Custodial symmetry, Georgi-Machacek model, and other scalar extensions

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Abstract

In an SU(2) gauge theory, if the gauge bosons turn out to be degenerate after spontaneous symmetry breaking, obviously these mass terms are invariant under a global SU(2) symmetry that is unbroken. The pure gauge terms are also invariant under this symmetry. This symmetry is called the custodial symmetry (CS). In SU(2) × U(1) gauge theories, CS implies a mass relation between the W and the Z bosons. The Standard Model (SM), as well as various extensions of it in the scalar sector, possess such a symmetry. In this paper, we critically examine the notion of CS and show that there may be three different classes of CS, depending on the gauge couplings and self-couplings of the scalars. Among old models that preserve CS, we discuss the Two-Higgs Doublet Model and the one doublet plus two triplet model by Georgi and Machacek. We show that for two-triplet extensions, the Georgi-Machacek model is not the most general possibility with CS. Rather, we find, as the most general extension, a new model with more parameters and hence a richer phenomenology. Some of the consequences of this new model have also been discussed.

1 Introduction

The concept of custodial symmetry (CS) is one of the cornerstones of the Standard Model (SM). CS explains why the ρ-parameter, defined as $M_W^2/M_Z^2 \cos^2 \theta_W$ where $\theta_W$ is the Weinberg angle, equals unity at the tree-level in the SM, and also what might be responsible for a departure from that value when loop corrections to the gauge boson self-energy are added.

To be precise, the Yukawa interaction in the SM breaks the CS and once the radiative corrections are incorporated, the ρ-parameter shifts from unity, the dominant contribution coming from the top quark loop and being proportional to $m_t^2$. One defines the parameter $\rho_0$ to absorb this correction so that in the SM, $\rho_0$ is unity by definition. The experimental value is close to unity [1]:

$$\rho_0 = 1.00038 \pm 0.00020,$$

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so CS must be close to an exact symmetry of nature. This has led to the construction of possible extensions of the scalar sector of the SM that respect CS. The Two-Higgs doublet model (2HDM) [2] is a well-known example. It is also possible to include larger multiplets of scalars in a way that CS is respected. The most famous example of this genre is the Georgi-Machacek (GM) model [3], where a complex triplet and a real triplet of scalars engender $\rho = 1$ at the tree-level. Effects and consequences of CS in 2HDM [4,5], multi-Higgs doublet models [6], and GM model [7–10] have been discussed in the literature. Examples of more CS-conserving models may be found in Ref. [11].

When we talk about symmetry, it is necessary to specify its nature. While the consequence of the CS is to keep $\rho_0 = 1$, one may like to know whether this is a symmetry of the entire action, or of only the gauge-covariant kinetic terms. For the first case, the members of every scalar multiplet under the custodial SU(2) should be degenerate in mass, which is not true for the second case. It is also possible to have a model where CS applies to a small part of the Lagrangian that includes the gauge boson mass terms so that the relation $\rho_0 = 1$ is satisfied, but the scalar fields cannot be grouped into CS multiplets. These are the cases that we will discuss and give examples.

We will, in particular, focus on the GM model. After discussing the nature of CS in the GM model, we will show that this is not the most general model with two triplets keeping $\rho_0 = 1$. We construct the most general model, which we call the extended Georgi-Machacek (eGM) model, which involves three more parameters in the scalar sector.

The eGM model, naturally, has a richer phenomenology than the GM model. In the second half of the paper, we point out some consequences of the eGM model. The two major features that may be interesting for collider physicists are:

- The scalar multiplets are no longer degenerate and therefore new decay modes for the scalars open up;
- The singly charged scalars all develop a nonzero doublet component and hence can decay to a fermion-antifermion pair.

This paper, thus, is organised as follows. In Section 2 we explain the meaning of the custodial symmetry (CS) and carefully investigate the link between $\rho_0 = 1$ and CS, focussing on the SM, 2HDM, and the GM model. In Section 3, we propose and describe the extended GM model, with special attention to the scalar potential and the scalar mass spectrum. Section 4 enlists the possible theoretical and experimental constraints on the eGM model. In Section 5 we provide a cursory look at the phenomenological aspects of this model, highlighting the possible signatures that may differentiate this from the conventional GM model, and accommodating hints of several new scalars within the eGM framework. Section 6 summarizes and concludes the paper.

2 Custodial Symmetry

First, let us define what custodial symmetry (CS) means, and then we will see how to ensure CS, as well as how the scalar multiplets should behave under any such CS. We will go through this in a bit detail as there have been confusing statements about CS in the literature, e.g., whether its role is limited to the gauge boson masses, or whether it is a symmetry of the scalar sector too.
2.1 The definition and the ingredients

The CS should be defined in the context of an SU(2) gauge theory. It is well-known that, if such a theory is spontaneously broken by the vacuum expectation value (VEV) of a complex scalar doublet, the three gauge bosons acquire equal mass. We can say that the gauge boson mass terms possess a global SU(2) symmetry under which the three gauge bosons form a triplet. It is trivial to see that the pure gauge terms are also invariant under this symmetry. This is called the custodial symmetry. Inspired by this example, we formulate the following general definition:

If an SU(2) gauge symmetry undergoes spontaneous symmetry breaking (SSB) in a way that the neutral and the charged gauge bosons remain mass-degenerate at the tree-level, there is an SU(2) symmetry in the gauge boson mass terms. The same global symmetry is present in the pure gauge terms even in the broken phase. This symmetry will be called the custodial symmetry.

If we talk about the SM gauge group SU(2) × U(1), there will be an extra neutral gauge boson $B$, in addition to the $W_0$ that comes from SU(2).\footnote{The electric charges will be shown as subscripts.} CS would imply that the $W_0-W_0$ term in the neutral gauge boson mass matrix will be equal to mass-squared value of the charged gauge bosons. The other two independent elements of the matrix are fixed \cite{12} by demanding that the matrix should have one null eigenvalue, and by calling the angle in the diagonalizing matrix $\theta_W$. These conditions imply that the non-zero mass eigenvalues would satisfy the relation

$$\rho_{\text{tree}} = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1.$$  \hspace{1em} (2)

From now on, we will talk only about tree-level expressions and relations, unless mentioned otherwise explicitly. Thus, $\rho_0$ and $\rho_{\text{tree}}$ become identical, and we will denote this by $\rho$. There is a reason for confining ourselves to the tree-level results; CS for the GM model holds only at the tree-level when U(1)$_Y$ is gauged \cite{7}. We will, therefore, confine ourselves only to the SU(2) part of the electroweak interaction, and the consequence of CS will be the mass degeneracy of the three weak gauge bosons.

Our definition of the CS may be used to test the presence of CS in extended models as well. Of course any extension of the SM will not satisfy the mass relation of Eq. (2). However, among the models that do, at the tree level at least, there are different kinds, depending on exactly which parts of the Lagrangian after SSB respect the CS. This is the main focus of the present Section.

First, let us write down the condition that ensures $\rho = 1$. Suppose we have a number of scalar multiplets, to be denoted collectively as $\Phi$. The dimensionality of each $\Phi$ is $2T + 1$, and the component that obtains a VEV equal to $v$ has $T_3$-value equal to $n$. Then, with an SU(2) theory, one obtains the following masses of the gauge bosons:\footnote{In this convention, for SM one obtains $M_W = gv/\sqrt{2}$, so that $v$ is 173 GeV.}

$$M_{W_\pm}^2 = g^2 \sum c(v^2 + a^2)v^2,$$  \hspace{1em} (3a)

$$M_{W_0}^2 = 2g^2 \sum cn^2v^2,$$  \hspace{1em} (3b)

where the sum is over all scalar multiplets present in the theory, with the complexity factor

$$c = \begin{cases} 
1 & \text{if the multiplet is complex}, \\
\frac{1}{2} & \text{if the multiplet is real}. 
\end{cases} \hspace{1em} (4)$$
The quantities $a_{\pm}$ that appear in Eq. (3a) are given by

$$a_{\pm} = \sqrt{\frac{1}{2}[T(T+1) - n(n \pm 1)]}.$$  \hspace{1cm} (5)

There are a couple of points associated with Eq. (3) that are worth mentioning here. First, it is obvious that the two terms in Eq. (3a) come from $|gW_{-T}(\Phi)|^2$ and $|gW_{T}(\Phi)|^2$. The extra factor of $\frac{1}{2}$ that appears within the square root sign of Eq. (5) occurs because of our definition of $T_{\pm}$ given in Eqs. (A.7) and (A.8), which differ by a factor from what is encountered in usual textbooks of quantum mechanics. Thus, adding the two terms, we can write

$$M_{W_{\pm}}^2 = g^2 \sum c[T(T+1) - n^2] v^2.$$  \hspace{1cm} (6)

While either $a_{+}$ or $a_{-}$ vanishes if the VEV occurs in one of the extreme components of the scalar multiplet, i.e., if $n = T$ or $n = -T$, Eq. (6) still remains valid, which can be easily verified by putting $n = \pm T$. Second, the factor $c$ is redundant in Eq. (3b) because real multiplets develop VEV only with $n = 0$ and therefore do not contribute to the $W_0$-mass. So, Eq. (2) will be satisfied if

$$\sum c [T(T+1) - n^2] v^2 = 2 \sum n^2 v^2,$$  \hspace{1cm} (7)

In the literature, there are examples of many models satisfying this criterion [3,11].

Even if the relation of Eq. (2) is realized by a choice of multiplets through Eq. (7), it will not be respected when higher order corrections are included. In particular, it is easy to see that the fermions will not respect CS. The fermion mass terms, obtained after SSB, are bilinear in fields: the left-chiral one being a doublet and the right-chiral one a singlet. This cannot be invariant under an SU(2). Therefore, in the SM, the CS is realised only on the parts of the Lagrangian without the fermions. As mentioned before, the Yukawa terms introduce a deviation of $n_{\text{tree}}$ from unity, that goes mostly as $m_t^2$. In view of this, in the rest of this article we will completely disregard the fermions, i.e., deal only with the scalars and gauge bosons. We will see that even the remaining part of the Lagrangian does not behave in the same way towards CS, and accordingly different kind of situations might arise regarding masses of different particles.

We will stick to only the SU(2) part of the SM gauge group because only this part is relevant for the discussion of CS. Without fermions, the Lagrangian density for a set of scalar multiplets $\{\Phi\}$ can be written as

$$\mathcal{L} = -\frac{1}{4} W_{\mu \nu} W^{\mu \nu} + \mathcal{L}_\text{kin} - V(\{\Phi\}),$$  \hspace{1cm} (8)

where $W_{\mu \nu}$ is the SU(2) gauge field tensor and $V(\{\Phi\})$ is the scalar potential. The gauge-covariant kinetic term of the scalars is

$$\mathcal{L}_\text{kin} = \sum c (D^\mu \Phi)^\dagger (D_\mu \Phi),$$  \hspace{1cm} (9)

where the factor $c$ was defined in Eq. (4), the sum is over all scalar multiplets in the theory, and the gauge-covariant derivative is defined as

$$D_\mu \Phi = \partial_\mu \Phi +igt \cdot W_\mu \equiv \partial_\mu \Phi + iG_\mu \Phi,$$  \hspace{1cm} (10)
being the SU(2) generator matrices in the representation that $\Phi$ belongs to. After SSB, the quantum fields $\Phi_0$ with electric charge equal to zero will need redefinition, for which we will use the notation

$$
\Phi_0 = \begin{cases}
  v + \frac{1}{\sqrt{2}}(\Phi' + i\phi'') & \text{if the multiplet } \Phi \text{ is complex}, \\
  v + \Phi' & \text{if the multiplet } \Phi \text{ is real},
\end{cases}
$$

(11)

Charged fields will not need redefinition.

Clearly, the kinetic term of the gauge bosons is unaffected by SSB, and will therefore be invariant under a global SU(2) symmetry. The $L_{\text{kin}}$ part of the Lagrangian will produce different kinds of terms after SSB:

$$
L_{\text{kin}} = L_{\text{mass}} + L_{\text{quad}} + L_{\text{mixed}} + L_{\text{deriv}} + L_{\text{cubic}} + L_{\text{quartic}}.
$$

(12)

We now explain the notation, mentioning their relationship with the CS.

$L_{\text{mass}}$: These are the mass terms of the gauge bosons. These terms define CS. Thus, we are talking only about models where these terms respect CS.

$L_{\text{quad}}$: These are the kinetic terms of the scalar fields, $c(\partial^{\mu}\Phi)^{\dagger}(\partial_{\mu}\Phi)$. They have the symmetry of the corresponding free field theory. They need not be, and will not be, discussed further.

$L_{\text{mixed}}$: These are the terms which contain one power of a scalar field and one power of a gauge boson field, i.e., are of the form $(\partial^{\mu}\Phi)^{\dagger}(iG_{\mu}\langle\Phi\rangle) + \text{h.c.}$. These are the terms which identify the unphysical Goldstone fields. We will see that these terms are also CS invariant, with suitably defined CS multiplets.

$L_{\text{deriv}}$: These are terms with derivative interactions, of the form $(\partial^{\mu}\Phi)^{\dagger}(iG_{\mu}\langle\Phi\rangle) + \text{h.c.}$, which contain two powers of scalar fields and one power of gauge boson fields.

$L_{\text{cubic}}$: These are non-derivative cubic couplings which are of the form $(G^{\mu}\Phi)^{\dagger}(G_{\mu}v) + \text{h.c.}$

$L_{\text{quartic}}$: These terms contain two powers of gauge fields and two powers of scalar fields, and are of the form $(G^{\mu}\Phi)^{\dagger}(G_{\mu}\Phi)$.

To summarize, the first kind defines CS and the next two kinds respect it for sure. Different types of CS will be distinguished by whether the remaining terms of Eq. (12), as well as the scalar potential, respect CS after SSB. We list three types, with suitable examples.

CS-1: CS is a symmetry of all terms shown in Eq. (8). Therefore, no correction to $\rho$ is generated even in loops, remembering of course that we need to confine ourselves only to the Lagrangian of Eq. (8), disregarding the fermions. We will show later that this is the case for the SM with only one scalar doublet. The canonical GM model is also a CS-1 type model.

CS-2: In this case, all terms in $L_{\text{kin}}$ respect CS after symmetry breaking, but the terms in the scalar potential do not. We will propose a model of this sort in Section 3. The 2HDM, with the most general scalar potential, also falls in this category.

CS-3: In this case, even $L_{\text{kin}}$ is not fully invariant under the CS. We mentioned before that $L_{\text{mass}}, L_{\text{quad}}$ and $L_{\text{mixed}}$ obey CS. However, the rest of the terms do not.
2.2 Some general considerations

The unphysical Goldstone bosons can be identified by the bilinear terms containing one scalar field and one gauge boson field, which arise from $\mathcal{L}_{\text{kin}}$ after SSB. More precisely, after SSB there will be terms of the form

$$\mathcal{L}_{\text{mixed}} = iM_+ W^\mu_+ \partial_\mu h_- - iM_- W^\mu_- \partial_\mu h_+ - M_0 W^\mu_0 \partial_\mu h_0,$$

(13)

where $h_+$ and $h_0$ are the unphysical Goldstone fields. We will assume, everywhere in this article, that all VEVs are real. Then,

$$h_+ = N \sum c v \left( a_+ \Phi_{(n+1)} - a_- \Phi^*_{(n-1)} \right),$$

(14a)

$$h_0 = \sqrt{2} N \sum n v \Phi_0^\prime,$$

(14b)

$$h_- = h^*_+.$$ (14c)

where $\Phi_{(n+1)}$ and $\Phi_{(n-1)}$ denote the components of the multiplet $\Phi$ just above and below the neutral component (with $T_3$ eigenvalue equal to $n$), which have electric charges $+1$ and $-1$ respectively. If the neutral component of a multiplet happens to be an extreme one, then one of the terms in Eq. (14a) would be zero, and comments made after Eq. (6) apply for that case. The normalization constant $N$ in Eq. (14) is obvious, and equal in the two equations owing to the CS relation, Eq. (7). Note that $h_0$ does not contain the neutral component of any real multiplet, since these components have $n = 0$ and therefore their VEVs do not break the neutral generator.

We are considering models with CS in which all the $W$-masses are equal and the three $W$’s transform as a triplet, with components

$$W = (-W_+, W_0, W_-).$$ (15)

If we now define the unphysical Goldstone fields as part of a triplet

$$h = (i h_+, h_0, i h_-),$$ (16)

then it is trivial to see that Eq. (13) can be written as

$$\mathcal{L}_{\text{mixed}} = -M_W W \cdot \partial h.$$ (17)

Here and henceforth, we employ the following notation for combinations of two CS multiplets denoted by $a$ and $b$:

$$a \cdot b : \text{ singlet combination;}$$

$$a \times b : \text{ triplet combination;}$$

$$a \otimes b : \text{ 5-plet combination.}$$ (18)

The expression in Eq. (17) therefore proves that $\mathcal{L}_{\text{mixed}}$ is always invariant under CS, an assertion that was made earlier. Therefore, among the terms in Eq. (12), we need to check only the status of $\mathcal{L}_{\text{deriv}}, \mathcal{L}_{\text{cubic}}$ and $\mathcal{L}_{\text{quartic}}$.

A couple of points can be made irrespective of the scalar multiple content. First, the fields associated with the VEVs must be singlets of CS, since the VEVs are so. These are the real parts of the neutral field in any multiplet. Second, suppose we have a collection of scalar multiplets in which the highest electric charge in any component is $Q_{\text{max}}$. Then it must be a part of a CS multiplet of dimension $2Q_{\text{max}} + 1$. Numbers of other CS multiplets can be determined by counting the number of fields of different charges at our disposal.

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3The minus sign in one of the components in Eq. (15), as well as the factors of $i$ in Eq. (16), and similar such factors in the definitions of various SU(2) multiplets later in the text, owe their origin to the spherical basis employed. For details, see Appendix A.
2.3 CS in the Standard Model

Let us first discuss the well-known case of the SM for the groundwork. In the SM, there is only one Higgs multiplet. It is a doublet of SU(2), and will be denoted by $\phi$:

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}. \quad (19)$$

From Eq. (14), we see that the unphysical Higgs fields are

$$h_\pm = \phi_\pm, \quad h_0 = \phi''_0. \quad (20)$$

The only left-over field must be a singlet of the CS. Let us call it by the suggestive name

$$S = \phi'_0. \quad (21)$$

It is now easy to see that, since the VEV is in the direction of $\phi'_0$ in the field space, there is an SO(3) symmetry in the remaining directions. This means that, in the symmetry broken state,

$$\phi^\dagger \phi = \phi_\phi + \phi_0^2 + \frac{1}{2} (\phi''_0)^2 = \frac{1}{2} \left[ h \cdot h + (S + \sqrt{2}v)^2 \right]. \quad (22)$$

This shows that $\phi^\dagger \phi$, and hence the scalar potential, is CS invariant.

We now look at the terms in Eq. (12). As explained before, only the last three kinds of terms need to be examined. It is easy to see that one can write

$$\mathcal{L}_{\text{deriv}} = \frac{1}{2} g \left[ -SW \cdot \partial h + (\partial S)W \cdot h - iW \cdot (h \times \partial h) \right], \quad (23)$$

$$\mathcal{L}_{\text{cubic}} = \frac{1}{2 \sqrt{2}} g^2 vSW \cdot W. \quad (24)$$

The form written here makes it obvious that this part is invariant under the global CS. Similarly, the quartic terms can be written as

$$\mathcal{L}_{\text{quartic}} = \frac{1}{8} g^2 (W \cdot W) \left[ h \cdot h + S^2 \right], \quad (25)$$

which is also invariant under CS. Thus, the SM action, even after SSB, is CS invariant if we neglect the fermions, and it falls under the CS-1 category.

2.4 CS in 2HDMs

We now extend our discussion to 2HDM involving two scalar doublets $\phi_1$ and $\phi_2$, with real VEVs $v_1$ and $v_2$ respectively. The unphysical fields are now given by

$$h_\pm = \cos \beta \phi_{1\pm} + \sin \beta \phi_{2\pm}, \quad (26a)$$

$$h_0 = \cos \beta \phi''_{10} + \sin \beta \phi''_{20}, \quad (26b)$$

where $\tan \beta = v_2/v_1$. As described earlier, these fields constitute a triplet of a prospective CS. The orthogonal combinations$^4$,

$$H_\pm = -\sin \beta \phi_{1\pm} + \cos \beta \phi_{2\pm}, \quad (27a)$$

$$H_0 = -\sin \beta \phi''_{10} + \cos \beta \phi''_{20}, \quad (27b)$$

$^4$The neutral component, the CP-odd scalar, is conventionally called $A_0$. We use a different notation to display the multiplet structure in a transparent way.
are then candidates for a triplet of physical Higgs bosons under CS. In addition, there are two singlets of CS, which are some combinations of $\phi'_{10}$ and $\phi'_{20}$. It will be convenient to define the linear combinations

$$
S_h = \cos \beta \phi'_{10} + \sin \beta \phi'_{20},
S_H = -\sin \beta \phi'_{10} + \cos \beta \phi'_{20},
$$

(28)
even though these may not turn out to be the mass eigenstates.

It can be seen easily that all terms in $\mathcal{L}_{\text{kin}}$ are invariant under the CS. In particular, we obtain

$$
\mathcal{L}_{\text{deriv}} = \frac{1}{2} g \left[ -S_h W \cdot \partial h - S_H W \cdot \partial H + (\partial S_h) W \cdot h + (\partial S_H) W \cdot H \
- iW \cdot (h \times \partial h) - iW \cdot (H \times \partial H) \right],
$$

(29)

$$
\mathcal{L}_{\text{cubic}} = \frac{1}{2\sqrt{2}} g^2 \sqrt{v_1^2 + v_2^2} S_h W \cdot W,
$$

(30)

$$
\mathcal{L}_{\text{quartic}} = \frac{1}{8} g^2 (W \cdot W) \left[ h \cdot h + H \cdot H + S_h^2 + S_H^2 \right].
$$

(31)

In order to check what kind of CS is realized in this model, we need to look at the scalar potential. For this, let us first recall the most general gauge invariant scalar potential of this model consistent with a discrete symmetry $\phi_1 \to -\phi_1$ in the quartic sector:

$$
V = m_{11}^2 \phi_1^\dagger \phi_1 + m_{22}^2 \phi_2^\dagger \phi_2 - m_{12}^2 (\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1) + \frac{\lambda_1}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda_2}{2} (\phi_2^\dagger \phi_2)^2 \\
+ \lambda_3 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 + \lambda_4 \phi_1^\dagger \phi_2 \phi_2^\dagger \phi_1 + \frac{\lambda_5}{2} (\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_1)^2,
$$

(32)

This potential has been studied in great detail by many authors [2]. Their results show that the physical states $H_\pm$ and $H_0$ are degenerate if $\lambda_4 = \lambda_5$. Unless that condition is fulfilled, we therefore have a CS-2 type symmetry only.

However, if the condition is fulfilled, the Higgs potential can be written as

$$
V = m_{11}^2 \phi_1^\dagger \phi_1 + m_{22}^2 \phi_2^\dagger \phi_2 - m_{12}^2 (\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1) + \frac{\lambda_1}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda_2}{2} (\phi_2^\dagger \phi_2)^2 \\
+ \lambda_3 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 + \frac{1}{2} \lambda_4 [\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1]^2,
$$

(33)

Let us now write the different combinations appearing here in terms of the fields defined in Eqs. (26), (27) and (28) in the Lagrangian after SSB:

$$
\phi_1^\dagger \phi_1 = \frac{1}{2} \cos^2 \beta \ h \cdot h + \frac{1}{2} \sin^2 \beta \ H \cdot H - \sin \beta \cos \beta \ h \cdot H + \frac{1}{2} (\phi'_{10} + \sqrt{2}v_1)^2
$$

$$
\phi_2^\dagger \phi_2 = \frac{1}{2} \sin^2 \beta \ h \cdot h + \frac{1}{2} \cos^2 \beta \ H \cdot H + \sin \beta \cos \beta \ h \cdot H + \frac{1}{2} (\phi'_{20} + \sqrt{2}v_2)^2
$$

$$
\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1 = \frac{1}{2} \sin 2\beta \ (h \cdot h - H \cdot H) + \cos 2\beta \ h \cdot H + (\phi'_{10} + \sqrt{2}v_1)(\phi'_{20} + \sqrt{2}v_2).
$$

(34)
From these forms, it is obvious that the potential of Eq. (33) is invariant under CS, which is reflected in the mass degeneracy of $H_\pm$ and $H_0$. Therefore, 2HDM, just like SM, is a CS-1 type model, provided there is a discrete symmetry $\phi_1 \rightarrow -\phi_1$ in the potential, and $\lambda_4 = \lambda_5$. If $\lambda_4 \neq \lambda_5$, or if more terms are admitted in the potential by sacrificing the discrete symmetry, the model is of CS-2 type.

2.5 CS in Georgi-Machacek model

In the GM model [3], the scalar sector consists of a complex doublet $\phi$ as in the SM, a real triplet $\xi$ and a complex triplet $\chi$ [10]. The electric charges of component scalar fields are as follows:

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi^+ \\ \chi_+ \\ \chi_0 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_+ \\ \xi_0 \\ -\xi_- \end{pmatrix},$$  \tag{35}

with $\xi_- = \xi_+^*$. Adopting the notation

$$\langle \phi_0 \rangle = v, \quad \langle \chi_0 \rangle = u, \quad \langle \xi_0 \rangle = w,$$  \tag{36}

for the VEVs, one obtains

$$\rho = \frac{v^2 + 2(u^2 + w^2)}{v^2 + 4u^2}.$$  \tag{37}

Thus, $\rho = 1$ in the GM model requires $u = w$.

In order to get into a discussion on CS in the scalar potential and scalar gauge interactions, let us first identify the unphysical Goldstone fields. These are

$$h_\pm = \cos \beta \phi_\pm + \frac{1}{\sqrt{2}} \sin \beta (\chi_\pm + \xi_\pm),$$  \tag{38a}

$$h_0 = \cos \beta \phi_0'' + \sin \beta \chi_0'',$$  \tag{38b}

where

$$\tan \beta = \frac{2u}{v}.$$  \tag{39}

As explained earlier, these three fields must transform like a triplet of CS. There will also be a triplet of physical Higgs bosons, given by [3]

$$H_\pm = -\sin \beta \phi_\pm + \frac{1}{\sqrt{2}} \cos \beta (\chi_\pm + \xi_\pm),$$  \tag{40a}

$$H_0 = -\sin \beta \phi_0'' + \cos \beta \chi_0''.$$  \tag{40b}

The doubly charged scalars will be part of a 5-plet [3]. In our notation, its components are

$$F_{\pm \pm} = \chi_{\pm \pm},$$  \tag{41a}

$$F_\pm = \frac{1}{\sqrt{2}} (\chi_{\pm} - \xi_{\pm}),$$  \tag{41b}

$$F_0 = \frac{1}{\sqrt{3}} (\chi_0' - \sqrt{2} \xi_0').$$  \tag{41c}
There are also two combinations which are singlets under CS:

\[ S_1 = \phi_0, \quad S_2 = \frac{1}{\sqrt{3}}(\sqrt{2} \chi'_0 + \xi'_0). \]  

(42)

There are three multiplets, and yet only two CS singlets. This is because there are only two independent VEVs. Note that the combination \( \chi_0 - \xi_0 \) has zero VEV. Therefore, nothing prevents the real part of \( \chi_0 - \xi_0 \) to be part of a non-trivial CS multiplet. Indeed, \( \text{Re}(\chi_0 - \xi_0) = \frac{1}{\sqrt{2}} \chi'_0 - \xi'_0 \) is the combination that appears as \( F_0 \), the neutral component of the CS 5-plet.

We first show explicitly that CS is obeyed by the entire \( \mathcal{L}_{\text{kin}} \). As argued earlier, the first three kinds of terms on the right side of Eq. (12) are always CS invariant. The other terms can be written as

\[
\mathcal{L}_{\text{deriv}} = -\frac{1}{2} g \cos \beta \left[ S_1 W \cdot \partial h - (\partial S_1) W \cdot h \right] + \frac{1}{2} g \sin \beta \left[ S_1 W \cdot \partial H - (\partial S_1) W \cdot H \right] - \sqrt{\frac{2}{3}} g \sin \beta \left[ S_2 W \cdot \partial h - (\partial S_2) W \cdot h \right] - \sqrt{\frac{2}{3}} g \cos \beta \left[ S_2 W \cdot \partial H - (\partial S_2) W \cdot H \right]
\]

\[
- \frac{i}{\sqrt{2}} g \left[ W \cdot (h \times \partial h + H \times \partial H) \right] + \frac{\sqrt{5}}{6} g \sin \beta W \cdot [F \times \partial h - h \times \partial F] + \frac{\sqrt{5}}{6} g W \cdot (F \times \partial F),
\]

(43)

\[
\mathcal{L}_{\text{cubic}} = \frac{g^2 v}{\sqrt{2}} S_1 W \cdot W + \frac{2 g^2 u}{\sqrt{3}} S_2 W \cdot W + g^2 u \left( W \otimes W \right) \cdot F,
\]

(44)

\[
\mathcal{L}_{\text{quartic}} = g^2 (W \cdot W) \left[ \frac{1}{8} S_1^2 + \frac{1}{3} S_2^2 + \frac{1}{2} \ F \cdot F + \left( \frac{1}{8} \cos^2 \beta + \frac{1}{3} \sin^2 \beta \right) h \cdot h \right]
\]

\[
+ \left( \frac{1}{8} \sin^2 \beta + \frac{1}{3} \cos^2 \beta \right) H \cdot H + \frac{5}{24} \sin 2 \beta h \cdot H \]

\[
+ g^2 (W \otimes W) \cdot \left[ \frac{1}{\sqrt{3}} S_2 F + \frac{\sqrt{2}}{4 \sqrt{3}} F \otimes F + \frac{\sqrt{3}}{4} \sin \beta F \otimes h + \frac{\sqrt{3}}{4} \cos \beta F \otimes H \right] - \frac{1}{4} \sin 2 \beta h \otimes H - \frac{1}{4} \sin^2 \beta h \otimes h - \frac{1}{4} \cos^2 \beta H \otimes H \right].
\]

(45)

where the symbol \( a \otimes b \) was explained in Eq. (18). We follow a certain sign convention for the Clebsch-Gordan coefficients. With a different convention, the overall signs might be different the terms where the combinations have been used.

Let us now turn to the scalar potential. The most general scalar potential that is invariant under \( \text{SU}(2)_L \times \text{U}(1)_Y \) has 16 parameters [10]. We write it down here, with some changes in notation which will be helpful in understanding symmetries of the theory:

\[
V(\phi, \chi, \xi) = -m^2_c \phi^\dagger \phi - m^2_\chi \chi^\dagger \chi - m^2_\xi \xi^\dagger \xi
\]

\[
+ \mu_1 (\chi^\dagger t_\alpha \chi) \xi_\alpha + \mu_2 (\phi^\dagger \tau_\alpha \phi) \xi_\alpha + \mu_3 \left( (\phi^\dagger \epsilon_\tau_\alpha \phi) \bar{\chi}_\alpha + \text{h.c.} \right)
\]

\[
+ \lambda_\phi (\phi^\dagger \phi)^2 + \lambda_\xi (\xi^\dagger \xi)^2 + \lambda_\chi (\chi^\dagger \chi)^2 + \lambda_\bar{\chi} |\bar{\chi}^\dagger \chi|^2
\]

\[
+ \lambda_{\phi \chi} (\phi^\dagger \phi) (\chi^\dagger \chi) + \lambda_{\phi \xi} (\phi^\dagger \phi) (\xi^\dagger \xi) + \lambda_{\chi \xi} (\chi^\dagger \chi) (\xi^\dagger \xi)
\]

\[
+ \kappa_1 |\xi^\dagger \chi|^2 + \kappa_2 (\phi^\dagger t_\alpha \phi) (\chi^\dagger t_\alpha \chi) + \kappa_3 \left( (\phi^\dagger \epsilon_\tau_\alpha \phi) (\xi^\dagger t_\alpha \xi) + \text{h.c.} \right),
\]

(46)
where \[
\tilde{\chi} = \begin{pmatrix} \chi_0^* \\ -\chi^- \\ \chi_{--} \end{pmatrix}, \tag{47}
\]
which also transforms like a triplet of SU(2), with the U(1) charge opposite to that of \(\chi\). The 2-dimensional and 3-dimensional representations of the SU(2) generators are denoted by \(\frac{1}{2}\tau_a\) and \(t_a\) respectively. One may note that these generators are in the spherical basis, as discussed in Appendix A, Eqs. (A.3), (A.7) and (A.8), and all of them are not hermitian. Their hermiticity property is summarized in Eq. (A.9). A comparison with earlier notations, e.g. in Ref. [10], has been presented in Appendix B. Other possible gauge-invariant combinations of four scalar multiplets can be written as linear combinations of those appearing in Eq. (46).

This potential will be useful for us for subsequent discussion, but it does not guarantee equal VEVs of \(\chi\) and \(\xi\) and therefore the \(W\)-mass terms are not degenerate. In order to have a CS-invariant potential, Georgi and Machacek used a curtailed potential with 9 parameters, obtained by putting 7 conditions on the parameters of Eq. (46). We divide these conditions into two categories, for reasons to be explained in Section 3. In the first category, there are four constraints:

\begin{align*}
\mu_2 &= \sqrt{2}\lambda_3, \\
\lambda_\chi &= 2\lambda_\xi + \frac{1}{2}\lambda_{\chi \xi}, \\
\lambda_{\phi \chi} - 2\lambda_{\phi \xi} &= \kappa_2 + \sqrt{2}\kappa_3. 
\end{align*}

The other category has three constraints, which are:

\begin{align*}
\kappa_2 + \sqrt{2}\kappa_3 &= 0, \\
\tilde{\lambda}_\chi = \frac{1}{2}\kappa_1 &= 2\lambda_\xi - \frac{1}{2}\lambda_{\chi \xi}. 
\end{align*}

With these identifications, the GM potential can be written in the form

\[
V = \frac{1}{2}m_\xi^2 \text{Tr}(\Phi^\dagger \Phi) + \frac{1}{2}m_3^2 \text{Tr}(X^\dagger X) \\
- M_1 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau_b) X_{ab} - M_2 \text{Tr}(X^\dagger t^a_b X t^b_a) X_{ab} \\
+ \lambda_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \lambda_2 (\text{Tr} X^\dagger X)^2 + \lambda_3 \text{Tr}(X^\dagger X X^\dagger X) \\
+ \lambda_4 (\text{Tr} \Phi^\dagger \Phi) \text{Tr}(X^\dagger X) - \lambda_5 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau_b) \text{Tr}(X^\dagger t^a_b X t^b_a), \tag{50}
\]

where \(\Phi\) and \(X\) are two matrices defined as

\[
\Phi = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_- & \phi_0 \end{pmatrix}, \\
X = \begin{pmatrix} \chi_0^* & \xi_+ & \chi_{++} \\ -\chi_- & \xi_0 & \chi_+ \\ \chi_{--} & -\xi_- & \chi_0 \end{pmatrix}. \tag{51}
\]

The parameters of Eq. (50) can easily be identified in terms of the parameters of Eq. (46) subject to the conditions of Eqs. (48) and (49). Such correspondences have been shown in Appendix B.
The question of CS then boils down to checking the CS invariance of the combinations of fields that appear in Eq. (50). For example, by inverting Eqs. (38), (40) and (41), it is straightforward to see that
\[
\phi^\dagger \phi = \frac{1}{2} \cos^2 \beta \mathbf{h} \cdot \mathbf{h} + \frac{1}{2} \sin^2 \beta \mathbf{H} \cdot \mathbf{H} - \sin \beta \cos \beta \mathbf{h} \cdot \mathbf{H} + \frac{1}{2} (S_1 + \sqrt{2} v)^2 ,
\]
which is CS invariant. As for the other bilinear terms, we find
\[
\begin{align*}
\chi^\dagger \chi & = F^{++} F^{--} + \frac{1}{2} \left| \sin \beta h_+ + \cos \beta H_+ + F_+ \right|^2 \\
& + \frac{1}{2} \left( \sin \beta h_0 + \cos \beta H_0 \right)^2 + \frac{1}{2} \left( \sqrt{\frac{1}{3}} F_0 + \sqrt{\frac{2}{3}} S_2 + \sqrt{2} u \right)^2 , \\
\xi^\dagger \xi & = \left| \sin \beta h_+ + \cos \beta H_+ - F_+ \right|^2 + \left( - \sqrt{\frac{2}{3}} F_0 + \sqrt{\frac{1}{3}} S_2 + u \right)^2 .
\end{align*}
\]
None of these expressions is CS invariant. However, the combination that appears in Eq. (50) is
\[
\chi^\dagger \chi + \frac{1}{2} \xi^\dagger \xi = \frac{1}{2} \left[ \mathbf{F} \cdot \mathbf{F} + \sin^2 \beta \mathbf{h} \cdot \mathbf{h} + \cos^2 \beta \mathbf{H} \cdot \mathbf{H} + \sin 2 \beta \mathbf{h} \cdot \mathbf{H} + S_2^2 \right],
\]
leaving out constant terms which are irrelevant and terms linear in the VEV, since they vanish through the potential minimization condition. This is obviously CS invariant. Similarly CS invariance can be checked for all other terms in the potential. The GM model is therefore of type CS-1.

2.6 CS-3 type models

Georgi and Machacek argued that, as long as the bosonic Lagrangian is invariant under an SU(2)_L ∋ SU(2)_R symmetry, and all scalar multiplets belong to \((N, N)\) representations for some \(N\), there will be a custodial symmetry. In fact, this symmetry can be realized in the CS-1 mode. We have seen that the scalar potential might spoil this symmetry, forcing the model to be of the CS-2 variety, as it happens for 2HDM with the most general gauge invariant potential.

As a matter of curiosity, we show here that CS-3 type models are also possible. In the SM, only one Higgs multiplet is needed to ensure CS. There are higher representations where also just one multiplet suffices. One example is a 7-plet with \(T = 3\) and \(n = -2\), whose components are
\[
\Sigma = (\Sigma_5, \Sigma_4, \Sigma_3, \Sigma_2, \Sigma_1, \Sigma_0, \Sigma_-) ,
\]
where the subscripts indicate the electric charges. Note that here \(\Sigma_-\) is not the charge conjugate of \(\Sigma_1\): they are independent complex fields. It is easily seen that a VEV of \(\Sigma_0\) would satisfy the condition of Eq. (7), which means that \(\mathcal{L}_{\text{mass}}\) will have a CS. So will \(\mathcal{L}_{\text{mixed}}\), as has been argued on general terms. The unphysical Goldstone multiplet appearing in \(\mathcal{L}_{\text{mixed}}\) will involve 3 real fields out of the 14 that appear in Eq. (55). The real part of \(\Sigma_0\) must be a CS-singlet. But \(\Sigma_5\) will have to be part of an 11-plet under CS, and there are not enough fields left after setting aside the multiplets mentioned already. Therefore, we cannot have CS in other terms of \(\mathcal{L}_{\text{kin}}\). The same conclusion holds if the scalar sector holds only a single multiplet higher than doublet, as there will not be enough neutral fields to complete the CS multiplet.

Of course this multiplet is phenomenologically not interesting since it cannot couple to fermions. However, even after adding a doublet, the CS in \(\mathcal{L}_{\text{mass}}\) is maintained, but there is no SU(2)_L ∋ SU(2)_R symmetry in the Lagrangian.
3 The extended Georgi-Machacek Model

The GM model, obviously, implies degeneracy among the components of the different multiplets of physical Higgs bosons. In particular, the masses of $F_{++}$, $F_+$ and $F_0$ turn out to be equal, and so do the masses of $H_+$ and $H_0$, because of the CS.

However, one can have the CS in the CS-2 mode with the particle content of the GM model. This means $\rho = 1$ but no degeneracy in the scalar masses. To our knowledge, this point has not been noticed in the earlier literature. This model will be called the extended Georgi-Machacek model (eGM), and this is the most general model with one SU(2) doublet, one real triplet, and one complex triplet scalar leading to $\rho = 1$. In the rest of the paper, we present this model and discuss its phenomenological consequences.

3.1 Requirements of the model

To understand the existence of such a scenario, we start from the most general gauge invariant potential given in Eq. (46). With the notation for the VEVs given in Eq. (36) and the assumption that they are all real, the minimization conditions of the potential are as follows:

\[ m_\phi^2 = -\mu_2 w - 2\sqrt{2}\mu_3 u + 2\lambda_\phi v^2 + (\lambda_\chi - \kappa_2) u^2 + \lambda_\phi\xi w^2 + 2\sqrt{2}\kappa_3 uv, \]

\[ um_\chi^2 = \mu_1 uw - \sqrt{2}\mu_3 v^2 + 2\lambda_\chi u^2 + (\lambda_\phi - \kappa_2) v^2 u + \lambda_\chi w^2 u + \sqrt{2}\kappa_3 v^2 w, \]

\[ 2wm_\xi^2 = \mu_1 u^2 - \mu_2 v^2 + 4\lambda_\chi w^2 + 2\lambda_\phi\xi v^2 w + 2\lambda_\chi u^2 w + 2\sqrt{2}\kappa_3 v^2 u. \]

As said before, we need $u = w$ in order to obtain $\rho = 1$. Putting $u = w$ in these equations and equating the coefficients of $u$, $w^3$, $v^2$ and $w^2$, we obtain the necessary conditions for obtaining the equality of $u$ and $w$. It turns out that these are just the conditions given in Eq. (48). The other conditions, shown in Eq. (49), are not really necessary for obtaining $\rho = 1$.

We can therefore consider a model with the scalar content of the GM model, but where the most general Higgs potential is constrained only by the conditions of Eq. (48). This is what we call the “extended GM model” (eGM), and discuss in some detail. This has 12 free parameters in the scalar potential, compared to 9 for the GM model, as the three constraint conditions in Eq. (49) are not used. This is a CS-2 type model. Since the gauge sector is the same as that of GM, it is obvious that $L_{\text{kin}}$ is invariant under CS. However, the scalar potential does not respect the symmetry.

From a purely gauge theoretical point of view, there is not much difference in the symmetries of CS-1, CS-2 or CS-3 type models. In all cases, the symmetry is not obeyed by the entire Lagrangian with a realistic fermion spectrum. Thus, the symmetry is realized in a limited sector of the theory, and is broken by loop corrections in all cases. Since the major consequence of CS, the mass relation of Eq. (2), is found to be consistent with experiments, one should explore different scenarios in which this relation can be obtained. This is what we do in the rest of this paper.

3.2 Mass terms in the general potential

To understand what kind of masses will be observed for the physical Higgs bosons, it is necessary to have a discussion of the masses that arise from the most general gauge invariant potential, Eq. (46), assuming no equality among the VEVs.
First, we give the mass of the doubly charged scalar:

$$M_{++}^2 = -2\mu_1 w + \sqrt{2(\mu_3 - \kappa_3 w) v^2 / u} + 4\lambda \chi u^2 + 2\kappa_2 v^2. \quad (57)$$

Next comes the mass squared matrix of the singly charged fields, in the basis $\phi_+ - \chi_+ - \xi_+$:

$$M_{++}^2 = \begin{pmatrix}
2\mu_2 w + 2\sqrt{2(\mu_3 - \kappa_3 w) u} & -2\mu_3 v - \sqrt{2\kappa_2 w} & -\sqrt{2}\mu_2 v + 2\kappa_3 w \\
-2\mu_3 v - \sqrt{2}\kappa_2 w & -\mu_1 w + \sqrt{2(\mu_3 - \kappa_3 w) v^2 / u} + \kappa_1 w^2 + \kappa_2 v^2 & \mu_1 u - \kappa_1 u w + \sqrt{2}\kappa_3 v^2 \\
-\sqrt{2}\mu_2 v + 2\kappa_3 w & \mu_1 u - \kappa_1 u w + \sqrt{2}\kappa_3 v^2 & \kappa_1 u^2 + \frac{1}{w}(-\mu_1 u^2 + \mu_2 v^2 - 2\sqrt{2}\kappa_3 v^2 u)
\end{pmatrix} \quad (58)$$

There are some instructive features of this mass matrix, which we describe now.

1. There is a zero eigenvalue. The corresponding eigenvector is proportional to $v\phi_+ + \sqrt{2}u\chi_+ + \sqrt{2}w\xi_+$. This is the unphysical charged Goldstone boson. Note that in the CS limit this combination reduces to the $h_+$ shown in Eq. (38).

2. If all the $\mu$-type and $\kappa$-type couplings are zero, $M_{++}^2 = 0$, i.e., there are 3 singly-charged massless scalars. This is expected since the potential in this case has a $[SU(2) \times U(1)]^3$ symmetry corresponding to separate transformations on the $\phi$, the $\chi$, and the $\xi$, and the VEVs break all of them. In fact, this is why our notation for using different letters for various quartic couplings is helpful: the masses in this sector do not depend on the $\lambda$-type couplings, as seen in Eq. (58).

3. Suppose only $\mu_1$ and $\kappa_1$ are nonzero among all $\mu$-type and $\kappa$-type couplings. In this case, the potential has an $[SU(2) \times U(1)]$ symmetry corresponding to the $\phi$ field only, and another $[SU(2) \times U(1)]$ encompassing $\chi$ and $\xi$. Sure enough, the mass matrix reduces to

$$M_{++}^2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\mu_1 w + \kappa_1 w^2 & \mu_1 u - \kappa_1 u w \\
0 & \mu_1 u - \kappa_1 u w & \kappa_1 u^2 - \mu_1 u^2 / w
\end{pmatrix}, \quad (59)$$

which has two zero modes, $\phi_+$ and $u\chi_+ + w\xi_+$.

4. Now suppose, among the $\mu$-type and $\kappa$-type couplings, only $\mu_3$ and $\kappa_2$ are non-zero. Now there is an independent $SU(2)$ for $\xi$, and

$$M_{++}^2 = \begin{pmatrix}
2\sqrt{2}\mu_3 u + 2\kappa_2 u^2 & -2\mu_3 v - \sqrt{2}\kappa_2 w & 0 \\
-2\mu_3 v - \sqrt{2}\kappa_2 w & 2\sqrt{2}\mu_3 v^2 / u + \kappa_2 v^2 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (60)$$

5. Similarly, if only $\mu_2$ is non-zero, $\chi_+$ will be massless, and so will be the combination $v\phi_+ + \sqrt{2}w\xi_+$.
There are two CP-odd neutral scalar fields: $\phi_0''$ and $\chi_0''$. Their mass matrix, choosing the basis in that order, is:

$$M_{\text{odd}}^2 = \sqrt{2}(\mu_3 - \kappa_3 w) \begin{bmatrix} 4u & -2v \\ -2v & v^2/u \end{bmatrix}$$  \hspace{1cm} (61)$$

Since real fields are involved here, the actual terms in the Lagrangian come with a factor of 1/2. The zero eigenmode is $v\phi_0'' + 2u\chi_0''$, which is the unphysical Goldstone mode shown in Eq. (38b). In various limits discussed in context of the singly charged scalar matrix, the behaviour of this matrix also follows the pattern expected from symmetry.

Finally, there are the CP-even neutral fields, the real parts of the neutral component of each multiplet. With the general 16-parameter potential and different VEVs as indicated in Eq. (36), these terms are as follows, in the basis $\phi_0' - \chi_0' - \xi_0'$:

$$M_{\text{even}}^2 = \begin{bmatrix} 4\lambda_\phi v^2 & 2\sqrt{2}(\mu_3 + \kappa_3 w)v & -\sqrt{2}\mu_2 v + 4\kappa_3 uv \\ 2\sqrt{2}(\mu_3 + \kappa_3 w)v & +2(-\kappa_2 + \lambda_\phi\chi)uv & +2\sqrt{2}\lambda_\phi\xi w \\ -\sqrt{2}\mu_2 v + 4\kappa_3 uv & +2\sqrt{2}\lambda_\phi\xi w & +8\lambda_\xi w^2 \end{bmatrix}.$$  \hspace{1cm} (62)$$

We will discuss the masses that come out of these expressions after putting the constraints applicable to the eGM model.

3.3 Masses in the eGM model

In the eGM model, we impose the four constraints of Eq. (48), which ensure $u = w$, as shown earlier. We will henceforth denote the physical states by a prime, so for the doubly charged scalar, $F_{++} = F_{++}'$. Its mass is obtained from Eq. (57) by putting the equality of the two VEVs:

$$M_{++}^2 = -2\mu_1 u + \sqrt{2}\mu_3 v^2 / u + 4\lambda_\chi u^2 + (2\kappa_2 - \sqrt{2}\kappa_3)v^2.$$  \hspace{1cm} (63)$$

The mass matrix of the singly charged scalars was given in Eq. (58), where we now put $u = w$. In the basis spanned by $h_+ - H_+ - F_+$ defined in Eqs. (38), (40) and (41), the mass matrix reduces to

$$M_{+}^2 = \frac{\sqrt{2}}{\sin 2\beta} \begin{bmatrix} 0 & 0 \\ 0 & (A - C) \\ 0 & (A + C) \cos \beta \end{bmatrix} + \sqrt{2}B \sin 2\beta,$$  \hspace{1cm} (64)$$

where $\tan \beta = 2u/v$ as defined earlier, and we have used the shorthands

$$A = 2\mu_3 v + \sqrt{2}\kappa_2 uv,$$

$$B = -\mu_1 u + \kappa_1 u^2 - \sqrt{2}\kappa_3 v^2,$$

$$C = -2\mu_3 v + 2\kappa_3 uw.$$  \hspace{1cm} (65a,b,c)
The zeroes in the first row and the first column are not surprising: they just confirm that \( h_+ \) is indeed a massless mode. The other two eigenvectors are

\[
\begin{align*}
H_+ &= \cos \theta_+ H_+ - \sin \theta_+ F_+ , \\
F'_+ &= \sin \theta_+ H_+ + \cos \theta_+ F_+ ,
\end{align*}
\]  

(66)

with

\[
\tan 2\theta_+ = \frac{2(A + C) \cos \beta}{\sqrt{2B \sin 2\beta - (A - C) \sin^2 \beta}}.
\]  

(67)

We now discuss the neutral fields. There is only one CP-odd neutral scalar, which we call \( H'_0 \). It is easily seen, from Eq. (61), that this is the combination proportional to \( 2u\phi''_0 - v\chi''_0 \), and its mass is given by

\[
M^2_{\text{odd}} = \sqrt{2(\mu_3 - \kappa_3 u)(4u^2 + v^2)}/u ,
\]  

(68)

the only non-zero eigenvalue of the matrix.

It is instructive to see the GM limit from these results. Note that Eq. (65) tells us that \( A + C = (\sqrt{2}\kappa_2 + 2\kappa_3)uv \), which vanishes in the GM limit, as seen from Eq. (49a). Thus \( \theta_+ = 0 \), implying that \( H_+ \) and \( F_+ \) are truly the mass eigenstates. Moreover, note that in this limit, the mass square of \( H_+ \) is the middle diagonal element of the matrix of Eq. (64), which is

\[
\sqrt{2} \sin 2\beta (A - C) = \sqrt{2(\mu_3 - \kappa_3 u)(4u^2 + v^2)}/u .
\]  

(69)

This is equal to the value that appears in Eq. (68), which is expected since \( H_+ \) and \( H_0 \) form a CS triplet. However, in the eGM, this equality is not maintained, because of extra terms that appear for \( \theta_+ \neq 0 \).

Similarly, we note from Eq. (64) that \( (M^2_{\text{even}})_{33} \) is given by

\[
\frac{\cot \beta}{\sqrt{2}} (A - C) + 2B = -2\mu_1 u + \sqrt{2}\mu_3 v^2/u + 2\kappa_1 u^2 + \frac{1}{2}(\kappa_2 - 5\sqrt{2}\kappa_3)v^2 .
\]  

(70)

If we put the extra GM conditions given in Eq. (49), this expression equals the expression given in Eq. (63) for the doubly charged Higgs boson, which happens because in GM model the fields are part of a degenerate 5-plet.

Finally, let us discuss the masses of the CP-even neutral scalars. In the basis \( \phi'_0 - \chi'_0 - \xi'_0 \) (which are defined in Eq. (11), and not to be confused with the primed mass eigenstates), the mass terms appearing in the Lagrangian can be written in the form of the matrix

\[
M^2_{\text{even}} = \begin{bmatrix}
4\lambda\phi v^2 & -2\sqrt{2}\mu_3 v & -2\mu_3 v \\
-2\sqrt{2}\mu_3 v & +4(\lambda\phi + \sqrt{2}\kappa_3)uv & -(\sqrt{2}\mu_3 v + 2\lambda\phi + 4\kappa_3)uv \\
+4(\lambda\phi + \sqrt{2}\kappa_3)uv & 4\lambda\chi u^2 + (\mu_3 - \kappa_3 u)\sqrt{2}v^2 / u & \sqrt{2}\mu_1 u + 2\kappa_3 v^2 & +4\sqrt{2}(\lambda\chi - 2\lambda\xi)u^2 \\
-2\mu_3 v & \sqrt{2}\mu_1 u + 2\kappa_3 v^2 & -\mu_1 u + \mu^2 v / u & -\mu_1 u + \mu_2 v^2 / u \\
+(2\sqrt{2}\lambda \phi + 4\kappa_3)uv & +4\sqrt{2}(\lambda\chi - 2\lambda\xi)u^2 & +8\lambda\xi u^2 - 2\sqrt{2}\kappa_3 v^2 & \end{bmatrix}
\]  

(71)
We have put the restriction \( u = w \), as well as the constraints shown in Eq. (48) that are necessary for obtaining the equality of the VEVs, thereby eliminating the couplings \( \mu_2, \kappa_2, \) and \( \lambda_\chi \xi \). But we have not used the conditions of Eq. (49).

It can be easily seen that the combination \( F_0 \), shown in Eq. (41), is an eigenvector of this matrix, and the eigenvalue is

\[
M_{F_0}^2 = -2\mu_1 u + \sqrt{2}\mu_3 \frac{v^2}{u} + 4(4\lambda_\xi - \lambda_\chi)u^2 - 3\sqrt{2}\kappa_3 v^2,
\]

so that the mass of \( F_0 \), in the GM limit with Eq. (49), is the same to that of \( F_+ \) and \( F_{++} = F_{+++} \), confirming the degeneracy of the 5-plet of CS for the GM model.

It is important to note that there was no need for using the constraints of Eq. (49) in order to show that \( F_0 \) is an eigenstate of the mass matrix. It means that this conclusion is valid for eGM as well, in presence of some extra terms in the potential which were not present in the GM model. The physics of the eGM model can therefore be better understood if we write the matrix of Eq. (71) in the basis \( S_1 - S_2 - F_0 \), defined in Eqs. (41) and (42). We impose the conditions of Eq. (48) on the couplings which are necessary for obtaining \( \rho = 1 \), and the resulting matrix is

\[
M_{\text{even}}^2 = \begin{bmatrix}
4\lambda_\phi v^2 & -2\sqrt{3}\mu_3 v + 2\sqrt{6}\lambda_\phi \xi uv & 0 \\
-2\sqrt{3}\mu_3 v + 2\sqrt{6}\lambda_\phi \xi uv & \mu_1 u + \sqrt{2}\mu_3 \frac{v^2}{u} & 0 \\
+4\sqrt{3}\kappa_3 uv & -8\lambda_\xi u^2 + 8\lambda_\chi u^2 & -2\mu_1 u + \sqrt{2}\mu_3 \frac{v^2}{u} \\
0 & 0 & +4(4\lambda_\xi - \lambda_\chi)u^2 - 3\sqrt{2}\kappa_3 v^2
\end{bmatrix}.
\]  

(73)

So the eigenstates of this matrix may be called \( S_1' \), \( S_2' \) and \( F_0' = F_0 \), where the first two are combinations of the CS singlets shown in Eq. (42):

\[
\begin{pmatrix}
S_1' \\
S_2'
\end{pmatrix} = \begin{pmatrix}
\cos \theta_0 & -\sin \theta_0 \\
\sin \theta_0 & \cos \theta_0
\end{pmatrix} \begin{pmatrix}
S_1 \\
S_2
\end{pmatrix},
\]

(74)

where

\[
\tan 2\theta_0 = \frac{2(-2\sqrt{3}\mu_3 v + 2\sqrt{6}\lambda_\phi \xi uv + 4\sqrt{3}\kappa_3 uv)}{\mu_1 u + \sqrt{2}\mu_3 \frac{v^2}{u} - 8\lambda_\xi u^2 + 8\lambda_\chi u^2 - 4\lambda_\phi v^2}.
\]

(75)

For small values of the mixing angle \( \theta_0 \), \( S_1' \) is dominantly the doublet and we expect its mass to be around 125 GeV.

### 4 Constraints on the eGM potential

Once we establish the eGM model as a more general extension of the conventional GM model with \( \rho = 1 \) at the tree-level ensured, one may now look at the possible phenomenological signatures. Similar signatures for a CS-conserving general Higgs sector have been discussed in Ref. [13]. For our case, however, most of the signatures are quite similar to the GM model, like the scalar production through gauge boson fusion mechanism, or \( S_1 \to S_2 V \) and \( S \to V_1 V_2 \) decay channels.
(where $S$ stands for any generic scalar and $V$ for any generic gauge boson). There exist a few notable differences, which we mention here, and plan to explore later in more detail.

To set some benchmark points, we first need to enlist the constraints on the potential. The theoretical constraints are quite standard, and may be summarized as follows:

- The potential must be bounded from below, i.e., there is no direction in the potential space that leads to an unbounded minimum. The constraints, for the most general 16-parameter model given by Eq. (46), are as follows:
  
  \[ \lambda_\phi, \lambda_\xi, \lambda_\chi, \lambda_\chi + \tilde{\lambda}_\chi > 0, \quad (76a) \]
  
  \[ 2\sqrt{\lambda_\phi \lambda_\xi} + \lambda_\phi \xi > 0, \quad (76b) \]
  
  \[ 2\sqrt{\lambda_\phi \lambda_\chi} + \lambda_\phi \chi > |\kappa_2|, \quad (76c) \]
  
  \[ 2\sqrt{\lambda_\phi (\lambda_\chi + \tilde{\lambda}_\chi) + \lambda_\phi \chi > 0}, \quad (76d) \]
  
  \[ \lambda_\chi \xi > -2\sqrt{\lambda_\chi \lambda_\xi} \quad \text{for } \tilde{\lambda}_\chi > 0, \kappa_1 > 0, \quad (76e) \]
  
  \[ \lambda_\chi \xi > -2\sqrt{\lambda_\chi \lambda_\xi} + \frac{1}{2} |\kappa_1| \quad \text{for } \tilde{\lambda}_\chi > 0, \kappa_1 < 0, \quad (76f) \]
  
  \[ \lambda_\chi \xi > -2\sqrt{(\lambda_\chi + \tilde{\lambda}_\chi) \lambda_\xi} \quad \text{for } \tilde{\lambda}_\chi < 0, \kappa_1 > 0, \quad (76g) \]
  
  \[ \lambda_\chi \xi > -2\sqrt{(\lambda_\chi + \tilde{\lambda}_\chi) \lambda_\xi + |\kappa_1|} \quad \text{for } \tilde{\lambda}_\chi < 0, \kappa_1 < 0. \quad (76h) \]

This shows that apart from $\lambda_\phi$, $\lambda_\chi$, and $\lambda_\xi$, none of the quartic couplings need to be positive. The constraint on $\kappa_3$ is more complicated, but once we go to the CS limit by imposing the relations shown in Eq. (48), $\kappa_3$ gets related with $\lambda_\phi \chi$, $\lambda_\phi \xi$, and $\kappa_2$. Also, in the CS limit, one finds $\lambda_\chi > \lambda_\xi$ if $\tilde{\lambda}_\chi, \kappa_1 > 0$, and for all other cases one has to replace $\lambda_\chi \xi$ with $2\lambda_\chi - 4\lambda_\xi$.

- The partial wave amplitudes for any scattering process must not violate the unitarity bounds:
  
  \[ |\text{Re}(a_\ell)| \leq \frac{1}{2}, \quad (77) \]

where $a_\ell$ is the $\ell$-th partial wave amplitude in any channel. This leads to the partial wave unitarity constraints, shown for the conventional GM model in Refs. [14, 15]. While we deviate from the conventional GM model by not imposing the conditions of Eq. (49), quartic couplings with magnitudes of the order of unity are expected to saturate the unitarity bounds.

- The oblique parameters, particularly $S$ and $T$, are within the experimental limits. This is expected, as $\alpha T = \rho - 1$ is already zero at the tree-level, and $S$ receives only logarithmic corrections from scalars.

- One must also ensure that the Landau pole is not hit too soon, or, in other words, none of the couplings blow up at a sufficiently low energy. We would like to demand all the couplings to be well-behaved below 40 TeV. To be rigorous, since we use one-loop renormalization group equations (RGE) to calculate the evolution, the demand is on the perturbative nature of the couplings. The RGEs are taken from Ref. [10] and displayed in Appendix C. For our benchmark points, and indeed over almost the entire parameter space, the potential never becomes unstable with RGE evolution.
• We have also ensured that over the allowed parameter space, all the scalar mass eigenstates are positive.

The experimental bounds on the scalar masses depend on the triplet VEV, and may be found in Refs. [16–18]. However, one may note that these are bounds in the GM model. We discuss below where eGM differs from GM.

## 5 A cursory analysis

In this Section, we would like to highlight some crucial differences between the GM and the eGM models. For this, we choose two benchmark points, which have been dubbed BP-1 and BP-2 in Table 1. The points have been chosen in a way that they satisfy the constraints enlisted in Section 4. While BP-1 has been chosen to highlight the differences between GM and eGM, BP-2 has been chosen in such a way as to reproduce some hints of new scalars found at the LHC, like a neutral scalar at 151 GeV, a pseudoscalar at about 400 GeV, and another heavy scalar at about 660 GeV [19–21]. In addition, we put some extra conditions, based on the following considerations.

1. The lightest CP-even scalar is at 125 GeV, and is dominantly a doublet. To ensure consistency with the LHC data, we demand that the doublet component is at least 90%. Note that triplet admixture suppresses the branching fractions to the fermionic channels, and enhances the same to the di-gauge channels.

2. The model must be valid till, at least, 40 TeV. This is a rather conservative estimate, even a 10 TeV upper limit would have easily passed all the experimental constraints. In other words, none of the couplings should blow up below that energy scale. This ensures that the pure quartic couplings $\lambda_\phi$, $\lambda_\chi$, and $\lambda_\xi$ cannot have large positive values to start with, at the electroweak scale. In fact, for BP-1, the model remains well-behaved well beyond 100 TeV, while for BP-2, the Landau pole is hit just beyond 40 TeV, because of the large values of the couplings that we have started with. We also find, as expected, that the $\rho$ parameter deviates from unity at a high scale (as the VEVs are functions of couplings which are in turn energy-dependent), but that is not a serious concern as it has been and will be measured only at the electroweak scale. The GM model keeps $\rho = 1$ at all energy scales, with the well-known caveat of divergent oblique parameters when the U(1)$_Y$ part is gauged.

3. If the doubly charged scalar $F_{++}'$ is below 1 TeV, the triplet VEV cannot be too large, or that would result in an unacceptably large rate for $F_{++}' \rightarrow W_\pm W_\pm$ at the LHC. We have

| Benchmark | $\mu_1$ | $\mu_3$ | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\lambda_\phi$ |
|-----------|---------|---------|------------|------------|------------|---------------|
| BP-1      | -9.13   | 29.03   | 0.12       | -1.41      | -1.09      | 0.140         |
| BP-2      | -2.56   | 0.55    | 0.50       | 3.80       | -3.10      | 0.134         |

| Benchmark | $\lambda_\chi$ | $\tilde{\lambda}_\chi$ | $\lambda_{\phi\chi}$ | $\lambda_{\chi\xi}$ | $v$ | $u$ |
|-----------|-----------------|------------------------|----------------------|--------------------|-----|-----|
| BP-1      | 0.76            | 1.19                   | 1.78                 | 0.45               | 173 | 10  |
| BP-2      | 1.50            | 0.70                   | 3.85                 | 0.10               | 174 | 1   |

Table 1: The benchmark points, with $\mu_1$, $\mu_3$, $u$ and $v$ in GeV.
\[ F' + H' + \delta \kappa = \kappa^2 + \sqrt{2}\kappa^3 \]

Singly charged scalar mass (GeV) 

| Mass (GeV) | \( \phi \)-comp. | Mass (GeV) | \( \phi \)-comp. |
|------------|-----------------|------------|-----------------|
| 125        | 0.995           | 125        | 0.964           |
| 352        | 0.097           | 155        | 0.265           |
| 512        | 0               | 649        | 0               |
| 414        | 0.115           | 395        | 0.011           |
| 329        | 0.106           | 643        | 4 \times 10^{-4} |
| 486        | 0.043           | 384        | 0.011           |
| 293        | 0               | 622        | 0               |

Table 2: Masses of various scalars, and their doublet admixture, for the two benchmark points chosen in Table 1. By \( \phi \)-component of any physical field \( X' \), we mean \( \left| \langle X' | \phi \rangle \right| \).
In Fig. 1, we show how the masses of the physical singly charged scalars, $H'_+$ and $F'_+$, vary with $\delta_\kappa \equiv \kappa_2 + \sqrt{2}\kappa_3$, which is zero in the GM limit. All the parameters of the scalar potential are fixed at the benchmark point BP-1, except $\kappa_3$, and $\lambda_\phi$ has been slightly adjusted to keep the mass of one of the CP-even neutral scalars, $S'_1$, at 125 GeV. The lower limit of $\delta_\kappa$ is kept at about $-\sqrt{4\pi}$, whereas the upper limit ($\sim -0.1$) is where the potential develops a saddle point instead of a true local minimum; the mass squared terms of the 5-plet scalars become negative. We also show the doublet admixture in $F'_+$ and $H'_+$ as a function of $\delta_\kappa$. Note that in the GM limit, $F'_+$ does not have any doublet component. The crossover, as can be seen in the right-hand panel of Fig. 1, occurs at $\kappa_3 = 0$.

In Fig. 2, we show the mass splitting among the members of the triplet and those of the 5-plet, as a function of $\delta_\kappa$, again varied around BP-1 as mentioned before. The nature of the plots changes if we vary one or more of the other parameters, or probe the violation of other GM constraints shown in Eq. (49), but such a detailed scan is not our goal.

Because of these features, the eGM model has a richer phenomenology, as commented earlier. Here, we list some of the new consequences to be expected from the eGM model, without going into a detailed analysis.

- The members of the physical scalar multiplets are not degenerate. This leads to the decays like $F'_0 \to F'_+ W^*$, $F'_+ \to F'_{++} W^*$, $H'_0 \to H'_+ W^*$, and their CP-conjugate modes. If the splitting is not too large, the virtual $W$ will either lead to a soft pion or to a pair of soft jets or leptons. However, the final states should be observable at the LHC if the boost is significant. On the other hand, for typical benchmark points as shown above, the splittings are large, and one may expect isolated hard leptons, over most of the parameter space. How to observe this signal over the SM background is a different issue.

- The neutral and singly-charged scalars belonging to the physical triplet and 5-plet contain some nontrivial component of the SU(2) doublet $\phi$. This leads to fermion-antifermion final states that never come from pure SU(2) triplets. The production cross-section will, of course, be suppressed by the mixing angle squared, and such signals may be interesting only at HL-LHC.

- Hints of several scalar resonances have been found at the LHC, e.g., a neutral scalar at 151 GeV, a pseudoscalar close to 400 GeV, and a heavy scalar at about 660 GeV [19]. BP-2 shows that it is possible to generate the spectrum in eGM. However, a detailed collider
study, matching theoretical predictions with corresponding signal strengths, lies beyond the scope of this paper.

6 Conclusion

In this paper, we critically discuss the concept of custodial symmetry. Defining it as something that keeps the SU(2) gauge bosons mass degenerate (in the limit $\cos \theta_W = 1$), we classify CS in three categories, depending on which parts of the action remain invariant under this. For example, we explicitly demonstrate that while the SM action respects CS, it is not the case for the most general 2HDM.

After this groundwork, we focus our attention to the Georgi-Machacek model with one scalar doublet and two triplets, which, by construction, keeps $\rho = 1$. The phenomenology and collider signatures of the GM model have been well explored, so we try to find whether this is the most general model with such scalar multiplet assignment to respect CS. We realize that this is not so; one may extend this model by introducing three more parameters in the potential and still keep $\rho = 1$ at the tree-level. This, of course, changes the type of CS that the GM model respects. Instead of 9 for the GM model, the scalar potential of this extended model, which we call extended GM or eGM, is specified by 12 independent parameters and we show that this is the most general extension of the scalar sector with one doublet and two triplets keeping CS intact.

This, in turn, affects the corresponding phenomenology. While our focus in this paper is not on the collider signals, we point out two important differences between GM and eGM models that should be relevant for a collider study. First, the CS multiplets, the triplet and the 5-plet, are no longer degenerate in the eGM model, and secondly, both the physical singly charged scalars have a doublet component which leads to a fermion-antifermion final state, and hence may change the search strategies. We will take up this study in detail in a subsequent publication.

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A SU(2) invariants in spherical basis

When we write the components of an SU(2) multiplet, e.g., in Eq. (35), we usually employ the “spherical” notation, meaning that each component has a well-defined value of $T_3$, the diagonal generator of SU(2). When we write the SU(2) generators, on the other hand, we tend to use the Cartesian ones. This mixed notation was employed in writing the GM potential in the literature [3, 11]. In this paper, we have used a spherical notation for everything. In this Appendix, we show the correspondence between the two notations and explain the way we write Eq. (50), for example.

In terms of the Cartesian components, the SU(2) generators in the 3-dimensional represen-
tation are as follows:
\[ t_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (A.1)

We can easily find a unitary matrix \( U \) such that \( Ut_z U^\dagger \equiv t_0 \) is diagonal. This matrix is

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}, \] (A.2)

and this gives

\[ t_0 = \text{diag} \ (1, 0, -1). \] (A.3)

So, for a vector \( A \), the spherical components and the Cartesian components would be related by

\[ A_{\text{sph}} = U A_{\text{Car}}, \] (A.4)

which gives

\[ A_{(+)} = \frac{1}{\sqrt{2}} (-A_x + iA_y), \]
\[ A_{(0)} = A_z, \]
\[ A_{(-)} = \frac{1}{\sqrt{2}} (A_x + iA_y). \] (A.5)

Note that the indices in parentheses denote the \( T_3 \)-eigenvalues of the components. This can differ from the electric charge when the \( U(1) \) quantum number is non-zero, \( e.g., \) for the complex triplet \( \chi \). When we write the charges, we do not use the parentheses. For the generators also, we do not use the parentheses since there is no scope of confusion there. The invariant combination of the two vectors will be given by

\[ A \cdot B = A_x B_x + A_y B_y + A_z B_z \]
\[ = -A_{(+)} B_{(-)} - A_{(-)} B_{(+)} + A_{(0)} B_{(0)}. \] (A.6)

The minus signs might look odd in this notation, but they are exactly in correspondence with the Clebsch-Gordan coefficients obtained while constructing a spin-0 combination from two spin-1 particles.

In the spherical notation, then, we should take the generators as \( t_0 \) given in Eq. (A.3), and

\[ t_+ = \frac{1}{\sqrt{2}} U (-t_x + it_y) U^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \]
\[ t_- = \frac{1}{\sqrt{2}} U (t_x + it_y) U^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \] (A.7)
where \( t_x \) and \( t_y \) have been read from Eq. (A.1). For the 2-dimensional representation, similarly, we will use the matrices

\[
\begin{align*}
\tau_+ &= \frac{1}{\sqrt{2}}(-\tau_x + i\tau_y) = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \\
\tau_0 &= \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\tau_- &= \frac{1}{\sqrt{2}}(\tau_x + i\tau_y) = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix},
\end{align*}
\]

(A.8)

where \( \tau_x, \tau_y \) and \( \tau_z \) are the usual Pauli matrices. Although the SU(2) generators are really half of the Pauli matrices, we have used, just for the ease of writing while writing the potential, the Pauli matrices themselves. This adjustment is inconsequential as far as the potential is concerned: it is just a matter of redefinition of the concerned coupling constant.

Note that, in this notation,

\[
T^\dagger_\pm = -T^-\pm, \quad T^\dagger_0 = T_0,
\]

(A.9)

and

\[
[T_0, T_\pm] = \mp T_\pm,
\]

(A.10)

where \( T \) stands for the generators in any representation. Also, note that the \( T_\pm \) generators differ from the usual definition of the ladder operators in quantum mechanics texts.

The advantage of working with the spherical components should be obvious. The components correspond to fields of definite electric charge in the SU(2) \( \times \) U(1) model. To see how it helps the notation, let us look at the term with the coefficient \( M_1 \) in Eq. (50), and examine one term whose trace appears there:

\[
\text{Tr}(\Phi^\dagger_\mp \Phi \tau^-) = \text{Tr} \left[ \left( \begin{array}{cc} \phi_0 & -\phi_+ \\ \phi_- & \phi_0^* \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ \sqrt{2} & 0 \end{array} \right) \left( \begin{array}{cc} \phi_0^* & \phi_+ \\ -\phi_- & \phi_0 \end{array} \right) \right] = 2\phi_0^* \phi_0^*.
\]

(A.11)

Considering the U(1) charge of this combination, it is clear that it can couple only to \( \chi_0 \), which is exactly the component \( X_{(-)} \) of the matrix \( X \), because the rows and columns are supposed to be marked by the eigenvalues +1, 0, -1 of \( t_0 \), in that order. Similarly, other components can be checked, in the same term as well as in other terms. This is much more intuitive than the notation used in the earlier literature which involves the Cartesian generators, and therefore the matrix \( U \), given in Eq. (A.2), appears explicitly in the expression for the potential.

Any SU(2) representation is self-conjugate. It means that, for any multiplet \( \Psi \) with \( N \) components, there is a matrix \( C_N \) such that \( C_N \Psi^* \) transforms exactly the same way as \( \Psi \). The matrix \( C \) should satisfy the relation

\[
CT^*_\alpha = -T^*_\alpha C
\]

(A.12)

on the hermitian generators \( T_x \), \( T_y \) and \( T_z \). In this article, only 2, 3 and 5 dimensional representations of SU(2) occur. For these representations, the conjugation matrices are

\[
C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(A.13)
For the doublet defined in Eq. (19), the conjugate doublet has a minus sign in one of the components, which explains the minus sign appearing in the first column of the matrix $\Phi$ in Eq. (51). The conjugate triplet of $\chi$ has been defined as

$$\tilde{\chi} = -C_3\chi^*,$$  \hspace{1cm} (A.14)

whose components are shown in Eq. (47). The minus sign in front of $C_3$ is purely a matter of convention. This $\tilde{\chi}$ appears in the leftmost column of the matrix $X$ in Eq. (35).

Let us now look at real multiplets. An arbitrary real triplet $\Sigma$ should satisfy the condition

$$\Sigma = C_3\Sigma^*.$$ \hspace{1cm} (A.15)

Equivalently, we can say

$$\Sigma^\dagger = \Sigma^\top C_3^\dagger = \Sigma^\top C_3.$$ \hspace{1cm} (A.16)

Therefore, the scalar product of two real vectors $\Sigma$ and $\Xi$ can be written as

$$\Sigma^\dagger \Xi = \Sigma^\top C_3 \Xi (A.17)$$

which exactly agrees with the expression given in Eq. (A.6).

The condition of Eq. (A.15) implies that the middle component of a real multiplet must be real. For the other two components, there are two ways that this condition can be realized. One is to take

$$\Sigma_{(-)} = -\Sigma_{(+)}^*,$$ \hspace{1cm} (A.18)

as suggested by Eq. (A.5). The other possibility is to define a $\Sigma_+$ (whose subscript is not parenthesized) by the relations

$$\Sigma_{(+)} = i\Sigma_+, \quad \Sigma_{(-)} = i\Sigma_+^*.$$ \hspace{1cm} (A.19)

In either case, we will write $\Sigma_+^*$ as $\Sigma_-$, so that the first kind of triplet will be written with components $(\Sigma_+, \Sigma_0, -\Sigma_-)$, and the second kind with $(i\Sigma_+, \Sigma_0, i\Sigma_-)$, where the unparenthesized indices indicate the electric charge of the concerned field. For the 5-dimensional representation of the custodial SU(2) of the GM model, the components are $(F_{++}, iF_+, F_0, iF_-, F_{--})$ where $F_0$ is real, $F_- = F_+^*$ and $F_{--} = F_{++}^*$.

For various CS multiplets appearing in our discussion, we have used Eq. (A.18)-type notation for the gauge boson triplet, i.e., we have denoted the CS triplet with components $(-W_+, W_0, W_-)$ with $W_+^* = W_-$. A similar notation has been used for the $\xi$-triplet in Eq. (35). On the other hand, for the unphysical Goldstone bosons, which also form a CS triplet, we use the reality condition as in Eq. (A.19), i.e., we have denoted the components as $(ih_+, h_0, ih_-)$, with $h_+^* = h_-$. 

**B  Correspondence between parameters in GM, eGM and the full potential**

The full gauge-invariant potential of the model with a doublet, a real triplet and a complex triplet of Higgs bosons was given in Eq. (46). Our notation differs considerably from the notation used by earlier authors [10, 22, 23]. The motivation for the change of notation was explained in
The terms of Eq. (46): notations in Table 3. Second, we give the full potential in component form, fully expanding the previous notation, we do two things here. First, we provide a translation chart between different the text: our use of two different kinds of quartic couplings, with two different letters, helps understand the symmetries of the mass matrices. In order to bridge between our notation and previous notation, we do two things here. First, we provide a translation chart between different notations in Table 3. Second, we give the full potential in component form, fully expanding the terms of Eq. (46):

\[
V(\phi, \chi, \xi) = -m_\phi^2(\phi_+\phi_+ + \phi_0^*\phi_0) - m_\xi^2(\chi_+\chi_- + \chi_+\chi_- + \chi_0^*\chi_0) - m_\xi^2(2\xi_+\xi_- + \xi_0^2) \\
- \mu_1 \left[ - (\chi_0\chi_- + \chi_+\chi_-)\xi_+ + (\chi_+\chi_- - \chi_0^*\chi_0)\xi_0 - (\chi_+\chi_0^* + \chi_+\chi_-)\xi_- \right] \\
- \mu_2 \left[ \sqrt{2}\phi_+\phi_0\xi_+ + (\phi_0^*\phi_0 - \phi_+\phi_-)\xi_0 + \sqrt{2}\phi_+\phi_0^*\xi_- \right] \\
- \mu_3 \left[ \sqrt{2}\phi_0\phi_0\chi_0^* + 2\phi_+\phi_0\chi_- + \sqrt{2}\phi_+\phi_+\chi_- + \text{h.c.} \right] \\
+ \lambda_0 (\phi_+\phi_- + \phi_0^*\phi_0)^2 + \lambda_\chi (\chi_+\chi_- + \chi_+\chi_- + \chi_0^*\chi_0)^2 \\
+ \lambda_\xi (2\xi_+\xi_- + \xi_0^2)^2 + \tilde{\lambda}_\chi \left[ 2\chi_0^*\chi_- - \chi_-\chi_- \right]^2 \\
+ \lambda_{\phi\chi} (\phi_+\phi_- + \phi_0^*\phi_0) (\chi_+\chi_- + \chi_+\chi_- + \chi_0^*\chi_0) \\
+ \lambda_{\phi\xi} (\phi_+\phi_- + \phi_0^*\phi_0) (2\xi_+\xi_- + \xi_0^2) + \lambda_{\chi\xi} (\chi_+\chi_- + \chi_+\chi_- + \chi_0^*\chi_0) (2\xi_+\xi_- + \xi_0^2) \\
+ \kappa_1 [\xi_0^* - \xi_0\chi_- - \xi_+\chi_-]^2 \\
+ \kappa_2 \left[ - \sqrt{2}\phi_+\phi_0(\chi_+\chi_0^* + \chi_+\chi_-) + (\phi_+\phi_- - \phi_0^*\phi_0)(\chi_0^*\chi_0 - \chi_+\chi_-) \right] \\
- \sqrt{2}\phi_+\phi_0^*(\chi_0\chi_- + \chi_+\chi_-) \right]
\]

| Full potential | GM                | eGM                  |
|----------------|-------------------|----------------------|
| This paper     | Ref. [10]         | as in Eq. (50)       |
| \( m_\phi^2 \) | \( -\mu_2^2 \)   | \( m_\phi^2 \)       |
| \( m_\chi^2 \) | \( -m_3^2 \)      | \( m_\chi^2 \)       |
| \( m_\xi^2 \)  | \( -\mu_3^2 \)   | \( m_\xi^2 \)        |
| \( \mu_1 \)    | \( -6M_2 \)       | \( \mu_1 \)          |
| \( \mu_2 \)    | \( \frac{1}{2}M_1 \) | \( 2M_1 \)          |
| \( \mu_3 \)    | \( \frac{1}{2\sqrt{2}}M'_1 \) | \( \sqrt{2}M_1 \) |
| \( \lambda_\phi \) | \( \lambda_1 \)  | \( \lambda_\phi \)   |
| \( \lambda_\chi \) | \( \lambda_7 \)  | \( 4\lambda_2 + 2\lambda_3 \) |
| \( \lambda_\xi \) | \( \lambda_8 \)  | \( \lambda_\xi \)   |
| \( \lambda_\chi \) | \( \lambda_5 \)  | \( 4\lambda_3 \)    |
| \( \lambda_{\phi\chi} \) | \( \lambda_6 \)  | \( \lambda_{\phi\chi} \) |
| \( \lambda_{\phi\xi} \) | \( \lambda_{10} \) | \( 4\lambda_5 \)    |
| \( \lambda_{\chi\xi} \) | \( \lambda_{9} \) | \( 4\lambda_3 \)    |
| \( \kappa_1 \) | \( \frac{1}{2}\lambda_3 \) | \( \lambda_{\phi\chi} - 2\lambda_{\phi\xi} - \sqrt{2}\kappa_3 \) |
| \( \kappa_2 \) | \( -\sqrt{2}\lambda_5 \) | \( \kappa_3 \)       |
| \( \kappa_3 \) | \( -2\sqrt{2}\lambda_5 \) | \( \kappa_3 \)       |

Table 3: Correspondence between different notations. Columns 1 and 2 show the notations of the 16-paramter potential. Column 3 shows the parametrization of the GM model potential as given in Eq. (50) of this paper. Column 4 shows the restrictions that lead to the eGM model proposed in this paper.
\[ +\kappa_3 \left[ \sqrt{2}\phi_0\phi_0(\xi_0\chi_\sigma + \xi_+\chi_-) + 2\phi_+\phi_0(\xi_\sigma\chi_\sigma + \xi_+\chi_-) \right. \\
\left. -\sqrt{2}\phi_+\phi_+(-\xi_-\chi_+ + \xi_0\chi_-) \right] + \text{h.c.} \], \quad \text{(B.1)}

C Renormalization Group Equations

The RGEs for the trilinear and quartic couplings for the most general 16-parameter model are taken from Ref. [10], with a translation to our convention, as shown in Appendix B. The eGM RGEs are easily obtained by imposing the necessary relationships among the couplings. We use the shorthand notation \( \beta_A = 16\pi^2 dA/dt \) with \( t = \ln q \), where \( q \) is the relevant energy scale. Note that the U(1)\(_Y\) gauge coupling \( g_1 \) is not GUT normalized.

\[
\begin{align*}
\beta_{\lambda_\phi} &= \lambda_\phi \left( 24\lambda_\phi + 12y_t^2 - \frac{9}{5} g_1^2 - 9g_2^2 \right) - 6y_t^4 + \frac{27}{200} g_1^4 + \frac{9}{8} g_2^4 + \frac{9}{20} g_1^2 g_2^2 \\
&\quad + 2\kappa_2^2 + 8\kappa_3^2 + 3\lambda_\phi^2 + 6\lambda_\phi^2, \\
\beta_{\lambda_\chi} &= \lambda_\chi \left( 16\lambda_\chi + 28\lambda_\chi - \frac{36}{5} g_1^2 - 24 g_2^2 \right) + \frac{54}{25} g_1^4 + 9g_2^4 + \frac{36}{5} g_1^2 g_2^2 \\
&\quad + 16\lambda_\chi^2 + 2\lambda_\phi^2 + \kappa_1^2 + 2\kappa_2^2 + 2\lambda_\chi^2 (3\lambda_\chi + 2\kappa_1), \\
\beta_{\lambda_\chi} &= 12\lambda_\chi \left( \lambda_\chi + 2\lambda_\chi - \frac{3}{5} g_1^2 - 2g_2^2 \right) + 3g_1^4 - \frac{36}{5} g_1^2 g_2^2 - 2\kappa_2^2 + \kappa_1^2, \\
\beta_{\lambda_\xi} &= 8\lambda_\xi \left( 11\lambda_\xi - 3g_2^2 \right) + 3g_2^4 + 2\lambda_\phi^2 + \kappa_1 (\kappa_1 + 2\lambda_\chi^2) + 3\lambda_\xi^2, \\
\beta_{\lambda_\phi\chi} &= \lambda_\phi \left( 4\lambda_\phi \chi + 12\lambda_\phi + 8\lambda_\chi + 16\lambda_\chi + 6y_t^2 - \frac{9}{2} g_1^2 - \frac{33}{2} g_2^2 \right) \\
&\quad + \frac{27}{25} g_1^4 + 6g_2^4 + 8\kappa_2^2 + 16\kappa_3^2 + 4\lambda_\phi\kappa_1 + 12\lambda_\phi\lambda_\chi^2, \\
\beta_{\lambda_\phi\xi} &= \lambda_\phi \left( 8\lambda_\phi + 12\lambda_\phi + 40\lambda_\xi + 6y_t^2 - \frac{9}{10} g_1^2 - \frac{33}{2} g_2^2 \right) + 3g_1^4 + 16\kappa_3^2 + 2\lambda_\phi\kappa_1 + 6\lambda_\phi\lambda_\chi^2, \\
\beta_{\lambda_\chi\xi} &= 2\lambda_\chi \left( -\frac{9}{5} g_1^2 - 12g_2^2 + 4\lambda_\chi + 8\lambda_\chi + 20\lambda_\xi + 4\lambda_\chi \xi \right) + 6g_4^2 \\
&\quad + 8\kappa_3^2 + 2\kappa_2^2 + 4\lambda_\phi\lambda_\phi + 4\kappa_1 (\lambda_\chi + 2\lambda_\xi), \\
\beta_{\kappa_1} &= 2\kappa_1 \left( 5\kappa_1 + 4\lambda_\chi + 2\lambda_\chi + 8\lambda_\xi + 8\lambda_\chi^2 - \frac{9}{5} g_1^2 - 12g_2^2 \right) + 6g_4^4 - 8\kappa_3^2, \\
\beta_{\kappa_2} &= \kappa_2 \left( 4\lambda_\phi - 8\lambda_\chi + 8\lambda_\phi + 4\lambda_\chi - \frac{9}{2} g_1^2 - \frac{33}{2} g_2^2 + 6y_t^2 \right) - \frac{18}{5} g_1^2 g_2^2 - 8\kappa_3^2, \\
\beta_{\kappa_3} &= \kappa_3 \left( 4\lambda_\phi - 4\kappa_2 + 4\lambda_\phi + 8\lambda_\phi - 2\kappa_1 + 4\lambda_\chi^2 + 6y_t^2 - \frac{27}{10} g_1^2 - \frac{33}{2} g_2^2 \right), \\
\beta_{\mu_1} &= \mu_1 \left( -8\lambda_\chi + 4\lambda_\chi - 4\kappa_1 + 8\lambda_\chi - \frac{18}{5} g_1^2 - 18g_2^2 \right) + 4\mu_2\kappa_2 - 16\mu_3\kappa_3, \\
\beta_{\mu_2} &= \mu_2 \left( 6y_t^2 - \frac{9}{10} g_1^2 - \frac{21}{2} g_2^2 + 4\lambda_\phi + 8\lambda_\phi \xi \right) + 4\mu_1\kappa_2 + 32\mu_3\kappa_3, \\
\beta_{\mu_3} &= \mu_3 (6y_t^2 - \frac{27}{10} g_1^2 - \frac{21}{2} g_2^2 + 4\lambda_\phi - 8\kappa_2 + 4\lambda_\phi \chi) + 4k_3 (2\mu_2 - \mu_1). \quad \text{(C.1)}
\end{align*}
\]
References

[1] P. A. Zyla et al. Review of Particle Physics. *PTEP*, 2020(8):083C01, 2020.

[2] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, Marc Sher, and Joao P. Silva. Theory and phenomenology of two-Higgs-doublet models. *Phys. Rept.*, 516:1–102, 2012.

[3] Howard Georgi and Marie Machacek. Doubly charged Higgs bosons. *Nucl. Phys. B*, 262:463–477, 1985.

[4] B. Grzadkowski, M. Maniatis, and Jose Wudka. The bilinear formalism and the custodial symmetry in the two-Higgs-doublet model. *JHEP*, 11:030, 2011.

[5] Howard E. Haber and Deva O’Neil. Basis-independent methods for the two-Higgs-doublet model III: The CP-conserving limit, custodial symmetry, and the oblique parameters S, T, U. *Phys. Rev. D*, 83:055017, 2011.

[6] M. Aa Solberg. Conditions for the custodial symmetry in multi-Higgs-doublet models. *JHEP*, 05:163, 2018.

[7] J. F. Gunion, R. Vega, and J. Wudka. Naturalness problems for $\rho = 1$ and other large one loop effects for a standard model Higgs sector containing triplet fields. *Phys. Rev. D*, 43:2322–2336, 1991.

[8] Cheng-Wei Chiang and Kei Yagyu. Testing the custodial symmetry in the Higgs sector of the Georgi-Machacek model. *JHEP*, 01:026, 2013.

[9] Simone Blasi, Stefania De Curtis, and Kei Yagyu. Effects of custodial symmetry breaking in the Georgi-Machacek model at high energies. *Phys. Rev. D*, 96(1):015001, 2017.

[10] Ben Keeshan, Heather E. Logan, and Terry Pilkington. Custodial symmetry violation in the Georgi-Machacek model. *Phys. Rev. D*, 102(1):015001, 2020.

[11] Heather E. Logan and Vikram Rentala. All the generalized Georgi-Machacek models. *Phys. Rev. D*, 92(7):075011, 2015.

[12] See, e.g., Palash B. Pal, *An introductory course of Particle Physics*, (CRC Press, 2014), Section 19.5.

[13] Anirban Kundu and Biswarup Mukhopadhyaya. A General Higgs sector: Constraints and phenomenology. *Int. J. Mod. Phys. A*, 11:5221–5244, 1996.

[14] Mayumi Aoki and Shinya Kanemura. Unitarity bounds in the Higgs model including triplet fields with custodial symmetry. *Phys. Rev. D*, 77(9):095009, 2008. [Erratum: Phys.Rev.D 89, 059902 (2014)].

[15] Katy Hartling, Kunal Kumar, and Heather E. Logan. The decoupling limit in the Georgi-Machacek model. *Phys. Rev. D*, 90(1):015007, 2014.

[16] Albert M Sirunyan et al. Search for Charged Higgs Bosons Produced via Vector Boson Fusion and Decaying into a Pair of W and Z Bosons Using pp Collisions at $\sqrt{s} = 13$ TeV. *Phys. Rev. Lett.*, 119(14):141802, 2017.
[17] Albert M Sirunyan et al. Search for charged Higgs bosons produced in vector boson fusion processes and decaying into vector boson pairs in proton–proton collisions at $\sqrt{s} = 13\,\text{TeV}$. *Eur. Phys. J. C*, 81(8):723, 2021.

[18] Ameen Ismail, Ben Keeshan, Heather E. Logan, and Yongcheng Wu. Benchmark for LHC searches for low-mass custodial fiveplet scalars in the Georgi-Machacek model. *Phys. Rev. D*, 103(9):095010, 2021.

[19] Richard, François, *Global interpretation of LHC indications within the Georgi-Machacek Higgs model*, Talk presented at the International Workshop on Future Linear Colliders (LCWS2021), 15-18 March 2021, arXiv:2103.12639.

[20] Andreas Crivellin, Yaquan Fang, Oliver Fischer, Abhaya Kumar, Mukesh Kumar, Elias Malwa, Bruce Mellado, Ntsoko Rapheeha, Xifeng Ruan, and Qiyu Sha. Accumulating Evidence for the Associate Production of a Neutral Scalar with Mass around 151 GeV. arXiv:2109.02650.

[21] Richard, François, *A Georgi-Machacek Interpretation of the Associate Production of a Neutral Scalar with Mass around 151 GeV*, ILC Workshop on Potential Experiments, arXiv:2112.07982.

[22] Dipankar Das and Ipsita Saha. Cornering variants of the Georgi-Machacek model using Higgs precision data. *Phys. Rev. D*, 98(9):095010, 2018.

[23] Avik Banerjee, Gautam Bhattacharyya, and Nilanjana Kumar. Impact of Yukawa-like dimension-five operators on the Georgi-Machacek model. *Phys. Rev. D*, 99(3):035028, 2019.