Non-Hermitian Floquet Chains as Topological Charge Pumps

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We show that non-Hermiticity enables topological phases with unidirectional transport in one-dimensional Floquet chains. The topological signatures of these phases are non-contractible loops in the spectrum of the Floquet propagator that are separated by an imaginary gap. Such loops occur exclusively in non-Hermitian Floquet systems. We define the corresponding topological invariant as the winding number of the Floquet propagator relative to the imaginary gap. To relate topology to transport, we first introduce the concept of regularization of the Floquet propagator, and then establish that the charge transferred over one period equals the winding number. We illustrate these theoretical findings with a Floquet chain that features a topological phase transition. In the non-trivial phase, this chain acts as a topological charge pump which, in fundamental difference to the situation for static or Hermitian chains, implements quantized unidirectional transport.

Quantum Hall systems [1, 2] and topological insulators [3, 4] are manifestations of a fundamental connection between topology and transport. Topological transport is distinguished from conventional transport by two characteristic properties: It is quantized and robust [5, 6]. Ultimately, quantization and robustness are consequences of the bulk-boundary correspondence, which relates transport via chiral (or helical) boundary states to the topological properties of an insulating bulk [2, 3]. Importantly, topological transport requires boundary or surface states. One-dimensional systems can exhibit non-trivial topology [7, 8], but cannot support directed transport without external sources or fields.

Recent research has shown that non-Hermiticity considerably extends this picture [9–24]. While the new non-Hermitian topological phases, which arise from imaginary and point gaps that occur in addition to the real gaps of the Hermitian case [9–11], have been classified at least for static systems, conclusive results on non-Hermitian topological transport are still rare. Even the status of a non-Hermitian bulk-boundary correspondence remains debatable, since transport and boundary states can be modified outside of the constraints the correspondence imposes on Hermitian systems [13–16]. Notably, non-Hermitian systems are not a theoretical construct but appear naturally in experiments [19–23].

The subject of this work is topological transport in one-dimensional non-Hermitian chains. We show that, contrary to the Hermitian case, these chains can act as topological charge pumps if—but only if—we consider Floquet chains with a time-periodic Hamiltonian \( H(t+T) = H(t) \). To obtain this result we identify a topological phase that occurs exclusively in non-Hermitian Floquet systems, and lies outside of the established classification [9–11] for static non-Hermitian systems.

An overview of the different topological scenarios for non-Hermitian chains is given in Fig. 1, starting from the standard example (St) of a static non-Hermitian chain with directional hopping [9, 23]. In this and later examples, the parameter \( J \) specifies the strength, the parameter \( \gamma \) the directionality of hopping (see the supplement material for specific). The spectrum \( E(k) = J(e^{-ik} + e^{ik}) \) of the Hamiltonian, given as a function of momentum \( k \), is a complex ellipse with real (imaginary) semi-axis \( 2J \cosh \gamma \) (\( 2J \sinh \gamma \)). The elliptical loop \( k \rightarrow E(k) \) is contractible. The non-trivial topology of a non-contractible loop could be enforced by declaring a point gap inside of the loop [9–11], but we will see below that a point gap does not provide us with a notion of topological transport. Splitting the hopping spatially, the chain (Sp) supports two loops separated by a real (G) and imaginary (i\( \Gamma \)) line gap. Right column: By splitting the hopping temporally, the Floquet chain (Fl) supports loops that traverse the complex quasienergy zone with its \( \varepsilon \rightarrow \varepsilon + 2\pi \) periodicity.

FIG. 1. Conceptual overview of the topological scenarios for non-Hermitian chains. Left column: The dispersion of a static chain (St) with directional nearest-neighbor hopping is an elliptical loop in the complex energy plane. Central column: By splitting the hopping spatially, the chain (Sp) supports two loops separated by a real (G) and imaginary (i\( \Gamma \)) line gap. Right column: By splitting the hopping temporally, the Floquet chain (Fl) supports loops that traverse the complex quasienergy zone with its \( \varepsilon \rightarrow \varepsilon + 2\pi \) periodicity.
wrap around the quasiequity of energy and thus are non-contractible. Note that the loops appear with opposite chirality, given by the sign of $\pm 2\pi$. We will now identify these non-contractible loops as the signatures of the topological phase of non-Hermitian Floquet chains, and later also as the origin of topological transport.

To appreciate the specific topology of the one-dimensional non-Hermitian setting we should consider the spectrum of the Floquet-Bloch propagator $\hat{U}(k) \equiv U(T,k)$ (see Fig. 2), which is the solution of the Schrödinger equation $i\partial_t U(t,k) = H(t,k)U(t,k)$ after one period $t = T$. The eigenvalues $\xi_m(k) = e^{-i\varepsilon_m(k)}$ of $\hat{U}(k)$ lie in the punctured complex plane $\mathbb{C} \backslash \{0\}$. Omission of the origin corresponds to invertibility of the propagator, which is guaranteed by the relation $U(t)^{-1} = U(-t)$ also in the non-Hermitian setting.

Static chains can be embedded into the Floquet picture through introduction of an artificial period $T$. Since the Floquet propagator of a static chain is $U = \exp(-iTH)$, a one-to-one correspondence $\varepsilon_m(k) \equiv TE_m(k) \mod 2\pi$ between quasienergies and energies holds for sufficiently small $T$, as long as $\text{Re} E_m(k) \in (-\pi/T, \pi/T)$.

Fig. 2 provides us with three topological insights. First, non-contractible loops of a Floquet chain wind around the origin. Conceptually, the origin serves as a natural point gap for the Floquet propagator (but not for the Hamiltonian). Second, an imaginary gap $i\Gamma$ partitions the spectrum into an inner and outer part, separated by the circle $\phi \mapsto e^{-i\phi+i\Gamma}$. A non-trivial imaginary gap requires a non-Hermitian Hamiltonian with a non-unitary propagator, where the spectrum is not restricted to the unit circle. Third, a real gap $G$, which corresponds to a radial line $r \mapsto re^{-i\phi}$ ($r \in \mathbb{R}_+$), prohibits non-contractible loops and implies trivial topology.

We here observe a fundamental distinction between the topological character of static and Floquet chains. For a static chain (without symmetries), real and imaginary gaps are equivalent through multiplication $H \mapsto iH$ of the Hamiltonian by the imaginary unit $i$. For a Floquet chain, real and imaginary gaps are strictly inequivalent. In consequence, non-Hermitian Floquet chains can support topological phases that lie outside of the established classification for static chains.

Translating the topological concepts of Fig. 2 into an invariant, we are led to the $\mathbb{Z}$-valued winding number

$$W(\Gamma) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \xi_m(k)^{-1} \partial_k \xi_m(k) \, dk$$

for $\xi_m(k) \equiv \exp\{\sum \varepsilon_m(k)\}$, where the map $\Gamma \mapsto \varepsilon_m(k)$ is guaranteed by the relation $U(t)^{-1} = U(-t)$ also in the non-Hermitian setting.

For the trivial imaginary gap $\Gamma = -\infty$, where the entire spectrum of $U$ contributes, we obtain the total winding number $W(-\infty) = 0$. This result follows because the spectrum of $U$ cannot pass through the origin, such that non-contractible loops appear with opposite chirality.

Hermitian chains, where loops cannot be separated by a non-trivial imaginary gap, and static chains, which have real gaps and thus only contractible loops, necessarily have zero winding number $W(\Gamma) = 0$.

The appearance of a non-zero winding number is nicely illustrated with the topological phase transition in the Floquet chain (Fl) (see Fig. 2). Study of the eigenvalues $\xi_{1,2}(k)$ of the Floquet propagator

$$\hat{U}_F(k) = \begin{pmatrix} e^{2-s^2 e^{-i k+2 \gamma}} & -2isc \cos(k/2 + i \gamma) \\ -2isc \cos(k/2 + i \gamma) & e^{2-s^2 e^{ik-2\gamma}} \end{pmatrix}$$

(2)

using the abbreviations $c \equiv \cos J$, $s \equiv \sin J$, reveals that a topological phase transition occurs at the critical value

$$\gamma_c(J) = \arccosh(1/\sin |J|)$$

(3)

(see the supplemental material for details).

Below the transition, for $|\gamma| < \gamma_c$, the spectrum consists of a single loop with periodicity $k \mapsto k + 4\pi$. The winding number necessarily is zero. At the transition, for $|\gamma| = \gamma_c$, the spectrum possesses an exceptional point at $k = 0$. Starting from the exceptional point, the spectrum splits into two loops above the transition, for $|\gamma| > \gamma_c$. The loops occur with opposite chirality, and are separated by an imaginary gap $i\Gamma$ at $\Gamma = 0$. The associated winding number is non-zero, with $W(\Gamma) = 1$ for $\gamma > 0$ (as shown in Fig. 2) and $W(\Gamma) = -1$ for $\gamma < 0$. 

FIG. 2. Top row: Spectrum of the (Floquet) propagator for the chains (Sp), (Fl) from Fig. 1. Only the Floquet chain (Fl) can exhibit non-contractible loops with non-zero winding number. Bottom row: Topological phase transition in the Floquet chain (Fl) with $J = \pi/3$. The transition at $\gamma = \gamma_c \approx 0.55$ (central panel) separates the trivial ($\gamma = 0.2$, left panel) from the non-trivial ($\gamma = 0.6$, right panel) phase.
While the appearance of a topological phase with non-contractible loops should remind us of the anomalous phase of two-dimensional Hermitian Floquet insulators [26–35], the topological phase observed here is specific to one-dimensional non-Hermitian Floquet chains. Formally, it requires a non-trivial imaginary gap. Physically, the non-contractible loops that appear here are not associated with boundary states as in the Floquet insulator, but with the spectrum of the infinite chain.

The second major aspect of non-Hermitian chains is transport, which can be quantified with the charge

$$C(n) = \text{tr}_\mathbb{Z}(U^\dagger [P_n, U])$$

transferred over one period from the left to the right of the chain, through a fictitious layer between sites $n - 1$ and $n$ (see the supplemental material for the origin of this and later expressions). Here, $P_n$ is the projection onto sites $i \geq n$, that is $P_n |i\rangle = |i\rangle$ for $i \geq n$ and $P_n |i\rangle = 0$ for $i < n$ in bra-ket-notation. Note that $C(n) \in \mathbb{R}$ although the operator in the trace generally is not Hermitian.

For a Hermitian chain, evaluation of the trace directly shows that the transferred charge is $C(n) = 0$. Note that this is true only in dimension one: The analogous expression in two dimensions gives the charge carried by chiral boundary states, and can be non-zero [36, 37].

For a non-Hermitian chain, $C(n) \neq 0$ becomes possible. In addition, a charge $c(n) = \langle n | [U, U^\dagger] | n \rangle$ is accumulated at site $n$. As a consequence, $C(n)$ is site-dependent, with

$$C(n) - C(m) = \text{tr}_\mathbb{Z}(U, U^\dagger)(P_n - P_m) = \sum_{l=m}^{n-1} c(l).$$

To remove the site dependence for a translationally invariant chain, we average the transferred charge over a unit cell of $L$ sites, to get $\bar{C} = (1/L) \sum_{n=0}^{L-1} C(n)$. Then, we have the expression

$$\bar{C} = \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{tr}_\mathbb{L}(\tilde{U}(k) \partial_k \tilde{U}(k)) \, dk,$$

where the trace $\text{tr}_\mathbb{L}$ runs over the $L$ sites of the unit cell. By itself, the transferred charge $\bar{C}$ has no topological meaning, and can be non-zero in any non-Hermitian chain, whether static or Floquet (see Fig. 3). The functional dependence of $\bar{C}$ does not reveal much about the nature of transport in these chains. However, if we probe transport by means of wave packet dynamics, propagating an initial state over several (here: $n_p = 40$) periods, we observe significant differences: The wave packet is strongly broadened in the static chain (Sp), but propagates almost coherently in the Floquet chain (Fl) above the topological phase transition (row "(Fl)" in Fig. 3).

To identify the difference between the transport mechanisms realized in static or Floquet chains, and thus to relate transport to topology, we have to introduce the concept of a regularized propagator. The regularization procedure we propose here is analogous to the procedures in Refs. [11, 27], with two essential differences: (i) our regularization applies to a Floquet propagator, not to a static Hamiltonian as the “flattening” in Ref. [11], (ii) our regularization is constructed relative to an imaginary gap $\Gamma$ for a non-unitary Floquet propagator, not relative to a real gap for a unitary propagator as in Ref. [27].

To regularize the propagator, we continuously move the eigenvalues $\xi_m(k)$ of $U(k)$ in- or outwards, relative to a imaginary gap $\Gamma$, as illustrated in Fig. 4. After the deformation, the regularized propagator $\tilde{U}_T(k)$ assumes the canonical form of a partial isometry: First, its eigenvalues $\xi_m(k)$ have modulus zero or one. Starting inside the imaginary gap, with $|\xi_m(k)| < e^{\Gamma}$, we deform $\xi_m(k) \rightarrow 0$. Starting outside of the gap, we deform $\xi_m(k) \rightarrow \xi_m(k)$ with $|\xi_m(k)| = 1$. Second, we require that the eigenvectors of $\tilde{U}_T(k)$ are orthogonal. This requirement can be expressed by the normality condition $[\tilde{U}_T(k), \tilde{U}_T(k) \dagger] = 0$ known from linear algebra [38]. Importantly, regularization affects the eigenvectors of the propagator, which is necessary (only) in the non-Hermitian setting.

As a partial isometry, the restriction of $\tilde{U}_T(k)$ to the orthogonal complement of its null space is unitary [35], but $\tilde{U}_T(k)$ generally is not. Only regularization with respect to the trivial gap $\Gamma = -\infty$ results in a unitary $\tilde{U}_T(k)$.

Regarding topological properties, details of the regularization procedure are not relevant (two techniques are described in the supplemental material). The central requirement is that the imaginary gap stays open during regularization, such that topological properties of the propagator are preserved: We have

$$W(\Gamma)|u = W(0^-)|\tilde{U}_T,$$

where $0^-$ denotes the imaginary gap of $\tilde{U}_T$ below $\Gamma = 0$.

Regularization of the propagator allows us to quantify the topological contribution to transport, without the secondary complications arising from non-Hermiticity. The crucial condition is the normality $[\tilde{U}_T(k), \tilde{U}_T(k) \dagger] = 0$ of the (regularized) propagator. Under this condition, no charge is accumulated ($c(n) = 0$), such that the trans-
ferred charge $C(n)$ is site-independent. Furthermore, we can diagonalize the propagator with a unitary transformation, and express the charge

$$\tilde{C} = \frac{i}{2\pi} \sum_{m=1}^{L} \int_{-\pi}^{\pi} \xi_m(k)^* \partial_k \xi_m(k) \, dk$$  \hspace{1cm} (7)$$

entirely in terms of the eigenvalues $\xi_m(k)$ of $U(k)$. Note that this expression is wrong if $[U, U^\dagger] \neq 0$.

Eq. (7) has a geometric interpretation as the area (with orientation) enclosed by the eigenvalue loops $k \mapsto \xi_m(k)$. This interpretation explains why a point gap does not provide us with a notion of topological transport. Even though the point gap enforces non-contractible loops, the enclosed area can be made arbitrarily small by continuous deformation, allowing for $\tilde{C} \to 0$. As seen in Figs. 2, 4, this is prevented only by an imaginary gap.

For the regularized propagator $\tilde{U}_\Gamma$, which is normal by construction, Eq. (7) applies. Furthermore, the modulus $|\xi_m(k)|$ of each eigenvalue is either zero or one. Under these conditions, Eq. (7) reduces to Eq. (1) for the winding number, and we obtain the fundamental relation

$$\tilde{C}|_{\tilde{U}_\Gamma} = W(\Gamma)$$  \hspace{1cm} (8)$$

between transport and topology. In particular, the transferred charge $\tilde{C}|_{\tilde{U}_\Gamma}$ is quantized, and vanishes in a static or Hermitian chain where $W(\Gamma) = 0$ (see Fig. 4).

For the Floquet chain (Fl), the regularized propagator can be specified explicitly. Below the topological phase transition, where only the trivial gap $\Gamma = -\infty$ exists, regularization corresponds to letting $\gamma \to 0$. This results in a unitary propagator without directed transport (see Fig. 5). Above the topological phase transition, regularization with respect to the imaginary gap at $\Gamma = 0$ corresponds to letting $\gamma \to \pm \infty$, such that exactly one of the loops $e^{\pi ik(\pm \gamma)}$ survives in the spectrum. The regularized propagator is obtained as $\tilde{U}_\Gamma = \lim_{\gamma \to \pm \infty} (e^{2i|\gamma|/\delta^2}U)$, which results in the right ($\gamma > 0$) or left ($\gamma < 0$) shift

$$S_+ = \sum_{n=0}^{\infty} |n+2\rangle \langle n|, \quad S_- = \sum_{n=-\infty}^{\infty} |n-2\rangle \langle n|,$$  \hspace{1cm} (9)$$

acting on either of the two types of sites in Fig. 1.

Shift operators are prototypical instances of regularized propagators, which describe quantized transport by an integer number of sites (see Fig. 5). The fundamental relation (8) is satisfied: We have $\tilde{C} = W = \pm 1$ for $S_\pm$. Notably, regularization is a physical concept: the shift operators are explicitly realized in the Floquet chain (Fl).

The intrinsic difference between transport in static and Floquet chains, which we anticipated in Fig. 3, is now evident. Transport in static chains is simply the by-product of $k$-dependent damping or amplification [24]. In contrast, Floquet chains with non-zero winding number act as charge pumps that implement truly topological, and thus quantized and robust, transport.

In conclusion, we present a theory of topology and transport in non-Hermitian chains. Regularization of the propagator connects both aspects. While transport occurs in any non-Hermitian chain, only Floquet chains possess non-trivial topology and act as topological charge pumps. These findings have several implications. First, they suggest that research into non-Hermitian systems should focus on the relation between topology and transport. Second, the non-Hermitian Floquet phase observed here shows that a complete topological classification of Floquet systems will reveal new phases with interesting transport and dynamical properties. Third, further investigation of the regularization of non-unitary propagators, extending the present approach to situations with disorder, defects, or interfaces, should be rewarding also with a view towards the experiment, where inaccuracies of the implementation need to be controlled. Fourth, already the present results open up new avenues for experiments on non-Hermitian topological systems.

Even the simple Floquet chain (Fl) introduced here allows for implementation of exact shift operators via continuous time evolution, e.g., in photonic waveguide systems [21, 32, 39, 40].
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Supplemental material for:
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The supplemental material (i) specifies the Hamiltonians of the three chains (St), (Sp), (Fl), (ii) derives the topological phase transition in the Floquet chain (Fl), (iii) derives the different expressions for the transferred charge \(C(n)\) and \(\bar{C}\), and (iv) specifies the regularization procedure for the propagator.

THE STATIC CHAINS

To specify the Hamiltonians, we use standard bra-ket notation, where \(|n\rangle\) denotes the state of a particle at site \(n \in \mathbb{Z}\). For the chains (Sp), (Fl) in Fig. 1 in the main text, we identify ‘filled’ sites ⋄ with even \(n\), and ‘open’ sites ○ with odd \(n\).

The Hamiltonian of the chain (St),

\[ H_{\text{St}} = J \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |n+1\rangle \langle n| + e^{-\gamma} |n\rangle \langle n+1| \right), \tag{1} \]

includes directional hopping between nearest neighbors. For \(\gamma > 0\) (\(\gamma < 0\)) hopping to the right (left) is enhanced. The Hamiltonian is Hermitian only for \(\gamma = 0\), when it reduces to that of a tight-binding chain with non-directional hopping.

The Hamiltonian \(H_{\text{St}}\) is invariant under translations by \(L = 1\) sites, and has a scalar Bloch Hamiltonian

\[ \hat{H}_{\text{St}}(k) = J(e^{-ik+\gamma} + e^{ik-\gamma}) \cdot \tag{2} \]

This gives the elliptical quasienergy loop \(E_{\text{St}}(k) \equiv H_{\text{St}}(k)\) shown in Fig. 1 in the main text.

The chain (Sp) essentially consists of two identical copies of the chain (St), placed either on the ‘filled’ or ‘open’ sites. To separate the two copies, we include a staggered potential \(\Delta \in \mathbb{C}\). To couple the two copies, we allow for hopping \(\lambda \in \mathbb{R}\) between the ‘filled’ and ‘open’ sites. This results in the Hamiltonian

\[ H_{\text{Sp}} = J \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |n+2\rangle \langle n| + e^{-\gamma} |n\rangle \langle n+2| \right) \]
\[ + \Delta \sum_{n \in \mathbb{Z}} \left( |2n\rangle \langle 2n| - |2n+1\rangle \langle 2n+1| \right) \]
\[ + \lambda \sum_{n \in \mathbb{Z}} \left( |n\rangle \langle n+1| + |n+1\rangle \langle n| \right), \tag{3} \]

with translational invariance by \(L = 2\) sites.

The corresponding Bloch Hamiltonian is a \(2 \times 2\)-matrix

\[ \hat{H}_{\text{Sp}}(k) = \begin{pmatrix} E_{\text{Sp}}(k) + \Delta & 2\lambda \cos(k/2) \\ 2\lambda \cos(k/2) & E_{\text{Sp}}(k) - \Delta \end{pmatrix}, \tag{4} \]

with eigenvalues

\[ E_{\text{Sp}}(k) = E_{\text{St}}(k) \pm \sqrt{4\lambda^2 \cos^2(k/2) + \Delta^2}. \tag{5} \]

For sufficiently large \(\Delta\) this gives two separate quasienergy loops, as sketched in Fig. 1 in the main text. Note that we normalize the Brillouin zone to the interval \([\pm \pi]\), independently of the size of the unit cell.

In Figs. 3, 4 in the main text we use the parameters \(J = \pi/3, \Delta = 3i\), and \(\lambda = 0.5\) for the chain (Sp). This gives two quasienergy loops separated by an imaginary gap at \(\Gamma = 0\). The gap exists for all \(\gamma \in [0, 1]\), such that the regularization of the chain (Sp) in Fig. 4 in the main text is well defined.

TOPOLOGICAL PHASE TRANSITION IN THE FLOQUET CHAIN

The Hamiltonian of the Floquet chain (Fl) consists of two periodically alternating steps

\[ H_{\text{Fl}}(t) = \begin{cases} H_{\text{Fl}}^{(1)} \text{ for } n_pT \leq t < (n_p + \frac{1}{2})T, \\ H_{\text{Fl}}^{(2)} \text{ for } (n_p + \frac{1}{2})T \leq t < (n_p + 1)T \end{cases}, \tag{6} \]

in each period \((n_p \in \mathbb{Z})\) of length \(T\), where

\[ H_{\text{Fl}}^{(1)} = \frac{2J}{T} \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |2n+1\rangle \langle 2n| + e^{-\gamma} |2n\rangle \langle 2n+1| \right), \tag{7a} \]
\[ H_{\text{Fl}}^{(2)} = \frac{2J}{T} \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |2n-1\rangle \langle 2n-1| + e^{-\gamma} |2n-1\rangle \langle 2n| \right). \tag{7b} \]

As for the chain (Sp), the Hamiltonian \(H_{\text{Fl}}(t)\) possesses translational invariance by \(L = 2\) sites.

The Floquet propagator in real space is

\[ U_{\text{Fl}} = e^{-iH_{\text{Fl}}^{(1)}T/2} e^{-iH_{\text{Fl}}^{(2)}T/2} = \begin{cases} \cosh |n\rangle \langle n| - s^2 \sum_{n \in \mathbb{Z}} \left( e^{2\gamma} |2n+2\rangle \langle 2n| + e^{-2\gamma} |2n-1\rangle \langle 2n-1| \right) \\ -2isc \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |2n+1\rangle \langle 2n-1| + e^{-\gamma} |2n-1\rangle \langle 2n| \right) \end{cases} \]

\[ -2isc \sum_{n \in \mathbb{Z}} \left( e^{\gamma} |2n+1\rangle \langle 2n-1| + e^{-\gamma} |2n-1\rangle \langle 2n| \right), \tag{8} \]
with the abbreviations $c \equiv \cos J$, $s \equiv \sin J$. The Floquet-Bloch propagator in momentum space is

$$
\hat{U}_{\text{Fl}}(k) = \left( \begin{array}{cc}
 c^2 - s^2e^{-ik+2\gamma} & -2isc\cos(k/2 + i\gamma) \\
 -2isc\cos(k/2 + i\gamma) & c^2 - s^2e^{2\gamma-i2k} 
\end{array} \right),
$$

which is Eq. (2) in the main text. The eigenvalues of $\hat{U}_{\text{Fl}}(k)$ are

$$
\xi_{1,2}(k) = 1 - 2s^2\cos^2(k/2 + i\gamma) \pm 2s\cos(k/2 + i\gamma)\sqrt{s^2\cos^2(k/2 + i\gamma) - 1},
$$

from which we can obtain the quasienergies $\varepsilon_{1,2}(k)$ via the relation $\xi_{1,2}(k) = e^{-i\varepsilon_{1,2}(k)}$. Eigenvalues and quasienergies are shown in Fig. 1.

The topological phase transition occurs when the square root in Eq. (10) vanishes, which happens at $\gamma = \pm \gamma_c$ for the critical value

$$
\gamma_c(J) = \text{arcosh}(1/\sin |J|).
$$

The spectrum consists of a single loop for $|\gamma| < \gamma_c$, and of two loops for $|\gamma| > \gamma_c$ (see Fig. 1). The loops are separated by an imaginary gap at $\Gamma = 0$, and have (necessarily) opposite winding number. Note that at the transition, the spectrum possesses an exceptional point at momentum $k = 0$ (and $\xi_1 = \xi_2 = -1$).

For the special value $J = \pi/2$, the spectrum of $\hat{U}_{\text{Fl}}(k)$ consists of two perfectly circular loops $\xi_{1,2}(k) = e^{\pm(ik-2\gamma)}$. The Floquet propagator in real space (see Eq. (8)) is a weighted sum $\bar{U}_{\text{Fl}} = -e^{2\gamma}S_+ - e^{-2\gamma}S_-$ of the right and left shift operator $S_+$, $S_-$ introduced in Eq. (9) in the main text. The transferred charge $\bar{C} = 2\sinh\gamma_\pi$, which is not quantized. In the Hermitian case $\gamma = 0$, it is $\bar{C} = 0$. In the non-Hermitian setting, the two loops of $\hat{U}_{\text{Fl}}(k)$ are separated by the imaginary gap at $\Gamma = 0$ for any $\gamma \neq 0$, since $|\gamma| > \gamma_c = 0$. Regularization can be expressed as the limit $\hat{U}_\Gamma = \lim_{\gamma \to \pm\infty} e^{-2i\gamma}U_{\text{Fl}}$. Only one of the two shift operators survives, with $\hat{U}_\Gamma = S_+$ for $\gamma > 0$, and $\hat{U}_\Gamma = S_-$ for $\gamma < 0$. Regularization thus results in the quantized charge $\bar{C} = \lim_{\gamma \to \pm\infty} e^{-2i\gamma}C = \text{sgn} \gamma$.

### The Transferred Charge

The transferred charge $C(n)$, defined in Eq. (4) in the main text, measures the net amount of particles moving from the left part (sites $i < n$) to the right part (sites $i \geq n$) of a chain.

After one period, the wave function of a particle starting at site $|i\rangle$ has evolved into $U|i\rangle$, with the Floquet propagator $U \equiv U(T)$ acting on real space. The amount of particles starting in the left part and ending in the right part thus is $\sum_{j \geq n} \langle j |U|i\rangle|^2$. Analogously, the amount of particles moving right to left is $\sum_{i \geq n} |\langle i |U|i\rangle|^2$. The net amount of particles is given by

$$
C(n) = \sum_{i \leq n, j \geq n} |\langle j |U|i\rangle|^2 - \sum_{i \geq n, j < n} |\langle j |U|i\rangle|^2
$$

$$
= \sum_{i \in \mathbb{Z}} \langle i |U^\dagger P_n U(1 - P_n) - U^\dagger (1 - P_n) U P_n |i\rangle
$$

$$
= \sum_{i \in \mathbb{Z}} \langle i |U^\dagger P_n U - U^\dagger U P_n |i\rangle
$$

$$
= \text{tr}_Z \left( U^\dagger [P_n, U] \right),
$$

where the projection $P_n$ gives $P_n|i\rangle = \Theta(i - n)|i\rangle$ with $\Theta(x) = 1$ for $x \geq 0$, $\Theta(x) = 0$ for $x < 0$. The trace is $\text{tr}_Z A = \sum_{i \in \mathbb{Z}} \langle i |A|i\rangle$. This is Eq. (4) in the main text. It generalizes the expressions for the Hermitian case (see, e.g., Eq. (3.3) in Ref. [1]) to the non-Hermitian setting.

Note that the operator $U^\dagger [P_n, U]$ in the trace in Eq. (12) is not Hermitian, but since $\sum_{i \in \mathbb{Z}} \langle i |U^\dagger P_n U - U^\dagger U P_n |i\rangle = \sum_{i \in \mathbb{Z}} \langle i |U^\dagger P_n U - P_n U^\dagger U |i\rangle$ we have $C(n) \in \mathbb{R}$.

Similarly, we find the charge accumulated at site $n$ as the difference

$$
C(n) = \sum_{i \in \mathbb{Z}} |\langle n |U|i\rangle|^2 - \sum_{i \in \mathbb{Z}} |\langle i |U|n\rangle|^2
$$

$$
= \langle n |[U, U^\dagger]|n\rangle
$$

between particles moving to or away from the site.

The accumulated charge gives the difference between
$C(n)$ at different sites. We have, for $n \geq m$,
\[
C(m) - C(n) = \sum_{i \in \mathbb{Z}} \langle i | U^\dagger (P_m - P_n) U - U^\dagger U (P_m - P_n) | i \rangle 
\]
\[
= \sum_{l=m}^{n-1} \sum_{i \in \mathbb{Z}} \langle i | U^\dagger | l \rangle | l \rangle \langle i | U^\dagger U | l \rangle | l \rangle i \rangle
\]
\[
= \sum_{l=m}^{n-1} c(l)
\]
(14)

as stated in the main text.

Now assume that $H$, and thus also $U$, is invariant under translations by $L$ sites, that is $U_{m+n+L} = U_{mn}$ for the matrix elements $U_{mn} = \langle m | U | n \rangle$ of the propagator. We define the Fourier transform as
\[
\hat{U}_{mn}(k) = \sum_{l \in \mathbb{Z}} e^{-i(k/l)(mL-n)} U_{ml,nL}(k)
\]
(15)
which is an $L \times L$ matrix with indices $m,n \in \mathbb{L} := \{0, \ldots, L-1\}$, parametrized by momentum $k$. With this definition, we have $2\pi$-periodicity $\hat{U}(k + 2\pi) = \hat{U}(k)$ independently of $L$. The inverse Fourier transform is
\[
U_{mn} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k/l)(mL-n)} \hat{U}_{[mL],[nL]}(k) dk
\]
(16)
where $[m]_L, [n]_L \in \mathbb{L}$ denotes the remainder after division by $L$.

By definition of $\bar{C}$ and $C(n)$, we have
\[
\bar{C} = \frac{1}{L} \sum_{n=0}^{L-1} C(n) 
\]
(17)
\[
= \sum_{m,l \in \mathbb{L}} U_{ml}^\dagger U_{lm} \frac{1}{L} \sum_{n \in \mathbb{L}} (\Theta(l-n) - \Theta(m-n))
\]

To proceed, we use the translational invariance $U_{lm} = U_{l-m+[m]_L,[m]_L}$ of $U$, and equivalently for $U_{ml}^\dagger$. Then, a standard Fourier computation gives
\[
\bar{C} = \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{tr}_L \left( \hat{U}^\dagger(k) \partial_k \hat{U}(k) \right) dk
\]
(18)
Here, $\text{tr}_L A = \sum_{n=0}^{L-1} A_{nn}$ sums over the indices of the Fourier transform, which correspond to the $L$ sites of a unit cell. This is Eq. (5) in the main text.

In contrast to the Hermitian case, the expression (18) is not invariant under $k$-dependent unitary transformations. If we replace $\hat{U}(k)$ by $Q(k) \hat{U}(k) Q(k)\dagger$, Eq. (18) changes into
\[
\bar{C} = \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{tr}_L \left( \hat{U}^\dagger(k) \partial_k \hat{U}(k) \right.
\]
\[
+ \left[ \hat{U}(k), \hat{U}(k)^\dagger \right] Q(k) \partial_k Q(k) \right)
\]
(19)
(we have used $(\partial_k Q(k)^\dagger) Q(k) = -Q(k)^\dagger \partial_k Q(k)$ for unitary $Q(k)$). Through the additional term in the second line, $\bar{C}$ depends explicitly on the eigenvectors of $\hat{U}(k)$. Therefore, we cannot diagonalize $\hat{U}(k)$ and express Eq. (18) entirely in terms of its eigenvalues. This complication is intrinsic to the non-Hermitian setting.

If, however, $[\hat{U}(k),\hat{U}(k)^\dagger] = 0$, the additional term drops out of Eq. (19). This condition is known from linear algebra [2], where it expresses the normality of the matrix $\hat{U}(k)$. Under this condition we can diagonalize the propagator with a unitary transformation as in the Hermitian case, write $\hat{U}(k) = Q(k) D(k) Q(k)^\dagger$ with a diagonal matrix $D(k)$ that contains the eigenvalues $\xi_1(k), \ldots, \xi_L(k)$ of $U(k)$, and arrive at
\[
\bar{C} = \frac{i}{2\pi} \sum_{m=1}^{L} \int_{-\pi}^{\pi} \xi_m(k)^* \partial_k \xi_m(k) dk
\]
(20)
This is Eq. (7) in the main text.

Since we know that $\bar{C} \in \mathbb{R}$, we can rewrite this equation as
\[
\bar{C} = -\frac{1}{2\pi} \sum_{m=1}^{L} \int_{-\pi}^{\pi} \text{Im}(\xi_m(k)^* \partial_k \xi_m(k)) dk
\]
(21)
which suggests the geometric interpretation of the transferred charge mentioned in the main text, namely, that $\bar{C}$ is the area in the $\xi$-plane enclosed by the loops $k \mapsto \xi_m(k)$. The sign of $\bar{C}$ is such that a clockwise loop gives a positive contribution.

**REGULARIZATION OF THE PROPAGATOR**

Regularization starts with a Floquet-Bloch propagator $\hat{U}(k)$ with imaginary gap $i\Gamma$. Note that all matrices in this section depend on momentum $k$, and we take it for granted that matrix functions $k \mapsto \hat{U}(k)$ etc. are (at least) continuous. In the following equations, we occasionally drop the $k$-dependence to allow for simpler notation.

For a unit cell of $L$ sites, $\hat{U}(k)$ is an $L \times L$ matrix. If the dimension of the eigenspace to eigenvalues $|\xi_m(k)| \geq e^\Gamma$ outside of the imaginary gap is $l$, with $1 \leq l \leq L$, the dimension of the eigenspace to eigenvalues $|\xi_m(k)| < e^\Gamma$ inside of the imaginary gap is $L-l$. Note that $l$ does not depend on $k$.

A direct approach to regularization of the propagator is to use the Schur decomposition [2], where we write the propagator as $\hat{U}(k) = Q(k) A(k) Q(k)^\dagger$ with unitary $Q(k)$ and triangular $A(k)$. One reason to use the Schur decomposition instead of the spectral decomposition is that the latter fails to exist at exceptional points.

Write $A(k) = D(k) + N(k)$, where the diagonal part $D(k)$ contains the eigenvalues $\xi_1(k), \ldots, \xi_L(k)$ of $\hat{U}(k)$, and $N(k)$ is strictly upper triangular. Now we can deform the matrix $A(k)$ as required for regularization. The
eigenvalues on the diagonal of $A(k)$ are moved radially towards the unit circle or the origin, and the strictly upper triangular part above the diagonal is sent to zero. To be specific, define the function

$$f_{\Gamma}(z, s) = \begin{cases} (1 - s)z & \text{if } |z| < e^{f_{\Gamma}}, \\ (1 - s + \frac{s}{|z|})z & \text{if } |z| > e^{f_{\Gamma}}. \\ \end{cases}$$

We have $f_{\Gamma}(z, 0) = z$, and $f_{\Gamma}(z, 1) = 0$ for $|z| < e^{f_{\Gamma}}$ but $f_{\Gamma}(z, 1) = z/|z|$ for $|z| > e^{f_{\Gamma}}$. Now we define a continuous deformation

$$A(s) = f_{\Gamma}(D, s) + (1 - s)N \tag{23}$$

with parameter $s \in [0, 1]$. Reinserting into the Schur decomposition gives a continuous deformation $\hat{U}(k, s)$ of the propagator that implements the regularization procedure as $s \to 1$. By properties of the function $f_{\Gamma}$, an imaginary gap stays open during the deformation.

This approach, while being direct and straightforward, relies on the assumption that the matrix functions $k \mapsto Q(k)$ and $k \mapsto A(k)$ in the Schur decomposition have $2\pi$ periodicity. This assumption is not valid in general (the single $k \mapsto \xi(k)$ loop in Fig. 2 in the main text is a counterexample). The reason is that the Schur decomposition is not unique, and we can only assume that $A(0)$ and $A(2\pi)$ are related by a unitary transformation (if all eigenvalues differ, this reduces to a permutation matrix).

If the functions $k \mapsto Q(k)$, $k \mapsto A(k)$ do not have $2\pi$ periodicity, regularization via the Schur decomposition can fail. The problem is that the deformation of $A(k)$ depends explicitly on $N$, and is not invariant under unitary transformations. Conceptually, for $N \neq 0$ the deformation in Eq. (23) involves a choice on the orthogonalization of the eigenvectors of $A(k)$ (hence of $\hat{U}(k)$), which depends on the specific form of $N$. For a normal propagator ($[\hat{U}, \hat{U}^\dagger] = 0$), where $N = 0$, this problem does not arise.

Incidentally, for the examples in the present paper, where $l = L - l = 1$, this problem also does not arise. Here, the Schur decomposition is essentially unique since the one-dimensional eigenspaces inside and outside of the imaginary gap remain separated for all $k$. Therefore, we used the Schur decomposition for Fig. 4 in the main text.

The complication with the Schur decomposition requires us to consider a second alternative approach to regularization. Let $P_{>}(k)$ denote the projection onto the eigenspace to eigenvalues $|\xi(k)| > e^{f_{\Gamma}}$ outside of the imaginary gap, and $P_{<}(k)$ the projection onto the eigenspace to eigenvalues $|\xi(k)| < e^{f_{\Gamma}}$ inside of the imaginary gap. Note that we cannot assume that the projections are orthogonal, unless $U$ is normal.

Using the projections, split $\hat{U}(k)$ as

$$\hat{U}(k) = P_{>}\hat{U}P_{>} + P_{<}\hat{U}P_{<} \tag{24}$$

$$= \hat{U}_{>} + \hat{U}_{<}.$$

We now deform $\hat{U}_{>}$ to a partial isometry, and let $\hat{U}_{<} \to 0$.

By assumption, $\hat{U}_{>}$ is a matrix with rank $l$. Therefore, it can be written in the form

$$\hat{U}_{>} = QBQ^\dagger, \tag{25}$$

where $Q$ is a $L \times l$ matrix with orthogonal columns (such that $Q^\dagger Q = 1_l$), and $B$ an $l \times l$ matrix. The spectrum of $B$ is equal to the spectrum of $\hat{U}_{>}$, in particular, $B$ has full rank $l$.

Note that here no problem arises with the periodicity of the function $k \mapsto Q(k)$, since we do not assume a specific form of $B$ as we did in the Schur decomposition for the matrix $A$. The function $k \mapsto Q(k)$ can always be chosen with periodicity $2\pi$.

Now we use a polar decomposition [2]

$$B = RS, \tag{26}$$

with a unitary $l \times l$ matrix $R$ and a Hermitian and positive definite $l \times l$ matrix $S$ (the latter is guaranteed since $B$ has full rank $l$). Using the polar decomposition, which is unique for full rank $B$, avoids the complications arising from the non-uniqueness of the Schur decomposition.

Since $S$ is positive definite, we can write $S = e^X$ with Hermitian $X$, and define a continuous deformation

$$\hat{U}(s) = QRe^{(1-s)X}Q^\dagger + g(s)\hat{U}_{<} \tag{27}$$

of $\hat{U}$ with parameter $s \in [0, 1]$. Here, $g(s)$ is a (largely arbitrary) function with $g(s) > 0$, $g(0) = 1$, and $g(1) = 0$. Since the smallest eigenvalue of the matrix $QRe^{(1-s)X}Q^\dagger$ is bounded away from zero for $0 \leq s \leq 1$, we can always achieve that an imaginary gap stays open during the deformation by letting $g(s)$ approach zero sufficiently fast.

As $s \to 1$, $\hat{U}(s)$ deforms the propagator into the regularized propagator $\hat{U}_T = \hat{U}(1) = QQR_{T}^\dagger$, which is a partial isometry. For the trivial imaginary gap at $\Gamma \to -\infty$, we have $l = L$ and $\hat{U}_T$ is unitary, as it must.

Note that the deformation (27) does not move eigenvalues strictly along radial lines as in the deformation (23). However, for normal $U$, we can choose a diagonal $B$ in Eq. (25). Inspection of the polar decomposition shows that in this case the deformation indeed moves eigenvalues along radial lines, and with $R = f_{\Gamma}(B, 1)$ we recover the regularized propagator of the previous approach.

Having presented two different approaches to regularization we recall that, regarding topological properties, details of the regularization procedure are irrelevant. Which approach to regularization will prove most convenient in practice is a matter of future experience.

[1] G. M. Graf and C. Tauber, Ann. Henri Poincaré 19, 709 (2018).

[2] G. H. Golub and C. F. Van Loan, Matrix Computations, 4th ed. (The Johns Hopkins University Press, 2013).