POPULAR PRODUCTS AND CONTINUED FRACTIONS

NIKOLAY MOSHCHEVITIN, BRENDAN MURPHY, AND ILYA SHKREDOV

Abstract. We prove bounds for the popularity of products of sets with weak additive structure, and use these bounds to prove results about continued fractions. Namely, we obtain a nearly sharp upper bound for the cardinality of Zaremba’s set modulo $p$.

1. Introduction

This paper is about a variation of the sum-product problem, and the application of such results to problems on continued fractions.

1.1. The sum-product problem. The sum-product problem is to show quantitatively that a finite subset of a ring cannot be approximately closed under addition and multiplication, unless it is approximately a subring. Originally, Erdős and Szemerédi [17] considered a finite set $A$ of integers and asked if $A$ must grow under either addition or multiplication. More precisely, they considered the sum set $A + A = \{a + a': a, a' \in A\}$ and product set $AA = \{aa': a, a' \in A\}$ and asked if we must have

$$\max(|A + A|, |AA|) \gg |A|^{1+\delta}$$

for some $\delta > 0$.

We study a related phenomenon: if $A$ is a subset of $\mathbb{F}_p$ and $A + B$ is small for some set $B$, which may be much smaller than $A$, then for any non-zero element $x \in AA$, the number of ways to write $x = aa'$ with $a, a' \in A$ is $o(|A|)$. That is, if $A$ is almost invariant under addition by a smaller set, then $AA$ contains no popular products.

1.2. Summary of results. Our first type of result shows that if the sumset of $A$ and $B$

$$A + B = \{a + b: a \in A, b \in B\}$$

is small, then the sumset of the set of reciprocals $A^{-1}$ with any other set $C$

$$A^{-1} + C = \{a^{-1} + c: a \in A, c \in C\}$$

must be large. These results work when $B$ and $C$ are much smaller than $A$. See Theorem and Theorem.
We use these results to show that if $A + B$ is small, where $B$ may be much smaller than $A$, then $A$ does not have any popular products. That is, for all $\rho \neq 0$,

\[ |A \cap \rho A^{-1}| = |\{(a, a') \in A \times A: aa' = \rho\}| = o(|A|). \]

See Corollaries 2, 3, and 5.

We use these bounds on popular products to bound the number of integers $1 \leq a \leq p - 1$ such that the continued fraction expansion of $a/p$ has partial quotients bounded by a fixed number $M$. See Theorem 6.

1.3. Methods. To prove lower bounds for $\max(|A + B|, |A^{-1} + C|)$, we consider a set $S$ of linear fractional transformations that map at least $|A|$ element of $A^{-1} + C$ to $A + B$. If both $A + B$ and $A^{-1} + C$ are not much larger than $A$, then $S$ is a set of rich linear fractional transformations of $Y = (A + B) \cup (A^{-1} + C)$. This is related to Elekes’ geometric proof [16, 15] of a lower bound for $\max(|A + B|, |AC|)$; since we need $B$ and $C$ to be much smaller than $A$, our methods of proof are closer to that of the asymmetric sum-product theorem [4, 56, 49, 48].

We use the $\ell^2$-flattening method of [6] to prove asymptotic estimates for the number of rich linear fractional transformations. See [57] for similar results and methods. In addition, a related result was proved by Bourgain [5], framed as an incidence bound for Cartesian product point sets and hyperbolas (corresponding to graphs of linear fractional transformations.)

1.4. Notation. Given two sets of finite subsets $A$ and $B$ of a commutative ring, we use $A \pm B$ to denote the sum set and difference set of $A$ and $B$

\[ A \pm B := \{a \pm b: a \in A, b \in B\} \]

and $AB$ to denote the product set of $A$ and $B$

\[ AB := \{ab: a \in A, b \in B\} \]

If the elements of $A$ are invertible, we use $A^{-1}$ to denote the set of inverses of elements of $A$. The ratio set of $A$ and $B$ is $A/B = A(\{0\})^{-1}$. If $\rho \neq 0$, we use $\rho A$ to denote the set of dilates of elements of $a$ by $\rho$

\[ \rho A := \{\rho a: a \in A\}. \]

All logarithms are base 2.

We use the standard Vinogradov symbols $\gg$ and $\ll$:

\[ f \ll g \iff \exists C > 0 \ f \leq Cg, \]

and $f \gg g$ if and only if $g \ll f$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$. A subscript in the asymptotic notation, such as $f \ll_M g$, means that the implicit constant $C$ depends on the variable $M$. We have used little-o notation in the introduction for brevity; we give precise statements below.

For a real number $x$, we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to $x$, and we use $\lceil x \rceil$ to denote the least integer greater than or equal to $x$. Thus, $\lfloor x \rfloor \leq x < \lceil x \rceil + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

We use vertical bars to denote the cardinality of a set, for instance $|A|$.
If $G$ is a group acting on a set $X$, $f: G \to \mathbb{C}$ has finite support, and $\phi: X \to \mathbb{C}$, then we define the \textit{convolution} $f * \phi: X \to \mathbb{C}$ by
\[
(f * \phi)(x) := \sum_{g \in G} f(g)\phi(g^{-1}x).
\]

A special case of this is when $X = G$ and $G$ acts on itself by left-translation.

1.5. Organization. This paper is organized as follows.

- In Section 2 we state lower bounds for $\max(|A + B|, |A^{-1} + C|)$ and use these bounds to derive popular product bounds for sets that are almost invariant under addition with a smaller set.
- In Section 3 we apply the results from the previous section to a problem in continued fractions.
- In Section 4 we prove bounds for the number of “rich” linear fractional transformations; this is the tool we use to prove bounds in Section 2.
- In Section 5 we prove the bounds for sums of reciprocals stated in Section 2, using the results in Section 4.
- In Section 6 we prove a $\ell^2$-flattening result for linear fractional transformations acting on the projective line.
- In Sections 7 and 8 we prove the results used to prove the rich linear fractional transformations results in Section 4.

2. Bounds for sums of reciprocals and popular products

In this section we state two lower bounds (Theorems 1 and 3) for sums of a set and its reciprocals, and then derive bounds for popular products. The proofs of Theorems 1 and 3 are in Section 5 since they require technical results stated in Section 4.

\textbf{Theorem 1.} Let $A, B,$ and $C$ be subsets of $\mathbb{F}_p$, and let $\rho$ be a non-zero element of $\mathbb{F}_p$.

There is a constant $b_0 > 1$ such that for all $\varepsilon > 0$ and all $\delta \leq \frac{1}{4}b_0^{-1/\varepsilon}$, if $\min(|B|, |C|) = p^\varepsilon$, then for all sufficiently large $p$ we have
\[
|A + B| + |\rho A^{-1} + C| \gg \min(p|A|, |A|p^\delta).
\]

In fact, if we write $W = (A + B) \cup (\rho A^{-1} + C)$, then we have
\[
|A| \leq \frac{|W|^2}{p} + C_* |W|p^{-\delta(k)},
\]
where $C_* \geq 6$ is an absolute constant and $\delta(k) = 2^{-(k+2)}$, where $k \gg \log_{\min(|B|, |C|)} p$.

Similar results were proved in [57], and other results about sums of reciprocals were proved in [1] and [44].

Theorem 1 implies a bound for popular products.
Corollary 2. There is a constant $b_0 > 1$ such that the following holds for all $0 < \kappa < 1$, $\varepsilon > 0$, and $\delta \leq \frac{1}{b_0^{1/\varepsilon}}$.

Suppose that $A \subseteq X + B$, where $A, X, B \subseteq \mathbb{F}_p$, $|B + B| \leq \sigma |B|$, $|B| \geq p^\varepsilon$, and $|X| \ll \frac{|A| |B|^\kappa}{|B|}$. If $\rho \neq 0$, then

$$|A \cap \rho A^{-1}| \ll \max \left( \frac{\sigma^2 |A|^2 |B|^{2\kappa}}{p}, \frac{|A| |B|^\kappa}{p^\delta} \right).$$

Proof. Put $A_* = A \cap \rho A^{-1}$. Then

$$|A_* + B| \leq |A + B| \leq |X + B + B| \leq |X||B + B| \leq \sigma |X||B| \ll \sigma |A||B|^\kappa.$$ 

Since $A_* = \rho A_*^{-1}$, we have

$$|A_* + B| + |\rho A_*^{-1} + B| \ll \sigma |A||B|^\kappa.$$ 

By Theorem 1

$$|A_* + B| + |\rho A_*^{-1} + B| \gg \min(\sqrt{p|A_*|}, |A_*|^p),$$

where $\delta \leq b_0^{1/\varepsilon}$. Combining the last two equations, we have the desired upper bound for $|A_*|$. \hfill \Box

Corollary 3. There is a constant $b_0 > 1$ such that the following holds for all $\varepsilon > 0$ and $\delta \leq \frac{1}{b_0^{1/\varepsilon}}$.

Suppose that $A, B \subseteq \mathbb{F}_p$, $|A + B| \leq \sigma |A|$, and $|B| \geq p^\varepsilon$. If $\rho \neq 0$, then

$$|A \cap \rho A^{-1}| \ll \max \left( \frac{\sigma^2 |A|^2}{p}, \frac{\sigma |A|}{p^\delta} \right).$$

Theorem 4. Fix $0 < \tau < 1/8$. Let $A, B,$ and $C$ be subsets of $\mathbb{F}_p$ such that $B = \{1, \ldots, M\}, C = \{1, \ldots, N\}$, and $1 \leq |A| \leq p^{1-\delta}$, where $\delta = 0.25 b_0^{-1/\tau}$ for an absolute constant $b_0 > 1$.

If $p \gg 1$ and $11 \leq \min(|B|, |C|) \leq p^{\tau}$ then

$$|A + B| + |A^{-1} + C| \geq \frac{|A|}{2} \left( \frac{\min(|B|, |C|)}{2} \right)^{\delta/\tau}.$$ 

Theorem 4 is proved in Section 5 using Theorem 11 stated below. Our motivation for proving Theorem 4 is the following corollary.

Corollary 5. Suppose that $A \subseteq X + B$, where $A, X, B \subseteq \mathbb{F}_p$, $1 \leq |A| \leq p^{1-\kappa}$, $B = \{1, \ldots, |B|\}$, and $|X| \leq \frac{|A|}{|B|} |B|^\kappa$. If $p \gg 1$ and $11 \leq |B| \leq p^{\tau}$, then for $0 < \tau < 1/8$ and $\kappa = 0.25 b_0^{-1/\tau}$ we have

$$|A \cap A^{-1}| \leq 2^{4+\kappa/\tau} |A||B|^{\kappa(1-1/\tau)}.$$
Proof. Let \( A_* = A \cap A^{-1} \). We have \( |A_* + B| \leq 2|A||B|^\kappa \).

Since \( A_* = A_*^{-1} \), by Theorem 4 with \( \delta = \kappa \) and \( 0 < \tau < 1/8 \) we have

\[
\frac{|A_*|}{4} \left( \frac{|B|}{2} \right)^{\kappa/\tau} \leq |A_* + B| + |A_*^{-1} + B| \leq 4|A||B|^\kappa.
\]

Hence

\[
|A_*| \leq 2^{4+\kappa/\tau} |A||B|^{\kappa(1-1/\tau)}.
\]

\[\square\]

3. Application to continued fractions with bounded partial quotients

Here we discuss some problems of representing rational numbers by finite continued fractions. By the Euclidean algorithm, a rational \( a/q \in [0,1], (a, q) = 1 \) can be uniquely represented as a regular continued fraction

(2) \[ \frac{a}{q} = [0; b_1, \ldots, b_s] = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{\ddots + \frac{1}{b_s}}}}, \quad b_s \geq 2. \]

Assuming \( q \) is known, we use \( b_j(a), j = 1, \ldots, s = s(a) \), to denote the partial quotients of \( a/q \); that is,

\[ \frac{a}{q} = [0; b_1(a), \ldots, b_s(a)]. \]

3.1. Zaremba’s conjecture. Zaremba’s famous conjecture [65] posits that there is an absolute constant \( \varepsilon \) with the following property: for any positive integer \( q \) there exists \( a \) coprime to \( q \) such that in the continued fraction expansion (2) all partial quotients are bounded:

\[ b_j(a) \leq \varepsilon, \quad 1 \leq j \leq s = s(a). \]

In fact, Zaremba conjectured that \( \varepsilon = 5 \). For large prime \( q \), even \( \varepsilon = 2 \) should be enough, as conjectured by Hensley.

Korobov [39] showed that for prime \( q \) there exists \( a, (a, q) = 1 \), such that

\[ \max_{\nu} b_\nu(a) \ll \log q. \]

Such a result is also true for composite \( q \). Moreover, Rukavishnikova [52] proved that Korobov’s bound holds with positive probability:

\[ \frac{1}{\varphi(q)} \left\{ a \in \mathbb{Z} : 1 \leq a \leq q, \ (a, q) = 1, \ \max_{1 \leq j \leq s(a)} b_j(a) \geq T \right\} \ll \frac{\log q}{T}. \]

The main results of Rukavishnikova’s papers [52, 53] deal with the typical values of the sum of partial quotients of fractions with a given denominator: she proves an analog of the law of large numbers.

It is clear that Zaremba’s conjecture is true when \( q = F_n \) is the \( n \)-th Fibonacci number. Niederreiter [51] proved that Zaremba’s conjecture is
true for $q = 2^\alpha, 3^\alpha, \alpha \in \mathbb{Z}_+$ with $\ell = 3$, and for $q = 5^\alpha$ with $\ell = 4$. By means of a quite similar argument Yodphotong and Laohakosol showed [64] that Zaremba’s conjecture is true for $q = 6^\alpha$ and $\ell = 5$. Komatsu [38] proved that Zaremba’s conjecture is true for $q = 7^2r^r, r = 1, 3, 5, 7, 9, 11$ and $\ell = 3$. Kan and Krotkova [36] obtained lower bounds for the number $f = |\{a \pmod{p^n} : a/p^n = [0; b_1, \ldots, b_s], b_j \leq p^n\}|$ of fractions with bounded partial quotients and the denominator of the form $p^n$. In particular they proved a bound of the form

$$f \geq C(n)m^\lambda, \quad C(n), \lambda > 0.$$  

Recently Bourgain and Kontorovich [10, 11] made significant progress on Zaremba’s conjecture. Consider the set

$$\mathfrak{Z}_k(N) := \{q \leq N : \exists a \text{ such that } (a, q) = 1, \ a/q = [0; b_1, \ldots, b_s], \ b_j \leq k\}$$

(so Zaremba’s conjecture means that $\mathfrak{Z}_k(N) = \{1, 2, \ldots, N\}$). In a wonderful paper [11] Bourgain and Kontorovich proved that for $k$ large enough there exists positive $c = c(k)$ such that for $N$ large enough one has

$$|\mathfrak{Z}_k(N)| = N - O(N^{1-c/\log \log N}).$$

For example, it follows from this result that for $k$ large enough the set $\bigcup_N \mathfrak{Z}_{50}(N)$ contains infinitely many prime numbers.

Another result from [11] states that for $k = 50$ the set

$$\bigcup_N \mathfrak{Z}_{50}(N)$$

has positive density in $\mathbb{Z}_+$, that is

$$|\mathfrak{Z}_{50}(N)| \gg N.$$

This result was improved by Frolenkov and Kan [35, 18, 32, 33, 34], Huang [30], and Magee, Oh, and Winter [41]. In particular, in [33] Kan proved that the set (3) has positive density in $\mathbb{Z}_+$ for $k = 4$.

### 3.2. Real numbers with bounded partial quotients.

By $F_M(Q)$ we denote the set of all rational numbers $\frac{u}{v}, (u, v) = 1$ from $[0, 1]$ with all partial quotients in (2) not exceeding $M$ and with $v \leq Q$:

$$F_M(Q) = \left\{ \frac{u}{v} = [0; b_1, \ldots, b_s] : (u, v) = 1, 0 \leq u \leq v \leq Q, b_1, \ldots, b_s \leq M \right\}.$$  

By $F_M$ we denote the set of all irrational real numbers from $[0, 1]$ with partial quotients less than or equal to $M$. From [29] we know that the Hausdorff dimension $w_M$ of the set $F_M$ satisfies

$$w_M = 1 - \frac{6}{\pi^2} - \frac{72 \log M}{\pi^4 M^2} + O\left(\frac{1}{M^2}\right), \quad M \to \infty,$$

however here we need simpler result from [27], which states that

$$1 - w_M \asymp \frac{1}{M}.$$
with absolute constants in the sign $\propto$. Explicit estimates for $\dim F_M$ for certain values of $M$ can be found in [31]. In the papers [27, 28] Hensley gives the bound

$$|F_M(Q)| \propto M Q^{2w_M}. \quad (6)$$

For a fixed $N$ we consider the set

$$Z_M(N) := \left\{ a \in \{1, 2, ..., N - 1\} : (a, N) = 1, \max_{1 \leq j \leq s} b_j(a) \leq M \right\}$$

of all positive integers $a$ less than $N$ so that the partial quotients of $a/N$ are all bounded by $M$. For instance, Zaremba’s conjecture is that for $M = 5$ and all $N$, we have $|Z_M(N)| > 0.$

In [15], the first author used Hensley’s bounds to show that

$$|Z_M(p)| \ll_M p^{w_M}. \quad (7)$$

Certain upper bounds for $|Z_M(p)|$ were obtained recently in [12] by means of Dynamical Systems. In the next subsection we improve on (7) in the case when $N = p$ is a prime number, and give an upper bound that is close to optimal.

### 3.3. New results.

For a prime $p$, we consider the set

$$Z_M(p) = \left\{ a \in \{1, \ldots, p - 1\} : \max_{1 \leq j \leq s(a)} b_j(a) \leq M \right\}$$

Our main new result is the following theorem.

**Theorem 6.** Given positive $\varepsilon$ there exists $M_0 = M_0(\varepsilon)$ such that for all $M \geq M_0$ one has

$$|Z_M(p)| \ll_M p^{2w_M - 1 + \varepsilon(1 - w_M)}. \quad (8)$$

For large values of $M$, the exponent here is close to the optimal exponent $2w_M - 1$ that was conjectured in lecture [17]. One can see that Theorem 6 improves the bound (7) from [15]. Some related problems are discussed in the preprint [16].

Before proving Theorem 6, we introduce some auxiliary sets.

Recall that if

$$a \equiv [0; b_1, \ldots, b_n],$$

then the $k$th convergent to $a/q$ is $[0; b_1, \ldots, b_k]$. We use $u_k$ and $v_k$ to denote coprime integers such that

$$\frac{u_k}{v_k} = [0; b_1, \ldots, b_k].$$

When $q$ is understood, we will write $u_k(a)$ and $v_k(a)$ for the convergents $u_k(a)/v_k(a)$ to $a/q$.

The integers $u_k$ and $v_k$ satisfy the following recursion relations: $u_0 = 0, u_1 = 1$, and for $k \geq 1$

$$u_{k+1} = b_{k+1}u_k + u_{k-1},$$
and \(v_0 = 1, v_1 = b_1\), and for \(k \geq 1\)

\[ v_{k+1} = b_{k+1}v_k + v_{k-1}. \]

In addition, we have the following error bound for approximating \(a/q\) by its convergents:

\[
\left| \frac{a}{q} - \frac{u_k}{v_k} \right| < \frac{1}{v_kv_{k+1}} < \frac{1}{v_k^2}.
\]

See [24, Chapter X] or [61, Chapter 1] for further properties of continued fractions and convergents.

Let

\[
A = A_M(p) = \{ a \in \{1, \ldots, p-1\} : b_k(a) \leq M \text{ for all } k \text{ such that } v_k(a) \leq \sqrt{p} \}.
\]

That is, \(A\) is the set of \(a\) such that the partial quotients of all convergent fractions \(u/v\) to \(a/p\) with \(v \leq \sqrt{p}\) are at most \(M\). Note that \(Z_M(p) \subseteq A_M(p)\), and that every convergent \(u/v\) to \(a/p\), with \(a \in A_M(p)\) and \(v \leq \sqrt{p}\) is contained in \(Z_M(\sqrt{p})\).

Further, the set \(A\) has an involution defined by \(a \mapsto a^\ast\), where \(aa^\ast \equiv 1 \pmod{p}\), so when we consider \(A\) as a subset of \(\mathbb{F}_p\), we have \(A = A^{-1}\).

More precisely, if

\[
\frac{a}{p} = [0; b_1, b_2 \ldots, b_s],
\]

with \(b_s \geq 2\) then for the inverse element \(a^\ast\) modulo \(p\) defined by \(aa^\ast \equiv 1 \pmod{p}\) we have [52, 53]:

\[
\frac{a^\ast}{p} = [0; b_s, b_{s-1} \ldots, b_1] \quad \text{if } s \text{ is even}
\]

\[
\frac{a^\ast}{p} = [0; 1, b_s - 1, b_{s-1} \ldots, b_1] \quad \text{if } s \text{ is odd.}
\]

Now we take \(\beta\) from the range

\[ 0 < \beta \leq \frac{1}{2} \]

and consider the set

\[ A_\beta = \{ a : \exists \frac{u}{v} \in F_M(p^\beta) \text{ such that } a = \lfloor \frac{p^\beta u}{v} \rfloor \}. \]

(Recall that \(\frac{u}{v} \in F_M(p^\beta)\) if \(0 \leq u \leq v \leq p^\beta, (u, v) = 1\), and all partial quotients of \(\frac{u}{v}\) are less than \(M\).)

**Lemma 7.** For \(0 < \beta \leq 1/2\), the map \(\frac{u}{v} \mapsto \lfloor \frac{p^\beta u}{v} \rfloor\) from \(F_M(p^\beta)\) to \(A_\beta\) is bijective. Hence \(|A_\beta| = |F_M(p^\beta)| \approx_M p^{2\beta \omega M}\).

**Proof.** By definition, the map \(\frac{u}{v} \mapsto \lfloor \frac{p^\beta u}{v} \rfloor\) from \(F_M(p^\beta)\) to \(A_\beta\) is surjective. For \(0 < \beta \leq 1/2\), this map is also injective, since for distinct \(\frac{u}{v}, \frac{u'}{v'} \in F_M(p^\beta)\) we have

\[
\left| \frac{u}{v} - \frac{u'}{v'} \right| \geq \frac{1}{v'v} \geq \frac{1}{p},
\]

hence different \(\frac{u}{v}\) give different \(a\).
It follows immediately that $|A_\beta| = |F_M(p^\beta)|$. By (1), $|F_M(p^\beta)| \asymp_M p^{2\beta\omega_M}$.

Now we define the set of consecutive integers
$$\mathcal{B}_\beta = \left\{0, \pm 1, \pm 2, \ldots, \pm \left\lfloor (M + 1)^2 p^{1-2\beta} + 1 \right\rfloor \right\}.$$  

**Lemma 8.** For $A, A_\beta$, and $B_\beta$ defined as above, we have $A \subseteq A_\beta + B_\beta$.

**Proof.** The denominators of convergents $u/v$ satisfy the relation $v_\nu < v_{\nu+1} = b_{\nu+1}v_\nu + v_{\nu-1} \leq (b_{\nu+1} + 1)v_\nu$.

So for any rational $a/p$ with partial quotients $\leq M$ and for any $\lambda$ from the interval $M + 1 \leq \lambda \leq p$ there exists a convergent fraction $u/v$ to $a/p$ with $v \geq v/\lambda$ and for this convergent fraction one has
$$\left|\frac{a}{p} - \frac{u}{v}\right| \leq \frac{1}{v^2} \leq \frac{(M + 1)^2}{p^{2\beta}}.$$  

This observation implies
$$\left|a - \left\lfloor \frac{p}{v} u \right\rfloor \right| \leq (M + 1)^2 p^{1-2\beta} + 1,$$  

which leads to the desired inclusion $A \subseteq A_\beta + B_\beta$. \qed

**Proof of Theorem 6**. Recall that $|A_\beta| \asymp_M p^{2\beta\omega_M}$ and $|B_\beta| = 2 \left\lfloor (M + 1)^2 p^{1-2\beta} + 1 \right\rfloor + 1$. 

Since $A \subseteq A_\beta + B_\beta$ and $$B_\beta + B_\beta \subseteq \left\{0, \pm \left\lfloor (M + 1)^2 p^{1-2\beta} + 1 \right\rfloor \right\} + B_\beta,$$  

we have
$$|A + B| \leq |A_\beta + B_\beta + B_\beta| \leq 3|A_\beta + B_\beta| \leq 3|A_\beta||B_\beta| \ll_M p^{1-2\beta(1-\omega_M)}.$$  

Since $A = A^{-1}$, we have
$$|A + B| + |A^{-1} + B_\beta| \ll_M p^{1-2\beta(1-\omega_M)}.$$  

By Theorem 4 with $\tau = 1 - 2\beta + 2\log_p(M + 1)$ and $\delta = 1 - \omega_M$, we have
$$p^{(1-2\beta)(1-\omega_M)/\tau} |A| \ll_M p^{1-2\beta(1-\omega_M)}$$  

provided that
$$\delta \leq \frac{1}{4} b_0^{-1/\tau}.$$  

Thus
$$|A| \ll_M p^{\omega_M + (1-2\beta)(1-\omega_M)(1-\tau^{-1})}.$$  

Now we choose $\beta = \frac{1-\delta}{2}$, so that
$$|A| \ll_M p^{\omega_M + (1-\omega_M)(1-1/(\epsilon + 2\log_p(M+1)))} \ll_M p^{2\omega_M - 1 + \epsilon(1-\omega_M)},$$
provided that
\[ 1 - \omega_M \leq \frac{1}{4} b_0^{-1/(\varepsilon + 2 \log_p (M+1))}. \]
For \( p \) sufficiently large, it suffices to take \( \varepsilon > 0 \) so that
\[ \frac{1}{M} \ll 1 - \omega_M \leq \frac{1}{4} b_0^{-0.9/\varepsilon}, \]
which is roughly \( \varepsilon \gg \frac{1}{\log M} \).

\[ \square \]

4. Bounds for rich linear fractional transformations

We begin with some basic facts on subgroups and quotients of the group \( GL_2(\mathbb{F}) \) of \( 2 \times 2 \) invertible matrices with entries in \( \mathbb{F} \). The special linear group \( SL_2(\mathbb{F}) \) consists of elements of \( GL_2(\mathbb{F}) \) with unit determinant.

The group \( GL_2(\mathbb{F}) \) acts on the projective line \( \mathbb{P}^1(\mathbb{F}) \) by linear fractional transformations. Informally, \( \mathbb{P}^1(\mathbb{F}) = \mathbb{F} \cup \{\infty\} \) is the affine line \( \mathbb{F} \) plus a point at infinity. A linear fractional transformation is a map of the form
\[
(10) \quad x \mapsto \frac{ax + b}{cx + d},
\]
where
\[
(11) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
is an element of \( GL_2(\mathbb{F}) \). If \( x = \infty \), then \( x \mapsto a/c \). By abuse of notation, we may use the matrix in equation (11) to denote the transformation in (10).

Clearly we may restrict the action (10) to \( SL_2(\mathbb{F}) \). A transformation acts trivially if and only if it is in the center \( Z = \{\lambda I : \lambda \in \mathbb{F}^*\} \) of \( GL_2(\mathbb{F}) \). The projective general linear group \( PGL_2(\mathbb{F}) = GL_2(\mathbb{F})/Z \) is the automorphism group of \( \mathbb{P}^1(\mathbb{F}) \) and the projective special linear group \( PSL_2(\mathbb{F}) = SL_2(\mathbb{F})/\{\pm I\} \) is a subgroup of \( PGL_2(\mathbb{F}) \) [3, Section 10.8]. If every element of \( \mathbb{F}^* \) is a square then \( PSL_2(\mathbb{F}) = PGL_2(\mathbb{F}); \) otherwise, the index of \( PSL_2(\mathbb{F}) \) in \( PGL_2(\mathbb{F}) \) is 2.

The group \( PGL_2(\mathbb{F}) \) acts simply 3-transitively on \( \mathbb{P}^1(\mathbb{F}) \), meaning that for every pair of triples \((x, y, z)\) and \((x', y', z')\) of distinct points in \( \mathbb{P}^1(\mathbb{F}) \), there is a unique transformation \( g \in PGL_2(\mathbb{F}) \) such that
\[
g(x, y, z) = (x', y', z').
\]
The first proof of this for a general field \( \mathbb{F} \) is due to Grothendieck, see [3, Section 10.8]. By a direct computation, one can show that \( PSL_2(\mathbb{F}) \) acts doubly transitively on \( \mathbb{P}^1(\mathbb{F}) \).

The graphs of linear fractional transformations define hyperbolas in \( \mathbb{F} \times \mathbb{F} \):
\[
(12) \quad y = \frac{ax + b}{cx + d} \iff cx\gamma + ax + cy + d = 0.
\]
If $g$ is the linear fractional transformation corresponding to the left-hand side of (12), let $\Gamma_g$ denote the curve in $\mathbb{F} \times \mathbb{F}$ defined by $cxy + ax + cy + d = 0$. If $S \subseteq \text{PSL}_2(\mathbb{F}_p)$ and $Y \subseteq \mathbb{F}_p$, we may define the number of incidences between $P = Y \times Y$ and the set of hyperbolas $\Gamma_g$ with $g$ in $S$ by

$$I(Y \times Y, S) := \{|(x, y, g) \in Y \times Y \times S : (x, y) \in \Gamma_g\}|.$$  

Note that

$$I(Y \times Y, S) = \sum_{g \in S} |Y \cap gY|.$$

The following theorem can be thought of as a bound for the number weighted incidences between a set of hyperbolas and a Cartesian product point set.

**Theorem 9.** Let $\nu$ be a probability measure on $G = \text{SL}_2(\mathbb{F}_p)$ such that

1. $\|\nu\|_\infty \leq K^{-1}$
2. for all $g \in G$ and all proper subgroups $\Gamma \leq G$, we have $\nu(g\Gamma) \leq K^{-1}$.

Then for any set $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ and any element $z \in \text{GL}_2(\mathbb{F}_p)$, there are absolute constants $c_* \in (0, 1)$ and $C_* \geq 6$ such that

$$\left|\sum_g (\delta_z * \nu)(g) |Y \cap gY| - \frac{|Y|^2}{p}\right| \leq C_*|Y| p^{-\delta},$$

where $\delta_z(x) = 1$ if $x = z$ and $\delta_z(x) = 0$ otherwise,

$$k = \frac{3 \log p}{c_* \log K},$$

and

$$\delta = \delta(k) = \frac{1}{2^{k+2}}.$$

As a corollary, we have the following incidence bound, originally proved by Bourgain [5] and used by Bourgain, Gamburd, and Sarnak to prove that a certain graph related to Markov triples is connected [9, 8].

**Corollary 10** (Bourgain’s hyperbola incidence bound). Given $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ and $S \subseteq G = \text{PSL}_2(\mathbb{F}_p)$ such that

- $|S| \geq p^\epsilon$,
- for all $g \in G$ and all proper subgroups $\Gamma \leq G$ we have $|S \cap g\Gamma| \leq |S|^{1-\eta}$,

we have

$$\left|I(Y \times Y, S) - \frac{|S||Y|^2}{p}\right| \leq C_*|Y||S| p^{-\delta},$$

where $\delta = 2^{-(k+2)}$ and $k = 3(c_*\eta\epsilon)^{-1}$.

**Proof.** Apply Theorem 9 with $K = p^{\eta \epsilon}$. \qed

The following bound applies when we know more structural information about the set of linear fractional transformations.
Theorem 11. There is an absolute constant $b_0 > 1$ such that the following holds for all $0 < \alpha < 1$, all sufficiently large primes $p \gg 1$, and all $0 < \tau \leq 1/8$.

Let $B = \{1, \ldots, M\}$ and let $C = \{1, \ldots, N\}$. Suppose that $5 \leq \min(M, N) \leq p^\tau$.

Set

$$S = \left\{ \begin{pmatrix} 1 & -b \\ c & 1 - bc \end{pmatrix} : b \in B, c \in C \right\}.$$

Let $\delta = 0.25b_0^{-1/\tau}$ and let $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ be a subset of size $1 \leq |Y| \leq p^{1-\delta}$.

If $|Y \cap gY| \geq \alpha|Y|$ for all $g$ in $S$, then

$$\min(M, N) \leq 2 \left( \frac{2}{\alpha} \right)^{\tau/\delta} + 1.$$

5. Proofs of bounds for sums of reciprocals

In this section, we prove Theorems 1 and 4 using the results from the previous section.

5.1. Proof of Theorem 1. Before proving Theorem 1, we state some classification results for the subgroups of $SL_2(\mathbb{F}_p)$, then state a key lemma, which states that the matrices relevant to Theorem 1 do not concentrate in subgroups.

5.1.1. Subgroups of $SL_2(\mathbb{F}_p)$. Let $B$ denote the standard Borel subgroup of $SL_2(\mathbb{F}_p)$:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{F}_p, ad = 1 \right\}.$$

We use $B'$ to denote the projection of $B$ to $PSL_2(\mathbb{F}_p)$.

Dickson [13, 14] classified the subgroups of $SL_2(\mathbb{F}_p)$ and $PSL_2(\mathbb{F}_p)$, see [58, Theorem 6.17, Theorem 6.25].

Theorem 12 (Dickson). Let $p \geq 5$ be a prime. Every proper subgroup of $PSL_2(\mathbb{F}_p)$ is isomorphic to one of the following groups:

1. the dihedral groups of order $p \pm 1$ and their subgroups,
2. the standard Borel subgroup $B'$ of $PSL_2(\mathbb{F}_p)$ and its subgroups,
3. $A_4, S_4, A_5$.

Further, every proper subgroup of $SL_2(\mathbb{F}_p)$ is isomorphic to one of the following groups:

1. the dihedral groups of order $2(p \pm 1)$ and their subgroups,
2. the dicyclic groups of order $4p, 4(p \pm 1)$ and their subgroups,
3. the standard Borel subgroup $B$ of upper triangular matrices, and its subgroup,
4. a finite group of order at most 120.
Thus every proper subgroup of $\text{PSL}_2(\mathbb{F}_p)$ containing more than 60 elements is solvable. See [58, Section 3.6] for a proof of the classification of subgroups of $\text{SL}_2(\mathbb{F})$ when $\mathbb{F}$ is an arbitrary field of characteristic $p$.

**Lemma 13.** Every cyclic subgroup of $\text{SL}_2(\mathbb{F}_p)$ is conjugate (by matrices in $\text{SL}_2(\mathbb{F}_p)$) to a subgroup of $B$ or to a subgroup of the following form:

$$K_\varepsilon := \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} : x, y \in \mathbb{F}_p, x^2 - \varepsilon y^2 = 1 \right\},$$

where $\varepsilon$ is a non-square.

**Proof.** Suppose

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

generates a cyclic subgroup $H$ of $\text{SL}_2(\mathbb{F}_p)$. If $\text{tr}(g)^2 - 4$ is a square over $\mathbb{F}_p$, then $g$ is conjugate (over $\mathbb{F}_p$) to a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$ 

Now, if $H$ is a subgroup of $\text{SL}_2(\mathbb{F}_p)$ that is isomorphic to a subgroup of the standard Borel subgroup $B$, then $H$ is conjugate to a subgroup of $B$ by an element of $\text{SL}_2(\mathbb{F}_p)$ [3, Proposition 16.6].

Otherwise, if $\text{tr}(g)^2 - 4$ is not a square, we can write

$$\begin{pmatrix} 1 & 0 \\ (d-a)/2b & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (d-a)/2b & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix},$$

where

$$x = \frac{a + d}{2}, \quad y = b, \quad \text{and} \quad \varepsilon = \frac{(a + d)^2 - 4}{4b^2}.$$ 

□

See also [58, Section 6, (6.3)] and [19, Section 5.2].

5.1.2. Non-concentration in subgroups. For subsets $B, C \subseteq \mathbb{F}_p$, let

$$S = S_\rho = S_\rho(B, C) := \left\{ \begin{pmatrix} -\rho^{-1}c & -1 + \rho^{-1}bc \\ 1 & -b \end{pmatrix} : b \in B, c \in C \right\}.$$ 

Since $S/Z$ has the same cardinality as $S$, we may consider $S$ as a subset of $\text{PSL}_2(\mathbb{F}_p)$.

**Lemma 14.** Let $S = S_\rho(B, C)$ be defined as in (14). Then for any $g_1, g_2 \in \text{PSL}_2(\mathbb{F}_p)$ one has

$$|g_1B'g_2 \cap S| \leq \max\{|B|, |C|\}.$$ 

In particular, if $B = C$, then

$$|g_1Bg_2 \cap S| \leq |S|^{1/2}.$$ 

Moreover, for any dihedral subgroup $\Gamma$ one has

$$|g_1\Gamma g_2 \cap S| \leq 8.$$
Proof. We will consider $S$ as a subset of $SL_2(F_p)$; projection to $PSL_2(F_p)$ cannot increase the size of the intersection of $S$ with subgroups.

First, we consider the number of elements of $S$ that are contained in a coset of a Borel subgroup. Since all Borel subgroups are conjugate to the standard Borel subgroup $B$, we consider the equation

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} r & q \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} -\rho^{-1}c & -1 + \rho^{-1}bc \\ 1 & -b \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

with $xw - yz = XW - YZ = 1$; that is,

$$\begin{pmatrix} xr & qx + y/r \\ zr & qz + w/r \end{pmatrix} = \begin{pmatrix} -\rho^{-1}c(X - bZ) - Z & -\rho^{-1}c(Y - bW) - W \\ X - bZ & Y - bW \end{pmatrix}. \tag{18}$$

Either $x$ or $z$ is non-zero, since $xz - yz = 1$. Suppose that $z \neq 0$. Then

$$x z r = -z(\rho^{-1}c(X - bZ) + Z)$$

so substituting $z r = X - bZ$, we have

$$x(X - bZ) = -z(\rho^{-1}c(X - bZ) + Z).$$

Thus

$$c = \frac{-bxZ + (xX + zZ)}{-z\rho^{-1}(X - bZ)} = \frac{(-xZ)b + (xX + zZ)}{(z\rho^{-1})b + (-z\rho^{-1}X)}.$$

Since

$$\det \begin{pmatrix} -xZ & xX + zX \\ \rho^{-1}zX & -\rho^{-1}zX \end{pmatrix} = -\rho^{-1}z^2Z^2,$$

c is determined uniquely by $b$ if $Z \neq 0$. On the other hand, if $Z = 0$, then $X \neq 0$ since $XW - YZ = 1$, so $c = -\rho x r / X$ and there are at most $|B|$ solutions to (18). If we assume $x \neq 0$, then a similar situation occurs, and in general there are at most $\max(|B|, |C|)$ solutions to (18).

Next, we consider the number of elements of $S$ contained in a dihedral or dicyclic subgroup of $SL_2(F_p)$. The number of such elements is at most four times the number of elements contained in a cyclic subgroup of $SL_2(F_p)$; by Lemma 13, every cyclic subgroup is conjugate either to a subgroup of the standard Borel subgroup $B$, in which case the previous analysis applies, or to a subgroup of the form

$$K_\varepsilon := \left\{ \begin{pmatrix} u & \varepsilon v \\ v & u \end{pmatrix} : u, v \in F_p, u^2 - \varepsilon v^2 = 1 \right\},$$

where $\varepsilon$ generates $F_p^\times$. Thus we consider the equation

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} u & \varepsilon v \\ v & u \end{pmatrix} = \begin{pmatrix} -\rho^{-1}c & -1 + \rho^{-1}bc \\ 1 & -b \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

with $xw - yz = XW - YZ = 1$; that is,

$$\begin{pmatrix} xu + yv & \varepsilon xu + yu \\ zu + wv & \varepsilon zu + wu \end{pmatrix} = \begin{pmatrix} -\rho^{-1}c(X - bZ) - Z & -\rho^{-1}c(Y - bW) - W \\ X - bZ & Y - bW \end{pmatrix}. \tag{19}$$
From (19) we derive
\[ (20) \quad -Z = (x + \rho^{-1}cz)u + (y + \rho^{-1}cw)v = Au + Bv. \]
Since \(xw - yz = 1\), either \(x + \rho^{-1}cz \neq 0\) or \(y + \rho^{-1}cw \neq 0\). Solving (20) for \(u\) or \(v\) and substituting into \(u^2 - \varepsilon v^2 = 1\) yields
\[(Bv + Z)^2 - \varepsilon A^2v^2 = A^2 \implies (B^2 - \varepsilon A^2)v^2 + 2BZv + Z^2 - A^2 = 0\]
or
\[B^2u^2 - \varepsilon(Au + Z)^2 = B^2 \implies (B^2 - \varepsilon A^2)u^2 - 2\varepsilon AZu - (Z^2 + B^2) = 0.\]
In both cases, the leading coefficient is non-zero since \(\varepsilon\) is not a square, so there are at most two solutions for \(u\) or \(v\); since \(u\) and \(v\) determine one another by (20), there are at most two pairs \((u, v)\) such that (19) holds.

The pair \((u, v)\) determines the left-hand side of equation (19). Since either \(Z \neq 0\) or \(W \neq 0\), and either \(X - bZ \neq 0\) or \(Y - bW \neq 0\), once \((u, v)\) is fixed, \(b\) and \(c\) are determined. Thus there are at most 2 elements of \(gK_\varepsilon g'\) contained in \(S\), and hence at most 4 elements of a coset of a dihedral group contained in \(S\) or at most 8 elements of a coset of a dicyclic group contained in \(S\).

5.1.3. **Proof of Theorem 7**

**Proof of Theorem 7** Suppose \(|A + B| + |\rho \cdot A^{-1} + C| \leq M|A|\). Let \(Y = (A + B) \cup (\rho \cdot A^{-1} + C)\), so that \(|Y| \leq M|A|\). Let
\[ g_{b,c}(x) = \frac{\rho}{x - b} + c \]
and let \(S\) denote the set of matrices in \(GL_2(\mathbb{F}_p)\) corresponding to the linear fractional transformations \(g_{b,c}\) with \(b \in B\) and \(c \in C\):
\[ S = \{ \begin{pmatrix} c & \rho - bc \\ 1 & -b \end{pmatrix} : b \in B, c \in C \}. \]
Let
\[ z^{-1} = \begin{pmatrix} -\rho^{-1} & 0 \\ 0 & 1 \end{pmatrix} \]
and let \(S' = z^{-1}S\) so that \(S' \subseteq SL_2(\mathbb{F}_p)\).

By Lemma 14, we have \(|g\Gamma \cap S'| \leq \max(|B|, |C|)\) for any proper subgroup \(\Gamma \leq SL_2(\mathbb{F}_p)\), assuming that \(\max(|B|, |C|) \geq 4\), which holds for \(p\) sufficiently large. Let \(K = \min(|B|, |C|)\) and let \(\nu\) be the uniform measure on \(S'\). Then \(||\nu||_\infty = |S'|^{-1} \leq K^{-1}\) and
\[ \nu(g\Gamma) = \frac{|g\Gamma \cap S'|}{|S'|} \leq K^{-1} \]
for all proper subgroups \(\Gamma \leq SL_2(\mathbb{F}_p)\). It follows from Theorem 9 that
\[ (21) \quad \sum_g (\delta_g * \nu)(g)|Y \cap gY| \leq \frac{|Y|^2}{p} + C_*|Y|p^{-\delta}, \]
where
\[ \delta = \frac{1}{4} b_0 \log_K p. \]
Since \( K = \min(|B|, |C|) = p^\varepsilon \), we have \( \delta \approx 2^{-O(1/\varepsilon)} \).

On the other hand, for all \( g \in zS' = S \) we have \( |Y \cap gY| \geq |A| \geq \frac{1}{M} |Y| \), hence
\[
(22) \quad \sum_g (\delta \ast \nu)(g)|Y \cap gY| = \frac{1}{|S|} \sum_{g \in g \in S} |Y \cap gY| \geq \frac{1}{M} |Y|.
\]
If \( |Y| \leq p^{1-\delta} \), then equations (21) and (22) yield
\[
M \geq \frac{1}{(C* + 1)} p^\delta,
\]
as claimed. \qed

5.2. Proof of Theorem 4. The proof of Theorem 4 uses the same idea as the proof of Theorem 1, but we use Theorem 11 in place of Theorem 9. In particular, Theorem 4 does not require the non-concentration results from the previous proof. (However, the proof of Theorem 11 does use non-concentration.)

Proof of Theorem 4. Suppose \(|A + B| + |A^{-1} + C| \leq \alpha^{-1} |A| \). Let \( Y = -(A + B) \cup (A^{-1} + C)^{-1} \), so that \(|Y| \leq \alpha^{-1} |A| \). Let
\[
1 \quad \frac{1}{g_{bc}(x)} = \frac{-1}{x-b} + c
\]
and let \( S \) denote the set of matrices in \( SL_2(\mathbb{F}_p) \) corresponding to the linear fractional transformations \( g_{bc} \) with \( b \in B \) and \( c \in C \). For each \( g \in S \), we have \( |Y \cap gY| \geq \alpha |A| \).

The elements of \( S \) have the form
\[
\begin{pmatrix} 1 & -b \\ c & 1 - bc \end{pmatrix},
\]
where \( b \in \{1, \ldots, M\} \) and \( c \in \{1, \ldots, N\} \).

By Theorem 11 we have
\[
\min(|B|, |C|) \leq 2 \left( \frac{2}{\alpha} \right)^{\tau/\delta} + 1,
\]
provided that \( \min(|B|, |C|) \geq 11 \).

It follows that
\[
\alpha^{-1} \geq \frac{1}{2} \left( \frac{\min(|B|, |C|)}{2} \right)^{\delta/\tau},
\]
which implies
\[
|A + B| + |A^{-1} + C| \geq \frac{|A|}{2} \left( \frac{\min(|B|, |C|)}{2} \right)^{\delta/\tau}.
\]
\qed
6. $\ell^2$-flattening/higher energies

For a probability measure $\mu$ on $SL_2(\mathbb{F}_p)$, let $\mu^{(\ell)}$ denote the $\ell$-fold convolution of $\mu$ with itself; that is, $\mu^{(1)} = \mu$ and $\mu^{(\ell+1)} = \mu * \mu^{(\ell)}$. The adjoint $\mu^\sim$ of a finitely supported measure $\mu$ is defined by $\mu^\sim(x) = \mu(-x)$.

The following theorem combines the “middle-game” and “end-game” steps of Bourgain and Gamburd’s proof of uniform expansion for Cayley graphs of $SL_2(\mathbb{F}_p)$ [7]. See [63] and [54] for an overview of the three steps of the proof of the main theorem from [7].

**Theorem 15.** Let $\mu$ be a symmetric probability measure on $SL_2(\mathbb{F}_p)$ such that for some parameter $K \geq 1$

- $\mu(g\Gamma) \leq K^{-1}$ for any proper subgroup $\Gamma \leq SL_2(\mathbb{F}_p)$ and element $g \in SL_2(\mathbb{F}_p)$, and
- $\|\mu\|_\infty \leq K^{-1}$.

Then for any integer $k$

$$\left| \left\| \mu^{(2^k)} \right\|_2^2 - \frac{1}{|SL_2(\mathbb{F}_p)|} \right| \leq C_* K^{-c_* k}$$

where $c_* \in (0, 1)$ and $C_* > 1$ are absolute constants.

Before proving Theorem 15 we state some preliminaries: a “quasi-randomness” bound for convolution on $PSL_2(\mathbb{F}_p)$, and results from arithmetic combinatorics.

The following bound is due to Gowers [23] and Babai, Nikolov, and Pyber [2].

**Theorem 16.** Let $\mu$ be a probability measure on $PSL_2(\mathbb{F}_p)$ and let $f : PSL_2(\mathbb{F}_p) \to \mathbb{C}$ have mean zero: $\sum_g f(g) = 0$. Then

$$\|\mu * f\|_2 \leq p \|\mu\|_2 \|f\|_2.$$ 

The following version of Balog-Szemerédi-Gowers theorem can be found in [48] or derived from arguments in [59]. Recall that the multiplicative energy of a finite subset $A$ of a multiplicative group is defined by

$$E(A) := \{|(a_1, a_2, a_3, a_4) \in A^4 : a_1 a_2 = a_3 a_4|\}.$$

**Lemma 17 (Balog-Szemerédi-Gowers theorem).** If $A$ is a finite subset of a group $G$ and $E(A) \geq \zeta |A|^3$, then there exists a set $S \subseteq G$ and an element $a$ in $A$ such that $S \subseteq a^{-1} A$, $|S| \gg \zeta^C |A|$, and $|S^3| \ll \zeta^{-C} |S|$, where $C > 0$ is an absolute constant.

The following lemma allows us to reduce a statement about measures whose self-convolutions have large $\ell^2$ norm to a statement about multiplicative energy.

**Lemma 18 (Weighted Balog-Szemerédi-Gowers [60, Lemma 1.4.1]).** Let $\nu$ be a finitely supported function on a multiplicative group with $\|\nu\|_1 \ll 1$. 


Suppose that $\|\nu*\nu\|_2^2 \geq M^{-1}\|\nu\|_2^2$ for some $M > 1$. Then there exists a set $A \subseteq \text{supp} (\nu)$ such that

$$\frac{1}{M\|\nu\|_2^2} \ll |A| \ll \frac{M^2}{\|\nu\|_2^2},$$

(24)

$$|\nu(g)| \gg \frac{\|\nu\|_2^2}{M^2}$$

for all $g \in A$, and

$$E(A) \gg M^{-3}\|\nu\|^{-6} \gg M^{-9}|A|^3.$$  

(25)

We prove Lemma 18 in the Appendix.

The final ingredient in the proof of Theorem 15 is Helfgott’s product theorem for $SL_2(\mathbb{F}_p)$ [25]. We quote the version from [40].

**Theorem 19** (Growth in $SL_2(\mathbb{F}_p)$). For $p$ prime and $A \subseteq SL_2(\mathbb{F}_p)$, if $A$ generates $SL_2(\mathbb{F}_p)$, then either

$$\left(A \cup A^{-1} \cup \{e\}\right)^3 = SL_2(\mathbb{F}_p)$$

or

$$|(A \cup A^{-1} \cup \{e\})^3| \geq |A|^{1+\delta},$$

(27)

$$|(A \cup A^{-1} \cup \{e\})^3| \geq |A|^{1+\delta},$$

(28)

where $\delta = \frac{1}{3024}$.

Since [26, Equation (3.2)]

$$|(A \cup A^{-1} \cup \{e\})^3| \leq \left(\frac{|A^3|}{|A|}\right)^3 |A|,$$

equation (28) implies

$$|A^3| \geq \frac{1}{3}|A|^{1+\delta/3}.$$  

(29)

**Proof of Theorem 15.** Let $\gamma = |SL_2(\mathbb{F}_p)|^{-1}$ and set $f(x) = \mu(x) - \gamma$. By induction, one can show that $f^{(n)}(x) = \mu^{(n)}(x) - \gamma$. Thus

$$\|f^{(n)}\|_2^2 = \|\mu^{(n)}\|_2^2 - \gamma.$$

Thus to prove Theorem 15 it suffices to show that

$$\|f^{(2\ell)}\|_2^2 \leq \frac{1}{M}\|f^{(\ell)}\|_2^2,$$

(30)

where $M^{-1} = C_* K^{-\epsilon_*}$ and $\ell$ is a dyadic integer. By Theorem 16

$$\|f^{(2\ell)}\|_2^2 \leq p^2\|f^{(\ell)}\|_2^4,$$

(31)

so we may assume that

$$\|f^{(\ell)}\|_2^2 \geq \frac{1}{p^2M},$$

(32)

Proof of Theorem 15. Let $\gamma = |SL_2(\mathbb{F}_p)|^{-1}$ and set $f(x) = \mu(x) - \gamma$. By induction, one can show that $f^{(n)}(x) = \mu^{(n)}(x) - \gamma$. Thus

$$\|f^{(n)}\|_2^2 = \|\mu^{(n)}\|_2^2 - \gamma.$$
otherwise we are done.

Let \( r(x) = f^{(\ell)}(x) \). Then

\[
\|r\|_1 \leq 1 + \|\mu^{(\ell)}\|_1 = 2.
\]

Thus if (30) is false, then we may apply Lemma 18 with \( \nu = r \) to find a subset \( P \subseteq \text{supp}(r) \) such that

\[
E(P) \gg M^{-9}|P|^3,
\]

\[
M^{-1}\|r\|_2^2 \ll |P| \ll M^2\|r\|_2^2,
\]

and

\[
|r(g)| \gg M^{-2}\|r\|_2^2
\]

for all \( g \) in \( P \).

Equations (32) and (35) imply that

\[
|P| \ll M^2\|r\|_2^2 \ll M^3p^2.
\]

We have a lower bound on \( |r(x)| = |f^{(\ell)}(x)| = |\mu^{(\ell)}(x) - \gamma| \) and would like a lower bound on \( |\mu^{(\ell)}(x)| \). If \( \mu^{(\ell)}(x) < 2\gamma \), then \( |r(x)| < \gamma \); however by (32) and (36) this implies that

\[
\frac{1}{p^3 - p} = \gamma > |r(x)| \gg \frac{\|r\|_2^2}{M^2} \geq \frac{1}{p^2M^3},
\]

hence \( M \gg p^{1/3} \). Choosing, say \( M \leq p^{1/4} \), for \( p \) sufficiently large, we have

\[
\mu^{(\ell)}(x) \gg M^{-2}\|r\|_2^2
\]

for all \( x \) in \( P \).

Now we apply Lemma 17 to \( P \) to find a subset \( S \subseteq g^{-1}P \) for some \( g \) in \( P \) such that \( |S| \gg M^{-C}|P| \) and \( |S^3| \ll M^C|S| \) for an absolute constant \( C > 0 \). By (37) and \( M \leq p^{1/4} \), we may apply Theorem 19 with \( \delta < 1/4 \) to find that either \( S \) is contained in a proper subgroup \( \Gamma \leq SL_2(\mathbb{F}_p) \) or

\[
|S|^{1+\varepsilon} \ll_{\delta} |S^3| \ll M^C|S|,
\]

for some \( \varepsilon = \varepsilon(\delta) > 0 \). (We may assume \( \delta \) is fixed, say \( \delta = 1/5 \).)

We will choose our parameters so that (39) cannot happen. Equation (39) implies that

\[
M^{-C\varepsilon}|P|^{\varepsilon} \ll |S|^\varepsilon \ll M^C,
\]

hence

\[
|P| \ll M^{C(1+1/\varepsilon)}.
\]

On the other hand, by the assumption \( \|\mu\|_\infty \leq K^{-1} \), we have

\[
\|r\|_\infty \leq \gamma + \|\mu^{(\ell)}\|_\infty \leq \gamma + \|\mu^{(\ell-1)}\|_1\|\mu\|_\infty \leq \frac{2}{K},
\]

since we may assume \( K \leq \gamma^{-1} = p^3 - p \). Thus

\[
\|r\|_2^2 \ll K^{-1}\|r\|_1 \ll K^{-1}.
\]
By equations (35) and (42), it follows that
\[ |P| \gg M^{-1}K \gg K^{1-c_*}. \]
Combining (40) and (43) yields a contradiction if \(c_*\) is sufficiently small, depending on \(\varepsilon\) (hence on \(\delta\)).

Thus we may assume that (39) does not hold, and hence \(S\) is a contained in a proper subgroup \(\Gamma \subseteq SL_2(\mathbb{F}_p)\). Again, we will derive a contradiction.

Since \(S \subseteq g^{-1}P\), we have
\[ |g\Gamma \cap P| \geq |S| \gg M^{1-c_*}|P|. \]
By (38) and (35), it follows that
\[ \mu^{(\ell)}(g\Gamma) \geq \frac{|g\Gamma \cap P|}{M^2\|r\|^{-2}} \gg \frac{M^{-C}|P|}{M^3|P|} = M^{-C-3}. \]
However, by assumption we have
\[ \mu^{(\ell)}(g\Gamma) \leq \|\mu^{(\ell-1)}\|_1 \sup_x \mu(xg\Gamma) \leq \frac{1}{K}. \]
If \(c_*\) is sufficiently small, this contradicts (45).

It follows that (30) must hold, and the proof is complete. \(\square\)

7. Proof of Theorem 9

Now we prove Theorem 9 which we recall here.

**Theorem 9.** Let \(\nu\) be a probability measure on \(G = SL_2(\mathbb{F}_p)\) such that
1. \(\|\nu\|_\infty \leq K^{-1}\)
2. for all \(g \in G\) and all proper subgroups \(\Gamma \subseteq G\), we have \(\nu(g\Gamma) \leq K^{-1}\).

Then for any set \(Y \subseteq \mathbb{P}^1(\mathbb{F}_p)\) and any element \(z \in GL_2(\mathbb{F}_p)\)
\[ \left| \sum_g (\delta_z \ast \nu)(g)|Y \cap gY| - \frac{|Y|^2}{p} \right| \leq C_*|Y|p^{-\delta(k)}, \]
where
\[ k = \frac{3\log p}{c_* \log K}; \]
and
\[ \delta(k) = \frac{1}{2k+2} \]
for absolute constants \(c_* \in (0,1)\) and \(C_* \geq 6\).

The proof of Theorem 9 requires pseudo-randomness bounds for the action of \(PSL_2(\mathbb{F}_p)\) on the projective line \(\mathbb{P}^1(\mathbb{F}_p)\).

Let \(G\) be a group acting on a set \(X\), let \(\mu: G \to \mathbb{C}\) and let \(f: X \to \mathbb{C}\). We define the convolution of \(\mu\) and \(f\) by
\[ \mu * f(y) := \sum_{g y = x} \mu(g)f(y). \]
Proposition 20. Suppose $G$ is a finite group that acts doubly transitively on a set $X$. Suppose $\mu : G \to \mathbb{C}$ and $f, h : X \to \mathbb{C}$ satisfy $\sum_x f(x) = \sum_x h(x) = 0$. Then

$$|\langle \mu \ast f, h \rangle| \leq \sqrt{\frac{|G|}{|X| - 1} \|f\|_2 \|h\|_2}.$$

We give an elementary proof of Proposition 20 in the Appendix, but it also follows from a result of Gill [22, Proposition 1.4] and a lower bound on the degree of doubly-transitive permutation representations [55, Exercise 2.6].

Since $G = PSL_2(\mathbb{F}_p)$ acts doubly-transitively on $\mathbb{P}^1(\mathbb{F}_p)$ we have the following bound.

Corollary 21. Let $\mu$ be a probability measure on $PSL_2(\mathbb{F}_p)$ and let $f$ be a function on $\mathbb{P}^1(\mathbb{F}_p)$ with mean zero. Then

$$\|\mu \ast f\|_2 \leq p \|\mu\|_2 \|f\|_2.$$

Proof. Apply Proposition 20 with $h = \mu \ast f$. $\Box$

The following theorem is better than Corollary 21 when $|Y|$ is small. A second bound is more useful if $|Y|$ is small.

Theorem 22. Let $W$ be a subset of $\mathbb{P}^1(\mathbb{F}_p)$ and let $\mu$ be a probability measure on $PSL_2(\mathbb{F}_p)$. Then either $\langle \mu \ast W, W \rangle < 4$ or

$$\langle \mu \ast W, W \rangle \leq 2 \|\mu\|_\infty^{1/3} \left( |W| - \frac{|W|}{p + 1} \right)^2. \tag{46}$$

If $f_W(x) := W(x) - |W|/(p + 1)$ is the balanced function of $W$, then either $\langle \mu \ast f_W, f_W \rangle \leq 8$ or

$$\langle \mu \ast f_W, f_W \rangle \leq 4 \|\mu\|_\infty^{1/3} \left( |W| - \frac{|W|}{p + 1} \right)^2. \tag{47}$$

We prove Theorem 22 in the appendix.

Proof of Theorem 9. For convenience, write $\nu_0 = \delta_x \ast \nu$. Let

$$\sigma \ast \sum_x Y(x)(\nu_0 \ast Y)(x)$$

and let $f_Y(x) = Y(x) - |Y|/(p + 1)$, so that

$$\sigma = \frac{|Y|^2}{p + 1} + \sum_x Y(x)(\nu_0 \ast f_Y)(x) = \frac{|Y|^2}{p + 1} + \sigma_*.$$

By Cauchy-Schwarz, we have

$$\left| \sigma - \frac{|Y|^2}{p + 1} \right|^2 = \sigma_*^2 \leq |Y| \sum_x f_Y(x)(\nu_0 \ast \nu_0 \ast f_Y)(x).$$

(Recall that $\nu_0^\ast(x) = \nu_0(x^{-1})$.)
Set \( \eta(x) = (\nu_0^* \ast \nu_0)(x) = (\nu^* \ast \nu)(x) \), so that
\[
\sigma^2 \leq \sum_x f_Y(x)(\eta \ast f_Y)(x).
\]

Note that \( \eta \) is a symmetric probability measure on \( SL_2(\mathbb{F}_p) \) satisfying \( \| \eta \|_\infty \leq \| \nu \|_\infty \) and
\[
\eta(g\Gamma) \leq \| \nu \|_1 \max_x \nu(x^{-1} g\Gamma) \leq \frac{1}{K}
\]
for all \( g \in G \) and proper subgroups \( \Gamma \leq G \).

Iterating Cauchy-Schwarz, we have
\[
\sigma^{2k+1} \leq \sum_x f_Y(x)(\eta^{(2^k)} \ast f_Y)(x).
\]

Thus by Corollary 21 we have
\[
\sigma^{2k+1}_* \leq p|Y|^{2^{k+1}-1} \sum_x f_Y(x)(\eta^{(2^k)} \ast f_Y)(x).
\]

By (48), we may apply Theorem 15 with \( \mu = \eta \) to find
\[
\| \eta^{(2^k)} \|_2 \leq \frac{1}{p^3 - p} + \frac{C^k}{K c_* K}.
\]

Choosing \( k \) such that \( K^{c_* k} = p^3 \), we have
\[
\| \eta^{(2^k)} \|_2 \leq \frac{\sqrt{C^k + 2} / p^{3/2}}{p^3 - p}.
\]

Combining (52) with (50), we have
\[
\sigma^{2k+1}_* \leq (C^k_* + 2)^{1/2} p^{-1/2} |Y|^{2^{k+1}}.
\]

Hence
\[
\sigma \leq \frac{|Y|^2}{p} + C_k \frac{|Y|}{p^\delta},
\]
where \( \delta = 2^{-(k+2)} \) and
\[
C_k^{2k+1} = \sqrt{C^k + 2}.
\]
If \( C_* \geq 6 \), then a calculation shows that \( C_k \leq C_* \).

Since \( K^{c_* k} = p^3 \), we have
\[
k = \frac{3 \log p}{c_* \log K}.
\]

Corollary 23. Let \( \nu \) be a probability measure on \( G = SL_2(\mathbb{F}_p) \) such that

1. \( \| \nu \|_\infty \leq K^{-1} \)
2. for all \( g \in G \) and all proper subgroups \( \Gamma \leq G \), we have \( \nu(g\Gamma) \leq K^{-1} \).
Then for any set $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ and any element $z \in \text{GL}_2(\mathbb{F}_p)$

$$\left| \sum_g (\delta_z * \nu)(g) |Y \cap gY| - \frac{|Y|^2}{p} \right| \leq C_* |Y|^{1-\delta}$$

where

$$\delta = \frac{1}{2^{k+1}} \left( \frac{c_*(k-1) \log K}{3 \log |Y|} - 1 \right),$$

and $c_* \in (0,1)$ and $C_* \geq 6$ are absolute constants.

Proof. Starting from (49) and applying Theorem 22, we have

$$\sigma_*^{2k+1} \leq 4|Y|^{2k+1+1} \|\eta^{(2k)}\|_{1/3}.$$  

By Cauchy-Schwarz and Theorem 15, recalling that $\gamma = 1/(p^3 - p)$, we have

$$\|\eta^{(2k)}\|_\infty = \|\eta^{(2k-1)} * \eta^{(2k-1)}\|_{\infty} \leq \|\eta^{(2k-1)}\|_2^2 \leq \gamma + C_*^{k-1} K - c_*(k-1).$$

Assuming $K = c_*(k-1) \leq p^3$, by (56) and (57), we have

$$\sigma_*^{2k+1} \leq 8C_*^{(k-1)/3}|Y|^{2k+1+1} K - c_*(k-1)/3.$$  

Choosing

$$K = |Y|^{3(1+2k+1)/c_*(k-1)},$$

we have

$$\sigma_* \leq (8C_*^{(k-1)/3})^{2^{-k-1}} |Y|^{1-\delta},$$

hence

$$\sum_x Y(x)(\delta_z * \nu * Y)(x) - \frac{|Y|^2}{p+1} \leq (8C_*^{(k-1)/3})^{2^{-k-1}} |Y|^{1-\delta},$$

where

$$\delta = \frac{1}{2^{k+1}} \left( \frac{c_*(k-1) \log K}{3 \log |Y|} - 1 \right).$$

For instance, if $k = 1 + 3 \log |Y|/c_* \log K$, we have $\delta = 2^{-k}$.  \qed

8. Proof of Theorem 11

8.1. Notation and statement of main lemmas. For a group $G$ and a finite subset $S \subseteq G$, the Cayley graph $\Gamma = \text{Cay}(G,S)$ has vertex set $G$ and edges defined by $\{x,sx\}$ with $x \in G$ and $s \in S \cup S^{-1}$; it is $|S \cup S^{-1}|$-regular. The girth of a graph is the length of its shortest cycle; we introduce a related quantity for Cayley graphs. Let $d(\mathcal{G})$ be the smallest positive integer such any two distinct paths in $\mathcal{G}$ of length $\leq d(\mathcal{G})$ starting at the identity have distinct end points. Since $\mathcal{G}$ is vertex-transitive, the girth of $\mathcal{G}$ is either $2d(\mathcal{G})$ or $2d(\mathcal{G}) - 1$. Hence a lower bound for $d(\mathcal{G})$ is equivalent to a lower bound for the girth of $\mathcal{G}$. 
Theorem 24. For all $0 < \alpha < 1$ the following holds for all sufficiently large primes $p$:

Let $S \subseteq G = \text{PSL}_2(\mathbb{F}_p)$ be a set of transformations such that

$$d(\text{Cay}(G, S)) \geq \tau_0 \log|S|(p)$$

for some $\tau_0 > 0$.

Let $\delta = 0.25 \cdot b_0^{-1/\tau}$ where $0 < \tau < \tau_0/2$ and $b_0 > 1$ is an absolute constant.

Let $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ be a subset of size $1 \leq |Y| \leq p^{1-\delta}$.

If $5 \leq |S| \leq p^\tau$ and there is an element $z$ in $\text{GL}_2(\mathbb{F}_p)$ such that

$$\sum_{g \in zS} |Y_g| \geq \alpha |Y||S|,$$

then

$$|S| \leq \left( \frac{2}{\alpha} \right)^{\tau/\delta}.$$

The girth condition (62) is satisfied by random subsets (asymptotically almost surely) \cite{21} and projections of generators of non-elementary subgroups of $\text{SL}_2(\mathbb{Z})$ \cite{7}.

Theorem 25. Let $N \geq 5$ be a positive integer and let $T$ denote the following set of elements of $\text{PSL}_2(\mathbb{F}_p)$:

$$T = \left\{ \begin{pmatrix} 1 & -2j \\ 2j & 1-4j^2 \end{pmatrix} : 1 \leq j \leq N \right\}.$$

Then for all $p \gg 1$

$$d(\text{Cay}(\text{PSL}_2(\mathbb{F}_p), T)) \geq \frac{1}{4} \log N p.$$

Theorem 11, which we recall here, follows from Theorems 24 and 25.

Theorem 11. There is an absolute constant $b_0 > 1$ such that the following holds for all $0 < \alpha < 1$, all sufficiently large primes $p \gg 1$, and all $0 < \tau \leq 1/8$.

Let $B = \{1, \ldots, M\}$ and let $C = \{1, \ldots, N\}$. Suppose that $11 \leq \text{min}(M, N) \leq p^\tau$.

Set

$$S = \left\{ \begin{pmatrix} 1 & -b \\ b & 1-bc \end{pmatrix} : b \in B, c \in C \right\}.$$

Let $\delta = 0.25b_0^{-1/\tau}$ and let $Y \subseteq \mathbb{P}^1(\mathbb{F}_p)$ be a subset of size $1 \leq |Y| \leq p^{1-\delta}$.

If $|Y \cap gY| \geq \alpha |Y|$ for all $g$ in $S$, then

$$\text{min}(M, N) \leq 2 \left( \frac{2}{\alpha} \right)^{\tau/\delta} + 1.$$
Proof. Let \( N_0 = \lfloor \min(M, N)/2 \rfloor \). If \( T \) is defined as in (64) with \( N = N_0 \), then \( T \subseteq S \), so \( |Y \cap gY| \geq \alpha |Y| \) for all \( g \) in \( T \). By Theorem 25, we have \( d(\text{Cay}(G, T)) \geq \frac{1}{4} \log|T| \), thus by Theorem 24 with \( \tau_0 = 1/4 \), we have
\[
\min(\lfloor M/2 \rfloor, \lfloor N/2 \rfloor) \leq \left( \frac{2}{\alpha} \right)^{\tau/\delta}
\]
for all \( 0 < \tau < 1/8 \), provided that \( 5 \leq |T| \leq p^{\tau} \). \( \square \)

8.2. Proof of Theorem 24. Throughout this section, \( G = PSL_2(\mathbb{F}_p) \), \( S \) is a subset of \( G \), and \( k = |S| \).

The girth bound (62) implies that the products \( S^m \) of \( S \) grow as quickly as possible for \( m \leq \gamma := d(\text{Cay}(G, S)) \). The following lemma is an immediate consequence of the definition of \( \gamma \).

Lemma 26 (Girth bound implies locally free). For \( m \leq \gamma \), the ball of radius \( m \) about the identity in \( \text{Cay}(G, S) \) is isomorphic to the ball of radius \( m \) about the identity in the Cayley graph of the free group \( F_k \) on \( k \) generators.

If \( \gamma \geq 2 \), then \( S \cap S^{-1} = \emptyset \), so \( \mu(x) = \frac{1}{2k} 1_{S \cup S^{-1}}(x) \) is the uniform measure on \( S \cup S^{-1} \). Recall that the \( m \)-fold convolution of \( \mu \) with itself is defined by
\[
\mu^{(m)}(x) = \sum_{y_1 \cdots y_m = x} \mu(y_1) \cdots \mu(y_m).
\]
For \( m \geq 1 \), the measure \( \mu^{(m)} \) is a symmetric probability measure on \( G \).

Lemma 27 (Bounds for convolutions of the uniform measure on \( S \)). For \( \gamma \geq 2 \) and \( m \leq \gamma \), we have
\[
\sum_{g \in G} |\mu^{(m)}(g)|^2 \leq \left( \frac{2}{k} \right)^m.
\]

Proof. The claimed bound is trivial if \( k = 1 \), so without loss of generality, assume that \( k \geq 2 \).

By Lemma 26 when \( m \leq \gamma \), \( \mu^{(m)}(x) \) is equal to the probability \( p^{(m)}(e, x) \) of arriving at \( x \) after \( m \) steps from the identity in the uniform random walk on \( F_k \); see [7, p. 637]. (By abuse of notation, we will use \( x \) to denote an element of \( F_k \) as well as the corresponding element in the ball of radius \( m \) about the identity in \( G \).)

Since \( \mu \) is symmetric, we have
\[
\mu^{(m)}(x) = \mu^{(m)}(x^{-1}) = p^{(m)}(e, x^{-1}) = p^{(m)}(e, x).
\]
Thus the probability of return to the identity in \( 2m \) steps is
\[
p^{(2m)}(e, e) = \sum_{x \in G} p^{(m)}(e, x)p^{(m)}(x, e) = \sum_{x \in G} |\mu^{(m)}(x)|^2.
\]
By \cite{62} Lemma 1.9, \( p^{(2m)}(e, e) \leq \rho^{2m} \), where \( \rho \) is the spectral radius of \( F_k \).

Kesten \cite{37} proved that if \( k \geq 2 \) then
\[
\rho \leq \left( \frac{2k - 1}{k^2} \right)^{1/2} \leq \left( \frac{2}{k} \right)^{1/2},
\]
which completes the proof. \( \square \)

**Lemma 28** (Non-concentration in proper subgroups). Let \( H \) be a proper subgroup of \( G \) and let \( g \) be an element of \( G \).

For \( 2 \leq m \leq \gamma/2 \) we have
\[
|\text{supp}(\mu^{(m)}) \cap gH| \leq m^6.
\]

*Proof.* If \( m \leq \gamma/2 \), then the support \( S \) of \( \mu^{(m)} \) is isomorphic to the ball \( B_{m/2} \) of radius \( m/2 \) in \( F_k \), hence \( S^{-1}S \) is isomorphic to \( B_m \).

Since \( |S^{-1} \cap Hg^{-1}| = |S \cap gH| \), in particular, \( S^{-1} \cap Hg^{-1} \) is non-empty (otherwise we are done), hence
\[
|S \cap gH| \leq |(S^{-1} \cap Hg^{-1})(S \cap gH)| \leq |S^{-1}S \cap H|.
\]

By Theorem 12 if \( |H| > 60 \) then \( H \) is two step solvable, hence (66)
\[
[[g_1, g_2], [g_3, g_4]] = 1
\]
for all \( g_1, \ldots, g_4 \) in \( H \). By \cite{7} Proposition 8, the number of elements in \( B_m \) satisfying (66) is at most \( m^6 \).

If \( |H| \leq 60 \), then the bound still holds, since \( m^6 \geq 64 \). \( \square \)

*Proof of Theorem 24.* Let \( \mu \) be the uniform measure on \( S \cup S^{-1} \). The hypothesis (63) translates to
\[
\sum_g (\delta_z * \mu)(g)|Y_g| \geq \frac{\alpha|Y||S|}{|S \cup S^{-1}|} \geq \frac{\alpha}{2}|Y|.
\]

By Cauchy-Schwarz and the inclusion
\[
Y_g \cap Y_{g'} = Y \cap gY \cap g'Y \subseteq gY \cap g'Y = gY_{g^{-1}g'},
\]
we have
\[
\left( \frac{\alpha}{2} \right)^2 |Y| \leq \sum_{g, g'} (\delta_z * \mu)(g)(\delta_z * \mu)(g')|Y_g \cap Y_{g'}|
\leq \sum_g ((\delta_z * \mu)(\delta_z * \mu))\mu(g)|Y \cap Y_g| = \sum_g \mu^{(2)}(g)|Y_g|,
\]
where \( f^\sim(x) := f(-x) \) is the adjoint of function \( f \).

Iterating this, we find that
\[
(\frac{\alpha}{2})^{2^j} |Y| \leq \sum_g \mu^{(2^j)}(g)|Y_g|.
\]

Let \( m \) denote a dyadic integer less than or equal to \( \gamma/2 \). (Recall \( \gamma = d(Cay(G, S)) \geq \tau_0 \log|S| p \).) We will choose \( m \) presently.
Let $\nu = \mu^{(m)}$. By Lemma 27

$$(68) \quad \|\nu\|_\infty \leq \|\mu^{(m/2)}\|_2 \leq \left( \frac{2}{|S|} \right)^{m/2}.$$ 

If $g \in G$ and $\Gamma$ is a proper subgroup of $G$ then by Lemma 28

$$(69) \quad \nu(g\Gamma) \leq |g\Gamma \cap \text{supp} (\mu^{(m)})| \|\nu\|_\infty \leq m^6 \left( \frac{2}{|S|} \right)^{m/2}.$$ 

Define $K^{-1} = |S|^{-m/4}$ and define $\tau$ by $m = \tau \log |S| p$. We want $K^{-1} \geq m^6(2/|S|)^{m/2}$, so that the hypotheses of Theorem 9 are satisfied. Thus we need

$$(70) \quad \frac{1}{|S|^{m/4}} \geq m^6 \left( \frac{2}{|S|} \right)^{m/2} \quad \text{or} \quad |S|^{m/2} \geq m^{12} 2^m.$$ 

By the definition of $\tau$, (70) is equivalent to

$$(71) \quad p^{\tau/2} \geq (\tau \log |S| p)^{12} p^{\tau \log |S|} \quad \text{or} \quad p^{\tau (1 - \log |S|)} \geq (\tau \log |S| p)^{24}.$$ 

If $|S| \geq 5$, then (71) is satisfied for $p \gg 1$.

Since

$$(72) \quad \left( \frac{\alpha}{2} \right)^m \leq \frac{1}{|Y|} \sum_g \nu(g)|Y \cap gY| \leq \frac{|Y|}{p} + C_* p^-\delta,$$

where $\delta = 2^{-(k+2)}$ and $k = 3 \log p / (c_\tau \log K)$.

By the definition of $\tau = m / \log |S| p$ we have

$$(73) \quad k = \frac{3 \log p}{c_\tau \log K} = \frac{12 \log p}{c_\tau m \log |S|} = \frac{12}{c_\tau} \tau^{-1}.$$ 

Suppose that $|Y| \leq p^{1-\delta}$. Then

$$(74) \quad \left( \frac{\alpha}{2} \right)^m \leq \frac{C_* + 1}{p^\delta}.$$ 

Since

$$(75) \quad \log |S| \leq \frac{\tau}{\delta} \log (2\alpha^{-1}) \implies |S| \leq \left( \frac{2}{\alpha} \right)^{\tau/\delta}.$$ 

By (73), we have

$$\delta = 2^{-(k+2)} = \frac{1}{4} b_0^{-1/\tau}$$
for $b_0 = 2^{12/c_4}$. Since $m \leq \gamma/2$, we can take $0 < \tau \leq \tau_0/2$. Finally, since $m \geq 1$, we need $1 \leq \tau \log_{|S|} p$, which follows from $|S| \leq p^\tau$. \qed

8.3. Proof of Theorem 25. Given a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbb{Z})$, we use $\|g\|$ to denote its norm as an operator on $\ell^2(\mathbb{R}^2)$:

$$\|g\| := \sup_{|x|_2 = 1} |gx|_2,$$

where $|(x_1, x_2)|_2 = \sqrt{x_1^2 + x_2^2}$. For a finite collection of matrices $S \subseteq SL_2(\mathbb{Z})$, we define

$$n(S) := \max_{g \in S} \|g\|.$$

If $S \subseteq PSL_2(\mathbb{Z})$, we define $n(S) = n(S')$, where $S' \subseteq SL_2(\mathbb{Z})$ is some collection of matrices representing the elements of $S$. Since $\|g\| = \|g\|$, this is well defined.

If $S \subseteq PSL_2(\mathbb{F}_p)$, we define

$$n(S) := \min \{n(\tilde{S}'): \tilde{S}' \subseteq PSL_2(\mathbb{Z}), \tilde{S}' \equiv S \mod p \}.$$

We will use the notation $\tilde{G} = PSL_2(\mathbb{Z})$ and $\tilde{S}$ for subsets of $\tilde{G}$; the map $\phi_p: \tilde{G} \to G = PSL_2(\mathbb{F}_p)$ is defined by reduction of the entries of matrices representing elements of $\tilde{G}$ modulo $p$. Thus $S = \phi_p(\tilde{S})$ in the above definition of $n(S)$.

A direct computation shows that

$$\frac{1}{2} \sqrt{a^2 + b^2 + c^2 + d^2} \leq \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| \leq \sqrt{a^2 + b^2 + c^2 + d^2},$$

thus $|S| \ll n(S)^4$.

The following theorem of Margulis [43, Section 6] gives a lower bound for $d(G)$ (and hence the girth) of the Cayley graph $G = Cay(G, S)$ in terms of the norm of $S$. See also [20, Section 2].

Theorem 29 (Girth bound for projections of free groups). If the group $\Lambda$ generated by $\tilde{S} \subseteq \tilde{G}$ is free, then

$$d(Cay(\phi_p(\Lambda), \phi_p(\tilde{S}))) \geq \log_{n(\tilde{S})} \left( \frac{p}{2} \right).$$

Hence

$$\text{Girth}(Cay(\phi_p(\Lambda), \phi_p(\tilde{S}))) \geq 2 \log_{n(\tilde{S})} \left( \frac{p}{2} \right) - 1.$$

Let $F_2 = \langle a, b \rangle$ be the free group on two generators $a$ and $b$. In general, let $F_n$ denote the free group on $n$ generators; we say that $n$ is the rank of $F_n$. If $S$ is a set of elements in a group that generates a free group $F_n$ with $n = |S|$, we say that $S$ freely generates $F_n$.

Theorem 30. For $n \geq 1$, let $S_n = \{ab, a^2b^2, \ldots, a^n b^n\} \subseteq F_2$. Then $S_n$ freely generates a subgroup of $F_2$ isomorphic to $F_n$. 

Proof. This is Exercise 12 in Section 1.4 of [42]. □

The free group $F_2$ is relevant to our problem because it is a subgroup of $PSL_2(\mathbb{Z})$. Let $\Gamma(2) \leq PSL_2(\mathbb{Z})$ be the kernel of the homomorphism defined by reduction mod 2:

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) : a, d \equiv 1 \pmod{2}, b, c \equiv 0 \pmod{2} \right\}.$$

It is known [42] that $\Gamma(2)$ contains an index two free subgroup $\Lambda$ on two generators $u$ and $v$ given by

$$u = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \quad (77)$$

Let

$$T = \{ v^{-j}u^j : 1 \leq j \leq N \}.$$

It follows from Theorem [30] that $T$ generates a free subgroup of $\Lambda$ of rank $N$.

**Corollary 31.** If

$$T = \left\{ \begin{pmatrix} 1 & -2j \\ 2j & 1-4j^2 \end{pmatrix} : 1 \leq j \leq N \right\},$$

then $T$ generates a free subgroup of $PSL_2(\mathbb{Z})$ of rank $|T| = N$.

**Proof.** Since

$$\begin{pmatrix} 1 & -2j \\ 2j & 1-4j^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2j & 0 \end{pmatrix} \begin{pmatrix} 1 & -2j \\ 0 & 1 \end{pmatrix} = v^j u^{-j}$$

we have $T = \{ v^j u^{-j} : 1 \leq j \leq N \}$, where $u$ and $v$ are the matrices in (77) that generate a subgroup of $PSL_2(\mathbb{Z})$ isomorphic to $F_2$. Since $v$ and $u^{-1}$ also generate the same subgroup, it follows by Theorem [30] that $T$ generates a free subgroup of rank $|T|$. (This is because $T$ is the set $S_N$ from Theorem [30] with $u$ replaced by $u^{-1}$.) □

**Proof of Theorem [29]** Let $\tilde{T} \subseteq PSL_2(\mathbb{Z})$ be such that $\phi_p(\tilde{T}) = T$; we may take $\tilde{T}$ to be the same set of matrices in $T$, but with coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}/p\mathbb{Z}$. By Corollary [31] $\tilde{T}$ generates a free subgroup $\tilde{\Lambda}$ of $PSL_2(\mathbb{Z})$ of rank $|T|$, so by Theorem [29]

$$d(Cay(\phi_p(\tilde{\Lambda})), \phi_p(\tilde{T})) \geq \log_{n(\tilde{T})} \left( \frac{p}{2} \right).$$

By (76), we have

$$\left\| \begin{pmatrix} 1 & -2j \\ 2j & 1-4j^2 \end{pmatrix} \right\| \leq \sqrt{2 + 16j^2} \leq 5j^2,$$

so $n(\tilde{T}) \leq 5|T|^2 = 5N^2 \leq N^3$ if $N \geq 5$. Thus

$$d(Cay(\phi_p(\tilde{\Lambda})), \phi_p(\tilde{T})) \geq \frac{1}{3} \log_N \left( \frac{p}{2} \right),$$

which proves the claimed bound if $p$ is sufficiently large. □
Appendix A. Analytic Lemmas

In this Appendix, we prove some technical lemmas quoted above.

A.1. Proof of Lemma 18. Recall Lemma 18.

Lemma 18. Let \( \nu \) be a finitely supported function on a multiplicative group with \( \| \nu \|_1 \ll 1 \). Suppose that \( \| \nu * \nu \|_2 \geq M^{-1} \| \nu \|_2^2 \) for some \( M > 1 \). Then there exists a set \( A \subseteq \text{supp}(\nu) \) such that

\[
\frac{1}{M \| \nu \|_2^2} \ll |A| \ll \frac{M^2}{\| \nu \|_2^2},
\]

\[
|\nu(g)| \gg \frac{\| \nu \|_2^2}{M^2}
\]

for all \( g \in A \), and

\[
E(A) \gg M^{-3} \| \nu \|_2^{-6} \gg M^{-9} |A|^3.
\]

The proof of Lemma 18 follows the proof of Lemma 1.4.1 in [60].

Proof of Lemma 18. Suppose \( G \) is a group, \( \nu : G \to \mathbb{C} \) has finite support, \( \| \nu \|_1 \ll 1 \), and

\[
\| \nu \|_2^2 \geq \frac{1}{M} \| \nu \|_2^2.
\]

We wish to find a subset \( A \subseteq \text{supp}(\nu) \) with \( |A| \ll 1/\| \nu \|_2^2 \) and \( |\nu(x)| \gg \| \nu \|_2^2 \) for all \( x \in A \) such that \( A \) has large additive energy.

Without loss of generality, we may replace \( \nu \) by its absolute value, so we will assume that \( \nu \) is non-negative.

Write \( \nu = \nu_1 + \nu_2 + \nu_3 \) where

\[
\nu_1 := \nu \cdot 1_{\{x : \nu(x) < \lambda \| \nu \|_2^2\}},
\]

\[
\nu_3 := \nu \cdot 1_{\{x : \nu(x) > \Lambda \| \nu \|_2^2\}},
\]

and

\[
\nu_2 := \nu - \nu_1 - \nu_3.
\]

We want a lower bound for \( \| \nu_2 * \nu_2 \|_2^2 \).

We have

\[
\| \nu_1 \|_2^2 \leq \lambda \| \nu \|_2^2 \| \nu_1 \|_1 \ll \lambda \| \nu \|_2^2
\]

and

\[
\| \nu_3 \|_1 \leq \frac{1}{\Lambda \| \nu \|_2^2} \| \nu_3 \|_2^2 \leq \frac{1}{\Lambda}.
\]

By Young’s inequality,

\[
\| \nu_1 * \nu \|_2 \leq \| \nu_1 \|_2 \| \nu \|_1 \ll \lambda^{1/2} \| \nu \|_2
\]

and

\[
\| \nu_3 * \nu \|_2 \leq \| \nu_3 \|_1 \| \nu \|_2 \leq \frac{1}{\Lambda} \| \nu \|_2.
\]
It follows that
\[ \|\nu_2 \ast \nu_2 - \nu \ast \nu\|_2 \leq \|\nu_2 \ast (\nu_1 + \nu_3)\|_2 + \|\nu \ast (\nu_1 + \nu_3)\|_2 \leq 2\|\nu \ast \nu_1\|_2 + 2\|\nu \ast \nu_3\|_2 \ll (\lambda^{1/2} + \Lambda^{-1})\|\nu\|_2. \]

Choosing \( \lambda \approx 1/M \) and \( \Lambda \approx M^{1/2} \), we have
\[ \|\nu_2 \ast \nu_2\|_2 \geq \|\nu \ast \nu\|_2 - \|\nu_2 \ast \nu_2 - \nu \ast \nu\|_2 \gg \frac{1}{M^{1/2}}\|\nu\|_2. \]

Let \( A := \{x: \nu(x) \geq \lambda\|\nu\|_2^2\} \). Then
\[ \|A \ast A\|_2 \geq \frac{1}{\Lambda^2\|\nu\|_2^2}\|\nu_2 \ast \nu_2\|_2 \gg \frac{1}{M^{3/2}\|\nu\|_2^2}, \]

hence
\[ E(A) \gg \frac{1}{M^{3}\|\nu\|_2^6}. \]

On the other hand, by Markov’s inequality and \( \|\nu\|_1 \ll 1 \),
\[ |A| \ll \frac{1}{\Lambda^2\|\nu\|_2^2} \ll \frac{M^2}{\|\nu\|_2^2}, \]

so
\[ E(A) \gg \frac{|A|^3}{M^9}. \]

The lower bound on \( |A| \) in Equation (24) follows from
\[ |A|^3 \geq E(A) \gg \frac{1}{M^3\|\nu\|_2^6}. \]

\[ \square \]

A.2. Proof of Proposition 20.

**Proposition 20.** Suppose \( G \) is a finite group that acts doubly transitively on a set \( X \). Suppose \( \mu: G \to \mathbb{C} \) and \( f, h: X \to \mathbb{C} \) satisfy \( \sum_x f(x) = \sum_x h(x) = 0 \). Then
\[ |\langle \mu \ast f, h \rangle| \leq \sqrt{|G|/|X| - 1}\|\mu\|_2\|f\|_2\|h\|_2. \]

The proof of Proposition 20 is a completion argument, similar to the arguments in [50].

**Proof of Proposition 20.** The proof is a completion argument using Cauchy-Schwarz:
\[ |\langle \mu \ast f, h \rangle| \leq \sum_{g \in G} |\mu(g)| \left| \sum_{x \in X} f(g^{-1}x)\overline{h(x)} \right| \leq \|\mu\|_2 \left( \sum_{g \in G} \left| \sum_{x \in X} f(g^{-1}x)\overline{h(x)} \right|^2 \right)^{1/2}. \]
Since $G$ acts transitively on $X$ and non-diagonal pairs in $X \times X$, we have

$$
\sum_{g \in G} \left| \sum_{x \in X} f(g^{-1}x)h(x) \right|^2 = \sum_{g \in G} \sum_{x, y \in X} f(g^{-1}x)f(g^{-1}y)h(x)h(y)
$$

$$
= \sum_{g \in G} \sum_{x \in X} |f(g^{-1}x)|^2 |h(x)|^2 + \sum_{g \in G, x \neq y \in X} f(g^{-1}x)f(g^{-1}y)h(x)h(y)
$$

$$
= \frac{|G|}{|X|} \sum_{x' \in X} |f(x')|^2 |h(x')|^2 + \frac{|G|}{|X|(|X| - 1)} \sum_{x \neq y, x' \neq y'} f(x')f(y')h(x)h(y)
$$

$$
= \frac{|G|}{|X|} \|f\|^2 \|h\|^2 + \frac{|G|}{|X|(|X| - 1)} \left( \sum_{x} |f(x)|^2 - \|f\|^2 \right) \left( \sum_{x} h(x)^2 - \|h\|^2 \right)
$$

$$
= \frac{|G|}{|X|} \|f\|^2 \|h\|^2 + \frac{|G|}{|X|(|X| - 1)} \|f\|^2 \|h\|^2
$$

$$
= \frac{|G|}{|X| - 1} \|f\|^2 \|h\|^2.
$$

\[\text{□}\]

A.3. Proof of Theorem 22

**Theorem 22.** Let $W$ be a subset of $\mathbb{P}^1(\mathbb{F}_p)$ and let $\mu$ be a probability measure on $PSL_2(\mathbb{F}_p)$. Then either $\langle \mu * W, W \rangle < 4$ or

$$
\langle \mu * W, W \rangle \leq 2\|\mu\|^{1/3} |W|^2.
$$

If $f_W(x) := W(x) - |W|/(p+1)$ is the balanced function of $W$, then either $\langle \mu * f_W, f_W \rangle \leq 8$ or

$$
\langle \mu * f_W, f_W \rangle \leq 4\|\mu\|^{1/3} \left( |W| - \frac{|W|}{p+1} \right)^2.
$$

**Proof of Theorem 22.** Since $PGL_2(\mathbb{F}_p)$ acts simply 3-transitively on $\mathbb{P}^1(\mathbb{F}_p)$ and $PSL_2(\mathbb{F}_p) \leq PGL_2(\mathbb{F}_p)$, for any pair of distinct triples of points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ in $\mathbb{P}^1(\mathbb{F}_p)^3$ there is at most one element $g \in PSL_2(\mathbb{F}_p)$ such that

$$
g(x_1, y_1, z_1) = (x_2, y_2, z_2).
$$

Since the function $x \mapsto \binom{x}{3}$ is convex and

$$
\langle \mu * W, W \rangle = \sum_g \mu(g) \langle \delta_g * W, W \rangle,
$$

we have

$$
\langle \mu * W, W \rangle \leq 2\|\mu\|^{1/3} |W|^2.
$$

If $f_W(x) := W(x) - |W|/(p+1)$ is the balanced function of $W$, then

$$
\langle \mu * f_W, f_W \rangle \leq 8.
$$

Since

$$
\langle \mu * f_W, f_W \rangle \leq 4\|\mu\|^{1/3} \left( |W| - \frac{|W|}{p+1} \right)^2
$$

we have

$$
\langle \mu * f_W, f_W \rangle \leq 8.
$$

By the convexity of $\binom{x}{3}$, we have

$$
\langle \mu * f_W, f_W \rangle \leq 8.
$$

Thus, either $\langle \mu * f_W, f_W \rangle \leq 8$ or

$$
\langle \mu * f_W, f_W \rangle \leq 4\|\mu\|^{1/3} \left( |W| - \frac{|W|}{p+1} \right)^2.
$$

□
we have
\[
3! \left( \frac{\langle \mu \ast W, W \rangle}{3} \right) = 3! \left( \sum_g \mu(g) \frac{\langle \delta_g \ast W, W \rangle}{3} \right)
\leq 3! \sum_g \mu(g) \left( \frac{\langle \delta_g \ast W, W \rangle}{3} \right)
\leq 3! \|\mu\|_\infty \sum_g \left( \frac{\langle \delta_g \ast W, W \rangle}{3} \right).
\]

By (80), the right-hand side of the last line is at most \(\|\mu\|_\infty\) times the number of pairs of distinct triples in \(W^3\), thus
\[
(81) \quad 3! \left( \frac{\langle \mu \ast W, W \rangle}{3} \right) \leq (3!)^2 \|\mu\|_\infty \left( \frac{|W|}{3} \right)^2.
\]

If \(\langle \mu \ast W, W \rangle \geq 4\), then the left-hand side of (81) is at least \(\langle \mu \ast W, W \rangle \rangle^3 / 4\), so
\[
\langle \mu \ast W, W \rangle^3 \leq 4 \|\mu\|_\infty |W|^6,
\]
which proves (46).

To prove (47), we decompose \(f_W\) into its positive and negative parts:
\[
(82) \quad f_W(x) = (1 - \alpha)W(x) - \alpha W^c(x)
\]
where \(\alpha = |W|/(p + 1)\) and \(W^c = \mathbb{P}^1 \setminus W\) is the complement of \(W\). It follows that
\[
(83) \quad \langle \mu \ast f_W, f_W \rangle \leq (1 - \alpha)^2 \langle \mu \ast W, W \rangle + \alpha^2 \langle \mu \ast W^c, W^c \rangle.
\]
By the first part of the theorem, we have either \(\langle \mu \ast W, W \rangle < 4\) or
\[
(1 - \alpha)^2 \langle \mu \ast W, W \rangle \leq (1 - \alpha)^2 2\|\mu\|^{1/3}_\infty |W|^2 = 2\|\mu\|^{1/3}_\infty \|f_W\|_2^2,
\]
and either \(\langle \mu \ast W^c, W^c \rangle < 4\) or
\[
\alpha^2 \langle \mu \ast W^c, W^c \rangle \leq \alpha^2 2\|\mu\|^{1/3}_\infty (p + 1 - |W|)^2 = 2\|\mu\|^{1/3}_\infty \|f_W\|_2^2.
\]

Thus by the above equations and (83) we have
\[
\langle \mu \ast f_W, f_W \rangle \leq \max \left( 4\|\mu\|^{1/3}_\infty \|f_W\|_2^2, 2\|\mu\|^{1/3}_\infty \|f_W\|_2^2 + 4, 8 \right).
\]
The maximum is only achieved by the middle term when all terms are equal to 8, so we have
\[
\langle \mu \ast f_W, f_W \rangle \leq \max \left( 4\|\mu\|^{1/3}_\infty \|f_W\|_2^2, 8 \right),
\]
as claimed. \(\square\)
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Academic Press, New York, 1972.

(Nikolay Moshchevitin) Lomonosov Moscow State University, Division of Mathematics, Moscow, Russia
E-mail address: moshchevitin@gmail.com

(Brendan Murphy) School of Mathematics, University of Bristol, Bristol, UK, and Heilbronn Institute of Mathematical Research
E-mail address: brendan.murphy@bristol.ac.uk

(Ilya Shkredov) Steklov Mathematical Institute, ul. Gubkina, 8, Moscow, Russia, 119991, IITP RAS, Bolshoy Karetny per. 19, Moscow, Russia, 127994, and MIPT, Institutskii per. 9, Dolgoprudny, Russia, 141701
E-mail address: ilya.shkredov@gmail.com