GENERALIZED GRASSMANN ALGEBRAS AND ITS CONNECTION TO THE EXTENDED SUPERSYMMETRIC MODELS

A.P. ISAEV*, Z. POPOWICZ** and O. SANTILLAN*

* Bogoliubov Laboratory of Theoretical Physics, JINR, 141 980 Dubna, Moscow Region, Russia
** Institute of Theoretical Physics, University of Wrocław pl. Maxa Borna 9, 50–204 Wrocław, Poland

ABSTRACT

It is shown that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators. The 2,3 and 4 dimensional Heisenberg-Weyl algebra is explicitly given in terms of the fractional generators. These algebras are used for the formulation of the $N = 2, 3, 4$ extended supersymmetry. As an example we reformulate the Lax approach of the supersymmetric Korteweg-de Vries equation in terms of the generators of the generalized Grassmann algebra.

* E-mail: isaevap@thsun1.jinr.ru
** E-mail: ziemek@ift.uni.wroc.pl
1. INTRODUCTION

Generalized Grassmann algebras have been introduced firstly in [1] in the framework of the 2D conformal field theories. Later these algebras were rediscovered in the contexts of quantum groups [2] and fractional generalizations of supersymmetric quantum mechanics [3]. Next, these algebras were used for the generalization of the supersymmetry to the fractional supersymmetry [4], [5], [6], [7].

Interestingly the generalized Grassmann algebras (or their intimate analogs) have found applications in the construction of the finite dimensional (cyclic) representations of the quantum groups, covariant quantum algebras and zero-mode algebras in WZNW models (see e.g. [2], [8], [9], [6], [10]). The generalized Grassmann algebras should be also useful for the construction of finite dimensional representations of the quadratic diffusion algebras [11] which are employed in the theory of one-dimensional stochastic processes with exclusion. The fractional extensions of the Virasoro algebra [4] utilizes generalized Grassmann algebras as well. Recently generalized Grassmann algebras appeared in the investigations of the particle systems on linear lattices with periodic boundary conditions [12] (see also [9]). In this paper we find the irreducible representations of the Clifford algebras by realizing them in terms of the generalized Grassmann algebras.

The main idea of the introduction of the generalized Grassmann algebras is to replace the usual fermionic condition $\theta^2 = 0$ by the more general nilpotence condition $\theta^{p+1} = 0$. The $p$ number in the physical literature is usually called the order of parastatistics. Due to the big interest to such generalization it is tempting to try to use these generalized algebras to the supersymmetric solitonic equations.

In this paper we propose the method of the reformulations, in terms of the generalized Grassmann variables, the usual extended supersymmetric models. We concentrate our attention to the supersymmetric Korteweg-de Vries (SKdV) equation only. However our approach could be used to any $N = 1, 2, 3, 4$ supersymmetric model as well.

We explicitly show that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators $\theta, \partial$ where the order of parastatistics $p$ is $2^N - 1$. We explicitly construct the $2, 3, 4$-dimensional Heisenberg-Weyl algebras in terms of the generalized Grassmann algebras with $p + 1 = 4, 8, 16$. It appears that this Heisenberg-Weyl algebra is closely related to the Clifford algebras in Euclidean spaces of the dimensions $D = 4, 6, 8$. One can use this observation to formulate the $N = 2, 3, 4$ extended supersymmetry. All of this allowed us to formulate the $N = 2$ SKdV equations in terms of the generalized Grassmann algebras for $p + 1 = 4$.

The paper is organized as follows: In Section 2 we recall the basic facts about generalized Grassmann algebras. In Section 3 the Weyl construction for the Heisenberg-Weyl and Clifford algebras is discussed. In Section 4 we construct fermionic $N = 2, 3, 4$ Heisenberg-Weyl algebras in terms of the generators of the $Z_4$, $Z_8$ and $Z_{16}$ generalized Grassmann algebras. In Section 5 we discuss the example of the $N = 1, 2$ SKdV equations rewritten in terms of $Z_2$ and $Z_4$ graded algebra respectively. The Section 6 contains conclusions.
2. GENERALIZED GRASSMANN ALGEBRA

Consider a deformed oscillator algebra with two generators $\theta$ and $\partial$ satisfying

$$\partial \theta - q \theta \partial = I, \quad (2.1)$$

$$\theta^{p+1} = \partial^{p+1} = 0 \Rightarrow q^{p+1} = 1, \quad (2.2)$$

where $p$ is an integer. Note that the complete basis of this algebra is given by the $(p + 1)^2$ elements $(\theta^n \partial^m) (0 \leq n, m \leq p)$ and this algebra is isomorphic to the algebra of matrices $\text{Mat}(p + 1)$ [2], [17]. The grading in this algebra is

$$\text{deg}(\theta^n \partial^m) = n - m. \quad (2.3)$$

It is convenient to introduce the grading operator

$$\omega = \partial \theta - \theta \partial \Rightarrow \omega^{p+1} = 1, \quad (2.4)$$

which defines the automorphisms

$$\omega \theta \omega^{-1} = q \theta, \quad \omega \partial \omega^{-1} = q^{-1} \partial. \quad (2.5)$$

Using (2.1) and (2.4) we obtain useful relations

$$\theta \partial = \frac{\omega - 1}{q - 1}, \quad \partial \theta = \frac{q \omega - 1}{q - 1}. \quad (2.6)$$

The algebra only defined by (2.1), but not (2.2), has been firstly considered in [13]. This algebra is closely connected to the algebra of Macfarlane - Biedenharn quantum oscillators [14]. For the nontrivial case, when $q$ is root of unity, this connection has been discussed in [9].

A characteristic equation $\omega^{p+1} = 1$ for the grading operator (2.4) can be rewritten in the form

$$(\omega - 1) (\omega - q) \ldots (\omega - q^p) = \prod_{n=0}^{p} (\omega - q^n) = 0, \quad (2.7)$$

and one can introduce the projection operators

$$P_k = \prod_{n=0,n\neq k}^{p} \frac{(\omega - q^n)}{(q^k - q^n)}, \quad (k = 0, \ldots, p). \quad (2.8)$$

These projectors satisfy the following identities

$$\sum_{k=0}^{p} P_k = 1, \quad P_n P_k = \delta_{nk} P_k, \quad \omega P_k = P_k \omega = q^k P_k, \quad (2.9)$$

$$P_0 \theta = 0, \quad \theta P_{k-1} = P_k \theta, \quad \theta P_p = 0, \quad \partial P_0 = 0, \quad P_{k-1} \partial = \partial P_k, \quad P_p \partial = 0.$$  

In view of the nilpotence conditions (2.2) one can deduce

$$\theta^{p+1-n} \partial^{p+1-n} \partial^n = 0 \Rightarrow \prod_{k=0}^{p-n} (\omega - q^k) \partial^n = 0. \quad (2.10)$$

$$\theta^n \theta^{p+1-n} \partial^{p+1-n} = 0 \Rightarrow \theta^n \prod_{k=0}^{p-n} (\omega - q^k) = 0.$$
It means that
\[ \theta^n P_m = 0 = P_m \partial^n \quad (p - n < m \leq p). \quad (2.11) \]

A generic function \( u \), with \( \theta \) as an argument, can be expanded in power series as
\[ u(\theta) = u_0 + \theta u_1 + \ldots + \theta^p u_p. \quad (2.12) \]
where \( u_i \) are in general noncommutative coefficients. The left action of \( \theta \) on \( u(\theta) \) is given by
\[ \theta u(\theta) = \theta u_0 + \theta^2 u_1 + \ldots + \theta^p u_{p-1}. \quad (2.13) \]

To define the left action of the operator \( \partial \) on the function \( u(\theta) \) (2.12) we find, using the relation (2.1), that
\[ \partial \theta^n = (1 + q + q^2 + \ldots + q^{n-1}) \theta^{n-1} + q^n \theta^n \partial = (n)_q \theta^{n-1} + q^n \theta^n \partial, \quad (2.14) \]
where the q-number \( (n)_q \) is
\[ (n)_q = \frac{(1 - q^n)}{1 - q}. \quad (2.15) \]

It is clear that \( \partial \) can be considered as a generalization of the ordinary derivative and using (2.14) we obtain
\[ \partial (u(\theta)) = u_1 + (2)_q \theta u_2 + \ldots + (p)_q \theta^{p-1} u_p. \quad (2.16) \]
The equation \( q^{p+1} = 1 \) is a consequence of the nilpotence condition \( \theta^{p+1} = 0 \) (2.2) and relation (2.14) for \( n = p + 1 \). If in addition we require that \( \partial(\theta^n) \neq 0 \) for \( n < p + 1 \), then \( q \) should be a primitive root of unity (\( q^n \neq 1 \) for all \( n < p + 1 \)) and one can choose \( q = \exp(2\pi i/(p + 1)) \). In this case, generators \( \theta \) and \( \partial \) (2.1), (2.2) are reduced to the ordinary fermionic \( (p = 1) \) and bosonic \( (p \to \infty) \) creation and annihilation operators.

Equations (2.13) and (2.16) give a matrix representation for \( \theta \) and \( \partial \). Indeed, arbitrary function \( u(\theta) \) (2.12) can be realized as \( (p + 1) \) dimensional vector
\[ u = \begin{pmatrix} u_0 \\ \vdots \\ u_p \end{pmatrix}, \quad u_k = <k| u >. \quad (2.17) \]

To reproduce the actions of \( \theta \) and \( \partial \) on the function \( u(\theta) \) (2.13) and (2.16) we need the following matrix representation of \( \theta \) and \( \partial \)
\[ \theta_{km} = \begin{pmatrix} 0 & \ldots & 0 \\ 1 & 0 & \ldots \\ \ldots & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = <k| \theta|m>, \quad (2.18) \]
\[ \partial_{km} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \ldots & 0 & (2)_q & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & (p)_q \end{pmatrix} = <k| \partial|m>. \]
The corresponding matrix representations of $\omega$ (2.4) and projectors (2.8) are

$$
\omega_{km} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & q & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & q^p
\end{pmatrix}
= < k|\omega|m >, \\
(2.19)
$$

and one can obtain the relations

$$
\omega^T = \omega, \quad \theta^T = (\omega - \frac{1}{q - 1}) \theta, \quad \theta^T = \partial \left( \sum_{k=1}^{p} \frac{1}{(k)_q} P_k \right), \\
(2.20)
$$

where $T$ is a transposition of the matrices. The last equation in (2.20) has been obtained with the help of projectors (2.8).

In eqs. (2.17) – (2.19) we have introduced the ladder of $p + 1$ states $|m> (m = 0, \ldots, p)$ defined by: $\partial|0> = 0, \theta|m> = |m + 1>, \partial|m> = (m)_q|m - 1>$. The dual states $< k|$ satisfy the orthogonality condition $< k|m> = \delta_{km}$. The matrix $\theta_{km}$ has the form $\delta_{k,m+1}$ and in general $(\theta^n)_{km} = \delta_{k,m+n}$, while for the matrix $\partial$ we have $(\partial^n)_{km} \sim \delta_{k+n,m}$. The form of the matrices $(\theta^n), (\partial^n)$ and $\omega$ corresponds to our choice of $Z_{p+1}$ grading (2.3) which is naturally called in [2] as ”along diagonal” grading. We use the matrix representations (2.18), (2.19) in our calculations below.

Note that the algebra (2.1), with additional relations (2.2), represents a special case of the more general algebra

$$
\mathcal{D} \Theta - q \Theta \mathcal{D} = I, \\
(2.21)
$$

$$
\Theta^{p+1} = c_1, \quad \mathcal{D}^{p+1} = c_2 \quad \Leftrightarrow \quad q^{p+1} = 1, \\
(2.22)
$$

where operators $c_i$ are central elements, $\text{deg}(\Theta) = 1$ and $\text{deg}(\mathcal{D}) = -1$. For all fixed values of $c_i$ these algebras are isomorphic to the algebra of matrices $\text{Mat}(p + 1)$ and, therefore, the generators $\Theta, \mathcal{D}$ can be expressed in terms of the elements $\theta, \partial$ of the generalized Grassmann algebra (2.1), (2.2). In the case $c_i = 0$ we have $\Theta = \theta, \mathcal{D} = \partial$. Let $c_i \neq 0$ (the case $c_2 \neq 0$ is considered below) and operators $\Theta$ and $\mathcal{D}$ are

$$
\Theta = \theta + \frac{1}{(p)_q} \partial^p c_1, \quad \mathcal{D} = \Theta^{-1} \frac{z \omega - 1}{q - 1}, \\
(2.23)
$$

where $z$ is a function of central elements $c_i$ which is fixed below, $(n)_q! := (1)_q \cdot (2)_q \cdots (n)_q$. To find the expression for $\mathcal{D}$ we have used (2.21) and the identity

$$
\mathcal{D} \Theta - \Theta \mathcal{D} = z \omega. \\
$$

Note that the operator $\Theta$ in the matrix representation has the familiar form of the cyclic matrix:

$$
\Theta_{km} = \begin{pmatrix}
0 & 0 & c_1 \\
1 & 0 & 0 \\
\cdot & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}. \\
$$
Operators $\Theta, \mathcal{D}$ (2.23) automatically satisfy conditions (2.21), (2.22) except eq. $\mathcal{D}^{p+1} = c_2$ which connects the parameter $z$ with the central elements $c_i$:

\[
c_1 c_2 (1 - q)^{p+1} = \prod_{k=0}^{p} (1 - z q^{-k} \omega) = 1 - z^{p+1},
\]

where we have used that $\omega^{p+1} = 1 = q^{p+1}$.

The case $c_2 \neq 0$ can be considered analogously and one can take

\[
\mathcal{D} = \partial + \frac{1}{(p)q!} \Theta^p c_2, \quad \Theta = \frac{z \omega - 1}{q - 1} \mathcal{D}^{-1},
\]

instead of (2.23) with the same relation (2.24) on the elements $c_i$ and $z$. This relation is rewritten in the form of Fermat curve $y^{p+1} + z^{p+1} = 1$ where $y = (c_1 c_2)^{1/(p+1)} (1 - q)$. At the end of this Section we note that the differential calculus on the generalized Grassmann algebras [2], [17] is closely related to the differential calculus on the finite groups $\mathbb{Z}_n$ [15].

3. THE HEISENBERG-WEYL AND CLIFFORD ALGEBRAS

The $N$-dimensional Heisenberg-Weyl algebra is generated by $2N$ operators $\theta_i$ and $\partial_i$ ($i = 1, 2, ..., N$) satisfying the following commutation relation

\[
\theta_\mu \theta_\nu + \theta_\nu \theta_\mu = 0, \\
\partial_\mu \partial_\nu + \partial_\nu \partial_\mu = 0, \\
\partial_\mu \theta_\nu + \theta_\nu \partial_\mu = \delta_{\mu\nu}.
\]

(3.26)

It is straightforward to check that

\[
\gamma_\mu = \partial_\mu + \theta_\mu, \quad \gamma_{N+\mu} = i(\partial_\mu - \theta_\mu),
\]

(3.27)

where $\mu = 1, 2, ..., N$ is a realization of $2N$-dimensional Euclidean Clifford algebra $[\gamma_A, \gamma_B]_+ = 2\delta_{AB}$. For our purposes it is convenient to recall the Weyl construction [16] of the generators $\theta_\mu, \partial_\mu$ of the algebra (3.26)

\[
\theta_\mu = \hat{\omega} \otimes \hat{\omega} \otimes ... \otimes \hat{\omega} \otimes I \otimes ... \otimes I, \\
\partial_\mu = \hat{\omega} \otimes \hat{\omega} \otimes ... \otimes \hat{\omega} \otimes \hat{\partial} \otimes I \otimes ... \otimes I,
\]

(3.28)

via the generators $\hat{\theta}$ and $\hat{\partial}$ of the 1-dimensional fermionic Heisenberg - Weyl algebra (2.1), (2.2) (for $p = 1$)

\[
\hat{\theta} \hat{\partial} + \hat{\partial} \hat{\theta} = 1, \quad \hat{\theta}^2 = 0 = \hat{\partial}^2, \quad \hat{\omega} = [\hat{\partial}, \hat{\theta}].
\]

(3.30)

Using relations (3.28) and (3.29) one can construct the irreducible representation of the HW algebra (3.26) and even dimensional Clifford algebras using the representation (2.18), (2.19) of the generalized Grassmann algebra for $p = 1$:

\[
\hat{\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\partial} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(3.31)
Recall that the tensorial product (in particular the tensorial products in (3.28), (3.29)) of two matrices $A$ and $B$ ($n \otimes n$ and $m \otimes m$ respectively) is defined by the following $n \cdot m \otimes n \cdot m$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ldots & a_{nn}B \end{pmatrix}.$$  \hfill (3.32)

The matrices (3.28), (3.29) are $2N \times 2N$ dimensional matrices. Our task is to embed the $2N$ dimensional Heisenberg-Weyl algebra (3.26) into the generalized Grassmann algebra (2.1), (2.2) for some fixed $p$. It is clear (by considering the dimensions of the matrix representations (2.18), (2.19) and (3.28), (3.29)) that the Heisenberg-Weyl algebra can be realized in terms of the q-operators only if $p + 1 = 2^N$ and, thus, $p$ must be odd.

**Remark 1.** Matrix representations of the generators $\theta_\mu$ and $\partial_\mu$ (3.28), (3.29), (3.31) satisfy conditions

$$\theta_\mu^T = \partial_\mu, \quad \partial_\mu^T = \theta_\mu, \quad \theta_\mu^* = \theta_\mu, \quad \partial_\mu^* = \partial_\mu \quad \text{(3.33)}$$

where $T$ is a transposition and $*$ is a complex conjugation of the matrices. It means that eqs. (3.27) define the real representation of the Clifford algebras.

**Remark 2.** The Weyl construction (3.28), (3.29) can be generalized for the case when the algebra $\{\hat{\theta}, \hat{\partial}\}$ is taken to be $Z_{p+1}$ algebra (2.1), (2.2) for arbitrary $p$. In this way one can construct multidimensional generalized Grassmann algebras (see [17], [9] for details) and the algebras of covariant quantum oscillators [4].

4. $N = 2, 3, 4$ HEISENBERG-WEYL ALGEBRAS VIA $Z_{4,8,16}$ GENERALIZED GRASSMANN ALGEBRAS

4.1 $N = 2$ Heisenberg-Weyl algebra

To understand what we mean with embedding of algebras, let us consider the case $N = 2$ (or $p = 3$). We will consider only primitive roots $q$. It means that for $q^4 = 1$, we have to put $q^2 = -1$. The Weyl representation (3.28), (3.29) gives

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{\hfill (4.34)}$$

$$\partial_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{\hfill (4.35)}$$

Note that the non zero elements of $\theta_1$ and $\partial_1$ are in the sites $(k, k-2)$ and $(k-2, k)$, while the non zero elements of $\theta_2$ and $\partial_2$ are in the sites $(k, k-1)$ and $(k-1, k)$. It means that
the generators \( \theta_i \) and \( \partial_i \) can be expressed (according with the along diagonal grading (2.3)) in terms of the generators of the generalized Grassmann algebra as

\[
\theta_1 = t_0 \theta^2 + t_1 \theta^3 \partial, \quad \theta_2 = t_0' \theta + t_1' \theta^2 \partial + t_2' \theta^3 \partial^2 .
\]

(4.36)

\[
\partial_1 = d_0 \partial^2 + d_1 \theta \partial^3, \quad \partial_2 = d_0' \partial + d_1' \theta \partial^2 + d_2' \theta^2 \partial^3 .
\]

(4.37)

Our aim is to determine the coefficients \( t_i \), \( t'_i \) and \( d_i \), \( d'_i \) in (4.36) and (4.37) in such a way that generators \( \theta_\mu \), \( \partial_\nu \) obey the defining relations (3.26). According with the representation (2.19) we find that

\[
\theta_1 = \theta^2, \quad \theta_2 = \theta (P_0 - P_2) = (P_1 - P_3) \theta .
\]

(4.38)

Since the generators \( \partial_\mu \) are related to the generators \( \theta_\mu \) by the transpositions (3.33) and using (2.20) one can deduce from (4.38) the expressions

\[
\partial_1 = \frac{1}{(2)_q} \partial^2 \left( P_2 + \frac{1}{(3)_q} P_3 \right), \quad \partial_2 = \partial \left( P_2 - \frac{1}{(3)_q} P_3 \right) .
\]

(4.39)

Since \( q \) is a primitive root of the equation \( q^4 = 1 \) the following identities hold:

\[
q^2 = -1, \quad (2)_q = 1 + q, \quad (3)_q = 1 + q + q^2 = q
\]

\[
downarrow
\]

\[
(2)_q (3)_q = (1 + q) q = q - 1,
\]

(4.40)

and using relations (2.1), (2.6) one can deduce another representation for the generators of the \( N = 2 \) Heisenberg-Weyl algebra

\[
\theta_1 = \theta^2, \quad \theta_2 = \theta - \theta^2 \partial + q \theta^3 \partial^2 ,
\]

\[
\partial_1 = \frac{1}{2} (1 - q) \partial^2 - \theta \partial^3, \quad \partial_2 = \partial - \theta \partial^2 + (1 + q) \theta^2 \partial^3 .
\]

One can express these generators in terms of the operator \( \omega \) using (4.38), (4.39) and (2.10)

\[
\theta_1 = \theta^2, \quad \partial_1 = \left( \frac{\omega - 1 - q}{1 - q} \right) \partial^2, \quad \theta_2 = \theta \left( \frac{\omega - q}{1 - q} \right)^2 , \quad \partial_2 = \omega \left( \frac{\omega - q}{1 - q} \right) \partial.
\]

4.2 \( N = 3 \) Heisenberg-Weyl algebra

Using Weyl construction (3.28), (3.29) one can obtain the following matrix representations for the generators \( \theta_\mu \) (\( \mu = 1, 2, 3 \)):

\[
\theta_1 = \sum_{m=0}^{3} e_{m+4,m}, \quad \theta_2 = \sum_{m=0}^{1} (e_{m+3,m} - e_{m+6,m+3}), \quad \theta_3 = e_{1,0} - e_{3,2} - e_{5,4} + e_{7,6},
\]

(4.41)

where the matrix units \( e_{ij} \) have been introduced

\[
(e_{ij})_{km} = \delta_{ik} \delta_{jm}, \quad e_{ij} e_{kl} = \delta_{jk} e_{il} .
\]

(4.42)
According with the representation (2.19) we find that
\[ \theta_1 = \theta^4, \quad \theta_2 = \theta^2 (P_0 + P_1 - P_4 - P_5) = (P_2 + P_3 - P_6 - P_7) \theta^2, \quad \theta_3 = \theta (P_0 - P_2 - P_4 + P_6) = (P_1 - P_3 - P_5 + P_7) \theta. \]  
(4.43)

Applying the transpositions (3.33) and using (2.20) one can deduce from (4.43) the expressions for generators \( \partial_\mu \):
\[
\begin{align*}
\partial_1 &= \partial^4 \left( \sum_{k=4}^{7} \frac{1}{(k-3)q(k-2)q(k-1)q(k)q} P_k, \right) \\
\partial_2 &= \partial^2 \left( \sum_{k=2,3} - \sum_{k=6,7} \frac{1}{(k)q} P_k, \right) \\
\partial_3 &= \partial \left( \sum_{k=1,7} - \sum_{k=3,5} \frac{1}{(k)q} P_k, \right)
\end{align*}
\] 
(4.44)

4.3 \( N = 4 \) Heisenberg-Weyl algebra

Using Weyl construction (3.28), (3.29) one can obtain the following matrix representations for the generators \( \theta_\mu \ (\mu = 1, 2, 3) \):
\[
\begin{align*}
\theta_1 &= \sum_{m=0}^{7} e_{m+8, m}, \\
\theta_2 &= \left( \sum_{m=0}^{7} - \sum_{m=8}^{11} \right) e_{m+4, m}, \\
\theta_3 &= \left( \sum_{m=0, 1, 12, 13} - \sum_{m=4, 5, 8, 9} \right) e_{m+2, m}, \\
\theta_4 &= \left( \sum_{m=0, 6, 10, 12} - \sum_{m=2, 4, 8, 14} \right) e_{m+1, m}.
\end{align*}
\] 
(4.45)

According with the representation (2.19) we find that
\[
\begin{align*}
\theta_1 &= \theta^8, \\
\theta_2 &= \left( \sum_{m=4}^{7} - \sum_{m=12}^{15} \right) P_m \theta^4, \\
\theta_3 &= \left( \sum_{m=2, 3, 14, 15} - \sum_{m=6, 7, 10, 11} \right) P_m \theta^2, \\
\theta_4 &= \left( \sum_{m=1, 7, 11, 13} - \sum_{m=3, 5, 9, 15} \right) P_m \theta.
\end{align*}
\] 
(4.46)

Applying the transpositions (3.33) and using (2.20) one can deduce from (4.43) the expressions for generators \( \partial_\mu \):
\[
\begin{align*}
\partial_1 &= \partial^8 \left( \sum_{k=8}^{15} \frac{(k-6)q^l}{(k)q} P_k, \right) \\
\partial_2 &= \partial^4 \left( \sum_{k=4}^{7} - \sum_{k=12}^{15} \frac{(k-4)q^l}{(k)q} P_k, \right) \\
\partial_3 &= \partial^2 \left( \sum_{k=2, 3, 14, 15} - \sum_{k=6, 7, 10, 11} \right) \frac{(k-2)q^l}{(k)q} P_k, \\
\partial_4 &= \partial \left( \sum_{k=1, 7, 11, 13} - \sum_{k=3, 5, 9, 15} \right) \frac{1}{(k)q} P_k.
\end{align*}
\] 

5. \( \text{EXAMPLE: } N = 2 \) SKDV EQUATIONS IN TERMS OF \( Z_4 \) GENERALIZED GRASSMANN ALGEBRA

We start with the discussion of the \( N = 1 \) super-extension of the KdV equation
\[
\frac{\partial \Phi}{\partial t} = -\partial_x \left( (\partial_x^2 \Phi) + 3 \Phi(D\Phi) \right). 
\] 
(5.47)

It is known (see e.g. [20]) that the Lax operator \( L \) with the Lax pair representation in this case can be taken in the form
\[
L = \partial_x^2 + \Phi D \quad ; \quad \frac{\partial L}{\partial t} = [L, \frac{D}{t \geq 0}],
\] 
(5.48)
where $\geq 0$ denotes the projection onto the purely (super)differential part of the operator, 
$\partial_x \equiv \frac{\partial}{\partial x}$ and we have used the odd superfield $\Phi(x, \theta) = \psi(x) + \hat{\theta} u(x)$ and covariant super-derivative $D = \hat{\theta} + \hat{\theta} \partial_x$.

The operators $\hat{\theta}, \hat{\partial}$ are the same as in (3.30), (3.31). Since the fermionic field $\psi(x)$ anticommutes with $\hat{\theta}$ and $\hat{\partial}$ (3.31) we should represent it in the form

$$\psi(x) = \hat{\omega} \hat{\psi}(x) = \begin{pmatrix} \psi & 0 \\ 0 & -\hat{\psi} \end{pmatrix},$$

which guarantees the anticommutation of $\psi$ with $\hat{\theta}$ and $\hat{\partial}$. Substitution of the matrix representations (3.31) and (5.49) in (5.48) gives the matrix representation for the $N = 1$ SKdV Lax operator

$$L = \partial_x^2 + \begin{pmatrix} \hat{\psi} & 0 \\ u & -\hat{\psi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \partial_x & 0 \end{pmatrix} = \partial_x^2 + \begin{pmatrix} 0 & \hat{\psi} \\ -\hat{\psi}\partial_x & u \end{pmatrix}.$$

Here the field $\hat{\psi}(x)$ should be considered as a fermionic field while $u(x)$ is a bosonic Virasoro field. Thus, we represent the Lax operator $L$ for super-KdV equation in two different but absolutely equivalent forms. The first formula (5.48) gives the representation of the $L$-operator in terms of $Z_2$ graded algebra (2.1), (2.2), while the second formula (5.50) gives corresponding graded matrix realization.

Now the discussion of the $N = 2$ SKdV Lax operators is in order. It is well known that there are three different completely integrable $N = 2$ supersymmetric extensions of the Korteweg-de Vries equations [18], [19]. All these extensions could be written as

$$\frac{\partial \Phi}{\partial t} = \partial_x \left( - (\partial_x^2 \Phi) + 3 \Phi (D_1 D_2 \Phi) + \frac{1}{2} (\alpha - 1) (D_1 D_2 \Phi^2) + \alpha \Phi^3 \right),$$

where $\alpha = 4, -2, 1$ and $\Phi(x, \theta_1, \theta_2) = w(x) + \theta_1 \psi_1(x) + \theta_2 \psi_2(x) + \theta_2 \theta_1 u(x)$. We would like to show that the Lax operators for these generalizations could be written in terms of the $Z_4$ generalized Grassmann algebra. Let us notice that the Lax operators for the $\alpha = 1$ case is

$$L_1 = \partial_x - \partial_x^{-1} D_1 D_2 \cdot \Phi(x, \theta_1, \theta_2),$$

where the superderivatives $D_i$ are

$$D_1 = \partial_1 + \theta_1 \partial_x , \quad D_2 = \partial_2 + \theta_2 \partial_x.$$

The supersymmetric KdV equation is obtained from the Lax pair representation as

$$\frac{\partial L_1}{\partial t} = \left[ L_1, (L_1^2)_{\geq 0} \right].$$

The Lax operators for the $\alpha = 4, -2$ are described in terms of the previous operator $L_1$ as

$$L_{-2} = L_1^\dagger L_1 \quad , \quad L_4 = (L_1^\dagger L_1)_{\geq 0},$$

where $\dagger$ denotes the hermitian conjugation. From that reason we consider therefore the case $\alpha = 1$ only.
The component $u(x)$ is a Virasoro field and variables $\theta_i$ and $\psi_j$ anticommute with each other. It means that we can represent $\psi_i$ in the form

$$\psi_i = \hat{\omega} \otimes \hat{\omega} \cdot \hat{\psi}_i = \begin{pmatrix} \hat{\psi}_i & 0 & 0 & 0 \\ 0 & -\hat{\psi}_i & 0 & 0 \\ 0 & 0 & -\hat{\psi}_i & 0 \\ 0 & 0 & 0 & \hat{\psi}_i \end{pmatrix}.$$ 

$$[\theta_i, \hat{\psi}_j] = 0 = \{\hat{\psi}_i, \hat{\psi}_j\}.$$ 

In matrix notations we have

$$D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_x & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \partial_x & 0 & 0 & 0 \\ 0 & 0 & -\partial_x & 0 \end{pmatrix},$$

$$\Phi = \begin{pmatrix} w & 0 & 0 & 0 \\ \hat{\psi}_2 & w & 0 & 0 \\ \hat{\psi}_1 & 0 & w & 0 \\ -u & -\hat{\psi}_1 & \hat{\psi}_2 & w \end{pmatrix}.$$ (5.56)

Finally we write the Lax operator (5.52) in the matrix form

$$L = \partial_x \mathbf{1} - \begin{pmatrix} 0 & 0 & 0 & -\partial_x^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \partial_x & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} w & 0 & 0 & 0 \\ \hat{\psi}_2 & w & 0 & 0 \\ \hat{\psi}_1 & 0 & w & 0 \\ -u & -\hat{\psi}_1 & \hat{\psi}_2 & w \end{pmatrix}. \quad (5.57)$$

The superfield $\Phi$ and covariant derivatives $D_i$ in terms of $Z_4$ variables are represented in the form

$$\Phi = w(x) + \theta (P_0 + P_2) \hat{\psi}_2(x) + \theta^2 (P_0 - P_1) \hat{\psi}_1(x) - \theta^3 u(x). \quad (5.58)$$

$$D_1 = \frac{1}{(3)_q} \partial^2 \theta (P_2 + \frac{1}{(3)_q} P_3) + \theta^2 \partial_x,$$

$$D_2 = \partial_\theta \left( P_1 - \frac{1}{(3)_q} P_3 \right) + \theta (P_0 - P_2) \partial_x,$$ \quad (5.59)

and using these relations we obtain

$$\partial_x^{-1} D_1 D_2 = \frac{(q + 1)}{2} \partial_\theta \partial_x^{-1} + \frac{(q - 1)}{2} \partial_\theta P_2 + \theta P_1 + \theta^3 \partial_x.$$ \quad (5.60)

Let us introduce the $Z_4$ covariant derivative

$$D = \partial_\theta + \frac{1}{(3)_q} \theta^3 \partial_x = \partial_\theta - \frac{1}{2} (q + 1) \theta^3 \partial_x, \quad D^4 = \partial_x,$$ \quad (5.61)
where the operators $\theta$ and $\partial_\theta$ generate the $Z_4$ algebra (2.1), (2.2) for $p = 3$ and parameter $q$ satisfy $q^2 = -1$. Now we express all $\partial^k_\theta$ via operator $\theta$ and $Z_4$-covariant derivative $\mathcal{D}$:

\[
\partial_\theta = \mathcal{D} + \frac{1}{2} (q + 1) \theta^3 \partial_x ,
\]

\[
\partial^2_\theta = \mathcal{D}^2 + \left( q \mathcal{D} \theta^3 + \frac{(q+1)}{2} \theta^2 \right) \partial_x ,
\]

\[
\partial^3_\theta = \mathcal{D}^3 + \left( \frac{(q-1)}{2} \mathcal{D}^2 \theta^3 + \frac{(q-1)}{2} \mathcal{D} \theta^2 + q \theta \right) \partial_x .
\]

Using this relation the differential operator (5.60) is rewritten as

\[
\left\{ \frac{q + 1}{2} \partial_x^{-1} \mathcal{D}^3 - \frac{1}{2} \mathcal{D}^2 \theta^3 + \mathcal{D} \left( \frac{(q-1)}{2} P_2 - \frac{1}{2} \theta^2 \right) + \theta \left( P_1 + \frac{(q-1)}{2} \right) + \theta^3 \partial_x \right\} .
\]

Finally we obtain the expression for the Lax operator $L$ (5.57) as a pseudodifferential operator with respect to the fractional covariant derivative $\mathcal{D}$:

\[
L = \partial_x + \mathcal{D}^{-1} \cdot u_{-1} + u_0 + \mathcal{D} \cdot u_1 + \mathcal{D}^2 \cdot u_2 + \partial_x \cdot u_4 ,
\]

(5.62)

where the fractional superfields $u_k(x, \theta)$ are defined in terms of the unique fractional prepotential $\Phi$ (5.58).

\[
u_{-1} = \frac{1}{q-1} \Phi , \quad u_0 = \theta \left( \frac{1}{q+1} - P_1 \right) \Phi ;
\]

\[
u_1 = \left( \frac{1-q}{2} P_2 + \frac{1}{2} \theta^2 \right) \Phi , \quad 2 u_2 = -u_4 = \theta^3 \Phi .
\]

6. CONCLUSION

In this paper we have shown that the fermionic Heisenberg-Weyl algebra with $2N = D$ fermionic generators is equivalent to the generalized Grassmann algebra with two fractional generators of the order of the parastatistic: $p = 2^N - 1$. The 2,3 and 4 dimensional Heisenberg-Weyl algebras (they are related to the 4,6 and 8 dimensional Clifford algebras) were explicitly given. The $N = 2, 3, 4$ extended supersymmetries can be described in terms of these algebras. We reformulate the Lax approach for the supersymmetric Korteweg-de Vries equation in terms of the generators of the generalized Grassmann algebra. Note that the theory of the pseudodifferential operators on the $(1|1)$ superline \cite{20} can be directly generalized to the theory of pseudodifferential operators on the fractional superline $(x, \theta)$ (we plan to return to these problems in one of the forthcoming papers). On the other hand, in the similar manner, one can construct the $Z_{p+1}$ graded extensions of the KdV and Kadomtsev-Petviashvily hierarchies. These extensions should be probably related to the graded matrix generalizations of the KdV and KP equations.

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