Geometric Formulation of the Averaging Operation

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Abstract

A general framework for different averaging procedures is introduced. We motivate the existence of this framework through three examples: 1. The Averaging Principle that appears in Classical Mechanics, which is on the basis of Perturbation Theory, 2. The integration along the fiber leading to the Thom isomorphism theorem in Algebraic Topology and 3. The averaging of some linear connections in some pull-back bundles. The resulting averaged connections are affine connections on the tangent bundle of the manifold M. The second motivation comes from the problem of, given vector bundle automorphism, to define a push-forward vector bundle automorphism. We will see that a definition of averaging exists such that it solves this problem and contains the three examples of averaging procedure described before. After this, we explain the notion of convex invariance for the case of orientable Riemannian vector bundles. As a consequence, we consider a new approach to Finsler Geometry.

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1 Motivation

The averaging operation can be written formally as a weighted integral,

\[ <A>(x) = \frac{\int_U d\mu(y) A(x, y)}{\int_U d\mu(y)}, \]

with \( y \in U \). This way of defining the average requires a positive measure \( d\mu \) defined on the set \( U \) and such that \( \int_U 1 d\mu < \infty \).

We consider three examples of averaging:

1. Averaging Method in Classical Mechanics [1].

   Let us consider the trivial fiber bundle \( T^k \times U \). The local coordinates on the \( k \)-dimensional torus \( T^k \) are \( \{ \Phi = (\phi_1, ..., \phi_k), \ \phi_i \in [0, 2\pi] \} \) and they are solutions of the “perturbed” system of differential equations:

   \[ \dot{\phi}_k = w(\vec{I}) + \epsilon f(\vec{I}, \vec{\phi}), \quad \dot{\phi} = \frac{d\phi}{dt}, \]  

   \[ \dot{\vec{I}} = \epsilon \vec{g}(\vec{I}, \vec{\phi}), \]  

   where \( \epsilon \) is a “small” perturbation parameter. The averaging principle consists on substituting the system of differential equations (1.1) by the averaged system:

   \[ \tilde{\vec{J}}(\tilde{\vec{J}}) = \epsilon \vec{g}, \quad \vec{g} = (2\pi)^{-k} \int_0^{2\pi} \cdots \int_0^{2\pi} \vec{g}(\vec{J}, \vec{\phi}) d\phi_1 \cdots d\phi_k. \]  

   Then, it is assumed (or proved in some cases) that the equation (1.3) is a good approximation to the original equation (1.2). Therefore one observes that the averaging operation in Classical Mechanics is an integration along the fiber, which is the \( k \)-torus.

   Usually the averaged system is simpler to analyze than the original one and sometimes there is a perturbation theory which allows to obtain approximate solutions of the original system of differential equations from the solutions of the system of averaged differential equations, using Perturbation Theory. In this sense, the averaged system of equations retains fundamental information from the original one.

2. Integration along the fiber in Algebraic Topology [2].

   The theorem of R. Thom in Algebraic Topology relates the compact de Rham cohomology \( H^*(\mathcal{E}) \) of a vector bundle \( \mathcal{E} \rightarrow M \) of finite rank
with the cohomology $H^*_{cv}(M)$ of the base manifold $M$. This is achieved using Poincaré’s lemma for compact vertical de Rham cohomology. This lemma is constructed through an integration along the fiber. If the dimension of the fiber is $k$ and a local trivialization of the vector bundle $E \longrightarrow M$ has coordinates $(x, t)$ and $\Phi$ is a $p$-form on $M$, the integration along the fiber is defined by

\begin{align*}
(a) \quad & \pi^*(\Phi(x)) f(x, t_1, ..., t_n) dt_1 \wedge \cdots \wedge dt_r \mapsto 0, \quad r < k. \\
(b) \quad & \pi^*(\Phi(x)) f(x, t_1, ..., t_n) dt_1 \wedge \cdots \wedge dt_k \mapsto \Phi(x) \int_{\mathbb{R}^k} f(x, t_1, ..., t_k) dt_1 \wedge \cdots \wedge dt_k.
\end{align*}

There are two basic properties of this operation:

(a) With the above operations, it is possible to define the averaging of any form on the vector bundle $E$.

(b) The integration along the fiber commutes with the exterior differential, $\pi^*d_{E}\phi := d_{M}(\pi^*\phi)$, where $d_{E}$ and $d_{M}$ are the corresponding exterior differential operators on $E$ and $M$. Therefore, $\pi^*$ defines a map between cohomology groups.

One can prove the following:

**Proposition 1.1 (Poincaré’s Lemma for differential forms with compact vertical support)**

Integration along the fiber produces the isomorphism:

$$\pi^* : H^*_{cv}(M \times \mathbb{R}^k) \longrightarrow H^{*-k}(M).$$

The global version of the proposition is the Thom isomorphism theorem:

**Theorem 1.2 (Thom’s Isomorphism)**

If the vector bundle $\pi : E \longrightarrow M$ of finite type and is orientated with rank $k$, then there exists the following isomorphism:

$$H^*_{cv}(E) \simeq H^{*-k}(M).$$

The fact that we want to emphasize here is the existence of an averaging operation on forms, which induces an homomorphism on the cohomology classes and the fact that the averaged forms still carry topological information of the original spaces.
3. **Average of a Family of Linear Operators on $\Gamma(\pi^*\mathcal{T}_M)$**.

In ref. [3] it was presented a way of averaging operators acting on sections of certain pull-back vector bundles and in particular on $\pi^*\mathcal{T}^{(p,q)}M$. We assume a Finsler structure $(M, F)$ on $M$ [4], [5]. Let us consider the pull-back bundle $\pi_2 : \pi^*\mathcal{T}^{(p,q)}M \to I$, defined by the commutative diagram

$$
\begin{array}{ccc}
\pi^*\mathcal{T}^{(p,q)}M & \xrightarrow{\pi_2} & \mathcal{T}^{(p,q)}M \\
\pi_1 \downarrow & & \downarrow \pi \\
I & \xrightarrow{\pi} & M.
\end{array}
$$

$I$ is the indicatrix bundle over $M$,

$$I := \bigsqcup_{x \in I} I_x := \{ y \in T_xM \mid F(x, y) = 1 \}.$$

Each of the indicatrix $I_x$ is a compact sub-manifold of $T_xM$.

Let us consider a family of operators between the corresponding tensor spaces,

$$A_w := \{ A_w : \pi^*_w \mathcal{T}^{(p,q)}M \to \pi^*_w \mathcal{T}^{(p,q)}M \},$$

with $w \in \pi^{-1}(x)$. The average of this family of operators is defined to be the operator

$$< A > (x) : \mathcal{T}_x^{(p,q)}M \to \mathcal{T}_x^{(p,q)}M,$$

with $x \in M$, given by the action:

$$< A_w > := < \pi_2|_u A_w \pi^*_u >_u S_x = \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} d\mu(u) \pi_2|_u A_w \pi^*_u \right) S_x,$$

$$\text{vol}(I_x) := \int_{I_x} 1 d\mu, \quad u \in \pi^{-1}(x), \quad S_x \subset \mathcal{T}_x^{(p,q)}M. \quad (1.4)$$

$d\mu$ is the standard volume form induced on the indicatrix $I_x$ from the Riemannian volume of the Riemannian structure $(T_xM \setminus \{0\}, g_x)$, where the metric is $g_x := g_{ij}(x, y)dy^i \otimes dy^j$, with fixed $x \in M$ and $y \in T_xM \setminus \{0\}$.

The above operation is pointwise. In order to define an averaged operator acting on sections, we proceed as follows. Let us denote by $\{ A_w, w \in I_x, z \in U \subset M \}$ and $U \ni x$ open an operator acting in local sections
of $\Gamma_{\pi^{-1}U}(\pi^*T^{(p,q)}M)$. Then, we define the average of the family to be an operator

$$< A > : \Gamma_U(TM) \longrightarrow \Gamma_U(TM)$$

$$(< A > S)|_x := \frac{1}{vol(I_x)} \left( \int_{I_x} d\mu(w) \pi_2|wA_w\pi^*_wS_z \right)|_x.$$

In this way, one defines pointwise a new section on the open neighborhood $U$, repeating this operation on each point $z \in U$.

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The second motivation to look for a geometric averaging operation of geometric objects comes as follows. Let us consider the following vector bundle automorphism

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\phi} & \mathcal{E} \\
\pi_1 & & \pi_1 \\
N & \xrightarrow{\phi} & N,
\end{array}$$

where the diagram (1.4) is commutative. Then, consider the bundle morphism defined by the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\tilde{\beta}} & TM \\
\pi_1 & & \pi \\
N & \xrightarrow{\beta} & M.
\end{array}$$

For a general vector bundle morphism $(\tilde{\beta}, \beta)$ there is not a natural vector bundle automorphism

$$\begin{array}{ccc}
TM & \xrightarrow{\tilde{\lambda}} & TM \\
\pi & & \pi \\
M & \xrightarrow{\lambda} & M
\end{array}$$

induced from (1.4) and (1.5). Therefore, we can formulate the following problem:

**Problem**

*When is it possible to formulate a natural push-forward bundle morphism of the auto-morphism (1.4)? How one can formulate the morphism?*
The definition of geometric averaging that we will propose provides a solution to this problem in some particular cases. It consists on the specification of a vector bundle morphism such that the diagram:

\[
\begin{array}{ccc}
TM & \xrightarrow{\langle \cdot, \cdot \rangle} & E \\
\pi & & \pi_1 \\
M & \xrightarrow{\langle \cdot, \cdot \rangle} & N
\end{array}
\]  

commutes and such that \((\tilde{\lambda}, \lambda)\) is defined using the vector bundle morphisms \((\tilde{\beta}, \beta), (\tilde{\Phi}, \Phi)\) and \((\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle)\).

### 2 Averaging Operation

It seems that one can define the averaging operation in different categories. However, we formulate it in the category of smooth manifolds and smooth maps. Let us consider the category of smooth vector bundles with vector morphisms \(\text{Vec}_R\). Let \(\pi : E \to N\) and \(\tilde{\pi}_1 : \tilde{E} \to \tilde{N}\) two elements of this category. Then, consider the vector bundle morphism:

\[
\begin{array}{ccc}
E & \xrightarrow{\tilde{\beta}} & \tilde{E} \\
\pi_1 & & \tilde{\pi}_1 \\
N & \xrightarrow{\beta} & N
\end{array}
\]  

Let us assume the existence of a smooth map \(\iota : N \to \tilde{E}\) such that

\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\iota} & N \\
\tilde{\pi}_1 & & \pi_1
\end{array}
\]

is a commutative diagram. We also assume the existence of some measures:

1. On each fiber \(\tilde{\pi}_1^{-1}(x) \subset \tilde{E}\), there is a real valued measure

\[
\mu : \sigma \to R, \quad \sigma \subset \tilde{\pi}_1^{-1}(x)
\]

such that \(\pi^{-1}(x) \supset \iota(u)\) and that \(\mu(\iota(u)) < \infty\) for each \(x \in \tilde{N}\).
2. There is a vector valued measure $\mu_V$, which takes values on the fiber:

$$\mu_V : \sigma \mapsto \int_{i(\sigma)} \iota(\xi) d\mu \in \tilde{\pi}_1^{-1}(x), \quad \sigma \subset \tilde{\pi}_1^{-1}(x).$$

Both measures have compact support on the fiber $\pi^{-1}(x) \subset \tilde{E}$.

Let us consider the vector bundle morphism (1.4). We will define an associated vector bundle auto-morphism on the bundle $\tilde{\pi}_1 : \tilde{E} \to \tilde{N}$,

\begin{equation}
\tilde{E} \xrightarrow{\lambda} \tilde{E} \xrightarrow{\pi_1} \tilde{N} \xrightarrow{\phi = Id} \tilde{N}.
\end{equation}

Combining the above commutative diagrams, we obtain the following diagram:

\begin{equation}
\tilde{E} \xrightarrow{\beta^{-1}} \tilde{E} \xrightarrow{\delta} \tilde{E} \xrightarrow{\tilde{\beta}} \tilde{E} \xrightarrow{\pi_1} \tilde{N} \xrightarrow{\phi = Id} \tilde{N}.
\end{equation}

Then, the let us consider the following composition (which is not a vector bundle morphism):

\begin{equation}
\tilde{E} \xrightarrow{\tilde{\pi}_1} \tilde{N} \xrightarrow{\beta^{-1}} \tilde{N} \xrightarrow{\iota} \tilde{E} \xrightarrow{\tilde{\beta}^{-1}} \tilde{E} \xrightarrow{\delta} \tilde{E} \xrightarrow{\tilde{\beta}} \tilde{E}.
\end{equation}

This composition is not a vector bundle morphism and this is in fact the problem to solve.

**Definition 2.1** The averaging operation is the following map,

\begin{equation}
S(\xi) \mapsto \frac{1}{\int_{i(\beta^{-1}(x))} d\mu} \int_{i(\beta^{-1}(x))} d\mu_V S(\xi), \quad S(\xi) \in \tilde{\pi}_1^{-1}(\tilde{x}) \subset \tilde{E}.
\end{equation}

**Remark**. Since the averaging operation depends on the measures $(\mu, \mu_V)$, one has to speak of the averaging operation, depending on the measure $(\mu, \mu_V)$.
From the composition (2.5), we can construct the following morphism:

\[ \tilde{E} \xrightarrow{\tilde{\pi}_1} \tilde{N} \xrightarrow{\beta^{-1}} N \xrightarrow{\iota} \tilde{E} \xrightarrow{\tilde{\beta}^{-1}} E \xrightarrow{\tilde{\phi}} E \xrightarrow{\tilde{\beta}} \tilde{E} \xrightarrow{\int_{\mu V}} \tilde{E}. \]  

(2.7)

Note that \( \beta^{-1}(x) \) and \( \tilde{\beta}^{-1}(u) \) are not maps but sub-sets of the corresponding manifolds.

We define the bundle push-forward automorphism \((\tilde{\lambda}, \lambda)\) to be the composition (2.7):

\[ \tilde{\lambda} : E \longrightarrow \tilde{E} \]

\[ \tilde{\lambda}(w) \mapsto \frac{1}{\mu(\lambda(w))} \cdot \int_{\tilde{\pi}_1^{-1}(\tilde{x})} \tilde{\beta} \circ \tilde{\Phi} \circ \tilde{\beta}^{-1} \circ \iota \circ \beta^{-1} \circ \tilde{\pi}_1(w) d\mu_V, \]

\[ w \in \pi^{-1}(\tilde{x}) \subset \tilde{E}, \quad \tilde{x} \in \tilde{N}. \]  

(2.8)

The bundle on the bases manifold is given by

\[ \lambda : \tilde{N} \longrightarrow \tilde{N} \]

\[ \tilde{x} \mapsto \tilde{x}. \]  

(2.9)

Therefore, we have proved the following

**Theorem 2.2** Given the vector bundle morphism (2.1) and the vector bundle automorphism (1.4), there is an induced vector bundle automorphism (2.3) defined by the map (2.7), the formula (2.8) and (2.9).

**Remark.** We do not need to consider \( \Phi = Id; \Phi \) can be also a surjective map. However, then one has to consider an extra map

\[ \begin{array}{c}
E \xrightarrow{\pi_2} \tilde{E} \\
\pi_1 \downarrow \downarrow \tilde{\pi}_1 \\
M \xrightarrow{\pi} \tilde{M} \end{array} \]  

(2.10)

**Definition 2.3** The commutative diagram of the form

\[ \begin{array}{c}
E \xrightarrow{\pi_2} \tilde{E} \\
\pi_1 \downarrow \downarrow \tilde{\pi}_1 \\
M \xrightarrow{\pi} \tilde{M} \end{array} \]  

(2.11)

will be called fundamental diagram.
3 Examples

The three examples of section 1 are re-considered here in the framework discussed in section 2.

1. Averaging Operation in Classical Mechanics

In this case, we make the following identifications \( N := U \times T^k \), where \( T^k \) is the \( k \)-dimensional torus and \( U \) is the \( k \)-dimensional real ball, \( \tilde{N} = U, \tilde{E} = TU, \phi \) is the canonical projection \( \pi : U \times T^k \to U \). Then \( \mathcal{E} = \pi^*TU \), while the map \( \iota : U \times T^k \to TU \) is an embedding. We have the following diagram,

\[
\begin{array}{ccc}
\pi^*TU & \xrightarrow{\pi_2} & TU \\
\downarrow{\pi_1} & \xrightarrow{\iota} & \downarrow{\tilde{\pi}_1} \\
U \times T^k & \xrightarrow{\pi} & U.
\end{array}
\]  

(3.1)

We assume that the support of the measure \( \mu_V \) lives on \( \iota(U \times T^k) \).

2. Integration along the fiber in Algebraic Topology

In this case, the identification is the following \( M = \tilde{N} \) is a manifold, \( N = E \) is a vector bundle over \( M \) of rank \( k \) of finite type. \( \Gamma^{p+k}E \) and \( \Gamma^pM \) are the vector bundles of compact support \( p+k \)-forms and \( p \)-forms on \( E \) and \( M \) of compact support on the fiber respectively. The measure is the usual measure given on each fiber. Therefore, the fundamental diagram is:

\[
\begin{array}{ccc}
\Gamma(\Lambda^{p+k}E) & \xrightarrow{\pi_2} & \Gamma(\Lambda^pM) \times \mathbb{R}^k \\
\downarrow{\pi_1} & \xrightarrow{\iota} & \downarrow{\tilde{\pi}_1} \\
\mathcal{E} & \xrightarrow{\pi} & M.
\end{array}
\]  

(3.2)

Note that in this case \( \iota \) is not an embedding, but a local embedding. Also note that the second fiber bundle contains a factor \( \mathbb{R}^k \). Essentially what one makes is to eliminate the dependence on the fiber by integration.

3. Average of Dynamical Connections

In this case, the diagram (1.6) is trivial, with \((\tilde{\phi}, \phi) = (Id, Id)\). The
fundamental diagram is

\[
\pi^* \mathbf{T}M \xrightarrow{\pi_2} \mathbf{T}M \xrightarrow{\pi_1} \mathbf{I} \xrightarrow{\iota} \mathbf{M}. \tag{3.3}
\]

The measures are defined by equation (1.3); \( \iota : \mathbf{N} \rightarrow \mathbf{T}M \) is the natural embedding of the indicatrix bundle on the tangent bundle.

Therefore our theory captures three interesting examples of averaging procedure.

4 Convex Invariance in Finsler Geometry

In this section is described an example of convex invariance which appears in Finsler Geometry in ref. [3].

In the case of Riemannian vector bundles, there is always a natural one. Consider a Riemannian vector bundle with fiber metric \( g_x \) (which is obtained from the fundamental tensor of the Finsler structure, but with \( x \) fixed). The local coordinates on the fiber are \((t^1, \ldots, t^k)\) and let us consider the following measure on each fiber:

\[
d\mu(x,t) = \sqrt{g} dt^1 \wedge \cdots dt^k. \tag{4.1}
\]

In the specific case of Finsler structures, the tensor \( g_x := g(x, \cdot) \) defines a Riemannian structure on the fibers of \( \pi^* \mathbf{T}M \).

Let us consider the \((k+1)\)-form \( w^i(x,y) \wedge d\mu \) associated to the linear connection 1-forms \( w^i \) of the Chern connection. The result are the 1-forms \( \langle w^i(x,y) \rangle \) on the bundle \( \mathbf{T}M \rightarrow \mathbf{M} \) with vector values on \( \mathbf{T}_x \mathbf{M} \). Then, we have the following:

**Proposition 4.1** Let the connection 1-forms \( w^i \) on the pull-back bundle \( \pi^* \mathbf{T}M \). Then \( \pi^* \langle w^i \rangle \) are the 1-forms of a linear connection on \( \pi^* \mathbf{T}M \).

**Proof:** Consider the convex sum of linear forms

\[
t_1 w^i_1(x, y_1) + \cdots + t_p w^i_p(x, y_p), \quad t_1 + \cdots + t_p = 1.
\]

Since we fix the vector dependence to be \( y_i \), each \( w^i_p(x, y_i) \) defines a 1-form over \( \mathbf{M} \), since the action of the form \( w^i(x, y) \) on a section \( \pi^* Z \in \Gamma \pi^* \mathbf{T}M \) is

\[
w^i_p(x, y_i)(\pi^* Z) = \Gamma^i l_{(x,y)}(\pi^* Z)^i.
\]
If one fix $y$, it defines a linear connection on $M$ by the relation

$$y^i_j(x)(Z) = w^i_j|_{(x,y)}(\pi^*Z)^j.$$  

Therefore, the above convex combination defines a connection on $M$, because any of these 1-forms corresponds to a linear connection on the tangent bundle $TM$ and convex combination of connection 1-forms is a connection 1-form.

Let us consider the manifold $\Sigma_x \subset \pi^{-1}(x) \subset N$ and a set of connection 1-forms $w^i(x,y)$ on $M$ labeled by points on $\Sigma$. Instead of considering finite convex combinations, we consider the averaged operation 2.1. This can be considered as a limit of a convex sums of linear 1-forms. Since all the operations are continuous and the limit exists, in the limit the 1-forms $\{<w^i_{(x,y)}>\}$ defines also the linear connection 1-forms of a linear connection on $M$. Then we only need to pull-back these one forms, obtaining $\pi^*<w^i>$. □

Now, we identify the 1-form $<w^i>$ with the averaged connection 1-form discussed in reference [3]. In order to show this, let us recall briefly the construction of [3]. Let us consider the fiber bundle $\pi : N \rightarrow M$ and the pull-back vector bundle $\pi^*TM \rightarrow N$. Let $\{e_1, ..., e_n\}$ be a local frame basis for the sections of $\Gamma TM$ Then $\{\pi^*e_1, ..., \pi^*e_n\}$ is a basis for the fiber $\pi_u^{-1} \subset \pi^*TM$, $u \in N$; $\{h_1, ..., h_n\}$ is a basis of the horizontal distribution $\mathcal{H}_u \subset T_uN$, while $\{v_1, ..., v_n\}$ is a basis frame for the vertical distribution $\mathcal{V}_u \subset T_uN$.

Given the non-linear connection $\nabla$ on $TN \rightarrow N$, there are several distinguished linear connections on the pull-back bundle $\pi^*TM \rightarrow N$. One of these linear connections is the Chern connection, which is a linear connection on $\pi^*TM \rightarrow N$ characterized by:

1. $\nabla$ on $\pi^*TM \rightarrow N$ is a symmetric connection in the sense that

$$\nabla_X \pi^*Y - \nabla_Y \pi^*X - \pi^*[X,Y] = 0,$$  \hspace{1cm} (4.2)

where $X, Y \in \Gamma TM$, $\tilde{X}, \tilde{Y} \in \Gamma TN$ are arbitrary lifts of $X, Y \in \Gamma TM$.

2. The covariant derivative along vertical directions of sections of $\pi^*TM$ is zero:

$$\nabla_{v_j} \pi^*e_k = 0.$$  \hspace{1cm} (4.3)

3. The covariant derivative of the metric along vertical directions is zero:

$$\nabla_{h_j}(\pi^*g) = 0.$$  \hspace{1cm} (4.4)
4. The covariant derivative of the metric along vertical connections is given in terms of the Cartan tensor [4]:

\[ \nabla_{F} \frac{\partial}{\partial j} (\pi^* g) = A(F \frac{\partial}{\partial j}, \cdot, \cdot). \] (4.5)

5. By definition the covariant derivative of a function \( f \in \mathcal{F}(\mathbb{N}) \) is given by

\[ \nabla_{\tilde{X}} f := \tilde{X}(f), \quad \forall \tilde{X} \in T_u \mathbb{N}. \]

The 1-forms \( w^i(x, y) > (x) \) are associated with the covariant derivative \( \langle \nabla \rangle \). To verify this fact, one needs to check that the connection corresponding to the 1-forms \( w^i > (x) \) and the averaged connection \( \langle \nabla \rangle \) have the same coefficients in a given local frame. The result for the connection coefficients of both connections are

\[ < \Gamma^i_{jk} > = \frac{1}{\text{vol}(\Sigma_x)} \cdot \int_{\Sigma_x} \Gamma^i_{jk}(x, y) d\mu(y). \] (4.6)

Similar results hold for any arbitrary linear connection on the vector bundle \( \pi^* \mathbf{TM} \).

**Theorem 4.2** There exists a homotopy between each linear connection on the bundle \( \pi^* \mathbf{TM} \rightarrow \mathbf{I} \) and the pull-back of its averaged connection.

**Proof:** Let us consider the connection 1-forms \( w^i \). Then, due to the form of the average (for instance, the formula (4.6)), the following convex homotopy property holds:

\[ < ((1 - t)w^i + t \pi^* w^i) > = < w^i > \Rightarrow \]

\[ \pi^* < ((1 - t)w^i + t \pi^* w^i) > = \pi^* < w^i > . \]

Since the parameter \( t \in [0, 1] \), we have proved that \( \phi \) defines a retraction between \( \{ (1 - t)w^i + t \pi^* w^i \} \) and \( \pi^* < w^i > \). Therefore there is a retraction between \( w^i \) and \( \pi^* < w^i > \).

This result suggests the following conjecture on the nature of Finsler Geometry:
Finsler Geometry is characterized by the following data:

1. An Affine Body, determined by the averaged connections, obtained averaging Chern, Cartan or Berwald connections, for instance. Note that the average of the Chern, Cartan and Berwald connection are in principle different.

2. Riemannian properties. They are obtained through averaging the fundamental tensor. The averaging of the fundamental tensor of a Finsler structure provides a Riemannian structure.

3. Non-convex invariant properties, in particular non-reversibility of the metric function. These properties are lost after averaging.

In principle, there is not relation between both averages. The non-Riemannian and non-affine properties are encoded in two tensors. The first tensor is the difference between the Finsler and the averaged structure,

\[ \delta g := (g_{ij} - \langle g_{ij} \rangle) \pi^* dx^i \otimes \pi^* dx^j. \]  
(4.7)

The second tensor depends on the difference between the initial and averaged connection and measures the degree of non-affine geometry of the Finsler geometry,

\[ \delta \Gamma := (\Gamma^i_{jk} - \langle \Gamma^i_{jk} \rangle) \pi^* \partial_i \otimes \pi^* dx^j \otimes \pi^* dx^k. \]  
(4.8)

There is a third tensor which measures how much different are the average of the metric and the average of the connection. Let us consider the Levi-Civita connection \( h^i \Gamma_{jk} \) of the Riemannian metric \( h = \langle g_{ij} \rangle dx^i \otimes dx^j \) and its Levi-Civita connections \( h^i \nabla^j_{lk} \)

\[ T := (h^i \Gamma^j_{lk} - \langle h^i \Gamma^j_{lk} \rangle) \pi^* \partial_i \otimes \pi^* dx^j \otimes \pi^* dx^k. \]  
(4.9)

This last tensor lives directly on the manifold \( M \). Therefore it is a convex invariant quantity.

There are some direct consequences about the above tensors:

**Proposition 4.3** The following relations hold:

1. \( T \) is convex invariant, \( \pi^* < T > = T \).
2. \( < \delta \Gamma > = 0 \).
3. \( < \delta g > = 0 \).
A Berwald space is a Finsler space such that the Chern connection defines an affine connection directly on $M$.

**Proposition 4.4** Let $(M, F)$ a Finsler structure. The following holds,

1. $\delta \Gamma = 0$ iff the structure $(M, F)$ is Berwald [4].
2. $\delta g = 0$ iff the structure $(M, F)$ is Riemannian.

It is difficult to state the general condition which implies $T = 0$. It seems that Landsberg spaces is the category to solve that problem [3, section 7].

Therefore, we can think a Finsler space as characterized by a set $(<g>, <\Gamma>, \delta g, \delta \Gamma, T)$ such that $<g>$ is a Riemannian metric, $<\Gamma>$ is an affine connection, $\delta g$ a symmetric tensor with norm (using $<g>$ less than 1 and $T^i_{jk}$ a symmetric tensor in $jk$.

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