ON A CLASS OF RATIONAL AND MIXED
SOLITON-RATIONAL SOLUTIONS OF TODA
LATTICE

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Abstract

A class of rational solutions of Toda lattice satisfying certain Backlund
transformations and a class of mixed rational-soliton solutions (quasisoli-
tons) in wronskian form are obtained using the method of Ablowitz and
Satsuma. Also an extended class of rational solutions are found using an
appropriate recursion relation. They are also solutions of Boussinesq equa-
tion and it is conjectured that there is a larger class of common solutions
of both equations.

1 Introduction

The class of rational solutions was firstly investigated for KdV equation [1], [2].
A very simple way to find them was developed by Ablowitz and Satsuma (AS)
[3] and consists in taking the ”long wave limit” in the multisoliton solution.
The method was used successfully to obtain the rational solutions also for other
nonlinear completely integrable systems [4], [5].

Mixed rational-soliton solutions (quasisolitons) of KdV equation were dis-
covered by Ablowitz and Cornille [4] and later were studied by H.Airault and
M.J.Ablowitz [6]. They were also found using the limiting procedure of AS [7].

In spite of the large popularity of Toda lattice (TL) as the most known and
studied nonlinear integrable lattice model [8] these types of solutions were very
little investigated. In a previous paper following AS procedure we obtained the
expressions for the first rational and mixed rational-soliton solutions [9].

Let us summarize the main results of [9]:

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• one rational solution:

\[ f_n^{(1)} = (n - \epsilon t) \] (1)

where \( \epsilon = \pm 1 \) depending on the propagation direction of the soliton.

• two rational solution:

\[ f_n^{(2)} = n^2 - t^2 - 1/4 \] (2)

if the two solitons are moving in opposite directions, and

\[ f_n^{(2)} = 4(n - \epsilon t)^3 - (n - \epsilon t) + 2\epsilon t \] (3)

if they move in the same direction.

• mixed 1-rational 1-soliton solution:

\[ f_n^{(1,1)} = (n - \epsilon t) + (n - \epsilon t + B) \exp(-2\eta) \] (4)

where

\[ \eta = Pn - \epsilon' t \sinh P + \eta^{(0)} \] (5)

and

\[ B = \begin{cases} \text{coth } P/2 & \text{if } \epsilon \epsilon' = 1 \\ \text{tanh } P/2 & \text{if } \epsilon \epsilon' = -1 \end{cases} \]

In all these expressions the soliton velocity was taken equal with unity and the solutions are written in Hirota’s formalism.

The aim of this paper is to investigate a larger class of similar solutions. In the second paragraph starting from a certain Backlund transformation and its solutions found by Hirota and Satsuma [10], a class of complex and implicitly nonsingular solutions is determined. In the next one a general form of 1-rational N-soliton solution is written down in a wronskian form [11]. In the last section starting from a nonlinear superposition formula [8] [12] the class of rational solutions is enlarged, some of them being different from those found using AS limiting procedure. It is observed that they satisfy also the Boussinesq equation and it is conjectured that there is a larger class of common rational solutions of both equations. This fact was somehow foreseen by Gibbon [13] in his study of the poles of Toda lattice and the connection with other Hamiltonian N-body systems and continuum limits as KdV and Boussinesq equations.
2 Rational Solutions

The adimensional form of Toda equation (10) is given by:

$$\ddot{y}_n(t) = \exp[-(y_n - y_{n-1})] - \exp[-(y_{n+1} - y_n)]$$  \hspace{1cm} (6)

where $y_n(t)$ is the displacement of the $n$-th particle from its equilibrium position. If we set:

$$\exp[-(y_n - y_{n-1})] = \frac{d^2}{dt^2} \ln f_n(t)$$  \hspace{1cm} (7)

and introducing in (6) Hirota form for Toda equation is obtained:

$$\left[D_t^2 - 4 \sinh^2 \left(\frac{D_n}{2}\right)\right] f_n(t) f_n(t) = 0$$  \hspace{1cm} (8)

where $D_t$ and $D_n$ are the well known Hirota bilinear operators.

Hirota and Satsuma [10] have introduced the following Backlund transformations (BT) for Toda lattice and equivalent systems:

$$D_t f'_n f_n + 2\alpha \sinh \left(\frac{D_n}{2}\right) g'_n g_n = 0$$  \hspace{1cm} (9)

$$D_t g'_n g_n + 2\alpha^{-1} \sinh \left(\frac{D_n}{2}\right) f'_n f_n = 0$$  \hspace{1cm} (10)

$$\left[\beta_1 \sinh \left(\frac{D_n}{2}\right) + \cosh \left(\frac{D_n}{2}\right)\right] g'_n g_n = 0$$  \hspace{1cm} (11)

$$\left[\beta_2 \sinh \left(\frac{D_n}{2}\right) + \cosh \left(\frac{D_n}{2}\right)\right] f'_n f_n = 0$$  \hspace{1cm} (12)

where $\alpha$, $\beta_1$ and $\beta_2$ are constants satisfying the relation:

$$\alpha^{-1}(\beta_1^2 - 1) = \alpha(\beta_2^2 - 1)$$

$(f_n, f'_n)$ and $(g_n, g'_n)$ are pairs of solutions of Toda equation of the same type.

The 1-soliton and 2-soliton solutions obtained from these BT are given by:

$$f_n = 1 + \exp (2\eta + \phi)$$
$$f'_n = 1 + \exp (2\eta + \phi')$$
$$g_n = 1 + \exp (2\eta + \psi)$$
$$g'_n = 1 + \exp (2\eta + \psi')$$  \hspace{1cm} (13)

and

$$f_n = 1 + \exp (2\eta_1 + \phi_1) + \exp (2\eta_2 + \phi_2) + \exp (2(\eta_1 + \eta_2) + \phi_1 + \phi_2 + A_{12})$$
$$f'_n = 1 + \exp (2\eta_1 + \phi'_1) + \exp (2\eta_2 + \phi'_2) + \exp (2(\eta_1 + \eta_2) + \phi'_1 + \phi'_2 + A_{12})$$
$$g_n = 1 + \exp (2\eta_1 + \psi_1) + \exp (2\eta_2 + \psi_2) + \exp (2(\eta_1 + \eta_2) + \psi_1 + \psi_2 + A_{12})$$
$$g'_n = 1 + \exp (2\eta_1 + \psi'_1) + \exp (2\eta_2 + \psi'_2) + \exp (2(\eta_1 + \eta_2) + \psi'_1 + \psi'_2 + A_{12})$$  \hspace{1cm} (14)
Here $\epsilon_i = \pm 1$ depending on the propagation direction of the soliton.

\[
\eta_i = \Omega_i t - P_i n + \eta_i^{(0)}
\]

\[
\Omega_i = \epsilon_i \sinh P_i
\]

\[
\exp A_{12} = \left[ \frac{\epsilon_1 \exp P_1 - \epsilon_2 \exp P_2}{1 - \epsilon_1 \epsilon_2 \exp (P_1 + P_2)} \right]^2
\]

and

\[
\exp \phi_i = \epsilon_i \alpha^{-1} \beta_1 + \beta_2 \cosh P_i - \sinh P_i
\]

\[
\exp \phi'_i = \epsilon_i \alpha^{-1} (\epsilon_i \alpha \beta_2 + \beta_1 \cosh P_i - \sinh P_i)
\]

\[
\exp \psi_i = \epsilon_i \alpha^{-1} (\epsilon_i \alpha \beta_2 + \beta_1 \cosh P_i + \sinh P_i)
\]

A class of complex solutions are obtained with a very simple choice of $\alpha$, $\beta_1$ and $\beta_2$ namely:

\[
\alpha = i
\]

\[
\beta_1 = -\beta_2 = 1
\]

In the limiting procedure of Ablowitz and Satsuma\[3\] the N-soliton solution is expanded in power series of $P$, $P_i = \delta P_i'$, with $\delta \to 0$ and the phase shifts $\exp 2\eta_i^{(0)}, i = 1,...N$ are determined cancelling all the terms of order $O(P^k)$ with $k$ at least smaller than $N$. Then the N-rational solution is found from the first nonvanishing term of order $O(P^k)$ with $k \geq N$.

In this way starting from the 1-soliton solutions (13) and taking $\exp 2\eta^0 = (1 \pm i)^{-1}$ the 1-rational solutions are found from $O(P)$ terms and are given by:

\[
f_n = n - (1 - i\epsilon)/4 - \epsilon t
\]

\[
f'_n = n + (1 - i\epsilon)/4 - \epsilon t
\]

\[
g_n = n + (1 + i\epsilon)/4 - \epsilon t
\]

\[
g'_n = n - (1 + i\epsilon)/4 - \epsilon t
\]

They satisfy the following symmetry relations:

\[
f_n = g_{n-1/2}, f'_n = g'_{n+1/2}
\]

\[
f_n = g_n^*, f'_n = g_n
\]

(16)

where (*) means the complex conjugation. It is easily seen that they verify the BT (14) and the Toda equation (8). In fact any expression

\[
f_n = n \pm \epsilon t + z
\]

\[1\]We have to take into account also that $f$ and $f \exp (\alpha t + \beta)$ are two equivalent solutions with $\alpha, \beta$ does not depending on $t$
with $z$ a complex number is a solution of (8).

In the case of 2-rational solutions two different expressions are obtained depending if the two solitons are moving in opposite direction ($\epsilon_1\epsilon_2 = -1$) or in the same direction ($\epsilon_1\epsilon_2 = 1$). Thus if $\epsilon_1\epsilon_2 = -1$ taking the limit $P_1 \to 0, P_2 \to 0$ and choosing:

$$\exp 2\eta^{(0)}_1 = (1 + i)^{-1}, \exp 2\eta^{(0)}_2 = (1 - i)^{-1}$$

the $O(1)$ and $O(P)$ terms cancel and the following 2-rational solutions are found from $O(P^2)$ terms:

$$f_n = (n - 1/4)^2 - (t - i/4)^2 - 1/4$$
$$f'_n = (n + 1/4)^2 - (t + i/4)^2 - 1/4$$
$$g_n = (n + 1/4)^2 - (t - i/4)^2 - 1/4$$
$$g'_n = (n - 1/4)^2 - (t + i/4)^2 - 1/4$$

They satisfy the symmetry relations (16) and verify the BT (11) and Toda equation (8).

When the two solitons are moving in the same direction $\epsilon_1\epsilon_2 = 1$ the rational solutions are obtained from the $O(P^3)$ terms. As in our previous calculations the phase shifts have to be $P$ dependent and with the choice:

$$\exp 2\eta^{(0)}_1 = \frac{-1}{1 + i\epsilon} P_1 + P_2 - \frac{i\epsilon}{4} P_1 P_2$$
$$\exp 2\eta^{(0)}_2 = \frac{1}{1 + i\epsilon} P_1 + P_2 - \frac{i\epsilon}{4} P_1 P_2$$

the terms of order $O(1)$, $O(P)$ and $O(P^2)$ are vanishing and from the $O(P^3)$ ones we get:

$$f_n = 4 \left[ n - (1 - i\epsilon)/4 - \epsilon t \right]^3 - \left[ n - (1 + i\epsilon)/4 - \epsilon t \right]^3 - 2\epsilon t + i\epsilon/2$$
$$f'_n = 4 \left[ n + (1 - i\epsilon)/4 - \epsilon t \right]^3 - \left[ n + (1 + i\epsilon)/4 - \epsilon t \right]^3 + 2\epsilon t - i\epsilon/2$$
$$g_n = 4 \left[ n + (1 + i\epsilon)/4 - \epsilon t \right]^3 - \left[ n + (1 + i\epsilon)/4 - \epsilon t \right]^3 + 2\epsilon t + i\epsilon/2$$
$$g'_n = 4 \left[ n - (1 + i\epsilon)/4 - \epsilon t \right]^3 - \left[ n - (1 + i\epsilon)/4 - \epsilon t \right]^3 + 2\epsilon t - i\epsilon/2$$

Again the symmetry relations (16) are satisfied and by straightforward calculations one checks that they are solutions of BT and Toda equations.

Higher order rational solutions can be found in the same way but the calculations become more and more difficult.

Beside the nice symmetry properties and their explicit form these solutions are no more singular, although their physical significance is not yet clear.
3 Mixed rational-soliton solution

We shall study now the class of mixed rational-soliton solutions also known as quasisolitons. As mentioned in the Introduction they were investigated firstly for KdV [5], [6] using different methods. Previously we have shown that they exist also for Toda lattice [9]. The method used was the same "long wave limit". In the present section we shall extend these calculations and we shall present an explicit form for the 1-rational N-soliton solution. The starting point is the expression of the \((N+1)\) soliton solution written as a wronskian determinant. Following Nimmo’s [11] notations we have:

\[
f_n^{(N+1)}(t) = \begin{vmatrix}
E_1^+ + E_1^- & \frac{d}{dt}(E_1^+ + E_1^-) & \cdots & \frac{d^N}{dt^N}(E_1^+ + E_1^-) \\
E_2^+ + E_2^- & \frac{d}{dt}(E_2^+ + E_2^-) & \cdots & \frac{d^N}{dt^N}(E_2^+ + E_2^-) \\
\vdots & \vdots & \ddots & \vdots \\
E_{N+1}^+ + E_{N+1}^- & \frac{d}{dt}(E_{N+1}^+ + E_{N+1}^-) & \cdots & \frac{d^N}{dt^N}(E_{N+1}^+ + E_{N+1}^-)
\end{vmatrix}
\]

where

\[E_i^\pm = a_i^\pm \exp \left[ \pm P_i n + \epsilon_i \exp (\pm P_i) \right], \quad i = 1, \ldots, N+1\]

Here \(a_i^\pm\) are functions of the initial phase \(\eta_i^0\) and \(\epsilon_i = \pm 1\) depending on the propagation direction of the \(i\)-th soliton. The 1-rational N-soliton is obtained in the limit \(P_{N+1} \to 0\). This affects only the last line of the determinant. Dropping the index \(N+1\) we have to expand in power series of \(P\) and keeping only the linear terms we get:

\[
\frac{d^j}{dt^j}(E_{N+1}^+ + E_{N+1}^-) \to \exp (\epsilon t) e^j \left[ (a^+ + a^-) + P(n + j + \epsilon t)(a^+ - a^-) + O(P^2) \right]
\]

The vanishing of \(O(1)\) terms gives:

\[a^+ + a^- = 0\]

and the 1-rational N-soliton solution is obtained from the \(O(P)\) terms, namely:

\[
F_{1,N}(t) = \begin{vmatrix}
E_1^+ + E_1^- & \frac{d}{dt}(E_1^+ + E_1^-) & \cdots & \frac{d^N}{dt^N}(E_1^+ + E_1^-) \\
\vdots & \vdots & \ddots & \vdots \\
E_{N}^+ + E_{N}^- & \frac{d}{dt}(E_{N}^+ + E_{N}^-) & \cdots & \frac{d^N}{dt^N}(E_{N}^+ + E_{N}^-) \\
n + \epsilon t & \cdots & \epsilon^j(n + j + \epsilon t) & \epsilon^n(n + N + \epsilon t)
\end{vmatrix}
\]

Higher order rational-N soliton solutions can be found starting from Wronskian determinants of order \((N+2), (N+3)\) and taking the limit \(P_{N+1} \to 0, P_{N+2} \to 0\).
4 Extended class of rational solutions

One of the main advantages of using the method of Backlund transformations is the possibility to derive a nonlinear superposition formula [12]. For Toda lattice starting from a given solution $y_n^{(0)}$ through two distinct BT the solutions $y_n^{(1)}$ and $y_n^{(2)}$ can be constructed. Interchanging the BT and assuming the permutable of the Bianchi diagram a new solution $y_n^{(1,2)}$ is expressed in terms of $y_n^{(0)}, y_n^{(1)}, y_n^{(2)}$ by a simple algebraic relation [12]:

$$\exp \left( y_n^{(1,2)} - \gamma^{(1,2)} + y_n^{(0)} - \gamma^{(0)} \right) =$$

$$= \exp \left( y_{n+1}^{(1)} - \gamma^{(1)} + y_{n+1}^{(2)} - \gamma^{(2)} \right) \frac{z_1 \exp \left( y_n^{(2)} - \gamma^{(2)} \right) - z_2 \exp \left( y_n^{(1)} - \gamma^{(1)} \right)}{z_1 \exp \left( y_{n+1}^{(2)} - \gamma^{(2)} \right) - z_2 \exp \left( y_{n+1}^{(1)} - \gamma^{(1)} \right)}$$

(18)

Here

$$\gamma^{(0)} = y_n^{(0)} - \infty, \gamma^{(1)} = y_n^{(1)}$$

$$\gamma^{(2)} = y_n^{(2)} - \infty, \gamma^{(1,2)} = y_n^{(1,2)}$$

and

$$z_1 = A_1 \exp \left( \gamma^{(0)} - \gamma^{(1)} \right) = A_1 \exp \left( \gamma^{(2)} - \gamma^{(1,2)} \right)$$

$$z_2 = A_2 \exp \left( \gamma^{(0)} - \gamma^{(2)} \right) = A_2 \exp \left( \gamma^{(1)} - \gamma^{(1,2)} \right)$$

where $A_1, A_2$ are constants.

Then we have

$$\exp \left( y_n^{(-i)} - y_n^{(i)} \right) = \prod_{k \leq n} \exp \left( y_{k-1}^{(i)} - y_k^{(i)} \right) = \frac{f_n^{(i)} f_{n+1}^{(i)}}{f_n^{(i)}} \lim_{k \to -\infty} \frac{f_n^{(i)}}{f_{n-k+1}^{(i)}}$$

It is convenient to consider:

$$\lim_{k \to -\infty} \frac{f_n^{(i)}}{f_{n-k+1}^{(i)}} = 1$$

Then the superposition formula (18) becomes:

$$\frac{f_n^{(1,2)}}{f_{n+1}^{(1,2)}} = \frac{z_1 f_n^{(2)} f_{n+1}^{(2)} - z_2 f_n^{(1)} f_{n+1}^{(2)} f_{n+2}^{(0)}}{z_1 f_{n+1}^{(2)} f_{n+2}^{(1)} - z_2 f_{n+1}^{(2)} f_{n+2}^{(1)} f_{n+2}^{(0)}}$$

(19)

It can be used not only to generate higher order soliton solutions but also other types of solutions. We shall apply it to find a larger class of 2-rational solutions starting from $f_n^{(0)} = \text{const.}$. It is convenient and possible to use a simplified form of (19) namely

$$f_n^{(1,2)} = (z_1 f_n^{(2)} f_{n+1}^{(2)} - z_2 f_n^{(1)} f_{n+1}^{(2)}) / f_{n+1}^{(0)}$$

(20)
Taking $f^{(1)}_n$ and $f^{(2)}_n$ as the 1-rational solution

$$f^{(1)}_n = n + t + \lambda_1, f^{(2)}_n = n - t + \lambda_2$$

and considering $z_1 + z_2 = 0$ one obtains:

$$f^{(1,2)}_n = n^2 - t^2 + (\lambda_1 + \lambda_2 + 1)n + (\lambda_1 - \lambda_2)t + (\lambda_1\lambda_2 + \frac{\lambda_1 + \lambda_2}{2}) \quad (21)$$

It is easily shown that for any complex numbers $\lambda_1, \lambda_2$, $f^{(1,2)}_n$ is a solution of Toda lattice and with an appropriate choice of $\lambda_1$ and $\lambda_2$ namely $\lambda_1 = \lambda_2 = -1/2$ the already known 2-rational solution(2) obtained by the "long wave limit" method of Ablowitz and Satsuma is found.

In fact any polynomial form $n^2 - t^2 + an + bt + c$ such that $a^2 - b^2 = 4c + 1$ is a solution for Toda lattice where $a, b$ and $c$ are complex numbers. The solution(21) verifies identically these conditions. One can see that for

$$\lambda_1, \lambda_2 \in \{(i - 3)/4, -(i - 3)/4, (i - 1)/4, -(i - 1)/4\}$$

we can recover the set of complex 2 rational solutions(17). The above superposition formula is not useful to find also rational solutions written as 3-degree polynomials in $n$ and $t$ starting from 1-rational solutions. Perhaps more complicated higher order rational solutions can be found using it in the form of ratios of polynomials, but soon the calculations become very tedious.

On the other hand it is worthwhile to mention the similarity of these solutions with the corresponding rational solutions of Boussinesq equation:

$$(D_t^2 - D_x^2 - 1/12D_x^4)f(x,t)f(x,t) = 0$$

For this variant of Boussinesq equation the rational solutions are:

$$x \pm t$$
$$x^2 - t^2 - 1/4$$
$$4(x \pm t)^3 - (x \pm t) \mp 2t$$

They have been found by Ablowitz and Satsuma using the "long wave limit" method [3]. Also, we can see that:

$$x \pm t + \lambda$$
$$x^2 - t^2 + ax + bt + c$$

with

$$a^2 - b^2 = 4c + 1$$

are solutions for Boussinesq equation. As this equation is one of the continuum limit of the Toda lattice we can ask ourselves if there are more identical rational solutions of both equations. We conjecture that this happens, although they are not necessary polynomials in Hirota formalism.
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