Global surfaces of section for Reeb flows in dimension three and beyond

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Suppose that a homeomorphism $f : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathbb{R}/\mathbb{Z} \times [0, 1]$ preserves orientation, area and boundary components.

Choose a lift

\[
\begin{array}{ccc}
\mathbb{R} \times [0, 1] & \xrightarrow{F} & \mathbb{R} \times [0, 1] \\
\downarrow & & \downarrow \\
\mathbb{R}/\mathbb{Z} \times [0, 1] & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \times [0, 1]
\end{array}
\]

and consider rotation numbers

\[
\rho_0 = \lim_{n \to \infty} \frac{p_1 \circ F^n(x, 0)}{n} \quad \rho_1 = \lim_{n \to \infty} \frac{p_1 \circ F^n(x, 1)}{n}
\]

Here $p_1 : \mathbb{R} \times [0, 1] \to \mathbb{R}$ denotes projection onto the first component.

**Twist condition:** $\rho_0 \neq \rho_1$
Poincaré and Birkhoff

**Theorem**

If $\frac{p}{q}$ is between $\rho_0$ and $\rho_1$ then there are two points $P, P' \in \mathbb{R} \times [0, 1]$ satisfying

\[ F^q(P) = P + (p, 0) \quad F^q(P') = P' + (p, 0) \]

whose orbits project to distinct periodic orbits of $f$.

**Corollary**

*The twist condition forces $f$ to have infinitely many periodic points.*
Poincaré and Birkhoff

Poincaré and the planar circular restricted 3-body problem

After regularizing collisions with the heavy primary – low energies and small mass ratio – the dynamics are governed by a flow on the 3-sphere with a pair of orbits that bound an annulus-like global surface of section.

The associated first return map satisfies the twist condition.
Global surfaces of section

Ideal scenario for the dynamicist, since it opens the way for 2-dimensional methods to help in understanding a 3-dimensional flow... and also for the geometer.
Birkhoff’s Theorem

\[ K > 0 \quad \Rightarrow \quad \text{Birkhoff’s annulus is a global surface of section} \]
Sharp systolic inequalities

Birkhoff’s annulus-like global surface of section encodes:

- The total area as the Calabi invariant of the return map.
- The length $\ell_{\text{min}}$ of the shortest closed geodesic as a spectral invariant of the generating function of the return map.

With this dictionary between dynamics and geometry one gets applications to systolic geometry.

Theorem (Abbondandolo, Bramham, Hryniewicz, Salomão)

If a Riemannian metric on $S^2$ is $\delta$-pinched for some $\delta > \frac{4+\sqrt{7}}{8}$ then

$$\ell_{\text{min}}^2 \leq \pi \ Area$$

Equality holds if, and only if, the metric is Zoll.
Consider a closed geodesic $\gamma$ of length $L$ on an oriented Riemannian surface. Jacobi equation along $\gamma$ is written as

$$y''(t) + K(\gamma(t)) y(t) = 0$$

where $K$ denotes the Gaussian curvature.

**Poincaré's inverse rotation number** is defined as

$$\rho(\gamma) = \lim_{t \to +\infty} \frac{L}{2\pi} \frac{\theta(t)}{t}$$

$$y'(t) + iy(t) = r(t)e^{i\theta(t)}$$

**Theorem**

If $K > 0$ and $\rho(\gamma) \neq 1$ ($\gamma$ embedded) then the return map to Birkhoff's annulus satisfies the twist condition. In particular, there are $\infty$-many closed geodesics.
Angenent’s satellites

**Theorem (Angenent – Ann. of Math. 2005)**

*If an embedded closed geodesic $\gamma$ on a Riemannian two-sphere satisfies $\rho(\gamma) \neq 1$ then for every $\frac{p}{q}$ between 1 and $\rho(\gamma)$ some $(p, q)$-satellite about $\gamma$ is realized as a closed geodesic.*

The main idea is to use the curve-shortening flow. Length is a Lyapunov function. One constructs Conley pairs with non-trivial index associated to rational numbers between 1 and $\rho(\gamma)$.

**This is a Poincaré-Birkhoff theorem for flows!**
A Poincaré-Birkhoff Theorem for Reeb flows

Consider $\mathbb{R}^4$ with coordinates $(x_1, y_1, x_2, y_2)$ and its standard symplectic structure

$$\omega_0 = d\lambda_0 = \sum_{j=1}^2 dx_j \wedge dy_j$$

where

$$\lambda_0 = \frac{1}{2} \sum_{j=1}^2 x_j dy_j - y_j dx_j.$$ 

The standard contact structure of $S^3 = \{\sum_{j=1}^2 x_j^2 + y_j^2 = 1\}$ is

$$\xi_{std} = \{ v \in TS^3 \mid \lambda_0(v) = 0 \}$$

**Fact A.** Reeb flows on $(S^3, \xi_{std})$ are exactly the Hamiltonian flows on star-shaped energy levels in $\mathbb{R}^4$, up to time-reparametrization.

**Fact B.** Double covers of Finsler geodesic flows on $S^2$ are very special Reeb flows on $(S^3, \xi_{std})$.

**Fact C.** For reversible metrics, an embedded closed geodesic lifts to a pair of periodic orbits forming a Hopf link (up to transverse isotopy).
A Poincaré-Birkhoff Theorem for Reeb flows

A Hopf link! Very much like in the situations encountered by Poincaré!
A Poincaré-Birkhoff Theorem for Reeb flows

\[ \gamma = T\text{-periodic contractible Reeb orbit} \]

Transverse rotation number
A Poincaré-Birkhoff Theorem for Reeb flows

Theorem (Hryniewicz, Momin, Salomão)

Consider a Reeb flow on \((S^3, \xi_{\text{std}})\) admitting periodic orbits \(P_0, P_1\) that form a Hopf link.

Let \(r_0, r_1\) be the associated rotation numbers computed with respect to disks.

If \(p, q\) are relatively prime integers satisfying

\[
(r_0, 1) < (p, q) < (1, r_1) \quad \text{or} \quad (1, r_1) < (p, q) < (r_0, 1)
\]

then there exists a periodic Reeb orbit \(P \subset S^3 \setminus (P_0 \cup P_1)\) satisfying

\[
\text{link}(P, P_0) = p \quad \text{link}(P, P_1) = q.
\]
A Poincaré-Birkhoff Theorem for Reeb flows
A Poincaré-Birkhoff Theorem for Reeb flows

\[ p = 7 \quad q = 1 \]
An application

By Grayson’s work, the curve shortening flow provides a simple closed geodesic $\gamma$ on any Riemannian two-sphere satisfying

$$\text{Morse}(\gamma) + \text{Nullity}(\gamma) \geq 3.$$ 

This implies

$$\rho(\gamma) \geq 1.$$ 

Two alternatives:

- If $\rho(\gamma) = 1$ then Hingston’s results provide $\infty$-many closed geodesics.
- If $\rho(\gamma) > 1$ then Angenent’s result (or HMS independently) provides $\infty$-many closed geodesics.

Theorem (Bangert, Franks)

*Every Riemannian metric on $S^2$ admits infinitely closed geodesics.*
Systems of transversal sections

When global surfaces of section are not available: another approach to study dynamics
$p_c$ is critical point of $H$ with Morse index 1.

Examples in Physical Chemistry, Celestial Mechanics etc.

Let us assume first that $W_E \simeq S^3 = S^3 \# S^3$ and the Hamiltonian flow on $W_E$ corresponds to the Reeb flow on the tight 3-sphere.
Finite energy foliations

Theorem (Hofer, Wysocki, Zehnder – Ann. of Math. 2003)

The Hamiltonian flow on a generic star-shaped hypersurface in $\mathbb{R}^4$ admits a system of transversal sections with finitely many binding orbits having Conley-Zehnder index in $\{1, 2, 3\}$.

- The regular leaves are punctured spheres, transverse to the Reeb vector field and foliate the complement of the binding orbits. At each puncture, they are asymptotic to a binding orbit.

This foliation is constructed using pseudo-holomorphic curves in symplectizations.
Possible regular leaves of a system of transversal sections

- weakly convex case: all periodic orbits have Conley-Zehnder index $\geq 2$. 

![Diagram of possible regular leaves](image-url)
Example: a 3-2-3 Foliation

- There exists an unknotted periodic orbit $P_3$ in the interior of the 3-ball which bounds a 1-parameter family of planes.
- There exists a cylinder from $P_3$ to $P_2$.
- The picture is the same in the other side of the dividing sphere.
Existence of 3-2-3 foliations

Theorem (de Paulo, Salomão)

If $H : \mathbb{R}^4 \to \mathbb{R}$ is real-analytic and both $S_0, S'_0 \subset H^{-1}(0)$ are strictly convex then for every $E > 0$ sufficiently small $W_E$ admits a 3-2-3 foliation.

- The 3-2-3 foliation implies infinitely many homoclinics to the Lyapunoff orbit and infinitely many periodic orbits.
Let us describe the 3-2-3 foliation...

- The dividing sphere on $W_E$ contains a binding orbit $P_2$ (called Lyapunov orbit in Celestial Mechanics).
- Its open hemispheres are transverse to the flow.
3-2-3 Foliation
Some dynamical consequences of the 3-2-3 foliation

- Either the stable and unstable manifolds of $P_2$ coincide, or there exist transverse homoclinics.
The Euler problem of two fixed centers in the plane

\[ H_\mu = \frac{p_x^2 + p_y^2}{2} - \frac{\mu}{\sqrt{(x + 1)^2 + y^2}} - \frac{1 - \mu}{\sqrt{(x - 1)^2 + y^2}} + (yp_x - xp_y), \]

where \( 0 < \mu < 1 \).

\[ \frac{\mu}{\sqrt{(x + 1)^2 + y^2}} - \frac{1 - \mu}{\sqrt{(x - 1)^2 + y^2}} \]
Critical value: $c_{\text{crit}} = -\frac{1}{2} - \sqrt{\mu - \mu^2}$
Hill’s region after regularization (elliptic coordinates)

\[ c > c_{\text{crit}} \Rightarrow S^1 \times S^2 \xrightarrow{2:1} \mathbb{RP}^3 \# \mathbb{RP}^3. \]
Rational 3-2-3 foliations

Theorem (de Paulo, Hryniewicz, Salomão)

For energies slightly above the critical value $c_{\text{crit}}$, the regularized component $W_E \cong \mathbb{R}P^3 \# \mathbb{R}P^3$ admits a rational 3-2-3 foliation.

Conjecture. The PCR3BP also admits 3-2-3 foliations.