Frames and weak frames for unbounded operators

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Abstract

In 2012, Găvruța introduced the notions of $K$-frame and of atomic system for a linear bounded operator $K$ in a Hilbert space $\mathcal{H}$, in order to decompose its range $\mathcal{R}(K)$ with a frame-like expansion. In this article, we revisit these concepts for an unbounded and densely defined operator $A : \mathcal{D}(A) \to \mathcal{H}$ in two different ways. In one case, we consider a non-Bessel sequence where the coefficient sequence depends continuously on $f \in \mathcal{D}(A)$ with respect to the norm of $\mathcal{H}$. In the other case, we consider a Bessel sequence and the coefficient sequence depends continuously on $f \in \mathcal{D}(A)$ with respect to the graph norm of $A$.

Keywords A-frames · Weak A-frames · Atomic systems · Reconstruction formulas · Unbounded operators

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1 Introduction

The notion of frame in Hilbert spaces dates back to 1952 when it was introduced in the pioneeristic paper of J. Duffin and A.C. Schaffer [21], and was resumed in 1986 by I. Daubechies, A. Grossman and Y. Meyer in [19]. This notion is a generalization of that of orthonormal bases. Indeed, let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$, a frame is a sequence in $\mathcal{H}$ that allows every element of $\mathcal{H}$ to be written as a stable, potentially infinite, linear combination of the elements of the sequence. The uniqueness of the decomposition is lost, in general, and this gives a
certain freedom in the choice of the coefficients in the expansion which is in fact a good quality in applications.

L. Găvruţa introduced in [23] the notion of atomic system for a linear, boundedorator K defined everywhere on \( \mathcal{H} \). This notion generalizes frames and also atomic systems for subspaces in [22]. More precisely, \( \{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) is an atomic system for \( K \) if there exists \( \gamma > 0 \) such that for every \( f \in \mathcal{H} \) there exists \( a_f = \{a_n(f)\}_{n \in \mathbb{N}} \in \ell^2 \), the usual Hilbert space of complex sequences, such that \( \|a_f\| \leq \gamma \|f\| \) and

\[
Kf = \sum_{n=1}^{\infty} a_n(f)g_n.
\]

This notion turns out to be equivalent to that of \( K \)-frame [23], i.e., a sequence \( \{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) satisfying

\[
\alpha \|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, g_n \rangle|^2 \leq \beta \|f\|^2, \quad \forall f \in \mathcal{H}, \tag{1.1}
\]

for some constants \( \alpha, \beta > 0 \), where \( K^* \) is the adjoint of \( K \). The main theorem in [23] states that if \( \{g_n\}_{n \in \mathbb{N}} \) is a \( K \)-frame, then there exists a Bessel sequence \( \{h_n\}_{n \in \mathbb{N}} \) in \( \mathcal{H} \), i.e., \( \sum_{n=1}^{\infty} |\langle f, h_n \rangle|^2 \leq \gamma \|f\|^2 \) for all \( f \in \mathcal{H} \) and some \( \gamma > 0 \), such that

\[
Kf = \sum_{n=1}^{\infty} \langle f, h_n \rangle g_n, \quad \forall f \in \mathcal{H}.
\]

This generalization of frames allows to write every element of \( \mathcal{R}(K) \), the range of \( K \), which need not be closed, as a superposition of the elements \( \{g_n\}_{n \in \mathbb{N}} \) which do not necessarily belong to \( \mathcal{R}(K) \). A question can arise at this point: why develop a theory of \( K \)-frames since there already exists a well-studied theory of frames that reconstruct the entire space \( \mathcal{H} \)? The answer is that if in a specific situation we are looking for sequences with some properties, then we may not find any possible frame, but we may find a \( K \)-frame because this notion is weaker and we could want to decompose just \( \mathcal{R}(K) \).

Let us see a concrete example: let \( \mathcal{H} = L^2(\mathbb{R}) \), \( \phi \in L^2(\mathbb{R}) \) and consider the translation system \( \{\phi_n(x)\}_{n \in \mathbb{Z}} := \{\phi(x-cn)\}_{n \in \mathbb{Z}} \) and the Gabor system \( G(\phi, a, b) = \{\phi_{m,n}(x)\}_{m,n \in \mathbb{Z}} := \{e^{2\pi ima}I_{\phi}(x-na)\} \) with \( a, b, c > 0 \). As it is known [16], there is no hope to have \( \{\phi_{n}\}_{n \in \mathbb{Z}} \) (or \( \{\phi_{m,n}\}_{m,n \in \mathbb{Z}} \) with \( ab > 1 \)) as a frame, whatever \( \phi \) is in \( L^2(\mathbb{R}) \). But if \( K \) is a bounded operator on \( L^2(\mathbb{R}) \) and \( \mathcal{R}(K) \neq \mathcal{H} \), then we might find \( \phi \) such that one of the previous sequences is a \( K \)-frame.

We have taken inspiration to [31, Example 1] for the following simple example.
We write \( \hat{f} \) for the Fourier transform of \( f \), which is defined for \( f \in L^1(\mathbb{R}) \) as \( \hat{f}(\gamma) := \int_{\mathbb{R}} f(x)e^{-2\pi i x \gamma} dx, \gamma \in \mathbb{R} \), and it is extended to \( f \in L^2(\mathbb{R}) \) in a standard way. Let \( PW_{1/4} = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset [-\frac{1}{4}, \frac{1}{4}]\} \). If \( \phi \in L^2(\mathbb{R}) \) is such that

\[
\hat{\phi}(\gamma) = \begin{cases} 1 & \text{for } |\gamma| \leq \frac{1}{4} \\ \text{decays to zero continuously} & \text{for } \frac{1}{4} \leq |\gamma| < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq |\gamma|, \end{cases}
\]
then we have for $f \in PW_{1/4}^1$
\[ \hat{f} = \hat{\phi} \hat{f} = \phi \sum_{n \in \mathbb{Z}} \langle \hat{f} | e_n \rangle e_n = \sum_{n \in \mathbb{Z}} \langle \hat{f} | e_n \rangle \hat{\phi} e_n = \sum_{n \in \mathbb{Z}} \langle \hat{f} | f_n \rangle \hat{\phi} e_n, \]
where
\[ e_n(\gamma) = \begin{cases} e^{2\pi i n \gamma} & \text{for } |\gamma| \leq \frac{1}{2} \\ 0 & \text{for } |\gamma| > \frac{1}{2}. \end{cases} \]
and
\[ f_n(\gamma) = \begin{cases} e^{2\pi i n \gamma} & \text{for } |\gamma| \leq \frac{1}{4} \\ 0 & \text{for } |\gamma| > \frac{1}{4}. \end{cases} \]

Thus, $f = \sum_{n \in \mathbb{Z}} \langle f | \psi_n \rangle \phi_n$ for $f \in PW_{1/4}^1$ where $\phi_n$ is the inverse Fourier transform of $\hat{\phi} e_n$, i.e., $\phi_n(x) = \phi(x - n)$, and $\psi_n := \hat{f} \hat{-n}$ is the inverse Fourier transform of $f_{-n}$, i.e.,
\[ \psi_n(x) = \hat{f} \hat{-n}(x) = \begin{cases} 4 \sin\left(\frac{\pi}{2} (x - n)\right) & \text{if } x \neq n \\ 2 & \text{if } x = n. \end{cases} \]

If $P$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $PW_{1/4}^1$, then we can write
\[ Pf = \sum_{n \in \mathbb{Z}} \langle Pf | \psi_n \rangle \phi_n = \sum_{n \in \mathbb{Z}} \langle f | \psi_n \rangle \phi_n, \quad \forall f \in L^2(\mathbb{R}) \]
since $\psi_n \in PW_{1/4}^1$. In conclusion, $\{\phi_n\}_{n \in \mathbb{Z}}$ is a $K$-frame with $K = P$ (it fulfills (1.1) as one can easily see by taking the Fourier transform) but of course $\{\phi_n\}_{n \in \mathbb{Z}}$ is not contained in $\mathcal{R}(P) = PW_{1/4}^1$. Moreover, it is not even a frame sequence, i.e., a frame for its closed span (indeed $\{\phi_n\}_{n \in \mathbb{Z}}$ does not satisfy [16, Theorem 9.2.5]).

In the literature, there are many further studies or variations of $K$-frames (see for example [24, 26, 29, 32, 33, 36] and the references therein).

In this paper, we deal with two different generalizations of [23] which involve a closed densely defined operator $A$ on $\mathcal{H}$. When the operator is bounded, all definitions do coincide with those in [23]. To justify our two different approaches, let us consider a Bessel sequence $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ and assume that, for $f \in \mathcal{D}(A)$, the domain of $A$, we have a decomposition
\[ Af = \sum_{n=1}^{\infty} a_n(f) g_n, \]
for some $a_f := \{a_n(f)\}_{n \in \mathbb{N}} \in \ell^2$; in particular, this situation appears when $\{g_n\}_{n \in \mathbb{N}}$ is a frame. If $A$ is unbounded, then the coefficients sequence $a_f$ can not depend continuously on $f$, i.e., it can not exists $\gamma > 0$ such that $\|a_f\| \leq \gamma \|f\|$ for every $f \in \mathcal{D}(A)$; this fact may represent another issue when we want to decompose $\mathcal{R}(A)$ by a frame.

For these reasons, we develop two approaches where either the sequence $\{g_n\}_{n \in \mathbb{N}}$ or the coefficients sequence $a_f$ is what represents the unboundedness of $A$. To go into more details, in the first case, we consider a non-Bessel sequence $\{g_n\}_{n \in \mathbb{N}}$ but the coefficients depend continuously on $f \in \mathcal{D}(A)$. In the second case, we take a Bessel sequence $\{g_n\}_{n \in \mathbb{N}}$ and coefficients depending continuously on $f \in \mathcal{D}(A)$ only in the graph topology of $A$, which is stronger than the one of $\mathcal{H}$ when $A$ is unbounded.
The paper is organized as follows. After some preliminaries, see Section 2, we introduce in Section 3, the notions of weak $A$-frame and weak atomic system for $A$ (Definitions 3.1 and 3.6, respectively), where $A$ is a, possibly unbounded, densely defined operator. The word weak is due to the fact that the decomposition of $\mathcal{R}(A)$, with $A$ also closable, holds only in a weak sense, in general; i.e., we find a Bessel sequence $\{t_n\}_{n \in \mathbb{N}}$ of $H$ such that

$$\langle Af | u \rangle = \sum_{n=1}^{\infty} \langle f | t_n \rangle \langle g_n | u \rangle \quad \forall f \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

see Theorem 3.10. Like in the bounded case (see [33, Lemma 2.2]), we have also

$$A^* u = \sum_{n=1}^{\infty} \langle u | g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*),$$

and thus we note a change of the point of view: a weak $A$-frame may be used to get a strong decomposition of $A^*$ rather than $A$.

In Section 4, we face our second approach, giving the general notions of atomic system for $A$ and $A$-frame, see Section 4.1, where $A$ is a, possibly unbounded, closed densely defined operator. Denote by $\langle \cdot | \cdot \rangle_A$ the inner product which induces the graph norm $\| \cdot \|_A$ of $A$. The resulting decomposition is

$$Af = \sum_{n=1}^{\infty} \langle f | k_n \rangle_A g_n \quad \forall f \in \mathcal{D}(A),$$

for some Bessel sequence $\{k_n\}_{n \in \mathbb{N}}$ of the Hilbert space $\mathcal{D}(A)[\| \cdot \|_A]$, see Corollary 4.8. Actually, this second approach is a particular case of $K$-frames, in the Găvruţa-like sense, where $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ is a bounded operator between two different Hilbert spaces $\mathcal{J}$ and $\mathcal{H}$, see Section 4. Indeed, for a densely defined closed operator $A$ on $\mathcal{H}$, we take $K = A$ and $\mathcal{J} = \mathcal{D}(A)[\| \cdot \|_A]$, see Corollary 4.8.

Throughout the paper, we give some examples of weak $A$-frames or $A$-frames that can be obtained from frames or that involve Gabor or wavelets systems.

## 2 Preliminaries

In the paper, we consider an infinite dimensional Hilbert space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$. The term operator is used for a linear mapping. Given an operator $F$, we denote its domain by $\mathcal{D}(F)$, its range by $\mathcal{R}(F)$, and its adjoint by $F^*$, if $F$ is densely defined. By $\mathcal{B}(\mathcal{H})$, we denote the set of bounded operators with domain $\mathcal{H}$ and we indicate by $\| F \|$ the usual norm of the operator $F \in \mathcal{B}(\mathcal{H})$. In some examples, we need the usual Hilbert spaces $L^2(0, 1)$, $L^2(\mathbb{R})$ and the Sobolev spaces, denoted with standard notations, $H^1(0, 1)$, $H^1_0(0, 1)$, $H^1(\mathbb{R})$, see [34, Section 1.3]. As usual, we will indicate by $\ell^2$ the Hilbert space consisting of all sequences $x := \{x_n\}_{n \in \mathbb{N}}$ satisfying $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ with norm $\| x \|_2 = \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$.

We will say that a series $\sum_{n=1}^{\infty} g_n$, with $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$, is convergent to $g$ in $\mathcal{H}$ if $\lim_{n \to \infty} \| \sum_{k=1}^{n} g_k - g \| = 0$. We will write $\{g_n\}$ to mean a sequence $\{g_n\}_{n \in \mathbb{N}}$ of
elements of $\mathcal{H}$. For the following definitions, the reader could refer, e.g., to [1, 3, 15, 16, 25, 27].

A sequence $\{g_n\}$ of elements in $\mathcal{H}$ is a Bessel sequence of $\mathcal{H}$ if any of the following equivalent conditions are satisfied, see [16, Corollary 3.2.4]

(i) there exists a constant $\beta > 0$ such that $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2$, for all $f \in \mathcal{H}$;

(ii) the series $\sum_{n=1}^{\infty} c_n g_n$ converges for all $c = \{c_n\} \in \ell^2$.

A sequence $\{g_n\}$ of elements in $\mathcal{H}$ is a lower semi-frame for $\mathcal{H}$ with lower bound $\alpha > 0$ if $\alpha \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2$, for every $f \in \mathcal{H}$. Note that the series on the right hand side may diverge for some $f \in \mathcal{H}$.

A sequence $\{g_n\}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist $\alpha, \beta > 0$ such that

$$\alpha \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2, \quad \forall f \in \mathcal{H}.$$ 

We now recall some operators which are classically used in the study of sequences, see [1–3, 15]. Let $\{g_n\}$ be a sequence of elements of $\mathcal{H}$. The analysis operator $C : D(C) \subseteq \mathcal{H} \rightarrow \ell^2$ of $\{g_n\}$ is defined by

$$D(C) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty \right\}$$

$$Cf = \{\langle f | g_n \rangle\}, \quad \forall f \in D(C).$$

The synthesis operator $D : D(D) \subseteq \ell^2 \rightarrow \mathcal{H}$ of $\{g_n\}$ is defined on the dense domain $D(D) := \left\{ \{c_n\} \in \ell^2 : \sum_{n=1}^{\infty} c_n g_n \text{ is convergent in } \mathcal{H} \right\}$

by

$$D\{c_n\} = \sum_{n=1}^{\infty} c_n g_n, \quad \forall \{c_n\} \in D(D).$$

The frame operator $S : D(S) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ of $\{g_n\}$ is defined by

$$D(S) := \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} \langle f | g_n \rangle g_n \text{ is convergent in } \mathcal{H} \right\}$$

$$Sf = \sum_{n=1}^{\infty} \langle f | g_n \rangle g_n, \quad \forall f \in D(S).$$

The main properties of these operators are summarized below.

**Proposition 2.1** [3, Prop. 3.3] Let $\{g_n\}$ be a sequence of $\mathcal{H}$. The following statements hold.

(i) $C = D^*$ and therefore $C$ is closed.

(ii) $D$ is closable if and only if $C$ is densely defined. In this case, $D \subseteq C^*$.

(iii) $D$ is closed if and only if $C$ is densely defined and $D = C^*$.

(iv) $S = DC$. 
A sequence \( \{g_n\} \) is a Bessel sequence if and only if one, and then all, the operators \( C, D, \) and \( S \) are bounded. Moreover, if \( \{g_n\} \) is a frame, then \( S \) is invertible with bounded inverse and the following reconstruction formula holds

\[
f = \sum_{n=1}^{\infty} \langle f|h_n \rangle g_n, \quad f \in \mathcal{H},
\]

(2.1)

where \( \{h_n\} \) is a frame for \( \mathcal{H} \) called a dual of \( \{g_n\} \). A choice of \( \{h_n\} \), which is always possible, is \( \{S^{-1}g_n\} \), called the canonical dual of \( \{g_n\} \), but it can be different if \( \{g_n\} \) is overcomplete, i.e., \( \{g_n\} \) is not a basis. As a consequence of (2.1), the Hilbert space \( \mathcal{H} \) must be separable.

Now we spend some words on non-Bessel sequences and reconstruction formulas. In general, if \( \{g_n\} \) is a lower semi-frame, then by [14, Proposition 3.4] or [18, Sect. 4], there exists a Bessel sequence \( \{h_n\} \) such that

\[
h = \sum_{n=1}^{\infty} \langle h|g_n \rangle h_n, \quad \forall h \in \mathcal{D}(C).
\]

Hence, a reconstruction formula holds in weak sense as

\[
\langle f|h \rangle = \sum_{n=1}^{\infty} \langle f|h_n \rangle \langle g_n|h \rangle, \quad f \in \mathcal{H}, h \in \mathcal{D}(C).
\]

(2.2)

Moreover, if \( \mathcal{D}(C) \) is dense, then one can take \( h_n = T^{-1}g_n \), where \( T := |C|^2 = C^*C \), a self-adjoint operator with bounded inverse on \( \mathcal{H} \), see [17, 18]. The “weakness” of the formula (2.2) is a consequence of the fact that the synthesis operator \( D \) is not closed, in general. If \( \{g_n\} \) is a lower semi-frame, \( \mathcal{D}(C) \) is dense and the synthesis operator \( D \) of \( \{g_n\} \) is closed, then \( D = C^* \), by Proposition 2.1. Thus, \( S = C^*C \) and the strong reconstruction formula again holds

\[
f = SS^{-1}f = \sum_{n=1}^{\infty} \langle f|S^{-1}g_n \rangle g_n, \quad \forall f \in \mathcal{H}.
\]

Remark 2.2 In the light of (2.2), we compare the pair \( \{(g_n), (h_n)\} \) with reproducing pairs [5, 6, 10, 11], weakly dual pairs [30], also called pairs of pseudoframes for \( \mathcal{H} \), and pairs of pseudoframes for subspaces [31]. If in (2.2) the formula holds for every \( h \in \mathcal{H} \), then by definition \( \{(g_n), (h_n)\} \) is a weakly dual pair. In (2.2), if in addition \( \mathcal{D}(C) \) is dense, the pair \( \{(g_n), (h_n)\} \) is a reproducing pair if and only if it is a weakly dual pair. In order the pair \( \{(g_n), (h_n)\} \) in (2.2) to be a pseudoframe for \( \mathcal{D}(C) \), this space has to be closed and \( \{g_n\} \) and \( \{h_n\} \) have to be Bessel sequences for \( \mathcal{D}(C) \) and
\( H \), respectively, so the nature of \( \{g_n\} \) and \( \{h_n\} \) in (2.2) is very different from the setting of pseudoframe for subspace, in general.

Now we recall the two notions we will generalize in the present paper. Let \( K \in \mathcal{B}(H) \). A sequence \( \{g_n\} \subset H \) is an atomic system for \( K \) [23] if the following statements hold

(i) \( \{g_n\} \) is a Bessel sequence of \( H \);
(ii) there exists \( C > 0 \) such that for every \( f \in H \), there exists \( a_f = \{a_n(f)\} \in \ell^2 \) such that \( \|a_f\| \leq C\|f\| \) and \( Kf = \sum_{n=1}^{\infty} a_n(f)g_n \).

In [23, Theorem 3], the author proves the following

**Theorem 2.3** Let \( K \in \mathcal{B}(H) \) and \( \{g_n\} \) a sequence of \( H \). The following statements are equivalent.

(i) \( \{g_n\} \) is an atomic system for \( K \).
(ii) there exist constants \( \alpha, \beta > 0 \) such that

\[
\alpha\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \leq \beta\|f\|^2, \quad \forall f \in H.
\]

(iii) there exists a Bessel sequence \( \{h_n\} \) of \( H \) such that

\[
Kf = \sum_{n=1}^{\infty} \langle f|h_n \rangle g_n, \quad \forall f \in H.
\]

Due to the inequalities in (ii) above, a sequence satisfying any of the conditions in Theorem 2.3 is also called a \( K \)-frame for \( H \).

Lastly, we will use the next lemma that can be obtained by Lemma 1.1 and Corollary 1.2 in [13].

**Lemma 2.4** Let \( H \) and \( K \) be Hilbert spaces. Let \( W : D(W) \subset K \rightarrow H \) be a closed densely defined operator with closed range \( \mathcal{R}(W) \). Then, there exists a unique \( W^\dagger \in \mathcal{B}(H, K) \) such that

\[
\mathcal{N}(W^\dagger) = \mathcal{R}(W)^\perp, \quad \overline{\mathcal{R}(W^\dagger)} = \mathcal{N}(W)^\perp, \quad WW^\dagger f = f, \quad f \in \mathcal{R}(W).
\]

The operator \( W^\dagger \) is called the pseudoinverse of \( W \).

## 3 Weak A-frames and weak atomic systems for A

In this section, we introduce our first generalization of the notion of \( K \)-frames to a densely defined operator on a Hilbert space \( H \).
**Definition 3.1** Let $A$ be a densely defined operator on $\mathcal{H}$. A weak $A$-frame for $\mathcal{H}$ is a sequence $\{g_n\} \subset \mathcal{H}$ such that

$$\alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty,$$  

(3.1)

for every $f \in D(A^*)$ and some $\alpha > 0$.

By [27, Theorem 7.2], if $A \in \mathcal{B}(\mathcal{H})$ then $\{g_n\}$ is a weak $A$-frame if and only if it is an $A$-frame in the sense of [23].

**Remark 3.2** As it is clear from (3.1), the property of being a weak $A$-frame does not depend on the ordering of the sequence.

**Remark 3.3** Let $A$ be a closable densely defined operator and $\{g_n\}$ a weak $A$-frame. The domain $D(C)$ of the analysis operator $C$ of $\{g_n\}$ contains $D(A^*)$. It is therefore dense and the synthesis operator $D$ is closable. Moreover,

$$\alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 = \|C f\|^2 = \|\frac{1}{2} f\|^2, \quad \forall f \in D(A^*),$$

where $T = C^* C$. This shows that the series in (3.1) is also bounded from above by the norm of a self-adjoint operator acting on $f \in D(A^*)$.

**Example 3.4** Let $A$ be a densely defined operator on a separable Hilbert space $\mathcal{H}$. Then a weak $A$-frame for $\mathcal{H}$ always exists. Indeed, let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$ contained in $D(A)$ (there always exists such a one, by [37, Ch. 1, Corollary 1] and the Gram-Schmidt orthonormalization process), it suffices to take $g_n = Ae_n$, because for every $f \in D(A^*)$, $\|A^* f\|^2 = \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2$, by the Parseval identity.

**Example 3.5** Let $A$ be a densely defined operator on a separable Hilbert space $\mathcal{H}$. A more general example of weak $A$-frame is obtained by taking a frame $\{f_n\} \subset D(A)$ for $\mathcal{H}$. In this case, in fact, there exist $\alpha, \beta > 0$ such that

$$\alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle A^* f | f_n \rangle|^2 \leq \beta \|A^* f\|^2, \quad \forall f \in D(A^*).$$

Therefore, $\{Af_n\}$ is a weak $A$-frame for $\mathcal{H}$.

Now we generalize the notion of atomic system to the case of an unbounded operator.

**Definition 3.6** Let $A$ be a densely defined operator on $\mathcal{H}$. A weak atomic system for $A$ is a sequence $\{g_n\} \subset \mathcal{H}$ such that

(i) $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty$ for every $f \in D(A^*)$;

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(ii) there exists $\gamma > 0$ such that, for every $h \in \mathcal{D}(A)$, there exists $a_h = \{a_n(h)\} \in \ell^2$ satisfying $\|a_h\| \leq \gamma \|h\|$ and

$$\langle Ah|u \rangle = \sum_{n=1}^{\infty} a_n(h)\langle g_n|u \rangle, \quad \forall u \in \mathcal{D}(A^*). \quad (3.2)$$

**Remark 3.7** If $\{g_n\}$ is a weak atomic system for $A$ then the series in (3.2) is unconditionally convergent. Indeed, it is absolutely convergent: fix any $h \in \mathcal{D}(A)$, $u \in \mathcal{D}(A^*)$, then $\sum_{n=1}^{\infty} |a_n(h)\langle g_n|u \rangle| \leq \|a_h\| \left(\sum_{n=1}^{\infty} |\langle g_n|u \rangle|^2\right)^{1/2} < \infty$.

The following lemma, which is a variation of [20, Theorem 2], will be useful in Theorem 3.10 for a characterization of weak atomic systems for $A$ and weak $A$-frames.

**Lemma 3.8** Let $(\mathcal{H}, \| \cdot \|)$, $(\mathcal{H}_1, \| \cdot \|_1)$ and $(\mathcal{H}_2, \| \cdot \|_2)$ be Hilbert spaces and $T_1 : \mathcal{D}(T_1) \subseteq \mathcal{H}_1 \rightarrow \mathcal{H}$, $T_2 : \mathcal{D}(T_2) \subseteq \mathcal{H} \rightarrow \mathcal{H}_2$ densely defined operators. Denote by $T_1^* : \mathcal{D}(T_1^*) \subseteq \mathcal{H} \rightarrow \mathcal{H}_1$ and $T_2^* : \mathcal{D}(T_2^*) \subseteq \mathcal{H}_2 \rightarrow \mathcal{H}$ the adjoint operators of $T_1$, $T_2$, respectively. Assume that

(i) $T_1$ is closed;
(ii) $\mathcal{D}(T_1^*) = \mathcal{D}(T_2)$;
(iii) $\|T_1^* f\|_1 \leq \lambda \|T_2 f\|_2$ for all $f \in \mathcal{D}(T_1^*)$ and some $\lambda > 0$.

Then there exists an operator $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1 = T_2^* U$.

**Proof** Define an operator $J$ on $R(T_2) \subseteq \mathcal{H}_2$ as $J T_2^* f = T_1^* f \in \mathcal{H}_1$. Then $J$ is a well-defined bounded operator by (iii). Now we extend $J$ to the closure of $R(T_2)$ by continuity and define it to be zero on $R(T_2)^\perp$. Therefore, $J \in B(\mathcal{H}_2, \mathcal{H}_1)$ and $J T_2 = T_1^*$, i.e., $T_1 = T_2^* J^*$ and the statement is proved by taking $U = J^*$. \qed

For the characterization in Theorem 3.10, we need the following definition.

**Definition 3.9** Let $A$ be a densely defined operator and $\{g_n\}$ a sequence on $\mathcal{H}$, then a sequence $\{t_n\}$ of $\mathcal{H}$ is called a weak $A$-dual of $\{g_n\}$ if

$$\langle Ah|u \rangle = \sum_{n=1}^{\infty} \langle h|t_n \rangle\langle g_n|u \rangle, \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*). \quad (3.3)$$

**Theorem 3.10** Let $\{g_n\} \subset \mathcal{H}$ and $A$ a closable densely defined operator on $\mathcal{H}$. Then the following statements are equivalent.

(i) $\{g_n\}$ is a weak atomic system for $A$;
(ii) $\{g_n\}$ is a weak $A$-frame;
(iii) $\sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 < \infty$ for every $f \in \mathcal{D}(A^*)$ and there exists a Bessel weak $A$-dual $\{t_n\}$. 

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Proof (i) ⇒ (ii) Let \( f \in \mathcal{D}(A^*) \). Then \( \|A^*f\| = \sup_{h \in \mathcal{H}, \|h\|=1} |\langle A^*f|h \rangle| \) and by the density of \( \mathcal{D}(A) \) in \( \mathcal{H} \)

\[
\|A^*f\| = \sup_{h \in \mathcal{D}(A), \|h\|=1} |\langle A^*f|h \rangle| = \sup_{h \in \mathcal{D}(A), \|h\|=1} |\langle f|Ah \rangle|
\]

\[
= \sup_{h \in \mathcal{D}(A), \|h\|=1} \left| \sum_{n=1}^{\infty} a_n(h) \langle f|g_n \rangle \right|
\]

\[
\leq \sup_{h \in \mathcal{D}(A), \|h\|=1} \left( \sum_{n=1}^{\infty} |a_n(h)|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \right)^{1/2}
\]

\[
\leq \gamma_A \left( \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \right)^{1/2},
\]

taking into account that \( \|a_h\| \leq \gamma_A \|h\| \) for some \( \gamma_A > 0 \) and every \( h \in \mathcal{D}(A) \).

(ii) ⇒ (iii) Let \( \{e_n\} \) be an orthonormal basis of \( \ell^2 \). Consider the densely defined operator \( B : \mathcal{D}(A^*) \to \ell^2 \) given by \( Bf = \{\langle f|g_n \rangle\} \) which is a restriction of the analysis operator \( C : \mathcal{D}(C) \to \ell^2 \). Since \( C \) is closed, \( B \) is closable.

We apply Lemma 3.8 to \( T_1 := \overline{A} \) and \( T_2 := B \) noting that \( \|Bf\|^2 = \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \). There exists \( M \in \mathcal{B}(\mathcal{H}, \ell^2) \) such that \( A = B^*M \). This implies that for \( h \in \mathcal{D}(A), u \in \mathcal{D}(A^*) = \mathcal{D}(B) \)

\[
\langle Ah|u \rangle = \langle B^*Mh|u \rangle = \langle Mh|Bu \rangle = \sum_{n=1}^{\infty} \langle Mh|e_n \rangle \langle g_n|u \rangle
\]

\[
= \sum_{n=1}^{\infty} \langle h|t_n \rangle \langle g_n|u \rangle,
\]

taking \( \{t_n\} = \{M^*e_n\} \) which is a Bessel sequence by [3, Proposition 4.6].

(iii) ⇒ (i) It suffices to take \( a_h = \{a_n(h)\} = \{(h|t_n)\} \) for all \( h \in \mathcal{D}(A) \). Indeed, for some \( \gamma_A > 0 \), we have \( \sum_{n=1}^{\infty} |a_n(h)|^2 = \sum_{n=1}^{\infty} |\langle h|t_n \rangle|^2 \leq \gamma_A \|h\|^2 \) since \( \{t_n\} \) is a Bessel sequence and \( \langle Ah|u \rangle = \sum_{n=1}^{\infty} a_n(h) \langle g_n|u \rangle \), for \( u \in \mathcal{D}(A^*) \).

The term “weak” of weak A-frame and of weak atomic system is due to the fact that (3.3) holds whereas, in general, the same decomposition in strong sense \( Ah = \sum_{n=1}^{\infty} \langle h|t_n \rangle g_n \) may fail, unlike the case of A-frame where \( A \in \mathcal{B}(\mathcal{H}) \), see [23, Theorem 3]. We show this with the following example.

Example 3.11 Suppose that \( \mathcal{H} \) is separable. Let \( \{e_n\} \) be an orthonormal basis for \( \mathcal{H} \) and \( \{g_n\} \) the sequence defined by \( g_1 = e_1 \) and \( g_n = n(e_n - e_{n-1}) \) for \( n \geq 2 \). We denote by \( C, D \) the analysis and synthesis operators of \( \{g_n\} \), respectively. As it is shown in [15], \( C \) is densely defined and \( D \) is a proper restriction of \( C^* \). In particular, \( \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \in \mathcal{D}(C^*) \setminus \mathcal{D}(D) \). Let \( T \) be the analysis operator of \( \{e_n\} \). Obviously it is a
bijection in $B(\mathcal{H}, \ell^2)$. Now consider the sesquilinear form

$$\Omega(f, u) = \sum_{n=1}^{\infty} \langle f \mid e_n \rangle \langle g_n \mid u \rangle,$$

which is defined on $\mathcal{H} \times \mathcal{D}(C)$. Moreover, $\Omega(f, u) = \langle I f \mid C u \rangle$ for all $f \in \mathcal{H}, u \in \mathcal{D}(C)$. Therefore, $\Omega(f, u) = \langle C^* I f \mid u \rangle$ for all $f \in \mathcal{D}(C^* I), u \in \mathcal{D}(C)$.

This suggests to define $A := C^* I$ which is a densely defined closed operator. The adjoint $A^*$ is equal to $I^* C$ and then it has $\mathcal{D}(C)$ as domain. Thus

$$\langle Af \mid u \rangle = \sum_{n=1}^{\infty} \langle f \mid e_n \rangle \langle g_n \mid u \rangle, \quad \forall f \in \mathcal{D}(A), u \in \mathcal{D}(A^*),$$

i.e., $\{g_n\}$ is a weak $A$-frame by Theorem 3.10. But the relation

$$Af = \sum_{n=1}^{\infty} \langle f \mid e_n \rangle g_n, \quad \forall f \in \mathcal{D}(A)$$

does not hold. Indeed, the element $f := \sum_{n=1}^{\infty} \frac{1}{n} e_n$ belongs to $\mathcal{D}(A)$ and the sum $\sum_{k=1}^{n} \langle f \mid e_k \rangle g_k = e_n$ for $n \in \mathbb{N}$ does not converge in $\mathcal{H}$.

**Example 3.12** In general, for a weak $A$-frame $\{g_n\}$ for $\mathcal{H}$ a Bessel weak $A$-dual $\{t_n\}$ is not unique. For all examples, we have considered we give here a possible choice of $\{t_n\}$.

(i) If $\{g_n\} := \{Ae_n\}$, where $\{e_n\} \subset \mathcal{D}(A)$ is an orthonormal basis for $\mathcal{H}$, then one can take $\{t_n\} = \{e_n\}$.

(ii) If $\{g_n\} := \{Af_n\}$, where $\{f_n\} \subset \mathcal{D}(A)$ is a frame for $\mathcal{H}$, then one can take for $\{t_n\}$ any dual frame of $\{f_n\}$.

**Remark 3.13** Let $A$ be a densely defined operator, $\{g_n\}$ a weak $A$-frame and $\{t_n\}$ a Bessel weak $A$-dual of $\{g_n\}$, then for $h \in \mathcal{D}(A)$ and $u \in \mathcal{D}(A^*)$

$$\langle A^* u \mid h \rangle = \langle u \mid Ah \rangle = \sum_{n=1}^{\infty} \langle h \mid t_n \rangle \langle g_n \mid u \rangle = \sum_{n=1}^{\infty} \langle u \mid g_n \rangle \langle t_n \mid h \rangle.$$

Since the sequence $\{t_n\}$ is Bessel, the series $\sum_{n=1}^{\infty} \langle u \mid g_n \rangle t_n$ is convergent. Therefore,

$$\langle A^* u \mid h \rangle = \left( \sum_{n=1}^{\infty} \langle u \mid g_n \rangle t_n \right) h, \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

and by the density of $\mathcal{D}(A)$, we obtain

$$A^* u = \sum_{n=1}^{\infty} \langle u \mid g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*). \quad (3.4)$$

In conclusion, it is worth noting that in this setting, surprisingly, from condition (3.1), the strong decomposition of $A^*$ follows, whereas for $A$, we have just a weak decomposition, in general. If $A$ is symmetric, i.e., $A \subset A^*$, then clearly from (3.4), we have a decomposition of $A$ in strong sense. If $\{g_n\}$ is also a Bessel sequence,
then $A$ is bounded on its domain, thus closable, and condition (3.1) gives us decom-
positions in strong sense for both the closure $\overline{A}$ and $A^*$ (see [23, Theorem 3] and
[33, Lemma 2.2]).

Remark 3.14 One could ask whether a weak $A$-dual $\{t_n\}$ of a weak $A$-frame $\{g_n\}$ is a
weak $A^*$-frame, with $A$ a closable densely defined operator. The answer is negative,
in general. Indeed, if $\{t_n\}$ is a Bessel sequence, an inequality as

$$\alpha \|Af\|^2 \leq \sum_{n=1}^{\infty} |(f|t_n)|^2, \quad \forall f \in D(A)$$

with $\alpha > 0$ implies that $A$ is bounded on its domain.

Under further assumption of $A$, weak $A$-frames can be used to decompose the
domain of $A^*$.

Theorem 3.15 Let $A$ be a densely defined closed operator with $\mathcal{R}(A) = \mathcal{H}$ and
$(A^\dagger)^* \in \mathcal{B}(\mathcal{H})$ the adjoint of the pseudoinverse $A^\dagger$ of $A$. Let $\{g_n\}$ be a weak $A$-
frame and $\{t_n\}$ a Bessel weak $A$-dual of $\{g_n\}$. Then, the sequence $\{h_n\}$, with $h_n := (A^\dagger)t_n \in \mathcal{H}$ for every $n \in \mathbb{N}$, is Bessel and

$$u = \sum_{n=1}^{\infty} \langle u|g_n\rangle h_n, \quad u \in D(A^*)$$

Proof First observe that, since $A$ is onto, $f = AA^\dagger f$, for every $f \in \mathcal{H}$. Let $\{g_n\}$, $\{t_n\}$, and $\{h_n\}$ be as in the statement. Then, by (3.3), we have that for $f \in \mathcal{H}, u \in D(A^*)$

$$\langle f|u \rangle = \langle AA^\dagger f|u \rangle = \sum_{n=1}^{\infty} \langle A^\dagger f|t_n \rangle \langle g_n|u \rangle = \sum_{n=1}^{\infty} \langle f|h_n \rangle \langle g_n|u \rangle$$

and for some $\gamma > 0$

$$\sum_{n=1}^{\infty} |\langle f|h_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle A^\dagger f|t_n \rangle|^2 \leq \gamma \|A^\dagger f\|^2 \leq \gamma \|A^\dagger\|^2 \|f\|^2$$

since $\{t_n\}$ is Bessel for $\mathcal{H}$ and $A^\dagger$ is bounded. Hence, $\{h_n\}$ is a Bessel sequence of
$\mathcal{H}$. Finally, for any $f \in \mathcal{H}, u \in D(A^*)$, we have $\langle u|f \rangle = \sum_{n=1}^{\infty} \langle \langle u|g_n\rangle h_n |f \rangle$. Since
the sequence $\{h_n\}$ is Bessel, the series $\sum_{n=1}^{\infty} \langle \langle u|g_n\rangle h_n |f \rangle$ is convergent and we conclude
that $u = \sum_{n=1}^{\infty} \langle u|g_n\rangle h_n$, for all $u \in D(A^*)$. □

Now we give another theorem of characterization for weak $A$-frames involving
the synthesis operator.

Theorem 3.16 Let $A$ be a closed densely defined operator, $\{g_n\} \subset \mathcal{H}$ and $D : \mathcal{D}(D) \subset \ell^2 \to \mathcal{H}$ the synthesis operator of $\{g_n\}$. The following statements are equivalent.

(i) The sequence $\{g_n\}$ is a weak $A$-frame for $\mathcal{H}$;
(ii) there exists a densely defined, closed extension $R$ of $D$ such that $A = RQ$ with some $Q \in \mathcal{B}(\mathcal{H}, \ell^2)$;

(iii) there exists a closed densely defined operator $L : \mathcal{D}(L) \subset \ell^2 \to \mathcal{H}$ such that and $\mathcal{D}(A^*) \subset \mathcal{D}(L^*)$, $g_n = L_n'$ where $\{e'_n\} \subset \mathcal{D}(L)$ is an orthonormal basis for $\ell^2$ and $A = LU$ for some $U \in \mathcal{B}(\mathcal{H}, \ell^2)$.

**Proof** (i) $\Rightarrow$ (ii) Following the proof of Theorem 3.10, $A = B^*M$. Then the statement is proved taking $Q = M$ and $R = B^*$, since $B^* \supseteq C^* \supseteq D$.

(ii) $\Rightarrow$ (iii) Since $R$ is an extension of the syntesis operator $D$, it suffices to take $L = R$, $U = M$, and $\{e'_n\}$ the canonical orthonormal basis of $\ell^2$.

(iii) $\Rightarrow$ (i) For every $f \in \mathcal{D}(A^*)$, the adjoint of $L$ is given by

$$L^* f = \sum_{n=1}^{\infty} \langle f | g_n \rangle e'_{n}.$$ 

Indeed, for $c \in \ell^2$

$$\langle L^* f | c \rangle = \langle L^* f | \sum_{n=1}^{\infty} c_n e'_n \rangle = \sum_{n=1}^{\infty} \overline{c_n} \langle f | Le'_n \rangle = \sum_{n=1}^{\infty} \langle e'_n | c \rangle \langle f | g_n \rangle = \sum_{n=1}^{\infty} \langle f | g_n \rangle e'_n | c \rangle.$$ 

Moreover, $\{g_n\}$ is a weak $A$-frame because for every $f \in \mathcal{D}(A^*)$, we have $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 = \|L^* f\|^2 < \infty$ and $\|A^* f\|^2 \leq \|U^* L^* f\|^2 \leq \|U\|^2 \|L^* f\|^2$. 

We conclude this section with some concrete examples.

**Example 3.17** Let us consider the differential operator $Af = -if'$ with domain $H^1(0, 1)$ which is a densely defined closed operator on $\mathcal{H} = L^2(0, 1)$, see [34, Section 1.3]. The sequence $\{g_n\}_{n \in \mathbb{Z}} = \{e_{nb}\}_{n \in \mathbb{Z}}$, where $0 < b \leq 1$ and $e_{nb}(x) = e^{2\pi inbx}$ for $x \in (0, 1)$, is a frame for $L^2(0, 1)$, see [16, Section 9.8]. Therefore, $Ag_n = \{2\pi n e_{nb}\}$ is a weak $A$-frame for $L^2(0, 1)$ by Example 3.5. The canonical dual frame of $\{e_{nb}\}$ is $\{\frac{1}{b} e_{nb}\}$, then according to Example 3.12, we can take $\{\frac{1}{b} e_{nb}\}$ as weak $A$-dual of $\{g_n\}$. The adjoint $A^*$ is the operator $A^* f = -if'$ with $\mathcal{D}(A^*) = H^1_0(0, 1)$, see again [34, Section 1.3]. Note that $A^* \subset A$. Hence, the decomposition in weak sense of Theorem 3.10 reads as

$$\langle -if' | h \rangle = \langle Af | h \rangle = \sum_{n \in \mathbb{Z}} 2\pi n \langle f | e_{nb} \rangle \langle e_{nb} | h \rangle, \quad \forall f \in H^1(0, 1), h \in H^1_0(0, 1).$$

Finally, we have also a strong decomposition of $A^*$ by (3.4):

$$-if' = A^* f = \sum_{n \in \mathbb{Z}} 2\pi n \langle f | e_{nb} \rangle e_{nb}, \quad \forall f \in H^1_0(0, 1).$$

**Example 3.18** Let $\mathcal{H} := L^2(\mathbb{R})$ and denote by $A$ the selfadjoint operator $Af = -if'$ with domain $\mathcal{D}(A) = H^1(\mathbb{R})$. Let $g : \mathbb{R} \to \mathbb{C}$ be a continuous and differentiable function with support $[0, L]$, more generally, one can take a function $g \in H^1(\mathbb{R})$.
such that $g \in W$ where $W$ is the Wiener space, see e.g. [16, Section 11.5] for the definition of $W$.

Let $y \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $T_y, M_\omega : \mathcal{H} \to \mathcal{H}$ be the translation and modulation operators defined, for $f \in \mathcal{H}$, by $(T_y f)(x) = f(x - y)$ and $(M_\omega f)(x) = e^{2\pi i \omega x} f(x)$, respectively. Consider the Gabor system $G(g, a, b)$. By the hypothesis, $\{g_{m,n}\}_{m,n \in \mathbb{Z}} \subseteq \mathcal{D}(A)$. Assume in particular that $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, a necessary and sufficient condition is given in [25, Theorem 6.4.1]. Then, by Example 3.5, $\{A_{g_{m,n}}\}_{m,n \in \mathbb{Z}}$ is a weak $A$-frame, i.e., for some $\gamma > 0$

$$\gamma \|A^* f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f | A_{g_{m,n}} \rangle|^2 < \infty \quad \forall f \in \mathcal{D}(A^*) = \mathcal{D}(A) = H^1(\mathbb{R}).$$

Explicitly,

$$A_{g_{m,n}}(x) = 2\pi b n e^{2\pi i b n x} g(x - a m) - i e^{2\pi i b n x} g'(x - a m) = 2\pi b n (M_{b n} T_{a m} g)(x) - i (M_{b n} T_{a m} g')(x).$$

For the decomposition of $A$, we can use the canonical dual of the Gabor frame $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$ which is a Gabor frame $\{h_{m,n}\}_{m,n \in \mathbb{Z}}$ with some window $h \in L^2(0, 1)$. Since $A$ is selfadjoint, we can write directly a decomposition in strong sense of $A$ according to (3.4)

$$-i f' = Af = \sum_{m,n \in \mathbb{Z}} \langle f | M_{b n} T_{a m} (2\pi b n g - i g') \rangle M_{b n} T_{a m} h, \quad \forall f \in H^1(\mathbb{R}).$$

Once more we point out that the property of being a weak $A$-frame does not depend on the ordering of the sequence $\{M_{b n} T_{a m} (2\pi b n g - i g')\}_{m,n \in \mathbb{Z}}$, see Remark 3.2.

**Example 3.19** Let us consider the same space $\mathcal{H} := L^2(\mathbb{R})$ and the operator $A f = f'$ with domain $\mathcal{D}(A) = H^1(\mathbb{R})$. Let $\phi \in H^1(\mathbb{R})$ and the shift-invariant system $\{\phi_k(x)\}_{k \in \mathbb{Z}} := \{\phi(x - c k)\}_{k \in \mathbb{Z}}$, with $c > 0$. Then $\{(A \phi_k)(x)\}_{k \in \mathbb{Z}} = \{\phi'(x - c k)\}_{k \in \mathbb{Z}}$. However, we cannot apply Example 3.5 to say that $\{A \phi_k\}$ is a weak $A$-frame. Indeed, as it is known [16], $\{\phi_k\}$ is never a frame for $L^2(\mathbb{R})$.

Consider instead the wavelet system $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} := \{a^{-m/2} \phi(a^{-m} x - nb)\}_{m,n \in \mathbb{Z}}$ with $a, b > 0$. We have $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} \subset H^1(\mathbb{R})$ and

$$\{(A \phi_{m,n})(x)\}_{m,n \in \mathbb{Z}} = \{a^{-m/2} \phi'(a^{-m} x - nb)\}_{m,n \in \mathbb{Z}}.$$ The sequence we obtained is nothing but the wavelet system $\{\phi'_{m,n}\}_{m,n \in \mathbb{Z}}$ generated by the derivative $\phi'$ multiplied by the scalars $\{a^{-m}\}_{m \in \mathbb{Z}}$.

When $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}}$ is a frame for $\mathcal{H}$, $\{A \phi_{m,n}\}_{m,n \in \mathbb{Z}}$ is a weak $A$-frame. In particular, by [25, Theorem 10.6 (c)], for any $k \in \mathbb{N}$, there exists a function $\phi$ with compact support and continuous derivatives up to order $k$ such that $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} := \{2^{-m/2} \phi(2^{-m} x - n)\}_{m,n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ and hence, $\{A \phi_{m,n}\}_{m,n \in \mathbb{Z}}$ is a weak $A$-frame.

**Example 3.20** Let $A$ be a closed and densely defined on $\mathcal{H}$. The domain $\mathcal{D}(A)$ of $A$ can be turned into a Hilbert space if endowed with the graph norm $\| \cdot \|_A$. Denote it
by $\mathcal{H}_A$ and by $\mathcal{H}_A^\times$ its conjugate dual and construct the rigged Hilbert space $\mathcal{H}_A \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_A^\times$, where $\hookrightarrow$ means that the embeddings $\mathcal{H}_A \subset \mathcal{H} \subset \mathcal{H}_A^\times$ are continuous with dense range, see e.g. [4, Chapter 10]. Since the sesquilinear form $B(\cdot, \cdot)$ that puts $\mathcal{H}_A$ and $\mathcal{H}_A^\times$ in duality is an extension of the inner product of $\mathcal{H}$, we write $B(\xi, f) = \langle \xi | f \rangle$ for the action of $\xi \in \mathcal{H}_A^\times$ on $f \in \mathcal{H}_A$.

Now let $\{g_n\} \subset \mathcal{H}$. Then $\{g_n\}$ can be regarded as a sequence in $\mathcal{H}_A^\times$. Assume that it is a Bessel-like sequence in the sense of [12, Definition 2.10], i.e., for every bounded subset $\mathcal{M} \subset \mathcal{H}_A$,

$$\sup_{f \in \mathcal{M}} \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty.$$ 

Then, by [12, Proposition 2.11], $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty$ and the operator $F : \mathcal{H}_A \rightarrow \ell^2$ given by $Ff := \{\langle f | g_n \rangle\}$ is bounded. If $F$ is also injective, e.g., if $\{g_n\}$ is dense in $\mathcal{H}$ and has closed range, then $\{g_n\}$ is a weak $A^\ast$-frame since

$$c \|Af\|^2 \leq c \|f\|_A^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 = \|Ff\|^2 < \infty, \quad \forall f \in \mathcal{D}(A)$$

and for some $c > 0$.

## 4 Atomic systems for bounded operators between different Hilbert spaces

In this section, we will give another generalization of the notions and results in [23] to unbounded closed densely defined operators in a Hilbert space. If $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ is a closed and densely defined operator, then it can be seen as a bounded operator $A : \mathcal{H}_A \rightarrow \mathcal{H}$ between two different Hilbert spaces, where by $\mathcal{H}_A$ we indicate the Hilbert space $\mathcal{D}(A)[\| \cdot \|_A]$ with $\| \cdot \|_A$ the graph norm.

Thus, before going forth, we reproduce the main definitions and results in [23] for bounded operators from a Hilbert space $\mathcal{J}$ into another, say $\mathcal{H}$, omitting the proofs since they are very similar to the standard ones where $\mathcal{J} = \mathcal{H}$, [23, 33]. We will come back to the operator $A : \mathcal{H}_A \rightarrow \mathcal{H}$ in Section 4.1.

Let $\langle \cdot | \cdot \rangle_{\mathcal{H}}, \langle \cdot | \cdot \rangle_{\mathcal{J}}$ be the inner products and $\| \cdot \|_{\mathcal{H}}, \| \cdot \|_{\mathcal{J}}$ the norms of $\mathcal{H}$ and $\mathcal{J}$, respectively. We denote by $\mathcal{B}(\mathcal{J}, \mathcal{H})$ the set of bounded linear operators from $\mathcal{J}$ into $\mathcal{H}$.

**Definition 4.1** Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. An atomic system for $K$ is a sequence $\{g_n\} \subset \mathcal{H}$ such that

1. $\{g_n\}$ is a Bessel sequence,
2. there exists $\gamma > 0$ such that for all $f \in \mathcal{J}$ there exists $a_f = \{a_n(f)\} \in \ell^2$, with $\|a_f\| \leq \gamma \|f\|_{\mathcal{J}}$ and $Kf = \sum_{n=1}^{\infty} a_n(f)g_n$.

Clearly the previous notion reduces to that of atomic system in [23] when $\mathcal{J} = \mathcal{H}$.
Example 4.2 Let $\mathcal{H}$ be separable and $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. Every frame $\{g_n\}$ for $\mathcal{H}$ is an atomic system for $K$. Indeed, if $\{v_n\}$ is a dual frame of $\{g_n\}$, then

$$Kf = \sum_{n=1}^{\infty} \langle Kf | v_n \rangle \mathcal{H} g_n, \quad \forall f \in \mathcal{J}$$

and the definition is satisfied by taking $a_f = \{ \langle Kf | v_n \rangle \mathcal{H} \}$ for $f \in \mathcal{J}$.

Example 4.3 Let $\mathcal{J}$ be separable, $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ and $\{f_n\}$ a frame for $\mathcal{J}$ with dual frame $\{h_n\} \subset \mathcal{J}$, then for all $f \in \mathcal{J}$

$$f = \sum_{n=1}^{\infty} \langle f | h_n \rangle \mathcal{J} f_n, \quad \text{hence} \quad Kf = \sum_{n=1}^{\infty} \langle f | h_n \rangle \mathcal{J} Kf_n.$$ 

Thus, the sequence $\{g_n\} = \{Kf_n\}$ is an atomic system for $K$, taking $a_f = \{a_n(f)\} = \{\langle f | h_n \rangle \mathcal{J} \}$.

For $L \in \mathcal{B}(\mathcal{J}, \mathcal{H})$, we denote by $L^* \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ its adjoint. We now give a characterization of the atomic systems for operators in $\mathcal{B}(\mathcal{J}, \mathcal{H})$ similar to that obtained by Găvruţa in [23, Theorem 3].

Theorem 4.4 Let $\{g_n\} \subset \mathcal{H}$ and $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. Then the following are equivalent.

(i) $\{g_n\}$ is an atomic system for $K$;

(ii) there exist $\alpha, \beta > 0$ such that for every $f \in \mathcal{H}$

$$\alpha \|K^* f\|_{\mathcal{J}}^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle_{\mathcal{H}}|^2 \leq \beta \|f\|_{\mathcal{H}}^2; \quad \text{(4.1)}$$

(iii) $\{g_n\}$ is a Bessel sequence of $\mathcal{H}$ and there exists a Bessel sequence $\{k_n\}$ of $\mathcal{J}$ such that

$$Kf = \sum_{n=1}^{\infty} \langle f | k_n \rangle \mathcal{J} g_n, \quad \forall f \in \mathcal{J}. \quad \text{(4.2)}$$

Definition 4.5 Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. A sequence $\{g_n\} \subset \mathcal{H}$ is called a $K$-frame for $\mathcal{H}$ if the chain of inequalities (4.1) holds true for all $f \in \mathcal{H}$ and some $\alpha, \beta > 0$.

By (4.2), the range $\mathcal{R}(K)$ must be a separable subspace of $\mathcal{H}$, which may be non separable. As in [33, Definition 2.1], a sequence $\{k_n\} \subset \mathcal{J}$ as in (4.2) is called a $K$-dual of the $K$-frame $\{g_n\} \subset \mathcal{H}$.

Example 4.6 As in Section 3, we remark that, in general, a $K$-dual $\{k_n\} \subset \mathcal{J}$ of a $K$-frame $\{g_n\} \subset \mathcal{H}$ is not unique. Then, for the $K$-frames $\{g_n\}$ considered in Examples 4.2 and 4.3, we give possible $K$-duals.

(i) If $\{g_n\} := \{f_n\}$, with $\{f_n\} \subset \mathcal{H}$ a frame for $\mathcal{H}$, then one can take $\{k_n\} = \{K^* v_n\}$ where $\{v_n\}$ is any dual frame of $\{f_n\}$. 

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(ii) If \( \{g_n\} := \{Kf'_n\} \), with \( \{f'_n\} \subset \mathcal{J} \) a frame for \( \mathcal{J} \), then one can take for \( \{k_n\} \) any dual frame of \( \{f'_n\} \).

Once at hand a \( K \)-frame \( \{g_n\} \), the Bessel sequence \( \{k_n\} \subset \mathcal{J} \) in Theorem 4.4 is a \( K^* \)-frame, see [33, Lemma 2.2] for the case \( \mathcal{J} = \mathcal{H} \).

We now give a characterization of \( K \)-frames involving the synthesis operator. The equivalence of the first two sentences is an easy generalization of [23, Theorem 4] and the other ones are straightforward.

**Theorem 4.7** Let \( K \in \mathcal{B}(\mathcal{J}, \mathcal{H}) \), \( \{g_n\} \subset \mathcal{H} \) and \( D : \mathcal{D}(D) \subseteq \ell^2 \rightarrow \mathcal{H} \) the synthesis operator of \( \{g_n\} \). The following statements are equivalent.

(i) \( \{g_n\} \) is a \( K \)-frame for \( \mathcal{H} \);
(ii) there exists \( L \in \mathcal{B}(\ell^2, \mathcal{H}) \) such that \( g_n = Le'_n \) where \( \{e'_n\} \) is an orthonormal basis for \( \ell^2 \) and \( \mathcal{R}(K) \subset \mathcal{R}(L) \);
(iii) \( D \in \mathcal{B}(\ell^2, \mathcal{H}) \) and \( \mathcal{R}(K) \subset \mathcal{R}(D) \);
(iv) \( D \in \mathcal{B}(\ell^2, \mathcal{H}) \) and there exists \( M \in \mathcal{B}(\mathcal{J}, \ell^2) \) such that \( K = DM \).

From Theorem 4.7 (iii), it follows that a \( K \)-frame is not necessarily a frame sequence, indeed the range of the synthesis operator may be not closed, see [16, Corollary 5.5.2].

### 4.1 Atomic systems for unbounded operators \( A \) and \( A \)-frames

As announced at the beginning of this section, we come back to our original aim to generalize \( K \)-frames, with \( K \in \mathcal{B}(\mathcal{H}) \), in the context of unbounded closed and densely defined operator \( A \) on a Hilbert space \( \mathcal{H} \). Here, for simplicity, we denote again by \( \langle \cdot | \cdot \rangle \) and \( \| \cdot \| \) the inner product and the norm of \( \mathcal{H} \), respectively.

From now on, we will consider \( A \) as a bounded operator in \( \mathcal{B}(\mathcal{H}_A, \mathcal{H}) \), where \( \mathcal{H}_A \) is the Hilbert space obtained endowing the domain \( \mathcal{D}(A) \) with the graph norm \( \| \cdot \|_A \), induced by the graph inner product \( \langle \cdot | \cdot \rangle_A \). Let \( A^Z : \mathcal{H} \rightarrow \mathcal{H}_A \) be the adjoint of \( A : \mathcal{H}_A \rightarrow \mathcal{H} \), different from \( A^* \) the adjoint of the unbounded operator \( A \).

For the reader’s convenience, we rewrite the definitions of atomic system for \( A \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}) \) and of \( A \)-frame. A sequence \( \{g_n\} \subset \mathcal{H} \) is said to be

(i) an **atomic system** for \( A \) if \( \{g_n\} \) is a Bessel sequence and there exists \( \gamma > 0 \) such that for all \( f \in \mathcal{D}(A) \), there exists \( a_f = \{a_n(f)\} \in \ell^2 \), with \( \|a_f\| \leq \gamma \|f\|_A \)
and \( Af = \sum_{n=1}^{\infty} a_n(f)g_n \), with respect to the norm of \( \mathcal{H} \);
(ii) an **\( A \)-frame** if there exist \( \alpha, \beta > 0 \) such that for every \( f \in \mathcal{H} \)

\[
\alpha \|A^zf\|_A^2 \leq \sum_{n=1}^{\infty} |\langle f |g_n\rangle|^2 \leq \beta \|f\|^2.
\]

Hence, Theorem 4.7 can be rewritten as follows.

**Corollary 4.8** Let \( \{g_n\} \subset \mathcal{H} \) and \( A \) a closed densely defined operator on \( \mathcal{H} \). Then the following are equivalent.
(i) \( \{g_n\} \) is an atomic system for \( A \);
(ii) \( \{g_n\} \) is an \( A \)-frame;
(iii) \( \{g_n\} \) is a Bessel sequence of \( \mathcal{H} \) and there exists a Bessel sequence \( \{k_n\} \) of \( \mathcal{H}_A \) such that
\[
Af = \sum_{n=1}^{\infty} \langle f|k_n \rangle_A g_n, \quad \forall f \in \mathcal{D}(A) \tag{4.3}
\]
with respect to the norm of \( \mathcal{H} \).
(iv) the synthesis operator \( D \) of \( \{g_n\} \) is bounded and everywhere defined on \( \ell^2 \) and \( \mathcal{R}(A) \subset \mathcal{R}(D) \);
(v) the synthesis operator \( D \) of \( \{g_n\} \) is bounded and everywhere defined on \( \ell^2 \) and there exists \( M \in \mathcal{B}(\mathcal{H}_A, \ell^2) \) such that \( A = DM \).

Note also that if \( A \in \mathcal{B}(\mathcal{H}) \), then the graph norm of \( A \) is defined on \( \mathcal{H} \) and it is equivalent to \( \| \cdot \| \); thus, our notion reduces to that of [23].

Remark 4.9 The expansion in (4.3) of \( Af \) in terms of \( \{g_n\} \) involves the inner product \( \langle \cdot|\cdot \rangle_A \). One might ask if there exists also a sequence \( \{t_n\} \subset \mathcal{H} \) such that
\[
Af = \sum_{n=1}^{\infty} \langle f|t_n \rangle g_n, \quad \forall f \in \mathcal{D}(A)
\]
like for atomic systems for \( A \in \mathcal{B}(\mathcal{H}) \), see [23, Theorem 3]. The answer, in general, is negative if \( A \) is unbounded. Indeed, let \( \{e_n\} \) be an orthonormal basis for a separable Hilbert space \( \mathcal{H} \) and \( A \) an unbounded closed and densely defined operator in \( \mathcal{H} \). Assume in particular that \( \{e_n\} \not\subset \mathcal{D}(A^*) \), such an orthonormal basis for \( \mathcal{H} \) can always be found. Clearly, \( \{e_n\} \) is an \( A \)-frame. Suppose that there exists a sequence \( \{t_n\} \subset \mathcal{H} \) such that \( Af = \sum_{n=1}^{\infty} (f|t_n)e_n \), for all \( f \in \mathcal{D}(A) \). Then \( \langle Af|e_n \rangle = \langle f|t_n \rangle \) for all \( f \in \mathcal{D}(A) \) and \( n \in \mathbb{N} \). But this leads to the contradiction that \( \{e_n\} \subset \mathcal{D}(A^*) \).

We conclude by showing an example of an \( A \)-frame which is not a frame.

Example 4.10 Let \( \mathcal{H} = L^2(\mathbb{R}) \), \( \{\alpha_k\}_{k \in \mathbb{Z}} \) be a complex sequence and \( A \) the closed and densely defined operator on \( L^2(\mathbb{R}) \) defined as
\[
(Af)(x) = \begin{cases} \alpha_k f(x) & x \in [2k, 2k + 1[ \\ \alpha_k f(x - 1) & x \in [2k + 1, 2k + 2[ 
\end{cases}
\]
where \( k \) varies in \( \mathbb{Z} \), with natural domain
\[
\mathcal{D}(A) = \left\{ f \in L^2(\mathbb{R}) : \sum_{k \in \mathbb{Z}} |\alpha_k|^2 \int_{2k}^{2k+1} |f(x)|^2 \, dx < \infty \right\}.
\]
The operator \( A \in \mathcal{B}(L^2(\mathbb{R})) \) if and only if \( \{\alpha_k\}_{k \in \mathbb{Z}} \) is bounded.

Now let \( g \in L^2(\mathbb{R}) \) be bounded with support \([0, 2]\) and let the essential infimum of \( |g| \) on \([0, 2]\) be positive, \( \text{essinf}_{x \in [0,2]} |g(x)| > 0 \). Consider the Gabor system \( \mathcal{G}(g, a, b) := \mathcal{G}(g, 2, 1) = \{e^{2\pi imx} g(x-2n)\}_{m,n \in \mathbb{Z}} \); it is Bessel because \( g \) is bounded.
and compactly supported, but it is not a frame since \( ab = 2 > 1 \). However, we show that it is an \( A \)-frame. Indeed, the range of the synthesis operator of \( G(g, 1, 2) \) is

\[
\mathcal{R}(D) = \{ f \in L^2(\mathbb{R}) : f(x) = f(x - 1), \forall x \in [2k + 1, 2k + 2[, \forall k \in \mathbb{Z} \}
\]

and contains \( \mathcal{R}(A) \). Therefore, by Corollary 4.8, \( G(g, 2, 1) \) is an \( A \)-frame.

## 5 Conclusions

In conclusion, we make some remarks to highlight the novelty and potential applications of the notion of weak \( A \)-frame. If \( \{ f_n \} \subset \mathcal{H} \) is a frame for \( \mathcal{H} \) and \( \{ h_n \} \subset \mathcal{H} \) is a dual frame of \( \{ f_n \} \), then a closable densely defined operator \( A \) in \( \mathcal{H} \) can be decomposed as follows:

\[
Af = \sum_{n=1}^{\infty} (Af|h_n) f_n, \quad \forall f \in \mathcal{D}(A).
\]

However, in this decomposition, the action of the operator \( A \) still appears. On the contrary, if \( \{ g_n \} \subset \mathcal{H} \) is a weak \( A \)-frame, then by Theorem 3.10, there exists a Bessel sequence \( \{ t_n \} \subset \mathcal{H} \) such that

\[
\langle Ah|u \rangle = \sum_{n=1}^{\infty} \langle h|t_n \rangle \langle g_n|u \rangle, \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*)
\]

and the action of the operator \( A \) does not appear in the decomposition. Since we have also

\[
A^*u = \sum_{n=1}^{\infty} \langle u|g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*)
\]

weak \( A \)-frames are clearly connected to multipliers that have been recently object of many studies, refer, e.g., to the survey [35]. However, few works were directed to unbounded multipliers, so our study could give a contribution in this direction, actually it is what we did in Examples 3.17 and 3.18 for some specific operators.

We want to mention [7–9, 28] where some unbounded multipliers have been defined as model of non-self-adjoint Hamiltonians. Let us focus on [8] for a connection with weak \( A \)-frames. Fixed a complex sequence \( \alpha = \{ \alpha_n \} \) and a Riesz basis \( \phi = \{ \phi_n \} \) with dual \( \psi = \{ \psi_n \} \), one can construct the operator

\[
H_{\alpha, \psi} f = \sum_{n=1}^{\infty} \alpha_n (f|\psi_n) \phi_n
\]

with \( \mathcal{D}(H_{\alpha, \psi}^\alpha) \) being the greatest subspace where (5.1) converges. Then \( \{ \alpha_n \phi_n \} \) is a weak \( H_{\alpha, \psi}^\alpha \)-frame, indeed by [8, Proposition 2.1]

\[
\mathcal{D}(H_{\alpha, \psi}^\alpha \psi^*) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} |(f|\alpha_n \phi_n)|^2 < \infty \right\}
\]

and thus, Theorem 3.10 (iii) is satisfied.
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