Hessenberg Input Normal Representations

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Abstract

Every stable controllable input pair \((\tilde{A}, \tilde{B})\) is equivalent to an input pair which is in Hessenberg form and is input normal \((AA^* + BB^* = I)\). \((A, B)\) is represented as a submatrix of the minimal number of Givens rotations. The representation is shown to be generically identifiable. This canonical form allows for fast state vector updates and improved conditioning of system identification problems.

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1 INPUT NORMAL PAIRS

The goal of this paper is to propose representations of system pairs \((A, B)\) that are both well conditioned for system identification and are fast and convenient for numerical computation. Here the advance matrix, \(A\), is a real \(n \times n\) matrix and the control matrix, \(B\), is a real \(n \times d\) matrix. The computational advantages of Hessenberg form for state space systems are well known. \([5, 16]\). Recently, studies have shown that input normal form results in significant improvement in the conditioning of system identification and filter design problems \([11, 7, 9]\). In this paper, we show that every stable controllable input pair are equivalent to an input pair with both properties. By placing the additional requirement that \(A_{i+1,i} \geq 0\), this Hessenberg input normal (HIN) form is generically unique.

We parameterize the representation using Givens rotations. Only \(nd\) Givens rotations are needed, so state vector advances take \(O(4nd)\) multiplications. Furthermore the mapping between the Givens angles and the HIN forms is generically unique. Thus this representation results in generically identifiable problems of system identification.

Let \((B | A)\) denote the concatenation of \(B\) with \(A\). Our parameterizations of the \((A, B)\) by the product of Givens matrices is of the form:

\[
(B | A) = \left[ \prod_{i=1}^{nd} G_{j(i),k(i)}(\theta_i) \right]_{1:n,1:(n+d)}. \tag{1.1}
\]

Here \(G_{j,k}\) is a Given’s rotation in the \(i\)th and \(j\)th coordinate by \(G_{ij}: g_{i,j} = g_{j,i} = \cos(\theta), g_{i,j} = -g_{j,i} = \sin(\theta)\) and \(g_{k,m} = \delta_{k,m}\) otherwise, where \(g_{k,m}\) are the elements of \(G_{ij}\).

Let \((A, B)\) be stable and controllable. We define the controllability Grammian, \(P_{A,B}\) by

\[
P_{A,B} - AP_{A,B}A^* = BB^*. \tag{1.2}
\]

Two systems pairs, \((A, B)\) and \((\tilde{A}, \tilde{B})\) are equivalent when \(\tilde{A} \equiv T^{-1}AT\), and \(\tilde{B} \equiv T^{-1}B\) for some invertible \(T\). If \(T\) is orthonormal, we say they are orthogonally equivalent. We use the freedom of choosing \(T\) to choose an equivalent representation of \((A, B)\) which satisfies

\[
AA^* + BB^* = I_n. \tag{1.3}
\]
where $I_n$ is $n \times n$ identity matrix. If $(A, B)$ satisfies (1.3), we say $(A, B)$ is input normal [10]. If $(A, B)$ is stable and controllable, (1.3) is equivalent to requiring that the controllability Grammian, $P_{A,B}$, equal the identity matrix.

Input normal pairs have superior conditioning and roundoff properties. As an consider the common problem of estimating the observability matrix, $C$ given a stable, controllable input pair, $(A, B)$ and a set of noisy measurements, $\{y_t\}$. The state vector evolves according to $z_{t+1} = Az_t + B\epsilon_t$, where $\epsilon_t$ are known random variables. The standard recursive least squares (RLS) estimate of $C$ calculates $\hat{P}_t$ and $\hat{d}_t$:

$$
\hat{P}_t = \frac{1}{t} \sum_{i=1}^{t} z_i z_i^*, \quad \hat{d}_t = \frac{1}{t} \sum_{i=1}^{t} z_i y_i^* .
$$

The unknown coefficients, $\hat{C}$, are estimated by solving $\hat{P}_t \hat{C} = \hat{d}_t$ [6, 7]. As time increases, $\hat{P}_t$ converges to the controllability Grammian [2] [7] when the forcing noise, $\epsilon_t$, is white. When $(A, B)$ is input normal, $\hat{P}_t \xrightarrow{t \to \infty} \text{constant} \times I_n$. Thus the regression estimate of $C$ is well-conditioned. For more complete analysis of conditioning in system identification, we refer the reader to [9].

Similarly, IN filter structures are resistant to roundoff error [11]. Note that input normal representations will time asymptotically satisfy the ansatz need by least mean squares (LMS) identification algorithms. This leads to a second advantage of IN filters: Gradient algorithms such as the least mean squares (LMS) algorithm often perform well enough in certain applications to obviate the need for more complicated and computationally intensive RLS algorithms.

Our representations include the banded orthogonal filters of [11] as a special case of $d = 1$. Together with A. Mullhaupt [10], we have proposed several other representations of input normal pairs, where $(A, B)$ is the product of $nd$ Givens rotations.

Our approach has similarities to that of embedded lossless systems (ELS) [3, 14, 15]. While embedded lossless systems are of interest in a few applications, we believe that it is simpler to parameterize $(A, B)$ as a HIN pair. This allows $(C, D)$ to be determined by linear or pseudolinear regression, and this regression is very well conditioned since the expected value of controllability Grammian is the identity matrix. In contrast, ELS satisfy $AA^* + BB^* + B_{\text{ext}} B_{\text{ext}}^* = I_n$ where $B_{\text{ext}}$ is an artificial term for the embedding. Thus the ELS controllability Grammian for the actual system pair can be ill-conditioned. The construction of the Givens matrix representation of [15] parallels our representation in Section 3. One can interpret our results as a simplification of [15] for those who prefer input normal representations to embedded lossless systems. Another difference between our treatment and the analyses of [3, 14, 15] is that we describe when redundant representations can occur.

**Notation:** By $A_{i,j,k,m}$, we denote the $(j-i+1) \times (m-k+1)$ subblock of $A$ from row $i$ to row $j$ and from column $k$ to column $m$. The direct sum of matrices is denoted by $\oplus$. We denote the matrix transpose of $A$ by $A^*$ with no complex conjugation since we are interested in the real system case.

## 2 Hessenberg Input Normal Pairs

Hessenberg form is a canonical form where $A$ is Hessenberg with the additional restriction that $B_{1,1} \geq 0, B_{j,1} = 0$ for $j > 1$.

**Definition 2.1.** The input pair $(A,B)$ is in Hessenberg form if $A$ is a Hessenberg matrix, $B_{1,1} \geq 0$, and $B_{j,1} = 0$ for $j > 1$. A Hessenber pair is nondegenerate if $|B_{1,1}| < 1$. A Hessenberg pair is unreduced if $A_{i+1,i} \neq 0$ for $1 \leq i < n$ and $B_{1,1} \neq 0$. A Hessenberg pair is standard if $A_{i+1,i} \geq 0$ for $1 \leq i < n$, $0 \leq B_{1,1} \leq 1$. A Hessenberg pair is strict if it is unreduced and standard.

If $(A, B)$ is a HIN pair, so is $(EAE^{-1}, EB)$ where $E$ is an arbitrary signanture matrix: $|E_{i,j}| = \delta_{i-j}$. Thus we seek a representation of standard HIN pairs to eliminate the multiplicity of equivalent representations when $(A, B)$ is unreduced.
Theorem 2.2. [5, 16] Any observable input pair is orthogonally equivalent to a system in Hessenberg form.

The standard proof of Theorem 2.2 begins by transforming $B$ to its desired form and then defines Givens rotations which zero out particular elements in $A$ in successive rows or columns [4, 5].

For nondegenerate HIN pairs, we find that the set of standard input normal pairs has a bijective representation as an easy to parameterize subset of Givens product representations. The existence and uniqueness results in this section originated in the unpublished technical report [10]. The proof of Theorem 2.3 is also in [15]. We reproduce them for completeness.

Theorem 2.3. [10] Every stable, real controllable output pair $(A, B)$, is similar to a standard HIN pair.

Proof: The unique solution, $P_{A,B}$, of Stein equation, (1.2), is strictly positive definite. Let $L$ be the unique Cholesky lower triangular factor of $P$ with positive diagonal entries: $P = LL^*$. We set $T = L^{-1}$. Let $U$ be orthogonal transformation that takes $(T^{-1}AT, T^{-1}B)$ to the Hessenberg form as described in [5, 16]. We now choose the signature matrix, $E$ such that $T \equiv EUL^{-*}$ makes $(T^{-1}AT, T^{-1}B)$ a standard HIN pair. □

Degenerate HIN pairs correspond to the direct sum of an identity matrix and a nondegenerate HIN system:

Lemma 2.4. Every stable, real controllable input pair $(A, B)$, is similar to a HIN pair which is the direct sum of the identity matrix and a nondegenerate HIN pair: $(B|A) = (\hat{I}_m \oplus \hat{Q})$ for some $m \leq d$, where $\hat{Q}$ is a $(n-m) \times (n+d-m)$ row orthogonal matrix.

There are two main ways in which one of our system representations can fail to parameterize linear time invariant systems in a bijective fashion. First, there may be a multiplicity of equivalent HIN systems. Second, Givens product representation such as (1.1) may have multiple (or no) parameterizations of the same input pair. We now show when that if $(A, B)$ is a strict HIN pair, there are no equivalent strict HIN pairs. The uniqueness results are based on the following lemma that generalizes the Implicit Q theorem [4] to HIN pairs:

Lemma 2.5. Let $(A, B)$ and $(\tilde{A}, \tilde{B})$ be orthogonally equivalent standard nondegenerate HIN pairs ($\tilde{A} \equiv T^{-1}AT, \tilde{B} \equiv T^{-1}B$ where $T$ is orthogonal). Let $A_{k+1,k} = 0$, $B_{11} > 0$ and $A_{j+1,j} > 0$ for $j < k$, then $T = I_k \oplus U_{n-k}$, where $U_{n-k}$ is an $(n-k) \times (n-k)$ orthogonal matrix. Furthermore, $k > 1$ and $\tilde{\hat{A}}_{k+1,k} = 0$.

Since $B_{j,j} = \tilde{B}_{j,j} = 0$, $j > 1$, $T_{j,1} = \delta_{j,1}$. The result follows from the Implicit Q theorem [4]. □

Corollary 2.6. If $(A, B)$ is a strict nondegenerate HIN pair, then there are no other equivalent strict HIN pairs.

When $A_{k+1,k} = 0$, there are many different equivalent HIN pairs. This degeneracy may be reduced or eliminated if additional restrictions are imposed on $(A, B)$. In [10], it is proposed that $A_{k+1,n,k+1:n}$ be placed in real Schur form with a given ordering of the eigenvalues and standardization of the $2 \times 2$ diagonal subblocks. An alternative restriction is to require $B_{k+1,2} \geq 0$ and $B_{k+2,n,2} = 0$ when $d > 1$.

3 Givens Representation of Hessenberg input normal form.

In this section, we give representation results for HIN pairs. The first column of $B$ satisfies $0 < B_{1,1} \leq 1$, $B_{j,1} = 0$ for $j > 1$. We use Givens rotations to zero out the lower diagonal of $A$ and the row 2 through row $d$ of $B$. For each row of the $(B|A)$, we use $d$ Givens rotations. to zero out the lower diagonal of $A$ and columns 2 through $d$ of $B$. The following lemma shows how this works on a single row of $(B|A)$.

Lemma 3.1. Let $X$ be the set of real $d+1$ tuples, $x = (x_1, \ldots, x_d, x_{d+1})$ where $\sum x_j^2 = 1$ and $x_d \geq 0$. 

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Let \( g(\theta) \) denote the \( d + 1 \) tuple \((0 \ldots 0,1)G_{d+1,d}(\theta_d)G_{d,d-1}(\theta_{d-1}) \ldots G_{3,2}(\theta_2)G_{2,1}(\theta_1), \) where \( G_{i,i+1} \) are the Givens rotations in \( R^{d+1} \). Let \( \Theta \) denote the restriction of the Givens rotation angles to \( 0 \leq \theta_i \leq \pi, -\pi/2 < \theta_{j} \leq \pi/2 \) for \( 1 < j \leq d \). The mapping from the \( \Theta \), the domain of the Givens angles, is onto \( X \). The mapping is one-to-one for the set of \( x \) where \( |x_1| > 0 \).

**Proof:** Clearly, the first Givens angle is uniquely determined by the last component of \( x \). If \( x_{d+1} = \pm 1 \), the rest of the representation, \( \{\theta_2 \ldots \theta_d\} \) is arbitrary. The \( k \)th component of \( g(\theta) \) is \( x_k = g(\theta)_k = \cos(\theta_k) \prod_{k=1}^{d} \sin(\theta_i) \). If \( x_{k-1} \) is nonzero, we determine \( \theta_k \) uniquely by \( g(\theta)_k = x_k \) and \( \text{sign}(g(\theta)_k) = \text{sign}(x_{k-1}) \). If \( x_{k-1} = 0 \), we require \( \text{sign}(g(\theta)_k) = \text{sign}(x_j) \) where \( j \) is the largest index, \( 1 \leq j < k \) with \( x_j \neq 0 \). This determination of the Givens angles is unique unless \( x_1 \) vanishes. 

Note that \( \theta_j = -\pi/2 \) is not necessary. We can make the mapping of \( g(\theta) : \Theta \rightarrow X \) globally one to one if we require that whenever \( \sum_{i<j} |x_i| = 0 \) for some \( j \), that \( \theta_i = 0 \) for \( i < j \) and that if \( j < d \) then \( \theta_j = 0 \). If \( j = d \) then \( \theta_j = \pm \pi \).

**Theorem 3.2.** Every real HIN pair has the representation:

\[
(B|A) = (0_n,d | I_n)U^{(1)}(\theta_1 : \theta_d)V^{(2)}(\theta_{d+1} : \theta_{2d}) \cdots V^{(n-1)}V^{(n)}(\theta_{(n-1)d+1} : \theta_{nd})
\]  

(3.1)

where \( 0_{n,d} \) is the \( n \times d \) matrix of zeros. Here \( U^{(1)} \) and \( V^{(k)} \) are \((n + d) \times (n + d)\) matrices which are the product of \( d \) Givens rotations:

\[
V^{(k)}(\theta_{(k-1)d+1} : \theta_{kd}) = G_{k+d,k+d-1}G_{n+d-1,d}G_{n+d-2,d-1}G_{n+d-3,d-2} \cdots G_{4,3}(\theta_{k-1d+1})G_{3,2}(\theta_{kd})
\]  

(3.2)

\[
U^{(1)}(\theta_1 : \theta_d) = G_{d+1,1}(\theta_1)G_{1,2}(\theta_2) \cdots G_{d-2,d-1}(\theta_{d-1})G_{d-1,d}(\theta_d)
\]  

(3.3)

Every matrix of the form of the righthand side of (3.1) is a HIN pair.

We successively determine the Givens angles, \( \theta_k \). At the \( k \)th stage, \( \theta_{(n-k-1)d+1} \ldots \theta_{(n-k)d} \) are determined to zero out the \( d \) of the \( d + 1 \) nonzero entries in the \( k \)th row. By orthogonality, the other entries in the \( n - k \)th column must be zero.

**Proof:** Let

\[
\Gamma^{(k)}(\theta_{kd+1} : \theta_{nd}) \equiv (B|A)V^{(n)}V^{(n-1)} \cdots V^{(k+1)}.
\]  

(3.4)

Assume that \( \Gamma^{(k)} \) has its last \( k \) rows satisfying \( \Gamma^{(k)}_{i,j} = \delta_{i-j-d} \). Since \( \Gamma^{(k)} \) has orthonormal rows, the last \( k \) columns satisfy \( \Gamma^{(k)}_{i,j} = \delta_{i-j+n} \). Select \( \theta_{kd+1} \) through \( \theta_{k+1d} \) such \( \Gamma^{(k+1)}_{1,d,k+1} = 0 \) and \( \Gamma^{(k+1)}_{n-k+2,d,n-k+1} = 0 \). Then \( \Gamma^{(k+1)}_{i,j} = \delta_{i-j+d} \) for the last \( k + 1 \) rows and the last \( k + 1 \) columns.

To show all Givens products of the form in (3.1) generate HIN pairs, consider

\[
X^{(k)}(\theta_1 \ldots \theta_{kd}) = (0_n | I_n)U^{(1)}(\theta_1 : \theta_d)V^{(2)}(\theta_{d+1} : \theta_{2d}) \cdots V^{(k)}(\theta_{(k-1)d+1} : \theta_{kd})
\]  

(3.5)

\( X^{(k)} \) is in Hessenberg form for each \( k \).

For \( d = 1 \) and \( B_{1,1} = 0 \), (3.1) is the well-known expression of an unitary Hessenberg matrix as a product of \( n \) Givens rotations \[11\]. The fast filtering methods of \[16\[7\] may be further sped up when \((B|A) \) is a submatrix product of \( nd \) Givens rotations.

**Theorem 3.3.** Let \( \Theta \) be the set \( 0 \leq \theta_{kd+1} \leq \pi, -\pi/2 < \theta_{kd+j} \leq \pi/2 \), where \( 1 < j \leq d \). The representations of Theorem 3.2 maps \( \Theta \) onto the set of standard HIN pairs. The mapping is one-to-one for \( d = 1 \). For \( d > 1 \), the mapping is one-to-one on the set where \( B_{1,d} > 0, B_{1,2} > 0 \) for \( j > 1 \). Let \( B^{\text{cat}} \) be the concatenation of \( B_{2,d} \) with the vector \((B_{1,1}, A_{2,1} \ldots A_{n-1,n}) \). If whenever \( \sum_{i<j} |B^{\text{cat}}_{i,j}| = 0 \), we impose the constraint that \( \theta_{(k-1)d+i} = 0 \) for \( i < j \) and for \( j < d \) the additional constraint that \( \theta_{(k-1)d+j} > 0 \), then there is a one to one correspondence between strict HIN pairs and the parameterization of Theorem 3.2.

**Proof:** We repeatedly apply Lemma 3.1 to each row of \( X^{(k)} \).

Theorem 3.3 does not address the multiplicity of equivalent HIN pairs when \( A_{i+1} \) vanishes. The representation of Theorem 3.2 is nonuniform because the order of the Givens rotations in \( U^{(1)} \) differ
from that of $V^{(k)}$. If we require $B_{1,d} \geq 0$ and $B_{j,d} = 0$ instead of $B_{1,1} \geq 0$ and $B_{j,1} = 0$, the representation is more uniform in selecting the order of the Givens rotations.

The disadvantage of the Givens angle restrictions is that the representation is discontinuous when $(A, B)$ corresponds to a boundary value of $\Theta$. An example of this behavior is that if $B_{n,d}$ is near zero, $\theta_1$ varies from $\pi/2$ to $-\pi/2$. For this reason, it may be preferable to eliminate the angular restrictions on $\theta$ while performing a numerical optimization to determine the Givens angles.

4 Summary.

We have examined the uniqueness/identifiability of system pairs when $(A, B)$ is simultaneously in Hessenberg form and input normal. Since controllability Grammian is the identity matrix, estimation of the observability matrix is well-conditioned. These system pairs have good roundoff properties. To transform a specific input pair to IN form, the Stein equation (1.2) must be solved for $P_{A,B}$. The numerical conditioning of this problem can be quite poor [13, 9]. Thus it is much better to start with an explicit representation of $(A, B)$ in HIN form than to transform a non-input normal system into HIN form. To make IN pair attractive for applications, we describe a concise representation with fast state vector updates.

In this case, the representation of Theorem 3.2 is particularly convenient. $(A, B)$ has a representation as a submatrix of the product of $n$ $d$ Givens matrices. We have shown how to place restrictions on the parameters such that the Givens representation is in one to one correspondence with the set of standard HIN pairs. For statistical estimation, it is sometimes desirable to eliminate redundancy in the parameterization. We address redundancy in two ways. First, we categorize when two distinct HIN pairs are equivalent. Second, we impose constraints on the parameters in (1.1) to eliminate redundant parameterizations of the same HIN pair generically. However the resulting constraint set, $\Theta$ has the property that small perturbations of $(A, B)$ can cause large changes in the Givens angles near the boundary of $\Theta$. Thus it may be better to not restrict the Givens angles in numerical optimizations.

We do not explicitly store or multiply by $V^{(k)}$ in (3.1). Instead we store only the Givens parameters and we perform the matrix multiplication implicitly. Thus vector multiplication by $A$ and by $\frac{d}{d\theta}A$ require $O(4nd)$ and $O(8nd)$ operations. Similar Givens product representations of output normal $(A, C)$ when $A$ is in real Schur form or $(A, C)$ or in Hessenberg observer form may be found in is in [10].

In summary, HIN representations offer the best possible conditioning while having a convenient representation with fast matrix multiplication.

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