Aitken’s $\Delta^2$ method extended

Shirley B. Pomeranz*

**Abstract:** Aitken’s $\Delta^2$ method is used to accelerate convergence of sequences, e.g. sequences obtained from iterative methods. An explicit assumption in deriving Aitken’s $\Delta^2$ method and establishing acceleration (for linearly convergent sequences) is that consecutive error iterates (or their approximations) have the same sign or have an alternating sign pattern. We extend the standard Aitken’s $\Delta^2$ method to the cases in which consecutive pairs of error iterates in the sequence have alternating signs. Under suitable restrictions, acceleration of convergence is proved. Implementation of our extended method is described. Numerical examples demonstrate the process. An example is included relating our results to results obtained from Richard extrapolation.

**Subjects:** Applied Mathematics; Mathematics & Statistics; Science

**Keywords:** Aitken’s $\Delta^2$ method; convergence acceleration; deferred corrections

**AMS subject classifications:** 65B05; 40A25

1. Introduction

Aitken’s $\Delta^2$ method is generally presented as a method for accelerating convergence of iterative techniques in which the consecutive error terms eventually have the same sign. Methods have been developed to generalize Aitken’s $\Delta^2$ method and enhance it for certain classes of sequences, e.g.

---

**ABOUT THE AUTHOR**

Shirley B. Pomeranz earned her BA in physics from Barnard College of Columbia University, MS in physics from New York University, MS in Mathematics from the University of Connecticut, and PhD in Mathematics from the University of Massachusetts. She joined the Department of Mathematics at The University of Tulsa in 1987, where she is currently a tenured associate professor of mathematics. She is a member of the Editorial Advisory Board for The International Journal of Engineering Education (IJEE), and was an American Society for Engineering Education (ASEE) National Board of Directors Member for 2005-2007. In 2005 she received the ASEE Mathematics Division Distinguished Educator and Service Award. She was the 2015 recipient of The University of Tulsa College of Engineering and Natural Sciences Kermit E. Brown Award for Teaching Excellence. She is a member of the Center for Boundary Integral Methods at The University of Tulsa. Her publication and research areas include boundary element and finite-element methods.

---

**PUBLIC INTEREST STATEMENT**

Aitken’s $\Delta^2$ method is a method used to accelerate convergence of sequences, e.g. sequences of numbers obtained from iterative methods. An explicit assumption in deriving Aitken’s $\Delta^2$ method and establishing acceleration (for linearly, relatively slowly convergent sequences) is that consecutive error iterates (or their approximations) have the same sign or have an alternating sign pattern. This paper presents an extension of the standard Aitken’s delta-squared method to the cases in which consecutive pairs of error iterates in the sequence have alternating signs. Under suitable restrictions, acceleration of convergence is proved. Implementation of this extended method is described. Numerical examples demonstrate the process. An example is included applying this work to results obtained from Richard extrapolation.
Bumbariu (2012) and Buoso, Karapiperi, and Pozza (2015), among others. In this paper, we present new work in which we extend the method to apply to sequences exhibiting a pattern in which the \((n + 2)\)nd error iterate has the opposite sign from the \(n\)th error iterate.

In Section 2 the basic Aitken’s \(\Delta^2\) method is reviewed. Our extension of the basic method is developed in Section 3. Numerical examples illustrate our extended method in Section 4. A summary of results is given in Section 5.

2. Aitken’s \(\Delta^2\) method acceleration

If \(\{q_n\}_{n=0}^{\infty}\) is a sequence linearly converging to the limit \(q\),

\[
0 < \lim_{n \to \infty} \left| \frac{q_{n+1} - q}{q_n - q} \right| = \lambda < 1,
\]

and the signs of \(q_n - q\), \(q_{n+1} - q\), and \(q_{n+2} - q\) are the same (or have a pattern of alternating signs), then for \(n\) sufficiently large, the following derivation of the standard Aitken’s \(\Delta^2\) method holds for accelerating the convergence this sequence (Burden, Faires, & Burden, 2016, Sec. 2.5; Issacson & Keller, 1966, pp. 103–107; Reich, 1970; Stoer & Bulirsch, 2010, Sec. 5.10).

\[
q_{n+1} = q_n - \frac{q_{n+1} - q}{q_{n+2} - 2q_{n+1} + q_n}
\]

Solving for \(q\) yields

\[
q \approx \frac{q_{n+2} q_n - q_{n+1}^2}{q_{n+2} - 2q_{n+1} + q_n} = q_n - \frac{(q_{n+1} - q_n)^2}{q_{n+2} - 2q_{n+1} + q_n}.
\]

The resulting Aitken’s \(\Delta^2\) approximation is denoted \(\hat{q}_n\) and defined as

\[
\hat{q}_n \equiv q_n - \frac{(q_{n+1} - q_n)^2}{q_{n+2} - 2q_{n+1} + q_n}.
\]

Under the above assumptions, the Aitken’s \(\Delta^2\) formula provides faster convergence:

**Theorem 2.1 (Aitken’s \(\Delta^2\) Method Acceleration Theorem)** If \(\{q_n\}_{n=0}^{\infty}\) is a sequence that converges linearly to the limit \(q\) with

\[
0 < \lim_{n \to \infty} \left| \frac{q_{n+1} - q}{q_n - q} \right| = \lambda < 1,
\]

and the signs of \(q_n - q\) are the same for each \(n\) sufficiently large, then the Aitken’s \(\Delta^2\) sequence \(\{\hat{q}_n\}_{n=0}^{\infty}\) converges to \(q\) faster than \(\{q_n\}_{n=0}^{\infty}\) in the sense that

\[
\lim_{n \to \infty} \frac{\hat{q}_n - q}{q_n - q} = 0.
\]

**Proof** Assume that

\[
\frac{q_{n+1} - q}{q_n - q} = \lambda + \delta_n, \quad n = 1, \ldots,
\]

where

\[
\lim_{n \to \infty} \delta_n = 0.
\]

This implies that
\[ q_{n+1} = q + (q_n - q)(\lambda + \delta_n) \]

and

\[ q_{n+2} = q + (q_{n+1} - q)(\lambda + \delta_{n+1}). \]

We have

\[
\frac{\dot{q}_n - q}{q_n - q} = \frac{q_n - q - \frac{(q_{n+1} - q_n)^2}{(q_n - q)(q_{n+2} - 2q_{n+1} + q_n)}}{q_n - q} = 1 - \frac{(q_{n+1} - q + q - q_n)^2}{(q_n - q)(q_{n+2} - 2q_{n+1} + q_n - q)}
\]

\[
= 1 - \frac{(q_{n+1} - q + q - q_n)^2}{(q_n - q)(\lambda + \delta_n - 1)^2 + 2(\lambda + \delta_n) + 1}.
\]

Taking the limit as \( n \to \infty \),

\[
\lim_{n \to \infty} \frac{\dot{q}_n - q}{q_n - q} = \lim_{n \to \infty} 1 - \frac{(\lambda + \delta_n - 1)^2}{(\lambda + \delta_n)(\lambda + \delta_n) - 2(\lambda + \delta_n) + 1} = 1 - \frac{(\lambda - 1)^2}{\lambda^2 - 2\lambda + 1} = 1 - \frac{(\lambda - 1)^2}{(\lambda - 1)^2} = 0,
\]

since the asymptotic error constant satisfies \( 0 < \lambda < 1 \). \[\square\]

A similar proof holds if the signs of the consecutive error iterates alternate.

3. Extension of Aitken’s \( \Delta^2 \) method

The Aitken’s \( \Delta^2 \) formula was derived under the assumption that for \( n \) sufficiently large, the consecutive error iterates have the same sign (or have a pattern of alternating signs). But what if this is not the case? We extend acceleration results to some cases in which the two pairs of error iterates \( q_n - q \) and \( q_{n+2} - q \) have opposite signs for \( n \) sufficiently large. The error iterate \( q_{n+1} - q \) may have either sign, positive or negative. For example, \( q_n - q > 0 \) and \( q_{n+1} - q > 0 \) (or \( q_{n+1} - q < 0 \)) but \( q_{n+2} - q < 0 \). Therefore, we can begin with the assumption that for \( n \) sufficiently large,

\[
\frac{q_{n+1} - q}{q_n - q} \approx \frac{q_{n+2} - q}{q_{n+1} - q}.
\]

This gives

\[(q_{n+1} - q)^2 \approx -(q_{n+2} - q)(q_n - q),\]

for \( n \) sufficiently large. Solving the resulting quadratic equation (approximation) for \( q \) yields

\[
q \approx \frac{1}{4} \left( q_n + 2q_{n+1} + q_{n+2} \pm \sqrt{(-q_n - 2q_{n+1} - q_{n+2})^2 - 8(q_{n+1}^2 + q_n q_{n+2})}\right).
\]
There are two cases to consider with respect to the sign choice preceding the square root. For one case, use the plus sign, denoted as:

\[ \tilde{q}_{+n} = \frac{1}{4} \left( q_n + 2q_{n+1} + q_{n+2} + \sqrt{(-q_n - 2q_{n+1} - q_{n+2})^2 - 8\left(q_{n+1}^2 + q_n q_{n+2}\right)} \right); \]

and for the other case, use the minus sign:

\[ \tilde{q}_{-n} = \frac{1}{4} \left( q_n + 2q_{n+1} + q_{n+2} - \sqrt{(-q_n - 2q_{n+1} - q_{n+2})^2 - 8\left(q_{n+1}^2 + q_n q_{n+2}\right)} \right). \]

Each of these two cases has two (sub)cases, as described in Theorem 3.1. An extension of the Aitken's \( \Delta^2 \) method acceleration theorem for these four cases is as follows:

**Theorem 3.1 (Extension of Aitken’s \( \Delta^2 \) method)** If \( \{q_n\}_{n=0}^{\infty} \) is a sequence that converges linearly to the limit \( q \),

\[ 0 < \lim_{n \to \infty} \left| \frac{q_{n+1} - q}{q_n - q} \right| = \lambda < 1, \]

and the pair of error iterates \( q_n - q \) and \( q_{n+2} - q \) have opposite signs for \( n \) sufficiently large, then the extended Aitken’s \( \Delta^2 \) sequences \( \{\tilde{q}_{+n}\}_{n=0}^{\infty} \) and \( \{\tilde{q}_{-n}\}_{n=0}^{\infty} \)

\[ \tilde{q}_{+n} = \frac{1}{4} \left( q_n + 2q_{n+1} + q_{n+2} + \sqrt{(-q_n - 2q_{n+1} - q_{n+2})^2 - 8\left(q_{n+1}^2 + q_n q_{n+2}\right)} \right) \]

and

\[ \tilde{q}_{-n} = \frac{1}{4} \left( q_n + 2q_{n+1} + q_{n+2} - \sqrt{(-q_n - 2q_{n+1} - q_{n+2})^2 - 8\left(q_{n+1}^2 + q_n q_{n+2}\right)} \right), \]

have subsequences with \( n = 4m + 3, 4m + 4, 4m + 2, \) or \( 4m + 1, m = 0, 1, 2, \ldots \) that converge to \( q \) faster than \( \{q_n\}_{n=0}^{\infty} \) in the sense that

\[ \lim_{n \to \infty} \frac{\tilde{q}_{+n} - q}{q_n - q} = 0 \]

or

\[ \lim_{n \to \infty} \frac{\tilde{q}_{-n} - q}{q_n - q} = 0. \]

**Proof** There are four cases to consider. For the first case, use the formula for \( \tilde{q}_{+n} \) with

\[ \frac{q_{n+1} - q}{q_n - q} = \lambda + \delta_n, \]

\[ \frac{q_{n+2} - q}{q_{n+1} - q} = \lambda + \delta_{n+1}, \quad n = 4m + 3, \quad m = 0, 1, 2, \ldots, \]

where

\[ \lim_{n \to \infty} \delta_n = 0. \]

Therefore, rearranging directly from Equation (1),
\[
\frac{q_n - q}{q_n} = \frac{q_n - q + 2(q_{n+1} - q) + (q_{n+2} - q)}{4(q_n - q)}
\]
\[
- \frac{1}{4(q_n - q)} \left( (q_n - q) + 4(q_{n+1} - q)(q_n - q) - 4(q_{n+1} - q)^2 \right)
\]
\[
= \frac{q_n - q + 2(q_{n+1} - q) + (q_{n+2} - q)}{4(q_n - q)}
\]
\[
- \frac{1}{4(q_n - q)} \left( (q_n - q)^2 + 4(q_{n+1} - q)(q_n - q) - 4(q_{n+1} - q)^2 \right)
\]
\[
-6(q_{n+2} - q)(q_n - q) + 4(q_{n+1} - q)(q_{n+2} - q) + (q_{n+2} - q)^2 \right)^{1/2},
\]
in which the algebra manipulations involve adding zero (and multiplying by one in the following algebra) in order to put expressions into more useful forms. Taking the limit as \( n \to \infty \),

\[
\lim_{n \to \infty} \frac{q_n - q}{q_n - q} = \lim_{n \to \infty} \frac{1}{4} \left( \frac{q_n - q + 2q_{n+1} - q + q_{n+2} - q}{q_n - q - q_n - q} \right)
\]
\[
- \frac{1}{4} \left( (q_n - q) + 4(q_{n+1} - q)(q_n - q) - 4(q_{n+1} - q)^2 \right)
\]
\[
= \frac{1}{4} \left( (\lambda + \delta_n) (\lambda + \delta_n + 1) - (\lambda^2 + 2\lambda + 1) \right)
\]
\[
= 0,
\]
since \( | - \lambda^2 + 2\lambda + 1 | > 0 \), for \( 0 < \lambda < 1 \).

A corresponding proof for the second case uses the formula for \( q_{n, \text{op}} \) with

\[
- \frac{q_{n+1} - q}{q_n - q} = \lambda + \delta_n,
\]
\[
\frac{q_{n+2} - q}{q_{n+1} - q} = \lambda + \delta_{n+1},
\]
\( n = 4m + 4, \quad m = 0, 1, 2, \ldots \),

where

\[
\lim_{n \to \infty} \delta_n = 0.
\]

The analogous computations yield the result that

\[
\lim_{n \to \infty} \frac{q_{n, \text{op}} - q}{q_n - q} = \frac{1}{4} \left( (\lambda^2 + 2\lambda + 1) + | - \lambda^2 - 2\lambda + 1 | \right)
\]
\[
= 0,
\]
provided that √2 − 1 ≤ 𝜆 < 1, since −λ² − 2λ + 1 ≤ 0, for √2 − 1 ≤ 𝜆 < 1. We do not obtain completely analogous results for this case with ̃q. This result for ̃q while not holding for all 0 < 𝜆 < 1, does pertain to the more important situations, those which have larger values of 𝜆, i.e. slower original rates of convergence. If 0 < 𝜆 < √2 − 1, the rate of convergence does not worsen, but there is no guaranteed acceleration.

For the third case, using ̃q with

\[
\frac{q_{n+1} - q}{q_n - q} = \lambda + \delta_n,
\]

\[
\frac{q_{n+2} - q}{q_{n+1} - q} = \lambda + \delta_{n+1}, \quad n = 4m + 1, 2, \ldots,
\]

where

\[
\lim_{n \to \infty} \delta_n = 0,
\]

we obtain the result that

\[
\lim_{n \to \infty} \frac{q_m - q}{q_n - q} = \frac{1}{4} \left( (-\lambda^2 - 2\lambda + 1) - | - \lambda^2 - 2\lambda + 1 | \right)
\]

\[
= 0,
\]

provided that 0 < 𝜆 ≤ √2 − 1. Otherwise, the rate of convergence does not worsen, but there is no guaranteed acceleration.

For the fourth case, using ̃q with

\[
\frac{q_{n+1} - q}{q_n - q} = \lambda + \delta_n,
\]

\[
-\frac{q_{n+2} - q}{q_{n+1} - q} = \lambda + \delta_{n+1}, \quad n = 4m + 1, 2, \ldots,
\]

where

\[
\lim_{n \to \infty} \delta_n = 0,
\]

gives the result that

\[
\lim_{n \to \infty} \frac{q_m - q}{q_n - q} = \frac{1}{4} \left( (-\lambda^2 + 2\lambda + 1) + | - \lambda^2 + 2\lambda + 1 | \right)
\]

\[
= \frac{1}{2}(-\lambda^2 + 2\lambda + 1) \in \left( \frac{1}{2}, 1 \right),
\]

for 0 < 𝜆 < 1. Therefore, the rate of convergence does not worsen, but there is no guaranteed acceleration.

These four cases apply to subsequences/subsets of the sequences that are of interest (see Section 4). Also, note that Theorem 3.1 gives sufficient conditions, which may not be necessary. The proof of Theorem 3.1 does not conflict with the numerical results given in Section 4, but does not give a complete explanation of the method. This is because some information is lost in the process of taking the limit, in particular the signs of the three relevant error terms preceding the square root. This aspect of the method is discussed in Section 4.3.

We need to decide which is the appropriate choice at each iteration. Recall that
\[ \tilde{q}_{n+1} \equiv \frac{1}{4} (q_n + 2q_{n+1} + q_{n+2}) + \frac{1}{4} \sqrt{q_n^2 + 4q_{n+1}q_n - 4q_{n+1}^2 - 6q_{n+2}q_n + 4q_{n+1}q_{n+2} + q_{n+2}^2} \]

and

\[ \tilde{q}_{n-1} \equiv \frac{1}{4} (q_n + 2q_{n+1} + q_{n+2}) - \frac{1}{4} \sqrt{q_n^2 + 4q_{n+1}q_n - 4q_{n+1}^2 - 6q_{n+2}q_n + 4q_{n+1}q_{n+2} + q_{n+2}^2}. \]

If we have a suitable sign pattern, for example, as given in Theorem 3.1, do we use \( \tilde{q}_{n+1} \) or \( \tilde{q}_{n-1} \) so as to minimize the \( n \)th iteration absolute error? In an actual application, the exact limit value \( q \) is unknown. However, if it were known, then to minimize the \( n \)th iteration absolute error,

\[ |\tilde{q}_n - q| = \left| \frac{1}{4} (q_n - q + 2(q_{n+1} - q) + (q_{n+2} - q)) \pm \frac{1}{4} \sqrt{q_n^2 + 4q_{n+1}q_n - 4q_{n+1}^2 - 6q_{n+2}q_n + 4q_{n+1}q_{n+2} + q_{n+2}^2} \right| \] (2)

use \( \tilde{q}_{n+1} \) if \( q_n - q + 2(q_{n+1} - q) + (q_{n+2} - q) > 0 \); alternatively, use \( \tilde{q}_{n-1} \) if \( q_n - q + 2(q_{n+1} - q) + (q_{n+2} - q) \leq 0 \). Since the exact value, \( q \), is unknown in applications, \( q \) can be approximated by \( q_{n+3} \) or other information about the behavior of the sequence can be used (see Section 4). In applying the standard Aitken’s \( \Delta^2 \) method, the signs of the error terms are similarly unknown.

4. Numerical results

Three numerical examples are presented. The first example considers a sublinearly convergent sequence, the second example uses a linearly convergent sequence and the third example compares methods.

4.1. Example 1

For Example 1, we have the following sequence converging to zero:

\[ q_n = (-1)^{\lfloor n/2 \rfloor} \frac{1}{n}, \quad n = 1, \ldots. \]

The notation \( \lfloor n \rfloor \) denotes the ceiling function, defined as

\[ \lfloor x \rfloor = \text{smallest integer greater than or equal to } x. \]

This sequence converges sublinearly, not linearly, but, even so, the generalization of Aitken’s \( \Delta^2 \) method accelerates the convergence, as will now be described. Since the sequence limit is zero, the errors are the same as the iterates. Some numerical data are presented in Table 1, and the iterates are represented in Figure 1.

From Table 1 it can be observed that the use of \( \{ \tilde{q}_{n+1} \}_{n=1}^{\infty} \) accelerates convergence compared to the rate of convergence of the original sequence \( \{ q_n \}_{n=1}^{\infty} \), provided that the following choices are made (see Table 1, Example 1 data in bold font): \( \tilde{q}_{1}, \tilde{q}_{3}, \tilde{q}_{5}, \tilde{q}_{9}, \tilde{q}_{13}, \tilde{q}_{17} \) etc. That is, for \( n > 1 \), alternate pairs of iterates, using \( \tilde{q}_{n} \) for two consecutive iterates, then \( \tilde{q}_{n} \) for the next two consecutive iterates, and continue with this pattern until sufficient accuracy is attained. The justification for this choice of iterates is discussed in Section 4.3.

4.2. Example 2

For example 2, we used the sequence:

\[ q_n = (-1)^{\lfloor n/2 \rfloor} 0.9^n + 0.5^n, \quad n = 1, \ldots. \]
The accelerated convergence to the limit, \( q = 0 \), is presented in Table 2, and typical iterates are represented in Figure 2.

Again, as with Example 1, the use of \( \tilde{q}_{\pm n} \) accelerates convergence compared to the rate of convergence of the sequence \( \{ q_n \} \), provided that the following choices are made (see Table 2, data in bold font): \( \tilde{q}_{1+}, \tilde{q}_{2-}, \tilde{q}_{3+}, \tilde{q}_{4-}, \tilde{q}_{5+} \), etc. i.e. for \( n \geq 1 \), alternate pairs of iterates, using \( \tilde{q}_{+} \) for two consecutive iterates, then \( \tilde{q}_{-} \) for next two consecutive iterates, and continue with this pattern until sufficient accuracy is attained.

### Table 1. Example 1 data

| \( n \) | \( q_n \) | \( \tilde{q}_{+n} \) | \( \tilde{q}_{-n} \) |
|--------|-------|----------------|------------------|
| 1      | -1.000000 | 0.047314 | -0.880647 |
| 2      | -0.500000 | 0.231565 | -0.029232 |
| 3      | 0.333333  | 0.323114  | -0.006448  |
| 4      | 0.250000  | 0.005099  | -0.163432  |
| 5      | -0.200000 | 0.002012  | -0.197250  |
| 6      | -0.166667 | 0.123742  | -0.001718  |
| 7      | 0.142857  | 0.141748  | -0.000875  |
| 8      | 0.125000  | 0.000776  | -0.099387  |
| 9      | -0.111111 | 0.000457  | -0.110558  |
| 10     | -0.100000 | 0.082991  | -0.000415  |
| 11     | 0.090909  | 0.090594  | -0.000268  |
| 12     | 0.083333  | 0.000247  | -0.071128  |
| 13     | -0.076923 | 0.000171  | -0.076727  |
| 14     | -0.071429 | 0.062361  | -0.000159  |
| 15     | 0.066667  | 0.066537  | -0.000115  |
| 16     | 0.062500  | 0.000108  | -0.055460  |
| 17     | -0.058824 | 0.000081  | -0.058733  |
| 18     | -0.055556 | 0.049231  | -0.000077  |
| 19     | 0.052632  | 0.052566  | -0.000060  |
| 20     | 0.050000  | 0.000057  | -0.045403  |
4.3. Example 3
For Example 3, the data were obtained from Pomeranz (2011, Sec. 5.1). The nodes used are some of the boundary nodes from a square domain, and the numerically obtained outward normal flux values, \( q \), were computed using the boundary element method. A test problem with known exact flux was used. A Mathematica (Wolfram Research, 2016) computer program was run for three different grid spacings, a coarse grid, an intermediate grid, and a relatively fine grid. Richardson extrapolation was performed at each node using the three flux values computed at that node. The order of the

| \( n \) | \( q_n \)   | \( \hat{q}_{n+1} \) | \( \hat{q}_{n-1} \) |
|-------|----------|-----------------|-----------------|
| 1     | -0.4000000 | 0.0377603       | -0.3707603      |
| 2     | -0.5600000 | 0.6996998       | 0.2336002       |
| 3     | 0.8540000  | 0.8429691       | 0.0230109       |
| 4     | 0.7186000  | 0.0563201       | -0.5141681      |
| 5     | -0.5592400 | 0.0051883       | -0.5575696      |
| 6     | -0.5158160 | 0.4311883       | 0.0141998       |
| 7     | 0.4861094  | 0.4833487       | 0.0013458       |
| 8     | 0.4343735  | 0.0035400       | -0.3456716      |
| 9     | -0.3854674 | 0.0003328       | -0.3836190      |
| 10    | -0.3477019 | 0.2808990       | 0.0008858       |
| 11    | 0.3142989  | 0.3127074       | 0.0000835       |
| 12    | 0.2826737  | 0.0002214       | -0.2273025      |
| 13    | -0.2540645 | 0.0000209       | -0.2527992      |
| 14    | -0.2287069 | 0.1841715       | 0.0000553       |
| 15    | 0.2059216  | 0.2048908       | 5.2 \times 10^{-4} |
| 16    | 0.1853173  | 0.0000138       | -0.1491648      |
| 17    | -0.1667642 | 1.3 \times 10^{-6} | -0.1659307      |
| 18    | -0.1500908 | 0.1208270       | 3.5 \times 10^{-4} |
| 19    | 0.1350871  | 0.1344116       | 3.3 \times 10^{-2} |
| 20    | 0.1215776  | 8.6 \times 10^{-7} | -0.0978690      |
The dominant error term in the flux computation (referred to as a ‘p value’) was approximated numerically (Pomeranz, 2011, Table 1, p. 2321; Smith, 2010, p. 249).

Richardson flux extrapolation in which the order of convergence was unknown, and which accordingly required results from three different grid spacings, gave results that were identical to the standard Aitken's $\Delta^2$ method results, provided that the three errors (from the coarse, intermediate, and fine grids) had the same sign (Pomeranz, 2011, Sec. 3). Richardson extrapolation does not apply at a given node if the error signs from each of the three grid spacings are not the same. Eight boundary nodes that appeared to present numerical difficulties were selected based on the criterion of having “bad p values”, as described in Pomeranz (2011, Sec. 5.1). The behavior of the errors and acceleration of convergence is now examined in detail for this example (Example 3). The difficulties in convergence behavior occurred because the outward normal flux error signs were not as required, and consequently the standard Aitken’s formula did not apply. However, when the standard Aitken’s

### Table 3. Example 3 data

| Node no. | $q_1$ coarse grid flux | $q_2$ medium grid flux | $q_3$ fine grid flux |
|----------|------------------------|------------------------|----------------------|
| 1        | -11.296351             | -9.990429              | -10.129195           |
| 9        | -9.925163              | -8.678436              | -8.813033            |
| 17       | -10.278317             | -10.319775             | -10.313804           |
| 18       | -12.960135             | -11.626693             | -11.715375           |
| 19       | 9.802039               | 8.606147               | 8.670163             |
| 21       | 7.868038               | 7.863586               | 7.864025             |
| 27       | 0.551749               | 0.765593               | 0.738812             |
| 28       | -2.469050              | -1.921923              | -1.968516            |

### Table 4. Example 3 comparison of methods

| Node no. | $\bar{q}$ | $\bar{q}^+$ | $\bar{q}^-$ | Exact $q$ |
|----------|-----------|-------------|-------------|-----------|
| 1        | -10.115866| -10.152011  | -10.551191  | -10.139897|
| 9        | -8.799918 | -8.835749   | -9.211785   | -8.823836 |
| 17       | -10.314556| -10.303817  | -10.312019  | -10.313108|
| 18       | -11.709845| -11.722848  | -12.241600  | -11.715044|
| 19       | 8.666910  | 9.167968    | 8.674279    | 8.667106  |
| 21       | 7.863985  | 7.865528    | 7.864089    | 7.864173  |
| 27       | 0.741993  | 0.732298    | 0.679575    | 0.734541  |
| 28       | -1.964860 | -1.973993   | -2.166713   | -1.972111 |

### Table 5. Example 3 best choices for nodal flux

| Node no. | Sign of $e_1 \cdot e_3$ | Sign of $e_1 + 2e_2 + e_3$ | Best choice | Value of best choice |
|----------|--------------------------|-----------------------------|-------------|----------------------|
| 1        | -                        | -                           | $\bar{q}^+$ | -10.152011           |
| 9        | -                        | -                           | $\bar{q}^+$ | -8.835749            |
| 17       | -                        | +                           | $\bar{q}^-$ | -10.312019           |
| 18       | +                        |                             | $\bar{q}$   | -11.709845           |
| 19       | +                        |                             | $\bar{q}^-$ | 8.666910             |
| 21       | -                        | +                           | $\bar{q}^-$ | 7.864089             |
| 27       | -                        | -                           | $\bar{q}^+$ | 0.732298             |
| 28       | -                        | -                           | $\bar{q}^+$ | -1.973993            |
\[ \Delta^2 \text{ formula (and Richardson's method formula) did not apply, one of our two extended formulas did improve results (see Theorem 3.1).} \]

The eight indicated boundary nodes, nodes numbered 1, 9, 17, 18, 19, 21, 27, and 28 were selected because these were the nodes at which complex values (with non-zero imaginary parts) for \( p \), the order of dominant flux error term, were obtained numerically (Pomeranz, 2011, Table 1). This is an indicator of numerical difficulties, since the order of convergence should be a positive real number at each node.

In Example 3, by observing the signs of the outward normal boundary flux errors, we can predict which variant of Aitken's \( \Delta^2 \) method should be applied. Also, we can confirm that the numerical results agree with the results predicted from the theory/discussion following Equation (2). The data are given in Table 3. The best convergence (values which minimize the absolute error) is indicated by the bold font values in Table 4.

The choice of which approximation is best in the sense of minimizing the absolute error is determined as follows. In this discussion, the exact value for \( q \), the outward normal boundary flux, is known. This would not be the case in practice, so some other information about the expected behavior of the iterates could be used, the next iterate could be used to approximate \( q \), or, in some cases, observed values/pattern of the iterates might suggest which choices give fastest convergence.

Let the error at the \( n \)th iteration be defined and denoted as

\[ e_n \equiv q_n - q, \quad n = 1, \ldots. \]

The first decision is whether to use \( \hat{q} \), the standard Aitken's \( \Delta^2 \) method value, or one of two values from the extension of the standard method, \( \hat{q}^+ \) or \( \hat{q}^- \), as developed in this paper. At a specific node and for iteration \( n \),

1. If \((q_n - q) \cdot (q_{n+2} - q) \equiv e_n \cdot e_{n+2} > 0\), use the standard Aitken's method iterate, \( \hat{q}_n^+ \).
2. If \((q_n - q) \cdot (q_{n+2} - q) \equiv e_n \cdot e_{n+2} < 0\), use an extended Aitken's method iterate, either \( \hat{q}_n^+ \) or \( \hat{q}_n^- \).

   (i) If \( e_n + 2e_{n+1} + e_{n+2} \equiv (q_n - q) + 2(q_{n+1} - q) + (q_{n+2} - q) \leq 0\), use \( \hat{q}_{n+1}^+ \).

   (ii) If \( e_n + 2e_{n+1} + e_{n+2} \equiv (q_n - q) + 2(q_{n+1} - q) + (q_{n+2} - q) > 0\), use \( \hat{q}_{n+1}^- \).

The choice of (i) or (ii) indicates which iterate, \( \hat{q}_{n+1}^+ \) or \( \hat{q}_{n+1}^- \), respectively, minimizes the absolute error, given in Equation (2). These values are displayed in Table 5. The drawback is that to use these formulas directly, the sequence limit, \( q \), is needed. This is not known in practice. Our intent here is to show

| Node no. | \( q \) error | \( \hat{q}^+ \) error | \( \hat{q}^- \) error |
|---------|---------------|----------------|----------------|
| 1       | 0.0240308     | −0.0121136     | −0.411294      |
| 9       | 0.0239181     | −0.0119132     | −0.38795       |
| 17      | −0.00144829   | 0.00929111     | 0.0010883      |
| 18      | 0.00519885    | −0.00780405    | −0.526556      |
| 19      | −0.00019554   | 0.500862       | 0.00717336     |
| 21      | −0.000187804  | 0.00135481     | −0.0000841862  |
| 27      | 0.00745177    | −0.00224339    | −0.0549658     |
| 28      | 0.00725122    | −0.00188214    | −0.194602      |
how the theory and numerical results agree for a specific sample test problem. The theory and the numerical results do agree, as is shown in Tables 5 and 6. In Table 6 we compare the error magnitudes at these eight nodes for each of the three possible formulas for the variants of Aitken’s $\Delta^2$ method, $\hat{q}$, $q+$, and $q−$. The minimum error magnitudes for the flux values at these eight nodes are, respectively, given by formulas $\hat{q}+$, $q+$, $q−$, $\hat{q}$, $\hat{q}−$, $\hat{q}+$, and $q+$, in agreement with the predictions from the theoretical analysis (see bold font values in Tables 4 and 6).

5. Conclusions
We have developed an extension of Aitken’s $\Delta^2$ method which, although very specialized, accelerates the convergence of iterative sequences of the form described in Sections 1 and 3. A proof for this extension was developed and two numerical examples illustrating application of the method were given. A third numerical example comparing results from this extended method and Richardson extrapolation was presented.

Funding
This work was supported by The University of Tulsa.

Author details
Shirley B. Pomeranz1
E-mail: pomeranz@utulsa.edu
1 Department of Mathematics, College of Engineering and Natural Sciences, The University of Tulsa, 800 S. Tucker Drive, Tulsa 74104OK, USA.

Citation information
Cite this article as: Aitken’s $\Delta^2$ method extended, Shirley B. Pomeranz, Cogent Mathematics (2017), 4: 1308622.

References
Bumbariu, O. (2012). A new Aitken type method for accelerating iterative sequences. Applied Mathematics and Computation, 219, 78–82.
Buoso, D., Karapiperi, A., & Pozza, S. (2015). Generalizations of Aitken’s process for a certain class of sequences. Applied Numerical Mathematics, 90, 38–54.
Burden, R., Faires, D., & Burden, A. (2016). Numerical analysis (10th ed.). Boston, MA: Cengage Learning.
Issacson, E., & Keller, H. B. (1966). Analysis of numerical methods. New York, NY: John Wiley & Sons.
Pomeranz, S. (2011). Richardson extrapolation applied to boundary element method results in a Dirichlet problem for the Laplace equation. International Journal of Computer Mathematics, 88, 2306–2330.
Reich, S. (1970). On Aitken’s $\Delta$-method. The American Mathematical Monthly, 77, 283–284. Retrieved August 12, 2015, from http://www.jstor.org/stable/2317712
Smith, G. D. (2010). Numerical solution of partial differential equations: Finite difference methods (3rd ed.). New York, NY: Oxford University Press.
Stoer, J., & Bulirsch, R. (2010). Introduction to numerical analysis (3rd ed.). New York, NY: Springer-Verlag.
Wolfram Research. (2016). Mathematica. Retrieved from http://www.wolfram.com/