Renormalization of Black Hole Entropy and of the Gravitational Coupling Constant

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ABSTRACT

The quantum corrections to black hole entropy, variously defined, suffer quadratic divergences reminiscent of the ones found in the renormalization of the gravitational coupling constant (Newton constant). We consider the suggestion, due to Susskind and Uglum, that these divergences are proportional, and attempt to clarify its precise meaning. We argue that if the black hole entropy is identified using a Euclidean formulation, including the necessary surface term as proposed by Gibbons and Hawking, then the proportionality is, up to small identifiable corrections, a fairly immediate consequence of basic principles – a low-energy theorem. Thus in this framework renormalizing the Newton constant renders the entropy finite, and equal, in the limit of large mass, to its semiclassical value. As a partial check on our formal arguments we compare the one loop determinants, calculated using heat kernel regularization. An alternative definition of black hole entropy relates it to behavior at conical singularities in two dimensions, and thus to a suitable definition of geometric entropy. A definition of geometric entropy, natural from the point of view of heat kernel regularization, permits the same renormalization, but it does not yield an intrinsically positive quantity. The possibility, for scalar fields, of non-minimal coupling to background curvature allows one to consider test the framework in a continuous family of theories, and crucially involves a subtle sensitivity of geometric entropy to curved space couplings. Fermions and gauge fields are considered as well. Their functional determinants are closely related to the determinants for non-minimally coupled scalar fields with specific values for the curvature coupling, and pose no further difficulties.
1. Introduction

It has been proposed that the divergences of the entropy in black hole thermodynamics have the same origin as, and indeed are proportional to, the divergences of the gravitational coupling constant in naïve perturbative quantum gravity [1]. The possibility of such a connection is certainly appealing, but several objections have been raised to it [1–3]. For one thing, the divergences arising in renormalization of $G$ are certainly sensitive to non-minimal couplings of the matter fields to curvature, whereas the relevant entropy can be identified in flat space. Also, the divergent renormalizations, at one loop, can have either sign depending on the spin and curvature coupling of the field involved, whereas the entropy would appear to be intrinsically positive by definition. Moreover, since both sides of the proposed equality are infinite, and the precise definition of one side (the black hole entropy) is notoriously controversial, clearly some non-trivial questions of interpretation are involved. In this paper we shall propose an interpretation in which the claim is both precise and true, as a low-energy theorem. We shall also discuss the tensions that arise in other interpretations, and show that at least some of these – specifically, the two mentioned above – are less severe than appears at first sight.

We will first consider the definition of black hole entropy proposed by Gibbons and Hawking [4– 5], within their Euclidean approach to quantum gravity. If we accept that framework for considering black hole entropy, then this entropy arises from a surface term in the effective action, whose coefficient is related in a precise numerical fashion to the bulk Einstein-Hilbert term. The coefficient of the Einstein-Hilbert term, of course, in turn defines the observed Newton constant $G$. Thus one obtains, in the limit of large black holes, a low-energy theorem for the black hole entropy, expressing it directly in terms of the fully renormalized Newton constant. This result relies only on the structure of the action, so it is valid upon the rather mild assumption that the effective quantum action has the same structure as the classical one. In this regard, note that to treat large black holes in the Euclidean formalism one need only consider smooth manifolds of uniformly small curvature.
Unfortunately these arguments are of course purely formal, since there are serious problems with the ultraviolet behavior of the underlying theory, and all the quantities involved are infinite unless regulated. Within the Euclidean framework the divergences in the entropy and in the quantum corrections to Newton's constant have a common origin in local vacuum polarization. Heat kernel methods provide an appropriate way to regulate such divergences for the one-loop contribution of matter fields, while maintaining symmetry and locality [6–9]. Using this regularization, we calculate the leading cutoff dependence explicitly in a uniform manner for various spins and statistics (and for minimal or non-minimal coupling).

The Gibbons–Hawking definition of black hole entropy does not on the face of it offer a satisfactory understanding of this entropy based on the same principles as conventional definitions of entropy in statistical physics. Thus it is not superfluous to consider other definitions of entropy that are closer in spirit to the conventional definitions. Recently geometric entropy, which has considerable intrinsic interest, has been extensively discussed in this context. [10–16] We show that the geometric entropy relevant to black hole physics features corrections, in the form of winding modes, that are non-local from the point of view of particle paths. These are responsible for divergences which, on the face of it, appear to have a very different origin than those arising in renormalizing the Newton constant. We find that, with natural definitions, the divergences in the geometric entropy and the gravitational coupling nevertheless coincide. This result emerges for reasons that are, at least to us, rather less transparent than in the alternative (by no means obviously equivalent) Euclidean framework. It is particularly interesting, for reasons mentioned already, to consider the effects of non-minimal curvature couplings. We show that such non-minimal couplings reflect themselves even in flat space, where they dictate the form of the local energy-momentum tensor. Specifically, they control total divergence terms, which do not affect the integrated energy-momentum but do affect the regulation of modes on a half-space, and thereby sneak into the calculation of the properly regulated geometric entropy.

While we were in the final stages of preparing this paper we received an impor-
tant paper by Kabat [17], which overlaps in part with our discussion of geometric entropy, while featuring quite different techniques and emphases. We shall make some more detailed comparisons below.

2. Low-Energy Theorem for Gibbons-Hawking Entropy

There are several definitions of the black hole entropy in common use, whose equivalence is not manifest. In this section we consider the entropy defined using a Euclidean path integral for quantum gravity, following Gibbons and Hawking [4-5]. This definition came very early historically, but has not been so prominent in the recent literature. Since it is the definition which makes the non-renormalization almost obvious, we will give an elementary, self-contained presentation of it in this Section. We also take this opportunity to fix conventions and notations.

Within ordinary quantum field theories one can define the canonical ensemble, fixing the temperature $T$ of the system, by a Euclidean path integral over all configurations subject to the constraint that they are periodic (antiperiodic for fermions) in imaginary time $\beta \equiv 1/T$. The “imaginary time” is a real variable, of course; the integrals are defined by analytic continuation, not substitution, from the real-time integrals. This prescription is essentially the same as the Kubo-Martin-Schwinger boundary condition, which can be derived from basic principles of axiomatic field theory. For several reasons these basic principles do not apply cleanly to general relativistic systems, so there is a leap of faith involved in the use of the Euclidean path integral for gravity. In this chapter we shall take the leap, and consider the path integral over Euclidean space-times as defining the partition function. (The Euclidean formalism does not give rise to any obviously problematic results for the kinds of calculations we do in this paper, involving the quantum mechanics of matter fields in simple curved spaces. However if we attempted to calculate the vacuum polarization due to gravitons – the spin-two functional determinant – we would meet with the notorious difficulties associated with the non-positivity of the action for the conformal factor [5].)
It is very important, in attempting to access the thermodynamics of a black hole, to consider a finite volume. Indeed the black hole contribution to thermodynamic functions is finite rather than extensive in the volume, and it will always be swamped, in a large enough volume, by the contribution of the ambient thermal bath. In setting up the finite volume integral, one must complement the standard Einstein-Hilbert action with a surface term. This arises because higher derivatives occur in the Einstein-Hilbert action, whose presence would invalidate the variational principle in its usual form. Fortunately the offending terms can be isolated, after an integration by parts, and subtracted off as a boundary term, thus yielding an action amenable to conventional path integral treatment. The partition function then becomes

\[ Z = \int Dg \, e^{-\frac{1}{\hbar L} \left( \int dV R - 2 \int d\sigma K \right)} \]

where \( K \) denotes the extrinsic curvature. This is to be evaluated as a function of \( \beta \) and the geometry of the bounding surface.

The semiclassical approximation is implemented by evaluating the action on some classical solution. In the zero angular momentum vacuum sector, the unique solutions are given by the Euclidean Schwarzschild metrics

\[ ds^2 = \left( 1 - \frac{2MG}{r} \right) dt^2 + \left( 1 - \frac{2MG}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 . \] (2.2)

Introducing the coordinate \( \tilde{r} = \sqrt{8MG(r - 2MG)} \), one finds that the metric close to \( r = 2MG \) takes the form

\[ ds^2 = \frac{\tilde{r}^2}{16M^2G^2} dt^2 + d\tilde{r}^2 + (2MG)^2 d\Omega^2 . \] (2.3)

The equations of motion require the scalar curvature to vanish everywhere. In the preceding form of the metric it is apparent that there is a conical singularity at the “horizon” \( \tilde{r} = 0 \) unless \( t \) is interpreted as an angular variable with period \( 8\pi MG \).
Since \( R = 0 \) for a solution of the equations of motion, the value of the action is entirely determined by the surface term. We are interested in the contribution of the non-trivial black hole topologies relative to the flat space topologies, so we normalize the path integral by a flat space solution with the same interior geometry on the boundary surface. Thus we obtain

\[
\ln Z = -\frac{1}{8\pi G \hbar} \int d\sigma [K] = -\frac{\beta^2}{16\pi G \hbar} ; \quad [K] = K^{BH} - K^{vac}
\]

The second equality require a small calculation using \( \int d\sigma K = \frac{\partial}{\partial n} \int d\sigma \), which we sketch in the following paragraph. Here \( \hat{n} \) is the inward pointing unit normal.

Choose the bounding surface to be \( r = r_0 \). We demand that at this surface the geometry be the product of a circle of length \( \beta \) and a sphere of area \( 4\pi r_0^2 \). For the Euclidean Schwarzschild solution, this fixes \( M \) in terms of \( \beta \) according to

\[
(1 - \frac{2MG}{r_0})^{1/2} 8\pi MG = \hbar \beta .
\]

The area of the 2-sphere as a function of \( r \) is simply \( 4\pi r^2 \) while the length of the imaginary time circle is \( (1 - \frac{2MG}{r})^{1/2} (1 - \frac{2MG}{r_0})^{-1/2} \beta \). To calculate \( K^{BH} \) we must take the derivative with respect to the unit vector, \( i.e. \) \( -(1 - \frac{2MG}{r})^{1/2} \frac{\partial}{\partial r} \) of the product of these factors, evaluated at \( r = r_0 \). For flat space one easily finds \( K^{vac} = 8\pi \beta r_0 \). In the difference the term which grows with \( r_0 \) cancels, and the term independent of \( r_0 \) gives the quoted result for the free energy. Higher order terms involving \( \beta/r_0 \) are neglected.

Now, from the standard thermodynamic formula

\[
S = -\beta^2 \frac{\partial}{\partial \beta} \frac{1}{\beta} \ln Z
\]

we obtain

\[
S = \hbar \frac{\beta^2}{16\pi G} . \quad (2.4)
\]

Then finally using the relationship between \( \beta \) and \( M \), we arrive at the celebrated
result

\[ S = \frac{4\pi GM^2}{\hbar} = \frac{1}{4G\hbar} A, \quad (2.5) \]

where \( A = 4\pi (2GM)^2 \) is the area of the event horizon for a black hole of mass \( M \).

A non-renormalization theorem for black hole entropy, as defined operationally following the Gibbons-Hawking procedure outlined above, is an immediate corollary of the structure of the calculation, given minimal assumptions regarding the locality and general covariance of the effective Lagrangian. Indeed, the effective action will contain a term of the Einstein-Hilbert form, which (in the absence of cosmological term) is the operator of lowest mass dimension and dominates the long-distance behavior for weak fields. The coefficient of this term for nearly flat space defines the renormalized Newton’s constant. The surface term at large distances, which as we have seen is responsible for the entropy, is uniquely determined – including its numerical coefficient – from the bulk term. The asymptotic form of the metric at infinity, up to the order in \( 1/r \) used in the calculation, is again determined in terms of the coefficient of the Einstein-Hilbert term. Thus, assuming only the validity of a general covariant, local Lagrangian description of the dynamics at weak fields, we arrive at (2.4) in a form involving only renormalized quantities. Then from the thermodynamic formula \( M = E = F + S/\beta \) we find \( \beta = 8\pi MG \), again in terms of renormalized quantities, and thereby the first equality in (2.5). These results are low-energy theorems, in the sense familiar from quantum field theory.

A minor but interesting point is that the Newton’s constant is not quite the empty-space Newton’s constant, but rather that appropriate to temperature \( \beta^{-1} \). These may differ by finite quantities, as one integrates through mass thresholds or (taking into consideration \( e.g. \ |\phi|^2 R \) terms, where \( \phi \) is a Higgs field) condensation scales.

The discussion so far has been extremely formal, in the sense that neither side of the claimed equality is properly defined, in view of various convergence problems.
in the quantum theory. To some extent this limitation is unavoidable, since no satisfactory overarching theory of quantum gravity is currently available in usable form. Nevertheless some partial calculations can be done as a consistency check. Specifically, we can use a heat kernel regularization of one-loop diagrams involving various quantum fields, to get explicit results for the renormalization constants. Since this regulator is local and general covariant, it embodies the conditions for the non-renormalization theorem.

For simplicity, let us consider at this point the contribution of a minimally coupled scalar field. The one-loop effective action $W$ is defined through

$$e^{-W} = \int D\phi \, e^{-\frac{1}{8\pi^2} \int \phi^2 (\Delta + m^2)} = [\det(\Delta + m^2)]^{-\frac{1}{2}}$$

We define the heat kernel

$$D(\tau) = \text{Tr} e^{-\tau \Lambda} = \sum_i e^{-\tau \lambda_i}$$

where $\lambda_i$ are the eigenvalues of $\Lambda = -\Delta + m^2$. A short calculation shows

$$W = \frac{1}{2} \ln \det \Lambda = \frac{1}{2} \ln \lambda_i = -\frac{1}{2} \int d\tau \frac{D(\tau)}{\tau}$$

The integral over $\tau$ does not converge for small $\tau$, so we replace the lower limit with a non-zero cut-off $\epsilon^2$ and obtain a well-defined expression. The leading divergence is related to the short time behaviour of the heat kernel, which is independent of mass. This problem is treated abundantly in the literature [6], so we only recall the main features of the calculation. The heat kernel is given by

$$D(\tau) = \int dx \, G(x, x, \tau)$$

where the Green’s function $G$ satisfies the differential equation

$$\left( \frac{\partial}{\partial \tau} - \Delta_x \right) G(x, x', \tau) = 0 \ , \ G(x, x', 0) = \delta(x - x')$$
In flat space

\[ G_0(x, x', \tau) = \left(\frac{1}{4\pi \tau}\right)^\frac{3}{2} e^{-\frac{1}{4\pi} (x-x')^2} \]

but in general we must expand in the Laplacian in local coordinates and expand for small curvatures. The result is [7]

\[ D(\tau) = \left(\frac{1}{4\pi \tau}\right)^\frac{3}{2} \left[ \int dV \pm \sqrt{\frac{\pi \tau}{2}} \int d\sigma + \frac{\tau}{6} \left( \int dVR - 2 \int d\sigma K \right) + O(\tau^\frac{5}{2}) \right] (2.8) \]

with Dirichlet and Neumann boundary conditions corresponding to the upper and lower sign, respectively.

Integrating over \( \tau \) we find the effective action for the scalar field. The first two terms in the heat kernel can be interpreted as a bulk cosmological constant and surface cosmological constant, respectively. They contribute neither to the gravitational coupling nor to the entropy. They cancel in the latter, because the intrinsic geometry of the Euclidean Schwarzschild bounding surface is the same as that of the flat space boundary, whose action is subtracted from it. The subsequent two terms translate directly into renormalisation of gravitational coupling and entropy, respectively. Explicitly, defining the regularized Newton’s constant

\[ \frac{1}{16\pi G_{\text{ren}}} = \frac{1}{16\pi} \left( \frac{1}{G_{\text{bare}}} + \frac{a_G}{\epsilon^2} \right) \]

and similarly the regularized entropy, one finds

\[ a_G = a_S = \frac{1}{12\pi} \frac{1}{\epsilon^2}. \]

A mass for the scalar field does not change the leading divergence, but contributes a logarithmic divergence and finite terms. All such terms are equal for the coupling constant and the entropy. Hence, in conclusion, we find the renormalized Gibbons-Hawking entropy

\[ S = \frac{G_{\text{ren}} M^2}{\hbar}, \]

even after quantum corrections.
The non-renormalization of the Gibbons-Hawking entropy follows directly from the explicit, known form of the heat kernel, which only contains the Ricci scalar and the extrinsic curvature of the boundary in the same combination $\int dVR - 2\int d\sigma K$ as in the tree level action. This is as we anticipated above on very general grounds. At the risk of belaboring the point, let us restate the general argument in a different language more adapted to the spirit of the explicit calculation. Upon making variations in the metric, no boundary term remains in the combination $\int dVR - 2\int d\sigma K$. Physically, this is precisely the condition that the energy-momentum tensor contains no boundary term. While physical membranes can be endowed with surface tension, the boundary that we consider here is a freely movable mathematical abstraction, and it had better not have such a tension. To insure the validity of the variational principle, Lagrangeans on manifolds with boundary are constructed such that there is no energy–momentum on the boundary. It should therefore come as no surprise that, even after integrating the scalar field out, no boundary energy–momentum is found. This argument appears to be very general, applying for example to fields of any spin. Moreover, it works entirely in the framework of smooth manifolds, and weak curvature. We assume only that variations in the metric can be performed either before or after the integration over fields, with concordant results. A failure of this assumption, would imply that local Lorentz invariance suffers an anomaly.

3. Geometric Entropy and Cone Geometry

3.1. Generalities, and the Minimally Coupled Scalar

In considering other possible definitions of the black hole entropy, an important distinction must be drawn. For on–shell definitions, which consider only regular geometries, the preceding arguments apply. An example of such a definition is the microcanonical version (fix $M$, not $\beta$ ) of the Gibbons–Hawking entropy [18]. We implicitly relied on the equivalence of different on–shell definitions, which is a purely formal result, in the thermodynamic manipulations of the previous section.
By contrast, in off–shell definitions one allows $\beta$ and $M$ to vary independently. As we have discussed, this will generally introduce, in the Euclidean formulation, a conical singularity at the horizon. We must reconsider the preceding arguments, for geometries of this kind.

Geometric entropy [10–16] is an example of an entropy implicitly defined off-shell. It is explicitly defined in terms of microstates, as $S = -\text{tr} \rho \ln \rho$, where $\rho$ is the density matrix obtained by tracing over some region of space. In the limit of interest for very large black holes the curvature is small at the event horizon and the angular variables decouple, so we are led to consider the trace over a half space. In that case, the geometric entropy is conveniently expressed using the replica–trick,

$$S_{\text{geom}} = (1 - n \frac{d}{dn})_{n=1} \ln Z(n)$$

where $Z(n)$ is the unnormalized partition function of the field theory in question, as calculated on the disc covered $n$ times. As $n$ is required in the neighborhood of flat space, $n = 1$, $Z(n)$ can be thought of as the partition function on a weakly singular cone.

The conical singularity prevents us from using the small curvature expansion, but we can still use the functional determinant (2.7) to calculate the partition function using the heat kernel. On a product space the heat kernel is equal to product of the heat kernels of each of the two spaces. The heat kernel on the transverse (angular) space is $A^{4\pi \tau}$ and we are left to evaluate the heat kernel of the cone in 2 dimensions. It is conformally equivalent to the plane so we can use techniques from conformal field theory, or others, to calculate the heat kernel exactly [12, 15]. It is

$$D(\tau) = \frac{A}{4\pi \tau} \frac{1}{12} \left( \frac{1}{n} - n \right).$$

(3.1)

In this expression we have retained the transverse dimensions and omitted terms proportional to the spacetime volume.
To find the geometric entropy from the effective action, we must apply the operator \((1 - n \partial n)\). The result is

\[
S_{\text{geom}} = \frac{A}{4} \frac{1}{12\pi} \frac{1}{\epsilon^2}.
\]  

(3.2)

This result is precisely such that if we interpret it as the first quantum correction to the classical entropy for a large black hole, we have the non-renormalization

\[
S_{\text{total}} = S_{\text{classical}} + S_{\text{quantum}} = \frac{A}{4} \left( \frac{1}{G_{\text{bare}} \hbar} + \frac{1}{12\pi} \frac{1}{\epsilon^2} \right) = \frac{A}{4G_{\text{ren}} \hbar},
\]  

(3.3)

as for the Gibbons–Hawking entropy.

### 3.2. Role of Winding Paths

It is instructive compare this treatment of the cone partition function with what might be inferred from the small curvature expansion of the heat kernel, (2.8). On the cone there are no boundaries, but the conical singularity contributes a pointlike, delta-function curvature. Let us consider approaching this situation by first smoothing the singularity, then taking the limit. The Gauss-Bonnet theorem in two dimensions is

\[
\frac{1}{4\pi} (\int R - 2 \int K) = \chi = 2 - 2h - b
\]  

(3.4)

where \(h\) denotes the number of handles and \(b\) the number of boundaries. It gives us the relevant integral over curvature, regardless of the details of the smoothing procedure, and we find

\[
D(\tau) \simeq \frac{A}{4\pi \tau} \frac{1}{6} (1 - n) .
\]  

(3.5)

Thus the small curvature expansion does not correctly capture the heat kernel in a conical background. At large \(n\), it differs from the exact result (by a factor of 2), and at small \(n\), it completely misses the term including an inverse power of \(n\).
which plays a crucial role in the evaluation of the entropy. Indeed the term linear in $n$, which in smooth backgrounds is responsible for coupling constant renormalization, does not contribute to the entropy at all! In physical terms, this may be interpreted as a consequence of the fact that local vacuum polarisation affects only the normalization of the density matrix, but not the entropy of entanglement.

The weak-curvature expansion of the heat kernel is derived from purely local considerations, using Riemann normal coordinates [7]. In the language of the underlying diffusion process, it does not take into account paths that go around the cone. We can roughly indicate the effect of these winding paths using the following procedure. Consider the cone as the union of infinitesimal slices with width $d\bar{r}$. Each slice is a product manifold of a small linear direction, and a periodic variable. At fixed $\bar{r}$, the coordinate $\theta = 2\pi\bar{r} \frac{t}{8\pi MG}$ is periodic with period $2\pi\bar{r} n$. The appropriate Greens function is

$$G_0 = \frac{d\bar{r}}{4\pi\tau} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{4\tau}(\theta-\theta'+2\pi\bar{r}nk)^2}$$

The terms with non-zero $k$ represent field configurations that wind around the cone. These terms are non-perturbative in $\tau$ and were, correctly, ignored in the small curvature expansion. They are not inconsequential, however. We take $\theta = \theta'$ and integrate over space, to find the heat kernel. It is

$$D(\tau) = \frac{A}{(4\pi\tau)^2} \int d\bar{r}(2\pi\bar{r} n) \sum_{k \neq 0} e^{-\frac{1}{4\tau}(2\pi\bar{r}nk)^2} = \frac{A}{4\pi\tau} \frac{1}{12} n$$

This reproduces the first term of the exact result. A more refined version of this calculation, using the exact kernel in polar coordinates, can be used to obtain (3.1) in its entirety.

From this perspective, then, the divergence in geometric entropy appears completely unrelated to perturbative coupling constant renormalization. Nevertheless, as we have seen, the numerical evaluations agree. One might rationalize this to
some extent by noting that the entropy calculation requires the partition function only very near to flat space, where the smoothing procedure is least drastic and the distinction between winding around a weak pointlike curvature singularity and passing through a region of high curvature is least distinct. This line of argument is successfully pursued in [8].

3.3. Non-minimal Couplings

In general, the Lagrangean of a massless scalar field is

\[ \mathcal{L} = \frac{1}{8\pi} \phi(-\Delta + \xi R)\phi \] (3.6)

So far we have assumed minimal coupling $\xi = 0$ for simplicity, but in general we should include all dimension four operators on an equal footing.

From the perturbative expansion of the heat kernel

\[ D(\tau) = \left( \frac{1}{4\pi \tau} \right)^{\frac{d}{2}} \left[ \int dV + \tau \left( \frac{1}{6} - \xi \right) \int dV R \right] \] (3.7)

on a manifold without any boundary, we find the renormalized gravitational coupling

\[ \frac{1}{G_{\text{ren}}} = \frac{1}{G_{\text{bare}}} + \frac{1}{2\pi} \left( \frac{1}{6} - \xi \right) \frac{1}{\epsilon^2} \] (3.8)

The heat kernel on the cone with $n = \frac{\beta}{8\pi M_G}$ is

\[ D(\tau) = \frac{1}{12} \left( \frac{1}{n} - n \right) - \xi (1 - n) \] (3.9)

Since $\frac{1}{4\pi} \int R = 1 - n$, the second term is the perturbative result, which is expected to apply. Indeed, write

\[ D(\tau) = \text{Tr} e^{-\tau(-\Delta + \xi R)} \sim \text{Tr} e^{\tau\Delta} [1 + (e^{-\tau\xi R} - 1)] \]

The $\xi$-dependent term is explicitly suppressed by one power of $\tau$ relative to the leading term. For the leading term, the most singular term (the volume term) is
given accurately by its perturbative form, the winding-modes affecting only the subsequent order. Similarly, for the \( \xi \)-dependent term, it is sufficient to use perturbative modes to evaluate this term to the leading order. A rigorous calculation with the same result will be presented later in this section.

Now, from the heat kernel on the cone we find the appropriate effective action in four dimensions, which in turn leads to the geometric entropy

\[
S_{\text{geom}} = \frac{A}{4} \frac{1}{2\pi} \left( \frac{1}{6} - \xi \right) \frac{1}{e^2}
\]

Hence, for the coupling constant and the entropy alike, the quadratic divergences are proportional to \( \frac{1}{6} - \xi \) and otherwise independent of \( \xi \). Therefore our conclusions from the previous sections remain in the case of non-minimal coupling. This result, simple as it appears, is somewhat surprising. The geometric entropy, defined as \( S = -\text{tr} \rho \ln \rho \), is positive definite for any finite matrix \( \rho \). As we see now, the proper definition of the formal expression does not in general lead to a positive definite quantity.

The coupling to curvature enters the present calculation of heat-kernel calculation of geometric entropy because of the singular curvature on the cone. However, geometric entropy is intrinsically defined in a flat background where, one might think, the coupling to background curvature is inconsequential. What is going on here?

A related phenomenon, reviewed later in this section, occurs in two dimensional conformal field theory. In that context, addition of a term \( \gamma R \phi \) to the Lagrangean changes the energy-momentum tensor – even in the limit of flat space – without destroying conformal invariance. The central charge is changed to \( c = 1 + 12\gamma^2 \). The geometric entropy is proportional to the central charge so, even in flat space, the coupling to background curvature affects the result for geometric entropy.

The physics behind these somewhat paradoxical results can be understood qualitatively as follows. The geometric entropy for a sharply sliced half-space
diverges, and only regulated forms of it can appear in physical results. If we wish to regulate it in a way that is both local and covariant, it is more or less inevitable that we must damp the contributions of high-frequency modes. This also corresponds to the realistic circumstance that the entropy associated with arbitrarily high-frequency modes is not easily accessible to observation. Now to identify the high-frequency modes, we must know the energy-momentum tensor. Indeed, it is important that we know the local energy-momentum tensor, so that we can identify these modes near the boundary (where the divergences arise). The local energy-momentum tensor is ambiguous up to a total divergence, if we merely demand that its integral yield the conserved quantities. However in extending the theory in curved space, and demanding covariance, we must remove this ambiguity: the true local energy-momentum tensor can be identified by varying the action with respect to the metric, according to

\[ T_{\mu\nu}(x) = -\frac{4\pi \delta}{\sqrt{g}} \int L \sqrt{g} \delta g^{\mu\nu} . \]  

(3.11)

Following this procedure, one arrives at a regulator which implicitly depends upon how one extends the theory into curved space-time. Thus even apparently flat-space quantities, such as geometric entropy, that need regulation can become connected – through the demands of locality and covariance – to parameters governing the behavior of the theory in curved space-time.

Now let us consider concretely the effect of the coupling \( \xi R \phi^2 \). This interaction features a dimensionless coupling constant in any dimension. For our purposes it is sufficient to consider two dimensions, where the coupling destroys conformal invariance. As discussed above, the energy-momentum tensor depends on the coupling to background curvature even in flat space. It is

\[ T \equiv T_{zz} = -\frac{1}{2} (\partial \phi)^2 + \xi ( (\partial \phi)^2 + \phi \partial^2 \phi) \]

and the propagator is \( \langle \phi(z) \phi(0) \rangle = -\log z \) for all \( \xi \). The \( \xi \)-dependent term is a total derivative. Under a conformal transformation \( z \to f(z) \) the energy–momentum
tensor transforms as

\[ T(z) \rightarrow (f'(z))^2 T(f(z)) + A_f(z) \]

For \( \xi = 0 \) the extension term

\[ A_f(z)^{\xi=0} \equiv \frac{1}{12} S_f(z) = \frac{1}{12} \frac{f''' f' - \frac{3}{2} (f'')^2}{(f')^2} \]

satisfies the associativity property

\[ A_y(z) = (x')^2 A_y(x) + A_x(z) ; \quad z = z(x) \]

Then successive transformations on \( T \) gives the same transformed tensor as one combined transformation, i.e. the energy–momentum tensor realizes the conformal group. In fact, the form of \( A_f(z) \) is determined by this property. In the general case of non-minimal coupling, \( \xi \neq 0 \), we expect conformal invariance to be broken, and \( A_f(z) \) to be of a more general form.

To calculate the extension term we proceed as in [19]. Noting that the formal expression contains singular products of operators at the same point we regulate by defining

\[ (\partial \phi)^2 \equiv [(\partial \phi(z + \frac{1}{2}d) \partial \phi(z - \frac{1}{2}d) + \frac{1}{d^2}]_{d \rightarrow 0} \]

\[ \phi \partial^2 \phi \equiv [\phi(z + \frac{1}{2}d) \partial^2 \phi(z - \frac{1}{2}d) - \frac{1}{d^2}]_{d \rightarrow 0} \]

The regulator \( d \) is kept fixed under a conformal transformation. After the transformation, a new regulator \( d' = f(z + \frac{1}{2}d) - f(z - \frac{1}{2}d) \) is appropriate, however. It is the difference of the two regulators that is the subtle origin of the extension
term. Explicitly,

\[ A_f(z)^{\xi=0} = \frac{1}{2} \frac{f'(z + \frac{1}{2}d) f'(z - \frac{1}{2}d)}{(f(z + \frac{1}{2}d) - f(z - \frac{1}{2}d))^2} - \frac{1}{d^2} d \to 0 = \frac{1}{12} \frac{f''' f' - \frac{3}{4} (f'')^2}{(f')^2} \]

Similar treatment of \( \phi \partial^2 \phi \) leads to

\[
A_f(z) = (1 - 2\xi) A_f(z)^{\xi=0} + \xi \left[ \frac{f''(z - \frac{1}{2}d)}{f(z + \frac{1}{2}d) - f(z - \frac{1}{2}d)} + \frac{f'(z - \frac{1}{2}d)^2}{(f(z + \frac{1}{2}d) - f(z - \frac{1}{2}d))^2} \right] d \to 0
\]

\[ = \frac{1}{12} (1 - 6\xi) S_f(z) - \frac{\xi}{4} \left( \frac{f''}{f'} \right)^2 \]

after a small calculation. The extra \( \left( \frac{f''}{f'} \right)^2 \) term violates the associativity property of \( S_f(z) \) so conformal invariance is not realized for \( \xi \neq 0 \). This is no surprise, since the energy–momentum tensor derives from a curved space action that explicitly breaks conformal invariance.

As the extension term appears as a consequence of the definition of the quantum operator products, it is usually an anomaly which is not present in the classical theory. Here, however, the operator \( (\partial \phi)^2 + \phi \partial^2 \phi \) is regular a short distances even though the individual terms are not. Therefore, the \( \xi \) dependent terms are not really anomalies, but rather explicit breaking of conformal invariance. The complete extension term is the sum of the anomalous (\( \xi \) independent) and explicit (\( \xi \)-dependent) contributions.

From the extension term we can calculate the constant term in the heat kernel. It is related, through the effective action, to the finite part of the trace of the energy–momentum tensor

\[
D = -\frac{1}{2\pi} \int \langle T^\mu_\mu \rangle d^2 r = -\left( \int z \langle T(z) \rangle \frac{dz}{2\pi i} + h.c. \right)
\]

If we insist on \( \langle T(z) \rangle = 0 \) in flat space, then \( \langle T(z) \rangle = A_f(z) \) on the cone, where \( f = z^{\frac{1}{\pi}} \) maps flat space on to a cone. Noting that in these coordinates \( z \) traverses
an angle $2\pi n$ around the singularity, we find

$$D = \frac{1}{12} \left( \frac{1}{n} - n \right) - \xi (1 - n)$$

as derived heuristically above. While the $(\frac{f''}{f'})^2$-term is proportional to $(1 - n)^2$, so that it does not contribute to the entropy, it is exactly such as to change the $\xi$-dependent term from the non-perturbative $(\frac{1}{n} - n)$-form to the perturbative $(1 - n)$-form. In this calculation, performed without any reference to curved space, we see how the total derivative in the energy–momentum tensor enters the final result.

To round out the discussion, we will now briefly analyze another possible form of nonminimal coupling. Consider the Lagrangean

$$\mathcal{L} = \frac{1}{8\pi} [((\nabla \phi)^2 + 2\gamma R\phi] . \quad (3.12)$$

The parameter $\gamma$ is dimensionless in two dimensions and has the dimension of mass in four dimensions. In flat space the $\gamma R\phi$ term vanishes but the energy–momentum tensor

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}} \frac{\delta \mathcal{L} \sqrt{g}}{\delta g^{\mu\nu}} = -\frac{1}{2} \partial_\mu \phi \nabla_\nu \phi + \frac{1}{4} \eta_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + \gamma (\eta_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) \phi$$

acquires a surface term.

In two dimensions, the holomorphic part of the energy–momentum tensor is

$$T = -\frac{1}{2} (\partial\phi)^2 - \gamma \partial^2 \phi. \quad \text{We use the propagator } \langle \phi(z)\phi(0) \rangle = -\log z \text{ to calculate the operator product expansion}$$

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T}{z} \text{ regular}$$

where $c = 1 + 12\gamma^2$. The form of the operator product expansion shows that the energy–momentum tensor defines a conformal field theory. This is by no means trivial. For example, the non-minimal coupling $\xi R\phi^2$ would give rise to a term
so that theory is too singular to realize conformal symmetry. For conformal field theories in two dimensions the geometric entropy is proportional to $c$ [12]. Thus a $\gamma R \phi$ interaction, and the ensuing surface term in the energy–momentum tensor, directly affect the geometric entropy in two dimensions.

In higher dimensions we proceed differently. Because the the non-minimal term is linear in the field $\phi$, rather than quadratic, we can not use the heat kernel to reduce the problem effectively to two dimensions, as we did before. Instead, we perform a Gaussian integral to compute the effective action

$$W_{\text{div}}(\gamma) - W_{\text{div}}(\gamma = 0) = \frac{\gamma^2}{8\pi} \int dxdx' R(x)G(x,x')R(x')$$

where $G = \frac{1}{\sqrt{g}}$ is the propagator. To facilitate comparison with previous results we need a local form of this result. We readily calculate

$$(T^\mu_\mu)_{\text{div}} = -4\pi g^{\mu\nu} \delta W_{\text{div}} \delta g^{\mu\nu} = (d - 1)\gamma^2 R + (T^\mu_\mu)_{\text{div}}^{\gamma=0} = -4\pi \frac{\partial W_{\text{div}}}{\partial \ln \frac{1}{\xi^2}}$$

In two dimensions, we use the heat kernel result for the $\gamma = 0$ case to find $(T^\mu_\mu)_{\text{div}}^{\gamma=0} = \frac{1}{12} R$, and thereby recover the anomaly calculated in conformal theory. In four dimensions the $\gamma^2$-term also contributes a logarithmic term to the effective action. However the leading divergence in the renormalization of Newton’s constant is quadratic; thus this particular form of non-minimal coupling does not contribute to the leading divergence. A closer analysis shows that in fact it contributes only a finite renormalization.

### 3.4. Geometric Entropy and Fermions

In this section and the next, we calculate the geometric entropy contributed by matter fields with spin. As we shall see, these calculations rapidly reduce to the calculations already done for non-minimally coupled scalars, with particular choices for the parameter $\xi$. 

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The geometric entropy of a number of noninteracting species of particles is equal to the entropy of each species by itself. For fermions the contribution is positive, as shown by an explicit calculation in [16]. However, since loops of virtual bosons and fermions differ by an explicit sign, reflecting the quantum statistics, one might expect that spin-0 and spin-1/2 contribute with opposite sign to the renormalization of the gravitational coupling. However upon closer scrutiny this heuristic argument is incomplete – and indeed, its conclusion is false – due to the fact that for spin-1/2 one has an effective non-minimal coupling to curvature.

Indeed consider the effective action

\[ e^{-W} = \int D\psi D\bar{\psi} e^{-\int \bar{\psi}i\gamma^\mu D_\mu \psi} = \det (i\gamma^\mu D_\mu) = [\det (-\gamma^\mu D_\mu)\gamma^\nu D_\nu]^\frac{1}{2}. \]  

(3.14)

The positive power of the functional determinant here, as contrasted with the negative power in (2.6), reflects the aforementioned sign. In a frame where the connection \( \Gamma = 0 \), we have

\[ -\gamma^\mu D_\mu \gamma^\nu D_\nu = -\gamma^\mu \partial_\mu \gamma^\nu \partial_\nu - \frac{i}{2} \gamma^\mu \gamma^\nu \partial_\mu \Gamma_\nu^{\alpha \beta} \Sigma_{\alpha \beta} \; ; \; \Sigma_{\alpha \beta} = \frac{i}{4} [\gamma_\alpha, \gamma_\beta]. \]

(3.15)

To extract the most divergent contribution, it suffices to take the Dirac trace at this early stage, and one finds

\[ e^{-W} = \det [-\Delta + \frac{1}{4}R]^\frac{1}{2} \]

(3.16)

where \( s \) is the dimension of fermion representation, \textit{e.g.} \( s = 2 \) for a Weyl fermion in four dimensions. Using now the heat kernel corresponding to non-minimally coupled scalars, we find

\[ W(\text{fermion}) = -sW(\text{boson}, \xi = \frac{1}{4}) \]

\[ = -s\frac{1}{4} - \frac{1}{6} \; W(\text{boson}, \xi = 0) \]

(3.17)

\[ = \frac{s}{2} W(\text{boson}, \xi = 0). \]

Thus each spin-1/2 fermionic degree of freedom contributes half as much to the renormalization of the gravitational coupling constant as a spin-0 bosonic one.
Nothing in the calculation prevents us from repeating it in a conical background, finding the same relation for the geometric entropies. In two dimensions this follows from relation between the corresponding conformal anomalies [12]. In heat kernel regularization the transverse dimensions decouple; thus the result holds in four dimensions as well. We see that for spin 1/2, as for spin 0, the divergences in the gravitational coupling agree with those of the geometric entropy in sign as well as in magnitude.

3.5. Gauge Fields

Now let us consider gauge fields. We write

\[ e^{-W} = \int D A \ e^{-\frac{1}{16\pi} \int F^2} = \int D A \ e^{-\frac{1}{8\pi} \int A_\mu (-g^{\mu\nu} \Delta + \nabla^\mu \nabla^\nu + R_{\mu\nu}) A_\nu} \]

\[ = \left[ \det(-g^{\mu\nu} \Delta + R_{\mu\nu}) \right]^{-\frac{1}{2}} V^{1/2} \det(-\Delta) \quad (3.18) \]

The final equality indicates we are working in Feynman gauge, for which the gauge fixing term cancels the \( \nabla^\mu \nabla^\nu \) term. The scalar determinant derives from the Jacobian involved in the gauge fixing, which can be expressed in terms of a Fadeev-Popov ghost field, i.e. a complex spin-0 field which is quantized as a fermion. A careful derivation of this result, in a more general context, can be found in [20].

To the accuracy we need, take the trace over the vector representation (V) first and find

\[ e^{-W} = \left[ \det(-\Delta + \frac{1}{d} R) \right]^{-\frac{d}{2}} S^{1/2} \det(-\Delta)_S \]

i.e. the different polarizations decouple. We immediately obtain

\[ W_{\text{gauge field}} = d \ W_{\text{boson}}(\xi = \frac{1}{d}) - 2W_{\text{boson}}(\xi = 0) \quad (3.19) \]

The preceding calculations apply equally to the conical backgrounds relevant to geometric entropy, and to the almost flat smooth background relevant to renormalization of the Newton constant. Thus we conclude for a spin-1 gauge field, as for
spin 0 and spin-\frac{1}{2}, the gravitational coupling constant and the geometric entropy have equal divergences. Quantitatively

$$\frac{1}{G_{\text{ren}}\hbar} = \frac{1}{G_{\text{bare}}\hbar} + \frac{1}{2\pi} \left( \frac{(d - 2)}{6} - 1 \right) \frac{1}{\epsilon^2}.$$  (3.20)

For \(d = 4\) this agrees with Kabat [17], who performed a very careful and explicit mode analysis, and earlier work [21]. Our derivation appears to highlight the basic similarity of different spins. In particular, gauge invariance specifies a unique non-minimal coupling to the background curvature, but plays no other role. This unique coupling leads to a negative value for the leading divergent contribution to the covariantly regulated geometric entropy. This is at first sight a startling phenomenon, but as we have seen it can occur already, and for similar reasons, in the context of non-minimally coupled scalars.

4. Discussion

- For the preceding analysis it has been crucial to use a local, covariant regulator throughout. The heat kernel method supplies a convenient regulator of this source, and in addition leads to very simple calculations, as we have seen. A pioneering analysis of divergences in the entropy [22], which (with minor variations) has been widely adopted, employed instead a physically motivated scheme in which one first evaluates local entropy density, and then integrates over space to find the total entropy. The divergence of the latter integral expresses the divergence in local temperature close to the horizon. A regulating cutoff is imposed at some specified physical distance from the horizon. Clearly there is considerable arbitrariness in the choice of distance, and it is difficult to compare this regulation of the black hole entropy to any convenient regulator for quantum fluctuations in smooth geometries. While the choice of regulator scheme cannot ultimately affect physical results, an unfortunate choice may obscure the physics by requiring complicated conspiracies among counterterms. Our choice has the virtue of allowing straightforward comparison between the regulated Newton’s constant and Gibbons-Hawking...
or geometric entropies, as we have seen. Similar issues were recently emphasized in [3].

- In calculating the entropy of a large black hole it seems natural to include the classical entropy known from black hole thermodynamics together with the geometric entropy of quantum origin. However, this procedure is somewhat *ad hoc*, and it is not entirely clear how it should be formulated for a finite black hole, or even for pure vacuum (recall that in the Gibbons-Hawking calculation the divergent flat-space piece was simply subtracted off).

Alternatively, one could entertain the hypothesis that gravity itself induces a natural cut-off which makes geometric entropy, without the addition of a separate classical term, take on the finite value $\frac{A}{4G_R\hbar}$. The equality of entropy and coupling renormalization discussed here, if it still applies, would then seem to indicate the renormalized gravitational coupling constant arises entirely from quantum corrections. This, of course, is the basic hypothesis of induced gravity [23].

How might such a cut-off conceivably arise? The gravitational coupling apparently suffers no ultraviolet divergences in string theory [24]. The connection discussed in this paper suggests that black hole entropy and the geometric entropy must likewise be finite in string theory. Unfortunately our understanding of large black holes and of space-times with boundary in string theory remains primitive, and it appears very challenging at present to substantiate these suggestions, or even to make them completely precise.

An alternative possibility, which does not necessarily contradict the previous one, is that properly implementing the constraints of gravity drastically reduces the number of states, and itself renders the various entropies finite. Bekenstein has forcefully advocated this possibility on a variety of physical grounds [25]. It appears amenable to investigation at the semiclassical level [26].

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