Inclusion of generalized Bessel functions in the Janowski class

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Abstract. Sufficient conditions on \( A, B, p, b \) and \( c \) are determined that will ensure the generalized Bessel functions \( u_{p,b,c} \) satisfies the subordination \( u_{p,b,c}(z) \prec (1 + Az)/(1 + Bz) \). In particular this gives conditions for \((-4\kappa/c)(u_{p,b,c}(z) - 1) \), \( c \neq 0 \) to be close-to-convex. Also, conditions for which \( u_{p,b,c}(z) \) to be Janowski convex, and \( zu_{p,b,c}(z) \) to be Janowski starlike in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) are obtained.

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) defined in the open unit disk \( D = \{ z : |z| < 1 \} \) normalized by the conditions \( f(0) = 0 = f'(0) - 1 \). If \( f \) and \( g \) are analytic in \( D \), then \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \), if there is an analytic self-map \( w \) of \( D \) satisfying \( w(0) = 0 \) and \( f = g \circ w \). For \(-1 \leq B < A \leq 1 \), let \( P[A, B] \) be the class consisting of normalized analytic functions \( p(z) = 1 + c_1 z + \cdots \) in \( D \) satisfying \( p(z) \prec 1 + Az/(1 + Bz) \).

For instance, if \( 0 \leq \beta < 1 \), then \( P[1 - 2\beta, -1] \) is the class of functions \( p(z) = 1 + c_1 z + \cdots \) satisfying \( \text{Re} \, p(z) > \beta \) in \( D \).

The class \( S^*[A, B] \) of Janowski starlike functions consists of \( f \in \mathcal{A} \) satisfying

\[
\frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B].
\]

For \( 0 \leq \beta < 1 \), \( S^*[1 - 2\beta, -1] := S^*(\beta) \) is the usual class of starlike functions of order \( \beta \); \( S^*[1 - \beta, 0] := S^*_\beta = \{ f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta \} \), and \( S^*[-\beta, \beta] := S^*[\beta] = \{ f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1| \} \).

These classes have been studied, for example, in [1, 2]. A function \( f \in \mathcal{A} \) is said to be close-to-convex of order \( \beta \) if \( \text{Re} \, (zf'(z)/g(z)) > \beta \) for some \( g \in S^* := S^*(0) \).
This article studies the generalized Beesel function $u_p(z) = u_{p,b,c}(z)$ given by the power series

$$u_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!} z^k,$$

(1)

where $\kappa = p + \frac{(b + 1)}{2} \neq 0, -1, -2, -3 \cdots$. The function $u_p(z)$ is analytic in $\mathbb{D}$ and solution of the differential equation

$$4z^2 u''(z) + 4\kappa zu'(z) + cz u(z) = 0,$$

(2)

if $b, p, c \in \mathbb{C}$, such that $\kappa = p + \frac{(b + 1)}{2} \neq 0, -1, -2, -3 \cdots$ and $z \in \mathbb{D}$. This normalized and generalized Bessel function of the first kind of order $p$, also satisfy the following recurrence relation

$$4\kappa u_p'(z) = -cu_{p+1}(z),$$

(3)

which is an useful tool to study several geometric properties of $u_p$. There has been several works \cite{3, 4, 15, 16, 5, 6} studying geometric properties of the function $u_p(z)$, such as on its close-to-convexity, starlikeness, and convexity, radius of starlikeness and convexity.

In Section 2 of this paper, sufficient conditions on $A, B, c, \kappa$ are determined that will ensure $u_p$ satisfies the subordination $u_p(z) \prec \frac{(1 + Az)}{(1 + Bz)}$. It is to be understood that a computationally-intensive methodology with shrewd manipulations is required to obtain the results in this general framework. The benefits of such general results are that by judicious choices of the parameters $A$ and $B$, they give rise to several interesting applications, which include extending the results of previous works. Using this subordination result, sufficient conditions are obtained for $(-4\kappa/c)u'(z) \in \mathcal{P}[A, B]$, which next readily gives conditions for $(-4\kappa/c)(u_p(z) - 1)$ to be close-to-convex. Section 3 gives emphasis to the investigation of $u_p(z)$ to be Janowski convex as well as of $zu_p(z)$ to be Janowski starlike.

The following lemma is needed in the sequel.

**Lemma 1.1.** \cite{11,12} Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfy

$$\Psi(\rho, \sigma; z) \not\in \Omega$$

whenever $z \in \mathbb{D}$, $\rho$ real, $\sigma \leq -(1 + \rho^2)/2$. If $p$ is analytic in $\mathbb{D}$ with $p(0) = 1$, and $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\text{Re } p(z) > 0$ in $\mathbb{D}$.

In the case $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, then the condition in Lemma 1.1 generalized to

$$\Psi(\rho, \sigma, \mu + i\nu; z) \not\in \Omega$$

$\rho$ real, $\sigma + \mu \leq 0$ and $\sigma \leq -(1 + \rho^2)/2$.}

**2. Close-to-convexity of the Bessel function**

In this section, one main result on the close-to-convexity of the generalized Bessel function with several consequences are discussed in details.
Theorem 2.1. Let $-1 \leq B \leq 3 - 2\sqrt{2} \approx 0.171573$. Suppose $B < A \leq 1$, and $c, \kappa \in \mathbb{R}$ satisfy

$$
k - 1 \geq \max \left\{ 0, \frac{(1+B)(1+A)}{4(A-B)} \mid c \right\}. \tag{4}
$$

Further let $A$, $B$, $\kappa$ and $c$ satisfy either the inequality

$$
(k-1)^2 + \frac{(k-1)(A+B)}{(1-B)} - \frac{k-1}{2} (A+B) c + \frac{(1+B)^2(1+A)}{4(A-B)(1-B)} c^2 \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \tag{5}
$$

whenever

$$
\left| 2(k-1)(1-B)(A+B)c + (1+B)^2(1+A)c \right| \geq \frac{1}{2} (A-B)(1-B)c^2, \tag{6}
$$
or the inequality

$$
\frac{(k-1)(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c^2 \
\leq \frac{c^2}{4} \left( (k-1)^2 + \frac{(k-1)(A+B)}{1-B} - \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \right) \tag{7}
$$

whenever

$$
\left| 2(k-1)(1-B)(A+B)c + (1+B)^2(1+A)c \right| \leq \frac{1}{2} (A-B)(1-B)c^2. \tag{8}
$$

If $(1+B)u_p(z) \neq (1+A)$, then $u_p(z) \in \mathcal{P}[A, B]$.

Proof. Define the analytic function $p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$
p(z) = -\frac{(1-A)-(1-B)u_p(z)}{(1+A)-(1-B)u_p(z)},
$$

Then, a computation yields

$$
u_p(z) = \frac{(1-A)(1+B)p(z)}{(1-B)(1+B)p(z)}, \tag{9}
$$

$$
u_p(z) = \frac{2(A-B)p'(z)}{(1-B)(1+B)p(z)}z, \tag{10}
$$

and

$$
u''_p(z) = \frac{2(1+B)(1-B)p'(z)(zp'(z))^2 + \kappa z p'(z)}{(1-B)(1+B)p(z)} \tag{11}
$$

Thus, using the identities (9)–(11), the Bessel differential equation (2) can be rewrite as

$$
z^2p''(z) - \frac{2(1+B)}{(1-B)+(1+B)p(z)}(zp'(z))^2 + \kappa z p'(z) \tag{12}
$$

$$
+ \frac{((1-B)+(1+B)p(z))(1-A)+(1+A)p(z))}{8(A-B)} cz = 0.
$$

Assume $\Omega = \{0\}$, and define $\Psi(r, s, t; z)$ by

$$
\Psi(r, s, t; z) := t - \frac{2(1+B)}{(1-B)+(1+B)r}s^2 + \kappa s + \frac{((1-B)+(1+B)r)((1-A)+(1+A)r)}{8(A-B)} cz.
$$

It follows from (12) that $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. To ensure $\text{Re } p(z) > 0$ for $z \in \mathbb{D}$, from Lemma [11], it is enough to establish $\text{Re } \Psi(ip, \sigma, \mu + iv; z) < 0$ in $\mathbb{D}$ for any real $\rho$, $\sigma \leq -(1+\rho^2)/2$, and $\sigma + \mu \leq 0$.

With $z = x + iy \in \mathbb{D}$ in (13), a computation yields

$$
\text{Re } \Psi(ip, \sigma, \mu + iv; z) = \mu - \frac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2} \sigma^2 + \kappa \sigma - \frac{\rho(1-AB)}{4(A-B)} cy
$$

On the Bessel function
Since $\sigma \leq -(1 + \rho^2)/2$, and $B \in [-1, 3 - 2\sqrt{2}]$,
\[
\frac{2(1-B^2)}{(1-B)^2 + (1+B)^2 \rho^2} \sigma^2 \geq \frac{2(1-B^2)}{(1-B)^2 + (1+B)^2 \rho^2} \frac{1+\rho^2}{4} \geq \frac{1+B}{2(1-B)}.
\]
Thus
\[
\text{Re } \Psi(i\rho, \sigma, \mu + i\nu; z) \leq (\kappa - 1)\sigma - \frac{1+B}{2(1-B)}\rho(1-AB) + \frac{1-B}{2(1-B)}\rho(1-AB) \frac{cx}{(A-B)}
\]
\[
\leq -\frac{1}{2}(\kappa - 1)(1+\rho^2) - \frac{1+B}{2(1-B)}\rho(1-AB) + \frac{1-B}{2(1-B)}\rho(1-AB) \frac{cx}{(A-B)}
\]
\[
= p_1\rho^2 + q_1\rho + r_1 := Q(\rho),
\]
where
\[
p_1 = -\frac{1}{2}(\kappa - 1) - \frac{(1+B)(1+A)cx}{8(A-B)},
\]
\[
q_1 = -\frac{1-AB}{4(A-B)} \frac{cy}{c},
\]
\[
r_1 = -\frac{1}{2}(\kappa - 1) + \frac{(1-B)(1-A)cx}{8(A-B)} - \frac{1+B}{2(1-B)}.
\]
Condition (4) shows that
\[
p_1 = -\frac{1}{2}(\kappa - 1) - \frac{(1+B)(1+A)cx}{8(A-B)}
\]
\[
< -\frac{1}{2} \left( (\kappa - 1) - \frac{(1+B)(1+A)cx}{4(A-B)} \right) |c| < 0.
\]
Since $\max_{\rho \in \mathbb{R}} \{p_1\rho^2 + q_1\rho + r_1\} = (4p_1r_1 - q_1^2)/(4p_1)$ for $p_1 < 0$, it is clear that $Q(\rho) < 0$ when
\[
\frac{(1-AB)^2 c^2 y^2}{16(A-B)^2} < 4 \left( -\frac{1}{2}(\kappa - 1) - \frac{(1+B)(1+A)cx}{8(A-B)} \right) \times \left( -\frac{1}{2}(\kappa - 1) + \frac{(1-B)(1-A)cx}{8(A-B)} - \frac{1+B}{2(1-B)} \right),
\]
with $|x|, |y| < 1$. As $y^2 < 1 - x^2$, the above condition holds whenever
\[
\frac{(1-AB)^2 c^2}{16(A-B)^2} (1 - x^2)
\]
\[
\leq \left( (\kappa - 1) + \frac{(1+B)(1+A)cx}{4(A-B)} \right) \left( (\kappa - 1) - \frac{(1-B)(1-A)cx}{4(A-B)} + \frac{1+B}{1-B} \right),
\]
that is, when
\[
\frac{c^2 x^2}{16} + \left( (\kappa - 1) - \frac{(A+B)}{2(A-B)}c + \frac{(1+B)(1+A)}{4(1-B)(A-B)}c \right) x
\]
\[
+ (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - \frac{(1-AB)^2 c^2}{16(A-B)^2} \geq 0. \tag{15}
\]
To establish inequality (15), consider the polynomial $R$ given by
\[
R(x) := mx^2 + nx + r, \quad |x| < 1,
\]

\[
+ \frac{(1-B)(1-A) - (1+B)(1+A)\rho^2}{8(A-B)} cx.
\]

\[
\frac{1+B}{2(1-B)}.
\]
where
\[ m := \frac{c^2}{16}, \quad n := (\kappa - 1) \left( \frac{A+B}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right) \]
\[ r := (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - \frac{(1-AB)^2}{16(A-B)^2} c^2. \]

The constraint (6) yields \(|n| \geq 2|m|\), and thus \(R(x) \geq m + r - |n|\). Now inequality (5) readily implies that
\[ R(x) \geq m + r - |n| \]
\[ = \frac{c^2}{16} + (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - \frac{(1-AB)^2}{4(A-B)} c^2 \]
\[ - |(\kappa - 1) \left( \frac{A+B}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right) | \]
\[ = (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - |(\kappa - 1) \left( \frac{A+B}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right) | \]
\[ - \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \]
\[ \geq 0. \]

Now considers the case of the constraint (8), which is equivalent to \(|n| < 2m\). Then the minimum of \(R\) occurs at \(x = -n/(2m)\), and (7) yields
\[ R(x) \geq \frac{4mr-n^2}{4m} \geq 0. \]

Evidently \(\Psi\) satisfies the hypothesis of Lemma 1.1 and thus \(\text{Re } p(z) > 0\), that is,
\[ -\frac{(1-A)-(1-B)u_p(z)}{(1-A)-(1+B)u_p(z)} < \frac{1+z}{1-z}. \]
Hence there exists an analytic self-map \(w\) of \(\mathbb{D}\) with \(w(0) = 0\) such that
\[ -\frac{(1-A)-(1-B)u_p(z)}{(1+A)-(1+B)u_p(z)} = \frac{1+w(z)}{1-w(z)}, \]
which implies that \(u_p(z) < (1 + Az)/(1 + Bz)\). \(\Box\)

Theorem 2.4 gives rise to simple conditions on \(c\) and \(\kappa\) to ensure \(u_p(z)\) maps \(\mathbb{D}\) into a half-plane.

**Corollary 2.1.** Let \(c \leq 0\) and \(2\kappa \geq 2 + c^2\). Then \(\text{Re } u_p(z) > c/(c-1)\).

**Proof.** Choose \(A = -(c+1)/(c-1),\) and \(B = -1\) in Theorem 2.4. Then both the conditions (4) and (6) are equivalent to \(\kappa \geq 1\) which clearly holds for \(\kappa \geq 1 + c^2/2\). The proof will complete if the hypothesis (5) holds, i.e.,
\[ (\kappa - 1)^2 \geq \frac{1}{2} (\kappa - 1) c^2. \] (16)
Since \(\kappa \geq 1 + c^2/2\), it follows that
\[ (\kappa - 1)^2 - \frac{1}{2} (\kappa - 1) c^2 = (\kappa - 1) \left( \kappa - 1 - \frac{c^2}{2} \right) \geq 0, \]
which establishes (16). \(\Box\)
Corollary 2.2. Let $c, \kappa$ be real such that
\[
\kappa \geq \begin{cases} 
1, & c \leq 0 \\
1 + \frac{c}{2}, & c \geq 0.
\end{cases}
\]
Then $\Re u_p(z) > 1/2$.

Proof. Put $A = 0$ and $B = -1$ in Theorem 2.1. The condition (4) reduces to $\kappa \geq 1$, which holds in all cases. It is sufficient to establish conditions (6) and (5), or equivalently,
\[
4(\kappa - 1) - c \geq 0, \quad (17)
\]
and
\[
(\kappa - 1)^2 - \frac{1}{2}(\kappa - 1)c \geq 0. \quad (18)
\]

For the case when $c \leq 0$, both the inequality (17) and (18) hold as $\kappa \geq 1$.

Finally it is readily established for $c \geq 0$ and $\kappa - 1 \geq c/2$ that $4(\kappa - 1) - c \geq 0$, and $(\kappa - 1)^2 - \frac{1}{2}(\kappa - 1)c \geq (\kappa - 1)(\kappa - 1 - \frac{c}{2}) \geq 0$. $\square$

It is known that for $b = 2$ and $c = \pm 1$, the generalized Bessel functions $u_{p,2,1}(z) = j_p(z)$ and $u_{p,2,-1}(z) = i_p(z)$ respectively gives the spherical Bessel and modified spherical Bessel functions. This specific choice of $b$ and $c$, Corollary 2.2 yield $\Re(i_p(z)) > 1/2$ for $p \geq -1/2$, and $\Re(j_p(z)) > 1/2$, for $p \geq 0$. Since $i'_p(0) = 1/(4p + 6)$ for $p \geq -1/2$, following inequalities can be obtain with the aid of results in [9].

Corollary 2.3. For $p \geq -1/2$, the modified spherical Bessel functions $i_p$ satisfy the following inequalities.

\[
|i_p(z)| \leq \frac{4p + 6 + |z|}{2(2p + 3)(1 - |z|^2)}, \quad (19)
\]
\[
\Re(i_p(z)) \geq \frac{p + 6 + |z|}{4p + 6 + 2|z| + 2(2p + 3)|z|^2}, \quad (20)
\]
\[
|i'_p(z)| \leq \frac{2 \Re(i_p(z)) - 1}{2(1 - |z|^2)} \frac{|z|^2 + 4(2p + 3)|z| + 1}{(2p + 3)|z|^2 + |z| + (2p + 3)}. \quad (21)
\]

Next theorem gives the sufficient condition for close-to-convexity when $B \geq 3 - 2\sqrt{2}$.

Theorem 2.2. Let $3 - 2\sqrt{2} \leq B < A \leq 1$ and $c, \kappa \in \mathbb{R}$ satisfy
\[
\kappa - 1 \geq \max \left\{ 0, \frac{(1+B)(1+A)}{4(A-B)} |c| \right\}. \quad (22)
\]
Suppose $A, B, \kappa$ and $c$ satisfy either the inequality
\[
(\kappa - 1)^2 + 16(\kappa - 1) \frac{B(1-B)}{(1+B)^2} - \left| \frac{(\kappa-1)(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right|
\]
\[
\geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \quad (23)
\]
whenever

\[ |(\kappa - 1)(1 + B)^3(A + B)c + 8B(1 - B^2)(1 + A)c| \geq \frac{c^2}{4}(A - B)(1 + B)^3, \]

(24)

or the inequality

\[
\left((\kappa - 1) \frac{(A + B)}{2(A - B)}c + \frac{4B(1 - B^2)(1 + A)}{(1 + B)^3(A - B)}c\right)^2
\leq \frac{c^2}{4} \left((\kappa - 1)^2 + 16(\kappa - 1) \frac{B(1 - B)}{(1 + B)^3} - \frac{(1 - AB)^2}{16(A - B)^2}c^2\right)
\]

(25)

whenever

\[ |((\kappa - 1)(A + B)(1 + B)^3 + 8B(1 - B^2)(1 + A)c| < \frac{c^2}{4}(A - B)(1 + B)^3. \]

(26)

If \((1 + B)u_p(z) \neq (1 + A),\) then \(u_p(z) \in \mathcal{P}[A, B].\)

**Proof.** First, proceed similar to the proof of Theorem 2.1 and derive the expression of \(\text{Re} \, \Psi(i\rho, \sigma, \mu + iv; z)\) as given in (14). Now for \(\sigma \leq -(1 + \rho^2)/2, \rho \in \mathbb{R},\) and \(B \geq 3 - 2\sqrt{2},\)

\[
\frac{2(1-B^2)}{(1-B)^2+(1+B)^2}\rho^2 \geq \frac{2(1-B^2)}{(1-B)^2+(1+B)^2}\rho^2 \geq \frac{8B(1 - B)}{(1 + B)^3},
\]

and then with \(z = x + iy \in \mathbb{D},\) and \(\mu + \sigma < 0,\) it follows that

\[
\text{Re} \, \Psi(i\rho, \sigma, \mu + iv; z)\]

\[\leq -\frac{1}{2}(\kappa - 1)(1 + \rho^2) - \frac{(1+B)(1+A)x}{8(A-B)}c x - \frac{\rho(1-AB)}{4(A-B)}cy\]

\[+ \frac{(1-B)(1-A)}{8(A-B)}cx - \frac{8B(1-B)}{(1+B)^3}cy\]

\[= p_2\rho^2 + q_2\rho + r_2 := Q_1(\rho),\]

where

\[p_2 = -\frac{1}{2}(\kappa - 1) - \frac{(1+B)(1+A)}{8(A-B)}cx,\]

\[q_2 = -\frac{(1-AB)cy}{4(A-B)},\]

\[r_2 = -\frac{1}{2}(\kappa - 1) + \frac{(1-B)(1-A)}{8(A-B)}cx - \frac{8B(1-B)}{(1+B)^3}cy.\]

Observe that the inequality (22) implies that \(p_2 < 0.\) Thus \(Q_1(\rho) < 0\)

for all \(\rho \in \mathbb{R}\) provided \(q_2^2 \leq 4p_2r_2,\) that is, for \(|x|, |y| < 1,\)

\[
\frac{(1-AB)^2}{16(A-B)^2}c^2y^2 \leq \left((\kappa - 1) + \frac{(1+B)(1+A)}{4(A-B)}cx\right) \left((\kappa - 1) - \frac{(1-B)(1-A)}{4(A-B)}cx + \frac{16B(1-B)}{(1+B)^3}\right).
\]

With \(y^2 < 1 - x^2,\) it is enough to show for \(|x| < 1,\)

\[
\frac{(1-AB)^2}{16(A-B)^2}c^2(1-x^2) \leq \left((\kappa - 1) + \frac{(1+B)(1+A)}{4(A-B)}cx\right) \left((\kappa - 1) - \frac{(1-B)(1-A)}{4(A-B)}cx + \frac{16B(1-B)}{(1+B)^3}\right).
\]
which is equivalent to
\[
R_1(x) := m_1 x^2 + n_1 x + r_1 \geq 0,
\]
where
\[
m_1 := \frac{c^2}{16},
n_1 := \left( (\kappa - 1) \left( \frac{c(A+B)}{2(A-B)} \right) + \frac{4B(1-B^2)(1+A)}{(A-B)(1+B)^3} c \right),
\]
\[
r_1 := (c-1)^2 + (c-1) \frac{16B(1-B)}{(1+B)^4} - \frac{\sigma^2 (1-AB)^2}{(A-B)^2}.
\]
If (24) holds, then \(|n_1| \geq 2|m_1|\). Since \(R_1\) is increasing, then \(R_1(x) \geq m_1 + r_1 - |n_1|\), which is nonnegative from (23). On the other hand, if (26) holds, then \(|n_1| < 2|m_1|\), \(R_1(x) \geq (4m_1 r_1 - n_1^2)/4m_1\), and (26) implies \(R_1(x) \geq 0\). Either case establishes (27).

**Theorem 2.3.** Let \(-1 \leq B \leq 3 - 2\sqrt{2} \approx 0.171573\). Suppose \(B < A \leq 1\), \(c, \kappa \in \mathbb{R}\) with \(c \neq 0\) and satisfying
\[
\kappa \geq \max \left\{ 0, \frac{(1+B)(1+A)}{4(A-B)} |c| \right\}.
\]
Further let \(A, B, \kappa\) and \(c\) satisfy either
\[
\kappa^2 + \kappa \frac{1+14}{1-B} - \kappa \frac{(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2
\]
whenever
\[
|2\kappa(1-B)(A+B)c + (1+B)^2(1+A)c| \geq \frac{1}{2}(A-B)(1-B)c^2,
\]
or the inequality
\[
\left( \kappa \frac{(A+B)}{2(A-B)} + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right)^2 \leq \frac{c^2}{4} \left( \kappa^2 + \frac{\kappa(1+B)}{1-B} - \frac{(1-AB)^2}{16(A-B)^2} c^2 \right)
\]
when
\[
|2\kappa(1-B)(A+B)c + (1+B)^2(1+A)c| < \frac{1}{2}(A-B)(1-B)c^2.
\]
If \((1+B)u_p(z) \neq (1+A), \text{ then } (-4\kappa/c)u_p'(z) \in \mathcal{P}[A,B] \).

**Theorem 2.4.** Let \(3 - 2\sqrt{2} < B < A \leq 1\). Suppose \(c, \kappa \in \mathbb{R}\), \(a \neq 0\), such that
\[
\kappa \geq \max \left\{ 0, \frac{(1+B)(1+A)}{4(A-B)} |c| \right\}.
\]
Suppose \(A, B, \kappa\) and \(c\) satisfy either
\[
\kappa^2 + 16\kappa \frac{B(1-B)}{(1+B)^3} - \kappa \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2
\]
whenever
\[
|\kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c| \geq \frac{c^2}{4}(A-B)(1+B)^3,
\]
or the inequality
\[
\left( \kappa \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right)^2 \leq \frac{c^2}{4} \left( \kappa^2 + \frac{16\kappa(B(1-B))}{(1+B)^3} - \frac{(1-AB)^2}{16(A-B)^2} c^2 \right)
\]
when
\[
|2\kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c| < \frac{c^2}{4}(A-B)(1+B)^3.
\]
If \((1 + B)u_p(z) \neq (1 + A)\), then \((-4\kappa/c)u_p'(z) \in \mathcal{P}[A, B]\).

**Corollary 2.4.** Let \(c \leq -1\), and
\[
\kappa \geq \max \left\{ \frac{z(c+1)}{2}, \frac{c}{2(c+1)} \right\}.
\]
Then \((-4\kappa/c)(u_p(z) - 1)\) is close-to-convex of order \((c+1)/c\) with respect to the identity function.

**Corollary 2.5.** Let \(c\) be a nonzero real number, and \(\kappa \geq |c|/2\). Then
\[
\text{Re}(-4\kappa/c)u_p'(z) > 1/2.
\]

### 3. Janowski starlikeness of generalized Bessel functions

This section contributes to find conditions to ensure a normalized and generalized Bessel functions \(zu_p(z)\) in the class of Janowski starlike functions. For this purpose, first sufficient conditions for \(u_p(z)\) to be Janowski convex is determined, and then an application of relation [3] yields conditions for \(zu_p(z) \in S^*[A, B]\).

**Theorem 3.1.** Let \(c, \kappa \in \mathbb{R}\) be such that \((A - B)u_p'(z) \neq (1 + B)zu_p''(z), -1 \leq B \leq 0 < A \leq 1\). Suppose
\[
\kappa(1 + B) \geq \frac{(1+B)^2}{4(A-B)} |c| - (1 + A - B).
\]
Further let \(A, B, \kappa\) and \(c\) satisfy
\[
(1 + A - B + \kappa(1 + B))(1 - A + B + \kappa(1 - B)) \geq \frac{(1-B^2)^2z^2}{16(A-B)^2}c^2 - \left| \frac{B-(A-B)(1+B^2)+(1-B^2)B\kappa}{2(A-B)}c \right|.
\]
If \(0 \notin u_p'(\mathbb{D}), 0 \notin u_p''(\mathbb{D})\), then
\[
1 + \frac{zu_p''(z)}{u_p'(z)} < \frac{1 + Az}{1 + Bz}
\]

**Proof.** Define an analytic function \(p : \mathbb{D} \to \mathbb{C}\) by
\[
p(z) := \frac{(A-B)u_p'(z)+(1-B)zu_p''(z)}{(A-B)u_p'(z)-(1+B)zu_p''(z)}.
\]
Then
\[
\frac{zu_p''(z)}{u_p'(z)} = \frac{(A-B)(p(z)-1)}{(p(z)+1)+B(p(z)-1)},
\]
and
\[
\frac{z^2u_p'''(z)+zu_p''(z)}{zu_p''(z)} - \frac{zu_p''(z)}{u_p'(z)} = \frac{zp'(z)}{(p(z)-1)} - \frac{(1+B)zp'(z)}{(p(z)+1)+B(p(z)-1)} = \frac{zp'(z)((p(z)+1)+B(p(z)-1)-(1+B)(p(z)-1))}{(p(z)-1)((p(z)+1)+B(p(z)-1))}.
\]
A rearrangement of [32] yields
\[
\frac{zu_p''(z)}{u_p'(z)} = \frac{2zp'(z)}{(p(z)-1)((p(z)+1)+B(p(z)-1))} - 1 + \frac{zu_p''(z)}{u_p'(z)}.
\]
Thus,
\[
\left( \frac{zu_p'''}{u_p''} \right) (z) + \frac{zu_p''}{u_p'} (z) = \frac{2(A-B)z}{{\big((p(z)+1)B(p(z)-1)\big)}^2} - \frac{(A-B)(p(z)-1)}{(p(z)+1)B(p(z)-1)} + \frac{(A-B)^2(p(z)-1)^2}{{\big((p(z)+1)B(p(z)-1)\big)}^2}.
\]  
(33)

Now a differentiation of (2) leads to
\[
4z^2u_p''''(z) + 4(\kappa + 1)z u_p'''(z) + c u_p''(z) = 0,
\]
which give
\[
\left( \frac{zu_p'''(z)}{u_p''(z)} \right) + \left( \frac{zu_p''(z)}{u_p'(z)} \right) + (\kappa + 1) \frac{zu_p'(z)}{u_p(z)} + \frac{c}{4} z = 0. 
\]  
(34)

Using (31) and (33), (34) yields
\[
\frac{2(A-B)z p'(z)}{{\big((p(z)+1)B(p(z)-1)\big)}^2} + \frac{(A-B)^2(p(z)-1)^2}{{\big((p(z)+1)B(p(z)-1)\big)}^2} + \frac{(A-B)(p(z)-1)\kappa}{{\big((p(z)+1)B(p(z)-1)\big)}^2} + \frac{c}{4} z = 0,
\]
equivalently
\[
z p'(z) + \left( \frac{A-B}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)} \right) (p(z))^2 - \left( A - B + \kappa B - \frac{c(1-B^2)}{4(A-B)} z \right) p(z) + \left( \frac{A-B}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)} \right) = 0.
\]  
(35)

Define,
\[
\Psi(p(z), z p'(z), z) := z p'(z) + F_1 (p(z))^2 + F_2 p(z) + F_3,
\]
where
\[
F_1 = \frac{(A-B)}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)},
F_2 = -(A-B) - \kappa B + \frac{c(1-B^2)}{4(A-B)} z, 
F_3 = \frac{(A-B)}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)}.
\]
Thus, (35) yields \(\Psi(p(z), z p'(z), z) \in \Omega = \{0\}\). Now with \(z = x + iy \in \mathbb{D}\), let
\[
G_1 := \text{Re}(F_1) = \frac{(A-B)}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)} 
= \frac{1}{2} \left( A - B + \kappa(1 + B) + \frac{cz(1+B)^2}{4(A-B)} \right),
\]
\[
G_2 := \text{Re}(iF_2) = -\frac{c(1-B^2)}{4(A-B)} y, 
G_3 := \text{Re}(F_3) = \frac{(A-B)}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)} 
= \frac{1}{2} \left( A - B - \kappa(1 - B) + \frac{c(1-B)^2}{4(A-B)} x \right).
\]
For \(\sigma \leq -(1 + \rho^2)/2, \rho \in \mathbb{R}\),
\[
\text{Re} \Psi(i\rho, \sigma, z) = \sigma - G_1 \rho^2 + G_2 \rho + G_3 
\leq -\frac{1+2G_1}{2} \rho^2 + G_2 \rho + \frac{2G_3+1}{2} := Q(\rho).
\]
Note that condition (28) implies \((1 + 2G_1)/2 > 0\). In this case, \(Q\) has a maximum at \(\rho = G_2/(1 + 2G_1)\). Thus \(Q(\rho) < 0\) for all real \(\rho\) provided
\[
G_2^2 \leq (1 + 2G_1)(1 - 2G_3), \quad |x|, |y| < 1.
\]
Since \(y^2 < 1 - x^2\), it is left to show that
\[
\frac{(1-B)^2}{16(A-B)^2} e^2 (1 - x^2)
\]
\[
\leq \left(1 + A - B + \kappa(1 + B) + \frac{c(1+B)^2}{4(A-B)}x\right) (1 - A + B - \kappa(-1 + B)
\]
\[\quad - \frac{c(1-B)^2}{4(A-B)} x),
\]
\(|x| < 1\). The above inequality is equivalent to
\[
H(x) := h_2(A, B)x + h_3(A, B) \geq 0,
\]
where
\[
h_2(A, B) = -\frac{(B-(A-B)(B^2+1)+(1-B^2)B\kappa)c}{2(A-B)},
\]
\[
h_3(A, B) = \left(1 + A - B + \kappa(1 + B)\right) (1 - A + B - \kappa(B - 1)) - \frac{(1-B)^2}{16(A-B)^2}c^2.
\]
Since \(|x| < 1\), the left-hand side of the inequality (36) satisfy
\[
h_2(A, B)x + h_3(A, B) \geq -|h_2(A, B)| + h_3(A, B).
\]
Now it is evident from (29) that \(H(x) \geq 0\) which establish the inequality (36).

Thus \(\Psi\) satisfies the hypothesis of Lemma 1.1 and hence \(Re p(z) > 0\), or equivalently
\[
\frac{(A-B)u' + (1-B)zu''}{(A-B)u' - (1+B)zu''} < \frac{1+z}{1-z}.
\]
By definition of subordination, there exists an analytic self-map \(w\) of \(D\) with \(w(0) = 0\) and
\[
\frac{(A-B)u'_p(z) + (1-B)zu''_p(z)}{(A-B)u'_p(z) - (1+B)zu''_p(z)} = \frac{1+w(z)}{1-w(z)}.
\]
A simple computation shows that
\[
1 + \frac{zu''_p(z)}{u'_p(z)} = \frac{1+Aw(z)}{1+Bw(z)},
\]
and hence
\[
1 + \frac{zu''_p(z)}{u'_p(z)} < \frac{1+Az}{1+Bz}.
\]

The relation (3) also shows that
\[
\frac{z}{zu_p(z)} = 1 + \frac{zu''_{p-1}(z)}{u'_{p-1}(z)}.
\]
Together with Theorem 3.1 it immediately yields the following result for \(zu_p(z) \in S^*[A, B]\).
Theorem 3.2. Let $c$ and $\kappa$ be real numbers such that $(A - B)u_p'(-1)(z) \neq (1 + B)zu_p''(z)$, $-1 \leq B < A \leq 1$. Suppose
\[\kappa(1 + B) \geq \frac{(1+B)^2}{4(A-B)} |c| - (A - 2B). \quad (37)\]
Further let $A$, $B$, $\kappa$ and $c$ satisfy
\[(A - 2B + \kappa(1 + B))(2B - A + \kappa(1 - B)) \geq \frac{(1-B^2)^2}{16(A-B)}c^2 + \left| \frac{B^3-(A-B)(1+B^2)+(1-B^2)B\kappa}{2(A-B)} c \right| .\]
Then $zu_p(z) \in S^*[A, B]$.

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