Adaptive test for large covariance matrices with missing observations

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Abstract

We observe \( n \) independent \( p \)-dimensional Gaussian vectors with missing coordinates, that is each value (which is assumed standardized) is observed with probability \( a > 0 \). We investigate the problem of minimax nonparametric testing that the high-dimensional covariance matrix \( \Sigma \) of the underlying Gaussian distribution is the identity matrix, using these partially observed vectors. Here, \( n \) and \( p \) tend to infinity and \( a > 0 \) tends to 0, asymptotically.

We assume that \( \Sigma \) belongs to a Sobolev-type ellipsoid with parameter \( \alpha > 0 \). When \( \alpha \) is known, we give asymptotically minimax consistent test procedure and find the minimax separation rates \( \tilde{\phi}_{n,p} = (a^2 n \sqrt{p})^{-\frac{2\alpha}{4\alpha + 1}} \), under some additional constraints on \( n, p \) and \( a \).

We show that, in the particular case of Toeplitz covariance matrices, the minimax separation rates are faster, \( \hat{\phi}_{n,p} = (a^2 n p)^{-\frac{2\alpha}{4\alpha + 1}} \). We note how the "missingness" parameter \( a \) deteriorates the rates with respect to the case of fully observed vectors (\( a = 1 \)).

We also propose adaptive test procedures, that is free of the parameter \( \alpha \) in some interval, and show that the loss of rate is \( (\ln \ln(a^2 n \sqrt{p}))^{\alpha/(4\alpha + 1)} \) and \( (\ln \ln(a^2 n p))^{\alpha/(4\alpha + 1)} \) for Toeplitz covariance matrices, respectively.

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1 Introduction

Recently, problems related to high-dimensional data became more popular. In particular, in many areas such as genetics, meteorology and others, the generated data sets are high-dimensional and incomplete, in the sense that they contain missing values. In this paper we investigate the problem of testing large covariance matrices from a sample of partially observed vectors.

Let $X_1, \ldots, X_n$, be $n$ independent and identically distributed $p$-vectors following a multivariate normal distribution $\mathcal{N}_p(0, \Sigma)$, where $\Sigma = [\sigma_{ij}]_{1 \leq i,j \leq p}$ is the normalized covariance matrix, with $\sigma_{ii} = 1$, for all $i = 1$ to $p$. Let us denote $X_k = (X_{k,1}, \ldots, X_{k,p})^\top$ for all $k = 1, \ldots, n$. Let $\{\varepsilon_{k,j}\}_{1 \leq k \leq n, 1 \leq j \leq p}$ be a sequence of i.i.d. Bernoulli random variables with parameter $a \in (0, 1)$, $\mathcal{B}(a)$, and independent from $X_1, \ldots, X_n$. We observe $n$ i.i.d. random vectors $Y_1, \ldots, Y_n$ such that

$$Y_k = (\varepsilon_{k,1} \cdot X_{k,1}, \ldots, \varepsilon_{k,p} \cdot X_{k,p})^\top \quad \text{for all} \quad k = 1, \ldots, n.$$ 

Each component of the vector $Y_k$ is observed with probability equal to $a$ and this is the context of missing observations. We denote by $P_{a,\Sigma}$ the probability distribution of the random vector $Y_k$ when $X_k \sim \mathcal{N}_p(0, \Sigma)$ and $\varepsilon_{k,j} \sim \mathcal{B}(a)$. We also denote by $\mathbb{E}_{a,\Sigma}$ and $\text{Var}_{a,\Sigma}$ the expected value and the variance associated to $P_{a,\Sigma}$. Given the partially observed vectors $Y_1, \ldots, Y_n$, we want to test the null hypothesis

$$H_0 : \Sigma = I$$

against a composite alternative hypothesis

$$H_1 : \Sigma \in Q(\mathcal{G}(\alpha), \varphi)$$

where $\alpha > 0$ and $\mathcal{G}(\alpha)$ is either

$$\mathcal{F}(\alpha) = \{\Sigma > 0, \ \text{symmetric}; \frac{1}{p} \sum_{1 \leq i<j \leq p} \sigma_{ij}^2 |i-j|^{2\alpha} \leq 1 \ \text{for all} \ p \geq 1 \ \text{and} \ \sigma_{ii} = 1 \ \text{for all} \ 1 \leq i \leq p\}$$

in the general case or

$$\mathcal{T}(\alpha) = \{\Sigma > 0, \ \text{symmetric,} \Sigma \ \text{is Toeplitz}; \sum_{j \geq 1} \sigma_{jj}^2 2^{\alpha} \leq 1 \ \text{and} \ \sigma_0 = 1\}$$

for the case of Toeplitz matrices. Thus, we define the following $\ell_2$ ellipsoids with $\ell_2$ balls removed:

$$Q(\mathcal{F}(\alpha), \varphi) = \left\{ \Sigma \in \mathcal{F}(\alpha) \quad \text{such that} \quad \frac{1}{p} \sum_{1 \leq i<j \leq p} \sigma_{ij}^2 \geq \varphi^2 \right\}$$

and

$$Q(\mathcal{T}(\alpha), \varphi) = \left\{ \Sigma \in \mathcal{T}(\alpha) \quad \text{such that} \quad \sum_{j \geq 1} \sigma_{jj}^2 \geq \varphi^2 \right\}.$$
Typically, the test procedures depend on the parameter $\alpha$ and it is therefore useful to construct a test procedure that is adaptive to $\alpha$ in some interval. Here we propose minimax and adaptive procedures for testing in the context of missing observations.

The problem of estimating a covariance matrix of partially observed vectors was investigated several times in the literature. The simplest method to deal with missing data is to ignore the missing values and restrict the study to a subset of fully observed variables. This method is not always reliable mainly when the number of missing values is relatively high. Hence, in order to treat this problem, methods based on filling in the missing values were developed, in particular the Expectation-Maximization(EM) algorithm see [12]. Recently, [11] proposed an estimating procedure that does not need imputation of the missing values. Instead, the setup with missing values is treated as an inverse problem. We will also follow this approach for the test problem.

The problem of testing large covariance matrices was considered only in the case of complete data. Out of the large amount of results in the literature on this latter problem, we mention only the most related papers where procedures to test the null hypothesis $H_0$ in (1) are derived. We refer to [1], [8] and [15], where test procedures based on the likelihood ratio are proposed, and to [10], [13], [5] and [4], where test statistics based on the quadratic loss function $tr(\Sigma - I)^2$ are used. Note that in [2] and [3] asymptotically consistent test procedures where given in order to test (1) against (2), when the covariance matrices belongs to (3) and to (4), respectively. They describe the minimax and sharp minimax separation rates. Here, we give the minimax separation rates when assuming that we have partially observed vectors. We describe how the "missingness" parameter $\alpha$ deteriorates the minimax rates in this context. Moreover we develop consistent test procedures free of the class parameter $\alpha$, via an aggregation procedure of tests.

Missing observations appeared recently in random matrix theory, see [9]. They show that the sequence of the spectral measures of sample covariance matrices with missing observations converge weakly to a sequence of non random measures. Also they studied the limits of the extremes eigenvalues in the same context.

In this paper, we describe the minimax separation rate for testing $H_0$ given in (1) against the composite alternative $H_1$ in (2), when the data contains missing values. For a test procedure $\Delta$ we define the type I error probability by $\eta(\Delta) = P_I(\Delta = 1)$, the maximal type II error probability by $\beta(\Delta, Q(\mathcal{G}(\alpha), \varphi)) = \sup_{\Sigma \in Q(\mathcal{G}(\alpha), \varphi)} P_{\Sigma}(\Delta = 0)$ and the total error probability by

$$
\gamma(\Delta, Q(\mathcal{G}(\alpha), \varphi)) = \eta(\Delta) + \beta(\Delta, Q(\mathcal{G}(\alpha), \varphi)).
$$

Moreover, we define the minimax total error probability over the class $Q(\mathcal{G}(\alpha), \varphi)$ by

$$
\gamma := \inf_{\Delta} \gamma(\Delta, Q(\mathcal{G}(\alpha), \varphi)).
$$
where the infimum is taken over all possible test procedures. We define the minimax separation rate \( \tilde{\varphi}_\alpha \). On the one hand, we construct a test procedure \( \Lambda \) and derive the conditions on \( \varphi \) for which \( \gamma(\Lambda, Q(\alpha, \varphi)) \to 0 \). The test \( \Lambda \) will be called asymptotically minimax consistent. On the other hand we give the conditions on \( \varphi \) for which \( \gamma \to 1 \). The previous conditions together allow us to determinate the minimax separations rate \( \tilde{\varphi}_\alpha \), such that there exists the test \( \Lambda \) with

\[
\gamma(\Lambda, Q(G(\alpha), \varphi)) \to 0 \quad \text{if} \quad \frac{\varphi}{\tilde{\varphi}_\alpha} \to +\infty,
\]

and

\[
\gamma = \inf_{\Delta} \gamma(\Delta, Q(G(\alpha), \varphi)) \to 1 \quad \text{if} \quad \frac{\varphi}{\tilde{\varphi}_\alpha} \to 0.
\]

In other words, when \( \varphi \gg \tilde{\varphi}_\alpha \) there exists an asymptotically minimax consistent test procedure and when \( \varphi \ll \tilde{\varphi}_\alpha \), there is no asymptotically consistent test procedure which can distinguish between the null and the alternative hypothesis.

We also consider the problem of adaptation with respect to the parameter \( \alpha \). To treat this problem we first assume that \( \alpha \in A \), for \( A \) an interval, and define a larger class of matrices under the alternative than \( Q(\alpha) \). The testing problem we are interested in now, is to test \( H_0 \) in (1) against

\[
H_1 : \Sigma \in \bigcup_{\alpha \in A} Q(F(\alpha), C\psi_\alpha),
\]

where \( \psi_\alpha = \rho_{n,p}/\tilde{\varphi}_\alpha \), and \( \tilde{\varphi}_\alpha \) is the minimax separation rate of testing \( H_0 \) given in (1) against \( H_1 \) in (2) for a known \( \alpha \). Our aim is to construct a test procedure \( \Delta_{ad} \) and to find the loss \( \rho_{n,p} \) such that for a large enough constant \( C > 0 \):

\[
\gamma(\Delta_{ad}, \bigcup_{\alpha \in A} Q(F(\alpha), C\psi_\alpha)) \to 0.
\]

In this case we say that \( \Delta_{ad} \) is an asymptotically adaptive consistent test procedure.

The paper is structured as follows: in section 2 we solve the case of general covariance matrices in \( F(\alpha) \) and in section 3 the particular case of Toeplitz covariance matrices in \( T(\alpha) \). In section 2.1, we study the test problem with alternative hypothesis \( Q(F(\alpha), \varphi) \). We construct an asymptotically minimax consistent test procedure based on the data with missing observations and show that the minimax separation rate is

\[
\tilde{\varphi}_\alpha(F) = (a^2 n \sqrt{p})^{-\frac{2a}{2a+1}}.
\]

In section 2.2 we propose a test procedure adaptive to the unknown parameter \( \alpha \). In section 3 we study the problem with alternative hypothesis \( Q(T(\alpha), \varphi) \) and derive analogous results. The minimax separation rate is

\[
\tilde{\varphi}_\alpha(T) = (a^2 np)^{-\frac{2a}{2a+1}}.
\]
We can view the vectors \( X_k \) in this case as a sample of size \( p \) from a stationary Gaussian process. However, due to the missing data, this is not true anymore for vectors \( Y_k \). Minimax and adaptive rates of testing are faster by a factor \( \sqrt{p} \) over classes \( T(\alpha) \) than over the classes \( F(\alpha) \). Note that the adaptive procedure attains the rates \((\sqrt{\ln(a^2np)}/(a^2n\sqrt{p}))^{2\alpha/(4\alpha+1)} \) and \((\sqrt{\ln(a^2np)}/(a^2np))^{2\alpha/(4\alpha+1)} \), respectively. However, the parameter \( a \) describing the probability of a missing coordinate appears similarly in both cases. It actually deteriorates the rates with respect to the case \( a = 1 \) of fully observed data. Proofs are given in section 4.

Note that, for the rest of the paper asymptotics will be taken when \( n \to +\infty \), \( p \to +\infty \) and \( a \) is either fix or tends to 0 under further constraints.

2 Test for covariance matrices

We want to test from the data with missing coordinates \( Y_1, \ldots, Y_n \) the null hypothesis (1) against the alternative (2) that we recall here:

\[
H_1 : \Sigma \in Q(F(\alpha), \varphi)
\]

where \( Q(F(\alpha), \varphi) \) is given in (3). This testing problem is treated in [2], for the case of fully observed data, which correspond to \( a = 1 \) in our case. For the sake of clarity, let us recall that in [2], the following weighted U-statistic was studied

\[
\hat{D}_{n,p} = \frac{1}{n(n-1)p} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} w_{ij}X_{k,i}X_{k,j}X_{l,i}X_{l,j}.
\]

The test based on \( \hat{D}_{n,p} \) was shown to achieve minimax and sharp minimax separation rates, i.e. asymptotic equivalents of the type II error and the total error probabilities are also given when \( \varphi \asymp \varphi_\alpha \). The weights \( \{w_{ij}\}_{1 \leq i < j \leq p} \) depend on the parameter \( \alpha \) and are chosen as solution of the following optimization problem:

\[
\sup_{\{w_{ij} \geq 0; \sum_{i < j} w_{ij}^2 =\frac{1}{2}\} \Sigma \in Q(F(\alpha), \varphi)} \mathbb{E}_{\Sigma}(\hat{D}_{n,p}) = \sup_{\{w_{ij} \geq 0; \sum_{i < j} w_{ij}^2 =\frac{1}{2}\} \Sigma \in Q(F(\alpha), \varphi)} \inf_{p} \frac{1}{p} \sum_{i < j} w_{ij}\sigma_{ij}^2 =: b(\varphi),
\]

where \( b(\varphi) \sim C^\frac{1}{2}(\alpha)\varphi^{2+\frac{1}{2\alpha}} \) with \( C(\alpha) = (2\alpha + 1)/((4\alpha + 1)^{1+\frac{1}{2\alpha}}) \), if \( \varphi \to 0 \) such that \( p\varphi^{1/\alpha} \to +\infty \).

In the next section we introduce a simpler U-statistic for the case of partially observed vectors and give the asymptotic minimax separation rates, then we aggregate these tests in order to construct a procedure free of the parameter \( \alpha \).
2.1 Test procedure and minimax separation rate

Let us introduce the asymptotically minimax consistent test procedure with simpler form than ̂D_{n,p} defined above. For an integer m ∈ N large enough, such that it verifies

\[ D \leq m^\alpha \cdot \varphi \leq K^{-2\alpha} \text{ for some constants } D > 1 \text{ and } K > 0, \]

we define the following test statistic

\[
\hat{D}_{n,p,m} = \frac{1}{n(n-1)p} \cdot \frac{1}{\sqrt{2m}} \sum_{1 \leq k \neq i \leq n} \sum_{1 \leq i < j \leq p} \frac{|i-j| < m}{|i-j| 
\]

where, for m → +∞ such that m/p → 0 and that (6) holds,

\[
T_1 \leq p \cdot a^4 (1 + o(1)) + p \cdot \mathbb{E}_{a,\Sigma} (\hat{D}_{n,p,m}) \cdot O(a^2 m \sqrt{m}),
\]

\[
T_2 \leq p \cdot \mathbb{E}_{a,\Sigma} (\hat{D}_{n,p,m}) \cdot O(a^2 \sqrt{m}) + p^{3/2} \left( \mathbb{E}_{a,\Sigma} (\hat{D}_{n,p,m}) O(a^2 m^{3/4}) + \mathbb{E}_{a,\Sigma} (\hat{D}_{n,p,m}) \cdot o(a^4) \right).
\]

**Proposition 1** Under the null hypothesis, the test statistic \( \hat{D}_{n,p,m} \) in (7) is a centered random variable with variance \( \text{Var}_{a,\Sigma} (\hat{D}_{n,p,m}) = a^4/(n(n-1)p) \). Moreover,

\[
\frac{n\sqrt{p}}{a^2} \cdot \hat{D}_{n,p,m} \to N(0, 1) \text{ under } P_I \text{- probability}
\]

Under the alternative hypothesis, for all \( \Sigma \in \mathcal{F}(\alpha) \) with \( \alpha > 1/2 \),

\[
\mathbb{E}_{a,\Sigma} (\hat{D}_{n,p,m}) = \frac{a^4}{p \cdot \sqrt{2m}} \sum_{1 \leq i < j \leq p} |i-j| \leq m \sigma_{ij}^2 \text{ and } \text{Var}_{a,\Sigma} = \frac{T_1}{n(n-1)p^2} + \frac{T_2}{np^2}
\]

Proof of Proposition 1. The proof follows the same steps as the proof of Proposition 1 of [2]. We use repeatedly the independence of \((\varepsilon_{k,i})_{k,i}\) and \((X_{k,\cdot})_k\) and obvious properties of the Bernoulli random variables.

Now, we propose the following test procedure

\[
\Delta_m = \Delta_m(t) = 1(\hat{D}_{n,p,m} > t), \quad t > 0
\]

where \( \hat{D}_{n,p,m} \) is the test statistic defined in (7).

**Theorem 1**

**Upper bound:** let m → +∞ such that m/p → 0 and that (6) holds. If \( \alpha > 1/2 \) and if

\[
\varphi \to 0 \text{ and } a^2 n \sqrt{p} \varphi^{2+1/2} \rightarrow +\infty
\]
the test procedure defined in \([8]\) with \(t \leq c \cdot a^4\varphi^2 + \frac{1}{a^2}\) for some constant \(c < K(1 - D^{-2})/\sqrt{2}\) and with \(n\sqrt{p}t/a^2 \to +\infty\), is asymptotically minimax consistent, that is

\[
\gamma(\Delta_m(t), Q(F(\alpha), \varphi)) \to 0.
\]

**Lower bound:** if \(\alpha > 1/2\) and if

\[
a^2n \to +\infty, \quad p = o(1) \cdot (a^2n)^{4\alpha - 1} \quad \text{and} \quad a^2n\varphi^2 + \frac{1}{a^2} \to 0
\]

then

\[
\gamma = \inf_{\Delta} \gamma(\Delta, Q(F(\alpha), \varphi)) \to 1
\]

**Proof of Theorem 1.** The proof is given in section 4. ■

From the previous theorem we deduce that the minimax separation rate is given by:

\[
\tilde{\varphi}_\alpha = \left(\frac{a^2n\varphi}{a^2n\sqrt{p}}\right)^{-\frac{2\alpha}{4\alpha + 1}}
\]

Thus the separation \(\tilde{\varphi}_\alpha\) obtained for the observations with missing values is slower by the \(a^2\) factor than the separation rate obtained in the case of fully observed vectors.

Note that the conditions on \(t\), the threshold of \(\Delta_m(t)\) in \([8]\), are compatible. Indeed, \(a^2/(n\sqrt{p}) \ll c \cdot a^4\varphi^2 + 1/(2\alpha)\) is equivalent to our assumption that \(a^2n\varphi^2 + 1/(2\alpha) \to \infty\).

### 2.2 Adaptation

In this section we construct an asymptotically adaptive consistent test procedure \(\Delta_{ad}\) free of the parameter \(\alpha \in A := [\alpha_s, \alpha_{n,p}^*] \subset ]1/2, +\infty[\), with \(\alpha_{n,p}^* \to +\infty\) and \(\alpha_{n,p}^* = o(1)\ln(a^2n\sqrt{p})\), to test \(H_0\) given in \([1]\) against the large alternative

\[
H_1 : \Sigma = \bigcup_{\alpha \in A} \{ F(\alpha) : \frac{1}{2} \sum_{i<j} \sigma_{ij}^2 \geq (C\Phi_\alpha)^2 \},
\]

where \(C > 0\) is some constant and

\[
\Phi_\alpha = \left(\frac{\sqrt{\ln(a^2n\sqrt{p})}}{a^2n\sqrt{\varphi}}\right)^{\frac{2\alpha}{4\alpha + 1}}.
\]

For each \(\alpha \in [\alpha_s, \alpha_{n,p}^*]\), there exists \(l \in \mathbb{N}^*\) such that

\[
2^{l-1} \leq (\Phi_\alpha)^{-\frac{1}{2\alpha}} < 2^l, \quad \text{it suffices to take} \quad l \sim \frac{2}{4\alpha + 1} \ln(a^2n\sqrt{p})/\ln(2).
\]

Let \(L_s, L^* \in \mathbb{N}^*\) be defined by

\[
L_s = \left(\frac{2}{(4\alpha_{n,p}^* + 1)\ln 2}\right) \ln(a^2n\sqrt{p}) \quad \text{and} \quad L^* = \left(\frac{2}{(4\alpha_s + 1)\ln 2}\right) \ln(a^2n\sqrt{p}).
\]
We see that $L_\ast, a, d L_\ast$ tend to infinity. We define the adaptive test procedure as follows
\[
\Delta_{ad} = \max_{L_\ast \leq t \leq L_\ast} I(D_{n,p,2t} > t)
\]  
(9)
where $D_{n,p,2t}$ is the test statistic defined in (7), with $m$ replaced by $2^l$.

**Theorem 2** The test procedure $\Delta_{ad}$ defined in (9), with $t_l = a^2 \sqrt{C_\ast \ln l} / n \sqrt{p}$, verifies:

Type I error probability : $\eta(\Delta_{ad}) \to 0$, for $C_\ast > 4$.

Type II error probability : if
\[
a^2 n \sqrt{p} \to +\infty, 2^{L_\ast/p} \to 0, \ln(a^2 n \sqrt{p})/n \to 0 \text{ and } C^2 > 1 + 4 \sqrt{C_\ast}
\]
we get
\[
\beta(\Delta_{ad}, \cup_{\alpha \in A} Q(F(\alpha), C\Phi_\alpha)) \to 0.
\]

Note that the condition $2^{L_\ast/p} \to 0$ is equivalent to $a^2 n \ll p^{2\alpha_\ast}$.

**Proof of Theorem 2.** The proof of this theorem is similar to the proof of the Theorem 4 which is given in section 4.

3 Toeplitz covariance matrices

In this section we assume that the covariance matrix $\Sigma$ is Toeplitz. In this case, we are interested to test (1) against the following alternative

\[
H_1 : \Sigma \in Q(T(\alpha), \phi)
\]  
(10)
where $Q(T(\alpha), \phi)$ is defined in (4) for $\phi$ instead of $\varphi$. This testing problem is treated in [3], for the particular case $a = 1$, where a weighted U-statistic $\hat{A}_{n,p}$ of order 2 is used to construct an asymptotically consistent test procedure that achieve the sharp separation rates. Similarly to the previous setup, we construct an asymptotically consistent test procedure with constant weights. Recall that in [2] the weights are defined as solution of the following optimization problem:

\[
\sup_{\{w_j \geq 0, \sum_j w_j^2 = \frac{1}{2}\}} \inf_{\Sigma \in Q(T(\alpha), \phi)} \left\{ \sum_{j \geq 1} w_j^2 \sigma_j^2 = \inf_{\Sigma \in Q(T(\alpha), \phi)} \sum_{j \geq 1} \sigma_j^4 = C^{1/2}(\alpha) \phi^{2+\frac{1}{2\alpha}} \right\}
\]  
(11)
Remark that the optimization problems given in (5) and (11) have the same solution when $\phi \to 0$ such that $p\phi^{1/\alpha} \to +\infty$. 

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3.1 Test procedure and separation rates

Take $m \in \mathbb{N}$ such that $m \to +\infty$ and $m$ verifies (6) for $\phi$ instead of $\varphi$, we define the following test statistic:

\[ \hat{A}_{n,p,m} = \frac{1}{n(n-1)(p-m)^2} \cdot \frac{1}{\sqrt{2m}} \sum_{1 \leq k \neq l \leq n} \sum_{j=1}^{m} \sum_{m+1 \leq i_1 \leq i_2 \leq p} Y_{k,i_1} Y_{k,i_1-j} Y_{l,i_2} Y_{l,i_2-j} \]  

(12)

The main difference between the two test statistic $\hat{D}_{n,p,m}$ and $\hat{A}_{n,p,m}$ is that in this latter we take into consideration the fact that, we have repeated information on the same diagonal elements.

Now, we give bounds on the moments of this test statistic:

**Proposition 2** Under the null hypothesis $\hat{A}_{n,p,m}$ is a centered random variable whose variance is $\text{Var}_{a,I}(\hat{A}_{n,p,m}) = a^4/(n(n-1)(p-m)^2)$. Moreover, we have that $(n(p-m)/a^2) \cdot \hat{A}_{n,p,m} / \to \mathcal{N}(0,1)$. Under the alternative hypothesis, for all $\Sigma \in T(\alpha)$,

\[ \mathbb{E}_{a,\Sigma}(\hat{A}_{n,p,m}) = (a^4/\sqrt{2m}) \sum_{j=1}^{m} \sigma_j^2 \text{ and } \text{Var}_{a,\Sigma} = \frac{R_1}{n(n-1)(p-m)^4} + \frac{R_2}{n(p-m)^2}, \]

where

\[
\begin{align*}
R_1 & \leq (p-m)^2 \cdot \{a^4(1 + o(1)) + \mathbb{E}_{\Sigma}(\hat{A}_{n,p,m}) \cdot (O(a^2m) + O(a^3m^{3/2-2\alpha})) \\
& \quad + \mathbb{E}^2_{\Sigma}(\hat{A}_{n,p,m}) \cdot O(m^2) \} \\
R_2 & \leq (p-m) \cdot \{a^2 \cdot \mathbb{E}_{\Sigma}(\hat{A}_{n,p,m}) \cdot o(1) + \mathbb{E}^3_{\Sigma}(\hat{A}_{n,p,m}) \cdot (O(a \cdot m^{1/4}) + O(a^2m^{3/4-\alpha})) \\
& \quad + \mathbb{E}^2_{\Sigma}(\hat{A}_{n,p,m}) \cdot O(m) \}.
\end{align*}
\]

It is easy to show that, since $m$ verifies (6), we have for all $\Sigma \in T(\alpha)$

\[ \mathbb{E}_{a,\Sigma}(\hat{A}_{n,p,m}) \geq a^4 B \cdot \phi^{2+\frac{1}{4\alpha}} \]

where $B$ is given in (16).

To test (1) against (10), we define the following test procedure based on the statistic defined in (12):

\[ \Delta^T_m = \Delta^T_m(t) = 1(\hat{A}_{n,p,m} > t), \quad t > 0. \]

(13)

**Theorem 3**

*Upper bound:* let $m \to +\infty$, such that $m/p \to 0$ and that (6) holds. If $\alpha > 1/4$ and if

\[ \phi \to 0 \text{ and } a^2np\phi^{2+\frac{1}{4\alpha}} \to +\infty \]

the test procedure defined in (8) with $t \leq \kappa \cdot a^4 \phi^{2+\frac{1}{4\alpha}}$ for some constant $\kappa \leq B$ such that $np t/a^2 \to +\infty$ is consistent, that is $\gamma(\Delta^T_m(t), Q(T(\alpha), \phi)) \to 0$. 

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Lower bound: if \( \alpha > 1/2 \) and if
\[
a^2np \to +\infty \quad \text{and} \quad a^2np\phi^{2+\frac{1}{2\alpha}} \to 0
\]
then
\[
\gamma = \inf_{\Delta} \gamma(\Delta, Q(T(\alpha), \phi)) \to 1.
\]
The main consequence of Theorem 3 is that the separation rate is given as follows:
\[
\tilde{\phi} = \left( a^2np \right)^{-\frac{2\alpha}{4\alpha+1}}
\]

**Proof of Theorem 3.** The proof follows the same steps as the proof of Theorem 1, we therefore omit it. The most significant difference is that in order to show the lower bound in this case, we consider a sub-class of Toeplitz matrices:
\[
Q_T = \{ \Sigma U : [\Sigma U]_{ij} = 1_{(i=j)} + u_{i-j}\sigma 1_{(|i-j|<T)} \text{ for all } 1 \leq i, j \leq p , U \in U \},
\]
where \( \sigma \) and \( T \) are defined in (17) and where
\[
U = \{ U = [u_{i-j}]_{1 \leq |i-j| \leq p-1} : u_{ii} = 0, \forall i \text{ and } u_{i-j} \pm 1 \cdot 1_{(|i-j|<T)}, \text{ for } i \neq j \}.
\]
Indeed, the signs are randomized but constant on each diagonal. We re-write the terms of \( L_{n,p} \) taking into consideration the fact that the matrices are Toeplitz see, for example, the proof of lower bound in [3].

**Remark 1** Remark that the conditions on \( m \) imply that \( m \) is of order of \( \phi^{-\frac{1}{2}} \) in the case of Toeplitz covariance matrices and of order of \( \varphi^{-\frac{1}{2}} \) in the case of general covariance matrices.

### 3.2 Adaptation

In this section, it is always assumed that the covariance matrices are Toeplitz. Our goal is to construct a consistent test procedure independent of the parameter \( \alpha \in A := [\alpha_s, \alpha_{n,p}^*] \subset [1/4, +\infty[, \) such that \( \alpha_{n,p}^* \to +\infty \) and \( \alpha_{n,p}^* = o(1) \ln(a^2np), \) to test \( H_0 \) given in (1) against the large alternative
\[
H_1 : \Sigma \in \bigcup_{\alpha \in A} \left\{ T(\alpha) ; \sum_{j=1}^{p-1} \sigma_j^2 \geq (C\psi_\alpha)^2 \right\}, \quad (14)
\]
where \( C > 0 \) is some constant and
\[
\psi_\alpha = \left( \frac{\sqrt{\ln \ln(a^2np)}}{a^2np} \right)^{\frac{2\alpha}{4\alpha+1}}.
\]
First, see that \( \forall \alpha \in [\alpha_s, \alpha_{n,p}^*], \exists l \in \mathbb{N}^* \) such that
\[
2^{l-1} \leq (\psi_\alpha)^{-\frac{1}{2}} < 2^l, \quad \text{it suffices to take } l \sim \frac{2}{4\alpha+1} \frac{\ln(a^2np)}{\ln(2)}
\]
Let $L_*, L^* \in \mathbb{N}^*$ be defined by
\[ L_* = \left( \frac{2}{(4\alpha_{n,p}^* + 1) \ln 2} \right) \ln(a^2 np) \text{ and } L^* = \left( \frac{2}{(4\alpha_0 + 1) \ln 2} \right) \ln(a^2 np) \]

We aggregate tests for all given values of $l$ from $L_*$ to $L^*$ giving the following test procedure free of the parameter $\alpha$:
\[ \Delta_{ad} = \max_{L_* \leq l \leq L^*} \mathbb{I}(A_{n,p,2^l} > t_l), \]
where $A_{n,p,2^l}$ is the test statistic defined in (12), with $m$ replaced by $2^l$.

**Theorem 4** The test procedure $\Delta_{ad}$ defined in (15), with $t_l = a^2 \sqrt{C^* \ln l} / (n(p - 2^l))$, verifies:

- Type I error probability: $\eta(\Delta_{ad}) \to 0$, for $C^* > 4$.
- Type II error probability: if $a^2 np \to +\infty$, $2L^*/p \to 0$, $\ln(a^2 np)/n \to 0$ and $C^2 \geq 1 + 4\sqrt{C^*}$

we get
\[ \beta(\Delta_{ad}, \bigcup_{\alpha \in A} Q(T(\alpha), C^*_\alpha)) \to 0. \]

**Proof of Theorem 4.** The proof is given in section 4. □

Remark that the condition $2L^*/p$ gives that $a^2 n \ll p^{2\alpha^* - \frac{1}{2}}$ and $\ln(a^2 np)/n \to 0$ implies that $a^2 np \ll e^n$. Together, these conditions are mild as they give $a^2 np \ll \min\{p^{2\alpha^* + \frac{1}{2}}, e^n\}$.

4 Proofs

**Proof of Theorem 1** Upper bound We use the asymptotic normality of $\hat{D}_{n,p,m}$ to show that, the type I error probability
\[ \eta(\Delta_m(t)) = P_{a,I}(\Delta_m(t) = 1) = P_{a,I}(\hat{D}_{n,p,m} > t) = \Phi \left( -\frac{n\sqrt{p}t}{a^2} \right) + o(1) = o(1) \]
as soon as $n\sqrt{p}t/a^2 \to +\infty$. In order to control the maximal type II error probability, we use the Markov inequality to get that for all $\Sigma$ in $Q(F(\alpha), \varphi)$:
\[ P_{a,\Sigma}(\Delta_m(t) = 0) = P_{a,\Sigma}(\hat{D}_{n,p,m} < t) \leq P_{a,\Sigma} \left( |\hat{D}_{n,p,m} - E_{a,\Sigma}(\hat{D}_{n,p,m})| < E_{a,\Sigma}(\hat{D}_{n,p,m}) - t \right) \]
\[ \leq \frac{\text{Var}_{a,\Sigma}(\hat{D}_{n,p,m})}{(E_{a,\Sigma}(\hat{D}_{n,p,m}) - t)^2} = \frac{T_1 + (n-1)T_2}{n(n-1)p^2(E_{a,\Sigma}(\hat{D}_{n,p,m}) - t)^2} \]
for $t$ properly chosen. In order to bound the previous quantity uniformly in $\Sigma$ over $Q(F(\alpha), \varphi)$ we need to control
\[ \inf_{\Sigma \in Q(F(\alpha), \varphi)} E_{a,\Sigma}(\hat{D}_{n,p,m}) = \inf_{\Sigma \in Q(F(\alpha), \varphi)} \frac{a^4}{p \cdot \sqrt{2m}} \sum_{1 \leq i < j \leq p, |i-j| \leq m} \sigma_{ij}^2. \]
For all $\Sigma \in Q(F(\alpha), \varphi)$, we have

\[
E_{a,\Sigma}(\tilde{D}_{n,p,m}) = \frac{a^4}{p \cdot \sqrt{2m}} \sum_{1 \leq i < j \leq p} \sigma_{ij}^2 = \frac{a^4}{p \cdot \sqrt{2m}} \left( \sum_{1 \leq i < j \leq p} \sigma_{ij}^2 - \sum_{1 \leq i < j \leq p} \sigma_{ij}^2 \right) \\
\geq \frac{a^4}{\sqrt{2m}} \left( \varphi^2 - \frac{1}{p} \sum_{i < j} |i-j|^2 \sigma_{ij}^2 \right) \geq \frac{a^4 \varphi^2}{\sqrt{2m}} \left( 1 - \frac{1}{m^{2\alpha} \varphi^2} \right).
\]

We use [6] to get that, for all $\Sigma \in Q(F(\alpha), \varphi)$

\[
E_{a,\Sigma}(\tilde{D}_{n,p,m}) \geq \frac{a^4 K}{\sqrt{2}} \cdot \varphi^{2+\frac{1}{2\alpha}} \left( 1 - \frac{1}{D^2} \right) =: a^4 B \cdot \varphi^{2+\frac{1}{2\alpha}}, \text{ where } B = \frac{K}{\sqrt{2}} (1 - D^{-2}).
\]

Therefore, take $t \leq c \cdot a^4 \varphi^{2+\frac{1}{2\alpha}}$ for $c < B$ and use (16) to obtain that

\[
\frac{T_1}{n(n-1)p^2(E_{a,\Sigma}(\tilde{D}_{n,p,m}) - t)^2} \leq \frac{1 + o(1)}{a^4 n(n-1)p \varphi^{4+\frac{1}{\alpha}} (B - c)^2} + \frac{a^2 \cdot O(m\sqrt{m})}{a^4 n(n-1)p \varphi^{2+\frac{1}{2\alpha}} (1 - c/B)^2} = o(1),
\]

if $a^4 n(n-1)p \varphi^{4+1/\alpha} \to +\infty$ and for all $\alpha > 1/2$. Indeed, $a^2 \cdot m\sqrt{m} \varphi^{2+1/(2\alpha)} \asymp a^2 \varphi^{2-\frac{1}{\alpha}} = o(1)$.

Similarly we show that under the previous conditions the term $T_2/np^2(E_{a,\Sigma}(\tilde{D}_{n,p,m}) - t)^2$ tends to $0$.

**Lower bound** To show the lower bound we first restrict the class $Q(F(\alpha), L)$ to the class

\[
Q := \{ \Sigma_U : [\Sigma_U]_{ij} = 1_{(i=j)} + u_{ij} \sigma \mathbf{1}_{(|i-j| < T)} \text{ for all } 1 \leq i, j \leq p, U \in \mathcal{U} \},
\]

where

\[
\sigma = \varphi^{1+\frac{1}{2\alpha}}, \quad T \asymp \varphi^{-\frac{1}{\alpha}},
\]

and

\[
\mathcal{U} = \{ U = [u_{ij}]_{1 \leq i, j \leq p} : u_{ii} = 0, \forall i \text{ and } u_{ij} = u_{ji} = \pm 1 \cdot \mathbf{1}_{(|i-j| < T)}, \text{ for } i \neq j \},
\]

Denote by $\varepsilon_k = (\varepsilon_{k,1}, \ldots, \varepsilon_{k,p})^T$ the random vector with i.i.d. entries $\varepsilon_{k,i} \sim B(a)$, for all $1 \leq k \leq n$. Moreover denote by $P_\varepsilon$ and by $P_{\varepsilon_k}$ the distributions of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and of $\varepsilon_k$, respectively. Recall that the observations $Y_1, \ldots, Y_n$ verify $Y_k = \varepsilon_k \ast X_k$ for all $1 \leq k \leq n$, where $\ast$ designates the Schur product.

Under the null hypothesis $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(0, I)$, thus the conditional random vectors $Y_k|\varepsilon_k$, are independent Gaussian vectors such that, for all $1 \leq k \leq n$, $Y_k|\varepsilon_k \sim N(0, I * (\varepsilon_k \varepsilon_k^T))$. We denote respectively by $P_I$ and by $P_I^{(\varepsilon)}$ the distributions of $(Y_1, \ldots, Y_n)$ and of $(Y_1, \ldots, Y_n)|(\varepsilon_1, \ldots, \varepsilon_n)$ under the null hypothesis. Under the alternative hypothesis, for
We denote by $t$ional distribution ($\varepsilon$ Gaussian vectors such that $Y \sim \mathcal{N}(0, \Sigma_U)$ for all $1 \leq k \leq n$, where

$$(\Sigma_U \ast (\varepsilon_k \varepsilon_k^T))_{ij} = \begin{cases} 
\varepsilon_{ik} & \text{for } i = j \\
\varepsilon_{ik} \varepsilon_{jk} \cdot \sigma & \text{if } 1 < |i - j| < T \\
0 & \text{otherwise.}
\end{cases}$$

We denote by $P_U = P_{\Sigma_U}$ and $P_U^{(\varepsilon)} = P_{\Sigma_U}^{(\varepsilon)}$ the distributions of $(Y_1, \ldots, Y_n)$ and of the conditional distribution $(Y_1, \ldots, Y_n)(\varepsilon_1, \ldots, \varepsilon_n)$ respectively, when $X_1, \ldots, X_n \sim \mathcal{N}(0, \Sigma_U)$.

We define the average distribution over $Q$ by

$$P_\pi = \frac{1}{2^{p(p-1)/2}} \sum_{U \in \mathcal{U}} P_U.$$

It is known (see [14]) that the minimax total error probability satisfies

$$\gamma \geq 1 - \frac{1}{2} \|P_I - P_\pi\|_1$$

In order to prove that $\gamma \rightarrow 1$, we bound from above the $L_1$ distance by the Kullback-Leibler divergence (see [14])

$$\|P_I - P_\pi\|_1^2 \leq \frac{1}{2} \cdot K(P_I, P_\pi), \quad \text{where } K(P_I, P_\pi) := \mathbb{E}_I \log \left( \frac{dP_I}{dP_\pi} \right).$$

Therefore, to complete the proof, it is sufficient to show that $K(P_I, P_\pi) \rightarrow 0$. In order to prove this we use the conditional likelihoods as follows:

$$K(P_I, P_\pi) = \mathbb{E}_I \log \left( \frac{d(P_I \otimes P_{\Sigma_U}^{(\varepsilon)})}{d(P_\pi \otimes P_{\Sigma_U}^{(\varepsilon)})} \right) = \mathbb{E}_\varepsilon \mathbb{E}_I^{(\varepsilon)} \log \left( \frac{dP_I^{(\varepsilon)}}{dP_\pi^{(\varepsilon)}} \right).$$

Let $\varepsilon(w)$ be a realization of $\varepsilon$, we denote by $S_k \in \{1, \ldots, p\}$ the support of $\varepsilon_k(w)$ that is $\varepsilon_{k,i}(w) = 1$ if and only if $i \in S_k$. Also we denote by $d_k = \text{Card}(S_k)$, $\Sigma_{U_k}^{(\varepsilon)(w)}$ the positive matrix $\in \mathbb{R}^{d_k \times d_k}$, defined as the sub-matrix of $\Sigma_U$ obtained by removing all the $i$-th rows and columns corresponding to $i \notin S_k$ and $X_{\varepsilon_k(w)}$ the sub vector of $X_k$ of dimension $d_k$ in which we retain the coordinate with indices in $S_k$. Thus,

$$L((Y_1, \ldots, Y_n)|\varepsilon(w)) := \mathbb{E}_I^{(\varepsilon(w))} \log \left( \frac{dP_I^{(\varepsilon(w))}}{dP_\pi^{(\varepsilon(w))}}(Y_1, \ldots, Y_n) \right) = \mathbb{E}_I^{(\varepsilon(w))} \left( -\log \mathbb{E}_U \exp \left( -\frac{1}{2} \sum_{k=1}^n \left( X_{\varepsilon_k(w)}^T ((\Sigma_{U_k}^{\varepsilon(k)}(w))^{-1} - I_{\varepsilon_k}) X_{\varepsilon_k(w)} + \log \det(\Sigma_{U_k}^{\varepsilon(k)}(w)) \right) \right) \right)$$

Therefore we have

$$\mathbb{E}_\varepsilon \mathbb{E}_I^{(\varepsilon)} \left( -\log \mathbb{E}_U \exp \left( -\frac{1}{2} \sum_{k=1}^n \left( X_{\varepsilon_k}^T ((\Sigma_{U_k}^{\varepsilon(k)})^{-1} - I_{\varepsilon_k}) X_{\varepsilon_k} + \log \det(\Sigma_{U_k}^{\varepsilon(k)}) \right) \right) \right)$$
Denote by
\[
L_{n,p} := \log \mathbb{E} U \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \left( X_{\varepsilon_k}^T \left( (\Sigma_{\varepsilon_k})^{-1} - I_{\varepsilon_k} \right) X_{\varepsilon_k} + \log \det (\Sigma_{\varepsilon_k}) \right) \right) \tag{18}
\]

We define \( \Delta_{\varepsilon_k}^U = \Sigma_{\varepsilon_k} - I_{\varepsilon_k} \), for all \( U \in \mathcal{U} \) and any realization of \( \varepsilon_k \), we have \( tr(\Delta_{\varepsilon_k}^U) = 0 \) and \( \|\Delta_{\varepsilon_k}^U\| = O(\varphi^{1-\frac{1}{2n}}) \) which is \( o(1) \), as soon as \( \varphi \to 0 \) and \( \alpha > 1/2 \). In fact, by the Gershgorin’s theorem we have
\[
\|\Delta_{\varepsilon_k}^U\| \leq \max_{i \in S_k} \sum_{j \in S_k, j \neq i} |(\Delta_{\varepsilon_k}^U)_{ij}| \leq \max_{i \in S_k} \sum_{j \in S_k, 1 < |i-j| < T} |u_{ij} \sigma| \leq 2T \cdot \sigma = O(\varphi^{1-\frac{1}{2n}}).
\]

For all \( x \in [-\frac{1}{2}, +\frac{1}{2}] \) we have the following inequalities

\[
\begin{align*}
    x - x^2 + x^3 - 2x^4 & \leq -\left(\frac{1}{1+x} - 1\right) \leq x - x^2 + x^3 \\
    -x + \frac{x^2}{2} - \frac{x^3}{3} & \leq -\log(1 + x) \leq -x + \frac{x^2}{2} - \frac{x^3}{3} + x^4.
\end{align*}
\]

Applying these inequalities to the eigenvalues of \( \Delta_{\varepsilon_k}^U \) we get
\[
\Delta_{\varepsilon_k}^U - (\Delta_{\varepsilon_k}^U)^2 + (\Delta_{\varepsilon_k}^U)^3 - 2(\Delta_{\varepsilon_k}^U)^4 \leq -((\Sigma_{\varepsilon_k})^{-1} - I_{\varepsilon_k}) \leq \Delta_{\varepsilon_k}^U - (\Delta_{\varepsilon_k}^U)^2 + (\Delta_{\varepsilon_k}^U)^3
\]
\[
\frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^2 - \frac{1}{3} tr(\Delta_{\varepsilon_k}^U)^3 \leq -\log(\Sigma_{\varepsilon_k}) \leq \frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^2 - \frac{1}{3} tr(\Delta_{\varepsilon_k}^U)^3 + \frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^4,
\]

for \( \varphi \) small enough such that \( \|\Delta_{\varepsilon_k}^U\| \leq 1/2 \). Thus we can bound \( L_{n,p}, L_{n,p} \leq L_{n,p} \leq \bar{L}_{n,p} \) where
\[
\begin{align*}
    L_{n,p} := & \log \mathbb{E} U \exp \left( \frac{1}{2} \sum_{k=1}^{n} X_{\varepsilon_k}^T (\Delta_{\varepsilon_k}^U - (\Delta_{\varepsilon_k}^U)^2 + (\Delta_{\varepsilon_k}^U)^3 - 2(\Delta_{\varepsilon_k}^U)^4) X_{\varepsilon_k} \right. \\
& \left. + \frac{1}{2} \sum_{k=1}^{n} \left( \frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^2 - \frac{1}{3} tr(\Delta_{\varepsilon_k}^U)^3 \right) \right), \text{ and } \\
\bar{L}_{n,p} := & \log \mathbb{E} U \exp \left( \frac{1}{2} \sum_{k=1}^{n} X_{\varepsilon_k}^T (\Delta_{\varepsilon_k}^U - (\Delta_{\varepsilon_k}^U)^2 + (\Delta_{\varepsilon_k}^U)^3) X_{\varepsilon_k} \right. \\
& \left. + \frac{1}{2} \sum_{k=1}^{n} \left( \frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^2 - \frac{1}{3} tr(\Delta_{\varepsilon_k}^U)^3 + \frac{1}{2} tr(\Delta_{\varepsilon_k}^U)^4 \right) \right).
\end{align*}
\]

Now we develop the terms of \( \bar{L}_{n,p} \)
\[
tr(\Delta_{\varepsilon_k}^U)^2 = 2\sigma^2 \sum_{1 < |i-j| < T} \varepsilon_{k,i} \varepsilon_{k,j}, \quad tr(\Delta_{\varepsilon_k}^U)^3 = 3\sigma^3 \sum_{1 < i_1 < i_2} u_{i_1 i_2} u_{i_1 i_2} \varepsilon_{k,i_1} \varepsilon_{k,i_2} \varepsilon_{k,i_1} \varepsilon_{k,i_2}.
\]
and
\[
\text{tr}(\Delta_{U}^{\varepsilon_{k}})^{4} = \sigma^{4} \sum_{i, i_{1}, i_{2}, i_{3}} u_{ii_{1}} u_{i_{1} i_{2}} u_{i_{2} i_{3}} u_{i_{3} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, i_{3}} \varepsilon_{k, i} \\
= \sigma^{4} \sum_{1 < |i - i_{1}|, |i_{1} - i_{2}| < T} \varepsilon_{k, i} \varepsilon_{k, i_{1}} + 2\sigma^{4} \sum_{1 < |i - i_{1}|, |i_{1} - i_{2}| < T} \varepsilon_{k, i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \\
+ 4! \sigma^{4} \sum_{1 < |i_{1} - i_{2}| < T} u_{i_{1} i_{2}} u_{i_{2} i_{3}} u_{i_{3} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, i_{3}} \varepsilon_{k, i}.
\]

Moreover, we have (recall that \(u_{ij}^{2} = 1\) and \(\varepsilon_{ij}^{2} = \varepsilon_{ij}\))
\[
\sum_{k=1}^{n} X_{\varepsilon_{k}}^{T} \Delta_{U}^{\varepsilon_{k}} X_{\varepsilon_{k}} = 2\sigma \sum_{1 < |i - j| < T} u_{ij} \sum_{k=1}^{n} \varepsilon_{k, i} \varepsilon_{k, j} X_{k, i} X_{k, j} \\
\sum_{k=1}^{n} X_{\varepsilon_{k}}^{T} (\Delta_{U}^{\varepsilon_{k}})^{2} X_{\varepsilon_{k}} = \sigma^{2} \sum_{k=1}^{n} \sum_{i, j} X_{\varepsilon_{k, i}}^{T} X_{\varepsilon_{k, j}} \sum_{1 < |i_{1} - i| < T} u_{i_{1} i_{2}} u_{i_{2} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, i} \\
+ 2\sigma^{2} \sum_{k=1}^{n} \sum_{i < j} X_{\varepsilon_{k, i}} X_{\varepsilon_{k, j}} \sum_{1 < |i_{1} - i| < T} u_{i_{1} i_{2}} u_{i_{2} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, i},
\]

and
\[
\sum_{k=1}^{n} X_{\varepsilon_{k}}^{T} (\Delta_{U}^{\varepsilon_{k}})^{3} X_{\varepsilon_{k}} = \sigma^{3} \sum_{k=1}^{n} \sum_{i, j} X_{\varepsilon_{k, i}}^{T} X_{\varepsilon_{k, j}} \sum_{1 < |i - i_{1}|, |i_{1} - i_{2}|, |i_{2} - j| < T} u_{i_{1} i_{2}} u_{i_{2} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, i} \\
+ 2\sigma^{3} \sum_{k=1}^{n} \sum_{i < j} X_{\varepsilon_{k, i}}^{T} X_{\varepsilon_{k, j}} \left( u_{i_{1} i_{2}}^{3} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, j} + 2 \sum_{1 < |i_{1} - i| < T} u_{i_{1} i_{2}} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, j} \right) \\
+ 2\sigma^{3} \sum_{k=1}^{n} \sum_{i < j} X_{\varepsilon_{k, i}} X_{\varepsilon_{k, j}} \sum_{1 < |i_{1} - i| < T} u_{i_{1} i_{2}} u_{i_{2} i} \varepsilon_{k, i_{1}} \varepsilon_{k, i_{2}} \varepsilon_{k, j}.
\]
In consequence, $\bar{L}_{n,p}$ can be written as follows:

$$
\bar{L}_{n,p} = \log \mathbb{E}_U \exp \left\{ \sum_{1<i<j}^{n} u_{ij} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} X_{k,i} X_{k,j} \left( \sigma^3 + 1 + \sum_{1<i<j}^{n} \varepsilon_{k,i} \varepsilon_{k,j} \sigma^2 \left( \sum_{1<i<j}^{n} \varepsilon_{k,i} \varepsilon_{k,j} \right) \right) \right\}
$$

$$
- \sum_{1<i_1<j}^{n} u_{i_1 j} \sum_{k=1}^{n} X_{k,i} X_{k,j} \varepsilon_{k,i} \varepsilon_{k,j} \varepsilon_{k,1} \varepsilon_{k,j} \sigma^2
$$

$$
+ \sum_{1<i_1<i_2}^{n} u_{i_1 i_2} \sum_{k=1}^{n} \varepsilon_{k,i_1} \varepsilon_{k,i_2} \varepsilon_{k,i_3} \varepsilon_{k,i_4} \sigma^3 \left( 3X_{k,i}^2 - 1 \right)
$$

$$
+ \sum_{1<i_1<i_2}^{n} u_{i_1 i_2} \sum_{k=1}^{n} X_{k,i} X_{k,j} \varepsilon_{k,i} \varepsilon_{k,j} \varepsilon_{k,i_2} \varepsilon_{k,j} \sigma^3
$$

$$
+ 6 \sum_{1<i_1<i_2<i_3}^{n} u_{i_1 i_2 i_3} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,i_1} \varepsilon_{k,i_2} \varepsilon_{k,i_3} \sigma^4 \right\}
$$

$$
\frac{\sigma^2}{4} \sum_{1<i<j}^{n} e_{k,i} e_{k,j} - \frac{\sigma^2}{2} \sum_{1<i_1<j}^{n} \sum_{k=1}^{n} X_{k,i}^2 e_{k,i} e_{k,i_1}
$$

$$
\frac{\sigma^4}{4} \sum_{1<i_1<j}^{n} \sum_{k=1}^{n} e_{k,i} e_{k,i_1} + \frac{\sigma^4}{2} \sum_{1<i_1<i_2}^{n} \sum_{k=1}^{n} e_{k,i} e_{k,i_1} e_{k,i_2}
$$

We have that $\{u_{ij}\}_{1<i<j}\leq p$ is a sequence of i.i.d Rademacher random variables. Note that sequences composed of finite products of i.i.d Rademacher random variables, for example the sequences $\{u_{ir} u_{r_s} u_{s_j}\}_{1<i\neq r\neq j}$ and $\{u_{ir} u_{r_s} u_{s_j}\}_{1<i\neq r\neq s\neq j}\leq p$, form sequences of i.i.d Rademacher random variables. Moreover they are mutually independent and independent from the initial sequence $\{u_{ij}\}_{i<j}$. Now we explicit in $\bar{L}_{n,p}$ the expected value with respect to the i.i.d Rademacher random variables and get
\[
\begin{align*}
\bar{L}_{n,p} &= \sum_{i<j} \log \cosh \left( \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} X_{k,i} X_{k,j} \left( \sigma + \sigma^3 (1 + 2 \sum_{i_1} \varepsilon_{k,i_1}) \right) \right) \\
&\quad + \frac{\sigma^2}{2} \sum_{i<j} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} - \sigma^2 \sum_{i<i_1} \sum_{k=1}^{n} X_{\varepsilon_{k,i}} \varepsilon_{k,i} \varepsilon_{k,i_1} \\
&\quad + \sum_{1<i-i_1<j} \log \cosh \left( \sum_{k=1}^{n} X_{\varepsilon_{k,i}} X_{\varepsilon_{k,j}} \varepsilon_{k,i} \varepsilon_{k,j} \sigma^2 \right) \\
&\quad + \sum_{1<i-i_1<j} \log \cosh \left( 3\sigma^3 \sum_{k=1}^{n} X_{\varepsilon_{k,i}} \varepsilon_{k,i} \varepsilon_{k,i_1} \varepsilon_{k,i_2} - \sigma^3 \right) \\
&\quad + \sum_{1<i-i_1<j} \log \cosh \left( \sum_{k=1}^{n} X_{\varepsilon_{k,i}} X_{\varepsilon_{k,j}} \varepsilon_{k,i} \varepsilon_{k,j} \varepsilon_{k,i_2} \sigma^3 \right) \\
&\quad + \sum_{1<i-i_1<j} \log \cosh \left( 6 \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,i_1} \varepsilon_{k,i_2} \varepsilon_{k,i_3} \sigma^4 \right) + \frac{\sigma^4}{4} \sum_{i,i_1} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,i_1} \\
&\quad + \frac{\sigma^2}{2} \sum_{1<i-i_1<j} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,i_1} \varepsilon_{k,i_2}.
\end{align*}
\]

We use the inequality \( \frac{x^2}{2} - \frac{x^4}{12} \leq \log \cosh(x) \leq \frac{x^2}{2} \) for all \( x \in \mathbb{R} \). Thus,

\[
\begin{align*}
\bar{L}_{n,p,1} &:= \sum_{i<j} \log \cosh \left( \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} X_{k,i} X_{k,j} \left( \sigma + \sigma^3 (1 + 2 \sum_{i_1} \varepsilon_{k,i_1}) \right) \right) \\
&\quad + \frac{\sigma^2}{2} \sum_{i<j} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} - \sigma^2 \sum_{i<i_1} \sum_{k=1}^{n} X_{\varepsilon_{k,i}} \varepsilon_{k,i} \varepsilon_{k,i_1} \\
&\quad \leq \frac{1}{2} \sum_{1<i-i_1<j} \left( \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} X_{k,i} X_{k,j} \left( \sigma + \sigma^3 (1 + 2T) \right) \right)^2 \\
&\quad + \frac{\sigma^2}{2} \sum_{1<i-i_1<j} \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} - \sigma^2 \sum_{i<i_1} \sum_{k=1}^{n} X_{\varepsilon_{k,i}} \varepsilon_{k,i} \varepsilon_{k,i_1}. \quad (19)
\end{align*}
\]
Therefore,

\[ E_x E^{(e)}_I (\tilde{L}_{n,p,1}) \leq \frac{na^2}{2} \sum_{i<j}^{1<|i-j|<T} \left( \sigma + \sigma^3(1 + 2T) \right)^2 + \frac{\sigma^2}{2} \sum_{i<j}^{1<|i-j|<T} na^2 - \sigma^2 \sum_{1<|i-\tilde{i}|<T} na^2 \]

\[ = \frac{na^2}{2} \sum_{i<j}^{1<|i-j|<T} \left( 2\sigma^4(1 + 2T) + \sigma^6(1 + 2T)^2 \right) = O(na^2pT^2\sigma^4) + O(na^2p^3\sigma^6) \]

\[ = O(a^2np^4) + O(a^2np^6) = O\left((a^2n)^{\frac{4a+1}{3a+1}} p^{\frac{1}{3a+1}} \right) = o(1), \quad (20) \]

as soon as \( p = o(1)(a^2n)^{4\alpha-1} \). Similarly, we show that

\[ E_x E^{(e)}_I (\tilde{L}_{n,p,1}) \geq -\frac{1}{12} E_x E^{(e)}_I \sum_{i<j}^{1<|i-j|<T} \left( \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} X_{k,i} X_{k,j} \left( \sigma + \sigma^3(1 + 2 \sum_{1<|i-\tilde{i}|<T} \varepsilon_{k,i}) \right) \right)^4 + o(1) \]

\[ = \frac{-a^2n}{12} \cdot E^{(e)}_I (X_{1,i}^2 X_{1,j}^2) \sum_{i<j}^{1<|i-j|<T} \left( \sigma + \sigma^3(1 + 2T) \right)^4 + o(1) \]

\[ - \frac{a^4(n-1)}{12} \cdot 3E^{(e)}_I (X_{1,i}^2 X_{1,j}^2) E^{(e)}_I (X_{2,i}^2 X_{2,j}^2) \sum_{i<j}^{1<|i-j|<T} \left( \sigma + \sigma^3(1 + 2T) \right)^4. \]

See that the first term was already bounded from above in the previous display and that

\[ \frac{a^4(n-1)}{12} \cdot 3E^{(e)}_I (X_{1,i}^2 X_{1,j}^2) E^{(e)}_I (X_{2,i}^2 X_{2,j}^2) \sum_{i<j}^{1<|i-j|<T} \left( \sigma + \sigma^3(1 + 2T) \right)^4 \]

\[ \leq O(a^4n^2) \cdot \left( pT^4 + pT^2\sigma^6 + pT^3\sigma^8 + pT^4\sigma^{10} + pT^5\sigma^{12} \right) = O(a^4n^2p^{4+\frac{1}{a}}) = o(1), \]

as soon as \( a^4n^2p^{4+\frac{1}{a}} \to 0 \) and \( \alpha > 1/2 \). We deduce that \( E_x E^{(e)}_I (\tilde{L}_{n,p,1}) \geq o(1) \). As consequence \( E_x E^{(e)}_I (\tilde{L}_{n,p,1}) = o(1) \). Now we treat the second term of \( \tilde{L}_{n,p} \):

\[ \tilde{L}_{n,p,2} := \sum_{1<|i-\tilde{i}|,|i-j|<T} \log \cosh \left( \sum_{k=1}^{n} X_{\varepsilon_{k,i}} X_{\varepsilon_{k,j}} \varepsilon_{k,i} \varepsilon_{k,j} \sigma^2 \right) \]

\[ \leq \sum_{1<|i-\tilde{i}|,|i-j|<T} \left( \sum_{k=1}^{n} X_{\varepsilon_{k,i}} X_{\varepsilon_{k,j}} \varepsilon_{k,i} \varepsilon_{k,j} \sigma^2 \right)^2. \]
So,
\[
E_x E_I^{(c)}(\bar{X}_{n,p,2}) \leq E_x E_I^{(c)} \sum_{i,j} \left( \sum_{k=1}^{n} X_{\varepsilon_{k,i}} X_{\varepsilon_{k,j}} \varepsilon_{k,i} \varepsilon_{k,j} \sigma^2 \right)^2
\]
\[
= \sum_{i,j} n a^3 \sigma^4 \leq a^3 n p T \sigma^4 = O(a^3 np \varphi^{4+\frac{1}{\alpha}}) = o(1).
\]

Using the bound from below of log cosh inequality, we show that \( E_x E_I^{(c)}(\bar{X}_{n,p,2}) \) is bounded from below by a quantity that tends to zero. Therefore we get \( E_x E_I^{(c)}(\bar{X}_{n,p,2}) = o(1) \). In a similar way we show that the expected value of the remaining terms with log cosh in \( \bar{X}_{n,p} \) tend to 0. Finally we have
\[
E_x E_I^{(c)}(\bar{X}_{n,p,2}) = O(a^3 np T n) + O(a^3 np T^2 n) = O(a^3 np \varphi^{4+\frac{1}{\alpha}}) = o(1),
\]
under the previous conditions. As consequence if \( p = o(1)(a^2 n)^{4\alpha-1} \) and if \( a^4 n^2 p \varphi^{4+\frac{1}{\alpha}} \to 0 \), then
\[
E_x E_I^{(c)}(\bar{X}_{n,p,2}) = o(1).
\]

To achieve the proof, we show in a similar way that \( E_x E_I^{(c)}(\bar{L}_{n,p}) = o(1) \). \( \blacksquare \)

**Proof of Theorem 4** To control the type I error probability, we derive an inequality of Berry-Essen type for \( A_{n,p,2} \). For any fixed \( l \) in \( \mathbb{N}^+ \) we denote by \( v_{n,p,l} := \text{Var}_{a,l}(A_{n,p,2}) \), which gives \( v_{n,p,l} \sim a^4/(n^2(p-2^l)^2) \) by Proposition 2. Next, we rewrite \( A_{n,p,2} \) as follows :
\[
A_{n,p,2} = \sum_{1 \leq k < \ell \leq n} H(Y_k,Y_\ell)
\]
where,
\[
H(Y_k,Y_\ell) = \frac{\sqrt{2}}{n(n-1)(p-2^l)^2} \cdot \frac{1}{\sqrt{2^l}} \sum_{j=1}^{2^l} \sum_{i_1 < i_2} Y_{k,i_1} Y_{k,i_1-j} Y_{\ell,i_2} Y_{\ell,i_2-j}.
\]

For \( 2 \leq k, h \leq n \), define
\[
Z_k = \frac{1}{\sqrt{v_{n,p,l}}} \sum_{\ell=1}^{k-1} H(Y_k,Y_\ell), \quad \text{and} \quad S_h = \sum_{k=2}^{h} Z_k.
\]

Remark that \( \{S_h\}_{n \geq 2} \) is a centered martingale with respect to the filtration \( \{\mathcal{F}_h\}_{n \geq 2} \) where \( \mathcal{F}_h \) is the \( \sigma \)-field generated by the random vectors \( \{X_1, \ldots, X_h\} \). Note that \( A_{n,p,2} = \sqrt{v_{n,p,l}} \cdot S_n \).
and let $V_n^2 = \sum_{k=2}^n \mathbb{E}_{a,I}(Z_k^2/F_{k-1})$. We fix $0 < \delta \leq 1$ and define

$$J_n = \sum_{k=2}^n \mathbb{E}_{a,I}(Z_k)^{2+2\delta} + \mathbb{E}_{a,I}|V_n^2 - 1|^{1+\delta}.$$ 

We use the Skorokhod representation and Lemma 3.3 in [6] to obtain that, for any $0 < \varepsilon < 1/2$ and any $x \in \mathbb{R}$, there exists a positive constant $C$ depending only on $\delta$ such that

$$|P_{a,I}(A_{n,p,2l} \leq x) - \Phi\left(x/\sqrt{V_{n,p,l}}\right)| = \left|P_{a,I}\left(S_n \leq \frac{x}{\sqrt{V_{n,p,l}}}\right) - \Phi\left(x/\sqrt{V_{n,p,l}}\right)\right| \leq 16\varepsilon^{1/2} \exp\left(-\frac{x^2}{4V_{n,p,l}}\right) + C \cdot \varepsilon^{-1-\delta} J_n.$$ 

Then using that $1 - \Phi(u) \leq (1/u) \exp(-u^2/2)$ for all $u > 0$, we obtain

$$P_{a,I}(A_{n,p,2l} > x) \leq \left(1 - \Phi\left(x/\sqrt{V_{n,p,l}}\right)\right) + 16\varepsilon^{1/2} \exp\left(-\frac{x^2}{4V_{n,p,l}}\right) + C \cdot \varepsilon^{-1-\delta} J_n$$

$$\leq \left(\frac{1}{x/\sqrt{V_{n,p,l}}} + 16\varepsilon^{1/2}\right) \exp\left(-\frac{x^2}{4V_{n,p,l}}\right) + C \cdot \varepsilon^{-1-\delta} J_n. \quad (23)$$

Choose $\delta = 1$, then

$$J_n = \sum_{k=2}^n \mathbb{E}_{a,I}(Z_k)^{4} + \mathbb{E}_{a,I}|V_n^2 - 1|^2.$$ 

We can show that

$$\sum_{k=2}^n \mathbb{E}_{a,I}(Z_k)^{4} = O\left(\frac{1}{n}\right) \quad \text{and} \quad \mathbb{E}_{a,I}|V_n^2 - 1|^2 = O\left(\frac{1}{n}\right) + O\left(\frac{1}{2^l}\right) \quad (24)$$

Take $t_l = a^2 \sqrt{\mathbb{C}^* \ln l}/n(p-2l)$, we use (23) and (24) to bound from above the type I error probability:

$$P_{a,I}(\Delta_{ad} = 1) = P_{a,I}(\exists l \in \mathbb{N}, L_* \leq l \leq L^*; A_{n,p,2l} > t_l) \leq \sum_{L_* \leq l \leq L^*} P_{a,I}(A_{n,p,2l} > t_l)$$

$$\leq \sum_{l \geq L_*} \left(\frac{a^2}{n(p-2l)t_l} + 16\varepsilon^{1/2}\right) \exp\left(-\frac{t_l^2}{4V_{n,p,l}}\right) + O\left(\frac{1}{\varepsilon^2}\right) + \frac{1}{n} + \frac{1}{2^l}\right)$$

$$\leq \sum_{l \geq L_*} \left(\frac{1}{\sqrt{\mathbb{C}^* \ln l}} + 16\varepsilon^{1/2}\right) \exp\left(-\frac{C^* \ln l}{4}\right) + O\left(\frac{L^* - L_*}{n\varepsilon^2} + \frac{O(1)}{\varepsilon^2} \sum_{l \geq L_*} \frac{1}{2^l}\right)$$

$$\leq \sum_{l \geq L_*} \left(\frac{1}{\sqrt{\mathbb{C}^* \ln l}} + 16\varepsilon^{1/2}\right) l^{-C^*/4} + O\left(\frac{L_*}{n\varepsilon^2} + \frac{O(1)}{\varepsilon^2} = o(1),

for $C^* > 4$ and since $L_*$ and $L^*$ both tend to infinity, such that $\ln(a^2 n \sqrt{p})/n$ tends to 0.

Now, we control the type II error probability. Assume that $\Sigma \in T(\alpha)$ and that $\alpha$ is such that there exists $l_0 \in \{L_*, \ldots, L^*\}$ such that $2^{l_0 - 1} \leq (\psi_\alpha)^{-\frac{1}{2}} < 2^{l_0}$, thus

$$\mathbb{E}_{a,\Sigma}(\widehat{D}_{n,p,2l_0}) = \frac{a^4}{\sqrt{2 \cdot 2^{l_0}}} \left(\sum_{1 \leq j < p} \sigma_j^2 - \sum_{2^{l_0} < j < p} \sigma_j^2\right)$$

$$\geq \frac{a^4}{2(\psi_\alpha)^{1/2}} \left(C^2 \psi_\alpha^2 - \sum_{j} j^{2\alpha} \sum_{2^{l_0} < j < p} \sigma_j^2\right) \geq (\psi_\alpha)^{2+\frac{1}{2\alpha}} \cdot \frac{a^4}{2} \left(C^2 - 1\right).$$
We assumed that \( a^2 np(\psi_\alpha)^{2 + \frac{1}{2a}} = \sqrt{\ln(a^2np)} \). Moreover, we have
\[
t_{t_0} \leq \frac{a^2 \sqrt{C^* \ln L^*}}{n(p - 2t_0)} \leq \frac{a^2 \sqrt{C^* \ln(a^2np)}}{n(p - 2t_0)} \leq 2\sqrt{C^*}a^4 (\psi_\alpha)^{2 + \frac{1}{2a}}.
\]
Thus, we have \( \mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0} \geq a^4 (\psi_\alpha)^{2+1/(2a)}(C^2 - 1 - 4\sqrt{C^*})/2 \) by our assumption that \( C^2 > 1 + 4\sqrt{C^*} \). Therefore we get
\[
P_{a,\Sigma}(\Delta_{ad} = 0) = P_{a,\Sigma}(\forall l \in \{L_1, \ldots, L^*\} : A_{n,p,2l} < t_l) \leq P_{a,\Sigma}(A_{n,p,2t_0} < t_{t_0}) \\
\leq P_{a,\Sigma}(|A_{n,p,2t_0} - \mathbb{E}_{a,\Sigma}(A_{n,p,2t_0})| > \mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0}).
\]
Now, we use Markov inequality and get :
\[
P_{a,\Sigma}(\Delta_{ad} = 0) \leq \frac{\text{Var}_{a,\Sigma}(A_{n,p,2t_0})}{(\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2} \\
\leq \frac{R_1 + (n - 1)(p - 2t_0)^2 R_2}{n(n - 1)(p - 2t_0)^4 (\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2}.
\]
First, we bound from above the first term of (25), using Proposition 2
\[
S_1 := \frac{R_1}{n(n - 1)(p - 2t_0)^4 (\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2} \\
= \frac{a^4(1 + o(1)) + \mathbb{E}_{\Sigma}(A_{n,p,2t_0}) \cdot (O(a^2 \sqrt{2t_0}) + O(a^3 (2t_0)^{3/2 - 2a}))}{n(n - 1)(p - 2t_0)^2 (\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2} \\
+ \frac{\mathbb{E}_{\Sigma}^2 (A_{n,p,2t_0}) \cdot O(m^2/a)}{n(n - 1)(p - 2t_0)^2 (\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2}
\]
We decompose \( S_1 \) as sum of three terms: the first one
\[
S_{1,1} := \frac{a^4(1 + o(1))}{n(n - 1)(p - 2t_0)^2 (\mathbb{E}_{a,\Sigma}(A_{n,p,2t_0}) - t_{t_0})^2} \\
\leq \frac{a^4(1 + o(1))}{n(n - 1)(p - 2t_0)^2 a^8 (\psi_\alpha)^{4 + \frac{1}{2a}} (C^2 - 1 - 4\sqrt{C^*})^2} \\
= O\left(\frac{1}{\ln \ln(a^2 np)}\right) = o(1), \text{ as soon as } a^2 np \to +\infty.
\]
Now we show that the second term of $S_1$ also tends to 0. Recall that $2^{l_0} \asymp (\psi_\alpha)^{1/2}$, therefore

\[
S_{1,2} := \frac{O(a^2 2^{l_0}) + O(a^3(2^{l_0})^{3/2-2\alpha})}{n(n-1)(p-2^{l_0})^2E_{n,n}(A_{n,p,2^{l_0}})(1 - t_{l_0}/E_{n,n}(A_{n,p,2^{l_0}}))^2}
\leq \frac{(O(a^2 2^{l_0}) + O(a^3(2^{l_0})^{3/2-2\alpha}))}{n(n-1)(p-2^{l_0})^2E_{n,n}(A_{n,p,2^{l_0}})(1 - \frac{4\sqrt{C^*}}{C^2 - 1})^2}
\leq \frac{O(2^{l_0}) + O(a(2^{l_0})^{3/2-2\alpha})}{n(n-1)(p-2^{l_0})^2a^2(\psi_\alpha)^{2+\frac{1}{2\alpha}}(1 - \frac{4\sqrt{C^*}}{C^2 - 1})^2}
\leq \frac{O(\sqrt{2^{l_0}} \cdot (\psi_\alpha)^{2+\frac{1}{2\alpha}}) + O((2^{l_0})^{3/2-2\alpha} \cdot (\psi_\alpha)^{2+\frac{1}{2\alpha}})}{\ln \ln(a^2 np)} = o(1),
\]

since $2^{l_0} \cdot (\psi_\alpha)^{2+\frac{1}{2\alpha}} \asymp (\psi_\alpha)^{2-\frac{1}{2\alpha}} = o(1)$ and $(2^{l_0})^{3/2-2\alpha} \cdot (\psi_\alpha)^{2+\frac{1}{2\alpha}} \asymp (\psi_\alpha)^{4-\frac{1}{\alpha}} = o(1)$ for all $\alpha > 1/4$. Finally,

\[
S_{1,3} := \frac{\mathbb{E}_\Sigma^2(\tilde{A}_{n,p,2^{l_0}}) \cdot O((2^{l_0})^2)}{n(n-1)(p-2^{l_0})^2E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2}
= \frac{O(2^{l_0})^2}{n(n-1)p^2} = o(1).
\]

Now, we bound from above the second term of (25).

\[
S_2 = \frac{R_2}{n(p-2^{l_0})^2(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2}
= \frac{\mathbb{E}_\Sigma(\tilde{A}_{n,p,2^{l_0}}) \cdot o(1)}{n(p-2^{l_0})^2(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2} + \frac{\mathbb{E}_\Sigma^2(\tilde{A}_{n,p,2^{l_0}}) \cdot O(2^{l_0})}{n(p-2^{l_0})^2(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2}
+ \frac{\mathbb{E}_\Sigma^{3/2}(\tilde{A}_{n,p,2^{l_0}}) \cdot (O(a \cdot (2^{l_0})^{1/4}) + O(a^2(2^{l_0})^{3/4-a}))}{n(p-2^{l_0})(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2}.
\]

We bound from above each term of $S_2$. For the first term,

\[
S_{2,1} := \frac{a^2 \cdot \mathbb{E}_\Sigma(\tilde{A}_{n,p,2^{l_0}}) \cdot o(1)}{n(p-2^{l_0})^2(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2} \leq \frac{o(1)}{n(p-2^{l_0})a^2(\psi_\alpha)^{2+\frac{1}{2\alpha}}}
= \frac{o(1)}{\sqrt{\ln \ln(a^2 np)}} = o(1).
\]

For the second term we have,

\[
S_{2,2} := \frac{\mathbb{E}_\Sigma^2(\tilde{A}_{n,p,2^{l_0}}) \cdot O(2^{l_0})}{n(p-2^{l_0})(E_{n,n}(A_{n,p,2^{l_0}}) - t_{l_0})^2} \leq \frac{O(2^{l_0})}{np} = o(1)
\]

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Finally for the last term,

$$S_{2,3} := \frac{E^{3/2}(\widehat{A}_{n,p,t_0}) \cdot (O(a \cdot (2^{t_0})^{1/4}) + O(a^2(2^{t_0})^{3/4-\frac{1}{10}}))}{n(p-2^{t_0})(E_{0,\Sigma}(\widehat{A}_{n,p,t_0}) - t_0)^2} \leq \frac{O((2^{t_0})^{1/4}) + O(a \cdot (2^{t_0})^{3/4-\frac{1}{10}})}{n(p-2^{t_0})a^{1+\frac{1}{10}}} \leq \frac{O((2^{t_0})^{1/4})}{\sqrt{n(p-2^{t_0}) (\ln(\ln(a^2np)))^{\frac{1}{8}}} + \frac{O(a^2 \cdot \psi^{1+\frac{1}{10}} \cdot (2^{t_0})^{3/4-\frac{1}{10}})}{\sqrt{\ln(\ln(a^2np))}} = o(1),$$

as $$a^2 \cdot \psi^{1+\frac{1}{10}} \cdot (2^{t_0})^{3/4-\frac{1}{10}} \lesssim \psi^{2-\frac{1}{20}} = o(1).$$

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