Extension of distributions, scalings and renormalization of QFT on Riemannian manifolds.

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Abstract

Let $M$ be a smooth manifold and $X \subset M$ a closed subset of $M$. In this paper, we introduce a natural condition of moderate growth along $X$ for a distribution $t$ in $\mathcal{D}'(M \setminus X)$ and prove that this condition is equivalent to the existence of an extension of $t$ in $\mathcal{D}'(M)$ generalizing previous results of Meyer and Brunetti–Fredenhagen. When $X$ is a closed submanifold of $M$, we show that the concept of distributions with moderate growth coincides with weakly homogeneous distributions of Meyer which can be intrinsically defined. Then we renormalize products of distributions with functions tempered along $X$ and finally, using the whole analytical machinery developed, we give an existence proof of perturbative quantum field theories on Riemannian manifolds.

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Introduction.

Let us start with the following example which is discussed in [26, Example 9 p. 140] and actually goes back to Hadamard. We denote by $\Theta$ the Heaviside function (the indicator function of $\mathbb{R} \geq 0$), consider the function $x^{-1}\Theta(x)$ viewed as a distribution in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$. Obviously, the linear map

$$\varphi \mapsto \int_0^{\infty} \frac{\varphi(x)}{x} \, dx$$

is ill-defined if $\varphi(0) \neq 0$ since the integral $\int_0^{\infty} \frac{dx}{x}$ diverges.
However, the integral \( \int_0^\infty dx x^{-1} \varphi(x) \) converges if \( \varphi(0) = 0 \) and an elementary estimate shows that \( x^{-1} \Theta(x) \) defines a linear functional on the ideal of functions \( x \mathcal{D}(\mathbb{R}) \) vanishing at 0. A test function \( \varphi \in \mathcal{D}(\mathbb{R}) \) being given, note that the following expression
\[
\lim_{\varepsilon \to 0} \int_\varepsilon^1 dx \frac{(\varphi(x) - \varphi(0))}{x} + \int_1^\infty dx \frac{\varphi(x)}{x}
\]
converges.

We thus define a renormalized distribution:
\[
x_+ = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} dx x^{-1} + \log(\varepsilon) \delta
\]
where we subtracted the distribution \( \log(\varepsilon) \delta \) supported at 0, which becomes singular when \( \varepsilon \to 0 \), called local counterterm. The renormalized distribution \( x_+ \in \mathcal{D}'(\mathbb{R}) \), called finite part of Hadamard, extends the linear functional \( x^{-1} \Theta(x) \in (x \mathcal{D}(\mathbb{R}))' \). Our example shows the most elementary situation where we can extend a distribution by an additive renormalization.

In what follows, \( M \) will always denote a smooth, paracompact and oriented manifold. In our paper, we investigate the following problem which has simple formulation: we are given a manifold \( M \) and a closed subset \( X \subset M \). We define a natural growth condition on \( t \in \mathcal{D}'(M \setminus X) \) which measures the singular behaviour near \( X \) and we address the following problems:

1. can we find a distribution \( \mathcal{T} \in \mathcal{D}'(M) \) s.t. the restriction of \( \mathcal{T} \) on \( M \setminus X \) coincides with \( t \),
2. can we construct a linear extension operator \( \mathcal{R} \), eventually give explicit formulas for \( \mathcal{R} \),
3. can we classify the different extension operators.

In general, the extension problem has no positive answer for a generic distribution \( t \) in \( \mathcal{D}'(M \setminus X) \) unless \( t \) has moderate growth when we approach the singular subset \( X \).

Distributions having moderate growth along a closed subset \( X \subset M \). If \( P \) is a differential operator with smooth coefficients on \( M \), and \( K \subset U \) a compact subset, we denote by \( \| \varphi \|_K \) (resp. \( \| \varphi \|_P \)) the seminorm \( \sup_{x \in K} |P \varphi(x)| \) (resp. \( \sup_{x \in U} |P \varphi(x)| \)). We also denote by \( d \) some arbitrary distance function induced by some choice of smooth metric on \( M \). For every open set \( V \subset M \), we denote by \( \mathcal{T}_{M \setminus X}(V) \) the set of distributions in \( \mathcal{D}'(V \setminus X) \) with moderate growth along \( X \) defined as follows:

**Definition 0.1.** A distribution \( t \in \mathcal{D}'(V \setminus X) \) has moderate growth along \( X \) if for all open relatively compact \( U \subset V \), there is a seminorm \( \| \cdot \|_P \) and a pair of constant \( (C, s) \in \mathbb{R}^2_{>0} \) such that
\[
|t(\varphi)| \leq C(1 + d(\text{supp } \varphi, X)^{-s}) \| \varphi \|_P.
\]
for all \( \varphi \in \mathcal{D}(U \setminus X) \).

**Remark:** If \( t \) were in \( \mathcal{D}'(M) \), we would have the same estimate without the divergent factor \( (1 + d(\text{supp } \varphi, X)^{-s}) \).

The space \( \mathcal{T}_{M \setminus X} \) is intrinsically defined since all metrics on \( M \) are locally equivalent. The first main theorem we shall prove is

**Theorem 0.1.** The three following claims are equivalent:

1. \( t \) has moderate growth along \( X \),
2. \( t \in \mathcal{D}'(M \setminus X) \) is extendible,
3. there is a family of functions \( (\beta_\lambda)_{\lambda \in (0, 1)} \subset C^\infty(M \setminus X) \), \( \beta_\lambda = 0 \) in a neighborhood of \( X \), \( \beta_\lambda \to 1 \) as \( \lambda \to 0 \) and a family of distributions \( (c_\lambda)_{\lambda \in (0, 1)} \) supported on \( X \) such that
\[
\lim_{\lambda \to 0} t \beta_\lambda - c_\lambda
\]
exists and defines an extension of \( t \) in \( \mathcal{D}'(M) \).
Our moderate growth condition is weaker than the hypothesis of [14, Lemma 3.3] and Theorem 0.1 can also be viewed as generalizations of Theorem [22, Thm 2.1 p. 48] and [4, Thm 5.2 p. 645] which only treat the extension problem in the case of a point. When \( X \) is a vector subspace of \( M = \mathbb{R}^n \), we prove in Theorem 0.1 that weakly homogeneous distributions in the sense of Meyer have moderate growth and are therefore extendible. In [10, Chapter 1], we proved that weakly homogeneous distributions along some vector subspace \( X \) are invariant by diffeomorphisms preserving \( X \) which implies that weakly homogeneous distributions along a submanifold \( X \subset M \) can be intrinsically defined.

In the third part of our paper, we apply our extension techniques to establish in Theorem 3.1 that the product of distributions in \( D'(M) \) with functions which are tempered along \( X \) (see definition 3.1 for the algebra \( M(X,M) \) of tempered functions) is renormalizable which implies that the space of extendible distributions or equivalently of distributions in \( T_{M\setminus X} \) is a left \( M(X,M) \)-module (Corollary 3.1).

Finally we apply our analytic machinery to the study of perturbative QFT on Riemannian manifolds. In QFT, one is interested in making sense of correlation functions denoted by \( \langle \phi^{i_1}(x_1) \cdots \phi^{i_n}(x_n) \rangle \) which are objects living in the configuration space \( M^n \) that can be expressed formally, using the Feynman rules, in terms of products of the form \( \prod_{1 \leq i < j \leq n} G(x_i, x_j)^{n_{ij}} \) where \( G \) is the Green function of \( \Delta_g + m^2 \) and is depicted pictorially by a graph with \( n \) labelled vertices \( \{1, \ldots, n\} \) where the vertices \( i \) and \( j \) are connected by \( n_{ij} \) lines. In the second main Theorem (Thm 4.2) of our paper, we prove that all Feynman amplitudes are renormalizable by a collection of extension maps \( (\mathcal{R}_n)_{n \in \mathbb{N}} \) where every map \( \mathcal{R}_n \) extends Feynman amplitudes living on the configuration space \( M^n \) minus all diagonals to distributions on \( M^n \) and the maps \( (\mathcal{R}_n)_{n \in \mathbb{N}} \) satisfy some axioms (definition 4.1) which are due to N. Nikolov [23]. This gives a different approach to Costello’s existence Theorem [8,9] for perturbative QFT on Riemannian manifolds.

Related works. In the literature, the idea to consider extendible distributions really goes back to Lojasiewicz [18] and tempered functions already appear in the work of B. Malgrange [19,20]. However, the first general definition of a tempered distribution on any open set \( U \) requires in some manifold \( M \) is due to M. Kashiwara, a distribution is tempered if it is extendible on \( \overline{U} \) where the following property holds (Theorem 0.1): these distributions are in \( T_{M\setminus \partial U} \) i.e. have moderate growth along \( \partial U \). His approach was then extended in [13,16,17]. Tempered functions and distributions were also recently studied in the context of real algebraic geometry [1,6] with applications in representation theory. More recently, a different approach to the extension problem in terms of scaling was developed by Meyer in his book [22], his purpose was to study the singular behaviour at given points of irregular functions with applications in multifractal analysis [13]. Our goal in this paper is to revive some techniques in analysis originally developed by H. Whitney [34] which were then improved by Malgrange and Lojasiewicz, to compare these techniques with the approach by scaling of Meyer [10,22] and finally show their relevance in solving the problem of constructing a perturbative quantum field theory on a Riemannian manifold.

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1 The extension of distributions.

1.1 Proof of Theorem 0.1

Localization on open charts by a partition of unity. We shall reduce the proof of (1) \( \Leftrightarrow \) (2) in Theorem 0.1 to the case where \( M = \mathbb{R}^n \), \( X \) is a compact set contained in a larger compact \( K \) and \( t \in D'(\mathbb{R}^n \setminus X) \) vanishes outside \( K \), this condition reads \( t \in D'_K(\mathbb{R}^n \setminus X) \). The first step is to localize the problem by a partition of unity. Choose a locally finite cover of \( M \) by relatively compact open charts \( (U_i)_i \) and a subordinated partition of unity \( (\varphi_i)_i \) s.t. \( \sum_i \varphi_i = 1 \). Denote by \( t_i \) the restriction \( t \big|_{U_i} \) and \( K_i = \text{supp} \ \varphi_i \subset U_i \). For all \( \varphi \in D(U), t \in D'(U \setminus X) \) has moderate growth implies the same property for \( t\varphi \in D'(U \setminus X) \); therefore each \( t\varphi_i \big|_{U_i \setminus X} \) is in \( D'_K(U_i \setminus (X \cap K_i)) \), \( t\varphi_i \) vanishes outside \( K_i \) and has
moderate growth along $X$. Hence it suffices to extend $t\varphi|_{U_i\setminus X}$ in each $U_i$ in such a way that the extension is supported by $K_i$. Call $t\varphi_i$ such extension in $\mathcal{E}'(U_i)$ then the locally finite sum $\tilde{t} = \sum_t t\varphi_i \in \mathcal{D}'(M)$ is a well defined extension of $t$.

**Working on $\mathbb{R}^n$.** The second step is to use local charts to work on $\mathbb{R}^n$. On every open set $(U_i)$, let $\psi_i : U_i \mapsto V \subset \mathbb{R}^n$ denote the corresponding chart then the pushforward $\psi_* (t\varphi_i)$ is in $\mathcal{D}'(\psi_i(K_i) \setminus \psi_i(X \cap K_i))$. Actually the compact set $\psi_i(X \cap K_i)$ is in the interior of $V$, since $(K_i \cap X) \subset \text{int}(U_i)$ and $\psi_i$ is a diffeomorphism. Therefore the distribution $\psi_* (t\varphi_i)$ is an element of $\mathcal{D}'_{K_i}((\mathbb{R}^n \setminus \psi_i(X \cap K_i))$ and we may reduce the proof of our theorem to the case where we have a distribution $t \in \mathcal{D}'_{K}((\mathbb{R}^n \setminus X)$ with moderate growth along $X$ where $X \subset K$ are **compact subsets** of $\mathbb{R}^n$. In the sequel, we use the seminorms $\|\varphi\|_m = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m} |\partial^\alpha_x \varphi(x)|$ and $\|\varphi\|_{K} = \sup_{x \in K, |\alpha| \leq m} |\partial^\alpha_x \varphi(x)|$ where $K$ runs over the compact subsets of $\mathbb{R}^n$. Let $\mathcal{I}(X, \mathbb{R}^n) = \{ \varphi \text{ s.t. supp } \varphi \cap X = \emptyset \} \subset C^\infty(\mathbb{R}^n)$, since $t$ vanishes outside some compact set $K$, the moderate growth condition now reads

$$\exists (C, s) \in \mathbb{R}_{\geq 0} \times ||K|| \text{ s.t. } \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |t(\varphi)| \leq C(1 + d(\text{supp } \varphi, X)^{-s}))||\varphi||_{K}.$$  

(6)

**Theorem 1.1.** Let $X \subset K$ be compact subsets of $\mathbb{R}^n$, then $t \in \mathcal{D}'_{K}((\mathbb{R}^n \setminus X)$ is extendible in $\mathcal{D}'_{K}((\mathbb{R}^n)$ if and only if $t$ has moderate growth along $X$.

**Proof.** We first prove a weaker equivalence: $t$ is extendible iff the estimate (6) holds with $s = 0$.

Assume the problem is solved and that we could find an extension $\tilde{t} \in \mathcal{D}'_{K}((\mathbb{R}^n)$ of $t$. Observe that $\forall \varphi \in \mathcal{I}, t(\varphi) = \tilde{t}(\varphi)$ then by definition $\tilde{t}$ is a linear continuous functional on $C^\infty(\mathbb{R}^n)$ equipped with the Fréchet topology, thus it induces a linear continuous map on the vector subspace $\mathcal{I}(X, \mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$:

$$\exists C \in \mathbb{R}_{\geq 0}, ||K|| \text{ s.t. } \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |t(\varphi)| = |\tilde{t}(\varphi)| \leq C||\varphi||_{K}. $$

Therefore, if $t$ is extendible then estimate (6) is satisfied with $s = 0$ and $t$ has moderate growth along $X$.

Conversely, if $\exists C \in \mathbb{R}_{\geq 0}, ||K|| \text{ s.t. } \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |t(\varphi)| \leq C||\varphi||_{K} \text{, then by the Hahn–Banach theorem [21] Thm 6.4. p. 46], we can extend $t$ as a linear continuous mapping $\tilde{t}$ on $C^\infty(\mathbb{R}^n)$ which satisfies the above estimate hence $\tilde{t} \in \mathcal{D}'_{K}((\mathbb{R}^n)$). Therefore to prove that $t$ has moderate growth implies that $t$ is extendible in $\mathcal{D}'_{K}((\mathbb{R}^n)$, it suffices to show that

$$\exists C \in \mathbb{R}_{\geq 0}, ||K|| \text{ s.t. } \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |t(\varphi)| \leq C(1 + d(\text{supp } \varphi, X)^{-s}))||\varphi||_{K}. $$

(6)

Let us admit the following central technical Lemma whose proof will be given later:

**Lemma 1.1.** For every integer $d \in \mathbb{N}$, let $\mathcal{I}^{m+d} (X, \mathbb{R}^n)$ denote the closed ideal of functions of regularity $C^{m+d}$ which vanish at order $m + d$ on $X$. Then there is a function $\chi_\lambda \in C^\infty(\mathbb{R}^n)$ parametrized by $\lambda \in (0,1]$ s.t. $\chi_\lambda = 1$ (resp $\chi_\lambda = 0$) when $d(x, X) \leq \frac{1}{\lambda}$ (resp $d(x, X) \geq \lambda$)

$$\exists \tilde{C}, \forall \lambda \in (0,1], \forall \varphi \in \mathcal{I}^{m+d}(X, \mathbb{R}^n), ||\chi_\lambda \varphi||_{m} \leq \tilde{C} \lambda^d ||\varphi||_{m+d}$$

(7)

where the constant $\tilde{C}$ does not depend on $\varphi, \lambda$.

If $s = 0$, then we know that there is an extension by Hahn Banach therefore we shall treat the case where $s > 0$. Our idea is to absorb the divergence by a dyadic decomposition:

$$\forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), \exists N \text{ s.t. } \chi_2^{-N} \varphi = 0$$

$$\Rightarrow t(\varphi) = t((1 - \chi_2^{-N}) \varphi)$$

$$\Rightarrow t(\varphi) = \sum_{j=0}^{N-1} t((\chi_2^{-j} - \chi_2^{-j+1}) \varphi) + t((1 - \chi_1) \varphi)$$

4
We easily estimate \( l((1 - \chi_1)\varphi) \): \( \forall \varphi \in C^\infty(\mathbb{R}^n), \|l((1 - \chi_1)\varphi)\| \leq C\|\varphi\|_m \) for some constant \( C \) since the support of \( 1 - \chi_1 \) does not meet \( X \). Choose \( d \in \mathbb{N}^* \) such that \( d > s \), then:

\[
|l(\chi_1\varphi)| \leq \sum_{j=0}^{N-1} |l((\chi_{2^{-j}} - \chi_{2^{-j-1}})\varphi)| \\
\leq C \sum_{j=1}^{N} (1 + d(\text{supp } \varphi(\chi_{2^{-j}} - \chi_{2^{-j-1}}), X)^{-s}) \|\varphi\|^{K_m}, \text{ by moderate growth} \\
\leq C \sum_{j=1}^{N} (1 + 2^{s(j+4)})(2^{-jd} + 2^{-(j+1)d})\tilde{C}\|\varphi\|^{K_{m+d}}, \text{ by the technical Lemma} \\
\leq C'\|\varphi\|^{K_{m+d}}
\]

for \( C' = \tilde{C}(1 + 2^{-d}) \sum_{j=1}^{\infty} 2^{-jd}(1 + 2^{(j+4)s}) < +\infty \) which is independent of \( N \) and \( \varphi \).

We now prove Lemma 1.1.

**Proof.** Choose \( \phi \geq 0 \) s.t. \( \int \phi = 1, \phi = 0 \) if \( |x| \geq \frac{3}{2} \) then set \( \phi_{\lambda} = \lambda^{-n}\phi(\lambda^{-1} \cdot) \) and \( \alpha_{\lambda} \) to be the characteristic function of the set \( \{ x \text{ s.t. } d(x, X) \leq \frac{1}{\lambda} \} \) then the convolution product \( \phi_{\lambda} \ast \alpha_{\lambda}(x) = 1 \) if \( d(x, X) \leq \frac{1}{\lambda} \) and equals 0 if \( d(x, X) > \lambda \). Since by Leibniz rule one has \( \partial^\alpha(\chi_{\lambda}\varphi)(x) = \sum_{|k| \leq |\alpha|} \binom{\alpha}{k} \partial^k \chi_{\lambda}\partial^{\alpha-k}\varphi(x) \), it suffices to estimate each term \( \partial^k \chi_{\lambda}\partial^{\alpha-k}\varphi(x) \) of the above sum.

For all multi-index \( k \), there is some constant \( C_k \) such that \( \forall x \in \mathbb{R}^n \setminus X, |\partial^k \chi| \leq \frac{C_k}{\lambda^{|k|}} \) and \( \text{supp } \partial^k \chi \subset \{ d(x, X) \leq \lambda \} \). Therefore for all \( \varphi \in \mathcal{I}^{m+d}(X, \mathbb{R}^n) \), for all \( x \in \text{supp } \partial^k \chi \partial^{\alpha-k}\varphi, y \in X \text{ such that } d(x, X) = |x - y|, \) we find that \( \partial^{\alpha-k}\varphi \) vanishes at \( y \) at order \( |k| + d \) therefore:

\[
\partial^{\alpha-k}\varphi(x) = \sum_{|\beta| = |k| + d} (x - y)^\beta R_\beta(x)
\]

where the right hand side is just the integral remainder in Taylor’s expansion of \( \partial^{\alpha-k}\varphi \) around \( y \). Hence:

\[
|\partial^k \chi_{\lambda}\partial^{\alpha-k}\varphi(x)| \leq \frac{C_k}{\lambda^{|k|}} \sum_{|\beta| = |k| + d} |(x - y)^\beta R_\beta(x)|.
\]

It is easy to see that \( R_\beta \) only depends on the Jets of \( \varphi \) of order \( \leq m + d \). Hence

\[
|\partial^k \chi_{\lambda}\partial^{\alpha-k}\varphi(x)| \leq C_k \lambda^d \sup_{x \in K, d(x, X) \leq \lambda, |\beta| = |k| + d} \sum |R_\beta(x)|
\]

and the conclusion follows easily.

Our partition of unity argument together with the result of Theorem 1.1 imply that (1) \( \iff \) (2) in Theorem 0.1.

### 1.2 Renormalizations and the Whitney extension Theorem.

The goal of this subsection is to replace the use of Hahn Banach theorem by a more constructive argument. First, we discuss a particular case of extension where there is some canonical choice for \( \mathcal{I} \).
Remark on the extension of positive measures with locally finite mass. The following proposition is inspired by some results of Skoda [32]. Let \( \mu \) be a positive measure in \( M \setminus X \), then we say that \( \mu \) has locally finite mass if:

\[
\forall K \subset M \text{ compact }, \exists C_K, \forall \varphi \in \mathcal{D}_K(M \setminus X), \varphi \geq 0, 0 \leq \mu(\varphi) \leq C_K \| \varphi \|_0.
\]

**Proposition 1.1.** Let \( \mu \) be a positive measure in \( M \setminus X \). If \( \mu \) has locally finite mass then \( \mu \) has a canonical extension in the space of positive measures.

**Proof.** By an obvious regularization argument, we can extend \( \mu \) to the space \( C^0_0(M \setminus X) \) of compactly supported functions of regularity \( C^0_0 \). Choose a family \( \chi_\lambda \) as in the main technical Lemma [11] which satisfies \( \chi_\lambda \geq 0, \chi_\lambda = 1 \) if \( d(x,X) \leq \frac{1}{\lambda} \) and \( \chi_\lambda = 0 \) when \( d(x,X) \geq \lambda \). Then for all \( \varphi \in C^0_0(M), \varphi \geq 0 \), the sequence \( \mu((1 - \chi_\lambda)\varphi)_n \) is increasing and bounded by \( C_K \| \varphi \|_0 \) where \( K \) is any compact set which contains the support of \( \varphi \). Therefore for each \( \varphi \geq 0 \), \( \lim_{n \to \infty} \mu((1 - \chi_\lambda)\varphi) \) exists. It is easy to conclude using the fact that \( C^0_0(M) \) is spanned by non negative functions.

**Constructive extension operator instead of Hahn Banach.** Recall we denote by \( \mathcal{I}(X,\mathbb{R}^n) \) the smooth functions vanishing in some neighborhood of \( X \). In the proof of Theorem [11] we showed that if \( t \) were extendible equivalently if \( t \) satisfies the moderate growth condition then:

\[
\exists (C,m), \forall \varphi \in \mathcal{I}(X,\mathbb{R}^n), |t(\varphi)| \leq C \| \varphi \|_m^K \tag{8}
\]

Therefore \( t \) defines a linear functional on \( \mathcal{I}(X,\mathbb{R}^n) \) for the induced topology of \( C^\infty(\mathbb{R}^n) \) and can be extended by Hahn Banach which is a non constructive argument and does not imply the existence of a linear extension operator \( t \in \mathcal{D}_K^0(M \setminus X) \mapsto \mathcal{I}_t \in \mathcal{D}_K^0(\mathbb{R}^n) \).

Denote by \( T^m(X,\mathbb{R}^n) \) the space of \( C^m \) functions which vanish on \( X \) together with all their derivatives of order less than \( m \). \( T^m(X,\mathbb{R}^n) \) is a closed ideal in \( C^m(\mathbb{R}^n) \). To construct a linear extension operator, we have to prove first that \( t \) extends by continuity to some element \( t_m \) in the topological dual \( T^m(X,\mathbb{R}^n)' \) of \( T^m(X,\mathbb{R}^n) \subset C^m(\mathbb{R}^n) \).

**Lemma 1.2.** \( t \) satisfies [5] if and only if \( t \) uniquely extends by continuity to an element \( t_m \) in \( T^m(X,\mathbb{R}^n)' \):

\[
\forall \varphi \in T^m(X,\mathbb{R}^n), t_m(\varphi) = \lim_{\lambda \to 0} \lim_{\varepsilon \to 0} t((1 - \chi_\lambda)\phi_\varepsilon \ast \varphi) \tag{9}
\]

for the family of cut-off functions \((\chi_\lambda)_\lambda \) defined in Lemma [11] and a mollifier \( \phi_\varepsilon \).

**Proof.** It suffices to prove that the space of \( C^\infty \) functions whose support does not meet \( X \) is dense in \( T^m(X,\mathbb{R}^n) \) in the \( C^\infty \) topology. In fact, we prove more, let \( \phi_\varepsilon \) be a smooth mollifier, then by a classical regularization argument, we have \( \lim_{\lambda \to 0}(1 - \chi_\lambda)\phi_\varepsilon \ast \varphi = (1 - \chi_\lambda)\varphi \in C^\infty(\mathbb{R}^n) \) for all \( \varphi \in C^m(\mathbb{R}^n) \) and \( \lim_{\lambda \to 0}(1 - \chi_\lambda)\varphi \to \varphi \) in \( T^m(X,\mathbb{R}^n) \). By the technical Lemma [11] (see [20] p. 11), we have

\[
\forall \varphi \in T^m(X,\mathbb{R}^n), \| \chi_\lambda \varphi \|_m^K \leq \tilde{C} \| \varphi \|_m^{K \cap \{d(x,X) \leq \lambda\}} \to 0
\]

when \( \lambda \to 0 \) therefore \( \varphi = \lim_{\lambda \to 0}(1 - \chi_\lambda)\varphi \) in the \( C^\infty \) topology. Finally this proves \( T^m(X,\mathbb{R}^n) \) is the closure in \( C^m(\mathbb{R}^n) \) of the space of \( C^\infty \) functions whose support does not meet \( X \).}

Set \( \beta_\lambda = 1 - \chi_\lambda \), from the above Theorem we can make a notation abuse and say that \( \lim_{\lambda \to 0} t_{\beta_\lambda} \in T^m(X,\mathbb{R}^n)' \) if \( t \) satisfies the estimate [8] (we just forget about the mollifier). The idea is to compose \( \lim_{\lambda \to 0} t_{\beta_\lambda} \) with a continuous projection \( I_m : C^m(\mathbb{R}^n) \to T^m(X,\mathbb{R}^n) \) so that \( \lim_{\lambda \to 0} t_{\beta_\lambda} \circ I_m \) defines an extension of \( t \). Dually, every compactly supported distribution of order \( m \) induces by restriction a linear functional on \( T^m(X,\mathbb{R}^n) \), in other words we have a surjective linear map \( p : \mathcal{E}_m'(\mathbb{R}^n) \to T^m(X,\mathbb{R}^n)' \). We want to construct a linear extension operator \( \mathcal{R} \) called a renormalization map from \( T^m(X,\mathbb{R}^n)' \) to \( \mathcal{E}_m'(\mathbb{R}^n) \) such that \( p \circ \mathcal{R} : T^m(X,\mathbb{R}^n)' \to T^m(X,\mathbb{R}^n)' \) is the identity map. Then it is immediate to note that the transpose of \( \mathcal{R} \) is the projection \( I_m \).

Denote by \( \mathcal{E}^m(X) \) the space of differentiable functions of order \( m \) in the sense of Whitney [20], Definition 2.3 p. 3, [2] p. 146.
Theorem 1.2. There is a bijection between:

- the space of renormalization maps
- the space of decompositions of $C^m(\mathbb{R}^n)$ in direct sum $C^m(\mathbb{R}^n) = \mathcal{I}^m(X, \mathbb{R}^n) \oplus B$ where $B$ is a closed subspace of $C^m$ which we call renormalization scheme
- the space of continuous linear splittings of the exact sequence

\[
0 \rightarrow \mathcal{I}^m(X, \mathbb{R}^n) \rightarrow C^m(\mathbb{R}^n) \stackrel{\delta}{\rightarrow} E^m(X) \rightarrow 0.
\]  \hfill (10)

Proof. The exactness of (10) and the existence of linear continuous splittings of (10) is a consequence of the Whitney extension theorem (see [21], p. 10, [2], Thm 2.3 p. 146). Since (10) is a continuous exact sequence of Fréchet spaces, the dual sequence:

\[
0 \rightarrow E'_m,\mathcal{X}(\mathbb{R}^n) \rightarrow E'_m(\mathbb{R}^n) \stackrel{\delta}{\rightarrow} \mathcal{I}'^m(X, \mathbb{R}^n)' \rightarrow 0
\]  \hfill (11)

is exact [21] Prop 26.4 p. 308].

$T$ is a linear splitting of (10)

- $\psi T \circ q$ is a continuous projector on the closed subspace $B = \text{ran}(T)$
- $C^m(\mathbb{R}^n) = B \oplus \mathcal{I}^m(X, \mathbb{R}^n)$ where the projection $I = T \circ q$ on $\mathcal{I}^m(X, \mathbb{R}^n)$ is denoted by $I_m$
- $\Rightarrow \mathcal{R} = I_m$ splits the dual exact sequence (11).

\[\square\]

The Whitney extension Theorem, formal neighborhoods and extendible distributions. Let us give several interpretations of the result of Theorem 1.2. First, the reader can think of the direct sum decomposition as a way to decompose a $C^m$ function as a sum of a “Taylor remainder” which vanishes at order $m$ on $X$ and a “Taylor polynomial” in $B$. If $X$ were a point, $E^m(X)$ is isomorphic to the space $\mathbb{R}_m[X_1, \ldots ,X_n]$ of polynomials of degree $m$ in $n$ variables, we can choose $B = \mathbb{R}_m[x_1, \ldots , x_n]$ and the decomposition $B + \mathcal{I}^m$ is given by Taylor’s formula. For $\varphi \in C^m(\mathbb{R}^n)$, one can think of $q(\varphi) \in E^m(X) \simeq C^m(\mathbb{R}^n)/\mathcal{I}^m(X, \mathbb{R}^n)$ as the restriction of $\varphi$ to the infinitesimal neighborhood of $X$ of order $m$. More generally, let $\mathcal{I}^\infty(X, \mathbb{R}^n)$ be the closed ideal of functions in $C^\infty(\mathbb{R}^n)$ which vanish at infinite order on $X$, this is a nuclear Fréchet space since it is a closed subspace of the nuclear Fréchet space $C^\infty(\mathbb{R}^n)$. We can think of the space $E(X)$ of $C^\infty$ functions in the sense of Whitney as some sort of $\infty$-jets in “the transverse directions” to $X$ since by the Whitney extension theorem, we have a continuous exact sequence of nuclear Fréchet spaces:

\[
0 \rightarrow \mathcal{I}^\infty(X, \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n) \rightarrow E(X) \rightarrow 0
\]  \hfill (12)

which implies that $E(X)$ is the quotient space $C^\infty(\mathbb{R}^n)/\mathcal{I}^\infty(X, \mathbb{R}^n)$. When $X$ is a submanifold of $\mathbb{R}^n$, it is interesting to think of $E(X)$ as smooth functions restricted to the formal neighborhood of $X$. And the formal neighborhood of $X$ is then defined as the topological dual of $E(X)$ which is nothing but the space of distributions $E_X(\mathbb{R}^n)$ with compact support contained in $X$ and fits in the continuous dual exact sequence of DNF spaces [6] appendix A:

\[
0 \rightarrow E'_X(\mathbb{R}^n) \rightarrow E'(\mathbb{R}^n) \rightarrow E(X)/E'_X(\mathbb{R}^n) \rightarrow 0
\]  \hfill (13)

where the quotient space $E(X)/E'_X(\mathbb{R}^n)$ should be interpreted as the space of distributions in $\mathcal{D}'(\mathbb{R}^n \setminus X)$ which are extendible in $E(X)$ and the continuous map $E'(\mathbb{R}^n) \rightarrow E(X)/E'_X(\mathbb{R}^n)$ is in fact the transpose of the inclusion map $\mathbb{R}^n \setminus X \hookrightarrow \mathbb{R}^n$. Another nice consequence of the theory of nuclear Fréchet spaces is that the space of extendible distributions is a DNF space.

The renormalization group. We also define the renormalization group $G$ as the collection of linear, continuous, bijective maps from $C^m(\mathbb{R}^n)$ to itself preserving $\mathcal{I}^m(X, \mathbb{R}^n)$. Note that $g \in G \Rightarrow g^{-1}$ is continuous by the open mapping theorem hence $G$ is well defined as a group. Let $\mathcal{R}$ be a renormalization map corresponding to a projection $I_m$. For any element $g \in G$, we define the action of $g$ on $\mathcal{R}$ as follows: $\forall t \in \mathcal{I}^m(X, \mathbb{R}^n)'$, $g.Rt(\varphi) = Rt(g(\varphi)) = t(I_m \circ g(\varphi))$ where $Rt(g) \in E'(\mathbb{R}^n)$ is an extension of $t \in \mathcal{I}^m(X, \mathbb{R}^n)'$ since $g$ preserves $\mathcal{I}^m(X, \mathbb{R}^n)$. 
Renormalization as subtraction of counterterms. Assume we choose a renormalization scheme. We denote by $P_n = 1d - I_m$ the projection from $C^m$ to the closed subspace $B \subset C^m$ which plays the role of the Taylor polynomials. From the above theorem and recall $\beta_\lambda = 1 - \chi_\lambda$ where $\chi_\lambda$ is the function of Lemma 1.1.

Proposition 1.2. If $t$ satisfies the estimate \[ \exists \] then:

$\forall \varphi \in C^\infty(\mathbb{R}^n), \mathcal{T}(\varphi) = \lim_{\lambda \to 0} t(\beta_\lambda I_m \varphi) = \lim_{\lambda \to 0} t(\beta_\lambda \varphi) - t(\beta_\lambda P_m \varphi)$ (14)

is a well defined extension of $t$.

We call such extension a renormalization. The divergences of $t(\beta_\lambda \varphi)$ come from the fact that $\varphi \notin I^m(X, \mathbb{R}^n)$, however these divergences are local in the sense they can be subtracted by the counterterm $t(\beta_\lambda P_m \varphi)$ which becomes singular when $\lambda \to 0$ and only depends on the restriction to $X$ of the $m$-jets of $\varphi$ (since $\varphi$ vanishes near $X$ implies that $\varphi \in I^m \implies P_m \varphi = 0$). By construction, the renormalization group $G$ acts on the space of all renormalizations of $t$.

1.3 Going back to the manifold case.

Difference between two extensions. Following the notations of [11] recall that $(U_i)$, was our locally finite open cover of $M$ by relatively compact sets. On each open set $U_i$, we defined a chart $\psi_i : U_i \mapsto V \subset \mathbb{R}^n$ and we considered a partition of unity $(\varphi_i)_i$ subordinated to $(U_i)_i$. Let $t \in D'(M \setminus X)$ be a distribution with moderate growth, then by Theorem 1.1 we may assume that:

$\forall U_i, \exists m_i \in \mathbb{N}, \exists C_i > 0, \forall \varphi \in C^\infty(\mathbb{R}^n \setminus X \cap \text{supp } \varphi_i), |\psi_i(t \varphi_i)(\varphi)| \leq C_i \| \varphi \|_m$. (15)

By Theorem 1.1 we may find an extension $\widetilde{t} = \sum_i t \varphi_i \in D'(M)$ in such a way that for every $i$, $t \varphi_i U_i$ has order $m_i$. If we prescribe the order of the extensions on every $U_i$, to be equal to $m_i \in \mathbb{N}$, then two extensions $t_1, t_2$ will differ on each $U_i$ by a distribution $t_1 - t_2 |_{U_i}$ of order $m_i$ supported on $X \cap U_i$.

How to renormalize in the manifold case? On each chart $\psi_i : U_i \mapsto V \subset \mathbb{R}^n$, we can extend $\psi_i(t \varphi_i) \in D'(V \setminus \psi_i(X \cap \text{supp } \varphi_i))$ by renormalization. In other words, by Proposition 1.2 there is a family of functions $\beta_\lambda(i) \in C^\infty(\mathbb{R}^n)$, $\beta_\lambda(i) \to 1$ and counterterms $c_\lambda(i) \in E^0_{\psi_i(X \cap \text{supp } \varphi_i)}(\mathbb{R}^n)$ such that $\lim_{\lambda \to 0} \psi_i(t \varphi_i) \beta_\lambda(i) - c_\lambda(i)$ is an extension of $\psi_i(t \varphi_i)$ in $E'(\mathbb{R}^n)$. Then setting

$\beta_\lambda = \sum_i \varphi_i \psi_i^* \beta_\lambda(i) \text{ and } c_\lambda = \sum_i \psi_i^* c_\lambda(i)$, (16)

we find that:

$t \beta_\lambda - c_\lambda = \sum_i t \varphi_i \psi_i^* \beta_\lambda(i) - \psi_i^* c_\lambda(i)$ (17)

converges to some extension of $t$ when $\lambda \to 0$. This proves (1) $\iff$ (3) in Theorem 1.1.

2 Moderate growth and scaling.

In this section, we compare two approaches that were developed to measure the singular behaviour of a distribution along a closed subset $X$: the moderate growth condition and the one used in [10, 22] in terms of scaling. We show that both approaches are equivalent when $X$ is a submanifold of $M$.

2.1 Weakly homogeneous distributions have moderate growth.

In this subsection, we work on $\mathbb{R}^n$ viewed as a product $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $n = n_1 + n_2$ and we adopt the following splitting of variables $x \in \mathbb{R}^n = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Here we establish the relationship between our definition of moderate growth and the one used by Yves Meyer [22] and the author [10] in terms
of scaling. First we scale in the transverse directions to a vector subspace $X = \mathbb{R}^{n_1} \times \{x_2 = 0\}$ of $\mathbb{R}^n$ with the maps $\Phi^\lambda : (x_1, x_2) \mapsto (\lambda x_1, x_2)$. By definition, the scalings act on $\mathcal{D}'(\mathbb{R}^n)$ by duality $(\Phi^\lambda t)(\varphi) = \lambda^{n_2} t(\Phi^{-1} \varphi)$. A distribution $t \in \mathcal{D}'(\mathbb{R}^n \setminus X)$ is said to be weakly homogeneous in $\mathcal{D}'(\mathbb{R}^n \setminus X)$ of degree $s$ if the family of distributions $\lambda^{-s} \Phi^\lambda t, \lambda \in (0, +\infty)$ is bounded in $\mathcal{D}'(\mathbb{R}^n \setminus X)$.

**Theorem 2.1.** If $t$ is weakly homogeneous of degree $s$ in $\mathcal{D}'(\mathbb{R}^n \setminus X)$ then $t$ has moderate growth along $X = \mathbb{R}^{n_1} \times \{x_2 = 0\}$. More precisely, for all compact subset $K \subset \mathbb{R}^n$ there is $(m, C) \in \mathbb{N} \times \mathbb{R}$ and a compact subset $B \subset \mathbb{R}^n$ containing $K$ s.t.

$$\forall \varphi \in \mathcal{D}_K(\mathbb{R}^n \setminus X), |t(\varphi)| \leq (1 + d(\text{supp } \varphi, X)^{s + n_2})\|\varphi\|^B_m. \quad (18)$$

It follows by Theorem [17] that such $t$ has an extension in $\mathcal{D}'(\mathbb{R}^n)$. Note that when $s + n_2 > 0$, we are in a trivial situation of moderate growth since the r.h.s. does not diverge.

**Proof.** The proof relies on the existence of a continuous partition of unity,

$$\int_0^\infty \frac{d\lambda}{\lambda} \psi(\lambda^{-1} x_2) = \int_0^\infty \frac{d\lambda}{\lambda} \Phi^{-1} \psi = 1$$

where $\psi(\lambda^{-1} x_2)$ is supported on the corona $\frac{1}{2} \leq |x_2| \leq 2\lambda$. Indeed, let $\chi \in C^\infty(\mathbb{R}^{n_2})$ be a function s.t. $\chi = 1$ (resp $\chi = 0$) when $|x_1| \leq \frac{1}{2}$ (resp $|x_1| \geq 2$) then set $\psi = -x_2 \frac{d\chi}{dx_2}$.

Fix a compact set $B = \{\sup_{i=1,2} |x_i| \leq L\}$, then for all test function $\varphi \in \mathcal{D}_B(\mathbb{R}^n \setminus X)$ we obviously have

$$\varphi = \int_\varepsilon^{2L} \frac{d\lambda}{\lambda} \left( \Phi^{\lambda^{-1}} \psi \right) \text{ for } \varepsilon \leq \frac{d(\text{supp } \varphi, X)}{2}.$$

since $\lambda \notin \left[ \frac{d(\text{supp } \varphi, X)}{2}, 2L \right] \Rightarrow \sup \left( \Phi^{\lambda^{-1}} \psi \right) \cap \sup (\varphi) = \emptyset$. Now it is obvious that

$$t(\varphi) = \int_{\frac{d(\text{supp } \varphi, X)}{2}}^{2L} \frac{d\lambda}{\lambda} t \left( \left( \Phi^{\lambda^{-1}} \psi \right) \varphi \right)$$

$$= \int_{\frac{d(\text{supp } \varphi, X)}{2}}^{2L} \frac{d\lambda}{\lambda} \lambda^{s + n_2} \left( \lambda^{-s} \Phi^\lambda t \right) \left( \psi \Phi^\lambda \varphi \right)$$

$$\Rightarrow |t(\varphi)| \leq \left( (2L)^{s + n_2} + \left( \frac{d(\text{supp } \varphi, X)}{2} \right)^{s + n_2} \right) \sup_{\lambda \leq 2L} \left| \lambda^{-s} \Phi^\lambda t \right| \left( \psi \Phi^\lambda \varphi \right)|$$

A simple calculation proves that $(\psi \Phi^\lambda \varphi)_{\lambda \leq 2L} \subset \mathcal{D}_\tilde{K}(\mathbb{R}^n \setminus X)$ for $\tilde{K} = \{(x_1, x_2) | |x_1| \leq L, \frac{1}{2} \leq |x_2| \leq 2\}$ and that:

$$\forall m \in \mathbb{N}, \exists C_m > 0, \forall \lambda, \|\psi \Phi^\lambda \varphi\|_m \leq C_m \|\varphi\|_m$$

therefore the family $(\psi \Phi^\lambda \varphi)_\lambda$ is bounded in the Fréchet space $\mathcal{D}_\tilde{K}(\mathbb{R}^n \setminus X)$.

The family $(\lambda^{-s} \Phi^\lambda t)_\lambda$ is weakly bounded in $(\mathcal{D}_\tilde{K}(\mathbb{R}^n \setminus X))'$ thus strongly bounded by the uniform boundedness principle since $\mathcal{D}_\tilde{K}(\mathbb{R}^n \setminus X)$ is Fréchet ([23] Thm 2.5 p. 44]):

$$\exists C' > 0, m \in \mathbb{N}, \forall \lambda, \forall \varphi \in \mathcal{D}_\tilde{K}(\mathbb{R}^n \setminus X), |(\lambda^{-s} \Phi^\lambda t) (\varphi)| \leq C' \|\varphi\|_m. \quad (19)$$

Therefore

$$\sup_{\lambda \leq 2L} \left| (\lambda^{-s} \Phi^\lambda t) (\psi \Phi^\lambda \varphi) \right| \leq C' \|\psi \Phi^\lambda \varphi\|_m$$

$$\leq C' C_m \|\varphi\|_m$$

$$\Rightarrow |t(\varphi)| \leq C(1 + d(\text{supp } \varphi, X)^{s + n_2}) \|\varphi\|_m$$

for some $C > 0$ independent of $\varphi \in \mathcal{D}_B(\mathbb{R}^n \setminus X)$.
3 Renormalized products.

Let $X \subset \mathbb{R}^n$ be some closed subset. In this section, we first define the class $\mathcal{M}(X, \mathbb{R}^n)$ of tempered functions along $X$:

**Definition 3.1.** $f$ is tempered along $X$ if

$$\forall m \in \mathbb{N}, \forall K \subset \mathbb{R}^n \text{ compact}, \exists (C_m, s) \in \mathbb{R}_+^2, \sup_{|x| \leq m} |\partial^s f(x)| \leq C(1 + d(x, X)^{-s}). \quad (20)$$

Tempered functions form an algebra by Leibniz rule. It is immediate that the definition can be generalized to some closed subset $X$ in a manifold $M$: we follow the notations of the partition of unity argument in [11.4] $f$ is tempered along $X$ i.e. $f \in \mathcal{M}(X, M)$ if in any local chart $\psi_i : U_i \subset M \mapsto V \subset \mathbb{R}^n$, $\psi_i^* \left( \phi_i f \right) \in \mathcal{M}(\psi_i(X), \mathbb{R}^n)$.

Then we establish a theorem about renormalized products:

**Theorem 3.1.** Let $M$ be a manifold and $X \subset M$ a closed subset. For all $f \in \mathcal{M}(X, M)$ and all $t \in D'(M)$, there exists a distribution $R(f t) \in D'(M)$ which coincides with the regular product $ft$ outside $X$.

Thanks to the partition of unity argument of [11.4] we may reduce to the case where $X$ is some closed subset of $M = \mathbb{R}^n$ hence $f \in \mathcal{M}(X, \mathbb{R}^n)$ and $t \in \mathcal{E}'(\mathbb{R}^n)$. By Theorem [11.4] distributions with moderate growth are extendible, therefore it suffices to prove that $ft$ has moderate growth along $X$ which is the content of the following proposition:

**Proposition 3.1.** Let $t \in \mathcal{D}'_K(\mathbb{R}^n \setminus X)$ and $f \in C^\infty(\mathbb{R}^n \setminus X)$ such that $(t, f)$ satisfy the estimates:

$$\exists (C, s_1) \in \mathbb{R}_+^2, \forall \phi \in \mathcal{I}(X, \mathbb{R}^n), |t|(|\phi|) \leq C(1 + d(\text{supp } \phi, X)^{-s_1})|||\phi||_m^K \quad (21)$$

$$\exists (C_m, s_2) \in \mathbb{R}_+^2, \forall x \in K \setminus X, \sup_{|x| \leq m} |\partial^s f(x)| \leq C_m(1 + d(x, X)^{-s_2}). \quad (22)$$

Then $ft$ satisfies the estimate:

$$\exists C', \forall \phi \in \mathcal{I}(X, \mathbb{R}^n), |ft|(|\phi|) \leq C'(1 + d(\text{supp } \phi, X)^{-s_1 + s_2})|||\phi||_m^K. \quad (23)$$

**Proof.** The claim follows from the estimate:

$$\forall \phi \in \mathcal{I}(X, \mathbb{R}^n), |ft|(|\phi|) \leq C(1 + d(\text{supp } \phi, X)^{-s_1})||f\phi||_m^K$$

$$\leq C C_m 2^m (1 + d(\text{supp } \phi, X)^{-s_1})(1 + d(\text{supp } \phi, X)^{-s_2})|||\phi||_m^K \quad \text{by Leibniz rule}$$

$$\leq 4 C C_m 2^m (1 + d(\text{supp } \phi, X)^{-s_1 + s_2})|||\phi||_m^K.$$}

**Example 3.1.** Our result shares some similarities with [22, Theorem 4.3 p. 85] where Meyer renormalizes the product of distributions $S_t \alpha$ at a point $x_0 \in \mathbb{R}^n$, $S_t \alpha = f p(x - x_0)^\gamma$ (Hadamard’s finite part), $t$ is a distribution which is weakly homogeneous of degree $\gamma$ at $x_0$ and $s + \gamma \notin \mathbb{N}$. He shows that the renormalized product $S_t \alpha$ is weakly homogeneous of degree $s + \gamma$ at $x_0$.

Let us recall that by Theorem [11.1] the space $\mathcal{T}_{\mathbb{R}^n \setminus X}(\mathbb{R}^n)$ of distributions with moderate growth along $X$ corresponds with the quotient space $\mathcal{D}'(\mathbb{R}^n)/\mathcal{D}'_X(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n \setminus X$ extendible on $\mathbb{R}^n$.

**Corollary 3.1.** $\mathcal{T}_M, X(M)$ is a left $\mathcal{M}(X, M)$ module.

This was also proved by Malgrange [19 Proposition 1 p. 4].

Let us consider a function $g \in C^\infty(\mathbb{R}^n)$, $X = \{ g = 0 \}$ and $g C^\infty(\mathbb{R}^n)$ is a closed ideal of $C^\infty(\mathbb{R}^n)$, then a result of Malgrange [20] inequality (2.1) p. 88 yields that $g$ satisfies the Lojasiewicz inequality:

$$\forall K \text{ compact }, \exists (C, s) \in \mathbb{R}_+^2, \forall x \in K, |g(x)| \geq C d(x, X)^s. \quad (24)$$

It follows by Leibniz rule that $f = g^{-1}$ must be tempered along $X$. We state and prove a specific case of "renormalized product" which is due to Malgrange [20 Thm 2.1 p. 100]:

10
Theorem 3.2. Let $M$ be a smooth paracompact manifold, let $f = g^{-1}$, $g \in C^\infty(M)$ such that the ideal $gC^\infty(M)$ is closed. Then

$$\forall T \in \mathcal{D}'(M), \exists S \in \mathcal{D}'(M) \text{ s.t. } gS = T$$

in particular, $S = fT$ outside $X$.

Beware that the renormalized product $S = fT$ is not uniquely defined, however it satisfies the equation $gS = T$ whereas without the closedness assumption on $gC^\infty(M)$, we would only have $gS = T$ modulo distributions supported by $X$.

Proof. By partition of unity, it suffices to prove that the linear map $m_g : t \in \mathcal{E}'(M) \mapsto gt \in \mathcal{E}'(M)$ is onto if $gC^\infty(M)$ is closed in $C^\infty(M)$. We will establish that $m_g$ has closed range and that $\text{ran}(m_g)$ is dense in $\mathcal{E}'(M)$.

$gC^\infty(M)$ is closed in $C^\infty(M)$ implies that the transposed map: $m_g^* : C^\infty(M) \to C^\infty(M)$ has closed range therefore $m_g$ has closed range since $C^\infty(M)$ is Fréchet and $\mathcal{E}'(M) = C^\infty(M)'$ (see [21] Thm 26.3 p. 307).

gC^\infty(M)$ is closed in $C^\infty(M)$ hence it is Fréchet. By the open mapping Theorem [21] Thm 8.5 p. 60, $m_g : C^\infty(M) \to gC^\infty(M)$ is a linear continuous, surjective map of Fréchet spaces hence $m_g$ is open. In terms of estimates, this implies that for any continuous seminorm $\|\|_m$ of $C^\infty(M)$, there is a continuous seminorm $\|\|_{m'}$ such that $\|f\|_m \leq \|g(f)\|_{m'}$ (see [20] inequality (2.2) p. 83), hence $g\varphi = 0 \implies \varphi = 0$. Then we conclude by the observation that $\text{ran}(m_g)^\perp = \{\varphi \in C^\infty(M) \text{ s.t. } \forall t \in \mathcal{E}'(M), gt(\varphi) = 0\} = \{\varphi \text{ s.t. } g\varphi = 0\} = \{0\} \implies \text{ran}(m_g)$ is everywhere dense in $\mathcal{E}'(\mathbb{R}^n)$. 

4 Renormalization of Feynman amplitudes in Euclidean quantum field theories.

4.1 Feynman amplitudes are extendible.

We give the main application of our extension techniques. Our approach to renormalization follows the philosophy of Brunetti–Fredenhagen [4, 5, 6]. Nikolov–Stora–Todorov [23] which goes back to [11, 12], and is based on the concept of extension of distributions. However, we will use the beautiful formalism of renormalization maps of N. Nikolov [23, 24] which is closest in spirit to the present paper. In what follows, we will always assume that $(M, g)$ is a smooth $d$-dimensional Riemannian manifold with Riemannian metric $g$. We denote by $\Delta_g$ the Laplace–Beltrami operator corresponding to $g$, and we consider the Green function $G \in \mathcal{D}'(M \times M)$ of the operator $\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. $G$ is the Schwartz kernel of the operator inverse of $\Delta_g + m^2$ ([31] Appendix 1) which always exists when $M$ is compact and $m^2 \notin \text{Spec}(\Delta_g)$. In the noncompact case, the general existence and uniqueness result for the Green function usually depends on the global properties of $\Delta_g$ and $(M, g)$. If $(M, g)$ has bounded geometry in the sense of [7] p. 33] and [27] (see also [31] Definition 1.1 Appendix 1, [50] Def 1.1 p. 3]), then one can find in [31] Appendix 1] conditions of spectral theoretic nature on $\Delta_g, m^2$ that imply the existence of an operator inverse $(\Delta_g + m^2)^{-1} : L^p(M) \to L^p(M), p \in (1, +\infty)$ whose Schwartz kernel is $G$.

However if $G$ exists, then we show a fundamental result about the asymptotics of $G$ near the diagonal:

Lemma 4.1. Let $(M, g)$ be a smooth Riemannian manifold and $\Delta_g$ the corresponding Laplace operator. If $G \in \mathcal{D}'(M \times M)$ is the fundamental solution of $\Delta_g + m^2$, then $G$ is tempered along $D_2 \subset M^2$.

Proof. Temperedness is a local property therefore it suffices to prove the Lemma for some compact domain $\Omega \times \Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ and $g$ is a Riemannian metric on $\mathbb{R}^d$. The differential operator $\Delta_g + m^2$ is elliptic with smooth coefficients, $G$ is a fundamental solution of $\Delta_g + m^2$ in particular it is a parametrix of $\Delta_g + m^2$ which implies it is an elliptic pseudodifferential operator with polyhomogeneous symbol ([29] Thm 2.7 p. 55]. Set $\mathcal{E}(x, z) = G(x, x + z)$, then by [29] Theorem 3.3 p. 58], there exists two sequences $(A_q(x, z))_q, (B_q(x, z))_q$ of functions smooth on $\Omega$ w.r.t. $x$ and real analytic on $\mathbb{S}^{d-1}$ w.r.t. $z$ such that $\mathcal{E}$
Definition 4.1. Along $\Omega$ runs over the open subsets of $M$ and $\partial$ runs over the same element. The configuration space $\mathcal{D}$ over the same element. The configuration space $\mathcal{D}$ along $\partial$ are extendible in $\mathcal{D}$. We introduce the vector space $\mathcal{O}$ generated by the Feynman amplitudes $G(x,z)$ and the corresponding big and small diagonals $D(I_{1, \ldots, n}), d(I_{1, \ldots, n})$ for simplicity.

**Theorem 4.1.** Let $(M,g)$ be a smooth Riemannian manifold, $\Delta_g$ the corresponding Laplace operator and $G$ the Green function of $\Delta_g + m^2$. Then all "Feynman amplitudes" of the form:

$$\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \in C^\infty(M^n \setminus D_n), n_{ij} \in \mathbb{N}$$

are extendible in $D'(M^n)$.

**Proof.** $d_{ij} \subset D_n \implies \forall s \geq 0, d(x, d_{ij})^{-s} \leq d(x, D_n)^{-s}$ together with the fact that $G(x_i, x_j)$ is tempered along $d_{ij}$ imply that $G(x_i, x_j) \in \mathcal{M}(D_n, M^n)$. Since $\mathcal{M}(D_n, M^n)$ is an algebra, $\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \in \mathcal{M}(D_n, M^n)$ and is therefore extendible on $M^n$ by Theorem 4.1.

4.2 Renormalization maps, locality and the factorization property.

**The vector subspace $\mathcal{O}(D_I, \cdot)$ generated by Feynman amplitudes.** In QFT, renormalization is not only extension of Feynman amplitudes in configuration space but our extension procedure should satisfy some consistency conditions in order to be compatible with the fundamental requirement of locality.

Recall that for any open subset $\Omega \subset M^I$, we denote by $\mathcal{M}(D_I, \Omega)$ the algebra of tempered functions along $D_I$. We introduce the vector space $\mathcal{O}(D_I, \Omega) \subset \mathcal{M}(D_I, \Omega)$ generated by the Feynman amplitudes

$$\mathcal{O}(D_I, \Omega) = \left\{ \left( \prod_{i < j \in I^2} G^{n_{ij}}(x_i, x_j) \right)^{n_{ij}} \right\}.$$  

**Axioms for renormalization maps: factorization property as a consequence of locality.** We define a collection of renormalization maps $(\mathcal{R}_{\Omega \subset M^I})_{\Omega \subset M^I}$, where $I$ runs over the finite subsets of $\mathbb{N}$ and $\Omega$ runs over the open subsets of $M$ which satisfy the following axioms which are simplified versions of those figuring in [21, 2.3 p. 12-14] [23, Section 5.33-35]:

**Definition 4.1.**

1. For every $I \subset \mathbb{N}, |I| < +\infty, \Omega \subset M^I, \mathcal{R}_{\Omega \subset M^I}$ is a linear extension operator:

$$\mathcal{R}_{\Omega \subset M^I} : \mathcal{O}(D_I, \Omega) \mapsto D'(\Omega).$$  

2. For all inclusion of open subsets $\Omega_1 \subset \Omega_2 \subset M^I$, we require that:

$$\forall f \in \mathcal{O}(D_I, \Omega_2), \forall \varphi \in D(\Omega_1)$$

$$\langle \mathcal{R}_{\Omega_2 \subset M^I}(f), \varphi \rangle = \langle \mathcal{R}_{\Omega_1 \subset M^I}(f), \varphi \rangle.$$
The renormalization maps satisfy the factorization property. If \((U, V)\) are disjoint open subsets of \(M\), and \((I, J)\) are disjoint finite subsets of \(\mathbb{N}\), \(\forall (f, g) \in \mathcal{O}(D_I, U^I) \times \mathcal{O}(D_J, V^J)\):

\[
\mathcal{R}_{(U^I \times V^J) \subset M^{I \cup J}} (f \otimes g) = \frac{\mathcal{R}_{(U^I \subset M^I}} (f) \otimes \mathcal{R}_{(V^J \subset M^J}} (g) \in \mathcal{D}'(U^I \times V^J)
\]

The most important property is the factorization property (3) which is imposed in [23, equation (2.2) p. 5].

**Remarks on the axioms of the Renormalization maps.** To define \(\mathcal{R}\) on \(M^I\), it suffices to define \(\mathcal{R}_{\Omega_i} \subset M^I\) for an open cover \((\Omega_i)_i\) of \(M^I\) (they do not necessarily coincide on the overlaps \(\Omega_i \cap \Omega_j\)) and glue the determinations by a partition of unity.

**Uniqueness property of renormalization maps.** The following Lemma is proved in [23, Lemmas 2.2, 2.3 p. 6] and tells us that if a collection of renormalization maps \((\mathcal{R}_{\Omega_i} \subset M^I)_{i,I}\) exists and satisfies the list of axioms then the restriction of \(\mathcal{R}_{M^n}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j))\) on \(M^n \setminus d_n\) would be uniquely determined by the renormalizations \(\mathcal{R}_{M^I}\) for all \(|I| < n\) because of the factorization axiom.

**Lemma 4.2.** Let \((\mathcal{R}_{\Omega_i} \subset M^I)_{i,I}\) be a collection of renormalization maps satisfying the axioms. Then for any Feynman amplitude \(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)\), the renormalization \(\mathcal{R}_{M^n\setminus d_n} M^I = \prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j))\) is uniquely determined by the renormalizations \(\mathcal{R}_{M^I}(\prod_{1 \leq i \in I^2} G^{n_{ij}}(x_i, x_j)))\) for all \(|I| < n\).

**Proof.** See [23, p. 6-7] for the detailed proof.

Beware that the above Lemma does not imply the existence of renormalization maps but only that they satisfy certain consistency conditions if they exist.

### 4.3 The existence Theorem for renormalization maps.

Now we give a short proof of the existence of renormalization maps on general Riemannian manifolds. Recall \((M, g)\) is a smooth Riemannian manifold, \(\Delta_g\) the corresponding Laplace operator, \(G\) the Green function of \(\Delta_g + n^2\) and for any open subset \(\Omega \subset M^I\), \(\mathcal{O}(D_I, \Omega)\) is the vector space generated by the Feynman amplitudes.

**Theorem 4.2.** There exists a collection of renormalization maps \((\mathcal{R}_{\Omega_i} \subset M^I)_{i,I}\) where \(I\) runs over the finite subsets of \(\mathbb{N}\) and \(\Omega\) runs over the open subsets of \(M^I\) which satisfies the axioms.

**Proof.** We proceed by induction on \(\mathbb{N}\).

It suffices to establish the existence of \(\mathcal{R}_{M^n}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j))\) for generic Feynman amplitudes \(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \in \mathcal{O}(D_n, M^n)\).

**Step 1.** We initialize our induction with \(\mathcal{R}_{M^I} : \mathcal{O}(D^I, M^2) \mapsto \mathcal{D}'(M^2)\) whose existence is guaranteed by Theorem 4.1.

**Step 2.** By Lemma 1.3 the complement \(M^n \setminus d_n\) of the small diagonal \(d_n\) in \(M^n\) is covered by open sets of the form \(C_{IJ} = M^n \setminus (\cup_{(i,j) \in I \times J} d_{ij})\) where \(I \cup J = \{1, \ldots, n\}\), \(I \cap J = \emptyset\). In the sequel, we write \(\mathcal{R}_{C_{IJ}}\) instead of \(\mathcal{R}_{C_{IJ} \subset M^n}\) for simplicity.

By factorization property, we also find that:

\[
\mathcal{R}_{C_{IJ}} \left( \prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j) \right) = \mathcal{R}_{M^I}(G_I) \mathcal{R}_{M^J}(G_J) \prod_{(i,j) \in I \times J} G^{n_{ij}}(x_i, x_j)
\]

\[
G_I = \prod_{(i < j) \in I^2} G^{n_{ij}}(x_i, x_j), \quad G_J = \prod_{(i < j) \in J^2} G^{n_{ij}}(x_i, x_j)
\]

therefore the renormalization map \(\mathcal{R}_{M^n \setminus d_n}\) is entirely determined by the renormalization maps \(\mathcal{R}_{M^I}\) for \(|I| \leq n - 1\) and the determination is in fact unique according to Lemma 4.2.
Step 3, in Lemma 4.3, we construct a partition of unity \((\chi_{IJ})_{IJ}\) of \(M^n \setminus d_n\) subordinated to the open cover \((C_{IJ})_{IJ}\), i.e. \(\text{supp} \chi_{IJ} \subset C_{IJ}\), \(\sum_{IJ} \chi_{IJ} = 1\) such that each \(\chi_{IJ}\) satisfies the essential property of being tempered along \(d_n\).

Step 4, the key idea is that the product \(\mathcal{R}_{IJ}(G_I)\mathcal{R}_{IJ}(G_J)\) is well defined in \(\mathcal{D}'(M^n)\) and the product \(\prod_{I,J} \mathcal{R}_{IJ}(G_I) \mathcal{R}_{IJ}(G_J)\) is tempered along \(\partial C_{IJ}\). Therefore

\[
\chi_{IJ}\mathcal{R}_{IJ}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)) = \chi_{IJ} \prod_{I,J} \mathcal{R}_{IJ}(G_I) \mathcal{R}_{IJ}(G_J) \in \mathcal{D}'(C_{IJ})
\]

is a product of tempered functions along \(\partial C_{IJ}\) with a distribution in \(\mathcal{D}'(M^n)\), therefore it has an extension \(\chi_{IJ}\mathcal{R}_{IJ}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)) \in \mathcal{D}'(M^n)\) by Theorem 3.1. By construction, \(\chi_{IJ}\) vanishes in some neighborhood of \(\partial C_{IJ} \setminus d_n\) in \(M^n \setminus d_n\) which implies that \(\chi_{IJ}\mathcal{R}_{IJ}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)) = 0\). Then we define \(\mathcal{R}_n(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j))\) to be the distribution

\[
\sum_{IJ} \chi_{IJ}\mathcal{R}_{IJ}(\prod_{1 \leq i < j \leq n} G^{n_{ij}}(x_i, x_j)).
\]

Covering lemma. The following Lemma is due to Popineau and Stora [23] Lemma 2.2 p. 6] [33] [25] and states that \(M^n \setminus d_n\) can be partitioned as a union of open sets on which the renormalization map \(\mathcal{R}_n\) can factorize.

Lemma 4.3. Let \(M\) be a smooth manifold and for all finite subsets \(I, J \subseteq \mathbb{N}\) s.t. \(I \cap J = \emptyset\), \(I \cup J = \{1, \ldots, n\}\), let \(C_{IJ} = \{(x_1, \ldots, x_n) \in M^n \mid \forall i, j \in I \times J, x_i \neq x_j\}\). Then

\[
\bigcup_{I \cap J = \emptyset, I \cup J = \{1, \ldots, n\}} C_{IJ} = M^n \setminus d_n.
\]

Proof. The key observation is the following, \((x_1, \ldots, x_n) \in d_n \iff \forall U\text{ neighborhood of } x_1, (x_1, \ldots, x_n) \in U^n\). On the contrary

\[
(x_1, \ldots, x_n) \notin d_n \implies \exists (U, V) \text{ open s.t. } U \cap V = \emptyset, \forall I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset \text{ s.t. } (x_1, \ldots, x_n) \in U^n \setminus V^n.
\]

It suffices to set \(\varepsilon = \inf_{1 \leq i \leq n} \{d(x_i, x_1) \text{ s.t. } d(x_i, x_1) > 0\}\) then let \(U = \{x \text{ s.t. } d(x, x_1) < \varepsilon/3\}\) and \(V = \{x \text{ s.t. } d(x, x_1) > 2\varepsilon/3\}\).

It follows that the complement \(M^n \setminus d_n\) of the small diagonal \(d_n\) in \(M^n\) is covered by open sets of the form \(C_{IJ} = M^n \setminus \{(x_1, \ldots, x_n) \mid I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset\}\)

Tempered partition of unity associated to the cover \((C_{IJ})_{JJ}\).

Lemma 4.4. Let \(M\) be a smooth manifold and let \((C_{IJ})_{IJ}\) be the cover of \(M^n \setminus d_n\) defined in Lemma 4.3 then there exists a partition of unity \((\chi_{IJ})_{IJ}\) subordinated to \((C_{IJ})_{IJ}\) such that every function \(\chi_{IJ}\) is tempered along \(d_n\).

Proof. We will first construct a partition of unity in some neighborhood \(\mathcal{N}\) of \(d_n\). Consider the normal bundle \(\mathcal{N}(d_n) \subset M^n\) of \(d_n\) in \(M^n\), by the tubular neighborhood Theorem, there is some neighborhood \(\mathcal{N}\) of \(d_n\) in \(M^n\) and a diffeomorphism \(\Phi : \mathcal{N} \rightarrow \Phi(\mathcal{N}) \subset \mathcal{N}(d_n) \subset M^n\) which identifies the neighborhood \(\Phi(\mathcal{N})\) of the zero section \(\bar{0} \subset \mathcal{N}(d_n) \subset M^n\) with \(\mathcal{N} \subset M^n\).

Let us denote by \(\Phi(\mathcal{N}) \setminus \bar{0}\) the neighborhood of the zero section \(\bar{0}\) deprived of \(\bar{0}\). Then \(\Phi(C_{IJ})_{IJ}\) forms a conical open cover of \(\Phi(\mathcal{N}) \setminus \bar{0}\). To see what happens in coordinates, let \(U_i\) be some cover of \(M\), we trivialize
our bundle over $U_i$, $\Psi_{U_i} : N(d_n \subset M)|_{U_i} \mapsto U_i \times \mathbb{R}^{(n-1)d}$ and we use local coordinates $(x, h_2, \ldots, h_n) \in U_i \times \mathbb{R}^{n(d-1)}$. The set $\Psi_{U_i} \circ \Phi(C_{IJ})$ has simple expression $\Psi_{U_i} \circ \Phi(C_{IJ}) = \bigcap_{(i,j) \in I \times J} \{ h_i - h_j \neq 0 \}$ and is therefore invariant under scalings $(x, h_2, \ldots, h_n) \mapsto (x, \lambda h_2, \ldots, \lambda h_n)$ and does not depend on $x \in U_i$. Therefore the sets $(\Psi_{U_i} \circ \Phi(C_{IJ}))_{i,j}$ is an open conical cover of $(\mathbb{R}^{d(n-1)} \setminus \{0\}) \times U_i$. Let $(\chi_{IJ})_{i,j}$ be the corresponding partition of unity, then we can choose every $\chi_{IJ}$ of the form $f_{IJ} \chi_{IJ}(\frac{h}{|h|})$, $|h| = \sqrt{\sum_{i=2}^{n} h_i^2}$, $f_{IJ} \in C^\infty(\mathbb{R}^{d(n-1)} \setminus \{0\})$ since $\Psi_{U_i} \circ \Phi(C_{IJ})$ is conical and does not depend on $x$. Therefore combining the Faà Di Bruno formula and the fact that $|\partial^n_h \chi_{IJ}(h)| \leq C_k(1 + |h|^{-|\alpha|})$ yields that:

$$|\partial^n_h \chi_{IJ}(h)| = |\partial^n_{f_{IJ}}(\frac{h}{|h|})| \leq C_k(1 + |h|^{-|\alpha|})$$

which implies that $\chi_{IJ}$ is tempered along $U_i \times \{0\}$ therefore $\Psi_{U_i}^* \chi_{IJ}$ is tempered along the zero section $\Psi_{U_i} \subset \mathbb{R}^{d(n-1)}$. Let $(\varphi_i)_i$ be a partition of unity subordinated to the cover $(U_i)_i$ of $M$, then the functions $(\sum_i \varphi_i \chi_{IJ}^i)_{i,j}$ form a partition of unity of $N(d_n \subset M^n)$ which is subordinated to the conic cover $\Phi(C_{IJ})_{i,j}$.

To go back to the configuration space $M^n$, choose a neighborhood $\mathcal{N}$ of $d_n$, s.t. $\mathcal{N}$ is a neighborhood of $\mathcal{N}'$, we have the inclusions $d_n \subset \mathcal{N}' \subset \mathcal{N}$. Let $\chi_1, \chi_2$ be a partition of unity subordinated to the cover $(\mathcal{N}, M^n \setminus \mathcal{N})$ and choose $(\chi_{IJ})_{i,j}$ to be an arbitrary partition of unity subordinated to the cover $C_{IJ} \subset \mathcal{N}$ of $M^n \setminus d_n$. Then set $\chi_{IJ} = \chi_1 \Phi^*(\sum_i \varphi_i \Psi_{U_i}^* \chi_{IJ}^i) + \chi_2 \chi_{IJ}$ and it follows by construction that every $\chi_{IJ}$ is tempered along $d_n$.

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