Dissipated Compacta*

Kenneth Kunen†‡

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Abstract

The dissipated spaces form a class of compacta which contains both the scattered compacta and the compact LOTSe (linearly ordered topological spaces), and a number of theorems true for these latter two classes are true more generally for the dissipated spaces. For example, every regular Borel measure on a dissipated space is separable.

A product of two compact LOTSe is usually not dissipated, but it may satisfy a weakening of that property. In fact, the degree of dissipation of a space can be used to distinguish topologically a product of $n$ LOTSe s from a product of $m$ LOTSe s.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As usual, a subset of a space is perfect iff it is closed and non-empty and has no isolated points, so $X$ is scattered iff $X$ has no perfect subsets.

There are many constructions in the literature which build a compactum $X$ as an inverse limit of metric compacta $X_{\alpha}$ for $\alpha < \omega_1$, with the bonding maps $\pi_{\alpha}^\beta : X_{\beta} \to X_{\alpha}$ for $\alpha < \beta < \omega_1$. In some cases, as in [7, 11, 12], the construction has the property that for each $\alpha, \beta$, $(\pi_{\alpha}^\beta)^{-1}\{x\}$ is a singleton for all but countably many $x \in X_{\alpha}$. We shall call such $\pi_{\alpha}^\beta$ tight maps; these are discussed in greater detail in Section 2. The spaces $X$ so constructed are examples of dissipated compacta; these are discussed in Section 3. Section 4 shows that the property of tightness is absolute for transitive models of set theory.

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†University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu
‡Author partially supported by NSF Grant DMS-0456653.
The precise definition of “dissipated” in Section 3 will be that there are “sufficiently many” tight maps onto metric compacta; so the definition will not mention inverse limits. Then, Section 6 will relate this definition to inverse limits.

Dissipated compacta include the scattered compacta, the metric compacta, and the compact LOTSe (totally ordered spaces with the order topology). Section 3 also describes the more general notion of \( \kappa \)-dissipated, which gets weaker as \( \kappa \) gets bigger; “dissipated” is the same as as “2–dissipated”, while “1–dissipated” is the same as “scattered”. Every regular Borel measure on a \( 2^{\aleph_0} \)-dissipated compactum is separable (see Section 5).

If \( X \) is the double arrow space of Alexandroff and Urysohn, then \( X \) is a non-scattered LOTS and hence is 2–dissipated but not 1–dissipated, while \( X^{n+1} \) is \( (2^n + 1) \)-dissipated but not \( 2^n \)-dissipated. Considerations of this sort can be used to distinguish topologically a product of \( n \) LOTSe from a product of \( m \) LOTSe; see Section 4.

2 Tight Maps

As usual, \( f : X \rightarrow Y \) means that \( f \) is a continuous map from \( X \) to \( Y \), and \( f : X \twoheadrightarrow Y \) means that \( f \) is a continuous map from \( X \) onto \( Y \).

**Definition 2.1** Assume that \( X,Y \) are compact and \( f : X \rightarrow Y \).

☞ A loose family for \( f \) is a disjoint family \( \mathcal{P} \) of closed subsets of \( X \) such that for some non-scattered \( Q \subseteq Y \), \( Q = f(P) \) for all \( P \in \mathcal{P} \).

☞ \( f \) is \( \kappa \)-tight iff there are no loose families for \( f \) of size \( \kappa \).

☞ \( f \) is tight iff \( f \) is 2–tight.

This notion gets weaker as \( \kappa \) gets bigger. \( f \) is 1–tight iff \( f(X) \) is scattered, so that “2–tight” is the first non-trivial case. \( f \) is trivially \( |X|^+ \)-tight. The usual projection from \([0,1]^2 \) onto \([0,1] \) is not \( 2^{\aleph_0} \)-tight.

Some easy equivalents to “\( \kappa \)-tight”:

**Lemma 2.2** Assume that \( X,Y \) are compact and \( f : X \rightarrow Y \). Then (1) \( \leftrightarrow \) (2). If \( \kappa \) is finite, then (1) \( \leftrightarrow \) (3); if also \( Y \) is metric, then all five of the following are equivalent:

1. There is a loose family of size \( \kappa \).
2. There is a disjoint family \( \mathcal{P} \) of perfect subsets of \( X \) with \( |\mathcal{P}| = \kappa \) and a perfect \( Q \subseteq Y \) such that \( Q = f(P) \) for all \( P \in \mathcal{P} \).
3. There are distinct \( a_i \in X \) for \( i < \kappa \) with all \( f(a_i) = b \in Y \) such that whenever \( U_i \) is a neighborhood of \( a_i \) for \( i < \kappa \), \( \bigcap_{i<\kappa} f(U_i) \) is not scattered.
4. For some metric $M$ and $\varphi \in C(X, M)$, $\{y \in Y : |\varphi(f^{-1}\{y\})| \geq \kappa\}$ is uncountable.

5. Statement (4), with $M = [0, 1]$.

Proof. (2) $\rightarrow$ (1) is obvious. Now, assume (1), and let $\mathcal{P}$ be a loose family of size $\kappa$, with $Q = f(P)$ for $P \in \mathcal{P}$. Let $Q'$ be a perfect subset of $Q$, and, for $P \in \mathcal{P}$, let $P'$ be a closed subset of $P \cap f^{-1}(Q')$ such that $f|P': P' \rightarrow Q'$ is irreducible. Then $\{P' : P \in \mathcal{P}\}$ satisfies (2).

From now on assume that $\kappa$ is finite.

(3) $\rightarrow$ (1) and (5) $\rightarrow$ (4) are obvious.

For (1) $\rightarrow$ (3), use compactness of $\prod_i P_i$ and the fact that a finite union of scattered spaces is scattered.

For (1) $\rightarrow$ (5): If $\mathcal{P} = \{P_i : i < \kappa\}$ is a loose family, with $Q = f(P_i)$, apply the Tietze Theorem to get $\varphi \in C(X, [0, 1])$ such that $\varphi(x) = \kappa$ for all $x \in P_i$.

Now, we prove (4) $\rightarrow$ (1) when $Y$ is metric. Fix $\varphi$ as in (4). We may assume that $M = \varphi(X)$, so that $M$ is compact. Let $\mathcal{B}$ be a countable base for $M$. Then we can find $B_i \in \mathcal{B}$ for $i < \kappa$ such that the $B_i$ are disjoint and such that $Q := \{y \in Y : \forall i < \kappa [\varphi^{-1}(\{y\}) \cap B_i \neq \emptyset]\}$ is uncountable, and hence not scattered (since $Y$ is metric). $Q$ is also closed. Let $P_i = f^{-1}(Q) \cap \varphi^{-1}(B_i)$. Then $\{P_i : i < \kappa\}$ is a loose family.

Lemma 2.3 If $X, Y$ are compact LOTSe's and $f : X \rightarrow Y$ is order-preserving $(x_1 < x_2 \rightarrow f(x_1) \leq f(x_2))$, then $f$ is tight.

Proof. If not, we would have $a_0 < a_1$ and $b$ as in (3) of Lemma 2.2. Let $U_0, U_1$ be open intervals in $X$ with disjoint closures such that each $a_i \in U_i$. But then $f(U_0) \cap f(U_1) = \{b\}$, a contradiction.

In many cases, the loose family will be defined uniformly via a continuous function, and we may replace the cardinal $\kappa$ in Definition 2.1 by some compact space $K$ of size $\kappa$.

Definition 2.4 Assume that $X, Y, K$ are compact spaces and $f : X \rightarrow Y$. Then a $K$-loose function for $f$ is a $\varphi : \text{dom}(\varphi) \rightarrow K$ such that: $\text{dom}(\varphi)$ is closed in $X$, and for some non-scattered $Q \subseteq Y$, $\varphi(f^{-1}(y)) = K$ for all $y \in Q$.

Note that we then have a loose family $\mathcal{P} = \{P_z : z \in K\}$ of size $|K|$, where $P_z = f^{-1}(Q) \cap \varphi^{-1}(\{z\})$. For finite $n$, we may view the ordinal $n$ as a discrete topological space, so an $n$-loose function is equivalent to a loose family $\mathcal{P} = \{P_i : i < n\}$, since $\varphi$ can map $P_i$ to $i \in n$. The same phenomenon holds for $\aleph_0$, but seems harder to prove:
Theorem 2.5 If $X, Y$ are compact and $f : X \to Y$, then there is an infinite loose family iff there is an $(\omega + 1)$–loose function.

This will be proved in Section 7. Beyond $\aleph_0$, there is no simple equivalence between the cardinal version and the topological version of looseness. At $2^{\aleph_0}$, we shall use the following terminology to avoid possible confusion between the Cantor set $2^\omega$ and the cardinal $c = 2^{\aleph_0}$:

Definition 2.6 Assume that $X, Y$ are compact and $f : X \to Y$.

☞ A strongly $c$–loose family for $f$ is a $K$–loose function $\varphi : \text{dom}(\varphi) \to K$, where $K$ is the Cantor set $2^\omega$.

☞ $f$ is weakly $c$–tight iff there is no strongly $c$–loose function for $f$.

In this paper, whenever we produce a loose family of size $2^{\aleph_0}$, it will usually be strongly $c$–loose. However, if we view $c + 1$ as a compact ordinal and let $X = Y \times (c + 1)$, then assuming that $Y$ is not scattered, the usual projection $f : X \to Y$ has an obvious loose family of size $c$ but no strongly $c$–loose family.

When $X, Y$ are both metric, the $\kappa$-tightness of $f$ is related to the sizes of the sets $f^{-1}\{y\}$ by:

Theorem 2.7 If $X, Y$ are compact metric and $f : X \to Y$, then $f$ is $\kappa$–tight iff $\{y \in Y : |f^{-1}\{y\}| \geq \kappa\}$ is countable. $f$ is weakly $c$–tight iff $f$ is $c$–tight.

In particular, if $f : X \to Y$, then $f$ is tight iff $f^{-1}\{y\}$ is a singleton for all but countably many $y$, as we said in the Introduction.

For both “iff”s, the $\leftarrow$ direction is trivial and is true for any $X, Y$. For $\kappa = 3$, say, the proof of the $\rightarrow$ direction will show that if there are uncountably many $y \in Y$ such that $f^{-1}\{y\}$ contains three or more points, then for some perfect $Q \subseteq Y$, we can, on $Q$, choose three of these points continuously, producing disjoint perfect $P_0, P_1, P_2 \subseteq X$ which $f$ maps homeomorphically onto $Q$, so $\{P_0, P_1, P_2\}$ is a loose family of size 3.

Since $X$ is second countable, each $f^{-1}\{y\}$ is either countable or of size $2^{\aleph_0}$, so it is sufficient to prove the theorem for the cases $\kappa \leq \aleph_0$ and $\kappa = 2^{\aleph_0}$. However, for $\kappa = 2^{\aleph_0}$, we can get more detailed results. For example, if there are uncountably many $y \in Y$ such that $f^{-1}\{y\}$ contains a Klein bottle, then we can choose the bottle continuously on a perfect set (see Theorem 2.9). This “continuous selector” result follows easily from standard descriptive set theory. First, observe:

Lemma 2.8 Suppose that $g : \Phi \to Y$, where $Y$ is a Polish space, $\Phi$ is an analytic subset of some Polish space, and $g(\Phi)$ is uncountable. Then there is a Cantor subset $C \subseteq \Phi$ such that $g$ is 1-1 on $C$. 
Proof. Let \( h : \omega^\omega \to \Phi \), apply the classical argument of Suslin to obtain a Cantor subset \( D \subseteq \omega^\omega \) such that \( g \circ h \) is 1-1 on \( D \), and let \( C = h(D) \).

**Theorem 2.9** Assume that \( X,Y,Z \) are compact metric, \( f : X \to Y \), and there are uncountably many \( y \in Y \) such that \( f^{-1}\{y\} \) contains a homeomorphic copy of \( Z \). Then there is a perfect \( Q \subseteq Y \) and a 1-1 map \( i : Q \times Z \to X \) such that \( f(i(q,u)) = q \) for all \( (q,u) \in Q \times Z \).

**Proof.** Assume that \( Z \neq \emptyset \). Fix metrics \( d_Z,d_X \) on \( Z,X \), and give \( C(Z,X) \) the usual uniform metric, which makes it a Polish space. Let \( \Phi \) be the set of all \( \varphi \in C(Z,X) \) such that \( \varphi \) is 1-1 and \( \varphi(Z) \subseteq f^{-1}\{y\} \) for some (unique) \( y \in Y \). Observe that \( \Phi \) is an \( F_{\sigma \delta} \) set, since the “\( \varphi \) is 1-1” can be expressed as:

\[
\forall \varepsilon > 0 \exists \delta > 0 \forall u,v \in Z [d_Z(u,v) \geq \varepsilon \rightarrow d_X(\varphi(u),\varphi(v)) \geq \delta]
\]

Define \( g : \Phi \to Y \) so that \( g(\varphi) \) is the \( y \in Y \) such that \( \varphi(Z) \subseteq f^{-1}\{y\} \). Using Lemma 2.8 let \( C \subseteq \Phi \) be a Cantor subset with \( g \) 1-1 on \( C \), let \( Q = g(C) \), and let \( i(g(\varphi),u) = \varphi(u) \).

**Proof of Theorem 2.7.** To prove the \( \rightarrow \) direction of the first “iff” in the three cases \( \kappa < \aleph_0 \), \( \kappa = \aleph_0 \), and \( \kappa = \mathfrak{c} \), apply Theorem 2.9 respectively with \( Z \) the space \( \kappa \) (with the discrete topology), \( \omega + 1 \), and \( 2^\omega \). This also yields the \( \rightarrow \) direction of the second “iff”.

Of course, we are using the fact that every uncountable metric compactum contains a copy of the Cantor set. One could also prove Theorem 2.7 using the following, plus the fact that every uncountable metric compactum maps onto \([0,1]\):

**Theorem 2.10** Assume that \( X,Y,K \) are compact metric with \( f : X \to Y \), and assume that for uncountably many \( y \in Y \), there is a closed subset of \( f^{-1}\{y\} \) which can be mapped onto \( K \). Then there is a \( K \)-loose function for \( f \).

**Proof.** Let \( H \) be the Hilbert cube, \([0,1]^\omega\). We may assume that \( K \subseteq H \). Then, for uncountably many \( y \in Y \), there is a \( \psi \in C(X,H) \) such that \( \psi(f^{-1}\{y\}) \supseteq K \). Let \( \Psi = \{(y,\psi) \in Y \times C(X,H) : \psi(f^{-1}\{y\}) \supseteq K \} \), and let \( g(y,\psi) = y \). Applying Lemma 2.8 let \( C \subseteq \Psi \) be a Cantor set on which \( g \) is 1-1, and let \( Q = g(C) \subseteq Y \). For \( (y,\psi) \in C \), let \( E_y = \{x \in X : \psi(x) \in K \} \). Define \( \varphi \) so that \( \text{dom}(\varphi) = \bigcup\{E_y : y \in Q\} \), and \( \varphi(x) = \psi(x) \) whenever \( x \in \text{dom}(\varphi) \) and \( (y,\psi) \in C \). Then \( \varphi \) is a \( K \)-loose function.

Theorems 2.7, 2.9, and 2.10 can fail when \( X \) is not metric; counter-examples are provided by the double arrow space and some related spaces described by:
Definition 2.11 \( I = [0, 1] \). If \( S \subseteq (0, 1) \), then \( I_S \) is the compact LOTS which results by replacing each \( x \in S \) by a pair of neighboring points, \( x^- < x^+ \). The double arrow space is \( I_{(0,1)} \).

\( I_S \) has no isolated points because \( 0, 1 \notin S \). The double arrow space is obtained by splitting all points other than \( 0, 1 \). \( I_0 = I \), and \( I_{\mathbb{Q}(0,1)} \) is homeomorphic to the Cantor set.

Lemma 2.12 For each \( S \subseteq (0,1) \), \( I_S \) is a compact separable LOTS with no isolated points. \( I_S \) is second countable iff \( S \) is countable.

Now, let \( Y = [0,1] \), let \( S \subseteq (0,1) \), let \( X = I_S \) and let \( f : X \to Y \) be the natural map. Then \( f \) is 2–tight by Lemma 2.13 but \( S = \{ y \in Y : |f^{-1}\{y\}| \geq 2 \} \) need not be countable, so Theorems 2.7, 2.9 and 2.10 fail here when \( S \) is uncountable (and hence \( X \) is not metric). However, one can apply these theorems in some generic extension, to get a (perhaps strange) alternate proof that \( f \) is 2–tight. Roughly, if \( V[G] \) makes \( S \) countable, then \( X, Y \) will both be compact metric in \( V[G] \), so Theorem 2.7 implies that \( f \) is 2–tight in \( V[G] \) (because \( S \) is countable); but then by absoluteness, \( f \) is 2–tight in \( V \). Absoluteness of tightness is discussed more precisely in Section 7.

The composition properties of tight maps are given by:

Lemma 2.13 Assume that \( X, Y, Z \) are compact, \( m, n \) are finite, \( f : X \to Y \), and \( g : Y \to Z \). Then:

1. If \( g \circ f \) is \( n \)–tight then \( g \) is \( n \)–tight.
2. If \( f \) and \( g \) are tight, then \( g \circ f \) is tight.
3. If \( f \) is \( m + 1 \)–tight and \( g \) is \( n + 1 \)–tight, then \( g \circ f \) is \( mn + 1 \)–tight.

Proof. (1) is trivial, and (2) is a special case of (3).

For (3), assume that \( f \) is \( m + 1 \)–tight, \( g \) is \( n + 1 \)–tight, and \( g \circ f \) is not \( mn + 1 \)–tight; we shall derive a contradiction. Fix disjoint closed \( P_0, P_1, \ldots, P_{mn} \subseteq X \) with \( g(f(P_0)) \cap g(f(P_1)) \cap \cdots \cap g(f(P_{mn})) \) not scattered. Shrinking \( X, Y, Z \), and the \( P_i \), we may assume WLOG that \( X = P_0 \cup P_1 \cup \cdots \cup P_{mn} \) and that \( g(f(P_i)) = Z \) for each \( i \). For each \( s \subseteq \{0,1,\ldots, mn\} \), let \( Q_s = \bigcap_{i \in s} f(P_i) \). Shrinking the \( P_i \), we may assume WLOG that each \( Q_s \subseteq Y \) is either empty or not scattered; to see this, for a fixed \( s \): if \( Q_s \) is scattered, then so is \( g(Q_s) \); if \( R \) is a perfect subset of \( Z \setminus g(Q_s) \), then we may replace \( Z \) by \( R \) and each \( P_i \) by \( P_i \cap f^{-1}(g^{-1}(R)) \).

Now, using compactness of \( P_0 \times P_1 \times \cdots \times P_{mn} \), as in the proof of Lemma 2.7 fix \( a_i \in P_i \) for \( i \leq mn \) such that \( g(f(U_0)) \cap \cdots \cap g(f(U_{mn})) \) is not scattered whenever each \( U_i \) is a neighborhood of \( a_i \). Then at least one of the following two cases holds:
Case I. Some \( n + 1 \) of the \( f(a_0), \ldots, f(a_{mn}) \) are different. WLOG, these are \( f(a_0), f(a_1), \ldots, f(a_n) \). Choose the \( U_i \) so that the \( f(U_0), f(U_1), \ldots, f(U_n) \) are all disjoint. But then \( g(f(U_0)) \cap \cdots \cap g(f(U_n)) \supseteq g(f(U_0)) \cap \cdots \cap g(f(U_{mn})) \) is not scattered, contradicting the \( n + 1 \)-tightness of \( g \).

Case II. Some \( m + 1 \) of the \( f(a_0), \ldots, f(a_{mn}) \) are the same. WLOG, \( f(a_0) = f(a_1) = \cdots = f(a_m) \). Let \( s = \{0, 1, \ldots, m\} \). Then \( Q_s \neq \emptyset \), so \( Q_s = \bigcap_{i \leq m} f(P_i) \) is not scattered, contradicting the \( m + 1 \)-tightness of \( f \). 

The “\( mn + 1 \)” in (3) cannot be reduced; for example, let \( Y = Z \times n \) and \( X = Y \times m \), with \( f, g \) the natural projection maps.

There is a similar result, with a similar proof, involving products:

**Lemma 2.14** Assume that for \( i = 0, 1 \): \( X_i, Y_i \) are compact, \( f_i : X_i \to Y_i \) is \((m_i + 1)\)-tight, \( m_i \leq n_i < \omega \), and \(|f_i^{-1}(y)| \leq n_i \) for all \( y \in Y_i \). Then \( f_0 \times f_1 : X_0 \times X_1 \to Y_0 \times Y_1 \) is \((\max(m_0n_1, m_1n_0) + 1)\)-tight.

**Proof.** Let \( L = \max(m_0n_1, m_1n_0) \), and let \( f = f_0 \times f_1 \). In view of Lemma 2.2, it is sufficient to fix any \( L + 1 \) distinct points \( a_0^0, a_1^0, \ldots, a_L^0 \in X_0 \times X_1 \) with all \( f(a_0^0) = b \in Y_0 \times Y_1 \), and show that one can find neighborhoods \( U_\alpha \) of \( a_\alpha \) for \( \alpha = 0, 1, \ldots, L \) such that \( \bigcap_\alpha f(U_\alpha) \) is scattered.

Let \( b = (b_0, b_1) \) and \( a_\alpha = (a_\alpha^0, a_\alpha^1) \).

Note that although the \( a_\alpha \) are all distinct points, the \( a_\alpha^0 \) need not be all different and the \( a_\alpha^1 \) need not be all different. However, \(| \{a_\alpha^0 : 0 \leq \alpha \leq L\} | \geq m_0 + 1 \): If not, then using \( f(a_\alpha^0) = b_0 \) and \(|f_0^{-1}(b_1)| \leq n_1 \), we would have \( L + 1 \leq m_0n_1 \), a contradiction. Likewise, \(| \{a_\alpha^1 : 0 \leq \alpha \leq L\} | \geq m_1 + 1 \).

Now, using Lemma 2.2 and the fact that each \( f_i : X_i \to Y_i \) is \((m_i + 1)\)-tight, choose neighborhoods \( U_i^\alpha \) of \( a_i^\alpha \) such that \( \bigcap_\alpha f_i(U_i^\alpha) \) is scattered for \( i = 0, 1 \). The \( U_i^\alpha \) can depend just on the value of \( a_i^\alpha \) (that is \( a_i^\alpha = a_i^\beta \to U_i^\alpha = U_i^\beta \)). Finally, let \( U^\alpha = U_0^\alpha \times U_1^\alpha \).

The bound on the \(|f_i^{-1}(y)| \) cannot be removed here. For example, for each cardinal \( \kappa \), one can find compact perfect LOTses \( X_0, X_1, Y_0, Y_1 \) with order-preserving \( f_i : X_i \to Y_i \) such that all point inverses have size at least \( \kappa \). Then the \( f_i \) are tight by Lemma 2.3, but \( f_0 \times f_1 \) is not \( \kappa \)-tight.

A variant of the product of maps is much simpler to analyze:

**Lemma 2.15** Assume that \( \ell \in \omega \) and \( f_i : X \to Y_i \) is \( \kappa \)-tight for each \( i < \ell \), where \( X \) and the \( Y_i \) are compact. Then the map \( x \mapsto (f_0(x), \ldots, f_{\ell-1}(x)) \) from \( X \) to \( \prod_{i<\ell} Y_i \) is also \( \kappa \)-tight.

We now consider the opposite of tight maps:
Definition 2.16 If $X, Y$ are compact and $f : X \to Y$, then $f$ is nowhere tight iff $f(X)$ is not scattered and there is no closed $P \subseteq X$ such that $f|P$ is tight and $f(P)$ is not scattered.

Note also that if $X, Y$ are metric compacta with $f : X \to Y$ and $Y$ not scattered, then there is a Cantor set $P \subseteq X$ such that $f|P$ is 1-1, so

Lemma 2.17 If $X, Y$ are compact and $f : X \to Y$ is nowhere tight, then $X$ is not second countable.

A further limitation on nowhere tight maps:

Lemma 2.18 If $f : X \to Y$ is nowhere tight, then $f$ is not weakly $c$-tight.

Proof. We shall get a non-scattered $Q \subseteq Y$ and disjoint non-scattered sets $P^k \subseteq X$ for $k \in 2^\omega$ so that each $f(P^k) = Q$. We shall build the $P^k$ and $Q$ by a tree argument. Each $P^k$ will be non-scattered because it will be formed using a Cantor tree of closed sets, so we shall actually get a doubly indexed family. So, we build $Q_s \subseteq Y$ for $s \in 2^{<\omega}$ and $P_s^t \subseteq X$ for $s, t \in 2^{<\omega}$ with $\text{lh}(s) = \text{lh}(t)$ satisfying:

1. $P_s^t$ is closed, $f(P_s^t) = Q_s$, and $Q_s$ is not scattered.
2. The sets $Q_{s_0}, Q_{s_1}$ are disjoint subsets of $Q_s$.
3. The sets $P_s^{t_0}, P_s^{t_1}, P_s^{t_0}, P_s^{t_1}$ are disjoint subsets of $P_s^t$.

We construct these inductively. $P^\emptyset$ and $Q_\emptyset$ exist (where $\emptyset$ is the empty sequence) because $f(X)$ is not scattered. Now, say we have $Q_s$ and $P_s^t$ for all $s, t$ with $\text{lh}(s) = \text{lh}(t) = n$. Fix $s$.

First, get disjoint closed non-scattered $\tilde{Q}_{s_0}, \tilde{Q}_{s_1} \subseteq Q_s$, and let $P_s^t = \bigcap_{s_0, t_0} f^{-1}(Q_{s_0})$ for each $t$ of length $n$ and each $\mu = 0, 1$. Then, use “nowhere tight” $2^n$ times to get $Q_s \subseteq P_s^t$ and $P_s^{t_0} \subseteq P_s^{t_1}$ for each $\nu = 0, 1$ and each $t$ of length $n$ so that each $f(P_s^{t_0}) = Q_s$ and each $Q_s$ is non-scattered.

For $h, k \in 2^{<\omega}$, define $Q_h = \bigcap_{n \in \omega} Q_{h|n}$ and $P_h^k = \bigcap_{n \in \omega} P_{h|n}$, let $Q = \bigcup \{Q_h : h \in 2^{<\omega}\}$, and let $P_h = \bigcup \{P_h^k : k \in 2^{<\omega}\}$ and $P^k = \bigcup \{P_h^k : h \in 2^{<\omega}\}$. Then $f(P_h) = Q_h$ and $f(P^k) = Q$, and the $\varphi$ of Definition 2.6 sends $P^k$ to $k \in 2^{<\omega}$, with $\text{dom}(\varphi) = \bigcup_k P_k$.

Corollary 2.19 If $X, Y$ are compact, $f : X \to Y$, $w(X) < c$, $Y$ is metric and not scattered, and $f$ is weakly $c$-tight, then $X$ has a Cantor subset.
Proof. Since \( f \) is not nowhere tight, we may assume, shrinking \( X \) and \( Y \), that \( f \) is tight. Let \( \kappa = w(X) \), and let \( \mathcal{B} \) be a base for \( X \) with \( |\mathcal{B}| = \kappa \). Whenever \( B_0, B_1 \in \mathcal{B} \) with \( B_0 \cap B_1 = \emptyset \), let \( S(B_0, B_1) = f(B_0) \cap f(B_1) \). Each \( S(B_0, B_1) \) is scattered, and hence countable, so at most \( \kappa \) points of \( Y \) are in some \( S(B_0, B_1) \), so there is a \( K \subseteq Y \) homeomorphic to the Cantor set with \( K \) is disjoint from all \( S(B_0, B_1) \). \( |f^{-1}\{y\}| = 1 \) for all \( y \in K \), so \( f^{-1}(K) \) is homeomorphic to \( K \).

Note that we have not yet given any examples of nowhere tight maps. The argument of Corollary 2.19 shows that one class of examples is given by:

**Example 2.20** If \( X, Y \) are compact, \( f : X \twoheadrightarrow Y \), \( w(X) < \mathfrak{c} \), \( Y \) is metric and not scattered, and \( X \) has no Cantor subset, then \( f \) is nowhere tight.

Of course, under CH, this class of examples is empty. More generally, the class is empty under MA (or just the assumption that \( \mathbb{R} \) is not the union of \( < \mathfrak{c} \) meager sets), since then every non-scattered compactum of weight less than \( \mathfrak{c} \) contains a Cantor subset (see [12]). However, by Dow and Fremlin [3], it is consistent to have a non-scattered compactum \( X \) of weight \( \aleph_1 < \mathfrak{c} \) with no convergent \( \omega \)–sequences, and hence with no Cantor subsets; in the ground model, CH holds, and \( X \) is any compact F-space (so \( w(X) \) can be \( \aleph_1 \)); then, the extension adds any number of random reals.

A class of ZFC examples of nowhere tight maps with \( w(X) = \mathfrak{c} \) is given by:

**Example 2.21** If \( X, Y \) are compact, \( f : X \twoheadrightarrow Y \), \( X \) is a compact F–space and \( Y \) is metric and not scattered, then \( f \) is nowhere tight.

**Proof.** Here, it is sufficient to prove that \( f \) is not tight, since any \( f\rvert P : P \twoheadrightarrow f(P) \) will have the same properties. Also, shrinking \( Y \), we may assume that \( Y \) has no isolated points.

First, choose a perfect \( Q \subseteq Y \) which is nowhere dense in \( Y \). Then, choose a discrete set \( D = \{d_n : n \in \omega\} \subseteq Y \setminus Q \) with \( \overline{D} = D \cup Q \) and each \( f^{-1}\{d_n\} \) not a singleton. Then, choose \( x_n, z_n \in f^{-1}\{d_n\} \) with \( x_n \neq z_n \). Now, since \( X \) is an F–space, \( \text{cl}\{x_n : n \in \omega\} \) and \( \text{cl}\{z_n : n \in \omega\} \) are two disjoint copies of \( \beta\mathbb{N} \) in \( X \) which map onto \( \overline{D} \).

### 3 Dissipated Spaces

Only a scattered compactum \( X \) has the property that all maps from \( X \) are tight: If \( X \) is not scattered, then \( X \) maps onto \([0, 1]^2 \); if we follow that map by the usual projection onto \([0, 1] \), we get a map from \( X \) onto \([0, 1] \) which is not even weakly \( \mathfrak{c} \)–tight.
The dissipated compacta have the property that unboundedly many maps onto metric compacta are tight:

**Definition 3.1** Assume that $X,Y,Z$ are compact, $f : X \to Y$, and $g : X \to Z$. Then $f \leq g$, or $f$ is finer than $g$, iff there is a $\Gamma \in C(f(X),g(X))$ such that $g = \Gamma \circ f$.

**Lemma 3.2** Assume that $X,Y,Z$ are compact, $f : X \to Y$, and $g : X \to Z$. Then $f \leq g$ iff $\forall x_1,x_2 \in X [f(x_1) = f(x_2) \to g(x_1) = g(x_2)]$.

**Proof.** For $\leftarrow$, let $\Gamma = \{(f(x),g(x)) : x \in X\} \subseteq f(X) \times g(X)$. 

**Definition 3.3** $X$ is $\kappa$-dissipated iff $X$ is compact and whenever $g : X \to Z$, with $Z$ metric, there is a finer $\kappa$-tight $f : X \to Y$ for some metric $Y$. $X$ is dissipated iff $X$ is $2$-dissipated. $X$ is weakly $c$-dissipated iff $X$ is compact and whenever $g : X \to Z$, with $Z$ metric, there is a finer weakly $c$-tight $f : X \to Y$ for some metric $Y$.

So, the $1$-dissipated compacta are the scattered compacta. Metric compacta are trivially dissipated because we can take $Y = X$, with $f$ the identity map. Besides the spaces from $[7, 11, 12]$, an easy example of a dissipated space is given by:

**Lemma 3.4** If $X$ is a compact LOTS, then $X$ is dissipated

**Proof.** Fix $g,Z$ as in Definition 3.3. On $X$, use $[x_1,x_2]$ for the closed interval $[\min(x_1,x_2),\max(x_1,x_2)]$, and define $x_1 \sim x_2$ iff $g$ is constant on $[x_1,x_2]$. Then $\sim$ is a closed equivalence relation, so define $Y = X/\sim$ with $f : X \to Y$ the natural projection. Then $Y$ is a LOTS and $f$ is order-preserving, so $f$ is tight by Lemma 2.3 and $f \leq g$ by Lemma 3.2. To see that $Y$ is metrizable, fix a metric on $Z$, and then, on $Y$, define $d(f(x_1),f(x_2)) = \text{diam}(g([x_1,x_2]))$.

By Corollary 2.19 if $w(X) < c$ and $X$ is $c$-dissipated and not scattered, then $X$ has a Cantor subset, while the double arrow space is an example of an $X$ with $w(X) = c$ which is $2$-dissipated and has no Cantor subset.

Note that just having one tight map $g$ from $X$ onto some metric compactum $Z$ is not sufficient to prove that $X$ is dissipated, since the tightness of $g$ says nothing at all about the complexity of a particular $g^{-1}\{z\}$. Trivial counter-examples are obtained with $|Z| = 1$ and $g$ a constant map. However, if all $g^{-1}\{z\}$ are scattered, then just one tight $g$ is enough:

**Lemma 3.5** Suppose that $g : X \to Z$ is $\kappa$-tight and all $g^{-1}\{z\}$ are scattered. Fix $f : X \to Y$ with $f \leq g$. Then $f$ is $\kappa$-tight. In particular, if $Z$ is also metric, then $X$ is $\kappa$-dissipated.
Proof. Fix \( \Gamma \in C(f(X), g(X)) \) such that \( g = \Gamma \circ f \). Suppose that \( \mathcal{P} \) were a loose family for \( f \) of size \( \kappa \); then we have \( Q \subseteq f(X) \) with \( Q = f(P) \) for all \( P \in \mathcal{P} \), and \( Q \) is not scattered. But \( \Gamma(Q) \) is scattered, since \( g \) is \( \kappa \)-tight and \( g(P) = \Gamma(f(P)) = \Gamma(Q) \) for all \( P \in \mathcal{P} \). It follows that we can fix \( z \in Z \) with \( Q \cap \Gamma^{-1}\{z\} \) not scattered. But then \( f(g^{-1}\{z\}) = \Gamma^{-1}\{z\} \) is not scattered, which is impossible, since \( g^{-1}\{z\} \) is scattered. \( \blacksquare \)

We next consider the degree of dissipation of products:

Lemma 3.6 Let \( X = A \times B \), where \( A, B \) are compact, \( B \) is not scattered, and assume that for each \( \varphi \in C(A, [0, 1]^\omega) \) there is a \( z \in [0, 1]^\omega \) with \( |\varphi^{-1}\{z\}| \geq \kappa \). Then \( X \) is not \( \kappa \)-dissipated. If for each \( \varphi \in C(A, [0, 1]^\omega) \) there is a \( z \) such that \( \varphi^{-1}\{z\} \) is not scattered, then \( X \) is not weakly \( c \)-dissipated.

Proof. Since \( B \) is not scattered, fix \( h : B \to [0, 1] \), and define \( g : X \to [0, 1] \) by \( g(a, b) = h(b) \). Now, fix any \( f : X \to Y \) with \( f \) finer than \( g \) and \( Y \) metric. We shall show that \( f \) is not \( \kappa \)-tight.

Define \( \hat{f} : A \to C(B, Y) \) by \( (\hat{f}(a))(b) = f(a, b) \). Since the range of \( \hat{f} \) is compact and hence embeddable in the Hilbert cube, we can fix \( \zeta \in C(B, Y) \) such that \( E := \{a : \hat{f}(a) = \zeta\} \) has size at least \( \kappa \). Let \( Q = \zeta(B) ; |Q| = c \) by \( f \leq g \), so \( Q \) is not scattered. For \( a \in E \), let \( P_a = \{a\} \times B \). Then \( \{P_a : a \in E\} \) is a loose family of size at least \( \kappa \).

The second assertion is proved similarly. \( \blacksquare \)

Note that \( A \) might be scattered; for example, \( A \) could be the ordinal \( \kappa + 1 \) (if \( \kappa \) is uncountable and regular) or the one point compactification of a discrete space of size \( \kappa \) (if \( \kappa \) is uncountable). \( B \) may be second countable; for example \( B \) can be the Cantor set.

A class of spaces \( A \) to which Lemma 3.6 applies is produced by:

Lemma 3.7 Suppose that \( f : \prod_{\alpha < \kappa} X_\alpha \to M \), where \( M \) is compact metric and, for each \( \alpha \), \( X_\alpha \) is compact and not metrizable. Then there are two-element sets \( E_\alpha \subseteq X_\alpha \) for each \( \alpha \) such that \( f \) is constant on \( \prod_{\alpha < \kappa} E_\alpha \).

Proof. For \( p \in \prod_{\alpha < \delta} X_\alpha \), define \( \hat{f}_p : \prod_{\alpha \geq \delta} X_\alpha \to M \) by: \( \hat{f}_p(q) = f(p \upharpoonright q) \). Then inductively choose \( E_\alpha \) so that for all \( \delta \leq \kappa \), the functions \( \hat{f}_p \) are the same for all \( p \in \prod_{\alpha < \delta} E_\alpha \). Say \( \delta < \kappa \) and we have chosen \( E_\alpha \) for \( \alpha < \delta \). Let \( g = \hat{f}_p \) for some (any) \( p \in \prod_{\alpha < \delta} E_\alpha \), and define \( g^* \in C(X_\delta, C(\prod_{\alpha \geq \delta} X_\alpha, M)) \) by: \( (g^*(x))(q) = g(x \upharpoonright q) \). Then \( g^* \) maps \( X_\delta \) into a metric space of functions, so \( \text{ran}(g^*) \) is a compact metric space, so \( g^* \) cannot be 1-1, so choose \( E_\delta \) of size 2 with \( g^* \) constant on \( E_\delta \). \( \blacksquare \)

Theorem 3.8 Assume that each \( X_k \) is compact:
1. If \( X_n \) is not scattered and \( X_k \), for \( k < n \), is not metrizable, then \( \prod_{k \leq n} X_k \) is not \( 2^n \)-dissipated.

2. If each \( X_k \) is not metrizable, then \( \prod_{k < \omega} X_k \) is not weakly \( \epsilon \)-dissipated.

**Proof.** For (1), apply Lemma 3.6 with \( A = \prod_{k < n} X_k \) and \( B = X_n \). For (2), apply Lemma 3.6 with \( A = \prod_{k < \omega} X_{2k} \) and \( B = \prod_{k < \omega} X_{2k+1} \).

In (1), if all \( X_k \) are scattered, then \( \prod_{k \leq n} X_k \) is scattered and hence dissipated.

As an example of (1) applied to LOTSes, if \( S \subseteq (0,1) \) is uncountable, then \( (I_S)^2 \) is not dissipated (2-dissipated), \( (I_S)^3 \) is not 4-dissipated, and \( (I_S)^4 \) is not 8-dissipated. By Theorem 3.9, these three spaces are, respectively, 3-dissipated, 5-dissipated, and 9-dissipated. However, Lemma 3.6 shows that for any \( \kappa \), we can find a product of two LOTSes which is not \( \kappa \)-dissipated.

The following theorem will often suffice to compute the degree of dissipation of a finite product of separable LOTSes:

**Theorem 3.9** Assume that \( n \) is finite and \( X_i \), for \( i \leq n \), is a compact separable LOTS. Then \( \prod_{i \leq n} X_i \) is \( (2^n + 1) \)-dissipated. Furthermore, if all the \( X_i \) are not scattered, and at most one of the \( X_i \) is second countable, then \( \prod_{i \leq n} X_i \) is not \( (2^n) \)-dissipated.

**Proof.** Let \( D_i \subseteq X_i \) be countable and dense. Choose \( f_i \in C(X_i, [0,1]) \) such that \( f_i \) is order-preserving and is 1-1 on \( D_i \) (such a function \( f_i \) exists; see the proof of Lemma 3.6 in [10]). Note that each \( |f_i^{-1}\{y\}| \leq 2 \), and, by Lemma 2.3, each \( f_i \) is 2-tight. Applying Lemma 2.14 and induction, \( \prod_{i \leq n} f_i \) is \( (2^n + 1) \)-tight. Then \( \prod_{i \leq n} X_i \) is \( (2^n + 1) \)-dissipated by Lemma 3.5.

The “furthermore” is by Theorem 3.8.

Next, we note that “dissipated” is a local property:

**Definition 3.10** Let \( \mathcal{R} \) be a class of compact spaces. \( \mathcal{R} \) is closed-hereditary iff every closed subspace of a space in \( \mathcal{R} \) is also in \( \mathcal{R} \). \( \mathcal{R} \) is local iff \( \mathcal{R} \) is closed-hereditary and for every compact \( X \): if \( X \) is covered by open sets whose closures lie in \( \mathcal{R} \), then \( X \in \mathcal{R} \).

Classes of compacta which restrict cardinal functions (first countable, second countable, countable tightness, etc.) are clearly local, whereas the class of compacta which are homeomorphic to a LOTS is closed-hereditary, but not local. To prove that “dissipated” is local, we use as a preliminary lemma:

**Lemma 3.11** Let \( X \) be an arbitrary compact space, with \( K \subseteq U \subseteq X \), such that \( U \) is open, \( K \) is closed, and \( \overline{U} \) is \( \kappa \)-dissipated. Fix \( g : \overline{U} \to Z \), with \( Z \) compact metric. Then there is an \( f : X \to Y \), with \( Y \) compact metric, \( f \) \( \kappa \)-tight, and \( f \upharpoonright K \leq g \upharpoonright K \).
Proof. Fix $\varphi : X \to [0, 1]$ with $\varphi(K) = \{0\}$ and $\varphi(\partial U) = \{1\}$. First get $f_0 : \overline{U} \to Y_0$, with $Y_0$ compact metric, $f_0 \κ$-tight, $f_0 \leq g$, and $f_0 \leq \varphi(\overline{U})$ (just let $f_0$ refine $x \mapsto (g(x), \varphi(x))$). Then $f_0(K) \cap f_0(\partial U) = \emptyset$. Let $Y = Y_0/f_0(\partial U)$, obtained by collapsing $f_0(\partial U)$ to a point, $p$. Let $f_1 : \overline{U} \to Y$ be the natural map, and extend $f_1$ to $f : X \to Y$ by letting $f_1(X \setminus U) = \{p\}$. 

Lemma 3.12 For any $\kappa$, the class of $\kappa$-dissipated compacta is a local class.

Proof. For closed-hereditary: Assume that $X$ is $\kappa$-dissipated and $K$ is closed in $X$. Fix $g : K \to Z$, with $Z$ metric. Then we may assume that $Z \subseteq I^{\omega}$, so that $g$ extends to some $\tilde{g} : X \to I^\omega$. Then there is a $\kappa$-tight $\tilde{f} : X \to Y$ for some metric $Y$, with $\tilde{f} \leq \tilde{g}$. If $f = f|K$, then $f$ is $\kappa$-tight and $f \leq g$.

For local: Assume that $X = \bigcup_{i<\ell} U_i$, where each $U_i$ is open and $\overline{U_i}$ is $\kappa$-dissipated. Fix $g : X \to Z$, with $Z$ metric. Choose closed $K_i \subseteq U_i$ such that $X = \bigcup_{i<\ell} K_i$. Then apply Lemma 3.11 and choose $f_i : X \to Y_i$, with $Y_i$ compact metric, $f_i \kappa$-tight, and $f_i|K_i \leq g|K_i$. Then the map $x \mapsto (f_0(x), \ldots, f_{\ell-1}(x))$ refines $g$, and is $\kappa$-tight by Lemma 2.15.

Many classes of compacta are closed under continuous images, but this is not true in general of the class of $\kappa$-dissipated spaces:

Example 3.13 There is a continuous image of a 3-dissipated space which is not $\omega$-dissipated.

Proof. Let $T = (D(\omega) \cup \{\infty\}) \times 2^{\omega}$, where $D(\omega) \cup \{\infty\}$ is the 1-point compactification of the ordinal $\omega$ with the discrete topology. Then $T$ is not $\omega$-dissipated by Lemma 3.6. Let $F_\alpha$, for $\alpha < \omega$, be disjoint Cantor subsets of $2^{\omega}$ such that for some $g : 2^{\omega} \to 2^{\omega}$, each $g(F_\alpha) = 2^{\omega}$. Let $X = \{\infty\} \times 2^{\omega} \cup \bigcup_{\alpha<\lambda} (\{\alpha\} \times F_\alpha) \subseteq T$. Then $X$ is 3-dissipated by Lemma 3.3 because the natural projection onto $2^{\omega}$ is 3-tight and all point inverses are scattered (of size $\leq 2$). But also, $T$ is a continuous image of $X$ via the map $\mathbb{1} \times g, (u, z) \mapsto (u, g(z))$.

Of course, the continuous image of a 1-dissipated (= scattered) compactum is 1-dissipated. We do not know about the dissipated (= 2-dissipated) spaces; perhaps 2 is a special case.

4 LOTS Dimension

We shall apply the results of Section 3 to products of LOTSes. Each $I^n$ has dimension $n$ under any standard notion of topological dimension, so that $I^{n+1}$ is not embeddable into $I^n$. Now, say we wish to prove such a result replacing $I$ by
some totally disconnected LOTS $X$. Then standard dimension theory gives all $X^n$ dimension 0. Furthermore, the result is false; for example, $X^{n+1} \cong X^n$ if $X$ is the Cantor set. However, if $X$ is the double arrow space, then $X^{n+1}$ is not embeddable into $X^n$. To study this further, we introduce a notion of LOTS dimension:

**Definition 4.1** If $X$ is any Tychonov space, then $\text{Ldim}_0(X)$ is the least $\kappa$ such that $X$ is embeddable into a product of the form $\prod_{\alpha<\kappa} L\alpha$, where each $L\alpha$ is a LOTS. Then $\text{Ldim}(X)$, the LOTS dimension of $X$, is the least $\kappa$ such that every point in $X$ has a neighborhood $U$ such that $\text{Ldim}_0(U) \leq \kappa$.

**Lemma 4.2** The class of compacta $X$ such that $\text{Ldim}(X) \leq \kappa$ is a local class.

If $X$ is any compact $n$–manifold, then $\text{Ldim}(X) = n < \text{Ldim}_0(X)$. We follow the usual convention that the empty product $\prod_{\alpha<0} L\alpha$ is a singleton, so that $\text{Ldim}(X) = 0$ iff $X$ is finite, although $\text{Ldim}_0(X) = 1$ if $1 < |X| < \aleph_0$.

**Lemma 4.3** If $X$ is compact, infinite, and totally disconnected, then $\text{Ldim}(X) = \text{Ldim}_0(X)$.

**Proof.** Use the fact that a disjoint sum of LOTSes is a LOTS. 

By Tychonov, $\text{Ldim}(X) \leq w(X)$, taking each $L\alpha = I$. In this section, we focus mainly on spaces whose LOTS dimension is finite, although this cardinal function might be of interest for other spaces. For example, $\text{Ldim}(\beta\mathbb{N}) = 2^{\aleph_0}$; this is easily proved using the theorem of Pospíšil that there are points in $\beta\mathbb{N}$ of character $2^{\aleph_0}$.

We shall show (Lemma 4.5) that $\text{Ldim}((I_S)^n) = n$ whenever $S$ is uncountable. When $S$ is countable, this is false if $S$ is dense in $I$ (then $(I_S)^n \cong I_S$ is the Cantor set) and true if $S$ is not dense in $I$ (by standard dimension theory; not by the results of this paper). More generally, we shall prove:

**Theorem 4.4** Let $Z_j$, for $1 \leq j \leq s$, be a compact LOTS. Assume that $s = r+m$, where $r, m \geq 0$. For $r+1 \leq j \leq s$, assume that $Z_j$ has either an increasing or decreasing $\omega_1$–sequence. For $1 \leq j \leq r$, assume that there is a countable $D_j \subseteq Z_j$ such that $D_j$ is not scattered, and assume that at most one of the $D_j$ is second countable. Then $\text{Ldim}(\prod_{j=1}^s Z_j) = s$.

The following lemma handles the case $r = s, m = 0$ if we replace each $Z_j$ by $L_j = \overline{D_j}$.

**Lemma 4.5** Assume that $n$ is finite and $L_j$, for $j < n$, is a compact separable LOTS. Also, assume that all the $L_j$ are not scattered, and that at most one of the $L_j$ is second countable. Then $\text{Ldim}(\prod_{j<n} L_j) = n$. 


4 LOTS DIMENSION

**Proof.** This is trivial if \( n \leq 1 \), so assume that \( n \geq 2 \). Clearly, \( \text{Ldim}(\prod_{j<n} L_j) \leq \text{Ldim}(\prod_{j<n} L_j) \leq \text{Ldim}(\prod_{j<n} L_j) \leq \text{Ldim}(\prod_{j<n} L_j) \leq 2^{n-1} \)-dissipated.

To see that \( \text{Ldim}(\prod_{j<n} L_j) \geq n \), assume that we could embed \( \prod_{j<n} L_j \) into \( \prod_{i<(n-1)} X_i \), where each \( X_i \) is a LOTS. Since the continuous image of a compact separable space is compact and separable, we may assume that each \( X_i \) is compact and separable, so that \( \prod_{i<(n-1)} X_i \) and \( \prod_{j<n} L_j \), are \( (2^{n-2} + 1) \)-dissipated by Theorem 3.3 a contradiction since \( 2^{n-2} + 1 \leq 2^{n-1} \).

Now, assume that \( \text{Ldim}(\prod_{j<n} L_j) < n \). Then we could cover \( \prod_{j<n} L_j \) by finitely many open boxes, each of the form \( \prod_{j<n} U_j \), with each \( U_j \) an open interval in \( L_j \), such that each open box satisfies \( \text{Ldim}(\prod_{j<n} U_j) < n \). But for at least one of these open boxes, the \( \prod U_j \) would satisfy all the same hypotheses satisfied by the \( L_j \), so that we would again have a contradiction.

In particular, if \( L \) is the double arrow space, then \( L^{n+1} \) is not embeddable into \( L^n \). Similar results were obtained by Burke and Lutzer [4] and Burke and Moore [5] for the Sorgenfrey line \( J \), which may be viewed as \( \{ z^+ : z \in (0,1) \} \subseteq L \). We do not see how to derive our results directly from [4, 5], since a map \( \phi : L^{n+1} \to L^n \) need not preserve order, so it does not directly yield a map from \( J^{n+1} \) to \( J^n \).

We now extend Lemma 4.5 to include LOTs which have an increasing or decreasing \( \omega_1 \)-sequence. First some preliminaries:

**Definition 4.6** \( [\alpha]^n = \{ (\alpha_1, \ldots, \alpha_n) : \alpha_1 < \cdots < \alpha_n \} \), where \( 1 \leq n < \omega \) and \( A \subseteq \omega_1 \). We give \( [A]^n \) the topology it inherits from \( (\omega_1)^n \). The club filter \( \mathcal{F}_n \) on \( [\omega_1]^n \) is generated by all the \( [C]^n \) such that \( C \) is club in \( \omega_1 \). \( \mathcal{I}_n \) is the dual ideal to \( \mathcal{F}_n \).

**Lemma 4.7** If \( B \subseteq [\omega_1]^n \) is a Borel set, then \( B \in \mathcal{F}_n \) or \( B \in \mathcal{I}_n \).

**Proof.** Since the \( \mathcal{I}_n \) and \( \mathcal{F}_n \) are countably complete, it is sufficient to prove this for closed sets \( K \). The case \( n = 1 \) is obvious, so we proceed by induction. We assume the lemma for \( n \), fix a closed \( K \subseteq [\omega_1]^{(n+1)+1} \), and show that \( K \in \mathcal{F}_{n+1} \) or \( K \in \mathcal{I}_{n+1} \). Applying the lemma for \( n \): For each \( \alpha_0 < \omega_1 \), choose \( \nu(\alpha_0) \in \{ 0, 1 \} \) and a club \( C_{\alpha_0} \subseteq (\alpha_0, \omega_1) \) such that for all \( (\alpha_1, \ldots, \alpha_n) \in [C_{\alpha_0}]^n \):

\[
\nu(\alpha_0) = 0 \rightarrow (\alpha_0, \alpha_1, \ldots, \alpha_n) \notin K \quad \nu(\alpha_0) = 1 \rightarrow (\alpha_0, \alpha_1, \ldots, \alpha_n) \in K \quad (*)
\]

Let \( C = \{ \delta : \delta \in \cap \{ C_{\alpha_0} : \alpha_0 < \delta \} \} \). Then \( C \) is club and \( (*) \) holds for all \( (\alpha_0, \alpha_1, \ldots, \alpha_n) \in [C]^{(n+1)+1} \). Also, \( D := \{ \alpha_0 \in C : \nu(\alpha_0) = 1 \} \) is closed because \( K \) is closed. \( [D]^{(n+1)+1} \subseteq K \), so if \( D \) is club, then \( K \in \mathcal{F}_{n+1} \). If \( D \) is bounded, then \( C \setminus D \) contains a club, and then \( K \in \mathcal{I}_{n+1} \).
Definition 4.8 If \( L \) is a LOTS, \( f \in C([\omega_1]^{m^*}, L) \), and \( \psi \in C([\omega_1]^{n^*}, L) \), then \( \psi \) is derived from \( f \) iff \( n \geq m \) and for some \( i_1, \ldots, i_m \): \( 1 \leq i_1 < \cdots < i_m \leq n \) and \( \psi(\alpha_1, \ldots, \alpha_m) = f(\alpha_{i_1}, \ldots, \alpha_{i_m}) \) for all \( (\alpha_1, \ldots, \alpha_m) \in [\omega_1]^{n^*} \). Then a set \( E \subseteq [\omega_1]^{n^*} \) is derived from \( f \) iff \( E \) is of the form \( \{ \vec{\alpha} : \psi_1(\vec{\alpha}) \leq \psi_2(\vec{\alpha}) \} \) or \( \{ \vec{\alpha} : \psi_1(\vec{\alpha}) < \psi_2(\vec{\alpha}) \} \), where \( \psi_1, \psi_2 \) are derived from \( f \).

Lemma 4.9 Suppose that \( f \in C([\omega_1]^{m^*}, L) \), where \( L \) is a compact LOTS. Then there is a club \( C \), a continuous \( g : C \to L \), and a \( j \in \{1, 2, \ldots, m\} \), such that for all \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in [\omega_1]^{m^*} \), we have \( f(\vec{\alpha}) = g(\alpha_j) \), and \( g \) is either strictly increasing or strictly decreasing or constant.

Proof. Applying Lemma 1.7 and then restricting everything to a club, we may make the following homogeneity assumption: for all \( n \geq m \) and all \( E \subseteq [\omega_1]^{n^*} \) which are derived from \( f \), either \( E = \emptyset \) or \( E = [\omega_1]^{n^*} \). Then, our club \( C \) will be all of \( \omega_1 \). We first consider the special cases \( m = 1 \) and \( m = 2 \).

For \( m = 1 \), we have \( f \in C(\omega_1, L) \). Applying homogeneity to the three derived sets \( \{ (\alpha, \beta) \in [\omega_1]^{2^*} : f(\alpha) \oplus f(\beta) \} \), where \( \oplus \) is one of \( <, > \), and =, we see that \( f \) is either strictly increasing or strictly decreasing or constant.

Likewise, for \( m > 1 \), if we succeed in getting \( f(\vec{\alpha}) = g(\alpha_j) \), then \( g \) must be either strictly increasing or strictly decreasing or constant.

Next, fix \( f \in C([\omega_1]^{2^*}, L) \). If \( \alpha < \beta < \gamma \to f(\alpha, \beta) = f(\alpha, \gamma) \), then \( f(\alpha, \beta) = g(\alpha) \), and we are done, so WLOG, assume \( \alpha < \beta < \gamma \to f(\alpha, \beta) < f(\alpha, \gamma) \). Let \( B_\alpha = \{ f(\alpha, \beta) : \alpha < \beta < \omega_1 \} \), which is a subset of \( L \) of order type \( \omega_1 \). Let \( h(\alpha) = \text{sup}(B_\alpha) \). Fix \( \alpha < \alpha' < \omega_1 \). There are now three cases; Cases II and III will lead to contradictions:

Case I. \( h(\alpha) = h(\alpha') \): By continuity of \( f \), there is a club \( C \subseteq (\alpha', \omega_1) \) such that \( f(\alpha, \beta) = f(\alpha', \beta) \) for all \( \beta \in C \). Applying homogeneity, we have \( \alpha < \alpha' < \beta \to f(\alpha, \beta) = f(\alpha', \beta) \), so \( f(\alpha, \beta) = g(\beta) \).

Case II. \( h(\alpha) < h(\alpha') \): Fix \( \beta \) such that \( \alpha < \alpha' < \beta \) and \( f(\alpha', \beta) > f(\alpha, \gamma) \) for all \( \gamma \). Then by homogeneity, \( \alpha < \alpha' < \beta < \gamma \to f(\alpha, \gamma) < f(\alpha', \beta) \) for all \( \alpha, \alpha', \beta, \gamma \). Let \( \alpha' \) be a limit and consider \( \alpha < \alpha' < \beta \): we get, by continuity, \( \alpha' < \beta < \gamma \to f(\alpha', \gamma) \leq f(\alpha', \beta) \), contradicting \( \alpha < \beta < \gamma \to f(\alpha, \beta) < f(\alpha, \gamma) \).

Case III. \( h(\alpha) > h(\alpha') \): Fix \( \beta \) such that \( \alpha < \alpha' < \beta \) and \( f(\alpha, \beta) > f(\alpha', \gamma) \) for all \( \gamma \). Then by homogeneity, \( \alpha < \alpha' < \beta < \gamma \to f(\alpha', \gamma) < f(\alpha, \beta) \) for all \( \alpha, \alpha', \beta, \gamma \). Letting \( \alpha < \alpha' \), we get a contradiction as in Case II.

Finally, fix \( m \geq 2 \) and assume that the lemma holds for \( m \). We shall prove it for \( m + 1 \), so fix \( f \in C([\omega_1]^{(m+1)^*}, L) \). Temporarily fix \( (\alpha_1, \ldots, \alpha_{m-1}) \in [\omega_1]^{(m-1)^*} \), and let \( \tilde{f}(\alpha_{m}, \alpha_{m+1}) = f(\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m}, \alpha_{m+1}) \); so \( \tilde{f} \in C([\omega_1]^{(m-1)^*}, L) \). Applying the \( m = 2 \) case, \( \tilde{f} \) is really just a function of one of its arguments, so that \( f \) just depends on an \( m \)-tuple (either \( (\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m+1}) \) or \( (\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m}) \) ), so we may now apply the lemma for \( m \). \( \square \)
It is easy to see from this lemma that $\text{Ldim}((\omega_1 + 1)^m) = m$, but we now want to consider products of $(\omega_1 + 1)^m$ with separable LOTSe{s}.

**Lemma 4.10** Suppose that $f \in C(X \times [\omega_1]^m, L)$, where $L$ is a compact LOTS and $X$ is compact, nonempty, first countable, and separable. Then there is a club $C \subseteq \omega_1$, a nonempty open $U \subseteq X$, a $g \in C(\overline{U} \times C, L)$, and a $j \in \{1, 2, \ldots, m\}$ such that $f(x, \vec{\alpha}) = g(x, \alpha_j)$ for all $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in [C]^m$ and all $x \in \overline{U}$, and such that either

1. for all $x \in \overline{U}$, the map $\vec{\alpha} \mapsto f(x, \vec{\alpha})$ is constant on $[C]^m$ or
2. for all $x \in \overline{U}$, the map $\xi \mapsto g(x, \xi)$ is strictly increasing on $C$, or
3. for all $x \in \overline{U}$, the map $\xi \mapsto g(x, \xi)$ is strictly decreasing on $C$.

**Proof.** First, let $K$ be the set of all $x$ such that $\vec{\alpha} \mapsto f(x, \vec{\alpha})$ is constant on some set in $\mathcal{F}_m$. Then $K$ is closed, since $X$ is first countable, so, replacing $X$ by some $\overline{U}$, we may assume that $K = X$ or $K = \emptyset$. If $K = X$, then intersecting the clubs for $x$ in a countable dense set, we get one club $C$ such that (1) holds.

Now, assume that $K = \emptyset$. Applying Lemma 4.9 for each $x \in X$ choose a club $C_x$, a $g_x \in C(C_x, L)$, and $j_x \in \{1, 2, \ldots, m\}$ and a $\mu_x \in \{-1, 1\}$ such that for all $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in [C_x]^m$, we have $f(x, \vec{\alpha}) = g_x(\alpha_{j_x})$, and each $g_x$ is either strictly increasing (when $\mu_x = 1$) or strictly decreasing (when $\mu_x = -1$).

For each $j, \mu$, let $H_j^\mu = \{x : j_x = j \& \mu_x = \mu\}$. Then the $H_j^\mu$ are disjoint, and they are also closed (since $K = \emptyset$). Since $\bigcup_{j, \mu} H_j^\mu = X$, $U$ can be any nonempty $H_j^\mu$. ✓

In situations (2) or (3), we shall apply:

**Lemma 4.11** Suppose that $g \in C(X \times (\omega_1 + 1), L)$, where $L$ is a compact LOTS and $X$ is compact, and suppose that $g(x, \xi) < g(x, \eta)$ for each $x \in X$ and each $\xi < \eta < \omega_1$. Let $h(x) = g(x, \omega_1)$. Then $h(X)$ is finite.

**Proof.** Assume that $h(X)$ is infinite. Then, choose $c_n \in X$ for $n \in \omega$ such that the sequence $\langle h(c_n) : n \in \omega \rangle$ is either increasing strictly or decreasing strictly. Let $c \in X$ be any limit point of $\langle c_n : n \in \omega \rangle$, and note that $h(c_n) \to h(c)$. Also note that $h(x) = \sup \{g(x, \xi) : \xi < \omega_1\}$ for every $x$. Consider the two cases:

Case I. $\langle h(c_n) : n \in \omega \rangle$ is increasing strictly. Then we can fix a large enough countable $\gamma$ such that $g(c_n, \omega_1) < g(c_{n+1}, \gamma)$ for all $n$. Then we have the $\omega$-sequence, $g(c_0, \gamma) < g(c_0, \omega_1) < g(c_1, \gamma) < g(c_1, \omega_1) < g(c_2, \gamma) < g(c_2, \omega_1) < \cdots$, whose limit must be $g(c, \gamma) = g(c, \omega_1)$, contradicting $g(c, \gamma) < g(c, \omega_1)$.

Case II. $\langle h(c_n) : n \in \omega \rangle$ is decreasing strictly. Then we can fix a large enough countable $\gamma$ such that $g(c_n, \gamma) > g(c_{n+1}, \omega_1)$ for all $n$. Then we have the $\omega$-sequence,
Lemma 4.13 to obtain $g(c, \omega_1) > g(c, \gamma) > g(c_1, \omega_1) > g(c_1, \gamma) > g(c_2, \omega_1) > g(c_2, \gamma) > \cdots$, whose limit must be $g(c, \omega_1) = g(c, \gamma)$, contradicting $g(c, \omega_1) > g(c, \gamma)$.

Now if $h(X)$ is finite, we can always shrink $X$ to a $U$ on which $h$ is constant. Then note that if $h(b) = h(c)$ and $\xi \mapsto g(x, \xi)$ is always an increasing function, then there is a club on which $g(b, \xi) = g(c, \xi)$. Putting these last two lemmas together, we get:

**Lemma 4.12** Suppose that $f \in C(X \times (\omega_1 + 1)^m, L)$, where $L$ is a compact LOTS and $X$ is compact, nonempty, first countable, and separable. Then there is a club $C \subseteq \omega_1$ and a nonempty open $U \subseteq X$ such that either:

1. For some $j \in \{1, 2, \ldots, m\}$ and some continuous $g : C \to L$: $f(x, \bar{\alpha}) = g(\alpha_j)$ for all $x \in U$ and all $\bar{\alpha} \in [C]^{m\uparrow}$ and $g$ is either strictly increasing or strictly decreasing, or

2. For some $h \in C(U, L)$: $f(x, \bar{\alpha}) = h(x)$ for all $x \in U$ and all $\bar{\alpha} \in [C]^{m\uparrow}$.

**Lemma 4.13** Assume that $X$ is compact, perfect, first countable, and separable, and $\text{Ldim}(X \times (\omega_1 + 1)^m) \leq n$. Then $n > m$ and there is a nonempty open $U \subseteq X$ such that $\text{Ldim}_0(U) \leq n - m$.

**Proof.** First, restricting everything to the closure of an open box, we may assume that $\text{Ldim}_0(X \times (\omega_1 + 1)^m) \leq n$.

Fix a continuous 1-1 $f : X \times (\omega_1 + 1)^m \to \prod_{r=1}^n L_r$, where each $L_r$ is a compact LOTS. Applying Lemma 4.12 to the projections, $f_r : X \times (\omega_1 + 1)^m \to L_r$, and permuting the $L_r$, we obtain a club $C$ and a $U$ such that on $U \times [C]^{m\uparrow}$:

$$f(x, \bar{\alpha}) = (g_1(\alpha_{j_1}), \ldots, g_p(\alpha_{j_p}), h_1(x), \ldots, h_q(x)),$$

where $p + q = n$. Then $\{j_1, \ldots, j_p\} = \{1, \ldots, m\}$, since $f$ is 1-1. Thus, $p \geq m$, so $q \leq n - m$, and for any fixed $\bar{\alpha}$, the map $x \mapsto (h_1(x), \ldots, h_q(x))$ embeds $U$ into $\prod_{i=1}^q L_{p+i}$.

**Proof of Theorem 4.4.** Let $n = \text{Ldim}(\prod_{j=1}^s Z_j)$. Clearly $n \leq s$. To prove that $n \geq s$, we may replace each $Z_j$ by a closed subset and assume that $Z_j = \omega_1 + 1$ when $r + 1 \leq j \leq s$, while $Z_j = \overline{D_j}$ when $1 \leq j \leq r$. We may also assume that whenever $Z_j = \overline{D_j}$ is not second countable, no open interval in $Z_j$ is second countable (since there is always a closed subspace with this property). Let $X = \prod_{j=1}^r Z_j$, and apply Lemma 4.13 to obtain $U \subseteq X$ with $\text{Ldim}(U) \leq n - m$. Since $\text{Ldim}(U) = r$ by Lemma 4.5 we have $r \leq n - m$, so $s = r + m \leq n$.

Note that this theorem does not cover all possible products of LOT Ses. For example, one can show by a direct argument that $\text{Ldim}((\omega + 1) \times I_s) = 2$ whenever...
S is uncountable, although \((\omega + 1) \times I_S\) is dissipated, so the methods used in the proof of Theorem 4.4 do not apply. Also, Theorem 4.4 says nothing about Aronszajn lines, which have neither an increasing or decreasing \(\omega_1\)-sequence, nor a countable subset whose closure is not second countable. In particular, it is not clear whether one can have a product of three compact Aronszajn lines which is embeddable into a product of two LOT5es.

In some sense, this “dimension theory” for products of totally disconnected LOT5es is more restrictive, not less restrictive, than the classical dimension theory for \(I^n\), since there is also a limitation on dimension-raising maps. For example, Peano \([18]\) shows how to map \(I\) onto \(I^2\), but his map has many changes of direction, so it does not define a map from \(I_S\) onto \((I_S)^2\). In fact, this is impossible:

**Proposition 4.14** If \(S\) is uncountable, then there is no compact LOT5 \(L\) such that \(L\) maps continuously onto \((I_S)^2\).

**Proof.** Say \(f : L \rightarrow (I_S)^2\). Replacing \(L\) by a closed subset, we may assume that \(f\) is irreducible. Then, \(L\) must be separable, since \((I_S)^2\) is separable. It follows (see Lutzer and Bennett \([17]\)) that \(L\) is hereditarily separable, which implies (by continuity of \(f\)) that \((I_S)^2\) is hereditarily separable, which is false.

We do not know whether, for example, one can map \(L^2\) onto \((I_S)^3\). Again, we may assume that \(L\) is separable, so that \(L^2\) is 3–dissipated, while \((I_S)^3\) is not even 4–dissipated. However, as we know from Example 3.13 a continuous image of a 3–dissipated space need not be even \(c\)–dissipated.

## 5 Measures, L-spaces, and S-spaces

As usual, if \(X\) is compact, a Radon measure on \(X\) is a finite positive regular Borel measure on \(X\), and if \(f : X \rightarrow Y\) and \(\mu\) is a measure on \(X\), then \(\mu f^{-1}\) denotes the induced measure \(\nu\) on \(Y\), defined by \(\nu(B) = \mu(f^{-1}(B))\). We shall prove some results relating \(\mu\) to \(\nu\) in the case that \(f\) is tight, and use this to prove that Radon measures on dissipated spaces are separable. We shall also make some remarks on compact L-spaces and S-spaces which are dissipated.

**Definition 5.1** For any space \(X\), \(\text{ro}(X)\) denotes the regular open algebra of \(X\). If \(B\) is any boolean algebra and \(b \in B\) with \(b \neq \emptyset\), then \(b\downarrow\) denotes the algebra \(\{x \in B : x \leq b\}\); so \(\mathbb{1}_{b\downarrow} = b\). A Suslin algebra is an atomless ccc complete boolean algebra which is \((\omega, \omega)\)-distributive.

So, there is a Suslin tree iff there is a Suslin algebra. We shall prove:
Theorem 5.2 If $X$ is compact, ccc, not separable, and $\aleph_0$–dissipated, then in ro($X$) there is a non-zero $b$ such that $b\downarrow$ is a Suslin algebra.

Of course, this is well-known in the case where $X$ is a LOTS, and is part of the proof that a Suslin line yields a Suslin tree. Since a Suslin line is a compact L-space and is 2–dissipated (by Lemma 3.4), we have

Corollary 5.3 There is an $\aleph_0$–dissipated compact L-space iff there is a Suslin line.

As usual, the support of a Radon measure $\mu$ is the smallest closed $H \subseteq X$ such that $\mu(H) = \mu(X)$. For this $H$, ro($H$) cannot be a Suslin algebra, so

Corollary 5.4 If $X$ is $\aleph_0$–dissipated, then the support of every Radon measure on $X$ is a separable topological space.

In these two corollaries, the “$\aleph_0$” cannot be replaced by “$\aleph_1$”, since the usual compact L-space construction shows the following (see Section 6 for a proof):

Proposition 5.5 CH implies that there is a compact L-space $X$ which is both $\mathfrak{c}$–dissipated and the support of a Radon measure $\mu$. Furthermore, $\mu$ is atomless, and, in $X$, the ideals of null subsets, meager subsets, and separable subsets all coincide.

Turning to compact S-spaces, the usual CH construction [14] yields one which is scattered, and hence dissipated. Less trivially, the construction of Fedorčuk [7] shows, under ♦, that there is a dissipated compact S-space with no isolated points and no non-trivial convergent $\omega$–sequences; see Section 6 for further remarks on this construction.

Proof of Theorem 5.2. Since $X$ is ccc, we may replace $X$ by some regular closed set and assume that $X$ is nowhere separable — that is, the closure of every countable subset is nowhere dense. Assume that in ro($X$) no $b\downarrow$ is Suslin, and we shall derive a contradiction.

Since $X$ is ccc, the fact that no $b\downarrow$ is Suslin implies that there are open $F_\sigma$ sets $V_n^j$ for $n, j \in \omega$ such that for each $n$, the $V_n^j$ for $j \in \omega$ are disjoint and $\bigcup_j V_n^j$ is dense, and such that for each $\varphi \in \omega^\omega$, $\bigcap_n V_n^{\varphi(n)}$ has empty interior. There is then a compact metric $Y$ and an $f : X \to Y$ such that $V_n^j = f^{-1}(f(V_n^j))$ for each $n, j$. Note that this implies that each $f(V_n^j)$ is open, since $f(V_n^j) = Y \setminus f(X \setminus V_n^j)$.

Replacing $f$ by a finer map, we may also assume that $f$ is $\aleph_0$–tight. Observe that $f^{-1}\{y\}$ is nowhere dense for each $y \in Y$, since either $f^{-1}\{y\} \subseteq \bigcap_n V_n^{\varphi(n)}$ for some $\varphi \in \omega^\omega$, or $f^{-1}\{y\} \subseteq X \setminus \bigcup_j V_n^j$ for some $n$.

Now, construct open $U_s \subseteq X$ and closed $K_s \subseteq X$ for $s \in 2^{<\omega}$ as follows: $U(\_\_ = X$, and each $\overline{U_{s-1}} \subseteq U_s \setminus K_s$, with $f(\overline{U_{s-0}}) \cap f(\overline{U_{s-1}}) = \emptyset$. Also, $K_s \subseteq \overline{U_s}$,
with $f(K_s) = f(U_s)$ and $f|K_s \to f(K_s)$ irreducible. Note that $K_s$ is separable and $U_s$ is nowhere separable, so that the construction can continue. More specifically, to choose $U_s=0$ and $U_s=1$: First, find $p_0, p_1 \in U_s \setminus K_s$ such that $f(p_0) \neq f(p_1)$; this is possible since otherwise we would have $f(U_s \setminus K_s) \subseteq \{y\}$, contradicting the fact that $f^{-1}\{y\}$ is nowhere dense. Next, find open $W_i \subseteq Y$ with $f(p_i) \in W_i$ and $W_0 \cap W_1 = \emptyset$. Then, choose $U_{s-i}$ with $U_{s-i} \subseteq U_{s-i} \subseteq (U_s \setminus K_s) \cap f^{-1}(W_i)$.

Let $Q_n = \bigcup\{f(K_s) : s \in 2^n\}$, and let $Q = \bigcap_n Q_n$, which is a non-scattered subset of $Y$. Let $P_n = f^{-1}(Q) \cap \bigcup\{K_s : s \in 2^n\}$. Then the $P_n$ are disjoint and each $f(P_n) = Q$, contradicting the $\aleph_0$-tightness of $f$. 

To study measures further, we use the following standard definitions:

**Definition 5.6** If $\mu$ is any finite measure on $X$, then $\text{ma}(\mu)$ denotes the measure algebra of $\mu$ — that is, the algebra of measurable sets modulo the null sets. If $f : X \to Y$, $\mu$ is a finite measure on $X$, and $\nu = \mu f^{-1}$, then $f^* : \text{ma}(\nu) \to \text{ma}(\mu)$ is defined by $f^*(([A]) = [f^{-1}(A)]$.

$\text{ma}(\mu)$ is a complete metric space with metric $d([A], [B]) = \mu(A\Delta B)$, where $[A], [B]$ denote the equivalence classes of the sets $A, B$. Note that we do not require $f$ to be onto here, although $Y\setminus f(X)$ is a $\nu$-null set. $f^*$ is an isometric isomorphism onto some complete subalgebra $f^*(\text{ma}(\nu)) \subseteq \text{ma}(\mu)$.

As usual, a measure $\mu$ on $X$ is separable iff $L^p(\mu)$ is a separable metric space for some (equivalently, for all) $p \in [1, \infty)$. Also $\mu$ is separable iff $\text{ma}(\mu)$ is a separable metric space iff $\text{ma}(\mu)$ is countably generated as a complete boolean algebra. Separability of $\mu$ is not related in any simple way to the separability of any topology that $X$ may have. Following [6]:

**Definition 5.7** $\text{MS}$ is the class of all compact spaces $X$ such that every Radon measure on $X$ is separable.

We shall prove:

**Theorem 5.8** If $X$ is a weakly $c$-dissipated space then $X$ is in $\text{MS}$.

In view of Lemma 3.4, Theorem 5.8 generalizes the result from [6] that every compact LOTS is in $\text{MS}$. Note that a space in $\text{MS}$ need not be $c$-dissipated. For example, $\text{MS}$ is closed under countable products (see [6]), but an infinite product of non-metric compacta is never weakly $c$-dissipated (see Theorem 3.8).

Theorem 5.8 will be an easy corollary of some general results about measures induced by weakly $c$-tight $f : X \to Y$, where $X, Y$ are compact. Say $\mu$ is a Radon measure on $X$, with $\nu = \mu f^{-1}$. Even if $f$ is tight (i.e., 2–tight), the separability of $\nu$ does not imply the separability of $\mu$; for example, $\nu$ may be a point mass
concentrating on \( \{y\} \), in which case \( \mu \) can be any measure supported on \( f^{-1}\{y\} \) with \( \mu(f^{-1}\{y\}) = \nu\{y\} \). However, if \( \nu \) is atomless, then the form of \( \nu \) will restrict the form of \( \mu \). There are really two kinds of ways that \( \nu \) might determine \( \mu \). We shall denote the stronger way as “\( X \) is skinny” and the weaker way as “\( X \) is slim”. We shall define “skinny” and “slim” also for arbitrary closed subsets of \( X \):

**Definition 5.9** Suppose that \( X,Y \) are compact, \( f : X \to Y \), \( \mu \) is a Radon measure on \( X \), and \( \nu = \mu f^{-1} \). Then:

- \( X \) is skinny with respect to \( f, \mu \) iff for all closed \( K \subseteq X \), \( \mu(K) = \nu(f(K)) \).
- \( X \) is slim with respect to \( f, \mu \) iff \( f^* : \text{ma}(\nu) \to \text{ma}(\mu) \) maps onto \( \text{ma}(\mu) \).

If \( H \) is a closed subset of \( X \), then we say that \( H \) is skinny (resp., slim) with respect to \( f, \mu \) iff \( H \) is skinny (resp., slim) with respect to \( f|H, \mu|H \).

Note that the equation \( \mu(K) = \nu(f(K)) \) shows that if \( X \) is skinny, then \( \nu \) determines \( \mu \); there is no Radon measure \( \mu' \neq \mu \) such that \( \nu = \mu' f^{-1} \).

**Lemma 5.10** If \( X \) is skinny with respect to \( f, \mu \), then \( X \) is slim.

**Proof.** If \( K \subseteq X \) is closed, then \( \mu(K) = \mu(f^{-1}(f(K))) \) implies that \( [K] = [f^{-1}(f(K))] = f^*([f(K)]) \) in \( \text{ma}(\mu) \). Thus, \( [K] \in \text{ran}(f^*) \) for all closed \( K \subseteq X \), which implies that \( f^* \) is onto. \( \spadesuit \)

The converse is false. For example, suppose that \( H \) is a closed subset of \( X \) such that \( \mu \) is supported on \( H \) and \( f|H \) is 1-1. Then \( X \) is slim, since \( \text{ma}(\mu) \cong \text{ma}(\mu|H) \), but \( X \) need not be skinny, since there may well be closed \( K \) disjoint from \( H \) with \( X = f^{-1}(f(K)) \); then \( \mu(K) = 0 \) but \( \nu(f(K)) = \mu(X) \). In this example, \( H \) is skinny with respect to \( f, \mu \). Some examples of skinny sets on which the function \( f \) is not 1-1 are given by:

**Lemma 5.11** Suppose that \( X,Y \) are compact, \( f : X \to Y \) is tight, \( \mu \) is a Radon measure on \( X \), and \( \nu = \mu f^{-1} \) is atomless. Then \( X \) is skinny with respect to \( f, \mu \).

**Proof.** If \( X \) is not skinny, fix a closed \( K \subseteq X \) with \( \mu(K) < \nu(f(K)) \), so that \( \mu(f^{-1}(f(K)) \setminus K) > 0 \). Then choose a closed \( L \subseteq f^{-1}(f(K)) \setminus K \) with \( \mu(L) > 0 \). Then \( K,L \) are disjoint in \( X \) and \( \nu(f(K) \cap f(L)) = \nu(f(L)) \geq \mu(L) > 0 \), so \( f(K) \cap f(L) \) cannot be scattered, since \( \nu \) is atomless, so \( f \) is not tight. \( \spadesuit \)

One cannot replace “tight” by “3–tight” here. For example, say \( X = Y \times \{0,1\} \), with \( f \) the natural projection, which is 2–tight. If \( \nu \) is any Radon measure on \( Y \), and on \( X \) we let \( \mu(E_0 \times \{0\} \cup E_1 \times \{1\}) = \frac{1}{2} (\nu(E_0) + \nu(E_1)) \), then \( X \) is not skinny (or even slim). Here, \( X \) is the union of two skinny subsets, and this situation generalized to:
Lemma 5.12 Suppose that \( X, Y \) are compact, \( f : X \to Y \) is \( \aleph_0 \)-tight and \( \mu \) is a Radon measure on \( X \) with \( \mu f^{-1} \) atomless. Then there is a countable family \( \mathcal{H} \) of disjoint skinny subsets of \( X \) such that \( \mu(X) = \sum \{ \mu(H) : H \in \mathcal{H} \} \).

Proof. If this fails, then the usual exhaustion argument lets us shrink \( X \) and assume that \( \mu(X) > 0 \) and there are no closed skinny \( H \subseteq X \) of positive measure. We now build an infinite loose family as follows:

Construct a tree of closed \( H_s \subseteq X \) for \( s \in 2^{<\omega} \); so \( H_{s_0}, H_{s_1} \) will be disjoint closed subsets of \( H_s \), and also \( f(H_{s_0}) \cap f(H_{s_1}) = \emptyset \). Each \( H_s \) will have positive measure. \( H(\cdot) \) can be \( X \).

Given \( H_s \): Since \( H_s \) is not skinny, we can choose a closed \( K_s \subseteq H_s \) with \( \mu(H_s \cap (f^{-1}(f(K_s)) \setminus K_s)) > 0 \). Then, since \( \mu \) is regular and \( \mu f^{-1} \) is atomless, we can choose closed \( H_{s_0}, H_{s_1} \subseteq H_s \cap (f^{-1}(f(K_s)) \setminus K_s) \) with each \( \mu(H_{s_0}) > 0 \) and \( f(H_{s_0}) \cap f(H_{s_1}) = \emptyset \).

Now, let \( Q_n = \bigcup \{ f(H_s) : s \in 2^n \} \) and let \( Q = \bigcap_n Q_n \); so, \( Q \) is non-scattered. Let \( P_n = f^{-1}(Q) \cap \bigcup \{ K_s : s \in 2^n \} \). Then \( \{ P_n : n \in \omega \} \) is a loose family.

It follows that the measure algebra of \( \mu \) is a countable sum of measure algebras isomorphic to algebras derived from measures on \( Y \). Note that the \( K_s \) in this proof may be null sets, so one cannot split them also to obtain a loose family of size \( \mathfrak{c} \), as we did in the proof of Lemma 2.18. In fact, the L-space of Proposition 5.5 shows that one cannot weaken “\( \aleph_0 \)-tight” to “\( \aleph_1 \)-tight” in this lemma. To see this, note that \( \mu \) is a separable measure on \( X \) by Theorem 5.8 so one can get an \( f : X \to Y \) such that \( Y \) is compact metric, \( \nu = \mu f^{-1} \) atomless, and \( f^*(\text{ma}(\nu)) = \text{ma}(\mu) \). Since \( X \) is \( \aleph_1 \)-dissipated, one can refine \( f \) and assume also that \( f \) is \( \aleph_1 \)-tight. Now, if \( H \) is skinny, let \( K \) be a closed subset of \( H \) such that \( f(K) = f(H) \) and \( f[K : K \to f(H) \) is irreducible. Then \( K \) is separable and hence null (by the properties of \( X \)), and \( \mu(H) = \mu(K) \) (since \( H \) is skinny), so \( \mu(H) = 0 \). Thus, there cannot be a family \( \mathcal{H} \) as in Lemma 5.12.

However, the analogous result with “slim” (Theorem 5.14) just uses \( \mathfrak{c} \)-tightness.

Definition 5.13 Suppose that \( X, Y \) are compact, \( f : X \to Y \), and \( \mu \) is a Radon measure on \( X \). Then \( X \) is simple with respect to \( f, \mu \) iff there is a countable disjoint family \( \mathcal{H} \) of slim subsets of \( X \) such that \( \sum \{ \mu(H) : H \in \mathcal{H} \} = \mu(X) \).

We shall prove:

Theorem 5.14 Suppose that \( X, Y \) are compact, \( f : X \to Y \), and \( \mu \) is a Radon measure on \( X \), with \( \nu = \mu f^{-1} \), and suppose that \( X \) is not simple with respect to \( f, \mu \). Then there is a \( \varphi : \text{dom}(\varphi) \to 2^\omega \), where \( \text{dom}(\varphi) \) is closed in \( X \), such that for some closed \( Q \subseteq Y \), \( \nu(Q) > 0 \) and \( \varphi(f^{-1}(y)) = 2^\omega \) for all \( y \in Q \). In particular, if \( \nu \) is atomless, then \( f \) is not weakly \( \mathfrak{c} \)-tight.
In proving this, the notion of conditional expectation (see [9, §48]) will be useful in comparing the induced measure \((\mu|S)f^{-1}\) to \(\nu\) for various \(S \subseteq X\):

**Definition 5.15** Suppose that \(f : X \to Y\), with \(X,Y\) compact, \(\mu\) is a measure on \(X\) and \(\nu = \mu f^{-1}\). If \(S\) is a measurable subset of \(X\), then the conditional expectation, \(E(S|f) = E_{\mu}(S|f)\), is the measurable \(\varphi : Y \to [0,1]\) defined so that \(\int_A \varphi(y)\,d\nu(y) = \mu(f^{-1}(A) \cap S)\) for all measurable \(A \subseteq Y\).

Of course, \(\varphi\) is only defined up to equivalence in \(L^\infty(\nu)\). Conditional expectations are usually defined for probability measures, but they make sense in general for finite measures; actually, \(E_{\mu}(S|f) = E_{c\mu}(S|f)\) for any non-zero \(c\). Note that \(\int_A \varphi(y)\,d\nu(y) = \int_{f^{-1}(A)} \varphi(f(x))\,d\mu(x)\). We may also characterize \(\varphi = E_{\mu}(S|f)\) by the equation:

\[
\int_S g(f(x))\,d\mu(x) = \int_X \varphi(f(x))\,g(f(x))\,d\mu(x) = \int_Y \varphi(y)\,g(y)\,d\nu(y).
\]

for \(g \in L^1(Y,\nu)\). \(\varphi\) is obtained either by the Radon-Nikodym Theorem, or, equivalently, by identifying \((L^1(Y,\nu))^*\) with \(L^\infty(Y,\nu)\), since \(\Gamma(g) := \int_S g(f(x))\,dx\) defines \(\Gamma \in (L^1(Y,\nu))^*\), with \(\|\Gamma\| \leq 1\).

Now, given \(\mu\) on \(X\) and \(f : X \to Y\), we shall consider various closed subsets \(H \subseteq X\) while studying the tightness properties of \(f\). When \(S \subseteq H \subseteq X\), one must be careful to distinguish \(E_{\mu}(S|f)\) (computed using \(\mu\) and \(f : X \to Y\)) from \(E_{\mu|H}(S \mid f \upharpoonright H)\) (computed using \(\mu|H\) and \(f \upharpoonright H : H \to Y\)). These are related by:

**Lemma 5.16** Suppose that \(f : X \to Y\), with \(X,Y\) compact, \(H\) is a closed subset of \(X\), and \(\mu\) is a Radon measure on \(X\). Let \(S\) be a measurable subset of \(H\). Then \(E_{\mu}(S|f) = E_{\mu}(H|f) \cdot E_{\mu|H}(S \mid f \upharpoonright H)\).

**Proof.** Let \(\varphi = E_{\mu}(S|f)\), \(\psi = E_{\mu}(H|f)\), and \(\gamma = E_{\mu|H}(S \mid f \upharpoonright H)\). We may take these to be bounded Borel-measurable functions from \(Y\) to \(\mathbb{R}\). For any bounded Borel-measurable \(g : Y \to \mathbb{R}\), we have

\[
\begin{align*}
\int_S g(f(x))\,d\mu(x) & = \int_X \varphi(f(x))\,g(f(x))\,d\mu(x) \\
\int_H g(f(x))\,d\mu(x) & = \int_X \psi(f(x))\,g(f(x))\,d\mu(x) \\
\int_S g(f(x))\,d\mu(x) & = \int_H \gamma(f(x))\,g(f(x))\,d\mu(x) = \int_X \psi(f(x))\,\gamma(f(x))\,g(f(x))\,d\mu(x),
\end{align*}
\]

which yields \(\varphi = \psi \gamma\). 

We now relate conditional expectations to slimness:
Lemma 5.17 Suppose that $X, Y$ are compact, $f : X \to Y$, and $\mu$ is a measure on $X$, with $\nu = \mu f^{-1}$. Let $S \subseteq X$ be measurable. Then $[S] \in \text{ran}(f^*)$ iff $\mathbb{E}(S|f] = [\chi_T]$ for some measurable $T \subseteq Y$, in which case $[S] = f^*([T])$.

Proof. For $\Rightarrow$: If $[S] = f^*([T])$ then $\mu(S \Delta f^{-1}(T)) = 0$, which implies $\mathbb{E}(S|f] = \mathbb{E}(f^{-1}(T)|f] = \chi_T$.

For $\Leftarrow$: If $\mathbb{E}(S|f] = [\chi_T]$ then $\mu(f^{-1}(A) \cap S) = \nu(A \cap T)$ for all measurable $A \subseteq Y$. Setting $A = Y \setminus T$, we get $\mu(S \setminus f^{-1}(T)) = 0$, so $[S] \leq [f^{-1}(T)]$. Setting $A = T$, we get $\mu(S \cap f^{-1}(T)) = \nu(T) = \mu(f^{-1}(T))$, so $[S] \geq [f^{-1}(T)]$. 

In particular, $X$ is slim with respect to $f, \mu$ iff every $\mathbb{E}(S|f]$ is the characteristic function of a set; this remark will be useful when applied also to $\mu|H$ for various $H \subseteq X$.

Lemma 5.18 Suppose that $X, Y$ are compact, $f : X \to Y$, and $\mu$ is a measure on $X$, with $\nu = \mu f^{-1}$, and suppose that $X$ is not slim with respect to $f, \mu$. Then there are disjoint closed $H_0, H_1 \subseteq X$ with $f(H_0) = f(H_1) = K$, such that $\nu(K) > 0$ and, for $i = 0, 1$, $0 < \mathbb{E}(H_i|f)(y) < 1$ for a.e. $y \in K$.

Proof. First, let $\tilde{H}_0 \subseteq X$ be closed with $[H_0] \notin \text{ran}(f^*)$. We can then, by Lemma 5.17 get a closed $\tilde{K} \subseteq f(\tilde{H}_0)$ with $\nu(\tilde{K}) > 0$ and $\mathbb{E}(\tilde{H}_0|f)(y) \in (0, 1)$ for a.e. $y \in \tilde{K}$. Then, choose a closed $\tilde{H}_1 \subseteq f^{-1}(\tilde{K}) \setminus \tilde{H}_0$ with $\mu(\tilde{H}_1) > 0$. Then, choose a closed $K \subseteq \tilde{f}(\tilde{H}_1)$ with $\nu(K) > 0$ and $\mathbb{E}(\tilde{H}_1|f)(y) > 0$ for a.e. $y \in K$, and let $H_i = \tilde{H}_i \cap f^{-1}(K)$. 

We now consider the opposite of slim:

Definition 5.19 $X$ is nowhere slim with respect to $f, \mu$ iff there is no closed $H \subseteq X$ with $\mu(H) > 0$ such that $H$ is slim with respect to $f, \mu$.

Lemma 5.20 Suppose that $X, Y$ are compact, $f : X \to Y$, and $\mu$ is a measure on $X$, with $\nu = \mu f^{-1}$, and suppose that $X$ is nowhere slim with respect to $f, \mu$. Fix $\varepsilon > 0$. Then there are disjoint closed $H_0, H_1 \subseteq X$ with $f(H_0) = f(H_1) = K$, such that $\nu(Y \setminus K) < \varepsilon$ and, for $i = 0, 1$, $0 < \mathbb{E}(H_i|f)(y) < 1$ for a.e. $y \in K$.

Proof. Fix $K$ such that

1. $K$ is a disjoint family of non-null closed subsets of $Y$.
2. For $K \in \mathcal{K}$, there are disjoint closed $H_0^K, H_1^K \subseteq X$ with $f(H_0^K) = f(H_1^K) = K$, and, for $i = 0, 1$, $0 < \mathbb{E}(H_i^K|f)(y) < 1$ for a.e. $y \in K$.
3. $\mathcal{K}$ is maximal with respect to (1)(2).
Then $K$ is countable. If $\nu(Y \setminus \bigcup K) = 0$, choose a finite $K' \subseteq K$ such that $\nu(Y \setminus \bigcup K') < \varepsilon$, set $K = \bigcup K'$, and set $H_i = \bigcup \{H_i^k : K \in K'\}$. If $\nu(Y \setminus \bigcup K) \neq 0$, choose a closed $E \subseteq Y \setminus \bigcup K$ with $\nu(E) > 0$, and use Lemma 5.18 to derive a contradiction from maximality of $K$ and the fact that $f^{-1}(E)$ is not slim.

We can now use a tree argument to prove Theorem 5.14.

**Proof of Theorem 5.14.** Since $f$ is not simple, there must be a closed $H \subseteq X$ such that $H$ is nowhere slim with respect to $\mu|H, f|H$. Restricting everything to $H$, we may assume that $X$ itself is nowhere slim. Also, WLOG $\mu(X) = \nu(Y) = 1$ and $f(X) = Y$. Now, get $P_s \subseteq X$ for $s \in 2^{\omega}$ and $Q_n \subseteq Y$ for $n \in \omega$ so that:

1. $P() = X$ and $Q_0 = Y$.
2. $P_s$ is closed in $X$ and $Q_n$ is closed in $Y$.
3. $Q_n = \cap \{f(P_s) : h(s) = n\}$.
4. $P_{s-0}$ and $P_{s-1}$ are disjoint subsets of $P_s$.
5. $\nu(f(P_s) \setminus f(P_{s-i})) \leq 6^{-n-1}$ when $h(s) = n$ and $i = 0, 1$.
6. $Q_{n+1} \subseteq Q_n$ and $\nu(Q_n \setminus Q_{n+1}) \leq 2^{n+1} \cdot 6^{-n-1} = 3^{-n-1}$.
7. $\mathbb{E}_\nu(P_s|f)(y) > 0$ for $\nu$-a.e. $y \in f(P_s)$.

Assuming that this can be done, let $Q = \cap_n Q_n$. $Q \subseteq f(P_s)$ for all $s \in 2^{\omega}$, so for $t \in 2^{\omega}$, let $P_t = f^{-1}(Q) \cap \cap_n P_{t[n]}$. Then the $P_t$ are disjoint and $f(P_t) = Q$ for all $t$. Also, $\mu(Q) \geq 1 - 1/3 - 1/9 - 1/27 - \cdots = 1/2$. Let $\varphi = \cup_t P_t$ with $\varphi(x) = t$ for $x \in P_t$.

Now, to do the construction, note first that (6) follows from (3)(4)(5). We proceed by induction on $hl(s)$, using (7) to accomplish the splitting. For $hl(s) = 0$, (1)(2)(3)(7) are trivial, since $\mathbb{E}(X|f)(y) = 1$ for a.e. $y \in Y$. Now fix $s$ with $hl(s) = n$. We obtain $P_{s-0}$ and $P_{s-1}$ by applying Lemma 5.20 with the $X, Y$ there replaced by $P_s, f(P_s)$; but then we must replace $\nu$ by $\lambda := (\mu|P_s)(f|P_s)^{-1}$ on $f(P_s)$. Let $\varphi = \mathbb{E}_\mu(P_f|f)$; then, by (7) for $P_s$, $\varphi(y) > 0$ for $\nu$-a.e. $y \in f(P_s)$; also $\varphi(y) = 0$ for a.e. $y \notin f(P_s)$, and $\int_A \varphi(y) \nu(y) = \mu(f^{-1}(A) \cap P_s) = \lambda(A)$ for all measurable $A \subseteq f(P_s)$. Fix $\delta > 0$ such that $\nu(y \in f(P_s) : \varphi(y) < \delta) \leq 6^{-n-1}/2$. Now apply Lemma 5.20 to get closed $P_{s-0}, P_{s-1}$ satisfying (4) with $K_s := f(P_{s-0}) = f(P_{s-1})$ so that, for $i = 0, 1$, $\mathbb{E}_\mu(P_s|f) > 0$ for $\lambda$-a.e. $y \in K_s$, and $\lambda(f(P_s) \setminus K_s) < \delta \cdot 6^{-n-1}/2$. Now, by Lemma 5.16, $\mathbb{E}_\mu(P_s|f) = \varphi \cdot \mathbb{E}_\mu(P_s|f)$, which yields (7) for $P_{s-1}$. To obtain (5), let $A = f(P_s) \setminus K_s$. We need $\nu(A) \leq 6^{-n-1}$, and we have $\int_A \varphi(y) \nu(y) = \lambda(A) < \delta \cdot 6^{-n-1}/2$. Let $A' = A' \cup A''$, where $\varphi < \delta$ on $A'$ and $\varphi \geq \delta$ on $A''$. Then $\nu(A') \leq 6^{-n-1}/2$ and $\nu(A'') \leq (1/\delta) \int_A \varphi(y) \nu(y) \leq 6^{-n-1}/2$, so $\nu(A') \leq 6^{-n-1}$. 


Corollary 5.21 Suppose that $X,Y$ are compact, $f : X \to Y$ is weakly $c$-tight, and $\mu$ is a Radon measure on $X$, with $\nu = \mu f^{-1}$ atomless and separable. Then $\mu$ is separable.

Proof. $X$ is simple with respect to $f, \mu$, by Theorem 5.14, which implies that $\text{ma}(\mu)$ is a countable disjoint sum of separable measure algebras. 

Proof of Theorem 5.8. Assume that $\mu$ is a non-separable Radon measure on $X$; we shall derive a contradiction. By subtracting the point masses, we may assume that $\mu$ is atomless.

First, fix a compact metric $Z$ and a $g : X \to Z$ such that $\mu g^{-1}$ is atomless. This is easily done by an elementary submodel argument. More concretely, one can assume that $X \subseteq [0,1]^\kappa$; then $g = \pi_d^\kappa$ for a suitably chosen countable $d \subseteq \kappa$.

We construct $d$ as $\bigcup d_i$, where the $d_i$ are finite and non-empty and $d_0 \subseteq d_1 \subseteq \cdots$. Given $d_i$, we have the space $Z_i = \pi_d^\kappa(X)$, with measure $\nu_i = \mu(\pi_d^\kappa)^{-1}$. Let $\{F^\ell_i : \ell \in \omega\}$ be a family of closed non-null subsets of $Z_i$ which is dense in the measure algebra, and make sure that for each $\ell$, there is some $j > i$ such that $Z_j$ contains a closed set $K \subseteq (\pi_d^j)^{-1}(F^\ell_i)$ with $\nu_j(K)/\mu_i(F^\ell_i) \in (1/3, 2/3)$.

Let $f : X \to Y$ be weakly $c$-tight, where $Y$ is metric and $f$ is finer than $g$. We then have $\Gamma \in C(Y,Z)$ such that $g = \Gamma \circ f$, so $\mu g^{-1} = (\mu f^{-1})\Gamma^{-1}$, so $\mu f^{-1}$ is atomless. Also, $\mu f^{-1}$ is separable because $Y$ is metric, contradicting Corollary 5.21.

6 Inverse Limits

Some compacta built as inverse limits in $\omega_1$ steps are dissipated. We avoid explicit use of the inverse limit by viewing $X$ as a subspace of some $M^{\omega_1}$, so the bonding maps in the inverse limit will be the projection maps.

Definition 6.1 For any space $M$ and ordinals $\alpha \leq \beta$: $\pi^\beta_\alpha : M^\beta \to M^\alpha$ denotes the natural projection.

Theorem 6.2 Let $M$ be compact metric, and suppose that $X$ is a closed subset of $M^{\omega_1}$. Let $X_\alpha = \pi^{\omega_1}_\alpha(X)$. Assume that for each $\alpha < \omega_1$, the map $\pi^{\omega_1}_{\alpha+1}|_{X_{\alpha+1}} : X_{\alpha+1} \to X_\alpha$ is tight. Then

1. For each $\alpha < \beta \leq \omega_1$, the map $\pi^\beta_\alpha|X_\beta : X_\beta \to X_\alpha$ is tight.
2. $X$ is dissipated.
Proof. For (1), fix $\alpha$ and induct on $\beta$. For successor stages, use Lemma 2.13. For limit $\beta > \alpha$, use the fact that if $P_0, P_1$ are disjoint closed subsets of $X_\beta$, then there is a $\delta$ with $\alpha < \delta < \beta$ and $\pi_\delta^\beta(P_0) \cap \pi_\delta^\beta(P_1) = \emptyset$.

For (2), observe that given $g : X \to Z$, with $Z$ metric, there is an $\alpha < \omega_1$ with $\pi_\alpha^{\omega_1}|X$ finer than $g$. Now, use the fact that all $\pi_\alpha^{\omega_1}|X$ are tight. \(\square\)

The proof of (2) did not actually require all $\pi_\alpha^{\omega_1}|X$ to be tight; we only needed unboundedly many. More generally, the definition of “dissipated” requires the family of tight maps to be unbounded, but it does not necessarily contain a club, although it does contain a club in the “natural” examples of dissipated spaces. We first point out an example where the tight maps do not contain a club. Then we shall formulate precisely what “contains a club” means.

Example 6.3 There is a closed $X \subseteq 2^{\omega_1}$ such that, setting $X_\alpha = \pi_\alpha^{\omega_1}(X)$:

a. $X$ is dissipated
b. For all $\alpha < \omega_1$, $\pi_\alpha^{\omega_1}|X : X \to X_\alpha$ is tight iff $\alpha$ is not a limit ordinal.

Proof. First note that (b) $\to$ (a) because whenever $g : X \to Z$, with $Z$ metric, there is always an $\alpha < \omega_1$ with $\pi_\alpha^{\omega_1}|X \leq g$. Then $\pi_\alpha^{\omega_1}|X \leq \pi_\beta^{\omega_1}|X \leq g$ and $\pi_\alpha^{\omega_1}|X$ is tight.

To prove (b), we use a standard inverse limit construction, building $X_\alpha$ by induction on $\alpha$. We shall have:

1. $X_\alpha$ is a closed subset of $2^\alpha$ for all $\alpha \leq \omega_1$, and $X = X_{\omega_1}$.
2. $X_\alpha = \pi_\alpha^\beta(X_\beta)$ whenever $\alpha \leq \beta \leq \omega_1$.
3. $X_\alpha = 2^\alpha$ for $\alpha \leq \omega$.
4. For $\alpha < \omega_1$: $X_{\alpha+1} = X_\alpha \times \{0\} \cup F_\alpha \times \{1\}$, where $F_\alpha$ is a closed subset of $X_\alpha$.
5. $F_\gamma$ is a perfect set for all limit $\gamma < \omega_1$.
6. $\pi_\delta^\alpha(F_\alpha)$ is finite whenever $\delta < \alpha < \omega_1$.
7. Whenever $\delta < \alpha < \omega_1$ and $\delta$ is a successor ordinal, there is an $n$ with $0 < n < \omega$ such that $\pi_{\alpha+n}^{\omega_1}(F_{\alpha+n}) = F_\delta \times \{0,1\}$.

Conditions (1)(2) imply that $X_\gamma$, for limit $\gamma$, is determined by the $X_\alpha$ for $\alpha < \gamma$; then, by (4), the whole construction is determined by the choice of the $F_\alpha \subseteq X_\alpha$; as usual, in stating (4), we are identifying $2^{\alpha+1}$ with $2^\alpha \times \{0,1\}$. By (3), $F_\alpha = X_\alpha$ when $\alpha < \omega$. By (6), $F_\alpha$ is finite for successor $\alpha$. Conditions (1)–(6) are sufficient to verify (b) of the theorem, but (7) was added to ensure that the construction can be carried out. Using (7), it is easy to construct $F_\gamma$ for limit $\gamma$ to satisfy (5)(6), and (7) itself is easy to ensure by a standard enumeration argument, since there are no further restrictions on the finite sets $F_{\alpha+n} \subseteq X_{\alpha+n}$ when $n > 0$. 
To verify $(b)$: If $\alpha < \omega_1$ is a limit ordinal, then $(4)(5)$ guarantee that $\pi_\alpha^{\omega_1}|X : X \to X_\alpha$ is not tight. Now, fix a successor $\alpha < \omega$. We prove by induction that $\pi_\alpha^{\omega_1}|X_\beta : X_\beta \to X_\alpha$ is tight whenever $\alpha \leq \beta \leq \omega_1$. This is trivial when $\beta = \alpha$. If $\beta > \alpha$ is a limit ordinal and $\pi_\alpha^{\omega_1}|X_\beta$ fails to be tight, then we have disjoint closed $P_0, P_1 \subset X_\beta$ with $Q = \pi_\alpha^{\omega_1}(P_0) = \pi_\alpha^{\omega_1}(P_1)$ and $Q$ not scattered; but then there is a $\delta$ with $\beta > \delta > \alpha$ such that $\pi_\beta^{\omega_1}(P_0) \cap \pi_\beta^{\omega_1}(P_1) = \emptyset$, and then the $\pi_\beta^{\omega_1}(P_i)$ refute the tightness of $\pi_\alpha^{\omega_1}$.

Finally, assume that $\alpha \leq \beta < \omega_1$ and that $\pi_\alpha^{\alpha+1}|X_\beta$ is tight. We shall prove that $\pi_\alpha^{\alpha+1}|X_\beta+1$ is tight. If $\beta$ is a successor, we note that $\pi^{\alpha+1}_\alpha|X_\beta+1$ is tight because $F_\beta$ is finite, so that $\pi^{\alpha+1}_\alpha|X^{\beta+1} = \pi^{\beta+1}_\alpha X_\beta \circ \pi^{\beta+1}_\alpha|X^{\beta+1}$ is tight by Lemma 2.13. Now, assume that $\beta$ is a limit (so $\alpha < \beta$) and that $\pi^{\alpha+1}_\alpha|X^{\beta+1}$ is not tight. Fix disjoint closed $P_0, P_1 \subset X_\beta+1$ with $Q = \pi^{\alpha+1}_\alpha(P_0) = \pi^{\alpha+1}_\alpha(P_1)$ and $Q$ not scattered. Since $\pi^{\alpha+1}_\beta(F_\beta)$ is finite, we may shrink $Q$ and the $P_i$ and assume that $Q \cap \pi^{\alpha+1}_\beta(F_\beta) = \emptyset$. Then $\pi^{\beta+1}_\beta(P_i) \cap F_\beta = \emptyset$, so that $\pi^{\beta+1}_\beta(P_0) \cap \pi^{\beta+1}_\beta(P_1) = \emptyset$, and the $\pi^{\beta+1}_\beta(P_i)$ contradict the tightness of $\pi^{\alpha+1}_\beta|X_\beta$.

There are various equivalent ways to formulate “contains a club”; the following is probably the quickest to state:

**Definition 6.4** The compact $X$ is wasted iff whenever $\theta$ is a suitably large regular cardinal and $M \prec H(\theta)$ is countable and contains $X$ and its topology, the natural evaluation map $\pi_M : X \to [0,1]^{\mathcal{C}(\mathcal{X},[0,1]) \cap M}$ is tight.

For the $X$ of Example 6.3, no $\pi_M$ is tight, since $\pi_M$ is equivalent to $\pi^{\omega_1}_\gamma$, where $\gamma = \omega_1 \cap M$. The $X$ of Theorem 6.2 is wasted, as is every compact LOTS. A notion intermediate between “dissipated” and “wasted” is obtained by requiring $\pi_M$ to be tight for a stationary set of $M \prec H(\theta)$.

In Theorem 6.2 since $X_{\alpha+1}$ and $X_\alpha$ are compact metric, the assumption that $\pi^{\alpha+1}_\alpha$ is tight is equivalent to saying that $\{y \in X_\alpha : |(\pi^{\alpha+1}_\alpha)^{-1}\{y\} \cap X_{\alpha+1}| > 1\}$ is countable (see Theorem 2.7). In the constructions of [7, 11, 12], this set is actually a singleton. In some cases, the spaces are also minimally generated in the sense Koppelberg [15] and Dow [4]:

**Definition 6.5** Let $X, Y$ be compact. Then $f : X \to Y$ is minimal iff $|f^{-1}\{y\}| = 1$ for all $y \in Y$ except for one $y_0$, for which $|f^{-1}\{y_0\}| = 2$.

We remark that this is the same as minimality in the sense that if $f = g \circ h$, where $h : X \to Z$ and $g : Z \to Y$, then either $g$ or $h$ is a bijection. Clearly, every minimal map is tight.

**Definition 6.6** $X$ is minimally generated iff $X$ is a closed subspace of some $2^\rho$, where, setting $X_\alpha = \pi^{\rho}_\alpha(X)$, all the maps $\pi^{\alpha+1}_\alpha|X_{\alpha+1} : X_{\alpha+1} \to X_\alpha$, for $\alpha < \rho$, are minimal.
Examples of such spaces are the Fedorčuk S-space [7], obtained under ♦ (here, \( \rho = \omega_1 \)), and the Efimov spaces obtained by Fedorčuk [8] and Dow [11], where \( \rho > \omega_1 \).

Clearly, if \( \rho = \omega_1 \), then \( X \) must be dissipated by Theorem 6.2, but this need not be true for \( \rho > \omega_1 \). For example, if \( A(\aleph_1) \) is the 1-point compactification of a discrete space of size \( \aleph_1 \), and \( X = A(\aleph_1) \times 2^\omega \), then \( X \) is not \( \aleph_1 \)-dissipated by Lemma 3.6, but \( X \) is minimally generated, with \( \rho = \omega_1 + \omega \).

Note that if we weaken “tight” to “3–tight” in Theorem 6.2, we get nothing of any interest in general. In fact, if \( M = 2 = \{0, 1\} \) and each \( X_\alpha = M^\alpha \), then all \( \pi_\alpha^{\alpha+1} \mid X_{\alpha+1} \) are 3–tight, but \( X \) is not weakly \( \mathfrak{c} \)-dissipated by Theorem 5.8. However, one can in some cases use an inverse limit construction build a space which is \( \aleph_0 \)-dissipated:

**Proof of Proposition 5.5.** We modify the standard construction of a compact \( L \)-space under \( \text{CH} \), following specifically the details in [16]; similar constructions are in Haydon [13] and Talagrand [19]. So, \( X \) will be a closed subset of \( 2^{\omega_1} \).

We inductively define \( X_\alpha \subseteq 2^\alpha \), for \( \omega \leq \alpha \leq \omega_1 \), along with an atomless Radon probability measure \( \mu_\alpha \) on \( X_\alpha \) such that the support of \( \mu_\alpha \) is all of \( X_\alpha \). Let \( X_\omega = 2^\omega \) with \( \mu_\omega \) the usual product measure. The measures will all cohere, in the sense that \( \mu_\alpha = \mu_\beta (\pi_\beta^\alpha)^{-1} \) whenever \( \alpha < \beta \). Along with the measures, we choose a countable family \( \mathcal{F}_\alpha \) of closed \( \mu_\alpha \)-null subsets of \( X_\alpha \) and a specific closed nowhere dense non-null \( K_\alpha \subseteq X_\alpha \). When \( \alpha < \beta < \omega_1 \), \( \mathcal{F}_\beta \) will contain \( (\pi_\beta^\alpha)^{-1}(F) \) for all \( F \in \mathcal{F}_\beta \), along with some additional sets. Since \( \mathcal{F}_\alpha \) is countable, we can choose a perfect \( C_\alpha \subseteq K_\alpha \) such that \( \mu_\alpha(C_\alpha) > 0 \), \( C_\alpha \) is the support of \( \mu_\alpha \mid C_\alpha \), and \( C_\alpha \cap F = \emptyset \) for all \( F \in \mathcal{F}_\alpha \). Then we let \( X_{\alpha+1} = X_\alpha \times \{0\} \cup C_\alpha \times \{1\} \). In the construction of [16], \( \mu_{\alpha+1} \) can be chosen arbitrarily to satisfy \( \mu_\alpha = \mu_{\alpha+1} (\pi_\alpha^{\alpha+1})^{-1} \), as long as all non-empty open subsets of \( C_\alpha \times \{1\} \) have positive measure; there is some flexibility here in distributing the measure on \( C_\alpha \) among its copies \( C_\alpha \times \{0\} \) and \( C_\alpha \times \{1\} \). In particular, depending on the choices made, the final measure \( \mu = \mu_\omega \) on \( X = X_\omega \) may be separable or non-separable. In any case, [16] shows that, assuming \( \text{CH} \), one may choose the \( \mathcal{F}_\alpha \) and \( K_\alpha \) appropriately to guarantee \( X \) is an \( L \)-space and that the ideals of null subsets, meager subsets, and separable subsets all coincide.

Now, always choose \( \mu_{\alpha+1} \) such that \( \mu_{\alpha+1}(C_\alpha \times \{0\}) = 0 \). This will guarantee that \( \mu \) on \( X \) is separable, with \( \text{ma}(\mu) \) isomorphic to \( \text{ma}(\mu_\omega) \) via \( (\pi_\omega^{\alpha+1})^* \). Also, put the set \( C_\alpha \times \{0\} \) into \( \mathcal{F}_{\alpha+1} \). Then, for all \( x \in X_\omega \), \( (\pi_\omega^{\alpha+1})^{-1}\{x\} \) is scattered (as is easy to verify), and hence countable (since \( X \) is HL). But then \( \pi_\omega^{\alpha+1} \mid X : X \to X_\omega \) is \( \aleph_1 \)-tight, so that \( X \) is \( \aleph_1 \)-dissipated by Lemma 3.5.

We remark that by Theorem 5.8, we know that the \( \mu \) of Proposition 5.5 must be separable, so it was natural to make \( \text{ma}(\mu) \) isomorphic to \( \text{ma}(\mu_\omega) \) in the construction.
7 Absoluteness

We shall prove here that tightness is absolute. This can then be applied in forcing arguments, but the absoluteness itself has nothing at all to do with forcing; it is just a fact about transitive models of ZFC, and is related to the absoluteness of \( \Pi^1_1 \) statements. Since we never need absoluteness of \( \Pi^1_2 \) (Shoenfield’s Theorem), we do not need the models to contain all the ordinals. So, we consider arbitrary transitive models \( M, N \) of ZFC with \( M \subseteq N \). If in \( M \), we have compacta \( X, Y \) and \( f : X \to Y \), we want to show that \( f \) is tight in \( M \) iff \( f \) is tight in \( N \).

To make this discussion precise, we must, in \( N \), replace \( X, Y \) by the corresponding compact spaces \( \tilde{X}, \tilde{Y} \). This concept was described by Bandlow \cite{1} (and later in \cite{4, 5, 6, 12}), and is defined as follows:

**Definition 7.1** Let \( M \subseteq N \) be transitive models of ZFC. In \( M \), assume that \( X \) is compact. Then \( \tilde{X} \) denotes the compactum in \( N \) characterized by:

1. \( X \) is dense in \( \tilde{X} \).
2. Every \( \varphi \in C(X, [0, 1]) \cap M \) extends to a \( \tilde{\varphi} \in C(\tilde{X}, [0, 1]) \) in \( N \).
3. The functions \( \tilde{\varphi} \) (for \( \varphi \in M \)) separate the points of \( \tilde{X} \).

If, in \( M \), \( X, Y \) are compact and \( f \in C(X, Y) \), then in \( N \), \( \tilde{f} \in C(\tilde{X}, \tilde{Y}) \) denotes the (unique) continuous extension of \( f \).

In forcing, \( \tilde{X} \) denotes the \( X \) of \( V[G] \), while \( \tilde{X} \) denotes the \( X \) of \( V[G] \).

**Theorem 7.2** Let \( M \subseteq N \) be transitive models of ZFC. In \( M \), assume that \( X, Y \) are compact, \( K \) is compact metric, and \( f : X \to Y \). Then the following are equivalent:

1. In \( M \): There is a \( K \)–loose function for \( f \).
2. In \( N \): There is a \( \tilde{K} \)–loose function for \( \tilde{f} \).

**Proof.** For (1) \( \to \) (2), just observe that if in \( M \), we have \( \varphi, Q \) satisfying Definition 2.4 (of \( K \)–loose), then \( \tilde{\varphi}, \tilde{Q} \) satisfy Definition 2.4 in \( N \).

For (1) \( \to \) (2), we shall define a partial order \( T \) in \( M \). We shall then prove that (1) implies the well-founded of \( T \) in \( M \), while the well-founded of \( T \) in \( N \) implies (2). The result then follows by the absoluteness of well-foundedness.

As in the proof of Theorem 2.10, let \( H = [0, 1]^\omega \), and assume that \( K \subseteq H \). Then the existence of a \( K \)–loose function is equivalent to the existence of a \( \varphi \in C(X, H) \) such that for some non-scattered \( Q \subseteq Y \) we have \( \psi(f^{-1}(y)) \supseteq K \) for all \( y \in Q \).

\( T \) is a tree of finite sequences, ordered by extension. \( T \) contains the empty sequence and all non-empty sequences

\[ \langle (E_0, \psi_0), (E_1, \psi_1), \ldots, (E_{n-1}, \psi_{n-1}) \rangle \]

satisfying:
a. Each \( \psi_i \in C(X, H) \).

b. Each \( \mathcal{E}_i \) is a disjoint family of \( 2^i \) non-empty closed subsets of \( Y \).

c. Whenever \( y \in E \in \mathcal{E}_i \) and \( z \in K \): \( d(z, \psi_i(f^{-1}\{y\})) \leq 2^{-i} \).

d. When \( i + 1 < n \): \( d(\psi_i, \psi_{i+1}) \leq 2^{i-1} \), and each \( E \in \mathcal{E}_i \) has exactly two subsets in \( \mathcal{E}_{i+1} \).

In \( M \), if \( T \) is not well-founded and \( (\mathcal{E}_0, \psi_0), (\mathcal{E}_1, \psi_1), \ldots \) is an infinite path through \( T \), then we get \( \varphi = \lim_i \psi_i \in C(X, H) \) using (a)(d) and \( Q = \bigcap_i \bigcup \mathcal{E}_i \), which is a non-scattered subset of \( Y \) using (b)(d), and (c)(d) implies that \( \varphi(f^{-1}\{y\}) \supseteq K \) for all \( y \in Q \), so (1) holds.

Now, suppose, in \( N \), that we have \( Q, \varphi \) for which (2) holds; then we construct a path through \( T \). To obtain the \( \psi_i \) (all of which must be in \( M \)), use the fact that \( \{ \overline{\psi} : \psi \in C(X, H)^M \} \) is dense in \( C(\overline{X}, \overline{H}) \). Likewise each \( E \in \mathcal{E}_i \) will be a closed set in \( M \) such that \( \overline{E} \cap Q \) is not scattered.

Note that Theorem 7.2 says that the existence of the \( \varphi \) and \( Q \) described in the proof Theorem 2.10 is absolute. The corresponding “absoluteness version” of Theorem 2.9 is false. For example, suppose that in \( V \), we have \( X = Y \times K \), where \( X, Y, K \) are compact and non-scattered, and in addition, \( K \) has no non-trivial convergent \( \omega \)-sequences. Then clearly in \( V \), there can be no perfect \( Q \subseteq Y \) and 1-1 map \( i : Q \times (\omega + 1) \rightarrow X \) such that \( f(i(q, u)) = q \) for all \( (q, y) \in Q \times (\omega + 1) \), whereas if \( V[G] \) collapses enough cardinals, it will contain such a \( Q, i \).

An application of the absoluteness result in Theorem 7.2 is:

**Proof of Theorem 2.5** Assume that in the universe, \( V \): \( X \) and \( Y \) are compact, \( f : X \rightarrow Y \), and we have an infinite loose family \( \{ P_i : i \in \omega \} \). Let \( V[G] \) be any forcing extension of \( V \) which makes the weights of \( X \) and \( Y \) countable, so that in \( V[G] \), we still have \( f : \overline{X} \rightarrow \overline{Y} \) and a loose family \( \{ \overline{P}_i : i \in \omega \} \), but \( \overline{X} \) and \( \overline{Y} \) are now compact metric, so that Theorem 2.10 gives us an \( (\omega + 1) \)-loose function in \( V[G] \). Hence, by absoluteness, there is one in \( V \).

A direct proof of this can be given without forcing, but it seems quite a bit more complicated, since one must embed into the proof the method of Suslin used in proving Lemma 2.8; one cannot just quote Suslin’s theorem, since the spaces are not Polish. Theorem 2.3 is needed for the \( \kappa = \omega \) part of:

**Corollary 7.3** Fix \( \kappa \leq \omega \). Let \( M, N \) be transitive models of ZFC, with \( M \subseteq N \). Assume that in \( M \) we have \( X, Y, f \) with \( X, Y \) compact and \( f : X \rightarrow Y \). Then \( M \models \text{"} f : X \rightarrow Y \text{ is } \kappa \text{-tight} \) iff \( N \models \text{"} \overline{f} : \overline{X} \rightarrow \overline{Y} \text{ is } \kappa \text{-tight} \).

Of course, the \( \leftarrow \) direction is trivial, and holds for all \( \kappa \) if we rephrase Definition 2.1 appropriately so that \( \kappa \) is not required to be a cardinal (since “cardinal” is not...
absolute). That is, if in $M$, we have a loose family $\{P_\alpha : \alpha < \kappa\}$, then $\{\tilde{P}_\alpha : \alpha < \kappa\}$ is loose in $N$. For a version of Corollary 7.3 for $\kappa = c$, we use the notion of “weakly $c$–tight” from Definition 2.6.

**Corollary 7.4** Fix $\kappa \leq \omega$. Let $M, N$ be transitive models of ZFC, with $M \subseteq N$. Assume that in $M$ we have $X,Y,f$ with $X,Y$ compact and $f : X \to Y$. Then $M \models "f : X \to Y \text{ is weakly } c\text{-tight}"$ iff $N \models "\tilde{f} : \tilde{X} \to \tilde{Y} \text{ is weakly } c\text{-tight}"$.

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