PARTIALLY UMBILIC SINGULARITIES OF
HYPERSURFACES OF $\mathbb{R}^4$

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Abstract. This paper establishes the geometric structure of the lines of principal curvature of a hypersurface immersed in $\mathbb{R}^4$ in a neighborhood of the set $S$ of its principal curvature singularities, consisting of the points at which at least two principal curvatures are equal. Under generic conditions defined by appropriate transversality hypotheses it is proved that $S$ is the union of regular smooth curves $S_{12}$ and $S_{23}$, consisting of partially umbilic points, where only two principal curvatures coincide. This curve is partitioned into regular arcs consisting of points of Darbouxian types $D_1$, $D_2$, $D_3$, with common boundary at isolated semi-Darbouxian transition points of types $D_{12}$ and $D_{23}$. The stratified structure of the partially umbilic separatrix surfaces, consisting of the boundary of the set of points through which the principal lines approach $S$, established in this work, extends to hypersurfaces in $\mathbb{R}^4$ the results of Darboux in [1] for umbilic points on analytic surfaces in $\mathbb{R}^3$, reformulated by Gutierrez and Sotomayor in [8], to describe the umbilic separatrix structures of the umbilic types $D_1$, $D_2$, $D_3$, and further developed by Garcia, Gutierrez and Sotomayor in [7], for their $D_{12}$ and $D_{23}$ generic bifurcations. This work complements results of Garcia [5] on the structure of principal curvature lines around the generic partially umbilic points of hypersurfaces in $\mathbb{R}^4$.

1. INTRODUCTION

Let $M^3$ be a $C^\infty$, oriented, compact, 3– dimensional manifold.

An immersion $\alpha$ of $M^3$ into $\mathbb{R}^4$ is a map such that $D\alpha_p : TM^3_p \to \mathbb{R}^4$ is one to one, for every $p \in M^3$. Denote by $Imm^k(M^3,\mathbb{R}^4)$ the set of $C^k$ -immersions of $M^3$ into $\mathbb{R}^4$ endowed with the $C^k$– topology, see [10].

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Associated to every \( \alpha \in \text{Imm}^k(M^3,\mathbb{R}^4) \) is defined the normal map \( N_\alpha : M^3 \to S^3 \):

\[
N_\alpha = (\alpha_1 \wedge \alpha_2 \wedge \alpha_3) / | \alpha_1 \wedge \alpha_2 \wedge \alpha_3 |,
\]

where \((u_1,u_2,u_3) : (M,p) \to (\mathbb{R}^3,0)\) is a positive chart of \( M^3 \) around \( p \), \( \wedge \) denotes the product of vectors determined by a once for all fixed orientation in \( \mathbb{R}^4 \). This space is endowed with the Euclidean norm \( | \cdot | = \langle \cdot, \cdot \rangle^{1/2} \). Also \( \alpha_1 = \frac{\partial \alpha}{\partial u_1}, \alpha_2 = \frac{\partial \alpha}{\partial u_2}, \alpha_3 = \frac{\partial \alpha}{\partial u_3} \).

Clearly, \( N_\alpha \) is well defined and of class \( C^{k-1} \) in \( M^3 \).

Since \( D\alpha_\alpha(p) \) has its image contained in that of \( D\alpha(p) \), the endomorphism \( \omega_\alpha : TM^3 \to TM^3 \) is well defined by

\[
D\alpha.\omega_\alpha = D\alpha_\alpha.
\]

It is well known that \( \omega_\alpha \) is a self adjoint endomorphism, when \( TM^3 \) is endowed with the metric \( \langle \cdot, \cdot \rangle_\alpha \) induced by \( \alpha \) from the metric in \( \mathbb{R}^4 \). See [19].

The opposite values of the eigenvalues of \( \omega_\alpha \) are called principal curvatures of \( \alpha \) and will be denoted by \( k_1 = k_1(\alpha) \leq k_2 = k_2(\alpha) \leq k_3 = k_3(\alpha) \).

The principal singularities of the immersion \( \alpha \) are defined as follows:

- **Umbilic Points**: \( U_\alpha = \{ p \in M^3 : k_1(p) = k_2(p) = k_3(p) \} \),
- **Partially Umbilic Points**: \( S_\alpha = S_{12}(\alpha) \cup S_{23}(\alpha) \), where
- \( S_{12}(\alpha) = \{ p \in M^3 : k_1(p) = k_2(p) < k_3(p) \} \),
- \( S_{23}(\alpha) = \{ p \in M^3 : k_1(p) < k_2(p) = k_3(p) \} \).

The eigenspaces associated to the principal curvatures, when simple, define three line fields \( L_i(\alpha) \), \( i = 1,2,3 \), mutually orthogonal in \( TM^3 \) (endowed with the metric \( \langle \cdot, \cdot \rangle_\alpha \)), called principal line fields of \( \alpha \). They are characterized by Rodrigues’ equations [18] [19] and [20].

\[
L_i(\alpha) = \{ v \in TM^3 : \omega_\alpha v + k_i v = 0, \ i = 1,2,3 \}.
\]

These line fields are well defined and smooth outside their respective sets of principal singularities, as follows: \( L_1(\alpha) \) is of class \( C^{k-2} \) outside \( U_\alpha \cup S_{12}(\alpha) \), \( L_3(\alpha) \) is of class \( C^{k-2} \) outside \( U_\alpha \cup S_{23}(\alpha) \), \( L_2(\alpha) \) is of class \( C^{k-2} \) outside \( U_\alpha \cup S_\alpha \).

This follows from the smooth dependence of simple eigenvalues and corresponding one-dimensional eigenspaces.

The integral curves of \( L_i \), \( i = 1,2,3 \) are called the principal foliations \( F_i(\alpha) \) of \( \alpha \).
Generically, for an open and dense set, in the space $Imm^k(M^3, \mathbb{R}^4)$, $U_\alpha = \emptyset$ and $S_\alpha$, when non empty, is a regular submanifold of codimension two, consisting of two pieces $S_{12}(\alpha)$ and $S_{23}(\alpha)$. This follows from the Transversality Theorem [9] together with the following well known facts: In the 6–dimensional space of $3 \times 3$ symmetric matrices, those with three equal eigenvalues, denoted $\Sigma^3$, has codimension 5 and those with only two equal eigenvalues, denoted $\Sigma^2$, has codimension 2. See Lax [14]. In Section 10 the symmetric matrices $\Sigma^3$ and $\Sigma^2$ will appear identified with the umbilic and partially umbilic 2–jets, $U^2$ and $(PU)^2$ and their extensions to the space of 4–jets, where the submanifolds of interest for this work naturally belong.

In this work will be assumed that the 2–jet extensions, $j^2_\alpha$, of the immersions $\alpha$ have their $\omega_\alpha$ endomorphism transversal to both manifolds, $\Sigma^3$ and $\Sigma^2$ described above. Coordinate expressions for the five types of generic partially umbilic points (of codimension $\leq 3$) and pertinent transversality conditions in terms of the 4–jets of $\alpha$ will be given in Definitions [1] [3] and [5] See also section [10].

Thus in this paper, when non empty, $S_\alpha$ will be a finite collection of closed regular curves of class $C^{k–2}$ called the partially umbilic curves of $\alpha$, the umbilic set $U_\alpha$ being empty.

At each point $p \in S_{12}$ is defined the partially umbilic plane $P_3(p)$ which is orthogonal to the unique well defined principal direction $L_3(p)$, eigenspace associated to the simple eigenvalue $-k_3(p)$. Analogous definition holds for $p \in S_{23}$, $-k_1(p)$ and $P_1(p)$.

Fig. 1 illustrates a partially umbilic curve $S_{12}$ and some of its possible contact behaviors with the distribution or field of planes $P_3$ in $M^3 \setminus S_{23}$. Similar illustration applies to $S_{23}$ and the distribution or field of planes $P_1$.

This is an initial schematic illustration. Specific configurations of principal curvature lines, which are the integral foliations $F_i(\alpha)$ of the line fields $L_i(\alpha)$, are established in this work for the generic case around partially umbilic points.

To make concrete this schematic picture the following must be added:
1. - Along the transversal arcs: sub-arcs of Darbouxian points $D_i$, $i = 1, 2$, are separated by $D_{1,2}$ transition points, as established in Theorem [2]

2.- At the quadratic tangential points the sub-arcs of Darbouxian $D_i$, $i = 2, 3$ points, are separated by $D_{23}$ transition points. See Theorem [3]
In this work will be established the *stratified structure* of the integral foliations $F_i(\alpha)$ near the generic singularities $S_\alpha$. See Mather [17]. To this end will be provided a new approach and proofs, based on the Lie - Cartan suspension, explained in section 6, which allows to improve the results given in Garcia [4] and [5].

See Theorem 1 in section 3, Theorem 2 in section 4 and Theorem 3 in section 5 for a synthesis of the main results of this work. The proofs will be given after some preliminaries on the differential equations of principal foliations and the basic properties of their Lie-Cartan suspensions presented in section 6.

Sections 7, 8 and 9, respectively, contain the proofs of Theorems 1, 2 and 3. See also the summarizing Theorem 4 in section 10.
Section 11 is devoted to a discussion of the relation of the results of this paper with previous and forthcoming ones. The Appendix 12 contains the coordinate expressions of the geometric functions essential for the calculations in the proofs of the main results of this paper.

This work can also be regarded as an extension to $\mathbb{R}^4$ the results about umbilic points and their generic bifurcations for surfaces immersed in $\mathbb{R}^3$. See [9], [8], [7], [11].

The structure of the global principal configurations on the ellipsoids in $\mathbb{R}^4$ has been established Garcia, Lopes and Sotomayor [15].

1.1. Definition of the Color Convention in Illustrations. The color convention for partially umbilic curves and the integral leaves of principal line fields, packed into strata, established in this paper is defined as follows.

- Black: integral curves of line field $L_1$,
- Red: integral curves of line field $L_2$,
- Blue: integral curves of line field $L_3$,
- Green: Partially umbilic arcs $S_{12}$,
- Light Blue: Partially umbilic arcs $S_{23}$.

2. Adapted Monge charts at Partially Umbilic Points

Let $p \in M^3$ be a partially umbilic point of an immersion $\alpha$ such that $k_1(p) = k_2(p) = k(p) < k_3(p)$. That is $p \in S_{12}(\alpha)$.

Let $(u_1, u_2, u_3) : M^3 \to \mathbb{R}^3$ be a local chart and $R : \mathbb{R}^4 \to \mathbb{R}^4$ be an isometry such that:

$$(R \circ \alpha)(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3))$$

where:

$$(1)\quad h = \frac{k}{2}(u_1^2 + u_2^2) + \frac{k_3}{2}u_3^2 + \frac{a}{6}u_1^3 + \frac{b}{2}u_1u_2^2 + \frac{c}{6}u_2^3 + \frac{q_{003}}{6}u_3^3 + \frac{q_{012}}{2}u_2u_3^2$$

$$+ q_{111}u_1u_2u_3 + \frac{q_{021}}{2}u_2^2u_3 + \frac{q_{102}}{2}u_1u_2^2 + \frac{q_{201}}{2}u_1^2u_3 + \frac{A}{24}u_1^4 + \frac{B}{6}u_1^3u_2$$

$$+ \frac{C}{4}u_1^2u_2^2 + \frac{D}{6}u_1u_3^3 + \frac{E}{24}u_4^3 + \frac{Q_{004}}{6}u_2u_3^3 + \frac{Q_{013}}{6}u_2^3u_3 + \frac{Q_{103}}{6}u_1u_3^3$$

$$+ \frac{Q_{022}}{4}u_2^2u_3^2 + \frac{Q_{202}}{6}u_1^3u_3 + \frac{Q_{112}}{2}u_1u_2u_2^2 + \frac{Q_{031}}{6}u_2^3u_3 + \frac{Q_{301}}{6}u_3^3u_3$$

$$+ \frac{Q_{121}}{2}u_1^2u_2u_3 + \frac{Q_{211}}{2}u_1u_2u_3 + h.o.t.$$  

Remark 1. The rotation $R$ was chosen to eliminate the coefficient of the term $u_1^2u_2$. In this sense the chart is said to be adapted by rotation.
Composing the immersion with a homothety, one of the coefficients \( k \) or \( k_3 \) in equation (1) can be taken to be 1 or \(-1\). After a reflection, 1 can always be chosen to be such coefficient. In this sense the chart is said to be adapted by homothety and reflection.

By means of an additional inversion the other coefficient (i.e. \( k_3 \) or \( k \)) may be assumed to be 0.

Some of the long expressions appearing in this work may be simplified by using adapted charts.

3. Darbouxian Partially Umbilic Points

Definition 1. The point \( p \) in \( S_{12}(\alpha) \) is called a Darbouxian partially umbilic point of type \( D_i \) if, in the notation of equation (1), the geometric transversality condition \( T \) and the discriminant condition \( D_i \) below hold.

\[
\begin{align*}
T & \quad b(b - a) \neq 0 \\
D_1 & \quad \frac{a}{b} > \left( \frac{c}{2b} \right)^2 + 2; \\
D_2 & \quad 1 < \frac{a}{b} < \left( \frac{c}{2b} \right)^2 + 2, \quad a \neq 2b; \\
D_3 & \quad \frac{a}{b} < 1.
\end{align*}
\]

The main result of this section is stated now.

Theorem 1. Let \( \alpha \in \text{Imm}^k(M^3, \mathbb{R}^4) \) and \( p \in S_{12}(\alpha) \). Then there is a neighborhood \( V_p \) of \( p \) where \( S = S_{12}(\alpha) \cap V_p \) is a smooth curve consisting of points \( D_i \) where it holds that:

i) For the case \( D_1 \), there exists a unique invariant separatrix surface \( W_1(S) \subset V_p, \partial W_1(S) = S \), of class \( C^{k-3} \), fibred over \( S \) whose fibers are leaves of \( F_1(\alpha) \). Only these leaves (the fibers) are asymptotic to the partially umbilic curve \( S \).

The set \( V_p \setminus W_1(S) \) is a hyperbolic sector of \( F_1(\alpha) \). See Fig. left.

ii) For the case \( D_2 \) there exist two invariant separatrix surfaces as described in item i) and exactly one wedge sector and one hyperbolic sector of \( F_1(\alpha) \). See Fig. center.

iii) For the case \( D_3 \), there exist three invariant separatrix surfaces as in item i) and exactly three hyperbolic sectors of \( F_1(\alpha) \). See Fig. right.

iv) The same conclusions hold for the foliation \( F_2(\alpha) \) which is orthogonal to \( F_1(\alpha) \) and singular in the set \( S_{12} \). See Fig. right.
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Definition 2. A separatrix surface as that in theorem 7 is called partially umbilic separatrix surface.

4. $D_{12}$ Partially Umbilic Points

Definition 3. Let $p$ be a partially umbilic point and $\alpha$ expressed as in equation (1). The point $p$ is called semi-Darbouxian of type $D_{12}$ if the following conditions are satisfied:

\[ D_{12} \quad bc(b - a) \neq 0, \quad \frac{a}{b} = 2 \quad \text{and} \quad \chi_{12} \neq 0, \]

where

\[
\chi_{12} = (k - k_3) \left[ (bq_{021} - bq_{201} - cq_{111}) B - bq_{111} C + b^2 Q_{211} \right] + q_{12} q_{201} b^2 + (2 q_{102} b^2 - bk^3 k_3 + bk^4) q_{111} - 3 cq_{111}^2 q_{201} - 3 bq_{111} q_{201} - 2b q_{111}^3 q_{201} q_{201} \]

(2)

Remark 2. When $\frac{a}{b} = \left(\frac{c}{2b}\right)^2 + 2$, $b(b - a) \neq 0$, another condition $\chi_{12}^* = \chi_{12}^*(j^4 h(0)) \neq 0$ characterizes a $D_{12}$ partially umbilic point, see equation

**Figure 2.** Principal Foliations $\mathcal{F}_1(\alpha)$ and partially umbilic separatrix surfaces in the neighborhood of the point $D_1$ (left), $D_2$ (center) and $D_3$ (right).

**Figure 3.** Principal Foliations $\mathcal{F}_2(\alpha)$ and partially umbilic separatrix surfaces in the neighborhood of the point $D_1$ (left), $D_2$ (center) and $D_3$ (right).
This condition can be obtained from that given in defining equation (2) by an appropriate rotation in the \((u,v)\)-plane in equation (1). In fact the two patterns of failure of the discriminant condition \(D_2\), \(\frac{a}{b} = (\frac{c}{2b})^2 + 2\) and \(a = 2b\), keeping the transversality condition \(b \neq a\), are permuted by a rotation.

The calculations of these conditions give shorter expressions working with \(\chi_{12}\).

**Remark 3.** The condition \(\chi_{12} \neq 0\) expresses the transversality, at \(D_{12}\), of the transition between types \(D_1\) and \(D_2\), as will follow from the analysis in section 8. As discussed in remark 1 the long expressions in (2) and (50) can be simplified by taking \(k_3 = 1\) and \(k = 0\) by applying to it an inversion and a homothety.

**Theorem 2.** Let \(p\) be a \(D_{12}\) partially umbilic point. Then it has a neighborhood \(V_p\) which intersects \(S_\alpha\) on a partially umbilic smooth curve \(S = S_1 \cup S_2 \cup \{p\}\), transversal to \(P_3(p)\), which consists on two arcs of points of types \(D_1\) and \(D_2\), having \(p\) as common boundary. It holds that:

i) There exists a partially umbilic separatrix surface of \(F_2(\alpha)\), \(W = W_1(S) \cup F_1 \cup W_2(S)\), stratified as follows:
- \(W_1(S)\) is a partially umbilic separatrix surface of an arc of partially umbilic points, \(S_1\), of type \(D_1\).
- \(W_2(S)\) is a partially umbilic separatrix surface of an arc \(S_2\) of partially umbilic points of type \(D_2\).
- \(F_1\) is a simple leaf of \(F_2(\alpha)\) asymptotic to the partially umbilic point \(p\) of type \(D_{12}\). See Fig. 4.

ii) There exists a partially umbilic separatrix surface \(W_3(S) \cup F_2\) of \(F_2(\alpha)\), stratified as follows:
- \(W_3(S)\) is a partially umbilic separatrix surface of an arc \(S_2\) of partially umbilic points of type \(D_2\).
- \(F_2\) is a double leaf of \(F_2(\alpha)\) asymptotic to the partially umbilic point \(p\) of type \(D_{12}\). See Fig. 4.

iii) There exists a three dimensional wedge sector \(W\) such that \(\partial W\) is a variety partitioned into strata of dimension two, one and zero, as follows:
- \(W_4(S) \cup W_3(S) \cup W_2(S)\) are the bi-dimensional strata. Moreover \(W_4(S)\) consists of leaves of \(F_2(\alpha)\) which are asymptotic to the partially umbilic point \(p\) of type \(D_{12}\). See Fig. 5.
- \(F_1 \cup F_2 \cup S_2\) are the one dimensional strata.
• \( p \) is the zero dimensional stratum.

iv) The same conclusions hold for the foliation \( \mathcal{F}_1(\alpha) \) which is orthogonal to \( \mathcal{F}_2(\alpha) \) and singular in the set \( \mathcal{S} \). See Fig. 4 (left).

The behavior of the foliations \( \mathcal{F}_1(\alpha) \) and \( \mathcal{F}_2(\alpha) \) a neighborhood \( V_p \) of \( p \) is illustrated in Fig. 4. The stratification of the wedge sector is illustrated in Fig. 5.

![Figure 4](image)

Figure 4. Principal foliations \( \mathcal{F}_1(\alpha) \) (Left) and \( \mathcal{F}_2(\alpha) \) (Right) in a neighborhood of the point \( D_{12} \).

![Figure 5](image)

Figure 5. Stratification of the wedge sector (strata of dimensions three, two and one) and umbilic separatrices of a \( D_2 \)– \( D_{12} \)– \( D_1 \) partially umbilic arc of \( \mathcal{F}_2(\alpha) \).

**Definition 4.** The leaf \( F_1 \) in the item i) of theorem 2 is called a simple \( D_{12} \) separatrix, and the leaf \( F_2 \) in the item ii) is called a double \( D_{12} \) separatrix. There are no leaves tangent to \( F_1 \) and asymptotic to the point \( D_{12} \). There are infinitely many leaves tangent to \( F_2 \) and asymptotic to the point \( D_{12} \).
5. $D_{23}$ Partially Umbilic Points

**Definition 5.** Let $p$ be a partially umbilic point and $\alpha$ parametrized as in equation (1). The point $p$ is called semi-Darbouxian of type $D_{23}$ if the following conditions hold:

$$D_{23}) \, b = a \neq 0, \, b(q_{201} - q_{021}) + cq_{111} \neq 0 \text{ and } \chi_{23} \neq 0$$

where,

$$\chi_{23} = (k - k_3)(bA + cB - bC - 2bk^3) + (3q_{201}^2 - q_{201}q_{021} - 2q_{111}^2)b + 3q_{111}q_{201}c$$

**Theorem 3.** Let $p$ be a $D_{23}$ partially umbilic point. Then there exists a neighborhood $V_p$ of $p$ where $\mathcal{S} \cap V_p = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{p\}$ is a smooth curve tangent to the partially umbilic plane, consisting of two arcs of partially umbilic points of types $D_2$ and $D_3$, separated by the $D_{23}$ point. The following holds in $V_p$.

i) There exists a regular surface $W_1(\mathcal{S}) \subset V_p$ containing the partially umbilic curve $\mathcal{S}$ with the following property: a connected component of $W_1(\mathcal{S}) \setminus \mathcal{S}$ is invariant by $F_1(\alpha)$ and the other is invariant by $F_2(\alpha)$. See Fig. 6.

ii) There exists a regular surface $W_2(\mathcal{S}) \subset V_p$ containing the partially umbilic curve $\mathcal{S}$ with the following property: one of the connected components of $W_2(\mathcal{S}) \setminus \mathcal{S}$ is invariant by $F_1(\alpha)$ and the other is invariant by $F_2(\alpha)$. See Fig. 6.

iii) There exists a surface $W_3(\mathcal{S}) \subset V_p$ containing the partially umbilic curve $\mathcal{S}$ with the following property: one of the connected components of $W_3(\mathcal{S}) \setminus \mathcal{S}$ is invariant by $F_1(\alpha)$ and the other is invariant by $F_2(\alpha)$.

Moreover $F_5 \subset W_3$ is a leaf of $F_2(\alpha)$ asymptotic to the $D_{23}$ partially umbilic point $p$. See Fig. 6.

iv) There exists a three dimensional wedge sector $W_1$ such that $\partial W_1$ is a variety (union of strata of dimension two, one and zero) partitioned as follows:

- $W_2(\mathcal{S}) \cup W_1(\mathcal{S})$ are the bi-dimensional strata. See Fig. 7.
- $F_1 \cup F_2 \cup S_1 \cup S_2$ are the one dimensional strata.
- $p$ is the one dimensional stratum.

v) There exists a three dimensional wedge sector $W_2$ such that $\partial W_2$ is a variety partitioned into strata of dimensions two, one and zero, as follows:
• $W_4(S) \cup W_2(S) \cup W_1(S)$ are the bi-dimensional strata. Moreover $W_4(S)$ consists of leaves of $\mathcal{F}_2(\alpha)$ which are asymptotic to the partially umbilic point $p$ of type $D_{23}$. See Fig. 8.

• $F_3 \cup F_4 \cup S_1 \cup S_2$ are the one dimensional strata.

• $p$ is the zero dimensional stratum.

vi) The behavior of the principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in the neighborhood of $p$ is as illustrated in Fig. 6 (left and right).

\[\text{Figure 6. Principal foliations } \mathcal{F}_1(\alpha) \text{ (Left, views from top and bottom) and } \mathcal{F}_2(\alpha) \text{ (Right, views from top and bottom) in the neighborhood of a point } D_{23}.\]

Remark 4. The condition $\chi_{23} \neq 0$ is equivalent to the quadratic contact, at $D_{23}$, of the partially umbilic curve $S_{12}$ with the plane field $\mathcal{P}_3$.

The analysis in section 8 shows that condition $\chi_{23} \neq 0$ gives also the saddle-node character of an equilibrium point which is essential to establish the principal configuration around the point $D_{23}$, exhibiting the transition between the types $D_2$ and $D_3$. 
Remark 5. The conditions that define the types $D_i$ and $D_{ij}$ are independent of the coordinate charts. This can be verified through a direct calculation,
similar to that performed in the two dimensional case, \([8, 10]\). The conditions above that define in sections \([7, 8]\) and \([8]\) the partially umbilic points studied in this work are closely related to those that define the Darbouxian and Semi-Darbouxian umbilic points in the case of two dimensional surfaces, see \([7]\) and \([8, 10, 11]\).

6. Differential Equations of the Lines of Principal Curvature and Lie-Cartan Suspension

In this section will be obtained the differential equation of the principal line fields \(L_i(\alpha)\).

Let \(\alpha : M^3 \to \mathbb{R}^4\) be an immersion of class \(C^k\), \(k \geq 6\), where \(M^3\) is compact and oriented manifold. The space \(\mathbb{R}^4\) is also oriented with the orientation fixed by the canonical basis \(\{E_1, E_2, E_3, E_4\}\).

In a local chart \((u_1, u_2, u_3) : M^3 \to \mathbb{R}^3\) the first and the second fundamental forms associated to the immersion \(\alpha\) are given, respectively, by \(I_\alpha = \sum g_{ij} du_i du_j\) and \(II_\alpha = \sum \lambda_{ij} du_i du_j\), where \(g_{ij} = \langle \partial \alpha / \partial u_i, \partial \alpha / \partial u_j \rangle\) and \(\lambda_{ij} = \langle \partial^2 \alpha / \partial u_i \partial u_j, N \rangle\) and \(N\) is the positive normal vector to the immersion \(\alpha\) which is defined by: \(N_\alpha(p) = (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)(p) / |(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)(p)|\).

The normal curvature at the point \(p\) in the direction \(v = (du_1, du_2, du_3)\) is defined by \(k_\alpha(p)(v) = (II_\alpha)(p)(v, v)\).

The directions at which \(k_\alpha\) assume critical values are the principal directions and the values of \(k_\alpha\) in these directions are the principal curvatures. They will be denoted by \(k_1(p) \leq k_2(p) \leq k_3(p)\).

To calculate the principal directions observe that the normal curvature can also defined by \(k_\alpha(p) = II_\alpha\) subject to the condition \(I_\alpha = 1\). Therefore using the method of Lagrange with parameter \(k\), it results:

\[
\begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{pmatrix}
\begin{pmatrix}
du_1 \\
du_2 \\
du_3
\end{pmatrix}
= k
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\begin{pmatrix}
du_1 \\
du_2 \\
du_3
\end{pmatrix}
\]

Therefore the principal directions are given by the following system of equations:

\[
(\lambda_{11} - k_1 g_{11}) du_1 + (\lambda_{12} - k_1 g_{12}) du_2 + (\lambda_{13} - k_1 g_{13}) du_3 = 0
\]

\[
(\lambda_{21} - k_2 g_{21}) du_1 + (\lambda_{22} - k_2 g_{22}) du_2 + (\lambda_{23} - k_2 g_{23}) du_3 = 0
\]

\[
(\lambda_{31} - k_3 g_{31}) du_1 + (\lambda_{32} - k_3 g_{32}) du_2 + (\lambda_{33} - k_3 g_{33}) du_3 = 0
\]
where \( k_i \) (\( i = 1, 2, 3 \)) are the principal curvatures, which are defined by the equation \( \det(\lambda_{ij} - kg_{ij}) = 0 \). This equation is useful only when the correspondent principal curvature is smooth. Near the partially umbilic points two principal curvatures are only continuous and it is more convenient to consider implicit differential equations.

Next will be obtained the Lie-Cartan vector field that will be used for the analysis of principal curvature line near the partially umbilic points in this work.

Consider the plane passing through \( q \in M \) having the principal direction \( e_3(q) \) as normal vector:

\[
\begin{align*}
(5) \quad & \mathcal{P}_3(q) = \{(du_1, du_2, du_3); \langle (du_1, du_2, du_3), G \cdot (e_3(q))^T \rangle = 0 \}, \\
& \text{where } G = [g_{ij}]_{3 \times 3} \text{ is the matrix of the first fundamental form.}
\end{align*}
\]

Using equation (5), solving the linear system it follows that the principal direction \( e_3(q) = (du_1, du_2, du_3) \) is given by

\[
\frac{du_1}{du_3} = \frac{U_1(u_1, u_2, u_3)}{W_1(u_1, u_2, u_3)}, \quad \frac{du_2}{du_3} = \frac{V_1(u_1, u_2, u_3)}{W_1(u_1, u_2, u_3)}
\]

where

\[
\begin{align*}
U_1 &= (g_{12}g_{23} - g_{22}g_{13}) k_3^2 + (-g_{12}\lambda_{23} - g_{23}\lambda_{12} + \lambda_{22}g_{13} + g_{22}\lambda_{13}) k_3 + \\
& \quad + \lambda_{23}\lambda_{12} - \lambda_{22}\lambda_{13} \\
V_1 &= (-g_{11}g_{23} + g_{13}g_{12}) k_3^2 + (\lambda_{11}g_{23} + g_{11}\lambda_{23} - \lambda_{13}g_{12} - g_{13}\lambda_{12}) k_3 + \\
& \quad - \lambda_{11}\lambda_{23} + \lambda_{13}\lambda_{12} \\
W_1 &= (g_{11}g_{22} - g_{12}^2) k_3^2 + (-\lambda_{11}g_{22} - g_{11}\lambda_{22} + 2\lambda_{12}g_{12}) k_3 + \\
& \quad + \lambda_{11}\lambda_{22} - \lambda_{12}^2.
\end{align*}
\]

Notice that \( W_1 \neq 0 \) in a neighborhood of the partially umbilic point. This follows from the calculations displayed in subsections 12.3 and 12.4 of the Appendix which give \( W_1(0, 0, 0) = (k - k_3)^2 \).

Therefore, from equation (5), the field of planes \( \mathcal{P}_3 \) is defined by the field of kernels of the differential one form

\[
\begin{align*}
(7) \quad & \omega = [g_{11}U_1 + g_{12}V_1 + g_{13}W_1]du_1 + [g_{12}U_1 + g_{22}V_1 + g_{23}W_1]du_2 \\
& \quad + [g_{13}U_1 + g_{23}V_1 + g_{33}W_1]du_3 = 0
\end{align*}
\]

**Remark 6.** The plane field \( \mathcal{P}_3 \) is in general not integrable. Therefore the analysis of the integral foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of the line fields \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) near a partially umbilic curve \( S_{12} \) is strictly three-dimensional. In other words it
cannot be reduced to a family of two-dimensional principal configurations on the integral leaves of the field of planes \( \mathcal{P}_3 \), obtained from Frobenius Theorem. See Spivak [19].

In fact, calculation in the Monge chart in equation (1), gives that \( \omega \wedge d\omega \) is expressed near 0 by:

\[
(\omega \wedge d\omega) = -(k - k_3)^2[q_{111}(a - b)u_1 + (b(q_{021} - q_{201}) - c q_{111})u_2 + \text{h.o.t.}]
\]

This shows that the condition for Frobenius integrability \( \omega \wedge d\omega = 0 \), identically, does not hold generically.

The principal directions \( e_1(q) \) and \( e_2(q) \) associated to \( k_1(q) \) and \( k_2(q) \) belong to the plane \( \mathcal{P}_3(q) \).

Consider the first and second fundamental forms of \( \alpha \) restricted to \( \mathcal{P}_3 \) and write,

\[
I_r(du_1, du_2) = I_\alpha \bigg|_{du_1 = \mathcal{U}(u_1, u_2, u_3)du_1 + \mathcal{V}(u_1, u_2, u_3)du_2} = E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2,
\]

\[
II_r(du_1, du_2) = II_\alpha \bigg|_{du_3 = \mathcal{U}(u_1, u_2, u_3)du_1 + \mathcal{V}(u_1, u_2, u_3)du_2} = e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2,
\]

where

\[
\mathcal{U} = -\frac{[g_{11}U_1 + g_{12}V_1 + g_{13}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]}, \quad \mathcal{V} = -\frac{[g_{12}U_1 + g_{22}V_1 + g_{23}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]},
\]

\[
E_r = \frac{\partial^2 I_r}{2\partial du_1^2}(0, 0), \quad F_r = \frac{\partial^2 I_r}{\partial du_1 \partial du_2}(0, 0), \quad G_r = \frac{\partial^2 I_r}{2\partial du_2^2}(0, 0),
\]

\[
e_r = \frac{\partial^2 II_r}{2\partial du_1^2}(0, 0), \quad f_r = \frac{\partial^2 II_r}{\partial du_1 \partial du_2}(0, 0), \quad g_r = \frac{\partial^2 II_r}{2\partial du_2^2}(0, 0).
\]

Write \( k^r_n(q; du_1, du_2) = \frac{II_r}{I_r}(q, du_1, du_2) \), where \( I_r(q) \) and \( II_r(q) \) are the first and second fundamental forms of \( \alpha \) restricted to the plane \( \mathcal{P}_3(q) \).

Let \( P = \frac{du_2}{du_1} \). Therefore the slopes of the principal directions \( e_1(q) \) and \( e_2(q) \) in the plane \( \mathcal{P}_3 \) are defined by the implicit differential equation:

\[
\mathcal{L} = L_r(u_1, u_2, u_3)P^2 + M_r(u_1, u_2, u_3)P + N_r(u_1, u_2, u_3) = 0
\]

\[
(8) \quad \omega = [g_{11}U_1 + g_{12}V_1 + g_{13}W_1]du_1 + [g_{12}U_1 + g_{22}V_1 + g_{23}W_1]du_2 + [g_{13}U_1 + g_{23}V_1 + g_{33}W_1]du_3 = 0,
\]
where

\[(9) \quad L_r = F_r g_r - f_r G_r, \quad M_r = E_r g_r - e_r G_r, \quad N_r = E_r f_r - e_r F_r.\]

**Remark 7.** The partially umbilic points \((k_1 = k_2)\) are defined by \(L_r(u_1, u_2, u_3) = 0\) and \(M_r(u_1, u_2, u_3) = 0\).

Let \(\mathcal{L}(u_1, u_2, u_3, P) = L_r(u_1, u_2, u_3)P^2 + M_r(u_1, u_2, u_3)P + N_r(u_1, u_2, u_3)\).

Consider in the \((u_1, u_2, u_3, P)\)-space the hypersurface

\[(10) \quad \mathcal{L} = \{(u_1, u_2, u_3, P) : \mathcal{L}(u_1, u_2, u_3; P) = 0\},\]
called Lie-Cartan hypersurface.

**Remark 8.** To cover the whole sub-bundle of the tangent projective bundle over \(M\), defined by the lines contained in the field of planes orthogonal to \(P_3\) one must consider also the equation \(L_r du_2^2 + M_r du_1 du_2 + N_r du_1^2 = 0\), with the projective coordinate \(Q = du_1/du_2\). Direct calculation shows that, in the cases considered in this paper, there are no singularities to analyze near the points with \(Q = 0\) which represent the points \(P = \infty\).

**Proposition 1.** The vector field \(X = X_\mathcal{L} = X_1 \frac{\partial}{\partial u_1} + X_2 \frac{\partial}{\partial u_2} + X_3 \frac{\partial}{\partial u_3} + X_4 \frac{\partial}{\partial P}\) where

\[(11) \quad X_1 = \mathcal{L}_P, \quad X_2 = P \mathcal{L}_P, \quad X_3 = (U + VP) \mathcal{L}_P, \quad X_4 = -(\mathcal{L}_{u_1} + P \mathcal{L}_{u_2} + \mathcal{L}_{u_3} (U + VP))\]
is of class \(C^{k-3}\), tangent to \(\mathcal{L}\) and the projections of the integral curves of \(X\) by \(\pi(u_1, u_2, u_3, P) = (u_1, u_2, u_3)\) are the principal lines of the two principal foliations \(\mathcal{F}_1\) and \(\mathcal{F}_2\) which are singular along the partially umbilic curve \(S_{12}\).

**Proof.** By direct verification the vector field \(X = X_\mathcal{L}\) is tangent to the hypersurface \(\mathcal{L}^{-1}(0)\) and its integral curves project to solutions of the implicit differential equation \(\mathcal{L}\). \(\square\)

7. **Proof of Theorem**

**Sketch of Proof.** The behavior of the curvature lines near a partially umbilic curve \(S\) will be described from the analysis of the lifted vector field \(X = X_1 \partial/\partial u_1 + X_2 \partial/\partial u_2 + X_3 \partial/\partial u_3 + X_4 \partial/\partial P\) given in equation \(11\).
The projections of the integral curves of $X$ are the curvature lines outside the umbilic and partially umbilic points. It will be shown that $X$ has lines of singularities contained in $\mathcal{L}$. More precisely, in the $D_1$ case, $X$ has a line of singularities $\beta_1$ normally hyperbolic such that the stable and unstable manifolds, $W^s_X(\beta_1) = \{ p : \omega(p) = \beta_1 \}$ and $W^u_X(\beta_1) = \{ p : \alpha(p) = \beta_1 \}$ are two dimensional smooth surfaces with $\partial W^s_X(\beta_1) = \partial W^u_X(\beta_1) = \beta_1$.

In the $D_2$ case, $X$ has three lines of singularities, $\beta_1$, $\beta_2$ and $\beta_3$, normally hyperbolic and such that $\dim W^s_X(\beta_1) = 3$ (repeller normally hyperbolic) and near $\beta_2$ and $\beta_3$ the behavior of $X$ is as in $D_1$ case.

Finally, in $D_3$ case, $X$ has three lines of singularities and near each line the behavior of $X$ is as in the $D_1$ case. See Figs. [12][13] and [11].

The partially umbilic separatrices are the diffeomorphic images of the bi-dimensional invariant manifolds associated to the normally hyperbolic singularities of $X$.

7.1. Local analysis of the Lie-Cartan vector field $X = X_1 \partial/\partial u_1 + X_2 \partial/\partial u_2 + X_3 \partial/\partial u_3 + X_4 \partial/\partial P$. First the singularities of $X$ will be determined.

In the Monge chart, see equation (11) and the Appendix 12.7 and 12.8 where the defining restricted functions in equation (9) are displayed, the vector field $X = X_L$ given in equation (11) is written as:

\begin{equation}
X_1 = (-2bu_2 - 2q_{111}u_3)P + (-a + b)u_1 + cu_2 + (-q_{201} + q_{021})u_3 + h.o.t
\end{equation}

\begin{align*}
X_2 &= PX_1 \\
X_3 &= \left( -\frac{q_{111}u_1 + q_{021}u_2 + q_{102}u_3}{k - k_3} - \frac{q_{201}u_1 + q_{111}u_2 + q_{102}u_3}{k - k_3} + h.o.t \right) X_1 \\
X_4 &= A_3(u_1, u_2, u_3)P^3 + A_2(u_1, u_2, u_3)P^2 + A_1(u_1, u_2, u_3)P + A_0(u_1, u_2, u_3)
\end{align*}

where

\begin{align*}
A_3(u_1, u_2, u_3) &= b + \left( C - k^3 + \frac{q_{111}^2 + q_{201}q_{021}}{k - k_3} \right) u_1 + \left( D + 3\frac{q_{111}q_{021}}{k - k_3} \right) u_2 + \\
&\quad + \left( Q_{121} + \frac{2q_{111}q_{021} + q_{102}q_{021}}{k - k_3} \right) u_3 + h.o.t \\
A_2(u_1, u_2, u_3) &= -c + \left( -D + 2B + \frac{6q_{111}q_{201} - 3q_{111}q_{021}}{k - k_3} \right) u_1 + \\
&\quad + \left( -E + k^3 + 2C + \frac{4q_{111}^2 - 3q_{201}q_{021} + 2q_{201}q_{021}}{k - k_3} \right) u_2 + \\
&\quad + \left( -Q_{031} + 2\frac{Q_{211}(2q_{201}q_{012} + 4q_{102}q_{111} - 3q_{012}q_{021})}{k - k_3} \right) u_3 + h.o.t
\end{align*}
\[ A_1(u_1, u_2, u_3) = a - 2b + \left(-2C + A - k^3 + \frac{-2q_{201}q_{021} - 4q_{111}^2 + 3q_{201}}{k - k_3}\right)u_1 + \left(-2D + B + \frac{3q_{111}q_{201} - 6q_{111}q_{021}}{k - k_3}\right)u_2 + \left(-2Q_{121} + Q_{301} + \frac{3q_{102}q_{201} - 2q_{102}q_{021} - 4q_{111}q_{012}}{k - k_3}\right)u_3 + \text{h.o.t} \]

\[ A_0(u_1, u_2, u_3) = \left(-B - 3q_{111}q_{201}/k - k_3\right)u_1 + \left(-C + k^3 - \frac{2q_{111}^2 + 2q_{021}q_{102}}{k - k_3}\right)u_2 + \left(-Q_{211} - \frac{q_{201}q_{012} + 2q_{102}q_{111}}{k - k_3}\right)u_3 + \text{h.o.t} \]

**Lemma 1.** Let \( S \) be a Darbouxian partially umbilic curve.

i) If \( S \) is of type \( D_1 \), then \( X \) has a line of singularities \( \gamma_1 \).

ii) If \( S \) is of type \( D_2 \) or \( D_3 \) then \( X \) has three lines of singularities \( \gamma_i \), \((i = 1, 2, 3)\).

**Proof.** The singular points of \( X = \mathcal{X}_L \) are given by

\[
\begin{cases}
L_r(u_1, u_2, u_3) = 0, & \text{(Partially Umbilic Points, see remark 7)} \\
M_r(u_1, u_2, u_3) = 0, \\
A_3(u_1, u_2, u_3)P^3 + A_2(u_1, u_2, u_3)P^2 + A_1(u_1, u_2, u_3)P + A_0(u_1, u_2, u_3) = 0
\end{cases}
\]

As \( \det \left( \frac{\partial (L_r, M_r)}{\partial (u_1, u_2)} \right)_{u_3=0} \neq 0 \) we can write \( u_1 = u_1(u_3) \) and \( u_2 = u_2(u_3) \) in \( L_r(u_1, u_2, u_3) = 0 \) and \( M_r(u_1, u_2, u_3) = 0 \).

Let \( C(u_3, P) = A_3(u_1(u_3), u_2(u_3), u_3)P^3 + A_2(u_1(u_3), u_2(u_3), u_3)P^2 + A_1(u_1(u_3), u_2(u_3), u_3)P + A_0(u_1(u_3), u_2(u_3), u_3) \). The discriminant of \( C(u_3, P) = 0 \) is given by

\[ D(u_3) = -\left( \frac{c^2}{4b^2} - \frac{a}{b} + 2 \right) \left( \frac{2a}{b} - 4 \right)^2 + O(1), \]

Therefore, for \( u_3 \) small it follows that:

Condition \( D_1 \) \( \Rightarrow \) \( D(u_3) > 0 \) \( \Rightarrow \) There exists an unique solution \( P_1(u_3) \) of \( C(u_3, P) = 0 \)

Conditions \( D_2 \) and \( D_3 \) \( \Rightarrow \) \( D(u_3) < 0 \) \( \Rightarrow \) There exist exactly 3 solutions of \( C(u_3, P) = 0 \) that will be denoted by \( P_1(u_3), P_2(u_3) \) and \( P_3(u_3) \).

As the equation \( C(0, P) = 0 \) has the solutions:
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$P_1(0) = 0; P_2(0) = \frac{c}{2b} + \sqrt{\frac{c^2}{4b^2} - \frac{a}{b} + 2}, P_3(0) = \frac{c}{2b} - \sqrt{\frac{c^2}{4b^2} - \frac{a}{b} + 2},$

it follows that:

• If $S$ is of type $D_1$ then there is the unique curve of singularities of $X$. The intersection of this curve with the axis $P$ is the point $P_1(0)$,
• If $S$ is of type $D_2$ or $D_3$ then $X$ has exactly 3 curves of singularities and these curves cross the axis $P$ at $P_1(0), P_2(0)$ and $P_3(0)$.

For $i = 1, 2, 3$, denote the curve of singularities of $X$, intersecting the axis $P$ at $P = P_i(0)$ by $\gamma_i(u_3)$.

\[\square\]

Lemma 2. Let $X = X_{\mathcal{L}}$ be the Lie-Cartan vector field restricted to the hypersurface $\mathcal{L}(u_1, u_2, u_3, P) = 0$.

Then:

i) Condition $D_1 \Rightarrow \gamma_1$ is normally hyperbolic of saddle type of $X_{\mathcal{L}}$.

ii) Condition $D_2 \Rightarrow$ for $i = 1, 2, 3$, $\gamma_i$ is normally hyperbolic of $X_{\mathcal{L}}$ and satisfy: one is attractor(or repeller) and the other two are of saddle type,

iii) Condition $D_3 \Rightarrow$ for $i = 1, 2, 3$, $\gamma_i$ is normally hyperbolic of saddle type of $X_{\mathcal{L}}$.

Proof. Let $X$ be the Lie-Cartan vector field. The linearisation of $X$ near the singularities will be analyzed below.

Let

(14) $\gamma_i(u_3) = (c_1(u_3), c_2(u_3), u_3, P_i(u_3)), \ i = 1, 2, 3,$

the curves of singularities given in lemma 1.

The characteristic polynomial of $DX(\gamma_i(u_3))$ is

(15) $p(\lambda) = \lambda^2 \cdot \left( \frac{\partial X_4}{\partial P} - \lambda \right) \cdot \left( \lambda - \frac{\partial X_3}{\partial u_3} - \frac{\partial X_2}{\partial u_2} - \frac{\partial X_1}{\partial u_1} \right)$

• Condition $D_1\left(\frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2\right)$:

Suppose that $b > 0$. The case $b < 0$ is similar. From equations (15) and (12) it follows that the eigenvalues of $DX(\gamma_1(u_3)) =$
\[ DX(c_1(u_3), c_2(u_3), u_3, P_1(u_3)) \] are \( \lambda_1(u_3) \equiv \lambda_2(u_3) = 0 \) and

\[
\lambda_3(u_3) = \frac{\partial X_1}{\partial P} = -2b + a + O(u_3),
\]

(16)

\[
\lambda_4(u_3) = \frac{\partial X_1}{\partial u_1} + \frac{\partial X_2}{\partial u_2} + \frac{\partial X_3}{\partial u_3} = b - a + O(u_3),
\]

For \( u_3 \) small, \( \lambda_3 > 0 \) and \( \lambda_4 < 0 \).

- Condition \( D_2 \) \( 1 < \frac{a}{b} < \left( \frac{c}{b} \right)^2 + 2 \): In this case, \( DX(\gamma_1(u_3)) \) has eigenvalues \( \lambda_1 = \lambda_2 \equiv 0 \) and \( \lambda_3, \lambda_4 \) defined by equation (16). In this case, for \( u_3 \) small it follows that:

(17)

\[
\lambda_4(u_3) < 0 \quad \text{and} \quad \begin{cases}
\lambda_3(u_3) > 0 & \text{if } a > 2b \\
\lambda_3(u_3) < 0 & \text{if } a < 2b
\end{cases}
\]

Suppose that \( a < 2b \) and \( b > 0 \). The other cases can be considered similarly. For \( i = 2, 3 \) the eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) of \( DX(\gamma_i(u_3)) = DX(c_1(u_3), c_2(u_3), u_3, P_i(u_3)) \) satisfy: \( \lambda_1 = \lambda_2 \equiv 0 \), \( \lambda_3 > 0 \) and \( \lambda_4 < 0 \) since for equations (15) and (12) it follows that:

(18)

\[
\lambda_3(u_3) = bP_i(0)^2 + 2b - a + O(1) > 0, \lambda_4(u_3) = -bP_i(0)^2 - b + O(1) < 0
\]

For \( a < 2b \) it follows that \( P_2(u_3) < 0 < P_3(u_3) \), for \( u_3 \) small.

- Condition \( D_3 \) \( \left( \frac{a}{b} < 1 \right) \): The eigenvalues of \( DX(\gamma_i(u_3)) \), for \( i = 1, 2, 3 \), satisfy:

(19)

\[
\lambda_1(u_3) = \lambda_2(u_3) = 0 \quad \text{and} \quad \lambda_3(u_3) \cdot \lambda_4(u_3) < 0
\]

since \( \frac{a}{b} < 1 \Rightarrow \frac{a}{b} < 2 \), see (17) and (18).

In the Monge chart we have that

\[
\frac{\partial L}{\partial u_2}(0, 0, 0, P_i(0)) = \frac{\partial L_r}{\partial u_2}(0, 0, 0)(P_i(0))^2 + \frac{\partial M_r}{\partial u_2}(0, 0, 0)P_i(0) + \frac{\partial N_r}{\partial u_2}(0, 0, 0)
\]

\[
= -b(P_i(0))^2 + cP_i(0) + b = \begin{cases} 
-\frac{b}{i = 1}, \\
\frac{a - b}{i = 2, 3}.
\end{cases}
\]

Therefore in a neighborhood of \((0, 0, 0, P_i(0)) \) \((i = 1, 2, 3)\), we can write \( u_2 = u_2(u_1, u_3, P) \) in the equation \( L(u_1, u_2, u_3, P) = 0 \). In the chart
(u₁, u₃, P) the Lie-Cartan vector field is given by

\[
X_L = \begin{cases}
\dot{u}_1 = X_1(u_1, u_2(u_1, u_3, P), u_3, P) \\
\dot{u}_3 = X_3(u_1, u_2(u_1, u_3, P), u_3, P) \\
\dot{P} = X_4(u_1, u_2(u_1, u_3, P), u_3, P)
\end{cases}
\]  

(20)

The linearisation of \(X_L\) in \((0, 0, 0, P_1(0))\) has one eigenvalue equal to zero and the other two are non zero and satisfy the same conditions of \(\lambda_3\) and \(\lambda_4\), given in equation (16), (17), (18) and (19). The eigenvector associated to \((0, 0, 0, P_1(0))\) is tangent to the curve of singularities. Therefore it follows that:

Condition \(D_1\) \(\Rightarrow\) \(\gamma_1(u_3)\) is normally hyperbolic of saddle type,

Condition \(D_2\) \(\Rightarrow\) if \(a < 2b\) and \(b > 0\) it follows that for \(u_3\) sufficiently small the curve \(\gamma_1(u_3)\) is normally hyperbolic of attracting type the curves \(\gamma_2(u_3)\) and \(\gamma_3(u_3)\) are normally hyperbolic of saddle type. The other cases are similar.

Condition \(D_3\) \(\Rightarrow\) \(\gamma_i(u_3)(i = 1, 2, 3)\) are normally hyperbolic of saddle type.

\[\Box\]

7.2. End of proof of Theorem 1. Let \(S\) be a Darbouxian partially umbilic curve. Suppose that \(S\) is of type \(D_1\). By lemma \(\ref{lem:curve_singularities}\) there exists a unique curve of singularities, \(\gamma_1(u_3)\) (see (14)), of the Lie-Cartan vector field. By lemma \(\ref{lem:curve_singularities}\) \(\gamma_1(u_3)\) is normally hyperbolic of saddle type.

By Invariant Manifold Theory (see [12, page 44] and [3]) in a neighborhood \(V_{\gamma_1}\) of \(\gamma_1(u_3)\) there are two dimensional invariant manifolds \(W^s_{\gamma_1}\) and \(W^u_{\gamma_1}\), of class \(C^{k-3}\), with \(W^u_{\gamma_1} \cap W^s_{\gamma_1} = \gamma_1\).

Claim: \(\Pi(W^u_{\gamma_1}) = S\), where \(\Pi(u_1, u_2, u_3, P) = (u_1, u_2, u_3)\). In fact,

- the axis \(P\) is invariant by \(X\)
- \((0, 0, 0, 1)\) is the eigenvector associated to the eigenvalue \(\frac{\partial X_4}{\partial P}(\gamma_1(u_3))\)
  - which is positive for \(u_3\) sufficiently small, see equation (16).

By the uniqueness of the invariant manifolds, it follows that in a neighborhood of \(\gamma_1\), \(W^u_{\gamma_1} = S \times (\text{axis } P)\), and therefore \(\Pi(W^u_{\gamma_1}) = S\). Let

\[V_S = \Pi(V_{\gamma_1})\] and \(W(S) = \Pi(W^s_{\gamma_1})\).

Therefore it follows that: if \(S\) is of type \(D_1\) then

- there exists a unique umbilic separatrix surface, \(W(S)\), of class \(C^{k-3}\),
- which is fibred over \(S\) and the fibers are the leaves of \(\mathcal{F}_1(\alpha)\),
• there exists a tubular neighborhood $V_S$ of $S$ such that the set $V_S \setminus W(S)$ is a hyperbolic sector of $F_1(\alpha)$.

See Fig. 9

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure9.png}
\caption{Lie-Cartan resolution of a partially umbilic point $D_1$.}
\end{figure}

Suppose that $S$ is of type $D_2$. By lemma 11 there exist three curves of singularities $\gamma_1(u_3)$, $\gamma_2(u_3)$ and $\gamma_3(u_3)$ of the Lie-Cartan vector field. By lemma 2 $\gamma_1(u_3)$ is normally hyperbolic (attractor); $\gamma_2(u_3)$ and $\gamma_3(u_3)$ are normally hyperbolic of saddle type.

By Invariant Manifold Theory (see [12, page 44] and [3]), it follows that

• For $i = 2, 3$, there are bi-dimensional invariant manifolds $W^s_{\gamma_i}$ (stable manifold) and $W^u_{\gamma_i}$ (unstable manifold), of class $C^{k-3}$, and $W^u_{\gamma_i} \cap W^s_{\gamma_i} = \gamma_i$.
• For $\gamma_1$, $W^u_{\gamma_1} = \emptyset$

Observe that $\Pi(W^s_{\gamma_i}) = \Pi(W^u_{\gamma_i}) = S$, $i = 2, 3$. Let $W_1 = \Pi(W^s_{\gamma_2})$ and $W_2 = \Pi(W^s_{\gamma_3})$. Then,

• there are two invariant manifolds (partially umbilic separatrix surfaces), $W_1 \cap W_2$, both of class $C^{k-3}$,
PARTIALLY UMBILIC SINGULARITIES OF HYPERSURFACES OF $\mathbb{R}^4$

- there exists exactly one hyperbolic sector and one wedge sector for of the principal foliation $\mathcal{F}_1(\alpha)$.

See Fig. 10.

![Figure 10. Lie-Cartan resolution of $D_2$ with $\frac{a}{b} < 2$.](image)

Suppose that $S$ is of type $D_3$. By lemma 1 there are three curves of singularities $\gamma_1(u_3)$, $\gamma_2(u_3)$ and $\gamma_3(u_3)$ for the Lie-Cartan vector field. By Invariant Manifold Theory (see [12] page 44) and [3], for $i = 1, 2, 3$, there are bi-dimensional invariant manifolds $W^s_{\gamma_i}$ and $W^u_{\gamma_i}$, of class $C^{k-3}$, with $W^u_{\gamma_i} \cap W^s_{\gamma_i} = \gamma_i$. Moreover, $\Pi(W^s_{\gamma_i}) = \Pi(W^u_{\gamma_i}) = S, i = 2, 3$. Let $W_1 = \Pi(W^u_{\gamma_1}), W_2 = \Pi(W^s_{\gamma_2})$ and $W_3 = \Pi(W^s_{\gamma_3})$. Then,

- there are three invariant surfaces (partially umbilic separatrix surfaces) $W_1$, $W_2$ e $W_3$, all of class $C^{k-3}$,
- there are exactly three hyperbolic sectors of the principal foliation $\mathcal{F}_1(\alpha)$.

See Fig. 11.
8. Principal foliations near a partially umbilic point $D_{12}$

**Lemma 3.** Let $S$ be the partially umbilic set. If $p \in S$ is of type $D_{12}$, then the Lie-Cartan vector field $X = X_L$ has two lines of singularities $\zeta_1$ and $\zeta_2$.

**Proof.** The singular points of $X$ are given by

$$L_r(u_1, u_2, u_3) = 0,$$

$$M_r(u_1, u_2, u_3) = 0,$$

$$A_3 P^3 + A_2 P^2 + A_1 P + A_0 = 0 \quad \text{(see equation (12))}$$

As $\det \left( \frac{\partial (L_r, M_r)}{\partial (u_1, u_2)} \right)_{u_1 = 0 = u_2 = u_3} = -b^2 \neq 0$ we can write $u_1 = u_1(u_3)$ and $u_2 = u_2(u_3)$ to solve $L_r(u_1, u_2, u_3) = 0$ and $M_r(u_1, u_2, u_3) = 0$. In the Monge chart, $S$ is parametrized by

$$u_1 = c_1(u_3) = \frac{b q_{021} - q_{201} b - q_{111} c}{b^2} u_3 + O(2),$$

$$u_2 = c_2(u_3) = -\frac{q_{111}}{b} u_3 + O(2),$$

Let $C(u_3, P) = A_3(c_1(u_3), c_2(u_3), u_3)P^3 + A_2(c_1(u_3), c_2(u_3), u_3)P^2 + A_1(c_1(u_3), c_2(u_3), u_3)P + A_0(c_1(u_3), c_2(u_3), u_3) = X_4(c_1(u_3), c_2(u_3), u_3, P)$. 

---

**Figure 11.** Lie-Cartan resolution of a partially umbilic point of type $D_3$. 

---

$W_2(S) = \Pi(W_{u_1}^{s})$

$W_1(S) = \Pi(W_{u_2}^{u})$

$W_3(S) = \Pi(W_{u_3}^{s})$

---

$W_{\gamma_2}^{s}$

$W_{\gamma_1}^{u}$

$W_{\gamma_3}^{s}$

---

$\gamma_2$

$\gamma_1$

$\gamma_3$
Direct calculation shows that

\begin{equation}
\frac{\partial C(u_3, P)}{\partial P} \bigg|_{u_3=0, P=\frac{c}{b}} = \frac{c^2}{b} \neq 0; \quad \frac{\partial C(u_3, P)}{\partial u_3} \bigg|_{u_3=0, P=0} = \chi_{12} \neq 0
\end{equation}

and

\begin{equation}
\frac{\partial C(u_3, P)}{\partial P} \bigg|_{u_3=0, P=0} = 0, \quad \frac{\partial^2 C(u_3, P)}{\partial P^2} \bigg|_{u_3=0, P=0} = -2c \neq 0,
\end{equation}

So, by equations (23) and (24), there exists a unique curve of singularities \( \zeta_1 \), transversal to the \( P \) axis, passing through \( u_3 = 0, \ P = c/b \); and there exists a unique curve, \( \zeta_2 \) tangent to the point \( P \) axis at the origin. \( \square \)

**Remark 9.** The discriminant of \( C(u_3, P) = 0 \) satisfies

\begin{equation}
D(u_3) = \chi_{12}u_3 + O(2),
\end{equation}

where \( \chi_{12} \) is given by equation 2.

In order to obtain the configuration shown in Fig. 4, it is sufficient to show that \( \zeta_1 \) is normally hyperbolic of saddle type and that \( \zeta_2 \) is of saddle-node type (non-hyperbolic).

**Lemma 4.** Let \( X_L \) be the Lie-Cartan vector field tangent to the Lie-Cartan hypersurface, and \( \zeta_1, \ z_2 \) the curves of singularities established in lemma 3. Then,

i) \( \zeta_1 \) is a curve of singularities normally hyperbolic of saddle type;

ii) there exists a two dimensional center manifold containing \( \zeta_2 \), and the phase portrait of \( X_L \) in a neighborhood of \( \zeta_2 \) is as shown in Fig. 12.

\( \square \)

**Proof.** For \( u_3 \) sufficiently small, the nonzero eigenvalues of \( DX(\zeta_1(u_3)) \) are:

\( \lambda_1(u_3) = -\frac{b^2 + c^2}{b} + O(1), \quad \lambda_2(u_3) = \frac{c^2}{b} + O(1). \)

Therefore, \( \zeta_1 \) is normally hyperbolic of saddle type in the neighborhood of \( \zeta_1(0) \) (see [12] and [3]).
Figure 12. Phase portrait of $X_L$ in a neighborhood of $\zeta_2$ and the center manifold $W^c_{\zeta_2}$.

Direct calculation shows that $\frac{\partial L}{\partial u_2}(0, 0, 0, 0) = b \neq 0$. The implicit solution $u_2 = u_2(u_1, u_3, P)$ of $L(u_1, u_2, u_3, P) = 0$, in a neighborhood of 0, is:

$$u_2(u_1, u_3, P) = -\frac{q_{111}}{b}u_3 - \frac{1}{2b} \left( B + 2\frac{q_{201}q_{111}}{k - k_3} \right) u_1^2 + \left( \frac{q_{201}}{b} - \frac{q_{021}}{b} + \frac{q_{111}C}{b^2} \right) u_3 P$$

$$+ \left( \frac{q_{201}q_{111}q_{021}}{b} - \frac{q_{201}q_{012}}{b} \right) \frac{1}{(k - k_3)b} - \frac{q_{111}k^3}{b^2} + \frac{q_{211}}{b} \right) u_1 u_3 + \left( -\frac{q_{111}^3q_{021}}{(k - k_3)b^2} + \frac{q_{111}q_{021}q_{102}}{(k - k_3)b^2} - \frac{1}{2} \frac{Dq_{111}^2}{b^3} \right) u_3^2 + O(3)$$

Near the origin, in the chart $(u_1, u_3, P)$ the vector field $X_L$ defined in equation (12), with $a = 2b$ and $u_2 = u_2(u_1, u_3, P)$, is given by:

$$X_L := \begin{cases} 
\dot{u}_1 &= X_1(u_1, u_2(u_1, u_3, P), u_3, P) \\
\dot{u}_3 &= X_3(u_1, u_2(u_1, u_3, P), u_3, P) \\
\dot{P} &= X_4(u_1, u_2(u_1, u_3, P), u_3, P) 
\end{cases}$$

(26)
where

\[ X_1 = \left( \frac{q_{021}b - cq_{111} - q_{201}b}{b} \right) u_3 + O(2) \]

\[ X_3 = \left( U(u_1, u_2(u_1, u_3, P), u_3) + PV(u_1, u_2(u_1, u_3, P), u_3) \right) X_1 \]

\[ X_4 = \left( -B - 3 \frac{q_{201}q_{111}}{k - k_3} \right) u_1 - bP + \]

\[ + \left( -Q_{211} + \frac{q_{111}C}{b} + k_3 \frac{k^3}{b} - \frac{2q_{102}q_{111}}{k - k_3} - \frac{q_{201}q_{102}}{k - k_3} \right) u_3 - \]

\[ + O(2) \]

Also \( DX_{L}(\zeta_2(P)) \) has one zero eigenvalue and the other ones are given by:

\[ \lambda_1(P) = 3bP^2 - 2cP + O(3) \]
\[ \lambda_2(P) = -b + cP - 2bP^2 + O(3). \]

It will be supposed that \( b, c > 0 \) without loss of generality.

As \( \lambda_2(P) < 0 \), for \( P \) sufficiently small, by Invariant Manifold Theory (see [12] page 44 and [3]) there exists an invariant manifold \( W^s(\zeta_2(P)) \), of class \( C^{k-3} \) where \( \zeta_2(P) \) is an attractor.

For \( u_1 = 0 \), there exists a two dimensional invariant center manifold, \( W^c_{\zeta_2} \), of class \( C^{k-3} \) that contains the curve of singularities \( \zeta_2 \) in a neighborhood of \( \zeta_2(0) \).

Below it will be shown that the phase portrait of \( X \) restricted to \( W^c_{\zeta_2} \) is as shown in Fig. 13.

\[ \text{Figure 13. Phase portrait of } X \text{ restricted to the center manifold } W^c_{\zeta_2} \text{ near the point } \zeta_2(0). \]
Perform the change of coordinates \((u_1, u_2, P) \rightarrow (u, w, \overline{P})\) such that the tangent plane to \(W^c_{\zeta_2}\) at zero is the plane \(w\overline{P}\). Let

\[
\begin{align*}
  u_1 &= \frac{(q_{021}b - q_{201}b - cq_{111})}{b^2}w + \frac{\chi_{12}}{b(Bk - Bk_3 + 3q_{201}q_{111})}\overline{P} \\
  u_3 &= w \frac{\chi_{12}}{b^2(k - k_3)}u + \frac{\chi_{12}}{b^3(k - k_3)} w - \frac{\chi_{12}}{b^3(k - k_3)}\overline{P}.
\end{align*}
\]  

(27)

Replacing (27) in (26) it follows that:

\[
Y_r := \begin{cases} 
  \dot{u} &= w + f(u, w, \overline{P}) \\
  \dot{w} &= g(u, w, \overline{P}) \\
  \dot{\overline{P}} &= -b\overline{P} + h(u, w, \overline{P}) 
\end{cases}.
\]

where \(f(0, 0, 0) = g(0, 0, 0) = h(0, 0, 0) = 0, \frac{\partial f}{\partial u}(0, 0, 0) = \frac{\partial f}{\partial w}(0, 0, 0) = 0, \frac{\partial g}{\partial u}(0, 0, 0) = \frac{\partial g}{\partial w}(0, 0, 0) = 0, \frac{\partial h}{\partial u}(0, 0, 0) = \frac{\partial h}{\partial w}(0, 0, 0) = 0, \frac{\partial h}{\partial \overline{P}}(0, 0, 0) = 0.
\]

The center manifold \(W^c_{\zeta_2}\), associated to \(Y_r\), can be parametrized by \(\overline{P} = \overline{P}(u, w)\). The restriction of \(Y_r\) to \(W^c_{\zeta_2}\) is given by:

\[
Y_r\big|_{W^c_{\zeta_2}} := \begin{cases} 
  \dot{u} &= U(u, w) \\
  \dot{\overline{P}} &= W(u, w) 
\end{cases}.
\]

(28)

where \(\frac{\partial U}{\partial w}(0, 0, 0) = 1\) and \(\frac{\partial^2 U}{\partial w^2}(0, 0, 0) = \frac{c\chi_{12}}{b^2(k - k_3)} \neq 0\).

So the phase portrait of \(Y_r\) restricted to the center manifold is given as in Fig. 13 (see [2], Theorem 3.5 pages 128 and 129).

Therefore, there exists invariant manifolds \(W^s_{\zeta_1(u_1)}\) and \(W^c_{\zeta_2}\) of \(X_L\) such that the phase portrait in the neighborhood of these manifolds is as illustrated in Fig. 12. \(\Box\)

8.1. **End of proof of Theorem**[2]. Let \(S\) be a partially umbilic curve and \(p \in S\) be of type \(D_{12}\). By lemma 3, the Lie-Cartan vector field has two curves of singularities \(\zeta_1\) and \(\zeta_2\). By lemma 4, \(\zeta_1\) is normally hyperbolic of saddle type, and the phase portrait of the Lie-Cartan vector field in a neighborhood of \(\zeta_2\) is as shown in Fig. 12. For \(\zeta_1\), there are bi-dimensional invariant manifolds \(W^s_{\zeta_1}\) (stable manifold) and \(W^s_{\zeta_1}\) (unstable manifold), of class \(C^{k-3}\), and \(W^s_{\zeta_1} \cap W^c_{\zeta_2} = \zeta_1\). For \(\zeta_2\), there are bi-dimensional invariant manifolds \(W^s_{\zeta_2}\) (stable manifold) and \(W^c_{\zeta_2}\) (center manifold).
Observe that $\Pi(W^u_{\zeta_1}) = \Pi(W^c_{\zeta_2}) = \mathcal{S}$. Define $W = \Pi(W^s_{\zeta_1})$ and $W_3 = \Pi(W^s_{\zeta_2})$.

This ends the proof of theorem 2. For an illustration see Fig. 14.

![Figure 14. Normally hyperbolic singularity $\zeta_1$ and center manifold $W^c_{\zeta_2}$ containing $\zeta_2$.](image)

9. Principal foliations near a partially umbilic point $D_{23}$

9.1. Local analysis of the Lie-Cartan vector field $X$.

**Lemma 5.** Let $S$ be the partially umbilic set. If $p \in S$ is of type $D_{23}$, then the Lie-Cartan vector field $X$ has three lines of singularities $\gamma_i$, $i = 1, 2, 3$.

**Proof.** The singular points of $X$ are given by

\begin{align}
\begin{cases}
L_r(u_1, u_2, u_3) = 0, \\
M_r(u_1, u_2, u_3) = 0, \\
A_3P^3 + A_2P^2 + A_1P + A_0 = 0
\end{cases} \quad \text{(Partially Umbilic Points)}
\end{align}

By lemma 1, $S$ is a regular curve. As $\det \left( \frac{\partial (L_r, M_r)}{\partial (u_2, u_3)} \right)_{u_1=0, u_2=0, u_3=0} = b(-q_{201} + q_{021}) + cq_{111} \neq 0$ we can write $u_2 = u_2(u_1)$ and $u_3 = u_3(u_1)$ in $L_r(u_1, u_2, u_3) = 0$ and $M_r(u_1, u_2, u_3) = 0$. In the Monge chart (restricted),

Figure 14. Normally hyperbolic singularity $\zeta_1$ and center manifold $W^c_{\zeta_2}$ containing $\zeta_2$. 
\( S \) is parametrized by
\[
(30) \quad u_2 = c_2(u_1) = -\frac{1}{2((q_{021} - q_{201}) b - cq_{111})} [q_{111} A + q_{111} C - (q_{021} - q_{201}) B

- 2 (q_{201} q_{021} - q_{111}^2) q_{111} (k - k_3)^{-1} - 2q_{111} k^3] u_1^2 + O(3)
\]
\[
u_3 = c_3(u_1) = \frac{1}{-2q_{201} b - 2cq_{111} + 42b} [bA - bC + cB - 2k^3b

+ (k - k_3)^{-1} (-2 (q_{111}^2 + q_{201}^2) b + 2q_{201} q_{111} c)] u_1^2 + O(3)
\]

From equation (12) with \( a = b \), the second equation of (29) is given by:
\[
(31) \quad (b + O(1)) P^3 + (-c + O(1)) P^2 + (-b + O(1)) P + O(1) = 0,
\]
The discriminant of the equation (31) is given by:
\[
(32) \quad D(u_1) = -\frac{1}{108} \frac{c^2 + 4b^2}{b^2} + O(1) < 0.
\]
Therefore, for \( u_1 \) sufficiently small the equation (31) has three solutions \( P_i(u_1)(i = 1, 2, 3) \) given by
\[
P_1(u_1) = 0 + O(1); P_2(u_1) = \frac{c + \sqrt{c^2 + 4b^2}}{2b} + O(1);
\]
\[
P_3(u_1) = \frac{c - \sqrt{c^2 + 4b^2}}{2b} + O(1).
\]
So \( X \) has three curves of singularities \( \gamma_1, \gamma_2 \) and \( \gamma_3 \). Moreover \( \Pi(\gamma_i) = S, i = 1, 2, 3 \).

In the space with coordinates \((u_1, u_2, u_3, P)\) the curves of singularities of \( X \) are given by:
\[
(34) \quad \gamma_i: \quad u_2 = c_2(u_1), u_3 = c_3(u_1) \quad \text{and} \quad P = P_i(u_1), \quad i = 1, 2, 3
\]
where \( c_2, c_3 \) and \( P_i \) \((i = 1, 2, 3)\) are given by equations (30) and (33), respectively.

\begin{lemma}
Let \( X_L \) be the Lie-Cartan vector field tangent to the Lie-Cartan hypersurface. The curves of singularities \( \gamma_i, i = 2, 3 \) are normally hyperbolic of saddle type of \( X_L \). Near the curve \( \gamma_1 \), the phase portrait of \( X_L \) is as shown in Fig. 15.
\end{lemma}


\textbf{Figure 15.} Phase portrait of $X_L$ in a neighborhood of $\gamma_1$.

\textit{Proof.} The linearisation $DX(\gamma_i(u_1)), i = 2, 3,$ of the Lie-Cartan vector field, has two zero eigenvalues and the non-zero ones are given by:

$$
\lambda_1(u_1) = -b(P_i(0)^2 + 1) + O(1), \quad \lambda_2(u_1) = b(P_i(0)^2 + 1) + O(1), \quad i = 2, 3.
$$

Therefore for $u_1$ sufficiently small it follows that $\lambda_1(u_1) < 0$ and $\lambda_2(u_1) > 0$, (assuming that $b > 0$).

Then the curves $\gamma_2$ and $\gamma_3$ are normally hyperbolic saddles of $X_L$.

In a neighborhood of $P = P_i(0), i = 2, 3$, the Lie-Cartan hypersurface is regular. In fact,

$$
\frac{\partial \mathcal{L}}{\partial u_3}(0, 0, 0, P_i(0)) = \frac{- (b(q_{021} - q_{201}) - cq_{111})(c + (-1)^i\sqrt{c^2 + 4b^2})}{2b^2} \neq 0.
$$

Next the analysis of $X$ near $\gamma_1(u_1)$, will be developed.

Direct calculation shows that $\frac{\partial \mathcal{L}}{\partial u_2}(0, 0, 0, 0) = b \neq 0$. The solution, $u_2 = u_2(u_1, u_3, P)$ of the implicit equation $\mathcal{L}(u_1, u_2(u_1, u_3, P), u_3, P) = 0$ near the $(0, 0, 0, 0)$ is given by

$$
u_2(u_1, u_3, P) = -\frac{q_{111}}{b}u_3 - \frac{1}{2b} \left( B + 2\frac{q_{201}q_{111}}{k - k_3} \right) u_1^2 + \left( \frac{q_{201}}{b} - \frac{q_{021}}{b} + \frac{cq_{111}}{b^2} \right) u_3P + \left( \frac{q_{021}q_{111}q_{201}}{(k - k_3)^3} - \frac{q_{201}q_{111}q_{021}}{(k - k_3)^3} b^2 \right) u_1u_3 + \left( -\frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} - \frac{1}{2} \frac{Dq_{111}^2}{b^3} \right) u_3^2 + O(3)$$

where

$$
B = \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} - \frac{1}{2} \frac{Dq_{111}^2}{b^3}
$$

and $\partial \mathcal{L}/\partial u_3 = - \frac{b(q_{021} - q_{201}) - cq_{111})(c + (-1)^i\sqrt{c^2 + 4b^2})}{2b^2} \neq 0$. The solution, $u_2 = u_2(u_1, u_3, P)$ of the implicit equation $\mathcal{L}(u_1, u_2(u_1, u_3, P), u_3, P) = 0$ near the $(0, 0, 0, 0)$ is given by

$$
u_2(u_1, u_3, P) = -\frac{q_{111}}{b}u_3 - \frac{1}{2b} \left( B + 2\frac{q_{201}q_{111}}{k - k_3} \right) u_1^2 + \left( \frac{q_{201}}{b} - \frac{q_{021}}{b} + \frac{cq_{111}}{b^2} \right) u_3P + \left( \frac{q_{021}q_{111}q_{201}}{(k - k_3)^3} - \frac{q_{201}q_{111}q_{021}}{(k - k_3)^3} b^2 \right) u_1u_3 + \left( -\frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} - \frac{1}{2} \frac{Dq_{111}^2}{b^3} \right) u_3^2 + O(3)$$

where

$$
B = \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} - \frac{1}{2} \frac{Dq_{111}^2}{b^3}
$$

and $\partial \mathcal{L}/\partial u_3 = - \frac{b(q_{021} - q_{201}) - cq_{111})(c + (-1)^i\sqrt{c^2 + 4b^2})}{2b^2} \neq 0$. The solution, $u_2 = u_2(u_1, u_3, P)$ of the implicit equation $\mathcal{L}(u_1, u_2(u_1, u_3, P), u_3, P) = 0$ near the $(0, 0, 0, 0)$ is given by

$$
u_2(u_1, u_3, P) = -\frac{q_{111}}{b}u_3 - \frac{1}{2b} \left( B + 2\frac{q_{201}q_{111}}{k - k_3} \right) u_1^2 + \left( \frac{q_{201}}{b} - \frac{q_{021}}{b} + \frac{cq_{111}}{b^2} \right) u_3P + \left( \frac{q_{021}q_{111}q_{201}}{(k - k_3)^3} - \frac{q_{201}q_{111}q_{021}}{(k - k_3)^3} b^2 \right) u_1u_3 + \left( -\frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} + \frac{q_{111}q_{012}}{b} - \frac{1}{2} \frac{Dq_{111}^2}{b^3} \right) u_3^2 + O(3)$$

where

$$
B = \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} + \frac{q_{111}^3}{(k - k_3)^3} - \frac{1}{2} \frac{Dq_{111}^2}{b^3}
$$

and $\partial \mathcal{L}/\partial u_3 = - \frac{b(q_{021} - q_{201}) - cq_{111})(c + (-1)^i\sqrt{c^2 + 4b^2})}{2b^2} \neq 0$.
Near the origin, in the chart \((u_1, u_3, P)\), the vector field \(X_L\) is given by

\begin{equation}
X_L := \begin{cases} 
\dot{u}_1 = X_1(u_1, u_2(u_1, u_3, P), u_3, P), \\
\dot{u}_3 = X_3(u_1, u_2(u_1, u_3, P), u_3, P), \\
\dot{p} = X_4(u_1, u_2(u_1, u_3, P), u_3, P)
\end{cases}
\end{equation}

where,

\begin{align*}
X_1 &= \frac{(q_{021}b - cq_{111} - q_{201}b)}{b} u_3 + O(2) \\
X_3 &= \left(\mathcal{U}(u_1, u_2(u_1, u_3, P), u_3) + PV(u_1, u_2(u_1, u_3, P), u_3)\right) X_1 \\
X_4 &= \left(-B - 3\frac{q_{201}q_{111}}{k - k_3}\right) u_1 - bP + \\
&\quad + \left(-Q_{211} + \frac{q_{111}C}{b} + \frac{k^3}{b} - \frac{2q_{102}q_{111}}{k - k_3} - \frac{q_{201}q_{012}}{k - k_3} b + \frac{2q_{111}^3}{(k - k_3) b} + \frac{q_{201}q_{021}q_{111}}{b} \right) u_3 - \\
&\quad + O(2)
\end{align*}

The eigenvalues of \(DX_L(\gamma_1(u_1))\) are given by:

\begin{align*}
\lambda_2(u_1) &= -b + O(1); \\
\lambda_3(u_1) &= \frac{q_{201}((q_{021} - q_{201})b - cq_{111})}{b(k - k_3)} u_1 + O(2).
\end{align*}

As \(\lambda_2(u_1) < 0\), for \(u_1\) sufficiently small, it follows by Invariant Manifold Theory, (see [12, page 44] and [3]) that there exists an invariant manifold \(W^s_{\gamma_1}\) (stable manifold), of class \(C^{k-3}\), see Fig. [15].

For \(u_1 = 0\), there exists a two dimensional invariant center manifold, \(W^c_{\gamma_1}\), of class \(C^{k-3}\) that contains the curve of singularities \(\gamma_1\) in a neighborhood of \(\gamma_1(0)\).

Below it will be shown that the phase portrait of \(X\) restricted to \(W^c_{\gamma_1}\) is as shown in Fig. [16].

Consider \(X_L\) restricted to \(W^c_{\gamma_1}\) and perform the change of coordinates such that the tangent plane to \(W^c_{\gamma_1}\) at zero is the plane \(uw\). Let,
From equation (36) it follows that $X_L$ is given by:

$$\dot{Y}_r := \begin{cases} 
\dot{u} &= w + f(u, w, \mathcal{P}) \\
\dot{w} &= g(u, w, \mathcal{P}) \\
\dot{\mathcal{P}} &= -b\mathcal{P} + h(u, w, \mathcal{P})
\end{cases}$$

where $f(0, 0, 0) = g(0, 0, 0) = h(0, 0, 0) = 0$, $\frac{\partial f}{\partial u}(0, 0, 0) = \frac{\partial f}{\partial w}(0, 0, 0) = \frac{\partial g}{\partial u}(0, 0, 0) = \frac{\partial g}{\partial w}(0, 0, 0) = 0$, $\frac{\partial h}{\partial u}(0, 0, 0) = \frac{\partial h}{\partial w}(0, 0, 0) = 0$.

The center manifold $W^c$, associated to $Y_r$ can be parametrized by $\mathcal{P} = \mathcal{P}(u, w)$. The restriction of $Y_r$ to $W^c$ is given by:

$$\left. Y_r \right|_{W^c} = \begin{cases} 
\dot{u} &= U(u, w) \\
\dot{w} &= W(u, w)
\end{cases}$$

where

$$U(u, w) = -\frac{1}{2b(k - k_3)}[(k - k_3)(-2bk^3 + Ab + cB - Cb)$$
$$+ (-2q_{111}^2 + 2q_{201}^2)b + 2q_{111}q_{201}c]u^2 + \frac{b(q_{201} - q_{201}) - cq_{111}}{b}w$$
$$+ A_{11}uw + A_{02}w^2 + O(3)$$

and

$$W(u, w) = \left[\frac{b(q_{201} - q_{201}) - cq_{111}}{b^2(k - k_3)}\right][q_{201}uw + (bq_{102} - q_{111}^2)w^2] + O(3)$$
Performing the change of variables $u_1 = u, w_1 = U(u, w)$ it is obtained the vector field

$$u_1' = w_1 \quad w_1' = w_1 \left(\frac{cq_{111} - b(q_{021} - q_{201})}{b^4(k - k_3)} \chi_{23} u_1 + A_w w_1 + O(2)\right)$$

The coefficients $A_{11}, A_{02}$ and $A_w$ have long expressions in terms of the other coefficients but are not relevant to determine the phase portrait in the center manifold.

So the phase portrait of $Y_r$ restricted to the center manifold is given as in Fig. 16 (see [2], Theorem 3.5, pages 128 and 129).

![Figure 16. Phase portrait of $Y_r \big|_{W^c}$](image)

Therefore, there exist invariant manifolds $W^s_{\gamma_1(u_1)}$ and $W^c_{\gamma_1(0)}$ of $X_L$ and the phase portrait in the neighborhood of these manifolds is as illustrated in Fig. 15.

9.2. **End of proof of Theorem 3**. Let $S$ be a partially umbilic curve and $p \in S$ of type $D_{23}$. By lemma 5 there exists three curves of singularities $\gamma_1(u_1), \gamma_2(u_1)$ and $\gamma_3(u_1)$ of the Lie-Cartan vector field. By lemma 6 the curves $\gamma_2$ and $\gamma_3$ are normally hyperbolic for the vector field $X_L$. Near $\gamma_1$ the phase portrait of $X_L$ is as in Fig. 15.

For $\gamma_1$, there are bi-dimensional invariant manifolds $W^s_{\gamma_1}$ (stable manifold) and $W^c_{\gamma_1}$ (center manifold).

For $i = 2, 3$, there are bi-dimensional invariant manifolds $W^s_{\gamma_i}$ (stable manifold) and $W^u_{\gamma_i}$ (unstable manifold), of class $C^{k-3}$, and $W^u_{\gamma_i} \cap W^s_{\gamma_i} = \gamma_i$.

Observe that $\Pi(W^u_{\gamma_2}) = \Pi(W^u_{\gamma_3}) = \Pi(W^s_{\gamma_1}) = S$, and define $W = \Pi(W^c_{\gamma_1})$, $W_1 = W_1(S) = \Pi(W^s_{\gamma_2})$ and $W_2 = W_2(S) = \Pi(W^s_{\gamma_3})$. 


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A connected component of $W_1 \setminus \mathcal{S}$ is invariant by $\mathcal{F}_1(\alpha)$ and the other is invariant by $\mathcal{F}_2(\alpha)$. The same conclusion for $W_2$.

The invariant surfaces $W_1$ and $W_2$ are regular and there exist four leaves $F_1$ and $F_4$ in $W_1$ and $F_2$ and $F_3$ in $W_2$ asymptotic to the partially umbilic $D_{23}$.

It follows that $F_1$ and $F_2$ are leaves of $\mathcal{F}_1(\alpha)$ and $F_3$ and $F_4$ are leaves of $\mathcal{F}_2(\alpha)$. See Figs. 6 and 17.

This ends the proof of Theorem 3.

Consider the space $\mathcal{J}^r(\mathbb{M}^3, \mathbb{R}^4)$ of $r$-jets of immersions $\alpha$ of $\mathbb{M}^3$ into $\mathbb{R}^4$, endowed with the structure of Principal Fiber Bundle. The base is $\mathbb{M}^3 \times \mathbb{R}^4$, the fiber is the space $\mathcal{J}^r(3, 4)$, where $\mathcal{J}^r(3, 4)$ is the space of $r$-jets of immersions of $\mathbb{R}^3$ to $\mathbb{R}^4$, preserving the respective origins. The structure group, $\mathcal{A}_r^+$, is the product of the group $\mathcal{L}_r^+(3, 3)$ of $r$-jets of diffeomorphisms of $\mathbb{R}^3$ preserving origin and the orientation, acting on the right by coordinate changes, and the group $\mathcal{O}_+(4, 4)$ of positive isometries; the action on the left consists of a positive rotation of $\mathbb{R}^4$.

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**Figure 17.** Projections of the integral curves of $X_\mathcal{L}$.

**10. A Summarizing Theorem**

Consider the space $\mathcal{J}^r(\mathbb{M}^3, \mathbb{R}^4)$ of $r$-jets of immersions $\alpha$ of $\mathbb{M}^3$ into $\mathbb{R}^4$, endowed with the structure of Principal Fiber Bundle. The base is $\mathbb{M}^3 \times \mathbb{R}^4$, the fiber is the space $\mathcal{J}^r(3, 4)$, where $\mathcal{J}^r(3, 4)$ is the space of $r$-jets of immersions of $\mathbb{R}^3$ to $\mathbb{R}^4$, preserving the respective origins. The structure group, $\mathcal{A}_r^+$, is the product of the group $\mathcal{L}_r^+(3, 3)$ of $r$-jets of diffeomorphisms of $\mathbb{R}^3$ preserving origin and the orientation, acting on the right by coordinate changes, and the group $\mathcal{O}_+(4, 4)$ of positive isometries; the action on the left consists of a positive rotation of $\mathbb{R}^4$. 
Each 4-jet of an immersion at a partially umbilic point is of the form 
\((p, \tilde{p}, w)\) with 
\((p, \tilde{p}) \in M^3 \times \mathbb{R}^4\) and \(w\) is in the orbit of a polynomial immersion 
\((u_1, u_2, u_3, h(u_1, u_2, u_3))\), where

\[ h(u_1, u_2, u_3) = \frac{k}{2}(u_1^2 + u_2^2) + \frac{k_2}{2}u_3^2 + \frac{a}{6}u_1^3 + \frac{b}{2}u_1u_2^2 + \frac{c}{6}u_2^3 + \frac{q_{003}}{6}u_3^3 + \]
\[ + \frac{q_{012}}{2}u_2u_3^2 + q_{111}u_1u_2u_3 + \frac{q_{021}}{2}u_2^2u_3 + \frac{q_{020}}{2}u_1u_3^2 + \frac{q_{011}}{2}u_1^2u_3 + \]
\[ + \frac{A}{24}u_1^4 + \frac{B}{6}u_1^3u_2 + \frac{C}{4}u_1^2u_2^2 + \frac{D}{6}u_1u_2^3 + \frac{E}{24}u_2^4 + \frac{Q_{004}}{24}u_3^4 + \]
\[ + \frac{Q_{014}}{6}u_3^4u_2 + \frac{Q_{104}}{6}u_3^3u_1 + \frac{Q_{022}}{4}u_2^2u_3^2 + \frac{Q_{020}}{4}u_2^2u_3^2 + \frac{Q_{003}}{6}u_3u_2^3 + \]
\[ + \frac{Q_{112}}{2}u_3^2u_1u_2 + \frac{Q_{011}}{6}u_3^2u_3 + \frac{Q_{121}}{2}u_3u_2^2u_1 + \frac{Q_{211}}{2}u_3u_2u_1^2 \]

The general quadratic part of \(\tilde{h}\), where \((u_1, u_2, u_3, \tilde{h})\) is in the orbit of \(h\), has the form

\[ k_{110}u_1u_2 + k_{101}u_1u_3 + k_{011}u_2u_3 + \frac{k_{200}}{2}u_1^2 + \frac{k_{020}}{2}u_2^2 + \frac{k_{002}}{2}u_3^2. \]

The manifold of partially umbilic jets, \((\mathcal{P}U)^4\), is defined by the condition that the symmetric matrix

\[
\begin{pmatrix}
  k_{200} & k_{110} & k_{101} \\
  k_{110} & k_{020} & k_{011} \\
  k_{101} & k_{011} & k_{002}
\end{pmatrix}
\]

has two equal eigenvalues. In [4], R. Garcia showed that \((\mathcal{P}U)^4\) is a submanifold of codimension 2 in \(\mathcal{J}^4(3, 4)\). See also Lax [14].

The manifold of umbilic jets, \(\mathcal{U}^4\), is defined by condition \(k_{200} = k_{020} = k_{002}\) and \(k_{110} = k_{101} = k_{011} = 0\). It is a closed submanifold of codimension 5 in \(\mathcal{J}^4(3, 4)\).

**Remark 10.** The expression in equation (38) is a representative of the orbit of partially umbilic jets where the term \(\frac{d}{2}u_1^2 u_2\) vanishes, as can be obtained by a rotation.

Define below a canonic stratification of \(\mathcal{J}^4(3, 4)\). The term canonic means that the strata are invariant under the action of the group \(A^4_+ = O_+(4, 4) \times L^4_+(3, 3)\). It is to the orbits of this actions that reference is made in the following definition.
Definition 6.  
(1) The Umbilic Jets: \( U^4 \) are those in the orbits of \( J^4(u_1, u_2, u_3, h) \), where \( h = h(u_1, u_2, u_3) \), is as in equation (38). It is a closed submanifold of codimension 5 in \( J^4(3, 4) \).

(2) The Partially Umbilic Jets: \( PU^4 \) are those in the orbits of \( J^4(u_1, u_2, u_3, h) \), where \( h = h(u_1, u_2, u_3) \), is as in equation (38). It is a submanifold of codimension 2 in \( J^4(3, 4) \).

(3) The Darbouxian Partially Umbilic Jets: \( D^4 = (D_1)^4 \cup (D_2)^4 \cup (D_3)^4 \) are those in the orbits of \( J^4(u_1, u_2, u_3, h) \), where \( h = h(u_1, u_2, u_3) \) is as in equation (38), where

(a)\( (D_1)^4: \left( \frac{c}{2b} \right)^2 - \frac{a}{b} + 2 < 0 \)
(b)\( (D_2)^4: \left( \frac{c}{2b} \right)^2 + 2 > \frac{a}{b} > 1, a \neq 2b \)
(c)\( (D_3)^4: \frac{a}{b} < 1 \).

(4) The Non-Darbouxian Partially Umbilic Jets: \( ND^4 \) are those in the orbits of \( J^4(u_1, u_2, u_3, h) \), where \( h = h(u_1, u_2, u_3) \) is as in equation (38) whose expression satisfy

\[ b(b - a) = 0 \text{ or } \]
\[ b(b - a) \neq 0 \text{ and } a - 2b = 0 \text{ or } c^2 - ab(a - 2b) = 0. \]

This manifold can be further partitioned into

(a)\( (D_{12})^4: \) Defined by the orbits of jets with
\[
\begin{align*}
& a = 2b \neq 0, c \neq 0, \text{ and } \chi_{12} \neq 0, \text{ or } \\
& c^2 - 4b(a - 2b) = 0 \text{ and } \chi_{12}^* \neq 0
\end{align*}
\]

See equation (2).

(b)\( (D_{23})^4: \) Defined by the orbits of jets with \( a = b \neq 0 \) and \( \chi_{23} \neq 0. \)

See definition (3).

The canonic stratification of \( J^4(3, 4) \) induces a canonic stratification of \( J^4(M^3, \mathbb{R}^4) \) whose strata are principal sub-bundles with codimension equal to that of their fibers, which are the canonic strata of \( J^4(3, 4) \), as defined above in items 1, 2, 3 and 4.

The collection of sub-bundles which stratify \( J^4(M^3, \mathbb{R}^4) \) will be called Partially Umbilic Stratification. The strata are: \( PU^4(M^3, \mathbb{R}^4) \) corresponding to \( (PU)^4; (D_i)^4(M^3, \mathbb{R}^4) \), \( i = 1, 2, 3 \), corresponding to the strata of Darbouxian partially umbilic jets \( (D_i)^4 \), \( i = 1, 2, 3 \), and so on, one bundle for each of the strata in definition (3).

Theorem 4. Let \( M^3 \) be a compact, oriented, smooth manifold. In the space of immersions \( Imm^r(M^3, \mathbb{R}^4) \), endowed with the \( C^4 \)-topology, the following properties define open and dense sets.
The set $S(\alpha)$ of partially umbilic points of $\alpha \in \text{Imm}^r(M^3, \mathbb{R}^4)$ is empty or it is a smooth submanifold of codimension 2, a curve, of $M^3$ stratified as follows:

i) The points $D_1, D_2$ and $D_3$ are distributed along a finite number of arcs along a connected component $S$.

ii) The points $D_{12}$ and $D_{23}$ are isolated along a connected component of $S$ and are the common border points of the arc in the previous item.

Proof. The density follows from Thom transversality theorem in the space of jets. See [13] and [9]. The openness in the compact case of $M^3$ is clear from the analysis in the proofs of the three main theorems of this paper. First treat the isolated points $D_{12}$ and $D_{23}$ with statements and proofs in sections 4, 8, and in sections 5 and 9, respectively. In fact the definitions of these points and their isolated character depend on the local openness of transversality, in its simplest implicit function theorem form, applied to mappings depending only up to fourth derivatives of the immersions, ([9]), and on the local openness of hyperbolicity and normal hyperbolicity and transversal saddle - nodes of equilibria of vector fields which, again, depend on the fourth order derivatives of the immersion. See [3], [12] and [16].

This analysis proofs that each such point $p_i$ has a neighboring closed arc $I_i$, where the conclusions of the theorems 2 and 3 hold for an open set of immersions. Then apply the same idea to each of the finite compact arcs $J_{ij}$ of Darbouxian arcs, which are complementary to the interior of arcs $I_i$.

The openness on the immersion invoked above hold also along the arcs $J_{ij}$. For this case the mappings and vector fields involved in the arguments depend only on the third order derivatives of the immersion. □

Remark 11. In the case of non-compact theorem has an obvious analogous replacing open and dense by residual and the standard $C^r$ – topology by the ph Whitney topology. See [9].

11. Concluding Remarks

In this work it is proved that the singularities of the principal line fields of a generic immersion of a compact oriented three-dimensional manifold $M^3$ into $\mathbb{R}^4$ consist of a regular curve $S$ of partially umbilic points, at which only two of the principal curvatures coincide.

The curve $S$ consists of arcs of transversal partially umbilic points, at which it crosses the partially umbilic planes.
These arcs have as common extremes a discrete set of semi-Darbouxian $D_{23}$ points, at which the contact of $S$ with the umbilic plane is quadratic. These points are the common extremes, or points of transition, between partially umbilic Darbouxian $D_2$ and $D_3$ arcs.

The transversal arcs contain a discrete set of semi-Darbouxian $D_{12}$ points. These points are the common extremes, or points of transition, between the partially umbilic Darbouxian $D_1$ and $D_2$ arcs.

The transversal structure of principal curvature lines, along the Darbouxian arcs $D_1$, $D_2$ and $D_3$ is reminiscent of the Darbouxian umbilics in surfaces of $\mathbb{R}^3$ as described by Darboux [1] and Gutierrez - Sotomayor [8]. At the transition points $D_{12}$ and $D_{23}$, it closely resembles the bifurcations of umbilic points appearing in generic one parameter families of surfaces $\mathbb{R}^3$ studied in the works of Gutierrez, Garcia and Sotomayor [7].

It must be emphasized, however, that the three dimensional analysis carried out in this paper is unavoidable. In fact, only on the highly non-generic case of Frobenius integrability of the plane distribution spanned by the principal line fields $L_1$ and $L_2$ around the partially umbilic curve $S_{12}$ the structure of the principal foliations $F_1$ and $F_2$ can be considered as those appearing in one-parameter families of surfaces. See remark 6. The same holds for the partially umbilic curve $S_{12}$ and the structure of the principal foliations $F_2$ and $F_3$.

The partition of the partially umbilic curve into the arcs $D_1$, $D_2$ and $D_3$ and the transition points $D_{23}$ and $D_{12}$, together with the stratified structure of the partially umbilic separatrix surfaces, consisting of all the principal lines approaching $S$, established in this work, constitute a natural extension to hypersurfaces in $\mathbb{R}^4$ of the results of Darboux for umbilic points for surfaces in $\mathbb{R}^3$, [1], as reformulated and extended by Gutierrez and Sotomayor in [8] and Gutierrez, Garcia and Sotomayor in [7].

The stratified structure of the invariant separatrix manifolds foliated by curvature lines approaching the partially umbilic curves, established in Theorems [1][2] and [3] being an invariant of each one dimensional foliation with singularities $F_i$, $(i = 1, 2, 3)$, under principal equivalence, is an important ingredient added to the original study of Garcia in [4] and [5].

It is a crucial structural step in order to formulate an improved genericity theorem stating that an immersion of $M^3$ into $\mathbb{R}^4$ has each one of its three principal foliations having all its separatrix strata intersecting pairwise transversally. By a principal equivalence is understood a homeomorphism of
M^3 which preserves the principal foliations with singularities \( F_i \), \( i = 1, 2, 3 \), one at the time. However, the non-integrability result established in remark 6 makes unfeasible the simultaneous principal structural stability considerations for pairs of principal foliations as was done in the case of surfaces. See \[8\] and \[11\]. Nevertheless, on the individual basis, additional elaboration of the normal hyperbolicity methods used in this work leads to the local principal structural stability of each of the foliations at the partial umbilic curves consisting of \( D_1, D_2, D_3, D_{12} \) and \( D_{23} \) points.

12. Appendix: Coordinate expressions for geometric functions appearing in this work

Consider a chart \((u_1, u_2, u_3)\) and an isometry \( R \) as in section 2 so that the immersion \( \alpha \) composed with \( R \) has the Monge form:

\[
(u_1, u_2, u_3) \rightarrow (u_1, u_2, u_3, h(u_1, u_2, u_3)),
\]

where \( h \) is given by equation \(11\). In this section will be obtained the coordinate expressions of functions that are essential for the calculations carried out in this work.

Direct calculation with \(11\) gives the expressions below:

\[
\frac{\partial h}{\partial u_1} = ku_1 + \frac{1}{2} au_1^2 + \frac{1}{2} bu_1^2 + \frac{2}{3} q_{102} u_3^2 + q_{201} u_1 u_3 + q_{111} u_2 u_3 + \frac{1}{6} Q_{103} u_3^3
\]
\[
+ \frac{1}{2} Bu_1^2 u_2 + \frac{1}{6} D u_2^3 + \frac{2}{3} Q_{202} u_1 u_3^2 + \frac{1}{2} Q_{301} u_1^2 u_3 + \frac{1}{2} Q_{112} u_2^3 u_2
\]
\[
+ Q_{211} u_3 u_2 u_3 + \frac{1}{6} Q_{121} u_3 u_2^2 + \frac{1}{2} A u_3^3 + \frac{1}{2} C u_1 u_2^2 + O(4),
\]

\[
\frac{\partial h}{\partial u_2} = ku_2 + \frac{1}{2} cu_1^2 + q_{021} u_2 u_3 + q_{021} u_1 u_2 + q_{111} u_1 u_3 + \frac{1}{2} D u_1^2 u_2 + \frac{1}{2} q_{012} u_3^2
\]
\[
+ \frac{1}{2} Bu_1^3 u_2 + \frac{1}{2} Q_{112} u_2^3 u_1 + \frac{1}{2} Q_{211} u_3 u_2^2 + Q_{121} u_3 u_2 u_1 + \frac{1}{2} Q_{022} u_2^3 u_2
\]
\[
+ \frac{1}{2} E u_3^3 + \frac{1}{6} Q_{013} u_3^3 + \frac{1}{2} Q_{031} u_3 u_3^2 + \frac{1}{2} C u_1^2 u_2 + O(4),
\]

\[
\frac{\partial h}{\partial u_3} = k_3 u_3 + \frac{1}{2} q_{021} u_2^2 + \frac{1}{2} q_{021} u_1^2 + q_{102} u_1 u_3 + q_{012} u_3 u_2 + q_{111} u_1 u_2 + \frac{1}{2} q_{003} u_3^2
\]
\[
+ \frac{1}{6} Q_{301} u_1^3 + Q_{112} u_3 u_2 u_3 + \frac{1}{2} Q_{013} u_3 u_3^2 + \frac{1}{6} Q_{031} u_3^2 + \frac{1}{2} Q_{202} u_3^3 u_3
\]
\[
+ \frac{1}{2} Q_{211} u_2^3 u_1 + \frac{1}{2} Q_{121} u_2^3 u_3 + \frac{1}{2} Q_{022} u_2^3 u_3 + \frac{1}{2} Q_{004} u_3^3 + \frac{1}{2} Q_{103} u_3^2 u_1
\]
\[+ O(4),\]
12.1. First Fundamental Form. The coefficients of the first fundamental form \((g_{ij})\) of \(\alpha\) in the Monge chart \((u_1, u_2, u_3)\) are:

\[
g_{11} = \langle \alpha_{u_1}, \alpha_{u_1} \rangle = 1 + k^2 u_1^2 + kau_1^3 + bku_2^2 u_1 + kq_{102} u_3^2 u_1 + 2kq_{111} u_2 u_3 u_1 \\
+ 2kq_{201} u_1^2 u_3 + h.o.t.,
\]

\[
g_{12} = \langle \alpha_{u_1}, \alpha_{u_2} \rangle = k^2 u_1 u_2 + kq_{111} u_3 u_1^2 + \frac{kc}{2} u_1 u_2 + (kq_{021} + q_{201} k) u_3 u_2 u_1 \\
+ k \left( b + \frac{a}{2} \right) u_1^2 u_2 + \frac{kq_{012}}{2} u_2^2 u_1 + \frac{bk}{2} u_2^3 + kq_{111} u_3 u_2^2 + q_{102} u_3^2 u_2 + h.o.t.,
\]

\[
g_{13} = \langle \alpha_{u_1}, \alpha_{u_3} \rangle = k_3 u_1 u_3 + \frac{kq_{201}}{2} u_1^3 + kq_{111} u_1^2 u_2 + \left( kq_{102} + \frac{ak_3}{2} \right) u_3 u_1^2 \\
+ \frac{kq_{021}}{2} u_2^2 u_1 + kq_{012} u_3 u_2 u_1 + \left( \frac{kq_{003}}{2} + q_{201} \right) u_3^2 u_1 + \frac{bk_3}{2} u_3 u_2^2 + q_{111} k_3 u_3^2 u_2 \\
+ \frac{q_{012} k_3}{2} u_3^3 + h.o.t.,
\]

\[
g_{22} = \langle \alpha_{u_2}, \alpha_{u_2} \rangle = 1 + k^2 u_2^2 + kcu_2^3 + 2kq_{021} u_3^2 u_2 + kq_{012} u_3^2 u_2 + 2kb u_1 u_2^2 \\
+ 2kq_{111} u_2 u_3 u_1 + h.o.t.,
\]

\[
g_{23} = \langle \alpha_{u_2}, \alpha_{u_3} \rangle = k k_3 u_2 u_3 + \frac{1}{2} k q_{201} u_2 u_1^2 + kq_{111} u_2^2 u_1 + (kq_{102} + bk_3) u_3 u_2 u_1 \\
+ q_{111} k_3 u_3 u_1 + \frac{1}{2} k q_{021} u_3^2 + \left( k q_{012} + \frac{ck_3}{2} \right) u_3 u_2^2 + \left( \frac{1}{2} k q_{003} + q_{201} k_3 \right) u_3 u_2 \\
+ \frac{k_3 q_{012}}{2} u_3^3 + h.o.t.,
\]

\[
g_{33} = \langle \alpha_{u_3}, \alpha_{u_3} \rangle = 1 + k_3 u_3^2 + q_{021} k_3 u_3^2 u_3 + q_{003} k_3 u_3^3 + 2q_{102} k_3 u_3 u_1^2 + q_{201} k_3 u_1^2 u_3 \\
+ 2k_3 q_{012} u_3^2 u_2 + 2k_3 q_{111} u_1 u_2 u_3 + h.o.t.,
\]

12.2. Normal Vector. Taylor expansion of the components of the unit normal of \(\alpha\), \(N = N_\alpha = (\alpha_{u_1} \wedge \alpha_{u_2} \wedge \alpha_{u_3}) / |\alpha_{u_1} \wedge \alpha_{u_2} \wedge \alpha_{u_3}|\), gives the following expressions for \(N = (n_1, n_2, n_3, n_4)\) in a neighborhood of \((0,0,0)\):
\( n_1 = -u_1 k - \frac{1}{2} q_{102} u_2^2 - \frac{1}{2} a u_1^2 - q_{111} u_2 u_3 - \frac{1}{2} b u_2^2 - q_{201} u_1 u_3 - \frac{1}{2} Q_{112} u_3^2 u_2 \\
- Q_{2111} u_1 u_2 u_3 - \frac{1}{6} Q_{103} u_3^3 - \frac{1}{2} Q_{1211} u_3 u_2 - \frac{1}{6} D u_3^3 + \left( \frac{1}{2} C + \frac{1}{2} k^3 \right) u_1 u_2^2 \\
+ \left( -\frac{1}{2} Q_{202} + \frac{1}{2} k k_3 \right) u_1 u_3^2 - \frac{1}{2} Q_{301} u_1 u_3^2 + \left( -\frac{1}{6} A + \frac{1}{2} k^3 \right) u_1^3 - B u_1^2 u_2 + h.o.t., \)

\( n_2 = -u_2 k - \frac{1}{2} q_{012} u_3^2 - b u_1 u_2 - \frac{1}{2} c u_2^2 - q_{111} u_1 u_3 - q_{021} u_2 u_3 - \frac{1}{2} Q_{112} u_3 u_2 u_1 \\
- \frac{1}{6} Q_{013} u_3^3 - Q_{031} u_3^2 u_2 - \frac{1}{2} C u_2^2 - \frac{1}{2} Q_{2111} u_3 u_2 - \frac{1}{2} D u_1 u_2^2 \\
+ \left( \frac{1}{2} C + \frac{1}{2} k^3 \right) u_2^2 u_2 + \left( -\frac{1}{2} Q_{202} + \frac{1}{2} k k_3 \right) u_2 u_3^2 + \left( -\frac{1}{6} E + \frac{1}{2} k^3 \right) u_3^3 \\
- \frac{1}{6} B u_1^3 + h.o.t., \)

\( n_3 = -k_3 u_3 - \frac{1}{2} q_{201} u_1^2 - \frac{1}{2} q_{003} u_3^2 - q_{102} u_1 u_3 - q_{111} u_1 u_2 - \frac{1}{2} q_{021} u_2^2 - q_{012} u_3 u_2 \\
- \frac{1}{2} Q_{2111} u_2 u_1 - \frac{1}{2} Q_{103} u_3^2 u_1 + \left( -\frac{1}{2} Q_{202} + \frac{1}{2} k^2 \right) u_1^2 u_3 - \frac{1}{2} Q_{013} u_1 u_2 \\
- \frac{1}{2} Q_{1211} u_2 u_2 + \left( -\frac{1}{2} Q_{202} + \frac{1}{2} k^2 \right) u_2 u_3 - \frac{1}{6} Q_{301} u_1^3 - Q_{112} u_3 u_1 u_2 \\
- \frac{1}{6} Q_{031} u_3^3 + \left( -\frac{1}{6} Q_{004} + \frac{1}{2} \right) u_3^3 + h.o.t., \)

\( n_4 = 1 - \frac{1}{2} u_2^2 k^2 - \frac{1}{2} u_1^2 k - \frac{1}{2} k_3 u_3^2 - \frac{1}{2} k u_3^3 + \left( -\frac{1}{2} k - 1 \right) q_{012} u_3^2 u_2 \\
+ \left( -\frac{1}{2} k - 1 \right) q_{102} u_1 u_3^2 - \frac{1}{2} q_{003} u_3^3 - \frac{3}{2} b k u_2^2 u_1 + \left( -2 k - 1 \right) q_{1111} u_1 u_3 u_2 \\
- \frac{1}{2} k c u_2^2 + \left( -k - \frac{1}{2} \right) q_{201} u_3 u_1^2 + \left( -k - \frac{1}{2} \right) q_{021} u_3 u_1^2 + h.o.t.. \)

12.3. Second Fundamental Form. The coefficients of the second fundamental form \((\lambda_{ij})\) are:

\[\lambda_{11} = \left( \frac{\partial^2 \phi}{\partial u_1^2}, N \right) = k + a u_1 + q_{201} u_3 + \left( -\frac{1}{2} k^3 + \frac{1}{2} A \right) u_1^2 + B u_1 u_2 + Q_{301} u_1 u_3 \\
+ \left( -\frac{1}{2} k^3 + \frac{1}{2} C \right) u_2^2 + Q_{2111} u_3 u_2 + \left( -\frac{k k_3}{2} + \frac{1}{2} Q_{202} \right) u_3^3 + h.o.t.,\]
\[ \lambda_{12} = \left( \frac{\partial^2 \alpha}{\partial u_1 \partial u_2}, N \right) = q_{111} u_3 + bu_2 + Q_{121} u_3 u_2 + \frac{1}{2} Q_{112} u_3^2 + Q_{211} u_3 u_1 + \frac{1}{2} Du_2^2 + Cu_1 u_2 + \frac{1}{2} Bu_1^2 + \text{h.o.t.}, \]

\[ \lambda_{13} = \left( \frac{\partial^2 \alpha}{\partial u_1 \partial u_3}, N \right) = q_{201} u_3 + q_{111} u_2 + q_{102} u_3 + \frac{1}{2} Q_{103} u_3^2 + Q_{112} u_3 u_2 + Q_{211} u_2 u_1 + \frac{1}{2} Q_{301} u_3^2 + \frac{1}{2} Q_{121} u_2^2 + \text{h.o.t.}, \]

\[ \lambda_{22} = \left( \frac{\partial^2 \alpha}{\partial u_2^2}, N \right) = k + cu_2 + q_{021} u_3 + bu_1 + \left( -\frac{1}{2} k^3 + \frac{1}{2} C \right) u_1^2 + Q_{121} u_3 u_1 + D u_1 u_2 + \left( -\frac{1}{2} k^3 + \frac{1}{2} E \right) u_2^2 + Q_{031} u_3 u_2 + \left( -\frac{k^2}{2} k + \frac{1}{2} Q_{002} \right) u_3^2 + O(3), \]

\[ \lambda_{23} = \left( \frac{\partial^2 \alpha}{\partial u_2 \partial u_3}, N \right) = q_{012} u_3 + q_{111} u_1 + q_{021} u_2 + \frac{1}{2} Q_{103} u_3^2 + Q_{112} u_3 u_1 + Q_{121} u_2 u_1 + \frac{1}{2} Q_{031} u_2^2 + Q_{222} u_3 u_2 + \frac{1}{2} Q_{211} u_2^2 + O(3), \]

\[ \lambda_{33} = \left( \frac{\partial^2 \alpha}{\partial u_3^2}, N \right) = k_3 + q_{012} u_2 + q_{102} u_1 + q_{003} u_3 + \left( -\frac{1}{k_3} k^2 + \frac{1}{2} Q_{020} \right) u_1^2 + Q_{112} u_1 u_2 + Q_{013} u_3 u_2 + \left( -\frac{k_3 k^2}{2} + \frac{1}{2} Q_{002} \right) u_2^2 + \left( \frac{1}{2} Q_{004} - \frac{k_3 k^2}{2} \right) u_3^2 + O(3), \]

12.4. **The principal curvature** \( k_3 \). The principal curvature \( k_3 \), which is smooth near the origin, is given by

\[
k_3(u_1, u_2, u_3) = k_3 + q_{012} u_1 + q_{012} u_2 + q_{003} u_3 - \frac{k_3 k^2 (k - k_3) - k Q_{202} + 2 q_{201}^2 + k_3 Q_{202} + 2 q_{111}^2 u_1^2}{k - k_3} + \frac{k Q_{112} - 2 q_{201} q_{111} - k_3 Q_{112} - 2 q_{111} q_{021}}{k - k_3} u_1 u_2 - \frac{k_3 k^2 (k - k_3) - k Q_{022} + 2 q_{111}^2 + 2 q_{021}^2 + k_3 Q_{022}}{k - k_3} u_2^2 + \frac{k Q_{103} - 2 q_{012} q_{111} - 2 q_{201} q_{02} - Q_{103} k_3}{k - k_3} u_2 u_3 + \frac{k Q_{013} - 2 q_{012} q_{111} - 2 q_{021} q_{02} - Q_{013} k_3}{k - k_3} u_2 u_3 + \frac{k Q_{004} - 3 k_3 k^2 + 3 k_3^2 - 2 q_{102}^2 - k_3 Q_{004} - 2 q_{012}^2}{k - k_3} u_3^2 + \text{h.o.t.}
\]
12.5. **The principal direction** \( e_3 \). The smooth principal direction \( e_3(q) = (du_1, du_2, du_3) \) is defined by

\[
\begin{align*}
\frac{du_1}{du_3} &= U_1(u_1, u_2, u_3), \quad \frac{du_2}{du_3} = V_1(u_1, u_2, u_3).
\end{align*}
\]

The functions \( U_1, V_1 \) and \( W_1 \) are given by solving the linear system \( \text{[1]} \), taking \( i = 3 \).

Using the equation \( \text{[3]} \) and the subsections \( \text{[12.1, 12.3, 12.4]} \) it is obtained:

\[
U_1 = (k_3 - k)q_{201}u_1 + (k_3 - k)q_{111}u_2 + (k_3 - k)q_{102}u_3 \\
+ (\frac{1}{2}Q_{301}(k_3 - k) - q_{201}(b + q_{102}))u_1^2 \\
+ (q_{201}b - cq_{111} + \frac{1}{2}(k_3 - k)Q_{121} + q_{111}q_{102})u_2^2 \\
+ (-q_{201}c + q_{201}q_{012} + q_{111}q_{102} + Q_{211}(k_3 - k))u_1u_2 \\
+ |q_{111} + (k_3^2 - Q_{202})(k - k_3) + q_{201}(q_{003} - q_{021}) + (q_{102} - b)q_{102}|u_1u_3 \\
+ (q_{012}b - q_{102}c + (-k + k_3)Q_{112} + q_{003}q_{111} + q_{012}q_{102})u_2u_3 \\
+ (q_{111}q_{102} - q_{102}q_{021} + \frac{1}{2}(k_3 - k)Q_{103} + q_{003}q_{102})u_3^2 + h.o.t.. \\
\]

\[
V_1 = (k_3 - k)q_{111}u_1 + (k_3 - k)q_{201}u_2 + (k_3 - k)q_{102}u_3 \\
+ (q_{111}(q_{102} - a) - \frac{1}{2}Q_{211}(k - k_3))u_1^2 + (q_{111}b + q_{012}q_{201} + \frac{1}{2}(k_3 - k)Q_{031})u_2^2 \\
+ |q_{111}q_{012} + q_{201}b + (k_3 - k)Q_{121} + q_{102}q_{021} - aq_{021}|u_1u_2 \\
+ (q_{003}q_{111} + q_{112}(q_{102} - a) + (k_3 - k)Q_{112})u_1u_3 \\
+ |q_{111}q_{102} + q_{112}q_{201} + q_{021}q_{003} + q_{012} + (k_3^2 - Q_{022})(k - k_3)|u_2u_3 \\
+ |\frac{1}{2}(k_3 - k)Q_{103} - q_{201}q_{102} + q_{111}q_{102} + q_{003}q_{102}|u_3^2 + h.o.t.. \\
\]

\[
W_1 = (k - k_3)^2 + (k - k_3)(a + b - 2q_{102})u_1 + (k - k_3)(c - 2q_{012})u_2 \\
+ (k - k_3)[2q_{003} - q_{021} - q_{021}]u_3 + h.o.t.
\]

12.6. **The field of planes** \( \mathcal{P}_3 \). The plane field \( \mathcal{P}_3 \) defined by \( \omega = 0 \) (see equation \( \text{[7]} \) can be written as \( du_3 = Ud_1 + Vdu_2 \) where

\[
U = -\frac{[g_{11}U_1 + g_{12}V_1 + g_{13}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]} \quad \text{and} \quad V = -\frac{[g_{12}U_1 + g_{22}V_1 + g_{23}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]}
\]
The Taylor expansions of $\mathcal{U}$ and $\mathcal{V}$ in a neighborhood of zero, are given by:

$$
\mathcal{U} = \frac{1}{2(k-k_3)^2} [2q_{201}(k-k_3)u_1 + 2q_{111}(k-k_3)u_2 + 2q_{102}(k-k_3)u_3
+ (-2q_{021}b + 2q_{111}q_{012} + kQ_{121} - k_3Q_{121})u_2^2 + (2q_{003}q_{102} - 2q_{201}q_{102}
+ (k-k_3)Q_{103} - 2q_{111}q_{012})u_3^2 + (2k^3 - 2q_{201}^2 - 2k_3Q_{202} - 2k_3^2)k^2
+ 2kQ_{202} + 2q_{102} + 2q_{111} - 2aq_{102} + 2q_{201}q_{003})u_3u_1
+ (-k_3Q_{301} + 2q_{201}q_{102} - 2q_{201}a + Q_{301}k)u_1^2 + (-2k_3Q_{112} + 2q_{012}q_{102}
- 2q_{111}q_{021} + 2q_{003}q_{111} - 2q_{112} + 2q_{201}q_{111} + 2kQ_{112})u_1u_2 + (-2aq_{111}
+ 2q_{102}q_{111} + 2q_{201}q_{012} - 2q_{111}b - 2k_3Q_{211} + 2Q_{211}k)u_1u_2] + h.o.t.
$$

$$
\mathcal{V} = \frac{1}{2(k-k_3)^2} [2q_{201}(k-k_3)u_2 + 2q_{111}(k-k_3)u_1 + 2q_{012}(k-k_3)u_3
+ (kQ_{031} - 2q_{111}b - 2q_{121}c - k_3Q_{031} + 2q_{012}q_{021})u_2^2 + (-2k_3^2)k^2
- 2q_{111}^2 + 2k_3k - 2q_{112} + 2kQ_{022} - 2k_3Q_{022} - 2q_{021}^2 - 2q_{012}b + 2q_{012}
+ 2q_{021}q_{003})u_3u_2 + (-2q_{111}q_{201} + 2q_{003}q_{111} + 2kQ_{112} - 2q_{111}q_{021}
+ 2q_{012}q_{102} - 2q_{102}b - 2k_3Q_{112} - 2q_{111}q_{012}
+ 2q_{102}q_{111} + 2kQ_{121} + 2q_{012}q_{021} - 2q_{121}b - 2k_3Q_{121}
+ Q_{211}k)u_1^2 + (kQ_{013} + 2q_{003}q_{012} - k_3Q_{013} - 2q_{112}q_{021} - 2q_{012}q_{011})u_3^2
+ (-2q_{111}c - 2q_{201}b + 2kQ_{021} + 2q_{102}q_{021} - 2q_{201}b - 2k_3Q_{121}
+ 2q_{111}q_{012})u_1u_2] + h.o.t.
$$

12.7. The first fundamental form restricted to the plane field $\mathcal{P}_3$. The first fundamental form $I = \sum g_{ij} du_i du_j$ restricted to the plane field $\mathcal{P}_3$ is given by:

$$
I_r(du_1, du_2) = I_{\alpha} \bigg|_{du_3 = \mathcal{U}du_1 + \mathcal{V}du_2} = E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2,
$$
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\[ E_r = 1 + \left( k^2 + \frac{q_{201}^2}{(k-k_3)^2} \right) u_1^2 + 2 \frac{q_{111}q_{201}}{(k-k_3)^2} u_1 u_2 + 2 \frac{q_{102}q_{201}}{(k-k_3)^2} u_1 u_3 \]

\[ + \frac{q_{111}^2}{(k-k_3)^2} u_2^2 + 2 \frac{q_{102}q_{111}}{(k-k_3)^2} u_2 u_3 + \frac{q_{102}^2}{(k-k_3)^2} u_3^2 + O(3); \]

\[ F_r = \frac{q_{201}q_{111} + q_{201}q_{012}}{(k-k_3)^2} u_3 u_1 + \frac{q_{102}q_{012}}{(k-k_3)^2} u_3^2 + \frac{q_{111}q_{021}}{(k-k_3)^2} u_2^2 \]

\[ + \frac{q_{201}q_{111}}{(k-k_3)^2} u_1^2 + O(3); \]

\[ G_r = 1 + \frac{q_{012}^2}{(k-k_3)^2} u_3^2 + \left( k^2 + \frac{q_{021}^2}{(k-k_3)^2} \right) u_2^2 + \frac{q_{111}^2}{(k-k_3)^2} u_1^2 \]

\[ + \frac{2q_{012}q_{012}}{(k-k_3)^2} u_2 u_3 + \frac{2q_{111}q_{021}}{(k-k_3)^2} u_1 u_2 + \frac{2q_{102}q_{111}}{(k-k_3)^2} u_1 u_3 + O(3); \]

12.8. The second fundamental form restricted to the plane field \( \mathcal{P}_3 \).

The second fundamental form \( II = \sum \lambda_{ij} du_i du_j \) restricted to the plane field \( \mathcal{P}_3 \) is given by:

\[ II_r(du_1, du_2) = II_{0, du_1 + q_{201} u_3 + \frac{A}{2} \left( 2k - k_3 \right) q_{201}^2 - \frac{k^3}{2}} du_1^2 + 2f_r du_1 du_2 + g_r du_2^2, \]

\[ e_r = k + au_1 + q_{201} u_3 + \left( \frac{A}{2} + \frac{2k - k_3}{(k-k_3)^2} \right) u_1^2 \]

\[ + \left( \frac{2k - k_3}{(k-k_3)^2} + C - \frac{k^3}{2} \right) u_2^2 + \left( \frac{Q_{202}}{2} + \frac{2k - k_3}{(k-k_3)^2} - \frac{k^3}{2} \right) u_3^2 \]

\[ + \left( B + \frac{4k - 2k_3}{(k-k_3)^2} \right) u_1 u_2 + \left( \frac{4k - 2k_3}{(k-k_3)^2} + Q_{201} \right) u_1 u_3 \]

\[ + \left( Q_{211} + \frac{4k - 2k_3}{(k-k_3)^2} \right) u_2 u_3 + O(3); \]
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\[(48)\]

\[f_r = bu_2 + q_{111}u_3 + \left( \frac{B}{2} + \frac{(2k - k_3)q_{201}q_{111}}{(k - k_3)^2} \right) u_1^2 \]

\[+ \left( \frac{(2k - k_3)q_{201}q_{111}}{(k - k_3)^2} + \frac{D}{2} \right) u_2^2 + \left( \frac{(2k - k_3)q_{102}q_{111}}{(k - k_3)^2} + \frac{Q_{112}}{2} \right) u_3^2 \]

\[+ \left( C + \frac{(2k - k_3)q_{021}q_{201}}{(k - k_3)^2} + \frac{(2k - k_3)q_{111}^2}{(k - k_3)^2} \right) u_1 u_2 \]

\[+ \left( Q_{211} + \frac{(2k - k_3)q_{012}q_{201}}{(k - k_3)^2} + \frac{(2k - k_3)q_{102}q_{111}}{(k - k_3)^2} \right) u_1 u_3 \]

\[+ \left( 2Q_{121} + \frac{(2k - k_3)q_{021}q_{111}}{(k - k_3)^2} + \frac{(2k - k_3)q_{021}q_{102}}{(k - k_3)^2} \right) u_2 u_3 + O(3); \]

\[(49)\]

\[g_r = k + bu_1 + cu_2 + q_{021}u_3 + \left( -\frac{k^3}{2} + \frac{C}{2} + \frac{(2k - k_3)q_{111}^2}{(k - k_3)^2} \right) u_1^2 \]

\[+ \left( -\frac{k^3}{2} + \frac{E}{2} + \frac{(2k - k_3)q_{201}^2}{(k - k_3)^2} \right) u_2^2 + \left( \frac{Q_{022}}{2} - \frac{kq_{111}^2}{2} + \frac{(2k - k_3)q_{012}^2}{(k - k_3)^2} \right) u_3^2 \]

\[+ \left( D + \frac{(-2k_3 + 4k)q_{021}q_{111}}{(k - k_3)^2} \right) u_1 u_2 + \left( Q_{121} + \frac{(-2k_3 + 4k)q_{012}q_{111}}{(k - k_3)^2} \right) u_1 u_3 \]

\[+ \left( Q_{031} + \frac{(-2k_3 + 4k)q_{012}q_{021}}{(k - k_3)^2} \right) u_2 u_3 + O(3) \]

12.9. **Coefficient $\chi_{12}^\ast$ in a Monge chart.** The two patterns for the failure of the discriminant condition $D_2$, $\frac{a}{b} = (\frac{c}{d})^2 + 2$ and $a = 2b$, keeping the transversality condition $b \neq a$, are permuted by a rotation in the $(u, v)$-plane preserving the form of equation (1).

When $\frac{a}{b} = (\frac{c}{d})^2 + 2$, $b(b - a) \neq 0$, the characterization of the partially umbilic point $D_{12}$ is as appearing in terms of $\chi_{12}$ in remark [9] and in terms of saddle node equilibrium in equation [28].
When \( \frac{\theta}{\delta} = (\frac{\theta}{2\delta})^{2} + 2 \), an analysis similar to that carried out in section 8 gives equations with a coefficient proportional to \( \chi_{12}^{*} \) given below, instead of \( \chi_{12} \).

\[
\chi_{12}^{*} = \chi_{11} + \chi_{22}
\]

where,

\[
\frac{\chi_{11}}{k - k_3} = 16 b^3 c \left( -b q_{201} + b q_{201} - q_{111} c \right) A \\
- 4 b^2 \left( 5 c^3 q_{111} - 8 b^3 q_{201} + 8 b^3 q_{021} - 4 b^2 c q_{111} - 4 b c^2 q_{201} + 4 b c^2 q_{201} \right) B \\
+ 4 b \left( 8 b^2 - 2 c^2 \right) \left( b^2 + c^2 \right) q_{111} - bc \left( 8 b^2 - c^2 \right) q_{201} + bc \left( 8 b^2 - c^2 \right) q_{201} C \\
+ c \left( -c q_{111} + 8 b^3 c q_{201} + 32 b q_{111} - 8 b^3 c q_{201} + 12 b^2 c^2 q_{111} \right) D \\
+ 2 b c q_{111} \left( 4 b^2 + c^2 \right) E - bc \left( 4 b^2 + c^2 \right) \left( 8 b^2 - c^2 \right) Q_{121} \\
- 4 b^2 \left( 4 b^2 + c^2 \right) \left( 2 b^2 - c^2 \right) Q_{211} - 2 b^2 c \left( 4 b^2 + c^2 \right) Q_{031} + 4 b^3 c \left( 4 b^2 + c^2 \right) Q_{301}
\]

\[
\chi_{22} = 4 b^2 c k^3 \left( 4 b^2 + c^2 \right) \left( k - k_3 \right) q_{201} - 4 b^2 c k^3 \left( 4 b^2 + c^2 \right) \left( k - k_3 \right) q_{201} \\
- 2 b k^3 \left( 4 b^2 - c^2 \right) \left( 4 b^2 + c^2 \right) \left( k - k_3 \right) q_{111} - 4 b^2 \left( 4 b^2 + c^2 \right) \left( 2 b^2 - c^2 \right) q_{201} q_{201} \\
- 6 b^2 c^2 \left( 4 b^2 + c^2 \right) q_{012} q_{201} - bc \left( 4 b^2 + c^2 \right) \left( 8 b^2 - c^2 \right) q_{201} q_{102} \\
- 8 b^2 \left( 4 b^2 + c^2 \right) \left( 2 b^2 - c^2 \right) q_{111} q_{102} + 12 b^3 c \left( 4 b^2 + c^2 \right) q_{102} q_{201} \\
- 2 b c \left( 4 b^2 + c^2 \right) \left( 8 b^2 - c^2 \right) q_{111} q_{012} - 48 b^4 c q_{201}^3 \\
+ 4 b^2 c \left( -17 c^2 + 28 b^2 \right) q_{111}^2 q_{201} + c \left( 44 b^2 c^2 + 32 b^4 - 3 c^4 \right) q_{111}^2 q_{201} \\
+ 16 b \left( 2 b - c \right) \left( 2 b + c \right) \left( b^2 + c^2 \right) q_{111}^3 - 8 b \left( 8 b^4 - 12 b^2 c^2 + c^4 \right) q_{201} q_{201} q_{111} \\
+ 96 b^3 \left( b - c \right) q_{201}^2 q_{111} + 6 b c^4 q_{201}^2 q_{111} - 4 b^2 c \left( 8 b^2 - c^2 \right) q_{201} q_{201}^2 \\
+ 4 b^2 c \left( -c^2 + 20 b^2 \right) q_{201}^2 q_{201}
\]

When \( q_{111} = 0 \) and \( q_{201} = q_{201} \) (necessary conditions for the integrability of the plane field \( \mathcal{P}_3 \)) the coefficient \( \chi_{12}^{*} \) is given by:

\[
\chi_{12}^{*} = - 4 b^2 \left( 4 b^2 + c^2 \right) \left( 2 b^2 - c^2 \right) \left( k - k_3 \right) Q_{211} \\
- bc \left( 4 b^2 + c^2 \right) \left( 8 b^2 - c^2 \right) \left( k - k_3 \right) Q_{121} \\
+ 4 b^3 c \left( 4 b^2 + c^2 \right) \left( k - k_3 \right) Q_{301} - 2 b^2 c^2 \left( 4 b^2 + c^2 \right) \left( k - k_3 \right) Q_{031} \\
- b q_{201} \left( 4 b^2 + c^2 \right)^2 \left( 2 q_{201} b - q_{102} c \right)
\]

Also the coefficient \( \chi_{12} \) is given by:

\[
\chi_{12} = b^2 \left( -k Q_{211} + k Q_{211} + q_{201} q_{201} \right)
\]
Remark 12. The long expressions appearing in this Appendix such as those in equations (42), (43), and (50), have been corroborated by Computer Algebra.

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