Mixed Vs stable anti-Yetter-Drinfeld contramodules.

Ilya Shapiro

Abstract

We examine the cyclic homology of the monoidal category of modules over a finite dimensional Hopf algebra, motivated by the need to demonstrate that there is a difference between the recently introduced mixed anti-Yetter-Drinfeld contramodules and the usual stable anti-Yetter-Drinfeld contramodules. Namely, we show that the Sweedler’s Hopf algebra provides an example where mixed complexes in the category of stable anti-Yetter-Drinfeld contramodules (classical) are not DG-equivalent to the category of mixed anti-Yetter-Drinfeld contramodules (new).

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1 Introduction.

Cyclic (co)homology was introduced independently by Boris Tsygan and Alain Connes in the 1980s. It has since been generalized, applied to many fields, and developed many flavours. Our investigations in this paper focus on the equivariant flavour that began with Connes-Moscovici and was generalized into Hopf-cyclic cohomology by Hajac-Khalkhali-Rangipour-Sommerhärzer and Jara-Stefan (independently). Roughly speaking, the original theory defines cohomology groups for an associative algebra that play the role of the de Rham cohomology in the noncommutative setting. The equivariant version considers an algebra with an action of a Hopf algebra and it turns out that just as in the de Rham cohomology, one has coefficients in the Hopf setting; it is an interesting fact that unlike the de Rham setting, Hopf-cyclic cohomology requires coefficients, i.e., there are no canonical trivial coefficients. These coefficients are now called stable anti-Yetter-Drinfeld modules, due to their similarity to the usual Yetter-Drinfeld modules. It turns out that the more natural, from a conceptual point of view, version of coefficients are stable anti-Yetter-Drinfeld contramodules [2]. It is the desire to understand the coefficients themselves that motivated a series of papers by the author of the present one, of which this is a natural next step.

This paper is a descendant of [11] where it is shown that the classic stable anti-Yetter-Drinfeld contramodules are simply objects in the naïve cyclic homology category of $\mathcal{H} \mathcal{M}$, the monoidal category of modules over the Hopf algebra. It is furthermore conjectured that the correct coefficients (generalizing the classical ones) are obtained from the true cyclic homology category; this makes exact the analogy between de Rham and Hopf-cyclic
coefficients since the former are shown to be so in [1]. More precisely, in [11], a category of mixed anti-Yetter-Drinfeld contramodules is defined by analogy with the derived algebraic geometry case of [1]. This new generalization is conceptual, and furthermore allows the expression of the Hopf-cyclic cohomology of an algebra \( A \) with coefficients in \( M \) as an \( \text{Ext} \) (in this category) between \( ch(A) \), the Chern character object associated to \( A \), and \( M \) itself. Even if one takes \( M \) to be a classical stable anti-Yetter-Drinfeld contramodule, the object \( ch(A) \) is truly a mixed anti-Yetter-Drinfeld contramodule. It is conjectured that mixed anti-Yetter-Drinfeld contramodules are exactly the true cyclic homology category of \( _HM \).

The comparison in [11] between anti-Yetter-Drinfeld contramodules and the cyclic homology category of \( _HM \) involves a monad on \( _HM \) with a central element \( \sigma \) (giving the \( S^1 \)-action). It is this description that allows us here to reduce the investigations into the differences between the classical and the new Hopf-cyclic cohomology to the analysis of modules categories over two differential graded algebras (DGAs). Namely, in the notation of the paper, we have an algebra \( \hat{D}(H) \) whose modules are the anti-Yetter-Drinfeld contramodules, we have a DGA \( \hat{D}(H)[\theta] \) with \( d\theta = \sigma - 1 \) that yields the new mixed anti-Yetter-Drinfeld contramodules, and we have a DGA \( \hat{D}(H)/(\sigma - 1)[\theta] \) with \( d\theta = 0 \) that yields the classical setting, i.e., the mixed complexes in stable anti-Yetter-Drinfeld contramodules. Thus, it suffices for our purposes to compare the DG categories of modules over these two DGAs; we concentrate on finite dimensional \( H \) and show that if \( S^2 = Id \) then the DG categories coincide (Proposition 2.4), while if we consider the Sweedler’s Hopf algebra (the simplest case of \( S^2 \neq Id \)) then they do not (Proposition 3.6).

**Conventions:** All algebras \( A \) in monoidal categories are assumed to be unital associative. Our \( H \) is a Hopf algebra over some fixed algebraically closed field \( k \), of characteristic 0, and \( \text{Vec} \) denotes the category of \( k \)-vector spaces. For the purposes of this paper we are only interested in finite dimensional Hopf algebras. We use the following version of Sweedler’s notation: for \( h \in H \) we denote the coproduct \( \Delta(h) \in H \otimes H \) by \( h^1 \otimes h^2 \). Finally, DG stands for differential graded.

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## 2 Twisted Drinfeld double.

Let \( H \) be a Hopf algebra. From [11] we see that the study of the Hochschild and cyclic homologies of \( _HM \), the monoidal category of \( H \)-modules, reduces to the study of modules over a certain natural, from the considerations there, monad on \( _HM \). Recall that the consideration of Hochschild and cyclic homologies of monoidal categories is motivated by their recently discovered role in the understanding of Hopf-cyclic theory coefficients.

Briefly, we have the monad

\[
\text{Hom}_k(H, -) : _HM \rightarrow _HM
\]
with the $x \in H$ acting on $\varphi \in \text{Hom}_k(H, V)$ as

$$x \cdot \varphi = x^2 \varphi(S(x^3) - x^1).$$

The unit $1_V : Id(V) \to \text{Hom}_k(H, V)$ is

$$1_V(v)(h) = \epsilon(h)v$$

and a crucial central element (responsible for the $S^1$-action) $\sigma_V : Id(V) \to \text{Hom}_k(H, V)$ is

$$\sigma_V(v)(h) = hv.$$

The anti-Yetter-Drinfeld contramodules then coincide with modules over this monad, while the stable ones consist of those for which the action of $\sigma$ agrees with that of 1, and the mixed ones introduced in [11] are the homotopic version of this on the nose requirement.

In this section we will define an explicit DG-algebra that will yield the mixed anti-Yetter-Drinfeld (aYD) contramodules (for $H$ finite dimensional) as its DG-modules. The construction of the twisted convolution algebra below is analogous to the classical Drinfeld double $D(H)$ and its anti-version $D_a(H)$ [4] (we review these in Section 4 where we expand upon this comparison).

**Definition 2.1.** Let $H$ be a Hopf algebra, define a multiplication on $\hat{D}(H) := \text{End}(H)$ via

$$(f \ast g)(h) = f(h^1)g(S(f(h^1)^3)h^2f(h^1)^1),$$

thus the multiplicative identity 1 is

$$\epsilon(-)1,$$

and the central element $\sigma(h) = h$ is invertible with inverse $S^{-1}$.

The following lemma is immediate.

**Lemma 2.2.** Let $H$ be a finite dimensional Hopf algebra. Then

- anti-Yetter-Drinfeld contramodules over $H$ are the same as $\hat{D}(H)$-modules.
- The stable aYD contramodules are modules over $A := \hat{D}(H)/(\sigma - 1)$.
- The mixed aYD contramodules are DG-modules over $B := \hat{D}(H)[\theta]$ where $\theta$ is a degree $-1$ graded commutative variable and $d\theta = \sigma - 1$.

*Proof.* In the finite dimensional case $\text{End}(H)$ with (2.1) is the quotient of the free algebra, generated by $H^*$ and $H$, by the relation:

$$h\chi = \chi(S(h^3) - h^1)h^2,$$

where $h \in H, \chi \in H^*$. Thus modules over the algebra are both $H$-modules and $H$-contramodules (same as $H^*$-modules for $H$ finite dimensional) and the actions satisfy the requisite compatibility conditions. \qed

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Our main goal is to examine when, and more interestingly when not, the category of $B$-modules is DG-equivalent to $A[\theta]$-modules with $d\theta = 0$, i.e., to compare the category of mixed aYD contramodules to the category of mixed complexes of stable aYD contramodules. The study of Hopf-cyclic cohomology has thus far only concerned itself with the latter.

The following simple lemma takes care of a lot of cases.

**Lemma 2.3.** Let $H$ be a finite dimensional Hopf algebra and suppose that the action of $\sigma - 1$ on $\hat{D}(H)$ is diagonalizable. Then the categories of mixed complexes in stable aYD-contramodules and mixed aYD-contramodules are DG-equivalent.

**Proof.** Since the former category is formed by modules over $A[\theta]$ and the latter by $B$-modules (in the notation of Lemma 2.2), and since the action is diagonalizable, we have that $B$ is quasi-isomorphic to its cohomology which is $A[\theta]$. \qed

**Proposition 2.4.** Let $H$ be a finite dimensional Hopf algebra such that the square of the antipode is identity, i.e., $S^2 = Id$. Then the categories of mixed complexes in stable aYD-contramodules and mixed aYD-contramodules are DG-equivalent.

**Proof.** Since $S^2 = Id$, so $H$ is semi-simple \[1, 8\], so $D(H)$ (its Drinfeld double) is semi-simple \[9\]. By Section 4 we know that it follows that $\hat{D}(H)$ is semi-simple, and so the action of the central element $\sigma - 1$ is diagonalizable. Thus we are done by Lemma 2.3. \qed

In light of the above we need to consider an example of $H$ with $S^2 \neq Id$. It turns out that the simplest such example suffices.

### 3 Taft Hopf algebras.

Let $\xi$ be a primitive $p$th root of unity in $k$, where $p$ is a prime. The Taft Hopf algebra (sometimes called the quantum $sl_2$ Borel algebra) \[13\] $T_p(\xi)$ is generated as a $k$-algebra by $g$ and $x$ with the relations

\begin{align*}
g^p &= 1, \\
x^p &= 0, \\
gx &= \xi x g.
\end{align*}

Thus it is $p^2$ dimensional over $k$. Furthermore, the coalgebra structure is

\begin{align*}
\Delta(g) &= g \otimes g, \\
\Delta(x) &= x \otimes 1 + g \otimes x
\end{align*}

with $\epsilon(g) = 1$, $\epsilon(x) = 0$, and thus $S(g) = g^{-1}$, while $S(x) = -g^{-1}x$. Note that

\[S^2(x) = \xi^{-1}x \neq x,\]

making $T_2(-1)$ the smallest Hopf algebra with $S^2 \neq Id$. The Taft algebra $T_2(-1)$ is somewhat different from the other $T_p(\xi)$ and has its own name: Sweedler’s Hopf algebra.
Note that \( T_p(\xi) \) is not isomorphic to \( T_{p'}(\xi') \) unless \( p = p' \) and \( \xi = \xi' \). Moreover, as Hopf algebras
\[
T_p(\xi)^* \simeq T_p(\xi)
\]
which will be explored in greater detail presently.

**Remark 3.1.** Compare what we do below to the exercise in [3], though their Taft algebra is slightly different from ours: the Drinfeld double of the Taft algebra is \( u_q(sl_2) \otimes \mathbb{C}/p \) for \( p \) odd. Similarly, see [6], where the Taft algebra is called the quantum \( sl_2 \) Borel algebra, and its Drinfeld double is computed. In light of Section 4 our analysis of \( \hat{D}(H) \) can be interpreted as that of \( D(H) \), with \( \sigma \) being a new ingredient.

### 3.1 The identification with the dual.

We need some explicit formulas establishing the isomorphism \( T_p(\xi) \simeq T_p(\xi)^* \) and vice versa. Let \( \omega \) denote a \( p \)-th root of unity, let \( (n)_{\omega} = 1 + \cdots + \omega^{n-1} \) and \( (n)_{\omega}! = (n)_{\omega} \cdots (1)_{\omega} \).

The verification of the following is left to the reader; key details can be found in [10].

**Lemma 3.2.** Consider a natural basis of \( T_p(\xi) \):
\[
\{ g^i x^j \}_{i,j=0}^{p-1}
\]
so that \( \{ (g^i x^j)^* \} \) denotes the dual basis of \( T_p(\xi)^* \). Then the isomorphism of algebras and its inverse are given by
\[
g^i x^j \mapsto (j)_{\xi^{-1}}! \sum_l \xi^{(j+l)}(g^l x^j)^*
\]
and
\[
(g^i x^j)^* \mapsto \frac{1}{(j)_{\xi^{-1}}!} \sum_l \xi^{-l(i+j)}g^l x^j.
\]

**Corollary 3.3.** The twisted double \( \hat{D}(T_p(\xi)) \) is the quotient of
\[
k\langle x, x', g, g' \rangle
\]
by the relations
\[
x^p = x'^p = g^p = 1 = g'^p = 0,
\]
\[
 gg' = g' g, \quad gx = \xi x g, \quad g' x' = \xi x' g', \quad gx' = \xi^{-1} x g', \quad g' x = \xi^{-1} x g',
\]
and
\[
xx' - \xi^{-1} x' x = 1 - \xi^{-1} g'^{-1} g.
\]
The actions of \( g' \) and \( g \) on \( V \), a \( \hat{D}(T_p(\xi)) \)-module, yields a \( \mathbb{Z}/p^2 \)-grading on \( V \) by their eigenspaces, i.e., \( g', g \) act on \( V_{ij} \) by \( \xi^i, \xi^j \) respectively. Thus \( x \) and \( x' \) have degrees \((-1, 1)\) and \((1, -1)\) respectively. The \( S^1 \)-action of \( \sigma \) on \( V_{ij} \) is
\[
\sum_{l=0}^{p-1} \frac{\xi^{l(i-l)(j+l)}}{(l)_{\xi^{-1}}!} x^n x'.
\]
Proof. We use the identifications $\hat{D}(T_p(\xi)) = (T_p(\xi))^* \otimes T_p(\xi) \simeq T_p(\xi) \otimes T_p(\xi)$. We let $x' = x \otimes 1, x = 1 \otimes x, g' = g \otimes 1, \text{and } g = 1 \otimes g$. To derive the rest of the relations we apply (2.2). The action of $\sigma = \sum_{i,j} (g^i x^j)^* \otimes g^i x^j$ is computed on the graded components directly.

Observe that $gg' \in \hat{D}(T_p(\xi))$ is central and its action on $\hat{D}(T_p(\xi))$ is diagonalizable with eigenvalues $\xi^s, s \in \mathbb{Z}/p$. Thus as an algebra

$$\hat{D}(T_p(\xi)) = \bigoplus_s \hat{D}(T_p(\xi))/\langle gg' - \xi^s \rangle \tag{3.5}$$

so that it suffices to understand $\hat{D}(T_p(\xi))/\langle gg' - \xi^s \rangle$. There are two cases: $p = 2$ and $p > 2$. We will begin by briefly discussing the latter (though without addressing the $S^1$-action), and then concentrate our attention on the former (with examining the $S^1$-action) to achieve the goal set out in the abstract.

3.2 The case of $p \geq 2$.

Let $p > 2$, then

$$\hat{D}(T_p(\xi))/\langle gg' - \xi^s \rangle \simeq u_q(sl_2) \tag{3.6}$$

where $q$, which exists since 2 is invertible modulo $p$, is a $p$th root of unity such that

$$q^2 = \xi^{-1}.$$

More precisely, let

$$E = \frac{q^{s+1}}{q - q^{-1}} x', \quad F = xg', \quad \text{and } \quad K = q^{s+1} g,$$

so that $\hat{D}(T_p(\xi))/\langle gg' - \xi^s \rangle$ is generated by $E, F, K$ subject to

$$E^p = F^p = K^p - 1 = 0,$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad KEK^{-1} = q^2 E, \quad \text{and } \quad KFK^{-1} = q^{-2} F.$$

So that, unsurprisingly (see Appendix and Remark 3.1):

$$\hat{D}(T_p(\xi)) \simeq u_q(sl_2) \otimes \mathcal{O}_{\mathbb{Z}/p} \simeq u_q(sl_2) \otimes k\mathbb{Z}/p.$$

As we will see below the case of $p = 2$ is very different, in particular as $s$ varies, the algebra will change significantly whereas here it does not (3.6).

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3.3 The case of $p = 2$.

We need to describe the algebra $\hat{D}(T_2(-1))$ in greater detail, paying particular attention to the element $\sigma$.

By (3.5) we deal with the cases: $s = 0$ and $s = 1$. Thus $\hat{D}(T_2(-1))[\theta]$-modules is a product category: $\mathcal{C}_0 \times \mathcal{C}_1$ that will be dealt with in turn.

Observe that by the Corollary 3.3 we have that the anti-commutator is

$$\{x, x'\} = 1 + (-1)^s,$$

so that

$$(x')^2 = (1 + (-1)^s)x'x. \quad (3.7)$$

We see from (3.7) that the minimal polynomial of $x'x$ depends only on $s$; this is exclusive to $p = 2$ and makes this case tractable. Note that

$$\sigma|_{V_{00}} = 1 - x'x \quad \sigma|_{V_{11}} = -1 + x'x \quad \text{and} \quad \sigma|_{V_{01}} = \sigma|_{V_{10}} = 1 + x'x. \quad (3.8)$$

Let

$$A_s = \hat{D}(T_2(-1))/(\sigma^s - (1)^s).$$

We begin with $s = 0$:

**Lemma 3.4.** The category $\mathcal{C}_0$ consists of the usual mixed complexes (see [5] for the definition).

**Remark 3.5.** Note that not only does $\mathcal{C}_0$ consists of the usual mixed complexes but it also does not provide any evidence of the need for the mixed aYD-contramodules (see proof).

**Proof.** The action of $\sigma - 1$ on $A_0$ is diagonalizable (by (3.7) and (3.8)) and so by the proof of Lemma 2.3 the algebras $A_0[\theta]$ (recall that $d\theta = \sigma - 1$ here) and $A_0/(\sigma - 1)[\theta]$ (recall that $d\theta = 0$ here) are quasi-isomorphic. Note that $A_0/(\sigma - 1)$-modules are just vector spaces Vec (so that $A_0/(\sigma - 1)[\theta]$-modules are mixed complexes). Indeed, let $y = x'/2$ then modding out by $\sigma - 1$ yields that $yx|_{V_{00}} = 0 \implies xy|_{V_{00}} = 1$ while $yx|_{V_{11}} = 1$ which together with $x^2 = y^2 = 0$ establishes the claim.

Moving on to $s = 1$ we find that things change for the better. More precisely, we have that $A_1$-modules coincide with $k[x, x']_{s\text{Vec}}$-mod. More precisely, we denote by $k[x, x']$ the free supercommutative algebra in the category of super vector spaces that is generated by the odd variables $x$ and $x'$; then modules over this algebra inside $s\text{Vec}$ agree with those over $A_1$ inside $\text{Vec}$. Note that the action of $\sigma - 1$ on $A_1$-mod is given by $x'x \in k[x, x']$ on $k[x, x']_{s\text{Vec}}$-mod. The following proposition is almost immediate:

**Proposition 3.6.** Let $H = T_2(-1)$, then the mixed complexes in the category of stable anti-Yetter-Drinfeld contramodules are not DG-equivalent to the category of mixed anti-Yetter-Drinfeld contramodules.
Proof. By the preceding discussion it suffices to show that the categories \( k[x, x']\theta \) and \( k[x, x']/\theta_{\text{ Vec}^c} \) are not DG-equivalent. We note that both algebras (giving the categories) are commutative and so we need only show that they are not quasi-isomorphic. Indeed, they do not even have isomorphic cohomology algebras. Namely, we need to compare the algebra/module pairs \((k[x, x']/\theta, M)\) and \((k[x, x']/(x'x), k[x, x']/(x'x)), \) where \( M = \ker(x'x : k[x, x'] \to k[x, x']). \) Let \((f, g) : (k[x, x']/\theta, M) \to (k[x, x']/(x'x), M)\) be such an isomorphism. We see that \( f : x \mapsto \alpha x + \beta x' \) while \( g : 1 \mapsto ax'x. \) We are done since \( x \cdot 1 \neq 0 = (\alpha x + \beta x') \cdot x'x. \)

4 Appendix.

Our purpose in this section is to compare \( \hat{D}(H) \) of Definition 2.1 to the more familiar Drinfeld double \( D(H) \) in the case of a finite dimensional Hopf algebra \( H. \) Since our convention differs from the usual ones let us spell out the definitions (for \( H \) finite dimensional):

**Definition 4.1.** The algebra \( D(H) \) is generated by \( H \) and \( H^* \) subject to the relations:

\[
\chi h = h^2 \chi (h^3 - S^{-1}(h^1)),
\]

for \( \chi \in H^* \) and \( h \in H. \) Thus \( D(H) = H \otimes H^* \) as vector spaces.

**Definition 4.2.** The algebra \( D_a(H) \) is generated by \( H \) and \( H^* \) subject to the relations:

\[
\chi h = h^2 \chi (h^3 - S(h^1)),
\]

for \( \chi \in H^* \) and \( h \in H. \) Thus \( D_a(H) = H \otimes H^* \) as vector spaces. The central element \( \sigma = e_i \otimes e^i, \) where \( e_i \) is any basis of \( H \) and \( e^i \) is its dual basis of \( H^*. \)

Note that modules over \( D_a(H) \) as specified in Definition 4.2 are what is classically called left-right anti-Yetter-Drinfeld modules, i.e., left modules and right comodules. Recall that if \( H \) is finite dimensional then we have an \( S^1 \)-equivariant equivalence between \( \hat{D}(H) \)-modules and \( D_a(H)\)-modules [12]. We will thus focus on the comparison between \( D_a(H) \) and \( D(H). \) It is well known that in general they give very different categories of modules. It is usually pointed out that if \( S^2 = Id \) then it is immediate from the definitions that the algebras in fact coincide. In this section we extend that observation slightly so as to cover our case of Taft algebras where we do not have \( S^2 = Id, \) but instead we get

\[
S^2(h) = uhu^{-1}
\]

for some group-like element \( u \in H, \) i.e., \( \Delta(u) = u \otimes u \) (in this case the category of finite dimensional \( H \)-modules is pivotal). For \( T_p(\xi) \) we have \( S^2(a) = g^{-1}ag \) with \( \Delta g^{-1} = g^{-1} \otimes g^{-1}, \) so that \( u = g^{-1}. \)

The following lemma is a straightforward computation:
Lemma 4.3. Let $H$ be a finite dimensional Hopf algebra and suppose that there exists a $u \in H$ with $\Delta(u) = u \otimes u$ such that $S^2(h) = uh^{-1}$ for all $h \in H$. Then

$$D(H) \to D_u(H)$$

$$h \otimes \chi \mapsto h \otimes \chi(-u)$$

is an isomorphism of algebras.

Thus for Taft algebras, Drinfeld doubles can play the role of $\hat{D}(H)$, as long as we are careful to remember about the $\sigma$.

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Department of Mathematics and Statistics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario N9B 3P4, Canada

_E-mail address:_ ishapiro@uwindsor.ca