Differential equation and inequalities of the generalized $k$-Bessel functions

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Abstract
In this paper, we introduce and study a generalization of the $k$-Bessel function of order $\nu$ given by

$$W_{\nu,c}^k(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)!} \left(\frac{x}{2}\right)^{2r + \frac{\nu}{k}}.$$ 

We also indicate some representation formulae for the function introduced. Further, we show that the function $W_{\nu,c}^k$ is a solution of a second-order differential equation. We investigate monotonicity and log-convexity properties of the generalized $k$-Bessel function $W_{\nu,c}^k$, particularly, in the case $c = -1$. We establish several inequalities, including a Turán-type inequality. We propose an open problem regarding the pattern of the zeroes of $W_{\nu,c}^k$.

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1 Introductions
Motivated with the repeated appearance of the expression

$$x(x + k)(x + 2k) \cdots (x + (n - 1)k)$$

in the combinatorics of creation and annihilation operators [13, 14] and the perturbative computation of Feynman integrals (see [12]), a generalization of the well-known Pochhammer symbols is given in [15] as

$$(x)_{n,k} := x(x + k)(x + 2k) \cdots (x + (n - 1)k),$$

for all $k > 0$, calling it the Pochhammer $k$-symbol. Closely associated functions that have relation with the Pochhammer symbols are the gamma and beta functions. Hence it is useful to recall some facts about the $k$-gamma and $k$-beta functions. The $k$-gamma function, denoted as $\Gamma_k$, is studied in [15] and defined by

$$\Gamma_k(x) := \int_0^\infty t^{x-1} e^{-\frac{t}{k}} \, dt \quad (1.1)$$

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for \( \text{Re}(x) > 0 \). Several properties of the \( k \)-gamma functions and applications in generalizing other related functions like \( k \)-beta and \( k \)-digamma functions can be found in [15, 27, 28] and references therein.

The \( k \)-digamma functions defined by \( \Psi_k := \Gamma'_k / \Gamma_k \) are studied in [28]. These functions have the series representation

\[
\Psi_k(t) := \frac{\log(k) - \gamma_1}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk + t)},
\]

where \( \gamma_1 \) is the Euler–Mascheroni constant.

A calculation yields

\[
\Psi_k'(t) = \sum_{n=0}^{\infty} \frac{1}{(nk + t)^2}, \quad k > 0 \text{ and } t > 0.
\]

Clearly, \( \Psi_k \) is increasing on \((0, \infty)\).

The Bessel function of order \( p \) given by

\[
J_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + p + 1) \Gamma(k + 1)} \left( \frac{x}{2} \right)^{2k+p}
\]

is a particular solution of the Bessel differential equation

\[
x^2 y''(x) + xy'(x) + \left(x^2 - p^2\right)y(x) = 0.
\]

Here \( \Gamma' \) denotes the gamma function. A solution of the modified Bessel equation

\[
x^2 y''(x) + xy'(x) - \left(x^2 + \nu^2\right)y(x) = 0,
\]

is the modified Bessel function

\[
I_\nu(x) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \nu + 1) \Gamma(k + 1)} \left( \frac{x}{2} \right)^{2k+\nu}.
\]

The Bessel function has several generalizations (see, e.g., [9, 10]) and is notably investigated in [1, 17]. In [1], a generalized Bessel function is defined in the complex plane, and sufficient conditions for it to be univalent, starlike, close-to-convex, or convex are obtained. This generalization is given by the power series

\[
W_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k \left( \frac{x}{2} \right)^{2k+p+1}}{\Gamma(k+1) \Gamma(k + p + b + c)}, \quad p, b, c \in \mathbb{C}.
\]

In this paper, we consider the function defined by the series

\[
W^k_{\nu,c}(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_r(k + \nu + k)} \left( \frac{x}{2} \right)^{2r+\nu},
\]
where \( k > 0, \nu > -1 \), and \( c \in \mathbb{R} \). As \( k \to 1 \), the \( k \)-Bessel function \( W^k_{\nu,1} \) is reduced to the classical Bessel function \( I_\nu \), whereas \( W^k_{\nu,-1} \) coincides with the modified Bessel function \( I_\nu \).

Thus, we call the function \( W^k_{\nu,c} \) the generalized \( k \)-Bessel function. Basic properties of the \( k \)-Bessel and related functions can be found in recent works [8, 19–21].

Turán [30] proved that the Legendre polynomials \( P_n(x) \) satisfy the determinantal inequality
\[
\begin{vmatrix}
P_n(x) & P_{n+1}(x) \\
P_{n+1}(x) & P_{n+2}(x)
\end{vmatrix} \leq 0, \quad -1 \leq x \leq 1,
\]
where \( n = 0, 1, 2, \ldots \), and the equality occurs only for \( x = \pm 1 \). The inequalities similar to (1.10) can be found in the literature [2, 3, 5, 11, 16, 25] for several other functions, for example, ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, and the polygamma function. Karlin and Szegő [24] named determinants in (1.10) as Turánians. More details about Turánians can be found in [5, 11, 18, 22, 23, 29].

The aim of this paper is to investigate the influence of the \( \Gamma_k \) functions on the properties of the \( k \)-Bessel function defined in (1.9). It is shown that the properties of the classical Bessel functions can be extended to the \( k \)-Bessel functions. Moreover, we investigate the effects of \( \Gamma_k \) instead of \( \Gamma \) on the monotonicity and log-convexity properties and related inequalities of the \( k \)-Bessel functions. The outcomes of our investigation are presented as follows.

In Section 2, we derive representation formulae and some recurrence relations for \( W^k_{\nu,c} \). More importantly, the function \( W^k_{\nu,c} \) is shown to be a solution of a certain differential equation of second order, which contains (1.5) and (1.6) for the particular case \( k = 1 \) and for particular values of \( c \). At the end of Section 2, we give two types of integral representations for \( W^k_{\nu,c} \).

Section 3 is devoted to the investigation of monotonicity and log-convexity properties of the functions \( W^k_{\nu,c} \) and to relation between two \( k \)-Bessel functions of different order. As a consequence, we deduce Turán-type inequalities.

In Section 4, we give concluding remarks and list two tables for the zeroes of \( W^k_{\nu,c} \), leading to an open problem for future studies.

2 Representations for the \( k \)-Bessel function

2.1 The \( k \)-Bessel differential equation

In this section, we find differential equations corresponding to the functions \( W^k_{\nu,c} \).

**Proposition 2.1** Let \( k > 0 \) and \( \nu > -k \). Then the function \( W^k_{\nu,c} \) is a solution of the homogeneous differential equation
\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \frac{1}{k^2} \left( cx^2 - v^2 \right) y = 0.
\]

**Proof** Differentiating both sides of (1.9) with respect to \( x \), it follows that
\[
\frac{d}{dx} W^k_{\nu,c}(x) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \frac{\nu}{k})}{\Gamma_k(rk + \nu + k)!} \left( x^{2r+\frac{\nu}{k}-1} \right).
\]
This implies

\[ x \frac{d}{dx} W_{k, c}^\nu (x) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \frac{\nu}{2})}{\Gamma_k (r k + \nu + k)} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \]  

(2.2)

Now differentiating (2.2) with respect to \( x \) and then using the property \( \Gamma_k (z + k) = z \Gamma_k (z) \) of the \( k \)-gamma function yield

\[
\begin{align*}
    x^2 \frac{d^2}{dx^2} W_{k, c}^\nu (x) &+ x \frac{d}{dx} W_{k, c}^\nu (x) \\
    &= \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \frac{\nu}{2})^2}{\Gamma_k (r k + \nu + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &= \sum_{r=1}^{\infty} \frac{(-c)^r 4r (r + \frac{\nu}{2})}{\Gamma_k (r k + \nu + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &\quad + \frac{\nu^2}{k^2} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + \nu + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &= \frac{4}{k} \sum_{r=1}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + \nu) (r - 1)!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &\quad + \frac{\nu^2}{k^2} W_{k, c}^\nu (x) \\
    &= -\frac{c x^2}{k} W_{k, c}^\nu (x) + \frac{\nu^2}{k^2} W_{k, c}^\nu (x).
\end{align*}
\]

A further simplification leads to the differential equation (2.1).

\[ \Box \]

2.2 Recurrence relations

From (2.2) we have

\[
\begin{align*}
    x \frac{d}{dx} W_{k, c}^\nu (x) &= \frac{1}{k} \sum_{r=0}^{\infty} \frac{(-c)^r (2r k + \nu)}{\Gamma_k (r k + \nu + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &= \frac{1}{k} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + \nu + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &\quad + 2 \sum_{r=1}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + \nu + k) (r - 1)!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &= \frac{1}{k} W_{k, c}^\nu (x) + 2 \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + \nu + 2k) r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{2}} \\
    &= \frac{1}{k} W_{k, c}^\nu (x) - x c W_{k, c}^\nu (x).
\end{align*}
\]

Thus we have the difference equation

\[ x \frac{d}{dx} W_{k, c}^\nu (x) = \frac{1}{k} W_{k, c}^\nu (x) - x c W_{k, c}^\nu (x). \]  

(2.3)
Again, rewrite (2.2) as

\[
x \frac{d}{dx} x^v \psi_{v,c}^k(x) = \frac{1}{k} \sum_{r=0}^{\infty} \frac{(-c)^r (2r k + 2v)}{\Gamma_k (r k + v + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{v}{k}}
\]

\[
= -\frac{v}{k} x \sum_{r=0}^{\infty} \frac{(-c)^r (2r k + 2v)}{\Gamma_k (r k + v + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{v}{k}} + 2 \sum_{r=0}^{\infty} \frac{(-c)^r (r k + v)}{\Gamma_k (r k + v + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{v}{k}}
\]

\[
= -\frac{v}{k} \psi_{v,c}^k (x) + \frac{x}{k} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + v - k + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{v}{k}}
\]

\[
= -\frac{v}{k} \psi_{v,c}^k (x) + \frac{x}{k} \psi_{v-k,c}^k (x).
\]

This gives us the second difference equation

\[
x \frac{d}{dx} x^v \psi_{v,c}^k (x) = \frac{x}{k} \psi_{v-k,c}^k (x) - \frac{v}{k} \psi_{v,c}^k (x).
\]

(2.4)

Thus (2.3) and (2.4) lead to the following recurrence relations.

**Proposition 2.2** Let \( k > 0 \) and \( v > -k \). Then

\[
2v \psi_{v,c}^k (x) = x \psi_{v-k,c}^k (x) + x c k \psi_{v+k,c}^k (x),
\]

(2.5)

\[
\psi_{v-k,c}^k (x) = \frac{2}{x} \sum_{r=0}^{\infty} (-1)^r (v + 2r k) \psi_{v+2r k,c}^k (x),
\]

(2.6)

\[
\frac{d}{dx} \left( x^v \psi_{v,k}^c (x) \right) = \frac{x^v}{k} \psi_{v-k,c}^k (x),
\]

(2.7)

\[
\frac{d}{dx} \left( x^v \psi_{v,c}^k (x) \right) = -c x^v \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k (r k + v + k) r!} \left( \frac{x}{2} \right)^{2r + \frac{v}{k}},
\]

(2.8)

\[
\frac{d^m}{dx^m} \left( \psi_{v,c}^k (x) \right) = \frac{1}{2^m m!} \sum_{n=0}^{m} \binom{m}{n} c^m x^v \psi_{v-m k + 2nk,c}^k (x) \quad \text{for all } m \in \mathbb{N}.
\]

(2.9)

**Proof** Relation (2.5) follows by subtracting (2.4) from (2.3).

Next to establish (2.6), let us rewrite (2.5) as

\[
\psi_{v-k,c}^k (x) + c k \psi_{v+k,c}^k (x) = 2 \frac{v}{x} \psi_{v,c}^k (x).
\]

(2.10)

Now multiply both sides of (2.10) by \(-c k\) and replace \( v \) by \( v + 2k \). Then we have

\[
-c k \psi_{v+k,c}^k (x) - c^2 k^2 \psi_{v+3k,c}^k (x) = -2 k x \frac{v + 2k}{x} \psi_{v+2k,c}^k (x).
\]

(2.11)

Similarly, multiplying both sides of (2.10) by \( c^2 k^2 \) and replacing \( v \) by \( v + 4k \) give

\[
c^2 k^2 \psi_{v+3k,c}^k (x) + c^3 k^3 \psi_{v+5k,c}^k (x) = 2 c^2 k^2 \frac{v + 4k}{x} \psi_{v+4k,c}^k (x).
\]

(2.12)

Continuing and adding them lead to (2.6).
From definition (1.9) it is clear that
\[ x^{\nu} W_{\nu}^{k}(x) = \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)^{2r + \frac{2r}{k}}} (x)^{2r + \frac{2r}{k}}. \] (2.13)

The derivative of (2.13) with respect to \( x \) is
\[
\frac{d}{dx} \left( x^{\nu} W_{\nu}^{k}(x) \right) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \frac{2r}{k})}{\Gamma_k(rk + \nu + k)^{2r + \frac{2r}{k} + 1}} (x)^{2r + \frac{2r}{k} + 1}
\]
\[
= \frac{x^{\nu}}{k} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)^{2r + \frac{2r}{k} + 1}} \left( \frac{x}{2} \right)^{2r + \frac{2r}{k} + 1}
\]
\[
= \frac{x^{\nu} W_{\nu}^{k}}{k} (x).
\]

Similarly,
\[
\frac{d}{dx} \left( x^{\nu} W_{\nu}^{k}(x) \right) = \sum_{r=1}^{\infty} \frac{(-c)^r 2r}{\Gamma_k(rk + \nu + k)^{2r + \frac{2r}{k} + 1}} (x)^{2r - 1}
\]
\[
= x^{\nu} \frac{x^{\nu}}{k} \sum_{r=1}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)(r - 1)(r)} \left( \frac{x}{2} \right)^{2r - 1}
\]
\[
= x^{\nu} \frac{x^{\nu}}{k} \sum_{r=0}^{\infty} \frac{(-c)^{r+1}}{\Gamma_k(rk + \nu + 2k)^{2r + \frac{2r}{k} + 1}} \left( \frac{x}{2} \right)^{2r + \frac{2r}{k} + 1}
\]
\[
= -cx^{\nu} W_{\nu}^{k}(x).
\]

Identity (2.9) can be proved by using mathematical induction on \( m \). Recall that
\[
\binom{r}{r} = \binom{r}{0} = 1
\]
and
\[
\binom{r}{n} + \binom{r}{n-1} = \binom{r+1}{n}.
\]

For \( m = 1 \), the proof of identity (2.9) is equivalent to showing that
\[
2k \frac{d}{dx} W_{\nu}^{k}(x) = W_{\nu-k}^{k}(x) - cW_{\nu-k}^{k}(x).
\] (2.14)

This relation can be obtained by simply adding (2.3) and (2.4). Thus, identity (2.9) holds for \( m = 1 \).

Assume that identity (2.9) also holds for any \( m = r \geq 2 \), that is,
\[
\frac{d^r}{dx^r} W_{\nu}^{k}(x) = \frac{1}{2^m k^r} \sum_{n=0}^{r} (-1)^n \binom{r}{n} c^r k^n W_{\nu-rk + 2nk}^{k}(x).
\]
This implies, for \( m = r + 1 \),

\[
\frac{d^{r+1}}{dx^{r+1}}(W^K(x)) = \frac{1}{2^r r!} \sum_{n=0}^{r} (-1)^n \left( \begin{array}{c} r \\ n \end{array} \right) c^{n}x^n \frac{d^n}{dx^n}I^K_{x-rk+2nk_c}(x)
\]

\[
= \frac{1}{2^{r+1} (r+1)!} \sum_{n=0}^{r} (-1)^n \left( \begin{array}{c} r+1 \\ n \end{array} \right) c^{n}x^n (W^K_{x-(r+1)k+2nk_c}(x) - cK \psi^K_{x-(r+1)k+2nk_c}(x))
\]

\[
= \frac{1}{2^{r+1} (r+1)!} \sum_{n=0}^{r} (-1)^n \left( \begin{array}{c} r+1 \\ n \end{array} \right) c^{n}x^n W^K_{x-(r+1)k+2nk_c}(x)
\]

Hence, identity (2.9) is concluded by the mathematical induction on \( m \).

\[\square\]

### 2.3 Integral representations of \( \kappa \)-Bessel functions

Now we will derive two integral representations of the functions \( W^K_{x,c} \). For this purpose, we need to recall the \( \kappa \)-Beta functions from [15]. The \( \kappa \) version of the beta functions is defined by

\[
B^K(x, y) = \frac{\Gamma^K(x) \Gamma^K(y)}{\Gamma^K(x + y)} = \frac{1}{\kappa} \int_{0}^{1} t^{x-1}(1-t)^{y-1} \, dt.
\]

(2.15)

Substituting \( t \) by \( t^2 \) on the integral in (2.15), it follows that

\[
B^K(x, y) = \frac{2}{\kappa} \int_{0}^{1} t^{x-1}(1-t)^{y-1} t^{2-1} \, dt.
\]

(2.16)
Let \( x = (r + 1)\alpha \) and \( y = \nu \). Then from (2.15) and (2.16) we have

\[
\frac{1}{\Gamma_k(r\kappa + v + k)} = \frac{2}{\Gamma_k((r + 1)\kappa)\Gamma_k(v)} \int_0^1 t^{2r+1}(1 - t^2)^{\frac{\nu}{2} - 1} \, dt. \quad (2.17)
\]

According to [15], we have the identity \( \Gamma_k(kx) = k^x\Gamma(x) \). This gives

\[
\frac{1}{\Gamma_k(r\kappa + v + k)} = \frac{2}{k\Gamma(r + 1)\Gamma_k(v)} \int_0^1 t^{2r+1}(1 - t^2)^{\frac{\nu}{2} - 1} \, dt. \quad (2.18)
\]

Now (1.9) and (2.18) together yield the first integral representation

\[
W^k_{\nu,c}(x) = \frac{2}{\Gamma_k(v)} \left(\frac{x}{2}\right)^{\frac{\nu}{2}} \int_0^1 t(1 - t^2)^{\frac{\nu}{2} - 1} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r + 1)r!} \left(\frac{xt}{2\sqrt{k}}\right)^{2r} \, dt
\]

\[
= \frac{2}{\Gamma_k(v)} \left(\frac{x}{2}\right)^{\frac{\nu}{2}} \int_0^1 t(1 - t^2)^{\frac{\nu}{2} - 1} W_{0,1/2,r}(\frac{xt}{\sqrt{k}}) \, dt, \quad (2.19)
\]

where \( W_{p,b,c} \) is defined in (1.8).

For the second integral representation, substitute \( x = r + \kappa/2 \) and \( y = \nu + \kappa/2 \) into (2.16). Then (2.17) can be rewritten as

\[
\frac{1}{\Gamma_k(r\kappa + v + k)} = \frac{2}{\Gamma_k((r + \frac{1}{2})\kappa)\Gamma_k(v + \frac{1}{2})} \int_0^1 t^{2r}(1 - t^2)^{\frac{\nu}{2} - 1} \, dt. \quad (2.20)
\]

Again, the identity \( \Gamma_k(kx) = k^x\Gamma(x) \) yields

\[
\Gamma_k\left(r + \frac{1}{2}\right) = k^{r-\frac{1}{2}}\Gamma\left(r + \frac{1}{2}\right). \quad (2.21)
\]

Further, the Legendre duplication formula (see [4, 6])

\[
\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \quad (2.22)
\]

shows that

\[
\Gamma\left(r + \frac{1}{2}\right)r! = r\Gamma\left(r + \frac{1}{2}\right)\Gamma(r) = \frac{\sqrt{\pi}(2r)!}{2^{2r}}.
\]

This, together with (2.20) and (2.21), reduces the series (1.9) of \( W^k_{\nu,c} \) to

\[
W^k_{\nu,c}(x) = \frac{2\sqrt{\kappa}}{\Gamma_k(v + \frac{1}{2}} \left(\frac{x}{2}\right)^{\frac{\nu}{2}} \int_0^1 (1 - t^2)^{\frac{\nu}{2} - 1} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r + 1)r!} \left(\frac{xt}{2\sqrt{k}}\right)^{2r} \, dt
\]

\[
= \frac{2\sqrt{\kappa}}{\sqrt{\pi}\Gamma_k(v + \frac{1}{2}} \left(\frac{x}{2}\right)^{\frac{\nu}{2}} \int_0^1 (1 - t^2)^{\frac{\nu}{2} - 1} \sum_{r=0}^{\infty} \frac{(-c)^r}{(2r)!} \left(\frac{xt}{\sqrt{k}}\right)^{2r} \, dt. \quad (2.23)
\]

Finally, for \( c = \pm\alpha^2, \alpha \in \mathbb{R} \), representation (2.23) respectively leads to

\[
W^k_{\nu,\alpha^2}(x) = \frac{2\sqrt{\kappa}}{\sqrt{\pi}\Gamma_k(v + \frac{1}{2}} \left(\frac{x}{2}\right)^{\frac{\nu}{2}} \int_0^1 (1 - t^2)^{\frac{\nu}{2} - 1} \cos\left(\frac{\alpha xt}{\sqrt{k}}\right) \, dt \quad (2.24)
\]
and
\[ W_{\nu,\alpha^2}(x) = \frac{2\sqrt{k}}{\sqrt{\pi \Gamma_k(\nu + \frac{2}{2})}} \left( \frac{x}{2} \right)^{\nu} \int_0^1 (1 - t^2)^{\frac{\nu-1}{2}} \cosh \left( \frac{\alpha xt}{\sqrt{k}} \right) dt. \] (2.25)

**Example 2.1** If \( \nu = k/2 \), then from (2.24) computations give the relation between sine and generalized \( k \)-Bessel functions by
\[ \sin \left( \frac{\alpha x}{\sqrt{k}} \right) = \frac{\alpha}{k} \sqrt{\frac{\pi}{x}} W_{\nu,\alpha^2}(x). \]

Similarly, the relation
\[ \sinh \left( \frac{\alpha x}{\sqrt{k}} \right) = \frac{\alpha}{k} \sqrt{\frac{\pi}{x}} W_{\nu,\alpha^2}(x) \]
can be derived from (2.25).

### 3 Monotonicity and log-convexity properties

This section is devoted to discuss the monotonicity and log-convexity properties of the modified \( k \)-Bessel function \( W_{\nu,\alpha^2} \). As consequences of those results, we derive several functional inequalities for \( I_{\nu}^k \).

The following result of Biernacki and Krzyż [7] will be required.

**Lemma 3.1** ([7]) Consider the power series \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \), where \( a_k \in \mathbb{R} \) and \( b_k > 0 \) for all \( k \). Further, suppose that both series converge on \( |x| < r \). If the sequence \( \{a_k/b_k\}_{k \geq 0} \) is increasing (or decreasing), then the function \( x \mapsto f(x)/g(x) \) is also increasing (or decreasing) on \( (0, r) \).

The lemma still holds when both \( f \) and \( g \) are even or both are odd functions.

We now state and prove our main results in this section. Consider the functions
\[ I_{\nu}^k(x) := \left( \frac{2}{x} \right)^{\frac{\nu}{2}} \Gamma_k(\nu + k) I_{\nu}^k(x) = \sum_{r=0}^{\infty} f_r(\nu)x^{2r}, \] (3.1)
where
\[ I_{\nu}^k(x) = W_{\nu,\alpha^2}(x) = \sum_{r=0}^{\infty} \frac{1}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r} \]
and
\[ f_r(\nu) = \frac{\Gamma_k(\nu + k)}{\Gamma_k(rk + \nu + k)4^r r!}. \] (3.2)

Then we have the following properties.

**Theorem 3.1** Let \( k > 0 \). The following results are true for the modified \( k \)-Bessel functions:
(a) If \( \nu \geq \mu = -k \), then the function \( x \mapsto I_{\nu}^k(x)/I_{\mu}^k(x) \) is increasing on \( \mathbb{R} \).
(b) The function \( \nu \mapsto I_{\nu + k}^k(x)/I_{\nu}^k(x) \) is increasing on \( (-k, \infty) \), that is, for \( \nu \geq \mu > -k \),
\[ I_{\nu + k}^k(x)/I_{\nu}^k(x) \geq I_{\mu}^k(x)/I_{\mu + k}^k(x) \] (3.3)
for any fixed \( x > 0 \) and \( k > 0 \).

(c) The function \( v \mapsto I^k_v(x) \) is decreasing and log-convex on \((-\kappa, \infty)\) for each fixed \( x > 0 \).

Proof (a) From (3.1) it follows that

\[
\frac{I^k_v(x)}{I^k_\mu(x)} = \frac{\sum_{r=0}^{\infty} f_r(v) x^{2r}}{\sum_{r=0}^{\infty} f_r(\mu) x^{2r}}.
\]

Denote \( w_r := f_r(v)/f_r(\mu) \). Then

\[
w_r = \frac{\Gamma_k(v + k) \Gamma_k(rk + \mu + k)}{\Gamma_k(\mu + k) \Gamma_k(rk + v + k)}.
\]

Now, using the property \( \Gamma_k(y + k) = y \Gamma_k(y) \), we can show that

\[
w_{r+1} = \frac{\Gamma_k(v + k + 1) \Gamma_k(rk + (\mu + 1) + k)}{\Gamma_k(\mu + k + 1) \Gamma_k(rk + v + k + 1)} = \frac{r k + \mu + 1}{r k + v + 1} \leq 1
\]

for all \( v \geq \mu > -k \). Hence, conclusion (a) follows from the Lemma 3.1.

(b) Let \( v \geq \mu > -k \). It follows from part (a) that

\[
\frac{d}{dx} \left( \frac{I^k_v(x)}{I^k_\mu(x)} \right) \geq 0
\]

on \((0, \infty)\). Thus

\[
(I^k_v(x))' \left( I^k_\mu(x) \right) - (I^k_\mu(x))' \left( I^k_v(x) \right) \geq 0.
\]

(3.4)

It now follows from (2.8) that

\[
\frac{x}{2} \left( I^k_{v+k}(x) I^k_\mu(x) - I^k_\mu(x) I^k_{v+k}(x) \right) \geq 0,
\]

whence \( I^k_{v+k}/I^k_v \) is increasing for \( v > -k \) and for some fixed \( x > 0 \), which concludes (b).

(c) It is clear that, for all \( v > -k \),

\[
f_r(v) = \frac{\Gamma_k(v + k)}{\Gamma_k(rk + v + k) 4^r r!} > 0.
\]

A logarithmic differentiation of \( f_r(v) \) with respect to \( v \) yields

\[
\frac{f'_r(v)}{f_r(v)} = \Psi_k(v + k) - \Psi_k(rk + v + k) \leq 0
\]

since \( \Psi_k \) are increasing functions on \((-k, \infty)\). This implies that \( f_r(v) \) is decreasing.

Thus, for \( \mu \geq v > -k \), it follows that

\[
\sum_{r=0}^{\infty} f_r(v) x^{2r} \geq \sum_{r=0}^{\infty} f_r(\mu) x^{2r},
\]
which is equivalent to say that the function \( \nu \mapsto I_k \nu \) is decreasing on \((-k, \infty)\) for some fixed \( x > 0 \).

The twice logarithmic differentiation of \( f_r(\nu) \) yields

\[
\frac{\partial^2}{\partial \nu^2} (\log(f_r(\nu))) = \Psi_k'(\nu + k) - \Psi_k'(r\nu + k)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{(nk + v + k)^2} - \frac{1}{(nk + r\nu + v + k)^2} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{r\nu(2nk + r\nu + 2v + 2k)}{(nk + v + k)^2(nk + r\nu + v + k)^2} \geq 0
\]

for all \( k > 0 \) and \( \nu > -k \). Since, a sum of log-convex functions is log-convex, it follows that \( \nu \mapsto I_k \nu \) is log-convex on \((-k, \infty)\) for each fixed \( x > 0 \).

\[\square\]

**Remark 3.1** One of the most significant consequences of the Theorem 3.1 is the Turán-type inequality for the function \( I_k \nu \). From the definition of log-convexity it follows from Theorem 3.1(c) that

\[
I_{k,\alpha}(x) := \frac{2x^{\nu}}{\Gamma(1+\alpha)x^{\nu}} \Gamma_k(\nu + k)J_k(\nu + k) = \sum_{r=0}^{\infty} g_r(\nu)x^{2r}, \tag{3.7}
\]

where

\[
J_k(\nu) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_k(\nu + k)r!} \left( \frac{x}{2} \right)^{2r+k}
\]

and

\[
g_r(\nu) = \frac{(-1)^r \Gamma_k(\nu + k)}{\Gamma_k(\nu + k)r!}, \tag{3.8}
\]

Our final result is based on the Chebyshev integral inequality [26, p. 40], which states the following: suppose \( f \) and \( g \) are two integrable functions and monotonic in the same sense (either both decreasing or both increasing). Let \( q : (a, b) \to \mathbb{R} \) be a positive integrable function. Then

\[
\left( \int_a^b q(t)f(t) \, dt \right) \left( \int_a^b q(t)g(t) \, dt \right) \leq \left( \int_a^b q(t) \, dt \right) \left( \int_a^b q(t)f(t)g(t) \, dt \right). \tag{3.6}
\]

Inequality (3.6) is reversed if \( f \) and \( g \) are monotonic in the opposite sense.

The following function is required:

\[
J_k(\nu) := \left( \frac{2}{x} \right)^{\nu} \Gamma_k(\nu + k)J_k(\nu) = \sum_{r=0}^{\infty} g_r(\nu)x^{2r}, \tag{3.7}
\]

where

\[
J_k(\nu) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_k(\nu + k)r!} \left( \frac{x}{2} \right)^{2r+k}
\]

and

\[
g_r(\nu) = \frac{(-1)^r \Gamma_k(\nu + k)}{\Gamma_k(\nu + k)r!}, \tag{3.8}
\]
Theorem 3.2 Let $\kappa > 0$. Then, for $\nu \in (-3k/4, -k/2] \cup [k/2, \infty)$,

$$I_{\nu}^{k}(x)I_{\nu+\frac{k}{2}}^{k}(x) \leq \frac{\sqrt{\kappa}}{x} \sin \left(\frac{x}{\kappa}\right)J_{2\nu+\frac{k}{2}}^{k}(x) \quad (3.9)$$

and

$$J_{\nu}^{k}(x)J_{\nu+\frac{k}{2}}^{k}(x) \leq \frac{\sqrt{\kappa}}{x} \sin \left(\frac{x}{\kappa}\right)J_{2\nu+\frac{k}{2}}^{k}(x). \quad (3.10)$$

Inequalities (3.9) and (3.10) are reversed if $\nu \in (-k/2, k/2)$.

Proof Define the functions $q, f,$ and $g$ on $[0, 1]$ as

$$q(t) = \cos \left(\frac{xt}{\sqrt{\kappa}}\right), \quad f(t) = (1 - t^2)^{\frac{\nu-1}{2}}, \quad g(t) = (1 - t^2)^{\frac{\nu+1}{2}}.$$  

Then, for any $x \geq 0$,

$$\int_{0}^{1} q(t) dt = \int_{0}^{1} \cos \left(\frac{xt}{\sqrt{\kappa}}\right) dt = \frac{\sqrt{\kappa}}{x} \sin \left(\frac{x}{\kappa}\right),$$

$$\int_{0}^{1} q(t)f(t) dt = \int_{0}^{1} \cos \left(\frac{xt}{\sqrt{\kappa}}\right)(1 - t^2)^{\frac{\nu-1}{2}} dt = I_{\nu}^{k}(x) \quad \text{if} \ \nu \geq -k,$$

$$\int_{0}^{1} q(t)g(t) dt = \int_{0}^{1} \cos \left(\frac{xt}{\sqrt{\kappa}}\right)(1 - t^2)^{\frac{\nu+1}{2}} dt = I_{\nu+\frac{k}{2}}^{k}(x) \quad \text{if} \ \nu \geq -2k,$$

$$\int_{0}^{1} q(t)f(t)g(t) dt = \int_{0}^{1} \cos \left(\frac{xt}{\sqrt{\kappa}}\right)(1 - t^2)^{\frac{3\nu}{2}} dt = J_{2\nu+\frac{k}{2}}^{k}(x) \quad \text{if} \ \nu \geq -\frac{3k}{4}.$$  

Since the functions $f$ and $g$ both are decreasing for $\nu \geq k/2$ and both are increasing for $\nu \in (-3k/4, -k/2]$, inequality (3.6) yields (3.9). On the other hand, if $\nu \in (-k/2, k/2)$, then the function $f$ is increasing, but $g$ is decreasing, and hence inequality (3.9) is reversed.

Similarly, inequality (3.10) can be derived from (3.6) by choosing

$$q(t) = \cosh \left(\frac{xt}{\sqrt{\kappa}}\right), \quad f(t) = (1 - t^2)^{\frac{\nu-\frac{1}{2}}{2}}, \quad g(t) = (1 - t^2)^{\frac{\nu+\frac{1}{2}}{2}}. \quad \square$$

4 Conclusion

It is shown that the generalized $k$-Bessel functions $W_{\nu}^{k}$ are solutions of a second-order differential equation, which for $k = 1$ is reduced to the well-known second-order Bessel differential equation. It is also proved that the generalized modified $k$-Bessel function $I_{\nu}^{k}$ is decreasing and log-convex on $(-\kappa, \infty)$ for each fixed $x > 0$. Several other inequalities, especially the Turán-type inequality and reverse Turán-type inequality for $I_{\nu}^{k}$ are established.

Furthermore, we investigate the pattern for zeroes of $W_{\nu}^{k,1}$ in two ways: (i) with respect to fixed $k$ and variation of $\nu$ and (ii) with respect to fixed $\nu$ and variation of $k$.

From the data in Table 1 and Table 2, we can observe that the zeroes of $W_{\nu}^{k,1}$ are increasing in both cases. However, we have no any analytical proof for this monotonicity of the zeroes of $W_{\nu}^{k,1}$. As there are several works on the zeroes of the classical Bessel functions,
Table 1 Positive zeroes of $W_{k,1}^\nu$ for fixed $\nu$ and different $k$

| $k$   | 0.5  | 1    | 1.5  | 2    | 2.5  |
|-------|------|------|------|------|------|
| $\nu = -0.4$ and $c = 1$ |
| 1st zero | 0.662422 | 1.75098 | 2.42334 | 2.95334 | 3.40423 |
| 2nd zero | 2.96686  | 4.87852 | 6.24148 | 7.3588  | 8.32849 |
| 3rd zero | 5.2018   | 8.01663 | 10.0812 | 11.7913 | 13.2836 |
| $\nu = 0.5$ and $c = 1$ |
| 1st zero | 2.70943  | 3.14159 | 3.55493 | 3.93277 | 4.28026 |
| 2nd zero | 4.96077  | 6.28319 | 7.38858 | 8.35255 | 9.21757 |
| 3rd zero | 7.19373  | 9.42478 | 11.2315 | 12.7879 | 14.1752 |

Table 2 Positive zeroes of $W_{k,1}^\nu$ for different $\nu$ and $k$

| $\nu$   | -0.4 | -0.3 | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 |
|---------|------|------|---|-----|---|-----|---|-----|
| $k = 0.5$ and $c = 1$ |
| 1st zero | 0.662422 | 0.97534 | 1.70047 | 2.70943 | 3.63143 | 4.51146 | 5.36577 | 6.20238 |
| 2nd zero | 2.96686  | 3.21271 | 3.90328 | 4.96077  | 5.95189 | 6.90209 | 7.82393 | 8.72471 |
| 3rd zero | 5.2018   | 5.43751 | 6.11911 | 7.19373  | 8.21647 | 9.20314 | 10.1629 | 11.1017 |
| $k = 1$ and $c = 1$ |
| 1st zero | 1.75098  | 1.92285 | 2.40483 | 3.14159  | 3.83171 | 4.49341 | 5.13562 | 5.76346 |
| 2nd zero | 4.87852  | 5.04213 | 5.52008 | 6.28319  | 7.01559 | 7.72525 | 8.41724 | 9.09501 |
| 3rd zero | 8.01663  | 8.17785 | 8.65373 | 9.42478  | 10.1735 | 10.9041 | 11.6198 | 12.3229 |
| $k = 1.5$ and $c = 1$ |
| 1st zero | 2.42334  | 2.55767 | 2.9453  | 3.55493  | 4.13426 | 4.69286 | 5.2362  | 5.76774 |
| 2nd zero | 6.24148  | 6.37291 | 6.76069 | 7.38858  | 7.9979  | 8.5923  | 9.1744  | 9.74613 |
| 3rd zero | 10.0812  | 10.2116 | 10.5986 | 11.2315  | 11.8513 | 12.4599 | 13.0587 | 13.6488 |
| $k = 2$ and $c = 1$ |
| 1st zero | 2.95334  | 3.06754 | 3.40094 | 3.93277  | 4.44288 | 4.93703 | 5.41885 | 5.8908 |
| 2nd zero | 7.3588   | 7.47176 | 7.80657 | 8.35255  | 8.88577 | 9.40825 | 9.92154 | 10.4269 |
| 3rd zero | 11.7913  | 11.9037 | 12.2382 | 12.7879  | 13.3286 | 13.8616 | 14.3875 | 14.907 |

The zeroes of $W_{k,1}^\nu$ would be an interesting topic for future investigations. The monotonicity of the zeroes of $W_{k,c}^\nu$ with respect to $c$ and fixed $k, \nu$ will be another open problem for further studies.

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Availability of data and materials
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Competing interests
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Authors’ contributions
Both authors contributed to each part of this work equally, and they both read and approved the final manuscript.

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