Traveling Waves for a Microscopic Model of Traffic Flow

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Abstract

We consider the follow-the-leader model for traffic flow. The position of each car \( z_i(t) \) satisfies an ordinary differential equation, whose speed depends only on the relative position \( z_{i+1}(t) \) of the car ahead. Each car perceives a local density \( \rho_i(t) \). We study a discrete traveling wave profile \( W(x) \) along which the trajectory \( (\rho_i(t), z_i(t)) \) traces such that \( W(z_i(t)) = \rho_i(t) \) for all \( i \) and \( t > 0 \); see definition 2.2. We derive a delay differential equation satisfied by such profiles. Existence and uniqueness of solutions are proved, for the two-point boundary value problem where the car densities at \( x \to \pm \infty \) are given. Furthermore, we show that such profiles are locally stable, attracting nearby monotone solutions of the follow-the-leader model.

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1 Introduction and Preliminaries

We consider a microscopic model for traffic flow. Let \( \ell \) be the length of all the cars, and let \( z_i(t) \) be the position of \( i \)th car at time \( t \). We order the indices for the cars such that

\[
z_i(t) \leq z_{i+1}(t) - \ell \quad \text{for every } i \in \mathbb{Z}.
\]

(1.1)

For a car with index \( i \), we define the local density

\[
\rho_i(t) \triangleq \frac{\ell}{z_{i+1}(t) - z_i(t)}.
\]

(1.2)

Note that if \( \rho_i = 1 \), then the two cars with indices \( i \) and \( i+1 \) will be bumper-to-bumper. Thus \( 0 \leq \rho_i \leq 1 \) for all \( i \).

We assume that the speed of the car with index \( i \) depends solely on the local density \( \rho_i \), i.e.,

\[
\dot{z}_i(t) = V \cdot \phi(\rho_i(t)) = V \cdot \phi \left( \frac{\ell}{z_{i+1}(t) - z_i(t)} \right).
\]

(1.3)

Here \( V \) is the speed limit, and the function \( \phi(\rho) \), defined on \( \rho \in [0, 1] \), satisfies

\[
\phi(1) = 0, \quad \phi(0) = 1, \quad \phi'(\rho) \leq -\hat{c}_0 < 0 \quad \text{for all } \rho \in [0, 1].
\]

(1.4)

We remark that a popular choice for \( \phi(\cdot) \) is the Lighthill-Whitham model [17]:

\[
\phi(\rho) = 1 - \rho.
\]

(1.5)
Given an initial distribution of car positions \( \{ z_i(0) \} \), the system (1.3) depicts the “follow-the-leader” behavior of each car. We refer to this model as the FtL model.

Note that (1.3) can be rewritten as a system of ODEs for the discrete density functions \( \rho_i(t) \),

\[
\dot{\rho}_i(t) = -\frac{\ell (\dot{z}_{i+1} - \dot{z}_i)}{(z_{i+1} - z_i)^2} = \frac{V}{\ell} \rho_i^2 \cdot (\phi(\rho_i) - \phi(\rho_{i+1})).
\] (1.6)

If one uses (1.5), then (1.6) becomes

\[
\dot{\rho}_i(t) = \frac{V}{\ell} \rho_i^2 (\rho_{i+1} - \rho_i).
\] (1.7)

Given the initial positions of the cars \( z_i(0) \) and the speed of the leader as \( i \to \infty \), the existence of solution for the ODE system (1.3) is established in the literature [8, 14]. We can define a piecewise constant function \( \rho^\ell(t, x) \) from the discrete densities \( \{ \rho_i \} \) as

\[
\rho^\ell(t, x) = \rho_i(t), \quad \text{for} \quad x \in [z_i(t), z_{i+1}(t)).
\] (1.8)

As \( \ell \to 0 \) and the number of the cars tends to \( \infty \), under suitable assumptions one has the convergence \( \rho^\ell(t, x) \to \rho(t, x) \), where the limit function \( \rho(t, x) \) provides a weak solution for the scalar conservation law

\[
\rho_t + f(\rho)_x = 0, \quad \text{where} \quad f(\rho) = V\rho \cdot \phi(\rho).
\] (1.9)

See [8] for a proof using direct properties of the solutions of (1.3), and some more recent works [14, 15] where the same results are achieved utilizing a Lagrangian formulation and the properties of monotone numerical schemes. For other related works including model derivations, analysis, and treatment of various conditions, we refer to [1–7, 11, 19] and the references therein.

It is well-known that, in the solutions of the nonlinear conservation law (1.9), discontinuities can form in finite time even with smooth initial data. Such discontinuities are known as shocks. In the literature for traffic flow, the flux \( f(\rho) \) is typically concave such that

\[
f(0) = f(1) = 0, \quad f''(\rho) \leq -c_0 < 0, \quad f'(\rho^*) = 0 \quad \text{for a unique} \quad \rho^* \in (0, 1). \] (1.10)

Note that (1.10) holds with (1.5), where \( f(\rho) = V\rho(1-\rho) \). In general, (1.10) leads to additional assumptions on \( \phi(\rho) \), besides (1.4), i.e.,

\[
\phi''(\rho) \leq -\frac{1}{\rho} \left[ 2\phi'(\rho) + c_0/V \right], \quad \rho \in [0, 1].
\] (1.11)

Since \( f'' < 0 \), only upward jumps are admissible in the solutions of (1.9). The solution for the Riemann problem with initial data

\[
\rho(0, x) = \begin{cases} 
\rho_-, & x < 0, \\
\rho_+, & x > 0,
\end{cases} \quad \text{and} \quad \rho_- < \rho_+
\]

results in a single shock which travels with the Rankine-Hugoniot jump speed:

\[
\sigma = \frac{f(\rho_-) - f(\rho_+)}{\rho_- - \rho_+}.
\]
The shock is stationary when \( f(\rho_-) = f(\rho_+) \).

In this work we seek a “discrete traveling wave profile” for the FtL model, as a corresponding approximation to the shock waves for the conservation law (1.9). To fix the idea, we start with a monotone stationary profile \( W(x) \) such that the position of the point \((z_i(t), \rho_i(t))\) traces along the graph of the function \( W(x) \) as time \( t \) evolves. To be precise, we require

\[
W(z_i(t)) = \rho_i(t), \quad \forall t \geq 0, \quad \forall i. \tag{1.12}
\]

We remark that for general traveling waves with speed \( \sigma \neq 0 \), the profile will be stationary in the shifted coordinate \( x \mapsto \xi = x - \sigma t \), see the discussion in Section 5.

Differentiating both sides of (1.12) in \( t \), and using (1.3) and (1.6), one gets

\[
W'(z_i) = \frac{\dot{\rho}_i}{\dot{z}_i} = \frac{\rho_i^2}{\ell \cdot \phi(\rho_i)} \left[ \phi(\rho_i) - \phi(\rho_{i+1}) \right] = \frac{W^2(z_i)}{\ell \cdot \phi(W(z_i))} \left[ \phi(W(z_i)) - \phi(W(z_{i+1})) \right].
\]

Note that \( z_{i+1} = z_i + \frac{\ell}{\rho_i} = z_i + \frac{\ell}{W(z_i)} \).

Since \( z_i \) is randomly chosen, we write \( x \) for \( z_i \), and obtain the following equation:

\[
W'(x) = \frac{W^2(x)}{\ell \cdot \phi(W(x))} \cdot \left[ \phi(W(x)) - \phi(W(x + \frac{\ell}{W(x)})) \right]. \tag{1.13}
\]

If \( W(x) = 0 \) for some \( x \), then we set

\[
W(x + \frac{\ell}{W(x)}) = W(+\infty).
\]

Equation (1.13) is a Delay Differential Equation (DDE). Furthermore, (1.13) is autonomous since the righthand side does not depend on \( x \) explicitly.

Once the “initial data” is given on an interval \([\hat{x}, \infty)\) for any \( \hat{x} \), the DDE (1.13) can be solved backwards in \( x \), and the profile \( W(x) \) can be obtained for all \( x \leq \hat{x} \). This is in agreement with the following-the-leader principle.

In this paper we study in detail the DDE (1.13). In particular, we study the “two-point-boundary-value” problem. To be specific, we seek solutions of (1.13) that satisfies the boundary conditions at the infinities:

\[
\lim_{x \to +\infty} W(x) = \rho_\pm, \quad 0 \leq \rho_- \leq \rho_+ \leq 1. \tag{1.14}
\]

In the case of the stationary profile \( W(x) \), \( \rho_\pm \) must further satisfy

\[
f(\rho_-) = f(\rho_+) = \bar{f}, \quad 0 \leq \rho_- \leq \rho^* \leq \rho_+ \leq 1 \quad \text{where} \quad f'(\rho^*) = 0. \tag{1.15}
\]

Note that any horizontal shift of the profile \( W(x) \) is again a profile. Thus, a unique profile can be achieved by requiring a “location-fixing” condition at \( x = 0 \), say

\[
W(0) = \rho^*.
\]

We show that, for any given \( \rho_\pm \) satisfying (1.15), there exists a profile \( W(x) \), unique up to horizontal shifts. Furthermore, such traveling waves are local attractors for the solutions of the FtL model (1.3).
In the literature, solutions for the conservation law (1.9) are approximated by various approaches. These include the viscous equations, kinetic models with relaxation terms, and various numerical approximations. For many of the approximate solutions, the study of traveling wave profiles is one of the key techniques in the analysis. In this paper, we consider the microscopic “particle” model and its traveling waves, filling a missing piece in the literature.

We mention also a study on traveling waves for a non-standard integro-differential equation modeling slow erosion [12], where uniqueness and local stability are achieved.

The rest of the paper is organized as follows. For stationary profiles \( W(x) \), in Section 2 we prove several technical Lemmas. These results are utilized in Section 3 where we prove the existence and uniqueness of the profile. Furthermore, such profiles are local attractors for solutions of the FtL model (1.3), proved in Section 4. Extension to general traveling waves with non-zeros speed is outlined in Section 5. Finally, concluding remarks and further open problems are discussed in Section 6.

2 Technical Lemmas; Properties of the Stationary Profile

We consider the stationary profile \( W(x) \), satisfying the DDE (1.13) with boundary conditions (1.14)-(1.15).

We first provide a formal argument which makes connection between the profile \( W(x) \) and the viscous shock for the conservation law (1.9). Assuming that \( \ell/W(x) > 0 \) is very small, by Taylor expansion we have

\[
\phi\left(W(x + \frac{\ell}{W(x)})\right) - \phi(W(x)) = \frac{\ell}{W(W)}(\phi(W))_x + \frac{1}{2} \left(\frac{\ell}{W}\right)^2 (\phi(W))_{xx} + O\left(\left(\frac{\ell}{W}\right)^3\right). 
\]  

(2.1)

Dropping the higher order terms, the DDE (1.13) is approximated by

\[
W_x = \frac{W}{\phi(W)} \cdot \left[-(\phi(W))_x - \frac{\ell}{2W}(\phi(W))_{xx}\right]
\]

This equation can be manipulated into:

\[
\phi(W)W_x + W(\phi(W))_x = -\frac{\ell}{2}(\phi(W))_{xx},
\]

and then

\[
(W \cdot \phi(W))_x = \frac{1}{V} f(W)_x = -\frac{\ell}{2}(\phi(W))_{xx}.
\]

We conclude

\[
f(W)_x = \left(-\frac{V\ell}{2}\phi(W)W_x\right)_x.
\]  

(2.2)

Now we consider the viscous conservation law

\[
\rho_t + (f(\rho))_x = \varepsilon \rho_{xx}.
\]

Stationary viscous shock waves \( \bar{\rho}(x) \) must satisfy the ODE

\[
f(\bar{\rho})_x = \varepsilon \bar{\rho}_{xx} = (\varepsilon \bar{\rho}_x)_x.
\]  

(2.3)
We observe that the ODEs (2.2) and (2.3) are connected through the relation:

\[ \varepsilon \approx - \frac{V \ell}{2} \phi'(W(x)). \]

For the case \( \phi(\rho) = 1 - \rho \) where \( \phi'(\rho) = -1 \), we have the connection \( \varepsilon \approx \frac{V \ell}{T} \).

For any given profile \( W(x) \), one can generate a distribution of car positions \( \{z_i\} \), and vice versa. We make the following definitions.

**Definition 2.1.** Let the function \( x \mapsto W \in (0, 1] \) be given for \( x \in \mathbb{R} \). We call a sequence of car positions \( \{z_i\} \) a *distribution generated by* \( W(x) \), if

\[
z_{i+1} - z_i = \frac{\ell}{W(z_i)}, \quad \forall i \in \mathbb{Z}. \tag{2.4}
\]

If one imposes \( z_0 = 0 \), then the distribution is unique.

**Definition 2.2.** Given a profile \( W(x) \) and a distribution of car positions \( \{z_i(t)\} \). Let \( \{\rho_i(t)\} \) be the corresponding discrete densities for the cars, computed as (2.2). We say that \( \{z_i(t)\} \) *traces along* \( W(x) \), if

\[
W(z_i(t)) = \rho_i(t) = \frac{\ell}{z_{i+1}(t) - z_i(t)}, \quad \forall i, t.
\]

The following Lemma is an immediate consequence of these definitions.

**Lemma 2.1.** Let \( W(x) \) be a given profile and \( \{z_i(0)\} \) be a distribution generated by \( W(x) \). Let \( \{z_i(t)\} \) be the solution of (1.3) with initial data \( \{z_i(0)\} \). Then, \( W(x) \) satisfies the DDE (1.13) if and only if \( \{z_i(t)\} \) traces along \( W(x) \).

Our first theorem states existence and uniqueness of monotone solutions of (1.13) as an initial value problem, under suitable assumptions on the initial data.

**Theorem 2.1.** Fix an \( \hat{x} \). Let \( \psi(x) \in (0, 1) \) be a continuous monotone function defined on the interval \( x \geq \hat{x} \) such that \( \psi'(x) > 0 \) for all \( x \geq \hat{x} \). Let \( W(x) \) be the solution of the DDE (1.13) on \( x < \hat{x} \), solved backwards in \( x \), with initial data \( \psi(x) \) given on \( x \geq \hat{x} \). Then, there exists a unique positive solution \( W(x) \), which is monotone increasing such that

\[
W'(x) > 0, \quad W(x) > 0, \quad \text{for } x \leq \hat{x}. \tag{2.5}
\]

**Proof.** The existence and uniqueness of the solution \( W(x) \) for the initial value problem follows from an iteration argument. It is understood that the derivative \( W'(x) \) in (1.13) is the left derivative. We clearly have

\[
W'('\hat{x}-') = \frac{\psi(\hat{x})}{\phi(\psi(\hat{x}))} \cdot \frac{\phi(\psi(\hat{x})) - \phi(\psi(\hat{x} + \ell/\psi(\hat{x})))}{\ell/\psi(\hat{x})} > 0.
\]

Now consider the interval \( x \in I_1 = [\hat{x} - \ell, \hat{x}] \). We claim that, if \( W(x) \) exists on \( I_1 \), then \( W(x) \geq 0 \). Indeed, the lower bound \( W(x) \geq 0 \) is clear since 0 is a critical point. Assuming that \( W(x) \) becomes negative on some subset of \( I_1 \), then there exists a point \( x_0 \in I_1 \) such that

\[
W(x_0) = 0, \quad W'(x_0) > 0.
\]
But this is not possible because by (1.13) we have
\[ W'(x_0) = \frac{W^2(x_0)}{\ell \phi(W(x_0))} [\phi(W(x_0)) - \phi(\rho_+)] = 0, \]
where \( \rho_+ = \lim_{x \to \infty} \psi(x) \),
a contradiction.

We further claim that, if \( W(x) \) exists on \( I_1 \), then it is monotonically increasing. We prove
by contradiction. Assume that \( W(x) \) is not monotone on \( I_1 \). Then there exists a value \( \tilde{x} \), with
\( \tilde{x} < \hat{x} \), where \( W' \) changes sign, such that
\[ W'(\tilde{x}) = 0, \quad \text{and} \quad W'(x) > 0, \quad x > \tilde{x}. \] (2.6)
However, this would imply
\[ W'(\tilde{x}) = \frac{W^2(\tilde{x})}{\ell \phi(W(\tilde{x}))} \cdot \left[ \phi(W(\tilde{x})) - \phi \left( W(\tilde{x} + \frac{\ell}{W(\tilde{x})}) \right) \right] = 0, \]
thus
\[ \phi(W(\tilde{x})) - \phi \left( W \left( \tilde{x} + \frac{\ell}{W(\tilde{x})} \right) \right) = 0 \quad \Rightarrow \quad W(\tilde{x}) = W \left( \tilde{x} + \frac{\ell}{W(\tilde{x})} \right), \]
a contradiction to (2.6).

Thus, we deduce that
\[ x + \ell/W(x) > \hat{x}, \quad \text{for every} \quad x \in I_1, \]
which means,
\[ W(x + \ell/W(x)) = \psi(x + \ell/W(x)), \quad \text{for every} \quad x \in I_1. \]
Then, the equation (1.13) reduces to an ODE of the form
\[ W'(x) = \mathcal{F}(W, \psi(x + \ell/W)) = \frac{W^2(x)}{\ell \phi(W(x))} [\phi(W(x)) - \phi(\psi(x + \ell/W(x)))] . \]
Since \( \mathcal{F} \) is Lipschitz in both arguments, \( \psi \) is continuous and Lipschitz for \( W > 0 \), by standard
ODE theory, the solution \( W(x) \) exists and is unique on \( I_1 \).

One can the iterate the argument on the intervals \( I_k = [\hat{x} - (k+1)\ell, \hat{x} - k\ell] \), for \( k = 1, 2, \cdots \),
completing the proof.

**Remark 2.1.** For general references on standard theory for delay differential equations, see [9, 10]. We remark that our equation (1.13) does not fall into the standard setting, therefore we provide a simple proof for existence and uniqueness of solutions. We further note that, if the initial condition shall be monotonically decreasing, such that \( \psi'(x) < 0 \) for \( x > \hat{x} \), the global existence of solution \( W(x) \) on \( x \in (-\infty, \hat{x}] \) fails. One simply observes that \( W'(x) < 0 \), so \( W(x) \) increases as \( x \) decreases, and \( W'(x) \) blows up to infinity as \( W(x) \) approaches 1.

Next Lemma describes the asymptotic behavior at the limits as \( x \to \pm \infty \).

**Lemma 2.2.** (Asymptotic Limits.) Assume that \( W(x) \) is a solution of (1.13) that satisfies
the boundary condition
\[ W(-\infty) = \rho_-, \quad W(+\infty) = \rho_. \]
Then, we have the followings.
• As $x \to \infty$, $W(x)$ can approach $\rho_+$ at an exponential rate only if $\rho_+ > \rho^*$. The exponential rate $\lambda_+$ satisfies the estimate

$$
\lambda_+ > \frac{2\rho_+}{\ell} \cdot \ln \left( 1 - \frac{f'(\rho_+)\rho_+}{f(\rho_+)} \right). \quad (2.7)
$$

• Similarly, as $x \to -\infty$, $W(x)$ can approach $\rho_-$ at an exponential rate only if $\rho_- < \rho^*$. The exponential rate $\lambda_-$ satisfies the estimates

$$
-\frac{\rho_-}{\ell} \cdot \ln \left( 1 - \frac{f'(\rho_-)\rho_-}{f(\rho_-)} \right) < \lambda_- < -\frac{2\rho_-}{\ell} \cdot \ln \left( 1 - \frac{f'(\rho_-)\rho_-}{f(\rho_-)} \right). \quad (2.8)
$$

**Proof. Step 1.** Consider the asymptotic behavior as $x \to +\infty$. By assumption we have $W'(x) \to 0$ as $x \to \infty$. Now, for $x$ large, we write

$$
W(x) = \rho_+ + \eta(x)
$$

where $\eta(x)$ is the first order perturbation. Plugging this into (1.13), and neglecting the higher order terms, we obtain the following linearized equation for $\eta(x)$:

$$
\eta'(x) = -\phi'(\rho_+) \cdot \frac{\rho_+^2}{\ell \cdot \phi(\rho_+)} \left[ \eta(x + \ell \rho_+) - \eta(x) \right]. \quad (2.9)
$$

Denoting the positive constants as

$$
a = \frac{\ell}{\rho_+}, \quad b = -\phi'(\rho_+) \cdot \frac{\rho_+}{\phi(\rho_+)}, \quad (2.10)
$$

we can write

$$
\eta'(x) = \frac{b}{a} \cdot (\eta(x + a) - \eta(x)). \quad (2.11)
$$

This is a linear delay differential equation, which can be solved explicitly using the characteristic equation. Seeking solution of the form

$$
\eta(x) = Me^{-\lambda x} \quad (2.12)
$$

where $M$ is an arbitrary constant (which could be both negative or positive), the rate $\lambda$ satisfies the characteristic equation

$$
G(\lambda) \doteq b \cdot \left( e^{-a\lambda} - 1 \right) + a\lambda = 0. \quad (2.13)
$$

We locate all the zeros for the function $G(\lambda)$, in particular the positive ones. We observe

$$
G(0) = 0, \quad (2.14)
$$

$$
G'(\lambda) = -ab \ e^{-a\lambda} + a, \quad G'(0) = a(1 - b), \quad (2.15)
$$

$$
G''(\lambda) = a^2 b \ e^{-a\lambda} > 0, \quad G''(0) = a^2 b, \quad (2.16)
$$

$$
G'''(\lambda) = -a^3 b \ e^{-a\lambda} < 0. \quad (2.17)
$$

Thus, $G(\cdot)$ is a convex function which goes through the origin. Typical graphs of $G(\lambda)$ for different values of $b$ are illustrated in Figure \[I\]. We have:

- If $b = 1$, then there is only one zero $\lambda_0 = 0$. 

• If $b < 1$, then there are two zeros $\{\lambda_0, \lambda_-\}$, where $\lambda_- < 0$.
• If $b > 1$, then there are two zeros $\{\lambda_0, \lambda_+\}$, where $\lambda_+ > 0$.

Recall that
\[ f(\rho) = V\rho \cdot \phi(\rho), \quad f'(\rho^*) = 0, \quad f''(\rho) < 0, \]
we have
\[ b = \frac{-\phi'(\rho_+)\rho_+}{\phi(\rho_+)} = 1 - \frac{f'(\rho_+)\rho_+}{f(\rho_+)} : \quad \begin{cases} = 1, & \text{if } \rho_+ = \rho^*, \\ > 1, & \text{if } \rho_+ > \rho^*, \\ < 1, & \text{if } \rho_+ < \rho^*. \end{cases} \]

Thus we conclude:
• If $\rho_+ = \rho^*$, then $\lambda = 0$ and we have the trivial solution $\eta(x) \equiv 0$, and thus $W(x) \equiv \rho^*$.
• If $\rho_+ < \rho^*$, then the other rate $\lambda_- < 0$ indicates exponential growth of $\eta(x)$, which is not valid. The only possible solution is the trivial one.
• If $\rho_+ > \rho^*$, then the other zero $\lambda_+ > 0$ indicates exponential decay of $\eta(x)$ in the limit as $x \to \infty$. This is the valid case.

Therefore, if $\rho_+ > \rho^*$, we can have the asymptotic limit as $x \to \infty$,
\[ W(x) \to \rho_+ + Me^{-\lambda_+x}. \]

Finally, we derive an estimate on the rate $\lambda_+$. Let
\[ \lambda_* = \frac{1}{a} \ln(b), \quad \text{where} \quad G'(\lambda_*) = 0. \]
By the properties (2.14)-(2.17), we conclude the estimate
\[ \lambda_+ > 2\lambda_* = \frac{2}{a} \ln(b), \]
proving (2.7).

**Step 2.** A similar computation can be carried out for $x \to -\infty$. We write
\[ W(x) = \rho_- + \zeta(x) \]
where $\zeta(x)$ is a small perturbation. The linearized equation for $\zeta(x)$ becomes

$$
\zeta'(x) = -\frac{\phi'(\rho_-)\rho_-^2}{\ell\phi(\rho_-)} \left[ \zeta(x + \ell/\rho_-) - \zeta(x) \right].
$$

Denoting the positive constants

$$
\hat{a} \doteq \ell/\rho_-, \quad \hat{b} \doteq -\frac{\phi'(\rho_-)\rho_-}{\phi(\rho_-)} = 1 - \frac{f'(\rho_-)\rho_-}{f(\rho_-)},
$$

and seeking solutions of the form

$$
\zeta(x) = \hat{M} e^{\lambda x},
$$

we arrive at the characteristic equation

$$
H(\lambda) \doteq \hat{b}(e^{\hat{a}\lambda} - 1) - \hat{a}\lambda = 0.
$$

To seek positive zeros of $H(\cdot)$, we first observe that $H(0) = 0$. Furthermore, we have

$$
H'(\lambda) = \hat{a}\hat{b}e^{\hat{a}\lambda} - \hat{a}, \quad H'(0) = \hat{a}(\hat{b} - 1),
$$

$$
H''(\lambda) = \hat{a}^2\hat{b}e^{\hat{a}\lambda} > 0, \quad H'''(\lambda) = \hat{a}^3\hat{b}e^{\hat{a}\lambda} > 0.
$$

Thus, positive rate $\lambda = \lambda_-$ exists only for the case when $\hat{b} < 1$, i.e., when $\rho_- < \rho^*$.

A similar computation as for (2.7) leads to an estimate with both upper and lower bounds:

$$
-\frac{1}{\hat{a}} \ln(\hat{b}) < \lambda_- < -\frac{2}{\hat{a}} \ln(\hat{b}),
$$

proving (2.8) \hfill \square

**Remark 2.2.** Lemma 2.2 indicates that the boundary conditions at $x \to \pm \infty$ are valid only when $\rho_- \leq \rho^* \leq \rho_+$. Together with Theorem 2.1, we conclude that stationary profiles of $W(x)$, if they exist, are monotonically increasing. This corresponds to the upward jumps of the admissible shocks for the conservation law (1.9).

Consequently, the constant $M$ in (2.12) is now negative, and $\hat{M}$ in (2.19) is positive.

**Remark 2.3.** If $\rho_- = \rho_+ = \rho^*$, then $\lambda = 0$, and one has the trivial solution $W(x) \equiv \rho^*$.

**Remark 2.4.** The estimates (2.7)-(2.8) can be expressed in different ways. Indeed, (2.7) implies

$$
\lambda_+ > \frac{2\rho_+}{\ell} \cdot \ln \left( 1 + \frac{c_0\rho^*}{f(\rho^*)}(\rho^* - \rho_-) \right),
$$

where $-c_0$ is an upper bound for $f''$, see (1.10). Similarly, if $\rho_- > 0$ is close to $\rho^*$, the estimate (2.8) implies

$$
-\frac{\rho_-}{\ell} \cdot \ln \left( 1 - \frac{c_0\rho^*}{f(\rho^*)}(\rho^* - \rho_-) \right) < \lambda_- < -\frac{2\rho_-}{\ell} \cdot \ln \left( 1 - \frac{c_0\rho^*}{f(\rho^*)}(\rho^* - \rho_-) \right),
$$

On the other hand, if $\rho_- \approx 0$, we have a different estimate

$$
-\frac{\rho_-}{\ell} \cdot \ln (\hat{c}_0\rho_-) < \lambda_- < -\frac{2\rho_-}{\ell} \cdot \ln (\hat{c}_0\rho_-),
$$

where $-\hat{c}_0$ is the upper bound for $\phi'$, see (1.4).
Remark 2.5. By the estimate (2.21), as $\rho_+ \to 1$ we have $\lambda_+ \to +\infty$. This indicates that if $\rho_+ = 1$, $W(x)$ approaches $\rho_+$ instantly. Thus, for some $\hat{x}$, we must have $W(x) = 1$ for $x \geq \hat{x}$.

On the other end, by estimate (2.23), as $\rho_- \to 0$, we have $\lambda_- \to \infty$. Thus, if $\rho_- = 0$, we must have $W(x) = 0$ for $x < \hat{x}$, for some $\hat{x}$.

One concludes that, if $\rho_- = 0$, $\rho_+ = 1$, the only possible profile is a step function with the jump located at some $\hat{x}$. This represents the scenario where cars are bumper-to-bumper on $x > \hat{x}$, and the road is empty for $x < \hat{x}$.

The next lemma is most interesting. It shows that, if $W(x)$ is a stationary monotone profile such that the solutions $\{z_i(t)\}$ of (1.3) traces along, then the distribution $\{z_i(t)\}$ demonstrates a “periodic” pattern.

Lemma 2.3. (Periodicity.) Let $W(x)$ be a monotone profile, and let $\{z_i(0)\}$ be the initial positions of cars generated by the profile $W(x)$. Let $\{z_i(t)\}$ be the solution of the FtL model (1.3) with this initial data. Then, the followings are equivalent.

(E1) $W(x)$ satisfies the DDE (1.13).

(E2) The solutions $\{z_i(t)\}$ exhibit the following periodic behavior. There exist a “period” $t_p$, such that after the period each car takes over the initial position of its leader, i.e.,

\[ z_i(t + t_p) = z_{i+1}(t), \quad \forall i, \quad \forall t \geq 0. \tag{2.24} \]

Proof. Step 1. We first prove that (E2) $\Rightarrow$ (E1). Without loss of generality, we consider a car initially located at $z_0(0) = x$ for some $x$, and its leader, initially located at

\[ z_1(0) = x + \frac{\ell}{W(x)}. \]

Thus, the evolution of $z_0(t)$ satisfies the ODE

\[ \frac{dz_0}{dt} = V \phi(W(z_0)), \quad z_0(0) = x, \quad z_0(t_p) = z_1(t) = x + \ell/W(x). \]

This is a separable equation. (E2) implies the following identity

\[ \int_{x}^{x+\ell/W(x)} \frac{1}{V \phi(W(z))} dz = \int_{0}^{t_p} dt = t_p = \text{constant}, \quad \forall x. \tag{2.25} \]

Differentiating (2.25) in $x$, we deduce

\[ \left( 1 - \frac{\ell W'(x)}{W(x)^2} \right) \cdot \frac{1}{\phi(W(x + \ell/W(x)))} - \frac{1}{\phi(W(x))} = 0, \tag{2.26} \]

which easily leads to (1.13), proving (E1).

Step 2. To prove the indication (E1) $\Rightarrow$ (E2), assume that $W(x)$ satisfies the DDE (1.13). For a given time $t$, let $\{z_i(t)\}$ be a distribution of cars generated by $W(x)$, as in Definition 2.1. We write now

\[ x = z_i(t), \quad z_{i+1}(t) = x + \frac{\ell}{W(x)}. \]
Since $W(x)$ solves (1.13), it satisfies (2.26). The time it takes for the $i$th car to reach the original position of its leader $z_{i+1}$ is

$$t_p(x) = \int_x^{x+\ell/W(x)} \frac{1}{V \phi(W(z))} \, dz.$$  \hspace{1cm} (2.27)

By (2.26) we immediately deduce that $t'_p(x) = 0$ for all $x$, thus $t_p(x) \equiv \text{constant.}$

The next Lemma connects the period $t_p$ to the limit values of $W(x)$ at $x \to \pm \infty$.

**Lemma 2.4.** Let $W(x)$ and $\{z_i(t)\}$ be given as in the setting of Lemma 2.3. Let $\rho_-, \rho_+$ be two states that satisfy

$$f(\rho_-) = f(\rho_+) = \bar f, \quad \rho_- \leq \rho^* \leq \rho_+.$$ \hspace{1cm} (2.28)

Then, the following additional properties are equivalent.

(E3) $W(x)$ satisfies the boundary conditions

$$\lim_{x \to \pm \infty} W(x) = \rho_\pm.$$ \hspace{1cm} (2.29)

(E4) The period $t_p$ for $\{z_i(t)\}$ is given as

$$t_p = \frac{\ell}{f} = \frac{\ell}{f(\rho_-)} = \frac{\ell}{f(\rho_+)}.$$ \hspace{1cm} (2.30)

**Proof.** **Step 1.** We first prove that (E3) $\Rightarrow$ (E4). Assume (E3) holds, such that $W(x)$ is a monotone profile satisfies (2.29). Let $\epsilon > 0$, and consider the limit as $x \to +\infty$. There exists an $M$ such that for all $x \geq M$ we have

$$\rho_+ - \epsilon < W(x) \leq \rho_+.$$  

Since $\phi'(\cdot) < 0$, we have, for all $x \geq M$,

$$0 < \frac{1}{\phi(\rho_+ - \epsilon)} < \frac{1}{\phi(W(x))} < \frac{1}{\phi(\rho_+)}.$$  

Integrating this inequality over $[x, x + \ell/W(x)]$, one has, for all $x \geq M$,

$$\int_x^{x+\ell/W(x)} \frac{1}{\phi(\rho_+ - \epsilon)} \, dz < \int_x^{x+\ell/W(x)} \frac{1}{\phi(W(z))} \, dz \leq \int_x^{x+\ell/W(x)} \frac{1}{\phi(\rho_+)} \, dz.$$  

This gives

$$\frac{\ell}{\rho_+} \cdot \frac{1}{\phi(\rho_+ - \epsilon)} < V \cdot t_p \leq \frac{\ell}{\rho_+ - \epsilon} \cdot \frac{1}{\phi(\rho_+)}.$$  

Taking the limit $\epsilon \to 0$, we get

$$t_p = \frac{\ell}{V \cdot \rho_+ \phi(\rho_+)} = \frac{\ell}{f(\rho_+)}.$$

The other limit $x \to -\infty$ can be treated in a completely similar way, proving (E4).

**Step 2.** The implication (E4) $\Rightarrow$ (E3) follows by contradiction. Assuming that (E4) holds, but

$$\lim_{x \to \infty} W(x) = \hat \rho_\pm, \quad \hat \rho_- < \rho^* < \hat \rho_+, \quad f(\hat \rho_-) = \hat f \neq \bar f.$$  

By the proof in Step 1 we have the contradiction $t_p = \ell/\hat f \neq \ell/\bar f$. 

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3 Approximate Sequence; Existence and Uniqueness of Traveling Wave Profiles

We now construct approximate solutions to $W(x)$ as a two-point-boundary-value problem, and prove their convergence, thus establishing the existence of traveling wave profiles.

**Theorem 3.1. (Existence.)** Given $\ell, V, \rho_- , \rho_+$, with
\[
0 \leq \rho_- \leq \rho^* \leq \rho_+ \leq 1, \quad f(\rho_-) = f(\rho_+),
\]
there exists a monotone stationary profile $W(x)$ which satisfies the DDE (1.13) and the “boundary” values
\[
\lim_{x \to -\infty} W(x) = \rho_-, \quad \lim_{x \to +\infty} W(x) = \rho_+.
\]

**Proof.** By Remarks 2.3-2.5, we rule out the trivial cases. For the rest of the proof, we consider
\[
0 < \rho_- < \rho^* < \rho_+ < 1. \tag{3.3}
\]
The proof takes a few steps.

(1). We first construct the sequence of approximate solutions. Let a sequence $\hat{x}_n$ be given such that
\[
\hat{x}_n > 0, \quad \hat{x}_n < \hat{x}_{n+1}, \quad \lim_{n \to \infty} \hat{x}_n = \infty.
\]
We define the function
\[
\psi(x) = \rho_+ - e^{-\lambda_+ x},
\]
where $\lambda_+$ is the rate computed in Lemma 2.2. Here we set the constant $M = -1$, since different values of it would only lead to a horizontal shift of the profile. Given $n$, let $\psi(x)$ be the boundary condition for the DDE (1.13) on $x \in [\hat{x}_n, \infty)$, and denote the corresponding solution as $W_n(x)$, for $x \leq \hat{x}_n$.

From Theorem 2.1, $W_n(x)$ is monotonically increasing and bounded below as $x \to -\infty$. Denoting that
\[
\rho_{-,n} = \lim_{x \to -\infty} W_n(x),
\]
\[
\lim_{n \to \infty} \rho_{-,n} = \rho_-.
\]

(2). We derive an estimate for $\rho_{-,n}$. Given an $n$ such that $\hat{x}_n$ is sufficiently large, so
\[
\psi(x) = \rho_+ - e^{-\lambda_+ x} \approx \rho_+, \quad \text{for} \quad x > \hat{x}_n.
\]
Consider the solution $W_n(x)$, given on $x \leq \hat{x}_n$. Let $\{z_i^n\}$ be a distribution of car positions generated by $W_n(x)$, with $z_i^n = \hat{x}_n$, for $i = 0, -1, -2, -3, \ldots$. Let $\{z_i^n(t)\}$ be the solution of the system of ODEs (1.3), for index $i < 0$, and the leader $z_0^n(t)$ traces along the initial condition $\psi(x)$ on $x > \hat{x}_n$.

By Lemma 2.3, $\{z_i^n(t)\}$ ($i < 0$) demonstrates periodic behavior. We denote the period by $t_{p,n}$. Note that once $t_{p,n}$ is given, we obtain the unique value of $\rho_{-,n}$ by the relation
\[
\rho_{-,n} \approx \rho^*, \quad f(\rho_{-,n}) = \frac{\ell}{t_{p,n}}.
\]
Thus, it suffices to show that
\[
\lim_{n \to \infty} t_{p,n} = t_p = \frac{\ell}{f(\rho_\pm)}.
\] (3.6)

We further observe that, thanks to the periodic behavior of \(\{z_i^n(t)\}, (i < 0)\), we only need to get an estimate of the time for car located at \(z_{-1}^n\), to reach \(z_0^n = \hat{x}_n\). Here \(z_{-1}^n\) is uniquely defined by the implicit relation
\[
z_{-1}^n + \frac{\ell}{W(z_{-1}^n)} = z_0^n.
\]

(3). By the set up, \(W_n(x)\) is very close to \(\rho_+\) on the interval \([z_{-1}^n, z_0^n]\), therefore an estimate on \(t_{p,n}\) can be obtained by linearization. By Lemma 2.2, we have the first order approximation of \(W_n(x)\) on \([z_{-1}^n, z_0^n]\), denoted as
\[
W_n(x) \approx \rho_+ - e^{-\lambda_+x}, \quad x \in [z_{-1}^n, z_0^n].
\]
To simplify the notation, we denote the magnitude of the perturbation by
\[
\delta_n \triangleq e^{-\lambda_+z_0^n} = e^{-\lambda_+\hat{x}_n}.
\] (3.7)
The first order approximation for \(W_n(x)\) can be written as
\[
W_n(x) = \rho_+ - \delta_ne^{-\lambda_+(x-z_0^n)}.
\]
The corresponding distance \(z_0^n - z_{-1}^n\) is computed approximately as
\[
z_0^n - z_{-1}^n = \frac{\ell}{\rho_+ - \delta_ne^{\lambda_+(z_0^n-z_{-1}^n)}} = \frac{\ell}{\rho_+} + \delta_n \cdot \frac{\ell}{\rho_+^{\ell/\rho_+}} e^{\lambda_+\ell/\rho_+} + O(\delta_n^2),
\] (3.8)
We can compute \(t_{p,n}\), using a first order approximation in \(\delta_n\), as
\[
t_{p,n} = \frac{1}{V} \int_{z_{-1}^n}^{z_0^n} \frac{1}{\phi(W(z))} dz = \frac{1}{V} \int_{z_{-1}^n}^{z_0^n} \frac{1}{\phi(\rho_+ - \delta_ne^{-\lambda_+(z-z_0^n)})} dz
\]
\[
= \frac{1}{V} \int_{z_{-1}^n}^{z_0^n} \frac{1}{\phi(\rho_+) - \delta_ne^{-\lambda_+(z-z_0^n)}\phi'(\rho_+)} dz
\]
\[
= \frac{1}{V \cdot \phi(\rho_+)} \int_{z_{-1}^n}^{z_0^n} \left[ 1 + \delta_n e^{-\lambda_+(z-z_0^n)} \frac{\phi'(\rho_+)}{\phi(\rho_+)} \right] dz
\]
\[
= \frac{1}{V \cdot \phi(\rho_+)} \left[ (z_0^n - z_{-1}^n) + \delta_n \cdot \frac{\phi'(\rho_+)}{\lambda_+ \phi(\rho_+)} \left[ e^{\lambda_+(z_0^n-z_{-1}^n)} - 1 \right] \right].
\]
Using (3.8), we get
\[
t_{p,n} = \frac{1}{V \cdot \phi(\rho_+)} \left[ \frac{\ell}{\rho_+} + \delta_n \cdot \left\{ \frac{\ell}{\rho_+^{\ell/\rho_+}} e^{\lambda_+\ell/\rho_+} + \frac{\phi'(\rho_+)}{\lambda_+ \phi(\rho_+)} \left[ e^{\lambda_+\ell/\rho_+} - 1 \right] \right\} \right]
\]
\[
= t_p + M \delta_n,
\] (3.9)
where \(M\) is a positive constant, depending on the data \(\rho_+, \phi, V, \ell\), but not on \(\delta_n\). As \(n \to \infty\), \(\delta_n \to 0\), and we conclude (3.6), completing the proof.
Figure 2: Numerical simulations for the approximate sequence $W_n(x)$ for various values of $\hat{x}_n$. We use $\rho_- = 0.3, \rho_+ = 0.7, \ell = 0.5, V = 1, \phi(\rho) = 1 - \rho$. The solid curve is the graph of $\psi(x) = \rho_+ - 0.2e^{\lambda_+ x}$, plotted on the interval $0 \leq x \leq 2$. The dotted curves are plots for $W_n(x)$ with $\hat{x}_n = 0, 0.1, 0.25, 0.5, 1$.

In Figure 2 we present some numerical simulations of the approximate solutions $W_n(x)$. We observe the convergence as $\hat{x}_n \to \infty$. Furthermore, from (3.9) it holds that $t_{p,n} > t_p$, indicating $\rho_{-n} < \rho_-$, which is consistent with the simulation results.

Once the existence of the profile $W(x)$ is proved, we establish the uniqueness of the solution for the “two-point-boundary-value-problem” for the DDE [1.13].

**Theorem 3.2. (Uniqueness.)** Consider the settings of Theorem 3.1. The solution $W(x)$ is unique up to a horizontal shift.

*Proof.* We first consider the trivial cases. If $\rho_- = \rho_+ = \rho^*$, the only monotone graph is $W(x) \equiv \rho^*$. If $\rho_- = 0$ and $\rho_+ = 1$, then $t_p \to \infty$ and nothing moves, so the flux must be 0 everywhere. The only monotone solution is a unit step function. In the rest of the proof, we assume

$$0 < \rho_- < \rho^* < \rho_+ < 1.$$  

We prove by contradiction. Consider the settings of Theorem 3.1, and let $W(x), \bar{W}(x)$ be two solutions which are different. Assume that, after some horizontal shift, the graphs of $W(x)$ and $\bar{W}(x)$ intersect at a point $\hat{x}$ so that

$$W(\hat{x}) = \bar{W}(\hat{x})$$

and

$$W(x) > \bar{W}(x), \quad \text{for} \quad \hat{x} < x < \hat{x} + \ell/W(\hat{x}).$$

Then, by the periodical property, we have

$$t_p = \int_{\hat{x}}^{\hat{x} + \ell/W(\hat{x})} \frac{1}{V\phi(W(z))} dz > \int_{\hat{x}}^{\hat{x} + \ell/W(\hat{x})} \frac{1}{V\phi(W(z))} dz = t_p,$$

a contradiction.
This means that, if the graphs of \( W \) and \( \bar{W} \) cross each other at \( \bar{x} \), then they must cross each other at least one more time on \((\bar{x}, \bar{x} + \ell/W(\bar{x}))\). Thus, they must cross each other infinitely many times for \( x \in \mathbb{R} \).

We now freeze the graph of \( W(x) \), and shift the graph of \( \bar{W}(x) \) to the right until they touch each other only at \( \bar{x} \) tangentially, such that

\[
W(\bar{x}) = \bar{W}(\bar{x}), \quad W(x) > \bar{W}(x) \quad \text{for} \quad \bar{x} < x < \bar{x} + \ell/W(\bar{x}).
\]

Again, by periodicity, we get

\[
t_p = \int_{\bar{x}}^{\bar{x} + \ell/W(\bar{x})} \frac{1}{V\phi(W(z))} \, dz > \int_{\bar{x}}^{\bar{x} + \ell/W(\bar{x})} \frac{1}{V\phi(W(z))} \, dz = t_p,
\]

a contradiction.

We conclude that the graphs of \( W \) and \( \bar{W} \) either completely coincide or never cross each other. Then they must be horizontal shifts of each other, proving the uniqueness.

**Numerical simulations.** Various profiles of \( W(x) \) are plotted in Figure 3 for various values of \( \rho_{\pm} \). Here we use \( \ell = 0.1 \) and \( \phi(\rho) = 1 - \rho \), such that \( \rho^* = 0.5 \). We plot the graphs of \( W(x) \) that connect the following pairs of limit values of \( (\rho_- , \rho_+) \)

\[
(0.4, 0.6), \quad (0.3, 0.7), \quad (0.2, 0.8), \quad (0.1, 0.9).
\]

The profiles are simulated numerically, obtained as the limits of the approximate sequences described in Theorem 3.1. The profiles are further shifted horizontally such that \( W(0) = 0.5 \).

![Figure 3: Numerical simulations of stationary profiles \( W(x) \) with various values of \( (\rho_-, \rho_+) \).](image)

We make a couple of observations.

1. For smaller values of \( (\rho_+ - \rho_-) \), the profiles \( W(x) \) has a smaller value of \( W'(0) \). We provide a formal argument. For small \( \ell \), \( W(x) \) can be approximated by a linear function

\[
W(x) = W(0) + \sigma x, \quad \sigma \doteq W'(0),
\]
for $x$ close to 0. Writing out only the first order approximations, we have

$$z_0 = W(0) = \rho^* = 0.5, \quad z_1 = \frac{\ell}{W(0)} = \frac{\ell}{\rho^*} = 2\ell, \quad W(z_1) = \rho^* + \sigma \frac{\ell}{\rho^*} = 0.5 + 2\ell\sigma,$$

and

$$f^* = f(\rho^*) = \frac{V}{4}, \quad \bar{f} = f(\rho_\pm) = V\rho_\pm(1 - \rho_\pm).$$

The periodic property gives

$$t_p = \frac{\ell}{f} = \frac{1}{V} \int_0^{z_1} \frac{1}{1 - W(x)} dx = \frac{1}{V} \int_{\rho^*}^{\rho^* + \sigma \frac{\ell}{\rho^*}} \frac{dW}{\sigma(1 - W)}.$$

Working out the integration, we get

$$\frac{V\ell}{f} = -\frac{1}{\sigma} \ln \frac{1 - \rho^* - \sigma \frac{\ell}{\rho^*}}{1 - \rho^*}.$$ Multiplying both sides by $-\sigma$ and then taking the exponential function on both sides, we obtain

$$e^{-V\ell\sigma/f} = \frac{1 - \rho^* - \sigma \frac{\ell}{\rho^*}}{1 - \rho^*} = 1 - \frac{\ell\sigma}{1 - \rho^*} = 1 - \frac{V\ell\sigma}{f^*}.$$ Moving everything to the right hand side, it gives the equation

$$K(\sigma) \doteq 1 - \frac{V\ell\sigma}{f^*} - e^{-V\ell\sigma/\bar{f}} = 0.$$ Recall that $f^* = \frac{V}{4}$ is a constant. The function $K(\sigma)$ has the following properties:

$$K(0) = 0, \quad K'(0) = V\ell(1/\bar{f} - 1/f^*) \geq 0, \quad K''(\sigma) = -(V\ell/\bar{f})^2 e^{-V\ell\sigma/\bar{f}} < 0.$$ If $\bar{f} = f^*$, the only zero for $K(\sigma)$ is $\sigma = 0$. When $\bar{f} < f^*$, then $K'(0) > 0$, and there exists another positive zero $\sigma_+$ for $K(\sigma)$. One can easily verify that $W'(0) = \sigma_+$ decreases as $\bar{f}$ increases to the value $f^*$, in correspondence to our simulation result.

(2). The profile $W(x)$ is not symmetric about $x = 0$ in the sense that the asymptotic limit $\rho_+$ is approached much faster than the limit $\rho_-$. Recall Lemma 2.2. For this simulation, $\rho_-$ and $\rho_+$ locate symmetrically around $\rho^* = 0.5$ such that $\rho_- + \rho_+ = 1$. The estimates (2.7) and (2.8) give

$$\lambda_+ > \frac{2\rho_+}{\ell} \ln(\rho_+/\rho_-), \quad \lambda_- < -\frac{2\rho_-}{\ell} \ln(\rho_-/\rho_+).$$ Since $\rho_- < \rho_+$, we have $\lambda_- < \lambda_+$.

We remark that this is different from a viscous shock profile, where the diffusion is uniform and the profile is odd symmetric about the location of the shock.

4 Stability of the discrete traveling waves

We now show that the traveling wave profiles are local attractors for the solution of the FtL model.
**Theorem 4.1. (Local Stability.)** Let $W(x)$ be the unique stationary profile established in Theorem 3.1 with
\[ W(0) = \rho^*, \quad \text{where} \quad f'(\rho^*) = 0. \]
Let \( \{z_i(t), \rho_i(t)\} \) be the solution of the ODEs (1.3) with the initial data \( \{z_i(0), \rho_i(0)\} \). Assume that there exist values $h_+, h_-$ with $h_+ < h_-$, such that the initial data satisfies
\[ W(z_i(0) - h_+) \geq \rho_i(0) \geq W(z_i(0) - h_-), \quad \forall i \in \mathbb{Z}. \tag{4.1} \]
Then, there exists a constant \( \bar{h} \) such that
\[ \lim_{t \to \infty} W(z_i(t) - \bar{h}) - \rho_i(t) = 0, \quad \forall i \in \mathbb{Z}. \tag{4.2} \]

Theorem 4.1 implies that as $t \to \infty$, \( \{z_i(t), \rho_i(t)\} \) approaches asymptotically a distribution generated by the profile $W(x - \bar{h})$.

**Proof.** Since $W(x)$ is monotonically increasing, with $\lim_{x \to \pm \infty} W(x) = \rho_\pm$, then for any point $(z, \rho)$ with $\rho_- < \rho < \rho_+$, there exists a unique value $h$ such that $\rho = W(z - h)$. We define the function
\[ H(z, \rho) = h, \quad \text{where} \quad \rho = W(z - h) \quad \text{and} \quad \rho_- < \rho < \rho_. \]
Then, for $t \geq 0$ and for each $i$, let
\[ h_i(t) = H(z_i(t), \rho_i(t)), \quad \text{where} \quad \rho_i(t) = W(z_i(t) - h_i(t)). \tag{4.3} \]
Denote
\[ y_i(t) = z_i(t) - h_i(t), \quad \text{so} \quad \rho_i(t) = W(y_i(t)). \]
Differentiating in $t$, we get
\[ \dot{\rho}_i = W'(y_i) \dot{y}_i = W'(y_i)(\dot{z}_i - \dot{h}_i). \]
Using that
\[
\dot{z}_i = V\phi(\rho_i), \\
\dot{\rho}_i = \frac{V}{\ell} \rho_i^2 (\phi(\rho_i) - \phi(\rho_{i+1})), \\
W'(y_i) = \frac{\rho_i^2}{\ell \phi(\rho_i)} [\phi(\rho_i) - \phi(W(y_i + \ell/\rho_i))],
\]
we get
\[
\dot{h}_i = \dot{z}_i - \frac{\dot{\rho}_i}{W'(y_i)} = V\phi(\rho_i) - V\phi(\rho_i) \frac{\phi(\rho_i) - \phi(\rho_{i+1})}{\phi(\rho_i) - \phi(W(y_i + \ell/\rho_i))} = \frac{V\phi(\rho_i)}{[\phi(W(y_i)) - \phi(W(y_i + \ell/\rho_i))]} \cdot [\phi(\rho_{i+1}) - \phi(W(y_i + \ell/\rho_i))].
\]
Then, if $h_i(t) < h_{i+1}(t)$, we have
\[ W(y_i + \ell/\rho_i) > \rho_{i+1}. \]
Since $\phi$ is a monotonically decreasing function, we have
\[ \phi(W(y_i + \ell/\rho_i)) < \phi(\rho_{i+1}) \quad \text{and} \quad \phi(W(y_i)) > \phi(W(y_i + \ell/\rho_i)), \]

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which implies that $\dot{h}_i(t) > 0$. Similarly, if $h_i(t) > h_{i+1}(t)$ then we have $\dot{h}_i(t) < 0$.

Now we define
\[
h^{\sharp}(t) \doteq \min_i h_i(t), \quad h^{\flat}(t) \doteq \max_i h_i(t). \tag{4.4}
\]

It suffices to show that
\[
\lim_{t \to \infty} \left[ h^{\flat}(t) - h^{\sharp}(t) \right] = 0. \tag{4.5}
\]

Indeed, for any given $t \geq 0$, from the previous discussion we have the following.

- Let $h_j(t)$ be a maximum. If $h_j(t) > h_{j+1}(t)$, then $\dot{h}_j < 0$; if $h_j(t) = h_{j+1}(t)$, then $\dot{h}_j = 0$ and $h_{j+1}(t)$ is also a maximum;
- Let $h_k(t)$ be a minimum. If $h_k(t) < h_{k+1}(t)$, then $\dot{h}_k > 0$; if $h_k(t) = h_{k+1}(t)$, then $\dot{h}_k = 0$ and $h_{k+1}(t)$ is also a minimum.

Then,
\[
\frac{d}{dt} \left[ h^{\flat}(t) - h^{\sharp}(t) \right] \leq 0,
\]
and the limit
\[
\lim_{t \to \infty} \left[ h^{\flat}(t) - h^{\sharp}(t) \right]
\]
exists and is non-negative. To show that the limit must be 0, we use contradiction and assume the opposite, such that
\[
\lim_{t \to \infty} \left[ h^{\flat}(t) - h^{\sharp}(t) \right] = d_h > 0,
\]
and let $\{\tilde{z}_i, \tilde{\rho}_i\}$ be the asymptotic car distribution, with the corresponding values of $\{\tilde{h}_i\}$. Now, take $\{\tilde{z}_i, \tilde{\rho}_i\}$ as the initial data and solve the system of ODEs (4.3). There exists an index $j$ where $\tilde{h}_j$ is the maximum with $\tilde{h}_j > \tilde{h}_{j+1}$. By the previous discussion we have $\frac{d}{dt} \tilde{h}_j < 0$. If this is the isolated maximum, then $\frac{d}{dt} h_j^\flat < 0$, a contradiction. If $\tilde{h}_{j-1}$ is also a maximum, then after an arbitrarily small amount of time we have $\frac{d}{dt} \tilde{h}_{j-1} < 0$ so $\frac{d}{dt} h^\flat < 0$, still a contradiction. Thus, we conclude (4.5), completing the proof.

5 Extension to general traveling waves

One can extend the analysis to traveling waves with speed different from 0, by a simple coordinate shift. Let $V\sigma$ be the wave speed and let $\xi = x - V\sigma t$ be the shifted space coordinate. Let $\zeta_i(t)$ be the position of the $i$th car in the shifted coordinate, and $\rho_i$ the discrete density. We have
\[
\dot{\zeta}_i = \dot{z} - V\sigma = V\phi(\rho_i) - V\sigma = V(\phi(\rho_i) - \sigma).
\]

Since the density is not affected by a horizontal shift, the ODE for $\rho_i$ is unchanged.

Consider a traveling wave profile $W(\xi) = W(x - V\sigma t)$. We must have
\[
W(\zeta_i(t)) = \rho_i(t), \quad \forall t > 0.
\]
This leads to the DDE:
\[
W'(\xi) = \frac{W^2(\xi)}{\ell(\phi(W(\xi)) - \sigma)} \left[ \phi(W(\xi)) - \phi(W(\xi + \ell/W(\xi))) \right].
\]
The corresponding conservation law is
\[ \rho_t + f(V, \rho)\xi = 0, \quad \text{where} \quad f(V, \rho) = V\rho(1 - \rho - \sigma). \]

The analysis for the stationary traveling wave can be applied here with minimal modifications.

6 Concluding Remark

In this paper we study traveling wave profiles of a particle model for traffic flow, i.e., the follow-the-leader (FtL) ODE models for car positions. Given any densities \( \rho_\pm \) at \( x \to \pm\infty \), with \( \rho_- < \rho_+ \), we prove that there exists a unique traveling wave profile for the FtL model. Furthermore, such profiles are locally stable which attract nearby solutions of the FtL model. In the limit as \( \ell \to 0 \), the traveling waves converge to admissible shocks for the solution of the conservation law (1.9). Our results fill a gap in existing theory on traveling waves. The admissible conditions derived from our result are in accordance to the counter part for the viscous equation
\[ \rho_t + f(\rho)_x = \varepsilon \rho_{xx}, \]
where stable viscous shocks only exist for upward jumps.

It’s interesting and also non-trivial to study the same particle model on a road with rough conditions. For example, let \( \kappa(x) \) denote the speed limit (which reflects the road condition), and assume that it is a piecewise constant function with a jump at \( x = 0 \). One would like to seek stationary traveling waves for the FtL model, around \( x = 0 \). The corresponding macroscopic model
\[ \rho_t + f(\rho, \kappa(x))_x = 0 \]
is a scalar conservation law with discontinuous flux. In existing literature, admissibility conditions on the jump at \( x = 0 \) are derived through the viscous model
\[ \rho_t + f(\rho, \kappa(x))_x = \varepsilon \rho_{xx} \]
and take the vanishing viscosity limit \( \varepsilon \to 0^+ \). However, our preliminary analysis shows a rather different scenario for limits of the FtL model, as \( \ell \to 0 \), where many of the vanishing viscosity limits are actually not admissible. Details are in a forthcoming work [22].

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