Symmetry and Symmetry Restoration of Lattice Chiral Fermion in the Overlap Formalism

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Three aspects of symmetry structure of lattice chiral fermion in the overlap formalism are discussed. By the weak coupling expansion of the overlap Dirac operator, the axial anomaly associated to the chiral transformation proposed by Lüscher is evaluated and is shown to have the correct form of the topological charge density for perturbative backgrounds. Next we discuss the exponential suppression of the self-energy correction of the lightest mode in the domain-wall fermion/truncated overlap. Finally, we consider a supersymmetric extension of the overlap formula in the case of the chiral multiplet and examine the symmetry structure of the action.

1. Introduction

The overlap formalism of the chiral determinant [1] provides a well-defined lattice regularization of chiral determinant. It can reproduce the known features of the chiral determinant in the continuum theory. This suggests that the overlap formalism is a promising building block for the construction of lattice chiral gauge models.

When applied to vector-like theories like QCD, the overlap formalism also provides a satisfactory description of massless Dirac fermion. It has been shown by Neuberger [2] that the overlaps for the massless Dirac fermion can be written as a single determinant of a Dirac operator, which explicit form is defined by

$$aD = 1 + X \frac{1}{\sqrt{X^\dagger X}} = 1 + \gamma_5 \frac{H}{\sqrt{H^2}}$$

where $X$ is the Wilson-Dirac operator,

$$X = \left\{ \frac{1}{2} \gamma_\mu \left( \nabla_\mu - \nabla_\mu^\dagger \right) + \frac{a}{2} \gamma_\mu \nabla_\mu^\dagger - \frac{1}{a} m_0 \right\}$$

(0 < m_0 < 2) and $H = \gamma_5 X$. The remarkable point about this Dirac operator is that it satisfies the Ginsparg-Wilson relation [3]. It implies that the symmetry breaking is restricted to unphysical local terms. This is the clue to escape the Nielsen-Ninomiya theorem. Moreover, as shown by Lüscher [4], the fermion action has exact symmetry under the lattice chiral transformation. For the flavor-singlet chiral transformation, the functional measure is not invariant and causes the anomaly. This anomaly is given exactly by the index of the overlap Dirac operator [5-7].

$$-a \text{Tr} \gamma_5 D = 2N_f \text{index}(D) = -N_f \text{Tr} \left( \frac{H}{\sqrt{H^2}} \right) .$$

In the overlap formalism, it is measured by the spectrum asymmetry of the Hamiltonian $H$, which has been considered as the topological charge by Narayanan and Neuberger [8].

2. Weak coupling expansion of overlap Dirac operator

In the weak coupling expansion, we have calculated the anomaly explicitly and have shown that it has the correct form of the topological charge density for perturbative background [9].

It is straightforward to obtain the weak coupling expansion of the overlap Dirac operator, Eq. (1). For the explicit expressions of the expansion, the author refers the reader to [10]. At the second order, it contains the following term

$$\int \frac{d^4 k}{(2\pi)^4} \frac{[\omega(q) + \omega(k) + \omega(p)]}{\omega(q)\omega(k)\omega(p)} \times$$
tor symmetric in contribute to the anomaly. This term has a fac-
more than three gamma matrices, which could

where

\[ \omega(p) = \sqrt{\sin^2 p_\mu + \left( \sum_\mu (1 - \cos p_\mu) - m_0 \right)^2}. \]

The Wilson-Dirac operator \( X \) is expanded up to the second orders in the momentum space as

\[ X = X_0 + X_1 + X_2 + \mathcal{O}(g^3). \]

Only this term has more than three gamma matrices, which could contribute to the anomaly. This term has a factor symmetric in \( \omega \)'s. If we recall the following identity,

\[ 1 \left( \omega_1 + \omega_2 + \omega_3 \right) \left( \omega_1 + \omega_3 \right) \omega_1 \omega_2 \omega_3 \]

\[ = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi (\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)(\omega^2 + \omega_3^2)}. \]

this structure suggests that the coefficient of the anomaly can be expressed in terms of a five-
dimensional propagator.

\[ X_5(\omega, p) \equiv i\gamma_5 \omega + X_0(p). \]

In fact, the vertex function of the anomaly can be expanded in terms of the external momenta without encountering any IR divergence. The coefficient of the leading operator of dimension four can be expressed as a winding number of a map defined by \( X_5(\omega, k) \), which is a function from \( T^4 \times R \)
to \( S^5 \). Then we see that the correct axial anomaly is reproduced in the covariant form:

\[ \lim_{a \to 0} \left( -a \sum_n \text{tr} \{ \alpha_n \gamma_5 D_{\alpha n} \} \right) \]

\[ = \frac{g^2}{32\pi^2} N_f \int d^4x \alpha(x) \epsilon_{\lambda\mu\sigma\nu} F_{\lambda\mu}^\alpha(x) F_{\sigma\nu}^\alpha(x). \]

The relation between this axial anomaly and the covariant anomaly discussed in the context of the overlap formalism in \[ \square \square \square \] will be discussed in more detail elsewhere.

3. **Exponentially suppressed self-energy correction in the domain-wall fermion / truncated overlap**

The next topic is about the exponentially suppressed self-energy correction in the domain-wall fermion / truncated overlap. The Shamir's variant of the domain-wall fermion can be regarded as a collection of a finite number, say \( N \), of Wilson-Dirac fermions. As discussed by Neuberger, it becomes more clear if we perform the chirally-asymmetric parity transformation in the flavor space. In this basis, the mass matrix becomes hermitian (N=4):

\[ M_{st}^H(n, m) = M_{st}(n, m) P = PM_{st}^\dagger(n, m) \]

where \( P \) stands for the parity operator acting on the flavor space.

At the tree level, this hermitian mass matrix is diagonalized in the momentum space by the Tchebycheff polynomials, \( u_s(x) = \sin(s\omega) / \sin(N\omega) \). Then, the mass eigenvalues are given by the following formula:

\[ m(p) = \frac{1}{u_{N+1}(x)} \]

\[ = \sin(N+1)\omega \]

\[ = \frac{1}{\sum_p (1 - \cos p_\mu) + M_0}. \]

For

\[ \frac{1}{\sum_p (1 - \cos p_\mu) + M_0} < 1 + \frac{1}{N} \]

a single mode with an imaginary \( \omega \) appears and it gives an exponentially small eigenvalue,

\[ m_0(p) = \frac{\sinh \lambda}{\sinh(N+1)\lambda} \simeq \exp(-N\lambda). \]

The question here is that this exponentially small mass eigenvalue survives the radiative correction due to the gauge field or not. Since there is no reason based on symmetry at finite flavor \( N \), we need to check it explicitly, at least in perturbation theory. Aoki and Taniguchi have calculated the self-energy correction at one-loop for asymptotically large flavor \( N \). Neuberger has examined the underlying mechanism for generating almost massless fermion \[ \square \square \square \square \]. We have shown that the calculation at one-loop can be done keeping the number of flavor \( N \) finite in the diagonal
basis of the leading mass matrix \([13]\). Thus we can check explicitly the exponential suppression of the self-energy correction to the lightest mode at one-loop. For more detail of the calculation at finite \(N\), the author refers the reader to \([13]\).

4. Chiral symmetry and supersymmetry in the overlap formalism

Finally, we discuss a supersymmetric extension of the overlap formalism in the case of the free chiral multiplet \([14]\). In the free theory, the overlap Dirac operator may be written in the following form.

\[
D = \gamma_\mu \nabla_\mu^X + M^X,
\]

where

\[
\nabla_\mu^X = \frac{1}{2} (\partial_\mu - \partial_\mu^\dagger) \frac{1}{\sqrt{X_0^4 X_0}},
\]

\[
M^X = 1 + \frac{1}{\sqrt{X_0^4 X_0}} \left( \frac{1}{2} \partial_\mu \partial_\mu^\dagger - m_0 \right).
\]

These difference operators satisfy the following identity, which is another expression of the Ginsparg-Wilson relation:

\[
2M^X = -(\nabla_\mu^X)^2 + (M^X)^2.
\]

Using these kinetic and mass operators, we can write a local action of the two-component Weyl fermion and the complex boson, which possesses manifest supersymmetry as follows:

\[
\sum_n \left\{ \bar{\psi}_n \sigma_\mu \nabla_\mu^X \psi_n - \phi_n^\dagger (\nabla_\mu^X)^2 \phi_n - F_n^* F_n \right\}
\]

\[
+ \sum_n \left\{ \frac{1}{2} \left( \psi_n^T M^X \psi_n - \bar{\psi}_n M^X \bar{\psi}_n^T \right) \right\}
\]

\[
+ \sum_n \left\{ \phi_n M^X F_n + F_n^* M^X \phi_n^\dagger \right\}.
\]

In addition to the manifest supersymmetry, the fermionic action possesses the chiral symmetry under the following transformation in terms of the two-component Weyl fermion field,

\[
\delta \psi_n = -\psi_n + \frac{1}{2} (M^X \psi_n - \sigma_\mu^\dagger \nabla_\mu^X \bar{\psi}_n^T),
\]

\[
\delta \bar{\psi}_n = +\bar{\psi}_n - \frac{1}{2} (\bar{\psi}_n M^X + \psi_n^T \sigma_\mu^\dagger \nabla_\mu^X).
\]

Moreover, the bosonic action has two \(U(1)\) symmetries. One of them is the bosonic extension of the lattice chiral symmetry of Lüscher:

\[
\delta \phi_n = \phi_n - \frac{1}{2} (M^X \phi_n - F_n^*),
\]

\[
\delta F_n = F_n - \frac{1}{2} (M^X F_n - (\nabla_\mu^X)^2 \phi_n^\dagger).
\]

This result implies that all the symmetries of the target continuum theory of the free chiral multiplet can be manifest on the lattice in the overlap formalism. The introduction of the supersymmetric cubic interaction seems hard, because of the breakdown of the Leibniz rule on the lattice. We need further considerations in this direction.

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