The nonrelativistic limit of the relativistic point coupling model

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Abstract

We relate the relativistic finite range mean-field model (RMF-FR) to the point-coupling variant and compare the nonlinear density dependence. From this, the effective Hamiltonian of the nonlinear point-coupling model in the nonrelativistic limit is derived. Different from the nonrelativistic models, the nonlinearity in the relativistic models automatically yields contributions in the form of a weak density dependence not only in the central potential but also in the spin-orbit potential. The central potential affects the bulk and surface properties while the spin-orbit potential is crucial for the shell structure of finite nuclei. A modification in the Skyrme-Hartree-Fock model with a density-dependent spin-orbit potential inspired by the point-coupling model is suggested.

Key words: Skyrme-Hartree-Fock model, relativistic mean-field model, nonrelativistic limit
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1 Introduction

Relativistic point-coupling (RMF-PC) models have proven to deliver predictions for nuclear ground-state observables which are of comparable quality as the ones from the well-established finite-range relativistic mean-field (RMF-FR) model (for a review see [4]) and the Skyrme-Hartree-Fock (SHF) approach.
(for a review see [6]) [1,2]. Besides opening up the way to relativistic Hartree-Fock calculations (see Ref. [7] for a recent application) numerically similar to Hartree calculations with the use of Fierz relations for the exchange terms (up to fourth order, see Ref. [3]) and to the study of the role of naturalness [8,9] in effective theories for nuclear structure related problems, the RMF-PC approach also provides an opportunity to study the interrelations between nonrelativistic and relativistic point-coupling models, i.e., between the RMF-PC and the SHF approach. In Ref. [22] Rusnak and Furnstahl have shown the profitability to apply the concepts of effective field theory such as naturalness to point-coupling models, where besides the version which will be used in our analysis, they consider also tensor terms, the mixing among the densities in the nonlinear terms, and nonlinear derivative terms. This investigation is motivated by the fact that up till now the role and the importance of the various terms in the RMF-PC ansatz is not completely understood, not speaking about terms which might yet be missing. There appear systematic differences in the model predictions which could not yet be mapped onto the corresponding features of the models. A comparison between the nonrelativistic limit of the RMF-PC model and the SHF model may help to clarify these questions.

One problem to face when comparing relativistic finite range with point-coupling nonrelativistic models is that two limits have to be taken: (a) the limit of letting the range of the mesons shrink to zero and (b) the expansion in powers of \( v/c \) (nonrelativistic reduction). The connection between the Skyrme-Hartree-Fock model and the RMF-FR model was done by several authors, without [10,11] and with nonlinear terms [4,12] (employed as self-interactions of the \( \sigma \)-meson), but they did not take into account tensor contributions. The role of the tensor coupling of the isoscalar vector meson to the nucleon in the framework of effective field theories was investigated in Ref. [13].

The nonlinear density dependence is introduced in much different fashion for Skyrme-Hartree-Fock as compared to the Walecka model which employs nonlinear meson self-couplings. This is a hindrance for a direct comparison [4,12]. On the other hand, nonlinear terms in RMF models are important, because only relativistic models with nonlinear terms can reproduce experimental data with acceptable accuracy [4,14,1]. We avoid the problems if we use the RMF-PC model [1], because in this model the nonlinear terms are explicitly density dependent, similar as in Skyrme-Hartree-Fock. Therefore it is worthwhile to
study the connection between RMF-FR and RMF-PC on one hand and between the RMF-PC model and the nonrelativistic Skyrme-Hartree-Fock model on the other hand. In this context, we can study the role not only of the linear terms but also the nonlinear ones of both models in the nonrelativistic limit.

The paper is outlined as follows: Section 2 evaluates the expansion of the finite range meson propagators of the RMF-FR into point-couplings. The nonrelativistic limit is then discussed in section 3 from the RMF-PC as starting point. The part discussing in particular the emerging structure of the spin-orbit functional is taken up in section 4. And a few general comments on exchange are finally made in section 5.

2 From RMF-FR to RMF-PC

The RMF-PC model can be considered as the mediator between RMF-FR and SHF. The effects of finite range have nothing to do with the nonrelativistic limit, so we study them first by comparing the zero-range limit of RMF-FR with the point-coupling ansatz while remaining at the level of RMF. Having done this, we proceed in the subsequent section with the derivation of the nonrelativistic limit of the RMF-PC model.

The covariant formulation of the RMF is based on a Lagrangian density. It is given for the RMF-FR in appendix A.2. For the stationary case, it can equally well be formulated as a Hamiltonian density. This reads

\[ \mathcal{H} = \mathcal{H}_{\text{free}}^{\text{nuc}} + \mathcal{H}_S + \mathcal{H}_V + \mathcal{H}_R, \]

\[ \mathcal{H}_{\text{free}}^{\text{nuc}} = \sum_{\alpha} \bar{\Psi}_\alpha (-i \gamma \cdot \nabla + m_B) \Psi_\alpha, \]

\[ \mathcal{H}_S = \frac{1}{2} ( (\nabla \Phi)^2 + m_0^2 \Phi^2 ) + g_S \Phi \rho_S + \frac{1}{3} b_2 \Phi^3 + \frac{1}{4} b_3 \Phi^4, \]

\[ \mathcal{H}_V = -\frac{1}{2} \left( \nabla V^\mu \nabla V_\mu + m_0^2 V^\mu V_\mu \right) + g_V \rho_0 V_0 - \frac{f_V}{2m_B} \rho_T V_0, \]

\[ \mathcal{H}_R = -\frac{1}{2} \left( \nabla R^\tau_\tau \nabla R_{\mu,\tau} + m_0^2 R^\mu_\tau R_{\mu,\tau} \right) + g_R \rho_0 R_{\tau_0} R_{\tau_0} - \frac{f_R}{2m_B} \rho_T R_{\tau_0}. \]

The \( g_i, f_i \) are coupling constants and the indices \( i \) denote scalar (S), vector (V), tensor (T) and isovector (\( \tau \)). \( \Phi, V_0 \) and \( R_{\tau_0} \) are the isoscalar-scalar and the
zero components of the isoscalar-vector and isovector-vector meson fields, respectively. The densities are defined as corresponding local densities:

isoscalar-scalar: \( \rho_S(\vec{r}) = \sum_{\alpha} \bar{\phi}_\alpha(\vec{r}) \phi_\alpha(\vec{r}), \)

isoscalar-vector: \( \rho_0(\vec{r}) = \sum_{\alpha} \bar{\phi}_\alpha(\vec{r}) \gamma_0 \phi_\alpha(\vec{r}), \)

isovector-vector: \( \rho_{\tau_0}(\vec{r}) = \sum_{\alpha} \bar{\phi}_\alpha(\vec{r}) \tau_3 \gamma_0 \phi_\alpha(\vec{r}), \)

isoscalar-tensor: \( \rho_T(\vec{r}) = -i \sum_{\alpha} \vec{\nabla} \cdot (\bar{\phi}_\alpha(\vec{r}) \vec{\alpha} \phi_\alpha(\vec{r})), \)

isovector-tensor: \( \rho_{\tau T}(\vec{r}) = -i \sum_{\alpha} \vec{\nabla} \cdot (\bar{\phi}_\alpha(\vec{r}) \tau_3 \vec{\alpha} \phi_\alpha(\vec{r})). \)

Now the Hamilton density will be expressed exclusively in terms of the densities (2). To this end, the meson fields are eliminated by inserting the solution of the meson field equation. This is straightforward for linear coupling. We exemplify it here for the isoscalar-vector field. The meson field equation is

\[ (m^2_V - \Delta) V_0 = g_V \rho_0 - \frac{f_V}{2m_B} \rho_T. \]

This is solved for \( V_0 \) by expansion in orders of \( \Delta^n \) going up to first order:

\[ g_V V_0 = \frac{1}{1 - \Delta/m^2_V} \left[ \frac{g^2_V}{m^2_V} \rho_0 - \frac{g_V f_V}{2m_B m^2_V} \rho_T + \frac{g^2_V}{4m^4_V} \Delta \rho_0 \right] \approx \alpha_V \rho_0 - \frac{\theta_T}{2} \rho_T + \frac{g^2_V}{m^4_V} \Delta \rho_0. \]

Reinserting that into the Hamiltonian densities (1d) yields

\[ H_V = \frac{1}{2} \alpha_V \rho_0^2 + \frac{1}{2} \delta_V \rho_0^2 \Delta \rho_0 - \frac{\theta_T}{2} \rho_T \rho_0 + \frac{1}{2} \kappa_T \rho_T^2, \]

with recoupled strengths

\[ \alpha_V = \frac{g^2_V}{m^2_V}, \]
\[ \delta_V = \frac{g^2_V}{m^4_V}, \]
\[ \theta_T = \frac{g_V f_V}{m_B m^2_V}, \]
\[ \kappa_T = \frac{f^2_V}{4m^2_B m^2_V}. \]

The form of the recoupled strengths is similar for the (linear) isovector-vector term.
The expansion is more complicated for the nonlinear isoscalar-scalar term. It involves a combination of expansion and iteration: first, the zero-range expansion as above, and second, an iteration of the nonlinearity. We start from the equation determining the scalar field

\[ g_S \Phi = \frac{1}{1 - \Delta/m_S^2} \left( \alpha_S \rho_S + \tilde{b}_2 \Phi^2 + \tilde{b}_3 \Phi^3 \right), \quad (4) \]

where \( \alpha_S \) is equal to \( -g_S^2/m_S^2 \) and \( \tilde{b}_k \) is equal to \( -b_k g_S/m_S^2 \). The meson propagator is expanded to \( \approx 1 + \Delta/m_S^2 \) as in the vector case. The nonlinearity is resolved by an iteration process. We can also obtain the \( \Phi \) by using another way, for example see Appendix A.3. We obtain the structure

\[ \mathcal{H}_S = \frac{1}{2} \alpha_S \rho_S^2 + \frac{1}{3} \delta_S \rho_S^3 \Delta \rho_S + \frac{1}{2} b_S \rho_S^3 \Delta \rho_S + \frac{1}{4} \gamma_S \rho_S^4 + \zeta_S \rho_S \Delta \rho_S + \zeta'_S \rho_S^2 \Delta \rho_S + \ldots \quad (5a) \]

with coefficients

\[ \beta_S = -b_2 \frac{g_S^3}{m_S^6}, \quad (5b) \]
\[ \gamma_S = (b_3 \frac{g_S^4}{m_S^8} - 2b_2 \frac{g_S^2}{m_S^6} - \frac{g_S^2}{m_S^6}), \quad (5c) \]
\[ \zeta_S = -3b_2 \frac{g_S^3}{m_S^6}, \quad (5d) \]
\[ \zeta'_S = (5b_3 \frac{g_S^4}{m_S^{10}} - 16b_2 \frac{g_S^2}{m_S^8} - \frac{16}{3} b_2 \frac{g_S^4}{m_S^{12}}). \quad (5e) \]

At this point, we can discuss the formal structure of the emerging effective Hamiltonian in comparison with the point-coupling model (6). We see that all the terms of RMF-PC are nicely generated by the above expansion. These are the terms with the coefficients \( \alpha_m, \beta_S, \gamma_S, \delta_m, \) and \( \theta_m \). The expansion generates some more terms not contained in the RMF-PC model. There is the term in \( \kappa_m \), the finite range correction for the tensor term. It can be assumed to be small because the tensor coupling as such is already a small correction. And there are the many further terms generated by the expansion of the nonlinear \( \Phi \) coupling plus finite-range corrections thereof, i.e. the terms in \( \zeta_S, \zeta'_S \) etc. They require a more quantitative consideration.

The mapping of the coefficients for known parametrisations can be seen in table 1. The first three columns show the effective point-coupling parameters
from the RMF-FR forces NL1 [4], NL3 [5] and NL-Z2 [21]. They are compared with the parameters of two genuine point-coupling forces PC-LA [20] and PC-F1 [2]. The “simple” parameters $\alpha_m$ and $\delta_m$ agree nicely amongst the

| Parameter | NL3  | NL1  | NL-Z2 | PC-F1 | PC-LA |
|-----------|------|------|-------|-------|-------|
| $\alpha_S$ | -15.74 | -16.52 | -16.45 | -14.94 | -17.55 |
| $\delta_S$ | -2.37 | -2.65 | -2.65 | -0.63 | -0.64 |
| $\alpha_V$ | 10.53 | 10.85 | 10.66 | 10.10 | 13.34 |
| $\delta_V$ | 0.67 | 0.67 | 0.68 | -0.18 | -0.17 |
| $\alpha_{rS}$ | 0 | 0 | 0 | 0 | 0.029 |
| $\delta_{rS}$ | 0 | 0 | 0 | 0 | 0 |
| $\alpha_{rV}$ | 1.34 | 1.65 | 1.39 | 1.35 | 1.27 |
| $\delta_{rV}$ | 0.09 | 0.11 | 0.11 | -0.06 | 0 |
| $\beta_S$ | 38.11 | 52.64 | 58.74 | 22.99 | 3.32 |
| $\gamma_S$ | -347 | -590.77 | -706.63 | -66.76 | 131.80 |
| $\zeta_S$ | 17.23 | 25.37 | 28.21 | 0 | 0 |
| $\zeta'_S$ | -196.78 | -348.92 | -408.00 | 0 | 0 |
| $\gamma_V$ | 0 | 0 | 0 | -8.92 | -100.84 |

Table 1
Comparison of the point-coupling parameters between RMF-FR and RMF-PC

various models. Comparing the $\delta_m$ between the two models, we notice that their absolute value is smaller in the point-coupling approach. Furthermore, as was already pointed out in Ref. [2], the $\delta_m$ values for the vector channels (both isoscalar and isovector) have a different sign than the ones from the RMF-FR variant. This strongly indicates that their role goes beyond the expansion of the propagators.

The parameters associated with nonlinearity show large deviations. Moreover, one sees that the expansion of the nonlinearity within the RMF-FR approach is slowly converging. The expansion was done here around $\rho_S = 0$. One may hope that other expansion points, as e.g. bulk equilibrium density, lead to better convergence. We have checked that and find that the situation remains as bad. A detailed explanation can be found in appendix A.3. A quick check is to take a typical value for the scalar density in bulk, $\rho_S \approx 0.14 \text{fm}^{-3}$ and to multiply each term with its power of $\rho_S$. We have then a sequence of 0.32, 0.14, 0.22 for NL1 and similar for NL-Z2. The terms in $\zeta_S$ are not small ei-
ther. This demonstrates that the parametrization of nonlinearity is different in its structure in both models. On the other hand, NL-Z2 and PC-F1 produce very similar results for a broad range of observables in existing nuclei [2]. Actual observables explore only a small range of densities around bulk equilibrium density, and they do not suffice to assess the underlying differences in nonlinearity.

To summarize: the expansion of the meson propagator of the RMF-FR into derivative couplings in RMF-PC works fairly well for the leading parameters. The resulting derivative couplings from RMF-FR are different in strength and sign from those of the RMF-PC which hints that there are other, more genuine, sources for gradient terms (quite similar to the density functional theory for electrons [26]). The worst case is the expansion of the non-linearity. The forms are so different in RMF-FR and RMF-PC that we could not find a simple mapping. In order to make these differences visible in practical applications, one needs yet to look for observables which are sensitive to very low densities. Halo nuclei could be a promising tool in that respect [27].

3 From RMF-PC to SHF

In this section we study the nonrelativistic reduction starting from the RMF-PC model including tensor terms and nonlinear terms in both isoscalar-scalar and isoscalar-vector densities. As starting point we use the energy density of the RMF-PC model

\[ H = H_{\text{free}} + H_{S}^{(PC)} + H_{V}^{(PC)} + H_{R}^{(PC)}, \]  

\[ H_{S}^{(PC)} = \frac{1}{2} \alpha_s \rho_s^2 + \frac{1}{2} \beta_s \rho_s^2 \Delta \rho_s + \frac{1}{3} \gamma_s \rho_s^3 + \frac{1}{4} \delta_s \rho_s^4, \]

\[ H_{V}^{(PC)} = \frac{1}{2} \alpha_v \rho_0^2 + \frac{1}{2} \beta_v \rho_0^2 \Delta \rho_0 - \frac{\theta_T}{2} \rho_T \rho_0, \]

\[ H_{R}^{(PC)} = \frac{1}{2} \alpha_r \rho_{rT}^2 + \frac{1}{2} \beta_r \rho_{rT}^2 \Delta \rho_{rT} - \frac{\theta_r T}{2} \rho_{rT}. \]  

This Hamiltonian contains, besides the tensor terms (isoscalar and isovector), the isovector-vector term which appeared to be the most important one in former investigations [1,2]. The parameters \( \alpha, \beta, \gamma, \theta \) are usually determined in a \( \chi^2 \) adjustment to finite nuclear observables. The tensor terms can either be put in by hand in a Hartree theory (like in [22]) or be thought of emerging
We now want to derive the nonrelativistic limit of that energy density following the procedure as described in [4]. In the first round, we consider only isoscalar fields and drop the Coulomb interaction to keep notations simple. Isovector contributions will be taken into account later in the calculation of the spin-orbit terms, because only in this sector the effect is significant. The isoscalar part of this Hamiltonian density leads to the stationary Dirac-equation

$$[-i\vec{\gamma} \cdot \vec{\nabla} + m_B + S + \gamma_0 V_0 + i\vec{\alpha} \cdot \vec{T}] \Psi_\alpha = \epsilon_\alpha \gamma_0 \Psi_\alpha, \quad (7a)$$

where

$$S = \alpha_S \rho_S + \delta_S \rho^2_S + \gamma_S \rho^3_S, \quad (7b)$$
$$V_0 = \alpha_V \rho_0 + \delta_V \rho_0 - \frac{\theta_T}{2} \rho_T, \quad (7c)$$
$$\vec{T} = -\frac{\theta_T}{2} \vec{\nabla} \rho_0. \quad (7d)$$

The details of the nonrelativistic expansion are given in appendix A.4. The nonrelativistic expansion in orders $v/c \propto p/m$ up to $(p/m)^2$ requires, of course, small $p/m$. One also needs to assume $\epsilon_\alpha \approx m_B$ which, however, is related to the first assumption of small momenta. The result of the expansion is that the normal nuclear density is associated with the zeroth component of the vector density $\rho_0$. We use henceforth the identification $\rho_0 = \rho$. The scalar density and the vector density are expressed in terms of this nuclear density $\rho$ and further densities and currents as follows:

$$\rho_S = \rho - 2B_0^2 \left( \tau - \vec{\nabla} \cdot \vec{J} + \rho \vec{T}^2 - 2\vec{T} \cdot \vec{J} + \vec{T} \cdot \vec{\nabla} \rho \right), \quad (8a)$$
$$\rho_T = -\vec{\nabla} \cdot (B_0 \vec{\nabla} \rho) + 2\vec{\nabla} \cdot (B_0 \vec{J}) - 2\vec{\nabla} \cdot (B_0 \rho \vec{T}), \quad (8b)$$
$$B_0 = [2m_B + S - V_0]^{-1}. \quad (8c)$$

The nonrelativistic densities $\rho$, $\tau$ and $\vec{J}$ are defined as

$$\rho = \sum_\alpha W_\alpha \varphi^{\text{cl}}_\alpha \varphi^{\text{cl}}, \quad (8d)$$
$$\tau = \sum_\alpha W_\alpha (\vec{\nabla} \varphi^{\text{cl}}_\alpha) \cdot (\vec{\nabla} \varphi^{\text{cl}}_\alpha), \quad (8e)$$
$$\vec{J} = -\frac{i}{2} \sum_\alpha W_\alpha \left[ \varphi^{\text{cl}}_\alpha (\vec{\nabla} \times \vec{\sigma} \varphi^{\text{cl}}) - (\vec{\nabla} \times \vec{\sigma} \varphi^{\text{cl}})^\dagger \varphi^{\text{cl}} \right]. \quad (8f)$$
where $\varphi^{cl}$ is the nonrelativistic single-nucleon wavefunction.

Finally, we insert the expanded $\rho_S$ and $\rho_T$ into the energy density (6) keeping again terms only up to second order. This yields the nonrelativistically mapped energy density as

$$H^{(cl)} = \frac{1}{2}(\alpha_S + \alpha_V)\rho^2 + \frac{1}{3}\beta_S \rho^3 + \frac{1}{4}\gamma_S \rho^4 + \frac{1}{2}(\delta_S + \delta_V)\rho \Delta \rho - \left(\alpha_S \rho + \beta_S \rho^2 + \gamma_S \rho^3\right) T - \frac{\theta_T}{2}\rho \left(-\vec{\nabla} \cdot (B_0 \vec{\nabla} \rho) + 2\vec{\nabla} \cdot (B_0 \vec{J}) + \theta_T \vec{\nabla} \cdot (B_0 \rho \vec{\nabla} \rho)\right), \quad (9)$$

where

$$T = 2B_0^2 \left(\tau - \vec{\nabla} \cdot \vec{J} + \rho \vec{T}^2 - 2\vec{T} \cdot \vec{J} + \vec{T} \cdot \vec{\nabla} \rho\right) = 2B_0^2 \left(\tau - \vec{\nabla} \cdot \vec{J} + \frac{\theta_T^2}{4} \rho (\vec{\nabla} \rho)^2 + \theta_T \vec{\nabla} \rho \cdot \vec{J} - \frac{\theta_T}{2} (\vec{\nabla} \rho)^2\right). \quad (10)$$

For better comparison, the Hamiltonian density is cast into a general form

$$H^{(cl)} = \frac{C_1}{2}\rho^2 + \frac{C_2}{2}\rho \Delta \rho + C_3 \rho \tau + C_4 \rho \nabla \cdot \vec{J} + \delta \mathcal{H} \quad (11)$$

where basic structures are singled out with a separate coefficient each and the coefficients all may depend on the density, i.e. $C_i = C_i(\rho)$. Less simple forms are lumped together in $\delta \mathcal{H}$. This form can be directly compared with the standard nonrelativistic mean field model, the Skyrme-Hartree-Fock (SHF) energy functional, for a recent review see [24]. The energy functional of SHF is given in appendix A.1. It is obvious that it has the structure of the functional (11). Table 2 compares the coefficients for the here derived nonrelativistic limit of RMF-PC with SHF. The table shows the similarities and the differences between the two models. The striking similarity consists in the fact that all $C_i$ terms appear in both models. A difference appears in the relation between $\tau$- and $\nabla J$-term. The RMF-PC without tensor coupling predicts $C_3 = C_4$ while SHF has two separate (and practically different) coefficients for that. It is interesting to note that the tensor coupling in RMF-PC also allows separate adjustment of these two terms. A basic difference appears with respect to the density dependence of the coefficients. All coefficients of RMF-PC carry a more or less involved density dependence. Only $C_1$ is density dependent for SHF and
\[ \begin{align*} 
C_1 &= \alpha_s + \alpha_V + \frac{2}{3} \beta_S \rho + \frac{1}{2} \gamma_S \rho^4 \\
C_2 &= \delta_S + \delta_V + \theta_T B_0 - \frac{\theta_T^2}{2} B_0 \rho \\
C_3 &= -2B_0^2 (\alpha_S + \beta_S \rho + \gamma_S \rho^2) \\
C_4 &= -C_3 - \theta_T B_0 \\
\delta H &= \frac{\theta_T}{2} \left\{ (\vec{\nabla} B_0) \cdot (\vec{\nabla} \rho) - 2(\vec{\nabla} B_0) \cdot \vec{J} - \theta_T (\vec{\nabla} B_0 \rho) \cdot (\vec{\nabla} \rho) \\
&\quad - 2B_0^2 \vec{\nabla} \rho \cdot \left( 2\vec{J} + \frac{\theta_T}{2} \rho \vec{\nabla} \rho - \vec{\nabla} \rho \right) (\alpha_s + \beta_S \rho + \gamma_S \rho^4) \right\} 
\end{align*} \]

Table 2

The coefficients of the energy density (11) for the nonrelativistic limit of RMF-PC compared with the corresponding terms of SHF. The RMF-PC is grouped in two columns. The first column collects the terms of the standard model. The second column adds terms stemming from tensor coupling. Terms of the RMF-PC which do not fit into the form (11) are collected in the last two rows. They are all related to tensor coupling. This table shows only isoscalar terms in either model.

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Even here, the form of density dependence differs. Both models yield a very similar description of a broad range of nuclear observables in practice. This hints that the differences in density dependence are all somehow compensated by choosing appropriate effective strengths. We urgently need observables for more extreme densities to pin down the differences of the models. All terms in the non-simple part \( \delta H \) come from the relativistic tensor coupling. It is very hard to assess their practical importance and relative weight. One can estimate that they are at most of the order of the \( \Delta \rho \) terms. A detailed analysis remains a task for future research. Last not least, it ought to be mentioned that some SHF functionals carry a term \( \propto J^2 \) which is not present in the above form. It would appear in the nonrelativistic limit of RMF-PC only in the next higher order.

To summarize: The non-relativistic limit of RMF-PC recovers the basic structure of terms in SHF. The latter is more general in that the kinetic and spin-orbit terms have independent parameters while they are more or less linked in RMF. The RMF-PC, on the other hand, adds density dependence to each one of the terms while SHF uses it only in the leading term.
4 Density dependent spin-orbit terms

This section continues the discussions of the non-relativistic limit with a particular emphasis on the spin-orbit potential. It has the general structure

$$\tilde{W}_q = b_4 \vec{\nabla} \rho + b'_4 \vec{\nabla} \rho_q + c_1 \vec{J} + c'_1 \vec{J}_q,$$  \hspace{1cm} (12)

where $q = p, n$ for proton or neutron density. The last two terms are not found in the non-relativistic reduction. They are discarded in the following discussion. The parameters $b_4$ and $b'_4$ are generally density dependent. The derivation from the RMF-PC model (with $\mathcal{H}_R$ taken into account) yields in detail

$$b_4 = -\frac{A(\rho_0)}{(2m_q + A'(\rho_0)\rho_0 + B\rho_q)^2},$$

$$b'_4 = -\frac{B}{(2m_q + A'(\rho_0)\rho_0 + B\rho_q)^2},$$

with

$$A(\rho_0) = (\alpha_S - \alpha_V + \alpha_{T,V}) + 2\beta_S \rho_0 + 3(\gamma_S - \gamma_V)\rho_0^2,$$

$$A'(\rho_0) = (\alpha_S - \alpha_V + \alpha_{T,V}) + \beta_S \rho_0 + (\gamma_S - \gamma_V)\rho_0^2,$$

$$B = -\alpha_{r,V}.$$  \hspace{1cm} (13)

A tensor term (as it appears for example in the model of Rusnak and Furnstahl [22]) will induce a further correction in $b_4$ and $b'_4$.

The spin-orbit term which is usually employed in Skyrme energy functionals exhibits two significant differences compared to the nonrelativistic limit of relativistic models, namely (a) the restricted isovector dependence and (b) the lack of nonlinear terms. An extension of the Skyrme model with an enhancement in $\tilde{W}$ by isospin contributions has already been done (SKI3-4) [17]. By introducing different isospin contributions into the spin-orbit potential, SKI3-4 do reproduce the isotope shift of the rms radii in the heavy Pb isotopes, but these parameter sets still yield different shell closures in superheavy nuclei than the RMF models [16]. On the other hand, in relativistic point-coupling models, the role of the density dependence has proven to be important to reproduce acceptable single particle spectra and spin-orbit splittings [2]. Therefore the enhancement of SKI3-4 with nonlinear terms (density dependence) might result in an improvement of their shell structure predictions.
In Fig. 1, we show the neutron spin-orbit potential for the two RMF forces PC-F1 (RMF-PC) and NL-Z2 (RMF-FR) as well as for the Skyrme interactions SkI3 and SkI4. (The spin-orbit potential for the RMF was obtained from mapping the Dirac equation into an effective Schrödinger equation.) Shown are the results for the four doubly-magic systems $^{16}$O, $^{48}$Ca, $^{132}$Sn and $^{208}$Pb. We recognize that the potentials of the two RMF models have a similar radial dependence, the potential of NL-Z2 being a bit deeper in three cases. The spin-orbit potential of the Skyrme forces, however, is both shifted to larger radii and also deeper in comparison with the RMF results. As could be shown in Ref. [2], the RMF-PC model with PC-LA has similarly a too deep potential peaked at a too large radius. It suffers from the same wrong trend with mass of spin-orbit splittings as do SkI3 and SkI4 (splittings get too large with increasing mass). The reason there are the actual values of the nonlinear parameters, leading to a density dependence of the mean fields that lead to this situation. This issue has been cured with the introduction of PC-F1 which has a predictive power of spin-orbit splittings comparable to the best RMF-FR forces. The Skyrme force SkI3 has a spin-orbit term that mimics the isospin-dependence of the RMF model, while SkI4 has an additional free parameter. However, the spin-orbit potential of SkI4 lies closer to the RMF results. A quantitative calculation using the Skyrme model not only with isospin terms but also with density dependent terms in the spin-orbit potential is strongly suggested. A density-dependent ansatz in the spin-orbit potential has been introduced in Ref. [23], but unfortunately this reference does not give a parameter set for this type of Skyrme model. The form of the spin-orbit potential, which is rather an ad hoc ansatz, is quite different from the one presented here. The spin-orbit part of the energy density which reproduces the spin-orbit potential predicted by our analysis is given by

$$
\varepsilon_{ls} \equiv b_{4}\rho \vec{\nabla} \vec{J} + b'_4(\rho_p \vec{\nabla} \vec{J}_p + \rho_n \vec{\nabla} \vec{J}_n) \\
+ \frac{1}{2}(W_1\rho + W_2\rho^2)(\vec{J}_n \cdot \vec{\nabla} \rho_p + \vec{J}_p \cdot \vec{\nabla} \rho_n + \sum_q \vec{J}_q \cdot \vec{\nabla} \rho_q). 
$$

(14)

The values of above parameters can be illustrated by taking the PC-F1 parameter set, using $\rho_{eq}^\text{PC} = \rho_{nm}$ (PC-F1). We obtain $b_4 = 122.93$ MeV fm$^5$, $b'_4 = 7.01$ MeV fm$^5$, $W_1\rho_{eq}/2 = 36.04$ MeV fm$^5$, and $W_2(\rho_{eq}/2)^2 = -10.26$ MeV fm$^5$. 

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Fig. 1. Neutron spin-orbit potential for the nuclei $^{16}$O, $^{48}$Ca, $^{132}$Sn, and $^{208}$Pb calculated with the forces as indicated.

5 Modifications due to exchange terms

Taking into account the exchange correction in the nonlinear terms (for more details about the calculation of these terms see Refs. [3,19]), after a straightforward calculation in the effective Hamiltonian except $C_2$, all $C_i$ appear to be density dependent (since the explicit form does not provide particular insight, we omit presenting it here). If we calculate them by using the model of Rusnak and Furnstahl [22] with the derivative nonlinear terms taken into account, we will obtain $C_2$ also to be density dependent. If we take into account the exchange corrections, we will have not only modifications in $b_4$ and $b'_4$, but $c_1$ and $c'_1$ also are not zero, due to the contribution of the tensor terms which come from the Fierz transformation [3,19], these corrections are small, however.

The explicit treatment of exchange terms, due to Fierz transformations both in Iso and Dirac space, gives birth to a variety of additional isovector terms without introducing new parameters. This is quite interesting, since there is still a problem in the isovector channel of the RMF model: on one hand, it
seems to be not flexible enough to account for isovector data, on the other hand, modern fitting strategies fail to fix additional terms corresponding to extensions in the isovector channel [2]. Thus, modifications governed by the treatment of exchange might be a cure.

6 Conclusions

We have performed a nonrelativistic reduction of the relativistic point-coupling model and compared it to both the Skyrme-Hartree-Fock model and the non-relativistic limit of the linear RMF model with meson exchange. The motivation was to gain more understanding about the interrelations of these different approaches, which, though looking quite different at first sight, appear to be very similar in the nonrelativistic limit.

We found that there are some significant differences in the models, namely (a) there is a difference in the parametrization of the density dependence of the mean-field, (b) there is no explicit density dependence in the spin-orbit term in the SHF model. We have written down an ansatz for a Skyrme energy functional containing these extensions which should be studied numerically in the future to see if it can improve the spectral features of the SHF model.

A complete treatment of exchange terms in the RMF-PC model would lead to a variety of additional isoscalar and isovector terms, which, in turn, would strongly influence the nonrelativistic limit. Relativistic Hartree-Fock calculations open the possibility to a close investigation of nonrelativistic versus relativistic kinematics and their relevance in effective models for nuclear structure calculations.

A Appendix

A.1 Skyrme Hartree-Fock Model (SHF)

The energy density functional of standard SHF is

\[ \varepsilon^0 = \varepsilon^0_{\text{kin}} + \varepsilon^0_{\text{Sk}}(\rho, \tau, \vec{j}, \vec{J}), \]  

(A.1)
the kinetic term is
\[ \varepsilon^{0}_{\text{kin}} = \frac{\hbar^2}{2m \tau}. \] (A.2)

The Skryme part reads
\[
\begin{align*}
\varepsilon^{0}_{\text{SK}} &= b_0 \rho^2 - b'_0 \sum_q \rho_q^2 + \frac{b_3}{3} \rho^{a+2} - \frac{b'_3}{3} \rho^a \sum_q \rho_q^2 \\
&\quad + b_1 (\rho \tau - \vec{j}^2) - b'_1 \sum_q (\rho_q \tau_q - \vec{j}_q^2) - \frac{b_2}{2} \rho \Delta \rho + \frac{b'_2}{2} \sum_q \rho_q \Delta \rho_q \\
&\quad - b_4 (\rho \vec{\nabla} \cdot \vec{j} + \sigma \cdot (\vec{\nabla} \times \vec{j}) + \sum_q [\rho_q (\vec{\nabla} \cdot \vec{j}_q) + \sigma_q \cdot (\vec{\nabla} \times \vec{j}_q)]]). 
\end{align*}
\] (A.3)

Additionally, the Coulomb energy has to be added. The densities not already defined previously are
\[
\vec{j}_q = -\frac{i}{2} \sum_i [\psi_i^\dagger(q) \vec{\nabla} \psi_i(q) - (\vec{\nabla} \psi_i(q))^\dagger \psi_i(q)], \quad (A.4)
\]
\[
\vec{\sigma}_q = \sum_i \psi_i^\dagger(q) \hat{\sigma} \psi_i(q), \quad (A.5)
\]
but these densities are zero in spherically symmetric systems due to time-reversal invariance.

A.2 Walecka Model (RMF-FR)

The Lagrangian density for RMF-FR is
\[ \mathcal{L} = \mathcal{L}_{\text{free nucleon}}^\text{free} + \mathcal{L}_{\text{free meson}}^\text{free} + \mathcal{L}_{\text{lin coupl}}^\text{lin} + \mathcal{L}_{\text{nonlin coupl}}^\text{nonlin}, \] (A.6)
where
\[ \mathcal{L}_{\text{free nucleon}}^\text{free} = \bar{\psi} (i \gamma_\mu \partial^\mu - m_B) \psi, \] (A.7)
\[
\begin{align*}
\mathcal{L}_{\text{meson}}^\text{free} &= \frac{1}{2} (\partial_\mu \Phi \partial^\mu \Phi - m_\Phi^2 \Phi^2) \\
&\quad - \frac{1}{2} \left( \frac{1}{2} G_{\mu\nu} G^{\mu\nu} - m_\varphi^2 V_\mu V^\mu \right) \\
&\quad - \frac{1}{2} \left( \frac{1}{2} \overline{B}_{\mu\nu} \cdot \overline{B}^{\mu\nu} - m_\overline{B}^2 \overline{R}_\mu \cdot \overline{R}^\mu \right) \\
&\quad - \frac{1}{4} \overline{F}_{\mu\nu} \overline{F}^{\mu\nu}, 
\end{align*}
\] (A.8)
\[ \mathcal{L}_{\text{lin coupl}} = -g_S \Phi \bar{\psi} \gamma^\mu \gamma^\nu \psi - g_R \bar{R}_\mu \gamma^\mu \gamma^\nu \psi - \frac{i f_R}{2m_B} \partial_\nu V_\mu \gamma^\mu \gamma^\nu \psi - \frac{i f_R}{4m_B} \partial_\nu \bar{R}_\mu \gamma^\mu \gamma^\nu \psi - e A_\mu \bar{\psi} \frac{1 + \tau_3}{2} \gamma^\mu \psi, \]  

(A.9)

\[ \mathcal{L}_{\text{nonlin coupl}} = -\frac{1}{3} b_2 \Phi^3 - \frac{1}{4} b_3 \Phi^4. \]  

(A.10)

and

\[ \mathcal{L}_{\text{nonlin}} = -\frac{1}{3} b_2 \Phi^3 - \frac{1}{4} b_3 \Phi^4. \]  

(A.11)

### A.3 Nonlinear Scalar Meson Equation

Here, we give an alternative approach [25] to arrive at Eq. (5a). In this approach it becomes transparent that the convergence of the nonlinear expansion of the scalar density depends solely on \( g_s \) and \( b_i \) and not on the expansion value of the scalar density. The equation for the scalar meson can be written as

\[ (\nabla - m_s^2) \Phi = g_s \rho_s + b_2 \Phi^2 + b_3 \Phi^3. \]  

(A.13)

One way to solve this equation is by using an iteration procedure as follows: we start by choosing an initial \( \Phi \):

\[ \Phi_0(r_0) = -g_s \int D(|r_0 - r_1|) \rho_s(r_1) d^3r_1, \]  

(A.14)

where \( D(|r_0 - r_1|) \) is the Greens function satisfying \( (\nabla - m_s^2) \Phi_0 = g_s \rho_s \), and substitute this \( \Phi \) into

\[ (\nabla - m_s^2) \Phi_n = g_s \rho_s + b_2 \Phi_n^{n-1} + b_3 \Phi_n^{n-1}. \]  

(A.15)

We can say that the iteration procedure terminates if \( \Phi_k = \Phi_{k+1} \). The result is

\[ \Phi(r) = -g_s \int D(|r - r_1|) \rho_s(r_1) d^3r_1 \]

\[ + \int \int f_2(r, r_1, r_2) \rho_s(r_1) \rho_s(r_2) d^3r_1 d^3r_2 \]

\[ + \int \int \int f_3(r, r_1, r_2, r_3) \rho_s(r_1) \rho_s(r_2) \rho_s(r_3) d^3r_1 d^3r_2 d^3r_3 \]

\[ + \cdots \]  

(A.16)

where
\[ f_2(\vec{r}, \vec{r}_1, \vec{r}_2) = -g_s^2 b_2 \int D(|\vec{r} - \vec{r}'|)D(|\vec{r}' - \vec{r}_1|)D(|\vec{r}' - \vec{r}_2|)d^3r' \]  
(A.17)

\[ f_3(\vec{r}, \vec{r}_1, \vec{r}_2, \vec{r}_3) = g_s^3 b_3 \int D(|\vec{r} - \vec{r}'|)D(|\vec{r}' - \vec{r}_1|)D(|\vec{r}' - \vec{r}_2|)d^3r' \\
- g_s^2 b_2^2 \int \int D(|\vec{r} - \vec{r}''|)D(|\vec{r}'' - \vec{r}'|)D(|\vec{r}'' - \vec{r}_1|)D(|\vec{r}' - \vec{r}_2|) \\
\cdot D(|\vec{r}'' - \vec{r}_3|)d^3r'd^3r'', \]  
(A.18)

... is the contribution from terms with \( \rho_s \) more than three. It is clear from Eq. (A.16) that the fast convergence in the nonlinearity (in the power of \( \rho_s \)) can only be obtained if the coupling constants \( g_s \) and \( b_i \) are small, basically independent of the point of expansion.

Because \( D(|\vec{r}'' - \vec{r}'|) \) is a distribution function, we can expand it into a delta function and its derivative:

\[ D(\vec{r}, \vec{r}') = \frac{1}{m^2} \delta^3(\vec{r} - \vec{r}') + \frac{\nabla}{m^2} \delta^3(\vec{r} - \vec{r}') + ... \]  
(A.19)

If we choose to expand only up to the second term, insert it into \( \Phi \) and insert \( \Phi \) into the scalar Hamiltonian, we obtain the same result as Eq. (5a). This delta expansion, if integrated with density, has a connection with the Taylor expansion of the density. It can be easily understood from an artificial 1 dimensional (1D) illustration as follows: in 1D, the propagator can be written as

\[ D(x) = -\frac{g_s}{m^2} \delta(x) + \frac{g_s}{m^2} \delta'(x) + .. + g_s \frac{(-1)^{n+1}}{m^2(2n+1)} \delta^{(n)}(x), \]

\[ = f_0 \delta(x) + f_1 \delta'(x) + .. + f_n \delta^{(n)}(x) \]  
(A.20)

where \( \delta^{(n)}(x) = d^n/dx^n \delta(x) \). Then \( \Phi_0(0) \) is

\[ \Phi_0(0) = \int_{-\infty}^{\infty} \rho(x)D(x)dx = f_0 \rho(0) - f_1 \rho'(0) + ... + (-1)^n \rho^{(n)}(0). \]  
(A.21)

Now if we expand \( \rho(x) \) around \( x=0 \) in a Taylor series as

\[ \rho(x) = \rho(0) + \rho'(0)x + ... + \frac{1}{n!} \rho^{(n)}(0), \]  
(A.22)
we obtain
\[
\Phi_0(0) = \rho(0) \int_{-\infty}^{\infty} D(x) dx + \rho'(0) \int_{-\infty}^{\infty} x D(x) dx + \ldots + \rho^{(n)}(0) \int_{-\infty}^{\infty} \frac{1}{n!} x^n D(x) dx.
\]
\[(A.23)\]

If we compare both \(\Phi_0\), we have an alternate representation of \(D(x)\):
\[
D(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^{(n)}(x) \int_{-\infty}^{\infty} \frac{1}{n!} x^n D(x') dx'.
\]
\[(A.24)\]

It is clear that the choice of proper origin for the expansion in \(x\) has an effect on how far we need to expand \(\rho(x)\) to obtain a good approximation. If we for example find \(x=0\) as a good position, so that \(\rho(x) \approx \rho(0) + \rho'(0)x\), it means that for every \(f_n, n>1\) gives contribution zero. The last equation tells us that it is nothing else than \(D(x) \approx -\frac{m^2}{m^3} \delta(x) + \frac{m^4}{m^3} \delta'(x)\). We suspect a similar behavior to happen in the real word (3D).

### A.4 Details of the nonrelativistic reduction

This appendix provides a brief outline of the nonrelativistic expansion of the scalar and tensor densities. For simplicity, the wavefunctions are used without the index for the state \(\alpha\).

The Dirac equation (7a) decomposes into
\[
(m - \epsilon + S + V_0) \varphi^{(up)} + \sigma \cdot (p + iT) \varphi^{(dw)} = 0, \quad (A.25a)
\]
\[
(m + \epsilon + S - V_0) \varphi^{(dw)} - \sigma \cdot (p - iT) \varphi^{(up)} = 0. \quad (A.25b)
\]

The scalar and vector densities are
\[
\rho_s = \left| \varphi^{(up)} \right|^2 - \left| \varphi^{(dw)} \right|^2, \quad (A.26a)
\]
\[
\rho_0 = \left| \varphi^{(up)} \right|^2 + \left| \varphi^{(dw)} \right|^2. \quad (A.26b)
\]

The “normal” (baryon) density is the vector density \(\rho_0\). In the following, we eliminate the lower component thereby carrying forth only terms up to second order in \(p/m\).
The lower component can be expressed through the upper component

\[
\varphi^{(dw)} = B_0 \sigma \cdot (p - iT) \varphi^{(up)} ,
\]

\[
B_0 = \frac{1}{m + \epsilon + S - V_0} \approx \frac{1}{2m + S - V_0} .
\]

The upper component is not yet the nonrelativistic wavefunction because as such it is not normalized. The classical wavefunction is introduced through

\[
\varphi^{(up)} = \hat{I}^{-1/2} \varphi^{(cl)} ,
\]

\[
\hat{I} = 1 + \sigma \cdot (p + iT) B_0^2 \sigma \cdot (p - iT) .
\]

This yields the desired result for the vector density

\[
\rho_0 = \left| \varphi^{(cl)} \right|^2 .
\]

More involved terms appear for the scalar density:

\[
\rho_s = \left| \varphi^{(up)} \right|^2 - \left| \varphi^{(dw)} \right|^2 ,
\]

\[
= \varphi^{(cl)^+} \hat{I}^{-1} \varphi^{(cl)} - \varphi^{(cl)^+} \hat{I}^{-1/2} \sigma \cdot (p + iT) B_0^2 \sigma \cdot (p - iT) \hat{I}^{-1/2} \varphi^{(cl)}
\]

\[
= \rho_0 - 2 \varphi^{(cl)^+} \sigma \cdot (p + iT) B_0^2 \sigma \cdot (p - iT) \varphi^{(cl)}
\]

\[
= \rho_0 - 2 B_0^2 \left( \tau - \nabla \cdot J + \rho_0 T^2 - 2T \cdot J + T \cdot \nabla \rho \right) .
\]

The tensor density is expanded as

\[
\rho_T = \nabla \varphi^+ \begin{pmatrix} 0 & -i \sigma \\ i \sigma & 0 \end{pmatrix} \varphi
\]

\[
= -\nabla \varphi^{(up)^+} i \sigma B_0 \sigma \cdot (p - iT) \varphi^{(up)} + \nabla \varphi^{(up)^+} \cdot \sigma (p + iT) B_0 i \sigma \varphi^{(up)}
\]

\[
\approx -\nabla (B_0 \nabla \rho_0) + 2 \nabla (B_0 J) - \nabla (B_0 T \rho_0) .
\]

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