Modules Over Affine Lie Superalgebras

JIANG-BEI FAN and MING YU

Institute of Theoretical Physics, Academia Sinica
P.O.Box 2735, Beijing 100080, P.R.China
Fax: 086-1-2562587

Abstract

Modules over affine Lie superalgebras $\mathcal{G}$ are studied, in particular, for $\mathcal{G} = \hat{OSP}(1,2)$. It is shown that on studying Verma modules, much of the results in Kac-Moody algebra can be generalized to the super case. Of most importance are the generalized Kac-Kazhdan formula and the Malikov-Feigin-Fuchs construction, which give the weights and the explicit form of the singular vectors in the Verma module over affine Kac-Moody superalgebras. We have also considered the decomposition of the admissible representation of $\hat{OSP}(1,2)$ into that of $\hat{SL}(2) \otimes$Virasoro algebra, through which we get the modular transformations on the torus and the fusion rules. Different boundary conditions on the torus correspond to the different modings of the current superalgebra and characters or super-characters, which might be relevant to the Hamiltonian reduction resulting in Neveu-Schwarz or Ramond superconformal algebras. Finally, the Felder BRST complex, which consists of Wakimoto modules by the free field realization, is constructed.
1 Introduction

Some recent work has been focused on the Hamiltonian reduction of the super-group valued WZNW theory which gives rise to a super-Toda and (extended) superconformal field theory $^{27, 17, 13, 21, 23}$. Such procedures might be the only accessible way of quantizing the super-Liouville field theory, since their discretized version, i.e. the super-symmetrized matrix model does not exist yet. In ref.$^{1, 22, 23}$, it has been shown that by combining the matter sector with the Liouville sector in a non-critical string theory one obtains a 2D topological field theory, which is equivalent to the $SL(2,\mathbb{R})/SL(2,\mathbb{R})$ gauged WZNW model. A striking feature of the non-critical string theory, as well as the $G/G$ model, is the appearance of infinitely many copies of the physical states with non-standard ghost numbers $^{36, 37, 7, 8}$. We expect similar structure exists when $G$ is a supergroup. Namely, there must be a close relation between the non-critical fermionic string and the $G/G$ gauged supergroup valued WZNW theory. The present paper is a preparation toward such a consideration.

The essential ingredients of the WZNW theory is encoded in its current algebra, the Kac-Moody algebra. It is also clear that it is the structure of the algebra modules that determine the physical states in $G/G$ WZNW theory. The general structure of the affine Kac-Moody modules has been extensively studied $^{22, 21, 14, 15}$. However, concerning its generalization, the contragradient super-algebra module, less results can be found in the literature $^{12, 43, 29, 30}$. In this paper, we try to shed some light on the structure of modules over superalgebras, which might be relevant to our understanding of the (gauged) supergroup valued WZNW theory. As a consequent problem, the analysis of the BRST semi-infinite cohomology of the $G/G$ WZNW theory, where $G$ is in general a supergroup, will be done in a separate paper $^{19}$. Our main results are the generalized Kac-Kazhdan formula, eq.(80) and the generalized MFF $^{38}$ construction. To give a clear idea what we have done in this paper, let us recall some essential facts about our knowledge of the infinite dimensional Lie algebras. In $^{32}$, the structure of Verma module over an infinite dimensional Lie algebra was studied; Kac-Kazhdan formula tells whether a Verma module is reducible, and gives the weights of the singular vectors in a Verma module. In $^{38}$, the explicit form of such a vector in a Verma module was constructed, in a way we might call it MFF construction. Moreover, the Wakimoto modules (namely, when restricted to Virasoro algebra, the Feigin-Fuchs modules $^{16}$), were extensively studied $^{14, 17}$.

Our paper is organized as follows. In section 3, we review some fundamental knowledge about Lie superalgebra. In section 4 we study a hidden Virasoro algebra in $\mathcal{U}(\hat{OSP}(1, 2))$ through GKO construction $^{20, 21}$ of $OS\hat{P}(1, 2)/SL(2)$. We get the decomposition of the representation of $\hat{OS\hat{P}}(1, 2)$ into $SL(2)\otimes$ Virasoro, and concentrate on the so called admissible representations $^{33, 18}$. Later we deal with the $S$ modular transformation and the fusion rules. Fusion rules of the the admissible representations are given through Verlinde formula $^{15, 34, 3}$. Similar to those of $SL(2)$, there are negatives integers appearing in the fusion rules, which might be explained as that there are lowest weight states appearing...
in the fusion of two highest weight states. It should be noted that the decomposition, eq. (45), is first given in [33] in a sophisticated way. We get the $S$-modular transformation of $\widehat{OSP}(1, 2)$ through that of the Virasoro and $\widehat{SL}(2)$, which are already known. The characters of the admissible $\widehat{OSP}(1, 2)$-modules are obtained by combining that of Virasoro and $\widehat{SL}(2)$.

In section 4, we study the $\widehat{OSP}(1, 2)$-module in general, and give out the structure of all $\widehat{OSP}(1, 2)$-modules. The result is analogous to the classification of Feigin and Fuchs on Virasoro-module [16] and that of Feigin-Frenkel on $\widehat{SL}(2)$-module [15]. The Kac-Kazhdan formula [32] and MFF construction [38] is generalized to superalgebra, as it had been mentioned above.

Section 5 is devoted to the construction of Wakimoto modules [46], which is the free field realization of the current superalgebra [3]. The Felder cohomology [17, 4] is analyzed in detail for $\mathcal{U}(\mathcal{G}) = \widehat{OSP}(1, 2)$.

In conclusion, we speculate that for a general affine superalgebra $\mathcal{G}$, the coset space construction for $\mathcal{G}/\mathcal{G}_0$, where $\mathcal{G}_0$ is the even part of $\mathcal{G}$, will result in a $W_N$ algebra. In appendix we deal with some technical details.

2 Lie superalgebra and Current superalgebra

In this section we review some fundamental properties of the Lie superalgebras, mainly to make our paper self-contained. Detailed discussion can be found in [12, 29, 30]. The representation theory of $OSP(1, 2)$ has been studied by the authors of ref. [42, 43].

2.1 Lie superalgebra

Lie superalgebra, namely, graded Lie algebra in mathematical terms, can be written as $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1$. $\mathcal{G}_0$, the even part of $\mathcal{G}$, is by itself a Lie algebra.

Denote the generators of $\mathcal{G}$ by $\tau^\alpha$, $\alpha = 1, \ldots, d_{\mathcal{G}_0} + d_{\mathcal{G}_1}$ (or $\infty$, when $\mathcal{G}$ infinite dim.), where $d_{\mathcal{G}_i}$ is the dimension of $\mathcal{G}_i$. The commutators are

$$[\tau^\alpha, \tau^\beta] = f^{\alpha\beta\gamma} \tau^\gamma,$$

where $f^{\alpha\beta\gamma}$ are the structure constants. The Lie bracket $[\ ,\ ]$ is defined to be

$$[a, b] = ab - (-1)^{deg(a)deg(b)}ba,$$

where $deg(a) = 0$ (resp. 1) for $a \in \mathcal{G}_0$ (resp. $\mathcal{G}_1$). From Jacobi identity

$$[\tau^\alpha, [\tau^\beta, \tau^\gamma]] = [[\tau^\alpha, \tau^\beta], \tau^\gamma] + (-1)^{d(\alpha)d(\beta)}[\tau^\beta, [\tau^\alpha, \tau^\gamma]],$$

(3)
we get
\[ f_{\alpha \sigma}^\alpha f_{\beta \rho}^\beta = f_{\rho \sigma}^\rho f_{\alpha \beta}^\alpha + (-1)^{d(\alpha)d(\beta)} f_{\rho \sigma}^\rho f_{\alpha \beta}^\beta \]  
(4)
where \( d(\alpha) = 0 \) (resp. 1) for \( \tau^\alpha \in G_0 \) (resp. \( G_1 \)).

In [30], Kac gave out a classification of all finite growth contragradient Lie superalgebras, each associated with a generalized indecomposable \( n \times n \) Cartan matrix and a subset of \( \{1 \ldots n\} \).

Let \( A = (a_{ij}) \) be an \( n \times n \) generalized Cartan matrix; \( \tau \subseteq I = \{1 \ldots n\} \). \( a_{ij} \) satisfy
\[
\begin{align*}
    a_{ii} &= 2, & a_{ij} &\leq 0 & \forall i \neq j, & i, j \in I, \\
    a_{ij} &= 0 \leftrightarrow a_{ji} = 0. & a_{ij} & \text{is even} & \forall i \in \tau.
\end{align*}
\]
(5)
The Chevalley basis \( \{e_i, f_i, h_i, i = 1 \ldots n\} \) satisfy
\[
\begin{align*}
    [e_i, f_j] &= \delta_{ij}h_i; & [h_i, h_j] &= 0; \\
    [h_i, e_j] &= a_{ij}e_j; & [h_i, f_j] &= -a_{ij}f_j; \\
    (ade_i)^{-a_{ij}+1}e_j &= (adf_i)^{-a_{ij}+1}f_j = 0 & \forall i \neq j, & i, j \in I;
\end{align*}
\]
(7)
where the \( Z_2 \)-degrees of \( e_i, f_i, h_i \) are
\[
\begin{align*}
    \deg(h_i) &= 0, & \forall i \in I; \\
    \deg(e_i) &= \deg(f_i) = 0, & \forall i \not\in \tau; \\
    \deg(e_i) &= \deg(f_i) = 1, & \forall i \in \tau.
\end{align*}
\]
(8)

Remark: (super)Virasoro algebra is not a contragradient superalgebra.

## 2.2 Finite dim. Lie superalgebras

In this subsection, we shall restrict our discussion the cases that \( \dim(G) \) is finite. Similar to Lie algebra, the adjoint representation of the finite dimensional Lie superalgebra takes the form
\[
(F^\alpha)^\beta = f^\alpha \gamma.
\]
(9)
Define the supertrace,
\[
\str(F) = \sum_{\alpha} (-1)^{d(\alpha)} F_{\alpha}^\alpha.
\]
(10)
There exists a metric tensor
\[
h^{\alpha \beta} = \str(F^\alpha F^\beta),
\]
(11)
with relation
\[
h^{\alpha \beta} = (-1)^{d(\alpha)d(\beta)} h^{\beta \alpha}.
\]
(12)
We would like to restrict our discussion to the semisimple Lie superalgebra, (which follows from the definition in ref. [42]), i.e. \( G \overline{G} \) is semisimple and
\[
det | h^{\alpha \beta} | \neq 0. \tag{13}
\]
Define the inverse of \( h \)
\[
h_{\alpha \beta} h^{\beta \gamma} = \delta^\gamma_\alpha. \tag{14}
\]
It can be verified [42] that
\[
C = \tau^\alpha h_{\alpha \beta} \tau^\beta \tag{15}
\]
is a Casimir operator. In the adjoint representation
\[
C_{\text{ad}} = 1. \tag{16}
\]
To make our discussion more concrete we consider a simple example, the \( \text{OSP}(1, 2) \) superalgebra,
\[
\{ j^+, j^- \} = 2 J^3; \quad \{ j^\pm, j^\pm \} = \pm 2 J^\pm; \\
[J^3, j^\pm] = \pm \frac{1}{2} j^\pm; \quad [J^\pm, j^\mp] = -j^\pm; \\
[J^+, J^-] = 2 J^3; \quad [J^3, J^\pm] = \pm J^\pm, \tag{17}
\]
while other (anti)commutators vanish. We see that the even generators, \( J^\pm, J^3 \) constitute a \( \text{SL}(2) \) subalgebra. The irreducible representation of \( \text{OSP}(1, 2) \) with highest weight \( j \) (except for \( j = 0 \)) can be decomposed into the irreducible representations of \( \text{SL}(2) \), with isospin \( j \) and \( j - 1/2 \) respectively. The HWS of the \( \text{OSP}(1, 2) \) is defined as
\[
j^+ |j, j, j\rangle = 0; \quad J^3 |j, j, j\rangle = j |j, j, j\rangle, \tag{18}
\]
where the first index in \( |i, j, k\rangle \) labels the \( \text{OSP}(1, 2) \) representation and the last two indices refer to the \( \text{SL}(2) \) isospin and its third component. By eq.(18), \( |j, j, j\rangle \) is also a HWS for \( \text{SL}(2) \) with isospin \( j \). Then
\[
|j, j - 1/2, j - 1/2\rangle = j^- |j, j, j\rangle \tag{19}
\]
satisfies
\[
J^+ |j, j - 1/2, j - 1/2\rangle = 0, \tag{20}
\]
and generates another \( \text{SL}(2) \) irreducible module of highest weight \( j - 1/2 \). The lowest dimensional faithful representation of \( \text{OSP}(1, 2) \) is given by \( j = 1/2 \), for which
\[
\text{str}\{ J^+ J^- \} = \text{str}\{ J^- J^+ \} = 1; \quad \text{str}\{ j^+ j^- \} = -\text{str}\{ j^- j^+ \} = 2; \\
\text{str}\{ J^3 J^3 \} = 1/2. \tag{21}
\]
The adjoint representation is of $j = 1$, by eqs. (11) the metric can be worked out explicitly,

\[
h^{\pm} = h^{-1} = 3; \quad h^{\pm} = -h^{\pm}; \quad h^{33} = 3/2.
\]

(22)

By eq. (13), the second Casimir operator of $OSP(1,2)$ is

\[
C = 1/3\{J^+J^- + J^-J^+ + 2J^3J^3 - 1/2j^+j^- + 1/2j^-j^+\}.
\]

(23)

For an irreducible representation labeled by isospin $j$

\[
C = j(2j + 1)/3 \cdot 1.
\]

(24)

### 2.3 Current superalgebras

We now turn to the current superalgebra $J^\alpha$ associated with a finite dimensional Lie superalgebra $G$.

The currents satisfy the short distance operator product expansion (OPE),

\[
J^\alpha(z_1)J^\beta(z_2) = \frac{\partial h^\alpha\beta}{z_{12}^2} + \frac{f^\alpha\beta\gamma J^\gamma(z_2)}{z_{12}}.
\]

(25)

By Sugawara construction, we get the stress-energy tensor

\[
T(z) = \frac{J^\alpha h^\alpha\beta J^\beta}{2\hat{k} + 1}, \quad c = \frac{2\hat{k}\text{sdim}(G)}{2\hat{k} + 1},
\]

(26)

where $c$ is the central extension of the Virasoro algebra and

\[
\text{sdim}(G) = d_{\bar{\psi}} - d_{\bar{\varphi}}.
\]

(27)

The level of the current superalgebra, eq. (22), is

\[
k = 2\hat{k}\hat{h}_G,
\]

(28)

where $\hat{h}_G$ is the dual Coexter number for Lie algebra, 3/2 for $OSP(1,2)$. For an integrable representation of $OSP(1,2)$, $k$ is an integer.

### 3 (Super)characters and Modular Transformations

The affine Kac-Moody superalgebra associated with $OSP(1,2)$, eq. (22), assumes the following form,

\[
\{j_+^+, j_-^-\} = 2J_+^3 + 2r\delta_{r+s,0}; \quad \{j_+^-, j_-^-\} = \pm 2J_-^3;
\]

\[
[J_+^\alpha, J_-^\beta] = \pm \frac{1}{2}J_+^\alpha \delta_{\alpha,\beta}; \quad [J_+^\alpha, J_+^\beta] = -J_-^\beta;
\]

\[
[J_+^\alpha, J_-^\beta] = 2J_+^3 + nk\delta_{n+m,0}; \quad [J_3^3, J_-^m] = \pm J_3^\pm;
\]

\[
[d, J_n^\alpha] = nJ_n^\alpha.
\]

(29)
Here, in general, the odd generators could be integral or half-integral moded. However, in the case of $\hat{OSP}(1,2)$, it is clear that the two modings are isomorphic. Unless specified we shall assume the integral moding for all the generators. In subsection (3.3) we shall come to this point again.

The highest weight state of $\hat{OSP}(1,2)$ is an eigenstate of $J_3$ and annihilated by $J_n^a$, $j_n^\alpha$, $n \in \mathbb{N}$ and $j_0^+$. The highest weight representation of $\hat{OSP}(1,2)$ is generated by acting on the highest weight state with the lowering operators. It is useful to notice that the even part of $\hat{OSP}(1,2)$ is itself a $\hat{SL}(2)$ Kac-Moody algebra. So in general we can decompose the representation space of $\hat{OSP}(1,2)$ into a tensor product of two spaces,

$$\hat{OSP}(1,2) \sim \hat{SL}(2) \otimes \hat{OSP}(1,2)/\hat{SL}(2).$$

Such decomposition is carried out explicitly in subsection (3.1). Similarly, the (super) characters of $\hat{OSP}(1,2)$ can be written in terms of that of $\hat{SL}(2)$ and its branching functions, as what is done in subsection (3.2). In section 4 we shall show that these results can be rederived from the pure algebraic approach.

### 3.1 The Decomposition

The even part of $\hat{OSP}(1,2)$ is the $\hat{SL}(2)$ subalgebra. The energy momentum tensor for $SL(2)$ WZNW theory is, via Sugawara construction,

$$T^{SL(2)}(z) = \frac{J^3J^3 + \frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+}{k + 2}.$$

By GKO construction, we get a Virasoro algebra for the coset space $V = \hat{OSP}(1,2)/\hat{SL}(2)$,

$$T^V(z) = T^{\hat{OSP}(1,2)}(z) - T^{\hat{SL}(2)}(z).$$

It is easy to verify that

$$[T^V, T^{\hat{SL}(2)}] = 0;

\begin{align*}
\omega^{\hat{OSP}(1,2)} &= \frac{2k}{2k+3}; \\
\omega^{\hat{SL}(2)} &= \frac{3k}{k+2}; \\
\omega^V &= 1 - 6\frac{(k+1)^2}{(2k+3)(k+2)}.
\end{align*}$$

So far we are interested in the rational conformal field theory (RCFT), for which the possible sets of the characters can be classified. For $\hat{SL}(2)_k$ RCFT \cite{33,40}, the level $k$ satisfies

$$k + 2 = \frac{p}{q},$$

where $p, q$ are coprime positive integers. Then by eqs.\cite{32,33}\cite{42,43},

$$\omega^V = 1 - 6\frac{(p - \tilde{q})^2}{pq}.$$
where

\[ \tilde{q} = 2p - q. \]  (35)

We see that \( \gcd(p, \tilde{q}) = 1 \). So the Virasoro algebra for the coset space \( V = OSP(1, 2)/SL(2) \) is in the \((p, \tilde{q})\) minimal series, provided

\[ p, q, 2p - q > 0. \]  (36)

Similar consideration has been taken by Kac and Wakimoto \[33\] in the study of the admissible representation of \( \hat{OSP}(1, 2) \). Notice that the unitarity of \( \hat{SL}(2) \) demands \( q = 1, \ p \geq 2 \), which leads to a non-unitary Virasoro minimal series in the \( V \) sector except for the trivial case, \( c^V = 0 \).

Remark: \( \hat{OSP}(1, 2) \) WZNW theory is not a unitary theory except for the trivial case, \( k = 0 \).

The admissible representation of \( \hat{SL}(2) \) \[33 \ 40\] is classified as

\[ 2j + 1 = r - s \frac{p}{q}; \ \text{r} = 1, \ldots, p - 1 \quad s = 0, \ldots, q - 1, \]  (37)

where \( j \) is the isospin of the HWS. Now we consider an HWS in \( \hat{OSP}(1, 2) \) with the level and the isospin as in eqs.\[33 \ 37\]. It is easy to see that such a HWS is a tensor product of the HWS’s in \( \hat{SL}(2) \) and \( V \) sector separately,

\[ |HWS\rangle_{\hat{OSP}(1,2)} = |HWS\rangle_{\hat{SL}(2)} \otimes |HWS\rangle_{V} \]

Its conformal weight in the \( V \) sector is

\[ h^V = h^\hat{OSP}(1,2) - h^\hat{SL}(2) \]
\[ = \frac{j(j + 1/2)}{k + 3/2} - \frac{j(j + 1)}{k + 2} = \frac{j(j - k - 1)}{(2k + 3)(k + 2)} \]
\[ = \frac{(mp - r\tilde{q})^2 - (p - \tilde{q})^2}{4p\tilde{q}} \]
\[ = h_{r,m}, \]  (38)

where

\[ m = 2r - s - 1, \]  (39)

Again we find that \( h^V \) is indeed in the set of conformal weights of the minimal \((p, \tilde{q})\) Virasoro series provided we restrict the possible values of \( r, s \) to be

\[ 0 < r < p, \quad 0 < m < \tilde{q}. \]  (40)

Soon after we shall see that upon this restriction the super-characters form a representation of the modular group on torus.
To decompose an irreducible $\widehat{OSP}(1, 2)$ module, let us proceed to find out all the states $|\phi\rangle$ in the $\widehat{OSP}(1, 2)$ module which are HWS's for both $\widehat{SL}(2)$ and Virasoro $(p, \tilde{q})$ algebra,

$$|\phi\rangle = |\tilde{r}, \tilde{s}\rangle_{SL(2)} \otimes |l, \tilde{m}\rangle_V.$$

(41)

Assuming that such a state appears in the $N$th level of the $\widehat{OSP}(1, 2)$ module, we try to solve the following equations,

$$\begin{cases} \frac{j(j+1/2)}{k+3/2} + N = \frac{\tilde{j}(\tilde{j}+1)}{k+2} + h_{l, \tilde{m}}, N \in \mathbb{Z}_+, \\
\tilde{j} = \frac{1}{2}(\tilde{r} - 1) - \frac{1}{2} \frac{\tilde{r}}{q} = j + n/2, n \in \mathbb{Z}, \end{cases}$$

(42)

which has infinitely many solutions

$$N = \frac{1}{2} n(n+1) + t^2 pq + t(r + n)\tilde{q} - tmp,$$

(43)

$$\tilde{j} = j + n/2, \quad l = r + n + 2tp, \quad \tilde{m} = m,$$

or

$$N = \frac{1}{2} n(n+1) + t^2 pq - t(r + n)\tilde{q} - tmp + ml,$$

(44)

$$\tilde{j} = j + n/2, \quad l = 2tp - (r + n), \quad \tilde{m} = m,$$

where $n, t \in \mathbb{Z}$ such that $N > 0$, or $N = 0, n < 0$. Because we are considering the irreducible representations of $\widehat{OSP}(1, 2)$, some solution should be excluded. We see that the state $|\pm l + 2t(p, s)_{SL(2)} \otimes |\pm l + 2t'(p, m)_{V}\rangle$ is a singular state in the Verma module $M(|l, s\rangle_{SL(2)} \otimes M(|l, m\rangle_{V})$, if $0 \leq l < p$. So in the decomposition we should restrict ourselves to the irreducible $SL(2)$ and Virasoro modules. After taking this into account, we get

$$L^{OSP(1, 2)}_{m,s} = \sum_{l=1}^{p-1} L^{SL(2)}_{l,s} \otimes L^{V}_{l,m}, \quad m + s \in \text{odd},$$

(45)

where $L^{OSP(1, 2)}_{m,s}$ is the irreducible representation for $\widehat{OSP}(1, 2)$ with the HWS of level $k$ and isospin $j_{m,s}$

$$k + 3/2 = \frac{\tilde{q}}{2q}, \quad q + \tilde{q} \in \text{even}, \quad gcd(q, \frac{q + \tilde{q}}{2}) = 1;$$

$$4j_{m,s} + 1 = m - s \frac{\tilde{q}}{q}, \quad m = 1, \cdots, \tilde{q} - 1, \quad s = 0, \cdots, q - 1.$$

(46)

It is easy to see that the isospin $j$ in eq.(46) is exactly that in eq.(37). However it may be convenient to adopt the $(q, \tilde{q})$ and $(m, s)$ notation for the case of $\widehat{OSP}(1, 2)$ (instead of the labels $(p, q)$ and $(r, s)$ for $\widehat{SL}(2)$).

### 3.2 Characters and Supercharacters on Torus

We note that for a $Z_2$ graded algebra, the module should also be $Z_2$ graded in consistence with that of the algebra. For convenience we assume that the $Z_2$ degree of the HWS of a HW module is even. The
character $\chi$ and the supercharacter $S\chi$ of a $G$-module $R$ are defined as
\[
\chi = trRE^{i2\pi r(L_0 - \tau^2) + i2\pi zJ_0^3}, \quad S\chi = strRE^{i2\pi r(L_0 - \tau^2) + i2\pi zJ_0^3}
\]  
(47)

The characters and supercharacters so defined correspond to the different boundary conditions along $\tau$ direction. Notice that up to a phase factor the supercharacter can be obtained from the character by letting $z \to z + 1$.

Using the decomposition, eq.(45), we have
\[
\chi_{m,s}^{OSP(1,2)}(z, \tau) = \sum_{l=1}^{q-1} \chi_{l,s}^{SL(2)}(z, \tau)^{V}, \quad S\chi_{m,s}^{OSP(1,2)}(z, \tau) = \sum_{l=1}^{q-1} (-1)^{l+r} \chi_{l,s}^{SL(2)}(z, \tau)^{V},
\]  
(48)

\[m + s \text{ is odd, } m = 1, \ldots, \tilde{q} - 1, \ s = 0, \ldots, p - 1.\]

$\chi_{l,m}^{V}(\tau)$'s so defined are called the $(OSP(1,2) \supset SL(2))$ branching functions.

Modular transformations of the characters of the $SL(2)$ admissible representations and Virasoro minimal models have been studied in detail in ref.[11, 26, 40]. Then the modular transformations of the $OSP(1,2)$ (super)characters can be obtained by using the decomposition formula eq.(45).

Under $S : \tau \to -1/\tau, z \to z/\tau$
\[
\chi_{l,s}^{SL(2)} = \sum_{l' = 1}^{p-1} \sum_{q' = 0}^{q-1} S_{l,s}^{l',q',SL(2)} \chi_{l',q'}^{SL(2)},
\]
\[
\chi_{l,m}^{V} = \sum_{l' = 1}^{p-1} \sum_{m' = 0}^{q-1} S_{l,m}^{l',m',V} \chi_{l',m'}^{V},
\]  
(49)

where
\[
S_{l,s}^{l',q',SL(2)} = \sqrt{\frac{2}{pq}}(-1)^{l'+s'+q'}e^{-i\pi ss'/q} \sin \frac{\pi qll'}{p};
\]
\[
S_{l,m}^{l',m',V} = \sqrt{\frac{2}{pq}}(-1)^{l'+m'+1} \sin \frac{\pi mm'}{\tilde{q}} \sin \frac{\pi qll'}{p}.
\]  
(50)

From eqs.(49,51), we get the $S$ modular transformation of supercharacters $S\chi_{m,s}^{OSP(1,2)}$, when $1 \leq m = 2r - s - 1 < \tilde{q}$, after a lengthy calculation.
\[
S\chi_{m,s}^{OSP(1,2)} = \sum_{m' = 1}^{\tilde{q}-1} \sum_{s' = 0}^{q-1} S_{m,s}^{m',s',OSP(1,2)} S\chi_{m',s'}^{OSP(1,2)};
\]
\[
S_{m,s}^{m',s',OSP(1,2)} = \sqrt{\frac{2}{pq}}(-1)^{(m-s+s'-m')/2}e^{-i\pi ss'/q} \sin (\pi mm'/\tilde{q}).
\]  
(51)

We see that when $1 \leq m < 2p - q = \tilde{q}, 0 \leq s < q, m + s \in \text{odd}$, $S\chi_{m,s}^{OSP(1,2)}$ form a representation of the modular group. So we get a rational conformal field theory. On the contrary, the characters are not closed under $S$ modular transformations. This issue will be addressed again in the next subsection.
Generally one can figure out the fusion rules through the $S$ matrix \[45\], The relation between the fusion rules and the $S$ matrix is conjectured by Verlinde \[45\], later proved in \[39\].

$$N_{ij}^k = \sum_n S_{ni} S_{nj}^{*n} S_{nk}^n$$  \hspace{1cm} (52)

However from the relation eq.(52), we get $-1$ for some $N_{ij}^k$'s, as it also happens in the case of $SL(2)$ \[4, 3\]. This can be interpreted as that there are lowest weight states appearing in the fusion of two HWS's. The fusion rules so obtained are

1. when $s_1 + s_2 < q$

$$[\varphi^{OSP(1,2)}_{m_1, s_1}] [\varphi^{OSP(1,2)}_{m_2, s_2}] = \sum_{m_3 = |m_1 - m_2| + 1, \begin{array}{c} m_3 + m_1 + m_2 \in odd \end{array}} \min [m_1 + m_2, 2q - m_1 - m_2] - 1 [\varphi^{OSP(1,2)}_{m_3, s_1 + s_2}]$$  \hspace{1cm} (53)

2. when $s_1 + s_2 \geq q$

$$[\varphi^{OSP(1,2)}_{m_1, s_1}] [\varphi^{OSP(1,2)}_{m_2, s_2}] = \sum_{m_3 = |m_1 - m_2| + 1, \begin{array}{c} m_3 + m_1 + m_2 \in odd \end{array}} \min [m_1 + m_2, 2q - m_1 - m_2] - 1 [\varphi^{OSP(1,2)}_{m_3, s_1 + s_2 - q}]$$  \hspace{1cm} (54)

Here, $[\varphi^{OSP(1,2)}_{m, s}]$ denotes the conformal block corresponding to the lowest weight representation.

The $T$ matrix for an irreducible module under : $\tau \rightarrow \tau + 1$, is diagonal,

$$S\chi \rightarrow e^{i\pi/4(\frac{(j+1)^2}{2k} - 1)}S\chi.$$  \hspace{1cm} (55)

Here $j$ and $k$ are the isospin and the level resp.. For the admissible representations(c.f. eq.(46)), we rewrite the matrix element as,

$$T_{m', s'} = e^{i\pi/4(\frac{(am - \tilde{q}m')^2}{q\tilde{q}} - 1)} \delta_{m, m'} \delta_{s, s'}$$  \hspace{1cm} (56)

$$m' + s', \begin{array}{c} m + s \in odd \end{array}$$

It is easy to verify that the following identities hold,

$$S^* S = T^* T = S^4 = T^{4[q, \tilde{q}]} = (ST)^3 = 1,$$  \hspace{1cm} (57)

where $[q, \tilde{q}]$, defined as the least common multiple of $q, \tilde{q}$, equals $q\tilde{q}$ $(q\tilde{q}/2)$ , when $gcd(q, \tilde{q}) = 1$ (2).

In \[33, 40\], the authors pointed out that the characters of $\hat{SL}(2)$ ($s \neq 0$) have a simple pole at $z = 0$, and that the residue contains a Virasoro characters as a factor. However, here we find that the supercharacters of $\hat{OSP}(1, 2)$ for the admissible representations are regular while the characters singular at $z = 0$. In fact, this is due to different locations of the poles of the modular functions in the $z$ complex plane.
Now let us calculate the characters of the $OSP(1,2)$ modules in a more explicit form. It is useful to consider the following $\vartheta$ function,

$$\vartheta \begin{bmatrix} \frac{b}{a} \\ 0 \end{bmatrix} (z, \alpha \tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \sigma(n+b/a)^2 + i 2 \pi z(n+b/a)}$$

(58)

The following product expansion of two $\vartheta$ functions is used many times in our calculation,

$$\vartheta \begin{bmatrix} \frac{a_1}{n_1} \\ 0 \end{bmatrix} (z_1, n_1 \tau) \vartheta \begin{bmatrix} \frac{a_2}{n_2} \\ 0 \end{bmatrix} (z_2, n_2 \tau) = \sum_{d \in \mathbb{Z}n_1+n_2} \vartheta \begin{bmatrix} n_1d+n_1+n_2 \\ n_1+n_2 \end{bmatrix} (z_1 + z_2, (n_1 + n_2) \tau) \vartheta \begin{bmatrix} n_1n_2d+n_2a_1-n_1a_2 \\ n_1n_2(n_1+n_2) \end{bmatrix} (n_2z_1 - n_1z_2, n_1n_2(n_1 + n_2) \tau),$$

(59)

from which we get

$$\vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (3z/2, 3\tau) - \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (3z/2, 3\tau) \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (3z/2, 3\tau) = e^{i \pi/3} \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z + 1/2, \tau) \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (1/2, 3\tau).$$

(60)

For a $OSP(1,2)$ Verma module labeled by isospin $j$ and level $k$, we have

$$\chi_{M_j} = e^{i 2 \pi z(j+1/4) + i 2 \pi \tau(j+1/4)^2} \left\{ \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (3/2z, 3\tau) - \vartheta \begin{bmatrix} -1/6 \\ 0 \end{bmatrix} (3/2z, 3\tau) \right\}^{-1},$$

$$S \chi_{M_j} = e^{i \pi/2 + i 2 \pi z(j+1/4) + i 2 \pi \tau(j+1/4)^2} \left\{ \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (3/2z + 3/2, 3\tau) - \vartheta \begin{bmatrix} -1/6 \\ 0 \end{bmatrix} (3/2z + 3/2, 3\tau) \right\}^{-1}. $$

(61)

**Proposition 1** For the admissible representations $L^{OSP(1,2)}_{m,s}$,

$$\chi^{OSP(1,2)}_{m,s} = e^{-i 3 \pi} \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z/2, \tau) \left\{ \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (1/2, 3\tau) \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z - 1/2, \tau) \right\}^{-1},$$

$$S \chi^{OSP(1,2)}_{m,s} = e^{-i 3 \pi} \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \left( z + \frac{1}{2}, \tau \right) \left\{ \vartheta \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} (1/2, 3\tau) \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z + 1/2, \tau) \right\}^{-1},$$

(62)
Proof. We need only to prove the equation valid for the character. Using eq.(59) and the character formula for $SL(2)$ and Virasoro algebra [10]

$$\chi_{V_{l,m}} = \vartheta \left[ \begin{array}{c} \frac{ql - pm}{2pq} \\ 0 \end{array} \right] (0, 2p\tilde{q}\tau) - \vartheta \left[ \begin{array}{c} \frac{ql + pm}{2pq} \\ 0 \end{array} \right] (0, 2p\tilde{q}\tau)$$

(63)

$$\chi_{SL(2)_{l,s}} = \vartheta \left[ \begin{array}{c} \frac{ql - ps}{2pq} \\ 0 \end{array} \right] (pz, 2pq\tau) - \vartheta \left[ \begin{array}{c} \frac{ql + ps}{2pq} \\ 0 \end{array} \right] (pz, 2pq\tau)$$

where

$$\eta(\tau) = q^{1/24} \prod (1 - q^n) = e^{-i\pi/6} \vartheta \left[ \begin{array}{c} 1/6 \\ 0 \end{array} \right] (1/2, 3\tau), \text{ for } q = e^{i2\pi\tau};$$

$$\pi(\tau, z) = e^{-i\pi/2} \vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (z + 1/2, \tau) = e^{i\pi/2} \vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (z - 1/2, \tau).$$

(64)

Moreover, when $l = q, 0, \text{ or } m = \tilde{q}, 0$,

$$\vartheta \left[ \begin{array}{c} \frac{ql - pm}{2pq} \\ 0 \end{array} \right] (0, 2p\tilde{q}\tau) - \vartheta \left[ \begin{array}{c} \frac{ql + pm}{2pq} \\ 0 \end{array} \right] (0, 2p\tilde{q}\tau) = 0.$$ 

(65)

Using eqs.(59, 60), the character formula, eq.(62), is obtained.

We see that $z = 0$ is a first order zero of the functions $\vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] ((z + 1)/2, \tau)$ and $\vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (z + 1/2, \tau)$, so the supercharacter of $OSP(1, 2)$ is not singular as $z \to 0$.

From the character of the Verma module and that of the irreducible module we can draw some information about the structure of these module. Notice that

$$\frac{\chi^{OSP(1, 2)}_{L_{m,s}}}{\chi^{OSP(1, 2)}_{M_{m,s}}} = \sum_n e^{i\pi\tau (q^2n^2 + n(qm - \tilde{q}s) + i\pi\tilde{q}zn)} - \sum_n e^{i\pi\tau (q^2n^2 - n(qm + \tilde{q}s + ms)) + i\pi\tilde{q}zn - n}. $$

(66)

The equation is very similar to that of $SL(2)$ and Virasoro minimal series [14, 33, 40, 9]. We may conjecture that the singular vectors in the admissible representations are of isospin $4j + 1 = 2\tilde{q}n \pm m - s^2_{\tilde{q}}$. Indeed this can be verified in the section 4.

### 3.3 NS and R type (Super-)Characters

Similar to the super-Virasoro algebras, there are two types of $OSP(1, 2)$ algebras. One is of Neveu-Schwarz( NS ) type, another of Ramond( R ) type. The two types of $OSP(1, 2)$ are in fact isomorphic while the corresponding Virasoro algebras by Sugawara constructions are somewhat different. The generators of R type $OSP(1, 2)$ are all integer moded while the fermionic generators of NS type $OSP(1, 2)$ are half-integer moded.
integer moded. Physically, the difference corresponds to the different boundary conditions, which might be seen from the modular transformations of the characters and the supercharacters. The subscript $r$ of $j^r_n$ in eq. (31) satisfies $r \in \mathbb{Z}(\mathbb{Z} + 1/2)$, resp. for R type (NS type, resp.) $OSP(1,2)$. There exists an isomorphism map from R type $OSP(1,2)$ to NS type $OSP(1,2)$

$$j^\pm_{n+1/2} \rightarrow \pm j^\mp_{n+1/2}; \quad J^\pm_n \rightarrow J^\mp_n;$$

$$J^3_n \rightarrow -J^3_n + k/2\delta_{n,0}; \quad k \rightarrow k;$$

$$d \rightarrow d + J^3_0,$$

which sends $G_\pm$ to $G_\mp$, $G_0$ to $G_0$.

For convenience, we add a superscript NS or R on the generators to distinguish the two types of superalgebras, for example,

$$J^{3,NS}_0 = \frac{k}{2} - J^{3,R}_0.$$  \hspace{1cm} (68)

The Virasoro generators by Sugawara construction, $\{L^NS_n\}$ and $\{L^R_n\}$, are also related,

$$L^R_0 = L^NS_0 - J^{3,NS}_0 + k/4;$$

$$L^R_n = L^NS_n - J^{3,NS}_n, \quad n \neq 0.$$  \hspace{1cm} (69)

By the isomorphism, eq.(67), a highest weight module over R type $OSP(1,2)$ is mapped to a highest weight module over NS type $OSP(1,2)$. In the following discussion, we adopt the notation in the previous subsection except for the superscript NS or R. Now we have two kinds of characters $\chi^NS_V$, $\chi^R_V$ and supercharacters $S\chi^NS_V$, $S\chi^R_V$ of a module $V$.

For a highest weight module with highest weight $(j^R, k)$, there exists the following relations

$$\chi_V(z + 1, \tau) = S\chi_V(z, \tau)e^{i2\pi j}.$$  \hspace{1cm} (70)

From eqs.(68, 69), we also have

$$\chi^R_V(z, \tau) = \chi^{NS}_V(-z - \tau, z, \tau)e^{i\pi k(z + \tau/2)}.$$  \hspace{1cm} (71)

The eqs. (70, 71) make our study of these (super)characters easier. What we are interested are the modular transformation properties of $S\chi^R$, the modular transformation of the others can be easily obtained. Under the $S$ transformation: $z \rightarrow z/\tau, \tau \rightarrow -1/\tau$, we have

$$\chi^R_{m,s}(z/\tau, -1/\tau) = e^{i\pi k(-z/\tau - 1)}S\chi^NS_{m,s'}(z, \tau);$$

$$S\chi^R_{m,s}(z/\tau, -1/\tau) = S\chi^R_{m,s'}(z, \tau);$$

$$S\chi^{NS}_{m,s}(z/\tau, -1/\tau) = e^{i\pi k/(z - 1)}S\chi^{R}_{m,s'}(z, \tau);$$

$$\chi^{NS}_{m,s}(z/\tau, -1/\tau) = e^{-i\pi k(z + \tau/2 + 1/(2\tau) - z/\tau + i2\pi j_{m,s}}.$$  \hspace{1cm} (72)

$$S\chi^{NS}_{m,s'}(z, \tau);$$
where $S^{m',s'}$ is given as in eq.[51]. Under $T$ transformation: $\tau \rightarrow \tau + 1$,

$$
\begin{align*}
\chi^R_V(z,\tau + 1) &= e^{2\pi h R} \chi^R_V(z,\tau); \\
S\chi^R_V(z,\tau + 1) &= e^{2\pi h R} S\chi^R_V(z,\tau); \\
\chi^{NS}_V(z,\tau + 1) &= e^{2\pi h} S\chi^{NS}_V(z,\tau); \\
S\chi^{NS}_V(z,\tau + 1) &= e^{2\pi h} S\chi^{NS}_V(z,\tau);
\end{align*}
$$

(73)

We see that under $S$ transformation

$$
S\chi^R \rightarrow S\chi^R, \quad \chi^{NS} \rightarrow \chi^{NS}, \quad \chi^R \leftrightarrow S\chi^{NS},
$$

and under $T$ transformation

$$
S\chi^R \rightarrow S\chi^R, \quad \chi^R \rightarrow \chi^R, \quad \chi^{NS} \leftrightarrow S\chi^{NS},
$$

which is the reminiscence of the free fermion theory on torus with various boundary conditions. The correspondence between the (super)characters and the boundary conditions on torus can be illustrated by fig. 1, where $P$ (A, resp.) stands for the periodic (antiperiodic, resp.) boundary condition.

![Figure 1: Boundary conditions on the torus](image)

4 Structure of the Verma Module

In section 3, we have formulated the characters via coset construction for the admissible representations of $\widehat{OSP}(1,2)$, from which we conjecture intuitively the structure of the corresponding Verma module.

In this section, we study the Verma module over a contragradient superalgebra via a quite different approach. The structure of the Verma module is analysed by generalizing the Kac-Kazhdan formula [32] to the case of superalgebra. Applying this result to $\widehat{OSP}(1,2)$, we reproduce the (super)character in section 4 for the admissible representation. The two important tools, the Jantzen filtration and the Casimir operator, are carried over to the super case. Finally, we give a explicit form for the construction of the singular vectors in the Verma module of $\widehat{OSP}(1,2)$, generalizing MFF’s result in ref.[38].
4.1 Structure of Verma module

Let us list some notations useful for our later discussion.

Notation:

$H$: the abelian diagonalizable subalgebra;

$\Delta$: the set of all roots;

$\Delta^+$: the set of all positive roots;

$\Delta_0$: the set of all even roots;

$\Delta_1$: the set of all odd roots;

$\Delta_0^+ \cap \Delta_0$;

$\Delta_1^+ \cap \Delta_1$;

$G_\alpha$: root space with root $\alpha$;

$(,)$: the nondegenerate symmetric bilinear form on $G$;

$e_\alpha^i$: basis of $G_\alpha$, $i = 1, \ldots, \dim G_\alpha$, such that $(e_\alpha^i, e_\beta^j) = \delta_{\alpha+\beta,0}\delta_{i,j}$, for $\alpha \in \Delta_+$;

$h_\alpha \in H$, $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha}) h_\alpha$;

$F(,)$: the symmetric bilinear form on $U(G)$;

$F_\eta(,)$: the matrix of $F(,)$ restricted to $U(G)_{-\eta} \otimes U(G)_\eta$ with a basis of $U(G)$, $\eta \in \Gamma^+$;

$M(\lambda), W(\lambda), L(\lambda)$: Verma module, Wakimoto module, and the irreducible module with HW $\lambda$, the definition see, for example [31, 46, 44, 7];

$\rho \in H^*$ such that $\rho(h_i) = 1$.

We introduce a Casimir operator on a $G$-module.

**Lemma 1** Let $V$ be a $G$-module, $\Omega$ is an operator on $V$,

$$\Omega(v) = (\mu + 2\rho, \mu) + 2 \sum_{\alpha \in \Delta^+} \sum_i e^{(i)}_{-\alpha} e^{(i)}_\alpha (v), \forall v \in V_\mu;$$

then (a).

$$[\Omega, g] = 0, \forall g \in G;$$

(b). if $V = M(\lambda), W(\lambda)$ or $L(\lambda)$,

$$\Omega = (\lambda + 2\rho, \lambda) \mathbf{1}.$$

Proof. See proposition 2.7 in ref. [30]; it follows from direct computation.

For an affine Kac-Moody superalgebra, up to a constant, $\Omega = (L_0 + d)(k + g)$, where $L_0$ is the zero-mode of the Virasoro algebra constructed by the Sugawara construction. Sometimes we might use $L_0$ instead of $-d$, when no confusions are made.

Since in a representation $\pi$ of $\mathcal{G}$, up to a constant, $\pi(e_{2\alpha}) = \pi(e_\alpha)^2$, when $\alpha \in \Delta_1$, we can identify $(e_\alpha)^2$ with $e_{2\alpha}$ in $U(\mathcal{G})$. So we select basis of $U(\mathcal{G})_-$, which take the following form

$$e^{n_{\alpha_1}}_{\alpha_1} e^{n_{\alpha_2}}_{\alpha_2} \cdots e^{n_{\alpha_k}}_{\alpha_k}$$

(77)
where $n_{\alpha_i} = 0, 1$, for $\alpha \in \Delta_1^-$, $n_{\alpha_i} \in Z_+$, for $\alpha \in \Delta_0^\pm$.

**Definition:** for $\eta \in \Gamma_+$, a partition of $\eta$ is a set of non-negative integers $\{n_{\alpha_i}\}$, where $n_{\alpha_i} = 0, 1$, for $\alpha_i \in \Delta_1^-$, $n_{\alpha_i} \in Z_+$, for $\alpha_i \in \Delta_0^\pm$, which satisfy

$$\sum_{\alpha_i} n_{\alpha_i} \alpha_i = \eta.$$  \hfill (78)

**Definition:** partition function $P(\eta)$ is the number of all partitions of $\eta$.

Now consider the leading term of $\det F(\eta)$, by which we mean the monomial term in $\det F(\eta)$ with the maximal power of $h_\alpha$’s. We get the following lemma which is proven in the appendix A.

**Lemma 2** (c.f. [32], Lemma 3.1) Let $\eta \in \Gamma_+$, then up to a constant factor, the leading term of $\det F_\eta$ is

$$\prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} h_\alpha^{P(\eta-n\alpha)} / \prod_{\alpha \in \Delta_1^-} \prod_{n=1}^{\infty} h_\alpha^{2P(\eta-2n\alpha)} = \prod_{\alpha \in \Delta^+_1, \alpha/2 \notin \Delta_1^-} \prod_{n=1}^{\infty} h_\alpha^{P(\eta-n\alpha)} \prod_{\alpha \in \Delta_1^-} \prod_{n=1}^{\infty} h_\alpha^{2P(\eta-2n\alpha)},$$  \hfill (79)

where the roots are taken with their multiplicities.

So far we have formulated the Casimir operator and the leading term in $\det F_\eta$. Then in the same consideration as over Lie algebra [32], we get

**Theorem 1** (c.f. [32], Theorem 3.1) Generalized Kac-Kazhdan Formula:

Let $\mathcal{G} = \mathcal{G}(A)$ be a contragradient superalgebra with a Cartan matrix $A$ defined in section 2.1. Then up to non-zero constant factor

$$\det F_\eta = \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \Phi_n(\alpha)^{P(\eta-n\alpha)} / \prod_{\alpha \in \Delta_1^-} \prod_{n=1}^{\infty} \Phi_{2n}(\alpha)^{2P(\eta-2n\alpha)}$$

$$= \prod_{\alpha \in \Delta^+_1, \alpha/2 \notin \Delta_1^-} \prod_{n=1}^{\infty} \Phi_n(\alpha)^{P(\eta-n\alpha)} \prod_{\alpha \in \Delta_1^-} \prod_{n=1}^{\infty} \Phi_n(\alpha)^{P(\eta-n\alpha)},$$  \hfill (80)

where

$$\Phi_n(\alpha) = h_\alpha + \rho(h_\alpha) - n/2 (\alpha, \alpha),$$  \hfill (81)

and the roots are taken with their multiplicities.

Proof is parallel to that of Kac-Kazhdan [32], by using lemmas 1, 2 and the Jantzen filtration.
4.2 Verma module over $\widehat{OSP}(1,2)$

In this subsection we shall apply the results of the last subsection to a particular case, namely, the $\widehat{OSP}(1,2)$ superalgebra. The results obtained in section 3 by the decomposition of $\widehat{OSP}(1,2)$ module are rederived, now, from the pure algebraic relations. Further more, all the $\widehat{OSP}(1,2)$ Verma modules are completely classified.

The Chevalley bases of $\widehat{OSP}(1,2)$ are

\[
e_0 = \sqrt{2}j_0^+, \quad f_0 = \sqrt{2}j_0^-, \quad h_0 = 4j_0^3,
\]
\[
e_1 = J_1^-, \quad f_1 = J_{-1}^+, \quad h_1 = -2j_0^3 + k
\]

The Cartan matrix is

\[
A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}
\]

and $e_0, f_0 \in G_1$, with the Abelian sub-algebra $H = \{h_0, h_1, d\}$. As in the Kac-Moody Lie algebra [30, 31], we can define a bilinear invariant form in $\widehat{OSP}(1,2)$ and $H^*$. Rewrite $A$ as

\[
A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1 \\ -1 & 2 \end{pmatrix} = \text{diag}(\epsilon_1, \epsilon_2) (h_{ij}).
\]

The bilinear form on $H$,

\[
(h_0, h_0) = 8, \quad (h_1, h_1) = 2, \quad (h_0, h_1) = -4, \quad (d, h_i) = (d, d) = 0
\]

On $H^*$,

\[
(\alpha_0, \alpha_0) = 1/2, \quad (\alpha_1, \alpha_1) = 2, \quad (\alpha_0, \alpha_1) = -1.
\]

The fundamental dominant weights $\Lambda_0, \Lambda_1$ satisfy

\[
\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0
\]

We see that $\Lambda_1 = 2\Lambda_0 + \alpha_1/2$ and

\[
(\Lambda_0, \Lambda_0) = 0, \quad (\alpha_0, \Lambda_0) = 1/4, \quad (\alpha_1, \Lambda_1) = 1
\]

\[
\rho = \Lambda_0 + \Lambda_1
\]

\[
\Delta^+ : \{\alpha = n_0\alpha_0 + n_1\alpha_1, \quad 2n_1 - n_0 = \pm 2, \pm 1, 0, \quad n_1 \geq 1 \text{ or } n_1 = 0, \quad n_0 = 1, 2 \}
\]

\[
\Delta_1^+ : \{\alpha = n_0\alpha_0 + n_1\alpha_1, \quad 2n_1 - n_0 = \pm 1, \quad n_1 \geq 1 \text{ or } n_1 = 0, n_0 = 1 \}
\]

A highest weight with level $k$, isospin $j$, is

\[
\Lambda = 4j\Lambda_0 + (k - 2j)\Lambda_1
\]
From

\[(\lambda + \rho, \alpha) = \frac{1}{4}(n_0 - 2n_1)(4j + 1) + (k + 3/2)n_1,\]
\[(\alpha, \alpha) = \frac{1}{2}(n_0 - 2n_1)^2,\] (92)

we have

\[\Phi_{\alpha,n} = (\lambda + \rho, \alpha) - \frac{n}{2}(\alpha, \alpha)\]
\[= \frac{1}{4}(n_0 - 2n_1)(4j + 1) + (k + 3/2)n_1 - \frac{n}{2}(n_0 - 2n_1)^2\] (93)

After direct computation, we get

**Lemma 3** \(M_j\) is reducible if and only if

\[4j + 1 = m - s(2k + 3), \text{ assuming } 2k + 3 \neq 0,\] (94)

for some \(m, s \in \mathbb{Z}\), \(m + s \in \text{ odd}, m < 0, s < 0\) or \(m > 0, s \geq 0\). If eq.(94) holds, then there exists a singular vector in \(M_j\) with isospin \(j - m, s\).

**Theorem 2** (c.f. [15] Theorem 4.1) Let \(k + 3/2 \neq 0\) then the structure of Verma module \(M(j)\) is described by one of the following diagram.

\[
\begin{align*}
&v_0 \quad v_0 \quad v_0 \quad v_0 \quad v_0 \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&v_1 \quad v_1 \quad v_1 \quad v_1 \quad v_1 \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&v_2 \quad v_2 \quad v_2 \quad v_2 \quad v_2 \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&v_3 \quad v_3 \quad v_3 \quad v_3 \quad v_3 \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&v_{i-1} \quad v_{i-1} \quad v_{i-1} \quad v_{i-1} \quad v_{i-1} \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&\vdots \quad \vdots \quad \vdots \quad v_i \quad \vdots \\
&I \quad II \quad III_- \quad III_+ \quad III_+^0 \quad III_+^0
\end{align*}
\] (95)

where \(v_i\)'s are the singular vectors in \(M(j)\). An arrow or a chain of arrows, goes from a vector to another iff the second vector is in the sub-module generated by the first one.

Proof follows from lemma [3] and the generalized Kac-Kazhdan formula eq.(80). It is analogous to that in ref.[4, 15].
If \( k \notin \mathbb{Q} \), then case I and case II occurs. If \( k \in \mathbb{Q} \), we write \( 2k + 3 = \tilde{q}/q \), where \( \tilde{q} + q \in \) even, \( \gcd(\tilde{q}, (q + \tilde{q})/2) = 1 \). If eq.(94) holds, then for \( 2k + 3 > 0(<0) \), case III (III) occurs. If \( j = j_{0,s} \) for some \( s \in \) odd, then \( III^- \) or \( III^+ \) occurs. Note that

\[
\Phi_{\alpha,n}(\lambda) = -\Phi_{\alpha,n}(-2\rho - \lambda + n\alpha). \tag{96}
\]

**Proposition 2** A singular vector \( v_1 \) with weight \( \lambda_1 \) is in a Verma module generated by \( v_2 \) with weight \( \lambda_2 \) iff a singular vector \( v'_{2} \) with weight \( \lambda'_{2} = -2\rho - \lambda_2 \) is in the Verma module generated by \( v'_{1} \) with weight \( \lambda'_{1} = -2\rho - \lambda_1 \).

**Remark:** The above duality relation is similar to that of Virasoro algebra, where we have the following duality, \((h,c) \leftrightarrow (1-h,26-c)\).

We note that case \( III^-,III^+ \) can be described by a subdiagram of diagram 1 and diagram 2 (resp.)

\[
v_{m,s} \rightarrow v_{-m+2\tilde{q},s} \rightarrow v_{m+2\tilde{q},s} \rightarrow \cdots \rightarrow v_{m+2n\tilde{q},s} \rightarrow v_{-m+2(n+1)\tilde{q},s} \rightarrow \]

\[
v_{0,s} \rightarrow v_{2\tilde{q},s} \rightarrow v_{-2\tilde{q},s} \rightarrow \cdots \rightarrow v_{2n\tilde{q},s} \rightarrow v_{-2n\tilde{q},s} \rightarrow \]

Diagram 1

\[
v_{-m,s} \leftarrow v_{m-2\tilde{q},s} \leftarrow v_{m-2\tilde{q},s} \leftarrow \cdots \leftarrow v_{m-2n\tilde{q},s} \leftarrow v_{m-2(n+1)\tilde{q},s} \leftarrow \]

\[
v_{0,s} \leftarrow v_{-2\tilde{q},s} \leftarrow v_{2\tilde{q},s} \leftarrow \cdots \leftarrow v_{-2n\tilde{q},s} \leftarrow v_{2(n+1)\tilde{q},s} \leftarrow \]

Diagram 2

Here \( 0 < m < \tilde{q}, 0 \leq s < q, \) and the sum of the two subscripts is always odd. From the above theorem, we get the following corollary about the relation between the irreducible module and the Verma module.

**Corollary 1** Let \( k+3/2 \neq 0, \) for any irreducible module \( L_{v_0} \), there exists a sequence which is a resolution of \( L_{v_0} \),

\[
\cdots \xrightarrow{\partial_{-1}} M^i \xrightarrow{\partial_{1}} M^{i+1} \xrightarrow{\partial_{i+1}} \cdots \xrightarrow{\partial_{2}} M^{-1} \xrightarrow{\partial_{i+1}} M^0 = M_{v_0} \rightarrow 0, \tag{99}
\]

where \( M^i \) is direct sum of Verma modules.
Proof. (i) If $M_{v_0}$ is in the case (I), then simply let $M^i = 0$, for $i < 0$.

(ii) If $M_{v_0}$ is in the case (II) or (III$^0$), let $M^i = 0$, for $i < -1; M^{-1} = M_{v_1}$, $\partial_{-1}$ is an embedding.

(iii) If $M_{v_0}$ is in the case (III$^-$), let $M^i = M_{v_i} \oplus M_{v_{i-1}}$, for $i < 0$, $\partial_i : (x, y) \to (x - y, x - y)$, for $i < -1; \partial_{-1} : (x, y) \to x - y$.

(iv) If $M_{v_0}$ is in the case (III$^+$), let $M^i = M_{v_i} \oplus M_{v_{i-1}}$, for $-n < i < 0 M^{-n} = M_{v_n}$. $\partial_i : (x, y) \to (x - y, x - y)$, for $-n < i < -1 \partial_{-1} : (x, y) \to x - y$, $\partial_{-n} : x \to (x, x)$.

Then the corollary is easily verified.

### 4.3 MFF Construction

We take steps after ref.[38], where a singular vector in a Verma module over $\widehat{SL(2)}$ is given explicitly. The crucial point is that the authors of [38] generalized polynomially commutators between generators in $U(\mathcal{G})$ to commutators between those with complex exponents. Similarly we also can generalize this procedure to contragradient Lie superalgebra. However we only discuss it on $OSP(1, 2)$.

Let $M_{f_{m,s}}$ be a Verma module as that in the lemma[3], then

1. if $m > 0$, $s \geq 0$

$$| - m, s \rangle = f_0^{m+s,q/q} f_1^{1/2[m+(s-1)q/q]} f_0^{m+(s-2)q/q} f_1^{1/2[m-(s-1)q/q]} f_0^{m-s,q/q} |m, s \rangle \quad (100)$$

2. if $m < 0$, $s < 0$

$$| - m, s \rangle = f_1^{-1/2[m+(s+1)q/q]} f_0^{-[m+(s+2)q/q]} f_1^{-1/2[m+(s+3)q/q]} \cdots$$

$$\cdots f_0^{-[m-(s+2)q/q]} f_1^{-1/2[m-(s+1)q/q]} |m, s \rangle \quad (101)$$

is a null vector. The action of a multiplying factor in the r. h. s. of the eqs.$(100,101)$ on the weight space is as the corresponding fundamental Weyl reflection. We see that $m + s \in \text{odd}$, $m$, $s < 0$, or $m > 0$, $s \geq 0$ is equivalent to the following restraints,

1. the sum of all the exponents of a fixed Chevalley generator is an non-negative integer.

2. the sum of all the exponents of a fixed odd Chevalley generator plus the times it appears in the expression is odd.

The second restraint is of great importance for the contragradient superalgebras. More exactly, we should rewrite the multiplying factor $f_0^2$ as $f_0(f_0^2)^{(x-1)/2}$ in the r. h. s. of the above equations. Define operator

$$F(m, s, t) = f_0(f_0^2)^{[m+st-1]/2} f_1^{[m-(s-1)t]/2} f_0(f_0^2)^{[m+(s-2)t-1]/2} \cdots f_1^{[m-(s-1)t]/2} f_0(f_0^2)^{[m-st-1]/2},$$

(102)
where \( m > 0, \ s \geq 0, \ m + s \in \text{odd}, \ t = \tilde{q}/q \). We assume that \((f_{2}^{3})^{x}\) is always an even operator. The sum of all the exponents of \( f_{2}^{3} \) is \((m - 1)(s + 1)/2\), a non-negative integer. To prove that \( | - m, s \rangle \) in eq. (101) is a singular vector in the Verma module, firstly, we have to show that that \( F(m, s, t) \) defined in eq. (102) is in the enveloping algebra. To do that we introduce the following generalized commutators \[38\],

\[
[g_{1}, g_{2}^{\gamma}] = \sum_{i=1}^{\infty} \left( \begin{array}{c}
\gamma \\
i
\end{array} \right) g_{2}^{-i} \cdots \left[ \left[ g_{1}, g_{2}, g_{2} \right], \cdots g_{2} \right]
\]

\[
= - \sum_{i=1}^{\infty} (-1)^{i} \left( \begin{array}{c}
\gamma \\
i
\end{array} \right) \cdots \left[ \left[ g_{1}, g_{2}, g_{2} \right], \cdots g_{2}^{\gamma-i} \right], \quad g_{2} \in \mathcal{G}, \tag{103}
\]

\[
[g_{1}^{\gamma_{1}}, g_{2}^{\gamma_{2}}] = \sum_{j_{1}=1}^{\infty} \sum_{j_{2}=1}^{\infty} \left( \begin{array}{c}
\gamma_{1} \\
j_{1}
\end{array} \right) \left( \begin{array}{c}
\gamma_{2} \\
j_{2}
\end{array} \right) Q_{j_{1}, j_{2}}(g_{1}, g_{2}) g_{2}^{\gamma_{2}} g_{1}^{\gamma_{1}-j_{1}}, \quad g_{1}, \ g_{2} \in \mathcal{G}, \tag{104}
\]

where \( Q_{j_{1}, j_{2}} \)'s are independent of \( \gamma_{1}, \ \gamma_{2} \) and given by induction,

\[
Q_{j_{1}, j_{2}}(g_{1}, g_{2}) = [g_{1}, Q_{j_{1}-1, j_{2}}] + \sum_{i=0}^{j_{2}-1} (-1)^{j_{2}-i-1} \left( \begin{array}{c}
j_{2} \\
i
\end{array} \right) Q_{j_{1}-1, i} \cdots \left[ \left[ g_{1}, g_{2}, g_{2} \right], \cdots g_{2} \right], \tag{105}
\]

and \( Q_{0,0} = 1, \ Q_{0,v} = 0, \forall v > 0 \). By repeatedly using eq. (103) and (104), we can rewrite \( F(m, s, t) \) in the following form,

\[
F(m, s, t) = \sum_{j_{0}=1}^{\infty} \sum_{j_{1}=1}^{\infty} P_{j_{0}, j_{1}}(f_{0}, f_{1}) f_{0}^{m(s+1)-j_{0}} f_{1}^{\frac{ms}{2}-j_{1}}, \tag{106}
\]

where \( P_{j_{0}, j_{1}} \in \mathcal{U}([\mathcal{G}, \mathcal{G}_{0}], \mathcal{G}_{0}) \), depends on \( t \) polynomially, with the power of \( t \) less than \( j_{0} + j_{1} + 1 \) and that of \( f_{0} \ (f_{1}) \) equal to \( j_{0} \ (j_{1}) \). We see that \( F(m, s, t) \) is in the enveloping algebra \( \mathcal{U}(\mathcal{G}) \) if and only if

\[
P_{j_{0}, j_{1}} = 0, \quad \text{for} \ t \not\in \mathcal{C}, \ \text{or} \ j_{1} \not\in \mathcal{C} \tag{107}
\]

To see that eq. (107) is indeed satisfied, let us check the cases for which \( t \in \text{odd} \). In that case, each exponent in the r.h.s. of the eq. (102) is an integer. By the analogy of the discussion in ref. \[38\], it can be shown that \( F(m, s, t) \) is well defined. \( F(m, s, t) \) is in the enveloping algebra \( \mathcal{U}(\mathcal{G}) \), and gives rise to the singular vector in the Verma module \( M(j_{m,s}) \) with isospin \( j_{m,s} \) and level \( k = (t-3)/2 \) when acting on the HWS \( |m, s \rangle \). Since eq. (107) valids for infinitely many \( t \)'s, i.e. for those \( t \in \text{odd} \), it can be deduced that so does it for all \( t \in \mathcal{C} \), noting that \( P_{j_{0}, j_{1}} \)'s depend on \( t \) polynomially. Now we come to the conclusion that \( F(m, s, t) \) is in the enveloping algebra \( \mathcal{U}(\mathcal{G}) \) for all \( t \in \mathcal{C} \). Secondly, it remains to check that

\[
e_{i}F(m, s, t)|m, s \rangle = 0, \quad i = 0, 1, \tag{108}
\]

which is equivalent to say that \( F(m, s, t)|m, s \rangle \) is singular vector in \( M(j_{m,s}) \) with isospin \( j_{m,s} \). Eq. (108) can be verified by contracting \( e_{i} \)'s on the r.h.s. of eq. (100). To make our discussion more concrete, in appendix B an example MFF construction is given for \( m = 2, \ s = 1 \).

So far we have illustrated that the r. h. s. of eq. (100) is well defined, the similar discussion can be taken over to the eq. (100) for \( m, s < 0, \ m + s \in \text{odd} \).
5 Wakimoto Module Over $\hat{OSp}(1,2)$

In this section we study the Wakimoto Module for $\hat{OSp}(1,2)$ in the free field representation [3]. Again we obtain results analogous to that of $\hat{SL}(2)$ [4]. The admissible representation and the Wakimoto module are related by the Felder [17, 4] BRST operator. More exactly the admissible representation is the zero degree cohomology of the Felder complex while other degree cohomology vanish. The crucial points for the BRST operator are the screening operator [4, 5] (or interwining operator in [15]) and again the Jantzen filtration [28].

Wakimoto module $W(\lambda)$ over $\hat{SL}(2)$ was first studied in ref. [46], later over affine Kac-Moody algebra generally in ref. [14, 15]. It admits a free field realization [15]. For superalgebra $\hat{OSp}(1,2)$, we also have the free field representation [3].

\[
\begin{align*}
J^+ &= -\beta; \\
J^- &= \beta \gamma^2 - i\alpha_+ \gamma \partial \phi + \gamma \psi \psi^+ - k \partial \gamma + (k + 1) \psi \partial \psi; \\
J^3 &= -\beta \gamma + i\alpha_+ / 2 \partial \phi - \frac{1}{2} \psi \psi^+; \\
J^- &= \gamma (\psi^+ - \beta \psi) + i\alpha_+ \partial \phi + (2k + 1) \partial \psi; \\
J^+ &= \psi^+ - \beta \psi;
\end{align*}
\]

where $\alpha_+ = \sqrt{2k+3}$, $(\beta, \gamma)$ are bosonic fields with conformal isospin $(1, 0)$, $(\psi^+, \psi)$ are fermionic fields with conformal isospin $(1, 0)$. More concretely, expand them in Laurent power series:

\[
\begin{align*}
\beta(z) &= \sum_n \beta_n / z^{n+1}; & \gamma(z) &= \sum_n \gamma_n / z^n; \\
\psi^+(z) &= \sum_n \psi^+_n / z^{n+1}; & \psi(z) &= \sum_n \psi_n / z^{n+1}; \\
i \partial \phi(z) &= \sum_n \phi_n / z^{n+1}. 
\end{align*}
\]

(110)

which is equivalent to the commutators

\[
\begin{align*}
[\beta_n, \gamma_m] &= \delta_{n+m,0}; & \{\psi^+_n, \psi_m\} &= \delta_{n+m,0}; \\
[\phi_n, \phi_m] &= n \delta_{n+m,0}.
\end{align*}
\]

(112)

while other commutators vanish.

The HWS in Wakimoto module is annihilated by all positive modes of these field as well as $\beta_0, \psi^+_0$, and is an eigenstate of $\phi_0$,

\[
\phi_0 |HWS\rangle = 2j/\alpha_+ |HWS\rangle.
\]

(113)

The Wakimoto module is the Fock space generated by $\beta_n, \gamma_n, \psi_n, \psi^+_n, n < 0$, and $\gamma_0, \psi_0$. 

\[
\begin{align*}
J^+ &= -\beta; \\
J^- &= \beta \gamma^2 - i\alpha_+ \gamma \partial \phi + \gamma \psi \psi^+ - k \partial \gamma + (k + 1) \psi \partial \psi; \\
J^3 &= -\beta \gamma + i\alpha_+ / 2 \partial \phi - \frac{1}{2} \psi \psi^+; \\
J^- &= \gamma (\psi^+ - \beta \psi) + i\alpha_+ \partial \phi + (2k + 1) \partial \psi; \\
J^+ &= \psi^+ - \beta \psi;
\end{align*}
\]
Proposition 3  Let

\[ V(z) = \sum_n V_n/z^{n+1} = (\psi^+ + \beta \psi)e^{i\alpha-\phi(z)}, \quad \alpha_- = -1/\alpha_+, \quad (114) \]

(i) \( V(z) \) is a screening operator. i.e. the OPE of the \( \text{OSP}(1,2) \) currents and \( V(z) \) are total derivatives [3].  (ii) \( V(z) \) induces an \( \text{OSP}(1,2) \)-module homomorphism \( W_j \rightarrow W_{j-1/2} \).

Proof. (i) follows from direct computation.

(ii) Note that the Fock space is characterized by the fact that it is an eigenspace of \( \phi_0 \) with eigenvalue \( 2j/\alpha_+ \), i.e.

\[ \forall |v\rangle \in F_j, \quad \phi_0 |v\rangle = 2j/\alpha_+ |v\rangle. \quad (115) \]

Since

\[ i\partial \phi(z_1)V(z_2) = -1/\alpha_+ \frac{V(z_2)}{z_1 z_2}, \quad [\phi_0, V_n] = -1/\alpha_+. \quad (116) \]

From (i) we have

\[ [J^a_n, V_0] = 0. \quad (117) \]

So we can deduce that \( V_0 : F_j \rightarrow F_{j-1/2} \) is a homomorphism.

Define

\[ Q_m = \oint \oint \ldots \oint V(z_1)V(z_2)\ldots V(z_m). \quad (118) \]

Then \( Q_m \) is an \( \text{OSP}(1,2) \)-module homomorphism,

\[ Q_m : F_j \rightarrow F_{j-1/2}. \quad (119) \]

The proof is as in ref. [4, 15] for the case of \( \text{SL}(2) \) current algebra.

Proposition 4  Let \( 2k + 3 = \tilde{q}/q \neq 0, \quad 4j_{m,s} + 1 = m - s(2k+3), \quad m, s \in \mathbb{Z}, \quad m+s \text{ odd}, \) then

(i) if \( \tilde{q} > m > 0, \quad s \geq 0, \) in Wakimoto module \( W_{j_{m,s}}, \) there exists one and only one cosingular vector \( |w_{-m,s}\rangle \) with the isospin \( j_{-m,s}, \) and under homomorphism map (up to a nonzero constant)

\[ Q_m : W_{j_{m,s}} \rightarrow W_{j_{m,s}}, \quad Q_m |w_{-m,s}\rangle = |j_{-m,s}\rangle, \quad (120) \]

where \( |j_{-m,s}\rangle \) is the vacuum vector in \( W_{j_{m,s}}. \)

(ii) if \( -\tilde{q} < m < 0, \quad s < 0, \) in Wakimoto module \( W_{j_{m,s}}, \) there exists one and only one singular vector \( |w_{m,s}\rangle \) with the isospin \( j_{m,s}, \) and under homomorphism map (up to a nonzero constant)

\[ Q_{-m} : W_{j_{m,s}} \rightarrow W_{j_{m,s}}, \quad Q_{-m} |j_{m,s}\rangle = |w_{m,s}\rangle, \quad (121) \]

where \( |j_{m,s}\rangle \) is the vacuum vector in \( W_{j_{m,s}}. \)
Proof see appendix C.

By the above proposition, we can work out the weights of singular vectors and cosingular vectors in a Wakimoto module $W_j$. By using the techniques such as the Jantzen filtration as those used in studying the structure of Wakimoto modules over $\widetilde{SL}(2)$ \[4, 15\] and Feigin-Fuchs modules over Vir. \[16\], we reach the following theorem.

**Theorem 3** (cf.\[14\], theorem 4.2) Let $k + 3/2 \neq 0$, then the structure of Wakimoto module $W_j$ can be described by one of the following diagram.

\[
\begin{array}{cccccccc}
  & w_0 & w_0 & w_0 & w_0 & w_0 & w_0 & w_0 \\
  \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow \\
  w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 & w_1 \\
  \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
  w_2 & w_2 & w_2 & w_2 & w_2 & w_2 & w_2 & w_2 \\
  \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow \\
  w_3 & w_3 & w_3 & w_3 & w_3 & w_3 & w_3 & w_3 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
  w_{n-1} & w_{n-1} & w_{n-1} & w_{n-1} & w_{n-1} & w_{n-1} & w_{n-1} & w_{n-1} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  w_n & w_n & w_n & w_n & w_n & w_n & w_n & w_n \\
\end{array}
\]

$\begin{array}{cccccccc}
  & I & II(+) & II(−) & III_0(+) & III_0(−) & III_{0+}(+) & III_{0−}(−) \\
  \end{array}$

\[
\begin{array}{cccccccc}
  & w_0 & w_0 & w_0 & w_0 & w_0 & w_0 & w_0 \\
  \nw_1 & \nw_1 & \nw_1 & \nw_1 & \nw_1 & \nw_1 & \nw_1 & \nw_1 \\
  \nw_2 & \nw_2 & \nw_2 & \nw_2 & \nw_2 & \nw_2 & \nw_2 & \nw_2 \\
  \nw_3 & \nw_3 & \nw_3 & \nw_3 & \nw_3 & \nw_3 & \nw_3 & \nw_3 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \nw_{n-1} & \nw_{n-1} & \nw_{n-1} & \nw_{n-1} & \nw_{n-1} & \nw_{n-1} & \nw_{n-1} & \nw_{n-1} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \nw_n & \nw_n & \nw_n & \nw_n & \nw_n & \nw_n & \nw_n & \nw_n \\
\end{array}
\]

$\begin{array}{cccccccc}
  & I - & II(+) & II(−) & III_0(+) & III_0(−) & III_{0+}(+) & III_{0−}(−) \\
  \end{array}$

(122)
where \( w_i \)'s are the singular vectors in \( W_j \) or in its subquotient corresponding to \( v_i \) in the Verma modules, i.e. they have the same weight. An arrow or a chain of arrows, goes from the vector to another iff the second vector is in the sub-module generated by the first one.

Having worked out the structure of the Wakimoto modules, we get the a resolution of the admissible module in terms of Wakimoto modules.

**Corollary 2** The following sequence is a resolution of admissible module \( L_{m,s} \),

\[
\cdots \xrightarrow{Q^m} W_{-m+2\bar{q},s} \xrightarrow{Q^{-m}} W_{m+2(n-1)\bar{q},s} \xrightarrow{Q^m} \cdots \xrightarrow{Q^{-m}} W_{-m+2\bar{q},s} \xrightarrow{Q^m} W_{m,s} \xrightarrow{Q^{-m}} W_{-m,s} \xrightarrow{Q^m} \cdots, \tag{123}
\]

The proof of the nilpotency and that the resolution is the irreducible module is similar to that of ref.[4] for the admissible modules over \( \hat{SL}(2) \) and ref.[17] for the irreducible modules in the minimal models over Virasoro algebra.

### 6 Conclusions and Conjectures

In this paper, we have studied the modules over affine Kac-Moody superalgebras in general. A naive approach to find the structure of superalgebra module is by coset construction \( G = G_0 \otimes G/G_0 \), where \( G_0 \) is generated by the even generators. Such decomposition is analysed in detail for \( \hat{OSP}(1,2) \), for which \( G_0 = \hat{SL}(2) \) in section 3. Our conjecture is that for a more general affine Lie superalgebra, the coset space \( G/G_0 \) will be generated by the \( W_N \) algebra.

Besides the coset decomposition, a more general procedure of classifying the superalgebra modules is provided by generalizing the Kac-Kazhdan formula to the super case. Indeed it is possible as we have done it in section 4. The corresponding MFF construction of null vectors is generalized in a similar fashion.

In parallel, we have also analysed the Wakimoto module for the affine superalgebra. All these results will be relevant for our analysis of the \( G/G \) gauged WZNW model on a supergroup manifold, which is the subject of our forthcoming paper([19]).

**Acknowledgement:** We are grateful to H.Y. Guo, H.L. Hu, K. Wu and R.H. Yue for useful discussions and suggestions. This work is supported in part by the National Science Foundation of China and the National Science Committee of China.
A The Leading Term of the $Det F_\eta$

Now we prove the lemma 2 in section 4. First, we prove several propositions.

**Definition:**

$$P_{\alpha_i}(\eta) = \sum_{\{n_{\alpha_j}\text{ partition of } \eta \atop \alpha_i \in \{\alpha_j\}}} n_{\alpha_i}.$$  

**Proposition 5**

$$\sum_{\eta \in \Delta_+} P(\eta)e^{-\eta} = \prod_{\alpha \in \Delta_0^+} \frac{1}{1-e^{-\alpha}} \prod_{\beta \in \Delta_1^+} (1+e^{-\beta}).$$  

(124)

Proof. It is obtained by direct computation.

**Proposition 6**

$$P_{\alpha_i}(\eta) = \begin{cases} \sum_{n=1} P(\eta-n\alpha_i), & \text{if } \alpha_i \in \Delta_0; \\ \sum_{n=1} P(\eta-n\alpha_i)(-1)^n, & \text{if } \alpha_i \in \Delta_1 \end{cases}$$  

(125)

Proof. Using the generating function,

$$\sum_{\eta} P_{\alpha_i}(\eta)e^{-\eta} = \prod_{\eta \setminus \eta_{\alpha_i}} \sum_{\{n_{\alpha_j}\text{ partition of } \eta \atop \alpha_j \neq \alpha_i, \alpha_j \in \Delta_0^+}} n_{\alpha_j} e^{-\sum_{\alpha_j} n_{\alpha_j} \alpha_j}$$  

(126)

$$= \prod_{\alpha_j \neq \alpha_i, \alpha_j \in \Delta_0^+} (1+e^{-\alpha_j}) \sum_{n_{\alpha_i}} n_{\alpha_i} e^{-n_{\alpha_i} \alpha_i};$$  

(127)

$$= \sum_{n_{\alpha_i}} n_{\alpha_i} e^{-n_{\alpha_i} \alpha_i} = \begin{cases} \frac{e^{-\alpha_i}}{(1-e^{-\alpha_i})^2}, & \text{if } \alpha_j \in \Delta_0; \\ e^{-\alpha_i}, & \text{if } \alpha_j \in \Delta_1 \end{cases}$$  

(128)

by proposition 6, we have

$$\sum_{\eta} P_{\alpha_i}(\eta)e^{-\eta} = \begin{cases} \sum_{\eta} P(\eta)e^{-\eta} \sum_{n=1} e^{-n\alpha_i}, & \text{when } \alpha_i \in \Delta_0; \\ \sum_{\eta} P(\eta)e^{-\eta} \sum_{n=1} e^{-n\alpha_i}(-1)^{n-1}, & \text{when } \alpha_i \in \Delta_0. \end{cases}$$  

(129)

By using eqs.(128,129) and comparing the coefficients of the term $e^{-\eta}$ in both sides of eq.(127), we get the proposition.

Now selecting the basis of $U(G)_\eta$ as in eq.(77), where $\{n_{\alpha_i}\}$ is a partition of $\eta$. Note that the leading term is the product of the leading terms in the diagonal elements of the matrix $F_\eta$, which is proportional to

$$\prod_{\alpha_i} h_{\alpha_i} P_{\alpha_i}(\eta).$$  

(130)

Moreover, note that $h_{2\alpha_i} = 2h_{\alpha_i}$, when $\alpha_i \in \Delta_1$, which completes the proof of the lemma.
B  The singular vector $|−2, 1\rangle$

In section 4, it is shown that the explicit form of the singular vector in a Verma module over $\hat{OSP}(1,2)$ can be constructed by the use of eqs. (100, 101). To elucidate such a procedure, we consider a simple case, i.e. $m = 2, s = 1$.

By definition,

$$F(2,1,t) = f_0(f_0^3)^\frac{1+t}{2}f_1f_0(f_0^3)^\frac{1+t}{2}.$$  

(131)

Here, for $\hat{OSP}(1,2)$ (see eq.(82)),

$$f_0 = \sqrt{2}j_0, \quad f_0^2 = -2J_0^-, \quad f_1 = J^+$$

$$[\cdots [f_1, f_0^2], \cdots, f_0^2] = 0, \quad \forall i \geq 3.$$  

(132)

By using eq.(103), eq.(131) can be rewritten as,

$$F(2,1,t) = f_0f_1f_0^3 - \frac{t+1}{2} f_0[f_1, f_0^3]f_0 + \frac{t^2-1}{8}[[f_1, f_0^2], f_0^2].$$  

(133)

More concretely,

$$F(2,1,t) = 4J^+_3(J_0^-)^2 - 4j_{-1}^+j_0^-J_0^- - 4(t+1)J_3^+J_0^- + 2(t+1)j_{-1}^-j_0^- - (t^2-1)J_{-1}^-.$$  

(134)

So it is obvious that $F(2,1,t) \in U(\mathcal{G})$. After the direct computation, we can get

$$[e_0, F(2,1,t)] = (f_1f_0^3 - f_0f_1f_0^3)((-2 + h_0)^\frac{1-t}{2} - \frac{t^2-1}{2}) + (f_0^3f_1f_0 - f_0^3f_0)(\frac{t+1}{2}h_0 + \frac{t^2-1}{2})$$

$$[e_1, F(2,1,t)] = f_0^3(-t+2+h_1).$$  

(135)

For the HWS $|2, 1\rangle$, $t = 2k + 3$,

$$h_0|2, 1\rangle = 4J_0^3|2, 1\rangle = (1 - t)|2, 1\rangle$$

$$h_1|2, 1\rangle = (-2J_0^3 + k)|2, 1\rangle = (t - 2)|2, 1\rangle.$$  

(136)

Combining eqs.(135, 136), we see that

$$e_iF(2,1,t)|2, 1\rangle = 0, \quad i = 0, 1.$$  

(137)

So $F(2,1,t)|2, 1\rangle$ is a singular vector $|2, 1\rangle$ in $M_{j_2, 1}$.

C  The Non-vanishing of $Q_m|w_{-m,s}\rangle$ and $Q_{-m}|j_{-m,s}\rangle$

In this appendix, we manage to prove the proposition $[8]$.

$$Q_m|m, -s\rangle \neq 0, \quad \text{if } 0 < m < \tilde{q}, \quad s > 0, \quad s + m \in \text{ odd.}$$  

(138)
Due to the presence of the fermionic operator in the screening operator $V(z)$, it takes much effort to complete our proof, in contrast to the analogous conclusions [17, 4] on the study of the Feigin-Fuchs modules over Virasoro algebra (or Wakimoto modules over $\widehat{SL}(2)$).

Now consider

$$Q_m = \oint \, dz \oint \prod_{i=1}^{m} dz_i V(z) \prod_{i=1}^{m} V(z_i),$$

(139)

where $V(z) = (\psi^+ + \beta \psi)e^{\alpha - \phi}(z)$, and the integration contour is depicted in fig. 2.

Figure 2: The integration contour in $Q_m$.

For convenience, let

$$z_1 = z, \quad X(\{z_i\}) = \prod_{i=1}^{m} (\psi^+ + \beta \psi)(z_i),$$

$$k = [m/2], \quad i\partial\phi_\alpha(z) = \sum_{n<0} \phi_n/z^{n+1},$$

(140)

where $[x]$ is the maximal integer no bigger than $x$. The most singular term in the OPE of $X(\{z_i\})$ is

$$\frac{1}{2^k k!} \sum_{P \in S(m)} (-1)^{\pi(P)} \prod_{i=1}^{k} \frac{\beta(z_{P_{2i-1}}) + \beta(z_{P_{2i}})}{z_{P_{2i-1}} - z_{P_{2i}}}(\psi^+(z_{P_m}) + \beta \psi(z_{P_m}))^{m-2k},$$

(141)

where $P$ is a permutation of the set $\{1, 2, \ldots, m\}$, $\pi(P)$ its $Z_2$ degree.

$$Q_m|m, -s\rangle = \oint \prod_{i=1}^{m} dz_i \prod_{1 \leq i < j \leq m} (z_i - z_j)^{\alpha} \prod_{i=1}^{m} z_i^{\alpha} X(\{z_i\}) e^{\sum_{i=1}^{m} \phi_n(z_i)} |m, -s\rangle.$$  

(142)

Let $n = m(s-2)/2 + k$, and

$$\langle \phi | = \langle -m, -s | \sum_{n=0}^{\infty} \phi_n n/\alpha, \quad \text{if } n \neq 0;$$

$$\langle \phi | = \langle -m, -s | \psi_n^{m-2k} m, \quad \text{if } n = 0.$$  

(143)

Consider the inner product,

$$\langle \phi |Q_m|m, -s\rangle = 1/2^k \oint \prod_{i=1}^{m} dz_i \prod_{1 \leq i < j \leq m} (z_i - z_j)^{\alpha} \prod_{i=1}^{m} z_i^{\alpha} \sum_{P \in S(m)} (-1)^{\pi(P)} (z_{P_{2i-1}} - z_{P_{2i}})^{-1} \sum_{i=1}^{m} z_i^n$$

(144)
To see that the integration on the r.h.s. of the eq.(144) does not vanish in general, it is more convenient to recast the integrand in a more suitable form.

Lemma 4 Let $m > 1$,

\[
A_m(z_1, z_2, \cdots, z_m) = \prod_{1 \leq i < j \leq m} (z_i - z_j),
\]

\[
f_m(z_1, z_2, \cdots, z_m) = \frac{A(z_1, z_2, \cdots, z_m)}{2^k k!} \sum_{P \in S(m)} (-1)^{\pi(P)} \prod_{i=1}^k (zp_{2i-1} - zp_{2i})^{-1},
\]

(145)

\[
g_m(z_1, z_2, \cdots, z_m) = \frac{A^2(z_1, z_2, \cdots, z_m)}{2^k (m-k)!} \sum_{P \in S(m)} (-1)^{\pi(P)} \prod_{i=1}^k \prod_{j=k+1}^m (zp_i - zp_j)^{-2},
\]

then

\[
f_m(z_1, z_2, \cdots, z_m) = g_m(z_1, z_2, \cdots, z_m), \quad \forall m > 1
\]

(146)

Proof. If $m = 2, 3$, eq. (146) can be verified by direct computation.

Let $m > 3$, now we prove the lemma by induction on $m$. Assume that

\[
f_k(z_1, z_2, \cdots, z_k) = g_k(z_1, z_2, \cdots, z_k), \quad \forall 1 < k < m.
\]

(147)

Let us consider a particular case for which $z_{m-1} = z_m$. Then

\[
f_m(z_1, z_2, \cdots, z_{m-1}, z_m) = \prod_{i=1}^{m-2} (z_i - z_{m-1})^2 f_{m-2}(z_1, z_2, \cdots, z_{m-2}),
\]

\[
g_m(z_1, z_2, \cdots, z_{m-1}, z_m) = \prod_{i=1}^{m-2} (z_i - z_{m-1})^2 g_{m-2}(z_1, z_2, \cdots, z_{m-2}).
\]

(148)

By induction

\[
f_{m-2}(z_1, z_2, \cdots, z_{m-2}) = g_{m-2}(z_1, z_2, \cdots, z_{m-2}).
\]

(149)

From eq.(148) and (149) we see that

\[
f_m(z_1, z_2, \cdots, z_m) - g_m(z_1, z_2, \cdots, z_m) = 0, \quad \text{if } z_{m-1} = z_m.
\]

(150)

So

\[
(z_{m-1} - z_m)(f_m(z_1, z_2, \cdots, z_m) - g_m(z_1, z_2, \cdots, z_m)).
\]

(151)

Secondly, from the fact that $f_m(\{z_i\}) - g_m(\{z_i\})$ is a symmetric homogeneous polynomial of $(z_1, z_2, \cdots, z_m)$, we have

\[
A_m(z_1, z_2, \cdots, z_m)(f_m(z_1, z_2, \cdots, z_m) - g_m(z_1, z_2, \cdots, z_m)).
\]

(152)
However, the degree of $A_m(z_1, z_2, \cdots, z_m)$ is $m(m-1)/2$, larger than that of $f_m(\{z_i\}) - g_m(\{z_i\})$, if $f_m \neq g_m$. So the only possibility is that

$$f_m(z_1, z_2, \cdots, z_m) = g_m(z_1, z_2, \cdots, z_m).$$

(153)

This completes the proof of the lemma by the induction rule.

By lemma 4, eq. (144) can be rewritten as

$$\langle \phi | Q_m | m, -s \rangle = 2\pi i \oint_{C_i} \prod_{i=2}^{m} du_i S(u_2, \cdots, u_m) \times \prod_{2 \leq i < j \leq m} (u_i - u_j)^{\alpha_i^2 - 1} \prod_{i=2}^{m} u_i^{-2\alpha_i^2 j m - s} (1 - u_i)^{\alpha_i^2 - 1},$$

(154)

where $z_i = z_1 u_i$, $i = 2, \ldots, m$, the integration over variable $z_1$ is completed and

$$S(u_2, \cdots, u_m) = k! g_m(z_1, z_2, \cdots, z_m) \sum_{i=1}^{m} z_i^n z_1^{-m(m+s-3)/2},$$

(155)

is a symmetric polynomial function of $(u_2, \ldots, u_m)$. To evaluate the integral on the r.h.s. of eq. (154), let us first consider a more general case,

$$\oint_{C_i} \prod_{i=1}^{n} u_i^a (1 - u_i)^b \prod_{1 \leq i < j \leq n} (u_i - u_j)^c S(u_1, u_2, \cdots, u_n),$$

(156)

with the integral contour depicted as in fig. 3, where $S(\{u_i\})$ is a symmetric polynomial of $\{u_i\}$.

Figure 3: The integration contour for $\prod_i du_i$.

Now we deform the integral contour to get an integration over $(1, -\infty)$. For convenience, we introduce a notation.

$$J_r = \int_{c_1'} \cdots \int_{c_{r-1}'} \oint_{c_r} \prod_{j=1}^{n} du_j (u_j - 1)^b \prod_{j=r}^{n} (1 - u_j)^b \prod_{j=1}^{n} u_j^a \prod_{1 \leq i < j \leq n} (u_i - u_j)^c S(u_1, u_2, \cdots, u_n),$$

(157)
where the contours $C'_i$, $C_j$ are depicted in fig. 4. When $a, b, c$ take general values in the complex plane, $J_r$ should be considered as its analytic continuation.

Figure 4: The deformed contour.

As in the same approach to the calculation of Dotsenko-Fateev integration [12], we get an inductive relation between $J_r$'s.

$$J_r = e^{-\pi b}(1 - e^{2\pi/(n-r)c+b+a} + e^{(r-1)c})J_{r+1}, \quad (158)$$

It is easy to get that

$$J_{n+1} = \int_{C'_i} \prod_{i=1}^{n} du_i \prod_{j=1}^{n} (u_j - 1)^b \prod_{j=1}^{n} u_j^a \prod_{1\leq i<j\leq n} (u_i - u_j)^c S(u_1, u_2, \ldots, u_n)$$

$$= \frac{1}{n!} \prod_{j=1}^{n} (1 + e^{-\pi c} + \cdots + e^{-\pi c(j-1)}) J$$, \quad (159)

where

$$J = \int_1^{\infty} \cdots \int_1^{\infty} \prod_{j=1}^{n} du_j \prod_{j=1}^{n} u_j^a (u_j - 1)^b \prod_{1\leq i<j\leq n} |u_i - u_j|^c S(u_1, u_2, \ldots, u_n). \quad (160)$$

Combining eq.(158) and eq.(159) we have

$$J_1 = \frac{(-i)^n}{n!} e^{i\pi c(n-1)/2 + i\pi n a} \left( \prod_{j=1}^{n} \sin \pi ((n-j-1)c+1+b+a) \sin \pi c/2 \right) J$$, \quad (161)

In our case

$$a = -\frac{(m-1)\alpha^2 + s}{2}, \quad b = c = \alpha^2 - 1$$, \quad (162)

The integrand in $J$ is always positive definite except for a measure zero set. It is easy to see that $J \neq 0$. So

$$\langle \phi | Q_m | m, -s \rangle = \frac{2\pi i (-2i)^{m-1}}{(m-1)!} e^{-\pi (m-1)c/2} \left( \prod_{j=1}^{m-1} \sin^2 \pi j c/2 \sin \pi c/2 \right) J$$ \quad (163)

is not zero provided $0 < m < \tilde{q}$. Then the state $Q_m | m, -s \rangle$ is nonvanishing in eq.(144).

Similarly another part of the proposition can be proved in the same way.
References

[1] Aharony, O., Ganor, O., Sonnenschein, T., Yankielowicz, S., Sochen, N.: Physical states in $G/G$ models and 2D gravity. TAUP-1961-92

[2] Alvarez-Gaumé, L., Sierra, G.: Topics in conformal field theory. Phys. and Math. of Strings (Knizhnik, V.G., memorial volume). World Scientific, 1989

[3] Awata, H., Yamada, Y.: Fusion rules for the fractional level $\widehat{SL}(2)$ algebra. KEK-TH-316, KEK Preprint 91-209

[4] Bernard, D., Felder, G.: Fock representations and BRST cohomology in $SL(2)$ current algebra. Commun. Math. Phys. 127, 145-168(1990)

[5] Bershadsky, M., Ooguri, H.: Hidden $OSP(N,2)$ symmetries in superconformal field theories. Phys. Lett. B 229, 374-378(1989)

[6] Bershadsky, M., Ooguri, H.: Hidden $SL(n)$ symmetry in conformal field theory. Commun. Math. Phys. 126, 49-83(1989)

[7] Bouwknegt, P., McCarthy, J., Pilch, K.: BRST analysis of physical states for 2D gravity coupled to $c \leq 1$ matter. Commun. Math. Phys. 145, 541-560(1992)

[8] Bouwknegt, P., McCarthy, J., Pilch, K.: Ground ring for the two-dimensional NSR string. Nucl. Phys. B 377, 541-570(1992)

[9] Bouwknegt, P., McCarthy, J., Pilch, K.: Semi-infinite cohomology in conformal field theory and 2d gravity. CERN-TH 6646/92, to be published in the proceedings of XXV Karpacz winter school of theoretical physics, Karpacz 17-27 February 1992

[10] Capelli, A., Itzykson, C., Zuber, J.B.: Modular invariant partition functions in two dimensions. Nucl. Phys. B 280, 445-465(1987)

[11] Cardy, J.L.: Operator content of two-dimensional conformally invariant theories. Nucl. Phys. B 270, 186-204(1986)

[12] Dostenko, Vl.S., Fateev, V.A.: four-point correlation functions and operator algebra in 2d conformal invariant theories with central charge $\leq 1$. Nucl. Phys. B 251, 691-734(1985)

[13] Delius, G.M., Grisaru, M.T., Van Nieuwenhuizen, P.: Induced $(N,0)$ supergravity as a constrained $Osp(N|2)$ WZNW model and its effective action. CERN-TH. 6458/92

[14] Feigin, B.L., Frenkel, E.V.: Affine Kac-Moody algebras and semi-infinite flag manifolds. Commun. Math. Phys. 128, 161-189(1990)
[15] Feigin, B.L., Frenkel, E.V.: Representations of affine Kac-Moody algebras and bosonization. Phys. and Math. of Strings (Knizhnik, V.G. memorial volume). World Scientific, 271(1989)

[16] Feigin, B.L., Fuchs, D.B., Representations of the Virasoro Algebra. Seminar on Supermanifolds no. 5, Leites, D. (ed.) 1986

[17] Felder, G.: BRST approach to minimal models. Nucl. Phys. B 317, 215-236(1989)

[18] Frenkel, E., Kac, V., Wakimoto, M.: Characters and fusion rules for W-algebras via quantum Drinfeld-Sokolov reduction. Commun. Math. Phys. 147, 295-328(1992)

[19] Fan, J. B., Yu, M., in preparation

[20] Goddard, P., Kent, A., Olive, D.: Virasoro algebras and coset space models. Phys. Lett. B 152, 88-92(1985)

[21] Goddard, P., Kent, A., Olive, D.: Unitary representations of the Virasoro and super-Virasoro algebras. Commun. Math. Phys. 103, 105-119(1986)

[22] Hu, H. L., Yu, M.: On the equivalence of non-critical strings and $G_k/G_k$ topological field theories. Phys. Lett. B 289, 302-308(1992)

[23] Hu, H. L., Yu, M.: On BRST cohomology of $SL(2,R)_{\xi-2}/SL(2,R)_{\xi-2}$ gauged WZNW models. AS-ITP-92-32, Nucl. Phys. B to appear.

[24] Ito, K., Madsen, J.O., Petersen, J.L.: Free field representations of extended superconformal algebra. NBI-HE-92-42 (July 1992).

[25] Ito, K., Madsen, J.O., Petersen, J.L.: Extended superconformal algebras from classical and quantum Hamiltonian reduction. NBI-HE-92-81, to appear in the proceedings of the International Workshop on “String Theory, Quantum Gravity and the Unification of the Fundamental Interactions”, Rome, September 21-26, 1992.

[26] Itzykson, C., Zuber, J.B.: Two-dimensional conformal invariant theories on a torus. Nucl. Phys. B 275, 580-616(1986)

[27] Inami, T., Izawa, K.-I.: Super-Toda theory from WZNW theories. Phys. Lett. B 255, 521-527(1991)

[28] Jantzen, J., Moduln mit einem höchsten Gewicht, Lect. Notes in Math. 750(1979)

[29] Kac, V.G.: Lie superalgebras. Adva. Math. 26,8-96(1977)

[30] Kac, V.G.: Infinite-dimensional algebras, Dedekind’s $\eta$-function, classical Möbius function and the very strange formula. Adva. Math. 30,85-136(1979)

[31] Kac, V.G., Infinite Dimensional Lie Algebras. Cambridge Univ. Press, Cambridge, U.K. (1985)
[32] Kac, V.G., Kazhdan, D.A.: Structure of representations with highest weight of infinite dimensional Lie algebras. Adv. Math. 34, 97-108(1979)

[33] Kac, V.G., Wakimoto, M.: Modular invariant representations of infinite dimensional Lie algebras and superalgebras. Proc. Natl. Acad. Sci. USA 85 4956-4960(1986)

[34] Koh, I.G., Yu, M.: Non-Abelian bosonization in higher genus Riemann surfaces. Phys. Lett. B203, 263-268(1988)

[35] Lian, B.H., Zuckerman, G.J.: BRST cohomology and highest weight vectors. I. Commun. Math. Phys. 135, 547-580(1991)

[36] Lian, B.H., Zuckerman, G.J.: New selection rules and physical states in 2D gravity; conformal gauge. Phys. Lett. B254, (3, 4), 417-423(1991)

[37] Lian, B.H., Zuckerman, G.J.: 2D gravity with c = 1 matter. Phys. Lett. B266, 21-28(1991)

[38] Malikov, F.G., Feigin, B.L., and Fuchs, D.B.: Singular vector in Verma modules over Kac-Moody algebras. Funkt. Anal. Prilozhen 20 No. 2 25-37(1986) (in Russian)

[39] Moore, G., Seiberg, N.: Polynomial equations for rational conformal field theories. IASSNS-HEP-88/18

[40] Mukhi, S., Panda, S.: Fractional level current algebras and the classification of characters. Nucl. Phys. B 338, 263-282(1990)

[41] Mumford, D., Tata Lectures on Theta Functions. (Progress in Mathematical; Vol. 28) Birkhäuser, Boston, Inc. (1983)

[42] Pais, A., Rittenberg, V.: Semisimple graded Lie algebras. J. Math. Phys., Vol. 16, No. 10, 2062-2073(1975)

[43] Scheunert, M., Nahm, W., Rittenberg, V.: Irreducible representations of the osp(2,1) and spl(2,1) graded Lie algebra. J. Math. Phys., Vol. 18, No. 1, 155-162(1977)

[44] Rocha-Caridi, A.: Vacuum vector representation of the Virasoro algebra, in Vertex Operators in Mathematics and Physics, 451(1985) ed. by J.Lepowsky, S. Mandelstam, and J. Singer, Publ. Math. Sciences Res. Inst. #3, Springer Verlag, New York

[45] Verlinde, E.: Fusion rules and modular transformations in 2d conformal field theory. Nucl. Phys. B 300, 360-375(1988)

[46] Wakimoto, M.: Fock representations of the affine Kac-Moody algebra $A_1^{(1)}$. Commun. Math. Phys. 104, 605-609(1986)
[47] Zhang, Y.Z.: N-extended super-Liouville theory from $OSP(N|2)$ WZNW model. Phys. Lett. B 283, 237-242(1992)

**Figure Caption**

Fig.1: Boundary conditions on the torus.
Fig.2: The integration contour in $Q_m$.
Fig.3: The integration contour for $\prod_i du_i$.
Fig.4: The deformed contour.