ON THE EXISTENCE OF NON-GEOMETRIC SECTIONS OF ARITHMETIC FUNDAMENTAL GROUPS

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Abstract. We show the existence of group-theoretic sections of the "étale-by-geometrically abelian" quotient of the arithmetic fundamental group of hyperbolic curves over p-adic local fields relative to a proper and flat model which are non-geometric, i.e., which do not arise from rational points.

§0. Introduction/Statement of the Main Result. This note is motivated by the p-adic analog of the Grothendieck anabelian section conjecture, which asks if group-theoretic sections of arithmetic fundamental groups of hyperbolic curves over p-adic local fields all arise from rational points. (See [Saïdi] for more details on the statement of this conjecture). For the time being it is not known if this conjecture holds or not (cf. loc. cit. for a conditional proof of this conjecture).

Recently, examples were found of group-theoretic sections of certain quotients of arithmetic fundamental groups of curves over p-adic local fields which are non-geometric, i.e., which do not arise from rational points. Hoshi has provided examples of sections of the geometrically pro-p quotient of arithmetic fundamental groups of curves over p-adic local fields which are non-geometric (cf. [Hoshi]). (Actually, Hoshi’s example arises from group-theoretic sections of geometrically pro-p fundamental groups of hyperbolic curves over number fields (cf. loc. cit.)). In [Saïdi1] we provided examples of group-theoretic sections of geometrically prime-to-p fundamental groups of hyperbolic curves over p-adic local fields which are non-geometric (cf. loc. cit. §3). The existence of these examples is crucial for our understanding of the p-adic section conjecture. Indeed, if the p-adic version of the section conjecture holds true then it may possibly hold true even for smaller quotients of the arithmetic fundamental group, and one would like to know these quotients in this case. On the other hand, more elaborate examples of non-geometric sections as above may lead to a counterexample for the p-adic version of the section conjecture.

In this note we provide examples of sections of certain quotients of arithmetic fundamental groups of curves over p-adic local fields which are non-geometric. (Our examples, including the quotients of arithmetic fundamental groups involved, are quite different from those considered in [Hoshi]). The quotients of arithmetic fundamental groups we consider are roughly speaking the "étale-by-geometrically abelian" quotient of the arithmetic fundamental group of a curve over a p-adic local field relative to a proper and flat model over the ring of integers of the base field (cf. the discussion before Theorem A for a precise definition of this quotient).

Next, we fix some notations and state our main results. Let

\[ 1 \to H' \to H \to G \to 1 \]
be an exact sequence of profinite groups. We will refer to a continuous homomorphism $s : G \to H$ such that $\text{pr} \circ s = \text{id}_G$ as a (group-theoretic) **section** of the above sequence, or simply a section of the projection $\text{pr} : H \to G$.

In this paper $p \geq 2$ is a prime integer. Let $k$ be a $p$-adic local field; i.e., $k/\mathbb{Q}_p$ is a finite extension, with ring of integers $\mathcal{O}_k$, and residue field $F$. Thus, $F$ is a **finite field** of characteristic $p$. Let $k$ be a $p$-adic local field; i.e., $k/\mathbb{Q}_p$ is a finite extension, with ring of integers $\mathcal{O}_k$, and residue field $F$. Thus, $F$ is a finite field of characteristic $p$. Let $X \to \text{Spec} k$ be a proper, smooth, and geometrically connected **hyperbolic** (i.e., $\text{genus}(X) \geq 2$) curve over $k$, and $\mathcal{X} \to \text{Spec} \mathcal{O}_k$ a proper, and flat model of $X$ over $\mathcal{O}_k$. We have a commutative diagram with cartesian squares

\[
\begin{array}{ccc}
X & \longrightarrow & \text{Spec} k \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \text{Spec} \mathcal{O}_k \\
\uparrow & & \uparrow \\
\mathcal{X}_s & \longrightarrow & \text{Spec} F
\end{array}
\]

where the right vertical maps are the natural ones, and $\mathcal{X}_s \overset{\text{def}}{=} \mathcal{X} \times_{\mathcal{O}_k} F$ is the special fibre of $\mathcal{X}$.

Let $\eta$ be a geometric point of $X$ above the generic point of $X$. Then $\eta$ determines naturally an algebraic closure $\overline{k}$ of $k$, and a geometric point $\overline{\eta}$ of $\overline{X} \overset{\text{def}}{=} X \times_k \overline{k}$. There exists a canonical exact sequence of profinite groups (cf. [Grothendieck], Exposé IX, Théorème 6.1)

\[
1 \to \pi_1(X, \overline{\eta}) \to \pi_1(X, \eta) \to G_k \to 1.
\]

Here, $\pi_1(X, \eta)$ denotes the arithmetic étale fundamental group of $X$ with base point $\eta$, $\pi_1(X, \overline{\eta})$ the étale fundamental group of $\overline{X} \overset{\text{def}}{=} X \times_k \overline{k}$ with base point $\overline{\eta}$, and $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$ the absolute Galois group of $k$.

Let $\xi$ be a geometric point of $\mathcal{X}_s$ above the generic point of some irreducible component $X_{i_0}$ of $\mathcal{X}_s$. Then $\xi$ determines naturally an algebraic closure $\overline{F}$ of $F$, and a geometric point $\overline{\xi}$ of $\overline{\mathcal{X}_s} \overset{\text{def}}{=} \mathcal{X}_s \times_F \overline{F}$. There exists a canonical exact sequence of profinite groups (cf. loc. cit.)

\[
1 \to \pi_1(\overline{\mathcal{X}_s}, \overline{\xi}) \to \pi_1(\mathcal{X}_s, \xi) \to G_F \to 1.
\]

Here, $\pi_1(\mathcal{X}_s, \xi)$ denotes the arithmetic étale fundamental group of $\mathcal{X}_s$ with base point $\xi$, $\pi_1(\overline{\mathcal{X}_s}, \overline{\xi})$ the étale fundamental group of $\overline{\mathcal{X}_s} \overset{\text{def}}{=} \mathcal{X}_s \times_F \overline{F}$ with base point $\overline{\xi}$, and $G_F \overset{\text{def}}{=} \text{Gal}(\overline{F}/F)$ the absolute Galois group of $F$.

In what follows we assume that the gcd of the multiplicities of the irreducible components of $\mathcal{X}_s$ equals 1. Then, after a suitable choice of the base points $\xi$ and $\eta$, there exists a natural commutative diagram
where the vertical and horizontal sequences are exact, $I_k$ (resp. $I_X$, and $I_{\overline{X}}$) are defined so that the right (resp. the middle, and left) vertical sequence is exact ($I_k$ is the inertia subgroup of $G_k$); as follows easily from the specialisation theory for fundamental groups of Grothendieck (cf. [Grothendieck], Exposé X, §2, and [Raynaud], Proposition 6.3.5, for the surjectivity of the left lower vertical map under our assumption on the gcd of the multiplicities of the components of $X_s$, and the fact that $F$ is perfect). Also, consider the following commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(\overline{X}, \eta) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\overline{X_s}, \xi) & \longrightarrow & \pi_1(X_s, \xi) & \longrightarrow & G_F & \longrightarrow & 1 \\
\end{array}
\]

(0.1)

where the horizontal sequences are exact, the upper and lower horizontal sequences are the middle and lower, respectively, horizontal sequences in diagram (0.1), the lower right square is cartesian, and all vertical maps are surjective. Thus, $\pi_1(X_s, \xi)$ is the pullback of $\pi_1(X_s, \xi)$ via the natural projection $G_k \rightarrow G_F$. We shall refer to the quotient

\[
\pi_1(X, \eta) \rightarrow \pi_1(X_s, \xi)
\]

of $\pi_1(X, \eta)$ as the étale quotient of $\pi_1(X, \eta)$ relative to the model $X$. We have an exact sequence (cf. diagram (0.1), and diagram (0.2))

\[
1 \rightarrow I_{\overline{X}} \rightarrow \pi_1(X, \eta) \rightarrow \pi_1(X_s, \xi) \rightarrow 1.
\]

Write $I_{\overline{X}}^{ab}$ for the maximal abelian quotient of $I_{\overline{X}}$, which is a characteristic quotient. Consider the following pushout diagrams
which defines a natural quotient \( \Delta_{et,ab}^{et,ab} \) of \( \pi_1(X, \bar{\eta}) \), and

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & I_X & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & I_X^{ab} & \longrightarrow & \Delta^{et,ab}_X & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & 1 \\
\end{array}
\]

which defines a natural quotient \( I_X^{ab} \) of \( I_X \). Let

\[
R_X \overset{\text{def}}{=} \text{Ker}(I_X \rightarrow I_X^{ab}) = \text{Ker}(I_X \rightarrow I_X^{ab}) = \text{Ker}(\pi_1(X, \bar{\eta}) \rightarrow \Delta^{et,ab}_X).
\]

Note that \( R_X \) is a normal subgroup of \( \pi_1(X, \eta) \). Indeed, this follows from the fact that \( I_X \) is the kernel of the natural surjective homomorphism \( \pi_1(X, \eta) \rightarrow \hat{\pi}_1(X, \bar{\eta}, \xi) \), and \( I_X^{ab} \) is a characteristic quotient of \( I_X \). Write

\[
\Pi_{X}^{(et,ab)} \overset{\text{def}}{=} \pi_1(X, \eta)/R_X.
\]

Note that \( I_X^{(ab)} \) is the image of \( I_X \) in \( \Pi_{X}^{(et,ab)} \) (cf. diagram (0.3.2)).

We have a natural commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta^{et,ab}_X & \longrightarrow & \Pi_{X}^{(et,ab)} & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\]

We will refer to \( \Pi_{X}^{(et,ab)} \) as the \textbf{étale-by-geometrically abelian} quotient of \( \pi_1(X, \eta) \) \textbf{relative to the model} \( X \). Note that a rational point \( x \in X(k) \) gives rise naturally (by the factoriality of \( \pi_1 \)) to (a conjugacy class of) a group-theoretic section

\[
s_x : G_k \rightarrow \pi_1(X, \eta)
\]

of the upper horizontal exact sequence in diagram (0.4), hence gives also rise to a section

\[
s_x : G_k \rightarrow \Pi_{X}^{(et,ab)}
\]

of the natural projection \( \Pi_{X}^{(et,ab)} \rightarrow G_k \) (cf. diagram (0.4)). We will refer to such a section as a \textbf{geometric} section of \( \Pi_{X}^{(et,ab)} \). A \textbf{non-geometric} section of the projection \( \Pi_{X}^{(et,ab)} \rightarrow G_k \) is a group-theoretic section which doesn’t arise from a \( k \)-rational point \( x \) as above. In this paper we investigate sections of \( \Pi_{X}^{(et,ab)} \rightarrow G_k \) for a given model \( X \) of \( X \). Our main results provide \textit{examples of group-theoretic sections of the natural projection} \( \Pi_{X}^{(et,ab)} \rightarrow G_k \) which are \textit{non-geometric} (cf. Theorem B, and Theorem C).
Definition 0.1. We use the same notations as above. Let \( \mathcal{X} \rightarrow \text{Spec} \, \mathcal{O}_k \) be a proper and flat model of \( X \) over \( \mathcal{O}_k \). We say that the model \( \mathcal{X} \) satisfies the condition \((\ast)\) if the following holds.

(a) The scheme \( \mathcal{X} \) is regular.

(b) There exists an irreducible component \( X_{i_0} \) of the special fibre \( \mathcal{X}_s \equiv \mathcal{X} \times \mathcal{O}_k \mathcal{F} \) of \( \mathcal{X} \) with the following properties.

(i) \( X_{i_0} \) is reduced. In particular, \( X_{i_0} \) is geometrically reduced since \( \mathcal{F} \) is perfect, and the gcd of the multiplicities of the irreducible components of \( \mathcal{X}_s \) equals 1.

(ii) \( X_{i_0} \) is geometrically irreducible. Thus, \( X_{i_0} \) is geometrically integral since \( \mathcal{F} \) is perfect.

(iii) \( X_{i_0} \) is geometrically unibranch, i.e., for each finite extension \( \mathcal{F}'/\mathcal{F} \) the morphism of normalisation \( X_{i_0,\mathcal{F}'} \rightarrow X_{i_0,\mathcal{F}} \equiv X_{i_0} \times \mathcal{F} \mathcal{F}' \) is a homeomorphism.

(iv) The normalisation \( X_{i_0,\mathcal{F}}^{\text{nor}} \) of \( X_{i_0} \) is hyperbolic.

Remark 0.2. (i) If the curve \( X \) has good reduction over \( \mathcal{O}_k \), i.e., if \( X \rightarrow \text{Spec} \, k \) extends to a proper and smooth relative curve \( \mathcal{X} \rightarrow \text{Spec} \, \mathcal{O}_k \) over \( \mathcal{O}_k \), then the smooth model \( \mathcal{X} \) of \( X \) satisfies the condition \((\ast)\).

(ii) If the curve \( X \) satisfies the condition \((\ast)\), then the index of \( X \) equals 1 (cf. [Liu], exercise 9.1.9, and the reference therein).

Our main results in this paper are the following.

Theorem A. Let \( X \) be a proper, smooth, and geometrically connected hyperbolic curve over the \( p \)-adic local field \( k \). Let \( \mathcal{X} \rightarrow \text{Spec} \, \mathcal{O}_k \) be a proper and flat model of \( X \) over \( \mathcal{O}_k \). Assume that the model \( \mathcal{X} \) satisfies the condition \((\ast)\) (cf. Definition 0.1). Then the corresponding exact sequence

\[
1 \rightarrow \Delta^{\text{et,ab}}_{\mathcal{X}} \rightarrow \Pi^{(\text{et,ab})}_{\mathcal{X}} \rightarrow G_k \rightarrow 1,
\]

where \( \Pi^{(\text{et,ab})}_{\mathcal{X}} \) is the \( \text{"etale-by-geometrically abelian quotient of the arithmetic fundamental group of} \ X \) relative to the model \( \mathcal{X} \) (cf. diagram (0.4)) is a split exact sequence of profinite groups.

As a corollary of Theorem A we obtain the following.

Theorem B. There exists a \( p \)-adic local field \( k \), and a smooth, projective, geometrically connected, and hyperbolic curve \( X \) over \( k \), such that the following holds. There exists a proper and smooth model \( \mathcal{X} \rightarrow \text{Spec} \, \mathcal{O}_k \) of \( X \) (i.e., the curve \( X \) has good reduction over \( \mathcal{O}_k \)), and a group-theoretic section \( s : G_k \rightarrow \Pi^{(\text{et,ab})}_{\mathcal{X}} \) of the corresponding exact sequence \( 1 \rightarrow \Delta^{\text{et,ab}}_{\mathcal{X}} \rightarrow \Pi^{(\text{et,ab})}_{\mathcal{X}} \rightarrow G_k \rightarrow 1 \) (cf. diagram (0.4), and the discussion before Theorem A) which is non-geometric, i.e., which doesn’t arise from a \( k \)-rational point of \( X \).

Proof. Indeed, this follows from Theorem A, Remark 0.2 (i), and the fact that there exists a \( p \)-adic local field \( k \) and a \( k \)-curve \( X \) satisfying the assumptions in Theorem B such that \( X(k) = \emptyset \) (cf. [Sa"idi1], Proof of Proposition 3.2.1). \( \square \)

More generally, we obtain the following.
Theorem C. There exists a $p$-adic local field $k$, and a smooth, projective, geometrically connected, and hyperbolic curve $X$ over $k$ such that the following holds. The curve $X$ has bad semi-stable reduction over $\mathcal{O}_k$, there exists a proper and flat semi-stable regular model $\mathcal{X} \to \text{Spec} \mathcal{O}_k$ of $X$ over $\mathcal{O}_k$, and a group-theoretic section $s : G_k \to \Pi_{\mathcal{X}}^{(et,ab)}$ of the corresponding exact sequence $1 \to \Delta_{\mathcal{X}}^{et,ab} \to \Pi_{\mathcal{X}}^{(et,ab)} \to G_k \to 1$ (cf. diagram (0.4), and the discussion before Theorem A) which is non-geometric, i.e., which doesn’t arise from a $k$-rational point of $X$.

§1. Group theoretic sections of the étale-by-geometrically abelian quotient $\Pi^{(et,ab)}$. We use the notations introduced in §0. In this section we investigate the group-theoretic splittings of the exact sequence (cf. §0, diagram (0.4))

$$1 \to \Delta_{\mathcal{X}}^{et,ab} \to \Pi_{\mathcal{X}}^{(et,ab)} \to G_k \to 1.$$

Let $k$, $X \to \text{Spec} k$, and $\mathcal{X} \to \text{Spec} \mathcal{O}_k$, be as in the discussion before Theorem A. Thus,

$$X \to \text{Spec} k$$

is a smooth, proper, and geometrically connected hyperbolic curve over the $p$-adic local field $k$, and

$$\mathcal{X} \to \text{Spec} \mathcal{O}_k$$

is a proper, and flat model of $X$ over the ring of integers $\mathcal{O}_k$ of $k$, such that the gcd of the multiplicities of the irreducible components of $\mathcal{X}_s$ equals 1. It follows from the various definitions that we have a commutative diagram of exact sequences

(1.1)

\[
\begin{array}{ccc}
1 & \to & 1 \\
\downarrow & & \downarrow \\
\Delta_{\mathcal{X}}^{et,ab} & \to & \Pi_{\mathcal{X}}^{(et,ab)} \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{X}_s, \bar{\xi}) & \to & \Pi^{(et)} = \hat{\pi}_1(\mathcal{X}_s, \xi) \\
\downarrow & & \downarrow \\
1 & \to & 1 \\
\end{array}
\]

where we noted

$$\Delta^{et} \overset{\text{def}}{=} \Delta_{\mathcal{X}}^{et,ab} \overset{\text{def}}{=} \pi_1(\mathcal{X}_s, \bar{\xi}), \quad \Delta_{et,ab} \overset{\text{def}}{=} \Delta_{\mathcal{X}}^{et,ab},$$

and

$$\Pi^{(et)} \overset{\text{def}}{=} \Pi_{\mathcal{X}}^{(et,ab)} \overset{\text{def}}{=} \pi_1(\mathcal{X}_s, \xi), \quad \Pi^{(et,ab)} \overset{\text{def}}{=} \Pi_{\mathcal{X}}^{(et,ab)}.$$


The profinite group $\Delta^{et}$ is **finitely generated** (as follows from the well-known finite generation of the profinite group $\pi_1(X, \eta)$ which projects onto $\Delta^{et}$). Let $\{\Delta^{et,i}\}_{i \geq 1}$ be a countable system of **characteristic open** subgroups of $\Delta^{et}$ such that
\[
\Delta^{et,i+1} \subseteq \Delta^{et,i}, \quad \Delta^{et,1} \overset{\text{def}}{=} \Delta^{et}, \quad \text{and} \quad \bigcap_{i \geq 1} \Delta^{et,i} = \{1\}.
\]
Write $\Delta_i \overset{\text{def}}{=} \Delta^{et}/\Delta^{et,i}$. Thus, $\Delta_i$ is a **finite, characteristic** quotient of $\Delta^{et}$, and we have a *pushout* diagram of exact sequences
\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{et} & \longrightarrow & \Pi^{(et)} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_i & \longrightarrow & \Pi_i & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\] (1.2)
which defines a *(geometrically finite)* quotient $\Pi_i$ of $\Pi^{(et)}$.

Note that the upper horizontal sequence in diagram (1.2) **splits**. Indeed, the lower horizontal exact sequence in diagram (0.2) splits since the Galois group $G_F$ of $F$ is pro-free ($F$ being a finite field), hence the middle horizontal sequence in diagram (0.2); which is by definition the sequence $1 \rightarrow \Delta^{et} \rightarrow \Pi^{(et)} \rightarrow G_k \rightarrow 1$, splits since the lower right square in that diagram is cartesian. Let $s : G_k \rightarrow \Pi^{(et)}$ be a **section** of the upper sequence in diagram (1.2), which naturally induces a section $s_i : G_k \rightarrow \Pi_i$ of the lower sequence in diagram (1.2), for each $i \geq 1$. Write
\[
\hat{\Pi}^i \overset{\text{def}}{=} \Pi^i[s] \overset{\text{def}}{=} \Delta^{et,i}.s(G_k).
\]
Thus, $\hat{\Pi}^i \subseteq \Pi^{et}$ is an **open** subgroup which contains the image $s(G_k)$ of $s$. Write $\Pi^i$ for the **inverse image** of $\hat{\Pi}^i$ in $\pi_1(X, \eta)$. Thus, $\Pi^i \subseteq \pi_1(X, \eta)$ is an **open** subgroup which corresponds to an étale cover
\[
X_i \rightarrow X_1 \overset{\text{def}}{=} X
\]
defined over $k$ (since $\Pi^i$ maps onto $G_k$ via the natural projection $\pi_1(X, \eta) \twoheadrightarrow G_k$, by the very definition of $\Pi^i$). Moreover, it follows from the various definitions that the étale cover $X_i \rightarrow X$ extends to an étale cover
\[
X_i \rightarrow \mathcal{X},
\]
defined over $O_k$.

Note that the étale cover $\overline{X}_i \overset{\text{def}}{=} X_i \times_k \overline{k} \rightarrow \overline{X}$ is Galois with Galois group $\Delta_i$, and we have a commutative diagram of étale covers
\[ X_i \longrightarrow \longrightarrow X \]
\[ \downarrow \quad \downarrow \]
\[ X_i \longrightarrow \longrightarrow X \]

where \( X_i \rightarrow X \) is Galois with Galois group \( \Pi_i \), and \( \overline{X}_i \rightarrow X_i \) is Galois with Galois group \( s_i(G_k) \). Moreover, we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \Delta_i & = & \pi_1(\overline{X}_i, \overline{\eta}) & \longrightarrow & \Pi_i = \pi_1(X_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \parallel & & 1 \\
1 & \longrightarrow & \pi_1(X, \overline{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\]

where \( \Delta_i \) is the inverse image of \( \Delta^{et,i} \) in \( \pi_1(X, \overline{\eta}) \), and the equalities \( \Delta_i = \pi_1(\overline{X}_i, \overline{\eta}) \), \( \Pi_i = \pi_1(X_i, \eta) \), are natural identifications; the base points \( \eta \) (resp. \( \overline{\eta} \)) of \( X_i \) (resp. \( \overline{X}_i \)) are those induced by the base points \( \eta \) (resp. \( \overline{\eta} \)) of \( X \) (resp. \( \overline{X} \)). Note that \( \Pi^{i+1} \subseteq \Pi^i \) and \( \Delta^{i+1} \subseteq \Delta^i \) as follows from the various definitions.

**Lemma 1.1.** With the same notations as above, the following holds:

\[ I_X = \bigcap_{i \geq 1} \Delta^i, \quad \text{and} \quad I_X = \bigcap_{i \geq 1} \Pi^i. \]

**Proof.** Follows from the various definitions. \( \square \)

For each integer \( i \geq 1 \), write \( \Delta^{i,ab} \) for the maximal abelian quotient of \( \Delta^i \); which is a characteristic quotient. Consider the natural pushout diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \Delta^i & = & \pi_1(\overline{X}_i, \overline{\eta}) & \longrightarrow & \Pi^i = \pi_1(X_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \parallel & & 1 \\
1 & \longrightarrow & \Delta^{i,ab} & \longrightarrow & \Pi^{i,ab} & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\]

Thus, \( \Pi^{i,ab} \) is the geometrically abelian fundamental group of \( X_i \). Write \( J_i \stackrel{def}{=} J_{X_i} \) for the jacobian variety of \( X_i \), and \( \overline{J}_i \stackrel{def}{=} J_i \times_k \overline{k} \) for the jacobian variety of \( \overline{X}_i \). Let \( J^1_i \) be the \( J_i \)-torsor \( \text{Pic}^0_X \). We have \( J^1_i(k) \neq \emptyset \) if \( X_i \) has index 1. In this case we have an identification \( J_i \sim \overline{J}_i \). Moreover, \( \Delta^{i,ab} \) is naturally identified with the Tate module \( T\overline{J}_i \) of \( \overline{J}_i \) as \( G_k \)-modules.

**Lemma 1.2.** We use the above notations. The exact sequence

\[ 1 \rightarrow \Delta^{i,ab} \rightarrow \Pi^{i,ab} \rightarrow G_k \rightarrow 1 \]

is a split exact sequence of profinite groups if the index of \( X_i \) equals 1.

**Proof.** Indeed, this exact sequence is naturally identified with the exact sequence

\[ 1 \rightarrow \pi_1(\overline{X}_i, \overline{\eta})^{ab} \rightarrow \pi_1(X_i, \eta)^{(ab)} \rightarrow G_k \rightarrow 1, \]

where \( \pi_1(\overline{X}_i, \overline{\eta})^{ab} \) is the maximal
Consider the natural pullback commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & I_{\mathcal{X}}^{ab} & \longrightarrow & \mathcal{H}_X & \overset{\text{def}}{=} & \mathcal{H}_X[s] & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & s & & \\
1 & \longrightarrow & I_{\mathcal{X}}^{ab} & \longrightarrow & \Pi^{(\text{et},ab)} & \longrightarrow & \Pi^{(\text{et})} & \longrightarrow & 1
\end{array}
\]

(1.3)

where the right square is cartesian. Thus, (the group extension) \(\mathcal{H}_X\) is the pullback of (the group extension) \(\Pi^{(\text{et},ab)}\) via the section \(s : G_k \rightarrow \Pi^{(\text{et})}\).

**Lemma 1.3.** We have natural identifications \(I_{\mathcal{X}}^{ab} \xrightarrow{\sim} \lim_{i \geq 1} \Delta^{i,ab}\), and \(\mathcal{H}_X \xrightarrow{\sim} \lim_{i \geq 1} \Pi^{(i,ab)}\).

**Proof.** Follows from the various definitions. More precisely, recall the étale Galois cover \(\overline{X}_i \rightarrow \mathcal{X}\) with Galois group \(\Delta_i\). For each integer \(j \geq 1\), let \(\Delta_{i,j} \overset{\text{def}}{=} \Delta^{i,ab}/j.\Delta^{i,ab}\), which is a characteristic quotient of \(\Delta^{i,ab}\), and \(\overline{X}_{i,j} \rightarrow \overline{X}_i\) the corresponding étale abelian cover with Galois group \(\Delta_{i,j}\). Then the étale cover \(\overline{X}_{i,j} \rightarrow \overline{X}\) is Galois with Galois group \(\overline{\Delta}_{i,j}\), which inserts in the following exact sequence \(1 \rightarrow \Delta_{i,j} \rightarrow \overline{\Delta}_{i,j} \rightarrow \Delta_i \rightarrow 1\). The \(\{\overline{\Delta}_{i,j}\}_{(i,j)}\), where the set of pairs \((i,j)\) is endowed with the product order, form a cofinal system of finite quotients of \(\Delta^{et,ab}\). From this it follows that the \(\{\Delta_{i,j}\}_{(i,j)}\) form a cofinal system of finite quotients of \(I_{\mathcal{X}}^{ab}\). Thus, \(I_{\mathcal{X}}^{ab} \xrightarrow{\sim} \lim_{(i,j)} \Delta_{i,j} \xrightarrow{\sim} \lim_{(i,j)} \lim_{j \geq 1} \Delta_{i,j} \xrightarrow{\sim} \lim_{i \geq 1} \Delta^{i,ab}\). The identification \(\mathcal{H}_X \xrightarrow{\sim} \lim_{i \geq 1} \Pi^{(i,ab)}\) follows immediately. \(\square\)

**Lemma 1.4.** The exact sequence \(1 \rightarrow \Delta^{et,ab} \rightarrow \Pi^{(et,ab)} \rightarrow G_k \rightarrow 1\) splits if the exact sequence \(1 \rightarrow I_{\mathcal{X}}^{ab} \rightarrow \mathcal{H}_X \rightarrow G_k \rightarrow 1\) splits.

**Proof.** Follows immediately from diagram (1.3). \(\square\)

**Proposition 1.5.** Suppose that the exact sequence \(1 \rightarrow \Delta^{i,ab} \rightarrow \Pi^{(i,ab)} \rightarrow G_k \rightarrow 1\) splits (this is the case for example if the curve \(X_i\) has index 1 (cf. Lemma 1.2)), for each integer \(i \geq 1\). Then the exact sequence \(1 \rightarrow I_{\mathcal{X}}^{ab} \rightarrow \mathcal{H}_X \rightarrow G_k \rightarrow 1\) splits.

**Proof.** We will show the existence of a section \(\tilde{s} : G_k \rightarrow \mathcal{H}_X\) of the natural projection \(\mathcal{H}_X \rightarrow G_k\). For each integer \(i \geq 1\) let

\[\{\Delta^{i,ab}_j \overset{\text{def}}{=} j.\Delta^{i,ab}_{j}\}_{j \geq 1},\]

which is a system of open characteristic subgroups of \(\Delta^{i,ab}\) such that

\[\Delta^{i,ab}_{j'} \subseteq \Delta^{i,ab}_j\] if \(j/j'\), \(\Delta^{i,ab}_1 \overset{\text{def}}{=} \Delta^{i,ab}\), and \(\bigcap_{j \geq 1} \Delta^{i,ab}_j = \{1\}\).
Write $\Delta_{i,j} \overset{\text{def}}{=} \Delta_{i,\text{ab}} / \Delta_{j,\text{ab}}$, and $\Pi_{i,j} \overset{\text{def}}{=} \Pi_{i,\text{ab}} / \Delta_{j,\text{ab}}$. Thus, we have a natural exact sequence

$$1 \to \Delta_{i,j} \to \Pi_{i,j} \to G_k \to 1,$$

and a projective system $\{\Pi_{i,j}\}_{(i,j)}$, where the the set of pairs of integers $(i, j)$ is endowed with the product order. Note that we have natural identifications $\Delta_{(i,\text{ab})} \sim \lim_{j \geq 1} \Delta_{i,j}$, and $\Pi_{(i,\text{ab})} \sim \lim_{j \geq 1} \Pi_{i,j}$. Moreover, we have a natural identification $\mathcal{H}_X \sim \lim_{(i,j)} \Pi_{i,j}$ (cf. Lemma 1.3, and the above discussion). In particular, the set $\text{Sec}(G_k, \mathcal{H}_X)$ of group-theoretic sections of the natural projection $\mathcal{H}_X \to G_k$ is naturally identified with the projective limit $\lim_{(i,j)} \text{Sec}(G_k, \Pi_{i,j})$ of the sets $\text{Sec}(G_k, \Pi_{i,j})$ of group-theoretic sections of the natural projection $\Pi_{i,j} \to G_k$.

For each pair of integers $(i, j)$ the set $\text{Sec}(G_k, \Pi_{i,j})$ is non-empty, since $\Pi_{i,j}$ is a quotient of $\Pi_{(i,\text{ab})}$ (cf. the assumption that (the group extension) $\Pi_{(i,\text{ab})}$ splits). Moreover, the set $\text{Sec}(G_k, \Pi_{i,j})$ is, up to conjugation by the elements of $\Delta_{(i,\text{j})}$, a torsor under the group $H^1(G_k, \Delta_{i,j})$ which is finite since $k$ is a $p$-adic local field (cf. [Neukirch-Schmidt-Winberg], (7.1.8) Theorem (iii)). Thus, $\text{Sec}(G_k, \Pi_{i,j})$ is a nonempty finite set. Hence the set $\text{Sec}(G_k, \mathcal{H}_X)$ is nonempty being the projective limit of nonempty finite sets. This finishes the proof of Proposition 1.5. 

In fact, we proved the following more precise statement.

**Proposition 1.6.** Let $s : G_k \to \Pi_{\mathcal{X}}^{(\text{et})}$ be a section of the natural projection $\Pi_{\mathcal{X}}^{(\text{et})} \to G_k$. Suppose that for each open subgroup $H \subseteq \Pi_{\mathcal{X}}^{(\text{et})}$ such that $s(G_k) \subset H$, and the corresponding étale cover $\mathcal{X}_H \to \mathcal{X}$, it holds that the index of the $k$-curve $\mathcal{X}_H \times_{\mathcal{X}} k$ equals 1. Then there exists a section $\tilde{s} : G_k \to \Pi_{\mathcal{X}}^{(\text{et,ab})}$ of the natural projection $\Pi_{\mathcal{X}}^{(\text{et,ab})} \to G_k$ which lifts the section $s$, i.e., such that we have a commutative diagram

$$
\begin{array}{ccc}
G_k & \xrightarrow{\tilde{s}} & \Pi_{\mathcal{X}}^{(\text{et,ab})} \\
\text{id} \downarrow & & \downarrow \\
G_k & \xrightarrow{s} & \Pi_{\mathcal{X}}^{(\text{et})}
\end{array}
$$

where the right vertical map is the map in diagram (1.1).

**Proof.** Follows from the above discussion, Proposition 1.4, Proposition 1.5, the commutative diagram (1.3), and Lemma 1.2. 

**Remark 1.7.** Proposition 1.6 remains valid if instead of assuming the curves $\mathcal{X}_H \times_{\mathcal{X}} k$ to have index 1 one assumes that if $J_H$ denotes the jacobian of $\mathcal{X}_H \times_{\mathcal{X}} k$ and $J_H^1$ is the corresponding torsor Pic$^1$, then the class $[J_H^1]$ of $J_H^1$ in $H^1(G_k, J_H)$ lies in the maximal divisible subgroup of $H^1(G_k, J_H)$, as Lemma 1.2 is still valid under a similar assertion (cf. proof of Lemma 1.2 and the reference therein).

§2. **Proof of Theorem A and Theorem C.** The rest of this paper is devoted to proving Theorem A and Theorem C. We use the notations introduced in §0 and §1.

**Proof of Theorem A.** Assume that $\mathcal{X}$ satisfies the condition $(\ast)$. Thus, $\mathcal{X}$ is regular, the special fibre $\mathcal{X}_s \overset{\text{def}}{=} \mathcal{X} \times_{\text{Spec} O_k} \text{Spec} F$ has an irreducible component $X_{i_0}$.
which is \textit{(geometrically) reduced, geometrically irreducible, geometrically unibranch}, and \textit{its normalisation $X_{i_0}^{\text{nor}}$ is hyperbolic}. In order to prove Theorem A it suffices to construct a group-theoretic section $s: G_k \to \Pi^{(\text{et})}$ of the natural projection $\Pi^{(\text{et})} \to G_k$, with corresponding open subgroups $\tilde{\Pi}^i[s] \subset \Pi^{(\text{et})}$, such that in the corresponding étale covers $X_i \to X$ the index of $X_i$ equals 1, for each integer $i \geq 1$ (cf. Proposition 1.5 and Lemma 1.4).

Let

$$D_{X_{i_0}} \subset \Pi^{(\text{et})} \overset{\text{def}}{=} \hat{\pi}_1(X_{i_0}, \xi)$$

be a \textit{decomposition group} associated to the irreducible component $X_{i_0}$ of $\mathcal{X}$. More precisely, $D_{X_{i_0}}$ is the decomposition group of a connected component $\tilde{X}_{i_0}$ of the fibre of $X_{i_0}$ in (the special fibre of) the universal pro-étale cover of $\mathcal{X}$, corresponding to the quotient $\Pi^{(\text{et})}$ of $\pi_1(X, \eta)$, and $D_{X_{i_0}}$ is only defined up to conjugation. Note that $\tilde{X}_{i_0}$ is an \textit{irreducible pro-curve} (i.e., is the projective limit of irreducible curves) since the morphism $\tilde{X}_{i_0} \to X_{i_0}$ is pro-étale, $X_{i_0}$ is geometrically unibranch, and $\tilde{X}_{i_0}$ is connected.

We have a commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & D_{X_{i_0}} & \longrightarrow & D_{X_{i_0}} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\tilde{X}_{i_0}, \tilde{\xi}) & \longrightarrow & \pi_1(X_{i_0}, \xi) & \longrightarrow & G_F & \longrightarrow & \\
\end{array}
$$

(2.1)

where $D_{X_{i_0}}$ is defined so that the upper sequence is exact, $\pi_1(X_{i_0}, \xi)$ is the \textit{arithmetic fundamental group} of $X_{i_0}$, the right vertical map is the natural projection, and the right square is \textit{cartesian}. In particular, the left vertical map in the above diagram is an isomorphism.

Let

$$\tilde{s}_{i_0}: G_F \to \pi_1(X_{i_0}, \xi)$$

be a group-theoretic section of the lower sequence in diagram (2.1) (this sequence splits since $G_F$ is pro-free), which induces naturally (cf. above cartesian square in diagram (2.1)) a section

$$s_{i_0}: G_k \to D_{X_{i_0}}$$

of the natural projection $D_{X_{i_0}} \to G_k$, hence induces also a \textbf{section}

$$s \overset{\text{def}}{=} s_{i_0}: G_k \to D_{X_{i_0}} \subset \Pi^{(\text{et})}$$

of the upper sequence in diagram (1.2) (cf. above discussion).

Recall the above étale cover $X_i \to \mathcal{X}$ (cf. discussion after diagram (1.2)), which corresponds to the open subgroup $\tilde{\Pi}^i[s] = \Delta^{i, \text{et}, s}(G_k)$. Let $X_{i,i_0}$ be the image of $\tilde{X}_{i_0}$ in $X_i$, and $X_{i,i_0} \to X_{i_0}$ the corresponding étale cover. Thus, the étale cover $X_{i,i_0} \to X_{i_0}$ corresponds to the open subgroup $\tilde{\Pi}^i[s] \cap D_{X_{i_0}} \subset D_{X_{i_0}}$ of $D_{X_{i_0}}$. Then $X_{i,i_0}$ is an $F$-\textit{curve} and the cover $X_{i,i_0} \to \tilde{X}_{i_0}$ is a morphism of $F$-\textit{curves}, since the open subgroup $\tilde{\Pi}^i[s] \cap D_{X_{i_0}} \subset D_{X_{i_0}}$ contains $s_{i_0}(G_k)$ (hence projects onto $G_k$ via the projection $D_{X_{i_0}} \to G_k$), $X_{i,i_0}$ is \textit{reduced} (since it is an étale cover of $X_{i_0}$ which is reduced), hence is also geometrically reduced since $F$
is perfect, and \(X_{i,\mathcal{O}_k}\) is geometrically irreducible (since \(\overline{X}_{i,\mathcal{O}_k}\) is irreducible (cf. above discussion)). More precisely, \(X_{i,\mathcal{O}_k}\) is geometrically unibranch; being an étale cover of \(X_{\mathcal{O}_k}\) which is geometrically unibranch, and is geometrically connected. Moreover, its normalisation \(\overline{X}_{i,\mathcal{O}_k}\) is hyperbolic, since \(\overline{X}_{i,\mathcal{O}_k}\) dominates \(\overline{X}_{\mathcal{O}_k}\). In particular, the index of \(X_i \equiv \mathcal{X}_i \times_{\Spec \mathcal{O}_k} \Spec k\) equals 1 (cf. Remark 0.2 (ii)).

This finishes the proof of Theorem A. □

Proof of Theorem C. Let \(F\) be a finite field, and \(C \to \Spec F\) a singular stable curve with arithmetic genus \(g(C) > 1\) such that each double point of \(C\) is \(F\)-rational and lies on two distinct irreducible components of \(C\). For example let \(X_0 \to \Spec F\) be a proper, smooth, and geometrically connected hyperbolic curve such that \(X_0(F) = \{x_i\}_{i \in I} \neq \emptyset\). For each \(i \in I\), let \(E_i\) be an elliptic curve over \(F\), and identify the origin of \(E_i\) with the rational point \(x_i\) of \(X_0\) into an ordinary \(F\)-rational double point \(x_i\). We obtain a singular stable curve \(C = X_0 + \sum_{i \in I} E_i\), whose configuration is tree like, where \(X_0\) intersects \(E_i\) at the rational double point \(x_i\), and \(E_i \cap E_j = \emptyset\) for \(i \neq j\). Let \(k\) be a \(p\)-adic local field with residue field \(F\) and ring of integers \(\mathcal{O}_k\). Using formal patching techniques one can construct a proper and stable relative curve \(\mathcal{X} \to \Spec \mathcal{O}_k\) such that \(\mathcal{X}\) is regular, and \(\mathcal{X} \times_{\mathcal{O}_k} F = C\) (compare with [Saïdi2], Proposition 3.6). Let \(X \equiv \mathcal{X} \times_{\mathcal{O}_k} k\). The morphism \(x_i : \Spec F \to C\) gives rise naturally to a (conjugacy class of) section \(s_{x_i} : G_F \to \pi_1(C)\) of the natural projection \(\pi_1(C) \to G_F\), which naturally induces a section \(s : G_k \to \Pi^{(\et)} \equiv \Pi^{(\et)}_\mathcal{X}\) of the natural projection \(\Pi^{(\et)} \to G_k\) (cf. diagram 0.2). Let \(H \subseteq \Pi^{(\et)}\) be an open subgroup such that \(s(G_k) \subset H\), and \(\mathcal{X}_H \to \mathcal{X}\) the corresponding étale cover. Thus, \(\mathcal{X}_H \to \Spec \mathcal{O}_k\) is a relative stable curve, and \(\mathcal{X}_H\) is regular. Let \(X_H \equiv \mathcal{X}_H \times_{\mathcal{O}_k} k\), then the index of \(X_H\) equals 1. Indeed, if \(C'\) is an irreducible component of \(C\) which contain the (double) point \(x_i\), and \(C'_H\) is an irreducible component of \(\mathcal{X}_H\) above \(C'\) containing an \(F\)-rational (double) point above \(x_i\) (cf. the construction of the section \(s\)), then \(C'_H\) is geometrically integral (cf. Remark 0.2 (ii)).

The above section \(s\) satisfies the assumption of Proposition 1.6. Thus, there exists a section \(\tilde{s} : G_k \to \Pi^{(\et, \text{ab})}\) of the natural projection \(\Pi^{(\et, \text{ab})} \to G_k\) which lifts the section \(s\) (cf. loc. cit.). The section \(\tilde{s}\) is not geometric. Indeed, if \(\tilde{s}\) is geometric and arises from a rational point \(x \in X(K)\), then \(s\) is also geometric and arises from the rational point \(x\). Then, by construction of the section \(s\), the point \(x\) would specialise in the double point \(x_i\) as is easily verified. But this is not possible since \(\mathcal{X}\) is regular and \(x\) should specialise in a smooth point of \(C\) (cf. [Liu], Corollary 9.1.32).

This finishes the proof of Theorem C. □

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