{recovery probability} \]
\[ Q = \begin{bmatrix} -2g & 2c + g & 2(1-c)g & 0 & 0 \\ 0 & -b & 0 & b & 0 \\ 0 & 0 & -a & a & 0 \\ d & 0 & 0 & -(g+d) & g \\ 0 & 0 & 0 & d & -d \end{bmatrix}; \quad \# \] 1
\[ p = \text{ctmc}(Q); \]

that is \( p = (9.9839 \times 10^{-1}, 2.9952 \times 10^{-6}, 6.6559 \times 10^{-6}, 1.5974 \times 10^{-3}, 1.2779 \times 10^{-6}) \). From these values we can derive several availability metrics; for example, the average time spent over one year in states \( RC, RB \) and 0 is:

\[
\begin{align*}
\text{p(2)*yy/mm} & \ # \ \text{minutes/year spent in RC} \\
\# & \Rightarrow 1.5743 \\
\text{p(3)*yy/mm} & \ # \ \text{minutes/year spent in RB} \\
\# & \Rightarrow 3.4984 \\
\text{p(5)*yy/mm} & \ # \ \text{minutes/year spent in 0} \\
\# & \Rightarrow 0.67169
\end{align*}
\]

that is, over a year, the system is unavailable for about 1.57 minutes due to reconfigurations, 3.50 minutes due to reboots and 0.67 minutes due to failure of both processors.

The Mean Time Between Failures (MTBF) is the average duration of continuous system operation. We assume that the system starts in state 2, and we consider the system operational also when in the reconfiguration state. Therefore, the set of states that we consider operational is \( \{2, 1, RC\} \). If we make states 0 and \( RB \) absorbing by removing all their outgoing transitions, the MTBF is the mean time to absorption of the (modified) CTMC:

\[
\begin{align*}
Q(3,:) &= Q(5,:) = 0; \quad \# \ \text{make states \{0, RB\} absorbing} \\
p0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \# \ \text{initial state occupancy prob.} \\
\text{MTBF} &= \text{ctmcmtta}(Q, p0)/\text{yy} \ # \ \text{MTBF (years)} \\
\# & \Rightarrow 2.8376
\end{align*}
\]

that yields a MTBF of approximately 2.84 years.

4 SINGLE-STATION QUEUEING SYSTEMS

A single-station queueing system, also called service center, consists of one or more servers connected to a shared queue. An infinite stream of requests (jobs) is generated outside the system and put into the queue. Jobs are extracted according to some queueing policy (e.g., First-Come-First-Served) and processed by one of the available servers. Once service completes, a job leaves the system permanently.

The following information is required to fully describe a single-station queueing system: (i) the nature of the arrival process; (ii) the distribution of service times; (iii) the number of servers; (iv) the size of the queue; (v) the queueing discipline, i.e., the policy used by the server(s) to extract requests from the queue.

Kendall’s notation [22] can be used to specify of queueing system. It consists of five symbols \( A/S/m/K/D \), where \( A \) denotes the type of arrival process, \( S \) the service time distribution, \( m \geq 1 \) the number of servers, \( K \geq m \) the maximum system capacity, and \( D \) the queueing discipline.
Several types of arrival processes $A$ and service time distributions $S$ have been studied in the literature, and assigned specific symbols: $M$ (exponential distribution), $D$ (deterministic distribution), $G$ (general distribution), $\text{Hyp}_{k}$ (hyperexponential distribution with $k$ phases), and others.

Queueing disciplines include First-Come First-Served (FCFS), Last-Come-First-Served (LCFS), Service In Random Order (SIRO), and Processor Sharing (PS). In the PS discipline all jobs are served at the same time (i.e., there is no queue), that is equivalent to round-robin scheduling with infinitesimally small time slice.

A commonly used arrival process and service time distribution are the Poisson point process and exponential distribution, respectively; both are denoted with the letter $M$ in Kendall’s notation.

Let $(\lambda t)^n/n!e^{-\lambda t}$ be the number of requests arriving at the queueing system during a time interval of length $t$; $A(t)$ is a Poisson point process if the probability $\Pr\{A(t) = n\}$ that there are $n$ arrivals is:

\[
\Pr\{A(t) = n\} = \frac{(\lambda t)^n}{n!}e^{-\lambda t}
\]

where $\lambda > 0$ is the expected number of arrivals for unit of time. It can be shown [24] that the stochastic variable $T$ representing the time between two successive arrivals (interarrival time) follows an exponential distribution with mean $1/\lambda$:

\[
\Pr\{T \leq t\} = 1 - e^{-\lambda t}
\]

The inter-arrival and service time distributions of a $M/M/\infty$ queue are therefore fully specified by the arrival rate $\lambda$ of requests and the throughput $\mu$ of each server. $M/M/\infty$ systems have the useful property that the Probability Mass Function (PMF) $\pi_k$ that there are $k \geq 0$ requests in the system$^1$ has a simple form allowing stationary performance measures to be expressed easily [7, 24].

Figure 3 shows a graphical representation of some of the single-station queueing system types supported by the queueing package. $M/M/m$ systems have $m \geq 1$ identical servers, so that up to $m$ requests can be serviced at the same time. Once a server becomes idle, it fetches the next request from the queue (if any) and processes it. The system is stable, i.e., the average queue length is finite, if $\lambda < m\mu$. Special cases of the $M/M/m$ system are the $M/M/1$ service center, where there is a single server, and the $M/M/\infty$ center where there are infinitely many identical servers, and therefore requests do not need to wait before receiving service. $M/M/\infty$ stations are also called Infinite

$^1$We adopt the widely used convention of using the same symbol $\pi_k$ for both the state occupancy probability of a queueing system and the state of a Markov chain.
Server (IS) nodes or delay centers, since they essentially delay incoming requests by an average duration $1/\mu$. IS nodes are always stable, irrespective of the arrival and service rates.

The $M/M/m/K$ system is a finite-capacity variants of the $M/M/m$ queueing center. The parameter $K \geq m$ represents the maximum number of jobs in the system, including those being served; therefore, there are $(K - m)$ slots in the queue. Finite-capacity centers are always stable, since each request that tries to join a full system is discarded.

Non-Markovian queues are used in some contexts, such as modeling of telecommunication networks. The asymmetric $M/M/m$ systems consists of $m$ exponential servers with possibly different service rates $\mu = (\mu_1, \ldots, \mu_m)$. At most $m$ requests can be served concurrently; if multiple servers are available, the next request receives service from a randomly chosen one. This system is stable if $\lambda < \sum_{i=1}^{m} \mu_i$. In the $M/\text{Hyper}_m/1$ system the server has $m$ different service rates $\mu = (\mu_1, \ldots, \mu_m)$ that are selected with probabilities $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\sum_{i=1}^{m} \alpha_i = 1$. Non-Markovian queueing systems are harder to analyze; the queueing package uses the approximation techniques described in [24], where both asymmetric $M/M/m$ and $M/\text{Hyper}_m/1$ queues are treated as $M/G/1$ systems.

Performance measures of single-station queueing systems include the following quantities:

- $U$: Utilization: mean fraction of time the servers are busy. In general, $U \in [0, 1]$: for example, for a stable $M/M/m$ system the utilization is $U = \lambda/(m\mu)$. In the case of the $M/M/\infty$ system, $U$ is defined as the traffic intensity $U = \lambda/\mu$ and can be also greater than one, since the system is always stable.
- $R$: Response time: average time spent by a request inside the system, i.e., the mean duration of the interval between a request arrival in the queue and its departure after completing service.
- $Q$: Mean queue length.
- $X$: Throughput: average number of requests that complete service in a unit of time. If the system is stable, then the throughput is equal to the arrival rate ($X = \lambda$).

The performance measures above can be derived from the steady-state probability $\pi_k$, although in most cases there are simpler closed-form expressions that do not require the explicit computation of $\pi_k$. However, of particular interest is the probability $\pi_0$ that the system is empty, and the rejection probability $\pi_K$ for a finite-capacity systems where at most $K$ jobs are allowed. The queueing package can compute the value of $\pi_k$ for Markovian queues for any given $k$.

Table 3 lists the functions provided by the queueing package to analyze the supported types of queueing systems. Note that $M/M/1$ and $M/M/\infty$ systems are handled separately from $M/M/m$ queues, since simpler formulas for the special cases $m = 1$ and $m = \infty$ are used.

| Function   | Description                                      |
|------------|--------------------------------------------------|
| qsmm1()    | $M/M/1$ system                                   |
| qsmm()     | $M/M/m$ system with $m$ identical servers        |
| qsmminf()  | $M/M/\infty$ system (delay center)               |
| qsmm1k()   | $M/M/1/K$ finite-capacity system                 |
| qsmmmk()   | $M/M/m/K$ finite-capacity system ($K \geq m$)    |
| qsmmm()    | Asymmetric $M/M/m$                               |
| qsmh1()    | $M/\text{Hyper}_m/1$ queue with hyper-exponential service time distribution |
| qsmg1()    | $M/G/1$ queue with general service time distribution |

Table 3. Supported single-station queueing systems
Example

Let us consider a $M/M/m$ center with arrival rate $\lambda$ and service rates $\mu$. Assuming stability ($\lambda < m\mu$), the steady state probability $\pi_{k,M/M/m}$ that there are $k \geq 0$ requests in the system is [7]:

$$
\pi_{k,M/M/m} = \begin{cases} 
\pi_{0,M/M/m} \frac{(m\rho)^k}{k!} & 0 \leq k \leq m \\
\pi_{0,M/M/m} \frac{\rho^k m^m}{m!} & k > m
\end{cases}
$$

(3)

where $\rho = \lambda/(m\mu)$ is the individual server utilization, and the steady-state probability $\pi_{0,M/M/m}$ that there are no requests in the system is:

$$
\pi_{0,M/M/m} = \left[ \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1 - \rho} \right]^{-1}
$$

(4)

The limit of (3) as $m$ tends to infinity is the steady state probability $\pi_{k,M/M/\infty}$ that there are $k$ request in a $M/M/\infty$ Infinite Server node:

$$
\pi_{k,M/M/\infty} = \lim_{m \to \infty} \pi_{k,M/M/m} = \frac{1}{k!} \left( \frac{\lambda}{\mu} \right)^k e^{-\lambda/\mu}
$$

The following fragment of GNU Octave code uses the functions `qsmmm()` and `qsmminf()` to compute the steady state probability that there are $k$ requests in the system, $k = 0, \ldots, 20$, for an $M/M/4$, $M/M/5$ and $M/M/\infty$ system.

```octave
lambda = 4; mu = 1.2; k = 0:20;
pi_mm4 = qsmmm(lambda, mu, 4, k);
pi_mm5 = qsmmm(lambda, mu, 5, k);
pi_mminf = qsmminf(lambda, mu, k);
```

Note that `qsmmm()`, `qsmmm()` and `qsmminf()` like other functions in the queueing package, support vector arguments. In these cases a vector of results is returned. Also, the queueing package relies on Horner’s rule

$$
\sum_{k=0}^{n} \frac{a^k}{k!} = 1 + a \left( 1 + \frac{a}{2} \left( 1 + \frac{a}{3} \left( 1 + \frac{a}{n} \cdots \right) \cdots \right) \right)
$$

to evaluate the summation of Eq. (4) more accurately.

Figure 4 shows that the marginal probabilities $\pi_{k,M/M/m}$ tend indeed to $\pi_{k,M/M/\infty}$ as the number of servers $m$ grows.

5 QUEUEING NETWORKS

AQN consists of $K \geq 1$ service centers (nodes) and a population of requests (jobs) that visit the servers in some order. Several types of QNs have been studied, depending on the type of population of requests. In open networks there is an infinite stream of jobs that originate outside the system and eventually leave the system forever (Figure 5a). In closed networks there is a fixed population of jobs that never leave the system (Figure 5b). Requests can be all of the same type (single-class networks) or of multiple types (multiclass models). In a multiclass QN, different types of requests can visit the service centers in a different order or have different service demands (the service demand is the average time spent by requests on a given node, see below). In mixed networks, open and closed classes of requests can coexist (Figure 5c).
QN analysis consists of computing steady-state performance measures such as throughput or average queue length of the service centers. These measures can be derived from the equilibrium state probability $\Pr(S)$ that the system is in state $S$ for each valid state, where the exact nature of $S$ is model-dependent.
Some classes of QNs enjoy product-form solution, meaning that $\Pr(S)$ has the relatively simple form

$$
\Pr(S) = \frac{1}{G(S)} d(S) \prod_{i=1}^{K} f_i(x_i)
$$

where $x_i$ is the configuration of the $i$-th service center, $f_i$ is a function that depends on the type of service center, $d(S)$ is a model-dependent function of the global state, and $G$ is a normalization constant. A QN with product-form solution can be analyzed efficiently by considering each node in isolation and combining the partial results.

The first class of product-form open networks was identified by Jackson [21]. Later, Gordon and Newell extended product-form solution to certain classes of closed networks [18]. These results were further extended by Baskett, Chandy, Muntz and Palacios [5] to include open, closed and mixed networks with multiple customer classes (since then known as BCMP networks). Other types of networks have been shown to possess product-form solution; the interested reader is referred to [2] for a review.

The queueing package supports a subset of BCMP networks that satisfy the following constraints:

- The network can consist of open or closed job classes (or both).
- The following queueing disciplines are allowed: FCFS, PS, Last-Came First-Served, Preemptive Resume (LCFS-PR) and IS.
- Service times for FCFS nodes are exponentially distributed and class-independent. For PS, LCFS-PR and IS nodes, different classes of customers can have different service times.
- The service rate of a FCFS node can depend on the number of jobs at this node (load-dependent service centers).
- In open networks two kinds of arrival processes are allowed: (i) Poisson arrival process with arrival rate $\lambda$. (ii) $C$ independent Poisson arrival streams where the $C$ job sources are assigned to the $C$ chains.

The constraints above allow a considerable simplification of the algorithms implemented, and at the same time include the types of networks that are most frequently used in practice.

Table 4 lists the main functions for QN analysis provided by the queueing package; more details are provided in the rest of this section.

**Single-class models.** In single class models, service centers do not differentiate the requests that they process. This means that, for example, the mean time spent by a request in a given server (service time) will depend only on the server, not on the type of request.

A single-class QN can be fully specified by the following parameters:

- $K$ Number of service centers.
- $\lambda_i$ (Open networks only) External arrival rate to center $i \in \{1, \ldots, K\}$.
- $N$ (Closed networks only) Total number of requests in the system.
- $Z$ (Closed networks only) Optional external delay ("think time") spent by each request outside the system after each interaction.
- $S_i$ Mean service time at any server inside center $i$ for each visit (not including the time spent waiting in the queue). For general load-dependent service centers, the service time is a vector where $S_i(n)$ is the service time when there are $n$ requests in center $i$.
- $P_{i,j}$ Probability that a request completing service at center $i$ is routed to center $j$. For open networks, the probability that a request leaves the system after completing service at center $i$ is $\left(1 - \sum_{j=1}^{K} P_{i,j}\right)$.
Mean number of visits to center \( i \) (also called visit ratio or relative arrival rate).

For open, single class networks the visit ratios \( V_i \) satisfy the following equations:

\[
V_i = P_{0,i} + \sum_{j=1}^{K} V_j P_{j,i} \quad i = 1, \ldots, K
\]  

(6)

where \( P_{0,i} \) is the probability that an external request goes to center \( i \). If we denote with \( \lambda_i \) the external arrival rate to center \( i \), and \( \lambda = \sum_i \lambda_i \) is the overall external arrival rate, then \( P_{0,i} = \lambda_i / \lambda \).

For closed networks, the visit ratios satisfy the following equation:

\[
\begin{cases}
V_i = \sum_{j=1}^{K} V_j P_{j,i} & i = 1, \ldots, K, \ i \neq r \\
V_r = 1 & \text{for a selected reference station } r \in \{1, \ldots, K\}
\end{cases}
\]  

(7)

The second condition ensures that the values \( V_i \) are uniquely defined. A job that returns to the reference station (default \( r = 1 \)) is assumed to have completed one interaction with the system. The product \( D_i = S_i V_i \) of the average service time per visit \( S_i \) and the mean number of visits \( V_i \) is called service demand, and can be understood as the total service time requested by a job during one interaction with the system. The service center with the larger service demand is the bottleneck of the system.

Most of the algorithms in the queueing package rely on the visit ratios \( V_i \); if only the routing matrix \( \mathbf{P} \) is available, functions \texttt{qnosvisits()} and \texttt{qncsvisits()} can be used to compute the \( V_i \) using Eq. (6) or (7), respectively.

The following performance results for single-class models are computed:
Algorithm 1 MVA algorithm without load-dependent service centers

Require: \( K, N, Z, S_i, V_i, i = 1, \ldots, K \)
Ensure: \( Q_i, R_i, U_i, X_i \)

for \( i \leftarrow 1, \ldots, K \) do
  \( Q_i \leftarrow 0 \)
for \( n \leftarrow 1, \ldots, N \) do
  for \( i \leftarrow 1, \ldots, K \) do
    \( R_i \leftarrow \begin{cases} S_i & \text{if center } i \text{ is } M/M/\infty \\ S_i(1 + Q_i) & \text{if center } i \text{ is } M/M/1 \end{cases} \)
  \( R \leftarrow \sum_{i=1}^{M} R_i V_i \)
  \( X = \frac{Z + R}{n} \)
  for \( i \leftarrow 1, \ldots, K \) do
    \( Q_i \leftarrow XV_i R_i \)
for \( i \leftarrow 1, \ldots, K \) do
  \( X_i \leftarrow XV_i \)
  \( U_i \leftarrow XV_i S_i \)

\[ U_i \quad \text{Utilization of service center } i; \]
\[ R_i \quad \text{Response time of service center } i; \]
\[ Q_i \quad \text{Average number of requests at center } i, \text{ including the request(s) being served;} \]
\[ X_i \quad \text{Throughput of service center } i; \]

From the values above, global performance measures can be derived:

\[ X \quad \text{System throughput, } X = X_i/V_i \text{ for any } i \text{ for which } V_i > 0; \]
\[ R \quad \text{System response time, } R = \sum_{i=1}^{K} R_i V_i; \]
\[ Q \quad \text{Average number of requests in the system, } Q = \sum_{i=1}^{K} Q_i \]

The MVA [31] and convolution [8] algorithms are the most widely used techniques to compute stationary performance measures of closed product-form networks. The convolution algorithm computes the normalization constant \( G \) in Eq. (5); all other performance measures are derived from \( G \). MVA relies on the fact that, in a closed network with \( N \) requests, the response time \( R_i(N) \) of center \( i \) can be expressed as [31]

\[ R_i(N) = S_i (1 + Q_i(N - 1)) \quad (8) \]

where \( Q_i(N - 1) \) is the mean queue length at center \( i \) if one request is removed from the system. In the case of a single-class network with \( M/M/1 \) center or \( M/M/\infty \) IS nodes only, MVA assumes the simple form shown in Algorithm 1.

A single-class closed networks with \( N \) requests and \( K \) service centers of type \( M/M/1 \) or \( M/M/\infty \) can be analyzed in time \( O(NK) \) by either the MVA or the convolution algorithms. Unfortunately, if multiple-server nodes or general load-dependent service centers are present, both algorithms suffer from numerical instabilities. In the case of MVA, Eq. (8) is no longer sufficient to compute the response time at center \( i \), since adding a new request may alter the (load dependent) service time \( S_i \). It is therefore necessary to compute the marginal probabilities \( p_i(j|n) \) that there are \( j \) requests at center \( i \), given that the total number of requests in the system is \( n \). At the end of each
iteration, the MVA algorithm computes the probability \( p_i(0|n) \) that center \( i \) is idle as

\[
p_i(0|n) = 1 - \sum_{j=1}^{n} p_i(j|n)
\]  

(9)

Eq. (9) is the source of numerical errors [30], especially if there are many requests (\( n \) is large) and/or there are servers whose utilization is close to 1.

So far, no numerically stable variant of the MVA and convolution algorithms exist, although stable approximations have been proposed [37]. The queueing package provides an implementation of the Conditional MVA (CMVA) algorithm [10], a numerically stable variant of MVA. Unfortunately, CMVA only supports a single load-dependent service center, and is therefore less general than MVA.

**Multiple-class models.** The MVA and convolution algorithms can be extended to handle QNs with multiple job classes. In a multiclass QN there are \( C \) customer classes; open and closed classes of requests may be present at the same time. Since a request may change class after service completion, the concept of *chain* needs to be introduced. Chains induce a partition of the set of classes: class \( c_1 \) and \( c_2 \) belong to the same chain if a job of class \( c_1 \) can eventually become a job of class \( c_2 \). A chain can contain multiple classes, but cannot contain both an open and a closed class. This prevents jobs from closed classes to enter open classes, or the other way around.

A multiclass network can be described using the same parameters as those used for single class models, with additional subscripts required to take classes into account:

\[
\begin{align*}
\lambda_{c,i} & \quad \text{(Open networks only) External arrival rate of class } c \text{ requests to service center } i. \\
N_c & \quad \text{(Closed networks only) Total number of class } c \text{ requests in the system.} \\
Z_c & \quad \text{(Closed networks only) External delay (also called “think time”) spent by each class } c \text{ request outside the system after one round of interaction with the service centers is completed. See below.} \\
S_{c,i} & \quad \text{Mean service time of class } c \text{ requests at center } i; \text{ product-form requires that service times at FCFS queues be class-independent, while service times at IS or PS nodes can vary on a per-class basis.} \\
P_{r,i,s,j} & \quad \text{Probability that a class } r \text{ request that completes service at center } i \text{ is routed to class } j \text{ as a class } s \text{ request.} \\
V_{c,i} & \quad \text{Mean number of visits of class } c \text{ requests to center } i.
\end{align*}
\]

Similarly, performance results (utilization, response times, and so on) are computed for each service center and class, e.g., \( X_{c,i} \) denotes the throughput of class \( c \) requests at center \( i \).

The queueing package analyzes product-form multiclass closed networks are using the multiclass MVA algorithm. Let \( N = (N_1, \ldots, N_C) \) be the population vector, i.e., the vector where \( N_c \) is the number of class \( c \) requests in the system, \( c = 1, \ldots, C \). Let \( 1_c \) be the vector of length \( C \) where the \( c \)-th element is one and all other elements are zero. For closed networks with only fixed-rate (\( M/M/1 \)) and IS (\( M/M/\infty \)) nodes, the BCMP theorem [5] states that the response time \( R_{c,i}(N) \) of class \( c \) requests at center \( i \) is:

\[
R_{c,i}(N) = S_{c,i} \left( 1 + Q_i(N - 1_c) \right)
\]

(10)

where \( Q_i(N - 1_c) \) is the mean queue length at center \( i \) with one class \( c \) customer removed (if \( n_c = 0 \) we let \( Q_i(N - 1_c) = 0 \)). Eq. (10) is similar to (8), and is the core of the multiclass MVA Algorithm shown in 2.

Multiclass MVA allows all performance measures to be computed starting from the queue lengths \( Q_i(0) = 0 \) of the network with no jobs. Specifically, Algorithm 2 computes the mean response time \( R_{c,i} \) of
In situations where accurate computation of performance measures is impractical, bound analysis can be used to provide upper/lower limits on the system throughput $X$ and response time $R$. Performance bounds on QNs can be computed quickly, and are useful for example in scenarios involving on-line performance tuning of systems [26–28]. The \texttt{queueing} package provides an implementation of Bard and Schweitzer’s iterative approximation scheme [4, 25, 32] through function \texttt{qncmmvabs()}.

**Algorithm 2 Multiclass MVA without load-dependent service centers**

\begin{algorithm}
\KwRequire{$K, C, N_c, Z_c, S_{c,i}, V_{c,i}, c = 1, \ldots, C, i = 1, \ldots, K$}
\KwEnsure{$Q_i, R_{c,i}, X_c$}
\For{$i \leftarrow 1, \ldots, K$}{
$Q_i(0) \leftarrow 0$
}
\For{$n \leftarrow 1, \ldots, \sum_{c=1}^{C} N_c$}{
\For{each feasible population $n = (n_1, \ldots, n_C)$ with $n$ total requests}{
\For{$c \leftarrow 1, \ldots, C$}{
$R_{c,i} \leftarrow \begin{cases} S_{c,i} & \text{if center } i \text{ is } M/M/\infty \\
S_{c,i} (1 + Q_i(n - 1_c)) & \text{if center } i \text{ is } M/M/1 
\end{cases}$
}
}
\For{$c \leftarrow 1, \ldots, C$}{
$X_c = \frac{n_c}{Z_c + \sum_{i=1}^{K} V_{c,i} R_{c,i}}$
}
\For{$i \leftarrow 1, \ldots, K$}{
$Q_i(n) \leftarrow \sum_{c=1}^{C} X_c V_{c,i} R_{c,i}$
}
\end{algorithm}

class $c$ requests at center $i$, the mean queue length $Q_i(N)$ at center $i$, and the global throughput $X_c$ of class $c$ requests. The other performance measures can be derived easily [7, 25]:

\begin{align*}
X_{c,i} &= X_c V_{c,i} & \text{class } c \text{ throughput at center } i \quad (11) \\
U_{c,i} &= X_c S_{c,i} V_{c,i} & \text{class } c \text{ utilization at center } i \quad (12) \\
Q_{c,i} &= X_c R_{c,i} & \text{mean number of class } c \text{ requests at center } i \quad (13)
\end{align*}

The multiclass MVA algorithm generates all feasible populations $n = (n_1, \ldots, n_C)$; we say that $n$ is feasible with respect to the population vector $N = (N_1, \ldots, N_C)$ if $0 \leq n_c \leq N_c$ for all $c = 1, \ldots, C$. It can be easily seen that there are $\prod_{c=1}^{C} (N_c + 1)$ feasible population vectors; therefore, for a closed, multiclass network with $K$ fixed-rate or IS nodes, $C$ customer classes and population vector $N$, multiclass MVA requires time $O\left(CK \prod_{c=1}^{C} (N_c + 1)\right)$ and space $O\left(K \prod_{c=1}^{C} (N_c + 1)\right)$.

Due to its computational cost, multiclass MVA is appropriate for networks with small population and limited number of classes. For larger networks, approximations based on the MVA have been proposed in the literature. The \texttt{queueing} package provides an implementation of Bard and Schweitzer’s iterative approximation scheme [4, 25, 32] through function \texttt{qncmmvabs()}. Bard-Schweitzer approximation requires space $O(CK)$, that compares favorably with that of standard multiclass MVA. Being an iterative scheme that stops as soon as a convergence criterion is met, the execution time of Bard-Schweitzer approximation depends on the network being analyzed, but is generally much lower than multiclass MVA (see the example at the end of this section). Unfortunately, the drawback is that there is no known way to estimate the accuracy of the results provided by the Bard-Schweitzer algorithm.

**Bound analysis.**
allows the computation of several classes of bounds: Asymptotic Bounds (AB), Balanced System Bounds (BSB), Composite Bounds (CB) and Geometric Bounds (GB).

Asymptotic Bounds [15] rely on the simplifying assumption that the service demand of a request at a service center is independent from the number of requests in the system and their exact location. Under this assumption (that is not true in general) it is possible to bound the system’s performance by considering the extreme situations of lowest and highest possible loads. ABs for a single-class network with \( K \) service centers can be computed in time \( O(K) \); for multiclass networks with \( C \) customer classes, the computational complexity is \( O(CK) \).

Balanced System Bounds [36] provide tighter bounds that are computed by forcing the service demands of the network under consideration to be all the same. BSB have the same computational complexity as AB, both for single and multiclass models. Composite Bounds [23] and Geometric Bounds [13] are yet different bounding techniques that, in many cases, produce even better bounds with the same computational cost.

Queueing networks with blocking. The queueing package provides limited support for analyzing closed, single-class networks with blocking. In blocking networks, queues have a finite capacity: a request joining a full queue will block until one slot becomes available. Apart from very few exceptions, queueing networks with blocking do not satisfy the conditions for product-form solution [3], and are therefore difficult to analyze.

The qnsmvablo() function implements the MVABLO algorithm [1] that is based on an extension of MVA. MVABLO provides approximate performance measures for closed, single-class networks with Blocking After Service (BAS) blocking. According to the BAS discipline, a request completing service at center \( i \) that wants to move to center \( j \) blocks the source server \( i \) until one slot is available at the destination.

Networks with a different type of blocking are handled by the qnmarkov() function. This function supports single-class, open or closed networks where all queues have (possibly different) finite capacity. The blocking discipline is Repetitive Service with Random Destination (RS-RD): when a request terminates service at center \( i \) and wants to move to a saturated center \( j \), the request is put back in the queue of center \( i \) so that it will eventually receive another round of service from \( i \). Each time the request completes a new round of service, it is routed to a possibly different, randomly chosen server. In the case of open networks, external arrivals to a saturated servers are discarded. The qnmarkov() function computes performance measures by building and analyzing the underlying Markov chain; this makes the function unsuited for even moderate networks due to the combinatorial explosion of the size of the Markov chain.

**Example**

We now demonstrate the use of the queueing package for analyzing the closed multiclass network shown in Figure 6, that represents a simple model of a scientific compute farm. The system has three classes of jobs that process data stored on disk servers; occasionally, data must be retrieved from tape libraries and copied to the disk servers. Each job spends some amount of time on CPU-intensive computations and then accesses data on external storage. Data resides on three tape libraries (nodes 1–3). Four disk servers (nodes 4–7) act as a cache for data copied from the tape libraries. Tape libraries and disk servers are modeled as \( M/M/1 \) service centers. We assume that the number of CPU cores is not a limiting factor, i.e., each job starting a CPU burst always finds a CPU core available. Therefore, the CPU farm can be represented as an IS node (\( M/M/\infty \) center).

In the model of Figure 6, the IS node represents the “think time” of jobs, a term that originated from batch systems where IS nodes were the terminals where users spend some time “thinking” before submitting new commands to the system.
Fig. 6. Multiclass closed network model of a scientific compute farm.

| Param | Description | Class 1 | Class 2 | Class 3 |
|-------|-------------|---------|---------|---------|
| $D_{c,1}$ | Tape Server | 100 | 180 | 280 |
| $D_{c,2}$ | Tape Server | 140 | 10 | 160 |
| $D_{c,3}$ | Tape Server | 200 | 70 | 150 |
| $D_{c,4}$ | Disk Server | 30 | 10 | 90 |
| $D_{c,5}$ | Disk Server | 50 | 90 | 20 |
| $D_{c,6}$ | Disk Server | 20 | 130 | 50 |
| $D_{c,7}$ | Disk Server | 10 | 30 | 18 |
| $Z_c$ | Cpu farm | 2400 | 1800 | 2100 |

Table 5. Parameters for the model in Figure 6

Let $N$ be the total number of jobs. We denote with $\beta = (\beta_1, \beta_2, \beta_3)$ the population mix of the network, where $\beta_c$ is the fraction of class $c$ jobs, $0 \leq \beta_c \leq 1$ and $\beta_1 + \beta_2 + \beta_3 = 1$. Thus, the number of class $c$ jobs is $N_c = \beta_c N$ rounded to the nearest integer. Let $D_{c,i}$ be the service demand of class $c$ requests at center $i$ (recall that the service demand is the product of the mean service time and the number of visits, $D_{c,i} = S_{c,i} V_{c,i}$). Let $Z_c$ be the average duration of a CPU burst of a class $c$ job. The parameter values are shown on Table 5.

We consider $N = 300$ jobs, and we want to study how different population mixes $\beta$ affect the system throughput $X$. For example, the following GNU Octave code computes the per-class utilizations $U_{c,i}$, response times $R_{c,i}$, mean queue lengths $Q_{c,i}$ and throughput $X_{c,i}$ when $\beta = (0.2, 0.3, 0.5)$:

```
N = 300; # total n. of jobs
S = [100 140 200 30 50 20 10; # service demands
     180 10 70 10 90 130 30;
     280 160 150 90 20 50 18];
Z = [2400 1800 2100]; # mean duration CPU bursts
V = ones(size(S)); # n. of visits
m = ones(1,columns(S)); # n. of servers in nodes
beta = [0.2, 0.3, 0.5]; # population mix
```
Fig. 7. Approximate system throughput as a function of the population mix $\beta = (\beta_1, \beta_2, 1 - \beta_1 - \beta_2)$; the values have been computed using Bard-Schweitzer approximation. Contour lines show the regions of equal throughput. Irregularities towards the center are caused by rounding the population to the nearest integer. (Best viewed in color)

```
pop = round(N*beta); pop(3) = N - pop(1) - pop(2);
[U R Q X] = qncmmva(pop, S, V, m, Z);
X_sys = sum(X(:,1) ./ V(:,1));  # System throughput
```

Note that `qncmmva()` expects as parameters the mean service times $S_{c,i}$ and the mean number of visits $V_{c,i}$. Since we know the service demands, we let $S_{c,i} = D_{c,i}$ and set all visits to one.

The system throughput of a multiclass network is $X_{sys} = \sum_c X_c$, where $X_c$ is the class $c$ throughput. The values of $X_c$ can be computed from the individual servers throughput $X_{c,i}$ that are returned by `qncmmva()`, using Eq. (11) with $i = 1$ (actually, any valid value for $i$ will do). In the example above we get $X_{sys} = 0.0053793$.

Even on such a small network, `qncmmva()` requires about 170s of CPU time on an Intel i7-4790 CPU at 3.60GHz running Ubuntu Linux 18.04 with GNU Octave 5.1; this makes the multiclass MVA algorithm impractical for this type of study, since analyzing many population mixes would require a prohibitive amount of time. We therefore resort to the much faster Bard-Schweitzer approximation, realized by function `qncmmvabs()`.

Figure 7 shows the system throughput $X$ for $(50 \times 50)/2$ different population mixes. Each square corresponds to a combinations of $\beta_1, \beta_2$, from which $\beta_3 = N - \beta_1 - \beta_2$. Contour lines show the regions of the parameter space of equal throughput; the population mixes that result in the optimal throughput are those towards the center of the image.

The whole Figure 7 can be computed in about 5s using `qncmmvabs()` on the same system above, i.e., orders of magnitude faster than the time that would be required by the multiclass MVA implementation from function `qncmmva()`.
6 CONCLUSIONS

In this paper we described the queueing package, a GNU Octave package for QNs and Markov chains analysis. The queueing package includes functions for transient and stationary analysis of discrete and continuous Markov Chains, and for stationary analysis of single-station queueing systems and product-form QNs. The queueing package can handle open, closed and mixed QNs with one or multiple classes of requests. Exact and approximate performance metrics can be obtained, as well as different types of bounds.

Research on efficient solution techniques for QN models is still an active topic. The queueing package will therefore be extended to include some of the newer algorithms. Furthermore, we plan to include support for more types of non product-form networks.

The latest version of the queueing package is available at https://octave.sourceforge.io/queueing/ and can be used, modified and distributed under the terms of the GNU General Public License (GPL) version 3 or later.

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