(p, q, t)-CATALAN CONTINUED FRACTIONS, GAMMA EXPANSIONS AND PATTERN AVOIDANCES

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ABSTRACT. We introduce a kind of (p, q, t)-Catalan numbers of Type A by generalizing the Jacobian type continued fraction formula, we prove that the corresponding expansions could be expressed by the polynomials counting permutations on $S_n(321)$ by various descent statistics. Moreover, we introduce a kind of (p, q, t)-Catalan numbers of Type B by generalizing the Jacobian type continued fraction formula, we prove that the Taylor coefficients and their $\gamma$-coefficients could be expressed by the polynomials counting permutations on $S_n(3124, 4123, 3142, 4132)$ by various descent statistics. Our methods include permutation enumeration techniques involving variations of bijections from permutation patterns to labeled Motzkin paths and modified Foata-Strehl action.

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1. Introduction

It is well-known that Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \) has the following Jacobi-type continued fraction expansion (cf. [5, 9, 13]):

\[
\sum_{n \geq 0} C_n z^n = \frac{1}{1 - z - \frac{1 \cdot z^2}{1 - z - \frac{1 \cdot z^2}{1 - \ldots}}}
\]

In this paper we define the following \((p, q, t)\)-Catalan continued fraction of Type A as

\[
\sum_{n \geq 0} C_n(p, q, t) z^n = \frac{1}{1 - \frac{tp \cdot z^2}{1 - \frac{(p + q) \cdot z - \frac{tp^2 q \cdot z^2}{1 - \frac{(p^2 + q^2) \cdot z - \frac{tp^3 q^2 \cdot z^2}{1 - \frac{(p^3 + q^3) \cdot z - \frac{tp^4 q^3 \cdot z^2}{1 - \ldots}}}}}}}}
\]

Where \( C_n(p, q, t) \) is the \((p, q, t)\)-Catalan numbers of type A as the Taylor coefficients in the above continued fraction.

In particular, let \( \bar{C}_n(q) := C_n(1, q, 1) \) and \( \tilde{C}_n(q) := C_n(q, 1, 1) \). We need a standard contraction formula for continued fractions, see [14, Lemma 5.1].

Lemma 1.1 (Contraction formula). The following holds

\[
\frac{1}{1 - c_1 z} = \frac{1}{1 - c_1 z - \frac{c_1 c_2 z^2}{1 - (c_2 + c_3) z - \frac{c_3 c_4 z^2}{1 - (c_4 + c_5) z - \frac{c_5 c_6 z^2}{1 - \ldots}}}}
\]

By contraction we derive the two q-Catalan numbers

\[
\sum_{n=0}^{\infty} \bar{C}_n(q) z^n = \frac{1}{z}, \quad \text{(1.2)}
\]

\[
\sum_{n=0}^{\infty} \tilde{C}_n(q) z^n = \frac{1}{1 - \frac{z}{1 - \frac{q z}{1 - \frac{q^2 z}{1 - \ldots}}}}
\]
Figure 1. The first values of $\bar{C}_n(q)$ (left) and $\tilde{C}_n(q)$ (right) for $0 < n \leq 5$. and

\[
\sum_{n=0}^{\infty} \tilde{C}_n(q)z^n = \frac{1}{1 - \frac{qz}{z}} \frac{1}{1 - \frac{q^2z}{z}} \frac{1}{1 - \frac{q^3z}{z}} \cdots \frac{1}{1 - \frac{q^kz}{z}} \frac{1}{1 - \cdots}.
\]

We give the first few terms for $\bar{C}_n(q)$ and $\tilde{C}_n(q)$ see Fig 1 and note that neither $\bar{C}_n(q)$ nor $\tilde{C}_n(q)$ is registered in the OEIS.

Type B Catalan numbers $B_n = \binom{2n}{n}$ has the following Jacobi-type continued fraction expansion (cf. [9, 22])

\[
\sum_{n \geq 0} B_n z^n = \frac{1}{1 - 2z - \frac{2z^2}{1 - 2z - \frac{1z^2}{1 - \cdots}}}. \tag{1.4}
\]

We define the following $(p, q, t)$-Catalan continued fraction of Type B as

\[
\sum_{n=0}^{\infty} \mathcal{B}_n(p, q, t)z^n = \frac{1}{1 - (1 + t)z - \frac{(p + q)t^2z^2}{1 - p(1 + t)z - \frac{p^3t^2z^2}{1 - p^2(1 + t)z - \frac{p^5t^2z^2}{\cdots}}}}. \tag{1.5}
\]

Where $\mathcal{B}_n(p, q, t)$ is the $(p, q, t)$-Catalan numbers of Type B as the Taylor coefficients in the above continued fraction.

The combinatorial constructions behind the proof of Theorem 1.2 and Theorem 1.4, two of our main results, originated from the fundamental work of Flajolet [9] for the lattice path interpretation of the formal continued fractions, and a bijection between sets of certain weighted Motzkin paths and permutations due to Françon-Viennot [12], see also [8, 11, 19, 20],
Remark 1.5. When we define now. Denote by $S_{B}$ and Ruškuc in [1].

The goal of this paper is to establish combinatorial interpretations for $C_{n}(p,q,t)$ and $B_{n}(p,q,t)$ as well as their corresponding $\gamma$-expansions, using pattern avoiding permutations, which we define now. Denote by $\mathfrak{S}_{n}$ the set of permutations of length $n$. Given two permutations $\pi \in \mathfrak{S}_{n}$ and $p \in \mathfrak{S}_{k}$, $k \leq n$, we say that $\pi$ avoids the pattern $p$ if there does not exist a set of indices $1 \leq i_{1} < i_{2} < \cdots < i_{k} \leq n$ such that the subsequence $\pi(i_{1})\pi(i_{2})\cdots\pi(i_{k})$ of $\pi$ is order-isomorphic to $p$. For example, the permutation 15324 avoids 231. The set of permutations of length $n$ that avoid patterns $p_{1}, p_{2}, \ldots, p_{m}$ is denoted as $\mathfrak{S}_{n}(p_{1}, p_{2}, \cdots, p_{m})$.

Many polynomials with combinatorial meanings have been shown to be unimodal; see the recent survey of Brändén [7]. Recall that a polynomial $h(t) = \sum_{i=0}^{d} h_{i}t^{i}$ of degree $d$ is said to be unimodal if the coefficients are increasing and then decreasing, i.e., there is an index $c$ such that $h_{0} \leq h_{1} \leq \cdots \leq h_{c} \geq h_{c+1} \geq \cdots \geq h_{d}$. Let $p(t) = a_{r}t^{r} + a_{r+1}t^{r+1} + \cdots + a_{s}t^{s}$ be a real polynomial with $a_{r} \neq 0$ and $a_{s} \neq 0$. It is called palindromic (or symmetric) of center $n/2$ if $n = r + s$ and $a_{r+i} = a_{s-i}$ for $0 \leq i \leq n/2$. For example, polynomials $1 + t$ and $t$ are palindromic of center 1/2 and 1, respectively. Any palindromic polynomial $p(t) \in \mathbb{Z}[t]$ can be written uniquely \cite{7,21} as

$$p(t) = \sum_{k=r}^{\left\lceil \frac{n}{2} \right\rceil} \gamma_{k}t^{k}(1 + t)^{n-2k},$$

where $\gamma_{k} \in \mathbb{Z}$. If $\gamma_{k} \geq 0$ then we say that it is $\gamma$-positive of center $n/2$. It is clear that the $\gamma$-positivity implies palindromicity and unimodality. For further $\gamma$-positivity results and problems, the reader is referred to the excellent exposition by Petersen \cite{16} and the most recent survey by Athanasiadis \cite{2}.

Now we give the first three main results of this paper, with the definitions of permutation statistics, permutation patterns postponed to the next section.

**Theorem 1.2.** The $(p,q,t)$-Type A Catalan numbers $C_{n}(p,q,t)$ have the following interpretation

$$C_{n}(p,q,t) = \sum_{\sigma \in \mathfrak{S}_{n}(321)} p(2^{13})^{\sigma} q(312)^{\sigma} t^{\text{des} \sigma}.$$

**Remark 1.3.** Barnabei, Bonetti and Silimbani \cite{4} Theorem 6] gave an ordinary generating function formula for $t^{n}C_{n}(1,1,1/t)$ by exploiting Krattenthaler’s bijection between 123-avoiding permutations and Dyck paths (see OEIS: A166073).

**Theorem 1.4.** The $(p,q,t)$-Type B Catalan numbers $B_{n}(p,q,t)$ have the following interpretation

$$B_{n}(p,q,t) = \sum_{\sigma \in \mathfrak{S}_{n+1}(3124,4123,3142,4132)} p(2^{13})^{\sigma} q(312)^{\sigma} t^{\text{des} \sigma}.$$

**Remark 1.5.** When $p = q = t = 1$, Albert, Linton and Ruškuc in \cite{1} Section 7.4 gave the above combinatorial interpretation for Type B Catalan numbers.
Theorem 1.6. For $n \geq 1$, the following $\gamma$-expansions formula holds true

$$\mathcal{B}_n(p, q, t) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \gamma_{n+1,k}(p, q)t^k(1+t)^{n-2k},$$

(1.6)

where

$$\gamma_{n+1,k}(p, q) := \sum_{\sigma \in \mathfrak{S}_{n+1,k}} p^{(2-13)\sigma} q^{(31-2)\sigma},$$

(1.7)

and

$$\mathfrak{S}_{n+1,k} := \{ \sigma \in \mathfrak{S}_{n+1}(3124, 4123, 3142, 4132) : \text{dd}(\sigma) = 0, \text{val} \sigma = \text{des} \sigma = k \}.$$

For example, the first few expansions of $\mathcal{B}_n(p, q, t)$ read as follows:

$$\mathcal{B}_1(p, q, t) = 1 + t;$$
$$\mathcal{B}_2(p, q, t) = (1 + t)^2 + (p + q)t;$$
$$\mathcal{B}_3(p, q, t) = (1 + t)^3 + (p + 2)(p + q)t(1 + t);$$
$$\mathcal{B}_4(p, q, t) = (1 + t)^4 + (p + q)(p^2 + 2p + 3)t(1 + t)^2 + t^2(p^3 + p + q)(p + q);$$
$$\mathcal{B}_5(p, q, t) = (1 + t)^5 + (p + q)(p^3 + 2p^2 + 3p + 4)t(1 + t)^3$$
$$+ t^2(1 + t)(p + q)(p^4 + 2p^3 + 2p^2 + (2q + 3)p + 3q).$$

Remark 1.7. Postnikov, Reiner and Williams [17, Proposition 11.15] showed

$$\gamma_{n+1,k}(1, 1) = \binom{n}{k, k, n-2k}$$

by the corresponding $h$-polynomials and standard quadratic transformations of hypergeometric series. However, we give $\gamma_{n,k}(p, q)$ by continued fraction theory and bijection between permutation patterns and weighted Motzkin paths.

The rest of this paper is organized as follows. In Section 2 we give most of the definitions and provide the previously known results, which will be used to prove Theorems 1.2 and 1.4 in Section 3 and Section 4 respectively. In Section 5 we introduce Brändén’s modified Foata-Strehl action and prove the theorem 1.6. We consider a variation of $(q, t)$-Catalan numbers in Section 6.

2. Definitions and Preliminaries

2.1. Permutation statistics. For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention $\sigma(0) = \sigma(n + 1) = 0$, a value $\sigma(i)$ $(1 \leq i \leq n)$ is called

- a peak if $\sigma(i - 1) < \sigma(i)$ and $\sigma(i) > \sigma(i + 1)$;
- a valley if $\sigma(i - 1) > \sigma(i)$ and $\sigma(i) < \sigma(i + 1)$;
- a double ascent if $\sigma(i - 1) < \sigma(i)$ and $\sigma(i) < \sigma(i + 1)$;
- a double descent if $\sigma(i - 1) > \sigma(i)$ and $\sigma(i) > \sigma(i + 1)$.

And we call a permutation $\sigma$ has a descent at $i$ with $1 \leq i < n$ if $\sigma(i) > \sigma(i + 1)$. The set of peaks (resp. valleys, double ascents, double descents, descents) of $\sigma$ is denoted by

$$P_k \sigma \text{ (resp. Val} \sigma, \text{Da} \sigma, \text{Dd} \sigma, \text{Des} \sigma).$$
Let $p_k \sigma$ (resp. $v_{\text{val}} \sigma$, $d_{\text{da}} \sigma$, $d_{\text{dd}} \sigma$, $d_{\text{des}} \sigma$) be the number of peaks (resp. valleys, double ascents, double descents, descents) of $\sigma$. For $i \in [n] := \{1, \ldots, n\}$, we introduce the following statistics:

$$
(31-2)_i \sigma = \#\{ j : 1 < j < i \text{ and } \sigma(j) < \sigma(i) < \sigma(j - 1) \} \quad (2.1) \\
(2-31)_i \sigma = \#\{ j : i < j < n \text{ and } \sigma(j + 1) < \sigma(i) < \sigma(j) \} \quad (2.2) \\
(2-13)_i \sigma = \#\{ j : i < j < n \text{ and } \sigma(j) < \sigma(i) < \sigma(j + 1) \} \quad (2.3) \\
(13-2)_i \sigma = \#\{ j : 1 < j < i \text{ and } \sigma(j - 1) < \sigma(i) < \sigma(j) \} \quad (2.4)
$$

and define four statistics:

$$
(31-2) = \sum_{i=1}^{n} (31-2)_i, \quad (2-31) = \sum_{i=1}^{n} (2-31)_i, \quad (2-13) = \sum_{i=1}^{n} (2-13)_i, \quad (13-2) = \sum_{i=1}^{n} (13-2)_i.
$$

For $\sigma \in S_n$ with convention $\sigma(0) = 0$ and $\sigma(n + 1) = \infty$, let

$$
L_{p_k} \sigma \quad \text{(resp. } L_{v_{\text{val}}} \sigma, L_{d_{\text{da}}} \sigma, L_{d_{\text{dd}}} \sigma)$$

be the set of peaks (resp. valleys, double ascents and double descents) of $\sigma$ and denote the corresponding cardinality by $l_{p_k} \sigma$ (resp. $l_{v_{\text{val}}} \sigma$, $l_{d_{\text{da}}} \sigma$, $l_{d_{\text{dd}}} \sigma$).

### 2.2. Insertion encoding of permutations.

Every permutation can be generated from the empty permutation by successive insertion of a new maximum element. We call this procedure the *evolution* of a permutation. This observation is used as a basis for the generating tree methodology. In this methodology, a partially constructed permutation belonging to some specified pattern class is viewed as having a number of active sites—points at which a new maximum could be inserted while remaining in the pattern class. In [1], Albert, Linton and Ruškuc modified this viewpoint slightly in that for them an active site in the construction of a permutation is one in which a new maximum element will eventually be inserted.

At each preliminary stage we index the slots which are present from left to right beginning with 1. Then each insertion is determined by the following:

- The index of the slot in which it occurs.
- The way in which the slot is affected by the insertion.

There are four different types of insertion of a new maximum element $n$ within a slot:

- $\bullet \quad \bullet \rightarrow \bullet n \bullet \rightarrow \bullet$ represented by $m$ (for middle)
- $\bullet \quad \bullet \rightarrow \bullet n \bullet \rightarrow \bullet$ represented by $l$ (for left)
- $\bullet \quad \bullet \rightarrow \bullet n \bullet \rightarrow \bullet$ represented by $r$ (for right)
- $\bullet \quad \bullet \rightarrow \bullet n \bullet \rightarrow \bullet$ represented by $f$ (for fill)

If we subscript the insertion type with the slot on which it operates, we obtain a uniquely defined encoding of any permutation. For instance, the insertion encoding of 423615 is $m_1 m_2 f_1 f_2 f_1$.

For clarity, we sometimes use negative subscripts in counting slots from the right. That is $m_{-1}$ represents a middle insertion into the rightmost slot and $r_{-2}$ represents an insertion on the right hand end of the second slot from the right. There are also situations where $a$
modification of the insertion encoding is desirable. Suppose that we are interested in studying a collection of permutations \( C \) with the property that for any \( \pi, \tau \in C \) the permutation \( \pi \oplus \tau \) which has an initial segment of small values order isomorphic to \( \pi \) followed by a segment of larger values order isomorphic to \( \tau \) is also in \( C \) (for instance, \( 21 \oplus 312 = 21534 \)).

Then it is natural to retain a "free slot" on the right hand end of any configuration. We enforce this by simply forbidding the \( f \) and \( r \) operations in the final slot.

In [1], the authors described several pattern classes by insertion encoding and gave the following properties.

**Proposition 2.1.** [1, Proposition 4] The permutations in \( S_n(312) \) are precisely those whose insertion encoding uses only the symbols \( f_1, l_1, r_1, \) and \( m_1 \).

In [1], the authors used the modification of the insertion encoding to give grammar describes for \( S_n(321) \). Here, we re-describe \( S_n(321) \) in the flavor of Proposition 2.1.

**Proposition 2.2.** The permutations in \( S_n(321) \) are precisely those whose insertion encoding uses only the symbols \( m_{-1} \) and \( l_{-1} \) if there is only one free slot, and uses only the symbols \( f_1, l_1, l_{-1} \) and \( m_{-1} \) if there are more than one free slot.

**Proof.** In this modified form, it is obviously that the only letters which may occur when only one slot is free are \( m_{-1} \) and \( l_{-1} \). If there are more than one free slot, insert \( f_1, l_1, l_{-1} \) and \( m_{-1} \) will not create a 321 pattern and also keep a free slot at the end of sequence. But, if we insert other operations, either they eliminate the free slot at the end of the figure or they will create a 321 pattern since there is a smaller element at the right side of this operation and there have at least one free slot, once an element is added to the first slot of this sequence, a 321 pattern is created. \( \square \)

**Proposition 2.3.** [1, Section 7.4] The permutations in \( S_n(3124, 4123, 3142, 4132) \) are precisely those whose insertion encoding uses only the symbols \( m_1, l_1, r_1 \) and \( f_1 \), but if there are two free slots then the insertion encoding can also use \( f_2 \).

### 2.3. Laguerre histories as permutation encodings.

A Laguerre history (resp. restricted Laguerre history) of length \( n \) is a pair of \((s, p)\), where \( s \) is a 2-Motzkin path \( s_1 \ldots s_n \) and \( p = (p_1, \ldots, p_n) \) with \( 0 \leq p_i \leq h_{i-1}(s) \) (resp. \( 0 \leq p_i \leq h_{i-1}(s) - 1 \) if \( s_i = L_r \) of \( D \)) with \( h_0(s) = 0 \). Let \( \mathcal{LH}_n \) (resp. \( \mathcal{LH}^*_n \)) be the set of Laguerre histories (resp. restricted Laguerre histories) of length \( n \). There are several well-known bijections between \( \mathcal{S}_n \) and \( \mathcal{LH}^*_n \) and \( \mathcal{LH}_{n-1} \), see [8][15].

### 2.4. Françon-Viennot bijection.

We recall a version of Françon and Viennot’s bijection \( \psi_{FV} : \mathcal{S}_{n+1} \rightarrow \mathcal{LH}_n \). Given \( \sigma \in \mathcal{S}_{n+1} \), the Laguerre history \( \psi_{FV}(\sigma) = (s, p) \) is defined as follows:

\[
   s_i = \begin{cases} 
   U & \text{if } i \in \text{Val} \sigma \\
   D & \text{if } i \in \text{Pk} \sigma \\
   L_b & \text{if } i \in \text{Da} \sigma \\
   L_r & \text{if } i \in \text{Dd} \sigma 
   \end{cases} 
\]

\[ (2.5) \]

and \( p_i = (2-13)_i \sigma \) for \( i = 1, \ldots, n \).
2.5. **Restricted Françon-Viennot bijection.** We recall a restricted version of Françon and Viennot's bijection $\phi_{FV} : S_n \to \mathcal{LH}_n$. Given $\sigma \in S_n$, the Laguerre history $(s, p)$ is defined as follows:

$$s_i = \begin{cases} 
U & \text{if } i \in \text{Lval } \sigma \\
D & \text{if } i \in \text{Lpk } \sigma \\
L_b & \text{if } i \in \text{Lda } \sigma \\
L_r & \text{if } i \in \text{Ldd } \sigma 
\end{cases} \quad (2.6)$$

and $p_i = (2-31)_i \sigma$ for $i = 1, \ldots, n$.

3. **Proof of Theorem 1.2**

A *Motzkin path* of length $n$ is a sequence of points $\omega := (\omega_0, \ldots, \omega_n)$ in the integer plane $\mathbb{Z} \times \mathbb{Z}$ such that

- $\omega_0 = (0, 0)$ and $\omega_n = (n, 0)$,
- $\omega_i - \omega_{i-1} \in \{(1, 0), (1, 1), (1, -1)\}$,
- $\omega_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N}$ for $i = 0, \ldots, n$.

In other words, a Motzkin path of length $n \geq 0$ is a lattice path in the right quadrant $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$ and ending at $(n, 0)$, each step $s_i = \omega_i - \omega_{i-1}$ is a rise $U = (1, 1)$, fall $D = (1, -1)$ or level $L = (1, 0)$. A 2-Motzkin path is a Motzkin path with two types of level-steps $L_b$ and $L_r$. Let $\mathcal{MP}_n$ be the set of 2-Motzkin paths of length $n$. Clearly we can identify 2-Motzkin paths of length $n$ with words $w$ on $\{U, L_b, L_r, D\}$ of length $n$ such that all prefixes of $w$ contain no more $D$'s than $U$'s and the number of $D$'s equals the number of $D$'s. The height of a step $s_i$ is the ordinate of the starting point $\omega_{i-1}$.

Let $a = (a_i)_{i \geq 0}$, $b = (b_i)_{i \geq 0}$, $b' = (b'_i)_{i \geq 0}$ and $c = (c_i)_{i \geq 1}$ be four sequences of indeterminates; we will work in the ring $\mathbb{Z}[a, b, b', c]$. To each Motzkin path $\omega$ we assign a weight $W(\omega)$ that is the product of the weights for the individual steps, where an up-step (resp. down-step) at height $i$ gets weight $a_i$ (resp. $c_i$), and a level-step of $L_b$ (resp. $L_r$) at height $i$ gets weight $b_i$ (resp. $b'_i$). The following result of Flajolet [9] is well-known.

**Lemma 3.1** (Flajolet). We have

$$\sum_{n=0}^{\infty} \left( \sum_{\omega \in \mathcal{MP}_n} W(\omega) \right) x^n = \frac{1}{1 - (b_0 + b'_0)x - \frac{a_0 c_1 x^2}{1 - (b_1 + b'_1)x - \frac{a_1 c_2 x^2}{1 - (b_2 + b'_2)x - \cdots}}}.$$

For $\sigma \in S_n(321)$, we append $\sigma(0) = 0, \sigma(n + 1) = n + 1$. And we also introduce a generalized $(2-13)$ statistic $(\widehat{(2-13)})$ on $\sigma$ by

$$\widehat{(2-13)}_i \sigma := \# \{ j : i < j \leq n \text{ and } \sigma(j) < \sigma(i) < \sigma(j + 1) \}$$

and

$$\widehat{(2-13)} = \sum_{i=1}^{n} \widehat{(2-13)}_i,$$
Let
\[ C_n(p, q, t, u, w) := \sum_{\pi \in S_n(321)} p^{(2-13) \pi} q^{(31-2) \pi} t^{\text{des} \pi} u^{\text{lda} \pi} w^{\text{val} \pi}, \]
then we have the following Theorem.

**Theorem 3.2.** We have
\[ \sum_{n \geq 0} C_n(p, q, t, u, w) z^n = \]
\[ \frac{1}{1 - wtp \cdot z^2} \frac{wtp^2 q \cdot z^2}{1 - (p + q)u \cdot z - \frac{wtp^3 q^2 \cdot z^2}{1 - (p^3 + q^3)u \cdot z - \frac{wtp^4 q^3 \cdot z^2}{1 - \cdots}} \]
(3.1)
(3.2)

It is easy to see that Theorem 3.2 is a generalization of Theorem 1.2. In this section we will give a proof for Theorem 3.2.

**Definition 3.3.** A path of diagram of type A of length \( n \) is a pair \((\omega', \xi')\), where \( \omega' \) is a Motzkin path of length \( n \) with conditions that at height 0, the type of steps can only be \( U \) or \( L_b \) and in other heights, the type of steps can only be \( D \), \( L_b \) and \( U \). \( \xi' = (\xi'_1, \xi'_2, \ldots, \xi'_n) \) is an integer sequence satisfying that if the \( k \)-th step of \( \omega' \) is at height \( h \) and the step is \( D \), then \( \xi'_k = h - 1 \). If the \( k \)-th step at height \( h \) is \( L_b \) then \( \xi'_k = 0 \) or \( \xi'_k = h \) if \( \xi' \geq 1 \). For the other cases, \( \xi'_k = 0 \). We denote by \( \mathcal{PDA}_n \) the set of path diagrams of type A of length \( n \).

**Remark 3.4.** Following Proposition 2.2, each \( \sigma \in S_n(321) \) can be constructed by inserting element one by one from 1 to \( n \), then we obtain the following observations for \((\omega', \xi') = \phi_{\text{FV}}(\sigma)\).

1. When the \( k \)-th step of \( \omega' \) is at height \( h = 0 \), then
   \[ \widehat{(2-13)}_k \sigma = (31-2)_k \sigma = 0. \]
2. When the \( k \)-th step of \( \omega' \) is \( D \) step at height \( h \geq 1 \), then
   \[ \widehat{(2-13)}_k \sigma = h \] and \( (31-2)_k \sigma = 0. \]
3. When the \( k \)-th step of \( \omega' \) is \( L_b \) step at height \( h \geq 1 \), then
   \[ \widehat{(2-13)}_k \sigma = \xi'_k \] and \( (31-2)_k \sigma = h - \xi'_k. \]
4. When the \( k \)-th step of \( \omega' \) is \( U \) step at height \( h \geq 1 \), then
   \[ \widehat{(2-13)}_k \sigma = 0 \] and \( (31-2)_k \sigma = h. \]

If we restrict \( \phi_{\text{FV}} \) on \( S_n(321) \) and denote as \( \Phi_1 \) for the restriction, then we have the following result.

**Lemma 3.5.** \( \Phi_1 \) is a bijection from \( S_n(321) \) to \( \mathcal{PDA}_n \).
**Proof.** By Proposition 2.2 and comparing the definition of $\phi_{FV}$ with Definition 3.3, we have the result.

**Proof of Theorem 3.2.** For $\pi \in S_n(321)$, we derive that

$$p^{(2-13)}q^{(31-2)}\sigma^{\text{des}}u^{\text{da}}\sigma^{\text{val}} = \prod_{i=1}^{n} W(s_i, \xi'_i) = W((\omega', \xi')),$$

with $h$ being the height of the steps $s_i$,

$$W(s_i, \xi'_i) = \begin{cases} q^h \cdot w & \text{if } s_i = U; \\ p^h \cdot t & \text{if } s_i = D; \\ p^h \cdot q^h - \xi'_i \cdot u & \text{if } s_i = L_b. \end{cases}$$

Therefore, the corresponding polynomial and weights become

$$C_n(p, q, t, u, w) = \sum_{\sigma \in S_n(321)} p^{(2-13)}q^{(31-2)}\sigma^{\text{des}}u^{\text{da}}\sigma^{\text{val}}$$

$$= \sum_{(\omega', \xi') \in \mathcal{PDB}_n} W((\omega', \xi')).$$

By Lemma 3.1, we derive the corresponding continued fraction for the generating function of $C_n(p, q, t, u, w)$.

### 4. Proof of Theorem 1.4

We give the following definition. Let

$$B_n(p, q, t, u, v, w) := \sum_{\sigma \in S_{n+1}(3124,4123,3142,4132)} p^{(2-13)}q^{(31-2)}\sigma^{\text{des}}u^{\text{da}}v^{\text{dd}}\sigma^{\text{val}}.$$

**Theorem 4.1.**

$$\sum_{n=0}^{\infty} B_n(p, q, t, u, v, w)z^n = \frac{1}{1 - (u+tv)z - \frac{(p+q)twz^2}{1 - p(u+tv)z - \frac{p^3twz^2}{1 - p^2(u+tv)z - \frac{p^5twz^2}{\ddots}}}}.$$  

(4.1)

It is easy to see that Theorem 4.1 is a generalization of Theorem 1.4. In this section we will give a proof for Theorem 4.1.

**Definition 4.2.** A path diagram of type B of length $n$ is a pair $(\omega'', \xi'')$, where $\omega''$ is a Motzkin path of length $n$, $\xi'' = (\xi''_1, \ldots, \xi''_n)$ is an integer sequence satisfying that if the $k$-th step of $\omega''$ is $D$ at height $1$, then $\xi''_k = 0$ or $\xi''_k = 1$, otherwise $\xi''_k = h$, where $h$ is the height of the $k$-th step of $\omega''$. We denote by $\mathcal{PDB}_n$ the set of path diagrams of type B of length $n$.

**Remark 4.3.** Following Proposition 2.3, each $\sigma \in S_n(3124,4123,3142,4132)$ can be constructed by inserting element one by one from 1 to $n$, then we obtain that if the $k$-th step of $\omega''$ is $D$ step at height $h = 1$, then $(31-2)_k \sigma = 1 - \xi''_k$, otherwise $(31-2)_k \sigma = 0$. 

If we restrict $\psi_{FV}$ on $\mathfrak{S}_n(3124,4123,3142,4132)$ and denote as $\Phi_2$ for the restriction, then we have the following result.

**Lemma 4.4.** $\Phi_2$ is a bijection from $\mathfrak{S}_{n+1}(3124,4123,3142,4132)$ to $PDB_n$.

**Proof.** By Proposition 2.3 and comparing the definition of $\phi_{FV}$ and Definition 4.2 we have the result. □

**Proof of Theorem 4.1.** For $\sigma \in \mathfrak{S}_n(3124,4123,3142,4132)$, we derive that

$$p^{(2-13)}q^{(31-2)}\ell^\text{des} u^\text{da} v^\text{dd} w^\text{val} = \prod_{i=1}^{n} W(s_i, \xi''_i) = W((\omega'', \zeta'')),$$

with $h$ being the height of the steps $s_i$,

$$W(s_i, \xi''_i) = \begin{cases} p^{\xi''}_i \cdot t \cdot w & \text{if } s_i = U; \\ p^{\xi''}_i \cdot q^{h-\xi''} & \text{if } s_i = D; \\ p^{\xi''}_i \cdot u & \text{if } s_i = L_b; \\ p^{\xi''}_i \cdot v \cdot t & \text{if } s_i = L_r. \end{cases} \quad (4.3)$$

Therefore, the corresponding polynomial and weights become

$$B_n(p, q, t, u, v, w) = \sum_{\sigma \in \mathfrak{S}_{n+1}(3124,4123,3142,4132)} p^{(2-13)}q^{(31-2)}\ell^\text{des} u^\text{da} v^\text{dd} w^\text{val} \quad (4.4)$$

$$= \sum_{(\omega'', \zeta'') \in PDB_n} W((\omega'', \zeta'')) \quad (4.5)$$

By Lemma 3.1 we derive the corresponding continued fraction for the generating function of $B_n(p, q, t, u, v, w)$.

5. **Proof of Theorem 1.6**

**Theorem 5.1.** We have

$$B_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n+1,k}(p, q)(tw)^k(u + tv)^{n-2k}. \quad (5.1)$$

It is easy to see that Theorem 5.1 is a generalization of Theorem 1.6. In this section we will give two different proofs for Theorem 5.1.

5.1. **Continued fraction proof.**

**Proof of Theorem 5.1.** Let

$$B_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n+1,k}(p, q, t, u, v)w^k, \quad (5.2)$$

so

$$a_{n+1,k}(p, q, t, u, v) = \sum_{\sigma \in \mathfrak{G}_{n+1,k}(3124,4123,3142,4132)} p^{(2-13)}q^{(31-2)}\ell^\text{des} u^\text{da} v^\text{dd} \quad (5.3)$$
where $\tilde{G}_{n+1,k}(3124, 4123, 3142, 4132) = \{\sigma \in G_n(3124, 4123, 3142, 4132) : \text{val} \sigma = k\}$.

Substituting $z \leftarrow \frac{z}{u+tv}$, $w \leftarrow \frac{w(u+tv)^2}{t}$ in (4.14), we obtain

$$\sum_{n=0}^{\infty} B_n(p, q, t, u, v, \frac{w(u+tv)^2}{t}) \cdot \frac{z^n}{(u+tv)^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n+1,k}(p, q, t, u, v) \frac{w^k(u+tv)^{2k}}{t^k} \cdot \frac{z^n}{(u+tv)^n}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{1 - z - \frac{(p+q)wz^2}{1 - pz - \frac{p^3wz^2}{1 - p^2z} - \frac{p^5wz^2}{1 - p^3z}} \cdots}. \tag{5.4}$$

Since (5.7) is free of variables $t$, $u$ and $v$. The coefficient of $w^kz^n$ in (5.6) is a polynomial in $p$ and $q$ with nonnegative integral coefficients, if we denote this coefficient by

$$P_{n+1,k}(p, q) := \frac{a_{n+1,k}(p, q, t, u, v)}{t^k(u+tv)^{n-2k}},$$

then we have

$$P_{n+1,k}(p, q) = a_{n+1,k}(p, q, 1, 1, 0), \tag{5.5}$$

thus

$$a_{n+1,k}(p, q, t, u, v) = a_{n+1,k}(p, q, 1, 1, 0)t^k(u+tv)^{n-2k}. \tag{5.6}$$

Compare with (5.3), we have $a_{n+1,k}(p, q, 1, 1, 0) = \sum_{\sigma \in \mathcal{G}_{n+1,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}$, with (5.9), which proved Theorem 5.1.

\[ \square \]

5.2. Group action proof. Besides the patterns mentioned in the introduction, we shall also consider the so-called vincular patterns [3]. The number of occurrences of vincular patterns 31-2-4, 31-4-2, 41-2-3, and 41-3-2 are defined by

\[
(31-2-4) \sigma = \# \{(i, j, k) : i + 1 < j < k \leq n \text{ and } \sigma(i+1) < \sigma(j) < \sigma(i) < \sigma(k)\}, \tag{5.10}
\]

\[
(31-4-2) \sigma = \# \{(i, j, k) : i + 1 < j < k \leq n \text{ and } \sigma(i+1) < \sigma(k) < \sigma(i) < \sigma(j)\}, \tag{5.11}
\]

\[
(41-2-3) \sigma = \# \{(i, j, k) : i + 1 < j < k \leq n \text{ and } \sigma(i+1) < \sigma(j) < \sigma(k) < \sigma(i)\}, \tag{5.12}
\]

\[
(41-3-2) \sigma = \# \{(i, j, k) : i + 1 < j < k \leq n \text{ and } \sigma(i+1) < \sigma(k) < \sigma(j) < \sigma(i)\}. \tag{5.13}
\]

For convenience, we also denote

\[
(31-2-4 + 31-4-2 + 41-2-3 + 41-3-2) \sigma := (31-2-4) \sigma + (31-4-2) \sigma + (41-2-3) \sigma + (41-3-2) \sigma.
\]
Lemma 5.2. For \( n \geq 1 \), we have

\[
\mathcal{S}_n(3124, 3142, 4123, 4132) = \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2). \tag{5.14}
\]

Proof. It is easy to see \( \mathcal{S}_n(3124, 3142, 4123, 4132) \subseteq \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \). We only need to show \( \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \subseteq \mathcal{S}_n(3124, 3142, 4123, 4132) \).

(1) Suppose \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \) contain the pattern 3124, i.e., there exists a subsequence \( i < j < k < l \) such that \( \sigma(l) > \sigma(i) > \sigma(k) > \sigma(j) \). It is easy to see there exist \( (i', i' + 1) \) such that \( \sigma(i') > \sigma(k) > \sigma(i' + 1) \) and \( i < i' < i' + 1 < j \), where \( i' \) and \( i' + 1 \) might be \( i \) and \( j \), respectively.

(a) if \( \sigma(i') < \sigma(l) \), it is easy to see \( \sigma(l) > \sigma(i) > \sigma(k) > \sigma(i' + 1) \) for \( i' < i' + 1 < k < l \), which is in contradiction with \( \sigma \in \mathcal{S}_n(31-2-4) \).

(b) if \( \sigma(i') > \sigma(l) \), it is easy to see \( \sigma(i') > \sigma(k) > \sigma(i' + 1) \) for \( i' < i' + 1 < k < l \), which is in contradiction with \( \sigma \in \mathcal{S}_n(41-2-3) \).

Above all, we have \( \sigma \in \mathcal{S}_n(3124) \) for \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \). The proof of \( \sigma \in \mathcal{S}_n(4123) \) for \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \) is similar and omitted.

(2) Suppose \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \) contain the pattern 3142, i.e., there exists a subsequence \( i < j < k < l \) such that \( \sigma(k) > \sigma(i) > \sigma(l) > \sigma(j) \). It is easy to see there exist \( (i', i' + 1) \) such that \( \sigma(i') > \sigma(l) > \sigma(i' + 1) \) and \( i < i' < i' + 1 < j \), where \( i' \) and \( i' + 1 \) might be \( i \) and \( j \), respectively.

(a) if \( \sigma(i') < \sigma(k) \), it is easy to see \( \sigma(k) > \sigma(i') > \sigma(l) > \sigma(i' + 1) \) for \( i' < i' + 1 < k < l \), which is in contradiction with \( \sigma \in \mathcal{S}_n(31-4-2) \).

(b) if \( \sigma(i') > \sigma(k) \), it is easy to see \( \sigma(i') > \sigma(k) > \sigma(l) > \sigma(i' + 1) \) for \( i' < i' + 1 < k < l \), which is in contradiction with \( \sigma \in \mathcal{S}_n(41-3-2) \).

Above all, we have \( \sigma \in \mathcal{S}_n(3142) \) for \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \). The proof of \( \sigma \in \mathcal{S}_n(4132) \) for \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \) is similar and omitted.

Then for \( \sigma \in \mathcal{S}_n(31-2-4, 31-4-2, 41-2-3, 41-3-2) \), we have \( \sigma \in \mathcal{S}_n(3124, 3142, 4123, 4132) \), which completes the proof. \( \square \)

The main tool to proof \((5.1)\) in this Section, is the well-known modified Foata-Strehl action. See Foata and Strehl \([10]\), Shapiro, Woan, and Getu \([18]\) and Brändén \([6]\). First, we recall some definitions from \([10]\). Let \( \sigma \in \mathcal{S}_n \) with boundary condition \( \sigma(0) = 0 \) and \( \sigma(n + 1) = 0 \), for \( x \in [n] \) the \( x \)-factorization of \( \sigma \) is defined by

\[
\sigma = w_1 w_2 x w_3 w_4, \tag{5.15}
\]

where \( w_2 \) (resp. \( w_3 \)) is the maximal contiguous subword immediately to the left (resp. right) of \( x \) whose letters are all larger than \( x \). Note that \( w_1, \ldots, w_4 \) may be empty. For instance, if \( x \) is a double ascent (resp. double descent), then \( w_2 = \emptyset \) (resp. \( w_3 = \emptyset \)), and if \( x \) is a peak then \( w_2 = w_3 = \emptyset \). Foata and Strehl \([10]\) considered a mapping \( \varphi_x \), i.e., Foata-Strehl action on permutations by exchanging \( w_2 \) and \( w_3 \) in \((5.1)\):

\[
\varphi_x(\sigma) = w_1 w_3 x w_2 w_4.
\]
We denote $\phi$ of $\sigma$ instead, then $\phi$ becomes a double ascent and will be not fixed by $\phi$. Also, we have $\phi'(\sigma) = \sigma$ if $x$ is a peak, valley, otherwise $\phi'(\sigma)$ exchanges $w_2$ and $w_3$ in the $x$-factorization of $\sigma$, which is equivalent to moving $x$ from a double ascent to a double descent or vice versa. Then $\phi'$'s are involutions and commute. Hence, for any subset $S \subseteq [n]$ we can define the map $\phi'_S : S \rightarrow S$ by

$$
\phi'_S(\sigma) = \prod_{x \in S} \phi'_x(\sigma).
$$

In other words, the group $\mathbb{Z}_2^n$ acts on $S$ via the mapping $\phi'_S$ with $S \subseteq [n]$. For example, let $\sigma = 472589316 \in S_9$. If $S = \{3, 4, 5\}$, we have $\phi'_S(\sigma) = 742389516$, see Fig. 2 for an illustration.

For entries $k, l \in [n]$ with $k < l$, we define the following refined statistics on $\sigma \in S_n$,

$$(31-2-4)\{k, l\}(\sigma) := \# \{ i : 1 < i < \sigma^{-1}(k) < \sigma^{-1}(l) \text{ and } \sigma(i) < k < \sigma(i-1) < l \},$$

$$(31-4-2)\{k, l\}(\sigma) := \# \{ i : 1 < i < \sigma^{-1}(l) < \sigma^{-1}(k) \text{ and } \sigma(i) < k < \sigma(i-1) < l \},$$

$$(41-2-3)\{k, l\}(\sigma) := \# \{ i : 1 < i < \sigma^{-1}(k) < \sigma^{-1}(l) \text{ and } \sigma(i) < k < l < \sigma(i-1) \},$$

$$(41-3-2)\{k, l\}(\sigma) := \# \{ i : 1 < i < \sigma^{-1}(l) < \sigma^{-1}(k) \text{ and } \sigma(i) < k < l < \sigma(i-1) \}.$$ 

We denote

$$(31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\sigma)$$

$$:=(31-2-4)\{k, l\}(\sigma) + (31-4-2)\{k, l\}(\sigma) + (41-2-3)\{k, l\}(\sigma) + (41-3-2)\{k, l\}(\sigma),$$
then the following Lemma holds.

**Lemma 5.3.** For \( \sigma \in S_n \) and \( k, l \in [n] \) with \( k < l \), the statistic \((31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}\) is invariant under the group action \( \varphi'_S \) with \( S \subset [n] \), i.e.,

\[
(31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\sigma) = (31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\varphi'_S(\sigma)).
\]

\((5.16)\)

**Proof.** An alternative description of \((31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\sigma)\) is the number pairs \((\sigma(i), \sigma(j))\) such that \(1 \leq i < j < \min\{\sigma^{-1}(k), \sigma^{-1}(l)\}\) and there exists \(i \leq m < j\) such that \(\sigma(m + 1) < k < \sigma(m) < l\) or \(\sigma(m + 1) < k < l < \sigma(m)\), where \((\sigma(i), \sigma(j))\) is a pair of consecutive peak and valley. By consecutive we mean that there are no other peaks and valleys in between \(\sigma(i)\) and \(\sigma(j)\). The number of such pairs is invariant under the action since \(\sigma(i)\) and \(\sigma(j)\) can not move and \(k, l\) can not move over the peak \(\sigma(i)\). \(\Box\)

Since we have

\[
(31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\sigma = \sum_{k, l \in [n]} (31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\sigma),
\]

\((5.18)\)

then we have following Lemma.

**Lemma 5.4.** For \( \sigma \in S_n \) the triple permutation statistic \((pk, val, 31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\) is invariant under the group action \( \varphi'_S \) with \( S \subset [n] \), i.e.,

\[
(pk, val, 31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\sigma = (pk, val, 31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\varphi'_S(\sigma).
\]

\((5.19)\)

**Proof.** Since the statistics \(pk\) and \(val\) are invariant under MFS-action, and Lemma 5.3 with \((5.18)\) ensure that \((31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\) is invariant under MFS-action, which proves the Lemma. \(\Box\)

![Figure 3](image.png)

**Figure 3.** Valley-hopping on \((31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\{k, l\}(\sigma)\)

Note that in Fig. 2 we have

\[
(31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\sigma = 9 = (31-2-4 + 31-4-2 + 41-2-3 + 41-3-2)\varphi'_S(\sigma).
\]

**Lemma 5.5.** The MFS-action \( \varphi'_S \) is closed on the subset \( S_n(3124, 3142, 4123, 4132) \).
Proof. This follows directly form Lemmas 5.2 and 5.4. □

Lemma 5.6. [13, Theorem 3.8] The statistics (2-13) and (31-2) are constant on any orbit under the MFS-action.

Proof of Theorem 5.1. For any permutation \( \sigma \in S_n \), let \( \text{Orb}(\sigma) = \{ g(\sigma) : g \in \mathbb{Z}_n^2 \} \) be the orbit of \( \sigma \) under the MFS-action. The MFS-action divides the set \( S_n \) into disjoint orbits. Moreover, for \( \sigma \in S_n \), if \( x \) is a double descent of \( \sigma \), then \( x \) is a double ascent of \( \varphi'_x(\sigma) \). Hence, there is a unique permutation in each orbit which has no double descent. Now, let \( \bar{\sigma} \) be such a unique element in \( \text{Orb}(\sigma) \), then
\[
\text{da}(\bar{\sigma}) = n - \text{pk}(\bar{\sigma}) - \text{val}(\bar{\sigma});
\]
\[
\text{des}(\bar{\sigma}) = \text{pk}(\bar{\sigma}) - 1 = \text{val}(\bar{\sigma}).
\]

And for any other \( \sigma' \in \text{Orb}(\pi) \), it can be obtained from \( \bar{\pi} \) by repeatedly applying \( \varphi'_x \) for some double ascent \( x \) of \( \bar{\sigma} \). Each time this happens, \( \text{des} \) and \( \text{dd} \) increases by 1, \( \text{da} \) decreases by 1, and \( \text{pk} \) and \( \text{val} \) keep unchanged. We have
\[
\sum_{\sigma' \in \text{Orb}(\sigma)} t^{\text{des} \sigma'} u^{\text{da} \sigma'} v^{\text{dd} \sigma'} w^{\text{val} \sigma'} = (tw)^{\text{val} \sigma}(u + tv)^{\text{da} \sigma}
\]
\[
= (tw)^{\text{des} \sigma}(u + tv)^{n - 2\text{des} \sigma - 1}.
\]

Therefore, by Lemma 5.6, we obtain (5.1). □

6. Another kind of \((q,t)\)-Catalan numbers

In this section, we introduce another kind of \((q,t)\)-Catalan numbers. Let
\[
\tilde{C}_n(q,t,u,v,w) := \sum_{\sigma \in S_{n+1}(312)} q^{(2-13) \sigma} t^{\text{des} \sigma} u^{\text{da} \sigma} v^{\text{dd} \sigma} w^{\text{val} \sigma}
\]

**Theorem 6.1.** We have
\[
\sum_{n \geq 0} \tilde{C}_n(q,t,u,v,w) z^n = \frac{1}{1 - (u + vt) \cdot z - \frac{wtq \cdot z^2}{1 - q(u + vt) \cdot z - \frac{wtq^3 \cdot z^2}{1 - q^2(u + vt) \cdot z - \frac{wtq^5 \cdot z^2}{1 - \cdots}}}}
\]

**Remark 6.2.** When \((q,t,u,v,w) = (q,t,1,1,1)\), Eq (6.2) reduces to the third interpretation of [13, Theorem 1]. The above theorem could be derived by [20, Eq. (28)] and [13].
Lemma 2.8], which was proved by the restricted version of continued fraction on $S_n$. However, we compute the continued fraction by using insertion encoding and Françon-Viennot’s bijection on $S_n(321)$.

**Definition 6.3.** A path diagram of type C of length $n$ is a pair $(\omega, \xi)$, where $\omega$ is a Motzkin path of length $n$, $\xi = (\xi_1, \ldots, \xi_n)$ is integer sequence satisfying that if the $k$-th step of $\omega$ is at height $h$, then $\xi_k = h$. We denote by $PDC_n$ the set of path diagrams of type C of length $n$.

For $\sigma \in S_n(312)$, we construct the path diagram $\Phi_3(\sigma) = (\omega, \xi)$, where $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ is defined as follows. For $j \in [n]$, $s_j = \omega_j - \omega_{j-1}$,

$$s_j = \begin{cases} U & \text{if } j \in \text{Val}\sigma \\ D & \text{if } j \in \text{Pk}\sigma \\ L_b & \text{if } j \in \text{Da}\sigma \\ L_r & \text{if } j \in \text{Dd}\sigma \end{cases} \quad (6.3)$$

As we have $\omega$, then $\xi$ can be obtained by the height of each step of $\omega$.

**Lemma 6.4.** The mapping $\Phi_3 : \sigma \mapsto (\omega, \xi)$ is a bijection.

**Proof.** By Proposition [2.1] we can see that $\Phi_3$ is actually a restriction of Françon-Viennot bijection on $S_n(312)$.

**Proof of Theorem 6.1.** For $\sigma \in S_n(312)$, we derive that

$$q^{(2-13)}\pi \leq\text{des}\pi u \leq\text{da}\pi v \leq\text{dd}\pi w \leq\text{val}\pi = \prod_{i=1}^{n+1} W(s_i, \xi_i) = W((\omega, \xi)), \quad (6.4)$$

with $h$ being the height of the steps $s_i$,

$$W(s_i, \xi_i) = \begin{cases} q^{\xi_i} \cdot w & \text{if } s_i = U; \\ q^{\xi_i} \cdot t & \text{if } s_i = D; \\ q^{\xi_i} \cdot u & \text{if } s_i = L_b; \\ q^{\xi_i} \cdot v \cdot t & \text{if } s_i = L_r. \end{cases} \quad (6.5)$$

Therefore, the corresponding polynomial and weights become

$$\tilde{C}_n(q, t, u, v, w) = \sum_{\pi \in S_n(312)} q^{(2-13)}\pi \leq\text{des}\pi u \leq\text{da}\pi v \leq\text{dd}\pi w \leq\text{val}\pi = \sum_{(\omega, \xi) \in PDC_n} W((\omega, \xi)). \quad (6.6)$$

By Lemma [3.1] we derive the corresponding continued fraction for the generating function of $\tilde{C}_n(q, t, u, v, w)$. 


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