Fractional plasticity with anisotropic yielding

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Abstract. Granular soil in the field may have an extent of anisotropy, due to natural sedimentation process. To capture the anisotropic constitutive behaviour of such granular soil, an anisotropic fractional-order plasticity approach based on an anisotropic yielding surface is developed in this study. By calculating the stress gradient on the anisotropic yielding surface, a state-dependent stress-dilatancy relation considering material anisotropy is obtained. Then, several different triaxial test results of anisotropically consolidated granular soil are simulated and compared by the model. It is found that the proposed anisotropic fractional plasticity model can reasonably simulate the stress-strain behaviour of silty sand Merriespruit tailing. However, more effort needs to be carried out to enhance the model for more complicated loading condition.

1. Introduction
Geomaterials, such as the consolidated soft clays, sandy soils and rock, usually exhibit a certain degree of anisotropy, due to either field condition or forming process [1]. Traditional isotropic models cannot well capture the anisotropic response of these materials, without necessary modification. Therefore, a number of different approaches considering material anisotropy have been proposed [2, 3], among which the most common way was to use an rotated yielding surface to capture the isotropic and distortional hardening of geomaterials. For example, Dafalias [2] developed a general framework for modelling soil anisotropy, from the perspective of energy dissipation. This work was then extended by many researchers [4] to capture the anisotropic response of different soils, including clay and sand. However, when modelling the constitutive behaviour of sand and other granular soils, an additional plastic potential and an empirical state parameter were usually needed, due to the density and pressure dependences of the stress-dilatancy response. To remedy this limitation, a new nonassociative fractional-order ($\alpha$) plasticity approach considering the dependence of material state was developed [5]. But, the proposed approach assumed an isotropic yielding of granular soil, before reaching critical state [6], which has limitations when applied to anisotropic soils.

The aim of this study is to develop an extended $\alpha$-plasticity [5] model to account for the anisotropic behaviour of granular soil. Unlike previous study assuming isotropic yielding, a rotated yielding surface considering the evolution of anisotropy is used in this study. There are four main sections of this study, i.e., Section 2 where the basic stress-strain notations and definitions of the constitutive relation, etc., are presented, Section 3 where an extended $\alpha$-plasticity approach considering anisotropic yielding and the associated derivations of an anisotropic flow rule is provided, Section 4 that provides details for identifying the model parameters and validating the model against different sets of experimental results of granular soils are simulated, and Section 5 where the conclusions and outlook of this study are provided.
2. Elastoplastic relation

2.1. Basic Notations for Stress and Strain

According to the convention in critical state soil mechanics (CSSM) [6], the extensive strain and stress can be defined as negative, whereas the compressive ones can be defined as positive. All the stress components in this study are effective stress unless otherwise specified. As this field in fractional plasticity were rarely reached, only triaxial loading condition is considered in this study, for simplicity.

Firstly, the elastic strain increment \(\varepsilon^e\) defined by the classic Hooke’s law is provided, such that:
\[
\varepsilon^e = C^e : \sigma^e
\]
where the increment is indicated by using a superimposed dot (\(\dot{\ }\)), while the superscript (e) means elastic component; the elastic strain increment \(\varepsilon^e\), stress increment \(\sigma^e\) and elastic compliance \(C^e\) are defined respectively as:
\[
\varepsilon^e = \begin{bmatrix} \varepsilon^e_1, \varepsilon^e_2, \varepsilon^e_3 \end{bmatrix}^T
\]
\[
\sigma^e = \begin{bmatrix} \dot{p}^e, \dot{q}^e \end{bmatrix}^T
\]
\[
C^e = \begin{bmatrix} 1/K & 0 & 0 \\ 0 & 1/(3G) & 0 \\ 0 & 0 & 1/(3G) \end{bmatrix}
\]
in triaxial loading condition the volumetric strain can be calculated as: \(\varepsilon_v = \varepsilon_1 + 2\varepsilon_3\), while the generalised shear strain can be calculated as: \(\varepsilon_s = 2(\varepsilon_1 - \varepsilon_3)/3\). In addition, the mean effective principal stress \(p' = (\sigma'_1 + 2\sigma'_3)/3\) and the deviator stress \(q = \sigma'_1 - \sigma'_3\). Note that \(\sigma'_i\) and \(\varepsilon_i\) \((i = 1, 3)\) are the principal stress and strain components. The bulk modulus \((K)\) and the shear modulus \((G)\) can be respectively defined as:
\[
K = \frac{(2+2\nu)G}{3(1-2\nu)}
\]
\[
G = G_0 \left(\frac{(2.97 - e)}{1 + e}\right)^2 \sqrt{p_a}
\]
where \(p_a = 100\) kPa, is the atmospheric pressure; the elastic constant, the Poisson’s ratio and the current void ratio are indicated by using \(G_0\), \(\nu\) and \(e\), respectively. Furthermore, the plastic strain increment \((\dot{\varepsilon}^p)\) can be obtained as:
\[
\dot{\varepsilon}^p = \frac{1}{H} n^T m \dot{\sigma}^e
\]
where \(m\), \(n\) and \(H\) represent the plastic loading direction, flow direction and hardening modulus, respectively, which will be provided later in the next section.

2.2. Caputo’s fractional derivative

According to the previous studies [7, 8], the Caputo’s fractional derivatives [9] are used here due to its relative simplicity in terms of the analytical solution:
\[
\dot{\sigma}^e D_{\sigma^e}^{\alpha} f(\sigma') = \frac{1}{\Gamma(n-\alpha)} \int_{\sigma_c^e}^{\sigma'} (\chi^{n-\alpha}) f^{(n)}(\chi) d\chi, \quad \sigma' > \sigma_c^e
\]
\[
\dot{\sigma}^e D_{\sigma^e}^{\alpha} f(\sigma') = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{\sigma_c^e}^{\sigma'} (\chi^{n-\alpha}) f^{(n)}(\chi) d\chi, \quad \sigma'_c > \sigma_c^e
\]
in which the partial derivation is indicated by $\partial^\alpha / \partial \sigma'^\alpha$. $\alpha \in (n-1, n)$, is the fractional order, describing the extent of non-associativity and state-dependence of the stress-strain response of soil [5]. $n > 0$, is an integer. $\Gamma$ is the gamma function. It can be found that the fractional derivatives have nonlocal definition. Therefore, the choice of the upper and lower limits for integration is of critical importance. Different choices can be used by different researchers. For capturing the state dependence, the current effective stress component is designated as $\sigma'_c = (p'_c$ or $q_c$), whereas the corresponding critical-state stress component is designated as $\sigma'_c = (p'_c$ or $q_c$). It should be noted that $p'_c$ and $q_c$ are two independent critical-state stress components, respectively, at the critical state line (CSL). As explained in Sun et al. [5], the application of $\sigma'_c$ for integral limit was owing to the basic experimental observation in CSSM: when subjected to external loading, soil specimen would deform continuously until reaching the final critical state. According to CSSM, the final critical state can be expressed by using two CSLs, respectively, in the $e - p'$ plane and $p' - q$ plane. Accordingly, the critical-state stresses can be defined as:

$$p'_c = p_a \left( e_e - e \right)^{\frac{1}{\lambda}}$$ (10)

$$q'_c = q + M_c (p' - p'_c)$$ (11)

where $e_e$ is the critical-state void ratio; $e$, $\lambda$ and $\xi$ are the material parameters, characterising the CSL in the $e - p'$ plane. $M$ is the critical-state stress ratio, which is also the slope of the CSL in the $p' - q$ plane.

Note that Eq. (10) was originally proposed by Li and Wang [10], for better characterisation of the CSL of state-dependent material, e.g., sand. Eq. (11) was proposed in Sun et al. [5], by evaluating the relative geometric position between the current stress point and the CSL in the $p' - q$ plane. According to CSSM [6], there can be three relative positions of the current stress point (i.e., above, on, or below the CSL) in relation to the CSL, before reaching critical state. How the current stress point locates with regarding to the CSL determines whether Eq. (8) or Eq. (9) can be used. Specific speaking, when $p' > p'_c$ or $q > q_c$, Eq. (8) would be used; whilst $p' < p'_c$ or $q < q_c$, Eq. (9) should be used. However, as will be demonstrated, even if distinct equations are used, a unique nonassociative anisotropic stress-dilatancy equation will be obtained, with no use of an additional plastic potential and an empirical state index.

3. Anisotropic fractional model

3.1. Plastic loading

To characterise the anisotropic plastic loading behaviour of granular soil, the following anisotropic yielding function ($f$) [2] is used:

$$f = (q - \beta p')^2 - (M^2 - \beta^2)(p'_0 - p')p' = 0$$ (12)

where $p'_0$ and $\beta$ define the location (or size hardening) and anisotropic extent of the yielding surface, respectively. $\beta \approx \eta_{K_o} + (\eta_{K_o} - \eta_{K_o}^c) / 3$, where $\eta_{K_o} = 3(1 - K_o)/(1 + 2K_o)$, is defined as the stress ratio obtained at the $K_0$-consolidation state [3]. $M_c$ can be defied as:

$$M = \frac{6 \sin \phi}{3t - \sin \phi}$$ (13)

where $\phi_c$ is the friction angle obtained at the critical state of granular soil; for triaxial compression case, $M_c = M$ and $t = +1$, whereas for triaxial extension case, $M_c = M$ and $t = -1$. It can be found that Eq.
(12) can degrade to the original Modified Cam-clay (MCC) yielding function [6] when $\beta = 0$ . According to Dafalias et al. [3], $\beta$ may not be a fixed value if subjected to shearing with volumetric deformation. However, $\beta$ is approximately constant during undrained loading where $\dot{e}_v \approx \dot{e}_v^G$ [2, 3]. Due to the complexity in solving fractional derivatives, this study only deals with the undrained loading condition, where $\beta$ is assumed to have a fixed value during model simulation.

As the loading direction ($m$) is perpendicular to the yielding surface, it can be derived from the first-order derivative of the yielding surface. Accordingly, one can has:

$$d_f = \frac{M^2 - \eta^2}{2(\eta - \beta)}$$

$$m = \left[\frac{d_f}{\sqrt{1 + d_f^2}}, \frac{1}{\sqrt{1 + d_f^2}}\right]^T$$

Through Eq. (14), one can assess the influence of $\beta$ on the anisotropic loading behaviour of granular soil. As clearly shown in Fig. 1(a), a clockwise rotation of the loading curve can be indicated with a decreasing $\beta$ . Eq. (15) can degrade to the traditional MCC model when $\beta$ decreases to zero.

![Fig. 1 Effects of $\beta$ and $\alpha$ on (a) loading behaviour and (b) flow behaviour [11]](image)

3.2. Plastic flow
Following the previous studies [5, 8], the nonassociative plastic flow vector ($n$) is non-orthogonal to the yielding surface, thus can be derived through the fractional derivatives of the yielding surface, such that:

$$n = \left[\frac{d_g}{\sqrt{1 + d_g^2}}, \frac{1}{\sqrt{1 + d_g^2}}\right]^T$$

in which $d_g$ indicates the anisotropic stress-dilatancy ratio considering state dependence. In the framework of CSSM, $d_g$ has a variety of empirical definitions. For example, the $\psi$-based Cam-clay expression [12], the $I_\rho$-based elliptic expression [13], where $\psi$ and $I_\rho$ are two empirical state parameters proposed by Been and Jefferies [14] and Wang et al. [15], respectively. Even though, these formulae were proven to be of significant efficiency in simulating the state-dependent dilatancy characteristic of soil, the basic mathematic principles were absent. In this study, an anisotropic stress-dilatancy ratio considering state dependence can be analytically derived by analogous to Sun et al. [5], where $d_g$ can be obtained by substituting Eq. (12) into Eqs. (8) and (9), such that:
where the plastic flow is found to be influenced by the components of the current stress \( (p', q) \), the components of the critical-state stress \( (p'_c, q_c) \) and most importantly the distances from the current state to the corresponding critical state \( (p'_c, q_c) \). By evaluating Eq. (17), one can find that the dilatancy ratio equals to zero at the critical state where \( p' = p'_c \) and \( q = q_c \), where the incremental plastic volumetric strain becomes zero. Moreover, \( d_v = 0 \) at the phase transformation state can be guaranteed by using an appropriate \( \alpha \). Rearranging the numerator of Eq. (17) at \( d_v = 0 \), one has:

\[
\alpha = \frac{p_v \exp[e_t - e_r]/\lambda - p'_v + p'_c - \beta (\beta p'_v - 2q_c)/M^2]}{p_v \exp[e_t - e_r]/\lambda - p'_v/2 + \beta (\beta p'_v - 2q_c)/M^2}
\]  

(18)

with \( e_v \), the void ratio, \( p'_c \) the mean effective principal stress, \( p'_v \) the size of the yielding surface, and \( q_c \) the deviator stress, respectively, at the phase transformation state. Therefore, it can be summarised that unlike classical anisotropic plasticity models where the effect of material state on plastic flow requires an empirical state parameter (e.g., \( \psi \)), an anisotropic plastic flow vector considering the dependence of material state is theoretically developed in this study. Further assessment of Eq. (17) can be carried out and shown in Fig. 1(b). It is found that the stress-dilatancy curve exhibits an anti-clockwise rotation as \( \alpha \) decreases.

3.3. Material hardening

The material hardening of granular soil can be characterised through a proper hardening modulus. According to Dafalias and Manzari [16], it can be expressed as:

\[
H = h_t G \frac{M_b - \eta}{\eta - \eta_m}
\]

(19)

where \( M_b = M \exp[n(e_t - e - \lambda (p'/p_s)^2)] \), is the bounding stress ratio; \( h_t = h_1 - h_c e \), is the hardening function; \( n, h_1 \) and \( h_c \) are material constants. \( \eta_m \) is the initial stress ratio of the sample before being subjected to triaxial shearing.

4. Model validation

4.1. Parameter Identification

According to the derived constitutive equations, the proposed anisotropic \( \alpha \)-plasticity model needs the calibration of ten parameters \( (\phi_c, \lambda, e_r, \xi, \alpha, n, h_1, h_2, G_0, \nu) \). All of the model parameters can be obtained by analysing the triaxial test results. Details for identifying model parameters can be found as follows.

Specifically speaking, \( \phi_c, \lambda, e_r \), and \( \xi \) are identical to the ones used in Cam-clay serial models, for instance, the one developed in Li and Dafalias [17]. The critical-state friction angle \( (\phi) \) can be determined from the slope \( (M_c) \) of the CSL, where

\[
\phi_c = \arcsin \left( \frac{3M_c}{6 + M_c} \right)
\]

(20)

\( \lambda, e_r \) and \( \xi \) can be obtained by using curve fitting of the data points at critical state in the \( e - \ln p' \) plane, The fractional order, \( \alpha \), defines the extent of state dependence and nonassociativity. As
explained before, it can be calculated by using the test data obtained at the phase transformation state \((d_g = 0)\), as shown in Eq. (18). The hardening parameters, \(n\), can be calculated by using the test data obtained at the peak state during triaxial compression, where \(H = 0\). Hence,

\[
    n = \frac{\ln(M_c/\eta_f)}{e_f - e_c + \lambda(p'_f/p_o)^{\xi}} \tag{21}
\]

with \(e_f\) the void ratio, \(p'_f\) the effective mean principal stress and \(\eta_f\) the stress ratio, respectively, at peak state. Furthermore, the parameters, \(h_1\) and \(h_2\), can be determined by trial and error. The elastic constants \(G_0\) and \(\nu\), define the elastic behaviour of the sample, which can be obtained from the initial loading stage by using \(G_0 \approx (1 + \epsilon)G/[(2.97 - \epsilon)^2(p'_f p_o)^{0.5}]\) and \(\nu \approx -\epsilon_3/\epsilon_1\). Table 1 lists the detailed values of each model parameter used for carrying out the further model simulation.

| Soil type                  | \(G_0\) | \(\nu\) | \(\phi_c\) (°) | \(\lambda\) | \(e_f\) | \(\xi\) | \(\alpha\) | \(n\) | \(h_1\) | \(h_2\) |
|---------------------------|---------|---------|----------------|-------------|---------|--------|-----------|------|------|-------|
| Silty sand [18]            | 60      | 0.20    | 33.4           | 0.045       | 0.57    | 0.65   | 1.25      | 3.0  | 0.3  | 0.05  |
| Merriespruit tailing [19]  | 50      | 0.20    | 32.7           | 0.152       | 0.97    | 0.5    | 1.08      | 1.0  | 1.0  | 0.1   |

4.2. Model performance

In this section, the undrained triaxial test data of two different granular soils, i.e., the silty sand [18] and Merriespruit tailings [19], are used to verify the model.

Fig. 2 shows the comparison between the model simulations and the experimental results of the silty sand [18] subjected to \(K_0\)-consolidation. The silty sand was reported to be taken from the Permian age at Mount Compass in South Australia. It was poorly graded and contained around ten percentage of fines. The specimens (height, diameter = 100 mm) were prepared by moist tamping of samples in ten layers. The initial void ratios of each sample were 0.562, 0.556 and 0.555, corresponding to the mean effective principal stresses of 108 kPa, 201 kPa and 278 kPa, respectively. One can find from Fig. 2 that the model can provide satisfactory predictions of the undrained anisotropic behaviour of silty sand subjected to different levels of initial pressures. All the model predictions reach the final steady state flow upon sufficient shearing, which agrees well with the corresponding laboratory observations.

Fig. 3 shows the model simulations of the undrained triaxial behaviour of \(K_0\)-consolidated Merriespruit tailings [19]. The material was reported to be poorly graded and contain 30% quartz fines. Moist-tamping method was adopted for specimen preparation. The initial void ratios were 0.973, 0.863 and 0.935, corresponding to the mean effective principal stresses of 115 kPa, 146 kPa and 170 kPa, respectively. It can be observed in Fig. 3 that the developed model can have good representations of the undrained anisotropic behaviour of \(K_0\)-consolidated Merriespruit tailings. The liquefaction instability can be well predicted.
5. Conclusions
To capture the nonassociative anisotropic behaviour of granular soil, a new anisotropic fractional elastoplastic model was proposed. The main findings of this study can be summarized as:

(1) The original isotropic fractional plasticity model was extended to characterise the anisotropic behaviour of granular soils, by using an anisotropic yielding surface. The developed model had eleven parameters. Details for identify each model parameter were provided.

(2) A nonassociative anisotropic dilatancy equation considering the dependence of material state was proposed, by using the fractional derivatives of the yielding function. Unlike classical approaches, none of the empirical state parameters or plastic potentials were required in the derived stress-dilatancy equation.

(3) The proposed model was validated against different sets of undrained experimental results of two granular soils, where the extended anisotropic fractional plastic model was observed to have satisfactory predictions of the stress-strain behaviour of different granular soils subjected to a variety of initial pressures and densities.
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