Bohr-Sommerfeld tori and relative Poincaré series on a complex hyperbolic space

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Abstract

Automorphic forms on a bounded symmetric domain $D = G/K$ can be viewed as holomorphic sections of $L^\otimes k$, where $L$ is a quantizing line bundle on a compact quotient of $D$ and $k$ is a positive integer.

Let $\Gamma$ be a cocompact discrete subgroup of $SU(n, 1)$ which acts freely on $SU(n, 1)/U(n)$. We suggest a construction of relative Poincaré series associated to loxodromic elements of $\Gamma$. In complex dimension 2 we describe the Bohr-Sommerfeld tori in $\Gamma\backslash SU(n, 1)$ associated to hyperbolic elements of $\Gamma$ and prove that the relative Poincaré series associated to hyperbolic elements of $\Gamma$ are not identically zero for large values of $k$.

1 Introduction

1.1 General definitions

We shall start with a brief review of the general concept of an automorphic form. Let $G$ be a connected non-compact real semi-simple Lie group, $K$ be a maximal compact subgroup of $G$, $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma\backslash G$ has a finite volume. Let $V$ be a finite-dimensional vector space, $\rho : K \to GL(V)$ be a representation of $K$. A smooth $Z(g)$-finite function $f : G \to V$ is called an automorphic form on $G$ for $\Gamma$ if

$$f(\gamma g k) = f(g)\rho(k)$$

(1)

for any $\gamma \in \Gamma$, $g \in G$, $k \in K$, and there are a positive constant $C$ and a non-negative integer $m$ such that

$$|f(g)| \leq C||g||^m$$

(2)

for any $g \in G$, here $|.|$ is the norm corresponding to a $\rho(K)$-invariant Hilbert structure on $V$, $||g|| = tr(g^*g)$ taken in the adjoint representation of $G$. An automorphic form $f$ is called a cusp form if $f \in L^\infty(\Gamma\backslash G)$.

The automorphy law (1) means geometrically that $f$ defines a $\Gamma$-invariant section of the vector bundle $G \times_K V \to G/K$ associated to the principal bundle $G \to G/K$, here $G \times_K V = G \times V/\sim$, and the equivalence relation is given by the representation $\rho : (g, v) \sim (gk, v\rho(k))$.  

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The growth condition (2) is automatically satisfied with \( m = 0 \) in the case when \( \Gamma \backslash G \) is compact and in this case any automorphic form is a cusp form.

Recall also that a function \( f : G \to V \) is said to be \( Z(g) \)-finite if it is annihilated by an ideal \( I \) of \( Z(g) \) of a finite codimension, here \( Z(g) \) is the center of the universal enveloping algebra \( U(g) \).

\( U(g) \) can be identified with the algebra \( D(G) \) of all left-invariant differential operators on \( G \): to \( Y \in g \) is associated a differential operator
\[
Yf(g) = \frac{d}{dt} f(ge^{tY})|_{t=0},
\]
this establishes a linear map \( g \to D(G) \) which extends to an isomorphism \( U(g) \to D(G) \). \( Z(g) \) can be viewed as the subalgebra of all bi-invariant differential operators, it is isomorphic to a polynomial ring in \( l \) letters where \( l \) is the rank of \( G \). A useful example to have in mind is \( G = \text{SL}(2, \mathbb{R}) \) and \( \text{codim} \ I = 1 \), then we have: \( l = 1 \), \( Z(g) \) is generated by the Casimir operator \( \mathcal{C} \), and saying that a function \( f \) is \( Z(g) \)-finite is equivalent to stating that \( f \) is an eigenfunction of \( \mathcal{C} \).

A well-known construction of an automorphic form on \( G \) is Poincaré series
\[
\sum_{\gamma \in \Gamma} q(\gamma g),
\]
where the function \( q : G \to V \) is \( Z(g) \)-finite and \( K \)-finite on the right (i.e. the set of its right translates under elements of \( K \) is a finite-dimensional vector space), and \( q \in L^1(G) \). One can also consider relative Poincaré series
\[
\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q(\gamma g),
\]
where \( q : G \to V \) is \( Z(g) \)-finite, \( K \)-finite on the right, \( \Gamma_0 \)-invariant, and \( q \in L^1(\Gamma_0 \backslash G) \).

Let us explain now how to construct an automorphic form on \( G/K \).

An automorphy factor is a map \( \mu : \Gamma \times G/K \to GL(V) \) such that \( \mu(g_1g_2,x) = \mu(g_1, g_2x)\mu(g_2, x) \). It allows to define an automorphic form on \( G/K \) as a function \( f : G/K \to V \) such that
\[
f(\gamma x)\mu(\gamma, x) = f(x)
\]
for any \( \gamma \in \Gamma, x \in G/K \). Notice that then the function \( F(g) = f(g(0))\mu(g, 0) \), where \( g \in G, x = g(0) \in G/K \), satisfies (3) with \( \rho(k) = \mu(k, 0) \). Here 0 is the fixed point of \( K \) in \( G/K \). If \( f \) is holomorphic then \( F \) is \( Z(g) \)-finite.

In particular, for a smooth function \( q \in L^1(G/K) \) the Poincaré series on \( G/K \) is
\[
\sum_{\gamma \in \Gamma} q(\gamma x)\mu(\gamma, x). \tag{3}
\]

Similarly for a smooth \( \Gamma_0 \)-invariant function \( q \in L^1(\Gamma_0 \backslash G/K) \) the relative Poincaré series is
\[
\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q(\gamma x)\mu(\gamma, x). \tag{4}
\]
1.2 Automorphic forms on bounded symmetric domains and quantization

Consider a classical system \((M, \omega)\), where \(M\) is a manifold, and \(\omega\) is a symplectic form on \(M\). The main problem of quantization is to associate a quantum system \((\mathcal{H}, \mathcal{O})\) to \((M, \omega)\), where \(\mathcal{H}\) is a Hilbert space and \(\mathcal{O}\) consists of symmetric operators on \(\mathcal{H}\).

The map \(f \mapsto \hat{f}\), where \(f \in C^\infty(M)\) and \(\hat{f} \in \mathcal{O}\), should satisfy the following requirements: 1) it is \(\mathbb{R}\)-linear, 2) if \(f = \text{const}\) then \(\hat{f}\) is the corresponding multiplication operator, 3) if \(\{f_1, f_2\} = f_3\) then \(\hat{f}_1 \hat{f}_2 - \hat{f}_2 \hat{f}_1 = -i\hbar \hat{f}_3\).

How do automorphic forms appear in the context of quantization?

Suppose that \(M\) is a compact Kähler manifold which is a quotient of a bounded symmetric domain \(D = G/K\) by the action of a discrete subgroup \(\Gamma\), i.e. \(M = \Gamma \backslash D\). Then the quantum phase space \(\mathcal{H}\) consists of holomorphic automorphic forms on \(D\) for \(\Gamma\). More precisely, let us consider the well-known quantization scheme for compact Kähler manifolds via Toeplitz operators (it is related to the standard scheme of geometric quantization with Kähler polarization). Then automorphic forms are holomorphic sections of \(L^{\otimes k}\), where the canonical line bundle \(L = \Lambda^n T^* M\) is the quantizing line bundle on \(M\), here \(n = \dim_{\mathbb{C}} M\) and \(k\) is a positive integer which determines the weight of an automorphic form, and \(\hbar = \frac{1}{k}\).

We also notice that the automorphic forms (3) and (4) are sums of coherent states associated to holomorphic discrete series representations of \(G\).

Let us describe all this in a bit more details. Let \(D = G/K\) be a bounded symmetric domain, it is a Hermitian symmetric space of non-compact type (so the Riemannian metric on \(D\) is given by the real part of the hermitian form, and the symplectic form, which is a Kähler form in this case, is given by the imaginary part of the hermitian form, all these forms are \(G\)-invariant, of course). The irreducible Hermitian spaces of non-compact type are

I) \(SU(p,q)/S(U(p) \times U(q))\),

II) \(Sp(p,\mathbb{R})/U(p)\),

III) \(SO^*(2p)/U(p)\),

IV) \(SO_0(p,2)/SO(p) \times SO(2)\)

(and let us omit the case of an exceptional Lie algebra). So we have a metric

\[ ds^2 = g_{ij} dz^i d\bar{z}^j, \quad (5) \]

the corresponding Kähler form is \(\omega = g_{ij} dz^i \wedge d\bar{z}^j = i\partial \partial^* \ln K(z, \bar{z})\), where \(K(z, \bar{w})\) is the Bergman kernel of the domain \(D\). Recall that \(K(z, w) = K(w, z)\) and

\[ K(\gamma z, \gamma w) = [\det J(\gamma, z)]^{-1}[\det J(\gamma, w)]^{-1} K(z, w). \]

The Poisson bracket is

\[ \{f, g\} = ig^{ij} \left( \frac{\partial f}{\partial z^j} \frac{\partial g}{\partial \bar{z}^i} - \frac{\partial g}{\partial z^j} \frac{\partial f}{\partial \bar{z}^i} \right). \]

The quantizing line bundle \(L \to M = \Gamma \backslash D\) can be defined as a line bundle such that the curvature of its natural connection is the Kähler form.
\( \omega \) on \( M \). Denoting the canonical line bundle by \( L \) we see that the potential 1-form corresponding to the natural connection on \( L \) is \( \Theta = i\partial\ln(s, s) = -i\partial\ln K(z, z) \), hence the curvature \( d\Theta = -i\partial\partial\ln K(z, z) = \omega \) and this is indeed a quantizing line bundle for \( M \).

A holomorphic function \( f : D \to \mathbb{C} \) is called an **automorphic form of weight** \( k \) if

\[
(f(z) [\det J(\gamma, z)]^k = f(z)
\]

for any \( z \in D, \gamma \in \Gamma \); here \( J(\gamma, z) \) is the Jacobi matrix of transformation \( \gamma \) at point \( z \). In the context of \ref{1.1} the automorphy forms of weight \( k \) form the complex inner product space \( H^0(M, L^{\otimes k}) \) of holomorphic sections of \( L^{\otimes k} \).

Now we consider a family of maps \( p_k \), here \( k \) is a positive integer, such that \( p_k(f) = T_f^{(k)} \), where \( f \) belongs to the Poisson algebra of smooth real-valued functions on \( M \) and \( T_f^{(k)} \) is the Toeplitz operator on \( H^0(M, L^{\otimes k}) \) obtained from multiplication operator \( M_f^{(k)}(g) = fg \) on \( L^2(M, L^{\otimes k}) \) by the orthogonal compression to the closed subspace \( H^0(M, L^{\otimes k}) \), i.e. \( T_f^{(k)} = \Pi^{(k)} \circ M_f^{(k)} \circ \Pi^{(k)} \), where \( \Pi^{(k)} \) is the orthogonal projection from \( L^2(M, L^{\otimes k}) \) to \( H^0(M, L^{\otimes k}) \).

In the Berezin scheme of quantization for each \( \hbar = \frac{1}{k} \) we consider the space \( \mathcal{F}_k \) of functions holomorphic in \( D \) and satisfying \[ ] \) with the scalar product defined by

\[
(f, g) = \text{const}(\hbar) \int_M f(z)\overline{g(z)}[K(z, z)]^k \, d\mu(z),
\]

where \( d\mu(z) = \omega^n \) is the \( G \)-invariant volume form on \( D \) corresponding to the metric \[ ] \). It is clear that \( \mathcal{F}_k \) is naturally identified with \( H^0(M, L^{\otimes k}) \). For the sake of completeness let us also explain briefly how the operator \( \hat{A} \) corresponding a classical observable \( A = A(z) \), is defined. First, we consider an analytic continuation \( \hat{A}(z, w) \) of the function \( A(z) \) to \( D \times D \). The covariant symbol \( A(z, w) \) of \( \hat{A} \) is defined as the diagonal value of the function

\[
A(z, w) = \frac{\int_M \hat{A}(z, w)[K(z, u)]^k d\mu(u)}{\int_M [K(z, u)]^k d\mu(u)},
\]

and

\[
(\hat{A}f)(z) = \text{const}(\hbar) \int_M A(z, w)f(w)[K(z, w)]^k \, d\mu(w).
\]

So we end up with the algebra \( A_\hbar \) of covariant symbols of bounded operators acting in \( \mathcal{F}_k \). The \( * \)-product in \( A_\hbar \) is given by

\[
A_1 * A_2 (z, z) = \text{const}(\hbar) \int_M A_1(z, w)A_2(w, z)\frac{K(z, w)K(w, z)}{K(z, z)K(w, w)} \, d\mu(w).
\]

In conclusion let us discuss the Poincaré series \[ ] \) and \[ ] \). Consider the unitary representation of \( G \) in \( \mathcal{F}_k \) given by the operators

\[
[k^k(g)(f)](z) = [\det J(g^{-1}, z)]^k f(g^{-1}z).
\]
It can be regarded as a subrepresentation of the left regular representation of \( G \) in \( L^2(\Gamma \setminus G) \). Fix \( f \in \mathcal{F}_{\hbar} \), then the set \( \{ \pi^k(g)(f) | g \in G \} \) is called a system of coherent states (strictly speaking, we should regard two coherent states \( \pi^k(g_1)(f) \) and \( \pi^k(g_2)(f) \) as equivalent if \( \pi^k(g_1)(f) = e^{i\alpha} \pi^k(g_2)(f) \)). Now it is clear that (\ref{eq:coherent_states}) and (\ref{eq:coherent_states_2}) are just sums of coherent states corresponding to \( f = q \) and the representation described above.

1.3 Comments on the subject of the present paper

In \cite{9} and in the present paper we consider holomorphic automorphic forms on \( D = \mathbb{H}_{\mathbb{C}}^n = SU(n, 1)/U(n) \). In \cite{9} we construct a set of relative Poincaré series generating the graded algebra of \( \mathbb{C} \)-valued cusp forms on a finite volume quotient of \( D \). In the present paper we regard holomorphic \( \mathbb{C} \)-valued automorphic forms on \( \mathbb{H}_{\mathbb{C}}^n \), as holomorphic sections of the line bundle \( L^{\otimes k} \to \Gamma \setminus \mathbb{H}_{\mathbb{C}}^n \), where \( L \) is a quantizing line bundle on \( \Gamma \setminus \mathbb{H}_{\mathbb{C}}^n \), \( k \) is an integer, and \( \Gamma \) is a discrete cocompact subgroup of \( SU(n, 1) \). We construct relative Poincaré series associated to loxodromic elements of \( \Gamma \) and we address an interesting problem which is not resolved for Poincaré series in general: is it true that these series are not identically zero? We restrict ourselves to the case of complex dimension 2 and answer “yes” to this question going through the following steps: 1) to each hyperbolic element of \( \Gamma \) we associate a certain Legendrian submanifold of the unit circle bundle in \( L^* \) such that the corresponding Lagrangian submanifold of \( \Gamma \setminus SU(2, 1)/U(2) \) satisfies a Bohr-Sommerfeld condition, 2) following the method of \cite{6} we conclude that the relative Poincaré series associated to hyperbolic elements are not zero for large values of \( k \) (i.e. in semi-classical limit \( \hbar = \frac{1}{k} \to 0 \)).

2 Preliminaries

2.1 Complex hyperbolic space

Consider the complex hyperbolic space

\[
\mathbb{H}_{\mathbb{C}}^n = SU(n, 1)/S(U(n) \times U(1)) = \mathbb{P}(\{ z \in \mathbb{C}^{n+1} \mid \langle z, z \rangle < 0 \}) \approx B^n,
\]

here \( B^n \) is the open unit ball in \( \mathbb{C}^n \), \( \langle \cdot, \cdot \rangle \) is the Hermitian product on \( \mathbb{C}^{n+1} \) given by \( \langle z, w \rangle = z_1\bar{w}_1 + ... + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1} \) for \( z = \begin{pmatrix} z_1 \\ ... \\ z_{n+1} \end{pmatrix} \in \mathbb{C}^{n+1} \), \( w = \begin{pmatrix} w_1 \\ ... \\ w_{n+1} \end{pmatrix} \in \mathbb{C}^{n+1} \).

A non-zero vector \( z \in \mathbb{C}^{n+1} \) is called negative (null, positive) if the value of \( \langle z, z \rangle \) is negative (null, positive).

For \( z, w \in B^n \) the corresponding vectors in \( \mathbb{C}^{n+1} \) are \( z = \begin{pmatrix} z_1 \\ ... \\ z_n \\ 1 \end{pmatrix} \) and \( w = \begin{pmatrix} w_1 \\ ... \\ w_n \\ 1 \end{pmatrix} \).
w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \\ 1 \end{pmatrix}, \text{ and } (z, w) = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n - 1.

The group of biholomorphic automorphisms of \( \mathbb{H}^n_\mathbb{C} \) is \( PU(n, 1) = SU(n, 1)/\text{center} \). The group \( SU(n, 1) \) acts on \( B^n \) by fractional-linear transformations: for \( \gamma \) we have:

\[
\gamma = \begin{pmatrix} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & b_n \\ c_1 & \ldots & c_n & d \end{pmatrix}
\]

we have:

\[
\gamma z = \gamma(z_1, \ldots, z_n) = (a_{11}z_1 + \ldots + a_{1n}z_n + b_1, \ldots, a_{n1}z_1 + \ldots + a_{nn}z_n + b_n, c_1 z_1 + \ldots + c_n z_n + d, z_1 \ldots z_n + d).
\]

Notice that \( \det J(\gamma, z) = (c_1 z_1 + \ldots + c_n z_n + d)^{-n+1} \), here \( J(\gamma, z) \) denotes the Jacobi matrix of transformation \( \gamma \) at point \( z \in B^n \).

An automorphism is called \textit{loxodromic} if it has no fixed points in \( B^n \) and fixes two points in \( \partial B^n \). Notice that the fixed points of the automorphisms correspond to the eigenvectors of the corresponding matrices in \( U(n, 1) \). A loxodromic automorphism is called \textit{hyperbolic} if it has a lift to \( U(n, 1) \) all of whose eigenvalues are real.

A loxodromic element \( \gamma_0 \in SU(n, 1) \) has \( n-1 \) positive eigenvectors and two null eigenvectors.

Let \( v_1, \ldots, v_{n-1} \) be the positive eigenvectors of \( \gamma_0 \), \( \tau_1, \ldots, \tau_{n-1} \) - the corresponding eigenvalues. Then \( |\tau_j| = 1 \), \( 1 \leq j \leq n-1 \).

Let \( X, Y \) be the null eigenvectors of \( \gamma_0 \). Then the corresponding eigenvalues are \( \lambda \) and \( \lambda^{-1} \) for some \( \lambda \in \mathbb{C} \), \( |\lambda| > 1 \).

A loxodromic transformation can always be represented by a matrix in \( U(n, 1) \) with eigenvalues \( \tau_1, \ldots, \tau_{n-1}, \lambda, \lambda^{-1} \) where \( \lambda \in \mathbb{R} \), \( |\lambda| > 1 \).

The geodesic connecting \( X \) and \( Y \) (it is an arc of a circle orthogonal to \( \partial B^n \) or a diameter) is \( \gamma_0 \)-invariant and is called the \textit{axis} of \( \gamma_0 \). The complex line containing \( X \) and \( Y \) (complex geodesic) is \( \gamma_0 \)-invariant too.

### 2.2 Automorphic forms and geometry of the quotient

Let \( \Gamma \) be a discrete cocompact subgroup of \( SU(n, 1) \) such that the quotient \( X := \Gamma \backslash B^n \) is smooth.

A holomorphic function \( f : B^n \rightarrow \mathbb{C} \) satisfying the automorphy law

\[
f(\gamma z)(\det J(\gamma, z))^k = f(z)
\]

for any \( \gamma \in \Gamma \) is a cusp form of weight \( (n+1)k \) for \( \Gamma \). The corresponding automorphic form on \( SU(n, 1) \) is given by \( F(g) = f(g(0))\zeta^k \), where \( \zeta = \det J(g, 0) \), and the automorphy law on the group is

\[
F(\gamma g) = F(g)
\]

for any \( \gamma \in \Gamma \). Notice that \( \gamma : \zeta \rightarrow \zeta \det J(\gamma, z) \) for any \( \gamma \in SU(n, 1) \).
We shall denote the space of cusp forms of weight \((n+1)k\) for \(\Gamma\) on \(B^n\) by \(S_{(n+1)k}(\Gamma)\) and the corresponding space of cusp forms on \(SU(n,1)\) by \(\tilde{S}_{(n+1)k}(\Gamma)\). The inner product on \(S_{(n+1)k}(\Gamma)\) and \(\tilde{S}_{(n+1)k}(\Gamma)\) is given by

\[
(f,g) = (f(z)\zeta^k, g(z)\zeta^k) = i^n \int_{\Gamma\backslash B^n} f\bar{g}(-\langle z,z \rangle)^{(n+1)k} \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(-\langle z,z \rangle)^{n+1}}.
\]

Given a subgroup \(\Gamma_0\) of \(\Gamma\) and a holomorphic function \(q(z) \in L^2(\Gamma\backslash B^n)\) satisfying \((7)\) for all \(\gamma \in \Gamma_0\), the relative Poincaré series for \(\Gamma_0\) is defined as

\[
\Theta(z) = \sum_{\gamma \in \Gamma_0\backslash \Gamma} q(\gamma z)(\det J(\gamma,z))^k,
\]

this series is converges absolutely and uniformly on compact sets and belongs to the space \(S_{(n+1)k}(\Gamma)\).

The Bergman kernel for the domain \(B^n\) is

\[
K(z,w) = \frac{1}{(-\langle z,w \rangle)^{n+1}}
\]

and the Kähler form on \(X\) is

\[
\Phi_\kappa = 2\kappa i \langle z,z \rangle^2 \frac{\sum_{j=1}^n dz_j \wedge d\bar{z}_j - \langle dz,z \rangle \wedge \langle z,dz \rangle}{(-\langle z,z \rangle)^n}
\]

where \(\kappa\) is a positive constant, it is an \(SU(n,1)\)-invariant Kähler form on \(B^n\).

**Remark 2.1** We will set \(\kappa = \frac{n+1}{2}\). Then the holomorphic sectional curvature on \(B^n\) is \(-\frac{2}{n+1}\) and the sectional curvature is pinched between \(-\frac{4}{n+1}\) and \(-\frac{1}{n+1}\) \(([10] II.2.2, III.1.5)\).

Consider the canonical line bundle \(L := \bigwedge^n T^*X\) and the dual bundle \(L^* = \bigwedge^n TX\). The potential 1-form \(\theta\) on \(L^*\) is characterized by

\[
\nabla s = -i\theta s,
\]

where \(\nabla\) is a connection on \(L^*\) and \(s\) is the unit section. The potential 1-form corresponding to the natural connection on \(L^*\) is

\[
\theta = i\partial \ln\langle s,s \rangle = i\partial \ln((-\langle z,z \rangle)^{n+1}).
\]

Here \(d = \partial + \bar{\partial}\). The curvature on \(L^*\) is \(d\theta = -\Phi_{\kappa+1} \frac{\kappa}{n+1} \), hence \(L\) is the natural quantizing line bundle for \(X\).

Let

\[
E_k = H^0(X,L^\otimes k)
\]

be the complex inner-product space of holomorphic sections of the \(k\)-th tensor power of \(L\). Consider the unit circle bundle \(P \subset L^*\), a point of \(P\) can be described as \((z,\zeta)\), where \(z \in B^n\) and \(\zeta\) is the coordinate on the fiber, \(|\zeta| = (-\langle z,z \rangle)^{n+1}\). We have:

\[
E_k = \tilde{S}_{(n+1)k}(\Gamma) = \{ f(z)\zeta^k \mid (z,\zeta) \in P, f \text{ is holomorphic on } B^n, f(\gamma z)(\det J(\gamma,z))^k = f(z) \text{ for any } \gamma \in \Gamma\}.
\]

Denote also \(\tilde{L} = \bigwedge^n T^*B^n\), \(\tilde{P}\) - the unit circle bundle in \(\tilde{L}^*\).
The connection form $\alpha : TP \to \mathbb{R}$ on $P$ is
\[ \alpha = \theta + \frac{d\zeta}{\zeta}. \]
It serves as a contact form on $\tilde{P}$ and $P$.

A Lagrangian submanifold $\Lambda_0 \subset X$ satisfies a Bohr-Sommerfeld condition if
\[ \frac{k}{2\pi} \int_C \theta \in \mathbb{Z} \]
for any closed curve $C \subset \Lambda_0$. The constant $\frac{1}{k}$ plays role of the Planck constant.

The unit disk bundle $W$ in $L^*$ is a compact, strictly pseudoconvex domain with smooth boundary. Let us consider the Hardy space $E \subset L^2(P)$ and the Szegő projector $\Pi : L^2(P) \to E$ given by the orthogonal projection of $L^2$ onto $E$.

We shall denote the corresponding orthogonal projection by $\tilde{\Pi} : L^2(\tilde{P}) \to \tilde{E}$.

### 3 Construction of relative Poincaré series associated to certain loxodromic elements of a discrete cocompact subgroup of $SU(n,1)$

Consider a loxodromic automorphism of $B^n$, represented it by a matrix $\gamma_0 \in U(n,1)$ with eigenvalues $\tau_1, ..., \tau_{n-1}, \lambda, \lambda^{-1}, |\tau_j| = 1, j = 1, ..., n-1$, $\lambda \in \mathbb{R}$, $|\lambda| > 1$, denote the corresponding eigenvectors by $v_1, ..., v_{n-1}, X, Y$ ($v_1, ..., v_{n-1}$ are positive, $X, Y$ are null). Notice that if each $\tau_j$ is a root of 1 then some power of $\gamma_0$ is a hyperbolic element.

**Assumption 3.1** Assume that 1 is among the eigenvalues of $\gamma_0$.

**Remark 3.2** If $g \in U(n,1)$ is hyperbolic then $g^2$ is a hyperbolic element of $SU(n,1)$ which satisfies Assumption 3.1 and has the same eigenvectors as $g$. 

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Generalizing the construction suggested in [9], for any collection, w.l.o.g. $v_1, \ldots, v_m, m \leq n - 1$, of positive eigenvectors corresponding to eigenvalue 1 we construct a relative Poincaré series

$$
\Theta_{\gamma_0, l, k} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q_l(\gamma z)(\det J(\gamma z))^{2k} \in S_{2(n+1)k},
$$

where $\Gamma_0 = \left< \gamma_0 \right>$, $q_l(z) = \frac{\langle z, v_1 \rangle^{l_1} \ldots \langle z, v_m \rangle^{l_m}}{((z, X)(z, Y))^{n+1} l_1 + \ldots + l_m}$. $l_1, \ldots, l_m$ are positive integers such that $l_1 + \ldots + l_m$ is even, $l = (l_1, \ldots, l_m)$. The series converges absolutely in $B^n$ and uniformly on the compact sets by the Theorem [9] for $k \geq 1$ (see the Appendix).

In dimension 2 the loxodromic elements of $\Gamma$ satisfying Assumption 3.1 are exactly the hyperbolic elements of $\Gamma$. The relative Poincaré series associated to a hyperbolic element $\gamma_0 \in \Gamma$ is

$$
\Theta_{\gamma_0, l, k} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q_l(\gamma z)(\det J(\gamma z))^{2k} \in S_{6k},
$$

where $\Gamma_0 = \left< \gamma_0 \right>$, $q_l(z) = \frac{(z, v)^{2l}}{((z, X)(z, Y))^{3k+l}}$, and $l$ is a positive integer.

**Remark 3.3** Let $\gamma_1$ and $\gamma_2$ be hyperbolic elements of $\Gamma$. If $\gamma_1 = \gamma_2^N$ for a positive integer $N$, then $\Theta_{\gamma_1, l, k} = \Theta_{\gamma_2, l, k}$. If $\gamma_1$ and $\gamma_2$ are conjugate in $\Gamma$ then $\Theta_{\gamma_1, l, k} = \Theta_{\gamma_2, l, k}$.

### 4 Bohr-Sommerfeld tori

Consider a hyperbolic element $\gamma_0 \in \Gamma$, denote its null eigenvectors by $X$, $Y$, denote its positive eigenvector by $v$, then the corresponding eigenvalues are $\lambda, \lambda^{-1}, 1$, for $\lambda \in \mathbb{R}, |\lambda| > 1$.

We choose $v$ so that

$$
A = \begin{bmatrix} v & X \\ \langle X, Y \rangle & \frac{X}{2} \end{bmatrix} + \begin{bmatrix} Y \\ \frac{X}{2} \end{bmatrix} = \begin{bmatrix} Y \\ \frac{X}{2} \end{bmatrix} \in SU(2,1).
$$

The transformation $A^{-1}$ maps the complex line containing $X$ and $Y$ to the complex line $\{z_1 = 0\}$ and maps the geodesic connecting $X$ and $Y$ to the geodesic $\tilde{C}$ connecting $(0, -1)$ and $(0, 1)$. The following loxodromic element of $SU(2,1)$ preserves $\tilde{C}$ and the line $\{z_1 = 0\}$:

$$
\gamma := A^{-1} \gamma_0 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix},
$$

where $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}$.
\[ a = \frac{\lambda^2 + 1}{2\lambda}, \quad b = \frac{\lambda^2 - 1}{2\lambda}, \quad a^2 - b^2 = 1. \]

Denote \( w = (w_1, w_2) = A^{-1}z, \) \( w_1 = A^{-1}z_1, \) \( w_2 = A^{-1}z_2, \)
and apply change of variables
\[ w_2 = \frac{re^{i\phi} - i}{re^{i\phi} + i}, \quad w_1 = \sqrt{1 - w_2w_2}Re^{i\phi}, \]
\[ 0 < \phi < \pi, \quad 0 < r < +\infty, \quad 0 < R < 1, \quad 0 \leq \Theta < 2\pi. \]

**Proposition 4.1** Any 2-cylinder \( C_{\phi,R} = \{ \phi = \text{const}, \ R = \text{const} \} \) is \( \gamma \)-invariant.

**Remark 4.2** The coordinates \((r, \Theta)\) on \( C_{\phi,R} \) are the “radial” and the “angular” coordinates respectively.

**Proof.**
\[ w_2 \rightarrow \frac{aw_2 + b}{bw_2 + a} = \frac{aw_2 + b}{bw_2 + a} = \frac{r(a + b)e^{i\phi} - i(a - b)}{r(a + b)e^{i\phi} + i(a - b)} = \frac{r^2a + b}{r^2a - b}e^{i\phi} - \frac{i}{r^2a - b}e^{i\phi}, \]
so
\[ r \rightarrow \frac{a + b}{a - b}, \quad \phi \rightarrow \phi, \]
also
\[ \frac{|w_1|}{\sqrt{1 - w_2w_2}} \rightarrow \frac{|w_1|}{\sqrt{1 - \frac{(aw_2 + b)^2}{bw_2 + a}}} = \frac{|w_1|}{\sqrt{1 - w_2w_2}}, \]
so \( R \rightarrow R. \)

For a positive integer \( l \) consider the following submanifold of \( \tilde{P} \):
\[ \tilde{T}(l) := \{(w, -\langle w, w \rangle)^{\frac{2}{k}}e^{i\psi} \mid w \in \{ \phi = \frac{\pi}{2}, \ R = \sqrt{\frac{l}{3k + l}} \}, \ \psi = -\frac{l}{k}\Theta \}. \]

Denote \( T(l) := < \gamma > \tilde{T}(l), \ \Lambda(l) := AT(l) \) and \( \Gamma_0 := < \gamma_0 >. \)

**Proposition 4.3** \( \Lambda(l) := AT(l) = \Gamma_0 \setminus \Lambda(l) \) is a compact Legendrian submanifold of \( P. \)

**Proof.** \( T(l) \) and \( \Lambda(l) \) are compact submanifolds of \( P. \)

Let us prove that \( \Lambda(l) \) is Legendrian. Submanifolds \( T(l) \) and \( \Lambda(l) \) have dimension 2. The restriction of \( \alpha \) onto \( \tilde{T}(l) \) is
\[ -3i\frac{dw}{\langle w, w \rangle} + \frac{d\zeta}{\zeta} = -3i\frac{dw}{\langle w, w \rangle} - d\psi + i\frac{d(-\langle w, w \rangle)^{\frac{2}{3}}}{(-\langle w, w \rangle)^{\frac{2}{3}}} = \]
\[ -3i\frac{dw}{\langle w, w \rangle} - d\psi - \frac{3}{2}\frac{dw}{\langle w, w \rangle} + \frac{3}{2}\frac{d\langle w, dw \rangle}{(-\langle w, w \rangle)^{\frac{2}{3}}}(-\langle w, w \rangle)^{\frac{2}{3}} = \]
\[ -3i\frac{dw}{\langle w, w \rangle} - d\psi + \frac{3}{2}\frac{dw}{\langle w, w \rangle} + \frac{3}{2}\frac{d\langle w, dw \rangle}{\langle w, w \rangle} = \]
Proposition 4.4

Let $\Lambda$ be a closed curve in $\mathbb{R}^3$. Then $\Lambda$ is a Legendrian submanifold of $\mathbb{R}^3$.

Proof. Let $\Lambda = \{ (z, \zeta) \in \mathbb{R}^3 \}$ be a closed curve in $\mathbb{R}^3$. Then $\Lambda$ is a Legendrian submanifold of $\mathbb{R}^3$ if and only if

$$\int_C \partial \ln(-\langle z, z \rangle) = -3i \int_C \partial \ln(-\langle Aw, Aw \rangle) = 0,$$

where $A$ is a $2 \times 2$ complex matrix.

We showed that $\Theta(l)$ is a Legendrian submanifold of $P$. To prove that $AT(l)$ is Legendrian too it is enough to show that $\alpha$ is $SU(2,1)$-invariant.

Let $M \in SU(2,1)$,

$$M : (z, \zeta) \rightarrow (Mz, \zeta \det J(M, z)) = (Mz, \zeta^3),$$

where $c = c(z) = (m_{31}z_1 + m_{32}z_2 + 1)^{-1}$.

$$\alpha = i \frac{dC}{\zeta} - 3i \frac{dz}{z} = i \frac{dC}{\zeta} - 3i \partial \ln(-\langle z, z \rangle).$$

We have:

$$i \frac{d(c^3 \zeta)}{c^3 \zeta} - 3i \partial \ln(-\langle Mz, Mz \rangle) = i \frac{c^3 d\zeta + 3c^2 \zeta dc}{c^3 \zeta} - 3i \partial \ln(-\langle z, z \rangle c\bar{c}) =$$

$$i \frac{dC}{\zeta} + 3i \frac{dc}{c} - 3i \partial \ln(-\langle z, z \rangle) - 3i \partial \ln(c\bar{c}) = i \frac{dC}{\zeta} + 3i \frac{dc}{c} - 3i \partial \ln(-\langle z, z \rangle) - 3i \frac{dc}{c} =$$

$$i \frac{dC}{\zeta} - 3i \partial \ln(-\langle z, z \rangle).$$

The natural projection $\Lambda_0(l)$ of $\Lambda(l)$ onto $X$ is a compact Lagrangian submanifold of $X$.

Proposition 4.4 $\Lambda_0(l)$ satisfies a Bohr-Sommerfeld condition.

Proof. Let $\tilde{T}_0(l)$ be the natural projection of $\tilde{T}(l)$ onto $B^2$, and let $T_0(l)$ be the natural projection of $T(l)$ onto $X$, $AT_0(l) = \Lambda_0(l)$. If $C \subset \Lambda_0(l)$ is a closed curve then $A^{-1}C \subset T_0(l)$ is also closed. Let $z \in \Lambda_0(l), w \in T_0(l), c = c(w) = (a_{31}w_1 + a_{32}w_2 + a_{33})^{-1}$, we have:

$$\int_C \theta = -3i \int_C \partial \ln(-\langle z, z \rangle) = -3i \int_{A^{-1}C} \partial \ln(-\langle Aw, Aw \rangle) =$$

$$-3i \int_{A^{-1}C} \partial \ln(-\langle w, w \rangle c\bar{c}) =$$
Proposition 4.5

The orthogonal projection of the delta function at \((w, \eta) \in \tilde{P}\) into \(\tilde{E}_k\) is the function

\[
\Psi_{(w, \eta)}(z, \zeta) := \tilde{\Pi}_k(\delta_{(w, \eta)}) = \frac{(3k - 1)(3k - 2)}{4\pi^2} \frac{\zeta^k \eta^k}{(z, w)^{3k}}.
\]

Remark 4.6

The orthogonal projection of the delta function at \((w, \eta) \in \tilde{P}\) into \(\tilde{E}_k\) is the coherent state in \(\tilde{E}_k\) associated to the point \((w, \eta) \in \tilde{P}\), by definition \(g \Psi_{(w, \eta)} = \Psi_{g(w, \eta)}\) for \(g \in SU(2, 1)\).

Proof. The fact that \(\Psi_{(w, \eta)} = \tilde{\Pi}_k(\delta_{(w, \eta)})\) is equivalent to the reproducing property:

\[
F(w, \eta) = \int_{\tilde{P}} \Psi_{(w, \eta)}(z, \zeta) F(z, \zeta) dV \wedge d\psi
\]

for all \(F \in \tilde{E}_k\). Given any orthonormal basis \(\{F_{l, k}\}\) for \(\tilde{E}_k\), we can write the reproducing kernel as the series \(\Psi_{(w, \eta)}(z, \zeta) = \sum_l \overline{F_{l, k}(w, \eta)} F_{l, k}(z, \zeta)\) which converges absolutely and uniformly on compact sets.
Using the basis

\[ F_{l,m,k}(z, \zeta) = \frac{1}{2\pi} \sqrt{\frac{(3k+l+m-1)!}{l!m!(3k-3)!}} z_1^l z_2^m \zeta^k, \]

which is orthonormal with respect to the inner product

\[ (f(z)\zeta^k, g(z)\zeta^k) = i^2 \int_{B^2} f \bar{g}(-\langle z, w \rangle)^{3k-3} dz_1 \wedge dz_2 \wedge dz_2, \]

we obtain:

\[ \Psi_{(w,\eta)}(z, \zeta) = \sum_{l,m} F_{l,m,k}(w, \eta) F_{l,m,k}(z, \zeta) = \]

\[ \sum_{l,m} \frac{1}{(2\pi)^2} \frac{(3k+l+m-1)!}{l!m!(3k-3)!} \bar{w}_1^l w_2^m \bar{z}_1^l z_2^m \zeta^k \eta^k = \]

\[ \frac{\zeta^k \eta^k}{4\pi^2(3k-3)!} \sum_{l,m} \frac{(3k+l+m-1)!}{l!m!} (\bar{w}_1 z_1)^l (\bar{w}_2 z_2)^m. \]

To calculate

\[ \sum_{l,m} \frac{(3k+l+m-1)!}{l!m!} \frac{(3k+l+m-1)!}{l!} x^l y^m = \sum_m \frac{y^m}{m!} \sum_l \frac{(3k+l+m-1)!}{l!} x^l \]

let us find first \( \sum_l \frac{(N+l)!}{(t+l)!} t^l \). Integrating once we get \( \sum_{l=0}^{\infty} \frac{(N+l)!}{(l+1)!} t^l \); integrating twice we get \( \sum_{l=0}^{\infty} \frac{(N+l)!}{(l+2)!} t^l \); integrating \( N \) times we get:

\[ \sum_{l=0}^{\infty} \frac{(N+l)!}{(l+N)!} t^l = \sum_{l=0}^{\infty} t^{l+N} = \frac{t^N}{1-t}, \]

differentiating this expression \( N \) times we get:

\[ \sum_l \frac{(N+l)!}{l!} t^l = (\frac{t^N}{1-t})^{(N)} = (\frac{t^N-1}{1-t})^{(N)} = \]

\[ \left( \frac{t-1}{t} \right)^{(N)} = \frac{N!}{(1-t)^{N+1}}, \]

hence

\[ \sum_m \frac{y^m}{m!} \sum_l \frac{(3k+l+m-1)!}{l!} x^l = \sum_m \frac{y^m}{m!} (3k+m-1)! \]

\[ \frac{1}{(1-x)^{3k}} \sum_m \frac{(3k+m-1)!}{m!} \left( \frac{y}{1-x} \right)^m = \frac{1}{(1-x)^{3k}} \frac{(3k-1)!}{(1-x-\frac{y}{1-x})^{3k}} = \frac{(3k-1)!}{(1-x-y)^{3k}}, \]

so

\[ \Psi_{(w,\eta)}(z, \zeta) = \frac{\zeta^k \eta^k}{4\pi^2(3k-3)!} \frac{(3k-1)!}{(1-\bar{w}_1 z_1 - \bar{w}_2 z_2)^{3k}} = \frac{(3k-1)(3k-2)}{4\pi^2} \zeta^k \eta^k \frac{1}{(\langle z, w \rangle)^{3k}}. \]
For weight $6k$ we have:

$$\Psi_{(u,\eta)}(z,\zeta) = \Pi_{2k}(\delta_{(u,\eta)}) = \frac{(6k-1)(6k-2)}{4\pi^2} \frac{e^{2k}\eta^{2k}}{\langle z, u \rangle^{6k}}.$$  

We omit the weight in the notation $\Psi_{(u,\eta)}(z,\zeta)$ but further exposition will be for weight $6k$ so this will not lead to any confusion.

To get the orthogonal projection of the delta function at $[(u,\eta)] \in P = \Gamma \setminus \tilde{P}$ (by $[(u,\eta)]$ we denote the equivalence class of $(u,\eta)$ into $E_{2k}$ we average over the action of $\Gamma$:

$$\Pi_{2k}(\delta_{(u,\eta)}) = \sum_{g \in \bar{\Gamma}} g \Psi_{(u,\eta)}. \quad (8)$$

The function $\Psi_{(u,\eta)}$ belongs to $\tilde{E}_{2k}$, hence the series $(8)$ converges absolutely and uniformly on compact sets by Theorem 9.1.

Following the method of $(6)$, to the submanifold $\Lambda(l) \subset P$ with a half-form $\nu$ we associate a function

$$\Phi := \int_{\Lambda(l)} \Pi_{2k}(\delta_{(u,\eta)}) \nu = \sum_{g \in \bar{\Gamma}} \int_{\Lambda(l)} g \Psi_{(u,\eta)} \nu =$$

$$\sum_{g \in \bar{\Gamma}/\Gamma_0} \sum_{m=-\infty}^{+\infty} \int_{\Lambda(l)} g_{\gamma_0}^m \Psi_{(u,\eta)}(z,\zeta) \nu = \sum_{g \in \bar{\Gamma}/\Gamma_0} \sum_{m=-\infty}^{+\infty} \int_{\Lambda(l)} g \Psi_{(u,\eta)}(z,\zeta) \nu =$$

$$\sum_{g \in \bar{\Gamma}/\Gamma_0} g \int_{\Lambda(l)} \Psi_{(u,\eta)}(z,\zeta) \nu = \sum_{g \in \bar{\Gamma}/\Gamma_0} g^{-1} \int_{\Lambda(l)} \Psi_{(u,\eta)}(z,\zeta) \nu.$$  

**Proposition 4.7**

$$\int_{\bar{\Lambda}(l)} \langle z, \nu \rangle^2 = \frac{\langle z, X \rangle^2}{\langle z, Y \rangle^2},$$

where the constant $C$ is given by

$$C = 2^{3k+l-2} \frac{i}{(2l)!} \frac{(6k+2l-1)!}{(6k-3)!} \frac{(3k)^{3k+l}}{(3k+1)^{3k+l}} \left( \frac{\langle Y, X \rangle}{\langle Y, X \rangle} \right)^{3k+l-1} \sum_{j=0}^{3k+l-1} \frac{(-1)^j (3k+l-1)!}{j! (6k+2l-1-j)!}.$$

and we take

$$\nu = \frac{d(A^{-1}u_2)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1 - (A^{-1}u_2)^2}.$$  

**Remark 4.8** The half-form $\nu$ on $\bar{\Lambda}(l)$ is $\gamma_0$-invariant and in properly chosen coordinates $(r, \Theta)$ on $\Lambda_0(l)$ it is expressed as $\nu = \frac{i}{2} d\Theta \wedge \frac{dr}{r}$.  

**Proof.** Let $u \in \bar{\Lambda}(l)$, $w = A^{-1}u \in \tilde{T}(l)$, then

$$\int_{\bar{\Lambda}(l)} \Psi_{(u,\eta)}(z,\zeta) \nu = \frac{(6k-1)(6k-2)}{4\pi^2} \int_{\bar{\Lambda}(l)} \frac{\zeta^{2k} \eta^{2k}}{\langle z, u \rangle^{6k}} \frac{d(A^{-1}u_1)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1 - (A^{-1}u_2)^2} =$$

$$\frac{(6k-1)(6k-2)}{4\pi^2} \zeta^k \int_{\bar{\Lambda}(l)} \frac{\eta^{2k}}{\langle z, u \rangle^{6k}} \frac{d(A^{-1}u_1)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1 - (A^{-1}u_2)^2} =$$

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\[(6k - 1)(6k - 2) \frac{2k}{4\pi^2} \int_{T(l)} \frac{(-\langle w, w \rangle)^2 e^{-i\theta} \det \tilde{J}(A, w)^{2k}}{(z, Aw)^{6k}} \frac{dw_1}{w_1} \wedge \frac{dw_2}{1 - w_2^2} =
\]
\[(6k - 1)(6k - 2) \frac{2k}{4\pi^2} \int_{T(l)} \frac{(-\langle w, w \rangle)^{3k} e^{-i2k\theta} \det \tilde{J}(A, w)^{2k}}{(A^{-1}z, w)^{3k} \det J(A^{-1}, z)^{2k}} \frac{dw_1}{w_1} \wedge \frac{dw_2}{1 - w_2^2},
\]

let \( A^{-1}z = \binom{v_1}{v_2} \), then we get:
\[(6k - 1)(6k - 2) \frac{2k}{4\pi^2} \zeta^{2k} (\det J(A^{-1}, z))^{2k} \int_{T(l)} \frac{(-\langle w, w \rangle)^{3k} e^{-i2k\theta} \det \bar{J}(w, w)^{2k}}{(v_1 w_1 + v_2 w_2 - 1)^{6k}} \frac{dw_1}{w_1} \wedge \frac{dw_2}{1 - w_2^2},
\]
on \( T(l) \)
\[
w_2 = r - 1, \quad w_1 = \sqrt{1 - w_2^2}e^{i\theta} = \frac{2\sqrt{r}}{r + 1} e^{i\theta},
\]
\[-\langle w, w \rangle = (1 - R^2)(1 - w_2^2) = (1 - R^2) \frac{4r}{(r + 1)^2},
\]
so we have:
\[
\frac{(6k - 1)(6k - 2)}{4\pi^2} \frac{1}{4(1 - R^2)^{3k}} \int_{T(l)} \frac{(-\langle w, w \rangle)^{3k} e^{-i2k\theta}}{(v_1 \frac{2\sqrt{r}}{r + 1} e^{i\theta} + v_2 \frac{r - 1}{r + 1} - 1)^{6k} \frac{i}{2}} d\theta \wedge \frac{dr}{r} =
\]
\[
\frac{(6k - 2)(6k - 2)}{4\pi^2} \zeta^{2k} (\det J(A^{-1}, z))^{2k} \int_{T(l)} \frac{(-\langle w, w \rangle)^{3k} e^{-i2k\theta} \det \bar{J}(w, w)^{2k}}{(v_1 \frac{2\sqrt{r}}{r + 1} e^{i\theta} + v_2 (r - 1) - r - 1)^{6k} \frac{i}{2}} d\theta \wedge \frac{dr}{r} =
\]

The integral
\[
\int_{|w| = 1} \frac{1}{(Aw + B)^{2k} w^{2k+1}} \left| \frac{B}{A} \right| > 1
\]
is equal to
\[
\frac{2\pi i}{(2l)!} \frac{d^{2l}}{dw^{2l}} \frac{1}{(Aw + B)^{2k} w^{2k+1}} \bigg|_{w=0} = \frac{2\pi i}{(2l)!} \frac{(6k + 2l - 1)!}{(6k - 1)!} \frac{A^{2l}}{B^{6k+2l}}.
\]

Let \( w = e^{-i\theta}, A = v_1 2\sqrt{r} R, B = v_2 (r - 1) - r - 1. \) Let us check that \( \left| \frac{\alpha}{\beta} \right| > 1. \)
\[
\frac{|v_2 (r - 1) - r - 1|}{v_1 2 \sqrt{r} R} = \frac{|v_2 w_2 - 1|}{v_1 R \sqrt{1 - w_2^2}} > \frac{|v_2 w_2 - 1|}{\sqrt{1 - v_2^2} R \sqrt{1 - w_2^2}} \geq 1
\]
because
\[
0 \leq |v_2 - w_2|^2 = (\bar{v}_2 - w_2)(v_2 - w_2) = v_2 \bar{v}_2 - \bar{v}_2 w_2 - w_2 v_2 + w_2^2 =
\]
\[-v_2 w_2 - w_2 v_2 + v_2 \bar{v}_2 w_2^2 + 1 + v_2 \bar{v}_2 + w_2^2 - v_2 \bar{v}_2 w_2^2 - 1 =
\]

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We get:

\[
\frac{(6k-1)(6k-2)}{4\pi^2} \zeta^{2k} (4(1-R^2))^{3k} \frac{1}{2(2l)!} \frac{2\pi i}{(6k+2l-1)!} \left( \frac{2\pi i}{(6k-1)!} \right)^2 (\det J(A^{-1}, z))^{2k} 
\]

\[
\int_0^\infty \frac{r^{3k-1+l}}{(v_2(r-1)-r-1)^{6k+2l}} dr = \pi i \frac{1}{4\pi^2} \zeta^{2k} (4(1-R^2))^{3k} \frac{1}{2(2l)!} \frac{v_2^{2l} R^{2l}}{(v_2-1)^{6k+2l}} (\det J(A^{-1}, z))^{2k} \int_0^\infty \frac{r^{3k-1+l}}{(r^2+1)^{6k+2l}} dr.
\]

The integral \( \int_0^\infty \frac{r^{3k-1+l}}{(r+\beta)^{6k+2l}} dr = -r e_{s,\beta} f(z) \) where \( f(z) = \frac{z^{3k-1}}{(z-a)^{6k+2l}} \).

(29.6). Let us calculate this residue:

\[
re_{s,\beta} f(z) = \frac{1}{(6k+2l-1)!} \frac{d^{6k+2l-1}}{dz^{6k+2l-1}} z^{3k-1} \ln z |_{z=a} = \]

\[
\frac{1}{(6k+2l-1)!} \sum_{j=0}^{6k+2l-1} C_{6k+2l-1}^j \left( \frac{z^{3k-1}}{(6k+2l-1)!} \right) (\ln z)^{(6k+2l-1-j)} |_{z=a} = \]

\[
\frac{1}{(6k+2l-1)!} \sum_{j=0}^{3k+l-1} C_{6k+2l-1}^j \left( \frac{z^{3k-1}}{(6k+2l-1)!} \right) \left( \frac{1}{2} \right) (6k+2l-2-j) |_{z=a} = C_1 \frac{1}{a^{3k+l}},
\]

where \( C_1 = \sum_{j=0}^{3k+l-1} (\frac{1}{2}) (\frac{3k+l-1)!}{(6k+2l-1-j)!} \). Hence

\[
\int_0^\infty \frac{r^{3k+l-1}}{(r^2+1)^{6k+2l}} dr = -C_1 \left( \frac{v_2-1}{v_2+1} \right)^{3k+l}.
\]

and we get:

\[
-\frac{\pi i}{(2l)!} \frac{1}{4\pi^2} \zeta^{2k} (4(1-R^2))^{3k} \frac{1}{2(2l)!} \frac{2\pi i}{(6k+2l-1)!} \frac{v_2^{2l} R^{2l}}{(v_2-1)^{6k+2l}} (\det J(A^{-1}, z))^{2k} \left( \frac{v_2-1}{v_2+1} \right)^{3k+l} = \]

\[
i \frac{1}{4\pi^2} \zeta^{2k} (4(1-R^2))^{3k} \frac{1}{2(2l)!} \frac{v_2^{2l} R^{2l}}{(v_2-1)^{6k+2l}} (\det J(A^{-1}, z))^{2k} \left( \frac{1}{(A^{-1}z)^2} \right)^{3k+l} = \]

\[
2^{6k+2l-2} \frac{i}{(2l)!} \frac{1}{(6k+2l-1)!} \frac{C_1 (3k+l)^{3k+l} (\frac{1}{2}) (Y, X)^{-3k-l} (z, v)^{2l}}{(v_2+1)^{3k+l} \zeta^{2k}}.
\]

\( \Phi(z, \zeta) = C \zeta^{2k} \sum_{g \in \Gamma_0 \backslash \Gamma} q(gz) (\det J(g, z))^{2k} \in E_k. \)
where
\[ q_l(z) = \frac{(z,v)^{2l}}{(\langle z,X \rangle \langle z,Y \rangle)^{3k+l}} \]
and the relative Poincaré series associated to \( \Lambda_0(l) \) is
\[ \Theta_{\gamma_0,l,k}(z) := C \sum_{g \in \Gamma_0 \setminus \Gamma} q_l(gz)(\det J(g,z))^{2k}. \]

From the results of [6] (Theorem 3.2, Corollary 3.3) it follows that \( \Theta_{\gamma_0,l,k} \) is non-vanishing for sufficiently large values of \( k \).

**APPENDIX.**

We shall prove the following theorem modifying the proof of convergence of Poincaré series contained in [4] and [3].

**Theorem 9** Let \( \varphi \) be a function on \( G = SU(n,1) \). Assume that
1) \( \varphi \) is \( Z \)-finite
2) \( \varphi \in L^1(\Gamma_0 \setminus G) \),
3) \( \varphi \) is \( K \)-finite on the right
Let \( p_\varphi(x) = \sum_{g \in \Gamma_0 \setminus \Gamma} \varphi(gx) \).
Then \( p_\varphi \) converges absolutely and uniformly on compact sets.

**Proof.**
By Lemma 9.2 [4] there exists \( \alpha \in C^\infty_c(G) \) satisfying \( \alpha(k^{-1}xk) = \alpha(x) \), \( k \in K \), \( x \in G \), such that \( \varphi = \varphi * \alpha \). such that \( U^{-1} = U \), the closure of \( U \) is compact, and \( U \supset \text{supp } \alpha \). We have:
\[ \varphi(\gamma x) = (\varphi * \alpha)(\gamma x) = \int_G \varphi(\gamma xy)\alpha(y^{-1})dy = \int_U \varphi(\gamma xy)\alpha(y^{-1})dy, \]

hence
\[ |\varphi(\gamma x)| \leq ||\alpha||_\infty \int_U |\varphi(\gamma xy)|dy = ||\alpha||_\infty \int_{xU} |\varphi(y)|dy \]

Here \( ||\alpha||_\infty = \sup_{y \in U} |\alpha(y)| \).

Fix a compact subset \( C \) of \( G \). We want to prove absolute and uniform convergence on \( C \). The closure of \( CU \) is compact. \( CU \) is covered by \( N \) copies of a "fundamental domain" of \( \Gamma \) in \( G \), \( N \) is a positive integer (because \( \Gamma \) is discrete). Denote these domains by \( F_1, \ldots, F_N \). By a "fundamental domain" of \( \Gamma \) in \( G \) I mean a connected set of representatives of \( \Gamma \setminus G \).

Let \( x \in C \). Then
\[ |\varphi(\gamma x)| \leq ||\alpha||_\infty \int_{xU} |\varphi(y)|dy \leq ||\alpha||_\infty \int_{CU} |\varphi(y)|dy \]
and we get
\[ \sum_{\gamma \in \Gamma \setminus \Gamma} ||\alpha||_\infty \int_{CU} |\varphi(y)|dy = ||\alpha||_\infty \sum_{\gamma \in \Gamma \setminus \Gamma} \int_{CU} |\varphi(y)|dy \leq \]

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\[ ||\alpha||_\infty \sum_{\gamma \in \Gamma_0 \setminus \Gamma} (\int_{F_1} |\varphi(\gamma y)|dy + ... + \int_{F_N} |\varphi(\gamma y)|dy) = \]
\[ ||\alpha||_\infty (\sum_{\gamma \in \Gamma_0 \setminus \Gamma} \int_{F_1} |\varphi(\gamma y)|dy + ... + \sum_{\gamma \in \Gamma_0 \setminus \Gamma} \int_{F_N} |\varphi(\gamma y)|dy) = \]
\[ N||\alpha||_\infty \int_{\Gamma_0 \setminus G} |\varphi(y)|dy < \infty. \]

So we proved that

\[ |\varphi(\gamma x)| \leq c_\gamma := ||\alpha||_\infty \int_{CU} |\varphi(\gamma y)|dy \]

and that the numerical series \( \sum_{\gamma \in \Gamma_0 \setminus \Gamma} c_\gamma \) converges, hence by Weierstrass theorem the series \( \sum_{\gamma \in \Gamma_0 \setminus \Gamma} \varphi(\gamma x) \) converges absolutely and uniformly on \( C \). Q.E.D.

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