POLYNOMIAL IDEALS AND DIRECTED GRAPHS

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Abstract. In this paper it is shown that it is possible to associate several polynomial ideals to a directed graph $D$ in order to find properties of it. In fact by using algebraic tools it is possible to give appropriate procedures for automatic reasoning on cycles and directed cycles of graphs.

Keywords: ideals, digraphs, algorithms.

1. Introduction

The aim of this paper is to study some problems in graph theory with commutative algebra tools. Connections between simplicial complexes and polynomial rings were first studied in ([23]). There were also studies on the connections between ideals and undirected graphs, as in Simis, Vasconcelos and Villarreal ([21] and [22]). In 1995 Villarreal ([25]) and in 1998 Hibi and Ohsugi ([15]) associated a toric ideal, called the edge ideal, to an undirected graph. They found some relations between ideal properties and even closed walks of the associated graph. Other studies of undirected graphs with the edge ideal can be found in Villarreal ([26]) and Herzog and Hibi ([13]). By using these papers the authors found other ideals associated to a graph and other properties on cycles and minimal vertex covers ([5]).

In this paper we extend the paper [5] to the case of directed graphs by founding a toric ideal, such that its generators are in one to one correspondence with directed and undirected cycles. The existence of such ideal was proved in different way in
2005 by Reyes ([20]) and in 2006 by Gitler, Reyes and Villarreal ([9]. Some binomial ideals associated to a digraph can be also found in [18] and [1], but there is no characterization of cycles. Furthermore we introduce the notion of sink and source covers and we find characteristic conditions for directly bipartite graphs. Many properties of a digraph $D$ are studied through some corresponding properties of two undirected graphs associated to $D$. Relations between Cohen-Macaulay property of such undirected graphs and properties of the digraph will be investigated in another paper.

Digraphs are very useful for many applications like in computational molecular biology ([6]), in the study of phylogenetic trees ([7] and [19]) and in the minimum cost flow problem in networks, which has many physical applications ([11]). By using the packages Groebner and networks of Maple 10 we have procedures for automatic deduction in graph theory in order to find bases of directed and undirected cycles of the given digraph and to check the existence of sink and source covers.

2. Preliminary tools

2.1. Gröbner Bases. In this section we introduce some basic notions and properties of toric ideals. Let $\mathbb{N}_0=\{0,1,2,\ldots,n,\ldots\}$ and let $X_1,\ldots,X_n$ be $n$ variables. Let $K$ be a field of characteristic zero. Let $A = K[X_1,\ldots,X_n]$ and let $PP(X_1,\ldots,X_n) = \{X_1^{a_1}\cdots X_n^{a_n}: (a_1,\ldots,a_n) \in \mathbb{N}_0^n\}$ be the set of power products in $\{X_1,\ldots,X_n\}$, that is equal to the set $T_A$ of terms of $A$.

**DEFINITION 1.** A term ordering $\sigma$ on $T_A$ is a total order such that:

(i) $1 \prec t$ for all $t \in T_A \setminus \{1\}$;

(ii) $t_1 \prec t_2$ implies $t_1 t' < \sigma t_2 t'$ for all $t' \in T_A$. 

If $\sigma$ is a term ordering on $T_A$ and $f \in A$, then $M_\sigma(f)$ is the monomial $c_j t_j$ iff $t_i <_\sigma t_j$ for all $i \neq j$, $i = 1, \ldots, r$. $M_\sigma(f)$ is the leading monomial of $f$. $t_j = T_\sigma(f)$ is the leading term of $f$, while $M_\sigma(f)$ is the leading monomial of $f$.

**DEFINITION 2.** Let $f$ be a polynomial in $A$, let $F = (f_1, \ldots, f_r)$ be a finite subset of $A$ and let $\sigma$ be a term ordering on $T_A$. If $f = \sum_{i=1}^s c_i t_i \in A$, then $f$ is reduced with respect to $F$ iff $t_i \neq tT_\sigma(f_h)$ for all $t \in T_A$, all $i = 1, \ldots, s$ and all $h = 1, \ldots, r$.

**DEFINITION 3.** Let $\sigma$ be a term ordering on $T_A$ and let $I$ be an ideal in $A$. The monomial ideal $M_\sigma(I) = (M_\sigma(f) : f \in I)$ is the initial ideal of $I$.

**DEFINITION 4.** (4) Let $I$ be an ideal in $A$ and let $\sigma$ be a term ordering on $T_A$. If $I = (f_1, \ldots, f_r)$, then $\{f_1, \ldots, f_r\}$ is a Gröbner basis of $I$ with respect to $\sigma$ on $T_A$ iff $M_\sigma(I) = (M_\sigma(f_1), \ldots, M_\sigma(f_r))$.

$\{f_1, \ldots, f_r\}$ is a reduced Gröbner basis iff $M_\sigma(f_h) = T_\sigma(f_h)$ and $f_h$ is reduced with respect to $F \setminus f_h$ for all $h = 1, \ldots, r$.

Every ideal $I$ in $A$ has a Gröbner basis and a reduced Gröbner basis with respect to a given term ordering $\sigma$ (4).

**DEFINITION 5.** Let $I$ be a nonzero ideal in $A$. The universal Gröbner basis of $I$ is the union of all reduced Gröbner bases of $I$.

### 2.2. Toric Ideals.

Here we introduce the notion and some properties of a toric ideal. Let $K$ be a field and let $A = K[X_1, \ldots, X_n]$ be as above. Let $B = K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ be the Laurent polynomial ring in the indeterminates $\{X_1, \ldots, X_n\}$ and let $EPP(X_1, \ldots, X_n) = \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{Z}^n\}$
be the set of power products in \( \{ X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1} \} \), that is equal to the set \( ET_A \) of extended terms of \( A \).

**Definition 6.** ([24]) Let \( M = (m_{ij})_{i=1, \ldots, m, j=1, \ldots, n} \) be a \((m, n)\)-matrix with \( m_{ij} \) in \( \mathbb{Z} \) for all \( i, j \). Let \( \pi_M : \mathbb{N}_0^m \rightarrow \mathbb{Z}^m \) be the semigroup homomorphism defined by \( \pi_M(u_1, \ldots, u_n) = (\sum_{j=1, \ldots, n} u_j m_{1j}, \ldots, \sum_{j=1, \ldots, n} u_j m_{mj}) \).

Let \( \exp_n : \mathbb{N}_0^m \rightarrow PP(X_1, \ldots, X_n) \) be the semigroup isomorphism defined by \( \exp_n(u_1, \ldots, u_n) = \prod_{j=1, \ldots, n} X_j^{u_j} \) and let \( \exp_Z : \mathbb{Z}^m \rightarrow EPP(t_1, \ldots, t_m) \) be the semigroup isomorphism defined by \( \exp_Z(a_1, \ldots, a_m) = \prod_{i=1, \ldots, m} t_i^{a_i} \).

Let \( \pi : PP(X_1, \ldots, X_n) \rightarrow EPP(t_1, \ldots, t_m) \) be the semigroup homomorphism, that is induced by \( \pi_M \) and it is defined by \( \pi(X_j) = \prod_{i=1, \ldots, m} t_i^{m_{ij}} \) for all \( j = 1, \ldots, n \). \( \pi \) extends uniquely to the homomorphism of semigroup algebras \( \pi' : K[X_1, \ldots, X_n] \rightarrow K[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}] \) defined by \( \pi'(X_j) = \pi(X_j) \) for all \( j = 1, \ldots, n \). \( I_M = \ker(\pi') \) is an ideal in \( A = K[X_1, \ldots, X_n] \), that is called the toric ideal of \( M \).

**Remark 1.** Let \( M \) be a matrix as above and let \( I_M \) be the toric ideal of \( M \). It is not too hard to show that \( I_M \) is an elimination ideal. More precisely, \( I_M = (X_j - \prod_{i=1, \ldots, m} t_i^{m_{ij}}, t_i^{-1} - 1 : j = 1, \ldots, n, i = 1, \ldots, m) \cap K[X_1, \ldots, X_n] \) ([24]).

If we put \( z_i = t_i^{-1} \) for all \( i = 1, \ldots, n \), then \( I_M \) is equal to the toric ideal \( I_{M'} \), where \( M' = (m'_{ij})_{i=1, \ldots, 2m, j=1, \ldots, n} \) is a \((2m, n)\)-matrix with \( m'_{ij} \) in \( \mathbb{N} \) for all \( i, j \). Moreover, \( m'_{ij} = m_{ij} \) for all \( i = 1, \ldots, m \) with \( m_{ij} \in \mathbb{Z}^+ \), \( m'_{ij} = 0 \) for all \( i = 1, \ldots, m \) with \( m_{ij} \in \mathbb{Z}^- \), \( m'_{ij} = -m_{i(m-j)} \) for all \( i = m+1, \ldots, 2m \) with \( m_{ij} \in \mathbb{Z}^- \) and \( m'_{ij} = 0 \) for all \( i = m+1, \ldots, 2m \) with \( m_{ij} \in \mathbb{Z}^+ \).
It is well known that every toric ideal is a prime binomial ideal. Other properties of binomial ideals can be found in ([8]), while properties of toric ideals can be found in ([24]) and ([KR05], chap.2).

**THEOREM 1.** ([24]). $I_M$ is generated by binomials of the type $X^{u^+} - X^{u^-}$, where $u^+, u^- \in \mathbb{Z}^n$ are non negative with disjoint support.

**DEFINITION 7.** A binomial $X^{u^+} - X^{u^-}$ is primitive if there exists no other binomial $X^{v^+} - X^{v^-} \in I_M$ such that $X^{v^+}$ divides $X^{u^+}$ and $X^{v^-}$ divides $X^{u^-}$. A binomial $X^{u^+} - X^{u^-}$ in $I_M$ is a circuit if $\text{supp}(u^+ - u^-)$ is minimal with respect to inclusion and the coordinates of $u^+ - u^-$ are relatively prime.

$Gr_M = \{ \text{primitive binomials in } I_M \}$ is the Graver basis of $I_M$.

**THEOREM 2.** ([24]). Let $U_M$ be the universal Gröbner basis of $I_M$ and let $C_M$ be the set of all circuits in $I_M$. Then:

(i) every binomial in $U_M$ is primitive.

(ii) $C_M \subseteq U_M \subseteq Gr_M$ for every $M$.

2.3. **Graphs and digraphs.** In this paper $G = (V(G), E(G))$ will be a finite graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Furthermore $[v_i, v_j]$ denotes the oriented edge from $v_i$ to $v_j$, while every non oriented edge between $v_i$ and $v_j$ is denoted by $\{v_i, v_j\}$.

The underlying graph $G_u$ of a directed graph $G$ is the undirected graph with $V(G_u) = V(G)$ and the same undirected edges of $G$. Often $D$ will denote a directed graph, that will be also called a digraph. All graphs in this paper will be simple, i.e. without multiple edges.
DEFINITION 8. Let $D$ be a digraph. A vertex $v_i$ in $V(D)$ is called a source if no edge is directed into $v_i$. A vertex $v_i$ in $D$ is called a sink if every adjacent edge is directed into $v_i$.

DEFINITION 9. Let $D$ be a digraph. A walk of length $n$ from a vertex $v_i$ to a vertex $v_j$ in $D$ is a sequence of vertices $v_i = v_{i(1)}, \ldots, v_j = v_{i(n+1)}$, such that either $[v_{i(h)}v_{i(h+1)}]$ or $[v_{i(h+1)}v_{i(h)}]$ is in $E(D)$ for all $h = 1, \ldots, n$. A walk is called a directed walk if $[v_{i(h)}v_{i(h+1)}] \in E(D)$ for all $h$. If $v_{i(1)} = v_{i(n+1)}$ in a directed walk, then it is called a directed cycle. A walk is called simple if there are no repeated edges, while it is called elementary whenever there are no repeated vertices.

DEFINITION 10. Let $G$ be an undirected graph. $G$ is bipartite if its vertices can be divided in two sets, such that no edge connects vertices in the same set. Equivalently, $G$ is bipartite iff all cycles in $G$ are even.

$G$ is acyclic if it has no cycle, while it is a tree if it is connected and acyclic.

Some generalizations of such definitions in the directed case are the following ones.

DEFINITION 11. Let $D$ be a digraph. $D$ is called directly bipartite whenever its underlying graph $D_u$ is bipartite with bipartition sets $W$ and $W'$, such that every $w \in W$ is a source in $D$, and every $w' \in W'$ is a sink in $D$. $D$ is called a directed acyclic graph, DAG for short, when there are no directed cycles.

3. Binomial ideals arising from a digraph

Here we will introduce the binomial ideals, associated to a digraph, that we will use in the paper. First of all we introduce some binomial ideals associated to the edges of a digraph.
DEFINITION 12. The binomial extended diedge ideal of a digraph $D$ is the ideal $I(D,E)=\langle e_h - z_i v_j, z_i v_j - 1: e_h = [v_i, v_j] \in E(D), i = 1, \ldots, n \rangle$ in $K[e_1, \ldots, e_m, v_1, \ldots, v_n, z_1, \ldots, z_n]$. The ideal $I(E)_D = I(D,E) \cap K[e_1, \ldots, e_m]$ is the binomial diedge ideal of the digraph $D$.

The definitions as above extend the analogous ones given in the case of undirected graphs as in [15] and [5].

REMARK 2. $I(E)_D$ is the toric ideal of the matrix $IM(D)^t$, that is the transpose of the incidence matrix $IM(D) = (a_{ih})_{i=1,\ldots,n;h=1,\ldots,m}$ of $D$ defined by $a_{ih} = -1$ if $e_h$ leaves $v_i$, $a_{ih} = 1$ if $e_h$ arrives to $v_i$ and $a_{ih} = 0$ if $v_i \not\in e_h$ for every $v_i \in V(E)$ and $e_h \in E(D)$.

EXAMPLE 1. Let $D_1$ be the digraph with $V(D_1) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(D_1) = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_3, v_1], e_4 = [v_1, v_4], e_5 = [v_3, v_4], e_6 = [v_4, v_5]\}$. The binomial extended diedge ideal is $I(D_1,E) = \langle e_1 - z_1v_2, e_2 - z_2v_3, e_3 - z_3v_1, e_4 - z_1v_4, e_5 - z_3v_4, e_6 - z_3v_6, z_1v_1 - 1, z_2v_2 - 1, z_3v_3 - 1, z_4v_4 - 1, z_5v_5 - 1 \rangle$. The binomial diedge ideal of $D_1$ is $I(E)_{D_1} = \langle e_1 e_2 e_3 - 1, e_3 e_4 - e_5, e_2 e_1 e_5 - e_4 \rangle$.

EXAMPLE 2. Let $D_2$ be the digraph with $V(D_2) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(D_2) = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_1, v_3], e_4 = [v_4, v_3], e_5 = [v_3, v_5]\}$. The binomial extended diedge ideal of $D_2$ is $I(D_2,E) = \langle e_1 - z_1v_2, e_2 - z_2v_3, e_3 - z_1v_3, e_4 - z_4v_3, e_5 - z_3v_5, z_1v_1 - 1, z_2v_2 - 1, z_3v_3 - 1, z_4v_4 - 1, z_5v_5 - 1 \rangle$ while the binomial diedge ideal of $D_2$ is $I(E)_{D_2} = \langle e_1 e_2 - e_3 \rangle$. 
4. The undirected graph $H_D$ associated to $D$

Here we associate an undirected bipartite graph $H_D$ to a digraph $D$. The introduction of such graph allows to prove properties of the cycles of $D$ through the properties of the cycles of $H_D$.

**DEFINITION 13.** Let $D$ be a digraph. Let $H_D$ be the undirected graph with $V(H_D) = V(D) \cup \{z_1, \ldots, z_n\}$ and $E(H_D) = \{e = \{z_i, v_j\}: \{v_i, v_j\} \in E(D)\} \cup \{f_i = \{z_i, v_i\}: i = 1, \ldots, n\}$. Let $R = K[v_1, z_1, f_1, \ldots, v_n, z_n, f_n, e_1, \ldots, e_m]$ and let $\pi : R \to R/(f_1 - 1, \ldots f_n - 1)$ be the canonical ring homomorphism defined by $\pi(v_i) = v_i$, $\pi(z_i) = z_i$, $\pi(f_i) = 1$, $\pi(e_j) = e_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

If $I(H_D, E)$ is the binomial extended edge ideal of the undirected graph $H_D$ as in [5], then $\pi(I(H_D, E)) = I(D, E)$ by definition of $\pi$.

The graph $H_D$ has some properties, that are independent on the properties of the graph $D$.

**DEFINITION 14.** A matching of an undirected graph $G$ is a subset of independent edges of $G$ (i.e. edges that do not share vertices). A matching is called perfect if its cardinality is maximal.

The following proposition shows the existence of a unique perfect matching of $H_D$.

**THEOREM 3.** Let $D$ be a digraph. The undirected graph $H_D$ is bipartite and it has a perfect matching.

**Proof.** Let $V = \{v_1, \ldots, v_n\}$ and $Z = \{z_1, \ldots, z_n\}$. $V$ and $Z$ are disjoint subsets of $V(H_D)$, whose union is exactly $V(H_D)$. By definition of $E(H_D)$ every edge in
$H_D$ connects a vertex in $V$ with a vertex in $Z$. So $H_D$ is bipartite with bipartition sets $V$ and $Z$, by definition of bipartite graph. Finally the edges $f_i = \{z_i, v_i\}$ for $i = 1, \ldots, n$ are independent and they are $n$ in a graph with $2n$ vertices. So the set $M = \{f_1, \ldots, f_n\}$ is a perfect matching for $H_D$ by its own definition. □

Now, given a bipartite graph $G$, we want to find a digraph $D$, with $H_D = G$. The following theorem shows some results in this direction.

**THEOREM 4.** Let $G$ be a bipartite undirected graph with $V(G) = \{v_1, \ldots, v_n, z_1, \ldots, z_n\}$. Then the following propositions are equivalent:

1. $G$ has a perfect matching;
2. there exists a digraph $D$, with $V(D) = \{v_1, \ldots, v_n\}$ such that $H_D = G$.

**Proof.** (1) $\Rightarrow$ (2). If $G$ has a perfect matching, then it has $n$ edges, that share no vertices. We can relabel the vertices in such a way as the $n$ independent edges are $f_i = \{z_i, v_i\}$, for $i = 1, \ldots, n$. So $V = \{v_1, \ldots, v_n\}$ and $Z = \{z_1, \ldots, z_n\}$ are the two bipartition sets. Otherwise either a vertex $z_i$ should be in a bipartition set with some $v_j$ (or a vertex $v_i$ should be in a bipartition set with some $z_j$). This fact would imply that the edge $f_i$ has two vertices in the same bipartition set, against the definition of bipartite graph. Now it is sufficient to define $D$ in such a way as $V(D) = \{v_1, \ldots, v_n\}$ and $[v_i, v_j] \in E(D)$ whenever $\{z_i, v_j\} \in E(G)$.

(2) $\Rightarrow$ (1). It follows from theorem. □

5. **Cycles in $D$ and $H_D$**

Let $G$ be an undirected graph. There exists a binomial ideal $I(G, E)$ in the vertices and edges associated to $G$ ([5]), while there exists a binomial ideal $I(E)_E \subseteq$
$I(G, E)$ in the edges associated to the even closed walks in $G$ \cite{15}. Given an even closed walk $C = (e_{i_1} = \{v_{i_1}, v_{i_2}\}, \ldots, e_{i_{2q-1}} = \{v_{i_{2q-1}}, v_{i_{2q}}\}, e_{i_{2q}} = \{v_{i_{2q}}, v_{i_1}\})$ of $G$, let $f_C = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$ be the corresponding binomial in $I(E)_G$.

The following theorem is a well known result about even closed walks in an undirected graph.

**THEOREM 5.** \cite{25}, \cite{15}

If $G$ is an undirected graph, then the toric ideal $I(E)_G$, associated to its incidence matrix, is generated by all binomials $f_C$, where $C$ is an even closed walk of $G$.

Now we want to extend this result to the case of digraphs.

**THEOREM 6.** Let $D$ be a digraph. The toric ideal $I(E)_D$ is generated by all binomials $f_C$, where $C$ is a cycle of $D$.

**Proof.** By looking at the undirected graph $H_D$ associated to $D$, then the ideal $I(E)_{H_D} = I(H_D, E) \cap K[e_1, \ldots, e_m, f_1, \ldots, f_n]$ is generated by all even closed walks in $H_D$ and then by all even cycles, because $H_D$ is bipartite by theorem \cite{3}. Now $I(D, E) = \pi(I(H_D, E))$ and the binomial edge ideal $I(E)_D = I(D, E) \cap K[e_1, \ldots, e_m]$ is equal to $\pi(I(H_D, E)) \cap K[e_1, \ldots, e_m, f_1, \ldots, f_n]$. It follows that $I(E)_D$ is generated by binomials $f = \prod_{i \in I} f_i \prod_{i' \in I'} e_{i'} - \prod_{j \in J} f_j \prod_{j' \in J'} e_{j'}$ with $I, J \subseteq \{1, \ldots, n\}$, $I', J' \subseteq \{1, \ldots, m\}$, $|I| + |I'| = |J| + |J'|$ and $f_i = f_j = 1$ for all $i, j = 1, \ldots, n$. If $C$ is a direct cycle in $D$, then we can relabel the vertices and the edges is such a way as $C=\{e_1 = [v_1, v_2], \ldots, e_{q-1} = [v_{q-1}, v_q], e_q = [v_q, v_1]\}$ and $C'=\{e_1, f_2, e_2, f_3, \ldots, e_{q-1}, f_q, e_q, f_1\}$ is an even cycle in $H_D$. So the binomial $f_{C'} = \prod_{h=1}^{q} e_h - \prod_{h=1}^{q} f_h$ is in $I(H_D, E) \cap K[e_1, \ldots, e_m, f_1, \ldots, f_n]$ and
the binomial $f_C = \prod_{h=1,\ldots,q} e_h - 1$ is in $I(D,E) \cap K[e_1, \ldots, e_m]$. Now let $C$ be an undirected cycle in $D$. Once again we can relabel the vertices and the edges in such a way as $C = \{\{v_1, v_2\}, \ldots, \{v_{q-1}, v_q\}, \{v_{q+1} = v_1\}\}$. Let $C'$ be the path in $H_D$ given in the following way: $C' = g_1, \ldots, g_q$, where $g_h = e_h f_{h+1}$ if $e_h = \{z_h, v_{h+1}\}$ and $[v_h, v_{h+1}] \in E(D)$, while $g_h = f_{h+1} e_h f_h$ if $e_h = \{z_{h+1}, v_h\}$ and $[v_{h+1}, v_h] \in E(D)$. $C' = \{z_1, v_2, z_3, \ldots, z_{q-1}, v_q, z_q, v_1\}$ is an even cycle in $H_D$. The corresponding binomial $f_{C'}$ is in the ideal $I(H_D, E) \cap K[e_1, \ldots, e_m, f_1, \ldots, f_n]$, while its image in $K[e_1, \ldots, e_m]$ is the binomial $f_C$ corresponding to $C$.

Conversely given a cycle $C'$ in $H_D$, then it is an even cycle, because $H_D$ is bipartite. So $C' = \{z_{i(1)}, v_{i(2)}, z_{i(3)}, v_{i(4)}, \ldots, z_{i(q-3)}, v_{i(q-2)}, z_{i(q-1)}, v_{i(q)}, z_{i(q+1)} = z_{i(1)}\}$. Let $e_h = \{z_{i(h)}, v_{i(h+1)}\}$ whenever $i(h) \neq i(h+1)$ and let $f_h = \{z_{i(h)}, v_{i(h+1)}\}$ whenever $i(h) = i(h+1)$. $C'$ determines the cycle $C = \{v_{i(1)}, v_{i(2)}, \ldots, v_{i(q)}, v_{i(q+1)} = v_{i(1)}\}$, where $v_{i(h)} = v_{i(h+1)}$, whenever $i(h) = i(h+1)$ and $e_{i(h)} = \{v_{i(h)}, v_{i(h+1)}\}$ is in $E(D)$, whenever $i(h) \neq i(h+1)$. □

**REMARK 3.** The existence of the toric ideal as above was proved in different way by using circuits associated to the transpose of the incidence matrix in 2005 by Reyes ([20]) and in 2006 by Gitler, Reyes and Villarreal ([19]).

**COROLLARY 1.** Let $G$ be an undirected graph and let $G^d$ be a directed graph, whose underlying graph is $G$ (e.g. let $G^d$ be a directed graph such that $G_{\text{u}}^d = G$). Then the ideal $I(G^d, E)$ is generated by all binomials $f_C$, such that $C$ is a cycle of $G$.

**Proof.** Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_{i,j} = \{v_i, v_j\}, i, j \in \{1, \ldots, n\}\}$. It is possible to construct such directed graph $G^d$ by setting randomly a direction
in each edge. Let $V(G^d) = V(G)$ and let $E(G^d) = \{e_{i,j} = [v_i, v_j]\}$, such that $\{v_i, v_j\} \in E(G)$. By theorem as above $I(G^d, E)$ is generated by binomial $f_C = \prod_{i \in I} e_i - \prod_{j \in J} e_j$, where $C$ is a cycle of $G^d$ and $J = \emptyset$ whenever $C$ is a directed cycle. Since every cycle in $G^d$ is a cycle in $G$, then we have the thesis. □

The theorem as above gives also decision procedures for the existence of directed and undirected cycles in a digraph as in the following remark.

**REMARK 4.** The proof of the theorem as above allows to check whether a given digraph $D$ has cycles and whether the cycles are either directed or undirected. By using Maple 10 we implemented the corresponding decision procedures. The algorithm works as follows: given a digraph $D$ by using the package networks we find the incidence matrix $M$ of $D$ and then by using the package Groebner we get a Gröbner basis $B$ of the toric ideal $I(E)_D$ of $M$. If $B$ contains a polynomial of the form $f = \prod_{i \in I} e_i - 1$, then in $D$ there is a directed cycle of length $|I|$. Furthermore, if $B$ contains a polynomial of the form $g = \prod_{i \in I} e_i - \prod_{j \in J} e_j$, then in $D$ there is an undirected cycle of length $|I| + |J|$. Of course the binomials associated to even and odd cycles in a digraph are in the edge toric ideal, while in the undirected case only even cycles appear in the edge toric ideal associated to the graph. By using corollary 1 we are able to check whether an undirected graph $G$ has even and odd cycles. We can construct $G^d$ by simply choosing randomly a direction for each edge and we can find the toric ideal $I(G^d, E)$. Such ideal is generated by binomials in the form either $f = \prod_{i \in I} e_i - 1$ or $g = \prod_{i \in I} e_i - \prod_{j \in J} e_j$, that are in one to one correspondence with cycles in $G$.

By using the algorithm sketched in the last remark it is possible to deduce properties about the digraphs $D_1$ and $D_2$ in examples [1] and [2].
EXAMPLE 3. The toric ideal basis of the ideal \( I(E)_{D_1} \), with respect to the lexicographic term order \( \sigma_1 \) with \( v_1 >_{\sigma_1} z_1 >_{\sigma_1} v_2 >_{\sigma_1} z_2 >_{\sigma_1} v_3 >_{\sigma_1} z_3 >_{\sigma_1} v_4 >_{\sigma_1} z_4 >_{\sigma_1} v_5 >_{\sigma_1} z_5 >_{\sigma_1} e_3 >_{\sigma_1} e_1 >_{\sigma_1} e_4 >_{\sigma_1} e_2 >_{\sigma_1} e_5 >_{\sigma_1} e_6 \) is 
\[ (e_1e_2e_3 - 1, e_1e_2e_5 - e_4, e_3e_4 - e_5). \] So the digraph \( D_1 \) has a directed cycle with the edges \( e_1, e_2, e_3 \) and two undirected cycles with the edges \( e_3, e_4, e_5 \) and \( e_1, e_2, e_4, e_5 \).

EXAMPLE 4. The digraph \( D_2 \) is a DAG, in fact the toric ideal basis of the ideal \( I(E)_{D_2} \), with respect to the lexicographic term order \( \sigma_2 \) with 
\[ v_1 >_{\sigma_2} z_1 >_{\sigma_2} v_2 >_{\sigma_2} z_2 >_{\sigma_2} v_3 >_{\sigma_2} z_3 >_{\sigma_2} v_4 >_{\sigma_2} z_4 >_{\sigma_2} v_5 >_{\sigma_2} z_5 >_{\sigma_2} e_1 >_{\sigma_2} e_2 >_{\sigma_2} e_3 >_{\sigma_2} e_4 >_{\sigma_2} e_5 \] is \( (e_1e_2 - e_3) \) and the only cycle with the edges \( e_1, e_2, e_3 \) is undirected.

Now we study a property of digraphs, that can be easily checked by using theorem 6.

DEFINITION 15. Given a digraph \( D \), a vertex \( v \) is reachable from another vertex \( u \) if there is a directed path that starts from \( u \) and ends at \( v \). If \( v \) is reachable from \( u \), then \( u \) is a predecessor of \( v \) and \( v \) is a successor of \( u \).

DEFINITION 16. Let \( D \) be a digraph and let \( v \) and \( u \) be vertices in \( D \). \( D \) is a UPD (Unique Path Digraph) iff whenever \( v \) is a successor of \( u \), then there is a unique elementary path between \( u \) and \( v \).

It is easy to show that a cycle of a UPD is directed. The following theorem is useful for a characterization of a UPD.

THEOREM 7. Let \( D \) be a digraph and let \( C_1 \) and \( C_2 \) be two directed cycles in \( D \), such that \( C_1 \) and \( C_2 \) are not edge-disjoint. Then \( C = (C_1 \cup C_2) \setminus (C_1 \cap C_2) \) is an undirected cycle.
Proof. Let $I(E)_D$ be the edge ideal associated to $D$. By theorem 6 the binomials representing the two directed cycles $C_1$ and $C_2$ lie in $I(E)_D$. Let $f_1 = \prod_{i \in I} e_i - 1$ be the binomial associated to $C_1$ and let $f_2 = \prod_{j \in J} e_j - 1$ be the binomial associated to $C_2$. Since $C_1 \cap C_2 \neq \emptyset$, then $K = I \cap J \neq \emptyset$ and $f_1 = (\prod_{k \in K} e_k \cdot \prod_{i \in I \setminus K} e_i) - 1$ while $f_2 = (\prod_{k \in K} e_k \cdot \prod_{j \in J \setminus K} e_j) - 1$. Let $\sigma$ be a lexicographic term ordering in $\mathbb{K}[e_1, \ldots, e_m]$, where $m$ is the number of edges in $D$. The S-polynomial between $f_1$ and $f_2$ with respect to the term ordering $\sigma$ is the binomial $f = -\prod_{j \in J \setminus K} e_j + \prod_{i \in I \setminus K} e_i$. By remark 4 the cycle $C$ corresponding to $f$ is an undirected cycle lying in $D$ and $C = (C_1 \cup C_2) \setminus (C_1 \cap C_2)$.

\[ \square \]

COROLLARY 2. Let $D$ be a digraph. $D$ is a UPD if and only if its cycles are directed with at most a vertex in common.

Proof. If $D$ is a tree or a forest there is nothing to prove. Otherwise all cycles have to be directed and by theorem 7 they can have at most a vertex in common.

\[ \square \]

REMARK 5. The UPD property of a digraph $D$ can be easily checked by using theorem 6 and corollary 7. In fact it is easy to show that $D$ is UPD if and only if either the binomial edge ideal $I(E)_D = (0)$, e.g. $D$ has no cycles, or $I(E)_D$ is generated by binomials $f_h = \prod_{j=1,\ldots,k(h)} e_j - 1$ with $\prod_{j=1,\ldots,k(h)} e_j$ and $\prod_{j=1,\ldots,k(h')} e_j$ coprime for all $h \neq h'$.

6. LINEAR IDEALS ARISING FROM DIGRAPHS

Here we will introduce some linear ideals, that we can associate to a digraph and we can use in decision procedures. By using algorithms from linear algebra, we
could lose some property of a digraph. Anyway such algorithms are very useful, because they are fast.

Let $D$ be a digraph and let $H_D$ be the associated undirected graph defined as before. In [5] the extended linear edge ideal (respectively the linear edge ideal) can be associated to $H_D$ and relations between the extended linear edge ideal and the extended binomial edge ideal (respectively the linear edge ideal and the binomial edge ideal) are shown.

Let $R = K[v_1, z_1, f_1, \ldots, v_n, z_n, f_n, e_1, \ldots, e_m]$ and let $\pi' : R \rightarrow R/(f_1, \ldots, f_n)$ be the canonical ring homomorphism defined by $\pi'(v_i) = v_i, \pi'(z_i) = z_i, \pi'(f_i) = 0, \pi'(e_j) = e_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

**DEFINITION 17.** The ideal $LI(D, E) = \pi'(LI(H_D, E))$ in $K[e_1, \ldots, e_m, v_1, \ldots, v_n]$ is the linear extended edge ideal of $D$. The ideal $LI(E)_D = LI(D, E) \cap K[e_1, \ldots, e_m]$ is the linear edge ideal of $D$.

**REMARK 6.** It is easy to show that the ideal $LI(D, E) = (e_h + v_i - v_j : e_h = [v_i, v_j]$ in $E(D))$ in $K[e_1, \ldots, e_m, v_1, \ldots, v_n]$ by definition of $\pi'$.

By repeating the proof as in [5] if we take the matrix $M = IM(D)$ and the linear homomorphism of semigroup algebras $\psi : K[e_1, \ldots, e_m] \rightarrow K[v_1, \ldots, v_n, v_1^{-1}, \ldots, v_m^{-1}]$ defined by $\psi(e_h) = v_j - v_i$ whenever $e_h = [v_i, v_j]$ for all $h = 1, \ldots, m$, then $LI(D, E)$ coincides with the ideal $LI_M = ker(\psi)$ in $K[e_1, \ldots, e_m]$, that is called the linear ideal of $M$.

**EXAMPLE 5.** Let $D_1$ be the graph as in example [1] and [3] The linear extended edge ideal of $D_1$ is $LI(D_1, E) = (e_1 + v_1 - v_2, e_2 + v_2 - v_3, e_3 + v_3 - v_1, e_4 + v_1 - \ldots$
\(v_4, e_5 + v_3 - v_4, e_6 + v_3 - v_5\), while the linear edge ideal of \(D_1\), is \(LI(E)_{D_1} = (e_3 + e_4 - e_5, e_1 + e_5 + e_2 - e_4)\).

**Example 6.** Let \(D_2\) be the graph as in example 2 and 4. The linear extended edge ideal of \(D_2\) is \(LI(D_2, E) = (e_3 + v_1 - v_2, e_2 + v_2 - v_3, e_3 + v_1 - v_3, e_4 + v_4 - v_3, e_5 + v_3 - v_5)\), while the linear edge ideal of \(D_2\), is \(LI(E)_{D_2} = (e_1 - e_3 + e_2)\).

In order to show the relations between the linear ideal and the toric ideal associated to the incidence matrix \(IM(D)\) we need the following definition.

**Definition 18.** (24) Let \(M = (m_{ij})_{i=1,\ldots,m; j=1,\ldots,n}\) be a \((m,n)\)-matrix with \(m_{ij}\) in \(\mathbb{N}_0\) for all \(i, j\) and let \(\pi_M : \mathbb{N}_n^m \rightarrow \mathbb{Z}^m\) and \(\pi'_M : \mathbb{Z}^n \rightarrow \mathbb{Z}^m\) be the semigroup homomorphisms as in definitions 6. \(\text{Ker}(M)\) is the kernel of the semigroup homomorphism \(\pi'_M\). Moreover if a finite set \(F\) generates \(\text{Ker}(M)\) as \(\mathbb{Z}\)-module, then \(I(F) = (e^u - e^{-u}, u \in F)\) is a lattice ideal associated to \(M\).

**Remark 7.** \(\phi^{-1}_n(\text{Ker}(M))\) is a \(\mathbb{Z}\)-submodule of \(\bigoplus_{j=1,\ldots,n} \mathbb{Z} X_j\) and it is the kernel of the \(\mathbb{Z}\)-module homomorphism \(\phi_n \pi'_M \phi_m\) by definition of the homomorphisms \(\pi'_M, \phi_n\) and \(\phi_m\). Finally since \(\bigoplus_{j=1,\ldots,n} \mathbb{Z} X_j\) is a \(\mathbb{Z}\)-submodule of \(A = K[X_1, \ldots, X_n]\), then \(\phi^{-1}_n(\text{Ker}(M)) = \text{ker}(\psi) \cap \bigoplus_{j=1,\ldots,n} \mathbb{Z} X_j\).

The following definition is also useful and related to the notion of saturation.

**Definition 19.** Let \(I\) be an ideal in the polynomial ring \(A\) and let \(f \in A\). \(I : f^\infty = (g : gf^m \in I \text{ for some } m \in \mathbb{N}_0)\).

Of course \(I \subseteq I : f^\infty\) for all \(f \in A\).
The relation between the ideals $LI(D,E)$ and $I(D,E)$ is given by the following facts. First of all in [14] and ([24], p.114) it is shown that if a finite set $B$ generates $Ker(IM(D))$ as $\mathbb{Z}$-module, $J_0=I(B)=(e_i^+ - e_i^-, u \in ker(IM(D))$ and $J_i=(J_{i-1} : e_i^\infty)$ for all $i = 1,\ldots,m$, then $J_m = I(D)_{E(D)}$.

Furthermore the generators of $J_0$ are in one to one correspondence with the elements of $F$ and then in one to one correspondence with the elements of $\phi^{-1}_n(F)$, that generate the $\mathbb{Z}$-module $\phi^{-1}_n(Ker(IM(D)))=LI(IM(D))\cap \bigoplus_{j=1,\ldots,n} \mathbb{Z}e_j = LI(D)_{E} \cap \bigoplus_{j=1,\ldots,n} \mathbb{Z}e_j$ by remark 7.

**Remark 8.** The algorithm in [14] called the saturation algorithm is one of the existing algorithms for finding such ideal. Another well known algorithm is the Lift-and-Project Algorithm in [2] and recently the algorithm in [12].

Now the relation between the ideals $LI(D,E)$ and $I(D,E)$ is the same as the relation between the ideals $LI(G,E)$ and $I(G,E)$ when $G$ is an undirected graph as in [5]. The relations between binomial and linear ideals associated to a digraph give also the following theorem.

**Theorem 8.** Let $D=(V(D),E(D))$ be a simple digraph.

(i) The directed cycle $C=(e_1,\ldots,e_k)$ is in $D$ iff the linear polynomial $h_C=\sum_{h=1,\ldots,k} e_h$ is in $LI(D,E)$.

(ii) The undirected cycle $C=(e_1,\ldots,e_k)$ is in $D$ iff the linear polynomial $h_C=\sum_{i \in I} e_i - \sum_{j \in J} e_j$, with $|I| + |J| = k$ is in $LI(D)_E$.

**Proof.** The proof of (i) and (ii) is the same as the proof of theorems about the corresponding binomial ideals.
REMARK 9. The theorem as above shows that the hypothesis on the characteristic of the field $K$ is necessary. In fact the theorem does not hold when the characteristic of $K$ is different from 0.

EXAMPLE 7. Let $D_1$ be the graph as in examples 1 and 3. Then the Gröbner basis of $LI(D_1, E)$ with respect to $\sigma_1$ is $(e_4 - e_5 + e_3, -e_4 + e_5 + e_1 + e_2, -e_5 + v_4 + e_6 - v_5, e_6 + v_3 - v_5, v_2 + e_2 - v_5 + e_6, v_1 - v_5 + e_4 + e_6 - e_5)$.

EXAMPLE 8. Let $D_2$ be the graph as in examples 2 and 4. The Gröbner basis of the ideal $LI(D_2, E)$ with respect to $\sigma_2$ is $(e_1 - e_3 + e_2, e_5 - v_5 + e_4 + v_4, e_3 + v_1 + e_5 - v_5, e_5 - v_5 + v_3, v_2 - v_5 + e_2 + e_3)$.

By using linear ideals, the procedures as above are very fast. Now theorem 8 establishes only that a polynomial $p$, that corresponds to a directed cycle, is in the ideal. There is no proof that $p$ is in some Gröbner basis of $LI(E)_D$ as in example 7 with $p = e_1 + e_2 + e_3$ and then we cannot decide if $D$ is a DAG. Anyway we can use the linear ideals, once we want to check the existence of cycles.

The same results can be obtained by using the classical tools of computational graph theory applied to cycle spaces and cycle bases as in [16] and [17].

DEFINITION 20. Let $D=(V, E)$ be a simple digraph and let $C$ be cycle in $D$. The incidence vector of $C$ is a vector in $\{-1, 0, 1\}^{|E|}$ defined as follow. For every $e \in E$ $C(e)$ is equal to 1 if $e = [v_i, v_j] \in C$, $C(e)$ is equal to $-1$ if $-e = [v_j, v_i] \in C$ and $C(e)$ is equal to 0 if $e \notin C$. The cycle space of $D$ is the vector space over $\mathbb{Q}$ spanned by the incidence vectors of its cycles. A cycle basis of $D$ is a basis of the cycle space.
Remark 10. If $D$ is connected, then the cycle space of $D$ has dimension $d = |E| - |A| + 1$.

It is well known ([10] and [3]) that if $D$ is a connected digraph, then the cycle space of $D$ is the kernel of the linear map from the edge space $E(D)$ to the vertices space $V(D)$ defined by the incidence matrix $IM(D)$ and a basis of such vector space is a cycle basis of $D$. Furthermore if $D$ is strongly connected, i.e. for every pair of vertices $u$ and $v$ in $D$, there are directed paths both from $u$ to $v$ and from $v$ to $u$, then it is possible to find a cycle basis that contains all cycles of length 2 and all directed cycles.

Now let $D = V(D), E(D)$ be a simple digraph. Let $n = |V(D)|$ and let $m = |E(D)|$. By looking at the ideal $LI(D, E)$, then it is generated by the kernel of the linear map associated to the matrix $(m, n + m)$-matrix $(IM(D)^t, -I(m))$, while $LI(E)_D$ is generated by a cycle basis of $D$, when we identify the incidence vector of a cycle $C$ with the corresponding linear polynomial in the edges in $C$. The results in [10] shows that we can know all directed cycles in a digraph when some other hypothesis are satisfied, (in particular digraphs with double edges, i.e. connecting two vertices in both directions, are allowed) while a basis of the toric ideal associated to the transpose of the incidence matrix, which is also an ideal in the edges containing the lattice ideal associated to the same matrix, contains a minimal set of directed cycles in $D$. In fact in general the set of all directed cycles in a digraph $D$ is not a basis of a vector space, because the sum of two directed cycles is not well defined.

There are many digraphs that are not strongly connected and we cannot apply neither the algorithm in [10] nor our theorem 8. So we have to use theorem 6.
Finally in [16] the authors give a fast algorithm to compute a cycle basis of minimal weight in undirected graphs.

7. Sink and source covers of a digraph

Here we introduce another undirected graph $K_D$, that we can associate to a digraph $D$. We can prove some properties of $D$ through the properties of such graph.

**DEFINITION 21.** Let $D$ be a digraph. Let $K_D$ be the undirected subgraph of $H_D$ with $V(K_D)=V(D) \cup \{z_1,\ldots,z_n\}$ and $E(K_D)=\{e=\{z_i,v_j\}: [v_i,v_j] \in E(D)\}$. $K_D$ is called the sink-source undirected graph associated to $D$.

In [5] it is introduced the extended vertex ideal of the undirected graph $G_u I(G_u, V(G_u))=(v_i-\prod e_h: v_i \text{ is in } e_h \in E(G_u))$.

**DEFINITION 22.** Let $D$ be a digraph and let $D^*=K_D \setminus L$, where $L$ is the set of isolated vertices. The extended divertex ideal of $D$ is equal to $I(D^*, V)$.

The following result can be found in [5].

**THEOREM 9.** Let $G=(V(G), E(G))$ be a simple undirected connected graph without isolated vertices and let $I(G,V)$ be the extended vertex ideal of $G$. Then the ideal $I(V)_G = I(G,V) \cap K[v_1,\ldots,v_n]$ contains an irreducible polynomial $p$ of the form $p=\prod_{j \in J} v_j - \prod_{k \in K} v_k$, if and only if $G$ is bipartite. Moreover the partition sets are $V’=\{v_j : j \in J\}$ and $V”=\{v_k : k \in K\}$.

A possible generalization of the last theorem to the digraph uses the notion of directly bipartite graph.
THEOREM 10. Let \( D = (V(D), E(D)) \) be a digraph and let \( I(V)_D \) be the divertex ideal of \( D \). Then \( D \) is directly bipartite if and only if \( I(V)_D \) contains an irreducible polynomial of the form \( p = \prod_{i \in I} z_i - \prod_{j \in J} v_j \), with \( I \cap J = \emptyset \), \( I \cup J = \{1, \ldots, n\} \).

Proof. \( D \) is directly bipartite if and only if \( V(D) = V_I \cup V_J \), with \( V_I = \{v_i : i \in I\} \), \( V_J = \{v_j : j \in J\} \), \( I \cap J = \emptyset \), \( I \cup J = \{1, \ldots, n\} \) and every edge in \( E(D) \) goes from a vertex in \( V_I \) to a vertex in \( V_J \). By definition of \( K_D \) this last fact is equivalent to say that \( K_D \) is bipartite with partition sets \( Z = \{z_i : i \in I\} \) and \( V = \{v_j : j \in J\} \) and with the \( n \) isolated vertices \( \{z_j : j \in J\} \cup \{v_i : i \in I\} \). Now by theorem 9 this fact is equivalent to say that the binomial \( p = \prod_{i \in I} z_i - \prod_{j \in J} v_j \) is in the extended vertex ideal of \( K_D \). Finally, this last ideal coincides with the extended vertex ideal of \( D \) by definition. \( \square \)

Let us recall that a vertex cover \( W \) of an undirected graph \( G = (V, E) \) is a subset of \( V \), such that every edge in \( E \) is incident with at least a vertex in \( W \). A vertex cover \( W \) is called minimal if every subset of \( W \) is not a vertex cover. It is possible to generalize the concept of minimal vertex cover for digraphs, in the following way:

DEFINITION 23. Let \( G = (V, E) \) be a digraph. A source cover of \( G \) is a vertex cover \( V' \) of \( G \), such that every edge in \( G \) leaves every vertex in \( V' \). A source cover \( V' \) of \( G \) is called minimal if no subset of \( V' \) is a source cover of \( G \).

A sink cover of \( G \) is a vertex cover \( V' \) of \( G \), such that every edge in \( G \) does not leave any vertex in \( V' \). A sink cover \( V' \) of \( G \) is called minimal if no subset of \( V' \) is a sink cover of \( G \).

The following proposition is the key result for our purposes.


**PROPOSITION 1.** Let $K[v] = K[v_1, ..., v_n]$ be a polynomial ring over a field $K$ and let $G$ be an undirected graph. If $P$ is the ideal of $K[v]$ generated by $A = \{v_1, ..., v_r\}$, then $P$ is a minimal prime ideal containing the edge ideal $I(G)_E$ if and only if $A$ is a minimal vertex cover of $G$.

By using the undirected graph $K_G$ it is possible to find source and sink covers of a digraph $G$, according to the following proposition.

**PROPOSITION 2.** Let $G$ be a directed graph and let $K_G$ be the associated source-sink undirected graph. Let $V'$ be a vertex cover of $K_G$. $V' = \{v_{i(1)}, ..., v_{i(l)}\}$ is a source cover of $G$ if and only if $\{z_{i(1)}, ..., z_{i(l)}\}$ is a vertex cover of $K_G$. In similar way $V' = \{v_{i(1)}, ..., v_{i(l)}\}$ is a sink cover of $G$ if and only if $\{v_{i(1)}, ..., v_{i(l)}\}$ is a vertex cover of $K_G$.

**Proof.** Let $V' = \{v_{i(1)}, ..., v_{i(l)}\}$ be a source cover of $G$. So $V'$ is a vertex cover of $G$, such that each edge of $G$ leaves at least one vertex in $V'$. It follows that $\{z_{i(1)}, ..., z_{i(l)}\}$ is a vertex cover of $K_G$ by its own definition. In similar way if $V'$ is a sink cover of $G$, then $V'$ is a vertex cover of $G$, such that each edge of $G$ does not leave any vertex in $V'$. So $V'$ is a vertex cover of $K_G$ by its own definition.

Conversely, let $Z'=\{z_{i(1)}, ..., z_{i(l)}\}$ be a vertex cover of $K_G$. So each edge in $K_G$ is incident with at least one vertex in $Z'$. It follows that each edge in $G$ is incident with at least one vertex in $V' = \{v_{i(1)}, ..., v_{i(l)}\}$ and it leaves such vertex by definition of $K_G$. So $V'$ is a source cover of $G$. In similar way if $V' = \{v_{i(1)}, ..., v_{i(l)}\}$ is a vertex cover of $K_G$, then it is a vertex cover of $G$. Moreover $V'$ is a sink cover of $G$ by definition of $K_G$. $\square$
EXAMPLE 9. Let $D_1$ be the graph as in example 1.  
$\{z_1, z_2, z_3\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$ are minimal vertex covers of $K_{D_1}$.  $\{v_1, v_2, v_3\}$ is a source cover of $D_1$ and $\{v_1, v_2, v_3, v_4, v_5\}$ is a sink cover of $D_1$.  $D_1$ is not directly bipartite.

EXAMPLE 10. Let $D_2$ be the graph as in example 2.  
$\{z_1, z_2, z_3, z_4\}$ and $\{v_2, v_3, v_5\}$ are minimal vertex covers of $K_{D_2}$.  $\{v_1, v_2, v_3, v_4\}$ is a source cover of $D_2$ and $\{v_2, v_3, v_5\}$ is a sink cover of $D_2$.  $D_2$ is not directly bipartite.

EXAMPLE 11. Let $D_4$ be the digraph with vertex set $V(D_4) = \{v_1, \ldots, v_5\}$ and edge set $E(D_4) = \{e_1 = [v_1, v_2], e_2 = [v_3, v_2], e_3 = [v_1, v_4], e_4 = [v_3, v_4], e_5 = [v_3, v_5]\}$.  $D_4$ is directly bipartite and it has a sink cover and a source cover.  In fact the divertex ideal of $D_4$, obtained by intersecting the ideal $(z_1 - e_1 e_3, v_2 - e_1 e_2, z_3 - e_2 e_4 e_5, v_4 - e_3 e_4, v_5 - e_5)$ with $K[z_1, \ldots, z_5, v_1, \ldots, v_5]$, is $I(V)_{D_4} = (v_5 v_2 v_4 - z_1 z_3)$.  So $D_4$ is directly bipartite, $\{z_1, z_3\}$ is a source cover and $\{v_2, v_4, v_5\}$ is a sink cover.  The same result can be obtained by finding directly sink and source covers of $D_4$ through the vertex cover of the undirected graph $K_{D_4}$.

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