Mixtures of Skewed Matrix Variate Bilinear Factor Analyzers

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Abstract

Clustering is the process of finding and analyzing underlying group structure in data. In recent years, data as become increasingly higher dimensional and, therefore, an increased need has arisen for dimension reduction techniques for clustering. Although such techniques are firmly established in the literature for multivariate data, there is a relative paucity in the area of matrix variate or three way data. Furthermore, the few methods that are available all assume matrix variate normality, which is not always sensible if cluster skewness or excess kurtosis is present. Mixtures of bilinear factor analyzers models using skewed matrix variate distributions are proposed. In all, four such mixture models are presented, based on matrix variate skew-t, generalized hyperbolic, variance gamma and normal inverse Gaussian distributions, respectively.

Keywords: Factor analysis; skewed matrix variate distribution; mixture models.

1 Introduction

Classification is the process of finding and analyzing underlying group structure in heterogeneous data. This problem can be framed as the search for class labels of unlabelled observations. In general, some (non-trivial) proportion of observations have known labels. A special case of classification, known as clustering, occurs when none of the observations have known labels. One common approach for clustering is mixture model-based clustering,
which makes use of a finite mixture model. In general, a $G$-component finite mixture model assumes that a multivariate random variable $\mathbf{X}$ has density

$$f(\mathbf{x} \mid \boldsymbol{\vartheta}) = \sum_{g=1}^{G} \pi_g f_g(\mathbf{x} \mid \boldsymbol{\theta}_g),$$

(1)

where $\boldsymbol{\vartheta} = (\pi_1, \pi_2, \ldots, \pi_G, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_G)$, $f_g(\cdot)$ is the $g$th component density, and $\pi_g > 0$ is the $g$th mixing proportion such that $\sum_{i=1}^{G} \pi_g = 1$. Note that the notation used in (1) corresponds to the multivariate case and, save for Appendix A, $\mathcal{X}$ will hereafter represent a matrix variate random variable with $\mathbf{X}$ denoting its realization.

McNicholas (2016) traces the relationship between mixture models and clustering to Tiedeman (1955), who uses a component of a mixture model to define a cluster. The mixture model was first used for clustering by Wolfe (1965) who considered a Gaussian mixture model. Other early uses of Gaussian mixture models for clustering can be found in Baum et al. (1970) and Scott & Symons (1971). Although the Gaussian mixture model is attractive due to its mathematical properties, it is problematic when dealing with outliers and asymmetry in the data and thus there has been an interest in non-Gaussian mixtures for the multivariate case. Some examples of mixtures of symmetric distributions that parameterize tail weight include the $t$ distribution (Peel & McLachlan 2000, Andrews & McNicholas 2011, 2012, Lin et al. 2014) and the power exponential distribution (Dang et al. 2015). There has also been work in the area of skewed distributions such as the skew-$t$ distribution, (Lin 2010, Vrbik & McNicholas 2012, 2014, Lee & McLachlan 2014, Murray, Browne & McNicholas 2014, Murray, McNicholas & Browne 2014), the normal-inverse Gaussian (NIG) distribution (Karlis & Santourian 2009), the shifted asymmetric Laplace (SAL) distribution (Morris & McNicholas 2013, Franczak et al. 2014), the variance-gamma distribution (McNicholas et al. 2017), the generalized hyperbolic distribution (Browne & McNicholas 2015), the hidden truncation hyperbolic distribution (Murray et al. 2017a), and the joint generalized hyperbolic distribution (Tang et al. 2018).

There has also been an increased interest in model-based clustering of matrix variate data such as multivariate longitudinal data and images. Such examples include the work of Viroli (2011) and Anderlucci et al. (2015), who consider mixtures of matrix variate normal distributions for clustering. More recently, Gallaugher & McNicholas (2018a) looked at
mixtures of four skewed matrix distributions, namely the matrix variate skew-$t$, generalized hyperbolic, variance gamma and normal inverse Gaussian (NIG) distributions and considered classification of greyscale Arabic numerals. Melnykov & Zhu (2018) also considered modelling skewness by means of transformations.

The main problem with all of the aforementioned methods, for both the multivariate and matrix variate cases, arises when the dimensionality of the data increases. Although the problem of dealing with high-dimensional data has been thoroughly addressed in the case of multivariate data, there is relative paucity of work for matrix variate data. In the matrix variate case, matrix variate bilinear probabilistic principal component analysis was developed by Zhao et al. (2012). More recently, Gallaugher & McNicholas (2018b) considered the closely-related mixture of matrix variate bilinear factor analyzers (MMVBFA) model for clustering. The MMVBFA model can be viewed as a matrix variate generalization of the mixture of factor analyzers model (Ghahramani & Hinton 1997) in the multivariate case. Although these methods allow for simultaneous dimension reduction and clustering, they both assume matrix variate normality which is not sensible if cluster skewness or heavy tails are present. Herein we present an extension of the MMVBFA model to skewed distributions, specifically the matrix variate skew-$t$, generalized hyperbolic, variance-gamma, and NIG distributions.

2 Background

2.1 Generalized Inverse Gaussian Distribution

The generalized inverse Gaussian distribution has two different parameterizations, both of which will be utilized herein. A random variable $Y$ has a generalized inverse Gaussian (GIG) distribution parameterized by $a, b$ and $\lambda$, denoted by GIG$(a, b, \lambda)$, if its probability density function can be written as

$$f(y|a, b, \lambda) = \frac{(a/b)^{\lambda/2} y^{\lambda-1}}{2K_{\lambda}(\sqrt{ab})} \exp \left\{ -\frac{ay + b}{2} \right\},$$

for $y > 0$, $a, b \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$, where
\[
K_\lambda(u) = \frac{1}{2} \int_0^\infty y^{\lambda-1} \exp \left\{ -\frac{u}{2} \left( y + \frac{1}{y} \right) \right\} dy
\]
is the modified Bessel function of the third kind with index \( \lambda \). Expectations of some functions of a GIG random variable have a mathematically tractable form, e.g.:

\[
\mathbb{E}(Y) = \sqrt{\frac{b}{a}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})},
\]

\[
\mathbb{E}(1/Y) = \sqrt{\frac{b}{a}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} - \frac{2\lambda}{b},
\]

\[
\mathbb{E}(\log Y) = \log \left( \sqrt{\frac{b}{a}} \right) + \frac{1}{\sqrt{ab}} \frac{\partial}{\partial \lambda} K_\lambda(\sqrt{ab}).
\]

Although this parameterization of the GIG distribution will be useful for parameter estimation, the alternative parameterization given by

\[
g(y|\omega, \eta, \lambda) = \frac{(w/\eta)^{\lambda-1}}{2\eta K_\lambda(\omega)} \exp \left\{ -\frac{\omega}{2} \left( \frac{w}{\eta} + \frac{\eta}{w} \right) \right\},
\]

where \( \omega = \sqrt{ab} \) and \( \eta = \sqrt{a/b} \), is used when deriving the generalized hyperbolic distribution (see [Browne & McNicholas 2015]). For notational clarity, we will denote the parameterization given in (5) by \( I(\omega, \eta, \lambda) \).

### 2.2 Matrix Variate Distributions

As in the multivariate case, the most mathematically tractable matrix variate distribution is the matrix variate normal. An \( n \times p \) random matrix \( \mathbf{X} \) follows an \( n \times p \) matrix variate normal distribution with \( n \times p \) location matrix \( \mathbf{M} \) and scale matrices \( \mathbf{\Sigma} \) and \( \mathbf{\Psi} \), of dimensions \( n \times n \) and \( p \times p \), respectively, denoted by \( \mathcal{N}_{n \times p}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi}) \) if the density of \( \mathbf{X} \) is

\[
f(\mathbf{X} | \mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{np/2} | \mathbf{\Sigma} |^{1/2} | \mathbf{\Psi} |^{1/2}} \exp \left\{ -\frac{1}{2} \text{tr} \left( \mathbf{\Sigma}^{-1}(\mathbf{X} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X} - \mathbf{M})' \right) \right\}.
\]

The matrix variate normal distribution is related to the multivariate normal distribution, as discussed in [Harrar & Gupta 2008], via \( \mathbf{X} \sim \mathcal{N}_{n \times p}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi}) \iff \text{vec}(\mathbf{X}) \sim \mathcal{N}_{np}(\text{vec}(\mathbf{M}), \mathbf{\Psi} \otimes \mathbf{\Sigma}) \), where \( \mathcal{N}_{np}(\cdot) \) is the multivariate normal density with dimension \( np \), \( \text{vec}(\mathbf{M}) \) is the vectorization of \( \mathbf{M} \), and \( \otimes \) denotes the Kronecker product. Although the
matrix variate normal distribution is popular, there are other well-known examples of matrix variate distributions. For example, the Wishart distribution (Wishart 1928) is the distribution of the sample covariance matrix from a multivariate normal sample. There are also a few formulations of a matrix variate skew normal distribution (Chen & Gupta 2005, Domínguez-Molina et al. 2007, Harrar & Gupta 2008).

More recently, Gallaugher & McNicholas (2017, 2019) derived a total of four skewed matrix variate distributions using a variance mean matrix variate mixture approach. This assumes that a random matrix $\mathcal{X}$ can be written as

$$\mathcal{X} = \mathbf{M} + W\mathbf{A} + \sqrt{W} \mathcal{Y},$$

where $\mathbf{M}$ and $\mathbf{A}$ are $n \times p$ matrices representing the location and skewness, respectively, $\mathcal{Y} \sim \mathcal{N}_{n \times p}(0, \Sigma, \Psi)$, and $W > 0$ is a random variable with density $h(w|\theta)$. Gallaugher & McNicholas (2017), show that the matrix variate skew-$t$ distribution, with $\nu$ degrees of freedom, arises from (7) with $W^{ST} \sim \text{IGamma}(\nu/2, \nu/2)$, where $\text{IGamma}(\cdot)$ denotes the inverse-gamma distribution with density

$$f(y | a, b) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp\left\{ -\frac{b}{y} \right\},$$

for $y > 0$ and $a, b \in \mathbb{R}^+$. The resulting density of $\mathcal{X}$ is

$$f_{\text{MVST}}(X | \vartheta) = \frac{2^{\frac{d}{2}}}{(2\pi)^{np} |\Sigma|^\frac{d}{2} |\Psi|^{\frac{d}{2}} \Gamma(\frac{\nu}{2})} \left( \frac{\delta(X; \mathbf{M}, \Sigma, \Psi) + \nu}{\rho(\mathbf{A}, \Sigma, \Psi)} \right)^{\frac{\nu+np}{2}}$$

$$\times K_{\frac{\nu+np}{2}} \left( \sqrt{[\rho(\mathbf{A}, \Sigma, \Psi)] \left[ \delta(X; \mathbf{M}, \Sigma, \Psi) + \nu \right]} \right),$$

where $\delta(X; \mathbf{M}, \Sigma, \Psi) = \text{tr}(\Sigma^{-1}(X - \mathbf{M})\Psi^{-1}(X - \mathbf{M})')$, $\rho(\mathbf{A}; \Sigma, \Psi) = \text{tr}(\Sigma^{-1} \mathbf{A} \Psi^{-1} \mathbf{A}')$ and $\nu > 0$. For notational clarity, this distribution will be denoted by MVST($\mathbf{M}, \mathbf{A}, \Sigma, \Psi, \nu$).

In Gallaugher & McNicholas (2019), one of the distributions considered is a matrix variate generalized hyperbolic distribution. This again arises from (7) with $W^{GH} \sim \text{I}(\omega, 1, \lambda)$. This distribution will be denoted by MVGH($\mathbf{M}, \mathbf{A}, \Sigma, \Psi, \lambda, \omega$), and the density is

$$f_{\text{MVGH}}(X | \vartheta) = \frac{\exp \left\{ \text{tr}(\Sigma^{-1}(X - \mathbf{M})\Psi^{-1}(X - \mathbf{M})') \right\}}{(2\pi)^{np} |\Sigma|^\frac{d}{2} |\Psi|^{\frac{d}{2}} \Gamma(\frac{\nu}{2})} \left( \frac{\delta(X; \mathbf{M}, \Sigma, \Psi) + \omega}{\rho(\mathbf{A}, \Sigma, \Psi) + \omega} \right)^{\frac{\lambda-\nu+np}{2}}$$

$$\times K_{\frac{\lambda-\nu+np}{2}} \left( \sqrt{[\rho(\mathbf{A}, \Sigma, \Psi) + \omega] \left[ \delta(X; \mathbf{M}, \Sigma, \Psi) + \omega \right]} \right),$$

for $\lambda > 0$ and $\nu, \omega \in \mathbb{R}^+$. The resulting density of $\mathcal{X}$ is

$$f_{\text{MVGH}}(X | \vartheta) = \frac{2^{\frac{d}{2}}}{(2\pi)^{np} |\Sigma|^\frac{d}{2} |\Psi|^{\frac{d}{2}} \Gamma(\frac{\nu}{2})} \left( \frac{\delta(X; \mathbf{M}, \Sigma, \Psi) + \nu}{\rho(\mathbf{A}, \Sigma, \Psi) + \omega} \right)^{\frac{\nu+np}{2}}$$

$$\times K_{\frac{\nu+np}{2}} \left( \sqrt{[\rho(\mathbf{A}, \Sigma, \Psi) + \omega] \left[ \delta(X; \mathbf{M}, \Sigma, \Psi) + \omega \right]} \right).$$
where $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{R}^+$. The matrix variate variance-gamma distribution, also derived in Gallaugher & McNicholas (2019) and denoted $\text{MVVG}(M, A, \Sigma, \Psi, \gamma)$, arises from (7) with $W_{\text{VG}} \sim \text{gamma}(\gamma, \gamma)$, where $\text{gamma}(\cdot)$ denotes the gamma distribution with density $f(y \mid a, b) = \frac{b^a}{\Gamma(a)} y^{a-1} \exp\{-by\}$, for $y > 0$ and $a, b \in \mathbb{R}^+$. The density of the random matrix $X$ with this distribution is

$$f_{\text{MVVG}}(X \mid \vartheta) = \frac{2\gamma^\gamma}{(2\pi)^{n(p/2)}|\Sigma|^{p/2}|\Psi|^{p/2}\Gamma(\gamma)} \exp\{\text{tr}(\Sigma^{-1}(X - M)\Psi^{-1}A')\} \left(\frac{\delta(X; M, \Sigma, \Psi)}{\rho(A, \Sigma, \Psi) + 2\gamma}\right)^{(\gamma-\text{np}/2)} \times K_{\gamma-\text{np}/2}\left(\sqrt{[\rho(A, \Sigma, \Psi) + 2\gamma]} \left[\delta(X; M, \Sigma, \Psi) + 1\right]\right),$$

where $\gamma > 0$.

Finally, the matrix variate NIG distribution arises when $W_{\text{NIG}} \sim \text{IG}(1, \tilde{\gamma})$, where $\text{IG}(\cdot)$ denotes the inverse-Gaussian distribution with density $f(y \mid \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} y^{-3/2} \exp\{-\frac{1}{2} \left(\frac{\delta^2}{y} + \gamma^2 y\right)\}$, for $y > 0$, $\delta, \gamma \in \mathbb{R}^+$. The density of $X$ is

$$f_{\text{MVNIG}}(X \mid \vartheta) = \frac{2}{(2\pi)^{np/2} |\Sigma|^{p/2} |\Psi|^{p/2}} \exp\{\text{tr}(\Sigma^{-1}(X - M)\Psi^{-1}A') + \tilde{\gamma}\} \left(\frac{\delta(X; M, \Sigma, \Psi) + 1}{\rho(A, \Sigma, \Psi) + \tilde{\gamma}^2}\right)^{-(1+np)/4} \times K_{-(1+np)/2}\left(\sqrt{[\rho(A, \Sigma, \Psi) + \tilde{\gamma}^2]} \left[\delta(X; M, \Sigma, \Psi) + 1\right]\right),$$

where $\tilde{\gamma} > 0$. This distribution is denoted by $\text{MVNIG}(M, A, \Sigma, \Psi, \tilde{\gamma})$.

### 2.3 Matrix Variate Factor Analysis

Readers who may benefit from the context provided by the mixture of factor analyzers model should consult the appendix. Xie et al. (2008) and Yu et al. (2008) consider a matrix variate extension of probabilistic principal components analysis (PPCA) and assumes an $n \times p$ random matrix $X$ can be written

$$X = M + \Lambda \Psi \Delta' + \xi,$$
where $M$ is an $n \times p$ location matrix, $\Lambda$ is an $n \times q$ matrix of column factor loadings, $\Delta$ is a $p \times r$ matrix of row factor loadings, $\mathcal{U} \sim \mathcal{N}_{q \times r}(0, I_q, I_r)$, and $\mathcal{E} \sim \mathcal{N}_{n \times p}(0, \sigma^2 I_n, \sigma^2 I_p)$. It is assumed that $\mathcal{U}$ and $\mathcal{E}$ are independent of each other. The main disadvantage of this model is that, in general, $\mathcal{X}$ does not follow a matrix variate normal distribution.

Zhao et al. (2012) present bilinear probabilistic principal component analysis (BPPCA) which extends (8) by adding two projected error terms. The resulting model assumes $\mathcal{X}$ can be written $\mathcal{X} = M + \Lambda \mathcal{U} \Delta' + \Lambda \mathcal{E}^B + \mathcal{E}^A \Delta' + \mathcal{E}$, where $\mathcal{U}$ is the same as in (8), $\mathcal{E}^B \sim \mathcal{N}_{q \times p}(0, I_q, \sigma B I_p)$, $\mathcal{E}^A \sim \mathcal{N}_{n \times r}(0, \sigma A I_n, I_r)$. In this model, it is assumed that $\mathcal{U}$, $\mathcal{E}^B$, $\mathcal{E}^A$, and $\mathcal{E}$ are all independent of each other. Gallaugher & McNicholas (2018b) further extend this to matrix variate factor analysis and consider a mixture of matrix variate bilinear factor analyzers (MMVBFA) model. For MMVBFA, Gallaugher & McNicholas (2018b) altered BPPCA by removing the isotropic constraints so that $\mathcal{E}^B \sim \mathcal{N}_{q \times p}(0, I_q, \Psi)$, $\mathcal{E}^A \sim \mathcal{N}_{n \times r}(0, \Sigma, I_r)$ and $\mathcal{E} \sim \mathcal{N}_{n \times p}(0, \Sigma, \Psi)$, where $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, with $\sigma_i > 0$, and $\Psi = \text{diag}\{\psi_1, \psi_2, \ldots, \psi_p\}$, with $\psi_i > 0$. With these slight modifications, it can be shown that $\mathcal{X} \sim \mathcal{N}_{n \times p}(M, \Lambda \Lambda' + \Sigma, \Delta \Delta' + \Psi)$, similarly to its multivariate counterpart (Appendix A).

It is important to note that the term “column factors” refers to reduction in the dimension of the columns, which is equivalent to the number of rows, and not a reduction in the number of columns. Likewise, the term “row factors” refers to the reduction in the dimension of the rows (number of columns). As discussed by Zhao et al. (2012), the interpretation of the terms $\mathcal{E}^B$ and $\mathcal{E}^A$ are the row and column noise, respectively, whereas the final term $\mathcal{E}$ is the common noise.

### 3 Mixture of Skewed Matrix Variate Bilinear Factor Analyzers

#### 3.1 Model Specification

We now consider a mixture of skewed bilinear factor analyzers according to one of the four skewed distributions discussed previously. Each random matrix $\mathcal{X}_i$ from a random sample
distributed according to one of the four distributions can be written as

\[ \mathcal{X}_i = M_g + W_{ig}A_g + \mathcal{Y}_{ig} \]

with probability \( \pi_g \) for \( g \in \{1, 2, \ldots, G\} \), \( \pi_g > 0 \), \( \sum_{g=1}^{G} \pi_g = 1 \), where \( M_g \) is the location of the \( g \)th component, \( A_g \) is the skewness, and \( W_{ig} \) is a random variable with the density \( h(w_{ig}|\theta_g) \). This will be dependent on the distribution in question, i.e., skew-t, generalized hyperbolic, variance-gamma or NIG. Assume also that \( \mathcal{Y}_{ig} \) can be written as

\[ \mathcal{Y}_{ig} = \Lambda_g \mathcal{U}_{ig} \Delta'_g + \Lambda_g \mathcal{E}_{ig}^B + \mathcal{E}_{ig}^A \Delta'_g + \mathcal{E}_{ig}, \]

where \( \Lambda_g \) is a \( n \times q \) matrix of column factor loadings, \( \Delta_g \) is a \( p \times r \) matrix of row factor loadings, and

\[ \begin{align*}
\mathcal{U}_{ig} | w_{ig} & \sim \mathcal{N}_{q\times r}(0, w_{ig} I_q, I_p), \\
\mathcal{E}_{ig}^B | w_{ig} & \sim \mathcal{N}_{q\times p}(0, w_{ig} I_q, \Psi_g), \\
\mathcal{E}_{ig}^A | w_{ig} & \sim \mathcal{N}_{n\times r}(0, w_{ig} \Sigma_g, I_r), \\
\mathcal{E}_{ig} | w_{ig} & \sim \mathcal{N}_{n\times p}(0, w_{ig} \Sigma_g, \Psi_g).
\end{align*} \]

Note that \( \mathcal{U}_{ig}, \mathcal{E}_{ig}^B, \mathcal{E}_{ig}^A \) and \( \mathcal{E}_{ig} \) are all independently distributed and independent of each other.

To facilitate clustering, introduce the indicator \( z_{ig} \), where \( z_{ig} = 1 \) if observation \( i \) belongs to group \( g \), and \( z_{ig} = 0 \) otherwise. Then, it can be shown that

\[ \mathcal{X}_i \mid z_{ig} = 1 \sim D_{n\times p}(M_g, A_g, \Sigma_g + \Lambda_g \Lambda'_g, \Psi_g + \Delta_g \Delta'_g, \theta_g), \]

where \( D \) is the distribution in question, and \( \theta_g \) is the set of parameters related to the distribution of \( W_{ig} \).

As in the matrix variate normal case, this model has a two stage interpretation given by

\[ \begin{align*}
\mathcal{X}_i &= M_g + W_{ig}A + \Lambda_g \mathcal{Y}_{ig}^B + \mathcal{R}_{ig}^B, \\
\mathcal{Y}_{ig}^B &= \mathcal{U}_{ig} \Delta'_g + \mathcal{E}_{ig}^B, \\
\mathcal{R}_{ig}^B &= \mathcal{E}_{ig}^A \Delta'_g + \mathcal{E}_{ig},
\end{align*} \]

and

\[ \begin{align*}
\mathcal{X}_i &= M_g + W_{ig}A + \mathcal{Y}_{ig}^A \Delta'_g + \mathcal{R}_{ig}^A, \\
\mathcal{Y}_{ig}^A &= \Lambda_g \mathcal{U}_{ig} + \mathcal{E}_{ig}^A, \\
\mathcal{R}_{ig}^A &= \Lambda_g \mathcal{E}_{ig}^B + \mathcal{E}_{ig},
\end{align*} \]

which will be useful for parameter estimation.
3.2 Parameter Estimation

Suppose we observe the $N$ $n \times p$ matrices $X_1, X_2, \ldots, X_N$ distributed according to one of the four distributions. We assume that this data is incomplete and employ an alternating expectation conditional maximization (AECM) algorithm \cite{Meng & van Dyk 1997}. This algorithm is now described after initialization.

**AECM 1st Stage** The complete-data in the first stage consists of the observed data $X_i$, the latent variables $W_i = (W_{i1}, W_{i2}, \ldots, W_{iG})'$ and the unknown group labels $z_i = (z_{i1}, z_{i2}, \ldots, z_{iG})'$ for $i = 1, 2, \ldots, N$. In this case, the complete-data log-likelihood is

$$
\ell_{C1} = C + \sum_{i=1}^{N} \sum_{g=1}^{G} \log \pi_g + \log h(w_{ig} | \theta_g) - \frac{1}{2} \text{tr} \left\{ \frac{1}{W_{ig}} \Sigma_g^{-1} (X_i - M_g) \Psi^{-1} (X_i - M_g)' \right\} - \Sigma_g^{-1} (X_i - M_g) \Psi^{-1} \Lambda_g' - \Sigma_g^{-1} \Lambda_g \Psi^{-1} (X_i - M_g)' + W_{ig} \Sigma_g^{-1} A_g \Psi^{-1} \Lambda_g' \right\},
$$

where $\Sigma_g = \Omega_g + \Omega_g \Lambda_g', \Psi_g = \Omega_g + \Delta_g \Delta_g'$ and $C$ is constant with respect to the parameters.

In the E-step, we calculate the following conditional expectations:

$$
\hat{z}_{ig} = \frac{\pi_g f(X_i | \hat{\theta}_g)}{\sum_{h=1}^{G} \pi_h f(X_i | \hat{\theta}_h)}, \quad a_{ig} = \mathbb{E}(W_{ig} | X_i, z_{ig} = 1, \hat{\theta}_g),
$$

$$
b_{ig} = \mathbb{E} \left( \frac{1}{W_{ig}} \bigg| X_i, z_{ig} = 1, \hat{\theta}_g \right), \quad c_{ig} = \mathbb{E}(\log W_{ig} | X_i, z_{ig} = 1, \hat{\theta}_g).
$$

As usual, all expectations are conditional on current parameter estimates; however, to avoid cluttered notation, we do not use iteration-specific notation. Although these expectations are dependent on the distribution in question, it can be shown that

$$
W_{ig}^{ST} | X_i, z_{ig} = 1 \sim \text{GIG} \left( \rho(A_g, \Sigma_g, \Psi_g), \delta(X; M_g, \Sigma_g, \Psi_g) + \nu_g, -\nu_g + np/2 \right),
$$

$$
W_{ig}^{GH} | X_i, z_{ig} = 1 \sim \text{GIG} \left( \rho(A_g, \Sigma_g, \Psi_g) + \omega_g, \delta(X; M_g, \Sigma_g, \Psi_g) + \omega_g, \lambda_g - np/2 \right),
$$

$$
W_{ig}^{VG} | X_i, z_{ig} = 1 \sim \text{GIG} \left( \rho(A_g, \Sigma_g, \Psi_g) + 2\gamma_g, \delta(X; M_g, \Sigma_g, \Psi_g), \gamma_g - np/2 \right),
$$

$$
W_{ig}^{NIG} | X_i, z_{ig} = 1 \sim \text{GIG} \left( \rho(A_g, \Sigma_g, \Psi_g) + s_g^2, \delta(X; M_g, \Sigma_g, \Psi_g) + 1, -(1 + np)/2 \right).
$$

Therefore, the exact updates are obtained by using the expectations given in (2)–(4) for appropriate values of $\lambda, a,$ and $b$. 

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In the M-step, we update $\pi_g, M_g, A_g, \theta_g$:

$$\hat{\pi}_g = \frac{N_g}{N}, \quad \hat{M}_g = \frac{\sum_{i=1}^N \hat{z}_{ig} X_i (\tilde{a}_g b_{ig} - 1)}{\sum_{i=1}^N \hat{z}_{ig} \tilde{a}_g b_{ig} - N_g}, \quad \hat{A} = \frac{\sum_{i=1}^N \hat{z}_{ig} X_i (\tilde{b}_g - b_{ig})}{\sum_{i=1}^N \hat{z}_{ig} \tilde{a}_g b_{ig} - N_g},$$

where

$$N_g = \sum_{i=1}^N \hat{z}_{ig}, \quad \tilde{a}_g = \frac{\sum_{i=1}^N \hat{z}_{ig} a_{ig}}{N_g}, \quad \tilde{b}_g = \frac{\sum_{i=1}^N \hat{z}_{ig} b_{ig}}{N_g}.$$

The update for $\theta_g$ is dependent on the distribution and are identical to those given in Gallaugher & McNicholas (2018a).

**AECM Stage 2**  In the second stage, the complete-data consists of the observed data $X_i$, the latent variables $W_i$, the unknown group labels $z_i$ and the latent matrices $\nu_i^B = (\nu_i^B, \nu_i^B, \ldots, \nu_i^B)$ for $i = 1, 2, \ldots, N$. The complete-data log-likelihood in this stage is

$$\ell_{C2} = C + \sum_{i=1}^N \sum_{g=1}^G z_{ig} \left[ \log \pi_g + \log h(W_{ig}|\nu_g) + \log \phi_{q\times p}(\nu_i^B|0, W_{ig} I_q, \Psi_g^*) \right] + \frac{1}{2} \log |\Sigma_g| + \text{tr} \left\{ \frac{1}{W_{ig}} \Sigma_g^{-1}(X_i - M_g) \Psi_g^{-1}(X_i - M_g)' - \Sigma_g^{-1}(X_i - M_g) \Psi_g^{-1} \nu_i^B A_g' - \Sigma_g^{-1} A_g \Psi_g^{-1} \nu_i^B (X_i - M_g)' + W_{ig} \Sigma_g^{-1} A_g \Psi_g^{-1} A_g' + \Sigma_g^{-1} A_g \Psi_g^{-1} \nu_i^B A_g' - \frac{1}{W_{ig}} \Sigma_g^{-1} A_g \nu_i^B \Psi_g^{-1} (X_i - M_g)' + \Sigma_g^{-1} A_g \nu_i^B \Psi_g^{-1} A_g' + \frac{1}{W_{ig}} \Sigma_g^{-1} A_g \nu_i^B \Psi_g^{-1} \nu_i^B A_g' \right\}.$$
and so we can calculate the expectations

\[ E_{1ig}^{(2)} := \mathbb{E}[(\Psi^B)_{ig} | \hat{\varphi}, X_i, z_{ig} = 1] = L_g(X_i - \hat{M}_g - a_{ig}\hat{A}_g) \]

\[ E_{2ig}^{(2)} := \mathbb{E} \left[ \frac{1}{W_{ig}} \Psi^B | \hat{\varphi}, X_i, z_{ig} = 1 \right] = L_g(b_{ig}(X_i - \hat{M}_g) - \hat{A}_g) \]

\[ E_{3ig}^{(2)} := \mathbb{E} \left[ \frac{1}{W_{ig}} \Psi^B \Psi^{-1} \Psi^B' \bigg| \hat{\varphi}, X_i, z_{ig} = 1 \right] \]

\[ = p(I_q + \hat{A}_g \hat{\Sigma}_g^{-1} \hat{A}_g)^{-1} + b_{ig}L_g(X_i - \hat{M}_g)\Psi^{-1}_g(X_i - \hat{M}_g)'L'_g \]

\[ - L_g((X_i - \hat{M}_g)\Psi^{-1}_g \hat{A}_g' + \hat{A}_g\Psi^{-1}_g(X_i - \hat{M}_g)'L'_g + a_{ig}L_g \hat{A}_g \Psi^{-1}_g \hat{A}_g' L'_g, \]

where \( L_g = (I_q + \hat{A}_g \hat{\Sigma}_g^{-1} \hat{A}_g)^{-1} \hat{A}_g \hat{\Sigma}_g^{-1} \).

In the M-step, the updates for \( A_g \) and \( \Sigma_g \) are calculated. These updates are given by

\[ \hat{A}_g = \sum_{i=1}^{N} \hat{z}_{ig} \left( (X_i - \hat{M}_g)\hat{\Psi}_g^{-1} - \hat{A}_g \hat{\Psi}_g^{-1} E_{1ig}^{(2)} \right) \left( \sum_{i=1}^{N} \hat{z}_{ig} E_{1ig}^{(2)} \right)^{-1} \]

and \( \hat{\Sigma}_g = \text{diag}(S^L_g) \),

\[ S^L_g = \frac{1}{N_g p} \sum_{i=1}^{N} \hat{z}_{ig} \left[ b_{ig}(X_i - \hat{M}_g)\Psi^{-1}_g(X_i - \hat{M}_g)' - (\hat{A}_g + \hat{A}_g E_{2ig}^{(2)} \hat{\Psi}_g^{-1}(X_i - \hat{M}_g) \right] \]

\[ - (X_i - \hat{M}_g)\hat{\Psi}_g^{-1} \hat{A}_g' + a_{ig} \hat{A}_g \hat{\Psi}_g^{-1} \hat{A}_g' + \hat{A}_g E_{2ig}^{(2)} \hat{\Psi}_g^{-1} \hat{A}_g' \]

\[ - (X_i - \hat{M}_g) \Psi^{-1}_g \hat{A}_g' + \hat{A}_g \Psi^{-1}_g E_{2ig}^{(2)} \hat{\Psi}_g^{-1} \hat{A}_g' + \hat{A}_g E_{2ig}^{(2)} \hat{\Psi}_g^{-1} \hat{A}_g' \].

**AECM Stage 3** In the third stage, the complete-data consists of the observed data \( X_i \), the latent variables \( W_i \), the labels \( z_i \) and the latent matrices \( \Psi_i^A = (\Psi^A_1, \Psi^A_2, \ldots, \Psi^A_G) \) for \( i = 1, 2, \ldots, N \).

\[ \ell_{C3} = C + \sum_{i=1}^{N} \sum_{g=1}^{G} z_{ig} \left[ \log \pi_g + \log h(W_{ig}\nu_g) + \log \phi_{N x p}(\Psi_{ig}^A | 0, W_{ig} \Sigma^*_g, I_p) \right] \]

\[ + \log \phi_{n x p}(X_i | M_g + W_{ig} A_g + \Psi_{ig}^A \Delta_g, W_{ig} \Sigma^*_g, \Psi^B_g) \]

\[ = C + \sum_{i=1}^{N} \sum_{g=1}^{G} \left( - \frac{1}{2} \hat{z}_{ig} \right) \left[ - n \log |\Psi_g| + \text{tr} \left( \frac{1}{W_{ig}} \Psi^{-1}_g(X_i - M_g)' \Sigma_g^{-1}(X_i - M_g) \right) \right. \]

\[ - \Psi^{-1}_g(X_i - M_g)' \Sigma_g^{-1} A_g \left] \right) \right) - \frac{1}{W_{ig}} \Psi^{-1}_g(X_i - M_g)' \Sigma_g^{-1} \Psi^A_{ig} \Delta_g' - \Psi^{-1}_g A_g \Sigma_g^{-1}(X_i - M_g) \]

\[ + W_{ig} \Psi^{-1}_g A_g' \Sigma_g^{-1} A_g + \Psi^{-1}_g A_g' \Sigma_g^{-1} \Psi^A_{ig} \Delta_g' - \frac{1}{W_{ig}} \Psi^{-1}_g \Delta_g \Psi^A_{ig} \Sigma_g^{-1}(X_i - M_g) \]

\[ + \Psi^{-1}_g \Delta_g \Psi^A_{ig} \Sigma_g^{-1} A_g + \frac{1}{W_{ig}} \Psi^{-1}_g \Delta_g \Psi^A_{ig} \Sigma_g^{-1} \Psi^A_{ig} \Delta'_g \].
In the E-step, it can be shown that

\[ \mathcal{Y}_i | X_i, W_{ig}, z_{ig} = 1 \sim \mathcal{N}_{n \times r}(\mathbf{X}_i - \mathbf{M}_g - W_{ig}\mathbf{A}_g)\Psi_g^{-1}\Delta_g(\mathbf{I}_r + \Delta'_g\Psi_g^{-1}\Delta_g)^{-1}, W_{ig}\Sigma'_g, (\mathbf{I}_r + \Delta'_g\Psi_g^{-1}\Delta_g)^{-1}) \]

and so we can calculate the expectations

\[ E_{1ig}^{(3)} := \mathbb{E}[\mathcal{Y}_i | \hat{\theta}, \mathbf{X}_i, z_{ig} = 1] = (\mathbf{X}_i - \hat{\mathbf{M}}_g - a_{ig}\hat{\mathbf{A}}_g)D_g, \]
\[ E_{2ig}^{(3)} := \mathbb{E} \left[ \frac{1}{W_{ig}} \mathcal{Y}_i | \hat{\theta}, \mathbf{X}_i, z_{ig} = 1 \right] = (b_{ig}(\mathbf{X}_i - \hat{\mathbf{M}}_g) - \hat{\mathbf{A}}_g)D_g, \]
\[ E_{3ig}^{(3)} := \mathbb{E} \left[ \frac{1}{W_{ig}} \mathcal{Y}_i \hat{\Psi}_g^{-1} \mathcal{Y}_i | \hat{\theta}, \mathbf{X}_i, z_{ig} = 1 \right] \]

\[ = n(\mathbf{I}_r + \Delta'_g\hat{\Psi}_g^{-1}\hat{\Delta}_g)^{-1} + b_{ig}D'_g(\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}(\mathbf{X}_i - \hat{\mathbf{M}}_g)D_g \]
\[ - D'_g((\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}\hat{\Delta}_g + \hat{\Delta}'_g\hat{\Sigma}_g^{-1}(\mathbf{X}_i - \hat{\mathbf{M}}_g))D_g + a_{ig}D'_g\hat{\Delta}'_g\hat{\Sigma}_g^{-1}\hat{\Delta}_gD_g, \]

where \( D_g = \hat{\Psi}_g^{-1}\Delta_g(\mathbf{I}_r + \Delta'_g\hat{\Psi}_g^{-1}\hat{\Delta}_g)^{-1} \).

In the M-step, the updates for \( \Delta_g \) and \( \Psi_g \) are calculated. These updates are given by

\[ \hat{\Delta}_g = \sum_{i=1}^{N} \tilde{z}_{ig}[(\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}E_{2ig}^{(3)} - \hat{\Delta}'_g\hat{\Sigma}_g^{-1}E_{1ig}^{(3)}](\sum_{i=1}^{N} z_{ig}E_{3ig}^{(3)})^{-1} \]

and \( \hat{\Psi}_g = \text{diag}(S_g^D) \), where

\[ S_g^D = \frac{1}{N_gp} \sum_{i=1}^{N} \tilde{z}_{ig}[b_{ig}(\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}(\mathbf{X}_i - \hat{\mathbf{M}}_g) - (\hat{\Delta}'_g + \hat{\Delta}_gE_{2ig}^{(3)})'\hat{\Sigma}_g^{-1}(\mathbf{X}_i - \hat{\mathbf{M}}_g) \]
\[ - (\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}\hat{\Delta}_g + a_{ig}\hat{\Delta}'_g\hat{\Sigma}_g^{-1}\hat{\Delta}_g + \hat{\Delta}_gE_{1ig}^{(3)}\hat{\Sigma}_g^{-1}\hat{\Delta}_g \]
\[ - (\mathbf{X}_i - \hat{\mathbf{M}}_g)'\hat{\Sigma}_g^{-1}E_{2ig}^{(3)}\hat{\Delta}_g' + \hat{\Delta}'_g\hat{\Sigma}_g^{-1}E_{1ig}^{(3)}\hat{\Delta}_g' + \hat{\Delta}_gE_{3ig}^{(3)}\hat{\Delta}_g']. \]

### 3.3 Selection, Convergence and Performance Evaluation Criteria and Initialization

In general the number of components, row factors and column factors are not known \textit{a priori} and therefore need to be selected. In our simulations and analyses, the Bayesian information criterion (BIC; \cite{Schwarz1978}) and is given by

\[ \text{BIC} = 2\ell(\hat{\theta}) - \rho \log N, \]

where \( \rho \) is the number of free parameters.
A simple convergence criterion is lack of progress where the algorithm is terminated when \(l^{(t+1)} - l^{(t)} < \epsilon\), where \(\epsilon > 0\) is a small number. Oftentimes, however, the likelihood can plateau before increasing again, thus using lack of progress would terminate the algorithm prematurely. Another option, and what is used for our analyses, is based on the Aitken acceleration criterion, (Aitken 1926). The acceleration at iteration \(t\) is

\[
a^{(t)} = \frac{l^{(t+1)} - l^{(t)}}{l^{(t)} - l^{(t-1)}},
\]

where \(l^{(t)}\) is the observed likelihood at iteration \(t\). We then define

\[
l^{(t+1)}_{\infty} = l^{(t)} + \frac{(l^{(t+1)} - l^{(t)})}{1 - a^{(t)}},
\]

(refer to Böhning et al. 1994, Lindsay 1995). This quantity is an estimate of the observed log likelihood after many iterations at iteration \(t + 1\). As in McNicholas et al. (2010), the algorithm is terminated when \(l^{(k+1)}_{\infty} - l^{(k)} \in (0, \epsilon)\). It should be noted that the tolerance for convergence should be set based on the magnitude of the likelihood. Therefore, after five iterations of the AECM algorithm in our analyses, we set the tolerance \(\epsilon\) to be three orders of magnitude lower than the likelihood.

To assess classification performance, the adjusted Rand index (ARI; Hubert & Arabie 1985) is used. The ARI is the Rand index (Rand 1971) corrected for chance agreement. The ARI compares two different classifications, in this case the predicted and true classifications, and takes a value of 1 if there is perfect agreement and its expected value is 0 when randomly assigning labels.

Finally, there is the issue of initialization. In our simulations and data analyses, we initialize the group membership at random and use soft classifications. Using these initial soft group memberships, we initialize the location matrices using

\[
\hat{M}_g = \sum_{i=1}^{N} \hat{z}_{ig} X_i \hat{N}_g.
\]

The skewness matrices is set to a matrix with all entries equal to 0.1. This can not be set to the zero matrix, since the component densities are not defined. The diagonal scale matrices, \(\Sigma_g\) and \(\Psi_g\) are initialized as follows
\[
\hat{\Sigma}_g = \text{diag} \left\{ \frac{\sum_{i=1}^{N} \hat{z}_{ig}(X_i - \hat{M}_g)(X_i - \hat{M}_g)'}{pN_g} \right\},
\]

and
\[
\hat{\Psi}_g = \text{diag} \left\{ \frac{\sum_{i=1}^{N} \hat{z}_{ig}(X_i - \hat{M}_g)'(X_i - \hat{M}_g)}{nN_g} \right\}.
\]

The factor loadings are initialized randomly.

### 3.4 Reduction in Number of Free Covariance Parameters

The reduction in the number of free parameters for each of these models is equivalent to the Gaussian case discussed in Gallaugher & McNicholas (2018b). The reduction in the number of free covariance parameters for the row covariance matrix is
\[
\frac{1}{2} n(n + 1) - nq - n + \frac{1}{2} q(q - 1) = \frac{1}{2} [(n - q)^2 - (n + q)],
\]
which is positive for \((n-q)^2 > n+q\). Likewise for the column covariance matrix the reduction in the number of parameters is
\[
\frac{1}{2} p(p + 1) - pr - p + \frac{1}{2} r(r - 1) = \frac{1}{2} [(p - r)^2 - (p + r)],
\]
which is positive for \((p-r)^2 > p+r\).

In applications herein, the model is fit for a range of row factors and column factors. If the number of factors chosen by the BIC is the maximum in the range, the number of factors would be increased so long as the above conditions are met.

### 3.5 Semi-Supervised Classification

Each of these four models presented herein may also be used in the context of semi-supervised classification. Suppose \(N\) matrices are observed and \(K\) of these observations have known labels from one of \(G\) classes. Following McNicholas (2010), and without loss of generality, the matrices are ordered so that the first \(K\) have known labels and the remaining observations
have unknown labels. The observed likelihood is then

\[ L(\vartheta) = \prod_{i=1}^{K} \prod_{g=1}^{G} \left[ \pi_g \varphi_{n \times p}(X_i | M_g, \Sigma_g + A_g A'_g, \Psi_g + B_g B'_g) \right]^{\hat{z}_{ig}} \times \prod_{j=K+1}^{N} \sum_{h=1}^{H} \pi_h \varphi_{n \times p}(X_i | M_h, \Sigma_h + A_h A'_h, \Psi_h + B_h B'_h). \]

It is possible for \( H \neq G \); however, for the analyses herein we assume that \( H = G \). Parameter estimation then proceeds in a similar manner for the clustering scenario. For more information on semi-supervised classification refer to McNicholas (2010, 2016).

### 3.6 Computational Issues

One situation that needs to be addressed for all four of these distributions, but particularly the variance-gamma distribution, is the infinite likelihood problem. This occurs as the result of the update for \( \hat{M}_g \) becoming very close and in some cases equal to an observation \( X_i \) when the algorithm gets close to convergence. A similar situation occurs in the multivariate case for the mixture of SAL distributions described in Franczak et al. (2014) and we follow a similar procedure when faced with this issue. While iterating the algorithm, when the likelihood becomes numerically infinite, we set the estimate of \( \hat{M}_g \) to the previous estimate which we will call \( \hat{M}_g^* \). We then update \( \hat{A}_g \) according to

\[ \hat{A}_g^* = \frac{\sum_{i=1}^{N} \hat{z}_{ig} (X_i - \hat{M}_g^*)}{\sum_{i=1}^{N} \hat{z}_{ig} a_{ig}}. \]

The updates for all other parameters remain the same. As mentioned in Franczak et al. (2014), this solution is a little naive; however, it does generally work quite well.

It is not surprising that this problem is particularly prevalent in the case of the variance-gamma distribution. This is because the SAL distribution also arises from a variance mean mixture with the mixing variable being distributed as an exponential random variable with rate 1; therefore, in the multivariate case, the SAL distribution is a special case of the variance gamma distribution. Therefore the close relationship between these two distributions would also be similar in the matrix variate case.

Another computational concern in the evaluation of the Bessel functions. In our computations it may be the case, especially with increase in dimension, that in the computation
of the generalized inverse Gaussian expected values and the computation of the component densities, that the argument is far larger than the magnitude of the index. Therefore, in these situations, the result is computationally equivalent to zero which causes issues with other computations. In such a situation, we calculate the exponentiated version of the Bessel function, i.e. we calculate \( \exp(u)K_\lambda(u) \) and subsequent calculations can be easily adjusted.

4 Simulation Study

A simulation study was performed for each of the four models presented herein. For each of the four distributions, we consider \( d \times d \) matrices with \( d \in \{10, 30\} \) and, for each value of \( d \), we consider datasets coming from a mixture with two components and \( \pi_1 = \pi_2 = 0.5 \). The datasets have sample sizes \( N \in \{100, 200, 400\} \) and the following parameters are used for all four distributions for each combination of \( d \) and \( N \). We take \( \mathbf{M}_1 = \mathbf{0} \) and \( \mathbf{M}_2 = \mathbf{M}_1 + \mathbf{C} \), where \( \mathbf{C} \) is a matrix with all entries equal to \( c \) for \( c \in \{1, 2, 4\} \). All other parameters are held constant. We take \( \mathbf{A}_1 = \mathbf{A}_2 = \mathbf{1} \), where \( \mathbf{1} \) is a matrix of 1’s, \( \mathbf{\Sigma}_1 = 2\mathbf{I}, \mathbf{\Sigma}_2 = \mathbf{I}, \mathbf{\Psi}_1 = \mathbf{I}, \mathbf{\Psi}_2 = 2\mathbf{I} \). Three column factors and two row factors are used with their values being taken randomly between \(-1\) and \(1\). The distribution-specific parameters are given in Table 1.

Table 1: Distribution-specific parameters used for the simulations, where the acronyms all take the form MMVDFA and denote “mixture of matrix variate D factor analyzers” with \( D \in \{\text{skew-t (ST)}, \text{generalized hyperbolic (GH)}, \text{variance-gamma (VG)}, \text{NIG}\} \).

|                | Component 1 | Component 2 |
|----------------|-------------|-------------|
| MMVSTFA        | \( \nu_1 = 4 \) | \( \nu_2 = 20 \) |
| MMVGHFA        | \( \omega_1 = 4, \lambda_1 = -4 \) | \( \omega_2 = 10, \lambda_2 = 4 \) |
| MMVVGFA        | \( \gamma_1 = 4 \) | \( \gamma_2 = 10 \) |
| MMVNIGFA       | \( \tilde{\gamma}_1 = 2 \) | \( \tilde{\gamma}_2 = 4 \) |

We fit the models for \( G \in \{1, 2, 3, 4\} \), and \( q, r \in \{1, 2, 3, 4, 5\} \). In Tables 2 and 3, we show the number of times that the BIC correctly chooses the number of groups, row factors and column factors. In Table 4, the average ARI and respective standard deviation for each setting is shown. As expected, as \( N \) increases, the classification performance generally gets better for all models. Also, in the case of \( d = 10 \), it is interesting to note that the number
Table 2: Number of datasets for which the BIC correctly chose the number of groups, row factors, and column factors \((d = 10)\).

| \(c\) | \(N\) | MMVSTFA \(G\) \(q\) \(r\) | MMVGHFA \(G\) \(q\) \(r\) | MMVVGFA \(G\) \(q\) \(r\) | MMVNIIGFA \(G\) \(q\) \(r\) |
|---|---|---|---|---|---|
|100 |18 |15 |19 |25 |16 |24 |18 |10 |12 |21 |21 |20 |
|1 |200 |23 |18 |21 |25 |25 |25 |21 |11 |8 |24 |24 |24 |
|400 |25 |21 |21 |25 |25 |25 |22 |17 |15 |24 |24 |25 |
|100 |18 |14 |17 |25 |9 |22 |16 |7 |4 |17 |18 |19 |
|2 |200 |24 |18 |19 |25 |22 |22 |23 |10 |2 |23 |23 |24 |
|400 |25 |23 |23 |25 |25 |25 |19 |20 |19 |25 |24 |25 |
|100 |8 |13 |14 |23 |5 |10 |23 |17 |11 |0 |24 |2 |7 |
|4 |200 |22 |9 |16 |25 |4 |16 |21 |8 |8 |24 |7 |24 |
|400 |25 |12 |21 |25 |22 |12 |17 |10 |19 |21 |0 |14 |

Table 3: Number of datasets for which the BIC correctly chose the number of groups, row factors, and column factors \((d = 30)\).

| \(c\) | \(N\) | MMVSTFA \(G\) \(q\) \(r\) | MMVGHFA \(G\) \(q\) \(r\) | MMVVGFA \(G\) \(q\) \(r\) | MMVNIIGFA \(G\) \(q\) \(r\) |
|---|---|---|---|---|---|
|100 |24 |11 |12 |25 |15 |18 |25 |12 |12 |25 |20 |21 |
|1 |200 |25 |17 |18 |25 |22 |23 |25 |21 |20 |25 |23 |25 |
|400 |25 |22 |23 |25 |25 |24 |25 |25 |20 |25 |25 |25 |
|100 |24 |15 |17 |25 |17 |18 |25 |13 |11 |25 |22 |23 |
|2 |200 |25 |22 |19 |25 |19 |22 |25 |20 |22 |25 |23 |25 |
|400 |25 |19 |20 |25 |22 |24 |25 |23 |24 |25 |24 |25 |
|100 |24 |17 |17 |25 |12 |14 |25 |17 |14 |25 |23 |16 |
|4 |200 |25 |18 |20 |25 |18 |23 |25 |21 |22 |25 |21 |21 |
|400 |25 |15 |15 |25 |20 |24 |25 |19 |20 |25 |21 |22 |

of correct choices for the row and column factors generally decreases as we increase the separation. This is particularly prevalent for the variance-gamma distribution. However, when \(d\) is increased to 30, the overall performance in choosing the correct number of groups, row factors and column factors increases for all degrees of separation. Overall the classification performance in all cases is fairly good.
Table 4: Average ARI values over 25 runs for each setting with standard deviations in parentheses.

| c | N  | MMVSTFA d = 10 | MMVSTFA d = 30 | MMVGHFA d = 10 | MMVGHFA d = 30 | MMVVGFA d = 10 | MMVVGFA d = 30 | MMVNIG d = 10 | MMVNIG d = 30 |
|---|---|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 100 |    | 0.91(0.08) 0.96(0.01) | 0.97(0.03) 0.97(0.02) | 0.90(0.1) 0.97(0.02) | 0.98(0.05) 1.00(0.00) |
| 1 | 200 | 0.98(0.03) 0.99(0.009) | 1.00(0.006) 1.00(0.007) | 0.97(0.03) 0.99(0.01) | 1.00(0.007) 1.00(0.00) |
| 400 | 1.00(0.005) 1.00(0.004) | 1.00(0.00) 1.00(0.00) | 0.99(0.03) 1.00(0.00) | 1.00(0.006) 1.00(0.00) |
| 100 |    | 0.94(0.03) 0.96(0.02) | 0.96(0.03) 0.97(0.03) | 0.88(0.1) 0.98(0.02) | 0.96(0.07) 1.00(0.00) |
| 2 | 200 | 0.98(0.02) 0.99(0.009) | 1.00(0.00) 1.00(0.00) | 0.97(0.05) 1.00(0.007) | 0.99(0.02) 1.00(0.00) |
| 400 | 1.00(0.005) 1.00(0.003) | 1.00(0.00) 1.00(0.00) | 0.98(0.05) 1.00(0.00) | 1.00(0.00) 1.00(0.00) |
| 100 |    | 0.84(0.08) 0.97(0.03) | 0.94(0.04) 0.97(0.02) | 0.92(0.1) 1.00(0.01) | 0.98(0.08) 1.00(0.00) |
| 4 | 200 | 0.98(0.02) 1.00(0.004) | 1.00(0.00) 1.00(0.00) | 0.97(0.05) 1.00(0.00) | 0.99(0.02) 1.00(0.00) |
| 400 | 1.00(0.004) 1.00(0.002) | 1.00(0.00) 1.00(0.00) | 0.98(0.03) 1.00(0.00) | 1.00(0.03) 1.00(0.00) |

5 MNIST Digits

Similar to Gallaugher & McNicholas (2018a,b), we consider the MNIST digits dataset, specifically looking at digits 1 and 7 because they are very similar in appearance. This dataset consists of 60,000 training images of Arabic numerals 0 to 9. We consider different levels of supervision and perform either clustering or semi-supervised classification. Specifically we look at 0% (clustering), 25% and 50% supervision. For each level of supervision, 25 datasets consisting of 200 images each of digit 1 and digit 7 are taken. Each of the four models presented here, as well as the MMVBFA model, is fitted for 10 to 17 row and column factors. As discussed in Gallaugher & McNicholas (2018a), because of the lack of variability in the outlying rows and columns, random noise is added to ensure non-singularity of the covariance matrices. In Table 5, the average ARI and misclassification rate (MCR) are presented for each model and each level of supervision.

In the completely unsupervised case, i.e., clustering, it is clear that the skewed models greatly outperform the normal case. In particular, the MMVVGFA model performs the best with an average misclassification rate of around 2%. At 25% supervision, the skewed models still notably outperform the MMVBFA model; however, at 50% supervision, the performance of all models is more similar.

In Figure 1, heat maps of the estimated location matrices for the MMVBFA and MMVVGFA models are shown for one of the datasets in the unsupervised case. It is clear that the images
Table 5: Average ARI and MCR values for the MNIST dataset for each level of supervision, with respective standard deviations in parentheses.

| Supervision | MMVSTFA | MMVGHFA | MMVVGFA | MMVNIGFA | MMVBFA |
|-------------|---------|---------|---------|----------|--------|
| 0% (clustering) | ARI 0.88(0.13) | 0.88(0.14) | 0.91(0.1) | 0.90(0.1) | 0.46(0.09) |
| | MCR 0.033(0.04) | 0.034(0.04) | 0.023(0.03) | 0.028(0.03) | 0.16(0.03) |
| 25% | ARI 0.93(0.03) | 0.93(0.03) | 0.93(0.03) | 0.93(0.03) | 0.81(0.15) |
| | MCR 0.017(0.009) | 0.017(0.008) | 0.017(0.009) | 0.018(0.01) | 0.052(0.04) |
| 50% | ARI 0.93(0.04) | 0.93(0.03) | 0.93(0.03) | 0.92(0.04) | 0.91(0.06) |
| | MCR 0.018(0.009) | 0.018(0.008) | 0.018(0.009) | 0.019(0.01) | 0.023(0.01) |

for the MMVVGFA models are far more clear. Moreover, although faint, the image for the 7 in the normal case appears to have an extra tail due to the misclassification of the 1s. The results are similar for each of the 25 runs.

Figure 1: Heat maps of estimated location matrices for the MMVBFA and MMVVGFA models for each class in the unsupervised case.
6 Discussion

The MMVBFA model has been extended to four skewed distributions; specifically, the matrix variate skew-$t$, generalized hyperbolic, variance-gamma and NIG distributions. AECMs algorithm was developed for parameter estimation and the approaches were illustrated on real and simulated data. In the simulation study, the models generally exhibited good performance under various scenarios. For the lower dimensional cases, the selection of the row and column factors was a little troublesome for the variance-gamma distribution, especially for a smaller sample size. Moreover, when the separation increased, the selection of row and column factors decreased in accuracy for all four models. This, however, was not the case when we increased the dimensionality. In the real data example, all four of the skewed models performed better than the MMVBFA model with. In the clustering case, this improvement was very notable. However, as the level of supervision increased, there was not a substantial improvement in the skewed models over the MMVBFA model.

Software is currently being developed for public release. Other topics of future work include considering a family of models similar to the parsimonious Gaussian mixture models of [McNicholas & Murphy (2008, 2010)]. Another area of future work would be to consider the compare this method of directly modelling skewness to using transformations such as those found in [Melnykov & Zhu (2018)]. It might also be of interest to consider matrix variate data of mixed type — in applications such as multivariate longitudinal data, the variables might be of mixed type, and hence these types of models would not be applicable. Finally, this methodology could be extended to multidimensional tensors, which would be useful for studying coloured images or black and white movie clips.

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A Mixture of Factor Analyzers Model

Because of data becoming increasingly higher dimensional, dimension reduction techniques are becoming more important. In the multivariate case, the mixture of factor analyzers model is widely used. Reverting back to the notation where \( X_i \) represents a \( p \)-dimensional random vector, with \( x_i \) as its realization, the factor analysis model for \( X_1, \ldots, X_n \) is given by

\[
X_i = \mu + \Lambda U_i + \epsilon_i,
\]

where \( \mu \) is a location vector, \( \Lambda \) is a \( p \times q \) matrix of factor loadings with \( q < p \), \( U_i \sim \mathcal{N}_q(0, I) \) denotes the latent factors, \( \epsilon_i \sim \mathcal{N}_p(0, \Psi) \), where \( \Psi = \text{diag}(\psi_1, \psi_2, \ldots, \psi_p) \), and \( U_i \) and \( \epsilon_i \) are each independently distributed and independent of one another. Under this model, the marginal distribution of \( X_i \) is \( \mathcal{N}_p(\mu, \Lambda \Lambda' + \Psi) \). Probabilistic principal component analysis
(PPCA) arises as a special case with the isotropic constraint $\Psi = \psi I$ (Tipping & Bishop 1999b).

Ghahramani & Hinton (1997) develop the mixture of factor analyzers model, which is a Gaussian mixture model with covariance structure $\Sigma_g = \Lambda_g \Lambda'_g + \Psi$. A small extension was presented by McLachlan & Peel (2000), who utilize the more general structure $\Sigma_g = \Lambda_g \Lambda'_g + \Psi_g$. Tipping & Bishop (1999a) introduce the closely-related mixture of PPCAs with $\Sigma_g = \Lambda_g \Lambda'_g + \psi_g I$. McNicholas & Murphy (2008) constructed a family of eight parsimonious Gaussian models by considering combinations of the constraints $\Lambda_g = \Lambda$, $\Psi_g = \Psi$ and $\Psi_g = \psi_g I$. There has also been work on extending the mixture of factor analyzers to other distributions, such as the skew-$t$ distribution (Murray, Browne & McNicholas 2014, Murray et al. 2017b), the generalized hyperbolic distribution (Tortora et al. 2016), the skew-normal distribution (Lin et al. 2016), the variance-gamma distribution (McNicholas et al. 2017) and others (e.g., Murray et al. 2017b).