Boundedness of operators generated by fractional semigroups associated with Schrödinger operators on Campanato type spaces via $T1$ theorem

Zhiyong Wang$^1$ · Pengtao Li$^1$ · Chao Zhang$^2$

Received: 8 May 2021 / Accepted: 21 August 2021 / Published online: 6 September 2021
© Tusi Mathematical Research Group (TMRG) 2021

Abstract
Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator, where the nonnegative potential $V$ belongs to the reverse Hölder class $B_q$. By the aid of the subordinative formula, we estimate the regularities of the fractional heat semigroup, $\{e^{-t\mathcal{L}^\alpha}\}_{t>0}$, associated with $\mathcal{L}$. As an application, we obtain the BMO$_x$-boundedness of the maximal function, and the Littlewood–Paley $g$-functions associated with $\mathcal{L}$ via $T1$ theorem, respectively.

Keywords Schrödinger operators · Fractional heat semigroups · $T1$ theorem · Campanato type space

Mathematics Subject Classification 35J10 · 42B20 · 42B30

1 Introduction

In the research of harmonic analysis and partial differential equations, the maximal operators and Littlewood–Paley $g$-functions play an important role and were investigated by many mathematicians extensively. For any integrable function $f$ on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator is defined as...
\[ M(f)(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|dy, \]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \). For \( f \in \text{BMO}(\mathbb{R}^n) \), Bennett–DeVore–Sharpley proved in [1] that \( M(f) \) is either infinite or belongs to \( \text{BMO}(\mathbb{R}^n) \). The boundedness result in [1] can be extended to other maximal operators. For example, let \(-\Delta\) be the Laplace operator: \( \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \). Denote by \( M_\Delta \) and \( g \) the maximal operator and Littlewood–Paley \( g \)-function generated by the heat semigroup \( \{e^{-t(-\Delta)}\}_{t>0} \), respectively, i.e.,

\[
\begin{align*}
M_\Delta(f)(x) &:= \sup_{t>0} |e^{-t(-\Delta)}(f)(x)|; \\
g(f)(x) &:= \left( \int_0^\infty |e^{-t(-\Delta)}(f)(x)|^2 \frac{dt}{t^{n+1}} \right)^{1/2}.
\end{align*}
\]

(1)

Due to the mean value on “large” cubes may be infinite, the \( \text{BMO}(\mathbb{R}^n) \)-boundedness of \( M_\Delta \) or \( g \) holds if \( M_\Delta(f) < \infty \) or \( g(f) < \infty \) for \( f \in \text{BMO}(\mathbb{R}^n) \).

However, if the Laplacian \(-\Delta\) is replaced by other second-order differential operators, the situation becomes different. Consider the Schrödinger \( \mathcal{L} = -\Delta + V \) in \( \mathbb{R}^n \), \( n \geq 3 \), where \( V \) is a nonnegative potential belonging to the reverse Hölder class \( B_q \) for some \( q > n/2 \). Here a nonnegative potential \( V \) is said to belong to \( B_q \) if there exists \( C > 0 \) such that for every ball \( B \),

\[
\left( \frac{1}{|B|} \int_B V^q(x)dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x)dx.
\]

In [1], the authors pointed out that for \( f \in \text{BMO}(\mathbb{R}^n) \) a supremum of averages of \( f \) over “large” cubes may be infinite, see [1, page 610]. In 2005, Dziubański et al. [9] proved the square functions associated with Schrödinger operators are bounded on the \( \text{BMO} \) type space \( \text{BMO}_\mathcal{L}(\mathbb{R}^n) \) related with \( \mathcal{L} \) which is distinguished from the case of \( \text{BMO}(\mathbb{R}^n) \). See also [13, 18] for similar results in the setting of Heisenberg groups and stratified Lie groups.

Let \( H = -\Delta + |x|^2 \) be the harmonic oscillator. In [2], Betancor et al. introduced a \( T1 \) criterion for Calderón–Zygmund operators related to \( H \) on the \( \text{BMO} \) type space \( \text{BMO}_H(\mathbb{R}^n) \). Later, Ma et al. [15] generalized the \( T1 \) criterion to the case of Campanato type spaces \( \text{BMO}_\mathcal{L}^r(\mathbb{R}^n) \) related with \( \mathcal{L} \). As applications, the authors in [15] proved that the maximal operators associated with the heat semigroup \( \{e^{-t\mathcal{L}}\}_{t>0} \) and with the generalized Poisson operators \( \{P^\sigma_t\}_{t>0}(0 < \sigma < 1) \), the Littlewood–Paley \( g \)-functions given in terms of the heat and the Poisson semigroups are bounded on \( \text{BMO}_\mathcal{L}^r(\mathbb{R}^n) \).

Notice that for \( \sigma \in (0, 1) \), the generalized Poisson operator \( \{P^\sigma_t\}_{t>0} \) is expressed as

\[
P^\sigma_t(f)(x) := \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{r^2}{4} - e^{-r\mathcal{L}}(f)} \frac{dr}{r^{\frac{n+1}{2} + \sigma}}.
\]

(2)

Specially, for \( \sigma = 1/2 \), \( \{P^{1/2}_t\}_{t>0} \) is corresponding to the Poisson semigroup \( \{e^{-t\mathcal{L}^{1/2}}\}_{t>0} \) associated with \( \mathcal{L} \). The main purpose of this paper is to derive
the pointwise estimate and regularity properties of the fractional heat semigroup \( \{e^{-tL^\alpha}\}_{t>0}, \alpha > 0 \), to prove the boundedness of the maximal function and the Littlewood–Paley \( g \)-functions generated by \( \{e^{-tL^\alpha}\}_{t>0} \) on \( \text{BMO}^f_L(\mathbb{R}^n) \), \( 0 < \gamma < \min\{2\alpha, \delta_0, 1\} \), via T1 theorem, respectively.

When \( L = -\Delta \), the kernels of the fractional heat semigroup \( \{e^{-t(-\Delta)^\alpha}\}_{t>0} \) can be defined via the Fourier transform, i.e.,

\[
K_{\alpha,t}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi - t|\xi|^{2\alpha}} d\xi. \tag{3}
\]

For \( L = -\Delta + V \) with \( V > 0 \), the kernels of fractional heat semigroups \( \{e^{-tL^\alpha}\}_{t>0}, \alpha \in (0, 1) \), can not be defined via (3). However, for \( \alpha > 0 \), the subordinative formula (cf. [10]) indicates that

\[
K_{\alpha,t}^L(x,y) = \int_0^\infty \eta_t^\alpha(s)K_s^L(x,y)ds, \tag{4}
\]

where \( \eta_t^\alpha(\cdot) \) is a continuous function on \((0, \infty)\) satisfying (19) below. In [12], the identity (4) was applied to estimate \( K_{\alpha,t}^L(\cdot, \cdot) \) via the heat kernel \( K_t^L(\cdot, \cdot) \), see Proposition 2. Specially, for \( \alpha = 1/2 \), the estimates of \( K_{\alpha,t}^L(\cdot, \cdot) \) goes back to those of the Poisson kernel \( P_t^L(\cdot, \cdot) \), see [5, Lemma 3.9].

We point out that, compared with the case of \( \{P_t^\gamma\}_{t>0} \), some new regularity estimates should be introduced to prove the \( \text{BMO}^f_L \)-boundedness of the maximal function and Littlewood–Paley \( g \)-functions generated by \( \{e^{-tL^\alpha}\}_{t>0} \). Let \( E = L^\infty((0, \infty), dr) \). It follows from (2) and the Minkowski integral inequality that

\[
\|P_t^\gamma(f)\|_E \leq C_\sigma \int_0^\infty t^{2\sigma} e^{-\pi t^2} \|e^{-tL^\alpha}(f)\|_E \frac{dr}{r^{1+\sigma}}.
\]

The fact that

\[
\int_0^\infty t^{2\sigma} e^{-\pi t^2} \frac{dr}{r^{1+\sigma}} < \infty
\]

ensures that the \( \text{BMO}^f_L \)-boundedness of the maximal function \( \sup_{t>0} |P_t^\gamma f(x)| \) can be deduced from that of the heat maximal function \( \sup_{t>0} |e^{-tL^\alpha} f(x)| \), see [15, Proposition 4.7]. However, we can see from the identity (4) that this method is not applicable to the case \( \{e^{-tL^\alpha}\}_{t>0} \).

In this paper, we get the following results:

- In Sect. 3.1, let \( V_x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n) \). By the perturbation theory for semigroups of operators, we deduce the pointwise estimates and the Hölder type estimates of the kernels:
In Sect. 3.2, we use (3) to obtain the corresponding estimates for
\[
\begin{cases}
\partial^m_t(K_t(x-y) - K^L_t(x,y)), \\
\n\end{cases}
\]
see Lemmas 8, 10 and Proposition 1, respectively.

- In Sect. 3.2, we use (3) to obtain the corresponding estimates for
\[
\begin{cases}
[K^L_{a,d}(x,y) - K_{a,d}(x-y)], \\
[t^{1/2a} \nabla x(K_{a,d}(x-y) - K^L_{a,d}(x,y))], \\
[t^m \partial^m_t(K_{a,d}(x-y) - K^L_{a,d}(x,y))], \\
\end{cases}
\]
see Propositions 5–10, respectively.

- In Sect. 4, as applications of the regularity estimates obtained in Sect. 3, we use the \(T_1\) criterion established in [15] to prove the boundedness on Campanato type spaces \(BMO_{\gamma}^\alpha(\mathbb{R}^n), 0 < \gamma < \min\{2\alpha, \delta_0, 1\}\), of the following maximal operator and \(g\)-functions:

\[
\begin{align*}
M^t_{\alpha}f(x) &:= \sup_{t > 0} |e^{-tL^\alpha}f(x)|; \\
\delta^t_{\alpha}(f)(x) &:= \left( \int_0^\infty |D_{a,t}^t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}; \\
\tilde{\delta}^t_{\alpha}f(x) &:= \left( \int_0^\infty |\tilde{D}_{a,t}^t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\end{align*}
\]

see Theorems 3–5, respectively, where \(D_{a,t}^t\) and \(\tilde{D}_{a,t}^t\) are the operators with the integral kernels
\[
\begin{cases}
D_{a,t}^t(x,y) := t^m \partial^m_t K^L_{a,d}(x,y), \\
\tilde{D}_{a,t}^t(x,y) := t^{1/2a} \nabla x K^L_{a,d}(x,y),
\end{cases}
\]
respectively.

**Notations** We will use \(c\) and \(C\) to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By \(B_1 \sim B_2\), we mean that there exists a constant \(C > 1\) such that \(C^{-1} \leq B_1/B_2 \leq C\).

### 2 Preliminaries

#### 2.1 Schrödinger operators and function spaces

In this paper, let \(\delta_0 = 2 - n/q\). At first, we list some properties of the potential \(V\) which will be used in the sequel.
Lemma 1 [16, Lemma 1.2]

(i) For $0 < r < R < \infty$,
\[ \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left( \frac{r}{R} \right)^{\delta} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy. \]

(ii) $r^{2-n} \int_{B(x,r)} V(y)dy = 1$ if $r = \rho(x)$, $r \sim \rho(x)$ if and only if $r^{2-n} \int_{B(x,r)} V(y)dy \sim 1$.

Lemma 2 [16, Lemma 1.4]

(i) There exist $C > 0$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^n$,
\[ C^{-1} \rho(x)(1 + |x - y|/\rho(x))^{-k_0} \leq \rho(y) \leq C \rho(x)(1 + |x - y|/\rho(x))^{k_0/(1+k_0)}. \]
In particular, $\rho(y) \sim \rho(x)$ if $|x - y| < C \rho(x)$.

(ii) There exists $l_0 > 1$ such that
\[ \int_{B(x,R)} \frac{V(y)}{|x - y|^{n-2}} dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C \left( 1 + \frac{R}{\rho(x)} \right)^{l_0}. \]

Lemma 3 [7, Corollary 2.8] For every nonnegative Schwarz function $\omega$, there exist $\delta > 0$ and $C > 0$ such that
\[ \int_{\mathbb{R}^n} r^{-n/2} \omega(|x - y|/\sqrt{t}) V(y) dy \leq \begin{cases} C r^{-1}(\sqrt{t}/\rho(x))^\delta, & t < \rho(x)^2; \\ C r^{-1}(\sqrt{t}/\rho(x))^{l_0}, & t \geq \rho(x)^2, \end{cases} \]
where $l_0$ is the constant given in Lemma 2.

It is well known that the classical Hardy space $H^1(\mathbb{R}^n)$ can be defined via the maximal function $\sup_{t > 0} |e^{-t(-\Delta)} f(x)|$ (cf. [17]). In this sense, we can say that the Hardy space $H^1(\mathbb{R}^n)$ is the Hardy space associated with $-\Delta$. Since 1990s, the theory of Hardy spaces associated with operators on $\mathbb{R}^n$ has been investigated extensively. In [6], Dziubański and Zienkiewicz introduced the Hardy space $H^1_L(\mathbb{R}^n)$ related to Schrödinger operators $L$ and obtained the atomic characterization and the Riesz transform characterization of $H^1_L(\mathbb{R}^n)$ via local Hardy spaces. By the aid of Campanato type spaces, the spaces $\bar{H}^p_L(\mathbb{R}^n)$ ($0 < p \leq 1$) were introduced by Dziubański and Zienkiewicz [7]. In recent years, the results of [6, 7] have been extended to other second-ordered differential operators, and various function spaces associated to operators have been established. For further information, we refer the reader to [3, 18–20] and the references therein.

For a Schrödinger operator $L$, let $\{e^{-itL}\}_{t > 0}$ be the heat semigroup generated by $L$ and denote by $K^t_L(\cdot, \cdot)$ the integral kernel of $e^{-itL}$. Because the potential $V \geq 0$, the Feynman-Kac formula implies that
0 \leq K^L_t(x, y) \leq K_t(x - y) := (4\pi t)^{-n/2}e^{-|x-y|^2/(4t)}.

The Hardy type spaces $H^p_L(\mathbb{R}^n)$, $0 < p \leq 1$, are defined as follows (cf. [7]):

**Definition 1** For $0 < p \leq 1$, the Hardy type space $H^p_L(\mathbb{R}^n)$ is defined as the completion of the space of compactly supported $L^1(\mathbb{R}^n)$-functions such that the maximal function

$$M_L(f)(x) := \sup_{t > 0} |e^{-tL}(f)(x)|$$

belongs to $L^p(\mathbb{R}^n)$. The quasi-norm in $H^p_L(\mathbb{R}^n)$ is defined as $\|f\|_{H^p_L} := |M_L(f)|_{L^p}$.

Let $f$ be a locally integrable function on $\mathbb{R}^n$ and $B = B(x, r)$ be a ball. Denote by $f_B$ the mean of $f$ on $B$, i.e., $f_B = |B|^{-1} \int_B f(y)dy$. Let

$$f(B, V) := \begin{cases} f_B, & r < \rho(x); \\ 0, & r \geq \rho(x), \end{cases}$$

where the auxiliary function $\rho(\cdot)$ is defined as

$$\rho(x) := \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, y)} V(y)dy \leq 1 \right\}.$$

**Definition 2** Let $0 < \gamma \leq 1$. The Campanato type space $\text{BMO}^\gamma_L(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ satisfying

$$\|f\|_{\text{BMO}^\gamma_L} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|^{1+\gamma/n}} \int_B |f(x) - f(B, V)|dx \right\} < \infty.$$

The dual space of $H^{n/(n+\gamma)}_L(\mathbb{R}^n)$, $0 \leq \gamma < 1$, is the Campanato type space $\text{BMO}^\gamma_L(\mathbb{R}^n)$ (cf. [14, Theorem 4.5]).

### 2.2 The $T_1$ criterion on Campanato type spaces

We denote by $L^p_L(\mathbb{R}^n)$ the set of functions $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, whose support $\text{supp}(f)$ is a compact subset of $\mathbb{R}^n$.

**Definition 3** Let $0 \leq \beta < n$, $1 < p \leq q < \infty$ with $1/q = 1/p - \beta/n$. Let $T$ be a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad f \in L^p_L(\mathbb{R}^n) \text{ and a.e. } x \not\in \text{supp}(f).$$

We shall say that $T$ is a $\beta$-Schrödinger–Calderón–Zygmund operator with regularity exponent $\delta > 0$ if there exists a constant $C > 0$ such that
\( \frac{C}{|x - y|^{n-\beta}} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-N} \) for all \( N > 0 \) and \( x \neq y \);

(ii) \( |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y - z|^{\delta}}{|x - y|^{n-\beta+\delta}} \) when \( |x - y| > 2|y - z| \).

The following \( T_1 \) type criterions on Campanato type spaces were established by Ma et al. [15].

**Theorem 1** [15, Theorem 1.1] Let \( T \) be a \( \beta \)-Schrödinger–Calderón–Zygmund operator, \( \beta \geq 0, 0 < \beta + \gamma < \min\{1, \delta\} \), with smoothness exponent \( \delta \). Then \( T \) is a bounded operator from \( \text{BMO}^\gamma_L(\mathbb{R}^n) \) into \( \text{BMO}^{\gamma+\beta}_L(\mathbb{R}^n) \) if and only if there exists a constant \( C \) such that

\[
\left( \frac{\rho(x)}{r} \right)^{\gamma} \frac{1}{|B|^{1+\beta/n}} \int_B |T(y) - (T1)_B| dy \leq C
\]

for every ball \( B(x, r), x \in \mathbb{R}^n \) and \( 0 < r \leq \rho(x)/2 \).

When \( \gamma = 0 \), the authors in [15] also proved

**Theorem 2** [15, Theorem 1.2] Let \( T \) be a \( \beta \)-Schrödinger–Calderón–Zygmund operator, \( 0 \leq \beta < \min\{1, \delta\} \), with smoothness exponent \( \delta \). Then \( T \) is a bounded operator from \( \text{BMO}^\gamma_L(\mathbb{R}^n) \) into \( \text{BMO}^{\beta}_L(\mathbb{R}^n) \) if and only if there exists a constant \( C \) such that

\[
\log \left( \frac{\rho(x)}{r} \right) \frac{1}{|B|^{1+\beta/n}} \int_B |T(y) - (T1)_B| dy \leq C
\]

for every ball \( B(x, r), x \in \mathbb{R}^n \) and \( 0 < r \leq \rho(x)/2 \).

**Lemma 4** [15, Remark 4.1] Theorems 1 and 2 can be also stated in a vector-valued setting. If \( Tf \) takes values in a Banach space \( \mathbb{B} \) and the absolute values in the conditions are replaced by the norm in \( \mathbb{B} \), then both results hold.

3 Regularity estimates

3.1 Regularities of heat kernels

By the fundamental solutions of Schrödinger operators, Dziubański and Zienkiewicz proved that the heat kernel \( K^{\mathcal{L}}_t(\cdot, \cdot) \) satisfies the following estimates, see also [11].
Lemma 5

(i) ([6, Theorem 2.11]) For any $N > 0$, there exist constants $C_N, c > 0$ such that

$$|K_t^L(x, y)| \leq C_N t^{-n/2} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$  

(ii) ([8, Theorem 4.11]) Assume that $0 < \delta \leq \min\{1, \delta_0\}$. For any $N > 0$, there exist constants $C_N, c > 0$ such that for all $|h| < \sqrt{t}$,

$$|K_t^L(x + h, y) - K_t^L(x, y)| \leq C_N \left(\frac{|h|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$  

Lemma 6 [7, Proposition 2.16] There exist constants $C, c > 0$ such that for $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|K_t^L(x, y) - K_t^L(x - y)| \leq C \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} t^{-n/2} e^{-c|x-y|^2/t}.$$  

In [5], under the assumption that $V \in B_q, q > n$, Duong et al. obtained the following regularity estimate for the kernel $K_t^L(\cdot, \cdot)$.

Lemma 7 [5, Lemma 3.8] Suppose that $V \in B_q$ for some $q > n$. For any $N > 0$, there exist constants $C > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|\nabla_x K_t^L(x, y)| \leq C t^{-(n+1)/2} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$  

(6)

By the perturbation theory for semigroups of operators,

$$K_t(x - y) - K_t^L(x, y) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(w - x)V(w)K_s^L(w, y)dwds$$

$$= \int_0^{t/2} \int_{\mathbb{R}^n} K_{t-s}(w - x)V(w)K_s^L(w, y)dwds$$

$$+ \int_{t/2}^t \int_{\mathbb{R}^n} K_s(w - x)V(w)K_{t-s}^L(w, y)dwds.$$  

(7)

Similar to [7, Proposition 2.16], we can prove the following lemma.

Lemma 8 Suppose that $V \in B_q$ for some $q > n$. There exist constants $C, c > 0$ such that
\[
|\nabla_x K_t(x-y) - \nabla_x K^\xi_t(x,y)\n| \leq C t^{-(n+1)/2} e^{-c|x-y|^2/t} \min \left\{ \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0}, \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0} \right\}.
\]

**Proof** If \( t \geq \rho(y)^2 \), it is easy to see that
\[
\left| \nabla_x K_t(x-y) - \nabla_x K^\xi_t(x,y) \right| \leq C t^{-(n+1)/2} e^{-c|x-y|^2/t} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0}.
\]

If \( t < \rho(y)^2 \), by (7), we get
\[
\left| \nabla_x K_t(x-y) - \nabla_x K^\xi_t(x,y) \right| = \left| \int_0^t \int_{\mathbb{R}^n} \nabla_x K_{t-s}(x-w) V(w) K^\xi_s(w,y) dw ds \right| \leq I_1 + I_2,
\]
where
\[
\begin{align*}
I_1 & := \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla_x K_{t-s}(x-w)| V(w) K^\xi_s(w,y) dw ds; \\
I_2 & := \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla_x K_{s}(x-w)| V(w) K^\xi_{t-s}(w,y) dw ds.
\end{align*}
\]

For \( I_1 \), it follows from Lemmas 7 and 3 that
\[
I_1 = \int_0^{t/2} \int_{|w-y| < |x-y|/2} |\nabla_x K_{t-s}(x-w)| V(w) K^\xi_s(w,y) dw ds \\
+ \int_0^{t/2} \int_{|w-y| \geq |x-y|/2} |\nabla_x K_{t-s}(x-w)| V(w) K^\xi_s(w,y) dw ds \\
\leq C t^{-(n+1)/2} e^{-c|x-y|^2/t} \int_0^{t/2} \int_{|w-y| < |x-y|/2} V(w) s^{-n/2} e^{-c|w|^2/s} dw ds \\
+ C t^{-(n+1)/2} \int_0^{t/2} \int_{|w-y| \geq |x-y|/2} V(w) s^{-n/2} e^{-c(|w|^2+|x-y|^2)/s} dw ds \\
\leq C t^{-(n+1)/2} e^{-c|x-y|^2/t} \int_0^{t/2} \frac{1}{s} \left( \frac{\sqrt{s}}{\rho(y)} \right)^{\delta_0} ds \\
= C t^{-(n+1)/2} e^{-c|x-y|^2/t} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0}.
\]

Similar to \( I_1 \), for the term \( I_2 \), we can obtain
\[ I_2 \leq Cr^{-\frac{n+1}{2}}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}. \]

It follows from (i) of Lemma 2 that
\[
\frac{\sqrt{t}}{\rho(x)} \leq C\left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{\delta_0} \frac{\sqrt{t}}{\rho(y)} \leq C'e^{|x-y|^2/t}\frac{\sqrt{t}}{\rho(y)},
\]
where \(\epsilon > 0\) is an arbitrary small constant. Hence
\[ I_2 \leq Cr^{-\frac{n+1}{2}}e^{-c|x-y|^2/t}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}. \]

\(\square\)

**Lemma 9** [7, Proposition 2.17] Let \(0 < \delta < \min\{1, \delta_0\}\). For every \(C' > 0\) there exists a constant \(C\) such that for every \(z, x, y \in \mathbb{R}^n, |y-z| \leq |x-y|/4, |y-z| \leq C'\rho(x)\) we have
\[
\left| (K_t^c(x, y) - K_t(x-y)) - (K_t^c(x, z) - K_t(x-z)) \right| \leq C\left(\frac{|y-z|}{\rho(y)}\right)^{\delta} t^{-n/2} e^{-c|x-y|^2/t}.
\]

**Lemma 10** Suppose that \(V \in B_q\) for some \(q > n\). Let \(\delta_1 = 1 - n/q\) and \(0 < \delta' < \delta_1\). For every \(C' > 0\) there exists a constant \(C\) such that for every \(u, x, y \in \mathbb{R}^n, |u| \leq |x-y|/4, |u| \leq C'\rho(x)\) we have
\[
\left| (\nabla_x K_t^c(x, y) - \nabla_x K_t(x-y)) - (\nabla_x K_t^c(x+u, y) - \nabla_x K_t(x+u-y)) \right| \\
\leq C't^{-\frac{n+1}{2}}e^{-c|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta'}. \]

**Proof** We prove this lemma by the same argument as Lemma 8. It is enough to verify that
\[
\left| (\nabla_x K_t^c(x, y) - \nabla_x K_t(x-y)) - (\nabla_x K_t^c(x+u, y) - \nabla_x K_t(x+u-y)) \right| \\
\leq C't^{-\frac{n+1}{2}}e^{-c|x-y|^2/t}\left(\frac{|u|}{\rho(y)}\right)^{\delta_1}, \tag{9}
\]
where \(\epsilon > 0\) is an arbitrary small constant. In fact, under the condition \(|u| < |x-y|/4\), it is easy to see that \(|x-y| \sim |x+u-y|\). We can deduce from Lemma 7 that
\[
\left| \left( \nabla_x K_t^\xi(x, y) - \nabla_x K_t(x - y) \right) - \left( \nabla_x K_t^\xi(x + u, y) - \nabla_x K_t(x + u - y) \right) \right|
\leq \left| \nabla_x K_t^\xi(x, y) \right| + \left| \nabla_x K_t(x - y) \right| + \left| \nabla_x K_t^\xi(x + u, y) \right| + \left| \nabla_x K_t(x + u - y) \right|
\leq Ct^{-(n+1)/2}e^{-c|x-y|^2/t}.
\]

Then, for \( \delta' \in (0, \delta_1) \), it follows from (9) and (10) that
\[
\left| \left( \nabla_x K_t^\xi(x, y) - \nabla_x K_t(x - y) \right) - \left( \nabla_x K_t^\xi(x + u, y) - \nabla_x K_t(x + u - y) \right) \right|
\leq C \left\{ t^{-(n+1)/2}e^{c|x-y|^2/t} \left( \frac{|u|}{\rho(y)} \right)^{\delta'/\delta_1} \left\{ t^{-(n+1)/2}e^{-c|x-y|^2/t} \right\}^{1-\delta'/\delta_1} \right\},
\]
which gives the desired estimate.

Now we prove (9). Since the case for \( |u| \geq \rho(y) \) is trivial, we may assume \( |u| < \rho(y) \). If \( t \leq 2|u|^2 \), the required estimate follows from Lemma 8. Hence we consider the case \( t > 2|u|^2 \) only. Recall that for the classical heat kernel \( K_t(\cdot) \), it holds
\[
\left| \nabla^2_x K_t(x) \right| \leq C t^{-n/2}e^{-c|x|^2/t}.
\]

A direct computation gives
\[
\left| \nabla_x K_t(x + u) - \nabla_x K_t(x) \right| \leq C |u| t^{-n/2-1},
\]
and for \( |u| \leq |x|/2 \),
\[
\left| \nabla_x K_t(x + u) - \nabla_x K_t(x) \right| \leq C |u| t^{-n/2-1}e^{-c|x|^2/t}.
\]

Similar to (8), we split
\[
\left| \left( \nabla_x K_t^\xi(x, y) - \nabla_x K_t(x - y) \right) - \left( \nabla_x K_t^\xi(x + u, y) - \nabla_x K_t(x + u - y) \right) \right| \leq J_1 + J_2,
\]
where
\[
\begin{aligned}
J_1 &:= \int_0^{t/2} \int_{\mathbb{R}^n} \left| \nabla_x K_{t-s}(w - (x + u)) - \nabla_x K_{t-s}(w - x) \right| V(w)K_s^\xi(w, y)dwds, \\
J_2 &:= \int_0^{t/2} \int_{\mathbb{R}^n} \left| \nabla_x K_s(w - (x + u)) - \nabla_x K_s(w - x) \right| V(w)K_{t-s}^\xi(w, y)dwds.
\end{aligned}
\]

For \( J_1 \), if \( t < 2\rho(y)^2 \), using Lemma 3 and (11), we get
\[
J_1 \leq C|u|t^{-n/2-1} \int_0^{t/2} \int_{\mathbb{R}^n} V(w)s^{-n/2}e^{-c|y-w|^2/s} dw ds \\
\leq C|u|t^{-n/2-1} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0} \\
\leq Ct^{-(n+1)/2} \left( \frac{|u|}{\rho(y)} \right)^{\delta_1}.
\]

If \( t \geq 2\rho(y)^2 \), applying Lemmas 3 and 5, we have

\[
J_1 \leq C|u|t^{-n/2-1} \int_0^{t/2} \int_{\mathbb{R}^n} V(w)s^{-n/2}e^{-c|y-w|^2/s} \left( 1 + \frac{\sqrt{s}}{\rho(y)} \right)^{-N} dw ds \\
\leq C|u|t^{-n/2-1} \left\{ \int_0^{\rho(y)^2} \frac{1}{s} \left( \frac{\sqrt{s}}{\rho(y)} \right)^{\delta_0} ds + \int_{\rho(y)^2}^{t/2} \frac{1}{s} \left( \frac{\sqrt{s}}{\rho(y)} \right)^{\delta_0} ds \right\} \\
\leq C|u|t^{-n/2-1} \left( \frac{|u|}{\rho(y)} \right)^{\delta_1},
\]

where \( N \) is chosen large enough satisfying \( N > l_0 \).

To estimate \( J_2 \), we use Lemma 5 and write \( J_2 \leq C(J_{2,1} + J_{2,2} + J_{2,3}) \), where

\[
J_{2,1} := t^{-n/2} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \int_0^{[u]^2} \int_{\mathbb{R}^n} \left| \nabla_x K_s(w - (x + u)) - \nabla_x K_s(w - x) \right| V(w) dw ds; \\
J_{2,2} := t^{-n/2} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \int_0^{[u]^2} \int_{[w-x]<2|u|} \left| \nabla_x K_s(w - (x + u)) - \nabla_x K_s(w - x) \right| V(w) dw ds; \\
J_{2,3} := t^{-n/2} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \int_0^{[u]^2} \int_{[w-x]<2|u|} \left| \nabla_x K_s(w - (x + u)) - \nabla_x K_s(w - x) \right| V(w) dw ds.
\]

Notice that \( \rho(x + u) \sim \rho(x) \) as \(|u| \leq \rho(x)\). It holds

\[
J_{2,1} \leq Ct^{-n/2} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \int_0^{[u]^2} \frac{1}{s^{3/2}} \left( \frac{\sqrt{s}}{\rho(x)} \right)^{\delta_0} ds \\
\leq Ct^{-n/2} [u]^{-1} \left( \frac{|u|}{\rho(y)} \right)^{-1} \left( \frac{\sqrt{t}}{|u|} \right)^{-1} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N+1} \left( \frac{|u|}{\rho(y)} \right)^{\delta_0} \left( \frac{\rho(y)}{\rho(x)} \right)^{\delta_0} \\
\leq Ct^{-(n+1)/2} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N+1} \left( \frac{|u|}{\rho(y)} \right)^{\delta_1} \left( \frac{\rho(y)}{\rho(x)} \right)^{\delta_0}.
\]
where in the last inequality we have used the fact that $\delta_0 = 2 - n/q$. By Lemmas 1 and 2, we apply (11) to get

\[
J_{2,2} \leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2}^{t/2} \int_{|w-x|<2|u|} |u|s^{-n/2-1}V(w)dwds
\]

\[
\leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2}^{t/2} |u|^{n-1}s^{-n/2-1} \left(\frac{|u|}{\rho(x)}\right) \delta_0 ds
\]

\[
\leq Ct^{-n/2} |u|^{-1} \left(\frac{|u|}{\rho(y)}\right) \delta_0 \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \left(\frac{\rho(y)}{\rho(x)}\right) \delta_0
\]

\[
\leq Ct^{-(n+1)/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N+1} \left(\frac{|u|}{\rho(y)}\right) \delta_1 \left(\frac{\rho(y)}{\rho(x)}\right) \delta_0.
\]

For $J_{2,3}$, if $t \leq 2\rho(x)^2$, it can be deduced from Lemma 3 and (12) that

\[
J_{2,3} \leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2}^{t/2} \int_{|w-x|\geq2|u|} |u|s^{-n/2-1}e^{-c|w-x|^2/s}V(w)dwds
\]

\[
\leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2}^{t/2} \frac{1}{s^2} \left(\frac{\sqrt{s}}{\rho(x)}\right) \delta_0 ds
\]

\[
\leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} |u|^{-1} \left(\frac{|u|}{\rho(y)}\right) \delta_0 \left(\frac{\rho(y)}{\rho(x)}\right) \delta_0
\]

\[
\leq Ct^{-(n+1)/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N+1} \left(\frac{|u|}{\rho(y)}\right) \delta_1 \left(\frac{\rho(y)}{\rho(x)}\right) \delta_0.
\]
If $t > 2\rho(x)^2$, then
\[
J_{2,3} \leq Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2} \int_{|w-x| \geq |u|} |\nabla sK_x(w - (x + u)) - \nabla sK_x(w - x)| V(w)dwds
\]
\[
+ Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2} \int_{|w-x| \geq |u|} |u|s^{-n/2-1}e^{-c|w-x|^2/s}V(w)dwds
\]
\[
\leq Ct^{-(n+1)/2} \left(\frac{|u|}{\rho(y)}\right)^{\delta_t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N+1} \left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}
\]
\[
+ Ct^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \int_{|u|^2} \frac{|u|}{s} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{l_0} ds
\]
\[
\leq Ct^{-(n+1)/2} \left(\frac{|u|}{\rho(y)}\right)^{\delta_t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N+1} \left(\frac{\rho(y)}{\rho(x)}\right)^{\delta_0}
\]
\[
+ Ct^{-(n+1)/2} \left(\frac{|u|}{\rho(y)}\right)^{\delta_t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N+1+\delta_0} \left(\frac{\rho(y)}{\rho(x)}\right)^{l_0},
\]
(13)
where in (13) we have used the estimate obtained for $t \leq 2\rho(x)^2$ and Lemma 3 for $s \geq \rho(x)^2$.

By Lemma 2,
\[
\frac{\rho(y)}{\rho(x)} \leq C \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{\epsilon m_0} \leq C e^{c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{m_0},
\]
where $\epsilon > 0$ is an arbitrary small constant. Choosing $N$ large enough in the estimates of $J_{2,1}, J_{2,2}$ and $J_{2,3}$, we obtain (9) and hence Lemma 10 is proved. \(\Box\)

We can obtain the following estimates, which generalize [4, Lemmas 3.7 and 3.8]. We also refer to [13, (57)] for the case $m = 1$ in the setting of Heisenberg groups.

**Proposition 1**

(i) There exist constants $C, c > 0$ such that
\[
|t^m \partial_t^m K^c_t(x,y) - t^m \partial_t^m K_t(x-y)| \leq Ct^{-n/2}e^{-c|x-y|^2/t} \min \left\{\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}, \left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0}\right\}.
\]
(ii) Let $0 < \delta < \min\{1, \delta_0\}$. For every $C' > 0$ there exist constants $C$ and $c$ such that for every $z, x, y \in \mathbb{R}^n, |y - z| \leq |x - y|/4, |y - z| \leq C' \rho(x)$ we have

$$
\left| \left( t^m \partial_t^n K_t^C(x, y) - t^m \partial_t^n K_t(x - y) \right) - \left( t^m \partial_t^n K_t^C(x, z) - t^m \partial_t^n K_t(x - z) \right) \right| 
\leq C \left( \frac{|y - z|}{\rho(y)} \right)^\delta t^{-n/2} e^{-c|x-y|^2/t}.
$$

**Proof** For $t > 0$ and $m \in \mathbb{Z}_+$, define $Q_t^C(x, y) := t^m \partial_t^n K_t^C(x, y)$ and $Q_t(x, y) := t^m \partial_t^n K_t(x - y)$. The proof of (i) is similar to [18, Lemmas 4.10 and 4.11], so we omit the details.

For (ii), by (7), we get

$$
\begin{align*}
(K_t(x + u - y) - K_t^C(x + u, y)) - (K_t(x - y) - K_t^C(x, y))
&= \int_0^{t/2} \int_{\mathbb{R}^n} (K_{t-s}(w - (x + u)) - K_{t-s}(w - x)) V(w) K_t^C(w, y) dw ds \\
&\quad + \int_0^{t/2} \int_{\mathbb{R}^n} (K_s(w - (x + u)) - K_s(w - x)) V(w) K_{t-s}^C(w, y) dw ds.
\end{align*}
$$

Similar to [18, Proposition 4.8], we can use a direct calculus to deduce:

1. If $m$ is even with $m \geq 2$, there exists a sequence of coefficients $\{C_{m,j}\}_{m \geq 2, 2j \leq m/2}$ such that

$$
\begin{align*}
&\frac{d^m}{dt^m} \left\{ \left( K_t(x + u - y) - K_t^C(x + u, y) \right) - \left( K_t(x - y) - K_t^C(x, y) \right) \right\} \\
&= \frac{m+1}{2} (E_1 + E_2) + \sum_{j=2}^{m/2} (C_{m-1,j-1} + C_{m-1,j}) \left( E_{3,1}^j + E_{3,2}^j \right) + E_4 + E_5. \quad (14)
\end{align*}
$$

2. If $m$ is odd with $m \geq 3$, there exists a sequence of coefficients $\{C_{m,j}\}_{m \geq 2, 2j \leq [m/2]}$ such that

$$
\begin{align*}
&\frac{d^m}{dt^m} \left\{ \left( K_t(x + u - y) - K_t^C(x + u, y) \right) - \left( K_t(x - y) - K_t^C(x, y) \right) \right\} \\
&= \frac{m+1}{2} (E_1 + E_2) + E_4 - E_5 + \sum_{j=2}^{[m/2]} (C_{m-1,j-1} + C_{m-1,j}) \left( E_{3,1}^j + E_{3,2}^j \right) \\
&\quad + 2C_{m,[m/2]} \frac{d^{[m/2]}}{dt^{[m/2]}} \left( K_{t/2}(w - (x + u)) - K_{t/2}(w - x) \right) V(w) \frac{d^{[m/2]}}{dt^{[m/2]}} K_{t/2}^C(w, y). \quad (15)
\end{align*}
$$

Here in the above (14) and (15),
We divide such that there exist constants \( Z. \) Wang et al. (64)\[ E < \) for \( |Q| \geq \frac{1}{2} \) |

\[
\begin{align*}
E_1 &= \int_{\mathbb{R}^n} t \left( r^{n-1} \frac{d^{n-1}}{dr^{n-1}} (K_{ij}^c(w - x - u) - K_{ij}^c(w - x)) \right) V(w) K_{ij}^c(w, y) \, dw; \\
E_2 &= \int_{\mathbb{R}^n} t (K_{ij}^c(w - x - u) - K_{ij}^c(w - x)) V(w) \left( r^{n-1} \frac{d^{n-1}}{dr^{n-1}} K_{ij}^c(w, y) \right) \, dw; \\
E_{3,1}^i &= \int_{\mathbb{R}^n} t \left( r^{m-j} \frac{d^{m-j}}{dr^{m-j}} (K_{ij}^c(w - x - u) - K_{ij}^c(w - x)) \right) V(w) \left( r^{m-j} \frac{d^{m-j}}{dr^{m-j}} K_{ij}^c(w, y) \right) \, dw; \\
E_{3,2}^i &= \int_{\mathbb{R}^n} t \left( r^{m-j} \frac{d^{m-j}}{dr^{m-j}} (K_{ij}^c(w - x - u) - K_{ij}^c(w - x)) \right) V(w) \left( r^{m-j} \frac{d^{m-j}}{dr^{m-j}} K_{ij}^c(w, y) \right) \, dw; \\
E_4 &= \int_{0}^{t/2} \int_{\mathbb{R}^n} \left( r^m \frac{d^m}{dr^m} (K_{ij}(w - x - u) - K_{ij}(w - x)) \right) V(w) K_{ij}^c(w, y) \, dw \, dr; \\
E_5 &= \int_{0}^{t/2} \int_{\mathbb{R}^n} (K_{ij}(w - x - u) - K_{ij}(w, x)) V(w) \left( r^m \frac{d^m}{dr^m} K_{ij}^c(w, y) \right) \, dw \, ds.
\end{align*}
\]

Below, for the sake of simplicity, we only estimate \( E_1, E_4, E_5. \) The estimations for \( E_2, E_{3,1}^i, E_{3,2}^j \) are similar, and so we omit the details. By the mean value theorem, we know that there exist constants \( C, c \) such that

\[
|Q_t(x + u) - Q_t(x)| \leq \begin{cases} 
C |u| t^{-(n+1)/2}, & \forall x, u \in \mathbb{R}^n, \ t \in (0, \infty); \\
C |u| t^{-(n+1)/2} e^{-c|x|^2/t}, & |u| \leq |x|/2, \ t > 0.
\end{cases}
\]

We divide \( E_1 \) as \( E_1 \leq E_{1,1} + E_{1,2}, \) where

\[
\begin{align*}
E_{1,1} &= \int_{|w-x| < 2u} \left| Q_{t/2, m-1}(w - (x + u)) - Q_{t/2, m-1}(w - x) \right| V(w) K_{ij}^c(w, y) \, dw; \\
E_{1,2} &= \int_{|w-x| \geq 2u} \left| Q_{t/2, m-1}(w - (x + u)) - Q_{t/2, m-1}(w - x) \right| V(w) K_{ij}^c(w, y) \, dw.
\end{align*}
\]

If \( t < 2\rho(y)^2, \) for \( E_{1,1}, \) by (16), Lemma 3 (i) and Lemma 5, we obtain

\[
E_{1,1} \leq C t^{1-n/2} \int_{|w-x| < 2u} |u| t^{-(n+1)/2} V(w) e^{-c|w-y|^2/t} \, dw
\]

\[
\leq C t^{1-n/2} |u| \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0}
\]

\[
= C t^{-n/2} |u| \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0} \leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right)^{\delta}.
\]

Similar to \( E_{1,1}, \) by (16) again, we get
If \( t > 2\rho(y)^2 \), for \( E_{1,1} \), by (16), Lemma 3 (ii) and Lemma 5, we obtain

\[
E_{1,1} \leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right) \delta_0 \int_{|w-x|<2|u|} |u|^{-(n+1)/2} e^{-c|w-y|^2/t} \, V(w) \, dw
\]

\[
\leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right) \delta_0 \int_{|w-x|<2|u|} |u|^{-(n+1)/2} V(w) \, dw
\]

\[
\leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right) \delta_0.
\]

Similar to \( E_{1,1} \), we can also choose \( N \) large enough such that

\[
E_{1,2} \leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right)^{-N} \int_{|w-x|\geq2|u|} |u|^{-(n+1)/2} V(w) \, dw
\]

\[
\leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right)^{-N} \int_{|w-x|\geq2|u|} \frac{|u|}{t^{3/2}} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{l_0}
\]

\[
\leq C t^{-n/2} \left( \frac{|u|}{\rho(y)} \right)^{-N+l_0} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N+l_0}.
\]

For \( E_4 \) and \( E_5 \), similar to [13, Lemma 10], by Lemma 5 and (16), we can obtain

\[
|E_4 + E_5| \leq C e t^{-n/2} e^{c|x-y|^2/t} \left( \frac{|u|}{\rho(y)} \right)^{\delta},
\]

where \( \epsilon > 0 \) is an arbitrary small constant. Hence Proposition 1 is proved.

\( \square \)

### 3.2 Fractional heat kernels associated with \( \mathcal{L} \)

In the following, we will derive some regularity estimates for the fractional heat kernels related with \( \mathcal{L} \). For \( \alpha \in (0, 1) \), the fractional power of \( \mathcal{L} \), denoted by \( \mathcal{L}^\alpha \), is defined as

\[
\mathcal{L}^\alpha := \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left( e^{-\sqrt{t}f(x)} - f(x) \right) \frac{dt}{t^{1+2\alpha}} \quad \forall f \in L^2(\mathbb{R}^n).
\]

We use the subordinative formula to express the integral kernel \( K_{\alpha,t}(\cdot, \cdot) \) of \( e^{-t\mathcal{L}^\alpha} \) as (cf. [10])
\[ K_{a,t}^{L}(x,y) = \int_{0}^{\infty} \eta_{t}^{a}(s)K_{s}^{L}(x,y)\mathrm{d}s, \]

where \( \eta_{t}^{a}(\cdot) \) satisfies

\[
\begin{align*}
\eta_{t}^{a}(s) &= 1/t^{1/a} \eta_{1}^{a}(s/t^{1/a}); \\
\eta_{t}^{a}(s) &\leq t/s^{1+a} \quad \forall \ s, t > 0; \\
\int_{0}^{\infty} s^{-r} \eta_{t}^{a}(s)\mathrm{d}s &< \infty, \ r > 0; \\
\eta_{t}^{a}(s) &\simeq t/s^{1+a} \quad \forall \ s \geq t^{1/a} > 0.
\end{align*}
\] (19)

By the subordinative formula (4) and Lemma 5, Li et al. [12] proved the following estimates for \( K_{a,t}^{L}(\cdot, \cdot) \).

**Proposition 2** [12, Propositions 3.1 and 3.2] Let \( 0 < \alpha < 1 \).

(i) For any \( N > 0 \), there exists a constant \( C_{N} > 0 \) such that

\[
\left| K_{a,t}^{L}(x,y) \right| \leq \frac{C_{N}t}{(t^{1/a} + |x - y|)^{n+2a}} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-N}.
\]

(ii) Let \( 0 < \delta \leq \min\{1, \delta_{0}\} \). For any \( N > 0 \), there exists a constant \( C_{N} > 0 \) such that for all \( |h| \leq t^{1/a} \),

\[
\left| K_{a,t}^{L}(x+h,y) - K_{a,t}^{L}(x,y) \right|
\leq C_{N}\left( \frac{|h|}{\sqrt{t^{1/a}}} \right)^{\delta} \frac{t}{(\sqrt{t^{1/a}} + |x - y|)^{n+2a}} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-N}.
\]

For the kernels \( \widetilde{D}_{a,t}^{L}(\cdot, \cdot) \) and \( D_{a,t}^{L,m}(\cdot, \cdot), m \in \mathbb{Z}_{+}, t > 0 \), defined by (5), the following regularity estimates were obtained by Li et al. [12].

**Proposition 3** [12, Proposition 3.3] Let \( 0 < \alpha < 1 \).

(i) For every \( N > 0 \), there is a constant \( C_{N} > 0 \) such that

\[
\left| D_{a,t}^{L,m}(x,y) \right| \leq \frac{C_{N}t}{(\sqrt{t^{1/a}} + |x - y|)^{n+2a}} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-N}.
\]

(ii) Let \( 0 < \delta < \min\{2\alpha, \delta_{0}, 1\} \). For every \( N > 0 \), there exists a constant \( C_{N} > 0 \) such that, for all \( |h| < \sqrt{t^{1/a}} \),
\[ \left| D_{a,t}^{L,m}(x+h,y) - D_{a,t}^{L,m}(x,y) \right| \leq C_N \left( \frac{t}{\sqrt{t^{1/\alpha}} (\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}} \right)^N. \]

(iii) There exists a constant \( C_N > 0 \) such that
\[ \left| \int_{\mathbb{R}^n} D_{a,t}^{L,m}(x,y) dy \right| \leq C_N \left( \frac{t}{\sqrt{t^{1/\alpha}} + |x-y|} \right)^N. \]

**Proposition 4** [12, Propositions 3.6, 3.9 and 3.10] Suppose that and \( V \in B_q \) for some \( q > n \).

(i) Let \( \alpha \in (0, 1) \). For every \( N \), there is a constant \( C_N > 0 \) such that
\[ \left| \tilde{D}_{a,t}^c(x,y) \right| \leq C_N t \left( \frac{t}{\sqrt{t^{1/\alpha}} + |x-y|} \right)^N. \]

(ii) Let \( \alpha \in (0, 1) \) and \( \delta_1 = 1 - n/q \). For every \( N > 0 \), there exists a constant \( C_N > 0 \) such that for all \( |h| < \sqrt{t^{1/\alpha}} \),
\[ \left| \tilde{D}_{a,t}^c(x+h,y) - \tilde{D}_{a,t}^c(x,y) \right| \leq C_N \left( \frac{t}{\sqrt{t^{1/\alpha}} + |x-y|} \right)^N. \]

(iii) Let \( \alpha \in (0, 1/2 - n/2q) \). There exists a constant \( C_N > 0 \) such that
\[ \left| \int_{\mathbb{R}^n} \tilde{D}_{a,t}^c(x,y) dy \right| \leq C_N \min \left\{ \left( \frac{t}{\sqrt{t^{1/\alpha}} + |x-y|} \right)^N, \left( \frac{t}{\sqrt{t^{1/\alpha}} + |x-y|} \right)^N \right\}. \]

To establish the \( \text{BMO}_l \)-boundedness of operators via \( T1 \) type theorem, we need the following propositions.

**Proposition 5** There exists a constant \( C > 0 \) such that
\[ \left| K_{a,t}(x,y) - K_{a,t}(x-y) \right| \leq \begin{cases} C \left( \frac{|x-y|}{\rho(x)} \right)^{\delta_0} \frac{t}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \leq |x-y|; \\ C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0} \frac{t}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \geq |x-y|. \end{cases} \]
Proof By the subordinative formula (4) and Lemma 6, we obtain

\[
|K_{a,t}^C(x,y) - K_{a,t}(x-y)| \leq \int_0^\infty \eta_t^a(s)|K_s^C(x,y) - K_s(x-y)|ds \\
\leq C \int_0^\infty \eta_t^a(s) \left( \frac{\sqrt{s}}{\rho(x)} \right) \delta_0 s^{-n/2}e^{-|x-y|^2/s}ds.
\]

On the one hand, letting \( s = t^{1/a}u \), we can get

\[
|K_{a,t}^C(x,y) - K_{a,t}(x-y)| \\
\leq C \int_0^\infty \frac{t}{s^{1+a}} \left( \frac{\sqrt{u}}{\rho(x)} \right) \delta_0 s^{-n/2}e^{-|x-y|^2/s}ds \\
\leq C \int_0^\infty \frac{t}{(t^{1/a}u)^{1+a}} \left( \frac{\sqrt{t^{1/a}u}}{\rho(x)} \right) \delta_0 (t^{1/a}u)^{-n/2}e^{-|x-y|^2/(t^{1/a}u)}t^{1/a}du \\
\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right) \delta_0 t^{-n/(2a)} \int_0^\infty u^{-1-a-n/2+\delta_0/2}e^{-|x-y|^2/(t^{1/a}u)}du.
\]

Applying the change of variables: \(|x-y|^2/(t^{1/a}u) = r^2\), we deduce that

\[
|K_{a,t}^C(x,y) - K_{a,t}(x-y)| \\
\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right) \delta_0 t^{-n/(2a)} \int_0^\infty \left( \frac{|x-y|^2}{t^{1/a}r^2} \right)^{-1-a+\delta_0/2-n/2}e^{-r^2 |x-y|^2/t^{1/a}r^3}dr \\
\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right) \delta_0 |x-y|^{-2a+\delta_0-n}t^{1-\delta_0/(2a)} \int_0^\infty e^{-r^2}r^{2a-\delta_0+n+1}dr \\
\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right) \delta_0 t^{1-\delta_0/(2a)} |x-y|^{2a+n-\delta_0}.
\]

On the other hand, taking \( \tau = s/t^{1/a} \), we obtain
There exists a constant $C > 0$ such that

$$\left| K_{a,x}^t(x,y) - K_{a,y}(x-y) \right| \leq C \int_0^\infty \left( \frac{\sqrt{s}}{\rho(x)} \right)^{\delta_0} s^{-n/2} t^{-1/a} \eta_1^n(s/t^{1/a}) ds$$

$$\leq C \int_0^\infty \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} (t^{1/a} \tau)^{-n/2} \eta_1^n(\tau) d\tau$$

$$\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} t^{-n/2a} \int_0^\infty \eta_1^n(\tau) \tau^{\delta_0/2-n/2} d\tau$$

$$\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} t^{-n/2a}.$$  

If $\sqrt{t^{1/a}} \leq |x-y|$, then

$$\left| K_{a,x}^t(x,y) - K_{a,y}(x-y) \right| \leq C \left( \frac{|x-y|}{\rho(x)} \right)^{\delta_0} \frac{t}{|x-y|^{2a+n}}$$

$$\leq C \left( \frac{|x-y|}{\rho(x)} \right)^{\delta_0} \frac{t}{(|x-y| + \sqrt{t^{1/a}})^{2a+n}}.$$  

If $\sqrt{t^{1/a}} > |x-y|$, we can see that

$$\left| K_{a,x}^t(x,y) - K_{a,y}(x-y) \right| \leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} \frac{t}{m/(2a)+1}$$

$$\leq C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} \frac{t}{(|x-y| + \sqrt{t^{1/a}})^{2a+n}}.$$  

Let $\tilde{D}_{a,x}(\cdot) = t^{1/(2a)} \nabla_x e^{-t(-\Delta)^{a/2}}(\cdot)$. Similar to the proof of Proposition 5, by (4) and Lemma 8, we have

**Proposition 6** There exists a constant $C > 0$ such that

$$|\tilde{D}_{a,x}^t(x,y) - \tilde{D}_{a,y}(x-y)| \leq \begin{cases} C \left( \frac{|x-y|}{\rho(x)} \right)^{\delta_0} \frac{t}{|x-y|^{2a+n}} & , \quad \sqrt{t^{1/a}} \leq |x-y|; \\ C \left( \frac{\sqrt{t^{1/a}}}{\rho(x)} \right)^{\delta_0} \frac{t}{(|x-y| + \sqrt{t^{1/a}})^{2a+n}} & , \quad \sqrt{t^{1/a}} \geq |x-y|. \end{cases}$$

Let $D_{a,x}^m(\cdot) = t^m \partial_t^m e^{-t(-\Delta)^{a/2}}(\cdot)$. We have
Proposition 7 There exists a constant $C > 0$ such that

$$\left| D_{a,t}^{L,m}(x,y) - D_{a,t}^{n}(x-y) \right| \leq \begin{cases} C \left( \frac{|x-y|}{\rho(x)} \right)^{\delta_0} \frac{t}{(|x-y| + \sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \leq |x-y|; \\
C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0} \frac{t}{(|x-y| + \sqrt{t^{1/\alpha}})^{n+2\alpha}}, & \sqrt{t^{1/\alpha}} \geq |x-y|. \end{cases}$$

Proof The proposition can be proved by Proposition 1 and (4). Since the argument is similar to that of Proposition 5, the details is omitted. \qed

Proposition 8 Let $0 < \delta < \min\{2\alpha, \delta_0\}$. For every $C' > 0$ there exists a constant $C$ such that for every $z, x, y \in \mathbb{R}^n, |y - z| \leq |x - y|/4, |y - z| \leq C' \rho(y)$ we have

$$\left| \left( K_{a,t}^{L}(x,y) - K_{a,t}(x-y) \right) - \left( K_{a,t}^{L}(x,z) - K_{a,t}(x-z) \right) \right| \leq C \left( \frac{|y-z|}{\rho(x)} \right)^{\delta} \frac{t}{(\sqrt{t^{1/\alpha}} + |x-y|)^{n+2\alpha}}.$$

Proof By the subordinative formula (4), we can use Lemma 9 to deduce

$$\left| \left( K_{a,t}^{L}(x,y) - K_{a,t}(x-y) \right) - \left( K_{a,t}^{L}(x,z) - K_{a,t}(x-z) \right) \right| \leq \int_0^\infty \eta^a_t(s) \left| \left( K_{a,t}^{L}(x,y) - K_{a,t}(x-y) \right) - \left( K_{a,t}^{L}(x,z) - K_{a,t}(x-z) \right) \right| ds \leq C \int_0^\infty \eta^a_t(s) \left( \frac{|y-z|}{\rho(x)} \right)^{\delta} s^{-n/2} e^{-|x-y|^2/s} ds.$$

On the one hand, taking $s = t^{1/\alpha}u$, we obtain

$$\left| \left( K_{a,t}^{L}(x,y) - K_{a,t}(x-y) \right) - \left( K_{a,t}^{L}(x,z) - K_{a,t}(x-z) \right) \right| \leq \int_0^\infty \frac{Ct}{s^{1+a}} \left( \frac{|y-z|}{\rho(x)} \right)^{\delta} s^{-n/2} e^{-|x-y|^2/s} ds \leq C \int_0^\infty \frac{t}{(t^{1/\alpha}u)^{1+a}} \left( \frac{|y-z|}{\rho(x)} \right)^{\delta} e^{-|x-y|^2/(t^{1/\alpha}u)^{1/\alpha}} du \leq C \left( \frac{|y-z|}{\rho(x)} \right)^{\delta} \int_0^\infty e^{-|x-y|^2/(t^{1/\alpha}u)} u^{-1-a-n/2} du.$$

Let $\frac{|x-y|^2}{(t^{1/\alpha}u)} = r$. We can see that
\[
\left| K_{a,t}^L(x,y) - K_{a,t}(x-y) \right| - \left| K_{a,t}^L(x,z) - K_{a,t}(x-z) \right| \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t^{-n/2}}{\tau^{1/2}} \int_0^\infty e^{-\tau} \left( \frac{|x - y|^2}{t^{1/2}} \right)^{-1-a-n/2} \frac{|x - y|^2}{t^{1/2} r^2} \, dr \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t}{|x - y|^{2a+n}}.
\]

On the other hand, letting \( \frac{s}{t^{1/2}} = \tau \), we obtain
\[
\left| K_{a,t}^L(x,y) - K_{a,t}(x-y) \right| - \left| K_{a,t}^L(x,z) - K_{a,t}(x-z) \right| \\
\leq \int_0^\infty \frac{1}{t^{1/2}} \eta_1^a(s/t^{1/2}) \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t^{-n/2}}{s^{n/2}} \, ds \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \int_0^\infty \eta_1^a(\tau(t^{1/2} \tau)^{-n/2} \, d\tau \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \tau^{-n/2}.
\]

**Case 1:** \( \sqrt{t^{1/2}} \leq |x - y| \). We obtain
\[
\left| K_{a,t}^L(x,y) - K_{a,t}(x-y) \right| - \left| K_{a,t}^L(x,z) - K_{a,t}(x-z) \right| \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t}{(\sqrt{t^{1/2}} + |x - y|)^{n+2a}}.
\]

**Case 2:** \( \sqrt{t^{1/2}} > |x - y| \). We can see that
\[
\left| K_{a,t}^L(x,y) - K_{a,t}(x-y) \right| - \left| K_{a,t}^L(x,z) - K_{a,t}(x-z) \right| \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t}{\rho^{n/2a+1}} \leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t}{(\sqrt{t^{1/2}} + |x - y|)^{n+2a}}.
\]

\( \square \)

**Proposition 9** Let \( 0 < \delta < \min\{2a, \delta_0\} \). For every \( C' > 0 \) there exists a constant \( C \) such that for every \( z, x, y \in \mathbb{R}^n, |y - z| \leq |x - y|/4, |y - z| \leq C' \rho(y) \) we have
\[
\left| D_{a,t}^L(x,y) - D_{a,t}^L(x-y) \right| - \left| D_{a,t}^L(x,z) - D_{a,t}^L(x-z) \right| \\
\leq C \left( \frac{|y - z|}{\rho(x)} \right)^\delta \frac{t}{(\sqrt{t^{1/2}} + |x - y|)^{n+2a}}.
\]
Proof The proof is similar to that of Proposition 8, so we omit the details. \qed

Proposition 10 Suppose that $V \in B_q$ for some $q > n$. Let $0 < \delta' < \delta_1 := 1 - n/q$. For every $C' > 0$ there exists a constant $C$ such that for every $z, x, y \in \mathbb{R}^n$, $|y - z| \leq |x - y|/4$, $|y - z| \leq C'\rho(y)$ we have

$$\left| \left( \widetilde{D}^C_{\alpha, j}(x, y) - \widetilde{D}^C_{\alpha, j}(x, y - z) \right) \right| \leq C \left( \frac{|y - z|}{\rho(x)} \right)^{\delta'} \frac{t}{(\sqrt[\gamma]{t})^n + |x - y|^{n+2\alpha}}.$$  

Proof The proof is similar to that of Proposition 8, so we omit the details. \qed

4 BMO$^\gamma_L$-boundedness via T1 theorem

4.1 Maximal operators for fractional heat semigroups

Definition 4 Let $0 < \gamma \leq 1$. The Campanato type space $\text{BMO}^\gamma_{L,L^\infty((0,\infty),dt)}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ satisfying

$$\|f\|_{\text{BMO}^\gamma_{L,L^\infty((0,\infty),dt)}} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|^{1+\gamma/n}} \int_B \|f(x, t) - f(B, V)\|_{L^\infty((0,\infty),dt)} dx \right\} < \infty.$$  

To prove that $M_0^\alpha$ is bounded from $\text{BMO}^\gamma_L(\mathbb{R}^n)$, $0 < \gamma < \min\{2\alpha, \delta_0, 1\}$, into itself, we give a vector-valued interpretation of the operator and apply Lemma 4. Indeed, it is clear that $M_0^\alpha f = \|e^{-tL^\alpha f}\|_{L^\infty((0,\infty),dt)}$. Hence, it is enough to show that the operator $\Lambda(f) := \{e^{-tL^\alpha f}\}_{t>0}$ is bounded from $\text{BMO}^\gamma_L$ into $\text{BMO}^\gamma_{L,L^\infty((0,\infty),dt)}$. By the spectral theorem, $\Lambda$ is bounded from $L^2(\mathbb{R}^n)$ into $L^2_{L^\infty((0,\infty),dt)}(\mathbb{R}^n)$. The desired result can be then deduced from the following theorem.

Theorem 3 Assume that the potential $V \in B_q$ with $q > n/2$. Let $x, y, z \in \mathbb{R}^n$.

(i) For any $N > 0$, there exists a constant $C_N$ such that

$$\left\| K^C_{\alpha, j}(x, y) \right\|_{L^\infty((0,\infty),dt)} \leq C_N \left( 1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-N}.$$  

(ii) For $|x - y| > 2|y - z|$ and any $0 < \delta < \min\{2\alpha, \delta_0\}$, there exists a constant $C > 0$ such that
\[\left\| K_{a,t}^C(x,y) - K_{a,t}^C(x,z) \right\|_{L^\infty((0,\infty),dt)} + \left\| K_{a,t}^C(y,x) - K_{a,t}^C(z,x) \right\|_{L^\infty((0,\infty),dt)} \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}.\] (20)

(iii) There exists a constant \( C \) such that for every ball \( B = B(x,r) \) with \( 0 < r \leq \rho(x)/2 \),

\[
\log \left( \frac{\rho(x)}{r} \right) \frac{1}{|B|} \int_B \left\| e^{-tL^a} 1(y) - (e^{-tL^a} 1)_B \right\|_{L^\infty((0,\infty),dt)} dy \leq C,
\]
and, if \( \gamma < \min\{2\alpha, 1, \delta_0\} \) then

\[
\left( \frac{\rho(x)}{r} \right) \gamma \frac{1}{|B|} \int_B \left\| e^{-tL^a} 1(y) - (e^{-tL^a} 1)_B \right\|_{L^\infty((0,\infty),dt)} dy \leq C.
\]

**Proof** For (i), from (i) of Proposition 2, we can get

\[
|K_{a,t}^C(x,y)| \leq C_N \min \left\{ \frac{t^{1+N/a}}{|x-y|^{n+2\alpha+2N}}, t^{-n/(2\alpha)} \right\} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-N}.
\]

If \( \sqrt{t^{1/a}} > |x-y| \), then

\[
|K_{a,t}^C(x,y)| \leq \frac{C_N}{(\sqrt{t^{1/a}})^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \leq \frac{C}{|x-y|^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}.
\]

If \( \sqrt{t^{1/a}} \leq |x-y| \), we obtain

\[
|K_{a,t}^C(x,y)| \leq \frac{C_N t^{1+N/a}}{|x-y|^{n+2\alpha+2N}} \left( \frac{\sqrt{t^{1/a}}}{|x-y|} \right)^{-N} \left( \frac{|x-y|}{\sqrt{t^{1/a}}} + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \leq \frac{C_N (\sqrt{t^{1/a}})^{2\alpha+N}}{|x-y|^{n+2\alpha+2N}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \leq \frac{C_N}{|x-y|^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}.
\]

For (ii), from (ii) of Proposition 2, we obtain

\[
|K_{a,t}^C(x,y) - K_{a,t}^C(x,z)| \leq C_N \min \left\{ \frac{t^{1+N/a} |y-z|^\delta}{|x-y|^{2\alpha+n+2N+\delta}}, t^{-n/(2\alpha)} \right\} \left( \frac{|y-z|}{\sqrt{t^{1/a}}} \right)^\delta.
\]
If $\sqrt{t^{1/\alpha}} > |x - y|$, then
\[
\left| K_{a,t}^\alpha(x, y) - K_{a,t}^\alpha(x, z) \right| \leq \frac{C}{|x - y|^\delta} \left( \frac{|y - z|}{|x - y|} \right)^\delta \leq \frac{C|y - z|^\delta}{|x - y|^{\alpha + \delta}}.
\]
If $|x - y| > \sqrt{t^{1/\alpha}}$, we can also get
\[
\left| K_{a,t}^\alpha(x, y) - K_{a,t}^\alpha(x, z) \right| \leq \frac{C|x - y|^{2\alpha + 2n}}{|x - y|^{2\alpha + n + 2n + \delta}} |y - z|^{\delta} \leq \frac{C|y - z|^\delta}{|x - y|^{\alpha + \delta}}.
\]
The symmetry of the kernel $K_{a,t}^\alpha(\cdot, \cdot)$ gives the conclusion of (ii).

For (iii), letting $B = B(x, r)$ with $0 < r \leq \rho(x)/2$, the triangle inequality gives
\[
\left\| e^{-tL_\alpha} 1(y) - (e^{-tL_\alpha} 1)_B \right\|_{L^\infty((0, \infty), dt)} \leq \frac{1}{|B|} \int_B \left\| e^{-tL_\alpha} 1(y) - e^{-tL_\alpha} 1(z) \right\|_{L^\infty((0, \infty), dt)} \, dz.
\]
We estimate $\left\| e^{-tL_\alpha} 1(y) - e^{-tL_\alpha} 1(z) \right\|_{L^\infty((0, \infty), dt)}$. Because $y, z \in B$, $\rho(y) \sim \rho(z) \sim \rho(x)$.

By Proposition 5, we split $|e^{-tL_\alpha} 1(y) - e^{-tL_\alpha} 1(z)| \leq S_1 + S_2$, where
\[
\begin{aligned}
S_1 &:= \int_{\mathbb{R}^n} \left| K_{a,t}^\alpha(y, w) - K_{a,t}^\alpha(z, w) \right| \, dw; \\
S_2 &:= \int_{\mathbb{R}^n} \left| K_{a,t}^\alpha(z, w) - K_{a,t}^\alpha(z, w) \right| \, dw.
\end{aligned}
\]

For $S_1$, if $|y - w| \leq \sqrt{t^{1/\alpha}}$, we obtain
\[
S_1 \leq C \int_{\mathbb{R}^n} \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0} \frac{t}{(\sqrt{t^{1/\alpha}} + |y - w|)^{\alpha + 2n}} \, dw \leq C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0}.
\]
If $|y - w| > \sqrt{t^{1/\alpha}}$, we can see that
\[
\begin{aligned}
S_1 &\leq C \int_{\mathbb{R}^n} \left( \frac{|y - w|}{\rho(x)} \right)^{\delta_0} \frac{t}{(\sqrt{t^{1/\alpha}} + |y - w|)^{\alpha + 2n}} \, dw \\
&\leq \frac{C}{\rho(x)^{\delta_0}} \int_0^\infty \frac{|y - w|^{\delta_0 + n - 1} t}{|y - w| + \sqrt{t^{1/\alpha})^{n + 2n}}} \, |y - w| \, dy \\
&\leq C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0} \int_0^\infty u^{\delta_0 + n - 1} (1 + u)^{-n - 2} \, du \\
&\leq C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0},
\end{aligned}
\]
where we have chosen $\delta_0 < 2\alpha$ since $\delta_0 = 2 - n/q, q > n/2$. The proof of the term $S_2$ is similar to that of the term $S_1$, so we omit it. Then we can get
\[ |e^{-tL^a} 1(y) - e^{-tL^a} 1(z)| \leq C \left( \frac{\sqrt{t^{1/\alpha}}}{\rho(x)} \right)^{\delta_0}, \]

which shows that if \( \sqrt{t^{1/\alpha}} \leq 2r \),

\[ |e^{-tL^a} 1(y) - e^{-tL^a} 1(z)| \leq C \left( \frac{r}{\rho(x)} \right)^{\delta_0}. \]

If \( \sqrt{t^{1/\alpha}} > 2r \), then \( |y - z| \leq 2r < \sqrt{t^{1/\alpha}} \). Hence, Proposition 2 implies that for \( 0 < \delta < \delta_0 \),

\[ |e^{-tL^a} 1(y) - e^{-tL^a} 1(z)| \leq \int_{\mathbb{R}^n} \left| K^c_{a,t}(y,w) - K^c_{a,t}(z,w) \right| dw \]

\[ \leq C \int_{\mathbb{R}^n} \left( \frac{|y - z|}{\sqrt{t^{1/\alpha}}} \right)^{\delta} \frac{t}{(\sqrt{t^{1/\alpha}} + |y - w|)^{n+2\alpha}} dw \leq C \left( \frac{|y - z|}{\sqrt{t^{1/\alpha}}} \right)^{\delta} \leq C \left( \frac{r}{\sqrt{t^{1/\alpha}}} \right)^{\delta}. \]

Therefore, if \( \sqrt{t^{1/\alpha}} > \rho(x) \), (21) gives

\[ |e^{-tL^a} 1(y) - e^{-tL^a} 1(z)| \leq C \left( \frac{r}{\rho(x)} \right)^{\delta}. \]

When \( 2r < \sqrt{t^{1/\alpha}} < \rho(x) \), we have \( |e^{-tL^a} 1(y) - e^{-tL^a} 1(z)| = I + II + III \), where

\[ I := \int_{|w - y| > \rho(y) > 4|y - z|} \left| K^c_{a,t}(y,w) - K^c_{a,t}(y,w) - (K^c_{a,t}(z,w) - K^c_{a,t}(z,w)) \right| dw; \]

\[ II := \int_{4|y - z| < |w - y| < \rho(y)} \left| (K^c_{a,t}(y,w) - K^c_{a,t}(y,w)) - (K^c_{a,t}(z,w) - K^c_{a,t}(z,w)) \right| dw; \]

\[ III := \int_{|w - y| < 4|y - z|} \left| (K^c_{a,t}(y,w) - K^c_{a,t}(y,w)) - (K^c_{a,t}(z,w) - K^c_{a,t}(z,w)) \right| dw. \]

Notice that the estimate (20) is valid for the classical fractional heat kernel. For I, by (20), we can get

\[ I \leq C \int_{|w - y| > \rho(y) > 4|y - z|} \frac{|y - z|^\delta}{|w - y|^{n+\delta}} dw \leq C \left( \frac{r}{\rho(x)} \right)^{\delta}. \]

For II, we apply Proposition 8 and the fact that \( \rho(w) \sim \rho(y) \) in the region of integration to deduce that

\[ II \leq C |y - z|^\delta \int_{4|y - z| < |w - y| < \rho(y)} \frac{t dw}{\rho(w)^{\delta(\sqrt{t^{1/\alpha}} + |w - y|)^{n+2\alpha}}} \leq C \left( \frac{r}{\rho(x)} \right)^{\delta}. \]

For III, since \( |y - z| \leq 2r < \sqrt{t^{1/\alpha}} \), we have \( |w - y| < C \sqrt{t^{1/\alpha}} \). For \( n - \delta_0 > 0 \), by Proposition 5, we obtain
Thus, when \( 2r < \sqrt{t^{1/\alpha}} < \rho(x) \),

\[
\left| e^{-tL^\alpha}1(y) - e^{-tL^\alpha}1(z) \right| \leq C \left( \frac{r}{\rho(x)} \right)^\delta.
\]

Combining the above estimates, we can get

\[
\| e^{-tL^\alpha}1(y) - e^{-tL^\alpha}1(z) \|_{L^\infty((0,\infty),dt)} \leq C \left( \frac{r}{\rho(x)} \right)^\delta. \tag{22}
\]

Therefore, it holds

\[
\log \left( \frac{\rho(x)}{r} \right) \cdot \frac{1}{|B|} \int_B \| e^{-tL^\alpha}1(y) - (e^{-tL^\alpha}1)_B \|_{L^\infty((0,\infty),dt)} \, dy \leq C \left( \frac{r}{\rho(x)} \right)^\delta \log \left( \frac{\rho(x)}{r} \right) \leq C,
\]

which is the first conclusion of (iii).

For the second estimate of (iii), take \( \delta \in (\gamma, \min\{2\alpha, 2 - n/q\}) \). By (22), we have

\[
\left( \frac{\rho(x)}{r} \right)^\gamma \cdot \frac{1}{|B|} \int_B \| e^{-tL^\alpha}1(y) - (e^{-tL^\alpha}1)_B \|_{L^\infty((0,\infty),dt)} \, dy \leq C \left( \frac{r}{\rho(x)} \right)^{\delta - \gamma} \leq C.
\]

\[
\]

\[
\]

### 4.2 Boundedness of the Littlewood–Paley \( g \)-function \( g^L_\alpha \)

Similar to Sect. 4.1, we introduce the following function space:

**Definition 5** Let \( 0 < \gamma \leq 1 \). The Campanato type space \( \text{BMO}^\gamma_{L,L^2((0,\infty),dt/t)}(\mathbb{R}^n) \) is defined as the set of all locally integrable functions \( f \) satisfying

\[
\]
\[ \|f\|_{\text{BMO}^q_{-L^2((0,\infty),dt)}} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|^{1+\gamma/n}} \int_{B} \|f(x,t) - f(B,V)\|_{L^2((0,\infty),dt)} \, dx \right\} < \infty. \]

The functional calculus and the spectral theorem imply that \( g^{\mathcal{L}}_\alpha \) is an isometry on \( L^2(\mathbb{R}^n) \). As before, to get the boundedness of \( g^{\mathcal{L}}_\alpha \) on \( \text{BMO}^q_{\mathcal{L}}(\mathbb{R}^n) \), \( 0 < \gamma < \min\{2\alpha, \delta_0, 1\} \), it is sufficient to prove the following result.

**Theorem 4** Assume that the potential \( V \in B_q \) with \( q > n/2 \). Let \( x, y, z \in \mathbb{R}^n \) and \( N > 0 \).

(i) For any \( N > 0 \), there exists a constant \( C_N \) such that

\[ \left\| D^N_{\alpha,t} (x,y) \right\|_{L^2((0,\infty),\frac{dt}{t^\gamma})} \leq C_N \frac{|x-y|}{|x-y|^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}. \]

(ii) If \( |x-y| > 2|y-z| \) and \( 0 < \delta < \min\{2\alpha, \delta_0, 1\} \), there exists a constant \( C \) such that

\[ \left\| D^m_{\alpha,t} (x,y) - D^m_{\alpha,t} (x,z) \right\|_{L^2((0,\infty),\frac{dt}{t^\gamma})} + \left\| D^m_{\alpha,t} (y,x) - D^m_{\alpha,t} (z,x) \right\|_{L^2((0,\infty),\frac{dt}{t^\gamma})} \leq C \frac{|y-z|}{|x-y|^n+\delta}. \] (23)

(iii) There exists a constant \( C \) such that for every ball \( B = B(x_0,r) \) with \( 0 < r \leq \rho(x)/2 \),

\[ \log \left( \frac{\rho(x)}{r} \right) \frac{1}{|B|} \int_{B} \| t^m \partial_t e^{-t\mathcal{L}^a} 1(y) - (t^m \partial_t e^{-t\mathcal{L}^a} 1)_B \|_{L^2((0,\infty),\frac{dt}{t^\gamma})} \, dy \leq C, \]

and, if \( \gamma < \min\{2\alpha, \delta_0, 1\} \), then

\[ \left( \frac{\rho(x)}{r} \right)^{\gamma} \frac{1}{|B|} \int_{B} \| t^m \partial_t e^{-t\mathcal{L}^a} 1(y) - (t^m \partial_t e^{-t\mathcal{L}^a} 1)_B \|_{L^2((0,\infty),\frac{dt}{t^\gamma})} \, dy \leq C. \]

**Proof** For (i), from Proposition 3, we have

\[ \left\| D^N_{\alpha,t} (x,y) \right\| \leq C_N \min \left\{ \frac{t^{1+n/2}}{|x-y|^{n+2\alpha+2n}}, \frac{t^{-n/2}}{|x-y|^{n+2\alpha+2n}} \right\} \left( 1 + \frac{\sqrt{t^{1/2}}}{\rho(x)} + \frac{\sqrt{t^{1/2}}}{\rho(y)} \right)^{-N}. \]
If $\sqrt{t^{1/a}} \leq |x - y|$, we obtain
\[
\left\| D_{a,t}^{L,m}(x, y) \cdot \chi_{\{t^{1/2a} \leq |x-y|\}} \right\|_{L^2((0, \infty), \frac{dt}{t})}^2 
\leq C_N \int_0^{\frac{|x-y|^{2a}}{t}} \frac{t^{2+2N/a}}{|x-y|^{2a+4N}} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-2N} \frac{dt}{t}
\leq C_N \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} \frac{CN}{|x-y|^{2a}} \int_0^{\frac{|x-y|^{2a}}{t}} \left( \frac{\sqrt{t^{1/a}}}{|x-y|} \right)^{4a+2N} \frac{dt}{t}.
\]

Let $\sqrt{t^{1/a}}/|x - y| = u$. We can see that
\[
\left\| D_{a,t}^{L,m}(x, y) \cdot \chi_{\{t^{1/2a} \leq |x-y|\}} \right\|_{L^2((0, \infty), \frac{dt}{t})}^2 
\leq \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} C_N \int_0^1 u^{a+2N-1} du
\leq \frac{C_N}{|x-y|^{2a}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N}.
\]

If $\sqrt{t^{1/a}} \geq |x - y|$, we can get
\[
\left\| D_{a,t}^{L,m}(x, y) \cdot \chi_{\{t^{1/2a} \geq |x-y|\}} \right\|_{L^2((0, \infty), \frac{dt}{t})}^2 
\leq C_N \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} \int_{\frac{|x-y|^{2a}}{t}}^{\infty} t^{-n/a-1} dt
\leq \frac{C_N}{|x-y|^{2a}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N}.
\]

For (ii), by Proposition 3, we have
\[
\left\| D_{a,t}^{L,m}(x, y) - D_{a,t}^{L,m}(x, z) \right\|_{L^2((0, \infty), \frac{dt}{t})}^2 
\leq \int_0^{\infty} \frac{Ct}{(\sqrt{t^{1/a}} + |x-y|^{2a+4N})^{2\delta}} \left( \frac{|y-z|}{\sqrt{t^{1/a}} + |x-y|^{2a+4N}} \right)^{2\delta} dt
\leq C|y-z|^{2\delta} \int_0^{\infty} \frac{((\sqrt{t^{1/a}})^{2a-2\delta}}{(\sqrt{t^{1/a}} + |x-y|^{2a+4N})} dt.
\]
Let \( \sqrt{t^{1/a}}/|x-y| = u \). We obtain
\[
\left\| D_{a,t}^{\mathcal{L},m}(x,y) - D_{a,t}^{\mathcal{L},m}(x,z) \right\|_{L^2((0,\infty), \frac{dt}{t})}^2 \leq C |y-z|^{2\delta} |x-y|^{-2\delta-2\beta} \int_0^\infty \frac{u^{4\alpha - 2\delta - 1}}{(1 + u)^{2n+4\alpha}} \, du
\]
\[
\leq C |y-z|^{2\delta} |x-y|^{2\beta+2\delta}.
\]

The symmetry of the kernel \( D_{a,t}^{\mathcal{L},m}(\cdot, \cdot) \) gives the conclusion of (ii).

For (iii), let us fix \( y, z \in B = B(x_0, r), 0 < r \leq \rho(x_0)/2 \). Similar to Theorem 3, we must handle
\[
\left\| t^{\beta} \partial_t^\beta e^{-t \mathcal{L}^a} 1(y) - t^{\beta} \partial_t^\beta e^{-t \mathcal{L}^a} 1(z) \right\|_{L^2((0,\infty), \frac{dt}{t})}
\]
\[
= M_1 + M_2 + M_3,
\]
where
\[
\left\{
\begin{array}{l}
M_1 := \int_0^{(2\gamma)^{2\alpha}} \left| \int_{\mathbb{R}^n} \left( D_{a,t}^{\mathcal{L},m}(x,y) - D_{a,t}^{\mathcal{L},m}(x,z) \right) \, dx \right|^2 \frac{dt}{t}; \\
M_2 := \int_{(2\gamma)^{2\alpha}}^{\rho(x_0)^{2\alpha}} \left| \int_{\mathbb{R}^n} \left( D_{a,t}^{\mathcal{L},m}(x,y) - D_{a,t}^{\mathcal{L},m}(x,z) \right) \, dx \right|^2 \frac{dt}{t}; \\
M_3 := \int_{\rho(x_0)^{2\alpha}}^{\infty} \left| \int_{\mathbb{R}^n} \left( D_{a,t}^{\mathcal{L},m}(x,y) - D_{a,t}^{\mathcal{L},m}(x,z) \right) \, dx \right|^2 \frac{dt}{t}.
\end{array}
\right.
\]

Since \( y, z \in B \subset B(x_0, \rho(x_0)) \), it follows that \( \rho(y) \sim \rho(x_0) \sim \rho(z) \). By Proposition 3 (iii),
\[
M_1 \leq C \int_0^{(2\gamma)^{2\alpha}} \left( \frac{t^{1/a}}{\rho(x_0)^{2\delta}} \right)^{2\delta} \, dt \leq C \int_0^{(2\gamma)^{2\alpha}} \left( \frac{t^{1/a}}{\rho(x_0)} \right)^{2\delta} \, dt = C \left( \frac{r}{\rho(x_0)} \right)^{2\delta}.
\]

Also, by Proposition 3(ii),
\[
M_3 \leq C \int_{\rho(x_0)^{2\alpha}}^{\infty} \left( \frac{|y-z|}{\sqrt{t^{1/a}}} \right)^{2\delta} \left| \int_{\mathbb{R}^n} \frac{t}{(t^{1/a} + |x-y|)^{n+2\alpha}} \, dx \right| \frac{dt}{t}
\]
\[
\leq C \int_{\rho(x_0)^{2\alpha}}^{\infty} \left( \frac{|y-z|}{\sqrt{t^{1/a}}} \right)^{2\delta} \frac{dt}{t} \leq C \left( \frac{r}{\rho(x_0)} \right)^{2\delta}.
\]

It remains to estimate the term \( M_2 \). In this case, \( |y-z| \leq 2r \leq \sqrt{t^{1/a}} \leq \rho(x_0) \). Then we can use the methods in Theorem 3 to obtain
\[ M_2 = \int_{(2r)^2} \left| M_{2,1} + M_{2,2} + M_{2,3} \right|^2 \frac{dt}{t}, \]

where

\[
\begin{aligned}
M_{2,1} &:= \int_{|x-y|<\rho(y)>4|y-z|} \left( D_{a,j}^m(x,y) - D_{a,j}^m(x-y) \right) - \left( D_{a,j}^m(x,z) - D_{a,j}^m(x-z) \right) dx; \\
M_{2,2} &:= \int_{4|y-z|<|x-y|<\rho(y)} \left( D_{a,j}^m(x,y) - D_{a,j}^m(x-y) \right) - \left( D_{a,j}^m(x,z) - D_{a,j}^m(x-z) \right) dx; \\
M_{2,3} &:= \int_{|x-y|<4|y-z|} \left( D_{a,j}^m(x,y) - D_{a,j}^m(x-y) \right) - \left( D_{a,j}^m(x,z) - D_{a,j}^m(x-z) \right) dx.
\end{aligned}
\]

For \( M_{2,1} \), similar to prove (23), we can also get

\[
\left| D_{a,j}^m(x,y) - D_{a,j}^m(x,z) \right| \leq \frac{C|y-z|^\delta}{|x-y|^{n+\delta}},
\]

which is valid to \( D_{a,j}^m(\cdot) \). So we obtain

\[
|M_{2,1}| \leq C \int_{|x-y|<\rho(y)>4|y-z|} \frac{|y-z|^\delta}{|x-y|^{n+\delta}} dx \leq C \left( \frac{r}{\rho(x_0)} \right)^\delta.
\]

For \( M_{2,2} \), by Proposition 9 and the fact that \( \rho(x) \sim \rho(y) \) in the region of integration.

\[
|M_{2,2}| \leq C|y-z|^\delta \int_{4|y-z|<|x-y|<\rho(y)} \frac{t}{\rho(x)^\delta(\sqrt{t^{1/n}} + |x-y|)^{n+2\alpha}} dx \leq C \left( \frac{r}{\rho(x_0)} \right)^\delta.
\]

For \( M_{2,3} \), since \( |y-z| \leq 2r < \sqrt{t^{1/n}} \), we have \( |x-y| < C\sqrt{t^{1/n}} \). For \( n - \delta_0 > 0 \), by Proposition 7, we obtain

\[
|M_{2,3}| \leq C \left( \frac{\sqrt{t^{1/n}}}{\rho(x_0)} \right)^\delta_0 \left( \int_{|x-y|<4|y-z|} \frac{rdx}{(\sqrt{t^{1/n}} + |x-y|)^{n+2\alpha}} \right) + \left( \int_{|x-y|<4|y-z|} \frac{rdx}{(\sqrt{t^{1/n}} + |x-z|)^{n+2\alpha}} \right) \leq C \left( \frac{r}{\rho(x_0)} \right)^\delta_0 \leq C \left( \frac{r}{\rho(x_0)} \right)^\delta.
\]

The estimates for \( M_{2,i}, i = 1, 2, 3 \), imply that

\[
M_2 \leq \int_{(2r)^2} \left( \frac{r}{\rho(x_0)} \right)^{2\delta} \frac{dt}{t} = C \left( \frac{r}{\rho(x_0)} \right)^{2\delta} \log \left( \frac{\rho(x_0)}{r} \right).
\]
Finally, we can get
\[
\left\| t^n \partial_t^n e^{-tL^\alpha} 1(y) - t^n \partial_t^n e^{-tL^\alpha} 1(z) \right\|_{L^2((0, \infty), \frac{dx}{t})} \leq C \left( \frac{r}{\rho(x_0)} \right)^\delta \left( \log \left( \frac{\rho(x_0)}{r} \right) \right)^{1/2}.
\]
Thus (iii) readily follows. \(\square\)

### 4.3 Boundedness of Littlewood–Paley \(g\)-function \(\tilde{g}_a^C\)

By the \(L^2\)-boundedness of Riesz transforms \(\nabla_x L^{-1/2}\), we can see that
\[
\left\| \tilde{g}_a^C f \right\|_{L^2}^2 \leq C \int_0^\infty \left( \int_{\mathbb{R}^n} |t^{1/2} L^{1/2} e^{-tL^\alpha} f(x)|^2 \, dx \right) \frac{dt}{t}.
\]
Then by the spectral theorem, we know that \(\tilde{g}_a^C\) is bounded from \(L^2(\mathbb{R}^n)\) into \(L^2(\mathbb{R}^n)\).

**Theorem 5** Assume that the potential \(V \in B_q\) with \(q > n\). Let \(x, y, z \in \mathbb{R}^n\).

(i) For any \(N > 0\), there exists a constant \(C_N\) such that
\[
\left\| \tilde{D}_a^C(x, y) \right\|_{L^2((0, \infty), \frac{dx}{t})} \leq \frac{C_N}{|x - y|^n} \left( 1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-N};
\]

(ii) Let \(|x - y| > 2|y - z|\) and \(0 < \delta < \min\{2\alpha, \delta_1, 1\}\). There exists a constant \(C\) such that
\[
\left\| \tilde{D}_a^C(x, y) - \tilde{D}_a^C(x, z) \right\|_{L^2((0, \infty), \frac{dx}{t})} \leq C \frac{|y - z|\delta}{|x - y|^{\alpha + \delta}};
\]

(iii) There exists a constant \(C\) such that for every ball \(B = B(x_0, r)\) with \(0 < r \leq \rho(x)/2\),
\[
\log \left( \frac{\rho(x)}{r} \right) \frac{1}{|B|} \int_B \left\| t^{1/(2\alpha)} \nabla_x e^{-tL^\alpha} 1(y) - (t^{1/(2\alpha)} \nabla_x e^{-tL^\alpha} 1)B \right\|_{L^2((0, \infty), \frac{dx}{t})} \, dy \leq C,
\]
and, if \(\gamma < \min\{2\alpha, \delta_1, 1\}\) then
\[
\left( \frac{\rho(x)}{r} \right)^{\gamma} \frac{1}{|B|} \int_B \left\| t^{1/(2a)} \nabla_y e^{-t L^a} \mathbf{1}(y) - (t^{1/(2a)} \nabla_y e^{-t L^a} \mathbf{1})_B \right\|_{L^2((0,\infty), \mu_t^a)} dy \leq C.
\]

**Proof** For (i), from Proposition 4, we have

\[
\left| \widehat{D}^L_{a,t}(x,y) \right| \leq C_N \min \left\{ \frac{t^{1+N/a+1/(2a)}}{|x-y|^{n+2a+2N+1}}, t^{-n/(2a)} \right\} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-N}.
\]

If \( \sqrt{t^{1/a}} \leq |x-y| \), we obtain

\[
\left\| \widehat{D}^L_{a,t}(x,y) \cdot \chi_{\{|t^{1/2a}| \leq |x-y|\}} \right\|_{L^2((0,\infty), \mu_t^a)}^2 
\leq C_N \int_0^{|x-y|^{2a}} \frac{t^{2+2N/a+1/a}}{|x-y|^{2a+4a+4N+2}} \left( 1 + \frac{\sqrt{t^{1/a}}}{\rho(x)} + \frac{\sqrt{t^{1/a}}}{\rho(y)} \right)^{-2N} \frac{dt}{t} 
\leq C_N \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} \int_0^{|x-y|^{2a}} \frac{(\sqrt{t^{1/a}})^{4a+2+4N+2}}{|x-y|^{2a+4a+4N+2}} \left( \frac{\sqrt{t^{1/a}}}{|x-y|} \right)^{-2N} \frac{dt}{t} 
\leq \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} C_N \int_0^{|x-y|^{2a}} \frac{\left( \frac{\sqrt{t^{1/a}}}{|x-y|} \right)^{4a+2+4N+2}}{t^{2a}} \frac{dt}{t}.
\]

Let \( \sqrt{t^{1/a}}/|x-y| = u \). We can see that

\[
\left\| \widehat{D}^L_{a,t}(x,y) \cdot \chi_{\{|t^{1/2a}| \leq |x-y|\}} \right\|_{L^2((0,\infty), \mu_t^a)}^2 
\leq \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} C_N \int_0^1 u^{4a+2N+1} du
\leq \frac{C_N}{|x-y|^{2a}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N}.
\]

If \( \sqrt{t^{1/a}} \geq |x-y| \), we can get

\[
\left\| \widehat{D}^L_{a,t}(x,y) \cdot \chi_{\{|t^{1/2a}| \geq |x-y|\}} \right\|_{L^2((0,\infty), \mu_t^a)}^2 
\leq C_N \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N} \int_{|x-y|^{2a}}^\infty t^{-n/a-1} dt
\leq \frac{C_N}{|x-y|^{2a}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-2N},
\]

which proves (i).

For (ii), by Proposition 4, we have
\[
\left\| \tilde{D}_{a,t}^c(x,y) - \tilde{D}_{a,t}^c(x,z) \right\|_{L^2(\mathbb{R}, \frac{\omega}{\pi})}^2 \leq \int_0^\infty \frac{Ct}{(\sqrt{t^{1/a}} + |x-y|^{2a+4})} \left( \frac{|y-z|}{\sqrt{t^{1/a}}} \right)^{2\delta} dt \\
\quad \leq C|y-z|^{2\delta} \int_0^\infty \frac{(\sqrt{t^{1/a}})^{2a-2\delta}}{(\sqrt{t^{1/a}} + |x-y|^{2a+4})} dt.
\]

Let \( \sqrt{t^{1/a}} / |x-y| = u \). We obtain
\[
\left\| \tilde{D}_{a,t}^c(x,y) - \tilde{D}_{a,t}^c(x,z) \right\|_{L^2(\mathbb{R}, \frac{\omega}{\pi})}^2 \leq C|y-z|^{2\delta} |x-y|^{-2\delta} \int_0^\infty \frac{u^{4a-2\delta-1}}{(1+u)^{2n+4\alpha}} du \\
\quad \leq \frac{C|y-z|^{2\delta}}{|x-y|^{2n+2\delta}}.
\]

The symmetry of the kernel \( D_{a,t}^c(\cdot, \cdot) \) gives the desired conclusion of (ii).

For (iii), fix \( y, z \in B = B(x_0, r) \) with \( 0 < r \leq \rho(x_0)/2 \). Similar to Theorem 4, we need to deal with the term
\[
\left\| t^{1/(2a)} \nabla_x e^{-tL^a} 1(y) - t^{1/(2a)} \nabla_x e^{-tL^a} 1(z) \right\|_{L^2(\mathbb{R}, \frac{\omega}{\pi})}^2 = G_1 + G_2 + G_3,
\]
first. Write
\[
\left\| t^{1/(2a)} \nabla_x e^{-tL^a} 1(y) - t^{1/(2a)} \nabla_x e^{-tL^a} 1(z) \right\|_{L^2(\mathbb{R}, \frac{\omega}{\pi})}^2 = G_1 + G_2 + G_3,
\]
where
\[
\begin{align*}
G_1 &:= \int_0^{(2\pi)^2} \int_{\mathbb{R}^n} \left( \tilde{D}_{a,t}^c(x,y) - \tilde{D}_{a,t}^c(x,z) \right) dx \frac{dr}{t}; \\
G_2 &:= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \tilde{D}_{a,t}^c(x,y) - \tilde{D}_{a,t}^c(x,z) \right) dx \frac{dr}{t}; \\
G_3 &:= \int_0^{(2\pi)^2} \int_{\mathbb{R}^n} \left( \tilde{D}_{a,t}^c(x,y) - \tilde{D}_{a,t}^c(x,z) \right) dx \frac{dr}{t};
\end{align*}
\]

Since \( y, z \in B \subset B(x_0, \rho(x_0)) \), then \( \rho(y) \sim \rho(x_0) \sim \rho(z) \). It follows from Proposition 4 (iii) that
\[
G_1 \leq C \int_0^{(2\pi)^2} (\sqrt{t^{1/a}} / \rho(x_0))^{1+2a} \frac{dr}{t} \leq C \int_0^{(2\pi)^2} \left( \frac{\sqrt{t^{1/a}}}{\rho(x_0)} \right)^{2\delta} \frac{dr}{t} = C \left( \frac{r}{\rho(x_0)} \right)^{2\delta}.
\]

Also, we apply Proposition 4 (ii) to deduce that
Then for $G_2$, following the procedure of the treatment for $M_2$ in Theorem 4, we obtain

$$G_2 \leq \int_{(2r)^a} \left( \frac{r}{\rho(x_0)} \right)^{2\delta} \frac{dt}{t} = C \left( \frac{r}{\rho(x_0)} \right)^{2\delta} \log \left( \frac{\rho(x_0)}{r} \right).$$

From the above estimates, we can get

$$\left\| t^{1/(2a)} \nabla_x e^{-t\mathcal{L}^u} 1(y) - t^{1/(2a)} \nabla_x e^{-t\mathcal{L}^u} 1(z) \right\|_{L^2(0,\infty)}^2 \leq C \left( \frac{r}{\rho(x_0)} \right)^{\delta} \log \left( \frac{\rho(x_0)}{r} \right)^{1/2} \log \left( \frac{\rho(x_0)}{r} \right).$$

Thus (iii) readily follows. \qed

Acknowledgements P. Li was financially supported by the National Natural Science Foundation of China (nos. 12071272, ZR2020MA004, ZR2017JL008). C. Zhang was supported by the National Natural Science Foundation of China (no. 11971431), the Zhejiang Provincial Natural Science Foundation of China (Grant no. LY18A010006) and the first Class Discipline of Zhejiang-A(Zhejiang Gongshang University-Statistics).

References

1. Bennett, C., DeVore, R.A., Sharpley, R.: Weak-$L^\infty$ and BMO. Ann. Math. 113, 601–611 (1981)
2. Betancor, J., Crescimbeni, R., Fariña, J., Stinga, P., Torrea, J.: A $T1$ criterion for Hermite–Calderón–Zygmund operators on the $\text{BMO}_g(\mathbb{R}^n)$ space and applications. Ann. Sc. Norm. Sup. Pisa Cl. Sci. 12, 157–187 (2013)
3. Bui, H., Duong, X., Yan, L.: Calderón reproducing formulas and new Besov spaces associated with operators. Adv. Math. 229, 2449–2502 (2012)
4. Chao, Z., Torrea, J.: Boundedness of differential transforms for heat semigroups generated by Schrödinger operators. Can. J. Math. 73, 622–655 (2021)
5. Duong, X., Yan, L., Zhang, C.: On characterization of Poisson integrals of Schrödinger operators with BMO traces. J. Funct. Anal. 266, 2053–2085 (2014)
6. Dziubański, J., Zienkiewicz, J.: Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev. Mat. Iberoam. 15, 279–296 (1999)
7. Dziubański, J., Zienkiewicz, J.: $H^p$ Spaces for Schrödinger operators. Fourier Anal. Relat. Top. Banach Cent. Publ. 56, 45–53 (2002)
8. Dziubański, J., Zienkiewicz, J.: $L_p$ spaces associated with Schrödinger operators with potentials from reverse Hölder classes. Colloq. Math. 98, 5–36 (2003)
9. Dziubański, J., Garrigós, G., Martínez, T., Torrea, J., Zienkiewicz, J.: BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. Math. Z. 249, 329–356 (2005)
10. Grigor’yan, A.: Heat kernels and function theory on metric measure spaces. Contemp. Math. 338, 143–172 (2002)
11. Kurata, K.: An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials. J. Lond. Math. Soc. 62, 885–903 (2000)
12. Li, P., Wang, Z., Qian, T., Zhang, C.: Regularity of fractional heat semigroup associated with Schrödinger operators. Preprint available at arXiv:2012.07234
13. Lin, C., Liu, H.: $\text{BMO}_p(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators. Adv. Math. 228, 1631–1688 (2011)
14. Ma, T., Stinga, P., Torrea, J., Zhang, C.: Regularity properties of Schrödinger operators. J. Math. Anal. Appl. 338, 817–837 (2012)
15. Ma, T., Stinga, P., Torrea, J., Zhang, C.: Regularity estimates in Hölder spaces for Schrödinger operators via $T1$ theorem. Ann. Mat. Pur. Appl. 193, 561–589 (2014)
16. Shen, Z.: $L^p$ estimates for Schrödinger operators with certain potentials. Ann. Inst. Fourier. 45, 513–546 (1995)
17. Stein, E.: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Mathematics Series, vol. 43. Princeton University Press, Princeton (1993)
18. Wang, Y., Liu, Y., Sun, C., Li, P.: Carleson measure characterizations of the Campanato type space associated with Schrödinger operators on stratified Lie groups. Forum Math. 32, 1337–1373 (2020)
19. Yang, D., Yang, D., Zhou, Y.: Localized Campanato-type spaces related to admissible functions on RD-spaces and applications to Schrödinger operators. Nagoya Math. J. 198, 77–119 (2010)
20. Yang, D., Yang, D., Zhou, Y.: Localized BMO and BLO spaces on RD-spaces and applications to Schrödinger operators. Commun. Pure Appl. Anal. 9, 779–812 (2010)