Generalized semi-Markovian dividend discount model: risk and return

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Abstract: The article presents a general discrete time dividend valuation model when the dividend growth rate is a general continuous variable. The main assumption is that the dividend growth rate follows a discrete time semi-Markov chain with measurable space \((E, \mathcal{E})\). The paper furnishes sufficient conditions that assure finiteness of fundamental prices and risks and new equations that describe the first and second order price-dividend ratios. Approximation methods to solve equations are provided and some new results for semi-Markov reward processes with Borel state space are established.

The paper generalizes previous contributions dealing with pricing firms on the basis of fundamentals.

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1 Introduction

One of the main approaches to calculate the value of a firm is by means of fundamentals. Fundamental analysis consists in discounting of future cash flows by using a required rate of return on the stock and in considering the value of the stock equal to the present value of this stream of payments, see e.g. Kettel (2002). The rate of return includes a risk premium for the shareholders to compensate for the uncertainty associated with the future evolution of cash flows.

Stockholders expect to receive cash flows from the firm in term of dividends therefore, usually cash flows are represented by dividend streams. Nevertheless, it is possible to replace dividends with other financial variables like sales (see, e.g. Damodaran 1994) or earnings and payout ratios (see e.g. Sharpe and Alexander 1990). Since generally, stockholder receives only dividends from the firm the majority of the models are based on dividends, for this reason henceforth, we will assume that the firm pays dividends.

Almost all studies in literature impose sufficient structures on the dividend growth variable to permit computable expressions of the present value of future dividends. The seminal paper by Gordon and Shapiro (1956) considers a constant dividend growth rate. Many variants of the Gordon and Shapiro model have been suggested in literature. These variants impose ever less stringent assumptions on the dividend process. For example, the papers by Brooks and Helms (1990) and Barsky and DeLong (1993) consider multistage models with dividend growth rates changing deterministically among the stages. Models based on Markov chains were proposed in Hurley and Johnson (1994;1998), Yao (1997) and Ghezzi and Piccardi (2003) and, in general, regime switching in the dividend process were advanced in Gutierrez and Vasquez (2004). The results of these articles were encompassed into a semi-Markovian framework as provided by D’Amico
(2013) where the semi-Markov hypothesis was advanced and validated. Mainly due to their generality and flexibility, recently semi-Markov processes have acquired even more interest in many areas of financial modeling ranging from credit rating dynamics (see, e.g. D’Amico et al. (2005), Vasileiou and Vassiliou (2006), Vassiliou and Vasileiou (2013) and Vassiliou (2014)) to high frequency finance (see D’Amico and Petroni (2011) and D’Amico and Petroni (2012)).

More recent contributions on dividend valuation models rely on one side on the adoption of Geometric and Additive Bernoulli Processes where the dividend process assumes a continuous set of values, see Hurley (2013) and on the other side on the derivation of formulas for the variance of the fundamental prices obtained in a binomial based model, see Agosto and Moretto (2013).

These two recent contributions necessitate a unifying treatment under the adoption of more general stochastic models that overcome the strong simplifying assumption of Bernoulli process. Indeed, it is necessary to avoid imposing strong hypotheses on the dividend process in favor of more general models that let data to speak for themselves without imposing stringent assumptions.

Therefore the aim of this paper is to generalize all these contributions by assuming that the growth dividend rate is a discrete time semi-Markov chain with Borel state space. Then, we advance a more formal and abstract dividend valuation model that contains as particular cases all the previous cite works, D’Amico (2013) included.

The second feature of this study is that we derive sufficient conditions that assure the finiteness of fundamental prices and that satisfy the transversality condition that avoid the presence of speculative bubbles within this general semi-Markov environment. Moreover we extend these results by computing the second order moment of the price process which reveals to be useful for measuring the
riskiness of the stock. To do this we determine the fundamental formula of the
risk (second order moment of the price process) and we describe methods for
computing it.

It is important to remark on the presence of contributions that generalize the
Gordon and Shapiro model by allowing arbitrary dividend growth and discount
rate processes, see Donaldson and Kamstra (1996). The Donaldson and Kam-
stra’s procedure involves a Monte Carlo simulation and numerical integration of
the possible paths followed by the joint processes of dividend growth and discount
rates.

The paper is organized as follows: first, in Section 2, we give minimal defini-
tions on discrete time semi-Markov chains with Borel state space. Next, Section 3
presents generalities on the dividend valuation model, on the semi-Markov stock
model and the relevant results of the proposed dividend valuation model. Con-
tinuing, Section 4 presents some concluding remarks. Finally, an appendix that
contains the proofs of the results exposed in Section 3 concludes the paper.

2 Semi-Markov chain with Borel state space

The following is the minimal set of definitions covering discrete time semi-Markov
chains with Borel state space necessary for the reading of the article. It should be
remarked that very few studies deal with general Borel state space semi-Markov
process and they concern mainly reliability models in continuous time, see e.g.
Limnios and Oprișan (2001), D’Amico (2011) and Limnios (2012).

Let \((E, \mathcal{E})\) be a measurable space such that for all \(x \in E\) it results that
\(
\{x\} \in \mathcal{E}.
\)

**Definition 2.1 (Sub-Markov Kernel).** A function \(p(x, A)\), \(x \in E\), \(A \in \mathcal{E}\), is
called a sub-Markov kernel on \((E, \mathcal{E})\) if:

i) for every \(x \in E\), \(p(x, \cdot)\) is a measure on \(\mathcal{E}\) such that \(p(x, E) \leq 1\);

ii) for every \(A \in \mathcal{E}\), \(p(\cdot, A)\) is a Borel measurable function.

**Definition 2.2 (Semi-Markov Kernel).** A function \(Q(x, A, t), x \in E, A \in \mathcal{E}, t \in \mathbb{N}\) is called a discrete time semi-Markov kernel (SMK) on \((E, \mathcal{E})\) if:

i) \(Q(x, A, \cdot)\) for all \(x \in E, A \in \mathcal{E}\) is a nondecreasing discrete real function such that \(Q(x, A, 0) = 0\);

ii) \(Q(\cdot, \cdot, t)\) for all \(t \in \mathbb{N}\) is a sub-Markov kernel on \((E, \mathcal{E})\);

iii) \(p(\cdot, \cdot) = Q(\cdot, \cdot, \infty)\) is a Markov transition probability function from \((E, \mathcal{E})\) to itself.

For each \((x, s) \in E \times \mathbb{N}\), there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P}_{(x,s)})\) and a sequence of r.v. \((J_n, T_n)\) such that:

\[
\mathbb{P}_{(x,s)}[J_0 = x, T_0 = s] = 1;
\]

\[
\mathbb{P}_{(x,s)}[J_{n+1} \in A, T_{n+1} \leq t \mid \mathcal{F}_n] = \mathbb{P}_{(x,s)}[J_{n+1} \in A, T_{n+1} \leq t \mid J_n, T_n] = Q(J_n, A, t - T_n).
\]

Thus \((J_n, T_n)_{n \in \mathbb{N}}\) is a Markov process with state space \(E \times \mathbb{N}\) and transition probability function given by the semi-Markov kernel \(Q(x, A, t)\).

The sequence \((J_n, n \geq 0)\) gives the successive states of \(E\) in time and the sequence \((T_n, n \geq 0)\) gives the times at which transitions occur.

Set \(N(0) = 0\) and define the process counting the number of transitions that occur time \(t\):

\[
N(t) = \sup\{n \in \mathbb{N} : T_n \leq t\}, t > 0.
\]
Definition 2.3 (Semi-Markov chain). The process \( \{Z(t), t \in \mathbb{N}\} \), defined by

\[
Z(t) = J_{N(t)}
\]

is a discrete time semi-Markov chain with arbitrary state space \( E \) and kernel \( Q(x, A, t) \).

Let denote by

\[
p(x, A) := \mathbb{P}[J_{n+1} \in A \mid J_n = x].
\]

It is simple to show that

\[
p(x, A) = \lim_{t \to \infty} Q(x, A, t).
\]

Define for any state \( x \in E \) the probability

\[
H(x, t) := \mathbb{P}[T_{n+1} - T_n \leq t \mid J_n = x] = Q(x, E, t).
\]

It denotes the probability to leave state \( x \) within time \( t \) with a transition toward any other state of \( E \). Consequently, the quantity \( 1 - H(x, t) \) denotes the survival probability in state \( x \in E \).

Inasmuch, \( \forall x \in E, A \in \mathcal{E}, t \in \mathbb{N} \)

\[
Q(x, A, t) \leq p(x, A),
\]

the measure \( Q(x, \cdot, t) \) is absolutely continuous with respect to \( p(x, \cdot) \), then, according to the Radon-Nikodym theorem, it exists a \( \mathcal{E} \)-measurable function \( F(x, A, \cdot) \) such that

\[
Q(x, A, t) = \int_A F(x, y, t)p(x, dy).
\]
The function $F$ can be chosen as a distribution function expressing the probability:

$$F(x, y, t) = \mathbb{P}[T_{n+1} - T_n \leq t \mid J_n = x, J_{n+1} = y].$$  \hspace{1cm} (2.1)$$

If the distribution function (2.1) is geometrically distributed, then $Z(t)$ is a Markov chain with measurable space $(E, \mathcal{E})$. In contrast, the semi-Markov chain environment allows the possibility to use any type of distribution function; as a result, semi-Markov chains supplies a much more general framework which is closer to the reality when compared with Markov chain based models.

We denote by

$$q(x, A, t) = \mathbb{P}[J_{n+1} \in A, T_{n+1} - T_n = t \mid J_n = x],$$ \hspace{1cm} (2.2)$$

the probability to visit with next transition the set $A$ with a sojourn time length $t$ in the state $x$. Obviously it results that

$$q(x, A, t) = \begin{cases} 
Q(x, A, t) - Q(x, A, t - 1) & \text{if } t > 0, \\
0 & \text{if } t = 0.
\end{cases}$$

**Definition 2.4** (n-fold convolution of SMK). *Given the SMK $Q(x, A, t)$, define its n-fold convolution as follows:*

$$Q^{(n)}(x, A, t) = \begin{cases} 
\int_E \sum_{s=1}^{t} Q(x, dy, s)Q^{(n-1)}(y, A, t - s) & \text{if } n \geq 1, t > 0, \\
0 & \text{if } t \leq 0,
\end{cases}$$ \hspace{1cm} (2.3)$$

where $Q^{(0)}(x, A, t) = \chi(x \in A)\chi(t > 0)$ and $\chi(L)$ is the indicator function of event $L$. 
It is simple to show that \( \forall h \in \mathbb{N} \)

\[
Q^{(n)}(x, A, t) = \mathbb{P}[J_{n+h} \in A, T_{n+h} - T_h \leq t \mid J_h = x].
\]

The function

\[
R(x, A, t) := \sum_{n=0}^{\infty} Q^{(n)}(x, A, t) = \mathbb{E}[N(t) \mid J_0 = x, T_0 = 0],
\]

is called Markov Renewal Function.

**Definition 2.5** (Normal SMK). The SMK \( Q(x, A, t) \) is said to be normal if 
\( R(x, A, t) < \infty \ \forall x \in E, A \in \mathcal{E}, t \in \mathbb{N} \).

It is important in semi-Markov modeling to introduce the following auxiliary stochastic process:

**Definition 2.6** (Backward recurrence time process). For each \( t \in \mathbb{N} \) the process 
\( \{B(t), t \in \mathbb{N}\} \) defined by

\[
B(t) = t - T_{N(t)},
\]

is the backward recurrence time process

The backward process denotes the time since the last transition and is an important tool for detecting and quantifying the so called duration effect, namely the fact that the time the system is in a state influence its transition probabilities, see e.g. D’Amico et al. (2011).

### 3 The dividend valuation model

Let denote by \( P(k) \) the random variable denoting the fundamental price of the stock at time \( k \in \mathbb{N} \), by \( D(k) \) we denote the dividend at the same time and by
one plus the required rate of return on the stock. We assume that \( r \) is known and focus only on the other elements involved in the valuation of the stock.

The fundamental equation affirms that \( p(k) := \mathbb{E}_k[P(k)] \) obey the following equation

\[
p(k) = \frac{\mathbb{E}(k)[D(k + 1) + P(k + 1)]}{r}, \tag{3.1}
\]

where \( \mathbb{E}(k) \) is the conditional expectation with respect to the information available up to time \( k \).

As it is well known, see for example Samuelson [21], if we assume that

\[
\lim_{i \to +\infty} \frac{\mathbb{E}(k)[P(k + i)]}{r^i} = 0, \tag{3.2}
\]

the unique solution of (3.1) is expressed by the series

\[
p(k) = \sum_{i=1}^{+\infty} \frac{\mathbb{E}(k)[D(k + i)]}{r^i}. \tag{3.3}
\]

Formula (3.3) valuates the stock as a function of expected future dividend stream and discount rates.

If condition (3.2) is not assumed, Blanchard and Watson [2] proved that there can exist different solutions of the fundamental equation revealing the presence of bubbles in the stock market.

The uncertainty in the dividend process propagates in the price process. To quantify this effect, it is possible to analyze the second order moment of the price process. To this end we denote by

\[
P^2(k) = \left( \frac{D(k + 1) + P(k + 1)}{r} \right)^2. \tag{3.4}
\]
Successive substitutions for the future prices with future dividends in (3.4) yield

\[ P^2(k) = \sum_{i=1}^{N} \frac{D^2(k + i)}{r^{2i}} + 2 \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{D(k + i)D(k + j)}{r^{i+j}} + 2 \sum_{i=1}^{N} \frac{D(k + i)P(k + N)}{r^{i+N}} + \frac{P^2(k + N)}{r^{2N}}. \]  

Therefore

\[ p^2(k) := \mathbb{E}[P^2(k)] = \sum_{i=1}^{N} \frac{\mathbb{E}[D^2(k + i)]}{r^{2i}} + 2 \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{\mathbb{E}[D(k + i)D(k + j)]}{r^{i+j}} + 2 \sum_{i=1}^{N} \frac{\mathbb{E}[D(k + i)P(k + N)]}{r^{i+N}} + \frac{\mathbb{E}[P^2(k + N)]}{r^{2N}}. \]  

Formula (3.6) says that the risk (measured by the second order moment) of the price process is related to all future risks of the dividend process, to their product moments plus two additional terms that are related to future price and risk of the stock.

If we assume that

\[ \lim_{N \to +\infty} \frac{\mathbb{E}_{(k)}[P^2(k + N)]}{r^{2N}} = 0, \]  

and

\[ \lim_{N \to +\infty} \sum_{i=1}^{N} \frac{\mathbb{E}[D(k + i)P(k + N)]}{r^{i+N}} = 0, \]  

then the unique solution of equation (3.6) is given by

\[ \mathbb{E}[P^2(k)] = \sum_{i=1}^{+\infty} \frac{\mathbb{E}[D^2(k + i)]}{r^{2i}} + 2 \sum_{i=1}^{+\infty} \sum_{j>i}^{+\infty} \frac{\mathbb{E}[D(k + i)D(k + j)]}{r^{i+j}}. \]  

We call expression (3.9) the fundamental formula of the risk (second order moment of the price process). This formula expresses the risk as a function of the second order and product moments of the dividend process. The transversality
conditions (3.7) and (3.8) eliminate the dependence of $p^2(k)$ on its own temporal path ($P^2(k + N)$). This means that conditions (3.7) and (3.8) are necessary in order to avoid rising in the risk ($p^2(k)$) due to an expected increase in the future risk ($P^2(k + N)$) and prices ($P(k + N)$) without any relation to the intrinsic risk of the dividend process. Therefore these conditions avoid the influence of future prices and risks on the present price and risk that are explained only in reference to the level of dividend and risk of the dividend process.

### 3.1 The semi-Markov stock model

As discussed in the introduction, a large literature deals with the issue of evaluating formula (3.3) under appropriate assumptions about the dividend dynamic. Here we improve even further this point advancing a more general model which imposes weaker assumptions on the dividend process that allow effective computation of the fundamental formulas (3.3) and (3.9) and the fulfillment of the transversality conditions (3.2), (3.7) and (3.8).

Let us assume that the dividend process \{$D(k)\}_{k \in \mathbb{N}}$ obeys the difference equation

$$ D(k + 1) = G(k + 1)D(k), \quad (3.10) $$

where the dividend growth factor \{$G(k)\}$ is described by a normal homogeneous semi-Markov chain with Borel state space $(E, \mathcal{E})$ and kernel $Q(x, A, t)$. It should be noted that model based on binomial, Markovian dynamics of the dividend are all embedded in the semi-Markov framework, therefore all the results we will prove have particular cases that may be of practical interest.

The model described by equation (3.10) is discrete in time because dividends must be declared periodically by the firms board of directors and therefore the
problem has a natural description in terms of discrete times. On the contrary, the growth factor \( \{ G(k) \} \) can assume any real value and this is the reason for the choice of the general state space instead of the discrete one considered in D’Amico (2013).

Inasmuch the semi-Markov process is time homogeneous, we can fix the current time \( k = 0 \) without having any loss of generality. It is easy to verify that the process \( (D(t), G(t), B(t)) \) is a Markov process, then the fundamental equation becomes (see D’Amico (2013))

\[
p(D(0), G(0), B(0)) = \frac{\mathbb{E}_{(D(0), G(0), B(0))}[D(1) + P(D(1), G(1), B(1))]}{r}.
\]  

(3.11)

Therefore the current price depends on the current value of dividends, on the value of the dividend growth process at that time and on the duration in this state.

Let us fix the initial condition at time 0, \( \{ D(0) = d, G(0) = g, B(0) = v \} \); the solution to equation (3.11) under the transversality condition

\[
\lim_{t \to +\infty} \frac{\mathbb{E}_{(d,g,v)}[P(D(t), G(t), B(t))]}{r^t} = 0,
\]  

(3.12)

produces the results that the price equals the expected present value of future dividends, i.e.

\[
p(d, g, v) = \sum_{t=1}^{+\infty} \frac{\mathbb{E}_{(d,g,v)}[D(t)\prod_{j=1}^{t} G(j)]}{r^t} d.
\]  

(3.13)

Similarly, the fundamental formula of the second order moment of the price
process $[3.9]$ becomes:

$$p^2(d, g, v) = \sum_{i=1}^{+\infty} \left( \frac{\mathbb{E}(d, g, v)[\prod_{j=1}^{i} G^2(j)]}{r^{2i}} \right) d^2 + 2 \sum_{l=1}^{+\infty} \sum_{w>l} \left( \frac{\mathbb{E}(d, g, v)[\prod_{j=1}^{l} G^2(j) \prod_{j=l+1}^{w} G(j)]}{r^{l+w}} \right) d^2,$$

as long as the following conditions are satisfied:

$$\lim_{N \to +\infty} \frac{\mathbb{E}(d, g, v)[P^2(N)]}{r^{2N}} = 0,$$  \hfill (3.15) \\
$$\lim_{N \to +\infty} \sum_{i=1}^{N} \frac{\mathbb{E}(d, g, v)[D(i)P(N)]}{r^{i+N}} = 0.$$  \hfill (3.16) \\

The computation and the convergence of formulas (3.13) and (3.14) needs the study of the (product) growth dividend process.

**Definition 3.1** (The growth dividend process). The process \( \{ A_{g,v}^{(k)}(t), t \in \mathbb{N}, g \in E, v \in \mathbb{N}, k \in \mathbb{N} \} \), defined for \( t > 0 \) by

$$A_{g,v}^{(k)}(t; \omega) = \begin{cases} 
\prod_{j=1}^{t} G^k(j; \omega) & \text{if } \omega \in \Omega_{g,v}, \\
0 & \text{otherwise},
\end{cases} \quad (3.17)$$

and for \( t = 0 \) by

$$A_{g,v}^{(k)}(0; \omega) = \begin{cases} 
1 & \text{if } \omega \in \Omega_{g,v}, \\
0 & \text{otherwise},
\end{cases} \quad (3.18)$$

where

$$\Omega_{g,v} = \{ \omega \in \Omega : G(0, \omega) = g, B(0; \omega) = v \},$$

is called the growth dividend process.

Thus, \( \forall t > 0 \) and \( \forall a \in \mathbb{R} \) the random variable \( A_{g,v}^{(k)}(t) \) has the following
distribution function

\[
\mathbb{P}[A_{g,v}^{(k)}(t) \leq a] = \mathbb{P}[\prod_{j=1}^{t} G^k(j) \leq a \mid G(0) = g, B(0) = v].
\] (3.19)

**Definition 3.2** (The product growth dividend process). The process \( A_{g,v}^{(k,w)}(t, s), s > t, g \in E, v \in \mathbb{N}, k, w \in \mathbb{N} \), defined for \( t > 0 \) by

\[
A_{g,v}^{(k,w)}(t, s) = A_{g,v}^{(k)}(t) \cdot A_{G(t),B(t)}^{(w)}(s)
\] (3.20)

is called the product growth dividend process.

Thus, \( \forall s > t > 0 \) and \( \forall a \in \mathbb{R} \) the random variable \( A_{g,v}^{(k,w)}(t, s) \) has the following distribution function

\[
\mathbb{P}[A_{g,v}^{(k,w)}(t, s) \leq a] = \mathbb{P}[\prod_{j=1}^{t} G^k(j) \prod_{j=t+1}^{s} G^w(j) \leq a \mid G(0) = g, B(0) = v].
\] (3.21)

We denote the corresponding expectations by

\[
M_{g,v}^{(k)}(t) = \mathbb{E}_{(d,g,v)}[A_{g,v}^{(k)}(t)],
\]

\[
M_{g,v}^{(k,w)}(t, s) = \mathbb{E}_{(d,g,v)}[A_{g,v}^{(k)}(t)A_{G(t),B(t)}^{(w)}(s)].
\]

**Proposition 3.3** (Moments of the product growth dividend process). For all \( g \in E, v, t, s \in \mathbb{N} \), the product moment \( M_{g,v}^{(k,w)}(t, s) \) satisfies the following equation:

\[
M_{g,v}^{(k,w)}(t, s) = \left(1 - \frac{H(g, v + s)}{1 - H(g, v)}\right) (g)^{w v s + t (k - w)} + \sum_{\theta = t+1}^{s} \int_{E} \hat{q}(g, y, \theta + v) \frac{1}{1 - H(g, v)} (g)^{w \theta + t (k - w) - w} y^w M_{g,0}^{(w)}(s - \theta) dy
\]

\[
+ \sum_{\theta = 1}^{t} \int_{E} \hat{q}(g, y, \theta + v) \frac{1}{1 - H(g, v)} (g)^{k (\theta - 1)} y^k M_{g,0}^{(w)}(t - \theta, s - \theta) dy
\] (3.22)
Corollary 3.4 (Moments of the growth dividend process). For all $g \in E$, $v,t \in \mathbb{N}$, the $k$-order moment of the dividend growth-product process satisfies the following equation:

\[
M_{g,v}^{(k)}(t) = \left( \frac{1 - H(g,v + t)}{1 - H(g,v)} \right)^{(g)kt} + \sum_{\theta=1}^{t} \int_{E} \hat{q}(g,y,\theta + v) \left( \frac{1}{1 - H(g,v)} \right)^{(g)k(\theta-1)} y^k M_{g,0}^{(k)}(t - \theta) dy. \tag{3.23}
\]

\[
M_{g,v}^{(k)}(0) = 1.
\]

Proof. See the appendix. □

Equations (3.22) and (3.23) are of recursive type and therefore can recursively be solved as discussed later in the subsection on computational methods. The only unknown parameters of these equations are $M_{g,v}^{(k,w)}(t, s)$ and $M_{g,v}^{(k)}(t)$.

Equation (3.22) is divided in three different parts, the first corresponds to the event that no transition in the growth dividend process is made, the second the event that next transition is made at $\theta \in \{t + 1, \ldots, s\}$, the third term includes the event that next transition of the growth dividend process is made at $\theta \in \{1, \ldots, t\}$.

Moreover notice that the solution of equation (3.22) necessitates before the resolution of equation (3.23) because the second term of (3.22) contains the factor $M_{g,v}^{(w)}(s - \theta)$ which is the unknown of equation (3.23).

Proposition 3.3 and Corollary 3.4 provide the following representations of prices and risks:

\[
p(d, g, v) = \sum_{t=1}^{+\infty} \frac{M_{g,v}^{(1)}(t)}{r^t} d. \tag{3.24}
\]
\[ p^2(d, g, v) = \sum_{t=1}^{+\infty} \frac{M_{g,v}(t)}{r^{2t}} d^2 + 2 \sum_{t=1}^{+\infty} \sum_{s>t} \frac{M_{g,v}(t,s)}{r^{t+s}} d^2. \] (3.25)

Major questions are about the convergence of these series and on computational methods.

### 3.2 Finiteness of prices and risks of the semi-Markov stock model

Let us consider now the question of the convergence of the series (3.24) and (3.25).

To this purpose, let denote by

\[ \bar{g}(v) = \sup_{y \in E} \left( \frac{1 - H(y, v + 1)}{1 - H(y, v)} + \int_{E} \frac{\dot{q}(y, x, v + 1)}{1 - H(y, v)} dx \right), \]

and by

\[ \bar{g} = \sup_{v \in \mathbb{N}} \bar{g}(v). \]

with the convention \( \frac{0}{0} = 0. \)

All the work in the rest of the article will be done under the assumption:

\[ A1 : \quad \bar{g} < r. \] (3.26)

**Theorem 3.5.** (Finiteness of prices) Under assumption A1 it results that:

i) the series \( p(d, g, v) = \sum_{t=1}^{+\infty} \frac{\mathbb{E}_{(d,g,v)}[D(t)]}{r^{t}} \) converges

ii) it meets the asymptotic condition

\[ \lim_{t \to +\infty} \frac{\mathbb{E}_{(d,g,v)}[P(D(t), G(t), B(t))]}{r^{t}} = 0. \] (3.27)

**Proof.** See the appendix. □
This theorem is the generalization of Theorem 2 in D’Amico (2013) where the dividend process was described by a finite state space semi-Markov chain. The result can be further generalized to cover the second order moment of the price process if an additional assumption, namely A2, is formulated:

\[ A2 : \quad \overline{g}^{(2)} < r^2, \quad (3.28) \]

where \( \overline{g}^{(2)} = \sup_{v \in \mathbb{N}} \overline{g}^{(2)}(v) \) and \( \overline{g}^{(2)}(v) = \sup_{y \in E} \left( y^2 \frac{1-H(y,v+1)}{1-H(y,v)} + \int_E x^2 \frac{2(y,x,v+1)}{1-H(y,v)} \, dx \right) \).

**Theorem 3.6.** (Finiteness of risks) Under assumptions A1 and A2 the series

\[
p^2(d, g, v) = \sum_{i=1}^{+\infty} \frac{\mathbb{E}_{(d,g,v)}[D^2(i)]}{r^{2i}} + 2 \sum_{i=1}^{+\infty} \sum_{j>i} \frac{\mathbb{E}_{(d,g,v)}[D(i)D(j)]}{r^{i+j}},
\]

converges and meets the asymptotic conditions:

\[
\lim_{N \to +\infty} \frac{\mathbb{E}_{(d,g,v)}[P^2(N)]}{r^{2N}} = 0,
\quad (3.29)
\]

\[
\lim_{N \to +\infty} \sum_{i=1}^{N} \frac{\mathbb{E}_{(d,g,v)}[D(i)P(N)]}{r^{i+N}} = 0.
\quad (3.30)
\]

**Proof.** See the appendix. \(\square\)

These two theorems present the assumptions under which the transversality conditions (3.27), (3.29), (3.30) are satisfied. This avoids the presence of speculative bubbles and therefore permits the representation of prices and risks as series that are shown to be convergent and that depend only on the fundamental variables (the dividend process).

Assumption A2 is necessary in determining the finiteness of risks because it controls the second order moment of the growth dividend process. The only assumption A1 is unable to guarantee the finiteness of both prices and risks because
it does not imply the A2. To fix this point we can consider a clarifying example where the dividend growth process obeys a two state discrete time Markov chain with transition probability matrix

\[
P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix},
\]

and state space \( E = \{1, 1.5\} \). Set \( g = [1, 1.5]' \), \( r = 1.41 \) and compute \( P \cdot g = [1.2, 1.4]' \). Then \( \bar{g} = 1.4 \) and the assumption A1 is satisfied. Anyway \( g^2 = [1, 2.25]' \) and \( P \cdot g^2 = [1.5, 2]' \) which implies \( \bar{g}^{(2)} = 2 \). A simple comparison shows that although A1 is satisfied, A2 is not.

### 3.3 Computational methods

In this subsection we propose two methods to compute the fundamental price and the fundamental risk as represented by formulas (3.24) and (3.25).

The first method is a direct computation of (3.24) and (3.25) based on Proposition (3.3) and Corollary (3.4) based on numerical approximation of the considered integral equations. This method was originally considered by Janssen and Manca (2004) for computing the transition probability function of a continuous time finite state space semi-Markov process. Here we modify it to be useful in evaluating the moments of the (product) growth dividend process.

Let us consider before the equation (3.23). As already remarked, this is a recursive equation which can be solved as explained here below.

Consider a state space grid with discretization step \( h \):

\[
\omega = \{x(i) = ih\} \tag{3.31}
\]
where \( i = d, d + 1, \ldots, N \), \( dh = \inf E \), \( Nh = \sup E \).

Now consider a generic quadrature formula to approximate the integral on the grid \( \omega \) to be applied \( \forall v \in N, \forall t \in N \):

\[
\int_E \frac{\dot{q}(g, y, \theta + v)}{1 - H(g, v)} (g)^{k(\theta-1)} y^k M_{y,0}^{(k)}(t-\theta) dy \\
\approx \sum_{l=d}^{N} w_{dN}(l) (ah)^{k(\theta-1)} (lh)^k M_{h,0}^{(k)}(t-\theta) \frac{\dot{q}(ah, lh, \theta + v)}{1 - H(ah, v)},
\]

where \( ah \) is the approximate value of \( g \) on the grid \( 3.31 \), \( \dot{q}(ah, lh, \theta + v) \) is the discrete derivative of \( q \) and \( w_{dN}(\cdot) \) are the weights relative to the quadrature formula.

Therefore by substitution we can approximate the equation \( 3.23 \) with the discrete equation:

\[
M_{ah,v}^{(k)}(t) = \frac{1 - H(ah, v + t)}{1 - H(ah, v)} (ah)^{kt} \\
+ \sum_{\theta=1}^{t} \sum_{l=d}^{N} w_{dN}(l) (ah)^{k(\theta-1)} (lh)^k M_{h,0}^{(k)}(t-\theta) \frac{\dot{q}(ah, lh, \theta + v)}{1 - H(ah, v)}.
\]

The top system of equations can also be written more compact using matrix notation. Let \( M_v^{(k)}(t) \) be the \( |\omega| \times 1 \) column vector of elements

\[
M_v^{(k)}(t) = [M_{ah,v}^{(k)}(t), M_{(d+1)h,v}^{(k)}(t), \ldots, M_{Nh,v}^{(k)}(t)]^t,
\]

which stores the discretized values of the \( k \)-order moment of the growth dividend process. The quantity \( |\omega| \) denotes the cardinality of the grid \( 3.31 \).
Let $D_v(t)$ be the diagonal matrix of dimension $|\omega| \times |\omega|$ with elements:

$$D_v(t) = \begin{bmatrix}
\frac{1-H(dh,v+t)}{1-H(dh,v)} & 0 & \ldots & 0 \\
0 & \frac{1-H((d+1)h,v+t)}{1-H((d+1)h,v)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1-H(Nh,v+t)}{1-H(Nh,v)}
\end{bmatrix}.$$ 

Define by $A^{(k)}(t)$ the diagonal matrix with elements:

$$A^{(k)}(t) = \begin{bmatrix}
(dh)^{kt} & 0 & \ldots & 0 \\
0 & ((d + 1)h)^{kt} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & (Nh)^{kt}
\end{bmatrix},$$

and by $B_v(t)$ the $|\omega| \times |\omega|$ matrix of generic element $B_v(t) = \left( w_{dN}(l) \frac{d l}{n, l, t + v}{1-H(n, v)} \right)_{ab, l h \in \omega}.$

Then the recursive equations (3.32) can be written in our new matrix-notation as,

$$M_v^{(k)}(t) = D_v(t) \cdot A^{(k)}(t) \cdot 1_{|\omega|}$$

$$+ \sum_{\theta=1}^{t} (B_v(\theta) \Join A^{(k)}(t)) \cdot ((A^{(k)}(1) \cdot 1_{|\omega|}) \Join M_v^{(k)}(t - \theta)),$$

where $1_{|\omega|}$ is the $|\omega| \times 1$ vector with all components equal 1, the symbol $\cdot$ is the ordinary row by column matrix product and $\Join$ is the Hadamard or element-wise multiplication of matrices.

The solution of (3.34) is straightforward because for $t = 0$ we know from Corollary (3.4) that $M_v^{(k)}(0) = 1_{|\omega|}$ $\forall v \in \mathbb{N}$ and $\forall k \in \mathbb{N}$. Then set $t = 1$ to have
\[ \forall v \in \mathbb{N} \]

\[
M_v^{(k)}(1) = D_v(1) \cdot A^{(k)}(1) \cdot 1_{|\omega|} + (B_v(1) \triangle A^{(k)}(1)) \cdot (A^{(k)}(1) \cdot 1_{|\omega|}). \tag{3.35}
\]

All the terms on the right hand side are known and then it is possible to compute \(M_v^{(k)}(1)\). Once we know \(M_v^{(k)}(1)\) we can proceed to compute \(M_v^{(k)}(2)\) and in general knowing \(M_v^{(k)}(0), M_v^{(k)}(1),..., M_v^{(k)}(t - 1)\) it is possible to compute \(M_v^{(k)}(t)\).

Similar arguments can be used to solve equation (3.22). Once the equations (3.23) and (3.22) are solved it is possible to evaluate the prices and the risks through formulas (3.24) and (3.25), respectively. Anyway here below we do not report all the necessary computation because we propose an alternative procedure for computing prices and risks since the direct computation is not the most convenient way to proceed. The above valuation formulas (3.24) and (3.25) can be conveniently represented by introducing two auxiliary functions.

**Definition 3.7** (price-dividend ratio). For all \(g \in E, v \in \mathbb{N}\) the price-dividend ratio is the function defined by:

\[\psi_1(v, g) = \sum_{t=1}^{+\infty} \frac{M_{g,v}^{(1)}(t)}{r^t}. \tag{3.36}\]

Hence, according to equation (3.24), the former definition provides a compact representation of the price:

\[p(d, g, v) = \psi_1(v, g)d. \tag{3.37}\]

**Definition 3.8** (second order price-dividend ratio). For all \(g \in E, v \in \mathbb{N}\) the
The second order price-dividend ratio is the function defined by:

$$\psi_2(v, g) = \left( \sum_{t=1}^{+\infty} \frac{M_{g,v}^{(2)}(t)}{r^{2t}} + 2 \sum_{t=1}^{+\infty} \sum_{s>t} \frac{M_{g,v}^{(2,1)}(t,s)}{r^{t+s}} \right).$$  \hspace{1cm} (3.38)

Hence, according to equation (3.25), the former definition provides a compact representation of the risk:

$$p^2(d, g, v) = \psi_2(v, g)d^2.$$  \hspace{1cm} (3.39)

**Proposition 3.9** (price-dividend ratio equation). Under assumption A1 the price dividend ratio satisfies for all $g \in E$ and $v \in \mathbb{N}$ the following system of equations:

$$1 - \frac{H(g, v + 1)}{1 - H(g, v)} g\psi_1(v + 1, g) - r\psi_1(v, g)$$

$$+ \int_E \psi_1(0, x)x^2 \frac{\dot{q}(g, x, v + 1)}{1 - H(g, v)} dx = -E(g, v).$$  \hspace{1cm} (3.40)

where $E(g, v) = \mathbb{E}_{(d, g, v)}[G(1)].$

**Proof.** See the appendix. \hspace{1cm} $\Box$

**Proposition 3.10** (second order price-dividend ratio equation). Under assumptions A1 and A2 the second order price-dividend ratio satisfies for all $g \in E$ and $v \in \mathbb{N}$ the following system of equations:

$$1 - \frac{H(g, v + 1)}{1 - H(g, v)} g^2\psi_2(v + 1, g) - r^2\psi_2(v, g)$$

$$+ \int_E \psi_2(0, x)x^2 \frac{\dot{q}(g, x, v + 1)}{1 - H(g, v)} dx + \frac{1 - H(g, v + 1)}{1 - H(g, v)} g^2\psi_1(v + 1, g)$$

$$+ \int_E \psi_1(0, x)x^2 \frac{\dot{q}(g, x, v + 1)}{1 - H(g, v)} dx = -E^2(g, v).$$  \hspace{1cm} (3.41)

where $E^2(g, v) = \mathbb{E}_{(d, g, v)}[G^2(1)].$
**Proof.** See the appendix.

The above propositions can be used to compute the fundamental price and the fundamental risk. We describe the methodology only for the price because similar considerations hold also for the risk.

As a first step it is necessary to evaluate

\[ \psi_1(0, x) = \sum_{t=1}^{+\infty} \frac{M_{x,0}(t)}{r^t}, \]

this can be done by using the truncated series development. This is possible because we proved that the series \( \sum_{t=0}^{+\infty} \frac{M_{x,0}(t)}{r^t} \) is convergent and then in principle we can control the error of approximation. Therefore, fixed an error \( \epsilon > 0 \) we can find an integer \( T \) such that

\[ \left| \sum_{t=1}^{+\infty} \frac{M_{x,0}(t)}{r^t} - \sum_{t=1}^{T} \frac{M_{x,0}(t)}{r^t} \right| < \epsilon. \]

The computation of \( \sum_{t=1}^{T} \frac{M_{x,0}(t)}{r^t} \) requires the solution of (3.34) only for \( v = 0 \).

As a second step we approximate the integral

\[ \int_E \psi_1(0, x) x \frac{\dot{g}(g, x, v + 1)}{1 - H(g, v)} dx \]

inside equation (3.40) by replacing to \( \psi_1(0, x) \) its truncation approximation. This can be accomplished as follows:

\[ \int_E \psi_1(0, x) x \frac{\dot{g}(g, x, v + 1)}{1 - H(g, v)} dx \approx \sum_{t=1}^{T} \int_E \frac{M_{x,0}(t)}{r^t} x \frac{\dot{g}(g, x, v + 1)}{1 - H(g, v)} dx \]

\[ \approx \sum_{t=1}^{T} \sum_{l=0}^{k} w(l) \frac{M_{l,0}(t)}{r^t} \cdot lh \cdot \frac{\dot{g}(g, l, v + 1)}{1 - H(g, v)} =: K(g, v), \]

(3.42)

where the last approximation is obtained by using a general quadrature formula after discretization of the phase space \( E \) with a discretization step length \( h \). It is important to note that the quadrature formula is applied here to evaluate a quantity \( K(g, v) \) that is independent of \( t \). On the contrary, the direct method requires the computation of the quadrature formula for all times \( t \).
The third step consists in substituting the quantity $K(g,v)$ inside equation (3.40) and to rearrange terms to obtain the following first order difference equation:

$$\psi_1(v + 1, g) = z(v, g, r)\psi_1(v, g) + \gamma(v, g, r),$$

(3.43)

where

$$z(v, g, r) = \left[ \frac{r(1 - H(g, v))}{g(1 - H(g, v + 1))} \right],$$

$$\gamma(v, g, r) = \frac{(K(g, v) + E(g, v))(1 - H(g, v))}{g(1 - H(g, v + 1))}.$$

The solution of the inhomogeneous first order linear difference equation (3.43) is given by

$$\psi_1(v + 1, g) = \left( \prod_{j=0}^{v} z(j, g, r) \right) \left( \psi_1(0, g) + \sum_{k=0}^{v} \frac{\gamma(k, g, r)}{\prod_{j=0}^{v} z(j, g, r)} \right).$$

Formula (3.37) provides the price-dividend ratio for all possible states $g$ and backward values $v$ and therefore gives the possibility to recover prices simply through a multiplication with the value of the dividend process:

$$p(d, g, v) = \left( \prod_{j=0}^{v} z(j, g, r) \right) \left( \psi_1(0, g) + \sum_{k=0}^{v} \frac{\gamma(k, g, r)}{\prod_{j=0}^{v} z(j, g, r)} \right) d.$$

4 Conclusions

Many studies deal with pricing firms on the basis of fundamentals. This paper shows how to compute prices and risks when the dividend dynamic is driven by a discrete time semi-Markov chain with general state space. Transversality conditions that avoids bubbles in the market and that guarantees the finiteness
of price and risk are established. Moreover different computational methods are supplied for the implementation of the model.

The proposed model encompasses the majority of the discounted dividend valuation models and therefore furnishes very general results in this field.

Possible avenues for future developments of our model include:

a) the application of the model to real dividend data which necessitates the developing of techniques for the estimation of the model parameters;

b) Estimation of the required rate of return by adopting an appropriate stochastic model.

Appendix: proofs

Proof of Proposition (3.3)

Let us assume, without loss of generality, that \( N(0) = 0 \). Then, because the event \( \{ T_1 > s \} \), \( \{ T_1 \in (t, s] \} \) and \( \{ T_1 \leq t \} \) are disjoint it results that:

\[
M_{g,v}^{(k,w)}(t, s) = \mathbb{E}[AP_{g,v}^{(k,w)}(t, s)1_{\{T_{N(0)+1} > s\}}] + \mathbb{E}[AP_{g,v}^{(k,w)}(t, s)1_{\{t < T_{N(0)+1} \leq s\}}] + \mathbb{E}[AP_{g,v}^{(k,w)}(t, s)1_{\{T_{N(0)+1} \leq t\}}].
\]  

(4.1)

Let us evaluate the three expectations above.

In the case \( T_{N(0)+1} > s \) there will be any transition up to time \( s \) and therefore relation (3.20) modifies as follows:

\[
AP_{g,v}^{(k,w)}(t, s) = A_{g,v}^{(k)}(t) \cdot A_{g(t),B(t)}^{(w)}(s) = \prod_{i=1}^{t} g^k \cdot \prod_{j=t+1}^{s} g^w = g^{w_s+t(k-w)}.
\]  

(4.2)
The event $T_{N(0)+1} > s$ occurs with probability

$$
P[T_{N(0)+1} > s \mid J_{N(0)} = g, T_{N(0)} = -v, T_{N(0)+1} > 0] = \frac{P[T_{N(0)+1} > s \mid J_{N(0)} = g, T_{N(0)} = -v]}{P[T_{N(0)+1} > 0 \mid J_{N(0)} = g, T_{N(0)} = -v]} = \frac{1 - H(g, s + v)}{1 - H(g, v)}. \quad (4.3)
$$

Then it results that

$$
\mathbb{E}[AP_{g,v}^{(k,w)}(t, s)1\{T_{N(0)+1} > s\}] = \mathbb{E}[A_{K,v}^{(k)}(t)A_{G(t),B(t)}^{(w)}(s)1\{T_{N(0)+1} > s\}] = \frac{1 - H(g, s + v)}{1 - H(g, v)} g^{ws + t(k - w)}. \quad (4.4)
$$

In the case when $T_{N(0)+1} \in (t, s]$ since $AP_{g,v}^{(k,w)}(t, s) = A_{g,v}^{(k)}(t) \cdot A_{G(t),B(t)}^{(w)}(s)$, and because any transition occurs until time $t$, we have

$$
A_{g,v}^{(k)}(t) = \prod_{j=1}^{t} G^{k}(j) = \prod_{j=1}^{t} g^{k} = g^{kt}. \quad (4.5)
$$

To evaluate $A_{G(t),B(t)}^{(w)}(s)$ consider also the state occupied with next transition, say $J_{N(0)+1}$. Thus

$$
A_{G(t),B(t)}^{(w)}(s) = \prod_{j=t+1}^{s} G^{w}(j) = \prod_{j=t+1}^{T_{N(0)+1} - 1} G^{w}(j) \cdot J_{N(0)+1}^{w} \cdot A_{J_{N(0)+1},0}^{(w)}(s - T_{N(0)+1}) = g^{w(T_{N(0)+1} - t - 1)} \cdot J_{N(0)+1}^{w} \cdot A_{J_{N(0)+1},0}^{(w)}(s - T_{N(0)+1)}. \quad (4.6)
$$

Multiply (4.5) and (4.6) to obtain:

$$
AP_{g,v}^{(k,w)}(t, s) = g^{ws + t(k - w) - w} \cdot J_{N(0)+1}^{w} \cdot A_{J_{N(0)+1},0}^{(w)}(s - T_{N(0)+1}). \quad (4.7)
$$
The expectation of the top process can be computed once the probability of the conditional event

$$\{J_{N(0)+1} = y, T_{N(0)+1} = \theta \in \{t + 1, \ldots, s\} \mid J_{N(0)} = g, T_{N(0)} = -v, T_{N(0)} > 0\}$$

is known. Thus,

$$\mathbb{P}[J_{N(0)+1} \in (y, y + dy), T_{N(0)+1} = \theta \in \{t + 1, \ldots, s\} \mid J_{N(0)} = g, T_{N(0)} = -v, T_{N(0)+1} > 0]$$

$$= \mathbb{P}[J_{N(0)+1} \in (y, y + dy), T_{N(0)+1} = \theta, T_{N(0)+1} > 0 \mid J_{N(0)} = g, T_{N(0)} = -v]$$

$$= \frac{\mathbb{P}[T_{N(0)+1} > 0, T_{N(0)} = -v \mid J_{N(0)} = g]}{\mathbb{P}[T_{N(0)+1} - T_{N(0)} > v \mid J_{N(0)} = g]}$$

$$= \frac{\dot{q}(g, y, \theta + v)dy}{1 - H(g, v)}.$$  \hspace{1cm} (4.8)

Notice that the random variable $A_{J_{N(0)+1}, \theta}^{(w)}(s - T_{N(0)+1})$ is independent of the joint random variable $(J_{N(0)+1}, T_{N(0)+1})$ because the product process $A_{J_{N(0)+1}, \theta}^{(w)}(s - T_{N(0)+1})$ has the Markov property at transition times and consequently once the state $J_{N(0)+1}$ and the time $T_{N(0)+1}$ are known, its behavior does not depend on the distribution of $(J_{N(0)+1}, T_{N(0)+1})$. Then, by taking the expectation of (4.7) we get:

$$\mathbb{E}[AP_{g,v}^{(k,w)}(t, s)1_{\{t < T_{N(0)+1} \leq s\}}] = \sum_{\theta=t+1}^{s} \int_{E} \frac{\dot{q}(g, y, \theta + v)}{1 - H(g, v)} g^{w\theta+(k-w)-w}y^wM_{y,0}^{(w)}(s-\theta)dy.$$  \hspace{1cm} (4.9)
The final case is when $T_{N(0)+1} < t$. The product growth dividend process becomes:

$$AP_{g,v}^{(k,w)}(t, s) = A_{g,v}^{(k)}(t) \cdot A_{G(t),B(t)}^{(w)}(s)$$

$$= \prod_{j=1}^{T_{N(0)+1}-1} g^k \cdot J_{N(0)+1}^k \cdot A_{J_{N(0)+1},0}^{(k)}(t - T_{N(0)+1}) \cdot A_{G(t-T_{N(0)+1}),B(t-T_{N(0)+1})}^{(w)}(s - T_{N(0)+1})$$

$$= g^{k(T_{N(0)+1}-1)} \cdot J_{N(0)+1}^k \cdot AP_{J_{N(0)+1},0}^{(k,w)}(t - T_{N(0)+1}, s - T_{N(0)+1}). \quad (4.10)$$

The expectation of (4.10) can be computed considering the probability of the conditional event

$$\{J_{N(0)+1} \in (y, y + dy), T_{N(0)+1} = \theta \leq t \mid J_{N(0)} = g, T_{N(0)} = -v, T_{N(0)} > 0\}$$

that coincides with $\hat{q}(g,y,\theta+v)dy$ and the fact that the random variable $AP_{J_{N(0)+1},0}^{(k,w)}(t - T_{N(0)+1}, s - T_{N(0)+1})$ is independent of the joint random variable $(J_{N(0)+1}, T_{N(0)+1})$ and therefore,

$$\mathbb{E}[A_{g,v}^{(k,w)}(t, s)1_{\{T_{N(0)+1}\leq t\}}] = \sum_{\theta=1}^{t} \int_E \frac{\hat{q}(g,y,\theta+v)}{1 - H(g,v)} (g)^{k(\theta-1)} y^k M_{y,0}^{(k,w)}(t - \theta, s - \theta) dy. \quad (4.11)$$

A substitution of (4.4), (4.9) and (4.11) into (4.1) concludes the proof.

Proof of Corollary (3.1)

By convention, for any sequence $\{a(t)\}_{t \in \mathbb{N}}$ we have that $\prod_{j=t+1}^{t} a(t) = 1$.

The product growth dividend process for $k \in \{1, 2\}$, $w = 1$ and $s = t$ is

$$AP_{g,v}^{(k,w)}(t, s) = A_{g,v}^{(k)}(t) \cdot A_{G(t),B(t)}^{(w)}(s)$$

$$= \prod_{j=1}^{T_{N(0)+1}-1} g^k \cdot J_{N(0)+1}^k \cdot A_{J_{N(0)+1},0}^{(k)}(t - T_{N(0)+1}) \cdot A_{G(t-T_{N(0)+1}),B(t-T_{N(0)+1})}^{(w)}(s - T_{N(0)+1})$$

$$= g^{k(T_{N(0)+1}-1)} \cdot J_{N(0)+1}^k \cdot AP_{J_{N(0)+1},0}^{(k,w)}(t - T_{N(0)+1}, s - T_{N(0)+1}). \quad (4.10)$$

The expectation of (4.10) can be computed considering the probability of the conditional event

$$\{J_{N(0)+1} \in (y, y + dy), T_{N(0)+1} = \theta \leq t \mid J_{N(0)} = g, T_{N(0)} = -v, T_{N(0)} > 0\}$$

that coincides with $\hat{q}(g,y,\theta+v)dy$ and the fact that the random variable $AP_{J_{N(0)+1},0}^{(k,w)}(t - T_{N(0)+1}, s - T_{N(0)+1})$ is independent of the joint random variable $(J_{N(0)+1}, T_{N(0)+1})$ and therefore,

$$\mathbb{E}[A_{g,v}^{(k,w)}(t, s)1_{\{T_{N(0)+1}\leq t\}}] = \sum_{\theta=1}^{t} \int_E \frac{\hat{q}(g,y,\theta+v)}{1 - H(g,v)} (g)^{k(\theta-1)} y^k M_{y,0}^{(k,w)}(t - \theta, s - \theta) dy. \quad (4.11)$$

A substitution of (4.4), (4.9) and (4.11) into (4.1) concludes the proof.

□

Proof of Corollary (3.1)

By convention, for any sequence $\{a(t)\}_{t \in \mathbb{N}}$ we have that $\prod_{j=t+1}^{t} a(t) = 1$.

The product growth dividend process for $k \in \{1, 2\}$, $w = 1$ and $s = t$ is
\[ AP^{(k,1)}_{g,v}(t,t) = A^{(k)}_{g,v}(t) \cdot A^{(1)}_{G(t),B(t)}(t). \]

According to definition (3.1) we have
\[
A^{(1)}_{G(t),B(t)}(t; \omega) = \begin{cases} 
\prod_{j=t+1}^{t} G(j; \omega) = 1 & \text{if } \omega \in \Omega_{G(t),B(t)}, \\
0 & \text{otherwise,}
\end{cases} \tag{4.12}
\]

Then \( AP^{(k,1)}_{g,v}(t,t) = A^{(k)}_{g,v}(t) \) and \( \mathbb{E}[AP^{(k,1)}_{g,v}(t,t)] = \mathbb{E}[A^{(k)}_{g,v}(t)] \). Therefore to get the result it is sufficient to take the equation (3.22) and to set \( w = 1 \) and \( s = t \).

\[ \square \]

**Proof of Theorem (3.5)**

The proof can be accomplished by using similar arguments as those of Theorem (3.6) and therefore is omitted.

\[ \square \]

**Proof of Theorem (3.6)**

In formula (3.14) we established that
\[
p^2(d, g, v) = \sum_{i=1}^{\infty} \left( \frac{\mathbb{E}_{(d,g,v)}[\prod_{j=1}^{t} G^2(j)]}{r^{2t}} \right) d^2 \\
+ 2 \sum_{t=1}^{\infty} \sum_{w>t} \left( \frac{\mathbb{E}_{(d,g,v)}[\prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j)]}{r^{t+w}} \right) d^2. \tag{4.13}
\]

Let us consider the term
\[
\mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j) \right] \tag{4.14}
\]

\[
= \mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w-1} G(j) \mathbb{E}_{(D(w-1),G(w-1),B(w-1))}[G(w)] \right] \\
= \mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w-1} G(j) M_{G(w-1),B(w-1)}^{(1)} \right].
\]
From Corollary 3.4 we know that

\[
M_{G(w-1),B(w-1)}^{(1)}(1) = \left( \frac{1 - H(G(w-1), B(w-1) + 1)}{1 - H(G(w-1), B(w-1))} \right) (G(w-1))
+ \int_E \frac{\dot{q}(G(w-1), y, B(w-1) + 1)}{1 - H(G(w-1), B(w-1))} y \, dy,
\]

then from assumption (A1) we obtain that \(M_{G(w-1),B(w-1)}^{(1)}(1) < \bar{g}\). As a direct consequence, by iteration we obtain:

\[
M_{g,v}^{(1)}(t) < (\bar{g})^t, \tag{4.15}
\]

and therefore

\[
\mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j) \right] \leq \mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \right] (\bar{g})^{w-t}. \tag{4.16}
\]

Working analogously on the terms \(G^2(j), j = 1, \ldots, t\) we have:

\[
\mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j) \right] \leq \mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t-1} G^2(j) \mathbb{E}_{(D(t-1),G(t-1),B(t-1))} [G^2(t)] \right] (\bar{g})^{w-t}
= \mathbb{E}_{(d,g,v)} \left[ \prod_{j=1}^{t-1} G^2(j) M_{G(t-1),B(t-1)}^{(2)}(1) \right] (\bar{g})^{w-t}. \tag{4.17}
\]

From Corollary 3.4 we know that

\[
M_{G(t-1),B(t-1)}^{(2)}(1) = \left( \frac{1 - H(G(t-1), B(t-1) + 1)}{1 - H(G(t-1), B(t-1))} \right) (G(t-1))^2
+ \int_E \frac{\dot{q}(G(t-1), y, B(t-1) + 1)}{1 - H(G(t-1), B(t-1))} y^2 \, dy,
\]
then from assumption (A2) we obtain that \( M_{G(t-1),B(t-1)}^{(2)}(1) < \bar{g}^{(2)}. \) As a direct consequence, by iteration we obtain

\[
M_{g,v}^{(2)}(t) < (\bar{g}^{(2)})^t, \tag{4.18}
\]

and in turn we get

\[
E_{(d,g,v)} \left[ \prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j) \right] \leq (\bar{g}^{(2)})^t \cdot (\bar{g})^{w-t}. \tag{4.19}
\]

The inequalities (4.19), (4.15), (4.18) together with the assumptions A1 and A2 imply the finiteness of prices:

\[
p^2(d, g, v) = \sum_{i=1}^{+\infty} \left( \frac{E_{(d,g,v)}[\prod_{j=1}^{t} G^2(j)]}{r^{2t}} \right) d^2
\]

\[
+ 2 \sum_{t=1}^{+\infty} \sum_{w>t} \left( \frac{E_{(d,g,v)}[\prod_{j=1}^{t} G^2(j) \prod_{j=t+1}^{w} G(j)]}{r^{t+w}} \right) d^2 \tag{4.20}
\]

\[
\leq \sum_{i=1}^{+\infty} \left( \frac{(\bar{g}^{(2)})^t}{r^{2t}} \right) d^2 + 2 \sum_{t=1}^{+\infty} \sum_{w>t} \left( \frac{(\bar{g}^{(2)})^t (\bar{g})^{w-t}}{r^{t+w}} \right) d^2 \leq +\infty.
\]

Now let us prove the asymptotic condition (3.29). To this end we define

\[
\bar{\psi}_2 = \sup_{v \in \mathbb{N}} \sup_{y \in E} \psi_2(v, y)
\]

\[
= \sup_{v \in \mathbb{N}} \sup_{y \in E} \left( \sum_{t=1}^{+\infty} \frac{M_{g,v}^{(2)}(t)}{r^{2t}} + 2 \sum_{t=1}^{+\infty} \sum_{s>0} \frac{M_{g,v}^{(2,1)}(t, s)}{r^{2t+s}} \right), \tag{4.21}
\]
then it results that

\[ 0 \leq \mathbb{E}_{(d,g,v)}[P^2(D(N), G(N), B(N))] \]

\[ = \mathbb{E}_{(d,g,v)} \left[ \sum_{s=N+1}^{+\infty} \frac{M_{G(t),B(t)}^{(2)}(s)}{r^{2s}} D^2(N) + 2 \sum_{s=N+1}^{+\infty} \sum_{w>s} M_{G(t),B(t)}^{(2,1)}(N,w) \frac{D^2(N)}{r^{N+w}} \right] \]

\[ \leq \psi_2 \mathbb{E}_{(d,g,v)}[D^2(N)]. \]

(4.22)

Consequently we have

\[ \mathbb{E}_{(d,g,v)} \left[ \frac{P^2(D(N), G(N), B(N))}{r^{-2N}} \right] \leq \psi_2 \mathbb{E}_{(d,g,v)} \left[ \frac{D^2(N)}{r^{-2N}} \right] = \psi_2 \frac{M_{(g,v)}^{(2)}(N)}{r^{2N}}. \]  

(4.23)

Now take the limit of (4.23) as \( N \to +\infty \) and observe that (4.18) implies that

\[ \lim_{N \to +\infty} \frac{M_{(g,v)}^{(2)}(N)}{r^{2N}} = 0, \]

thus

\[ \lim_{N \to +\infty} \frac{\mathbb{E}_{(d,g,v)}[P^2(D(N), G(N), B(N))]}{r^{-2N}} = 0. \]  

(4.24)

It remains to prove the validity of the asymptotic condition (3.30). Applying the Cauchy-Schwartz inequality we have

\[ \lim_{N \to +\infty} \sum_{t=1}^{N} \frac{\mathbb{E}_{(d,g,v)}[D(t)P(N)]}{r^{t+N}} \]

\[ \leq \lim_{N \to +\infty} \sum_{t=1}^{N} \left( \frac{\mathbb{E}_{(d,g,v)}[D^2(t)]}{r^{2t}} \right)^{\frac{1}{2}} \cdot \left( \frac{\mathbb{E}_{(d,g,v)}[P^2(N)]}{r^{2N}} \right)^{\frac{1}{2}} \]

\[ = \lim_{N \to +\infty} \left( \frac{\mathbb{E}_{(d,g,v)}[P^2(N)]}{r^{2N}} \right)^{\frac{1}{2}} \cdot \lim_{N \to +\infty} \sum_{t=1}^{N} \left( \frac{\mathbb{E}_{(d,g,v)}[D^2(t)]}{r^{2t}} \right)^{\frac{1}{2}}. \]

(4.25)

Now it is sufficient to note that

\[ \lim_{N \to +\infty} \left( \frac{\mathbb{E}_{(d,g,v)}[P^2(N)]}{r^{2N}} \right)^{\frac{1}{2}} = 0, \]  

(4.26)
from (4.24) and that due to the finiteness of \( p^2(d, g, v) \) we have that

\[
\lim_{N \to +\infty} \sum_{t=1}^{N} \left( \frac{\mathbb{E}_{(d,g,v)}[D^2(t)]}{r^{2t}} \right)^{\frac{1}{2}} < +\infty.
\] (4.27)

Formulas (4.26) and (4.27) imply

\[
\lim_{N \to +\infty} \sum_{t=1}^{N} \frac{\mathbb{E}_{(d,g,v)}[D(t)P(N)]}{r^{t+N}} = 0.
\] (4.28)

\[\Box\]

**Proof of Proposition 3.9**

The proof can be accomplished by using similar arguments as those of Proposition (3.10) and therefore is omitted.

\[\Box\]

**Proof of Proposition 3.10**

Formula (3.39) implies that

\[
\psi_2(g, v) = \frac{p^2(d, g, v)}{d^2}.
\]

We have also seen that the second order moment of the price process at current time \( k = 0 \) is given by

\[
p^2(d, g, v) = \mathbb{E}_{(d,g,v)} \left[ \left( \frac{D(1) + P(1)}{r} \right)^2 \right] \\
\mathbb{E}_{(d,g,v)} \left[ \frac{D^2(1)}{r^2} \right] + \mathbb{E}_{(d,g,v)} \left[ \frac{P^2(1)}{r^2} \right] + 2 \mathbb{E}_{(d,g,v)} \left[ \frac{D(1) + P(1)}{r^2} \right].
\] (4.29)
Now let us compute these three expectations.

\[
\mathbb{E}_{(d,g,v)} \left[ \frac{D^2(1)}{r^2} \right] = \frac{1}{r^2} \mathbb{E}_{(d,g,v)}[G^2(1)D^2(0)] = \frac{d^2}{r^2} \mathbb{E}_{(d,g,v)}[G^2(1)]
\]

\[
= \frac{d^2}{r^2} \left( \frac{1 - H(g,v+1)}{1 - H(g,v)} \right) g^2 + \int_E x^2 \frac{\dot{q}(g,x,v+1)}{1 - H(g,v)} dx, \quad (4.30)
\]

\[
\mathbb{E}_{(d,g,v)} \left[ \frac{P^2(1)}{r^2} \right] = \frac{1}{r^2} \mathbb{E}_{(d,g,v)}[\psi_2(G(1),B(1))D^2(1)] = \frac{d^2}{r^2} \mathbb{E}_{(d,g,v)}[\psi_2(G(1),B(1))G^2(1)]
\]

\[
= \frac{d^2}{r^2} \left( \frac{1 - H(g,v+1)}{1 - H(g,v)} \right) \psi_2(g,v+1)g^2 + \int_E \psi_2(x,0)x^2 \frac{\dot{q}(g,x,v+1)}{1 - H(g,v)} dx. \quad (4.31)
\]

It remains to compute the third expectation:

\[
\frac{2d}{r^2} \mathbb{E}_{(d,g,v)}[D(1)P(1)] = \frac{2d}{r^2} \mathbb{E}_{(d,g,v)}[G(1)D(0)P(1)] = \frac{2d}{r^2} \mathbb{E}_{(d,g,v)}[G(1)P(1)]
\]

\[
= \frac{2d}{r^2} \mathbb{E}_{(d,g,v)}[G(1)\psi_1(G(1),B(1))D(1)] = \frac{2d}{r^2} \mathbb{E}_{(d,g,v)}[G(1)\psi_1(G(1),B(1))G(1)D(0)]
\]

\[
= \frac{2d^2}{r^2} \left( \frac{1 - H(g,v+1)}{1 - H(g,v)} \right) \psi_1(g,v+1)g^2 + \int_E \psi_1(x,0)x^2 \frac{\dot{q}(g,x,v+1)}{1 - H(g,v)} dx. \quad (4.32)
\]

A substitution of (4.30), (4.31) and (4.32) into (4.29) gives

\[
p^2(d,g,v) = \psi_2(g,v)d^2 = \frac{d^2}{r^2} \mathbb{E}(g,v) + \frac{d^2}{r^2} \left( \frac{1 - H(g,v+1)}{1 - H(g,v)} \right) \psi_2(g,v+1)g^2
\]

\[
+ \int_E \psi_2(x,0)x^2 \frac{\dot{q}(g,x,v+1)}{1 - H(g,v)} dx, \quad + \frac{2d^2}{r^2} \left( \frac{1 - H(g,v+1)}{1 - H(g,v)} \right) \psi_1(g,v+1)g^2
\]

\[
+ \int_E \psi_1(x,0)x^2 \frac{\dot{q}(g,x,v+1)}{1 - H(g,v)} dx. \quad (4.33)
\]

A simple rearrangement of the terms gives equation (3.41)
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