ArXiv Identifier: 1606.00435v2

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We study a testable dark matter (DM) model outside of the standard WIMP paradigm in which the observed ratio \( \Omega_{\text{dark}} \approx \Omega_{\text{visible}} \) is insensitive to the axion mass which may assume any value within the observationally allowed window \( 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV} \). We also estimate the baryon to entropy ratio \( \eta \equiv n_B / n_s \sim 10^{-10} \) within this scenario. Finally, we comment on implications of these results to the axion search experiments, including microwave cavity and the Orpheus experiments.

I. INTRODUCTION

The origin of the observed asymmetry between matter and antimatter is one of the largest open questions in cosmology. The nature of the dark matter (DM) is another open question in cosmology. In this paper we advocate an idea that these two, apparently unrelated, problems are in fact two sides of the same coin. Furthermore, both mysterious effects are originated at one and the same cosmological epoch from one and the same QCD physics. Normally, it is assumed that the majority of dark matter is represented by a new fundamental field coupled only weakly to the standard model particles, these models may then be tuned to match the observed dark matter properties. We take a different perspective and consider the possibility that the dark matter is in fact composed of well known quarks and antiquarks but in a new high density phase, similar to the Witten’s strangelets, see original work [1] and some related studies [2].

There are few new crucial elements in proposal [3, 4], in comparison with previous studies [1, 2]. First of all, the nuggets could be made of matter as well as antimatter in our framework as a result of separation of charges, see few comments below. Secondly, the stability of the DM nuggets is provided by the axion domain walls with extra pressure, in contrast with original studies when stability is assumed to be achieved even in vacuum, at zero external pressure. Finally, an overall coherent baryon asymmetry in the entire Universe is a result of the strong CP violation due to the fundamental \( \theta \) parameter in QCD which is assumed to be nonzero at the beginning of the QCD phase transition. This source of strong CP violation is no longer available at the present epoch as a result of their common QCD origin when both types of matter (DM and visible) are formed at the QCD phase transition and both are proportional to \( \Lambda_{\text{QCD}} \). Instead of conventional “baryogenesis” mechanism we advocate a paradigm when the “baryogenesis” is actually a charge separation process which always occur in the presence of the CP odd axion field \( a(x) \). In this scenario the global baryon number of the Universe remains zero, while the unobserved anti-baryon charge is hidden in form of heavy nuggets, similar to Witten’s strangelets and compromise the DM of the Universe.

In the present work we study in great details a possible formation mechanism of such macroscopically large heavy objects. We argue that the nuggets will be inevitably produced during the QCD phase transition as a result of Kibble-Zurek mechanism on formation of the topological defects during a phase transition. Relevant topological defects in our scenario are the closed bubbles made of the \( N_{\text{DW}} = 1 \) axion domain walls. These bubbles, in general, accrete the baryon (or anti baryon) charge, which eventually result in formation of the nuggets and anti-nuggets carrying a huge baryon (anti-baryon) charge. A typical size and the baryon charge of these macroscopically large objects is mainly determined by the axion mass \( m_a \). However, the main consequence of the model, \( \Omega_{\text{dark}} \approx \Omega_{\text{visible}} \) is insensitive to the axion mass which may assume any value within the observationally allowed window \( 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV} \). We also estimate the baryon to entropy ratio \( \eta \equiv n_B / n_s \sim 10^{-10} \) within this scenario. Finally, we comment on implications of these results to the axion search experiments, including microwave cavity and the Orpheus experiments.

II. THE FORMATION OF AXION FIeld AND THE QUARK NUGGETS

In the model [3, 4] baryogenesis occurs at the QCD phase transition. Both quarks and antiquarks are thermally abundant in the primordial plasma but, in addition to forming conventional baryons, some fraction of them are bound into heavy nuggets of quark matter in a colour superconducting phase. Nuggets of both matter and antimatter are formed as a result of the dynamics of the axion domain walls as originally proposed in refs. [3, 4]. A number of very hard dynamical questions in strongly coupled QCD which are related to the nuggets’s formation have not been studied in any details in the original papers. The main goal of the present work is to make the first step in the direction to address these hard questions.

If fundamental \( \theta \) parameter were identically zero at the
QCD phase transition in early universe, an equal number of nuggets made of matter and antimatter would be formed. It would result in vanishing of the visible baryon density at the present epoch. However, the fundamental CP violating processes associated with the \( \theta \) term in QCD (which is assumed to be small, but still non-zero at the very beginning of the QCD phase transition) result in the preferential formation of anti-nuggets over the nuggets. This preference is essentially determined by the dynamics of coherent axion field \( \theta(x) \) at the initial stage of the nugget’s formation. The resulting asymmetry is not sensitive to a small magnitude of the axion field \( \theta \) at the QCD phase transition as long as it remains coherent on the scale of the Universe, see section VII for the details.

The remaining antibaryons in the plasma then annihilate away leaving only the baryons whose antimatter counterparts are bound in the excess of anti-nuggets and thus unavailable to annihilate. All asymmetry effects are of first order, irrespectively to the magnitude of \( \theta \) thus unavailable to annihilate. All asymmetry effects are counterparts are bound in the excess of anti-nuggets and details.

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The remaining antibaryons in the plasma then annihilate away leaving only the baryons whose antimatter counterparts are bound in the excess of anti-nuggets and thus unavailable to annihilate. All asymmetry effects are of first order, irrespectively to the magnitude of \( \theta \) at the moment of formation. This is precisely the main reason of why the visible and dark matter densities must be the same order of magnitude

\[
\Omega_{\text{dark}} \approx \Omega_{\text{visible}} \tag{1}
\]
as they both proportional to the same fundamental \( \Lambda_{\text{QCD}} \) scale, and they both are originated at the same QCD epoch. In particular, if one assumes that the nuggets and anti-nuggets saturate the dark matter density than the observed matter to dark matter ratio \( \Omega_{\text{dark}} \approx 5 \cdot \Omega_{\text{visible}} \) corresponds to a specific proportion when number of anti-nuggets is larger than number of nuggets by a factor of \( \sim 3/2 \) at the end of nugget’s formation. This would result in a matter content with baryons, quark nuggets and antiquark nuggets in an approximate ratio

\[
|B_{\text{visible}}| : |B_{\text{nuggets}}| : |B_{\text{antinuggets}}| \approx 1 : 2 : 3, \tag{2}
\]
with no net baryonic charge. If these processes are not fundamentally related the two components \( \Omega_{\text{dark}} \) and \( \Omega_{\text{visible}} \) could easily exist at vastly different scales.

Though the QCD phase diagram at \( \theta \neq 0 \) as a function of \( T \) and \( \mu \) is basically unknown, it is well understood that \( \theta \) is in fact the angular variable, and therefore supports various types of the domain walls, including the so-called \( N_{DW} = 1 \) domain walls when \( \theta \) interpolates between one and the same physical vacuum state \( \theta \rightarrow \theta + 2\pi \). Furthermore, it is expected that the closed bubbles made of these \( N_{DW} = 1 \) axion domain walls are also produced during the QCD phase transition with a typical wall tension \( \sigma_a \sim m_a^{-1} \) where \( m_a \) is the axion mass. Precisely this scale determines the size and the baryon charge of the nuggets, see (3), (4) below.

The collapse of these close bubbles is halted due to the Fermi pressure acting inside of the bubbles. The crucial element which stops the collapse of the bubbles from complete annihilation is the presence of the QCD substructure inside the axion domain wall. This substructure forms immediately after the QCD phase transition as discussed in [3]. The equilibrium of the obtained system has been analyzed in [3] for a specific axion domain wall tension within the observationally allowed window \( 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV} \) consistent with the recent constraints [8–15]. It has been also argued in [3] that the equilibrium is typically achieved when the Fermi pressure inside the nuggets falls into the region when the colour superconductivity (CS) indeed sets in\(^1\).

The size and the baryon charge of the nuggets scale with the axion mass as follows

\[
\sigma_a \sim m_a^{-1}, \quad R \sim \sigma_a, \quad B \sim \sigma_a^3. \tag{3}
\]

Therefore, when the axion mass \( m_a \) varies within the observationally allowed window \( 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV} \) the nuggets parameters also vary as follows

\[
10^{-6} \text{cm} \lesssim R \lesssim 10^{-3} \text{cm}, \quad 10^{23} \lesssim B \lesssim 10^{32}, \tag{4}
\]

where the lowest axion mass \( m_a \approx 10^{-6} \text{eV} \) approximately\(^2\) corresponds to the largest possible nuggets with \( \langle B \rangle \geq 10^{32} \). Variation of the axion mass by three orders of magnitude results in variation of the nugget’s baryon charge by nine orders of magnitude according to relation (3). The corresponding allowed region is essentially unconstrained by present experiments, see details in section II below.

The fact that the CS may be realized in nature in the cores of neutron stars has been known for sometime [16, 17]. A new element which was advocated in proposal [3] is that a similar dense environment can be realized in nature due to the axion domain wall pressure playing a role of a “squeezer,” similar to the gravity pressure in the neutron star physics.

Another fundamental ratio (along with \( \Omega_{\text{dark}} \approx \Omega_{\text{visible}} \) discussed above) is the baryon to entropy ratio at present time

\[
\eta \equiv \frac{n_B - n_{\bar{B}}}{n_\gamma} \sim \frac{n_B}{n_\gamma} \sim 10^{-10}. \tag{5}
\]

If the nuggets were not present after the phase transition the conventional baryons and anti-baryons would continue to annihilate each other until the temperature reaches \( T \approx 22 \text{ MeV} \) when density would be 9 orders of magnitude smaller than observed (5). This annihilation catastrophe, normally thought to be resolved as a result

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\(^1\) There is no requirement on the first order phase transition (in contrast with original proposal [1]) for the bubble formation in this framework because the \( N_{DW} = 1 \) axion domain walls are formed irrespectively to the order of the phase transition. Needless to say that the phase diagram in general and the order of the phase transition in particular at \( \theta \neq 0 \) are still unknown because of the longstanding “sign problem” in the QCD lattice simulations at \( \theta \neq 0 \), see few comments and related references in Conclusion.

\(^2\) There is no one-to-one correspondence between the axion mass \( m_a \) and the baryon charge of the nuggets \( B \) because for each given \( m_a \) there is an extended window of stable solutions describing different nugget’s sizes [3].
of “baryogenesis” as formulated by Sakharov [18], see also review [19]. In this framework the ratio (5) is highly sensitive to many specific details of the models such as the spectrum of the system in general and the coupling constants and the strength of CP violation in particular, see e.g. review [19].

In our proposal (in contrast with conventional frameworks on baryogenesis) this ratio is determined by a single parameter with a typical QCD scale, the formation temperature $T_{\text{form}}$. This temperature is defined by a moment in evolution of the Universe when the nuggets and anti-nuggets basically have completed their formation and not much annihilation would occur at lower temperatures $T \leq T_{\text{form}}$. The exact magnitude of temperature $T_{\text{form}} \approx T_{\text{QCD}}^3$ in our proposal is determined by many factors: transmission/reflection coefficients, evolution of the nuggets, expansion of the universe, cooling rates, evaporation rates, viscosity of the environment, the dynamics of the axion domain wall network, etc. All these effects are, in general, equally contribute to $T_{\text{form}}$, at the QCD scale. Technically, the corresponding effects are hard to compute from the first principles as even basic properties of the QCD phase diagram at nonzero $\theta \neq 0$ are still unknown $^3$. We plot three different conjectured cooling paths on Fig. 1.

However, the estimate of $T_{\text{form}}$ up to factor 2 is quite a simple exercise as $T_{\text{form}}$ must be proportional to the gap $\Delta \sim 100$ MeV when colour superconducting (CS) phase sets in inside the nuggets. The observed ratio (5) corresponds to $T_{\text{form}} \approx 40$ MeV, see [4] for the details. This temperature indeed represents a typical QCD scale, slightly below the critical temperature $T_{\text{CS}} \approx 0.6\Delta \approx 60$ MeV, according to standard estimates on colour superconductivity, see reviews [16, 17].

Unlike conventional dark matter candidates, such as WIMPs (Weakly Interacting Massive Particles) the dark-matter/antimatter nuggets are strongly interacting but macroscopically large. They do not contradict any of the many known observational constraints on dark matter or antimatter for three main reasons [20]:

- They carry a huge (anti)baryon charge $|B| \gtrsim 10^{25}$, and so have an extremely tiny number density;
- The nuggets have nuclear densities, so their effective interaction is small $\sigma/M \sim 10^{-10}$ cm$^2$/g, well below the typical astrophysical and cosmological limits which are on the order of $\sigma/M < 1$ cm$^2$/g;
- They have a large binding energy $\sim \Delta$, such that baryon charge in the nuggets is not available to participate in big bang nucleosynthesis (BBN) at $T \approx 1$ MeV.

To reiterate: the weakness of the visible-dark matter interaction is achieved in this model due to the small geometrical parameter $\sigma/M \sim B^{-1/3}$ rather than due to a weak coupling of a new fundamental field with standard model particles. In other words, this small effective interaction $\sim \sigma/M \sim B^{-1/3}$ replaces a conventional requirement of sufficiently weak interactions of the visible matter with WIMPs.

As we already mentioned, this model when DM is represented by quark and antiquark nuggets is consistent with fundamental astrophysical constraints as highlighted above. Furthermore, there is a number of frequency bands where some excess of emission was observed, but not explained by conventional astrophysical sources. Our comment here is that this model may explain some portion, or even entire excess of the observed radiation in these frequency bands. This phenomenological part of the proposal is the key ingredient in our advocacy of the model, and may play very important role for interpretation of the present and future observations. Therefore, we devote next section II to review

\[ ^3 \text{The basic consequence (1) as well as (5) of this proposal are largely insensitive to the absolute value of initial magnitude of the } \theta \text{ parameter. In other words, a fine tuning of the initial } \theta \text{ parameter is not required in this mechanism. The same comment (on “insensitivity” of the initial conditions) also applies to efficiency of the nugget’s formation. This is because the baryon density at the present time is } 10 \text{ orders of magnitude lower than the particle density at the QCD phase transition epoch according to the observations (5). Therefore, even a sufficiently low efficiency of the nugget’s formation (still larger than } 10^{-7}, \text{ see estimates in section VII C) cannot drastically modify the generic relations (1), (5) due to a long evolution which eventually washes out any sensitivity to the initial conditions. The only crucial parameter which determines the final outcome (1), (5) is the formation temperature } T_{\text{form}} \text{ as estimated below.} \]
the original results [21–30] where predictions of the model have been confronted with the observations in specific frequency bands covering more than eleven orders of magnitude, from radio frequency with $\omega \sim 10^{-4}$ eV to $\gamma$ rays with $\omega \sim 10$ MeV. We also mention in section II some interesting results [31–35] which presently perfectly consistent with the model. However, in future, the same studies with modest improvements will provide a powerful test of the viability of the quark nugget dark matter model.

One should emphasize here that the corresponding analysis [21–30] is determined by conventional physics, and as such all effects are calculable from the first principles. In other words, the model contains no tuneable fundamental parameters, except for a single mean baryon number of a nugget $\langle B \rangle \sim 10^{25}$ which enters all the computations [21–30] as a single normalization factor. At the same time, the crucial assumptions of the model, such as specific mechanisms on the baryon charge separation and dynamics of the nugget formation, etc., have never been explored in our previous studies.

We believe that the phenomenological success [21–30] of the model warrants further theoretical studies of this framework, in spite of its naively counter-intuitive nature. Therefore, the present work should be considered as the first step in this direction where we attempt to develop the theoretical framework to address (and hopefully answer) some of the hardest questions about a possible mechanism for the nugget’s formation during the QCD phase transition in strongly coupled regime when even the phase diagram at $\theta \neq 0$ as a function of the chemical potential $\mu$ and temperature $T$ is still unknown, see footnote 1.

The structure of this work is as follows. In section II we briefly review the observational constraints on the model. In section III we highlight the basic assumptions and ingredients of this framework, while in sections IV and V we present some analytical estimates which strongly substantiate the idea that such heavy objects indeed can be formed and survive until the present epoch during the QCD phase transition in early Universe. Section VI as well as Appendices A and B are devoted to a number of technical details which support our basic claim.

In section VII we argue that there will be the preferential formation of one species of nuggets over another. This preference is determined by the dynamics of the axion field $\theta(x)$ which itself is correlated on the scales of the Universe at the beginning of the nugget’s formation. Finally, in section VIII we comment on implications of our studies to direct axion search experiments.

To conclude this long introduction: the nuggets in our framework play the dual role: they serve as the DM candidates and they also explain the observed asymmetry between matter and antimatter. These two crucial elements of the proposal lead to very generic consequence of the entire framework expressed by eq. (1). This basic generic result is not very sensitive to any specific details of the model, but rather, entirely determined by two fundamental ingredients of the framework:

- the contribution to $\Omega$ for both types of matter (visible and dark) are proportional to one and the same fundamental scale $\sim \Lambda_{\text{QCD}}$;
- the preferential formation of one species of nuggets over another is correlated on large cosmological scales where $CP$ violating axion phase $\theta(x)$ remains coherent just a moment before the QCD phase transition.

The readers interested in the cosmological consequences, rather than in technical computational details may directly jump to section III where we formulate the basics ingredients of the proposal, to section VII B where we explain the main model-independent consequence (1) of this framework, and to section VIII where we make few comments on implications to other axion search experiments, including microwave cavity [8–10, 13] and the Orpheus experiments [14].

II. QUARK (ANTI) NUGGET DM

CONFRONTING THE OBSERVATIONS

While the observable consequences of this model are on average strongly suppressed by the low number density of the quark nuggets $\sim B^{-1/3}$ as explained above, the interaction of these objects with the visible matter of the galaxy will necessarily produce observable effects. Any such consequences will be largest where the densities of both visible and dark matter are largest such as in the core of the galaxy or the early universe. In other words, the nuggets behave as a conventional cold DM in the environment where density of the visible matter is small, while they become interacting and emitting radiation objects (i.e. effectively become visible matter) when they are placed in the environment with sufficiently large density.

The relevant phenomenological features of the resulting nuggets are determined by properties of the so-called electro-sphere as discussed in original refs. [21–30]. These properties are in principle, calculable from first principles using only the well established and known properties of QCD and QED. As such the model contains no tuneable fundamental parameters, except for a single mean baryon number $\langle B \rangle$ which itself is determined by the axion mass $m_a$ as we already mentioned.

A comparison between emissions with drastically different frequencies from the centre of galaxy is possible because the rate of annihilation events (between visible matter and antimatter DM nuggets) is proportional to the product of the local visible and DM distributions at the annihilation site. The observed fluxes for different frequencies from the centre of galaxy is possible because the rate of annihilation events (between visible matter and antimatter DM nuggets) is proportional to the product of the local visible and DM distributions at the annihilation site. The observed fluxes for different frequencies from the centre of galaxy is possible because the rate of annihilation events (between visible matter and antimatter DM nuggets) is proportional to the product of the local visible and DM distributions at the annihilation site. The observed fluxes for different frequencies from the centre of galaxy is possible because the rate of annihilation events (between visible matter and antimatter DM nuggets) is proportional to the product of the local visible and DM distributions at the annihilation site. The observed fluxes for different frequencies from the centre of galaxy is possible because
tween DM and visible matter. As \( n_{DM} \sim B^{-1} \) the effective interaction is strongly suppressed \( \sim B^{-1/3} \) as we already mentioned in the Introduction. The parameter \( (B) \sim 10^{25} \) was fixed in this proposal by assuming that this mechanism saturates the observed 511 keV line [21, 22], which resulted from annihilation of the electrons from visible matter and positrons from anti-nuggets. It has been also assumed that the observed dark matter density is saturated by the nuggets and anti-nuggets. It corresponds to an average baryon charge \( \langle B \rangle \sim 10^{25} \) for typical density distributions \( n_{\text{visible}}(r), n_{DM}(r) \) entering (6). Other emissions from different bands are expressed in terms of the same integral (6), and therefore, the relative intensities are completely determined by internal structure of the nuggets which is described by conventional nuclear physics and basic QED. We present a short overview of these results below.

Some galactic electrons are able to penetrate to a sufficiently large depth of the anti-nuggets. These events no longer produce the characteristic positronium decay spectrum (511 keV line with a typical width of order no longer produce the characteristic positronium decay spectrum) but a direct non-resonance \( e^-e^+ \rightarrow 2\gamma \) emission spectrum. The transition between the resonance positronium decays and non-resonance regime is determined by conventional physics and allows us to compute the strength and spectrum of the MeV scale emissions relative to that of the 511 keV line [23, 24]. Observations by the COMPTEL satellite indeed show some excess above the galactic background consistent with our estimates.

Galactic protons incident on the anti-nugget will penetrate some distance into the quark matter before annihilating into hadronic jets. This process results in the emission of Bremsstrahlung photons at x-ray energies [25]. Observations by the CHANDRA observatory apparently indicate an excess in x-ray emissions from the galactic centre.

Hadronic jets produced deeper in the nugget or emitted in the downward direction will be completely absorbed. They eventually emit thermal photons with radio frequencies [26, 27]. Again the relative scales of these emissions may be estimated and is found to be in agreement with observations.

These apparent excess emission sources have been cited as possible support for a number of dark matter models as well as other exotic astrophysical phenomenon. At present however they remain open matters for investigation and, given the uncertainties in the galactic spectrum and the wide variety of proposed explanations are unlikely to provide clear evidence in the near future. Therefore, it would be highly desirable if some direct detection of such objects is found, similar to direct searches of the weakly interacting massive particles (WIMPs).

While direct searches for WIMPs require large sensitivity, a search for very massive dark matter nuggets requires large area detectors. If the dark matter consists of quark nuggets at the \( B \sim 10^{25} \) scale they will have a flux of

\[
\frac{dN}{dA \, dt} = n \, e \approx \left( \frac{10^{25}}{B} \right) \text{km}^{-2} \text{yr}^{-1}.
\]  

Though this flux is far below the sensitivity of conventional dark matter searches it is similar to the flux of cosmic rays near the GZK limit. As such present and future experiments investigating ultrahigh energy cosmic rays may also serve as search platforms for dark matter of this type.

It has been suggested that large scale cosmic ray detectors may be capable of observing quark (anti-) nuggets passing through the earth’s atmosphere either through the extensive air shower such an event would trigger [28] or through the geosynchrotron emission generated by the large number of secondary particles [29], see also [30] for review.

It has also been estimated in [31] that, based on Apollo data, nuggets of mass from \( \sim 10 \) kg to 1 ton (corresponding to \( B \sim 10^{28-30} \)) must account for less than an order of magnitude of the local dark matter. While our preferred range of \( B \sim 10^{25} \) is somewhat smaller and is not excluded by [31], we still believe that \( B \geq 10^{28} \) is not completely excluded by the Apollo data, as the corresponding constraints are based on specific model dependent assumptions about the nugget mass-distribution.

It has also been suggested that the ANITA experiment may be sensitive to the radio band thermal emission generated by these objects as they pass through the antarctic ice [32]. These experiments may thus be capable of adding direct detection capability to the indirect evidence discussed above, see Fig.2 taken from [32] which reviews these constraints.

It has also been suggested recently [33] that the interactions of these (anti-) nuggets with normal matter in the Earth and Sun will lead to annihilation and an associated neutrino flux. Furthermore, it has been claimed [33] that the antiquark nuggets cannot account for more than 20% of the dark matter flux based on constraints for the neutrino flux in 20-50 MeV range where sensitivity of the underground neutrino detectors such as SuperK have their highest signal-to-noise ratio.

However, the claim [33] was based on assumption that the annihilation of visible baryons with antiquark nuggets generate the neutrino spectrum similar to conventional baryon-antibaryon annihilation spectrum when the large number of produced pions eventually decay to muons and consequently to highly energetic neutrinos in the 20-50 MeV energy range. Precisely these highly energetic neutrinos play the crucial role in analysis [33]. However, in most CS phases the lightest pseudo Goldstone mesons (the pions and Kaons) have masses in the 5-20 MeV range [16, 17] in huge contrast with hadronic confined phase where \( m_{\pi} \sim 140 \) MeV. Therefore, such light pseudo Goldstone mesons in CS phase cannot produce highly energetic neutrinos in the 20-50 MeV energy range and thus are not subject to the SuperK constraints [35].
ically to this value of the axion mass $m$.

VIII. Orpheus experiment "B" is designed to be sensitive for the direct axion search experiments as discussed in section VIII. Orpheus experiment "B" is designed to be sensitive exactly to this value of the axion mass $m$ according to the scaling relation (3). The corresponding constraints expressed in terms of $\theta$ correspond to the axion mass $m_a \simeq 10^{-4}$ eV, see Fig. 3.

- We conclude this brief overview on observational constraints of the model with the following remark. This model which has a single fundamental parameter (the mean baryon number of a nugget $\langle B \rangle \sim 10^{65}$), corresponding to the axion mass $m_a \simeq 10^{-4}$ eV), and which enters all the computations is consistent with all known astrophysical, cosmological, satellite and ground based constraints as highlighted above. Furthermore, in a number of cases the predictions of the model are very close to the presently available limits, and very modest improving of those constraints may lead to a discovery of the nuggets. Even more than that: there is a number of frequency bands where some excess of emission was observed, and this model may explain some portion, or even entire excess of the observed radiation in these frequency bands.

In the light of this (quite optimistic) assessment of the observational constraints of this model it is quite obvious that further and deeper studies of this model are worthwhile to pursue. The relevant developments may include, but not limited, to such hard problems as formation mechanisms during the QCD phase transition in early Universe, even though many key elements for proper addressing those questions at $\theta \neq 0, \mu \neq 0, T \neq 0$ are still largely unknown in strongly coupled QCD as shown on Fig. 1. This work is the first step in the direction to explore a possible mechanism of formation of the nuggets.

III. FORMATION OF THE NUGGETS: THE CRUCIAL INGREDIENTS OF THE PROPOSAL.

1. First important element of this proposal is the presence of the topological objects, the axion domain walls [36]. As we already mentioned the $\theta$ parameter is the angular variable, and therefore supports various types of the domain walls, including the so-called $N_{DW} = 1$ domain walls when $\theta$ interpolates between one and the same physical vacuum state with the same energy $\theta \rightarrow \theta + 2\pi n$. The axion domain walls may form at the same moment when the axion potential get tilted, i.e. at the moment $T_a$ when the axion field starts to roll due to the misalignment mechanism. The tilt becomes much more pronounced at the phase transition when the chiral condensate forms at $T_c$. In general one should expect that the $N_{DW} = 1$ domain walls form once the axion potential is sufficiently tilted, i.e. anywhere between $T_a$ and $T_c$.

One should comment here that it is normally assumed that for the topological defects to be formed the Peccei-Quinn (PQ) phase transition must occur after inflation. This argument is absolutely correct for a generic type of domain walls with $N_{DW} \neq 1$. The conventional argument is based on the fact that few physically different vacua with the same energy must be present inside of the same horizon for the domain walls to be formed. The $N_{DW} = 1$ domain walls are unique and very special in the sense that $\theta$ interpolates between one and the same physical vacuum state. Such $N_{DW} = 1$ domain walls can be formed even if the PQ phase transition occurred before inflation and a unique physical vacuum occupies entire Universe, see some elaboration of this point at the end of this section.

It has been realized many years after [36] that the walls, in general, demonstrate a sandwich-like substructure on the QCD scale $\Lambda_{QCD}^{-1}$ fm. The arguments supporting the QCD scale substructure inside the axion domain walls are based on analysis [37] of QCD in the large $N$ limit with inclusion of the $\eta'$ field and independent analysis [38] of supersymmetric models where a similar $\theta$ vacuum structure occurs.

One should remark here that the described structure is classically stable configuration. In particular, the $\eta'$ field cannot decay to $2\gamma$ simply due to the kinematical reasons when the $\eta'$ field is off-shell, and cannot be expressed as a superposition of on-shell free particles. It can only decay through the tunnelling, and therefore, such $N_{DW} = 1$
domain walls are formally metastable rather than absolutely stable configurations.

2. Second important element is that in addition to this known QCD substructures \([37–39]\) of the axion domain walls expressed in terms of the \(\eta^\prime\) and gluon fields, there is another substructure with a similar QCD scale which carries the baryon charge. Precisely this novel feature of the domain walls which was not explored previously in the literature will play a key role in our proposal because exactly this new effect will be eventually responsible for the accretion of the baryon charge by the nuggets. Both, the quarks and anti-quarks can accrete on a given closed domain wall making eventually the quark nuggets or anti-nuggets, depending on the sign of the baryon charge. The sign is chosen randomly such that equal number of quark and antiquark nuggets are formed if the external environment is CP even, which is the case when fundamental \(\theta = 0\). One can interpret this phenomenon as a local spontaneous symmetry breaking effect, when on the scales of order the correlation length \(\xi\) the nuggets may acquire the positive or negative baryon charge with equal probability, as discussed in great details in next section IV.

3. Next important ingredient of the proposal is the Kibble-Zurek mechanism which gives a generic picture of formation of the topological defects during a phase transition, see original papers \([40]\), review \([41]\) and the textbook \([42]\). In our context the Kibble-Zurek mechanism suggests that once the axion potential is sufficiently tilted the \(N_{\text{DW}} = 1\) domain walls form. The potential becomes much more pronounced when the chiral condensate forms at \(T_c\). After some time after \(T_a\) the system is dominated by a single, percolated, highly folded and crumpled domain wall of very complicated topology. In addition, there will be a finite portion of the closed walls (bubbles) with typical size of order correlation length \(\xi(T)\), which is defined as an average distance between folded domain walls at temperature \(T\). It is known that the probability of finding closed walls with very large size \(R \gg \xi\) is exponentially small. Furthermore, numerical simulations suggest \([42]\) that approximately \(87\%\) of the total wall area belong to the percolated large cluster, while the rest is represented by relatively small closed bubbles with sizes \(R \sim \xi\).

The key point for our proposal is there existence of these finite closed bubbles made of the axion domain walls\(^5\). One should remark here that these closed bubbles had been formed sometime after \(T_a\) when original \(\theta\) parameter has not settled yet to its mininum value. It implies that the domain wall evolution starts at the time when \(\theta\) parameter is not yet zero\(^6\). Normally it is assumed that these closed bubbles collapse as a result of the domain wall pressure, and do not play any significant role in dynamics of the system. However, as we already mentioned in Introduction the collapse of these closed bubbles is halted due to the Fermi pressure acting inside of the bubbles. Therefore, they may survive and serve as the dark matter candidates.

The percolated network of the domain walls will decay to the axion in conventional way as discussed in \([43, 44, 46]\). Those axions (along with the axions produced by the conventional misalignment mechanism \([44, 47]\)) will contribute to the dark matter density today. The corresponding contribution to dark matter density is highly sensitive to the axion mass as \(\Omega_{\text{dark}} \propto m_a^{-1}\). It may saturate the observed dark matter density if \(m_a \approx 10^{-6} \text{ eV}\) \([8–15]\), while it may contribute very little to \(\Omega_{\text{dark}}\) if the axion mass is slightly heavier than \(m_a \approx 10^{-6} \text{ eV}\). In contrast, in our framework an approximate relation \(\Omega_{\text{dark}} \approx \Omega_{\text{visible}}\) holds irrespectively to the axion mass \(m_a\).

We shall not elaborate on the production and spectral properties of these axions in the present work. Instead, the focus of the present paper is the dynamics of the closed bubbles, which is normally ignored in computations of the axion production. Precisely these closed bubbles, according to this proposal, will eventually become the stable nuggets and may serve as the dark matter candidates.

As we already mentioned the nugget’s contribution to \(\Omega_{\text{dark}}\) is not very sensitive to the axion mass, but rather, is determined by the formation temperature \(T_{\text{form}}\) as explained in Introduction, see also footnote 3 with few important comments on this. The time evolution of these nuggets after their formation is the subject of section V.

4. There existence of CS phase in QCD represents the next crucial element of our scenario. The CS has been an active area of research for quite sometime, see review papers \([16, 17]\) on the subject. The CS phase is realized when quarks are squeezed to the density which is few times nuclear density. It has been known that this regime may be realized in nature in neutron stars interiors and in the violent events associated with collapse of massive stars or collisions of neutron stars, so it is important for astrophysics.

The force which squeezes quarks in neutron stars is gravity; the force which does an analogous job in early universe during the QCD phase transition is a violent collapse of a bubble of size \(R \sim \xi(T)\) formed from the moment when the domain walls form. It is not exactly the same value as the misalignment angle which normally enters all the computations due to the conventional misalignment mechanism \([44, 47]\). This is because the temperature when the domain walls form and the temperature \(T_a\) when the axion field starts to roll do not exactly coincide though both effects are due to the same axion tilted potential. The crucial point is that the \(\theta\) parameter, as defined above, could be numerically small, nevertheless, it preserves its coherence over entire Universe, see item 5 below and section VII for the details.

\(^5\) The presence of such closed bubbles in numerical simulations in context of the axion domain wall has been mentioned in \([10]\), where it was argued that these bubbles would oscillate and emit the gravitational waves. However, we could not find any further details on the fate of these closed bubbles in the literature.

\(^6\) This \(\theta\) parameter in our work is defined as the value of \(\theta\) at the
axion domain wall as described in item 3 above. If number density of quarks trapped inside of the bubble (in the bulk) is sufficiently large, the collapse stops due to the internal Fermi pressure. In this case the system in the bulk may reach the equilibrium with the ground state being in a CS phase. As we advocate in section V this is very plausible fate of a relatively large size bubbles of size \( R \sim \xi(T) \) made of the axion domain walls which were produced after the QCD phase transition.

5. If \( \theta \) vanishes, then equal number of nuggets and anti-nuggets would form. However, the \( \mathbb{CP}_1 \) violating \( \theta \) parameter (the axion field), which is defined as value of \( \theta \) at the moment of domain wall formation generically is not zero, though it might be numerically quite small. Precisely the dynamics of the coherent axion field \( \theta(x) \) leads to preferences in formation of one species of nuggets, as discussed in section VII. This sign-preference is correlated on the scales where the axion field \( \theta(x) \) is coherent, i.e. on the scale of the entire Universe at the moment of the domain wall formation. In other words, we assume that the PQ phase transition happened before inflation. One should emphasize that this assumption on coherence of the axion field on very large scales is consistent with formation of \( N_{DW} = 1 \) domain walls, see item 1 above. This coherence obviously cannot be satisfied for a generic type of the domains walls with \( N_{DW} \neq 1 \) when \( N_{DW} \) physically distinct vacuum states with the same energy must be present in the system.

There are few arguments supporting this claim. First of all, one should remind that the axion domain wall with \( N_{DW} = 1 \) corresponds to the configuration when \( \theta \) field interpolates between \( \theta = 0 \) and \( \theta = 2\pi \). It implies that the axion field, describing the domain wall, interpolates between topologically distinct but physically identical and unique vacuum state. We present few strong arguments below suggesting that the topological sectors must be always present in the system everywhere in space, and inflation does not remove different topological sectors from the system. Therefore \( N_{DW} = 1 \) can be formed even if the PQ phase transition happened before inflation.

The simplest way to explain this claim is to analyze the expression for vacuum energy \([48, 49]\) in the limit of large number of colours \( N_c \to \infty \), though, it is known that the arguments still hold for finite \( N_c \) as well\(^7\). The main point is that the vacuum energy as a function of \( \theta \) assumes the following form

\[
E_{\text{vac}}(\theta) \sim \min_k (\theta + 2\pi k)^2 + O\left(\frac{1}{N_c}\right), \tag{8}
\]

where \( \theta \) in the present context plays the role of the axion field. This formula explicitly shows that for each given \( \theta \) the vacuum state is unique. However, there is a number of different branches, classified by parameter \( k \) such that when \( \theta = \pm \pi \) the system becomes double degenerate, and one branch replaces another branch at \( \theta = \pm \pi \). Precisely this pattern provides the required \( 2\pi \) periodicity of the system. This picture of the \( \theta \) dependence is commonly accepted by the community, and in fact emerges in many different gauge theories where exact computations can be carried out, including the holographic description \([49]\).

The key point in these arguments is the presence of \( k \) different branches which must be present in the system everywhere in space in order to provide the \( 2\pi \) periodicity of the vacuum energy (8). There is only one physical vacuum in the system, which however always accompanied by \( k \) different branches. Inflation cannot remove different \( k \) branches outside the horizon because they are inherent elements of the system at each point in space. The domain wall solution with \( N_{DW} = 1 \) corresponds to interpolation between different topological sectors \( k = 1 \) and \( k = 0 \) which always present in the system inside the same horizon.

Another argument which leads to the same conclusion goes as follows. The \( N_{DW} = 1 \) is formed as a result of twisting of the axion field in configurational space when the axion field returns to its initial physical vacuum state after making a full circle as explained above. Topologically it is identical to creation of solitons in two dimensional Sine-Gordon model \( \sim \cos \phi \) in condensed matter physics when the \( \phi \) field interpolates between one and the same physical, but topologically distinct states \( \phi = 0 \) and \( \phi = \pm 2\pi \). In the dual picture the Sine-Gordon solitons can be thought as \( \psi \) fermions, see section IV for references on this duality relation. In this dual picture the production of solitons corresponds to production of the fermi \( \psi \) fields. It is quite obvious that the production of the \( \psi \) fields is perfectly allowed process at \( T \neq 0 \) irrespectively whether inflation happened before or after the PQ symmetry breaking occurred. Formally, the mere existence of \( \psi \) field in the system is due to \( k \) topological sectors in the theory when \( \phi \) enters the Lagrangian in combination \( (\phi + 2\pi k) \). The inflation obviously cannot remove \( k \) sectors from the system because it would violate the fundamental properties of the theory, such as duality between \( \psi \) and \( \phi \) descriptions.

For our system it implies that the \( N_{DW} = 1 \) corresponding to interpolation between \( k = 1 \) and \( k = 0 \) is allowed configuration, irrespectively to inflation, as all \( k \) topological sectors must be present in the system in every point of space-time.

To conclude this section: as we argue below the generic consequence of this framework (1) is not very sensitive to an absolute value of \( \theta \) at the moment of the domain wall formation, see comment in footnote 3 on this matter. One can say that the coherent axion field \( \theta(x) \neq 0 \), being numerically small, plays the role of the \( \mathbb{CP}_1 \) violating catalgyst which determines a preferred direction for separation of the baryon charges on the Universe scale. This role of \( \mathbb{CP}_1 \) violation in our proposal is quite differ-
ent from the role it plays in conventional “baryogenesis” mechanisms.

**IV. FORMATION OF THE NUGGETS:**

**ACCRETION OF THE BARYON CHARGE**

From now on and until section VII we focus on the dynamics of a single closed bubble produced during the domain wall formation as described in item 3 in section III. The correlation length $\xi(T)$ is defined as an average distance between folded domain walls at temperature $T$. We assume that initial size of the bubble $\xi(T)$ is sufficiently large, few times larger than the axion domain wall width $\sim m_a^{-1}$, such that one can locally treat the surface of the closed bubble being flat.

The main goal of this section is to demonstrate that such a bubble will generically acquire a baryon (or anti-baryon) charge in very much the same way as the $\eta'$ field was dynamically accreted as originally discussed in [37] and briefly explained above as item 2 in section III. In other words, we shall argue in this section that the baryon (or anti-baryon) charge in very much the same way as the $\eta'$ field does not exist. However, the singlet field which accompanied the quark field still present in the system. The coefficient $m$ in this phase can be computed using the instanton liquid model. At very high temperature the parameter $m$ is proportional to the quark masses and indeed very small. When temperature decreases the instanton contribution grows very fast. At this point parameter $m$ is proportional to the vacuum expectation value of the 't Hooft determinant. When temperature further decreases the parameter $m$ is proportional to the quark condensate in CS phase or the chiral condensate in the hadronic phase, see Fig.1. We shall not elaborate along this line by assuming $m$ is negligible for all our estimates which follow.

Parameter $m$ in eq.(9) should not be literally identified with the quark mass, nor with the nucleon mass. Instead, this dimensional parameter $m \sim \Lambda_{QCD}$ should be thought as an effective coupling in our model when parameter $m$ effectively describes the interaction with fermi field $\Psi$ in all phases during the formation time, including the quark gluon plasma as well as hadronic and CS phases. The same comment also applies to a numerical value of the chemical potential $\mu$; it vanishes during initial time and becomes very large when CS phase sets in inside the nugget.

The strategy is to break (9) into two $1 + 1$ dimensional components by setting $\partial_z = \partial_y = 0$ (this is the approximation that the physics in the $z$ direction decouples from the physics in the $x$-$y$ plane) and then by manipulating the system of equations that result.

First, we introduce the following chiral components of the Dirac spinors:

$$\Psi_+ = \frac{1}{\sqrt{8}} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \Psi_- = \frac{1}{\sqrt{8}} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad (10)$$

$$\Psi = \frac{1}{\sqrt{2S}} \begin{pmatrix} \chi_1 + \xi_1 \\ \chi_2 + \xi_2 \\ \chi_1 - \xi_1 \\ \chi_2 - \xi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} (\Psi_+ + \Psi_-), \quad (11)$$

where $S$ is the area of the wall. This normalization factor cancels the degeneracy factor proportional to $S$ added in the text below.

8 In quark gluon phase the colour singlet $\eta'$ field does not exist. However, the singlet phase which accompanied the quark field still present in the system. The coefficient $m$ in this phase can be computed using the instanton liquid model. At very high temperature the parameter $m$ is proportional to the quark masses and indeed very small. When temperature decreases the instanton contribution grows very fast. At this point parameter $m$ is proportional to the vacuum expectation value of the 't Hooft determinant. When temperature further decreases the parameter $m$ is proportional to the quark condensate in CS phase or the chiral condensate in the hadronic phase, see Fig.1. We shall not elaborate along this line by assuming $m$ is negligible for all our estimates which follow.

9 We are using the standard representation here:

$$\gamma_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
The associated Dirac equation is

\[\begin{pmatrix}
-m e^{i(\phi - \theta)} & i(\partial_t + \partial_z - \mu) \\
(i(\partial_t - \partial_z) - \mu & -m e^{-i(\phi - \theta)}
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\xi_1
\end{pmatrix} = 0,
\quad (12a)
\]

\[\begin{pmatrix}
-m e^{i(\phi - \theta)} & i(\partial_t + \partial_z - \mu) \\
(i(\partial_t + \partial_z) - \mu & -m e^{-i(\phi - \theta)}
\end{pmatrix}
\begin{pmatrix}
\chi_2 \\
\xi_2
\end{pmatrix} = 0.
\quad (12b)
\]

where we decouple the \( z \) coordinates from \( x \) and \( y \) by setting \( \partial_x = \partial_y = 0 \). Remember that we are looking for a two-dimensional Dirac equation, thus we want the kinetic terms to look the same. For this reason we should flip the rows and columns of the second equation. Doing this and defining the two-dimensional spinors

\[\Psi_1 = \begin{pmatrix}
\chi_1 \\
\xi_1
\end{pmatrix}, \quad \Psi_2 = \begin{pmatrix}
\chi_2 \\
\xi_2
\end{pmatrix},
\quad (13)
\]

the equations have the following structure:

\[\begin{pmatrix}
i\gamma^\mu \partial_\mu - m e^{i(\theta - \phi)} \gamma^5 - \mu \gamma_0
\end{pmatrix} \Psi_1 = 0
\quad (14a)
\]

\[\begin{pmatrix}
i\gamma^\mu \partial_\mu - m e^{-i(\theta - \phi)} \gamma^5 - \mu \gamma_0
\end{pmatrix} \Psi_2 = 0
\quad (14b)
\]

where the index \( \nu \in \{t, z\} \), the Lorentz signature is \( (1, -1) \) and we define the following two-dimensional version of the gamma matrices:

\[\gamma_t = \sigma_1, \quad \gamma_z = -i \sigma_2, \quad \gamma_5 = \sigma_3.
\]

These satisfy the proper two-dimensional relationships \( \gamma_5 = \gamma_t \gamma_z \) and \( \gamma_\mu \gamma_\nu = g_{\mu \nu} + \epsilon_{\mu \nu \rho} \gamma_\rho \). We can reproduce equation (14) from the following effective two-dimensional Lagrangian density,

\[\mathcal{L}_2 = \Psi_1 \left( i\gamma^\mu \partial_\mu - m e^{i(\theta - \phi)} \gamma^5 - \mu \gamma_0 \right) \Psi_1 + \Psi_2 \left( i\gamma^\mu \partial_\mu - m e^{-i(\theta - \phi)} \gamma^5 - \mu \gamma_0 \right) \Psi_2,
\quad (15)\]

where two different species of fermion with opposite chiral charge interact with the axion domain wall background determined by \( \theta(z) \) and \( \phi(z) \) fields. Note that, due to the normalization factor \( 1/\sqrt{S} \) we introduced above, the two-dimensional fields \( \Psi_{(i)} \) have the correct canonical dimension \( 1/2 \).

We have thus successfully reduced our problem to a two-dimensional fermionic system. It is known that for several systems in \( 1 + 1 \) dimensions, the fermionic representation is equivalent to a \( 1 + 1 \) dimensional bosonic system through the following equivalences \[51, 52]:

\[\begin{align*}
\bar{\Psi}_{(j)} i\gamma^\mu \partial_\mu \Psi_{(j)} & \to -\frac{1}{2}(\partial_\mu \theta_j)^2, \\
\bar{\Psi}_{(j)} \gamma_\mu \Psi_{(j)} & \to \frac{1}{\sqrt{\pi}} \epsilon_{\mu \nu} \partial^\nu \theta_j,
\end{align*}\]

\[\begin{align*}
\bar{\Psi}_{(j)} \Psi_{(j)} & \to -m_0 \cos(2\sqrt{\pi} \theta_j), \\
\bar{\Psi}_{(j)} i\gamma_\mu \Psi_{(j)} & \to -m_0 \sin(2\sqrt{\pi} \theta_j).
\end{align*}\]

The constant \( m_0 \) in the last two equations is a dimensional parameter of order \( m_0 \sim m \sim \Lambda_{QCD} \). The exact coefficient of this factor depends on renormalization procedure and is only known for few exactly solvable systems but in all cases, is of order unity.

After making these replacements, we are left with the following two-dimensional bosonic effective Lagrangian density describing the two fields \( \theta_1 \) and \( \theta_2 \) in the domain wall background determined by \( \phi(z) \) and \( \theta(z) \):

\[\mathcal{L}_2 = \frac{1}{2}(\partial_\mu \theta_1)^2 + \frac{1}{2}(\partial_\mu \theta_2)^2 - U(\theta_1, \theta_2) + \frac{m}{\sqrt{\pi}} \frac{\partial \theta_2 + \theta_1}{\partial z}
\quad (17)\]

where the effective potential is

\[U(\theta_1, \theta_2) = -m m_0 \left[ \cos(2\sqrt{\pi} \theta_1 - \phi + \theta) \right]
\quad (18)\]

The conventional procedure to study the system (17) is to add the kinetic terms for the axion and the \( \eta' \) field \( \phi \) into (17) and study a resulting solution depending on four dynamical fields by specifying all possible boundary conditions when the potential energy (18) assumes its minimal value \[10\]. In other words, one should take into account the dynamics of the \( \theta \) and \( \phi \) fields together with \( \theta_1, \theta_2 \) because the typical scales for \( \phi, \theta_1, \theta_2 \) are roughly the same order of magnitude and of order of \( \Lambda_{QCD} \). Recapitulate it: one cannot study the dynamics of \( \theta_1, \theta_2 \) field by neglecting their back reaction on the background axion and \( \phi \) fields.

For our present purposes, however, we do not really need an explicit profile functions for large number of different domain walls determined by various boundary conditions controlled by (18). The only important element relevant for our future discussions is the observation that some of the domain walls may carry the baryon (anti baryon) charge. Indeed, the domain walls which satisfy the boundary conditions

\[2\sqrt{\pi} \theta_1(z = +\infty) - 2\sqrt{\pi} \theta_1(z = -\infty) = 2\pi n_1
\quad (19)\]

\[2\sqrt{\pi} \theta_2(z = +\infty) - 2\sqrt{\pi} \theta_2(z = -\infty) = 2\pi n_2
\quad (20)\]

carry the following baryon charge \( N \) defined for one particle Dirac equation

\[N = \int d^3x \Psi_0 \Psi = \int dz \left( \Psi_1 \gamma_0 \Psi_1 + \Psi_2 \gamma_0 \Psi_2 \right)
\quad (20)\]

where we express the final formula in terms of the auxiliary two-dimensional fields \( \theta_1 \) and \( \theta_2 \) and corresponding boundary conditions given by eq. (19). Factor \( S \) also cancels with our normalization for four dimensional \( \Psi \) field.

\[10\] In fact it was precisely the procedure which has been adopted in [37] for a similar problem of computing of the profile functions of the axion, \( \pi \)-meson and \( \eta' \)-domain wall described by \( \theta - \pi - \eta' \) fields.
To complete the computations for four dimensional baryon charge $B$ accumulated on the domain wall we need to multiply (20) by the degeneracy factor in vicinity of the domain wall which can be estimated as follows

$$B = N \cdot g \cdot \int \frac{d^2 x \cdot d^2 k}{(2\pi)^2} \frac{1}{\exp\left(\frac{\mu}{T}\right) + 1},$$  \hspace{1cm} (21)

where $g$ is appropriate degeneracy factor, e.g. $g \approx N_c N_f$ in CS phase. We note that an additional degeneracy factor $2$ due to the spin is already accounted for by parameter $N$ defined in eq. (20). For high chemical potential $\mu \gg T$ corresponding to CS phase the baryon charge per unit area accreted in vicinity of the domain wall can be approximated as

$$\frac{B}{S} \simeq N \cdot \frac{g \mu^2}{4\pi}. \hspace{1cm} (22)$$

In the opposite limit of high temperature $\mu \ll T$ which corresponds to the quark gluon plasma phase, the corresponding magnitude can be estimated as follows

$$\frac{B}{S} \simeq N \cdot \frac{g \pi T^2}{24}. \hspace{1cm} (23)$$

It is instructive to compare the estimate (23) with number density $N/V$ of all degrees of freedom in vicinity of the domain wall. Assuming that the baryon charge in the domain wall background is mainly concentrated on distances of order $m^{-1}$ from the center of the domain wall we arrive to the following estimate for the ratio of the baryon number density bound to the wall in comparison with the total number density of all degrees of freedom responsible for the thermodynamical equilibrium in this phase

$$r \sim \frac{(B/S) \cdot m}{N/V} \sim N \left(\frac{m}{T}\right) \left(\frac{\pi^3 g}{18 \xi(3) g^*}\right). \hspace{1cm} (24)$$

where effective degeneracy factor $g^*$ for quark gluon plasma is $g^* \approx [\frac{1}{4} N_c N_f + 2(N_c^2 - 1)]$ and $\xi(3) \approx 1.2$ is the Riemann zeta function. Ratio (24) shows that the accreted quark density bounded to the domain wall at high temperature represents parametrically small contribution to all thermodynamical observables mainly because of a small parameter $m/T \ll 1$ in this phase. The situation drastically changes as we discuss in next section V when the temperature slowly decreases due to expansion of the Universe and the system enters the hadronic or CS phase, as shown on Fig. 1. At this point the baryon charge accumulation in the domain wall background becomes the major player of the system, which eventually leads to the formation of the CS nuggets or anti-nuggets when quarks (anti-quarks) fill entire volume of the nuggets (anti-nuggets).

We conclude this section with the following important comments. First, we argued that the domain walls in general accrete the baryon (or anti baryon) charge in vicinity of the centre of the domain wall. The effect in many respects is similar to fractional charge localization on domain walls, while the rest of the charge is de-localized in the rest of volume of the system as discussed in original paper [53]. The effect is also very similar to previously discussed phenomenon on dynamical generation of the $\eta'$ field in the domain wall background. The key point is that at sufficiently high temperature the $N_{DW} = 1$ domain walls form by the usual Kibble-Zurek mechanism as explained in section III. The periodic fields $\theta, \phi, \theta_1, \theta_2$ may assume physically identical but topologically distinct vacuum values (20) on opposite sides of the walls. When the system cools down the corresponding fields inevitably form the domain wall structure, similar to analysis in hadronic [37] and CS phases [39].

We advocate the picture that the closed bubbles will be also inevitably formed as discussed in section III. The collapse of these bubbles halts as a result of Fermi pressure due to the quarks accumulated inside the nugget during the evolution of the domain wall network. Next section V is devoted precisely the question on time evolution of these closed bubbles made of $N_{DW} = 1$ domain wall.

- The most important lesson of this section is that there is a variety of acceptable boundary conditions determined by potential (18) when the energy assumes its vacuum values. Some of the domain walls will carry zero baryon charge when the combination $(n_1 + n_2)$ vanishes according to (20). However, generically the domain walls will acquire the baryon or anti-baryon charge. This is because the domain wall tension is mainly determined by the axion field while corrections due to QCD substructure will lead to a small correction of order $\sim m/f_s \ll 1$, similar to studies of the (axion - $\eta'$ - $\pi$) domain wall [37]. Therefore, the presence of the QCD substructure with non vanishing $(n_1 + n_2) \neq 0$ increases the domain wall tension only slightly. In other words, accumulation of the baryon charge in vicinity of the wall does not lead to any suppression during the formation stage. Consequently, this implies that the domain closed bubbles carrying the baryon or anti baryon charge will be copiously produced during the phase transition as they are very generic configurations of the system. Furthermore, the baryon charge cannot leave the system during the evolution as it is strongly bound to the wall due to the topological reasons. The corresponding binding energy per quark is order of $\mu$ and increases with time as we discuss in next section.

This phenomenon of “separation of the baryon charge” can be interpreted as a local version of spontaneous symmetry breaking of the baryon charge. This symmetry breaking occurs not in the entire volume in the ground state determined by the potential (18). Instead, the symmetry breaking occurs on scale $\xi(T)$ in vicinity of the field configurations which describe the interpolation between physically identical but topologically distinct vacuum states (19). One should add that a similar phenomenon occurs with accumulation of the $\eta'$ field in vicinity of the axion domain wall as described in [37]. However, one could not term that effect as a “local spontaneous violation” of the $U(1)_A$ symmetry because the
$U(1)_A$ symmetry is explicitly broken by anomaly, in contrast with our present studies when the baryon charge is the exact symmetry of QCD. Nevertheless, the physics is the same in a sense that the closed bubble configurations generically acquire the axial as well as the baryon charge. This phenomenon is as generic as formation of the topological domain walls themselves when the periodic fields $\phi, \theta_1, \theta_2$ may randomly assume physically identical but topologically distinct vacuum values on the correlation lengths of order $\xi$.

Finally, one should also mention here that very similar effect of the “local $CP$ violation” can be experimentally tested in heavy ion collisions in event by event basis where the so-called induced $\theta_{\text{ind}}$-domain with a specific sign in each given event can be formed. This leads to the “charge separation effect” which can be experimentally observed in relativistic heavy ion collisions [54]. This “charge separation effect” in all respects is very similar to the phenomenon discussed in the present section. In fact, the main motivation for one of the authors (AZ) for studies [54] was a possibility to test the ideas advocated in this work by performing a specific analysis in the controllable “little Bang” heavy ion collision experiments, in contrast with “Big Bang” which happened billion of years ago. This field of research initiated in [54] became the hot topic in recent years as a result of many interesting theoretical and experimental advances, see recent review papers [55–57] on the subject.

V. FORMATION OF THE NUGGETS: TIME EVOLUTION

We assume that a closed $N_{DW} = 1$ domain wall has been formed as discussed in previous section III. Furthermore, we also assume that this domain wall is classified by non-vanishing baryon number $(n_1 + n_2)$ according to eq. (20). Our goal now is to study the time evolution of the obtained configuration. As we argue below the contraction of the bubbles halts as a result of the Fermi pressure due to baryon charge accreted during the evolution. As a result, the system comes to the equilibrium at some temperature $T_{\text{form}}$ when the nuggets complete their formation. We want to see precisely how it happens, and what are the typical time scales relevant for these processes.

We start with the following effective Lagrangian describing the time evolution of the closed spatially symmetric domain wall of radius $R(t)$,

$$L = \frac{4\pi\sigma R^2(t)}{2} R^2(t) - 4\pi\sigma R^2(t) \left[ 1 - \frac{1}{3} \frac{d^2R(t)}{dt^2} \right] + \frac{4\pi R^2(t)}{3} [P_{\text{in}}(\mu) - P_{\text{out}}(t)] + [\text{other terms}].$$

(25)

Here $\sigma$ is the key dimensional parameter, the domain wall tension $\sigma \sim f_\pi^2 a_\pi f_a \sim m_a^{-1}$ as reviewed in Introduction, see eq. (3). The tension $\sigma$, in principle, is also time-dependent parameter because the axion mass depends on time, but for qualitative analysis of this section we ignore this time dependence for now. We return to this question later in the text. Parameters $P_{\text{in}}(\mu(t))$ and $P_{\text{out}}(t)$ represent the pressure inside and outside the bubble. The outside pressure in QGP phase at high temperature can be estimated as

$$P_{\text{out}} \approx \frac{\pi^2 g_{\text{out}}^4}{90}, \quad T_{\text{out}} \approx T_0 \left( \frac{t_0}{t} \right)^{1/2},$$

$$g_{\text{out}}^4 \approx \left( \frac{7}{8} 4N_c N_f + 2(N_c^2 - 1) \right).$$

(26)

where $g_{\text{out}}^4$ is the degeneracy factor, while $T_0 \approx 100$ MeV and $t_0 \sim 10^{-4}$s represents initial temperature and time determined by the cosmological expansion. We also assume that the thermodynamical equilibrium is maintained at all times between inside and outside regions such that the temperature inside the bubble approximately follows the outside temperature $T_{\text{out}}(t) \approx T_{\text{in}}(t)$. Very quick equilibration indeed is known to take place even in much faster processes such as heavy ion collisions. The fast equilibration in our case can be justified because the heat transport between the phases is mostly due to the light NG bosons which can easily penetrate the domain wall with little on no interaction, in contrast with quarks and baryons discussed in the previous section. This assumption will be justified posteriori, see (55) on flux exchange rate between interior and exterior regions. Therefore, we believe our approximation $T_{\text{out}}(t) \approx T_{\text{in}}(t)$ is sufficiently good, at least for qualitative estimates which is the main goal of this work.

The expression for the pressure inside the bubble $P_{\text{in}}(t)$ depends on a number of quite nontrivial features of QCD such as the bag vacuum energy, corrections due to the gap in CS phase and many other phenomena, to be discussed later in the text.

The equation of motion which follows from (25) is

$$\sigma \frac{dR(t)}{dt} = \frac{2\sigma}{R(t)} + \frac{\sigma \frac{d^2R(t)}{dt^2}}{R(t)} + \Delta P(\mu) - 4\eta \frac{dR(t)}{dt},$$

(27)

where $\Delta P(\mu(t)) \equiv \left[ P_{\text{in}}(\mu(t)) - P_{\text{out}}(t) \right]$. We also inserted an additional term (which cannot be expressed in the Lagrangian formulation (25)), the shear viscosity $\eta$ to the right hand side of the equation, which effectively describes the “friction” of the system when the domain wall bubble moves in “unfriendly” environment\textsuperscript{11}. On the microscopical level this term effectively accounts for

\textsuperscript{11} We use conventional normalization factor of $4\eta R(t)/R(t)$ for the viscous term. This normalization factor is the same which appears in the Rayleigh-Plesset equation in the classical hydrodynamics when the viscous term, the surface tension term $2\sigma/R(t)$ and the pressure term $\Delta P$ enter the equation in a specific combination as presented in (27). One should emphasize that our equation (27) describes the dynamics of the 2d surface characterized by the same surface tension $\sigma$ in contrast with classical equation
a large number of different effects which do occur during the time evolution. Such processes include, but not limited to different scattering process by quarks, gluons or Nambu Goldstone Bosons in different phases. All these particles and quasiparticles interact between themselves and also with a moving domain wall. Furthermore, the annihilation processes which take place inside the bubble and which result in production of a large number of strongly interacting quasi-particles also contribute to $\eta$.

Having discussed an expression for $P_{\text{out}}(T)$ and viscous term $\sim \eta$ we now wish to discuss the structure of the internal pressure $P_{\text{in}}(\mu)$ which enters (27). It has a number of contributions which are originated from very different physics. We represent $P_{\text{in}}(\mu)$ as a combination of three terms to be discussed one by one in order,

$$P_{\text{in}}(\mu) \simeq P_{\text{in}}^{(\text{Fermi})}(\mu) + P_{\text{in}}^{(\text{bag constant})}(\mu) + P_{\text{in}}^{(\text{others})}. \quad (28)$$

In this formula $P_{\text{in}}^{(\text{Fermi})}(\mu)$ can be represented as follows

$$P_{\text{in}}^{(\text{Fermi})}(\mu) = \frac{E}{3V} = \frac{g^{\text{in}}}{6\pi^2} \int_0^{\infty} \frac{k^3dk}{\left[\exp\left(\frac{\epsilon(k) - \mu}{T}\right) + 1\right]}, \quad (29)$$

where we assume that quarks are massless and the chemical potential $\mu(t)$ implicitly depends on time as a result of the bubble’s evolution (shrinking). The degeneracy factor in this formula is

$$g^{\text{in}} \simeq 2N_cN_f, \quad (30)$$

where we keep only the quark contribution by neglecting the antiquarks. In other words, we simplify the problem by ignoring the time dependence of the degeneracy factor $g^{\text{in}}(t)$ which effectively varies as a result of $\mu(t)$ variation.

Now we are in position to discuss $P_{\text{in}}^{(\text{bag constant})}$ from (28) which can be represented as follows

$$P_{\text{in}}^{(\text{bag constant})}(\mu) \simeq -E_B \cdot \theta [\mu - \mu_1] \left[1 - \frac{\mu_1^2}{\mu^2}\right], \quad (31)$$

where positive parameter $E_B \sim (150 \text{ MeV})^3$ is the famous “bag constant” from MIT bag model, see [3] for references and numerical estimates for this parameter in the given context of the nugget structure. The bag constant can be expressed in terms of the gluon and quark condensates in QCD. We shall not elaborate on this problem in the present work by referring to [3] with relevant studies in the given context.

The bag “constant” $E_B$ describes the differences of vacuum energies (and therefore, vacuum pressure) in the interior and exterior regions of the nuggets. This difference occurs in our context because the phases realized outside and inside of the nugget are drastically distinct. For example, at the end of formation the outside region of the nugget is in cold hadronic phase, while inside region is in CS phase. The vacuum energies in these two phases are known to be drastically different. This term works as a “squeezer”, similar to the role it plays in the MIT bag model, when the vacuum energy outside of the nugget is lower than the vacuum energy inside the nugget. Therefore it enters with the same sign minus as the domain wall pressure.

A specific $\mu$-dependence used in (31) is an attempt to model a known feature of QCD that the absolute value of the vacuum energy decreases when the chemical potential increases. This feature is well established and tested in conventional nuclear matter physics, and it was analytically derived in simplified version of QCD with number of colours $N_c = 2$, see [3] for references and details. Our parametrization (31) corresponds to the behaviour when $P_{\text{in}}^{(\text{bag constant})}(\mu) = 0$ for small chemical potentials $\mu \leq \mu_1$ when the vacuum energy inside and outside of the nuggets approximately equal. This term becomes very important “squeezer” at large chemical potential at $\mu \geq \mu_1$ when the system outside is in the hadronic vacuum state while inside it is in a CS phase. The numerical value for parameter $\mu_1$ can be estimated as $\mu_1 \sim 330 \text{ MeV}$ [3] when the baryon density is close to the nuclear matter density.

The last term entering (28) and coined as $P_{\text{in}}^{(\text{others})}(\mu)$ is due to a large number of other effects which we ignore in present work. In particular, there is a conventional contribution due to the boson degrees of freedom which cancels the corresponding portion of $g^{\text{out}}$ from (26) at high temperature, $T \gg \mu$. It does not play any important role in our analysis because we are mainly concerned with analysis of fermion degrees of freedom and building the chemical potential inside the bubble. Another effect which worth to be mentioned is the formation of the gap in CS phase due to the quark pairing, similar to formation of the gap in conventional superconductors. The generation of the gap obviously decreases the energy of the system. There are many other phenomena which are known to occur in CS phase [16]. However, we expect that these effects are less important in comparison with the dominating contributions which are explicitly written down, (29) and (31).

The equation (27) can be numerically solved for $R(t)$ if time variation of the chemical potential $\mu(t)$ entering (29) and (31) is known. To study the corresponding time evolution for the chemical potential $\mu(t)$ we use expression (21) for the baryon charge bounded to the domain wall. We assume that the thermodynamical equilibrium is maintained between internal and external parts of the nugget such that $T_{\text{in}}(t) \simeq T_{\text{out}}(t)$. This assumption will be justified a posteriori, see discussions after eq. (5). At the same time the chemical potential is quickly increasing with time inside the nugget due to decreasing of the nugget’s size. We also assume a fast equilibration for the chemical potential within the nugget in its entire volume.
In other words, we describe the system using one and the same chemical potential in vicinity of the wall and deep inside the bubble. Justification for this assumption will be given later in the text.

With this picture in mind, we proceed by differentiating eq.(21) with respect to time to arrive at the following implicit equation relating \( \mu(t) \) and \( R(t) \) at fixed temperature \( T \),

\[
\dot{B} = \frac{Ng}{4\pi^2} \dot{S}(t) \int \frac{d^2k_\perp}{[\exp(-\mu(t)/T) + 1]} \exp(-\mu(t)/T) + 1 \quad \text{(32)}
\]

\[+ \frac{NgS}{4\pi^2} \frac{\dot{\mu}(t)}{T} \int \frac{d^2k_\perp}{\exp(\mu(t)/T)}\left[\exp(-\mu(t)/T) + 1\right] = (\text{fluxes}) = 0,
\]

where term “fluxes” in (32) describes the loss of baryonic matter due to annihilation and other processes describing incoming and outgoing fluxes, to be discussed later in the text. The relation (32) gives an implicit relation between \( \mu(t) \) and \( R(t) \) which can be used for numerical studies of our equation (27) describing the time evolution of the system.

We shall discuss the physics related to incoming and outgoing fluxes in Appendix A. If we neglect this term which describes the loss of baryonic matter we can analytically solve (32) for small \( \mu \ll T \) when one can use the Taylor expansion of the integrals entering (32). The result is

\[
(\mu(t) - \mu_0) \approx \frac{\pi^2 T}{6 \ln 2} \ln \left(\frac{R(t)}{R_0}\right), \quad \text{(33)}
\]

where \( R_0 \) is initial size of the system at \( t = t_0 \) while \( \mu_0 \approx 0 \) is initial chemical potential. One can explicitly see that the chemical potential builds in very fast when the nugget reduces its size only slightly. This formula (33) is only justified for very small \( \mu(t) \). For larger values of \( \mu \) one should use exact formula (32).

Finally, one should note that at the end of formation at time \( t \to \infty \) when temperature \( T \ll \mu \) the evolution stops, in which case all derivatives vanish, \( \dot{R}_{\text{form}} = \ddot{R}_{\text{form}} = \dot{\mu}_{\text{form}} = 0 \). At this point the nugget assumes its final configuration with size \( R \approx R_{\text{form}} \), and the equation (27) assumes the form

\[
\frac{2\sigma}{R_{\text{form}}} = P_m = \frac{g^{\text{in}} \mu^4}{24\pi^2} - E_B \left(1 - \frac{\mu_1^2}{\mu^2}\right), \quad \mu \geq \mu_1. \quad \text{(34)}
\]

This condition is precisely the equilibrium condition studied in [3] with few neglected contributions (such as the quark-quark interaction leading to the gap). This is of course expected result as the time evolution, which is the subject of the present work, must lead to the equilibrium configuration when the free energy assumes its minimum determined by (34).

One should recall that analysis of the equilibrium presented in ref. [3] with typical QCD parameters strongly suggests that the system indeed falls into CS phase when the axion domain wall pressure \( \sigma \) assumes its conventional value. At the same time, the equilibrium is not likely to emerge with the same typical QCD parameters without an additional external pressure related to the axion domain wall. In this sense the axion domain wall with extra pressure due to \( \sigma \neq 0 \) plays the role of an additional “squeezer” stabilizing the nuggets.

The key element of this section is equation (32) which is the direct consequence of a spontaneous accretion of the baryon (or antibaryon) charge in the domain wall background as discussed in section IV. Precisely this equation explicitly shows that the chemical potential \( \mu(t) \) grows very fast when the domain wall shrinks as a result of the domain wall pressure \( \sigma \). The presence of a nonvanishing chemical potential in the vicinity of the domain wall obviously implies the generation of the binding forces between the fermions and the domain wall, such that a typical bound energy of a single fermion to the domain wall is of order of \( \mu \).

A generic solution of equations (27) and (32), as we discuss in next section, shows an oscillatory behaviour of \( R(t) \) with a slow damping of the amplitude such that the system eventually settles down at the equilibrium point (34). However, even the very first oscillation with initial \( \mu_0 \approx 0 \) leads to very fast growth of the chemical potential \( \mu(t) \approx T \) as analytical estimates represented by eq.(33) shows. In next section we develop a quantitative framework which allows us to analyze our basic equation (27) for \( R(t) \) where time dependence \( \mu(t) \) is implicitly expressed in terms of the same variable \( \mu(t) \) as determined by (32).

VI. FORMATION OF THE NUGGETS.

QUALITATIVE ANALYSIS.

Our goal in this section is to solve for \( R(t) \) and therefore \( \mu(t) \) by solving (27) and (32), which implicitly relate both variables. We shall observe that a nugget experiences large number of oscillations during its evolution with slow damping rate, and eventually settles down at the equilibrium point (34). This behaviour of the system will be coined as “underdamped oscillations”. In next section VIA we formulate some assumptions and present the technical details, while the interpretation of the obtained results will be presented in section VI B. We want to make a number of simplifications in our analysis in present section to demonstrate the generic features of these oscillations. The numerical studies presented in Appendices A, B and C support our basic picture of oscillatory behaviour advocated in this section.

A. Assumptions, approximations, simplifications

Exact analytical analysis of either (27) or (32) can be obtained only during the first moment of the initial stage of evolution of the system when \( \mu \) is sufficiently small
indeed quite small. Hence, (32) is now simplified to:

\[ \dot{B} = \frac{d}{dt} \left\{ \frac{N g}{4\pi^2 S} \int \frac{d^2 k_\perp \exp(-\frac{\mu}{T})}{1 + 1} \right\} = 0 \]  

(35)

which means in this approximation, the baryonic charge is roughly conserved in the domain wall background at all times during the evolution of the system.

As our second simplification we neglect the mass of the fermions in comparison with temperature \( T \) and the chemical potential \( \mu \), i.e. we use the following dispersion relation \( \epsilon = \sqrt{k_\perp^2 + m^2} \approx k_\perp \) in vicinity of the domain wall. This approximation is somewhat justified in QGP and CS phases, and therefore along the path 3 as shown on Fig. 1. It is not literally justified for paths 1 and 2 as in the hadronic phase where the mass term should be identified with the so-called “constituent” quark mass which is proportional to the chiral condensate. Nevertheless, to simplify the problem we neglect the mass \( m(T) \) for all paths in our qualitative analysis of the time evolution as we do not expect any drastic changes in our final outcome as a result of this technical simplification. With these assumptions we can approximate the integral entering eq. (35) as follows,

\[ \int_0^\infty \frac{dk_\perp \cdot k_\perp}{e^{\frac{\mu}{T} - \frac{k_\perp^2}{2}} + 1} = T^2 \cdot I(\frac{\mu}{T}) \]  

(36)

where the omitted terms \( \sim \frac{\mu}{T} e^{-\mu/T} \) will be neglected thereafter, as they will never dominate in neither small nor large limit of \( \mu \). One can numerically check that approximation (36) describes the relevant integral \( I(\frac{\mu}{T}) \) sufficiently well in the entire parametrical space of \( \mu/T \), see Appendix C with corresponding analysis. As a quick test of this approximation one can check that approximate expression (36) reproduces an exact (in the small \( \mu \) limit) expression (33) with accuracy of order 15%, which is more than sufficient for our qualitative studies of this section.

As mentioned above, if flux term (32) is neglected, the curly-bracket term in (35) is a conserved quantity. Equating it to its initial values where \( S(t = 0) = 4\pi R_0^2 \), \( \mu(t = 0) = \mu_0 \geq 0 \) one arrives to

\[ T^2 R_0^2 \left( \frac{\pi^2}{6} + \frac{1}{2} \left( \frac{\mu}{T} \right)^2 - \frac{\pi^2}{12} e^{-\mu/T} \right) = \frac{\pi^2}{12} T_0^2 R_0^2. \]  

(37)

In what follows we assume that the thermodynamical equilibration is established very quickly such that one can approximate \( T \approx T_0 \) during the time evolution as we already discussed in previous section V. To simplify further the system we wish to represent the equation relating \( R \) and \( \mu/T \) in the following form

\[ f(R) = \frac{\pi^2}{6} \left[ \frac{1}{2} \left( \frac{R_0}{R} \right)^2 - 1 \right] = \frac{1}{2} \left( \frac{\mu}{T} \right)^2 - \frac{\pi^2}{12} e^{-\mu/T}, \]  

(38)

where we introduced function \( f(R) \) for convenience of the analysis which follows. Essentially, the idea here is to simplify the basic equation (27) as much as possible to express the \( \mu(t) \)- dependent terms entering through the pressure (28) in terms of \( R(t) \) such that the equation (27) would assume a conventional differential equation form for a single variable \( R(t) \).

Our next step is to simplify the expression for the Fermi pressure (29) entering (28) using the same procedure we used to approximate formula (36), i.e.

\[ P_{\text{in}}^{(\text{Fermi})} = \frac{g_{\text{in}}}{6\pi^2} \int_0^\infty \frac{k^3 dk}{\exp\left(\frac{\epsilon(k) - \mu}{T}\right) + 1} \]  

\[ \simeq \frac{g_{\text{in}} T^4}{6\pi^2} \left\{ \frac{7\pi^4}{60} + \frac{\pi^2}{2} \left( \frac{\mu}{T} \right)^2 - \frac{7\pi^4}{120} e^{-\mu/T} + \frac{1}{4} \left( \frac{\mu}{T} \right)^4 \right\} \]  

\[ + O\left( \frac{\mu}{T} e^{-\mu/T} \right) \]  

\[ \simeq \frac{g_{\text{in}} T^4}{6} \left\{ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu}{T} \right)^2 - \frac{\pi^2}{12} e^{-\mu/T} \right\} + \frac{1}{4} \left( \frac{\mu}{T} \right)^4 \]  

\[ + \frac{g_{\text{in}} T^4}{6} \left\{ \frac{\pi^2}{40} e^{-\mu/T} + O\left( \frac{\mu}{T} e^{-\mu/T} \right) \right\}. \]  

In what follows we neglect the last line in eq. (39). The justification for this procedure is the same as before: it produces a small contribution in entire region of \( \mu \) in comparison with accounted terms. The technical advantage for this procedure is the possibility to rewrite (39) in terms of function of \( R(t) \), rather than \( \mu(t) \) using our relation (38).

The formula in the square bracket in (39) is just \( f(R) \) defined by (38). The remaining \( \left( \frac{\mu}{T} \right)^4 \) term can be also expressed in terms of \( R \) by taking square of (38):

\[ f^2(R) = \left[ \frac{1}{2} \left( \frac{\mu}{T} \right)^2 - \frac{\pi^2}{12} e^{-\mu/T} \right]^2 \]  

\[ \simeq \frac{1}{4} \left( \frac{\mu}{T} \right)^4 + \frac{\pi^2}{12} - \frac{\pi^2}{144} \]  

(40)

where the correction term \( \sim O(\mu e^{-\mu/T}) \) will be dropped in what follows, as before. Thus, we approximate \( P_{\text{in}}^{(\text{Fermi})} \) in terms of \( R(t) \) as follows

\[ P_{\text{in}}^{(\text{Fermi})} \simeq \frac{g_{\text{in}} T^4}{6} \left[ \frac{7\pi^2}{60} + f(R) + f^2(R) - \frac{\pi^2}{144} \right]. \]  

(41)

The expression for the Fermi pressure \( P_{\text{in}}^{(\text{Fermi})}(R) \) now is expressed in terms of \( R \) rather than in terms of \( \mu \) as in the original expression (29).
We wish to simplify the expression for $P_{in}^{(bag\,\text{const})}(\mu)$ entering (28) in a similar manner to express $P_{in}^{(bag\,\text{const})}$ in terms of $R$. This contribution becomes important as discussed after eq. (31) only for sufficiently large $\mu$. In this region $f(R)$ can be well approximated as
\[
f(R) \simeq \frac{1}{2} \left( \frac{\mu}{T} \right)^2, \quad \mu \gg T \quad (42)
\]
so that we have
\[
P_{in}^{(bag\,\text{const})} \simeq -E_B \cdot \theta \left( \sqrt{2f(R)} - \frac{\mu_1}{T} \right) \left( 1 - \frac{\mu_1^2}{2T^2f(R)} \right). \quad (43)
\]
As a result of these simplifications and approximations the pressure term which enters the basic equation (27), $\Delta P(\mu) \equiv [P_{in}(\mu) - P_{out}(t)]$ which was initially formulated in terms of the chemical potential $\mu$ inside the bubble can be now written entirely in terms of a single variable, the size of the bubble $R(t)$:
\[
\Delta P[f(R)] \simeq \frac{g_{in}^2 \pi^2}{6} T^4 \left[ \frac{79}{720} \frac{g_{out}^2}{g_{in}^2} + f(R) \frac{T^2}{\pi^2} + \frac{f^2(R)}{\pi^4} \right] -E_B \cdot \theta \left( \sqrt{2f(R)} - \frac{\mu_1}{T} \right) \left( 1 - \frac{\mu_1^2}{2T^2f(R)} \right), \quad (44)
\]
where $f(R)$ is defined by eq. (38). With these technical simplifications the basic equation (27) can now be written as a second order differential equation entirely in terms of $R(t)$ rather than $\mu$:
\[
\sigma \dot{R}(t) = -\frac{2\sigma}{R} - \frac{\sigma R^2}{R} + \Delta P[f(R)] - 4\eta \frac{\dot{R}}{R}, \quad (45)
\]
with $\Delta P[f(R)]$ determined by eq. (44).

This equation can be solved numerically. In fact, it is precisely the subject of Appendix B. However, the most important quantitative features of the obtained solution can be understood without any numerical studies, but rather using a simplified analytical analysis, which is precisely the subject of the next section.

### B. Time evolution. Qualitative analysis.

As we already mentioned a nugget assumes its final form at $t \to \infty$ when all time derivatives vanish and the equation for the equilibrium is given by (34) at $T = 0$. In this section we generalize this equation for the equilibrium by defining $R_{\text{form}}(T)$ as the solution of eq.(46), see below, at $T \neq 0$. In other words, the starting point of the present analysis at $T \neq 0$ is the equilibrium condition when the “potential” energy assumes its minimal value. The corresponding minimum condition is determined by equation
\[
2\sigma = \Delta P(R_{\text{form}}), \quad (46)
\]
where $\Delta P(R_{\text{form}})$ is defined by eq.(44). This condition obviously reduces to eq. (34) at $t \to \infty$ when $\mu \gg T$.

We follow the conventional technique and expand (45) around the equilibrium value $R_{\text{form}}(T)$ to arrive to an equation for a simple damping oscillator:
\[
\frac{d^2 (\delta R)}{dt^2} + \frac{2}{\tau} \frac{d(\delta R)}{dt} + \omega^2 (\delta R) = 0, \quad (47)
\]
where $\delta R \equiv [R(t) - R_{\text{form}}]$ describes the deviation from the equilibrium position, while new parameters $\tau$ and $\omega$ describe the effective damping coefficient and frequency of the oscillations. Both new coefficients are expressed in terms of the original parameters entering (45) and are given by
\[
\tau = \frac{\sigma}{2\eta} R_{\text{form}} \quad (48a)
\]
\[
\omega^2 = \frac{1}{\sigma} \left. \frac{d\Delta P(R)}{dR} \right|_{R_{\text{form}}} - \frac{2}{R_{\text{form}}^2}. \quad (48b)
\]
The expansion (47) is justified, of course, only for small oscillations about the minimum determined by eq.(46), while the oscillations determined by original equation (45) are obviously not small. However, our simple analytical treatment (47) is quite instructive and gives a good qualitative understanding of the system. Our numerical studies presented in Appendix B fully support the qualitative picture presented below.

We start our qualitative analysis with estimates of the parameter $\omega$ which depends on $\frac{d\Delta P(R)}{dR}$ computed at $R = R_{\text{form}}$ according to (48b). First of all, in this qualitative analysis we neglect the bag constant term $P_{in}^{(bag\,\text{constant})}$ because it only starts to play a role for sufficiently large $\mu \geq \mu_1 \sim 330$ MeV, when formation is almost completed. This term obviously cannot change the qualitative behaviour of the system discussed below. Our numerical studies presented in Appendix B (where the bag constant term $\sim E_B$ is included in the analysis) support this claim.

The key element for our simplified analysis is the observation that the ratio $(R_0/R_{\text{form}})^2 \geq 14$ is expected to be numerically large number. This expectation will be soon confirmed a posteriori. This observation considerably simplifies our qualitative analysis because in this case $\Delta P(R_{\text{form}})$ defined by (44) can be approximated by a single term $\sim f^2(R)$ in square brackets in (44) as this term essentially saturates $\Delta P(R_{\text{form}})$. This is because the function $f(R)/\pi^2$ becomes numerically large in the relevant region $f(R)/\pi^2 \sim (R_0/R_{\text{form}})^2$ according to (38).

With these simplifications we can now estimate $\omega^2$ as follows
\[
\omega^2 \approx \left( \frac{g_{in}^2 \pi^2}{216} \right) \left( \frac{T^4}{\sigma R_{\text{form}}} \right) \left( \frac{R_0}{R_{\text{form}}} \right)^4 - \frac{2}{R_{\text{form}}^2}. \quad (49)
\]
To simplify analysis further one can represent the last term as
\[
\left( \frac{2}{R_{\text{form}}^2} \right) = \left( \frac{1}{R_{\text{form}}} \right) \left( \frac{\Delta P(R_{\text{form}})}{\sigma} \right), \quad (50)
\]
and keep the leading term \( \sim f^2(R) \) in expression for \( \Delta P(R_{\text{form}}) \). One can easily convince you that \( \omega^2 > 0 \) is always positive in this approximation such that the condition for a desired underdamped oscillations assumes a simple form

\[
\frac{f(R_{\text{form}})}{\pi^2} \gtrsim 1 \quad \Rightarrow \quad \left( \frac{R_0}{R_{\text{form}}} \right) \gtrsim \sqrt{14} \tag{51}
\]

when \( \Delta P(R_{\text{form}}) \) defined by (44) is dominated by a single term \( \sim \left( \frac{\omega}{2} \right)^2 \), which itself can be approximated by the leading quadratic term \( \sim \left( \frac{R_0}{R} \right)^2 \) according to (38). Our numerical studies presented in Appendix B support the numerical estimate (51).

One can also check that if condition (51) is not satisfied than system shows an “over-damped” behaviour when very few oscillations occur before complete collapse of the system, in which case the nuggets obviously do not form. These short-lived bubbles will never get to a stage when the temperature drops below the critical value \( T_{\text{CS}} \). Therefore, a CS phase cannot form in these “short-lived” bubbles. It should be contrasted with “long-lived” bubbles with much longer formation-time of order \( \tau \), see comments below.

The condition (51) is extremely important for our analysis. It essentially states that the initial size of a closed bubble \( R_0 \) must be sufficiently large for a successful formation of a nugget of size \( R_{\text{form}} \). On other hand, a formation of very large closed bubbles is strongly suppressed \( \sim \exp[-(R_0/\xi)^2] \) by the KZ mechanism as reviewed in section III. This constraint will be important in our estimation of a suppression factor in section VII C due to necessity to form a sufficiently large bubble (51) during the initial stage of formation.

Assuming that condition (51) is satisfied we estimate a typical frequency oscillations as follows

\[
\omega \sim \frac{1}{R_{\text{form}}} \sim m_a, \quad t_{\text{osc}} \sim \omega^{-1} \sim m_a^{-1} \tag{52}
\]

where we used the scaling properties (3) to relate the nugget’s size \( R_{\text{form}} \) with the axion mass \( m_a \). One should emphasize that the estimate (52) is not sensitive to any approximations and simplifications we have made in our qualitative treatment of the time evolution in this section. In fact, all parameters entering relation (52) are expressible in terms of the QCD scale \( \Lambda_{\text{QCD}} \) and a single “external” parameter, the axion mass \( m_a \), which we keep unspecified at this point. Of course we always assume that the axion mass may take any value from the observationally allowed window \( 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV} \).

We now turn our attention to the damping coefficient defined in terms of the original parameters by eq. (48a).

It is convenient to estimate the dimensionless combination \( \omega \tau \) as follows

\[
\omega \tau \approx \frac{1}{R_{\text{form}}} \cdot \left( \frac{\sigma}{2\eta} R_{\text{form}} \right) \approx \frac{\sigma}{2\eta} \sim \frac{m_\pi}{m_a} \sim 10^{11}, \tag{53}
\]

where we substituted \( \omega \sim R_{\text{form}}^{-1} \) according to (52) and assumed that \( \eta \sim m_a^2 \) has conventional QCD scale of order \( \text{fm}^{-3} \) while the wall tension \( \sigma \) can be approximated with high accuracy as \( \sigma \approx m_a^2/m_a \). This relation implies that the damping is extremely slow on the QCD scales. Therefore, the solution describing the time evolution of a “long-lived” bubble can be well approximated as follows

\[
R(t) = R_{\text{form}} + (R_0 - R_{\text{form}}) e^{-t/\tau} \cos \omega t \tag{54}
\]

which is obviously a solution of the approximate equation (47). This solution represents an “under-damped” oscillating \( R(t) \) with frequency \( \omega \sim \frac{1}{R_{\text{form}}} \) and damping time \( \tau \sim \frac{\sigma}{2\eta} R_{\text{form}} \). Precisely these “long-lived” bubbles will eventually form the DM nuggets.

The time scale (53) is very suggestive and implies that the damping term starts to play a role on very large scales when the cosmological expansion of the Universe with the typical scale \( t_0 \approx 10^{-4} \text{s} \) must be taken into account. We have not included the corresponding temperature variation in our studies because on the QCD scales (which is the subject of the present studies) the corresponding variations are negligible. However, the estimate (53) shows that for a proper analysis of the time scales \( \tau \) the expansion of the Universe (and related to the expansion the temperature variation) must be included. The corresponding studies are beyond the scope of the present work. However, the important comment we would like to make here is that the emergent large time scale (53) is fully consistent with our anticipation that the temperature of the Universe drops approximately by a factor of \( \sim 3 \) or so when a CS phase forms in interior of the nugget during the formation period. It is quite obvious that if the time scale (53) were considerably shorter than the cosmological time scale \( t_0 \approx 10^{-4} \text{s} \) than the temperature \( T \sim t^{-1/2} \) inside the nugget could not drop sufficiently deep into the region where CS sets in as plotted on Fig.1. Fortunately, the timescale (53) is long enough and automatically satisfies this requirement.

Now we want to elaborate on one more element of the dynamics which is also important for a successful formation of the nuggets. To be more specific, we want to discuss the flux of particle exchange, which was ignored in our qualitative analysis in this section and which is estimated in Appendix A. This flux describes the rate of number of particle flowing between inside and outside the system, which can be appreciably large even if the net baryonic flux is negligibly small. To be more precise, there are two kinds of fluxes, both investigated in Appendix A, that we are discussing in this paper: the net flux of baryonic charge \( \Delta \Phi \equiv \Phi_{\text{in}} - \Phi_{\text{out}} \), and the average flux of particle number \( \langle \Phi \rangle \equiv \frac{1}{2} (\Phi_{\text{in}} + \Phi_{\text{out}}) \). The first one corresponds to the flux term entering eq. (32); while the latter is important in understanding what is the typical time scale for a complete “refill” of the particles during the time evolution. The last question is important for understanding of the time scale for thermal equilibration.

We start our analysis with discussions of an average flux \( \langle \Phi \rangle \) at small chemical potential. It is estimated to be \( \langle \Phi \rangle \approx 1\text{ fm}^{-3} \) according to Appendix A. The magnitude of this flux can be fully appreciated by computing the
total number of particle exchange per one cycle of the oscillation
\[ \frac{2\pi}{\omega} \cdot 4\pi R_{\text{form}}^2 \cdot (\langle \Phi \rangle \sim R_{\text{form}}^2 \text{fm}^{-3} \sim |\mathcal{B}|, \] (55)

where \( \omega \) is a typical frequency oscillation estimated in (52) while \( |\mathcal{B}| \) is the total number of particles (quarks and antiquark) stored in the nugget. The physical meaning of this estimate is that a nugget can in principle entirely refill its interior with “fresh” particles within a few cycles of exchange. Similar estimate for the net baryon flux which includes \( \Delta \Phi \) is suppressed, see Appendix A.

The main reason for emergence of this large scale in expression (55) is a long time scale of a single cycle (52) which is determined by the axion mass \( m_a \) rather than by QCD physics. Nevertheless, estimate (55) is quite remarkable and shows that even very low rate of chemical potential accretion of (anti)quarks being tracked per oscillation, the high exchange rate (55) is still sufficient enough to turn a baryonically neutral nugget into one completely filled with (anti)quarks. When the quarks become effectively massive as it happens in hadronic and CS phases, the flux for the exchange of the baryon charge is drastically decreased by a factor \( \sim \exp(-m/T). \)

The same estimate (55) essentially holds for exchange of almost massless Nambu-Goldstone bosons for sufficiently high temperature. In fact, the lightest degrees of freedom play the crucial role in cooling processes of the interior of the nugget as these particles can easily penetrate the sharp domain wall structure. Therefore, the high exchange rate between exterior and interior of a nugget essentially implies that the thermal equilibrium is maintained in our system with very high precision due to a huge rate per cycle (55) when large number of degrees of freedom \( \sim B \) have a chance of order one to interact with “fresh” particles from the exterior during a single cycle. Therefore, our assumption on thermal equilibrium between interior and exterior is justified a posteriori.

We conclude this section with few important comments. The most important result of this section is that the nuggets can be formed during the QCD phase transition provided the initial size of the nuggets is sufficiently large as stated in eq.(51), in which case they survive the evolution. The key role in this successful formation plays, of course, the effect of “local spontaneous violation” of the baryon symmetry as we discuss below.

VII. BARYON CHARGE SEPARATION. CORRELATION ON COSMOLOGICAL SCALES.

Until this section we mostly concentrated on the time evolution of a single nugget (or anti-nugget). The main lesson of our previous discussions is that such nuggets can be formed, remain stable configurations, and therefore, can serve as the dark matter candidates. In other words, the focus of our previous studies was a problem of a local separation of charges on small scales of order nugget’s size. The key element of that separation of charges is eq. (19) which can be thought as a local version of spontaneous symmetry breaking of the baryon charge as explained in section IV. However, on a larger scale it is quite obvious that equal number of nuggets and anti-nuggets will be formed as a result of an exact symmetry as we discuss below.

This symmetry, however, does not hold anymore on large scales if the axion \( \mathcal{CP} \)-odd coupling is included into consideration, which eventually leads to very generic, essentially insensitive to most parameters, consequence of this framework represented by eq.(1), which is the subject of next subsections VIIA, VIIIB. The subsection VIIIC is devoted to some more specific and model-dependent consequences of this framework. In particular, we want to estimate a suppression factor related to a necessary to form a large size bubble (51) in KZ mechanism.

A. Coherent axion field as the source of \( \mathcal{CP} \) violation

First of all, let us show that the baryon charge hidden in nuggets on average is equal to the baryon charge hidden in anti-nuggets, of course with sign minus. Indeed, the analysis of the anti-nuggets can be achieved by flipping the sign of the chemical potential in eq. (17), i.e. \( \mu \rightarrow -\mu \). One can restore the original form of the \( \mu \) term in Lagrangian (17) by replacing \( \theta_1 \rightarrow -\theta_1 \) and \( \theta_2 \rightarrow -\theta_2 \). Finally, one should change the signs for the axion \( \theta \) and the pseudo-scalar singlet \( \eta' \) meson represented by \( \phi \) field in the interaction term (18) to restore the original form of the Lagrangian. These symmetry arguments imply that as long as the pseudo-scalar axion field fluctuates around zero as conventional pseudo-scalar fields (as \( \pi, \eta' \) mesons, for example), the theory remains invariant under \( \mathcal{P} \) and \( \mathcal{CP} \) symmetries. Without this symmetry the number density and size distribution of the nuggets and
anti-nuggets could be drastically different\(^\text{12}\).

Therefore, the symmetry arguments suggest that on average an equal number of nuggets and anti-nuggets would form if the axion field is represented by a conventional quantum fluctuating field oscillating around zero point. If it were the case, the baryons and anti-baryons would continue to annihilate each other as well as annihilate with the nuggets and anti-nuggets in our framework. Eventually it would lead to the Universe with large amount of dark matter in form of nuggets and anti-nuggets (they are far away from each other, therefore they do not annihilate each other) and no visible matter. However, the axion dynamics which is determined by the axion field correlated on the scale of the entire Universe leads to a preferential formation of a specific type of nuggets on the same large scales where the axion field is correlated as we argue below. Such coherent axion field emerges if the PQ phase transition occurs before or during inflation as discussed in items 1 and 5 in section III.

First of all we want to argue that the time dependent axion field implies that there is an additional coupling to fermions (56). Indeed, by making the time-dependent \( U(1)_A \) chiral transformation in the path integral one can always represent the conventional \( \theta \) term in the following form

\[
\Delta \mathcal{L}_4 = \mu_5(t) \bar{\Psi} \gamma_5 \Psi \quad \mu_5 \equiv \dot{\theta}.
\]  

In this formula \( \mu_5 \equiv \dot{\theta} \) can be thought as the chiral chemical potential. Many interesting properties emerge in the systems if \( \mu_5 \) is generated. In fact, it has been an active area of research in recent years, mostly due to very interesting experimental data suggesting that the \( \mu_5 \) term can be generated in heavy ion collisions, see original paper [54] and recent reviews [55–57] for the details. In the present context the \( \mu_5 \) term is generated as a result of the axion dynamics. As a matter of fact, the original studies [54] were motivated by the proposal that the separation of the baryon charges which may occur in early Universe, as advocated in this paper, could be tested in laboratory experiments with heavy ion collisions.

Now we are prepared to formulate the main claim of this section which can be stated as follows. When interaction (18), (56) is introduced into the system there will be a preferential evolution in the system of the nuggets versus anti-nuggets provided that nuggets and anti-nuggets had been already formed and chemical potential \( \mu \) had been already generated locally inside the nuggets as described in the previous section VI. As we already explained earlier, the generation of \( \mu \) can be interpreted as a “local violation” of \( C \) invariance in the system.

This preferential evolution is correlated with the \( CP \)-odd parameter on the scales where the axion field \( \theta(x) \) is coherent. In our arguments presented below we make a standard assumption that the initial value of \( \theta(x) \) and its time derivative \( \dot{\theta}(x) \) are correlated on the entire observable Universe, such that \( \mu_5 \equiv \dot{\theta} \) is also correlated on the same large scale. Such large scale correlation is guaranteed if the PQ phase transition occurs before inflation, see items 1 and 5 in section III with details. This is the standard assumption in most studies on axion physics when one computes the present density of axions due to the misalignment mechanism, see refs [8–15].

For our present studies the key element is that the dynamics of the axion field until the QCD phase transition is determined by the coherent state of axions at rest such that [8–15]:

\[
\theta(t) \sim \frac{C}{t^{3/4}} \cos \int^t dt' \omega_a(t'), \quad \omega_a^2(t) = m_a^2(t) + \frac{3}{16\pi^2},
\]

where \( C \) is a constant, and \( t = \frac{1}{\sqrt{\theta}} \) is the cosmic time. This formula suggests that for \( m_a(t) t \gg 1 \) when the axion potential is sufficiently strongly tilted the chiral chemical potential is essentially determined by the axion mass at time \( t \)

\[
\mu_5(t) = \dot{\theta}(t) \sim \omega_a(t) \approx m_a(t).
\]

The crucial point is that \( \theta(t) \) is one and the same in the entire Universe as it is correlated on the Universe size scale. Another important remark is that the axion field \( \theta(t) \) continues to oscillate with frequency (58) until the QCD phase transition at \( T_c \), though its absolute value \( |\theta(t)| \approx 0.01 \) might be few orders of magnitude lower at \( T_c \approx 170 \text{ MeV} \) than its original value \( \theta_0 \) at \( T \approx 1 \text{ GeV} \) when the axion field only started to roll, see e.g. [44]. As we discuss below, the relevant physics is not very sensitive to an absolute value of \( |\theta(t)| \) in this regime, and therefore, we do not elaborate further on this rather technical and computational element of the axion dynamics, see footnote 13 below with comments on this matter.

In the context of the nugget’s evolution (accretion of the baryon charge) this claim implies that on the entire Universe size scale with one and the same sign of \( \theta(t) \) a specific single type of nuggets will prevail in terms of the number density and sizes. Indeed, one can present the same arguments (see the beginning of this section) with flipping the sign \( \mu \rightarrow -\mu \) with the only difference is that the interaction (18) prevents us from making the variable change \( \theta(i) \leftrightarrow -\theta(i) \) for a given \( \theta(t) \) because it changes its form under \( \theta(i) \leftrightarrow -\theta(i) \). In other words, slow varying (on the QCD scale) \( CP \) violating terms (18), (56) lead to a preferential evolution of the system for a specific species of the nuggets with a given sign of \( \mu \).

Indeed, it has been known for quite sometime, see e.g. [58, 59] that in the presence of \( \theta \neq 0 \) a large number of different \( CP \) violating effects take place. In particular, the Nambu-Goldstone bosons become a mixture of pseudo-scalar and scalar fields, their masses are drastically different from \( \theta = 0 \) values. Furthermore, the

\(^{12}\) If \( \pi \) meson condensation were occur in nuclear matter it would unambiguously imply that the \( CP \) invariance is broken in such a phase. Some of the phases in CS systems indeed break the \( CP \) invariance as a result of condensation of a pseudo-scalar Nambu-Goldstone bosons.
quark chiral $\langle \bar{\psi}\psi \rangle$ and the gluon $(G^2)$ condensates become the superposition with their pseudo-scalar counterparts $\langle \bar{\psi}\gamma_5\psi \rangle$ and $(GG)$ such that entire hadron spectrum and their interactions modify in the presence of $\theta \neq 0$. All these strong effects, of course, are proportional to $\theta$, and therefore numerically suppressed in case consideration (57) by a factor $|\theta/\theta_0| \sim 10^{-2}$ in the vicinity of the QCD phase transition. Naively, this small numerical factor $|\theta/\theta_0| \sim 10^{-2}$ may lead only to minor effects $\sim 10^{-2}$. However, the crucial point is that while coupling (18) of the axion background field with quarks is indeed relatively small on the QCD scales, it is nevertheless effectively long-ranged and long-lasting in contrast with conventional QCD interactions. As a result, this coherent $\mathcal{CP}$ odd coupling may produce large effects of order of one as we argue below.

Indeed, as we discussed in previous section VI a typical oscillation time $t_{osc}$ when the baryon charge accretes on the wall is of order $t_{osc} \sim m_a^{-1}$ according to eq. (52). But this time scale $t_{osc} \sim m_a^{-1}$ is precisely the time scale when $\dot{\theta} = m_a(t)$ varies according to (58). Therefore, while the dynamical fermi fields $\theta_1, \theta_2$ defined by (16) fluctuate with typical scale of order $\Delta_{QCD} \gg m_a$, the coherent variation of these fields will occur during a long (on the QCD scales) coherent process when a nugget makes a single cycle. These coherent corrections are expected to be different for nuggets (positive $\mu$) and anti-nuggets (negative $\mu$) as a result of many $\mathcal{C}$ and $\mathcal{CP}$ violating effects such as scattering, transmission, reflection, annihilation, evaporation, mixing of the scalar and pseudo-scalar condensates, etc which are all responsible for the accretion of the baryon charge on a nugget during its long evolution.

Important comment here is that each quark experiences a small difference in interacting with the domain wall surrounding nuggets or anti-nuggets during every single QCD event (mentioned above) with typical QCD time scale $\Delta_{QCD}$. However, the number of the coherent QCD events $n_{coherent}$ during a long single cycle is very large

$$n_{coherent} \sim \Delta_{QCD} t_{osc} \sim \frac{\Delta_{QCD}}{m_a} \sim 10^{10} \gg 1. \quad (59)$$

Therefore, a net effect during every single cycle will be order of one, in spite of the fact that each given QCD event is proportional to the axion field $\theta(t)$ and could be quite small.

The argument presented above holds as long as the axion field remains coherent, see also a comment at the very end of this subsection. In other words, a small but non vanishing coherent $\mathcal{CP}$ violating parameter $\theta(t)$ plays the role of catalyst which determines a preferred direction for separation of the baryon charges on the Universe scale, see few comments in section III on justification of this assumption. This role of $\mathcal{CP}$ violation in our framework is very different from conventional “baryogenesis” mechanisms when $\mathcal{CP}$ violating parameter explicitly enters the final expression for the baryon charge production.

The corresponding large coherent corrections during a single cycle $t_{osc}$ imply that the fast fluctuating fields $\theta_1, \theta_2$ (which effectively describe the dynamics of the fermions living on the wall according to (16)) receive large corrections during every single cycle

$$\Delta \theta_1(t) \sim \Delta \theta_2(t) \sim 1. \quad (60)$$

These changes of order one of the strongly interacting $\theta_1, \theta_2$ fields lead to modification of the accreted baryon charge per single cycle per single degree of freedom

$$\Delta N \sim (\Delta \theta_1 + \Delta \theta_2) \sim 1 \quad (61)$$
on the nuggets according to (20). One should emphasize that the corrections (61) are expected to be different for nuggets and anti-nuggets because the interaction (56), (18) which is responsible for these corrections (61) breaks the symmetry between nuggets and anti-nuggets when $\mu \rightarrow -\mu$ as discussed above.

Precise computations of these coherent $\mathcal{CP}$ violating effects are hard to carry out explicitly as it requires a solution of many-body problem of the coherent wall fermions with surrounding environment in the background of axion field (57) when a large number of $\mathcal{C}$ and $\mathcal{CP}$ violating effects take place and drastically modify evolution of nuggets versus anti-nuggets. A large number of cycles of every individual nugget (anti-nugget) also introduces a huge uncertainty in computations of $\Delta N$ during the time evolution when a single cycle leads to the effect of order one, with possible opposite sign for a consequent cycle. In other words, it is very hard to predict what would be the final outcome of the system after a large number of cycles when each cycle produces the effect of order 1. We expect that the final result would be again of order one. Such a computation is beyond the scope of the present work. Therefore, in what follows we introduce a phenomenological parameter $c(T)$ of order one to account for these effects. All the observables will be expressed in terms of this single phenomenological parameter $c(T) \sim 1$, see eq. (62).

Our final comment in this subsection is as follows. The charge separation effect on largest possible scales is only possible when the axion field (57) is coherent on the scales of the Universe. This coherence is known to occur in conventional studies on the dynamics of the axion field in the vicinity of the QCD phase transition if the PQ phase transition occurs before inflation, see few comments in section III on this matter. At the same time, soon after the QCD phase transition the dominant part of the axion field transfers its energy to the free propagating on-shell axions (which is the subject of axion search experiments [8–13, 15]). These randomly distributed free axions are not in coherent state anymore. Therefore, the coherent accumulation effect which leads to a preferential formation of one species of nuggets, as discussed above, ceases to be operational at the moment of decoherence $t_{dec}$ when the description in terms of the coherent axion field (57) breaks down. The baryon asymmetry we

\[ \text{The decoherence time } t_{dec} \text{ is not entirely determined by absolute} \]
observe today in this framework is a result of accumulation of the charge separation effect from the beginning of the nugget’s formation until this very last “freeze-out” moment determined by $t_{\text{dec}}$. 

B. Nuggets vs anti-nuggets on the large scale. 

Generic consequences.

As we already mentioned to make any precise dynamical computations of $\Delta N \sim 1$ due to the coherent axion field (57) is a hard problem of strongly coupled QCD at $\theta \neq 0$. In order to effectively account for these coherent effects one can introduce an unknown coefficient $c(T)$ of order one as follows

$$B_{\text{nuggets}} = c(T) \cdot B_{\text{nuggets}}, \text{ where } |c(T)| \sim 1, \quad (62)$$

where $c(T)$ is obviously a negative constant of order one. We emphasize that the main claim of this section represented by eq. (62) is not very sensitive to the axion mass $m_a(T)$ nor to the magnitude of $\theta(T)$ at the QCD phase transition when the bubbles start to oscillate and slowly accrete the baryon charge. The only crucial factor in our arguments is that the typical variation of $\theta(t)$ is determined by the axion mass (58), which is the same order of magnitude as $t_{\text{osc}}^{-1}$, and furthermore, this variation is correlated on the scale where the axion field (57) can be represented by the coherent superposition of the axions at rest.

The key relation of this framework (62) unambiguously implies that the baryon charge in form of the visible matter can be also expressed in terms of the same coefficient $c(T) \sim 1$ as follows

$$B_{\text{visible}} = -B_{\text{antinuggets}} - B_{\text{nuggets}}, \quad (63)$$

Using eq. (62) it can be rewritten as

$$B_{\text{visible}} \equiv (B_{\text{baryons}} + B_{\text{antibaryons}}) = - (1 + c(T)) B_{\text{nuggets}} = - \left[1 + \frac{1}{c(T)} \right] B_{\text{antinuggets}}. \quad (64)$$

The same relation can be also represented in terms of the measured observables $\Omega_{\text{visible}}$ and $\Omega_{\text{dark}}$ at later times when only the baryons (and not anti-baryons) contribute to the visible component

$$\Omega_{\text{dark}} \approx \left(\frac{1 + |c(T)|}{1 + c(T)}\right) \cdot \Omega_{\text{visible}} \text{ at } T \leq T_{\text{form}}, \quad (65)$$

One should emphasize that the relation (64) holds as long as the thermal equilibrium is maintained, which we assume to be the case. Another important comment is that each individual contribution $|B_{\text{baryons}}| \sim |B_{\text{antibaryons}}|$ entering (64) is many orders of magnitude greater than the baryon charge hidden in the form of the nuggets and anti-nuggets at earlier times when $T_c > T > T_{\text{form}}$. It is just their total baryon charge which is labeled as $B_{\text{visible}}$ and representing the net baryon charge of the visible matter is the same order of magnitude (at all times) as the net baryon charge hidden in the form of the nuggets and anti-nuggets according to (63).

The baryons continue to annihilate each other (as well as baryon charge hidden in the nuggets) until the temperature reaches $T_{\text{form}}$ when all visible anti baryons get annihilated, while visible baryons remain in the system and represent the visible matter we observe today. It corresponds to $c(T_{\text{form}}) \approx -1.5$ as estimated below if one neglects the differences in gaps in CS and hadronic phases, see footnote 14. After this temperature the nuggets essentially assume their final form, and do not loose or gain much of the baryon charge from outside. The rare events of the annihilation between anti-nuggets and visible baryons continue to occur. In fact, the observational excess of radiation in different frequency bands, reviewed in section II, is a result of these rare annihilation events at present time.

The generic consequence of this framework represented by eqs. (62), (64), (65) takes the following form at this time $T_{\text{form}}$ for $c(T_{\text{form}}) \approx -1.5$ which corresponds to the case when the nuggets saturate entire dark matter density:

$$B_{\text{visible}} \approx \frac{1}{2} B_{\text{nuggets}} = - \frac{1}{3} B_{\text{antinuggets}}, \quad (66)$$

$$\Omega_{\text{dark}} \approx 5 \cdot \Omega_{\text{visible}}$$

which is identically the same relation (2) presented in Introduction. The relation (66) emerges due to the fact that all components of matter, visible and dark, proportional to one and the same dimensional parameter $\Lambda_{\text{QCD}}$, see footnote 14 with a comment on this approximation. In formula (66) $B_{\text{nuggets}}$ and $B_{\text{antinuggets}}$ contribute to $\Omega_{\text{dark}}$, while $B_{\text{visible}}$ obviously contributes to $\Omega_{\text{visible}}$. The coefficient $\sim 5$ in relation $\Omega_{\text{dark}} \approx 5 \cdot \Omega_{\text{visible}}$

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14 In eq. (65) we neglect the differences (due to different gaps) between the energy per baryon charge in hadronic and CS phases to simplify notations. The corresponding corrections in energy per baryon charge in hadronic and CS phases, in principle, can be explicitly computed from the first principles. However, we ignore these modifications in the present work. This correction obviously does not change the main claim of this proposal stating that $\Omega_{\text{visible}} \approx \Omega_{\text{dark}}$. 

value of amplitude of the axion field (57). In fact, the amplitude could be quite small, but the field remains coherent on large scales. The computation of the decoherence time $t_{\text{dec}}$ is a hard problem of QFT, similar to a problem in quantum optics when initially coherent light becomes de-coherent superposition of uncorrelated photons.
is obviously not universal, but relation (1) is universal, and very generic consequence of the entire framework, which was the main motivation for the proposal [3, 4].

For example, if $c(T_{\text{form}}) \simeq -2$ then the corresponding relation (65) between the dark matter and the visible matter would assume the form $\Omega_{\text{dark}} \simeq 3 \cdot \Omega_{\text{visible}}$. Such a relation implies that there is a plenty of room for other types of dark matter to saturate the observed ratio $\Omega_{\text{dark,observed}} \simeq 5 \cdot \Omega_{\text{visible,observed}}$. This comment will be quite important in our discussions in section VIII where we comment on implications of this framework for other axion search experiments.

One should emphasize once again that the generic consequences of the framework represented by (1), (65) are not sensitive to any specific parameters such as efficiency of the domain wall production or the magnitude of $\theta$ at the QCD phase transition, which could be quite small, see footnote 13 with few comments on that. Nevertheless, precisely the coupling with the coherent $CP$ odd axion field plays a crucial role in generation of $|c(T)| \neq 1$, i.e. the axion plays the role of catalyst in the baryon charge separation effect on the largest possible scales. Some other observables which are sensitive to the dynamical characteristics (e.g. efficiency of the domain wall production) will be discussed below.

C. $n_B/n_\gamma$ ratio. Model dependent estimates.

The time evolution of the dark matter within this framework is amazingly simple. The relations (62), (63), (64) hold at all times. The baryon charges of the nuggets and anti-nuggets vary until its radius $R(T)$ assumes its equilibrium value as described in sections V, VI. It happens approximately at time when the CS phase forms in interior of the nuggets, which can be estimated as $T_{CS} \simeq 0.6 \Delta \simeq 60$ MeV, where $\Delta \simeq 100$ MeV is the gap of the CS phase. After this temperature the nuggets essentially assume their final form, with very little variation in size (and baryon charge). The rare events of the annihilation of course continue to occur even for lower temperaeures. In fact, the observational consequences reviewed in section II is a result of these annihilation events at present time.

The variation of the visible matter $B_{\text{visible}}$ demonstrates much more drastic changes after the QCD phase transition at $T_c$ because the corresponding number density is proportional to $\exp(-m_N/T)$ such that at the moment of formation $T_{\text{form}} \approx 40$ MeV the baryon to entropy ratio assumes its present value (5) which we express as follows

$$\eta \equiv \frac{n_B}{n_\gamma} \simeq \frac{B_{\text{visible}}/V}{n_\gamma} \sim 10^{-10}, \quad n_B \equiv \frac{B_{\text{visible}}}{V}. \quad (67)$$

If the nuggets and anti-nuggets were not present at this temperature the conventional baryons and anti-baryons would continue to annihilate each other until the density would be 9 orders of magnitude smaller than observed (67) when the temperature will be around $T \simeq 22$ MeV. Conventional baryogenesis resolves this “annihilation catastrophe” by producing extra baryons in early times, see e.g. review [19], while in our framework extra baryons and anti baryons are hidden in form of the macroscopically large nuggets.

In our framework the ratio (67) can be rewritten in terms of the nugget’s density as the baryon charge in form of the visible matter and in form of the nuggets are related to each other according to (64). This relation allows us to infer what efficiency is required for the bubbles to be formed and survive until the present time when observed ratio is measured (67).

One should emphasize that any small factors which normally enter the computations in conventional baryogenesis (such as $C$ and $CP$ violating parameters) do not enter in the estimates presented below in our framework as result of two effects. First, the $C$ violation enters the computation as a result of generation of the chemical potential $\mu$ as described in section IV. It is expressed in terms of spontaneous accretion of the baryon charge on the surface of the nuggets as given by eq. (20) which effectively generates the chemical potential (35), which can be thought as the local violation of the symmetry on the scale of a single nugget. Secondly, the $CP$ violation enters the computation in form of the coupling with the coherent axion field (56). Precisely this coupling as we argued above leads to removing of the degeneracy between nuggets and anti-nuggets formally expressed as $c(T) \sim 1$ in eq. (62). Therefore, the only small parameter we anticipate in our estimates below is due to some suppression of the closed bubbles which must be formed with sufficiently large sizes during the QCD phase transition.

We cannot compute the probability for the bubble formation as it obviously requires the numerical simulations, which is beyond the scope of the present work. Instead, we go backward and ask the question: What should be the efficiency of the bubble formation at the QCD phase transition in order to accommodate the observed ratio (67)?

With these comments in mind we proceed with our estimates as follows. First, from (64), (66) we infer that the baryon charge hidden in the nuggets and anti-nuggets is the same order of magnitude as the baryon charge of the visible baryons at $T_{\text{form}}$ at the end of the formation, i.e.

$$\frac{B_{\text{nuggets}}/V}{n_\gamma} \gtrsim 10^{-10}, \quad (68)$$

where we use sign $\gtrsim$ instead of $\approx$ used in eq (67) to emphasize that there is long time for equilibration between the moment $T_{CS} \simeq 0.6 \Delta \simeq 60$ MeV when CS phase forms in interior of the nuggets and $T_{\text{form}} \simeq 40$ MeV when all anti baryons of the visible matter get annihilated, corresponding to the present observed value (67). During this period the equilibrium between the visible matter and the baryons from nuggets is maintained, and some por-
tion of the nugget’s baryon charge might be annihilated by the visible matter. It explains our sign $\geq$ used in eq. (68).

The relation (68) implies that the number density of nuggets and anti-nuggets can be estimated as

$$\frac{\langle B \rangle n_{\text{nuggets}}}{n_{\gamma}} \geq 10^{-10}, \quad \langle B \rangle n_{\text{nuggets}} \equiv \frac{B_{\text{nuggets}}}{V}, \quad (69)$$

where $\langle B \rangle$ is the average baryon charge of a single nugget at $t_{\text{form}}$.

Now we want to estimate the same ratio (69) using the Kibble-Zurek (KZ) mechanism[40–42] reviewed in section III. The basic idea of the KZ mechanism is that the total area of the crumpled, twisted and folded domain wall is proportional to the volume of the system, and can be estimated as follows:

$$S_{\text{(total DW)}} = \frac{V}{\xi(T)}, \quad (70)$$

where $\xi(T)$ is the correlation length which is defined as an average distance between crumpled domain walls at temperature $T$. Largest part of the wall belongs to the percolated large cluster. It is known that some closed walls (bubbles) with typical size $\xi(T)$ will be also formed. These bubbles with sufficiently large size $R \sim \xi(T)$ will eventually become nuggets. We introduce parameter $\gamma$ to account for the suppression related to the closed bubble formation. In other words, we define

$$S_{\text{nuggets}} = \gamma S_{\text{(total DW)}} = \frac{\gamma V}{\xi(T)}, \quad \gamma \ll 1. \quad (71)$$

At the same time total area of the nuggets $S_{\text{nuggets}}$ can be estimated as

$$S_{\text{nuggets}} = 4\pi R_0^2(T) [V \cdot n_{\text{nuggets}}], \quad (72)$$

where $R_0$ is the size of a nuggets at initial time, while $[V \cdot n_{\text{nuggets}}]$ represents the total number of nuggets in volume $V$. Comparison (71) with (72) gives the following estimate for the nugget’s density when bubbles just formed,

$$n_{\text{nuggets}} \simeq \frac{\gamma}{4\pi R_0^2 \xi}, \quad (73)$$

The last step in our estimates is the computation of the average baryon charge of a nugget at $T_{\text{CS}}$ when CS sets inside the nugget. The corresponding estimates have been worked out long ago [3] and reproduced in section V in the course of the time evolution by taking $t \to \infty$, see (34). The baryon number density inside the nuggets depends on a model being used [3], but typically it is few times the nuclear saturation density $n_0 \simeq (108 \text{ MeV})^3$ which is consistent with conventional computations for the baryon density in CS phases. Therefore, we arrive to

$$\langle B \rangle \simeq (2 - 6)n_0 \cdot \frac{4\pi R_{\text{form}}^3}{3}, \quad (74)$$

where $R_{\text{form}}$ is the final size of the nuggets. By substituting (74) and (73) to (69) we arrive to the following constraint on efficiency of the bubble formation represented by parameter $\gamma$

$$(2 - 6) \cdot \frac{\gamma}{3} \left( \frac{R_{\text{form}}}{R_0} \right) \left( \frac{R_{\text{form}}}{R_0} \right)^2 \left( \frac{n_0}{n_\gamma} \right) \geq 10^{-10}, \quad (75)$$

where expression for $n_\gamma(T)$ should be taken at the formation time

$$n_\gamma = \frac{2\xi(T)}{\pi^2 R_{\text{form}}^3}, \quad \xi(T) \simeq 1.2, \quad (76)$$

while the correlation length $\xi(T)$ should be evaluated at much earlier times, close to $T_c$ when domain wall network only started to form. Typically bubbles form with $R_0 \sim \xi$. However, the bubbles shrink approximately 3-5 times according to (51) before they reach equilibrium during the time evolution as discussed in section VI. Therefore, to be on a safe side, we make very conservative assumption that

$$\frac{R_{\text{form}}}{R_0} \sim 0.1, \quad R_0 \sim \xi. \quad (77)$$

To proceed with numerical estimates, it is convenient to separate $\gamma$ on two pieces,

$$\gamma \equiv \gamma_{\text{formation}} \cdot \gamma_{\text{evolution}}, \quad \gamma_{\text{formation}} \sim 0.1, \quad (78)$$

where the first part, $\gamma_{\text{formation}} \sim 0.1$ has been estimated using numerical simulations, see textbook [42] for review. The second suppression factor $\gamma_{\text{evolution}}$ is unknown, and includes a large number of different effects. In particular, many small closed bubbles with $R_0 \leq \xi$ are very likely to be formed but may not survive the evolution as we discussed in section VI. Furthermore, there are many effects such as evaporation, annihilation inside the nuggets which may also lead to collapse of relatively small nuggets. Furthermore, the formation probability of large closed bubbles with $R_0 \gg \xi$ (which most likely to survive) is highly suppressed $\sim \exp(-R_0^2/\xi^2)$. All these effects are included in unknown parameter $\gamma_{\text{evolution}}$. Our constraint (from observations on $n_B/n_\gamma$ within our mechanism) can be inferred from (75)

$$\gamma_{\text{evolution}}(T_{\text{form}}) \simeq 10^{-7}. \quad (79)$$

One suppression factor which obviously contributes to suppression (79) is related to necessity to produce a sufficiently large initial bubble for successful nugget formation as given by eq. (51).

Now we can interpret the estimate (79) in two complimentary ways. First interpretation of estimate (79) is as follows. Small numerical value (79) implies that only sufficiently large nuggets survive the evolution in unfriendly environment mentioned above. It is hard to

\[15\] We are thankful to anonymous referee who hinted on possibility of the first interpretation. The second interpretation of estimate (79) is our original and preferable interpretation.
estimate all the QCD effects mentioned above (evaporation, annihilation inside the nuggets etc), but the dominant suppression factor is related to formation suppression $\sim \exp(-R_0^2/\xi^2)$. The observed abundance (67) can be interpreted in this case as a specific value for the formation size $R_0$ which satisfies the constraint (79). There is an exponential sensitivity to $R_0$ within this interpretation. In particular if $R_0 \sim (3 - 4)\xi$,
\begin{equation}
\exp\left(-\frac{R_0^2}{\xi^2}\right) \sim (10^{-4} - 10^{-7}).
\end{equation}

This estimate is consistent with the observational constraint as unaccounted QCD effects mentioned above may saturate (79).

We do not consider this sensitivity to $R_0$ as a fine tuning problem. Indeed, in many cases the physics is highly sensitive to some parameters of the theory, which however cannot be interpreted as a fine tuning problem. In particular, in the context of this paper the conventional formula (81) for the dark matter density resulted form the misalignment mechanism is highly sensitive to the axion mass $m_a$. However, we do not interpret this dependence as a fine tuning problem.

Our second (and preferable) interpretation can be explained as follows. The observed ratio (67) is highly sensitive to $T_{\text{form}} \approx 40$ MeV due to exponential dependence of the baryon number density $\sim \exp(-m_N/T)$. This formation temperature in our framework is defined as the temperature when the nuggets complete their formation. This value for the temperature is very reasonable as it lies slightly below $T_{\text{CS}}$ when the CS phase sets in inside the nuggets. Obviously, we do not interpret this sensitivity to the formation temperature $T_{\text{form}} \approx 40$ MeV as a fine tuning problem.

The crucial point here is that the saturation of the observed ratio (67) can be interpreted in terms of $T_{\text{form}}$, or it can be interpreted in terms of $R_0$ which is the key parameter in our first interpretation. Small variation of $T_{\text{form}}$ can be thought as small variations of $R_0$, which however lead to very large changes of the observed ratio (67) due to the exponential sensitivity. In other words, small increase of $T_{\text{form}}$ when nuggets complete the formation can be interpreted as small decrease of survival size $R_0$ in our first interpretation given above.

We do not call this effect as a fine tuning. This is because the equilibration of the baryon charge from the nuggets with the visible baryons always lead to the result (1), (64) when all contributions are the same order of magnitude. A small observed ratio (67) is determined by a precise and specific moment in evolution of the Universe when the nuggets complete their formation at temperature $T_{\text{form}} \sim \Lambda_{\text{QCD}}$, which is again, perfectly consistent with the main paradigm of the entire framework that all dimensional parameters are order of $\Lambda_{\text{QCD}}$. This $T_{\text{form}}$ corresponds to a very specific value $R_0$ for nuggets to complete their formation at time $T_{\text{form}}$.

How one can understand the result (79) which essentially states that even very tiny probability of the formation of the closed bubbles is still sufficient to saturate the observed ratio (67)? The answer lies in the observation that the baryon density $n_B \approx n_\bar{B}$ was 10 orders of magnitude larger at the moment of the bubble formation. Therefore, even a tiny probability at the moment of formation of a closed bubble with sufficiently large size will lead to effects of order one at the moment when the baryon number density drops 10 order in magnitude. Another reason why very tiny probability of the formation of the closed bubbles nevertheless is sufficient to saturate the observed ratio (67) is that typical “small factors” which normally accompany the conventional baryogenesis mechanisms such as $C\bar{P}$ and $C$ odd couplings do not appear in estimate (79) due to the reasons already explained after eq (67).

- We conclude this section with the following comment: The basic consequences of this framework represented by eqs. (1), (64), (65) are very generic. These features are not very sensitive to efficiency of the closed domain wall formation nor to the absolute value of $\theta$ as long as coherence is maintained, see footnote 13. These generic features hold for arbitrary value of the axion mass $10^{-6}$eV $\leq m_a \leq 10^{-3}$eV, in contrast with conventional treatment of the axion as the dark matter candidate, when $\Omega_{\text{DM}}$ can be saturated by the axions only when the axion mass assumes a very specific and definite value $m_a \sim 10^{-6}$eV, see next section with details.

The derivation of the observed ratio (67) from the first principles (which is determined by parameter $R_0$ in first interpretations or parameter $T_{\text{form}}$ in the second interpretation) is a hard computational problem of strongly coupled QCD when all elements such as cooling rate, annihilation rate, charge separation rate, damping rate, evaporation rate and many other effects are equally contribute to $T_{\text{form}}$. However, it is important that the “observational” value $T_{\text{form}} \approx 40$MeV lies precisely in the region where it should be: $T_{\text{form}} < T_{\text{CS}}$, i.e. slightly below the temperature where CS sets in. Therefore, any fine-tuning procedures have never been required in this framework to accommodate the observed ratio presented by eq.(1).

**VIII. IMPLICATIONS FOR THE AXION SEARCH EXPERIMENTS**

The goal of this section is to comment on relation of our framework and the direct axion search experiments [8–15]. We start with the following comment we made in section II: this model which has a single fundamental parameters (a mean baryon number of a nugget (B) $\sim 10^{25}$ entering all the computations) is consistent with all known astrophysical, cosmological, satellite and ground based constraints as reviewed in section II. For discussions of this section it is convenient to express this single normalization parameter $\langle B \rangle \sim 10^{25}$ in terms of the axion mass $m_a \sim 10^{-4}$ eV as these two parameters directly related according to the scaling relations (3). The corresponding relation between these two parameters occur
because the axion mass $m_a$ determines the wall tension $\sigma \sim m_a^{-1}$ which itself enters the expression for the equilibrium value of the size of the nuggets, $R_{\text{form}}$ at the end of the formation. One should emphasize that it is quite nontrivial that the cosmological constraints on the nuggets as shown on Fig. 2 and formulated in terms of $\langle B \rangle$ are compatible with known upper limit on the axion mass $m_a < 10^{-3}\text{eV}$ within our framework. One could regard this compatibility as a nontrivial consistency check for this proposal.

The lower limit on the axion mass, as it is well known, is determined by the requirement that the axion contribution to the dark matter density does not exceed the observed value $\Omega_{\text{dark}} \approx 0.23$. There is a number of uncertainties in the corresponding estimates. We shall not comment on these subtleties by referring to the review papers[8–15]. The corresponding uncertainties are mostly due to the remaining discrepancies between different groups on the computations of the axion production rates due to the different mechanisms such as misalignment mechanism versus domain wall/string decays. In what follows to be more concrete in our estimates we shall use the following expression for the dark matter density in terms of the axion mass resulted from the misalignment mechanism [15]:

$$\Omega_{(\text{DM axion})} \simeq \left(\frac{6 \cdot 10^{-6}\text{eV}}{m_a}\right)^2$$  \hspace{1cm} (81)

This formula essentially states that the axion of mass $m_a \sim 2 \times 10^{-5}\text{eV}$ saturates the dark matter density observed today, while the axion mass in the range $m_a \geq 10^{-4}\text{eV}$ contributes very little to the dark matter density. This claim, of course, is entirely based on estimate (81) which accounts only for the axions directly produced by the misalignment mechanism suggested originally in [47].

There is another mechanism of the axion production when the Peccei-Quinn symmetry is broken after inflation. In this case the string-domain wall network produces a large number of axions such that the axion mass $m_a \sim 10^{-4}\text{eV}$ may saturate the dark matter density, see relatively recent estimates [44–46] with some comments and references on previous papers. The corresponding formula from refs. [44–46] is also highly sensitive to the axion mass with $m_a$- dependence being very similar to eq. (81).

The main lesson to be learnt from the present work is that in addition to these well established mechanisms previously discussed in the literature there is an additional contribution to the dark matter density also related to the axion field. However, the mechanism which is advocated in the present work contributes to the dark matter density through formation of the nuggets, rather than through the direct axion production. The corresponding mechanism as argued in section VII B always satisfies the relation $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$, and, in principle, is capable to saturate the dark matter density $\Omega_{\text{dark}} \approx 3\Omega_{\text{visible}}$ by itself for arbitrary magnitude of the axion mass $m_a$ as the corresponding contribution is not sensitive to the axion mass in contrast with conventional mechanisms mentioned above. A precise coefficient in ratio $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$ is determined by a parameter of order one, $|c(T)| \sim 1$, which unfortunately is very hard to compute from the first principles, as discussed in section VII B.

Our choice for $m_a \sim 10^{-4}\text{eV}$ which corresponds to $\langle B \rangle \sim 10^{25}$ is entirely motivated by our previous analysis of astrophysical, cosmological, satellite and ground based constraints as reviewed in Section II. As we mentioned in Section II there is a number of frequency bands where some excess of emission was observed, and this model may explain some portion, or even entire excess of the observed radiation in these frequency bands. Our normalization $\langle B \rangle \sim 10^{25}$ was fixed by eq.(6) with assumption that the observed dark matter is saturated by the nuggets. The relaxing this assumption obviously modifies the coefficient $c(T)$ as well as $\langle B \rangle$.

Interestingly enough, this range of the axion mass $m_a \sim 10^{-4}\text{eV}$ is perfectly consistent with recent claim [60],[61] that the previously observed small signal in resonant S/N/S Josephson junction [62] is a result of the dark matter axions with the mass $m_a \sim 1.1 \times 10^{-4}\text{eV}$. Furthermore, it has been also claimed that similar anomalies have been observed in other experiments [63–65] which all point towards an axion mass $m_a \approx 1.1 \times 10^{-4}\text{eV}$ if interpreted within framework [60],[61]. The only comment we would like to make here is that if the interpretation [60],[61] of the observed anomalies [62–65] is indeed due
to the dark matter axions, then the corresponding axion mass is perfectly consistent with our estimates (based on cosmological observations) of the average baryon charge of the nuggets $\langle B \rangle \simeq 10^{25}$ as reviewed in section II.

We conclude this section on optimistic note with a remark that the most interesting region of the parametric space corresponding to the nuggets with mean baryon charge $\langle B \rangle \simeq 10^{25}$ might be tested by the Orpheus axion search experiment [14] as shown on Fig. 3.

**CONCLUSION. FUTURE DIRECTIONS.**

First, we want to list the main results of the present studies, while the comments on possible future developments will be presented at the end of this Conclusion.

1. First key element of this proposal is the observation (20) that the closed axion domain walls are copiously produced and generically will acquire the baryon or antibaryon charge. This phenomenon of “separation of the baryon charge” can be interpreted as a local version of spontaneous symmetry breaking. This symmetry breaking occurs not in the entire volume of the system, but on the correlation length $\langle T \rangle \sim m_{a}^{-1}$ which is determined by the folded and crumpled axion domain wall during the formation stage. Precisely this local charge separation eventually leads to the formation of the nuggets and anti-nuggets serving in this framework as the dark matter component $\Omega_{\text{dark}}$.

2. Number density of nuggets and anti-nuggets will not be identically the same as a result of the coherent (on the scale of the Universe) axion CP-odd field. We parameterize the corresponding effects of order one by phenomenological constant $c(T) \sim 1$. It is important to emphasize that this parameter of order one is not fundamental constant of the theory, but, calculable from the first principles. In practice, however, such a computation could be quite a challenging problem when even the QCD phase diagram is not known. The fundamental consequence of this framework, $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$, which is given by (1) is universal, and not sensitive to any parameters as both components are proportional to $\Lambda_{\text{QCD}}$. The observed ratio (2), (66) corresponds to a specific value of $c(T_{\text{form}}) \simeq -1.5$ as discussed in section VIIB.

3. Another consequence of the proposal is a natural explanation of the ratio (5) in terms of the formation temperature $T_{\text{form}} \simeq 40$ MeV, rather than in terms of specific coupling constants which normally enter conventional “baryogenesis” computations. This observed ratio is expressed in our framework in terms of a single parameter $T_{\text{form}}$ when nuggets complete their formation. This parameter is not fundamental constant of the theory, and as such is calculable from the first principles. In practice, however, the computation of $T_{\text{form}}$ is quite a challenging problem as explained in section VII C. Numerically, the observed ratio (5) corresponds to $T_{\text{form}} \simeq 40$ MeV which is indeed slightly below the critical temperature $T_{\text{CS}} \simeq 60$ MeV where the colour superconductivity sets in.

The relation $T_{\text{form}} \lesssim T_{\text{CS}} \sim \Lambda_{\text{QCD}}$ is universal in this framework as both parameters are proportional to $\Lambda_{\text{QCD}}$. As such, the universality of this framework is similar to the universality $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$ mentioned in previous item. At the same time, the ratio (5) is not universal itself as it is exponentially sensitive to precise value of $T_{\text{form}}$ due to conventional suppression factor $\sim \exp(-m_{a}/T)$. 

4. The only new fundamental parameter of this framework is the axion mass $m_{a}$. Most of our computations (related to the cosmological observations, see section II and Fig. 2), however, are expressed in terms of the mean baryon number of nuggets $\langle B \rangle$ rather than in terms of the axion mass. However, these two parameters are unambiguously related according to the scaling relations (3). Our claim is that all universal properties of this framework listed above still hold for any $m_{a}$. In other words, there is no any fine tuning in the entire construction with respect to $m_{a}$. The constraints (and possible cosmological observations) from section II strongly suggest $\langle B \rangle \simeq 10^{25}$ which can be translated into preferred value for the axion mass $m_{a} \approx 10^{-4}$ eV.

5. This region of the axion mass $m_{a} \approx 10^{-4}$ eV corresponding to average size of the nuggets $\langle B \rangle \approx 10^{25}$ can be tested in the Orpheus axion search experiment [14] as shown on Fig. 3.

We conclude with few thoughts on future directions within our framework. It is quite obvious that future progress cannot be made without a much deeper understanding of the QCD phase diagram at $\theta \neq 0$. In other words, we need to understand the structure of possible phases along the third dimension parametrized by $\theta$ on Fig 1.

Presently, very few results are available regarding the phase structure at $\theta \neq 0$. First of all, the phase structure is understood in simplified version of QCD with two colours, $N_{c} = 2$ at $T = 0, \mu \neq 0$, see [66]. In fact, the studies [66] were mostly motivated by the subject of the present work and related to the problem of formation of the quark nuggets during the QCD phase transition in early Universe with non vanishing $\theta$. With few additional assumptions the phase diagram can be also conjectured for the system with large number of colours $N_{c} = \infty$ at non vanishing $T, \mu, \theta$, see [67, 68].

Due to the known “sign problem”, see footnote 1, the conventional lattice simulations cannot be used at $\theta \neq 0$. The corresponding analysis of the phase diagram for non vanishing $\theta$ started just recently by using some newly invented technical tricks [69–72].

Another possible development from the “wish list” is a deeper understanding of the closed bubble formation. Presently, very few results are available on this topic. The most relevant for our studies is the observation made in [10] that a small number of closed bubbles are indeed observed in numerical simulations. However, their detail properties (their fate, size distribution, etc) have not been studied yet. A number of related questions such as an estimation of correlation length $\langle T \rangle$, the genera-
tion of the structure inside the domain walls, the baryon charge accretion on the bubble, etc, hopefully can be also studied in such numerical simulations.

One more possible direction for future studies from the “wish list” is a development some QCD-based models where a number of hard questions such as: evolution of the nuggets, cooling rates, evaporation rates, annihilation rates, viscosity of the environment, transmission/reflection coefficients, etc in unfriendly environment with non-vanishing $T, \mu, \theta$ can be addressed, and hopefully answered. All these and many other effects are, in general, equally contribute to our parameters $T_{\text{form}}$ and $c(T)$ at the $\Lambda_{QCD}$ scale in strongly coupled QCD. Precisely these numerical factors eventually determine the coefficients in the observed relations: $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$ given by eq. (65) and $n_B/n_c$ expressed by eq. (67).

Last but not least: the discovery of the axion in the Orpheus experiment [14] would conclude a long and fascinating journey of searches for this amazing particle conjectured almost 40 years ago. Such a discovery would be a strong motivation for related searches of “something else” as the axion mass $m_a \simeq 10^{-4}$ is unlikely to saturate the dark matter density observed today. We advocate the idea that “something else” is the “quark nuggets” (where the axion plays the key role in entire construction) which could provide the principle contribution to dark matter of the Universe as the relation $\Omega_{\text{dark}} \approx \Omega_{\text{visible}}$ in this framework is not sensitive to the axion mass.

**ACKNOWLEDGMENTS**

This work was supported in part by the National Science and Engineering Research Council of Canada.

**Appendix A: Estimation of fluxes**

The main goal of this Appendix is to argue that the approximation in eq. (32) which was adopted in the text by neglecting extra term “fluxes” is justified, at least on qualitative level. In other words, while these “flux” terms obviously present in the system, they, nevertheless, do not drastically change a key technical element (an implicit relation between $R(t)$ and $\mu(t)$) which this equation provides. Precisely this implicit relation between $R(t)$ and $\mu(t)$ eventually allows us to express the $\mu$-dependent pressure $\Delta P[\mu]$ in terms of $R$ dependent function $\Delta P[f(R)]$ such that the basic equation (45) describing the time evolution of the nuggets is reduced to a differential equation on a single variable $R(t)$.

Our starting point is the observation that the relevant flux which enters equation (32) is $\Delta \Phi = (\Phi_\infty - \Phi_c)$, counting the net baryon charge transfer and sensitive to the chemical potential difference, rather than total flux $\langle \Phi \rangle$ which counts the exchange of all the particles, including bosons\textsuperscript{16}. In fact, if the average flux $\langle \Phi \rangle$ were entering equation (32) one could explicitly check that this term would be the same order of magnitude as two other terms of the equation. However, the key point is that the baryon charge transfer $\Delta \Phi$ is numerically suppressed, i.e. $\Delta \Phi \ll \langle \Phi \rangle$. In fact, $\Delta \Phi$ identically vanishes for $\mu = 0$. Furthermore, one can use the same technique which has been used in section VI A to argue that $\Delta \Phi \ll \langle \Phi \rangle$ in entire region of $\mu$. Numerical analysis supports this claim.

To reiterate this claim: while a typical flux defined as

$$\Phi = \frac{g_{\text{in}}}{(2\pi)^3} \int \frac{v_z d^3k}{\exp\left(\frac{k^2 - \mu}{T}\right) + 1} + (\text{bosons}) \sim (\text{fm})^{-3} \quad (A1)$$

assumes a conventional QCD value, the net baryonic flux $\Delta \Phi \cdot S$ through surface $S$ is numerically suppressed, and can be neglected in eq. (32).

One can explain this result as follows. Consider a single oscillation of the domain wall evolution. To be more specific, consider a squeezing portion of this evolution when $R(t)$ decreases. During this process the chemical potential (and the baryon charge density) locally grow as we discussed in section VI A. The major portion of this growth is resulted from the baryon charge which was already bound to the domain wall, rather than from the baryon charge which enters the system as a result of the baryonic flux transfer.

On an intuitive level the dominance of the bound charges (accounted in eq. (32)) in comparison with flux-contribution (neglected in eq. (32)) can be explained using pure geometrical arguments. Indeed, the chemical potential increases very fast as a result of rapid shrinking of the bubble with speed $v \simeq c$. The corresponding contraction of a bubble leads to proportionally rapid increase of the chemical potential on the domain wall. This happens because the baryon charges strongly bound to the wall, and cannot leave the system due to the topological reasons as the boundary conditions effectively lock the charge to the macroscopically large domain wall. As a result of this evolution the binding energy of a quark $\sim \mu$ increases when the bubble contracts. This process represents a highly efficient mechanism of very rapid growth of the chemical potential due to the domain wall dynamics. It is very hard to achieve a similar efficiency with the flux-contribution when probability for a reflection from the domain wall is typically much higher than probability for a transmission. Furthermore, a non-vanishing quark mass make suppression even stronger $\sim \exp(-m/T)$.

To conclude: we do expect that an accounting for the flux-contribution modifies our equations relating $\mu(t)$

\textsuperscript{16} The dominant contribution to the fluxes normally comes from the lightest degrees of freedom which are the Nambu-Goldstone bosons in hadronic and CS phases. These contributions are crucial for maintaining the thermodynamical equilibrium between exterior and interior, but they do not play any role in the baryon fluxes which enter eq.(32).
and \( R(t) \) as expressed by eqs. (32), (38). However, we do not expect that this modification may drastically change the basic qualitative features of eqs. (32), (38) which have been heavily employed in this work.

### Appendix B: Formation of the nuggets: numerical analysis

This appendix is devoted to exact numerical computation in contrast with analytical qualitative arguments presented in section VI. The basic lesson of this Appendix is that a number of simplifications which have been made in section VI are justified, at least, on a qualitative level.

Before we proceed with numerical computations we want to make few comments on parameters entering the basic dynamical equation (45). In the previous sections, \( \sigma \) was treated as a constant in order to simplify the analysis. This approximation is justified as long as a typical curvature of the domain wall is much smaller than the width of the domain wall, i.e. \( R \gg m_a^{-1} \). This condition is only marginally justified as a typical radius of the bubble is of order \( m_a^{-1} \), which is the same order of magnitude as the width of the wall. At the same time, the width of the QCD substructure of the domain wall (including the \( q \) substructure and the baryon substructure) is very small in comparison with the curvature, and it does satisfy the criteria of a thin wall approximation as \( m^{-1} \ll R \sim m_a^{-1} \). Precisely this QCD substructure plays a crucial role in our analysis in section IV where we studied the “local violation” of the baryon charge in the presence of the domain walls. The broad structure of the domain wall due to the axion field with the width \( m_a^{-1} \) does not play any role. However, precisely this structure determines the large tension \( \sigma \sim m_a^{-1} \) of the domain wall.

We want to effectively account for this physics by assuming that \( \sigma(R) \) effectively depends on the radius of the bubble \( R \). On the physical grounds we expect that \( \sigma(R) \) approaches its asymptotic value at large \( R \) when the domain wall is almost flat, \( \sigma(R \to \infty) \to \sigma_0 \), while \( \sigma \) reduces its value at smaller \( R \), and eventually vanishes at some cutoff \( R_{\text{cut}} \). A natural choice is \( R_{\text{cut}} \approx 0.24 R_0 \) which corresponds to large \( \mu_{\text{cut}} \lesssim 500 \text{MeV} \) from (38), when the chemical potential assumes its typical CS value. To introduce such an infrared cutoff smoothly, it is convenient to parametrize \( \sigma \) as follows

\[
\sigma(R) = \sigma_0 e^{-r_0/2(R-R_{\text{cut}})} \tag{B1}
\]

where \( \sigma_0 \approx 9 f_a^2 m_a \) is the conventional domain wall tension, see e.g. [10], while \( r_0 \) is a free phenomenological parameter, \( 0 < r_0 \lesssim R_0 \) as we expect \( \sigma(R_0) \approx \sigma_0 \). In our numerical studies we verified that the physical results (such as formation size \( R_{\text{form}} \)) are not very sensitive to parameter \( r_0 \).

Another parameter which requires some comments is the viscosity \( \eta \). In the context of the present work, the viscosity accounts for a number of different QCD effects which lead to dissipation and “friction”. Such effects include, but not limited to different scattering processes by quarks, gluons or Nambu Goldstone Bosons in different phases. Furthermore, the annihilation processes which take place inside the bubble and which result in production of a large number of strongly interacting quasi-particles also contribute to \( \eta \). The viscosity can be computed from the first principles in weakly coupled quark-gluon phase [73]. However, we are more interested in behaviour of \( \eta \) below \( T_c \). In this case the computations [74] based on chiral perturbation theory suggest that \( \eta \sim m_a^2 \). This numerical value is quite reasonable in all respects, and consistent with simple dimensional arguments. It is also known that \( \eta(T) \) depends on temperature [74]. However, we neglect this dependence in our estimates which follow.

Now we can proceed with our numerical studies. Since \( \sigma(R) \) is a function of \( R \) as explained above, we should start with a modified differential equation for \( R(t) \):

\[
\sigma(R)\ddot{R}(t) = \frac{2\sigma(R)}{R} - \frac{\sigma(R)\dot{R}^2}{R} + \Delta P(R) \tag{B2}
\]

This equation is not identically the same as equation (45) discussed in section VI. This is due to the fact that the tension \( \sigma(R) \) is now become \( R \) dependent function as we discussed above. The equation (B2) has been solved numerically using parameters listed in Table I. The numerical values of these parameters can be obviously somewhat modified. However, the basic qualitative features presented in section VI do not drastically change when the QCD parameters are varied within reasonable parametrical region. Our numerical studies, as we discuss below, do support the analytical qualitative results presented in section VI.

We start our short description with Fig.4. It shows a typical evolution of a bubble with time. The frequencies of oscillations are determined by the axion mass \( m_a^{-1} \), while typical damping time is determined by parameter \( \tau \) as discussed in section VI. To make the pattern of oscillations visible, the viscosity has been rescaled\(^\dagger\). At large times, \( t \to \infty \), the solution settles at \( R_0/R_{\text{form}} \approx 2.9 \) and \( \mu_{\text{form}} \approx 330 \text{MeV} \sim \mu_1 \), consistent with qualitative analysis of section VI.

We now describe Fig. 5 where we zoom-in first few oscillations of a typical solution shown on previous plot Fig.

\(^\dagger\) In this plot we use \( \eta \approx 10^8 \mu_0 \), which is eight orders of magnitude larger than \( \eta_0 \approx m_a^2 \). We did it on purpose: First, it simplifies the numerics. Indeed, the \( \eta \) parameter determines the damping time scale (53) which is many orders of magnitude longer than any other scales of the problem. Secondly, we use \( \eta \approx 10^8 \mu_0 \) for the demonstration purposes. Indeed, a typical oscillation time \( \omega^{-1} \) and the damping time \( \tau \) are characterized by drastically different scales. If we take \( \eta \) according to its proper QCD value than the time scale on plots Fig. 4 would be eight orders of magnitude longer than shown.
We want to emphasize that a seeming cusp singularity is actually a smooth function near \( R_{\text{min}} \). It looks “cuspy” as a result of a large time scale on Fig. 4. The “cusp” is relatively narrow comparing to macroscopic period of oscillation \((\delta t_{\text{cusp}} \sim 10^{-3} R_0)\). Nevertheless it actually lasts much longer in comparison with a typical QCD scale \((\delta t_{\text{cusp}} \gg \Lambda_{\text{QCD}}^{-1})\).

On Fig. 6 we demonstrate a (non)sensitivity of the system to parameter \( r_0 \) introduced in eq. (B1). One can explicitly see that the initial evolution is indeed quite sensitive to ad hoc parameter \( r_0 \). However, the final stage of the evolution is not sensitive to \( r_0 \). In other words, the physical parameters \( R_{\text{form}} \) and \( \tau \) are not sensitive to ad hoc parameter \( r_0 \). Note that estimation of damping time \( \tau \) and period of oscillation \( t_{\text{osc}} \) agree well with qualitative estimations presented in section VI.

Appendix C: Evaluation of Fermi-Dirac integrals

The main goal of this Appendix is to present some supporting arguments suggesting that the approximation we have used in section VIA and which involves the Fermi-Dirac integrals are qualitatively justified. Indeed, the

![FIG. 7. Comparison of \( I_n^{(0)} \) to \( I_n \) with different values of \( b \).](image)
relevant integrals which enter eqs. (36), (39) have the form

\[ I_n(b) = \int_0^\infty \frac{dx}{e^{x-b} + 1}, \quad b = \frac{\mu}{T} > 0, \]  

where \( n = 2 \) appears in integral (36), while \( n = 4 \) appears in (39). We will hence focus on evaluation of \( I_2 \) and \( I_4 \) in this appendix. They can be exactly evaluated as

\[ I_2(b) = \frac{\pi^2}{6} + \frac{1}{2} b^2 + \text{Li}_2(-e^{-b}) \]  

\[ I_4(b) = \frac{7\pi^4}{60} + \frac{\pi^2}{2} b^2 + \frac{1}{4} b^4 + 6 \text{Li}_4(-e^{-b}) \]

where \( \text{Li}_2(-z) \) and \( \text{Li}_4(-z) \) are the polylogarithm functions of order 2 and 4, respectively. Polylogarithm functions are commonly known to represent the Fermi-Dirac integrals. The Polylogarithm functions are defined as

\[ \text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{k^n}. \]  

Note that \(|z| = e^{-b} \leq 1 \) for any positive \( b \). In this case \( \text{Li}_n(-z) \) is evidently fast-converging, so that we can efficiently estimate it by extracting the leading exponent \( e^{-b} \) then using the Taylor expansion for the remaining piece:

\[ \text{Li}_2(-e^{-b}) \approx e^{-b} \left( -\frac{\pi^2}{12} + (\ln 2 - \frac{\pi^2}{12}) b + O(b^2) \right) \]

\[ \text{Li}_4(-e^{-b}) \approx e^{-b} \left( -\frac{\pi^4}{720} + \frac{\zeta(3)}{4} - \frac{7\pi^4}{720} b + O(b^2) \right), \]

where \( \zeta(3) \approx 1.202 \) is the Riemann zeta function. Neglecting the terms of order \( O(be^{-b}) \) which are small in both limits, at large and small chemical potentials, one can approximate \( I_2 \) and \( I_4 \) as follows

\[ I_2^{(0)} \approx \frac{\pi^2}{6} + \frac{1}{2} b^2 - \frac{\pi^2}{12} b^2 + O(be^{-b}) \]

\[ I_4^{(0)} \approx \frac{7\pi^4}{60} + \frac{\pi^2}{2} b^2 + \frac{1}{4} b^4 - \frac{7\pi^4}{120} b^4 + O(be^{-b}). \]

We test our approximation by comparing our approximate expressions (C5a), (C5b) with exact formulae (C2a), (C2b). As one can see from Fig. 7, our approximation shown in blue \( (I_2^{(0)}/I_2) \) and orange \( (I_4^{(0)}/I_4) \) is very good with errors less than 3% in the entire range of \( b > 0 \).

On the same plot we also show the approximation \( \tilde{I}_4^{(0)} \) for approximate expression \( I_4^{(0)} \) used in the main text in eq. (39)

\[ \tilde{I}_4^{(0)} \approx \frac{7\pi^4}{60} + \frac{\pi^2}{2} b^2 + \frac{1}{4} b^4 - \frac{7\pi^4}{12} b^4. \]

The error for \( \tilde{I}_4^{(0)} \) is quite large for very small chemical potential \( b \ll 0.5 \), on the level of 40%, shown in green. The error becomes much smaller after short period of time when \( b = \mu/T \geq 0.5 \) becomes sufficiently large. To conclude: the approximations of the integrals in section VIA are sufficiently good for qualitative analysis presented in that section.

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