Optimizing the eigenvector computation algorithm with diffusion approach

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ABSTRACT
In this paper, we apply the ideas of the matrix column based diffusion approach to define a new eigenvector computation algorithm of a stationary probability of a Markov chain.

Categories and Subject Descriptors
G.1.3 [Mathematics of Computing]: Numerical Analysis—Numerical Linear Algebra; G.2.2 [Discrete Mathematics]: Graph Theory—Graph algorithms

General Terms
Algorithms, Performance

Keywords
Numerical computation; Iteration; Fixed point; Eigenvector.

1. INTRODUCTION
In this paper, we assume that the readers are already familiar with the idea of the fluid diffusion associated to the D-iteration [3] to solve the equation:
\[ X = P.X + B \]
and its application to PageRank equation [4].

For the general description of alternative or existing iteration methods, one may refer to [2, 8].

2. ALGORITHM DESCRIPTION
2.1 Notation
We recall that the D-iteration is defined by the couple \((P, B) \in \mathbb{R}^{N \times N} \times \mathbb{R}^{N}\) and exploits two state vectors: \(H_n\) (history) and \(F_n\) (residual fluid):
\[
F_0 = B \\
F_n = (I_d - J_{i_n} + P J_{i_n}) F_{n-1}. \tag{2}
\]

where \(I_d\) is the identity matrix, \(J_k\) a matrix with all entries equal to zero except for the \(k\)-th diagonal term: \((J_k)_{kk} = 1\), \(i_n\) the \(n\)-th node selected for the diffusion and
\[
H_n = \sum_{k=1}^{n} J_{i_k} F_{k-1} \tag{3}
\]
\[= (I_d - J_{i_n} (I_d - P)) H_{n-1} + J_{i_n} B. \tag{4}
\]

The diffusion of a node \(i\) containing \((F_n)_i = f\) means the following operation:

\[
(H_n)_i = f: \text{this defined } H_{n+1};
\]
\[
(F_n)_i = 0 \text{ and } (F_n)_i' = f \times p_{ij}: \text{this defines } F_{n+1}.
\]

Here, we’ll use the equation satisfied by the state vectors (cf. [3]):
\[ H_n + F_n = P.H_n + B. \tag{5} \]

We’ll use here the notation \(P.X\) meaning the usual matrix-vector product \(P \times X\).

We define \(\sigma : \mathbb{R}^N \to \mathbb{R}\), by \(\sigma(X) = \sum_{i=1}^{N} x_i.\) We define the \(L_1\) norm, \(|X| = \sum_{i=1}^{N} |x_i|\).

Below, we denote by \(e\) the normalized unit column vector \(1/N(1, \ldots, 1)^t\).

2.2 Assumption
We will assume in this paper that \(P\) is a stochastic matrix (we took the notation of the stochastic matrix per column), i.e. \(\sum_i p_{ij} = 1\) for each \(i\). We define below the optimal algorithm to find \(X\) such that \(X = P.X\). For the sake of simplicity, we will assume that \(P\) is irreducible and ergodic, so that \(P^n\) converges to a unique solution. In fact, our approach finds a solution as soon as \(P\) is stochastic and with no empty columns or rows in \(P\), or a bit more generally if \(P^n.e\) is convergent.

2.3 Algorithm description
We first apply a full product \(P.e\) then compensate by subtracting \(e\) so that \(\sigma(P.e - e) = 0\). Then we set \(F_0 = P.e - e\) and apply the diffusion iteration on \((P, F_0)\) (cf. [3, 5]). This means that we solve \(H\) such that \(H = P.H + P.e - e\) and \(X\) is obtained by \(H + e\). Note that \(H + e\) is not necessarily normalized to 1.

REMARK 1. The results we present here are in fact independent of the choice of \(e\) and we could use any other vector.

2.4 Some intuition on the optimality
The benchmark test to existing methods will be addressed in a future paper. From the intuition point of view, the D-iteration method was initially used to solve \(X = P.X + B\) when the spectral radius of \(P\) is strictly less than 1, so that the fluid \(B\) converges to zero exponentially. For the problem \(X = P.X\), the direct application of the diffusion would keep the same amount of fluids in the system and the convergence can be obtained by Cesaro averaging (or multiplicative normalization), but this would lead a very slow convergence [3]. The algorithm above starts by creating an initial vector in the kernel of \(\sigma\) in which we have positive fluids compensating
exactly all negative fluids. Then the convergence is obtained by the fact that positive fluids meeting negative fluids just
vanishes.

Now, intuitively, this is clearly optimal, because with the
initial setting we are killing all cycles and the diffusion be-
hind kills all remaining forwarding diffusions.

3. CONVERGENCE

**Theorem 1.** There exists a choice of the sequence of nodes
such that the diffusion applied on $(P, F_0)$ converges to $X - e$.

**Proof.** Let’s first prove that the usual iteration method:
$H_{n+1} = P.H_n + F_0$ is convergent. By induction, it is straight-
forward to obtain: $H_n = P^n.e - e$. Since $P^n.e ightarrow X$, we have
$H_n = H = X - e$.

For the diffusion process, from the equations (2) and (3),
we can easily prove by induction that: $\sigma(F_1) = \sigma(H_n) = 0$.
Then we prove that $|F_n|$ is non-increasing function: when
the diffusion is applied on a positive fluid, the same amount
is distributed; when they meet negative fluid, a part of them
vanishes so that $|F_n|$ is decreased, if not $|F_n|$ is not modified.
For a more rigorous proof: set $j = i_{n+1}$ and $f = (F_{i_{n+1}})$,
then:

$$|F_{n+1}| = \sum_{i \neq j} |F_{n+1}i| + |(F_{n+1})j|$$
$$= \sum_{i \neq j} |F_{n}i| + fp_{ij} + |f_{pj}|.$$

Let’s call $\Delta$ the set of nodes $i$ such that $(F_{n})i$ has a sign
opposed to $f$. Then,

$$|F_{n+1}| = \sum_{i \neq j} |F_{n}i| + |f_{pij}| + |f_{pj}|$$
$$+ \sum_{i \in \Delta} |(F_{n}i) + fp_{ij} - |(F_{n})i| - |f_{pij}|$$
$$= |F_{n}| + \sum_{i \in \Delta} |(F_{n}i) + fp_{ij} - |F_{n})i| - |f_{pij}|.$$

Now, we use $|x + y| \leq |x| + |y|$ to get $|F_{n+1}| \leq |F_{n}|$.
Therefore, $|F_{n}|$ is convergent. The limit is necessarily equal
to zero, because of the irreducibility of $P$: there exists a
path (diffusion sequence) such that positive and negative
fluids necessarily meet each others. For all $i, j$, there ex-
ists $n$ such that $(P)^n_{ij} > 0$ implying there is a path
$i_1 = j, i_2, ..., i_n = i$ with strictly positive weight $w(i, j) = p_{i_1,i_2} \times \ldots \times p_{i_{n-1},i_n} > 0$. Applying the diffusion successively on the
nodes $i_1, i_2$ from $F_n = F$, we are sure to cancel at least
$\min(|F_{ij}|, w(i, j) \times |F_{ij}|)$ (taking $F_{i} \times (F_{j}) < 0$). Now
taking $w = \min_{j} w(i, j)$, we are sure to cancel in less than
$N$ diffusions at least $w \times \max_{i} |F_{ij}|$, which means an ex-
ponential convergence, and this guarantee the convergence of
$H_n$.

This theorem allows us to apply heuristic policy to opti-
imize the sequence of nodes for the diffusion.

4. APPLICATION TO PAGERANK EQUA-
TION

The PageRank equation can be written as:

$$X = P.X \quad (6)$$

where $P = dP_g + (1 - d)/NJ$, where $J$ is the matrix with all
entries equal to 1 and $P_g$ is the completed matrix from the
initial $P_0$ (corresponding to the web graph) by $e$ on columns
associated to dangling nodes (nodes with no outgoing links).

If we apply the same method than above, we get:

$$H = dP_g H + dP_g e - e.$$ 

We can rewrite $dP_g e - e$ as $F_0 = dP_g e - d\sigma(P_g)e$.
In this case, because of the factor $d$ and the presence of
dangling nodes, the iteration does not maintain $H_n, F_0$ in
the kernel of $\sigma$. However, the system is dominated by a
system which is exponentially decreasing ($d$) and it is easy
to prove its convergence for D-iteration.

The limit of $D$-iteration on $(dP_g, F_0)$ satisfies:

$$H = dP_g H + F_0$$

and

$$H = \sum_{i > 0} dP_g F_0$$
$$= \sum_{i > 0} dP_g \times [(dP_g e - e) + (1 - d + df_1)e]$$
$$= -e + \sum_{i > 0} dP_g (1 - d + df_1)e,$$

where $f_1 = 1 - \sigma(P_g,e)$.

The limit of $D$-iteration on $(dP_g, F_0)$ satisfies:

$$H' = dP_g H' + F_0$$
$$= dP_g H' + d\sigma(H' - P_g H')e + F_0$$
$$= dP_g H' + (dP_g e - e) + (1 - d + df_2 + df_1)e$$

where $f_2 = \sigma(H' - P_g H')$ and $f = f_2 + f_1$, and

$$H' = \sum_{i > 0} dP_g F_0$$
$$= \sum_{i > 0} dP_g \times [(dP_g e - e) + (1 - d + df_1)\sigma]$$
$$= -e + \sum_{i > 0} dP_g (1 - d + df_1)e.$$

Therefore, we have:

$$X = H' + e$$
$$= \frac{1 - d + df_1}{1 - d + df_1} (H + e).$$

Then $X$ can be computed from $H$ by normalizing $H + e$ to
one.

5. EXTENSION TO POSITIVE MATRIX

Now, if $P$ is a positive irreducible matrix, Perron-Frobenius
theorem says that $(\rho^{-1}P)^n.e$ is convergent ($\rho$ is the spectral
radius of $P$) to the unique eigenvector which is strictly pos-
itive. Therefore, to solve $P.X = \rho X$, we can apply the
method above and solve $H$ such that $H = P'.H + P.e - e$ with $P' = \rho^{-1}P$. We would have $H_n = P^n.e - e$ which
converges to $X - e$.

**Theorem 2.** There exists a choice of the sequence of nodes
such that the diffusion applied on $(P', P'e - e)$ converges to
$X - e$. 

Table 1: Extracted graph: \( N = 10^3 \) to \( 10^6 \).

| \( N \) | L/N | D/N | E/N | O/N | max\(_{in} \) | max\(_{out} \) |
|---|---|---|---|---|---|---|
| \( 10^3 \) | 12.9 | 0.041 | 0.032 | 0.236 | 716 | 130 |
| \( 10^4 \) | 12.5 | 0.008 | 0.145 | 0.114 | 7982 | 751 |
| \( 10^5 \) | 31.4 | 0.027 | 0.016 | 0.175 | 34764 | 3782 |
| \( 10^6 \) | 41.2 | 0.046 | 0 | 0 | 403441 | 4655 |

Proof. The proof is similar to the stochastic matrix case, except that we replace the \( L \) norm by the norm \( |X| = \sum |x_i| \) where \( V \) is the left eigenvector of \( P \) (which has all entries strictly positive). Then, \( F_n \) is such that \( \sigma_V(F_n) = \sum x_i \times v_i = 0 \) and \( F_n \) is non-increasing function for the norm \( |.|_V \). Of course, the beauty of this result is that we don’t need the explicit expression of \( V \).

6. FIRST CONVERGENCE COMPARISON

6.1 Error

For the PageRank equation we will consider below, the distance to the limit is computed as follows:

- for the power iteration (PI), the distance to the limit is bounded by \( |X_{n+1} - X_n| \times d/(1-d) \); if \( d = 1 \), we use an estimate of the distance to the limit by \( |X_{n+1} - X_n| \);

- for the initial D-iteration method (DI), the distance to the limit is given exactly by the \( L \) norm of the residual fluid divided by \( 1 - d \): \( |F_n|/(1-d) \);

- for DI+, the distance to the limit is bounded by \( |F_n|/(1-d) \); for \( d = 1 \), we use an estimate of the distance to the limit by \( |X_{n+1} - X_n| \).

6.2 Data set

For the evaluation purpose, we used the web graph imported from the dataset uk-2007-05@1000000 (available on [1]) which has 41,247,159 links on 1,000,000 nodes.

Below we vary \( N \) from \( 10^3 \) to \( 10^6 \) extracting from the dataset the information on the first \( N \) nodes. Some graph properties are summarized in Table 1.

- \( L \): number of non-null entries (links) of \( P \);
- \( D \): number of dangling nodes (0 out-degree nodes);
- \( E \): number of 0 in-degree nodes: the 0 in-degree nodes are defined recursively: a node \( i \), having incoming links from nodes that are all 0 in-degree nodes, is also a 0 in-degree node; from the diffusion point of view, those nodes are those who converged exactly in finite steps;
- \( O \): number of loop nodes (\( p_{ii} \neq 0 \));
- \( \max_{in} = \max \#in_i \) (maximum in-degree, the in-degree of \( i \) is the number of non-null entries of the \( i \)-th line vector of \( P \));
- \( \max_{out} = \max \#out_i \) (maximum out-degree, the out-degree of \( i \) is the number of non-null entries of the \( i \)-th column vector of \( P \)).

| \( d \) | nb iter | gain | time (s) | gain |
|---|---|---|---|---|
| \( d = 0.5 \) | PI | 7 | x | 0.01 | x |
| DI | 4.8 | \( \times 1.5 \) | 0 | \( > x \) | 1 |
| DI+ | 3.8 | \( \times 1.8 \) | 0 | \( > x \) | 1 |

| \( d = 0.85 \) | PI | 20 | x | 0.02 | x |
| DI | 15.8 | \( \times 1.3 \) | 0 | \( > x \) | 2 |
| DI+ | 11.8 | \( \times 1.7 \) | 0 | \( > x \) | 2 |

| \( d = 0.99 \) | PI | 469 | x | 0.14 | x |
| DI | 202 | \( \times 2.3 \) | 0.02 | \( \times 7 \) |
| DI+ | 193 | \( \times 2.4 \) | 0.02 | \( \times 7 \) |

| \( d = 1 \) | PI | 5425 | x | 1.47 | x |
| DI | 1379 | \( \times 3.9 \) | 0.2 | \( \times 7.4 \) |
| DI+ | 1282 | \( \times 4.2 \) | 0.17 | \( \times 8.6 \) |

Table 2: \( N = 10^3 \): Comparison of the runtime for a target error of 1/N. Gain: speed-up gain factor w.r.t. PI.

To guarantee the continuity to \( d = 1 \) and to meaningfully consider all nodes, the above graph has been completed with one random outgoing link for all dangling nodes and with one random incoming link for all nodes having no incoming links.

Note that for \( d < 1 \), there exists a unique solution (stationary probability). For \( d = 1 \), DI is not defined, PI may not converge (\( P \) may not be aperiodic) and DI+ converges to the solution which is the limit of \( d \rightarrow 1 \).

6.3 Comparison

For the evaluation of the computation cost, we used Linux (Ubuntu) machines: Intel(R) Core(TM)2 CPU U7600, 1.20GHz, cache size 2048 KB (Linux1, \( g++ -4.4 \)). The algorithms that we evaluated are:

- PI: Power iteration (equivalent to Jacobi iteration), using row vectors; for \( d = 1 \), to force the convergence, we used relaxation idea with parameter 0.5;
- DI: D-iteration with node selection, if \( (F_n)_i > r_n \times \#out_i/L \), where \( \#out_i \) is the out-degree of \( i \) and \( r_n \) is computed per cycle \( n' \);
- DI+: proposed solution. Initialization to \( P \cdot e = e \) followed by D-iteration with node selection, if \( |(F_n)_i| > r_n \times \#out_i/L \), where \( \#out_i \) is the out-degree of \( i \) and \( r_n \) is computed per cycle \( n' \).

7. CONCLUSION

We proposed a new algorithm to accelerate the computation of the dominant eigenvector of non-negative matrix inspired from the D-iteration diffusion vision. To show its potential, first evaluation results are included.
| nb iter | gain | time (s) | gain |
|---------|------|----------|------|
|         |      |          |      |
| $d = 0.5$ |      |          |      |
| PI      | 9 x  | 0.04 x   |      |
| DI      | 5.3 × 1.7 | 0.01 × 4 |      |
| DI+     | 5.1 × 1.8 | 0 × ×4   |      |
|         |      |          |      |
| $d = 0.85$ |      |          |      |
| PI      | 42 x | 0.15 x   |      |
| DI      | 15 × 2.8 | 0.02 × 8 |      |
| DI+     | 13.6 × 3.1 | 0.01 × 15 |      |
|         |      |          |      |
| $d = 0.99$ |      |          |      |
| PI      | 541 x | 1.73 x   |      |
| DI      | 120 × 4.5 | 0.16 × 11 |      |
| DI+     | 117 × 4.6 | 0.15 × 12 |      |
|         |      |          |      |
| $d = 0.999$ |      |          |      |
| PI      | 7739 x | 24.4 x   |      |
| DI      | 919 × 8.4 | 1.34 × 18 |      |
| DI+     | 880 × 8.8 | 1.25 × 20 |      |
|         |      |          |      |
| $d = 1$ |      |          |      |
| PI      | 393 x | 1.26 x   |      |
| DI      | undefined × undefined | undefined × 18 |      |
| DI+     | 53 × 7.4 | 0.07 × 18 |      |

Table 3: $N = 10^4$: Comparison of the runtime for a target error of $1/N$. Gain: speed-up gain factor w.r.t. PI.

| nb iter | gain | time (s) | gain |
|---------|------|----------|------|
|         |      |          |      |
| $d = 0.5$ |      |          |      |
| PI      | 12 x | 1.1 x    |      |
| DI      | 5.7 × 2.1 | 0.14 × 8 |      |
| DI+     | 5.9 × 2.0 | 0.14 × 8 |      |
|         |      |          |      |
| $d = 0.85$ |      |          |      |
| PI      | 53 x | 4.6 x    |      |
| DI      | 17.4 × 3.0 | 0.44 × 10 |      |
| DI+     | 16.6 × 3.2 | 0.41 × 11 |      |
|         |      |          |      |
| $d = 0.99$ |      |          |      |
| PI      | 834 x | 72 x    |      |
| DI      | 163 × 5.1 | 4.6 × 16 |      |
| DI+     | 156 × 5.3 | 4.3 × 17 |      |
|         |      |          |      |
| $d = 0.999$ |      |          |      |
| PI      | 8253 x | 710 x   |      |
| DI      | 936 × 8.0 | 32 × 22 |      |
| DI+     | 897 × 9.2 | 29 × 24 |      |
|         |      |          |      |
| $d = 1$ |      |          |      |
| PI      | 15546 x | 1357 x  |      |
| DI      | undefined × undefined | undefined × 45 |      |
| DI+     | 931 × 17 | 30 × 45 |      |

Table 4: $N = 10^5$: Comparison of the runtime for a target error of $1/N$. Gain: speed-up gain factor w.r.t. PI.

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