A Self-Correcting Quantum Memory in a Thermal Environment

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The ability to store information is of fundamental importance to any computer, be it classical or quantum. Identifying systems for quantum memories which rely, analogously to classical memories, on passive error protection (‘self-correction’) is of greatest interest in quantum information science. While systems with topological ground states have been considered to be promising candidates, a large class of them was recently proven unstable against thermal fluctuations. Here, we propose new two-dimensional (2D) spin models unaffected by this result. Specifically, we introduce repulsive long-range interactions in the toric code and establish a memory lifetime polynomially increasing with the system size. This remarkable stability is shown to originate directly from the repulsive long-range nature of the interactions. We study the time dynamics of the quantum memory in terms of diffusing anyons and support all our analytical results with extensive numerical simulations. Our findings demonstrate that self-correcting quantum memories can exist in 2D at finite temperatures.

I. INTRODUCTION

Quantum computers cannot be realized without the help of error correction. By encoding quantum information into logical states and designing correction circuits working on them, computations and information can in principle be protected from decoherence. However, the need of such an active control mechanism poses a major challenge for any physical implementation. It is therefore of greatest interest to look for passively protected systems which are intrinsically stable against the destructive influence of a thermal environment. For this reason, the idea to encode quantum information in a topologically ordered ground state $|\Psi_0\rangle$ of a suitable Hamiltonian has attracted a lot of interest.

Important candidates among such topological models are stabilizer Hamiltonians, which are given by a sum of mutually commuting many-body Pauli operators. The advantage of such Hamiltonians is that the full energy spectrum is known and error correction schemes are readily derived. However, very recent results show that in one and two spatial dimensions no stabilizer Hamiltonian with finite-range interactions (including the well-known Kitaev model) can serve as a self-correcting quantum memory due to the errors induced by a thermal environment. In other words, increasing the size of such a system does not prolong the protection of its ground-state space from decoherence. These negative results point towards the fundamental question whether topologically ordered quantum states, and hence self-correcting quantum memories, can exist at all on a macroscopic scale. In the following, we will demonstrate that self-correcting properties of 2D stabilizer Hamiltonians can indeed be established when we allow for long-range repulsive interactions between the elementary excitations (anyons). While the purpose of the present work is of principal nature, we note that such interacting models can be expected to be realized in physical systems. We discuss this issue in greater detail at the end, where we also show how tunable repulsive long-range interactions could be mediated via photons in an optical cavity.

II. MODEL

Let us consider an $L \times L$ square lattice with periodic boundary conditions (a ‘torus’), and place a spin-$\frac{1}{2}$ on each of its $2L^2$ edges. Starting from Kitaev’s ‘toric code’ model, we consider the more general stabilizer Hamiltonian

$$H_0 = \frac{1}{2} \sum_{pp'} U_{pp'} n_p n_{p'} + \frac{1}{2} \sum_{ss'} V_{ss'} n_s n_{s'},$$

where $n_p = (1 - \prod_{i \in \ell} \sigma_{z,i})/2$, $n_s = (1 - \prod_{i \in \ell} \sigma_{x,i})/2$, and $\sigma_{x,i}, \sigma_{z,i}$ denote the usual single-spin $x$ and $z$ Pauli operators applied to spin $i$. The indices $p$ and $p'$ run over all ‘plaquettes’ (including the four spins on the edges of a unit cell), whereas $s$ and $s'$ run over all ‘stars’ (including the four spins around a corner of a unit cell), see Fig. 1. The operator $n_p (n_s)$ has eigenvalues 0, 1 and counts the number of plaquette- (star-) anyons at site $p$ ($s$). The fourfold degenerate energy levels encode two qubits with logical operators given by $Z_i = \prod_{k \in \ell_i} \sigma_{z,k}$ and $X_i = \prod_{k \in \ell'_i} \sigma_{x,k}$, $i = 1, 2$, where $\ell_i$ and $\ell'_i$ are strings of spins topologically equivalent to single loops around the torus (see Fig. 1 for an example). These operators commute with all $n_p$ and $n_s$ and obey themselves the usual spin commutation relations. Note that by specializing to $U_{pp'} = 2J \delta_{pp'}$ and $V_{ss'} = 2J \delta_{ss'}$, where $J > 0$ is the single-anyon excitation energy, Kitaev’s original toric code model is recovered.

Since all $n_p$ and $n_s$ are mutually commuting, the Hamiltonian Eq. (1) describes two independent lattice gases of plaquettes and stars, respectively. Without loss of generality, we can thus restrict our analysis to the dynamics of plaquettes and their influence on one of the $Z_i$ operators, say $Z_1 \equiv Z$. A corresponding logical operator $Z_{ec}$ is defined by the error correction procedure (see Fig. 1 and Appendix A). Consequently, we set $V_{ss'} = 0$ for all stars while assuming the plaquette interactions $U_{pp'}$ to be of the generic form

$$U_{pp'} = 2J \delta_{pp'} + \frac{A}{(r_{pp'})^\alpha} (1 - \delta_{pp'}),$$

where $\alpha > 0$ is a tunable parameter.
This rate equation reads

\[ \dot{p}_m = \sum_i \left[ \gamma(-\omega_i(m))p_{x,i}(m) - \gamma(\omega_i(m))p_m \right], \]

where we have defined \( x_i(m) = \) the state \( m \) with an additional \( \sigma_x \)-error applied to spin \( i \), and \( \omega_i(m) = \epsilon_m - \epsilon_{x,i}(m) \) is the energy difference between the states \( m \) and \( x_i(m) \). The time evolution of the probabilities \( p_m \) determines the decay of the expectation values \( \langle Z_{\text{ec}} \rangle = \sum_m p_m \langle \Psi_m | Z_{\text{ec}} | \Psi_m \rangle \).

The rates \( \gamma(\omega) \) describe the transition probabilities between states with energy difference \( \omega \). A standard expression for \( \gamma(\omega) \) can be obtained from a spin-boson model and reads:

\[ \gamma(\omega) = 2g_n \left| \frac{\omega}{\lambda} \right| e^{-|\omega|/\omega_c}. \]

Here, \( \beta = 1/T \), with \( T \) being the temperature of the bath (we set Boltzmann’s constant to one). For simplicity, we assume in the following a large cut-off frequency \( \omega_c \to \infty \) and choose appropriate time units for which \( g_n = 1 \). For \( n = 1 \), the bath is called ‘Ohmic’, whereas for \( n \geq 2 \) it is called ‘super-Ohmic’. We find in this work that \( n \) has a strong influence on the decay times of the encoded states.

III. NON-INTERACTING CASE

We first briefly discuss the features of a non-interacting system, i.e., \( A = 0 \). In this case, the relevant rates entering Eq. (3) are \( \gamma(0) \) (rate for an anyon to hop to a free neighboring site), \( \gamma(-2J) \) (rate to create an anyon pair) and \( \gamma(2J) = \gamma(-2J)e^{2J\beta} \) (rate to annihilate a pair of adjacent anyons, obtained from the detailed balance condition). The diffusive motion of the anyons causes the bare and error-corrected logical operators \( \langle Z \rangle \) and \( \langle Z_{\text{ec}} \rangle \), respectively, to decay with time (see Fig. 2) and

FIG. 1: (Color online.) Quantum memory based on the toric code. Illustrated is an 8 \( \times \) 8 lattice (periodic boundary conditions) with a total of 128 spins (grey dots) on its edges. The four-body plaquette and star operators are indicated in transparent grey and orange, respectively. A particular choice for all logical operators \( X_1, Z_1, X_2, \) and \( Z_2 \) is shown, although we will focus only on the decay of \( Z_1 \equiv Z \) (see main text). A number of spins is affected by \( \sigma_x \)-errors (green and red dots), leading to excited plaquettes, or ‘plaquette anyons’ (solid green and red plaquettes). Measuring the plaquette operators yields the positions of the excited plaquettes, but reveals no information about how they were originally paired or which path (indicated by the plaquettes framed in green and red) they took. A minimum-weight error correction procedure (see Appendix A) applies \( \sigma_x \)-operators to the spins marked by orange circles. While the green anyons are annihilated ‘properly’ (with a trivial loop of errors remaining from the top pair and no error from the bottom pair), the red pair is connected around a topologically non-trivial loop on the torus. Although the red pair is annihilated as well, an uncorrected \( \sigma_x \)-error remains on the logical \( Z \) string, having thereby introduced a logical error in the state stored in the memory.

where \( r_{pp'} \) denotes the shortest distance on the torus between the centers of plaquettes \( p \) and \( p' \), see Fig. 1. The strength of the repulsive plaquette interaction is given by the energy \( A \geq 0 \), and the interaction is long-range for \( 0 \leq \alpha < 2 \) (see below). The model is also equivalent to a long-range Ising model, see Appendix B.

We model the interaction of the system with a thermal environment by coupling each spin to a bath which can introduce \( \sigma_x \)-errors in the initial state |Ψ0\rangle. From a standard master equation approach in the weak coupling limit, we derive a rate equation for the probabilities \( p_m \) of the system to be in state \( \Psi_m = \prod_i \sigma_{x,i} |\Psi_0\rangle \), where \( \{m\} \) is the set of all possible patterns of \( \sigma_x \)-errors. This rate equation reads

\[ \dot{p}_m = \sum_i \left[ \gamma(-\omega_i(m))p_{x,i}(m) - \gamma(\omega_i(m))p_m \right], \]

and

\[ \langle Z_{\text{ec}} \rangle = \sum_m p_m \langle \Psi_m | Z_{\text{ec}} | \Psi_m \rangle \]

FIG. 2: (Color online.) Decay of the logical \( Z \) operator in a non-interacting toric code. The simulation data is obtained for grid sizes \( L \) increasing by powers of two from 16 (blue) to 512 (red). All curves are ensemble averages over 10\(^4\) runs. The main plot displays \( \langle Z_{\text{ec}} \rangle \), which is the average value of \( Z \) one would find if an error correction scheme would be applied at the readout time \( t \). The inset shows the expectation value of the bare (uncorrected) logical \( Z \) operator. We use the parameters \( J = 1, T = 0.3, \) and \( \gamma(0) = \gamma(2J) = 1 \). See Appendix A for further details on the simulation.
Appendix C). We can relate this process to a diffusion constant \( D \) which is generally given by \( D = \gamma(0) \). However, we additionally find \( D = 4\gamma(-2J) \) due to ‘indirect hopping’ (motion of an anyon due to a neighboring pair creation, see Appendix D). This rate becomes relevant, e.g., for a super-Ohmic bath, where \( \gamma(0) = 0 \).

We obtain the lifetime as follows. The error correction fails when the fraction \( f \) of spins with \( \sigma_z \)-error is larger than some critical value \( f_c \). Assuming \( N \) diffusing anyons present in the system, the accumulated number of \( \sigma_z \)-errors after a time \( t \) is roughly given by \( NDt \). This gives \( f \simeq NDt/2L^2 \), from which we determine the lifetime \( \tau \) as

\[
\tau \simeq 2f_c - \frac{\beta J + 1}{\max\{\gamma(0), 4\gamma(-2J)\}}.
\]  

Here, we have replaced the factor \( N/L^2 \) by the Fermi equilibrium density \( \langle n_p \rangle = 1/(\beta \rho + 1) \). In Ref. 3 an upper bound \( f_c < 0.11 \) was obtained by assuming an independent error model. This value yields \( \tau \simeq 5.8 \) for the same parameters as used in Fig. 2. Equation (5) gives reasonable values of \( \tau \) and is in agreement with lifetimes obtained from other derivations.

Note that \( \tau \) is independent of the system size \( L \), consistent with previous findings. This fact is also confirmed by our simulations, as shown in Fig. 2, where \( \langle Z_{cc} \rangle \) clearly approaches a step-function with increasing \( L \). We also see that the bare expectation value \( \langle Z \rangle \) decays even faster with larger \( L \). Indeed, at sufficiently short times \( t \ll 1/\max\{\gamma(0), 4\gamma(-2J)\} \), when anyon pairs have not yet diffused apart from each other (the ‘nonsplit-pair’ regime, indicated by an asterisk), we obtain \( \langle Z \rangle = (1 - 1/L)^{N^*/2} \simeq e^{-N^*/2L} \). By using \( N^* \simeq 4L^2\gamma(-2J)t \), it follows that \( \langle Z \rangle \) decays exponentially with \( L \).

IV. INTERACTING CASE

We now turn to the interacting case \( A > 0 \). We first determine the equilibrium number of anyons \( N \) within a mean-field treatment (mean-field values will be indexed with a subscript ‘mf’). This becomes accurate in the relevant limit of large \( L \). We obtain the single-particle energy at plaquette \( p \) as \( \epsilon_p = \delta H_0/\delta n_p = J + \sum_{p' \neq p} U_{pp'} n_{p'} \). Replacing \( n_{p'} \) by the average value \( n_{mf} = N_{mf}/L^2 \) and taking the continuum limit, we find the mean-field value for \( \epsilon_p \) to be

\[
\epsilon_{mf} = J + n_{mf} \int_{L \times L} \frac{A}{p^a} \, dp = J + n_{mf} TL_{\alpha},
\]  

where we use the notation \( L_{\alpha} = c_{\alpha} \beta AL^{2-\alpha} \). The constant \( c_{\alpha} \) is a geometrical factor of order one, given by the integration of \( 1/p^{\alpha} \) on a unit square centered at the origin. In particular, \( c_0 = 1 \). On the other hand, we have \( n_{mf} = 1/(\beta \rho + 1) \) since the occupation numbers \( n_p \) can only assume the values 0 or 1. By using Eq. 6 to calculate \( n_{mf} \), we find the self-consistent equation

\[
n_{mf} = \frac{1}{\epsilon_{3f^t + n_{mf} \rho L_{\alpha} + 1}},
\]  

with the following expansion at large \( L_{\alpha} \)

\[
n_{mf} = \frac{1}{L_{\alpha}} [\ln L_{\alpha} - \ln \ln L_{\alpha} - \beta J + \ldots].
\]  

Higher order terms in the square parenthesis are small if \( \ln L_{\alpha} \gg \beta J, |\ln \ln L_{\alpha}| \). For fixed temperature \( T \) and interaction strength \( A \), these conditions are always satisfied at sufficiently large \( L \) since \( L_{\alpha} \propto L^{2-\alpha} \). We have confirmed the validity of the mean-field approximation by Monte Carlo simulations, see Appendix E.

From Eq. 8 we obtain that, even though the number of anyons \( N_{mf} \) grows with the system size \( L \), the anyon density \( n_{mf} \) goes to zero for long-range repulsive interactions with \( 0 \leq \alpha < 2 \). Hence, the population of anyons is increasingly diluted and the system is essentially frozen in the ground state at large system size. This remarkable effect can be attributed to the divergence of the excitation energy \( \epsilon_{mf} \simeq T \ln L_{\alpha} \), which is self-consistently determined from the anyon population in the whole sample due to the long-range nature of the interactions. Note also that, despite the fact that \( \epsilon_{mf} \) is diverging, the total excitation energy density \( n_{mf} \epsilon_{mf}/2 \) goes to zero for large \( L \).

Secondly, the divergence of \( \epsilon_{mf} \) leads to a vanishing anyon pair creation rate at large \( L \),

\[
\gamma(-2\epsilon_{mf}) \simeq T n \left( \frac{2 \ln L_{\alpha}}{2L_{\alpha}^2} \right)^{n+2}.
\]  

This fact allows us to revise the lifetime for the non-interacting memory Eq. 6, simply by substituting \( J \) with the equilibrium value \( \epsilon_{mf} \), yielding

\[
\tau \simeq \frac{2f_c}{\max\{\gamma(0), 4\gamma(-2\epsilon_{mf})\}}.
\]  

From this we obtain the lifetime of an interacting memory in case of an Ohmic (\( n = 1 \)) or super-Ohmic (\( n > 1 \)) bath as

\[
\tau \simeq \begin{cases} \frac{f_c L_{\alpha}}{T \ln L_{\alpha}}, & \text{Ohmic} \\ \frac{2f_c L_{\alpha}^3}{T n (2 \ln L_{\alpha})^{n+3}}, & \text{super-Ohmic} \end{cases}
\]

in the limit of large grid size [see after Eq. 8]. It is clear from these expressions that the memory lifetime is diverging with \( L \), in strong contrast to the non-interacting case where it was bounded by a constant. In the Ohmic case, this divergence of \( \tau \) is entirely due to the vanishing density, since \( \gamma(0) = 2T \) is non-zero. In the super-Ohmic case, however, an additional divergence due to the vanishing of \( \gamma(-2\epsilon_{mf}) \) is obtained, see Eq. 9. Since the energy
gap grows logarithmically with \( L \), \( \tau \) grows polynomially, but with a rather favorable power. For instance, constant interaction (\( \alpha = 0 \), see also below) leads to \( \tau \propto L^2/\ln L \) in the Ohmic case and to \( \tau \propto L^9/\ln^5 L \) in the super-Ohmic (\( n = 2 \)) case.

Eq. (10) is valid in the mean-field limit and does not include fluctuations which give rise to additional errors due to the long-range interactions. However, the memory remains self-correcting both for an Ohmic and for a super-Ohmic bath. Indeed, for an Ohmic bath, we can neglect the effect of the repulsive force if the change of energy \( \omega \) in a diffusive step is smaller in magnitude than \( T \) (see Eq. (1)), so that we can approximate \( \gamma(\omega) \approx \gamma(0) \).

In particular, for a single pair of anyons at distance \( r \), we have \( |\omega| \lesssim A/r^{\alpha+1} \), which defines a critical radius

\[
r_c = (\alpha A \beta)^{1/\alpha+1} ,
\]

(12)
beyond which the fluctuations become negligible. For \( \alpha = 0 \) one has \( r_c = 0 \). For \( \alpha > 0 \), since the average distance \( \sim 1/\sqrt{\text{anyons} \times L} \) between anyons grows with \( L \) while \( r_c \) is independent of \( L \), the fluctuations also become negligible. The validity of Eq. (10) for the Ohmic case is confirmed by numerical simulations both for \( \alpha = 0 \) (see Fig. 3) and \( \alpha > 0 \) (see Fig. 4).

Concerning the super-Ohmic case, we expect deviations from Eq. (10) if the fluctuations of \( \omega \approx 0 \) are more effective for the anyon motion than the indirect diffusion mechanism, proportional to the rate \( \epsilon \). However, due to the decreasing interaction strength, such fluctuations in \( \omega \) become small at large \( L \) and still result in a vanishing diffusion coefficient. Furthermore, at \( \alpha = 0 \) direct hopping is impossible and Eq. (10) is valid (see Fig. 3).

V. NUMERICAL SIMULATIONS

We turn now to the numerical simulations of our model, Eq. (1), and focus first on constant long-range interactions (\( \alpha = 0 \)). In this case, the total energy \( E_N = NJ + J/2N(N-1) \) depends only on the number of anyons \( N \), but not on their position. This simplifies the numerical treatment considerably. Our results are displayed in Fig. 5. The numerical data show a clear increase of the memory lifetime \( \tau \) with \( L \). Note that this holds already for the bare \( Z \). Like in the non-interacting case (see Fig. 2), the beneficiary effect of the error correction at read-out is to prolong the lifetime by maintaining \( \langle Z_{ec} \rangle \) close to one (see inset of Fig. 3).

Our analytical results describe the numerical data remarkably well. By fitting \( f_c \) in Eq. (10) to the simulation data, excellent agreement is found for an Ohmic bath (top panel of Fig. 3), while for a super-Ohmic bath (lower panel), analytics and numerics agree well for \( L \gtrsim 64 \). Further, the fit yields values for \( f_c \) of about 0.01 – 0.02, which is reasonable in comparison to the upper bound \( f_c = 0.11 \) found for a model of uncorrelated errors (dashed-dotted lines in Fig. 5). See also Appendix F for an extended discussion.

The lifetime \( \tau \) can be compared to the physical time scales of single spin flips, \( 1/\gamma(0) \) and \( 1/\gamma(-2J) \). For instance, for the \( L = 256 \) super-Ohmic case in Fig. 3 we obtain \( \tau \gamma(-2J) \approx 5 \times 10^3 \), i.e., already for a moderate system size the lifetime \( \tau \) of the memory is about a million times longer than the single-spin lifetime. For quantum dots, the latter is typically in the range of milliseconds to seconds at about 100 mK.

For non-constant long-range interaction (\( 0 < \alpha < 2 \)), simulating the time dynamics of the memory is numerically more costly due to an \( O(L^2) \) overhead coming from recalculating all spin flip rates. Nevertheless, we were
able to study the (more tractable) case of an Ohmic bath. The results are presented in Fig. 4 for α = 0.5 and α = 1. Clearly, the memory lifetime is still increasing with L, proving the memory to be self-correcting also for α ≠ 0. Furthermore, the data is in perfect agreement with the analytically calculated lifetime Eq. (10). The super-Ohmic case for α > 0 is more difficult due to the increased memory lifetime, and will be examined elsewhere.

We consider now in greater detail the super-Ohmic case at α = 0, which has the most favorable scaling. On a sufficiently short time scale (in the nonsplit-pair regime), the rate equation

\[
\frac{dN^*_m}{dt} = 4L^2\gamma(-2\epsilon_m^*) - N^*_m\gamma(2\epsilon_m^*),
\]

(13)
describes the initial time-evolution of the system well, since in this non-diffusive regime only pair creation and annihilation takes place. We confirm this by comparing a numerical integration of Eq. (13) with a direct simulation, presented in Fig. 5. After a rapid initial ‘build-up’ phase, \(N^*_m\) saturates to a value determined by the self-consistent condition \(N^*_m = 4L^2e^{-2(J+AN^*_m)/\gamma}\), obtained by setting \(dN^*_m/dt = 0\) in Eq. (13). In this state, the excitation energy is diverging with L, since we have \(\epsilon_m^* \approx AN^*_m \approx AN^*_m/2 \propto \ln L\). This effectively suppresses the indirect diffusion of anyons. Therefore, the system remains in a quasi-stationary state which evolves to the final anyon density on a time scale also diverging with L. In this regime of nonsplit pairs, one has \(\langle Z \rangle \approx e^{-N^*_m/2L}\). This leads to the quasi-stationary value

\[
\langle Z \rangle \approx e^{-\ln L/2},
\]

which approaches one for large L (see Fig. 5).

VI. DISCUSSION AND CONCLUSIONS

We now discuss various physical aspects arising in the context of our model. We first remark that super-Ohmic baths, which provide the best scaling of the lifetime with L, are not uncommon and emerge, e.g., for quantum dot spins in contact with phonons. As for the periodic boundary conditions, these are not an essential ingredient to a topological stabilizer code. Concerning the many-body nature of the interactions involved, general n-body couplings can in principle be engineered from short-range 2-body interactions. For example, toric codes interacting anyons are derived in. A systematic procedure to construct such effective low-energy Hamiltonians can be rigorously founded on the Schrieffer-Wolff transformation. A promising candidate system to realize topological models are ultracold atoms or molecules in optical lattices.

In the same way, physical long-range interactions of the type considered in this work could also be generated perturbatively. A well-known example is the RKKY interaction, e.g., for a 2D Kondo-lattice of nuclear spins. Alternatively, constant interactions (α = 0) can be realized for qubits coupled to photon modes in QED-cavities. The interaction range is determined by the wavelength of the photon and can reach macroscopic distances, in particular in superconducting cavity striplines. As an illustrative example let us consider a two-photon coupling (see also Appendix C) of the form

\[
H_{\text{int}} = \sum_{i=1}^{2} \omega_i a_i^\dagger a_i + \sum_p g_p n_p (a_1^\dagger a_2 + a_1 a_2^\dagger).
\]

Here, \(\omega_i\) are the photon frequencies, and \(g_p\) is the coupling strength of plaquette \(p\) to the modes. Note that \(g_p\) is spatially constant over the photon wavelength \(\lambda\).

Performing a Schrieffer-Wolff transformation (see also Appendix C) yields an effective Hamiltonian containing the desired constant anyon interaction,

\[
H_{\text{eff}} = \frac{a_1^\dagger a_1 - a_2^\dagger a_2}{\omega_1 - \omega_2} \left( \sum_p g_p n_p \right)^2 + \sum_{i=1}^{2} \omega_i a_i^\dagger a_i.
\]

The strength and sign of the interaction is tunable via the difference in frequencies and occupation numbers of the modes, and can consequently be made repulsive in a steady-state regime. For typical stripline cavities with \(\lambda \sim \text{cm}\) and for typical lattice constants of 100 nm (e.g. quantum dots), we see that the anyon interaction stays constant over system sizes L as large as \(10^5\). With the super-Ohmic scaling \((\tau \sim L^6)\), and a single-spin lifetime \(1/\gamma(-J) \sim 1\mu s\), the memory lifetime \(\tau\) can be made sufficiently long for any practical purposes.
In conclusion, we have demonstrated the existence of 2D stabilizer quantum memories at finite temperatures. The stability of the memory results from the repulsive long-range nature of the interaction we have introduced. We expect that similar systems in the presence of such interactions also prove useful as self-correcting quantum memories.

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Appendix A: Numerical simulation

1. Time evolution

In order to achieve a time evolution in accordance with Eq (3), each iteration of a simulation consists of the following steps. (i) We record the relevant parameters of the system. (ii) We calculate the total spin flip rate $R = \sum_s \gamma (\epsilon_s - \epsilon_{x(s)})$, where $s$ is the current state of the system. (iii) We draw the time $\Delta t$ it takes for the next spin to flip from an exponential distribution, $\Delta t \sim \text{Exp}(1/R)$, and then add this to the current total time. (iv) We calculate all individual spin flip probabilities $p_i = \gamma (\epsilon_s - \epsilon_{x(s)})/R$ and flip a spin at random accordingly. After some initially specified time has been reached, we stop and have obtained a single ‘run’. The final data presented in this work is then generated by averaging over many (typically several thousand) runs.

2. Error correction

We have seen that, for an interacting system, already the bare logical $Z$ operator becomes stable with increasing $L$. Nevertheless, it is useful to apply an error correction scheme once the memory is being read out. By $\langle Z_{ec} \rangle(t)$, we denote in this work the average value of $Z$ we would have obtained if we had performed error correction at time $t$. The goal here is to properly annihilate corresponding anyons (by applying $\sigma_x$-operations), thereby reverting the undesired operations performed by anyon paths crossing the logical operator strings. However, since only the positions of the anyons are known, this correspondence has to be guessed. We do this by choosing the pairing with the minimal sum of connection path lengths using Blossom V, which is the latest improvement on Edmonds’ minimal-weight perfect matching algorithm. If many anyons are present, using the complete graph as the input to this algorithm is numerically infeasible. In excellent approximation, we therefore replace the complete graph by a Delaunay triangulation.

Appendix B: Mapping from lattice gas to Ising model

Note that $H_0$ in Eq. (1) has the general form of two independent lattice gases, which is also equivalent to two Ising spin lattices. We explicitly perform the transformation in the plaquette sector by identifying the Ising variables $s_p \equiv 1 - 2 n_p$, yielding

$$H_0 = - \sum_p \left( \frac{J}{2} + \sum_{p'} \frac{U_{pp'}}{4} s_p s_{p'} \right) + \frac{1}{8} \sum_{p,p'} U_{pp'} s_p s_{p'} + \ldots ,$$

(B1)

where $U_{pp'}$ is given in Eq. (3) and the primes in the summations indicate $p' \neq p$. We have used $U_{pp} = 2J$ and

![Graph of short-time dynamics of the interacting memory in a super-Ohmic bath.](image)
$U_{pp'} = U_{p'p}$. The noninteracting Kitaev model corresponds to noninteracting spins in an external magnetic field. The ground state corresponds to the fully polarized state $s_p = 1$ for all $p$, where no anyon is present. However, for $T > 0$ a finite density of anyons emerges and is sufficient to destroy the information stored in the memory.

If a short-range ferromagnetic interaction is introduced, ordering of the system is spontaneously favored below some critical temperature. A higher magnetization corresponds to a lower population of anyons and improves the lifetime. However, short range interactions do not improve the scaling of the lifetime with the system size, since a residual density of anyons is left at any finite temperature. As in the noninteracting case, a finite density of excited plaquettes efficiently destroys the stored quantum information, in agreement with the general analysis of Appendix C. Instead, repulsive long-range interactions lead to a fully polarized system at a given temperature for sufficiently large system size $L$.

Appendix C: Lifetime in the presence of a single pair

The decay of the bare and logical $Z$ operators is most simply illustrated by assuming only a single anyon pair in the memory. We set $\gamma(2J) = 0$, so that pair creation and annihilation are not allowed. If no anyons were present, the initial values $(Z) = (Z_{cc}) = 1$ would be stable. We apply one $\sigma_x$-operation at a randomly chosen site and thereby create two neighboring anyons at $t = 0$. This causes a partial decay of the bare logical operator already at $t = 0$, since we might have chosen to flip a spin on the logical $Z$ operator, yielding $(Z) = 1 - \frac{1}{L}$. This has been used in the main text in the discussion of the nonsplit-pair regime.

We now study the decay for $t > 0$ in the continuum limit and therefore neglect the $1/L$ correction at $t = 0$. We consider a single pair of diffusing anyons with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ created at the origin. We then assume that the probability to find an anyon at position $r$ is described by the probability density

$$p(r) = \frac{1}{4\pi\gamma(0)t} e^{-\frac{r^2}{4\gamma(0)t}}.$$  

We represent the torus as an infinite plane with the points $(x, y)$ and $(x + mL, y + nL)$ being equivalent $(m, n \in \mathbb{Z})$. The logical $Z$ operator is then represented by parallel lines at $y_2 = L/2 + nL$. The two anyons diffuse along $y$ with probability density $p(y_1 - y_0) = e^{-(y_1 - y_0)^2/4\gamma(0)t}/\sqrt{4\pi\gamma(0)t}$, where $i = 1, 2$ and the initial (random) coordinate satisfies $-L/2 \leq y_0 < L/2$. The average of the logical operator at time $t$ is

$$\langle Z \rangle = \int_{-L/2}^{L/2} dy_0 \int dy_1 dy_2 p(y_1 - y_0) p(y_2 - y_0) z(y_1, y_2),$$  

where $z(y_1, y_2)$ gives the sign of $Z$ if the two anyons have diffused to the coordinates $y_1$ and $y_2$. Since $Z$ changes sign each time an anyon crosses the lines at $y_2$, we have

$$z(y_1, y_2) = z(y_1) z(y_2)$$

when $z(y) = 1$ if $-L/2 + 2nL \leq y < L/2 + 2nL$ and $-1$ otherwise $(n \in \mathbb{Z})$. Therefore we can write

$$\langle Z \rangle = \int_{-L/2}^{L/2} dy_0 f(z_0)^2,$$

where we have made the change of variables $y_0 = Lz_0$, such that

$$f(z_0) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} (-1)^n \left[ \frac{\text{erf} \left( \frac{2z_0 + 2n + 1}{4\sqrt{\gamma(0)t/L^2}} \right)}{\frac{2z_0 + 2n - 1}{4\sqrt{\gamma(0)t/L^2}}} \right].$$

We now consider the average of the error-corrected logical operator $Z_{cc}$. In this case, only the distance $y_{12} = y_1 - y_2$ between the two anyons is important since the value of $Z_{cc}$ is $1$ if $-L/2 + 2nL \leq y_{12} < L/2 + 2nL$, and $-1$ otherwise. The probability distribution for $y_{12}$

$$\langle Z_{cc} \rangle = \sum_{n=-\infty}^{+\infty} (-1)^n \text{erf} \left( \frac{2n + 1}{2\sqrt{2\gamma(0)t/L^2}} \right).$$  

Both functions (C3) and (C5) are plotted in Fig. 6 and show perfect agreement with the numerical simulation. An important feature of the above analytical expressions is that the time dependence only enters through the combination $\gamma(0)t/L^2$, which makes it possible to scale curves from different system sizes and diffusion constants onto each other.

![FIG. 6: (Color online.) Decay of the bare and corrected expectation value of $Z$ due to a single pair of anyons in the memory. The dots show numerical data (averaged over $10^4$ samples) while the two curves are the continuum limit expressions Eq. (C3) and (C5) for $(Z_{cc})$ (solid) and $(Z)$ (dashed). The numerical data has been obtained for $L = 32$ with $\gamma(0) = 1, 5, 10$ and $L = 64, 128$ with $\gamma(0) = 1$. All points collapse onto each other when plotted as a function of $\gamma(0)t/L^2$.](image-url)
Appendix D: Indirect diffusion of anyons

To determine the rate $D$ associated with indirect diffusion, we consider an isolated anyon in the lattice and its probability $p_{i,j}$ to be at site $(i,j)$. In the Ohmic case, we have $\gamma(0) \neq 0$, and direct hopping to neighboring sites is thus allowed. In the continuum limit, a standard diffusion equation $\frac{dp_{i,j}}{dt} = D \nabla^2 p(r)$ with $D = \gamma(0)$ is obtained. In the non-interacting super-Ohmic case, however, $\gamma(0) = 0$. We assume $2\beta J \gg 1$, such that, since $\gamma(2J) = e^{2\beta J} \gamma(-2J)$, the recombination of a pair of anyons is essentially instantaneous. Hopping from the site $(i,j)$ to, e.g., $(i,j+2)$ is possible by creation of an anyon pair occupying sites $(i,j+1)$ and $(i,j+2)$, an event occurring with rate $\gamma(-2J)$. Since the intermediate state can decay back to the initial state, the actual rate for the indirect hopping process is $\gamma(-2J)/2$. Similar considerations hold for all other sites. Accounting for all these, we write

$$\frac{dp_{i,j}}{dt} = \gamma(-2J) \left( -12p_{i,j} + p_{i+2,j} + p_{i-2,j} + p_{i,j+2} + p_{i,j-2} + 2p_{i+1,j+1} + 2p_{i+1,j-1} + 2p_{i-1,j+1} + 2p_{i-1,j-1}, \right)$$

which in the continuum limit yields $D = 4\gamma(-2J)$. We can expect that the properties of the memory improve by lowering the value of $\gamma(0)$, but only as long as $\gamma(0) \gtrsim 4\gamma(-2J)$. In the interacting case, $J$ is replaced by an appropriate excitation energy (e.g., $\epsilon_{\text{mf}}$ at equilibrium).

Appendix E: Equilibrium density of interacting anyons

The equilibrium number $N$ of excited plaquettes can be approximated with arbitrary accuracy by using the Metropolis algorithm to sample the probability distribution $\propto e^{-\beta E \sum p_r U_{p_r} p_{p_r}^\prime}$, see Eq. (1). This can be used to study the accuracy of the mean-field value $N_{\text{mf}} = n_{\text{mf}} L^2$ [see Eq. (2)], in particular for values $\alpha \neq 0$.

Due to the long-range nature of the interaction, we find that $N_{\text{mf}}$ compares very well to the equilibrium value of $N$ at generic values of the temperature and interaction exponent $\alpha$. This is illustrated in Fig. 5, which shows a satisfactory agreement already at moderate values of $L$.

We also note that, for the case of constant interaction ($\alpha = 0$), the average number can be calculated directly from the grand-canonical partition function

$$\sum_{2k \leq L^2} \left( \frac{L^2}{2k} \right) e^{-\beta E_{2k}} = \frac{1}{2} \sum_{k \leq L^2} (2k \frac{L^2}{2k}) e^{-\beta E_{2k}},$$

since the energy of a given anyon configuration does not depend on the positions of the anyons, but only on their total number $N = \sum p_r n_{p_r}$. In the presence of an appreciable anyon interaction and at low temperature, the number of excited plaquettes is much smaller than $L^2$. Therefore, one can restrict the sum to the first relevant terms.

In a similar way, the exact quasi-stationary number of paired anyons $N^\ast$, displayed in Fig. 5 (crosses), can be calculated from the partition function

$$\sum_{k \leq L^2} \left( \frac{2L^2}{k} \right)^{2k} e^{-\beta E_{2k}}.$$
FIG. 8: (Color online.) Average of the corrected operator $Z_{ec}$ for a model with independent $\sigma_x$-errors occurring with probability $f$ at each spin. The solid curves refer to our numerical simulations with lattice sizes $L = 40, 100, 200$. The error correction fails at a value $f_c \approx 0.1$, which is slightly smaller than the value 0.11 from Ref. 10 (dotted line). In the inset, we plot the value of $f_c$ from simulations of the non-interacting toric code in contact with a bath at temperature $T$ ($\gamma(0) = \gamma(2J) = 1$). The fraction $f_c$ is extracted at the time $\tau$ when $\langle Z_{ec} \rangle$ decays to zero in the limit of large $L$ (see Fig. 3). This value is always smaller than $f = 0.11$ and depends on $T$.

Appendix F: Critical fraction of errors

We discuss here in greater detail the role of $f_c$ in the memory lifetime Eq. (5). An analogous result can be obtained based on the following different reasoning. The distance between two anyons of a pair after a time $\tau$ is of order $\Delta \ell = \sqrt{D\tau}$ and this is required to be much smaller than the average anyon separation $\sim \sqrt{L^2/N}$. This gives

$$\tau \ll \frac{e^{3J} + 1}{\max\{\gamma(0), 4\gamma(-2J)\}}.$$  \hspace{1cm} (F1)

Interestingly, this upper bound coincides with the right-hand side of Eq. (5) if the probability for each spin to be flipped is $f_c = \frac{1}{2}$. Hence, this should be an upper bound on the values of $f_c$ we determine numerically. In the following, we demonstrate that this is indeed the case, and that the values we obtain are meaningful.

We first determine $f_c$ for an independent error model. By mapping the toric code to a random-bond Ising model, a phase transition at $f_c = 0.11$ was obtained in Ref. 10. Numerically, we find in this case the value $f_c \approx 0.1$, see Fig. 8. This shows that our minimum-weight error correction scheme works close to optimal.

In the simulations of the non-interacting model, we observe a sharp transition in time similar to Fig. 8 (see Fig. 2) but, not surprisingly, we find at the transition point a relative fraction of errors different from 0.1. We attribute this fact to the statistical properties of the distribution of errors generated by the time evolution. Clearly, the errors created by the anyons in their diffusive motion have strong spatial correlations, rather than being independent and uniformly distributed across the sample. We find that such correlations yield values of $f_c$ strictly smaller than 0.1 but still of the order of a few percent, see Fig. 5.

We mention here a second uncertainty in Eq. (5), namely the estimate $NDt$ of the total number of errors. An isolated anyon can have either one or three $\sigma_x$-errors at its plaquette spins. In the first case it contributes to the error rate with $2D = 3D - D$ since for three hopping processes one error is created and for the fourth a pre-existing error is removed. If three $\sigma_x$-errors are present, an opposite rate $-2D$ is obtained but three-error plaquettes are rare in the initial time evolution (they only arise from the crossing of two anyon paths). This gives a total error rate from diffusion of order $D$. Furthermore, a typical number of anyons $N$ is determined by the equilibrium density, but the actual value is always smaller (since there are initially no anyons present in the memory). Therefore, $NDt \approx \frac{\ell^2Dt}{1 + e^{\beta J}}$ differs from the actual number of errors by an unknown prefactor.

From the previous discussion it is clear that $f_c$ in Eq. (5) is generally different from the value $f_c = 0.11$ known in the literature. This justifies our practice of using $f_c$ as a fitting parameter to reproduce the functional dependence of the lifetime, e.g., as a function of $L$ or $T$. Remarkably, the asymptotic dependence on $L$ is well described by Eq. (5), both in the interacting and non-interacting case (see Figs. 2 and 5). An example of the temperature dependence of $\tau$ in the non-interacting case is shown in Fig. 9 and is also described by Eq. (5) very well. Note that fitting the data always yield values of $f_c$ smaller than $f_c = 0.11$, and of the order of a few percent. This is consistent with their interpretation as a critical fraction of errors.
Appendix G: Effective Hamiltonian via Schrieffer-Wolff transformation

In order to find an effective Hamiltonian for (14), we write \( H = H_0 + V \), where \( H_0 = \sum_{i=1}^{2} \omega_i a_i^\dagger a_i \) and \( V = \sum_p g_p n_p (a^\dagger_2 a_1^\dagger + a_1 a^\dagger_2) \), and treat \( V \) as a small perturbation. The general expression for the Schrieffer-Wolff transformation of \( H \) up to second order in \( V \) reads

\[
H_{\text{eff}} = H_0 + \frac{i}{2} \lim_{\varepsilon \to 0} \int_0^\infty dt e^{-\varepsilon t} [V, V(t)] + O(V^3),
\]

(G1)

where \( V(t) = \exp(iH_0 t)V \exp(-iH_0 t) \), which yields in our case

\[
V(t) = \sum_p g_p n_p \left( e^{i(\omega_1 - \omega_2)t} a_1^\dagger a_2 + e^{-i(\omega_1 - \omega_2)t} a^\dagger_2 a^\dagger_1 \right).
\]

(G2)

With this, the commutator in Eq. (G1) evaluates to

\[
[V, V(t)] = 2i \sum_p g_p n_p^2 (a^\dagger_2 a^\dagger_2 - a^\dagger_1 a^\dagger_1) \sin(\omega_1 - \omega_2)t.
\]

(G3)

Inserting this into Eq. (G1) and performing the integral yields Eq. (15).

A two-photon coupling of the type \( a_1^\dagger a_2 \) can emerge from a quadratic coupling of spin to electric (or magnetic) cavity fields such as \( E_x E_y \). Now, the toric code model can be obtained from Kitaev’s honeycomb model by perturbation theory. Indeed, in leading order one obtains \( J \propto J_x^2 J_y^2 / J_z^3 \), where \( J_k \) are the exchange couplings in the honeycomb model. In multiferroic materials electric fields can couple to the spin (-texture) via a modification of the exchange interaction like \( J_k \to J_k + \gamma_k E_k \) (with \( \gamma_{x,y,z} \) some coupling constants). Thus, if e.g. one \( J_x \) and one \( J_y \) occurring in \( J \propto J_x^2 J_y^2 / J_z^3 \) get modified in this way (by locally modifying the corresponding links), we end up with a coupling of the desired form. By going to resonance \( \omega_1 \approx \omega_2 \), the quadratic term \( E_x E_y \) can be made dominant over the linear ones (which are non-resonant).

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