We consider the kinetic Wigner — Vlasov — Boltzmann equation \[28:\]
\[\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} = -\frac{ie}{\hbar} W[U, f] + B[f, f].\]

Here \( e \) is the charge of electron, \( W[U, f] \) is the functional of Wigner — Vlasov for the scalar potential \( U \),
\[W[U, f] = \frac{1}{(2\pi)^3} \int \left[U \left( r + \frac{\hbar b}{2}, t \right) - U \left( r - \frac{\hbar b}{2}, t \right)\right] \times f(r, p', t) \exp(i b(p' - p)) \, d^3 b \, d^3 p',\]  

(1.1)
\( b = \{b_x, b_y, b_z\} \) is the vector, \( f = f(r, p, t) \) is the Wigner function for electrons, \( \hbar \) is the Planck constant, \( B[f, f] \) is the collision integral.

II. SOLUTION OF THE KINETIC EQUATION

Dielectric permeability of quantum plasma is widely used also for studying the screening of the electric field and Friedel oscillations (see, for example, [19] - [21]). In the work [27] screening of the Coulomb fields in magnetised electronic gas has been studied.

In theory of quantum plasma there exist two essentially various possibilities of construction of the relaxation kinetic equation in \( \tau \) — approximation: in the space of impulses (in the space of Fourier images of the distribution function) and in the space of coordinates. On the basis of the relaxation kinetic equations in the space of momentum Mermin [23] has carried out consistent derivation of the dielectric permeability for quantum collisional plasma in 1970 for the first time.

In the present work expression for the longitudinal dielectric permeability with use of the relaxation equations in space of coordinates is deduced. If in the received expression we make Planck constant converge to zero \( (\hbar \rightarrow 0) \), we will receive exactly classical expression of dielectric permeability of degenerate plasma. Various limiting cases of the dielectric permeability are investigated. Comparison with Mermin’s result is carried out also.

I. INTRODUCTION

In the present work formulas for conductivity and for dielectric permeability of quantum electronic plasma are deduced.

Dielectric permeability is one of the major plasma characteristics. This quantity is necessary for description of process of propagation and attenuation of the plasma oscillations, skin effect, the mechanism of electromagnetic waves penetration in plasma [1 — 3], and for analysis of other problems in plasma physics.

Dielectric permeability in the collisionless quantum gaseous plasma was studied by many authors (see, for example, [4 — 9]). In work [7], where the one-dimensional case of the quantum plasma is investigated, importance was marked. The present work is devoted to performance of this problem.

In the present work for a derivation of dielectric permeability quantum kinetic Wigner — Vlasov — Boltzmann equation (WVB-equation) with collision integral in the form of \( \tau \)-models is applied. Such collision integral is named BGK-collision integral.

The WVB-equation is written for Wigner function, which is analogue of distribution function of electrons for quantum plasma (see [11] and [12]).

The most widespread method of investigation of quantum plasmas is the method of Hartree — Fock or a method equivalent to it, namely, the method of Random Phase Approximation [17, 18]. In work [22] this method has been applied to receive expression for dielectric permeability of quantum plasma in \( \tau \) — approximation. However, in work [24] it is shown, that expression received in [22] is noncorrect, as does not turn into classical expression under a condition, when quantum amendments can be neglected. Thus in work [24] empirically corrected expression for dielectric permeability of quantum plasma, free from the specified lack has been offered. By means of this expression authors investigated quantum amendments to optical properties of metal [25], [26].

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Dielectric permeability of the degenerate electronic gas has been applied to receive expression for dielectric permeability with use of the relaxation equation in the momentum space has been received by Mermin.

Keywords: Degenerate Electron Gas, Dielectric Permeability and Conductivity, Collision Integral, Lindhard Function, Kohn’s Singularities.

PACS numbers: 50, 52.25.Dg Plasma kinetic equations, 52.25.-b Plasma properties

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II. SOLUTION OF THE KINETIC EQUATION

Dielectric permeability of the degenerate electronic gas for the collisional plasmas is found. The kinetic equation of Wigner — Vlasov — Boltzmann with integral of collisions in relaxation form in coordinate space is used. We will notice that dielectric permeability with using of the relaxation equation in the momentum space has been received by Mermin.
Collision integral for quantum plasma in general case can have rather complex form. In particular, it can be non-local by coordinates as well. A limiting case of such quantum non-locality is considered in [23]. In the present work the case when it is possible to present collision integral in a local form is considered. Particularly, we will consider collision integral representation in a form of standard model BGK–collision integral (Bhatnagar — Gross — Krook) [29]. Then the previous equation will be written in the following form:

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = -\frac{ie}{\hbar} W[U, f] + \nu \left[ f_{eq}(r, p, t) - f(r, p, t) \right]. \tag{1.2}
\]

This equation describes behaviour of the collisional degenerate quantum plasma.

Here \( \nu \) is the effective scattering frequency of electrons (in particular, on impurities), \( f_{eq} \) is the equilibrium Fermi — Dirac distribution function of electrons. Further we will consider the case of degenerate quantum plasma. Then the equilibrium distribution function can be expressed in terms of Heaviside function

\[ f_{eq} = \Theta(\mathcal{E}_{eq}(r, t) - \mathcal{E}), \]

the function \( \Theta(x) \) is the function of Heaviside,

\[ \Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \]

\( \mathcal{E} \) is the kinetic energy of electrons, \( \mathcal{E} = \frac{mv^2}{2} = \frac{p^2}{2m} \),

\[ \mathcal{E}_F(r, t) = \frac{m v_F^2(r, t)}{2} = \frac{p_F^2(r, t)}{2m}, \]

is the perturbed Fermi energy of the electrons, \( p = m v \) is the momentum of the electron, \( p_F \) is the momentum of the electron on the Fermi surface. We assume that Fermi surface is spherical.

Let’s consider, that distribution electron function depends on one spatial coordinate \( x \), time \( t \) and momentum \( p \), and the electric potential depends on one spatial coordinate \( x \) and time \( t \). Then the equations (1.1) and (1.2) can be written in a form:

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = -\frac{ie}{\hbar} W[U, f] + \nu \left[ f_{eq}(x, p, t) - f(x, p, t) \right]. \tag{1.3}
\]

\[
W[U, f] = \frac{1}{(2\pi)^3} \int \left[ U \left( x + \frac{hb}{2}, t \right) - U \left( x - \frac{hb}{2}, t \right) \right] \times \nonumber \times f(x, p', t) \exp(i(p' - p)) \, d^3b \, d^3p'. \tag{1.4}
\]

We will carry out linearization of the equations (1.3) and (1.4). Unperturbed absolute Fermi — Dirac distribution function for degenerate plasma has the form

\[ f_r(p) = \Theta(\mathcal{E}_F - \mathcal{E}), \]

where \( \mathcal{E}_F \) is the kinetic energy of electron on the Fermi surface,

\[ \mathcal{E}_F(p) = \frac{mv_F^2}{2} = \frac{p_F^2}{2m}. \]

In linear approximation in expression (1.4) instead of \( f \) it is necessary to take the absolute Fermi — Dirac distribution function \( f_r \). Our linearization of the Wigner function for electrons and the equilibrium distribution function leads to equalities:

\[ f = f_r(p) + U_0 e^{i(kx-\omega t)} f_1(p), \quad \tag{1.5} \]

\[ f_{eq} = f_r(p) + \delta(\mathcal{E}_F - \mathcal{E}) \delta \mathcal{E}_F(x, t), \quad \tag{1.6} \]

where \( f_1(p) \) is a new unknown function, \( \delta(x) \) is the Dirac delta–function,

\[ \delta \mathcal{E}_F(x, t) = \mathcal{E}_F(x, t) - \mathcal{E}_F, \]

\( U_0 \) is the potential amplitude. We assume that has the form of the traveling wave

\[ U(x, t) = U_0 e^{i(kx-\omega t)}. \quad \tag{1.7} \]

The quantity \( \delta \mathcal{E}_F(x, t) \) describes local change of Fermi’s energy of the electronic gas, caused by change of its density. Presence of this term in collisions integral provides realization of the particle number conservation law for electrons.

Let’s substitute (1.5) and (1.6) in the equation (1.3). We receive the following equation:

\[
f_1(p) \left[ \nu - i\omega + ikv_x \right] U_0 e^{i(kx-\omega t)} = \nonumber \times -\frac{ie}{\hbar} W[U, f_r] + \nu \delta(\mathcal{E}_F - \mathcal{E}) \delta \mathcal{E}_F(x, t). \tag{1.8}
\]

The Wigner — Vlasov functional has the following form in linear approximation:

\[
W[U, f_r] = \frac{1}{(2\pi)^3} \int \left[ U \left( x + \frac{hb}{2}, t \right) - U \left( x - \frac{hb}{2}, t \right) \right] \times \nonumber \times f_r(p') \exp(i(p' - p)) \, d^3b \, d^3p', \tag{1.9}
\]

We derive following expression for potential:

\[ U(x + \frac{hb}{2}, t) - U(x - \frac{hb}{2}, t) = \nonumber \]
\[ \Phi_{i \omega}(\nu) = \int \Theta_{i \omega}(\nu - i) \Phi_{i \omega}(\nu) \, d\nu \]

(1.10)

We will integrate in (1.9) by \( d^3b \). Considering (1.10), we deduce:

\[
\frac{1}{(2\pi)^3} \int \left[ U \left( x + \frac{\hbar b}{2} \right) - U \left( x - \frac{\hbar b}{2} \right) \right] e^{i(p' - p) \cdot b} \, d^3b = \]

\[
= U(x, t) \int \left[ \exp\left( i\frac{\hbar b_{y}}{2} \right) - \exp\left( i\frac{\hbar b_{z}}{2} \right) \right] e^{-i(p' - p) \cdot b} \, d^3b = \]

\[
= U(x, t) \delta(p'_{y} - p_{y}) \delta(p'_{z} - p_{z}) \times \]

\[
\times \left[ \delta(p'_{x} - p_{x} + \frac{\hbar k}{2}) - \delta(p'_{x} - p_{x} - \frac{\hbar k}{2}) \right] = \]

\[
= U(x, t) \delta(p_{y} - p'_{y}) \delta(p_{z} - p'_{z}) \times \]

\[
\times \left[ \delta(p_{x} - p'_{x} - \frac{\hbar k}{2}) - \delta(p_{x} - p'_{x} + \frac{\hbar k}{2}) \right]. \]

It is necessary to integrate by momentums:

\[
W[U, f] = U(x, t) \int \delta(p_{y} - p'_{y}) \delta(p_{z} - p'_{z}) \times \]

\[
\times \left[ \delta(p_{x} - p'_{x} - \frac{\hbar k}{2}) - \delta(p_{x} - p'_{x} + \frac{\hbar k}{2}) \right] \times \]

\[
\times \Theta\left( \frac{p_{y}^{2}}{2m} - \frac{p'_{y}^{2}}{2m} \right) \, dp_{y} \, dp_{x} \, dp_{z}. \]

As a result of integration by momentums we obtain:

\[
W[U, f] = U_{0} e^{i(k_{x}x - \omega t)} \times \]

\[
\times \left[ \Theta\left( \frac{p_{y}^{2}}{2m} - \frac{p'_{y}^{2}}{2m} - \left( \frac{p_{z} - \frac{\hbar k}{2}}{2m} \right)^{2} \right) - \right. \]

\[
- \Theta\left( \frac{p_{y}^{2}}{2m} - \frac{p'_{y}^{2}}{2m} - \left( \frac{p_{z} + \frac{\hbar k}{2}}{2m} \right)^{2} \right) \right], \]

or

\[
W[U, f] = U_{0} e^{i(k_{x}x - \omega t)} \times \]

\[
\times \left\{ \Theta\left( v_{F}^{2} - \left( \frac{v_{x} - \frac{\hbar k}{2m}}{2m} \right)^{2} - \left( v_{y}^{2} + v_{z}^{2} \right) \right) - \right. \]

\[
- \Theta\left( v_{F}^{2} - \left( \frac{v_{x} + \frac{\hbar k}{2m}}{2m} \right)^{2} - \left( v_{y}^{2} + v_{z}^{2} \right) \right) \right\} = \]

\[
= U_{0} e^{i(k_{x}x - \omega t)} \left[ \Theta_{+}(\nu) - \Theta_{-}(\nu) \right], \]

where

\[
\Theta_{+}(\nu) = \Theta\left( v_{F}^{2} - \left( \frac{v_{x} - \frac{\hbar k}{2m}}{2m} \right)^{2} - \left( v_{y}^{2} + v_{z}^{2} \right) \right), \]

\[
\Theta_{-}(\nu) = \Theta\left( v_{F}^{2} - \left( \frac{v_{x} + \frac{\hbar k}{2m}}{2m} \right)^{2} - \left( v_{y}^{2} + v_{z}^{2} \right) \right). \]

So, the Wigner — Vlasov’s functional is equal to:

\[
W[U, f_{M}] = U_{0} e^{i(k_{x}x - \omega t)} \left[ \Theta_{+}(\nu) - \Theta_{-}(\nu) \right]. \quad (1.11) \]

The quantity \( \delta\mathcal{E}_{F}(x, t) \) we will find from the conservation law of particle number:

\[
\int \nu(f_{eq} - f) \, d\Omega_{F} = 0, \]

where

\[
d\Omega_{F} = \frac{2 \, d^{3}p}{(2\pi\hbar)^{3}}. \quad (1.12) \]

According to the equality (1.12) we get the following equation:

\[
\int \nu(f_{eq} - f) \, d^{3}v = \]

\[
= \int \left[ \delta\mathcal{E}_{F}(x, t) \delta(\mathcal{E}_{F} - \mathcal{E}) - U_{0} e^{i(k_{x}x - \omega t)} f_{1}(p) \right] \, d^{3}v = 0. \]

From this equality we obtain:

\[
\delta\mathcal{E}_{F}(x, t) = U_{0} e^{i(k_{x}x - \omega t)} \frac{\int f_{1}(p) \, d^{3}v}{\int \delta(\mathcal{E}_{F} - \mathcal{E}) \, d^{3}v}. \quad (1.13) \]

The denominator of the expression (1.13) is equal to the following:

\[
\int \delta(\mathcal{E}_{F} - \mathcal{E}) \, d^{3}v = \frac{1}{mv_{F}} \int \delta(v_{F} - v) \, d^{3}v = \frac{4\pi v_{F}}{m}. \]

According to the equality (1.13) we have:

\[
\delta\mathcal{E}_{F}(x, t) = U_{0} e^{i(k_{x}x - \omega t)} \frac{m}{4\pi v_{F}} \int f_{1}(p) \, d^{3}v. \quad (1.14) \]

Substituting the expression (1.14) in the equation (1.8), we obtain:

\[
U_{0} e^{i(k_{x}x - \omega t)} f_{1}(p) \left( \nu - \frac{i\omega}{m} + ikv_{x} \right) = -\frac{i\nu}{\hbar} W[U, f] + \]

\[
+ U_{0} e^{i(k_{x}x - \omega t)} \frac{m\delta(\mathcal{E}_{F} - \mathcal{E})}{4\pi v_{F}} \int f_{1}(p) \, d^{3}v. \quad (1.15) \]
Let’s rewrite the equation (1.15) with the help of (1.11) in a form:

\[ f_1(p) (\nu - i \omega + ikv_x) = -\frac{ie}{\hbar} \left[ \Theta_+ (v) - \Theta_- (v) \right] + \]
\[ + \frac{mv\delta(E_F - E)}{4\pi v_F} \int f_1(p) dv^3. \]  

(1.16)

Taking into account the following equality

\[ \delta(E_F - E) = \frac{1}{mv_F} \delta(v_F - v) \]

from the equation (1.16) we obtain:

\[ f_1(p) = -\frac{ie}{\hbar} \frac{\Theta_+ (v) - \Theta_- (v)}{\nu + i(kv_x - \omega)} + \]
\[ + \frac{Av}{4\pi v_F^2} \frac{\delta(v_F - v)}{\nu + i(kv_x - \omega)}. \]  

(1.17)

III. LONGITUDINAL PERMEABILITY AND CONDUCTIVITY

Let’s designate:

\[ A = \int f_1(p) dv^3. \]  

(2.1)

Substituting (1.17) in the relationship (2.1), we get:

\[ A = -\frac{ie}{\hbar} \int \frac{\Theta_+ (v) - \Theta_- (v)}{\nu + i(kv_x - \omega)} dv^3 + \]
\[ + \frac{Av}{4\pi v_F^2} \int \frac{\delta(v_F - v) dv^3}{\nu + i(kv_x - \omega)}. \]  

(2.2)

The last integral in (2.2) is easily calculated with the use of spherical coordinates:

\[ \int \frac{\delta(v_F - v) dv^3}{\nu + i(kv_x - \omega)} = \frac{2\pi}{\nu + i(kv_F \omega - \omega)} = \]
\[ = 2\pi v_F^2 \int^1_{-1} \frac{d\mu}{\nu + i(kv_F \mu - \omega)} = 2\pi v_F^2 \frac{i}{kv_F} \ln \frac{\omega + iv + kv_F}{\omega + iv - kv_F}. \]

Let’s designate further:

\[ g_0(\omega, k, \nu) = \frac{iv}{2kv_F} \ln \frac{\omega + iv + kv_F}{\omega + iv - kv_F}. \]

Now from the equation (2.2) we obtain a relationship:

\[ A = -\frac{ie}{\hbar} \frac{1}{1 - g_0(\omega, k, \nu)} \int \frac{\Theta_+ (v) - \Theta_- (v)}{\nu + i(kv_x - \omega)} dv^3. \]  

(2.3)

Let’s consider the integral from (2.3)

\[ J(\omega, k, \nu) = \int \frac{\Theta_+ (v) - \Theta_- (v)}{\nu + i(kv_x - \omega)} dv^3. \]

According to definition of Heaviside function we have:

\[ \Theta_\pm (v) = \begin{cases} 1, & (v_x \pm \frac{\hbar k}{2m})^2 + v_y^2 + v_z^2 \leq v_F^2, \\ 0, & (v_x \pm \frac{\hbar k}{2m})^2 + v_y^2 + v_z^2 > v_F^2. \end{cases} \]

Hence, this integral is equal to the following:

\[ J(\omega, k, \nu) = J^+(\omega, k, \nu) - J^-(\omega, k, \nu), \]

where

\[ J^\pm (\omega, k, \nu) = \int \frac{dv_x dv_y dv_z}{S^3} \frac{dv_x dv_y dv_z}{\nu + i(kv_x - \omega)} \]

Here \( S^3_\pm \) is a sphere with the centre in the point \((\pm \frac{h k}{2m}, 0, 0)\). The radius of this sphere is equal to electron velocity on Fermi’s surface,

\[ S^3_\pm = \left\{(v_x, v_y, v_z) : \left( v_x \pm \frac{\hbar k}{2m} \right)^2 + v_y^2 + v_z^2 \leq v_F^2 \right\}. \]

After obvious replacement of a variable \( v_x \pm \frac{h k}{2m} \rightarrow v_x \) we receive for integrals \( J^\pm \):

\[ J^\pm (\omega, k, \nu) = \int_{S^3(0)} \frac{dv_x dv_y dv_z}{\nu + i(kv_x - \omega) - i\omega}, \]

where \( S^3 \) is the Fermi’s sphere with the centre in the beginning of coordinates,

\[ S^3(0) = S^3(0, 0, 0) = \{(v_x, v_y, v_z) : v_x^2 + v_y^2 + v_z^2 \leq v_F^2 \}. \]

Fermi’s sphere \( S^3(0) \) we will present in the form:

\[ S^3(0) = \bigcup_{v_x = -v_F}^{v_x = v_F} S^2_{v_F - v_x}(0, 0). \]

Here \( S^2_{v_F - v_x}(0, 0) \) there is a circle of the following form:

\[ S^2_{v_F - v_x}(0, 0) = \{(v_y, v_z) : v_y^2 + v_z^2 < v_F^2 - v_x^2 \}. \]
Now we will calculate integrals \( J^\pm \) as repeated:

\[
J^\pm(\omega, k, \nu) = \int_{-v_F}^{v_F} \frac{dv_x}{\nu + ik(v_x \pm \frac{\hbar k}{2m} - i\omega} \\
\times \int \int_{s^2_{v_F} - v^2} dv_y \; dv_z = \pi \int_{-v_F}^{v_F} \frac{(v_F^2 - v^2)dv_x}{\nu + ik(v_x \pm \frac{\hbar k}{2m} - i\omega}.
\]

Now these integrals can be calculated easily:

\[
J^\pm(\omega, k, \nu) = \frac{2i\pi \nu v_F}{k^2}(\omega_+ + i\nu) - \frac{i\pi \nu}{k^3}[(\omega_+ + i\nu)^2 - k^2v_F^2] \ln \frac{\omega_+ + i\nu + kv_F}{\omega_+ + i\nu - kv_F} - \frac{i\pi \nu}{k^3}[(\omega_- + i\nu)^2 - k^2v_F^2] \ln \frac{\omega_- + i\nu + kv_F}{\omega_- + i\nu - kv_F}. \tag{2.4}
\]

Let’s present a difference (2.4) in the form:

\[
J(\omega, k, \nu) = -\frac{2i\pi \nu v_F}{m} \left[ 1 - \frac{g(\omega_+, k, \nu) + g(\omega_-, k, \nu)}{} \right]. \tag{2.5}
\]

where

\[
g(\omega_\pm, k, \nu) = \frac{m[(\omega_\pm + i\nu)^2 - k^2v_F^2]}{2\hbar k^3v_F} \ln \frac{\omega_\pm + i\nu + kv_F}{\omega_\pm + i\nu - kv_F}. \tag{2.6}
\]

Thus, the quantity \( A \) according to (2.3) and (2.5) is equal to:

\[
A = -\frac{ie}{\hbar} \frac{J(\omega, k, \nu)}{1 - g_0(\omega, k, \nu)}.
\]

Hence, according to both (1.17) and (2.5) function \( f_1(p) \) is constructed also:

\[
f_1(p) = -\frac{ie}{\hbar} \left[ \frac{\Theta_+(v) - \Theta_-(v)}{\nu + i(kv_x - \omega)} + \frac{J(\omega, k, \nu)}{4\pi v_F^2(1 - g_0(\omega, k, \nu))} \cdot \frac{\delta(v_F - \nu)}{\nu + i(kv_x - \omega)} \right]. \tag{2.7}
\]

Let’s consider a relationship between electric field and potential

\[
\mathbf{E}(x, t) = -\text{grad} \; U(x, t),
\]

or

\[
\mathbf{E}(x, t) = -\left\{ \frac{\partial U(x, t)}{\partial x}, 0, 0 \right\},
\]

and the equation of a continuity for current and charge densities:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} = 0.
\]

Here according to definition of dielectric conductivity we may represent the current density in the form:

\[
j_x = \sigma_1 E_x = -\sigma_1 \frac{\partial U}{\partial x} = -\sigma_1 U_0 ike^{i(kx - \omega t)} = -\sigma_1 ikU(x, t).
\]

Hence,

\[
\frac{\partial j_x}{\partial x} = \sigma_1 k^2 U(x, t).
\]

Taking into account obvious equality for charge density

\[
\rho = e \int f d\Omega_F = e \int \left[ f_0(\mathbf{E}) + U_0 e^{i(kx - \omega t)} f_1 \right] d\Omega_F,
\]

we obtain:

\[
\frac{\partial \rho}{\partial t} = -i e \omega U(x, t) \int f_1 d\Omega_F.
\]

Substituting last two equalities in the continuity equation, we find the general formula for calculation of longitudinal conductivity:

\[
\sigma_1 = \frac{ie \omega}{k^2} \int f_1 d\Omega_F = \frac{2ie \omega \Theta(2\pi \hbar)^3 k^2}{(2\pi \hbar)^3 k^2} \int f_1 d^3v. \tag{2.8}
\]

Substituting (2.7) in (2.8), we get:

\[
\sigma_1 = \frac{2e^2 \tau \omega m^3}{(2\pi \hbar)^3 k^2} \left[ \int \frac{\Theta_+(v) - \Theta_-(v)}{\nu + i(kv_x - \omega)} d^3v + \frac{J(\omega, k, \nu)}{4\pi v_F^2(1 - g_0(\omega, k, \nu))} \int \frac{\delta(v_F - \nu)}{\nu + i(kv_x - \omega)} d^3v \right].
\]

and, using formulas

\[
\int \frac{(\Theta_+(v) - \Theta_-(v))d^3v}{\nu + i(kv_x - \omega)} = J(\omega, k, \nu),
\]
\[ \int \delta(v_F - v) d^3v = \frac{4\pi e^2 J(\omega, k, \nu)}{\nu + ikv_x - i\omega} g_0(\omega, k, \nu), \]

we derive:
\[ \sigma_l = \frac{i\pi e^2 \omega m^3 v_F}{2\pi^3 m^3 k^2} \frac{1}{1 - g_0(\omega, k, \nu)}. \]

With the help of the formula for numerical electron density of degenerate plasma
\[ \left( \frac{m v_F}{\hbar} \right)^3 = 3\pi^2 N, \]

following expression for calculation of longitudinal conductivity is obtained:
\[ \sigma_l = -\frac{3ie^2 N\omega}{2mk^2v_F} \frac{1 - g(\omega_+, k, \nu) + g(\omega_-, k, \nu)}{1 - g_0(\omega, k, \nu)}, \quad (2.9) \]
or, with the use of classical conductivity \( \sigma_0 = \frac{e^2 N}{m\nu} \), this formula will be written in the form:
\[ \sigma_l = \sigma_0 \left( \frac{-3i}{2} \right) \cdot \frac{\omega \nu}{(kv_F)^2} \cdot \frac{1 - g(\omega_+, k, \nu) + g(\omega_-, k, \nu)}{1 - g_0(\omega, k, \nu)}. \quad (2.9') \]

Using definition of dielectric permeability
\[ \epsilon_l = 1 + \frac{4\pi i}{\omega} \sigma_l, \]

with the help of (2.9) we will get the following representation for longitudinal dielectric permeability of plasma:
\[ \epsilon_l(\omega, k, \nu) = 1 + \frac{3\omega_p^2}{2k^2v_F} \frac{1 - g(\omega_+, k, \nu) + g(\omega_-, k, \nu)}{1 - g_0(\omega, k, \nu)}, \quad (2.10) \]

where \( \omega_p \) is the electron plasma frequency,
\[ \omega_p = \frac{4\pi e^2 N}{m}. \]

Let's enter dimensionless parameters:
\[ z = x + iy, \quad x = \frac{\omega}{kv_F}, \quad y = \frac{\nu}{kv_F}, \]
\[ x_p = \frac{\omega_p}{kv_F}, \quad q = \frac{k}{k_F}, \]

where \( k_F = \frac{mv_F}{\hbar} \) is the Fermi wave number.

The dimensionless dielectric function of these parameters has the the following form:
\[ \epsilon_l(x, y, q) = 1 + \frac{3}{2} x_p^2 \frac{1 - g(z, +q) + g(z, -q)}{1 - g_0(x, y)} \quad (2.11) \]

Here
\[ g_0(x, y) = i\frac{y}{2} \ln \frac{x + iy + 1}{x + iy - 1}, \quad z = x + iy, \]
\[ g(z, \pm q) = \frac{(z \pm q/2)^2 - 1}{2q} \ln \frac{z \pm q/2 + 1}{z \pm q/2 - 1}. \]

Let \( \nu = 0 \), i.e. plasma is collisionless; then from the expression (2.10) the following classical formula for the dielectric permeability follows:
\[ \epsilon_l(\omega, k) = 1 + \frac{3\omega_p^2}{2k^2v_F} \left[ 1 - g(\omega_-, k) + g(\omega_+, k) \right], \quad (2.12) \]

where
\[ g(\omega_\pm, k) = \frac{m(\omega_\pm^2 - k^2v_F^2)}{2hk^2v_F} \ln \frac{\omega_\pm + kv_F}{\omega_\pm - kv_F}. \]

This formula is called (see, for example, [19 - 21]) dielectric Lindhard’s function [22] in the literature. It is deduced by the method of random phases approximation.

In dimensionless variables dielectric Lindhard’s function has the following form:
\[ \epsilon_l(x, q) = 1 + \frac{3}{2} x_p^2 \left[ 1 - g(x, +q) + g(x, -q) \right], \quad (2.13) \]

where
\[ g(x, \pm q) = \frac{(x \pm q/2)^2 - 1}{2q} \ln \frac{x \pm q/2 + 1}{x \pm q/2 - 1}. \]

FIG. 1: Kohn’s singularity, \( x_p = 1, \ x = 0 \); curves of 1, 2, 3 correspond to parameters which are values of dimensionless collision frequencies \( y = 0, 0.005, 0.01 \).

On Fig. 1 Kohn’s singularity in a case, when \( x_p = 1, \ x = 0 \) is represented; curves of 1, 2, 3 correspond to parameters which are values of dimensionless collision frequencies \( y = 0, 0.005, 0.01 \).

On Fig. 2 Kohn’s singularities in a case, when \( x_p = 10, \ x = 0 \) are represented, curves of 1, 2, 3 correspond to the values of parameter \( y = 0; 0.01; 0.02 \). On
In the formula (2.11) we will write new dimensionless variables:

\[
z = \frac{\omega + i\nu}{k_F v_F} = x + iy, \quad x = \frac{\omega}{k_F v_F}, \quad y = \frac{\nu}{k_F v_F}, \quad q = \frac{k}{k_F}.
\]

At such replacement of variables we obtain the following formula for the longitudinal dielectric permeability:

\[
\varepsilon_l(x, y, q) = 1 + \frac{3x_p^2}{2q^2} \frac{1 - g_+(z, q) + g_-(z, q)}{1 - g_0(x, y, q)},
\]

where

\[
x_p^2 = \frac{\omega_p^2}{k_F v_F},
\]

\[
g_0(x, y, q) = \frac{iy}{2q} \ln \frac{x + iy + q}{x + iy - q},
\]

\[
g_+(z, q) = \frac{(z + q^2/2)^2 - q^2}{2q^3} \ln \frac{z + q + q^2/2}{z - q + q^2/2},
\]

\[
g_-(z, q) = \frac{(z - q^2/2)^2 - q^2}{2q^3} \ln \frac{z + q - q^2/2}{z - q - q^2/2}.
\]

Kohn’s singularities are determined from four equations:

\[
q^2 \pm 2q \pm 2z = 0.
\]

These equations at \(y = 0; (\nu = 0)\) define four Kohn’s singularities, two of which at \(\omega \neq 0\) lay in neighbourhood of point \(q = 2\):

\[
q_{1,2} = 1 \pm \sqrt{1 \pm 2x},
\]

and two others lay in point neighbourhood \(q = -2\):

\[
q_{3,4} = -1 - \sqrt{1 \pm 2x}.
\]

In the case of infinitesimal \(x\) we have from these formulas:

\[
q_{1,2} \approx 2 \pm x \approx 2 \pm \frac{\omega}{k_F v_F}, \quad q_{3,4} \approx -2 \pm x \approx -2 \pm \frac{\omega}{k_F v_F}.
\]

In terms of dimensional Fermi wave number the Kohn’s singularities are determined by equalities:

\[
k_{1,2} = k_F + \sqrt{k_F^2 \pm 2 \frac{k_F \omega}{v_F}},
\]

and

\[
k_{3,4} = -k_F - \sqrt{k_F^2 \pm 2 \frac{k_F \omega}{v_F}}.
\]

Besides that, these formulas may be rewritten in a form:

\[
k_{1,2} = \frac{mv_F}{\hbar} \left(1 + \sqrt{1 \pm \frac{\hbar \omega}{mv_F^2}}\right) = \frac{p_F}{\hbar} \left(1 + \sqrt{1 \pm \frac{\hbar \omega}{E_F}}\right)
\]

and

\[
k_{3,4} = \frac{mv_F}{\hbar} \left(-1 - \sqrt{1 \pm \frac{\hbar \omega}{mv_F^2}}\right) = \frac{p_F}{\hbar} \left(1 + \sqrt{1 \pm \frac{\hbar \omega}{E_F}}\right).
\]

Here \(E_F\) is the electron kinetic energy on a Fermi’s surface

\[
E_F = \frac{mv_F^2}{2}.
\]

Thus, into collisionless plasma (\(\nu = 0\)) at \(\omega \neq 0\) there is a splitting of Kohn’s singularities.
III. COMPARISON WITH MERMIN’S RESULT

Mermin (see Mermin N.D. [23]) has obtained the following expression of dielectric function:

\[
\varepsilon^M(\omega, k) = 1 + \frac{(\omega + iv)}{(\omega + iv)\varepsilon^\circ[(\omega + iv, k) - 1]}.
\]

(3.1)

The formula (3.1) is obtained on the basis of the kinetic equation for one–partial density matrix \( \rho \) in momentum space.

In the formula (3.1) the function \( \varepsilon^\circ(\omega, k) \) is the so-called Lindhard’s dielectric function, i.e. the dielectric function obtained for collisionless plasma, and defined by the equality (2.12):

\[
\varepsilon^\circ(\omega, k) \equiv \varepsilon_1(\omega, k) = 1 + \frac{3\omega_p^2}{2k^2v_F^2}\left[1 - g(\omega_+ + iv, k) + g(\omega_- + iv, k)\right].
\]

(3.2)

Expression \( \varepsilon^\circ(\omega + iv, k) \) means that arguments of dielectric Lindhard function \( \omega_\pm \) are replaced formally on \( \omega_\pm + iv \), i.e.

\[
\varepsilon^\circ(\omega + iv, k) - 1 = \frac{3\omega_p^2}{2k^2v_F^2}\left[1 - g(\omega_+ + iv, k) + g(\omega_- + iv, k)\right].
\]

(3.3)

So the function \( \varepsilon^\circ(0, k) \) has the form

\[
\varepsilon^\circ_1(0, k) - 1 = \frac{3\omega_p^2}{2k^2v_F^2}\left[1 - g(\omega_+, k) + g(\omega_-, k)\right].
\]

(3.4)

With the help of (3.2)–(3.4) Mermin formula (3.1) will be written in our designations in the following form:

\[
\varepsilon^M = 1 + \frac{3\omega_p^2}{2k^2v_F^2}\times\frac{1 - g(\omega_+ + iv, k) + g(\omega_- + iv, k)}{1 - g(0_+, k) + g(0_-, k)}.
\]

(3.5)

From formulas (2.10) and (3.5) one can see, that in the case \( \nu \to 0 \) the formula deduced in this work and Mermin formula turn into the same expression for dielectric function for quantum collisionless plasma that is Lindhard dielectric function (2.12).

IV. CONCLUSION

It is interesting to notice, that in the case of low-frequency limit, i.e. at \( \omega = 0 \) Mermin dielectric function does not depend on collision frequency \( \nu \). Indeed, assuming \( \omega = 0 \) in the formula (3.1), we obtain

\[
\varepsilon^M(0, k, \nu) = \varepsilon^\circ(0, k) = 1 + \frac{3\omega_p^2}{2k^2v_F^2}\left[1 - g(0_+, k) + g(0_-, k)\right].
\]

(4.1)

or, in dimensionless variables,

\[
\varepsilon^M(0, k, \nu) = 1 + \frac{3\omega_p^2}{2k^2v_F^2}\left[1 - \frac{w^2 - 1}{2w}\ln\frac{w + 1}{w - 1}\right].
\]

(4.2)

From the formula obtained in this work (2.10) we have another formula in a low–frequency limit:

\[
\varepsilon(0, y, w) = 1 + \frac{3x_p^2}{2}\left[1 - \frac{iy}{2}\ln\frac{iy + 1}{iy - 1}\right]^{-1} \times
\]

\[
\times\left[1 - \frac{(iy + w)^2 - 1}{4w}\ln\frac{iy + w + 1}{iy - w - 1}\right] + \frac{(iy - w)^2 - 1}{4w}\ln\frac{iy - w + 1}{iy - w - 1}.
\]

(4.3)

The formula (4.3) transforms into the formula (4.2) at \( y = 0 \).

So, in the present work analytical expression for the longitudinal quantum dielectric permeability of degenerate electron plasma is derived. Kinetic Wigner — Vlasov — Boltzmann equation with collision integral in the form of relaxation \( \tau \) — model in coordinate space is used.
It is shown, that in a limiting case, when Planck constant tends to zero, the expression obtained is transformed in the classical formula for the longitudinal dielectric permeability of degenerate plasma.

Static limits ($\omega \to 0$) for the dielectric permeability for collisionless, and for collisional plasma have been found.

Splitting of Kohn singularities in collisionless plasma is marked.

Comparison with classical Mermin’s result for dielectric permeability has been carried out. We will notice, that Mermin formula was obtained with use of the relaxation kinetic equation in the momentum space. For collisionless plasma the formula deduced in this work, and the Mermin formula as well can be transformed into the same Lindhard formula.

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