YAMABE FLOW: STEADY SOLITONS AND TYPE II SINGULARITIES

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Abstract. We study the convergence of complete non-compact conformally flat solutions to the Y amabe flow to Y amabe steady solitons. We also prove the existence of Type II singularities which develop at either a finite time $T$ or as $t \to +\infty$.

1. Introduction

Let $(M, g_0)$ be a Riemannian manifold without boundary of dimension $n \geq 3$. If $g = u^{\frac{4}{n-2}} g_0$ is a metric conformal to $g_0$, the scalar curvature $R$ of $g$ is given in terms of the scalar curvature $R_0$ of $g_0$ by

$$R = u^{-1} \left( - \tilde{e}_n \Delta_{g_0} u^{\frac{n+2}{n-2}} + R_0 u^{\frac{n}{n-2}} \right)$$

where $\Delta_{g_0}$ denotes the Laplace Beltrami operator with respect to $g_0$ and $\tilde{e}_n = 4(n-1)/(n-2)$.

In 1989 R. Hamilton introduced the Y amabe flow

$$\frac{\partial g}{\partial t} = -R g$$

as an approach to solve the Y amabe problem on manifolds of positive conformal Yamabe invariant.

In the case where $M$ is compact, the long time existence and convergence of Yamabe flow is well understood. Hamilton [H1] himself showed the existence of the normalized Yamabe flow (which is the re-parametrization of (1.1) to keep the volume fixed) for all time; moreover, in the case when the scalar curvature of the initial metric is negative, he showed the exponential convergence of the flow to a metric of constant scalar curvature. Chow [Ch] showed the convergence of the flow, under the conditions that the initial metric is locally conformally flat and of positive Ricci curvature. The convergence of the flow for any locally conformally flat initially metric was shown by Ye [Ye].

Schwetlick and Struwe [SS] obtained the convergence of the Yamabe flow on a general compact manifold under a suitable Kazdan-W arner type of condition that rules out the formation of bubbles and this condition is verified (via the positive mass Theorem) in dimensions $3 \leq n \leq 5$. The convergence result, in its full generality, was established by Brendle [B1] and [B2] (up to a technical assumption, in dimensions $n \geq 6$, on the rate of vanishing of Weyl tensor at the points at which it vanishes): starting with any smooth metric on a compact manifold, the normalized Yamabe flow converges to a metric of constant scalar curvature.

Although the Yamabe flow on compact manifolds is well understood, the complete non-compact case is unsettled. Even though the analogue of Perelman’s monotonicity formula is still lacking for the Yamabe flow,
one expects that gradient Yamabe soliton solutions model finite and infinite time singularities. These are special solutions $g = g_{ij}$ of the Yamabe flow (1.1) for which there exist a potential function $P(x, t)$ so that

$$(R - \rho)g_{ij} = \nabla_i \nabla_j P, \quad \rho \in \{1, -1, 0\}$$

where the covariant derivatives on the right hand side are taken with respect to metric $g(\cdot, t)$. Depending on the sign of the constant $\rho$, a Yamabe soliton is called a Yamabe shrinker, a Yamabe expander or a Yamabe steady soliton if $\rho = 1, -1$ or $0$ respectively.

The classification of locally conformally flat Yamabe solitons with positive sectional curvature was established in [DS2] (see also [CSZ] and [CMM]). It is shown in [DS2] that such solitons are globally conformally equivalent to $\mathbb{R}^n$ and correspond to self-similar solutions of the fast-diffusion equation

$$(1.2) \quad u_t = \frac{n-1}{m} \Delta u^m, \quad m = \frac{n-2}{n+2}$$

satisfied by the conformal factor defined by $g_{ij} = u^{\frac{4}{n+2}} \delta_{ij}$. A complete description of those solutions is given in [DS2]. In [CSZ] the assumption of positive sectional curvature was relaxed to that of nonnegative Ricci curvature.

The works [DKS, DS2] address the singularity formation of complete non-compact solutions to the conformally flat Yamabe flow whose conformal factors have cylindrical behavior at infinity. It was shown in these works that the singularity profiles of such solutions are Yamabe solitons which are determined by the second order asymptotics at infinity of the initial data which is matched with that of the corresponding self-similar solution. The solutions may become extinct at the extinction time $T$ of the cylindrical tail or may live longer than $T$. In the first case, the singularity profile is described by a Yamabe shrinker that becomes extinct at time $T$. This result can be seen as a stability result around the Yamabe shrinkers with cylindrical behavior at infinity. In the second case, the flow develops a singularity at time $T$ which is described by a singular Yamabe shrinker slightly before $T$ and by a matching Yamabe expander slightly after $T$. All such singularities are of type I.

In this paper, we address singularities which are modeled on Yamabe steady solitons. In Theorem 3.1, we find a condition on a conformally flat initial data $g_0 = u_0^{\frac{4}{n+2}} \delta_{ij}$ under which the Yamabe flow converges, as $t \to +\infty$, to a steady gradient soliton. In Theorem 3.6, we study a more general class of non-smooth initial data. In the Section 4, we provide conditions on a complete non-compact and conformally flat initial data $g_0 = u_0^{\frac{4}{n+2}} \delta_{ij}$ which guarantee that the Yamabe flow will form a type II singularity. We show the existence of both finite time and infinite time type II singularities in Theorems 4.1 and 4.2 respectively. To our knowledge, this is the first time that a type II singularity has been shown to exist in the Yamabe flow.

In what follows we will simply say that a metric $g_{ij}$ is conformally flat if it is globally conformally flat over $\mathbb{R}^n$, namely $g_{ij} = u^{\frac{4}{n+2}} \delta_{ij}$ for a conformal factor $u$ defined on $\mathbb{R}^n$, and we will often use the notation $(\mathbb{R}^n, g_{ij})$ to denote such a metric.
It was shown in [DS2] and [H1], that if \( g_{ij} = u^4 \delta_{ij} \) is a conformally flat Yamabe steady gradient soliton with positive sectional curvature, then \( u \) is a smooth entire and rotationally symmetric solution of the elliptic equation

\[
\frac{n-1}{m} \Delta u^m + \beta x \cdot \nabla u + \gamma u = 0, \quad \text{on } \mathbb{R}^n
\]

for parameters

\[
\beta \geq 0 \quad \text{and} \quad \gamma = \frac{2\beta}{1-m}.
\]

Moreover, it follows by the results in [DS2] that for each \( \beta > 0 \) and \( \gamma = \frac{2\beta}{1-m} \), the equation (1.3) admits one parameter family of rotationally symmetric solutions \((u_{\beta, \lambda})_{\lambda > 0}\), satisfying the asymptotic behavior

\[
u_{\beta, \lambda}^{1-m} \sim O\left( \frac{\ln |x|}{|x|^2} \right), \quad \text{as } |x| \to +\infty
\]

which are uniquely determined by their value at the origin, that is

\[
u_{\beta, \lambda}(0) = \lambda.
\]

It should be noted that for a fixed \( \beta > 0 \), \( u(t) = u_{\beta, e^{-\gamma t}} \), for \( t \in (-\infty, \infty) \) is a solution of the Yamabe flow and hence \( \lambda \) is just a time dilation parameter. Moreover, all \((\mathbb{R}^n, u_{\beta, \lambda}^{1-m} \delta_{ij})\) are isometric to each other by conformal changes \( x \to ax \), \( a > 0 \).

Hsu, in [H1] obtained the first order decay rate at infinity of a Yamabe steady soliton \( u_{\beta, \lambda}\). Namely, it was shown that

\[
\lim_{|x| \to \infty} \frac{|x|^2 u_{\beta, \lambda}^{1-m}}{\ln |x|} = \frac{(n-1)(n-2)}{\beta}.
\]

In order to study the stability around these solitons, it is necessary to establish their second order asymptotics at infinity. In Section 2 we establish such asymptotics showing that second order decay depends on the parameter \( \lambda \).

In Section 3 we prove that if an initial conformally flat metric is asymptotically close to a steady soliton \( u_{\beta, \lambda} \) up to the second order, namely

\[
u_0^{1-m} = \frac{1}{r^2} \left( \frac{(n-2)(n-1)}{\beta} \ln r + K + o(1) \right)
\]

for some \( K \in \mathbb{R} \), then the rescaled solution \( \tilde{u}(x, t) = e^{\gamma t} u(e^{\beta t} x, t) \) converges, as \( t \to +\infty \), to \( u_{\beta, \lambda} \). The constant \( \lambda \) is determined by \( K \) through the second order decay rate at infinity of \( u_{\beta, \lambda} \), namely \( K = \frac{2 \ln \lambda}{n+2} + \frac{\ln \beta}{2} + \kappa(n) \), for some universal constant \( \kappa = \kappa(n) \).

Finally, in Section 4 we construct examples of complete noncompact and globally conformally flat solutions of the Yamabe flow which develop type II singularities. It has been observed in [DKS] that a conformally
flat Yamabe gradient shrinker $v_{\beta,\lambda}$ which vanishes at time $T$, satisfies the asymptotic behavior
\[
|x|^2 v_{\beta,\lambda}^{1-m}(x) = (n-1)(n-2) T - B \cdot |x|^{-\gamma} + o(|x|^{-\gamma}), \quad \text{as } |x| \to +\infty.
\]

The key point is that the decay rate $\gamma > 0$ depends only on $\beta > 0$, which is related with the scalar curvature at the tip (where the maximum scalar curvature occurs) and this $\gamma := \gamma(\beta) \to 0$, as $\beta \to \infty$. Thus, one may guess that a solution may develop a type II singularity, if its initial data has slower second order decay rate than any Yamabe shrinkers. We will therefore choose, for any given $T > 0$, an initial data $g_0 = u_0^{1-m} \delta_{ij}$ such that the tail of $|x|^2 u_0^{1-m}(x) - (n-1)(n-2) T$ decays slower than any power $|x|^{-\gamma}$, with $\gamma > 0$, and prove that the solution with initial data $g_0$ will develop a type II singularity at its extinction time $T$. This idea is similar to that in [HR] where Hamel and Roques found an accelerating fast front propagation for the KPP type equation $u_t = u_{xx} + f(u)$ and for slowly decaying initial data. We will also find a class of initial data for which the Yamabe flow develops a type II singularity, as $t \to +\infty$.

2. Lower Order Asymptotics

In this section, we will derive the second and third order asymptotics of conformally flat radial steady gradient solitons $u_{\beta,\lambda}$, as $r = |x| \to +\infty$. As we saw in the introduction these are solutions of the elliptic equation
\[
\frac{n-1}{m} \Delta u + \beta \cdot \nabla u + \gamma u = 0, \quad \text{on } \mathbb{R}^n, \quad m = \frac{n-2}{n+2}
\]
with parameters
\[
\beta \geq 0 \quad \text{and} \quad \gamma = \frac{2\beta}{1-m}
\]
and for each $\beta > 0$ they are uniquely determined by their value at the origin $\lambda := u_{\beta,\lambda}(0)$.

For the remaining of the section we fix $\beta > 0$ and $\lambda > 0$ and set for simplicity $u(r) := u_{\beta,\lambda}(x)$, $r = |x|$. It is convenient to work in cylindrical coordinates $s = \log r$. Using this change, the radial metric $g = u(r)^{1-m} \, dx^2$ is expressed as
\[
g = u(r)^{1-m}(dr^2 + r^2 g_{S^{n-1}}) = w(s)(ds^2 + g_{S^{n-1}})
\]
where
\[
w(s) = r^2 u(r)^{1-m}.
\]

Using that $m = \frac{n-2}{n+2}$, we find by direct calculation that (2.7) translates into the following equation for $w$
\[
w_{ss} = \frac{6 - n}{4} w_s^2 + \left( n - 2 - \frac{\beta}{n-1} w_s \right) w, \quad \text{for } s \in (-\infty, \infty).
\]

We recall in the next Proposition previous results regarding the first order asymptotics of $w$ which were shown in [DS2] and [H1].
Proposition 2.1 ([DS2], [H1]). For a conformally flat and radially symmetric steady gradient soliton \( w \), we have \( w > 0 \) and \( w_s > 0 \) for all \( s \in \mathbb{R} \). Moreover, there are positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq w_s \leq C_2, \text{ for } s \geq 0 \text{ and } \tag{2.9}
\]

\[
(2.10)
\]

\[
(2.11)
\]

\[
(2.12)
\]

(2.8)

Proof. Recall that \( w = r^2 u^{1-m} > 0 \). Differentiating in \( s \) gives that \( w_s = r w_r = 2 r^2 u^{1-m} + (1-m) r^3 u^m u_r \).

Using that \( \lim_{r \to 0} u(r) = 0 \) and \( \lim_{r \to 0} u_r(r) = 0 \), we conclude \( \lim_{s \to -\infty} w_r \to 0 \) and moreover we can check

The signs of \( h_{ss} = w_{ss} \) and \( h_s = w_s - \frac{(n-1)(n-2)}{\beta} \) can be determined, depending on dimension \( n \) as shown next.

Proposition 2.2. For \( n \geq 6 \), \( h_{ss} = w_{ss} > 0 \) and \( h_s < 0 \) for all \( s \in (-\infty, \infty) \). For \( 3 \leq n < 6 \), there exist \( s_0 > 0 \) such that \( h_{ss} = w_{ss} < 0 \) and \( h_s > 0 \) for all \( s > s_0 \).

Proof. Assume now that \( n > 6 \). We will show that \( w_{ss} > 0 \) is preserved for all \( s \in \mathbb{R} \). To this end, we differentiate (2.8) in \( s \) and obtain

\[
\frac{w_{ss}}{w} = \frac{6-n}{4} \left( \frac{2 w_r w_{ss}}{w} - \frac{w_s}{w^2} \right) + \left( n - 2 - \frac{\beta}{n-1} w_s \right) w_s - \frac{\beta}{n-1} w_s w.
\]

Suppose \( w_{ss} = 0 \) at some \( s = \bar{s} \). From (2.8), using \( 6 - n < 0 \), \( w > 0 \) and \( w_s > 0 \), we get

\[
0 < w_s(\bar{s}) < \frac{(n-2)(n-1)}{\beta}.
\]

Plugging this bound into (2.12), gives \( (w_{ss})(\bar{s}) > 0 \). Hence, \( w_{ss} > 0 \) for all \( s \in \mathbb{R} \), implying that \( h_{ss} = w_{ss} > 0 \).

Since, from Proposition 2.1, we have

\[
h_s = w_s - \frac{(n-2)(n-1)}{\beta} \to 0, \quad \text{as } s \to \infty
\]

we also conclude that \( h_s < 0 \).

When \( n = 6 \), equation (2.8) could be viewed as a 1st order linear equation of \( w_s \) assuming \( w \) is given. Hence we can integrate from \( s = -\infty \) and use \( \lim_{s \to -\infty} w_s = 0 \) to obtain

\[
w_s(s) = \frac{(n-1)(n-2)}{\beta} \left( 1 - e^{-\phi} \int_{w(0)}^{w(s)} dt \right)
\]
from which the bounds  
\[ w_{ss} > 0 \quad \text{and} \quad w_s < \frac{(n-1)(n-2)}{\beta}, \quad \text{for all} \quad s \in \mathbb{R} \]
readily follow. We conclude in this case that \( h_{ss} = w_{ss} > 0 \) and \( h_s < 0 \) for all \( s \in \mathbb{R} \).

Finally, assume that \( 3 < n < 6 \). By (2.11) we have \( \lim_{n \to -\infty} w_{ss} / w = 4 > 0 \). If we have \( w_{ss}(s_0) = 0 \) at some \( s = s_0 \), then by (2.8) and because \( 3 < n < 6 \) at this time, we have \( w_s(s_0) > \frac{(n-2)(n-1)}{\beta} \) and hence \( (w_{ss})_s(s_0) < 0 \) by (2.12). Thus, \( w_{ss}(s) < 0 \), for all \( s > s_0 \). On the contrary, if we don’t have such a point \( s_0 \), then \( h_{ss} = w_{ss} > 0 \) for all \( s \in \mathbb{R} \). Thus (2.9) implies that \( w_s < \frac{(n-2)(n-1)}{\beta} \) and hence \( w_{ss} > \frac{6 - n w_s^2}{4 w} \) from equation (2.8). But since \( w_s \to \frac{(n-1)(n-2)}{\beta} \), as \( s \to +\infty \), there is some \( c > 0 \)
\[ w_{ss} > \frac{6 - n w_s^2}{4 w} > \frac{c}{s}, \quad \text{for} \quad s \gg 1 \]
implying that \( w_s \to \infty \), a contradiction. We conclude that \( w_{ss} < 0 \), for \( s \geq s_0 \), for some \( s_0 \in \mathbb{R} \), implying that \( h_{ss} < 0 \) for all \( s \geq s_0 \). Since, \( w_{ss} < 0 \) and (2.9) holds, we must also have \( h_s > 0 \) for all \( s > s_0 \). This finishes the proof of the proposition. \( \square \)

Employing the previous Proposition, we can now prove the following.

**Proposition 2.3.** For all \( n \geq 3 \), we have
\[
\lim_{s \to +\infty} s^2 h_s = \frac{(6 - n)(n-1)}{4\beta}.
\]

**Proof.** Begin by observing that by Proposition 2.1, we have \( h_s \to 0 \) as \( s \to \infty \), implying that \( |h| = o(s) \). For \( n \geq 6 \), we showed in Proposition 2.2 that \( h_s < 0 \). Therefore, it follows from (2.10), that for all \( \epsilon > 0 \), there exists \( s_0 > 0 \) such that for \( s \geq s_0 \)
\[
L_n h := h_{ss} + ((n-2) - \epsilon) s h_s = -\epsilon s h_s - \frac{\beta}{n-1} h h_s + \frac{6 - n w_s^2}{4 w} h_s + \frac{6 - n w_s^2}{4 w} h_s \geq \frac{6 - n w_s^2}{4 w} h_s.
\]
Multiplying by \( \exp\left(\frac{\epsilon}{2} ((n-2) - \epsilon)\right) \) and integrating from \( s_0 \) to \( s \), we get
\[
\left[ h_s(l) \exp\left(\frac{\epsilon}{2} ((n-2) - \epsilon)\right)\right]_{l=s_0}^s \geq \int_{s_0}^s \frac{6 - n w_s^2}{4 w} \exp\left(\frac{\epsilon}{2} ((n-2) - \epsilon)\right) dl.
\]
Setting \( I(s) := s^2 \exp\left(\frac{-\epsilon}{2} ((n-2) - \epsilon)\right) \) and taking \( \liminf_{s \to +\infty} \), on the LHS of (2.14), gives
\[
\liminf_{s \to +\infty} [I(s) \text{LHS (2.15)}] = \liminf_{s \to +\infty} s^2 h_s.
\]
For the RHS of (2.14), we can apply L’Hôpital’s rule to obtain
\[
\lim_{s \to +\infty} [I(s) \text{RHS (2.15)}] = \lim_{s \to +\infty} s^2 \frac{6 - n w_s^2}{4 w} \exp\left(\frac{\epsilon}{2} ((n-2) - \epsilon)\right)
\]
\[= \frac{(6 - n)(n-2)(n-1)}{4\beta} ((n-2) - \epsilon)^{-1}.\]


In the last equality, we used that \( w_s \to \frac{(n-1)(n-2)}{\beta} \) and \( \frac{w}{s} \to \frac{(n-1)(n-2)}{\beta} \) as \( s \to \infty \). Combining both sides, gives

\[
\lim_{s \to \infty} s^2 h_s \geq \frac{(6-n)(n-2)(n-1)}{4\beta} ((n-2) - \epsilon)^{-1}.
\]

By taking \( \epsilon \downarrow 0 \), we obtain \( \lim_{s \to \infty} s^2 h_s \geq \frac{(6-n)(n-1)}{4\beta} \). If we chose \( \epsilon < 0 \) in the beginning, then we get the reversed inequality in (2.14) and the same argument, yields \( \lim_{s \to \infty} \text{sup} s^2 h_s \leq \frac{(6-n)(n-1)}{4\beta} \). We conclude that (2.13) holds. For the remaining cases \( 3 \leq n < 6 \), once we choose \( s_0 \) so that \( h_{s_0} > 0 \) on \([s_0, \infty)\), again a similar argument leads to the same conclusion. \( \square \)

**Corollary 2.4.** There exists a constant \( K = K(n, \beta, \lambda) \) such that \( h(s) = K + \frac{(n-6)(n-1)}{4\beta} \frac{1}{s} + o\left(\frac{1}{s}\right) \) as \( s \to +\infty \). It follows that

\[
w(s) = \frac{(n-2)(n-1)}{\beta} s + K + \frac{(n-6)(n-1)}{4\beta} \frac{1}{s} + o\left(\frac{1}{s}\right).
\]

**Proof.** The convergence of \( h(s) \to K(n, \beta, \lambda) \) readily follows from the result in Proposition 2.3. Define \( \tilde{h}(s) := h(s) - K - \frac{(n-6)(n-1)}{4\beta} \frac{1}{s} \) and integrate again Proposition 2.3 we obtain \( |\tilde{h}(s)| \leq c_1 \int_{s_0}^{\infty} |\tilde{h}| \, ds \). Hence, \( \lim_{s \to \infty} s^2 \tilde{h}_s = 0 \) yields \( \tilde{h}(s) = o(s^{-1}) \).

**Remark 2.5.** In the special case \( n = 6 \), it is easy to see that \( h_{s_0} \) decays exponentially as \( s \to +\infty \). Indeed, by (2.15) and \( h_{s_0} < 0 \), it follows that for each small \( \epsilon > 0 \), there exist two constant \( c, C > 0 \) such that

\[
c \exp\left(-\frac{s^2}{2}(4 + \epsilon)\right) \leq |h_{s_0}| \leq C \exp\left(-\frac{s^2}{2}(4 - \epsilon)\right)
\]

for large \( s \geq s_0 \).

We will next use the rich scaling properties of our equation (2.7) to determine the value of the constant \( K = K(n, \beta, \lambda) \) in Corollary 2.4 up to an additive constant that depends only on the dimension \( n \) and obtain the main result in this section which describes the asymptotic behavior for any steady soliton \( u_{\beta, \lambda} \) up to third order.

**Proposition 2.6.** For \( \beta > 0, \lambda > 0 \), let \( u_{\beta, \lambda} \) denote the unique radially symmetric solution of equation (2.7) with \( u_{\beta, \lambda}(0) = \lambda \). Then there exists a constant \( \kappa = \kappa(n) \in \mathbb{R} \) depending only on dimension \( n \) such that

\[
u^{1-m}_{\beta, \lambda}(r) = \frac{(n-1)(n-2)}{\beta r^2} \left( \ln r + \frac{2 \ln \lambda}{n+2} + \frac{\ln \beta}{2} + \kappa(n) \right) + \frac{(n-6)}{4(n-2) \ln r} + o\left(\frac{1}{\ln r}\right)
\]

**Proof.** For a radial solution \( u \) of (1.3), the rescaling \( \tilde{u}(x) = Au(Bx) \) with \( A, B > 0 \) becomes again radial solution of (1.3) with the same \( \beta \) and \( \gamma \), if and only if \( B = A^{-\frac{1}{n-2}} \). These solutions are uniquely determined by their value at the origin. Hence, we have

\[
u_{\beta, \lambda}(r) = \frac{A_1}{A_2} u_{\beta, \lambda} \left( r \left( \frac{A_1}{A_2} \right)^{-\frac{1}{n-2}} \right)
\]
Similarly, by plugging into the equation (2.7), the uniqueness again implies that

\[(2.18)\quad u_{\beta_1, \beta_2}^{\gamma} (r) = \left(\frac{\beta_2}{\beta_1}\right)^\frac{\gamma}{m} u_{\beta_2, \beta_1}^{\gamma} (r).\]

Combining the two scalings above, gives

\[(2.19)\quad u_{\beta, \lambda} (x) = \left(\frac{\lambda}{\beta}\right) u_{\beta_1, \beta_2}^{\gamma} (r) = \lambda u_{\beta_1, \beta_2}^{\gamma} (r).\]

By Corollary 2.4, there is some \(\kappa = \kappa(n) \in \mathbb{R}\) for which

\[(3.20)\quad u_{1, m} (x) = \frac{1}{|x|^2} (A \ln |x| + B + o(1)) \quad \text{as} \quad |x| \to +\infty, \quad \text{for some} \quad A > 0, \quad B \in \mathbb{R}\]

the Yamabe flow \(g(t) = u(\cdot, t)^{1-m} \delta_{ij}\) with initial data \(g_0\) would converge, as \(t \to +\infty\), and after rescaling to the unique steady soliton \(g_{\beta, \lambda} : = u_{1, m}^{\beta, \lambda} \delta_{ij}\) having the same asymptotics of (3.20).

In what follows we will show that this is indeed true. This will be done in two steps: In Theorem 3.1 we will establish the \(L^1_{\text{loc}}\) convergence of the flow, under the assumption that \(u_0 \in L^1_{\text{loc}}\) and satisfies (3.20). In Theorem 3.6 we will provide an extra condition on \(u_0\), namely that \(u_0\) belongs to the local Marcinkiewicz space \(M^1_{\text{loc}}(1-m/n)\), which guarantees the smooth convergence of the rescaled metric. While smooth globally conformally flat metrics are included in this space, it also allows certain singularities and degeneracies in the metric. In particular, certain cylindrical ends can be added at those singularity points and the flow starting with this locally conformally flat metric also converges to a steady gradient soliton after those ends pinch off in a finite time.

For a solution \(u\) of (1.2) we consider the rescaled solution

\[(3.21)\quad \bar{u}(x, t) := e^{\gamma t} u(e^{\beta t} x, t), \quad \beta > 0, \quad \gamma = \frac{2\beta}{1-m}.\]

A direct calculation shows that \(\bar{u}\) satisfies the equation

\[(3.22)\quad \bar{u}_t = \frac{n-1}{m} \Delta \bar{u}^m + \beta \bar{x} \cdot \nabla \bar{u} + \gamma \bar{u}, \quad \text{on} \quad \mathbb{R}^n, \quad m = \frac{n-2}{n+2}.\]

The following result holds.

3. Long Time Stability
Theorem 3.1. Assume that $g = u_{1-m} \delta_{ij}$ is a solution of the Yamabe flow (1.1) with nonnegative initial data $u_0 \in L^1_{loc}(\mathbb{R}^n)$ which has the decomposition $u_0 = \phi + \psi$ with $\psi \in L^1(\mathbb{R}^n)$ and

$$\phi_{1-m} = \frac{1}{r^2} \frac{(n-2)(n-1)}{\beta} (\ln r + K + o(1)) \quad \text{for some } \beta > 0 \text{ and } K \in \mathbb{R}. \tag{3.23}$$

Then, the rescaled solution $\bar{u}(x, t) := e^{nt} u(e^\beta x, t)$ converges, as $t \to +\infty$, to $u_{\beta, \lambda}$ in $L^1_{loc}(\mathbb{R}^n)$, for some $\lambda > 0$. Moreover, the number $\lambda$ is uniquely determined by the coefficient $K$ in the asymptotic behavior of $u_0$, namely

$$K = \frac{2 \ln \lambda}{n+2} + \frac{\ln \beta}{2} + \kappa(n), \quad \text{for some universal constant } \kappa = \kappa(n).$$

The proof of Theorem 3.1 will be based on the following $L^1$-contraction property between two rescaled solutions $\bar{u}_1$ and $\bar{u}_2$ of equation (1.2).

Lemma 3.1. If $u_1$ and $u_2$ are solutions of equation (1.2) and $\bar{u}_1$ and $\bar{u}_2$ are the rescaled solutions, respectively, then

$$\int_{\mathbb{R}^n} |\bar{u}_1(x, t) - \bar{u}_2(x, t)| \, dx \leq e^{(\gamma - n\beta)t} \int_{\mathbb{R}^n} |\bar{u}_1(x, 0) - \bar{u}_2(x, 0)| \, dx. \tag{3.24}$$

Note that $\gamma - n\beta = (\frac{2}{1-m} - n)\beta = 2 - \beta < 0$, for $n \geq 3$.

Proof. It is well known that any two solutions $u_1$ and $u_2$ of (1.2) satisfy the contraction principle

$$\int_{\mathbb{R}^n} |u_1(x, t) - u_2(x, t)| \, dx \leq \int_{\mathbb{R}^n} |u_1(x, 0) - u_2(x, 0)| \, dx.$$ 

Hence, (3.24) follows by direct calculation. \qed

Proof of Theorem 3.1. By Lemma 3.1 it suffices to prove the result when $u_0 = \phi$. Consider the self-similar solution $u_{\beta, \lambda}$ satisfying the asymptotic behavior (2.10) with $\lambda$ determined by $K = \frac{2 \ln \lambda}{n+2} + \frac{\ln \beta}{2} + \kappa(n)$. It follows from (2.10) and the given asymptotics of initial data (3.23), that for each $\epsilon > 0$, there exists $R_{\epsilon} > 1$ such that

$$u_{\beta, \lambda, \epsilon} (x) \leq u_0 \leq u_{\beta, \lambda, \epsilon}, \quad \text{for } |x| \geq R_{\epsilon}. \tag{3.21}$$

Hence, we have

$$\min(u_0, u_{\beta, \lambda, \epsilon}) - u_{\beta, \lambda, \epsilon} \in L^1(\mathbb{R}^n) \quad \text{and} \quad \max(u_0, u_{\beta, \lambda, \epsilon}) - u_{\beta, \lambda, \epsilon} \in L^1(\mathbb{R}^n).$$

Let $w, h$ denote the solutions to equation (1.2) with initial data $\min(u_0, u_{\beta, \lambda, \epsilon})$, $\max(u_0, u_{\beta, \lambda, \epsilon})$ respectively, and denote by $\bar{w}, \bar{h}$ the rescaled solutions defined by (3.21). The comparison principle then implies the inequality

$$\bar{w} \leq \bar{u} \leq \bar{h}, \quad \text{for } t > 0$$

and by Lemma 3.1 we have

$$\bar{w} \to u_{\beta, \lambda, \epsilon} \quad \text{and} \quad \bar{h} \to u_{\beta, \lambda, \epsilon} \quad \text{in } L^1(\mathbb{R}^n), \quad \text{as } t \to \infty.$$
For any compact set $K \subset \mathbb{R}^n$, we have

$$\int_K |\bar{u} - u_{\beta, \ell}| \leq \int_K |\bar{h} - u_{\beta, \ell}| \leq \int_K |\bar{h} - u_{\beta, \ell + 1}| + \int_K |u_{\beta, \ell + 1} - u_{\beta, \ell}|.$$

Doing the same computation for $\int_K |u - u_{\beta, \ell}|$ and taking $\limsup_{t \to \infty}$ yields

$$\limsup_{t \to \infty} \int_K |u - u_{\beta, \ell}| \leq \int_K |u_{\beta, \ell + 1} - u_{\beta, \ell}| + \int_K |u_{\beta, \ell + 1} - u_{\beta, \ell}|.$$

Taking $\epsilon \to 0$, the right hand side of above equality converges to 0 and this finishes the proof.

By the Arzela-Ascoli theorem, the $L^1_{loc}$ convergence in the previous result can be directly improved to $C_{loc}$ convergence, when $\bar{u}(t)$ is locally equicontinuous for large $t$. It is well known, that for solutions $\bar{u}$ of (3.22), an $L^\infty_{loc}$ bound implies equicontinuity (see in Section 1.5 in [DK]). Thus, if we knew for instance that

$$(3.25) \quad u_0 \leq u_{\beta, \lambda_0}, \quad \text{for some } \lambda_0 > 0$$

then we would know that $\bar{u}(t) \leq u_{\beta, \lambda_0}$ for all $t > 0$ and as a consequence $\bar{u}(t)$ would converge to $u_{\beta, \lambda_0}$ as $t \to +\infty$, in $C_{loc}$. Then, standard regularity theory for uniformly parabolic equations would imply $C_{loc}^\infty$ convergence. Condition (3.25) certainly holds if $u_0 \in L^\infty$. Thus the following follows from our discussion above.

**Corollary 3.2.** If $u_0 \in L^\infty$ and $u_0^{1-m} = \frac{1}{\beta} \left( \frac{(n-2)(n-1)}{\beta} \ln R + C + o(1) \right)$, there is some $\lambda_0 > 0$ such that $u_0 \leq u_{\beta, \lambda_0}$.

**Proof.** For fixed $R > 0$, since $u_{\beta, \lambda}$ is decreasing in $|x|$, we have

$$\inf_{|x| \leq R} u_{\beta, \lambda}^{1-m}(x) = u_{\beta, \lambda}(R) = \lambda^{1-m} u_{\beta, \lambda}^{1-m}(\lambda^{\frac{1}{1-m}} R) = \frac{(n-2)(n-1)}{R^2 \beta} \ln \lambda^{\frac{1}{1-m}} + O(1) \to \infty$$

as $\lambda \to \infty$, i.e. $u_{\beta, \lambda}$ blow up on every compact sets as $\lambda \to \infty$. $L^\infty$ bound and decay asymptotics of initial data $u_0$ imply existence of a large $\lambda_0$ with $u_0 \leq u_{\beta, \lambda_0}$.

Condition (3.25) is too restrictive and in particular does not allow any singularities or degeneracies in our initial metric. The object in the rest of this section is to give a condition on initial data which would guarantee that for some $t_0$ large we have

$$(3.26) \quad \bar{u}(\cdot, t_0) \leq u_{\beta, \lambda_0}, \quad \text{for some } \lambda_0 > 0 \quad \text{and} \quad t_0 >> 1$$

and hence imply smooth convergence on compact sets.

Next, we will show that (3.26) holds for a certain class of locally conformally flat and possibly singular initial data. The extra condition we will assume is that $u_0$ belongs to the Marcinkiewicz space $M_{loc}^{p^*}$, with $p^* = (1-m)\frac{\beta}{2} = \frac{4n}{2n+2}$.

To establish that (3.26) holds, we need an estimate which shows that a solution with non smooth, singular initial data becomes bounded and smooth. Such smoothing estimates of the fast diffusion equation are well studied. If $u_0 \in L^q_{loc}$, with $q > (1-m)\frac{\beta}{2} = p^*$, then we have that $u_0 \in L^q$ due to our asymptotics and the
Remark 3.5. It is known that $L^p$ such that we assumed. Finally, a typical function $f \in M^p_{loc}$, with $p^* = (1 - m)\frac{n}{2}$, then $u(t)$ eventually becomes in $L^\infty$ for some $t_0 > 0$ but it takes some time to get there. For the convenience of the reader, we next define the space $M^p_{loc}$ referring to Chapter 1 and 6 of [V] for further related preliminaries and details.

**Definition 3.3 (Marcinkiewicz Space).** For an open set $\Omega \subset \mathbb{R}^n$

\begin{equation}
(3.27) \quad M^p(\Omega) := \{ f \in L^1_{loc}(\Omega) \mid \exists C \text{ s.t. } \int_K |f| \, dx \leq C |K|^{(p-1)/p} \text{ for all } |K| < \infty \}
\end{equation}

\begin{equation}
(3.28) \quad \|f\|_{M^p(\Omega)} = \sup \{|K|^{(1-p)/p} \int_K |f| \, dx : K \subset \Omega, 0 < |K| < \infty \}
\end{equation}

\begin{equation}
(3.29) \quad M^p_{loc}(\mathbb{R}^n) := \{ f \in L^1_{loc}(\mathbb{R}^n) \mid f \in M^p(\Omega) \text{ for every bounded open set } \Omega \}
\end{equation}

The following fundamental result was shown.

**Theorem 3.4 ([V] Theorem 6.1).** Let $u_0 \geq 0$ to be in the space $M^{p^*} + L^\infty$. Then there is a time $T > 0$ after which the solution $u$ of (1.3) becomes bounded and continuous. More precisely, there is a constant $c = c(n)$ such that

\begin{equation}
(3.30) \quad T < c N^{1-m},
\end{equation}

where $N = N_p(u_0) := \lim_{A \to \infty} \|([f] - A)_{+}\|_{M^{p^*}}$.

**Remark 3.5.** It is known that $L^p \subset M^p$ and $N = N_p(u_0) = 0$ if $u_0 \in L^{p^*} + L^\infty$ hence in this case $L^\infty$ bound is immediate for $t > 0$. Next, $M^{p^*} + L^\infty \subset M^p_{loc}$, but they are the same under the decay condition of $u_0$ we assumed. Finally, a typical function $f \in M^{p^*}$, but not in $L^{p^*}$ is $f(x) = \left(\frac{1}{n}\right)^{\frac{1}{p^*}}$. In terms of metric this corresponds to a cylindrical end and the delayed regularity result describes a situation this cylinder shrinks and becomes extinct in a finite time.

We will prove the following result.

**Theorem 3.6.** Assume that $g = u^{1-m} \delta_{ij}$ is a solution of the Yamabe flow (1.1) with nonnegative initial data $u_0 \in M^{p^*}_{loc}(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$, $p^* = (1 - m)\frac{n}{2}$, such that

\begin{equation}
(3.31) \quad \limsup_{|x| \to \infty} \left[|x|^2 u_0^{1-m} - \frac{(n - 2)(n - 1)}{\beta} \ln |x| \right] < \infty
\end{equation}

for some $\beta > 0$. Assume in addition that $u_0$ has a decomposition $u_0 = \phi + \psi$ with $\psi \in L^1(\mathbb{R}^n)$ and $\phi$ satisfying (3.28). Then, the rescaled solution $\tilde{u}(x,t) := e^{\frac{m}{n}}u(e^{\beta t}x,t)$ converges as $t \to +\infty$, smoothly on compact sets of $\mathbb{R}^n$, to $u_{A,\beta}$ which is the unique radial entire solution of (2.7) satisfying (3.23).
The crucial step in the proof of Theorem 3.6 is to show that the upper bound (3.26) holds for some time \( t_0 \) after delayed regularity. For this, we will need to prove that the asymptotics (3.31) of our initial data will not deteriorate but evolve according to the Yamabe flow. We achieve this by constructing barriers outside of compact balls. We will use the notation \( f \sim g \) as \( r \to \infty \) to indicate that \( \lim_{r \to \infty} f / g = 1 \).

**Proposition 3.7** (Barrier construction). There is \( R = R(n, \beta, \lambda) > 0 \) such that for any \( h > R \), the functions

\[
\bar{v} := \left( \frac{r^2}{r^2 - h^2} \right)^{\frac{1}{m}} u_{\beta, \lambda} \quad \text{and} \quad \underline{v} := \left( \frac{r^2 - h^2}{r^2} \right)^{\frac{1}{m}} u_{\beta, \lambda}
\]

are a supersolution and subsolution, respectively, of the equation

\[
(3.32) \quad u_r = \frac{n - 1}{m} \Delta u^m + \beta x \cdot \nabla u + \gamma u, \quad \text{on } ||x|| > h \times (-\infty, \infty).
\]

**Proof.** In the proof of this proposition, we may fix \( \lambda = 1, \beta = 1 \) and show that the proposition holds for \( R = R(n) \) from the scaling shown in eq (2.19). However, we will not use this since it does not makes the proof easy in a significant way.

We need the following claim.

**Claim 3.8.** The solution radially symmetric solution \( u(x) := u_{\beta, \lambda}(x) = u_{\beta, \lambda}(||x||) \) of (3.32) satisfies

\[
- \frac{n - 1}{m} \Delta u^m = \beta \left( ru_r + \frac{2}{1 - m} u \right) \sim \frac{\beta}{(1 - m) \ln r} u, \quad \text{as } r \to \infty
\]

and hence

\[
r u_r \sim \frac{2}{1 - m} u, \quad \text{as } r \to \infty.
\]

**Proof of Claim.** As in the previous section, we set \( w_s := r^2 u^{1-m}(r) \) and \( s = \ln r \). Then,

\[
w_s = r w_r = r (2ru_{r}^{1-m} + r^2 (1-m)u_{r}^{m}u_r) = r^2 u^m \left( ru_r + \frac{2}{1 - m} u \right).
\]

The claim readily follows from \( \lim_{r \to \infty} u_r^{1-m} \frac{r^2}{\ln r} = \frac{(n - 1)(n - 2)}{\beta} \) and \( \lim_{s \to \infty} w_s = \frac{(n - 1)(n - 2)}{\beta} \). \( \square \)

Denote for simplicity \( f := \left( \frac{r^2}{r^2 - h^2} \right)^{\frac{1}{m}} \) and \( u(x) := u_{\beta, \lambda}(x) \) so that \( \bar{v} = uf \). We have

\[
(3.33) \quad \frac{n - 1}{m} \Delta \bar{v}^m + \beta (ru_r + \frac{2}{1 - m} \bar{v}) = \frac{n - 1}{m} (f^m \Delta u^m + u^m \Delta f^m + 2u_r^m f_r^m) + \beta (ru_r + \frac{2}{1 - m} u)f + \beta u_r f_r + \beta u f_r,
\]

Meanwhile,

\[
(3.34) \quad u_r^m f_r^m = m^2 u^{m-1} u_r f^{m-1} f_r
\]
We want to bound all other terms by first negative term $\beta u f_r$. In that purpose, we compute

$$
\begin{align*}
&f = \left(1 + \frac{h^2}{r^2 - h^2}\right)^{-\frac{1}{2}}, \quad f - f^m = (f^{1-m} - 1)f^m = \frac{h^2}{r^2 - h^2}f^m \\
r_f = \frac{r}{1 - m} \left(1 + \frac{h^2}{r^2 - h^2}\right)^{-\frac{m}{2}} = \frac{-2h^2 r^2}{1 - m (r^2 - h^2)^2}, \\
r^2f^{m-2}f_r^2 = f^{m-2} \frac{1}{(1 - m)^2} f^2 \frac{4h^4}{(r^2 - h^2)^2} = \frac{f^m}{1 - m} \frac{4h^4}{(r^2 - h^2)^2} \\
f^{m-1}rf_r = \frac{1}{1 - m} f^m \frac{-2h^2}{(r^2 - h^2)^2}
\end{align*}
$$

and

$$
\begin{align*}
r^2f^{m-1}f_{rr} &= (r^2 - h^2)f_{rr} \\
&= \frac{m(r^2 - h^2)}{(1 - m)^2} \left(\frac{r^2}{r^2 - h^2}\right)^{m-1} \frac{4h^4}{(r^2 - h^2)^3} + \frac{r^2 - h^2}{1 - m} \left(\frac{r^2}{r^2 - h^2}\right)^{m} \left(\frac{2h^2 r^2 + 2h^4}{(r^2 - h^2)^3}\right) \\
&= \frac{1}{1 - m} f^m \left(\frac{m}{1 - m (r^2 - h^2)^2} + \frac{2(r^2 h^2 + h^4)}{(r^2 - h^2)^2}\right).
\end{align*}
$$

This shows there is some $C = C(n) > 0$, which, in particular, independent of $h$, such that for all $h > 1$ on $\{r > h\}$,

$$
|f - f^m|, \ |r^2f^{m-2}f_r^2|, \ |r^2f^{m-1}f_{rr}|, \ |rf^{m-1}f_r| \leq -C r f_r.
$$

Using the asymptotics in Claim 5.8, we have

$$
\begin{align*}
\frac{-\Delta u^m}{u} &\sim \frac{m\beta}{(n - 1)(n - 1 - m) \ln r}, \quad \frac{u^{m-1}}{r^2} \sim \frac{\beta}{(n - 1)(n - 2) \ln r}, \quad \frac{ru_r}{u} \sim \frac{2}{1 - m} \\
as \; r \to \infty.
\end{align*}
$$

Combining 5.35, 5.37 and 5.38, shows that there exists $R = R(\beta, n, \lambda)$ such that for $\bar{v} = u \left(\frac{r^2}{r^2 - h^2}\right)^{\frac{1}{m}}$ with $h > R$, we have

$$
\frac{n}{m} \Delta \bar{v}^m + \beta x \cdot \nabla \bar{v} + \gamma \bar{v} \leq \frac{\beta}{2} r u f_r < 0 \quad \text{on} \; \{|r| > h\}.
$$

This proves that $\bar{v}$ is a supersolution of (3.32) in the considered region. For the $\bar{v} = u \left(\frac{r^2}{r^2 - h^2}\right)^{\frac{1}{m}} = u g$,
equations (3.33), (3.34), (3.35), and (3.36) are the same except \( \tilde{u} \) and \( f \) changed by \( \nu \) and \( g \). We compute,

\[
\begin{align*}
g &= \left( 1 - \frac{h^2}{r^2} \right)^{-1}, \\
g - g^m &= (g^{1-m} - 1)g^m = -\frac{h^2}{r^2}g^m
\end{align*}
\]

\[
rg_r = r \frac{1}{1-m} \left( 1 - \frac{h^2}{r^2} \right)^{\frac{m}{2}} \frac{2h^2}{r^3} = \frac{1}{1-m} g^m \frac{2h^2}{r^2}, \\
g^{m-1}rg_r = \frac{1}{1-m} g^m \frac{2h^2}{r^2 - h^2}
\]

and

\[
r^2g^{m-1}g_{rr} = \frac{r^4}{r^2 - h^2}g_{rr}
\]

\[
= r^4 \frac{m}{r^2 - h^2 (1-m)^2} \left( \frac{r^2 - h^2}{r^2} \right)^{\frac{m}{2} - 1} \frac{4h^4}{r^6} + \frac{r^4}{r^2 - h^2} \frac{1}{1-m} \left( \frac{r^2 - h^2}{r^2} \right)^{\frac{m}{2}} (-\frac{6h^2}{r^4})
\]

\[
= \frac{1}{1-m} \left( m \frac{4h^2}{r^2 - h^2} - \frac{6h^2}{r^2 - h^2} \right).
\]

Since \( \left( \frac{h^2}{r^2 - h^2} \right)^2 \) in \( r^2g^{m-2}g_{rr} \) dominates all other terms appearing above, namely \( \left( \frac{h^2}{r^2 - h^2} \right), \frac{h^2}{r^2} \) and \( \frac{h^4}{r^6} \) near \( r = h \), we may combine (3.36) and (3.38) to find \( R_1 = R_1(\beta, \lambda, \delta) \) such that for \( h > R_1 \) and \( h < r \leq (1 + \delta)h \),

\[
\frac{n-1}{m} \Delta u^m + \beta x \cdot \nabla u + \gamma u \geq \frac{n-1}{2} u^{m-1}_{rr} > 0 \quad \text{on } h < r \leq (1 + \delta)h.
\]

On the remaining region \( r > (1 + \delta)h \), there is \( C = C(n, \delta) > 0 \) such that

\[
(3.40) \quad |g - g^m|, |r^2g^{m-2}g_{rr}|, |r^2g^{m-1}g_{rr}|, |rg^{m-1}g_r| \leq C r g_r.
\]

Combining again (3.36) and (3.38), it follows that for each \( \delta > 0 \) there is \( R_2 = R_2(\beta, n, \lambda, \delta) \) such that for \( \nu = \frac{u}{\left( \frac{r^2 - h^2}{r^2} \right)} \) with \( h > R_2 \),

\[
\frac{n-1}{m} \Delta \nu^m + \beta x \cdot \nabla \nu + \gamma \nu \geq \frac{\beta}{2} u^{m-1}g_r > 0 \quad \text{on } \{ r > (1 + \delta)h \}.
\]

Setting \( R := \max(R_1, R_2) \), it follows that \( \nu \) is a subsolution on the region \( \{ r > h \} \), for \( h > R \), concluding the proof of the proposition. \( \square \)

Using the previous barrier construction we will now show that the Yamabe flow preserves the asymptotic behavior of our initial data \( u_0 \) as in Theorem 3.6.

**Proposition 3.9.** Let \( u_0 \in L^1_{loc}(\mathbb{R}^n) \) satisfying \( \limsup_{r \to \infty} \left[ r^2 u_0^{1-m} - \frac{(n-2)(n-1)}{\beta} \ln r \right] = K_1 < \infty. \) Then, the solution \( u \) of (1.3) with initial data \( u_0 \) satisfies

\[
\limsup_{r \to \infty} \left[ r^2 u^{1-m} - \frac{(n-2)(n-1)}{\beta} \right] \leq K_1 - (n-1)(n-2)r, \quad \text{for } t \geq 0.
\]
Also if \( \liminf_{r \to \infty} \left[ r^2 u_0^{1-m} - \frac{(n-2)(n-1)}{\beta} \ln r \right] = K_2 > -\infty \), then
\[
K_2 - (n-1)(n-2)t \leq \liminf_{r \to \infty} \left[ r^2 u^{1-m} - \frac{(n-2)(n-1)}{\beta} \right], \quad \text{for } t \geq 0.
\]

**Proof.** By Proposition 2.6 there exists \( \lambda_1 > 0 \) such that
\[
\lim_{r \to \infty} \left[ r^2 \bar{u}_{\beta,\lambda}^{1-m} - \frac{(n-2)(n-1)}{\beta} \ln r \right] = K_1.
\]
For each \( \epsilon > 0 \) there exists \( h > R(n, \beta, \lambda_1) \), where \( R(n, \beta, \lambda_1) \) is taken by Proposition 3.7 such that
\[
u_0 \leq \nu_{\beta,\lambda_1+\epsilon} \left( \frac{r^2}{r^2 - \frac{h^2}{\beta}} \right) := \nu_{\beta,\lambda_1+\epsilon}, \quad \text{on } [r > h].
\]
Since \( \nu_{\beta,\lambda_1+\epsilon} := \nu_{\beta,\lambda_1+\epsilon} \left( \frac{r^2}{r^2 - \frac{h^2}{\beta}} \right) \to \infty \) as \( r \to h^+ \), the comparison gives us that \( \bar{u}(x, t) := e^t u(e^\beta x, t) \leq \nu_{\beta,\lambda_1+\epsilon} \) on \( r > h \) and \( t > 0 \). Also, since \( \left( \frac{r^2}{r^2 - \frac{h^2}{\beta}} \right) \to 1 \), as \( r \to +\infty \), using (2.16) we conclude that
\[
\lim_{r \to \infty} \left[ r^2 \nu_0^{1-m} - \frac{(n-2)(n-1)}{\beta} \ln r \right] = K_1 + \frac{2(n-1)(n-2)}{(n+2)\beta} \ln(1 + \frac{\epsilon}{\lambda_1}).
\]
This translates into the following asymptotics of \( u(x, t) \) which holds for for each \( t > 0 \)
\[
\limsup_{r \to \infty} \left[ r^2 u^{1-m} - \frac{(n-2)(n-1)}{\beta} \ln r \right] \leq K_1 + \frac{2(n-1)(n-2)}{(n+2)\beta} \ln(1 + \frac{\epsilon}{\lambda_1}) - (n-1)(n-2)t.
\]
Taking the limit \( \epsilon \to 0^+ \) we reach our conclusion. The other side inequality can be done similarly by comparison with the constructed subsolution \( \nu_{\beta,\lambda_1-\epsilon} := \nu_{\beta,\lambda_1-\epsilon} \left( \frac{r^2 - \frac{h^2}{\beta}}{r^2} \right) \).

We will now conclude the proof of Theorem 3.6

**Proof of Theorem 3.6** We have seen in Theorem 3.1 that the rescaled solution \( \bar{u}(x, t) := e^{\frac{\epsilon}{\beta t}} u(e^{\beta} x) \) converges in \( L^\infty_{\text{loc}} \) as \( t \to +\infty \), to the steady soliton \( u_{\beta,\lambda} \). By the discussion following Theorem 3.1 to establish the \( C^\infty_{\text{loc}} \) convergence, it suffices to show that \( \bar{u}(t) \) satisfies a uniform in time \( L^\infty_{\text{loc}} \) bound for \( t \geq T \), for some \( T > 0 \).

Indeed, by Theorem 3.4 there is a finite time \( T > 0 \) such that \( ||u(t)||_{L^\infty(\mathbb{R}^n)} < \infty \) for \( t \geq T \). At \( t = T \), Proposition 3.5 implies
\[
\limsup_{|x| \to \infty} \left[ |x|^2 u(T)^{1-m} - \frac{(n-2)(n-1)}{\beta} |x| \right] < \infty.
\]
We may now combine the \( L^\infty \) bound on \( u(T) \) and this asymptotic behavior (similarly as in the proof of Corollary 3.2) to show that there exists \( \lambda_2 > 0 \) for which
\[
u_0 \leq \nu_{\beta,\lambda_2} \left( \frac{r^2 - \frac{h^2}{\beta}}{r^2} \right) \leq u_{\beta,\lambda_2}(e^{\beta t} x).
\]
This implies the bound \( \bar{u}(x, t) \leq u_{\beta,\lambda_2}(x) \) for \( t \geq T \), from which the \( L^\infty_{\text{loc}} \) bound on \( \bar{u} \) readily follows. This concludes the proof of the theorem. \( \square \)
4. Examples of Type II Singularity

In this last section, we will construct noncompact conformally flat solutions $g = u \delta_{ij}$ of the Yamabe flow \((1.1)\) which admit type II singularities both in a finite time and infinite time. Before we start, let us fix the following notation.

**Notation.** For any fixed $\beta > 0$ and $\lambda > 0$, we denote by

- $u_{\beta, \lambda}$ (gradient Yamabe steady soliton) to be the unique radial solution of equation \((1.3)\) with $\gamma = \frac{2\beta}{1-m}$ and $u_{\beta, \lambda}(0) = \lambda$, and
- $v_{\beta, \lambda}$ (gradient Yamabe shrinker soliton) to be the unique radial solution with $\beta, \lambda = \frac{2\beta_x}{1-m}$ and $v_{\beta, \lambda}(0) = \lambda$.

**Definition 4.1.** Suppose that a solution $g(t)$ to Yamabe flow \((1.1)\) on $t \in [0, T]$ has a singularity at $t = T < \infty$; this finite time singularity is called type I if

$$\sup_{M \times [0, T)} |Rm(x, t)|(T - t) < \infty,$$

and is called type II if

$$\sup_{M \times [0, T)} |Rm(x, t)|(T - t) = \infty.$$

**Definition 4.2.** A solution to Yamabe flow \((1.1)\) on $t \in [0, \infty)$ is called type I if

$$\sup_{M \times [0, \infty)} |Rm(x, t)| < \infty,$$

and is called type II if

$$\sup_{M \times [0, \infty)} |Rm(x, t)| = \infty.$$

Before we proceed, we begin with the next simple observation.

**Lemma 4.3.** Let $g(t) = u^{1-m} \delta_{ij}$ on $t \in [0, T)$ be a solution of the Yamabe flow \((1.1)\) such that the scalar curvature satisfies $R(x, t) \leq f(t) \in L^1_{loc}((0, T))$. Then the conformal factor $u$ satisfies a pointwise estimate

$$u(x, t)^{1-m} \geq u_0(x)^{1-m} e^{-\int_0^t f(s)ds} \text{ for } t \in [0, T).$$

In particular, $R(x, t) \leq \frac{K}{1-m}$ implies that $u(x, t) \geq u_0(x)(T - t)^{\frac{1}{1-m}}$ and $R(x, t) \leq K$ that $u(x, t) \geq u_0(x) e^{-\frac{K}{1-m}t}$.

**Proof.** This is a straightforward ODE estimate. For each fixed $x \in M$, the function $\phi(x, t) := u^{1-m}(x, t)$ satisfies $\phi_t = -R \phi$, hence $(\log \phi)_t = -R \geq -f$. Integrating in time gives the result. \[\square\]

**Theorem 4.1.** Suppose $0 < u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and satisfies the bound $|x|^2 u_0^\frac{2}{1-m} < (n-1)(n-2)T$, for some $T > 0$, and the asymptotic behavior

$$\lim_{|x| \to \infty} \left[ |x|^2 u_0^\frac{2}{1-m} - (n-1)(n-2)T \right] = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \left[ |x|^2 u_0^\frac{2}{1-m} - (n-1)(n-2)T \right] = \infty, \forall \epsilon > 0.$$
Then, the solution \( g = u^{1-m} \delta_{ij} \) of the Yamabe flow (1.1) with initial data \( g_0 = u_0^{1-m} \delta_{ij} \) becomes extinct at time \( T \) and develops a type II singularity at \( t = T \).

**Proof.** The fact that the unique smooth solution \( u \) of (1.3) with initial data \( u_0 \) exists at least up to time \( t = T \), can be easily seen by comparing with family of Barrenblatt solutions which extinct at \( T - \epsilon \), as in Lemma 4.2 in [DKS]. On the other hand because of our initial bound \( |x|^2 u_0^{\frac{2}{n}} < (n-1)(n-2)T \), by comparing with the shrinking cylinder which vanishes at \( t = T \), we know that our solution becomes extinct at \( t = T \).

It suffices to prove that the singularity is of type II. We argue by contradiction and suppose that it is of type I, which means that there is \( K > 0 \) such that

\[
\frac{|R|}{T-t} \leq C_n \frac{|Rm|}{T-t} \leq C_n K.
\]

Let us fix \( \beta > 0 \) so that

\[
\gamma = \frac{2\beta + 1}{1 - m} > \frac{C_n K}{1 - m}.
\]

Then, for this choice of \( \beta \), there is an one parameter family \( \{v_{\lambda}\}_{\lambda > 0} \) of radial solutions (shrinkers) of (1.3), with \( v_{\lambda}(0) = \lambda \).

**Claim 4.4.** There exist a large \( \lambda_0 > 0 \) such that \( T^\gamma v_{\lambda_0}(xT^{\beta}) \geq u_0(x) \), for all \( x \in \mathbb{R}^n \).

**Proof of Claim 4.4.** By a scaling argument, we can assume \( T = 1 \). First, choose any \( \lambda_1 \) with \( v_{\lambda_1}(0) = \lambda_1 > u_0(0) \). If the claim holds for this \( \lambda_1 \), we are done. If not, we first recall asymptotics at infinity for the conformally flat shrinker \( g_1 := v_{\lambda_1} \delta_{ij} \) shown in [DKS], namely

\[
|x|^2 v_{\lambda_1}^{1-m}(x) = (n-1)(n-2) - B |x|^{-\gamma} + o(|x|^{-\gamma})
\]

as \( |x| \to \infty \) for some \( 0 < B = B(n,\beta,\lambda_1) \) and \( 0 < \gamma = \gamma(n,\beta) \). Our assumed conditions on the initial data \( u_0 \) and (4.41) imply that \( \mathcal{K} := \{x \in \mathbb{R}^n | u_0 \geq v_{\lambda_1}\} \) is a compact set which doesn’t contain the origin (since \( v_{\lambda_1}(0) = \lambda_1 > u_0(0) \)). Next, we can observe that

\[
|x|^2 v_{\lambda_1}(x)^{1-m} = ((\lambda/\lambda_1)^{\frac{2}{n}} |x|) ^2 v_{\lambda_1}(\lambda/\lambda_1)^{\frac{2}{n}} |x|
\]

This and (4.41) imply \( |x|^2 v_{\lambda_1}(x)^{1-m} \to (n-1)(n-2) \), as \( \lambda \to \infty \), uniformly on \( \mathcal{K} \) while \( |x|^2 u_0(x) < (n-1)(n-2) \) on \( \mathcal{K} \). Using this uniform convergence, therefore, we may find some \( \lambda_0 > \lambda_1 \) such that \( |x|^2 v_{\lambda_0}(x) > |x|^2 u_0(x) \) on \( \mathcal{K} \), namely \( v_{\lambda_0}(x) > u_0(x) \) on \( \mathcal{K} \). On the other hand, the monotonicity of \( v_{\lambda_1} \) with respect to \( \lambda \) implies that \( v_{\lambda_0} > v_{\lambda_1} > u_0 \) on \( \mathbb{R}^n \setminus \mathcal{K} \), concluding that \( v_{\lambda_0} > u_0 \) on \( \mathbb{R}^n \).

We will now conclude the proof of the theorem. By the comparison principle, \( (T-t)^\gamma v_{\lambda_0}(x(T-t)^{\beta}) \geq u(x,t) \), on \( t \in [0, T) \). On the other hand, by Lemma 3.3 \( u(x,t) \geq u_0(x)(T-t)^{\frac{2}{n}} \). In particular, at \( x = 0 \), we have

\[
(T-t)^\gamma v_{\lambda_0}(0) \geq u(0,t) \geq u_0(0)(T-t)^{\frac{2}{n}}.
\]
Since $\gamma > \frac{C_n K}{1-m} > 0$ and $u_0(0) > 0$, there must be some $t < T$ close to $T$ so that above inequality fails to hold, leading to a contradiction. We conclude that the singularity must be of type II.

\[ \Box \]

**Theorem 4.2.** Suppose $0 < u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and satisfies

\[
\lim_{|x| \to \infty} |x|^2 u_0^\frac{4}{n+2} = \infty \quad \text{and} \quad \lim_{|x| \to \infty} \frac{|x|^2}{\ln |x|} u_0^\frac{4}{n+2} = 0.
\]

Then, the solution $g = u^{1-m} \delta_{ij}$ of the Yamabe flow (1.1) with initial data $g_0 = u_0^{1-m} \delta_{ij}$ exist globally on $0 \leq t < +\infty$ and develops a type II singularity as $t \to \infty$.

**Proof.** The proof is very similar to that of Theorem 4.1 where the shrinkers $v_\lambda$ are replaced by the steady soliton $u_\lambda$. The global in time existence with such initial condition is well known, for instance in Theorem 1.1 in [H2], hence it suffices to prove that the solution develops a type II singularity at $t = \infty$. Suppose it is type I and so that there is $K > 0$ such that $|\mathcal{R}_m| \leq K$ and hence

\[ |\mathcal{R}| \leq C_n |\mathcal{R}_m| \leq C_n K. \]

Let us choose any $\beta > 0$ such that $\gamma = \frac{2\beta}{1-m} > \frac{C_n K}{1-m}$. For this fixed $\beta > 0$, there is an one parameter family $\{u_\lambda\}_{\lambda > 0}$ of radial solutions of (1.3) with $u_\lambda(0) = \lambda$.

Using the asymptotics of a steady soliton and the observation that $\inf_\mathcal{K} u_\lambda \to \infty$, as $\lambda \to \infty$, for each compact $\mathcal{K} \subset \mathbb{R}^n$ (Corollary 3.2), we may find large $\lambda_0 > 0$ such that $u_{\lambda_0}(x) > u_0(x)$ for all $x \in \mathbb{R}^n$. Thus $e^{-\gamma t} u_{\lambda_0}(xe^{-\beta t}) \geq u(x, t)$ on $t \in [0, \infty)$, by the comparison principle. On the other hand, by Lemma 4.3, $u(x, t) \geq u_0(x) e^{-\frac{C_n K}{(1-m)t}}$. In particular, at $x = 0$, we have

\[ e^{-\gamma t} u_{\lambda_0}(0) \geq u(0, t) \geq u_0(0) e^{-\frac{C_n K}{(1-m)t}}. \]

Since $\gamma > \frac{C_n K}{1-m} > 0$, there must be some $t$ large so that above inequality fails, leading to a contradiction. We conclude that the singularity of the solution $u$ must be type II. \[ \Box \]

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