On Classical and Quantum MDS-Convolutional BCH Codes

Giuliano G. La Guardia

Abstract—Several new families of multi-memory classical convolutional Bose-Chaudhuri-Hocquenghem (BCH) codes as well as families of unit-memory quantum convolutional codes are constructed in this paper. Our unit-memory classical and quantum convolutional codes are optimal in the sense that they attain the classical (quantum) generalized Singleton bound. The constructions presented in this paper are performed algebraically and not by computational search.

Index Terms — convolutional codes, quantum convolutional codes, MDS codes, cyclic codes

I. INTRODUCTION

Several works available in the literature deal with constructions of quantum error-correcting codes (QECC) [4–9, 13, 17, 21, 22, 24, 33, 41, 42]. In contrast with this subject of research one has the theory of quantum convolutional codes [1–3], 12, 14–16, 34, 35, 43–45]. Ollivier and Tillich [34, 35] were the first to develop the stabilizer structure for these codes. Almeida and Palazzo Jr. construct an (4, 1, 3) (memory m = 3) quantum convolutional code [1]. Grassl and Rötteler [14–16] constructed quantum convolutional codes as well as they provide algorithms to obtain non-catastrophic encoders. Forney, in a joint work with Guha and Grassl, constructed rate 2, 14–16, 34, 35, 43–45] constructed quantum convolutional codes as well as the paper by Rosenthal and York [39].

The constructions proposed here deal with suitable properties of cyclotomic cosets, that will be specified throughout this paper. These nice properties of the cosets hold when considering classical convolutional codes of length n = q + 1 over the field \( F_q \) for all prime power \( q \), or even quantum convolutional codes of length \( n = q^2 + 1 \) over \( F_{q^2} \), where \( q = 2^t, t \geq 3 \) is an integer. In the quantum case, the corresponding classical codes are endowed with the Hermitian inner product.

The new families of classical convolutional MDS codes constructed have parameters

\[(n, n - 2i, 2; 1, 2i + 3)_{q}, \text{ where } 1 \leq i \leq \frac{n - 1}{2} - 1, q = 2^t, t \geq 3 \text{ is an integer, } n = q + 1 \text{ is the code length, } k = n - 2i \text{ is the code dimension, } \gamma = q \text{ is the degree of the code, } m = 1 \text{ is the memory and } d_f = 2i + 3 \text{ is the free distance of the code;}
\]

\[(n, n - 2i + 1; 2, 1, 2i + 2)_{q}, \text{ where } q = p^t, t \geq 2 \text{ is an integer, } p \text{ is an odd prime number, } n = q + 1 \text{ and } 2 \leq i \leq \frac{n - 1}{2} - 1.
\]

The multi-memory (classical) convolutional codes constructed here have parameters

\[(n, 2r + 1, 2m; m, d_f \geq n - 2[r + m])_{q}, \text{ where } q = p^t, t \geq 2 \text{ is an integer, } p \text{ is an odd prime number, } n = q + 1, r, m \text{ are integers with } r \geq 1, m \geq 2 \text{ and } 3 \leq r + m \leq \frac{n}{2} - 1.
\]

The new convolutional stabilizer MDS codes have parameters

\[[(n, n - 4i, 1; 2, 2i + 3)]_{q}, \text{ where } 2 \leq i \leq \frac{n - 1}{2} - 2, q = 2^t, t \geq 3 \text{ is an integer and } n = q^2 + 1. \text{ Here, } n \text{ is the frame size}.
\]

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size, \( k = n - 4i \) is the number of logical qubits per frame, \( m = 1 \) is the memory, \( \gamma = 2 \) is the degree and \( d_f = 2i + 3 \) is the free distance of the code.

Note that the order between the degree and the memory are changed when comparing the parameters of classical and quantum convolutional codes. This notation is adopted to keep the same notation utilized in [2].

Let us now give the structure of the paper. In Section II, we review basic concepts on cyclic codes. In Section III, a review of concepts concerning classical and quantum convolutional codes is given. In Section IV, we propose constructions of new families of classical MDS convolutional codes as well as families of multi-memory convolutional codes. In Section V we construct new optimal (MDS) quantum convolutional codes and, in Section VI, a brief summary of this work is described.

II. REVIEW OF CYCLIC CODES

Notation. Throughout this paper, \( p \) denotes a prime number, \( q \) is a prime power and \( F_q \) is a finite field with \( q \) elements. The code length is denoted by \( n \) and we always assume that \( \gcd(q, n) = 1 \). As usual, the multiplicative order of \( q \) modulo \( n \) is given by \( l = \text{ord}_n(q) \), \( \alpha \) denotes a primitive \( n \)-th root of unity, and the minimal polynomial (over \( F_q \)) of \( \alpha \) is the monic polynomial of smallest degree over \( F_q \) that is, \( \alpha \) is the \( n \)-th root of unity. Let \( \alpha \) be a cyclic code of length \( n \) over \( F_q \). Then there exists only one monic polynomial \( g(x) \) with minimal degree in \( C \).

Moreover, \( C = \{g(x)\} \), i.e., \( g(x) \) is a generator polynomial of \( C \) and \( g(x) \) is a factor of \( x^n - 1 \). The dimension of \( C \) equals \( n - r \), where \( r = \deg g(x) \).

Theorem 2.1: (The BCH bound)[32, pg. 201] Let \( \alpha \) be a primitive \( n \)-th root of unity. Let \( C \) be a cyclic code with generator polynomial \( g(x) \) such that, for some integers \( b \geq 0 \) and \( \delta \geq 1 \), and for \( \alpha \in F_q \), we have \( g(\alpha^b) = g(\alpha^{b+1}) = \ldots = g(\alpha^{b+\delta-2}) = 0 \), that is, the code has a sequence of \( \delta - 1 \) consecutive powers of \( \alpha \) as zeros. Then the minimum distance of \( C \) is at least, \( \delta \).

Definition 2.1: [32, pg. 202] Let \( q \) be a prime power and \( \alpha \) be a primitive \( n \)-th root of unity. A cyclic code \( C \) of length \( n \) over \( F_q \) \( (\gcd(q, n) = 1) \) is a BCH code with designed distance \( \delta \) if, for some integer \( b \geq 0 \), we have

\[ g(x) = \text{lcm}\{M(b)(x), M(b+1)(x), \ldots, M(b+\delta-2)(x)\}, \]

that is, \( g(x) \) is the monic polynomial of smallest degree over \( F_q \) having \( \alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+\delta-2} \) as zeros. Therefore, \( c \in C \) if and only if \( c(\alpha^b) = c(\alpha^{b+1}) = \ldots = c(\alpha^{b+\delta-2}) = 0 \). Thus the code has a string of \( \delta - 1 \) consecutive powers of \( \alpha \) as zeros. A parity check matrix for \( C \) is given by

\[
H_{\delta,b} = \begin{bmatrix}
1 & \alpha^b & \alpha^{2b} & \ldots & \alpha^{(n-1)b} \\
1 & \alpha^{(b+1)} & \alpha^{2(b+1)} & \ldots & \alpha^{(n-1)(b+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(b+\delta-2)} & \ldots & \ldots & \alpha^{(n-1)(b+\delta-2)}
\end{bmatrix},
\]

where each entry is replaced by the corresponding column of \( l \) elements from \( F_q \), where \( l = \text{ord}_n(q) \), and then removing any linearly dependent rows. The rows of the resulting matrix over \( F_q \) are the parity checks satisfied by \( C \).

Let \( B = \{b_1, \ldots, b_l\} \) be a basis of \( F_q^l \) over \( F_q \). If \( u = (u_1, \ldots, u_n) \in F_q^n \) then one can write the vectors \( u_i, 1 \leq i \leq n \), as linear combinations of the elements of \( B \), that is, \( u_i = u_{i1}b_1 + \ldots + u_{il}b_l \). Consider that \( u^{(j)} = (u_{j1}, \ldots, u_{jn}) \) are vectors in \( F_q^n \) with \( 1 \leq j \leq l \). Then, if \( v \in F_q^l \), one has \( v \cdot u = 0 \) if and only if \( v \cdot u^{(j)} = 0 \) for all \( 1 \leq j \leq l \).

From the BCH bound, the minimum distance of a BCH code is greater than or equal to its designed distance \( \delta \). If \( n = q^i - 1 \) then the BCH code is called primitive and if \( b = 1 \) it is called narrow-sense.

III. REVIEW OF CONVOLUTIONAL CODES

In this section we present a brief review of classical and quantum convolutional codes. For more details we refer the reader to [2, 3, 11, 19, 20, 36]. The following results can be found in [2, 3, 19, 20].

Recall that a polynomial encoder matrix \( G(D) \in F_q[D]^{k \times n} \) is called basic if \( G(D) \) has a polynomial right inverse. A basic generator matrix is called reduced (or minimal [19, 29, 40]) if the overall constraint length \( \gamma = \sum_{i=1}^n \gamma_i \) has the smallest value among all basic generator matrices (in this case the overall constraint length \( \gamma \) will be called the degree of the resulting code).

Definition 3.1: [3] A rate \( k/n \) convolutional code \( C \) with parameters \( (n, k, \gamma; m, d_f) \) is a submodule of \( F_q[D]^n \) generated by a reduced basic matrix \( G(D) = (g_{ij}) \in F_q[D]^{k \times n} \), that is, \( C = \{u(D)G(D) | u(D) \in F_q[D]^k \} \), where \( n \) is the length, \( k \) is the dimension, \( \gamma = \sum_{i=1}^n \gamma_i \) is the degree, where \( \gamma_i = \max_{1 \leq j \leq n} \{\deg g_{ij}\} \), \( m = \max_{1 \leq i \leq k} \{\gamma_i\} \) is the memory and \( d_f = \text{wt}(C) = \min\{\text{wt}(v(D)) | v(D) \in C, v(D) \neq 0\} \) is the free distance of the code.

In the above definition, the weight of an element \( v(D) \in F_q[D]^n \) is defined as \( \text{wt}(v(D)) = \sum_{i=1}^n \text{wt}(v_i(D)) \), where \( \text{wt}(v_i(D)) \) is the number of nonzero coefficients of \( v_i(D) \).

If one considers the field of Laurent series \( F_q((D)) \) whose elements are given by \( u(D) = \sum_{i=1}^r u_i D^i \), where \( u_i \in F_q \) and \( u_i = 0 \) for \( i \leq r \), for some \( r \in \mathbb{Z} \), we define the weight of \( u(D) = \text{wt}(u(D)) = \sum_i \text{wt}(u_i) \). A generator matrix \( G(D) \) is called catastrophic if there exists a \( u(D)^{k^2} \in F_q((D))^k \) of infinite Hamming weight such that \( u(D)^{k^2}G(D) \) has finite Hamming weight. Since a basic generator matrix is non-catastrophic, all the classical (quantum) convolutional codes constructed in this paper have non-catastrophic generator matrices.

Let us recall that the Euclidean inner product of two \( n \)-tuples \( u(D) = \sum_{i=1}^n u_i D^i \) and \( v(D) = \sum_{i=1}^n v_i D^i \) in \( F_q[D]^n \) is defined as \( \langle u(D) | v(D) \rangle = \sum_{i=1}^n u_i v_i \). If \( C \) is a convolutional code then the code \( C^\perp = \{u(D) \in F_q[D]^n | \langle u(D) | v(D) \rangle = 0 \text{ for all } v(D) \in C \} \) denotes its Euclidean dual.
Similarly, the Hermitian inner product is defined as \( \langle u(D) \mid v(D) \rangle_h = \sum_i u_i \cdot v_i^*, \) where \( u_i, v_i \in F_{q^2}^n \) and \( v_i^* = (v_{i_1}, \ldots, v_{i_m}). \) The Hermitian dual of the code \( C \) is defined by \( C^h = \{ u(D) \in F_{q^2}^n \mid \langle u(D) \mid v(D) \rangle_h = 0 \text{ for all } v(D) \in C \}. \)

A. Convolutional Codes Derived from Block Codes

In this subsection we recall some results shown in [2] that will be utilized in the proposed constructions.

We consider that \( [n, k, d_q] \) is a block code with parity check matrix \( H \) and then we split \( H \) into \( m+1 \) disjoint submatrices \( H_i \) such that

\[
H = \begin{bmatrix}
H_0 \\
H_1 \\
\vdots \\
H_m
\end{bmatrix},
\]

where each \( H_i \) has \( n \) columns, obtaining the polynomial matrix

\[
G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \ldots + \tilde{H}_m D^m,
\]

where the matrices \( \tilde{H}_i \), for all \( 1 \leq i \leq m \), are derived from the respective matrices \( H_i \) by adding zero-rows at the bottom in such a way that the matrix \( \tilde{H}_i \) has \( \kappa \) rows in total, where \( \kappa \) is the maximal number of rows among the matrices \( H_i \). As it is well known, the matrix \( G(D) \) generates a convolutional code with \( \kappa \) rows. Note that \( m \) is the memory of the resulting convolutional code generated by the matrix \( G(D) \).

**Theorem 3.1:** [2, Theorem 3] Suppose that \( C \subseteq F_q^n \) is a linear code with parameters \( [n, k, d_q] \) and assume also that \( H \in F_q^{(n-k)\times n} \) is a parity check matrix for \( C \) partitioned into submatrices \( H_0, H_1, \ldots, H_m \) as in eq. (1) such that \( \kappa = rkH_0 \) and \( rkH_i \leq \kappa \) for \( 1 \leq i \leq m \) and consider the polynomial matrix \( G(D) \) as in eq. (2). Then we have:

(a) The matrix \( G(D) \) is a reduced basic generator matrix;

(b) If \( C^h \subset C \) (resp. \( C^{Lh} \subset C \)), then the convolutional code \( V = \{ v(D) = u(D)G(D) \mid u(D) \in F_q^{n-k}[D] \} \) satisfies \( V \subseteq V^h \) (resp. \( V \subseteq V^{Lh} \));

(c) If \( d_f \) and \( d_f^+ \) denote the free distances of \( V \) and \( V^h \), respectively, \( d_f \) denote the minimum distance of the code \( C = \{ v \in F_q^n \mid \forall H_i^t v = 0 \} \) and \( d^+ \) is the minimum distance of \( C^{Lh} \), then one has \( \min\{d_0 + d_m, d^+\} \leq d_f \leq d \) and \( d_f \geq d^+ \).

B. Review of Quantum Convolutional Codes

We begin this subsection by describing briefly the concept of quantum convolutional codes. For more details the reader can consult [35].

A quantum convolutional code is defined by means of its stabilizer which is a subgroup of the infinite version of the Pauli group, consisting of tensor products of generalized Pauli matrices acting on a semi-infinite stream of qudits. The stabilizer can be defined by a stabilizer matrix of the form

\[
S(D) = (X(D) \mid Z(D)) \in F_{q^2}^{(n-k)\times 2n} \quad \text{satisfying} \quad X(D)Z(1/D)^t - Z(D)X(1/D)^t = 0 \quad \text{(symplectic orthogonality)}.
\]

More precisely, consider a quantum convolutional code \( C \) defined by a full-rank stabilizer matrix \( S(D) \) given above. Then \( C \) is a rate \( k/n \) code with parameters \( [(n, k, m; \gamma, d_f)]_q \), where \( n \) is the frame size, \( k \) is the number of logical qudits per frame, \( m = \max_{1\leq i\leq n-k, 1\leq j\leq m} \{ \max \{ \deg X_{ij}(D), \deg Z_{ij}(D) \} \} \) is the memory, \( d_f \) is the free distance and \( \gamma \) is the degree of the code. Similarly as in the classical case, the constraint lengths are defined as \( \gamma_i = \max_{1\leq i\leq n} \{ \max \{ \deg X_{ij}(D), \deg Z_{ij}(D) \} \} \), and the overall constraint length is defined as \( \gamma = \sum_{i=1}^{n-k} \gamma_i \).

On the other hand, a quantum convolutional code can also be described in terms of a semi-infinite stabilizer matrix \( S \) with entries in \( F_q \times F_q \) in the following way. If \( S(D) = \sum S_i D^i \), where each matrix \( S_i \) for all \( i = 0, \ldots, m \), is a matrix of size \( (n - k) \times n \), then the semi-infinite matrix is defined as

\[
S = \begin{bmatrix}
G_0 & G_1 & \ldots & G_m & 0 & \ldots & \ldots \\
0 & G_0 & G_1 & \ldots & G_m & 0 & \ldots \\
& & & & \vdots & & \vdots \\
& & & & \vdots & & \vdots
\end{bmatrix}.
\]

Next, let \( H = C^{\otimes n} = C^{\otimes n} \otimes \cdots \otimes C^{\otimes n} \) be the Hilbert space and \( |x> \) be the vectors of an orthonormal basis of \( C^n \), where the labels \( x \) are elements of \( F_q \). Consider \( a, b \in F_q \) and take the unitary operators \( X(a) \) and \( Z(b) \) in \( C^n \) defined by \( X(a)|x> = |x+a> \) and \( Z(b)|x> = e^{i\pi b x}|x> \), respectively, where \( w = \exp(2\pi i/p) \) is a primitive \( p \)-th root of unity, \( p \) is the characteristic of \( F_q \) and \( tr \) is the trace map from \( F_p \) to \( F_p \). Considering the error basis \( E = \{ X(a), Z(b) \mid a, b \in F_q \} \), one defines the set \( P_{\infty} \) (according to [3]) as the set of all infinite tensor products of matrices \( N \in \langle M \mid M \in E \rangle \), in which all but finitely many tensor components are equal to \( I \), where \( I \) is the \( q \times q \) identity matrix. Then one defines the weight \( wt \) of \( A \in P_{\infty} \) as its (finite) number of nonidentity tensor components. In this context, one says that a quantum convolutional code has free distance \( d_f \) if and only if it can detect all errors of weight less than \( d_f \), but cannot detect some error of weight \( d_f \).

The following lemma deals with the existence of convolutional stabilizer codes derived from classical convolutional codes:

**Lemma 3.2:** [2, Proposition 2] Let \( C \) be an \( (n, (n-k)/2; \gamma; m)_q \) convolutional code such that \( C \subseteq C^{Lh} \). Then there exists an \( [(n, k, m; \gamma, d_f)]_q \) convolutional stabilizer code, where \( d_f = wt(C^{Lh} \setminus C) \).

In [3], the authors derived the quantum Singleton bound for quantum convolutional codes as it is shown in the next theorem. Let \( C \) be an \( (n, k, m; \gamma, d_f)_q \) quantum convolutional code. Recall that \( C \) is a pure code if it does not exist errors of weight less than \( d_f \) in the stabilizer of \( C \).

**Theorem 3.3:** (Quantum Singleton bound) The free distance of an \( [(n, k, m; \gamma, d_f)]_q \) \( F_{q^2} \)-linear pure convolutional stabilizer code is bounded by

\[
d_f \leq \left\lfloor \frac{n-k}{2} \left\lfloor \frac{2\gamma}{n+k} + 1 \right\rfloor + \gamma + 1 \right\rfloor.
\]

**Remark 3.4:** When Klappenecker et al. introduced the generalized quantum Singleton bound (GQSB) (see [3]) they
developed an approach to convolutional stabilizer codes based on direct limit constructions. It seems that the direct limit structure behaves well with respect to the trace-alternant form. In this context they derived the GQSB. It is interesting to note that this is one of few bounds presenting in the literature concerning quantum convolutional codes.

IV. NEW CLASSICAL MDS-CONVOLUTIONAL CODES

Constructions of classical convolutional codes with good or even optimal parameters (where the latter class of codes is known as maximum-distance-separable or MDS codes, i.e., codes attaining the generalized Singleton bound according to [38]) is a difficult task [10, 18, 26–31, 36, 38, 40]. Due to this difficulty, most of methods available in the literature are based on computational search. Keeping in mind the discussion above, our purpose is to construct new families of classical and quantum MDS convolutional codes by applying algebraic methods.

The main results of this section are Theorem 4.2 and Theorem 4.6. They generate new families of optimal (in the sense that the codes attain the generalized Singleton bound [38]) convolutional codes of length \( n = q + 1 \), over \( F_q \) for all prime power \( q \). Before proceeding further, recall the well known result from [32]:

**Lemma 4.1:** [32, Theorem 9, Chapter 11] Suppose that \( q = 2^t \), where \( t \geq 2 \) is an integer, \( n = q + 1 \) and consider that \( a = \frac{n}{2} \). Then one has:

i) With exception of coset \( C_0 = \{0\} \), each one of the other \( q \)-ary cyclotomic cosets is of the form \( C_{a-i} = \{a-i, a+i+1\} \), where \( 0 \leq i \leq a-1 \);

ii) The \( q \)-ary cosets \( C_{a-i} = \{a-i, a+i+1\} \), where \( 0 \leq i \leq a-1 \), are mutually disjoint.

We are now able to show one of the main results of this section:

**Theorem 4.2:** Assume that \( q = 2^t \), where \( t \geq 3 \) is an integer, \( n = q + 1 \) and consider that \( a = \frac{n}{2} \). Then there exist classical MDS convolutional codes with parameters \( (n, n-2i, 2i+1, 2i+3)_{F_q}, \) where \( 1 \leq i \leq a-1 \).

**Proof:** We first note that \( \gcd(n, q) = 1 \) and \( \operatorname{ord}_n(q) = 2 \). The proof consists of two steps. The first one is the construction of suitable BCH (block) codes and the second step is the construction of convolutional BCH codes derived from the BCH (block) codes generated in the first step.

Let us begin the first step. Let \( C_2 \) be the BCH code of length \( n \) over \( F_q \) generated by the product of the minimal polynomials

\[ C_2 = \langle g_2(x) \rangle = \langle M^{(a-i)}(x)M^{(a-i+1)}(x) \cdots M^{(a-1)}(x)M(x) \rangle. \]

A parity check matrix of \( C_2 \) is obtained from the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & \alpha^{a} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \alpha^{a-i} & \alpha^{a-i+1} & \cdots & \alpha^{n-1}
\end{bmatrix}
\]

by expanding each entry as a column vector (containing 2 rows) with respect to some \( F_q \)-basis \( \beta \) of \( F_{q^2} \) and then removing any linearly dependent rows. This new matrix \( H_{C_2} \) is a parity check matrix of \( C_2 \) and it has \( 2i + 2 \) rows. Since the dimension of \( C_2 \) is equal to \( n - 2(i + 1) \) (as proved in the paragraph below), so there is no linearly dependent rows in \( H_{C_2} \).

From Lemma 4.1, each one of the \( q \)-ary cyclotomic cosets \( C_{a-i}, \) where \( 0 \leq i \leq a-1 \) (corresponding to the minimal polynomials \( M^{(a-i)}(x) \)), has two elements and they are mutually disjoint. Since the degree of the generator polynomial \( g_2(x) \) of the code \( C_2 \) equals the cardinality of its defining set, then one has \( \deg(g_2(x)) = 2(i + 1) \), so the dimension \( k_{C_2} \) of \( C_2 \) equals \( k_{C_2} = n - \deg(g_2(x)) = n - 2(i + 1) \). Moreover, the defining set of the code \( C_2 \) consists of the sequence \( \{a-i, a-i+1, \ldots, a, a+1, \ldots, a+i+1\} \) of \( 2i + 2 \) consecutive integers, so, from the BCH bound, the minimum distance \( d_{C_2} \) of \( C_2 \) satisfies \( d_{C_2} \geq 2i + 3 \). Thus, \( C_2 \) is a MDS code with parameters \([n, n-2i-2, 2i+3]_q\) and, consequently, its (Euclidean) dual code has dimension \( 2i + 2 \).

We next consider that \( C_1 \) is the BCH code of length \( n \) over \( F_q \) generated by the product of the minimal polynomials

\[ C_1 = \langle g_1(x) \rangle = \langle M^{(a-i+1)}(x)M^{(a-i+2)}(x) \cdots M^{(a-1)}(x)M^{(a)}(x) \rangle. \]

Similarly, \( C_1 \) has a parity check matrix derived from the matrix

\[
\begin{bmatrix}
\alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \cdots & \alpha^{(n-1)(a-i+1)} \\
\alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \cdots & \alpha^{(n-1)(a-i+2)} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\alpha^{a} & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix}
\]

by expanding each entry as a column vector (containing 2 rows) with respect to some \( F_q \)-basis \( \beta \) (already done, since \( H_{2i+1,a-i+1} \) is a submatrix of \( H_{2i+3,a-i+1} \)). After performing the expansion to all entries, such new matrix is denoted by \( H_{C_1} \) (\( H_{C_1} \) is a submatrix of \( H_{C_2} \)). Applying again Lemma 4.1 and proceeding similarly as above, it follows that \( C_1 \) is a MDS code with parameters \([n, n-2i, 2i+1]_q\).

To finish the first step, consider \( C \) be the BCH code of length \( n \) over \( F_q \) generated by the minimal polynomial \( M^{(a-i)}(x) \), that is,

\[ C = \langle M^{(a-i)}(x) \rangle. \]

\( C \) has parameters \([n, n-2, d \geq 2]_q \). A parity check matrix \( H_C \) of \( C \) is given by expanding each entry of the matrix

\[
\begin{bmatrix}
\alpha^{(a-i)} & \alpha^{2(a-i)} & \cdots & \alpha^{(n-1)(a-i)} \\
\alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{a} & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix}
\]

with respect to \( \beta \) (already done, since \( H_{2,a-i} \) is a submatrix of \( H_{2i+3,a-i+1} \)). Since \( C \) has dimension \( n-2 \), \( H_C \) has rank 2 (\( H_C \) is also a submatrix of \( H_{C_2} \)).
Next we describe the second step. We begin by rearranging the rows of $H_{C_2}$ in the form

$$H = \begin{bmatrix} 1 & \alpha^a & \cdots & \alpha^{(n-1)a} \\ 1 & \alpha^{(a-1)} & \cdots & \alpha^{(n-1)(a-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(a-i+1)} & \cdots & \alpha^{(n-1)(a-i+1)} \\ 1 & \alpha^{(a-i)} & \cdots & \alpha^{(n-1)(a-i)} \end{bmatrix},$$

(to simplify the notation we write $H$ in terms of powers of $\alpha$, although it is clear from the context that this matrix has entries in $F_q$, which are derived from expanding each entry with respect to the basis $\beta$ already performed).

Then we split $H$ into two disjoint submatrices $H_0$ and $H_1$ of the forms

$$H_0 = \begin{bmatrix} 1 & \alpha^a & \cdots & \alpha^{(n-1)a} \\ 1 & \alpha^{(a-1)} & \cdots & \alpha^{(n-1)(a-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(a-i+1)} & \cdots & \alpha^{(n-1)(a-i)} \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} 1 & \alpha^{(a-i)} & \cdots & \alpha^{(n-1)(a-i)} \end{bmatrix},$$

respectively, where $H_0$ is obtained from the matrix $H_{C_1}$ by rearranging rows and $H_1$ is derived from $H_{C_2}$ also by rearranging rows. Hence it follows that $\text{rk}H_0 \geq \text{rk}H_1$.

Then we form the convolutional code $V$ generated by the reduced basic (according to Theorem 3.1 Item (a)) generator matrix $G(D) = \tilde{H}_0 + \tilde{H}_1 D$, where $\tilde{H}_0 = H_0$ and $\tilde{H}_1$ is obtained from $H_1$ by adding zero-rows at the bottom such that $\tilde{H}_1$ has the number of rows of $H_0$ in total. By construction, $V$ is a unit-memory convolutional code of dimension $2i$ and degree $\delta_V = 2$.

Consider next the Euclidean dual $V^\perp$ of the convolutional code $V$. We know that $V^\perp$ has dimension $n - 2i$ and degree 2. Let us now compute the free distance $d_f^\perp$ of $V^\perp$. By Theorem 3.1 Item (c), the free distance of $V^\perp$ is bounded by $\min\{d_0 + d_1, d\} \leq d_f^\perp \leq d$, where $d_i$ is the minimum distance of the code $C_i = \{v \in F_q^n \mid v \tilde{H}_i = 0\}$. From construction one has $d = 2i + 3$, $d_0 = 2i + 1$ and $d_1 \geq 2$, so $V^\perp$ has parameters $(n, n - 2i, 2; 1, 2i + 3)$.

Recall that the generalized (classical) Singleton bound [40] of an $(n, k, \gamma; m, d_f)_q$ convolutional code is given by

$$d_f \leq (n - k)[\lfloor \gamma/k \rfloor + 1] + \gamma + 1.$$ 

Replacing the values of the parameters of $V^\perp$ in the above inequality one concludes that $V^\perp$ is a MDS convolutional code and the proof is complete.

**Remark 4.3:** Note that the new codes have degree $\gamma = 2$. The reason for this is as follows: in order to obtain codes with maximum minimum distances we have to construct codes (the notation is the same utilized in Theorem 4.2) satisfying the inequalities $\min\{d_0 + d_1, d\} \leq d_f^\perp \leq d$. Therefore one designs the code $C$ with parameters $[n, n - 2i, 2i + 3]$. Now, it is easy to see that the corresponding convolutional code $V^\perp$ has degree 2.

Let us now give an illustrative example.

**Example 4.1:** According to Theorem 4.2, let $q = 16$, $n = q + 1 = 17$ and $a = 8$. Assume $C_2$ is an $(17, 11, 7)_{16}$ (cyclic) MDS code generated by the product of the minimal polynomials $M^{(6)}(x)M^{(7)}(x)M^{(6)}(x)$. The corresponding cyclotomic cosets of $C_2$ are $\{8, 9\}, \{7, 10\}$ and $\{6, 11\}$. Consider $C_1$ be the (cyclic) MDS code generated by the product of the minimal polynomials $M^{(6)}(x)M^{(7)}(x); C_1$ has parameters $[17, 13, 5]_{16}$. Finally, suppose $C$ is the cyclic code generated by $M^{(6)}(x)$, where $C$ has parameters $[17, 15, d \geq 2]_{16}$. In this case we have $i = 2$. Then we can form the convolutional code $V$ with reduced basic generator matrix $G(D) = \tilde{H}_0 + \tilde{H}_1 D$, where $\tilde{H}_0 = H_0$ and $\tilde{H}_1$ is obtained from $H_1$ by adding zero-rows at the bottom such that $\tilde{H}_1$ has the number of rows of $H_0$. The matrix $H_0$ is the parity check matrix of $C_1$ (up to permutation of rows) and $H_1$ is the parity check matrix of $C$. $V$ has parameters $(17, 4, 2; 1, d_f^\perp)_{16}$. The Euclidean dual $V^\perp$ has parameters $(17, 13, 2; 1, d_f^\perp)_{16}$, where $\min\{d_0 + d_1, d\} \leq d_f^\perp \leq d$, where $d_0 = 5$, $d_1 \geq 2$ and $d = 7$. Therefore $V^\perp$ has parameters $(17, 13, 2; 1, 7)_{16}$. Applying the generalized Singleton bound one has $7 = 4([2/13] + 1)+2+1$, so $V^\perp$ is MDS.

It is well known (see for example [38]) that if a convolutional code $C$ is MDS then one can not guarantee that its dual also is MDS. Unfortunately in the above construction, although the codes $V^\perp$ are MDS, there is no guarantee that their duals $V$ are MDS.

**Corollary 4.4:** Assume $q = 2^t$, where $t \geq 3$ is an integer, $n = q + 1$ and consider that $a = \frac{n}{2}$. Then there exist classical convolutional codes with parameters $(n, 2i, 2; 1, d_f)_q$, where $1 \leq i \leq a - 1$ and $d_f \geq n - 2i - 1$.

**Proof:** Consider the same construction and notation used in Theorem 4.2. We know that $V$ has parameters $(n, 2i, 2; 1, d_f)_q$. Let us compute $d_f$. From Theorem 3.1 Item (b), $d_f \geq d_f^\perp$. We know that the matrix $H$ is obtained by rearranging the rows of $H_{C_2}$ and the code $C_f^\perp$ is a MDS code with parameters $[n, 2i + 2, n - 2i - 1]_q$. Thus $d_f \geq n - 2i - 1$ and $V$ has parameters $(n, 2i, 2; 1, d_f)_q$, where $d_f \geq n - 2i - 1$.

Theorem 4.6, given in the sequence, is the second main result of this section. More precisely, in such theorem, we construct new families of (classical) MDS convolutional codes over $F_q$ for all $q = p^t$, where $t \geq 2$ and $p$ is an odd prime number. In order to prove it, we need the following well known result:

**Lemma 4.5:** [32, Theorem 9, Chapter 11] Suppose that $q = p^t$, where $t \geq 2$ is an integer and $p$ is an odd prime number. Let $n = q + 1$ and consider that $a = \frac{n}{2}$. Then one has:

i) The $q$-ary coset $C_a$ has only one element, that is, $C_a = \{a\}$;

ii) With exception of cosets $C_0 = \{0\}$ and $C_a$, each one of the other $q$-ary cyclotomic cosets is of the form $C_{a-i} = \{0, 1, \ldots, a - 1\}$.
\{a - i, a + i\}, where 1 \leq i \leq a - 1;

iii) The \(q\)-ary cosets \(C_{a-i} = \{a - i, a + i\}\), where 1 \leq i \leq a - 1, are mutually disjoint and have two elements.

Let us now prove Theorem 4.6. Since its proof is analogous to that of Theorem 4.2, we only give a sketch of it.

**Theorem 4.6:** Assume that \(q = p^t\), where \(t \geq 2\) is an integer and \(p\) is an odd prime number. Consider that \(n = q + 1\) and \(a = \frac{q}{2}\). Then there exist classical MDS convolutional codes with parameters \((n, n - 2i + 1, 2; 1, 2i + 2)_q\), where \(2 \leq i \leq a - 1\).

**Proof:** Let \(C_2\) be the BCH code of length \(n\) over \(F_q\) generated by the product of the minimal polynomials
\[ C_2 = \langle g_2(x) \rangle = \langle M^{(a-i)}(x)M^{(a-i+1)}(x) \rangle, \]
\[ \ldots, M^{(a-i)}(x)M^{(a)}(x) \],
whose parity check matrix \(H_{C_2}\) is obtained from the matrix

\[
H_{2i+2,a-i} = \\
\begin{bmatrix}
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)} \\
1 & a^{(a+i)} & a^{2(a+i)} & \cdots & a^{(n-1)(a+i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
a & \cdots & \cdots & \cdots & a^{(n-1)a}
\end{bmatrix}
\]

by expanding each entry as a column vector over some \(F_q\)-basis \(\beta\) of \(F_{q^2}\) and removing one linearly dependent row, because \(H_{C_2}\) has rank \(2i + 1\) (computed below).

From Lemma 4.5, each one of the \(q\)-ary cyclotomic cosets \(C_{a-i}\), where \(2 \leq i \leq a - 1\), has two elements, they are mutually disjoint and the coset \(C_a\) has only one element. Thus the dimension \(k_{C_2}\) of \(C_2\) equals \(k_{C_2} = n - \deg(g_2(x)) = n - 2i - 1\). Moreover, since the defining set of the code \(C_2\) consists of the sequence \(\{a - i, a - i + 1, \ldots, a, a + 1, \ldots, a + i\}\) of \(2i + 1\) consecutive integers then the minimum distance \(d_{C_2}\) of \(C_2\) satisfies \(d_{C_2} \geq 2i + 2\). Hence, \(C_2\) is a MDS code with parameters \((n, n - 2i + 1, 2i + 2)_q\).

We next consider \(C_1\) as the BCH code of length \(n\) over \(F_q\) generated by the product of the minimal polynomials
\[ C_1 = \langle g_1(x) \rangle = \langle M^{(a-i+1)}(x)M^{(a-i+2)}(x) \rangle, \]
\[ \ldots, M^{(a)}(x)M^{(a)}(x) \],
whose parity check matrix \(H_{C_1}\) is derived from the matrix

\[
H_{2i,a-i} = \\
\begin{bmatrix}
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
1 & a^{(a+i+1)} & a^{2(a+i+1)} & \cdots & a^{(n-1)(a+i+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
a & \cdots & \cdots & \cdots & a^{(n-1)a}
\end{bmatrix}
\]

by expanding each entry as a column vector with respect to \(\beta\) of \(F_{q^2}\). Then it follows that \(C_1\) is a MDS code with parameters \([n, n - 2i + 1, 2i + 1]_q\) and \(H_{C_1}\) has rank \(2i + 1\).

Assume that \(\bar{C}\) is the BCH code generated by the minimal polynomial \(M^{(a-i)}(x)\). Then \(\bar{C}\) has parameters \([n, n - 2, d \geq 2\]_q. A parity check matrix \(H_C\) of \(C\) is given by expanding each entry of the matrix

\[
H_{2i,a-i} = \\
\begin{bmatrix}
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)} \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)}
\end{bmatrix}
\]

with respect to \(\beta\). \(H_C\) has rank 2.

Rearranging the rows of \(H_{C_2}\) we obtain the matrix

\[
H = \\
\begin{bmatrix}
1 & a^a & \cdots & \cdots & a^{(n-1)a} \\
1 & a^{(a-1)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)}
\end{bmatrix}
\]

where \(a = \frac{q}{2}\). Next we split \(H\) into two disjoint submatrices \(H_0\) and \(H_1\) (as in Theorem 4.2) of the form

\[
H_0 = \\
\begin{bmatrix}
1 & a^{a} & \cdots & \cdots & a^{(n-1)a} \\
1 & a^{(a-1)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)}
\end{bmatrix}
\]

and

\[
H_1 = \\
\begin{bmatrix}
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)} \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)}
\end{bmatrix}
\]

obtaining, in this way, the convolutional code \(V\) generated by the matrix

\[ G(D) = \tilde{H}_0 + \tilde{H}_1 D\]

with parameters \((n, 2i - 1, 2; 1, d_f)_q\). Proceeding similarly as in Theorem 4.2, one has a MDS convolutional code \(V^2\) with parameters \((n, n - 2i + 1, 2; 1, 2i + 2)_q\), for all \(2 \leq i \leq a - 1\).

In the next result, we construct memory-two convolutional codes:

**Theorem 4.7:** Assume that \(q = p^t\), where \(t \geq 2\) is an integer and \(p\) is an odd prime number. Consider that \(n = q + 1\) and \(a = \frac{q}{2}\). Then there exist convolutional codes with parameters \((n, 2i - 3, 4; 2, d_f)_q\), where \(3 \leq i \leq a - 1\).

**Proof:** Let \(C_3\) be the BCH code of length \(n\) over \(F_q\) generated by the product of the minimal polynomials
\[ C_3 = \langle g_3(x) \rangle = \langle M^{(a-i)}(x)M^{(a-i+1)}(x)M^{(a-i+2)}(x) \rangle, \]
\[ \ldots, M^{(a-i)}(x)M^{(a-i+1)}(x) \],
whose parity check matrix \(H_{C_3}\) is obtained from the matrix

\[
H_{2i+2,a-i} = \\
\begin{bmatrix}
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)} \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(a-i)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
a & \cdots & \cdots & \cdots & a^{(n-1)a}
\end{bmatrix}
\]
by expanding each entry as a column vector over some $F_q$-basis $\beta$ of $F_q^2$. We know that $C_3$ is a MDS code with parameters $[n, n - 2i - 1, 2i + 2]_q$ and $H_{C_3}$ has rank $2i + 1$.

We next consider $C_2$ as the BCH code of length $n$ over $F_q$ generated by the product of the minimal polynomials

$$C_2 = \langle g_2(x) \rangle = \langle M^{(a-1)}(x)M^{(a)}(x) \rangle,$$

whose parity check matrix $H_{C_2}$ is derived from the matrix

$$H_{2i-2,a-i+2} = \begin{bmatrix}
1 & \alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \cdots & \alpha^{(n-1)(a-i+2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix},$$

by expanding each entry as a column vector with respect to $\beta$ of $F_q^2$. Then it follows that $C_2$ is a code with parameters $[n, n - 2i + 3, 2i - 2]_q$.

Let $C_1$ be the BCH code of length $n$ over $F_q$ generated by $M^{(a+i+1)}(x)$ whose parity check matrix $H_{C_1}$ is given by expanding each entry of the matrix

$$H_{2,a-i+1} = \begin{bmatrix}
1 & \alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \cdots & \alpha^{(n-1)(a-i+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-i)} & \cdots & \cdots & \alpha^{(n-1)(a-i)}
\end{bmatrix},$$

with respect to $\beta$, and assume that $C$ is the BCH code generated by the minimal polynomial $M^{(a-i)}(x)$ with parity check matrix $H_C$ given by expanding each entry of the matrix

$$H_{2,a-i} = \begin{bmatrix}
1 & \alpha^{(a-i)} & \alpha^{2(a-i)} & \cdots & \alpha^{(n-1)(a-i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix},$$

with respect to $\beta$. We know that $C_1$ and $C$ has parameters $[n, n - 2, d \geq 2]_q$.

Rearranging the rows of $H_{C_3}$ we obtain the matrix

$$H = \begin{bmatrix}
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a} \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \cdots & \alpha^{(n-1)(a-i+2)} \\
1 & \alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \cdots & \alpha^{(n-1)(a-i+1)} \\
1 & \alpha^{(a-i)} & \alpha^{2(a-i)} & \cdots & \alpha^{(n-1)(a-i)}
\end{bmatrix},$$

Next we split $H$ into three disjoint submatrices $H_0$ and $H_1$ and $H_2$ (as in Theorem 4.2) of the form

$$H_0 = \begin{bmatrix}
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a} \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \cdots & \alpha^{(n-1)(a-i+2)} \\
1 & \alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \cdots & \alpha^{(n-1)(a-i+1)} \\
1 & \alpha^{(a-i)} & \alpha^{2(a-i)} & \cdots & \alpha^{(n-1)(a-i)}
\end{bmatrix},$$

and

$$H_2 = \begin{bmatrix}
1 & \alpha^{(a-i)} & \alpha^{2(a-i)} & \cdots & \alpha^{(n-1)(a-i)} \\
1 & \alpha^{(a-r+m)} & \alpha^{2(a-r+m)} & \cdots & \alpha^{(n-1)(a-r+m)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-r+1)} & \alpha^{2(a-r+1)} & \cdots & \alpha^{(n-1)(a-r+1)} \\
1 & \alpha^{(a-r)} & \alpha^{2(a-r)} & \cdots & \alpha^{(n-1)(a-r)} \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix},$$

by expanding each entry as a column vector over some $F_q$-basis $\beta$ of $F_q^2$. We know that $C$ is a MDS code with parameters $[n, n - 2r - 1, 2r + 2]_q$.

We next consider $C_0$ as the BCH code of length $n$ over $F_q$ generated by the product of the minimal polynomials

$$C_0 = \langle g_0(x) \rangle = \langle M^{(a-r)}(x) \cdots M^{(a-1)}(x)M^{(a)}(x) \rangle,$$

whose parity check matrix $H_{C_0}$ is derived from the matrix

$$H_{2r+2,a-r} = \begin{bmatrix}
1 & \alpha^{(a-r)} & \alpha^{2(a-r)} & \cdots & \alpha^{(n-1)(a-r)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix},$$

by expanding each entry as a column vector with respect to $\beta$ of $F_q^2$. We know that $C_0$ is a MDS code with parameters $[n, n - 2r - 1, 2r + 2]_q$. 

Theorem 4.7 can be easily generalized as one can see in the next result:

**Theorem 4.8:** Assume that $q = p^t$, where $t \geq 2$ is an integer and $p$ is an odd prime number. Consider that $n = q + 1$, $a = \frac{a}{2}$ and let $r, m$ integers with $r \geq 1$, $m \geq 2$ such that $3 \leq r + m \leq a - 1$. Then there exist convolutional codes with parameters $(n, 2r + 1, 2m; m, d_f \geq n - 2[r + m])_q$.

**Proof:** Let $C$ be the BCH code of length $n$ over $F_q$ generated by the product of the minimal polynomials

$$C = \langle g(x) \rangle = \langle M^{(a-\lfloor r\rfloor)}(x) \cdots M^{(a-\lfloor r+2\rfloor)}(x) \rangle,$$

$$\cdots \langle M^{(a-\lfloor r+m\rfloor)}(x) \rangle,$$

whose parity check matrix $H_C$ is obtained from the matrix

$$H_{2[r+m]+2,a-r+2} = \begin{bmatrix}
1 & \alpha^{(a-r+2)} & \alpha^{2(a-r+2)} & \cdots & \alpha^{(n-1)(a-r+2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-r+1)} & \alpha^{2(a-r+1)} & \cdots & \alpha^{(n-1)(a-r+1)} \\
1 & \alpha^{(a-r)} & \alpha^{2(a-r)} & \cdots & \alpha^{(n-1)(a-r)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(a-1)} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a}
\end{bmatrix},$$

by expanding each entry as a column vector over some $F_q$-basis $\beta$ of $F_q^2$. We know that $C_0$ is a MDS code with parameters $[n, n - 2r - 1, 2r + 2]_q$.
Let $C_i$ for all $1 \leq i \leq m$, be the BCH code of length $n$ over $F_q$ generated by $M^{(a-\{r+i\})}(x)$ whose parity check matrix $Hc_i$ is given by expanding each entry of the matrix

$$H_{2,a-\{r+i\}} = \begin{bmatrix} 1 & \alpha^{(a-\{r+i\})} & \alpha^{2(a-\{r+i\})} & \ldots & \alpha^{(n-1)(a-\{r+i\})} \end{bmatrix}$$

with respect to $\beta$. We know that $C_i$ has parameters $[n, n-2, d \geq 2]_q$.

Proceeding similarly as in the proof of Theorem 4.7, one obtains a convolutional code $V$ generated by the matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1D + \tilde{H}_2D^2 + \ldots + \tilde{H}_mD^m$$

with parameters $(n, 2r+1, 2m; m, d_f)_q$, where $d_f \geq n-2[r+m]$.

Remark 4.9: It is important to observe that the procedure adopted in Theorem 4.8 has several variants and, therefore, several more new families can be constructed straightforwardly based on our method.

Remark 4.10: Unfortunately if one considers $m > 1$, there is no guarantee that the corresponding convolutional codes are MDS.

V. NEW QUANTUM MDS-CONVOLUTIONAL CODES

As in the classical case, the construction of MDS quantum convolutional codes is a difficult task. This task is performed in [3, 12, 14, 16] but only in [3, 14] the constructions are made algebraically. Based on this view point, we propose the construction of more MDS convolutional stabilizer codes.

It is well known that convolutional stabilizer codes can be constructed from classical convolutional codes (see for example [2, Proposition 1 and 2]). In the first construction, one utilizes convolutional codes endowed with the Euclidean inner product and in the second one, the codes are endowed with the Hermitian inner product. Considering the $q$-ary cosets modulo $n = q+1$ as given in the previous section, it is easy to see that the dual-containing property with respect to the Euclidean inner product does not hold for (classical) convolutional codes derived from block codes with defining set of this type. However, when considering cyclic codes endowed with the Hermitian inner product one can show the existence of convolutional codes, derived from them, which are (Hermitian) self-orthogonal (see Lemma 5.1). This fact permits the construction of MDS quantum convolutional codes (in the sense that they attain the generalized quantum Singleton bound (Theorem 3.3) as it is shown in Theorem 5.2; given in the following. More precisely, we utilize the MDS-convolutional codes constructed in the previous section for constructing quantum MDS convolutional codes. Before proceeding further, we need the following result:

Lemma 5.1: Assume $q = 2^t$, where $t$ is an integer such that $t \geq 1$, $n = q^2 + 1$ and let $a = \frac{q^2}{2}$. If $C$ is the cyclic code whose defining set $Z$ is given by $Z = C_{a-1} \cup \ldots \cup C_a$, where $0 \leq i \leq \frac{q^2}{2} - 1$, then $C$ is Hermitian dual-containing.

Proof: See [23, Lemma 4.2].

Although Theorem 5.2 is a Corollary of Theorem 4.2, we consider it as a theorem because the resulting quantum convolutional codes are MDS.

Theorem 5.2: Assume $q = 2^t$, where $t \geq 3$ is an integer, $n = q^2 + 1$ and consider that $a = \frac{q^2}{2}$. Then there exist quantum MDS convolutional codes with parameters $[(n, n-4i, 1; 2, 2i+3)]_q$, where $2 \leq i \leq \frac{q^2}{2} - 2$.

Proof: We consider the same notation utilized in Theorem 4.2. We know that $\gcd(n, q^2) = 1$. From Theorem 4.2, there exists a classical convolutional MDS code with parameters $(n, n-2i, 2, 1; 2i+3)_q$, for each $2 \leq i \leq \frac{q^2}{2} - 2$. This code is the Euclidean dual $V^⊥$ of the convolutional code $V$ whose parameters are given by $(n, 2i, 2; 1, d_f)\_q$. The codes $V^⊥$ and $V^{⊥\_h}$ have the same degree as code (see the proof of Theorem 7 in [3]). Additionally, it is straightforward to check that $\wt(V^⊥) = \wt(V^{⊥\_h})$, so $V^{⊥\_h}$ has parameters $(n, n-2i, 2; m^*, 2i+3)\_q$. From Lemma 5.1 and from Theorem 4.2, there exists an $V\_q$ convolutional stabilizer code, for each $2 \leq i \leq \frac{q^2}{2} - 2$. Replacing the parameters of the previously constructed codes in the quantum generalized Singleton bound (Theorem 3.3) one has the equality $2i + 3 = 2i\left(\left\lfloor \frac{1}{2\cdot q^{t-1}} \right\rfloor + 1\right) + 2 + 1$. Therefore, there exist MDS-convolutional stabilizer codes with parameters $[(n, n-4i, 1; 2, 2i+3)]_q$, for each $2 \leq i \leq \frac{q^2}{2} - 2$.

Example 5.1: To illustrate the previous construction, assume that $q = 8$, $n = 65$ and $i = 2$. Applying Theorem 5.2 there exists an $[(65, 57, 1; 2, 7)]_8$ convolutional stabilizer code that attains the generalized quantum Singleton bound.

Considering $q = 16$, $n = 257$ and $i = 2, 3, 4, 5$ then one has quantum MDS codes with parameters $[(257, 249, 1; 2, 7)]_{16}$, $[(257, 245, 1; 2, 9)]_{16}$, $[(257, 241, 1; 2, 11)]_{16}$, $[(257, 237, 1; 2, 13)]_{16}$, respectively, and so on.

VI. SUMMARY

In this paper we have constructed several new families of multi-memory classical convolutional BCH codes. The families of unit-memory codes are optimal in the sense that they attain the classical generalized Singleton bound. Moreover, we also have constructed families of unit-memory optimal quantum convolutional codes in the sense that these codes attain the quantum generalized Singleton bound. All the constructions presented here are performed algebraically and not by exhaustively computational search.

ACKNOWLEDGMENT

I would like to thank the anonymous referees and the Associate Editor Prof. Dr. Alexei Ashikhmin for their valuable comments and suggestions that improve significantly the quality and the presentation of this paper. This research has been partially supported by the Brazilian Agencies CAPES and CNPq.

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