Let $X$ be a separable Banach space endowed with a non-degenerate centered Gaussian measure $\mu$. The associated Cameron–Martin space is denoted by $H$. Consider two sufficiently regular convex functions $U : X \to \mathbb{R}$ and $G : X \to \mathbb{R}$. We let $\nu = e^{-U}$ and $\Omega = G^{-1}(-\infty, 0]$. In this paper we are interested in the $W^{2,2}$ regularity of the weak solutions of elliptic equations of the type

$$\lambda u - L_{\nu, \Omega} u = f,$$

where $\lambda > 0$, $f \in L^2(\Omega, \nu)$ and $L_{\nu, \Omega}$ is the self-adjoint operator associated with the quadratic form $(\psi, \varphi) \mapsto \int_{\Omega} \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu$, $\psi, \varphi \in W^{1,2}(\Omega, \nu)$.

In addition we will show that if $u$ is a weak solution of problem (0.1) then it satisfies a Neumann type condition at the boundary, namely for $\rho$-a.e. $x \in G^{-1}(0)$

$$\langle \text{Tr}(\nabla_H u)(x), \text{Tr}(\nabla_H G)(x) \rangle_H = 0,$$

where $\rho$ is the Feyel–de La Pradelle Hausdorff–Gauss surface measure and $\text{Tr}$ is the trace operator.

1. Introduction

Let $X$ be a separable Banach space with norm $\|\cdot\|_X$, endowed with a non-degenerate centered Gaussian measure $\mu$. The associated Cameron–Martin space is denoted by $H$, its inner product by $\langle \cdot, \cdot \rangle_H$ and its norm by $|\cdot|_H$. The spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$ are the classical Sobolev spaces of the Malliavin calculus (see [8]).

In this paper we are interested in the study of maximal Sobolev regularity for the solution $u$ of the problem

$$\lambda u(x) - L_{\nu, \Omega} u(x) = f(x) \quad \mu$-a.e. $x \in \Omega,$

where $\lambda > 0$, $\Omega$ is a convex subset of $X$, $\nu$ is a measure of the form $e^{-U}$ with $U : X \to \mathbb{R}$ a convex function, $f \in L^2(\Omega, \nu)$ and $L_{\nu, \Omega}$ is the operator associated to the quadratic form $(\psi, \varphi) \mapsto \int_{\Omega} \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu$, $\psi, \varphi \in W^{1,2}(\Omega, \nu)$,

where $\nabla_H \psi$ is the gradient along $H$ of $\psi$ and $W^{1,2}(\Omega, \nu)$ is the Sobolev space on $\Omega$ associated to the measure $\nu$ (see Section 2).

We need to clarify what we mean by solution of problem (1.1). We say that $u \in W^{1,2}(\Omega, \nu)$ is a weak solution of problem (1.1) if

$$\lambda \int_{\Omega} u \varphi d\nu + \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu = \int_{\Omega} f \varphi d\nu \quad \text{for every } \varphi \in W^{1,2}(\Omega, \nu).$$

Notice that if the weak solution $u$ of problem (1.1) is unique, then $u = R(\lambda, L_{\nu, \Omega})f$, the resolvent of $L_{\nu, \Omega}$.

Results about existence, uniqueness and regularity of the weak solutions of problem (1.1), in domains with sufficiently regular boundary, are known in the finite dimensional case (see the classical books [24] and [27]).
for a bounded $\Omega$ and [7], [15], [29], [16] and [17] for an unbounded $\Omega$). If $X$ is infinite dimensional separable Hilbert space then some maximal Sobolev regularity results are known. See for example [3] and [4] where $U \equiv 0$ and [19] where $U$ is bounded from below.

When $\Omega = X$ more results are known, see for example [14], [30] and [28] if $X$ is finite dimensional, [18] if $X$ is a Hilbert space and [12] if $X$ is a separable Banach space. If $X$ is general separable Banach space and $\Omega \subset X$, then the only results about maximal Sobolev regularity are the ones in [11], where problem (1.1) was studied when $U \equiv 0$, namely when $L_{\nu,\Omega}$ is the Ornstein–Uhlenbeck operator on $\Omega$. We do not know of any $W^{2,2}$ regularity results for solutions of problem (1.1) in subsets of infinite dimensional Banach spaces in the case $U \neq 0$.

In order to state the main results of this paper we need some hypotheses on the set $\Omega$ and on the weighted measure $\nu$.

**Hypothesis 1.1.** Let $G : X \to \mathbb{R}$ be a $(2,r)$-precise version (see Section 2), for some $r > 1$, of a function belonging to $W^{2,q}(X,\mu)$ for every $q > 1$ and assume

1. $\mu(G^{-1}(-\infty,0]) > 0$ and $G^{-1}(-\infty,0]$ is closed and convex;
2. $\|\nabla H G\|^{-1}_H \in L^q(G^{-1}(-\infty,0],\mu)$ for every $q > 1$.

We set $\Omega := G^{-1}(-\infty,0]$.

We will recall the definition of $H$-closure in Section 2. All our results will be independent on our choice of a precise version of the function $G$ made in Hypothesis 1.1. This hypothesis is taken from [13] in order to define traces of Sobolev functions on level sets of $G$ and to make in Hypothesis 1.1 and the hypothesis contained in [13] are the requirements of closure and convexity of the set $G^{-1}(-\infty,0]$. It will be clear when these two additional assumptions will be useful.

**Hypothesis 1.2.** $U : X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function belonging to $W^{1,2}(X,\mu)$ for some $t > 3$. We set $\nu := e^{-U} \mu$.

The assumption $t > 3$ may sound strange, but it is needed to define the weighted Sobolev spaces $W^{1,2}(X,\nu)$. Indeed observe that, by [1, Lemma 7.5], $e^{-U}$ belongs to $W^{1,r}(X,\mu)$ for every $r < t$. Thus if $U$ satisfies Hypothesis 1.2, then it satisfies [21, Hypothesis 1.1]; namely $e^{-U} \in W^{1,s}(X,\mu)$ for some $s > 1$ and $U \in W^{1,s}(X,\mu)$ for some $r > s'$. Then following [21] it is possible to define the space $W^{1,2}(X,\nu)$ as the domain of the closure of the gradient operator along $H$ (see Section 2 for a in-depth discussion). We want to point out that Hypothesis 1.2 is different from [12, Hypothesis 1.1], since here we require just lower semicontinuity instead of continuity. We simply realized that the continuity hypothesis was not needed. This implies that in the case $\Omega = X$ the results of this paper are also a generalization of the results in [12].

Our main result is a generalization of the main results of both [19] and [11].

**Theorem 1.3.** Assume Hypotheses 1.1 and 1.2 hold. For every $\lambda > 0$ and $f \in L^2(\Omega,\nu)$ problem (1.1) has a unique weak solution $u \in W^{2,2}(\Omega,\nu)$. In addition the following inequalities hold

$$
\|u\|_{L^2(\Omega,\nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\Omega,\nu)}; \quad \|\nabla H u\|_{L^2(\Omega,\nu;H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(\Omega,\nu)};
$$

$$
\|\nabla^2 H u\|_{L^2(\Omega,\nu;H_2)} \leq \sqrt{2} \|f\|_{L^2(\Omega,\nu)},
$$

where $H_2$ is the space of Hilbert–Schmidt operators in $H$. See Definition 2.3 for the definition of the space $W^{2,2}(\Omega,\nu)$.

The idea of the proof of Theorem 1.3 is to approximate the solution of problem (1.1) by the solutions of penalized problems on the whole space. This method was already used in the papers [3], [4] and [19], where the authors used some properties of Hilbert spaces, namely the differentiability of the Moreau–Yosida approximations and of the square of the distance function. Due to the lack of differentiability, at least in general, of the natural norm of a separable Banach space these methods cannot be applied in our case. The idea behind this paper is to replace “differentiability” by “differentiability along $H$” which is sufficient for our goals, because we use a modification of the Moreau–Yosida approximations, which already appeared in [12] (see Section 4), and of the distance function (see Section 3).
The paper is organized in the following way: in Section 2 we recall some basic definitions and we fix the notations. In Section 3 we introduce the distance function \( d_H(\cdot, \mathcal{C}) : X \to \mathbb{R} \cup \{+\infty\} \) defined as
\[
d_H(x, \mathcal{C}) = \begin{cases} \inf \{|h|_H | h \in H \cap (x - \mathcal{C})\} & \text{if } H \cap (x - \mathcal{C}) \neq \emptyset; \\ +\infty & \text{if } H \cap (x - \mathcal{C}) = \emptyset, \end{cases}
\]
where \( \mathcal{C} \) is a Borel subset of \( X \). If \( \mathcal{C} \) is a convex and closed set, we will prove some basic properties of such function that will be useful in the proof of Theorem 1.3. In Section 4 we recall the definition and some properties of the Moreau–Yosida approximations along \( H \) (see [12]). We will also prove an important property of the gradient along \( H \) of the Moreau–Yosida approximations along \( H \). Section 5 is dedicated to the proof of Theorem 1.3, while in Section 6 we will show that if \( u \in W^{2,2}(\Omega, \nu) \) is a weak solution of equation (1.1), then it satisfies a Neumann type condition at the boundary:

**Theorem 1.4.** Assume that Hypotheses 1.1 and 1.2 hold. Let \( \lambda > 0 \), \( f \in L^2(\Omega, \nu) \) and let \( u \in W^{2,2}(\Omega, \nu) \) be the weak solution of problem (1.1), given by Theorem 1.3. Then for \( \rho \)-a.e. \( x \in G^{-1}(0) \)
\[
\langle \text{Tr}(\nabla H u)(x), \text{Tr}(\nabla H G)(x) \rangle_H = 0,
\]
where \( \rho \) is the Feyel–de La Pradelle Hausdorff–Gauss surface measure (see Section 2 for the definition) and \( \text{Tr} \) is the trace operator in the space \( W^{1,2}(X, \nu; H) \) defined in Section 2.

Finally in Section 7 we consider the Banach space \( \mathcal{C}[0, 1] \) of continuous functions on the closed interval \([0, 1]\) with the sup norm, endowed with the classical Wiener measure (see [8, Example 2.3.11 and Remark 2.3.13] for its construction). We study weights of the type
\[
U_1(f) = \Phi \left( \int_0^1 f(\xi)d\tau(\xi) \right), \quad U_2(f) = \int_0^1 \Psi(f(\xi), \xi)d\xi,
\]
where \( \tau \) is a finite Borel measure in \([0, 1]\), \( f \in \mathcal{C}[0, 1] \) and \( \Phi : \mathbb{R} \to \mathbb{R} \) and \( \Psi : \mathbb{R}^2 \to \mathbb{R} \) are sufficiently regular convex functions. In addition \( \Omega \) will be a closed halfspace or the set \( \{ f \in \mathcal{C}[0, 1] | \int_0^1 f^2(\xi)d\xi \leq r \} \) for some \( r > 0 \).

### 2. Notations and preliminaries

We will denote by \( X^* \) the topological dual of \( X \). We recall that \( X^* \subseteq L^2(X, \mu) \). The linear operator \( R_\mu : X^* \to (X^*)' \)
\[
R_\mu x^*(y^*) = \int_X x^*(x)y^*(x)d\mu(x)
\]
(2.1)
is called the covariance operator of \( \mu \). Since \( X \) is separable, then it is actually possible to prove that \( R_\mu : X^* \to X \) (see [8, Theorem 3.2.3]). We denote by \( X^*_\mu \) the closure of \( X^* \) in \( L^2(X, \mu) \). The covariance operator \( R_\mu \) can be extended by continuity to the space \( X^*_\mu \), still by formula (2.1). By [8, Lemma 2.4.1] for every \( h \in H \) there exists a unique \( g \in X^*_\mu \) with \( h = R_\mu g \), in this case we set
\[
\widehat{h} := g.
\]

Throughout the paper we fix an orthonormal basis \( \{ e_i \}_{i \in \mathbb{N}} \) of \( H \) such that \( \widehat{e}_i \) belongs to \( X^* \), for every \( i \in \mathbb{N} \). Such basis exists by [8, Corollary 3.2.8(ii)].

#### 2.1. Differentiability along \( H \)

We say that a function \( f : X \to \mathbb{R} \) is **differentiable along** \( H \) at \( x \) if there is \( v \in H \) such that
\[
\lim_{t \to 0} \frac{f(x + th) - f(x)}{t} = \langle v, h \rangle_H \quad \text{uniformly for } h \in H \text{ with } |h|_H = 1.
\]
In this case the vector \( v \in H \) is unique and we set \( \nabla_H f(x) := v \), moreover for every \( k \in \mathbb{N} \) the derivative of \( f \) in the direction of \( e_k \) exists and it is given by
\[
\partial_k f(x) := \lim_{t \to 0} \frac{f(x + te_k) - f(x)}{t} = \langle \nabla_H f(x), e_k \rangle_H.
\]
We denote by $\mathcal{H}_2$ the space of the Hilbert–Schmidt operators in $H$, that is the space of the bounded linear operators $A : H \to H$ such that $\| A \|_{\mathcal{H}_2}^2 = \sum |Ae_i|^2_H$ is finite (see [20]). We say that a function $f : X \to \mathbb{R}$ is two times differentiable along $H$ at $x$ if it is differentiable along $H$ at $x$ and $A \in \mathcal{H}_2$ exists such that

$$H\lim_{t \to 0} \frac{\nabla_h f(x + th) - \nabla_h f(x)}{t} = Ah \quad \text{uniformly for } h \in \mathbb{R} \text{ with } |h|_H = 1.$$ 

In this case the operator $A$ is unique and we set $\nabla_h f(x) := A$. Moreover for every $i, j \in \mathbb{N}$ we set

$$\partial_{ij} f(x) := \lim_{t \to 0} \frac{\partial_i f(x + te) − \partial_i f(x)}{t} = (\nabla_H f(x)e_j, e_i)_H.$$

2.2. Special classes of functions. For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{E}_k^A(X)$ the space of the cylindrical function of the type $f(x) = \varphi(x_1(x), \ldots, x_n(x))$ where $\varphi \in \mathcal{C}_k^A(\mathbb{R}^n)$ and $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. We remark that $\mathcal{E}_k^\infty(X)$ is dense in $L^p(X, \nu)$ for all $p \geq 1$ (see [21, Proposition 3.6]). We recall that if $f \in \mathcal{E}_k^\infty(X)$, then $\partial_{ij} f(x) = \partial_{ji} f(x)$ for every $i, j \in \mathbb{N}$ and $x \in X$.

If $Y$ is a Banach space, a function $F : X \to Y$ is said to be $H$-Lipschitz if $C > 0$ exists such that

$$\| F(x + h) - F(x) \|_Y \leq C|h|_H,$$

for every $h \in \mathbb{R}$ and $\mu$-a.e. $x \in X$ (see [8, Section 4.5 and Section 5.11]).

A function $F : X \to \mathbb{R}$ is said to be $H$-continuous, if $\lim_{h \to 0^+} F(x + h) = F(x)$, for $\mu$-a.e. $x \in X$.

2.3. Sobolev spaces. The Gaussian Sobolev spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$, with $p \geq 1$, are the completions of the smooth cylindrical functions $\mathcal{E}_k^\infty(X)$ in the norms

$$\| f \|_{W^{1,p}(X, \mu)} := \| f \|_{L^p(X, \mu)} + \left( \int_X \| \nabla_h f(x) \|^p_{H^*} \, d\mu(x) \right)^{\frac{1}{p}};$$

$$\| f \|_{W^{2,p}(X, \mu)} := \| f \|_{W^{1,p}(X, \mu)} + \left( \int_X \| \nabla_H^2 f(x) \|^p_{H^2} \, d\mu(x) \right)^{\frac{1}{p}}.$$ 

Such spaces can be identified with subspaces of $L^p(X, \mu)$ and the (generalized) gradient and Hessian along $H, \nabla_h f$ and $\nabla_H^2 f$, are well defined and belong to $L^p(X, \mu; H^*)$ and $L^p(X, \mu; \mathcal{H}_2)$, respectively. The spaces $W^{1,p}(X, \mu; H)$ are defined in a similar way, replacing smooth cylindrical functions with $H$-valued smooth cylindrical functions (i.e. the linear span of the functions $x \mapsto f(x)h$, where $f$ is a smooth cylindrical function and $h \in \mathbb{H}$). For more information see [8, Section 5.2].

Now we consider $\nabla_H : \mathcal{E}_k^\infty(X) \to L^p(X, \nu; H)$. This operator is closable in $L^p(X, \nu)$ whenever $p > \frac{n}{n-2}$ (see [21, Definition 4.3]). For such $p$ we denote by $W^{1,p}(X, \nu)$ the domain of its closure in $L^p(X, \nu)$. In the same way the operator $(\nabla_H, \nabla_H^2) : \mathcal{E}_k^\infty(X) \to L^p(X, \nu; H) \times L^p(X, \nu; \mathcal{H}_2)$ is closable in $L^p(X, \nu)$, whenever $p > \frac{n}{n-2}$ (see [12, Proposition 2.1]). For such $p$ we denote by $W^{2,p}(X, \nu)$ the domain of its closure in $L^p(X, \nu)$. The spaces $W^{1,p}(X, \nu; H)$ are defined in a similar way, replacing smooth cylindrical functions with $H$-valued smooth cylindrical functions.

We want to point out that if Hypothesis 1.2 holds, then $\frac{n}{n-2} - \frac{1}{2} < 2$. In particular the above discussion allows us to define the Sobolev spaces $W^{1,2}(X, \nu)$ and $W^{2,2}(X, \nu)$.

We shall use the integration by parts formula (see [21, Lemma 4.1]) for $\varphi \in W^{1,p}(X, \nu)$ with $p > \frac{n}{n-2}$:

$$\int_X \partial_k \varphi \, d\nu = \int_X \varphi (\partial_k U + \tilde{e}_k) \, d\nu \quad \text{for every } k \in \mathbb{N},$$

where $\tilde{e}_k$ is defined in formula (2.2).

Throughout the paper we will use the following simplified version of [8, Theorem 5.11.2] several times.

**Theorem 2.1.** Let $Y$ be either $\mathbb{R}$ or $H$, and let $F : X \to Y$ be a measurable $H$-Lipschitz mapping. Then $F \in W^{1,p}(X, \mu; Y)$ for every $p > 1$.
2.4. Capacities and versions. Let $L_p$ be the infinitesimal generator of the Ornstein–Uhlenbeck semigroup in $L^p(X,\mu)$

$$T(t)f(x) := \int_X f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)d\mu(y) \quad \text{for } t > 0.$$\n
The $C_{2,p}$-capacity of an open set $A \subseteq X$ is

$$C_{2,p}(A) := \inf \left\{ \|f\|_{L^p(X,\mu)} \left| (I-L_p)^{-1}f \geq 1 \mu\text{-a.e. in } A \right. \right\}.$$\n
For a general Borel set $B \subseteq X$ we let $C_{2,p}(B) = \inf \{C_{2,p}(A) | B \subseteq A \text{ open} \}$. Let $f \in W^{2,p}(X,\mu)$, $f$ is an equivalence class of functions and we call every element “version”. A version $\overline{f}$ of $f$ exists that is Borel measurable and $C_{2,p}$-quasicontinuous, i.e. for every $\varepsilon > 0$ there exists an open set $A \subseteq X$ such that $C_{2,p}(A) \leq \varepsilon$ and $\overline{f}|_{X-A}$ is continuous. See [8, Theorem 5.9.6]. Such a version is called a $(2,p)$-precise version of $f$. Two precise versions of the same $f$ agree outside sets with null $C_{2,p}$-capacity.

2.5. Sobolev spaces on sublevel sets. The proof of the results recalled in this subsection can be found in [13] and [21]. Let $G$ be a function satisfying Hypothesis 1.1. We are interested in Sobolev spaces on sublevel sets of $G$.

For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{F}C_b^k(\Omega)$ the space of the restriction to $\Omega$ of functions in $\mathcal{F}C_b^k(X)$. The spaces $W^{1,p}(\Omega,\mu)$ and $W^{2,p}(\Omega,\mu)$ for $p \geq 1$ are defined as the domain of the closure of the operators $\nabla H : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega,\mu;H)$ and $(\nabla H,\nabla^2 H) : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega,\mu;H) \times L^p(\Omega,\mu;H_2)$. See [13, Lemma 2.2] and [11, Proposition 1].

We remind the reader that the operator $\nabla H : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega,\mu;H)$ is closable in $L^p(\Omega,\nu)$, whenever $p > \frac{1}{1-\frac{1}{2}}$ (see [21, Proposition 6.1]). For such $p$ we denote by $W^{1,p}(\Omega,\nu)$ the domain of its closure in $L^p(\Omega,\nu)$ and we will still denote by $\nabla H$ the closure operator.

In order to define the spaces $W^{2,p}(\Omega,\nu)$ we need the closability of the operator $(\nabla H,\nabla^2 H)$ in $L^p(\Omega,\nu)$.

**Proposition 2.2.** Assume Hypotheses 1.1 and 1.2 hold. Then for every $p > \frac{1}{1-\frac{1}{2}}$, the operator $(\nabla H,\nabla^2 H) : \mathcal{F}C_b^\infty(\Omega) \to L^p(\Omega,\nu;H) \times L^p(\Omega,\nu;H_2)$ is closable in $L^p(\Omega,\nu)$. The closure will be still denoted by $(\nabla H,\nabla^2 H)$.

**Proof.** Let $(f_k)_{k \in \mathbb{N}} \subseteq \mathcal{F}C_b^\infty(\Omega)$ such that

$$\lim_{k \to +\infty} f_k = 0, \quad \text{in } L^p(\Omega,\nu);$$

$$\lim_{k \to +\infty} \nabla H f_k = F, \quad \text{in } L^p(\Omega,\nu;H);$$

$$\lim_{k \to +\infty} \nabla^2 H f_k = \Phi, \quad \text{in } L^p(\Omega,\nu;H_2).$$

We will assume that the functions $f_k$ are actually smooth cylindrical functions on the whole space $X$. By [21, Proposition 6.1] we have that $F(x) = 0$ for $\nu$-a.e. $x \in \Omega$. We want to prove that $\Phi(x) = 0$ for $\nu$-a.e. $x \in \Omega$. Since the restrictions to $\Omega$ of the elements of $\mathcal{F}C_b^\infty(X)$ are dense in $L^p(\Omega,\nu)$ (as a consequence of the density of $\mathcal{F}C_b^\infty(X)$ in $L^p(X,\nu)$, see [21, Proposition 3.6]), we just have to prove that

$$\int_\Omega \langle \Phi(x)e_j, e_i \rangle u(x)dv(x) = 0$$

holds for every $i,j \in \mathbb{N}$ and $u \in \mathcal{F}C_b^\infty(X)$.

Let $\eta : \mathbb{R} \to \mathbb{R}$ a smooth function such that $\|\eta\|_{\infty} \leq 1$, $\|\eta'\|_{\infty} \leq 2$ and

$$\eta(\xi) = \begin{cases} 0 & \xi \geq -1 \\ 1 & \xi \leq -2 \end{cases}$$

Let $\eta_n(\xi) := \eta(n\xi)$ and $u_n(x) = u(x)\eta_n(G(x))$. Observe that $u_n$ converges pointwise $\nu$-a.e. to $u$ in $\Omega$ and $|u_n| \leq |u|$ $\nu$-a.e., then by Lebesgue’s dominated convergence theorem

$$\lim_{n \to +\infty} \int_\Omega \langle \Phi(x)e_j, e_i \rangle u_n(x)dv(x) = \int_\Omega \langle \Phi(x)e_j, e_i \rangle u(x)dv(x).$$

For every $n \in \mathbb{N}$ we have $u_n \in W^{1,r}(X,\nu)$ with every $r > 1$, then

$$\partial_i u_n(x) = \partial_i u(x)\eta_n(G(x)) + u(x)\eta'_n(G(x))\partial_i G(x);$$
see [21, Proposition 4.5(5) and Proposition 4.6]. Observe that

$$\int_X u_n \partial_j f_k d\nu = \int_X \partial_j f_k u_n (\tilde{e}_i + \partial_i U) d\nu - \int_X (\eta_n \circ G) \partial_j f_k \partial_i u d\nu - \int_X (\eta_n \circ G) u \partial_j f_k \partial_i G d\nu,$$

and the following estimate holds:

$$\int_X \left| \partial_j f_k u_n - (\Phi e_j, e_i)_H u \right| d\nu \leq \int_X \left| \partial_j f_k \right| u_n - u \right| d\nu + \int_X \left| \partial_j f_k - (\Phi e_j, e_i)_H \right| u \right| d\nu \leq \left( \int_X \left| \partial_j f_k \right|^p d\nu \right)^{\frac{1}{p}} + \left( \int_X \left| \partial_j f_k - (\Phi e_j, e_i)_H \right|^p d\nu \right)^{\frac{1}{p}},$$

this implies $$\lim_{n \to +\infty} \lim_{k \to +\infty} \int_X u_n \partial_j f_k d\nu = \int_{\Omega} (\Phi e_j, e_i) u d\nu.$$ Furthermore for every $$n \in \mathbb{N}$$ we get

$$\int_X \left| (\eta_n \circ G) \partial_j f_k \partial_i u \right| d\nu \leq \int_X \left| \partial_j f_k \partial_i u \right| d\nu \leq \left( \int_X \left| \partial_j f_k \right|^p d\nu \right)^{\frac{1}{p}} \left( \int_X \left| \partial_i u \right|^p d\nu \right)^{\frac{1}{p}} \xrightarrow{k \to +\infty} 0.$$

Let $$s > 1,$$ then for every $$n \in \mathbb{N}$$

$$\int_X \left| (\eta_n \circ G) u \partial_j f_k \partial_i G \right| d\nu \leq 2n \|u\|_\infty \left( \int_X \left| \partial_j f_k \right|^p d\nu \right)^{\frac{1}{p}} \left( \int_X \left| \partial_i G \right|^p \right)^{\frac{1}{p}} \left( \int_X e^{-sU} d\mu \right)^{\frac{1}{p}} \xrightarrow{k \to +\infty} 0,$$

where the last limit follows from Hypothesis 1.2;

$$\int_X \left| \partial_j f_k u_n \tilde{e}_i \right| d\nu \leq \|u\|_{\infty} \left( \int_X \left| \partial_j f_k \right|^p d\nu \right)^{\frac{1}{p}} \left( \int_X \left| \tilde{e}_i \right|^p d\nu \right)^{\frac{1}{p}} \xrightarrow{k \to +\infty} 0;$$

Let $$r > 1,$$ then for every $$n \in \mathbb{N}$$ we have

$$\int_X \left| \partial_j f_k u_n \partial_i U \right| d\nu \leq \|u\|_{\infty} \left( \int_X \left| \partial_j f_k \right|^p d\nu \right)^{\frac{1}{p}} \left( \int_X e^{-rU} d\mu \right)^{\frac{1}{p}} \left( \int_X \left| \partial_i U \right|^p r' \right)^{\frac{1}{p}} \xrightarrow{k \to +\infty} 0,$$

and the last limit exists whenever $$p'r' \leq r.$$ 

\[ \square \]

We remark that in the proof of Proposition 2.2 we have not used the assumptions of convexity and closure of $$\Omega.$$

We are now able to define the Sobolev spaces $$W^{2,p}(\Omega, \nu).$$

**Definition 2.3.** Assume that Hypotheses 1.1 and 1.2 hold. For $$p > \frac{m}{m-1}$$ we denote by $$W^{2,p}(\Omega, \nu)$$ the domain of the closure of the operator $$\left( \nabla H, \nabla^2 H \right) : \mathcal{F} C^1_{\text{loc}}(\Omega) \to L^p(\Omega, \nu; H) \times L^p(\Omega, \nu; H_L)$$ in $$L^p(\Omega, \nu).$$

Finally we want to remark that if Hypotheses 1.1 and 1.2 hold, then $$\frac{m}{m-1} < 2.$$ In particular the above discussion allows us to define the Sobolev spaces $$W^{1,2}(\Omega, \nu)$$ and $$W^{2,2}(\Omega, \nu).$$

### 2.6. Surface measures.

For a comprehensive treatment of surface measures in infinite dimensional Banach spaces with Gaussian measures we refer to [23], [22] and [13]. We recall the definition of the Feyel–de La Pradelle Hausdorff–Gauss surface measure. If $$m \geq 2$$ and $$F = \mathbb{R}^m$$ equipped with a norm $$\|\cdot\|_F,$$ we define

$$d\theta^F(x) = \frac{1}{(2\pi)^{\frac{m}{2}}} e^{-\frac{\|x\|_F^2}{2}} dH_{m-1}(x),$$

where $$H_{m-1}$$ is the spherical $$(m-1)$$-dimensional Hausdorff measure in $$\mathbb{R}^m,$$ i.e.

$$H_{m-1}(A) = \liminf_{\delta \to 0} \left\{ \sum_{n \in \mathbb{N}} w_{m-1} r_{n}^{m-1} \left| A \subseteq \bigcup_{n \in \mathbb{N}} B(x_n, r_n), \ r_n < \delta, \ \text{for every} \ n \in \mathbb{N} \right. \right\},$$
where \( w_{m-1} = \pi^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right)^{-1} \). For every \( m \)-dimensional \( F \subseteq H \) we consider the orthogonal projection (along \( H \)) on \( F \):

\[
x \mapsto \sum_{n=1}^{m} \langle x, f_n \rangle_H f_n \quad x \in H
\]

where \( \{f_n\}_{n=1}^{m} \) is an orthonormal basis of \( F \). There exists a \( \mu \)-measurable projection \( \pi_F \) on \( F \), defined in the whole \( X \), that extends it (see [8, Theorem 2.10.11]). We denote by \( \hat{F} := \ker \pi_F \) and by \( \mu\hat{F} \) the image of the measure \( \mu \) on \( \hat{F} \) through \( I - \pi_F \). Finally we denote by \( \mu_F \) the image of the measure \( \mu \) on \( F \) through \( \pi_F \), which is the standard Gaussian measure on \( \mathbb{R}^m \) if we identify \( F \) with \( \mathbb{R}^m \). Let \( A \subseteq X \) be a Borel set and identify \( F \) with \( \mathbb{R}^m \), we set

\[
\rho^F(A) := \int_{\ker \pi_F} \theta^F(A_x) d\mu\hat{F}(x),
\]

where \( A_x = \{ y \in F \mid x + y \in A \} \). The map \( F \mapsto \rho^F(A) \) is well defined and increasing, namely if \( F_1 \subseteq F_2 \) are finite dimensional subspaces of \( H \), then \( \rho^{F_1}(A) \leq \rho^{F_2}(A) \) (see [2, Lemma 3.1] and [22, Proposition 3.2]). The Feyel–de La Pradelle Hausdorff–Gauss surface measure is defined by

\[
\rho(A) = \sup \{ \rho^F(A) \mid F \subseteq H, F \text{ is a finite dimensional subspace} \}.
\]

Finally we remind the reader of the following density result (see [11, Proposition 7]).

**Proposition 2.4.** Assume Hypotheses 1.1 and 1.2 hold and let \( p > 1 \). If \( g \in L^p(G^{-1}(0), \rho) \) is such that for every \( \varphi \in \mathcal{F}_b^\infty(X) \)

\[
\int_{G^{-1}(0)} \varphi g d\rho = 0
\]

then \( g(x) = 0 \) for \( \rho \)-a.e. \( x \in G^{-1}(0) \).

**2.7. Traces of Sobolev functions.** Traces of Sobolev functions in infinite dimensional Banach spaces are studied in [13] in the Gaussian case and in [21] in the weighted Gaussian case. Assume that Hypotheses 1.1 and 1.2 hold and let \( p > \frac{t-1}{t-2} \). If \( \varphi \in W^{1,p}(\Omega, \nu) \) we define the trace of \( \varphi \) on \( G^{-1}(0) \) as follows:

\[
\text{Tr} \varphi = \lim_{n \to +\infty} \varphi_n|_{G^{-1}(0)} \quad \text{in } L^q(G^{-1}(0), e^{-U} \rho) \quad \text{for every } q \in \left[ 1, p \frac{t-2}{t-1} \right],
\]

where \( \{\varphi_n\}_{n \in \mathbb{N}} \) is any sequence in \( \text{Lip}_b(\Omega) \), the space of bounded and Lipschitz functions on \( \Omega \), which converges in \( W^{1,p}(\Omega, \nu) \) to \( \varphi \). The definition does not depend on the choice of the sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \) in \( \text{Lip}_b(\Omega) \) approximating \( \varphi \) in \( W^{1,p}(\Omega, \nu) \) (see [21, Proposition 7.1]). In addition the following result holds.

**Proposition 2.5.** Assume Hypotheses 1.1 and 1.2 hold. The operator \( \text{Tr} : W^{1,p}(\Omega, \nu) \to L^q(G^{-1}(0), e^{-U} \rho) \) is continuous for every \( p > \frac{t-1}{t-2} \) and \( q \in \left[ 1, p \frac{t-2}{t-1} \right] \). Moreover if \( U \equiv 0 \), then the trace operator is continuous from \( W^{1,p}(\Omega, \mu) \) to \( L^q(G^{-1}(0), \rho) \) for every \( p > 1 \) and \( q \in [1, p) \) (see [13, Corollary 4.2] and [21, Corollary 7.3]).

We will still denote by \( \text{Tr} \Psi = \sum_{n=1}^{+\infty} (\text{Tr} \psi_n) e_n \) if \( \Psi \in W^{1,p}(\Omega, \nu; H) \), for \( p > \frac{t-1}{t-2} \), and \( \psi_n = \langle \Psi, e_n \rangle_H \). The main result of [21] is the following integration by parts formula.

**Theorem 2.6.** Assume Hypotheses 1.1 and 1.2 hold and let \( p > \frac{t-1}{t-2} \). For every \( \varphi \in W^{1,p}(\Omega, \nu) \) and \( k \in \mathbb{N} \) we have

\[
\int_{\Omega} \left( \partial_k \varphi - \varphi \partial_k U - \varphi \partial_k \right) d\nu = \int_{G^{-1}(0)} \text{Tr}(\varphi) \text{Tr} \left( \frac{\partial_k G}{|\nabla H G|_H} \right) e^{-U} d\rho.
\]
3. H-distance function

In this section we study some properties of the following function. Let \( x \in X \) and let \( C \subseteq X \) be a Borel set. We define

\[
d_H(x, C) := \begin{cases} 
\inf \left\{ |h|_H \mid h \in H \cap (x - C) \right\} & \text{if } H \cap (x - C) \neq \emptyset; \\
+\infty & \text{if } H \cap (x - C) = \emptyset.
\end{cases}
\]  

\( d_H \) can be seen as a distance function from \( C \) along \( H \). This function was already considered in [26], [33], [8, Example 5.4.10] and [25], but the results of this section are new. We remark that

\[
d_H(x, C) = \begin{cases} 
\inf \left\{ |x - w|_H \mid h \in C \cap (x + H) \right\} & \text{if } C \cap (x + H) \neq \emptyset; \\
+\infty & \text{if } C \cap (x + H) = \emptyset.
\end{cases}
\]

which agrees with [25, Definition 2.5].

The aim of this section is to prove that the function \( d_H^2(\cdot, C) \) is differentiable along \( H \) \( \mu \)-a.e., whenever \( C \) is a closed and convex subset of \( X \). The ideas of the proof are actually pretty similar to the classical arguments that can be found in [10, Section 5.1], but we need to pay special attention since \( d_H \) is not globally defined and behaves well just along the directions of \( H \).

For the rest of the paper we will denote by \( D(C) \) the set

\[
D(C) := \{ x \in X \mid H \cap (x - C) \neq \emptyset \},
\]

whenever \( C \subseteq X \). We recall that \( d_H(x, C) \) is a measurable function (see [25, Lemma 2.6]).

**Lemma 3.1.** Let \( C \) be a Borel subset of \( X \). Then \( D(C) \) is measurable and \( H \)-translation invariant, i.e. \( D(C) + H = D(C) \). Moreover \( C \subseteq D(C) \) and if \( \mu(C) > 0 \), then \( \mu(D(C)) = 1 \).

**Proof.** The measurability of \( D(C) \) follows from the measurability of \( d_H \). \( D(C) \subseteq D(C) + H \) and \( C \subseteq D(C) \) are obvious. Let \( x \in D(C) \) and \( h \in H \), then by the very definition of \( D(C) \), there exists \( k \in H \cap (x - C) \). We have that \( k + h \in H \cap (x + h - C) \). So \( x + h \in D(C) \). We recall that for \( H \)-translation invariant sets a zero-one law holds. Namely \( \mu(D(C)) = 0 \) or \( \mu(D(C)) = 1 \) (see [8, Corollary 2.5.4] and [25, Proposition 2.1]).  

**Proposition 3.2.** Let \( C \subseteq X \) be a closed convex set. For every \( x \in D(C) \), there exists a unique \( m(x, C) \in H \cap (x - C) \) such that

\[
|m(x, C)|_H = d_H(x, C).
\]

**Proof.** By [6, Proposition 11.14] there exists \( h \in H \cap (x - C) \) such that

\[
|\nabla| = d_H(x, C).
\]

Assume \( h_1, h_2 \in H \cap (x - C) \) are such that \( |h_1|_H = |h_2|_H = d_H(x, C) \). Observe that

\[
\frac{h_1 + h_2}{2} \in H \cap (x - C)
\]

and \( |h_1|_H \leq |2^{-1}(h_1 + h_2)|_H \leq 2^{-1}|h_1| + 2^{-1}|h_2| = |h_1|_H \). In particular \( |h_1|_H = |h_2|_H = |2^{-1}(h_1 + h_2)|_H \). So, by the strict convexity of \( |\cdot|_H \), we have \( h_1 = h_2 \).

**Proposition 3.3.** Let \( C \subseteq X \) be a closed convex set and \( x \in D(C) \). For \( m \in H \cap (x - C) \), we have \( m = m(x, C) \) if, and only if,

\[
\langle h - m, m \rangle_H \geq 0,
\]

for every \( h \in H \cap (x - C) \).

**Proof.** Let \( v_t = (1 - t)h + tm(x, C) \) for \( t \in (0, 1) \) and \( h \in H \cap (x - C) \). Since \( v_t \in H \cap (x - C) \) we have \( |m(x, C)|_H \leq |v_t|_H \). So

\[
|m(x, C)|_H^2 \leq |v_t|_H^2 = (1 - t)^2|h|_H^2 + 2t(1 - t)\langle h, m(x, C) \rangle_H + t^2|m(x, C)|_H^2.
\]

Dividing by \( 1 - t \) we get

\[
(1 + t)|m(x, C)|_H^2 \leq (1 - t)|h|_H^2 + 2t\langle h, m(x, C) \rangle_H.
\]
Letting $t \to 1^-$ we obtain  
$$\langle h - m(x, C), m(x, C) \rangle_H \geq 0.$$  
Now let $m \in H \cap (x - C)$ be an element satisfying inequality (3.2). For every $h \in H \cap (x - C)$ we have  
$$|m|_H^2 = \langle m - h, m \rangle_H + \langle h, m \rangle_H \leq \langle h, m \rangle_H \leq |h|_H |m|_H.$$  
Thus we get $|m|_H \leq |h|_H$ for every $h \in H \cap (x - C)$. □

**Proposition 3.4.** Let $C \subseteq X$ be a closed convex set and $x \in D(C)$. The map $m_{x, C} : H \to H$ defined as $m_{x, C}(h) := m(x + h, C)$ is well defined for every $h \in H$ and Lipschitz continuous, with Lipschitz constant less or equal than 1.

**Proof.** If $x \in D(C)$, then by Lemma 3.1 $x + h \in D(C)$ for every $h \in H$. So by Proposition 3.2 the element $m(x + h, C) \in H \cap (x + h - C)$ such that $d_H(x + h, C) = |m(x + h, C)|_H$ exists and it is unique. Thus the map $m_{x, C}(h) := m(x + h, C)$ is well defined for every $h \in H$.

By the very definition of the function $m(\cdot, C)$ (see Proposition 3.2) we get that for every $x \in X$ and $h \in H$, $m(x, C) + h \in H \cap (x + h - C)$ and $m(x, C) - h \in H \cap (x - C)$. By Proposition 3.3 we get

$$0 \leq \langle k - m(x, C), m(x, C) \rangle_H \quad \text{for every } k \in H \cap (x - C);$$

$$0 \leq \langle l - m(x + h, C), m(x + h, C) \rangle_H \quad \text{for every } l \in H \cap (x + h - C).$$

Set $k = m(x + h, C) - h$ and $l = m(x, C) + h$ and sum inequalities (3.3) and (3.4)

$$0 \leq \langle m(x + h, C) - h - m(x, C), m(x, C) \rangle_H + \langle m(x, C) + h - m(x + h, C), m(x + h, C) \rangle_H =$$

$$-|m(x + h, C) - m(x, C)|^2_H + \langle h, m(x + h, C) - m(x, C) \rangle_H.$$  

(3.5)

By the Cauchy–Schwarz inequality, we get $|m(x + h, C) - m(x, C)|_H \leq |h|_H$  

□

We remark that by inequality (3.5) we get  

$$\langle m(x + h, C) - m(x, C), h \rangle_H \geq 0$$  

(3.6)

for every $x \in D(C)$ and $h \in H$. This fact will come in handy in the next proposition.

**Proposition 3.5.** Let $C \subseteq X$ be a closed convex set. The function $d^2_H(\cdot, C)$ is differentiable along $H$ at every point $x \in D(C)$. Furthermore  

$$\langle \nabla_H d^2_H(\cdot, C)(x) \rangle = 2m(x, C).$$

**Proof.** By the very definition of the function $m(\cdot, C)$ (see Proposition 3.2) we get that for every $x \in X$ and $h \in H$, $m(x, C) + h \in H \cap (x + h - C)$ and $m(x, C) - h \in H \cap (x - C)$. So by definition in formula (3.1) and Proposition 3.2 we have

$$|m(x + h, C)|_H \leq |m(x, C) + h|_H \quad \text{and} \quad |m(x, C)|_H \leq |m(x + h, C) - h|_H.$$  

(3.7)

Now using the left hand side inequality in (3.7) we get

$$d^2_H(x + h, C) - d^2_H(x, C) - 2\langle m(x, C), h \rangle_H =$$

$$|m(x + h, C)|^2_H - |m(x, C)|^2_H - 2\langle m(x, C), h \rangle_H \leq$$

$$|m(x, C) + h|^2_H - |m(x, C)|^2_H - 2\langle m(x, C), h \rangle_H =$$

$$= |m(x, C)|^2_H + |h|^2_H + 2\langle m(x, C), h \rangle_H - |m(x, C)|^2_H - 2\langle m(x, C), h \rangle_H = |h|^2_H.$$  

(3.8)

In addition using the right hand side inequality in (3.7) and inequality (3.6) we get

$$d^2_H(x + h, C) - d^2_H(x, C) - 2\langle m(x, C), h \rangle_H =$$

$$|m(x + h, C)|^2_H - |m(x, C)|^2_H - 2\langle m(x, C), h \rangle_H \geq$$

$$|m(x + h, C)|^2_H - |m(x + h, C) - h|^2_H - 2\langle m(x, C), h \rangle_H =$$

$$|m(x + h, C)|^2_H - |m(x + h, C)|^2_H - |h|^2_H + 2\langle m(x + h, C) - m(x, C), h \rangle_H \geq -|h|^2_H.$$  

(3.9)

Combining inequalities (3.8) and (3.9) we get

$$-|h|^2_H \leq d^2_H(x + h, C) - d^2_H(x, C) - 2\langle m(x, C), h \rangle_H \leq |h|^2_H.$$  

(3.10)

Letting $|h|^2_H \to 0$ we get the assertions of our proposition. □
Proposition 3.6. Let \( \mathcal{C} \subseteq X \) be a closed convex set such that \( \mu(D(\mathcal{C})) = 1 \). The function \( d^2_H(\cdot, \mathcal{C}) \) is convex, \( H \)-continuous and belongs to \( W^{1,p}(X, \mu) \) for every \( p > 1 \).

Proof. Convexity follows from a standard argument: let \( \epsilon > 0 \), \( x, y \in D(\mathcal{C}) \) and choose \( h_\epsilon(x) \in H \cap (x - \mathcal{C}) \) and \( h_\epsilon(y) \in H \cap (y - \mathcal{C}) \) such that
\[
|h_\epsilon(x)|^2_H \leq d^2_H(x, \mathcal{C}) + \epsilon \quad \text{and} \quad |h_\epsilon(y)|^2_H \leq d^2_H(y, \mathcal{C}) + \epsilon.
\]
Observe that \( \lambda h_\epsilon(x) + (1 - \lambda) h_\epsilon(y) \in H \cap (\lambda x + (1 - \lambda)y - \mathcal{C}) \), then
\[
d^2_H(\lambda x + (1 - \lambda)y, \mathcal{C}) \leq |\lambda h_\epsilon(x) + (1 - \lambda) h_\epsilon(y)|^2_H \leq \lambda |h_\epsilon(x)|^2_H + (1 - \lambda) |h_\epsilon(y)|^2_H \leq \lambda d^2_H(x, \mathcal{C}) + (1 - \lambda)d^2_H(y, \mathcal{C}) + \epsilon.
\]
Letting \( \epsilon \to 0 \) we get the convexity of \( d^2_H(\cdot, \mathcal{C}) \). If neither \( x \in D(\mathcal{C}) \) nor \( y \in D(\mathcal{C}) \), then for every \( \lambda \in [0,1] \) we have
\[
d^2_H(\lambda x + (1 - \lambda)y, \mathcal{C}) \leq \lambda d^2_H(x, \mathcal{C}) + (1 - \lambda)d^2_H(y, \mathcal{C}) = +\infty.
\]
\( H \)-continuity follows from Proposition 3.2 and Proposition 3.4, since for every \( x \in D(\mathcal{C}) \) and \( k \in H \)
\[
|d_H(x + k, \mathcal{C}) - d_H(x, \mathcal{C})| = ||m(x + k, \mathcal{C})|_H - |m(x, \mathcal{C})|_H| \leq |m(x + k, \mathcal{C}) - m(x, \mathcal{C})|_H \leq |k|_H.
\]
So \( d^2_H(\cdot, \mathcal{C}) \) is the composition of a \( H \)-Lipschitz function and a continuous function, then it is \( H \)-continuous.

The functions \( d_H(\cdot, \mathcal{C}) \) and \( m(\cdot, \mathcal{C}) \) are \( H \)-Lipschitz. Since \( d^2_H(\cdot, \mathcal{C}) \) is the composition of a smooth function and a \( H \)-Lipschitz function and it has \( H \)-Lipschitz gradient, we get \( d^2_H(\cdot, \mathcal{C}) \in W^{1,p}(X, \mu) \) for every \( p > 1 \). \( \square \)

Proposition 3.7. Let \( \mathcal{C} \subseteq X \) be a closed convex set. Then \( d_H(x, \mathcal{C}) = 0 \) if, and only if, \( x \in \mathcal{C} \).

Proof. We remark that if \( x \in \mathcal{C} \), then \( 0 \in H \cap (x - \mathcal{C}) \) and \( d_H(x, \mathcal{C}) = 0 \).

We recall that \( H \) is continuously embedded in \( X \) (see \cite[Theorem 2.4.5]{8}), so there exists \( K > 0 \) such that
\[
\|h\|_X \leq K|h|_H \quad \text{for every} \quad h \in H.
\]
Let \( x \in X \) such that \( d_H(x, \mathcal{C}) = 0 \). By definition we have \( x \in D(\mathcal{C}) \). Then a sequence \( \{h_n\}_{n \in \mathbb{N}} \subseteq H \cap (x - \mathcal{C}) \) exists such that \( |h_n|_H \to 0 \). For every \( n \in \mathbb{N} \) there exists \( c_n \in \mathcal{C} \) such that \( h_n = x - c_n \), so
\[
\lim_{n \to +\infty} \|x - c_n\|_X = \lim_{n \to +\infty} \|h_n\|_X \leq K \lim_{n \to +\infty} |h_n|_H = 0.
\]
Thus, by the closure of \( \mathcal{C} \), \( x \in \mathcal{C} \). \( \square \)

4. A PROPERTY OF THE MOREAU–YOSIDA APPROXIMATIONS ALONG \( H \)

We start this section by recalling the definition and some basic properties of the subdifferential of a convex continuous function. If \( f : X \to \mathbb{R} \) is a proper, convex and lower semicontinuous function, we denote by \( \text{dom}(f) \) the domain of \( f \), namely \( \text{dom}(f) := \{ x \in X \mid f(x) < +\infty \} \), and by \( \partial f(x) \) the subdifferential of \( f \) at the point \( x \), i.e.,
\[
\partial f(x) := \begin{cases} \{x^* \in X^* \mid f(y) \geq f(x) + x^*(y - x) \text{ for every } y \in X\} & x \in \text{dom}(f); \\ \emptyset & x \notin \text{dom}(f). \end{cases}
\]
We recall that \( \partial f(x) \) is convex and weak-star compact for every \( x \in X \), but it may be empty even if \( x \in \text{dom}(f) \). Furthermore \( \partial f \) is a monotone operator, namely for every \( x, y \in X \), \( x^* \in \partial f(x) \) and \( y^* \in \partial f(y) \) the following inequality holds
\[
y^*(y - x) \geq x^*(y - x).
\]
For a classical treatment of monotone operators and subdifferential of convex functions we refer to \cite{31} and \cite{5}.

We recall that for \( \alpha > 0 \) the Moreau–Yosida approximation along \( H \) of a proper convex and lower semicontinuous function \( f : X \to \mathbb{R} \cup \{+\infty\} \) is
\[
f_\alpha(x) := \inf \left\{ f(x + h) + \frac{1}{2\alpha} |h|^2_H \mid h \in H \right\}.
\]
See [12, Section 3] for more details and [9] and [6, Section 12.4] for a treatment of the classical Moreau–Yosida approximations in Hilbert spaces, which are different from the one defined in (4.3). In the following proposition we recall some results contained in [12, Section 3].

**Proposition 4.1.** Let $x \in \mathbb{R}$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. The following properties hold:

1. the function $g_{a,x} : H \to \mathbb{R}$ defined as $g_{a,x}(h) := f(x + h) + \frac{1}{2a} \|h\|^2_H$, has a unique global minimum point $P(x,a) \in H$. Moreover $P(x,a) \to 0$ in $H$ as $a$ goes to zero.
2. $f_{a}(x) \not\geq f(x)$ as $\alpha \to 0^+$. In particular $f_{a}(x) \leq f(x)$ for every $\alpha > 0$ and $x \in X$;
3. $P_{x,a} : H \to H$ defined as $P_{x,a}(h) := P(x,a) + h - \frac{1}{\alpha} f'(P(x,a))$, for every $h \in H$;
4. the function $P_{x,a} : H \to H$ defined as $P_{x,a}(h) := P(x,a)/\alpha$ is Lipschitz continuous, with Lipschitz constant less or equal than $1$;
5. $f_{a}$ is differentiable along $H$ at every point $x \in X$. In addition, for every $x \in X$, we have $\nabla f_{a}(x) = -\alpha^{-1} P(x,a)$.
6. $f_{a}$ belongs to $W^{2,p}(X,\mu)$, whenever $f \in L^{p}(X,\mu)$ for some $1 \leq p < +\infty$.

We will dedicate this section to prove that for every $x \in X$, $\nabla f_{a}(x)$ converges to $\nabla f(x)$ as $a$ goes to zero. In order to obtain such result we need a couple of lemmata.

**Lemma 4.2.** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, belonging to $W^{1,p}(X,\mu)$ for some $p > 1$. Let $x \in \text{dom}(f)$ and $\alpha > 0$. If we let $F : H \to \mathbb{R}$ be the function defined as $F(h) := f(x + h)$, then $F$ is proper convex and lower semicontinuous. Moreover $\nabla f(x) \in \partial F(0)$ and $\nabla f_{a}(x) \in \partial F(P(x,a))$.

**Proof.** Convexity and properness are trivial. Let $H\lim_{n \to +\infty} h_n = h$. Since $H$ is continuously embedded in $X$, $X\lim_{n \to +\infty} h_n = h$. By the fact that $f$ is $\|\cdot\|_{X}$-lower semicontinuous, we get

\[ F(h) = f(x + h) \leq \lim_{n \to +\infty} f(x + h_n) = \lim_{n \to +\infty} F(h_n). \]

So $F$ is $\|\cdot\|_{H}$-lower semicontinuous. Since $x \in \text{dom}(f)$, then $0 \in \text{dom}(F)$. In addition, by Proposition 4.1(1) we have $f_{a}(x) = f(x + P(x,a)) + (2\alpha)^{-1}\|P(x,a)\|_{H}^2 \leq f(x)$, so

\[ F(P(x,a)) = f(x + P(x,a)) \leq f(x) - \frac{1}{2\alpha}\|P(x,a)\|_{H}^2 < +\infty. \]

This implies $P(x,a) \in \text{dom}(F)$. Let $h \in H$, then by [32, Proof of proposition 3.1] we get

\[ F(h) = f(x + h) \geq f(x) + \langle \nabla f(x), h \rangle_{H} = F(0) + \langle \nabla f(x), h \rangle_{H}. \]

Moreover by Proposition 4.1(3) and Proposition 4.1(5) we get

\[ F(h) = f(x + h) \geq f(x + P(x,a)) - \frac{1}{\alpha} \langle P(x,a), h - P(x,a) \rangle_{H} = F(P(x,a)) + \langle \nabla f_{a}(x), h - P(x,a) \rangle_{H}. \]

By formula (4.1), we get $\nabla f_{a}(x) \in \partial F(0)$ and $\nabla f_{a}(x) \in \partial F(P(x,a))$. □

**Lemma 4.3.** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, belonging to $W^{1,p}(X,\mu)$ for some $p > 1$. Let $x \in \text{dom}(f)$ and $\alpha, \beta > 0$. Then

\[ (f_{\alpha})_{\beta}(x) = f_{\alpha+\beta}(x). \]

In particular $\nabla f_{a}(x) = \nabla f_{a+\beta}(x)$ and

\[ |\nabla f_{a+\beta}(x)|_{H} \leq |\nabla f_{a}(x)|_{H}, \]

for every $x \in X$.\]
Proof. The proof of equality (4.4) is similar to the one in [6, Proposition 12.22], we give it just for the sake of completeness.

\[(f_\alpha)_\beta(x) = \inf_{h \in H} \left\{ f_\alpha(x + h) + \frac{1}{2\beta} |h|_H^2 \right\} = \]
\[= \inf_{h \in H} \left\{ \inf_{k \in H} \left\{ f(x + h + k) + \frac{1}{2\alpha} |k|_H^2 \right\} + \frac{1}{2\beta} |h|_H^2 \right\} = \]
\[= \inf_{h \in H} \left\{ \inf_{w \in H} \left\{ f(x + w) + \frac{1}{2\alpha} |w - h|_H^2 \right\} + \frac{1}{2\beta} |h|_H^2 \right\} = \]
\[= \inf_{w \in H} \left\{ f(x + w) + \frac{\alpha + \beta}{2\alpha\beta} \inf_{h \in H} \left\{ \frac{\beta}{2} |w - h|_H^2 + \frac{\alpha}{\alpha + \beta} |h|_H^2 \right\} \right\} = \]
\[= \inf_{w \in H} \left\{ f(x + w) + \frac{\alpha + \beta}{2\alpha\beta} \inf_{h \in H} \left\{ \frac{\alpha\beta}{(\alpha + \beta)^2} |w|_H^2 + \frac{\beta}{\alpha + \beta} |w - h|_H^2 \right\} \right\} = \]
\[= \inf_{w \in H} \left\{ f(x + w) + \frac{1}{2(\alpha + \beta)} |w|_H^2 \right\} = f_{\alpha + \beta}(x). \]

So \((f_\alpha)_\beta(x) = f_{\alpha + \beta}(x).\)

We will now prove inequality (4.5). Let \(x \in X\) and \(x, y > 0\). By Lemma 4.2 we get \(\nabla_H f_\alpha(x) \in \partial F_\alpha(0)\) and \(\nabla_H (f_\alpha)_\beta(x) \in \partial F_\alpha(P_\alpha(x, \beta))\), where \(F_\alpha(h) := f_\alpha(x + h)\) for \(h \in H\) and \(P_\alpha(x, \beta)\) is the unique minimum point of the function \(f_\alpha(x + h) + \frac{1}{2\alpha} |h|_H^2\). Such minimum exists by Proposition 4.1(1). By the monotonicity of the subdifferential (formula (4.2)), Proposition 4.1(3) and equality (4.4) we get

\[0 \leq \langle \nabla_H f_\alpha(x) - \nabla_H f_\alpha(x), -P_\alpha(x, \beta) \rangle_H \leq \beta \langle \nabla_H f_\alpha(x) - \nabla_H f_\alpha(x), -P_\alpha(x, \beta) \rangle_H \leq \]
\[\leq \beta \langle \nabla_H f_\alpha(x) - \nabla_H f_\alpha(x), -P_\alpha(x, \beta) \rangle_H = \beta \langle \nabla_H f_\alpha(x) - \nabla_H f_\alpha(x), -\nabla_H f_\alpha(x) + \nabla_H f_\alpha(x) \rangle_H. \]

So \(|\nabla_H f_\alpha(x)|_H^2 \leq \langle \nabla_H f_\alpha(x), \nabla_H f_\alpha(x) \rangle_H\), and \(|\nabla_H f_\alpha(x)|_H\leq |\nabla_H f_\alpha(x)|_H\). \(\square\)

Proposition 4.4. Let \(f : X \to \mathbb{R} \cup \{+\infty\}\) be a proper convex and lower semicontinuous function, belonging to \(W^{1,p}(X, \mu)\) for some \(p > 1\). Let \(x \in \text{dom}(f)\). Then \(|\nabla_H f_\alpha(x)|_H \leq |\nabla_H f(x)|_H\) as \(\alpha \to 0^+\) for \(\mu\)-a.e. \(x \in X\). In particular

\[|\nabla_H f_\alpha(x)|_H \leq |\nabla_H f(x)|_H \quad (4.6)\]

for \(\mu\)-a.e. \(x \in X\) and for every \(x > 0\).

Proof. Let \(x \in \text{dom}(f)\) and \(\alpha > 0\). By Lemma 4.2 we get \(\nabla_H f_\alpha(x) \in \partial F(P(x, \alpha))\), where \(F(h) := f(x + h)\) for \(h \in H\) and \(P(x, \alpha)\) is the unique minimum point of the function \(f(x + h) + \frac{1}{2\alpha} |h|_H^2\). Such minimum exists by Proposition 4.1(1). By the weak compactness of the subdifferential there exists a point of minimal norm \(h_0 \in H\) in \(\partial F(0)\).

By the monotonicity of the subdifferential (formula (4.2)) we have

\[0 \leq \langle P(x, \alpha), \nabla_H f_\alpha(x) - h_0 \rangle_H = \alpha \langle -\alpha^{-1} P(x, \alpha), h_0 - \nabla_H f_\alpha(x) \rangle_H = \alpha \langle \nabla_H f_\alpha(x), h_0 - \nabla_H f_\alpha(x) \rangle_H, \]

where the last equality follows from Lemma 4.1(5). By the Cauchy–Schwarz inequality we get

\[\langle \nabla_H f_\alpha(x) \rangle_H^2 \leq \langle h_0, \nabla_H f_\alpha(x) \rangle_H \quad (4.7)\]

Using inequality (4.7) we get

\[|\nabla_H f_\alpha(x)|_H \leq |h_0|_H; \quad (4.8)\]

\[|\nabla_H f_\alpha(x) - h_0|_H^2 = |\nabla_H f_\alpha(x)|_H^2 - 2\langle h_0, \nabla_H f_\alpha(x) \rangle_H + |h_0|_H^2 \leq |h_0|_H^2 - |\nabla_H f_\alpha(x)|_H^2. \quad (4.9)\]
By inequality (4.8) we get that the set \( \{ \nabla_H f_{\alpha}(x) \mid \alpha > 0 \} \) is bounded in \( H \). Let \((\alpha_n)_{n \in \mathbb{N}}\) be a sequence converging to zero. By weak compactness a subsequence, that we will still denote by \((\alpha_n)_{n \in \mathbb{N}}\), and \( y \in H \) exist such that \( \nabla_H f_{\alpha_n}(x) \) weakly converges to \( y \) as \( n \) goes to \( +\infty \). By inequality (4.8) and weakly lower semicontinuity of \( |\cdot|_H \) we have that
\[
|y|_H \leq \lim_{n \to +\infty} |\nabla_H f_{\alpha_n}(x)|_H \leq |h_0|_H. \tag{4.10}
\]
We claim that \( y \in \partial F(0) \), indeed recalling that \( \|P(x,\alpha_n)\|_H \to 0 \) as \( n \) goes to \( +\infty \) (Proposition 4.1(1)), \( \{\nabla_H f_{\alpha}(x) \mid \alpha > 0\} \) is bounded in \( H \) and \( f \) is lower semicontinuous we have
\[
\langle y, h \rangle_H = \lim_{n \to +\infty} \langle \nabla_H f_{\alpha_n}(x), h \rangle_H \leq \lim_{n \to +\infty} \langle f(x+h) - f(x) + P(x,\alpha_n), x \rangle_H \leq \limsup_{n \to +\infty} \langle f(x+h) - f(x) + P(x,\alpha_n), x \rangle_H \leq f(x+h) - \liminf_{n \to +\infty} f(x+P(x,\alpha_n)) + \lim_{n \to +\infty} \langle \nabla_H f_{\alpha_n}(x), P(x,\alpha_n) \rangle_H \leq f(x+h) - f(x) = F(h) - F(0)
\]
and since \( 0 \in \text{dom}(F) \), then \( y \in \partial F(0) \). By the fact that \( h_0 \) is an element of minimal norm in \( \partial F(0) \), then all the inequalities in (4.10) are actually equalities. So
\[
\lim_{n \to +\infty} |\nabla_H f_{\alpha_n}(x)|_H = |h_0|_H.
\]
So by inequality (4.9) we have \( \lim_{n \to +\infty} |\nabla_H f_{\alpha_n}(x) - h_0|_H = 0 \). Since \( f \) belongs to \( W^{1,p}(X,\mu) \) for some \( p > 1 \), \( f_{\alpha_n} \) converges to \( f \) in \( L^p(X,\mu) \) as \( n \) goes to \( +\infty \) (Proposition 4.1(2)) and \( f_{\alpha} \in W^{2,p}(X,\mu) \) (Proposition 4.1(6)), then \( h_0 = \nabla_H f(x) \) for \( \mu \)-a.e. \( x \in X \). So we get inequality (4.6) using inequality (4.8).

We have proved that for every sequence \( (\alpha_n)_{n \in \mathbb{N}} \) converging to zero, there exists a subsequence \( (\alpha_{n_k})_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to +\infty} \nabla_H f_{\alpha_{n_k}} = \nabla_H f
\]
in \( L^p(X,\mu;H) \). This implies that \( \nabla_H f_{\alpha} \) converges to \( \nabla_H f \) in \( L^p(X,\mu;H) \) as \( \alpha \) goes to zero. Monotonicity of the convergence of the norms follows from inequality (4.5). \( \Box \)

5. Sobolev regularity estimates

The purpose of this section is to prove Theorem 1.3 and in order to do so we will use a penalization method similar to the one used in [3], [4] and [19]. For \( \alpha > 0 \) let \( U_{\alpha} \) be the Moreau–Yosida approximation along \( H \) of \( U \), as defined in formula (4.3). We recall the following proposition (see [12, Proposition 5.12])

**Proposition 5.1.** Assume Hypothesis 1.2 holds and let \( \alpha \in (0,1] \). Then \( U_{\alpha} \) satisfies Hypothesis 1.2 and \( U_{\alpha} \) is differentiable along \( H \) at every \( x \in X \) with \( \nabla_H U_{\alpha} \mid_{H} \)-Lipschitz. Moreover \( e^{-U_{\alpha}} \in W^{1,p}(X,\mu) \), for every \( p \geq 1 \), and \( U_{\alpha} \in W^{2,\infty}(X,\mu) \), where \( t \) is given by Hypothesis 1.2.

We approach the problem in \( \Omega \) by penalized problems in the whole space \( X \), replacing \( U \) by
\[
V_{\alpha}(x) := U_{\alpha}(x) + \frac{1}{2\alpha} d_H^2(x,\Omega). \tag{5.1}
\]
for \( \alpha \in (0,1] \). Namely for \( \alpha \in (0,1] \), we consider the problem
\[
\lambda u_{\alpha} - L_{\nu_{\alpha}} u_{\alpha} = f \tag{5.2}
\]
where \( \lambda > 0 \), \( f \in L^2(X,\nu_{\alpha}) \), \( \nu_{\alpha} = e^{-V_{\alpha}} \mu \) and \( L_{\nu_{\alpha}} \) is the operator defined as
\[
D(L_{\nu_{\alpha}}) = \left\{ u \in W^{1,2}(X,\nu_{\alpha}) \mid \right\},
\]
there exists \( v \in L^2(X,\nu_{\alpha}) \) such that
\[
\int_X \langle \nabla_H u, \nabla_H \varphi \rangle d\nu_{\alpha} = -\int_X v \varphi d\nu_{\alpha} \text{ for every } \varphi \in \mathcal{C}_c^\infty(X),
\]
with \( L_{\nu_{\alpha}} u = v \) if \( u \in D(L_{\nu_{\alpha}}) \).

We set \( D(\Omega) = \{ x \in X \mid H \cap (x - \Omega) \neq \emptyset \} \), as in Section 3. We remark that \( \Omega \subseteq D(\Omega) \) and if Hypothesis 1.1 holds, then \( \mu(D(\Omega)) = 1 \) (Lemma 3.1).
Proposition 5.2. Assume Hypotheses 1.1 and 1.2 hold and let $\alpha \in (0,1]$. Then the following properties hold:

1. $V_\alpha$ is a convex and $H$-continuous function;
2. $V_\alpha$ is differentiable along $H$ at every point $x \in D(\Omega)$ with $\nabla_H V_\alpha$ $H$-Lipschitz;
3. $e^{-V_\alpha} \in W^{1,p}(X,\mu)$, for every $p \ge 1$;
4. $V_\alpha \in W^{2,t}(X,\mu)$, where $t$ is given by Hypothesis 1.2;
5. $\lim_{\alpha \to 0^+} V_\alpha(x) = \begin{cases} U(x) & x \in \Omega; \\ +\infty & x \notin \Omega. \end{cases}$

Proof. Proposition 3.6 says that $d^2(\cdot,\Omega)$ is convex and $H$-continuous, while from Proposition 5.1 we get convexity of $U_\alpha$. By Proposition 4.1(5) we get that the function $\Upsilon_x : H \to \mathbb{R} \cup \{+\infty\}$ defined as $\Upsilon_x(h) := U_\alpha(x+h)$ is Fréchet differentiable for every $x \in \text{dom}(U)$. By Hypothesis 1.2 we have $\mu(\text{dom}(U)) = 1$, so $U_\alpha$ is $H$-continuous. Therefore $V_\alpha$ is convex and $H$-continuous.

By Proposition 3.4, Proposition 3.5 we get that $d^2_H(\cdot,\Omega)$ is differentiable along $H$ at every point $x \in D(\Omega)$ with $H$-Lipschitz gradient. Proposition 5.1 says that $U_\alpha$ is differentiable along $H$ at every point $x \in X$ with $H$-Lipschitz gradient. Then $V_\alpha$ is differentiable along $H$ at every point $x \in D(\Omega)$ with $H$-Lipschitz gradient. Since

$$\int_X e^{-V_\alpha(x)} d\mu(x) \le \int_X e^{-U_\alpha(x)} d\mu(x),$$

for every $\alpha \in (0,1]$ and $\mu$-a.e. $x \in X$, applying Proposition 5.1 we get that $e^{-V_\alpha} \in L^p(X,\mu)$, for every $p \ge 1$. For every $x \in D(\Omega)$

$$\nabla_H e^{-V_\alpha(x)} = e^{-V_\alpha(x)} \left( \nabla_H U_\alpha(x) + \frac{1}{2\alpha} \nabla_H (d^2_H(\cdot,\Omega)(x)) \right).$$

By point (2) the right side of equality (5.3) is $H$-Lipschitz. By Theorem 2.1 we get $e^{-V_\alpha} \in W^{1,p}(X,\mu)$, for every $p \ge 1$. Using the same argument we get $V_\alpha \in W^{2,t}(X,\mu)$, where $t$ is given by Hypothesis 1.2, for every $\alpha \in (0,1]$.

Finally equality (5) follows from Proposition 3.7 and Proposition 4.1(2). \qed

By Proposition 5.2 we can apply [12, Theorem 5.10] to problem (5.2) and get the following maximal Sobolev regularity result.

Theorem 5.3. Assume Hypothesis 1.2 holds and let $\alpha \in (0,1]$, $\lambda > 0$ and $f \in L^2(X,\nu_\alpha)$. Equation (5.2) has a unique weak solution $u_\alpha$. Moreover $u_\alpha \in W^{2,2}(X,\nu_\alpha)$ and

$$\|u_\alpha\|_{L^2(X,\nu_\alpha)} \le \frac{1}{\lambda} \|f\|_{L^2(X,\nu_\alpha)}; \quad \|\nabla_H u_\alpha\|_{L^2(X,\nu_\alpha;H)} \le \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X,\nu_\alpha)}; \quad \|\nabla^2_H u_\alpha\|_{L^2(X,\nu_\alpha;H^2)} \le \sqrt{2} \|f\|_{L^2(X,\nu_\alpha)}.$$

In addition, for every $\alpha \in (0,1]$, there exists a sequence $\{u^{(n)}_\alpha\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_b^{\alpha}(X)$ such that $u^{(n)}_\alpha$ converges to $u_\alpha$ in $W^{2,2}(X,\nu_\alpha)$ and $\lambda u^{(n)}_\alpha - L_{\mu_\alpha} u^{(n)}_\alpha$ converges to $f$ in $L^2(X,\nu_\alpha)$.

Now we have all the ingredients necessary to prove Theorem 1.3. The proof is similar to the one in [19, Section 3].

Proof of Theorem 1.3. Let $f \in \mathcal{F}_b^{\infty}(X)$. By Theorem 5.3 we get that, for every $\alpha \in (0,1]$, equation (5.2) has a unique weak solution $u_\alpha \in W^{2,2}(X,\nu_\alpha)$ such that inequalities (5.4) and inequality (5.5) hold. Moreover for every $\varphi \in \mathcal{F}_b^{\infty}(X)$ we have

$$\lambda \int_X u_\alpha \varphi d\nu_\alpha + \int_X (\nabla_H u_\alpha, \nabla_H \varphi)_H d\nu_\alpha = \int_X f \varphi d\nu_\alpha.$$ 

By Proposition 3.7 and Proposition 4.1(2) we have

$$e^{-U(x)} \le e^{-U_\alpha(x)} = e^{-V_\alpha(x)} \quad x \in \Omega.$$

So we get the inclusion $W^{2,2}(\Omega,\nu_\alpha) \subseteq W^{2,2}(\Omega,\nu)$ for every $\alpha \in (0,1]$. 

Let \( \{ \alpha_n \}_{n \in \mathbb{N}} \) be a sequence converging to zero such that \( 0 < \alpha_n \leq 1 \) for every \( n \in \mathbb{N} \). By inequalities (5.4) and inequality (5.5) the set \( \{ u_{\alpha_n} | n \in \mathbb{N} \} \) is a bounded set in \( W^{2,2}(\Omega, \nu) \). By weak compactness a subsequence, that we will still denote by \( \{ \alpha_n \}_{n \in \mathbb{N}} \), exists such that \( u_{\alpha_n} \) weakly converges to an element \( u \in W^{2,2}(\Omega, \nu) \). Without loss of generality we can assume that \( u_{\alpha_n} \), \( \nabla_H u_{\alpha_n} \) and \( \nabla_H^2 u_{\alpha_n} \) converge pointwise \( \mu \)-a.e. respectively to \( u \), \( \nabla_H u \) and \( \nabla_H^2 u \).

By inequalities (5.4), for every \( n \in \mathbb{N} \), we have

\[
\int_X u_{\alpha_n} \varphi e^{-\alpha_n \nu} d\mu \leq \left( \int_X u_{\alpha_n}^2 e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \left( \int_X \varphi^2 e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \leq \frac{\| \varphi \|_\infty}{\lambda} \left( \int_X f^2 e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \left( \int_X e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \leq \frac{\| f \|_\infty \| \varphi \|_\infty}{\lambda} \int_X e^{-\nu} d\mu.
\]

By inequality (5.7), Proposition 5.2(5) and the Lebesgue dominated convergence theorem we get

\[
\lim_{n \to +\infty} \lambda \int_X u_{\alpha_n} \varphi d\nu_{\alpha_n} = \lambda \int_\Omega u \varphi d\nu.
\]

By inequalities (5.4), for every \( n \in \mathbb{N} \), we have

\[
\int_X \langle \nabla_H u_{\alpha_n}, \nabla_H \varphi \rangle_H e^{-\alpha_n \nu} d\mu \leq \int_X |\nabla_H u_{\alpha_n}|_H |\nabla_H \varphi|_H e^{-\alpha_n \nu} d\mu \leq \left( \int_X |\nabla_H u_{\alpha_n}|_H^2 e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \left( \int_X |\nabla_H \varphi|_H^2 e^{-\alpha_n \nu} d\mu \right)^{\frac{1}{2}} \leq \frac{\| f \|_\infty \| \varphi \|_\infty}{\sqrt{\lambda}} \int_X e^{-\nu} d\mu.
\]

By inequality (5.9), Proposition 5.2(5) and the Lebesgue dominated convergence theorem we get

\[
\lim_{n \to +\infty} \int_X \langle \nabla_H u_{\alpha_n}, \nabla_H \varphi \rangle_H d\nu_{\alpha_n} = \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu.
\]

Finally we have

\[
\int_X f \varphi e^{-\alpha_n \nu} d\mu \leq \| f \|_\infty \| \varphi \|_\infty \int_X e^{-\alpha_n \nu} d\mu \leq \| f \|_\infty \| \varphi \|_\infty \int_X e^{-\nu} d\mu.
\]

By inequality (5.11), Proposition 5.2(5) and the Lebesgue dominated convergence theorem we get

\[
\lim_{n \to +\infty} \int_X f \varphi d\nu_{\alpha_n} = \int_\Omega f \varphi d\nu.
\]

Inequality (5.8), inequality (5.10) and inequality (5.12) give us that \( u \) is a weak solution of equation (1.1), i.e. for every \( \varphi \in \mathcal{C}_0^\infty(X) \)

\[
\lambda \int_\Omega u \varphi d\nu + \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu = \int_\Omega f \varphi d\nu.
\]

By the lower semicontinuity of the norm of \( L^2(\Omega, \mu) \) and \( L^2(\Omega, \mu; H) \), inequalities (5.4), inequality (5.6), inequality (5.11) and the Lebesgue dominated convergence theorem we get

\[
\| u \|_{L^2(\Omega, \nu)} \leq \liminf_{n \to +\infty} \| u_{\alpha_n} \|_{L^2(\Omega, \nu)} \leq \liminf_{n \to +\infty} \| u_{\alpha_n} \|_{L^2(X, \nu_{\alpha_n})} \leq \frac{1}{\lambda} \liminf_{n \to +\infty} \| f \|_{L^2(X, \nu_{\alpha_n})} = \frac{1}{\lambda} \| f \|_{L^2(\Omega, \nu)};
\]

and

\[
\| \nabla_H u \|_{L^2(\Omega, \nu; H)} \leq \liminf_{n \to +\infty} \| \nabla_H u_{\alpha_n} \|_{L^2(\Omega, \nu; H)} \leq \liminf_{n \to +\infty} \| \nabla_H u_{\alpha_n} \|_{L^2(X, \nu_{\alpha_n}; H)} \leq \frac{1}{\sqrt{\lambda}} \liminf_{n \to +\infty} \| f \|_{L^2(X, \nu_{\alpha_n}; H)} = \frac{1}{\sqrt{\lambda} \lambda} \| f \|_{L^2(\Omega, \nu; H)}.
\]
In the same way by the lower semicontinuity of the norm of $L^2(\Omega, \mu; \mathcal{H}_2)$, inequality (5.5), inequality (5.6), inequality (5.11) and the Lebesgue dominated convergence theorem we get
\[
\|\nabla_H^2 u\|_{L^2(\Omega, \nu; \mathcal{H}_2)} \leq \liminf_{n \to +\infty} \|\nabla_H^2 u_{\alpha_n}\|_{L^2(\Omega, \nu; \mathcal{H}_2)} \leq \liminf_{n \to +\infty} \|\nabla_H^2 u_{\alpha_n}\|_{L^2(\Omega; \nu; \mathcal{H}_2)} \leq \\
\leq \liminf_{n \to +\infty} \|\nabla_H^2 u_{\alpha_n}\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \sqrt{2} \liminf_{n \to +\infty} \|f\|_{L^2(X, \nu; \mathcal{H}_2)} = \sqrt{2} \|f\|_{L^2(\Omega, \nu; \mathcal{H}_2)}.
\]
If $f \in L^2(\Omega, \nu)$, a standard density argument gives us the assertions of our theorem.

\[\square\]

6. The Neumann condition

We are now interested in proving Theorem 1.4. As in Section 5 we approach the problem in $\Omega$ by penalized problems in the whole space $X$, replacing $U$ by the functions $V_\alpha$ defined via equation (5.1).

We start by proving a technical lemma that we will use in the proof of Theorem 1.4.

**Lemma 6.1.** Assume Hypotheses 1.1 and 1.2 hold and let $\alpha \in (0, 1]$. Let $f \in \mathcal{F}C_b^\infty(X)$ and let $u_\alpha$ be a weak solution of equation (5.2). For every $\varphi \in \mathcal{F}C^\infty_b(X)$ the function
\[
F_\alpha(x) := \varphi(x) \left\langle \nabla_H u_\alpha(x), \frac{\nabla_H G(x)}{|\nabla_H G(x)|}_H \right\rangle e^{-V_\alpha(x)}
\]
belongs to $W^{1,r}(\Omega, \mu)$ for every $1 < r < 2$. Furthermore we have $\text{Tr} F_\alpha \in L^q(G^{-1}(0), \rho)$ for every $1 < q < 2$ and
\[
\lim_{\alpha \to 0^+} \text{Tr} F_\alpha = \varphi \left\langle \text{Tr}(\nabla_H u), \left( \frac{\nabla_H G}{|\nabla_H G|}_H \right) \right\rangle e^{-U},
\]
where the limit is taken in $L^q(G^{-1}(0), \rho)$, for every $1 < q < 2$.

**Proof.** We start by proving that $F_\alpha \in L^q(\Omega, \mu)$ for every $1 < r < 2$.
\[
\int_\Omega |F_\alpha|^r d\mu = \int_\Omega \left| \varphi \left( \nabla_H u_\alpha, \frac{\nabla_H G(x)}{|\nabla_H G(x)|}_H \right) \right|^r e^{-V_\alpha} d\mu \leq \|\varphi\|_\infty \int_\Omega |\nabla_H u_\alpha|^r e^{-V_\alpha} d\mu = \\
= \|\varphi\|_\infty \int_\Omega |\nabla_H u_\alpha|^r e^{-(r-1)V_\alpha} e^{-V_\alpha} d\mu.
\]
By using the Hölder inequality with $2/r$, for the measure $e^{-V_\alpha} \mu$, we get
\[
\int_\Omega |F_\alpha|^r d\mu \leq \|\varphi\|_\infty \left( \int_\Omega |\nabla_H u_\alpha|^2 d\mu \right)^{r/2} \left( \int_\Omega e^{-\frac{2}{r-1}V_\alpha} d\mu \right)^{2-r}.
\] (6.1)

By Proposition 5.2, Theorem 5.3 and the fact that $r/(2-r) > 1$ we have
\[
\int_\Omega |F_\alpha|^r d\mu \leq \|\varphi\|_\infty \left( \int_\Omega f^2 e^{-V_\alpha} d\mu \right)^{r/2} \left( \int_\Omega e^{-\frac{r}{r-1}V_\alpha} d\mu \right)^{2-r} \leq \\
\leq \frac{\|f\|_\infty^r \|\varphi\|_\infty^r}{\lambda^{2-r}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-\frac{r}{r-1}V_\alpha} d\mu \right)^{\frac{2-r}{2}} \leq \\
\leq \frac{\|f\|_\infty^r \|\varphi\|_\infty^r}{\lambda^{2-r}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-\frac{r}{r-1}V_\alpha} d\mu \right)^{\frac{2-r}{2}}.
\] (6.2)
So $F_\alpha \in L^q(\Omega, \mu)$ for every $1 < r < 2$.

Now we want to prove that $\nabla_H F_\alpha \in L^r(\Omega, \mu; \mathcal{H}_2)$, for every $1 < r < 2$. Observe that $\nabla_H F_\alpha$ exists by Hypotheses 1.1 and 1.2 and
\[
\nabla_H F_\alpha = \left\langle \nabla_H u_\alpha, \frac{\nabla_H G}{|\nabla_H G|}_H \right\rangle e^{-V_\alpha} \nabla_H \varphi + \varphi \frac{e^{-V_\alpha}}{|\nabla_H G|}_H \nabla_H^2 u_\alpha \nabla_H G + \\
+ \varphi \frac{e^{-V_\alpha}}{|\nabla_H G|}_H \nabla_H G \nabla_H^2 u_\alpha - \varphi \left\langle \nabla_H u_\alpha, \frac{\nabla_H G}{|\nabla_H G|}_H \right\rangle e^{-V_\alpha} \nabla_H^2 G \nabla_H G + \\
- \varphi \left\langle \nabla_H u_\alpha, \frac{\nabla_H G}{|\nabla_H G|}_H \right\rangle e^{-V_\alpha} \nabla_H G.
\] (6.3)
We will estimate each addend. We have
\[
\int_{\Omega} \left| \nabla H u_\alpha, \nabla H G \right| H \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H \varphi \right|^r \frac{d\mu}{H} \leq \| |\nabla H \varphi| H \|_\infty \int_{\Omega} |\nabla H u_\alpha| H e^{-r V_\alpha} d\mu .
\]
Repeating the same arguments as in inequality (6.1) and inequality (6.2) we get
\[
\int_{\Omega} \left| \nabla H u_\alpha, \nabla H G \right| H \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H \varphi \right|^r \frac{d\mu}{H} \leq \frac{\|f\|_\infty \| |\nabla H \varphi| H \|_\infty}{\lambda^\frac{r}{2}} \left( \int_X e^{-V_1} d\mu \right)^{\frac{2-r}{2}} \left( \int_X e^{-\frac{r}{2-r} V_1} d\mu \right)^{2-r} .
\]
(6.4)
Recalling Theorem 5.3 and repeating some arguments used in inequality (6.1) and in inequality (6.2) we have
\[
\int_{\Omega} \left| \varphi \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H \varphi \nabla H u_\alpha \right|^r \frac{d\mu}{H} \leq \| \varphi \|_\infty \int_{\Omega} \left| \nabla H u_\alpha \right| H \frac{e^{-r V_\alpha}}{H} d\mu \leq \| \varphi \|_\infty \left( \int_{\Omega} \left| \nabla H u_\alpha \right| H \frac{e^{-r V_\alpha}}{H} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} \leq 2 \pi \| \varphi \|_\infty \left( \int_{\Omega} f e^{-V_\alpha} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} ,
\]
(6.5)
Now we integrate the third addend of equality (6.3),
\[
\int_{\Omega} \left| \varphi \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H G \nabla H u_\alpha \right|^r \frac{d\mu}{H} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{e^{-V_\alpha}}{\nabla H G H} \left| \nabla H G \right| H \frac{e^{-r V_\alpha}}{H} d\mu \right)^{\frac{r}{2}} .
\]
Applying Hölder inequality with an exponent \( \beta > 1 \) such that \( r \beta < 2 \) we get
\[
\int_{\Omega} \left| \varphi \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H G \nabla H u_\alpha \right|^r \frac{d\mu}{H} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{\| \nabla^2 G \|_{H^2}}{\nabla H G H} \frac{r^\beta}{2} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{\| \nabla^2 G \|_{H^2}}{\nabla H G H} \frac{r^\beta}{2} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} .
\]
(6.6)
By Proposition 5.2, Theorem 5.3 and the fact that \( r \beta/(2 - r \beta) > 1 \) we get
\[
\int_{\Omega} \left| \varphi \frac{e^{-V_\alpha}}{\nabla H G H} \nabla H G \nabla H u_\alpha \right|^r \frac{d\mu}{H} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{\| \nabla^2 G \|_{H^2}}{\nabla H G H} \frac{r^\beta}{2} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{\| \nabla^2 G \|_{H^2}}{\nabla H G H} \frac{r^\beta}{2} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} ,
\]
(6.6)
Arguing as in inequality (6.6), then for the fourth addend of equality (6.3) we have
\[
\int_{\Omega} \left| \varphi \frac{\nabla H u_\alpha, \nabla H G \nabla H G}{\nabla H G H} \frac{e^{-V_\alpha}}{H} \nabla H G \nabla H G \right|^r \frac{d\mu}{H} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{e^{-V_\alpha}}{\nabla H G H} \left| \nabla H G \right| H \frac{e^{-r V_\alpha}}{H} d\mu \right)^{\frac{r}{2}} \leq \| \varphi \|_\infty \left( \int_{\Omega} \frac{\| \nabla^2 G \|_{H^2}}{\nabla H G H} \frac{r^\beta}{2} d\mu \right)^{\frac{r}{2}} \left( \int_X e^{-V_\alpha} d\mu \right)^{\frac{2-r}{2}} .
\]
(6.7)
Let $\beta > 1$ such that $r\beta < 2$. For the last addend of equality (6.3) we obtain
\[
\int_\Omega \left| \varphi \left( \nabla H u_\alpha, \nabla H \lambda \right)_H e^{-V_u} \nabla H \lambda \right|^r \, d\mu \leq \| \varphi \|_\infty^r \int_\Omega \left( |\nabla H u_\alpha| e^{-V_u} |\nabla H \lambda| \right)^r \, d\mu \leq \frac{\| f \|_\infty \| \varphi \|_\infty^r}{\lambda^r} \left( \int_\Omega |\nabla H \lambda|^{r\beta \alpha} \, d\mu \right)^{1/r} \left( \int_X e^{-V_i} \, d\mu \right)^{1/r} \left( \int_X e^{-\frac{r}{2-r\beta} V_i} \, d\mu \right)^{2-\frac{r\beta}{2}}.
\]
Proceeding as in inequality (6.6) and recalling that $\nabla H \lambda$ is $H$-Lipschitz (see Proposition 5.2 and Theorem 2.1) we have
\[
\int_\Omega \left| \varphi \left( \nabla H u_\alpha, \nabla H \lambda \right)_H e^{-V_u} \nabla H \lambda \right|^r \, d\mu \leq \frac{\| f \|_\infty \| \varphi \|_\infty^r}{\lambda^r} \left( \int_\Omega |\nabla H \lambda|^{r\beta \alpha} \, d\mu \right)^{1/r} \left( \int_X e^{-V_i} \, d\mu \right)^{1/r} \left( \int_X e^{-\frac{r}{2-r\beta} V_i} \, d\mu \right)^{2-\frac{r\beta}{2}}.
\]
Finally recalling that $|\nabla \lambda(x)|_H \leq |U(x)|_H$ for every $x \in \Omega$ (see Proposition 4.4) we get
\[
\int_\Omega \left| \varphi \left( \nabla H u_\alpha, \nabla H \lambda \right)_H e^{-V_u} \nabla H \lambda \right|^r \, d\mu \leq \frac{\| f \|_\infty \| \varphi \|_\infty^r}{\lambda^r} \left( \int_\Omega |\nabla H \lambda|^{r\beta \alpha} \, d\mu \right)^{1/r} \left( \int_X e^{-V_i} \, d\mu \right)^{1/r} \left( \int_X e^{-\frac{r}{2-r\beta} V_i} \, d\mu \right)^{2-\frac{r\beta}{2}}. \tag{6.8}
\]
By inequalities (6.4), (6.5), (6.6), (6.7) and (6.8) we get that $F_\alpha$ belongs to $W^{1,r}(\Omega, \mu)$, for every $1 < r < 2$.

Observe that the final estimate of the inequalities (6.4), (6.5), (6.6), (6.7) and (6.8) does not depend on $\alpha$. Then by Proposition 4.4, Proposition 5.2, the Lebesgue dominated convergence theorem and Proposition 2.5, we get the furthermore part of our statement. $\square$

We are now able to prove that if $u$ is a weak solution of problem (1.1), then $u$ satisfies a Neumann type condition at the boundary.

**Proof of Theorem 1.4.** By Theorem 1.3 we get that for every $\varphi \in \mathcal{C}^\infty_b(X)$
\[
\lambda \int_X u_\alpha \varphi \, d\nu + \int_\Omega \langle \nabla H \varphi, \nabla H u_\alpha \rangle_H \, d\nu = \int_\Omega f \varphi \, d\nu. \tag{6.9}
\]

Thanks to Proposition 5.2 and Theorem 5.3, equation (5.2) has a unique solution $u_\alpha \in W^{2,2}(X, \nu_\alpha)$, for every $\alpha \in (0, 1]$, such that inequalities (5.4) and inequality (5.5) hold. Moreover for every $\varphi \in \mathcal{C}^\infty_b(X)$ and $\alpha \in (0, 1]$ we have
\[
\lambda \int_X u_\alpha \varphi \, d\nu_\alpha + \int_X \langle \nabla H u_\alpha, \nabla H \varphi \rangle_H \, d\nu_\alpha = \int_X f \varphi \, d\nu_\alpha. \tag{6.10}
\]

In addition for every $\alpha \in (0, 1]$ there exists a sequence $(u_{\alpha}^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{C}^\infty_b(X)$ such that
\[
W^{2,2}(X, \nu_\alpha) - \lim_{n \to +\infty} u_{\alpha}^{(n)} = u_\alpha; \tag{6.11}
\]
\[
L^2(X, \nu_\alpha) - \lim_{n \to +\infty} \lambda u_{\alpha}^{(n)} = f. \tag{6.12}
\]

Finally $u_\alpha$ converges to $u$ in $W^{2,2}(\Omega, \nu)$. For every $n \in \mathbb{N}$ and $\alpha \in (0, 1]$ we set $f_{\alpha}^{(n)} := \lambda u_{\alpha}^{(n)} - L_{\nu_\alpha} u_{\alpha}^{(n)}$, then the following equality holds
\[
\lambda \int_X u_{\alpha}^{(n)} \varphi \, d\nu_\alpha - \int_X \varphi L_{\nu_\alpha} u_{\alpha}^{(n)} \, d\nu_\alpha = \int_X f_{\alpha}^{(n)} \varphi \, d\nu_\alpha \tag{6.13}
\]
for every $\varphi \in \mathcal{C}^\infty_b(X)$. By [21, Proposition 5.3] we get that if $\psi \in \mathcal{C}^\infty_b(X)$ then
\[
L_{\nu_\alpha} \psi = \sum_{i=1}^{+\infty} \partial_i \psi - \sum_{i=1}^{+\infty} (\partial_i V_{\alpha} + \tilde{\epsilon}_i) \partial_i \psi \tag{6.14}
\]
where the series converges in $L^2(X, \nu)$. Since $L^2(\Omega, \nu) \subseteq L^2(\Omega, \nu)$, the series (6.13) also converges in $L^2(\Omega, \nu)$. Using equality (6.13) and the integration by parts formula (Theorem 2.6) we get

$$
\int_\Omega \varphi L_{u_\alpha}^{(n)} d\nu_\alpha = \int_\Omega \varphi \left( \sum_{i=1}^{+\infty} \partial_i u_\alpha^{(n)} - (\partial_i V_{\alpha} + \tilde{c}_i) \partial_i u_\alpha^{(n)} \right) d\nu_\alpha = \\
= \sum_{i=1}^{+\infty} \int_\Omega \varphi \left( \partial_i u_\alpha^{(n)} - (\partial_i V_{\alpha} + \tilde{c}_i) \partial_i u_\alpha^{(n)} \right) d\nu_\alpha = \\
= \sum_{i=1}^{+\infty} \left( - \int_\Omega \varphi \partial_i u_\alpha^{(n)} d\nu_k + \int_{G^{-1}(0)} \varphi \text{Tr}(\partial_i u_\alpha^{(n)}) \text{Tr} \left( \frac{\partial_i G}{|\nabla H G|_H} \right) e^{-U_{\alpha}} d\rho \right) = \\
= - \int_\Omega \left( \varphi \nabla_H \varphi, \nabla_H u_\alpha^{(n)} \right)_H d\nu_\alpha + \int_{G^{-1}(0)} \varphi \left( \text{Tr} \left( \nabla_H u_\alpha^{(n)} \right), \text{Tr} \left( \frac{\nabla_H G}{|\nabla H G|_H} \right) \right)_H e^{-U_{\alpha}} d\rho.
$$

Arguing as in Lemma 6.1 and recalling (6.10) we get

$$
\lim_{n \to +\infty} \int_{G^{-1}(0)} \varphi \left( \text{Tr} \left( \nabla_H u_\alpha^{(n)} \right), \text{Tr} \left( \frac{\nabla_H G}{|\nabla H G|_H} \right) \right)_H e^{-U_{\alpha}} d\rho = \int_{G^{-1}(0)} \varphi \left( \text{Tr} \left( \nabla_H u_\alpha \right), \text{Tr} \left( \frac{\nabla_H G}{|\nabla H G|_H} \right) \right)_H e^{-U_{\alpha}} d\rho.
$$

By (6.11) and Proposition 2.5, letting $n \to +\infty$ in equality (6.12) we get

$$
\lambda \int_\Omega u_\alpha \varphi d\nu_\alpha + \int_\Omega \langle \nabla_H \varphi, \nabla_H u_\alpha \rangle_H d\nu_\alpha - \int_{G^{-1}(0)} \varphi \left( \text{Tr} \left( \nabla_H u_\alpha \right), \text{Tr} \left( \frac{\nabla_H G}{|\nabla H G|_H} \right) \right)_H e^{-U_{\alpha}} d\rho = \int_\Omega f \varphi d\nu_\alpha.
$$

By Theorem 5.3 we get

$$
\int_\Omega u_\alpha \varphi e^{-V_{\alpha}} d\mu \leq \|\varphi\|_\infty \left( \int_\Omega u_\alpha^2 e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \left( \int_\Omega e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \leq \\
\leq \frac{\|\varphi\|_\infty}{\lambda} \left( \int_\Omega \int_\Omega \varphi e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \left( \int_\Omega e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \leq \\
\leq \lambda \left( \int_\Omega \int_\Omega \varphi e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \left( \int_\Omega e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \leq \\
\leq \frac{\|\varphi\|_\infty \int_\Omega e^{-V_{\alpha}} d\mu}{\lambda} \int_\Omega e^{-V_{\alpha}} d\mu,
$$

and

$$
\int_\Omega \langle \nabla_H u_\alpha, \nabla_H \varphi \rangle_H e^{-V_{\alpha}} d\mu \leq \|\nabla_H \varphi\|_H \|\nabla_H u_\alpha\|_H \left( \int_\Omega \|\nabla_H u_\alpha\|^2_H e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \left( \int_\Omega e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \leq \\
\leq \frac{\|\nabla_H \varphi\|_H}{\lambda} \left( \int_\Omega \int_\Omega \varphi e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \left( \int_\Omega e^{-V_{\alpha}} d\mu \right)^{\frac{1}{2}} \leq \\
\leq \frac{\|\varphi\|_\infty \|\nabla_H \varphi\|_H}{\lambda} \int_\Omega e^{-V_{\alpha}} d\mu \leq \frac{\|\varphi\|_\infty \|\nabla_H \varphi\|_H}{\lambda} \int_\Omega e^{-V_{\alpha}} d\mu.
$$

Moreover we have

$$
\int_\Omega f \varphi e^{-V_{\alpha}} d\mu \leq \|f\|_\infty \|\varphi\|_\infty \int_\Omega e^{-V_{\alpha}} d\mu \leq \|f\|_\infty \|\varphi\|_\infty \int_\Omega e^{-V_{\alpha}} d\mu.
$$

By Lemma 6.1 the map

$$
x \mapsto \varphi(x) \langle \nabla_H u_\alpha(x), \nabla_H G(x) \rangle_H \frac{e^{-V_{\alpha}(x)}}{|\nabla H G(x)|_H} =: F_{\alpha}(x)
$$

belongs to $W^{1,r}(\Omega, \mu)$, for every $1 < r < 2$. In particular $\text{Tr} F_{\alpha} \in L^q(G^{-1}(0), \rho)$ for every $1 < q < 2$. Taking the limit $\alpha \to 0^+$ in equality (6.14), by Proposition 5.2 and the Lebesgue dominated convergence theorem we get

$$
\lambda \int_\Omega \varphi d\nu + \int_\Omega \langle \nabla_H \varphi, \nabla_H u \rangle_H d\nu - \int_{G^{-1}(0)} \varphi \left( \text{Tr} \langle \nabla_H u \rangle, \text{Tr} \left( \frac{\nabla_H G}{|\nabla H G|_H} \right) \right)_H e^{-U_{\alpha}} d\rho = \int_\Omega f \varphi d\nu.
$$
Taking into consideration equality (6.9), then equality (6.15) becomes
\[ \int_{G^{-1}(0)} \varphi \left( \text{Tr}(\nabla_H u), \text{Tr} \left( \frac{\nabla_H G}{\nabla_H G_\H} \right) \right)_\H e^{-U} d\rho = 0, \]
for every \( \varphi \in \mathcal{F} \mathcal{C}_b^\infty(X) \). By Proposition 2.4 we get \( \langle \text{Tr}(\nabla_H u)(x), \text{Tr}(\nabla_H G)(x) \rangle_H = 0 \) for \( \rho \)-a.e. \( x \in G^{-1}(0) \). \( \square \)

7. Examples

In this section we show how our theory can be applied to some examples. Let \( d\xi \) be the Lebesgue measure on \([0,1]\) and consider the classical Wiener measure \( P^W \) on \( \mathcal{C}[0,1] \) (see [8, Example 2.3.11 and Remark 2.3.13] for its construction). The Cameron–Martin space \( H \) is the space of the continuous functions \( f \) on \([0,1]\) such that \( f \) is absolutely continuous, \( f' \in L^2([0,1], d\xi) \) and \( f(0) = 0 \). Moreover \( |f|_H = ||f'||_{L^2([0,1], d\xi)} \) (see [8, Lemma 2.3.14]). An orthonormal basis of \( L^2([0,1], d\xi) \) is given by the functions
\[ e_n(\xi) = \sqrt{2} \sin \frac{\xi}{\sqrt{\lambda_n}} \quad \text{where} \quad \lambda_n = \frac{4}{\pi^2(2n-1)^2} \quad \text{for every} \quad n \in \mathbb{N}. \]
We recall that if \( f, g \in H \), then
\[ |f|^2_H = \sum_{i=1}^{\infty} \lambda_i^{-1} \langle f, e_i \rangle_{L^2([0,1], d\xi)}^2, \quad \langle f, g \rangle_H = \sum_{i=1}^{\infty} \lambda_i^{-1} \langle f, e_i \rangle_{L^2([0,1], d\xi)} \langle g, e_i \rangle_{L^2([0,1], d\xi)}. \]
Finally we remind the reader that an orthonormal basis for \( H \) is given by the sequence \( \{ \sqrt{\lambda_k} e_k \mid k \in \mathbb{N} \} \).

7.1. Admissible sets. Let
\[ G_{\sigma,c}(f) = \int_0^1 f(\xi) d\sigma(\xi) - c, \quad G^{(r)}(f) = \int_0^1 |f(\xi)|^2 d\xi - r^2, \]
where \( \sigma \) is a finite, non everywhere zero, Borel measure in \([0,1]\), \( f \in \mathcal{C}[0,1] \) and \( c, r \in \mathbb{R} \). Observe that the sets \( G_{\sigma,c}(-\infty, 0] \) are halfspaces, since \( G_{\sigma,c} \in (\mathcal{C}[0,1])^* \). Now we show that \( G_{\sigma,c} \) and \( G^{(r)} \) satisfy Hypothesis 1.1.

Easy calculations give
\[ \nabla_H G_{\sigma,c}(f) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \left( \int_0^1 e_i(\xi) d\sigma(\xi) \right) (\sqrt{\lambda_i} e_i), \quad (7.1) \]
\[ \nabla_H G^{(r)}(f) = 2 \sum_{i=1}^{\infty} \sqrt{\lambda_i} \left( \int_0^1 f(\xi) e_i(\xi) d\xi \right) (\sqrt{\lambda_i} e_i). \quad (7.2) \]
So
\[ |\nabla_H G_{\sigma,c}(f)|_H^2 = \sum_{i=1}^{\infty} \lambda_i \left( \int_0^1 e_i(\xi) d\sigma(\xi) \right)^2, \]
\[ |\nabla_H G^{(r)}(f)|_H^2 = 4 \sum_{i=1}^{\infty} \lambda_i \left( \int_0^1 f(\xi) e_i(\xi) d\xi \right)^2 \]
Since \( \sigma \) is non everywhere zero, then \( |\nabla_H G_{\sigma,c}(f)|_H \) is a non zero constant. So \( |\nabla_H G_{\sigma,c}|_H^{-1} \) belongs to every \( L^q(\mathcal{C}[0,1], P^W) \) for every \( q > 1 \). Now let \( q > 1 \) and fix an integer \( K \) bigger than a \( q \), then
\[ \int_{\mathcal{C}[0,1]} \frac{1}{|\nabla_H G^{(r)}(f)|_H^q} dP^W(f) = 2^{-q} \int_{\mathcal{C}[0,1]} \left( \sum_{i=1}^{\infty} \lambda_i \left( \int_0^1 f(\xi) e_i(\xi) d\xi \right)^2 \right)^{-\frac{q}{2}} dP^W(f) \leq \]
\[ \leq 2^{-q} \int_{\mathcal{C}[0,1]} \left( \sum_{i=1}^{K} \lambda_i \left( \int_0^1 f(\xi) e_i(\xi) d\xi \right)^2 \right)^{-\frac{q}{2}} dP^W(f). \]
Since the maps $T : f \mapsto (\int_0^1 f(\xi)e_1(\xi)d\xi, \ldots, \int_0^1 f(\xi)e_K(\xi)d\xi)$ is linear and continuous, we can use the change of variable formula (see [8, Formula (A.3.1)]) and obtain
\[
\int_{\mathbb{E}[0,1]} \frac{1}{\sqrt{H^W(\Phi(f))}} dP_W(f) \leq 2^{-q} \int_{\mathbb{R}^K} \left( \sum_{i=1}^K \lambda_i \eta_i^2 \right)^{-\frac{q}{2}} dP_K(\eta) \leq (4\lambda K)^{-\frac{q}{2}} \int_{\mathbb{R}^K} \|\eta\|^{-q} dP_K(\eta),
\]
where $P_K^W$ the centered $K$-dimensional Gaussian measure given by $P_K^W := P^K \circ T^{-1}$. The last integral in inequality (7.3) is finite, since we took $K > q$. Thus both $G_{\sigma,c}$ and $G^{(r)}$ satisfy Hypothesis 1.1(2). Checking Hypothesis 1.1(1) is trivial.

Finally we have for $f \in \mathbb{E}[0,1]$
\[
\begin{align*}
\nabla_H^2 G_{\sigma,c}(f) &= 0, \\
\nabla_H^2 G^{(r)}(f) &= 2^{+\infty} \sum_{i=1}^\infty \lambda_i \left( (\sqrt{\lambda_i}e_i) \otimes (\sqrt{\lambda_i}e_i) \right).
\end{align*}
\]
In particular $\|\nabla_H^2 G^{(r)}(f)\|_{H^2} = \sum_{i=1}^{+\infty} \lambda_i^2 = 1/6$. Then $G_{\sigma,c}$ and $G^{(r)}$ satisfy all the conditions of Hypothesis 1.1. We set $\Omega_{\sigma,c} := G_{\sigma,c}^{-1}(-\infty,0]$ and $\Omega^{(r)} := (G^{(r)})^{-1}(-\infty,0]$.

7.2. An example of admissible weight (1). Let $\tau$ be a finite positive Borel measure in $[0,1]$. Consider the function $U : \mathbb{E}[0,1] \to \mathbb{R}$ defined as
\[
U(f) = \Phi \left( \int_0^1 f(\xi)d\tau(\xi) \right),
\]
where $\Phi : \mathbb{R} \to \mathbb{R}$ is a $C^1$ convex function such that for $\xi \in \mathbb{R}$
\[
|\Phi'(\xi)| \leq C e^{\beta|\xi|},
\]
for some $C \geq 0$ and $\beta > 0$. Easy computations give that $U$ is a convex and continuous function. Using the fundamental theorem of calculus we get for every $\xi \in \mathbb{R}$
\[
|\Phi(\xi)| \leq |\Phi(0)| + \frac{C}{\beta} e^{\beta|\xi|}.
\]
So
\[
|U(f)| = \left| \Phi \left( \int_0^1 f(\xi)d\tau(\xi) \right) \right| \leq |\Phi(0)| + \frac{C}{\beta} e^{\beta\|f\|_{\infty}\|\tau\|_{(\mathbb{E}[0,1])^*}}.
\]
Therefore, by Fernique theorem, $U$ belongs to $L^1(\mathbb{E}[0,1], P^W)$ for every $t \geq 1$.

Observe that $U$ is Fréchet differentiable with continuous derivative, since it is the composition of a element of $(\mathbb{E}[0,1])^*$ and a $C^1(\mathbb{R})$ function. By the chain rule for every $f,g \in \mathbb{E}[0,1]$ we have
\[
U'(f)(g) = \Phi' \left( \int_0^1 f(\xi)d\tau(\xi) \right) \int_0^1 g(\xi)d\tau(\xi).
\]
So
\[
|\nabla_H U(f)|^2_H = \sum_{n=1}^{+\infty} |\partial_n U(f)|^2 = \sum_{n=1}^{+\infty} \left| U'(f)(\sqrt{\lambda_n}e_n) \right|^2 =
\]
\[
= \left( \Phi' \left( \int_0^1 f(\xi)d\tau(\xi) \right) \right)^2 \sum_{n=1}^{+\infty} \lambda_n \left( \int_0^1 e_n(\xi)d\tau(\xi) \right)^2 
\]
\[
\leq 2(\tau([0,1]))^2 \left( \Phi' \left( \int_0^1 f(\xi)d\tau(\xi) \right) \right)^2 \sum_{n=1}^{+\infty} \lambda_n = (\tau([0,1]))^2 \left( \Phi' \left( \int_0^1 f(\xi)d\tau(\xi) \right) \right)^2.
\]
By using inequality (7.4) we get
\[
|\nabla_H U(f)|^2_H \leq C^2(\tau([0,1]))^2 e^{2\beta\|f\|_{\infty}\|\tau\|_{(\mathbb{E}[0,1])^*}} \leq C^2(\tau([0,1]))^2 e^{2\beta\|f\|_{\infty}\|\tau\|_{(\mathbb{E}[0,1])^*}}.
\]
So, by Fernique’s theorem, we get that $U$ belongs to $W^{1,4}(\mathbb{E}[0,1], P^W)$ for every $t \geq 1$. This implies that $U$ satisfies Hypothesis 1.2, since checking convexity and continuity of $U$ is trivial.
Consider the problem
\[ \lambda u(f) - L_{e^{-v}p} \Omega_{\sigma,c} u(f) = g(f) , \] (7.5)
with data \( \lambda > 0 \) and \( g \in L^2(\Omega_{\sigma,c}, e^{-u}P^W) \). By using Theorem 1.3 we get that for every \( \lambda > 0 \) and \( g \in L^2(\Omega_{\sigma,c}, e^{-u}P^W) \) problem (7.5) has an unique weak solution \( u \in W^{2,2}(\Omega_{\sigma,c}, e^{-u}P^W) \). In addition the following inequalities hold
\[ \|u\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W)} \leq \frac{1}{\lambda} \|g\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W)} ; \]
\[ \|\nabla H u\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W ; H)} \leq \frac{1}{\sqrt{\lambda}} \|g\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W)} ; \]
\[ \|\nabla^2 H u\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W ; H_2)} \leq \sqrt{2} \|g\|_{L^2(\Omega_{\sigma,c}, e^{-u}P^W)} . \]

Furthermore by Theorem 1.4 we get that for \( \rho \text{-a.e. } f \in G^{-1}_\sigma(0) \)
\[ \langle \Tr(\nabla H u)(f), \Tr(\nabla H G_{\sigma,c})(f) \rangle_H = 0 , \]
then by equality (7.1) we get for \( \rho \text{-a.e. } f \in \mathcal{C}[0,1] \) with \( \int_0^1 f(\xi) d\sigma(\xi) = c \)
\[ \sum_{i=1}^{+\infty} \sqrt{\lambda_i} (\Tr \partial_i u(f)) \left( \int_0^1 e_i(\xi) d\sigma(\xi) \right) = 0 . \]

In a similar way by using Theorem 1.3 to the problem
\[ \lambda u(f) - L_{e^{-v}P} \Omega(r), e^{-u}P^W u(f) = g(f) , \] (7.6)
with data \( \lambda > 0 \) and \( g \in L^2(\Omega(r), e^{-u}P^W) \), we get that problem (7.6) has an unique weak solution \( u \in W^{2,2}(\Omega(r), e^{-u}P^W) \). In addition the following inequalities hold
\[ \|u\|_{L^2(\Omega(r), e^{-u}P^W)} \leq \frac{1}{\lambda} \|g\|_{L^2(\Omega(r), e^{-u}P^W)} ; \]
\[ \|\nabla H u\|_{L^2(\Omega(r), e^{-u}P^W ; H)} \leq \frac{1}{\sqrt{\lambda}} \|g\|_{L^2(\Omega(r), e^{-u}P^W)} ; \]
\[ \|\nabla^2 H u\|_{L^2(\Omega(r), e^{-u}P^W ; H_2)} \leq \sqrt{2} \|g\|_{L^2(\Omega(r), e^{-u}P^W)} . \]

Moreover by Theorem 1.4 we get that for \( \rho \text{-a.e. } f \in (G(\sigma))^{-1}(0) \)
\[ \langle \Tr(\nabla H u)(f), \Tr(\nabla H G_{\sigma,c})(f) \rangle_H = 0 , \]
then by equality (7.2) we get for \( \rho \text{-a.e. } f \in \mathcal{C}[0,1] \) with \( \|f\|_2 = r \)
\[ \sum_{i=1}^{+\infty} \sqrt{\lambda_i} (\Tr \partial_i u(f)) \left( \int_0^1 f(\xi) e_i(\xi) d\xi \right) = 0 . \]

### 7.3. An example of admissible weight (2).
Throughout this subsection we will assume that the following hypothesis holds.

**Hypothesis 7.1.** Let \( \Psi \in C^1([0,1]) \) be such that
\begin{enumerate}
  \item for every fixed \( r \in [0,1] \), the function \( \Psi(\cdot, r) \) is convex;
  \item for all \( s \in [0,1] \) we have
    \[ \left| \frac{\partial \Psi}{\partial s}(s, r) \right| \leq C(r) e^{\beta |s|} \]
\end{enumerate}
where \( \beta > 0 \) and \( C(\cdot) \) is a non-negative function belonging to \( L^2([0,1], d\xi) \).

We want to show that the weight
\[ U(f) := \int_0^1 \Psi(f(\xi), \xi) d\xi , \quad f \in \mathcal{C}[0,1] \]
satisfies Hypothesis 1.2. First we remark that
\[ |\Psi(s, r)| \leq |\Psi(0, r)| + C(r) \frac{e^{\beta |s|}}{\beta} . \]
So for every $f \in \mathcal{C}[0, 1]$ we get $|U(f)| \leq \|\Psi(0, \cdot)\|_\infty + \beta^{-1}\|C\|_{L^2([0, 1], d\xi)}\|e^\beta f\|_\infty$, and by Fernique’s theorem $U$ belongs to $L^t(\mathcal{C}[0, 1], P^W)$ for every $t \geq 1$.

Observe that $U$ is a Fréchet differentiable since it is the composition of a $\mathcal{C}^1$ function and a smooth function. In addition for every $f, g \in \mathcal{C}[0, 1]$ 

$$U'(f)(g) = \int_0^1 \frac{\partial \Psi}{\partial s}(f(\xi), \xi)g(\xi)d\xi.$$ 

So we get 

$$|\nabla_H U(f)|^2 = \sum_{n=1}^{\infty} |\partial_n U(f)|^2 = \sum_{n=1}^{\infty} \left|U'(f)\sqrt{\lambda_n e_n}(\xi)\right|^2 = \sum_{n=1}^{\infty} \int_0^1 \lambda_n e_n(\xi)\left|\frac{\partial \Psi}{\partial s}(f(\xi), \xi)\right|^2d\xi.$$ 

Then by Hypothesis 7.1(2) we get 

$$|\nabla_H U(f)|^2 \leq 2\int_0^1 C^2(\xi)e^{2\beta f(\xi)}d\xi \leq e^{2\beta f} - \|C\|^2_{L^2([0, 1], d\xi)}.$$ 

Therefore, by Fernique’s theorem, we get that $U$ belongs to $W^{1, t}(\mathcal{C}[0, 1], P^W)$ for every $t \geq 1$. So $U$ satisfies Hypothesis 1.2, since checking convexity and continuity is trivial.

Consider the problem 

$$\lambda u(f) - L_{e^{-U}P^W, \Omega, c} u(f) = g(f),$$  

with data $\lambda > 0$ and $g \in L^2(\Omega, e^{-U}P^W)$. By using Theorem 1.3 we get that for every $\lambda > 0$ and $g \in L^2(\Omega, e^{-U}P^W)$ problem (7.7) has an unique weak solution $u \in W^{2, 2}(\Omega, e^{-U}P^W)$, and the following inequality holds 

$$\|u\|_{W^{2, 2}(\Omega, e^{-U}P^W)} \leq \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2}\right)\|g\|_{L^2(\Omega, e^{-U}P^W)}.$$ 

Furthermore by Theorem 1.4 we get that for $\rho$-a.e. $f \in G^{-1}_1(0)$  

$$\langle \text{Tr}(\nabla_H u), \text{Tr}(\nabla_H G_{\sigma, c}(f)) \rangle_H = 0,$$ 

then by equality (7.1) we get for $\rho$-a.e. $f \in \mathcal{C}[0, 1]$ with $\int_0^1 f(\xi)d\sigma(\xi) = c,$ 

$$\sum_{i=1}^{\infty} \sqrt{\lambda_i}(\text{Tr}\partial_i u(f))\left(\int_0^1 e_i(\xi)d\sigma(\xi)\right) = 0.$$ 

In a similar way, by Theorem 1.3, the problem 

$$\lambda u(f) - L_{e^{-U}P^W, \Omega(r)} u(f) = g(f),$$  

has an unique weak solution $u \in W^{2, 2}(\Omega(r), e^{-U}P^W)$, whenever $\lambda > 0$ and $g \in L^2(\Omega(r), e^{-U}P^W)$. In addition 

$$\|u\|_{W^{2, 2}(\Omega, e^{-U}P^W)} \leq \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2}\right)\|g\|_{L^2(\Omega, e^{-U}P^W)}.$$ 

Moreover, by Theorem 1.4, if $u$ is the weak solution of (7.8), then for $\rho$-a.e. $f \in (G^{-1(r)})^{-1}(0)$  

$$\langle \text{Tr}(\nabla_H u)(f), \text{Tr}(\nabla_H G^{(r)}(f)) \rangle_H = 0,$$ 

then by equality (7.2) we get for $\rho$-a.e. $f \in \mathcal{C}[0, 1]$ with $\|f\|_2 = r,$ 

$$\sum_{i=1}^{r} \sqrt{\lambda_i}(\text{Tr}\partial_i u(f))\left(\int_0^1 f(\xi)e_i(\xi)d\xi\right) = 0.$$
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