An oriented Riemannian 4-manifold $(M^4, g)$ is said to be anti-self-dual if its self-dual Weyl curvature vanishes. For example, any oriented locally-conformally-flat 4-manifold is anti-self-dual.

A Kähler manifold of complex dimension two is anti-self-dual if and only if it is scalar-flat. Among anti-self-dual conformal classes, those that can be represented by Kähler metrics are highly non-generic. The authors illustrate this fact using the example $M = \Sigma \times \mathbb{CP}^1$ where $\Sigma$ is a closed Riemann surface of genus $g \geq 2$. The moduli space of anti-self-dual conformal classes on $M$ has dimension $30(g-1)$, but only a subspace of dimension $12(g-1)$ arises from conformal structures that contain scalar-flat Kähler metrics.

One then considers a larger class of almost Kähler anti-self-dual metrics to provide a possible model for general anti-self-dual metrics. Almost Kähler anti-self-dual metrics share many of the remarkable properties of the scalar-flat Kähler metrics. More importantly, the almost Kähler condition is open in the moduli space of anti-self-dual conformal classes.

This paper asks whether almost-Kähler condition is also closed in the anti-self-dual context and shows that the answer is no.

The proof is by constructing a family of metrics on the 4-manifold $M = \Sigma \times S^2$, where $\Sigma$ is a closed oriented Riemann surface of even large genus. Indeed, the authors show that the 4-manifold $M$ admits locally-conformally-flat conformal classes $[g]$ that cannot be represented by almost-Kähler metrics, but are deformations of scalar-flat Kähler metric on $M$. Such $(M, [g])$ are anti-self-dual 4-manifold arising via the hyperbolic ansatz from a quasi-Fuchsian 3-manifold of Bers type.

The examples described above are all locally-conformally-flat, and thus live on 4-manifolds of signature zero. A similar construction in this paper produces many explicit examples showing that the almost Kähler condition is not closed living on 4-manifolds with negative signature, and so are not locally conformally flat.

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