On the Poincaré’s generating function and the symplectic mid-point rule

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March 5, 2022

Abstract

The use of Liouvillian forms to obtain symplectic maps for constructing numerical integrators is a natural alternative to the method of generating functions, and provides a deeper understanding of the geometry of this procedure. Using Liouvillian forms we study the generating function introduced by Poincaré (1899) and its associated symplectic map. We show that in this framework, Poincaré’s generating function does not correspond to the symplectic mid-point rule, but to the identity map. We give an interpretation of this result based on the original framework constructed by Poincaré.

1 Introduction

From the available tools for constructing symplectic maps, the method of generating functions has been a cornerstone to understand the links between the geometry and topology of the phase space of Hamiltonian mechanical systems. In his famous Les méthodes nouvelles de la mécanique céleste [14], Poincaré develops the theory of integral invariants with applications to the study of periodic orbits in celestial mechanics. Poincaré constructed a locally exact differential 1-form defined on closed orbits with prescribed fixed period $T > 0$, such that its exterior differential gives the canonical symplectic form on the phase space. This 1-form is the differential of a function known as the Poincaré’s generating function [17, 10] which is defined locally on a Lagrangian submanifold where it coincides with a Liouvillian form. Since the orbit was periodic he considered a section\(^1\) (the Poincaré’s section) and a non-trivial map defined on such a section (the Poincaré’s map) such that the periodic orbit corresponds to a fixed point of the map. The imposed condition for the fixed point was that the first-return map must be non-reversing [14, 17].

From the numerical point of view, generating functions are used to construct numerical algorithms preserving the main geometrical properties of the phase space, in particular the symplectic form, naming the numerical algorithms as

\(^1\)The term section has a different meaning here than in homological algebra or vector bundles.
**Symplectic integrators.** Symplectic integrators are just the algorithmic numerical realization of symplectic maps close to the identity map, under some particular constraints. In the second half of the 80’s, the construction of symplectic integrators using generating functions was systematically studied by Feng Kang and co-workers [2, 9, 3, 4]. In those papers, the Poincaré’s generating function was associated to the symplectic mid-point integrator. However, there is no formal proof or construction of this correspondence. Recently, an alternative method for constructing symplectic maps has been developed using Liouvillian forms [7]. This method was developed for working with exterior differential forms in order to handle covariant objects. It becomes a natural alternative to the method of generating functions in the following way: A generating function \( S \) defines a Lagrangian submanifold \( \Lambda \subset M \), meanwhile a Liouvillian form \( \theta \) defines a codimension 1 coisotropic submanifold \( C \subset M \), such that if \( dS|_{\Lambda} = \theta|_{\Lambda} \) then \( \Lambda \subset C \) and \( T\Lambda \subset TC \subset TM \) are vector sub-bundles. In this case we have \( \ker \theta \subset \ker \pi^*dS \), for a suitable projection \( \pi : M \rightarrow \Lambda \). Instead of solving the Hamilton-Jacobi equation for \( \frac{1}{2} \dim(M) \) local constants, we use the 1 dimensional kernel of \( \theta \) for constructing a symplectic map, in fact we use the local Liouville vector field \( Z \) (dual to Liouvillian form \( \theta = iZ\omega \)) over the Hamiltonian flow for approximating the deviation of the numerical solution from the exact solution.

In contrast with other methods for constructing symplectic integrators, in the method of Liouvillian forms we can approach the continuous flow of a generic autonomous Hamiltonian system by a classical result relating Hamiltonian and Liouville vector fields [6, 12]. The application for symplectic integrators is given in [7].

We can approach the method of Liouvillian forms using special symplectic manifolds [16, 15]. Alternatively, we can define a quaternionic or hypercomplex structure [1] on the product manifold, which produces transparent definitions, giving more information about the symplectic map and a better interpretation of its geometry. The quaternionic structure induces three different symplectic forms, and consequently, three different families of Liouvillian forms, one for each symplectic form (see Sec 2). Symplectic maps obtained in the linear approximation coincide with those found by Feng Kang and collaborators using generating functions and matrix algebra [10]. These maps were used by Kang as the input for the Hamilton-Jacobi equation, and they depend on a Hamiltonian matrix \( b \in M_{2n \times 2n}(\mathbb{R}) \) without any particular interpretation. In our case, we have a \((1,1)\)-tensor, denoted by \( b \), which is related with the way the solution curves in the phase space. This tensor comes from the symmetric part of the Liouvillian form. When \( b \) is a constant tensor, it becomes the matrix \( b \) studied by Kang and collaborators. Moreover, in this case we have solutions with constant curvature corresponding to the flow of quadratic Hamiltonian functions.

A remarkable result is that we can construct symplectic integrators adapted to any well posed Hamiltonian problem, since we can associate a Liouville vector field to any regular level hypersurface of a Hamiltonian function. This is a classical problem in the interface of contact and symplectic geometry [6]. By symplectic duality, we can associate a Liouvillian form to the system, and
consequently a symplectic integrator. This association is locally defined on a prescribed hypersurface level and it defines a linear bundle whose dimension depends on the symmetries of the Hamiltonian. If the Hamiltonian system have not other first integral than the Hamiltonian function, then it is a real line bundle (dimension 1). In other case, we have a group of symmetries $G$ acting in a Hamiltonian way on the level surfaces and the bundle is related with the corresponding momentum map. This point of view has not been developed for the moment.

Liouvillian forms let us associate: 1) Hamiltonian systems, 2) Liouville vector fields, and 3) symplectic maps. In particular, we are interested in the relation between Liouvillian forms and symplectic maps. In contrast to the claim found in the papers of Kang and his collaborators, our results associate the differential of the Poincaré’s generating function to the identity map. This is not a surprise since this is the original goal of Poincaré. Moreover, the mid-point rule and the Poincaré’s differential form belong to different families of minimizers of the action integral. Indeed, the mid-point rule minimizes the action along a path with different fixed boundary points. Meanwhile Poincaré’s differential form minimizes the action integral along a periodic closed path with prescribed fixed period $T > 0$, characterized on a Poincaré’s section by a fixed point [14].

The goal of this communication is to understand the structure of Poincaré’s generating function in the canonical coordinates of the product manifold showing that its structure is completely different to the Liouvillian form producing the mid point rule.

2 Symplectic integrators from Liouvillian forms

For more detail concerning this section, we refer the reader to [7, 8]. Let $(M, \omega)$ be a $2n$-dimensional exact symplectic manifold with local coordinates $\{q_i, p_i\}_{i=1}^n \in M$, and $J_{2n}$ the canonical complex structure or canonical symplectic matrix given by

$$J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \quad I_n, 0_n \in M_{n \times n}(\mathbb{R}). \quad (1)$$

A symplectomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is defined as a diffeomorphism satisfying $\phi^* \omega_2 = \omega_1$. Consider the product manifold $\mathcal{P} = M_1 \times M_2$ and two differential forms induced by the canonical projections $\pi_i : \mathcal{P} \to M_i$, $i = 1, 2$ given by $\omega_{\mathcal{P}} = \pi_1^* \omega_1 - \pi_2^* \omega_2$ and $\theta_{\mathcal{P}} = \pi_1^* \theta_1 - \pi_2^* \theta_2$. The couple $(\mathcal{P}, \omega_{\mathcal{P}} = d\theta_{\mathcal{P}})$ is an exact symplectic manifold of dimension $4n$ [11], where the graph of $\phi$ defined by the set

$$\Gamma = \{(z, \phi(z)) | z \in M_1, \phi(z) \in M_2\},$$

represents a Lagrangian submanifold $\Gamma \subset (\mathcal{P}, \omega_{\mathcal{P}})$. Given canonical coordinates on the factor manifolds $(M_1, \omega_1)$, $(M_2, \omega_2)$, and an Euclidean structure $\langle \cdot, \cdot \rangle_{4n}$ on $T_m \mathcal{P}$, $m \in \mathcal{P}$, we define three complex structures

$$I = \begin{pmatrix} 0_{2n} & -I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}, \quad J_{\mathcal{O}} = \begin{pmatrix} J_{2n} & 0_{2n} \\ 0_{2n} & J_{2n}^T \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0_{2n} & J_{2n} \\ J_{2n} & 0_{2n} \end{pmatrix},$$
where $0_{2n}, I_{2n}, J_{2n} \in \mathbb{M}_{2n \times 2n}(\mathbb{R})$. The set $\{I_{4n}, \mathcal{I}, \mathcal{J}, \mathcal{K}\} \in \text{End}(\mathcal{T}P)$ induces an almost quaternionic or almost hypercomplex structure\footnote{We use $\mathcal{I} = -J_{2n} = J_{2n}^2$ for avoiding a negative sign in the quaternionic structure. The negative sign has been the source of different sign conventions between symplectic and complex geometries (see Remark 3.1.6 in [12]).}, and three different symplectic structures on $\mathcal{P}$ given by $\omega_{\mathcal{I}}(\cdot, \cdot) = \langle \cdot, \mathcal{I}\rangle$, $\omega_{\mathcal{J}}(\cdot, \cdot) = \langle \cdot, \mathcal{J}\rangle$ and $\omega_{\mathcal{K}}(\cdot, \cdot) = \langle \cdot, \mathcal{K}\rangle$, where $\omega_{\mathcal{J}} \equiv \omega_{\mathcal{I}}$.

In this framework, the $2n$-dimensional submanifolds $\Lambda \rightarrow \mathcal{P}$ adapted for constructing symplectic maps are those which are Lagrangian with respect to $\omega_{\mathcal{J}}$ and $\omega_{\mathcal{K}}$ and symplectic with respect to $\omega_{\mathcal{I}}$ [8]. The simplest non-trivial case is when a Liouvillian form $\theta := \theta_{\mathcal{I}} = \theta_{\mathcal{J}}$ has linear components and the structure $\theta = \pi^*_1\theta_1 - \pi^*_2\theta_2$. In canonical coordinates $\{z_0, z_h\}$ it is necessary that $\theta_1 = dz_0\left(\frac{1}{2}I_{2n} + S\right)z_0$ and $\theta_2 = dz_h\left(\frac{1}{2}I_{2n} - S\right)z_h$, where $S \in \mathbb{M}_{2n \times 2n}(\mathbb{R})$ is a symmetric Hamiltonian matrix. The symplectic map is given by $z_h = (I_{2n} - 2b)^{-1}\left(I_{2n} + 2b\right)z_0$, where $b = J_{2n}S$ is again a symmetric Hamiltonian matrix. This symplectic map is just the solution to the equation $\pi_\ast(\mathcal{I}(v)) = 0$, where $v = Z(z_0, z_h)$ is the element of the Liouville vector field $Z$ in the point $(z_0, z_h) \in \mathcal{P}$, and $Z$ is the dual of the Liouvillian form $\theta = i_Z\omega_{\mathcal{J}}$ [7, 8].

For constructing symplectic integrators we use the Liouville vector field $Z_H$ on $(\mathcal{M}, \omega)$ with equations $\dot{z} = Z_H(z)$, a first order approximation of the flow of $\dot{z}$ is given by the implicit Euler scheme

$$z_h = z_0 + hX_H(\bar{z}), \quad \bar{z}, z_h, z_0 \in \mathcal{M}, \quad 0 < h \ll 1. \quad (2)$$

This map is symplectic when the point $\bar{z}$ is given by [10, 7],

$$\bar{z} = \frac{1}{2}(z_0 + z_h) + b(z_h - z_0). \quad (3)$$

It corresponds to the expression $\bar{z} = \pi_\ast(v) = \pi_\ast(Z(z_0, z_h))$. Moreover, the implicit Euler scheme is both symmetric and symplectic if $\bar{z} = \frac{1}{2}(z_0 + z_h) + hb(z_h - z_0)$ [7].

All the one-step symplectic integrators are realizations of symplectic maps which arrive in this way. The three well-known one-step methods are the symplectic Euler methods $A$ and $B$ and the mid point rule. The case $b = 0_{2n}$ corresponds to the implicit mid-point $\bar{z}$ whose explicit symplectic map is the identity map. This is a degenerated case which corresponds to the flow of constant vector fields. On the other hand, the Euler methods have Hamiltonian matrices

$$b_A = \frac{1}{2} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad b_B = \frac{1}{2} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad b_A, b_B \in \mathbb{M}_{2n \times 2n}(\mathbb{R}).$$

Remark 1: $b_A$ and $b_B$ are exceptional matrices of rank $n$, i.e. det$(I_{2n} \pm 2b_a) = 0$ and they have not a well defined map of the form: $z_h = (I_{2n} - 2b)^{-1}(I_{2n} + 2b)z_0$. Their symplectic maps induce the only explicit one-step symplectic integrators.

The matrix $b$ is related with the curvature of the flow lines. However, this subject is out of the scope of this paper.
3 The Poincaré’s Generating Function

In the 3rd volume of *Les méthodes nouvelles de la mécanique céleste* [14], Poincaré introduced the 1-form

\[ dS = \frac{1}{2} \sum \left\{ (Q - q)d(P + p) - (P - p)d(Q + q) \right\} \quad (4) \]

looking for periodic orbits bifurcating from a prescribed periodic orbit of period \( T > 0 \). In expression (4) variables \((q, p)\) are positions and conjugate momenta in the phase space at time \( t \) and \((Q, P)\) are positions and conjugate momenta at time \( t + T \). We denote them by \( z_0 = (q, p) \) and \( z_h = (Q, P) \). This form was rediscovered by Feng Kang and his collaborators when they were studying the construction of symplectic integrators using generating functions. Kang’s group interprets the form (4) as the linear mapping

\[ pdq + PdQ \mapsto \frac{1}{2} \left[ (Q - q)d(P + p) - (P - p)d(Q + q) \right] \quad (5) \]

given by the matrix

\[ \alpha = \begin{pmatrix} -J_{2n} & J_{2n} \\ \frac{1}{2}I_{2n} & \frac{1}{2}I_{2n} \end{pmatrix}, \quad J_{2n}, I_{2n} \in \mathbb{M}_{2n \times 2n}(\mathbb{R}). \quad (6) \]

Given a generating function \( u : \Lambda \to \mathbb{R} \) on a Lagrangian submanifold \( \Lambda \subset P \) with local coordinates \( w = w(z_0, z_h) \), they systematically associate the numerical method

\[ z_h - z_0 = -J_0 \frac{\partial u}{\partial w} \left( \frac{z_h + z_0}{2} \right). \quad (7) \]

to the form (4) [2, 9, 3, 4], and consequently the symplectic mid-point rule with the Poincaré’s generating function. Note that the map (7) is a time-1 map.

Poincaré’s form (4) has linear components and it accepts a matrix representation \( \theta = dx \left( \frac{1}{2}J + \mathbf{R} \right) x^T = dx\mathbf{A}x^T \), where \( x = (q, p, Q, P) = (z_0, z_h) \in P \),

\[ \mathbf{R} = \frac{1}{2} \begin{pmatrix} 0_{2n} & J_{2n}^T \\ J_{2n} & 0_{2n} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \frac{1}{2} \begin{pmatrix} J_{2n} & J_{2n}^T \\ J_{2n} & J_{2n}^T \end{pmatrix}, \quad J_{2n}, I_{2n} \in \mathbb{M}_{2n \times 2n}(\mathbb{R}). \quad (8) \]

In order to obtain the implicit Euler scheme we follow the same procedure in [8]. Let \( Z \) be the Liouville vector field for \( \omega_{\mathcal{J}} \) dual to \( \theta \) and we compute \( v = Z(z_0, z_h) \). From (4) and (8) we have, in local coordinates, \( v = Z(z_0, z_h) = \mathcal{J}^T \mathbf{A}x^T \). A direct computation shows that the point \( \bar{z} \) is given by

\[ \bar{z} = \pi_* (v) = \pi_* (\mathcal{J} \circ \mathbf{A}x^T) = 0_{2n}. \quad (9) \]

This is the point where we evaluate the Hamiltonian system and we consider that \( \bar{z} = 0_{2n} \) is a fixed point \( X_H(0_{2n}) = 0_{2n} \) as Poincaré did, obtaining

\[ z_h = z_0 + hX_H(0_{2n}) = z_0, \quad (10) \]
which corresponds to the identity map for every $\hbar \in \mathbb{R}$. Alternatively, the expression (4) defines a Lagrangian subspace by the equation $dS = 0$ with solutions $p - P = 0$ and $Q - q = 0$, which produce the identity map $P = p$ and $Q = q$.

This is nothing else that Poincaré's original framework, since he constructed the generating function to be defined on periodic orbits of prescribed period $T > 0$ such that if $z_0 = (q, p)$, $z_h = (Q, P)$ then $z_h = z_0$ was a fixed point on a Poincaré's section but such that the orbit was a non-trivial one (see Figure 1). Moreover, this hypothesis considers that for $T$ variable, $\lim_{T \to 0} dS = 0$ and the function $S$ goes to a constant $S \to S_0$ which, for simplicity, he considered $S_0 = 0$. We have proven that our interpretation of the Poincaré's generating function using the framework of Liouvillian forms matches with the original framework from Poincaré.

Let $\theta := \pi^*(dS)$ be the Liouvillian form obtained by the pull-back of (4). Let $Z = \theta^\flat$ be the Liouville vector field dual to $\theta$ under $\omega_J$, i.e. $\theta = i_Z \omega_J$, and let $v = Z(z_0, z_h) \in T_{(z_0, z_h)} \mathcal{P}$ the corresponding vector of $Z$ at the point $(z_0, z_h) \in \mathcal{P}$.

**Proposition 3.1** The implicit map $\bar{z} = \pi_*(v)$ associated to the Poincaré's form (4) under the method of Liouvillian forms corresponds to the null map (9). If $\bar{z}$ is a fixed point of the Hamiltonian vector field $X_H(\bar{z}) = \bar{z}$, the generalized Euler scheme corresponds to the identity map.

It is well-known that symplectic maps close to the identity can be used to construct symplectic integrators [5], but this is not the case for maps obtained by Poincaré’s generating function. This is because the variational problem for which it was constructed assumes non-trivial periodic orbits with period $T > 0$ [14]. If the fixed point is $\bar{z} \neq 0$ then it concerns a structurally different Liouvillian form as showed in the previous section. In fact, the Liouvillian form determines the evaluation point $\bar{z}$. We will show that the symplectic map produced by Poincaré’s generating function is far from the set of symplectic maps producing regular symplectic integrators. For this, we construct a path of
Liouvillian forms connecting the symplectic Euler methods $A$ and $B$ with the linear form (4) using a loop of symplectic rotations.

**Lemma 3.2** The 1-parameter family of Liouvillian forms on $(\mathcal{P}, \omega_J)$, given by

$$\theta_\phi = (\cos \phi Q - \sin \phi q) d(\cos \phi P + \sin \phi p) - (\sin \phi P - \cos \phi p) d(\sin \phi Q + \cos \phi q).$$

connects Poincaré’s 1-form, to those associated with the symplectic Euler schemes $A$ and $B$.

**Proof.** It is a family of Liouvillian forms on $(\mathcal{P}, \omega_J)$ since $d \theta_\phi = \omega_J$ is independent of the parameter $\phi \in [0, 2\pi]$. To prove that this family contains both Euler schemes and the Poincaré’s 1-form, it is enough to compute $\theta_\phi$ for the values $\phi \in \{0, \pi/4, \pi/2\}$ obtaining

$$\theta_0 = pdq + QdP, \quad \theta_{\pi/2} = -qdP - PdQ, \quad (11)$$

$$\theta_{\pi/4} = \frac{1}{2} \{(Q - q)d(P + p) - (P - p)d(Q + q)\} \quad (12)$$

which corresponds to the forms associated with the symplectic Euler schemes $A$ and $B$ [7] and the Poincaré’s 1-form, respectively. □

Expand the family $\theta_\phi$ from Lemma 3.2 in order to recover the matrix representation of the family $\theta_\phi = dxA_\phi x^T$ where

$$A_\phi = \begin{pmatrix}
0 & \cos^2 \phi I_n & 0_n & -\cos \phi \sin \phi I_n \\
-\sin^2 \phi I_n & 0_n & \cos \phi \sin \phi I_n & 0_n \\
0_n & \cos \phi \sin \phi I_n & 0_n & -\sin^2 \phi I_n \\
-\cos \phi \sin \phi I_n & 0_n & \cos^2 \phi I_n & 0_n
\end{pmatrix}.$$  

Using trigonometric identities we rewrite $A_\phi = \frac{1}{2} (J + \cos 2\phi \ S_c + \sin 2\phi \ S_s)$, where

$$S_c = \begin{pmatrix}
0_n & I_n & 0_n & 0_n \\
I_n & 0_n & 0_n & 0_n \\
0_n & 0_n & I_n & 0_n \\
0_n & 0_n & 0_n & I_n
\end{pmatrix} \quad \text{and} \quad S_s = \begin{pmatrix}
0_{2n} & J_{2n}^T \\
J_{2n} & 0_{2n}
\end{pmatrix}.$$  

Elements of the family $\theta_\phi$ have the shape $\theta_\phi = \pi^1_\phi \theta_1 - \pi^2_\phi \theta_2$ when the matrix $A_\phi$ is block diagonal, and it happens if and only if $\sin 2\phi = 0$. This condition is satisfied in $[0, \pi/2]$, for $\phi = 0$ or $\phi = \pi/2$. Consequently the Hamiltonian matrices $b_A = J^T S_c$ and $b_B = J S_c$ are block diagonal matrices corresponding to the matrices of the symplectic Euler schemes $A$ and $B$ (see Figure 2), we have proven the following:

**Theorem 3.3** The family of Liouvillian forms $\theta_\phi$ from Lemma 3.2 renders the generalized implicit Euler method, symplectic if and only if $\phi = 0$ or $\phi = \pi/2$, equivalently if and only if it is one of the symplectic Euler schemes: $A$ or $B$.  

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Figure 2: The family $\theta_\phi$ and its projection on the subspace $(Q_1 \times Q_2)$. The set of Liouvillian forms whose projection gives symplectic integrators, reproduces the original positions and conjugated momenta on the diagonal $(q = Q, p = P)$, which is not the case for the Poincaré’s 1-form.

Then we cannot construct a well defined symplectic integrator by this formalism using Poincaré’s generating function. This negative result has a simple explanation in the framework of Liouvillian forms: the Liouville vector field induced by Poincaré’s generating function (4) is Liouville for $\omega_J$ but it is not Liouville for $\omega_I$. As a consequence, the submanifold $\Lambda \hookrightarrow \mathcal{P}$ associated to $dS$ is Lagrangian with respect to $\omega_J$ but not for $\omega_I$ nor $\omega_K$.

4 The Liouvillian form for the mid point rule

Looking for the most generic Liouvillian form on $(\mathcal{P}, \omega_J)$ which induces the mid point rule as discrete map, we consider the expression for $\bar{z}$ corresponding to

$$\bar{z} = \pi_*(v) = \pi_* \left\{ \left( \frac{1}{2} I_{4n} + J^T S \right) x^T \right\} = \frac{1}{2} (z_0 + z_h),$$

where $x = (z_0, z_h) \in \mathcal{P}$ and $S \in M_{4n \times 4n}(\mathbb{R})$ is a symmetric matrix. This expression is satisfied when $\pi_* \left( J^T S x^T \right) = 0_{2n}$. In behalf of simplicity, we consider $\mathcal{P} = \mathbb{R}^{4n}$ for avoiding curvature issues, and we write $S$ in $2n \times 2n$ blocks. We have

$$S = \begin{pmatrix} S_1 & G_1 \\ G_1^T & S_2 \end{pmatrix} \Rightarrow J^T S x^T = \left( J^T S \right) \begin{pmatrix} z_0 \\ z_h \end{pmatrix} = \begin{pmatrix} J^T_{2n} S_2 \begin{pmatrix} z_0 \\ z_h \end{pmatrix} + J^T_{2n} G_1^T z_h \\ J_{2n} G_1 \bar{z}_0 + J_{2n} S_2 \bar{z}_h \end{pmatrix},$$

where $S_1$ and $S_2$ are symmetric matrices and $G_1$ is a generic matrix in $M_{2n \times 2n}(\mathbb{R})$. Condition $\pi_* \left( J^T S x^T \right) = 0_{2n}$ holds if and only if $G_1 = S_1 = S_2$, and the matrix $S$ has the form

$$S = \begin{pmatrix} S_1 & S_1 \\ S_1 & S_1 \end{pmatrix}, \quad S_1 = S_1^T \in M_{2n \times 2n}(\mathbb{R}).$$
This corresponds to a family of \( n(2n + 1) \) free parameters producing the midpoint rule, which accepts a description in the form \( \theta = \pi_1^* \theta_1 - \pi_2^* \theta_2 \) if and only if \( S_1 \equiv 0_{2n} \), since \( S \in M_{4n \times 4n}(\mathbb{R}) \) must be a block diagonal matrix.

**Theorem 4.1** The Liouvillian form associated to the mid-point rule has a unique element of the type \( \theta = \pi_1^* \theta_1 - \pi_2^* \theta_2 \) corresponding to the basic Liouvillian form \( \theta_0 \). In local coordinates \( x \in (\mathcal{P}, \omega_J) \) we write \( \theta_0 = \frac{1}{2} d\mathcal{J}x \) or in extended form

\[
\theta_0 = \frac{1}{2} (pdq - qdp - PdQ + QdP), \quad x = (q, Q, p, P) \in (\mathcal{P}, \omega_J). \tag{13}
\]

Moreover, the Liouville vector field \( Z \) associated to the basic Liouvillian form \( \theta_0 \) is just the “expanding” or Euler vector field. If we consider local coordinates \( \{x_i\}_{i=0}^{4n} \) then \( Z = \frac{1}{2} \sum_i x_i \frac{\partial}{\partial x_i} \) is Liouville for the three symplectic forms \( \omega_I \), \( \omega_J \) and \( \omega_K \). This also implies that the mid-point rule is a degenerated case of Liouvillian forms for constructing symplectic maps, corresponding to the flow of constant (Hamiltonian) vector fields. This comes from the expression \( z_0 = (I_{2n} - 2b)^{-1}(I_{2n} + 2b)z_0 \) since the mid-point rule corresponds to \( b = 0_{2n} \).

The degeneracy is related with the dimension of the immersed submanifold \( \Lambda \). Fixing the almost quaternionic structure \( \{I_{4n}, I, J, K\} \) on \( \mathcal{P} \), an immersion \( j: \Lambda \hookrightarrow \mathcal{P} \) which is Lagrangian with respect to \( \omega_I \) and \( \omega_J \), must be symplectic with respect to \( \omega_K \). In the case of the mid-point rule, the Liouville vector fields for the different symplectic forms coincide, and consequently the immersion corresponds to an isotropic submanifold.

The argument for naming \( \theta_0 \) the basic Liouvillian form is based on the Hodge decomposition of differential forms on a differential manifold [13]. In this decomposition, every Liouvillian form on a symplectic manifold \( (\mathcal{M}, \omega) \) is given by \( \theta = d\eta + dF + \alpha \), where \( \alpha \) is a harmonic form, \( dF \) is the differential of a function \( F: \mathcal{M} \to \mathbb{R} \), and \( d\eta \) is the codifferential of a 2-form \( \eta \in \Omega^2(\mathcal{M}) \). Then \( \theta_0 = d\eta \) is the only contribution to the symplectic form \( \omega = d\theta_0 = d\delta \eta \) since \( d(dF + h) = 0 \). It is a different point of view than the geometrical interpretation of the Liouville form as a tautological form on a cotangent bundle.

## 5 Conclusions

In this paper we used Liouvillian forms [7] for studying the relation of the mid-point rule with Poincaré’s 1-form, which is the differential of the Poincare’s generating function (4), introduced in [14]. We showed that the classical association between these two objects is not the right one. This comes from the fact that Poincaré’s 1-form and mid-point rule are techniques applied to two different types of variational problems:

- Poincaré’s 1-form was designed for dealing with periodic orbits with prescribed period \( T > 0 \), it means, non-trivial loops or cycles (no boundary);
- the mid-point rule is the simplest approximation for problems with fixed values at the boundary (initial and final fixed points).
We showed that the structure of Poincaré’s 1-form differs drastically from the structure of Liouvillian forms generating the mid-point rule, and in general to those generating symplectic integrators. In order to better understand this discrepancy, we constructed two families of 1-forms. The first one is a one-parameter family (a path) joining the symplectic Euler maps $A$ and $B$ with Poincaré’s 1-form. The only elements in this family which generate symplectic integrators are the boundary points of the path corresponding to the Euler maps $A$ and $B$. The second family shows that the only Liouvillian form of type $\theta_3 = \pi_1^*\theta_1 - \pi_2^*\theta_2$ producing the mid-point rule on the product manifold, is the basic Liouvillian form $\theta_0$ which has null symmetric part.

Acknowledgements

This research was developed with support from the Fondation du Collège de France and Total under the research convention PU14150472, as well as the ERC Advanced Grant WAVETOMO, RCN 99285, Subpanel PE10 in the F7 framework.

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