ADVECTION-DOMINATED ACCRETION WITH INFALL AND OUTFLOWS

THOMAS BECKERT
Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138; Max-Planck-Institut für Radioastronomie, Auf dem Hügel 69, 53121 Bonn, Germany; tbeckert@mpifr-bonn.mpg.de

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ABSTRACT

We present self-similar solutions for advection-dominated accretion flows with radial viscous force in
the presence of outflows from the accretion flow or infall. The axisymmetric flow is treated in variables
integrated over polar sections and the effects of infall and outflows on the accretion flow are param-
etrized for possible configurations compatible with the self-similar solution. We investigate the
resulting accretion flows for three different viscosity laws and derive upper limits on the viscosity param-
eter \( \alpha \). In addition, we find a natural connection to nonrotating and spherical accretion with turbulent
viscosity, which is assumed to persist even without differential rotation. Positive Bernoulli numbers for
advection-dominated accretion allow a fraction of the gas to be expelled in an outflow, and the upper
limit on the viscosity predicts that outflows are inevitable for equations of state close to an ideal gas.

Subject headings: accretion, accretion disks — Galaxy: center — hydrodynamics

1. INTRODUCTION

Advection-dominated accretion flows (ADAFs) have been invented to explain low-luminosity black hole candidates such as Sgr A* (Narayan et al. 1998) in our Galactic center. The low luminosity of this model is achieved in an optically thin plasma, where most of the energy is stored in hot ions, while electrons as potential radiators are inefficiently coupled to the heat source and remain relatively cold. The electrons nonetheless become mildly relativistic close to the central black hole and the inevitable synchrotron radiation is observed from most ADAF candidates. The presence of magnetic fields not far from energy equipartition with the gas is indicative of their origin in MHD instabilities (Balbus & Hawley 1991) leading to turbulence in the accretion flow and subsequent generation of an effective viscosity. On larger scales, magnetic fields are likely to be responsible for the collimation of outflows from accretion disks into jets, seen in the cores of M87 (Reynolds et al. 1996) and NGC 4258 (Lasota et al. 1996; Herrnstein et al. 1998), which are prototypical ADAF candidates. Furthermore, the model can explain the accretion in some low-luminosity active galactic nuclei (AGNs) of elliptical galaxies (di Matteo et al. 1999) and in NGC 4258 (Gammie, Narayan, & Blandford 1999). A recent X-ray survey (Sambruna, Eracleous, & Mushotzky 1999), found a few more examples of low-luminosity cores in radio-loud AGNs, which are candidates for advection-dominated accretion flows in their central engines. This suggests that ADAFs can be found even in radio-loud AGNs and that jet formation is a common feature.

Outflow models for ADAFs have been investigated by Blandford & Begelman (1999) and applied to several candidates (di Matteo et al. 1999; Quataert & Narayan 1999). On the theoretical side, Igumenshchev & Abramowicz (1999) and Stone, Pringle, & Begelman (1999) have performed time-dependent, two-dimensional calculations of accretion flows, which in some cases resemble ADAFs for certain viscosity parameters \( \alpha \approx 0.1 \), but suggest the production of outflows for larger \( \alpha \) (Igumenshchev & Abramowicz 1999). It is found that the \((rr)\) stress tensor component, which was not included in the original description of vertically integrated models for accretions flows, is important in the cited calculations for ADAFs. The existence of self-similar solutions with a radial viscous force has been shown in previous work (Narayan & Yi 1995a) and discussed for two-dimensional solutions with a separation of variables.

This paper describes advection-dominated accretion flows in polar-integrated variables, including the radial viscous braking force, which either produce outflows or are formed by wind infall. The wind infall is assumed to consist of free-falling low angular momentum gas, which is cold with respect to the already existing accretion flow. One possible source for this gas are stellar winds from massive stars in young clusters as in the center of our Galaxy. In the following, we will refer to this wind infall into the accretion flow as infall. In our treatment, the main difference between infall and outflow is the sign of the mass infall rate, and we talk about winds if it is not necessary to distinguish between infall and outflow. We restrict the discussion to an extension of the self-similar solutions given by Narayan & Yi (1994) for the Newtonian limit.

In § 2 we present the equations, which describe the accretion flow, including the reaction to winds. The role of \((rr)\) stresses and bulk viscosity is emphasized. We discuss angular momentum transport and viscosity in § 3 and specify the possible equations of state and the resulting energy equation in § 4. General features of self-similar solutions are presented in § 5, and in § 6 detailed solutions for the \(\alpha\)-viscosity law are shown. Consequences of the alternative \(\beta\)-viscosity are discussed in § 7, and ADAFs with an intermediate shear-limited viscosity law follow in § 8. We compare the solutions and draw our conclusions in § 9.

2. STATIONARY ACCRETION WITH INFALL OR OUTFLOWS

Our first step is to establish the set of equations that describe the accretion flow. Advection-dominated flows, our interest here, are known to be quasi-spherical (Narayan & Yi 1995a); therefore, we will use spherical coordinates in our discussion. Consider a stationary, axisymmetric and rotating flow with angular velocity \( \Omega \) around a compact object of mass \( M \). Instabilities in the flow, either hydrodynamic or magnetohydrodynamic in origin, generate turbulence on small scales and lead to an effective viscosity much
larger than the microscopic one. The effective turbulent viscosity \( \nu \) in the flow will redistribute specific angular momentum \( \ell (r, \theta) = \sin^2 \theta \Omega \) by a local viscous force between neighboring rings or shells. Short-term evolution on scales smaller than the mean free path of eddies \( \lambda \) in the turbulent flow, which is related to the viscosity \( \nu = v_{\text{eddy}} \lambda \), is not resolved in this description. All quantities, such as density or accretion velocity, must be understood in a local time averaged sense with probably large short-term variations. Besides the time average, we will also discuss accretion flows as polar-averaged, one-dimensional flows with only a radial coordinate \( r \). The average is taken over sections of shells occupied by the flow. Consequently, polar motions cannot be traced in this treatment. We are left with the angular velocity \( \Omega = \Omega (r) \) and the radial component of the velocity, which is the accretion velocity \( u = u (r) \). Mass is accreted by the central object with a rate \( \dot{M} = -2 \pi r u \Sigma \). In this discussion, \( \Sigma \) is a suitable polar integral of the density \( \Sigma = 2 \int_{\pi/2 - \delta}^{\pi/2 + \delta} \sin \theta \, d \theta \, r \sin \theta \rho \), so that \( H = r \delta \) is the vertical thickness of the accretion flow and the integral in equation (1) is restricted to a region around the symmetry plane of the flow to make room for winds seen in Figure 1. Nonetheless, any vertical motion has a radial and polar component and the later is considered as a source or sink of mass, momentum, and energy. We regard these sources as external to our description of the flow and call it infall into the accretion flow or an outflow, respectively.

For a nonrelativistic flow, the conservation of mass implies that the change in mass flux at every radius is balanced by mass exchange with the wind in the stationary case

\[
\frac{1}{r} \frac{\partial (ru \Sigma)}{\partial r} = \Sigma_w ,
\]

where \( \Sigma_w \) is the mass per surface area added to the flow from the infall per time at a given radius. According to equation (1), this happens at a height \( r \delta \), but rapid mixing with the flow is assumed so that the equations averaged over a polar section occupied by the accretion flow remain valid. Angular momentum is a conserved quantity in accretion flows without winds and is redistributed by torques generated by viscous stresses \( \tau_{\phi} \)

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru \Sigma l - r^2 t_{\phi \rho}) = \Sigma_w \ell_w .
\]

Here \( \ell \) is a mass-weighted polar average of \( \ell (r, \theta) \). The infall contributes its own angular momentum to the flow \( \Sigma_w \ell_w \) and exerts an additional torque, which depends on the relative angular momentum of infall and flow \( \ell_w - \ell \).

The accretion velocity \( u \) follows from the imbalance of gravitational force, centrifugal barrier, and radial pressure gradient. In addition to that, we include the viscous braking force \( \tau_u \) from the \( \rho \rho \) component of the stress tensor and bulk viscosity

\[
\frac{1}{r} \frac{\partial (ru \Sigma u)}{\partial r} = \sum \left( \frac{\Omega^2 - GM}{r^3} - \frac{\partial \rho}{\partial r} + F_v + \Sigma_w u_w \right)
\]

with the integrated pressure \( \rho = \int \rho \, d \theta \rho \) and the radial velocity of the wind \( u_w \), which adds its momentum to the flow. The change in specific radial momentum induced by winds is proportional to the velocity difference \( u_w - u \). The viscous force (see Appendix)

\[
F_v = F_{rr} + F_{\text{bulk}}
\]

\[
F_{rr} = \eta_1 \left\{ \frac{4}{3} \frac{\partial}{\partial r} \left[ \frac{v \Sigma}{r^2} \frac{\partial u}{\partial r} \right] \right\} + 4 \nu \frac{\partial^2 u}{\partial r^2}
\]

\[
F_{\text{bulk}} = \eta \frac{r}{r^3} \frac{\partial}{\partial r} \left[ \frac{v \Sigma}{r^2} \frac{\partial (ru)^2}{\partial r} \right]
\]

has a contribution proportional the \( \rho \rho \) component of the shear tensor of the flow in equation (6) and the compression of the gas in equation (7). We make the crude assumption that the corresponding viscosities are the same for the \( \rho \rho \) and \( \rho \phi \) components of the shear stress and also for the bulk viscosity. Our ignorance is cast into the parameters \( \eta_1 \) and \( \eta \), which measure the strength of the \( \rho \rho \) and shear viscous force measures in the following. Finally, an energy equation must be given,

\[
\frac{u \Sigma}{r^2} = \frac{\partial}{\partial r} \left( \Sigma_w \frac{\partial (ru)^2}{\partial r} \right) + Q^+ - \Lambda + \Sigma_w \left[ \omega_w - \omega + \frac{(v_w - v)^2}{2} \right]
\]

which describes the change of specific internal energy \( e \) of the flow caused by compression (first term, right-hand side), viscous heating \( Q^+ \), radiative cooling \( \Lambda \), the enthalpy differ-
ence between flow \( \omega \) and wind \( \omega_w \), and the kinetic energy associated with the velocity difference, which has to be dissipated into heat. Here \( \mathbf{v} \) is the velocity vector of the flow and \( \mathbf{v}_w \) the corresponding vector for the wind.

3. ANGULAR MOMENTUM TRANSPORT AND VISCOSITY

The \((r\phi)\) component of the stress tensor is assumed to be proportional to the corresponding component of the shear tensor with the kinematic viscosity \( \nu = \nu_0 r \) as the factor of proportionality. For the viscosity \( \nu \), we will adapt three different representations. The first and obvious one is the \( \alpha \) prescription introduced by Shakura & Sunyaev (1973), and the resulting \( \alpha \)-distributions will be discussed in §6. The \( \alpha \)-viscosity is not a unique choice and effects of the so-called \( \beta \)-viscosity introduced by Duschl, Strittmatter, & Biermann (2000) are discussed as \( \beta \)-distributions in §7. For the \( \alpha \)-viscosity

\[
v = \alpha \frac{c_s^2}{\Omega} ,
\]

we avoided introducing the vertical scale height \( H \) of the disk because we believe it leads to the misunderstanding that angular momentum transport in the radial direction depends on the thickness of the disk. Only with the assumption of vertical hydrostatic equilibrium is it possible to introduce \( H = c_s \Omega / \kappa \) in equation (9).

The \( \alpha \)-viscosity can be recovered from the \( \beta \)-viscosity law in the case of shock-limited turbulence in a Keplerian disk. The \( \beta \)-viscosity assumes that a typical length scale in the direction of transport \( \Delta r \) and the typical velocity difference \( \Delta v_\phi \) between interacting shells, which exchange eddies, equal the mean free path of eddies and their typical velocities relative to the mean flow, respectively. A parametrization with a new constant \( \beta \) is suggested as

\[
v = \Delta v_\phi \Delta r = \beta v_\phi r .
\]

If the typical velocity of eddies in a differentially rotating disk is limited by the sound speed, then the typical length scale \( \Delta r \) of communication between differentially rotating rings is estimated by Duschl et al. (2000) to be

\[
\Delta r \leq \frac{c_s}{\Omega} .
\]

Only in disks with Keplerian rotation does this length scale equal the vertical scale height and the form \( v = \alpha c_s \) \( H \) is recovered. As a third possibility, we consider the shear-limited form of the \( \beta \)-viscosity with the length scale from equation (11), which has not been used for sub-Keplerian accretion flows before. This is reasonable, if the mean free path of eddies is not limited by the vertical scale height but, instead, determined from the distance between shells, for which the velocity difference caused by differential rotation equals the eddy velocity. This argument implies a larger effective viscosity for disks in sub-Keplerian rotation. The \((r\phi)\) component of the viscous stress tensor becomes, in any case,

\[
t_{r\phi} = v \Sigma r \frac{c_s^2}{\gamma} .
\]

We assume that the same viscosity prescription can be applied to the \((rr)\) component of the stress and the bulk viscosity. Their relative strength is scaled to the \((r\phi)\) viscosity by the force measures \( \eta_s \) and \( \eta \) introduced above. The heat generated by the described viscous forces has to be included in the energy equation and amounts to

\[
Q^+ = v \sum r \left( \frac{c_s^2}{\gamma} \right)^2 + \frac{4}{3} \eta_1 \nu \sum \left( \frac{\alpha}{\nu} \right)^2
\]

\[
+ \eta \sum \frac{\alpha(r^2 v^2)}{\gamma} r^2 .
\]

The internal energy is increased by viscous heat and compression and reduced by radiative cooling \( \Lambda \).

4. EQUATION OF STATE

In general accretion flows are expected to exist for all reasonable equations of state for the accreted gas. The self-similar solutions we will use in what follows, are restricted to rather simple equations of state. For ADAFs the gas should be well described by an ideal gas law with \( \gamma = 5/3 \) for ratio of specific heats, because the temperatures in ADAF solutions are so large (\( T > 10^7 K \) for most radii of interest) that the gas is completely ionized and only ions of a few heavy elements retain their highly bound inner electrons (see Narayan & Raymond 1999) for possible emission lines of these elements). The change of the equation of state caused by partial ionization can be neglected, but the contribution of magnetic to the total pressure must be considered for two reasons. The first is the observational evidence that at least some candidates (e.g., Sgr A*, see Narayan et al. 1998) for ADAFs show a significant contribution of synchrotron emission to the total luminosity, which requires magnetic fields close to pressure equipartition with the gas. Given their existence, we might try an MHD description of the accretion flow to separate the evolution of magnetic fields and gas, which might be different (Bisnovatyi-Kogan & Lovelace 1997) unless magnetic diffusivity (Heyvaerts, Priest, & Bardou 1996) and/or reconnection require an even more complex model. The easy way out is to assume that turbulence generates small-scale magnetic fields, which dominate the energy density in magnetic fields and produce an isotropic contribution to total pressure and energy. In doing so, one arrives at a hydrodynamic description of the accretion flow with an equation of state, which has to incorporate the magnetic pressure. Consider an equation of state of the quite general form

\[
P \sim \rho^\gamma T^{\gamma-1} ,
\]

with the exponents \( \chi_p \), \( \chi_T \) defined according to Cox & Giuli (1968). For the self-similar solutions, we have to restrict the internal energy per unit volume to be proportional to the total pressure

\[
P = \frac{\chi_p}{\chi_T} \left( \frac{\gamma}{\gamma - 1} \right) \Gamma_3 - 1 ,
\]

with \( \gamma = c_p/c_v \) being the ratio of specific heats and \( \Gamma_3 \) the adiabatic coefficient defined by Chandrasekhar (1939). The restriction from equation (15) is that \( \chi_p \), \( \chi_T \) and \( \Gamma_3 \) have to be constants, which requires that the relative strength of the magnetic pressure must be constant so that the effective equation of state is an ideal gas law with a ratio of specific heats less than 5/3. We define the isothermal sound speed \( c_s = \sqrt{\gamma P_0} / \Sigma \) from the integrated pressure and density. The sound speed used in the viscosity prescription is the total pressure divided by the mass density. We use thermodynamic relations (Cox & Giuli 1968) to get an equation for the temperature from equation (8) for stationary accretion
flows:

\[ u c_v \frac{\partial T}{\partial r} = \frac{\chi_p r c^2}{\Sigma} u \frac{\partial}{\partial r} \left( \frac{\Sigma}{r} \right) + \frac{Q^+ - \Lambda}{\Sigma} + \frac{\Sigma_w}{\Sigma} \left[ \omega_w - \omega + \frac{(v_w - v)^2}{2} \right]. \]  

(16)

Here \( c_v \) is the specific heat at constant volume

\[ c_v = \frac{\chi_T c_s^2}{\chi_T (\gamma - 1) T}, \]  

(17)

and if we assume that gas pressure contributes a fraction \( \delta \) to the total pressure and magnetic fields are responsible for the rest, we get

\[ \Gamma_3 - 1 = \frac{2\delta}{3\delta + 2(1 - \delta)} \]  

(18)

for the adiabatic exponent \( \Gamma_3 \), which is related to \( \gamma \) by equation (15). For constant \( \chi_p \) and \( \chi_T \), we can rewrite the temperature gradient in equation (16) as a gradient of the sound speed and search for solutions in terms of the sound speed.

5. SELF-SIMILAR ADAF SOLUTIONS WITH A WIND

The set of equations described in §§ 2 and 3 allow self-similar power-law solutions for accretion velocity, angular velocity, and sound speed in the way Narayan & Yi (1994) have shown:

\[ u = -u_0 r^{-1/2} c, \quad \Omega = \Omega_0 r^{-3/2} \frac{c}{r_G}, \quad c_s^2 = a^2 s^{-1} c^2, \]  

(19)

with the radial coordinate scaled to the gravitational radius of the central mass

\[ s = r; \quad r_G = \frac{GM}{c^2} \]  

(20)

and the speed of light \( c \). It is required that the cooling is either completely negligible or is a radius-independent fraction of the heating rate

\[ f = 1 - \frac{\Lambda}{Q^+} \]  

(21)

so that \( 1 - f \) is the cooling efficiency, which will be small, if electrons are inefficiently coupled to the heat source. This happens if ions are preferentially heated by viscous friction and the energy transfer rate between electrons and ions is smaller than the heating rate.

For the self-similar solution, the radial component of the wind velocity must be proportional to the accretion velocity and we get a measure \( \xi = u_w/u_0 \), which tells us the relative radial velocity of the wind, but we will neglect it in the following discussion and assume \( \xi = 0 \). This is motivated from the infall calculations of Coker, Melia, & Falcke (1999), which predict a very small radial velocity of the infalling material in the disk midplane for the infall from stellar winds in the Galactic center. If a fraction \( f \) larger than a few percent of the viscously generated heat is not radiated away but kept in the flow as internal energy, the disk scale height \( h = c_s \Omega_k \) is of the same order as the radius and the infall may have a significant radial velocity component. Nonetheless, we will neglect it in the following discussion. The same is true for the enthalpy of the infalling material, but here we have better reasons to believe that the gas joining the ADAF is cold compared with the gas in the ADAF, which is heated by viscous friction, while the infalling gas may be provided by stellar winds or the gas of the H II region Sgr A West, in case of the Galactic center, with temperatures of \( 6 \times 10^7 - 10^8 \) K (see Mezger, Duschl, & Zylka 1996 for a review of the Galactic center). This should be compared to typical temperatures of ADAFs of \( 10^7 \) K at \( 10^4 \) black hole radii. The situation with outflows might be different, where it is hard to imagine that the gas leaving the disk has a different temperature than the gas left in the flow. The gas might need some outward pointing radial momentum to get away from the ADAF, but, again, we will ignore it here. For infalling material, we expect it to be in free fall so that the total velocity is \( v_w = v_{\text{ff}} = 2cs^{-1/2} \). In the case of outflows, we require that they are able to escape the gravitational attraction of the central object and that their velocity must be at least the free-fall velocity \( v_{\text{ff}} \).

In the same way, the rotation of the infall has to be constant fraction \( \xi = \Omega_{\text{w}}/\Omega \) of the rotation of the accretion flow itself; again, calculations by Coker et al. (1999) for the Galactic center predict infall with only weak rotation. Outflows, on the one hand, might be expected to carry their initial angular momentum from the point of origin and therefore \( \xi = 1 \). On the other hand, magnetically driven outflows will exert a torque on the remaining accretion flow if the gas in the outflow has different angular velocity than the point in the flow to which it is connected by magnetic field lines. In this case, \( 0 < \xi < 1 \) would be expected.

The rate of mass added by the inflow per surface area of the flow in its central plane is constrained to steep radial profiles and introduces one free parameter \( p \) (Blandford & Begelman 1999) not to be confused with the pressure \( P \):

\[ \dot{\Sigma}_w = -\frac{\mu c}{r}, \quad \Sigma = \Sigma_0 r^{-1/2 + p} \]  

(22)

For infall into accretion flows, \( p \) has to be negative and the radial dependence of the infall rate must be steeper than \( r^{-2} \). Therefore, most of the mass is added at small radii and the total mass infall \( dM_w \sim dr r^{-1 + p} \) diverges in the center. This solution is consequently not valid for small radii, where the infall must deviate from the solution in equation (22). Positive values of \( p \) correspond to outflows generated by the accretion flow and, reversing the argument for the total mass now taken away from the ADAF, the solution is confined to small radii so that \( M_w \) is still smaller than the mass accretion rate supplied at the outer radius. The surface density of the flow implies a density \( \rho \sim r^{-3/2 + p} \), which restricts the specific equation of state (eq. [14]) to those with \( \chi_0 \Omega_T \) and \( \Gamma_3 \) being constants and the temperature as a function of radius follows

\[ T^{xy} \sim r^{-1 + (1 - \xi_0)(-3/2 + p)} \]  

(23)

so that the radial pressure gradient is independent of the thermodynamic exponents and \( P \sim r^{-5/2 + p} \). The gas density in self-similar solutions for advection-dominated accretion is unconstrained and only the required inefficient cooling restricts the solutions to small mass accretion rates below \( \sim 10^{-2} \dot{M}_{\text{Ed}} \) (Narayan & Yi 1995b). \( \dot{M}_{\text{Ed}} \) is the Eddington accretion rate for a radiation efficiency of 10% in terms of the rest mass of the accreted material.

6. \( \alpha \)-ADAFS

Whether the parametrization of the wind, which is necessary for the self-similar solution in § 5, is justified or not
depends on boundary conditions, which the wind has to meet at large distances, or on the internal physics of outflows. While outflows might naturally follow the self-similarity of ADAFs, the distribution of stellar wind sources will determine if a power-law dependence of the inflow rate is reasonable. The parametrization of the viscosity (Shakura & Sunyaev 1973) in equation (9) is based on dimensional arguments and, besides numerical simulations, which do not exist for ADAF conditions, there is no way to determine what values for \( \alpha \) are appropriate in the case of stationary ADAFs. Without information on the strength of the magnetic fields from MHD calculations, the adiabatic exponent \( \Gamma_3 \) is also undetermined. We will discuss the self-similar solutions as functions of \( \alpha \) and \( \Gamma_3 \), as well as for certain wind parameters and viscous force measures \( \eta \) and \( \eta_1 \).

The self-similar Ansatz in § 5 reduces the dynamical equations (3), (4), and (16) to a set of nonlinear algebraic equations for the coefficients of sound speed \( a \), angular velocity \( \Omega_0 \), and accretion velocity \( u_0 \). The accretion velocity is derived from equation (3) in terms of sound speed \( a \) and \( \alpha \):

\[
u_0 = (3/2)\alpha a^2 a', \quad \alpha = \frac{1 + 2p}{1 + 2p(1 - \xi)}
\]

with a constant \( \alpha \), which reflects the torque exerted by the wind on the flow. The combined wind parameter \( \alpha \) is 1 for a vanishing wind or a nonrotating wind. For rotating infall, the solution is restricted to moderate infall rates with \( p > -1/2 \). Otherwise accretion would not be possible. For a given sound speed and accretion velocity, the centrifugal barrier and, therefore, the angular velocity follows from the radial momentum equation (4), if we know the viscous force measures. We find for the square of the angular velocity

\[
\Omega_0^2 = c_s^2 - (5/2 - p)a^2 - (9/8)\alpha^2 a' \times \{ a' [1 - 2p(1 - \xi)] + \eta(5 - 2p) \\
+ (4/3)\eta_1 (1 + 2p) \} \frac{a^4}{c_s^2}.
\]

We are left with a quadratic equation for the square of the sound speed. One solution turns out to be irrelevant either by predicting \( a^2 \) to be negative or by leaving \( \Omega_0^2 \) in equation (25) negative. But even the second root does not always give reasonable solutions because with increasing \( \alpha \) and small changes in \( a^2 \), as seen in Figures 2 and 3, \( \Omega_0^2 \) decreases and becomes negative. Solutions for the accretion problem exist only for values of the viscosity parameter smaller than a critical \( \alpha_c \). The wind parameters for the solutions in Figures 2 and 3 have been chosen in a way that the infall or outflow is in free fall or leaves with the escape speed. No radial momentum of the wind is included and both infall and outflow are cold. While the infall has no proper angular momentum, the outflow rotates with the \( \Omega \) of the accretion flow. With the requirements of a cold outflow and total velocity of the wind being the escape speed, which both enter the energy equation (16) in the same way, we make a minimal energy assumption for the extraction of internal energy by the outflow. For the equation of state, we take the natural choice \( \chi_p = 1 \) so that the mix of gas and magnetic field behaves like an ideal gas. The ratio of specific heats equals \( \Gamma_3 \), if we assume \( \chi_T = 1 \). \( \chi_T \) is not a parameter of our solutions but determines the actual temperature of the gas through equation (23). For energy equipartition, we have \( \Gamma_3 = 1.4 \) from equation (18), and we use this value unless stated otherwise. For the cooling efficiency \( 1 - f_c \), we assume a low rate of 1%. For the dynamics of the flow, it is not the energy radiated away that matters but the energy left in the flow. The actual cooling efficiency is unimportant as long as it is small and \( f < 1 \).

For an ADAF without winds, it is possible to derive an analytic solution for the sound speed and determine the maximum allowed \( \alpha \) given by

\[
\alpha_c = \frac{[8/9 - 2(\Gamma_3 - 1)(7/3 - \Gamma_3) + \mathcal{B}]^{1/2}}{f(\Gamma_3 - 1)(3\eta + 4\eta_1)}
\]

and

\[
\mathcal{B} = 2[5\eta + (4/3)\eta_1] \times \{ f(\Gamma_3 - 1)[(7/3 - \Gamma_3) - f(5/3 - \Gamma_3)] \\
+ (32/3)\eta_1 f(\Gamma_3 - 1)(5/3 - \Gamma_3) \}.
\]

In that case, \( \alpha \) equals 1, and the accretion velocity is \( u_0 = (3/2) \alpha a^2 \). If the viscosity parameter is small, the limit \( \alpha \to 0 \) provides a good approximation for the sound speed

\[
a^2 \approx \frac{6f(\Gamma_3 - 1)}{3(\Gamma_3 - 1)(5\eta - 2\chi_p) + 10 - 6\chi_p}.
\]

and the angular velocity \( \Omega_0 = [c_s^2 - (5/2)\alpha a^2]^{1/2} \) as seen in Figures 2 and 3. From equations (26) and (27), we immediately see that \( 1 < \Gamma_3 < 5/3 \) is required for the no-wind case, and the familiar result arises that no solution exists for a nonrelativistic ideal gas. It is also obvious that the upper limit on \( \alpha \) increases and becomes irrelevant with vanishing radial viscous force. In other words, with increasing viscous force measures \( \eta, \eta_1 \), ADAF solutions are restricted to reasonably small values of \( \alpha < \alpha_c \).
Radial accretion velocity $-u$ in units of $a$ times the local free-fall velocity ($a$) and angular rotation velocity $\Omega$ of the flow relative to the local Keplerian rotation ($b$). Solutions are given for $\Gamma_3 = 1.4$ and different strength of the radial viscous force. $\eta$ measures the strength of bulk viscosity and $\eta_1$ the $(rr)$-shear stress. Also shown is the standard ADAF solution without winds and no radial viscous force $\eta = \eta_1 = 0$. The actual accretion velocity is almost linear in $a$ and the difference at $a \to 0$ is a difference in slope as a function of $a$. The rotation velocity $\Omega$ is shown only for ADAFs without a wind.

One additional criterion for the realization of accretion solutions is the Bernoulli number,

$$Be = -\frac{GM}{r} + r^2 \Omega^2 + u^2 + e + \frac{P}{\rho},$$

which was discussed by Narayan & Yi (1994) and Narayan & Yi (1995a) for ADAFs and by Blandford & Begelman (1999) for advection-dominated inflow-outflow solutions. While the total energy decreases inward, the Bernoulli number being the total specific energy plus $P/\rho$ is positive and increases inward for ADAFs. This changes with outflows because of their cooling effect by removing internal energy and reducing the Bernoulli number of the remaining flow. Correspondingly, an infall increases the Bernoulli number of the flow. The combinations of allowed values for

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**Fig. 3a**

**Fig. 3b**

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**Fig. 4**—Excluded, allowed, and marginally allowed regions of ADAF solutions with infall or outflows in the plane of wind parameter $p$ and ratio of specific heats, which equals $\Gamma_3$ for $\chi = \chi_1 = 1$, as mentioned in the text. $\Omega^2$ becomes negative in the forbidden region, and we can distinguish regions where the Bernoulli number is either positive, representing unbound flows, or negative. (a) shows the situation for corotation of wind and flow $\zeta = 1$. In (b), a nonrotating wind $\zeta = 0$ is assumed. The dividing line of allowed and forbidden regions depends on the strength of the radial viscous force, for which $\eta = 0$ and $\eta_1 = 1$ is assumed. For increasing $\chi$, the allowed region moves deeper into the outflow regime and dividing lines are shown for $x = 0.01, 0.1, 0.3, 0.6, 0.9$. 

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\( \alpha \) and \( \Gamma_3 \) as a function of the wind strength \( p \) are shown in Figure 4 for corotating (\( \zeta = 1 \)) and nonrotating winds. The statement that ADAFs exist for an ideal gas with negative Bernoulli numbers, if an outflow is present (Blandford & Begelman 1999), is confirmed, even when the radial viscous force is included. The minimal possible wind strength depends on the viscosity parameter \( \alpha \) as seen in Figure 4.

7. \( \beta \)-ADAFs with Winds

Following the arguments in Duschl et al. (2000), the viscosity law should not depend on the sound speed, if the actual velocity of turbulent eddies is smaller than the sound speed in the gas. In the \( \beta \)-viscosity law from equation (10), the eddy velocity is determined from the actual rotation velocity \( v_{\text{edd}} \approx \sqrt[3]{\beta r \Omega} \), with the lessening \( \sqrt[3]{\beta} \) equally distributed between velocity and length scale. The arguments given in \$3 \) suggest that \( \alpha \)-viscosity can be seen as the shock-limited case of the \( \beta \)-viscosity and that the \( \alpha \)-ADAFs discussed in \$6 \) can be checked for consistency if this hypothesis is correct. Specializing on the no-wind case in the limit of small \( \alpha \), we find the ratio
\[
\frac{c_s^2}{(r \Omega)^2} = \frac{f(\Gamma_3 - 1)}{(5/3 - \Gamma_3 \chi_{\rho})},
\]
and the \( \beta \)-viscosity can be applied for \( \beta < 1.5f \) with the adoption of \( \chi_{\rho} = 1 \) and \( \Gamma_3 = 1.4 \). Therefore the \( \beta \)-viscosity is valid for ADAFs with low cooling efficiencies, if the resulting flows are similar to \( \alpha \)-disks. The scale length of the \( \alpha \)-viscosity is the vertical scale height \( H = c_s/\sqrt{\beta r} \), which turns out to be \( H \approx 1.55r[(/1.6 + 6f)]^{1/2} \approx 0.6r \) for \( \alpha \)-ADAFs in the no-wind case and the limit \( \alpha \to 0 \). The length scale of the \( \beta \)-viscosity is \( \sqrt[3]{\beta r} \), and we expect the same sound speeds and angular velocities for \( \alpha \)- and \( \beta \)-ADAFs below \( \beta \leq 0.3 \). This is verified by comparing Figures 2 and 3 with Figure 5. The actual solution for \( \beta \)-ADAFs is derived in the same way as for \( \alpha \)-ADAFs in \$6 \), and we find the accretion velocity depending explicitly on \( \Omega \) and not on \( c_s^2 \).
\[
u_0 = -\frac{1}{2} \beta \Omega_0 \alpha^2 .
\] (31)

The constant \( \alpha^2 \) is defined in equation (24). The increased accretion velocity seen in Figure 6 compared to \( \alpha \)-ADAFs is explained by the larger value of \( \Omega_0 \) in equation (24) relative to the sound speed squared in case of the \( \alpha \)-viscosity. The angular velocity of the accretion flow can be given as
\[
\Omega_0^2 = 1 - \frac{5}{2} p a^2
\]
\[
\times \left[ \frac{9}{8} \left[ \frac{1}{2} (1 - 2p(1 - \zeta_1)) \right] \alpha^2 \right.
\]+ \left[ (5 - 2p) \eta + 4/[3(1 + 2p)\eta_1] \alpha^2 \right] \beta^2 + 1)^{-1} ,
\] (32)

including the reaction of the flow because of the presence of winds. Turning to the energy equation shows that two obvious solutions exist for the sound speed. First, a nonrotating and nonaccreting solution,
\[
a^2 = \frac{2c_s^2}{5 - 2p},
\] (33)

where the gravitational attraction is completely balanced by the pressure gradient. The gas is in hydrostatic equilibrium and constitutes an optically thin atmosphere. This is not a viable solution for a black hole because general relativity does not allow a pressure-supported atmosphere close to the horizon. The second solution is an accreting and rotating flow as shown in Figure 5. The gas is cooler than in the atmosphere solution and strong outflows in corotation with the flow or even nonrotating outflows reduce the sound speed further so that the accreted gas can be gravitationally bound to the central mass. For a corotating, cold outflow, which reaches infinity with the escape velocity from the radius of origin \( \zeta_2 = 2 \), and a simple equa-

![Fig. 5a](attachment:image.png)

![Fig. 5b](attachment:image.png)

**Fig. 5a**—Isothermal sound speed relative to the local free-fall velocity (a) and angular velocity in units of the local Keplerian value (b) for \( \beta \)-ADAFs. For the no-wind case \( p = 0 \), the radial force measures are \( \eta = \eta_1 = 1 \) for the thick solid line, \( \eta = 0 \), \( \eta_1 = 1 \) for the alternating long-and-short dashed line, and no radial viscous force \( \eta = \eta_1 = 0 \) for the dotted line. The inflow solution \( p = -0.4 \) has no rotation of the inflowing gas, and for the outflow solution with \( p = 0.6 \), the difference between a corotating outflow (thick solid line) and a nonrotating outflow is highlighted.
tions of state $\gamma_p = 1$, we can give the sound speed for a case of negligible radial viscosity $\eta = \eta_1 = 0$:

$$a^2 = \frac{f - [3(3/4 - p)\beta^2.a^2 + 1]p.\mathcal{A}}{(5/2 - p)f - \Xi} \quad (34)$$

$$\Xi = \frac{\mathcal{A}}{24} \left\{ 4(6 - p)(1 - 2p)(1 + \beta^2.a^2) + \beta^2.a^2 [3(1 + 4p)^2 - p(17 + 2p)] \right\} + \frac{p(1 - p)}{\Gamma_3 - 1} \left[ \frac{2}{3} + \frac{3}{4} \beta^2.a^2(1 - 2p) \right]. \quad (35)$$

For nonrotating and cold infall, which is in free fall for $p < 0$, or outflows barely reaching infinity ($\zeta_2 = 1$) with $p < 0.36$, the ratio of specific heats has to be smaller than a limiting value

$$\Gamma_3 - 1 < 2 \frac{1 - p}{3 - 9p + 2p^2}, \quad (36)$$

at which the rotation of the flow stops and no solutions exist for $\Gamma_3$ larger than that. This limit for $\Gamma_3$ equals 5/3 for ADAFs without winds or outflows and is the well-known result that ADAFs do not exist for an ideal equation of state. The consistency of $\beta$-ADAFs follows from the above predictions based on $a$ flows. With infall into the flow, the sound speed increases and the rotation velocity decreases. The $\beta$-viscosity law is therefore more likely to be applicable to ADAFs with infall than with outflows.

### 8. ADAFs WITH SHEAR-LIMITED VISCOSITY AND WINDS

The motivation for a shear-limited viscosity law is the observation that the standard $a$-viscosity agrees with the general $\beta$-viscosity law only for Keplerian accretion disks with a shock-limited eddy velocity. In the shear-limited viscosity

$$v = \frac{c_s^2}{\Omega}, \quad (37)$$

we assume that the eddy velocity is indeed shock limited and the mean free path of eddies is determined by the radial distance of differentially rotating shells, which allows the exchange of eddies along radial paths in a comoving frame of one shell. If the eddy velocity is the sound speed, the maximal distance of shells follows from a Taylor expansion of $\Omega$ in $\Delta\phi$:

$$\Delta r \leq \frac{c_s}{2\Omega}. \quad (38)$$

The length scale over which shells can interact is larger for sub-Keplerian rotation. This is the main difference of equation (37) to the original $a$-viscosity. The consequence for the accretion velocity,

$$u_0 = \frac{3}{2} \frac{a^2}{\Omega_0} \mathcal{A}, \quad (39)$$

is an explicit dependence on the inverse of $\Omega_0$ contrary to $\beta$-ADAFs in equation (31). The combined wind parameter $\mathcal{A}$ is defined in equation (24). The square of the angular velocity follows from a quadratic equation with two roots:

$$\Omega_0^2 = \frac{1}{2} \frac{5 - 2p}{4} a^2 \pm \sqrt{\left( \frac{1}{2} \frac{5 - 2p}{4} a^2 \right)^2 - \frac{2^2 a^4 \Psi}{4}}, \quad (40)$$

$$\Psi = 9/8.a^2 \{[1 - 2\mathcal{A}(1 - \zeta_1)]\mathcal{A} + (4/3)\eta_1(1 + 2p) + \eta(5 - 2p) \}. \quad (41)$$

---

**Fig. 6a**

Fig. 6a.—Accretion velocity (a) and Bernoulli number (b) for different $\beta$-ADAFs. The identification of solutions is the same as in Fig. 5.
In the limit of vanishing viscosity parameter $\tilde{\alpha}$, the angular velocity has two obvious solutions,

$$\tilde{\alpha} \to 0 \Rightarrow \Omega_0 \to \begin{cases} \sqrt{1 - (5/2 - p)a^2} \\ 0 \end{cases},$$

or (42)

For small $\tilde{\alpha}$, the rotation for the second solution is proportional to $\tilde{\alpha}$ and positive. The existence of these slowly or nonrotating ADAFs depends on nonvanishing force measures $\eta, \eta_1 \neq 0$. The solution disappears without a radial viscous force. The viscosity in nonrotating ADAFs tends to a finite value at $\tilde{\alpha} = 0$,

$$v = a \frac{1 - (5/2 - p)a^2}{a^2 \Psi} c \sqrt{\frac{\eta_0}{\theta}},$$

and so does the accretion velocity. If we interpret $\tilde{\alpha}c_{s}/\Omega$ as the viscous length scale, it does not diverge for small $\tilde{\alpha}$. The sound speed also tends to a positive value, as is seen in the upper solution branches for the sound speed in Figure 8 and for accretion velocity in Figure 7 corresponding to the lower branches for the angular velocity in Figure 7. The energy equation is itself quadratic in $a^2$ for either choices of $\Omega_0$ in equation (40). Only one solution satisfies the condition

$$a^2 \leq \frac{2}{5 - 2p}$$

for real values of $\Omega$. We end up with two solution branches, which join at a critical $\tilde{\alpha}_c$. Beyond $\tilde{\alpha}_c$, no physical solution exists for the accretion problem. Besides the slowly rotating solution discussed above, we find more familiar ADAFs, where the accretion velocity is linear in $\tilde{\alpha}$ as seen in Figure 7.

For small viscosity parameters, the solution is independent of the force measures $\eta, \eta_1$. Real solutions for the square of the sound speed in the no-wind case are only possible for

$$\tilde{\alpha} \leq \frac{(5/3 - \chi_p \Gamma_3)}{\sqrt{3f(\Gamma_3 - 1)\sqrt{\eta + (4/3)\eta_1}}}.$$  (45)

The situation does not differ from the standard $\alpha$-viscosity insofar as no upper limit for $\tilde{\alpha}$ exists in the limit of vanishing radial viscosity $\eta = \eta_1 = 0$ in the radial momentum equation (4) as long as the ratio of specific heats is in the range $1 < \Gamma_3 < 5/3$. In the limit of vanishing viscosity $\tilde{\alpha} \to 0$, the rotating solution coincides with the unique ADAF solution without winds and radial viscosity. For the simplest equation of state $\chi_p = 1$, the sound speed in the limit $\tilde{\alpha} \to 0$ is

$$a^2 = \frac{2\epsilon f}{2\epsilon^2 + 5\epsilon f}, \quad \epsilon = \frac{5/3 - \Gamma_3}{\Gamma_3 - 1}.$$  (46)

The critical $\tilde{\alpha}_c$, which sets the upper limit for the existence of ADAFs with shear-limited viscosity, depends not only on the force measures $\eta, \eta_1$, but also on the presence of winds. The same pattern as for the standard $\alpha$-viscosity appears, in which solutions with outflows are possible for a wider range of $\tilde{\alpha}$ than for infall solutions as seen in Figures 7 and 8. It turns out that the slowly rotating branch of solutions produces smaller Bernoulli numbers than the fast rotators for infall and the no-wind case $p = 0$. Only for outflow solutions is the fast rotor preferred with smaller Bernoulli numbers. But in all cases shown in Figure 8, the Bernoulli number is positive and no gravitationally bound flow is found. This changes if more energy is extracted by the wind and the fast rotating outbound solution shown in the figures.

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**Fig. 7a** — Accretion velocity ($a$) and rotation ($\Omega$) for ADAFs with the shear-limited viscosity law. The ratio of specific heats equals $\Gamma_3 = 1.4$, and wind parameters are $p = 0.2$ for the corotating outflow and $p = -0.1$ for infall with no angular momentum of the infalling material. The solution without radial viscous force is plotted as a solid thin line available for all values of $\tilde{\alpha}$. Solutions with different viscous force measures are shown for the no-wind case $p = 0$ with $\eta = 0, \eta_1 = 1$ as the thick solid line, $\eta = 1, \eta_1 = 1$ as the thin dash-dotted line, and $\eta = 0, \eta_1 = 0.5$ as the short-dashed line. For each solution, the branch with large radial velocity shows slower rotation and vice versa.

**Fig. 7b** — Accretion velocity ($a$) and rotation ($\Omega$) for ADAFs with the shear-limited viscosity law. The ratio of specific heats equals $\Gamma_3 = 1.4$, and wind parameters are $p = 0.2$ for the corotating outflow and $p = -0.1$ for infall with no angular momentum of the infalling material. The solution without radial viscous force is plotted as a solid thin line available for all values of $\tilde{\alpha}$. Solutions with different viscous force measures are shown for the no-wind case $p = 0$ with $\eta = 0, \eta_1 = 1$ as the thick solid line, $\eta = 1, \eta_1 = 1$ as the thin dash-dotted line, and $\eta = 0, \eta_1 = 0.5$ as the short-dashed line. For each solution, the branch with large radial velocity shows slower rotation and vice versa.
Fig. 8.—Isothermal sound speed and Bernoulli number for the two-branch ADAF model with shear-limited viscosity law. The identification of solutions is by line type and is the same as in Fig. 7. The branch of each solution with large sound speed corresponds also to large accretion velocities and slow rotation. The Bernoulli number for outflow solutions is smaller for the rapidly rotating branch with small sound speed as compared with that of the slowly rotating branch. In all other cases, the Bernoulli number of the slowly rotating solution is smaller.

has negative Bernoulli numbers, if the outflow is cold and reaches a terminal velocity equal to the escape speed at its origin. The allowed combinations of $\alpha$ and $\Gamma_3$, regardless of Bernoulli numbers, are shown in Figure 9. The dividing lines in the $(\alpha, \Gamma_3)$-plane are vertical without a radial viscous force and bend in the way drawn in the figure when the $(rr)$ component of the stress tensor is included.

9. CONCLUSIONS

We have investigated the importance of a radial viscous braking force for advection-dominated accretion flows (ADAFs) in the presence of infall or outflows. Under the assumption of almost isotropic turbulence, bulk viscosity and the $(rr)$ component of the viscous stress tensor provide an efficient brake of the rapid radial accretion of hot gas in ADAFs and produce additional heat caused by viscous friction. This opens a second channel for transfer of kinetic into internal energy and supports, in part, the radial motion so that the shear caused by differential rotation and the centrifugal barrier is reduced in ADAFs with radial viscous braking.

We derived self-similar solutions of ADAFs for three different viscosity laws. The standard $\alpha$-viscosity produces solutions that show more and more sub-Keplerian rotation with increasing $\alpha$. The solutions terminate at a critical $\alpha_c$, which depends strongly on the ratio of specific heats so that the greatest possible $\alpha$ tends to zero for a nonrelativistic ideal gas. At the critical $\alpha_c$, the rotation of the flow vanishes and a purely radial inflow appears, which is not only supported by a pressure gradient but also by the viscous braking force, provided the flow is still turbulent in the absence of differential rotation. This limit is reminiscent of Bondi accretion (Bondi 1952) in the presence of effective turbulent viscosity. The transition from ADAFs to Bondi accretion nonetheless affords a suspicious fine-tuning in $\alpha$.

The $\beta$-viscosity law of Duschl et al. (2000), based on geometrical arguments in the absence of shock-limited turbulence, allows ADAF solutions for all reasonable values of $\beta$. The estimates of $\beta$ as inverse of the critical Reynolds number of the flow suggests small $\beta$s, for which the solu-
tions do not differ from \( \alpha \)-ADAFs. No transition to a Bondi-like flow is possible in this case and no upper bound on \( \beta \) exists.

We showed that it is possible to derive a shear-limited viscosity law from the \( \beta \)-viscosity mentioned above, for which the transition to nonrotating Bondi-like accretion with turbulent viscosity occurs naturally. The now-familiar ADAF solutions with sub-Keplerian rotation at \( \dot{\alpha} \to 0 \) join with a second branch of solutions at a maximal sustainable \( \dot{\alpha} \). Similar to the \( \alpha \)-ADAFs, no accretion flow with larger \( \dot{\alpha} \) are possible. The second solution branch is a hot, slowly rotating, and rapidly accreting solution even for small values of \( \dot{\alpha} \). The transition to viscous Bondi accretion occurs from the slowly rotating to the nonrotating solution in the limit \( \dot{\alpha} \to 0 \), where \( \Omega \) is linear in \( \dot{\alpha} \). A finite viscous force is present in these flows even in the limit \( \dot{\alpha} = 0 \). In all cases, the connection of ADAFs to nonrotating flows depends on the presence of a radial viscous force.

The observation that ADAFs generally possess positive Bernoulli numbers led to the idea (Narayan & Yi 1995a; Blandford & Begelman 1999) that ADAFs are good candidates for the production of outflows. In that way, the accretion flow loses energy to the outflow and the remaining material is left with negative Bernoulli numbers and is gravitationally bound to the central accreting mass. We confirm that statement in the presence of a radial viscous force for \( \alpha \)- and \( \beta \)-ADAFs and show the back-reaction of outflows on ADAFs for different outflow characteristics. Most noticeable is the cooling effect and the increased accretion velocity of the remaining ADAF as angular momentum and internal energy is carried away. Outflows with a minimal energy assumption for the extracted energy have to be fairly massive, \( p \geq 0.35 \), to lower the Bernoulli number and leave a bound flow in case of the shear-limited viscosity. It is much easier to get a bound flow if the minimal energy assumption is violated and the terminal velocity of the outflow is the escape speed from the origin of the outflowing material.

If a natural choice of the viscosity parameter—either \( \alpha \) or \( \dot{\alpha} \)—exists and, provided the strength of the radial viscous force can be estimated from isotropic turbulence, then outflows are inevitable for certain equations of state with ratios for specific heats close to \( 5/3 \). This conclusion is independent of arguments based on the positiveness of Bernoulli numbers for ADAFs.

The scenario of thin-disk evaporation in binary systems (Liu et al. 1999) or the transition from cooling flows to ADAFs in low-luminosity cores of elliptical galaxies (Quataert & Narayan 2000) can explain the existence of ADAFs in these systems. The formation of an ADAF, which is the most promising model for the spectral energy distribution of Sgr A* in the Galactic center, cannot proceed in either way. We suggest that an ADAF in the Galactic center forms out of stellar wind infall (Coker et al. 1999) and the transfer of kinetic energy of the infall into internal energy of an advection-dominated flow. The even larger Bernoulli numbers produced in this way would give rise to subsequent outflows and reduce the mass accretion rate inferred from infall calculations by Coker et al. (1999) to the smaller accretion rates predicted from spectral fitting (Quataert & Narayan 1999) of ADAF models to Sgr A*.

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APPENDIX

THE RADIAL VISCOUS FORCE

The radial viscous force in spherical coordinates is derived from the tensor divergence of the shear stress tensor \( t_{ij} \) and the bulk viscous force from the divergence of bulk viscosity and compression. The stress tensor is assumed to be proportional to the shear tensor \( \mathcal{S} \), which is defined as

\[
\mathcal{S}_{ij} = \frac{4}{3} (v_{i,j} + v_{j,i}) - \frac{4}{3} g_{ij} v^l_l,
\]

with the kinematic viscosity \( \mu = \nu \rho \) as a scalar function relating shear and stress \( t_{ij} = \mu \mathcal{S}_{ij} \). Here \( v_i \) are velocity vector components, \( g_{ij} \) the metric tensor, and \( v_{i,j} \) implies covariant differentiation of the velocity. The radial viscous force derived from the shear tensor is

\[
f_r = (2\mu \mathcal{S}^r)^{ij}_{,ij} = 4 \left( \frac{\partial}{\partial r} \frac{\mu r}{\partial r} \left( \frac{u}{r} \right) \right) + 3 \mu \frac{\partial}{\partial r} \left( \frac{u}{r} \right) \right),
\]

where axial symmetry and \( v_\theta = 0 \) is assumed and \( u \) is the radial velocity as used throughout the paper. The corresponding heating rate caused by this viscous friction is

\[
q^+_{rr} = \frac{4}{3} \mu r^2 \left( \frac{\partial}{\partial r} \left( \frac{u}{r} \right) \right)^2.
\]

The contribution from compression and bulk viscosity \( \zeta \) to the radial viscous force is

\[
f^\text{bulk} = (\zeta v^l_l)_{,r} = \frac{\partial}{\partial r} \left[ \frac{\zeta}{r^2} \frac{\partial}{\partial r} (ur^2) \right],
\]

and the corresponding heating rate is

\[
q^\text{bulk}_{rr} = \left[ \frac{\zeta}{r^2} \frac{\partial}{\partial r} (ur^2) \right]^2.
\]
Multiplying by $r$ and performing polar integration of the force gives the radial viscous force used in equations (6) and (7),

$$F_{rr} = \frac{4r}{3} \left\{ \frac{\partial}{\partial r} \left[ \nu \Sigma \frac{\partial}{\partial r} \left( \frac{u}{r} \right) \right] + 3\nu \Sigma \frac{\partial}{\partial r} \left( \frac{u}{r} \right) \right\} \quad \text{(A6)}$$

and

$$F_{\text{bulk}} = r \frac{\partial}{\partial r} \left[ \frac{\nu \Sigma}{r^3} \frac{\partial}{\partial r} (ur^3) \right] \quad \text{(A7)}$$

with the replacement of the bulk viscosity by the effective turbulent viscosity $\nu$. The heating rates used in equation (13) are derived from equations (A3) and (A5) in the same way. The force measures $\eta$, $\eta_1$ used in the text have been omitted here.

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