REFLEXIVITY OF BANACH $C(K)$-MODULES VIA THE REFLEXIVITY OF BANACH LATTICES

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Abstract. We extend the well known criteria of reflexivity of Banach lattices due to Lozanovsky and Lotz to the class of finitely generated Banach $C(K)$-modules. Namely we prove that a finitely generated Banach $C(K)$-module is reflexive if and only if it does not contain any subspace isomorphic to either $l^1$ or $c_0$.

1. Introduction

Suppose $K$ is a compact Hausdorff space and $C(K)$ is the algebra of complex (or real) valued functions on $K$ with the supremum norm. Let $X$ be a Banach space and let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators on $X$ with the operator norm. Let $m : C(K) \rightarrow \mathcal{L}(X)$ be a contractive and unital algebra homomorphism. In general the kernel of $m$ is a closed ideal of $C(K)$. Therefore, by reducing to the quotient $C(K)/\text{Ker}(m)$, we may assume that $m$ is one-to-one. Then we will regard $X$ as a Banach $C(K)$-module with $ax = m(a)(x)$ for all $a \in C(K)$ and $x \in X$. When $x \in X$, we denote by $X(x) := \text{cl}(C(K)x)$ the cyclic subspace of $X$ generated by $x$. (Here ‘cl’ denotes closure in norm in $X$.) It is familiar that $X(x)$ is representable as a Banach lattice where the cone $X(x)_+ = \text{cl}(C(K)_+x)$ and $x$ is a quasi-interior point of $X(x)_+$ [20], [23]. (Here $C(K)_+$ denotes the non-negative continuous functions on $X$.) More generally by $X(x_1, x_2, \ldots, x_n)$ we denote the closed submodule of $X$ generated by the subset $\{x_1, x_2, \ldots, x_n\}$. If for such a subset we have $X = X(x_1, x_2, \ldots, x_n)$, we will say that $X$ is finitely generated.

It is a well known result of Lozanovsky [13], [20], [14] that a Banach lattice is reflexive if and only if it does not contain a copy of either $l^1$ or $c_0$. We say a Banach space $X$ contains a copy of a Banach space $Y$, if there is a subspace of $X$ that is isomorphic to $Y$. In this paper our purpose is to extend the result of Lozanovsky to finitely generated Banach $C(K)$-modules. Namely, a finitely generated Banach $C(K)$-module $X$ is reflexive if and only if $X$ does not contain a copy of either $l^1$ or $c_0$.

We will also use a theorem of Lotz [12], [20], [14] that in one direction is stronger then the Theorem of Lozanovsky. Lotz showed that a Banach lattice is reflexive if and only if it does not contain a copy of either $l^1$ or $c_0$ as a sublattice. That is, no sublattice of the Banach lattice is lattice isomorphic to $l^1$ or $c_0$. In the proof of Theorem 1 we will use the Theorem of Lotz as well.

Theorem 1. Let $X$ be a finitely generated Banach $C(K)$-module. Then the following are equivalent:

\begin{itemize}
  \item [1] $X$ is reflexive.
  \item [2] $X$ does not contain a copy of either $l^1$ or $c_0$.
\end{itemize}
(1) \( X \) is reflexive,
(2) \( X \) does not contain a copy of either \( l^1 \) or \( c_0 \),
(3) \( X' \) does not contain a copy of \( l^1 \),
(4) Each cyclic subspace of \( X \) does not contain a copy of either \( l^1 \) or \( c_0 \),
(5) Each cyclic subspace of \( X \) is reflexive.

Since one may regard any Banach space as a Banach module over \( \mathbb{C} \) (or \( \mathbb{R} \)), the well known example, the James Space [9], [14], or its variants show that in general we cannot drop the condition that the module is finitely generated.

2. Preliminaries

Initially we will recall some information concerning Banach \( C(K) \)-modules that we need. Suppose that the compact Hausdorff space \( K \) is Stonian in the sense that the closure of an open set in \( K \) is open (i.e., clopen). Then the characteristic functions of the clopen sets (i.e., the idempotents in \( C(K) \)) form a complete Boolean algebra of projections \( \mathcal{B} \) on \( X \) and the closed linear span of \( \mathcal{B} \) in \( \mathcal{L}(X) \) is equal to \( m(C(K)) \). A stronger condition is to require that \( \mathcal{B} \) is a Bade complete Boolean algebra of projections on \( X \) in the sense that in addition to being a complete Boolean algebra, \( \mathcal{B} \) has the property that whenever \( \{ \chi_\alpha : \alpha \in \Gamma \} \) is an increasing net of idempotents in \( \mathcal{B} \) with least upper bound \( \chi \in \mathcal{B} \) and \( x \in X \), then \( \chi_\alpha x \) converges to \( \chi x \) in \( X \) [2], [6, XVII.3], [20]. This is equivalent to that \( K \) is hyperstonian (i.e., \( C(K) \) is a dual Banach space) and that the homomorphism \( m \) is continuous with respect to the weak*-topology on \( C(K) \) and the weak operator topology on \( \mathcal{L}(X) \) [15]. It also implies that \( m(C(K)) \) is closed in the weak operator topology in \( \mathcal{L}(X) \). In such a case each cyclic subspace \( X(x) \), as a Banach lattice, has order continuous norm [20], [29] and its ideal center \( Z(X(x)) \) is given by \( m(C(K))|_{X(x)} \) [11], [17], that is, each operator in \( m(C(K)) \) is restricted to the subspace \( X(x) \).

For a general Banach \( C(K) \)-module \( X \), the weak operator closure of \( m(C(K)) \) in \( \mathcal{L}(X) \) is given by the range of a map \( \hat{m} : C(\tilde{K}) \to \mathcal{L}(X) \) where \( \tilde{K} \) is a compact Hausdorff space that contains \( K \) as a quotient and \( \hat{m} \) is a contractive, unital and one-to-one algebra homomorphism that extends \( m \) [1], [8]. The closed submodules of \( X \) are the same with respect to either module structure therefore we may assume that \( m(C(K)) \) is weak operator closed without any loss of generality. The weak operator closure of \( m(C(K)) \) is generated by a Bade complete Boolean algebra of projections if and only if \( m(C(K)) \) has weakly compact action on \( X \) in the sense that the mapping \( C(K) \to X \) defined by \( a \to ax \) for all \( a \in C(K) \) for a fixed \( x \in X \) is a weakly compact linear map for each \( x \in X \) [15]. For example if \( X \) does not contain any copy of \( c_0 \), then \( m(C(K)) \) has weakly compact action on \( X \) [18]. In general even when \( m(C(K)) \) is weak operator closed in \( \mathcal{L}(X) \), when restricted to a closed submodule \( Y \), \( m(C(K))|_Y \) need not be weak operator closed in \( \mathcal{L}(Y) \). An exception is when the weak operator closed algebra \( m(C(K)) \) is generated by a Bade complete Boolean algebra of projections on \( X \). In this case both \( m(C(K))|_Y \) and \( m(C(K))|_{X/Y} \) are generated by a Bade complete Boolean algebras of projections.

Lemma 1. Suppose that \( X \) is a Banach \( C(K) \)-module such that \( m(C(K)) \) is generated by a Bade complete Boolean algebra of projections on \( X \). Let \( Y \) be a closed submodule of \( X \). Then, when restricted to either \( Y \) or \( X/Y \), \( m(C(K)) \) is generated by a Bade complete Boolean algebra of projections.
Proof. Since \( m(C(K)) \) is generated by a Bade complete Boolean algebra of projections on \( X \), we have that \( K \) is hyperstonian. The Boolean algebra of the idempotents \( B \) in \( C(K) \) is the Bade complete Boolean algebra of projections that generates \( m(C(K)) \). That is whenever \( \{ \chi_\alpha : \alpha \in \Gamma \} \) is an increasing set of idempotents in \( B \) with least upper bound \( \chi \in B \) and \( x \in X \), then \( \chi_\alpha x \) converges to \( \chi x \) in \( X \). In general \( m \) will not be one-to-one when restricted to \( Y \). However that \( m \) is weak* to weak-operator continuous implies that the kernel of \( m \) in both cases will be a weak*-closed ideal in \( C(K) \). Here we will give the proof of the lemma in the case of \( X/Y \). In the case of \( Y \), a proof is given in [9], but a proof similar to the one below is also possible. Let \( \{ a_\alpha \} \) be a net in \( C(K) \) that converges to \( a \in C(K) \) in the weak* topology. Suppose that \( \{ a_\alpha \} \) is in the kernel of \( m \) when restricted to \( X/Y \). Then \( a_\alpha x \in Y \) for all \( x \in X \) and for all \( \alpha \) in the index set. Since \( Y \) is weakly closed, we have \( ax \in Y \). So the kernel of \( m \) is a weak* closed ideal and therefore a weak*-closed band in \( C(K) \). That is there exists an idempotent \( \pi \in C(K) \) such that \( m \) is one-to-one on \( \pi C(K) \) and the kernel of \( m \) is \( (1-\pi)C(K) \). Clearly \( \pi C(K) = C(S) \) for some clopen subset \( S \) of \( K \) and furthermore, since \( \pi \) is a weak*-continuous band projection on \( C(K) \), \( C(S) \) is also a dual Banach space.

For any \( y \in X/Y \) let \( [x] = x + Y \) in \( X/Y \). Then for the increasing net of idempotents above we have
\[
\|(\chi - \chi_\alpha)[x]\| \leq \|(\chi - \chi_\alpha)x\|
\]
where the right hand side of the inequality goes to 0 in \( X \). Hence \( \pi B \) is a Bade complete Boolean algebra of projections on \( X/Y \).

We also need some lemmas concerning Banach spaces.

Lemma 2. Let \( X \) be a Banach space and \( Y \) be a reflexive subspace of \( X \).

1. For any \( x \in X \setminus Y \) there is \( y \in Y \) such that \( ||x|| = ||x + y|| \).
2. If \( ||x|| \leq C \) for some \( x \in X \setminus Y \), then \( G = \{ y \in Y : ||x + y|| \leq C \} \) is a non-empty, convex and weakly compact subset of \( Y \).

Proof. Let \( x \in X \setminus Y \) and \( n \) be a positive integer. There is \( y_n \in Y \) such that
\[
||x + y_n|| < ||x|| + \frac{1}{n}.
\]
Then \( ||y_n|| < 2||x|| + \frac{1}{n} \) for each \( n = 1, 2, \ldots \). That is \( \{ y_n \} \) is a bounded sequence in the reflexive space \( Y \). Therefore it has a weakly convergent subsequence. Without loss of generality assume that \( \{ y_n \} \) converges weakly to some \( y \in Y \). Let \( f \in X' \) with \( ||f|| = 1 \). Given any \( \varepsilon > 0 \), we have
\[
|f(x + y) - f(x + y_n)| = |f(y - y_n)| < \varepsilon
\]
for sufficiently large \( n \). Hence, for sufficiently large \( n \),
\[
|f(x + y)| < ||x + y_n|| + \varepsilon < ||x|| + \varepsilon + \frac{1}{n}.
\]
Therefore \( ||x + y|| \leq ||x|| + \varepsilon \) for all \( \varepsilon > 0 \). Then \( ||x + y|| = ||x|| \| \) and \( ||y|| \leq ||x|| + C \). From this it follows that \( G \) is non-empty, bounded, and closed. It is easy to check that \( G \) is convex. Since \( Y \) is reflexive \( G \) is weakly compact.

Lemma 3. Let \( X \) be a Banach space such that its dual \( X' \) does not contain any copy of \( l^1 \). Then \( X \) does not contain any copy of either \( l^1 \) or \( c_0 \).

Proof. The proof depends on two well known main results of this topic. A result of Bessaga and Pelczynski states that if \( X' \) contains a copy of \( c_0 \) then \( X \) contains a complemented copy of \( l^1 \) [4], [10]. Since we assume that \( X' \) does not contain any copy of \( l^1 \), \( X'' \) does not contain any copy of \( c_0 \). Then, as a subspace of \( X'' \), \( X \) does
not contain any copy of \(c_0\). On the other hand, a result of Hagler states that \(X\) contains a copy of \(l^1\) if and only if \(X'\) contains a copy of \(L^1[0,1]\) [12]. Since \(L^1[0,1]\) contains copies of \(l^1\), our assumption on \(X'\) implies that \(X'\) does not contain any copy of \(L^1[0,1]\) and therefore \(X\) does not contain any copy of \(l^1\). \(\square\)

3. Proof of the main result

Proof. When \(X\) is reflexive then \(X'\) is reflexive and therefore \(X'\) can not contain any copy of \(l^1\). That is (1) implies (3). Lemma 3 yields that (3) implies (2). It is clear that (2) implies (4). Also, since for each \(x \in X\) the cyclic subspace \(X(x)\) may be represented as a Banach lattice, Lozanovsky’s Theorem implies that (4) if and only if (5). Hence the proof will be complete if we show (4) implies (1).

Initially observe that by (4) each cyclic subspace of \(X\) does not contain any copy of \(c_0\). Then, as noted in Section 2, \(m(C(K))\) restricted to the cyclic subspace has its weak closure generated by a Bade complete Boolean algebra of projections. Hence for each \(x \in X\), the map \(m(C(K)) \to X\) defined by \(a \to ax\) is a weakly compact operator. Therefore, if we assume without loss of generality that \(m(C(X))\) is weak operator closed in \(L(X)\), then, again as noted in Section 2, we have that \(m(C(K))\) is generated by a Bade complete Boolean algebra of projections. This means that \(K\) is hyperstonian and the idempotents in \(C(K)\) (which correspond to the characteristic functions of the clopen subsets of \(K\)) form the Bade complete Boolean algebra of projections that generate \(m(C(K))\) on \(X\).

Now we prove (4) implies (1) by induction on the number of generators of \(X\). Suppose \(X\) is generated by one element \(x_0 \in X\). Then the cyclic space \(X = X(x_0)\) may be represented as a Banach lattice. Hence, (4) and Lozanovsky’s Theorem imply that \(X\) is reflexive. Now suppose that whenever a finitely generated Banach \(C(K)\)-module has \(r \geq 1\) generators and satisfies (4) then it is reflexive. Suppose \(X\) is a Banach \(C(K)\)-module with \(r + 1\) generators and satisfies (4). Let \(\{x_0, x_1, \ldots, x_r\}\) be a set of generators of \(X\). Let \(Y\) be the closed submodule of \(X\) generated by \(\{x_1, x_2, \ldots, x_r\}\). Then \(Y\) satisfies (4) and therefore \(Y\) is reflexive by the induction hypothesis. Since we have that \(m(C(K))\) is generated by a Bade complete Boolean algebra of projections on \(X\), Lemma 1 implies that the same is true for the quotient \(X/Y\). Note that since \(X/Y\) is a cyclic space generated by \([x_0]\), it may be represented as a Banach lattice such that \([x_0]\) is a quasi-interior point and the ideal center \(Z(X/Y) = m(C(K))|_{X/Y}\). That \(m(C(K))\) is generated by a Bade complete Boolean algebra of projections on \(X/Y\) implies that as a Banach lattice \(X/Y\) has order continuous norm. This means in particular that each band in \(X/Y\) is the range of a band projection. Suppose the Banach lattice \(X/Y\) is not reflexive then, by Lotz’s Theorem, \(X/Y\) must contain a copy of either \(l^1\) or \(c_0\) as a sublattice [12].

First, assume that there is a sublattice of \(X/Y\) that is lattice isomorphic to \(l^1\). Let \(\{e_n\}\) be a sequence in \(X\) such that \(\{|e_n|\}\) corresponds to the basic sequence of \(l^1\) in the sublattice of \(X/Y\) that is lattice isomorphic to \(l^1\). That is \(\{|e_n|\}\) is a pairwise disjoint positive sequence in \(X/Y\) such that for some \(0 < d < D\) we have for each \(\{\xi_n\} \in l^1\)

\[
d\Sigma|\xi_n| \leq \|\Sigma \xi_n [e_n]\| \leq D \Sigma |\xi_n|.
\]

Here \(\Sigma \xi_n [e_n]\) represents the limit of the Cauchy sequence given by the partial sums of the series in \(X/Y\). Let \(\xi_n \in C(K)\) be the band projection onto the band
generated by \([e_n]\) in \(X/Y\). Since the elements of the sequence \(\{e_n\}\) are disjoint in \(X/Y\), the elements of the sequence of band projections \(\{\chi_n\}\) are also disjoint as idempotents in \(C(K)\). Since \([e_n] = \chi_n[e_n] = [\chi_n e_n]\), without loss of generality, we assume that \(e_n = \chi_n e_n\) for each \(n\). Furthermore again without loss of generality we may assume that for some \(\varepsilon > 0\) and for each \(n\), we have \(\|e_n\| \leq D(1 + \varepsilon)\) in \(X\).

Now we have for each \((\xi_n)\) in \(l^1\) and positive integer \(N\),

\[
d|\sum_{n=1}^N |\xi_n| \leq \sum_{n=1}^N |\xi_n[e_n]| \leq \sum_{n=1}^N |\xi_n| e_n \leq \sum_{n=1}^N |\xi_n||e_n| \leq D(1 + \varepsilon)\sum_{n=1}^N |\xi_n|.
\]

By passing to the limit, we have

\[
d\sum_{n=1}^\infty |\xi_n| \leq \sum_{n=1}^\infty |\xi_n e_n| \leq D(1 + \varepsilon)\sum_{n=1}^\infty |\xi_n|
\]

where \(\sum_{n=1}^\infty |\xi_n e_n|\) is now the limit of the Cauchy sequence given by the partial sums of the series in \(X\). Hence \(X\) has a subspace that is isomorphic to \(l^1\). Let \(y = \sum \frac{1}{n} e_n\). Then \(\chi_n y = \frac{1}{n} e_n\) for each \(n\). That is, the subspace of \(X\) that is isomorphic to \(l^1\) is contained in the cyclic subspace \(X(y)\) of \(X\). This contradicts (4) and thus \(X\) does not contain a copy of \(l^1\) as a sublattice.

It follows that if the Banach lattice \(X/Y\) is not reflexive, it must contain a copy of \(c_0\) as a sublattice. Let \(\{e_n\}\) be a sequence in \(X\) such that \(\{|e_n|\}\) corresponds to the basic sequence of \(c_0\) in the sublattice of \(X/Y\) that is lattice isomorphic to \(c_0\).

Let \(\chi_n \in C(K)\) be the band projection onto the band generated by \([e_n]\) in \(X/Y\). Since the elements of the sequence \(\{e_n\}\) are disjoint in \(X/Y\), the elements of the sequence of band projections \(\{\chi_n\}\) are also disjoint as idempotents in \(C(K)\). Since \([e_n] = \chi_n[e_n] = [\chi_n e_n]\), without loss of generality, we assume that \(e_n = \chi_n e_n\) for each \(n\). We will assume that \(0 < d < D\) are the constants that give the lattice isomorphism of \(c_0\) into \(X/Y\). That is

\[
d(\max |\xi_n|) \leq \|\sum |\xi_n[e_n]|\| \leq D(\max |\xi_n|)
\]

when \((\xi_n) \in c_0\) where \(\sum |\xi_n[e_n]|\) denotes the limit of the Cauchy sequence given by the partial sums of the series in \(X/Y\). Let \(z_n = e_1 + e_2 + \ldots + e_n\) and \(\zeta_n = \chi_1 + \chi_2 + \ldots + \chi_n\) for each \(n = 1, 2, \ldots\). Let \(G_n = \{y \in Y : \|z_n + y\| \leq D\}\) for each \(n\). By Lemma 2, \(G_n\) is a non-empty, convex and weakly compact subset of \(Y\). Let \(H_n = \zeta_n G_n\). Clearly, when \(n \geq m\), \(\zeta_m z_n = z_m\). Then, for each \(y \in G_n\),

\[
\|z_n + \zeta_n y\| = \|\zeta_n(z_n + y)\| \leq \|z_n + y\| \leq D.
\]

It follows that, for each \(n\), \(H_n \subset G_n\) and that \(H_n\) is itself a non-empty, convex and weakly compact subset of \(Y\). We choose a sequence \(\{y_i\}\) in \(Y\) with \(y_i \in H_i\). It follows from above that for each \(n\), the sequence \(\{\zeta_n y_i\}_{i \geq n}\) is in \(H_n\). Namely, if \(i \geq n\) then

\[
\|z_n + \zeta_n y_i\| = \|\zeta_n(z_i + y_i)\| \leq \|z_i + y_i\| \leq D.
\]

Now let \(\{i_n\}_{n \in \Gamma}\) be an ultra-subnet of the sequence of positive integers \(\{i\}\) and for each \(n\) let \(\Gamma_n = \{\alpha \in \Gamma : i_{\alpha} \geq n\}\). Then \(\{\zeta_n y_{i_n}\}_{n \in \Gamma_n}\) is an ultra-subnet of the sequence \(\{\zeta_n y_i\}_{i \geq n}\) in \(H_n\). Since \(H_n\) is weakly compact, the ultranet \(\{\zeta_n y_{i_n}\}_{n \in \Gamma_n}\) converges weakly to some \(w_n \in H_n\). Since for any positive integers with \(n \geq m\), we have \(\Gamma_n \subset \Gamma_m\) it follows from the definition of the sequences that \(\zeta_m w_n = w_m\) whenever \(n \geq m\). In particular, since \(\zeta_{n-1} w_n = w_{n-1}\), we have \(w_n = \chi_n w_n + w_{n-1}\) for all \(n \geq 2\). Hence, by induction, we have that \(w_n = \chi_1 w_1 + \chi_2 w_2 + \ldots + \chi_n w_n\) for all positive integers \(n\). Define a sequence \(\{u_n\}\) in \(X\) such that \(u_n = e_n + \chi_n w_n\) for each \(n\). Clearly

\[
d \leq \|\{e_n\}\| \leq \|u_n\|
\]
for each $n$. Also, since $u_1 + u_2 + \ldots + u_n = z_n + w_n$ we have

$$\|u_n\| \leq \|u_1 + u_2 + \ldots + u_n\| = \|z_n + w_n\| \leq D$$

for each $n$. Then it follows that the closed subspace spanned by $\{u_n\}$ in $X$ is lattice isomorphic to $c_0$. To see this, let $u = \sum_{n=1}^{\infty} u_n$ in $X$ and consider the cyclic subspace $X(u)$ of $X$. We have that $X(u)$ is a Banach lattice with quasi-interior point $u$ and ideal center $Z(X(u)) = m(C(K))_X$. Since $\chi_n u = u_n$ for each $n$, $\{u_n\}$ is a pairwise disjoint positive sequence in the Banach lattice $X(u)$. Then the closed sublattice of $X(u)$ generated by $\{u_n\}$ is lattice isomorphic to $c_0$, this follows directly from the $Z(X(u))$ module structure of $X(u)$, but also it follows by [14 Lemma 2.3.10]. This contradicts the assumption (4). Hence, the Banach lattice $X/Y$ cannot contain a copy of $c_0$ as a sublattice, as well as not containing a copy of $l^1$ as a sublattice. Then Lotz’s refinement of Lozanovsky’s Theorem implies that $X/Y$ is reflexive. Since by induction hypothesis $Y$ is reflexive, we have that $X$ is also reflexive. The reader will observe that this is the familiar three space property of reflexivity. Therefore (4) implies (1). \hfill $\square$

One of the questions naturally connected with the statement of Theorem 1 is the following one. Can we substitute condition (5) of this theorem with a weaker condition that for some set $\{z_1, \ldots, z_n\}$ of generators of the $C(K)$-module $X$ the cyclic subspaces $X(z_i), i = 1, \ldots, n$ are reflexive? The two examples below show that the answer to this question in general is negative.

**Example 1.** Let $E = L^2(0, 1) \oplus L^1(0, 1) \oplus L^1(0, 1)$ be an $L^\infty(0, 1)$-module with the norm $\|(f, g, h)\| = \|f\|_2 + \|g\|_1 + \|h\|_1$ for all $f \in L^2(0, 1)$ and $g, h \in L^1(0, 1)$. The module structure is carried over coordinatewise from the module structure of $L^2(0, 1)$ and $L^1(0, 1)$.

Let $X = \{(f, g, h) \in E : f + g + h = 0\}$. Then $X$ is a closed submodule of $E$ that has two generators. It is easy to see for example that $\{(-1, 1, 0), (1, 0, -1)\}$ generates $X$ where 1 is the identity in $L^\infty(0, 1)$. Then $X((-1, 1, 0))$ and $X((1, 0, -1))$ are both isomorphic to $L^2(0, 1)$. Thus the cyclic subspace generated by either of these vectors is reflexive.

But $X$ itself is not reflexive. Indeed $(0, 1, -1) \in X$ and $X((0, 1, -1))$ is isomorphic to $L^1(0, 1)$ and hence is not reflexive. Moreover, if we use the set of generators $\{(1, 0, -1), (0, 1, -1)\}$, it is straightforward to see that $X$ is isomorphic (but not isometric) to $L^2(0, 1) \oplus L^1(0, 1)$.

Now we will provide an even simpler example of a nonreflexive Banach $C(K)$-module $X$ which is the direct sum of two cyclic subspaces but also has two generators such that the corresponding cyclic subspaces are reflexive.

**Example 2.** Let $X = c_0 \oplus l^2$ be a Banach $l^\infty$-module with the norm $\|(x, y)\| = \|x\|_\infty + \|y\|_1$, for all $x \in c_0$ and $y \in l^2$. Like in Example 1 the $l^\infty$-module structure is carried over coordinatewise from the module structure of $c_0$ and $l^2$. It is straightforward to observe that if $x_n = 1/n, n \in \mathbb{N}$ then $x \in c_0 \cap l^2$ and the vectors $(x, x)$ and $(0, x)$ generate $X$. Moreover, $X(x, x)$ is isomorphic to $l^2$ and $X(0, x) = l^2$. Hence once again $X$ is not reflexive but has a pair of generators such that each of them generates a reflexive cyclic subspace.
4. Boolean algebras of projections of finite multiplicity

In this section we will consider extending the conclusions of Theorem 1 to the case where the Banach $C(K)$-module may be infinitely generated while staying close to being finitely generated. Let $X$ be a Banach $C(K)$-module where $K$ is hyperstonian and the homomorphism $m$ is weak* to weak-operator continuous. That is, as discussed in Section 2, $\mathcal{B}$ (the set of idempotents in $m(C(K))$) is a Bade complete Boolean algebra of projections on $X$. Throughout this section we will assume that $\mathcal{B}$ is a Bade complete Boolean algebra of projections on $X$. Then $\mathcal{B}$ is said to be of uniform multiplicity $n$ if there exists a disjoint family of projections $\{\chi_\alpha\}$ in $\mathcal{B}$ such that for each projection $\chi \in \mathcal{B}$ with $\chi \chi_\alpha = \chi$ one has $\chi X$ is generated by a minimum of $n$ elements in $\chi X$, and also $\sup \chi_\alpha = 1$ in $\mathcal{B}$.

If $\mathcal{B}$ is of uniform multiplicity one then it was shown by Rall [19], that $X$ is represented as a Banach lattice with order continuous norm and its ideal center is $m(C(K))$ (for a proof, see [16, Lemma 2]). The only difference from the cyclic case is that instead of a quasi-interior element one has a topological orthogonal system [20]. (For example consider $l^2(\Gamma)$ when $\Gamma$ is an uncountable discrete set.) To prove our next result, Corollary 1, we need the following Lemma.

**Lemma 4.** Let $X$ be a Banach space and let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$ that is of uniform multiplicity one. Then $X$ is reflexive if and only if no cyclic subspace of $X$ contains a copy of either $l^1$ or $c_0$.

**Proof.** It is necessary to prove only one direction. Namely, suppose $X$ has no cyclic subspace that contains a copy of either $l^1$ or $c_0$. By Rall’s Theorem, $X$ is represented as a Banach lattice with order continuous norm and its ideal center is $m(C(K))$ (for a proof, see [16, Lemma 2]). The only difference from the cyclic case is that instead of a quasi-interior element one has a topological orthogonal system [20]. (For example consider $l^2(\Gamma)$ when $\Gamma$ is an uncountable discrete set.) To prove our next result, Corollary 1, we need the following Lemma.

**Remark 1.** The special case of Lemma 4 when $X$ is cyclic was obtained by Tzafriri in [22].

Then the methods of Theorem 4 when $X$ is cyclic obtained by Tzafriri in [22].

**Corollary 1.** Let $X$ be a Banach space and let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$ that is of uniform multiplicity $n$. Then the conditions (i)-(v) of Theorem 4 are equivalent.

**Proof.** The part of the proof up to (iv) implies (i) is as in the proof of Theorem 4 and is clear. In the proof of (iv) implies (i), Lemma 4 shows the case $n = 1$ is true. The rest of the proof follows by induction on $n$ just as in the proof of Theorem 4.
We say $B$ is of finite multiplicity on $X$ if there exists a disjoint family of projections $\{\chi_n\}$ in $B$ such that $\chi_nX$ is generated by a minimum of $n_n$ elements in $\chi_nX$ and sup $\chi_n = 1$ in $B$. Then it follows that there exists a sequence $\{e_n\}$ of disjoint projections in $B$ such that $e_nB$ is of uniform multiplicity $n$ on $e_nX$ and sup $e_n = 1$ \cite[XVIII.3]{[6]}. In such a case it is clear that $X = \text{cl}(\text{span}\{e_nX : n = 1, 2, \ldots\})$.

**Theorem 2.** Let $X$ be a Banach space and let $B$ be a Banach complete Boolean algebra of projections on $X$ that is of finite multiplicity. Then the conditions (i)-(v) of Theorem 1 are equivalent.

**Proof.** Once again, (i) implies (iii), (iii) implies (ii), (ii) implies (iv), and (iv) if and only if (v) are clear. We need to show (v) implies (i). Suppose there exists a positive integer $n$ such that $e_m = 0$ for all $m > n$. Then, by the Corollary \[4\] $X$ is the direct sum of a finite collection of reflexive Banach spaces. Hence $X$ is reflexive. Therefore, without loss of generality, assume that $e_n \neq 0$ for all $n$. Note that, by (iv) and the Corollary \[4\] we have that the subspace $e_nX$ is reflexive for each $n$. We need to show that this extends to $X$. For each $n$, let $\chi_n = e_1 + e_2 + \ldots + e_n \in B$.

Initially motivated by an idea of James \cite[Theorem 1(b)]{[9]}, we will prove the following: (*) Suppose each cyclic subspace of $X$ does not contain any copy of $l^1$, then $\|f - \chi_n f\| \to 0$ for all $f \in X'$.

For some $f \in X'$ with $\|f\| = 1$, suppose that (*) is false. Since $\{\chi_n x\}$ converges to $x$ in norm for all $x \in X$, we have $\{\chi_n f\}$ converges to $f$ in the weak* topology in $X'$. Therefore that $\{\chi_n f\}$ does not converge to $f$ in norm implies that $\{\chi_n f\}$ does not converge in norm in $X'$.

That is, $\{\chi_n f\}$ is not a Cauchy sequence. Hence there exists an $\varepsilon > 0$ and a subsequence $\{\chi_{n_k}\}$ such that

$$\|\chi_{n_{k+1}} f - \chi_{n_k} f\| \geq \varepsilon$$

for all $n$ and sup $\chi_{n_k} = 1$. So there is a disjoint sequence of idempotents $\{\zeta_n\}$ in $B$ with sup $\zeta_n = 1$ such that (i) $\zeta_1 = \chi_{k(2)}$, (ii) $\zeta_n = \chi_{k(n+1)} - \chi_{k(n)}$ for $n \geq 2$, and (iii) $\|\zeta_n f\| \geq \varepsilon$ for all $n$. This implies that $X'$ contains a copy of $l^\infty$. In fact if $\{a_n\} \in l^\infty$, then $\sum a_n \zeta_n \in C(K)$. Here $\sum a_n \zeta_n$ represents the unique extension on $K$ of the densely defined bounded continuous function that is equal to $a_n$ on the clopen subset of $K$ with characteristic function $\zeta_n$ for each $n$.

Hence

$$\varepsilon \|\{a_n\}\|_\infty \leq \|\sum a_n \zeta_n f\| \leq \|\{a_n\}\|_\infty$$

for all $\{a_n\} \in l^\infty$. Clearly $\{\zeta_n f\}$ corresponds to the basis of $c_0$ and $f$ corresponds to the unit of $l^\infty$. Now in a standard manner one can show that $X$ contains a copy of $l^1$. Namely, for each $n$, choose $x_n \in \zeta_n X$ such that $\|x_n\| = 1$ and $f(x_n) > \frac{\varepsilon}{2}$. Given any $\{\xi_n\} \in l^1$, consider $\xi_n = |\xi_n| e^{i\theta_n}$ for some $\theta_n \in [0, 2\pi)$, for all $n$. Then

$$\frac{\varepsilon}{2} \|\xi_n\|_1 \leq f(\sum e^{-i\theta_1} \xi_n x_n) = \sum |\xi_n| f(x_n) \leq \|\sum \xi_n x_n\| \leq \|\xi_n\|_1.$$

Here $\sum \xi_n x_n$, as before, denotes the limit of the partial sums of the series in $X$. If we let $y = \sum \xi_n x_n \in X$, it is clear that the subspace of $X$ that is a copy of $l^1$, is contained in the cyclic subspace $X(y)$. But this is a contradiction. Therefore (*) is proved.

Now assume that $\{x_\lambda\}$ is an ultrafied in the unit ball of $X$. Then $\{e_n x_\lambda\}$ is an ultrafied in the weakly compact unit ball of the reflexive space $e_n X$. Therefore $\{e_n x_\lambda\}$ converges weakly to some $y_\lambda$ in the unit ball of $e_n X$. Clearly the closed submodule $Y$ of $X$ that is generated by $\{y_\lambda\}$ is of uniform multiplicity one. Therefore, by (iv) and the Corollary \[4\] $Y$ is reflexive. Also by Rall’s result $Y$ is a Banach
lattice. Therefore $Y$ is a KB-space \cite{20, 14}. Let $z_n = y_1 + y_2 + \ldots + y_n \in \chi_n X$, for each $n$. Then the ultrafilter $\{\chi_n x_\lambda\}$ converges weakly to $z_n$ in the unit ball of the reflexive space $\chi_n X$. But this means that $\{z_n\}$ is a positive, increasing sequence in the unit ball of the KB-space $Y$. Therefore there exists $z \in Y$ such that $z = \sup z_n$ and $\{z_n\}$ converges to $z$ in norm in $Y$. Hence, given $\varepsilon > 0$, we have

$$\|z - z_n\| < \frac{\varepsilon}{4}$$

for sufficiently large $n$. Also for any $f \in X'$ with $\|f\| = 1$, by (\ref{equation:star}), we have

$$\|f - \chi_n f\| < \frac{\varepsilon}{4}$$

for sufficiently large $n$. Then for some fixed $n$ that is sufficiently large, we have

$$|f(x\lambda - z)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + |\chi_n f(z_n - \chi_n x\lambda)|$$

for all $\lambda$. Since $\{\chi_n x_\lambda\}$ converges weakly to $z_n$ in $\chi_n X$, it follows that the ultrafilter $\{x_\lambda\}$ converges to $z$ weakly in the unit ball of $X$. But any net in $X$ has an ultra-subnet. That is we have proved that any net in the unit ball of $X$ has a weakly convergent subnet with limit point in the unit ball. Hence, the unit ball of $X$ is weakly compact and $X$ is reflexive. \hfill \Box

Dieudonné \cite{5} constructed the famous example of a Banach space $X$ and a Bade complete Boolean algebra of projections $B$ on $X$ that is of uniform multiplicity 2 and has the following property (\text{D}): for any $x, y \in X$ and any $e \in B \setminus \{0\}$, $eX$ is not equal (or even isomorphic) to the sum of the cyclic subspaces $eX(x)$ and $eX(y)$. The space $X$ in the Dieudonné’s example is not reflexive but a minor modification of his example outlined in Example 3 below provides a reflexive space with similar properties.

**Example 3.** (Dieudonné) Let the interval $[0, \gamma]$ and the functions $\omega_i, i = 1, 2, 3$, be as constructed in \cite{5} Section 6. Let $f$ be a Lebesgue measurable function on $[0, \gamma]$. As usual we denote the equimeasurable decreasing rearrangement of $|f|$ as $|f|^\ast$. Let $L(0, \gamma)$ be the space of equivalence classes of measurable functions on $[0, \gamma]$. Consider the Lorentz spaces

$$L^2_{\omega_i} = \{f \in L(0, \gamma) : \int_0^\gamma (|f|^\ast)^2 \omega_i dx < \infty\}, i = 1, 2, 3$$

with the norm $N_i(f) = \left(\int_0^\gamma (|f|^\ast)^2 \omega_i dx\right)^{1/2}$.

The spaces $L^2_{\omega_i}, i = 1, 2, 3$, are reflexive (see \cite{11}) Banach lattices with the ideal center $Z(L^2_{\omega_i}) = L^\infty(0, \gamma)$, $i = 1, 2, 3$. Dieudonné’s construction in \cite{5} shows that $\omega_i \omega_j \in L^1(0, \gamma)$ if and only if $i \neq j, (i, j = 1, 2, 3)$. Moreover, $\omega_i^{1/2} \in L^2_{\omega_i}$, if and only if $i \neq j, (i, j = 1, 2, 3)$. Consider the Banach $L^\infty(0, \gamma)$-module $E = L^2_{\omega_1} \oplus L^2_{\omega_2} \oplus L^2_{\omega_3}$ endowed with the norm $N(f, g, h) = N_1(f) + N_2(g) + N_3(h)$. Let $X = \{(f, g, h) \in E : f + g + h = 0\}$ and let $B$ be the algebra of all the idempotents in $L^\infty(0, \gamma)$. Then the proof given by Dieudonné in \cite{5} shows that $B$ is a Bade complete Boolean algebra of projections on $X$ that is of uniform multiplicity 2 with the property $D$.

**Remark 2.** In his study of multiplicity of Boolean algebras of projections, motivated by Dieudonné’s Example, Tzafriri (see \cite{21}) gave the following formal definition of
property $D$: Suppose $X$ is a Banach space and $B$ is a Bade complete Boolean algebra of projections on $X$ of uniform multiplicity $n$. $B$ has property $D$: if, for any $x_i \in X (i = 1, \ldots, n)$, any $e \in B$, and any $p, 1 \leq p < n$, $eX$ is not equal to the sum of $eX(x_1, \ldots, x_p)$ and $eX(x_{p+1}, \ldots, x_n)$.

Tzafriri showed in [21] that $B$ has property $D$ on $X$ if and only if any bounded projection on $X$ that commutes with $B$ is itself in $B$.

In connection with Example 3 one can consider the following question.

**Problem 1.** When is it possible to embed a reflexive (in particular, finitely generated) Banach $C(K)$-module into a reflexive Banach lattice as a closed subspace?

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