Cox Rings

Ivan Arzhantsev
Ulrich Derenthal
Jürgen Hausen
Antonio Laface

Department of Algebra, Faculty of Mechanics and Mathematics, Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia
E-mail address: arjantse@mccme.ru

Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstrasse 39, 80333 München, Germany
E-mail address: ulrich.derenenthal@mathematik.uni-muenchen.de

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: juergen.hausen@uni-tuebingen.de

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: alaface@udec.cl
This is the first chapter of an introductory text under construction; further chapters are available on the author’s web pages. Our aim is to provide an elementary access to Cox rings and their applications in algebraic and arithmetic geometry. We are grateful to Victor Batyrev and Alexei Skorobogatov for helpful remarks and discussions. Any comments and suggestions on this draft will be highly appreciated.
Contents

Chapter I. Basic concepts 5
   1. Graded algebras 5
      1.1. Monoid graded algebras 5
      1.2. Veronese subalgebras 8
   2. Gradings and quasitorus actions 10
      2.1. Quasitori 10
      2.2. Affine quasitorus actions 11
      2.3. Good quotients 15
   3. Divisorial algebras 18
      3.1. Sheaves of divisorial algebras 18
      3.2. The relative spectrum 19
      3.3. Unique factorization in the global ring 21
      3.4. Geometry of the relative spectrum 23
   4. Cox sheaves and Cox rings 25
      4.1. Free divisor class group 25
      4.2. Torsion in the class group 28
      4.3. Well-definedness 30
      4.4. Examples 32
   5. Algebraic properties of the Cox ring 34
      5.1. Integrity and Normality 34
      5.2. Localization and units 36
      5.3. Divisibility properties 37
   6. Geometric realization of the Cox sheaf 40
      6.1. Characteristic spaces 40
      6.2. Divisor classes and isotropy groups 43
      6.3. Total coordinate space and irrelevant ideal 44
      6.4. Characteristic spaces via GIT 45

Bibliography 49

Index 55
CHAPTER I

Basic concepts

In this chapter we introduce the Cox ring and, more generally, the Cox sheaf and its geometric counterpart, the characteristic space. Moreover, algebraic and geometric aspects are discussed. Section 1 is devoted to commutative algebras graded by monoids. In Section 2 we recall the correspondence between actions of quasitori (also called diagonalizable groups) on affine varieties and affine algebras graded by abelian groups, and we provide the necessary background on good quotients. Section 3 is a first step towards Cox rings. Given a normal variety $X$ and a finitely generated subgroup $K \subseteq \text{WDiv}(X)$ of the group of Weil divisors, we consider the associated sheaf of divisorial algebras

$$S = \bigoplus_{D \in K} \mathcal{O}_X(D).$$

We present criteria for local finite generation and consider the relative spectrum. A first result says that $\Gamma(X, S)$ is a unique factorization domain if $K$ generates the divisor class group $\text{Cl}(X)$. Moreover, we characterize divisibility in the ring $\Gamma(X, S)$ in terms of divisors on $X$. In Section 4 the Cox sheaf of a normal variety $X$ with finitely generated divisor class group $\text{Cl}(X)$ is introduced; roughly speaking it is given as

$$\mathcal{R} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{O}_X(D).$$

The Cox ring then is the corresponding ring of global sections. In the case of a free divisor class group well-definiteness is straightforward. The case of torsion needs some effort, the precise way to define $\mathcal{R}$ then is to take the quotient of an appropriate sheaf of divisorial algebras by a certain ideal sheaf. Basic algebraic properties and divisibility theory of the Cox ring are investigated in Section 5. Finally, in Section 6 we study the characteristic space, i.e., the relative spectrum $\tilde{X} = \text{Spec}_X \mathcal{R}$ of the Cox sheaf. It comes with an action of the characteristic quasitorus $H = \text{Spec} \mathbb{K}[\text{Cl}(X)]$ and a good quotient $\tilde{X} \to X$. We relate geometric properties of $X$ to properties of this action and give a characterization of the characteristic space in terms of Geometric Invariant Theory.

1. Graded algebras

1.1. Monoid graded algebras. We recall basic notions on algebras graded by abelian monoids. In this subsection, $R$ denotes a commutative ring with unit element.

**Definition 1.1.1.** Let $K$ be an abelian monoid. A $K$-graded $R$-algebra is an associative, commutative $R$-algebra $A$ with unit that comes with a direct sum decomposition

$$A = \bigoplus_{w \in K} A_w$$

into $R$-submodules $A_w \subseteq A$ such that $A_w \cdot A_{w'} \subseteq A_{w+w'}$ holds for any two elements $w, w' \in K$. 


We will also speak of a $K$-graded $R$-algebra as a monoid graded algebra or just as a graded algebra. In order to compare $R$-algebras $A$ and $A'$, which are graded by different abelian monoids $K$ and $K'$, we work with the following notion of a morphism.

**Definition 1.1.2.** A morphism from a $K$-graded algebra $A$ to a $K'$-graded algebra $A'$ is a pair $(\psi, \tilde{\psi})$, where $\psi: A \to A'$ is a homomorphism of $R$-algebras, $\tilde{\psi}: K \to K'$ is a homomorphism of abelian monoids and

$$\psi(A_w) \subseteq A_{\tilde{\psi}(w)}$$

holds for every $w \in K$. In the case $K = K'$ and $\tilde{\psi} = \text{id}_K$, we denote a morphism of graded algebras just by $\psi: A \to A'$ and also refer to it as a $(K')$-graded homomorphism.

**Example 1.1.3.** Given an abelian monoid $K$ and $w_1, \ldots, w_r \in K$, the polynomial ring $R[T_1, \ldots, T_r]$ can be made into a $K$-graded $R$-algebra by setting

$$R[T_1, \ldots, T_r]_w := \left\{ \sum_{\nu \in \mathbb{Z}_{\geq 0}^r} a_{\nu} T^\nu; \ a_{\nu} \in R, \nu_1 w_1 + \ldots + \nu_r w_r = w \right\}.$$ 

This $K$-grading is determined by $\deg(T_i) = w_i$ for $1 \leq i \leq r$. Moreover, $R[T_1, \ldots, T_r]$ comes with the natural $\mathbb{Z}_{\geq 0}$-grading given by

$$R[T_1, \ldots, T_r]_{\nu} := R \cdot T^\nu,$$

and we have a canonical morphism $(\psi, \tilde{\psi})$ from $R[T_1, \ldots, T_r]$ to itself, where $\psi = \text{id}$ and $\tilde{\psi}: \mathbb{Z}_{\geq 0}^r \to K$ sends $\nu$ to $\nu_1 w_1 + \ldots + \nu_r w_r$.

For any abelian monoid $K$, we denote by $K^\pm$ the associated group of differences and by $K_Q := K^\pm \otimes_{\mathbb{Z}} \mathbb{Q}$ the associated rational vector space. Note that we have canonical maps $K \to K^\pm \to K_Q$, where the first one is injective if $K$ admits cancellation and the second one is injective if $K^\pm$ is free. By an integral $R$-algebra, we mean an $R$-algebra $A$ without zero-divisors.

**Definition 1.1.4.** Let $A$ be an integral $K$-graded $R$-algebra. The **weight monoid** of $A$ is the submonoid

$$S(A) := \{ w \in K; A_w \neq 0 \} \subseteq K.$$

The **weight group** of $A$ is the subgroup $K(A) \subseteq K^\pm$ generated by $S(A) \subseteq K$. The **weight cone** of $A$ is the convex cone $\omega(A) \subseteq K_Q$ generated by $S(A) \subseteq K$.

We recall the construction of the algebra associated to an abelian monoid; it defines a covariant functor from the category of abelian monoids to the category of monoid graded algebras.

**Construction 1.1.5.** Let $K$ be an abelian monoid. As an $R$-module, the associated **monoid algebra** over $R$ is given by

$$R[K] := \bigoplus_{w \in K} R \cdot \chi^w$$

and its multiplication is defined by $\chi^w \cdot \chi^{w'} := \chi^{w + w'}$. If $K'$ is a further abelian monoid and $\tilde{\psi}: K \to K'$ is a homomorphism, then we have a homomorphism

$$\psi := R[\tilde{\psi}]: R[K] \to R[K'], \quad \chi^w \mapsto \chi^{\tilde{\psi}(w)}.$$ 

The pair $(\psi, \tilde{\psi})$ is a morphism from the $K$-graded algebra $R[K]$ to the $K'$-graded algebra $R[K']$, and this assignment is functorial.
Note that the monoid algebra $R[K]$ has $K$ as its weight monoid, and $R[K]$ is finitely generated over $R$ if and only if the monoid $K$ is finitely generated. In general, if a $K$-graded algebra $A$ is finitely generated over $R$, then its weight monoid is finitely generated and its weight cone is polyhedral.

**Construction 1.1.6** (Trivial extension). Let $K \subseteq K'$ be an inclusion of abelian monoids and $A$ a $K$-graded $R$-algebra. Then we obtain an $K'$-graded $R$-algebra $A'$ by setting
\[
A' := \bigoplus_{u \in K'} A'_u, \quad A'_u := \begin{cases} A_u & \text{if } u \in K, \\ \{0\} & \text{else.} \end{cases}
\]

**Construction 1.1.7** (Lifting). Let $G: \tilde{K} \rightarrow K$ be a homomorphism of abelian monoids and $A$ a $K$-graded $R$-algebra. Then we obtain a $\tilde{K}$-graded $R$-algebra
\[
\tilde{A} := \bigoplus_{u \in \tilde{K}} \tilde{A}_u, \quad \tilde{A}_u := A_{G(u)}.
\]

**Definition 1.1.8.** Let $A$ be a $K$-graded $R$-algebra. An ideal $I \subseteq A$ is called $(K\text{-})homogeneous$ if it is generated by $(K\text{-})homogeneous$ elements.

An ideal $I \subseteq A$ of a $K$-graded $R$-algebra $A$ is homogeneous if and only if it has a direct sum decomposition
\[
I = \bigoplus_{w \in K} I_w, \quad I_w := I \cap A_w.
\]

**Construction 1.1.9.** Let $A$ be a $K$-graded $R$-algebra and $I \subseteq A$ a homogeneous ideal. Then the factor algebra $A/I$ is $K$-graded by
\[
A/I = \bigoplus_{w \in K} (A/I)_w, \quad (A/I)_w := A_w + I.
\]
Moreover, for each homogeneous component $(A/I)_w \subseteq A/I$, one has a canonical isomorphism of $R$-modules
\[
A_w/I_w \rightarrow (A/I)_w, \quad f + I_w \mapsto f + I.
\]

**Construction 1.1.10.** Let $A$ be a $K$-graded $R$-algebra, and $\psi: K \rightarrow K'$ be a homomorphism of abelian monoids. Then one may consider $A$ as a $K'$-graded algebra with respect to the coarsened grading
\[
A = \bigoplus_{u \in K'} A_u, \quad A_u := \bigoplus_{\tilde{w}(u) = u} A_w.
\]

**Example 1.1.11.** Let $K = \mathbb{Z}^2$ and consider the $K$-grading of $R[T_1, \ldots, T_5]$ given by $\deg(T_i) = w_i$, where
\[
w_1 = (-1, 2), \quad w_2 = (1, 0), \quad w_3 = (0, 1), \quad w_4 = (2, -1), \quad w_5 = (-2, 3).
\]
Then the polynomial $T_1T_2 + T_3^2 + T_4T_5$ is $K$-homogeneous of degree $(0, 2)$, and thus we have a $K$-graded factor algebra
\[
A = R[T_1, \ldots, T_5]/(T_1T_2 + T_3^2 + T_4T_5).
\]
The standard $\mathbb{Z}$-grading of the algebra $A$ with $\deg(T_1) = \ldots = \deg(T_5) = 1$ may be obtained by coarsening via the homomorphism $\psi: \mathbb{Z}^2 \rightarrow \mathbb{Z}, (a, b) \mapsto a + b$.

**Proposition 1.1.12.** Let $A$ be a $\mathbb{Z}'$-graded $R$-algebra satisfying $ff' \neq 0$ for any two non-zero homogeneous $f, f' \in A$. Then the following statements hold.

(i) The algebra $A$ is integral.
(ii) If $gg'$ is homogeneous for $0 \neq g, g' \in A$, then $g$ and $g'$ are homogeneous.
(iii) Every unit $f \in A^*$ is homogeneous.
we obtain a morphism $(\pi, \alpha)$, defined by

The inverse image $\pi^{-1}(\alpha)$ of differences, we may assume that

Moreover, set

Lemma 1.2.3. Let $K$ be an abelian monoid admitting cancellation and let $L, M \subseteq K$ be finitely generated submonoids. Then $L \cap M$ is finitely generated.

Proof. Consider the embedding $K \subseteq K^\pm$ into the group of differences and define a homomorphism $\alpha: \mathbb{Z}^\tau \to K^\pm$ with $L, M \subseteq \alpha(\mathbb{Z}^\tau)$. Then $\alpha^{-1}(L)$ and $\alpha^{-1}(M)$ are finitely generated monoids; indeed, if $w_i = \alpha(v_i)$, where $1 \leq i \leq k$, generate $L$ and $u_1, \ldots, u_s$ is a basis for $\ker(\alpha)$, then $\alpha^{-1}(L)$ is generated by $v_1, \ldots, v_k$ and $\pm u_1, \ldots, \pm u_s$.

To prove the assertion, it suffices to show that the intersection $\alpha^{-1}(L) \cap \alpha^{-1}(M)$ is finitely generated. In other words, we may assume that $K = \mathbb{Z}^\tau$ holds. Then $L$ and $M$ generate convex polyhedral cones $\tau$ and $\sigma$ in $Q^\tau$, respectively. Consider $\omega := \tau \cap \sigma$ and the tower of algebras

Gordan's Lemma [110] Theorem 7.6 shows that $Q[\omega \cap \mathbb{Z}^\tau]$ is finitely generated over $Q$. Moreover, for every $v \in \tau \cap \sigma$, some positive integral multiple $k \cdot v$ belongs to $L \cap M$. Thus, $Q[\omega \cap \mathbb{Z}^\tau]$ is integral and hence finite over $Q[L \cap M]$. The Artin-Tate Lemma [66] page 144 tells us that $Q[L \cap M]$ is finitely generated over $Q$. Consequently, the weight monoid $L \cap M$ of $Q[L \cap M]$ is finitely generated. □

Proof of Proposition 1.2.2. According to Lemma 1.2.3 we may assume that $L$ is contained in the weight monoid $S(A)$. Moreover, replacing $K$ with its group of differences, we may assume that $K$ is a group. Fix homogeneous generators $f_1, \ldots, f_r$ for $A$ and set $w_i := \deg(f_i)$. Then we have an epimorphism

Moreover, set $B := R[T_1, \ldots, T_r]$ and endow it with the natural $\mathbb{Z}^\tau$-grading. Then we obtain a morphism $(\pi, \alpha)$ of graded $R$-algebras from $B$ to $A$, where $\pi$ is the epimorphism defined by

The inverse image $\alpha^{-1}(L) \subseteq \mathbb{Z}^\tau$ is a finitely generated monoid. By Lemma 1.2.3 the intersection $M := \alpha^{-1}(L) \cap \mathbb{Z}^\tau_{\geq 0}$ is finitely generated and hence generates a polyhedral convex cone $\sigma = \text{cone}(M)$ in $Q^\tau$. Consider the tower of $R$-algebras

\[ R \subseteq R[M] \subseteq R[\sigma \cap \mathbb{Z}^\tau]. \]
The $R$-algebra $R[\sigma \cap \mathbb{Z}']$ is finitely generated by Gordan’s Lemma [110, Theorem 7.6], and it is integral and thus finite over $R[M]$. The Artin-Tate Lemma [66, page 144] then shows that $R[M]$ is finitely generated over $R$. By construction, $\pi : B \to A$ maps $R[M] \subseteq B$ onto $A(L) \subseteq A$. This implies finite generation of $A(L)$.

**Proposition 1.2.4.** Suppose that $R$ is noetherian. Let $K$ be a finitely generated abelian group, $A$ a $K$-graded integral $R$-algebra and $L \subseteq K$ be a subgroup of finite index. Then the (unique, invertible) elements $\varphi : B_u \to A F(\sigma)$ are $K$-homogeneous zero divisors in $B$. Fix generators $u_1, \ldots, u_r$ for $K$. Then we may write $u_i = u_i^+ - u_i^-$ with $u_i^\pm \in S(A)$. Choose nontrivial elements $g_i^\pm \in A_{u_i^\pm}$. With $f_i := g_i^+/g_i^-$ we have

$$\text{Quot}(A) = \text{Quot}(A(L))(f_1, \ldots, f_r).$$

By our assumption, $A$ is contained in the integral closure of $A(L)$ in Quot($A$). Applying [20, Proposition 5.17] we obtain that $A$ is a submodule of some finitely generated $A(L)$-module. Since $R$ and hence $A(L)$ is noetherian, $A$ is finitely generated as a module over $A(L)$ and thus as an algebra over $R$. □

Putting Propositions 1.2.2 and 1.2.4 together, we obtain the following well known statement on gradings by abelian groups.

**Corollary 1.2.5.** Let $R$ be noetherian, $K$ a finitely generated abelian group, $A$ an integral $K$-graded $R$-algebra and $L \subseteq K$ a subgroup of finite index. Then the following statements are equivalent.

(i) The algebra $A$ is finitely generated over $R$.

(ii) The Veronese subalgebra $A(L)$ is finitely generated over $R$.

**Proposition 1.2.6.** Suppose that $R$ is noetherian. Let $L, K$ be abelian monoids admitting cancellation and $(\varphi, F)$ be a morphism from an $L$-graded $R$-algebra $B$ to an integral $K$-graded $R$-algebra $A$. Assume that the weight monoid of $B$ is finitely generated and $\varphi : B_u \to A F(\sigma)$ is an isomorphism for every $u \in L$. Then finite generation of $A$ implies finite generation of $B$.

**Proof.** We may assume that $K$ is an abelian group. In a first step we treat the case $L = \mathbb{Z}'$ without making use of finite generation of $S(B)$. Since $A$ is integral, there are no $\mathbb{Z}'$-homogeneous zero divisors in $B$ and thus $B$ is integral as well, see Proposition 1.1.12. Set $L_0 := \ker(F)$. By the elementary divisors theorem there is a basis $u_1, \ldots, u_r$ for $\mathbb{Z}'$ and $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 1}$ such that $a_1 u_1, \ldots, a_s u_s$ is a basis for $L_0$. Let $L_1 \subseteq \mathbb{Z}'$ be the sublattice spanned by $u_{s+1}, \ldots, u_r$. This gives Veronese subalgebras

$$B_0 := \bigoplus_{u \in L_0} B_u, \quad B_1 := \bigoplus_{u \in L_1} B_u, \quad C := \bigoplus_{u \in L_0 \oplus L_1} B_u.$$ Note that $\varphi$ maps $B_1$ isomorphically onto the Veronese subalgebra of $A$ defined by $F(L_1)$. In particular, $B_1$ is finitely generated. Moreover, $C$ is generated by $B_1$ and the (unique, invertible) elements $f_{i}^{1 \pm} \in B_{z_i a_i u_i}$ mapping to $1 \in A_0$. Thus, the Veronese subalgebra $C \subseteq B$ is finitely generated. Since $L_0 \oplus L_1$ is of finite index in $\mathbb{Z}'$, also $B$ is finitely generated, see Corollary 1.2.5.

We turn to the general case. Let $u_1, \ldots, u_r \in L$ generate the weight monoid of $B$. Consider the homomorphism $G : \mathbb{Z}' \to L^\pm$ to the group of differences sending the $i$-th canonical basis vector $e_i \in \mathbb{Z}'$ to $u_i \in L$ and the composition $G' := F^\pm \circ G$, where $F^\pm : L^\pm \to K$ extends $F : L \to K$. Regarding $B$ as $L^\pm$-graded, $G$ and $G'$
define us lifted algebras \( \tilde{B} \) and \( \tilde{A} \), see Construction\(^{1.1.7}\) fitting into a commutative diagram of canonical morphisms

\[
\begin{array}{ccc}
\tilde{B} & \longrightarrow & \tilde{A} \\
\downarrow & & \downarrow \\
B & \varphi & A
\end{array}
\]

The canonical morphism from \( \tilde{A} \) to \( A \) is as required in the first step and thus \( \tilde{A} \) is finitely generated. The weight monoid \( M \) of \( \tilde{B} \) is generated by the kernel of \( G \) and preimages of generators of the weight monoid of \( B \); in particular, \( M \) is finitely generated. Moreover, \( \tilde{B} \) maps isomorphically onto the Veronese subalgebra of \( \tilde{A} \) defined by \( M \subseteq \mathbb{Z}^r \); here we use that \( \varphi: B_u \to A_{F(u)} \) is an isomorphism for every \( u \in L \). By Proposition\(^{1.2.2}\) the algebra \( \tilde{B} \) is finitely generated. Finally, \( \tilde{B} \) maps onto \( B \) which gives finite generation of \( B \).

2. Gradings and quasitorus actions

2.1. Quasitori. We recall the functorial correspondence between finitely generated abelian groups and quasitori (also called diagonalizable groups). Details can be found in the standard textbooks on algebraic groups, see for example \cite{39}, Section 8], [88 Section 16], [123 Section 3.2.3] or [140 Section 2.5].

We work in the category of algebraic varieties defined over an algebraically closed field \( \mathbb{K} \) of characteristic zero. Recall that an (affine) algebraic group is an (affine) variety \( G \) together with a group structure such that the group laws

\[ G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2, \quad G \to G, \quad g \mapsto g^{-1} \]

are morphisms of varieties. A morphism of two algebraic groups \( G \) and \( G' \) is a morphism \( G \to G' \) of the underlying varieties which moreover is a homomorphism of groups.

A character of an algebraic group \( G \) is a morphism \( \chi: G \to \mathbb{K}^* \) of algebraic groups, where \( \mathbb{K}^* \) is the multiplicative group of the ground field \( \mathbb{K} \). The character group of \( G \) is the set \( \mathcal{X}(G) \) of all characters of \( G \) together with pointwise multiplication. Note that \( \mathcal{X}(G) \) is an abelian group, and, given any morphism \( \varphi: G \to G' \) of algebraic groups, one has a pullback homomorphism

\[ \varphi^*: \mathcal{X}(G') \to \mathcal{X}(G), \quad \chi' \mapsto \chi' \circ \varphi. \]

**Definition 2.1.1.** A quasitorus is an affine algebraic group \( H \) whose algebra of regular functions \( \Gamma(H, \mathcal{O}) \) is generated as a \( \mathbb{K} \)-vector space by the characters \( \chi \in \mathcal{X}(H) \). A torus is a connected quasitorus.

**Example 2.1.2.** The standard n-torus \( \mathbb{T}^n := (\mathbb{K}^*)^n \) is a torus in the sense of\(^{2.1.1}\) Its characters are precisely the Laurent monomials \( T^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n} \), where \( \nu \in \mathbb{Z}^n \), and its algebra of regular functions is the Laurent polynomial algebra

\[ \Gamma(\mathbb{T}^n, \mathcal{O}) = \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] = \bigoplus_{\nu \in \mathbb{Z}^n} \mathbb{K} \cdot T^\nu = \mathbb{K}[\mathbb{Z}^n]. \]

We now associate to any finitely generated abelian group \( K \) in a functorial way a quasitorus, namely \( H := \text{Spec} \mathbb{K}[K] \); the construction will show that \( H \) is the direct product of a standard torus and a finite abelian group.

**Construction 2.1.3.** Let \( K \) be any finitely generated abelian group. Fix generators \( w_1, \ldots, w_r \) of \( K \) such that the epimorphism \( \pi: \mathbb{Z}^r \to K, \ e_i \mapsto w_i \) has the kernel

\[ \ker(\pi) = \mathbb{Z}a_1 e_1 + \ldots + \mathbb{Z}a_s e_s \]
with \( a_1, \ldots, a_s \in \mathbb{Z}_{\geq 1} \). Then we have the following exact sequence of abelian groups

\[
0 \rightarrow \mathbb{Z}^s \xrightarrow{e_i \mapsto a_i e_i} \mathbb{Z}^r \xrightarrow{e_i \mapsto w_i} K \rightarrow 0
\]

Passing to the respective spectra of group algebras we obtain with \( H := \text{Spec} \mathbb{K}[K] \) the following sequence of morphisms

\[
1 \leftarrow T^s \leftarrow T^r \leftarrow H \leftarrow 1
\]

The ideal of \( H \subseteq T^r \) is generated by \( T_a^i - 1 \), where \( 1 \leq i \leq s \). Thus \( H \) is a closed subgroup of \( T^r \) and the sequence is an exact sequence of quasitori; note that

\[
H \cong C(a_1) \times \cdots \times C(a_s) \times T^{r-s}, \quad C(a_i) := \{ \zeta \in \mathbb{K}^* ; \ z_a^i = 1 \}.
\]

The group structure on \( H = \text{Spec} \mathbb{K}[K] \) does not depend on the choices made: the multiplication map is given by its comorphism

\[
\mathbb{K}[K] \rightarrow \mathbb{K}[K] \otimes \mathbb{K}[K], \quad \chi^w \mapsto \chi^w \otimes \chi^w,
\]

and the neutral element of \( H = \text{Spec} \mathbb{K}[K] \) is the ideal \( \langle \chi^w - 1 ; \ w \in \mathbb{K} \rangle \). Moreover, every homomorphism \( \psi : K \rightarrow K' \) defines a morphism

\[
\text{Spec} \mathbb{K}[\psi] : \text{Spec} \mathbb{K}[K'] \rightarrow \text{Spec} \mathbb{K}[K].
\]

**Theorem 2.1.4.** We have contravariant exact functors being essentially inverse to each other:

\[
\{ \text{finitely generated abelian groups} \} \leftrightarrow \{ \text{quasitori} \}
\]

\[
K \mapsto \text{Spec} \mathbb{K}[K], \quad \psi \mapsto \text{Spec} \mathbb{K}[\psi],
\]

\[
X(H) \leftrightarrow H, \quad \varphi^* \leftrightarrow \varphi.
\]

Under these equivalences, the free finitely generated abelian groups correspond to the tori.

This statement includes in particular the observation that closed subgroups as well as homomorphic images of quasitori are again quasitori. Note that homomorphic images of tori are again tori, but every quasitorus occurs as a closed subgroup of a torus.

Recall that a rational representation of an affine algebraic group \( G \) is a morphism \( \rho : G \rightarrow \text{GL}(V) \) to the group \( \text{GL}(V) \) of linear automorphisms of a finite dimensional \( \mathbb{K} \)-vector space \( V \). In terms of representations, one has the following characterization of quasitori, see e.g. [140, Theorem 2.5.2].

**Proposition 2.1.5.** An affine algebraic group \( G \) is a quasitorus if and only if any rational representation of \( G \) splits into one-dimensional subrepresentations.

### 2.2. Affine quasitorus actions.

Again we work over an algebraically closed field \( \mathbb{K} \) of characteristic zero. Recall that one has contravariant equivalences between affine algebras, i.e. finitely generated \( \mathbb{K} \)-algebras without nilpotent elements, and affine varieties:

\[
A \mapsto \text{Spec} A, \quad X \mapsto \Gamma(X, \mathcal{O}).
\]

We first specialize these correspondences to graded affine algebras and affine varieties with quasitorus action; here “graded” means graded by a finitely generated abelian group. Then we look at basic concepts such as orbits and isotropy groups from both sides.

A \( G \)-variety is a variety \( X \) together with a morphical action \( G \times X \rightarrow X \) of an affine algebraic group \( G \). A morphism from a \( G \)-variety \( X \) to \( G' \)-variety \( X' \) is a pair...
subrepresentations, see \([126,239]\).\]

Construction 2.2.3.\]

Hsitorus \(X\) into a \(X\), the decomposition of \(\Gamma(X)\) to an \(H\)-algebra \(A\) associated \(\phi\), again this construction is functorial. If \((\psi, \tilde{\psi})\) from a \(K\)-graded affine algebra \(A\) to \(K'\)-graded affine algebra \(A'\), we have a morphism \((\varphi, \tilde{\varphi})\) from the associated \(H'\)-variety \(X'\) to the \(H\)-variety \(X\), where \(\varphi = \text{Spec} \psi\) and \(\tilde{\varphi} = \text{Spec} \tilde{\psi}\).

For the other way round, i.e., from affine varieties \(X\) with action of a quasitorus \(H\) to graded affine algebras, the construction relies on the fact that the representation of \(H\) on \(\Gamma(X, \mathcal{O})\) is rational, i.e., a union of finite dimensional rational subrepresentations, see [140 Proposition 2.3.4] and [95 Lemma 2.5] for non-affine \(X\). Proposition 2.1.3 then shows that it splits into one-dimensional subrepresentations.

Construction 2.2.3. Let a quasitorus \(H\) act on a not necessarily affine variety \(X\). Then \(\Gamma(X, \mathcal{O})\) becomes a rational \(H\)-module by

\[ (h \cdot f)(x) := f(h \cdot x). \]

The decomposition of \(\Gamma(X, \mathcal{O})\) into one-dimensional subrepresentations makes it into a \(\mathcal{X}(H)\)-graded algebra:

\[ \Gamma(X, \mathcal{O}) = \bigoplus_{\chi \in \mathcal{X}(H)} \Gamma(X, \mathcal{O})_{\chi}, \quad \Gamma(X, \mathcal{O})_{\chi} := \{ f \in \Gamma(X, \mathcal{O}); f(h \cdot x) = \chi(h)f(x) \}. \]

Again this construction is functorial. If \((\varphi, \tilde{\varphi})\) is a morphism from an \(H\)-variety \(X\) to an \(H'\)-variety \(X'\), then \((\varphi^*, \tilde{\varphi}^*)\) is a morphism of the associated graded algebras.
Theorem 2.2.4. We have contravariant functors being essentially inverse to each other:

\[
\begin{align*}
\{\text{graded affine algebras}\} & \leftrightarrow \{\text{affine varieties with quasitorus action}\} \\
A & \mapsto \text{Spec } A, \\
(\psi, \tilde{\psi}) & \mapsto (\text{Spec } \psi, \text{Spec } \mathbb{K}[\tilde{\psi}]).
\end{align*}
\]

Under these equivalences the graded homomorphisms correspond to the equivariant morphisms.

We use this equivalence of categories to describe some geometry of a quasitorus action in algebraic terms. The first basic observation is the following.

Proposition 2.2.5. Let \(A\) be a \(K\)-graded affine algebra and consider the action of \(H = \text{Spec } \mathbb{K}[K]\) on \(X = \text{Spec } A\). Then for any closed subvariety \(Y \subseteq X\) and its vanishing ideal \(I \subseteq A\), the following statements are equivalent.

(i) The variety \(Y\) is \(H\)-invariant.

(ii) The ideal \(I\) is homogeneous.

Moreover, if one of these equivalences holds, then one has a commutative diagram of \(K\)-graded homomorphisms

\[
\begin{array}{ccc}
\Gamma(X, \mathcal{O}) & \xrightarrow{\cong} & A \\
\downarrow{f \mapsto f|_Y} & & \downarrow{f \mapsto f+I} \\
\Gamma(Y, \mathcal{O}) & \cong & A/I
\end{array}
\]

We turn to orbits and isotropy groups. First recall the following fact on general algebraic group actions, see e.g. [88, Section II.8.3].

Proposition 2.2.6. Let \(G\) be an algebraic group, \(X\) a \(G\)-variety, and let \(x \in X\). Then the isotropy group \(G_x \subseteq G\) is closed, the orbit \(G \cdot x \subseteq X\) is locally closed, and one has a commutative diagram of equivariant morphisms of \(G\)-varieties

\[
\begin{array}{ccc}
G & \xrightarrow{g \mapsto g \cdot x} & G \cdot x \\
\downarrow{\pi} & & \downarrow{g \mapsto g \cdot x} \\
G/G_x & \cong & G \cdot x
\end{array}
\]

Moreover, the orbit closure \(\overline{G \cdot x}\) is the union of \(G \cdot x\) and orbits of strictly lower dimension and it contains a closed orbit.

Definition 2.2.7. Let \(A\) be a \(K\)-graded affine algebra and consider the action of \(H = \text{Spec } \mathbb{K}[K]\) on \(X = \text{Spec } A\).

(i) The orbit monoid of \(x \in X\) is the submonoid \(S_x \subseteq K\) generated by all \(w \in K\) that admit a function \(f \in A_w\) with \(f(x) \neq 0\).

(ii) The orbit group of \(x \in X\) is the subgroup \(K_x \subseteq K\) generated by the orbit monoid \(S_x \subseteq K\).
**Proposition 2.2.8.** Let \( A \) be a \( K \)-graded affine algebra, consider the action of \( H = \text{Spec} \mathbb{K}[K] \) on \( X = \text{Spec} A \) and let \( x \in X \). Then there is a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K_x & \rightarrow & K & \rightarrow & K/K_x & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \mathbb{X}(H/H_x) & \rightarrow & \mathbb{X}(H) & \rightarrow & \mathbb{X}(H_x) & \rightarrow & 0
\end{array}
\]

where \( \iota : H_x \rightarrow H \) denotes the inclusion of the isotropy group and \( \pi : H \rightarrow H/H_x \) the projection. In particular, we obtain \( H_x \cong \text{Spec} \mathbb{K}[K/K_x] \).

**Proof.** Replacing \( X \) with \( H \cdot x \) does not change \( K_x \). Moreover, take a homogeneous \( f \in A \) vanishing along \( H \cdot x \setminus H \cdot x \) but not at \( x \). Then replacing \( X \) with \( X_f \) does not affect \( K_x \). Thus, we may assume that \( X = H \cdot x \) holds. Then the weight monoid of the \( H \)-variety \( H \cdot x \) is \( K_x \) and by the commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\pi} & H/H_x \\
\downarrow & & \downarrow \cong \\
H \cdot x & \xrightarrow{\iota \circ h \cdot x} & H \cdot x
\end{array}
\]

we see that \( \pi^*(\mathbb{X}(H/H_x)) \) consists precisely of the characters \( \chi^w \) with \( w \in K_x \), which gives the desired diagram. \( \square \)

**Proposition 2.2.9.** Let \( A \) be a \( K \)-graded affine algebra, consider the action of \( H = \text{Spec} \mathbb{K}[K] \) on \( X = \text{Spec} A \) and let \( x \in X \). Then the orbit closure \( H \cdot x \) comes with an action of \( H/H_x \), and there is an isomorphism \( H \cdot x \cong \text{Spec} \mathbb{K}[S_x] \) of \( H/H_x \)-varieties.

**Proof.** Write for short \( Y := H \cdot x \) and \( V := H \cdot x \). Then \( V \subseteq Y \) is an affine open subset, isomorphic to \( H/H_x \), and we have a commutative diagram

\[
\begin{array}{cccc}
\Gamma(Y, \mathcal{O}) & \cong & \mathbb{K}[K_x] \\
\| & & \| \\
\Gamma(V, \mathcal{O}) & \cong & \mathbb{K}[S_x]
\end{array}
\]

of graded homomorphisms, where the horizontal arrows send a homogeneous \( f \) of degree \( w \) to \( f(x)\chi^w \). The assertion is part of this. \( \square \)

**Proposition 2.2.10.** Let \( A \) be an integral \( K \)-graded affine algebra and consider the action of \( H = \text{Spec} \mathbb{K}[K] \) on \( X = \text{Spec} A \). Then there is a nonempty invariant open subset \( U \subseteq X \) with

\[
S_x = S(A), \quad K_x = K(A) \quad \text{for all } x \in U.
\]

**Proof.** Let \( f_1, \ldots, f_r \) be homogeneous generators for \( A \). Then the set \( U \subseteq X \) obtained by removing the zero sets \( V(X, f_i) \) from \( X \) for \( i = 1, \ldots, r \) is as wanted. \( \square \)

Recall that an action of a group \( G \) on a set \( X \) is said to be effective if \( g \cdot x = x \) for all \( x \in X \) implies \( g = e_G \).

**Corollary 2.2.11.** Let \( A \) be an integral \( K \)-graded affine algebra and consider the action of \( H = \text{Spec} \mathbb{K}[K] \) on \( X = \text{Spec} A \). Then the action of \( H \) on \( X \) is effective if and only if \( K = K(A) \) holds.
2.3. **Good quotients.** We summarize the basic facts on good quotients. Everything takes place over an algebraically closed field $\mathbb{K}$ of characteristic zero. Besides varieties, we consider more generally possibly non-separated prevarieties. By definition, a $(\mathbb{K},\cdot)$-prevariety is a space $X$ with a sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued functions covered by open subspaces $X_1, \ldots, X_r$, each of which is an affine $(\mathbb{K},\cdot)$-variety.

Let an algebraic group $G$ act on a prevariety $X$, where, here and later, we always assume that this action is given by a morphism $G \times X \to X$. Recall that a morphism $\varphi: X \to Y$ is said to be $G$-invariant if it is constant along the orbits. Moreover, a morphism $\varphi: X \to Y$ is called affine if for any open affine $V \subseteq Y$ the preimage $\varphi^{-1}(V)$ is an affine variety. When we speak of a reductive algebraic group, we mean a not necessarily connected affine algebraic group $G$ such that every rational representation of $G$ splits into irreducible ones.

**Definition 2.3.1.** Let $G$ be a reductive algebraic group $G$ act on a prevariety $X$. A morphism $p: X \to Y$ of prevarieties is called a **good quotient** for this action if it has the following properties:

(i) $p: X \to Y$ is affine and $G$-invariant,

(ii) the pullback $p^*: \mathcal{O}_Y \to (p_*\mathcal{O}_X)^G$ is an isomorphism.

A morphism $p: X \to Y$ is called a **geometric quotient** if it is a good quotient and its fibers are precisely the $G$-orbits.

**Remark 2.3.2.** Let $X = \text{Spec} A$ be an affine $G$-variety with a reductive algebraic group $G$. The finiteness theorem of Classical Invariant Theory ensures that the algebra of invariants $A^G \subseteq A$ is finitely generated [100, Section II.3.2]. This guarantees existence of a good quotient $p: X \to Y$, where $Y := \text{Spec} A^G$. The notion of a good quotient is locally modeled on this concept, because for any good quotient $p': X' \to Y'$ and any affine open $V \subseteq Y'$ the variety $V$ is isomorphic to $\text{Spec} \Gamma(p'^{-1}(V), \mathcal{O})^G$, and the restricted morphism $p'^{-1}(V) \to V$ is the morphism just described.

**Example 2.3.3.** Consider the $\mathbb{K}^*$-action $t \cdot (z, w) = (t^a z, t^b w)$ on $\mathbb{K}^2$. The following three cases are typical.

(i) We have $a = b = 1$. Every $\mathbb{K}^*$-invariant function is constant and the constant map $p: \mathbb{K}^2 \to \{\text{pt}\}$ is a good quotient.

(ii) We have $a = 0$ and $b = 1$. The algebra of $\mathbb{K}^*$-invariant functions is generated by $z$ and the map $p: \mathbb{K}^2 \to \mathbb{K}, (z, w) \mapsto z$ is a good quotient.

(iii) We have $a = 1$ and $b = -1$. The algebra of $\mathbb{K}^*$-invariant functions is generated by $zw$ and $p: \mathbb{K}^2 \to \mathbb{K}, (z, w) \mapsto zw$ is a good quotient.
Note that the general $p$-fiber is a single $K^*$-orbit, whereas $p^{-1}(0)$ consists of three orbits and is reducible.

**Example 2.3.4.** Let $A$ be a $K$-graded affine algebra. Consider a homomorphism $\psi: K \to L$ of abelian groups and the coarsified grading

$$A = \bigoplus_{u \in L} A_u, \quad A_u = \bigoplus_{w \in \psi^{-1}(u)} A_w.$$ 

Then the diagonalizable group $H = \text{Spec} \mathbb{K}[L]$ acts on $X = \text{Spec} A$, and for the algebra of invariants we have

$$A^H = \bigoplus_{w \in \ker(\psi)} A_w.$$ 

Note that in this special case, Proposition 1.2.2 ensures finite generation of the algebra of invariants.

**Example 2.3.5 (Veronese subalgebras).** Let $A$ be a $K$-graded affine algebra and $L \subseteq K$ a subgroup. Then we have the corresponding Veronese subalgebra

$$B = \bigoplus_{w \in L} A_w \subseteq \bigoplus_{w \in K} A_w = A.$$ 

By the preceding example, the morphism $\text{Spec} A \to \text{Spec} B$ is a good quotient for the action of $\text{Spec} \mathbb{K}[K/L]$ on $\text{Spec} A$.

We list basic properties of good quotients. The key to most of the statements is the following central observation.

**Theorem 2.3.6.** Let a reductive algebraic group $G$ act on a prevariety $X$. Then any good quotient $p: X \to Y$ has the following properties.

(i) *G-closedness:* If $Z \subseteq X$ is $G$-invariant and closed, then its image $p(Z) \subseteq Y$ is closed.

(ii) *G-separation:* If $Z, Z' \subseteq X$ are $G$-invariant, closed and disjoint, then $p(Z)$ and $p(Z')$ are disjoint.

**Proof.** Since $p: X \to Y$ is affine and the statements are local with respect to $Y$, it suffices to prove them for affine $X$. This is done in [100] Section II.3.2], or [130] Theorems 4.6 and 4.7].

As an immediate consequence, one obtains basic information on the structure of the fibers of a good quotient.

**Corollary 2.3.7.** Let a reductive algebraic group $G$ act on a prevariety $X$, and let $p: X \to Y$ be a good quotient. Then $p$ is surjective and for any $y \in Y$ one has:

(i) There is exactly one closed $G$-orbit $G \cdot x$ in the fiber $p^{-1}(y)$.

(ii) Every orbit $G \cdot x' \subseteq p^{-1}(y)$ has $G \cdot x$ in its closure.

The first statement means that a good quotient $p: X \to Y$ parametrizes the closed orbits of the $G$-prevariety $X$. 
Corollary 2.3.8. Let a reductive algebraic group $G$ act on a prevariety $X$, and let $p: X \to Y$ be a good quotient.

(i) The quotient space $Y$ carries the quotient topology with respect to the map $p: X \to Y$.

(ii) For every $G$-invariant morphism of prevarieties $\varphi: X \to Z$, there is a unique morphism $\psi: Y \to Z$ with $\varphi = \psi \circ p$.

Proof. The first assertion follows from Theorem 2.3.6 (i). The second one follows from Corollary 2.3.7, Property 2.3.1 (ii) and the first assertion. □

A morphism $p: X \to Y$ with the last property is also called a categorical quotient. The fact that a good quotient is categorical implies in particular, that the good quotient space is unique up to isomorphy. This justifies the notation $X \to X//G$ for good and $X \to X/G$ for geometric quotients.

Proposition 2.3.9. Let a reductive algebraic group $G$ act on a prevariety $X$, and let $p: X \to Y$ be a good quotient.

(i) Let $V \subseteq Y$ be an open subset. Then the restriction $p^{-1}(V) \to V$ is a good quotient for the restricted $G$-action.

(ii) Let $Z \subseteq X$ be a closed $G$-invariant subset. Then the restriction $p: Z \to p(Z)$ is a good quotient for the restricted $G$-action.

Proof. The first statement is clear and the second one follows immediately from the corresponding statement on the affine case, see [100] Section II.3.2. □

Example 2.3.10 (The Proj construction). Let $A = \oplus A_d$ be a $\mathbb{Z}_{\geq 0}$-graded affine algebra. The irrelevant ideal in $A$ is defined as

$$A_{>0} := \{ f; f \in A_d \text{ for some } d > 0 \} \subseteq A.$$ 

For any homogeneous $f \in A_{>0}$ the localization $A_f$ is a $\mathbb{Z}$-graded affine algebra; concretely, the grading is given by

$$A_f = \bigoplus_{d \in \mathbb{Z}} (A_f)_d, \quad (A_f)_d := \{ h/f^l \in A_f; \deg(h) - l \deg(f) = d \}.$$ 

In particular, we have the, again finitely generated, degree zero part of $A_f$; it is given by

$$A(f) := (A_f)_0 = \{ h/f^l \in A_f; \deg(h) = l \deg(f) \}.$$ 

Set $X := \text{Spec}(A)$ and $Y_0 := \text{Spec}(A_0)$, and, for a homogeneous $f \in A_{>0}$, set $X_f := \text{Spec} A_f$ and $U_f := \text{Spec} A_f$. Then, for any two homogeneous $f, g \in A_{>0}$, we have the commutative diagrams

$$
\begin{array}{ccc}
A_f & \to & A_f g \\
\downarrow \quad \quad & \quad & \downarrow \\
A(f) & \to & A(f g)
\end{array}
\quad
\begin{array}{ccc}
X_f & \to & X_f g \\
\downarrow \quad \quad & \quad & \downarrow \\
U_f & \to & U_f g
\end{array}
$$

where the second one arises from the first one by applying the Spec-functor. The morphisms $U_{f g} \to U_f$ are open embeddings and gluing the $U_f$ gives the variety $Y = \text{Proj}(A)$. With the zero set $F := V(X, A_{>0})$ of the ideal $A_{>0}$, we have canonical morphisms, where the second one is projective:

$$X \setminus F \xrightarrow{\pi} Y \xrightarrow{\pi_0} Y_0.$$
Geometrically the following happened. The subset $F \subseteq X$ is precisely the fixed point set of the $\mathbb{K}^*$-action on $X$ given by the grading. Thus, $\mathbb{K}^*$ acts with closed orbits on $W := X \setminus F$. The maps $X_f \to U_f$ are geometric quotients, and glue together to a geometric quotient $\pi : W \to Y$. Moreover, the $\mathbb{K}^*$-equivariant inclusion $W \subseteq X$ induces the morphism of quotients $Y \to Y_0$.

3. Divisorial algebras

3.1. Sheaves of divisorial algebras. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We will not only deal with varieties over $\mathbb{K}$ but more generally with prevarieties.

Let $X$ be an irreducible prevariety. The group of Weil divisors of $X$ is the free abelian group $\text{WDiv}(X)$ generated by all prime divisors, i.e., irreducible subvarieties $D \subseteq X$ of codimension one. To a non-zero rational function $f \in \mathbb{K}(X)^*$ one associates a Weil divisor using its order along prime divisors $D$; recall that, if $f$ belongs to the local ring $\mathcal{O}_{X,D}$, then $\text{ord}_D(f)$ is the length of the $\mathcal{O}_{X,D}$-module $\mathcal{O}_{X,D}/(f)$, and otherwise one writes $f = g/h$ with $g, h \in \mathcal{O}_{X,D}$ and defines the order of $f$ to be the difference of the orders of $g$ and $h$. The divisor of $f \in \mathbb{K}(X)^*$ then is

$$\text{div}(f) := \sum_{D \text{ prime}} \text{ord}_D(f) \cdot D.$$ 

The assignment $f \mapsto \text{div}(f)$ is a homomorphism $\mathbb{K}(X)^* \to \text{WDiv}(X)$, and its image $\text{PDiv}(X) \subseteq \text{WDiv}(X)$ is called the subgroup of principal divisors. The divisor class group of $X$ is the factor group

$$\text{Cl}(X) := \text{WDiv}(X) / \text{PDiv}(X).$$

A Weil divisor $D = a_1D_1 + \ldots + a_sD_s$ with prime divisors $D_i$ is called effective, denoted as $D \geq 0$, if $a_i \geq 0$ holds for $i = 1, \ldots, s$. To every divisor $D \in \text{WDiv}(X)$, one associates a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by defining its sections over an open $U \subseteq X$ as

$$\Gamma(U, \mathcal{O}_X(D)) := \{ f \in \mathbb{K}(X)^* ; (\text{div}(f) + D)|_U \geq 0 \} \cup \{0\},$$

where the restriction map $\text{WDiv}(X) \to \text{WDiv}(U)$ is defined for a prime divisor $D$ as $D|_U := D \cap U$ if it intersects $U$ and $D|_U := 0$ otherwise. Note that for any two functions $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$ the product $f_1f_2$ belongs to $\Gamma(U, \mathcal{O}_X(D_1 + D_2))$.

**Definition 3.1.1.** The sheaf of divisorial algebras associated to a subgroup $K \subseteq \text{WDiv}(X)$ is the sheaf of $K$-graded $\mathcal{O}_X$-algebras

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D, \quad \mathcal{S}_D := \mathcal{O}_X(D),$$

where the multiplication in $\mathcal{S}$ is defined by multiplying homogeneous sections in the field of functions $\mathbb{K}(X)$.

**Example 3.1.2.** On the projective line $X = \mathbb{P}_1$, consider $D := \{\infty\}$, the group $K := \mathbb{Z}D$, and the associated $K$-graded sheaf of algebras $\mathcal{S}$. Then we have isomorphisms

$$\varphi_n : \mathbb{K}[T_0, T_1]_n \to \Gamma(\mathbb{P}_1, \mathcal{S}_nD), \quad f \mapsto f(1, z),$$

where $\mathbb{K}[T_0, T_1]_n \subseteq \mathbb{K}[T_0, T_1]$ denotes the vector space of all polynomials homogeneous of degree $n$. Putting them together we obtain a graded isomorphism

$$\mathbb{K}[T_0, T_1] \cong \Gamma(\mathbb{P}_1, \mathcal{S}).$$
Fix a normal (irreducible) prevariety $X$, a subgroup $K \subseteq \text{WDiv}(X)$ on the normal prevariety $X$ and let $S$ be the associated divisorial algebra. We collect first properties.

**Remark 3.1.3.** If $V \subseteq U \subseteq X$ are open subsets such that $U \setminus V$ is of codimension at least two in $U$, then we have an isomorphism $$\Gamma(U, S) \to \Gamma(V, S).$$ In particular, the algebra $\Gamma(U, S)$ equals the algebra $\Gamma(U_{\text{reg}}, S)$, where $U_{\text{reg}} \subseteq U$ denotes the set of smooth points.

**Remark 3.1.4.** Assume that $D_1, \ldots, D_s$ is a basis for $K \subseteq \text{WDiv}(X)$ and suppose that $U \subseteq X$ is an open subset on which each $D_i$ is principal, say $D_i = \text{div}(f_i)$. Then, with $\deg(T_i) = D_i$ and $f_i^{-1} \in \Gamma(X, S_{D_i})$, we have a graded isomorphism $$\Gamma(U, \mathcal{O}) \otimes_{K} \mathbb{K}[T_1^{\pm 1}, \ldots, T_s^{\pm 1}] \to \Gamma(U, S), \quad g \otimes T_1^{\nu_1} \cdots T_s^{\nu_s} \mapsto gf_1^{-\nu_1} \cdots f_s^{-\nu_s}.$$  

**Remark 3.1.5.** If $K$ is of finite rank, say $s$, then the algebra $\Gamma(X, S)$ of global sections can be realized as a graded subalgebra of the Laurent polynomial algebra $\mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_s^{\pm 1}]$. Indeed, let $D_1, \ldots, D_s$ be a basis for $K$. Then we obtain a monomorphism $$\Gamma(X, S) \to \mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_s^{\pm 1}], \quad \Gamma(X, S_{a_1D_1 + \cdots + a_sD_s}) \ni f \mapsto fT_1^{a_1} \cdots T_s^{a_s}.$$ In particular, $\Gamma(X, S)$ is an integral ring and we have an embedding of the associated quotient fields $$\text{Quot}(\Gamma(X, S)) \to \mathbb{K}(X)(T_1, \ldots, T_s).$$  

For quasaffine $X$, we have $\mathbb{K}(X) \subseteq \text{Quot}(\Gamma(X, S))$ and for each variable $T_i$ there is a non-zero function $f_i \in \Gamma(X, S_{D_i})$. Thus, for quasaffine $X$, one obtains $$\text{Quot}(\Gamma(X, S)) \cong \mathbb{K}(X)(T_1, \ldots, T_s).$$

The support $\text{Supp}(D)$ of a Weil divisor $D = a_1D_1 + \cdots + a_sD_s$ with prime divisors $D_i$ is the union of those $D_i$ with $a_i \neq 0$. Moreover, for a Weil divisor $D$ on a normal prevariety $X$ and a non-zero section $f \in \Gamma(X, \mathcal{O}_X(D))$, we define the $D$-divisor and the $D$-localization $$\text{div}_D(f) := \text{div}(f) + D \in \text{WDiv}(X), \quad X_{D,f} := X \setminus \text{Supp}(\text{div}_D(f)) \subseteq X.$$ The $D$-divisor is always effective. Moreover, given sections $f, g \in \Gamma(X, \mathcal{O}_X(E))$ and $g \in \Gamma(X, \mathcal{O}_X(E))$, we have $$\text{div}_{D+E}(fg) = \text{div}_D(f) + \text{div}_E(g), \quad f^{-1} \in \Gamma(X_{D,f}, \mathcal{O}_X(-D)).$$

**Remark 3.1.6.** Let $D \in K$ and consider a non-zero homogeneous section $f \in \Gamma(X, S_D)$, then one has a canonical isomorphism of $K$-graded algebras $$\Gamma(X, S_D, f) \cong \Gamma(X, S)_f.$$ Indeed, the canonical monomorphism $\Gamma(X, S)_f \to \Gamma(X_{D,f}, S)$ is surjective, because for any $g \in \Gamma(X_{D,f}, S)$, we have $gf^m \in \Gamma(X, S_{mD+E})$ with some $m \in \mathbb{Z}_{\geq 0}$.

**3.2. The relative spectrum.** Again we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $X$ be a normal prevariety. As any quasicoherent sheaf of $\mathcal{O}_X$-algebras, the sheaf of divisorial algebras $S$ associated to a group $K \subseteq \text{WDiv}(X)$ of Weil divisors defines in a natural way a geometric object, its relative spectrum $\bar{X} := \text{Spec}_X S$. We briefly recall how to obtain it.

**Construction 3.2.1.** Let $S$ be any quasicoherent sheaf of reduced $\mathcal{O}_X$-algebras on a prevariety $X$, and suppose that $S$ is locally of finite type, i.e., $X$ is covered by open affine subsets $X_1, \ldots, X_r \subseteq X$ with $\Gamma(X_i, S)$ finitely generated. Cover each intersection $X_{ij} := X_i \cap X_j$ by open subsets $(X_i)_{f_{ijk}}$, where $f_{ijk} \in \Gamma(X_i, \mathcal{O})$. Set
\( \tilde{X}_i := \text{Spec} \Gamma(X_i, S) \) and let \( \tilde{X}_{ij} \subseteq \tilde{X}_i \) be the union of the open subsets \((\tilde{X}_i)_{I_{ijk}} \). Then we obtain commutative diagrams

\[
\begin{array}{cccc}
\tilde{X}_i & \xleftarrow{\alpha} & \tilde{X}_{ij} & \xrightarrow{\beta} & \tilde{X}_j \\
\downarrow & & \downarrow & & \downarrow \\
X_i & \xleftarrow{\gamma} & X_{ij} & \xrightarrow{\delta} & X_j
\end{array}
\]

This allows us to glue together the \( \tilde{X}_i \) along the \( \tilde{X}_{ij} \), and we obtain a prevariety \( \tilde{X} = \text{Spec}_X S \) coming with a canonical morphism \( p: \tilde{X} \to X \). Note that \( p_* (\mathcal{O}_{\tilde{X}}) = S \) holds. In particular, \( \Gamma(\tilde{X}, \mathcal{O}) = \Gamma(X, S) \). Moreover, the morphism \( p \) is affine and \( \tilde{X} \) is separated if \( X \) is so. Finally, the whole construction does not depend on the choice of the \( X_i \).

Before specializing this construction to the case of our sheaf of divisorial algebras \( S \) on \( X \), we provide two criteria for \( S \) being locally of finite type. The first one is an immediate consequence of Remark 3.1.6.

**Proposition 3.2.2.** Let \( X \) be a normal prevariety, \( K \subseteq \text{WDiv}(X) \) a finitely generated subgroup, and \( S \) the associated sheaf of divisorial algebras. If \( \Gamma(X, S) \) is finitely generated and \( X \) is covered by affine open subsets of the form \( X_{D,f} \), where \( D \in K \) and \( f \in \Gamma(X, S_D) \), then \( S \) is locally of finite type.

A Weil divisor \( D \in \text{WDiv}(X) \) on a prevariety \( X \) is called Cartier if it is locally a principal divisor, i.e., locally of the form \( D = \text{div}(f) \) with a rational function \( f \). The prevariety \( X \) is locally factorial, i.e., all local rings \( \mathcal{O}_{X,x} \) are unique factorization domains if and only if every Weil divisor of \( X \) is Cartier. Recall that smooth prevarieties are locally factorial. More generally, a normal prevariety is called \( \mathbb{Q} \)-factorial if for any Weil divisor some positive multiple is Cartier.

**Proposition 3.2.3.** Let \( X \) be a normal prevariety and \( K \subseteq \text{WDiv}(X) \) a finitely generated subgroup. If \( X \) is \( \mathbb{Q} \)-factorial, then the associated sheaf \( S \) of divisorial algebras is locally of finite type.

**Proof.** By \( \mathbb{Q} \)-factoriality, the subgroup \( K^0 \subseteq K \) consisting of all Cartier divisors is of finite index in \( K \). Choose a basis \( D_1, \ldots, D_s \) for \( K \) such that with suitable \( a_i > 0 \) the multiples \( a_iD_i \), where \( 1 \leq i \leq s \), form a basis for \( K^0 \). Moreover, cover \( X \) by open affine subsets \( X_1, \ldots, X_r \subseteq X \) such that for any \( D \in K^0 \) all restrictions \( D|_{X_i} \) are principal. Let \( S^0 \) be the sheaf of divisorial algebras associated to \( K^0 \). Then \( \Gamma(X_i, S^0) \) is the Veronese subalgebra of \( \Gamma(X_i, S) \) defined by \( K^0 \subseteq K \). By Remark 3.1.3, the algebra \( \Gamma(X_i, S^0) \) is finitely generated. Since \( K^0 \subseteq K \) is of finite index, we can apply Proposition 3.1.4 and obtain that \( \Gamma(X_i, S) \) is finitely generated.

**Construction 3.2.4.** Let \( X \) be a normal prevariety, \( K \subseteq \text{WDiv}(X) \) a finitely generated subgroup and \( S \) the associated sheaf of divisorial algebras. We assume that \( S \) is locally of finite type. Then, in the notation of Construction 3.2.1, the algebras \( \Gamma(X_i, S) \) are \( K \)-graded. This means that each affine variety \( \tilde{X}_i \) comes with an action of the torus \( H := \text{Spec} \mathbb{K}[K] \), and, because of \( S_0 = \mathcal{O}_X \), the canonical map \( \tilde{X}_i \to X_i \) is a good quotient for this action. Since the whole gluing process is equivariant, we end up with an \( H \)-prevariety \( \tilde{X} = \text{Spec}_X S \) and \( p: \tilde{X} \to X \) is a good quotient for the \( H \)-action.

**Example 3.2.5.** Consider once more the projective line \( X = \mathbb{P}_1 \), the group \( K := \mathbb{Z}D \), where \( D := \{ \infty \} \), and the associated sheaf \( S \) of divisorial algebras. For the
affine charts $X_0 = K$ and $X_1 = K^* \cup \{\infty\}$ we have the graded isomorphisms
\[
\mathbb{K}[T_0^{\pm 1}, T_1] \to \Gamma(X_0, \mathcal{S}), \quad \mathbb{K}[T_0^{\pm 1}, T_1] \ni f \mapsto f(1, z) \in \Gamma(X_0, \mathcal{S}_D),
\]
\[
\mathbb{K}[T_0, T_1^{\pm 1}] \to \Gamma(X_1, \mathcal{S}), \quad \mathbb{K}[T_0, T_1^{\pm 1}] \ni f \mapsto f(z, 1) \in \Gamma(X_1, \mathcal{S}_D).
\]
Thus, the corresponding spectra are $\mathbb{K}^2_0$ and $\mathbb{K}^2_1$. The gluing takes place along $(\mathbb{K}^*)^2$ and gives $\tilde{X} = \mathbb{K}^2 \setminus \{0\}$. The action of $\mathbb{K}^* = \text{Spec} \mathbb{K}[K]$ on $\tilde{X}$ is the usual scalar multiplication.

The above example fits into the more general context of sheaves of divisorial algebras associated to groups generated by a very ample divisor, i.e., the pullback of a hyperplane with respect to an embedding into a projective space.

**Example 3.2.6.** Suppose that $X$ is projective and $K = \mathbb{Z}D$ holds with a very ample divisor $D$ on $X$. Then $\Gamma(X, \mathcal{S})$ is finitely generated and thus we have the affine cone $\overline{X} := \text{Spec} \Gamma(X, \mathcal{S})$ over $X$. It comes with a $\mathbb{K}^*$-action and an attractive fixed point $x_0 \in \overline{X}$, i.e., $x_0$ lies in the closure of any $\mathbb{K}^*$-orbit. The relative spectrum $\tilde{X} = \text{Spec} X \mathcal{S}$ equals $\overline{X} \setminus \{x_0\}$.

**Remark 3.2.7.** In the setting of 3.2.4, let $U \subseteq X$ be an open subset such that all divisors $D \in K$ are principal over $U$. Then there is a commutative diagram of $H$-equivariant morphisms
\[
p^{-1}(U) \xrightarrow{\cong} H \times U \xrightarrow{p} U \xrightarrow{pr_U} H \times U
\]
where $H$ acts on $H \times U$ by multiplication on the first factor. In particular, if $K$ consists of Cartier divisors, e.g. if $X$ is locally factorial, then $\tilde{X} \to X$ is a locally trivial $H$-principal bundle.

**Proposition 3.2.8.** Situation as in Construction 3.2.4. The prevariety $\tilde{X}$ is normal. Moreover, for any closed $A \subseteq X$ of codimension at least two, $p^{-1}(A) \subseteq \tilde{X}$ is as well of codimension at least two.

**Proof.** For normality, we have to show that for every affine open $U \subseteq X$ the algebra $\Gamma(p^{-1}(U), \mathcal{O})$ is a normal ring. According to Remark 3.1.3 we have
\[
\Gamma(p^{-1}(U), \mathcal{O}) = \Gamma(p^{-1}(U_{\text{reg}}), \mathcal{O}).
\]
Using Remark 3.1.4 we see that the latter ring is normal. The supplement is then an immediate consequence of Remark 3.1.3. □

### 3.3. Unique factorization in the global ring.
Here we investigate divisibility properties of the ring of global sections of the sheaf of divisorial algebras $\mathcal{S}$ associated to a subgroup $K \subseteq \text{WDiv}(X)$ on a normal prevariety $X$. The key statement is the following.

**Theorem 3.3.1.** Let $X$ be a smooth prevariety, $K \subseteq \text{WDiv}(X)$ a finitely generated subgroup, $\mathcal{S}$ the associated sheaf of divisorial algebras and $\tilde{X} = \text{Spec}_X \mathcal{S}$. Then the following statements are equivalent.

(i) The canonical map $K \to \text{Cl}(X)$ is surjective.

(ii) The divisor class group $\text{Cl}(\tilde{X})$ is trivial.
We need a preparing observation concerning the pullback of Cartier divisors. Recall that for any dominant morphism \( \varphi : X \to Y \) of normal prevarieties, there is a pullback of Cartier divisors: if a Cartier divisor \( E \) on \( Y \) is locally given as \( E = \text{div}(g) \), then the pullback divisor \( \varphi^*(E) \) is the Cartier divisor locally defined by \( \text{div}(\varphi^*(g)) \).

**Lemma 3.3.2.** Situation as in Construction [3.2.4]. Suppose that \( D \in \mathcal{K} \) is Cartier and consider a non-zero section \( f \in \Gamma(X, \mathcal{S}_D) \). Then one has

\[
p^*(D) = \text{div}(f) - p^*(\text{div}(f)),
\]

where on the right hand side \( f \) is firstly viewed as a homogeneous function on \( \tilde{X} \), and secondly as a rational function on \( X \). In particular, \( p^*(D) \) is principal.

**Proof.** On suitable open sets \( U_i \subseteq X \), we find defining equations \( f_i^{-1} \) for \( D \) and thus may write \( f = h_i f_i \), where \( h_i \in \Gamma(U_i, \mathcal{S}_0) = \Gamma(U_i, \mathcal{O}) \) and \( f_i \in \Gamma(U_i, \mathcal{S}_D) \). Then, on \( p^{-1}(U_i) \), we have \( p^*(h_i) = h_i \) and the function \( f_i \) is homogeneous of degree \( D \) and invertible. Thus, we obtain

\[
p^*(D) = p^*(\text{div}(f) + D) - p^*(\text{div}(f)) = p^*(\text{div}(h_i)) - p^*(\text{div}(f)) = \text{div}(h_i) - p^*(\text{div}(f)) = \text{div}(h_i f_i) - p^*(\text{div}(f)) = \text{div}(f) - p^*(\text{div}(f)).
\]

\( \square \)

We are almost ready for proving the Theorem. Recall that, given an action of an algebraic group \( G \) on a normal prevariety \( X \), we obtain an induced action of \( G \) on the group of Weil divisors by sending a prime divisor \( D \subseteq X \) to \( g \cdot D \subseteq X \). In particular, we can speak about invariant Weil divisors.

**Proof of Theorem 3.3.1.** Suppose that (i) holds. It suffices to show that every effective divisor \( \tilde{D} \) on \( \tilde{X} \) is principal. We work with the action of the torus \( H = \text{Spec} \mathbb{K}[K] \) on \( \tilde{X} \). Choosing an \( H \)-linearization of \( \tilde{D} \), see [95] Section 2.4, we obtain a representation of \( H \) on \( \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{D})) \) such that for any section \( \tilde{f} \in \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{D})) \) one has

\[
\text{div}_{\tilde{D}}(h \cdot \tilde{f}) = h \cdot \text{div}_{\tilde{D}}(\tilde{f}).
\]

Taking a non-zero \( \tilde{f} \), which is homogeneous with respect to this representation, we obtain that \( \tilde{D} \) is linearly equivalent to the \( H \)-invariant divisor \( \text{div}_{\tilde{D}}(\tilde{f}) \). This reduces the problem to the case of an invariant divisor \( \tilde{D} \); compare also [9] Theorem 4.2. Now, consider any invariant prime divisor \( \tilde{D} \) on \( \tilde{X} \). Let \( D := p(\tilde{D}) \) be the image under the good quotient \( p : \tilde{X} \to X \). Remark 3.2.4 gives \( \tilde{D} = p^*(D) \). By assumption, \( D \) is linearly equivalent to a divisor \( D' \in \mathcal{K} \). Thus, \( \tilde{D} \) is linearly equivalent to \( p^*(D') \), which in turn is principal by Lemma 3.3.2.

Now suppose that (ii) holds. It suffices to show that any effective \( D \in \text{WDiv}(X) \) is linearly equivalent to some \( D' \in \mathcal{K} \). The pullback \( p^*(D) \) is the divisor of some function \( f \in \Gamma(\tilde{X}, \mathcal{O}) \). We claim that \( f \) is \( K \)-homogeneous. Indeed

\[
F : H \times \tilde{X} \to \mathbb{K}, \quad (h, x) \mapsto f(h \cdot x)/f(x)
\]

is an invertible function. By Rosenlicht’s Lemma [96] Section 1.1, we have \( F(h, x) = \chi(h)g(x) \) with \( \chi \in \mathcal{X}(H) \) and \( g \in \Gamma(\tilde{X}, \mathcal{O}^*) \). Plugging \((1, x)\) into \( F \) yields
g = 1 and, consequently, \( f(h \cdot x) = \chi(h)f(x) \) holds. Thus, we have \( f \in \Gamma(X, S_{D'}) \) for some \( D' \in K \). Lemma 3.3.2 gives
\[
p^*(D) = \text{div}(f) = p^*(D') + p^*(\text{div}(f)),
\]
where in the last term, \( f \) is regarded as a rational function on \( X \). We conclude \( D = D' + \text{div}(f) \) on \( X \). In other words, \( D \) is linearly equivalent to \( D' \in K \).

As an immediate consequence, we obtain factoriality of the ring of global sections provided \( K \to \text{Cl}(X) \) is surjective, see also [28, 68 and 15].

**Theorem 3.3.3.** Let \( X \) be a normal prevariety, \( K \subseteq \text{WDiv}(X) \) a finitely generated subgroup and \( S \) the associated sheaf of divisorial algebras. If the canonical map \( K \to \text{Cl}(X) \) is surjective, then the algebra \( \Gamma(X, S) \) is a unique factorization domain.

**Proof.** According to Remark 3.1.3, the algebra \( \Gamma(X, S) \) equals \( \Gamma(X_{\text{reg}}, S) \) and thus we may apply Theorem 3.3.1.

Divisibility and primality in the ring of global sections \( \Gamma(X, S) \) can be characterized purely in terms of \( X \).

**Proposition 3.3.4.** Let \( X \) be a normal prevariety, \( K \subseteq \text{WDiv}(X) \) a finitely generated subgroup projecting onto \( \text{Cl}(X) \) and let \( S \) be the associated sheaf of divisorial algebras.

(i) An element \( 0 \neq f \in \Gamma(X, S_D) \) divides an element \( 0 \neq g \in \Gamma(X, S_E) \) if and only if \( \text{div}_D(f) \leq \text{div}_E(g) \) holds.

(ii) An element \( 0 \neq f \in \Gamma(X, S_D) \) is prime if and only if the divisor \( \text{div}_D(f) \in \text{WDiv}(X) \) is prime.

**Proof.** We may assume that \( X \) is smooth. Then \( \tilde{X} = \text{Spec}_X S \) exists, and Lemma 3.3.2 reduces (i) and (ii) to the corresponding statements on regular functions on \( \tilde{X} \), which in turn are well known.

### 3.4. Geometry of the relative spectrum.

We collect basic geometric properties of the relative spectrum of a sheaf of divisorial algebras. We will use the following pullback construction for Weil divisors.

**Remark 3.4.1.** Consider any morphism \( \varphi: \tilde{X} \to X \) of normal prevarieties such that the closure of \( X \setminus \varphi(\tilde{X}) \) is of codimension at least two in \( X \). Then we may define a pullback homomorphism for Weil divisors
\[
\varphi^*: \text{WDiv}(X) \to \text{WDiv}(\tilde{X})
\]
as follows: Given \( D \in \text{WDiv}(X) \), consider its restriction \( D' \) to \( X_{\text{reg}} \), the usual pullback \( \varphi^*(D') \) of Cartier divisors on \( \varphi^{-1}(X_{\text{reg}}) \) and define \( \varphi^*(D) \) to be the Weil divisor obtained by closing the support of \( \varphi^*(D') \). Note that we always have
\[
\text{Supp}(\varphi^*(D)) \subseteq \varphi^{-1}(\text{Supp}(D)).
\]

If for any closed \( A \subseteq X \) of codimension at least two, \( \varphi^{-1}(A) \subseteq \tilde{X} \) is as well of codimension at least two, then \( \varphi^* \) maps principal divisors to principal divisors, and we obtain a pullback homomorphism
\[
\varphi^*: \text{Cl}(X) \to \text{Cl}(\tilde{X}).
\]

**Example 3.4.2.** Consider \( X = V(\mathbb{K}^4; T_1T_2 - T_3T_4) \) and \( \tilde{X} = \mathbb{K}^4 \). Then we have a morphism
\[
p: \tilde{X} \to X, \quad z \mapsto (z_1z_2, z_3z_4, z_1z_3, z_2z_4).
\]

For the prime divisor \( D = \mathbb{K} \times 0 \times \mathbb{K} \times 0 \) on \( X \), we have
\[
\text{Supp}(p^*(D)) = V(\tilde{X}; Z_4) \subsetneq V(\tilde{X}; Z_4) \cup V(\tilde{X}; Z_2, Z_3) = p^{-1}(\text{Supp}(D)).
\]
In fact, \( \tilde{X} \to X \) is the morphism determined by the sheaf of divisorial algebras associated to \( K = \mathbb{Z}D \).

We say that a prevariety \( X \) is of affine intersection if for any two affine open subsets \( U, U' \subseteq X \) the intersection \( U \cap U' \) is again affine. For example, every variety is of affine intersection. Note that a prevariety \( X \) is of affine intersection if it can be covered by open affine subsets \( X_1, \ldots, X_s \subseteq X \) such that all intersections \( X_i \cap X_j \) are affine. Moreover, if \( X \) is of affine intersection, then the complement of any affine open subset \( U \subseteq X \) is of pure codimension one.

**Proposition 3.4.3.** In the situation of Proposition 3.4.1, consider the pullback homomorphism \( p^* : \text{WDiv}(X) \to \text{WDiv}(\tilde{X}) \) defined in (3.4.4). Then, for every \( D \in K \) and every non-zero \( f \in \Gamma(X, S_D) \), we have

\[
\text{div}(f) = p^*(\text{div}_D(f)),
\]

where on the left hand side \( f \) is a function on \( \tilde{X} \), and on the right hand side a function on \( X \). If \( X \) is of affine intersection and \( X_{D,f} \) is affine, then we have moreover

\[
\text{Supp}(\text{div}(f)) = p^{-1}(\text{Supp}(\text{div}_D(f))).
\]

**Proof.** By Lemma 3.3.2 the first equation holds on \( p^{-1}(X_{\text{reg}}) \). By Proposition 3.2.8 the complement \( \tilde{X} \setminus p^{-1}(X_{\text{reg}}) \) is of codimension at least two and thus the first equation holds on the whole \( \tilde{X} \). For the proof of the second one, consider \( X_{D,f} = X \setminus \text{Supp}(\text{div}_D(f)), \quad \tilde{X}_f = \tilde{X} \setminus V(\tilde{X}, f) \).

Then we have to show that \( p^{-1}(X_{D,f}) \) equals \( \tilde{X}_f \). Since \( f \) is invertible on \( p^{-1}(X_{D,f}) \), we obtain \( p^{-1}(X_{D,f}) \subseteq \tilde{X}_f \). Moreover, Lemma 3.3.2 yields

\[
p^{-1}(X_{D,f}) \cap p^{-1}(X_{\text{reg}}) = \tilde{X}_f \cap p^{-1}(X_{\text{reg}}).
\]

Thus the complement \( \tilde{X}_f \setminus p^{-1}(X_{D,f}) \) of the affine subset \( p^{-1}(X_{D,f}) \subseteq \tilde{X}_f \) is of codimension at least two. Since \( p : \tilde{X} \to X \) is affine, the prevariety \( \tilde{X} \) inherits the property to be of affine intersection from \( X \) and hence \( \tilde{X}_f \setminus p^{-1}(X_{D,f}) \) must be empty. \( \square \)

**Corollary 3.4.4.** Situation as in Construction 3.2.4. Let \( \bar{x} \in \tilde{X} \) be a point such that \( H \cdot \bar{x} \subseteq \tilde{X} \) is closed, and let \( 0 \neq f \in \Gamma(X, S_D) \). Then we have

\[
f(\bar{x}) = 0 \iff p(\bar{x}) \in \text{Supp}(\text{div}_D(f)).
\]

**Proof.** Remark 3.3.1 and Proposition 3.4.3 show that \( p(\text{Supp}(\text{div}(f))) \) is contained in \( \text{Supp}(\text{div}_D(f)) \). Moreover, they coincide along the smooth locus of \( X \) and Theorem 3.3.6 ensures that \( p(\text{Supp}(\text{div}(f))) \) is closed. This gives

\[
p(\text{Supp}(\text{div}(f))) = \text{Supp}(\text{div}_D(f)).
\]

Thus, \( f(\bar{x}) = 0 \) implies \( p(\bar{x}) \in \text{Supp}(\text{div}_D(f)) \). If \( p(\bar{x}) \in \text{Supp}(\text{div}_D(f)) \) holds, then some \( \bar{x}' \in \text{Supp}(\text{div}(f)) \) lies in the \( p \)-fiber of \( \bar{x} \). Since \( H \cdot \bar{x} \) is closed, it is contained in the closure of \( H \cdot \bar{x}' \), see Corollary 3.3.4. This implies \( \bar{x} \in \text{Supp}(\text{div}(f)) \). \( \square \)

**Corollary 3.4.5.** Situation as in Construction 3.2.4. If \( X \) is of affine intersection and covered by affine open subsets of the form \( X_{D,f} \), where \( D \in K \) and \( f \in \Gamma(X, S_D) \), then \( \tilde{X} \) is a quasiaffine variety.

**Proof.** According to Proposition 3.4.3 the prevariety \( \tilde{X} \) is covered by open affine subsets of the form \( \tilde{X}_f \) and thus is quasiaffine. \( \square \)

**Corollary 3.4.6.** Situation as in Construction 3.2.4. If \( X \) is of affine intersection and \( K \to \text{Cl}(X) \) is surjective, then \( \tilde{X} \) is a quasiaffine variety.
Cover $X$ by affine open sets $X_1, \ldots, X_r$. Since $X$ is of affine intersection, every complement $X \setminus X_i$ is of pure codimension one. Since $K \to \text{Cl}(X)$ is surjective, we obtain that $X \setminus X_i$ is the support of the $D$-divisor of some $f \in \Gamma(X, S_D)$. The assertion thus follows from Corollary 3.4.5. □

**Proposition 3.4.7. Situation as in Construction 3.2.4.** For $x \in X, \lim K^0_x \subseteq K$ be the subgroup of divisors that are principal near $x$ and let $\tilde{x} \in p^{-1}(x)$ be a point with closed $H$-orbit. Then the isotropy group $H_{\tilde{x}} \subseteq H$ is given by $H_{\tilde{x}} = \text{Spec} K[K/K^0_{\tilde{x}}]$.

Proof. Replacing $X$ with a suitable affine neighbourhood of $x$, we may assume that $\tilde{X}$ is affine. By Proposition 2.2.8, the isotropy group $H_{\tilde{x}}$ is isomorphic to $\text{Spec} K[K/K_{\tilde{x}}]$ with the orbit group

$$K_{\tilde{x}} = \langle D \in K; f(\tilde{x}) \neq 0 \text{ for some } f \in \Gamma(X, S_D) \rangle \subseteq K.$$ Using Corollary 3.4.4, we obtain that there exists an $f \in \Gamma(X, S_D)$ with $f(\tilde{x}) \neq 0$ if and only if $D \in K^P_{\tilde{x}}$ holds. The assertion follows. □

**Corollary 3.4.8. Situation as in Construction 3.2.4.**

(i) If $X$ is locally factorial, then $H$ acts freely on $\tilde{X}$.
(ii) If $X$ is $\mathbb{Q}$-factorial, then $H$ acts with at most finite isotropy groups on $\tilde{X}$.

4. Cox sheaves and Cox rings

**4.1. Free divisor class group.** As before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We introduce Cox sheaves and Cox rings for a prevariety with a free finitely generated divisor class group. As an example, we compute in 4.1.6 the Cox ring of a non-separated curve, the projective line with a free prevariety with a free finitely generated divisor class group. As an example, we compute in 4.1.6 the Cox ring of a non-separated curve, the projective line with a free finitely generated divisor class group.

**Construction 4.1.1.** Let $X$ be a normal prevariety with free finitely generated divisor class group $\text{Cl}(X)$. Fix a subgroup $K \subseteq \text{WDiv}(X)$ such that the canonical map $c: K \to \text{Cl}(X)$ sending $D \in K$ to its class $[D] \in \text{Cl}(X)$ is an isomorphism. We define the **Cox sheaf** associated to $K$ to be

$$\mathcal{R} := \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \text{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasicoherent sheaf of normal integral $\mathcal{O}_X$-algebras and, up to isomorphism, it does not depend on the choice of the subgroup $K \subseteq \text{WDiv}(X)$. The **Cox ring** of $X$ is the algebra of global sections

$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}(X), \quad \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{O}_X(D)).$$

**Proof of Construction 4.1.1.** Given two subgroups $K, K' \subseteq \text{WDiv}(X)$ projecting isomorphically onto $\text{Cl}(X)$, we have to show that the corresponding sheaves of divisorial algebras $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic. Choose a basis $D_1, \ldots, D_s$ for $K$ and define a homomorphism

$$\eta: K \to \mathbb{K}(X)^*, \quad a_1D_1 + \ldots + a_sD_s \mapsto f_1^{a_1} \cdots f_s^{a_s},$$

where $f_1, \ldots, f_s \in \mathbb{K}(X)^*$ are such that the divisors $D_i - \text{div}(f_i)$ form a basis of $K'$. Then we obtain an isomorphism $(\psi, \tilde{\psi})$ of the sheaves of divisorial algebras $\mathcal{R}$ and $\mathcal{R}'$ by setting

$$\tilde{\psi}: K \to K', \quad D \mapsto -\text{div}(\eta(D)) + D,$n(\psi, \tilde{\psi}) \psi: \mathcal{R} \to \mathcal{R}', \quad \Gamma(U, \mathcal{R}_{[D]}) \ni f \mapsto \eta(D)f \in \Gamma(U, \mathcal{R}_{[\tilde{\psi}(D)]}).$$
Example 4.1.2. Let $X$ be the projective space $\mathbb{P}_n$ and $D \subseteq \mathbb{P}_n$ be a hyperplane. The class of $D$ generates $\text{Cl}(\mathbb{P}_n)$ freely. We take $K$ as the subgroup of $\text{WDiv}(\mathbb{P}_n)$ generated by $D$, and the Cox ring $\mathcal{R}(\mathbb{P}_n)$ is the polynomial ring $\mathbb{K}[z_0, z_1, \ldots, z_n]$ with the standard grading.

Remark 4.1.3. If $X \subseteq \mathbb{P}_n$ is a closed normal subvariety whose divisor class group is generated by a hyperplane section, then $\mathcal{R}(X)$ coincides with $\Gamma(X, \mathcal{O})$, where $\overline{X} \subseteq \mathbb{K}^{n+1}$ is the cone over $X$ if and only if $X$ is projectively normal.

Remark 4.1.4. Let $s$ denote the rank of $\text{Cl}(X)$. Then Remark 3.1.6 realizes the Cox ring $\mathcal{R}(X)$ as a graded subring of the Laurent polynomial ring:

$$\mathcal{R}(X) \subseteq \mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_s^{\pm 1}].$$

Using the fact that there are $f \in \mathcal{R}_{|D|}(X)$ with $X_{D,f}$ affine and Remark 3.1.6, we see that this inclusion gives rise to an isomorphism of the quotient fields

$$\text{Quot}(\mathcal{R}(X)) \cong \mathbb{K}(X)(T_1, \ldots, T_s).$$

Proposition 4.1.5. Let $X$ be a normal prevariety with free finitely generated divisor class group.

(i) The Cox ring $\mathcal{R}(X)$ is a unique factorization domain.

(ii) The units of the Cox ring are given by $\mathcal{R}(X)^* = \Gamma(X, \mathcal{O}^*)$.

PROOF. The first assertion is a direct consequence of Theorem 3.3.3. To verify the second one, consider a unit $f \in \mathcal{R}(X)^*$. Then $fg = 1 \in \mathcal{R}_0(X)$ with some unit $g \in \mathcal{R}(X)^*$. This can only happen, when $f$ and $g$ are homogeneous, say of degree $|D|$ and $-|D|$, and thus we obtain

$$0 = \text{div}_0(1) = \text{div}_D(f) + \text{div}_{-D}(g) = (\text{div}(f) + D) + (\text{div}(g) - D).$$

Since the divisors $(\text{div}(f) + D)$ and $(\text{div}(g) - D)$ are effective, we conclude that $D = -\text{div}(f)$. This means $|D| = 0$ and we obtain $f \in \Gamma(X, \mathcal{O}^*)$. □

Example 4.1.6. Compare Section 2. Take the projective line $\mathbb{P}_1$, a tuple $A = (a_0, \ldots, a_r)$ of pairwise different points $a_i \in \mathbb{P}_1$ and a tuple $n = (n_0, \ldots, n_r)$ of integers $n_i \in \mathbb{Z}_{\geq 1}$. We construct a non-separated smooth curve $\mathbb{P}_1(A, n)$ mapping birationally onto $\mathbb{P}_1$ such that over each $a_i$ lie precisely $n_i$ points. Set

$$X_{ij} := \mathbb{P}_1 \setminus \bigcup_{k \neq i} a_k, \quad 0 \leq i \leq r, \quad 1 \leq j \leq n_i.$$

Gluing the $X_{ij}$ along the common open subset $\mathbb{P}_1 \setminus \{a_0, \ldots, a_r\}$ gives an irreducible smooth prevariety $\mathbb{P}_1(A, n)$ of dimension one. The inclusion maps $X_{ij} \to \mathbb{P}_1$ define a morphism $\pi: \mathbb{P}_1(A, n) \to \mathbb{P}_1$, which is locally an isomorphism. Writing $a_{ij}$ for the point in $\mathbb{P}_1(A, n)$ stemming from $a_i \in X_{ij}$, we obtain the fibre over any $a \in \mathbb{P}_1$ as

$$\pi^{-1}(a) = \begin{cases} \{a_{i1}, \ldots, a_{in_i}\} & a = a_i \text{ for some } 0 \leq i \leq r, \\ \{a\} & a \neq a_i \text{ for all } 0 \leq i \leq r. \end{cases}$$

We compute the divisor class group of $\mathbb{P}_1(A, n)$. Let $K'$ denote the group of Weil divisors on $\mathbb{P}_1(A, n)$ generated by the prime divisors $a_{ij}$. Clearly $K'$ maps onto the divisor class group. Moreover, the group of principal divisors inside $K'$ is

$$K_0' := K' \cap \text{PDiv}(\mathbb{P}_1(A, n)) = \left\{ \sum_{0 \leq i \leq r} c_ia_{ij} ; \ c_0 + \ldots + c_0 = 0 \right\}.$$
One directly checks that \( K' \) is the direct sum of \( K_0' \) and the subgroup \( K \subseteq K' \) generated by \( a_{01}, \ldots, a_{0n_0} \) and the \( a_{i1}, \ldots, a_{in_i} \). Consequently, the divisor class group of \( \mathbb{P}_1(A, n) \) is given by
\[
\text{Cl}(\mathbb{P}_1(A, n)) = \bigoplus_{j=1}^{n_0} \mathbb{Z} \cdot [a_{0j}] \oplus \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n_i-1} \mathbb{Z} \cdot [a_{ij}].
\]

We are ready to determine the Cox ring of the prevariety \( \mathbb{P}_1(A, n) \). For every \( 0 \leq i \leq r \), define a monomial
\[
T_i := T_{1i} \cdots T_{in_i} \in \mathbb{K}[T_{ij}; \; 0 \leq i \leq r, \; 1 \leq j \leq n_i].
\]
Moreover, for every \( a_i \in \mathbb{P}_1 \) fix a presentation \( a_i = [b_i, c_i] \) with \( b_i, c_i \in \mathbb{K} \) and for every \( 0 \leq i \leq r - 2 \) set \( k = j + 1 = i + 2 \) and define a trinomial
\[
g_i := (b_j c_k - b_k c_j)T_i + (b_k c_j - b_j c_k)T_j + (b_c c_j - b_j c_i)T_k.
\]
We claim that for \( r \leq 1 \) the Cox ring \( \mathcal{R}(\mathbb{P}_1(A, n)) \) is isomorphic to the polynomial ring \( \mathbb{K}[T_{ij}] \), and for \( r \geq 2 \) it has a presentation
\[
\mathcal{R}(\mathbb{P}_1(A, n)) \cong \mathbb{K}[T_{ij}; \; 0 \leq i \leq r, \; 1 \leq j \leq n_i] / \langle g_i; \; 0 \leq i \leq r - 2 \rangle,
\]
where, in both cases, the grading is given by \( \text{deg}(T_{ij}) = [a_{ij}] \). Note that all relations are homogeneous of degree
\[
\text{deg}(g_i) = [a_{i1} + \ldots + a_{in_i}] = [a_{01} + \ldots + a_{0n_0}].
\]

Let us verify this claim. Set for short \( X := \mathbb{P}_1(A, n) \) and \( Y := \mathbb{P}_1 \). Let \( K \subseteq W \text{Div}(X) \) be the subgroup generated by all \( a_{ij} \in X \) different from \( a_{1n_1}, \ldots, a_{rn_r} \), and let \( L \subseteq W \text{Div}(Y) \) be the subgroup generated by \( a_0 \in Y \). Then we may view the Cox rings \( \mathcal{R}(X) \) and \( \mathcal{R}(Y) \) as the rings of global sections of the sheaves of divisorial algebras \( \mathcal{S}_X \) and \( \mathcal{S}_Y \) associated to \( K \) and \( L \). The canonical morphism \( \pi: X \to Y \) gives rise to injective pullback homomorphisms
\[
\pi^*: L \to K, \quad \pi^*: \Gamma(Y, \mathcal{S}_Y) \to \Gamma(X, \mathcal{S}_X).
\]
For any divisor \( a_{ij} \in K \), let \( T_{ij} \in \Gamma(X, \mathcal{S}_X) \) denote its canonical section, i.e., the rational function \( 1 \in \Gamma(X, \mathcal{S}_X_{a_{ij}}) \). Moreover, let \([z, w]\) be the homogeneous coordinates on \( \mathbb{P}_1 \) and consider the sections
\[
S_i := \frac{b_i w - c_i z}{b_0 w - c_0 z} \in \Gamma(Y, \mathcal{S}_Y_{a_0}), \quad 0 \leq i \leq r.
\]
Finally, set \( d_{in_i} := a_{01} + \ldots + a_{0n_0} - a_{i1} - \ldots - a_{in_i} \leq K \) and define homogeneous sections
\[
T_{in_i} := \pi^* S_i(T_{i1} \cdots T_{in_i-1})^{-1} \in \Gamma(X, \mathcal{S}_{X_{d_{in_i}}}), \quad 1 \leq i \leq r.
\]
We show that the sections \( T_{ij} \), where \( 0 \leq i \leq r \) and \( 1 \leq j \leq n_i \), generate the Cox ring \( \mathcal{R}(X) \). Note that we have
\[
\text{div}_{a_{ij}}(T_{ij}) = a_{ij}, \quad \text{div}_{d_{in_i}}(T_{in_i}) = a_{in_i}.
\]
Consider \( D \subseteq K \) and \( h \in \Gamma(X, \mathcal{S}_D) \). If there occurs an \( a_{ij} \) in \( \text{div}_D(h) \), then we may divide \( h \) in \( \Gamma(X, \mathcal{S}) \) by the corresponding \( T_{ij} \), use Proposition \( 3.3.4 \) (i). Doing this as long as possible, we arrive at some \( h' \in \Gamma(X, \mathcal{S}_D) \) such that \( \text{div}_D(h') \) has no components \( a_{ij} \). But then \( D' \) is a pullback divisor and hence \( h' \) is contained in
\[
\pi^*(\Gamma(Y, \mathcal{S}_Y)) = \mathbb{K}[\pi^* S_0, \pi^* S_1] = \mathbb{K}[T_{01} \cdots T_{0n_0}, T_{11} \cdots T_{1n_1}].
\]
Finally, we have to determine the relations among the sections \( T_{ij} \in \Gamma(X, \mathcal{S}_X) \). For this, we first note that among the \( S_i \in \Gamma(Y, \mathcal{S}_Y) \) we have the relations
\[
(b_j c_k - b_k c_j)S_i + (b_k c_i - b_i c_k)S_j + (b_i c_j - b_j c_i)S_k = 0, \quad j = i + 1, \quad k = i + 2.
\]
Given any nontrivial homogeneous relation \( F = \alpha_1 F_1 + \ldots + \alpha_l F_l = 0 \) with \( \alpha_i \in \mathbb{K} \) and pairwise different monomials \( F_i \) in the \( T_{ij} \), we achieve by subtracting suitable multiples of pullbacks of the above relations a homogeneous relation

\[
F' = \alpha'_1 F'_1^{*} S_0^{k_1} \pi^{*} S_1^{l_1} + \ldots + \alpha'_m F'_m^{*} S_0^{k_m} \pi^{*} S_1^{l_m} = 0
\]

with pairwise different monomials \( F'_j \), none of which has any factor \( \pi^{*} S_i \). We show that \( F' \) must be trivial. Consider the multiplicative group \( M \) of Laurent monomials in the \( T_{ij} \) and the degree map

\[
M \to \mathbb{K}, \quad T_{ij} \mapsto \deg(T_{ij}) = \begin{cases} a_{ij}, & i = 0 \text{ or } j \leq n_i - 1, \\ d_{in_i}, & i \geq 1 \text{ and } j = n_i. \end{cases}
\]

The kernel of this degree map is generated by the Laurent monomials \( \pi^{*} S_0^{k_1} \pi^{*} S_1^{l_1} + \ldots + \alpha'_m \pi^{*} S_0^{k_m} \pi^{*} S_1^{l_m} = 0 \).

This relation descends to a relation in \( \Gamma(X, S_Y) \), which is the polynomial ring \( \mathbb{K}[S_0, S_1] \). Consequently, we obtain \( \alpha'_1 = \ldots = \alpha'_m = 0 \).

### 4.2. Torsion in the class group

Again we work over an algebraically closed field \( \mathbb{K} \) of characteristic zero. We extend the definition of Cox sheaf and Cox ring to normal prevarieties \( X \) having a finitely generated divisor class group \( \text{Cl}(X) \) with torsion. The idea is to take a subgroup \( K \subseteq \text{WDiv}(X) \) projecting onto \( \text{Cl}(X) \), to consider its associated sheaf of divisorial algebras \( S \) and to identify in a systematic manner homogeneous components \( S_D \) and \( S_{D'} \), whenever \( D \) and \( D' \) are linearly equivalent.

**Construction 4.2.1.** Let \( X \) be a normal prevariety with \( \Gamma(X, \mathcal{O}^*) = \mathbb{K}^* \) and finitely generated divisor class group \( \text{Cl}(X) \). Fix a subgroup \( K \subseteq \text{WDiv}(X) \) such that the map \( c: K \to \text{Cl}(X) \) sending \( D \in K \) to its class \([D] \in \text{Cl}(X)\) is surjective. Let \( K^0 \subseteq K \) be the kernel of \( c \), and let \( \chi: K^0 \to \mathbb{K}(X)^* \) be a character, i.e. a group homomorphism, with

\[
\text{div}(\chi(E)) = E, \quad \text{for all } E \in K^0.
\]

Let \( S \) be the sheaf of divisorial algebras associated to \( K \) and denote by \( \mathcal{I} \) the sheaf of ideals of \( S \) locally generated by the sections \( 1 - \chi(E) \), where 1 is homogeneous of degree zero, \( E \) runs through \( K^0 \) and \( \chi(E) \) is homogeneous of degree \( -E \). The **Cox sheaf** associated to \( K \) and \( \chi \) is the quotient sheaf \( \mathcal{R} := S/\mathcal{I} \) together with the \( \text{Cl}(X) \)-grading

\[
\mathcal{R} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \pi \left( \bigoplus_{D' \in c^{-1}([D])} S_{D'} \right).
\]

where \( \pi: S \to \mathcal{R} \) denotes the projection. The Cox sheaf \( \mathcal{R} \) is a quasicoherent sheaf of \( \text{Cl}(X) \)-graded \( \mathcal{O}_X \)-algebras. The **Cox ring** is the ring of global sections

\[
\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}(X), \quad \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{R}_{[D]}).
\]

For any open set \( U \subseteq X \), the canonical homomorphism \( \Gamma(U, S)/\Gamma(U, \mathcal{I}) \to \Gamma(U, \mathcal{R}) \) is an isomorphism. In particular, we have

\[
\mathcal{R}(X) \cong \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I}).
\]

All the claims made in this construction will be verified as separate Lemmata in the next subsection. The assumption \( \Gamma(X, \mathcal{O}^*) = \mathbb{K}^* \) is crucial for the following uniqueness statement on Cox sheaves and rings.
Proposition 4.2.2. Let \(X\) be a normal prevariety with \(\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*\) and finitely generated divisor class group \(\text{Cl}(X)\). If \(K, \chi\) and \(K', \chi'\) are data as in Construction 4.2.1 then there is a graded isomorphism of the associated Cox sheaves.

Also this will be proven in the next subsection. The construction of Cox sheaves (and thus also Cox rings) of a prevariety \(X\) can be made canonical by fixing a suitable point \(x \in X\).

Construction 4.2.3. Let \(X\) be a normal prevariety with \(\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*\) and finitely generated divisor class group \(\text{Cl}(X)\). Fix a point \(x \in X\) with factorial local ring \(\mathcal{O}_{X,x}\). For the subgroup

\[ K^x := \{ D \in \text{WDiv}(X); \ x \notin \text{Supp}(D) \} \]

let \(S^x\) be the associated sheaf of divisorial algebras and let \(K^{x,0} \subseteq K^x\) denote the subgroup consisting of principal divisors. Then, for each \(E \in K^{x,0}\), there is a unique section \(f_E \in \Gamma(X, S_{-E})\), which is defined near \(x\) and satisfies

\[ \text{div}(f_E) = E, \quad f_E(x) = 1. \]

The map \(\chi^x: K^x \to \mathbb{K}(X)^*\) sending \(E\) to \(f_E\) is a character as in Construction 4.2.1. We call the Cox sheaf \(\mathcal{R}^x\) associated to \(K^x\) and \(\chi^x\) the canonical Cox sheaf of the pointed space \((X, x)\).

Example 4.2.4 (An affine surface with torsion in the divisor class group). Consider the two-dimensional affine quadric

\[ X := V(\mathbb{K}^3; T_1T_2 - T_3^2) \subseteq \mathbb{K}^3. \]

We have the functions \(f_1 := T_2\) on \(X\) and with the prime divisors \(D_1 := V(X; f_1)\) and \(D_2 := V(X; f_2)\) on \(X\), we have

\[ \text{div}(f_1) = 2D_1, \quad \text{div}(f_2) = 2D_2, \quad \text{div}(f_3) = D_1 + D_2. \]

The divisor class group \(\text{Cl}(X)\) is of order two; it is generated by \([D_1]\). For \(K := \mathbb{Z}D_1\), let \(S\) denote the associated sheaf of divisorial algebras. Consider the sections

\[ g_1 := 1 \in \Gamma(X, S_{D_1}), \quad g_2 := f_3^{-1} \in \Gamma(X, S_{D_1}), \]

\[ g_3 := f_1^{-1} \in \Gamma(X, S_{2D_1}), \quad g_4 := f_1 \in \Gamma(X, S_{-2D_1}). \]

Then \(g_1, g_2\) generate \(\Gamma(X, S_{D_1})\) as a \(\Gamma(X, S_0)\)-module, and \(g_3, g_4\) are inverse to each other. Moreover, we have

\[ f_1 = g_1^2g_4, \quad f_2 = g_2^2g_4, \quad f_3 = g_1g_2g_4. \]

Thus, \(g_1, g_2, g_3\) and \(g_4\) generate the \(\mathbb{K}\)-algebra \(\Gamma(X, S)\). Setting \(\deg(Z_i) := \deg(g_i)\), we obtain a \(K\)-graded epimorphism

\[ \mathbb{K}[Z_1, Z_2, Z_3^{\pm 1}] \to \Gamma(X, S), \quad Z_1 \mapsto g_1, \ Z_2 \mapsto g_2, \ Z_3 \mapsto g_3, \]

which, by dimension reasons, is even an isomorphism. The kernel of the projection \(K \to \text{Cl}(X)\) is \(K^0 = 2\mathbb{Z}D_1\) and a character as in Construction 4.2.1 is

\[ \chi: K^0 \to \mathbb{K}(X)^*, \quad 2nD_1 \mapsto f_1^n. \]

The ideal \(\mathcal{I}\) is generated by \(1 - f_1\), where \(f_1 \in \Gamma(X, S_{-2D_1})\), see Remark 4.3.2 below. Consequently, the Cox ring of \(X\) is given as

\[ \mathcal{R}(X) \cong \Gamma(X, S)/\Gamma(X, \mathcal{I}) \cong \mathbb{K}[Z_1, Z_2, Z_3^{\pm 1}]/(1 - Z_1^{-1}) \cong \mathbb{K}[Z_1, Z_2], \]

where the \(\text{Cl}(X)\)-grading on the polynomial ring \(\mathbb{K}[Z_1, Z_2]\) is given by \(\deg(Z_1) = \deg(Z_2) = [D_1]\).
4.3. Well-definedness. Here we prove the claims made in Construction 4.2.1 and Proposition 4.2.2. In particular, we show that, up to isomorphy, Cox sheaf and Cox ring do not depend on the choices made in their construction.

**Lemma 4.3.1.** Situation as in Construction 4.2.1. Consider the \( \text{Cl}(X) \)-grading of the sheaf \( \mathcal{S} \) defined by

\[
\mathcal{S} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{S}_{[D]}, \quad \mathcal{S}_{[D]} := \bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'}.
\]

Given \( f \in \Gamma(U, \mathcal{I}) \) and \( D \in K \), the \( \text{Cl}(X) \)-homogeneous component \( f_{[D]} \in \Gamma(U, \mathcal{S}_{[D]}) \) of \( f \) has a unique representation

\[
f_{[D]} = \sum_{E \in K^0} (1 - \chi(E))f_E, \quad \text{where } f_E \in \Gamma(U, \mathcal{S}_D) \text{ and } \chi(E) \in \Gamma(U, \mathcal{S}_{-E}).
\]

In particular, the sheaf \( \mathcal{I} \) of ideals is \( \text{Cl}(X) \)-homogeneous. Moreover, if \( f \in \Gamma(U, \mathcal{I}) \) is \( K \)-homogeneous, then it is the zero section.

**Proof.** To obtain uniqueness of the representation of \( f_{[D]} \), observe that for every \( 0 \neq E \in K^0 \), the product \(-\chi(E)f_E\) is the \( K \)-homogeneous component of degree \( D - E \) of \( f_{[D]} \). We show existence. By definition of the sheaf of ideals \( \mathcal{I} \), every germ \( f_x \in \mathcal{I}_x \) can on a suitable neighbourhood \( U_x \) be represented by a section

\[
g = \sum_{E \in K^0} (1 - \chi(E))g_E, \quad \text{where } g_E \in \Gamma(U_x, \mathcal{S}).
\]

Collecting the \( \text{Cl}(X) \)-homogeneous parts on the right hand side represents the \( \text{Cl}(X) \)-homogeneous part \( h \in \Gamma(U_x, \mathcal{S}_{[D]}) \) of degree \( [D] \) of \( g \in \Gamma(U_x, \mathcal{S}) \) as follows:

\[
h = \sum_{E \in K^0} (1 - \chi(E))h_E, \quad \text{where } h_E \in \Gamma(U_x, \mathcal{S}_{[D]}).
\]

Note that we have \( h \in \Gamma(U_x, \mathcal{I}) \) and \( h \) represents \( f_{[D],x} \). Now, developing each \( h_E \in \Gamma(U_x, \mathcal{S}_{[D]}) \) according to the \( K \)-grading gives representations

\[
h_E = \sum_{D' \in D + K^0} h_{E,D'}, \quad \text{where } h_{E,D'} \in \Gamma(U_x, \mathcal{S}_{D'}).
\]

The section \( h'_{E,D'} := \chi(D - D)h_{E,D'} \) is \( K \)-homogeneous of degree \( D \), and we have the identity

\[
(1 - \chi(E))h_{E,D'} = (1 - \chi(E + D - D'))h'_{E,D'} - (1 - \chi(D - D'))h'_{E,D'}.
\]

Plugging this into the representation of \( h \) establishes the desired representation of \( f_{[D]} \) locally. By uniqueness, we may glue the local representations. \( \square \)

**Remark 4.3.2.** Situation as in Construction 4.2.1. Then, for any two divisors \( E, E' \in K^0 \), one has the identities

\[
1 - \chi(E + E') = (1 - \chi(E)) + (1 - \chi(E'))\chi(E),
\]

\[
1 - \chi(-E) = (1 - \chi(E))(-\chi(E)).
\]

Together with Lemma 1.3.1, this implies that for any basis \( E_1, \ldots, E_s \) of \( K^0 \) and any open \( U \subseteq X \), the ideal \( \Gamma(U, \mathcal{I}) \) is generated by \( 1 - \chi(E_i) \), where \( 1 \leq i \leq s \).

**Lemma 4.3.3.** Situation as in Construction 4.2.1. If \( f \in \Gamma(U, \mathcal{S}) \) is \( \text{Cl}(X) \)-homogeneous of degree \([D]\) for some \( D \in K \), then there is a \( K \)-homogeneous \( f' \in \Gamma(U, \mathcal{S}) \) of degree \( D \) with \( f - f' \in \Gamma(U, \mathcal{I}) \).
Proof. Writing the Cl(X)-homogeneous $f$ as a sum of $K$-homogeneous functions $f_{D'}$, we obtain the assertion by means of the following trick:

\[
   f = \sum_{D' \in D + K^0} f_{D'} = \sum_{D' \in D + K^0} \chi(D' - D)f_{D'} + \sum_{D' \in D + K^0} (1 - \chi(D' - D))f_{D'}.
\]

\[\square\]

**Lemma 4.3.4.** *Situation as in Construction 
[4.2.1]* Then, for every $D \in K$, we have an isomorphism of sheaves $\pi _{|S_D} : S_D \to \mathcal{R}_{[D]}$.

**Proof.** Lemma [4.3.1] shows that the homomorphism $\pi _{|S_D}$ is stalkwise injective and from Lemma [4.3.3] we infer that it is stalkwise surjective. \[\square\]

**Lemma 4.3.5.** *Situation as in Construction 
[4.2.1]* Then, for every open subset $U \subseteq X$, we have a canonical isomorphism

\[
   \Gamma(U, S)/\Gamma(U, \mathcal{I}) \cong \Gamma(U, S/\mathcal{I}).
\]

**Proof.** The canonical map $\psi : \Gamma(U, S)/\Gamma(U, \mathcal{I}) \to \Gamma(U, S/\mathcal{I})$ is injective. In order to see that it is as well surjective, let $h \in \Gamma(U, S/\mathcal{I})$ be given. Then there are a covering of $U$ by open subsets $U_i$ and sections $g_i \in \Gamma(U_i, S)$ such that $h|_{U_i} = \psi(g_i)$ holds and $g_i - g_j$ belongs to $\Gamma(U_i \cap U_j, \mathcal{I})$. Consider the Cl(X)-homogeneous parts $g_i|_{[D]} \in \Gamma(U_i, S_{[D]})$ of $g_i$. By Lemma [4.3.1] the ideal sheaf $\mathcal{I}$ is homogeneous and thus also $g_j|_{[D]} - g_i|_{[D]}$ belongs to $\Gamma(U_i \cap U_j, \mathcal{I})$. Moreover, Lemma [4.3.3] provides $K$-homogeneous $f_i|_{D}$ with $f_i|_{D} - g_i|_{[D]}$ in $\Gamma(U_i, \mathcal{I})$. The differences $f_j|_{D} - f_i|_{D}$ lie in $\Gamma(U_i \cap U_j, \mathcal{I})$ and hence, by Lemma [4.3.1], vanish. Thus, the $f_i|_{D}$ fit together to $K$-homogeneous sections $f_D \in \Gamma(U, S)$. By construction, $f = \sum f_D$ satisfies $\psi(f) = h$. \[\square\]

**Proof of Proposition [4.2.2]** In a first step, we reduce to Cox sheaves arising from finitely generated subgroups of $\text{WDiv}(X)$. So, let $K \subseteq \text{WDiv}(X)$ and $\chi : K^0 \to \mathbb{K}(X)^*$ be any data as in [4.2.1]. Choose a finitely generated subgroup $K_1 \subseteq K$ projecting onto $\text{Cl}(X)$. Restricting $\chi$ gives a character $\chi_1 : K_1^0 \to \mathbb{K}(X)^*$. The inclusion $K_1 \to K$ defines an injection $S_1 \to S$ sending the ideal $\mathcal{I}_1$ defined by $\chi_1$ to the ideal $\mathcal{I}$ defined by $\chi$. This gives a $\text{Cl}(X)$-graded injection $\mathcal{R}_1 \to \mathcal{R}$ of the Cox sheaves associated to $K_1, \chi_1$ and $K, \chi$ respectively. Lemma [4.3.3] shows that every $\text{Cl}(X)$-homogeneous section of $\mathcal{R}$ can be represented by a $K_1$-homogeneous section of $S$. Thus, $\mathcal{R}_1 \to \mathcal{R}$ is also surjective.

Next we show that for a fixed finitely generated $K \subseteq \text{WDiv}(X)$, any two characters $\chi, \chi' : K^0 \to \mathbb{K}(X)^*$ as in [4.2.1] give rise to isomorphic Cox sheaves $\mathcal{R}'$ and $\mathcal{R}$. For this note that the product $\chi^{-1}\chi'$ sends $K^0$ to $\Gamma(X, \mathcal{O}^*)$. Using $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, we may extend $\chi^{-1}\chi'$ to a homomorphism $\vartheta : K \to \Gamma(X, \mathcal{O}^*)$ and obtain a graded automorphism $(\alpha, \text{id})$ of $S$ by

\[
   \alpha_D : S_D \to S_D, \quad f \mapsto \vartheta(D)f.
\]

By construction, this automorphism sends the ideal $\mathcal{I}'$ to the ideal $\mathcal{I}$ and induces a graded isomorphism from $S/\mathcal{I}'$ onto $S/\mathcal{I}$.

Now consider two finitely generated subgroups $K, K' \subseteq \text{WDiv}(X)$ both projecting onto $\text{Cl}(X)$. Then we find a homomorphism $\tilde{\alpha} : K \to K'$ such that the following diagram is commutative
This homomorphism $\tilde{\alpha}: K \to K'$ must be of the form $\tilde{\alpha}(D) = D - \text{div}(\eta(D))$ with a homomorphism $\eta: K \to \mathbb{K}(X)^*$. Choose a character $\chi': K^0 \to \mathbb{K}(X)^*$ as in Lemma 4.3.4, we see that the induced homomorphism $\chi': K^0 \to \mathbb{K}(X)^*$, Thus, $D$ equals the divisor of the function $\chi(D) := \chi'((\tilde{\alpha}(D))\eta(D))$. This defines a character $\chi: K^0 \to \mathbb{K}(X)^*$. Altogether, we obtain a morphism $(\alpha, \tilde{\alpha})$ of the sheaves of divisorial algebras $\mathcal{S}$ and $\mathcal{S}'$ associated to $K$ and $K'$ by

$$
\alpha_D: S_D \to S'_{\tilde{\alpha}(D)}, \quad f \mapsto \eta(D)f.
$$

By construction, it sends the ideal $\mathcal{I}$ defined by $\chi$ to the ideal $\mathcal{I}'$ defined by $\chi'$. Using Lemma 4.3.4 we see that the induced homomorphism $\mathcal{R} \to \mathcal{R}'$ is an isomorphism on the homogeneous components and thus it is an isomorphism. □

4.4. Examples. For a normal prevariety $X$ with a free finitely generated divisor class group, we obtained in Proposition 4.4.3 that the Cox ring is a unique factorization domain having $\Gamma(X, \mathcal{O}^*)$ as its units. Here we provide two examples showing that these statements need not hold any more if there is torsion in the divisor class group. As usual, $\mathbb{K}$ is an algebraically closed field of characteristic zero.

Example 4.4.1 (An affine surface with non-factorial Cox ring). Consider the smooth affine surface

$$
Z := V(\mathbb{K}^3; T_1^2 - T_2 T_3 - 1).
$$

We claim that $\Gamma(Z, \mathcal{O}^*) = \mathbb{K}^*$ and $\text{Cl}(Z) \cong Z^\times$ hold. To see this, consider $f_1 := T_1|Z$ and the prime divisors

$$
D_+ := V(Z; f_1 - 1, f_2) = \{1\} \times \{0\} \times \mathbb{K},
$$

$$
D_- := V(Z; f_1 + 1, f_2) = \{-1\} \times \{0\} \times \mathbb{K}.
$$

Then we have $\text{div}(f_2) = D_+ + D_-$. In particular, $D_+$ is linearly equivalent to $-D_-$. Moreover, we have

$$
Z \setminus \text{Supp}(\text{div}(f_2)) = Z_{f_2} \cong \mathbb{K}^* \times \mathbb{K}.
$$

This gives $\Gamma(Z, \mathcal{O}^*) = \mathbb{K}^*$, and shows that $\text{Cl}(Z)$ is generated by the class $[D_+]$. Now suppose that $n[D_+] = 0$ holds for some $n > 0$. Then we have $n D_+ = \text{div}(f)$ with $f \in \Gamma(Z, \mathcal{O})$ and $f^n = fh$ holds with some $h \in \Gamma(Z, \mathcal{O})$ satisfying $\text{div}(h) = n D_-$. Look at the $\mathbb{Z}$-grading of $\Gamma(Z, \mathcal{O})$ given by

$$
\text{deg}(f_1) = 0, \quad \text{deg}(f_2) = 1, \quad \text{deg}(f_3) = -1.
$$

Any element of positive degree is a multiple of $f_2$. It follows that in the decomposition $f_2^n = fh$ one of the factors $f$ or $h$ must be a multiple of $f_2$, a contradiction. This shows that $\text{Cl}(Z)$ is freely generated by $[D_+]$.

Now consider the involution $Z \to Z$ sending $z$ to $-z$ and let $\pi: Z \to X$ denote the quotient of the corresponding free $\mathbb{Z}/2\mathbb{Z}$-action. We claim that $\text{Cl}(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by the class of $D := \pi(D_+)$. Indeed, the subset

$$
X \setminus \text{Supp}(D) = \pi(Z_{f_2}) \cong \mathbb{K}^* \times \mathbb{K}
$$

is factorial and $2D$ equals $\text{div}(f_2^2)$. Moreover, the divisor $D$ is not principal, because $\pi^*(D) = D_+ + D_-$ is not the divisor of a $\mathbb{Z}/2\mathbb{Z}$-invariant function on $Z$. This verifies our claim. Moreover, we have $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$.

In order to determine the Cox ring of $X$, take $K = ZD \subseteq \text{WDiv}(X)$, and let $\mathcal{S}$ denote the associated sheaf of divisorial algebras. Then, as $\Gamma(X, \mathcal{S}_0)$-modules, $\Gamma(X, \mathcal{S}_D)$ and $\Gamma(X, \mathcal{S}_{-D})$ are generated by the sections

$$
a_1 := 1, \quad a_2 := f_1 f_2^{-1}, \quad a_3 := f_2^{-1} f_3 \in \Gamma(X, \mathcal{S}_D),
$$

respectively.
Thus, using the fact that \( f_2^2 \) define invertible elements of degree \(-2D\), we see that \( a_1, a_2, a_3, b_1, b_2, b_3 \) generate the algebra \( \Gamma(X, S) \). Now, take the character \( \chi: K^0 \to \mathbb{K}(X)^* \) sending \( 2nD \to f_2^{-n} \). Then, by Remark 4.3.2, the associated ideal \( \Gamma(X, \mathcal{I}) \) is generated by \( 1 - f_2^2 \). The generators of the factor algebra \( \Gamma(X, S) / \Gamma(X, \mathcal{I}) \) are

\[
Z_1 = a_2 + \mathcal{I} = b_1 + \mathcal{I}, \quad Z_2 = a_1 + \mathcal{I} = b_2 + \mathcal{I}, \quad Z_3 = a_3 + \mathcal{I} = b_3 + \mathcal{I}.
\]

The defining relation is \( Z_1^2 - Z_2 Z_3 = 1 \). Thus the Cox ring \( \mathcal{R}(X) \) is isomorphic to \( \Gamma(Z, \mathcal{O}) \). In particular, it is not a factorial ring.

**Example 4.4.2** (A surface with only constant invertible functions but non-constant invertible elements in the Cox ring). Consider the affine surface

\[
X := V(\mathbb{K}^3; T_1 T_2 T_3 - T_1^2 - T_2^2 - T_3^2 + 4).
\]

This is the quotient space of the torus \( \mathbb{T}^2 := (\mathbb{K}^*)^2 \) with respect to the \( \mathbb{Z} / 2\mathbb{Z} \)-action defined by the involution \( t \mapsto t^{-1} \); the quotient map is explicitly given as

\[
\pi: \mathbb{T}^2 \to X, \quad t \mapsto (t_1 + t_1^{-1}, t_2 + t_2^{-1}, t_1 t_2 + t_1^{-1} t_2^{-1}).
\]

Since every \( \mathbb{Z} / 2\mathbb{Z} \)-invariant invertible function on \( \mathbb{T}^2 \) is constant, we have \( \Gamma(X, \mathcal{O}^*) = \mathbb{K}^* \). Moreover, using [96] Proposition 5.1, one verifies

\[
\text{Cl}(X) \cong \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z}, \quad \text{Pic}(X) = 0.
\]

Let us see that the Cox ring \( \mathcal{R}(X) \) has non-constant invertible elements. Set \( f_i := T_i | X \) and consider the divisors

\[
D_{\pm} := V(X; f_1 \pm 2, f_2 \pm 3), \quad D := D_+ + D_-. \]

Then, using the relations \((f_1 \pm 2)(f_2 f_3 - f_1 \pm 2) = (f_2 \pm f_3)^2\), one verifies \( \text{div}(f_1 \pm 2) = 2D_{\pm} \). Consequently, we obtain

\[
2D = \text{div}(f_1^2 - 4).
\]

Moreover, \( D \) is not principal, because otherwise \( f_1^2 - 4 \) must be a square and hence also \( \pi^*(f_1^2 - 4) \) is a square, which is impossible due to

\[
\pi^*(f_1^2 - 4) = t_1^2 + t_1^{-2} - 4 = (t_1 + t_1^{-1} + 2)(t_1 + t_1^{-1} - 2).
\]

Now choose Weil divisors \( D_i \) on \( X \) such that \( D, D_2, D_3 \) form a basis for a group \( K \subseteq \text{WDiv}(X) \) projecting onto \( \text{Cl}(X) \), and let \( S \) be the associated sheaf of divisorial algebras. As usual, let \( K^0 \subseteq K \) be the subgroup consisting of principal divisors and fix a character \( \chi: K^0 \to \mathbb{K}(X)^* \) with \( \chi(2D) = f_1^2 - 4 \). By Remark 4.3.2, the associated ideal \( \Gamma(X, \mathcal{I}) \) in \( \Gamma(X, S) \) is generated by

\[
1 - \chi(2D), \quad 1 - \chi(2D_2), \quad 1 - \chi(2D_3),
\]

where \( \chi(2D) = f_1^2 - 4 \) lives in \( \Gamma(X, S_{\leq 2D}) \). Now consider \( f_1 \in \Gamma(X, S_0) \) and the canonical section \( 1_D \in \Gamma(X, S_D) \). Then we have

\[
(f_1 + 1_D)(f_1 - 1_D) = f_1^2 - 1_D^2 = 4 - 1_D^2 \cdot (1 - \chi(2D)) \in \mathbb{K}^* + \Gamma(X, \mathcal{I}).
\]

Consequently, the section \( f_1 + 1_D \in \Gamma(X, S) \) defines a unit in \( \Gamma(X, \mathcal{R}) \). Note that \( f_1 + 1_D \) is not \( \text{Cl}(X) \)-homogeneous.
5. Algebraic properties of the Cox ring

5.1. Integrity and Normality. As before, we work over an algebraically closed field $K$ of characteristic zero. The following statement ensures in particular that the Cox ring is always a normal integral ring.

**Theorem 5.1.1.** Let $X$ be a normal prevariety with only constant invertible functions, finitely generated divisor class group, and Cox sheaf $\mathcal{R}$. Then, for every open $U \subseteq X$, the ring $\Gamma(U, \mathcal{R})$ is integral and normal.

The proof is based on the geometric construction 5.1.4 which is also used later and therefore occurs separately. We begin with two preparing observations.

**Lemma 5.1.2.** Situation as in Construction 4.2.1. For any two open subsets $V \subseteq U \subseteq X$ such that $U \setminus V$ is of codimension at least two in $U$, one has the restriction isomorphism

$$\Gamma(U, \mathcal{R}) \rightarrow \Gamma(V, \mathcal{R}).$$

In particular, the algebra $\Gamma(U, \mathcal{R})$ equals the algebra $\Gamma(U_{\text{reg}}, \mathcal{R})$, where $U_{\text{reg}} \subseteq U$ denotes the set of smooth points.

**Proof.** According to Remark 3.1.4 the restriction $\Gamma(U, \mathcal{S}) \rightarrow \Gamma(V, \mathcal{S})$ is an isomorphism. Lemma 4.3.1 ensures that $\Gamma(U, \mathfrak{I})$ is mapped isomorphically onto $\Gamma(V, \mathfrak{I})$ under this isomorphism. By Lemma 4.3.3 we have $\Gamma(U, \mathcal{R}) = \Gamma(U, \mathcal{S}) / \Gamma(U, \mathfrak{I})$ and $\Gamma(V, \mathcal{R}) = \Gamma(V, \mathcal{S}) / \Gamma(V, \mathfrak{I})$, which gives the assertion. \(\square\)

**Lemma 5.1.3.** Situation as in Construction 4.2.1. Then for every open $U \subseteq X$, the ideal $\Gamma(U, \mathfrak{I}) \subseteq \Gamma(U, \mathcal{S})$ is radical.

**Proof.** By Lemma 4.3.3 the ideal $\Gamma(U, \mathfrak{I})$ is radical if and only if the algebra $\Gamma(U, \mathcal{R})$ has no nilpotent elements. Proposition 4.2.2 thus allows us to assume that $\mathcal{S}$ arises from a finitely generated group $K$. Moreover, by Remark 3.1.3 we may assume that $X$ is smooth and it suffices to verify the assertion for affine $U \subseteq X$. We consider $\tilde{U} = \text{Spec} \Gamma(U, \mathcal{S})$ and the zero set $\tilde{U} \subseteq \tilde{U}$ of $\Gamma(U, \mathfrak{I})$. Note that $\tilde{U}$ is invariant under the action of the quasitorus $\tilde{H} = \text{Spec} K[\text{Cl}(X)]$ on $\tilde{U}$ given by the $\text{Cl}(X)$-grading.

Now, let $f \in \Gamma(U, \mathcal{S})$ with $f^n \in \Gamma(U, \mathfrak{I})$ for some $n > 0$. Then $f$ and thus also every $\text{Cl}(X)$-homogeneous component $f_{[D]}$ of $f$ vanishes along $\tilde{U}$. Consequently, $f^n_{[D]} \in \Gamma(U, \mathfrak{I})$ holds for some $m > 0$. By Lemma 4.3.3 we may write $f_{[D]} = f_D + g$ with $f_D \in \Gamma(U, \mathcal{S}_D)$ and $g \in \Gamma(U, \mathfrak{I})$. We obtain $f^n_D \in \Gamma(U, \mathfrak{I})$. By Lemma 4.3.1 this implies $f^n_D = 0$ and thus $f_D = 0$, which in turn gives $f_{[D]} \in \Gamma(U, \mathfrak{I})$ and hence $f \in \Gamma(U, \mathfrak{I})$. \(\square\)

**Construction 5.1.4.** Situation as in Construction 4.2.1. Assume that $K \subseteq \text{WDiv}(X)$ is finitely generated and $X$ is smooth. Consider $\hat{X} := \text{Spec}_X \mathcal{S}$ with the action of the torus $H := \text{Spec} K[K]$ and the geometric quotient $p: \hat{X} \rightarrow X$ as in Construction 3.2.4. Then, with $\tilde{X} := V(\mathfrak{I})$ and $H_X := \text{Spec} K[\text{Cl}(X)]$, we have a commutative diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{q_X} & X \\
\downarrow_{H_X} & & \downarrow_{H} \\
\tilde{X} & & \tilde{X}
\end{array}
\]

The prevariety $\tilde{X}$ is smooth, and, if $X$ is of affine intersection, then it is quasiaffine. The quasitorus $H_X \subseteq H$ acts freely on $\tilde{X}$ and $q_X: \tilde{X} \rightarrow X$ is a geometric quotient.
for this action; in particular, it is an étale \( H_X \)-principal bundle. Moreover, we have a canonical isomorphism of sheaves
\[
\mathcal{R} \cong (q_X)_*(\mathcal{O}\hat{\mathcal{X}}).
\]

**Proof.** With the restriction \( q_X: \hat{X} \to X \) of \( p: \hat{X} \to X \) we obviously obtain a commutative diagram as above. Moreover, Lemma \[5.1.3\] gives us \( \mathcal{R} \cong q_*(\mathcal{O}\hat{\mathcal{X}}) \).

Since the ideal \( \mathcal{I} \) is \( \text{Cl}(X) \)-homogeneous, the quasitorus \( H_X \subseteq H \) leaves \( \hat{X} \) invariant. Moreover, we see that \( q_X: \hat{X} \to X \) is a good quotient for this action, because we have the canonical isomorphisms
\[
(q_X)_*(\mathcal{O}\hat{\mathcal{X}})_0 \cong \mathcal{R}_0 \cong \mathcal{O}_X \cong S_0 \cong p_*(\mathcal{O}\hat{\mathcal{X}})_0.
\]

Freeness of the \( H_X \)-action on \( \hat{X} \) is due to the fact that \( H_X \) acts as a subgroup of the freely acting \( H \), see Remark \[3.4.3\]. As a consequence, we see that \( q_X: \hat{X} \to X \) is a geometric quotient. Luna’s Slice Theorem \[105\] gives commutative diagrams
\[
\begin{array}{ccc}
H_X \times S & \overset{q_X}{\longrightarrow} & \hat{X} \\
\downarrow{\text{pr}_S} & & \downarrow{q_X} \\
S & \overset{q_X}{\longrightarrow} & X
\end{array}
\]
where \( U \subseteq X \) are open sets covering \( X \) and the horizontal arrows are étale morphisms. By \[109\] Proposition 1.3.17, étale morphisms preserve smoothness and thus \( \hat{X} \) inherits smoothness from \( X \). If \( X \) is of affine intersection, then \( \hat{X} \) is quasi-affine, see Corollary \[3.4.6\] and thus \( \hat{X} \) is quasi-affine. \( \square \)

**Lemma 5.1.5.** Let \( \mathbb{L} \) be a field of characteristic zero containing all roots of unity, and assume that \( a \in \mathbb{L} \) is not a proper power. Then, for any \( n \in \mathbb{Z}_{\geq 1} \), the polynomial \( 1 - at^n \) is irreducible in \( \mathbb{L}[t,t^{-1}] \).

**Proof.** Over the algebraic closure of \( \mathbb{L} \) we have \( 1 - at^n = (1-a_1t) \cdots (1-a_nt) \), where \( a_i = a \) and any two \( a_i \) differ by a \( n \)-th root of unity. If \( 1 - at^n \) would split over \( \mathbb{L} \) non-trivially into \( h_1(t)h_2(t) \), then \( a^k \) must be contained in \( \mathbb{L} \) for some \( k < n \). But then also \( a^k \) lies in \( \mathbb{L} \) for the greatest common divisor \( d \) of \( n \) and \( k \). Thus \( a \) is a proper power, a contradiction. \( \square \)

**Proof of Theorem 5.1.1** According to Proposition \[4.2.2\] and Lemma \[5.1.2\] we may assume that we are in the setting of Construction \[5.1.4\] where it suffices to prove that \( \hat{X} \) is irreducible. Since \( q_X: \hat{X} \to X \) is surjective, some irreducible component \( \hat{X}_1 \subseteq \hat{X} \) dominates \( X \). We verify that \( \hat{X}_1 \) equals \( \hat{X} \) by checking that \( q_{\hat{X}}^{-1}(U) \) is irreducible for suitable open neighbourhoods \( U \subseteq X \) covering \( X \).

Let \( D_1, \ldots, D_s \) be a basis of \( K \) such that \( n_1D_1, \ldots, n_kD_k \), where \( 1 \leq k \leq s \), is a basis of \( K^0 \). Enlarging \( K \), if necessary, we may assume that the \( D_i \) are primitive, i.e., no proper multiples. We take subsets \( U \subseteq X \) such that on \( U \) every \( D_i \) is principal, say \( D_i = \text{div}(f_i) \). Then, with \( \text{deg}(D_i) := D_i \), Remark \[5.1.4\] provides a \( K \)-graded isomorphism
\[
\Gamma(U,\mathcal{O}) \otimes_K \mathbb{K}[T_1^{\pm 1}, \ldots, T_s^{\pm 1}] \cong \Gamma(U,S), \quad g \otimes T_1^{\nu_1} \cdots T_s^{\nu_s} \mapsto g f_1^{-\nu_1} \cdots f_s^{-\nu_s}.
\]
In particular, this identifies \( p^{-1}(U) \) with \( U \times \mathbb{T}^s \), where \( \mathbb{T}^s := (\mathbb{K}^*)^s \). According to Remark \[4.3.2\] the ideal \( \Gamma(U,\mathcal{I}) \) is generated by \( 1 - \chi(n_iD_i) \), where \( 1 \leq i \leq k \). Thus \( q_{\hat{X}}^{-1}(U) \) is given in \( U \times \mathbb{T}^s \) by the equations
\[
1 - \chi(n_iD_i)f_i^{n_iT_i} = 0, \quad 1 \leq i \leq k.
\]
To obtain irreducibility of $q_X^{-1}(U)$, it suffices to show that each $1 - \chi(n_iD_i) f_i^n T_i^{n_i}$ is irreducible in $\mathbb{K}(X)[T_i^{\pm 1}]$. With respect to the variable $S_i := f_i T_i$, this means to verify irreducibility of

$$1 - \chi(n_iD_i) S_i^{n_i} \in \mathbb{K}(X)[S_i^{\pm 1}].$$

In view of Lemma 4.3.3, we have to show that $\chi(n_iD_i)$ is not a proper power in $\mathbb{K}(X)$. Assume the contrary. Then we obtain $n_iD_i = k_i \text{div}(h_i)$ with some $h_i \in \mathbb{K}(X)$. Since $D_i$ is primitive, $k_i$ divides $n_i$ and thus, $n_i/k_i D_i$ is principal. A contradiction to the choice of $n_i$.

The fact that each ring $\Gamma(U; \mathcal{R})$ is normal follows directly from the fact that it is the ring of functions of an open subset of the smooth prevariety $\tilde{X}$. □

### 5.2. Localization and units.

We treat localization by homogeneous elements and consider the units of the Cox ring $\mathcal{R}(X)$ of a normal prevariety $X$ defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The main tool is the divisor of a homogeneous element of $\mathcal{R}(X)$, which we first define precisely.

In the setting of Construction 4.2.1, consider a divisor $D \in K$ and a non-zero element $f \in \mathcal{R}_{[D]}(X)$. According to Lemma 4.3.3 there is a (unique) element $\tilde{f} \in \Gamma(X, \mathcal{S}_D)$ with $\pi(\tilde{f}) = f$, where $\pi: S \to \mathcal{R}$ denotes the projection. We define the $[D]$-divisor of $f$ to be the effective Weil divisor

$$\text{div}_{[D]}(f) := \text{div}_D(\tilde{f}) = \text{div}(\tilde{f}) + D \in \text{WDiv}(X).$$

**Lemma 5.2.1.** The $[D]$-divisor depends neither on representative $D \in K$ nor on the choices made in 4.2.1. Moreover, the following holds.

1. For every effective $E \in \text{WDiv}(X)$ there are $[D] \in \text{Cl}(X)$ and $f \in \mathcal{R}_{[D]}(X)$ with $E = \text{div}_{[D]}(f)$.
2. Let $[D] \in \text{Cl}(X)$ and $0 \neq f \in \mathcal{R}_{[D]}(X)$. Then $\text{div}_{[D]}(f) = 0$ implies $[D] = 0$ in $\text{Cl}(X)$.
3. For any two non-zero homogeneous elements $f \in \mathcal{R}_{[D_1]}(X)$ and $g \in \mathcal{R}_{[D_2]}(X)$, we have

$$\text{div}_{[D_1] + [D_2]}(fg) = \text{div}_{[D_1]}(f) + \text{div}_{[D_2]}(g).$$

**Proof.** Let $f \in \mathcal{R}_{[D]}(X)$, consider any two isomorphisms $\varphi_1: \mathcal{O}_X(D_i) \to \mathcal{R}_{[D]}$ and let $\tilde{f}_i$ be the sections with $\varphi_1(\tilde{f}_i) = f$. Then $\varphi_2^{-1} \circ \varphi_1$ is multiplication with some $h \in \mathbb{K}(X)^*$ satisfying $\text{div}(h) = D_1 - D_2$. Well-definedness of the $[D]$-divisor thus follows from

$$\text{div}_{D_1}(\tilde{f}_1) = \text{div}(h\tilde{f}_1) + D_2 = \text{div}_{D_2}(\tilde{f}_2).$$

If $\text{div}_{[D]}(f) = 0$ holds as in (ii), then, for a representative $\tilde{f} \in \Gamma(X, \mathcal{O}_X(D))$ of $f \in \mathcal{R}_{[D]}(X)$, we have $\text{div}_D(\tilde{f}) = 0$ and hence $D$ is principal. Observations (i) and (iii) are obvious. □

For every non-zero homogeneous element $f \in \mathcal{R}_{[D]}(X)$, we define the $[D]$-localization of $X$ by $f$ to be the open subset

$$X_{[D], f} := X \setminus \text{Supp}(\text{div}_{[D]}(f)) \subseteq X.$$

**Proposition 5.2.2.** Let $X$ be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox ring $\mathcal{R}(X)$. Then, for every non-zero homogeneous $f \in \mathcal{R}_{[D]}(X)$, we have a canonical isomorphism

$$\Gamma(X_{[D], f}; \mathcal{R}) \cong \Gamma(X, \mathcal{R})_f.$$
Proposition 5.2.3. Let $X$ be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox ring $\mathcal{R}(X)$.

(i) Every homogeneous invertible element of $\mathcal{R}(X)$ is constant.

(ii) If $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds, then every invertible element of $\mathcal{R}(X)$ is constant.

Proof. For (i), let $f \in \mathcal{R}(X)^*$ be homogeneous of degree $[D]$. Then its inverse $g \in \mathcal{R}(X)^*$ is homogeneous of degree $-[D]$, and $fg = 1$ lies in $\mathcal{R}(X)^*_0 = \mathbb{K}^*$. By Lemma 5.2.1 (iii), we have
\[ 0 = \text{div}_0(fg) = \text{div}_{[D]}(f) + \text{div}_{[-D]}(g). \]
Since the divisors $\text{div}_{[D]}(f)$ and $\text{div}_{[-D]}(g)$ are effective, they both vanish. Thus, Lemma 5.2.1 (ii) yields $[D] = 0$. This implies $f \in \mathcal{R}(X)^*_0 = \mathbb{K}^*$ as wanted.

For (ii), we have to show that any invertible $f \in \mathcal{R}(X)$ is of degree zero. Choose a decomposition $\text{Cl}(X) = K_0 \oplus K_t$ into a free part and the torsion part, and consider the coarsified grading $\mathcal{R}(X) = \bigoplus_{w \in K_0} R_w$, where $R_w := \bigoplus_{u \in K_t} \mathcal{R}(X)_{w+u}$.

Then, as any invertible element of the $K_0$-graded integral ring $\mathcal{R}(X)$, also $f$ is necessarily $K_0$-homogeneous of some degree $w \in K_0$. Decomposing $f$ and $f^{-1}$ into $\text{Cl}(X)$-homogeneous parts we get representations
\[ f = \sum_{u \in K_t} f_{w+u}, \quad f^{-1} = \sum_{u \in K_t} f_{-w+u}^{-1}. \]
Because of $ff^{-1} = 1$, we have $f_{w+u}f_{-w+u}^{-1} \neq 0$ for at least one $v \in K_t$. Since $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds, $f_{w+u}f_{-w+u}^{-1}$ must be a non-zero constant. Using Lemma 5.2.1 we conclude $w + v = 0$ as before. In particular, $w = 0$ holds and thus each $f_{w+u}$ has a torsion degree. For a suitable power $f_{w+u}^n$ we have $n\text{div}_{w+u}(f_{w+u}) = 0$, which implies $f_{w+u} = 0$ for any $u \neq 0$.

Remark 5.2.4. The affine surface $X$ treated in Example 4.4.2 shows that requiring $\Gamma(X, \mathcal{O})^* = \mathbb{K}^*$ is in general not enough in order to ensure that all units of the Cox ring are constant.

5.3. Divisibility properties. For normal prevarieties $X$ with a free finitely generated divisor class group, we saw that the Cox ring admits unique factorization. If we have torsion in the divisor class group this does not need to hold any more. However, restricting to homogeneous elements leads to a framework for a reasonable divisibility theory; the precise notions are the following.

Definition 5.3.1. Consider an abelian group $K$ and a $K$-graded integral $\mathbb{K}$-algebra $R = \bigoplus_{w \in K} R_w$.

(i) A non-zero non-unit $f \in R$ is $K$-prime if it is homogeneous and $f|gh$ with homogeneous $g, h \in R$ implies $f|g$ or $f|h$.

(ii) We say that $R$ is factorially graded if every homogeneous non-zero non-unit $f \in R$ is a product of $K$-primes.
(iii) An ideal \( a \triangleleft R \) is \( K \)-prime if it is homogeneous and for any two homogeneous \( f, g \in R \) with \( fg \in a \) one has either \( f \in a \) or \( g \in a \).

(iv) A \( K \)-prime ideal \( a \triangleleft R \) has \( K \)-height \( d \) if \( d \) is maximal admitting a chain \( a_0 \subset a_1 \subset \ldots \subset a_d = a \) of \( K \)-prime ideals.

Let us look at these concepts also from the geometric point of view. Consider a prevariety \( Y \) with an action of an algebraic group \( H \). Then \( H \) acts also on the group \( \text{WDiv}(Y) \) of Weil divisors via

\[
h \cdot \sum a_D D := \sum a_D(h \cdot D).
\]

By an \( H \)-prime divisor we mean a non-zero sum \( \sum a_D D \) with prime divisors \( D \) such that \( a_D \in \{0,1\} \) always holds and the \( D \) with \( a_D = 1 \) are transitively permuted by \( H \). Note that every \( H \)-invariant divisor is a unique sum of \( H \)-prime divisors. We say that \( Y \) is \( H \)-factorial if every \( H \)-invariant Weil divisor on \( Y \) is principal.

**Proposition 5.3.2.** Let \( H = \text{Spec} \mathbb{K}[K] \) be a quasitorus and \( W \) an irreducible normal quasiaffine \( H \)-variety. Consider the \( K \)-graded algebra \( R := \Gamma(W,\mathcal{O}) \) and assume \( R^* = \mathbb{K}^* \). Then the following statements are equivalent.

(i) Every \( K \)-prime ideal of \( K \)-height one in \( R \) is principal.

(ii) The variety \( W \) is \( H \)-factorial.

(iii) The ring \( R \) is factorially graded.

Moreover, if one of these statements holds, then a homogeneous non-zero non-unit \( f \in R \) is \( K \)-prime if and only if the divisor \( \text{div}(f) \) is \( H \)-prime, and every \( H \)-prime divisor is of the form \( \text{div}(f) \) with a \( K \)-prime \( f \in R \).

**Proof.** Assume that (i) holds and let \( D \) be an \( H \)-invariant Weil divisor on \( W \). Write \( D = D_1 + \ldots + D_r \) with \( H \)-prime divisors \( D_i \). Then the vanishing ideal \( a_i \) of \( D_i \) is of \( K \)-height one, and (i) guarantees that it is principal, say \( a_i = (f_i) \). Thus \( D_i = \text{div}(f_i) \) and \( D = \text{div}(f_1 \cdots f_r) \) hold, which proves (ii).

Assume that (ii) holds. Given a homogeneous element \( 0 \neq f \in R \setminus R^* \), write \( \text{div}(f) = D_1 + \ldots + D_r \) with \( H \)-prime divisors \( D_i \). Then \( D_i = \text{div}(f_i) \) holds, where, because of \( R^* = \mathbb{K}^* \), the elements \( f_i \) are homogeneous. One verifies directly that the \( f_i \) are \( K \)-prime. Thus we have \( f = \alpha f_1 \cdots f_r \) with \( \alpha \in \mathbb{K}^* \) as required in (iii).

If (iii) holds and \( a \) is a \( K \)-prime ideal of \( K \)-height one, then we take any homogeneous \( 0 \neq f \in a \) and find a \( K \)-prime factor \( f_1 \) of \( f \) with \( f_1 \in a \). This gives inclusions \( 0 \subsetneq (f_1) \subseteq a \) of \( K \)-prime ideals, which implies \( a = (f_1) \).

**Corollary 5.3.3.** Under the assumptions of Proposition 5.3.2 factoriality of the algebra \( R \) implies that it is factorially graded.

We are ready to study the divisibility theory of the Cox ring. Here comes the main result; it applies in particular to complete varieties, see Corollary 5.3.8.

**Theorem 5.3.4.** Let \( X \) be an irreducible normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. If the Cox ring \( \mathcal{R}(X) \) satisfies \( \mathcal{R}(X)^* = \mathbb{K}^* \), then it is factorially graded.

**Lemma 5.3.5.** In the situation of Construction 5.1.4, every non-zero element \( f \in \Gamma(X,\mathcal{R}^D) \) satisfies

\[
\text{div}(f) = q_X^*(\text{div}_D(f)),
\]

where on the left hand side \( f \) is a regular function on \( \tilde{X} \) and on the right hand side \( f \) is an element on \( \mathcal{R}(X) \).
5. Algebraic Properties of the Cox Ring

Proof. In the notation of 5.1.4 let $D \in K$ represent $|D| \in \text{Cl}(X)$, and let $\tilde{f} \in \Gamma(X, S_D)$ project to $f \in \Gamma(X, R_{|D|})$. The commutative diagram of 5.1.4 yields

$$\text{div}(f) = \iota^*(\text{div}(\tilde{f})) = \iota^*(p^*(\text{div}_D(\tilde{f}))) = q_X(\text{div}_D(f)),$$

where $\iota: \tilde{X} \rightarrow \tilde{X}$ denotes the inclusion and the equality $\text{div}(f) = p^*(\text{div}_D(\tilde{f}))$ was established in Lemma 5.3.2.

Lemma 5.3.6. In the situation of Construction 5.1.4 the prevariety $\tilde{X}$ is irreducible, smooth and $H$-factorial.

Proof. As remarked in Construction 5.1.4 the prevariety $\tilde{X}$ is smooth and due to Proposition 5.1.1 it is irreducible. Let $D$ be an invariant Weil divisor on $\tilde{X}$. Using, for example, the fact that $q_X: \tilde{X} \rightarrow X$ is an étale principal bundle, we see that $\tilde{D} = q_X^*(D)$ holds with a Weil divisor $D$ on $X$. Thus, we have to show that all pullback divisors $q_X^*(D)$ are principal. For this, it suffices to consider effective divisors $D$ on $X$, and these are treated by Lemmas 5.3.1 and 5.3.5.

Proof of Theorem 5.3.4. According to Lemma 5.1.2 we may assume that $X$ is smooth. Then $R(X)$ is the algebra of regular functions of the quasiaffine variety $\tilde{X}$ constructed in 5.1.4. Lemma 5.3.6 tells us that $\tilde{X}$ is irreducible, smooth and $H$-factorial. Thus, Proposition 5.3.2 gives the assertion.

Corollary 5.3.7. Let $X$ be a normal prevariety of affine intersection with $\Gamma(X, \mathcal{O}) = \mathbb{K}$ and finitely generated divisor class group. Then the Cox ring $R(X)$ is factorially graded.

Proof. According to Proposition 5.3.4 the assumption $\Gamma(X, \mathcal{O}) = \mathbb{K}$ ensures $R(X)^\ast = \mathbb{K}^\ast$. Thus Theorem 5.3.1 applies.

Corollary 5.3.8. Let $X$ be a complete normal variety with finitely generated divisor class group. Then the Cox ring $R(X)$ is factorially graded.

As in the torsion free case, see Proposition 5.3.4 divisibility and primality of homogeneous elements in the Cox ring $R(X)$ can be characterized in terms of $X$.

Proposition 5.3.9. Let $X$ be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. Suppose that the Cox ring $R(X)$ satisfies $R(X)^\ast = \mathbb{K}^\ast$.

(i) An element $0 \neq f \in \Gamma(X, R_{|D|})$ divides $0 \neq g \in \Gamma(X, R_{|E|})$ if and only if $\text{div}_{|D|}(f) \leq \text{div}_{|E|}(g)$ holds.

(ii) An element $0 \neq f \in \Gamma(X, R_{|D|})$ is $\text{Cl}(X)$-prime if and only if the divisor $\text{div}_{|D|}(f) \in \text{WDiv}(X)$ is prime.

Proof. According to Lemma 5.1.2 we may assume that $X$ is smooth. Then Construction 5.1.4 presents $X$ as the geometric quotient of the smooth quasiaffine $H_X$-variety $\tilde{X}$, which has $R(X)$ as its algebra of regular functions. The first statement follows immediately from Lemma 5.3.5 and, for the second one, we additionally use Proposition 5.3.2.

Remark 5.3.10. Let $X$ be a prevariety of affine intersection with only constant invertible functions, finitely generated divisor class group and a Cox ring $R(X)$ with only constant invertible elements. Then the assignement $f \mapsto \text{div}_{|D|}(f)$ induces an isomorphism from the multiplicative semigroup of homogeneous elements of $R(X)$ modulo units onto the semigroup $\text{WDiv}^+(X)$ of effective Weil divisors on $X$. The fact that $R(X)$ is factorially graded reflects the fact that every effective Weil divisor is a unique non-negative linear combination of prime divisors.
Remark 5.3.11. For the affine surface $X$ considered in Example 4.4.1 the Cox ring $\mathcal{R}(X)$ is factorially $\mathbb{Z}/2\mathbb{Z}$-graded but it is not a factorial ring.

6. Geometric realization of the Cox sheaf

6.1. Characteristic spaces. We study the geometric realization of a Cox sheaf, its relative spectrum, which we call a characteristic space. For locally factorial varieties, e.g. smooth ones, this concept coincides with the universal torsor introduced by Colliot-Thélène and Sansuc in [53], see also [56] and [142]. As soon as we have non-factorial singularities, the characteristic space is not any more a torsor, i.e. a principal bundle, as we will see later. As before, we work with normal prevarieties defined over an algebraically closed field of characteristic zero. First we provide two statements on local finite generation of Cox sheaves.

Proposition 6.1.1. Let $X$ be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then the Cox sheaf $\mathcal{R}$ is locally of finite type.

Proof. The assumption that $X$ is of affine intersection guarantees that it is covered by open affine subsets $X[D,f]$, where $[D] \in \text{Cl}(X)$ and $f \in \mathcal{R}_D(X)$. By Proposition 5.2.2 we have $\Gamma(X[D,f], \mathcal{R}) = \mathcal{R}(X)_f$, which gives the assertion. $\square$

Proposition 6.1.2. Let $X$ be a normal prevariety with only constant invertible functions and finitely generated divisor class group. If $X$ is $\mathbb{Q}$-factorial, then any Cox sheaf $\mathcal{R}$ is locally of finite type.

Proof. By definition, the Cox sheaf $\mathcal{R}$ is the quotient of a sheaf of divisorial algebras $\mathcal{S}$ by some ideal sheaf $\mathcal{I}$. According to Proposition 4.2.2 we may assume that $\mathcal{S}$ arises from a finitely generated subgroup $K \subseteq \text{WDiv}(X)$. Proposition 3.2.3 then tells us that $\mathcal{S}$ is locally of finite type, and Lemma 4.3.5 ensures that the quotient $\mathcal{R} = \mathcal{S}/\mathcal{I}$ can be taken at the level of sections. $\square$

We turn to the relative spectrum of a Cox sheaf. The following generalizes Construction 5.1.4, where the smooth case is considered.

Construction 6.1.3. Let $X$ be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = K^*$ and finitely generated divisor class group, and let $\mathcal{R}$ be a Cox sheaf. Suppose that $\mathcal{R}$ is locally of finite type, e.g., $X$ is $\mathbb{Q}$-factorial or $\mathcal{R}(X)$ is finitely generated. Taking the relative spectrum gives an irreducible normal prevariety

$$\tilde{X} := \text{Spec}_X(\mathcal{R}).$$

The Cl$(X)$-grading of the sheaf $\mathcal{R}$ defines an action of the diagonalizable group $H_X := \text{Spec} \mathbb{K}[\text{Cl}(X)]$ on $\tilde{X}$, the canonical morphism $q_X : \tilde{X} \to X$ is a good quotient for this action, and we have an isomorphism of sheaves

$$\mathcal{R} \cong (q_X)_*(\mathcal{O}_{\tilde{X}}).$$

We call $q_X : \tilde{X} \to X$ the characteristic space associated to $\mathcal{R}$, and $H_X$ the characteristic quasitorus of $X$.

Proof. Everything is standard except irreducibility and normality, which follow from Theorem 5.1.1 $\square$

The Cox sheaf $\mathcal{R}$ was defined as the quotient of a sheaf $\mathcal{S}$ of divisorial algebras by a sheaf $\mathcal{I}$ of ideals. Geometrically this means that the characteristic space comes embedded into the relative spectrum of a sheaf of divisorial algebras; compare 5.4.4 for the case of a smooth $X$. Before making this precise in the general case, we have to relate local finite generation of the sheaves $\mathcal{R}$ and $\mathcal{S}$ to each other.
Proposition 6.1.4. Let $X$ be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox sheaf $\mathcal{R}$. Moreover, let $\mathcal{S}$ be the sheaf of divisorial algebras associated to a finitely generated subgroup $K \subseteq \text{WDiv}(X)$ projecting onto $\text{Cl}(X)$ and $U \subseteq X$ an open affine subset. Then the algebra $\Gamma(U, \mathcal{R})$ is finitely generated if and only if the algebra $\Gamma(U, \mathcal{S})$ is finitely generated.

Proof. Lemma 4.3.5 tells us that $\Gamma(U, \mathcal{R})$ is a factor algebra of $\Gamma(U, \mathcal{S})$. Thus, if $\Gamma(U, \mathcal{S})$ is finitely generated then the same holds for $\Gamma(U, \mathcal{R})$. Moreover, Lemma 4.3.3 says that the projection $\Gamma(U, \mathcal{S}) \to \Gamma(U, \mathcal{R})$ defines isomorphisms along the homogeneous components. Thus, Proposition 1.2.6 shows that finite generation of $\Gamma(U, \mathcal{R})$ implies finite generation of $\Gamma(U, \mathcal{S})$. \hfill \Box

Construction 6.1.5. Let $X$ be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group, and let $K \subseteq \text{WDiv}(X)$ be a finitely generated subgroup projecting onto $\text{Cl}(X)$. Consider the sheaf of divisorial algebras $\mathcal{S} = \mathcal{S}/\mathcal{I}$ as constructed in 4.2.1, and suppose that one of these sheaves is locally of finite type. Then the projection $\mathcal{S} \to \mathcal{R}$ of $\mathcal{O}_X$-algebras defines a commutative diagram

\[ \begin{array}{ccc} 
\hat{X} & \xrightarrow{i} & \bar{X} \\
\downarrow q_X & & \downarrow p \\
X & & X
\end{array} \]

for the relative spectra $\hat{X} = \text{Spec}_X \mathcal{R}$ and $\bar{X} = \text{Spec}_X \mathcal{S}$. We have the actions of $H_X = \text{Spec} \mathbb{K}[\text{Cl}(X)]$ on $\hat{X}$ and $H = \text{Spec} \mathbb{K}[K]$ on $\bar{X}$. The map $i: \hat{X} \to \bar{X}$ is a closed embedding and it is $H_X$-invariant, where $H_X$ acts on $\hat{X}$ via the inclusion $H_X \subseteq H$ defined by the projection $K \to \text{Cl}(X)$. The image $i(\hat{X}) \subseteq \bar{X}$ is precisely the zero set of the ideal sheaf $\mathcal{I}$.

Proposition 6.1.6. Situation as in Construction 6.1.5.

(i) The inverse image $q_X^{-1}(X_{\text{reg}}) \subseteq \hat{X}$ of the set of smooth points $X_{\text{reg}} \subseteq X$ is smooth, the group $H_X$ acts freely on $q_X^{-1}(X_{\text{reg}})$ and the restriction $q_X: q_X^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ is an étale $H_X$-principal bundle.

(ii) For any closed $A \subseteq X$ of codimension at least two, $q_X^{-1}(A) \subseteq \hat{X}$ is as well of codimension at least two.

(iii) The prevariety $\hat{X}$ is $H_X$-factorial.

(iv) If $X$ is of affine intersection, then $\hat{X}$ is a quasiaffine variety.

Proof. For (i), we refer to the proof of Construction 5.1.2. To obtain (ii) consider an affine open set $U \subseteq X$ and $\tilde{U} := q_X^{-1}(U)$. By Lemma 6.1.2, the open set $\tilde{U} \setminus q_X^{-1}(A)$ has the same regular functions as $\tilde{U}$. Since $\hat{X}$ is normal, we conclude that $\tilde{U} \cap q_X^{-1}(A)$ is of codimension at least two in $\tilde{U}$. Now, cover $X$ by affine $U \subseteq X$ and Assertion (ii) follows. We turn to (iii). According to (ii) we may assume that $X$ is smooth. In this case, the statement was proven in Lemma 5.3.0. We show (iv). We may assume that we are in the setting of Construction 6.1.5. Corollary 3.4.6 then ensures that $\hat{X}$ is quasiaffine and Construction 6.1.5 gives that $\hat{X}$ is a closed subvariety of $\bar{X}$. \hfill \Box

The following statement relates the divisor of a $[D]$-homogeneous function on $\hat{X}$ to its $[D]$-divisor on $X$; the smooth case was settled in Lemma 5.3.5.
Proposition 6.1.7. In the situation of \[6.1.6\] consider the pullback homomorphism \(q_X^*: \text{WDiv}(X) \to \text{WDiv}(\tilde{X})\) defined in \[3.4.4\]. Then, for every \([D] \in \text{Cl}(X)\) and every \(f \in \Gamma(X, R_{[D]})\), we have

\[
\text{div}(f) = q_X^*(\text{div}_{[D]}(f)),
\]

where on the left hand side \(f\) is a function on \(\tilde{X}\), and on the right hand side a function on \(X\). If \(X\) is of affine intersection and \(X \setminus \text{Supp}(\text{div}_{[D]}(f))\) is affine, then we have moreover

\[
\text{Supp}(\text{div}(f)) = q_X^{-1}(\text{Supp}(\text{div}_{[D]}(f))).
\]

Proof. We may assume that we are in the setting of Construction \[6.1.6\]. Let the divisor \(D \in K\) represent the class \([D] \in \text{Cl}(X)\), and let \(\tilde{f} \in \Gamma(X, S_D)\) project to \(f \in \Gamma(X, R_{[D]})\). The commutative diagram of \[6.1.5\] yields

\[
\text{div}(f) = i^*(\text{div}(\tilde{f})) = i^*(p^*(\text{div}_D(\tilde{f}))) = q_X^*(\text{div}_{[D]}(f)),
\]

where \(i: \tilde{X} \to X\) denotes the inclusion and the equality \(\text{div}(\tilde{f}) = p^*(\text{div}_D(\tilde{f}))\) was established in Proposition \[3.4.3\]. Similarly, we have

\[
\text{Supp}(\text{div}(f)) = i^{-1}(\text{Supp}(\text{div}(\tilde{f})))
\]

\[
= i^{-1}(p^{-1}(\text{Supp}(\text{div}_D(f))))
\]

\[
= q_X^{-1}(\text{Supp}(\text{div}_{[D]}(f)))
\]

provided that \(X\) is of affine intersection and \(X \setminus \text{Supp}(\text{div}_{[D]}(f))\) is affine, because Proposition \[6.1.7\] then ensures \(\text{Supp}(\text{div}(\tilde{f})) = p^{-1}(\text{Supp}(\text{div}_D(\tilde{f})))\). \(\square\)

Corollary 6.1.8. Situation as in Construction \[6.1.3\]. Let \(\tilde{x} \in \tilde{X}\) be a point such that \(H_X \cdot \tilde{x} \subseteq \tilde{X}\) is closed, and let \(f \in \Gamma(X, R_{[D]})\). Then we have

\[
f(\tilde{x}) = 0 \iff q_X(\tilde{x}) \in \text{Supp}(\text{div}_{[D]}(f)).
\]

Proof. The image \(q_X(\text{Supp}(\text{div}(f)))\) is contained in \(\text{Supp}(\text{div}_{[D]}(f))\). By the definition of the pullback and Proposition \[6.1.7\], the two sets coincide in \(X_{\text{reg}}\). Thus, \(q_X(\text{Supp}(\text{div}(f)))\) is dense in \(\text{Supp}(\text{div}_{[D]}(f))\). By Theorem \[2.3.6\] the image \(q_X(\text{Supp}(\text{div}(f)))\) is closed and thus we have

\[
q_X(\text{Supp}(\text{div}(f))) = \text{Supp}(\text{div}_{[D]}(f)).
\]

In particular, if \(f(\tilde{x}) = 0\) holds, then \(q_X(\tilde{x})\) lies in \(\text{Supp}(\text{div}_{[D]}(f))\). Conversely, if \(q_X(\tilde{x})\) belongs to \(\text{Supp}(\text{div}_{[D]}(f))\), then some \(\tilde{x}' \in \text{Supp}(\text{div}(f))\) belongs to the fiber of \(\tilde{x}\). Since \(H_x \cdot \tilde{x}\) is closed, Corollary \[2.3.4\] tells us that \(\tilde{x}\) is contained in the orbit closure of \(\tilde{x}'\) and hence belongs to \(\text{Supp}(\text{div}(f))\). \(\square\)

Corollary 6.1.9. Situation as in Construction \[6.1.3\] and suppose that \(X\) is of affine intersection. For \(x \in X\), let \(\tilde{x} \in q_X^{-1}(x)\) such that \(H_X \cdot \tilde{x}\) is closed in \(\tilde{X}\). Then \(H \cdot \tilde{x}\) is closed in \(\tilde{X}\).

Proof. Assume that the orbit \(H \cdot \tilde{x}\) is not closed in \(\tilde{X}\). Then there is a point \(\tilde{x} \in p^{-1}(x)\) having a closed \(H\)-orbit in \(\tilde{X}\), and \(\tilde{x}\) lies in the closure of \(H \cdot \tilde{x}\). Since \(\tilde{X}\) is quasiaffine, we find a function \(\tilde{f} \in \Gamma(X, S_D)\) with \(\tilde{f}(\tilde{x}) = 0\) but \(\tilde{f}(\tilde{x}) \neq 0\). Corollary \[3.4.3\] gives \(p(\tilde{x}) \in \text{Supp}(\text{div}_D(\tilde{f}))\). Since we have \(q_X(\tilde{x}) = p(\tilde{x})\), this contradicts Corollary \[6.1.8\]. \(\square\)
6.2. **Divisor classes and isotropy groups.** The aim of this subsection is to relate local properties of a prevariety to properties of the characteristic quasitorus action on its characteristic space. Again, everything takes places over an algebraically closed field $\mathbb{K}$ of characteristic zero.

For a normal prevariety $X$ and a point $x \in X$, let $\text{PDiv}(X,x) \subseteq \text{WDiv}(X)$ denote the subgroup of all Weil divisors, which are principal on some neighbourhood of $x$. We define the local class group of $X$ in $x$ to be the factor group

$$\text{Cl}(X,x) := \text{WDiv}(X)/\text{PDiv}(X,x).$$

Obviously the group $\text{PDiv}(X)$ of principal divisors is contained in $\text{PDiv}(X,x)$. Thus, there is a canonical epimorphism $\pi_x : \text{Cl}(X) \to \text{Cl}(X,x)$. The Picard group of $X$ is the factor group of the group $\text{CDiv}(X)$ of Cartier divisors by the subgroup of principal divisors:

$$\text{Pic}(X) = \text{CDiv}(X)/\text{PDiv}(X) = \bigcap_{x \in X} \ker(\pi_x).$$

**Proposition 6.2.1.** Situation as in Construction 6.1.3. For $x \in X$, let $\hat{x} \in q_X^{-1}(x)$ be a point with closed $H_X$-orbit. Define a submonoid

$$S_x := \{[D] \in \text{Cl}(X) : f(\hat{x}) \neq 0 \text{ for some } f \in \Gamma(X, R[D]) \} \subseteq \text{Cl}(X),$$

and let $\text{Cl}_x(X) \subseteq \text{Cl}(X)$ denote the subgroup generated by $S_x$. Then the local class groups of $X$ and the Picard group are given by

$$\text{Cl}(X,x) = \text{Cl}(X)/\text{Cl}_x(X), \quad \text{Pic}(X) = \bigcap_{x \in X} \text{Cl}_x(X).$$

**Proof.** First observe that Corollary 6.1.8 gives us the following description of the monoid $S_x$ in terms of the $[D]$-divisors:

$$S_x = \{[D] \in \text{Cl}(X) : x \notin \text{div}_{[D]}(f) \text{ for some } f \in \Gamma(X, R[D]) \} = \{[D] \in \text{Cl}(X) : D \geq 0, x \notin \text{Supp}(D) \},$$

where the latter equation is due to the fact that the $[D]$-divisors are precisely the effective divisors with class $[D]$. The assertions thus follow from

$$\text{Cl}_x(X) = \{[D] \in \text{Cl}(X) : x \notin \text{Supp}(D) \} = \{[D] \in \text{Cl}(X) : D \text{ principal near } x \}.$$

$$\square$$

**Proposition 6.2.2.** Situation as in Construction 6.1.3. Given $x \in X$, let $\hat{x} \in q_X^{-1}(x)$ be a point with closed $H_X$-orbit. Then the inclusion $H_{X,\hat{x}} \subseteq H_X$ of the isotropy group of $\hat{x} \in \hat{X}$ is given by the epimorphism $\text{Cl}(X) \to \text{Cl}(X,x)$ of character groups. In particular, we have

$$H_{X,\hat{x}} = \text{Spec } \mathbb{K}[\text{Cl}(X,x)], \quad \text{Cl}(X,x) = \mathbb{K}(H_{X,\hat{x}}).$$

**Proof.** Let $U \subseteq X$ be any affine open neighbourhood of $x \in X$. Then $U$ is of the form $X_{[D],f}$ with some $f \in \Gamma(X, R[D])$ and $\hat{U} := q_X^{-1}(U)$ is affine. According to Proposition 6.2.2 we have

$$\Gamma(\hat{U}, \mathcal{O}) = \Gamma(U, \mathcal{O}) = \Gamma(X, \mathcal{R})_f = \Gamma(\hat{X}, \mathcal{O})_f.$$

Corollary 6.1.8 shows that the group $\text{Cl}_x(X)$ is generated by the classes $[E] \in \text{Cl}(X)$ admitting a section $g \in \Gamma(U, R|_{[E]})$ with $g(\hat{x}) \neq 0$. In other words, $\text{Cl}_x(X)$ is the orbit group of the point $\hat{x} \in \hat{U}$. Now Proposition 2.2.18 gives the assertion. $\square$
A point $x$ of a normal prevariety $X$ is called factorial if near $x$ every divisor is principal. Thus, $x \in X$ is factorial if and only if its local ring $\mathcal{O}_{X,x}$ admits unique factorization. Moreover, a point $x \in X$ is called $\mathbb{Q}$-factorial if near $x$ for every divisor some multiple is principal.

**Corollary 6.2.3.** Situation as in Construction [6.1.3]

(i) A point $x \in X$ is factorial if and only if the fiber $q_X^{-1}(x)$ is a single $H_X$-orbit with trivial isotropy.

(ii) A point $x \in X$ is $\mathbb{Q}$-factorial if and only if the fiber $q_X^{-1}(x)$ is a single $H_X$-orbit.

**Proof.** The point $x \in X$ is factorial if and only if $\text{Cl}(X,x)$ is trivial, and it is $\mathbb{Q}$-factorial if and only if $\text{Cl}(X,x)$ is finite. Thus, the statement follows from Proposition [6.2.2] and Corollary [2.3.7].

**Corollary 6.2.4.** Situation as in Construction [6.1.3]

(i) The action of $H_X$ on $\hat{X}$ is free if and only if $X$ is locally factorial.

(ii) The good quotient $q_X : \hat{X} \to \overline{X}$ is geometric if and only if $X$ is $\mathbb{Q}$-factorial.

**Corollary 6.2.5.** Situation as in Construction [6.1.3]. Let $\hat{H}_X \subseteq H_X$ be the subgroup generated by all isotropy groups $H_{X,\hat{x}}$, where $\hat{x} \in \hat{X}$. Then we have

$$\ker(\mathbb{X}(H_X) \to \mathbb{X}(\hat{H}_X)) = \bigcap_{\hat{x} \in \hat{X}} \ker(\mathbb{X}(H_X) \to \mathbb{X}(H_{X,\hat{x}}))$$

and the projection $H_X \to H_X / \hat{H}_X$ corresponds to the inclusion $\text{Pic}(X) \subseteq \text{Cl}(X)$ of character groups.

**Corollary 6.2.6.** Situation as in Construction [6.1.3]. If the variety $\hat{X}$ contains an $H_X$-fixed point, then the Picard group $\text{Pic}(X)$ is trivial.

### 6.3. Total coordinate space and irrelevant ideal

Here we consider the situation that the Cox ring is finitely generated. This allows us to introduce the total coordinate space as the spectrum of the Cox ring. As always, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

**Construction 6.3.1.** Let $X$ be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. Let $\mathcal{R}$ be a Cox sheaf and assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we have a diagram

$$
\begin{array}{ccc}
\text{Spec} \mathcal{R} & \xrightarrow{q_X} & \overline{X} \\
\downarrow & & \downarrow \\
X & & \text{Spec}(\mathcal{R}(X))
\end{array}
$$

where the canonical morphism $\hat{X} \to \overline{X}$ is an $H_X$-equivariant open embedding, the complement $\overline{X} \setminus \hat{X}$ is of codimension at least two and $\overline{X}$ is an $H_X$-factorial affine variety. We call the $H_X$-variety $\overline{X}$ the total coordinate space associated to $\mathcal{R}$.

**Proof.** Cover $X$ by affine open sets $X_{[D],f} = X \setminus \text{Supp}(\text{div}_{[D]}(f))$, where $[D] \in \text{Cl}(X)$ and $f \in \Gamma(X,\mathcal{R}_{[D]})$. Then, according to Proposition [6.1.7], the variety $\hat{X}$ is covered by the affine sets $\hat{X}_f = q_X^{-1}(X_{[D],f})$. Note that we have

$$\Gamma(\hat{X}_f,\mathcal{O}) = \Gamma(\hat{X},\mathcal{O})_f = \Gamma(\overline{X},\mathcal{O})_f = \Gamma(\overline{X},\mathcal{O}).$$

Consequently, the canonical morphisms $\hat{X}_f \to \overline{X}_f$ are isomorphisms. Gluing them together gives the desired open embedding $\hat{X} \to \overline{X}$. □
Definition 6.3.2. Situation as in Construction 6.3.1. The irrelevant ideal of the prevariety $X$ is the vanishing ideal of the complement $\overline{X} \setminus \hat{X}$ in the Cox ring:

$$\mathcal{J}_{\text{irr}}(X) := \{ f \in \mathcal{R}(X); f|_{\overline{X} \setminus \hat{X}} = 0 \} \subseteq \mathcal{R}(X).$$

Proposition 6.3.3. Situation as in Construction 6.3.1.

(i) For any section $f \in \Gamma(X, \mathcal{R}_{\{D\}})$, membership in the irrelevant ideal is characterized as follows:

$$f \in \mathcal{J}_{\text{irr}}(X) \iff \overline{X}_f = \hat{X}_f \iff \hat{X}_f \text{ is affine.}$$

(ii) Let $0 \neq f \in \Gamma(X, \mathcal{R}_{\{D\}})$. If the $\{D\}$-localization $X_{\{D\},f}$ is affine, then we have $f \in \mathcal{J}_{\text{irr}}(X)$.

(iii) Let $0 \neq f_i \in \Gamma(X, \mathcal{R}_{\{D\}})$, where $1 \leq i \leq r$ be such that the sets $X_{\{D\},f_i}$ are affine and cover $X$. Then we have

$$\mathcal{J}_{\text{irr}}(X) = \sqrt{(f_1, \ldots, f_r)}.$$

Proof. The first equivalence in (i) is obvious and the second one follows from the fact that $\overline{X} \setminus \hat{X}$ is of codimension at least two in $\overline{X}$. Proposition 6.1.7 tells us that for affine $X_{\{D\},f}$ also $\hat{X}_f$ is affine, which gives (ii). We turn to (iii). Proposition 6.1.7 and (ii) ensure that the functions $f_1, \ldots, f_r$ have $\overline{X} \setminus \hat{X}$ as their common zero locus. Thus Hilbert’s Nullstellensatz gives the assertion. $\square$

Corollary 6.3.4. Situation as in Construction 6.3.1. Then $X$ is affine if and only if $\hat{X} = \overline{X}$ holds.

Proof. Take $f = 1$ in the characterization 6.3.3(i). $\square$

Corollary 6.3.5. Situation as in Construction 6.3.1 and assume that $X$ is $\mathbb{Q}$-factorial. Then $0 \neq f \in \Gamma(X, \mathcal{R}_{\{D\}})$ belongs to $\mathcal{J}_{\text{irr}}(X)$ if and only if $X_{\{D\},f}$ is affine. In particular, we have

$$\mathcal{J}_{\text{irr}}(X) = \text{link}(f \in \Gamma(X, \mathcal{R}_{\{D\}}); [D] \in \text{Cl}(X), X_{\{D\},f} \text{ is affine}).$$

Proof. We have to show that for any $\{D\}$-homogeneous $f \in \mathcal{J}_{\text{irr}}(X)$, the $\{D\}$-localization $X_{\{D\},f}$ is affine. Note that $\hat{X}_f$ is affine by Proposition 6.3.3(i). The assumption of $\mathbb{Q}$-factoriality ensures that $q_X : \hat{X} \to X$ is a geometric quotient, see Corollary 6.2.4. In particular, all $H_X$-orbits in $\hat{X}$ are closed and thus Corollary 6.1.8 gives us $\hat{X}_f = q_X^{-1}(X_{\{D\},f})$. Thus, as the good quotient space of the affine variety $\hat{X}_f$, the set $X_{\{D\},f}$ is affine. $\square$

Recall that a divisor $D$ on a prevariety $X$ is called ample if it admits sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X(D))$ such that the sets $X_{D,f_i}$ are affine and cover $X$.

Corollary 6.3.6. Situation as in Construction 6.3.1. If $[D] \in \text{Cl}(X)$ is the class of an ample divisor, then we have

$$\mathcal{J}_{\text{irr}}(X) = \sqrt{\Gamma(X, \mathcal{R}_{\{D\}})).$$

6.4. Characteristic spaces via GIT. As we saw, the characteristic space of a prevariety $X$ of affine intersection is a quasiaffine variety $\hat{X}$ with an action of the characteristic quasitorus $H_X$ having $X$ as a good quotient. Our aim is to characterize this situation in terms of Geometric Invariant Theory. The crucial notion is the following.

Definition 6.4.1. Let $G$ be an affine algebraic group and $W$ a $G$-prevariety. We say that the $G$-action on $W$ is strongly stable if there is an open invariant subset $W' \subseteq W$ with the following properties:
(i) the complement $W \setminus W'$ is of codimension at least two in $W$,
(ii) the group $G$ acts freely, i.e. with trivial isotropy groups, on $W'$,
(iii) for every $x \in W'$ the orbit $G \cdot x$ is closed in $W$.

**Remark 6.4.2.** Let $X$ be a normal prevariety as in Construction 6.1.3 and consider the characteristic space $q_X: \widehat{X} \to X$ introduced there. Then Proposition 6.1.6 shows that the subset $q_X^{-1}(X_{reg}) \subseteq \widehat{X}$ satisfies the properties of 6.4.1.

Let $X$ and $q_X: \widehat{X} \to X$ be as in Construction 6.1.3. In the sequel, we mean by a characteristic space for $X$ more generally a good quotient $q: \mathcal{X} \to X$ for an action of a diagonalizable group $H$ on a prevariety $\mathcal{X}$ such that there is an equivariant isomorphism $(\mu, \bar{\mu})$ making the following diagram commutative

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mu} & \widehat{X} \\
\downarrow{q} & & \downarrow{q_X} \\
X & & 
\end{array}
$$

Recall that here $\mu: \mathcal{X} \to \widehat{X}$ is an isomorphism of varieties and $\bar{\mu}: H \to H_X$ is an isomorphism of algebraic groups such that we always have $\mu(hx) = \bar{\mu}(h)\mu(x)$. Note that a good quotient $q: \mathcal{X} \to X$ of a quasiaffine $H$-variety is a characteristic space if and only if we have an isomorphism of graded sheaves $R \to q_*(O_{\mathcal{X}})$, where $R$ is a Cox sheaf on $X$.

**Theorem 6.4.3.** Let a quasitorus $H$ act on a normal quasiaffine variety $\mathcal{X}$ with a good quotient $q: \mathcal{X} \to X$. Assume that $\Gamma(\mathcal{X}, O^+) = \mathbb{K}^*$ holds, $\mathcal{X}$ is $H$-factorial and the $H$-action is strongly stable. Then $X$ is a normal prevariety of affine intersection, $\Gamma(\mathcal{X}, O^+) = \mathbb{K}^*$ holds, $\Gamma(X)$ is finitely generated, the Cox sheaf of $X$ is locally of finite type, and $q: \mathcal{X} \to X$ is a characteristic space for $X$.

The proof will be given later in this section. First we also generalize the concept of the total coordinate space of a prevariety $X$ of affine intersection with finitely generated Cox ring $R(X)$: this is from now on any affine $H$-variety isomorphic to the affine $H_X$-variety $\overline{X}$ of Construction 6.1.4.

**Corollary 6.4.4.** Let $Z$ be a normal affine variety with an action of a quasitorus $H$. Assume that every invertible function on $Z$ is constant, $Z$ is $H$-factorial, and there exists an open $H$-invariant subset $W \subseteq Z$ with $\text{codim}_Z(Z \setminus W) \geq 2$ such that the $H$-action on $W$ is strongly stable and admits a good quotient $q: W \to X$. Then $Z$ is a total coordinate space for $X$.

A first step in the proof of Theorem 6.4.3 is to describe the divisor class group of the quotient space. Let us prepare the corresponding statement. Consider an irreducible prevariety $\mathcal{X}$ with an action of a quasitorus $H = \text{Spec} \, \mathbb{K}[M]$. For any $H$-invariant morphism $q: \mathcal{X} \to X$ to an irreducible prevariety $X$, we have the push forward homomorphism

$$
q_*: \text{WDiv}(\mathcal{X})^H \to \text{WDiv}(X)
$$

from the invariant Weil divisors of $\mathcal{X}$ to the Weil divisors of $X$ sending an $H$-prime divisor $D \subseteq \mathcal{X}$ to the closure of its image $q(D)$ if the latter is of codimension one and to zero else. By a homogeneous rational function we mean an element $f \in \mathbb{K}(\mathcal{X})$ that is defined on an invariant open subset of $\mathcal{X}$ and is homogeneous there. We denote the multiplicative group of non-zero homogeneous rational functions on $\mathcal{X}$ by $E(\mathcal{X})$ and the subset of non-zero rational functions of weight $w \in M$ by $E(\mathcal{X})_w$. 

**Proposition 6.4.5.** Let a quasitorus $H = \text{Spec} \mathbb{K}[M]$ act on a normal quasiaffine variety $X$ with a good quotient $q : X \to \tilde{X}$. Assume that $\Gamma(X, O^*) = \mathbb{K}^*$ holds, $X$ is $H$-factorial and the $H$-action is strongly stable. Then $X$ is a normal prevariety of affine intersection and there is an epimorphism

$$\delta : E(X) \to \text{WDiv}(X), \quad f \mapsto q_*(\text{div}(f)).$$

We have $\text{div}(f) = q^*(q_*(\text{div}(f)))$ for every $f \in E(X)$. Moreover, the epimorphism $\delta$ induces a well-defined isomorphism

$$M \to \text{Cl}(X), \quad w \mapsto [\delta(f)], \quad \text{with any } f \in E(X)_w.$$ 

Finally, for every $f \in E(X)_w$, and every open set $U \subseteq X$, we have an isomorphism of $\Gamma(U, O)$-modules

$$\Gamma(U, O_X(\delta(f))) \to \Gamma(q^{-1}(U), O_X)_w, \quad g \mapsto f q^*(g).$$

**Proof.** First of all note that the good quotient space $X$ inherits normality and the property to be of affine intersection from the normal quasiaffine variety $X$.

Let $\tilde{X}' \subseteq \tilde{X}$ be as in Definition 6.4.4. Then, with $X' := q(\tilde{X}')$, we have $q^{-1}(X') = X'$. Consequently, $X' \subseteq X$ is open. Moreover, $X \setminus X'$ is of codimension at least two in $X$, because $X' \setminus X'$ is of codimension at least two in $X$. Thus, we may assume that $X = X'$ holds, which means in particular that $H$ acts freely. Then we have homomorphisms of groups:

$$E(X) \xrightarrow{f \mapsto \text{div}(f)} \text{WDiv}(X)^H \xrightarrow{q_*} \text{WDiv}(X).$$

The homomorphism from $E(X)$ to the group of $H$-invariant Weil divisors $\text{WDiv}(X)^H$ is surjective, because $X$ is $H$-factorial. Moreover, $q^*$ and $q_*$ are inverse to each other, which follows from the observation that $q : X \to \tilde{X}$ is an étale $H$-principal bundle. This establishes the first part of the assertion.

We show that $\delta$ induces an isomorphism $M \to \text{Cl}(X)$. First we have to check that $[\delta(f)]$ does not depend on the choice of $f$. So, let $f, g \in E(X)_w$. Then $f/g$ is $H$-invariant, and hence defines a rational function on $X$. We infer well-definedness of $w \mapsto [\delta(f)]$ from

$$q_*(\text{div}(f)) - q_*(\text{div}(g)) = q_*(\text{div}(f) - \text{div}(g)) = q_*(\text{div}(f/g)) = \text{div}(f/g).$$

To verify injectivity, let $\delta(f) = \text{div}(h)$ for some $h \in K(X)^*$. Then we obtain $\text{div}(f) = \text{div}(q^*(h))$. Thus, $f/q^*(h)$ is an invertible homogeneous function on $X$ and hence is constant. This implies $w = \text{deg}(f/q^*(h)) = 0$. Surjectivity is clear, because $E(X) \to \text{WDiv}(X)$ is surjective.

We turn to the last statement. First we note that for every $g \in \Gamma(U, O_X(\delta(f)))$ the function $f q^*(g)$ is regular on $q^{-1}(U)$, because we have

$$\text{div}(f q^*(g)) = \text{div}(f) + \text{div}(q^*(g)) = q^*(\delta(f)) + \text{div}(q^*(g)) = q^*(\delta(f)) + \text{div}(g) \geq 0.$$ 

Thus, the homomorphism $\Gamma(U, O_X(\delta(f))) \to \Gamma(q^{-1}(U), O_X)_w$ sending $g$ to $f q^*(g)$ is well defined. Note that $h \mapsto h/f$ defines an inverse.

**Corollary 6.4.6.** Consider the characteristic space $q : \tilde{X} \to X$ obtained from a Cox sheaf $R$. Then, for any non-zero $f \in \Gamma(X, R[D])$ the push forward $q_*(\text{div}(f))$, equals the $[D]$-divisor $\text{div}_{[D]}(f)$.

**Proof.** Proposition 6.4.5 shows that $q^*(q_*(\text{div}(f)))$ equals $\text{div}(f)$ and Proposition 6.1.7 tells us that $q^*(\text{div}_{[D]}(f))$ equals $\text{div}(f)$ as well. 

□
Proof of Theorem 6.4.3. Writing $H = \text{Spec} \mathbb{K}[M]$ with the character group $\mathbb{H}$ of $H$, we are in the setting of Proposition 6.4.3. Choose a finitely generated subgroup $K \subseteq \text{WDiv}(X)$ mapping onto $\text{Cl}(X)$, and let $D_1, \ldots, D_s \in \text{WDiv}(X)$ be a basis of $K$. By Proposition 6.4.5, we have $D_i = \delta(h_i)$ with $h_i \in E(X)_w$. Moreover, the isomorphism $M \to \text{Cl}(X)$ given there identifies $w_i \in M$ with $[D_i] \in \text{Cl}(X)$. For $D = a_1D_1 + \ldots + a_sD_s$, we have $D = \delta(h_D)$ with $h_D = h_1^{a_1} \cdots h_s^{a_s}$.

Let $\mathcal{S}$ be the sheaf of divisorial algebras associated to $K$ and for $D \in K$, let $w \in M$ correspond to $[D] \in \text{Cl}(X)$. Then, for any open set $U \subseteq X$ and any $D \in K$, Proposition 6.4.3 provides an isomorphism of $\mathbb{K}$-vector spaces

$$\Phi_{U, D} : \Gamma(U, \mathcal{S}_D) \to \Gamma(q^{-1}(U), \mathcal{O}_w), \quad g \mapsto q^*(g) h_D.$$ 

The $\Phi_{U, D}$ fit together to an epimorphism of graded sheaves $\Phi : \mathcal{S} \to q_*(\mathcal{O}_X)$. Once we know that $\Phi$ has the ideal $\mathcal{I}$ of Construction 12.2 as its kernel, we obtain an induced isomorphism $\mathcal{R} \to q_* \mathcal{O}_X$, where $\mathcal{R} = \mathcal{S}/\mathcal{I}$ is the associated Cox sheaf; this shows that $\mathcal{R}$ is locally of finite type and gives an isomorphism $\mu : \mathcal{X} \to \tilde{X}$.

Thus we are left with showing that the kernel of $\Phi$ equals $\mathcal{I}$. Consider a $\text{Cl}(X)$-homogeneous element $f \in \Gamma(U, \mathcal{S})$ of degree $[D]$, where $D \in K$. Let $K^0$ be the kernel of the surjection $K \to \text{Cl}(X)$. Then we have

$$f = \sum_{E \in K^0} f_{D+E}, \quad \Phi(f) = \sum_{E \in K^0} q^*(f_{D+E}) h_{D+E}.$$ 

With the character $\chi : K^0 \to \mathbb{K}(X)^*$ defined by $q^* \chi(E) = h_E$, we may rewrite the image $\Phi(f)$ as

$$\Phi(f) = \sum_{E \in K^0} q^*(\chi(E) f_{D+E}) h_D = q^* \left( \sum_{E \in K^0} \chi(E) f_{D+E} \right) h_D.$$ 

So, $f$ lies in the kernel of $\Phi$ if and only if $\sum \chi(E) f_{D+E}$ vanishes. Now observe that we have

$$f = \sum_{E \in K^0} (1 - \chi(E)) f_{D+E} + \sum_{E \in K^0} \chi(E) f_{D+E}.$$ 

The second summand is $K$-homogeneous, and thus we infer from Lemma 4.3.1 that $f \in \mathcal{I}$ holds if and only if $\sum \chi(E) f_{D+E} = 0$ holds. \hfill $\square$

Remark 6.4.7. Consider the isomorphism $(\mu, \overline{\mu})$ identifying the characteristic spaces $q : \mathcal{X} \to X$ and $q_X : \tilde{\mathcal{X}} \to X$ in the above proof. Then the isomorphism $\overline{\mu}$ identifying the quasitori $H$ and $H_X$ is given by the isomorphism $M \to \text{Cl}(X)$ of their character groups provided by Proposition 6.4.5.
Bibliography

[1] A. A’Campo-Neuen: Note on a counterexample to Hilbert’s fourteenth problem given by P. Roberts. Indag. Math. (N.S.) 5 (1994), no. 3, 253–257.
[2] A. A’Campo-Neuen, J. Hausen: Quotients of toric varieties by the action of a subtorus. Tohoku Math. J. (2) 51 (1999), no. 1, 1–12.
[3] A. A’Campo-Neuen, J. Hausen: Toric prevarieties and subtorus actions. Geom. Dedicata 87 (2001), no. 1–3, 35–64.
[4] A. A’Campo-Neuen, J. Hausen, S. Scharer: Homogeneous coordinates and quotient presentations for toric varieties. Math. Nachr. 246/247 (2002), 5–19.
[5] V.A. Alekseev, V.V. Nikulin: Classification of log del Pezzo surfaces of index \(\leq 2\). Memoirs of the Mathematical Society of Japan, vol. 15 (2006), Preprint Version [arXiv:math.AG/0406536].
[6] V.A. Alexeev, V.V. Nikulin: Del Pezzo and K3-surfaces. MSJ Memoirs, 15. Mathematical Society of Japan, Tokyo, 2006.
[7] K. Altmann, J. Hausen: Polyhedral divisors and algebraic torus actions. Math. Ann. 334 (2006), no. 3, 557–607.
[8] K. Altmann, J. Hausen, H. Süss: Gluing affine torus actions via divisorial fans. Transformation Groups 13 (2008), no. 2, 215–242.
[9] D.F. Anderson: Graded Krull domains. Comm. Algebra 7 (1979), no. 1, 79–106.
[10] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris: Geometry of algebraic curves. Grundlehren der Mathematischen Wissenschaften 267, Springer-Verlag, New York, 1985.
[11] M. Artebani, J. Hausen, A. Laface: On Cox rings of K3-surfaces. To appear in Compositio Math., [arXiv:0901.0369].
[12] M. Artebani, A. Laface: Cox rings of surfaces and the anticanonical Iitaka dimension. Preprint, [arXiv:0909.1835].
[13] M. Artin: Some numerical criteria for contractability of curves on algebraic surfaces. Amer. J. Math. 84 (1962), 485–496.
[14] I.V. Arzhantsev: Projective embeddings of homogeneous spaces with small boundary. Izvestiya RAN: Ser. Mat. 73 (2009), no. 3, 3–22 (Russian); English transl.: Izvestiya Mathematics 73 (2009), no. 3, 437–453.
[15] I.V. Arzhantsev: On the factoriality of Cox rings. Mat. Zametki 85 (2009), no. 5, 643–651 (Russian); English transl.: Math. Notes 85 (2009), no. 5, 623–629.
[16] I.V. Arzhantsev, S.A. Gaifullin: Cox rings, semigroups and automorphisms of affine algebraic varieties. Mat. Sbornik 201 (2010), no. 1, 3–24 (Russian); English transl.: Sbornik Math. 201 (2010), no. 1, 1–21.
[17] I.V. Arzhantsev, J. Hausen: On embeddings of homogeneous spaces with small boundary. J. Algebra 304 (2006), no. 2, 950–988.
[18] I.V. Arzhantsev, J. Hausen: On the multiplication map of a multigraded algebra. Math. Res. Lett. 14 (2007), no. 1, 129–136.
[19] I.V. Arzhantsev, J. Hausen: Geometric Invariant Theory via Cox rings. J. Pure Appl. Algebra 213 (2009), no. 1, 154–172.
[20] M.F. Atiyah, I.G. Macdonald: Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1969.
[21] M. Audin: The topology of torus actions on symplectic manifolds. Prog. Math., 93. Birkhäuser Verlag, Basel, 1991.
[22] H. Bäker: Good quotients of Mori Dream Spaces. Proc. Amer. Math. Soc. ? (2010), no. ?, ??–??.
[23] W.P. Barth, K. Hulek, C.A.M. Peters, A. Van de Ven: Compact complex surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 4, Second edition, Springer-Verlag, Berlin, 2004.
[24] V.V. Batyrev: Quantum cohomology rings of toric manifolds. In: Journées de Géométrie Algébrique d’Orsay, Astérisques 218, 9–34 (1993).
[25] V.V. Batyrev, F. Haddad: The geometry of SL(2)-equivariant flips. Mosc. Math. J. 8 (2008), no. 4, 621–646.
[26] V.V. Batyrev, O.N. Popov: The Cox ring of a Del Pezzo surface. In: Arithmetic of higher-dimensional algebraic varieties, Progr. Math. 226, 85–103 (2004)
[27] A. Beauville: Complex algebraic surfaces. Second edition. London Mathematical Society Student Texts 34. Cambridge University Press, Cambridge, 1996
[28] F. Berchtold, J. Hausen: Homogeneous coordinates for algebraic varieties. J. Algebra 266 (2003), no. 2, 636–670.
[29] F. Berchtold, J. Hausen: Bunches of cones in the divisor class group – a new combinatorial language for toric varieties. Inter. Math. Research Notices 6 (2004), 261–302.
[30] F. Berchtold, J. Hausen: GIT-equivalence beyond the ample cone. Michigan Math. J. 54 (2006), no. 3, 483–515.
[31] F. Berchtold, J. Hausen: Cox rings and combinatorics. Trans. Amer. Math. Soc. 359 (2007), no. 3, 1205–1252.
[32] A. Białynicki-Birula: Algebraic quotients. In: Encyclopaedia of Mathematical Sciences 131. Invariant Theory and Algebraic Transformation Groups, II. Springer-Verlag, Berlin, 2002
[33] A. Białynicki-Birula, J. Święcicka: Complete quotients by algebraic torus actions. In: Group actions and vector fields (Vancouver, B.C., 1981), 10–22, Lecture Notes in Math. 956, Springer, Berlin, 1982
[34] A. Białynicki-Birula, J. Święcicka: Three theorems on existence of good quotients. Math. Ann. 307 (1997), 143–149.
[35] A. Białynicki-Birula, J. Święcicka: A recipe for finding open subsets of vector spaces with a good quotient. Colloq. Math. 77 (1998), no. 1, 97–114.
[36] F. Bien, A. Borel: Sous-groupes épimorphiques de groupes linéaires algébriques I. C. R. Acad. Sci. Paris, t. 315, Série I (1992), 649–653.
[37] F. Bien, A. Borel: Sous-groupes épimorphiques de groupes linéaires algébriques II. C. R. Acad. Sci. Paris, t. 315, Série I (1992), 1341–1346.
[38] F. Bien, A. Borel, J. Kollar: Rationally connected homogeneous spaces. Invent. Math. 124 (1996), no. 1-3, 103–127.
[39] A. Borel: Linear algebraic groups. Second edition. Graduate Texts in Mathematics 126, Springer-Verlag, New York, 1991
[40] N. Bourbaki: Commutative algebra. Chapters 1–7. Elements of Mathematics, Springer-Verlag, Berlin, 1998
[41] J. Boutot: Singularités rationnelles et quotients par les groupes réductifs. Invent. Math. 88, 65–68 (1987)
[42] H. Brenner: Rings of global sections in two-dimensional schemes. Beiträge Algebra Geom. 42 (2001), no. 2, 443–450.
[43] M. Brion: The total coordinate ring of a wonderful variety. J. Algebra 313 (2007), no. 1, 61–99.
[44] V.M. Buchstaber, T.E. Panov: Torus actions and their applications in topology and combinatorics. Univ. Lecture Series 24, Providence R.I., AMS, 2002
[45] A.-M. Castravet: The Cox ring of $\mathbb{M}_{0,6}$. Trans. Amer. Math. Soc. 361 (2009), no. 7, 3851–3878.
[46] A.-M. Castravet, E.A. Tevelev: Hilbert’s 14th problem and Cox rings. Compos. Math. 142 (2006), no. 6, 1479–1498.
[47] R. Chirivi, A. Muffei: The ring of sections of a complete symmetric variety. J. Algebra 261 (2003), no. 2, 310–326.
[48] F.R. Cossec and I.V. Dolgachev: Enriques surfaces I. Birkhäuser, Progress in Mathematics, Vol. 76, 1989
[49] D.A. Cox: The homogeneous coordinate ring of a toric variety. J. Alg. Geom. 4 (1995), no. 1, 17–50.
[50] D.A. Cox: Recent developments in toric geometry. Proc. Symp. Pure Math. 62 (1997), 389–436.
[51] D.A. Cox, J. Little, H. Schenck: Toric varieties, draft.
[52] A. Craw, D. Maclagan: Fiber fans and toric quotients. Discrete Comput. Geom. 37 (2007), no. 2, 251–266.
[53] J.L. Colliot-Thélène, J.-J. Sansuc: Torsors sous des groupes de type multiplicatif, applications à l’étude des points rationnels de certaines variétés algébriques. C. R. Acad. Sci. Paris Sér. A-B 282 (1976), no. 18, A1113–A1116.
[54] J.L. Colliot-Thélène, J.-J. Sansuc: Variétés de première descente attachées aux variétés rationnelles. C. R. Acad. Sci. Paris Sér. A-B 284 (1977), no. 16, A967–A970.
[55] J.L. Colliot-Thélène, J.-J. Sansuc: La descente sur une variété rationnelle définie sur un corps de nombres. C. R. Acad. Sci. Paris Sér. A-B 284 (1977), no. 19, A1215–A1218.
[56] J.L. Colliot-Thélène, J.-J. Sansuc: La descente sur les variétés rationnelles. II. Duke Math. J. 54 (1987), no. 2, 375–492.
[57] V.I. Danilov: The geometry of toric varieties. Uspekhi Mat. Nauk 33 (1978), no. 2(200), 85–134 (Russian); English transl: Russian Math. Surveys 33 (1978), no. 2, 97–154.
[58] J.A. De Loera, J. Ramban, F. Santos: Triangulations – Structures for Algorithms and Applications Series: Algorithms and Computation in Mathematics, Vol. 25. Springer Verlag, 2010.
[59] T. Delzant: Hamiltoniens périodiqües et images convexes de l’application moment. Bull. Soc. Math. France 116 (1988), no. 3, 315–339.
[60] U. Derenthal: Singular Del Pezzo surfaces whose universal torsors are hypersurfaces. Preprint, arXiv:math.AG/0604194
[61] U. Derenthal: Universal torsors of Del Pezzo surfaces and homogeneous spaces. Adv. Math. 213 (2007), no. 2, 849–864.
[62] U. Derenthal, Yu. Tschinkel: Universal torsors over Del Pezzo surfaces and rational points. In ”Equidistribution in Number theory, An Introduction”, (A. Granville, Z. Rudnick eds.), 169-196, NATO Science Series II, 237, Springer, 2007
[63] I.V. Dolgachev: Rational surfaces with a pencil of elliptic curves. Izv. Akad. Nauk SSSR Ser. Mat., 30 (1966), no. 5, 1073–1100 (Russian).
[64] I.V. Dolgachev, Y. Hu: Variation of geometric invariant theory quotients. With an appendix by Nicolas Ressayre. Inst. Hautes Etudes Sci. Publ. Math. 87 (1998), 5–56.
[65] D. Eisenbud, S. Popescu: The projective geometry of the Gale transform. J. Algebra 230 (2000), no. 1, 127–173.
[66] D. Eisenbud: Commutative Algebra with a view towards algebraic geometry. Graduate texts in Math. 150, Springer Verlag, New York, 1995
[67] E.J. Elizondo: Chow varieties, the Euler-Chow series and the total coordinate ring. In: Transcendental aspects of algebraic cycles, 3–43, London Math. Soc. Lecture Note Ser. 313, Cambridge Univ. Press, Cambridge, 2004
[68] E.J. Elizondo, K. Kurano, K. Watanabe: The total coordinate ring of a normal projective variety. J. Algebra 276 (2004), no. 2, 625–637.
[69] G. Ewald: Polygons with hidden vertices. Beiträge Algebra Geom. 42 (2001), no. 2, 439–442.
[70] W. Fulton: Introduction to Toric Varieties. Princeton Univ. Press, Princeton, 1993
[71] W. Fulton: Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 2. Springer-Verlag, Berlin, 1998
[72] W. Fulton, J. Harris: Representation theory. A first course. Graduate Texts in Mathematics, 129. Springer-Verlag, New York, 1991.
[73] S.A. Gaifullin: Affine toric SL(2)-embeddings. Mat. Sbornik 199 (2008), no. 3, 3–24 (Russian); English transl.: Sbornik Math. 199 (2008), no. 3-4, 319–339.
[74] C. Galindo, F. Monserrat: The total coordinate ring of a smooth projective surface. J. Algebra 284 (2005), no. 1, 91–101.
[75] S. Giuffrida, R. Maggioni: The global ring of a smooth projective surface. Matematiche (Catania) 55 (2000), no. 1, 133–159.
[76] F.D. Grosshans: Algebraic Homogeneous Spaces and Invariant Theory. LNM 1673, Springer-Verlag, Berlin, 1997
[77] N. Guay: Embeddings of symmetric varieties. Transformation Groups 6 (2001), no. 4, 333–352.
[78] R. Hartshorne: Algebraic Geometry. GTM 52, Springer-Verlag, 1977
[79] B. Hassett: Equations of universal torsors and Cox rings. Math. Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004, 135–143, Universitätsdrucke Göttingen, Göttingen (2004)
[80] B. Hassett, Yu. Tschinkel: Universal torsors and Cox rings. In: Arithmetic of higher-dimensional algebraic varieties, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 149–173 (2004)
[81] J. Hausen: Producing good quotients by embedding into toric varieties. Geometry of toric varieties, 193–212, Sémin. Congr., 6, Soc. Math. France, Paris, 2002.
[82] J. Hausen: Equivariant embeddings into smooth toric varieties. Canad. J. Math. 54 (2002), no. 3, 554–570
[83] J. Hausen: Geometric invariant theory based on Weil divisors. Compos. Math. 140 (2004), no. 6, 1518–1536.
[84] J. Hausen: Cox rings and combinatorics II. Mosc. Math. J. 8 (2008), no. 4, 711–757.
[85] J. Hausen, H. Süß: The Cox ring of an algebraic variety with torus action. To appear in Adv. Math., arXiv:0903.4789
[86] M. Hochster, J. Roberts: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. Math. 13, 115–175 (1974)
[87] Y. Hu, S. Keel: Mori dream spaces and GIT. Michigan Math. J. 48 (2000) 331–348.
[88] J.E. Humphreys: Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975
[89] T. Kajiwara: The functor of a toric variety with enough invariant effective Cartier divisors. Tohoku Math. J. 50 (1998), 139–157.
[90] M.M. Kapranov: Chow quotients of Grassmannians. I. Advances in Soviet Math. 16 (1993), Part 2, 29–110.
[91] A.D. King: Moduli of representations of finite-dimensional algebras. Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.
[92] F. Kirwan: Cohomology of quotients in symplectic and algebraic geometry. Math. Notes 31, Princeton University Press, Princeton, NJ, 1984
[93] P. Kleinschmidt: A classification of toric varieties with few generators. Aequationes Math. 35 (1988), no. 2-3, 254–266.
[94] F. Knop: Über Hilberts vierzehntes Problem für Varietäten mit Kompliziertheit eins. Math. Z. 213 (1993), no. 1, 33–36.
[95] F. Knop, H. Kraft, D. Luna, Th. Vust: Local properties of algebraic group actions. Algebraische Transformationsgruppen und Invariantentheorie, 63–75, DMV Sem., 13, Birkhäuser, Basel (1989)
[96] F. Knop, H. Kraft, Th. Vust: The Picard group of a G-variety. Algebraische Transformationsgruppen und Invariantentheorie, 77–87, DMV Sem., 13, Birkhäuser, Basel (1989)
[97] S. Kondō: Algebraic K3-surfaces with finite automorphism groups. Nagoya Math. J. 116, (1989), 1–15.
[98] S. Kovács: The cone of curves of a K3-surface. Math. Ann. 300 (1994), 681–691.
[99] M. Koras, P. Russell: Linearization problems. In Algebraic group actions and quotients, 91–107, Hindawi Publ. Corp., Cairo (2004)
[100] H. Kraft: Geometrische Methoden in der Invariantentheorie. Vieweg Verlag, Braunschweig, 1984
[101] H. Kraft, V.L. Popov: Semisimple group actions on the three dimensional affine space are linear. Comm. Math. Helv. 60 (1985), no. 3, 466–479.
[102] A. Laface, M. Velasco: A survey on Cox rings. Geom. Dedicata 139 (2009), 269–287.
[103] A. Laface, M. Velasco: Picard-graded Betti numbers and the defining ideals of Cox rings. J. Algebra 322 (2009), no. 2, 353–372.
[104] R. Lazarsfeld: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. A Series of Modern Surveys in Mathematics 48. Springer-Verlag, Berlin, 2004
[105] D. Luna: Slices étalés. Bull. Soc. Math. Fr., Suppl. Mém. 33 (1973), 81–105.
[106] D. Luna, Th. Vust: Plongements d’espaces homogènes. Comment. Math. Helv. 58 (1983), no. 2, 186–245.
[107] K. Matsuki: Introduction to the Mori program. Berlin, Heidelberg, New York, Springer-Verlag (2001)
[108] A.L. Mayer: Families of K3-surfaces. Nagoya Math. J. 48 (1972), 1–17.
[109] J.S. Milne: Étale Cohomology. Princeton University Press, Princeton, NJ, 1980
[110] E. Miller, B. Sturmfels: Combinatorial Commutative Algebra. Graduate Texts in Math. 227, Springer-Verlag, New York, 2005.
[111] S. Mori: Graded factorial domains. Japan J. Math. 3 (1977), no. 2, 223–238.
[112] D. Morrison: On K3-surfaces with large Picard number. Invent. Math. 75, (1984), no. 1, 105–121.
[113] D. Mumford: The red book of varieties and schemes. Springer-Verlag
[114] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete 34. Springer-Verlag, Berlin, 1994
[115] I.M. Musson: Differential operators on toric varieties. J. Pure Appl. Algebra 95 (1994), no. 3, 303–315.
[116] M. Mustaţă: Vanishing theorems on toric varieties. Tohoku Math. J. (2) 54 (2002), no. 3, 451–470.
[117] V.V. Nikulin: A remark on algebraic surfaces with polyhedral Mori cone. Nagoya Math. J. 157 (2000), 73–92.
[118] V.V. Nikulin: Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. J. Soviet Math. 22 (1983), 1401–1475.
[119] V.V. Nikulin: Surfaces of type K3 with a finite automorphism group and a Picard group of rank three. Trudy Mat. Inst. Steklov 165 (1984), 119–142 (Russian); English transl.: Proc. Steklov Math. Institute, Issue 3, 131–155 (1985)
[120] T. Oda: Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1988
[121] T. Oda: Problems on Minkowski sums of convex polytopes. The Oberwolfach Conference "Combinatorial Convexity and Algebraic Geometry", 26.10–02.11, 1997. [arXiv:0812.1418]
[122] T. Oda, H.S. Park: Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions. Tohoku Math. J. (2) 43 (1991), no. 3, 375–399.
[123] A.L. Onishchik, E.B. Vinberg: Lie Groups and Algebraic Groups. Springer-Verlag, Berlin Heidelberg, 1990
[124] T.E. Panov: Topology of Kempf-Ness sets for algebraic torus actions. Proc. Steklov Math. Inst. 263 (2008), 1–13.
[125] H.S. Park: The Chow rings and GKZ-decompositions for Q-factorial toric varieties. Tohoku Math. J. (2) 45 (1993), no. 1, 109–145.
[126] M. Perling: Graded rings and equivariant sheaves on toric varieties. Math. Nachr. 265 (2004), 87–107.
[127] L. Petersen, H. Süß: Torus invariant divisors. Preprint, [arXiv:0811.0517] to appear in Israel J. Math.
[128] I. Piatetskii-Shapiro, I.R. Shafarevich: A Torelli theorem for algebraic surfaces of type K3. Math. USSR Izv. 5 (1971), 547–587.
[129] O.N. Popov: The Cox ring of a Del Pezzo surface has rational singularities. Preprint, [arXiv:math.AG/0402154]
[130] V.L. Popov, E. Vinberg: Invariant Theory. Encyclopedia Math. Sciences 55, 123–185, Springer-Verlag, Heidelberg, 1994
[131] L. Renner: The cone of semi-simple monoids with the same factorial hull. Preprint, [arXiv:math.AG/0603222]
[132] N. Ressayre: The GIT-equivalence for G-line bundles. Geom. Dedicata 81 (2000), no. 1-3, 295–324.
[133] P. Russell: Gradings of polynomial rings. In Algebraic geometry and its applications (West Lafayette, IN, 1990), 365–373, Springer, New York (1994)
[134] B. Saint-Donat: Projective models of K3-surfaces. Amer. J. Math. 96 (1974), no. 4, 602–639.
[135] P. Samuel: Lectures on unique factorization domains. Tata Institute, Mumbay, 1964
[136] G. Scheja, U. Storch: Zur Konstruktion faktorieller graduierter Integritätsbereiche. Arch. Math. (Basel) 42 (1984), no. 1, 45–52.
[137] V.V. Serganova, A.N Skorobogatov: Del Pezzo surfaces and representation theory. Algebra Number Theory 1 (2007), no. 4, 393–419.
[138] V.V. Serganova, A.N Skorobogatov: On the equations for universal torsors over del Pezzo surfaces. J. Inst. Math. Jussieu 9 (2010), no. 1, 203-223.
[139] C.S. Seshadri: Quotient spaces modulo reductive algebraic groups. Ann. Math. (2) 95 (1972), 511–556.
[140] T.A. Springer: Linear algebraic groups. Reprint of the 1998 second edition. Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009
[141] A.N. Skorobogatov: On a theorem of Enriques-Swinnerton-Dyer. Ann. Fac. Sci. Toulouse Math. (6) 2 (1993), no. 3, 429–440.
[142] A.N. Skorobogatov: Torsors and rational points. Cambridge Tracts in Math. 144, Cambridge University Press, 2001
[143] R. Steinberg: Nagata’s example. Algebraic groups and Lie groups, 375–384, Austral. Math. Soc. Lect. Ser. 9, Cambridge Univ. Press, Cambridge, 1997
[144] M. Stillman, D. Testa, M. Velasco: Gröbner bases, monomial group actions, and the Cox rings of del Pezzo surfaces. J. Algebra 316 (2007), no. 2, 777–801.
[145] B. Sturmfels: Gröbner bases and convex polytopes. University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996
[146] B. Sturmfels, M. Velasco: Blow-ups of P^n−3 at n points and spinor varieties. Preprint, [arXiv:0906.5096]
[147] B. Sturmfels, Zh. Xu: SAGBI bases of Cox-Nagata rings. Preprint, [arXiv:0803.0892]
[148] H. Sumihiro: Equivariant completion. J. Math. Kyoto Univ. 14 (1974), 1–28.
[149] H. Süß: Canonical divisors on T-varieties. Preprint, [arXiv:0811.0620]
[150] J. Święcicka: Quotients of toric varieties by actions of subtori. Colloq. Math. 82 (1999), no. 1, 105–116.
[151] J. Święcicka: A combinatorial construction of sets with good quotients by an action of a reductive group. Colloq. Math. 87 (2001), no. 1, 85–102.
[152] D. Testa, A. Várilly-Alvarado, M. Velasco: Cox rings of degree one del Pezzo surfaces. Preprint, [arXiv:0803.0353]
[153] D. Testa, A. Várilly, M. Velasco: Big rational surfaces. Preprint, [arXiv:0901.1094]
[154] M. Thaddeus: Geometric invariant theory and flips. J. Amer. Math. Soc. 9 (1996), no. 3, 691–723.
[155] D.A. Timashev: Homogeneous spaces and equivariant embeddings. To appear in Encyclopaedia Math. Sci. 138, Springer-Verlag, 2010; [arXiv:math/0602228]
[156] E.B. Vinberg: Complexity of actions of reductive groups. Func. Anal. Appl. 20 (1986), no. 1, 1–11.
[157] E.B. Vinberg: On reductive algebraic semigroups. In: Lie Groups and Lie Algebras, E.B. Dynkin’s Seminar, Amer. Math. Soc. Transl. Ser. 2, vol. 169, Amer. Math. Soc., Providence, RI, 145–182 (1995)
[158] J. Włodarczyk: Embeddings in toric varieties and prevarieties. J. Alg. Geom. 2 (1993), no. 4, 705–726.
[159] D.-Q. Zhang: Quotients of K3-surfaces modulo involutions. Japan. J. Math. (N.S.) 24 (1998), no. 2, 335–366.
Index

D-divisor, 19  
D-localization, 19  
G-closedness, 16  
G-separation, 16  
G-variety, 11  
      morphism of, 11  
H-factorial, 38  
H-prime divisor, 38  
K-height, 37  
K-prime, 37  
[D]-divisor, 36  
[D]-localization, 36  
Q-factorial  
      point, 44  
      prevariety, 20  

action  
      effective, 14  
      strongly stable, 45  
affine algebraic group, 10  
affine intersection, 24  
affine morphism, 15  

algebra  
      graded, 5  
      integral, 6  
      algebraic group, 10  
      morphism of, 10  
      ample divisor, 45  

Cartier divisor, 20  
categorical quotient, 17  
character, 10  
character group, 10  
characteristic quasitorus, 40  
characteristic space, 40, 46  
coarsened grading, 7  
Cox ring, 25, 28  
      canonical, 29  
Cox sheaf, 25, 28  
      canonical, 29  

diagonal action, 12  
diagonalizable group, 10  

divisor  
      H-prime, 38  
      ample, 45  
      Cartier, 20  
      effective, 18  
      of a function, 18  
      prime, 18  
      principal, 18  

support of, 19  
Weil, 18  
divisor class group, 18  
divisorial algebra  
      sheaf of, 18  
effective  
      action, 14  
      effective divisor, 18  
equivariant morphism, 11  

factorial  
      point, 44  
      factorially graded, 37  
function  
      rational homogeneous, 46  
geometric quotient, 15  
good quotient, 15  
graded algebra, 5  
morphism of, 6  
graded homomorphism, 6  
grading, 5  
      coarsened, 7  
group  
      affine algebraic, 10  
      algebraic, 10  
      diagonalizable, 10  
      reductive, 15  

homogeneous ideal, 7  

ideal  
      homogeneous, 7  
      irrelevant, 17, 45  
integral algebra, 6  
invariant morphism, 15  
irrelevant ideal, 17, 45  

local class group, 43  
locally factorial  
      variety, 20  

monoid algebra, 6  
morphism  
      affine, 15  
      equivariant, 11  
      invariant, 15  
      of algebraic groups, 10  
      of graded algebras, 6  
orbit group, 13
orbit monoid, 13
order
    of a function, 18
Picard group, 43
point
    \( \mathbb{Q} \)-factorial, 44
    factorial, 44
prevariety, 15
prime divisor, 18
principal divisor, 18
Proj construction, 17
quasitorus, 10
    characteristic, 40
quotient
    categorical, 17
    geometric, 15
    good, 15
rational homogeneous function, 46
rational representation, 11, 12
reductive group, 15
relative spectrum, 19
representation
    rational, 11, 12
sheaf of divisorial algebras, 18
space
    characteristic, 46
    characteristic, 40
spectrum
    relative, 19
standard \( n \)-torus, 10
strongly stable action, 45
support
    of a divisor, 19

torus, 10
total coordinate space, 44, 46
Veronese subalgebra, 8, 16
weight cone, 6
weight group, 6
weight monoid, 6
Weil divisor, 18