Correspondence of multiplicity and energy distributions

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The evaluation of the number of ways we can distribute energy among a collection of particles in a system is important in many branches of modern science. In particular, in multiparticle production processes the measurements of particle yields and kinematic distributions are essential for characterizing their global properties and to develop an understanding of the mechanism for particle production. We demonstrate that energy distributions are connected with multiplicity distributions by their generating functions.

For the count probability distribution, \( P(N) \), the generating function \( G(z) \) is defined as:

\[
G(z) = \sum_{N=0}^{\infty} P(N) z^N.
\]

Thus far the dummy variable \( z \) of the generating function has been considered just as a technical auxiliary variable ("book keeping variable"). Only in the so-called method of collective marks one gives a probability interpretation for the variable \( z \). If we mark each of the \( N \) elements in the set independently with probability \( 1-z \) and leave it unmarked with probability \( z \), then \( G(z) \) is the probability that there is no mark in the whole set.

In this letter multiplicity distributions \( P(N) \) in quasi power-law ensembles and their generating functions \( G(z) \) are discussed. They are connected with the energy distributions \( F(E) \) of elements in the ensemble.

| \( P(N) \) | \( G(z) \) |
|---|---|
| Poisson (PD) | \( \frac{\lambda^N}{N!} \exp(-\lambda) \) | \( \exp(\lambda (z-1)) \) |
| Negative Binomial (NBD) | \( \frac{\Gamma(N+k)}{(N+1)\Gamma(k)} p^k (1-p)^N \left[ 1 - \frac{p}{1-p} (z-1) \right]^{-k} \) |
| Binomial (BD) | \( \frac{K^1}{N![K-N]!} p^N (1-p)^{K-N} \left[ 1 + p (z-1) \right]^K \) |

Note, that generating functions of NBD and BD (shown in Table 1) are in fact some quasi-power functions of \( z \) and as such can be written in the form of the corresponding Tsallis distributions [6–9].

\[
G(z) = \exp_q ([N] (1-z)), \quad (2)
\]

where \( q - 1 = 1/K \) for NBD, \( q - 1 = -1/K \) for BD, and \( q - 1 \to 0 \) for PD. For

\[
z = 1 - \frac{E}{U}, \quad (3)
\]

with the total available energy

\[
U = \sum_{i=1}^{N} E_i, \quad (4)
\]

the multiplicity generating function \( G(z) \) gives the energy distribution

\[
F(E) = G(z = 1 - E/U) = \left[ 1 + (q-1) \frac{E}{U} \right]^{1/q-1}. \quad (5)
\]

which is the well known Tsallis distribution [6], and which for \( q \to 1 \) becomes Boltzmann-Gibbs distribution. This distribution was first proposed in [10] as the simplest formula extrapolating exponential behavior observed for low transverse momenta to power law behavior at large transverse momenta. At present it is known as the QCD-inspired Hagedorn formula [12,13]. Function \( F(E) \) is usually interpreted in terms of the statistical model of particle production employing the Tsallis non-extensive statistics [6,8] and widely used in description of multiparticle production processes [14,15].

To explain the correspondence of multiplicity and energy distributions (schematically illustrated in Figure 1), let us consider a simple example. For fixed number of particles \( N \), energy distribution emerges directly from the calculus of probability for a situation known as induced partition [11]. In short: \( N-1 \) randomly chosen independent points \( \{U_1, \ldots, U_{N-1}\} \) split a segment \((0,U)\) into \( N \) parts, whose length is distributed according to:

\[
F(E|N) = \frac{N-1}{U} \left( 1 - \frac{E}{U} \right)^{N-2}. \quad (6)
\]

The length of the kth part corresponds to the value of energy \( E_k = U_{k+1} - U_k \) (for ordered \( U_k \)). Whereas for fixed \( N \) one have [6], then for \( N \) fluctuating according to \( P(N) \), the resulting energy distribution is

\[
F(E) = \sum_{N=2}^{\infty} P(N) F(E|N). \quad (7)
\]

For \( P(N) \) given by PD, BD, and NBD, equation (7) leads to Tsallis distribution given by equation (5). Relationships between Poissonian multiplicity distribution
and Boltzmann-Gibbs energy distribution are discussed in more detail in the Appendix.

Note that $P(N)$, defined for $N > 1$, describe multiplicity distribution in the full phase-space. In experiments, particle multiplicity is measured usually only within some window of phase-space. Let us assume that the detection process is a Bernoulli process described by the BD ($K = 1$ and $p = \alpha$ for a fixed experimental acceptance $\alpha < 1$). The number of registered particles is

$$M = \sum_{i=1}^{N} n_i,$$

where $n_i$ follows the BD with the generating function $G_{BD}(z)$ and $N$ comes from $P(N)$ with the generating function $G(z)$. The measured multiplicity distribution

$$P(M) = \left. \frac{1}{M!} \frac{d^M H(z)}{dz^M} \right|_{z=0}$$

is therefore given by generating function $H(z) = G(G_{BD}(z))$. Such rough procedure applied to NBD, BD or PD gives again the same distributions but with modified parameters: $p \to \alpha p/[1 - p(1 - \alpha)]$ for NBD, $p \to \alpha p$ for BD, and $\lambda \to \alpha \lambda$ for PD. The measured multiplicity distribution is given by

$$P(M) = \sum_{N=M}^{\infty} P(N) P(M|N)$$

with the acceptance function

$$P(M|N) = \frac{N!}{M!(N-M)!} \alpha^M (1-\alpha)^{N-M}$$

Detection process extend $P(M)$ distribution to multiplicities $M = 0$ and $M = 1$, namely: $P(0) = \sum_{N=2}^{\infty} P(N) (1-\alpha)^N$ and $P(1) = \sum_{N=2}^{\infty} P(N) N \alpha (1-\alpha)^{N-1}$.

The statistical properties of the energy division between a set of particles are completely characterized by the generating function $G(z)$. Despite correspondence between multiplicity and energy distributions, the multiplicity distribution gives in practice complementary information to the energy distribution, because $P(N)$ is defined by the $N^{th}$ derive of $G(z) = F(E)$ at $E = U$, i.e., in the region not available experimentally in measurements at collider experiments.

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**Appendix. Boltzmann-Gibbs energy distribution and Poissonian multiplicity distribution**

Suppose that one has $N$ independently produced particles with energies $\{E_1, ..., N\}$, distributed according to Boltzmann distribution,

$$F(E) = \frac{1}{T} \exp \left( -\frac{E}{T} \right)$$

with “temperature” parameter $T = \langle E \rangle$. The sum of energies, $U = \sum_{i=1}^{N} E_i$ is then distributed according to gamma distribution

$$F_N(U) = \frac{1}{T(N-1)!} \left( \frac{U}{T} \right)^{N-1} \exp \left( -\frac{U}{T} \right)$$

with distribuant equal to:

$$F_N(>U) = 1 - \sum_{i=1}^{N-1} \frac{1}{(i-1)!} \left( \frac{U}{T} \right)^{i-1} \exp \left( -\frac{U}{T} \right).$$

FIG. 1. Multiplicity distributions and corresponding energy distributions.
Looking for such \( N \) that \( \sum_{i=0}^{N} E_i \leq U \leq \sum_{i=0}^{N+1} E_i \) we find its distribution, which has known Poissonian form

\[
P(N) = \frac{(U/T)^N}{N!} \exp \left( -\frac{U}{T} \right)
\]

with \( \langle N \rangle = U/T \).

For the constrained systems (if the available energy is limited, \( U = \text{const} \)), whenever we have independent variables \( \{ E_1, \ldots, N \} \) taken from the exponential distribution \( (A.1) \), the corresponding multiplicity \( N \) has Poissonian distribution \( (A.4) \) \[17\]. However, if the multiplicity is limited, \( N = \text{const} \), the resulting conditional probability becomes:

\[
F(E|N) = \frac{F_1(E)F_{N-1}(U-E)}{F_N(U)} = \frac{N - 1}{U} \left( 1 - \frac{E}{U} \right)^{N-2}
\]

the same as given by equation \( (5) \), and only in the limit \( N \rightarrow \infty \) the energy distribution goes to the Boltzmann distribution \( (A.1) \). For fluctuating multiplicity according to Poisson distribution, the energy distribution is given by \( (A.1) \).

In the same way, as demonstrated in Ref. \[18\], Tsallis energy distribution is connected with the NBD of multiplicity.

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\[1\] The method of collective marks was originated by van Dantzig \[2\], and discussed in \[3\] and \[4\]. Recently, the collective marks method was used to find the probability generating function for first passage probabilities of Markov chains \[5\].

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\[13\] Actually this is the method of generating Poisson distribution in the numerical Monte Carlo codes.

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