Realization of $q$-deformed spacetime as star product by a Drinfeld twist

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Abstract. Covariance ties the noncommutative deformation of a space into a quantum space closely to the deformation of the symmetry into a quantum symmetry. Quantum deformations of enveloping algebras are governed by Drinfeld twists, inner automorphisms which relate the deformed to the undeformed coproduct. While Drinfeld twists naturally define a covariant star product on the space algebra, this product is in general not associative and does not yield a quantum space. It is reported that, nevertheless, there are certain Drinfeld twists which realize the quantum plane, quantum Euclidean 4-space, and quantum Minkowski space.

1. Introduction

From the beginnings of quantum field theory it had been argued that the pathological ultraviolet divergences should be remedied by limiting the precision of position measurements. This is one of the main motivations to study noncommutative geometries, which imply a space uncertainty in a natural and fundamental way. From experience we know that, if spacetime is noncommutative, the noncommutativity can only be small. This suggests to describe noncommutative spacetime as perturbative deformation of ordinary, commutative Minkowski space. The algebraic aspects of a deformation can be separated from the analytic questions of continuity and convergence by considering formal power series. In such a framework a noncommutative geometry is a formal deformation in the sense of Gerstenhaber [1] of the function algebra on the space manifold. Such formal deformations have appeared naturally in the context of gauge theories on noncommutative spaces [2,3].

Algebraically, physical spacetime is characterized by the Minkowski algebra of spacetime functions and covariance with respect to the Lorentz symmetry. The symmetry distinguishes Minkowski space from other 4-dimensional flat spaces, such as Euclidean 4-space. Covariance ties the deformation of the symmetry closely to the deformation of the space. Quantum deformations of the enveloping algebra which describes this symmetry are known to be governed by Drinfeld twists, inner automorphisms which relate the deformed to the undeformed coproduct [4,5]. Therefore, one ought to be able to use these twists in order to deform the space algebra into the according quantum space as it was suggested in [6]. It will be shown that for quantum Minkowski space this is indeed possible.

2. The problem

2.1. Covariant quantum spaces

Let $g$ be the Lie algebra of the symmetry group of a space and $\mathcal{R}$ be the function algebra of this space. The elements $g \in g$ of the Lie algebra act on $\mathcal{R}$ as derivations, $g \triangleright xy = \ldots$
an inner automorphism. That is, there is an invertible element $F$ such that the star product (6) can be written in the more familiar form [9]

$$x \star y := \mu_h(x \otimes y)$$

and $g_{(1)} \otimes g_{(2)} := \Delta_h(g)$. A large class of deformations which are covariant in this sense are quantum spaces.

2.2. Star products by Drinfeld twists

In the case of quantum spaces, the deformed coproduct belongs to the Drinfeld-Jimbo deformation $\mathcal{H}(\mathfrak{g})$ of the enveloping Hopf algebra $[7][8]$. Drinfeld has observed $[4][5]$ that as $\hbar$-adic algebras $\mathcal{H}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})[[\hbar]]$ are isomorphic and that the deformed coproduct $\Delta_h$ of the Hopf algebra $\mathcal{H}(\mathfrak{g}) \cong (\mathcal{U}(\mathfrak{g})[[\hbar]], \Delta_\hbar, \varepsilon_\hbar, S_\hbar)$ is related to the undeformed coproduct $\Delta$ by an inner automorphism. That is, there is an invertible element $\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ with $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\hbar)$, called Drinfeld twist, such that

$$\Delta_h(g) = \mathcal{F} \Delta(g) \mathcal{F}^{-1}.$$  

Comparing the covariance condition (2) of the deformed multiplication,

$$g \triangleright \mu_h(x \otimes y) = \mu_h(\Delta_h(g) \triangleright [x \otimes y]) = \mu_h(\Delta(\mathcal{F} \Delta(g) \mathcal{F}^{-1}) \triangleright [x \otimes y]) $$

with the covariance property (1) of the undeformed product, we see that Eq. (5) is naturally satisfied if we define the deformed product by

$$\mu_h(x \otimes y) := \mu(\mathcal{F}^{-1} \triangleright [x \otimes y]) \iff x \star y := (\mathcal{F}^{-1}_{[1]} \triangleright x)(\mathcal{F}^{-1}_{[2]} \triangleright y),$$

as it was observed in [6] (suppressing in a Sweedler like notation the summation of $\mathcal{F} = \sum_i \mathcal{F}_{i1} \otimes \mathcal{F}_{i2} \equiv \mathcal{F}_{[1]} \otimes \mathcal{F}_{[2]}$). Since the elements of the Lie algebra $\mathfrak{g}$ act on the undeformed space algebra $\mathbb{K}$ as derivations, $\mathcal{F}^{-1}$ acts as $\hbar$-adic differential operator on $\mathcal{K} \otimes \mathcal{K}$. Hence, writing out the $\hbar$-adic sum of $\mathcal{F}^{-1} = 1 \otimes 1 + \sum_i \hbar^i \mathcal{F}^{-1}_k$ we can define the bidifferential operators

$$B_k(x, y) := \mu(\mathcal{F}^{-1} \triangleright [x \otimes y]) = (\mathcal{F}^{-1}_{[1]} \triangleright x)(\mathcal{F}^{-1}_{[2]} \triangleright y),$$

such that the star product (6) can be written in the more familiar form [9]

$$x \star y := xy + \hbar B_1(x, y) + \hbar^2 B_2(x, y) + \ldots$$

2.3. The problem of associativity

Even though the twist $\mathcal{F}$ yields by Eq. (4) a coassociative coproduct, Eq. (6) will in general not define an associative product. The associativity condition $(x \star y) \star z = x \star (y \star z)$ for $\mu_h$ can be expressed with the Drinfeld coassociator

$$\Phi := (\Delta \otimes \text{id})(\mathcal{F}^{-1})\mathcal{F}^{-1} \otimes 1 \otimes \mathcal{F} (\text{id} \otimes \Delta) \mathcal{F}^{-1},$$
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\[(\Phi_1 \triangleright x)(\Phi_2 \triangleright y)(\Phi_3 \triangleright z) = xyz.\]  \hfill (10)

for all $x, y, z \in \mathcal{X}$.

For a given $\mathcal{U}_h(g)$-covariant quantum space, is there a Drinfeld twist $F$ which yields by Eq. (6) the associative product of the quantum space? We will answer this question positively for three important cases: the quantum plane, quantum Euclidean 4-space, and quantum Minkowski space.

3. Constructing covariant star products

3.1. The general approach

To our knowledge, no Drinfeld twist for the Drinfeld-Jimbo quantum enveloping algebra of a semisimple Lie algebra has ever been computed. This indicates that it will be rather difficult to answer this question on an algebraic level. The representations of Drinfeld twists, however, can be expressed by Clebsch-Gordan coefficients \[10, 11\]. Therefore, we propose the following approach, which tackles the problem on a representation theoretic level:

(i) Determine the irreducible highest weight representations of all possible Drinfeld twists from $\Delta$ to $\Delta_h$.

(ii) Determine the basis $\{T^j_m\}$ of the quantum space $\mathcal{X}_h$ which completely reduces $\mathcal{X}_h$ into (possibly degenerate) irreducible highest weight-$j$ representations of $\mathcal{U}_h(g)$.

(iii) Calculate the multiplication map $\mu_h$ of $\mathcal{X}_h$ with respect to this basis. The undeformed limit $\mu = \lim_{\hbar \to 0} \mu_h$ yields the commutative multiplication map with respect to this basis.

(iv) Check on the level of representations if one of the twists realizes the deformed multiplication by Eq. (6) as linear map with respect to this basis.

Since this procedure reduces the algebraic problem to a representation theoretic one, it works well for cases where the representation theory is well understood, such as for the quantum spaces of $\mathcal{U}_h(su_2)$, $\mathcal{U}_h(so_4)$, and $\mathcal{U}_h(sl_2(C))$.

3.2. Example: the quantum plane

The $\hbar$-adic quantum plane generated by $x$ and $y$ with commutation relations $xy = qyx$, $q := e^{\hbar}$, is a $\mathcal{U}_h(su_2)$-covariant space. Let us denote by $\rho^j$ the structure map of the spin-$j$ representation of $\mathcal{U}_h(su_2)$. The results of the proposed approach are:

(i) The irreducible representations of the Drinfeld twists can be expressed by the $q$-deformed and undeformed Clebsch-Gordan coefficients \[12\] as

\[(\rho^{j_1} \otimes \rho^{j_2})(\mathcal{F})^{m_1m_2m_1m_2} = \sum_{j, m} \eta(j_1, j_2, j) \left( j_1 j_2 | j m \right) \left( j_1 j_2 | j m \right). \hfill (11)\]

where $\eta(j_1, j_2, j) \in \mathbb{C}[\hbar]$ is some complex formal power series \[10, 11\].

(ii) A basis of the irreducible spin-$j$ $\mathcal{U}_h(su_2)$-subrepresentation of the quantum plane is

$$T^j_m = \left[ \begin{array}{c} 2j \cr j + m \end{array} \right] \frac{\hbar}{q - q^{-1}} x^j y^{j+m}, \text{ where } \left[ \begin{array}{c} j \cr k \end{array} \right] \text{ is the } q\text{-binomial coefficient.} \hfill (12)$$
(iii) The multiplication map with respect to this basis is

\[ \mu_{\bar{h}}(T_{m_1}^{j_1} \otimes T_{m_2}^{j_2}) = \left( \begin{array}{cc} j_1 & j_2 \\ m_1 & m_2 \end{array} \right) T_{m_1 + m_2}^{j_1 + j_2}. \]  

(13)

For the undeformed limit \( \mu_{\bar{h}} \rightarrow \mu \) the \( q \)-Clebsch-Gordan coefficient has to be replaced by the undeformed Clebsch-Gordan coefficient.

(iv) The twist \( \mathcal{F} \) which yields \( \mu_{\bar{h}} \) by (6) can now be read off using the orthogonality of the Clebsch-Gordan coefficients to be the one with \( \eta(j_1, j_2, j) = 1 \) in Eq. (11).

3.3. Quantum Minkowski space

It can be shown [11] that there is also a twist \( \mathcal{F}_{\text{so}_4} \) of \( \mathcal{U}_{\bar{h}}(\text{so}_4) \) which realizes quantum Euclidean 4-space and a twist \( \mathcal{F}_{\text{sl}_2(\mathbb{C})} \) of the quantum Lorentz algebra \( \mathcal{U}_{\bar{h}}(\text{sl}_2(\mathbb{C})) \) which realizes quantum Minkowski space by (6). These twists are composed out of the twist \( \mathcal{F} \) which realizes the quantum plane and the universal \( \mathcal{R} \)-matrix of \( \mathcal{U}_{\bar{h}}(\text{su}_2) \) as

\[ \mathcal{F}_{\text{so}_4} = \mathcal{F}_{13} \mathcal{F}_{24}, \quad \mathcal{F}_{\text{sl}_2(\mathbb{C})} = \mathcal{R}^{-1}_{23} \mathcal{F}_{13} \mathcal{F}_{24}, \]  

(14)

where we use tensor leg notation, \( \mathcal{F}_{13} = \mathcal{F}_{[1]} \otimes 1 \otimes \mathcal{F}_{[2]} \otimes 1 \), etc. These expressions are plausible, considering the fact that \( \mathcal{U}_{\bar{h}}(\text{so}_4) \) is the product of two copies of \( \mathcal{U}_{\bar{h}}(\text{su}_2) \) and that \( \mathcal{U}_{\bar{h}}(\text{sl}_2(\mathbb{C})) \) is \( \mathcal{U}_{\bar{h}}(\text{so}_4) \) twisted by the \( \mathcal{R} \)-matrix.

4. Conclusion

By definition, Drinfeld twists yield the deformation of an enveloping algebra into a quantum enveloping algebra. It was shown that out of all twists of the Drinfeld-Jimbo algebras \( \mathcal{U}_{\bar{h}}(\text{su}_2) \) and \( \mathcal{U}_{\bar{h}}(\text{so}_4) \), and the quantum Lorentz algebra \( \mathcal{U}_{\bar{h}}(\text{sl}_2(\mathbb{C})) \) there are certain twists which realize their fundamental covariant quantum spaces, the quantum plane, quantum Euclidean 4-space, and quantum Minkowski space, respectively, as covariant star products on the undeformed, commutative space algebras. In other words, these particular twists describe the deformation of space and symmetry completely.

Therefore, it can be expected that these twists also describe constructions which are solely based on this deformation, such as the realization of the quantum Minkowski space algebra within the undeformed Poincaré algebra or the formal equivalence of deformed and undeformed gauge theory which was conjectured by Seiberg and Witten [2].

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