A NOTE ON LOCAL WELL-POSEDNESS OF GENERALIZED KDV TYPE EQUATIONS WITH DISSIPATIVE PERTURBATIONS

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Abstract. In this note we report some local well-posedness results for the Cauchy problems associated to generalized KdV type equations with dissipative perturbation for given data in the low regularity $L^2$-based Sobolev spaces. The method of proof is based on the contraction mapping principle employed in some appropriate time weighted spaces.

1. Introduction

In this brief note we are interested in reporting well-posedness results for the Cauchy problems associated to generalized Korteweg-de Vries (KdV) type equations with dissipative perturbations, viz.,

\[
\begin{cases}
v_t + v_{xxx} + \eta Lv + (v^{k+1})_x = 0, & x \in \mathbb{R}, \ t \geq 0, \ k \in \mathbb{N}, \\
v(x, 0) = v_0(x),
\end{cases}
\]  

(1.1)

and

\[
\begin{cases}
u_t + v_{xxx} + \eta Lu + (u^{k+1})_x = 0, & x \in \mathbb{R}, \ t \geq 0, \ k \in \mathbb{N}, \\
u(x, 0) = u_0(x),
\end{cases}
\]  

(1.2)

where $\eta > 0$ is a constant; $u = u(x,t)$, $v = v(x,t)$ are real valued functions and the linear operator $L$ is defined via the Fourier transform by $\hat{L}f(\xi) = -\Phi(\xi)\hat{f}(\xi)$. For $p \in \mathbb{R}^+$, the Fourier symbol $\Phi(\xi) \in \mathbb{R}$ is measurable and has the form

\[
\Phi(\xi) = -|\xi|^p + \Phi_1(\xi),
\]

(1.3)

where $|\Phi_1(\xi)| \leq C(1 + |\xi|^q)$ with $0 \leq q < p$.

The models under consideration are posed for $t \geq 0$. In order to deal with some estimates involving semi-group, we extend these models for all $t \in \mathbb{R}$. For this, we define

\[
\eta(t) \equiv \eta \text{ sgn}(t) = \begin{cases} 
\eta & \text{if } t \geq 0, \\
-\eta & \text{if } t < 0,
\end{cases}
\]

(1.4)

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and write (1.1) and (1.2) respectively in the form
\[ \begin{align*}
v_t + v_{xxx} + \eta(t)Lv + (v^{k+1})_x &= 0, \quad x, t \in \mathbb{R}, \\
v(0) &= v_0, \end{align*} \tag{1.5} \]
and
\[ \begin{align*}
u_t + u_{xxx} + \eta(t)Lu + (u_x^{k+1}) &= 0, \quad x, t \in \mathbb{R}, \\
u(x,0) &= u_0(x). \end{align*} \tag{1.6} \]

This sort of extension to consider the models posed for all \( t \in \mathbb{R} \) has already been appeared in earlier works [3, 4]. In what follows we will study the well-posedness issues for the Cauchy

\[ \text{and introduce the Banach space} \ X \]

\[ \text{and introduce, accordingly, the Banach space} \ Y \]

Motivation behind the definition of these sort of spaces is the work of Dix [5] where the author uses space with time weight to prove sharp well-posedness result for the Cauchy problem associated to the Burgers’ equation in \( H^s(\mathbb{R}) \), \( s > -\frac{1}{2} \) by showing that uniqueness fails for data Sobolev regularity less than \( -\frac{1}{2} \). In our recent work [2], similar spaces are used to get sharp local well-posedness results for KdV type equations. We notice that the space used in [2] needs restriction of the Sobolev regularity of the initial data not only from below but also from above. To overcome the restriction of Sobolev regularity from above, we used extra norms in the definitions of \( X \) and \( Y \) above. Although we could remove the regularity restriction from above with the introduction of extra norms, we could not lower
the regularity requirement of the initial data as much as we desired. This question will be addressed elsewhere.

Now, we state the main results of this work.

**Theorem 1.1.** Let $\eta > 0$ be fixed, $k > 0$ and $\Phi(\xi)$ be as given by (1.3) with $p > \frac{3}{2}k + 1$ as the order of the leading term. Then the Cauchy problem (1.5) is locally well-posed for any data $v_0 \in H^s(\mathbb{R})$, whenever $s \geq -1$. Moreover, the map $v_0 \mapsto v$ is smooth from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R})) \cap X^s_T$ and $v \in C([-T, T]\{0\}; H^\infty(\mathbb{R}))$.

**Theorem 1.2.** Let $\eta > 0$ be fixed, $k > 0$ and $\Phi(\xi)$ be as given by (1.3) with $p > \frac{3}{2}k + 1$ as the order of the leading term. Then the Cauchy problem (1.6) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, whenever $s \geq 0$. Moreover, the map $u_0 \mapsto u$ is smooth from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R})) \cap Y^s_T$ and $u \in C([-T, T]\{0\}; H^\infty(\mathbb{R}))$.

**Remark 1.3.** The results in Theorems 1.1 and 1.2 improve the known well-posedness results for the Cauchy problems (1.5) and (1.6) proved in [3]. However, in view of our recent work [2], we believe that there is still room to get better results than the ones presented here. Due to the presence of higher order nonlinearity, the time weighted spaces introduced in [2] are not good enough in this case. So, one expects to introduce some new sort of spaces to cater higher order nonlinearity. Currently, we are working in this direction.

In earlier works [3] (see also [4]) we proved that the IVPs (1.5) and (1.6) (with $k = 1$) for given data in $H^s(\mathbb{R})$ are locally well-posed whenever $s > -\frac{3}{4}$ and $s > \frac{1}{2}$ respectively. The IVPs (1.5) and (1.6) with $k = 2, 3, 4$ were also considered in [3] to get local well-posedness results respectively for $s > \frac{1}{4}, -\frac{1}{6}, 0$ and $s > \frac{5}{4}, \frac{5}{6}, 1$. To obtain these results we followed the techniques developed by Bourgain [1] and Kenig, Ponce and Vega [7] (see also [11]). The main ingredients in the proof are estimates of the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$. The principal tool in [3] and [4] is the usual Bourgain space associated to the KdV equation instead of that associated to the linear part of the IVPs (1.5) and (1.6).

A detailed explanation about the particular examples that belong to the class considered in (1.5) and (1.6) and the known well-posedness results about them are presented in our earlier works [2, 3].

This paper is organized as follows: In Section 2, we prove some linear and nonlinear estimates. Section 3 is dedicated to prove the main results.

### 2. Basic estimates

In this section we derive some linear and nonlinear estimates that will be used to prove the main results of this work. Consider the Cauchy problem associated to the linear parts of
\[ w_t + w_{xxx} + \eta(t)Lw = 0, \quad x, t \in \mathbb{R}, \]
\[ w(0) = w_0. \]  
(2.1)

The solution to (2.1) is given by
\[ w(x, t) = V(t)w_0(x) \]
where the semi-group \( V(t) \) is defined as
\[ \hat{V}(t)w_0(\xi) = e^{it\xi^3 + \eta|t|\Phi(\xi)}\hat{w}_0(\xi). \]  
(2.2)

In what follows, we prove some estimates satisfied by the semi-group defined in (2.2). Without loss of generality, we suppose \( \eta = 1 \) from here onwards. We start with the following lemmas.

**Lemma 2.1** (Hausdorff-Young). Let \( p_1 \geq 2 \) and \( \frac{1}{p_1} + \frac{1}{q_1} = 1 \). Then
\[ \|f\|_{L^{p_1}} \leq C\|\hat{f}\|_{L^{q_1}}. \]  
(2.3)

**Lemma 2.2.** There exists a sufficiently large number \( M > 0 \) such that the following estimates
\[ \Phi(\xi) = -|\xi|^p + \Phi_1(\xi) < -1, \]  
(2.4)
\[ \frac{|\Phi_1(\xi)|}{|\xi|^p} \leq \frac{1}{2}, \]  
(2.5)

and
\[ |\Phi(\xi)| \geq \frac{|\xi|^p}{2}, \]  
(2.6)
hold true for all \( |\xi| \geq M \).

**Proof.** Note that,
\[ \lim_{|\xi| \to \infty} \frac{\Phi_1(\xi) + 1}{|\xi|^p} = 0 \quad \text{and} \quad \lim_{|\xi| \to \infty} \frac{|\Phi_1(\xi)|}{|\xi|^p} = 0. \]  
(2.7)

The estimates (2.4) and (2.5) follow respectively from the first and the second relation in (2.7).

For \( |\xi| \geq M \), using (2.4) and (2.5), one has that
\[ |\Phi(\xi)| = |\xi|^p - \Phi_1(\xi) \geq \frac{|\xi|^p}{2}, \]  
(2.8)
which proves (2.6). \( \square \)

**Lemma 2.3.** Let \( |t| \leq T \) and \( \Phi(\xi) \) be as defined in (1.3). Then \( \Phi(\xi) \) is bounded from above and
\[ \|e^{t|\Phi(\xi)|}\|_{L^\infty} \leq e^{TCM}. \]  
(2.9)

**Proof.** Using (2.4) in Lemma 2.2, we see that there is \( M > 1 \) large enough such that \( \Phi(\xi) < -1 \). Therefore,
\[ e^{t|\Phi(\xi)|} \leq e^{-|t|} \leq 1, \quad \forall \ |\xi| \geq M. \]  
(2.10)

For \( |\xi| < M \), it is easy to show that \( \Phi(\xi) < C_M \), and consequently
\[ e^{t|\Phi(\xi)|} \leq e^{TCM} \quad \forall \ |\xi| < M. \]  
(2.11)

Combining (2.10) and (2.11) we conclude the proof. \( \square \)
Lemma 2.4. Let \( v_0 \in H^s(\mathbb{R}) \) and \( V(t) \) be defined as in (2.2), then
\[
V(\cdot)v_0 \in C(\mathbb{R}; H^s(\mathbb{R})) \cap C(\mathbb{R} \setminus \{0\}; H^\infty(\mathbb{R})).
\]

Proof. It is enough to show that \( V(t)v_0 \in H^{s'}(\mathbb{R}) \) whenever \( s' > s \). We have,
\[
\|V(t)v_0\|_{H^{s'}} = \|\langle \xi \rangle^{s'} e^{it\xi^2 + |t|\Phi(\xi)}\tilde{v}_0(\xi)\|_{L^2}
\leq \|\langle \xi \rangle^{s'} \tilde{v}_0(\xi)\|_{L^2} \langle |t| \rangle \langle \xi \rangle^{s'} |t|^{1/p} \|\Phi(\xi)\|_{L^2}
\leq \|\langle \xi \rangle^{s'} e^{it\xi^2 + |t|\Phi(\xi)}\|_{L^2} \langle |t| \rangle \langle \xi \rangle^{s'} |t|^{1/p} \|\Phi(\xi)\|_{L^2}.
\] (2.12)

where we have used the notation \( \langle x \rangle := 1 + |x| \).

Let \( M > 1 \) be as in Lemma 2.2. Then one gets
\[
\|\langle \xi \rangle^{s'} e^{it\xi^2 + |t|\Phi(\xi)}\|_{L^\infty} \leq \|\langle \xi \rangle^{s'} e^{it\Phi(\xi)}\|_{L^\infty(\{\xi|\leq M\})} + \|\langle \xi \rangle^{s'} e^{it\Phi(\xi)}\|_{L^\infty(\{\xi| > M\})}
\leq C_M + \| e^{-\frac{|t|}{2}} |\xi|^{s'-s} \|_{L^\infty} < \infty, \quad t \in \mathbb{R} \setminus \{0\}.
\] (2.13)

One can use the dominated convergence theorem to prove continuity. \(\square\)

Lemma 2.5. Let \( s > -1 - \frac{1}{2(k+1)} \), \( |t| \leq T \leq 1 \) and \( p > 0 \), then we have
\[
|t|^{\frac{1}{p}} \|\partial_x D_k^s V(t)w_0\|_{L^{2(k+1)}} \leq C \|w_0\|_{H^s}.
\] (2.14)

and
\[
|t|^{\frac{1}{p}} \|V(t)w_0\|_{L^{2(k+1)}} \leq C \|w_0\|_{H^{-1}}.
\] (2.15)

Proof. By Hausdorff-Young and Hölder inequalities, we obtain
\[
\|\partial_x D_k^s V(t)w_0\|_{L^{2(k+1)}} \leq C \|e^{it\Phi(\xi)}\xi^s \tilde{w}_0\|_{L^{2(k+1)}} \leq C \|e^{it\Phi(\xi)}\xi^s \tilde{w}_0\|_{L^\infty} \|\xi^s \tilde{w}_0\|_{L^2}.
\] (2.16)

Let \( M \) as in Lemma 2.2. Then we have
\[
\frac{\|\xi^s e^{it\Phi(\xi)}\|_{L^{2(k+1)}}}{\|\xi^s\|_{L^{2(k+1)}}} \leq \frac{\|\xi^s e^{it\Phi(\xi)}\chi_{\{\xi| \leq M\}}\|_{L^{2(k+1)}}}{\|\xi^s\|_{L^{2(k+1)}}} + \frac{\|\xi^s e^{it\Phi(\xi)}\chi_{\{\xi| > M\}}\|_{L^{2(k+1)}}}{\|\xi^s\|_{L^{2(k+1)}}}
\leq C_M + \left( \int_{\mathbb{R}} |\xi|^{k(l+1)} e^{-|t| \left( \frac{1}{2(k+1)} \right)} d\xi \right)^\frac{1}{k(l+1)}.
\] (2.17)

Using a change of variable \( |t|^{-1/p} \xi := x \), we get
\[
\frac{\|\xi^s e^{it\Phi(\xi)}\|_{L^{2(k+1)}}}{\|\xi^s\|_{L^{2(k+1)}}} \leq C_M + \frac{1}{|t|^{\frac{1}{p}}} \left( \int_{\mathbb{R}} |x|^{k(l+1)} e^{-|t| \left( \frac{1}{2(k+1)} |x|^p \right)} dx \right)^\frac{1}{k(l+1)}.
\] (2.18)

Since the integral in (2.18) is finite, inserting (2.15) in (2.10) and multiplying by \( |t|^{\frac{1}{p}} \) we get the required estimate (2.22).

With a similar argument as above (considering \( s = 0 \) in (2.22)), one can obtain
\[
|t|^{\frac{1}{p}} \|D_2 V(t)w_0\|_{L^{2(k+1)}} \leq C \|w_0\|_{L^2}.
\] (2.19)
Lemma 2.8. Let
\[ |t|^{\frac{p}{2}} \|V(t)w_0\|_{L^2(\mathbb{R}^d)} \leq C\|w_0\|_{L^2}. \]  
Now, using (2.19) and (2.20), we obtain
\[ |t|^{\frac{p}{2}} \|(1 + D_x)V(t)w_0\| \leq C\|w_0\|_{L^2}. \]  
This final estimate (2.21) is equivalent to (2.23) and this completes the proof. \(\square\)

Corollary 2.6. Let \( s > -1 - \frac{k}{2(k+1)}, \ |\tau| \leq T, \ |t| \leq T \leq 1 \) and \( p > 0 \), then we have
\[ |t - \tau|^{\frac{p}{2}} \|\partial_x D_x^s V(t - \tau)f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C\|f(t, \cdot)\|_{H^s}. \]  
and
\[ |t - \tau|^{\frac{p}{2}} \|V(t - \tau)f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C\|f(t, \cdot)\|_{H^{s-1}}. \]  

Lemma 2.7. Let \( V(t) \) be as defined in (2.22), \(|t| \leq T \leq 1\) and \( p > 3/2k + 1 \) then for all \( s \geq -1 \), we have
\[ \|V(t)w_0\|_{X^s_T} \leq C\|w_0\|_{H^s}. \]  
and for all \( s \geq 0 \), we have
\[ \|V(t)w_0\|_{Y^s_T} \leq C\|w_0\|_{H^s}. \]  

Proof. We provide a detailed proof for (2.24), the proof of (2.25) follows similarly. We start with the first component of \( X^s_T \)-norm. First, note that
\[ \|V(t)w_0\|_{H^s} = \|s|\Phi(\xi)| w_0(\xi)\|_{L^2} \leq \|e|t|\Phi(\xi)|\|_{L^\infty}\|w_0\|_{H^s}. \]  
From (2.9) and (2.26), we can conclude
\[ \|V(t)w_0\|_{H^s} \leq e^{TC_M}\|w_0\|_{H^s}. \]  
(2.27)

For the next components of the \( X^s_T \)-norm, we use (2.22) and (2.23) from Lemma 2.5 to get, for \( s \geq -1 \)
\[ |t|^{\frac{p}{2}} \|D_x^s \partial_x V(t)w_0\|_{L^2(\mathbb{R}^d)} \leq C\|w_0\|_{H^s}. \]  
(2.28)

Combining (2.27) and (2.28) we get the required estimate (2.24). \(\square\)

Lemma 2.8. Let \( V(t) \) be as defined in (2.22), \(|t| \leq T \leq 1\) and \( p > 3/2k + 1 \), then the following estimates hold true.
\[ \left\| \int_0^t V(t - \tau)\partial_x (u^{k+1}(\tau))d\tau \right\|_{X^s_T} \leq T^{\omega_k}\|v\|_{X^s_T}^{k+1}, \ \forall s \geq -1, \]  
(2.29)
and
\[ \left\| \int_0^t V(t - \tau)(\partial_x u)^{k+1}(\tau)d\tau \right\|_{Y^s_T} \leq T^{\omega_k}\|u\|_{Y^s_T}^{k+1}, \ \forall s \geq 0, \]  
(2.30)
where \( \omega_k = \frac{2p-3k-2}{2p} > 0 \).
Proof. We start by proving (2.29). First consider the case when \( s \geq 0 \). Using the definition of \( X^p_s \)-norm and Lemma 2.5 for \( s \geq 0 \) (see (1.7)), we get, for the first component

\[
\sup_{t \in [-T,T]} \left\| \int_0^t V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau \right\|_{H^p_T} \leq \sup_{t \in [-T,T]} \int_0^t \left\| \partial_x(v^{k+1})(\tau) \right\|_{H^p}d\tau.
\]

(2.31)

To estimate the other components of \( X^p_s \)-norm, we use Minkowski’s integral inequality and Corollary 2.6. For the sake of brevity and clarity, we give details for the fourth component only, because others follow similarly.

\[
|t|^\frac{s}{p} \left\| D_x^s \partial_x \int_0^t V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau \right\|_{L^{2(k+1)}} = |t|^\frac{s}{p} \left\| \int_0^t D_x^s \partial_x V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau \right\|_{L^{2(k+1)}}
\]

\[
\leq C |t|^\frac{s}{p} \int_0^t \left\| D_x^s \partial_x V(t-\tau)\partial_x(v^{k+1})(\tau) \right\|_{L^{2(k+1)}}d\tau
\]

\[
\leq C |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \left\| \partial_x(v^{k+1})(\tau) \right\|_{H^p}d\tau.
\]

(2.32)

Combining (2.31), (2.32) and similar estimates for the other components, we obtain

\[
\left\| \int_0^t V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau \right\|_{X^p_T} \leq \sup_{t \in [-T,T]} \int_0^t \left( \left\| \partial_x(v^{k+1}) \right\|_{L^2} + \left\| D_x^s \partial_x(v^{k+1}) \right\|_{L^2} \right) d\tau
\]

\[
+ C \sup_{t \in [-T,T]} |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \left( \left\| \partial_x(v^{k+1}) \right\|_{L^2} + \left\| D_x^s \partial_x(v^{k+1}) \right\|_{L^2} \right) d\tau
\]

\[
\leq C \sup_{t \in [-T,T]} |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \left\| \partial_x(v^{k+1}) \right\|_{L^2}d\tau + \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \left\| D_x^s(v^{k+1}) \right\|_{L^2}d\tau
\]

\[=: I_1 + I_2,
\]

(2.33)

where we used \( \tilde{s} = 1 + s \) and the estimate

\[
1 = \frac{|t| - |\tau|}{|t-\tau|} \leq \frac{|t| + |\tau|}{|t-\tau|} \leq \frac{2|t|}{|t-\tau|}, \quad 0 \leq \tau < |t|.
\]

In what follows, we will obtain an estimate for \( I_2 \). Now, using fractional chain rule (see Tao [9] (A.15) page 338), for \( \tilde{s} \geq 0 \), we have

\[
I_2 \leq C \sup_{t \in [-T,T]} |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \left\| v^{k+1} \right\|_{L^{2(k+1)}} \left\| D_x^s v \right\|_{L^{2(k+1)}} d\tau
\]

\[
\leq C \left\| v \right\|_{X^p_T} \sup_{t \in [-T,T]} |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \frac{1}{|t-\tau|^\frac{2k}{p}} \frac{1}{|t-\tau|^\frac{2k}{p}} d\tau
\]

\[
\leq C \left\| v \right\|_{X^p_T} \sup_{t \in [-T,T]} |t|^\frac{s}{p} \int_0^t \frac{1}{|t-\tau|^\frac{2k}{p}} \frac{1}{|t-\tau|^\frac{2k}{p}} d\tau
\]

\[
\leq C_{p,k} \left\| v \right\|_{X^p_T} T^{\omega_k}.
\]

(2.34)
The estimate for $I_1$ will follow from the one of $I_2$ by considering $s = 0$. In fact,

$$I_1 \leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \|v^k\|_{L^{2(k+1)}} \|\partial_x v\|_{L^{2(t+1)}} d\tau$$

$$\leq C \|v\|^{k+1}_{C^T} \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \frac{1}{\tau^{\frac{3k+2}{2p}}} d\tau$$

$$\leq C_{p,k} \|v\|^{k+1}_{C^T} T^{\omega_k}. \quad (2.35)$$

Inserting estimates (2.34) and (2.35) in (2.33) we obtain the required estimate (2.29) in the case $s \geq 0$.

Next we consider the case $-1 \leq s < 0$. Using a similar argument as above, one gets

$$\left\| \int_0^t V(t - \tau) \partial_x (v^{k+1}) (\tau) d\tau \right\|_{Y^s_T} \leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \|\partial_x (v^{k+1})(\tau)\|_{H^s} d\tau$$

$$\leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \|D_x^s \partial_x (v^{k+1})\|_{L^2} d\tau$$

$$\leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \|D_x^s (v^{k+1})\|_{L^2} d\tau$$

$$\leq C_{p,k} \|v\|^{k+1}_{C^T} T^{\omega_k}, \quad (2.36)$$

where $\tilde{s} = 1 + s \geq 0$, for $s \geq -1$.

Now, we move to prove the estimate (2.30). By using (2.23) from Lemma 2.7 and fractional chain rule as in (2.34), for $s \geq 0$, we get

$$\left\| \int_0^t V(t - \tau) (\partial_x u)^{k+1}(\tau) d\tau \right\|_{Y^s_T}$$

$$\leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \left( \|u_x\|^{k+1}_{L^2} + \|D_x^s (u_x)^{k+1}\|_{L^2} \right) d\tau$$

$$\leq C \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \left( \|u_x\|^{k+1}_{L^{2(t+1)}} + \|u_x\|^{k}_{L^{2(t+k+1)}} \right) d\tau$$

$$\leq C \|u\|^{k+1}_{Y^s_T} \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \left[ \frac{1}{\tau} \frac{1}{\frac{k+1}{2p}} + \frac{1}{\tau} \frac{1}{\frac{k}{2p}} \right] d\tau$$

$$\leq C \|u\|^{k+1}_{Y^s_T} \sup_{t \in [-T, T]} |t|^{\frac{2k}{p}} \int_0^t \frac{1}{|t - \tau|^{\frac{2k}{p}}} \frac{1}{\tau} \frac{1}{\frac{k+2}{2p}} d\tau$$

$$\leq C_{p,k} \|u\|^{k+1}_{Y^s_T} T^{\omega_k}, \quad (2.37)$$

as required. In the last inequality of (2.37) the condition $p > \frac{3}{2}k + 1$ has been used. \hfill \Box

Now, we move to prove some more estimates that are useful in our analysis.
Lemma 2.9. Let \( \theta \geq 0, p > 0 \) and \( \tau \in [-1, 1] \setminus \{0\} \). Then we have
\[
\| \langle \xi \rangle^\theta e^{i|\tau|\Phi(\xi)} \|_{L^\infty} \lesssim \frac{1}{|\tau|^{\frac{p}{2}}}.
\] (2.38)

Proof. Considering \( M \) large as in Lemma [2.2] one can obtain
\[
\| \langle \xi \rangle^\theta e^{i|\tau|\Phi(\xi)} \|_{L^\infty} \leq \| \langle \xi \rangle^\theta e^{i|\tau|\Phi(\xi)} \|_{L^\infty(|\xi| \leq M)} + \| \langle \xi \rangle^\theta e^{i|\tau|\Phi(\xi)} \|_{L^\infty(|\xi| > M)} =: A_1 + A_2.
\] (2.39)

For \( \tau \in [-1, 1] \setminus \{0\} \) and \( \frac{\theta}{p} \geq 0 \), we have
\[
A_1 \leq C_M e^{i|\tau|C_M} \lesssim \frac{1}{|\tau|^{\frac{p}{2}}}.
\] (2.40)

In what follows, we obtain and estimate for the high frequency part \( A_2 \). From (2.6) we have
\[
A_2 \lesssim \frac{\| |\tau|^{\frac{1}{2}} \langle \xi \rangle^\theta e^{-|\tau|^{\frac{1}{2}}|\xi|^p} \|_{L^\infty(|\xi| > M)}}{\| \tau \|^p} \lesssim \frac{1}{|\tau|^{\frac{p}{2}}},
\] (2.41)

where in the last inequality \( x^\theta e^{-x} \leq C_\theta, \) if \( x \geq 0 \) has been used. The proof of the lemma follows inserting (2.40) and (2.41) in (2.39).

Proposition 2.10. Let \( s > -1, p > \frac{3k}{2} + 1, k \in \mathbb{N} \). There exists \( \mu := \mu(s, p, k) > 0 \) such that if
\[
\| f \|_{Z^s_T} := \sup_{t \in [-T, T] \setminus \{0\}} \left\{ \| f(t) \|_{H^s + t^{\frac{3k}{2}} \| f(t) \|_{L^{2(2k+1)}}} \right\} < \infty,
\] (2.42)

then the application
\[
t \mapsto \mathcal{L}(f)(t) := \int_0^t V(t - t')\partial_x(f^{k+1}((t')) dt', \quad 0 < |t| \leq T \leq 1,
\] (2.43)

is continuous from \([-T, T] \setminus \{0\}\) to \( H^{s+\mu} \).

Proof. We prove this proposition considering \( 0 < t \leq T \leq 1 \). The case \(-1 \leq -T \leq t < 0\) follows similarly. First we show that \( \mathcal{L}(f)(t) \in H^{s+\mu}(\mathbb{R}) \forall f \in Z^s_T \). We do this considering two different cases.

Case 1, \(-1 \leq s < p - 1\): Let \( 0 < t \leq T \leq 1 \) and \( f \in Z^s_T \). In this case, we have
\[
\| \mathcal{L}(f)(t) \|_{H^{s+\mu}} = \left\| \langle \xi \rangle^{s+\mu} \int_0^t e^{i(t-t')\xi^3 + |t-t'|\Phi(\xi)} i \xi f^{k+1}(\xi, t') dt' \right\|_{L^2}
\leq \int_0^t \| \langle \xi \rangle^{s+\mu} \xi e^{-|t-t'|\Phi(\xi)} \|_{L^\infty} \| f^{k+1}(\cdot, t') \|_{L^2} dt'
\leq \int_0^t \| \langle \xi \rangle^{s+\mu+1} e^{-|t-t'|\Phi(\xi)} \|_{L^\infty} \| f(\cdot, t') \|_{L^{2(k+1)}} dt'
\lesssim \| f \|_{Z^s_T} \int_0^t \frac{1}{|t - t'|^{\frac{1 + \mu + \frac{3k}{2}}{p}}} \frac{1}{|t - t'|^{\frac{3k + 2}{2p}}} dt' < \infty.
\] (2.44)
where $0 < \mu < p - 1 - s$, the definition of $Z_T^\mu$-norm, Minkowski’s inequality and inequality \[2.38\] from Lemma \[2.9\] are used. Note that, the condition $s \geq -1$ guarantees $s + \mu + 1 \geq 0$ to use \[2.38\].

**Case II, $s \geq p - 1$:** Similarly as above, using the fact that, in this case $H^s$ is an algebra, we obtain

\[
\|\mathcal{L}(f)(t)\|_{H^{s+\mu}} = \left\| (\xi)^{s+\mu} \int_0^t e^{i((t-t')\xi^3 + |t-t'|\Phi(\xi))} i\xi f^{k+1}(\xi, t')dt' \right\|_{L^2} \\
\leq \int_0^t \| (\xi)^{s+\mu} e^{i|t-t'|\Phi(\xi)} (J^s(f^{k+1})(\cdot, t'))^\wedge(\xi) \|_{L^2} dt' \\
\leq \int_0^t \| (\xi)^{s+\mu} \xi (e^{i|t-t'|\Phi(\xi)}) \|_{L^\infty} \| f^{k+1}(\cdot, t') \|_{H^{s+\mu}} dt' \\
\leq \| f \|_{Z_T^{s+\mu}} \int_0^t \frac{1}{|t-t'|^{\frac{1+\mu}{\mu}}} dt' < \infty.
\]

The last estimate in \[2.45\] follows by choosing $0 < \mu < p - 1$.

To prove the continuity part, let $t_0 \in (0, T]$ fixed, and let $f \in Z_T^\mu$. We will prove that

\[
\lim_{t \to t_0} \|\mathcal{L}(f)(t) - \mathcal{L}(f)(t_0)\|_{H^{s+\mu}} = 0. \tag{2.46}
\]

We divide the proof in two different cases.

**Case a, $0 < t \leq t_0$:** In this case, using \[2.43\] and the additive property of the integral, we obtain

\[
\|\mathcal{L}(f)(t) - \mathcal{L}(f)(t_0)\|_{H^{s+\mu}} = \left\| \int_0^{t_0} V(t_0 - t') \partial_x(f^{k+1})(t')dt - \int_0^t V(t - t') \partial_x(f^{k+1})(t')dt' \right\|_{H^{s+\mu}} \\
\leq \left\| \int_0^t (V(t_0 - t') - V(t - t')) \partial_x(f^{k+1})(t')dt' \right\|_{H^{s+\mu}} + \left\| \int_0^{t_0} V(t_0 - t') \partial_x(f^{k+1})(t')dt' \right\|_{H^{s+\mu}} \\
=: I_1(t, t_0) + I_2(t, t_0). \tag{2.47}
\]

We write

\[
I_1(t, t_0) = \left\| (\xi)^{s+\mu} \int_0^t \left( e^{i((t_0-t')\xi^3 + (t_0-t')\Phi(\xi))} - e^{i((t-t')\xi^3 + (t-t')\Phi(\xi))} \right) i\xi f^{k+1}(\xi, t')dt' \right\|_{L^2}, \tag{2.48}
\]

and note that

\[
(e^{i((t_0-t')\xi^3 + (t_0-t')\Phi(\xi))} - e^{i((t-t')\xi^3 + (t-t')\Phi(\xi))}) = \int_t^{t_0} (i\xi^3 + \Phi(\xi))e^{i(t - \tau)\xi^3 + (t - \tau)\Phi(\xi)} d\tau.
\]

We analyse $I_1$, considering two different cases.
Case a.1, $-1 \leq s < \min\{2(p - 2), p - 1\}$: In this case, we have

$$I_1(t, t_0) = \left\| \int_t^{t_0} (\xi)^{s+\mu}(\xi^3 + \Phi(\xi))e^{i(\tau - \xi')\xi'}|\Phi(\xi)|f^{k+1}(\xi, \tau) d\tau dt' \right\|_{L^2}$$

$$\leq \left\| \int_0^t \int_t^{t_0} (\xi)^{s+\mu+1}|\xi^3 + \Phi(\xi)|e^{i(\tau - \xi')\Phi(\xi)}|f^{k+1}(\xi, \tau)| d\tau dt' \right\|_{L^2}$$

$$\leq \int_0^t \int_t^{t_0} (|\xi|^{s+\mu+1}|\xi^3 + \Phi(\xi)|e^{i(\tau - \xi')\Phi(\xi)}|f^{k+1}(\xi, t')| d\tau dt'$$

(2.49)

$$\leq \int_0^t \int_t^{t_0} (|\xi|^{s+\mu+1} e^{i(\tau - \xi')\Phi(\xi)}|f^{k+1}(\xi, t')|) d\tau dt'$$

$$\leq \|f\|_{L^2}^{k+1} \int_0^t \int_t^{t_0} \frac{1}{|\tau - t'|^{\frac{\mu+s+1}{p}}} \frac{1}{|t'|^{\frac{\mu+s+1}{p}}} d\tau dt'$$

$$=: \|f\|_{L^2}^{k+1} J_1$$

where $r = \max\{3, p\} + 1$. In the domain of integration one has $0 \leq t' \leq t \leq \tau \leq t_0$, and consequently $|\tau - t'| = (\tau - t') \geq t - t' = |t - t'|$. Therefore, we can estimate $J$ as follows

$$J = \int_0^t \int_t^{t_0} \frac{1}{|\tau - t'|^{\frac{\mu+s+1}{p}}} \frac{1}{|t'|^{\frac{\mu+s+1}{p}}} d\tau dt'$$

(2.50)

$$\sim \int_0^t \int_t^{t_0} \frac{1}{|\tau - t'|^{\frac{\mu+s+1}{p}}} \left( \frac{1}{(t_0 - t')^{\alpha_\mu}} - \frac{1}{(t - t')^{\alpha_\mu}} \right) d\tau dt'$$

where

$$\alpha_\mu = \alpha_\mu(s, p) = \begin{cases} \frac{\mu+s+1}{p}, & \text{if } p \geq 3 \text{ and } 0 < \mu < p - 1 - s, \\ \frac{\mu+s+4-p}{p}, & \text{if } p \leq 3 \text{ and } 0 < \mu < 2(p - 2) - s. \end{cases}$$

(2.51)

Note that, for the choices of $p, s$ and $\mu$, we have $\alpha_\mu < 1$.

Now, making change of variables $t' = t_0 x$ and $t' = t x$ respectively in the first and second integrals in (2.50) and taking in account that $\alpha_\mu < 1$, we obtain

$$J \sim t_0^{1-\alpha_\mu-\frac{3\mu+2}{2p}} \int_0^{t_0} \frac{1}{|x|^{\frac{3\mu+2}{2p}}} \frac{1}{(1 - x)^{\alpha_\mu}} dx - t^{1-\alpha_\mu-\frac{3\mu+2}{2p}} \int_0^1 \frac{1}{|x|^{\frac{3\mu+2}{2p}}} \frac{1}{(1 - x)^{\alpha_\mu}} dx \to 0,$$

(2.52)

whenever $t \to t_0$.

Case a.2, $s \geq \min\{2(p - 2), p - 1\}$: In this case, with a similar argument as above, and the fact that $H^s$ is an algebra, one can get

$$I_1(t, t_0) \lesssim \|f\|_{L^2}^{k+1} \int_0^t \int_t^{t_0} \frac{1}{|\tau - t'|^{\frac{\mu+s}{p}}} d\tau dt'$$

(2.53)

$$:= \|f\|_{L^2}^{k+1} J.$$
Similarly to \( J \) in (2.52), we get
\[
\begin{align*}
\tilde{J} &\sim \int_0^t \left( \frac{1}{(t_0 - t')^{\alpha_\mu}} - \frac{1}{(t - t')^{\alpha_\mu}} \right) dt', \\
(2.54)
\end{align*}
\]
where
\[
\alpha_\mu = \alpha_\mu(s, p) = \begin{cases} \frac{\mu + 1}{p}, & \text{if } p \geq 3 \text{ and } 0 < \mu < p - 1, \\ \frac{\mu}{p}, & \text{if } p \leq 3 \text{ and } 0 < \mu < 2(p - 2). \end{cases} \\
(2.55)
\]
In this case too, for the choices of \( p, s \) and \( \mu \), we have that \( \alpha_\mu < 1 \).

As in (2.52), making change of variables and the fact that \( \alpha_\mu < 1 \), we obtain
\[
\tilde{J} \sim t_0^{1 - \alpha_\mu} \int_0^{t/t_0} \frac{1}{(1 - x)^{\alpha_\mu}} dx - t^{1 - \alpha_\mu} \int_0^1 \frac{1}{(1 - x)^{\alpha_\mu}} dx \to 0, \\
(2.56)
\]
whenever \( t \to t_0 \).

Therefore, in the light of (2.52) and (2.56), we get
\[
I_1(t, t_0) \to 0, \quad \text{if } t \to t_0. \\
(2.57)
\]
Analogously, since
\[
\int_0^{t_0} \| V(t - t') \partial_x(f^{k + 1})(t') dt' \|_{H^{s + \mu}} < \infty \\
\]
we get
\[
I_2(t, t_0) \to 0, \quad \text{if } t \to t_0. \\
(2.58)
\]

Hence, using (2.57) and (2.58) in (2.47), we conclude the proof of the proposition in this case.

**Case b, \( 0 < t_0 < t \):** In this case too, using (2.43) and the additive property of the integral, we obtain
\[
\| \mathcal{L}(f)(t) - \mathcal{L}(f)(t_0) \|_{H^{s + \mu}} = \left\| \int_0^{t_0} V(t_0 - t') \partial_x(f^{k + 1})(t') dt' \right\|_{H^{s + \mu}} \\
\leq \left\| \int_0^{t_0} (V(t_0 - t') - V(t - t')) \partial_x(f^{k + 1})(t') dt' \right\|_{H^{s + \mu}} + \left\| \int_{t_0}^t V(t - t') \partial_x(f^{k + 1})(t') dt' \right\|_{H^{s + \mu}} \\
=: I_1(t, t_0) + I_2(t, t_0). \\
(2.59)
\]
The rest follows analogously as in **Case a**, and this completes the proof of the proposition. □

**Proposition 2.11.** Let \( s \geq 0, p > \frac{3k}{2} + 1 \). There exists \( \mu := \mu(s, p, k) > 0 \) such that if
\[
\| f \|_{Z_T^k} := \sup_{t \in [-T, T] \setminus \{0\}} \left\{ \| f(t) \|_{H^{s + \mu}} + t^{\frac{2k}{p}} \| \partial_x f(t) \|_{L^2(\mathbb{R})} \right\} < \infty, \\
(2.60)
\]
then the application
\[
t \mapsto \tilde{\mathcal{L}}(f)(t) := \int_0^t V(t - t')(f_x)^{k + 1}(t') dt', \quad 0 < |t| \leq T \leq 1, \\
(2.61)
\]
is continuous from \([-T, T] \setminus \{0\}\) to \( H^{s + \mu} \).
Proof. The proof follows by using (2.38) and a similar procedure applied in the proof of Proposition 2.9.

3. Proof of the main results

Having the linear and nonlinear estimates at hand from the previous section, now we provide proof of the main results of this work.

Proof of Theorem 1.1. Consider the Cauchy problem (1.5) in its equivalent integral form

$$ v(t) = V(t)v_0 - \int_0^t V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau, $$

(3.1)

where \( V(t) \) is the semi-group associated with the linear part given by (2.2).

Let us define an application

$$ \Psi(v)(t) = V(t)v_0 - \int_0^t V(t-\tau)\partial_x(v^{k+1})(\tau)d\tau. $$

(3.2)

For \( s \geq -1, r > 0 \) and \( 0 < T \leq 1 \), let

$$ B_T^r = \{ f \in \mathcal{X}_T^s; \| f \|_{\mathcal{X}_T^s} \leq r \}, $$

be a ball in \( \mathcal{X}_T^s \) with center at origin and radius \( r \). We will show that there exists \( r > 0 \) and \( 0 < T \leq 1 \) such that the application \( \Psi \) maps \( B_T^r \) into \( B_T^r \) and is a contraction. For this, let \( v \in B_T^r \). Using the estimates (2.24) and (2.29), one can obtain

$$ \| \Psi(v) \|_{\mathcal{X}_T^s} \leq C \| v_0 \|_{H^s} + C_{p,k}T^{\omega_k} \| v \|^k_{\mathcal{X}_T^s}, $$

(3.3)

where \( \omega_k = \frac{2p-3k-2}{2p} > 0 \).

For \( v \in B_T^r \) let us choose \( r = 4c\|v_0\|_{H^s} \) in such a way that \( cT^{\omega_k}r^k = 1/4 \). For this choice, one can easily obtain

$$ \| \Psi(v) \|_{\mathcal{X}_T^s} \leq \frac{r}{4} + cT^{\omega_k}r^{k+1} \leq \frac{r}{2}. $$

(3.4)

From (3.3) we conclude that \( \Psi \) maps \( B_T^r \) into itself. A similar argument proves that \( \Psi \) is a contraction. Hence \( \Psi \) has a fixed point \( v \) which is a solution of the Cauchy problem (1.5) such that \( v \in C([0, T]; H^s(\mathbb{R})). \) The rest of the proof follows from standard argument, see for example [6].

The regularity part follows using Lemma 2.4 and Proposition 2.10 as in [2].

Proof of Theorem 1.2. Proof of this theorem is very similar to that of Theorem 1.1. In this case, we use the estimate (2.25) from Lemma 2.7 and estimate (2.30) from Lemma 2.8 to perform the contraction mapping argument.

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