Are Quantum-Classical Hybrids compatible with Ontological Cellular Automata?

Hans-Thomas Elze

Received: date / Accepted: date

Abstract Based on the concept of ontological states and their dynamical evolution by permutations, as assumed in the Cellular Automaton Interpretation (CAI) of quantum mechanics, we address the issue whether quantum-classical hybrids can be described consistently in this framework. We consider chains of ‘classical’ two-state Ising spins and their discrete deterministic dynamics as an ontological model with an unitary evolution operator generated by pair exchange interactions. A simple error mechanism is identified, which turns them into quantum mechanical objects, chains of qubits. Consequently, an interaction between a quantum mechanical and a ‘classical’ chain can be introduced and its consequences for this quantum-classical hybrid be studied. We find that such hybrid character of composites, generally, does not persist under interactions and, therefore, cannot be upheld consistently, or even as a fundamental notion à la Kopenhagen interpretation, within CAI.

Keywords Quantum-classical hybrid system · Cellular automaton · Ising spin · Qubit · Ontological state · Baker-Campbell-Hausdorff formula · Quantum mechanics

1 Introduction

Our aim presently is to apply recent results illustrating the Cellular Automaton Interpretation of quantum mechanics [1,2,3,4,5,6] and reconsider the status of Quantum-Classical Hybrids [7], in particular. Can such hybrids exist and consistently be described in a theoretical framework according to this interpretation? We summarize necessary ingredients here, as well as in the following sections, which will ultimately lead to a negative answer to this question.

Hans-Thomas Elze
Dipartimento di Fisica “Enrico Fermi”
Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italia
E-mail: elze@df.unipi.it
1.1 Quantum-Classical Hybrids – a reminder

Quantum-Classical Hybrids (QCH) have been of interest since the early days when quantum theory has obtained its canonical formulation, as represented in well known textbooks [8,9].

In the beginning, to separate a composite system into a classical part and a quantum mechanical part, as far as their respective states and dynamical behaviour are concerned, may have been mostly of practical interest as an approximation method for otherwise intractable situations [10] – which, to this day, continues to be essential in quantum chemical studies of complex molecules, even biomolecules, and their reactions; similarly, for example, in nuclear reactions, etc.

However, eventually this topic also gained attention from the perspective of foundational issues in quantum theory [11]. In particular, numerous and varied attempts to understand and resolve the infamous measurement problem of quantum mechanics (QM) have continued to flourish, which we shall not review here.\(^1\) they all, in one way or another, try to deal with the separation of measurement readings of a classical apparatus, i.e., corresponding to non-superposed pointer states, from the quantum evolution of the object under study. No consensus has been reached, how to resolve this conundrum within quantum theory – yet the Cellular Automaton Interpretation of QM does offer an elegant resolution, to be mentioned in Section 1.2.

Lately, questions of the existence (or not) of QCH have newly moved into focus, since the fundamentally important debate about the either classical or quantum mechanical nature of gravitation is soon to be accompanied by (possibly table-top) high precision experiments, see, e.g., [14,15,16,17,18].

Presently, we are interested to consider the consistency or inconsistency of QCH on the basis of the Cellular Automaton Interpretation of QM. In the past, there have been numerous attempts to provide a theoretically satisfactory description of QCH, trying to embed the formalisms of QM and classical mechanics into a common framework, especially along the lines of Ref. [19] (with references to earlier work there). However, interacting QCH present numerous features, where consistency can fail. An almost complete list of essential consistency checks has been presented and passed in Ref. [7]. Additional issues, however, have been discussed in Refs. [20,21,22,23], for example.

1.2 Cellular Automaton Interpretation of QM in a nutshell

According to the Cellular Automaton Interpretation (CAI) of QM, the evolution of the Universe is deterministic and happens in discrete steps [1].

Ontological States (OS) are the discrete physical states the Universe can be in. At each step of its evolution, the Universe is in a definite state, which

\(^1\) However, for a review and a recent study, see the Refs. [12,13].
we may denote, for example, by

$$|A⟩, |B⟩, |C⟩, \ldots ;$$

for simplicity, we assume a finite (or countably infinite) number $N$ of states.

No superpositions of the OS exist physically “out there”. It is only by permutations that OS evolve into OS, e.g.,

$$|A⟩ → |B⟩ → |C⟩ → \ldots .$$

Thus, the set of OS forms a preferred basis. We may declare this basis to be orthonormal and define an associated Hilbert space.

Diagonal operators on the OS are beables and their eigenvalues correspond to the labels $A, B, C, \ldots$ by which we labeled the ontological states.

Mathematically, by unitary transformations of the preferred basis we arrive at Quantum States QS as superpositions of OS. These constructs serve to do physics with the help of mathematical language, according to CAI.

An important property of QS follows immediately from the evolution of OS by permutations. Referring to the above example, a generic quantum state,

$$|Q⟩ := α|A⟩ + β|B⟩ + \ldots , \quad |α|^2 + |β|^2 + \ldots = 1 ,$$

(1)

evolves accordingly:

$$|Q⟩ → α|B⟩ + β|C⟩ + \ldots ,$$

(2)

where the amplitudes $α, β, etc.$ are conserved. This conservation law has been termed the conservation of ontology.

If we choose the amplitudes introduced in the QS of Eq. (1) to encode probabilities $|α|^2, |β|^2, etc.$ of the OS $|A⟩, |B⟩, etc.$, respectively, to present the initial state of the evolution step (2), then we have the Born rule of QM built right into the formalism. Notably, the conservation of ontology under evolution is an essential ingredient.

Completing this sketch of CAI, Classical States (CS) are considered as probabilistic distributions of OS. Typically, they describe physical macrosystems, which cannot be represented in terms of individual OS.

Experiments that “happen” repeatedly – like apparently repeating evolutions of a sufficiently but always incompletely isolated part of the Universe – pick up different initial conditions regarding OS. Therefore, the classical apparatus in an experiment must generally be expected to yield different pointer positions as outcomes. Due to the conservation of ontology, the probability of a particular outcome directly reflects the probability of having a particular OS as initial condition, since it evolves by permutations of elements of this preferred basis only, cf. (2).

Thus, according to CAI, the apparent reduction or collapse to a single pointer position as a measurement result arises due to the intermediary use of quantum mechanical (superposition) states, when describing what in reality are evolving OS that differ in different runs of an experiment.
Summarizing, the Cellular Automaton Interpretation suggests to build \textit{discrete deterministic dynamical models} with evolution generated by \textit{permutations of ontological states}. The periodicity of trajectories in the space of states, \textit{i.e.} without fusion or fission, turns them into QM models, as has been demonstrated in several studies before, see, \textit{e.g.}, Refs. \cite{2,3,21,25,26,27}.

We have seen how from rather parsimonious and simple assumptions about the underlying ontology one arrives at essential features of QM. While other aspects will be discussed in the following, we conclude here with a few remarks concerning open problems of CAI that need to be addressed in the future.

First of all, it turns out to be difficult to incorporate \textit{interactions} into previously noninteracting models. This problem, however, cannot be sidestepped, if anything like an interacting quantum field theory, especially the paradigmatic Standard Model, is to be explained by an ontological underpinning along the lines of CAI. Furthermore, the relation of the dynamics generated by permutations of ontological states to a \textit{Hamiltonian} that determines the unitary evolution operator as in QM is generally not straightforward to work out. Some glimpses of this will be encountered in what follows.

In Sections 2 and 3, we mainly reformulate results obtained in Refs. \cite{4,5,6} in such a way that our argument against the consistency of quantum-classical hybrids, according to CAI, follows easily, as given in Section 4. Conclusions are presented in Section 5.

\section*{2 Permutations of ontological states}

Let \(N\) objects, \(A_1, A_2, \ldots, A_N\) ("states"), be mapped in \(N\) steps onto one another, involving all states. Suitably arranging the sequence of states, this may always be represented by an \textit{unitary} \(N \times N\) matrix with one off-diagonal phase per column and row and zero elsewhere:

\[
\hat{U}_N := \begin{pmatrix}
0 & \cdots & 0 & e^{i\phi_N} \\
0 & \cdots & 0 & e^{i\phi_1} \\
0 & \cdots & 0 & e^{i\phi_2} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & e^{i\phi_{N-1}}
\end{pmatrix}.
\]

We associate with this particular representation a basis of vectors called the \textit{standard basis}.

It is easy to see that the matrix \(\hat{U}_N\) has the significant property:

\[
(\hat{U}_N)^N = e^{i\sum_{k=1}^{N} \phi_k} \mathbf{1}.
\]

This implies that the \textit{Hamiltonian}, defined by the relation \(\hat{H}_N = e^{-i\hat{H}_N T}\), can be immediately diagonalized to give the diagonal matrix elements (\(n = 2\) We mention that various demonstrations exist of how to arrive at QM with statistical arguments or with stochastic or dissipative modifications of \textit{classical deterministic} dynamics, \textit{i.e.} without postulating an underlying ontology, such as \cite{28,29,30,31,32,33,34,35}.}
Are Quantum-Classical Hybrids compatible with Ontological Cellular Automata? 5

Fig. 1 The cogwheel model of a system jumping periodically through \( N \) states, with \( T \) denoting the time it takes per jump.

\[
\hat{H}_N = \frac{1}{NT} \left( 2\pi (n - 1) - \sum_{k=1}^{N} \phi_k \right),
\]

which refer to the diagonal basis. In the following, the arbitrary phase angles \( \phi_k \) play no role, so we set them to zero.

Naturally, the diagonal and standard bases are unitarily related. The transformation between the bases can be obtained explicitly by a discrete Fourier transformation \([6]\). Using this result, in turn, one evaluates the Hamiltonian for the standard basis, to find the diagonal and off-diagonal matrix elements, respectively:

\[
(\hat{H}_N)_{nn} = \frac{\pi}{NT} (N - 1), \quad n = 1, \ldots, N,
\]

and,

\[
(\hat{H}_N)_{n\neq m} = \frac{\pi}{NT} \left( -1 + i \cot \left( \frac{\pi}{N} (n - m) \right) \right), \quad n, m = 1, \ldots, N,
\]

and \( \hat{H}_N = \hat{H}_N^\dagger \).

A cogwheel model describes the discrete deterministic dynamics by a unitary permutation matrix \( \hat{U}_N \) \([1]\). A sketch is shown in Fig. 1.

It may come as a surprise, however, in the limits \( N \to \infty, T \to 0 \), with \( \omega := 1/NT \) fixed, the simple cogwheel model describes the quantum harmonic oscillator or systems that can be related to it \([1,24,25]\). Cellular automata which lead to more general quantum models in the continuum limit are studied in Ref. \([26]\).

In the next sections we will apply the dynamics of cogwheel models to study the behaviour of particular Ising spin systems. We choose them as classical discrete deterministic many-body systems evolving by permutations of their states, in line with the picture of ontological cellular automata.
3 Ising spin chains

Let a one-dimensional spin chain with periodic boundary condition be composed of $2S + 1$ classical two-state spins, which represent $2^{2S}$ OS. We denote the spins and their values by, respectively:

$$s_k = \pm 1 = \uparrow, \downarrow, \; k = 1, \ldots, 2S + 1, \; s_{2S+1} \equiv s_1,$$

where the first and last spin are identified.

Permutations of the OS of the chain will be generated by spin exchange transpositions acting on pairs of spins:

$$\hat{P}_{ij}|s_i, s_j\rangle := |s_j, s_i\rangle, \; \hat{P}_{2S \; 2S+1} \equiv \hat{P}_{2S \; 1},$$

with $\hat{P}_{ij} \equiv \hat{P}_{ji}$; from here on we shall often use the ket notation familiar from QM to write down states of two or more Ising spins. These transpositions have the properties $\hat{P}^2 = 1$ and $[\hat{P}_{ij}, \hat{P}_{jk}] \neq 0$, for $i, j, k$ different from each other.

Furthermore, it is well known but crucial fact that spin exchange operators or transpositions can be expressed in terms of a vector $\hat{\sigma}$ formed by the three Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$:

$$\hat{P}_{ij} = \frac{(\hat{\sigma}_i \cdot \hat{\sigma}_j + 1)}{2}.$$

Finally, we define the dynamics of the spin chain by a particular choice of the unitary matrix $\hat{U}$ incorporating the transpositions that produce permutations of the OS:

$$\hat{U} := \prod_{k=1}^{S} \hat{P}_{2k-1 \; 2k} \prod_{l=1}^{S} \hat{P}_{2l \; 2l+1} =: \exp(-i\hat{H}T),$$

with the associated Hamiltonian $\hat{H}$ to be extracted next. This evolution operator respects several conservation laws, among them translation invariance, as discussed in Ref. [6].

We note that $\hat{U}$ distinguishes even/odd pairs $ij$, according to whether $i$ of $\hat{P}_{i<j}$ is even/odd. Following Eq. (11), all even pairs are updated first, which is followed by the update of all odd pairs, when $\hat{U}$ is applied to a given state once. All even pair transpositions $\hat{P}_{2l \; 2l+1}$ commute with each other but not with the odd ones and vice versa. Therefore, the effect of certain transposition is felt by neighbouring pairs only when $\hat{U}$ is applied a second time. This implements a final signal velocity for the propagation of perturbations along the chain.

3.1 Extracting the spin-exchange Hamiltonian

Consider a generic initial OS of a chain of $2S$ Ising spins,

$$|\psi\rangle := |s_1, s_2, s_3, s_4, s_5, \ldots, s_{2S-1}, s_{2S}\rangle,$$

where the first and last spin are identified.
where the periodic boundary condition means that \( s_1 \equiv s_{2S+1} \); such that the rightmost transposition in \( \hat{U} \) of Eq. (11), \( P_{2S+1} \) can meaningfully be applied to the state.

Applying \( \hat{U} \) once, we observe that the resulting evolution of the state is very simple, which reflects the conservation laws present. This is illustrated in Fig. 2. Consecutive updates lead to repetitions of this pattern of leftmovers and rightmovers.

If we follow the motion of a particular left- or rightmover, it will consecutively pass all odd or even sites, respectively, and continue along the chain periodically. Thus, we find in this model a composition of cogwheels, cf. Section 2, which is produced by nearest-neighbour transpositions in the classical spin chain. Since after \( S \) updates the initial OS of the chain is always recovered, the latter itself behaves like an \( S \)-state cogwheel.

In Ref. [6], we indicated how this motion can be encoded in two discrete two-component field equations. Which, in turn, can be mapped one-to-one on (1+1)-dimensional partial differential equations for bandwidth-limited classical fields with the help of Sampling Theory, e.g. similarly as in Ref. [26].

We now briefly outline how the Hamiltonian \( \hat{H} \) is extracted, once we have recognized that any OS of the chain evolves like a state in a \( S \)-state cogwheel, however, with an internal structure of \( 2S \) Ising spins and nearest-neighbour exchange interactions.

Since the overall behaviour is that of a \( S \)-state cogwheel, the Hamiltonian must be related to \( \hat{H}_{N=S} \), cf. Eqs. (6)–(7). Following simple examples treated explicitly in Refs. [4,5], it was shown in Ref. [6] how to ‘dress’ \( \hat{H}_{N=S} \), in order to obtain \( \hat{H} \). Namely, the latter has to act on the spin variables of a generic state \( |\psi\rangle \), Eq. (12), in one-to-one correspondence to the former. This results in:

\[
\hat{H} = \sum_{n=1}^{S} (\hat{H}_S)_n \hat{U}^{n-1}
\]

\[
= \frac{\pi}{T} \left( 1 + i \frac{\pi}{S} \sum_{n=1}^{S-1} \cot\left( \frac{\pi}{S} n \right) \hat{U}^{n} \right) , \text{ for } \hat{U}|\psi\rangle \neq |\psi\rangle
\]
Instead, for states with $\hat{U}|\psi_0\rangle = |\psi_0\rangle$, one consistently finds that $\hat{H}|\psi_0\rangle = 0$ for these static zero modes – examples of these have all spins either up or down or all leftmovers up (down) and all rightmovers down (up). Taking into account that $\hat{U}^{S-k} = (\hat{U}^\dagger)^k$, which follows from $\hat{U}^k\hat{U}^{S-k} = 1 = \hat{U}^k(\hat{U}^\dagger)^k$, for $1 \leq k \leq S$, and the symmetry of the cot-function, the final result is given in a manifestly self-adjoint form.

3.2 Remarks

In Ref. [6] several comments were made about the interpretation of the Hamiltonian $\hat{H}$ that acts on $OS$ of the Ising spin chain – this concerns physical properties (degeneracy and magnetization of states, translation invariance), relation to a discrete field theory, and that formally the results summarized in this section yield an interesting, especially terminating Baker-Campbell-Hausdorff formula:

$$\hat{U} = \exp \left( -i\pi \left( 1 + \frac{i}{S} \sum_{n=1}^{S-1} \cot\left( \frac{\pi}{S} n \right) \hat{U}^n \right) \right), \text{ for } \hat{U}|\psi\rangle \neq |\psi\rangle,$$

(16)

with $\hat{U}$ of Eq.(11).

We emphasize that the unitary operator $\hat{U}$, Eq. (11), with $\hat{H}$ of Eqs. (13)–(15), has been constructed to evolve in a discrete, deterministic way states of the classical Ising spin model with exchange interactions. Most importantly, these $OS$ evolve by permutations (transpositions) of the spin variables without ever forming superposition states, in accordance with CAI (cf. Section 1.2).

However, if numerical constants of the Hamiltonian, especially in Eq. (15), are only slightly perturbed, this becomes a genuine quantum mechanical operator. Generally, this will produce superposition states of $OS$, which would necessitate a corresponding Hilbert space of $2S$ 2-state quantum spins for its description, i.e. QM of qubits “by mistake” [6]. – Such perturbations could, for example, be caused by neglected or integrated out fast high-energy degrees of freedom, as suggested on other grounds recently [3].

In any case, small errors in $\hat{H}$, or $\hat{U}$, tend to have a QM effect. For illustration, consider a long chain, with $S \gg 1$, and approximate $\hat{H}$, Eq. (15), roughly by the leading terms:

$$\hat{H} \approx \frac{\pi}{T} \left( 1 + \frac{i}{\pi} (\hat{U} - \hat{U}^\dagger) \right).$$

(17)

Then, choosing a state, $|\psi_{\downarrow\downarrow}\rangle := |\ldots , \uparrow, \uparrow, \downarrow, \downarrow, \uparrow, \uparrow, \uparrow, \ldots \rangle$, where the two indicated down spins are located on neighbouring even and odd sites, we
obtain by definition of $\hat{U}$, Eq. (11):

$$
(\hat{U} - \hat{U}^\dagger)|\psi_{\downarrow\downarrow}\rangle = |\ldots, \uparrow, \uparrow, \uparrow, \uparrow, \downarrow, \uparrow, \ldots\rangle
- |\ldots, \uparrow, \downarrow, \uparrow, \uparrow, \uparrow, \uparrow, \ldots\rangle ,
$$

and correspondingly $\hat{H}|\psi_{\downarrow\downarrow}\rangle$, by Eq. (17). This reminds of an entangled Bell state; it is not an OS but a quantum mechanical superposition state instead.

### 4 Inconsistency of Quantum-Classical Hybrids in CAI

Next, we employ our observations, in order to study the (in)consistency of Quantum-Classical Hybrids (QCH), cf. Section 1.1, when confronted with CAI by way of a simple but representative example.

We consider two Ising spin chains for this purpose, one which has become quantum mechanical by introducing superpositions of its OS, e.g. by a mechanism similar to the one discussed above, and one which is classical in the form of a probability distribution of OS, cf. Section 1.2. In the simplest case, the latter may be sharply peaked on a single OS.

Assuming a temporary interaction, which acts in between two particular updates of both chains, it should be of the ontological kind, i.e. involve only permutations (transpositions) of the Ising spins of the two chains. To be definite, we choose the exchange interaction illustrated in Fig. 3.

Let the initial quantum state of one of the two chains be given by the superposition:

$$
|Q\rangle = \alpha|\ldots, a_3, a_4, a_5, a_6, \ldots\rangle + \beta|\ldots, b_3, b_4, b_5, b_6, \ldots\rangle ,
$$

with $|\alpha|^2 + |\beta|^2 = 1$ and $a_k, b_k$ denoting specific values of the respective Ising spins. Furthermore, let the initial classical state of the second chain be given sharply by an OS:

$$
|C\rangle = |\ldots, s_3', s_4', s_5', s_6', \ldots\rangle .
$$

Applying the interaction $\hat{I}$ depicted in Fig. 3 the QCH consisting of both chains is transformed into:

$$
\hat{I}|Q\rangle|C\rangle = \alpha|\ldots, a_3, s_5', a_6, \ldots\rangle|\ldots, s_3', a_4, a_5, a_6, \ldots\rangle
+ \beta|\ldots, b_3, s_5', b_6, \ldots\rangle|\ldots, s_3', b_4, b_5, b_6, \ldots\rangle .
$$

Thus, such an interaction is a bilinear map $\hat{I} : \{\text{OS}\} \times \{\text{OS}\}' \rightarrow \mathcal{T}$ on the direct product of spaces spanned by the preferred bases of ontological states of both chains. However, due to the linearity, linear superpositions of OS from the quantum state of one of the chains, e.g. Eq. (19), are transferred by the interaction into the target space $\mathcal{T}$, which becomes the Hilbert space generated by the tensor product of those bases, $\mathcal{T} = \{\text{OS}\} \otimes \{\text{OS}\}'$.

A curious case arises, if both states in the superposition [19] differ only in one (or two) $a, b$ pair(s) out of $a_4, a_5, b_4, b_5$, but are identical otherwise: then,
the feature of being in a quantum superposition state and in a classical state (OS), respectively, is swapped between both chains.

However, generally, the result of Eq. (21) presents an entangled quantum state involving both chains. In particular, the classical state $|C\rangle$ of the second chain is changed in such a way that it cannot be factored out anymore.

Thus, the interaction changes here the character of the composite of two Ising chains from being a quantum-classical hybrid to an overall quantum state. This result follows similarly, if the quantum state in Eq. (19) is replaced by a more complicated superposition or the sharp classical state of the second chain is more generally replaced by a probabilistic mixture of OS initially, cf. Section 1.2, which can be incorporated in a density matrix.

We conclude that in the framework of CAI quantum-classical hybrids are not a consistent construct. With hindsight, this was to be expected, since classical states are of ontological kind, while quantum states are mathematical constructs of epistemological character, according to the Cellular Automaton Interpretation of QM [1], see also Ref. [36].

Perhaps, the longstanding difficulties faced by all attempts to find a satisfactory description of quantum-classical composites in quantum theory, cf. Section 1.1 and see, e.g., Refs. [7,20,21,22,23] with further references, have indicated already that such attempts are fundamentally flawed. It is interesting that CAI reveals this in a very straightforward way, as we have seen.

5 Conclusions

We have reviewed the concept of and interest in Quantum-Classical Hybrids (QCH), i.e. composites of at least one classical and one quantum mechanical
part, followed by a sketch of the Cellular Automaton Interpretation (CAI) of quantum mechanics.

We elaborated some aspects of the latter in models of chains consisting of two-state Ising spins, which interact by transpositions of spins that is by permutations of the states of a chain. The latter are considered as ontological states (OS), existing “out there” in this model Universe. The resulting deterministic dynamics is generated by fine-tuned self-adjoint Hamiltonian operators, much like in QM.

Perturbations of such Hamiltonians, generally, necessitate an enlarged state space for the ensuing approximate description of the model. It is the Hilbert space that admits superposition states of the OS as an epistemic mathematical construct. This is where quantum states enter.

Based on these observations, the description of QCH can be embedded in CAI, as shown in Section 4. Then, the essential interaction between classical and quantum mechanical components can be introduced, as we have illustrated by a generic example, and its effect on the quantum-classical composite structure be analyzed.

This reveals that QCH of interacting quantum mechanical and ontological degrees of freedom are inconsistent: Unavoidably, interactions, \( \hat{I} : \{\text{OS}\} \times \{\text{OS}\}' \rightarrow \{\text{OS}\} \otimes \{\text{OS}\}' \), which bilinearly map OS of both chains to tensor product states in a resulting Hilbert space, transfer quantumness in the form of (entangled) superpositions also to the assumed classical sector of a QCH. Thus, the hybrid feature is lost through interactions and the concept of QCH looses its meaning within CAI, which is intended to address the hypothesis that the Universe is fundamentally of discrete deterministic character.

Several concluding remarks are in order here:

Since we have demonstrated inconsistency of quantum-classical hybrids from the perspective of the Cellular Automaton Interpretation of quantum mechanics, this should, of course, not detract from their practical importance in approximation schemes for otherwise still impossible studies of complex objects, cf. Section 1.1.

It may be obvious that cellular automaton models related to CAI fall into the class of superdeterministic dynamical models, which recently have found renewed interest – especially, in order to show that the verdict of Bell’s theorem against realistic local hidden variable theories can be circumvented \[1\]37,38,39\.

The interaction between two chains that was instrumental for our present argument, of course, has been chosen ad hoc as an example. It is left for future work to construct models along these lines that may approximate physically relevant continuum field theories. It would be very interesting to understand the relation between such interactions and computational gates.

Acknowledgements It is a pleasure to thank Louis Vervoort and Theo Raptis for correspondence and Ken Konishi for discussions on various related matters. The organizers of The Quantum & The Gravity 2021 are thanked for the kind invitation to present this work.
Conflict of interest and other statements

The author declares that he has no conflict of interest. – This work received no external funding and has been performed solely by the author.

References

1. ’t Hooft, G.: The Cellular Automaton Interpretation of Quantum Mechanics. *Fundamental Theories of Physics* **185**, Springer International Publishing (2016)
2. ’t Hooft, G: Deterministic Quantum Mechanics: the Mathematical Equations. *Frontiers in Phys.* 29 July 2020; doi.org/10.3389/fphy.2020.00253
3. ’t Hooft, G: Fast Vacuum Fluctuations and the Emergence of Quantum Mechanics. *Found. of Physics* **51**, 63 (2021)
4. Elze, H.-T.: Qubit exchange interactions from permutations of classical bits. *Int. J. Quant. Info. (IJQI)* **17**(8), 1941003 (2019)
5. Elze, H.-T.: A Baker-Campbell-Hausdorff formula for the logarithm of permutations. *Int. J. Geom. Meth. Mod. Phys. (IJGMMP)* **17**(4), 2050052 (2020)
6. Elze, H.-T.: Are quantum spins but small perturbations of ontological Ising spins? *Found. of Physics* **50**(12), 1875 (2020)
7. Elze, H.-T.: Linear dynamics of quantum-classical hybrids. *Phys. Rev. A* **85**, 052109 (2012)
8. Dirac, P.A.M.: THE PRINCIPLES OF QUANTUM MECHANICS. Third Edition. Oxford University Press (1947)
9. von Neumann, J.: MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS. Princeton University Press (1955)
10. Born, M., Oppenheimer, J.R.: Zur Quantentheorie der Molekeln. *Annalen der Physik* (in German) **389**(20), 457 (1927)
11. Sherry, T.N., Sudarshan, E.C.G.: Interaction between classical and quantum systems: A new approach to quantum measurement. I. *Phys. Rev. D* **18**, 4580 (1978); do. II. *Phys. Rev. D* **20**, 857 (1979)
12. Allahverdyan, A.E., Balian, R., Nieuwenhuizen, T.M.: Understanding quantum measurement from the solution of dynamical models. *Phys. Rep.* **525**, 1 (2013)
13. Konishi, K.: Quantum fluctuations, particles, entanglement and measurements. *Preprint arXiv:2111.14723 [quant-ph] (2021)
14. Lampo, A., Fratino, L., Elze, H.-T.: Mirror-induced decoherence in hybrid quantum-classical theory. *Phys. Rev. A* **90**, 042120 (2014)
15. Hu, B.L., Verdaguer, E.: Stochastic Gravity: Theory and Applications. *Living Rev. Relativity* **11**, 3 (2008)
16. Bassi, A., Grossardt, A., Ulbricht, H.: Gravitational Decoherence. *Class. Qu. Grav.* **34**, 193002 (2017)
17. Asprea, L., Bassi, A., Ulbricht, H., Gasbarri, G.: On the decoherence effect of a stochastic gravitational perturbation on scalar matter and the possibility of its interferometric detection. *Phys. Rev. Lett.* **126**, 200403 (2021)
18. Marletto, C., Vedral, V.: When can gravity path-entangle two spatially superposed masses? *Phys. Rev. D* **98**, 046001 (2018)
19. Heslot, A.: Quantum mechanics as a classical theory. *Phys. Rev. D* **31**, 1341 (1985)
20. Salcedo, L.L.: Absence of classical and quantum mixing. *Phys. Rev. A* **54**, 3657 (1996)
21. Elze, H.-T.: Four questions for quantum-classical hybrid theory. *J. Phys.: Conf. Ser.* **361**, 012004 (2012)
22. Diósi L.: Classical-Quantum Coexistence: a ‘Free Will’ Test. *J. Phys. Conf. Ser.* **361**, 012028-(7) (2012)
23. Diósi L.: Hybrid Quantum-Classical Master Equations. *Phys. Scr. T* **163**, 014004-(8) (2014)
24. Elze, H.-T., Schipper, O.: Time without time: a stochastic clock model. *Phys. Rev. D* **66**, 044020 (2002)
25. Elze, H.-T.: Emergent discrete time and quantization: relativistic particle with extradimensions. *Phys. Lett. A* **310**, 110 (2003)
26. Elze, H.-T.: Action principle for cellular automata and the linearity of quantum mechanics. *Phys. Rev. A* **89**, 012111 (2014)
27. Margolus, N.: Mechanical systems that are both classical and quantum. *Lect. Notes Comput. Sci.* **9099**, 169 (2015); see also *Preprint* [arXiv:0805.3357v2 [quant-ph]] (2008)
28. Haba, Z., Kleinert, H.: Towards a simulation of quantum computers by classical systems. *Phys. Lett. A* **294**, 139 (2002)
29. Khrennikov, A.: Generalizations of quantum mechanics induced by classical statistical field theory. *Found. Phys. Lett.* **18**, 637 (2005)
30. Blasone, M., Jizba, P., Scardigli, F., Vitiello, G.: Dissipation and quantization for composite systems. *Phys. Lett. A* **373**, 4106 (2009)
31. Sakellariadou, M., Stabile, A., Vitiello, G.: Noncommutative spectral geometry, algebra doubling and the seeds of quantization. *Phys. Rev. D* **84**, 045026 (2011)
32. Acosta, D., Fernandez de Cordoba, P., Isidro, J.M., Santander, J.L.G.: An entropic picture of emergent quantum mechanics. *Int. J. Geom. Meth. Mod. Phys.* **9**, 1250048 (2012)
33. Bisio, A., D’Ariano, G.M., Tosini, A.: Quantum field as a quantum cellular automaton: the Dirac free evolution in one dimension. *Ann. Phys.* **354**, 244 (2015)
34. Wetterich, C.: Quantum mechanics from classical statistics. *Ann. Phys.* **325**, 852 (2010)
35. Wetterich, C.: Probabilistic cellular automata for interacting fermionic quantum field theories. *Nucl. Phys. B* **903**, 115296 (2021)
36. Rovelli, C.: An argument against the realistic interpretation of the wave function. *Found. of Phys.* **46**(10), 1229 (2016)
37. Vervoort, L.: Bell’s Theorem: Two Neglected Solutions. *Found. of Phys.* **43**, 769 (2013)
38. Vervoort, L.: Probability theory as a physical theory points to superdeterminism. *Entropy* **21**(9), 848 (2019)
39. Nikolaev, V., Vervoort, L.: Aspects of Superdeterminism Made Intuitive. *Submitted for publication* (2022)