\[ \Gamma - \Sigma - C^0 - \text{determinacy of} \quad \Gamma - \Sigma - C^0 - \text{equivariant bifurcation problems with respect to} \quad \Gamma - \Sigma - C^0 - \text{BD and contact equivalence from the weighted point view} \]

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abstract

In this paper, \( C^0 \) finite determination of \( \Gamma \)–equivariant bifurcation problems in the relative case from the weighted point view is being discussed. Some criteria on the \( C^0 \) finite determination of \( \Gamma \)–equivariant bifurcation problems in the relative case are then obtained in terms of an analytic-geometric nondegeneracy condition, which generalize the result on the \( C^0 \) finite determination of bifurcation problems given by P.B.Percell and P.N.Brown.

**Keywords:** \( \Gamma - \Sigma - C^0 \)–contact equivalence, \( \Gamma - \Sigma - [C^r] \) BD equivalence,  
\( \Gamma - \Sigma - C^0 \)–determinacy, singular Riemann metric, vector field.

**MSC2000:** 37G40, 58K70, 58K40.

1 Introduction

Bifurcation phenomenon arise from a large number of nonlinear problems, the associated bifurcation problems are studied via a reduction ( such as the Lyapunov-Schmidt procedure ) to a finite dimensional local model \( G(u, \lambda) : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow \)

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This is viewed as a perturbation $G_\lambda(u) = G(u, \lambda)$ using parameters $\lambda \in \mathbb{R}^l$, of a germ $G_0(u) = G(u, 0)$. There are a variety of notions of equivalence for studying such perturbations. The feature of interest in the present content is the variation of the set of zeros of $G_\lambda(0)$ with the parameter $\lambda$.

A bifurcation problem is considered to be a family of maps

$$G(\cdot, \lambda) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$$

parameterized by $\lambda \in \mathbb{R}^l$ such that $G(0,0) = 0$, or, more compactly, a map

$$G(\cdot, \lambda) : (\mathbb{R}^n \times \mathbb{R}^l, 0) \longrightarrow (\mathbb{R}^p, 0).$$

The set $G^{-1}(0)$ is called the bifurcation diagram. Roughly, two bifurcation problems are equivalent if their bifurcation diagrams are locally homeomorphic in a neighbourhood of the origin, and a very interesting problem is to determine what terms from the Taylor expansion at some points in neighbourhood of 0 may be omitted without changing the topological type determined by $G$ and the value of the bifurcation parameter $\lambda$. It is concerned with determinacy of bifurcation problems.

There is an extensive literature related to determination of bifurcation problems.

$C^\infty$ theory of finite determinacy of bifurcation problems was systematically studied by M. Golubitsky using singularity theory and group-theoretic techniques (ref [4], [5]). $C^0$ theory of finite determination of bifurcation problems is explored by Peter B. Percell and Peter N. Brown in [6]. They have shown that $C^0$ finite determination of bifurcation diagrams follow from an analytic-geometric non-degeneracy condition which is modelled on a criterion of Kuo, rather than an algebraic condition of the type found in the $C^\infty$ theory. In [14], Z. Jiangcheng, S. Fuwei, S. Ruixia and L. Guofu have discussed the d-determination of bifurcation problems with respect to $C^0$ contact equivalence from the weighted point of view, a criterion is given to judge the d-determination of bifurcation problems with respect to $C^0$ contact equivalence. Bucher, Marsden and Schecter ([2]) have also obtained a criterion for $C^0$ finite determination of bifurcation diagrams using a blowing-up construction and techniques from algebraic geometry. Above works on determination of bifurcation problems only deal with in the case that perturbation $G_\lambda(u) = G(u, \lambda)$ with parameter $\lambda$ of a germ $G_0(u) = G(u, 0)$ has isolated bifurcation point in neighbourhood of the origin.

The case where a perturbation $G_\lambda(u) = G(u, \lambda)$ with parameter $\lambda$ of a germ $G_0(u) = G(u, 0)$ has non-isolated bifurcation points is more complicated. What we want to know is whether $G$ is determined up to equivalence by finite coefficients in their Taylor expansion at every point that belongs to the subset of non-isolated bifurcation points of $G_\lambda(u) = G(u, \lambda)$ with parameter $\lambda$ in neighbourhood of the origin.

The bifurcation diagram of $G$ actually bifurcate at $(x, \lambda)$ it is necessary

$$\text{rank } [\nabla_x G(x, \lambda)] < p,$$

where $\nabla_x G$ denotes partial derivative of $G$ with respect to the variable $x \in \mathbb{R}^n$.

Now if for a given closed set $\Sigma$ containing the origin in $(\mathbb{R}^n, 0)$, there exists some neighbourhood $V$ of $0 \in \mathbb{R}^n \times \mathbb{R}^l$ such that

$$\text{rank } [\nabla_x G(x, \lambda)] = p \quad (x, \lambda) \in V \setminus \Sigma \times \mathbb{R}^l,$$

(1.1)
then bifurcation points of $G$ only appear in set $\Sigma \times \mathbb{R}^l$ and the bifurcation diagram of $G$ has good behaviour away from $\Sigma \times \mathbb{R}^l$. This leads to one to propose (1.1) as the basic nondegeneracy condition for bifurcation diagrams. However, since (1.1) alone is not always adequate as a criterion for finite determination, therefore our full nondegeneracy condition ($K_{\Sigma, \omega}^{r, \delta}$), which contain and refine condition (1.1), will be stated in Section 2. It is a version of a condition of Kuo [15].

In this paper, we apply the idea of $r - \Sigma - C^0$-equivalence map jets relative to a given closed set $\Sigma$ in $(\mathbb{R}^n, 0)$, which is considered by Karim Bekka and Satoshi Koike [1] and B. Osińska-Ulrych, T. Rodak, G. Skalski in [13], to bifurcation theory, and introduce the $\Gamma - \Sigma - C^0$-bifurcation diagram and $\Gamma - \Sigma - C^0$-contact equivalence about $\Gamma$-equivariant bifurcation problems from the weighted point view and explore the finite determinacy of $\Gamma$-equivariant bifurcation problems with respect to above equivalences. Some criteria about determination of $\Gamma$-equivariant bifurcation problems with respect to the $\Gamma - \Sigma - C^0$-bifurcation diagram and $\Gamma - \Sigma - C^0$-contact equivalence are given.

**Theorem 1.1.** Let $\Gamma$ be a compact Lie group acting orthogonally on space on space $\mathbb{R}^n \times \mathbb{R}^l$ and space $\mathbb{R}^p$. $f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ be a $\Gamma$-equivariant $C^2$ map-germ. Suppose $f$ satisfies the condition ($K_{\Sigma, \omega}^{r, \delta}$) and $| \omega | + \delta \geq 1$, then $f$ is $\Gamma - \Sigma - BD$, or $\Gamma - \Sigma$-contact $r$-determined.

The assumptions in the above theorem are natural and cannot be essentially improved. Our method used in proof of above theorem is concretely offering a controlled vector field. This vector field provides integration. But in order to obtain integration for giving a $\Gamma - \Sigma - C^0$-locally homeomorphic in a neighbourhood of the origin, it is necessary for a key tool to use. This tool is the existence and uniqueness of solution about differential equation in a singular Riemann metric on $\mathbb{R}^n \times \mathbb{R}^l$ from the weighted point view, which is stated and proved in section 2.

When Lie group is a $\{e\}$, where $\{e\}$ is a unit element, and a singular Riemann metric transforms Euclidean metric, Theorem 1.1 implies

**Theorem 1.2.** Let $f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ be a $C^r$ map-germ and $p : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ be a $c^r$ map-germ which satisfies:

$$|p_i| = o(d(x, \Sigma)^r), \quad |\frac{\partial p_i}{\partial x_j}| = o(d(x, \Sigma)^{r-1}), \quad i = 1, \ldots, p; \quad j = 1, \ldots, n.$$ 

If $f$ satisfies the condition ($K_{\Sigma}^{r, \delta}$), then $f$ and $f + p$ is $\Sigma$-BD $r$-determined.

**Corollary 1.3.** Suppose $f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ be $C^{r+2}$ map-germ which satisfies the condition ($K_{\Sigma}^{r, \delta}$) and $g : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ be $C^{r+2}$ map-germ which satisfies the following condition: there exists a neighbourhood $U$ of $0$ in $\mathbb{R}^n \times \mathbb{R}^l$ such that, for every point $a \in (\Sigma \times \mathbb{R}^l) \cap U$, the jet $f^{(r+2)}$ of Taylor formula of degree $(r + 2)$ of $f$ at $a$ equals to the jet $g^{(r+2)}$ of Taylor formula of $(r + 2)$ of $g$ at $a$, then $f$ and $g$ is $\Sigma - C^0$-BD equivalent and $\Sigma - C^0$-contact equivalent.

Theorem 1.2 and Corollary 1.3 show to what terms from the Taylor expansion at
every point that belongs to a closed subset $\Sigma \times \mathbb{R}^l$ such that $(0, 0) \in \Sigma \times \mathbb{R}^l$ may be omitted without changing the topological type determined by $f$ and the value of the bifurcation parameter $\lambda$.

Finally, we show that $C^0$--finite determination of bifurcation problem, which is given by Peter B. Percell and Peter N. Brown In [6], is a corollary of Theorem 1.1.

**Theorem 1.4.** ([6], Theorem 3.1) Suppose $F \left( \mathbb{R}^n \times \mathbb{R}^l, 0 \right) \to \left( \mathbb{R}^p, 0 \right)$ is a $C^1$ map and $\nu = (\nu_1, \cdots, \nu_p)$ such that $F$ is ND($\nu$). Then $F$ is contact $| \nu |$--determined.

The rest of the paper is organized as follows. Section 2 contains necessary definitions, notation and technical preliminaries. Section 3 is devoted to proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.3 respectively. In section 4, we give the proof of Theorem 1.4.

## 2 Preliminaries

Let $\Gamma$ be a compact Lie group. It acts linearly on $\mathbb{R}^n \times \mathbb{R}^l$ by definition

$$\gamma(x, \lambda) = (\gamma x, \lambda), \quad \forall \gamma \in \Gamma, \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda \in \mathbb{R}^l.$$  

Meantime $\Gamma$ also acts linearly on $\mathbb{R}^p$, we say map-germ $G(\cdot, \lambda) : (\mathbb{R}^n \times \mathbb{R}^l, 0) \to (\mathbb{R}^p, 0)$ is $\Gamma$--equiviant if

$$G(\gamma x, \lambda) = \gamma G(x, \lambda), \quad \forall \gamma \in \Gamma, \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda \in \mathbb{R}^l$$

let $G, F : (\mathbb{R}^n \times \mathbb{R}^l, 0) \to (\mathbb{R}^p, 0)$ be $\Gamma$--equiviant continuous maps. We say $G$ and $F$ are $\Gamma$--equiviant BD equivalent ($BD$ for bifurcation diagram) if there are a neighbourhood $V \subset \mathbb{R}^n \times \mathbb{R}^l$ of the origin satisfying $\gamma \cdot V \subset V$, $\forall \gamma \in \Gamma$ and a $\Gamma$--equiviant map persevering parameter level

$$\phi : (V, 0) \to (\mathbb{R}^n \times \mathbb{R}^l, 0)$$

of the form

$$\phi(x, \lambda) = (\phi_1(x, \lambda), \lambda), \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^l$$

which is a homeomorphism $[C^n]$ diffeomorphism]onto its image such that

$$\phi(G^{-1}(0) \cap V) = F^{-1}(0) \cap \phi(V).$$

We call $G$ and $F$ are $\Gamma$--equiviant contact equivalent if there exist a neighbourhood $V \subset \mathbb{R}^n \times \mathbb{R}^l$ of the origin satisfying $\gamma \cdot V \subset V$, $\forall \gamma \in \Gamma$ and a $\Gamma$--equiviant map persevering parameter level

$$\phi : (V, 0) \to (\mathbb{R}^n \times \mathbb{R}^l, 0)$$

which is $\Gamma$--equiviant $BD$ equivalence between $G$ and $F$ and a continuous $[C^n]$ map

$$T : V \to GL(\mathbb{R}^p).$$
satisfying
\[ T(\gamma x, \lambda) = \gamma \cdot T(x, \lambda) \gamma^{-1}, \gamma \in \Gamma \]
such that
\[ G(u) = T(u) \cdot F(\phi(u)), \quad u = (x, \lambda). \]

Let us fix a system of positive numbers \( \omega = (\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+l}) \), call the weight of the variable \( x \) a map-germ \( G \) with respect to \( G \) where \( q \nabla \) is by definition orthogonal basis of (\( R^n \times R^l, < \cdot, \cdot > \)) be inner space. \( e_1, \ldots e_{n+l} \) is the weighted horn neighbourhood \( R^n \times R^l, < \cdot, \cdot > \) of degree \( d \), and \( (R^n \times R^l, < \cdot, \cdot >) \) be inner space. \( e_1, \ldots e_{n+l} \) is an orthogonal basis of (\( R^n \times R^l, < \cdot, \cdot > \)). For \( u = u_1 e_1 + u_2 e_2 + \ldots + u_{n+l} e_{n+l} = x_1 e_1 + \ldots + x_n e_n + \lambda_{n+l} e_{n+1} + \ldots + \lambda_{n+l} e_{n+l} \) we may introduce the function
\[ \rho = \rho(u) = \|u\|_\omega = (\sum_{i=1}^{n+l} u_i^{2q_i})^{1/2} = (\sum_{i=1}^{n} x_i^{2q_i} + \sum_{i=n+1}^{n+l} \lambda_i^{2q_i})^{1/2}, \]
where \( q_i = \frac{\omega_i}{\omega}, \quad 1 \leq i \leq n + l \) and \( q = \omega_1 \omega_2 \cdots \omega_{n+l} \).

**Remark 1.** ([13]) \( \rho(u)^{2q} \) satisfies a Lojasiewicz condition \( \rho(u)^{2q} \geq c \|u\|^{2\alpha} \) for some constants \( c \) and \( \alpha \).

**Definition 2.1** ([3]) Using this \( \rho \) we may introduce the singular Riemannian metric on \((R^n \times R^l, < \cdot, \cdot >)\), namely the Riemannian metric on \((R^n \times R^l \setminus \{0\})\) defined by the following bilinear form:
\[ \langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \rangle = \rho^{-2\omega_i}, \quad \langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \rangle = 0, \quad 1 \leq i, j \leq n + l, \quad i \neq j. \]
\[ \langle du_i \wedge \cdots \wedge du_k, du_i \wedge \cdots \wedge du_k \rangle = \|u\|_\omega^2 (\omega_1 + \cdots + \omega_k) = \rho^{2(\omega_1 + \cdots + \omega_k)}. \]

We denote by \( \nabla_\omega, \| \cdot \|_\omega \) the corresponding gradient and norm associated with this Riemannian metric. For a function-germ \( G, F : (R^n \times R^l, 0) \rightarrow (R, 0) \),
\[ \nabla_\omega G = \sum_{i=1}^{n+l} \rho^{\omega_i} \frac{\partial G}{\partial u_i}, \quad \|\nabla_\omega G\|_\omega = \sum_{i=1}^{n+l} (\rho^{\omega_i} \frac{\partial G}{\partial u_i})^2, \]
\[ \langle \nabla_\omega G, \nabla_\omega F \rangle = \sum_{i=1}^{n+l} \rho^{2\omega_i} \frac{\partial G}{\partial u_i} \frac{\partial F}{\partial u_i}. \]

If a map-germ \( G(\cdot, \lambda) : (R^n \times R^l, 0) \rightarrow (R^p, 0) \), then the gradient of component \( g_i \) of \( G \) with respect to \( x \) is
\[ \nabla_\omega x g_i = \sum_{i=1}^{n} \rho^{\omega_i} \frac{\partial g_i}{\partial x_j} \rho^{\omega_j} \frac{\partial}{\partial x_j}, \quad \|\nabla_\omega x g_i\|_\omega = \sum_{j=1}^{n} (\rho^{\omega_j} \frac{\partial g_i}{\partial x_j})^2, \]
We still denote by \( R^n \times R^l \) the inner linearly space \( R^n \times R^l \) with this singular metric. The weighted horn neighbourhood of degree \( d \) and width \( c > 0 \) of a variety \( G^{-1}(0) \) is by definition
\[ H_d(G, c) = \{ u \in R^n \times R^l | \|G(u)\|_{(R^p, < \cdot, \cdot >)} \leq c \rho^d \}. \]
Now let $\Sigma$ be a germ of a closed subset of $\mathbb{R}^n$ such that $0 \in \Sigma$. Then we denote by $\mathcal{R}_{\Sigma \times \mathbb{R}^n}^{fix}$ the group of germs of homeomorphisms $\phi: (\mathbb{R}^n \times \mathbb{R}^l, 0) \to (\mathbb{R}^n \times \mathbb{R}^l, 0)$ at $0 \in \mathbb{R}^n \times \mathbb{R}^l$ which fixes $\Sigma \times \mathbb{R}^l$ namely
\[
\phi(x, \lambda) = ((\phi_1(x, \lambda), \lambda) = (x, \lambda), \quad \forall x \in \Sigma, \quad \lambda \in \mathbb{R}^l
\]

We consider the following equivalence relation:

**Definition 2.2** ([6]) Let $\Gamma$ be a compact Lie group, $G, F: (\mathbb{R}^n \times \mathbb{R}^l, 0) \to (\mathbb{R}^p, 0)$ be $\Gamma$–equivariant continuous maps. We say that $G$ and $F$ are $\Gamma – \Sigma – [C^r]$ BD equivalent (BD for bifurcation diagram) if there are a neighborhood $V \subset \mathbb{R}^n \times \mathbb{R}^l$ of the origin satisfying $\gamma \cdot V \subset V, \forall \gamma \in \Gamma$ and a homeomorphism $\phi \in \mathcal{R}_{\Sigma \times \mathbb{R}^n}^{fix}$ $[C^r$ diffeomorphism] onto its image such that
\[
\phi(G^{-1}(0) \cap V) = F^{-1}(0) \cap \phi(V)
\]
and fixes $G^{-1}(0) \cap \Sigma$.

We call that $G$ and $F$ are $\Gamma – \Sigma – [C^r]$ contact equivalent if there exist a neigbourhood $V \subset \mathbb{R}^n \times \mathbb{R}^l$ of the origin satisfying $\gamma \cdot V \subset V, \forall \gamma \in \Gamma$ and a homeomorphism $\phi \in \mathcal{R}_{\Sigma \times \mathbb{R}^n}^{fix}$ $[C^r$ diffeomorphism] which is $\Gamma – \Sigma – [C^r]$ BD equivalence between $G$ and $F$ and a continuous $\Gamma – [C^r]$ equivariant map
\[
T: V \to GL(\mathbb{R}^p),
\]

namely
\[
T(\phi_1(x, \lambda), \lambda) = \gamma T(x, \lambda)\gamma^{-1}, \quad \forall \gamma \in \Gamma,
\]
such that
\[
G(u) = T(u) \cdot F(\phi(u)), \quad u = (x, \lambda), \quad u = (x, \lambda) \in V.
\]

Now we introduce the finite determination concepts.

**Definition 2.3** Let $\Gamma$ be a compact Lie group and $f: (\mathbb{R}^n \times \mathbb{R}^l, 0) \to (\mathbb{R}^p, 0)$ be $\Gamma$–equivariant continuous map. We say that $f$ is $\Gamma – \Sigma – [C^r]$ BD , or $\Gamma – \Sigma – [C^r]$ contact d–determined if $f$ and $f + p$ are $\Gamma – \Sigma – [C^r]$ BD , or $\Gamma – \Sigma – [C^r]$ contact equivalent for every $C^2[C^{max(r, 2)}] \Gamma–$ equivariant perturbation
\[
p: (\mathbb{R}^n \times \mathbb{R}^l) \to (\mathbb{R}^p, 0)
\]
such that, with $u = (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$ and
\[
|p_i| = o(d_\omega(x, \Sigma)^d + |\omega|), \quad |\frac{\partial p_i}{\partial x_j}| = o(d_\omega(x, \Sigma)^d), \quad i = 1, \ldots, p; \quad j = 1, \ldots, n, \quad (2.2)
\]
where
\[
d_\omega(x, \Sigma) = \inf_{y \in \Sigma} \{\|x – y\|_\omega\}
\]
and
\[
|\omega| = \max\{|\omega_i| : i = 1, \ldots, n + l\}.
\]
Let $\Sigma$ be a germ of closed set of $\mathbb{R}^n$ such that $0 \in \Sigma$. Let $G$ be a germ of $C^1$ vector field on $\mathbb{R}^n \times \mathbb{R}^l \setminus \Sigma \times \mathbb{R}^l$ which satisfies the relative Lipschitz condition: $\| G(x, t) \| \leq Cd(x, \Sigma)$.

For a fixed vector $v \in \{0\} \times \mathbb{R}^l$, we define

$$X(u, t) = \begin{cases} G(x, t) + v, & \text{if } x \in \mathbb{R}^n \setminus \Sigma. \\ v, & \text{if } x \in \Sigma \end{cases}$$

Then

**Lemma 2.4** ([1], Proposition 2.15) For $G(x, t)$ satisfying the preceding conditions, $X(u, t)$ is locally integrable in the sense that there are a neighbourhood $W$ of $(x_0, t_0)$ in $\mathbb{R}^n \times \mathbb{R}^l, \delta > 0$, and a family of homeomorphisms $\Phi_s$ defined on $W$ for $s < \delta$ so that $\Phi_0 = \text{id}$ and for $(x, t, s) \in W \times (-\delta, \delta)$,

$$\frac{\partial \Phi_s}{\partial s} = X \circ \Phi_s.$$ 

**Lemma 2.5** ([1], Lemma 2.16; [13], Lemma 2.13) Let $U$ be an open subset of $\mathbb{R}^n \setminus \Sigma$, let $0 \in (a, b)$ and let $G : U \times (a, b) \to \mathbb{R}^n$ be a continuous mapping which satisfies

$$\| G(x, t) \| \leq Cd(x, \Sigma)$$

for some $C > 0$ and $(x, t) \in U \times (a, b)$. Let $\varphi(\alpha, \beta) \to U$ be an integral solution of the system of differential equations $y' = G(y, t)$ with the initial condition $\varphi(0) = x_0$ where $x_0 \in U$ and $0 \in (\alpha, \beta) \subset (a, b)$. Then we have

$$d(x_0, \Sigma)e^{-C|t|} \leq d(\varphi(t), \Sigma) \leq d(x_0, \Sigma)e^{C|t|}$$

for $t \in (\alpha, \beta)$.

Next we provide the following key proposition which is used in the proof of Theorem 1.1.

**Proposition 2.6** (Key Proposition) For $G(x, t)$ satisfying

$$\| G(x, t) \| \leq Cd(x, \Sigma),$$

$X(u, t)$ is locally integrable in the sense that there are a neighbourhood $W$ of $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^l, \delta > 0$, and a family of homeomorphisms $\Phi_s$ defined on $W$ for $s < \delta$ so that $\Phi_0 = \text{id}$ and for $(x, t, s) \in W \times (-\delta, \delta)$,

$$\frac{\partial \Phi_s}{\partial s} = X \circ \Phi_s.$$ 

**Lemma 2.7** Let $U$ be an open subset of $\mathbb{R}^n \setminus \Sigma$, let $0 \in (a, b)$ and let $G : U \times (a, b) \to \mathbb{R}^n$ be a continuous mapping which satisfies

$$\| G(x, t) \| \leq Cd(x, \Sigma)$$

(2.4)
for some $C > 0$ and $(x, t) ∈ U \times (a, b)$. Let $φ(α, β) → U$ be an integral solution of the system of differential equations $y' = G(y, t)$ with the initial condition $φ(0) = x_0$ where $x_0 ∈ U$ and $0 ∈ (α, β) ⊂ (a, b)$. Then we have

$$d_ω(φ(0), Σ)e^{-CL|t|} < d_ω(φ(t), Σ) ≤ d_ω(φ(0), Σ)e^{CL|t|},$$

for $t ∈ (α, β)$, and some positive number $L$.

**Proof.** Owing to $φ(t) ∈ U$, for $y ∈ Σ$ and $t ∈ (α, β)$, let $∥φ_y(t)∥_ω = ∥φ(t) - y∥_ω > 0$ and $φ_y(t) = (φ_{y,1}(t), \cdots, φ_{y, n}(t))$

We can get the function $κ(φ_y(t)) = \frac{1}{2ω}ln \parallel φ_y(t) \parallel_{ω}^{2ω} = \frac{1}{2ω}lnρ(φ_y(t))^{2ω}$ for $t ∈ (α, β)$ and

$$\frac{dκ(φ_y(t))}{dt} = \frac{1}{2ω} \parallel φ_y(t) \parallel_{ω}^{2ω-1} \cdot \frac{d}{dt} \parallel φ_y(t) \parallel_{ω} = \frac{1}{∥φ_y(t)∥_ω} \cdot \sum_{i=1}^{n} \frac{∂ρ(x)}{∂x_i} \cdot φ_{y,i}(t)$$

$$= \frac{1}{∥φ_y(t)∥_ω} \cdot \sum_{i=1}^{n} \frac{∂ρ(x)}{∂x_i} \cdot G_i(φ_y(t), t)$$

for $t ∈ (α, β)$.

Since

$$\frac{∂ρ(x)}{∂x_i} = \frac{1}{2ω} \left( \sum_{i=1}^{n} \frac{2ω}{x_i} \right)^{\frac{ω}{2ω} - 1} \cdot \frac{2ω}{ω_i} \cdot \frac{2ω}{ω_i} - 1$$

$$= \frac{1}{ω_i} \cdot \frac{ρ \cdot x_i^{\frac{2ω}{ω_i}}}{\left( \sum_{i=1}^{n} x_i^{\frac{2ω}{ω_i}} \right) x_i}$$

$$\frac{∂ρ(x)}{∂x_i} \cdot ρ^{ω_i - 1} = \frac{1}{ω_i} \cdot \frac{x_i^{\frac{2ω}{ω_i}}}{\left( \sum_{i=1}^{n} x_i^{\frac{2ω}{ω_i}} \right) x_i} \cdot \left( \sum_{i=1}^{n} x_i^{\frac{2ω}{ω_i}} \right)^{\frac{ω_i}{2ω}}$$

Now let us observe that

$$\left| \frac{∂ρ(x)}{∂x_i} \right| ≤ \frac{L}{n} ρ^{1-ω_i} \text{ for some } L > 0 \text{ (also by (7)).}$$

(2.5)

and

$$| G_i (φ_y(t), t) | = \left[ \parallel G_i (φ_y(t), t) \parallel_{ω_i}^{\frac{1}{ω_i}} \right]^{ω_i} \leq \parallel G (φ_y(t), t) \parallel_{ω_i}^{ω_i} ≤ C \parallel φ_y(t) \parallel_{ω_i}^{ω_i} \text{ (by (2.2)).}$$

I.e.

$$| G_i (φ_y(t), t) | ≤ C \parallel φ_y(t) \parallel_{ω_i}^{ω_i}. \text{ (2.6)}$$
From the mean value theorem, for every \( t \in (0, \beta) \), there exists \( \theta \in (0, \beta) \) such that
\[
| \kappa(t) - \kappa(0) | \leq \left| \frac{d \kappa(\varphi_y(\theta))}{dt} \right| \cdot t = \frac{1}{\| \varphi_y(t) \|_\omega} \cdot \left| \sum_{i=1}^{n} \frac{\partial p(x)}{\partial x_i} \cdot G_i(\varphi_y(t), t) \right| \cdot t
\]
\[
\leq \frac{1}{\| \varphi_y(t) \|_\omega} \cdot \sum_{i=1}^{n} \left| \frac{\partial p(x)}{\partial x_i} \right| \cdot | G_i(\varphi_y(t), t) | \cdot t
\]
\[
\leq \frac{1}{\| \varphi_y(t) \|_\omega} \cdot \sum_{i=1}^{n} L \rho^{1-\omega_i} \cdot C [\| \varphi_y(t) \|_\omega]^{\omega_i} t
\]
\[
= C \sum_{i=1}^{n} \frac{1}{\| \varphi_y(t) \|_\omega^{1-\omega_i}} \cdot \frac{L \rho^{1-\omega_i}}{n} \cdot t = C \cdot L \cdot t,
\]
where \( \rho = \| \varphi_y(t) \|_\omega \).
Hence for every \( t \in (0, \beta) \),
\[
\kappa(0) - C L t \leq \kappa(t) \leq \kappa(0) + C L t.
\]
The above inequalities hold also for \( t = 0 \), i.e.
\[
\| \varphi_y(0) \|_\omega e^{-CL|t|} \leq \| \varphi_y(t) \|_\omega \leq \| \varphi_y(0) \|_\omega e^{CL|t|}.
\] (2.7)
Because \( d_\omega(\varphi(t), \Sigma) = \inf_{y \in \Sigma} \| \varphi_y(t) \|_\omega \) and (2.4), we obtain
\[
d_\omega(\varphi(0), \Sigma)e^{-CL|t|} < d_\omega(\varphi(t), \Sigma) \leq d_\omega(\varphi(0), \Sigma)e^{CL|t|}.
\] (2.8)

**Proof of Proposition 2.6.** The proof will be essentially the same as that of Proposition 2.15 of [1] using Lemma 2.7.

**Lemma 2.8.** ([1], Lemma 2.4) Let \( \Sigma \) be a germ at \( 0 \in \mathbb{R}^n \) of a closed subset, and \( f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0) \) be a \( C^k \) map-germ, \( k \geq 1 \), which satisfies the following condition:
there exists a neighbourhood \( U \) of \( 0 \) in \( \mathbb{R}^n \) such that, for every point \( a \in \Sigma \cap U \), the \( k \)-jet \( j^k f(a) \) of Taylor formula of degree \( k \) of \( f \) at \( a \) is \( 0 \). Then \( \| f(x) \| = o(d(k, \Sigma)^k) \).

Let map \( f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is a \( C^r \) map, we consider vectors
\[
N(f, j, u) = \nabla_x f_j(u, t) - q_j(u, t) \quad 1 \leq j \leq p,
\]
where \( q_j(u, t) \) is the projection of \( \nabla_x f_j(u, t) \) to the subspace \( V^j_u \) spanned by \( \{ \nabla_x f_i \}_{j \neq i} \).
The Kuo pseudo-distance \( d_{\omega, x} \nabla f \) is defined by
\[
d_{\omega, x} \nabla f = \min_{1 \leq i \leq p} \{ \| N(f, i, x) \|_\omega \}.
\]

**Definition 2.9.** (the relative Kuo condition \( (K^r) \)) Let map \( f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is a \( C^2 \) map, the map \( f \) satisfies the relative Kuo condition \( (K^r) \) if there is a strictly positive number \( C, \delta, \alpha \) and \( \overline{\nu} \) such that
\[
d_{\omega, x} \nabla f \geq C \cdot d(x, \Sigma)^{r-\delta}
\]
holds on \( u = (x, \lambda) \in H_r(f, \omega) \cap \{ \| u \| < \alpha \} \), where

\[
H_r^{\Sigma}(f, \omega) = \{ u \in \mathbb{R}^n \times \mathbb{R}^l : \| f(u) \| \leq \omega d(x, \Sigma)^r \}.
\]

**Definition 2.10.** (The relative Kuo condition \((K^{\gamma}_{\omega,r})_\Sigma))** Let \( \Gamma \) be a compact Lie group acting on space on space \( \mathbb{R}^n \times \mathbb{R}^l \) and space \( \mathbb{R}^p \). and \( \Sigma \) be a germ of closed set of \( \mathbb{R}^n \) such that \( 0 \in \Sigma \) and \( \gamma \cdot \Sigma \subset \Sigma, \forall \gamma \in \Gamma \). A \( C^2 \) \( \Gamma \)-equivariant map \( f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is said to satisfy the relative Kuo condition \((K^{\gamma}_{\omega,r})_\Sigma\) if there is a strictly positive number \( C, \delta, \alpha \) and \( \overline{\omega} \) such that

\[
d_{\omega} \nabla f \geq C \cdot d_{\omega}(x, \Sigma)^{r-\delta}
\]

holds on \( u = (x, \lambda) \in H^{\Sigma}_{\omega,r}(f, \omega) \cap \{ \| u \| < \alpha \} \), where

\[
H^{\Sigma}_{\omega,r}(f, \omega) = \{ u \in \mathbb{R}^n \times \mathbb{R}^l : \| f(u) \| \leq \overline{\omega} d_{\omega}(x, \Sigma)^r \}.
\]

**Remark 2.** Since \( \rho(u)^{2q} \geq c \| u \|^{2q} \) for some constants \( c \) and \( \alpha \) by Remark 1, then

\[
H^{\Sigma}_{\omega,q}(f, \overline{\omega} \gamma) \subset H^{\Sigma}_{\omega,r}(f, \omega).
\]

In fact, if \( u \in H^{\Sigma}_{\omega,q}(f, \overline{\omega} \gamma) \),

\[
\| f(u) \| \leq \overline{\omega} \gamma d(x, \Sigma)^{\alpha r} = \overline{\omega} \left( c d(x, \Sigma)^{2\alpha} \right)^{\frac{\alpha r}{2q}} \leq \overline{\omega} d_{\omega}(x, \Sigma)^r.
\]

### 3 The determinacy of \( \Gamma \)-equivariant bifurcation problems with respect to \( \Gamma - \Sigma - \text{BD} \) and \( \Gamma - \Sigma - \text{contact equivalence from the weighted point view} \)

In order to prove Theorem 1.1, we need to following Lemma.

**Lemma 3.1.** Let \( \Gamma \) be a compact Lie group acting orthogonally on space on space \( (\mathbb{R}^n \times \mathbb{R}^l, \langle \cdot, \cdot \rangle) \) and space \( (\mathbb{R}^p, \langle \cdot, \cdot \rangle) \), then,

1. The representation of \( \Gamma \) on the inner space \( (\mathbb{R}^n \times \mathbb{R}^l, \langle \cdot, \cdot \rangle) \) is orthogonal, i.e. \( \langle \gamma x, \gamma y \rangle_\Gamma = \langle x, y \rangle_\Gamma \).
2. \( \rho(u) = \| u \|_{\omega} = \| u \| = \rho(\gamma u), \forall \gamma \in \Gamma \).
3. If \( f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is \( \Gamma \)-equivariant \( C^r \) map-germ and \( \Sigma \) be a germ of closed set of \( \mathbb{R}^n \) such that \( 0 \in \Sigma \) and \( \gamma \cdot \Sigma \subset \Sigma, \forall \gamma \in \Gamma \), then, for \( \gamma \in \Gamma \), \( \gamma \cdot H^{\Sigma}_{r}(f, \omega) = H^{\Sigma}_{r}(f, \overline{\omega} \gamma) \) and \( \{ \| u \| < \alpha \} = \{ \| u \| < \alpha \} \) and \( \gamma \cdot H^{\Sigma}_{\omega,r}(f, \omega) = H^{\Sigma}_{\omega,r}(f, \overline{\omega} \gamma) \). i.e. \( H^{\Sigma}_{r}(f, \omega) \) and \( H^{\Sigma}_{\omega,r}(f, \omega) \) are \( \Gamma \)-invariant set.

**Proof.** (1) is obvious.

(2) In fact, for \( \gamma \in \Gamma \), if the representation of \( \gamma \) is orthogonal matrix \( A(\gamma) = (a_{ij}) \), then

\[
\gamma \cdot u = A(\gamma) \cdot u = (v_1, v_2, \cdots, v_{n+l}) = (\Sigma_{1 \leq i \leq n+l} a_{1i} u_i, \cdots, \Sigma_{1 \leq i \leq n+l} a_{n+l} u_i).
\]
By Definition 2.1,

\[ \langle dv_1 \wedge \cdots \wedge dv_k, dv_1 \wedge \cdots \wedge dv_k \rangle = \| \gamma \cdot u \|_\Sigma^{2(\omega_1 + \cdots + \omega_k)} = \rho(v)^{2(\omega_1 + \cdots + \omega_k)}. \]

So

\[ \langle dv_1 \wedge \cdots \wedge dv_{n+t}, dv_1 \wedge \cdots \wedge dv_{n+t} \rangle = \| \gamma \cdot u \|_\Sigma^{2(\omega_1 + \cdots + \omega_{n+t})} = \rho(v)^{2(\omega_1 + \cdots + \omega_{n+t})}. \]

Moreover

\[
\begin{align*}
\langle dv_1 \wedge \cdots \wedge dv_{n+t}, dv_1 \wedge \cdots \wedge dv_{n+t} \rangle &= \\
&= \langle | A(\gamma) | du_1 \wedge \cdots \wedge du_{n+t}, | A(\gamma) | du_1 \wedge \cdots \wedge du_{n+t} \rangle \\
&= \langle du_1 \wedge \cdots \wedge du_{n+t}, du_1 \wedge \cdots \wedge du_{n+t} \rangle \\
&= \| u \|_\Sigma^{2(\omega_1 + \cdots + \omega_{n+t})} = \rho(u)^{2(\omega_1 + \cdots + \omega_{n+t})}
\end{align*}
\]

So

\[ \rho(u)^{2(\omega_1 + \cdots + \omega_{n+t})} = \rho(\gamma \cdot u)^{2(\omega_1 + \cdots + \omega_{n+t})} \]

and \( \rho(u) = \rho(\gamma u) \).

(3) Since \( f \) is \( \Gamma \)-equivariant and \( \Gamma \) is orthogonally act on \( \mathbb{R}^n \times \mathbb{R}^l \) and \( \mathbb{R}^p \), then

\[ \| f(\gamma u) \| = \| \gamma f(u) \| = \| f(u) \|, \]

\[ d(\gamma x, \Sigma) = d(\gamma x, \gamma \Sigma) = d(x, \Sigma). \]

and

\[ d_\omega(\gamma x, \Sigma) = d_\omega(\gamma x, \gamma \Sigma) = d_\omega(x, \Sigma). \]

\[ \| f(u) \| \leq \bar{w}d(x, \Sigma)^{\gamma} \] if and only if \( \| f(\gamma \cdot u) \| \leq \bar{w}d(\gamma x, \Sigma)^{\gamma} = \bar{w}d(x, \Sigma)^{\gamma} \) and

\[ \| f(u) \| \leq \bar{w}d_\omega(x, \Sigma)^{\gamma} \] if and only if \( \| f(\gamma \cdot u) \| \leq \bar{w}d_\omega(\gamma x, \Sigma)^{\gamma} = \bar{w}d_\omega(x, \Sigma)^{\gamma} \).

Therefore, for \( \gamma \in \Gamma, \gamma \cdot H^\Sigma_r(f, \bar{w}) \subset H^\Sigma_r(f, \bar{w}) \) and \( \gamma \cdot H^\Sigma_{\omega,r}(f, \bar{w}) \subset H^\Sigma_{\omega,r}(f, \bar{w}). \)

Again since \( u = \gamma \cdot (\gamma^{-1} u), H^\Sigma_r(f, \bar{w}) \subset \gamma \cdot H^\Sigma_r(f, \bar{w}) \) and \( H^\Sigma_{\omega,r}(f, \bar{w}) \subset \gamma \cdot H^\Sigma_{\omega,r}(f, \bar{w}). \)

I.e. \( H^\Sigma_r(f, \bar{w}) \) and \( H^\Sigma_{\omega,r}(f, \bar{w}) \) are \( \Gamma \)-invariant set.

**Lemma 3.2.** Let \( \Gamma \) be a compact Lie group acting orthogonally on space on space \((\mathbb{R}^n \times \mathbb{R}^l, \langle \cdot, \cdot \rangle)\) and space \((\mathbb{R}^p, \langle \cdot, \cdot \rangle)\). Suppose \( f, p : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is \( \Gamma \)-equivariant \( C^r \) map-germ such that \( p(u) \) satisfies (2.1). Define

\[ F(u, t) = f(u) + tp(u), \ u = (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l, t \in I = [0, 1]. \]

If there is a strictly positive number \( C, \delta, \alpha \) and \( \bar{w} \) such that \( f \) satisfies the condition \((K^{\Sigma}_{\gamma, \delta})_\Gamma\) as above, then when

\[ (u, t) = (x, \lambda, t) \in H^\Sigma_{\omega,r}(f, \bar{w}) \cap \{ \| u \| < \alpha \} \times I, \]

there exists some positive constant \( C' \) such that

\[ d_\omega, \nabla F = d_\omega, (\nabla_\omega, F_1, \ldots, \nabla_\omega, F_p) \geq C'd_\omega(x, \Sigma)^{r-\delta} \]
holds.

Proof. First we prove
\[ \| \nabla_{\omega,x} F_i(u, t) - \nabla_{\omega,x} f_i(u) \|_\omega \leq o(d_\omega(x, \Sigma)^r) \] (3.9)

In fact, by the definition of the singular Riemannian metric,
\[ \| \nabla_{\omega,x} F_i(u, t) - \nabla_{\omega,x}^\prime f_i(u) \|_\omega = \| \nabla_{\omega,x}^\prime f_i(u) \|_\omega \]
\[ = \left( \sum_{j=1}^n (\rho_j^\omega \frac{\partial p_i}{\partial x_j})^2 \right)^{\frac{1}{2}} \]
\[ \leq |t| \left( \sum_{j=1}^n \rho_j^\omega \right) \frac{\partial p_i}{\partial x_j} \]
\[ = |t| \left( \sum_{j=1}^n \rho_j^\omega o(d_\omega(x, \Sigma)^r) \right) \]
\[ = o(d_\omega(x, \Sigma)^r). \]

So (3.7) holds.

Now for \((u, t) = (x, \lambda, t) \in \{ u \| < \alpha \} \Sigma \times \mathbf{R}^l \times I, \) the vectors \(\{ \nabla_{\omega,x} f_1, \ldots, \nabla_{\omega,x} f_p \}\) are linearly independent. Let \(V_j^i\) be the subspace spanned by the \(\{ \nabla_{\omega,x} f_k(u, t), k \neq j \}\). we consider vectors
\[ N(f, j, u) = \nabla_{\omega,x} f_j(u, t) - q_j(u, t) \quad 1 \leq j \leq p, \]
where \(q_j(u, t)\) is the projection of \(\nabla_{\omega,x} f_j(u, t)\) to the subspace \(V_j^i\). So there exist \(\alpha_{ji}\) such that
\[ N(f, j, x) = \nabla_{\omega,x} f_j - \sum_{i=1, i \neq j}^p \alpha_{ji} \nabla_{\omega,x} f_i(u). \]

When
\[ u = (x, \lambda) \in H_{\omega,x}^\Sigma(f, \Sigma) \bigcap \{ u \| < \alpha \} \Sigma \times \mathbf{R}^l, \]
by the condition \((K_{\Sigma}^{\alpha,\delta})\), if \(\lambda_k(u, t) \neq 0, \)
\[ \frac{\| \lambda_k (\nabla_{\omega,x} F_k(u, t) - \nabla_{\omega,x} f_k(u)) \|_\omega}{\| \sum_{j=1}^p \lambda_j \nabla_{\omega,x} f_j(u) \|_\omega} = \frac{\| \nabla_{\omega,x} F_k(u, t) - \nabla_{\omega,x} f_k(u) \|_\omega}{\| \nabla_{\omega,x} f_k + \sum_{j=1, j \neq k}^p \frac{\lambda_j}{\lambda_k} \| \nabla_{\omega,x} f_j(u) \|_\omega} \leq \frac{o(d_\omega(x, \Sigma)^r)}{C \cdot d_\omega(x, \Sigma)^{r-\delta}}. \]

Hence, for a enough small positive number \(\varepsilon, \)
\[ \| \lambda_k (\nabla_{\omega,x} F_k(u, t) - \nabla_{\omega,x} f_k(u)) \|_\omega \leq \varepsilon d_\omega(x, \Sigma)^\delta \sum_{j=1}^p \lambda_j \nabla_{\omega,x} f_j(u) \|_\omega. \] (3.10)

Again since
\[ \| \sum_{j=1}^p \lambda_j \nabla_{\omega,x} F_j(u) \|_\omega \geq \| \sum_{j=1}^p \lambda_j \nabla_{\omega,x} f_j(u) \|_\omega - \| \sum_{j=1}^p \lambda_j (\nabla_{\omega,x} F_j - \nabla_{\omega,x} f_j) \|_\omega \]
\[ \geq \| \sum_{j=1}^p \lambda_j \nabla_{\omega,x} f_j(u) \|_\omega - \sum_{j=1}^p \| \lambda_j (\nabla_{\omega,x} F_j - \nabla_{\omega,x} f_j) \|_\omega \]
So, by (3.9)
\[
\|\sum_{j=1}^{p} \lambda_j \nabla_{\omega,x} F_j(u)\|_\omega \geq \|\sum_{j=1}^{p} \lambda_j \nabla_{\omega,x} f_j(u)\|_\omega - \sum_{j=1}^{p} \varepsilon d_\omega(x, \Sigma) \|\sum_{j=1}^{p} \lambda_j \nabla_{\omega,x} f_j(u)\|_\omega
\]
\[
= (1 - p\varepsilon d_\omega(x, \Sigma)) \|\sum_{j=1}^{p} \lambda_j \nabla_{\omega,x} f_j(u)\|_\omega.
\]

Therefore
\[
\|\nabla_{\omega,x} F_k - \sum_{j=1, j \neq k}^{p} \lambda_j \nabla_{\omega,x} F_j(u)\|_\omega \geq (1 - p\varepsilon d_\omega(x, \Sigma)) \|\nabla_{\omega,x} f_k - \sum_{j=1, j \neq k}^{p} \lambda_j \nabla_{\omega,x} f_j(u)\|_\omega
\]
\[
\geq \left[1 - \varepsilon \cdot d_\omega(x, \Sigma)\right] N(f, k, x) \geq \left[1 - \varepsilon \cdot d_\omega(x, \Sigma)\right] C \cdot d(x, \Sigma)^{r-\delta} \geq C' \cdot d_\omega(x, \Sigma)^{r-\delta}
\]
is true for \(d_\omega(x, \Sigma) < 1\). I.e.
\[
d_\omega \nabla F = d_\omega \nabla F_1, \ldots, \nabla F_p = \min_{1 \leq i \leq p} \{\|N(F, i, x)\|_\omega\} \geq C' d_\omega(x, \Sigma)^{r-\delta}
\]
holds in a enough small neighborhood of 0.

**Lemma 3.3.** Let \(\Gamma\) be a compact Lie group acting orthogonally on space on space
\((\mathbb{R}^n \times \mathbb{R}^l, \langle \cdot, \cdot \rangle)\) and space \((\mathbb{R}^p, \langle \cdot, \cdot \rangle)\) and \(\Sigma\) be a germ of closed set of \(\mathbb{R}^n\) such that 
0 \(\in\Sigma\) and \(\gamma \cdot \Sigma \subseteq \Sigma\), \(\forall \gamma \in \Gamma\). Suppose \(f : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)\) is \(\Gamma\)-equivariant.
\(H^\Sigma_{\omega,r}(f, \overline{\omega})\) is a horn neighbourhood of \(f\), then there exists a \(C^\infty - \Gamma\)-invariant function \(\chi : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)\), such that ,with \(u \in (\mathbb{R}^n \times \mathbb{R}^l \setminus \{0\})\),
\[
0 \leq \chi(u) \leq 1, \quad \chi(u) = \begin{cases} 
 1, & \text{if } u \in H^\Sigma_{\omega,r}(f, \overline{\omega}), \\
 0, & \text{if } u \in \mathbb{R}^n \times \mathbb{R}^l \setminus H^\Sigma_{\omega,r}(f, \overline{\omega}).
\end{cases}
\]
and \(\chi(\gamma u) = \chi(u), \ \gamma \in \Gamma\).

**Proof.** Since \(\Gamma \subseteq O(n)\), then, for \(\gamma \in \Gamma\), 
\(\gamma \cdot H^\Sigma_{\omega,r}(f, \overline{\omega}) = H^\Sigma_{\omega,r}(f, \overline{\omega})\) by Lemma 3.2(3). Let a function \(\alpha : \mathbb{R} \rightarrow \mathbb{R}\) be defined by
\[
\alpha(t) = \begin{cases} 
 e^\frac{t}{2}, & \text{if } t > 0, \\
 0, & \text{if } t \leq 0.
\end{cases}
\]
\(\alpha\) is a \(C^\infty\) function. Again let a function \(\beta : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}\) be defined by
\[
\beta(x) = \frac{\alpha(1 - \|x\|)}{\alpha(1 - \|x\|) + \alpha(\|x\| - \frac{1}{2})}
\]
Owing to \(\langle \gamma x, \gamma x \rangle = \langle x, x \rangle\) for \(\gamma \in \Gamma\), then \(\beta(\gamma x) = \beta(x)\), \(\beta(x)\) is \(C^\infty - \Gamma\)-invariant function and \(0 \leq \beta(x) \leq 1\).
Moreover when \( \|x\| \leq \frac{1}{2}, \beta(x) = 1; \) when \( \|x\| > 1, \beta(x) = 0. \)

Now the function \( \chi : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow \mathbb{R} \) is defined by \( \chi(u) = \beta \left( \frac{f(u)}{\|u\|} \right) \) Then, for \( u \in \mathbb{R}^n \times \mathbb{R}^l \setminus \{0\}, \)

\[
0 \leq \chi(u) \leq 1, \quad \chi(u) = \begin{cases} 
1, & \text{if } u \in H^\Sigma_{\omega,r}(f, \overline{\pi}), \\
0, & \text{if } u \in \mathbb{R}^n \times \mathbb{R}^l \setminus H^\Sigma_{\omega,r}(f, \overline{\pi}). 
\end{cases}
\]

and \( \chi(\gamma u) = \chi(u) = \beta \left( \frac{f(\gamma u)}{\|\gamma u\|} \right) = \chi(u) = \beta \left( \frac{\gamma f(u)}{\|u\|} \right) = \chi(u), \gamma \in \Gamma. \)

**Proof of Theorem 1.1** Let \( t_0 \) be an arbitrary element of \( I = [0, 1] \). Define

\[
F(u, t) = f(u) + tp(u), \quad u = (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l, t \in I
\]

where \( p(u) \) satisfies

\[
| p_i | = o(d_\omega(x, \Sigma)^r|\omega|), \quad \left| \frac{\partial p_i}{\partial x_j} \right| = o(d_\omega(x, \Sigma)^r), \quad i = 1, \ldots, p; \quad j = 1, \ldots, n. \quad (3.11)
\]

we define in addition to the bilinear form on definition 1.1,

\[
\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial t} \right\rangle = 0, \quad 1 \leq i \leq n + l, \quad \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = 0.
\]

\[
\nabla_{\omega} F_i = \sum_{j=1}^{n} \rho^{\omega j} \left( \frac{\partial f_i}{\partial x_j} + t \frac{\partial p_i}{\partial x_j} \right) \frac{\partial}{\partial x_j} + \sum_{j=n+1}^{n+l} \rho^{\omega j} \left( \frac{\partial f_i}{\partial \lambda_j} + t \frac{\partial p_i}{\partial \lambda_j} \right) \frac{\partial}{\partial \lambda_j} + p(u) \frac{\partial}{\partial t}
\]

\[
= \nabla_{\omega,x} F_i + \sum_{j=n+1}^{n+l} \rho^{\omega i} \left( \frac{\partial f_i}{\partial \lambda_j} + t \frac{\partial p_i}{\partial \lambda_j} \right) \frac{\partial}{\partial \lambda_j} + p(u) \frac{\partial}{\partial t}. \]

we consider vectors

\[
N(F, j, u) = \nabla_{\omega,x} F_j(u, t) - Q_j(u, t) \quad 1 \leq j \leq p,
\]

where \( Q_j(u, t) \) is the projection of \( \nabla_{\omega,x} F_j(u, t) \) to the subspace \( W^j_u \), where \( W^j_u \) is the subspace spanned by the \( \{\nabla_{\omega,x} F_k(u, t); \ k \neq j\} \).

Since \( f \) satisfies the relative Kuo condition \( (K^r_{\Sigma})_r \) and using Lemma 3.2, there exist a positive number \( C' \) such that, for \( (u, t) \in W := H^\Sigma_{\omega,r}(f, \overline{\pi}) \cap \{\|u\| < \alpha\} \times I, \)

\[
d_\omega \nabla F = \min_{1 \leq i \leq p} \|N(F, i, x)\|_\omega \geq C' d_\omega(x, \Sigma)^{r-\delta}
\]

holds.

Firstly, we define a version the Kuo-vector field

\[
X_1(u, t) = \begin{cases} 
\frac{1}{\|N(F, j, u)\|_\omega^2} N(F, j, u), & (u, t) \in W \setminus \Sigma \times \mathbb{R}^l \times I \\
\frac{\partial}{\partial t}, & (u, t) \in W \cap \Sigma \times \mathbb{R}^l \times I
\end{cases}
\]
Now we give another form of $X_1(u, t)$.
In fact, since
\[
\langle N(F, j, u), \nabla_{\omega,x}F_k(u,t) \rangle = \langle \nabla_{\omega,x}F_j(u,t) - Q_j(u,t), \nabla_{\omega,x}F_k(u,t) \rangle = 0 \quad 1 \leq j \leq p, \ k \neq j,
\]
where $Q_j(u,t) = \sum_{i=1, j \neq k}^p \alpha_{ij} \nabla_{\omega,x}F_i(u)$. Let
\[
b_{hi} = \langle \nabla_{\omega,x}F_k(u,t), \nabla_{\omega,x}F_i(u,t) \rangle
\]
\[
\begin{align*}
\sum_{k=1, j \neq k}^p b_{kj} \alpha_{kj} &+ b_j \cdot (-1) = 0 \\
\vdots & \quad \vdots \\
\sum_{k=1, j \neq k}^p b_{kj} \hat{\alpha}_{kj} &+ \hat{b}_j \cdot (-1) = 0 \\
\vdots & \quad \vdots \\
\sum_{k=1, j \neq k}^p b_{kj} \alpha_{kj} &+ b_j \cdot (-1) = 0 \\
\sum_{k=1, j \neq k}^p \nabla_{\omega,x}F_k(u,t) \alpha_{kj} &+ Q_j(u,t) \cdot (-1) = 0
\end{align*}
\]
where the hat means omission.

This system of equations has non-zero solution $\alpha_{kj}, \cdots, -1, j \neq k$.

So
\[
\begin{vmatrix}
b_{11} & \cdots & b_{ij} & \cdots & b_{1p} & \nabla_{\omega,x}F_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{j-1} & \cdots & b_{j-1} & \cdots & b_{j-1} & \nabla_{\omega,x}F_{j-1} \\
b_{j+1} & \cdots & b_{j+1} & \cdots & b_{j+1} & \nabla_{\omega,x}F_{j+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{b_1} & \cdots & b_{b_1} & \cdots & b_{bp} & Q_j
\end{vmatrix} = 0.
\]

Now let $(D_{\omega,x}F) = (\nabla_{\omega,x}F_1, \cdots, \nabla_{\omega,x}F_p)^T$.

\[
(D_{\omega,x}F) (D_{\omega,x}F)^T =
\]
\[
= \begin{bmatrix}
\langle \nabla_{\omega,x}F_1, \nabla_{\omega,x}F_1 \rangle & \cdots & \langle \nabla_{\omega,x}F_1, \nabla_{\omega,x}F_p \rangle \\
\vdots & \vdots & \vdots \\
\langle \nabla_{\omega,x}F_p, \nabla_{\omega,x}F_1 \rangle & \cdots & \langle \nabla_{\omega,x}F_p, \nabla_{\omega,x}F_p \rangle \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
b_{11} & \cdots & b_{1p} \\
\vdots & \vdots & \vdots \\
b_{p1} & \cdots & b_{pp}
\end{bmatrix}
\]
If $M_{ij}$ denote the matrix obtained from $(D_{\omega,x} F) (D_{\omega,x} F)^T$ deleting the $i$th row and the $j$th column. Let $A_{ij} = (-1)^{i+j} det(M_{ij})$. then $Q_j$ equals to $$-A_{jj}$$ i.e. $$Q_j = - \sum_{i=1, i \neq j}^{p} \frac{A_{ji}}{A_{jj}} \nabla_{\omega,x} F_i.$$ Now $N(F, j, u) = \nabla_{\omega,x} F_j(u, t) - Q_j$ is equal to $$-A_{jj}$$ $$= \sum_{i=1}^{p} \frac{A_{ji}}{A_{jj}} \nabla_{\omega,x} F_i.$$ Owning to $$d_{\omega,x} \nabla F \geq C' d(x, \Sigma)^{r-\delta},$$ for $(u, t) = (x, \lambda, t) \in W \setminus \Sigma \times R^t \times I$, the vectors $\{\nabla_{\omega,x} F_1, \ldots, \nabla_{\omega,x} F_p\}$ are linearly independent. So $(D_{\omega,x} F) (D_{\omega,x} F)^T$ has rank $p$ and $$\left((D_{\omega,x} F) (D_{\omega,x} F)^T\right)^{-1} = \frac{1}{d} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{p1} \\ A_{12} & A_{22} & \cdots & A_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1p} & A_{2p} & \cdots & A_{pp} \end{bmatrix},$$ where $d = \det (D_{\omega,x} F) (D_{\omega,x} F)^T$.

We again compute $\|N(F, j, u)\|^2_\omega$.

$$\|N(F, j, u)\|^2_\omega = \nabla_{\omega,x} F_j \cdot N(F, j, u) = \left< \nabla_{\omega,x} F_j, \sum_{i=1}^{p} \frac{A_{ji}}{A_{jj}} \nabla_{\omega,x} F_i \right>$$ $$= \sum_{i=1}^{p} \frac{A_{ji}}{A_{jj}} \left< \nabla_{\omega,x} F_j, \nabla_{\omega,x} F_i \right> = \frac{d}{A_{jj}}.$$
Therefore, when

\[ \frac{N(F,1,u)}{\|N(F,1,u)\|^2_\omega}, \ldots, \frac{N(F,p,u)}{\|N(F,p,u)\|^2_\omega} \]

\[ \frac{A_{11}}{d}N(F,1,u), \ldots, \frac{A_{pp}}{d}N(F,p,u) \]

Since

\[ (D_{\omega,x}F)^T \left( (D_{\omega,x}F) (D_{\omega,x}F)^T \right)^{-1} = (\nabla_{\omega,x}F_1, \ldots, \nabla_{\omega,x}F_p) \frac{1}{d} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{p1} \\ A_{12} & A_{22} & \cdots & A_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1p} & A_{2p} & \cdots & A_{pp} \end{bmatrix} = \]

\[ \frac{1}{d} \left( \sum_{i=1}^{p} A_{1i} \nabla_{\omega,x}F_i \right), \ldots, \frac{1}{d} \left( \sum_{i=1}^{p} A_{pi} \nabla_{\omega,x}F_i \right) = \left( \frac{A_{11}}{d}N(F,1,u), \ldots, \frac{A_{pp}}{d}N(F,p,u) \right) \]

Then,

\[ \left( \frac{N(F,1,u)}{\|N(F,1,u)\|^2_\omega}, \ldots, \frac{N(F,p,u)}{\|N(F,p,u)\|^2_\omega} \right) = (D_{\omega,x}F)^T \left( (D_{\omega,x}F) (D_{\omega,x}F)^T \right)^{-1}. \quad (3.12) \]

Therefore, when \( u \in H^2_\Gamma(f, \bar{w}) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l, t \in I \),

\[ X_1(u, t) = \frac{\partial}{\partial t} + \sum_{i=1}^{p} \frac{p_i(u)}{\|N_i\|^2_\omega} N_i \]

\[ = \left( -\sum_{i=1}^{p} \frac{p_i(u)}{\|N_i\|^2_\omega} N_i, 0, 1 \right)^T \]

\[ = \left( - \left[ (D_{\omega,x}F)^T \left( (D_{\omega,x}F) (D_{\omega,x}F)^T \right)^{-1} p(u) \right]^T, 0, 1 \right)^T. \]

Using the formula of \( X_1(u, t) \), we show that \( X_1(u, t) \) is \( \Gamma \)-equivariant.

Owing to Lemma 3.1 and \( \Sigma \) is \( \Gamma \)-invariant, for \( \forall \gamma \in \Gamma, H^2_{\omega,x}(f, \bar{w}) \) is \( \Gamma \)-invariant and \( H^2_{\omega,x}(f, \bar{w}) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \) is \( \Gamma \)-invariant. \( F(\gamma u, t) = \gamma F(u, t) \) with \( \gamma \gamma^T = E \). Differentiating \( F_i(\gamma u, t) = \gamma F_i(u, t) \) with respect to \( x \) yields

\[ \left( \frac{\partial F_i}{\partial x_1}(\gamma u, t), \ldots, \frac{\partial F_i}{\partial x_n}(\gamma u, t) \right) \gamma = \gamma \left( \frac{\partial F_i(u, t)}{\partial x_1}, \ldots, \frac{\partial F_i(u, t)}{\partial x_n} \right). \quad (3.13) \]

By Lemma 3.1 and (3.11),

\[ \nabla_{\omega,x}F_i(\gamma u, t) = \left( \sum_{j=1}^{n} \rho(\gamma u)^{\omega_j} \frac{\partial F_i(\gamma u, t)}{\partial x_j} - \rho(\gamma u)^{\omega_j} \frac{\partial}{\partial x_j} \right) \gamma \]

\[ = \gamma \left( \sum_{j=1}^{n} \rho(u)^{\omega_j} \frac{\partial F_i(u, t)}{\partial x_j} - \rho(u)^{\omega_j} \frac{\partial}{\partial x_j} \right). \]
It implies $D_{\omega,x}F(\gamma u, t)\gamma = \gamma D_{\omega,x}F(u, t)$.

\[
(D_{\omega,x}F(\gamma u, t))^T \left( (D_{\omega,x}F(\gamma u, t)) (D_{\omega,x}F(\gamma u, t))^T \right)^{-1} \\
= \left( \gamma D_{\omega,x}F(u, t)\gamma^{-1} \right)^T \left( \left( \gamma D_{\omega,x}F(u, t)\gamma^{-1} \right)^T \left( D_{\omega,x}F(u, t) \right)^T \gamma \right)^{-1} \\
= (\gamma^T)^T (D_{\omega,x}F(u, t))^T \gamma^T (\gamma D_{\omega,x}F(u, t)\gamma (D_{\omega,x}F(u, t))^T)^{-1} \gamma^T \\
= \gamma (D_{\omega,x}F(u, t))^T \gamma^T (\gamma^T)^T (D_{\omega,x}F(u, t) (D_{\omega,x}F(u, t))^T)^{-1} \gamma^T \\
= \gamma (D_{\omega,x}F(u, t))^T (D_{\omega,x}F(u, t) (D_{\omega,x}F(u, t))^T)^{-1} \gamma^T
\]

\[
X_1(\gamma u, t) = \left( - \left[ (D_{\omega,x}F(\gamma u, t))^T \left( (D_{\omega,x}F(\gamma u, t)) (D_{\omega,x}F(\gamma u, t))^T \right)^{-1} p(\gamma u) \right]^T, 0, 1 \right)^T \\
= \left( - \left[ \gamma (D_{\omega,x}F(u, t))^T \left( D_{\omega,x}F(u, t) (D_{\omega,x}F(u, t))^T \right)^{-1} \gamma^{-1} \gamma p(u) \right]^T, 0, 1 \right)^T \\
= \gamma \left( - \left[ (D_{\omega,x}F)^T \left( (D_{\omega,x}F) (D_{\omega,x}F)^T \right)^{-1} p(u) \right]^T, 0, 1 \right)^T \\
= \gamma X_1(u, t).
\]

\[
X_1(\gamma u, t) = \left( - \left[ (D_{\omega,x}F(\gamma u, t))^T \left( (D_{\omega,x}F(\gamma u, t)) (D_{\omega,x}F(\gamma u, t))^T \right)^{-1} p(\gamma u) \right]^T, 0, 1 \right)^T \\
= \left( - \left[ \gamma (D_{\omega,x}F(u, t))^T \left( D_{\omega,x}F(u, t) (D_{\omega,x}F(u, t))^T \right)^{-1} \gamma^{-1} \gamma p(u) \right]^T, 0, 1 \right)^T \\
= \gamma \left( - \left[ (D_{\omega,x}F)^T \left( (D_{\omega,x}F) (D_{\omega,x}F)^T \right)^{-1} p(u) \right]^T, 0, 1 \right)^T \\
= \gamma X_1(u, t).
\]

Therefore $X_1(u, t)$ is $\Gamma$-equivariant in $W$.

Again for $u \in H^\Sigma_{\omega,p}(f, \eta, \eta^*) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l$, $t \in T$,

\[
p_j(u) = \Sigma_{i=1}^p p_i(u) \nabla_{\omega,x}F_j \cdot \frac{N(F, i, u)}{\|N(F, i, u)\|_2^2} = \\
p_j(u) - \Sigma_{i=1}^p p_i(u) (\nabla_{\omega,x}F_j \cdot N_i) \cdot \frac{1}{\|N_i\|_2} = 0, \quad j = 1, \ldots, p.
\]

So

\[
\nabla_{\omega} F_j \cdot X_1(u, t) = \langle \nabla_{\omega} F_j, X_1(u, t) \rangle = 0, \quad j = 1, \ldots, p.
\] (3.14)
Moreover

\[ \Lambda, X_1(u, t) > \omega = 0, \quad \Lambda \in \{0\} \times \mathbb{R}^l \times \{0\} \]

\[ < \frac{\partial}{\partial t}, X_1(u, t) >= 1 \tag{3.15} \]

and \( X_1(u, t) \) is \( C^1 \) on \( H^\Sigma_f, \| u \| < \alpha \) \( \setminus \Sigma \times \mathbb{R}^l, t \in T. \)

By Lemma 3.3, we extend the vector field \( X_1(u, t) \) to a vector field \( X(u, t) \) defined on a neighbourhood of zero.

Let \( X(u, t) \) be defined by

\[
X(u, t) = \begin{cases} 
\chi(u)X_1(u, t) + (1 - \chi(u))\frac{\partial}{\partial t}, & (u, t) \in \{\| u \| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \times I \\
\frac{\partial}{\partial t}, & (u, t) \in \{\| u \| < \alpha\} \cap \Sigma \times \mathbb{R}^l \times I
\end{cases}
\]

i.e. when \( (u, t) \in \{\| u \| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \times I, \)

\[
X(u, t) = \left(-\chi(u) \cdot \left[D_{\omega, x} F \right]^T \left((D_{\omega, x} F) (D_{\omega, x} F)^T \right)^{-1} p(u) \right]^T, 0, 1 \right)^T;
\]

when \( (u, t) \in \{\| u \| < \alpha\} \cap \Sigma \times \mathbb{R}^l \times I, \)

\[
X(u, t) = (0, 0, 1).
\]

Since

\[
\| X_1(u, t) - \frac{\partial}{\partial t} \| \omega = \| \sum_{j=1}^{p} \frac{p_j(u)}{\| N_i \|_\omega} N_i \| \omega
\]

\[
\leq \sum_{j=1}^{p} \left| \frac{p_j(u)}{\| N_i \|_\omega} \right| \omega
\]

\[
\leq \sum_{j=1}^{p} \left( \frac{d_{\omega(x, \Sigma)}^{r+|\omega|}}{C d_{\omega(x, \Sigma)}^{r-\delta}} \right)
\]

\[
\leq \sum_{j=1}^{p} o \left( \frac{d_{\omega(x, \Sigma)}^{r+|\omega|}}{C' d_{\omega(x, \Sigma)}^{r+|\omega|}} d_{\omega(x, \Sigma)}^{r+|\omega|} d_{\omega(x, \Sigma)}^{r+|\omega|} \right)
\]

\[
\leq \sum_{j=1}^{p} o \left( \frac{d_{\omega(x, \Sigma)}^{r+|\omega|}}{C' d_{\omega(x, \Sigma)}^{r+|\omega|}} d_{\omega(x, \Sigma)}^{r+|\omega|} \right),
\]

then, by assumption \( | \omega | + \delta - 1 > 0, \)

\[
\| X(u, t) - \frac{\partial}{\partial t} \| \omega = \| \chi(u) \left( X_1(u, t) - \frac{\partial}{\partial t} \right) \| \omega
\]
\[
\begin{align*}
&\leq \sum_{j=1}^{p} \frac{o(d_\omega(x, \Sigma)^{r+|\omega|})}{C d_\omega(x, \Sigma)^{r+|\omega|}} d_\omega(x, \Sigma)^{|\omega|+\delta-1} d_\omega(x, \Sigma) \\
&\leq \sum_{j=1}^{p} \frac{o(d_\omega(x, \Sigma)^{r+|\omega|})}{C' d_\omega(x, \Sigma)^{r+|\omega|}} d_\omega(x, \Sigma) \leq Md_\omega(x, \Sigma),
\end{align*}
\]

where \( M \) is a positive number and \( u \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \) with an enough small \( \alpha \) and \( d(x, \Sigma) \leq 1 \), i.e.

\[
\|X(u, t) - \frac{\partial}{\partial t}\|_\omega \leq Md_\omega(x, \Sigma), \quad \text{for } u \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l. \quad (3.16)
\]

Now we show \( X(u, t) \) is \( \Gamma \)-equivariant.

In fact,

\[
X(\gamma u, t) = \begin{cases} 
\chi(\gamma u)X_1(\gamma u, t) + (1 - \chi(\gamma u))\frac{\partial}{\partial t}, & u \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \\
\chi(u)\gamma X_1(u, t) + (1 - \chi(u))\frac{\partial}{\partial t}, & u \in \{\|u\| < \alpha\} \cap \Sigma \times \mathbb{R}^l
\end{cases}
\]

\[
= \begin{cases} 
-\chi(u) \cdot \gamma \cdot \left[ (D_x F)^T ((D_x F)(D_x F)^T)^{-1} p(u) \right]^T, & u \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \\
(0, 0, 1), & u \in \{\|u\| < \alpha\} \cap \Sigma \times \mathbb{R}^l
\end{cases}
\]

\[
= \begin{cases} 
\gamma \cdot (-\chi(u) \cdot \left[ (D_x F)^T ((D_x F)(D_x F)^T)^{-1} p(u) \right]^T, & u \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l \\
(0, 0, 1), & u \in \{\|u\| < \alpha\} \cap \Sigma \times \mathbb{R}^l
\end{cases}
\]

\[
= \gamma \cdot X(u, t).
\]

Therefore \( X(u, t) \) is \( \Gamma \)-equivariant.

Moreover \( X(u, t) \) satisfies

\[
< \Lambda, X(u, t) > = 0, \quad \Lambda \in \{0\} \times \mathbb{R}^l \times \{0\} \quad (3.17)
\]

\[
< \frac{\partial}{\partial t} X(u, t) > = 0 \quad (3.18)
\]

\( X(u, t) \) is \( C^1 \) on \( \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l = \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^l, \ t \in T. \) and satisfies

\[
\nabla_\omega F_j \cdot X(u, t) = < \nabla_\omega F_j, X(u, t) > = 0, \quad j = 1, \cdots, p. \quad (3.19)
\]
It implies $X(u, t)$ is perpendicular to $\nabla_{\omega, x} F$ at every $(u, t) \in \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I \times I$, hence $X(u, t)$ is tangent to the level surface $F = \text{constant}$.

In addition, if, for a enough small $\alpha$, $u \in F^{-1}(0) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I$, then

$$u \in H^\Sigma_{\omega, f}(f, w) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I.$$ 

In fact, by $F_i(u, t) = f_i(u) + t p_i(u) = 0,$

$$\frac{|f_i(u)|}{d\omega(x, \Sigma)^r} = \frac{|t p_i(u)|}{d\omega(x, \Sigma)^r} = \frac{t \cdot o(d\omega(x, \Sigma)^r + |\omega|)}{d\omega(x, \Sigma)^r} < o(d\omega(x, \Sigma)^r) d\omega(x, \Sigma) \leq \rho(u)^r.$$ 

So when $\alpha$ is enough small and

$$u \in F^{-1}(0) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I,$$

there is a enough small $\tilde{w}$ such that

$$\|f(u)\| \leq \tilde{w} d\omega(x, \Sigma)^r \leq \tilde{w} \|u\|_{\omega} \leq \tilde{w} \rho(u)^r.$$ 

Therefore when $\alpha$ is enough small and

$$u \in F^{-1}(0) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I,$$

we have

$$u \in H^\Sigma_{\omega, f}(f, w) \cap \{\|u\| < \alpha\} \setminus \Sigma \times \mathbb{R}^I.$$ 

In order to be able to define $\phi$, for a sufficient small $\alpha$, if $(u, t) \in \{\|u\| < \alpha\} \times I$, since (3.15), by Proposition 2.6, the following system of differential equations:

$$u' = X(u, t)$$

(3.20)

is integrable.

Now for $(u, t) \in W$ define $\gamma_{(u,t)}$ to be the maximal solution of (3.19) such that $\gamma_{(u,t)}(t) = u$. Let $H_0, \tilde{H}_0 : \{\|u\| < \alpha\} \times T \to \{\|u\| < \alpha\}$ be given by

$$H_0(u, t) = \gamma_{(u, t_0)}(t), \quad \tilde{H}_0(v, t) = \gamma_{(v, t)}(t_0),$$

where $T$ is a small neighbourhood of $t$ in $I$. By Proposition 2.6, the mappings $H_0, \tilde{H}_0$ are continuous mappings and uniqueness solutions of (3.19) and the property $\gamma_{(\xi + \eta, x)}(s + t) = \gamma_{(\eta, \gamma_{(\xi, x)}(s))}(t)$ of the flow, it is easy to check that for $(u, t) \in \{\|u\| < \alpha\} \times T$ we have

$$\tilde{H}_0(H_0(u, t), t) = u, \quad H_0(u, t_0) = u,$$
and
\[ H_0(\tilde{H}_0(v, t), t) = v, , \tilde{H}_0(u, t) = u. \]
\[ F(\gamma(u, t_0)(t, t) = F(u, t_0) \text{ for any } t \in T, \text{ namely we have } f(H_0(u, t)) + t \cdot p(H_0(u, t)) = F(x, t) \text{ for } (u, t) \in \{\|u\| < \alpha\} \times T. \]
In particular, by (3.17), (3.18) for all \( t, t' \in T \), The germ of \( F(u, t) = 0 \) and \( F(u, t') = 0 \) are \( \Sigma \)-homeomorphic (i.e. by a homeomorphism in \( R_{\Sigma}^{fix} \)). By \( X(u, t) \) is \( \Gamma \)-equivariant, the \( \Sigma \)-homeomorphic between the germ of \( F(u, t) = 0 \) and \( F(u, t') = 0 \) is \( \Gamma \)-equivariant.

Using compactness of \([0, 1] \), we obtain that a homeomorphism \( \psi \) which has the form
\[ \psi(u, t) = (\bar{\psi}(u, t), t) \in (R^n \times R^l) \times I. \]
We define \( \phi = \bar{\psi}(u, 1) \) which is \( \Gamma - \Sigma - \text{BD equivalence between } G = F(\cdot, 0) \) and \( \bar{G} = F(\cdot, 1) \).

Now we need to upgrading the equivalence between \( G \) and \( \bar{G} \) to a contact equivalence. We construct a matrix valued map \( \tau \) which completes a contact equivalence in the same way as Theorem 3.1 of [6].

Let
\[ P_\phi(u) = \bar{G}(\phi(u)) - G(u) \text{ for } u \in \{\|u\| < \alpha\} \]
\( P_\phi(u) \) is \( \Gamma \)-equivariant.

We shall need to the fact that
\[ \|d_\omega(x, \Sigma)^{-r}P_\phi(u)\| = o(1). \tag{3.21} \]
Let
\[ Q_\psi(u, s) = F(\psi(u, s)) - G(u) \]
and
\[ \sigma = d_\omega(\bar{\psi}(u, s), \Sigma \times R^l). \]
Then
\[ \frac{\partial}{\partial s}[Q_\psi(u, s)] = F'(\psi(u, s)) \cdot X(\psi(u, s)) \]
\[ = (\nabla_{\omega,x}F, \nabla_{\omega,y}F, p) \cdot X(\psi(u, s)) \]
\[ = (1 - \chi(\bar{\psi}(u, s)))p(\bar{\psi}(u, s)) \]
by (3.11).
So, since \( Q_\psi(u, 0) = 0 \),
\[ \|d_\omega(x, \Sigma)^{-r}Q_\psi(u, t)\| \leq \int_0^t \|d_\omega(x, \Sigma)^{-r}\frac{\partial}{\partial s}[Q_\psi(u, s)]\| ds \]
\[ \leq \int_0^t \|d_\omega(x, \Sigma)^{-r}P(\bar{\psi}(u, s))\| ds \]
\[ \leq \int_0^t |(\sigma/d_\omega(x, \Sigma))^r| \cdot |\sigma^{-r}P(\bar{\psi}(u, s))| ds. \]
Because \( d_\omega \left( u, \Sigma \times \mathbb{R}^l \right) = d_\omega \left( x, \Sigma \right) \), by Lemma 2.7, we have

\[
d_\omega(x, \Sigma) e^{-C|t|} \leq d_\omega \left( \bar{\psi}(u, s), \Sigma \times \mathbb{R}^l \right) \leq d_\omega(x, \Sigma) e^{C|t|}
\]  

(3.22)

for \( t \in (\alpha, \beta) \).

By (2.1) and (3.21),

\[
\| d_\omega(x, \Sigma)^{-r} Q_\psi(u, t) \| = o(1) \quad \text{uniformly in } s \in [0, 1].
\]

(3.23)

Because \( Q_\psi(u, 1) = P_\phi(u) \), this proves (3.20).

Again owing to

\[
F(\psi(u, s) = G(\bar{\psi}(u, s)) + sP \left( \bar{\psi}(u, s) \right),
\]

using (2.1), (3.21) and (3.22), if \( u \in H_\omega^\Sigma(G, \beta) \cap \{ \| u \| < \alpha \} \), then

\[
\| \sigma^{-r}G \left( \bar{\psi}(u, s) \right) \| = \| \sigma^{-r} \left[ F(\psi(u, s)) - sP \left( \bar{\psi}(u, s) \right) \right] \| \leq \leq (d_\omega(x, \Sigma)/\sigma)^r \cdot \left\{ d_\omega(x, \Sigma)^{-r} \| G(u) \| + d_\omega(x, \Sigma)^{-r} \| Q_\psi(u, s) \| \right\} + \| \sigma^{-r}P \left( \bar{\psi}(u, s) \right) \|
\]

\[
= (d_\omega(x, \Sigma)/\sigma)^r \cdot d_\omega(x, \Sigma)^{-r} \| G(u) \| + o(1) \leq e^{r^2} \beta + o(1)
\]

uniformly for \( s \in [0, 1] \). Therefore, we may choose \( V \subset \{ \| u \| < \alpha \} \) to be a neighbourhood of the origin small enough and \( \beta \) sufficiently small such that

\[
\psi(u, s) \in \left( H_\omega^\Sigma(G, \frac{\beta}{2}) \right) \cap \{ \| u \| < \alpha \} \times T \quad \text{when } (u, s) \in \left( H_\omega^\Sigma(G, \beta) \right) \cap V \times T.
\]

Hence, by (3.18), \( F \) is constant on the flow of \( X(u, t) \) which remain in \( \psi(u, s) \in \left( H_\omega^\Sigma(G, \frac{\beta}{2}) \right) \cap \{ \| u \| < \alpha \} \times T \), so, when \( u \in \left( H_\omega^\Sigma(G, \beta) \right) \cap V \),

\[
P_\phi(u) = F(\psi(u, 1)) - F(\psi(u, 0)) = 0.
\]

(3.24)

Finally we construct a \( \Gamma \)-equivariant matrix valued map \( \tau : (V, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^l, 0) \) which completes a contact equivalence For \( u \in V \) and \( v \in \mathbb{R}^p \), let

\[
\theta : V \rightarrow \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)
\]

be defined by

\[
\theta(u)(v) = \begin{cases} 
\frac{(G(u), v)}{\| G(u) \|^2} P_\phi(u), & G(u) \neq 0, \\
0, & G(u) = 0.
\end{cases}
\]

(3.25)

For \( \gamma \in \Gamma \),

\[
\theta(\gamma u)(v) = \begin{cases} 
\frac{(G(\gamma u), v)}{\| G(\gamma u) \|^2} P_\phi(\gamma u), & G(\gamma u) \neq 0, \\
0, & G(\gamma u) = 0.
\end{cases}
\]
\[\langle \gamma G(u), \gamma \gamma^t v \rangle - \|G(u)\|_2 \gamma P(\gamma u) \neq 0, \quad G(\gamma u) = 0\]

\[\langle G(u), \gamma \gamma^t v \rangle - \|G(u)\|_2 \gamma P(\gamma u) = 0, \quad G(\gamma u) = 0\]

Meantime,

\[
\gamma \cdot \theta(\gamma u) = \gamma \cdot \theta(\gamma u) \cdot \gamma^t = \gamma \cdot \theta(\gamma u) \cdot \gamma^{-1},
\]

i.e. \(\theta(u)\) is \(\Gamma\)-matrix valued map.

When \(G(u) \neq 0\) and \(P(\gamma u) \neq 0\), \(\theta(u)\) is just a rank one linear transformation designed so that

\[\theta(u) (G(u)) = P(\gamma u).\]

By (3.23) and (3.24), \(\theta(u) = 0\) for \(u \in (H_{\omega,t}^\Sigma(G, \beta)) \cap V\), which is a neighbourhood of \([G \setminus V \setminus \Sigma \times R]^t\)\(^{-1}(0)\), so \(\theta\) is continuous on \(V \setminus \Sigma \times R\). By (3.20) and (3.24),

\[\|\theta(u)\| = \|P(\gamma u)\|/\|G(u)\| \leq o(d_\omega(x, \Sigma)^r)/d_\omega(x, \Sigma)^r = o(1),\]

for \(u \in V \setminus (H_{\omega,t}^\Sigma(G, \beta)) \cap V\), so \(\theta\) is continuous at every point of \((\Sigma \times R) \cap V\). Because \(\theta\) is continuous on \(V\) and \(\theta(0) = 0\), let \(V\) be enough small so that

\[I + \theta(u) \in GL(R^p), \quad u \in V.\]

Let

\[\tau(u) = (I + \theta(u))^{-1}, \quad u \in V.\]

Then

\[\tau : V \rightarrow GL(R^p)\]

is clearly continuous and

\[G(u) = \tau(u \left(\tilde{G}(\phi(u))\right)),\]

since

\[(I + \theta(u))(G(u)) = G(u) + P(\gamma u) = \tilde{G}(\phi(u)).\]

**Remark 3.** If \(\Gamma\) be a compact Lie group acting linearly on space on space \((R^n \times R^l, \langle \cdot, \cdot \rangle)\) and space \((R^p, \langle \cdot, \cdot \rangle)\), then we can define a new inner on \(R^n \times R^l\) following([10]):

\[\langle u, v \rangle_{\Gamma} = \int_G \langle \gamma u, \gamma v \rangle d\gamma,\]
where $f$ is Haar integral on $\Gamma$, $u, v \in \mathbb{R}^n \times \mathbb{R}^l$. It is important that $\Gamma$ be acting orthogonally on new inner space $(\mathbb{R}^n \times \mathbb{R}^l, \langle \cdot \rangle_\Gamma)$ and new inner space $(\mathbb{R}^p, \langle \cdot \rangle_\Gamma)$.

Now we show to what terms from the Taylor expansion at every point that belongs to a closed subset of $\mathbb{R}^n$ such that $0 \in \Sigma$ may be omitted without changing the topological type determined by $G$ and the value of the bifurcation parameter $\lambda$.

**Proof of Theorem 1.2.** The proof will be similar to that given in Theorem 1.1. Let

$$F(u, t) = f(u) + tp(u).$$

In proof of Theorem 1.1, let $\Gamma = \{e\}$ and $\omega = \{\omega_1, \omega_2, \ldots, \omega_{n+1}\} = \{1, 1, \ldots, 1\}$, then $d_{\omega}(x, \Sigma)$ be substituted by $d(x, \Sigma) \cap \{\|u\| < \alpha\}$ and $f$ satisfies the condition $(K_{\Sigma}^r, \delta)$, we have $d_x \nabla F \geq C'd(x, \Sigma)^{r-\delta}$.

Again we use a version the Kuo-vector field

$$X_1(u, t) = \left\{ \frac{\partial}{\partial t} + \Sigma^p_{j=1} \frac{p(u)}{|N_i|}^2 N_i, \quad (u, t) \in W \setminus \Sigma \times T, \right. \quad \left. (u, t) \in W \cap \Sigma \times T. \right\}$$

where $W = H_{\Sigma}^n(f, \bar{w}) \cap \{u\} < \alpha$.

Moreover we have a vector field $X(u, t)$.

Finally, using Lemma 2.4, we may obtain a homeomorphism between $f$ and $f + p$.

**Proof of Corollary 1.3.** Let $h(u) = g(u) - f(u)$. Then $j^{r+2}h(u) = 0$ for $u = (x, \lambda) \in (\Sigma \times \mathbb{R}^l) \cap U$. By Lemma 2.8,

$$\|h(u)\| = \tilde{o}\left((d(u, \Sigma \times \mathbb{R}^l)^{r+2})\right) = \tilde{o}\left((d(x, \Sigma)^{r+2})\right).$$

It implies

$$|h_i| = \tilde{o}\left((d(x, \Sigma)^{d+1})\right), \quad \left|\frac{\partial h_i}{\partial x_j}\right| = \tilde{o}\left((d(x, \Sigma)^{d})\right), \quad i = 1, \ldots, p; \quad j = 1, \ldots, n.$$ 

From Theorem 1.1, we obtain that $f$ and $g$ is $\Sigma - C^0$-BD equivalent and $\Sigma - C^0$-contact equivalent.

## 4 $C^0$-Finite determination of the bifurcation diagram

In the section, we show that contact $|\nu|$—determination of the bifurcation problem is a corollary of Theorem 3.1.

We begin by presenting notation and concepts needed from [6].

For $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, let

$$\kappa(A) = \inf \| \alpha' A \|: \alpha' \in \mathbb{R}^p, \| \alpha' \| = 1.$$
When $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ and rank$(A) = p$, let

$$A^+ = A'(AA')^{-1}.$$  

Obvious $AA^+ = I$ and $\kappa(A) = \| A^+ \|^{-1}$ by [5]. For $\rho > 0$ and $\nu \in \mathbb{R}^p$, let

$$\rho'' = \text{diag}(\rho'^{\nu_1}, \cdots, \rho'^{\nu_p});$$  

$$\rho^{1-|\nu|} = \text{diag}(\rho^{1-|\nu_1|}, \cdots, \rho^{1-|\nu_p|}).$$

When $F : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0)$ is a $C^1$ map and $\nu \in \mathbb{R}^p$, we say that $F$ is ND($\nu$) if there exist $\varepsilon > 0, \delta > 0$ and a neighbourhood $U$ of the origin in $\mathbb{R}^n \times \mathbb{R}^l$ for which, with $\rho = \|u\|$,  

$$\kappa \left( \rho^{1-\nu} \nabla_x F(u) \right) \geq \varepsilon \text{ if } u = (x, \lambda) \in H_\nu(F, \delta) \cap U,$$  

(4.26)

where  

$$H_\nu(F, \delta) = \{ u \in \mathbb{R}^n \times \mathbb{R}^l : \| \rho^{1-\nu} F(u) \| \leq \delta \}; \quad \nabla_x F(u) = (\nabla_x F_1(u), \cdots, \nabla_x F_p(u)).$$

**Remark 4.** If $F$ is ND($\nu$), then $F$ is also ND($\| \nu \|$) by [6].

The following lemmas will be used to prove Theorem 1.4.

**Lemma 4.1.** ([11], Lemma A.9) The operator norm $\| \cdot \|$ of a submatrix is bounded by one of the whole matrix. More precisely, if $A \in C^{m \times n}$ has the form  

$$A = \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix}$$

for matrices $A^{(l)}$, then $\| A^{(l)} \| \leq\| A \|$ for $l = 1, 2, 3, 4$. In particular, any entry of $A$ satisfies $| A_{j,k} | \leq\| A \|$.

**Proof.** We give the proof for $A^{(l)}$. The other cases are analogous. Let $A^{(l)}$ be of size $m_1 \times n_1$. Then for the vector $x^{(1)} \in C^{m_1}$, we have

$$\| A^{(l)} x^{(1)} \|^2 \leq \| A^{(l)} x^{(1)} \|^2 + \| A^{(3)} x^{(1)} \|^2 = \| A^{(2)} x^{(1)} \|^2 = \| A \begin{pmatrix} x^{(1)} \\ O \end{pmatrix} \|^2.$$

The set $T_1$ of vectors $\begin{pmatrix} x^{(1)} \\ O \end{pmatrix} \in C^n$ with $\| x^{(1)} \| \leq 1$ is contained in the set $T = \{ x \in C^n : \| x \| \leq 1 \}$. Therefore, the supremum over $x^{(1)} \in T_1$ above is bounded by $\sup_{x \in T} \| A x \|^2 = \| A \|$. This concludes the proof.
Lemma 4.2. Suppose \( F : (\mathbb{R}^n \times \mathbb{R}^l, 0) \rightarrow (\mathbb{R}^p, 0) \) is a \( C^1 \) map and \( \nu = (\nu_1, \cdots, \nu_p) \) such that \( F \) is \( \text{ND}(\nu) \). Then
\[
 d_x \nabla F \geq Cd(x, 0)\|\nu\|^{-1} \quad \text{when } u = (x, \lambda) \in H(F, |\nu|, \delta) \cap U.
\]

Proof. By Remark 4, \( F \) is also \( \text{ND}(|\nu|) \). Then
\[
(\rho^{1-|\nu|} \nabla_x F)^+ = \left( \rho^{1-|\nu|} \nabla_x F \right)^t \left( \rho^{1-|\nu|} \nabla_x F \cdot \left( \rho^{1-|\nu|} \nabla_x F \right)^t \right)^{-1}
\]
\[
= \rho^{1-|\nu|} \left( \nabla_x F \right)^t \rho^{1-|\nu|} \left( \left( \nabla_x F \right)^t \right)^{-1} \rho^{1-|\nu|}
\]
\[
= \left( \nabla_x F \right)^t \left( \nabla_x F \right)^t \left( \nabla_x F \right)^t \cdot \rho^{1-|\nu|}
\]
\[
= \rho^{1-|\nu|} \cdot \rho^{1-|\nu|} = \rho^{2\rho^{1-|\nu|}}
\]

By (3.11),
\[
(\nabla_x F)^+ = \left( \frac{N(F, 1, u)}{\|N(F, 1, u)\|}, \cdots, \frac{N(F, p, u)}{\|N(F, p, u)\|} \right)
\]
\[
(\rho^{1-|\nu|} \nabla_x F)^+ = \rho^{1-|\nu|} \left( \frac{N(F, 1, u)}{\|N(F, 1, u)\|^2}, \cdots, \frac{N(F, p, u)}{\|N(F, p, u)\|^2} \right)
\]
where \( \rho^{1-|\nu|} = \text{diag} \left( \rho^{1-|\nu|}, \cdots, \rho^{1-|\nu|} \right) \), when \( \nu = 1, \rho^{1-|\nu|} = I. \)

For vectors
\[
\frac{\rho^{1-|\nu|} N(F, i, u)}{\|N(F, i, u)\|^2}, \quad i = 1, \cdots, p
\]
, By Lemma 4.1,
\[
\sum_{j=1}^{n+l} \left( \frac{\rho^{1-|\nu|} N(F, i, u)_j}{\|N(F, i, u)\|^2} \right)^2 = \frac{1}{\|N(F, i, u)\|^4} \sum_{j=1}^{n+l} \left( \rho^{1-|\nu|} N(F, i, u)_j \right)^2
\]
\[
\leq \sum_{j=1}^{n+l} \| \left( \rho^{1-|\nu|} \nabla_x F \right)^+ \|^2,
\]
where \( N(F, i, u)_j \) is the j-component of \( N(F, i, u) \), i.e.
\[
\rho^{2\rho^{1-|\nu|}} \frac{1}{\|N(F, i, u)\|^2} \leq (n + l) \| \left( \rho^{1-|\nu|} \nabla_x F \right)^+ \|^2.
\]

\[
\kappa \left( \rho^{1-|\nu|} \nabla_x F(u) \right) = \| \left( \rho^{1-|\nu|} \nabla_x F \right)^+ \|^2 \leq \left( \rho^{1-|\nu|} \frac{1}{\|N(F, i, u)\|} \right)^{-1} \frac{1}{(n + l)^{\frac{1}{2}}},
\]
i.e.
\[
\kappa \left( \rho^{1-|\nu|} \nabla_x F(u) \right) \leq \rho^{1-|\nu|} \|N(F, i, u)\| \frac{1}{|\nu|^{-1}}.
\]
Since
\[ d_x \nabla F = \min \{ \| N(F, i, u) \| : i = 1, \cdots, p \}, \]
and
\[ \kappa \left( \rho^{1-|\nu|} \nabla_x F(u) \right) \geq \varepsilon, \]
then
\[ d_x \nabla F \geq \varepsilon((n + l))^{\frac{1}{2}} \rho^{|\nu|-1} \geq C d(x, 0)^{|\nu|-1} \]
when \( u = (x, \lambda) \in H_{|\nu|}(F, \delta) \cap U. \) where \( C = \varepsilon((n + l))^{\frac{1}{2}}. \)

**Proof of Theorem 1.4.** In Theorem 1.1, let \( \Gamma = \varepsilon, \Sigma = \{ 0 \} \) and \( \omega = (1, 1, \cdots, 1). \) Since \( F \) is ND(\( \nu \)), \( F \) satisfies relative Kuo condition \( \left( K_{\{0\}}^{[\nu], 1}(\varepsilon) \right) \) by Lemma 4.2. Again Theorem 1.1, \( F \) is BD \( | \nu | - \) determined.

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