A SAT Encoding to Compute Aperiodic Tiling Rhythmic Canons

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Abstract. In Mathematical Music theory, the Aperiodic Tiling Complements Problem consists in finding all the possible aperiodic complements of a given rhythm $A$. The complexity of this problem depends on the size of the period $n$ of the canon and on the cardinality of the given rhythm $A$. The current state-of-the-art algorithms can solve instances with $n$ smaller than 180. In this paper we propose an ILP formulation and a SAT Encoding to solve this mathemusical problem, and we use the Maplesat solver to enumerate all the aperiodic complements. We validate our SAT Encoding using several different periods and rhythms and we compute for the first time the complete list of aperiodic tiling complements of standard Vuza rhythms for canons of period $n = \{180, 420, 900\}$.

Keywords: Mathematical models for music · Aperiodic tiling rhythms · SAT Encoding · Integer Linear Programming

1 Introduction

Mathematical Music Theory is the study of Music from a mathematical point of view. Many connections have been discovered, some of which have a long tradition, but they seem to be still offering new problems and ideas to researchers, whether they be music composers or computer scientists. The first attempt to produce music through a computational model dates back to 1957, when the authors composed a string quartet, also known as the Illiac Suite, through random number generators and Markov chains \cite{11}. Since then, a plethora of other works have explored how computer science and music can interact: to compose music \cite{20,22}, to analyse existing compositions and melodies \cite{7,8,9}, or even to represent human gestures of the music performer \cite{18}. In particular, Constraint Programming has been used to model harmony, counterpoint and other aspects of music (e.g., see \cite{3}), to compose music of various genres as described in the book \cite{3}, or to impose musical harmonization constraints in \cite{16}.

In this paper, we deal with Tiling Rhythmic Canons, that are purely rhythmic contrapuntal compositions. For a fixed period $n$, a tiling rhythmic canon is a...
couple of sets $A, B \subset \{0, 1, 2, \ldots, n - 1\}$ such that at every instant there is exactly one voice playing; $A$ defines the sequence of beats played by every voice, $B$ the instants at which voices start to play. If one of the sets, say $A$, is given, it is well-known that the problem of finding a complement $B$ has in general no unique solution. It is very easy to find tiling canons in which at least one of the set is periodic, i.e. it is built repeating a shorter rhythm. From a mathematical point of view, the most interesting canons are therefore those in which both sets are aperiodic (the problem can be equivalently rephrased as a research of tessellations of a special kind). Enumerating all aperiodic tiling canons has to face two main hurdles: on one side, the problem lacks the structure of other algebraic ones, such as ring or group theory; on the other side, the combinatorial size of the domain becomes enormous very soon. Starting from the first works in the 1940s, research has gradually shed some light on parts of the problem from a theoretical point of view, and several heuristics and algorithms that allow to compute tiling complements have been introduced, but a complete solution appears to still be out of reach.

Contributions. The main contributions of this paper are the Integer Linear Programming (ILP) model and the SAT Encoding to solve the Aperiodic Tiling Complements Problem presented in Section 3. Using a modern SAT solver we are able to compute the complete list of aperiodic tiling complements of a class of Vuza rhythms for periods $n = \{180, 420, 900\}$.

Outline. The outline of the paper is as follows. Section 2 reviews the main notions on Tiling Rhythmic Canons and defines formally the problem we tackle. In Section 3 we introduce an ILP model and a SAT Encoding of the Aperiodic Tiling Complements Problem expressing the tiling and the aperiodicity constraints in terms of Boolean variables. Finally, in Section 4 we include our computational results to compare the efficiency of the aforementioned ILP model and SAT Encoding with the current state-of-the-art algorithms.

2 The Aperiodic Tiling Complements Problem

We begin fixing some notation and giving the main definitions. In the following, we conventionally denote the cyclic group of remainder classes modulo $n$ by $\mathbb{Z}_n$ and its elements with the integers $\{0, 1, \ldots, n-1\}$, i.e. identifying each class with its least non-negative member.

**Definition 1.** Let $A, B \subset \mathbb{Z}_n$. Let us define the application

$$\sigma : A \times B \to \mathbb{Z}_n, (a, b) \mapsto a + b.$$ 

We set $A + B := \text{Im}(\sigma)$; if $\sigma$ is bijective we say that $A$ and $B$ are in direct sum, and we write

$$A \oplus B := \text{Im}(\sigma).$$

If $\mathbb{Z}_n = A \oplus B$, we call $(A, B)$ a tiling rhythmic canon of period $n$; $A$ is called the inner voice and $B$ the outer voice of the canon.
Remark 1. It is easy to see that the tiling property is invariant under translations, i.e. if $A$ is a tiling complement of some set $B$, also any translate $A + z$ of $A$ is a tiling complement of $B$ (and any translate of $B$ is a tiling complement of $A$). In fact, suppose that $A \oplus B = \mathbb{Z}_n$; for every $k, z \in \mathbb{Z}_n$ by definition there exists one and only one pair $(a, b) \in A \times B$ such that $k - z = a + b$. Consequently, there exists one and only one pair $(a + z, b) \in (A + z) \times B$ such that $k = (a + z) + b$, that is $(A + z) \oplus B = \mathbb{Z}_n$. In view of this, without loss of generality, we shall limit our investigation to rhythms containing 0 and consider equivalence classes under translation.

Example 1. We consider a period $n = 9$, and the two rhythms $A = \{0, 1, 5\} \subset \mathbb{Z}_9$ and $B = \{0, 3, 6\} \subset \mathbb{Z}_9$ in Figure 1 and Figure 2. They provide the canon $A \overset{\pm}{=} B \subset \mathbb{Z}_9$, since $t_{0, 1, 5} \overset{\pm}{=} t_{0, 3, 6}$. Notice that if $A$ is periodic of period $z$, $z$ must be a strict divisor of the period $n$ of the canon.

Tiling rhythmic canons can be characterised using polynomials, as follows.

Lemma 1. Let $A$ be a rhythm in $\mathbb{Z}_n$ and let $p_A(x)$ be the characteristic polynomial of $A$, that is, $p_A(x) = \sum_{k \in A} x^k$. Given $B \subset \mathbb{Z}_n$ and its characteristic polynomial $p_B(x)$, we have that

$$p_A(x) \cdot p_B(x) \equiv \sum_{k=0}^{n-1} x^k \pmod{x^n - 1}$$

if and only if $p_A(x), p_B(x)$ are polynomials with coefficients in $\{0, 1\}$ and $A \oplus B = \mathbb{Z}_n$.

Definition 2. A tiling rhythmic canon $(A, B)$ in $\mathbb{Z}_n$ is a Vuza canon if both $A$ and $B$ are aperiodic.
Remark 2. Note that a set $A$ is periodic modulo $z$ if and only if it is periodic modulo all the non-trivial multiples of $z$ dividing $n$. For this reason, when it comes to check whether $A$ is periodic or not, it suffices to check if $A$ is periodic modulo $m$ for every $m$ in the set of maximal divisors of $n$. We denote by $D_n$ this set:

$$D_n := \{ n/p \mid p \text{ is a prime factor of } n \}.$$ 

We also denote with $k_n$ the cardinality of $D_n$, so that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k_n}^{\alpha_{k_n}}$ is the unique prime factorization of $n$, where $\alpha_1, \ldots, \alpha_{k_n} \in \mathbb{N}^+$. 

For a complete and exhaustive discussion on tiling problems, we refer the reader to [2]. In this paper, we are interested in the following tiling problem.

**Definition 4.** Given a period $n \in \mathbb{N}$ and a rhythm $A \subset \mathbb{Z}_n$, the **Aperiodic Tiling Complements Problem** consists in finding all aperiodic complements $B$, i.e., subsets of $\mathbb{Z}_n$ such that $A \oplus B = \mathbb{Z}_n$.

Some problems very similar to the decision of tiling (i.e., the tiling decision problem DIFF in [13]) have been shown to be NP-complete; a strong lower bound for computational complexity of the tiling decision problem is to be expected, too.

## 3 A SAT Encoding

In this section, we present in parallel an ILP model and a new SAT Encoding for the Aperiodic Tiling Complements Problem that are both used to enumerate all complements of $A$. We define two sets of constraints: (i) the *tiling constraints* that impose the condition $A \oplus B = \mathbb{Z}_n$, and (ii) the *aperiodicity constraints* that impose that the canon $B$ is aperiodic.

**Tiling constraints.** Given the period $n$ and the rhythm $A$, let $\mathbf{a} = [a_0, \ldots, a_{n-1}]^T$ be its characteristic (column) vector, that is, $a_i = 1$ if and only if $i \in A$. Using vector $\mathbf{a}$ we define the circulant matrix $T \in \{0, 1\}^{n \times n}$ of rhythm $A$, that is, each column of $T$ is the circular shift of the first column, which corresponds to vector $\mathbf{a}$. Thus, the matrix $T$ is equal to

$$T = \begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}.$$ 

We can use the circulant matrix $T$ to impose the tiling conditions as follows. Let us introduce a literal $x_i$ for $i = 0, \ldots, n - 1$, that represents the characteristic vector of the tiling rhythm $B$, that is, $x_i = 1$ if and only if $i \in B$. Note that a literal is equivalent to a 0–1 variable in ILP terminology. Then, the tiling condition can be written with the following linear constraint:

$$\sum_{i \in \{0, \ldots, n-1\}} T_{ij} x_i = 1, \quad \forall j = 0, \ldots, n - 1. \quad (2)$$
Notice that the set of linear constraints (2) imposes that exactly one variable (literal) in the set \( \{ x_{n+i-j} \mod n \} \in A \) is equal to one. Hence, we encode this condition as an \textbf{Exactly-one} constraint, that is, exactly one literal can take the value one. The \textbf{Exactly-one} constraint can be expressed as the conjunction of the two constraints \textbf{At-least-one} and \textbf{At-most-one}, for which standard SAT encoding exist (e.g., see [6,17]). Hence, the tiling constraints (2) are encoded with the following set of clauses depending on \( i = 0, \ldots, n-1 \):

\[
\bigvee_{j \in A} \left( x_{n-(j-i)} \mod n \right) \wedge \bigwedge_{k,l \in A, k \neq l} \left( \neg x_{n-(k-i)} \mod n \vee \neg x_{n-(l-i)} \mod n \right). \tag{3}
\]

**Aperiodicity constraints.** In view of Definition 2, if there exists a \( b \in B \) such that \( (d+b) \mod n \neq b \), then the canon \( B \) is not periodic modulo \( d \). Notice that by Remark 2 we need to check this condition only for the values of \( d \in \mathcal{D}_n \).

We formulate the aperiodicity constraints introducing auxiliary variables \( y_{d,i}, z_{d,i}, u_{d,i} \in \{0,1\} \) for every prime divisor \( d \in \mathcal{D}_n \) and for every integer \( i = 0, \ldots, d-1 \). We set

\[
u_{d,i} = 1 \iff \left( \sum_{k=0}^{n/d-1} x_{i+kd} = \frac{n}{d} \right) \vee \left( \sum_{k=0}^{n/d-1} x_{i+kd} = 0 \right), \tag{4}\]

for all \( d \in \mathcal{D}_n, \ i = 0, \ldots, d-1 \), with the condition

\[
\sum_{i=0}^{d-1} u_{d,i} \leq d - 1, \quad \forall d \in \mathcal{D}_n. \tag{5}\]

Similarly to [5], the constraints (4) can be linearized using standard reformulation techniques as follows:

\[
0 \leq \sum_{k=0}^{n/d} x_{i+kd} - \frac{n}{d} y_{d,i} \leq \frac{n}{d} - 1 \quad \forall d \in \mathcal{D}_n, \ i = 0, \ldots, d-1, \tag{6}\]

\[
0 \leq \sum_{k=0}^{n/d} (1 - x_{i+kd}) - \frac{n}{d} z_{d,i} \leq \frac{n}{d} - 1 \quad \forall d \in \mathcal{D}_n, \ i = 0, \ldots, d-1, \tag{7}\]

\[
y_{d,i} + z_{d,i} = u_{d,i} \quad \forall d \in \mathcal{D}_n, \ i = 0, \ldots, d-1. \tag{8}\]

Notice that when \( u_{d,i} = 1 \) exactly one of the two incompatible alternatives in the right hand side of (4) is true, while whenever \( u_{d,i} = 0 \) the two constraints are false. Correspondingly, the constraint (8) imposes that the variables \( y_{d,i} \) and \( z_{d,i} \) cannot be equal to 1 at the same time. On the other hand, constraint (5) imposes that at least one of the auxiliary variables \( u_{d,i} \) be equal to zero.

Next, we encode the previous conditions as a SAT formula. To encode the if and only if clause, we make use of the logical equivalence between \( C_1 \iff C_2 \) and \( (\neg C_1 \vee C_2) \land (C_1 \vee \neg C_2) \). The clause \( C_1 \) is given directly by the literal \( u_{d,i} \).
The clause $C_2$, expressing the right hand side of (4), i.e. the constraint that the variables must be either all true or all false, can be written as

$$C_2 = \left( \bigwedge_{k=0}^{n/d} x_{i+kd} \right) \vee \left( \bigwedge_{k=0}^{n/d} \bar{x}_{i+kd} \right), \quad \forall d \in \mathcal{D}_n.$$ 

Then, the linear constraint (5) can be stated as the SAT formula:

$$\neg \left( u_{d,0} \land u_{d,1} \land \cdots \land u_{d,(d-1)} \right) = \bigvee_{i=0}^{d-1} \bar{u}_{d,i}, \quad \forall d \in \mathcal{D}_n.$$ 

Finally, we express the aperiodicity constraints using

$$\bigwedge_{i=0}^{d-1} \left[ (\neg C_2 \lor u_{d,i}) \land (C_2 \lor \bar{u}_{d,i}) \right] \land \bigvee_{i=0}^{d-1} \bar{u}_{d,i}, \quad \forall d \in \mathcal{D}_n. \quad (9)$$ 

Note that joining (2), (6)–(8) with a constant objective function gives a complete ILP model, which can be solved with a modern ILP solver such as Gurobi to enumerate all possible solutions. At the same time, joining (3) and (9) into a unique CNF formula, we get our complete SAT Encoding of the Aperiodic Tiling Complements Problem. (see Section 4 for computational results).

### 3.1 Existing solution approaches

For the computation of all the aperiodic tiling complements of a given rhythm the two most successful approaches already known are the Fill-Out Procedure [14] and the Cutting Sequential Algorithm [5].

The Fill-Out Procedure. The Fill-Out Procedure is the heuristic algorithm introduced in [14]. The key idea behind this algorithm is the following: given a rhythm $A \subseteq \mathbb{Z}_n$ such that $0 \in A$, the algorithm sets $P = \{0\}$ and starts the search for possible expansions of the set $P$. The expansion is accomplished by adding an element $\alpha \in \mathbb{Z}_n$ to $P$ according to the reverse order induced by a ranking function $r(x, P)$, which counts all the possible ways in which $x$ can be covered through a translation of $A$. This defines a new set, $\tilde{P} \supset P$, which is again expanded until either it can no longer be expanded or the set becomes a tiling complement. The search ends when all the possibilities have been explored. The algorithm finds also periodic solutions that must removed in post-processing, as well as multiple translations of the same rhythm.

The Cutting Sequential Algorithm (CSA). In [5], the authors formulate the Aperiodic Tiling Complements Problem using an Integer Linear Programming (ILP) model that is based on the polynomial characterization of tiling canons. The ILP model uses auxiliary 0–1 variables to encode the product $p_A(x) \cdot p_B(x)$ which characterizes tiling canons. The aperiodicity constraint is formulated analogously.
to what done above. The objective function is equal to a constant and does not impact the solutions found by the model. The ILP model is used within a sequential cutting algorithm that adds a no-good constraint every time a new canon $B$ is found to prevent finding solutions twice. In addition, the sequential algorithm sets a new no-good constraints for every translation of $B$; hence, in contrast to the Fill-Out Procedure, the CSA Algorithm does not need any post-processing.

4 Computational Results

First, we compare the results obtained using our ILP model and SAT Encoding with the runtimes of the Fill-Out Procedure and of the CSA Algorithm. We use the canons with periods 72, 108, 120, 144 and 168 that have been completely enumerated by Vuza [19], Fripertinger [10], Amiot [1], Kolountzakis and Matolesi [14]. Table 1 shows clearly that the two new approaches outperform the state-of-the-art, and in particular, that SAT provides the best solution approach. We then choose some periods $n$ with more complex prime factorizations, such as $n = p^2q^2r = 180$, $n = p^2qrs = 420$, and $n = p^2q^2r^2 = 900$. To find aperiodic rhythms $A$, we apply Vuza’s construction [19] with different choices of parameters $p_1, p_2, n_1, n_2, n_3$. Thus, having $n$ and $A$ as inputs, we search for all the possible aperiodic complements and then we filter out the solutions under translation. Since the post-processing is based on sorting canons, it requires a comparatively small amount of time. We report the results in Table 2: the solution approach based on the SAT Encoding is the clear winner (from a Music theory perspective, it is also noteworthy that this is the first time that all the tiling complements, whose number is reported in the last column of the two tables, of the studied rhythms are computed).

Implementation Details. We have implemented in Python the ILP model and in PySat [12] the SAT Encoding discussed in Section 3. We use Gurobi 9.1.1 as ILP solver and Maplesat [15] as SAT solver. The experiments are run on a Dell Workstation with a Intel Xeon W-2155 CPU with 10 physical cores at 3.3GHz and 32 GB of RAM. In case of acceptance, we will release the source code and the instances on GitHub.

Conclusions and Future Work. It is thinkable to devise an algorithm that, for a given $n$, finds all the pairs $(A, B)$ that give rise to a Vuza canon of period $n$. This could provide in-depth information on the structure of Vuza canons.

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Table 1: Aperiodic tiling complements for periods \( n \in \{72, 108, 120, 144, 168\} \).

| \( n \) | \( D_n \) | \( p_1 \) | \( n_1 \) | \( p_2 \) | \( n_2 \) | \( p_3 \) | \( n_3 \) | runtimes (s) | \#B |
|------|--------|------|------|------|------|------|------|----------|-----|
| 72   | \( \{24, 36\} \) | 2 | 2 | 3 | 3 | 2 | 1.59 | 0.10 < 0.01 | 0.03 | 6 |
| 108  | \( \{36, 54\} \) | 2 | 2 | 3 | 3 | 3 | 896.06 | 7.84 0.09 | 0.19 | 252 |
| 120  | \( \{24, 40, 60\} \) | 2 | 2 | 5 | 3 | 2 | 24.16 | 0.27 0.02 | 0.04 | 18 |
| 144  | \( \{48, 72\} \) | 4 | 2 | 3 | 3 | 2 | 82.53 | 2.93 0.02 | 0.11 | 36 |
| 168  | \( \{24, 56, 84\} \) | 2 | 2 | 7 | 3 | 2 | 461.53 | 17.61 0.04 | 0.20 | 54 |

Table 2: Aperiodic tiling complements for periods \( n \in \{180, 420, 900\} \).

| \( n \) | \( D_n \) | \( p_1 \) | \( n_1 \) | \( p_2 \) | \( n_2 \) | \( p_3 \) | \( n_3 \) | runtimes (s) | \#B |
|------|--------|------|------|------|------|------|------|----------|-----|
| 180  | \( \{36, 60, 90\} \) | 2 | 2 | 5 | 3 | 3 | 2.57 3.57 | 5.62 | 2052 |
| 420  | \( \{60, 84, 140, 210\} \) | 7 | 5 | 3 | 2 | 2 | 2.13 3.57 | 720 |
| 900  | \( \{180, 300, 450\} \) | 2 | 25 | 3 | 3 | 2 | 1.17 4.13 | 1120 |
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