Direct and inverse maximum theorems, and some applications

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Abstract

We deal with direct and inverse maximum theorems. Alternative versions to the Berge theorem are provided, by relaxing the compactness condition of the constraint correspondence. These variants allow us to generalize a result by Terazono and Matani. Also, inverse maximum theorems are introduced. These are of two types, according to their generality. First, we consider the framework consisting of topological spaces without linear structure and, on the other hand, the convex case, i.e., when the range space is a vector space is separately considered. By means of one of our inverse maximum theorem, we generalize a corresponding result by Koyima. In the field of applications, we prove the equivalences of some remarkable results existing in the literature.

Keywords: Inverse maximum theorem, Berge’s maximum theorem, Minimax inequality, generalized Nash game, Kakutani’s fixed point theorem.

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1 Introduction

Direct maximum theorems state conditions for objective functions such that correspondences associated to these functions have good properties. The Berge maximum theorem is a remarkable result
in this context. In this case, any continuous objective function defines an upper semicontinuous maximizers’ correspondence, even if the objective function is a continuous correspondence (constraint correspondence) composed with a continuous single-valued map. Moreover, the Berge theorem assures the value function is continuous. Conversely, an inverse maximum theorem seeks to find conditions on the maximizers’ correspondence for obtaining a continuous objective function defining it. In this work, we are interested in both. Indeed, first, we state versions and some generalizations of maximum theorems. In particular, we replace the compactness condition on the values of the constraint correspondence with a compactness condition on the maximizers’ correspondence. A number of authors have addressed these subjects. For instance, Feinberg, Kasyanov and Voorneveld in [17] also relax the compactness condition for the images of the constraint correspondence. However, they add additional requirements to the continuity of the objective function. Dutta and Mitra [13] weaken continuity of the objective function, but impose order theoretic and convexity assumptions on it. Tian and Zhou [34] provide a necessary and sufficient condition for the upper semi-continuity and the non-empty-valuedness of the argmax correspondence. In [24], Khan and Uyanik weaken continuity assumptions of the Berge maximum theorem, for both functions and correspondences. Recently, Neufeld and Sester, in [30], present a novel application of the Berge maximum theorem to the stability of the martingale optimal transport problem. We think the perspective of these authors, along with some results and ideas of the current paper, allow us to prove stability for more general transport problems. Although restricted to the finite-dimensional case, Terazono and Matani introduce a variant of the Berge theorem [33]. In this work, we extend their result for more general spaces. On the other hand, the study of inverse maximum theorems also has paid an important attention. Based on a result by Tulcea in [35], Komiya [25] prove an inverse theorem in the finite-dimensional case, without constraint correspondence. We extend, in this work, Komiya’s result to the infinite-dimensional setting. Another extension of this result is given by Park and Komiya in [31], by extending the result to the infinite-dimensional scenario, under the assumption that the maximizers are contained in the constraint correspondence. Other contributions to the study of an inverse maximum theorem are given by Aoyama in [2] and Yamahuchi in [37]. It is worth noting some key results, such as those given by Tulcea in [35], Haddad in [20], and Haddad and Lasry in [21] help prove the graphs of the minimizers’ correspondences are $G^s$ sets. This fact used to be a recurrent tool for proving inverse theorems and, this also applies to our work.

Among the main applications of the inverse maximum theorems, we mention the equivalences of some remarkable results. By making use of the finite dimensional inverse maximum theorem by Komiya [25], this author in its work proves the equivalence between the Kakutani theorem and an existence theorem for maximal elements by Yannelis and Prabhakar in [38]. Also, the same inverse maximum theorem is used by Yu, Wang and Yang [39] to prove that the Kakutani and Brouwer fixed point theorems can be obtained from the Nash equilibrium theorem. By taking advantage that the inverse maximum theorems, presented in the current work, are valid in the infinite-dimensional setting, we prove the Fan inequality, as stated in Theorem 1 in [16], is equivalent to the direct corollary following the mentioned inequality. Moreover, we prove the equivalence between the Kakutani-Fan-Glicksberg
theorem \cite{13} \cite{18} and the Debreu-Glicksberg-Fan theorem, which generalizes the equivalences proved by Yu, Wang and Yang \cite{39}. Also, we prove that these two results are, at the same time, equivalent to a infinite-dimensional version of the Arrow-Debreu result \cite{2}. Furthermore, since our direct and inverse maximum results include constraint correspondences, we also consider the situation of social equilibrium proposed by Debreu in \cite{12} and Arrow and Debreu in \cite{3}, also known as an abstract economy. However, this kind of problem has other names in the literature like pseudo-games or coupled constraint equilibrium problems, but here we prefer the name generalized Nash games according with \cite{22} \cite{14}. Additionally, it is known that generalized Nash games can be reformulate as quasi-variational inequality problems or quasi-equilibrium problems, see for instance \cite{8} \cite{11} \cite{10} \cite{9} \cite{5} and references therein. Recently, Bueno and Cotrina in \cite{6} showed the inverse formulation, that means some existence results for quasi-variational inequality problems and quasi-equilibrium problems can be deduced from a result for generalized Nash games by using an inverse maximum result on Banach spaces. To this respect, we prove that this infinite-dimensional version of the Arrow-Debreu theorem is also equivalent to the Kakutani-Fan-Glicksberg theorem. We remark that some authors have generalized some equilibrium theorem by relaxing the continuity assumption of the payoff functions, see for instance \cite{34} \cite{32} \cite{31} \cite{30} \cite{29}. However, their existence results are consequence of the Kakutani-Fan-Glicksberg theorem and, according to the equivalence proved in this work, they are equivalent.

Besides this introduction, this work is subdivided as follows. In Section 2 we provide some notations, definitions and facts. Sections 3 and 4 are devoted for the main results and applications of them are presented in Section 5. Finally, we summarize the major results of this work in Section 6.

2 Preliminaries

Let $X$ be a topological space. A function $f : X \to [-\infty, \infty]$ is said to be:

- **lower semicontinuous** if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \leq \lambda\}$ is closed on $X$;
- **upper semicontinuous** if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \geq \lambda\}$ is closed on $X$;
- **continuous** if it is both upper and lower semicontinuous.

It is clear that, a function $f$ is lower semicontinuous if, and only if, $-f$ is upper semicontinuous.

On the other hand, a function $f : C \to \mathbb{R}$ on a convex set $C$ in a vector space is said to be

- **quasi-convex** if for each $\lambda \in \mathbb{R}$ the set $\{x \in C : f(x) \leq \lambda\}$ is convex; and
- **quasi-concave** if for each $\lambda \in \mathbb{R}$ the set $\{x \in C : f(x) \geq \lambda\}$ is convex.

Clearly, $f$ is quasi-convex if, and only if, $-f$ is quasi-concave. Moreover, $f$ is quasi-concave if,
and only if, \( f(tx + (1-t)y) \geq \min\{f(x), f(y)\} \) for all \( x, y \in C \) and \( t \in [0, 1] \). More details about quasi-convex functions and quasi-convex optimization can be found in [4].

Let \( X \) and \( Y \) be two non-empty sets and \( \mathcal{P}(Y) \) be the family of all subsets of \( Y \). A correspondence or set-valued map \( T : X \rightrightarrows Y \) is an application \( T : X \to \mathcal{P}(Y) \), that is, for \( u \in X \), \( T(u) \subset Y \). The graph of \( T \) is defined as

\[
\text{gra}(T) = \{(u,v) \in X \times Y : v \in T(u)\}.
\]

We now recall the notion of continuity for correspondences. Let \( X \) and \( Y \) be two topological spaces. A correspondence \( T : X \rightrightarrows Y \) is said to be:

- **closed**, when \( \text{gra}(T) \) is a closed subset of \( X \times Y \);
- **lower semicontinuous** if the set \( \{x \in X : T(x) \cap G \neq \emptyset\} \) is open, whenever \( G \) is open;
- **upper semicontinuous** if the set \( \{x \in X : T(x) \cap F \neq \emptyset\} \) is closed, whenever \( F \) is closed; and
- **continuous** if it is both lower and upper semicontinuous.

It is straightforward to verify that a correspondence \( T \) is lower semicontinuous if, and only if, for all \( x \in X \) and any open set \( G \subset Y \), with \( T(x) \cap G \neq \emptyset \), there exists a neighborhood \( V_x \) of \( x \) such that \( V_x \) such that \( T(x') \cap G \neq \emptyset \) for all \( x' \in V_x \). In a similar way, \( T \) is upper semicontinuous if, and only if, for all \( x \in X \) and any open set \( G \), with \( T(x) \subset G \), there exists a neighborhood \( V_x \) of \( x \) such that \( T(V_x) \subset V \).

When \( (Y, \mathcal{U}_Y) \) is a uniform space, i.e., \( \mathcal{U}_Y \) is a uniformity on \( Y \), we consider this space endowed with the topology generated by the family \( \{U[a]\}_{a \in Y} \), where \( U[a] = \{y \in Y : (a, y) \in U\} \), for all \( a \in Y \). Also, for all \( A \subseteq Y \) and \( U \in \mathcal{U}_Y \), we denote \( U[A] = \bigcup_{a \in A} U[a] \). For all \( U, V \in \mathcal{U} \), we denote \( U^{-1} = \{(y, x) \in Y \times Y : (x, y) \in U\} \) and \( U \circ V = \{(x, y) \in Y \times Y : \exists z \in Y, (z, y) \in U \text{ and } (x, z) \in V\} \).

Finally, from now on, we will assume that any topological spaces is Hausdorff.

### 3 Direct versions of maximum theorems

The Berge maximum theorem can be stated as follows (see [1]).

**Theorem 1.** Let \( X, Y \) be two topological spaces, \( K : X \rightrightarrows Y \) be a continuous correspondence with non-empty compact values and \( \theta : \text{gra}(K) \to \mathbb{R} \) be a continuous function. Then, the function \( m : X \to \mathbb{R} \) defined as \( m(x) = \max_{y \in K(x)} \theta(x, y) \) is continuous, and the correspondence \( M : X \rightrightarrows Y \), defined as \( M(x) = \{y \in K(x) : \theta(x, y) = m(x)\} \) is upper semicontinuous and has non-empty compact values.
However, for the value function $m$ to be continuous, the constraint correspondence $K$ need not have compact values. Theorem 2 below gives an account of this fact.

**Theorem 2.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a continuous correspondence with non-empty values, $\theta : \text{gra}(K) \to \mathbb{R}$ be a continuous function and $m : X \to (-\infty, \infty]$ be defined as $m(x) = \sup_{y \in K(x)} \theta(x, y)$. Then, $m$ is continuous.

**Proof.** Let $x_0 \in X$. Since, for all $\alpha \in \mathbb{R}$, $\{ x \in X : m(x) > \alpha \}$ is open, we have $m$ is continuous at $x_0$, whenever $m(x_0) = \infty$. Suppose $m(x_0) < \infty$ and fix $\alpha \in \mathbb{R}$ such that $m(x_0) < \alpha$. Choose $\beta \in \mathbb{R}$ satisfies $m(x_0) < \beta < \alpha$ and define the correspondence $H : X \rightrightarrows \mathbb{R}$ as $H(x) = \{ \theta(x, y) : y \in K(x) \}$. We have $H = \theta \circ J$, where $J : X \rightrightarrows X \times Y$ is defined as $J(x) = \{ x \} \times K(x)$. Let $G$ be an open subset of $X \times Y$ such that $J(x_0) \subseteq G$. We easily see that there exists a neighborhood $U_1$ of $x_0$ and an open subset $V$ of $Y$ such that $J(x_0) \subseteq U_1 \times V \subseteq G$. From the upper semicontinuity of $K$, we have there exists a neighborhood $U_2$ of $x_0$ such that $K(x) \subseteq V$, for all $x \in U_2$. Hence, $J(x) \subseteq G$, for all $x \in U_1 \cap U_2$. This proves that $J$ is upper semicontinuous at $x_0$. By applying Theorem 17.23 in [1], we have $H$ is upper semicontinuous. Consequently, there exists a neighborhood $U$ of $x_0$ such that $m(x) = \sup_{y \in K(x)} \theta(x, y) \leq \beta < \alpha$, for all $x \in U$. This proves the upper semicontinuity of $m$. The lower semicontinuity of $m$ is obtained from Lemma 17.29 in [1]. Accordingly, $m$ is continuous and the proof is complete.

Also, it is worth noting that for the argmax correspondence $M$ to be upper semicontinuous, the constraint correspondence need not be compact-valued. However, in this case, we ask $M$ to be compact-valued. More precisely, we have the following result.

**Theorem 3.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a closed-valued and continuous correspondence with non-empty values, $\theta : \text{gra}(K) \to \mathbb{R}$ be a continuous function and $m : X \to (-\infty, \infty]$ be defined as $m(x) = \sup_{y \in K(x)} \theta(x, y)$. Suppose the correspondence $M : X \rightrightarrows Y$, defined as $M(x) = \{ y \in K(x) : \theta(x, y) = m(x) \}$, is compact-valued. Then, $M$ is upper semicontinuous.

**Proof.** By Corollary 17.18 in [1], it suffices to prove that $\text{gra}(M)$ is closed. Let $U$ be a filter in $\text{gra}(M)$ converging to a point $(x_0, y_0) \in X \times Y$. By the assumption and Theorem 2 the functions $\theta$ and $m$ are continuous. Hence, $\theta(U)$ and $(m \circ \pi_X)(U)$ converge to $\theta(x_0, y_0)$ and $(m \circ \pi_X)(x_0, y_0) = m(x_0)$, respectively, where $\pi_X : X \times Y \to X$ is the canonical projection, which is defined as $\pi_X(x, y) = x$. Consequently, $\theta(x_0, y_0) = m(x_0)$ and it only remains to prove that $(x_0, y_0) \in \text{gra}(K)$. Suppose $y_0 \notin K(x_0)$. By regularity, there exist a neighborhood $V_{y_0}$ of $y_0$ and an open set $G$ of $Y$ such that, $K(x_0) \subseteq G$ and $V_{y_0} \subseteq G = \emptyset$. Since $K$ is upper semicontinuous, there exists a neighborhood $U_{x_0}$ of $x_0$ such that $K(x) \subseteq G$, for all $x \in U_{x_0}$. On the one hand, $U_{x_0} \times V_{y_0} \subseteq X \times Y \setminus \text{gra}(K)$ and, on the other hand, there exists $A \in U$ such that $A \subseteq U_{x_0} \times V_{y_0}$, since $U_{x_0} \times V_{y_0}$ is a neighborhood of $(x_0, y_0)$. This is a contradiction, because, for all $A \in U$, $A \subseteq \text{gra}(K)$. Therefore, $\text{gra}(M)$ is closed, $M$ is upper semicontinuous and the proof is complete.
**Example**  Given a topological space $X$, we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $(X, \mathcal{B}(X))$ and by $\Pi(X)$ the set of all probability measures defined on $(X, \mathcal{B}(X))$. We endow the space $\Pi(X)$ with the weak topology. Let $X$ and $Y$ be two complete metric spaces and $c : X \times Y \to \mathbb{R}$ be a continuous and bounded function, the classical transport problem can be formulated as follows: find $\pi^* \in K(\mu, \nu)$ such that

$$
\int c(x, y)\pi^*(dx, dy) = \sup_{\pi \in K(\mu, \nu)} \int c(x, y)\pi(dx, dy),
$$

where $K : \Pi(X) \times \Pi(Y) \to \Pi(X \times Y)$ is the correspondence defined as $K(\mu, \nu)$ is the set of all couplings, $\pi$, with $X$-marginal $\mu$ and $Y$-marginal $\nu$, i.e., $\mu = \pi p_X^{-1}$ and $\nu = \pi p_Y^{-1}$, with $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the usual projections on $X$ and $Y$, respectively. Since the product measure $\mu \otimes \nu$, of $\mu$ and $\nu$, belongs to $K(\mu, \nu)$, we have $K$ has non-empty values and a classical result in optimal transport asserts that $\pi \in K(\mu, \nu)$, if and only if, the support of $\pi$ is $c$-cyclically monotone (see [30], for instance). Moreover, the function $\theta : \text{gra}(K) \to \mathbb{R}$ such that $\theta((\mu, \nu), \pi) = \int c(x, y)\pi(dx, dy)$ is clearly continuous. Accordingly to Theorem 2 in order to $m : \Pi(X) \times \Pi(Y) \to \mathbb{R}$, defined as $m(\mu, \nu) = \sup_{\pi \in K(\mu, \nu)} \theta((\mu, \nu), \pi)$, be continuous, it suffices that $K$ be continuous. Some authors, as in [19] and [30] for instance, refer to this fact as stability of the optimal transport problem. In [30], this point of view is applied to the particular case when $K(\mu, \nu)$ describes the set of martingale measures on $\mathbb{R}^2$, with fixed marginal measures $\mu$ and $\nu$. We conclude that the stability could prove in quite general optimal transport problems by proving continuity of the correspondence $K$. Moreover, in this case, the correspondence $M : \Pi(X) \times \Pi(Y) \to \Pi(X \times Y)$, defined as $M(\mu, \nu) = \{\pi \in K(\mu, \nu) : \theta((\mu, \nu), \pi) = m(\mu, \nu)\}$, would be upper semicontinuous, whenever $M$ was compact-valued.

By combining Theorems 2 and 3 we have the following alternative version to the Berge maximum theorem.

**Corollary 4.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a closed-valued and continuous correspondence with non-empty values, $\theta : \text{gra}(K) \to \mathbb{R}$ be a continuous function and $m : X \to (-\infty, \infty]$ be defined as $m(x) = \sup_{y \in K(x)} \theta(x, y)$. Suppose the correspondence $M : X \rightrightarrows Y$, defined as $M(x) = \{y \in K(x) : \theta(x, y) = m(x)\}$, is compact-valued. Then, $m$ is continuous and $M$ is upper semicontinuous.

**Remark.** In [17], the authors also relax the compactness condition for the images of the constraint correspondence. Even they drop the upper semicontinuity assumption for this correspondence. However, they add additional requirements to the continuity of the objective function, $\theta$. Indeed, among other conditions, they ask $\theta$ to satisfy $\{(x, y) \in \text{gra}(K) : \theta(x, y) \geq \alpha\}$ is compact, for all $\alpha \in \mathbb{R}$. When the graph of $K$ is not compact, it is easy to find continuous functions, $\theta$, not satisfying this condition. For instance, we consider $K : \mathbb{R} \rightrightarrows \mathbb{R}$ and $\theta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$
K(x) = \mathbb{R} \text{ and } \theta(x, y) = \begin{cases} 
-|y|, & y \leq 1 \\
-1, & y > 1
\end{cases}
$$

Clearly, $\theta$ is continuous but the set $\{(x, y) \in \text{gra}(K) : \theta(x, y) \geq -1\}$ is not compact. Since $M(x) = \{0\}$, for all $x \in \mathbb{R}$, $M$ is compact-valued and Corollary 3 applies. Thus, Corollary 3 is not a consequence.
of Theorem 1.4 in [17].

When the range of the constraint and argmax correspondences is a vector space, we have the following corollary.

**Corollary 5.** Let $X$ and $Y$ be two topological spaces, with $Y$ a vector space, $K : X \rightrightarrows Y$ be a closed-valued and continuous correspondence with non-empty and convex values, $\theta : \text{gra}(K) \to \mathbb{R}$ be a continuous and quasi-concave on $Y$ function and $m : X \to (-\infty, \infty]$ be defined as $m(x) = \sup_{y \in K(x)} \theta(x, y)$. Suppose the correspondence $M : X \rightrightarrows Y$, defined as $M(x) = \{ y \in K(x) : \theta(x, y) = m(x) \}$, is compact-valued. Then, $m$ is continuous and $M$ is an upper semicontinuous correspondence with convex values.

**Proof.** By Corollary 4, we only have to prove that $M$ has convex values. This fact follows due to $M(x) = \bigcap_{z \in K(x)} \{ y \in K(x) : \theta(x, y) \geq \theta(x, z) \}$, for all $x \in X$. 

The above corollary generalizes Theorem 3.1 in [33].

**Corollary 6** (Terazono and Matani, 2015). Let us assume that $K : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell$ is closed-valued, convex-valued, and continuous; $\theta : \text{gra}(K) \to \mathbb{R}$ is continuous and $\theta(x, \cdot)$ is quasi-concave, for each $x \in \mathbb{R}^k$; and $m : X \to (-\infty, \infty]$ be defined as $m(x) = \sup_{y \in K(x)} \theta(x, y)$. Suppose the correspondence $M : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell$, defined as $M(x) = \{ y \in K(x) : \theta(x, y) = m(x) \}$, is nonempty-valued and bounded-valued. Then, $m$ is continuous, and $M$ is convex-valued, compact-valued, and upper semicontinuous.

Below we state, without assumption of compactness for the constraint correspondence, a version of Theorem 3 in [34].

**Proposition 7.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a correspondence with non-empty values, $\theta : \text{gra}(K) \to \mathbb{R}$ and $m : X \to (-\infty, \infty]$ be two functions satisfying $m(x) = \sup_{y \in K(x)} \theta(x, y)$, and $M : X \rightrightarrows Y$ be the correspondence defined as $M(x) = \{ y \in K(x) : \theta(x, y) = m(x) \}$. Then, the following three conditions are equivalent:

(i) for every $(x, y) \in \text{gra}(K)$, the condition $\theta(x, z) > \theta(x, y)$, for some $z \in K(x)$, implies that there exists a neighborhood $V_{(x, y)}$ of $(x, y)$ such that, for any $(x', y') \in V_{(x, y)} \cap \text{gra}(K)$, there exists $z' \in K(x')$ satisfying $\theta(x', z') > \theta(x', y')$,

(ii) $\text{gra}(K) \setminus \text{gra}(M)$ is open in $\text{gra}(K)$, and

(iii) $\text{gra}(M)$ is closed in $\text{gra}(K)$.

If, in addition, $K$ is compact-valued and upper semicontinuous, then $M$ is upper semicontinuous.
Proof. The first part straightforward follows from the obvious equivalences between (i) and (ii) and between (ii) and (iii). By assuming that $M$ is a closed subcorrespondence of $K$ and, $K$ is compact-valued and upper semicontinuous, Corollary 17.18 in [1] implies that $M$ is upper semicontinuous, which completes the proof.

Remark. Proposition 7 does not generalize Theorem 3 in [34], due to $M(x)$ could be empty, for some $x \in X$. However, in [34], a necessary and sufficient condition is given to the correspondence of maximizers has non-empty values.

Lemma 8. Let $X$ and $Y$ be two topological spaces, $K : X \Rightarrow Y$ be a correspondence with non-empty and compact values, $\theta : \text{gra}(K) \to \mathbb{R}$ and $m : X \to \mathbb{R}$ be two functions satisfying $m(x) = \max_{y \in K(x)} \theta(x, y)$, and $M : X \Rightarrow Y$ be the correspondence defined as $M(x) = \{y \in K(x) : \theta(x, y) = m(x)\}$. Then, for each $x \in X$, the following two conditions are equivalent:

(i) $M(x)$ is non-empty, and

(ii) for every $y \in K(x)$, the condition $\theta(x, z) > \theta(x, y)$, for some $z \in K(x)$, implies that there exists $z' \in Y$ and a neighborhood $V_y$ of $y$ such that, for any $y' \in V_{(x,y)} \cap K(x)$, we have $\theta(x, z') > \theta(x, y')$ and $z' \in K(x)$.

Proof. It directly follows from Theorem 2 in [34].

By combining Proposition 7 and Lemma 8, we derive Theorem 3 in [34].

Corollary 9 (Tian and Zhou, 1995). Let $X$ and $Y$ be two topological spaces, $K : X \Rightarrow Y$ be a non-empty and compact-valued and closed correspondence and $\theta : X \times Y \to \mathbb{R}$ and $m : X \to \mathbb{R}$ be two functions satisfying $m(x) = \max_{y \in K(x)} \theta(x, y)$. Then, the correspondence $M : X \Rightarrow Y$, defined as $M(x) = \{y \in K(x) : \theta(x, y) = m(x)\}$, is non-empty compact-valued and closed, if and only if, $\theta$ satisfies condition (i) in Proposition 7 and condition (ii) in Lemma 8. If, in addition, $K$ is upper semicontinuous, then, $M$ is upper semicontinuous.

Remark. Let $K : X \Rightarrow Y$ be a correspondence and $\theta : \text{gra}(K) \to (-\infty, \infty]$ be a function. This function is said to be quasi-transfer upper continuous in $(x, y)$ with respect to $K$, whenever $\theta$ satisfies condition (i) in Proposition 7 and, $\theta$ is said to be transfer upper continuous in $y$ on $K$, whenever $\theta$ satisfies condition (ii) in Lemma 8 for all $x \in X$. We refer to [34] for more details with respect to these concepts.

4 Inverse maximum theorems

In this section, we present some inverse maximum theorems, which we divide into two subsections, according to their generality level. First, we consider the framework consisting of topological spaces
without linear structure. As a second subsection, we introduce the convex case, i.e., when the range space is a vector space.

4.1 The general case

**Theorem 10.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a continuous correspondence with normal graph and non-empty values, and $\hat{M}$ be a subset of $\text{gra}(K)$. Then, the following two conditions are equivalent:

(a) there exists a continuous function $\theta : \text{gra}(K) \to [0, 1]$, such that

$$\hat{M} = \left\{ (x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z) \right\},$$

and

(b) $\hat{M}$ is a closed and $G_\delta$ set.

**Proof.** Let $m : X \to (-\infty, \infty]$ be the function defined by $m(x) = \sup_{z \in K(x)} \theta(x, z)$ and suppose condition (a) holds. Theorem 2 implies that $m$ is continuous, and consequently, $\hat{M}$ is closed. Moreover,

$$\hat{M} = \bigcap_{n=1}^{\infty} \left\{ (x, y) \in \text{gra}(K) : \theta(x, y) > (1 - 1/n)m(x) \right\},$$

which proves that $\hat{M}$ is a $G_\delta$ set and hence condition (b) holds.

Next, suppose there exists a non-increasing sequence of open subsets of $X \times Y$, $\{U_n\}_{n \in \mathbb{N}}$, such that $\hat{M} = \bigcap_{n \in \mathbb{N}} U_n \cap \text{gra}(K)$. Since $\text{gra}(K)$ is normal, for each $n \in \mathbb{N}$, there exists a Urysohn function $\theta_n : \text{gra}(K) \to [0, 1]$ such that $\theta_n \equiv 1$ on $\hat{M}$ and $\theta_n \equiv 0$ on $\text{gra}(K) \setminus U_n$. Let $\theta : \text{gra}(K) \to [0, 1]$ be defined as

$$\theta(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \theta_n(x, y).$$

We have, $\theta$ is continuous, $\theta^{-1}(\{1\}) = \hat{M}$ and therefore

$$\hat{M} = \left\{ (x, y) \in \text{gra}(K) : \theta(x, y) = m(x) \right\},$$

which completes the proof. \qed

As a consequence of the previous result, we have the following two corollaries.

**Corollary 11.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a continuous correspondence with perfectly normal graph and non-empty values, and $\hat{M}$ be a non-empty subset of $\text{gra}(K)$. Then, the following two conditions are equivalent:
(a) there exists a continuous function \( \theta : \text{gra}(K) \to [0,1] \), such that

\[
\hat{M} = \left\{ (x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z) \right\},
\]

and

(b) \( \hat{M} \) is a closed set.

**Corollary 12.** Let \( X \) and \( Y \) be two topological spaces such that \( X \) is normal and \( Y \) is compact, \( K : X \Rightarrow Y \) be a continuous correspondence with non-empty and closed values, and \( \hat{M} \) be a non-empty subset of \( \text{gra}(K) \). Then, the following two conditions are equivalent:

(a) there exists a continuous function \( \theta : \text{gra}(K) \to [0,1] \), such that

\[
\hat{M} = \left\{ (x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z) \right\},
\]

(b) \( \hat{M} \) is a closed and \( G_\delta \) set.

**Proof.** By the closed graph theorem, \( \text{gra}(K) \) is closed, and, since \( X \times Y \) is normal, we have \( \text{gra}(K) \) is normal. Therefore, Theorem 10 applies and the proof is complete.

**Remarks.**  
- According to the Berge maximum theorem, the set \( \hat{M} \), in Theorem 10, is the graph of an upper semicontinuous correspondence \( M : X \Rightarrow Y \) with non-empty and compact values, whenever the two equivalent conditions hold.
- In each of the previous statements, the implication corresponding to obtain (a) from (b) can be considered as an inverse maximum theorem.

The following result gives necessary and sufficient conditions for the existence of an inverse maximum theorem.

**Theorem 13.** Let \( X \) and \( Y \) be two topological spaces such that \( X \) is normal and \( Y \) is compact, \( K : X \Rightarrow Y \) be a continuous correspondence with non-empty and closed values, and \( M : X \Rightarrow Y \) be a correspondence with non-empty values. Then, the following two conditions are equivalent:

(a) there exists a continuous function \( \theta : X \times Y \to [0,1] \) such that \( M(x) = \{ y \in Y : \theta(x,y) = \sup_{z \in Y} \theta(x,z) \}, \) for all \( x \in X \), and

(b) \( M \) is upper semicontinuous with compact values, and \( \text{gra}(M) \) is a \( G_\delta \) set.

**Proof.** By Corollary 12, condition (a) is equivalent to the following condition:

(c) \( \text{gra}(M) \) is a closed \( G_\delta \) set.
From the closed graph theorem, conditions (b) and (c) are equivalent. Therefore, conditions (a) and (c) are equivalent and the proof is complete.

**Remark.** It is worth noting that, in Theorem 13, to obtain (b) from (a), the product space $X \times Y$ need not be normal, due to, essentially, this implication is a consequence of the Berge maximum theorem. On the other hand, by the closed graph theorem, to prove (b) implies (a), we could assume $M$ has a closed graph, instead of $M$ is upper semicontinuous.

The following proposition gives sufficient conditions, to guarantee the existence of an inverse maximum theorem, without compactness condition for the restriction correspondence. Moreover, it explicitly gives a form for the objective function.

**Proposition 14.** Let $X$ and $Y$ be two topological spaces, $K : X \rightrightarrows Y$ be a correspondence with non-empty values, and $\hat{M}$ be a non-empty subset of $\text{gra}(K)$. Suppose there exists a non-increasing sequence, $\{U_n\}_{n \in \mathbb{N}}$, of open subsets of $\text{gra}(K)$, such that

(i) $U_0 = \text{gra}(K)$

(ii) $U_{n+1} \subseteq U_n$, for all $n \in \mathbb{N}$, and

(iii) $\hat{M} = \bigcap_{n=1}^{\infty} U_n$.

Then, the function $\theta : \text{gra}(K) \to [0, 1]$ defined as

$$\theta(x, y) = 1 - \inf \{t > 0 : (x, y) \in U_{n(t)}\} \land 1$$

is continuous, where $n(0) = 0$, $n(t) = \min\{n \in \mathbb{N} : 2^n t \geq 1\}$, for all $t > 0$, and $[a]$ stands for the integer part of $a \in \mathbb{R}$. Furthermore,

(iv) $\hat{M} = \left\{(x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z)\right\}$.

**Proof.** Let $D = \{n/2^n : n \in \mathbb{N}\}$ and $\tau : \text{gra}(K) \to [0, 1]$ be defined as $\tau(x, y) = \inf \{t > 0 : (x, y) \in U_{n(t)}\} \land 1$. Since $\text{gra}(K) = \bigcup_{t \in D} U_{n(t)}$ and $\overline{U_{n(t)}} \subseteq U_{n(t)}$, for all $s, t \in D$, such that $s < t$, Lemma 3, Chapter 4 in [23] implies that $\tau$ is continuous. By defining $\theta = 1 - \tau$, we have $\theta$ is continuous. Moreover, $\theta(x, y) = 1$, if and only if, $(x, y) \in \hat{M}$. Therefore, condition (iv) holds and the proof is complete.

4.2 The convex case

The conclusion of Proposition 14 can be improved when the range space of the correspondences is a vector space.
Proposition 15. Let $X$ and $Y$ be two topological spaces, with $Y$ a vector space, $K : X \rightrightarrows Y$ be a correspondence with non-empty values, and $\hat{M}$ be a non-empty subset of $\text{gra}(K)$. Suppose there exists a non-increasing sequence, $\{U_n\}_{n \in \mathbb{N}}$, of open subsets of $\text{gra}(K)$, such that

(i) $U_0 = \text{gra}(K)$,
(ii) $\overline{U}_{n+1} \subseteq U_n$, for all $n \in \mathbb{N}$,
(iii) $\hat{M} = \bigcap_{n=1}^{\infty} U_n$, and
(iv) $\{y \in Y : (x, y) \in U_n\}$ is convex, for all $n \in \mathbb{N}$ and $x \in X$.

Then, there exists a continuous function $\theta : \text{gra}(K) \to [0, 1]$ such that, for all $x \in X$, the following two conditions:

(v) $\hat{M} = \{(x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z)\}$, and
(vi) $\theta(x, \cdot)$ is quasi-concave.

Proof. Condition (v) is condition (iv) in Proposition [14] and, by applying Lemma 2, Chapter 4 in [23], for all $s \in (0, 1]$ and $x \in X$, we have

$$\{y \in Y : \theta(x, y) \geq s\} = \bigcap_{t \in (1-s, 1] \cap D} \{y \in Y : (x, y) \in U_{n(t)}\}.$$ 

Therefore, condition (vi) holds and the proof is complete.

Theorem 16. Let $X$ be a paracompact space, $Y$ be a locally convex topological space, $K : X \rightrightarrows Y$ be a correspondence with non-empty values and normal graph, and $M : X \rightrightarrows Y$ be a correspondence with convex and compact values (possibly empty), and non-empty closed $G_\delta$ graph contained in $\text{gra}(K)$. Then, there exists a continuous function, $\theta : \text{gra}(K) \to [0, 1]$, such that the following two conditions hold:

(i) $\text{gra}(M) = \{(x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z)\}$, and
(ii) $\theta(x, \cdot)$ is quasi-concave, for all $x \in X$.

Proof. Let us verify the assumptions of Proposition [14]. Since $\text{gra}(M)$ is a $G_\delta$ set, there exists a decreasing sequence, $\{G_n\}_{n \in \mathbb{N}}$, of open subsets of $X \times Y$ such that $\text{gra}(M) = \bigcap_{n \in \mathbb{N}} G_n$. By Lassonde [20] (Chapter 5, Theorem 2), for each $n \in \mathbb{N}$, there exists a correspondence, $W_n : X \rightrightarrows Y$, with convex values and open graph such that $\text{gra}(M) \subseteq \text{gra}(W_n) \subseteq G_n$, for all $n \in \mathbb{N}$. Let $U_n = \text{gra}(W_n)$ ($n \in \mathbb{N}$). We can assume $\{U_n\}_{n \in \mathbb{N}}$ is decreasing, $U_0 = \text{gra}(K)$ and, by normality, $\overline{U}_{n+1} \subseteq U_n$, for all $n \in \mathbb{N}$. The sequence $\{U_n\}_{n \in \mathbb{N}}$ satisfies the four conditions of Proposition [14]. Therefore, condition (i) and (ii) hold and the proof is complete.
As a consequence of Theorem 16, we have the following two corollaries.

**Corollary 17.** Let $X$ be a paracompact space, $Y_0$ be a compact and convex subset of a locally convex topological vector space $Y$, $K : X \rightrightarrows Y_0$ be an upper semicontinuous correspondence with non-empty closed values, and $M : X \rightrightarrows Y$ be an upper semicontinuous correspondence with convex and closed values (possibly empty), and non-empty $G_δ$ graph contained in $\text{gra}(K)$. Then, there exists a continuous function, $\theta : \text{gra}(K) \to [0, 1]$, such that the following two condition hold:

(i) $\text{gra}(M) = \{(x, y) \in \text{gra}(K) : \theta(x, y) = \sup_{z \in K(x)} \theta(x, z)\}$, and

(ii) $\theta(x, \cdot)$ is quasi-concave, for all $x \in X$.

**Proof.** By the closed graph theorem, $\text{gra}(K)$ and $\text{gra}(M)$ are closed sets. Since $X \times Y_0$ is normal, $\text{gra}(K)$ so is. Moreover, $M$ has convex and compact values. Therefore, Theorem 16 applies and the proof is complete. □

**Corollary 18.** Let $X$ be a paracompact space, $Y_0$ be a compact and convex subset of a locally convex topological vector space $Y$, and $M : X \rightrightarrows Y_0$ be an upper semicontinuous correspondence with convex and closed values, and non-empty $G_δ$ graph. Then, there exists a continuous function $\theta : X \times Y_0 \to [0, 1]$ such that, for all $x \in X$, the following two conditions hold:

(i) $\text{gra}(M) = \{y \in Y : \theta(x, y) = \sup_{z \in Y_0} \theta(x, z)\}$, and

(ii) $\theta(x, \cdot)$ is quasi-concave, for all $x \in X$.

The following two lemmas allows us to prove an extension of the inverse maximum theorem stated by Komiya in [25].

**Lemma 19.** Let $X$ be a topological space, $(Y, \mathcal{U}_Y)$ be a uniform space, and $K : X \rightrightarrows Y$ a non-empty compact-valued lower semicontinuous correspondence. Then, for all $U \in \mathcal{U}_Y$ and $x_0 \in X$, there exists a neighborhood, $V_{x_0}$, of $x_0$ such that $K(x_0) \subseteq U[K(x)]$, for all $x \in V_{x_0}$.

**Proof.** Let $U$ and $W$ be in $\mathcal{U}_Y$ such that $W^{-1} \circ W \subseteq U$ and $x_0 \in X$. Let $y_1, \ldots, y_r$ in $Y$ such that $K(x_0) \subseteq W[y_1] \cup \cdots \cup W[y_r]$ and $K(x_0) \cap W[y_i] \neq \emptyset$, for all $i \in \{1, \ldots, r\}$. Since $K$ is lower semicontinuous, there exists a neighborhood, $V_{x_0}$, of $x_0$ such that $K(x) \cap W[y_i] \neq \emptyset$, for all $x \in V_{x_0}$. Hence, $K(x_0) \subseteq U[K(x)]$, for all $x \in V_{x_0}$. Indeed, let $y \in K(x_0)$ and $x \in V_{x_0}$. There exists $y_i \in Y$ such that $y \in K(x_0) \cap W[y_i]$. Let $z \in K(x) \cap W[y_i]$. We have $(y_i, y) \in W$, $(y_i, z) \in W$ and hence $(z, y) \in W^{-1} \circ W \subseteq U$. Consequently $y \in U[K(x)]$, which completes the proof. □

**Lemma 20.** Let $X$ be a topological space, $Y$ be a topological vector space, $C$ be a convex and open neighbourhood of zero in $Y$, $K : X \rightrightarrows Y$ be a non-empty compact-valued lower semicontinuous correspondence, and $K_C : X \rightrightarrows Y$ the correspondence defined by $K_C(x) = K(x) + C$. Then, $K_C$ has open graph.
Proof. Let \((x_0, y_0) \in \text{gra}(K_C)\). Since \(y_0 \in K(x_0) + C\), there exists \(\delta \in (0, 1)\) such that \(y_0 \in K(x_0) + \delta C\).

Let \(W = \delta C\). From Lemma 15 there exists a neighborhood, \(V_{x_0}\), of \(x_0\) such that \(K(x_0) \subseteq K(x) + W\), for all \(x \in V_{x_0}\). We have \(V_{x_0} \times (y_0 + W) \subseteq \text{gra}(K_C)\). Indeed, let \((x, y) \in V_{x_0} \times (y_0 + W)\). Hence

\[
y \in y_0 + W \subseteq K(x_0) + \delta C + W \subseteq K(x) + W + \delta C + W \subseteq K_C(x).
\]

Therefore, \((x, y) \in \text{gra}(K_C)\) and the proof is complete. \(\square\)

The following theorem generalized the main result by Komiya in [25].

**Theorem 21.** Let \(X\) be a metric space, \(Y\) be a Frechet space and \(M : X \rightrightarrows Y\) be an upper semicontinuous correspondence with non-empty, convex and compact valued, such that the following condition holds:

(i) for any ball, \(B\), in \(X\), \(M(B)\) is relatively compact in \(Y\).

Then, there exists a continuous function \(\theta : X \times Y \to [0, 1]\) such that, for all \(x \in X\), the following two conditions hold:

(ii) \(M(x) = \{y \in Y : \theta(x, y) = \sup_{z \in Y} \theta(x, z)\}\), and

(iii) \(\theta(x, \cdot)\) is quasi-concave.

Proof. From Theorem 16 it suffices to prove that \(\text{gra}(M)\) is a \(G_\delta\) set. By Theorem A.II.2 in [21], there exists a non-increasing sequence \(\{A_n\}_{n \in \mathbb{N}}\), of continuous correspondences from \(X\) to \(Y\), such that \(M = \bigcap_{n \in \mathbb{N}} \text{gra}(A_n)\). For each \(n \in \mathbb{N}\), let \(B(0, \epsilon_n)\) be the ball in \(Y\), with center at zero and radius \(\epsilon_n\), where \(\{\epsilon_n\}_{n \in \mathbb{N}}\) is a non-increasing sequence of positive real numbers converging to zero. Lemma 19 implies that the correspondence \(K_n : X \rightrightarrows Y\), defined as \(K_n(x) = A_n(x) + B(0, \epsilon_n)\), has open graph, for all \(n \in \mathbb{N}\). Moreover, \(\text{gra}(M) = \bigcap_{n \in \mathbb{N}} \text{gra}(K_n)\). This proves that \(\text{gra}(M)\) is a \(G_\delta\) set, which completes the proof. \(\square\)

**Corollary 22** (Komiya, 1997). Let \(X\) be a subset of \(\mathbb{R}^l\), and \(M : X \rightrightarrows \mathbb{R}^m\) be a non-empty compact convex-valued upper semicontinuous correspondence. Then, there exists a continuous function \(\theta : X \times \mathbb{R}^m \to [0, 1]\) such that, for all \(x \in X\), the following two conditions hold:

(i) \(M(x) = \{y \in Y : \theta(x, y) = \sup_{z \in \mathbb{R}^m} \theta(x, z)\}\)

(ii) \(\theta(x, \cdot)\) is quasi-concave.

Proof. Since \(M\) is upper semicontinuous, for any relatively compact set, \(B\), we have \(M(B)\) is relatively compact. Consequently, condition (i) in Theorem 21 holds, which completes the proof. \(\square\)
5 Applications

In this section we present some remarks about the Ky Fan minimax inequality and generalized Nash games. More precisely, we will use inverse maximum theorems in order to establish some equivalence between some famous results.

5.1 Remarks on the Ky Fan minimax inequality

The following result is the famous Ky Fan minimax inequality [16].

**Theorem 23** (Fan, 1972). Let $X$ be a compact convex subset of a topological vector space. Let $\theta$ be a real-valued function defined on $X \times X$ such that

(i) for each $y \in X$, $\theta(\cdot, y)$ is lower semicontinuous;

(ii) for each $x \in X$, $\theta(x, \cdot)$ is quasi-concave.

Then, the minimax inequality

$$\min_{x \in X} \max_{y \in X} \theta(x, y) \leq \max_{x \in X} \theta(x, x)$$

holds.

As a direct consequence of Theorem 23 we have the following results.

**Corollary 24.** Let $X$ be a compact convex set in topological vector space. Let $\theta$ be a real-valued function defined on $X \times X$ such that

(i) $\theta$ is continuous;

(ii) for each $x \in X$, $\theta(x, \cdot)$ is quasi-concave.

Then, the minimax inequality

$$\min_{x \in X} \max_{y \in X} \theta(x, y) \leq \max_{x \in X} \theta(x, x)$$

holds.

In this section, our purpose is to illustrate how an inverse theorem allows to prove that this last corollary implies the Ky Fan inequality and, consequently, both results are equivalent. To this end, we need the following lemma.

**Lemma 25.** Let $X$ be a compact set of a locally convex topological vector space $E$ and $T : X \rightrightarrows X$ be a upper semicontinuous correspondence with compact values. Then the graph of $T$ is a $G_{\delta}$ set.
Proof. Let $P$ be a separating family of seminorms that generates the topology of $E$. For each $\rho \in P$, we consider the function $d_\rho : X \times X \to \mathbb{R}$ defined as

$$d_\rho(x, y) = \inf_{(u, v) \in \text{gra}(T)} \rho(x - u) + \rho(y - v),$$

which is continuous. Now, for each $n \in \mathbb{N}$ we also consider the set

$$O_\rho = \{(x, y) \in X \times X : d_\rho(x, y) < 1/n\}.$$

It is clear that $O_\rho$ is open. Thus the set $O_n = \bigcup_{\rho \in P} O_\rho$ is open. Moreover, it is not difficult to see that $\text{gra}(T) \subset O_n$. We affirm that $\text{gra}(T) = \bigcap_{n \in \mathbb{N}} O_n$. Indeed, let $(x, y)$ be an element of $\bigcap_{n \in \mathbb{N}} O_n$, which means that for each $n \in \mathbb{N}$ and $\rho \in P$, $d_\rho(x, y) < 1/n$. Letting $n$ tends to $+\infty$, we deduce $d_\rho(x, y) = 0$. Therefore, $(x, y) \in \text{gra}(T)$, which completes the proof. 

Proposition 26. Theorem 23 is consequence of Corollary 24.

Proof. Thanks to Lemma 25 and Corollary 17 there exists a continuous function $\theta : C \times C \to [0, 1]$ such that $\theta$ is quasi-concave in its second argument and

$$T(x) = \left\{ y \in C : \theta(x, y) = \max_{z \in C} \theta(x, z) \right\}.$$

We consider the function $g : C \times C \to \mathbb{R}$ defined as

$$g(x, y) = \theta(x, y) - \theta(x, x).$$

Clearly, $g$ vanishes on the diagonal of $C \times C$ and it satisfies assumptions of Corollary 24. Thus, there exists $x_0 \in C$ such that $g(x_0, y) \leq 0$, for all $y \in C$. That means, $x_0$ maximizes $g(x_0, \cdot)$.

On the other hand, it is clear that $T(x) = \{ y \in C : g(x, y) = \max_{z \in C} g(x, z) \}$. Therefore, we deduce that $x_0 \in T(x_0)$. This completes the proof. 

5.2 Remarks on generalized Nash games

A Nash game, [29, Nash 1951], consists of $p$ players, each player $i$ controls the decision variable $x_i \in C_i$ where $C_i$ is a subset of a locally convex topological vector space $E_i$. The “total strategy vector” is $x$ which will be often denoted by

$$x = (x_1, \ldots, x_i, \ldots, x_p).$$

Sometimes we write $(x_i, x_{-i})$ instead of $x$ in order to emphasize the $i$-th player’s variables within $x$, where $x_{-i}$ is the strategy vector of the other players. Player $i$ has a payoff function $\theta_i : E \to \mathbb{R}$ that depends on all player’s strategies, where $E = \prod_{i=1}^p E_i$. Given the strategies $x_{-i}$ of the other players, the aim of player $i$ is to choose a strategy $x_i$ solving the problem $P_i(x_{-i})$:

$$\max_{x_i} \theta_i(x_i, x_{-i}) \text{ subject to } x_i \in C_i.$$
A vector $\hat{x} \in C = \prod_{i=1}^{p} C_i$ is a Nash equilibrium if for all $i \in \{1, \ldots, p\}$, $\hat{x}_i$ solves $P_i(x_{-i})$.

In a generalized Nash game, each player’s strategy must belong to a set identified by the set-valued map $K_i : C^{-1} \rightrightarrows C_i$ in the sense that the strategy space of player $i$ is $K_i(x_{-i})$, which depends on the rival player’s strategies $x_{-i}$. Given the strategy $x_{-i}$, player $i$ chooses a strategy $x_i$ such that it solves the following problem

$$\min_{x_i} \theta_i(x_i, x_{-i}) \text{ subject to } x_i \in K_i(x_{-i}).$$

Thus, a generalized Nash equilibrium (GNEP) is a vector $\hat{x} \in C$ such that the strategy $\hat{x}_i$ is a solution of the problem $(GP_i(x_{-i}))$ associated to $\hat{x}_{-i}$, for any $i \in \{1, \ldots, p\}$.

It is clear that any Nash game is a generalized Nash game. Moreover, the last one is more complex due to the strategy set of each player depends on the strategy of his/her rivals.

The Kakutani fixed point theorem admits an extension to locally convex spaces, by means of the Kakutani-Fan-Glicksberg theorem (see [15, 18]), which we state below.

**Theorem 27** (Kakutani-Fan-Glicksberg). Let $C$ be a non-empty convex and compact subset of a Hausdorff locally convex topological vector space $Y$ and let $T : C \rightrightarrows C$ be a correspondence. If $T$ is upper semicontinuous with convex, closed and non-empty values, then the set $\{x \in C : x \in T(x)\}$ is non-empty.

This theorem allows to obtain the following well known result on the existence of Nash equilibria.

**Theorem 28** (Debreu-Glicksberg-Fan). Suppose $C_1, \ldots, C_p$ are compact, convex and non-empty, and $\theta_1, \ldots, \theta_p$ are continuous and quasi-concave in their first variable, then there exists at least one Nash equilibrium.

The authors in [39] showed that the previous result implies Kakutani’s fixed point theorem, Theorem 27, on finite dimensional spaces. We show that this is also true on locally convex topological spaces.

**Proposition 29.** Theorem 28 implies Theorem 27.

**Proof.** By Lemma 25, the graph of $T$ is a $G_\delta$ set. Thus, there exists a continuous function $f : X \times X \to [0, 1]$ such that $T(x) = \{y \in X : f(x, y) \geq f(x, z) \text{ for all } z \in X\}$ and $f(x, \cdot)$ is quasi-concave, for all $x \in X$, due to Corollary 17. Let $P$ be a separating family of seminorms that generates the topology of $E$. For each $\rho \in P$, we consider the set

$$F_\rho = \{(x, y) \in X \times X : f(x, y) \geq f(x, z) \text{ and } \rho(x - y) \leq \rho(w - y), \text{ for all } w, z \in X\}.$$

Clearly $F_\rho$ is compact. Furthermore, the family $\{F_\rho\}_\rho$ has the finite intersection property. Indeed, consider $\rho_1, \ldots, \rho_n \in P$ and consider the game the functions $\theta_1, \theta_2 : X \times X \to \mathbb{R}$ defined by

$$\theta_1(x, y) = -\sum_{i=1}^{n} \rho_i(x - y) \text{ and } \theta_2(x, y) = f(x, y).$$
Since each function $\theta_j$ is continuous and quasi-concave in $x_j$, from Theorem 28, we deduce the existence of a Nash equilibrium $(\hat{x}, \hat{y})$. That means

$$f(\hat{x}, \hat{y}) \geq f(\hat{x}, y) \text{ for all } y \in X$$

and

$$\sum_{i=1}^{n} \rho_i(\hat{x} - \hat{y}) \leq \sum_{i=1}^{n} \rho_i(x - \hat{y}) \text{ for all } x \in X.$$ 

In this last inequality for $x = y$ we obtain that $\rho_i(\hat{x} - \hat{y}) = 0$ for all $i \in \{1, 2, \ldots, n\}$. Consequently, $(\hat{x}, \hat{y}) \in \bigcap_{i=1}^{n} F_{\rho_i}$. Hence, there exists $(\hat{x}, \hat{y}) \in \bigcap_{\rho \in P} F_{\rho}$ and this implies $\rho(\hat{x} - \hat{y}) = 0$, for all $\rho \in P$. Thus $\hat{x} = \hat{y}$. Since $\hat{y}$ maximizes the function $f(\hat{x}, \cdot)$, we obtain that $\hat{x} \in T(\hat{x})$ and the proof is complete. \hfill \Box

The following result is due to Arrow and Debreu [3] in the finite dimensional setting, we state it as in [14], but on locally convex topological vector spaces.

**Theorem 30** (Arrow-Debreu, 1954). Suppose for each $i \in \{1, 2, \ldots, p\}$, $C_i$ is compact, convex and non-empty, and the following two conditions hold:

(i) the payoff function $\theta_i$ is continuous and quasi-concave in $x_i$, and

(ii) the set-valued map $K_i$ is lower and upper semicontinuous with convex, closed and non-empty values.

Then, there exists at least a generalized Nash equilibrium.

Clearly, Theorem 30 implies Theorem 28. However, the following result establishes that they are equivalent. Consequently, they are equivalent to Theorem 27.

**Theorem 31.** Theorem 28 implies Theorem 30.

**Proof.** For each $i \in \{1, 2, \ldots, p\}$, by Berge maximum theorem, Theorem 1, there exists an upper semicontinuous correspondence with compact, convex and nonempty values $M_i : C_{-i} \rightrightarrows C_i$ such that

$$M_i(x_{-i}) = \left\{ x_i \in K_i(x_{-i}) : \theta_i(x_i, x_{-i}) = \max_{z_i \in K_i(x_{-i})} \theta_i(z_i, x_{-i}) \right\}$$

Now, by Theorem 16 there exists a continuous function $\vartheta_i : C \to [0, 1]$ such that it is quasi-concave in $x_i$ and

$$M_i(x_{-i}) = \left\{ x_i \in C_i : \vartheta_i(x_i, x_{-i}) = \max_{z_i \in C_i} \vartheta_i(z_i, x_{-i}) \right\}.$$

Considering the Nash game defined by the functions $\vartheta_i$s and the strategy sets $C_i$. Theorem 28 guarantees the existence of a Nash equilibrium $\hat{x} \in C$, that means

$$\vartheta_i(\hat{x}) \geq \vartheta_i(x_i, \hat{x}_{-i}), \text{ for all } x_i \in C_i \text{ and all } i \in \{1, 2, \ldots, p\}.$$
In other words, \( \hat{x}_i \in M_i(\hat{x}_{-i}) \), for all \( i \). Therefore, \( \hat{x} \) is a generalized Nash equilibrium and the proof is complete. \( \square \)

**Remarks.**
- The previous result generalizes Theorem 5.6 in [7]. Moreover, we reformulate the generalized Nash equilibrium problem as a classical Nash game.
- In [5, 11, 28, 27, 34] the authors have generalized Theorem 28 and Theorem 30 where the continuity assumption of each payoff function is relaxed. However, their existence results are consequence of Theorem 27. Thus, thanks to Proposition 29 these results are equivalent to Theorem 28.

6 Conclusions

Motivated by the Berge maximum theorem and some inverse theorems associated to this result, we introduce some variants to the direct Berge maximum theorem, which allows us generalize, to more general spaces, a result by Terazono and Matani. This variant consists of relaxing the compactness condition for the values of the constraint correspondence. Under this weakening, we still can obtain continuity of the values function. This fact could allow us to prove stability for some optimal transport problems. If, in addition, the maximizers’ correspondence takes compact values, we obtain the upper semicontinuity of this correspondence. Our inverse maximum results are presented as equivalences, between the existence of continuous objective functions and the property that the maximizers’ correspondence has a \( G_\delta \) graph. This fact, along with results by Tulcea in [35], Haddad in [20], and Haddad and Lasry in [21] helped us prove the inverse maximum theorems introduced in this work. In particular, we extended the main result by Komiya in [25] to the infinite-dimensional setting. The last section of this paper was devoted to two applications. The first one consists of proving the equivalence between the Fan inequality with a direct consequence of it, while the second one shows that famous equilibrium theorems in the literature are equivalent to the well-known Kakutani-Fan-Glicksberg theorem.

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