Improved $\alpha_s$ from $\tau$ Decays

B. Malaescu

Laboratoire de l’Accélérateur Linéaire, IN2P3/CNRS et Université Paris-Sud 11 (UMR 8607), F–91405, Orsay Cedex, France

We present an update of the measurement of $\alpha_s(m_\tau^2)$ from ALEPH $\tau$ hadronic spectral functions. We report a study of the perturbative prediction(s) showing that the fixed-order perturbation theory manifests convergence problems not presented in the contour-improved calculation. Potential systematic effects from quark-hadron duality violations are estimated to be within the quoted systematic errors. The fit result is $\alpha_s(m_\tau^2) = 0.344 \pm 0.005 \pm 0.007$, where the first error is experimental and the second theoretical. After evolution, the $\alpha_s(m_Z^2)$ determined from $\tau$ data is the most precise one to date, in agreement with the corresponding $N^3LO$ value derived from $Z$ decays.

1 Introduction

The $\tau$ lepton, through its hadronic decays, provides a clean laboratory to perform precise studies of QCD. Invariant mass distributions obtained from long distance hadron data allow one to compute the spectral functions, which permit the study of short distance quark interactions. In particular, these spectral functions can be exploited to precisely determine the strong coupling constant at the $\tau$-mass scale, $\alpha_s(m_\tau^2)$. The present analysis is described in detail in ref[1].

2 Tau Hadronic Data and Spectral Functions

The nonstrange vector (axial-vector) spectral functions $v_1(a_1)$, for a spin 1 hadronic system, are obtained from the squared hadronic mass distribution, normalised to the hadronic branching fraction (with $R_{\tau,V/A} = \frac{B_{\tau\to V^-/A^- (\gamma)\nu\bar{\nu}_\tau}}{B_{\tau\to e^-\tau_\nu\bar{\nu}_\tau}}$), and divided by a factor exhibiting kinematics and spin characteristics

$$v_1(s)/a_1(s) \propto \frac{dN_{V/A}}{N_{V/A} ds} \frac{B_{\tau\to V^-/A^- (\gamma)\nu\bar{\nu}_\tau}}{B_{\tau\to e^-\tau_\nu\bar{\nu}_\tau}} \left[ \left( 1 - \frac{s}{m_\tau^2} \right)^2 \left( 1 + \frac{2s}{m_\tau^2} \right) \right]^{-1}.$$ (1)

where

$$B_{\tau\to V^-/A^- (\gamma)\nu\bar{\nu}_\tau} \propto \frac{1}{s} \int \frac{d^2 P}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \frac{d^2 p}{(2\pi)^2} \frac{1}{1 + \frac{s}{m_\tau^2} - \frac{2m_\tau^2}{s}} \frac{1}{1 + \frac{2m_\tau^2}{s}} \left( 1 - \frac{s}{m_\tau^2} \right)^2 \left( 1 + \frac{2s}{m_\tau^2} \right)$$

with

$$\int \frac{d^2 P}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \frac{d^2 p}{(2\pi)^2} \frac{1}{1 + \frac{s}{m_\tau^2} - \frac{2m_\tau^2}{s}} \frac{1}{1 + \frac{2m_\tau^2}{s}} \left( 1 - \frac{s}{m_\tau^2} \right)^2 \left( 1 + \frac{2s}{m_\tau^2} \right) = \frac{1}{\sqrt{\pi}} \Gamma\left( \frac{1}{2} \right)^4$$

where $s$ is the invariant mass squared.
The basis for comparing a theoretical description of strong interaction with hadronic data is provided by the optical theorem, which relates the imaginary part of the polarisation functions on the branch cut along the real axis, to the spectral functions: $2\pi \cdot \text{Im}\Pi^{V/A}_{V/A}(s) = v_1/a_1(s)$.

The total hadronic observable $R_\tau$ is obtained from measured leptonic branching ratios, or only from the electron channel assuming universality. The two determinations are in very good agreement, yielding $R_\tau = (1 - B_e - B_\mu)/B_e = 1/B_e^{\text{new}} - 1.9726 = 3.640 \pm 0.010$. One can identify in $R_\tau$ a component with net strangeness and two nonstrange vector(V) and axial-vector(A) components. Including the latest results from BABAR and Belle the value of the strange component is $R_{\tau,S} = 0.1615 \pm 0.0040$. The separation of the V and A components is straightforward for final states with only pions using G-parity. However, $K\bar{K}$ modes are generally not eigenstates of G-parity. The decay to $K^-\bar{K}^0$ is pure vector. The vector component of the $K\bar{K}$ mode is determined assuming CVC and using new measurements from the BABAR Collaboration\textsuperscript{\textcopyright} for the $e^+e^-$ annihilation to $K^+\bar{K}^-\pi^0$ and to $K^0\bar{K}^{\pm}\pi^{\mp}$. After integration one gets a clear dominance of axial-vector-component, $f_{A,CVC}(K\bar{K}) = 0.833 \pm 0.024$. For the $K\bar{K}\pi\pi$ rarer modes a conservative value $f_{A}(K\bar{K}\pi\pi) = 0.5 \pm 0.5$ is used. Finally, we get the components: $R_{\tau,V} = 1.783 \pm 0.011 \pm 0.002$ and $R_{\tau,A} = 1.695 \pm 0.011 \pm 0.002$, where the first errors are experimental and the second due to the $V/A$ separation.

### 3 Theoretical Prediction of $R_\tau$

The nonstrange ratio $R_{\tau,V/A}$ can be written as an integral of the spectral functions over the invariant mass-squared $s$ of the final state hadrons

$$R_{\tau,V/A}(s_0) \propto \int_{0}^{s_0} \frac{ds}{s_0} \left( 1 - \frac{s}{s_0} \right)^2 \left[ \left( 1 + 2 \frac{s}{s_0} \right) \text{Im}\Pi^{(1)}_{V/A}(s + i\varepsilon) + \text{Im}\Pi^{(0)}_{V/A}(s + i\varepsilon) \right].$$

(2)

The two point correlator can not be predicted by QCD in this region of the real axis. However, using Cauchy’s theorem, one can relate this expression to an integral on a circle in the complex plane. Then, the OPE yields

$$R_{\tau,V/A} \propto 1 + \delta^{(0)} + \delta^{(2,\text{mq})}_{\text{EW}} + \sum_{D=4,6,...} \delta^{(D)}_{\text{ud,V/A}},$$

(3)

with a massless perturbative contribution, a non-logarithmic electroweak correction, the dimension two perturbative quark-mass contribution and higher dimension nonperturbative condensates contributions respectively. The perturbative part reads $\delta^{(0)} = \sum_{n=1}^{\infty} \tilde{K}_n(\xi)A^{(n)}(\alpha_s)$, with the functions

$$A^{(n)}(\alpha_s) = \frac{1}{2\pi i} \oint_{|s| = s_0} ds \left( 1 - 2 \frac{s}{s_0} + 2 \left( \frac{s}{s_0} \right)^3 - \left( \frac{s}{s_0} \right)^4 \right) \left( \frac{\alpha_s(-\xi s)}{\pi} \right)^n,$$

(4)

where $\xi$ is a scale factor. A breakthrough was made recently\textsuperscript{\textcopyright}, so that the perturbative coefficients are now known up to $\tilde{K}_4$ (see\textsuperscript{\textcopyright} for the numerical values of the $\tilde{K}_n(\xi)$ coefficients).

#### 3.1 Perturbative Methods

The perturbative contribution to $R_\tau$ provides the main source of sensitivity to $\alpha_s(s_0)$. The value of the strong coupling in the complex plane can be computed assuming the validity of the renormalisation group equation (RGE) outside the real axis, and using a Taylor series of $\eta = \ln(s/s_0)$. In the fixed order perturbation theory (FOPT), at each integration step, the Taylor expansion is made around the physical value $\alpha_s(s_0)$. This may cause important problems as the
absolute value of $\eta$ gets large and the convergence speed of the series is reduced. In addition, a cut at a fixed order in $\alpha_s(s_0)$ is applied on the Taylor series and on the integration result in FOPT. Therefore, important known higher order terms are neglected, yielding additional systematic uncertainties. A better suited method is CIPT which, at each integration step, computes $\alpha_s(s)$ using the value found at the previous step. In this approach the Taylor expansion is always used for small absolute values of its parameter, hence excellent convergence properties.

In practice we have also used geometric growth estimations for the first unknown coefficients $\beta_4$, $K_5$ and $K_6$. We have tested that CIPT is less sensitive to changes of these coefficients, and it also exhibits a smaller scale dependence than FOPT. Numerically, the difference of the perturbative contributions computed with the two methods are about 15%. In fact this difference could have been much larger if not for the properties of the kernel in the integral (4) which has small absolute values in the region where the $\alpha_s(s)$ predictions of the two methods are rather different.

The CIPT method behaves better than FOPT and is to be preferred. The difference between the results obtained with the two approaches is not to be interpreted as a systematic theoretical error, but rather like a problem of FOPT.

### 3.2 Quark-Hadron Duality Violation

It is known that OPE describes only part of the nonperturbative effects. In order to estimate the impact of the missing contributions, we test two models based on resonances and on instantons. We add their contributions to the theoretical prediction, choosing parameters that provide a good matching to the V+A spectral function near the $\tau$ mass. For these models, we find corrections situated within our systematic uncertainties.

### 4 Combined Fit

In order to obtain additional experimental information, we use spectral moments defined as

$$R_{\tau,V/A}^{kl} = \int_0^{m_\tau^2} ds \left( 1 - \frac{s}{m_\tau^2} \right)^k \left( \frac{s}{m_\tau^2} \right)^l dR_{\tau,V/A} ds .$$

They allow one to better exploit the shape of the spectral functions and they suppress the region where OPE fails. The corresponding theoretical prediction is very similar to (3), with consequent perturbative and nonperturbative contributions. Due to strong correlations, we use only $R_\tau$ ($k = 0$ and $l = 0$) and four additional moments ($k = 1$ and $l = 0, 1, 2, 3$) to simultaneously fit $\alpha_s(m_\tau^2)$ and the leading $D = 4, 6, 8$ nonperturbative contributions.

In spite of the fact that the nonperturbative contributions fitted for the V and A spectral functions have opposite signs and they are one order of magnitude larger than those from V+A, we find an excellent agreement between the values found for $\alpha_s(m_\tau^2)$ from the three fits. The result of the fit to the V+A spectral moments reads

$$\alpha_s(m_\tau^2) = 0.344 \pm 0.005 \pm 0.007 ,$$

where the first error is experimental and the second is theoretical. When evolving this value to the Z scale (see Fig. 1) one gets

$$\alpha_s^{(\tau)}(m_Z^2) = 0.1212 \pm 0.0005 \pm 0.0008 \pm 0.0005 ,$$

where the first two errors are propagated from (6), and the last one summarises uncertainties in the evolution. The consistency between this result and the value found by a global fit to
Figure 1: Top: The evolution of $\alpha_S(m_\tau^2)$ to higher scales $\mu$ using the four-loop RGE and the three-loop matching conditions applied at the heavy quark-pair thresholds (hence the discontinuities at $2m_c$ and $2m_b$). The evolution is compared with independent measurements covering $\mu$ scales that vary over more than two orders of magnitude. Bottom: The corresponding $\alpha_S$ values evolved to $m_Z$. The shaded band displays the $\tau$ decay result within errors.

electroweak data at the Z-mass scale $\prod (\alpha_S^{(\tau)}(m_Z^2) - \alpha_S^{(Z)}(m_Z^2)) = 0.0021 \pm 0.0029$, provides the most powerful present test of the evolution of the strong interaction coupling over a range of $s$ spanning more than three orders of magnitude.

5 Conclusions

Motivated by some new results both on theoretical and experimental grounds, we have revisited the determination of $\alpha_S(m_\tau^2)$ from the ALEPH $\tau$ spectral functions. We have reexamined two common numerical methods: we have identified specific consistency problems of FOPT, which do not exist in CIPT. The $\tau$ measurement of $\alpha_S$ evolved to the Z scale is found to be in excellent agreement with the direct determination from Z decays. Both results are the only ones at $N^3LO$ order so far, confirming the running of $\alpha_S$ between 1.8 and 91 GeV, as predicted by QCD, with an unprecedented precision of 2.4%.

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References

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