Localization and Perron–Frobenius Theory for Directed Polymers

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Abstract

We consider directed polymers in a random potential given by a deterministic profile with a strong maximum at the origin taken with random sign at each integer time.

We study two main objects based on paths in this random potential. First, we use the random potential and averaging over paths to define a parabolic model via a random Feynman–Kac evolution operator. We show that for the resulting cocycle, there is a unique positive cocycle eigenfunction serving as a forward and pullback attractor. Secondly, we use the potential to define a Gibbs specification on paths for any bounded time interval in the usual way and study the thermodynamic limit and existence and uniqueness of an infinite volume Gibbs measure. Both main results claim that the local structure of interaction leads to a unique macroscopic object for almost every realization of the random potential.

1 Introduction

In this note, we consider two problems related to directed polymers with a localization property. Directed $d+1$-dimensional polymers are modeled by random walks in a random potential $\phi : \mathbb{Z}^d \times \mathbb{Z} \times \Omega \to \mathbb{R}$ defined for all times $n \in \mathbb{Z}$ on the lattice $\mathbb{Z}^d$ via

$$\phi_n(x) = \phi_n(x, \omega) = V(x)B_n(\omega).$$

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Here $V : \mathbb{Z}^d \to \mathbb{R}$ is a deterministic bounded function, and $B_n(\omega)$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values $\pm 1$ with probabilities $1/2$. It is convenient to suppose that $\Omega = \{-1, 1\}^\mathbb{Z}$, $\mathcal{F}$ is the cylindric $\sigma$-algebra, and $P$ is the 1/2-Bernoulli product measure on $\Omega$. The random variables $B_n$ coincide with coordinate maps: $B_n(\omega) = \omega_n$ for $\omega \in \Omega, n \in \mathbb{Z}$.

For any integers $n_1 \leq n_2$ the set of paths that we consider consists of all possible trajectories of the so called “lazy random walk”. Namely, we deal only with paths $\gamma : [n_1, n_2] \cap \mathbb{Z} \to \mathbb{Z}^d$ such that for any $n$, $\gamma_{n+1}$ equals either $\gamma_n$ or one of the $2d$ neighbors of $\gamma_n$ in $\mathbb{Z}^d$, i.e., $|\gamma_{n+1} - \gamma_n| \leq 1$.

We study two main objects based on paths in the random potential $\phi$. First, we use the random potential $\phi$ and averaging over paths to define a parabolic model via a random Feynman–Kac evolution operator. We show that for the resulting cocycle, there is a unique positive cocycle eigenfunction serving as a forward and pullback attractor. Secondly, we use $\phi$ to define a Gibbs specification on paths for any bounded time interval in the usual way and study the thermodynamic limit and a.s.-existence and uniqueness of an infinite volume Gibbs measure. Both main results claim that the local structure of interaction leads to a unique macroscopic object for almost every realization of the random potential.

The analogues of our results for a setting with a compact spin space replacing $\mathbb{Z}^d$ are based entirely on elementary contraction arguments. The non-compactness of our spin space $\mathbb{Z}^d$ is the crucial feature that makes the problem more interesting and difficult. We have to combine the contraction arguments with those based on localization. It is the localization that makes the evolution of the system essentially compact. The localization arguments we use are similar to those used by Sinai in his paper [Sin93] on Nechaev’s model. However, the mechanism of localization in Nechaev’s model is different from ours. The result on existence and uniqueness of a Gibbs measure is also related to the characterization of stationary Markov chains with finitely many states as Gibbs distributions with zero-range potential, see e.g. [Geo88, Chapter 3].

Localization for random walks in a random potential has been studied intensively in the past 10 years, see [CH02], [CSY03], [CSY04], [AS06], [Gia07], and references therein. In our setting, the localization properties are based on a specific structure of the spatial potential $V$. We assume that it has a large maximum (or minimum) located (without loss of generality) at the origin and has the following form: $V(x) = V_0(x) + \Lambda \delta_0(x)$. Here $V_0$ is an arbitrary bounded potential and the delta-potential $\delta_0$ takes value 1 at the origin and 0 everywhere else. All the results in this paper are proven...
under the assumption that Λ is large, the exact condition being expressed in terms of max|V₀(x)| and the dimension d. A special case when V₀(x) ≡ 0 was considered in [AS06], [Gia07]. It turn out that in dimension one and two localization occurs for an arbitrary Λ > 0.

In the theory of directed polymers it is often assumed that the dependence of the random potential on the space variable x is non-deterministic. Of course, we can also think that V₀ is a realization of a bounded stationary potential V₀ω. However, since our results hold for all such realizations we fix a particular realization and assume that V is deterministic.

In the physics literature, mostly the model with independent noise values Bₓ,n for all space-time points (x, n) has been considered. In the model that we suggest it is essential that one realization of the noise serves all lattice points x at once, thus modeling a spatially disordered structure represented by V and embedded into an external potential Bₙ fluctuating in time. The conditions we impose on the random potential φ can be significantly relaxed though, and the results we prove hold true for a wider class of potentials with localization properties.

For example, one can consider the following random potential:

\[ φ_n(x, ω) = \sum_{i=1}^{m} V_i(x) B_n^{(i)}(ω). \]

Here \((V_i)_{1≤i≤m}\) is a collection of bounded potentials and \((B^{(i)})_{1≤i≤m}\) is a family of m independent copies of the “random sign” process \(B_n(ω)\). To guarantee the localization it is sufficient to assume that one of the potentials \(V_i\) has a large maximum or minimum at the origin.

It is also possible to study the continuous time case where \(B_n(ω)\) is replaced by the white noise \(\dot{W}(t)\). However, this case is technically more complicated and it will be considered in a forthcoming publication.

The paper is organized as follows. In this section, we describe the setting in detail and state our two main results. The crucial localization lemma is proven in Section 2 and it is used to prove the main results in Sections 3 (cocycle eigenfunctions) and 4 (infinite volume Gibbs distributions) respectively. Section 5 is devoted to auxiliary technical lemmas.

1.1 The parabolic model

The Feynman–Kac operator associated with φ is defined for two integers \(n₁ < n₂\) and all bounded functions u as

\[ T^{n₁,n₂}u(x) = (T^{n₁,n₂}(ω)u)(x) = E e^{Φ_{n₁,n₂}(γ)} u(γ(n₁)), \]
where
\[ \Phi_{n_1,n_2}(\gamma) = \Phi_{n_1,n_2}(\gamma,\omega) = \sum_{n=n_1+1}^{n_2} \phi_n(\gamma(n),\omega) = \sum_{n=n_1+1}^{n_2} V(\gamma(n))\omega_n, \]
and the expectation \( E \) is taken with respect to the uniform distribution on the space of directed polymer realizations \( \Gamma_{n_1,n_2}(x) \) which is the set of admissible paths \( \gamma : [n_1,n_2] \to \mathbb{Z}^d \) with \( \gamma(n_2) = x \), i.e.
\[ T^{n_1,n_2}u(x) = \frac{1}{(2d+1)^{n_2-n_1}} \sum_{\gamma \in \Gamma_{n_1,n_2}(x)} e^{\Phi_{n_1,n_2}(\gamma)}u(\gamma(n_1)). \] (2)

We recall that throughout the paper, by admissible paths or, simply, paths we mean lazy random walk trajectories (see the Introduction), and by \([n_1,n_2]\) we always mean a discrete set \( \{n_1,\ldots,n_2\} \).

It is easy to see that \( T^{n_1,n_2} \) can be considered as a discrete time analogue of the Feynman-Kac kernel for the linear heat equation. The family of operators \( T^{n_1,n_2} \) defines a random dynamical system (cocycle) on \( L^\infty = L^\infty(\mathbb{Z}^d) \) (we shall denote the \( L^\infty \) norm by \( \| \cdot \| \)): for every \( \omega \in \Omega \)
\[ T^{n_2,n_3}(\omega)T^{n_1,n_2}(\omega) = T^{n_1,n_3}(\omega), \]
or, equivalently,
\[ T^{n_2}(\theta^{n_1}\omega)T^{n_1}(\omega) = T^{n_1+n_2}(\omega), \]
where \( T^n(\omega) = T^{0,n}(\omega) \) and the time shift \( \theta^k : \Omega \to \Omega \) is defined via
\[ \theta^k\omega(n) = \omega(k+n), \quad k,n \in \mathbb{Z}. \]

Our goal is to prove a Perron–Frobenius theorem for the cocycle \( T \) defined above. The classical notion of an eigenfunction has to be modified for the cocycle setting.

We say that an \( L^\infty \)-valued random variable \( u^\omega \) is an eigenfunction for the cocycle \( T \) if with probability 1 and for all \( n \in \mathbb{N} \), \( T^n(\omega)u^\omega \) is equal to \( u^{\theta^n\omega} \) up to a random scalar factor. It is sufficient to impose this requirement for \( n = 1 \). Obviously, \( u^\omega \) is defined up to a multiplication by an arbitrary \( c(\omega) \neq 0 \). Thus, it is convenient to introduce normalized eigenfunctions assuming \( \|u^\omega\| = 1 \). Then
\[ T^n(\omega)u^\omega = \kappa_n(\omega)u^{\theta^n\omega}, \quad n \in \mathbb{N}. \]
for some process $(\kappa_n(\omega))_{n \in \mathbb{N}}$. When working with normalized functions, it is useful to consider a normalized version of the cocycle $T^\omega$ defined by

$$
\overline{T}^n(\omega) \phi = \frac{T^n(\omega) \phi}{\|T^n(\omega) \phi\|}.
$$

For a positive $\lambda$, an eigenfunction $u^\omega(x)$ is called $\lambda$-localized if there is a random variable $c = c(\omega) > 0$ such that with probability 1, for all $x \in \mathbb{Z}^d$,

$$
|u^\omega(x)| \leq c(\omega)e^{-\lambda|x|},
$$

where $|x| = \max\{|x_1|, \ldots, |x_d|\}$.

Throughout the paper we shall require the following conditions on the potential function $V$: there are constants $M_0, M_1$ and $\lambda_0$ such that

1. $V(0) = M_0$, \hfill (3)
2. $|V(x)| \leq M_1$, $x \neq 0$, \hfill (4)
3. $\lambda_0 = \frac{1}{2}(M_0 - 3M_1) - \ln(2d + 1) > 0$, \hfill (5)

For our first main theorem we shall need one more condition:

$$
\lambda_1 \overset{\text{def}}{=} 2M_1 + \ln(2d + 1) < \lambda_0
$$

\textbf{Theorem 1} Suppose conditions (3) – (6) on $V$ are satisfied. Then there is a unique normalized, positive eigenfunction $u^\omega$ for the cocycle $T$ satisfying

$$
\limsup_{n \to -\infty} u^{\theta^n \omega}(0) > 0.
$$

This eigenfunction is $\lambda$-localized for any $\lambda \in (0, \lambda_0)$.

Moreover, almost surely, for any nonnegative function $v \in L^\infty$ not identically equal to zero,

$$
\lim_{n \to -\infty} \|u^{\theta^n \omega} - \overline{T}^n v\| = 0 \quad \text{ (forward attraction)},
$$

and

$$
\lim_{n \to -\infty} T^{-n, 0} v = u^\omega \quad \text{ (pullback attraction)}.
$$

A proof of this result is given in Section 3.
Remark 1 It follows straightforwardly from the ergodicity of the shift operator $\theta$ that there is a non-random Lyapunov exponent $\lambda_L$ so that with probability 1,
\[
\lim_{n \to \infty} \frac{\ln \kappa_n(\omega)}{n} = \lambda_L.
\]

Remark 2 Assumption (3) can be replaced by $|V(0)| = M_0$. We require $V(0)$ to be positive without loss of generality and only to simplify the notations in the proofs.

Remark 3 Condition (6) may be relaxed if the forward and pullback attraction hold for functions $v$ that satisfy certain localization conditions (belong to $L^1$, $L^2$, or have compact support).

1.2 Localized Gibbs distributions

Let $\Gamma_{n_1,n_2}(x_1,x_2)$ be the set of all admissible paths on $[n_1,n_2]$ with fixed endpoints $\gamma(n_1) = x_1$, $\gamma(n_2) = x_2$.

We say that a measure $\mu$ on $(\mathbb{Z}^d)^\mathbb{Z}$ (i.e. on paths $\alpha : \mathbb{Z} \to \mathbb{Z}^d$) is a Gibbs measure corresponding to a realization of the potential $\phi : \mathbb{Z} \times \mathbb{Z}^d \to \mathbb{R}$ if it satisfies the DLR condition with Gibbs specification given by

\[
\mu_{n_1,n_2}(\gamma|\, x_1,x_2) = \frac{1}{Z_{n_1,n_2}(x_1,x_2)} e^{\Phi_{n_1,n_2}(\gamma)}, \quad \gamma \in \Gamma_{n_1,n_2}(x_1,x_2),
\]

with

\[
Z_{n_1,n_2}(x_1,x_2) = \sum_{\gamma \in \Gamma_{n_1,n_2}(x_1,x_2)} e^{\Phi_{n_1,n_2}(\gamma)}.
\]

Namely, for any times $n_1 < n_2$ and any points $x_1, x_2 \in \mathbb{Z}$ the conditional distribution on paths in $\Gamma_{n_1,n_2}(x_1,x_2)$ defined by $\mu$ agrees with this specification: for any path $\gamma \in \Gamma_{n_1,n_2}(x_1,x_2)$,

\[
\mu \{ \alpha_{[n_1,n_2]} = \gamma \mid \alpha_{[n_1,n_2]} \in \Gamma_{n_1,n_2}(x_1,x_2) \} = \mu_{n_1,n_2}(\gamma|\, x_1,x_2),
\]

where $\alpha_{[n_1,n_2]}$ is the restriction of $\alpha$ on $[n_1,n_2]$.

Notice that $Z_{n_1,n_2}(x_1,x_2)$ can be expressed via the cocycle $T$:

\[
Z_{n_1,n_2}(x_1,x_2) = (2d+1)^{n_2-n_1} T^{n_1,n_2} \delta_{x_1}(x_2), \quad \text{(8)}
\]

where

\[
\delta_{x_1}(x) = \begin{cases} 1, & x = x_1, \\ 0, & x = 0. \end{cases}
\]
Theorem 2  There is a set $\Omega' \subset \Omega$ of probability 1 with the following properties:

1. For every $\omega \in \Omega'$, there is a Gibbs measure $\mu = \mu^\omega$ corresponding to the realization of the potential $\phi$.

2. For every $\omega \in \Omega'$, the measure $\mu$ is a unique Gibbs measure with the following property: for every $\varepsilon > 0$ there is a number $r_\varepsilon$ such that
   \[ \liminf_{n \to \pm \infty} \mu\{\alpha : |\alpha_n| > r_\varepsilon\} < \varepsilon. \] (9)

Moreover, for any $\lambda \in (0, \lambda_0)$, the measure $\mu$ is $2\lambda$-localized: there is a random variable $c = c(\omega) > 0$ such that for every $\omega \in \Omega'$, every $n \in \mathbb{Z}$,
   \[ \mu\{\alpha : \alpha_n = x\} < c(\theta^n \omega)e^{-2\lambda|x|}, \quad x \in \mathbb{Z}^d. \] (10)

Remark 4  Notice that the exponent in the r.h.s. of (10) is $2\lambda$ rather than $\lambda$ that appears in the statement of Theorem 1. This is due to two-sided estimates that we can use in the proof of Theorem 2.

A proof of Theorem 2 is given in Section 4.

2  Main localization lemma: the optimal path vs. entropy

This section is devoted to the estimate that plays the central role in the proofs given in forthcoming sections. We shall need the following “optimal” path $\gamma^*$:
   \[ \gamma^*_n(\omega) = \begin{cases} 0, & \omega_n = 1 \\ e_1, & \omega_n = -1. \end{cases} \] (11)
where $e_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^d$. Clearly,
   \[ \Phi_{n_1, n_2}(\gamma^*) \geq \sum_{n_1+1}^{n_2} \xi_m, \]
where the random variables $(\xi_m)_{m \in \mathbb{Z}}$ are defined by
   \[ \xi_m(\omega) = M_01_{\{\omega_m = 1\}} - M_11_{\{\omega_m = -1\}}. \]
**Lemma 1** Let \( \lambda \in (0, \lambda_0) \). Then there are random variables \( \nu^-(\omega) = \nu^-(\omega_1, \omega_2, \ldots) \) and \( \nu^+(\omega) = \nu^+(\omega_1, \omega_2, \ldots) \) such that with probability 1, for any \( k \geq \nu^-(\omega) \),

\[
e^{\Phi_{-k,0}(\gamma^*)} \geq e^{\sum_{m=-k+1}^{0} \xi_m} > (2d+1)^k e^{M_1 k e^{\lambda k}},
\]

and for any \( k \geq \nu^+(\omega) \),

\[
e^{\Phi_{0,k}(\gamma^*)} \geq e^{\sum_{m=1}^{k} \xi_m} > (2d+1)^k e^{M_1 k e^{\lambda k}}.
\]

In addition, \( P\{\nu^-(\omega) = 1\} > 0 \), and \( P\{\nu^+(\omega) = 1\} > 0 \).

**Proof:** Both \( \sum_{m=1}^{k} \xi_m \) and \( \sum_{m=-k+1}^{0} \xi_m \) are sums of i.i.d. random variables with mean equal to \( (M_0 - M_1)/2 \). Let

\[
\varepsilon = \frac{M_0 - M_1}{2} - M_1 - \ln(2d+1) - \lambda. \tag{12}
\]

Notice that \( \varepsilon > 0 \) due to (5). Now the strong law of large numbers implies the existence of random variables \( \nu^\pm \) such that for almost every \( \omega \) if \( k \geq \nu^-(\omega) \) then

\[
\sum_{m=-k+1}^{0} \xi_m > ((M_0 - M_1)/2 - \varepsilon)k.
\]

and if \( k \geq \nu^+(\omega) \) then

\[
\sum_{m=1}^{k} \xi_m > ((M_0 - M_1)/2 - \varepsilon)k.
\]

Therefore, for these values of \( k \), respectively,

\[
\frac{1}{(2d+1)^k} e^{\sum_{m=-k+1}^{0} \xi_m} \geq e^{((M_0 - M_1)/2 - \varepsilon - \ln(2d+1))k} = e^{M_1 k e^{\lambda k}},
\]

\[
\frac{1}{(2d+1)^k} e^{\sum_{m=1}^{k} \xi_m} \geq e^{((M_0 - M_1)/2 - \varepsilon - \ln(2d+1))k} = e^{M_1 k e^{\lambda k}},
\]

and the proof is complete. \( \square \)

### 3 Proof of Theorem 1

Our plan is to find, for any large value of \( r \in \mathbb{N} \), a sequence of numbers \((n_i)\) decreasing to \(-\infty\) and such that, \( T_{n_i+1,n_i} \) is essentially a contraction in
Hilbert projective metric on functions restricted to $B_r = [-r, r]^d$, the ball of radius $r$ in the sup-norm $|\cdot|$ on $\mathbb{Z}^d$.

The proof relies on several lemmas. The first part of the proof is devoted to finding an invariant and almost compact set essentially supporting the dynamics.

We begin with the main localization result. For any $c \geq 1$, we denote

$$F(c) = \{ \phi : \mathbb{Z}^d \to \mathbb{R}^+ : \| \phi \| \leq c\phi(0) \}.$$ 

**Lemma 2** Suppose $\lambda \in (0, \lambda_0)$. There is a number $K_1(\lambda) \geq 1$ such that if $c \geq 1$, $\phi \in F(c)$, $|y| \geq \nu^- (\omega)$, and $n \geq \nu^- (\omega)$, then

$$T^{-n,0} \phi(y) \leq K_1(\lambda) (e^{-\lambda |y|} + ce^{-\lambda n}) T^{-n,0} \phi(0).$$

The proof of this Lemma is given in Section 5.

For $\lambda \in (0, \lambda_0)$, $r \in \mathbb{N}$, and $c \geq 1$, we define

$$n_0(\lambda, r, c) = \frac{\ln c}{\lambda} + r + 1.$$ 

**Lemma 3** Suppose $\lambda \in (0, \lambda_0)$, $r \in \mathbb{N}$, and $c \geq 1$. If $n > n_0(\lambda, r, c)$, $r \geq \nu^- (\omega)$, and $\phi \in F(c)$, then

$$\| (T^{-n,0} \phi) 1_{B_r} \| \leq 2K_1(\lambda) e^{-\lambda (r+1)} T^{-n,0} \phi(0).$$

This lemma is a direct implication of Lemma 2 since its conditions automatically imply $n > \nu^- (\omega)$.

For the rest of the proof we fix $\lambda \in (0, \lambda_0)$. Lemma 1 implies that for any $r \in \mathbb{N}$, there is an event $A$ with $P(A) > 0$ such that for every $\omega \in A$ the following conditions hold true:

1. $\omega_1 = \ldots = \omega_r = 1$;
2. $\nu_+ (\theta^r \omega) = 1$;
3. $\omega_{-r+1} = \omega_{-r+2} = \ldots = \omega_0 = 1$;
4. $\nu_- (\theta^{-r} \omega) = 1$.

Therefore, with probability 1, we can choose a sequence $(n_i)_{i \in \mathbb{N}}$ (depending on $\omega$) decreasing to $-\infty$ and such that the following Conditions 1–5 are satisfied for each $i \in \mathbb{N}$:

1. $\omega_{n_i+1} = \ldots = \omega_{n_i+r} = 1$;
2. $\nu_+ (\theta^{n_i+r} \omega) = 1$;

3. $\omega_{n_i-r+1} = \omega_{n_i-r_0+2} = \ldots = \omega_{n_i} = 1$;

4. $\nu_- (\theta^{n_i-r} \omega) = 1$.

5. $n_{i-1} - n_i > 2n_0(\lambda, r, 2K_1(\lambda))$.

The sequence can be chosen in a measurable way. Notice that if it satisfies conditions 1–5 for some $r$ it also satisfies the same conditions with $r$ replaced by any nonnegative $r' < r$.

For $r \in \mathbb{N}$, we define

$$G(\lambda, r) = \{ \phi \in F(2K_1(\lambda)) : \| \phi 1_{B^c_r} \| \leq \| \phi 1_{B_r} \| \}.$$  

**Lemma 4** There is a nonrandom $r_0 \in \mathbb{N}$ such that

1. If $c \geq 1$, $r > r_0$, and $-n < n_i - n_0(\lambda, r, c)$, then

$$T^{-n, n_i} F(c) \subset G(\lambda, r).$$

2. If $r > r_0$, then for any $i \in \mathbb{N}$,

$$T^{n_i, n_i-1} G(\lambda, r) \subset G(\lambda, r).$$

**Proof:** The first part of the lemma follows from Lemma 3. The second part is a consequence of the first one and Condition 5. $\square$

**Lemma 5** For $r \in \mathbb{N}$, there is a number $K_2(\lambda, r) \geq 1$ such that if $\| \phi 1_{B^c_r} \| \leq \| \phi 1_{B_r} \|$ then for any $y_1$ and $y_2$ with $|y_1|, |y_2| \leq r$, and any $i \in \mathbb{N}$,

$$\frac{1}{K_2(\lambda, r)} \leq \frac{T^{n_i, n_i-1} \phi(y_1)}{T^{n_i, n_i-1} \phi(y_2)} \leq K_2(\lambda, r).$$

The proof is given in Section 5.

We introduce now

$$H(\lambda, r) = \left\{ \phi \in G(\lambda, r) : \| \phi \| = 1 \text{ and } \phi(y) \geq \frac{1}{K_2(\lambda, r)} \text{ for } |y| \leq r \right\}.$$  

**Lemma 6** 1. If $r > r_0$, then for any $i \in \mathbb{N}$,

$$T^{n_i, n_i-1} H(\lambda, r) \subset H(\lambda, r).$$
2. If $c > 0$, $r > r_0$, $-n < n_i - n_0(\lambda, r, c)$, then

$$T^{-n_i, n_{i-1}} F(c) \subset H(\lambda, r).$$

**Proof:** Lemmas 4 and 5 imply

$$T^{n_i, n_{i-1}} G(\lambda, r) \subset H(\lambda, r),$$

and the first part of the lemma follows since $H(\lambda, r) \subset G(\lambda, r)$. The second part of the lemma follows from the first one and Lemma 4. □

The Hilbert projective metric $\rho_r$ between two functions $\phi, \psi : B_r \rightarrow \mathbb{R}_+$ is defined by

$$\rho_r(\phi, \psi) = \ln \left( \max_{|x| \leq r} \phi(x) \cdot \max_{|x| \leq r} \psi(x) \right).$$

The Hilbert metric $\rho_r$ is not a true metric since it does not distinguish functions proportional to each other. However it does define a metric on normalized positive functions defined on $B_r$.

For a function $\phi : \mathbb{Z}^d \rightarrow \mathbb{Z}_+$, we denote its restriction onto $B_r$ by $\pi_r \phi$.

We notice that $\pi_r H(\lambda, r)$ is closed in $\rho_r$, and $\text{diam}_r(H(\lambda, r)) < \infty$, where for a set $A$ we denote

$$\text{diam}_r(A) = \sup \{ \rho_r(\pi_r \phi, \pi_r \psi) : \phi, \psi \in A \}.$$

The following is the main contraction estimate:

**Lemma 7** There are numbers $K_3(\lambda), K_4(\lambda)$ such that for any $r$, any set $A \subset H(\lambda, r)$, and any $i \geq 2$,

$$\text{diam}_r(T^{n_i, n_{i-1}} A) \leq (1 - K_3(\lambda)e^{-2\lambda_2 r})\text{diam}_r(A) + K_4(\lambda)e^{-2\lambda_2 r}.$$

The proof is given in Section 5.

We can apply this estimate recursively along the sequence $(n_i)$. We need the following lemma to make connection to time 0 which is not included in $(n_i)$:

**Lemma 8** There is a number $K_5(\lambda)$ such that for any $r \in \mathbb{N}$ and sufficiently large $i$, if $\phi, \psi \in F(2K_1(\lambda))$, then

$$\rho_r(\pi_r T^{n_i, 0} \phi, \pi_r T^{n_i, 0} \psi) \leq \rho_r(\pi_r \phi, \pi_r \psi) + K_5(\lambda)e^{-\lambda_2 r},$$

where

$$\lambda_2 = 2(M_0 - M_1 - \ln(2d + 1)) > 0.$$
The proof is given in Section 5.

Proof of Theorem 1: Lemma 6 implies that for sufficiently large $n$,
\[ T^{-n,0}F(c) \subset T^{m_{i-1},0}H(\lambda, r), \tag{13} \]
and if $i_1 < i_2$,
\[ T^{m_{i_2},0}H(\lambda, r) \subset T^{m_{i_1},0}H(\lambda, r). \tag{14} \]

Take any sequence $\phi = (\phi_n)_{n \in \mathbb{N}}$ in $F(c)$ and consider the sequence $(T^{-n,0}\phi_n)$. Relations (13) and (14) imply that for each $r$, $\pi_r T^{-n,0}\phi_n$ are uniformly bounded in $n$ and $\phi$. We consider the all pointwise limit points of $T^{-n,0}\phi_n$. Since pointwise limit points are not necessarily normalized in the uniform norm, we normalize them and denote the resulting set by $S_\phi$. We denote by $S$ the union of $S_\phi$ over all possible sequences $\phi$.

It follows from the classical diagonal method that $S$ is not empty. Let us show that $S$, in fact, consists of a single point. Suppose that on the contrary $S$ contains at least two different points $\psi$ and $\psi'$. Then there exists $\bar{\rho} > 0$ such that $\rho_r(\pi_r \psi, \pi_r \psi') \geq \bar{\rho}$ for all large enough $r$. Since $\psi$ and $\psi'$ are limiting points, there exist two sequences of functions $(\psi_k)$ and $(\psi'_k)$ and two sequences of times $(m_k)$ and $(m'_k)$ decaying to $-\infty$ such that for all large enough $k$,
\[ \rho_r(\pi_r T^{-m_k,0}\psi_k, \pi_r T^{-m'_k,0}\psi'_k) \geq \frac{\bar{\rho}}{2}. \tag{15} \]

On the other hand, for any $i$, if $k$ is large enough, then both $T^{-m_k,n_i}\psi_k$ and $T^{-m'_k,n_i}\psi'_k$ belong to $F(2K_1(\lambda))$. Lemmas 7 and 8 imply that
\[ \limsup_{i \to \infty} \text{diam}_r(T^{-n,0}F(2K_1(\lambda))) \leq d(\lambda, r) = \frac{K_4(\lambda) e^{-2\lambda r}}{K_3(\lambda) e^{-2\lambda_1 r}} + K_5(\lambda) e^{-\lambda_2 r}. \]

Taking $r$ large enough so that $d(\lambda, r) < \frac{\bar{\rho}}{2}$, we obtain a contradiction with (15). We conclude that $S_\infty$ cannot contain two distinct elements. Therefore, $S_\infty = \{\psi_\infty\}$ for some $\phi_\infty$, and it is easy to see that $\psi_\infty$ does not depend on $c$.

We now set $u^\omega = \psi_\infty$, where $\psi_\infty$ is the unique element of $S$. The uniqueness above ensures that $u$ is a positive cocycle eigenfunction satisfying (7). It is also obviously unique. The desired localization property follows from Lemma 2. The pullback attraction follows since for any $\varepsilon$, we can find $r_0$ and $n_0$ such that
\[ \|u_{B_{r_0}}\| < \varepsilon, \]
\[ \|(T^{-n,0}u)_{B_{r_0}}\| < \varepsilon, \quad n > n_0, \]
and
\[ \rho_{r_0} (T^{-n,0}_v, u) < \varepsilon, \quad n > n_0. \]
The forward attraction is proven similarly. \(\square\)

4 Proof of Theorem 2

For a fixed \(\omega\), we say that a measure \(\mu_{n_1,n_2}\) on \(\Gamma_{n_1,n_2}\) is a finite volume Gibbs distribution on \([n_1,n_2]\) for the realization of the potential \(\phi\) if for any points \(x_1,x_2\) and any path \(\gamma \in \Gamma_{n_1,n_2}(x_1,x_2)\),
\[
\mu_{n_1,n_2}(\{\gamma\} \mid \Gamma_{n_1,n_2}(x_1,x_2)) = \frac{e^{\Phi_{n_1,n_2} (\gamma)}}{Z_{n_1,n_2} (x_1,x_2)}.
\]

Let us introduce \(\nu = \max\{\nu^+,\nu^-\}\), where \(\nu^\pm\) are introduced in Section 2.

The proof of the following localization lemma for finite volume Gibbs distributions is given in Section 5:

**Lemma 9** For any \(\lambda \in (0,\lambda_0)\), there is a constant \(C > 0\) and a function \(N_1 : \mathbb{N} \to \mathbb{N}\) with the following property. Suppose \(\mu_{n_1,n_2}\) is a finite volume Gibbs measure on an interval \([n_1,n_2]\) for a realization of the potential \(\phi\). If \(r \in \mathbb{N}\) and \(n \in [n_1,n_2]\) satisfy
\[
\mu_{n_1,n_2}\{\alpha : \alpha_{n_1}, \alpha_{n_2} \in B_r\} = 1,
\]

\[
\nu(\theta^n \omega) < r,
\]

\[
n_2 - n > N_1(r),
\]

\[
n - n_1 > N_1(r),
\]

then
\[
\mu_{n_1,n_2}\{\alpha : |\alpha_n| > r\} < C e^{-2\lambda r}.
\]

Let us prove the existence first. We fix an \(\omega \in \Omega\) and for each \(m \in \mathbb{N}\) consider a unique measure \(\mu^m\) on \((\mathbb{Z}^d)^\mathbb{Z}\) such that

1. \(\mu^m\{\alpha : \alpha_k = 0\} = 1\) for all \(k \leq -m\) and all \(k \geq m\)

2. The projection of \(\mu^m\) on \([-m,m]\) is a finite volume Gibbs measure.

The following statement is a direct consequence of Lemma 9.
Lemma 10  For any $\lambda \in (0, \lambda_0)$, there is a random variable $c(\omega)$ such that for almost every $\omega \in \Omega$ and every $r$,

$$\mu^m\{\alpha : |\alpha_n| > r\} \leq c(\theta^n \omega)e^{-2\lambda r}.$$ 

Applying this result, we conclude that with probability 1, the sequence of measures $(\mu^m)$ is tight in $(\mathbb{Z}^d)^\mathbb{Z}$, and, due to the Prokhorov criterion, it contains a weakly convergent subsequence. Denoting the limit of this subsequence by $\mu$, one can easily verify that $\mu$ is a Gibbs measure satisfying (10).

To prove the uniqueness, we must show that any two Gibbs measures $\mu$ and $\mu'$ satisfying (9) coincide. The plan is as follows. We shall consider a sequence of restrictions of $\mu$ and $\mu'$ on intervals $[n_j^-, n_j^+]$, with $n_j^+ \to \infty$ and $n_j^- \to -\infty$ as $j \to \infty$. We shall iteratively estimate the proximity of these restrictions to each other in total variation, by showing that the restrictions on $[n_{j-1}^-, n_{j+1}^-]$ are (up to a small correction) closer to each other than the respective restrictions on $[n_j^-, n_j^+]$, by a multiplicative factor that is less than 1. The multiplier and the correction can be controlled by the choice of the sequences $(n_j^+)$ and $(n_j^-)$.

Lemma 11  There is a constant $c > 0$ and a function $N_2 : \mathbb{N} \to \mathbb{N}$ with the following property. Suppose $r \in \mathbb{N}$, and

$$\omega_{-N_2(r)+1} = \omega_{-N_2(r)+2} = \ldots = \omega_{N_2(r)} = 1.$$ 

If $n > N_2(r)$ and $\mu$ a finite volume Gibbs measure on $[-n, n]$ such that

$$\mu\{\gamma : \gamma_{-N_2(r)}, \gamma_{N_2(r)} \in B_r\} = 1,$$

then

$$\mu\{\gamma : \gamma_0 = 0\} > c\mu\{\gamma : \gamma_0 = x\}, \quad x \neq 0.$$ 

The proof is analogous to that of Lemma 2.

Lemma 12  Let $\lambda \in (0, \lambda_0)$. Then for almost every $\omega \in \Omega$ and for every $r > 0$, there a doubly infinite sequence $(n_i)_{i \in \mathbb{Z}}$ such that

$$\lim_{i \to +\infty} n_i = \infty, \quad \lim_{i \to -\infty} n_i = -\infty,$$

and for any $i$,

$$\nu^\mu(\theta^{-N_2(r)+n_i} \omega) = 1,$$

$$\omega_{-N_2(r)+1+n_i} = \omega_{-N_2(r)+2+n_i} = \ldots = \omega_{N_2(r)+n_i} = 1,$$
\[ \nu^+(\theta^{N_2(r) + n_i \omega}) = 1, \]

and for all \( i \),

\[ n_{i+1} - n_i > 2N_2(r). \]

**Proof:** Given \( r > 0 \).

\[
P\{ \nu - (\theta - N_2(r)) \omega = 1; \omega_{-N_2(r)+1} = \ldots = \omega_{N_2(r)} = 1; \nu^+(\theta^{N_2(r)} \omega) = 1 \} \]
\[
= P\{ \nu - (\theta - N_2(r)) \omega = 1 \} P\{ \omega_{-N_2(r)+1} = \omega_{-N_2(r)+2} = \ldots = \omega_{N_2(r)} = 1 \} \times P\{ \nu^+(\theta^{N_2(r)} \omega) = 1 \} > 0,
\]

so that the lemma follows from the Bernoulli property. \( \square \)

Now we return to the proof of the uniqueness in Theorem 2. For a measure \( \mu \) on \((\mathbb{Z}^d)^\mathbb{Z}\) and any set \( S \subset \mathbb{Z} \) we denote by \( \mu_S \) the measure induced by \( \mu \) on paths restricted to \( S \).

Let now \( \mu \) and \( \mu' \) be two Gibbs measures satisfying (9) for a given \( \omega \). We have to show that for any \( l > 0 \), the distributions induced by \( \mu \) and \( \mu' \) on trajectories defined on \([-l, l]\) coincide: \( \mu_{[-l, l]} = \mu'_{[-l, l]} \). Since \( \mu \) and \( \mu' \) are Gibbs measures with nearest neighbor interaction, it is sufficient to check that the two-dimensional boundary distributions coincide: \( \mu_{\{-l\}} = \mu'_{\{-l\}} \).

We fix an arbitrary \( \varepsilon > 0 \) and use (9) to find \( r_\varepsilon > 0 \) and sequences \( (m_k)_{k \in \mathbb{Z}}, (m'_k)_{k \in \mathbb{Z}} \) such that

\[
\mu\{ \alpha : |\alpha_{m_k}| > r_\varepsilon \} < \varepsilon, \quad k \in \mathbb{Z},
\]

and

\[
\mu'\{ \alpha : |\alpha_{m_k}| > r_\varepsilon \} < \varepsilon, \quad k \in \mathbb{Z}.
\]

For \( r \geq r_\varepsilon \) and \( k \in \mathbb{N} \) consider measures \( \mu^{k,r} \) and \( \mu'^{k,r} \) obtained from \( \mu \) and \( \mu' \) by conditioning, respectively, on \( \{ \alpha_{m_k^+} \leq r \} \) and \( \{ \alpha_{m'_k} \leq r \} \). Due to the arbitrary choice of \( \varepsilon \), it is sufficient to show that the total variation distance between \( \mu^{k,r}_{[-l,l]} \) and \( \mu'^{k,r}_{[-l,l]} \) can be made arbitrarily small by choosing sufficiently large \( r \) and \( k \).

Let us fix \( r \) and find the sequence \( (n_i)_{i \in \mathbb{Z}} \) provided by Lemma 12. For any given \( i_0 \in \mathbb{N} \), one can find \( k \) such that

\[ n_{i_0} + N(r) < \min\{m_k, m'_k\} \]

and

\[ n_{-i_0} - N(r) > \max\{m_{-k}, m'_{-k}\}. \]
Lemma 9 implies that if $|i| < i_0$, then
\[ \mu^{k,r}_{\{\alpha : |\alpha_n| > r\}} < Ce^{-2\lambda r}, \]  
\[ \mu^{k,r}_{\{\alpha : |\alpha_n+N_2(r)| > r\}} < Ce^{-2\lambda r}, \]  
\[ \mu^{k,r}_{\{\alpha : |\alpha_n-N_2(r)| > r\}} < Ce^{-2\lambda r}, \]  

and same estimates hold for $\mu'^{k,r}$. For $0 < i < i_0$, let us denote
\[ \mu_i = \mu^{k,r}_{\{n_i, n_i\}}, \quad \mu'_i = \mu'^{k,r}_{\{n_i, n_i\}}. \]

We are going to estimate the total variation distance $d_{TV}(\mu_{i-1}, \mu'_{i-1})$ via $d_{TV}(\mu_i, \mu'_i)$.

For any $i$ we introduce $\sigma_i$ to be the maximal measure minorizing both $\mu_i$ and $\mu'_i$ and concentrated on $B_r \times B_r$:
\[ \mu_i = \sigma_i + \rho_i + \delta_i \]
\[ \mu'_i = \sigma_i + \rho'_i + \delta'_i, \]

Here $\rho_i$ and $\rho'_i$ are mutually singular measures on $B_r \times B_r$, and measures $\delta_i, \delta'_i$ are supported on $\mathbb{Z}^d \times \mathbb{Z}^d \setminus B_r \times B_r$. We have
\[ d_{TV}(\mu_i, \mu'_i) \geq \rho_i(B_r), \]
and, due to (17), (18),
\[ \delta_i(\mathbb{Z}^d \times \mathbb{Z}^d) \leq 2Ce^{-2\lambda r}, \]
\[ \delta'_i(\mathbb{Z}^d \times \mathbb{Z}^d) \leq 2Ce^{-2\lambda r}. \]

Combining a basic coupling estimate based on Lemma 11, the fact that there are $(2r + 1)^2$ points in $B_r$, and the estimate (16), we obtain:
\[ d_{TV}(\mu_{i-1}, \mu'_{i-1}) \leq d_{TV}(\mu_i, \mu'_i) \left(1 - \left(\frac{c}{(2r + 1)^d - 1 + c}\right)^2\right) + 2Ce^{-2\lambda r}. \]

Since the total variation distance is always bounded by 1, applying this estimate iteratively, we obtain
\[ d_{TV}(\mu_i, \mu'_i) < \frac{C}{e^2}e^{-2\lambda r}((2r + 1)^d - 1 + c)^2, \quad i \in \mathbb{N}. \]

Since the choice of $r$ is arbitrary and the r.h.s. converges to 0 as $r \to \infty$, the proof is completed. $\Box$
5 Proofs of auxiliary lemmas

Proof of Lemma 2 Let us write
\[(2d + 1)^n T^{-n,0} \phi(y) \leq \sum_{k=|y|}^n \sum_{x} \phi(x) Z_{-n,-k}(x,0) \tilde{Z}_{-k,0}(0,0) + \tilde{Z}_{-n,0}(y) \|\phi\| \]
\[\leq \sum_{k=|y|}^n I_k + \bar{I}. \quad (19)\]

Here \(\tilde{Z}_{-k,0}(0,0)\) is the partition function over all paths on \([-k,0]\) connecting 0 to \(y\) and avoiding 0 after time \(-k\); \(\tilde{Z}_{-n,0}(y)\) is the partition function over paths on \([-n,0]\) ending up at \(y\) and never visiting 0.

Considering all paths that coincide with \(\gamma^*\) on \([-k+1,0]\), we can write
\[(2d + 1)^n T^{-n,0} \phi(0) \geq \sum_{x} \phi(x) Z_{-n,-k}(x,0) e^{\sum_{m=-k+1}^0 \xi_m} e^{-(M_0+M_1)},\]
where the factor of \(e^{-(M_0+M_1)}\) appears since it is possible that \(\gamma^*(0) \neq 0\).

We also notice that
\[\tilde{I} = \tilde{Z}_{-n,0}(y) \|\phi\| \leq (2d + 1)^n e^{M_1 n} \|\phi\| \leq (2d + 1)^n e^{M_1 n} e^{\lambda n} e^{-(M_0+M_1)},\]

On the other hand,
\[(2d + 1)^n T^{-n,0} \phi(0) \geq \phi(0) e^{\sum_{m=-n+1}^0 \xi_m} e^{-(M_0+M_1)} \]
\[\geq \phi(0)(2d + 1)^n e^{M_1 n} e^{\lambda n} e^{-(M_0+M_1)},\]

where the second inequality follows from Lemma 1 and condition \(k \geq |y| \geq \nu^-(\omega)\). Similarly, for \(\bar{I}\) we get:
\[\bar{I} = \bar{Z}_{-n,0}(y) \|\phi\| \leq (2d + 1)^n e^{M_1 n} \|\phi\| \leq (2d + 1)^n e^{M_1 n} c \phi(0).\]
where the second inequality follows from Lemma 1 and condition \( n \geq \nu(\omega) \).

Combining these estimates, we get

\[
\bar{I} \leq (2d + 1)^n e^{M_1 n} c \frac{(2d + 1)^n T^{-n/0} \phi(0)}{(2d + 1)^n e^{M_1 n} e^{\lambda n} e^{-(M_0 + M_1)}} \\
\leq c e^{M_0 + M_1} e^{-\lambda n} (2d + 1)^n T^{-n/0} \phi(0).
\]

(21)

The lemma now follows from (19), (20), and (21). \( \square \)

**Proof of Lemma 5** It is sufficient to prove the upper bound, so we write

\[
(2d + 1)^{n_i - n_i} T^{n_i, n_i - 1} \phi(y_1)
\]

\[
\leq \left( \sum_{n_i + r + 1 \leq k \leq l \leq n_i - r} Z_{n_i, n_i - 1}^{k,l}(y_1) + \bar{Z}_{n_i, n_i - 1}(y_1) \right) \| \phi \|, \tag{22}
\]

where \( Z_{n_i, n_i - 1}^{k,l}(y_1) \) is the partition function over all paths \( \gamma \) defined on interval \([n_i, n_i - 1]\) that terminate at \( y_1 \) and satisfy

\[
\min \{ m > n_i + r : \gamma(m) = 0 \} = k,
\]

\[
\max \{ m \leq n_i - r : \gamma(m) = 0 \} = l,
\]

and \( \bar{Z}_{n_i, n_i - 1}(y_1) \) is the partition function over all paths that terminate at \( y_1 \) and do not visit \( 0 \) between \( r \) and \( n - r \). We have

\[
Z_{n_i, n_i - 1}^{k,l}(y_1) \leq (2d + 1)^r e^{M_0 r} (2d + 1)^{k-(n_i+r)} e^{M_1 (k-(n_i+r)-1)} e^{M_0} \times Z_{k,l}(0,0) \right.
\]

\[
(2d + 1)^{n_i - n_i - r} T^{n_i, n_i - 1} + \bar{Z}_{n_i, n_i - 1}(y_1) \leq (2d + 1)^r e^{M_0 r} \sum_{m=n_i+r+1}^{n_i-1} \bar{Z}_{k,l}(0,0) e^{\lambda m} e^{-M_0} \phi(x^*)
\]

and, considering a point \( x^* \) such that \( |x^*| \leq r \) and \( \phi(x^*) = \| \phi \|,

\[
(2d + 1)^{n_i - n_i - r} T^{n_i, n_i - 1} \phi(y_2)
\]

\[
\geq e^{-M_0 r} e^{\lambda m} e^{-M_0} \sum_{m=n_i+r+1}^{n_i-1} e^{-\lambda m} e^{-M_0 r} \phi(x^*)
\]

so that

\[
\frac{Z_{n_i, n_i - 1}^{k,l}(y) \| \phi \|}{(2d + 1)^{n_i - n_i - r} T^{n_i, n_i - 1} \phi(y_2)} \\
\leq (2d + 1)^{2r+2} e^{M_0 (4r+2)} e^{-\lambda (k-(n_i+r)-1)} e^{-\lambda (n-r-1)}.
\]

(23)

Since

\[
\sum_{n_i + r + 1 \leq k \leq l \leq n_i - r} e^{-\lambda (k-(n_i+r)-1)} e^{-\lambda (n-r-1)} < \infty,
\]

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Lemma 14: positive functions \( \phi \)

Lemma 13 and estimate (24) imply that for any

\[
\rho_r(T_r^{n_i,n_i-1} \phi, T_r^{n_i,n_i-1} \psi) < \left(1 - \sqrt{K_6(\lambda)e^{-2\lambda}}\right) \rho_r(\phi, \psi).
\] 

\[ (25) \]

Now we begin preparations for the proof of the main contraction estimate, Lemma 7. For any \( i \) and \( r \), we define truncated operators

\[
T_r^{n_i,n_i-1} u(y) = \frac{1}{(2d + 1)^{n_i-n_i}} \sum_{|x| \leq r} Z_{n_i,n_i-1}(x,y)u(x) = T_r^{n_i,n_i-1}(u1_{B_r})(y),
\]

\[
T_r^{n_i,n_i-1} u(y) = \frac{1}{(2d + 1)^{n_i-n_i}} \sum_{|x| > r} Z_{n_i,n_i-1}(x,y)u(x) = T_r^{n_i,n_i-1}(u1_{B_r^c})(y).
\]

The restriction of \( T_r^{n_i,n_i-1} \) on \( B_r \) can be viewed as a linear finite-dimensional operator in \( \mathbb{R}^{B_r} \) given by a matrix \( (2d + 1)^{n_i-n_i} Z_{n_i,n_i-1}(x,y) \), \( x, y \in B_r \), with positive entries. Therefore, we can apply a classical estimate on contraction in Hilbert metric, see e.g., Theorem 3.12 in [Sen81]: for any functions \( \phi, \psi : \mathbb{Z}^d \rightarrow \mathbb{R}_+ \),

\[
\rho_r(T_r^{n_i,n_i-1} \phi, T_r^{n_i,n_i-1} \psi) \leq \frac{1 - \sqrt{L_r^{n_i,n_i-1}}}{1 + \sqrt{L_r^{n_i,n_i-1}}} \rho_r(\phi, \psi),
\]

where

\[
L_r^{n_i,n_i-1} = \min_{|x_1|,|x_2|,|y_1|,|y_2| \leq r} \left( \frac{Z_{n_i,n_i-1}(x_1,y_1)}{Z_{n_i,n_i-1}(x_2,y_1)} \cdot \frac{Z_{n_i,n_i-1}(x_2,y_2)}{Z_{n_i,n_i-1}(x_1,y_2)} \right).
\]

Lemma 13: There is a positive constant \( K_6(\lambda) \) such that for any \( i \) and \( r \),

\[
L_r^{n_i,n_i-1} \geq K_6(\lambda)e^{-4\lambda_1r}.
\]

Since the dynamics of the actual system is not restricted to \( B_r \) we have to estimate the influence of \( B_r^c \):

Lemma 14: There is a positive number \( K_7(\lambda) \) such that if \( |y| \leq r \), then

\[
\sum_{x:|x| > r} Z_{n_i,n_i-1}(x,y) \leq K_7(\lambda)e^{-2\lambda_1r} Z_{n_i,n_i-1}(0,y).
\]

We postpone the proof of Lemmas 13 and 14 till the end of this section.

Proof of Lemma 7: Lemma 13 and estimate (24) imply that for any positive functions \( \phi \) and \( \psi \),

\[
\rho_r(T_r^{n_i,n_i-1} \phi, T_r^{n_i,n_i-1} \psi) < \left(1 - \sqrt{K_6(\lambda)e^{-2\lambda_1}}\right) \rho_r(\phi, \psi).
\]
To estimate the full untruncated operators, we write:

$$
\rho_r(T^{n_i,n_{i-1}}\phi, T^{n_i,n_{i-1}}\psi) = \ln \left( \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\phi(y)}{T^{n_i,n_{i-1}}\psi(y)} \cdot \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\psi(y)}{T^{n_i,n_{i-1}}\phi(y)} \right)
$$

(26)

$$
\leq \ln \left( \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\phi(y) + T^{n_i,n_{i-1}}\psi(y)}{T^{n_i,n_{i-1}}\psi(y)} \right)
+ \ln \left( \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\psi(y) + T^{n_i,n_{i-1}}\phi(y)}{T^{n_i,n_{i-1}}\phi(y)} \right)
$$

Lemma 14 implies

$$
\hat{T}^{n_i,n_{i-1}}\phi(y) \leq \frac{1}{(2d + 1)^{n_{i-1} - n_i}} K_7(\lambda) e^{-2\lambda_0 r} Z_{n_i,n_{i-1}}(0, y) \| \phi 1_{B_r} \|
$$

$$
\leq K_7(\lambda) e^{-2\lambda_0 r} \frac{1}{(2d + 1)^{n_{i-1} - n_i}} Z_{n_i,n_{i-1}}(0, y) \cdot 2K_1(\lambda) \phi(0)
$$

$$
\leq 2K_1(\lambda) K_7(\lambda) e^{-2\lambda_0 r} T^{n_i,n_{i-1}}\phi(y).
$$

Analogously,

$$
\hat{T}^{n_i,n_{i-1}}\psi(y) \leq 2K_1(\lambda) K_7(\lambda) e^{-2\lambda_0 r} T^{n_i,n_{i-1}}\psi(y).
$$

Plugging the last two inequalities into (26), we get

$$
\rho_r(T^{n_i,n_{i-1}}\phi, T^{n_i,n_{i-1}}\psi) \leq \ln \left( \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\phi(y)(1 + 2K_1(\lambda) K_7(\lambda) e^{-2\lambda_0 r})}{T^{n_i,n_{i-1}}\psi(y)} \right)
+ \ln \left( \max_{|y| \leq r} \frac{T^{n_i,n_{i-1}}\psi(y)(1 + 2K_1(\lambda) K_7(\lambda) e^{-2\lambda_0 r})}{T^{n_i,n_{i-1}}\phi(y)} \right)
\leq \rho_r(T^{n_i,n_{i-1}}\phi, T^{n_i,n_{i-1}}\psi) + 4K_1(\lambda) K_7(\lambda) e^{-2\lambda_0 r}
$$

This estimate along with (25) implies the lemma with $K_3(\lambda) = \sqrt{K_6(\lambda)}$ and $K_4(\lambda) = 4K_1(\lambda) K_7(\lambda)$. □

**Proof of Lemma 13** Let us estimate the ratios in the r.h.s. of the definition of $L_{n_i,n_{i-1}}^k$. For $n_i + r < k \leq l \leq n - r$, we define $Z_{n_i,n_{i-1}}^{k,l}(x, y)$ as the partition function over all paths $\gamma$ on $[n_i,n_{i-1}]$ connecting $x$ to $y$ satisfying

$$
\min\{m > n_i + r : \gamma(m) = 0\} = k,
$$

$$
\max\{m \leq n_{i-1} - r : \gamma(m) = 0\} = l,
$$

$$
we also define \( \bar{Z}_{n_i,n_{i-1}}(x,y) \) to be the partition function over all paths on \([n_i, n_{i-1}]\) connecting \(x\) to \(y\) that do not visit 0 between \(n_i + r + 1\) and \(n_{i-1} - r\), so that

\[
Z_{n_i, n_{i-1}}(x, y) = \sum_{n_i + r < k \leq l \leq n_{i-1} - r} Z_{n_i, n_{i-1}}^{k,l}(x, y) + \bar{Z}_{n_i, n_{i-1}}(x, y). \tag{27}
\]

For \(n_i + r < k \leq l \leq n_{i-1} - r\) and any points \(x, y\) with \(|x|, |y| \leq r\),

\[
Z_{n_i, n_{i-1}}^{k,l}(x, y) \leq (2d + 1)^{|x| - 1} e^{M_1(|x| - 1)} (2d + 1)^{|y| - 1} e^{M_0(r - |y| + 1)} e^{M_1(k - 1 - (n_i + r))} e^{M_0} Z_{k,l}(0, 0) \times (2d + 1)^{n - r - l} e^{M_1(n - r - l)} \times (2d + 1)^{|y| - 1} e^{M_0(r - |y|)} (2d + 1)^{|y|} e^{M_1|y|}. \tag{28}
\]

On the other hand, considering all paths that start at \(x\), go straight to 0 (which takes \(|x|\) steps), stay at 0 until time \(n_i + r\) accumulating \(M_0\) at each time step, follow the optimal path \(\gamma^*\) from \(n_i + r + 1\) to \(k - 1\) (if \(k = n_i + r + 1\) this part is empty), at \(k\) visit 0, return to 0 at time \(l\), follow \(\gamma^*\) up to \(n_{i-1} - r\), stay at 0 up to \(n_{i-1} - |y|\), go straight to \(y\) where they terminate at time \(n_{i-1}\),

\[
Z_{n_i, n_{i-1}}(x, y) \geq e^{-M_1(|x| - 1)} e^{M_0(r - |x| + 1)} e^{\sum_{m=n_i+r+1}^{k-1} \xi_m} e^{-M_0} \times Z_{k,l}(0, 0) e^{\sum_{m=l+1}^{n_{i-1}-r} \xi_m} e^{M_0(r - |y|)} e^{-M_1|y|}. \tag{29}
\]

The definitions of \(\nu^+\) and \(\nu^-\) from Lemma [1] imply that, due to our assumptions on \(n_i\) and \(n_{i-1}\),

\[
e^{\sum_{m=n_i+r+1}^{k-1} \xi_m} \geq (2d + 1)^{k - 1 - (n_i + r)} e^{M_1(k - 1 - (n_i + r))} e^{\lambda(k - 1 - (n_i + r))}, \tag{30}
\]

and

\[
e^{\sum_{m=l+1}^{n_{i-1}-r} \xi_m} \geq (2d + 1)^{n_{i-1} - r - l} e^{M_1(n_{i-1} - r - l)} e^{\lambda(n_{i-1} - r - l)}, \tag{31}
\]

so that

\[
\frac{Z_{n_i, n_{i-1}}(x_2, y_1)}{Z_{n_i, n_{i-1}}(x_1, y_1)} \leq (2d + 1)^{2r + 1} e^{M_1(|x_1| + |x_2|)} e^{M_0(|x_1| - |x_2|)} e^{M_0 + M_1} \times e^{2M_1|y_1|} e^{-\lambda(k - r + (n - r - l) - 2)} \leq (2d + 1)^{2r + 1} e^{4M_1r} e^{M_0(|x_1| - |x_2| + 2)} \times e^{-\lambda((k - 1 - (n_i - r)) + (n_{i-1} - r - l))}. \tag{32}
\]
For $Z_{n_i,n_{i-1}}(x, y)$ we have

$$Z_{n_i,n_{i-1}}(x, y) \leq (2d + 1)^{|x|} e^{M_1(|x| - 1)} (2d + 1)^{|y| + 1} e^{M_0(|y| + 1)} \times (2d + 1)^{n_{i-1} - n_i - 2r} e^{M_1(n_{i-1} - n_i - 2r)} \times (2d + 1)^{|y|} e^{M_0(|y|)} (2d + 1)^{|y|} e^{M_1|y|},$$

and

$$Z_{n_i,n_{i-1}}(x, y) \geq e^{-M_1(|x| - 1)} e^{M_0(|x| - 1)} \sum_{i=n_i}^{n_{i-1}-2r} \xi_i e^{M_0(|y|)} e^{-M_1|y|},$$

so that

$$\frac{\tilde{Z}_{n_i,n_{i-1}}(x_2, y_1)}{Z_{n_i,n_{i-1}}(x_1, y_1)} \leq (2d + 1)^{2r + 1} e^{4M_1r} e^{M_0(|x_1| - |x_2|)}. \tag{33}$$

Plugging (32) and (33) into (27), we see that

$$\frac{Z_{n_i,n_{i-1}}(x_2, y_1)}{Z_{n_i,n_{i-1}}(x_1, y_1)} \leq (2d + 1)^{2r + 1} e^{4M_1r} e^{M_0(|x_1| - |x_2|)} K_8(\lambda),$$

for some $K_8(\lambda) > 0$. This inequality and its counterpart with $x_1, x_2,$ and, respectively, $y_1$ replaced by $x_2, x_1,$ and, respectively, $y_2$, immediately implies:

$$L^{n_i,n_{i-1}} \geq (2d + 1)^{-4r - 2} e^{-8M_1r} K_8^{-2}(\lambda),$$

and the lemma holds true with $K_6(\lambda) = (2d + 1)^{-2} K_8^{-2}(\lambda)$. \hfill \Box

Proof of Lemma 13. The following decomposition is analogous to (27):

$$\sum_{x:|x|>r} Z_{n_i,n_{i-1}}(x, y) \leq \sum_{n_i+r<k\leq n_{i-1}-r} Z_{n_i,n_{i-1}}^{k,l}(y) + \tilde{Z}_{n_i,n_{i-1}}(y), \tag{34}$$

where for $n_i + r < k \leq l \leq n_{i-1} - r$, $Z_{n_i,n_{i-1}}^{k,l}(y)$ is the partition function over all paths $\gamma$ on $[n_i, n_{i-1}]$ terminating at $y$ and satisfying

$$\min\{m : \gamma(m) = 0\} = k,$$

$$\max\{m \leq n_{i-1} - r : \gamma(m) = 0\} = l,$$

and $\tilde{Z}_{n_i,n_{i-1}}(y)$ is the partition function over all paths terminating at $y$ that do not visit 0 between $n_i$ and $n_{i-1} - r$. Analogously to (28),

$$Z_{n_i,n_{i-1}}^{k,l}(y) \leq (2d + 1)^r e^{M_1r} (2d + 1)^{k-1-(n_i+r)} e^{M_1(k-1-(n_i+r))} e^{M_0} \times Z_{k,l}(0, 0) (2d + 1)^{n_i-r-l} e^{M_1(n_{i-1} - r - l)} \times (2d + 1)^{|y|} e^{M_0(|y|)} (2d + 1)^{|y|} e^{M_1|y|}. \tag{35}$$
Analogously to (29),
\[
Z_{n_i,n_{i-1}}(0, y) \geq e^{M_0} e^{\sum_{m=n_i+1}^{n_{i-1}} \xi_m e^{-M_0}}
\times Z_k I(0, 0) e^{\sum_{m=n_i+1}^{n_{i-1}-r} \xi_m e^{M_0(r-|y|)}} e^{-M_1|y|}.
\] (36)

Dividing (35) by (36) and taking into account (30), (31), we obtain:
\[
\frac{Z_{n_i,n_{i-1}}^{k,l}(y)}{Z_{n_i,n_{i-1}}(0, y)} \leq e^{(3M_1-M_0)r} (2d+1)^{2r} e^{2M_0} e^{-\lambda(k-1-(n_i+r)+(n_{i-1}-r-l))}
\leq e^{-2\lambda_0 r} e^{2M_0} e^{-\lambda(k-1-(n_i+r)+(n_{i-1}-r-l))}.
\] (37)

For \(\bar{Z}_{n_i,n_{i-1}}(y)\) we have
\[
\bar{Z}_{n_i,n_{i-1}}(y) \leq (2d+1)^{r} e^{M_r} (2d+1)^{n_i-1-n_i-2r} e^{M_1(n_i-1-n_i-2r)} (2d+1)^r e^{M_r}
\]
and
\[
Z_{n_i,n_{i-1}}(y) \geq e^{M_0} e^{\sum_{m=n_i+1}^{n_{i-1}-r} \xi_m e^{M_0(r-|y|)}} e^{-M_1|y|},
\]
so that
\[
\frac{\bar{Z}_{n_i,n_{i-1}}(y)}{Z_{n_i,n_{i-1}}(0, y)} \leq (2d+1)^{2r} e^{(3M_1-M_0)r} e^{-\lambda(n_i-1-n_i-2r)} \leq e^{-2\lambda_0 r}.
\] (38)

Now the lemma follows from (34) and (37), (38). □

**Proof of Lemma 8.** Proceeding as above, one can use the fact that all paths contributing to \(\hat{T}_r^{n_i,0} \phi(y)\) do not visit 0 at first \(r\) steps, and show that for some \(K_9(\lambda)\) and sufficiently large \(i\),
\[
\hat{T}_r^{n_i,0} \phi(y) \leq K_9(\lambda)(2d+1)^r e^{(M_1-M_0)r} T_r^{0,n} \phi(y),
\]
whenever \(|y| \leq r\). Since (26) implies:
\[
\rho_r(T_r^{n_i,0} \phi, T_r^{n_i,0} \phi)
\leq \ln \left( \max_{|y| \leq r} \frac{T_r^{n_i,0} \phi(y)}{T_r^{n_i,0} \psi(y)} \left( 1 + \frac{\hat{T}_r^{n_i,0} \phi(y)}{T_r^{n_i,0} \psi(y)} \right) \right)
\times \left( \max_{|y| \leq r} \frac{T_r^{n_i,0} \psi(y)}{T_r^{n_i,0} \phi(y)} \left( 1 + \frac{\hat{T}_r^{n_i,0} \psi(y)}{T_r^{n_i,0} \phi(y)} \right) \right)
\leq \rho_r(\phi, \psi) + \max_{|y| \leq r} \frac{\hat{T}_r^{n_i,0} \phi(y)}{T_r^{n_i,0} \phi(y)} \times \max_{|y| \leq r} \frac{\hat{T}_r^{n_i,0} \psi(y)}{T_r^{n_i,0} \psi(y)},
\]

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the lemma follows with $K_5(\lambda) = K_9(\lambda)^2$. □

**Proof of Lemma 9** We can write

$$\mu_{n_1, n_2}\{\alpha_n > r\} = \sum_{x_1, x_2 \in B_r} \mu_{n_1, n_2}\{\alpha_{n_1} = x_1, \alpha_{n_2} = x_2\} \frac{Z_{n_1, n_2}(x_1, x_2)}{Z_{n_1, n_2}(x_1, x_2)}.$$ 

where $Z_{n_1, n_2}(x_1, x_2)$ denotes the partition function over all paths $\gamma$ defined on $[n_1, n_2]$ with $\gamma(n_1) = x_1$, $\gamma(n_2) = x_2$, and such that $|\gamma(n)| > r$. For any $x_1, x_2 \in B_r$,

$$\frac{\tilde{Z}_{n_1, n_2}(x_1, x_2)}{Z_{n_1, n_2}(x_1, x_2)} \leq \sum_{k_1, k_2: \text{restricted to } [x_1] \leq n_1 + k_1 \leq n - r \text{ and } n + r \leq n_1 + k_2 \leq n_2 - |x_2|} \frac{\tilde{Z}_{k_1, k_2}^{n_1, n_2}(x_1, x_2)}{Z_{n_1, n_2}(x_1, x_2)} + \tilde{Z}_{n_1, n_2}(x_1, x_2),$$

where $\tilde{Z}_{k_1, k_2}^{n_1, n_2}(x_1, x_2)$ and $\tilde{Z}_{n_1, n_2}(x_1, x_2)$ are partition functions taken over paths $\gamma$ contributing to $\tilde{Z}_{n_1, n_2}(x_1, x_2)$ and $Z_{n_1, n_2}(x_1, x_2)$ respectively, with the following restriction:

$$\sup\{k \leq n : \gamma_k = 0\} = k_1, \quad \inf\{k \geq n : \gamma_k = 0\} = k_2,$$

and $\tilde{Z}_{n_1, n_2}(x_1, x_2)$ is defined as the partition function over paths contributing to $\tilde{Z}_{n_1, n_2}(x_1, x_2)$ and never visiting the origin between $n_1$ and $n_2$. Using Lemma 5 to estimate the contribution of the optimal path $\gamma^*$ to the denominator, we can write:

$$\frac{\tilde{Z}_{k_1, k_2}^{n_1, n_2}(x_1, x_2)}{Z_{n_1, n_2}(x_1, x_2)} \leq \frac{e^{M_1(k_2-k_1)}(2d+1)^{k_2-k_1}}{(2d+1)^{k_2-k_1}e^{M_1}(k_2-k_1)e^{\lambda(k_2-k_1)}e^{-(M_0+M_1)}} \leq e^{-(M_0+M_1)}e^{-\lambda(k_2-k_1)}.$$

For the last term, we get:

$$\frac{\tilde{Z}_{n_1, n_2}(x_1, x_2)}{Z_{n_1, n_2}(x_1, x_2)} \leq \frac{e^{M_1(n_2-n_1)}(2d+1)^{n_2-n_1}}{e^{-M_1|x_1|}e^{-(M_0+M_1)}(2d+1)^{n_2-n_1-|x_1|-|x_2|}} \times \frac{1}{e^{M_1(n_2-n_1-|x_1|-|x_2|)}e^{\lambda(n_2-n_1-|x_1|-|x_2|)}e^{-M_1|x_2|}} \leq (2d+1)^{2r}e^{4M_1r}e^{M_0+M_1}e^{-\lambda(n_2-n_1-2r)}.$$

and the lemma follows by combining the estimates above. □
6 Acknowledgements

The authors would like to thank Leonid Koralov for reading the manuscript and suggesting several useful corrections. The research of Yuri Bakhtin is partially supported by NSF through CAREER grant DMS-0742424. The research of Konstantin Khanin is partially supported by NSERC.

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