FROM WMAP TO PLANCK: EXACT RECONSTRUCTION OF FOUR- AND FIVE-DIMENSIONAL INFLATIONARY POTENTIAL FROM HIGH-PRECISION COSMIC MICROWAVE BACKGROUND MEASUREMENTS

L. A. Popa\(^1\), N. Mandolesi\(^2\), A. Carame\(\text{ê}\)\(^1\), and C. Burgana\(^2\)

\(^1\) Institutul de Științe Spațiale București-Măgurele, Ro-077125, Romania; lpopa@venus.nipne.ro
\(^2\) INAF/IASF, Istituto di Astrofisica Spaziale e Fisica Cosmica Bologna, I-40129, Italy

Received 2009 July 31; accepted 2009 October 13; published 2009 November 9

ABSTRACT

We make a more general determination of the inflationary observables in the standard four-dimensional (4D) and five-dimensional (5D) single-field inflationary scenarios by the exact reconstruction of the dynamics of the inflation potential during the observable inflation with a minimal number of assumptions: the computation does not assume the slow-roll approximation and is valid in all regimes if the field is monotonically rolling down its potential. We address higher order effects in the standard and braneworld single-field inflation scenarios by fitting the Hubble expansion rate and subsequently the inflationary potential directly to WMAP5+SN+BAO and Planck-like simulated data sets. Making use of the Hamilton–Jacobi formalism developed for the 5D single-field inflation model, we compute the scale dependence of the amplitudes of the scalar and tensor perturbations by integrating the exact mode equation. The solutions in 4D and 5D inflation scenarios differ through the dynamics of the background scalar field and the number of e-folds assumed to be compatible with the observational window of inflation. We analyze the implications of the theoretical uncertainty in the determination of the reheating temperature after inflation on the observable predictions of inflation and evaluate its impact on the degeneracy of the standard inflation consistency relation. We find that the detection of tensor perturbations and the theoretical uncertainties in the inflationary observable represents a significant challenge for the future Planck cosmic microwave background measurements: distinguishing between the observational signatures of the standard and braneworld single-field inflation scenarios. This work has been done in the frame of Planck Core Team activities.

Key words: cosmic microwave background – cosmological parameters – cosmology: observations – early universe

Online-only material: color figures

1. INTRODUCTION

The primary goal of particle cosmology is to obtain a concordant description of the origin and early evolution of the universe, consistent with both unified field theory and astrophysical and cosmological measurements.

Inflation is the most simple and robust theory able to explain the astrophysical and cosmological observations, providing at the same time self-consistent primordial initial conditions (Starobinsky 1979; Guth 1981; Sato 1981; Albrecht & Steinhardt 1982; Linde 1983) and the mechanisms for quantum generation of the scalar and tensor perturbations (Mukhanov & Chibisov 1981; Hawking 1982; Starobinsky 1982; Guth & Pi 1982; Bardeen et al. 1983; Abbott & Wise 1984).

In the simplest class of inflationary models, inflation is driven by a single scalar field \(\phi\) (inflaton) with some potential \(V(\phi)\), minimally coupled to Einstein gravity. The perturbations are predicted to be adiabatic, nearly scale-invariant, and Gaussian distributed, resulting in an effectively flat universe. At the leading order in slow-roll approximation (Steinhardt & Turner 1984; Salopek & Bond 1990; Liddle & Turner 1994), the amplitudes of scalar and tensor perturbations on a specified comoving wavenumber \(k\), are related through the consistency equation

\[
\frac{A^2_S}{A^2_T} = -\frac{n_T}{2},
\]

where \(A^2_S \propto k^{n_S - 1}\) and \(A^2_T \propto k^{n_T}\) are the amplitudes of scalar and tensor perturbations, respectively, and \(n_S\) and \(n_T\) are their tilts. The consistency equation may be regarded as an independent test of single-field inflationary models as it does not depend on the specific functional form of the inflation potential.

Recent WMAP 5-year cosmic microwave background (CMB) measurements, alone (Dunkley et al. 2009) or complemented with other cosmological data sets (Komatsu et al. 2009), support the standard inflationary predictions of a nearly flat universe with adiabatic initial density perturbations.

In particular, the detected anti-correlations between temperature and \(E\)-mode polarization anisotropy on degree scales (Nolta et al. 2009) provide strong evidence for correlation on length scales beyond the Hubble radius.

Despite the successes of inflationary cosmology, recent proposals in theoretical physics motivated by the developments in superstring and M-theory (Horava & Witten 1996), suggest that our four-dimensional (4D) universe could lie on a brane embedded in higher dimensional spacetime (see, e.g., Rubakov 2001; Maartens 2004 and references therein). In particular, in the type II Randall–Sundrum model (RSII; Randall & Sundrum 1999a, 1999b) our 4D universe is a brane with positive tension \(\lambda\) embedded in a five-dimensional (5D) anti-de Sitter spacetime (AdS\(_5\)). At sufficiently low energies \((\rho \ll \lambda)\), the standard cosmic behavior is recovered and the primordial nucleosynthesis constraint is satisfied, provided that \(\lambda \gtrsim (1 \text{ MeV})^4\).

The simplest way to realize inflation in the RSII model is to have a single scalar field confined to the brane and only gravity in the bulk (Maartens et al. 2000). In this case, the Friedmann equation is modified so that the Hubble parameter \(H \propto \rho\) rather than \(H \propto \rho^{1/2}\) as in the 4D case, leading to significantly modifications of the amplitudes and scale dependences of scalar and tensor perturbations (Binetruy et al. 2000).
The observational constraints on inflationary parameters in 5D scenario, made in general by using the slow-roll (SR) approximation in the high-energy regime ($\rho \gg \lambda$), show that the leading order in SR the consistency equation has precisely the same form as in the standard 4D scenario, the relationship between inflationary observables being independent of the brane tension (Tsujikawa & Liddle 2004; Seery & Taylor 2005). The degeneracy of the consistency equation was associated with the fact that 5D inflationary observables smoothly approach their 4D counterparts as the brane decouples from the bulk approaching the low-energy regime ($\rho \ll \lambda$). The main assumption made by these works is that the back reaction due to metric perturbation in the bulk can be neglected. This assumption is valid to the leading order in the SR approximation, as the coupling between inflation fluctuations and metric perturbation vanishes.

Recently, it was shown (Koyama et al. 2004, 2005a, 2005b) that the sub-horizon inflation fluctuations on the brane excite an infinite ladder of Klauz–Klein modes of the bulk metric perturbations to second order in SR parameters. If the back reaction is taken into account, the amplitude of the scalar perturbations receives second-order SR corrections in addition to Stewart–Lyth corrections (Stewart & Lyth 1993) of the same order of magnitude (Koyama et al. 2008). The degeneracy of consistency equations does not hold when the second-order corrections in SR expansion for perturbations are included (Calcagni 2003, 2004; Ramirez & Liddle 2004; Seery & Taylor 2005).

One of the most anticipated results of forthcoming high-precision CMB experiments is the probing of the physics of inflation and in particular the reconstruction of the inflation potential. In order to have a robust interpretation of upcoming observations it is imperative to understand how the reconstruction process may be affected by the degeneracy of the inflationary observables. In this paper, we aim to make a more general determination of the inflationary observables in 4D and 5D inflationary scenarios by exact reconstruction of the dynamics of the inflation potential during the observable inflation with a minimal number of assumptions (Lesgourgues et al. 2008; Hamann et al. 2008).

Taking the advantage of the formalism developed for the standard single-field inflation (Peiris et al. 2003; Peiris & Easther 2006a, 2006b; Martin & Ringealv 2006; Lesgourgues et al. 2008; Alabidi & Lidsey 2008), we carry out similar calculations for 5D inflation models by fitting the Hubbble function, $H(\phi)$, and subsequently the inflationary potential, $V(\phi)$, directly to WMAP 5-year data (Dunkley et al. 2009; Komatsu et al. 2009) complemented with geometric probes from the Type Ia supernovae (SN) distance–redshift relation and the baryon acoustic oscillations (BAO) measurements and Planck-like CMB anisotropy simulated data.

Our specific goal is to address higher order effects in the standard and braneworld single-field inflation models and to analyze the sensitivity of the present and future CMB temperature and polarization measurements to discriminate between them.

The paper is organized as follows. In Section 2, we review the Hamilton–Jacobi formalisms for 4D and 5D single-field inflation models. In Section 3, we compute the scalar and tensor perturbation spectra for standard and braneworld single-field inflation models by using the exact mode equation. In Section 4, we present the implementation of the Markov Chain Monte Carlo methodology and describe the data sets involved in our analysis. Section 5 is dedicated to the analysis and the interpretation of our results: we present the derived bounds on the inflationary parameters, Hubble SR parameters and the magnitude, slope, and curvature of the inflationary potentials obtained from the fits of 4D and 5D single-field inflation models to our data sets and analyze the possibility of disentangling between standard and braneworld scenarios by using the future Planck high-precision CMB measurements. In Section 6, we draw our conclusions.

Throughout the paper, $m_4$ and $m_5$ denote the corresponding 4D and 5D Planck mass scales, and we have set $Gm_5^2 = \hbar = c = 1$. Also, we denote the derivative by dot with respect to the time and the derivative by prime with respect to the scalar field.

2. THE FOUR- AND FIVE-DIMENSIONAL HAMILTON–JACOBI FORMALISM

2.1. The 4D Single-field Inflation Case

The Hubble SR (HSR) formalism for the standard single-field inflation was set down in detail by Liddle et al. (1994).

The Friedmann equation in a zero-curvature universe is given by

$$H^2_{\text{4D}} = \frac{8\pi}{3m_4^2} \rho,$$

(2)

where $H \equiv \dot{a}/a$ is the Hubble parameter, $a$ is the cosmological scale factor, $\rho = V + \dot{\phi}^2/2$ is the total energy density, where $V(\phi)$ and $\dot{\phi}^2/2$ are the potential and kinetic energy density terms, respectively. Since the dark energy contribution is strongly suppressed by the exponential expansion during inflation (Maartens et al. 2000; Langlois et al. 2001), we set to zero the dark energy term in the above equation.

The equation of motion for the scalar field is given by

$$\ddot{\phi} + 3H\dot{\phi} = -V',$$

(3)

Equations (2) and (3) can be written in the Hamilton–Jacobi form, allowing us to consider inflation in terms of $H(\phi)$ rather than $V(\phi)$ (Liddle et al. 1994; Kinney 2002; Easther & Kinney 2003; Peiris et al. 2003; Kinney et al. 2004):

$$H'(\phi)a'(\phi) = -\frac{4\pi}{m_4^2} H(\phi)a(\phi),$$

(4)

$$\dot{\phi} = \frac{m_4^2}{4\pi} H'(\phi),$$

(5)

$$[H'(\phi)]^2 - \frac{12\pi}{m_4^2} H^2(\phi) = -\frac{32\pi^2}{m_4^2} V(\phi).$$

(6)

For any value of $H(\phi)$, Equation (6) can be used to find $V(\phi)$, while Equations (4) and (5) allow us to convert $\dot{\phi}$-dependence into time dependence.

In the standard 4D inflation, the first three HSR parameters are given (Liddle et al. 1994):

$$\epsilon H = \frac{m_4^2}{4\pi} H^2(\phi),$$

(7)

$$\eta_4 H(\phi) = \frac{m_4^2}{4\pi} H'(\phi),$$

(8)

$$\xi_4 H(\phi) = \frac{m_4^2}{16\pi^2} \frac{H'(\phi) H''(\phi)}{H(\phi)}.$$  

(9)
The dependence of \( V(\phi) \) on \( H(\phi) \) can be obtained by substituting \( \epsilon_H \) into Equation (6) leading to

\[
\frac{8\pi}{3m_d^2} V(\phi) = H^2(\phi) \left[ 1 - \frac{1}{3} \epsilon_H(\phi) \right].
\]  

(10)

The HSR formalism ensures that the condition for inflation to occur is precisely \( \epsilon_H < 1 \), and inflation ends exactly when \( \epsilon_H = 1 \).

2.2. The 5D Single-field Inflation Case

In the 5D inflation case, Equation (2) receives an additional term quadratic in energy density

\[
H_{5D}^2 = \frac{8\pi}{3m_d^2} \rho \left( 1 + \frac{\rho}{2\lambda} \right),
\]  

(11)

where \( \rho = V + \phi/2 \) is the total energy density, and \( \lambda \) is the brane tension. The scalar field \( \phi \) is assumed to obey the same equation of motion as in 4D standard inflation as given by Equation (3). In the low-energy regime \( (\rho \ll \lambda) \), the quadratic term in Equation (11) can be neglected, and one recovers the behavior of the 4D standard cosmology. In a high-energy regime \( (\rho \gg \lambda) \), the deviation from the standard expansion changes the amplitudes and scale dependence of cosmological perturbations.

Hereafter, we will make use of the approach developed by Hawkins & Lidsey (2001, 2003) who derived a general formalism for the 5D inflation case valid in all regimes, having many of the properties of the Hamilton–Jacobi formalism in 4D standard inflation. They defined a quantity \( y(\phi) \) with the same role as \( H(\phi) \) in the case of 4D standard inflation:

\[
y^2(\phi) = \frac{\rho/2\lambda}{1 + \rho/2\lambda},
\]  

(12)

with the inverse relation given by

\[
\rho = \frac{2\lambda y^2(\phi)}{1 - y^2(\phi)}.
\]  

(13)

In terms of \( y(\phi) \), the Friedmann Equation (11) reads as

\[
H_{5D}^2(y) = \frac{16\pi \lambda}{m_d^2} \frac{y^2(\phi)}{(1 - y^2(\phi))^2},
\]  

(14)

where the restriction \( y^2 < 1 \) is imposed, implying that \( y(\phi) \) is proportional to \( H(\phi) \) in the low-energy limit, \( y \to 0 (\rho/\lambda \to 0) \).

The Hamilton–Jacobi equations analogous to Equations (4)–(6) for 4D standard inflation are given by Hawkins & Lidsey (2003) and Ramirez & Liddle (2004)

\[
y'(\phi)u'(\phi) = -\frac{4\pi}{m_d^2} y(\phi)u(\phi),
\]  

(15)

\[
\dot{\phi} = -\left( \frac{\lambda m_d^2}{3\pi} \right)^{1/2} \frac{y'}{1 - y^2},
\]  

(16)

\[
H'(\phi) = -\frac{4\pi}{m_d^2} \frac{(1 + y^2)}{(1 - y^2)^2} \phi,
\]  

(17)

and the dependence of \( V(\phi) \) on \( y(\phi) \) can be obtained by combining Equations (16) and (17) leading to

\[
V(\phi) = \frac{2\lambda y^2}{1 - y^2} - \frac{\lambda m_d^2}{6\pi} \left( \frac{y'}{1 - y^2} \right).
\]  

(18)

The first three HSR parameters in terms of \( y(\phi) \) read as (Ramirez & Liddle 2004):

\[
\epsilon_H = \left( \frac{\lambda m_d^2}{3\pi} \right)^{1/2} \frac{y}{(1 + y^2)} \frac{H^2}{H^2},
\]  

(19)

\[
\eta_H = \left( \frac{\lambda m_d^2}{3\pi} \right)^{1/2} \left[ \frac{y}{(1 + y^2)} \frac{H''}{H^2} - \frac{4 y^2}{(1 + y^2)^2} \frac{H^2}{H^2} \right],
\]  

(20)

\[
\xi_H^2 = \frac{\dot{\phi}}{H^2\phi} - \eta_H^2.
\]  

(21)

The above definitions of HSR parameters are valid in all regimes, generalizing the previous ones, preserving at the same time many of the inflation key properties: they are obtained by demanding the condition for inflation to occur precisely for \( \epsilon_H < 1 \) and to end exactly when \( \epsilon_H = 1 \). Also, they are preserving the lowest order SR definitions of the scalar spectral index, \( n_S = 1 - 4\epsilon_H + 2\eta_H \), and of its running, \( dn_S/d\ln k = 5\epsilon_H \eta_H - 4\epsilon_H^2 - 2\xi_H^2 \).

3. THE 4D AND 5D EXACT MODE EQUATION

The scale dependence of the amplitudes on the scalar (S) and tensor (T) perturbations can be exactly obtained by integrating the mode equation (Mukhanov 1985, 1989):

\[
u_k'' + \left( k^2 - \frac{u_k'(S,T)}{\xi_{S,T}} \right) u_k = 0, \tag{22}
\]

where primes denote the second derivatives with respect to the conformal time.

The numerical evaluation of the spectra involves solving Equation (22) for each value of the wavenumber \( k \), the evolution of \( |u_k(S,T)|/\xi_{S,T} \) to a constant value defining the observable power spectra \( P_{S,T} \). The solutions differ through the evolution of the background scalar field and the prior on the number of e-folds assumed to be compatible with the observational window of inflation.

3.1. The 4D Single-field Inflation Case

We compute the amplitudes of scalar and tensor perturbations by using the standard inflation numerical module from Lesgourgues et al. (2008). For each wavenumber \( k \) in a given range, the code integrates Equation (22) in an observational inflationary window corresponding to a number of \( \Delta N \) e-folds, imposing that \( k \) grows monotonically to the wavenumber \( k_* \) that leaves the Hubble radius when \( \phi = \phi_* \), eliminating at the same time the models violating the condition for inflation (\( \epsilon_H < 1 \)).

For the purpose of present analysis, we reconstruct the Hubble expansion rate \( H(\phi - \phi_*) \) from the data by using the Taylor expansion up to the cubic term,

\[
H(\phi - \phi_*) = H_* + H'(\phi - \phi_*) \phi + \frac{1}{2} H''(\phi - \phi_*) \phi^2 + \frac{1}{6} H'''(\phi - \phi_*) \phi^3,
\]  

(23)

equivalent to keeping the first three HSR parameters. We consider wavenumbers in the range \( 5 \times 10^{-6} - 5 \) Mpc\(^{-1}\) needed
to numerically derive the CMB angular power spectra and the Hubble crossing scale \( k_* = 0.01 \text{ Mpc}^{-1} \).

The analysis, however, depends on the prior on the interval over which the dynamics of the background field is tracked. Actually, the standard inflation numerical module integrates Equation (22) from the time at which \( k/aH = 50 \) until \( d\ln P_{S,T}/d\ln a < 3 \times 10^{-3} \). This choice ensures that inflation started enough time before the observational range and ends (\( \epsilon_H = 1 \)) enough time after the smallest observable scale leaves the Hubble horizon scale \( k_* = 0.01 \text{ Mpc}^{-1} \), leading at the same time to an accuracy of \( \sim 0.1\% \) in final power spectra amplitudes, that is smaller than the expected sensitivity of CMB data (Hamann et al. 2008). For these reasons, we choose to keep this time integration window for our computation.

The power spectra of scalar and tensor perturbations are obtained as (Stewart & Lyth 1993; Copeland et al. 1994)

\[
P_S(k) = \frac{k^3}{2\pi^2} \left( \frac{H}{\phi} \right)^2 \left| \frac{\mu_k}{a} \right|^2,
\]

\[
P_T(k) = \frac{16k^3}{\pi m^2} \left| \frac{\mu_k}{a} \right|^2, \tag{24}
\]

where \( z_S = \frac{a\dot{\phi}}{H} \) for scalars, \( z_T = a \) for tensors and the temporal evolution of the scalar field is given by Equation (5).

3.2. The 5D Single-field Inflation Case

There is supporting evidence for the use of the exact mode Equation (22) in braneworld context, because its derivation does not involve the Friedmann equation (Liddle & Lyth 2000).

We modify the standard inflation module from Lesgourgues et al. (2008) to compute the power spectra of scalar and tensor perturbations for the single-field braneworld inflation. As in the case of 4D standard inflation, the Hubble expansion rate \( H(\phi - \phi_*) \) is obtained from the data by the Taylor expansion around the inflation field and the bulk metric perturbations.[1] The wavenumber \( k \) at the Hubble radius crossing \( k_* \) in both 4D and 5D inflationary scenarios, ensuring the same accuracy in the reconstruction of the inflationary potential (Kinney 2002). As in the 4D case, we impose the condition that each mode \( k \) grows monotonically to the wavenumber \( k_* \) and we eliminate those models violating the condition for inflation (\( \epsilon_H < 1 \)).

Taking \( z_S = aH/\phi \) for scalars, \( z_T = a \) for tensors and the evolution of the scalar field given by Equation (16), the power spectra of scalar and tensor perturbations are then obtained as (Ramirez & Liddle 2004; Koyama et al. 2008)

\[
P_S(k) = \frac{k^3}{25\pi^2} \left( \frac{H}{\phi} \right)^2 \left| \frac{|\mu_k|}{a} \right|^2[1 + K(\beta^2)],
\]

\[
P_T(k) = \frac{16k^3}{\pi m^2} \left| \frac{|\mu_k|}{a} \right|^2. \tag{25}
\]

The correction \( K(\beta^2) \) to \( P_S(k) \) is solely due to the coupling between the inflation field and the bulk metric perturbations (Koyama et al. 2005a, 2005b, 2008). For each wavenumber \( k \), we obtained \( K(\beta^2) \) by numerical computation, taking \( \beta^2 \) given by Koyama et al. (2008):

\[
\beta^2 = \frac{1}{3} \epsilon_H \left[ 1 + \left( \frac{H}{\mu} \right)^2 \right]^{-1/2}. \tag{26}
\]

One should note that, although the power spectra of the tensor perturbations in 4D and 5D inflation have the same form, they differ through their dependences on the cosmological scale factor \( a(\phi) \) and on the conformal time: \( \eta(a) = \int da/a H^2 \).

Defining the amplitudes of scalar and tensor power spectra as (Copeland et al. 1994; Koyama et al. 2005): \( A_S(k) = 2P_{S}(k)/5 \) and \( A_T(k) = P_T^{1/2}(k)/10 \), the scalar and tensor spectral indexes \( n_S,T \) and the running of scalar tilt \( \alpha_S \) at the Hubble radius crossing \( k = aH \) are defined as usual by

\[
n_S - 1 = \frac{d\ln A_S^2}{d\ln k} \bigg|_{k=aH}, \hspace{1cm} n_T = \frac{d\ln A_T^2}{d\ln k} \bigg|_{k=aH}, \hspace{1cm} \alpha_S = \frac{d^2\ln A_S^2}{d\ln^2 k} \bigg|_{k=aH}. \tag{27}
\]

4. THE MARKOV CHAIN MONTE CARLO METHODOLOGY

We use the Markov Chain Monte Carlo (MCMC) technique to reconstruct the inflationary potential and to derive constraints on the inflationary observables in the 4D inflation and 5D single-field inflation models by using the WMAP 5-year data (Dunkley et al. 2009; Komatsu et al. 2009) complemented with geometric probes from the Type Ia supernovae (SN) distance–redshift relation and the BAO.

The SN distance–redshift relation has been studied in detail in the recent unified analysis of the published heterogeneous SN data sets—the Union Compilation08 (Kowalski et al. 2008).

The BAO in the distribution of galaxies are extracted from the Sloan Digital Sky Surveys (SDSS) and Two Degree Field Galaxy Redshift Survey (2DFGRS; Percival et al. 2007). The CMB, SN, and BAO data (WMAP5+SN+BAO) are combined by multiplying the likelihoods. We decided to use these measurements especially because we are testing models deviating from the standard Friedmann expansion. These data sets properly enable us to account for any shift of the CMB angular diameter distance and of the expansion rate of the universe.

For the forecast from Planck-like simulated data, we use the CMB temperature (\( T \)) and polarization (\( P \)) power spectra of our fiducial cosmological model and the expected experimental characteristics of the Planck frequency channels given in Table 1 (Mandolesi et al. 2009; The Planck Consortia 2005). For each frequency channel, we consider a homogeneous detector noise

\[
\begin{array}{cccccc}
\text{Frequency (GHz)} & \text{FWHM (arcminutes)} & \Delta T & \Delta T & \Delta T
\end{array}
\]

| Frequency (GHz) | FWHM (arcminutes) | $\Delta T$ (\mu K) | $\Delta T$ (\mu K) | $\Delta T$ (\mu K) |
|-----------------|-------------------|--------------------|--------------------|--------------------|
| 70              | 13                | 23.48              | 33.21              |                   |
| 100             | 9.5               | 6.8                | 10.9               |                   |
| 143             | 7.1               | 6.0                | 11.4               |                   |

Note. $\Delta T$ and $\Delta T$ are the sensitivities per pixel for temperature and polarization maps.

\[\text{Table 1: The Expected Experimental Characteristics for the Planck Frequency Channels Considered in the Paper (Mandolesi et al. 2009; The Planck Consortia 2005)}\]

The normalization of $P_T(k)$ ensures that $A_S$ coincides precisely with the density contrast $\delta_H$ at the Hubble radius crossing as defined by Liddle & Lyth (2000). The normalization of $P_T(k)$ is then chosen so that $\epsilon_H = A_T^2/A_S^2$.\[\text{Note. $\Delta T$ and $\Delta T$ are the sensitivities per pixel for temperature and polarization maps.}\]
Figure 1. CMB angular power spectra of the fiducial ΛCDM cosmological model and the temperature (dashed red line) and polarization (dash-dotted blue line) noise power spectra obtained for the Planck experimental characteristics presented in Table 1, considering a coverage of the sky of 80%. (A color version of this figure is available in the online journal.)

with the power spectrum given by Perotto et al. (2006) and Popa & Vasile (2007):

\[ N_{\ell h}^v = (\theta_0 \Delta_0)^2 \exp \left( \frac{19+10y}{8} \ln^2 \frac{c}{(T, P)} \right) , \tag{28} \]

where \( v \) is the frequency of the channel, \( \theta_0 \) is the FWHM of the beam, and \( \Delta_0 \) are the corresponding sensitivities per pixel. The global noise of the experiment is obtained as

\[ N_{\ell}^c = \left[ \sum_v \left( N_{\ell, v}^c \right)^{-1} \right]^{-1} . \tag{29} \]

Our fiducial model is the standard ΛCDM cosmological model with the physical baryon density \( \Omega_b h^2 = 0.022 \), the physical dark matter density \( \Omega_{cdm} h^2 = 0.11 \), the ratio of the sound horizon distance to the angular diameter distance \( \theta_s = 1.04 \), the reionization optical depth \( \tau = 0.085 \), the scalar spectral index \( n_S = 0.96 \) and the curvature fluctuations amplitude \( \Lambda^2 = 2.28 \times 10^{-9} \) at pivot scale \( k = 0.01 \text{ Mpc}^{-1} \) (Dunkley et al. 2009; Komatsu et al. 2009). We present in Figure 1 the CMB angular power spectra of the fiducial ΛCDM cosmological model and the temperature and polarization noise power spectra obtained for the Planck experimental characteristics presented in Table 1 considering a coverage of the sky of 80%. The final evaluation of the systematic effects that could remain in the Planck data after data reduction affecting scientific exploitation will come from accurate in-flight analyses and extensive Monte Carlo simulations, and is out of the scope of this work. On the other hand, we include in this study also a degradation of Planck ideal sensitivity possibly introduced by residuals of systematic effects at low multipoles, where the cross-check for systematics possible at high multipoles comparing different sky areas is obviously not feasible. The most critical source of contamination will likely come from the stray light, e.g., the signal entering far side lobes (Sandri et al. 2004) at a large angular distance from the main beam. Two different sources mainly contribute to this effect: the CMB dipole (Burigana et al. 2004) and the Galactic emission (Burigana et al. 2006). The former affects only even multipoles, but it is larger in amplitudes at the considered frequencies, the latter is smaller in amplitudes, but affect all multipoles.

simple conservative toy model, based on the above studies, assumes an increasing of the noise power at low multipoles coming from residuals of the angular power spectra estimated for these systematic effects possible generated by a non-perfect subtraction of them, as in the case in which the properties of Planck optical response in the far side lobes are known only with an accuracy of about 30%. The uncertainty added to the instrumental (receiver) noise is clearly visible in Figure 1.

We evaluate the likelihood function for 4D and 5D inflationary models by using the public packages CosmoMC\(^4\) and CAMB\(^5\) (Lewis & Briddle 2000; Lewis et al. 1999) modified to enable us to include the corresponding Hamilton–Jacobi formalism as described in the previous section. We perform the analysis in the framework of the flat ΛCDM standard cosmological model.

For the 4D inflation case, the ΛCDM standard cosmological model is described by the following sets of parameters receiving uniform priors:

\[ \Omega_b h^2, \Omega_c h^2, \theta_s, \tau, A_S^2, H^2, H'' H, H' H', \]

where \( H', H'', \) and \( H''' \) are the derivatives of the Hubble expansion rate \( H \) with respect to the scalar field. As noted before by Lesgourgues et al. (2008), because the physical effects in the primordial power spectra depend on combinations of Hubble expansion rate derivatives, the basis of parameters receiving uniform priors should consists in functions of the above combinations or linear combinations of them, ensuring that Markov Chains can converge in a reasonable amount of time.

By analogy, we take for the 5D inflation case, the following basis of parameters receiving uniform priors:

\[ \Omega_b h^2, \Omega_c h^2, \theta_s, \tau, A_S^2, \frac{y^2}{y'}, \frac{y''}{y'}, \frac{y'''}{y'}, \mu, \]

where \( y', y'', \) and \( y''' \) are the derivatives with respect to the scalar field of the parameter \( y(\phi) \) defined by Equation (12). One should note that the 5D inflation case requires the additional parameter \( \mu \) that controls the hierarchy of 4D and 5D Planck mass scales through the brane tension \( \lambda \):

\[ \mu = \frac{m_S^2}{m_4} = \sqrt{\frac{4 \pi \lambda}{3}} . \tag{30} \]

We run 32 Monte Carlo chains per model and data set, imposing for each case the Gelman and Rubin convergence criterion (Gelman & Rubin 1992).

5. THE RESULTS: ANALYSIS AND INTERPRETATION

5.1. The 4D and 5D Inflationary Parameter Bounds

The parameter bounds derived from each set of chains are given in Table 2, while Figures 2 and 3 show the results of our fits of 4D and 5D inflationary models on WMAP5+SN+BAO data set and Planck-like simulated data set. All parameters are computed at the Hubble radius crossing \( k_s = 0.01 \text{ Mpc}^{-1} \).

From the fit of 4D inflation model to WMAP5+SN+BAO data set, we obtain bounds on \( n_S, A_S^2, \alpha_S, \) and \( R \) at \( k_s = 0.01 \text{ Mpc}^{-1} \) that translated into bounds at \( k_s = 0.002 \text{ Mpc}^{-1} \) show a good

4 http://cosmologist.info/cosmomc/
5 http://camb.info
Table 2

The Mean Values and 95% CL Lower and Upper Intervals of the Derived Parameters Obtained from the Fit of 4D and 5D Inflation Models to WMAP5+SN+BAO Data Set and Planck-like Simulated Data Set

| Parameter          | WMAP5+SN+BAO | 4D Inflation | 5D Inflation | Planck |
|--------------------|--------------|--------------|--------------|--------|
| \( \Omega_B h^2 \) | 0.021 ± 0.023 | 0.021 ± 0.023 | 0.022 ± 0.023 | 0.023 ± 0.023 |
| \( \Omega_c h^2 \) | 0.11 ± 0.117 | 0.11 ± 0.117 | 0.11 ± 0.117 | 0.112 ± 0.122 |
| \( \sigma_8 \)    | 0.95 ± 0.055 | 0.89 ± 0.056 | 0.88 ± 0.067 | 0.88 ± 0.073 |
| \( \theta_s \)    | 1.031 ± 0.034 | 1.039 ± 0.034 | 1.050 ± 0.031 | 1.051 ± 0.035 |
| \( \ln[10^{10} A_s^2] \) | 3.143 ± 0.201 | 3.061 ± 0.137 | 3.172 ± 0.194 | 3.130 ± 0.142 |
| \( \epsilon_H \)  | < 0.035      | < 0.024      | < 0.019      | < 0.017 |
| \( \eta_H \)     | 0.011 ± 0.051 | -0.008 ± 0.011 | -0.006 ± 0.013 | -0.006 ± 0.013 |
| \( \xi^2 \)      | 0.007 ± 0.017 | 0.001 ± 0.011 | -0.001 ± 0.011 | 0.001 ± 0.008 |
| \( n_S \)        | 0.956 ± 0.032 | 0.941 ± 0.009 | 0.950 ± 0.010 | 0.968 ± 0.010 |
| \( a_s \)        | -0.012 ± 0.021 | -0.006 ± 0.021 | -0.006 ± 0.022 | -0.006 ± 0.022 |
| \( n_T \)        | > -0.042      | > -0.055      | > -0.028      | > -0.043 |
| \( \ln[10^{10} A_T^2] \) | -1.174 ± 0.203 | -0.636 ± 0.177 | -2.687 ± 0.042 | -2.448 ± 0.208 |
| \( R \)          | < 0.556      | < 0.476      | < 0.278      | < 0.220 |
| \( V/\lambda \)  | ⋯ | 8.13 ± 0.623 | ⋯ | 3.196 ± 0.361 |
| \( 10^{13} \times M_4 \) | 1.51 ± 0.011 | 1.34 ± 0.007 | 0.440 ± 0.144 | 0.393 ± 0.131 |
| \( V'/V' m_4^2 \) | 0.897 ± 0.031 | 0.921 ± 0.037 | 0.422 ± 0.071 | 0.680 ± 0.122 |
| \( V''/V' m_4^2 \) | 0.619 ± 0.031 | -0.683 ± 0.105 | 0.356 ± 0.116 | -0.479 ± 0.074 |

Note. All parameters are computed at the Hubble radius crossing \( k_a = 0.01 \) Mpc\(^{-1}\).

hybrid field classes of inflation models (Kinney 2002; Liddle & Taylor 2002) overlaid with our constraints on their joint 68% and 95% confidence intervals. We see that all three classes of inflation models are allowed at 2σ level by the fit of 4D standard inflation to our data sets.

The joint marginalized distribution of \( \epsilon_H \) and \( \eta_H \) obtained from the fit of 5D inflation model shows that the large field and small field classes of inflationary models are allowed by WMAP5+SN+BAO and by Planck datasets at 2σ level, while the hybrid class of inflationary models seems to be disfavored by both data sets in the 5D single-field inflation scenario. The parameter values within each class of allowed inflationary models are tightly constrained by Planck data.

Of particular interest are the differences between the degeneracy directions in \( \epsilon_H - \eta_H \) plane found from the fit of 4D inflation model to WMAP5+SN+BAO data set and Planck data set that arise due to the dependence of \( \alpha_s \) on \( \eta_H \). The role of the \( \xi^2 \) in the dynamics of inflation is discussed in details in (Chongchitnan & Efstathiou 2005; Easther & Peiris 2006) and the accuracy of SR inflation models with significant running is probed by using Monte Carlo reconstruction in (Easther & Kinney 2003; Makarov 2005; Peiris & Easther 2006). Looking at Figure 4, one can see the preference of WMAP5+SN+BAO data set for large and positive \( \xi^2 \) values in the 4D standard inflation case that translates into large negative values of the running of scalar spectral index \( a_s \) and a larger degeneracy in the \( n_S - \alpha_s \) plane, when compared to the similar results obtained from the fit to the Planck data set (see Figure 2).

The differences between the degeneracy directions obtained in the 4D standard inflation case and 5D inflation arise via the dependence of HSR parameters on the dynamical equations driving inflation, which are different in 4D and 5D inflationary models.

5.2. Reconstruction of 4D and 5D Inflationary Potential

The aim in the reconstruction of the inflationary potential is to take the measurements of various inflationary observables corresponding to a particular wavenumber \( k \) and to use them to obtain the inflationary potential \( V(\phi) \) and its derivatives at the scalar-field value \( \phi_s \), when the scale \( k \) crosses the Hubble radius \( k_a \) during inflation.

In the general case of the single-field braneworld inflation, the slope and the curvature of the 5D inflationary potential as a function of inflationary observables \( R \) and \( n_s \), and on the combination \( V/\lambda \) are given by Liddle & Taylor (2002)

\[
\frac{V'}{V} = \sqrt{\frac{16\pi R}{m_4^2}} \left[ 1 + \frac{V}{2\lambda} \right] \]

(31)

\[
\frac{V''}{V} = \frac{4\pi}{m_4^2} \left( 1 + \frac{V}{2\lambda} \right) \left[ 6R \frac{1 + V/\lambda}{G^{2}(V/\lambda)} + (n_S - 1) \right] \]

(32)

where

\[
G(x) = \sqrt{1 + x^2 - x^2 \sinh^2 \frac{1}{x}} \]

(33)

In the high-energy limit (\( V \gg \lambda \)), the function \( G^2(V/\lambda) \to 3V/2\lambda \). In the low-energy limit (\( \lambda \gg V \)), \( G^2(V/\lambda) \to 1 \) and the scalar and tensor perturbation spectra of 4D standard inflation are recovered.

We compute the magnitude, the slope, and the curvature of the inflationary potential from the fits of 4D and 5D inflation
Figure 2. We show the results of the fits to the WMAP5+SN+BAO data set of 4D inflation model in red and 5D inflation model in blue. The results of the fits to the Planck-like simulated data set of the 4D inflation model are in magenta and of 5D in the inflation model are in cyan. The top plot in each column shows the probability distribution of different scalar inflationary observables, while the other plots show their joint 68% and 95% confidence intervals. All parameters are computed at the Hubble radius crossing $k = 0.01 \text{ Mpc}^{-1}$.

(A color version of this figure is available in the online journal.)

models to our data sets by using Equations (10) and (18), respectively. In Figure 5, we show the allowed regions of the recovered magnitude of the inflationary potential, its slope, and curvature from the fit of 4D and 5D inflation models to WMAP5+SN+BAO data set and Planck-like simulated data set. We also show the one-dimensional (1D) marginal distribution of the recovered inflationary potential, its slope, and curvature as a function of $V/\lambda$, obtained from the fit of 5D inflation model to the same data sets.

Figure 5 explicitly demonstrates the effect of the braneworld reconstruction of the inflationary potential. As $V/\lambda$ is increased, the magnitude and the curvature of the inflationary potential are decreased while its slope steepens. Also, the magnitude, the slope, and the curvature of the inflationary potential are increased in both 4D and 5D inflationary scenarios when $R$ increases.

The mean values of the magnitude, slope, and curvature of the inflationary potentials together with their 95% upper and lower intervals are given in Table 2.

In Figure 6, we show the dependence of the reconstructed regions of 4D and 5D inflationary potentials allowed by the same data sets (at 68% CL) in an observational inflationary window corresponding to $\Delta N = 11$ e-folds, as functions of the scalar field.

5.3. The 4D and 5D Single-field Inflation Consistency Relations

There is an infinite hierarchy of consistency equations of the single-field standard inflation (Lidsey et al. 1997; Song & Knox 2003; Chung et al. 2003; Chung & Romano 2006; Cortês & Liddle 2006). To the leading order in the SR approximation, the consistency relation of the standard scenario given in Equation (1) is degenerate. To next-to-leading order, this consistency relation receives corrections of the form (Copeland
Figure 3. We show the results of the fits to the WMAP5+SN+BAO data set of the 4D inflation model in red and of 5D in the inflation model in blue. The results of the fits to the Planck-like simulated data set of the 4D inflation model are in magenta and of 5D in the inflation model are in cyan. The top plot in each column shows the probability distribution of different tensorial inflationary observables, while the other plots show their joint 68% and 95% confidence intervals. All parameters are computed at the Hubble radius crossing $k_* = 0.01 \text{ Mpc}^{-1}$.

(A color version of this figure is available in the online journal.)

Figure 4. We show the bounds on the HSR parameters derived from the fits to WMAP5+SN+BAO data set of the 4D inflation model in red and of the 5D inflation model in blue. The results of the fits to the Planck-like simulated data set of the 4D inflation model are in magenta and of 5D in the inflation model are in cyan. The top plot in each column shows the probability distribution of different HSR parameters, while the other plots show their joint 68% and 95% confidence intervals. All parameters are computed at the Hubble radius crossing $k_* = 0.01 \text{ Mpc}^{-1}$. We show the division of $\epsilon_H - \eta_H$ plane into a large field ($-\epsilon_H < \eta_H < \epsilon_H$), small field ($\eta_H < -\epsilon_H$), and hybrid field ($0 < \epsilon_H < \eta_H$) classes of inflation models.

(A color version of this figure is available in the online journal.)
As the inflationary observables $n_S$, $n_T$, and $R$ are evaluated at the epoch of horizon crossing quantified by the number of $e$-folds $N$ before the end of the inflation at which our present Hubble scale equaled the Hubble scale during inflation, the uncertainties in the determination of $N$ translate to theoretical errors in the determination of the inflationary observables. Assuming that the ratio of the entropy per comoving interval today to that after reheating is negligible, the main uncertainty in the determination of $N$ is caused by our ignorance in the determination of the reheating temperature after inflation leading to an error of $\Delta N \sim 14$ (Kinney & Riotto 2006; Adshead & Easther 2008).

In order to test the observational signature that standard and braneworld inflationary scenarios may produce, we use the estimates of the inflationary parameters obtained from the fits to WMAP5+SN+BAO and Planck-like simulated data sets to compare the experimental difference between tensor spectral indexes, $n_T^{4D} - n_T^{5D}$, to the theoretical error in the tensor spectral index computed by using the consistency relation (Equation (34)). To the lowest order in SR parameters, the uncertainties $\Delta R$ and $\Delta n_S$ in terms of the uncertainty in the number of $e$-folds $\Delta N$ are given by (Kinney 2002; Kinney et al. 2004):

$$\frac{\Delta R}{\Delta N} = R \left[ (n_S - 1) + \frac{R}{8} \right],$$

$$\frac{\Delta n_S}{\Delta N} = -\frac{5}{16} R(n_S - 1) - \frac{3}{32} R^2 + 2 \xi^2.$$
The theoretical uncertainty on the tensor spectral index in the standard 4D inflation can be straightforward obtained from Equation (34) by using Equations (35) and (36): \[
\frac{\Delta n_T^n}{\Delta N} = \frac{1}{4} \left[ 1 - \frac{n_s}{A_T^2} + \frac{A_T^2}{A_S^2} \right] \Delta R + \frac{1}{8} \frac{A_T^2}{A_S^2} \frac{n_s}{A_T} \Delta N \tag{37}
\]

The estimate of $\Delta n_T^n / \Delta N$ should be compared to $\Delta n_T = n$. In Table 3, we present the mean values of the lowest order estimates of the theoretical errors $\Delta R / R$, $\Delta n_S / n_S$, and $\Delta n_T^n$ from the fit of 4D inflation model to WMAP5+SN+BAO and Planck data sets obtained by assuming $\Delta N = 14$ e-folds and the mean values of the difference between the experimental values of the tensor spectral indexes $\Delta n_T = n_T^{4D} - n_T^{5D}$, while in Figure 7, we show their 1D marginalized likelihood probability distributions. We also show in the same figure the 1D marginalized likelihood probability distribution of the lowest order estimates of the theoretical errors $\Delta n_T^n$, obtained by assuming $\Delta N = 14$, compared with the 1D marginalized likelihood probability distribution of the difference $\Delta n_T = n_T^{4D} - n_T^{5D}$ and their 2D joint allowed bounds (at 68% and 95% CL).

The analysis of the results presented in Figure 7 and Table 3 shows that the $\Delta n_T$ parameter space obtained from the fits of 4D and 5D inflation models to WMAP5+SN+BAO data set is dominated by the theoretical error $\Delta n_T^n$: the confidence interval corresponding to $\Delta n_T^n$ is smaller by a factor of 1.2 than that corresponding to $\Delta n_T$. The same parameter space is better constrained by the Planck data set: in this case, the confidence interval corresponding to $\Delta n_T^n$ is 3 times smaller than that corresponding to $\Delta n_T$. 

### Table 3

| Theoretical Errors | WMAP5+SN+BAO | Planck |
|--------------------|--------------|--------|
| $\Delta R / R$     | -0.115 -0.049 | 0.177 0.478 |
| $\Delta n_S / n_S$ | 0.104 0.512  | -0.006 0.140 |
| $\Delta n_T^n$     | -0.002 -0.003 | -0.004 0.009 |
| $\Delta n_T = n_T^{4D} - n_T^{5D}$ | -0.003 0.040 | 0.003 0.043 |

*Note.* The values are obtained from the fit of 4D inflation model to WMAP5+SN+BAO and Planck data sets by assuming $\Delta N = 14$, and the mean values and 95% CL lower and upper intervals of $\Delta n_T = n_T^{4D} - n_T^{5D}$ obtained from the fits of 4D and 5D inflation models to the same data sets.
We conclude that the detection of tensor perturbations and the theoretical uncertainties in the inflationary observable represents a significant challenge for the future Planck CMB measurements: distinguishing between the observational signatures of the standard and braneworld single-field inflation scenarios.

6. CONCLUSIONS

One of the most anticipated results of forthcoming Planck high-precision CMB measurements is the probing of the physics of inflation and in particular, the reconstruction of the inflation potential. On the other hand, the possibility that our 4D universe could lie on a brane embedded in a higher dimensional space has important consequences for the early universe and in particular for the cosmological implications of the theoretical uncertainty in the determination of the reheating temperature after inflation on the observable window of inflation.

We address higher order effects in the standard and braneworld single-field inflation scenarios by fitting the Hubble expansion rate $H(\phi)$ and subsequently the inflationary potential $V(\phi)$, directly to WMAP5+SN+BAO and Planck-like simulated data sets. One should note that our results refer to the initial scalar and tensor perturbation spectra and not to the braneworld effects on the subsequent evolution of the perturbations that is likely to be model dependent (Leong et al. 2002; Rhodes et al. 2003).

Assuming that the ratio of the entropy per comoving interval today to that after reheating is negligible, we analyze the implications of the theoretical uncertainty in the determination of the reheating temperature after inflation on the observable predictions of inflation.

We find that the detection of tensor perturbations and the theoretical uncertainties in the inflationary observables represents a significant challenge for the future Planck CMB measurements: distinguishing between the observational signatures of the standard and braneworld single-field inflation scenarios.

We acknowledge the use of the GRID computing system facility at the Institute for Space Sciences Bucharest and to the staff working there. L.P. and A.C. are partially supported by ESA/PECS Contract C98051 and CNCSIS Contract 539/2009.
Randall, L., & Sundrum, R. 1999a, Phys. Rev. Lett., 83, 4690
Randall, L., & Sundrum, R. 1999b, Phys. Rev. Lett., 83, 3370
Rhodes, C. S., van de Bruck, C., Brax, P., & Davis, A. C. 2003, Phys. Rev. D, 68, 083511
Rhodes, C. S., van de Bruck, C., Brax, P., & Davis, A. C. 2003, Phys. Rev. D, 68, 083511
Rhodes, C. S., van de Bruck, C., Brax, P., & Davis, A. C. 2003, Phys. Rev. D, 68, 083511
Rubakov, V. A. 2001, Phys. Usp., 44, 871
Salopek, D. S., & Bond, J. R. 1990, Phys. Rev. D, 42, 3936
Sandri, M., Villa, F., Nesti, R., Burigana, C., Bersanelli, M., & Mandolesi, N. 2004, A&A, 428, 299
Sato, K. 1981, MNRAS, 195, 467
Seery, D., & Taylor, A. 2005, Phys. Rev. D, 71, 063508
Song, Y. S., & Knox, L. 2003, Phys. Rev. D, 68, 043518
Starobinsky, A. A. 1979, JETP Lett., 30, 682
Starobinsky, A. A. 1982, Phys. Lett. B, 117, 175
Starobinsky, A. A. 1982, Phys. Lett. B, 117, 175
Steinhardt, P. J., & Turner, M. S. 1984, Phys. Rev. D, 29, 2162
Stewart, E. D., & Lyth, D. H. 1993, Phys. Lett. B, 302, 171
The Planck Consortia 2005, ESA-SCI, 1, arXiv:0604069
Tsujikawa, S., & Liddle, A. R. 2004, J. Cosmol. Astropart. Phys., JCAP03(2004)001