Phase Transition on the Degree Sequence of a Mixed Random Graph Process

Xian-Yuan Wu\textsuperscript{1}*, Zhao Dong\textsuperscript{2}†, Ke Liu\textsuperscript{2}‡ and Kai-Yuan Cai\textsuperscript{3}§

\textsuperscript{1}School of Mathematical Sciences, Capital Normal University, Beijing, 100037, China. Email: wuxy@mail.cnu.edu.cn
\textsuperscript{2}Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100190, China. Email: dzhao@amss.ac.cn; kliu@amss.ac.cn
\textsuperscript{3}Department of Automatic Control, Beijing University of Aeronautics and Astronautics, Beijing, 100083, China. Email: kycai@buaa.edu.cn

Abstract: This paper focuses on the problem of the degree sequence for a mixed random graph process which continuously combines the classical model and the BA model. Note that the number of step added edges for the mixed model is random and non-uniformly bounded. By developing a comparing argument, phase transition on the degree distributions of the mixed model is revealed: while the pure classical model possesses a exponential degree sequence, the pure BA model and the mixed model possess power law degree sequences. As an application of the methodology, phase transition on the degree sequence of another mixed model with hard copying is also studied, especially, in the power law region, the inverse power can take any value greater than 1.

1 Introduction and statement of the results

Graph theory \cite{5,20,22,24} is a rich research area that can be traced back to the problem on the seven bridges in Königsberg considered by celebrated mathematician Euler in 1736. In 1950s, Hungarian...
mathematicians Erdős and Rényi extended the graph theory into random environments and developed the classical theory of random graphs. In their paper [20], Erdős and Rényi define the random graph $G_{n,M}$ (ER model) which consists of $n$ nodes and $M$ randomly chosen edges of the all $\binom{n}{2}$ possible edges, and study the property of $G_{n,M}$ as $n \to \infty$, with $M = M(n)$ a function of $n$. At the time when Erdős and Rényi started their investigations of $G_{n,M}$, Gilbert [22] introduced a more fundamental random graph model $G_{n,p}$ as follows: Given $n$ nodes, each of the $\binom{n}{2}$ distinct couples of nodes is linked with an edge with probability $p$. For $M \sim p \binom{n}{2}$ as $n \to \infty$, the models $G_{n,M}$ and $G_{n,p}$ are almost interchangeable and are subsequently called the classical random graph models in the literature. Clearly, the generation mechanism of the classical random graph is featured with several characteristics. First, the number of nodes is given a priori and keeps constant during the process of graph generation. Second, the edges are generated in a random manner. Finally, each edge is generated in an equal probability.

On the other hand, in recent years complex networks have drawn a lot of attentions in disparate communities including statistical mechanics, computer networks, control theory, among others [1, 4, 8, 9, 26, 27]. Various models involving random factors have been proposed and investigated. Among them, the model proposed by A.-L. Barabási and R. Albert [7] (BA model) has been well received and can be described as follows. A graph with $n_0$ nodes and $m_0$ edges is given at the beginning. Then the graph starts to evolve. At each time a new node with several new edges is added to the graph. While all these new edges are linked with the new node, the other node that links an edge of these new edges is selected from the existing nodes according to the principle of preferential attachment. Suppose that there are $n$ nodes in the graph already, with $d_{x_i}$ being the degree of the $i^{th}$ node. The principle of preferential attachment asserts that the $i^{th}$ node is selected as the node that links one of the $m$ edges with probability $d_{x_i}/\sum_{i=1}^{n} d_{x_i}$. It is shown that the degree distribution of the resulting graph obeys a power law. Different from the generation mechanism of Erdős and Rényi, for a random graph, the generation mechanism of BA is featured with the following characteristics. First, the size of the graph in terms of the number of nodes and edges is varying during the process of graph generation. The graph tends to evolve. Second, the added new edges are generated with unequal probabilities according the principle of preferential attachment. Obviously, the BA model can hardly be treated as an extension of the ER model.

A natural question is how to reconcile the ER theory of random graphs and various models of complex networks and develop a coherent or modern theory of random graph and complex networks,
this forms the first motivation of the present paper. As a useful step, it should be interesting to combine the distinct features of the two graph generation mechanisms described above and investigate various properties of the resulting graph. In this paper we will first introduce an evolving classical random graph model and then modify this classical model according to the principle of preferential attachment.

The ER model can be easily modified in an evolving way as follows. Fix some constant $\mu > 0$. Let’s consider the following process which generates a sequence of simple graphs $\{G^0_t = (V_t, E_t), t \geq 1\}$:

**Time-Step 1.** Let $G^0_1$ consists of vertices $x_0, x_1$ and the edge $(x_0, x_1)$. In general, $(u, v)$ denotes the edge with endpoints $u, v$.

**Time-Step $t \geq 2$.** We add a vertex $x_t$ to $G^0_{t-1}$ and then add random edges incident with $x_t$: for any $0 \leq i \leq t-1$, edge $(x_i, x_t)$ is added independently with probability $(\mu \land t)/t$.

The process $\{G^0_t : t \geq 1\}$ defined above is called classical, for edges are added in an equal probability at any Time-Step, which coincides with the basic feature of ER model.

It is easily observed that the classical model $\{G^0_t : t \geq 1\}$ is not appropriate for studying real world networks also. Actually, model $\{G^0_t : t \geq 1\}$ can be farther modified to the following BA model $\{G_t = (V_t, E_t) : t \geq 1\}$, which fits the first motivation of us:

**Time-Step 1.** Let $G_1$ consists of vertices $x_0, x_1$ and the edge $(x_0, x_1)$.

**Time-Step $t \geq 2$.** We add a vertex $x_t$ to $G_{t-1}$ and then add random edges incident with $x_t$: for any $0 \leq i \leq t-1$, edge $(x_i, x_t)$ is added independently with probability $\mu d_{x_i}(t-1)/2e_{t-1} \land 1$, where $d_{x_i}(t-1)$ be the degree of $x_i$ in $G_{t-1}$ and $e_{t-1} = |E_{t-1}|$.

The second motivation for us to consider the above process $\{G_t : t \geq 0\}$ is to model the www-typed real-world networks properly. We say a real-world network is of www-typed, if the following holds

1. Excepting for all the isolated vertices (nodes), the network has only one connected component;
2. There is no loop and multi-edge in the network;
3. While a new vertex (node) is added, the number of added new edges (links) between it and the existing vertices is finite but unbounded; and
4. Edges (links) are added in the preferential attachment manner.

Actually, to model the real world networks by random complex graphs, many new models (deferring from the ER model) have already been introduced. By studying complex graphs, various topological
properties such as degree-distribution \([7,10,14,21]\), diameter \([2,4,13,30]\), clustering \([11,27]\), stability \([5,6,12]\) and spectral gap \([9]\) of these real-world networks have been presented. One of the most basic properties of real-world networks is the power law degree distribution, many new models have been introduced to explain the underlying causes for the emergence of power law degree distributions. This can be observed in the ‘LCD model’ \([13]\); the generalization of ‘LCD model’ due to Buckley and Osthus \([10]\); ‘copying’ models of Kumar et al. \([21]\); ‘hard copying’ models of Wu et al. \([28]\); the general models defined by Copper and Frieze \([15]\); the growth-deletion models of Copper, Frieze and Vera \([16]\), Chung and Lu \([17]\) and Wu et al. \([31]\) etc. The main difference between our model and those introduced in \([10,13,15,16,17,25]\) and \([31]\) is that, in our setting, the number of step added edges is random and non-uniformly bounded. Note that the ‘hard copying’ model introduced in \([28]\) is also a model with non-uniformly bounded edge addition. Obviously, the model \(\{G_t : t \geq 1\}\) seems to be a more proper candidate for modeling the www-typed real-world networks.

Now, let \(D_k(t)\) be the number of vertices with degree \(k \geq 0\) in \(G_t\) and let \(\overline{D}_k(t)\) be the expectation of \(D_k(t)\). Note that, in this paper, for any kind of random graph process, we always denote \(D_k(t)\) the number of vertices with degree \(k \geq 0\) and \(\overline{D}_k(t)\) its expectation.

The first result of this paper is about BA model \(\{G_t = (V_t, E_t) : t \geq 1\}\), it follows as

**Theorem 1.1** For any \(0 < \mu \leq 2\), there exists positive constants \(C_1\) and \(C_2\) such that

\[
C_1 k^{-3} \leq \liminf_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq C_2 k^{-3}
\]

for all \(k \geq 1\).

**Remark 1.1** In this paper, the condition \(0 < \mu \leq 2\) is purely technical, and it is conjectured that our results hold for any \(\mu > 0\).

By definition, excepting for the isolated vertices, \(G_t\) contains a unique connected component, we call it the giant component of \(G_t\). Denote by \(C_t\) the giant component. The following is our result on \(\mathbb{E}(|C_t|)\), the mean size of \(C_t\).

**Theorem 1.2** Assume that \(0 < \mu \leq 2\). Then for any small enough \(\nu > 0\), we have

\[
\mathbb{E}(|C_t|) = (1 - e^{-\nu})t + O(t^{\frac{\nu}{\mu}}).
\]

Note that the hidden constant in \(O(t^{\frac{\nu}{\mu}})\) only depends on \(\nu\).
Now, we present a mixed model which continuously combines the classical model \( \{G_t^0 : t \geq 1\} \) and the above BA model \( \{G_t : t \geq 1\} \). Fix some constants \( 0 \leq \alpha \leq 1 \) and \( \mu, \zeta > 0 \). Define random graph process \( \{G_t^\alpha = (V_t, E_t) : t \geq 1\} \) as follows.

**Time-Step 1.** Let \( G_1^\alpha \) consists of vertices \( x_0, x_1 \) and the edge \( \langle x_0, x_1 \rangle \).

**Time-Step** \( t \geq 2 \). We add a new vertex \( x_t \) to \( G_{t-1}^\alpha \) and then

1. with probability \( \alpha \), we add random edges incident with \( x_t \) in the preferential attachment manner: 
   for any \( 0 \leq i \leq t - 1 \), edge \( \langle x_i, x_t \rangle \) is added independently with probability \( \frac{\mu d_{x_i}^\alpha(t-1)}{2e_{t-1}} \wedge 1 \), where \( d_{x_i}^\alpha(t-1) \) be the degree of \( x_i \) in \( G_{t-1}^\alpha \);

2. with probability \( 1 - \alpha \), we add random edges incident with \( x_t \) in the classical manner: for any \( 0 \leq i \leq t - 1 \), edge \( \langle x_i, x_t \rangle \) is added independently with probability \( (\zeta \wedge t)/t \).

It is straightforward to generalize the approach developed for Theorem 1.1 to prove the following corollary for \( \{G_t^\alpha : t \geq 1\} \), \( 0 < \alpha < 1 \):

**Corollary 1.3** For any \( 0 < \alpha < 1 \), \( 0 < \mu \leq 2 \) and \( \zeta > 0 \), there exists positive constants \( C_1^\alpha \) and \( C_2^\alpha \) such that

\[
C_1^\alpha k^{-\beta} \leq \liminf_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq C_2^\alpha k^{-\beta}
\]

for all \( k \geq 1 \), where \( \beta = 1 + 2 \left( 1 + \frac{(1-\alpha)\zeta}{\alpha \mu} \right) \).

**Remark 1.2** At any Time-Step \( t > \zeta \), the mean number of added new edges is \( \xi := \alpha \mu + (1 - \alpha)\zeta \) and \( \frac{(1-\alpha)\zeta}{\alpha \mu} \) be the limit ratio of the number of the two kinds of edges in \( G_t^\alpha \).

In the case of \( \alpha = 0 \), we get the classical process \( \{G_t^0 : t \geq 1\} \) parameterized by \( \zeta > 0 \). Just as one expects, the model \( \{G_t^0 : t \geq 1\} \) possesses a classical (exponential) degree sequence as

**Corollary 1.4** For random graph process \( \{G_t^0 : t \geq 1\} \), there exists positive constants \( C_1^0 \) and \( C_2^0 \) such that

\[
C_1^0 \left( \frac{\zeta}{1+\zeta} \right)^k \leq \liminf_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq C_2^0 \left( \frac{\zeta}{1+\zeta} \right)^k
\]

for all \( k \geq 0 \).

Theorems 1.1 and Corollaries 1.3 and 1.4 exhibit a phase transition on the degree distributions of the mixed model \( \{G_t^\alpha : t \geq 1\} \) while \( \alpha \) varies from 0 to 1. Note that phase transition on degree...
distributions of random graph process is first studied in the recent work [31] of Wu et al. More precisely, [31] introduced a model with edge deletions and showed that, while a relevant parameter varies, the model exhibits power law degree distribution, a special degree distribution lying between power law and exponential, and exponential degree distribution in turn. A numerical investigation to phase transition on degree distributions of networks can be founded in reference [33].

The rest of the paper is organized as follows. In Section 2, we give some useful estimates to $e_t$, the number of edges in $G_t$. In section 3, we bound the maximum degree of vertex in $G_t$, and then prove Theorem 1.2. In section 4, we establish the recurrence for $D_k(t)$, then solve the recurrence by using a compare argument, and finally finish the proof of Theorem 1.1. In Section 5, we adopt the comparing argument developed in Section 4 to prove Corollaries 1.3 and 1.4. In Section 6, we apply the comparing argument to study the phase transition on the degree sequence of a mixed model with hard copying.

## 2 Estimates for $e_t$

In this section we give some lemmas for $e_t$, which will play important roles in the proofs of our main results.

We first consider the increments of $e_t$. Let $a_t = e_{t+1} - e_t$ and $\{\mathcal{F}_t : t \geq 1\}$ be the natural $\sigma$-flow generated by process $\{G_t : t \geq 1\}$. Then

**Lemma 2.1** For all $t \geq 1$, we have

$$E(a_t \mid \mathcal{F}_t) = \mu$$

(2.1)

and

$$E(a_k^t \mid \mathcal{F}_t) \leq (\mu \lor 1)^k k!$$

for $k \geq 2$.

**Proof:** Let $\{p_i : 1 \leq i \leq n\}$, $n \geq 2$, be a serial of positive numbers satisfying $p_i \leq \frac{1}{2}$, $\sum_{i=1}^{n} p_i = 1$, and let $\{X_i, 1 \leq i \leq n\}$ be the independent random variables with

$$P(X_i = 1) = \mu p_i = 1 - P(X_i = 0).$$

Let $X = \sum_{i=1}^{n} X_i$. Clearly, to prove the lemma, it suffices to prove that

$$E(X) = \mu \quad \text{and} \quad E(X^k) \leq (\mu \lor 1)^k \times k! \quad \forall \quad k \geq 2.$$
For \( k = 1 \), it is straightforward to see that \( E(X^k) = E(X) = \mu \leq \mu \lor 1 \). Assume that \( E(X^m) \leq (\mu \lor 1)^m \times m! \) for some \( m \geq 1 \), then

\[
E(X^{m+1}) = E \left( \sum_{i=1}^{n} X_i \right)^{m+1} = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \sum_{i_{m+1}=1}^{n} E(X_{i_1} \cdots X_{i_m} X_{i_{m+1}})
\]

\[
= \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \left( \sum_{i_{m+1} \in \{i_1, \ldots, i_m\}} E(X_{i_1} \cdots X_{i_m}) + \sum_{i_{m+1} \notin \{i_1, \ldots, i_m\}} E(X_{i_1} \cdots X_{i_m})E(X_{i_{m+1}}) \right)
\]

\[
\leq \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} (mE(X_{i_1} \cdots X_{i_m}) + \mu E(X_{i_1} \cdots X_{i_m}))
\]

\[
\leq (m + 1)(\mu \lor 1) \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} E(X_{i_1} \cdots X_{i_m})
\]

\[
= (m + 1)(\mu \lor 1)E(X^m) \leq (\mu \lor 1)^{m+1} \times (m + 1)!
\]

Thus we finish the proof by induction. \( \square \)

Now, define \( Y_t = e_t - \mu t \) for \( t \geq 1 \), then, by the definition of \( G_t \), \( \{Y_t : t \geq 1\} \) forms a martingale with respect to \( \{\mathcal{F}_t : t \geq 1\} \).

**Lemma 2.2** There exists some constant \( c_1 > 0 \) such that

\[
P( |e_t - \mu t| \geq t^{4/5} ) \leq c_1 t^{-3/5} \quad (2.3)
\]

for all \( t \geq 1 \).

**Proof:** By the property of martingale, first, we have

\[
E(Y_t - Y_1)^2 = E \left( \sum_{i=1}^{t-1} (Y_{i+1} - Y_i) \right)^2 = \sum_{i=1}^{t-1} E(Y_{i+1} - Y_i)^2 = \sum_{i=1}^{t-1} \text{Var}(a_i). \quad (2.4)
\]

Then, by Lemma 2.1

\[
E(Y_t - Y_1)^2 = \sum_{i=1}^{t-1} \text{Var}(a_i) \leq (2(\mu \lor 1)^2 - \mu^2) (t - 1). \quad (2.5)
\]

Finally, using the relation that \( E(Y_t^2) = E(Y_t - Y_1)^2 + (1 - \mu)^2 \) and the Markov’s inequality, we have

\[
P( |e_t - \mu t| \geq t^{4/5} ) \leq \frac{E(Y_t^2)}{t^{8/5}} \leq \frac{(2(\mu \lor 1)^2 - \mu^2)(t - 1) + (1 - \mu)^2}{t^{8/5}} \leq c_1 t^{-3/5}
\]

for some constant \( c_1 > 0 \). \( \square \)

7
Lemma 2.3 For any $\nu > 0$, there exists constants $c_2, c_3 > 0$ such that

$$P(|e_t - \mu t| \geq \nu t) \leq c_2 e^{-c_3 t}$$

for all $t \geq 1$.

Proof: By Lemma 2.1 for small $\lambda > 0$, we have

$$E(e^{\lambda \xi_t} | \mathcal{F}_t) = 1 + \lambda \mu + O(\lambda^2),$$

then

$$E(e^{\lambda \xi_t}) = E(E(e^{\lambda \xi_t} | \mathcal{F}_t)) = E(e^{\lambda \xi_t} E(e^{\lambda \xi_t} | \mathcal{F}_t)) = (1 + \lambda \mu + O(\lambda^2)) E(e^{\lambda \xi_t}).$$

This implies that

$$E(e^{\lambda \xi_t}) = (1 + \lambda \mu + O(\lambda^2))^{-1} E(e^{\lambda \xi_t}) = \frac{e^\lambda}{1 + \lambda \mu + O(\lambda^2)} \exp\{\ln(1 + \lambda \mu + O(\lambda^2)) \lambda t\}.$$ 

For given $\nu > 0$, take $\lambda > 0$ small enough such that

$$c_3' := (\mu + \nu) \lambda - \ln(1 + \lambda \mu + O(\lambda^2)) > 0.$$ 

Taking $c_2' = e^\lambda / (1 + \lambda \mu + O(\lambda^2))$, we have

$$P(e_t \geq (\mu + \nu) t) \leq E(e^{\lambda \xi_t}) e^{-(\mu + \nu) \lambda t} \leq c_2' e^{-c_3' t}. \quad (2.7)$$

Similarly, for some $c_2'', c_3'' > 0$, we have

$$P(e_t \leq (\mu - \nu) t) \leq e^{(\mu - \nu) \lambda t} E(e^{-\lambda \xi_t}) \leq c_2'' e^{-c_3'' t}. \quad (2.8)$$

The lemma follows from (2.7) and (2.8). \qed

3 Bounding the degree and the proof of Theorem 1.2

For times $s$ and $t$ with $0 \leq s \leq t$, $t \geq 1$, let $d_{x_s}(t)$ be the degree of vertex $x_s$ in $G_t$. In this section, we will concentrate on the upper bound of $d_{x_s}(t)$ and then prove Theorem 1.2.

We say an event happens quite surely (qs) if the probability of the complimentary set of the event is $O(t^{-K})$ for any $K > 0$.

The following is our bounding for $d_{x_s}(t)$. As noted in [31], our result will depend on Lemma 2.3, the exponential inequality for $e_t$. 

8
Lemma 3.1 For small $\nu > 0$ and $1 \leq s \leq t$, we have

$$d_{x_s}(t) \leq (t/s)^{1/\nu} (\log t)^3 \quad \text{qs.} \tag{3.1}$$

Proof: Let $X_\tau = d_{x_s}(\tau)$ for $\tau = s, s+1, \ldots, t$. Conditional on $X_\tau = x$ and $e_\tau$, we have

$$X_{\tau+1} = x + B \left( 1, \frac{\mu x}{2e_\tau} \right), \tag{3.2}$$

where $B \left( 1, \frac{\mu x}{2e_\tau} \right)$ be the $\{0,1\}$-valued random variable with $\mathbb{P} \left( B \left( 1, \frac{\mu x}{2e_\tau} \right) = 1 \right) = \frac{\mu x}{2e_\tau}$. Lemma 3.1 follows immediately from (2.6), (3.2) and a standard argument which can be found in the proof of Lemma 2.1 in [31]. □

Remark 3.1 Because $d_{x_0}(t)$ and $d_{x_1}(t)$ are same distributed, Lemma 3.1 implies that

$$d_{x_0}(t) \leq t^{1/\nu} (\log t)^3, \quad \text{qs.}$$

Now, based on Lemma 3.1 we prove Theorem 1.2 as follows.

Proof of Theorem 1.2: To prove Theorem 1.2, it suffices to show that

$$\mathbb{E}(|V_t \setminus C_t|) = e^{-\mu t} + O(t^{1/\nu}). \tag{3.3}$$

Denote by $\Delta_t$ the maximal degree in $G_t$. By Lemma 2.3 Lemma 3.1 and Remark 3.1, we have

$$\frac{\Delta_t}{e_t} \leq Lt^{1/\nu - 1}, \quad \text{qs} \tag{3.4}$$

where $L$ be a constant independent of $t$.

For large $t$, let’s consider the probability $\mathbb{P}(a_t = 0)$, recall that $a_t = e_{t+1} - e_t$ be the increment of $e_t$ at Time-Step $t+1$. By equation (3.4), we have

$$\mathbb{P}(a_t = 0) = \mathbb{E}(I_{a_t=0}) = \mathbb{E}(\mathbb{E}(I_{a_t=0} | \mathcal{F}_t))$$

$$= \mathbb{E} \left( \mathbb{E}(I_{a_t=0} | \mathcal{F}_t) \left| \frac{\Delta_t}{e_t} \leq Lt^{1/\nu - 1} \right. \right) \mathbb{P} \left( \frac{\Delta_t}{e_t} \leq Lt^{1/\nu - 1} \right)$$

$$+ \mathbb{E} \left( \mathbb{E}(I_{a_t=0} | \mathcal{F}_t) \left| \frac{\Delta_t}{e_t} > Lt^{1/\nu - 1} \right. \right) \mathbb{P} \left( \frac{\Delta_t}{e_t} > Lt^{1/\nu - 1} \right)$$

$$= \mathbb{E} \left( \mathbb{E}(I_{a_t=0} | \mathcal{F}_t) \left| \frac{\Delta_t}{e_t} \leq Lt^{1/\nu - 1} \right. \right) + O(t^{-10}). \tag{3.5}$$
The term $E(I_{a_t=0} \mid \mathcal{F}_t)$ can be expressed as
\[
E(I_{a_t=0} \mid \mathcal{F}_t) = \prod_{i=0}^{t} (1 - \frac{\mu d_x(t)}{2e_t}) = \exp \left\{ \sum_{i=0}^{t} \log \left( 1 - \frac{\mu d_x(t)}{2e_t} \right) \right\}
\]
\[
= \exp \left\{ -\sum_{i=0}^{t} \frac{\mu d_x(t)}{2e_t} + O \left( \sum_{i=0}^{t} \left( \frac{\mu d_x(t)}{2e_t} \right)^2 \right) \right\} = e^{-\mu} + O \left( \frac{\Delta_t}{e_t} \right),
\]
therefore,
\[
E \left( E(I_{a_t=0} \mid \mathcal{F}_t) \right) \leq L t^{\frac{1}{t-1}} - 1 = e^{-\mu} + O \left( t^{\frac{1}{t-1}} \right).
\]
Thus, (3.6) and (3.7) imply that
\[
P(a_t = 0) = e^{-\mu} + O \left( t^{\frac{1}{t-1}} \right).
\]

Now, by the definition of $G_t$, we have
\[
E(|V_t \setminus C_t|) = \sum_{s=2}^{t} P(d_s(t) = 0) = \sum_{s=1}^{t-1} P(a_s = 0),
\]
equation (3.3) follows immediately from (3.8) and (3.9).

**Remark 3.2** For any $t \geq 1$, we have
\[
P(a_t = 0) = E(E(I_{a_t=0} \mid \mathcal{F}_t)) = E \left( \prod_{s=0}^{t} \left( 1 - \frac{\mu d_x(t)}{2e_t} \right) \right)
\]
\[
\leq E \left( \prod_{s=0}^{t} \exp \left\{ -\frac{\mu d_x(t)}{2e_t} \right\} \right) = e^{-\mu}.
\]
Furthermore, equation (3.8) implies that $\lim_{t \to \infty} P(a_t = 0) = e^{-\mu}$.

For the probability $P(a_t = 1)$, using (3.3) again, the same arguments as in (3.6, 3.7) imply that
\[
\lim_{t \to \infty} P(a_t = 1) = \mu e^{-\mu}.
\]

### 4 The comparing Approach and The proof of Theorem 1.1

In this Section, we develop a comparing approach to prove Theorem 1.1. We first follow the basic procedures in [16] to establish the recurrence for $D_k(t)$. By the definition of $G_t$, first of all, we have $D_0(1) = 0, D_1(1) = 2$ and $D_k(t) = 0$ for all $k, t$ with $k > t \geq 1$. 

---

10
Now, put $D_{-1}(t) = 0$ for all $t \geq 1$. For $t + 1 \geq k \geq 0$ and $t \geq 1$, we have
\[
E(D_k(t+1) \mid \mathcal{F}_t) = D_k(t) + \left( -\frac{kD_k(t)}{2t} + \frac{(k-1)D_{k-1}(t)}{2t} \right) + E(I_{a_t = k} \mid \mathcal{F}_t).
\] Taking expectation and then using the basic inequality
\[
e_t \leq \sum_{s=1}^{t} s = \frac{t(t+1)}{2}
\] and the estimations given in Lemmas 2.2 and 2.3, (4.1) implies that
\[
T_k(t+1) = T_k(t) + \frac{k-1}{2} T_{k-1}(t) - \frac{k}{2} T_k(t) + O(t^{-1/5}) + f_k(t),
\] where $f_k(t) = \mathbb{P}(a_t = k)$. Note that term $O(t^{-1/5})$ is independent of $k$. We get the recurrence for $T_k(t)$ as:
\[
\begin{cases}
T_k(t+1) = T_k(t) + \frac{k-1}{2} T_{k-1}(t) - \frac{k}{2} T_k(t) + O(t^{-1/5}) + f_k(t), \\
t + 1 \geq k \geq 0, \ t \geq 1; \\
T_0(1) = 0; \ T_1(1) = 2; \ T_k(t) = 0, \ k > t \geq 1; \ T_{t+1}(t) = 0, \ t \geq 1.
\end{cases}
\] (4.3)

To solve the recurrence (4.3), we need a comparing argument. Note that the recurrence as (4.3) with $\{f_k(t)\}$ replaced by a serial of constants can be solved directly by the method developed in [15], [16], and [31]. Let
\[
F_k(t) := T_k(t+1) - T_k(t) - \frac{k-1}{2} T_{k-1}(t) + \frac{k}{2} T_k(t) - f_k(t).
\]
Obviously, $F_k(t)$ is a determined (or known!) function in $k$ and $t$ satisfying
\[
|F_k(t)| \leq R t^{-1/5}, \ \forall k \geq 0, \ t \geq 1.
\] (4.4)

For $k \geq 0$, define
\[
A_k(t) = \begin{cases}
F_k(t), & \text{if } t \geq k, \\
F_k(t) + f_k(t), & \text{if } t \leq k - 1;
\end{cases}
\]
and
\[
g_k(t) = \begin{cases}
f_k(t), & \text{if } t \geq k, \\
0, & \text{if } t \leq k - 1.
\end{cases}
\]

Then, (4.3) can be rewritten as
\[
\begin{cases}
T_k(t+1) = T_k(t) + \frac{k-1}{2} T_{k-1}(t) - \frac{k}{2} T_k(t) + A_k(t) + g_k(t), \\
t + 1 \geq k \geq 0, \ t \geq 1; \\
T_0(1) = 0; \ T_1(1) = 2; \ T_k(t) = 0, \ k > t \geq 1; \ T_{t+1}(t) = 0, \ t \geq 1.
\end{cases}
\] (4.5)
By the fact that \( f_k(t) = 0 \) for \( t \leq k - 2 \) and \( f_k(k - 1) = P(a_{k-1} = k) \leq \mu k^{-1} \) for \( k \geq 2 \), similar to (4.4), we have for some \( R_1 > 0 \)

\[
|A_k(t)| \leq R_1 t^{-1/5}, \quad \forall \ k \geq 0, \ t \geq 1.
\]

(4.6)

In the rest of this section, we will try to solve the recurrence (4.5) for any given function serial \( \{A_k(t)\} \) satisfying (4.6). The lack of the existence of such limit as \( \lim_{t \to \infty} f_k(t) \) makes it difficult to solve (4.5) directly. In fact, to solve (4.5) by the known argument developed in [15], [16] and [31], we not only need the existence of such limits, but also need a uniform speed faster than \( t^{-\varepsilon}, \varepsilon > 0 \), of the corresponding convergence. But this seems impossible (see the proof of Corollary 1.4), we have to develop a new method to study \( D_k(t) \).

By Remark 3.2, \( \lim_{t \to \infty} P(a_t = 0) = e^{-\mu} \), then, for some constant \( \rho > 0 \),

\[
P(a_t = 0) \geq \rho > 0, \quad \forall \ t \geq 1.
\]

(4.7)

For \( k \geq 0 \), let

\[
\psi_k = \begin{cases} 0, & k \geq 1, \\ \rho, & k = 0; \end{cases}, \quad \varphi_k = \begin{cases} Ck^{-4}, & k \geq 1, \\ e^{-\mu}, & k = 0, \end{cases}
\]

(4.8)

with \( C = (\mu \lor 1)^4 \times 4! \). Define

\[
\psi_k(t) = \begin{cases} 0, & k \geq 1, \ t \geq 1, \\ \psi_k, & k = 0, \ t \geq 1; \end{cases}, \quad \varphi_k(t) = \begin{cases} \varphi_k, & t \geq k, \\ 0, & 1 \leq t < k. \end{cases}
\]

(4.9)

By Lemma 2.1, equation (3.10) and the Markov’s inequality, we have

\[
\psi_k(t) \leq g_k(t) \leq \varphi_k(t), \quad \forall \ k \geq 0, \ t \geq 1.
\]

(4.10)

Now, with \( g_k(t) \) in (4.5) replaced by \( \psi_k(t) \) and \( \varphi_k(t) \) respectively, we get the following recurrences for \( \tilde{D}_k(t) \) and \( \hat{D}_k(t) \):

\[
\begin{align*}
\tilde{D}_k(t+1) &= \tilde{D}_k(t) + \frac{k-1}{2} \tilde{D}_{k-1}(t) + \frac{k}{2} \tilde{D}_k(t) + A_k(t) + \psi_k(t), \quad t \geq 1, \\
\tilde{D}_1(1) &= 2, \quad \tilde{D}_k(t) = 0, \quad k > t \geq 1.
\end{align*}
\]

(4.11)
\[
\begin{aligned}
\hat{D}_k(t+1) &= \hat{D}_k(t) + \frac{k-1}{2} \frac{\hat{D}_{k-1}(t)}{t} - \frac{k}{2} \frac{\hat{D}_k(t)}{t} + A_k(t) + \varphi_k(t), \quad t \geq k \geq 0, \quad t \geq 1; \\
\hat{D}_0(1) &= 0; \quad \hat{D}_1(1) = 2; \quad \hat{D}_k(t) = 0, \quad k > t \geq 1; \quad \hat{D}_{-1}(t) = 0, \quad t \geq 1.
\end{aligned}
\]

(4.12)

We first give the following comparing lemma to show that \(\hat{D}_k(t)\) and \(\hat{D}_k(t)\) are lower and upper bounds for \(D_k(t)\) respectively.

**Lemma 4.1** [Comparing Lemma] Assume that \(\hat{D}_k(t)\) and \(\hat{D}_k(t)\) be the solutions of (4.11) and (4.12) respectively. Then

\[
\hat{D}_k(t) \leq \bar{D}_k(t) \leq \hat{D}_k(t), \quad \forall \ k \geq -1, \ t \geq 1.
\]

(4.13)

**Proof:** We only prove the first inequality in (4.13), the situation for the second one is the same. Firstly, noticing that \(\hat{D}_{-1}(t) = \bar{D}_{-1}(t) = 0\) for all \(t \geq 1\), we have

\[
\hat{D}_0(t+1) = \hat{D}_0(t) + A_0(t) + \psi_0(t)
\]

and

\[
\hat{D}_0(t+1) = \hat{D}_0(t) + A_0(t) + g_0(t)
\]

for all \(t \geq 1\). This, together with the fact that \(\hat{D}_0(1) = \bar{D}_0(1) = 0\) and the inequality (4.10), implies

\[
\hat{D}_0(t) \leq \bar{D}_0(t), \quad \forall \ t \geq 1.
\]

(4.14)

Secondly, by the fact that \(\hat{D}_{k+1}(k) = \bar{D}_{k+1}(k) = \psi_{k+1}(k) = g_{k+1}(k) = 0\) for all \(k \geq 1\), we have

\[
\hat{D}_{k+1}(k+1) = \frac{1}{2} \hat{D}_k(k) + A_{k+1}(k)
\]

(4.15)

and

\[
\hat{D}_{k+1}(k+1) = \frac{1}{2} \bar{D}_k(k) + A_{k+1}(k)
\]

for all \(k \geq 1\). This, together with the initial condition \(\hat{D}_1(1) = \bar{D}_1(1) = 2\), implies that

\[
\hat{D}_k(k) = \bar{D}_k(k), \quad \forall \ k \geq 1.
\]

(4.16)

Suppose we have proved that for some \(m \geq 0\),

\[
\hat{D}_k(k+m) \leq \bar{D}_k(k+m), \quad \forall \ k \geq 1.
\]

(4.17)
If we can prove
\[
\hat{D}_k(k + (m + 1)) \leq \overline{D}_k(k + (m + 1)), \quad \forall \ k \geq 1,
\] (4.18)
then we get the lemma by induction.

By (4.10) and (4.17), (4.18) can be easily proved by induction. The details are omitted.

Now we begin to solve (4.11) and (4.12). We introduce two recurrences with respect to \( \{\psi_k\} \) and \( \{\varphi_k\} \) as follows:
\[
\begin{align*}
  \tilde{d}_k &= \frac{k - 1}{2} \tilde{d}_{k-1} - \frac{k}{2} \tilde{d}_k + \psi_k, \quad k \geq 0, \\
  \tilde{d}_{-1} &= 0;
\end{align*}
\] (4.19)
\[
\begin{align*}
  \hat{d}_k &= \frac{k - 1}{2} \hat{d}_{k-1} - \frac{k}{2} \hat{d}_k + \varphi_k, \quad k \geq 0, \\
  \hat{d}_{-1} &= 0.
\end{align*}
\] (4.20)
The following Lemma show that (4.19) and (4.20) are good approximation to (4.11) and (4.12) respectively.

**Lemma 4.2** Assume that \( \{\tilde{D}_k(t) : k \geq -1, t \geq 1\} \) (resp. \( \{\hat{D}_k(t) : k \geq -1, t \geq 1\} \)) be the solution of recurrence (4.11) (resp. (4.12)) and \( \{\tilde{d}_k : k \geq -1\} \) (resp. \( \{\hat{d}_k : k \geq -1\} \)) be the solution of (4.19) (resp. (4.20)). If \( \tilde{d}_k \leq C/k \) (resp. \( \hat{d}_k \leq C/k \)) for \( k > 0 \) and some constant \( C \), then there exists constant \( M_1 \) (resp. \( M_2 \)) such that
\[
|\tilde{D}_k(t) - t\tilde{d}_k| \leq M_1 t^{4/5} \quad \text{(resp. } |\hat{D}_k(t) - t\hat{d}_k| \leq M_2 t^{4/5})
\] (4.21)
for all \( k \geq -1 \) and \( t \geq 1 \).

**Proof of Lemma 4.2** By using the fact that \( \tilde{D}_k(t) = 0 \) (resp. \( \hat{D}_k(t) = 0 \)) for \( k > t \geq 1 \) and the condition \( \tilde{d}_k \leq C/k \) (resp. \( \hat{d}_k \leq C/k \)), it is straightforward to prove Lemma 4.2 by induction (in \( t \)). Note that our inductive hypothesis is
\[
\tilde{H}_t : \ |\tilde{\Theta}_k(t)| \leq M_1 t^{4/5} \quad \text{for all } k \geq -1. \quad \text{(resp. } \hat{H}_t : \ |\hat{\Theta}_k(t)| \leq M_2 t^{4/5} \quad \text{for all } k \geq -1.)
\]
For details, one may refer to [31] (the proof of Lemma 2.2).

Now, we finish the proof of Theorem 1.1 as follows.
Proof of Theorem 1.1 For any given constant number serial \( \{\phi_k : k \geq 0\} \), the recurrence in \( k \) with the form

\[
\begin{cases}
  d_k = \frac{k-1}{2} d_{k-1} - \frac{k}{2} d_k + \phi_k, & k \geq 0, \\
  d_{-1} = 0,
\end{cases}
\]

can be directly solved as: \( d_{-1} = 0, d_0 = \phi_0, d_1 = \frac{3}{4} \phi_1 \) and

\[
d_k = \sum_{j=1}^{k} \frac{2(j+1)}{k(k+1)(k+2)} \phi_j = \frac{1}{k(k+1)(k+2)} \sum_{j=1}^{k} 2(j+1) \phi_j, \quad \forall \ k \geq 2.
\]

(4.22)

Applied to \( \{\psi_k\} \) and \( \{\varphi_k\} \), the summation in the right hand side of equation (4.22) converges as \( k \to \infty \), thus, \( \hat{d}_k \) and \( \hat{d}_k \) decay as \( k^{-3} \). Clearly, \( \hat{d}_k \) and \( \hat{d}_k \) satisfy the requirement of Lemma 4.2 and for some constants \( C_1, C_2 \),

\[
C_1 k^{-3} \leq \hat{d}_k, \quad \hat{d}_k \leq C_2 k^{-3} \quad \forall \ k \geq 1.
\]

(4.23)

By Lemma 4.1, Lemma 4.2 and equation (4.23), we have

\[
C_1 k^{-3} \leq \tilde{d}_k = \lim_{t \to \infty} \frac{\text{D}_k(t)}{t} \leq \liminf_{t \to \infty} \frac{\text{D}_k(t)}{t} \leq \limsup_{t \to \infty} \frac{\text{D}_k(t)}{t} \leq \lim_{t \to \infty} \frac{\hat{D}_k(t)}{t} = \hat{d}_k \leq C_2 k^{-3}
\]

for all \( k \geq 1 \). \( \square \)

5 Proofs of Corollaries 1.3 and 1.4

In this section, we prove Corollaries 1.3 and 1.4. Because the basic approach is the same as we have used in the proof of Theorem 1.1, we only give out a sketch.

For the process \( \{G_t^\alpha : t \geq 1\}, 0 \leq \alpha < 1 \), denote by \( e_t \) the number of edges in \( G_t^\alpha \) and \( a_t = e_t + 1 - e_t \) none the less.

Sketch of the proof of Corollary 1.3 For simplicity, we only deal with the special case of \( \mu = \zeta \).

Firstly, it is straightforward to check that Lemmas 2.1, 2.2 and 2.3 hold for \( e_t \). Then the recurrence of \( \text{D}_k(t) \) can be derived as

\[
\begin{cases}
  \text{D}_k(t+1) = \text{D}_k(t) + \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \frac{\text{D}_{k-1}(t)}{t} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \frac{\text{D}_k(t)}{t} \\
  + A_k(t) + g_k^\alpha(t), \quad t + 1 \geq k \geq 0, \ t \geq 1; \\
  \text{D}_0(1) = 0; \quad \text{D}_1(1) = 2; \quad \text{D}_k(t) = 0, \ k > t \geq 1; \quad \text{D}_{-1}(t) = 0, \ t \geq 1.
\end{cases}
\]

(5.1)
where $A_k(t)$ satisfying (4.6), $g_\alpha^k(t) = 0$, $\forall t \leq k - 1$ and

$$g_\alpha^k(t) = \mathbb{P}(a_t = k)$$

$$= \alpha \mathbb{P} \left( \sum_{i=0}^{t-1} B \left( \frac{t, \frac{\mu d_\alpha^k(t)}{2e_t}}{1} \right) = k \right) + (1 - \alpha) \mathbb{P} \left( B \left( \frac{t + 1, \mu}{t + 1} \right) = k \right)$$

$$= \alpha f_\alpha^k(t) + (1 - \alpha) \bar{f}_k(t), \quad \forall t \geq k. \tag{5.2}$$

In the case of $0 \leq \alpha < 1$, we have

$$\liminf_{t \to \infty} g_\alpha^k(t) \geq (1 - \alpha) \lim_{t \to \infty} \bar{f}_k(t) = (1 - \alpha) \frac{\mu^k}{k!} e^{-\mu}, \quad \forall k \geq 0, \tag{5.3}$$

then, there exists some $\rho > 0$ such that (4.7) holds. Note that here we get such $\rho$ from (5.3), but in case of $\alpha = 1$, we get it from the existence of $\lim_{t \to \infty} \mathbb{P}(a_t = 0)$, which depends on the degree bounds given in Lemma 3.1.

In case of $\alpha > 0$, let $n(\alpha) = 3 + \lceil 2/\alpha \rceil$, where $\lceil 2/\alpha \rceil$ be the integer part of $2/\alpha$. It is straightforward to check that

$$g_\alpha^0(t) \leq e^{-\mu}, \quad \forall t \geq 1 \text{ and } g_\alpha^k(t) \leq \frac{(\mu \lor 1)^{n(\alpha)}}{k!} n(\alpha)!, \quad \forall k \geq 1, t \geq 1. \tag{5.4}$$

Define $\{\psi_k\}$ and $\{\phi_k\}$ as

$$\psi_0 = \begin{cases} 0, & k \geq 1, \\ \rho, & k = 0; \end{cases} \quad \text{and} \quad \phi_0 = \begin{cases} C(\alpha) k^{-n(\alpha)}, & k \geq 1, \\ e^{-\mu}, & k = 0, \end{cases}$$

with $C(\alpha) = (\mu \lor 1)^{n(\alpha)} n(\alpha)!$. Then define

$$\psi_k(t) = \begin{cases} \psi_k, & t \geq \left( \frac{\alpha k}{2} + (1 - \alpha) \mu \right) \lor k, \\ g_\alpha^k(t), & 1 \leq t < \left( \frac{\alpha k}{2} + (1 - \alpha) \mu \right) \lor k; \end{cases}$$

and

$$\phi_k(t) = \begin{cases} \phi_k, & t \geq \left( \frac{\alpha k}{2} + (1 - \alpha) \mu \right) \lor k, \\ g_\alpha^k(t), & 1 \leq t < \left( \frac{\alpha k}{2} + (1 - \alpha) \mu \right) \lor k. \end{cases}$$

Thus we have

$$\psi_k(t) \leq g_\alpha^k(t) \leq \phi_k(t), \quad \forall k \geq 0, t \geq 1. \tag{5.5}$$
Let \( \tilde{D}_k(t) \) and \( \hat{D}_k(t) \) be the solutions of the recurrences obtained from \( [5.1] \) with \( g_k^\alpha(t) \) substituted by \( \psi_k(t) \) and \( \varphi_k(t) \) respectively. Namely
\[
\begin{align*}
\tilde{D}_k(t+1) &= \tilde{D}_k(t) + \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \frac{\tilde{D}_{k-1}(t)}{t} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \frac{\tilde{D}_k(t)}{t} + A_k(t) + \psi_k(t), \quad t + 1 \geq k \geq 0, \quad t \geq 1; \\
\tilde{D}_0(1) &= 0; \quad \tilde{D}_1(1) = 2; \quad \tilde{D}_k(t) = 0, \quad k > t \geq 1; \quad \tilde{D}_{-1}(t) = 0, \quad t \geq 1;
\end{align*}
\]
and
\[
\begin{align*}
\hat{D}_k(t+1) &= \hat{D}_k(t) + \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \frac{\hat{D}_{k-1}(t)}{t} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \frac{\hat{D}_k(t)}{t} + A_k(t) + \varphi_k(t), \quad t + 1 \geq k \geq 0, \quad t \geq 1; \\
\hat{D}_0(1) &= 0; \quad \hat{D}_1(1) = 2; \quad \hat{D}_k(t) = 0, \quad k > t \geq 1; \quad \hat{D}_{-1}(t) = 0, \quad t \geq 1.
\end{align*}
\]
Then Lemma \( [4.1] \) holds and we have
\[
\tilde{D}_k(t) \leq \overline{D}_k(t) \leq \hat{D}_k(t), \quad \forall k \geq -1, \quad t \geq 1. \tag{5.6}
\]

Define the two recurrences with respect to \( \{\psi_k\} \) and \( \{\varphi_k\} \) respectively as
\[
\begin{align*}
\hat{d}_k &= \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \hat{d}_{k-1} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \hat{d}_k + \psi_k, \quad k \geq 0, \\
\hat{d}_{-1} &= 0;
\end{align*}
\]
and
\[
\begin{align*}
\hat{d}_k &= \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \hat{d}_{k-1} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \hat{d}_k + \varphi_k, \quad k \geq 0, \\
\hat{d}_{-1} &= 0.
\end{align*}
\]
Then Lemma \( [4.2] \) holds, namely, under the condition that \( \hat{d}_k \leq C/k \) (resp. \( \hat{d}_k \leq C/k \)) for some constant \( C \) and \( k \geq 1 \), there exists constant \( M_1 \) (resp. \( M_2 \)) such that
\[
|\tilde{D}_k(t) - t\hat{d}_k| \leq M_1 t^{4/5} \quad \text{(resp. } |\hat{D}_k(t) - t\hat{d}_k| \leq M_2 t^{4/5}) \tag{5.7}
\]
for all \( k \geq -1 \) and \( t \geq 1 \).

Finally, it suffices to solve the recurrence in \( k \) with the form
\[
\begin{align*}
\hat{d}_k &= \left( \frac{\alpha(k-1)}{2} + (1-\alpha)\mu \right) \hat{d}_{k-1} - \left( \frac{\alpha k}{2} + (1-\alpha)\mu \right) \hat{d}_k + \phi_k, \quad k \geq 0, \\
\hat{d}_{-1} &= 0, \tag{5.8}
\end{align*}
\]
where \( \{ \phi_k : k \geq 0 \} \) be a serial of nonnegative numbers. Clearly, recurrence (5.8) can be solved as:

\[
d_{t-1} = 0, \quad d_0 = \frac{2}{b\alpha} \phi_0 \quad \text{and} \quad d_k = \prod_{j=1}^{k} \left( 1 - \frac{\beta}{j+b} \right) \left( \sum_{i=1}^{k} \frac{1}{\prod_{j=1}^{i} (1 - \frac{\beta}{j+b})} \phi_i + \frac{2}{b\alpha} \phi_0 \right), \quad \text{for} \quad k \geq 1, (5.9)
\]

where \( \beta = 1 + 2/\alpha \) and \( b = 2/\alpha + 2(1 - \alpha)\mu/\alpha \). Applying to \( \{ \psi_k \} \) and \( \{ \varphi_k \} \), the summation term in the right side of equation (5.9) converges as \( k \to \infty \), this implies that \( \tilde{d}_k, \hat{d}_k \) decay as \( k^{-\beta} \). In particular, for some positive constants \( C_1^\alpha \) and \( C_2^\alpha \),

\[
C_1^\alpha k^{-\beta} \leq \tilde{d}_k, \quad \hat{d}_k \leq C_2^\alpha k^{-\beta}, \quad \forall \quad k \geq 1. (5.10)
\]

Corollary 1.3 follows immediately from (5.6), (5.7) and (5.10).

Sketch of the proof of Corollary 1.4: In the case of \( \alpha = 0 \), the recurrence of \( D_k(t) \) can be derived as

\[
\begin{aligned}
D_k(t+1) &= D_k(t) + \zeta \frac{D_{k-1}(t)}{t} - \frac{D_k(t)}{t} + \bar{A}_k(t) + g_k^0(t), \quad t + 1 \geq k \geq 0, \quad t \geq (\zeta - 1) \lor 1; \\
D_0(1) &= 0, \quad D_1(1) = 2; \quad D_k(t) = 0, k > t \geq 1; \quad D_{t-1}(t) = 0, t \geq 1; \\
D_{k'}(k) &= 0, \quad D_k(k) = (k+1), 0 \leq k' < k, 1 < k < \zeta,
\end{aligned}
\]

where

\[
\bar{A}_k(t) = \frac{\zeta (D_k(t) - D_{k-1}(t))}{t(t+1)}
\]

and \( g_k^0(t) = \bar{f}_k(t) \), which is given in (5.2) with the parameter \( \mu \) replaced by \( \zeta \). Note that the last line in (5.11) comes from the fact \( G_0^k \) is a complete graph while \( t < \zeta \).

It is clear that \( |\bar{A}_k(t)| \leq (2\zeta)/t \) and then satisfies (5.6), i.e., for some \( R_1 > 0 \),

\[
|\bar{A}_k(t)| \leq R_1 t^{-1/5}, \quad \forall \quad t \geq 1, k \geq 0.
\]

For the term \( g_k^0(t) = \bar{f}_k(t) \), we have

\[
\lim_{t \to \infty} \bar{f}_k(t) = \frac{\zeta^k}{k!} e^{-\zeta}, \quad \forall \quad k \geq 0;
\]

on the other hand,

\[
\bar{f}_k(t-1) = \binom{t}{k} \left( \frac{\zeta}{t} \right)^k \left( 1 - \frac{\zeta}{t} \right)^{t-k} \leq \frac{t(t-1) \cdots (t-k+1) \zeta^k}{(t-\zeta)^k} \frac{k!}{k} e^{-\zeta},
\]
for all \( t \geq \zeta \lor 2 \), and \( 1 \leq k \leq t \), this implies that

\[
\hat{f}_k(t) \leq C(0) \frac{\zeta^k}{k!} e^{-\zeta}, \quad \text{for all } k \geq 1 \text{ and } t \geq 1
\]

for some constant \( C(0) > 0 \).

Now, by (5.3), we choose \( \rho > 0 \) satisfying (4.7) and define \( \{\psi_k\}, \{\varphi_k\} \) as

\[
\psi_k = \begin{cases} 
0, & k \geq 1, \\
\rho, & k = 0;
\end{cases} \quad \varphi_k = \begin{cases} 
C(0) \frac{\zeta^k}{k!} e^{-\zeta}, & k \geq 1, \\
e^{-\zeta}, & k = 0.
\end{cases}
\]

Then, Corollary 1.4 follows from the comparing argument used above and the fact that

\[
\sum_{k=0}^{\infty} \left( \frac{1+\zeta}{\zeta} \right)^k \phi_k < \infty
\]

for \( \phi_k = \psi_k \) and \( \varphi_k \) respectively. \( \square \)

**Remark 5.1** To get the degree distribution by the standard argument introduced in [15] and [16], appropriate upper bounds for \( \Delta_t \), the maximum degree, are always necessary. We point out that no bounds for \( \Delta_t \) are used in our proofs of Corollaries 1.3 and 1.4.

### 6 Application to the Hard Copying Model

It is well known that, besides the BA mechanism, copying is another mechanism that may lead to power law degree sequence. The basic idea of copying comes from the fact that a new web page is often made by copying an old one. A kind of copying models was proposed in Kumar et al. [25] to explain the emergence of the degree power laws in the web graphs. These models are parameterized by a *copy factor* \( \alpha \in (0, 1) \) and a constant out-degree \( d \geq 1 \). At each time step, one vertex \( u \) is added and \( d \) out-links are generated for \( u \) as follows. First, an existing vertex \( p \) is chosen uniformly at random; then with probability \( 1 - \alpha \) the \( i^{th} \) out-link of \( p \) is taken to be the \( i^{th} \) out-link of \( u \), and with probability \( \alpha \) a vertex is chosen from the existing vertices uniformly at random to be the destination of the \( i^{th} \) out-link of \( u \). It is proved in [25] that the above copying models possess a power law degree sequence as \( d_k \sim Ck^{-(2-\alpha)/(1-\alpha)} \).

In this section, as an application of the comparing argument, we will introduce a new copying model, here we call it *hard copying* model. Note that another hard copying model is introduced in [28], which is a mixed model of BA mechanism and hard copying mechanism.
For fixed $0 \leq \alpha \leq 1$ and $\mu > 0$, define random graph process $\{\bar{G}_t^\alpha = (V_t, E_t) : t \geq 1\}$ as follows

**Time-Step 1.** Let $\bar{G}_1^\alpha$ consists of vertices $x_0, x_1$ and the edge $\langle x_0, x_1 \rangle$.

**Time-Step $t \geq 2$.**

1. with probability $\alpha$, we generate vertex $x_t$ by copying an existing vertex $x_i$, $0 \leq i \leq t - 1$ from $V_{t-1}$ uniformly at random. Note that in this case, all neighbors of $x_t$ are those of the copied vertex $x_i$;

2. with probability $1 - \alpha$, we add a new vertex $x_t$ to $\bar{G}_{t-1}^\alpha$ and then add random edges incident with $x_t$ in the classical manner: for any $0 \leq i \leq t - 1$, edge $\langle x_i, x_t \rangle$ is added independently with probability $(\mu \wedge t)/t$.

As calculated in [28], for the present model, we have

$$\mathbb{E}(e_t) = \mu t + O(t^{2\alpha}). \quad \text{(6.1)}$$

So $e_t = |E_t|$ increase super-linearly when $\alpha > 1/2$. This makes our model interesting and deferring from the model introduced in [25].

Another fact for the present model is that, in any case of $\alpha$,

$$\Delta_t \leq t, \quad \forall \ t \geq 1, \quad \text{(6.2)}$$

where $\Delta_t$ be the maximum degree of $\bar{G}_t^\alpha$.

In the case of $\alpha = 0$, $\{\bar{G}_t^0 : t \geq 1\}$ is just $\{G_t^0 : t \geq 1\}$ and its degree sequence is given in Corollary [12] (with $\mu$ in place of $\zeta$). In the case of $\alpha = 1$, we get a pure hard copying model and, using (6.2), the recurrence of $\overline{D}_k(t)$ can be derived as

$$\begin{cases}
\overline{D}_k(t+1) = \overline{D}_k(t) + (k-1) \left( \frac{\overline{D}_{k-1}(t)}{t+1} - \frac{\overline{D}_k(t)}{t+1} \right), & t+1 \geq k \geq 1; \\
\overline{D}_1(1) = 2; & \overline{D}_k(t) = 0, \ k > t \geq 1; \quad \overline{D}_0(t) = 0, \ t \geq 1.
\end{cases} \quad \text{(6.3)}$$

By (6.3), it is straightforward to prove by induction (in $t$) that, there exists some $M > 0$ such that

$$\overline{D}_k(t) \leq M(t+1)^{1/2}, \quad \forall \ k \geq 1 \quad \text{(6.4)}$$

for all $t \geq 1$. Thus we obtain a degenerated degree distribution as follows.
Proposition 6.1 For any $K \geq 1$, we have

\[
\lim_{t \to \infty} \frac{\sum_{k=1}^{K} \overline{D}_k(t)}{t+1} = 0; \tag{6.5}
\]

furthermore, for any $\epsilon > 0$, we have

\[
\lim_{t \to \infty} \mathbb{P}\left( \sum_{k=1}^{K} D_k(t) \geq \epsilon (t+1) \right) = 0. \tag{6.6}
\]

For the case of $0 < \alpha < 1$, using (6.2) again, the recurrence of $\overline{D}_k(t)$ can be derived as

\[
\begin{align*}
\overline{D}_k(t+1) &= \overline{D}_k(t) + \left[ \alpha(k-1) + (1-\alpha)\mu \right] \left( \frac{\overline{D}_{k-1}(t)}{t+1} - \frac{\overline{D}_k(t)}{t+1} \right) + (1-\alpha)\overline{f}_k(t), \\
& \quad t+1 \geq k \geq 0, \ t \geq (\mu-1) \lor 1;
\end{align*}
\tag{6.7}
\]

where $\overline{f}_k(t)$ is given in (5.2).

By the comparing argument developed in Section 4, we can solve (6.7) and obtain the following result.

Theorem 6.2 For any $0 < \alpha < 1$ and $\mu > 0$, there exists positive constants $\overline{C}_1^{\alpha}$ and $\overline{C}_2^{\alpha}$ such that

\[
\overline{C}_1^{\alpha} k^{-1/\alpha} \leq \liminf_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t \to \infty} \frac{\overline{D}_k(t)}{t} \leq \overline{C}_2^{\alpha} k^{-1/\alpha} \tag{6.8}
\]

for all $k \geq 1$.

Theorem 6.2 provides an interesting result: in the case of $e_t$ increasing super-linearly, i.e. $\alpha > 1/2$, the model processes power law degree sequence, furthermore, the inverse power lies in interval $(1,2]$, which is never considered in previous literature.

Remark 6.1 We note here that, except for (6.2), no bounds for $e_t$ and $\Delta_t$ are used in our proof of Theorem 6.2. Clearly, (6.2) holds for all models studied in this paper, and (6.2) implies

\[
\overline{D}_k(t) = 0, \ \forall \ k > t \geq 1. \tag{6.9}
\]

In fact, (6.9) is a key evidence to ensure Lemmas 4.1 and 4.2 in the comparing argument.
References

[1] R. Albert and A.-L. Barabási (2002) *Statistical Mechanics of Complex Networks*, Reviews of Modern Physics, 74, pp. 47-97.

[2] R. Albert, A. Barabási and H. Jeong (1999) *Diameter of the World Wide Web*. Nature, 401, pp. 103-131.

[3] W. Aiello, F. R. K. Chung and L. Lu (2002) *Random Evolution in Massive Graphs In Handbook on Massive Data Sets*, edited by James Abello et al., pp. 510-519. Norwood, MA: Kluwer Academic Publishers

[4] L. A. N. Amaral, A. Scala, M. Barthélémy and H. E. Stanley (2000) *Classes of Small-World Networks*, Proc Natl Acad Sci U S A. 2000 October 10; 97: pp. 11149-11152.

[5] B. Bollobás (1998) *Modern Graph Theory* Springer-Verlag New York

[6] B. Bollobás (2001) *Random Graph (second edition)*, Cambridge University Press

[7] A.-L. Barabási and R. Albert (1999) *Emergence of Scaling in Random Networks*, Science 286, pp. 509-512

[8] H. R. Bernard, P. D. Killworth, M. J. Evans, C. McCarty and G. A. Shelley (1988) *Studying Social Relations Cross-Culturally*, Ethnology 27, pp. 155-179

[9] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins and J. Wiener (2000) *Graph Structure in the Web In Proceedings of the 9th International World Wide Web Conference on Computer Networks*, pp. 309-320. Amsterdam: North-Holland Publishing Co.

[10] P. G. Buckley and D. Osthus (2004) *Popularity Based Random Graph Model Leading to a Scale-Free Degree Sequence*, Discrete Mathematics, 282, pp. 53-68.

[11] B. Bollobás and O. Riordan (2002) *Mathematical Results on Scale-Free Random Graphs*. In *Handbook of Graphs and Networks*, pp. 1-34. Berlin: Wiley-VCH.

[12] B. Bollobás and O. Riordan (2003) *Robustness and Vulnerability of Scale-Free Random Graph*, Internet Mathematics 1, pp.1-35
[13] B. Bollobás and O. Riordan (2004) The Diameter of a Scale-Free Random Graph, Combinatorica 4, pp. 5-34.

[14] B. Bollobás, O. Riordan, J. Spencer and G. Tusnády (2001) The Degree Sequence of a Scale-Free Random Graph Process Random Structure and Algorithms, 18, pp. 279-290.

[15] C. Cooper and A. Frieze (2003) A General Model of Undireted Web Graphs. Random Structures and Algorithms, 22, pp. 311-335.

[16] C. Cooper, A. Frieze and J. Vera (2004) Random Deletion in a Scale-Free Random Graph Process. Internet Mathematics 1, pp. 463-483

[17] F. Chung, L. Lu (2004) Coupling Online and Offline Analysis for Random Power Law Graphs Internet Mathematics 1, pp. 409-461

[18] M. E. Dieckmann, I. Lerche, P. K. Shukla and L. O. C. Drury (2007) Aspects of Self-Similar Current Distributions Resulting from the Plasma Filamentation Instability New Journal of Physics 9: Art. No. 10.

[19] S. N. Dorogovtsev and J. E. F. Mendes (2001) Scaling Properties of Scale-Free Evolving Networks: Continuous Approach. Physical Review E 63, 056125.

[20] P. Erdős and A. Rényi (1959) On Random Graphs I, Publicationes Mathematicae Debrecen 5, pp. 290-297.

[21] M. Faloutsos, P. Faloutsos and C. Faloutsos (1999) On Power-Law Relationships of the Internet Topology, In Proceedings of the Conference on Applications, Technologies, Architectures, and Protocols for Computer Communication, pp. 251-262. New York: ACM Press.

[22] E. N. Gilbert (1959) Random Graphs, Annals of Mathematical Statistics 30, pp. 1141-1144.

[23] K. I. Goh, B. Kahng and D. Kim (2005) Nonlocal Evolution of Weighted Scale-Free Networks, Physical Review E 72, 017103.

[24] B. Hayes (2000) Graph Theory in Practice: Part II, American Scientist 88, pp. 104-109.

[25] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins and E. Upfal (2000) Stochastic Models for the Web Graph, In 41st FOCS, pp. 57-65.
[26] S. Lehmann, B. Lautrup and A. D. Jackson (2003) *Citation Networks in High Energy Physics*, Phys. Rev. E (Statistical, Nonlinear, and Soft Matter Physics), 68: 026113

[27] M. E. J. Newman (2003) *The Structure and Function of the Complex Networks*, SIAM Review, 45, pp. 167-256.

[28] Gao-Rong Ning, Xuan-Yuan Wu, and Kai-Yuan Cai (2008) *The Degree Sequence of a Scale-Free Random Graph Process with Hard Copying*, to appear [arXiv:0807.2819v1/math.PR]

[29] A. F. J. V. Raan (2006) *Performance-Related Differences of Bibliometric Statistical Properties of Research Groups: Cumulative Advantages and Hierarchically Layered Networks*, Journal of the American Society for Information Science and Technology, 54, pp. 1919-1935.

[30] A. Scala, L. A.N. Amaral and M. Barthelemy (2001) *Small-World Networks and the Conformation Space of a Short Lattice Polymer Chain*, Europhys. Lett., 55, pp. 594-599.

[31] Xuan-Yuan Wu, Zhao Dong, Ke Liu and Kai-Yuan Cai (2008) *On the Degree Sequence and its Critical Phenomenon of an Evolving Random Graph Process*, to appear [arXiv:0806.4684v1/math.PR]

[32] S. Zhou and R. J. Mondragon (2004) *Accurately Modeling the Internet Topology*, Physical Review E 70, 066108.

[33] T. Zhou, Y.-D. Jin, B.-H. Wang, D.-R. He, P.-P. Zhang, Y. He, B.-B. Su, K. Chen and Z.-Z. Zhang (2005) *A General Model for Collaboration Networks*, [arXiv:cond-mat/0502253v2]