The computational complexity of the gear placement problem

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Abstract
In this paper, we analyze the complexity of the gear placement problem (GPP). In the GPP, we are given a rectangular plane, called a gearbox, on which a torque generator source and a set of gears, called target gears, are placed. The task is to find a placement of a set of gears called sub-gears, to connect every target gear to the torque generator source so that every target gear rotates in a given direction. The objective is to minimize the number of sub-gears to be used. We prove that the GPP is NP-hard by giving a reduction from the Hamiltonian path problem on 3-regular planar graphs, which is known to be NP-complete, to the GPP. We also present an upper bound for the number of sub-gears to be placed.

Keywords: Computational complexity, Gear placement problem, NP-hard, Reduction

1. Introduction
In this paper, we present a problem that arises in machine design and manufacturing. Given a gearbox, a torque generator, and some gears that need to be rotated in specified directions, the task is to place into the gearbox the minimum number of extra gears to transmit the torque from the source to the other gears. We call this problem the gear placement problem (GPP).

A gear often consists of a wheel with relatively small teeth around the edge and can be represented as a circle of radius equal to the nominal radius of the gear, and the gearbox can be represented by a rectangular container. Hence the GPP is a variant of the problem of packing circular objects into a rectangular container, with additional constraints with respect to connections between gears. The problem of packing circular objects into a container has been the target of several works in the literature, not only in discrete optimization, but also in continuous optimization (Hifi and M’hallah, 2009). One of the standard versions of the problem of packing circular objects into a container asks to pack as many possible uniform circles into a square container without overlap. Optimal solutions for this problem were found in Birgin and Gentil (2010) for instances with up to $n = 50$ unit radius circles. Variations of this problem include packing unit circles in triangular, rectangular, or circular containers (López and Beasley, 2011) and packing $n$ uniform/non-uniform circles into a unit square maximizing the circles radii (Castillo et al., 2008). Real-world applications may require static stability (i.e., each circle must be in contact with at least three other circles, and for every circle, it is not allowed to have all such contact points on a semicircle) (Goldberg, 1970). For further details, the authors recommend the comprehensive survey paper by Hifi and M’hallah (2009). Although the circle packing problem and its variants have been extensively studied, to the best of the knowledge of the authors, the problem of packing circular objects considering rotation direction and connectivity among circles has not been addressed in the literature.
In this paper, we give a proof that the GPP is NP-hard by giving a polynomial time reduction from the Hamiltonian path problem on 3-regular planar graphs, which is known to be NP-hard (Garey et al., 1976), to the GPP. We also present an upper bound on the number of additional gears required to rotate all gears in the desired directions.

2. The Gear Placement Problem

In this section, we formally define the gear placement problem. Given a rectangular gearbox of fixed width and height, a torque generator (called the root gear) and some gears that need to be rotated in specified directions (called target gears), where the sizes and the positions of the root and target gears are fixed, the task is to place into the gearbox the minimum number of extra gears (called sub-gears) of radius in the given interval \([r_{\text{min}}, r_{\text{max}}]\) to transmit the torque from the root to the target gears. We say that two gears mesh if the distance between their contact surfaces is within the tolerance \(\epsilon\) (that is, two gears with radii \(r_1\) and \(r_2\) mesh if the distance between their centers is between \(r_1 + r_2 - \epsilon\) and \(r_1 + r_2 + \epsilon\)). We also say that two gears are connected if there is a sequence of meshed gears between them (i.e., gears \(g_1\) and \(g_l (l \geq 2)\) are connected if there is a sequence of gears \(g_1, g_2, \ldots, g_l\) such that \(g_i\) meshes \(g_{i+1}\) for every \(i = 1, 2, \ldots, l - 1\)). We call such a sequence a connecting sequence, and we define its length to be the number of gears between the two gears (i.e., the length of a connecting sequence \(g_1, g_2, \ldots, g_l\) is \(l - 2\)). If a gear rotates clockwise (resp., anti-clockwise), all gears meshed to it rotates in the opposite direction, anti-clockwise (resp., clockwise); hence, if the specified rotation direction of a target gear is the same as (resp., opposite of) the root gear, the length of every connecting sequence between them must be odd (resp., even).

Note that if a gear meshes two gears rotating in different directions, the three gears are not able to properly rotate. To avoid this and for all gears to properly rotate, if there is a connecting sequence from the root gear to the target gears respecting the desired rotation directions, using \(\epsilon\), there are no such cycles in good layouts because one connecting sequence from the root gear suffices for every target gear. In a feasible placement of sub-gears for the GPP, all sub-gears are of radii in the specified range and are placed inside the given rectangular gearbox, none of the sub-gears overlaps with other gears, every target gear is connected to the root gear, and all target-gears properly rotate in the specified directions. Then the objective of the problem is to find a feasible placement of minimum number of sub-gears.

![Diagram of Gear Placement Problem](image)

Fig. 1 An example of input (left) and output (right) of the GPP

Figure 1 gives an example of input and output for the GPP. The left picture of Fig. 1 shows the bounding area (gearbox), the root gear and the target gears. The right picture represents a feasible placement of sub-gears, which connects the root gear to the target gears respecting the desired rotation directions. For a given input, there are many possible layouts, and to choose the most desirable one among them, various criteria are possible. In this paper, we focus on minimizing the total number of sub-gears to be used, which is important in reducing production costs. We note that, with this criterion, it is not necessary to distinguish the root gear from target gears (because all target gears are connected to the root gear if and only if all pairs of the given gears, including the root and target gears, are connected), so throughout the remainder of this paper we consider the root gear as a target gear.

In the next section, we analyze the complexity of the decision version of the GPP. In the decision version, we are given a gearbox, some target gears, their rotation directions, an allowable interval \([r_{\text{min}}, r_{\text{max}}]\) of the radius of sub-gears, and a constant \(s\) that specifies an upper bound on the number of sub-gears. Then, the problem is to decide if there is a feasible layout that connects all of the target gears while respecting the desired rotation directions, using \(s\) or less sub-gears of radii in the specified range.
3. NP-hardness Proof

In this section, we prove that the GPP is NP-hard by giving a polynomial time reduction from the Hamiltonian path problem on 3-regular planar graphs. In the proof, we use some important results that make it possible to draw a 3-regular planar graph as an orthogonal graph in the plane in polynomial time and polynomial size of the number of vertices.

3.1. Hamiltonian Path on 3-Regular Planar Graphs

This section describes the Hamiltonian path problem and its complexity on 3-regular planar graphs. For a graph, a Hamiltonian path is a path that visits every vertex in the graph exactly once. The Hamiltonian path problem is the problem of determining whether a given graph has a Hamiltonian path (see Fig. 2 (a)). The Hamiltonian path problem on general graphs is known to be NP-hard (Garey and Johnson, 1979). It has been shown that even when the graph is limited to be 3-regular and planar, the problem remains NP-hard (Garey et al., 1976).

![Fig. 2 (a): A Hamiltonian path on a 3-regular planar graph. (b): A planar representation of the graph on a square grid.](image)

**Theorem 1.** (Garey et al., 1976) The Hamiltonian path problem on 3-regular planar graphs is NP-complete.

3.2. Planar Orthogonal Graph Drawing

In this section, we summarize some results in planar orthogonal graph drawing. A planar orthogonal drawing of a planar graph is a representation in which each edge of the graph is a polygonal chain consisting of only horizontal and vertical line segments (Biedl, 1996) without overlap. A graph admits such a drawing if it has maximum degree 4 (Papakostas and Tollis, 1995). An example of the planar orthogonal representation of Fig. 2 (a) in a square grid is shown in Fig. 2 (b).

Two important measurements of the quality of a drawing are the grid size and the number of bends. Minimizing the number of bends was proved to be NP-hard (Garg and Tamassia, 2001). Several polynomial time algorithms with theoretical guarantees on such measurements have been developed, and an overview of the bounds provided by such algorithms is summarized in Biedl (1996).

In this work, we use the following property.

**Theorem 2.** (Papakostas and Tollis, 1995) There exists a polynomial time algorithm that draws any 3-regular planar graph into a 2-dimensional square grid. The required size of the grid is a polynomial size of $|V|$.

3.3. Outline of the Proof

In this section, we give an outline of the proof, presenting key properties that will be satisfied by the GPP instance generated for the reduction from the Hamiltonian path problem, and then, in the subsequent sections, we show the details of how we generate such a GPP instance.

Given a 3-regular planar graph $G = (V, E)$ as an instance of the Hamiltonian path problem, we give a polynomial time algorithm to construct an instance of the decision version of the GPP. Then, we prove that there is a Hamiltonian path on $G$ if and only if there is a feasible layout of sub-gears.

For a given 3-regular planar graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_{|V|}\}$, consider the following:

- For each vertex $v_i \in V$, we create a group of gears called a **node gadget** $NG_i$. A node gadget $NG_i$ is composed of 5 gears, 4 bigger ones called **corner gears** and 1 smaller one called the **center gear**, as shown in Fig. 3. We note that each gear in a node gadget has the same radius as the correspondents in other node gadgets.

- For each edge $e = (v_i, v_j) \in E$, we create a group of gears called an **edge gadget** $EG_{(i,j)}$, which consists of a polynomial number (with respect to the number of vertices $|V|$) of pairs of gears, distributed between $NG_i$ and $NG_j$. In Fig. 3, $EG_{(i,j)}$
is illustrated by the 12 smallest gears consisting of 6 vertically arranged pairs of gears between the node gadgets. Even though the shape of an edge gadget can be asymmetric as in Fig. 3, its functional property is symmetric, and for each undirected edge, the corresponding edge gadget can take either direction.

Using the target gears in the node gadgets and in the edge gadgets, we create a new instance, denoted \( I(G) \), for the GPP from graph \( G = (V, E) \) of the Hamiltonian path instance. Figure 4 shows an instance of the GPP created from the graph in Fig. 2. In order to connect every target gear in edge gadget \( EG_{(i,j)} \) to a target gear of a node gadget \( NG_i \) or \( NG_j \), we add sub-gears, which we call connection gears. In Fig. 5 (resp., Fig. 6), the connection gears are illustrated by the 9 (resp., 8) horizontally aligned gears between \( NG_i \) and \( NG_j \). As a result, all gears in \( EG_{(i,j)} \) are connected to \( NG_i \) or \( NG_j \). In the instance \( I(G) \), we allow only one size for sub-gears (i.e., we set \( r_{\min} = r_{\max} \)) in order to force feasible layouts to have limited possibilities such as those in Figs. 5 and 6. Figure 7 shows a yes certificate to the GPP instance in Fig. 4, that is, a feasible layout of sub-gears that connects all target gears, respecting the desired rotation directions. In Fig. 7, each edge contained in the Hamiltonian path of Fig. 2 corresponds to the sequence of sub-gears (solid-line circles) connecting the two node gadgets corresponding to the end vertices of the edge, and each edge not contained in the Hamiltonian path corresponds to the sequence of sub-gears (dashed-line circles) that does not connect the two node gadgets. The instance \( I(G) \) is created in such a way as to fulfill the following properties:

**Property 1:** For each edge gadget \( EG_{(i,j)} \), there is a unique layout of sub-gears that connect two corner gears from \( NG_i \) with other two from \( NG_j \) with the minimum number of sub-gears. Such a layout is called the outer connection layout and is denoted by \( OG_{(i,j)} \). An example is shown in Fig. 5.
Property 2: For each edge gadget $EG_{i,j}$, there is a unique layout of sub-gears that connect every gear in $EG_{i,j}$ with the center gear of either $NG_i$ or $NG_j$ with the minimum number of sub-gears. Such a layout is called the inner connection layout and is denoted by $IG_{i,j}$. Moreover, adding one more gear of the same size to the inner connection layout does not connect $NG_i$ with $NG_j$. An example is shown in Fig. 6.

Property 3: For each node gadget $NG_i$, let $EG_{i,j_1}$, $EG_{i,j_2}$ and $EG_{i,j_3}$ be the three edge gadgets incident to $NG_i$. When the gears are connected using the outer or inner connection layout, the center gear of $NG_i$ is connected to all of the corner gears of $NG_i$ if and only if at least one of the three edge gadgets $EG_{i,j_1}$, $EG_{i,j_2}$ and $EG_{i,j_3}$ is connected with the inner connection layout.

Property 4: For each $(i, j) \in E$, $|OG_{i,j}| = |IG_{i,j}| + 1$, where $|OG_{i,j}|$ and $|IG_{i,j}|$ denote the number of sub-gears in $OG_{i,j}$ and $IG_{i,j}$, respectively.

Property 5: The upper bound on the number of sub-gears $s$ that we can use is set to $\sum_{(i,j) \in E} |IG_{i,j}| + |V| - 1$.

Property 6: If there is a layout that connects all target gears using $s$ or less sub-gears, there also exists a layout consisting only of outer and inner connection layouts.

Property 7: For each edge gadget, the rotation direction constraints are satisfied for any combination of outer and inner connection layout.

The gearbox size of instance $I(G)$ is defined to be the minimum size such that all node and edge gadgets, as well as all inner and outer connection layouts, can be placed inside it.

![Fig. 5 An example of outer connection layout](image1)

![Fig. 6 An example of inner connection layout](image2)

Theorem 3. Given a 3-regular planar graph $G = (V, E)$ of an instance of the Hamiltonian path problem, suppose that the instance $I(G)$ created for the GPP satisfies Properties 1–7. Then, $G$ has a Hamiltonian path if and only if the answer of the GPP for $I(G)$ is yes.

Proof. Suppose that the answer of the GPP for $I(G)$ is yes. By Property 6, there exists a layout composed of only inner and outer connection layouts, connecting all target gears. Let us call this layout $W$. We also define $P_W \subseteq E$ to be the set of edges of $G$ corresponding to all gears in outer connection layouts. By Properties 1 and 2, in order to connect two node gadgets, it is necessary to connect them by the outer connection layout, and because all gears are connected in $W$, the graph consisting of vertex set $V$ and edge set $P_W$, denoted by $G[P_W] = (V, P_W)$, must be connected. By Property 4, $|OG_{i,j}| - |IG_{i,j}| = 1$, and, by Property 5, the number of gears used in the outer connection layouts of $W$ is at most $|V| - 1$ (i.e., $|P_W| \leq |V| - 1$), which implies that the graph $G[P_W]$ is a spanning tree of $G$. By Property 3, if none of the three edge gadgets incident to a node gadget $NG_i$ takes the inner connection layout, the center gear of $NG_i$ is not connected. This

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means that any vertex of $G[P_W]$ has degree 2 or less. In other words, $G[P_W]$ represents a Hamiltonian path (Fig. 2). This shows that, if the answer of the GPP for $I(G)$ is yes, then there exists a Hamiltonian path.

On the other hand, suppose that there exists a Hamiltonian path in $G$. By the above facts, one can guarantee that all target gears are connected by the layout that uses the outer connection layout for those gears corresponding to the edges in the Hamiltonian path and uses the inner connection layout for the other edges. The number of sub-gears used is $s$, and the correct rotation direction of any gear is guaranteed by Property 7. Hence, if there exists a Hamiltonian path in $G$, the answer of the GPP for $I(G)$ is yes.

3.4. Generation of Instance $I(G)$

In this section, we give the steps to generate from graph $G$, the instance $I(G)$ that satisfies Properties 1–7.

By Theorem 2, we can draw the given graph $G$ on a 2-dimensional square grid, and in such a drawing, the vertices of $G$ are placed on grid points, and edges of $G$ are drawn on grid lines without intersection. Then we place node gadgets on those grid points at which the vertices are placed, and we place edge gadgets along those grid lines on which the edges are drawn. We assume that all sub-gears have a unitary radius (i.e., $r_{\min} = r_{\max}$) and all target gears have to be rotated in the same direction. From Step 1 to Step 3, we assume that all numeric values are real numbers, and then, in Section 3.5, we transform the real values to rational values and prove that none of the important properties are lost.

**Step 1:** (Drawing the graph on the grid). Using the algorithm cited by Theorem 2, we draw the graph $G = (V, E)$ on the grid. The grid point corresponding to vertex $v_i \in V$ is written as $v_i^0$, and we define $V'$ to be the set consisting of all those grid points $v_i^0$ corresponding to $v_i \in V$. In the same way, the path (consisting of grid lines) corresponding to edge $e = (v_i, v_j) \in E$ is written as $e_{i,j}^0$, and the union of those paths is denoted by $E'$.

**Step 2:** (Construction of a node gadget). For an odd constant $A$ greater than or equal to 7, we set the height and width of a cell of the grid (i.e., the distance between two adjacent parallel grid lines) to be equal to $w = 2(A - 1) + 4\sqrt{2}$. For each $v_i \in V$, we create a node gadget $NG_i$ and place the center of the center gear at the grid point $v_i^0$ of the square grid (see Fig. 8 (a)). The radius and position of each gear in $NG_i$ are set to satisfy the following properties.

- The gears of $NG_i$ can be connected by unity radius sub-gears as in Figs. 8 (b) and 8 (c).
- In the configuration of Fig. 8 (b), the distance between the center of a sub-gear and the center of the center gear of $NG_i$ is $2\sqrt{2}$.
- The rotation directions of all five gears in a node gadget are the same and are set to be clockwise.

**Step 3:** (Construction of an edge gadget). For each edge $e = (v_i, v_j) \in E$, we create an edge gadget. We first calculate the placement of the gears in the outer and inner connection layout $OG_{i,j}$ and $IG_{i,j}$. Then we place the edge gadget $EG_{i,j}$ in such a way that this layout becomes unique.

![Fig. 7 A yes certificate for the instance in Fig. 4](image-url)
Consider a simple case in which $NG_i$ and $NG_j$ are aligned horizontally in a straight line as in Fig. 9. In this case, the corresponding outer connection layout $OG_{(i,j)}$ is the one in Fig. 10, and the corresponding inner connection layout $IG_{(i,j)}$ is the one in Fig. 11, which is obtained by removing a gear (any gear except the three gears at each end) from $OG_{(i,j)}$ and shifting all gears to the left (resp., right) of the removed gear toward the left (resp., right) until the gear at the end touches the center gear of the node gadget. To determine the positions of the gears in $EG_{(i,j)}$, we draw both of the inner and outer connection layouts as in Fig. 12. Then we place small gears (the gears in $EG_{(i,j)}$) above and below some intersection points of two gears, one from $OG_{(i,j)}$ and another from $IG_{(i,j)}$, as in Fig. 13, so that each gear in $EG_{(i,j)}$ meshes with one gear from $OG_{(i,j)}$ and another from $IG_{(i,j)}$.

In general, the path $e'_{(i,j)}$ in the grid corresponding to an edge $e = (v_i, v_j)$ is represented by several vertical and horizontal line segments on the grid. As a result, it is difficult to represent the layout using only the above described straight connection of gears. In this paper, we resolve this problem by slightly bending the path as in Fig. 14.

We split the construction of the edge gadget into three parts. In Step 3a (resp., Step 3b), we explain how to construct an outer (resp., inner) connection layout, and in Step 3c, we explain how to place the edge gadget.

**Step 3a:** (Outer connection layout) For each path $e'_{(i,j)} \in E'$, starting from node gadget $NG_i$ and the configuration shown in Fig. 8 (b), we add sub-gears following the path to $NG_j$ in the grid using Algorithm 1 (an example of the output can be seen in Fig. 14). We set this group of connected gears to be $OG_{(i,j)}$. We denote by $OG_{(i,j)} = \left(OG_{(i,j)}^{0}, OG_{(i,j)}^{1}, \ldots, OG_{(i,j)}^{\text{last}}\right)$ the sequence of connected gears in $OG_{(i,j)}$, where $OG_{(i,j)}^{0}$ represents the first gear connected directly to $NG_i$ and $OG_{(i,j)}^{\text{last}}$ the last gear connected directly to $NG_j$. In the same way, we denote by $e'_{(i,j)} = \left(e_{(i,j)}^{0}, \ldots, e_{(i,j)}^{\text{last}}\right)$ the grid segments (i.e., line segments).
Fig. 11  Inner connection layout $IG_{(i,j)}$

Fig. 12  Outer connection layout and inner connection layout

Fig. 13  Edge gadget $EG_{(i,j)}$
segments between two adjacent grid mechanical design) used in path $e'_{(i,j)}$ ($e'_{(i,j)}^{(l_{max})}$ represents the last grid segment, and one of its end points is the grid point $e'_{(i,j)}$).

**Algorithm 1 Outer connection layout**

1. for each $e'_{(i,j)}$ in $E'$
2. $OG_{(i,j)} := \emptyset$
3. insert the sequence $\left(OG_{(i,j)}^{0}, \ldots, OG_{(i,j)}^{A} \right)$ shown in Fig. 15 to the last position of $OG_{(i,j)}$
4. $l := A - 1; h := 1$
5. while ($h \leq h_{lim}$)
6. if ($e'_{(i,j)}^{(h-1)}$ and $e'_{(i,j)}^{(h)}$ are both horizontal or both vertical)
7. insert the sequence $\left(OG_{(i,j)}^{h-1}, \ldots, OG_{(i,j)}^{h+A} \right)$ shown in Fig. 16 to the last position of $OG_{(i,j)}$
8. $l := l + A + 3; h := h + 1$
9. }end if
10. else
11. insert the sequence $\left(OG_{(i,j)}^{h+1}, \ldots, OG_{(i,j)}^{h+A+1} \right)$ shown in Fig. 17 to the last position of $OG_{(i,j)}$
12. $l := l + A + 1; h := h + 1$
13. }end else
14. }end while
15. }end for
16. output: $OG_{(i,j)}$

**Step 3b:** (Inner connection layout). For each path $e'_{(i,j)} \in E'$, we obtain the inner connection layout by modifying $OG_{(i,j)}$. We first remove a sub-gear of $OG_{(i,j)}$ on the last grid segment $e'_{(i,j)}^{(l_{max})}$ of the path $e'_{(i,j)}$. To be more precise, we choose an odd number $B (3 \leq B \leq A - 3)$ and remove sub-gear $OG_{(i,j)}^{(A+1)b_{lim}-(B-2)}$. Then we shift the sub-gears $OG_{(i,j)}^{(0)}, OG_{(i,j)}^{(1)}, \ldots, OG_{(i,j)}^{(A+1)b_{lim}-(B-1)}$ in the direction of $NG_i$ and $OG_{(i,j)}^{(A+1)b_{lim}-(B-3)}$, $OG_{(i,j)}^{(A+1)b_{lim}-(B-4)}$, $\ldots, OG_{(i,j)}^{(l_{max})}$ in the direction of $NG_j$ until the first and last gears of $OG_{(i,j)}$ touch the center gear of $NG_i$ and $NG_j$, respectively (Figure 18 represents the layout obtained by applying Step 3b to the $OG_{(i,j)}$ illustrated in Fig. 14). At the places where the layout of the sub-gears bends, gears are placed as in Fig. 19. We denote by $IG_{(i,j)} = \left(I_{(i,j)}^{(0)}, \ldots, I_{(i,j)}^{(A+1)b_{lim}-(B-1)}, I_{(i,j)}^{(A+1)b_{lim}-(B-3)}, \ldots, I_{(i,j)}^{(l_{max})} \right)$ the sequence of sub-gears in $IG_{(i,j)}$, where $I_{(i,j)}^{(l)}$ corresponds to the sub-gear shifted from $OG_{(i,j)}^{(l)}$. As a result of removing sub-gear $OG_{(i,j)}^{(A+1)b_{lim}-(B-2)}$, the corresponding sub-gear $I_{(i,j)}^{(A+1)b_{lim}-(B-2)}$ is not in $IG_{(i,j)}$. An example can be seen in Fig. 20.

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Fig. 15 Construction of $OG_{(i,j)}$ for the first grid segment

Fig. 16 Construction of $OG_{(i,j)}$ for a grid segment in the same direction as the previous one

Fig. 17 Construction of $OG_{(i,j)}$ for a grid segment orthogonal to the previous one

Fig. 18 An example of $IG_{(i,j)}$
Step 3c: (Edge gadget). For each path $e^{(i,j)} \in E'$, the gear $OG_{(i,j)}$ of the outer connection layout $OG_{(i,j)}$ and the gear $IG_{(i,j)}$ of the inner connection layout $IG_{(i,j)}$ overlap. For each $l = 1, 2, \ldots, (A+1)h_{last}-(B-3), \ldots, l_{max}$, we add a pair of small gears as shown in Fig. 21, which we call $EG_{(i,j)}$. In the last grid segment $e^{(h_{last})}$, at the place where a gear in $OG_{(i,j)}$ is removed, we place the gears of $EG_{(i,j)}$ as shown in Fig. 22. The gears in the edge gadgets with even indices (i.e., $EG_{(2)}$, $EG_{(4)}$, $\ldots$) are set to have the same rotation direction as the center gear, and the gears in the edge gadgets with odd indices (i.e., $EG_{(1)}$, $EG_{(3)}$, $\ldots$) are set to have the opposite rotation direction of the center gear.

3.5. Transformation to Rational Numbers

We create a rational squared lattice in which the $x$- and $y$-coordinates of any crossing point of horizontal and vertical lattice lines are multiples of $1/M$ (as shown in Fig. 23), where $M$ is a positive integer such that $1/M$ is sufficiently small compared to the minimum distance $x_{min}$ between the contact surfaces of two gears that are not meshed among those generated in Steps 1–3 of Section 3.4, including all target gears and sub-gears in any layout obtainable by choosing one of the inner and outer connection layouts for each edge. Even though there are $O(2^{(d)})$ possible layouts with respect to the combinations of inner and outer connection layouts, $x_{min}$ depends only on local configurations of a small number of gears and can be computed in polynomial time. We move the centers of each target gear created in Steps 1–3 of Section 3.4 to the nearest crossing point in the lattice that has equal or smaller coordinates, and we decrease the radius of each target gear to the closest multiple of $1/M$. Then we set the tolerance $\varepsilon$ to be a sufficiently small multiple of $1/M$ that satisfies $\varepsilon \geq (\sqrt{2} + 2)/M$. This condition ensures that for every feasible layout with real coordinates (with tolerance 0), there is a
layout with rational coordinates (with tolerance $\epsilon$) having the same connection topology (i.e., the set of meshed pairs of gears). With the exception of one gear in each $OG_{(i,j)}$, all sub-gears must be connected to at least two target gears in every feasible layout with $s$ or less sub-gears; hence the tolerance error does not accumulate.

When the tolerance is 0, all gears need to be meshed with perfection, while when the tolerance is positive, there is an area in which a sub-gear can be placed without changing the connection topology. For example, the sub-gear in Fig. 24 meshes with the two target gears in the figure if its center $O_s$ is placed in the area enclosed by the four circular segments $\text{circ}(O_1, r_1 + r_s + \epsilon)$, $\text{circ}(O_2, r_2 + r_s + \epsilon)$, $\text{circ}(O_1, r_1 + r_s + \epsilon)$, and $\text{circ}(O_2, r_2 + r_s + \epsilon)$, where $\text{circ}(O, r)$ denotes the circle with radius $r$ whose center is placed at point $O$. It is not hard to show that, for a sufficiently small $\epsilon$, the center of the sub-gear can be placed anywhere in such an area so that no gears other than those meshed when the tolerance is 0 will become meshed (i.e., the longest line segment that can be placed in the enclosed area is less than or equal to $x_{\text{min}}/2$ for a sufficiently small $\epsilon$). Note that $x_{\text{min}}$ is a constant that does not depend on the size of $I(G)$ and depends only on local configurations. Furthermore, the shape of the enclosed area in Fig. 24 depends only on local configurations of target gears, and hence the length of the longest line segment that can be placed in such an area is a constant. As a result, neither $M$ nor $\epsilon$ depends on the size of the instance.

### 3.6. Correctness and Time Complexity

In this section, we prove that the time required to create the instance $I(G)$ from the graph $G = (V, E)$ is a polynomial
of $|V|$ (n.b., by the fact that $G = (V, E)$ is a 3-regular graph, we have $|E| = 3|V|/2 = O(|V|)$), and we also prove that instance $I(G)$ satisfies Properties 1–7.

First, we analyze the time required by each step.

By Theorem 2, the time required by Step 1 is a polynomial of $|V|$.

The time required to construct a node gadget is $O(1)$, and in Step 2, we create a node gadget for each $v_i \in V$; therefore the total time required for this step is $O(|V|)$.

By Theorem 2, the size of the grid created to draw the graph $G = (V, E)$ is a polynomial of $|V|$, and by the fact that all edges in $E$ are drawn on the grid without overlapping, we can say that for each grid segment, there exists at most one corresponding path. Moreover, for each grid segment in a path, the number of gears in an edge gadget is bounded by a constant $2(A + 3)$. As a result, the number of gears created by Step 3 is bounded by the number of grid segments; therefore the time required by Step 3 (including Steps 3a–3c) is a polynomial of $|V|$.

Next, we prove that the created instance $I(G)$ satisfies Properties 1–7.

(Property 1 and 2). In order to connect all target gears in each edge gadget $EG_{(i,j)}$ to $NG_i$ or $NG_j$ using the minimum number of sub-gears, we need to add gears without waste, connecting both target gears in $EG_{(i,j)}$. Consider the following patterns:

Pattern inner-inner (II): The same configuration as $IG_{(i,j)}$;

Pattern outer-outer (OO): The same configuration as $OG_{(i,j)}$, but without the gear represented by $OG_{((A+1)h_{1\text{max}}-(B-2))}$ at the disconnected point (see Fig. 20);

Pattern outer-inner, inner-outer (IO): The two configurations obtained by exchanging two connected parts before and after the disconnected point of II and OO.

It is clear that there is no other layout to connect all gears in the edge gadget other than the above four patterns. Among the four patterns, only the pattern OO can connect $NG_i$ and $NG_j$ by adding one extra sub-gear, satisfying Properties 1 and 2.

(Property 3). Possible positions of gears that can connect two gears in a node gadget are the four gears in Fig. 8 (b) and the four gears in Fig. 8 (c). Property 3 is satisfied by the fact that from the eight possible gears, only the four gears in Fig. 8 (c) can connect to the center gear of the node gadget, and the fact that the four corner gears can be connected by choosing three of the eight possible gears.

(Property 4). By construction, Property 4 immediately follows.

(Property 5). Property 5 only defines $s$, and therefore no proof is required.

(Property 6). It is almost trivial that in order to connect all target gears using $s$ or less sub-gears, we have to connect $|V| - 1$ edge gadgets using the outer connection layout. We can substitute the connection pattern of the remaining edge gadgets by the inner connection layout. Even after such modifications, all the gears would remain connected, satisfying Property 6.

(Property 7). By construction, an odd number of sub-gears are used in the outer connection layout between any two node gadgets, and hence, when all gears are connected, the rotation directions of all gears in all node gadgets are the same.
The rotation direction of any gear in \( EG_{i,j} \) does not change with either \( OG_{i,j} \) or \( IG_{i,j} \), because for each \( i \), the two gears in \( EG_{i,j} \) mesh the gear \( OG_{i,j}^{(1)} \) or \( IG_{i,j}^{(1)} \), both of which are connected to \( NG \), with the same number of sub-gears in the outer or inner connection layout. By the definition of the specified rotation directions of target gears, it is easy to see that Property 7 is satisfied.

**Theorem 4.** There exists a polynomial time reduction from the Hamiltonian path problem on 3-regular planar graphs to the gear placement problem.

**Proof.** The proof follows by Theorem 3 and the discussion in Section 3.6.

\[ \square \]

### 4. Upper Bound

In this section, we present an upper bound on the number of sub-gears that need to be placed to connect all target gears for the special case in which the gearbox is sufficiently large and there is no restriction on the size of sub-gears.

**Theorem 5.** Given an instance \( I \) of the GPP with \( n \) target gears, if \( r_{\min} = 0 \), \( r_{\max} = \infty \), \( s \geq 2(n - 1) \), and the gearbox is sufficiently large, then the answer to the GPP for \( I \) is yes.

**Proof.** We prove the theorem by showing that Algorithm 2 outputs a layout that uses \( 2(n - 1) \) or fewer sub-gears. The set \( \text{Sub} \) represents the sub-gears that will be placed into the gearbox, the set \( \text{Connect} \) represents the gears that are meshed by the algorithm, and the set \( T \setminus \text{Connect} \) represents the gears that still need to be connected.

Starting with set \( \text{Connect} \) consisting of an arbitrary target gear, the algorithm repeats until \( T \setminus \text{Connect} \) becomes empty, the process of meshing a gear \( g_a \in \text{Connect} \) with a gear \( g_b \in T \setminus \text{Connect} \) by adding a new sub-gear \( g_c \) (resp., two new sub-gears \( g_c \) and \( g_d \) of the same radius) whose center is (resp., centers are) on the line segment that connects the centers of \( g_a \) and \( g_b \), if \( g_a \) and \( g_b \) have the same (resp., opposite) rotation directions.

Below we prove that this is always possible by appropriately choosing \( g_a \) and \( g_b \). For the simplicity of the proof, suppose that gear \( g_a \) and gear \( g_b \) have the same rotation directions. A similar proof can be used for the case of opposite rotation directions.

Suppose that sub-gear \( g_c \) cannot be placed. Then there exists at least one gear \( g_c \in \text{Connect} \cup T \) overlapping with \( g_c \). We now prove that the distance of gear \( g_a \) to gear \( g_b \) and that of gear \( g_c \) to gear \( g_b \) are shorter than the distance of gear \( g_b \) to gear \( g_a \). We denote the radii of gears \( g_b \), \( g_a \), and \( g_c \) by \( r_{g_b} \), \( r_{g_a} \), and \( r_{g_c} \), respectively. Let \( d_{g_b,g_a} \) be the distance between the center of \( g_b \) and that of \( g_a \), \( d_{g_b,g_c} = d_{g_a,g_b} - r_{g_a} - r_{g_c} \) be the distance between the contact surfaces of \( g_a \) and \( g_b \), and \( d_{g_c,g_b} \) be the distance between \( g_c \) and \( g_b \), and \( \text{overlap} \geq 0 \) be the amount of overlapping between gear \( g_c \) and gear \( g_a \), if \( g_c \) is to be placed into the gearbox (illustrated in Fig. 25). By construction, we have \( r_{g_b} = d_{g_b,g_a}/2 \). The distance between the centers of gear \( g_c \) and gear \( g_a \) is \( d_{g_a,g_b} = 2 + r_{g_a} - \text{overlap} \). The distance between the centers of \( g_a \) and \( g_c \) is \( d_{g_a,g_c} = d_{g_b,g_c}^0 + r_{g_a} + r_{g_c} \), and the distance between the centers of \( g_b \) and \( g_c \) is \( d_{g_b,g_c}^0 + r_{g_a} \). By the triangle inequality, we have \( d_{g_a,g_b} \leq d_{g_a,g_c} + d_{g_c,g_b} \), and \( d_{g_a,g_c} \leq d_{g_a,g_b} + d_{g_b,g_c} \), and when \( \text{overlap} = 0 \) holds, these inequalities become strict for the following reason: The equality in the first triangle inequality can be dropped because, when \( \text{overlap} = 0 \), the equality can only be achieved if the gears \( g_a \), \( g_b \), and \( g_c \) are aligned in this order in a straight line with \( g_c \) meshing with both \( g_a \) and \( g_b \), and by construction \( g_a \), \( g_b \), and \( g_c \) are on the same line in this order with \( g_c \) meshing with both \( g_a \) and \( g_b \), implying that \( g_a \) and \( g_b \) overlap, which contradicts the fact that the current layout (before adding \( g_c \)) is feasible. The same can be shown for the second inequality. Rearranging the equations, we have \( d_{g_a,g_b} - \text{overlap} \leq d_{g_a,g_c} + d_{g_c,g_b} \) and \( d_{g_a,g_c} - \text{overlap} \leq d_{g_a,g_b} \). That is, the distance between the contact surfaces of \( g_a \) and \( g_b \), and that of \( g_a \) and \( g_c \), and that of \( g_c \) and \( g_b \) are less than \( d_{g_a,g_b} \).

Thus, if the gear \( g_a \) that connects \( g_b \) and \( g_c \) cannot be placed, there exists a pair \( (g_a, g_b) \) such that \( g_a \in \text{Connect} \), \( g_b \in T \setminus \text{Connect} \), and \( d_{g_a,g_b} < d_{g_a,g_c} \), or a pair \( (g_c, g_b) \) such that \( g_c \in T \setminus \text{Connect} \), \( g_b \in \text{Connect} \), and \( d_{g_a,g_b} < d_{g_a,g_c} \). Then we regard such a pair as the new \( (g_a, g_b) \). If the new \( g_b \) can be placed without overlap, we are done; otherwise, we can apply the same argument again. This repetition will stop because the number of candidates for \( (g_a, g_b) \) is \( O(|\text{Connect}| \cdot |T \setminus \text{Connect}|) \) and the distance between \( g_a \) and \( g_b \) is monotonically decreasing. Because Algorithm 2 examines all possible pairs for \( (g_a, g_b) \) (until an appropriate pair is found) in each iteration of the while loop, the algorithm can find a pair of gears \( (g_a, g_b) \) that can be connected by a sub-gear \( g_c \) or two sub-gears \( g_c \) and \( g_d \). Thus, in each iteration, the algorithm adds at least one target gear and at most two sub-gears to \( \text{Connect} \).

Because the number of target gears that still need to be connected (i.e., \( |T \setminus \text{Connect}| \)) is initially \( n - 1 \) and is decreased by at least one in each iteration, the number of iterations is at most \( n - 1 \). Moreover, in each iteration, at most two new sub-gears are added. Hence, the maximum number of sub-gears used by Algorithm 2 is \( 2(n-1) \).

\[ \square \]
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Algorithm 2 Find a feasible layout of sub-gears

1. **input**: $T :=$ The set of target gears
2. **Connect** := A set with an arbitrary target gear
3. **while** ($T \setminus \text{Connect} \neq \emptyset$)
4.  **for each** $g_a \in \text{Connect}$
5.      **for each** $g_b \in T \setminus \text{Connect}$
6.        **if** ($g_a$ and $g_b$ have the same rotation directions)
7.          Create gear $g_c$ that connects $g_a$ and $g_b$, and the centers of $g_a$, $g_b$, and $g_c$ are collinear
8.          **if** (gear $g_c$ can be placed in the gearbox)
9.            Insert $g_b$ and $g_c$ into Connect
10.         go to Line 3
11.      **end if**
12.  **end if**
13.      **else** ($g_a$ and $g_b$ have opposite rotation directions)
14.          Create gears $g_c$ and $g_d$ that have the same radius and connect $g_a$ and $g_b$, and the centers of $g_a$, $g_b$, $g_c$, and $g_d$ are collinear
15.          **if** (gear $g_c$ and $g_d$ can be placed in the gearbox)
16.            Insert $g_b$, $g_c$, and $g_d$ into Connect
17.          go to Line 3
18.      **end if**
19.  **end else**
20.  **end for**
21. **end for**
22. **end while**
23. **output**: Sub = Connect \ T

5. Conclusion

We presented the problem of finding a layout of gears that uses the minimum number of sub-gears to connect all the given target gears. Then, we showed that the gear placement problem (GPP) is NP-hard by giving a polynomial time reduction from the Hamiltonian path problem on 3-regular planar graphs to the GPP. We also gave an upper bound on the number of sub-gears required to connect all target gears for the case in which there is no restrictions on the size of the sub-gears and the gearbox. In future works, we aim to study more general cases, such as the GPP in a 3D space and/or that with an objective function considering the number of sub-gears between the root gear and the target gears,
which is important in maximizing the torque to be transferred. Another interesting generalization would be a model with size-dependent costs for sub-gears. We considered in this paper the minimization of the total number of sub-gears; this might closely related to the production costs when the range of sub-gear sizes \([r_{\text{min}}, r_{\text{max}}]\) is narrow, but when this range is wider, it would be more realistic to consider size-dependent costs for sub-gears. It would also be interesting to consider the case in which we can choose sub-gear sizes from a discrete set of candidates. Note that the GPP considered in this paper could be seen as a special case of many of such more general cases, to which the proof of hardness presented in this paper applies.

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