Novel Mechanism for Discrete Scale Invariance in Sandpile Models

Matt Lee$^1$ and Didier Sornette$^{1,3}$

$^1$ Institute of Geophysics and Planetary Physics
University of California, Los Angeles, California 90095
$^2$ Department of Earth and Space Science
University of California, Los Angeles, California 90095
$^3$ LPMC, CNRS UMR6622 and Université de Nice-Sophia Antipolis
B.P. 71, Parc Valrose, 06108 Nice Cedex 2, France

(March 24, 2022)

Numerical simulations and a mean-field analysis of a sandpile model of earthquake aftershocks in 1d, 2d and 3d euclidean lattices determine that the average stress decays in a punctuated fashion after a main shock, with events occurring at characteristic times increasing as a geometrical series with a well-defined multiplicative factor which is a function of the stress corrosion exponent, the stress drop ratio and the degree of dissipation. These results are independent of the discrete nature of the lattice and stem from the interplay between the threshold dynamics and the power law stress relaxation.

Discrete scale invariance (DSI) is the partial breaking of continuous scale invariance in which a system or an observable is invariant only under scaling ratios that are integer powers of a fundamental factor $\lambda$. DSI leads to complex critical exponents (or dimensions), i.e. to logarithmic corrections to scaling, which reflect the existence of a discrete self-similar spectrum of characteristic scales decorating the usual scale-free power law behavior.

Several mechanisms responsible for this partial breaking of the continuous scale symmetry have been expounded, which include build-in pre-existing hierarchy, intermittent diffusion in discrete euclidean lattices and cascades of ultra-violet instabilities in growth processes and rupture. Other situations are less well understood but can be traced back to special technical properties such as the non-unitary structure of the underlying field theory describing the coarse-grained properties of animals and of quenched disordered spin systems with long-range interactions. Another example is the gravitational collapse leading to critical black hole formation described by a system of partial differential equations possessing an asymptotic solution which can be understood from a renormalization group with a limit cycle having a single unstable mode.

Here, we present a novel scenario for DSI based on the interplay between the threshold dynamics characteristic of sandpile models and a scale-free relaxation process. Specifically, we study a conceptual sandpile model of earthquake aftershocks on a euclidean discrete $d$-dimensional cubic lattice with $L^d$ cells and periodic boundary conditions with $d = 1, 2, 3$. Each cell represents a region which is unloaded when an elementary fault is activated. We neglect the tensorial nature of the stress field and consider an anti-plane driving configuration in which loading and rupture are controlled by a single shear stress component $V(\vec{r})$.

There are two distinct temporal phases. First, the stress is uniformly increased at a very slow rate on all cells to mimic the tectonic loading. Due to the rupture and loading rules described below, the system self-organises into a statistical stationary state, characterised by a power law distribution of event sizes. Once this statistical stationarity state is established, we freeze the loading and the aftershock sequence starts, mimicking the aftermath of a great earthquake. The second phase is characterised by the fact that the aftershocks are not driven by the tectonic loading but by relaxation processes as described below.

An initial stress threshold $B(\vec{r})$ is assigned to each cell from a random uniform distribution in the interval $B_0[1-r, 1+r]$. We find similar results both for the annealed and quenched version of the model, in which either the thresholds are fixed or are resampled in the interval after each rupture. When the stress $V(\vec{r})$ in a cell at $\vec{r}$ becomes larger or equal to $B(\vec{r})$, the stress is re-distributed according to the rules

\begin{equation}
V(\vec{r})|_{\text{after}} = V(\vec{r})|_{\text{before}} (1 - \gamma),
\end{equation}

\begin{equation}
V(\vec{r})|_{\text{in}} = V(\vec{r})|_{\text{before}} + V(\vec{r})|_{\text{before}} \frac{(1 - \beta)\gamma}{2d}.
\end{equation}

Rule (3) applies to each of the $2d$ nearest neighbours (n.n.) of $\vec{r}$ carrying an initial stress $V(\vec{r})|_{\text{in}}$ which evolves into $V(\vec{r})|_{\text{after}}$. Because the toppling criterion depends only on $V$ and not on its gradient, the order of site toppling commutes.

$\gamma$ is the relative stress drop, with $\gamma = 1$ corresponding to a complete stress drop. $\beta$ is the dissipation where $1 - \beta$ quantifies the amount of stress drop transferred...
to n.n. and is known as the seismic efficiency. For \( \gamma = 1 \), (3) and (4) are identical to the rules used in the non-conservative sandpile model [4], motivated from the coupling of blocks to a rigid upper driving plate in the Burridge-Knopoff model. Here, the dissipation accounts for the loss of stress and of stored elastic energy due to an earthquake under constant displacement conditions at the boundaries [13].

In the second relaxation phase, the loading stops and the thresholds decay in time according to the law

\[
B(\vec{x}, t) = B(\vec{x}, t_0) - \frac{[V(\vec{x}, t)]^\alpha}{B(\vec{x}, t_0)} (t - t_0). \tag{3}
\]

This model incorporates the mechanism of sub-critical crack growth and stress corrosion [13], which has been proposed as a possible delay mechanism for aftershocks [13,14]. In absence of loading, events are triggered each time the thresholds decay below the local stresses. When this occurs, the stress redistribution obeys (3) and (2). The system eventually relaxes after a infinite time and in an intermittent manner to an equilibrium of zero stress on all elements of the lattice. A closed conservative spring-block system has also been found to relax in a infinite time and in an intermittent manner to the zero-stress equilibrium with self-organized critical behavior [18]. It is this complex relaxation that we study. It occurs via the triggering of what can be called aftershocks which exhibit remarkable properties. The results presented below are also found for a version of the model with continuous elasticity [19] derived from Ref. [17] and are probably robust features of the general interplay between threshold dynamics and relaxation phenomena.

We show here the simulations for 2d systems of \( 20 \times 20 \) elements (simulations have been performed with size up to \( 100 \times 100 \) with no change of results) and in the annealed case where, after each toppling, thresholds are reassigned from the uniform distribution \( B_0[1-r, 1+r] \) with \( r = 0.75 \). The system is up-dated by finding the site closest to rupture and incrementing time, so that this site reaches its threshold. Once a site becomes unstable due to either loading (in the first phase) or by the decay of the threshold (in the second phase), stress is distributed to n.n. according to (1) and (3). The n.n. may also become unstable, releasing their stress and the process continues until no further nodes are unstable, thus defining an event. During the rupture process, time is “freezed” to ensure a separation of time scales between rupture (fast) and loading/decay (slow). When no further sites are unstable, loading or decay is continued until the next event, when time “freezes” again. Typically \( 3 \cdot 10^3 \) to \( 15 \cdot 10^3 \) events were sampled in the decay regime. The results are robust with respect to heterogeneity level \( r \), open or closed boundary conditions, and to the size and dimension \( d = 1, 2, 3 \) of the lattice.

Figure 1 shows the rate of aftershocks \( n(t) \) as a function of time after the loading has ceased. The \( 1/t \) decay

is in full agreement with Omori’s law for real aftershock sequences [20] and is very robust over a wide range of parameters, i.e. \( \alpha > 0, \gamma > 0 \) and \( \beta < 1 \). Typical values for the earth are \( 10 \leq \alpha \leq 100, \gamma = 5 - 15\% \) (stress drop) and \( \beta \approx 99\% \) (corresponding to a seismic efficiency of 1%). The distribution of event sizes also follows a power law (Gutenberg-Richter for aftershocks), but in contrast to Omori’s law, the exponent continuously depends on the three parameters. Furthermore, for large dissipation \( \beta \), the power law extends only up to a maximum scale which decreases as \( \beta \rightarrow 1 \).

These results can be rationalized by the following mean field theory. From (4), we see that the time \( \Delta t \) needed for an isolated element to reach rupture is such that the threshold \( B(\vec{x}, t_0 + \Delta t) \) decreases to the stress level \( V(\vec{x}) \). The mean field argument simply assumes that we can extend this result over all elements of the lattice by replacing \( V(\vec{x}) \) by the average stress \( \langle V(t) \rangle_{\vec{x}} \) accounting for the influence of the possible loading by n.n. This approximation becomes better and better as the dissipation \( \beta \) increases and the dynamics of n.n. elements becomes increasingly uncoupled. This corresponds to a decreasing dependence on spatial inhomogeneities, which is exactly the underlying assumption of any mean-field approximation. We get

\[
\Delta t \approx B_0^2 / \langle V \rangle_{\vec{x}}^\alpha, \tag{4}
\]

where we have approximated \( B_0 - \langle V \rangle_{\vec{x}} \approx B_0 \), since for large times the mean stress level becomes very low compared to the thresholds that are healed back to a typical value in the interval \( B_0[1-r, 1+r] \) after each event. Expression (4) has the same form as obtained from a model of cracks undergoing sub-critical crack growth [9]. Over such a time interval, essentially one main event occurs on each site and, as a consequence, the average stress goes from \( \langle V \rangle_{\vec{x}} \) to \( (1 - \gamma) \langle V \rangle_{\vec{x}} \) corresponding to a typical stress decrease \( \gamma \langle V \rangle \). We can thus write

\[
- \gamma \langle V \rangle_{\vec{x}} \frac{d}{dt} \approx - \frac{\gamma}{B_0^2} \langle V \rangle_{\vec{x}}^{1+\alpha}, \tag{5}
\]
whose solution is
\[ \langle V(t) \rangle_x = (B_0^2/\alpha c)^{1/\alpha} (t + c)^{-1/\alpha} . \]  
(6)
c is a constant determined from the initial value of the average stress at the beginning of the aftershock relaxation sequence.

To get Omori’s law, we recognise that the rate \( n(t) \) of aftershocks is simply proportional to the rate with which the thresholds \( B(\mathcal{F}, t) \) reach the stress level. \( n(t) \) is also proportional to \( 1/\Delta t \). This yields \( n(t) \propto dB(\mathcal{F}, t)/dt \sim \langle V(\mathcal{F}, t) \rangle^\alpha \sim \langle (V(t))_x \rangle^\alpha \sim \frac{1}{(t + c)^{\alpha}} \), with \( p = 1 \). According to this mean field theory, Omori’s law is obtained with the universal exponent \( p = 1 \) independently of the value of the stress-corrosion exponent \( \alpha \), as long as it is positive. For \( \alpha = 0 \), the average stress level decays exponentially fast with time and the rate of aftershocks is constant.

The mean field theory also provides a prediction of log-periodicity. The stress redistribution laws (3) and (4) imply that, over a typical time \( \Delta t \) given by (1), the average stress undergoes the change \( \langle V \rangle_x \rightarrow \langle V \rangle_x/\mu \), where
\[ 1/\mu = [f(d)(1 - \gamma) + n_{\text{eff}}\gamma(1 - \beta)]/f(d). \]  
(7)
f(d) is a geometric factor counting the effective number of n.m. \( f(d) = 2d \) in the large dissipation limit \( \beta \rightarrow 1 \). The first contribution \( (1 - \gamma) \) in the r.h.s. of (7) is simply the initial stress minus the stress drop. The second contribution \( n_{\text{eff}}\gamma(1 - \beta)/f(d) \) results from the number \( n_{\text{eff}} \sim 1 \) of stress loading on a given element due to the earthquakes occurring on its neighbours.

Each time the average stress is decreased by a factor \( \mu \), we see from (4) that the time interval \( \Delta t \) is increased by a factor
\[ \lambda = \mu^{\alpha} . \]  
(8)
Since \( \mu > 1 \), the total time is essentially dominated by the last time interval between the two last cycles. This allows us to write an approximate scaling relation on the average stress \( \langle V \rangle \) :
\[ \langle V(t) \rangle_x = \mu \langle V(\lambda t) \rangle_x \]  
Since the aftershock rate \( n(t) \) is proportional to \( \langle V \rangle_x^\alpha \mu = \lambda \), this leads to \( n(t) = \lambda n(\lambda t) \), using (8). Looking for a power law solution \( n(t) \sim t^{-\beta} \), we retrieve Omori’s exponent \( p = 1 \). A more general solution is the power law \( t^{-\beta} \) multiplied by a periodic function of \( \ln t \),
\[ n(t) = t^{-\beta} P_1(\ln t/\ln \lambda), \]  
where \( P_1(x) \) is periodic with period one. Expanding this periodic function into its Fourier series gives
\[ n(t) = t^{-\beta} \sum_{k=-\infty}^{+\infty} a_k e^{2\pi ik/\ln \lambda}, \]  
with \( a_{-k} = a_k \), which defines the discrete spectrum of complex exponents \( p_k = 1 - i \frac{2\pi k}{\ln \lambda} \). The leading correction to the power Omori’s law gives the log-periodic expression
\[ n(t) = \frac{1}{\tau} (a_0 + a_1 \cos(2\pi \ln t/\ln \lambda)). \]  

To test this prediction, we find that the local exponent \( p(t) \), defined by \( d \ln n(t)/d \ln t = -p(t) \), gives the most sensitive measure of deviation from the \( 1/t \) Omori’s law. We estimate \( p(t) \) by a maximum likelihood estimator in a running window ending at \( t \) [21].

Defining the starting \( t \) and ending \( t_U \) times of a window and the average \( \langle \ln t_i \rangle \) of the logarithms of all \( N \) aftershock times within this window, the MLE is
\[ p(t) \approx 12(\ln \sqrt{P_U} - (\ln t_i))/((\ln (t_U/t)) + 1), \]  
with a variance \( \sigma^2 \approx 12(N - 1)^{-1}(\ln (t_U/t))^{-2} \). The estimation of \( p(t) \) is very robust over a large set of window sizes and have been tested thoroughly on synthetic Omori’s laws [21].

Figure 2 shows the local exponent \( p(t) \) as a function of time \( t \) estimated using a window size of 100 events. Clear log-periodic oscillations around \( p \approx 1 \) can be identified with a (log-)frequency of \( \approx 0.12 \) giving a preferred scale factor \( \lambda \approx 3500 \) in reasonable agreement with the theoretical value of 4000 calculated from (7,8). Increasing the window size with \( t \) or keeping a window with size fixed in time provides the same estimate.

To see this, we find that the local exponent \( p(t) \), defined by \( d \ln n(t)/d \ln t = -p(t) \), gives the most sensitive measure of deviation from the \( 1/t \) Omori’s law. We estimate \( p(t) \) by a maximum likelihood estimator in a running window ending at \( t \) [21].

Defining the starting \( t \) and ending \( t_U \) times of a window and the average \( \langle \ln t_i \rangle \) of the logarithms of all \( N \) aftershock times within this window, the MLE is
\[ p(t) \approx 12(\ln \sqrt{P_U} - (\ln t_i))/((\ln (t_U/t)) + 1), \]  
with a variance \( \sigma^2 \approx 12(N - 1)^{-1}(\ln (t_U/t))^{-2} \). The estimation of \( p(t) \) is very robust over a large set of window sizes and have been tested thoroughly on synthetic Omori’s laws [21].

Figure 2 shows the local exponent \( p(t) \) as a function of time \( t \) estimated using a window size of 100 events. Clear log-periodic oscillations around \( p \approx 1 \) can be identified with a (log-)frequency of \( \approx 0.12 \) giving a preferred scale factor \( \lambda \approx 3500 \) in reasonable agreement with the theoretical value of 4000 calculated from (7,8). Increasing the window size with \( t \) or keeping a window with size fixed in time provides the same estimate.

Figure 3 presents \( p(t) \) for the stress corrosion exponent \( \alpha \approx 25 \), a value estimated from a set of adjacent time-delayed multiple events in western Japan [16], for a stress drop \( \gamma = 5\% \) and a seismic efficiency of 1\%, i.e. \( \beta = 0.99 \). The measured scaling factor is now \( \lambda = 3.5 \) while the mean field prediction is \( \lambda = 3.6 \).
The comparisons between the numerical simulations and the predictions of the mean field theory are presented in figures 4-6. The mean field theory, while not perfect, accounts well for most of the behavior. We also have checked that $\lambda$ depends on the space dimension $d$ for $d = 1, 2, 3$ as predicted from (7,8).

Discrete scale invariance and its log-periodic signature is associated in this model to the threshold nature of the dynamics. The discrete scaling emerges from the fact that the thresholds are healed back to a value close to $B_0$ after each rupture and then have to decay down to the current stress threshold to trigger the next rupture. When this occurs, the stress jumps to a smaller value by a finite amount and can also latter be reloaded again by active neighbours. It is fundamentally these finite jumps in the stress proportional to the current stress (which are thus scale-free) which are at the origin of discrete scale invariance and log-periodicity. This novel mechanism of log-periodicity does not rely on a pre-existing discrete structural hierarchy of faults but is dynamical and reflects the existence of an approximately fixed stress drop together with the scale-free stress corrosion power law acting during inter-seismic phases.

This study suggests to search for log-periodic signatures in real aftershock sequences, with the potential bonus that log-periodicity would constrain the stress drop ratio, an elusive quantity to estimate by direct seismic measurements. A systematic analysis on more than thirty large aftershock sequences found some indications but was unable to conclude decisively yet due to a noise level which is comparable to the amplitude of the signal. Further studies are thus called for with better quality data.

We are especially grateful to A. Johansen and L. Knopoff for very stimulating discussions. M.L. thanks the Physics department at UCLA for hospitality.

[1] Sornette, D., Phys. Rep. 297, 239 (1998).
[2] Dubrulle, B., F. Graner and D. Sornette, eds., Scale invariance and beyond (EDP Sciences and Springer, Berlin, 1997).
[3] B. Derrida, L. De Seze and C. Itzykson, J. Stat. Phys. 33, 559 (1983); B. Derrida, C. Itzykson and J.M. Luck, Commun.Math.Phys. 94, 115 (1984).
[4] Bernasconi J. and W.R. Schneider, J.Phys.A15, L729 (1983); D. Stauffer and D. Sornette, Physica A 252, 271 (1998).
[5] Sornette, D. et al., Phys. Rev. Lett. 76, 251 (1996); Y. Huang, G. Ouillon, H. Saleur and D. Sornette, Phys. Rev. E 55, 6433 (1997); A. Johansen and D. Sornette, Int. J. Mod. Phys. C 9, 433 (1998).
[6] Saleur, H. and D. Sornette, J.Phys.I France 6, 327 (1996).
[7] Aharony A., Phys. Rev. B 12, 1049 (1975); Chen J.-H. and T.C. Lubensky, Phys. Rev. B 16, 2106 (1977); D.E. Khmelnitskii, Phys. Lett. A67,59 (1978); Phys. Rev. B 26, 154-170 (1982); A. Weinrib and B.I. Halperin, Phys. Rev. B 27, 413 (1983).
[8] M.W. Choptuik, Phys. Rev. Lett. 70, 9 (1993); C. Gundlach, Phys. Rev. D 55, 695 (1997).
[9] Lee, M.W., Unstable fault interactions and earthquake self-organization, PhD Thesis, UCLA (1999).
[10] Bak, P. and C. Tang, J. Geophys. Res. 94, 15635 (1989); Ito, K. and M. Matsuzaki, J. Geophys. Res. 95, 6853 (1990).
[11] Dhar, D., Phys. Rev. Lett. 64, 1613 (1990).
[12] Christensen, K. and Z. Olami, J. Geophys. Res. 97, 8729 (1992).
[13] Huang, Y., H. Saleur, C. G. Sammis, D. Sornette, Europhysics Letters 41, 43 (1998).
[14] Charles, R.J., J. Appl. Phys. 29, 1549 (1958); Atkinson, B.K., J. Geophys. Res. 89, 4077 (1984).
[15] Scholz, C.H., Seism. Soc. Am. Bull. 58, 1117 (1968).
[16] Das, S. and C.H. Scholz, J. Geophys. Res. 86, 6039 (1981); Shaw, B., Geophys. Res. Lett. 20, 907 (1993).
[17] Cowie, P., C. Vanneste and D. Sornette, J. Geophys. Res. 98, 21809 (1993); Sornette, D., P. Miltenberger and C. Vanneste, Pure and Applied Geophysics 142, 491 (1994).
[18] Leung, K.T., J. V. Andersen and D. Sornette, Phys. Rev. Lett. 80, 1916 (1998); Physica A 254, 85 (1998).
[19] L. Knopoff and M.W. Lee, The self-organization of earthquake aftershocks, to be published.
[20] Omori, F., J. Coll. Sci. Japan Imp. Univ. 7, 111 (1895); T. Utsu, Geophysical Magazine 30, 521 (1961).
[21] Huang, Y., A. Johansen, M. Lee, H. Saleur and D. Sornette, to be published.