The Fundamental Theorem of Perfect Simulation

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Abstract

Here several perfect simulation algorithms are brought under a single framework, and shown to derive from the same probabilistic result, called here the Fundamental Theorem of Perfect Simulation (FTPS). An exact simulation algorithm has output according to an input distribution $\pi$. Perfect simulations are a subclass of exact simulations where recursion is used either explicitly or implicitly. The FTPS gives two simple criteria that, when satisfied, give a correct perfect simulation algorithm. First the algorithm must terminate in finite time with probability 1. Second, the algorithm must be locally correct in the sense that the algorithm can be proved correct given the assumption that any recursive call used returns an output from the correct distribution. This simple idea is surprisingly powerful. Like other general techniques such as Metropolis-Hastings for approximate simulation, the FTPS allows for the flexible construction of existing and new perfect simulation protocols. This theorem can be used to verify the correctness of many perfect simulation protocols, including Acceptance Rejection, Coupling From the Past, and Recursive Bernoulli factories.

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1 Introduction

Perfect simulation algorithms generate random variates exactly from a target distribution using a random number of steps. They are typically used for the same types as problems where Markov chain Monte Carlo algorithms are employed, such as Bayesian posterior inference and approximation algorithms for \#P-complete problems. Perfect simulation algorithms are still the only methods known to give practical algorithms for exact simulation from high-dimensional examples such as the Ising model [13].

There exist many different protocols for building perfect simulation algorithms, including several variations of acceptance/rejection (AR) and coupling from the past (CFTP) [13]. There are also more specialized algorithms such as the Recursive Bernoulli factory [6].

The purpose of this work is to bring all of these methods under a common mathematical framework. Each of these methods can be individually proved to be correct. The proofs (as well as the algorithmic structure) of these protocols share common features, and the goal of this work is to identify the most important common feature. Once isolated, this notion is intuitively very compelling, and in fact it is not difficult to show that this feature gives correctness of the algorithm. With this notion in place, it becomes straightforward to show correctness of probabilistic recursive algorithms such as the Bernoulli factory.

As with Markov chain Monte Carlo approaches, the idea is simple, but the applications wide-ranging. What separates perfect simulation algorithms from other algorithms is the use of recursion. That is, after taking a step in the process, the algorithm typically calls itself again, perhaps with different parameter inputs.

In other words, at each step, the algorithm makes random choices and transforms the problem into one of simulating from a new distribution that depends on the random choices made. If the new distribution puts probability 1 on a single state, then call this a halting distribution, as the algorithm need merely output that single state and then terminate.

To have a correct perfect simulation algorithm for a target distribution \(\pi\), it must be necessary that the algorithm halts with probability 1. It turns out that this necessary condition is also sufficient: any recursive probabilistic algorithm that halts with probability 1 will output from \(\pi\) if the recursive distributions are chosen appropriately. This is the Fundamental Theorem of perfect simulation (FTPS).
Theorem 1.1 (Fundamental Theorem of perfect simulation). Consider the following algorithm

\begin{itemize}

\item \textbf{Perfect\_Simulation} \\
\textit{Input:} $\pi_i, i$

\item 1) Draw $U_i \leftarrow \nu_i$

\item 2) If $U_i \in A_i$

\item 3) Return $f_i(U_i)$

\item 4) Else

\item 5) Draw $X \leftarrow \text{Perfect\_Simulation}(\pi_{U_i}, i + 1)$

\item 6) Return $g_i(X, U_i)$

\end{itemize}

For all $i$ let

\begin{equation}
X_i = f_i(U_i) \mathbb{1}(U_i \in A_i) + g_i(Y_i, U_i) \mathbb{1}(U_i \notin A_i),
\end{equation}

where $Y_i \sim \pi_{U_i}$. If $X_i \sim \pi_i$ for all $i$ and the algorithm halts with probability 1, then the output of \textbf{Perfect\_Simulation}(\pi_0, 0) is exactly $\pi_0$.

Another way to view the FTPS is that when designing a recursive simulation algorithm, it is only necessary to verify that the algorithm is locally correct and halts with probability 1. In other words, you can assume that the recursive call correctly generates a draw from the desired distribution in proving that the algorithm works. In any other context this would be circular reasoning, but for recursive simulation algorithms that run in finite time (with probability 1), this is enough to guarantee global correctness of the algorithm.

This FTPS idea was first introduced in a text ([8]). Here we generalize the notion as first introduced and expand its application to several problems that do not appear in [8]. The remainder of the paper is organized as follows. Section 2 gives examples of perfect simulation protocols that fall into this framework, and uses the FTPS to show correctness. Section 3 then proves the FTPS and discusses various interpretations.

2 Perfect simulation protocols

This section shows how FTPS implies the correctness of several perfect simulation methodologies.
2.1 Acceptance/rejection

The acceptance/rejection (AR) protocol (also known as rejection sampling) was the first widely used perfect simulation method.

**Example 2.1** (AR for a five sided fair die). Suppose that it is possible to draw independently identically distributed samples from a fair six-side die, and the goal is to simulate uniformly from $\{1, 2, 3, 4, 5\}$.

| AR EXAMPLE |
|----------------------------------|
| 1) Draw $X \leftarrow \text{Unif}(\{1, 2, 3, 4, 5, 6\})$ |
| 2) If $X \in \{1, 2, 3, 4, 5\}$ |
| 3) return $X$ and halt |
| 4) Else |
| 5) $X \leftarrow \text{AR}_{\text{EXAMPLE}}$ |
| 6) return $X$ and halt |

Here $\text{Unif}(A)$ denotes the uniform distribution over the set $A$.

This is an example of a recursive algorithm: it might call itself in the course of execution. Note that the algorithm will halt with probability 1. Because the recursive call to the algorithm is the same as the original call, it is easy to prove correctness. For $i \in \{1, 2, 3, 4, 5\}$ and output $X$:

$$P(X = i) = 1/6 + (1/6)P(X = i).$$

Solving then gives $P(X = i) = 1/5$ as desired.

The FTPS can be applied to this example as follows. The target distribution is $\pi \sim \text{Unif}(\{1, \ldots, 5\})$. Using the notation of the pseudocode in Theorem 1.1, for all $i$, $\nu_i \sim \text{Unif}(\{1, \ldots, 6\})$, $A_i = \{1, \ldots, 5\}$, $\pi U_i \sim \pi$, $f_i(u) = u$ and $g_i(x, u) = x$. Since at each step the probability of halting is $5/6$, with probability 1 the algorithm terminates. Let

$$X_i = f_i(U_i)1(U_i \in A) + g_i(Y_i, U_i)1(U_i \notin A) = U_i1(U_i \leq 5) + Y_i1(U_i = 6)$$

where $Y_i \sim \pi$. For $i \in \{1, \ldots, 5\}$,

$$P(X_i = i) = P(U_i = i) + P(Y_i = i)P(U_i = 6) = (1/6) + (1/5)(1/6) = 1/5,$$

so the algorithm is locally correct. Global correctness follows immediately from the FTPS.
Any algorithm expressible in pseudocode can also be represented graphically as a branching process, and Figure 1 shows this representation for AR\textunderscore EXAMPLE. In the figure, $\delta(U)$ represents the Dirac delta function that puts all the probability mass on $U$. That is, for $[X|U] \sim \delta(U)$, $\mathbb{P}(X = U) = 1$.

$$U \sim \text{Unif}\{1, \ldots, 6\}$$

$$U \in \{1, \ldots, 5\} \rightarrow \delta(U)$$

$$\text{Unif}\{1, \ldots, 6\}$$

$$U = 6 \rightarrow \text{Unif}\{1, \ldots, 6\}$$

Figure 1: Branching process representation of AR\textunderscore EXAMPLE.

**Example 2.2** (General AR). Suppose that $\nu$ is a measure over $B$, and $A$ is a measurable subset of $B$ such that $\nu(A) > 0$. Then general AR samples $X \sim \nu$ conditioned to lie in $A$.

| GENERAL\textunderscore AR |
|---------------------------|
| 1) Draw $X \leftarrow \nu$
| 2) If $X \in A$
| 3) return $X$ and halt
| 4) Else
| 5) $X \leftarrow \text{GENERAL\textunderscore AR}$
| 6) return $X$ and halt |

Then as with the simple example, for all $i$, set $\nu_i \sim \nu$, $A_i = A$, $f_i(u) = u$, and $g_i(x, u) = x$. Then since the chance of halting at each step is $\nu(A) > 0$, the algorithm halts with probability 1, and FTPS gives correctness.

Careful use of AR can result in polynomial time algorithms even for high-dimensional examples. For instance, AR can be used to sample from weighted permutations for approximating the permanent of dense matrices \cite{3, 7} or satisfying assignments of disjunctive normal forms. For both these applications the associated counting problem is $\#P$-complete \cite{14, 15, 9}.
2.2 Coupling from the past

There do exist high-dimensional problems where the running time of basic AR grows exponentially with the distribution, thereby rendering the protocol impractical for these models.

A canonical example of this is the Ising model. This model takes a graph \((V, E)\), and assigns weight \(w(x)\) to \(x \in \{-1, 1\}^V\) of \(\exp(-\beta H(x))\), where \(\beta\) is a parameter of the model, and \(H(x) = -\sum_{(i,j) \in E} x(i)x(j)\) is called the Hamiltonian. The probability distribution then becomes,

\[
\pi(\{x\}) = \frac{w(x)}{Z_\beta}, \quad Z_\beta = \sum_{y \in \{-1, 1\}^V} w(y).
\]

For many decades, dealing with distributions like the Ising model through the use of Markov chains to generate approximately correct samples was the only method available. A Markov chain with a particular stationary distribution is implemented in a computer simulation via a stationary update function.

**Definition 2.1.** Call \(\phi: \Omega \times [0, 1] \rightarrow \Omega\) a stationary update function for distribution \(\pi\) over \(\Omega\) if for \(X \sim \pi\) and \(U \sim \text{Unif}([0, 1])\), \(\phi(X, U) \sim \pi\) as well.

If \(U_0, U_1, \ldots\) are iid \(\text{Unif}([0, 1])\), then setting \(X_0 = x_0\) and \(X_{t+1} = \phi(X_t, U_t)\) creates a Markov chain, and it is well known that under mild conditions the distribution of \(X_t\) will approach \(\pi\). Unfortunately, it is very difficult to determine bounds on how quickly the chain approaches distribution \(\pi\), called the mixing time of the Markov chain.

For some chains, there might exist positive probability that \(\phi(x, U)\) is the same state for all \(x \in \Omega\). When \(#\phi(\Omega, U) = 1\), say that the state space has completely coupled or coalesced.

**Example 2.3** (Coupling from the past). CFTP [13] is a perfect simulation protocol designed to use stationary update functions to generate samples exactly from the distribution \(\pi\). A recursive formulation of the algorithm is as follows. Here, suppose \(A\) is a set such that for \(u \in A\), \(\phi(\Omega, u)\) is a set that consists of only a single state.
CFTP

1) Draw $U \leftarrow \text{Unif}([0, 1])$
2) If $U \in A$
3) return the unique element of $\phi(\Omega, U)$ and halt
4) Else
5) $X \leftarrow \text{Doubling CFTP}$
6) return $\phi(X, U)$ and halt

This falls nicely in the FTPS framework. Here for all $i$, $\nu_i \sim \text{Unif}([0, 1])$, $A_i = A$, $f_i(u)$ is the unique element of $\phi(\Omega, u)$, and $g_i(x, u) = \phi(x, u)$. To use the FTPS, it must be true that $P(\#\phi(\Omega, U) = 1) > 0$. Then CFTP terminates with probability 1 and FTPS gives correctness.

Often, a single step in a Markov chain is not enough to have positive probability of coalescence. Note that for a fixed $t$, composing $\phi$ with itself $t$ times also gives a stationary update. Let $\phi_t = \phi \circ \phi \circ \cdots \circ \phi$ denote this $t$-fold composition. The hope is if $t$ is large enough, then $\phi_t$ might coalesce.

Like with the mixing time of the chain, finding $t$ exactly is not usually possible. Therefore, Propp and Wilson [13] suggested doubling $t$ at each recursive step.

Example 2.4 (Doubling CFTP). This method is described using the following pseudocode.

**Doubling_CFTP**

*Input:* $t$

1) Draw $U \leftarrow \text{Unif}([0, 1]^t)$
2) If $U \in A$
3) return the unique element of $\phi_t(\Omega, U)$ and halt
4) Else
5) $X \leftarrow \text{Doubling_CFTP}(2t)$
6) return $\phi(X, U)$ and halt

Here $\nu_t \sim \text{Unif}([0, 1]^{2^t})$, $f_i(u)$ is the unique element of $\phi(\Omega, u)$, $g_i(x, u) = \phi_{2^i}(\Omega, U)$. To prove the algorithm terminates with probability 1, it suffices for there to exist $t$ such that coalescence occurs using $\phi_t$ with positive probability.

With this condition in place, the FTPS says that to prove global correctness, one can assume that the recursive call in line 6 returns output from the
correct distribution. Assuming $X$ from line 6 is drawn from $\pi$, then for a measurable set $B$, the probability that the output of the algorithm is in $B$ is

$$
P(\phi_t(\Omega, U) \in B | U \in A)P(U \in A) + P(\phi_t(X, U) \in B | U \notin A)P(U \notin A) = P(\phi_t(X, U) \in B) = \pi(B)
$$

In [4], a more sophisticated scheme for altering $t$ was used to guarantee that the probability that the running time was much larger than the mean time decreased exponentially. Although the scheme was more complex, it also fits the framework of FTPS and so correctness immediately follows.

## 2.3 Recursive Bernoulli Factory

Bernoulli factories were introduced in [1] as a subroutine needed for perfect simulation from the stationary distribution of regenerative processes. Work on constructing efficient and practical Bernoulli factories has continued since [10, 12, 11, 5].

A Bernoulli factory works as follows. Suppose that an iid sequence of Bernoulli random variables $B_1, B_2, \ldots \sim \text{Bern}(p)$ are available but $p$ itself is unknown. The goal is to build a new random variable $X \sim \text{Bern}(p)$ as a function of the $\{B_i\}$ together with external randomness $U \sim \text{Unif}([0, 1])$ that uses as few coin flips as possible.

**Definition 2.2.** Given $p^* \in (0, 1]$ and a function $f : [0, p^*] \to [0, 1]$, a Bernoulli factory is a computable function $A$ that takes as input a number $u \in [0, 1]$ together with a sequence of values in $\{0, 1\}$, and returns an output in $\{0, 1\}$ where the following holds. For any $p \in [0, p^*], X_1, X_2, \ldots$ iid $\text{Bern}(p)$, and $U \sim \text{Unif}([0, 1])$, let $T$ be the infimum of times $t$ such that the value of $A(U, X_1, X_2, \ldots)$ only depends on the values of $X_1, \ldots, X_t$. Then

1. $T$ is a stopping time with respect to the natural filtration and $P(T < \infty) = 1$.
2. $A(U, X_1, X_2, \ldots) \sim \text{Bern}(f(p))$.

Call $T$ the running time of the Bernoulli factory.

Using the perfect simulation notation from earlier, a Bernoulli factory algorithm is a perfect simulation algorithm for $\text{Bern}(f(p))$ such that for all $i$, $\nu_i \in \text{Bern}(p)^k \times \text{Unif}([0, 1])$ for some nonnegative integer $k$. The state space for
Bern(f(p)), is \{0, 1\}, and so it holds that \( f_i(u) \in \{0, 1\} \) and \( g_i(x, u) \in \{0, 1\} \) for all \( i \).

In other words, all of the distributions employed by the perfect simulation algorithm also must be Bernoulli distributions. That means that to check (1), it suffices to show that the probability the left hand side equals 1 equals the probability the right hand side is 1, greatly simplifying the calculations.

**Example 2.5** (Von Neumann’s Bernoulli factory). Von Neumann [16] constructed a simple Bernoulli factory where \( f(p) = 1/2 \) for all \( p \). It utilized two flips of the coin at each level of recursion, and is represented graphically in Figure 2. At each level of the recursion there is a \( 2p(1-p) \) chance of halting,

\[
(U_1, U_2) \text{ are iid } \text{Bern}(p)
\]

\[
\begin{align*}
U_1 = 1, U_2 = 0 & \quad \text{Bern}(1) \\
U_1 = U_2 & \quad \text{Bern}(1/2) \\
U_1 = 0, U_2 = 1 & \quad \text{Bern}(0)
\end{align*}
\]

Figure 2: Branching process representation of the Von Neumann constant Bernoulli factory.

so for \( p \in (0, 1) \) the algorithm terminates in finite time with probability 1. Moreover,

\[
1/2 = p(1-p)(1) + [p^2 + (1-p)^2](1/2) + (1-p)p(0),
\]

so the local correctness condition is satisfied. The algorithm is therefore correct by the FTPS.

**Example 2.6** (Exponential Bernoulli factory). In [2] showed how to build a Bernoulli factory for \( f(p) = \exp(-p) \), which was needed as part of a method for perfectly simulating from diffusions. They created such a factory using a thinned Poisson process.
Consider here the slightly more general problem of drawing from \( f(p) = \exp(-Cp) \), where \( C \) is a known positive constant. Then using a single coin flip together with an exponential random variable, the algorithm for this factory is represented in Figure 3.

\[
U_1 \sim \text{Bern}(p), U_2 \sim \text{Exp}(C)
\]

\[
\begin{array}{c}
\text{Bern}(\exp(-Cp)) \\
\text{Bern}(1) \\
\text{Bern}(0)
\end{array}
\]

\[
\begin{array}{c}
U_2 \geq 1 \\
U_2 < 1, U_1 = 1 \\
U_2 < 1, U_1 = 0
\end{array}
\rightarrow
\begin{array}{c}
\text{Bern}(\exp(-C(1-U_2)p))
\end{array}
\]

Figure 3: Branching process representation of an exponential factory.

The probability that \( U_2 \geq 1 \) is \( \exp(-C) \). If \( U_2 < 1 \) and \( U_1 = 0 \), then the probability of a 1 becomes \( \exp(-C(1-U_2)p) \). Also, \( \mathbb{P}(U_1 = 0) = 1 - p \). Therefore, the right hand side of equation (1) is:

\[
\exp(-C) + (1-p) \int_{u_2 \in [0,1]} C \exp(-Cu_2) \exp(-C(1-u_2)p) 
\] 

which evaluates to \( \exp(-Cp) \) as desired.

At each recursive step, there is at least a \( \exp(-C) \) chance of terminating and so the overall algorithm terminates with probability 1. Therefore the FTPS immediately gives correctness.

**Example 2.7** (Linear Bernoulli factory). The original application of Asmussen et. al [1] required Bernoulli factories of the form \( f(p) = Cp \) for a constant \( p \).

Nacu and Peres [12] called a randomized algorithm with random running time \( T \) a fast simulation if there existed constants \( M > 0 \) and \( \rho < 1 \) such that \( \mathbb{P}(T > t) \leq M \rho^t \) for all \( t > 0 \). One of their results was that if \( 2p \) has a fast simulation, then any function \( f(p) \) that is real analytic over \( (0, 1) \) has a
fast simulation. The converse also holds: any \( f \) with a fast simulation is real analytic on any open subset of its domain.

For these reasons, the \( C_p \) Bernoulli factory is especially important. The first provably polynomial expected time Bernoulli factory for \( C_p \) coins was introduced in \([6]\), and was an explicitly recursive perfect simulation algorithm. It was shown there that the expected number of coin flips needed was bounded above by

\[
9.5C \epsilon^{-1}.
\]

It was also shown in \([6]\) that any Bernoulli factory that worked for all \( p \) and \( C \) such that \( C p \in [0, 1 - \epsilon] \) required at least

\[
0.004C \epsilon^{-1}
\]

flips of the coin on average. Hence the algorithm of \([6]\) is the best possible up to the constant.

The Bernoulli factory of \([6]\) actually solves the more general problem of flipping a \((Cp)^i\) coin for any integer \( i \). Of course, \( i = 1 \) is the case of actual interest, but the factory works for any integer \( i \geq 1 \).

This algorithm can be represented using three types of recursions. In the first recursion (Figure 4), a single \( p \)-coin is flipped. If it is heads, then the algorithm halts and outputs a 1, otherwise it changes the problem to flipping a \((C - 1)p/(1 - p)\) coin.

\[
U_1 \sim \text{Bern}(p)
\]

\[
\begin{array}{c}
U_1 = 1 \\
\text{Bern}((Cp)^{i-1})
\end{array}
\]

\[
\begin{array}{c}
U_1 = 0 \\
\text{Bern}((Cp)^{i-1}(C - 1)p/(1 - p))
\end{array}
\]

Figure 4: The first piece of the recursive Bernoulli factory.

When \( i = 0 \), the goal is just to flip a \( \text{Bern}(1) \)-coin, which is always 1, and so this is a halting state.
The second piece attempts to turn a \((C - 1)p/(1 - p)\)-coin flip problem back into a \(Cp\)-coin flip problem. This is done by flipping a \((C - 1)/C\)-coin. If heads, then it is necessary to flip one \(Cp\)-coin. Otherwise, it is necessary to flip both one \(Cp\)-coin and still one \((C - 1)p/(1 - p)\)-coin. This step can be repeated (a geometrically distributed number of times) until the \((C - 1)p/(1 - p)\)-coin flip is gone. This is represented in Figure 5.

\[
U_2 \sim \text{Bern}((C - 1)/C)
\]

\[
\begin{array}{c}
\text{Bern} \left( \frac{(Cp)^i(Cp-1)p}{1-p} \right) \\
U_1 = 1 \quad \text{Bern}((Cp)^{i+1}) \\
U_1 = 0 \quad \text{Bern} \left( \frac{(Cp)^{i+1}(C-1)p}{1-p} \right)
\end{array}
\]

Figure 5: The second piece of the recursive Bernoulli factory.

The third and final piece works for any function \(g(p)\) and parameter \(\alpha\) such that \(g(p) \leq \alpha\). It flips an \(\alpha\)-coin. If this is tails, then the overall output is tails. If it is heads, then a \(\alpha^{-1}g(p)\)-coin must be flipped.

These pieces are combined as follows. Begin with \(i = 1\), and use the first piece to either move to \(i = 0\) (which halts) or to a \((C - 1)p/(1 - p)\). Use the second piece to replace the \((C - 1)p/(1 - p)\)-coin with a geometric (with mean \(C/(C - 1)\)) number of \(Cp\)-coins. Continue until \(i = 0\) or \(i \geq 4.6\epsilon^{-1}\). At this point, for \(Cp \leq 1 - \epsilon\), \((Cp)^i \leq \alpha = 1/(1 + \epsilon/2)^i\), and so the third piece of the recursion can be employed. Reset \(\epsilon\) to be \(\epsilon/2\), \(C\) to be \(C(1 + \epsilon/2)\), and return to the earlier stage until once again \(i = 0\) or \(i \geq 4.6\epsilon^{-1}\). Continue until termination occurs. Theorem 1 of [6] showed that the expected running time of this algorithm was at most \(9.5C\epsilon^{-1}\).

**Lemma 2.1.** The algorithm is a correct Bernoulli factory.

**Proof.** To show global correctness, it suffices to first show local correctness for the three pieces of the recursion. Since Bernoulli distributions are determined
by their mean, that is equivalent to verifying

$$(Cp)^i = p(Cp)^{i-1} + (1-p)(Cp)^{i-1}(C-1)p/(1-p)$$

$$(Cp)^i(C-1)p \frac{1}{1-p} = \frac{C-1}{C}(Cp)^{i+1} + \frac{1}{C}(Cp)^{i-1}(C-1)p \frac{1}{1-p}$$

$$g(p) = \alpha \cdot \alpha^{-1}g(p) + (1-\alpha) \cdot 0$$

Each of these results is straightforward to verify.

Since the expected running time of the algorithm is finite, the algorithm terminates with probability 1, and so the FTPS immediately gives that the algorithm is correct. \qed

3 Proof of the FTPS

Let $X$ denote the output of $\text{Perfect Simulation}(\pi,0)$. Let $T$ denote the largest value of $i$ attained during recursive calls to the algorithm. Then the assumption that the algorithm halts with probability 1 is equivalent to saying that the probability $T$ is finite is 1. The following tells us how close the output distribution is to the target after a finite number of steps.

**Lemma 3.1.** Suppose equation (1) holds, and for all $i$ let $Y_i \sim \pi_{U_i}$. Then for all $i$ and measurable $C$,

$$\pi(C) = \mathbb{P}(X \in C, T < i) + \mathbb{P}(f_i(U_i) \in C, T = i) + \mathbb{P}(g_i(Y_i, U_i) \in C, T > i).$$

(2)
Proof. The proof proceeds by induction. Start with the $i = 0$ case. Then always $T \geq 0$, so the first term on the right hand side is 0. For $T = 0$, it must hold that $U_0 \in A_0$. Since $\pi_0 \sim \pi$, equation (2) becomes
\[
\pi_0(C) = \mathbb{P}(f(U_0) \in C, U_0 \in A_0) + \mathbb{P}(g_0(U_0, Y_0) \in C, U_0 \notin A_0) = \mathbb{P}(X_0 \in C).
\]
By equation (1) this holds.

Our induction hypothesis assumes (2) holds for $i$, and consider what happens with $i + 1$:
\[
\mathbb{P}(X \in C, T < i + 1) = \mathbb{P}(X \in C, T < i) + \mathbb{P}(X \in C, T = i)
\]
\[
= \mathbb{P}(X \in C, T < i) + \mathbb{P}(f_i(U_i) \in C, T = i)
\]
\[
= \pi(C) - \mathbb{P}(g_i(Y_i, U_i) \in C, T > i)
\]
where the last step is our induction hypothesis. Rearranging gives
\[
\pi(C) = \mathbb{P}(X \in C, T < i + 1) + \mathbb{P}(g_i(Y_i, U_i) \in C, T > i).
\]
To understand the second term on the right, note $\pi_{U_i} \sim \pi_{i+1}$, so by (2)
\[
Y_i \sim X_{i+1} = f_{i+1}(U_{i+1})1(U_{i+1} \in A_{i+1}) + g_{i+1}(Y_{i+1}, U_{i+1})1(U_{i+1} \notin A_{i+1}).
\]
That implies that
\[
\mathbb{P}(g_i(Y_i, U_i) \in C, T > i) = \mathbb{P}(f_{i+1}(U_{i+1}) \in C, T = i) + \mathbb{P}(g_{i+1}(Y_{i+1}, U_{i+1}) \in C, T > i),
\]
which completes the induction.

This leads to a simple bound on the output probabilities.

**Lemma 3.2.** For all measurable $C$ and $i$,
\[
\mathbb{P}(X \in C, T < i) \leq \pi(C) \leq \mathbb{P}(X \in C, T < i) + \mathbb{P}(T \geq i). \tag{3}
\]

**Proof.** The two rightmost terms in (2) are bounded below by 1, and above by $\mathbb{P}(T \geq i)$, which gives the bound.

With this bound in hand, the FTPS can now be proved.

**Proof of the FTPS.** Let $C$ be any measurable set. Simply take the limit as $i$ goes to infinity of (3). If $\mathbb{P}(T < \infty) = 1$, then by the Dominated Convergence Theorem, this gives
\[
\mathbb{P}(X \in C) \leq \pi(C) \leq \mathbb{P}(X \in C),
\]
which implies $\mathbb{P}(X \in C) = \pi(C)$. \qed

14
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