Black hole entropy, curved space and monsters

Stephen D. H. Hsu and David Reeb

Institute of Theoretical Science
University of Oregon, Eugene, OR 97403

We investigate the microscopic origin of black hole entropy, in particular the gap between the maximum entropy of ordinary matter and that of black holes. Using curved space, we construct configurations with entropy greater than the area $A$ of a black hole of equal mass. These configurations have pathological properties and we refer to them as monsters. When monsters are excluded we recover the entropy bound on ordinary matter $S < A^{3/4}$. This bound implies that essentially all of the microstates of a semiclassical black hole are associated with the growth of a slightly smaller black hole which absorbs some additional energy. Our results suggest that the area entropy of black holes is the logarithm of the number of distinct ways in which one can form the black hole from ordinary matter and smaller black holes, but only after the exclusion of monster states.

Black holes radiate \cite{1} and have entropy $S = A/4$, where $A$ is the surface area in Planck units \cite{2}. The nature of this entropy is one of the great mysteries of modern physics, especially due to its non-extensive nature: it scales as the area of the black hole in Planck units, rather than its volume. This peculiar property has led to the holographic conjecture \cite{3, 4} proposing that the number of degrees of freedom in any region of our universe grows only as the area of its boundary. The AdS/CFT correspondence \cite{5} is an explicit realization of holography.

The entropy of a thermodynamic system is the logarithm of the number of available microstates of the system, subject to some macroscopic constraints such as fixed total energy. For a black hole, this means all possible internal states with fixed total mass, charge and angular momentum. In certain string theory black holes, these states have been counted explicitly \cite{6, 7}. As a proxy for counting microstates, we might instead count the number of distinct ways of forming a black hole \cite{8}, since each distinct pre-configuration presumably corresponds to a unique microstate.

It is easy to see that gravitational collapse limits the entropy of physical systems. Information (entropy) requires energy, while gravitational collapse (formation of a horizon or black hole) restricts the amount of energy allowed in a finite region \cite{9}. ’t Hooft \cite{10} showed that if one excludes configurations whose energies are so large that they will inevitably undergo gravitational collapse, one obtains $S < A^{3/4}$. To deduce this result, ’t Hooft replaces the system under study with a thermal one. This is justified because, in the large volume limit, the entropy of a system with constant total energy $E$ (i.e., the logarithm of the phase space volume of a microcanonical ensemble) is given to high accuracy by that of a canonical ensemble whose temperature has been adjusted so that the average energies of the two ensembles are the same.

Given a thermal region of radius $R$ and temperature $T$, we have $S \sim T^3 R^3$ and $E \sim T^4 R^3$. Requiring $E < R$ (using the hoop conjecture \cite{10, 11}) then implies $T < R^{-1/2}$ and $S < R^{3/2} \sim A^{3/4}$. We stress that the use of temperature here is just a calculational trick – the result can also be obtained by directly computing the volume of phase space on a surface of fixed energy, as limited by the collapse condition.

In \cite{12}, it was shown that imposing the condition $\text{Tr} [\rho H] < R$ on a density matrix $\rho$ implies a similar bound $S_{vN} < A^{3/4}$ on the von Neumann entropy $S_{vN} = -\text{Tr} \rho \ln \rho$. For $\rho$ a pure state the result reduces to the previous Hilbert space counting.

We note that these bounds are more restrictive than the bound obtained from black hole entropy: $S < A/4$. Is there a gap between the maximum entropy of matter configurations and that of black holes? If so, it would imply that microstates of a large black hole are overwhelmingly dominated by those originating from a slightly smaller black hole \cite{13}. The $\exp(A^{3/4})$ matter configurations without horizons would be negligible compared to the $\exp(A/4 - \delta)$ slightly smaller black holes that might, upon the addition of a small amount of energy, have formed a given black hole of area $A$.

Packing it in

’t Hooft’s calculation described above is done in flat space, taking spatial volume to be proportional to $R^3$. The only appearance of general relativity is in the hoop conjecture. We now show that the $A^{3/4}$ and $A$ entropy bounds can be exceeded by matter configurations in curved space, in effect by changing the relationship between internal volume and surface area. A technical remark: in a general curved spacetime the “size” or “area” of a region is difficult to define in a coordinate-independent way. However, in the case of spherical symmetry, which we assume here, these issues do not arise \cite{14}. Moreover, what we are primarily interested in is the entropy of our configuration relative to the area of a black hole of equal mass, into which it will evolve.

We consider spherically symmetric, but not necessarily static, distributions of matter, using standard coordi-
\[ ds^2 = -g_{tt}(r,t)dt^2 + g_{rr}(r,t)dr^2 + r^2d\Omega^2 \ . \]

Further, we define
\[ \epsilon(r) = 1 - \frac{2M(r)}{r} , \]
with (“energy within radius \( r^\prime \)"
\[ M(r) = 4\pi \int_0^r dr' r'^2 \rho(r') \ , \]
where \( \rho(r) = \rho(r,t_0) \) is the proper energy density (i.e., as seen by a stationary observer at \( r \)) on the initial time slice \( t = t_0 \). Then, assuming the matter to be initially at rest w.r.t. our \((r,\theta,\phi)\) coordinates, the metric on that slice is fully determined by \(15\).

\[ g_{rr}(r,t_0) = \epsilon(r)^{-1} \ . \]

The total mass (or ADM energy) is simply \( M \equiv M(R) \) if \( R \) is the radius of the distribution. The total entropy is obtained as follows. First, assume the existence of a covariantly conserved entropy current \( j^\mu \), i.e. \( j^{\mu,\nu} = 0 \). (If entropy is not conserved, the second law requires that it always increases, which means our result is still a lower bound for any black hole produced.) From the Stokes theorem we have that
\[ S_\Sigma = \int_\Sigma d^3 x \sqrt{-g} s = \text{constant} \ , \]
where the integral is taken over a constant time slice \( \Sigma \) with induced metric \( \gamma \) and unit normal \( n^\mu \sim (\partial_t)^\mu \), and \( s = -j^\mu n_\mu \) is the proper entropy density (as seen by a stationary inertial observer). In our coordinates, \( s(r) = j^0(r,t_0)\partial_t(r,t_0)^{1/2} \) and the total entropy of the initial configuration on the time slice \( t = t_0 \) is given by
\[ S = 4\pi \int_0^R dr r^2 \epsilon(r)^{-1/2} s(r) \ . \]

For related discussion, see \[16\].

Note that the proper mass of our object is
\[ M_p = 4\pi \int_0^R dr r^2 \epsilon(r)^{-1/2} \rho(r) \ , \]
and the difference between \( M \) and \( M_p \) is the negative binding energy. As discussed below, the ratio \( M/M_p \) can be made as small as desired for large \( R \) \[17\].

To ensure that our object is not already a black hole, we require \( \epsilon(r) > 0 \) at all \( r \). Subject to this constraint, we attempt to maximize \( S \). The resulting entropy is a lower bound on the entropy of a black hole of radius \( R \). Here we of course refer to the internal state of the hole; from the outside they are identical.

We take \( s(r) \sim \rho(r)^{3/4} \), which is appropriate for relativistic matter. For thermal matter we would have \( s \sim T(r)^3 \) while \( \rho \sim T(r)^4 \), where \( T(r) \) is the temperature at radius \( r \). Note we do not assume our configuration is in thermal equilibrium; temperature is used here to count the number of initial configurations with the desired energy density profile \( \rho(r) \), as in the case of the flat space calculation. Another possibility is a relativistic Fermi gas, in which the energy scale is determined by the Fermi momentum.

The difference between the curved and flat space cases is due entirely to the factor of \( \epsilon(r)^{-1/2} \) in integrals like Eq. \[6\] and Eq. \[7\]. Consequently, the flat space bound of \( S < A^{3/4} \) can only be exceeded for configurations in which \( \epsilon(r) \) is close to zero (equivalently, \( 2M(r) \approx r \)) in a subregion containing significant entropy and energy density. In fact, for any configuration in which \( \epsilon(r) > \epsilon_0 \) for all \( r \), one can easily deduce that \( S < \epsilon_0^{-1/2} A^{3/4} \), since by removing \( \epsilon(r) \) from the integral in Eq. \[6\] one is left with the flat space entropy.

Some explicit examples are given below, in which curved space allows violation of both the \( A^{3/4} \) and \( A \) entropy bounds. (See \[18\] for a discussion of highly en- tropic objects and their effect on black hole thermodynamics.) Subsequently, we will show that configurations with significant energy density in regions with \( \epsilon(r) \approx 0 \) have pathological properties, and we will refer to them as monsters.

**Example 1: blob of matter**

As a simple example, consider an object with a small core of radius \( r_0 \) and mass \( M_0 \) and density profile
\[ \rho(r) = \rho_0 \left( \frac{r_0}{r} \right)^2 \quad (r_0 < r < R) \ . \]

Then
\[ M(r) = M_0 + 4\pi \rho_0 r_0^2 (r - r_0) \ . \]

We choose \( 8\pi \rho_0 r_0^2 = 1 \) so that
\[ \epsilon(r) = \epsilon_0 \left( \frac{r_0}{r} \right) \ , \]
where \( \epsilon_0 = 1 - 2M_0/r_0 \).

From Eq. \[6\], the total entropy of this object is (neglecting the small core region \( r < r_0 \))
\[ S \sim 4\pi \int_{r_0}^R dr r^2 \left( \frac{r}{r_0 \epsilon_0} \right)^{1/2} \rho^{3/4} \sim \frac{\rho_0^{3/4} r_0^2}{\sqrt{\epsilon_0}} R^2 \ . \]

Note that area scaling has been achieved. The overall entropy \( S \) can be made as large as desired by taking \( \epsilon_0 \) small. We can also obtain faster than area scaling by taking \( \epsilon(r) \) to approach zero faster than \( 1/r \).

**Example 2: thin shell**

Consider a thin shell of material with \( R < r < R + d \).
We first consider the class of models \( M(r) = (R + d) z^n / 2, \) where \( z = (r - R)/d \) and \( n > 0. \) In these models the mass of the shell increases smoothly to the maximum possible value as \( r \) approaches \( R + d. \)

We write the energy density \( \rho(r) \) as

\[
\rho(r) = \frac{M'(r)}{4\pi r^2},
\]

where prime denotes differentiation with respect to \( r. \) Then, the entropy of the shell is given by

\[
S = (4\pi)^{1/4} \int_{R}^{R + d} dr r^{1/2} \frac{M'(r)^{3/4}}{\epsilon(r)^{1/2}}.
\]

Taking \( d \) much less than \( R, \)

\[
S \sim R^{5/4} d^{1/4} \int_{0}^{1} dz \frac{n^{3/4} z^{3(n-1)/4}}{(1 - z^n)^{1/2}}.
\]

The \( z \) integral is convergent for \( n > 0, \) so the entropy scaling is given by \( S \sim R^{5/4} d^{1/4} \) or at most \( S \sim A^{3/4} \) if \( d \) scales at most as \( R. \)

However, we can also construct thin shell configurations with unbounded entropy. For example, take \( \epsilon(r) \) to decrease rapidly to some \( \epsilon_0 \) between \( R \) and \( R_1:\)

\[
M(r) = \frac{r - R}{2(R_1 - R)} R_1 (1 - \epsilon_0) \quad (R < r < R_1),
\]

and then hold \( \epsilon(r) = \epsilon_0 \) for \( r > R_1:\)

\[
M(r) = \frac{r}{2} (1 - \epsilon_0) \quad (R_1 < r < R + d).
\]

The entropy in the region \( R_1 < r < R + d \) can be made as large as desired by taking \( \epsilon_0 \) sufficiently small.

As demonstrated, curved space configurations can have greater entropy than their flat space counterparts of the same mass or size. This is because of their small \( \epsilon(r): \) the configurations have proper surface area \( A \sim M^2, \) but have internal proper volume much larger than \( A^{3/2}. \) Equivalently, they have very large proper mass \( M_p \) relative to mass \( M. \) It is easy to see that the ratio \( M/M_p \) can be made as small as desired if \( \epsilon(r) \) approaches zero for large \( r. \) The large negative gravitational binding energy allows us to pack substantially more proper mass into the region than suggested by a flat space analysis.

Regarding coordinate invariance of our results, we note that the total entropy \( S \) of the initial configuration on the time slice \( t = t_0 \) is, by construction in [19], coordinate-invariant. Also, the area \( A \) of a black hole formed by one of our configurations (by construction, on the verge of collapse) is simply a function of the ADM mass \( M, \) which is invariant. Of physical interest here is the entropy of our configuration compared to the area \( A \) of a black hole of equal mass.

Without a constraint on how close \( \epsilon(r) \) can get to zero, \( S \) can be made arbitrarily large. Invoking quantum effects, one might require that a Planck length uncertainty in the proper radial distance not cause horizon formation, i.e. that \( \epsilon(r) \) not become negative if the denominator in Eq. (2) is replaced by \( r \pm \epsilon(r)r^{1/2}. \) This implies \( \epsilon(r) > r^{-2}, \) and limits the entropy of configurations as in example 1 to \( S \sim R^{5/2}. \) This is still potentially problematic for the area entropy of black holes. A limit of \( S < A \) would require that \( \epsilon(r) > r^{-1}. \) This would be the consequence of the previous logic if one assumed a Planck length uncertainty in the radial coordinate \( r \) rather than the proper radial distance \( r \epsilon(r)r^{-1/2} \) (or equivalently an uncertainty in proper radial distance which grows as \( \epsilon(r)r^{-1/2}. \) This seems unphysical, but nevertheless cannot be excluded as a consequence of quantum gravity. For related ideas, see the stretched horizon in string theory [20].

Below, we discuss the pathological properties of the configurations which exceed the \( A^{3/4} \) bound.

**Destroy all monsters!**

To obtain entropy scaling faster than \( A^{3/4}, \) we must consider configurations in which \( \epsilon(r) \) is close to zero in regions containing significant entropy and energy density. We now show that such configurations have the following pathological properties.

I. They inevitably evolve into black holes, even in the absence of any outside perturbation.

II. Even their time-reversed evolution leads to black hole formation.

They are therefore neither ordinary black holes nor ordinary matter configurations. We refer to them as monsters.

To demonstrate I and II we consider the critical escape angle \( \theta_c \sim \epsilon(r)^{1/2} \) [21]. Only particles whose trajectories make an angle less than \( \theta_c \) with the outward radial direction can escape to infinity. All others follow orbits which bring them to smaller \( r. \) (This phenomenon also contributes to the persistence of a black hole atmosphere, or stretched horizon [22].) A highly entropic configuration – i.e., one in which individual particle states have nearly monodromy configurations. We refer to them as monsters.

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These conclusions apply as well to the time-reversed evolution, since the time-reversal of a highly entropic configuration is similar to the original. (If momenta of individual particles in the configuration are randomly distributed, then so are time-reversed momenta.) A small subset of configurations, with entropy density reduced by a factor of \( d^2 \sim \epsilon(r), \) can avoid I and II if their individual particle momenta are all nearly radial, or equivalently if all modes are nearly S-wave. However, the reduction in entropy density by \( \epsilon(r) \) implies that the total entropy of such configurations is less than that of flat space configurations.

In fact, if one defines a black hole as a region whose future does not include future null infinity, then most of
a monster configuration already comprises a black hole \[23\]. Even particles with exactly radial trajectories cannot escape if they are deep inside – the infall of modes closer to the surface will cause a horizon to form before they can escape. Roughly speaking, a configuration can have no more than a mass fraction of order \(\epsilon_0\) in a region with \(\epsilon(r) \sim \epsilon_0\) without eventually becoming a black hole.

One might argue that it is impossible to create a monster, since it turns into a black hole when evolved backwards in time: we would have to begin with a white hole. However, the argument is not conclusive: we could start with a normal configuration with the same quantum numbers (e.g., ADM mass, charge, etc.) as the monster, which tunnels or fluctuates quantum mechanically into the monster state. There must be a nonzero, albeit very small, probability for this if no conservation law is violated. Unless this process is forbidden by new physics, it implies at least \(\exp S\) black hole microstates, where \(S\) can grow faster than \(A\) and may even be unbounded. Note, though, that these states are inaccessible to observers outside the hole. They cannot affect aspects of black hole thermodynamics involving physics outside the horizon.

Clearly, monsters pose an interesting challenge to the interpretation of black hole entropy as the logarithm of the number of microstates. Nevertheless, the interpretation that \(S = A/4\) represents the number of ways to construct a hole out of ordinary matter and other (non-monster) black holes still seems self-consistent, as we discuss below.

**Growing a black hole**

If we exclude monsters from consideration, ordinary matter configurations have much less entropy than black holes of similar size or mass. Almost all of the entropy of a given black hole must result from a smaller black hole which has absorbed some additional mass. This is the picture that has been developed in the membrane paradigm \[22, 25\] within a quasistationary approximation.

Consider a black hole of area \(A\) that results from a hole of area \(A\) eating a small amount of energy \(m\). We must have \(\exp A' = \exp A \cdot \exp S\), where \(S\) is the matter entropy. There must exist matter configurations of mass \(m\) near a black hole horizon which have entropy \(S\) of order \(Mm\), since \(A' - A \sim (M + m)^2 - M^2 \sim Mm\).

One can construct thin shell examples with mass \(m\) and entropy \(Mm\), again taking advantage of curved space. Consider a shell of thickness \(d\) just outside the horizon, with energy density \(\rho(y) \sim m/dR^2\) and \(\epsilon(r) \sim y/R\). Its entropy is

\[
S \sim \int_0^d dy (R + y)^2 \left(\frac{R}{y}\right)^{1/2} \left(\frac{m}{d}\right)^{3/4} R^{-3/2} \sim \frac{Mm}{(md)^{1/4}}.
\]

(17)

If one requires that the energy density \(\rho\) be comprised of thermal modes with wavelength \(\lambda \sim \rho^{-1/4}\) less than \(\sqrt{Rd}\), the proper width of the shell, one obtains the constraint that \(md \sim 1\), so \(S \sim Mm\) as desired.

It is also worth noting that a single s-wave mode with energy \(m \sim 1/R \sim 1/M\) has entropy \(O(1)\), so satisfies \(S \sim Mm\). Thus, a black hole can move along the \(S \sim A\) curve by absorbing such modes. This is arguably the smallest amount of energy that can be absorbed by the hole, since otherwise the Compton wavelength of the mode is much larger than the horizon itself.

**Note added:** After this work was completed we were informed of related results obtained by Sorkin, Wald and Zhang \[26\]. Those authors investigated monster-like objects as well as local extrema of the entropy \(S\) subject to an energy constraint, which correspond to static configurations and obey \(A^{3/4}\) scaling. For example, in the case of a perfect fluid the local extrema satisfy the Tolman–Oppenheimer–Volkoff equation of hydrostatic equilibrium. In considering monster configurations, Sorkin et al. show that requiring a configuration to be no closer than a thermal wavelength \(\lambda \sim \rho^{-1/4}\) from its Schwarzschild radius imposes the bound \(S < A\). While this may be a reasonable criterion that must be satisfied for the assembly of an initial configuration, it does not seem to apply to states reached by quantum tunneling. Note that from a global perspective configurations with \(S > A^{3/4}\) are already black holes in the sense that the future of the interior of the object does not include future null infinity.

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