Solving the octic by iteration in six dimensions

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Abstract. The requirement for solving a polynomial is a means of breaking its symmetry, which in the case of the octic is that of the symmetric group $S_8$. Its eight-dimensional linear permutation representation restricts to a six-dimensional projective action. A mapping of complex projective 6-space with this $S_8$ symmetry can provide the requisite symmetry-breaking tool. This paper describes some of the $S_8$ geometry in $\mathbb{CP}^6$ as well as a special $S_8$-symmetric rational map in degree four. Several basins-of-attraction plots illustrate the map’s geometric and dynamical properties. The work culminates with an explicit algorithm that uses this map to solve a general octic. A concluding discussion treats the generality of this approach to equations in higher degree.

Received 13 August 2001; revised version accepted 15 January 2002

1. Overview

In Crass (2001), I develop a solution to the quintic that relies on a single iteration in three dimensions. Given almost any quintic $p$, there is a map on complex projective 3-space $\mathbb{CP}^3$ whose dynamics provides for a root of $p$. This approach is geometric: the map has the $S_5$ symmetry of the general quintic.

The present paper extends this work to the eighth-degree equation. At its core is an $S_8$-symmetric map on $\mathbb{CP}^6$ whose geometric behaviour is connected to a special configuration of lines.

Motivating this general project is a desire to develop solutions to equations that utilize symmetrical and geometrically elegant dynamical systems. By contrast, Newton’s method in one variable typically fails to possess the symmetries of an associated polynomial. Furthermore, its dynamical behaviour is usually not reliable in the sense defined below. This geometric emphasis also distinguishes my approach from numerical methods. I do not consider the numerical aspects of the algorithm. However, since the map converges very rapidly, numerical considerations might well be of interest. Indeed, the geometry associated with the map accounts for the rapid convergence.

In addition, the work establishes, for all degrees greater than four, the existence of a method analogous to the eighth-degree case. This involves showing that there is an infinite family of maps—one for each dimension greater than two—with special
geometric properties. Moreover, in each dimension, this geometry conforms to a pattern.

Finally, these maps add to the examples of complex dynamics in several dimensions. This recently active and difficult field seems to be in need of examples that are not concocted for purposes of illustration. While the paper treats some aspects of the dynamics of the octic-solving six-dimensional map, several challenging problems also arise. Can the symmetry and special geometry of these maps provide a means for investigation of global dynamics? For instance, can one profitably study their dynamics on the quotient space? Also, do these maps possess geometric and dynamical properties that suggest a more general theoretical treatment?

The work unfolds in four stages: (1) some background geometry, (2) a special map with $S_8$ symmetry, (3) a solution to the octic based on the preceding stages, (4) a consideration of whether the octic algorithm generalizes to higher degree equations.

Section 2: $S_8$ geometry. The setting here is $\mathbb{CP}^6$ upon which the symmetric group $S_8$ acts. Finding a map with special $S_8$ geometry requires some familiarity with this action. We shall consider some features associated with the map that emerges in the second stage. Indeed, the discovery of this map derives from an awareness of the algebraic and geometric surroundings:

- coordinate systems on $\mathbb{CP}^6$ that are sensitive to the $S_8$ action;
- the structure of certain special orbits of points, lines, planes, and hyperplanes;
- the system of $S_8$-invariant polynomials—the building blocks for maps that are $S_8$-symmetric.

Section 3: maps with $S_8$ symmetry. At this stage, we exploit our geometric understanding to discover empirically a map associated with the complete graph on eight vertices—an 8-point $S_8$-orbit. The discussion turns to its geometric and dynamical behaviour—empirical testing suggests that the 8-point orbit is the only attractor. However, whether it possesses this or another desired global dynamical property is not known. In light of substantial experimental and graphical evidence, I attribute these properties to the map in conjectures.

Section 4: dynamical solution to the octic. A special family of octics corresponds to a ‘rigid’ family $\mathcal{E}$ of $S_8$-symmetric maps on $\mathbb{CP}^6$. ‘Rigidity’ means that each member of $\mathcal{E}$ is conjugate to a single reference map $f$. Thus, associated with an octic $p$ is a map $g_p = \phi_p f \phi_p^{-1}$ that we iterate. Using $S_8$ tools, the dynamical output—conjecturally, a single $S_8$-orbit—provides for an approximate solution to $p = 0$. Since almost any octic $p$ transforms into the special family, the solution is general.

Note: up to this point, the exposition follows that of Crass (2001) which the reader can consult for details.

Section 5: generalization: solving the $n$th-degree equation by iteration in $n-2$ dimensions. The geometric and dynamical description of the octic-solving map has an analogue for each permutation-based $S_n$ action on $\mathbb{CP}^{n-2}$ with $n \geq 5$. Here, we can show that there is always a map with special properties related to the complete graph on $n$ vertices. Given such a map for which the $n$-point orbit is...
the attractor, the solution algorithm for the octic generalizes to one for the $n$th-degree equation. Indeed, solving the general case requires only straightforward substitutions that take account of the presence of more variables. It should be possible to develop a single procedure that is indexed by the degree of the polynomial.

2. $S_8$ acts on $\mathbb{CP}^6$

The search for a special $S_8$-symmetric map begins with a faithful $S_8$ action. Klein’s approach to the $n$th-degree equation was to look for the lowest dimensional faithful representation of $S_n$ or the alternating group $A_n$. For $n < 8$, there are special actions of either $S_n$ or $A_n$; that is, there are faithful representations that do not derive directly from permutations on $\mathbb{C}^n$. However, special geometry—at least for linear actions—ends at $n = 7$. When $n > 7$, the space of least dimension on which $S_n$ or $A_n$ acts faithfully is $\mathbb{CP}^{n-2}$ (Wiman 1899).

The permutation action of the symmetric group $S_8$ on $\mathbb{C}^8$ preserves the hyperplane

$$\mathcal{H}_x = \left\{ \sum_{k=1}^{8} x_k = 0 \right\} \simeq \mathbb{C}^7$$

and, thereby, restricts to a faithful seven-dimensional irreducible representation. This $\mathbb{C}^7$ action projects one-to-one to a group $\mathcal{G}_8!$ on $\mathbb{CP}^6$.

2.1. Coordinates

For many purposes, the most perspicuous geometric description of $\mathcal{G}_8!$ employs eight coordinates that sum to zero. One advantage is the simple expression of the $S_8$-duality between points and hyperplanes. In general, for a finite action $\mathcal{G}$ whose matrix representatives are unitary, a point $a$ is ‘$\mathcal{G}$-dual’ to the hyperplane

$$\mathcal{L} = \{ \bar{a} \cdot x = 0 \}.$$ 

Consequently, $a$ and $\mathcal{L}$ have the same stabilizer in $\mathcal{G}$. Since the action of $S_8$ on $\mathbb{C}^7$ is orthogonal, a point

$$a = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8] \sum a_k = 0 \in \mathbb{CP}^6$$

corresponds to the hyperplane

$$\{ a \cdot x = 0 \} = \left\{ \sum_{k=1}^{8} a_k x_k = 0 \right\}.$$ 

(Square brackets indicate homogeneous coordinates.)

A system of seven ‘hyperplane coordinates’ describes the hyperplane $\mathcal{H}_u$. It arises from the ‘hermitian’ change of variable
\[
H = \begin{pmatrix}
1 & \omega & i & \omega^3 & -1 & \omega^5 & -i & \omega^7 \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & \omega^3 & -i & \omega & -1 & \omega^7 & i & \omega^5 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & \omega^5 & i & \omega^7 & -1 & \omega & -i & \omega^3 \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & \omega^7 & -i & \omega^5 & -1 & \omega^3 & i & \omega
\end{pmatrix}
\]

where \( \omega = e^{\pi i/4} \) and the choice of scalar factor gives

\[
H H^T = I_7 \quad H^T H = (a_{ij}) \quad a_{ij} = \begin{cases} 
-\frac{7}{8} & i = j \\
\frac{1}{8} & i \neq j
\end{cases}, \quad i, j = 1, \ldots, 8. \tag{1}
\]

2.2. Invariant polynomials

According to the fundamental result on symmetric functions the \( n \) elementary symmetric functions of degrees 1 through to \( n \) generate the ring of \( S_n \)-invariant polynomials. Since the \( S_8 \) action on \( \mathbb{CP}^6 \) occurs where the degree-1 symmetric polynomial vanishes, there are seven generating \( \mathcal{G}_{8!} \)-invariants. By Newton's identities, the power sums

\[
F_k(x) = \sum_{\ell=1}^{8} x^k_{\ell} \quad k = 2, \ldots, 8
\]

also generate the \( \mathcal{G}_{8!} \)-invariants. In hyperplane coordinates, the ‘forms’ in degrees two and three are

\[
\Phi_2(u) = F_2(H^T u) = u_4^2 + 2u_3u_5 + 2u_2u_6 + 2u_1u_7
\]

\[
\Phi_3(u) = \frac{3}{2\sqrt{2}} (u_2u_3^2 + u_2^2u_4 + 2u_1u_3u_5 + 2u_1u_2u_5 + u_1^2u_6 + u_2^2u_6 + u_4u_6^2
\]

\[+ 2u_4u_5u_7 + 2u_3u_6u_7 + u_2u_7^2).
\]

The remaining five generating invariants \( \Phi_k(u) \) arise algebraically from the above two invariants. Classical techniques show that, given three \( \text{GL}_n(\mathbb{C}) \) invariants \( F, G, \) and \( J, \) a ‘relative invariant’—invariant up to a multiplicative character—results from taking the determinant of the ‘bordered hessian’
\[ BH(F, G, J) = \begin{pmatrix} H(F) & \cdots & \frac{\partial G}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial J}{\partial x_1} & \cdots & \frac{\partial J}{\partial x_n} & 0 \end{pmatrix} \]

where the \( H(F) \) is the \( n \times n \) hessian of \( F \).

**Proposition 1.** Given \( T \in \text{GL}_n(\mathbb{C}) \) and \( B(x) = \det(BH(F(x), G(x), J(x))) \) for \( \text{GL}_n(\mathbb{C}) \) invariants \( F, G, J \)

\[ B(Tx) = (\det T)^{-2} B(x). \]

For the permutation action of \( S_8 \), this results in an ‘absolute’ invariant that is expressible in terms of the generators \( \Phi_k \). The following result will serve a subsequent computational purpose. (Many of this work’s results arise from *Mathematica* computations.)

**Proposition 2.** With

\[ G_4 = \det(BH(\Phi_2, \Phi_3, \Phi_3)) \quad G_5 = \det(BH(\Phi_2, \Phi_3, \Phi_4)) \]
\[ G_6 = \det(BH(\Phi_2, \Phi_4, \Phi_4)) \quad G_7 = \det(BH(\Phi_2, \Phi_4, \Phi_5)) \]
\[ G_8 = \det(BH(\Phi_2, \Phi_5, \Phi_5)), \]

the ‘power-sum’ invariants are given by

\[ \Phi_4 = \frac{1}{576}(72\Phi_2^2 + G_4) \quad \Phi_5 = \frac{1}{768}(96\Phi_2\Phi_3 + G_5) \]
\[ \Phi_6 = \frac{1}{960}(120\Phi_2\Phi_4 + G_6) \quad \Phi_7 = \frac{1}{1280}(160\Phi_3\Phi_4 + G_7) \]
\[ \Phi_8 = \frac{1}{1600}(200\Phi_4^2 + G_8). \]

### 2.3. Special orbits

The six-dimensional \( S_8 \) action comes in both real and complex versions. This means that, in the standard \( x \) coordinates, \( \mathbb{G}_8 \) acts on \( \mathbb{R} \)—the \( \mathbb{R} \mathbb{P}^6 \) of points with real components. Table 1 enumerates some special orbits contained in \( \mathbb{R} \). For ease of expression, I refer to special points (or lines, planes, etc.) in terms of the orbit size: ‘8-points’ (28-lines, 56-planes, 28-hyperplanes). Also, these points receive a symbolic description in reference to orbit size (superscript) and coordinate expression (subscript).

Corresponding to each special point \( a \) is the hyperplane \( \{a \cdot x = 0\} \). In the case of the 28-points
there are the 28-hyperplanes

\[
\mathcal{L}^5_{28_{x2}} = \{ x_1 = x_2 \}, \ldots, \mathcal{L}^5_{28_{x8}} = \{ x_7 = x_8 \}.
\]

The involutions

\[
x_1 \leftrightarrow x_2, \ldots, x_7 \leftrightarrow x_8
\]

fix the respective hyperplanes point-wise. These 28 transpositions generate \( G_8 \), so that it acts as both a real and complex reflection group. (See Shephard and Todd (1954).) Various special planes and lines appear as intersections of the 28-hyperplanes. Tables 2 and 3 summarize the situation. Of particular dynamical significance is the collection of 28-lines \( \mathcal{L}^1_{28_i} \). This configuration forms the complete graph on the 8-points. (See figure 1 for two views.)

### Table 2. Some fundamental \( \mathbb{CP}^2 \) orbits

| Geometric definition | Descriptor | Set-wise stabilizer | Point-wise stabilizer | Restricted action |
|----------------------|------------|---------------------|----------------------|-------------------|
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_t} \cap \mathcal{L}^5_{28_m} \cap \mathcal{L}^5_{28_n} \) | \( \mathcal{L}^5_{28_{pq}} \) | \( S_5 \times S_3 \) | \( S_5 \) | \( S_3 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \) | \( \mathbb{Z}_2 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |

### Table 3. Special \( \mathbb{CP}^1 \) orbits

| Geometric definition | Descriptor | Set-wise stabilizer | Point-wise stabilizer | Restricted action |
|----------------------|------------|---------------------|----------------------|-------------------|
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( \mathcal{L}^5_{28_y} \cap \mathcal{L}^5_{28_x} \cap \mathcal{L}^5_{28_{sm}} \cap \mathcal{L}^5_{28_{mp}} \) | \( \mathcal{L}^1_{28_{pq}} \) | \( S_6 \times \mathbb{Z}_2 \) | \( S_6 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
3. Equivariant maps

The primary tool to be used in solving the general octic is a rational map $f$ with $S_8$ symmetry. In algebraic terms, this means that

$$f \circ T = T f \quad \text{for all } T \in G_8$$

while the geometric upshot is that $f$ sends $G_8$-orbits to $G_8!$-orbits. Furthermore, the map should have ‘reliable dynamics’: its attractor

1) is a ‘single’ $G_8!$-orbit;
2) has a corresponding basin with full measure in $\mathbb{CP}^6$ (‘strongly’ reliable);
2') alternatively, has a corresponding basin that is dense in $\mathbb{CP}^6$ (‘weakly’ reliable).

3.1. Basic maps

A finite group action $G$ on $\mathbb{C}^n$ induces an action on the associated exterior algebra. Moreover, $G$-invariant $(n-1)$-forms correspond to $G$-equivariant maps (see Crass (1999)).

For a reflection group, the number of generating 0-forms (that is, polynomials) is the dimension of the action (Shephard and Todd 1954, p. 282). From a result in complex reflection groups, this is also the number of generating 1-forms and $(n-1)$-forms (Orlik and Terao 1992, p. 232). Indeed, the generating 1-forms are exterior derivatives of the 0-forms while the generating $(n-1)$-forms are wedge products of 1-forms.

**Proposition 3.** With $X_1^k = -7x_i^k + \sum_{j \neq i} x_j^k$, the seven maps

$$f_k(x) = [X_1^k, X_2^k, X_3^k, X_4^k, X_5^k, X_6^k, X_7^k, X_8^k] \quad k = 1, \ldots, 7$$

generate the module of $G_8!$-equivariants over the ring of $G_8!$-invariants.
These maps are projections on to the hyperplane $\mathcal{H}_x$ along $[1, 1, 1, 1, 1, 1, 1]$ of the power maps

$$[x_1^k, x_2^k, x_3^k, x_4^k, x_5^k, x_6^k, x_7^k].$$

**Proposition 4.** Under an orthogonal action an invariant $F(x)$ gives rise to an equivariant $f(x)$ by means of a formal gradient

$$f(x) = \nabla_x F(x) = \left[ \frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x) \right].$$

**Proof.** See Crass (2001). \hfill $\square$

Note that the $G_{64}$-equivariant $f_k(x)$ is not equal to $\nabla_x F_{k+1}(x)$, but is a multiple of

$$\nabla_x F_{k+1}(x)|_{x_i^k \rightarrow x_i^1}.$$  

While this may be a source of confusion, it does not cause problems since we are working on the hyperplane $\mathcal{H}_x$. When using hyperplane coordinates on $\mathcal{H}_u$, the discrepancy manifests itself in the appearance of a constant $-8/(k + 1)$ for each map $\phi_k(u)$ (see below).

Recalling the change of coordinates from $\mathcal{H}_x$ to $\mathcal{H}_u$, a map on $\mathcal{H}_x$ expresses itself as a map

$$\phi(u) = Hf(\overline{H^Tu})$$

on $\mathcal{H}_u$. Having these maps in terms of the basic $u$-invariants $\Phi_k(u)$ will be useful.

**Definition 1.** Let

$$R = (r_{ij}) = \begin{cases} 
1 & i + j = 8 \\
0 & \text{otherwise}
\end{cases}$$

and $\nabla^u F(u) = R\nabla u F(u)$ represent the ‘reversed identity’ and ‘reversed gradient’.

**Proposition 5.** In $\mathcal{H}_u$ coordinates, the map $\phi(u) = Hf(\overline{H^Tu})$ is given by

$$\phi(u) = \nabla^u \Phi(u)$$

where $\Phi(u) = F(\overline{H^Tu}) = F(x)$ and $f(x) = \nabla_x F(x)$.

**Proof.** See Crass (2001). \hfill $\square$

Thus, the generating equivariants $\phi_k(u) = Hf_k(\overline{H^Tu})$ are

$$\phi_k(u) = -\frac{8}{k + 1} \nabla^u \Phi_{k+1}(u).$$

(2)

Although the factors $-8/(k + 1)$ have no projective effect on the maps individually, they do play a role when forming combinations of maps from parameterized families that do not have a common degree (see section 4.6). Explicit expressions for the maps of degrees one and two are

$$\phi_1(u) = -8[u_1, u_2, u_3, u_4, u_5, u_6, u_7]$$

$$\phi_2(u) = -2\sqrt{2}[2(u_4u_5 + u_3u_6 + u_2u_7), u_1^2 + u_2^2 + 2u_4u_6 + 2u_3u_7, 2(u_1u_2 + u_5u_6 + u_4u_7),$$

$$u_1^2 + u_1u_3 + u_6^2 + 2u_5u_7, 2(u_2u_3 + u_1u_4 + u_6u_7),$$

$$u_3^2 + 2u_2 u_4 + 2u_1 u_5 + u_7^2, 2(u_3 u_4 + u_2 u_5 + u_1 u_6)].$$
The lengthy results for the remaining maps are available at Crass (2000).

3.2. *A fixed point property*
For a $G_{8!}$-equivariant $f$ and a point $a$ that an element $T \in G_{8!}$ fixes

\[ Tf(a) = f(Ta) = f(a). \]

Hence, equivariants preserve fixed points of a group element.

Being point-wise fixed by the involution

\[ x_i \mapsto x_j, \]

a 28-hyperplane

\[ \mathcal{L}_{28}^5 = \{ x_1 - x_j = 0 \} \]

either maps to itself or collapses to its companion 28-point

\[ p_{ij}^{28} = \{ \ldots 0, 1, \ldots 0, \ldots -1, \ldots 0 \ldots \} \not\in \mathcal{L}_{28}^5. \]

In the former generic case, the map preserves the planes and lines that are intersections of 28-hyperplanes (see tables 2 and 3).

3.3. *Families of equivariants*
The $G_{8!}$-equivariants form a degree-graded module over the $G_{8!}$-invariants. This means that for an invariant $F_\ell$ and equivariant $g_m$ of degrees $\ell$ and $m$, the product

\[ F_\ell \cdot g_m \]

is an equivariant of degree $\ell + m$. When looking for a map in a certain degree $k$ with special geometric or dynamical properties, my approach is to express the entire family of ‘$k$-maps’ and, by manipulation of parameters, locate a subfamily with the desired behaviour.

3.4. *A special map in degree four*
In the configuration of 28-lines $\mathcal{L}_{28}^1$, each 8-point lies at the intersection of seven lines while each $\mathcal{L}_{28}^1$ contains the points $p_i^8$ and $p_j^8$ (see section 2.3). Moreover, these are the only intersections of 28-lines. We can attempt to exploit this structure by looking for a map with superattracting ‘pipes’ along the 28-lines: this means that, at each point on the line, the map is critical in all five ‘off-line’ directions. Under such a map, the 8-points would be superattracting in all directions. In addition, we want the map’s degree to be as small as possible. Degrees two and three avail us of too few parameters.

The family of 4-maps has (homogeneous) dimension three:

\[ \alpha_1 \Phi_3 \phi_1 + \alpha_2 \Phi_2 \phi_2 + \alpha_3 \phi_4. \]

Of course, choosing two parameter values determines a map on projective space. Obtaining a map $g_4$ for which the 28-lines are critical in the off-line directions requires two parameters. With the third parameter, we choose a lift of the map to $C^7$ that fixes the 8-points. The result is
\[ g_4 = -2\Phi_3\phi_1 - 9\Phi_2\phi_2 + 84\phi_4. \] (3)

The central dynamical role played by the 8-points suggests that a good choice of coordinates for this map places these points at
\[ [1, 0, 0, 0, 0, 0, 0], \ldots, [0, 0, 0, 0, 0, 0, 1], [1, 1, 1, 1, 1, 1, 1]. \]

Using \([v_1, \ldots, v_7]\) for this system, the map takes the form
\[ g_4(v) = [v_1T_1(v), \ldots, v_7T_7(v)] \]
where
\[ T_k(v) = 7v_k^3 - 4v_k^2S_1(\hat{v}_k) + 2v_kS_2(\hat{v}_k) - S_3(\hat{v}_k), \]
\(S_k\) is the degree-\(k\) elementary symmetric function in six variables, and
\(\hat{v}_k = (\ldots, v_{k-1}, v_{k+1}, \ldots)\).

In the \(v\)-coordinates, the equations
\[ v_i = x, v_j = y, v_k = 0 \quad \text{for } k \neq i, j \]
determine 21 of the 28-lines \(L_{28}^1\). For all \(k\), each term in \(S_2(\hat{v}_k)\) and \(S_3(\hat{v}_k)\) contains at least one \(v_\ell\) (\(\ell \neq i, j\)). Thus
\[ S_2(\hat{v}_k)|_{L_{28}^1} = 0 \quad \text{and} \quad S_3(\hat{v}_k)|_{L_{28}^1} = 0 \]
so that in the inhomogeneous coordinate \(z = x/y\) on the line the map restricts to
\[ z \rightarrow -\frac{7z - 4}{4z - 7} \]
while the pair of 8-points appear at 0 and \(\infty\).

**Proposition 6.** The 8-points \(p^8_i, p^8_j\) are the attractor for the restriction \(g = g_4|_{L_{28}^1}\). Furthermore, in the coordinates used above, the Julia set \(J_g\) is the unit circle.

**Proof.** Since the mobius transformation
\[ \frac{7z - 4}{4z - 7} \]
preserves the unit disc
\[ |g(z)| < |z|^3 \]
for \(|z| < 1\). If \(|z| < 1\), iteration yields
\[ |g^n(z)| < |g^{n-1}(z)|^3 < \cdots < |g(z)|^{3^{n-1}} < |z|^{3^n}. \]
Thus, every point in the disc belongs to the basin of the superattracting fixed point 0. Since \(g\) is symmetric under
\[ z \rightarrow \frac{1}{z}, \]
the basin of \(\infty\) is \(\{|z| > 1\}\).

The forward and backward invariance of \(\{|z| = 1\}\) implies that it contains and, indeed, is the map’s Julia set. \(\square\)

Thus, the basins of the 8-points contain the 28-lines excepting the \(\mathbb{RP}^1\) equator—the unit circle in the coordinates above. Along this circle \(C\), \(g_4\) has periodic, pre-
periodic, and what we might call chaotic saddle points. Attached to each point \( z \) on \( C \) is a five-dimensional stable manifold \( W_z \) consisting of points attracted to \( C \), that is, to the trajectory of \( z \). Locally, the stable manifolds are mutually disjoint and, collectively over the circle, give a stable manifold \( W_C \) of \( C \) whose real-dimension is 11 and that belongs to the basin boundaries of the pair of 8-points. We can see (real) two-dimensional slices of this stable manifold by plotting basins of attraction on spaces that are \( g_4 \)-invariant and intersect \( W_C \) (see the appendix).

By construction, \( g_4 \) self-maps each \( S_8 \)-symmetric 28-hyperplane \( \mathcal{L}_5^{28} \) and, hence, preserves each \( \mathbb{CP}^1 \) and \( \mathbb{CP}^2 \) intersection of hyperplanes. Denote these one- and two-dimensional spaces by \( \mathcal{L}_1^m \) and \( \mathcal{L}_2^n \). Furthermore, \( g_4 \) is equivariant under the anti-holomorphic transformation

\[
x \mapsto \bar{x}
\]

and, thereby, preserves \( \mathcal{R} \)—the \( S_8 \)-symmetric \( \mathbb{RP}^6 \). We can get a picture of the map’s ‘restricted dynamics’ by plotting basins of attraction on \( \mathcal{L}_1^m \)—a \( \mathbb{CP}^1 \)—and on the \( \mathbb{RP}^2 \) intersections

\[
\mathcal{L}_2^2 \cap \mathcal{R}.
\]

The basin portraits appear in the appendix. Graphical and experimental evidence support a claim of reliability for \( g_4 \).

**Conjecture 1.** The attractor for \( g_4 \) is the 8-point orbit.

**Conjecture 2.** Under \( g_4 \), the basins of the 8-points fill up \( \mathbb{CP}^6 \) in measure.

Finally, since \( g_4 \) has real coefficients (see (3)), it preserves the \( \mathbb{RP}^6 \) whose points have real \( u \) coordinates. This is not the \( S_8 \)-symmetric \( \mathcal{R} \). Rather it has the \( S_7 \) symmetry of \( p_1^8 \) which is \([1,1,1,1,1,1,1,1]\) in the \( u \) space. Accordingly, there are eight spaces of this type.

4. **Solving the octic**

To compute a root of a polynomial, one must overcome its symmetry. For a general equation of degree \( n \) the obstacle is \( S_n \). Klein described a means to this end: given values for an independent set of \( S_n \)-invariant homogeneous polynomials

\[
a_1 = G_1(x) \cdots a_m = G_m(x),
\]

find the \( S_n \) orbits of solutions \( x \) to these equations (Klein 1913, pp. 69ff). This task of inverting the \( G_k \) is the ‘form problem’ on \( S_n \). It also has an inhomogeneous manifestation: for \( m - 1 \) given values, invert \( m - 1 \) invariant rational functions of degree zero.

By iterating a reliable \( S_n \)-equivariant we can break the obstructing symmetry. In effect, the dynamics provides a mechanism for solving the form problem and, hence, the \( n \)th-degree equation.

4.1. **Parameters**

The \( G_{8!} \) rational form problem is to invert
K_1 = \frac{\Phi_3(u)^2}{\Phi_2(u)^3} \quad K_2 = \frac{\Phi_4(u)}{\Phi_2(u)^2} \quad K_3 = \frac{\Phi_5(u)}{\Phi_2(u)\Phi_3(u)}

K_4 = \frac{\Phi_6(u)}{\Phi_2(u)^3} \quad K_5 = \frac{\Phi_7(u)}{\Phi_2(u)\Phi_5(u)} \quad K_6 = \frac{\Phi_8(u)}{\Phi_2(u)^4}.

As functions, the K_i define the G_{8!} quotient map

\[ [K_1, K_2, K_3, K_4, K_5, K_6, 1] = [\Phi_2 \Phi_3 \Phi_5, \Phi_2^2 \Phi_3 \Phi_4 \Phi_5, \Phi_3^2 \Phi_5, \Phi_2 \Phi_3 \Phi_5 \Phi_6, \Phi_2 \Phi_3 \Phi_7, \Phi_3 \Phi_5 \Phi_8, \Phi_4 \Phi_5 \Phi_3 \Phi_5] \]
on \mathbb{CP}^6 \setminus \{\Phi_2 = \Phi_3 = \Phi_5 = 0\}. The generic fibre over points in \mathbb{CP}^6 is a G_{8!}-orbit given by

\{ \Phi_3^2 = K_1 \Phi_2^3 \} \cap \{ \Phi_4 = K_2 \Phi_2^3 \} \cap \{ \Phi_5 = K_3 \Phi_2 \Phi_3 \} \cap \{ \Phi_6 = K_4 \Phi_3^3 \} \cap \{ \Phi_7 = K_5 \Phi_2 \Phi_5 \} \cap \{ \Phi_8 = K_6 \Phi_2^4 \}.

Exceptional locations are

\{0, 0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, 0, 1\}, \{0, 0, 0, 0, 0, 0, 0\}

where the respective fibres are the hypersurfaces \{\Phi_3 = 0\}, \{\Phi_5 = 0\}, and \{\Phi_2 = 0\}. The parameters K_i forge a link between octic equations and G_{8!} actions. The connection consists in K-parameterizations of each regime.

4.2. A family of S_8 quintics

Let \mathcal{G}_v be a version of \mathcal{G}_{8!} that acts on a v-coordinatized \mathbb{CP}^6. This will be a parameter space—the coordinate v merely stands in for u. The linear polynomials

\[ X_k(x) = -7x_k + \sum_{i \neq k} x_i \quad (4) \]

form an orbit of size eight. Let

\[ L_k(u) = X_k(H^T u). \]

Then the rational functions

\[ \sigma_k(v) = \frac{\Phi_2(v)L_k(v)}{\Phi_3(v)} \]

also give an 8-orbit. Taking the \sigma_k as roots of a polynomial

\[ R_v(s) = \prod_{k=1}^{8}(s - \sigma_k(v)) = \sum_{k=0}^{8} C_k(v)s^{8-k} \]

creates a family of octics whose members generically have S_8 symmetry. Since \mathcal{G}_v permutes the \sigma_k(v), each coefficient C_k(v) is \mathcal{G}_v-invariant and hence, expressible in the basic forms \Phi_k(v). Ultimately, we can express each C_k in terms of the K_i. By direct calculation
Members of the family of octic ‘resolvents’

\[ R_K(s) = \sum_{n=0}^{8} C_n s^{8-n} \]

parameterized by \( K = (K_1, \ldots, K_6) \) are particularly well suited for an iterative solution that employs \( g_4 \). For chosen values of the \( K_i \), a solution to the resulting form problem yields a root of \( R_K \). We can use \( G_{8!} \) symmetry to find such a solution without explicitly inverting the \( K_i \) equations. Our attention will turn to this issue after we connect the general octic to the special family \( R_K \).

4.3. Reduction of the general octic to the \( G_{8!} \) resolvent

By means of a linear Tschirnhaus transformation

\[ x \rightarrow y = \frac{a_1}{8} \]

the general octic

\[ p(x) = x^8 + a_1 x^7 + a_2 x^6 + a_3 x^5 + a_4 x^4 + a_5 x^3 + a_6 x^2 + a_7 x + a_8 \]

becomes the standard 7-parameter ‘resolvent’

\[ q(y) = y^8 + b_2 y^6 + b_3 y^5 + b_4 y^4 + b_5 y^3 + b_6 y^2 + b_7 y + b_8 \]

where
\[ b_2 = \frac{-7a_1^2 + 16a_2}{16} \]
\[ b_3 = \frac{7a_1^3 - 24a_1a_2 + 32a_3}{32} \]
\[ b_4 = \frac{-105a_1^4 + 480a_1^2a_2 - 1280a_1a_3 + 2048a_4}{2048} \]
\[ b_5 = \frac{7a_1^5 - 40a_1^3a_2 + 160a_1^2a_3 - 512a_1a_4 + 1024a_5}{1024} \]
\[ b_6 = \frac{-35a_1^6 + 240a_1^4a_2 - 1280a_1^3a_3 + 6144a_1^2a_4 - 24576a_1a_5 + 65536a_6}{65536} \]
\[ b_7 = \frac{3a_1^7 - 24a_1^5a_2 + 160a_1^4a_3 - 1024a_1^3a_4 + 6144a_1^2a_5 - 32768a_1a_6 + 131072a_7}{131072} \]
\[ b_8 = \frac{-7}{16777216} (a_1^8 + 64a_1^6a_2 - 512a_1^5a_3 + 4096a_1^4a_4 - 32768a_1^3a_5 + 262144a_1^2a_6 - 2097152a_1a_7 + 16777216a_8). \]

Application of another linear Tschirnhaus transformation

\[ s \mapsto \frac{y}{\lambda} \]

converts the 6-parameter family \( R_K(s) \) into a \( G_{8!} \) resolvent

\[ \Sigma_{K,\lambda}(y) = \lambda^8 R_K \left( \frac{y}{\lambda} \right) = \sum_{n=0}^{8} \lambda^n C_n y^{8-n} \]

in the seven parameters \( K_1, \ldots, K_6 \), and the auxiliary \( \lambda \).

The functions

\[ b_n = \lambda^n C_n(K) \]

relate the coefficients of \( q \) and \( \Sigma_{K,\lambda} \). The \( b_n \) invert to

\[ K_1 = \frac{-9b_3^2}{8b_2^3} \]
\[ K_2 = \frac{b_2^2 - 2b_4}{2b_2^2} \]
\[ K_3 = \frac{5(b_2b_3 - b_5)}{6b_2b_3} \]
\[ K_4 = \frac{2b_2^3 - 3b_3^2 - 6b_2b_4 + 6b_6}{8b_2^3} \]
\[ K_5 = \frac{7(b_2^2b_3 - b_3b_4 - b_2b_5 + b_7)}{10b_2(b_2b_3 - b_3)} \]
\[ K_6 = \frac{b_2^4 - 4b_2b_3^2 - 4b_2^2b_4 + 2b_3^2 + 4b_3b_5 + 4b_2b_6 - 4b_8}{8b_2^4} \]
\[ \lambda = \frac{-3b_3}{16b_2} \]
Thus, almost any octic descends to a member of $R_K$. The reduction fails when

$$-7a_1^2 + 16a_2 = 16b_2 = 0 \text{ or } 7a_1^3 - 24a_1a_2 + 32a_3 = 32b_3 = 0$$

or

$$-105a_1^5 + 600a_1^3a_2 - 768a_1a_2^2 - 608a_1^2a_3 + 1024a_2a_3 + 512a_1a_4 - 1024a_5$$

$$= 1024(b_2b_3 - b_5) = 0.$$ 

A solution to the special resolvent $R_K$ then ascends to a solution to the general quintic.

4.4. A family of $S_8$ actions

With the basic $G_v$ forms and maps, we can define the ‘parameterized change of coordinates’

$$u = \tau_v w = \sum_{k=1}^{7} (\Phi_{9-k}(v) \phi_k(v)) w_k.$$ 

For a choice of parameter $v$

$$\tau_v : \mathbb{CP}^6_w \rightarrow \mathbb{CP}^6_u$$

is linear in $w$ and provides a parameterized family of $G_{8!}$ groups

$$G^v_w = \tau_v^{-1} G_v \tau_v.$$ 

A matrix form results from taking the $\phi_k(v)$ as column vectors:

$$\begin{pmatrix} u_1 \\ \vdots \\ u_7 \end{pmatrix} = (\Phi_8(v) \phi_1(v) \cdots \Phi_2(v) \phi_7(v)) \begin{pmatrix} w_1 \\ \vdots \\ w_7 \end{pmatrix}.$$ 

The set-up here is as follows.

- $G_u$ is a version of $G_{8!}$ that acts on a ‘reference space’ $\mathbb{CP}^6_u$.
- $G_v$ is a version of $G_{8!}$ that acts on a ‘parameter space’ $\mathbb{CP}^6_v$.
- $G_u$ and $G_v$ have identical expressions in their respective coordinates.
- $G^v_w$ are versions of $G_{8!}$ that act on $v$-parameterized spaces $\mathbb{CP}^6_w$.
- The iteration that solves octics $R_K$ takes place in $\mathbb{CP}^6_w$.

Each $G^v_w$ has its system of invariants and equivariants. From this point of view we can see, in the resolvents $R_v$ and $G^v_w$-equivariants, a connection between octics and dynamical systems. Furthermore, each $G^v_w$-invariant and equivariant is expressible in the $K_i$. This circumstance connects a resolvent $R_K$ with $G^v_w$-symmetric maps.

By construction, $\tau_v w$ possesses an equivariance property:

$$\tau_{A v} w = A \tau_v w \quad \text{for} \quad A \in G_v, G_u.$$ 

The determinant of $\tau_v$ will enter into upcoming calculations and so demands more attention. Since

$$\det \tau_{A v} = \det A \det \tau_v,$$
det $\tau_v$ is invariant under the $A_8$ subgroup $G_{8!}/2$ of $G_v$, but only relatively invariant under the full $S_8$ group $G_{8!}$. (The ‘even transformations’ have determinant 1 while the odd elements have determinant $-1$.) Furthermore

$$\det \tau_v = \left( \prod_{k=2}^{8} \Phi_k(v) \right) \det(\phi_1(v) \cdots \phi_7(v))$$

$$= \left( \prod_{k=2}^{8} \Phi_k(v) \right) \Psi_{28}(v)$$

where $\Psi_{28}$ is, according to a basic result in reflection group theory, a scalar multiple of the product of the 28 linear forms associated with the 28-hyperplanes that are fixed by the reflections that generate $G_v$. Furthermore, $\Psi_{28}$ is invariant under the group $G_{8!}/2$ (isomorphic to the alternating group $A_8$) but is relatively invariant under $G_{8!}$. Consequently, the degree-126 square of $\det \tau_v$ is $G_{8!}$-invariant with $K$-expression

$$(\det \tau_v)^2 = \Phi_{2}^{63}(v)t_K.$$ The explicit form of $t_K$ appears at Crass (2000).

### 4.5. A family of $S_8$-invariants

The equivariance in $v$ of $\tau_v w$ implies that $\Phi_2(\tau_v w)$ is $G_v$-invariant. Thus, each $w$ coefficient of $\Phi_2(\tau_v w)$ inherits the same invariance. Since

$$\deg_v \Phi_2(\tau_v w) = \deg_u \Phi_2(u) \cdot \deg_v \tau_v w = 2 \times 9 = 18,$$

the rational function

$$\frac{\Phi_2(u)}{\Phi_2(v)^9} = \frac{\Phi_2(\tau_v w)}{\Phi_2(v)^9}$$

is degree zero in $v$ and thereby, expressible in $K$. For each $w$ monomial, we can solve a system of linear equations whose dimension is that of the degree-18 $G_v$-invariants. The result is an explicit expression in $K$ for each $w$-coefficient of $\Phi_2(\tau_v w)$. Let

$$\Phi_2(v)^9 \Phi_{2x}(w) = \Phi_2(u)$$

(5)

define the basic degree-2 $G_v$-invariant $\Phi_{2x}(w)$. Similar considerations apply in degree three where

$$\Phi_2(v)^{12} \Phi_3(v) \Phi_{3x}(w) = \Phi_3(u).$$

(6)

The results appear at Crass (2000).

By Proposition 2, the degree-4 and degree-5 invariants derive from those in degrees two and three. The chain rule determines a transformation formula for the bordered hessian. Let $| \cdot |$ represent the determinant and $A^T$ the transpose of $A$.

**Proposition 7.** For $y = Ax$

$$BH_{x}(F(y), G(y), J(y)) = \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix} BH_{y}(F(y), G(y), J(y)) \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

where the subscript indicates the variable of differentiation. Thus
\[ |BH_{x}(F(y), G(y), J(y))| = |A|^2 |BH_{y}(F(y), G(y), J(y))|. \]

Applied to the parameterized change of variable \( w = \tau_{y}^{-1}u \), this formula along with (5) and (6) yields

\[
G_{4}(u) = |BH_{u}(\Phi_{2}(u), \Phi_{3}(u), \Phi_{3}(u))| = |BH_{u}(\Phi_{2}(v)^9 \Phi_{2k}(w), \Phi_{2}(v)^{12} \Phi_{3}(v)\Phi_{3k}(w), \Phi_{2}(v)^{12} \Phi_{3}(v)\Phi_{3k}(w))| \]
\[
= \frac{\Phi_{2}(v)^6 \Phi_{3}(v)^2}{|\tau_{y}|^2} |BH_{w}(\Phi_{2k}(w), \Phi_{3k}(w), \Phi_{3k}(w))| \]
\[
= \frac{\Phi_{2}(v)^{18} \Phi_{3}(v)^2}{t_{K}} |BH_{w}(\Phi_{2k}(w), \Phi_{3k}(w), \Phi_{3k}(w))| \]
\[
= \frac{\Phi_{2}(v)^{18} K_{1}}{t_{K}} |BH_{w}(\Phi_{2k}(w), \Phi_{3k}(w), \Phi_{3k}(w))| \]
\[
= \Phi_{2}(v)^{18} G_{4k}(w). \]

Using Proposition 2 we obtain

\[
\Phi_{4}(u) = \frac{1}{576} (72\Phi_{2}(v)^{18} \Phi_{2k}(w)^2 + \Phi_{2}(v)^{18} G_{4k}(w)) \]
\[
= \Phi_{2}(v)^{18} \Phi_{4k}(w) \]

so that

\[
G_{5}(u) = |BH_{u}(\Phi_{2}(u), \Phi_{3}(u), \Phi_{4}(u))| = |BH_{u}(\Phi_{2}(v)^9 \Phi_{2k}(w), \Phi_{2}(v)^{12} \Phi_{3}(v)\Phi_{3k}(w), \Phi_{2}(v)^{18} \Phi_{4k}(w))| \]
\[
= \Phi_{2}(v)^{21} \Phi_{3}(v)G_{5k}(w) \]

and

\[
\Phi_{5}(u) = \frac{1}{768} (96\Phi_{2}(v)^{9} \Phi_{2k}(w), \Phi_{2}(v)^{12} \Phi_{3}(v)\Phi_{3k}(w), \Phi_{2}(v)^{21} \Phi_{3}(v)G_{5k}(w)) \]
\[
= \Phi_{2}(v)^{21} \Phi_{3}(v)\Phi_{5k}(w). \]

Employed here are the obvious definitions

\[
G_{4k}(w) = \frac{K_{1}}{t_{K}} |BH_{w}(\Phi_{2k}(w), \Phi_{3k}(w), \Phi_{3k}(w))| \]
\[
G_{5k}(w) = \frac{1}{t_{K}} |BH_{w}(\Phi_{2k}(w), \Phi_{3k}(w), \Phi_{4k}(w))| \]

as well as natural definitions for \( \Phi_{4k}(w) \) and \( \Phi_{5k}(w) \).
4.6. A family of $S_8$-equivariants

Emerging from each $G_v^w$ action is a version $\tau_v^{-1}g_4(\tau_vw)$ of $g_4(u)$. Being $G_v$-invariant, these maps also admit parameterization by $K$. In this way, each octic $R_K$ enters into association with a dynamical system $g_K$ on $\mathbb{CP}^6$.

The reversed identity $R$ and gradient $\nabla^r = R\nabla$ appeared in the context of a change from eight $x$ coordinates to seven $u$ coordinates (see Definition 1). In the present setting, a ‘reversed transpose’ arises.

**Definition 2.** The ‘repose’ $A^r$ of an $n \times n$ matrix $A$ is its reflection through the ‘reversed diagonal’—the entries whose subscripts sum to $n + 1$. Alternatively

$$A^r = RA^TR.$$

**Proposition 8.** For a coordinate change $x = Ay$ and a function $F(y) = \tilde{F}(x)$, the reversed gradient map transforms by

$$\nabla^r_y F(y) = A^r \nabla^r_y \tilde{F}(x).$$

**Proof.** See Crass (2001), Proposition 4.2, but note that the coordinate change there should be $w = Au$. \hfill \Box

Using (2), the degree-1 $G_8$! map is

$$\phi_1(u) = -\frac{8}{2} \nabla^r_v \Phi_2(v)^9 \Phi_2_k(w)$$

$$= -4 \Phi_2(v)^9 (\tau_v^{-1})^r \nabla^r_0 \Phi_2_k(w)$$

$$= -4 \Phi_2(v)^9 \tau_v \tau_v^{-1}(\tau_v^{-1})^r \nabla^r_0 \Phi_2_k(w)$$

$$= -4 \tau_v \Phi_2(v)^9 (\tau_v^r \tau_v)^{-1} \nabla^r_0 \Phi_2_k(w).$$

Thus

$$\tau_v^{-1} \phi_1(\tau_vw) = -4 \Phi_2(v)^9 (\tau_v^r \tau_v)^{-1} \nabla^r_0 \Phi_3_k(w).$$

Using the description on the left-hand side, a straightforward calculation reveals this map to be invariant in $v$ so that the matrix $\tau_v^r \tau_v$ has entries that are degree-18 $G_v$-invariants. Hence, the matrix product has the $K$-expression

$$\tau_v^r \tau_v = \Phi_2(v)^9 T_K \quad \text{or} \quad (\tau_v^r \tau_v)^{-1} = \frac{T_K^{-1}}{\Phi_2(v)^9}. \quad (7)$$

(See Crass (2000) for the explicit form.) Also, note that

$$\det T_K = \frac{\det(\tau_v^r \tau_v)}{\Phi_2(v)^9} = \frac{(\det \tau_v)^2}{\Phi_2(v)^6} = t_K.$$ 

Making use of (7) to express the transformation of basic equivariants yields

$$\phi_1(u) = -4 \Phi_2(v)^9 \tau_v \frac{T_K^{-1}}{\Phi_2(v)^9} \nabla^r_0 \Phi_2_k(w)$$

$$= -4 \tau_v T_K^{-1} \nabla^r_0 \Phi_2_k(w).$$

As for the other relevant maps, similar calculations give
Finally, we can identify a $K$-parameterized 4-map $g_K(w)$ that is conjugate to $g_4(u)$. The map’s expression in basic terms appears after substitution into the formula found in section 3.4:

$$
\phi_2(u) = -\frac{8}{3} \Phi_2(v)^{12} \Phi_3(v) \tau_v \frac{T_K^{-1}}{\Phi_2(v)} \nabla_w \Phi_3_k(w)
$$

$$
= -\frac{8}{3} \Phi_2(v)^3 \Phi_3(v) \tau_v T_K^{-1} \nabla_w \Phi_3_k(w)
$$

$$
\phi_4(u) = -\frac{8}{4} \Phi_2(v)^{21} \Phi_3(v) \tau_v \frac{T_K^{-1}}{\Phi_2(v)} \nabla_w \Phi_5_k(w)
$$

$$
= -2 \Phi_2(v)^{12} \Phi_3(v) \tau_v T_K^{-1} \nabla_w \Phi_5_k(w).
$$

Thus, we have a $K$-parameterized family of 4-maps on $\mathbb{CP}^6$:

$$
g_K(w) = T_K^{-1} (21 \nabla_w \Phi_5_k(w) - 3 \Phi_2_k(w) \nabla_w \Phi_3_k(w) - 2 \Phi_3_k(w) \nabla_w \Phi_2_k(w)).
$$

whose relation to the reference 4-map is

$$
g_4(u) = -8 \Phi_2^{12}(v) \Phi_3(v) \tau_v g_K(w).
$$

4.7. Root selection

Being conjugate to $g_4(u)$ each $g_K(w)$ shares the former’s conjectured reliable dynamics. Accordingly, the attractor for each choice of $K$ is the 8-point orbit in the corresponding $\mathbb{CP}^6$ so that for almost every $w_0 \in \mathbb{CP}^6$

$$
g_K(w_0) \rightarrow \tau_v^{-1} p_i^8 \text{ for some 8-point } p_i^8 \in \mathbb{CP}^6.
$$

To solve the resolvent $R_K$, the output of the iteration must link with the roots of $R_K$. From here, we see that solving $R_K$ amounts to inverting $\tau_v$—the form problem in another guise. This is effectively what the dynamics of $g_K$ accomplishes with the assistance of a $G_{8!}$ tool that I now describe.

The quadratic $S_7$-invariants

$$
X_k^2(x) = -7x_k^2 + \sum_{i \neq k} x_i^2
$$

form a $G_{8!}$-orbit of size eight. Recall that

$$
L_k(u) = X_k(H^1 u)
$$

and let

$$
Q_k(u) = X_k^2(H^1 u)
$$

be the $H_u$ expression for $X_k^2$. Furthermore, each of the eight forms
\[ G_k(u) = Q_k(u) - \frac{3}{4}L_k(u)^2 \quad k = 1, \ldots, 8 \]

vanish at the 8-points \( p_l^k \) with \( l \neq k \) but not at \( p_k^8 \).

Now, consider the rational function

\[
J_v(w) = \alpha \sum_{k=1}^{8} \frac{G_k(\tau_v w) \Phi_2(v) L_k(v)}{\Phi_2(\tau_v w) \Phi_3(v)} = \alpha \sum_{k=1}^{8} \frac{G_k(\tau_v w) \sigma_k(v)}{\Phi_2(\tau_v w) \Phi_3(v)}
\]

where \( \alpha \) is a constant to be determined. Since the \( v \)-degree of the numerator and denominator is \( 21 = 2 \times 9 + 3 \) while the \( w \)-degree is 2, the function is rationally degree zero in both variables. At an 8-point \( \tau_v^{-1}p_l^k \) in \( \mathbb{CP}_w^6 \) seven of the eight terms in \( J_v \) vanish; this leaves

\[
\alpha \frac{G_l(p_l^k)}{\Phi_2(p_l^k)} \sigma_l(v).
\]

Choosing

\[
\alpha = \frac{\Phi_2(p_l^k)}{G_k(p_l^k)} = \frac{1}{48} \quad (k = 1, \ldots, 8)
\]

‘selects’ the root \( \sigma_l(v) \) of \( R_k(s) \). Since the iterative output of \( g_K(w) \) is a single 8-point in \( \mathbb{CP}_w^6 \), the dynamics produces one root.

To obtain a usable form of the root-selector \( J_v(w) \), let

\[
\Gamma_v(w) = \sum_{k=1}^{8} G_k(\tau_v w) L_k(v).
\]

Since \( G \) permutes its terms, \( \Gamma \) is invariant under the action and hence, expressible in \( K \):

\[
\Gamma_v(w) = \Phi_2(v)^8 \Phi_3(v) \Gamma_K(w).
\]

(The explicit form of \( \Gamma_K \) appears at Crass (2000).) Application of (5) yields

\[
J_v(w) = \frac{\Phi_2(v) \Gamma_v(w)}{48 \Phi_3(v) \Phi_2(\tau_v w)}
\]

\[
J_K(w) = \frac{\Gamma_K(w)}{48 \Phi_2(\tau_v w)}.
\]

4.8. \section{The procedure summarized}

At Crass (2000), there are \textit{Mathematica} data files and a notebook that implement the iterative solution to the octic.

(1) Select a general 8-parameter octic \( p(x) \).

(2) Tschirnhaus transform \( p(x) \) into a member \( R_k(s) \) of the 6-parameter family of \( G_{8l} \) octics—this determines values for \( K_1, \ldots, K_6 \) as well as the auxiliary parameter \( \lambda \).

(3) For the selected \( K \) values, compute the matrix \( T_K \), the invariants \( \Phi_{2k}(w) \) and \( \Phi_{3k}(w) \), the 4-map \( g_K(w) \), the form \( \Gamma_K(w) \), and the root-selector \( J_K(w) \).

(4) From an arbitrary initial point \( w_0 \) iterate \( g_K \) until convergence:
Conjecturally, the output $w_{\infty}$ is an 8-point in $\mathbb{CP}^6$.

5. Beyond the octic

5.1. The general case

In the following discussion the $S_n$ actions under consideration derive from permutation of coordinates on the $S_n$-invariant hyperplanes

$$H_n^{n-1} = \left\{ \sum_{k=1}^{n} x_k = 0 \right\}.$$ 

These irreducible representations of $S_n$ project to actions $G_n$ on $P\mathcal{H}^{n-1} \simeq \mathbb{CP}^{n-2}$.

For equations of degree $n \neq 8$, does the analogue of the octic-solving algorithm exist? Evidently, the reduction of the $n$th-degree polynomial to an $(n-2)$-parameter family of $G_n$ resolvents is general.

**Query 1.** Is there an $S_n$-equivariant 4-map on $P\mathcal{H}^{n-1}$ that, on an $\binom{n}{2}$-line $L^1$ (where all but two coordinates are equal):

- superattracts in the off-line directions and
- restricts to a reliable map whose attractor consists of the pair of $n$-points on $L^1$?

**Query 2.** If so, is the map expressible as

$$z \rightarrow z^3 \frac{az + b}{bz + a}, \quad a, b \in \mathbb{R} \quad \text{and} \quad |b| < |a|$$

when restricted to $L^1$?

The affine space $\{x_n \neq 0\}$ parameterized by

$$\left[ x_1, \ldots, x_{n-2}, -\sum_{k=1}^{n-2} x_k - 1, 1 \right]$$

is tangent to $P\mathcal{H}^{n-1}$ at the affine part of $L^1$ given by

$$[\zeta, \ldots, \zeta, (2-n) \zeta - 1, 1].$$

We can identify these spaces respectively with

$$\{(x_1, \ldots, x_{n-2})\} \simeq \mathbb{C}^{n-2} \quad \text{and} \quad \{\zeta, \ldots, \zeta\} \simeq \mathbb{C}.$$ 

**Definition 3.** Abusing notation, let $L^1$ be the affine part of an $\binom{n}{2}$-line. Use $\mathcal{T}_{L^1}$ to denote the tangent space to $P\mathcal{H}^{n-1}$ along $L^1$. Also

$$\mathcal{L}^1 = \left\{ \sum_{k=1}^{n-2} x_k = 0 \right\} \simeq \mathbb{C}^{n-3}$$

is the (euclidean) orthogonal complement in $\mathcal{T}_{L^1}$ to $L^1$. 

\[ g_K^n(w_0) \longrightarrow w_{\infty}. \]
The subgroup $S_{L^1}$ of $G_n!$ that stabilizes $T^1_L$ is isomorphic to $S_{n-2}$ and acts by permutations on $C^{n-2}$. The action of $S_{L^1}$ fixes $T^1_L$ and $L^1_\perp$ set-wise and fixes $L^1$ in a point-wise manner.

Remark 1. In the treatment of $L^1$ below, we need not worry about the point at infinity

$$[1, \ldots, 1, 2 - n, 0],$$

since this point is in the same $G_n!$-orbit as the affine point

$$[1, \ldots, 1, 0, 2 - n]$$

which we have identified with

$$\frac{1}{2 - n}(1, \ldots, 1).$$

For $n \geq 5$, the family of 4-maps on $P^{H^{n-1}}$ is, in homogeneous parameters,

$$g_\alpha = \alpha_1 f_4 + \alpha_2 F_2 f_2 + \alpha_3 F_3 f_1$$

where definitions of $F_k$ and $f_k$ are obvious extensions from the $G_8!$ case. (For $n < 5$, the family of 4-maps is not three-dimensional.) Do three parameters suffice to obtain a map $g_\alpha$ with jacobian matrix $g'_\alpha(z)$ at $z$ such that

$$g'_\alpha(z)L^1_\perp = 0 \text{ for all } z \in L^1?$$

First of all, symmetry demands that, for each $g_\alpha$ and all $z \in L^1$, there is only one ‘off-line’ eigenvalue of $g'_\alpha(z)$.

**Lemma 1.** For an action $G$ on $CP^n$ and a $G$-equivariant $f$ that is holomorphic at $a$, the jacobian $f'(a)$ is equivariant on the tangent space $T_a \simeq C^n$ under the stabilizer $S_a$ of $a$.

**Proof.** After treating several technical matters, the proof amounts to a simple calculation.

1. Take $a$ to be $[0, \ldots, 0, 1]$ and $T_a$ to be $\{x_{n+1} \neq 0\}$ and lift them to $\hat{a} = (0, \ldots, 0, 1)$ and $\hat{T}_a = \{x_{n+1} = 1\}$.

2. Strictly speaking, $S_a$ is a group of projective transformations. Here, we choose linear representatives $T$ of $S_a$ that, as maps on $C^{n+1}$, satisfy $Ta = a$. This group of linear transformations acts on $\hat{T}_a$.

3. For a homogeneous polynomial $G(x)$ of degree $r$

$$rG(x) = (\nabla_x G(x))^T x$$

is a familiar identity. Generalized to a rational map $g(x)$, the result is

$$rg(x) = g'(x)x.$$

Given $T \in G$, let $\hat{T}$ be a lift to $C^{n+1}$. Then for all $x \in C^{n+1}$

$$\hat{T}^1 x = g'(x)x.$$
\[(f'(\hat{T}x)\hat{T})x = (f'(\hat{T}x))\hat{T}x\]
\[= rf(\hat{T}x)\]
\[= \hat{T}(rf(x))\]
\[= \hat{T}(f'(x))x\]
\[= (\hat{T}f'(x))x.\]

Thus \(f'(\hat{T}x)\hat{T} = \hat{T}f'(x)\).

In particular, when \(x = a\) and \(\hat{T} \in S_a\)

\[f'(a)\hat{T} = f'(\hat{T}a)\hat{T} = \hat{T}f'(a).\]

\[\square\]

**Lemma 2.** A linear \(S_L^1\)-equivariant \(T\) preserves \(L_1^1\).

**Proof.** The transformation

\[A : T_{L_1} \longrightarrow T_{L_1}\]

that cyclically permutes coordinates according to \((12 \ldots (n-3)(n-2))\) has \(n-2\) eigenspaces, namely, the lines \(L_k^1\) given by

\[L_k^1 = \left\{ tw_k \mid t \in \mathbb{C}, w_k = \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-3)k} \end{pmatrix}, \omega = e^{2\pi i/(n-2)} \right\} \quad k = 1, \ldots, n-2.\]

Note that \(L_1^1 = L_{n-2}^1\) and \(\{w_k \mid k = 1, \ldots, n-3\}\) is a basis for \(L_1^1\). Moreover, each line \(L_k^1\) has the eigenvalue \(\omega^{(n-3)k}\).

Since \(A \in S_{L_1}\) and

\[A(Tw_k) = T(Aw_k) = \omega^{(n-3)k}Tw_k,\]

\(Tw_k\) is also an \(\omega^{(n-3)k}\)-eigenvector of \(A\). Thus, each \(L_k^1\) is an eigenspace of \(T\) so that \(L_1^1\) is \(T\)-invariant.

\[\square\]

**Proposition 9.** For all \(z \in L_1\), the eigenvectors of the Jacobian \(g_\alpha'(z)\) span \(L_1^1\). Moreover, all associated eigenvalues are equal, making \(L_1^1\) an eigenspace of \(g_\alpha'(z)\).

**Proof.** Recall that the stabilizer \(S_{L_1}\) is isomorphic to \(S_{n-2}\). (The \(n-2\) things that \(S_{L_1}\) permutes are the vectors \(w_k\) defined above.) By Lemmas 1 and 2, let \(v \in L_1^1\) be an eigenvector of \(g_\alpha'(z)\) with eigenvalue \(\lambda\). For \(A \in S_{L_1}\), Lemma 1 gives

\[g_\alpha'(z)Av = Ag_\alpha'(z)v = \lambda Av.\]

Hence, \(Av\) is also a \(\lambda\)-eigenvector. Clearly, \(\{Av \mid A \in S_{L_1}\}\) spans \(L_1^1\).

\[\square\]

The question now is whether there is always some parameter choice for which the eigenvalue of \(L_1^1\) vanishes for all \(z\).
Proposition 10. For \( n \geq 5 \), there is a 4-map \( g \) whose critical set includes the \( \binom{n}{2} \)-lines. Moreover, at each point on an \( \binom{n}{2} \)-line, \( g \) is critical in every direction away from the line.

Proof. To facilitate exposition, we work in the linear space \( \mathcal{H}^{n-1} \). For an arbitrary member of the family \( g_\alpha \), select a lift \( \tilde{g}_\alpha \) to \( \mathcal{H}^{n-1} \). The line \( L^1 \) lifts to the plane \( \tilde{L}^1 \) parameterized by

\[
(x + y, \ldots, x + y, (1 - n)x + y, x + (1 - n)y).
\]

(8)

Furthermore, the orthogonal complement \( \tilde{L}^1 \) in \( \mathcal{H}^{n-1} \) is

\[
\left\{ \sum_{k=1}^{n-2} x_k = 0, x_{n-1} = x_n = 0 \right\}.
\]

By symmetry, we can consider a single line. Using the parameterization above, the pair of \( n \)-points on \( \tilde{L}^1 \) correspond to the lines \( x = 0 \) and \( y = 0 \). Meanwhile, the line specified by

\[
x + y = 0
\]

determines an element in one of the special orbits of \( \binom{n}{2} \)-points.

Proposition 9 implies that the characteristic polynomial for the jacobian of \( \tilde{g}_\alpha \), when restricted to \( \tilde{L}^1 \), has the form

\[
\chi_{\tilde{g}_\alpha \mid \tilde{L}^1} = \det(tI_n - \tilde{g}_\alpha) = (t - A(x, y))^n - (t^2 + B(x, y)t + C(x, y))
\]

(Note that the factor \( t^2 + Bt + C \) is the characteristic polynomial of the jacobian of the map \( \tilde{g}_\alpha \mid \tilde{L}^1 \)). Hence,

\[
B(x, y) = B_1 x^3 + B_2 x^2 y + B_3 xy^2 + B_4 y^3
\]

where the \( B_i \) are linear in the parameters \( \alpha_j \). It follows that

\[
A(x, y) = A_1 x^3 + A_2 x^2 y + A_3 xy^2 + A_4 y^3
\]

where the \( A_j \) are linear in the \( \alpha_j \).

The polynomial \( A \) gives the eigenvalue in the off-line directions in \( \mathbf{PH}^{n-1} \). The remaining factor corresponds to behaviour along \( \tilde{L}^1 \). Our interest here is \( A \). In particular, we want to force it to vanish identically in \( x \) and \( y \).

Since \( \chi_{\tilde{g}_\alpha \mid \tilde{L}^1} \) is invariant under permutation of coordinates, its restriction to \( \tilde{L}^1 \) is invariant under the interchange \( x \leftrightarrow y \) (which corresponds to the transposition \( ((n - 1)n) \) in (8)). Accordingly, \( A_1 = A_4 \) and \( A_2 = A_3 \) so that

\[
A = A_1(x^3 + y^3) + A_2xy(x + y)
\]

\[
= (x + y)(A_1(x^2 - xy + y^2) + A_2xy).
\]

(9)

Thus, the \( \binom{n}{2} \)-points are automatically critical away from \( L^1 \).
We want to solve the linear equations \( A_1 = 0 \) and \( A_2 = 0 \) in the three parameters \( \alpha_1, \alpha_2, \alpha_3 \). Such a system has non-trivial solutions that give \( A = 0 \) for all \( x, y \).

**Remark 2.**

1. The price of \( A = 0 \) is at most two parameters. With the third parameter, we can only normalize the map. In the discussion that follows, we shall discover that the cost is two parameters so that the resulting map is unique.
2. Forcing \( A_1 = 0 \) is tantamount to making the pair of \( n \)-points on \( \mathcal{L}^1 \) critical. This ‘two-birds-with-one-stone’ effect is what makes the procedure successful.
3. Ostensibly, this argument is consistent with our obtaining a map that blows up at the \( n \)-points. At an \( n \)-point such a map would be critical in the ‘radial’ direction in which \( \mathcal{H}^{n-1} \) projects to \( \mathcal{P}\mathcal{H}^{n-1} \). Such a circumstance would force \( C = 0 \) when \( xy \neq 0 \). Can the maps be critical in other directions as well? The preceding results and proofs provide for explicit calculation of the special 4-map. As a consequence, we see that the map does not blow up at the \( n \)-points.

Furthermore, we can derive the form of the map on an \( \binom{n}{2} \)-line.

At first, we use the parameterization

\[
[x, \ldots, x, y, (2 - n)x - y]
\]

for \( \mathcal{L}^1 \). Restricting the basic invariants to \( \mathcal{L}^1 \) gives

\[
\tilde{F}_k = F_k|_{\mathcal{L}^1} = (n - 2)x^k + y^k + ((2 - n)x - y)^k.
\]  \hspace{1cm} (10)

As for basic maps, note that

\[
f_k = [(f_k)_1, \ldots, (f_k)_n] \quad \text{with} \quad (f_k)_\ell = F_k - nx_k^\ell.
\]  \hspace{1cm} (11)

Thus

\[
\tilde{\tilde{f}_k} = f_k|_{\mathcal{L}^1} = [\tilde{F}_k - nx_k, \ldots, \tilde{F}_k - nx_k, \tilde{F}_k - ny_k, \tilde{F}_k - n((2 - n)x - y)^k]
\]

and we can express the homogeneous map ‘on’ \( \mathcal{L}^1 \) as

\[
\phi_k : [x, y] \rightarrow [\tilde{F}_k - nx_k, \tilde{F}_k - ny_k].
\]

Restricting the family of 4-maps to \( \mathcal{L}^1 \) gives

\[
\tilde{g}_\alpha = \alpha_1 \tilde{f}_4 + \alpha_2 \tilde{f}_2^2 + \alpha_3 \tilde{f}_3
\]

which, as a map on the line, is

\[
\gamma_\alpha[x, y] = [\alpha_1(\tilde{F}_4 - nx^4) + \alpha_2(\tilde{F}_2^2 - nx^2) - n\alpha_3 x \tilde{F}_3, \alpha_1(\tilde{F}_4 - ny^4) + \alpha_2(\tilde{F}_2^2 - ny^2) - n\alpha_3 y \tilde{F}_3].
\]

We can now determine three linear conditions on the \( \alpha_1 \) that correspond to the following.

1. Normalizing the map so that the \( n \)-points are fixed in the affine case—hence, they are not blown up.
2. Making the map critical in every direction at the \( n \)-points.
3. Making the map critical in every off-line direction along the \( \binom{n}{2} \)-lines.

Consider the cases in turn.
Using the inhomogeneous coordinate

\[ x = \gamma_{a}[z, 1]_{1} \]

Solving (12), (13), and (14) yields a unique 4-map

\[ g_{a} = \frac{1}{(n - 4)n^{3}}, \]

\[ (n - 1)(\gamma_{a}[x + y, (1 - n)x + y])_{1} - (\gamma_{a}[x + y, (1 - n)x + y])_{2}, \]

Using the inhomogeneous coordinate \( z = x/y \) the map becomes
\[ z \rightarrow -\frac{(n - 1)z - 4}{4z - (n - 1)} \]

in agreement with section 3.4. As for dynamics the respective pair of \( n \)-points are 0 and \( \infty \). Since \( n - 1 \geq 4 \), the restricted map has 0 and \( \infty \) as its only attractor. The respective basins are \( \{|z| < 1\} \) and \( \{|z| > 1\} \).

5.2. Another description

Using \( n - 1 \) homogeneous coordinates, generalize to \( \mathbb{CP}^{n-1} \) the \( v \) coordinates that describe the \( G_{4!} \)-equivalent \( g_4 \). These place the \( n \)-points at

\[ p_1 = [1, 0, \ldots, 0], \ldots, p_{n-1} = [0, \ldots, 0, 1], p_n = [1, \ldots, 1]. \]

The coordinate change is given by

\[ x = P v, \quad v = Q x \]

where

\[ P = (p_{ij}) = \begin{cases} 1 - n & i = j \\ 1 & i \neq j \end{cases} \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq n - 1 \]

and \( Q \) is the ‘inverse’ of \( P \):

\[ Q = (q_{ij}) = \begin{cases} -1 & i = j \\ 0 & i \neq j, \quad j < n \quad \text{for } 1 \leq i \leq n - 1, 1 \leq j \leq n. \\ 1 & j = n \end{cases} \]

Using \([v_1, \ldots, v_{n-1}]\) for this system, we define \( S_k \) to be the \( k \)th elementary symmetric function in \( n - 2 \) variables and the coordinates

\[ \hat{v}_k = (\ldots, v_{k-1}, v_{k+1}, \ldots) \]

complementary to \( v_k \). The stabilizer \( G_{(n-1)!} \) of \( p_n \) is the \( S_{n-1} \) group of permutations of the \( v_k \). The order-2 transformation \( Z_n \) that exchanges \( p_1 \) and \( p_n \) while fixing the remaining \( p_k \) generates \( G_{n!} \) over \( G_{(n-1)!} \). Note that \( \binom{n - 1}{2} \) of the \( \binom{n}{2} \)-lines consist of points for which all but two coordinates vanish, while the remaining \( n \) lines have points with all but one coordinate equal.

Let

\[ g(x) = [g_1(x), \ldots, g_n(x)]. \]

To compute the special 4-map

\[ \gamma(v) = Q(g(Pv)) = [g_n(Pv) - g_1(Pv), \ldots, g_n(Pv) - g_{n-1}(Pv)] \]

in \( v \) coordinates, we need to find only the first component of \( \gamma \). Permutation symmetry in \( v \) tends to the remaining components. Note that
Evidently, the maps in (16) are \( S.Crass \)

\[
P_v = \begin{pmatrix}
(1 - n)v_1 + S_1(\tilde{v}_1) \\
v_1 + S_1(\tilde{v}_1) - nv_2 \\
\vdots \\
v_1 + S_1(\tilde{v}_1) - nv_{n-1} \\
v_1 + S_1(\tilde{v}_1)
\end{pmatrix} = \begin{pmatrix}
u_1 - nv_1 \\
u_1 - nv_2 \\
\vdots \\
u_1 - nv_{n-1} \\
u_1
\end{pmatrix}
\]

where \( u_1 = v_1 + S_1(\tilde{v}_1) \). Application of (11) gives

\[
\gamma_1(v) = g_n(P_v) - g_1(Pv)
\]

\[
= (\alpha_1(F_4(Pv) - nu_1^4) + \alpha_2(F_2(Pv)^2 - nF_2(Pv)u_1^2) + n\alpha_3F_3(Pv)u_1)
- (\alpha_1(F_4(Pv) - n(u_1 - nv_1)^4) + \alpha_2(F_2(Pv)^2 - nF_2(Pv)(u_1 - nv_1)^2)
+ n\alpha_3F_3(Pv)(u_1 - nv_1))
\]

\[
= n(\alpha_1((u_1 - nv_1)^4 - u_1^4) + \alpha_2F_2(Pv)((u_1 - nv_1)^2 - u_1^2)
+ \alpha_3F_3(Pv)((u_1 - nv_1) - u_1))
\]

\[
= -n^2v_1(\alpha_1((u_1 - nv_1)^3 + (u_1 - nv_1)^2u_1 + (u_1 - nv_1)u_1^2 + u_1^3)
+ \alpha_2F_2(Pv)(2u_1 - nv_1) + \alpha_3F_3(Pv)).
\]

Straightforward calculation yields

\[
F_2(Pv) = u_1^2 - nv_1^2 - nS_1(\tilde{v}_1)^2 + 2nS_2(\tilde{v}_1)
\]

\[
F_3(Pv) = 2u_1^3 - 3mu_1v_1^2 + n^2v_1^3 + 3mu_1S_1(\tilde{v}_1)^2 + 3n^2S_1(\tilde{v}_1)^3
+ 6mu_1S_2(\tilde{v}_1) - 3n^2S_1(\tilde{v}_1)S_2(\tilde{v}_1) + 3n^2S_3(\tilde{v}_1).
\]

Using (15), the substitution for \( u_1 \), and permutation in the \( v_j \), the map takes the form

\[
\gamma(v) = [v_1T_1(v), \ldots, v_{n-1}T_{n-1}(v)]
\]

(16)

where

\[
T_k(v) = v_k^3 - a_2v_k^2S_1(\tilde{v}_k) + a_3v_kS_2(\tilde{v}_k) - a_4S_3(\tilde{v}_k)
\]

and

\[
a_2 = \frac{4}{n - 1} \quad a_3 = \frac{12}{(n - 1)(n - 2)} \quad a_4 = \frac{24}{(n - 1)(n - 2)(n - 4)}.
\]

Evidently, the maps in (16) are \( G_{n^{-1}} \)-equivariant. They also satisfy

\[
Z_n \circ g = g \circ Z_n \quad \text{for all} \ n \geq 5.
\]

5.3. Revisiting the quintic

In solving the quintic, Crass (2001) harnesses the dynamics of a degree-6 map whose behaviour is similar to the maps treated in the present paper. The \( G_{5!} \)-equivariant 4-
map is also a good candidate for inclusion in a quintic-solving algorithm. Having lower degree, its global dynamics might be more tractable. For instance, this map has a kind of ‘critical finiteness’ that other \( \zeta_4 \) 4-maps do not share. I plan to examine this map in more dynamical detail in a forthcoming paper.

**Acknowledgements**

The National Science Foundation supported this work with an International Research Fellowship (Award 9901230) for study at the University of Warwick during 1999–2000. There I had the benefit of discussions with Stefano Luzzatto, Anthony Manning, and Sebastian van Strien.

As always, Peter Doyle contributed much to this research.

**Appendix. Basin portraits**

The plots that follow are productions of the program *Dynamics 2* (Hunt and Kostelich 1997) running on a Dell Dimension XPS with a Pentium II processor. Its BA and BAS routines produced the figures. (See Nusse and Yorke (1998).) Briefly, each procedure divides the screen into a grid of cells and then colours each cell according to which attracting point its trajectory approaches. If it finds no such attractor after a specified number of iterations—usually 60, the cell is black. The BA algorithm looks for an attractor whereas BAS requires the user to specify a candidate attracting set of points. Each portrait exhibits the highest resolution available—a 720 × 720 grid.

All of the images show \( g_4 \) restricted to either a \( \mathbb{CP}^1 \) or \( \mathbb{RP}^2 \) that it preserves. Some restricted maps have attracting sites that are not the 8-points. However, none of the detected ‘restricted attractors’ other than the 8-points themselves are ‘overall’ attractors with 6-dimensional basins.

*Figure A1.* When restricted to the line \( \mathcal{L}_{168}^1 \), \( g_4 \) has only trivial symmetry. Consequently, there is no natural choice of coordinates in which to express the restriction to the line. With an 8-point at 0 and a 28-point \( q_{28}^8 \) at \( \infty \), the map takes the form

\[
    z \mapsto -\frac{z^3(3z + 4)}{4(z^2 + 8z + 14)}.
\]

The two basins that appear are associated with the 8-point and 28-point. The latter is an ‘equatorial’ saddle point on the 28-line \( \mathcal{L}_{28}^8 \) whose basin in \( \mathcal{L}_{168}^1 \) is a slice of its five-dimensional stable manifold. The reflective symmetry that appears is due to the map’s antiholomorphic equivariance (see section 3.4).

Not pictured is the portrait for \( \mathcal{L}_{210}^1 \) on which the map takes the form

\[
    z \mapsto -z^2 \frac{3z + 1}{z + 3}.
\]

As in the case of a 28-line, this map has two basins: \( \{ |z| < 1 \} \) and \( \{ |z| > 1 \} \). The attracting points are of the types

\[
    [1, 1, -1, -1, 0, 0, 0, 0] \quad \text{and} \quad [1, 1, 1, 1, -1, -1, -1, -1].
\]
Overall, these behave as saddles. In fact, at the latter point, $g_4$ blows up on to the associated hyperplane

$$\{x_1 + x_2 + x_3 + x_4 = 0\} \cap \{x_5 + x_6 + x_7 + x_8 = 0\}.$$

**Figure A2.** This shows $g_4$ on $\mathcal{L}_{128}^1$ in the form

$$z \mapsto \frac{8z^3}{5z^4 + 6z^2 - 3}.$$ The basin at 0 is due to a 28-point $p_{g_4}^{28}$. This point belongs to $\mathcal{L}_{28}^1$ and is repelling on that line. Thus, the basin pictured here is a one-dimensional slice of the five-dimensional stable manifold attached to the point. At $\pm 1$ we see petals due to rationally indifferent points of type

$$[3, 3, 3, 3, 3, -5, -5, -5].$$

The indifferent local behaviour at these points is evident in this and several subsequent images.

**Figure A3.** On $\mathcal{M}_{280}^1$, $g_4$ is a polynomial map expressible by

$$z \mapsto -z(7z^2 - 5z + 1).$$

Here, an 8-point is at $\infty$ (the dark basin) and, once again, a rationally indifferent point of type

$$[3, 3, 3, 3, 3, -5, -5, -5]$$

appears at 0.
Figure A4. We see the $S_3$-symmetric restriction to the $\mathbb{RP}^2$ intersection of $L_{56}^2$ and $\mathcal{R}$. The three basins result from a triple of 8-points arranged symmetrically on the unit circle. A line of reflective symmetry passes through each of the 8-points and (0,0)—which corresponds to a point of type $\frac{1}{2}$. Solving the octic in six dimensions.
These lines are $\mathbb{RP}^1$ intersections of an $L_{168}^1$ with the $\mathbb{RP}^2$. Each $\mathbb{RP}^1$ contains points in the basin of an 8-point and as well as points in the boundary between the basins of the other two 8-points. This interval lies in the basin of a 28-point $q_{ij}^{28}$ as seen in figure A1.

**Figure A4.** Dynamics of $g_4$ on $L_{56}^2 \cap \mathcal{R}$.

$[3, 3, 3, 3, 3, -5, -5, -5]$.

**Figure A5.** The image displays the basins of the $S_4$-symmetric map on the intersection of $L_{105}^2$ and $\mathcal{R}$. Each $L_{105}^2$ is canonically associated with a $\mathbb{CP}^3$, $L_{105}^3$, whose points have coordinates that come in four mutually negative pairs:

$$\{x_i = x_j\} \cap \{x_k = x_\ell\} \longleftrightarrow \{x_i = -x_j\} \cap \{x_k = -x_\ell\}$$

$$\cap \{x_m = x_n\} \cap \{x_p = x_q\} \quad \cap \{x_m = -x_n\} \cap \{x_p = -x_q\}.$$  

Under $g_4$, $L_{105}^3$ blows down to $L_{105}^2$. Furthermore, each point in the orbit of $[1, -1, 1, -1, 1, -1, 1, -1]$ belongs to 24 of the $L_{105}^3$ so that the blowing down of each $\mathbb{CP}^3$ forces all coordinates of the image to be equal. Hence, they must all vanish.

To describe things explicitly, take the plane

$$L_{105}^2 = \{x_1 = x_2\} \cap \{x_3 = x_4\} \cap \{x_5 = x_6\} \cap \{x_7 = x_8\}$$

parameterized by

$$[x, y, z] \longrightarrow [x + y + z, x + y + z, x - y - z, x - y - z, -x - y + z, -x - y + z, -x + y - z, -x, +y - z].$$  

There is a 3-point orbit at
corresponding to the points

\[ [1,1,1,1,-1,-1,-1,-1], [1,1,-1,-1,-1,1,1], [1,1,1,1,1,1,-1,1,1] \]

and a 4-point orbit at \([\pm 1, \pm 1, 1]\) corresponding to the 28-points

\[ q^{28}_{12} = [-3,-3,1,1,1,1,1,1], q^{28}_{34} = [1,1,-3,-3,1,1,1,1] \]
\[ q^{28}_{56} = [1,1,1,1,-3,-3,1,1], q^{28}_{78} = [1,1,1,1,1,1,-3,-3]. \]

Associated with the 3-points are the lines

\[ \{x=0\}, \{y=0\}, \{z=0\} \]

corresponding to intersections with three of the \(L^3_{105}\):

\[ [u,u,-u,-u,v,v,-v,-v], [u,u,v,v,-v,-v,-u,-u], [u,u,v,-u,-u,-v,v]. \]

In these coordinates, the map has the expression

\[ [x,y,z] \rightarrow [yz(4x^2 + y^2 + z^2), xz(x^2 + 4y^2 + z^2), xy(x^2 + y^2 + 4z^2)]. \]

The attractor here seems to consist of four 28-points \(q^{28}_{ij}\) at \((\pm 1, \pm 1, \pm 1)\). The points \(q^{28}_{ij}\) are on \(L^1_{28}\), and, thus, repel in directions away from the picture plane. Here we see off-line directions relative to \(L^1_{28}\). Each \(q^{28}_{ij}\) lies on 15 of the \(L^2_{105}\) and their ‘restricted basins’ in the \(\mathbb{CP}^2\) belong to their stable manifolds as well as to the overall Julia set of \(g_4\). Hence, these pieces of the Julia set have zero measure in \(\mathbb{CP}^6\). As for real dynamics, the basins appear to be the four quadrants

\[ \{x > 0, y > 0\}, \{x < 0, y > 0\}, \{x < 0, y < 0\}, \{x > 0, y < 0\} \]
which are forward and, thus, totally invariant. Accordingly, the coordinate axes form the basin boundaries.

Finally, each coordinate axis blows down to its companion point while a 3-point blows up on to its associated axis.
Figure A6. For the $\mathbb{Z}_2^2$ map on the intersection of $L_{280}^2$ and $R$ the two prominent basins belong to a pair of 8-points at $(\pm 1,0)$. The other two attracted regions are associated with a pair of indifferent points (both eigenvalues are $-1$) of type $[3,3,3,3,3,-5,-5,-5]$.

Figure A7. On the $\mathbb{Z}_2$-symmetric restriction to the intersection of $L_{420}^2$ and $R$, two 8-points at $(\pm 1,0)$ and a $q_{ij}^{28}$ at $(0,-1)$ account for the basins.

Figure A8. As for the behaviour of the $\mathbb{Z}_2$-symmetric map on the intersection of $L_{540}^2$ and $R$, there is an 8-point at $(0,0)$ and a pair of 28-points $q_{ij}^{28}, q_{k\ell}^{28}$ at $(\pm 1,1)$. Trajectories starting in the thin vertical black strip near the bottom middle require too many iterations in order to converge to one of the three attractors for the BA routine to detect attraction. Experimental results suggest that this region is filled with rather small components of the three basins.

References
Crass, S., 1999, Solving the sextic by iteration: a study in complex geometry and dynamics. *Experimental Mathematics*, 8(3): 209–240.
Crass, S., 2000, *Mathematica* notebooks and supporting files for the solution of the octic. See http://www.csulb.edu/~scrass/Math/Octic/Solve.
Crass, S., 2001, Solving the quintic by iteration in three dimensions. *Experimental Mathematics*, 10(1): 1–24.
Hunt, B., and Kostelich, E., 1997, *Dynamics 2* computer program.
Klein, F., 1913, *Lectures on the Icosahedron and the Solutions of Equations of the Fifth Degree* (London: Kegan Paul). Reprint by Dover, New York. Translation by G. Morrice of *Vorlesungen über das Ikosaeder und die Auflösungen der Gleichungen vom fünften Grade* (Leipzig: Teubner, 1884).
Nusse, H., and York, J., 1998, *Dynamics: Numerical Explorations*, 2nd edn (Berlin: Springer-Verlag).
Orik, P., and Terao, H., 1992, *Arrangements of Hyperplanes* (Berlin: Springer-Verlag).
Shephard, G. C., and Todd, T. A., 1954, Finite unitary reflection groups. *Canadian Journal of Mathematics*, 6: 274–304.

Wiman, A., 1899. Ueber die Darstellung der symmetrischen und alternirenden Vertauschungsgruppen als Collineationsgruppen von möglichst geringer Dimensionzahl. *Mathematische Annalen*, 52: 243–270.