HIGGS BUNDLES AND REPRESENTATION SPACES ASSOCIATED TO MORPHISMS

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ABSTRACT. Let $G$ be a connected reductive affine algebraic group defined over the complex numbers, and $K \subset G$ be a maximal compact subgroup. Let $X, Y$ be irreducible smooth complex projective varieties and $f : X \to Y$ an algebraic morphism, such that $\pi_1(Y)$ is virtually nilpotent and the homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective. Define

$$R^f(\pi_1(X), G) = \{ \rho \in \text{Hom}(\pi_1(X), G) \mid A \circ \rho \text{ factors through } f_* \},$$

$$R^f(\pi_1(X), K) = \{ \rho \in \text{Hom}(\pi_1(X), K) \mid A \circ \rho \text{ factors through } f_* \},$$

where $A : G \to \text{GL}(\text{Lie}(G))$ is the adjoint action. We prove that the geometric invariant theoretic quotient $R^f(\pi_1(X, x_0), G)/G$ admits a deformation retraction to $R^f(\pi_1(X, x_0), K)/K$. We also show that the space of conjugacy classes of $n$ almost commuting elements in $G$ admits a deformation retraction to the space of conjugacy classes of $n$ almost commuting elements in $K$.

1. INTRODUCTION

Let $G$ be a connected reductive affine algebraic group defined over the complex numbers. Consider an algebraic morphism

$$f : X \to Y$$

where $X$ and $Y$ are irreducible smooth complex projective varieties, and let

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

be the induced morphism of fundamental groups, where $x_0 \in X$ is a base point. In certain situations, the representations

$$\rho : \pi_1(X, x_0) \to G$$

that factor through $f_*$ have special geometric properties. See [KP], where necessary and sufficient conditions for such a factorization are given in terms of the spectral curve of the $G$-Higgs bundle associated to $\rho$.

In this article, we are interested in the whole moduli space of representations that factor in a similar way, and in its topological properties. Under some assumptions on $f$ and $Y$, we provide a natural deformation retraction between two such representation spaces, described as follows.

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The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let $A : G \to \text{GL}(\mathfrak{g})$ be the homomorphism given by the adjoint action of $G$ on $\mathfrak{g}$. Fix a maximal compact subgroup $K \subset G$ and define:

$$\mathcal{R}^f(\pi_1(X, x_0), G) = \{ \rho \in \text{Hom}(\pi_1(X, x_0), G) \mid A \circ \rho \text{ factors through } f_* \} ,$$

$$\mathcal{R}^f(\pi_1(X, x_0), K) = \{ \rho \in \text{Hom}(\pi_1(X, x_0), K) \mid A \circ \rho \text{ factors through } f_* \} .$$

We note that the group $G$ (respectively, $K$) acts on $\mathcal{R}^f(\pi_1(X, x_0), G)$ (respectively, on $\mathcal{R}^f(\pi_1(X, x_0), K)$) via the conjugation action of $G$ (respectively, $K$) on itself. The quotient $\mathcal{R}^f(\pi_1(X, x_0), K)/K$ is contained in the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0), G)/G$.

We prove the following in Theorem 2.6:

Suppose that the fundamental group of $Y$ is virtually nilpotent, and the homomorphism $f_*$ is surjective. Then $\mathcal{R}^f(\pi_1(X, x_0), G)/G$ admits a deformation retraction to the subset $\mathcal{R}^f(\pi_1(X, x_0), K)/K$.

In Section 3, we consider spaces of almost commuting elements in $K$ and in $G$. Define:

$$\text{AC}^n(K) = \{(g_1, \cdots, g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_K \ \forall \ i, j \} ,$$

where $Z_K$ denotes the center of $K$. The moduli space of conjugacy classes:

$$\text{AC}^n(K) / K ,$$

where $K$ acts by simultaneous conjugation, was studied in [BFM], [KS], and plenty of information is known in the cases $n = 2$ and $n = 3$. For instance, the number of components of $\text{AC}^3(K) / K$ has been related in [BFM] to the Chern–Simons invariants associated to flat connections on a 3-torus.

In a similar fashion, we define $\text{AC}^n(G)/G$, the moduli space of conjugacy classes of $n$ almost commuting elements in $G$. For example, if $G$ has trivial center, then $\text{AC}^{2n}(G)/G$ coincides with

$$\text{Hom}(\pi_1(X, x_0), G)/G ,$$

where $X$ is an abelian variety of complex dimension $n$. In Proposition 3.1 we show that $\text{AC}^n(G) / G$ admits a deformation retraction to $\text{AC}^n(K) / K$, and that the same holds for $\text{AC}^n(G)$ and $\text{AC}^n(K)$, extending one of the main results in [FL] and [BF1].

2. Representation spaces associated to a morphism

Let $X$ be an irreducible smooth complex projective variety. Fix a point $x_0 \in X$. Let

$$f : X \to Y$$

be an algebraic morphism, where $Y$ is also an irreducible smooth complex projective variety, such that:

1. the fundamental group $\pi_1(Y, f(x_0))$ is virtually nilpotent, and
2. the homomorphism of fundamental groups induced by $f$

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$  (2.1)

is surjective.
Using the homomorphism $f_*$ in (2.1), we will consider $\pi_1(Y, f(x_0))$ as a quotient of the group $\pi_1(X, x_0)$.

Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let

$$A : G \rightarrow \text{GL}(\mathfrak{g}) \quad (2.2)$$

be the homomorphism given by the adjoint action of $G$ on $\mathfrak{g}$. The affine algebraic variety (not necessarily irreducible) of representations

$$\rho : \pi_1(X, x_0) \rightarrow G$$

will be denoted by $\text{Hom}(\pi_1(X, x_0), G)$.

**Definition 2.1.** Let $\rho \in \text{Hom}(\pi_1(X, x_0), G)$. We say that $A \circ \rho$ factors through $f_*$ in (2.1) (or that $A \circ \rho$ factors geometrically through $f : X \rightarrow Y$, see [KP]) if there exists a homomorphism $\rho' \in \text{Hom}(\pi_1(Y, f(x_0)), \text{GL}(\mathfrak{g}))$ such that

$$\rho' \circ f_* = A \circ \rho. \quad (2.3)$$

**Remark 2.2.**

1. Clearly, if $\rho$ itself factorizes as $\rho = \tilde{\rho} \circ f_*$ for some $\tilde{\rho} \in \text{Hom}(\pi_1(X, x_0), G)$, then $A \circ \rho$ factorizes through $f_*$ as in the definition; the converse is not always true.
2. It is clear that $A \circ \rho \in \text{Hom}(\pi_1(X, x_0), \text{GL}(\mathfrak{g}))$ factors through $f_*$ as in (2.3), if and only if $A \circ \rho$ is trivial on the kernel of $f_*$. Moreover, when $A \circ \rho$ factors through $f_*$, a homomorphism $\rho' \in \text{Hom}(\pi_1(Y, f(x_0)), \text{GL}(\mathfrak{g}))$ satisfying equation (2.3) is unique, because $f_*$ is surjective.

In the framework of non-abelian Hodge theory, there is a correspondence between semistable $G$-Higgs bundles over $X$ and representations in $\text{Hom}(\pi_1(X, x_0), G)$, [Si], [BG]. Denote by $(E_\rho, \theta_\rho)$ the semistable $G$–Higgs bundle on $X$ associated to $\rho$ under this correspondence. We note that $(E_\rho, \theta_\rho)$ is semistable with respect to every polarization on $X$.

**Lemma 2.3.** Let $\rho \in \text{Hom}(\pi_1(X, x_0), G)$ be such that $A \circ \rho$ factors through $f_*$. Then, the above principal $G$–bundle $E_\rho$ on $X$ is semistable.

**Proof.** Let

$$\text{ad}(E_\rho) := E_\rho \times^A \mathfrak{g} \rightarrow X$$

be the adjoint vector bundle of $E_\rho$. The Higgs field on $\text{ad}(E_\rho)$ induced by $\theta_\rho$ will be denoted by $\text{ad}(\theta_\rho)$.

Let $\rho' : \pi_1(Y, f(x_0)) \rightarrow \text{GL}(\mathfrak{g})$ be the unique homomorphism satisfying equation (2.3); the uniqueness of $\rho'$ is a consequence of the surjectivity of $f_*$ as remarked above. Let $(E', \theta')$ be the semistable Higgs vector bundle on $Y$ associated to this homomorphism $\rho'$. Since the fundamental group of $Y$ is virtually nilpotent, we know that the vector bundle $E'$ is semistable [BL2, Proposition 3.1]. Let $c_i(E')$, $i \geq 0$, be the sequence of Chern classes of the bundle $E'$. Then, $c_i(E') = 0$ for all $i > 0$ because the $C^\infty$ complex vector bundle underlying $E'$ admits a flat connection (it is isomorphic to the $C^\infty$ complex vector bundle underlying the flat vector bundle associated to $\rho'$). Therefore, by [BE, p. 39, Theorem 5.1], the vector bundle $E'$ admits a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = E'$$
of holomorphic subbundles such that each successive quotient $V_i/V_{i-1}$, $1 \leq i \leq \ell$, admits a flat unitary connection. Consider the pulled back filtration

$$0 = f^*V_0 \subset f^*V_1 \subset \cdots \subset f^*V_{\ell-1} \subset f^*V_\ell = f^*E'. \quad (2.4)$$

A flat unitary connection on $V_i/V_{i-1}$ pulls back to a flat unitary connection on $f^*V_i/(f^*V_{i-1}) = f^*(V_i/V_{i-1})$. Since each successive quotient for the filtration of $f^*E'$ in (2.4) admits a flat unitary connection, we conclude that the holomorphic vector bundle $f^*E'$ is semistable.

From (2.3) it follows that

$$(\text{ad}(E^\rho), \text{ad}(\theta^\rho)) = (f^*E', f^*\theta'). \quad (2.5)$$

Since $f^*E'$ is semistable, from (2.5) it follows that $\text{ad}(E^\rho)$ is semistable. This implies that the principal $G$–bundle $E^\rho$ is semistable [AB, p. 214, Proposition 2.10].

Lemma 2.3 has the following corollary:

**Corollary 2.4.** For any Higgs field $\theta$, the $G$–Higgs bundle $(E^\rho, \theta)$ is semistable.

Let

$$\rho^\lambda : \pi_1(X, x_0) \to G \quad (2.6)$$

be a homomorphism corresponding to the Higgs $G$–bundle $(E^\rho, \lambda \cdot \theta^\rho)$, which is semistable by Corollary 2.4. We note that although $\rho^\lambda$ is not uniquely determined by $(E^\rho, \lambda \cdot \theta^\rho)$, the point in the quotient space

$$\text{Hom}(\pi_1(X, x_0), G)/G$$

given by $\rho^\lambda$ does not depend on the choice of $\rho^\lambda$. In other words, any two different choices of $\rho^\lambda$ differ by an inner automorphism of the group $G$.

**Lemma 2.5.** For every $\lambda \in \mathbb{C}$, the homomorphism $A \circ \rho^\lambda$ factors through $f_*$, where $\rho^\lambda$ is defined in (2.6).

**Proof.** Let $(\text{ad}(E^\rho)^\lambda, \text{ad}(\theta^\rho)^\lambda)$ be the Higgs vector bundle associated to the homomorphism $A \circ \rho^\lambda$. We note that $(\text{ad}(E^\rho)^\lambda, \text{ad}(\theta^\rho)^\lambda)$ is isomorphic to $(f^*E', f^*(\lambda \cdot \theta'))$, because the Higgs bundle $(E', \theta')$ corresponds to $\rho'$, and (2.3) holds. We saw in the proof of Lemma 2.3 that $E'$ is semistable with $c_i(E') = 0$ for all $i > 0$. Since $(\text{ad}(E^\rho)^\lambda, \text{ad}(\theta^\rho)^\lambda)$ is isomorphic to the pullback of a semistable Higgs vector bundle on $Y$ such that all the Chern classes of positive degrees of the underlying vector bundle on $Y$ vanish, it can be deduced that $A \circ \rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$. In fact, if

$$\delta : \pi_1(Y, f(x_0)) \to \text{GL}(g)$$

is a homomorphism corresponding to the Higgs vector bundle $(E', \lambda \cdot \theta')$, then

- the homomorphism $A \circ \rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$, and
- the homomorphism $\pi_1(Y, f(x_0)) \to \text{GL}(g)$ resulting from $A \circ \rho^\lambda$ differs from $\delta$ by an inner automorphism of $\text{GL}(g)$.

This completes the proof. \qed
Fix a maximal compact subgroup
\[ K \subset G. \]
Define
\[
\mathcal{R}^f(\pi_1(X, x_0), G) = \{ \rho \in \text{Hom}(\pi_1(X, x_0), G) \mid A \circ \rho \text{ factors through } f_* \},
\]
\[
\mathcal{R}^f(\pi_1(X, x_0), K) = \{ \rho \in \text{Hom}(\pi_1(X, x_0), K) \mid A \circ \rho \text{ factors through } f_* \}.\]
Since \( \pi_1(X, x_0) \) is a finitely presented group, the affine algebraic structure of \( G \) produces an affine algebraic structure on \( \mathcal{R}^f(\pi_1(X, x_0), G) \). The group \( G \) acts on \( \mathcal{R}^f(\pi_1(X, x_0), G) \) via the conjugation action of \( G \) on itself. Let
\[
\mathcal{R}^f(\pi_1(X, x_0), G)\!/G
\]
be the corresponding geometric invariant theoretic quotient. We note that this geometric invariant theoretic quotient \( \mathcal{R}^f(\pi_1(X, x_0), G)\!/G \) is a complex affine algebraic variety. Let
\[
\mathcal{R}^f(\pi_1(X, x_0), K)\!/K
\]
be the quotient of \( \mathcal{R}^f(\pi_1(X, x_0), K) \) for the adjoint action of \( K \) on itself.

The inclusion of \( K \) in \( G \) produces an inclusion of \( \mathcal{R}^f(\pi_1(X, x_0), K) \) in \( \mathcal{R}^f(\pi_1(X, x_0), G) \), which, in turn, gives an inclusion
\[
\mathcal{R}^f(\pi_1(X, x_0), K)\!/K \hookrightarrow \mathcal{R}^f(\pi_1(X, x_0), G)\!/G. \tag{2.7}
\]
Instead of working with the Zariski topology on \( \mathcal{R}^f(\pi_1(X, x_0), G)\!/G \), we consider on it the Euclidean topology which is induced from an embedding of this space in a complex affine space. Indeed, such an embedding can always be obtained by considering a finite set of generators of the algebra of \( G \)-invariant regular functions on \( \mathcal{R}^f(\pi_1(X, x_0), G) \). Moreover, this topology is independent of the choice of such embedding, and compatible with the inclusion \( (2.7) \).

**Theorem 2.6.** The topological space \( \mathcal{R}^f(\pi_1(X, x_0), G)\!/G \) admits a deformation retraction to the above subset \( \mathcal{R}^f(\pi_1(X, x_0), K)\!/K \).

**Proof.** Two elements of \( \text{Hom}(\pi_1(X, x_0), G) \) are called equivalent if they differ by an inner automorphism of \( G \). Points of \( \mathcal{R}^f(\pi_1(X, x_0), G)\!/G \) correspond to the equivalence classes of homomorphisms \( \rho \in \text{Hom}(\pi_1(X, x_0), G) \) such that the action of \( \pi_1(X, x_0) \) on \( \mathfrak{g} \) given by \( A \circ \rho \) is completely reducible, meaning that \( \mathfrak{g} \) is a direct sum of irreducible \( \pi_1(X, x_0) \)-modules. Let \( (E_\rho, \theta_\rho) \) be the semistable \( G \)-Higgs bundle corresponding to the above homomorphism \( \rho \), and let \( (\text{ad}(E_\rho), \text{ad}(\theta_\rho)) \) be the semistable adjoint Higgs vector bundle associated to \( (E_\rho, \theta_\rho) \). The above condition that the action of \( \pi_1(X, x_0) \) on \( \mathfrak{g} \) given by \( A \circ \rho \) is completely reducible is equivalent to the condition that the semistable Higgs vector bundle \( (\text{ad}(E_\rho), \text{ad}(\theta_\rho)) \) is polystable.

Let
\[
\phi : (\mathcal{R}^f(\pi_1(X, x_0), G)\!/G) \times [0, 1] \longrightarrow \mathcal{R}^f(\pi_1(X, x_0), G)\!/G
\]
be the map defined by \( (\rho, \lambda) \longmapsto \rho^{1-\lambda} \) (defined in \( (2.6) \)), where \( \rho \in \text{Hom}(\pi_1(X, x_0), G) \) satisfies the condition that the action of \( \pi_1(X, x_0) \) on \( \mathfrak{g} \) given by \( A \circ \rho \) is completely reducible. It is easy to see that \( \phi \) is well-defined. We note that the point in the geometric invariant theoretic quotient \( \mathcal{R}^f(\pi_1(X, x_0), G)\!/G \) given by \( \rho \) lies in the subset \( \mathcal{R}^f(\pi_1(X, x_0), K)/K \) if and only if the Higgs field \( \theta_\rho \) on the principal \( G \)-bundle \( E_\rho \) vanishes identically (as before, \( (E_\rho, \theta_\rho) \) is the Higgs \( G \)-bundle corresponding to \( \rho \)).
The following are straightforward to check:

- \( \phi(z, 0) = z \) for all \( z \in \mathcal{R}^f(\pi_1(X, x_0), G)//G \),
- \( \phi(z, 1) \in \mathcal{R}^f(\pi_1(X, x_0), K)/K \) for all \( z \in \mathcal{R}^f(\pi_1(X, x_0), G)//G \), and
- \( \phi(z, \lambda) = z \) for all \( z \in \mathcal{R}^f(\pi_1(X, x_0), K)/K \) and \( \lambda \in [0, 1] \).

Therefore, the above map \( \phi \) produces a deformation retraction of \( \mathcal{R}^f(\pi_1(X, x_0), G)//G \) to \( \mathcal{R}^f(\pi_1(X, x_0), K)/K \).

**Remark 2.7.** Lemma 2.3 and Theorem 2.6 are also valid for morphisms \( f : X \to Y \) in the category of compact Kähler manifolds, under the same assumptions on \( Y \) and \( f_* \). The proofs of these results are analogous, by replacing semistability with the notion of pseudostability (see [BG], [BF2]).

3. **Deformation retraction of the space of almost commuting elements**

Again, let \( G \) be a connected complex reductive group, and \( K \) be a maximal compact subgroup. Let

\[ Z_G \subset G \]

be the center of \( G \) and let

\[ PG := G/Z_G \]

be the quotient group. We note that the center of \( PG \) is trivial. Let

\[ q : G \to PG \]

be the quotient map. The image

\[ PK := q(K) \subset PG \]

is a maximal compact subgroup of \( PG \). We have \( q^{-1}(PK) = K \).

Fix a positive integer \( n \). Define

\[ AC^n(G) = \{ (g_1, \ldots, g_n) \in G^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \ \forall \ i, j \} \]

It is a subscheme of the affine variety \( G^n \). The group \( G \) acts on \( AC^n(G) \) as simultaneous conjugation of the \( n \) factors. Let

\[ ACE^n(G) := AC^n(G)//G \]

be the geometric invariant theoretic quotient. Also, define

\[ AC^n(K) = \{ (g_1, \ldots, g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \ \forall \ i, j \} \]

So \( AC^n(K) = AC^n(G) \cap K^n \). Let

\[ ACE^n(K) := AC^n(K)/K \]

be the quotient for the simultaneous conjugation action of \( K \) on the \( n \) factors. Note that the inclusion of \( K \) in \( G \) produces an inclusion

\[ ACE^n(K) \hookrightarrow ACE^n(G) \]

**Proposition 3.1.** Let \( G \) be semisimple. Then, the topological space \( ACE^n(G) \) admits a deformation retraction to the above subset \( ACE^n(K) \).
Proof. When $G$ is semisimple, $Z_G$ is a finite subgroup of $G$, so that the map (3.1) is a Galois covering. Also, $Z_G \subset K$. Define $AC^n(PG)$ and $ACE^n(PG)$ by substituting $PG$ in place of $G$ in the above constructions. Note that $AC^n(PG)$ parametrizes commuting $n$ elements of $PG$ because the center of $PG$ is trivial. Similarly, define $AC^n(PK)$ and $ACE^n(PK)$ by substituting $PK$ in place of $K$. So $AC^n(PK)$ parametrizes commuting $n$ elements of $PK$. The projection

$$\beta : ACE^n(G) \longrightarrow ACE^n(PG)$$

(3.2)

constructed using the the projection $q$ in (3.1) is a Galois covering with Galois group $Z^n_G$. However it should be mentioned that $ACE^n(G)$ need not be connected. Let

$$\gamma : ACE^n(K) \longrightarrow ACE^n(PK)$$

be the projection constructed similarly using $q$. Clearly, $\gamma$ coincides with the restriction of $\beta$ to $ACE^n(K) \subset ACE^n(G)$.

There is a deformation retraction of $ACE^n(PG)$ to $ACE^n(PK)$

$$\varphi : ACE^n(PG) \times [0,1] \longrightarrow ACE^n(PG)$$

[FL, Theorem 1.1] (see also [BF1]). In particular, $\varphi|_{ACE^n(PG) \times \{0\}}$ is the identity map of $ACE^n(PG)$.

Applying the homotopy lifting property to the covering $\beta$ in (3.2), there is a unique map

$$\tilde{\varphi} : ACE^n(G) \times [0,1] \longrightarrow ACE^n(G)$$

such that

1. $\beta \circ \tilde{\varphi} = \varphi \circ (\beta \times \text{Id}_{[0,1]})$, and
2. $\tilde{\varphi}|_{ACE^n(G) \times \{0\}}$ is the identity map of $ACE^n(G)$.

This map $\tilde{\varphi}$ is a deformation retraction of $ACE^n(G)$ to $ACE^n(K)$, because $\varphi$ is a deformation retraction.

Proposition 3.1 remains valid in the more general situation when $G$ is reductive.

Theorem 3.2. Let $G$ be a connected reductive affine algebraic group over $\mathbb{C}$. Then, $ACE^n(G)$ admits a deformation retraction to the subset $ACE^n(K)$.

Proof. First, note that Proposition 3.1 is clearly valid if $G$ is a product of copies of the multiplicative group $\mathbb{C}^*$. Hence it remains valid for any $G$ which is a product of a semisimple group and copies of $\mathbb{C}^*$. For a general connected reductive group $G$, consider the natural homomorphism

$$\eta : G \longrightarrow PG \times (G/[G,G]).$$

It is a surjective Galois covering map, the quotient $PG := G/Z_G$ is semisimple, while the quotient $G/[G,G]$ is a product of copies of $\mathbb{C}^*$. As mentioned above Proposition 3.1 is valid for $PG \times (G/[G,G])$. Using this and the above homomorphism $\eta$ it follows that Proposition 3.1 is valid for $G$. □
3.1. Deformation retraction of the space of $n$ commuting elements. Finally, we note that the analogous result is also verified for the space of $n$ commuting elements, $AC^n(G)$.

Theorem 3.3. Let $G$ be a connected reductive affine algebraic group over $\mathbb{C}$. Then, the space $AC^n(G)$ admits a deformation retraction to the subset $AC^n(K)$.

Proof. Since $PG$ and $PK$ have trivial center, the spaces $AC^n(PG)$ and $AC^n(PK)$ consist of $n$ commuting elements: If $(g_1, \cdots, g_n) \in AC^n(PG)$, then $g_ig_j = g_jg_i$, for all $i, j \in \{1, \cdots, n\}$.

Therefore, it is known that $AC^n(PG)$ admits a deformation retraction to $AC^n(PK)$ [PS, p. 2514, Theorem 1.1]. In view of this, imitating the proof of Proposition 3.1 it follows that $AC^n(G)$ admits a deformation retraction to $AC^n(K)$. \qed

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