Some Remarks on the Operators’ Formalism for Nonlocal Poisson Brackets *

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Abstract

A common approach to the theory of nonlocal Poisson brackets, seen from the operatorial point of view, has been to keep implicit the sets on which these brackets act. In this paper we aim to explicitly define appropriate functional spaces underlying to the theory of 1 codimensional weakly nonlocal Poisson brackets, motivating the definitions, and to prove the validity in this context of some classical results in the field. We start by introducing the spaces for the local case, which will serve as building tools for those in the nonlocal one. The definition and the study of these nonlocal functionals are the core of this work; in particular we work out a characterization of the variational derivative of such objects. We then translate everything to the level of manifolds, defining a global version of the functionals, and introduce the notion nonlocal Poisson brackets in this context. We conclude by applying all the machinery to prove a theorem due to Ferapontov. This last application is the natural conclusion of our discussion and shows that the spaces we introduce are suitable objects to work with when studying topics in this theory.

Introduction

The theory of Poisson brackets over functional spaces has its roots in the work [1] of B.A. Dubrovin and S.P. Novikov, in which they studied the conditions for local brackets to be skew symmetric and to satisfy the Jacobi identity. A fundamental result of their work was understanding that these conditions have a differential geometric nature: such Poisson brackets acting on local functionals over a manifold $M$ are related to pseudo-Riemannian structures on the manifold. In [2], E.V. Ferapontov studied the same conditions for weakly nonlocal Poisson brackets (whose name comes from the work [3] of A.Ya. Maltsev and S.P. Novikov), finding an even richer bond with Riemannian geometry, that involves a link between the theory of these brackets and the theory of hypersurfaces of Euclidean spaces. In this context, computations were brought on without focusing much on specifying the functional spaces on which the theory was rooted, but with the intention of highlighting the links with differential geometry and the applications to mathematical physics. Here we try to fix a choice for such spaces. It’s definitely worth mentioning that there are alternative solutions to the problem of building a formal environment around the theory of these brackets. The most influential one comes from the work, completely based on abstract algebra, of A. De Sole and V.G. Kac (for example see [4]). For an overview of different computational

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techniques that can be used to prove the theorem of Ferapontov in this setting, two references are [5] and [6]. The aim of the present work is to show that it’s not necessary to pass through this more abstract algebraic formalism in order to study these objects from a rigorous point of view.

1 FIRST DEFINITIONS: LOCAL FUNCTIONALS

We first introduce a class of functionals that play a role in the theory of local Poisson brackets [1]. We will denote with \( \mathcal{S}_n \) the linear metric space of functions from \( \mathbb{R} \) to \( \mathbb{R}^n \) whose components are Schwartz functions. We’ll consider the linear integral operator \( I : \mathcal{S}_1 \to \mathbb{R} \) defined as

\[
I[h] = \int_{\mathbb{R}} h(x) \, dx
\]

It is clearly well defined and bounded.

**Definition 1.1.** Let \( N \) be a natural number. A function \( f : \mathcal{S}_n \to \mathcal{S}_1 \) is said to be \( N \)-local (or \( N \)-local) if there exists \( \phi \in C^\infty(\mathbb{R}^{1+n(N+1)}) \) such that

- \( f(u)(x) := \phi(x, u(x), u^{(1)}(x), ..., u^{(N)}(x)) \) \( \forall x \in \mathbb{R}, u \in \mathcal{S}_n \).
- Fixed a bounded \( B \subset \mathbb{R}^{n(N+1)} \), for each partial derivative \( \psi \) (of any order) of \( \phi \), we have
  \[
  \sup_{(x,y) \in \mathbb{R} \times B} |\psi(x,y)| < +\infty
  \]

For every \( i \in \{0, ..., N\} \), \( j \in \{1, ..., n\} \) we define

\[
\frac{\partial f}{\partial u_{i}^{(j)}} : \mathcal{S}_n \to C^\infty(\mathbb{R}, \mathbb{R}^n)
\]

such that:

\[
\frac{\partial f}{\partial u_{i}^{(j)}}(v)(x) := \frac{\partial \phi}{\partial u_{i}^{(j)}}(x, v(x), v'(x), ..., v^{(N)}(x))
\]

An analogous definition is given for derivatives of higher order. A functional \( F : \mathcal{S}_n \to \mathbb{R} \) is said to be \( N \)-local if \( F = I \circ f \) where \( f \) is an \( N \)-local function. In this case we will write \( F \in \mathcal{L}^N \).

For local functionals, the Gateaux differential exists and takes a particular well known form:

**Theorem 1 (Euler-Lagrange formula).** Pick \( v \in \mathcal{S}_n \), \( F \in \mathcal{L} \) and let \( f \) be the \( N \)-local function associated to \( F \). Consider the \( C^\infty(\mathbb{R}, \mathbb{R}^n) \) function defined by

\[
\frac{\delta F}{\delta v_{j}}(x) := \sum_{i=0}^{N} (-d/dx)^i(\frac{\partial f}{\partial u_{i}^{(j)}})(v)(x)
\]

(1)

called variational derivative of \( F \) in \( u \). \( F \) is \( G \)-differentiable and

\[
d_{G}F(u)[h] = \int_{\mathbb{R}} \frac{\delta F}{\delta u}(x)h(x) \, dx
\]

(2)

holds for each \( h \in \mathcal{S}_n \).
2 ADDING NONLOCALITY: WEAKLY NONLOCAL FUNCTIONALS

In order to introduce a slightly bigger class of functionals, we consider the linear operator \( d^{-1} : \mathcal{S} \to \mathcal{C}_0^\infty(\mathbb{R}) \) defined by

\[
d^{-1}(f)(x) := \frac{1}{2} \int_{-\infty}^{x} f(z) dz - \frac{1}{2} \int_{x}^{+\infty} f(z) dz
\]

(3)

This operator is well defined by convergence of the integrals, due to the basic properties of Schwartz functions. Let’s highlight three properties of this object:

- Let \( f \in \mathcal{S} \). Then \( d^{-1}(f) \) is an antiderivative of \( f \). More precisely, it’s the antiderivative that at \(-\infty\) tends to \(-\frac{1}{2} \int_{\mathbb{R}} f dx\).
- Let \( f, g \in \mathcal{S} \). Then \( \xi := d^{-1}(f)d^{-1}(g) \) is such that

\[
\lim_{x \to +\infty} \xi(x) - \lim_{y \to -\infty} \xi(y) = 0
\]

and

\[
\xi'(x) = d^{-1}(f)g + fd^{-1}(g)
\]

(4)

- Let \( f, g \in \mathcal{S} \). Then \( f \cdot d^{-1}(g) \in \mathcal{S} \).

The third property allows us to give the following definition.

**Definition 2.1.** Consider the linear subspaces of the set of functions \( \mathcal{S} \to \mathcal{S} \) defined inductively by

\[
\mathcal{D}_0 := \{ \text{local functions} \} \quad \text{and} \quad \mathcal{D}_m := \text{span}_{\mathbb{R}} \mathcal{D}_m
\]

where

\[
\mathcal{D}_m := \left\{ g \prod_{\alpha=1}^{A} d^{-1}(h_\alpha) \mid g, h_\alpha \in \mathcal{D}_{m-1} \quad A \in \mathbb{N} \right\}
\]

for each \( m > 0 \). Let \( \mathcal{D} \) be their union. We call weakly nonlocal (WNL) functional every functional of the form \( I \circ f \) where \( f \in \mathcal{D} \). We will write:

\[
\mathcal{W}_m := I \circ \mathcal{D}_m \quad ; \quad \mathcal{W}_m := I \circ \mathcal{D}_m \quad ; \quad \mathcal{W} := I \circ \mathcal{D}
\]

**Remark 1.** In the definition above, the case \( A = 0 \) is not excluded, hence we have \( \mathcal{D}_i \subset \mathcal{D}_j \) whenever \( i < j \).

**Remark 2.** For a functional \( F \in \mathcal{W}_m \) we can find an explicit representative for its density: it will be of the form

\[
g \prod_{\alpha=1}^{A} d^{-1} \left( \cdots (h_{a_1, \ldots, a_n} A_{a_1, \ldots, a_n} \prod_{\alpha_{n+1}=1}^{A} d^{-1} (h_{a_1, \ldots, a_{n+1}}) \cdots) \right)
\]

(5)

where \( g \) and all the \( h \)'s are local functions and some of the \( A \)'s can be zero.

We now want to extend the formula for the variational derivative to these new functionals. First of all, we consider the simplest nonlocal case: the one of \( \mathcal{W}_1 \). Let’s remark that the following version of the Leibniz rule holds as a consequence of the Taylor formula.
Lemma 1. Consider \( F \in \mathcal{F}_1 \). It’s \( G \)-differentiable and
\[
d_G F(u)[k] = \int_{\mathbb{R}} \left( \sum_{i=0}^{N} \frac{\partial g_i}{\partial u_i}(u) \cdot k^{(i)}(u) \right) \prod_{\alpha=1}^{A} (d^{-1} h_\alpha(u)) dx
+ \sum_{\alpha=1}^{A} \int_{\mathbb{R}} g(u) d^{-1} \left( \sum_{i=0}^{M} \frac{\partial h_\alpha}{\partial u_i}(u) \cdot k^{(i)}(u) \right) \prod_{\beta \neq \alpha} (d^{-1} h_\beta(u)) dx
\]

Proof. Let’s consider the case \( n = A = 1 \); the general case is proven analogously. We have \( F := I \circ (g \cdot d^{-1} h) \) and let \( \phi \in C^\omega(\mathbb{R}^{2+\mathcal{N}}) \), \( \psi \in C^\omega(\mathbb{R}^{2+\mathcal{M}}) \) the N-local and M-local functions associated to \( g \) and \( h \) respectively. Let’s write the limit defining the \( G \)-differential w.r.t. these two functions and develop the factors through the Taylor formula with Lagrangian reminder of the second order:

\[
\int_{\mathbb{R}} \left( \sum_{i=0}^{M} \frac{\partial g_i}{\partial u_i}(u) k^{(i)}(u) \right) d^{-1} h(u) dx + \int_{\mathbb{R}} g(u) d^{-1} \left( \sum_{i=0}^{N} \frac{\partial h}{\partial u_i}(u) k^{(i)}(u) \right) dx
+ \lim_{t \to 0} \frac{1}{2} \int_{\mathbb{R}} \left( \sum_{i,j=0}^{N} \frac{\partial^2 \phi}{\partial u_i \partial u_j}(x,y)(x) \right) k^{(i)}(u) k^{(j)}(u) d^{-1} h(u) dx
+ \lim_{t \to 0} \frac{1}{2} \int_{\mathbb{R}} g(u) d^{-1} \left( \sum_{i,j=0}^{M} \frac{\partial^2 \psi}{\partial u_i \partial u_j}(x,z)(x) \right) k^{(i)}(u) k^{(j)}(u) dx + ...
\]

The three dots hide four more terms that can be easily treated in the same way of the two explicitly written. To conclude the proof it’s enough to show that the two integrals in the limits are bounded by a constant when \( y_t \) and \( z_t \) vary. First of all notice that
\[
s^{ij}(u) := k^{(i)} k^{(j)} d^{-1} h(u)
\]
is an \( \mathcal{F}_1 \) function. The crucial fact is that \( y_t(x) \) and \( z_t(x) \), for each \( t \) and \( x \), always belong to the bounded set
\[
B := \prod_{i=0}^{N} \left( \text{Im}(u^{(i)}) + B_0(\|k\|_\omega) \right)
\]

So thanks to the boundedness property of derivatives of local functions there are \( M^{ij}(B) \) positive reals such that for each \( t \in (-1,1) \)
\[
\left| \sum_{i,j=0}^{N} \frac{\partial^2 \phi}{\partial u_i \partial u_j}(x,y) s^{ij}(u)(x) \right| \leq \sum_{i,j=0}^{N} M^{ij}(B) |s^{ij}(u)(x) |
\]
and the last function is in \( L^1(\mathbb{R}) \). This shows that the first limit is zero. The situation for the second one is very similar and can easily be recovered adapting the argument above. \( \square \)

This lemma brings us to the following result, which gives, combined with the Euler-Lagrange formula, the general form of the \( G \)-differential of a \( \mathcal{F}_1 \) functional (by linearity this extends to every element of \( \mathcal{F}_1 \)).
Theorem 2. Let $F \in \mathfrak{F}_1$ and consider $v \in \mathcal{S}_n$. Then $F$ is G-differentiable and defined 
\[ \frac{\delta F}{\delta v(x)} := R(x) + \sum_{\alpha=1}^{\beta} T_\alpha(x) \]
with
\[ R := \sum_{i=1}^{N} \left( -\frac{d}{dx} \right)^{i} \left[ \frac{\partial g}{\partial u^{(i)}}(v) \cdot \prod_{\alpha=1}^{A} d^{-1}(h_{\alpha}(v)) \right] \]
\[ T_\alpha := -\sum_{k=0}^{M_\alpha} \left( -\frac{d}{dx} \right)^{k} \left[ d^{-1}\left( g(v) \cdot \prod_{\beta \neq \alpha} d^{-1}(h_{\beta}(v)) \right) \frac{\partial h_{\alpha}}{\partial u^{(k)}}(v) \right] \]
we have
\[ d_{\alpha} F(u)[k] = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} k(x) dx \]
for each $k \in \mathcal{S}_n$. In this formula $M_\alpha$ is the order of the local function $h_{\alpha}$ and $N$ the one of $g$.

Proof. As we did before, we work out the proof for the case $n = 1$. This result follows as a consequence of the integration by parts of the integrals appearing in the statement of the previous lemma. From the first integral we quickly find $R$, so we consider the latter. Integrating it by parts we get (considering each addendum alone)
\[ \int_{\mathbb{R}} g(u) d^{-1}\left( \frac{\partial h_{\alpha}}{\partial u^{(i)}}(u) \cdot k^{(i)} \right) \prod_{\beta \neq \alpha} (d^{-1} \circ h_{\beta}(u)) dx \]
\[ = d^{-1}\left( \frac{\partial h_{\alpha}}{\partial u^{(i)}}(u) \cdot k^{(i)} \right) \prod_{\beta \neq \alpha} (d^{-1} \circ h_{\beta}(u)) \bigg|_{-\infty}^{+\infty} \]
\[ - \int_{\mathbb{R}} d^{-1}\left( g(u) \prod_{\beta \neq \alpha} (d^{-1} \circ h_{\beta}(u)) \right) \frac{\partial h_{\alpha}}{\partial u^{(i)}}(u) \cdot k^{(i)} dx \]
where the boundary term vanishes. Integrating by parts $i$-times lowering the order of the derivative of $k$ we get $T_\alpha$. \[ \square \]

With a completely analogous proof using the representation \( \delta \), one finds the G-differentiability of general WNL functionals and obtains a formula for their variational derivative. In order to keep a readable notation without losing any conceptual point, we write this formula only for functionals $F$ having density of the type
\[ g \prod_{\alpha=1}^{A} d^{-1} \left( h_{\alpha,1} d^{-1} \left( h_{\alpha,D_{a-1}} d^{-1} \left( h_{\alpha,D_{a}} \right) \right) \right) \]
for such a functional we obtain
\[ \frac{\delta F}{\delta v(x)} := R(x) + \sum_{\alpha=1}^{\beta} T_{\alpha}(x) \]
where (omitting all evaluations in $v$)
\[ R := \sum_{i=1}^{N} \left( -\frac{d}{dx} \right)^{i} \left[ \frac{\partial g}{\partial u^{(i)}}(v) \cdot \prod_{\alpha=1}^{A} H_{\alpha} \right] \]
\[ T_{\alpha} := (-1)^{\delta} \sum_{k=0}^{M_{\alpha}^{\delta}} \left( -\frac{d}{dx} \right)^{k} \left[ \frac{\partial H_{\alpha}}{\partial u^{(k)}}(v) \cdot \prod_{\beta \neq \alpha} H_{\beta} \right] \]
\[ \left. \frac{\partial h_{\alpha,\delta}}{\partial u^{(k)}}(v) \right|_{\delta+1} \]

5
where we have defined

\[ \hat{H}_{\delta}^{\gamma} := d^{-1} (h_{\alpha,\delta d^{-1} (h_{\alpha, D_{\alpha}}^{-1} (h_{D_{\alpha}})) \ldots}) \]

\[ \hat{H}_{\delta}^{\gamma} (\ast) := d^{-1} (h_{\alpha,\delta^{-1} d^{-1} (h_{\ast})} \ldots) \]

\[ H_{\alpha} := \hat{H}_{\alpha}^{1} \]

Here \( N \) is the order of \( g \) and \( M_{\delta}^{\gamma} \) the one of \( h_{\alpha, \delta} \). Thanks to these computations we get the following

**Corollary 1.** Let \( F \in \mathcal{W} \). Then its variational derivative w.r.t. every \( v \in S_{n} \) is bounded.

**Proof.** For simplicity, we will work out the proof for functionals having densities of the form (6) and for \( n = 1 \). In the whole computation we omit the evaluation at \( v \) of all the local functions. Consider first the part given by \( R \). We claim that

\[ \left( \frac{d}{dx} \right)^{i} \left[ \frac{\partial g}{\partial u_{(j)}} \prod_{\alpha=1}^{A} H_{\alpha} \right] \]

is bounded. We have by Leibniz rule that the expression above is equal to

\[ \sum_{j=0}^{i} \binom{i}{j} \left( \frac{\partial g}{\partial u_{(j)}} \prod_{\alpha=1}^{A} H_{\alpha} \right)^{(i-j)} \]

Now \( (\partial g/\partial u_{(j)})^{(j)} \), by chain rule, can be written as finite sum of partial derivatives of \( g \), some of which are multiplied by a derivative of \( v \). We have by definition that those partial derivative are bounded, and \( u \) is a Schwartz function, so the whole sum is bounded. On the other hand the term \( (\prod_{\alpha=1}^{A} H_{\alpha})^{(i-j)} \) is bounded too, as \( \prod_{\alpha=1}^{A} H_{\alpha} \) is a product of bounded functions with bounded derivatives of all orders. This holds because

\[ \frac{d}{dx} H_{\alpha} = h_{\alpha, 1} d^{-1} (\ldots (h_{\alpha, D_{\alpha}^{-1} d^{-1} (h_{D_{\alpha}}) \ldots) \right) \in \mathcal{S}_{n} \]

by the third property of \( d^{-1} \) highlighted after its definition. For what concerns \( T_{\delta}^{\gamma} \), the argument for proving that it’s bounded is essentially the same we used above for \( R \). \( \Box \)

### 3 FROM LOCAL TO GLOBAL: FUNCTIONALS ON MANIFOLDS AND POISSON BRACKETS

Let’s now give a global interpretation to these functionals. To do such a thing, we need to shift our attention a bit toward geometry, looking at manifolds modeled on infinite dimensional spaces. The theory of such manifolds is very rich and well studied [7], while in this work we just need some elementary constructions. So, we describe them explicitly in this section.

**Definition 3.1.** Let \( \Omega \in \mathbb{R}^{n} \) an open neighborhood of the origin. We define the open \( \mathcal{S} (\Omega) \subset \mathcal{S} \) of the Schwartz functions having image in \( \Omega \). We call local functions on \( \Omega \) the restrictions of local functions to \( \mathcal{S} (\Omega) \), and local functionals on \( \Omega \) as compositions of the integral functional \( I \) with a local function on \( \Omega \). Analogously are defined the WNL functionals on \( \Omega \) as before where all the local functions appearing in the previous definitions are now local functions on \( \Omega \). The spaces of these functionals will be denoted \( \mathcal{L} (\Omega) \) and \( \mathcal{W} (\Omega) \).
Let $M$ be a smooth connected manifold and fix $y \in M$. We want this $y$ to play the role of the origin in our manifold, following the approach outlined in [3]. Let $\mathcal{A} := \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$, the subset of the differential structure of $M$ such that $y \in U_\lambda$ and $\varphi_\lambda(y) = 0$ for each $\lambda \in \Lambda$.

**Remark 3.** We point out that the submanifold of $M$ covered by $\mathcal{A}$ is the the whole $M$ itself. This can be shown using tubular neighborhoods of the (embedded) path connecting each chosen point to $y$. So $\mathcal{A}$ is an atlas for our manifold.

First of all, some notation: we will write $\tilde{U}_\lambda$ instead of $\varphi_\lambda(U_\lambda)$ and $S_\lambda$ instead of $S(\tilde{U}_\lambda)$. In the same way we will write $L_\lambda := L(\tilde{U}_\lambda)$ and $W_\lambda := W(\tilde{U}_\lambda)$. Now consider the following gluing data:

1. We have some sets $\{S_\lambda\}_{\lambda \in \Lambda}$.
2. For each pair $\lambda, \mu \in \Lambda$ we have a subset $S_{\lambda \mu} := \{u \in S_\lambda : \varphi_\lambda^{-1} \circ u(\mathbb{R}) \subset U_\mu\} \subseteq S_\lambda$. Moreover there is a bijection $\varphi_{\lambda \mu} : S_{\lambda \mu} \to S_{\mu \lambda}$ s.t. $u \mapsto \varphi_\mu \circ \varphi_\lambda^{-1} \circ u$.

   Why is this map well defined? It’s easy to see that given $\lambda, \mu \in \Lambda$ and $u : \mathbb{R} \to U_\lambda \cap U_\mu$ such that $\varphi_\lambda \circ u \in S_\mu$, then $\varphi_\mu \circ u \in S_\lambda$. In fact, the composition of a Schwartz function with a diffeomorphism which fixes the origin is still a Schwartz function by chain rule.

   Moreover, the map above is a bijection because it has an inverse, namely $\varphi_{\mu \lambda}$.

3. These maps satisfy the cocycle conditions: for each triplet $\lambda, \mu, \kappa \in \Lambda$ we have $\varphi_{\lambda \lambda} = 1_{S_\lambda}$ and the following diagram commutes

$$
\begin{array}{ccc}
S_{\lambda \mu} \cap S_{\kappa \lambda} & \xrightarrow{\varphi_{\lambda \mu}} & S_{\kappa \lambda} \\
\varphi_{\mu \kappa} & \downarrow & \varphi_{\kappa \mu} \\
S_{\mu \kappa} \cap S_{\mu \lambda} & \xrightarrow{\varphi_{\mu \kappa}} & S_{\mu \lambda}
\end{array}
$$

This allows us to build a gluing of this data. Namely, we give the following definition:

$$M^* := \bigsqcup_{\lambda \in \Lambda} S_\lambda / \sim$$

where $(u)_\lambda \sim (w)_\mu$ if and only if $u \in S_{\lambda \mu}$, $w \in S_{\mu \lambda}$ and $w = \varphi_{\mu \lambda}(u)$.

**Remark 4.** The one we just defined is certainly an equivalence relation thanks to the properties written in the third point above. Notice that at this point we only have a structure of set on $M^*$ (even though the topology can be easily chosen to be the quotient topology of the disjoint union topological space w.r.t. our relation $\sim$), and this is all we will need in this paper.

Intuitively, our definition of “global functional” over $M$ will be the one of a function on the disjoint union defined above that passes through equivalence.

**Definition 3.2.** Consider a family of local functionals $\{F_\lambda \in L_\lambda\}_{\lambda \in \Lambda}$ such that the function

$$F : \bigsqcup_{\lambda \in \Lambda} S_\lambda \to \mathbb{R} \quad \text{s.t.} \quad u_\lambda \mapsto F_\lambda(u_\lambda)$$
passes through equivalence ∼. Then the quotient map \( \tilde{F} : M^* \to \mathbb{R} \) is called local functional on \( M \). WNL functionals on \( M \) are defined in a completely analogous way. The spaces of these functionals will be denoted by \( \mathcal{L}(M) \) and \( \mathcal{W}(M) \) respectively.

In what follows we will identify these functionals with the families that define them through equivalence.

**Remark 5.** Notice that \( \mathcal{L}(M) \) and \( \mathcal{W}(M) \) are two \( \mathbb{R} \)-linear spaces w.r.t. the operations defined at the level of the families that define the functionals by passing to equivalence.

**Definition 3.3.** We will call WNL Poisson bracket over \( M \) fixed by a map \( \{ \cdot, \cdot \} : \mathcal{W}(M) \times \mathcal{W}(M) \to \mathcal{W}(M) \) which is bilinear and satisfies the following two identities:

\[
\{ F, G \} = -\{ G, F \}
\]

\[
\{\{ F, G \}, H \} + \{\{ G, H \}, F \} + \{\{ H, F \}, G \} = 0
\]

for each \( F, G, H \in \mathcal{W}(M) \). Moreover, we require it to have the form

\[
\{ F, G \}_\lambda(u) := \int_{\mathbb{R}^n} \frac{\partial F}{\partial u_i(x)} \left( P_{ij}^\lambda(u) \frac{\partial G}{\partial u_j}(x) \right) dx
\]

(7)

where

- \( P_{ij}^\lambda(u) \), given \( u \in \mathcal{S}(\tilde{U}_\lambda) \), is the operator \( C^\infty(\mathbb{R}, \mathbb{R}^n) \to C^\infty_b(\mathbb{R}, \mathbb{R}^n) \) defined by

\[
P_{ij}^\lambda(u) := g_{ij}^\lambda(u)^{ik} \frac{d}{dx} - g_{ij}^\lambda(u)^{ik} \Gamma^l_{kl} u_k^l + w_{ij}^\lambda(u)^{ik} u_k^l - w_{ij}^\lambda(u)^{ik} u_k^l +
\]

- \( g_{ij}^\lambda, w_{ij}^\lambda \in C^\infty(\tilde{U}_\lambda, \mathbb{R}^{n \times n}) \), \( \Gamma^l_{kl} \in C^\infty(\tilde{U}_\lambda, \mathbb{R}^{n \times n \times n}) \) are such that the matrix \( g_{ij}^\lambda(u_1, \ldots, u_n) \) is in \( GL_n(\mathbb{R}) \) for each \( (u_1, \ldots, u_n) \in \tilde{U}_\lambda \).

for each \( \lambda \in \Lambda \).

**Remark 6.** The term "weakly nonlocal" comes from the work [3] of A.Ya. Maltsev and S.P. Novikov.

First of all, being the variational derivative of a WNL functional bounded, one obtains that the integrals in (7) are convergent. This means that fixed \( F, G \in \mathcal{W}(M) \) our bracket gives a well defined map

\[
\{ F, G \} : \bigsqcup_{\lambda \in \Lambda} S_\lambda \to \mathbb{R}
\]

Now we have to check which conditions on the elements \( (g, \Gamma, w) \) defining the bracket allow us to pass to equivalence. In the case of these brackets, the following well known geometric characterization holds:

**Proposition 1.** A family of maps of the form (7) defines a map \( \mathcal{W}(M) \times \mathcal{W}(M) \to \mathcal{W}(M) \) if and only if the families \( \{ g_{ij}^\lambda, \Gamma_{ij}^{kl}, w_{ij}^\lambda \}_{\lambda \in \Lambda} \) define on \( M \) a (2,0) tensor field, a connection and a (1,1) tensor field respectively.

The proof of this result is just a computation and is therefore omitted (see for example [2]).
4 FERAPONTOV’S THEOREM

The next part of this work is devoted to showing how this precise choice of the functional spaces allows us to prove in a simple way this theorem due to Ferapontov [2].

Theorem 3. A bracket of the form (7) defines a Poisson bracket if and only if its coefficients define on M a pseudometric g, its Levi Civita connection Γ and the Gauss and Peterson-Codazzi-Mainardi equations hold.

The nature of this topic is local, so we will assume to be working on a fixed $U_\lambda$ without specifying it anymore. We will denote with $\Omega$ the open $\tilde{U}_\lambda$. To simplify the notation a bit we’ll denote the derivation w.r.t. $x$ with $'$.

Lemma 2. Consider a bracket $\{\cdot, \cdot\}$ of the form (7) and assume the skew-symmetry and the Jacobi identity hold for local functionals of the form

$$F(u) := \int_\mathbb{R} \alpha_i(x)u'(x)dx$$

where $\alpha_i \in C^\infty_b(\mathbb{R})$. Then the bracket is a WNL Poisson bracket.

Proof. First of all, notice that applying the Euler-Lagrange formula to such an $F$ we get

$$\frac{\delta F}{\delta u_i(x)} = \alpha_i(x)$$

Let $F, G \in \mathcal{W}(\Omega)$ and fix $w \in \mathcal{L}(\Omega)$. If we define $\tilde{F}, \tilde{G} \in \mathcal{L}(\Omega)$ as

$$\tilde{F}(u) := \int_\mathbb{R} \frac{\delta F}{\delta w_j(x)}u'(x)dx$$

$$\tilde{G}(u) := \int_\mathbb{R} \frac{\delta G}{\delta w_j(x)}u'(x)dx$$

we have that $\tilde{F}, \tilde{G}$ are of the form (8) and

$$\{F, G\}[w] = \int_\mathbb{R} \frac{\delta F}{\delta w_j(x)}P^{ij}[w]\frac{\delta G}{\delta w_j(x)}dx$$

$$= \int_\mathbb{R} \frac{\delta \tilde{F}}{\delta w_j(x)}P^{ij}[w]\frac{\delta \tilde{G}}{\delta w_j(x)}dx = \{\tilde{F}, \tilde{G}\}[w]$$

the same argument holds for $\{G, F\}$, so

$$\{F, G\}[w] = \{\tilde{F}, \tilde{G}\}[w] = -\{\tilde{G}, \tilde{F}\}[w] = -\{G, F\}[w]$$

Being $w$ arbitrary the thesis for the skew-symmetry follows. For the Jacobi identity see [2].

We will use many times the following classical lemma, which we will state in a weak form.

Lemma 3 (Variational Lemma). Let $g \in C^0(\mathbb{R})$ and assume $\int_\mathbb{R} fg dx = 0$ for any $f \in C^\infty_b(\mathbb{R})$. Then $g = 0$.

The following result is an immediate application of what we have found above.

Theorem 4. A bracket of the form (7) is skew-symmetric iff g defines a pseudometric on M and the connection $\Gamma$ is compatible with g.
From now on we’ll denote with $F$, $G$ and $H$ functionals of the form

$$F(u) := \int_{\Omega} f_i(x) u^i(x) dx, \quad G(u) := \int_{\Omega} g_j(x) u^j(x) dx,$$

$$H(u) := \int_{\Omega} h_l(x) u^l(x) dx$$

where $f_i$, $g_j$ and $h_l$ belong to $C^\infty_p(\Omega)$. Moreover, we will use the following notation:

$$f := d^{-1} \left( w_k^i u^k_i f_i \right), \quad \tilde{g} := d^{-1} \left( w_k^i u^k_i g_i \right), \quad \tilde{h} := d^{-1} \left( w_k^i u^k_i h_i \right)$$

Thanks to our formula for the variational derivative of a $\hat{\mathcal{F}}_1$ functional, a straightforward computation gives the following result for a skew-symmetric bracket of the form (7):

$$\frac{\delta \{F,G\}}{\delta u^p} = f_i^j g^i_{kp} \Gamma^j_{ip} - f_i^j g^i_{kp} g_{jk} + \tilde{g}_i^j \Gamma^i_{jk} \Gamma^j_{ip} + \tilde{g}_i^j g_{jk} \Gamma^i_{jk} - \tilde{g}_i^j \Gamma^i_{jk} \Gamma^j_{ip} + f_i^j \left( \frac{\partial w_k^i}{\partial u^p} - \frac{\partial w_k^j}{\partial u^p} \right) g_{ij}$$

where $R^i_{jk} := g^{is} \left( \frac{\partial \Gamma^i_{jk}}{\partial u^s} - \frac{\partial \Gamma^i_{js}}{\partial u^k} + \Gamma^i_{js} \Gamma^j_{kp} - \Gamma^i_{jk} \Gamma^j_{ip} \right)$.

We are now ready to give a proof of Theorem 3. What remains to prove is that a skew-symmetric bracket satisfies the Jacobi identity iff

$$\Gamma^i_{jk} = \Gamma^j_{ki} \quad (9)$$

$$R^i_{jk} = w_p^i w_k^j - w_k^i w_p^j \quad (10)$$

$$w_p^i g^{ij} = w_p^j g^{ji} \quad (11)$$

$$\nabla_p w_k^i = \nabla_k w_p^i \quad (12)$$

where $R$ is the Riemann tensor of $(M,g)$ and $\nabla$ is its Levi Civita connection.

Proof. In this proof we will write $\partial_p$ instead of $\frac{\partial}{\partial u^p}$. With the symbol $\circ_{\alpha\beta\gamma}$ we’ll denote the sum over the cycles of $S_3$ applied to $(\alpha\beta\gamma)$. So the Jacobi identity is written $\circ_{FGH} \{\{F,G\},H\} = 0$. For brackets of the form (7) this identity translates to

$$\int_{\Omega} \circ_{FGH} \left[ \frac{\delta \{F,G\}}{\delta u^p} \left( g^{pi} h_i^l - g^{pi} \Gamma^i_{kl} u_k^l + w_k^i u^l_i h_l \right) \right] dx = 0$$

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Thanks to the previous calculation it's easy to compute the integrand above, which is

\[- f_i g_j h u \delta^i_k \delta^j_k + f_i g_j h u \delta^i_k b^j_k + f_i g_j h u \delta^i_k b^j_k + f_i g_j h u \delta^j_k b^i_k \]

\[+ \tilde{f}_i g_j h u \delta^i_k \delta^j_k + f_i \tilde{g}_j h u \delta^i_k \delta^j_k + f_i g_j \tilde{h} u \delta^i_k \delta^j_k \]

\[+ f_i \tilde{g}_j h u \delta^i_k \delta^j_k + f_i g_j h u \delta^j_k \delta^i_k \]

\[+ f_i \tilde{g}_j h u \delta^j_k \delta^i_k - f_i g_j h u \delta^j_k \delta^i_k \]

\[+ f_i \tilde{g}_j h u \delta^j_k \delta^i_k - f_i g_j \tilde{h} u \delta^j_k \delta^i_k \]

\[+ f_i \tilde{g}_j h u \delta^j_k \delta^i_k + f_i g_j h u \delta^j_k \delta^i_k \]

\[+ f_i \tilde{g}_j h u \delta^j_k \delta^i_k + f_i g_j h u \delta^j_k \delta^i_k \]

where

\[a^i_k := \left[ \partial_{ij} G \right]_{ij} G^{x} \left( \Gamma^i_{xp} \Gamma^j_{sk} - \Gamma^i_{xk} \Gamma^j_{sp} \right) \]

\[+ R^i_{pk} w^j_k + w^i_k w^j_p a^j_k \]

\[b^j_k := g^{ip} \left( \Gamma^i_{xp} \Gamma^j_{sk} - \Gamma^i_{xk} \Gamma^j_{sp} \right) g^{qj} \]

\[+ \left[ g^{ip} \left( \Gamma^i_{xp} \Gamma^j_{sk} - \Gamma^i_{xk} \Gamma^j_{sp} \right) + R^j_{pk} - w^j_k w^j_p \right] g^{ip} \]

\[c^j_k := \left( \partial_{wp} w^j_k - \partial_{wp} w^j_k \right) g^{ip} G^{i} \]

\[+ \left[ g^{ip} \left( \Gamma^i_{xp} \Gamma^j_{sk} - \Gamma^i_{xk} \Gamma^j_{sp} \right) + R^j_{pk} - w^j_k w^j_p \right] u^i_k w^j_p \]

\[d^j_k := g^{ip} \left( \partial_{wp} w^j_k - \partial_{wp} w^j_k \right) g^{ip} \]

\[e^j_k := w^j_k g^{ip} \Gamma^i_{ak} = \left( \partial_{wp} w^j_k - \partial_{wp} w^j_k \right) g^{ip} - g^{ip} \Gamma^i_{ak} \]

\[m^i_k := w^j_k g^{ip} + w^i_k w^j_p \]

In the whole computation we have omitted the evaluation in x and u(x). Then the proof follows from the following two claims:

1. For a skew symmetric bracket of the form we consider, the Jacobi identity holds iff

\[b^j_k(z) = d^j_k(z) = e^j_k(z) = m^j_k(z) = 0 \quad (13)\]

for each i, j, l, k \in \{1, \ldots, n\} and for each z \in \Omega.

2. The system (12) is equivalent to the system (9), (10), (11), (12).

Let's start from the second one: \((\Rightarrow)\) Using the symmetry of g and renaming two indices we can write 0 = d^j_k = g^{ip} \left( \Gamma^i_{ps} - \Gamma^i_{sp} \right) g^{pj}. In matricial form, defined A(l) := \left( \Gamma^i_{ps} - \Gamma^i_{sp} \right) p, s = 1, \ldots, n \in \mathbb{R}^{n \times n} this means gA(l)g = 0. By non degeneracy of the pseudo-metric g it follows A(l) = 0 for each l, which is (9). Now consider b^j_k; we can write it as

\[g^{ip} g^{pj} \left( \Gamma^{ij}_{sk}, \Gamma^{ij}_{sp} - \Gamma^{ij}_{sk}, \Gamma^{ij}_{sp} \right) + g^{ip} g^{aj} \left( \Gamma^{ij}_{ps} - \Gamma^{ij}_{sp} \right) \]

\[+ g^{ip} \left( R^{ip}_{pk} - w^j_k w^j_p \right) \]

\[+ g^{ip} \left( R^{ij}_{pk} - w^j_k w^j_p \right) \]
Consider the first two summands: renaming the indices so that \( g^{\alpha \beta} g^{\gamma \nu} \) is a common factor we get that, using (9), their sum is equal to

\[
g^{\alpha \beta} g^{\gamma \nu} \left( \Gamma^{j}_{ik} - \Gamma^{i}_{jk} \Gamma^{j}_{k\nu} \right) = g^{\alpha \beta} g^{\gamma \nu} \Gamma^{j}_{ik} - g^{\alpha \beta} g^{\gamma \nu} \Gamma^{i}_{jk} \Gamma^{j}_{k\nu}
\]

\[
= g^{\alpha \beta} g^{\gamma \nu} \Gamma^{j}_{ik} - g^{\alpha \beta} g^{\gamma \nu} \Gamma^{i}_{jk} \Gamma^{j}_{k\nu} = g^{\alpha \beta} \Gamma^{j}_{ik} \Gamma^{j}_{k\nu} (g^{\nu \gamma} - g^{\gamma \nu}) = 0
\]

So \( h^{ij}_k = 0 \) gives (10) by the usual non degeneracy of \( g \). Trivially \( m^{ij}_l = 0 \) is (11). Using this last equation and renaming a couple of indices we get that

\[
0 = e^{ij}_k = g^{lp} \left( \partial_p w^{ij}_k + \Gamma^{j}_{lp} w^{l}_k \right)
\]

\[
= g^{lp} \left[ \left( \partial_p w^{ij}_k + \Gamma^{j}_{lp} w^{l}_k - \Gamma^{l}_{kp} w^{l}_j \right) - \left( \partial_p w^{ij}_l + \Gamma^{l}_{kp} w^{l}_j - \Gamma^{l}_{jp} w^{l}_k \right) \right]
\]

which by non degeneracy of \( g \) is (12).

\((\Leftarrow)\) Is clear by looking at the definitions of \( b, d, e \) and \( m \).

Now let’s consider our first claim. \((\Leftarrow)\) This is the easiest implication of the two. It’s just a matter of checking that \( a^{ij}_k \) and \( c^{ij}_k \) are equal to zero for each \( u \). But this consists in doing computations completely analogous to the ones above, so we omit them.

\((\Rightarrow)\) If the Jacobi identity holds then the integral over \( \mathbb{R} \) of the function in the previous page has to vanish for any choice of \( f, g, h \in C^0_b(\mathbb{R}, \mathbb{R}^n) \) and \( u \in \mathcal{Q}(\Omega) \). First of all, let’s fix \( i, j, l \) and consider \( f, g, h \) having only one non zero component, respectively the \( i \)-th, \( j \)-th and \( l \)-th. So, we can erase the sum on those indices in the computation. In this part of the proof we assume that at least one of the functions \( w^{ij}_k \) is non zero; namely, there exists \( i, k \in \{1, \ldots, n \} \) and \( \bar{z} \in \Omega \) such that \( w^{ij}_k(\bar{z}) \neq 0 \). The case where the \( w \) are all zero is completely analogous, but simpler. We can regroup the terms and write the Jacobi identity in the following form:

\[
\int_{\mathbb{R}} (f \alpha + f' \beta + \tilde{f} \gamma) dx = 0 \quad \forall f \in C^0_b(\mathbb{R})
\]

Restricting to functions \( f \in C^0_b(\mathbb{R}) \) we can integrate by parts getting

\[
\int_{\mathbb{R}} f \left( \alpha - \beta' - w^{ij}_k u^i d^{-1}(\gamma) \right) dx = 0 \quad \forall f \in C^0_b(\mathbb{R})
\]

**Remark 7.** In the integration by parts of the third term we can neglect the boundary term by the second property of the operator \( d^{-1} \) we have highlighted just after its definition.

Now we apply the variational lemma:

\[
\alpha - \beta' - w^{ij}_k u^i d^{-1}(\gamma) = 0 \quad \forall g, h \in C^0_b(\mathbb{R}) \quad \forall u \in \mathcal{Q}(\Omega)
\]  

\((14)\)
Explicitly this equation is

\[-g_h u_k^{ij} \frac{\partial}{\partial z_k} + \left( g_h u_k^{ij} \right) + g'_h u_k^{ij} + g h' u_k^{ij}
- w_j u_l d^{-1} \left( g_h u_k^{ij} \right) + g_h u_k^{ij} + g h' u_k^{ij} \]

\[- \left( g' h d^{ij} \right) - \left( g h' d^{ij} \right) + g h' d^{ij} \]

\[- \left( g_h u_k^{ij} \right) + \left( g h' u_k^{ij} \right) + w_j u_l d^{-1} \left( g_h u_k^{ij} \right) \]

\[+ g_h u_k^{ij} - w_j u_l d^{-1} \left( g_h u_k^{ij} \right) - g h' u_k^{ij} \]

\[- \left( g' h' d^{ij} \right) - \left( g h' m^{ij} \right) - w_j u_l d^{-1} \left( g' h' m^{ij} \right) = 0 \]

Remark 8. In this part of the proof we will use the arbitrariness of \( u \in \mathcal{S}(\Omega) \), \( g, h \in C^\infty_b(\mathbb{R}) \) to choose particular functions that, plugged into (14), give us relations that will imply the claim. In particular, we will use that given any bounded subset \( B \subset \mathbb{R} \) and given any smooth function \( f : B \rightarrow \mathbb{R} \) there exists \( f^* \in \mathcal{A}_1 \subset C^\infty_b(\mathbb{R}) \) such that \( f^*_h = f \). This follows by the existence of bump functions.

From now on we consider a point \( z \) on our \( \Omega \). Consider \( u \in \mathcal{S}(\Omega) \) such that

\[ u(0) = z \quad , \quad u_x(0) = 0 \]  
\[ u(1) = \bar{z} \quad , \quad u_x(1) = \delta_k \quad \forall s \in \{1, ..., n\} \]

Plugging this \( u \) in (14) and evaluating at \( x = 0 \) we get

\[ g'' m^{ij} + g' h m^{ij} + g h' m^{ij} + g'' h d^{ij} + g' h' d^{ij} \]
\[+ g' h' d^{ij} + g h' d^{ij} - g h' d^{ij} = 0 \]

Now we choose \( g \) such that \( g(0) = g'(0) = 0 \) and \( g''(0) \neq 0 \). Then we have

\[ g'' m^{ij} + g h' m^{ij} + g'' h d^{ij} = 0 \]

What we have written until now holds for any choice of indices \( i, j, l \). Now choose \( l := l \). Then we can construct \( h \) such that \( \bar{h}(0) = h'(0) = h''(0) = 0 \) and \( \bar{h}(0) \neq 0 \) through the use of bump functions. For this choice of \( h \) our equation becomes \( m^{ij}(z) = 0 \) for each \( i, j \). Then the equation at the beginning of the page implies \( d^{ij} = 0 \) for every \( i, j, l \). Now the equation (14) is considerably simplified. Let’s fix \( k \) and consider another \( u \in \mathcal{S}(\Omega) \) such that

\[ u(0) = z \quad , \quad u_x(0) = 0 \]  
\[ u(1) = \bar{z} \quad , \quad u_x(1) = \delta_k \quad \forall s \in \{1, ..., n\} \]

The equation becomes

\[ g h b_k^{ij} + g h e_k^{ij} - g h e_k^{ij} = 0 \]

Fixing \( l := l \) we can choose \( h \) as above, so we get \( e_k^{ij}(z) = 0 \) for each \( i, j, k \), and hence \( b_k^{ij} = 0 \) for every \( i, j, l, k \).
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References

[1] B.A. Dubrovin & S.P. Novikov. *Hydrodynamics of weakly deformed soliton lattices*. Differential geometry and Hamiltonian theory. Russian Mathematical Surveys, Volume 44, pp 35, 1989.

[2] E.V. Ferapontov. *Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type*. Functional Analysis and its Applications, Volume 25, pp 195, 1991.

[3] A.Ya. Maltsev and S.P. Novikov. *On the local systems Hamiltonian in the weakly nonlocal Poisson brackets*. Physica D: Nonlinear Phenomena, Volume 156, pp 53-80, 2001.

[4] A. De Sole and V.G. Kac. *Non-local Poisson structures and applications to the theory of integrable systems*. Japanese Journal of Mathematics, Volume 8, pp 233, 2013.

[5] M. Casati, P. Lorenzoni and R. Vitolo. *Three computational approaches to weakly nonlocal Poisson brackets*. Preprint, [arXiv:1903.08204](https://arxiv.org/abs/1903.08204), 2019.

[6] P. Lorenzoni and R. Vitolo. *Weakly nonlocal Poisson brackets, Schouten brackets and supermanifolds*. Preprint, [arXiv:1909.07695](https://arxiv.org/abs/1909.07695), 2019.

[7] S. Lang. *Differential manifolds*. Springer, New York, NY, 1985.