Particular solutions to multidimensional PDEs represented in the form of one-dimensional flow

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Abstract

We represent an algorithm reducing the \((M+1)\)-dimensional nonlinear partial differential equation (PDE) representable in the form of one-dimensional flow \(u_t+w_{x_1}(u, u_x, u_{xx}, \ldots) = 0\), (where \(w\) is an arbitrary local function of \(u\) and its \(x_i\)-derivatives, \(i = 1, \ldots, M\)) to the family of \(M\)-dimensional nonlinear PDEs \(F(u, w) = 0\), where \(F\) is general (or particular) solution of a certain second order two-dimensional nonlinear PDE. Particularly, the \(M\)-dimensional PDE might be an ODE which, in some cases, may be integrated yielding the explicite solutions to the original \((M+1)\)-dimensional PDE. Moreover, the spectral parameter may be introduced into the function \(F\) which yields a linear spectral equation associated with the original PDE. Simplest examples of nonlinear PDEs with explicite solutions are given.

1 Introduction

It is well known that the method of characteristics \([1]\) allows one to integrate the first order nonlinear partial differential equations (PDEs) in arbitrary dimensions. This approach seemed out to be very useful in study of the \((1+1)\)-dimensional systems of hydrodynamic type, where a modification of this method (the holograph method) was developed \([2, 3, 4, 5]\). However, introduction of the higher order derivatives into the nonlinear PDE either destroys the integrability or requires a different integration algorithm. For instance, the \((1+1)\)-dimensional inviscid Bürgers equation is the simplest nonlinear PDE completely integrable by the method of characteristics:

\[
    u_t - uu_x = 0.
\]

Adding the second order derivative into this equation we obtain the well known viscous Bürgers equation:

\[
    u_t + uu_x - u_{xx} = 0. \tag{2}
\]

This equation is also completely integrable, although a different tool must be applied. Namely, it is linearizable by the Hopf-Cole substitution \([6, 7, 8]\). However, the higher-dimensional viscous Bürgers system may not be completely integrated through linearization and must be studied using a different approach. For instance, its solutions with the finite time blow up were studied by the renormalization group method in refs.\([9, 10]\). Next, introducing the third order derivative into eq.\((1)\) we obtain the Korteweg-de Vries equation (KdV)

\[
    u_t + uu_x - u_{xxx} = 0
\]

(3)
integrable by the inverse spectral transform method (ISTM) [11, 12, 13, 14, 15, 16] which is another productive tool for integration of a large class of nonlinear PDEs.

It is shown in ref.[17] that, deforming the characteristics of the (1+1)-dimensional PDE (1), we reveal a family of particular solutions to a large class of multidimensional higher order nonlinear evolutionary PDEs with the KdV-type nonlinearity described by \((M - 1)\)- (or \(M - 2)\) dimensional nonlinear PDEs. In particular, there is a class of (2+1)-dimensional PDEs with solutions satisfying the appropriate ODE, which, in turn, might have explicit solutions.

In this paper we extend the algorithm of ref.[17] to the PDEs writable in the form of one-dimensional flow

\[ u_t + w_{x_1}(u, u_x, u_{xx}, \ldots) = 0, \]  

where \(w\) is an arbitrary local function of \(u\) and its \(x_i\)-derivatives, \(i = 1, \ldots, M\), \(M\) is the number of variables \(x_i\) in the list \(x = \{x_1, x_2, \ldots, x_M\}\). We show that this equation possesses a large manifold of solutions described by the equation \(F(u, w) = 0\) (or \(F(x_1 + ut, w) = 0\)) where \(F\) satisfies a certain two-dimensional second order PDE. This PDE holds for any nonlinear PDE representable in the form (4). Simplest examples of PDEs together with solutions are represented.

The structure of this paper is following. In Sec.2, we consider the nonlinear PDE (4) with \(w = -\frac{1}{2}u^2 + f(u_x, u_{xx}, \ldots)\), whose particular solutions involve the characteristics of eq.(1). Eq.(4) with arbitrary \(w\) is considered in Sec.3. In this case solution of PDE does not depend on the characteristics of eq.(1). The second order PDE for the function \(F\) is discussed in Sec.4. The spectral problem for the nonlinear PDE (4) is derived in Sec.5, where the richness of the solution space is discussed as well. Particular cases of solvable equation \(F = 0\) are represented in Sec.6. Conclusions are given in Sec.7.

2 Equation \(u_t - uu_{x_1} + f_{x_1}(u_x, u_{xx}, \ldots) = 0\)

In this section we propose an algorithm for construction of a family of \(M\)-dimensional reductions to the nonlinear PDE writable in the form (4) with \(w\) having a particular form

\[ w = -\frac{1}{2}u^2 + f(u_x, u_{xx}, \ldots), \]  

which generalizes the equations considered in ref.[17]. Here \(f\) is an arbitrary function of any order derivatives of \(u\) with respect to \(x_i\), \(i = 1, \ldots, M\), but independent on \(u\). Since the total \(x_1\)-derivative \(w_{x_1}\) may be written as

\[ w_{x_1} = Qu_{x_1}, \]  

where \(Q\) is the differential operator

\[ Q = -u + \sum_{i=1}^{M} f_{u_{x_i}} \partial_{x_i} + \sum_{i_1,i_2=1}^{M} f_{u_{x_{i_1}x_{i_2}}} \partial_{x_{i_1}x_{i_2}} + \ldots, \]  

we may rewrite eq.(2) as

\[ E(u) = u_t + Qu_{x_1} = 0. \]
It is important that the function $w$ satisfies the linear equation
\[
E(w) = w_t + Qw_{x_1} = 0,
\] (9)
which may be shown directly. Further we need the following evident relation between $E(u)$ and $E(w)$:
\[
E(w) = QE(u),
\] (10)
which may be derived applying the operator $Q$ to eq.(9) and using relation (6) together with $Qu_t = w_t$. The family of $M$-dimensional reductions of eq.(4) with $w$ defined in eq.(5) may be constructed using the following theorem.

**Theorem 1.** Let the function $u$ be a solution of the following nonlinear $M$-dimensional PDE:
\[
F(\xi, w) = 0, \quad \xi = x_1 + tu,
\] (11)
where the function $F(\xi, w)$ satisfies the two-dimensional second order PDE:
\[
F_{\xi\xi} + F_{ww} \frac{F_{\xi}^2}{F_w^2} - 2F_{\xi}F_{w} \frac{F_{\xi}}{F_w} = M(F, \xi, w), \quad \xi = tu + x_1, \quad F_w|_{F=0} \neq 0.
\] (12)
Here $M$ is some function of $F$, $\xi$ and $w$ satisfying the condition
\[
M(F, \xi, w)|_{F=0} = 0.
\] (13)
Then the function $\psi$,
\[
\psi \equiv E(u) = u_t + w_{x_1},
\] (14)
is a solution of the linear PDE
\[
\left(F_{\xi}t + F_wQ\right)\psi = 0.
\] (15)
If, in addition, $\psi$ satisfies the zero initial-boundary conditions, then $u$ is a solution of nonlinear PDE (4) with $w$ given in form (5).

**Proof.** To derive the nonlinear PDE (11) from eq.(11), we first differentiate eq.(11) with respect to $x_i$ and $t$:
\[
E_{x_1} := F_{\xi}(1 + tu_{x_1}) + F_w w_{x_1} = 0,
\] (16)
\[
E_{x_i} := F_{\xi}(tu_{x_1}) + F_w w_{x_i} = 0, \quad i > 1,
\] (17)
\[
E_t := F_{\xi}(u + tu_t) + F_w w_t = 0.
\] (18)
If $F_{\xi}|_{F=0} = F_w|_{F=0} = 0$, then eqs.(16,18) become identities. We assume that $F_w|_{F=0} \neq 0$ (the case $F_{\xi}|_{F=0} \neq 0$ can be treated similarly). Then solving eqs. (16) and (17) with respect to $w_{x_1}$ and $w_{x_i}$ ($i > 1$) respectively we obtain:
\[
w_{x_i} = -\frac{F_{\xi}}{F_w}(\delta_{1i} + tu_{x_1}), \quad i \geq 1.
\] (19)
Let us consider the second derivatives of eq. (11) with respect to an arbitrary pair $x_i$ and $x_j$ and substitute eqs. (19) for the derivatives $w_{x_i}$, $i \geq 1$:

\[
F_\xi t u_{x_i x_j} + F_w w_{x_i x_j} + (\delta_1 + tu_{x_i})(\delta_1 + tu_{x_j}) \left( F_\xi + F_{ww} \frac{F_\xi}{F_w} - 2F_\xi w \frac{F_\xi}{F_w} \right) = 0.
\]  

(20)

If $F$ satisfies the nonlinear PDE (12), then the last term in eq. (20) vanishes reducing eq. (20) to

\[
F_\xi t u_{x_i x_j} + F_w w_{x_i x_j} = 0.
\]

(21)

Solving this equation with respect to $w_{x_i x_j}$, we obtain the relation between the second derivatives of the functions $w$ and $u$:

\[
w_{x_i x_j} = -\frac{F_\xi}{F_w} t u_{x_i x_j}.
\]

(22)

By induction, for any order derivative $D = \prod_{i=1}^{M} \partial_{n_i}$ (where $n_i$ are arbitrary integers), we obtain the following relation:

\[
F_\xi t D u + F_w D w = 0.
\]

(23)

Consequently, for the differential operator $\tilde{Q} = Q + u$ (with the operator $Q$ given by expression (7)) we may write

\[
E_{\tilde{Q}} := F_\xi t \tilde{Q} u_{x_1} + F_w \tilde{Q} w_{x_1} = 0.
\]

(24)

Now we consider the following combination of eqs. (16, 18, 24):

\[
E_t - u E_{x_1} + E_{\tilde{Q}} = 0,
\]

(25)

which reads

\[
F_\xi t E(u) + F_w E(w) = 0.
\]

(26)

In virtue of relation (10), we may write eq. (26) as eq. (15) with $\psi$ given by expression (14). If $\psi$ satisfies the zero initial-boundary conditions, then $\psi \equiv E(u) \equiv 0$, which is equivalent to eq. (15). □

Now we shall give several remarks.

1. Domain of variables $x_i$, $i = 1, \ldots, M$, might be either bounded or unbounded.

2. For the particular choice of $F$,

\[
F(\xi, w) = \xi + w,
\]

eq. (11) transforms to the form considered in [17].

3. If the function $u$ is a solution of eq. (15) and of eq. (11) with $F_w|_{F=0} \neq 0$, then it satisfies the equation

\[
u_t - \frac{F_\xi}{F_w} (1 + tu_{x_1}) = 0.
\]

(28)

In particular, if $F$ is taken in the form (27), then eq. (28) reduces to

\[
u_t = tu_{x_1} + 1,
\]

(29)

obtained in [17]. To derive eq. (28) we differentiate eq. (11) with respect to $x_1$, solve the resulting equation for $w_{x_1}$ and substitute it into eq. (4).
4. Since the differential operator $Q$ (7) is defined by the function $f$ in expression for $w$ (5), then the zero boundary conditions imposed on the function $\psi$ (14) are completely defined by the mentioned above function $f$ and does not depend on the particular function $F(\xi, w)$.

2.1 Example

As an example we consider the nonlinear PDE

$$u_t + u_{x_1 x_2} - uu_{x_1} + \alpha(u_{x_2}^2)_{x_1} = 0,$$

which may be viewed as a deformation of equation

$$u_t + u_{x_1 x_2} - uu_{x_1} = 0$$

considered in ref.[17]. In this case

$$w = u_{x_2} - \frac{u^2}{2} + \alpha u_{x_2}.$$  

Using eq.(27) as a simple solution of eq.(12) with $M = 0$, we write eq.(11) as

$$u_{x_2} + \alpha u_{x_2} = \frac{u^2}{2} - tu - x_1.$$  

Solving this equation for $u_{x_2}$ we obtain

$$u_{x_2} = - \frac{1}{2\alpha} \left( 1 \pm \sqrt{1 + 2\alpha(u^2 - 2(tu + x_1))} \right).$$

Integration of this ODE yields a solution $u$ in the following implicit form:

$$h_{\pm}(u, x_1, t) = C(x_1, t) + x_2,$$

where

$$h_{\pm}(u, x_1, t) = \frac{1}{2\sqrt{\eta}} \left( \ln \frac{\sqrt{\eta} - u + t}{\sqrt{\eta} + u - t} \pm \ln \frac{\sqrt{\eta} - u + t}{\sqrt{\eta} + u - t} \right) \pm \ln \frac{2\alpha(\eta + \sqrt{\eta}(u - t)) - \sqrt{2\alpha(u - t)^2 - 2\alpha\eta + 1} - 1}{2\alpha(\eta - \sqrt{\eta}(u - t)) - \sqrt{2\alpha(u - t)^2 - 2\alpha\eta + 1} - 1} \pm \sqrt{2\alpha \ln(\sqrt{2\alpha(u - t) + \sqrt{2\alpha(u - t)^2 - 2\alpha\eta + 1}})},$$

with

$$\eta = t^2 + 2x_1.$$  

Here the function $C$ may not be arbitrary but must provide the zero initial-boundary condition for the function $\psi$,

$$\psi = u_t + u_{x_1 x_2} - uu_{x_1} + \alpha(u_{x_2}^2)_{x_1}.$$

5
considered as a solution to eq. (15) with $F_\xi = F_w = 1$ and the first order one-dimensional differential operator $Q = ((1 + 2\alpha u_x x^2)\partial_{x^2} - u)$. Thus we need a single boundary condition at the boundary point, say $x_2 = 0$:

$$\psi|_{x_2=0} = 0 \quad \text{with} \quad u|_{x_2=0} = \chi(x_1, t). \quad (39)$$

Writing eq. (29) at the boundary point $x_2 = 0$ we obtain

$$\chi_t = t\chi_{x_1} + 1. \quad (40)$$

Thus

$$\chi = A(\eta) + t. \quad (41)$$

Substituting $\chi$ from eq. (41) into eqs. (35,36) with $x_2 = 0$ we conclude that $C$ is the function of $\eta$ only. This, in turn, allows us to consider $C(\eta)$ as an arbitrary function of $\eta$ in eq. (35), while the boundary condition (39) may be taken as an implicit definition of $A(\eta)$ in terms of the function $C(\eta)$:

$$C(\eta) = h_+(A(\eta) + t, x_1, t) \Rightarrow$$

$$C(\eta) = \frac{1}{2\sqrt{\eta}} \left( \ln \frac{\sqrt{\eta} - A(\eta)}{\sqrt{\eta} + A(\eta)} \mp \ln \frac{\sqrt{\eta} - A(\eta)}{\sqrt{\eta} + A(\eta)} \right) \pm$$

$$\ln \frac{2\alpha(\eta + \sqrt{\eta}A(\eta)) - \sqrt{2\alpha A(\eta)^2 - 2\alpha\eta + 1} - 1}{2\alpha(\eta - \sqrt{\eta}A(\eta)) - \sqrt{2\alpha A(\eta)^2 - 2\alpha\eta + 1} - 1} \pm$$

$$\pm \frac{\sqrt{2\alpha} \ln(\sqrt{2\alpha A(\eta)} + \sqrt{2\alpha A(\eta)^2 - 2\alpha\eta + 1})}{\sqrt{2\alpha}}. \quad (42)$$

If $\alpha \ll 1$, then expansion of (36) in powers of $\alpha$ (up to the linear in $\alpha$ term) yields

$$h_+ = \frac{1}{\sqrt{\eta}} \ln \frac{\sqrt{\eta} - u + t}{\sqrt{\eta} + u - t} + \sqrt{2\alpha} \ln 2 + \alpha(u - t) + o(\alpha), \quad (43)$$

$$h_- = -\alpha(u - t) - \sqrt{2\alpha} \ln 2 + o(\alpha). \quad (44)$$

Thus, the solution $u$ corresponding to $h_-$ in eq. (35) becomes singular as $\alpha \to 0$. The solution $u$ corresponding to $h_+$ can be written as

$$u = u_1 + \frac{\alpha}{2}(C_2(\eta) - u_1)(u_1^2 - \eta) + o(\alpha) \quad (45)$$

where $u_1$ is a solution of eq. (31) obtained in ref. [17]:

$$u_1 = t \pm \frac{1 + \tilde{C}(\eta)e^{\pm 2\sqrt{\eta}}}{1 - \tilde{C}(\eta)e^{\pm 2\sqrt{\eta}}}, \quad (46)$$

$$\tilde{C}(\eta) = e^{(C_1(\eta) - \sqrt{2\alpha \ln 2})/\sqrt{\eta}}.$$

and we represent an arbitrary function $C(\eta)$ of eq. (35) in the form $C(\eta) = C_1(\eta) + \alpha C_2(\eta)$. Considering solutions having finite asymptotics as $t \to \infty$, we note that $u_1 \to 0$ as $t \to \infty$. Thus, if $C_2(\eta) = \frac{C_2(\eta)}{\eta}$ and $\tilde{C}_2(\eta) \to c_2 = const$ as $t \to \infty$, then $u \to -\frac{c_2}{2}$ as $t \to \infty$. If $C_2(\eta) \to 0$ as $t \to \infty$, then $u \to 0$ as well (up to $o(\alpha)$).
3 Equation (4) with arbitrary \( w \)

In this section we modify an algorithm of Sec.2 to solve equation of more general form (4), where \( w \) is an arbitrary function of \( u \) and its derivatives with respect to \( x_i, i = 1, \ldots, M \). Now we may rewrite eq.(4) as eq.(8) where the linear differential operator \( Q \) reads:

\[
Q = w_u + \sum_{i=1}^{M} w_{ux_i} \partial_{x_i} + \sum_{i_1,i_2=1}^{M} w_{ux_{i_1}x_{i_2}} \partial_{x_{i_1}x_{i_2}} + \ldots . \tag{47}
\]

It is important that the function \( w \) satisfies the linear equation (9) with \( Q \) given by (47). Relation (10) holds as well.

The family of \( M \)-dimensional reductions for eq.(4) may be constructed using the following theorem, which is similar to Theorem 1 of Sec.2.

**Theorem 2.** Let the function \( u \) be a solution of the following nonlinear \( M \)-dimensional PDE:

\[
F(u, w) = 0, \tag{48}
\]

where the function \( F(u, w) \) satisfies the PDE:

\[
F_{uu} + F_{ww} \frac{F_u^2}{F_w} - 2F_{uw} \frac{F_u}{F_w} = M(F, u, w), \quad F_w|_{F=0} \neq 0 \tag{49}
\]

and \( M \) is some function of \( F, u \) and \( w \) satisfying the condition

\[
M(F, u, w)|_{F=0} = 0. \tag{50}
\]

Then the function \( \psi \),

\[
\psi \equiv E(u) = u_t + w_{x_1}, \tag{51}
\]

is a solution of the linear PDE

\[
\left( F_u + F_w Q \right) \psi = 0. \tag{52}
\]

If, in addition, \( \psi \) satisfies the zero initial-boundary conditions, then \( u \) is a solution of nonlinear PDE (4).

**Proof.** The proof of this theorem is quite similar to that of Theorem 1. To derive nonlinear PDE (4) from eq.(48) we differentiate eq.(48) with respect to \( x_i \) and \( t \):

\[
E_{x_i} := F_u u_{x_i} + F_w w_{x_i} = 0, \quad i > 1, \tag{53}
\]

\[
E_t := F_u u_t + F_w w_t = 0. \tag{54}
\]

If \( F_w|_{F=0} = F_u|_{F=0} = 0 \), then eqs.(53,54) become identities. Assume that \( F_w|_{F=0} \neq 0 \) (the case \( F_u|_{F=0} \neq 0 \) can be treated similarly). Then eq.(53) yields

\[
w_{x_i} = -\frac{F_u}{F_w} u_{x_i}, \quad i = 1, 2, \ldots . \tag{55}
\]
Let us consider the second derivatives with respect to an arbitrary pair \( x_i \) and \( x_j \) and substitute eqs.\(^{(55)}\) for \( w_{x_i} \), \( i \geq 1 \):

\[
F_u u_{x_i} x_j + F_w w_{x_i} x_j + u_{x_i} u_{x_j} \left( F_{uu} + F_{wu} \frac{F_u^2}{F_w^2} - 2F_{uw} \frac{F_u}{F_w} \right) = 0. \tag{56}
\]

If \( F \) satisfies the nonlinear PDE \(^{(49)}\), then eq.\(^{(56)}\) yields

\[
F_u u_{x_i} x_j + F_w w_{x_i} x_j = 0, \quad \Rightarrow \quad w_{x_i x_j} = -\frac{F_u}{F_w} u_{x_i} x_j, \tag{57}
\]

By induction, for any order derivative \( D = \prod_{i=1}^{M} \partial_{x_i}^{n_i} \) (where \( n_i \) are arbitrary integers) we obtain the following relation:

\[
F_u D u + F_w D w = 0. \tag{58}
\]

Consequently, for the differential operator \( Q \) \(^{(47)}\) we have

\[
E_Q := F_u Q u_{x_1} + F_w Q w_{x_1} = 0. \tag{59}
\]

Now we may consider the following combination of eqs.\(^{(54,59)}\):

\[
E_t + E_Q = 0, \tag{60}
\]

which reads

\[
F_u E(u) + F_w E(w) = 0. \tag{61}
\]

In virtue of relation \(^{(10)}\) we may write eq.\(^{(61)}\) as eq.\(^{(52)}\) with \( \psi \) given by expression \(^{(51)}\). If, in addition, \( \psi \) satisfies the zero initial-boundary conditions, then \( \psi \equiv E(u) \equiv 0 \), which is equivalent to eq.\(^{(4)}\). \( \square \)

Now we shall give several remarks similar to those given in Sec.2

1. Domain of variables \( x_i, \ i = 1, \ldots, M \), might be either bounded or unbounded.

2. Eq.\(^{(49)}\) is equivalent to eq.\(^{(12)}\) up to the re-notations \( \xi \leftrightarrow u \).

3. If the function \( u \) satisfies eqs.\(^{(4)}\) and \(^{(48)}\) with \( F_w |_{F=0} \neq 0 \), then it satisfies equation

\[
u_t - \frac{F_u}{F_w} u_{x_1} = 0. \tag{62}\]

In particular, if \( F \) is taken in the form \( F = u + w \), then eq.\(^{(28)}\) reduces to

\[
u_t = u_{x_1}. \tag{63}\]

To derive eq.\(^{(62)}\) we differentiate eq.\(^{(48)}\) with respect to \( x_1 \), solve the resulting equation for \( w_{x_1} \) and substitute it into eq.\(^{(4)}\).

4. Since the differential operator \( Q \) \(^{(47)}\) is defined by the function \( w \) in PDE \(^{(4)}\), then the zero boundary condition imposed on the function \( \psi \) \(^{(51)}\) is completely defined by the above function \( w \) and does not depend on the particular function \( F(u, w) \).
3.1 Example

As an example, we consider the nonlinear PDE

\[ u_t + u_{x_1x_2} - (u^3)_{x_1} = 0. \] (64)

In this case

\[ w = u_{x_2} - u^3. \] (65)

The simple solution to eq.(49) is following \((M = 0)\):

\[ F(u, w) = w - \gamma_1 u - \gamma_0, \] (66)

which corresponds to the reduction \(u_t = -\gamma_1 u_{x_1}\) in eq.(4). Then eq.(48) reads

\[ u_{x_2} = u^3 + \gamma_1 u + \gamma_0. \] (67)

Integration of ODE (67) yields

\[
\frac{1}{(u_1 - u_2)(u_1 - u_3)} \ln(u - u_1) + \frac{1}{(u_2 - u_1)(u_2 - u_3)} \ln(u - u_2) + \frac{1}{(u_3 - u_1)(u_3 - u_2)} \ln(u - u_3) = x_2 + C(x_1, t),
\] (68)

where \(u_1, u_2\) and \(u_3\) are roots of the polynomial equation

\[ u^3 + \gamma_1 u + \gamma_0 = 0. \] (69)

In particular, if \(\gamma_0 = 0, \gamma_1 = -1\), then integration of eq.(67) yields

\[ u^2 = \frac{1}{1 + C(x_1, t)e^{2x_2}}. \] (70)

The functions \(C\) (or \(\tilde{C}\)) may not be arbitrary but must provide the zero initial-boundary condition for the function \(\psi\),

\[ \psi = u_t + u_{x_1x_2} - (u^3)_{x_1}, \] (71)

as a solution to the first order linear PDE (52) with \(F_u = -\gamma_1, F_w = 1\) and \(Q = \partial_{x_2} - 3u^2\).

Thus we have a single boundary condition at the boundary point, say \(x_2 = 0\):

\[ \psi|_{x_2=0} = 0 \text{ with } u|_{x_2=0} = \chi(x_1, t). \] (72)

Eq.(62) with \(F\) given in eq.(66) at the boundary point \(x_2 = 0\) reads

\[ \chi_t + \gamma_1 \chi_{x_1} = 0. \] (73)

Thus

\[ \chi = A(\eta), \] (74)
\[ \eta = x_1 - \gamma_1 t. \]  

(75)

Substituting \( \chi \) from eq. (74) into eqs. (68) and (70) with \( x_2 = 0 \) we see that the function \( C \) (or \( \tilde{C} \)) must be a function of the single variable \( \eta \). Thus, \( C \) and \( \tilde{C} \) may be taken as arbitrary functions of \( \eta \) in eqs. (68) and (70), while boundary condition (72) should be considered as a definition of the function \( A(\eta) \) in terms of \( C(\eta) \) or \( \tilde{C}(\eta) \):

\[
C(\eta) = \frac{1}{(u_1 - u_2)(u_1 - u_3)} \ln(A(\eta) - u_1) + \frac{1}{(u_2 - u_2)(u_2 - u_3)} \ln(A(\eta) - u_2) + \frac{1}{(u_3 - u_1)(u_3 - u_2)} \ln(A(\eta) - u_3). \]

(76)

and

\[
A^2(\eta) = \frac{1}{1 + \tilde{C}(\eta)e^{2x_2}}. \]

(77)

If \( \tilde{C} \) is a bounded function of argument, then solution (70) is a bounded solution to eq.(64).

4 On solutions to nonlinear PDEs (12) and (49)

Thus we reduce \((M + 1)\)-dimensional PDE (11) to \(M\)-dimensional PDE (11) (if \( w \) is in the form (5)) or (48) (which holds for any \( w \)), where \( F \) satisfies the two-dimensional second order PDE, respectively, eq. (12) or (49). Equations (11,12) are equivalent to eqs.(48,49) up to the replacement \( \xi \leftrightarrow u \). We combine them in the following pair of equations:

\[
F(p,q) = 0, \]

(78)

and

\[
F_{pp}F_q^2 + F_{qq}F_p^2 - 2F_{pq}F_pF_q = \mathcal{M}(F,p,q), \]

(79)

which is symmetrical in the variables \( p \) and \( q \). The variable \( p \) must be taken for either \( \xi \) or \( u \), while the variable \( q \) must be taken for the variable \( w \). In addition, the function \( \mathcal{M} \) must satisfy the conditions

\[
\mathcal{M}(F,p,q)|_{F(p,q)=0} = 0. \]

(80)

Eq. (79) may be viewed as the compatibility condition of \((M + 1)\)-dimensional PDE (11) and \(M\)-dimensional PDE (78). In Secs 2 and 3 we consider examples of particular solutions to the nonlinear PDE (11) associated with the simplest solution to eq. (79). However, this equation possesses a rich manifold of solutions parametrized by two arbitrary functions of single variable and deserves the detailed study which is not represented in this paper. Below we consider only three particular solutions of eq.(79) in the form of a degenerate function of \( p \) and \( q \):

\[
F(p,q) = \sum_i f_i(p)g_i(q). \]

(81)
Example 1. A simplest solution reads:

$$F = \alpha_0 + \alpha_1 p + \alpha_2 q,$$

(82)

where $\alpha_i, i = 0, 1, 2$, are arbitrary constant parameters. Namely this solution is used in examples of Secs.2.1 ($\alpha_0 = 0$, $\alpha_1 = \alpha_2 = 1$) and 3.1 ($\alpha_0 = -\gamma_0$, $\alpha_1 = -\gamma_1$, $\alpha_2 = 1$). In this case $M = 0$ in eq.(79).

It is worthwhile to note that the local linear expansion of function $F(p,q)$ in the neighborhood of a fixed point $(p_0,q_0)$ reads (we assume that the first derivatives of $F$ exist in that point)

$$F(p,q) \approx F(p_0,q_0) + F_p(p_0,q_0)(p-p_0) + F_q(p_0,q_0)(q-q_0),$$

(83)

which coincides with expression (82) if $\alpha_0 = F(p_0,q_0) - F_p(p_0,q_0)p_0 - F_q(p_0,q_0)q_0$, $\alpha_1 = F'_p(p_0,q_0)$, $\alpha_2 = F'_q(p_0,q_0)$. Consequently, function (82) is responsible for the local solvability of PDE (4).

Example 2. Next, we suggest the following function:

$$F = e^{\alpha_p} + c_2 e^{\beta_p + c_1 q} + c_3 e^{\frac{\alpha q}{\alpha - \beta}},$$

(84)

where $\alpha, \beta, c_i, i = 1, 2, 3$, are arbitrary parameters. In this case

$$M = a_1 F + a_2 F^2,$$

(85)

$$a_1 = -\left(\frac{\alpha c_1}{\alpha - \beta}(\alpha - \beta)c_2 e^{\beta p + c_1 q} + \alpha c_3 e^{\frac{\alpha q}{\alpha - \beta}}\right)^2,$$

$$a_2 = (\alpha c_1)^2 \left(c_2 e^{\beta p + c_1 q} + \frac{\alpha^2 c_3}{(\alpha - \beta)^2} e^{\frac{\alpha q}{\alpha - \beta}}\right).$$

Example 3. Another example of solution to eq.(79) is following:

$$F = e^{2p} + e^p g(q) + e^{c_1} c_2,$$

(86)

where $c_1$ and $c_2$ are arbitrary constants and $g(w)$ satisfies the following second order ODE:

$$g'' + \frac{1}{4e^{c_1} c_2 - g^2} \left(g(g')^2 - e^{c_1} c_1 c_2 (4g' - c_1 g)\right) = 0.$$

(87)

It may be written in terms of variable $s = e^{\alpha c_1}$ and function $h(s) = g\left(\frac{\ln(s)}{c_1}\right)$ as

$$(s c_1)^2 \left(h'' + \frac{h((h')^2 s - hh' + c_2)}{s(4sc_2 - h^2)}\right) = 0.$$  

(88)

In this case

$$M = (a_1 + a_2 e^p) F + (a_3 + a_4 e^p) F^2,$$

(89)

$$a_1 = -d^{-1}4c_2 e^{c_1 q}(2c_1 c_2 e^{c_1 q} - gg')^2,$$

$$a_2 = -8c_1^2 c_2 g^2(2g' + c_1 g) - e^{c_1 q} c_2 g(4c_1^2 g^2 - 16c_1 gg' - 4(g')^2) - 4g^3 (g')^2,$$

$$a_3 = d^{-1}4(2c_1 c_2 e^{c_1 q} - gg')^2,$$

$$a_4 = d^{-1}4c_1 c_2 e^{c_1 q}(c_1 g - 4g') + 4g(g')^2,$$

$$d = 4c_2 e^{c_1 q} - g^2.$$
We may enrich the solution space adding arbitrary parameters (which might be called spectral parameters) in the function $F$. These parameters appear in solution (54) (parameters $\alpha$, $\beta$, $c_i$, $i = 1, 2, 3$) and in solution (56) (parameters $c_1, c_2$). In addition, two arbitrary functions of single variables (say, $C_i(q)$, $i = 1, 2$) appear in the course of integration of the second order two-dimensional nonlinear PDE (79). Let us denote the set of all arbitrary parameters by the vector parameter $\lambda$ and use it as follows.

Instead of the variable $w$ (which is a function of $u$ and its $x$-derivatives) in the list of parameters of the function $F$ we use another variable $W(\lambda, C_1, C_2)$ related with $w$ by the integral

$$w(u, u_x, u_{xx}, \ldots) = \int DC_1 DC_2 \int d\Omega(\lambda) W(\lambda, C_1, C_2),$$

(90)

where $\Omega$ is some measure in the space of the vector parameter $\lambda$ and $\int DC_1 DC_2$ means the functional integration with respect to the functions $C_i$, $i = 1, 2$. This equation must be considered as a PDE for the function $u$, which is independent on $\lambda$. Now we have to replace equations (11) and (48) with the following ones, respectively

$$F(\xi, W(\lambda, C_1, C_2), \lambda) = 0,$$

(91)

$$F(u, W(\lambda, C_1, C_2), \lambda) = 0.$$

(92)

These equations must be solved for $W$. Applying the same algorithm as in Secs. 2 and 3 we derive equations

$$F_t E(u) + F_w E(W) = 0,$$

(93)

$$F_u E(u) + F_w E(W) = 0$$

(94)

instead of eqs. (26) and (61) respectively. Here

$$E(W) = W_t + QW_{x_1}.$$

(95)

Now we divide eqs. (93) and (94) by $F_W$ and apply the operator $\int DC_1 DC_2 \int d\Omega(\lambda)$. In virtue of eqs. (90) and (10) we obtain

$$\left( \int DC_1 DC_2 \int d\Omega(\lambda) \frac{F_t}{F_W} + Q \right) \psi = 0,$$

(96)

$$\left( \int DC_1 DC_2 \int d\Omega(\lambda) \frac{F_u}{F_W} + Q \right) \psi = 0.$$  

(97)

Note that $Q$ and $\psi$ are different in eqs. (96) and (97), see eqs. (714) and (47511) respectively. Now we may use the zero boundary conditions discussed in Theorems 1 and 2. Finally, solving eq. (90) for $u$ with the above boundary conditions we obtain a solution to the original $(M+1)$-dimensional PDE.

Now we turn to eqs. (93) and (94). Since the constructed $u$ is a solution to the original nonlinear PDE, the first terms in eqs. (93) and (94) vanish. Consequently, since $F_W \neq 0$, we derive a linear PDE for the spectral function $W(\lambda)$:

$$W_t(\lambda, C_1, C_2) + QW_{x_1}(\lambda, C_1, C_2) = 0.$$  

(98)
This equation can be considered as a spectral equation for the nonlinear PDE (4). Emphasize that, unlike the ISTM, we have only one spectral problem associated with the given nonlinear PDE. Consequently, PDE (4) may not be viewed as the compatibility condition for some overdetermined system of linear PDEs.

5.1 On the richness of the solution space to the \((M+1)\)-dimensional nonlinear PDE (4)

The richness of solution space to general \((M+1)\)-dimensional nonlinear PDE (4) is defined by two sets of arbitrary functions. First set consists of two arbitrary functions of single variable appearing in the general solution of two-dimensional second order PDE (79) with the given function \(\mathcal{M}\). Moreover, the function \(\mathcal{M}\) is an arbitrary function of three arguments (restricted by the only condition (80)) which, in addition, might arbitrarily depend on the spectral parameter \(\lambda\). The second set appears in the general solution to either eq.(11), or (48), or (90), which are the \(M\)-dimensional (in general) nonlinear PDEs due to the fact that \(w\) is an \(M\)-dimensional differential expression of \(u\). Thus, its general solution depends on the arbitrary functions of \(M\) variables: \((M-1)\) variables from the list \(x\) supplemented by the variable \(t\). However, these arbitrary functions satisfy the zero boundary conditions for the function \(\psi = u_t + w_{x_1}\), which are discussed in Theorems 1 and 2 (see Secs.2 and 3 respectively). This fact imposes additional constraints on the above arbitrary functions. The unresolved problem is whether the combination of these two sets of arbitrary functions together with arbitrary function \(\mathcal{M}\) and spectral parameter may lead to the complete integrability of the nonlinear PDE (4). This problem is left for further study. The role of functional integration in equation (90) should be also clarified.

6 Solvability of eqs. (11) and (48)

After the function \(F(p,q)\) is constructed, we have to solve \(M\)-dimensional PDE, either (11) or (48), for the function \(u\). Thus we deal with the integrability of these equations. As a simple case we consider eq.(48) of Sec.3 with \(F(u,w) = u + w\):

\[ u + w = 0. \]  

(99)

This equation may be integrated, for instance, in the following cases.

6.1 Eq.(99) is integrable by the inverse spectral transform method: short-pulse equation

First, we consider the case when eq.(99) is integrable by ISTM. For instance, let eq.(99) be a short-pulse equation (SPE) [18, 19]

\[ u_{x_2x_3} + (u^3)_{x_2x_2} + u = 0, \]  

(100)

i.e.

\[ w = u_{x_2x_3} + (u^3)_{x_2x_2}. \]  

(101)

In this case eq.(41) reads

\[ u_t + (u_{x_2x_3} + (u^3)_{x_2x_2})x_1 = 0 \]  

(102)
and may be considered as an additional commuting flow to SPE (100). Remember, that not any solution of SPE satisfies eq. (102), but only those that satisfy the zero boundary condition for the function

$$\psi = u_t + (u_{x2x3} + (u^3)_{x2x2})_{x1}$$

(103)

as a solution to a 2-dimensional PDE (52) where \(F_u = F_w = 1\) and the differential operator \(Q\) is defined as

$$Q = 6(u^2_{x2} + uu_{x2x2}) + 12uu_x\partial_{x2} + 3u^2\partial_{x2x2} + \partial_{x2x3}.$$  

6.2 Linearizable eq. (99)

Now we consider the linearizable equation (99). For instance, if \(w = -\ln(u_{x3} + u_{x2x2} - u^2_{x2})\), then eq. (99) is linearizable by the substitution \(u = -\ln \varphi\) yielding

$$\ln(-\varphi_{x3} - \varphi_{x2x2}) = 0 \Rightarrow \varphi_{x3} + \varphi_{x2x2} + 1 = 0.$$  

(104)

The corresponding PDE (4) reads

$$u_t - \ln(x_1(u_{x3} + u_{x2x2} - u^2_{x2}) = 0.$$  

(105)

Therewith the function \(\psi = u_t - \ln(x_1(u_{x3} + u_{x2x2} - u^2_{x2})\) must satisfy the zero boundary condition for the PDE (52) with \(F_u = F_w = 1\) and \(Q = -\partial_{x3} + \partial_{x2x2} - 2u_{x2}\partial_{x2}.$$

6.3 Eq. (99) reducible to ODE

In this subsection we consider the case when eq. (99) itself can be represented in form (4):

$$u + w^{(1)}(u, u_{y1}, u_{y1y1}, \ldots) = 0$$

(106)

with some function \(w^{(1)}\) of \(u\) and its derivatives with respect to \(x_i, i = 2, 3, \ldots, M\). Hereafter in this section we use the lists \(y_i, i = 1, 2, \ldots, M - 1\), of variables \(x_i\) defined as \(y_i = \{x_{i+1}, \ldots, x_{M}\}\), and functions \(w^{(1)}(u, u_{y1}, u_{y1y1}, \ldots)\) depending on \(u\) and its derivatives with respect to the variables \(x_{i+1}, x_{i+2}, \ldots, x_M\). Thus, eq. (106) has solutions defined by equation of form (99):

$$u + w^{(1)} = 0.$$  

(107)

If, in turn, eq. (107) is representable in form (4),

$$u + w^{(1)} \equiv u_{x2} + w^{(2)}_{x3}(u, u_{y2}, u_{y2y2}, \ldots) = 0,$$  

(108)

then it possesses solutions defined by equation of form (99):

$$u + w^{(2)} = 0.$$  

(109)

and so on. If this process may be continued, then, on the \(i\)th step, we have the equation

$$u + w^{(i-1)} \equiv u_{x_{i+1}} + w^{(i)}_{x_{i+1}}(u, u_{y1}, u_{y1y1}, \ldots) = 0,$$  

(110)

which possesses solutions defined by equation of form (99):

$$u + w^{(i)} = 0.$$  

(111)
On the last step \(y_{M-1} = x_M\), so that we end up with ODE:

\[
 u + w^{(M-1)}(u, u_{xM}, u_{xMxM}, \ldots) = 0.
\]

(112)

On each step we have to provide a proper zero boundary conditions for the function \(\psi^{(i)} = u_{xi} + w^{(i)}_{x1} \) as a solution to the linear differential equation \((1 + Q^{(i)})\psi^{(i)} = 0\) with the differential operator \(Q^{(i)} = w^{(i)}_u + \sum_{j=1}^{M} w^{(i)}_{xj} \partial_{xj} + \sum_{j_1,j_2=1}^{M} w^{(i)}_{xj_1xj_2} \partial_{xj_1} \partial_{xj_2} + \ldots\). Remember that PDE (106) itself is supplemented by the boundary condition for the function \(\psi = u_t + w_{x1}\) as a solution to PDE \((1 + Q)\psi = 0\) with \(Q\) given by expression (17). The simplest equation of this form is following

\[
 u_t + (u_x - u + f(x))x_x \equiv u_t - u_x + u_{xx} + f_{xx}(u) = 0,
\]

(113)

where \(f\) is an arbitrary function of \(u\).

7 Conclusions

We represent an algorithm reducing a large class of \((M + 1)\)-dimensional nonlinear evolutionary PDE having the form of one-dimensional flow (4) to a family of \(M\)-dimensional nonlinear PDEs (48) (or (11) if \(w\) is in the form (5)). The compatibility of \((M + 1)\)-dimensional and \(M\)-dimensional PDEs leads to the two-dimensional second order PDE (79) for the function \(F\). It is remarkable, that this PDE holds for any \((M + 1)\)-dimensional PDE of the form (4). Moreover, a vector spectral parameter may be introduced into the PDE for \(F\). The presence of this parameter leads to the single linear spectral equation associated with the original \((M + 1)\)-dimensional nonlinear PDE. However, there is no second spectral equation which would compose the Lax representation, like in the case of the ISTM. The solvability of the equation for \(F\) is the first nontrivial step in construction of particular solutions since this equation is nonlinear one, which will be studied in a different paper. Here we represent several particular solutions to this equation in Sec.4. Another problem is solvability of the \(M\)-dimensional PDE \(F = 0\) for the function \(u\), which involves the function \(w\) from eq.(4) and must be studied in each particular case. In some cases, we may reduce the \((M + 1)\)-dimensional PDEs to ODEs, which, sometimes, might be explicitly solved, as is demonstrated in the examples of Secs.2.1 and 3.1, see also Sec.6.3.

It is obvious that there is a large manifold of the nonlinear PDEs from the derived class that have a physical application. We may refer to the nonlinear PDEs considered in ref.[17], which concern our case as well. A deformation of one of these equations is considered in Sec.2.1.

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