A new perturbed smooth penalty function for inequality constrained optimization

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Abstract

In this paper, we first propose a new smoothing approach to the nonsmooth penalty function for constrained optimization problems, and then show that an approximate optimal solution of the original problem can be obtained by solving an optimal solution of the smoothed optimization problem. Based on the perturbed smooth exact penalty function, we develop an algorithm respectively to finding an approximate optimal solution of the original constrained optimization problem and prove the convergence of the proposed algorithm. The effectiveness of the smoothed penalty function is illustrated through three examples, which show that the algorithm seems efficient.

Keywords: Exact penalty function, inequality constrained optimization, smoothing method, optimal solution.

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1. Introduction

Consider the following constrained optimization problem:

$$\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i \in I = \{1, 2, \ldots, m\}, \\
& \quad x \in \mathbb{R}^n,
\end{align*}$$

(P)

where $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are twice continuously differentiable functions. Let $X_0 = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\}$ be the feasible set of (P) and we assume that $X_0$ is not empty. This problem is widely applied in the fields such as mathematical programming, economy, transportation, network structures, etc, and it has received extensive attention on a related problem, for example, interval-valued optimization problems, equilibrium problem, bilevel programming, etc (see, e.g., [1, 8, 11, 15, 17–19]). Up

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to now, some efficient methods have been studied for solving general constrained optimization problem, including the penalty function methods. The penalty function methods have been proposed in order to transform a constrained optimization problem to an unconstrained optimization problem. Since exact penalty function method in solving (P) was proposed by Zangwill [24], it has attracted researchers and lots of penalty functions have been introduced in literature.

According to Zangwill [24], the $l_1$ exact penalty function has been defined by

$$\omega_1^\rho(x) = f(x) + \rho \sum_{i=1}^{m} \max\{0, g_i(x)\},$$

where $\rho > 0$ is a penalty parameter. It is proved that there exists a fixed constant $\rho_0 > 0$, for any $\rho > \rho_0$, any global solution of the exact penalty problem is also a global solution of the original problem. Therefore, the exact penalty function methods have been widely used for solving constrained optimization problems (see, e.g., [2, 7, 9, 10, 16, 23, 24]).

Recently, a class of lower order penalty functions has been investigated in [20] as the following form

$$\omega^k_\rho(x) = f(x) + \rho \sum_{i=1}^{m} \max\{0, g_i(x)\}^k, \quad k \in (0, 1). \quad (1.1)$$

Obviously, the lower order penalty problem and the original problem have the same set of global minima when the penalty parameter is sufficiently large. If $k = 1$ the lower order penalty function $\omega^1_\rho(x)$ is reduced to the $l_1$ exact penalty function. However, both the penalty function $\omega^k_\rho(x)$ ($0 < k < 1$) and the $l_1$ exact penalty function are non-smooth (non-Lipschitz). One of the important tools for solving these types of non-smooth problems is the smoothing approach. The smoothing approach is based on making some modification on the objective function or approximate the objective function by smooth functions. Thus smoothing approach for the exact penalty function have been proposed in the literature (see, e.g., [3–6, 12, 14, 21, 22, 25]). Pinar and Zenios [14] first proposed a smoothing method to the exact penalty functions, it was shown that an approximate solution of the original convex programming problem can be obtained by solving the smoothed penalty problem. Meng et al. [12] discussed two smoothing approximations to the lower order penalty function (1.1) with $k = \frac{1}{2}$. Wu et al. [21] proposed a quadratic smoothing approximation to the $l_1$ exact penalty function. It is shown that under certain conditions, any global minimizer of the smoothed penalty problem is a global minimizer of the original problem when the penalty parameter is sufficiently large. Wu et al. [20] and Binh [4] presented the $\epsilon$-smoothing of (1.1), and obtained a modified exact penalty function under some mild conditions. Binh et al. [5] also proposed two new perturbed smoothing approach to the lower order exact penalty function, which was proved to have good prospects in solving a global approximate solution to the constrained optimization problem.

In this paper, we introduce a new smoothing method for the low order penalty function and compare it with the existing methods. With a different segmentation method, we will give a new piecewise smooth function and propose a new method to smooth the lower order penalty function (1.1).

The rest of this paper is organized as follows. In Section 2, we propose a new smoothing method and prove some results for error estimates among the optimal objective function values of the nonsmooth penalty problem, smoothed penalty problem and original constrained optimization problem. In Section 3, we construct the minimization algorithm to finding an approximate optimal solution of the inequality constrained optimization problems. In Section 4, some numerical examples are given. Finally, conclusions are discussed in Section 5.

2. A new perturbed smoothing method

For any $k \in (0, 1)$, let:

$$q^k(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
t^k, & \text{if } t \geq 0.
\end{cases}$$
Then, we have the following low order penalty function:

\[ \omega^k_\rho(x) = f(x) + \rho \sum_{i=1}^{m} q^k(g_i(x)), \]  

(2.1)

and the corresponding penalty problem

\[ \min \omega^k_\rho(x), \quad \text{s.t. } x \in \mathbb{R}^n. \] 

\((P_\rho)\)

For the penalty problem \((P_\rho)\), in order to establish the global exact penalization, the following assumption is given in [20]. We will consider a new perturbed smoothing method under the following assumption.

**Assumption 2.1.**

(i) \(f(x)\) satisfies the following coercive condition:

\[ \lim_{\|x\| \to +\infty} f(x) = +\infty. \]

(ii) The set \(G(P)\) is a finite set, where \(G(P)\) is the set of global solutions of \((P)\).

Under Assumption 2.1, problem \((P)\) is equivalent to the following problem:

\[ \min f(x), \quad \text{s.t. } g_i(x) \leq 0, \quad i \in I, \quad x \in X \subset \mathbb{R}^n, \]  

\((P')\)

where \(X\) is a box with \(\text{int}(X) \neq \emptyset\).

For any \(k \in (0, 1)\), problem \((P_\rho)\) is equivalent to the following problem:

\[ \min \omega^k_\rho(x), \quad \text{s.t. } x \in X. \] 

\((P'_\rho)\)

Now, we consider the perturbed smooth exact penalty function. For \(\frac{1}{2} \leq k < 1\), \(\epsilon > 0\) and \(\rho > 0\), the function \(q^{k}_{\epsilon, \rho}(t)\) is defined as:

\[ q^{k}_{\epsilon, \rho}(t) = \begin{cases} 
0, & \text{if } t \leq -\left(\frac{\epsilon}{m \rho}\right)^k, \\
k \left( \frac{m \rho}{2 \epsilon} \right)^k t + \left( \frac{\epsilon}{m \rho} \right)^k, & \text{if } -\left(\frac{\epsilon}{m \rho}\right)^k < t < 0, \\
k \left( \frac{m \rho}{2 \epsilon} \right)^{k-1} t + k \left( \frac{\epsilon}{m \rho} \right)^{2k-1} - \left( \frac{\epsilon}{m \rho} \right)^k, & \text{if } t \geq 0.
\end{cases} \]

**Remark 2.2.** Obviously, \(q^{k}_{\epsilon, \rho}(t)\) has the following attractive properties:

(i) \(q^{k}_{\epsilon, \rho}(t)\) is continuously differentiable for \(\frac{1}{2} \leq k < 1\) on \(\mathbb{R}\), where

\[ [q^{k}_{\epsilon, \rho}(t)]' = \begin{cases} 
0, & \text{if } t \leq -\left(\frac{\epsilon}{m \rho}\right)^k, \\
k \left( \frac{m \rho}{2 \epsilon} \right)^k t + \left( \frac{\epsilon}{m \rho} \right)^k, & \text{if } -\left(\frac{\epsilon}{m \rho}\right)^k < t < 0, \\
k \left( \frac{m \rho}{2 \epsilon} \right)^{k-1} t, & \text{if } t \geq 0.
\end{cases} \]
(ii) \( \lim_{\epsilon \to 0^+} q^k_{\epsilon, \rho}(t) = q^k(t) \);

(iii) \( q^k(t) \leq q^k_{\epsilon, \rho}(t), \quad \forall t \in \mathbb{R} \).

The behavior of \( q^k(t) \) and \( q^k_{\epsilon, \rho}(t) \) is illustrated in Figure 1.

![Figure 1: The behavior of \( q^k(t) \) and \( q^k_{\epsilon, \rho}(t) \).](image)

Based on this, consider the perturbed smooth exact penalty function as follows:

\[
\omega^k_{\epsilon, \rho}(x) = f(x) + \rho \sum_{i=1}^{m} q^k_{\epsilon, \rho}(g_i(x)),
\]

where \( \epsilon > 0, \rho > 0 \). Clearly, \( \omega^k_{\epsilon, \rho}(x) \) is continuously differentiable at any \( x \in \mathbb{R}^n \). The corresponding smoothed optimization problem is:

\[
\min \omega^k_{\epsilon, \rho}(x), \quad \text{s.t. } x \in X.
\]

**(SP_{\epsilon, \rho})**

**Lemma 2.3.** For any \( x \in X, \epsilon > 0 \) and \( \rho > 0 \), then

\[
-k \frac{2}{k-1} e^{2k-1}(mp)^{2(1-k)} \leq \omega^k_{\rho}(x) - \omega^k_{\epsilon, \rho}(x) < e^{k(mp)^{1-k}}, \quad k \in \left[ \frac{1}{2}, 1 \right).
\]

**Proof.** For any \( x \in X \), we have

\[
\omega^k_{\rho}(x) - \omega^k_{\epsilon, \rho}(x) = \rho \sum_{i=1}^{m} (q^k(g_i(x)) - q^k_{\epsilon, \rho}(g_i(x))).
\]

Note that

\[
q^k(g_i(x)) - q^k_{\epsilon, \rho}(g_i(x))
\]
Theorem 2.4. Let \( \{\epsilon_j\} \to 0, \forall \epsilon_j > 0, \) and \( x_j \) be a solution of \((SP_{\epsilon_j, \rho})\) for \( \rho > 0, k \in \left[\frac{1}{2}, 1\right)\). Assume that \( x' \) is an accumulation point of \( \{x_j\} \). Then \( x' \) is an optimal solution of \((P_\rho)\).

Lemma 2.3 means that the gap between \( \omega^k_{\rho}(x) \) and \( \omega^k_{\epsilon, \rho}(x) \) can be arbitrarily small if the smoothing parameter \( \epsilon \) is sufficiently small.
Proof. Since $x_j$ is a solution of $(\text{SP}_{\epsilon,j,\rho})$, we have
\[ \omega_{\epsilon,j,\rho}^k(x_j) \leq \omega_{\epsilon,j,\rho}^k(x). \]
By Lemma 2.3, we have
\[ \omega_{\rho}^k(x_j) < \omega_{\epsilon,j,\rho}^k(x_j) + \epsilon_j^k(m\rho)^{1-k} \leq \omega_{\epsilon,j,\rho}^k(x) + \epsilon_j^k(m\rho)^{1-k} \leq \omega_{\rho}^k(x) + \frac{k}{2}\epsilon_j^{2k-1}(m\rho)^{2(1-k)} + \epsilon_j^k(m\rho)^{1-k}. \]
Since $\{\epsilon_j\} \to 0$ and $x'$ is an accumulation point of $\{x_j\}$, we obtain
\[ \omega_{\rho}^k(x') \leq \omega_{\rho}^k(x). \]
Thus $x'$ is an optimal solution of $(\text{P}_\rho)$.

**Theorem 2.5.** Let $x^*_\rho$ be an optimal solution of $(\text{P}'_\rho)$ and $x^*_\epsilon,\rho \in X$ be an optimal solution of $(\text{SP}_{\epsilon,\rho})$ for some $\rho > 0$ and $\epsilon > 0$. Then we have
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq \omega_{\rho}^k(x^*_\rho) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}, \quad k \in \left[\frac{1}{2}, 1\right]. \]

**Proof.** From Lemma 2.3, we obtain
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq \omega_{\rho}^k(x^*_\rho) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) \leq \omega_{\rho}^k(x^*_\rho) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}. \]
Therefore,
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq \omega_{\rho}^k(x^*_\rho) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}. \]

**Theorem 2.6.** Suppose that Assumption 2.1 holds, and for any $x^* \in G(\text{P})$, there exists a $\mu \in \mathbb{R}^m_+$ such that the pair $(x^*, \mu^*)$ satisfies the second-order sufficiency condition of problem (P) in [20]). Let $x^*$ be an optimal solution of (P) and $x^*_\epsilon,\rho \in X$ be an optimal solution of $(\text{SP}_{\epsilon,\rho})$. Then there exists $\rho_0 > 0$ such that for any $\rho > \rho_0$, it holds that
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq f(x^*) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}. \]

**Proof.** From [20, Corollary 2.3], we have that $x^*$ is an optimal solution of $(\text{P}'_\rho)$. By Theorem 2.5, we have the following:
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq \omega_{\rho}^k(x^*) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}. \]
Note that
\[ \omega_{\rho}^k(x^*) = f(x^*) + \rho \sum_{i=1}^m q^k(g_i(x^*)). \]
Since $\sum_{i=1}^m q^k(g_i(x^*)) = 0$, we have $\omega_{\rho}^k(x^*) = f(x^*)$. Thus, we obtain
\[ -\frac{k}{2}\epsilon^{2k-1}(m\rho)^{2(1-k)} \leq f(x^*) - \omega_{\epsilon,\rho}^k(x^*_\epsilon,\rho) < \epsilon^k(m\rho)^{1-k}. \]

**Definition 2.7.** A point $x^*_\epsilon \in X$ is called $\epsilon$-feasible solution of (P), if it satisfies $g_i(x^*_\epsilon) \leq \epsilon$, $\forall i \in I$. 

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Theorem 2.8. Let \( x^*_\rho \) be an optimal solution of \((P^*_\rho)\) and \( x^*_\epsilon, \rho \in X \) be an optimal solution of \((SP^*_\epsilon, \rho)\) for some \( \rho > 0 \) and \( \epsilon > 0 \). If \( x^*_\rho \) is a feasible solution of \((P)\), and \( x^*_\epsilon, \rho \) is an \( \epsilon \)-feasible solution of \((P)\), then we have
\[
-\frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)} \leq f(x^*_\rho) - f(x^*_\epsilon, \rho) < 2^{k} \epsilon^{k}(mp)^{1-k} + \frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)}.
\]

Proof. By Theorem 2.5, we have
\[
-\frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)} \leq f(x^*_\rho) + \rho \sum_{i=1}^{m} q^k (g_i(x^*_\rho)) - f(x^*_\epsilon, \rho) < 2^{k} \epsilon^{k}(mp)^{1-k} + \frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)}.
\]

Since \( x^*_\rho \) is a feasible solution of \((P)\), we have \( \sum_{i=1}^{m} q^k (g_i(x^*_\rho)) = 0 \). It follows that
\[
-\frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)} + \rho \sum_{i=1}^{m} q^k (g_i(x^*_\epsilon, \rho)) \leq f(x^*_\rho) - f(x^*_\epsilon, \rho) \quad \text{(2.2)}
\]
\[
< 2^{k} \epsilon^{k}(mp)^{1-k} + \rho \sum_{i=1}^{m} q^k (g_i(x^*_\epsilon, \rho)).
\]

Note that \( g_i(x^*_\epsilon, \rho) \leq \epsilon, \forall i \in I \). Thus, it follows from the definition of \( q^k (g_i(x^*_\epsilon, \rho)) \) that
\[
0 \leq \rho \sum_{i=1}^{m} q^k (g_i(x^*_\epsilon, \rho)) \leq 2^{k} \epsilon^{k}(mp)^{1-k} + \frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)} - \epsilon^{k}(mp)^{1-k}. \quad \text{(2.3)}
\]

Combining equations (2.2) and (2.3), we have
\[
-\frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)} \leq f(x^*_\rho) - f(x^*_\epsilon, \rho) < 2^{k} \epsilon^{k}(mp)^{1-k} + \frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)}.
\]

By Theorem 2.8, an optimal solution of \((P)\) can be controlled through the smoothing parameter \( \epsilon \), and the optimal solution of \((SP^*_\epsilon, \rho)\) is an approximate optimal solution of \((P)\) if \( x^*_\epsilon, \rho \) is an \( \epsilon \)-feasible solution of \((P)\).

Definition 2.9. A feasible solution \( x^* \) of \((P)\) is called a KKT point, if there exists a \( \mu^* \in \mathbb{R}^m \) such that the pair \( (x^*, \mu^*) \) satisfies the following conditions
\[
\nabla f(x^*) + \sum_{i=1}^{m} \mu^*_i \nabla g_i(x^*) = 0, \quad \text{(2.4)}
\]
\[
\mu^*_i g_i(x^*) = 0, \quad g_i(x^*) \leq 0, \quad \mu^*_i \geq 0, \quad i \in I. \quad \text{(2.5)}
\]

Theorem 2.10. Suppose the functions \( f, g_i \) \((i \in 1)\) in problem \((P)\) are convex. Let \( x^* \) be an optimal solution of \((P)\) and \( x^*_\epsilon, \rho \in X \) be an optimal solution of \((SP^*_\epsilon, \rho)\). If \( x^*_\epsilon, \rho \) is feasible of \((P)\), and there exists a \( \mu^* \in \mathbb{R}^m \) such that the pair \( (x^*_\epsilon, \rho, \mu^*) \) satisfies the conditions in equations (2.4) and (2.5), then we have
\[
0 \leq f(x^*_\epsilon, \rho) - f(x^*) < 2^{k} \epsilon^{k}(mp)^{1-k} + \frac{k}{2} \epsilon^{2k-1}(mp)^{2(1-k)}.
\]

Proof. Since \( f, g_i \) \((i \in 1)\) are continuously differentiable and convex, we see that
\[
f(x^*) \geq f(x^*_\epsilon, \rho) + \nabla f(x^*_\epsilon, \rho)^T (x^* - x^*_\epsilon, \rho), \quad \text{(2.6)}
g_i(x^*) \geq g_i(x^*_\epsilon, \rho) + \nabla g_i(x^*_\epsilon, \rho)^T (x^* - x^*_\epsilon, \rho), \quad i = 1, 2, \ldots, m. \quad \text{(2.7)}
\]
By Equations (2.1), (2.4), (2.5), (2.6) and (2.7), we have the following:

$$
\omega_{\rho}^k(x^*) = f(x^*) + \rho \sum_{i=1}^{m} q_i^k (g_i(x^*))
$$

$$
\geq f(x_{e,\rho}^*) + \nabla f(x_{e,\rho}^*)^T (x^* - x_{e,\rho}^*)
$$

$$
= f(x_{e,\rho}^*) - \sum_{i=1}^{m} \mu_i^k \nabla g_i(x_{e,\rho}^*)^T (x^* - x_{e,\rho}^*)
$$

$$
\geq f(x_{e,\rho}^*) - \sum_{i=1}^{m} \mu_i^k [g_i(x^*) - g_i(x_{e,\rho}^*)] = f(x_{e,\rho}^*) - \sum_{i=1}^{m} \mu_i^k g_i(x^*) \geq f(x_{e,\rho}^*).
$$

From Lemma 2.3, we obtain

$$
\omega_{\rho}^k(x^*) < \omega_{e,\rho}^k(x^*) + e^k(m\rho)^{1-k}.
$$

It follows that

$$
f(x_{e,\rho}^*) < \omega_{e,\rho}^k(x^*) + e^k(m\rho)^{1-k} = f(x^*) + \rho \sum_{i=1}^{m} q_i^k (g_i(x^*)) + e^k(m\rho)^{1-k}
$$

$$
\leq f(x^*) + \frac{k}{2} e^{2k-1}(m\rho)^{2(1-k)} + e^k(m\rho)^{1-k}.
$$

(2.8)

Since $x_{e,\rho}^*$ is feasible of (P), which is

$$
f(x^*) \leq f(x_{e,\rho}^*). \quad (2.9)
$$

Combining equations (2.8) and (2.9), we have

$$
f(x^*) \leq f(x_{e,\rho}^*) < f(x^*) + \frac{k}{2} e^{2k-1}(m\rho)^{2(1-k)} + e^k(m\rho)^{1-k},
$$

which is

$$
0 \leq f(x_{e,\rho}^*) - f(x^*) < \frac{k}{2} e^{2k-1}(m\rho)^{2(1-k)} + e^k(m\rho)^{1-k}.
$$

□

3. Algorithm for minimization procedure

In this section, by considering the above smoothed penalty function, we propose algorithm to find an approximate optimal solution of (P), defined as Algorithm 3.1.

**Algorithm 3.1** (Algorithm for solving problem (P)).

**Step 1:** Determine the initial point $x_0^0 \in X$ and a stopping tolerance $\epsilon > 0$. Determine $\epsilon_1 > \epsilon$, $\rho_1 > 0$, $0 < \gamma < 1$ and $N > 1$, let $j = 1$ and go to Step 2.

**Step 2:** Start from the point $x_0^j$ and solve the following problem:

$$
(SP_{\epsilon_j,\rho_j}) : \min_{x \in \mathbb{R}^n} \omega_{\epsilon_j,\rho_j}^k(x) = f(x) + \rho_j \sum_{i=1}^{m} q_i^k (g_i(x)).
$$

Let $x_{\epsilon_j,\rho_j}^j$ be an optimal solution of $(SP_{\epsilon_j,\rho_j})$. Here $x_{\epsilon_j,\rho_j}^j$ is obtained by the BFGS method given in [13].

**Step 3:** If $x_{\epsilon_j,\rho_j}^j$ is an $\epsilon$-feasible of (P), then the algorithm stops and $x_{\epsilon_j,\rho_j}^j$ is an approximate optimal solution of problem (P). Otherwise, determine $\rho_{j+1} = N\rho_j$, $\epsilon_{j+1} = \gamma \epsilon_j$, $x_0^{j+1} = x_{\epsilon_j,\rho_j}^j$, and $j = j + 1$, then go to Step 2.
Remark 3.2. Since $N > 1$, $0 < \gamma < 1$, let $N\gamma^{2k-1} < 1$, as $j \to +\infty$, the sequence $\{e_j\}$ converges to 0, the sequence $\{p_j\}$ converges to $+\infty$ and $\{p_j e_j^{2k-1}\}$ converges to 0, as $j \to +\infty$.

Theorem 3.3. For $\frac{1}{2} < k < 1$, suppose that for any $\epsilon \in (0, \epsilon_1]$, $\rho \in [\rho_1, +\infty)$, the set

$$\arg \min_{x \in \mathbb{R}^m} \omega^k_{e_i, \rho}(x) \neq \emptyset.$$ 

Let $\{x^*_i, p_j\}$ be the sequence generated by Algorithm 3.1 satisfying $N\gamma^{2k-1} < 1$. If the sequence $\{\omega^k_{e_i, \rho}(x^*_i, p_j)\}$ is bounded, and Assumption 2.1 holds, then $\{x^*_i, p_j\}$ is bounded and the limit point of $\{x^*_i, p_j\}$ is a solution of $(P)$.

Proof. First, we prove that $\{x^*_i, p_j\}$ is bounded. Note that

$$\omega^k_{e_i, \rho}(x^*_i, p_j) = f(x^*_i, p_j) + p_j \sum_{i=1}^{m} q^k_{e_i, \rho_j}(g_i(x^*_i, p_j)), \quad j = 0, 1, \ldots, \tag{3.1}$$

and by the definition of $p^k_{e, \rho}(t)$, we have

$$\rho_j \sum_{i=1}^{m} q^k_{e_i, \rho_j}(g_i(x^*_i, p_j)) \geq 0. \tag{3.2}$$

Suppose on the contrary that the sequence $\{x^*_i, p_j\}$ is unbounded. Without any loss of generality $\|x^*_i, p_j\| \to +\infty$ as $j \to +\infty$. Then, $\lim_{j \to +\infty} f(x^*_i, p_j) = +\infty$ by Assumption 2.1, and from equations (3.1) and (3.2), we have

$$\omega^k_{e_i, \rho}(x^*_i, p_j) \geq f(x^*_i, p_j) \to +\infty, \quad j = 0, 1, \ldots,$$

which contradicts with the sequence $\{\omega^k_{e_i, \rho}(x^*_i, p_j)\}$ being bounded. Thus, $\{x^*_i, p_j\}$ is bounded.

Next, we prove that the limit point of $\{x^*_i, p_j\}$ is the solution of $(P)$. Without loss of generality, we assume $x^*_i, p_j \to x^*$ as $j \to \infty$. To prove $x^*$ is an optimal solution of $(P)$, it is sufficient to show that $x^* \in X_0$, and $f(x^*) \leq f(x)$, $\forall x \in X_0$.

(i) To show that $x^* \in X_0$, we outline a proof by contradiction. Suppose $x^* \notin X_0$. Then, there exist $\theta_0 > 0$ and the subset $J \subset N$, such that $g_{i'}(x^*_i, p_j) \geq \theta_0 > e_j$ for any $j \in J$ and some $s' \in I$, where $N$ is the natural number set.

From the definition of $q^k_{e_i, \rho_j}(t)$ and $x^*_i, p_j$ is the optimal solution according $j$-th values of the parameters $e_j, p_j$ for any $x \in X_0$, we have

$$f(x^*_i, p_j) + p_j \left[\left(\theta_0 + \frac{e_j}{\rho_j}\right)^k + \frac{k}{2} \left(\frac{e_j}{\rho_j}\right)^{2k-1}\right] \leq \omega^k_{e_i, \rho_j}(x^*_i, p_j) \leq \omega^k_{e_i, \rho_j}(x) \leq f(x) + \frac{k}{2} \rho_j \left(\frac{e_j}{\rho_j}\right)^{2k-1}.$$ 

It follows that

$$f(x^*_i, p_j) + p_j \left[\left(\theta_0 + \frac{e_j}{\rho_j}\right)^k - \frac{e_j}{\rho_j}\right] \leq f(x) + (m - 1) \frac{k \rho_j e_j^{2k-1}}{2(\rho_j)^{2k-1}},$$

which contradicts with $p_j \to +\infty$, $e_j \to 0$, and $e_j^{2k-1} \to 0$, as $j \to \infty$. Then, we have that $x^* \in X_0$.

(ii) For any $x \in X_0$, it holds that

$$f(x^*_i, p_j) \leq \omega^k_{e_i, \rho_j}(x^*_i, p_j) \leq \omega^k_{e_i, \rho_j}(x) \leq f(x) + \frac{k \rho_j e_j^{2k-1}}{2(\rho_j)^{2k-1}}.$$ 

Letting $j \to \infty$ yields that

$$f(x^*) \leq f(x).$$

4. Numerical examples

In this section, we apply the Algorithm 3.1 to test problems. In each example, we let $\varepsilon = 10^{-6}$ is expected to get an $\varepsilon$-solution of (P) with Algorithm 3.1, $j$ be the number of iterations, $x_{\varepsilon_j, 1}^*$ be the optimal solution of the $j$-th iteration, $f(x_{\varepsilon_j, 1}^*)$ be the objective value at $x_{\varepsilon_j, 1}^*$, $g_l(x_{\varepsilon_j, 1}^*)$, $i \in I$ is a constrain value at $x_{\varepsilon_j, 1}^*$, and the numerical results are presented in the tables as following.

Example 4.1. Consider the following problem ([21], Example 3.3)

\begin{equation}
\min \quad f(x) = -2x_1 - 6x_2 + x_1^2 - 2x_1x_2 + 2x_2^2 \\
\text{s.t.} \quad g_1(x) = x_1 + x_2 - 2 \leq 0, \\
\quad g_2(x) = -x_1 + 2x_2 - 2 \leq 0, \\
\quad x_{1, 2} \geq 0. 
\end{equation}

Let $x^0_1 = (1, 1)$, $x^0_2 = 2$, $N = 8$, $\varepsilon_1 = 0.1$, $\gamma = 0.01$. With different $k$, the results of Algorithm 3.1 for solving problem 4.1 are shown in Tables 1, 2, and 3.

| Table 1: Results of Algorithm 3.1 with $k = \frac{2}{5}$ for problem 4.1. |
|---|
| $j$ | $\rho_j$ | $\varepsilon_j$ | $x_{\varepsilon_j, 1}^*$ | $f(x_{\varepsilon_j, 1}^*)$ | $g_1(x_{\varepsilon_j, 1}^*)$ | $g_2(x_{\varepsilon_j, 1}^*)$ |
| 1 | 2 | 0.1 | (2.000000, 1.993789) | -11.987500 | 1.993789 | -0.012422 |
| 2 | 16 | 0.001 | (1.217802, 1.337227) | -8.656528 | 0.555029 | -0.543348 |
| 3 | 128 | 0.00001 | (0.800000, 1.200000) | -7.200000 | -0.000000 | -0.400000 |

| Table 2: Results of Algorithm 3.1 with $k = \frac{3}{5}$ for problem 4.1. |
|---|
| $j$ | $\rho_j$ | $\varepsilon_j$ | $x_{\varepsilon_j, 1}^*$ | $f(x_{\varepsilon_j, 1}^*)$ | $g_1(x_{\varepsilon_j, 1}^*)$ | $g_2(x_{\varepsilon_j, 1}^*)$ |
| 1 | 2 | 0.1 | (4.059684, 2.997685) | -15.991521 | 5.057369 | -0.064313 |
| 2 | 16 | 0.001 | (0.820632, 1.322433) | -7.575229 | 0.143065 | -0.175767 |
| 3 | 128 | 0.00001 | (0.800002, 1.199998) | -7.199999 | -0.000000 | -0.400006 |

| Table 3: Results of Algorithm 3.1 with $k = \frac{4}{5}$ for problem 4.1. |
|---|
| $j$ | $\rho_j$ | $\varepsilon_j$ | $x_{\varepsilon_j, 1}^*$ | $f(x_{\varepsilon_j, 1}^*)$ | $g_1(x_{\varepsilon_j, 1}^*)$ | $g_2(x_{\varepsilon_j, 1}^*)$ |
| 1 | 2 | 0.1 | (2.826217, 2.402502) | -14.115896 | 3.228719 | -0.021214 |
| 2 | 16 | 0.001 | (0.808600, 1.191516) | -7.199961 | 0.000116 | -0.425569 |
| 3 | 128 | 0.00001 | (0.799693, 1.200307) | -7.199999 | -0.000000 | -0.399080 |

The results in Tables 1-3 show that, the convergence Algorithm 3.1 and the obtained approximate optimal solutions are similar. In [21], the obtained approximate optimal solution is $x^* = (0.8000, 1.2000)$ with objective function value $f(x^*) = -7.2000$. Numerical results obtained by our algorithm are similar to the results in [21].

Example 4.2. Consider the following problem ([12], Example 4.2)

\begin{equation}
\min \quad f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 \\
\text{s.t.} \quad g_1(x) = 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_4 - 5 \leq 0, \\
\quad g_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \leq 0, \\
\quad g_3(x) = x_1^2 + 2x_2^2 + x_3^2 + 2x_2^2 - x_1 - x_4 - 10 \leq 0. 
\end{equation}
For $k = \frac{7}{3}$, let $x_0^k = (5, 5, 5, 5)$, $p_1 = 10$, $N = 8$, $\epsilon_1 = 0.1$, $\gamma = 0.01$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 4.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j, \rho_j}^k$ | $f(x_{\epsilon_j, \rho_j}^k)$ | $g_1(x_{\epsilon_j, \rho_j}^k)$ | $g_2(x_{\epsilon_j, \rho_j}^k)$ | $g_3(x_{\epsilon_j, \rho_j}^k)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|----------------|
| 1   | 10        | 0.1          | (0.042482, 0.996409, 1.957354, -1.068548) | -43.979820 | -0.0159503 | 0.039639 | -0.871644 |
| 2   | 80        | 0.001        | (0.169560, 0.835531, 2.008634, -0.964876) | -44.233826 | -0.000004 | -0.000004 | -1.883130 |

For $k = \frac{4}{3}$, let $x_0^k = (7, 7, 7, 7)$, $p_1 = 10$, $N = 9$, $\epsilon_1 = 0.01$, $\gamma = 0.1$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 5.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j, \rho_j}^k$ | $f(x_{\epsilon_j, \rho_j}^k)$ | $g_1(x_{\epsilon_j, \rho_j}^k)$ | $g_2(x_{\epsilon_j, \rho_j}^k)$ | $g_3(x_{\epsilon_j, \rho_j}^k)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|----------------|
| 1   | 10        | 0.01         | (0.228405, 0.827990, 2.195021, -1.162955) | -47.791667 | 0.729868 | 1.666709 | -0.119102 |
| 2   | 90        | 0.001        | (0.168417, 0.836036, 2.009003, -0.964650) | -44.233824 | -0.000001 | -0.000002 | -1.880298 |

For $k = \frac{3}{4}$, let $x_0^k = (1, 1, 1, 1)$, $p_1 = 10$, $N = 8$, $\epsilon_1 = 0.1$, $\gamma = 0.1$. The results of Algorithm 3.1 for solving problem 4.2 are shown in Table 6.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j, \rho_j}^k$ | $f(x_{\epsilon_j, \rho_j}^k)$ | $g_1(x_{\epsilon_j, \rho_j}^k)$ | $g_2(x_{\epsilon_j, \rho_j}^k)$ | $g_3(x_{\epsilon_j, \rho_j}^k)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|----------------|
| 1   | 10        | 0.1          | (0.164525, 0.852259, 2.013672, -0.954093) | -44.276228 | 0.062573 | -0.001386 | -1.855209 |
| 2   | 80        | 0.01         | (0.169557, 0.835528, 2.008628, -0.964875) | -44.233725 | -0.000041 | -0.000041 | -1.883165 |

The results in Tables 4-6 show that, the convergence of Algorithm 3.1 and the obtained approximate optimal solutions are similar. By Table 4, an approximate optimal solution to problem 4.2 is obtained after 2 iterations with objective function value $f(x^*) = -44.233826$. In [4], the obtained approximate optimal solution is $x^* = (0.170446, 0.834248, 2.007853, -0.964559)$ with function value $f(x^*) = -44.233627$. In the paper [12], the obtained approximate optimal solution is $x^* = (0.169234, 0.835656, 2.008690, -0.964901)$ with function value $f(x^*) = -44.233582$. Numerical results obtained by our algorithm are slightly better than the results in [4, 12].

**Example 4.3.** Consider the following problem ([21], Example 3.2)

\[
\begin{align*}
\min \quad & f(x) = -x_1 - x_2 \\
\text{s.t.} \quad & g_1(x) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_1 - 2 \leq 0, \\
& g_2(x) = -4x_2^4 + 32x_3^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leq 0, \\
& 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 4.
\end{align*}
\]
Let $k = \frac{3}{4}$, $\rho_1 = 8$, $N = 6$, $\epsilon_1 = 0.4$, $\gamma = 0.1$. With different starting points, the results of Algorithm 3.1 for solving problem 4.3 are shown in Tables 7, 8, and 9.

Table 7: Results of Algorithm 3.1 with $x_0^0 = (0, 3)$ for problem 4.3.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j,\rho_j}^*$ | $f(x_{\epsilon_j,\rho_j}^*)$ | $g_1(x_{\epsilon_j,\rho_j}^*)$ | $g_2(x_{\epsilon_j,\rho_j}^*)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|
| 1   | 8         | 0.4          | (2.112066, 3.900188) | -6.012254       | 0.000021       | 0.000028       |
| 2   | 48        | 0.04         | (2.112096, 3.900107) | -6.012203       | -0.000012      | -0.000001      |

Table 8: Results of Algorithm 3.1 with $x_0^0 = (2, 1)$ for problem 4.3.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j,\rho_j}^*$ | $f(x_{\epsilon_j,\rho_j}^*)$ | $g_1(x_{\epsilon_j,\rho_j}^*)$ | $g_2(x_{\epsilon_j,\rho_j}^*)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|
| 1   | 8         | 0.4          | (2.112028, 3.900313) | -6.012341       | 0.000063       | 0.000085       |
| 2   | 48        | 0.04         | (2.112101, 3.900094) | -6.012195       | -0.000018      | -0.000004      |

Table 9: Results of Algorithm 3.1 with $x_0^0 = (3, 1)$ for problem 4.3.

| $j$ | $\rho_j$ | $\epsilon_j$ | $x_{\epsilon_j,\rho_j}^*$ | $f(x_{\epsilon_j,\rho_j}^*)$ | $g_1(x_{\epsilon_j,\rho_j}^*)$ | $g_2(x_{\epsilon_j,\rho_j}^*)$ |
|-----|-----------|--------------|----------------|-----------------|----------------|----------------|
| 1   | 8         | 0.4          | (2.112028, 3.900313) | -6.012341       | 0.000063       | 0.000085       |
| 2   | 48        | 0.04         | (2.112087, 3.900025) | -6.012112       | -0.000002      | -0.000099      |

The results in Tables 7-9 show that, the convergence Algorithm 3.1 and the obtained approximate optimal solutions are similar. That is to say, the numerical results of Algorithm 3.1 does not depend on the starting point $x^0$ for this example. By Table 7, an approximate optimal solution to problem 4.3 is obtained after 2 iterations with objective function value $f(x^*) = -6.012203$. In [21], the obtained global solution is $x^* = (2.3295, 3.1784)$ with objective function value $f(x^*) = -5.5080$. In the paper [22], the obtained approximate optimal solution is $x^* = (2.112103, 3.900086)$ with objective function value $f(x^*) = -6.012190$. Numerical results obtained by our algorithm are much better than the results in [21] and find the correct solutions as in [22].

5. Conclusions

In this paper, we proposed a new perturbed smooth penalty function for inequality constrained optimization. Furthermore, we proved that the algorithm based on the smoothed penalty functions is globally convergent under mild conditions. The numerical results given in Section 4 show that the Algorithm 3.1 has a good convergence for an approximate optimal solution.

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