Optimal distillation of quantum coherence with minimal waste of resources

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We present an optimal probabilistic protocol to distill quantum coherence. Inspired by a specific entanglement distillation protocol, our main result yields a strictly incoherent operation that produces one of a family of maximally coherent states of variable dimension from any pure quantum state. We also expand this protocol to the case where it is possible, for some initial states, to avert any waste of resources as far as the output states are concerned, by exploiting an additional transformation into a suitable intermediate state. These results provide practical schemes for efficient quantum resource manipulation.

INTRODUCTION

Over the past three decades quantum entanglement has been identified as one of the main resources that allows to overcome the intrinsic limits of classical information processing in a distributed setting [1]. It is therefore not surprising that entanglement manipulation is often seen as one of the fundamental tasks in the theory of quantum information. In several cases of practical interest, the goal is that of preparing a target state (e.g., maximally entangled) starting either from many i.i.d. copies of the same state [2, 3], or – probabilistically – from a single copy of a known pure state [4–6]. The problem of generating as much entanglement as possible from a given pure state by means of a probabilistic protocol using local operations and classical communication (LOCC) was considered in [7, 8]. Instead of aiming at a single output state, however, one can consider a discrete class of states as targets, namely that formed by all maximally entangled states of any possible local dimension \( q \). The protocol given in [7, 8] always succeeds in producing one of these states, and a failure occurs only when said local dimension takes the “trivial” value \( q = 1 \).

As entanglement of pure states is one of the manifestations of the superposition principle, one can more fundamentally regard the phenomenon of coherent superposition as a valuable resource in its own right. Quantum coherence plays in fact an essential role in applications to quantum algorithms, quantum metrology, and quantum biology [9]. To deal with this point of view, a resource theory of quantum coherence has been recently established [9–12]. Coherence distillation is a central task in the resource theory of quantum coherence, and is a subject of very active current investigation [13–16].

In this paper, we introduce an explicit protocol for coherence distillation via a single strictly incoherent operations where we originally have a \( d \)-level coherent input state, see Fig. 1. This strategy is a counterpart to the entanglement distillation given in [7, 8]. One of the most significant points of this single-step strategy, when we compare it to some common distillation protocols [4, 7, 17, 18], is that we can have any of all \( q \)-level \( (q = 2, 3, \ldots, d) \) maximally coherent pure states at the end of the measurement process. When compared with the previously available protocols [4], we see that the failure probability is thus relatively small, and a useful coherent state is almost always produced, unless the incoherent outcome \( (q = 1) \) is obtained. In particular, our protocol is optimal with respect to the distillation of \( d \)-level maximally coherent states, as the associated probability of success is maximal. We complement our analysis with a quantification of the coherence loss on average in our protocol, and comment on how and for which input states it is possible to modify our strategy, to avoid any waste of resources and always output a state with nonzero coherence.

OPTIMAL DISTILLATION PROTOCOL

To start with, we need to recall the basis-dependent notions of incoherent and coherent states followed by incoherent operations. Quantum states that are diagonal with respect to a fixed

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Fig. 1. Our strategy solves the coherence distillation problem as follows. We originally have a \( d \)-level coherent pure state \( |\psi\rangle \). We perform a strictly incoherent operation (SIO) on the particle and obtain any of all \( q \)-level \((q = 2, 3, \ldots, d)\) maximally coherent states \( |\Psi_q\rangle\), or an incoherent state \((q = 1)\). The explicit quantum operation used in the protocol is described in the main text.
orthonormal basis \( \{|i\rangle\}_{i=1,2,...,d} \) are defined as incoherent, and they constitute a set labeled by \( \mathcal{I} \). All incoherent states \( \rho \in \mathcal{I} \) are of the form

\[
\rho = \sum_{i=1}^{d} p_i |i\rangle \langle i|,
\]

where \( p_i \in [0,1] \) and \( \sum p_i = 1 \). In addition to this, a finite \( d \)-dimensional pure coherent state is given by

\[
|\psi\rangle = \sum_{j=1}^{d} e^{i\theta_j} |j\rangle, \quad (0 \leq \theta_j \leq \pi), \tag{2}
\]

where \( \{|j\rangle\}_{j=1,2,...,d} \) are non-negative real numbers, arranged in descending order \( |j| \geq |j+1| \geq 0 \), and satisfying \( \sum_{j=1}^{d} |j\rangle \langle j| = 1 \). Here, without loss of generality, we can take \( \theta_j = 0 \) as all these complex phases also can be eliminated by diagonal unitaries, e.g., \( \sum_{j=1}^{d} e^{-i\theta_j} |j\rangle \langle j| \). Throughout this paper we will take as input the state in Eq. (2) without complex phases, i.e., \( |\psi\rangle = \sum_{j=1}^{d} |j\rangle \).

We will focus on particular quantum operations for which measurement outcomes are retained as stated in [11]. These quantum operations are defined by Kraus operators \( \{K_j\} \) that map incoherent states into incoherent states, i.e., such that \( \sum K_j^\dagger K_j = I \) and, for all \( i \) and \( \rho \in \mathcal{I} \),

\[
\rho \rightarrow \rho_i = \frac{K_i \rho K_i^\dagger}{\text{Tr}[K_i \rho K_i^\dagger]} \in \mathcal{I}. \tag{3}
\]

Operations of the form as in Eq. (3) in which the Kraus operators satisfy the above condition are known as incoherent operations (IO) and can be adopted as the free operations in the context of the resource theory of coherence as defined in [11]. A relevant subset of IO is constituted by strictly incoherent operations (SIO), which are completely positive trace preserving maps whose Kraus operators \( K_i \) satisfy both \( K_i K_i^\dagger \leq I \) and \( K_i^\dagger K_i \subseteq \mathcal{I} \). We will demonstrate that, although SIO have a very limited coherence distillation power when mixed input states are concerned [16], they nonetheless suffice for our distillation protocol with pure input states.

We are now ready to analyze the task of coherence distillation, i.e., the probabilistic transformation of the input states given in Eq. (2) into maximally coherent ones via SIO in the asymptotic setting. The state given in Eq. (2) is a \( d \)-level maximally coherent state for \( \{|i\rangle\}_{i=1,2,...,d} = \{ \frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}} \} \), i.e.,

\[
|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle. \tag{4}
\]

An optimal local conversion strategy of bipartite entangled pure states was proposed by Vidal [8]. Adapting those results to the case of coherence distillation, one can obtain the maximal probability of transforming the coherent state \( |\psi\rangle \) in Eq. (2) to the maximally coherent state \( |\Psi_d\rangle \) in Eq. (4), which is given by

\[
p(\{\psi\} \rightarrow |\Psi_d\rangle) = \min_{k \in [1,d]} \left\{ \frac{d \sum_{i=k}^{d} |\psi\rangle^2}{d - k + 1} \right\} = d |\psi|^2. \tag{5}
\]

We construct the explicit Kraus operators to implement the transformations

\[
\sum_{i=1}^{d} \psi_i |i\rangle \xrightarrow{\text{SIO}} \left\{ \left( p_j, \frac{\sum_{i=1}^{d-j+1} |\psi\rangle^2}{\sqrt{d-j+1}} \right) \right\}_{j=1,2,...,d}, \tag{6}
\]

by means of SIO as defined above. These are given by

\[
K_j = \sqrt{p_j} \left( \sum_{i=1}^{d-j+1} |i\rangle \langle i| / \sqrt{d-j+1} \right), \tag{7}
\]

where

\[
p_1 := d |\psi|^2, \quad p_j := (d - j + 1) (|\psi|^2 d_{j+1} - |\psi|^2 d_{j+2}) , \quad j = 2,\ldots,d. \tag{8}
\]

Note that the operation identified by the above Kraus operators is not only incoherent but also strictly incoherent. Observe further that the above Kraus operators satisfy the normalization condition \( \sum_{j=1}^{d} K_j^\dagger K_j = I_d \), implying that they define a legitimate quantum channel. By construction, we have that

\[
K_j |\psi\rangle = \sqrt{p_j} |\Psi_{d-j+1}\rangle, \quad j = 1,\ldots,d, \tag{9}
\]

i.e., such channel implements the transformation in Eq. (6). Observe that the success probability of the transformation \( |\psi\rangle \rightarrow |\Psi_d\rangle \), denoted by \( p_1 \) and given by Eq. (8), achieves its maximal value as given by Eq. (5) (see also [5]). It is not difficult to verify that the probabilities in Eq. (9) correctly satisfy the completeness relation, that is \( \sum_{j=1}^{d} p_j = 1 \). As a result, we initially have the coherent state \( \sum_{i=1}^{d} \psi_i |i\rangle \), and after a single-step measurement process with the given Kraus operators in Eq. (7) we obtain any of all \( q \)-level \( (q = 2,\ldots,d) \) maximally coherent states with finite probabilities, see Eq. (9) and Eq. (8). This ensures minimal waste of resources in the distillation protocol, as a useful (albeit of smaller dimension) maximally coherent state is obtained even when the desired outcome is not recorded. Such a feature is explored in more quantitative detail in the following subsection.

**COHERENCE LOSS**

While we know that the degree of coherence can not increase under IO defined in Eq. (3), when quantified by suitable coherence monotones [9], one may wonder how much coherence is lost on average during our protocol. We adopt the \( l_1 \) norm of coherence [11], a proper quantifier of coherence fulfilling strong monotonicity under IO, for this study. The \( l_1 \) norm of coherence of the state \( |\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle \) is given by

\[
C_{l_1} (|\psi\rangle) = \left( \sum_{i=1}^{d} |\psi_i|^2 \right)^{1/2} - 1, \tag{10}
\]

and the \( l_1 \) norm of coherence of the (maximally coherent) state \( |\Psi_{d-j+1}\rangle = \sum_{i=1}^{d-j+1} |i\rangle \) is given by

\[
C_{l_1} (|\Psi_{d-j+1}\rangle) = d - j, \quad (j = 1,2,\ldots,d-1). \tag{11}
\]
Combining Eq. (8) with Eq. (11), we can obtain the average coherence for the output ensemble, given by
\[
\tilde{C}_l(\rho_{\text{out}}) = \sum_{j=1}^{d} p_j C_l(\rho_{d-j+1}) = 2 \sum_{i=1}^{d} (i-1) \psi_i^2,
\]
where $|\Psi_1\rangle = |1\rangle$ is an incoherent state, and therefore, $C_l\left(\rho_{\Psi_1}\right) = 0$. Monotonicity (under selective IO on average) yields that $C_l\left(\rho_{|\Psi\rangle}\right) \geq \tilde{C}_l\left(\rho_{\text{out}}\right)$, i.e., \(\sum_{i=1}^{d} \psi_i^2 - 1 \geq 2 \sum_{i=1}^{d} (i-1) \psi_i^2\). Thus, the average loss of coherence for our protocol is found to be
\[
C_l\left(\rho_{|\Psi\rangle}\right) - \tilde{C}_l\left(\rho_{\text{out}}\right) = \left(\sum_{i=1}^{d} \psi_i\right)^2 - 2 \sum_{i=1}^{d} i \psi_i^2 + 1. \tag{13}
\]

The quantity in Eq. (13) obviously vanishes when the input is already a $d$-dimensional maximally coherent state, in which case the protocol leaves it invariant with certainty. On the other hand, it can be interesting to investigate classes of states for which there is a large loss of coherence on average during the distillation protocol. One such a class is given by what we may refer to as ‘harmonic power states’, namely, input states $|\psi\rangle$ with coefficients
\[
\psi_i = \frac{1}{i^\alpha \sqrt{H_d^{(2\alpha)}}}, \tag{14}
\]
where $H_d^{(2\alpha)}$ is the $d$th harmonic number of order $2\alpha$, $H_d^{(2\alpha)} = \sum_{i=1}^{d} 1/i^{2\alpha}$, with $\alpha \in [0, \infty)$. These states nearly achieve the minimal $C_l\left(\rho_{\text{out}}\right)$ for a given $C_l\left(\rho_{|\Psi\rangle}\right)$, as plotted in Fig. 2 for dimension $d = 4$.

**NO WASTE OF RESOURCES**

The above strategy can be improved so as to avoid complete waste of resources with certainty, provided that the initial state satisfies some mild assumptions on the amount of coherence it contains. Namely, if $\psi_i^2 \leq 1/2$ it is possible to modify the described protocol in such a way as to make $p_d = 0$, where $p_d$ is the probability of outputting an incoherent state (the case $j = d$ in (9)). This can be accomplished by first transforming $|\psi\rangle$ into an appropriate intermediate state $|\chi\rangle$, and by finally applying the original protocol to $|\chi\rangle$. The required state $|\chi\rangle$ takes the form
\[
|\chi\rangle = \psi_1 \sum_{i=1}^{k} |i\rangle + \psi_{k+1}^2 |k+1\rangle + \sum_{i=k+2}^{d} \psi_i |i\rangle, \tag{15}
\]
where $k$ any integer such that $k \psi_1^2 + \psi_{k+1}^2 + \sum_{i=k+2}^{d} \psi_i^2 = 1$ is satisfied for some $\psi_{k+1}^2$ subjected to the constraints $\psi_1 \geq \psi_{k+1}^2 \geq \psi_{k+2} \geq \cdots \psi_d > 0$. Using the results in [19], we know that the transformation $|\psi\rangle \rightarrow |\chi\rangle$ can be performed deterministically. After attaining the temporary state $|\chi\rangle$, we apply the protocol defined by Eq. (7), which outputs the ensemble $|\chi\rangle \rightarrow \{(|p_j|, |\Psi_{d-j+1}\rangle)\}_{j=1,2,\ldots,(d-k+1)}$. The entire transformation is then given by
\[
|\psi\rangle \rightarrow |\chi\rangle \rightarrow \left\{ \left( p_j, \frac{\sum_{i=1}^{d-j+1} |i\rangle}{\sqrt{d-j+1}} \right) \right\}_{j=1,2,\ldots,(d-k+1)}. \tag{16}
\]

Let us discuss a simple example of the above procedure. Consider the initial state $|\psi\rangle = \sqrt{0.35} |1\rangle + \sqrt{0.3} |2\rangle + \sqrt{0.25} |3\rangle + \sqrt{0.1} |4\rangle$ in dimension $d = 4$. We can transform this into the temporary state $|\chi\rangle = \sqrt{0.35} |1\rangle + \sqrt{0.35} |2\rangle + \sqrt{0.2} |3\rangle + \sqrt{0.1} |4\rangle$ (k = 2) with unit probability. Then, the protocol in Eq. (7) yields the states $|\Psi_4\rangle$, $|\Psi_3\rangle$ and $|\Psi_2\rangle$ with probabilities 0.4, 0.3 and 0.3, respectively. Ultimately, by the help of a proper intermediate state $|\chi\rangle$ given in Eq. (15), we can obtain an ensemble of maximally coherent $q$-level ($q = 2, 3, \ldots, d$) states, guaranteeing that some coherence is always produced.

As one can easily notice, this strategy can also be adapted to the entanglement distillation by local operations and classical communication. For the initial bipartite pure entangled state $|\phi\rangle = \sum_{i=1}^{d} \phi_i |i\rangle$ (Schmidt coefficients are ordered in descending order as usual), provided that $\phi_i^2 \leq 1/2$ one can find an intermediate state $|\varphi\rangle$ such that
\[
|\varphi\rangle = \phi_1 |1\rangle + \phi_2 |2\rangle + \phi_3 |3\rangle + \sum_{i=4}^{d} \phi_i |i\rangle, \tag{17}
\]
where $\phi_1 \geq \phi_2 \geq \phi_3 \geq \cdots \phi_d$. Then, analogously to coherence distillation, using the results in [19] we can obtain the transformation $|\phi\rangle \rightarrow |\varphi\rangle$ deterministically in order to avoid generating a separable output with certainty. The complete
transformation is then given by

$$|\phi\rangle \rightarrow |\phi\rangle \rightarrow \left\{ (p_j, |\Phi_{d-j+1}\rangle) \right\}_{j=1,2,\ldots,(d-1)}, \quad (18)$$

where \( |\Phi_{d-j+1}\rangle = \frac{1}{\sqrt{d-j+1}} \sum_{i=1}^{d-j+1} |ii\rangle \) and the probability of obtaining the separable state \( |\Phi_1\rangle = |1\rangle \) is equal to zero. Here, while the probability of getting \( |\Phi_2\rangle \) and \( |\Phi_0\rangle \) decreases \((p_{d-2} = 3|\phi_1^2 - \phi_2^2|)\) and increases \((p_{d-1} = 2|\phi_1^2 - \phi_2^2|)\), respectively, the other probabilities \( p_{d-m+1} \) of getting \( |\Phi_m\rangle \) \((m = 4, 5, \ldots, d)\) remain unchanged. It is always possible to find an intermediate state \( |\phi_i\rangle \) of the form \([17]\) for the initial states such that \( \phi_3^2 - \phi_4^2 \geq \phi_1^2 - \phi_2^2 \). Therefore, if the initial entangled bipartite states \( \sum_{i=1}^{d} |i\phi_i\rangle \) satisfy this relation, our results ensure that both the transformation given in Eq. (18) can be implemented and hence that no waste of entanglement resources is achieved when only the set of the output states are considered.

**CONCLUSION**

In this paper, we presented a simple, practical and efficient strategy for optimal asymptotic distillation of quantum coherence from pure input states of arbitrary dimension. The key advantage of our protocol lies in its ability to provide a single map to obtain all \( q \)-level \((q = 2, 3, \ldots, d)\) maximally coherent pure states starting from a \( d \)-level coherent input pure state, as illustrated in Fig. [1]. In this way, useful degrees of coherence resource are “recycled” even when the maximally resourceful \( d \)-dimensional state is not produced.

The probability of success, defined by the outcome \( q = d \), is maximal, confirming optimality of the protocol. On the other hand, our protocol only fails when the trivial outcome \( q = 1 \) is obtained, in which case no resource is distilled. This makes our protocol preferable to conventional distillation protocols such as the one in [4], which has instead a higher failure probability and produces no useful output in case the desired maximally resourceful output is not obtained. We furthermore showed how to modify the protocol into a two-step strategy which completely nullifies the failure probability, leading to no waste of coherence in the outputs; this is possible for a subclass of input states that we characterize. Our strategy can also be adapted to entanglement distillation.

A further generalization of our scheme and of the seminal works in [7] to other quantum resource theories [20], beyond coherence and entanglement, would be a worthwhile direction for future investigation.

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