THE SOBOLEV INEQUALITIES ON REAL HYPERBOLIC SPACES AND EIGENVALUE BOUNDS FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

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Abstract. In this paper, we prove the uniform estimates for the resolvent \((\Delta - \alpha)^{-1}\) as a map from \(L^q\) to \(L^{q'}\) on real hyperbolic space \(\mathbb{H}^n\) where \(\alpha \in \mathbb{C} \setminus \{(n - 1)^2/4, \infty\}\) and \(2n/(n + 2) \leq q < 2\). In contrast with analogous results on Euclidean space \(\mathbb{R}^n\), the exponent \(q\) here can be arbitrarily close to 2. This striking improvement is due to two non-Euclidean features of hyperbolic space: the Kunze-Stein phenomenon and the exponential decay of the spectral measure. In addition, we apply this result to the study of eigenvalue bounds of the Schrödinger operator with a complex potential. The improved Sobolev inequality results in a better long range eigenvalue bound on \(\mathbb{H}^n\) than that on \(\mathbb{R}^n\).

1. Introduction

Let \(\mathbb{H}^n\) be the \(n\)-dimensional real hyperbolic space for \(n > 2\) and \(\Delta\) the Laplacian on \(\mathbb{H}^n\). We are concerned with the \((L^s, L^r)\) type estimates for the resolvent \((\Delta - \alpha)^{-1}\) with \(\alpha \in \mathbb{C}\), i.e. the norm estimates for
\[
\|(\Delta - \alpha)^{-1}\|_{L^s(\mathbb{H}^n) \to L^r(\mathbb{H}^n)}.
\]

This type of resolvent estimates traces back to the classical Sobolev inequality on Euclidean space \(\mathbb{R}^n\),
\[
\|\Delta^{-1}\|_{L^{2n/(n+2)}(\mathbb{R}^n) \to L^{2n/(n-2)}(\mathbb{R}^n)} < C.
\]
One can regard this inequality as the \(L^{2n/(n+2)} - L^{2n/(n-2)}\) boundedness of the resolvent at \(\alpha = 0\). More generally, Kenig-Ruiz-Sogge [20] extended this to the non-zero energies \(\alpha \neq 0\) and proved that for \(\alpha \in \mathbb{C} \setminus [0, \infty)\) and \(1 < s, r < \infty\),
\[
\text{with } \frac{1}{s} = \frac{1}{r} + \frac{2}{n}, \quad \min \left\{\left|\frac{1}{r} - \frac{1}{2}\right|, \left|\frac{1}{s} - \frac{1}{2}\right|\right\} > \frac{1}{2n},
\]
the following uniform Sobolev inequality holds,
\[
(1) \quad \|(\Delta_{\mathbb{R}^n} - \alpha)^{-1}\|_{L^s(\mathbb{R}^n) \to L^r(\mathbb{R}^n)} < C.
\]

The inequalities of \((L^q, L^{q'})\) type are of particular interests, where \(1/q + 1/q' = 1\). As is well-known, they are closely tied to a number of applications in PDEs. Let us mention the eigenvalue bounds for Schrödinger operators with complex potentials, the endpoint Strichartz estimates for Schrödinger equations, the Carleman inequalities to deduce relevant unique continuation theorem.

Guillarmou-Hassell [13] proved that,
\[
\|(\Delta_M - \alpha)^{-1}\|_{L^q(M) \to L^{q'}(M)} \leq C|\alpha|^{n(1/q-1/2)-1},
\]

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for $2n/(n+2) \leq q \leq 2(n+1)/(n+3)$ on $M$, a non-trapping $n$-dimensional asymptotically Euclidean space. We remark that the range of the exponent $q$ here is linked with the range of the exponent in the Stein-Tomas restriction estimates or essentially the following spectral measure estimates, studied in [15],

$$\left| \left( \frac{d}{d\lambda} \right)^j dE_{\sqrt{\Delta_M}}(\lambda) \right|(z,z') \leq C\lambda^{n-j}(1+\rho\lambda)^{-n/2+j},$$

where $\rho$ is the geodesic distance function on $M$ and the spectral measure $dE_{\sqrt{\Delta_M}}(\lambda)$ is defined through spectral theorem by

$$f(\sqrt{\Delta_M}) = \int_0^\infty f(\lambda)dE_{\sqrt{\Delta_M}}(\lambda) d\lambda,$$

for all bounded functions $f$. It is also worth pointing out that Knapp’s counterexample shows that $2(n+1)/(n+3)$ is the upper bound of the exponents for Stein-Tomas estimates on asymptotically Euclidean spaces. Therefore, one can not obtain uniform Sobolev estimates on $\mathbb{R}^n$ for $p$ close to 2.

In this paper, we shall study the uniform Sobolev inequalities of $(L^q, L^{q'})$ type on real hyperbolic spaces. More precisely, we prove

**Theorem 1.** Suppose $L = \Delta - (n-1)^2/4$ and $\alpha \in \mathbb{C} \setminus [0, \infty)$.

On the one hand, for high energies $|\alpha| > 1$, we have that

$$\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^{q'}(\mathbb{H}^n)} \leq C|\alpha|^{1/2-1/q},$$

if $2(n+1)/(n+3) \leq q < 2$;

$$\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^{q'}(\mathbb{H}^n)} \leq C|\alpha|^{n(1/q-1/2)-1},$$

if $2n/(n+2) \leq q \leq 2(n+1)/(n+3)$.

On the other hand, for low energies $|\alpha| < 1$, we have that for $2n/(n+2) \leq q < 2$,

$$\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^{q'}(\mathbb{H}^n)} \leq C.$$

We remark that the constant $C$ here is $q$-dependent and it blows up as $q$ goes to 2. Because all $q \in [2n/(n+2), 2)$ satisfy $1/2 - 1/q < 0$ and $n(1/q - 1/2) - 1 \leq 0$, Theorem 1 trivially implies

**Corollary 2.** For all $\alpha \in \mathbb{C} \setminus [0, \infty)$,

$$\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^{q'}(\mathbb{H}^n)} \leq C|\alpha|^{1/2-1/q},$$

if $2(n+1)/(n+3) \leq q < 2$;

$$\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^{q'}(\mathbb{H}^n)} \leq C|\alpha|^{n(1/q-1/2)-1},$$

if $2n/(n+2) \leq q \leq 2(n+1)/(n+3)$.

In contrast with the results on (asymptotically) Euclidean space, the exponent $q$ in the results on $\mathbb{H}^n$ can be arbitrarily close to 2. We can interpret this striking phenomenon as the consequences of two non-Euclidean features of $\mathbb{H}^n$: the Kunze-Stein phenomenon and the exponential decay of the spectral measure.

The Kunze-Stein phenomenon, proved by Cowling [6] on connected semi-simple Lie groups $G$ with finite center, says that the convolution inequality,

$$L^2(G) * L^p(G) \subset L^2(G),$$
holds for $p \in [1, 2)$. Note that above inequalities on Euclidean space, by Young’s equality, are only valid for $p = 1$. If $\mathbb{G}$ is in addition of real hyperbolic spaces, it can be refined in terms of Lorentz spaces. Cowling-Meda-Setti \cite{17} and Ionescu \cite{18} showed

$$L^{p,q}(\mathbb{G}^*) * L^{p,b}(\mathbb{G}) \subset L^{p,c}(\mathbb{G}),$$

provided

$$\begin{cases} 1/a + 1/b \geq 1 + 1/c, & \text{if } p \in (1, 2) \\ a = 1, b = 1, c = \infty, & \text{if } p = 2 \end{cases}.$$  

A useful corollary of this on $\mathbb{H}^n$, proved by Anker-Pierfelice-Vallarino \cite{2}, is that for a radial kernel $\kappa(\rho)$

$$\|f * \kappa\|_{L^q(\mathbb{H}^n)} \leq C q \|f\|_{L^n(\mathbb{H}^n)} \left( \int_0^{\infty} (\sinh \rho)^{n-1} (1 + \rho)e^{-(n-1)\rho^2/2} |\kappa(\rho)|^{q'/2} \, d\rho \right)^{2/q'},$$

with $1 < q \leq 2$. Applying (5) to the resolvent kernel $e^{\pm i\rho \lambda} / \sinh(\rho)$, with sufficiently small $|\text{Im } \lambda|$, on $\mathbb{H}^3$ immediately proves the $(L^q, L^q)$ Sobolev inequalities for any $6/5 < q < 2$.

The proof, by Kenig-Ruiz-Sogge, of uniform resolvent estimates is closely tied to the Stein-Tomas restriction theorem \cite{24, 25}, which says for any function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2(n + 1)/(n + 3)$ one can meaningfully restrict its Fourier transform $\hat{f}$ to the sphere $S^{n-1}$. By a $TT^*$ argument, one can rewrite the restriction theorem, in terms of the spectral measure, as

$$\|dE_{\sqrt{\Delta_n}}(\lambda)\|_{L^p(\mathbb{R}^n)} \to L^{p'}(\mathbb{R}^n) \leq C \lambda^{n(1/p-1/p')-1},$$

provided $1 \leq p \leq 2(n + 1)/(n + 3)$. This inequality was proved by Guillarmou-Hassell-Sikora \cite{15} (see also \cite{3} for a complete proof) on asymptotically Euclidean spaces via the pointwise estimates \cite{2} for the spectral measure.

Interestingly, all of these estimates at high energies on real hyperbolic spaces turn out to be better than on Euclidean spaces. Hassell and the author \cite{4} proved

**Theorem 3.** Let $\rho$ be the distance function between $z, z' \in \mathbb{H}^n$. For $\lambda \geq 1$, the spectral measure of $\sqrt{\Delta}$ satisfies the following pointwise estimates,

$$\left| \left( \frac{d}{d\lambda} \right)^j dE_{\sqrt{\Delta}}(\lambda) \right|(z, z') \leq \begin{cases} C \lambda^{n-1-j}(1 + \rho \lambda)^{-(n-1)/2-j}, & \text{for } \rho \leq 1 \\ C \lambda^{(n-1)/2} \rho e^{-(n-1)\rho^2/2}, & \text{for } \rho \geq 1. \end{cases}$$

Moreover, one has the Stein-Tomas estimates at high energies, for $1 \leq p < 2$,

$$\|dE_{\sqrt{\Delta}}(\lambda)\|_{L^p(\mathbb{H}^n)} \to L^{p'}(\mathbb{H}^n) \leq \begin{cases} C \lambda^{n(1/p-1/p')-1}, & 1 \leq p \leq \frac{2(n+1)}{n+3}, \\ C \lambda^{(n-1)(1/p-1/2)}, & \frac{2(n+1)}{n+3} \leq p < 2. \end{cases}$$

Here the exponential decay in space of the spectral measure was exploited to prove the broader Stein-Tomas estimates for $\frac{2(n+1)}{n+3} \leq p < 2$. Therefore it is natural to expect such better spectral measure estimates will lead to a larger set of exponents for uniform Sobolev inequalities.

Resolvent estimates on real hyperbolic spaces have been studied before. For exponents away from 2, Huang-Sogge \cite{17} generalized \cite{1} at high energies to $\mathbb{H}^{n+1}$. For $p = 2$, Melrose-Sá Barreto-Vasy \cite{22} established weighted $L^2$ estimates. The weight (or a compact cut-off) is essential for $p = 2$, since $\lim_{p \to 2^-} \|R(\lambda)\|_{L^p} \to L^{p'}$ blows up. However, we believe that Theorem 1 is the first result of uniform Sobolev inequalities with non-Euclidean features of real hyperbolic spaces.

Moreover, one should be able to generalize our results to general non-trapping asymptotically hyperbolic manifolds (even if there are pairs of conjugate points) by employing
the diagonal estimates for the spectral measure in [4] as well as adapting the off-diagonal arguments in [13].

As was mentioned, the Sobolev inequalities can be applied to tackling a number of problems. In the present paper, we are concerned about the eigenvalue bounds for Schrödinger operators. Specifically, given a complex-valued potential $V \in L^p(\mathbb{H}^n)$ with $p \geq n/2$, what can we say about the spectrum of the Schrödinger operator $\Delta + V$? Adapting previous work for abstract operators on Hilbert spaces, due to Frank [12], one can easily prove that the spectrum of $\Delta + V$ consists of the essential spectrum of $\Delta$ and isolated eigenvalues of finite algebraic multiplicity. The essential spectrum of $\Delta$ is the half real line $((n-1)^2/4, \infty) \subset \mathbb{R}$. Nonetheless, what do we know about those isolated eigenvalues from the potential $V$?

This problem has been intensively studied on Euclidean space $\mathbb{R}^n$, including the location of individual eigenvalues and the distribution of eigenvalues. Here we are interested in the location of individual eigenvalues, that is to prove the eigenvalues $\lambda$ are located in a ball, the size of which is controlled by the norm of the potential. In $\mathbb{R}^1$, Abramov-Aslanyan-Davies [1] proved that $|\lambda|^{1/2} \leq C \int_{\mathbb{R}} |V|$. In higher dimensions $\mathbb{R}^n$, we write $p = \gamma + n/2$ in connection with Lieb-Thirring inequalities. Frank [9], Frank-Sabin [10], Frank-Simon [11] studied the short range case $0 < \gamma \leq 1/2$ and showed that $|\lambda|^\gamma \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V|^\gamma + n/2$, for $n \geq 2$.

On the other hand, Frank [12] established the following long range result for $\gamma > 1/2$, $d(\lambda)^{-1/2} |\lambda|^{1/2} \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V|^\gamma + n/2$, where $d(\lambda) = \text{dist}(\lambda, [0, \infty))$. Furthermore, Guillarmou-Hassell-Krupchyk [14] generalized all of these results to a large class of non-trapping asymptotically Euclidean manifolds for $n \geq 3$. We also refer the reader to [12, 14] for a complete list of references. Furthermore, this also has been considered on $\mathbb{H}^2$. Hansmann [16] recently generalized these classical results to the hyperbolic plane.

In the present paper, we consider analogous eigenvalue bounds on $\mathbb{H}^n$. On the one hand, we prove similar short range results, aligned with the Sobolev inequality (7),

**Theorem 4.** For $0 < \gamma \leq 1/2$ and $n \geq 3$, we have $|\lambda|^\gamma \leq C_{\gamma,n} \int_{\mathbb{H}^n} |V|^\gamma + n/2$.

On the other hand, we, by means of the Sobolev inequality (6), obtain the following better long range results

**Theorem 5.** For $\gamma \geq 1/2$ and $n \geq 3$, we have $|\lambda|^{1/2} \leq C_{\gamma,n} \int_{\mathbb{H}^n} |V|^\gamma + n/2$.

In addition, eigenvalues do not exist if the potential is too 'small'.

**Theorem 6.** If $\|V\|_{L^{\gamma + n/2}(\mathbb{H}^n)}$ with $\gamma \geq 0$ is sufficiently small, the Schrödinger operator $\Delta + V$ has no eigenvalues.
The significant improvement in the long range case also results from the better Sobolev estimates on hyperbolic space. Due to lack of uniform Sobolev estimates for \(2(n+1)/(n+3) < p < 2\) on \(\mathbb{R}^n\), one has to interpolate the Stein-Tomas estimates on \(\mathbb{R}^n\) with the crude \(L^2\) estimates with the factor \(d(\lambda)\) from the spectral theorem, which causes the presence of the factor of \(d(\lambda)\) in the eigenvalue bounds on Euclidean space in the long range case. In contrast, we have uniform Sobolev inequalities on \(\mathbb{H}^n\) without the factor \(d(\lambda)\) for \(2(n+1)/(n+3) < p < 2\) on \(\mathbb{H}^n\). Consequently, we have the better eigenvalue bounds on \(\mathbb{H}^n\) in the long range case.

In summary, the following diagrams elucidate the relations between spectral measure estimates, Stein-Tomas estimates, Sobolev inequalities, and eigenvalue bounds.

![Diagram](image)

The paper is organized as follows. We shall review the microlocal descriptions of the resolvent and spectral measure on (asymptotically) hyperbolic spaces in Section 2. We break the main theorem into a few propositions and then prove them in Section 3-5. It is followed by the applications on eigenvalue bounds for Schrödinger operators with complex potentials.

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2. Resolvent and spectral measure on hyperbolic spaces

Consider the real hyperbolic space \(\mathbb{H}^n\) and its Laplacian \(\Delta\). There are several ways to view \(\mathbb{H}^n\). We choose the Poincaré upper half plane model. Specifically, \(\mathbb{H}^n\) is the following Riemannian manifold,

\[
\left\{ (x, y) | x > 0, y \in \mathbb{R}^{n-1}, \frac{dx^2 + dy^2}{x^2} \right\}.
\]

Then the Laplacian \(\Delta\) reads

\[
-(x\partial_x)^2 + (n-1)x\partial_x - \sum_{i=1}^{n-1} (x\partial_{y_i})^2.
\]

The spectrum of \(\Delta\) consists only of the half line \([n-1]^2/4, +\infty)\). So we denote \(L = \Delta - (n-1)^2/4\) through out this paper for simplicity.

We are concerned about the resolvent. Specifically, the resolvent kernel of \((L - \lambda^2)^{-1}\), if \(\lambda^2 \notin [0, \infty)\), reads

\[
(11) \quad \begin{cases} 
\frac{C}{\lambda} \left( \frac{1}{\sinh(\rho)} \right)^{(n-1)/2} e^{\pm i\lambda \rho} & \text{when } n \text{ is odd}; \\
C \int_{\rho}^{\infty} e^{\pm i\lambda s}(\cosh(s) - \cosh(\rho))^{- (n-1)/2} \sinh(s) \, ds & \text{when } n \text{ is even},
\end{cases}
\]
where $\rho$ is the distance function. The resolvent $(L - \lambda^2)^{-1}$, as a map between some weighted $L^2$ spaces, extends to be meromorphic in the whole complex plane, with finite number of poles when $n$ is even. Therefore, its kernel is a distribution meromorphically extendible in $\mathbb{C}$. In particular, the bottom of the spectrum, $\alpha = 0$, is neither an eigenvalue nor a resonance. Namely, it is analytic at this point.

Alternatively, Mazzeo-Melrose [21] microlocally constructed the resolvent kernel on asymptotically hyperbolic spaces and described it as

\[ \text{Ker}(\Delta - \zeta(n - 1 - \zeta))^{-1} = R_{nd} + e^{\pm i\lambda \rho} R_{od}. \]

Here $R_{nd}$ is the kernel of a pseudodifferential operator of degree $-2$ supported near the diagonal, whilst $R_{od}$ is a smooth function supported away from the diagonal.

The analyticity at the bottom of the spectrum readily implies that (see [4, Theorem 1.3]) for sufficiently small $|\lambda|$,

\[ dE_{\sqrt{L}}(\lambda)(x, y, x', y') = (x x')^{(n-1)/2} \lambda ((x x')^{1\lambda} a(\lambda) - (x x')^{-1\lambda} a(-\lambda)), \]

where $a \in C^\infty([-1, 1] \times \overline{\mathbb{H}^n} \times \overline{\mathbb{H}^n})$, $\overline{\mathbb{H}^n}$ is the compactification of $\mathbb{H}^n$, and $x$ is the boundary defining function of $\mathbb{H}^n$.

To understand the uniform Sobolev inequalities, it is also important to understand the asymptotic behaviour of the resolvent for large spectral parameters. Following [21], Melrose-Sá Barreto-Vasy [22] constructed the following high energy resolvent on Cartan-Hadamard asymptotically hyperbolic manifolds.

**Theorem 7.** The resolvent kernel $(L - \lambda^2)^{-1}$ is analytic in a neighbourhood of lower half plane $\{\text{Im} \lambda \leq 0\}$ and takes the form

\[ \text{Ker}(L - \lambda^2)^{-1} = R_0 + e^{-i\lambda \rho} R_\infty, \quad \text{for } |\lambda| > 1 \]

where $R_0$ is the kernel of a pseudodifferential operator of degree $-2$ supported near the diagonal, and $R_\infty$ is a smooth function pointwisely bounded by a multiple of

\[ \begin{cases} 
\rho^{-(n-1)/2} (1 + |\lambda|)^{n/2-3/2} & \text{for } \rho \leq C \\
 e^{-(n-1)\rho/2} (1 + |\lambda|)^{n/2-3/2} & \text{for } \rho \geq C.
\end{cases} \]

As is well-known in spectral theory, one can obtain the spectral measure of the operator $\sqrt{L}$ through the resolvent near the spectrum. Using Melrose-Sá Barreto-Vasy’s resolvent construction, the author and Hassell gave the microlocal description of the spectral measure at high energies and proved the bounds for the spectral measure in Theorem 3.

It is also useful to have the heat kernel. Davies-Mandouvalous [8] proved that the heat kernel $e^{-tL}$ is equivalent to

\[ t^{-n/2} e^{-(n-1)\rho/2 - \rho^2/(4t)} (1 + \rho + t)^{n/2-3/2} (1 + \rho), \]

where $\rho$ is the distance function. This has been further generalized to asymptotically hyperbolic spaces by Hassell and the author [5].

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1 This is also true on a broad class of asymptotically hyperbolic manifolds.

2 From the viewpoint of asymptotically hyperbolic manifolds, it is more convenient to understand the compactification in the Poincaré disc model. We can map the upper half plane $\mathbb{R}^n_+$ to the unit ball $B^n$ via the Cayley transform. The compactification $\overline{\mathbb{H}^n}$ we mean is just the closure $B^n$. 

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3. Sobolev Inequalities

We proceed from the simplest case.

**Lemma 8.** For \( \beta \leq 0 \),
\[
\| (L - \beta)^{-1} \|_{\mathcal{L}^q(\mathbb{H}^n) \to \mathcal{L}^{q'}(\mathbb{H}^n)} < C, \quad \text{with } 2n/(n + 2) \leq q < 2.
\]

**Proof.** We firstly assume \( \beta < 0 \) and invoke the heat kernel on \( \mathbb{H}^n \) via the Laplace transform
\[
(L - \beta)^{-1} = \mathcal{L}(e^{-L})(-\beta) = \int_0^\infty e^{-tL}e^{\beta t} dt.
\]

In order to simplify the heat kernel (13) a bit, we break into the following three cases:
\[
\text{Ker } e^{-tL} \approx \begin{cases} 
  t^{-n/2}e^{-(n-1)\rho/2-\rho^2/(4t)}(1 + \rho)^{(n-1)/2} & t < C \\
  t^{-n/2}e^{-(n-1)\rho/2-\rho^2/(4t)}(1 + \rho)^{(n-1)/2} & \rho > t > C \\
  t^{-3/2}e^{-(n-1)\rho/2-\rho^2/(4t)}(1 + \rho) & t > \rho, t > C.
\end{cases}
\]

Using this, together with the integral formula
\[
\int_0^\infty t^\nu e^{-\xi t} dt = 2\left(\frac{\xi}{\xi}\right)^{\nu/2} K_\nu(2\sqrt{\xi}), \quad \text{Re } \xi > 0, \text{Re } \zeta > 0
\]
where \( K_\nu \) is the modified Bessel functions, we obtain that \( |\text{Ker } (L - \beta)^{-1}| \) with \( \beta < 0 \) is bounded from above by a multiple of
\[
e^{-(n-1)\rho/2}(1 + \rho)^{(n-1)/2} \rho^{-(n-2)/2}(-\beta)^{(n-2)/4} K_{(n-2)/2}(\rho \sqrt{-\beta}) + (1 + \rho)^{1/2}(-\beta)^{1/4} K_{1/2}(\rho \sqrt{-\beta})
\]
As is well-known, the modified Bessel function obeys the following asymptotic behaviours:
\[
K_\nu(h) \approx \begin{cases} 
  C_\nu h^{-\nu}, & 0 < h < 1; \\
  C_\nu h^{-1/2} e^{-h}, & h > 1,
\end{cases}
\]
provided \( \nu > 0 \). Therefore, this yields that
\[
|\text{Ker } (L - \beta)^{-1}| \leq \begin{cases} 
  e^{-(n-1)\rho/2 - \rho \sqrt{-\beta}}(-\beta)^{(n-3)/4} + 1, & \rho \sqrt{-\beta} > 1, \\
  e^{-(n-1)\rho/2}(1 + \rho)^{(n-1)/2} \rho^{-(n-2)}, & \rho \sqrt{-\beta} < 1.
\end{cases}
\]

We further break up these bounds into six cases and consequently have the following upper bounds for the resolvent kernel
\[
|\text{Ker } (L - \beta)^{-1}| \leq \begin{cases} 
  C e^{-(n-1)\rho/2}, & \text{Case (i): } \rho > 1, \sqrt{-\beta} > 1, \\
  C e^{-(n-1)\rho/2}, & \text{Case (ii): } \rho > 1, \sqrt{-\beta} < 1, \rho \sqrt{-\beta} > 1, \\
  C e^{-\rho \sqrt{-\beta}}(-\beta)^{(n-3)/4}, & \text{Case (iii): } \rho < 1, \sqrt{-\beta} > 1, \rho \sqrt{-\beta} > 1, \\
  C \rho^{2-n}, & \text{Case (iv): } \rho < 1, \sqrt{-\beta} > 1, \rho \sqrt{-\beta} < 1, \\
  C e^{-(n-1)\rho/2 - \rho \sqrt{-\beta}}(-\beta)^{(n-3)/4} + 1, & \text{Case (v): } \rho > 1, \sqrt{-\beta} < 1, \rho \sqrt{-\beta} < 1, \\
  C \rho^{2-n}, & \text{Case (vi): } \rho < 1, \sqrt{-\beta} < 1.
\end{cases}
\]

Applying the Kunze-Stein phenomenon (8) gives for all \( \beta < 0 \),
\[
\| (L - \beta)^{-1} \|_{\mathcal{L}^q(\mathbb{H}^n) \to \mathcal{L}^{q'}(\mathbb{H}^n)} < C, \quad \text{with } 2n/(n + 2) < q < 2.
\]

In fact, it suffices to check if the integral
\[
\int_0^\infty (\sinh \rho)^{n-1}(1 + \rho)e^{-(n-1)\rho/2}|\text{Ker } (L - \beta)^{-1}|^{1+\varepsilon} d\rho
\]
is convergent uniformly in \( \beta < 0 \), where \( q' = 2 + 2\varepsilon \) and \( 0 < \varepsilon < 2/(n - 2) \).
For Case (i) and Case (ii), the integral (17) reduces to
\[ \int_{1}^{\infty} e^{(n-1)\rho}(1 + \rho)e^{-(n-1)\rho/2} \left| e^{-(n-1)\rho/2} \right|^{1+\varepsilon} d\rho, \]
which is obviously convergent.

For Case (iv) and Case (vi), we have to estimate the fractional integral
\[ \int_{0}^{1} \rho^{n-1} \rho^{(2-n)+\varepsilon(2-n)} d\rho, \]
which is convergent as long as \( \varepsilon < 2/(n-2) \).

For Case (iii), the integral is bounded above by
\[ \int_{(-\beta)^{-1/2}}^{1} \rho^{n-1} \left| e^{-\rho\sqrt{-\beta}}(\sqrt{-\beta})^{(n-3)/2} \right|^{1+\varepsilon} d\rho. \]
The range of \( \varepsilon \) guarantees the power of \( \rho \) is bigger than that of \( \sqrt{-\beta} \). Since \( \rho < 1 \) and
\[ e^{-\rho\sqrt{-\beta}}(\rho\sqrt{-\beta})^{(n-3)(1+\varepsilon)/2} < C, \]
we have the uniform convergence in \( \beta \).

For Case (v), we end up with
\[ \int_{1}^{\infty} e^{(n-1)\rho}(1 + \rho)e^{-(n-1)\rho/2} \left| e^{-(n-1)\rho/2} \right|^{1+\varepsilon} d\rho. \]
We will have \( e^{-(n-1)\rho/2} \) left in the integrand after trivial cancellations. This factor kills any polynomial growth in \( \rho \) and makes the integral uniformly convergent.

This inequality also holds at the endpoint \( q = 2n/(n + 2) \), which corresponds to \( \varepsilon = 2/(n-2) \). In cases (i) (ii) (iii) (v), the proof above works verbatim. But it fails in cases (iv) (vi) when \( \rho \) is small. To remedy this, one can apply the Hardy-Littlewood Sobolev inequality to the diagonal part of the resolvent kernel, since the diagonal part is not affected by the exponential volume growth on hyperbolic spaces.

Noting the operator norm bound is independent of \( \beta \), we have the inequality (14) in addition at \( \beta = 0 \). The proof is now complete. \( \square \)

This is further generalized to spectral parameters away from the positive real axis.

**Lemma 9.** For any \( \alpha \in \mathbb{C} \) with \( | \arg(\alpha) | \geq \theta > 0 \), we have
\[ (L - \alpha)^{-1} \bigg|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} < C, \quad \text{with} \quad 2n/(n + 2) \leq q < 2. \]

**Proof.** By duality, (14) implies that for \( \beta < 0 \)
\[ \left\| (L - \beta)^{-1/2} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} < C, \]
\[ \left\| (L - \beta)^{-1/2} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} < C. \]
For any complex spectral parameter \( \alpha = \beta + i\gamma \) with \( \gamma \neq 0 \), it follows that
\[ \left\| (L - \alpha)^{-1} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} < C \left\| (L - \beta)(L - \alpha)^{-1} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)}. \]
The right hand side is indeed bounded from above
\[ \left\| (L - \beta)(L - \alpha)^{-1} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} = \sup_{\lambda^2 > 0} \frac{|\lambda^2 - \beta|}{|\lambda^2 - \beta - i\gamma|} \leq 1. \]
Similarly, for \( \alpha = -\beta + i\gamma \) with \( | \arg \alpha | \geq \theta > 0 \), we have
\[ \left\| (L - \beta)(L - \alpha)^{-1} \right\|_{L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)} = \sup_{\lambda^2 > 0} \frac{|\lambda^2 - \beta|}{|\lambda^2 + \beta - i\gamma|} \leq C(1 + \frac{|\beta|}{|\gamma|}) \leq C. \]
Combining this with $L^2$ bound of the resolvent, we can prove

**Proposition 10.** If $\alpha \in \mathbb{C}$ is away from an open sector $\{ \alpha : |\arg(\alpha)| < \theta \}$ containing the positive real axis, we have

\[
\|(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^p(\mathbb{H}^n)} \leq C|\alpha|^{n(1/q - 1/2) - 1},
\]

for $2n/(n + 2) \leq q < 2$.

Alternatively, for any $\alpha \in \mathbb{C}$, we have

\[
\|(1 - \phi(L/|\alpha|))(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^p(\mathbb{H}^n)} \leq C|\alpha|^{n(1/q - 1/2) - 1},
\]

data**n**

provided that $\phi \in C_c^\infty(\mathbb{R})$ is a cut-off function supported around 1 and equal to 1 in a small compact neighbourhood of 1.

We remark that the insertion of the cut-off function $(1 - \phi(\cdot/|\alpha|))$ eliminates the restriction of the spectral parameter $\alpha$ but does not change the off-spectrum nature. The constraint of $\alpha$ being away from the spectrum in (19) is moved to the cut-off function in (20). The support of the cut-off function is determined by the sector $\{ \alpha : |\arg(\alpha)| < \theta \}$.

**Proof.** First of all, spectral theorem yields that for any $\alpha \in \mathbb{C}$ with $|\arg(\alpha)| \geq \theta > 0$

\[
\|(L - \alpha)^{-1}\|_{L^2(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)} \leq \sup_{\lambda^2 > 0} C/|\lambda^2 - \alpha| \leq C/|\alpha|.
\]

Then (19) is obtained by applying Riesz-Thorin interpolation to this $L^2$ bound and (18).

Noting that the operator $1 - \phi(L/|\alpha|)$ is commutative with the function of $L$, we can use the same argument for (18) to prove, for any $\alpha \in \mathbb{C}$,

\[
\|(1 - \phi(L/|\alpha|))(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^p(\mathbb{H}^n)} < C\|(1 - \phi(L/|\alpha|))(L + |\alpha|)(L - \alpha)^{-1}\|_{L^2(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)}
\]
\[
< C \sup_{\lambda^2 > 0} |1 - \phi(\lambda^2/|\alpha|)(\lambda^2 + |\alpha|)|/|\lambda^2 - \alpha|
\]
\[
< C \sup_{\mu \geq 0} |1 - \phi(\mu)(\mu + 1)|/|\mu - e^{i\arg(\alpha)}|
\]
\[
< C.
\]

The last inequality used the fact that $|\mu - e^{i\arg(\alpha)}|$ is bounded uniformly in $\alpha$ from below on the support of $1 - \phi(\mu)$.

Finally, the interpolation with

\[
\|(1 - \phi(L/|\alpha|))(L - \alpha)^{-1}\|_{L^2(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)} \leq C/|\alpha|
\]
yields (20). □

4. **Sobolev estimates near the spectrum**

It remains to consider the case when $\alpha$ is near the spectrum, say $|\arg(\alpha)| < \theta < \pi/4$. By choosing a cut-off function $\phi$ supported around 1 as above, we consider $\phi(L/|\alpha|)(L - \alpha)^{-1}$ instead. This spectral cut-off does not change the result but brings us some convenience to use the spectral measure estimates.

On the one hand, one can prove the following low energy results
Proposition 11. Suppose \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) with \( |\alpha| < 1 \) and \( |\text{arg}(\alpha)| < \theta < \pi/4 \). For \( 2n/(n+2) \leq q < 2 \),
\[
\|\phi(L/|\alpha|)(L - \alpha)^{-1}\|_{L^q(\mathbb{H}^n) \to L^q(\mathbb{H}^n)} \leq C.
\]

Proof. As is indicated in the description of Mazzeo-Melrose, the resolvent kernel near the diagonal is a pseudodifferential operator of degree \(-2\). Then (12) yields that
\[
\text{Ker}\left(\phi(L/|\alpha|)(L - \alpha)^{-1}\right) = O(\rho^{-n+2}), \quad \text{for small } \rho.
\]

If we live away from the diagonal, we rewrite this kernel in terms of the spectral theorem,
\[
\text{Ker}\left(\phi(L/|\alpha|)(L - \alpha)^{-1}\right) = \int \phi(\lambda^{2}/|\alpha|)(\lambda^{2} - \alpha)^{-1}dE_{\sqrt{\lambda}}(\lambda)d\lambda.
\]

Then (12) yields that
\[
\text{Ker}\left(\phi(L/|\alpha|)(L - \alpha)^{-1}\right)
= \left(xx'\right)^{-(n-1)/2} \int \phi(\lambda^{2}/|\alpha|) \frac{\lambda}{\lambda^{2} - \alpha} \left(e^{\lambda \log(xx')}a(\lambda) - e^{-\lambda \log(xx')}a(-\lambda)\right)d\lambda
= \left(xx'\right)^{-(n-1)/2} \int \phi(\lambda^{2}/|\alpha|) \left(\frac{1}{\lambda - \sqrt{\alpha}} + \frac{1}{\lambda + \sqrt{\alpha}}\right) \sum_{\pm} e^{\pm \lambda \log(xx')}a(\pm \lambda)d\lambda.
\]

For \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) with \( |\text{arg}(\alpha)| < \theta \), we write \( \sqrt{\alpha} = \beta + i\epsilon \). Shifting \( \lambda \) by \( \pm \beta \) gives
\[
\text{Ker}\left(\phi(L/|\alpha|)(L - \alpha)^{-1}\right)
= \left(xx'\right)^{-(n-1)/2} e^{i \log(xx')\beta} \int \frac{\phi((\lambda + \beta)^{2}/|\alpha|)}{\lambda - i\epsilon} e^{\lambda \log(xx')}a(\lambda + \beta)d\lambda
- \left(xx'\right)^{-(n-1)/2} \int \frac{\phi((\lambda + \beta)^{2}/|\alpha|)}{\lambda - i\epsilon} e^{-\lambda \log(xx')}a(-\lambda - \beta)d\lambda
+ \left(xx'\right)^{-(n-1)/2} e^{-i \log(xx')\beta} \int \frac{\phi((\lambda - \beta)^{2}/|\alpha|)}{\lambda + i\epsilon} e^{\lambda \log(xx')}a(\lambda - \beta)d\lambda
- \left(xx'\right)^{-(n-1)/2} \int \frac{\phi((\lambda - \beta)^{2}/|\alpha|)}{\lambda + i\epsilon} e^{-\lambda \log(xx')}a(-\lambda + \beta)d\lambda.
\]

Now we view the products \( \phi((\lambda \pm \beta)^{2}/|\alpha|)a(\pm \lambda \pm \beta) \) as compactly supported smooth functions in \( \lambda \). Then their Fourier transforms are Schwartz. The factors \( 1/(\lambda \pm i\epsilon) \) can be Fourier transformed by
\[
\mathcal{F}(e^{-cz}H(z))(\zeta) = \frac{1}{c + i\zeta},
\]
where \( c > 0 \) and \( H(\cdot) \) is the Heaviside unit step function. Hence above integrals are all convolutions of some Schwartz function and \( e^{-i|\cdot|}H(\cdot) \). Invoking \( \rho \sim -\log(xx') \) if \( \rho \) is large, we conclude that
\[
\left|\text{Ker}\left(\phi(L/|\alpha|)(L - \alpha)^{-1}\right)\right| \leq Ce^{-(n-1)\rho/2}.
\]

Finally, we complete the proof by applying (8) to the kernel bound of \( \phi(L/|\alpha|)(L - \alpha)^{-1} \).

On the other hand, we prove the following high energy results.
Sobolev Inequalities

Proposition 12. For \(2(n+1)/(n+3) \leq q < 2\) and \(|\alpha| > 1\) with \(|\text{arg}(\alpha)| < \theta\),
\[
\|\phi(L/|\alpha|)(L - \alpha)^{-1}\|_{L^q(H^s) \to L^q(H^s)} \leq C|\alpha|^{1/2-1/q}\).
\]
For \(2n/(n+2) \leq q \leq 2(n+1)/(n+3)\) and \(|\alpha| > 1\) with \(|\text{arg}(\alpha)| < \theta\),
\[
\|\phi(L/|\alpha|)(L - \alpha)^{-1}\|_{L^q(H^s) \to L^q(H^s)} \leq C|\alpha|^{n(1/q-1/2)-1}.
\]

Proof. The estimates (22) is the same with the Euclidean case in [13]. We can use the proof in there verbatim to show (22).

The strategy to prove (21) is to use the Stein’s complex interpolation [23] for the analytic family of operators \(H_{s,\alpha}(\sqrt{L/|\alpha|})\), where
\[
H_{s,\alpha}(x) = e^{s^2|\alpha|^2}\phi(x^2(1-x^2+a0)^8),
\]
provided \(\phi\) is a cut-off function supported around 1.

In particular, it suffices to prove \((L^q,L^q)\) estimates, with some \(2(n+1)/(n+3) \leq q < 2\), for
\[
H_{-1,\alpha}(\sqrt{L/|\alpha|}) = \epsilon \text{Ker}\phi(L/|\alpha|)(\alpha - L + i0)^{-1},
\]
for \(\alpha > 1\). Here we are only concerned about the estimates at the spectrum as the off-spectrum part is less singular.

By complex interpolation, it suffices to establish \((L^2,L^2)\) estimates for \(H_{s,\alpha}(\sqrt{L/|\alpha|})\) and \((L^1,L^\infty)\) estimates for \(H_{-j+1,\alpha}((\sqrt{L/|\alpha|})\) with any integer \(j > (n-1)/2\). As the former is trivial, it suffices to prove an upper bound of the kernel of \(H_{-j+1,\alpha}(\sqrt{L/|\alpha|})\).

We firstly rewrite the kernel of \(H_{-j+1,\alpha}(\sqrt{L/|\alpha|})\), in terms of the spectral measure, as
\[
e^{-j+1+it}e^{-j+1u-1} \int_0^\infty \phi\left(\frac{\lambda^2}{|\alpha|}\right) \left(1 - \frac{\lambda^2}{|\alpha|} + i0\right)^{-j+1+it} dE_{\sqrt{\lambda}}(\lambda) d\lambda.
\]
To make use of the spectral measure upper bound [21], we change coordinates and further rewrite it as
\[
H_{-j+1,\alpha}(\sqrt{L/|\alpha|}) = \int_{0}^{\infty} \phi(\lambda) d\lambda \int_{0}^{\infty} \frac{\lambda^{j-1}}{2\sqrt{\lambda}} (1 - \lambda + i0)^{-j+1+it} dE_{\sqrt{\lambda}}(\sqrt{\alpha}\sqrt{\lambda}) d\lambda.
\]
We now make integration by parts and estimate the kernel as follows.
\[
(-1)^{j-1} H_{-j+1,\alpha}(\sqrt{L/|\alpha|}) = \int_{0}^{\infty} \frac{\lambda^{j-2+it}}{2\sqrt{\lambda}} (1 - \lambda + i0)^{-2+it} d\lambda.
\]
Now we use a family of distributions \(\{x_+^{a} = x_+^{a}/\Gamma(a+1)\}\) defined on all \(a \in \mathbb{C}\), where \(\Gamma\) is the gamma function and
\[
x_+^{a} = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.
\]
It follows that $\chi_+^0(x) = H(x)$ and $\chi_-^k = \delta_0^{(k-1)}$, where $H$ is the Heaviside function and $\delta_0^{(k-1)}$ is the $k-1$-derivative of the delta function at 0. By [13] p.608, (26), the integral above obeys

$$\left| \int_0^\infty (1 - \lambda + t\alpha)^{-2+\epsilon} d\lambda \right| \le C(1 + |t|) e^{\pi t/2} \sup_{\sigma} \left| \left( \chi_+^1 * \frac{d^{j-1}}{d\lambda^{j-1}} \left( \frac{\phi(\lambda)}{2\sqrt{\lambda}} dE_{\mathcal{T}}(\sqrt{\alpha \sqrt{\lambda}}) \right) \right) (\sigma) \right|^{1/2}$$

where $\chi_+^1$ is the Heaviside function and $\phi(\lambda) = \lambda^\alpha$. Recall that $\phi$ is supported around 1. The spectral measure estimates [9] yields that

$$|H_{j-1+\epsilon,\alpha}(\sqrt{L/|\alpha|})| \le C_{j,t}|\alpha|^{-j-1/2+(n-1)/4+j/2} \le C_{j,t}|\alpha|^{-1/2}.$$ 

Noting that $\theta = 1/(j+1), q = 2(j+1)/(j+2)$ solves the elementary system of equations,

$$\begin{cases} 0(1 - \theta) + (j + 1)\theta = 1 \\ (1 - \theta)/2 + \theta = 1/q \end{cases},$$

we obtain that for any $1/2 < \epsilon < 1$,

$$\|\phi(L/|\alpha|)(L - \alpha + t\alpha)^{-1}\|_{L^q(\mathbb{H}^n) \rightarrow L^q(\mathbb{H}^n)} \le C|\alpha|^{1/2-1/q}.$$

\[ \square \]

5. Eigenvalue bounds for Schrödinger operators with complex potentials

Inspired by the work of Frank-Simon [11] [12], we apply the Sobolev inequalities on hyperbolic space to the study of eigenvalue bounds for a Schrödinger operator with complex potential. More precisely, for an $L^2$-eigenvalue $\lambda$ of a Schrödinger operator, $\Delta + V$, on $\mathbb{H}^{n+1}$ with a complex potential $V$, we want to understand the upper bound of $|\lambda|$ in terms of the Lebesgue norm of $|V|$. In addition, we will prove that if the norm of the potential is sufficiently small the Schrödinger operator has no eigenvalues.

Let $\lambda \in \mathbb{C}$ be an eigenvalue and $\psi \in H^1(\mathbb{H}^n)$ the corresponding eigenfunction of $L + V$, namely

$$(L + V)\psi = \lambda \psi.$$
Short range $0 < \gamma \leq 1/2$.

**Proof of Theorem 4.** Assume first that $\lambda \in \mathbb{C} \setminus [0, \infty)$. We write

$$\gamma + \frac{n}{2} = \frac{p}{2-p}.$$ 

This implies $2n/(n+2) < p \leq 2(n+1)/(n+3)$ and $2(n+1)/(n-1) \leq p' < 2n/(n-2)$. Proposition 13 shows that $\Delta + (V - n^2/4)$ is an m-sectorial operator with domain in $H^1(\mathbb{H})$. By Sobolev’s embedding, we further have $\psi \in L^{2n/(n-2)}(\mathbb{H})$, whence $\psi \in L^r(\mathbb{H})$ for $2 \leq r \leq 2n/(n-2)$. Additionally, we have

$$\psi = (L - \lambda)^{-1}(L - \lambda)^{-1}(V\psi).$$

Using Hölder’s inequality and uniform Sobolev inequalities (7), we obtain that

$$\|\psi\|_{L^p(H^n)} \leq \|(L - \lambda)^{-1}\|_{L^p(H^n) \to L^{p'}(H^n)} \|V\psi\|_{L^p(H^n)} \leq C|\lambda|^{n(1/p-1/p')-1} \|V\|_{L^{\gamma+n/2}(H^n)} \|\psi\|_{L^p(H^n)}.$$

Noting that

$$\frac{n}{2} \left(\frac{1}{p} - \frac{1}{p'}\right) - 1 = -\frac{\gamma}{\gamma + n/2},$$

we conclude that

\begin{equation}
|\lambda|^\gamma \leq C\|V\|^{\gamma+n/2}_{L^{\gamma+n/2}(H^n)}.
\end{equation}

If $\lambda \in (0, \infty)$, we instead consider

$$\psi_\epsilon = (L - \lambda - \epsilon)^{-1}(L - \lambda)^{-1}(L - \lambda- \epsilon)\psi = f_\epsilon(L)\psi, \quad \text{with } f_\epsilon(t) = (t - \lambda)/(t - \lambda - \epsilon) \text{ for } t > 0.$$

Then the spectral theorem yields that

$$\|\psi_\epsilon - \psi\|^2_{L^2(H^n)} = \|f_\epsilon(L)\psi - \psi\|^2_{L^2(H^n)} = \int |f_\epsilon(t) - 1|^2 d(E_L(t)\psi, \psi)_{L^2(H^n)},$$

where $E_L(t)$ is the spectral projection of $L$. In view of the fact that $f_\epsilon \to 1$ as $\epsilon \to 0$, the dominated convergence theorem yields that $\psi_\epsilon \to \psi$ in $L^2(\mathbb{H}^n)$ as $\epsilon \to 0$.

For $\psi_\epsilon$, we similarly obtain that

$$\|\psi_\epsilon\|_{L^{p'}(H^n)} \leq C|\lambda|^{n(1/p-1/p')-1/2} \|V\|_{L^{\gamma+n/2}(H^n)} \|\psi\|_{L^{p'}(H^n)}.$$

It follows that there exists $\tilde{\psi} \in L^{p'}(\mathbb{H}^n)$ such that $\psi_\epsilon \to \tilde{\psi}$ in the weak * topology of $L^{p'}(\mathbb{H}^n)$, whence $\psi = \tilde{\psi} \in L^{p'}(\mathbb{H}^n)$. Consequently, we have

$$\|\psi\|_{L^{p'}(H^n)} \leq \liminf_{\epsilon \to 0} \|\psi_\epsilon\|_{L^{p'}(H^n)} \leq |\lambda|^{n(1/p-1/p')-1} \|V\|_{L^{\gamma+n/2}(H^n)} \|\psi\|_{L^{p'}(H^n)},$$

which completes the proof for the short range case. \hfill \Box

**Long range $\gamma > 1/2$.**

**Proof of Theorem 5.** We still use the proof for the short range case but replace (7) by (6). This leads to that

$$\|\psi\|_{L^{p'}(H^n)} \leq \|(L - \lambda)^{-1}\|_{L^p(H^n) \to L^{p'}(H^n)} \|V\psi\|_{L^p(H^n)} \leq C|\lambda|^{1/2-1/p} \|V\|_{L^{\gamma+n/2}(H^n)} \|\psi\|_{L^{p'}(H^n)}.$$

Combing this together with

$$\frac{1}{p} = \frac{1 + \gamma + n/2}{2(\gamma + n/2)},$$

we conclude that,
\[ |\lambda|^{1/2} \leq C\|V\|_{L^{n/2}}^{n/2} \]

\[ \square \]

**Sufficiently ’small’ potential \( V \).**

_Proof of Theorem 6_ In the end, we prove, by a contradiction argument, that if the norm \( \|V\|_{L^{n/2}(\mathbb{H}^n)} \) is sufficiently small, the Schrödinger operator \( \Delta + V \) has no eigenvalues.

Let \( \lambda \) be an eigenvalue of \( L + V \) with \( \|V\|_{L^{n/2}(\mathbb{H}^n)} < c < 1 \). It follows, from the eigenvalue bounds we have proved, that \( |\lambda| < C e^{(\gamma + n/2)/\gamma} < C \) for \( \gamma \geq 0 \). When \( \lambda \notin [0, \infty) \), we have that for any eigenfunction \( \psi \) of \( \lambda \)
\[ \|\psi\|_{L^p(\mathbb{H}^n)} \leq \|(L - \lambda)^{-1}\|_{L^p(\mathbb{H}^n)\rightarrow L^p(\mathbb{H}^n)} \|V\|_{L^p(\mathbb{H}^n)} \leq C_1 \|V\|_{L^{n/2}(\mathbb{H}^n)} \|\psi\|_{L^p(\mathbb{H}^n)}, \]
where we invoked the low energy Sobolev inequalities \[5\). When \( \lambda \in (0, \infty) \), it is similar to deduce that
\[ \|\psi\|_{L^p(\mathbb{H}^n)} \leq \liminf_{\epsilon \to 0} \|\psi_\epsilon\|_{L^p(\mathbb{H}^n)} \leq \|(L - \lambda - \epsilon)^{-1}\|_{L^p(\mathbb{H}^n)\rightarrow L^p(\mathbb{H}^n)} \|V\|_{L^p(\mathbb{H}^n)} \leq C_2 \|V\|_{L^{n/2}(\mathbb{H}^n)} \|\psi\|_{L^p(\mathbb{H}^n)}, \]
where \( \psi_\epsilon = (L - \lambda - \epsilon)^{-1}(L - \lambda)\psi \) as above. If \( c < \min\{1/C_1, 1/C_2\} \), these inequalities fail to hold. Therefore, there are no eigenvalues. \[ \square \]

**APPENDIX A. M-SECTORIAL OPERATORS**

The purpose of this section is to prove that

**Proposition 13.** Given a complex potential \( V \in L^p(\mathbb{H}^n) \) with \( n/2 \leq p < \infty \), the Schrödinger operator \( \Delta + V \) is an m-sectorial operator with a domain contained in \( H^1(\mathbb{H}^n) \). The spectrum of \( \Delta + V \) consists of the essential spectrum of \( \Delta \) and isolated eigenvalues of finite algebraic multiplicity.

To begin with, we review the relevant definitions on unbounded operators and forms in Hilbert spaces. We refer the reader to \[19\] Chapter V-VI for more information. Suppose \( \mathbf{H} \) is a Hilbert space with an inner product \((\cdot, \cdot)\).

- An operator \( T \) in \( \mathbf{H} \) is said to be accretive if
  \[ \text{Re}(Tu, u) \geq 0 \quad \text{for all } u \in \text{Dom}(T). \]
- If \( T + \alpha \) for some scalar \( \alpha \) is accretive, we say \( T \) is quasi-accretive.
- If an accretive operator \( T \) is surjective and also obeys that for any \( \lambda \) with \( \text{Re} \lambda > 0 \) in the resolvent set,
  \[ \|(T + \lambda)^{-1}\| \leq (\text{Re} \lambda)^{-1}, \]
\( T \) is said to be \( m \)-accretive.
- If \( T + \alpha \) for some scalar \( \alpha \) is \( m \)-accretive, we say \( T \) is quasi-\( m \)-accretive.
- We say \( T \) is sectorially valued or simply sectorial with vertex \( \gamma \) and semi-angle \( \theta \), if
  \[ \{\text{Re}(Tu, u) : u \in \text{Dom}(T)\} \subset \{z \in \mathbb{C} : |\arg(z - \gamma)| \leq \theta < \pi/2\}. \]
- If \( T \) is sectorial and quasi-\( m \)-accretive, we say \( T \) is \( m \)-sectorial.
• We say a quadratic form $t$ in $H$ is sectorially bounded from the left or simply sectorial, if
\[ \{ t[u,u] : \|u\| = 1, u \in \Dom(t) \} \subset \{ z \in \mathbb{C} : |\arg(z - \gamma)| \leq \theta < \pi/2, \gamma \in \mathbb{R} \}. \]

• A sectorial form is said to be closed if that a sequence $\{u_n \in \Dom(t)\}$ converges to $u$ in $H$ and $t[u_n - u_m] \to 0$ as $n, m \to \infty$ implies that $u \in \Dom(t)$ and $t[u_n - u] \to 0$ as $n \to \infty$.

Let $H_0$ be a self-adjoint, non-negative operator in a Hilbert space $H$. In addition, we suppose $G_0$ and $G$ are operators from $H$ to another Hilbert space $G$ such that
\[ \Dom(H_0^{1/2}) \in \Dom(G_0) \cap \Dom(G), \]
but also
\[ (24) \quad G_0(H_0 + 1)^{-1/2} \text{ and } G(H_0 + 1)^{-1/2} \text{ are compact.} \]

Under such assumptions, Frank [12, Lemma B.1, Lemma B.2] proved that

**Lemma 14.** The quadratic form
\[ \|H_0^{1/2}u\|_H + (Gu,Gu_0)_G \]
with $\Dom(H_0^{1/2})$ is closed and sectorial. Moreover, it generates an $m$-sectorial operator $H = H_0 + G^*G_0$. The spectrum of $H$ consists of the essential spectrum of $H_0$ and isolated eigenvalues of finite algebraic multiplicity.

We want to apply this abstract lemma to the following quadratic form on $L^2(\mathbb{H}^n)$,
\[ \|\Delta^{1/2}u\|_{L^2(\mathbb{H}^n)} + (\sqrt{V}u,\sqrt{V}|u|)_{L^2(\mathbb{H}^n)}. \]

Then $\Delta + V$ would be an $m$-sectorial operator with a domain contained in $H^1(\mathbb{H}^n)$. It remains to prove (21) for $\Delta$, $\sqrt{V}$ and $\sqrt{|V|}$. We have

**Lemma 15.** Suppose $V \in L^p(\mathbb{H}^n)$ with $n/2 \leq p < \infty$ is a complex potential. The operator $\sqrt{|V|}(\Delta + 1)^{-1/2}$ is compact on $L^2(\mathbb{H}^{n+1})$.

**Proof.** Sobolev's embedding $H^1(\mathbb{H}^n) \subset L^{2n/(n-2)}(\mathbb{H}^n)$ and the mapping property $(\Delta + 1)^{-1/2} : L^2(\mathbb{H}^n) \to H^1(\mathbb{H}^n)$ yield that
\[ (\Delta + 1)^{-1/2} : L^2(\mathbb{H}^n) \to L^{2n/(n-2)}(\mathbb{H}^n). \]

It implies that
\[ (\Delta + 1)^{-1/2} : L^2(\mathbb{H}^n) \to L^q(\mathbb{H}^n), \quad \text{for any } 2 \leq q \leq 2n/(n-2). \]

From this, one can obtain that for any $W \in L^{2p}(\mathbb{H}^n)$,
\[ (25) \quad \|W(\Delta + 1)^{-1/2}f\|_{L^2(\mathbb{H}^n)} \leq C\|W\|_{L^{2p}(\mathbb{H}^n)}\|f\|_{L^2(\mathbb{H}^n)}. \]

We select a sequence $\{W_j \in C_0^\infty(\mathbb{H}^n)\}$ such that $W_j \to \sqrt{|V|}$ in $L^{2p}(\mathbb{H}^n)$. Then $W_j(\Delta + 1)^{-1/2}$ is an operator which maps $L^2(\mathbb{H}^n)$ to $H^1(\mathbb{H}^n)$. Since $W_j$ is compactly supported, the support of $W_j$ can be thought of as a compact manifold $M_j$ with $C^1$-boundary. The image of $W_j(\Delta + 1)^{-1/2}$ is contained in $H^1(M_j)$. Since Rellich-Kondrachov theorem implies $H^1(M_j) \subset L^2(M_j) \subset L^2(\mathbb{H}^n)$, we have that $W_j(\Delta + 1)^{-1/2}$ is a compact operator on $L^2(\mathbb{H}^n)$. On the other hand, (25) yields that
\[ W_j(\Delta + 1)^{-1/2} \to \sqrt{|V|}(\Delta + 1)^{-1/2} \quad \text{in } L^2(\mathbb{H}^n). \]

Therefore, $\sqrt{|V|}(\Delta + 1)^{-1/2}$ is also a compact operator.

□
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