Modular forms of orthogonal type and Jacobi theta-series

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Abstract In this paper, we study Jacobi forms of half-integral index for any even integral positive definite lattice $L$ (classical Jacobi forms from the book of Eichler and Zagier correspond to the lattice $A_1 = (2)$). We construct Jacobi forms of singular (respectively, critical) weight in all dimensions $n \geq 8$ (respectively, $n \geq 9$). We give the Jacobi lifting for Jacobi forms of half-integral indices and we obtain an additive lifting construction of new reflective modular forms which are natural generalizations to $O(2, n)$ ($n = 4, 5$ and $6$) of the Igusa modular form $\Delta_5$.

Keywords Jacobi forms for lattices and root systems · Theta-series and Weil representations · Modular forms of orthogonal type · Lifting of modular forms

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1 Introduction

The divisor of a strongly reflective modular form with respect to an integral orthogonal group of signature $(2, n)$ is determined by reflections. Such modular forms determine Lorentzian Kac–Moody Lie (super) algebras. The most famous reflective modular form is the Borcherds function $\Phi_{12}$ with respect to $O^+(I_{2,26})$ which determines the Fake Monster Lie algebra (see [1]). One can consider reflective modular forms as automorphic discriminants or multi-dimensional Dedekind $\eta$-functions (see [2, 3, 17–20]). Reflective modular forms play also an
important role in complex algebraic geometry (see [21] and [14]). All of them are Borcherds automorphic products and some of them can be constructed as additive (or Jacobi) lifting. If a reflective modular form can be obtained by a Jacobi lifting then one has a simple formula for its Fourier coefficients which determine the generators and relations of Lorentzian Kac–Moody algebras (see [17]).

In [14], the second author constructed the Borcherds-Enriques form $\Phi_4$, the automorphic discriminant of the moduli space of Enriques surfaces (see [2]), as Jacobi lifting, $\text{Lift}(\vartheta(\tau,z_1) \ldots \vartheta(\tau,z_8))$, of the tensor product of eight classical Jacobi theta-series (see [11] for the definition of Lift which provides a modular form on orthogonal group by its first Fourier–Jacobi coefficient). This new construction of $\Phi_4$ gives an answer to a problem formulated by K.-I. Yoshikawa ([33]) and to a question of J.A. Harvey and G. Moore ([24]) about the second Lorentzian Kac–Moody super Lie algebra determined by the Borcherds–Enriques form $\Phi_4$ and its quasi-pullbacks.

Another application of reflective modular forms of type Lift$(\vartheta(\tau,z_1) \ldots \vartheta(\tau,z_8))$ is the construction of new examples of modular varieties of orthogonal type of Kodaira dimension 0 (see the beginning of Sect. 2). The first two examples of this type of dimension 3 are related to reflective Siegel cusp forms of weight 3 and Siegel modular three-folds having compact Calabi–Yau models (see [5, 16] and [9]). In the case of dimension 4, the unique cusp form of weight 4 was defined in [14] as a Borcherds product but it can also be constructed as a lifting of a Jacobi form of half-integral index with a character of order 2 of the full modular group $\text{SL}_2(\mathbb{Z})$ (see Example 3.4 in Sect. 2). Jacobi forms of half-integral index in one variable are very important in the theory of Lorentzian Kac–Moody algebras of hyperbolic rank 3 corresponding to Siegel modular forms (see [17–19]). Moreover, they are very natural in the structure theory of classical Jacobi forms (in the sense of Eichler and Zagier [8]) and in applications to topology and string theory (see [7, 13]).

In this paper, we consider Jacobi forms of half-integral index for any positive definite lattice $L$ (Jacobi forms in [8] correspond to the lattice $A_1 = \langle 2 \rangle$). Jacobi forms in many variables naturally appeared in the theory of affine Lie algebras (see [25] and [26]). One can consider Jacobi forms as vector-valued modular forms in one variable. Vector-valued modular forms are used in the additive Borcherds lifting (see [3, Sect. 14]) which is a generalization of the Jacobi lifting of [11]. In this paper, we follow the general approach to Jacobi forms proposed in [11]. The first section contains all necessary definitions and basic results on Jacobi forms in many variables. It turns out that the order of the character of the integral Heisenberg group of such Jacobi forms is always at most 2 (see Proposition 2.3). Using the classical Jacobi theta-series, we give examples of Jacobi forms for the root lattices. We show, at the end of the first section (Examples 2.8–2.9), that the natural theta-products give all Jacobi forms of singular weight (or vector valued $\text{SL}_2(\mathbb{Z})$-modular forms of weight 0 related to the Weil representation) for the lattices $D_m$.

In Sect. 2, we give the Jacobi lifting for Jacobi forms of half-integral index with a possible character. This explicit construction has many advantages: one can see immediately a part of its divisor, the maximal modular group of the lifting, etc. (Compare our construction of $\Phi_4$ with the construction of S. Kondo in [28].)

We build many modular forms of singular, critical and canonical weights on orthogonal groups. In particular, we give in Example 3.4 the Jacobi lifting construction of a new strongly reflective modular form of singular weight on $O(2, 6)$. This modular forms gives a tower of four reflective modular forms based on the classical Igusa modular form $\Delta_5$.

In Sect. 3, we analyze Jacobi forms of singular (the minimal possible) and critical (singular weight $+\frac{1}{2}$) weights using the theta-products and their pullbacks (see Proposition 4.1–4.2). In this way, we construct Jacobi forms of singular and critical weights in all dimensions.
$n \geq 8$ (see Propositions 4.3–4.11). In particular, using this approach, we give a new explanation of theta-quarks in Corollary 4.4, which are the simplest examples of holomorphic theta-blocks (see [23]).

2 Jacobi group and Jacobi modular forms

In this section, we discuss Jacobi forms of orthogonal type. In the definitions below, we follow the papers [10, 11] where Jacobi forms were considered as modular forms with respect to a parabolic subgroup of an orthogonal group of signature $(2, n)$.

By a lattice we mean a free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ with values in $\mathbb{Z}$. A lattice is even if $(l, l)$ is even for all its elements. Let $L_2$ be a lattice of signature $(2, n_0 + 2)$. All the bilinear forms we deal with can be extended to $L_2 \otimes \mathbb{C}$ (respectively to $L_2 \otimes \mathbb{R}$) by $\mathbb{C}$-linearity (respectively by $\mathbb{R}$-linearity) and we use the same notations for these extensions. Let

$$D(L_2) = \{ [Z] \in \mathbb{P}(L_2 \otimes \mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) > 0 \}^+$$

be the $(n_0 + 2)$-dimensional bounded symmetric Hermitian domain of type IV associated to the lattice $L_2$ (here + denotes one of its two connected components). We denote by $O^+(L_2 \otimes \mathbb{R})$ the index 2 subgroup of the real orthogonal group preserving $D(L_2)$. Then $O^+(L_2)$ is the intersection of the integral orthogonal group $O(L_2)$ with $O^+(L_2 \otimes \mathbb{R})$. We use the similar notation $SO^+(L_2)$ for the special orthogonal group.

In this paper, we assume that $L_2$ is an even lattice of signature $(2, n_0 + 2)$ containing two hyperbolic planes

$$L_2 = U \oplus U_1 \oplus L(-1), \quad U \simeq U_1 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $L$ is a positive definite even lattice of rank $n_0$ and $L(-1)$ denotes its rescaling by $-1$. We fix a basis of the hyperbolic plane $U = \mathbb{Z}e \oplus \mathbb{Z}f$; $e \cdot f = (e, f) = 1$ and $e^2 = f^2 = 0$. Similarly $U_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. Let $F$ be the totally isotropic plane spanned by $f$ and $f_1$ and let $P_F$ be the parabolic subgroup of $SO^+(L_2)$ that preserves $F$. This corresponds to a 1-dimensional cusp of the modular variety $SO^+(L_2) \backslash D(L_2)$ (see [21]). We choose a basis of $L_2$ of the form $(e, e_1, \ldots, f_1, f)$ where $\ldots$ denote a basis of $L(-1)$. In this basis, the quadratic form associated to the bilinear form on $L_2$ has the following Gram matrix

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -S & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $S$ is a positive definite integral matrix with even entries on the main diagonal. We denote the positive definite even integral bilinear form on the lattice $L$ by $(\cdot, \cdot)$ and the bilinear form of signature $(1, n_0 + 1)$ on the hyperbolic lattice $U_1 \oplus L(-1)$ by $(\cdot, \cdot)_1$. Therefore, for any $v = ne_1 + l + mf_1 \in L_1$, we have $(v, v)_1 = 2nm - (l, l)$. 
The subgroup $\Gamma^J(L)$ of $PF$ of elements acting trivially on the sublattice $L$ is called the Jacobi group. The Jacobi group has a subgroup isomorphic to $\text{SL}_2(\mathbb{Z})$. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we denote

$$\{A\} := \begin{pmatrix} A^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} \in \Gamma^J(L), \quad \text{where} \quad A^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t A^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$  

(1)

The second standard subgroup of $\Gamma^J(L)$ is the Heisenberg group $H(L)$ acting trivially on the totally isotropic plane $F$. This is the central extension $\mathbb{Z} \rtimes (L \times L)$. More precisely, we define

$$H(L) = \{ [x, y; r] : x, y \in L, \ r \in \frac{1}{2} \mathbb{Z} \text{ with } r + \frac{1}{2}(x, y) \in \mathbb{Z} \}$$

where

$$[x, y; r] := \begin{pmatrix} 1 & 0 & \frac{y S}{2} (x, y)/r - \frac{(y, y)}{2} \\ 0 & 1 & \frac{x S}{2} (x, x)/r - \frac{(x, y)}{2} + r \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $S$ the positive definite Gram matrix of the quadratic form $L$ of rank $n_0$, $(x, y) = t x y$ and we consider $x$ and $y$ as column vectors. The multiplication in $H(L)$ is given by the following formula

$$[x, y; r] \cdot [x', y'; r'] = \left[ x + x', y + y'; r + r' + \frac{1}{2} \left( (x, y') - (x', y) \right) \right].$$

(3)

In particular, the center of $H(L)$ is equal to $\{[0, 0; r], \ r \in \mathbb{Z} \}$. We introduce a subgroup of $H(L)$

$$H_s(L) = \{ [x, 0; 0], [0, y; 0] | x, y \in L \}$$

with a smaller center and we call it minimal integral Heisenberg group of $L$. The group $\text{SL}_2(\mathbb{Z})$ acts on $H(L)$ by conjugation:

$$A \cdot [x, y; r] := \{A\}[x, y; r]\{A^{-1}\} = [dx - cy, -bx + ay; r].$$

(4)

Using (3) and (4), one can define the integral Jacobi group or its subgroup $\Gamma^J_s(L)$ as the semidirect product of $\text{SL}_2(\mathbb{Z})$ with the Heisenberg group $H(L)$ or $H_s(L)$

$$\Gamma^J(L) \simeq \text{SL}_2(\mathbb{Z}) \rtimes H(L) \quad \text{and} \quad \Gamma^J_s(L) \simeq \text{SL}_2(\mathbb{Z}) \rtimes H_s(L).$$

Extending the coefficients, we can define the real Jacobi group which is a subgroup of the real orthogonal group: $\Gamma^J(L \otimes \mathbb{R}) \simeq \text{SL}_2(\mathbb{R}) \rtimes H(L \otimes \mathbb{R})$.

In what follows, we need characters of Jacobi groups. Let $\chi : \Gamma^J(L) \rightarrow \mathbb{C}^\times$ be a character of finite order. Its restriction to $\text{SL}_2(\mathbb{Z})$, $\chi|_{\text{SL}_2(\mathbb{Z})}$, is an even power $v^D_\eta$ of the multiplier system of the Dedekind $\eta$-function and we have

$$\chi([A] \cdot [x, y; r]) = v^D_\eta(A) \cdot v([x, y; r]), \quad \text{where} \quad \chi|_{\text{SL}_2(\mathbb{Z})} = v^D_\eta, \ v = \chi|_{H(L)}.$$ 

(5)

If $D$ is odd then we obtain a multiplier system of the Jacobi group. The properties of the character of the Heisenberg group are clarified by the next proposition.
Proposition 2.1 1. Let $s(L) \in \mathbb{N}^*$ (resp. $n(L)$) denote the generator of the integral ideal generated by $(x, y)$ (resp. $(x, x)$) for $x$ and $y$ in $L$. Let $[H(L), H(L)]$ be the derivated group of $H(L)$. Then

$$[H(L), H(L)] = [H_s(L), H_s(L)] = \{0, 0; r \mid r \in s(L)\mathbb{Z}\}.$$ 

This subgroup is the center of $H_s(L)$. 

2. Let $\nu : H(L) \to \mathbb{C}^*$ be a character of finite order which is invariant with respect to the $\text{SL}_2(\mathbb{Z})$-action: $\nu(A[x, y; r]) = \nu([x, y; r])$. Then

$$\nu([x, y; r]) = e^{2\pi it(x, x) + (y, y) - (x, y) + 2r}$$

where $t \in \mathbb{Q}$ such that $t \cdot s(L) \in \mathbb{Z}$. The restriction $\nu|_{H_s(L)}$ is a binary character which is trivial if $t \cdot n(L) \in 2\mathbb{Z}$. 

Remark The constants $s(L)$ and $n(L)$ are called scale and norm of the integral lattice $L$. The scale $s(L)$ is the greatest common divisor of the entries of the Gram matrix $S$ of $L$. For any even lattice $L$ the norm $n(L)$ is an even divisor of $s(L)$. 

Proof The first property follows from the formula for the commutator of the elements of $H(L)$

$$[x, y; r] \cdot [x', y'; r'] \cdot [x, y; r]^{-1} \cdot [x', y'; r']^{-1} = [0, 0; (x, y) - (x', y)]$$

because $[x, y; r]^{-1} = [-x, -y; -r]$. 

Considering the restriction of the character to the center of $H(L)$, isomorphic to $\mathbb{Z}$, we get $\nu([0, 0; r]) = \exp(2\pi itr)$ with $t \in \mathbb{Q}$. The formula for the commutator (6) shows that $t \cdot s(L) \in \mathbb{Z}$. 

The invariance of the character with respect to $A = -I_2$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives us $\nu([x, y; r]) = \nu([-x, -y; r]) = \nu([-y, x; r]) = \nu([x, y - x; r])$. Therefore $\nu([x, 0; 0]) = \nu([-x, 0; 0]) = \nu([x, 0; 0])^{-1}$, $\nu([0, y; 0]) = \nu([y, 0; 0])$ and $\nu([x, 0; 0]) = \nu([-x, 0; 0])$. We have

$$[x, -x; 0] = [x, 0; 0] \cdot [0, -x; 0] \cdot \left[0, 0; \frac{1}{2}(x, x)\right].$$

Therefore $\nu([x, 0; 0]) = \nu([-x, 0; 0]) = e^{2\pi it(x, x)}$ and the final formula follows from the decomposition

$$[x, y; r] = \left[x, y; \frac{1}{2}(x, y)\right] \cdot \left[0, 0; r - \frac{1}{2}(x, y)\right] = [x, 0; 0] \cdot [0, y; 0] \cdot \left[0, 0; r - \frac{1}{2}(x, y)\right].$$

We see that $t \cdot (x, x) \in \mathbb{Z}$. Therefore the order of $\nu|_{H_s(L)}$ is equal to 1 or 2. 

In order to define Jacobi forms, we have to fix a tube realization of the homogeneous domain $\mathcal{D}(L_2)$ related to the 1-dimensional boundary component of its Baily–Borel compactification corresponding to the Jacobi group related to the isotropic flag $(f) \subset (f, f_1)$. Let $\mathcal{Z} = \langle \mathcal{X} + i\mathcal{Y} \rangle \in \mathcal{D}(L_2)$. Then 

$$(\mathcal{X}, \mathcal{Y}) = 0, \quad (\mathcal{X}, \mathcal{X}) = (\mathcal{Y}, \mathcal{Y}) \quad \text{and} \quad (\mathcal{Z}, \mathcal{Z}) = 2(\mathcal{Y}, \mathcal{Y}) > 0.$$
Using the basis \( \langle e, f \rangle_Z = U \) we write \( Z = z'e + \tilde{Z} + zf \) with \( \tilde{Z} \in L_1 \otimes \mathbb{C} \), where

\[
L_1 = U_1 \oplus L(-1)
\]

is the hyperbolic lattice of signature \((1, n_0 + 1)\) with the bilinear form \((\cdot, \cdot)_1\). We note that \( z \neq 0 \). (If \( z = 0 \) the real and imaginary parts of \( \tilde{Z} \) form two orthogonal vectors of positive norm in the hyperbolic lattice \( L_1 \otimes \mathbb{R} \).) Thus \([\tau, Z] = [(-\frac{1}{2}(Z, Z), \tau, 1)]\). Using the basis \( \langle e_1, f_1 \rangle_Z = U \) of the second hyperbolic plane in \( L \), we see that \( D(L_2) \) is isomorphic to the tube domain

\[
\mathcal{H}(L) = \mathcal{H}_{n_0+2}(L) = \left\{ Z = \begin{pmatrix} \omega \\ 3 \\ \tau \end{pmatrix} \mid \tau, \omega \in \mathbb{H}_1, \ 3 \in L \otimes \mathbb{C}, \ (\text{Im } Z, \text{Im } Z)_1 > 0 \right\}
\]

where

\[
(\text{Im } Z, \text{Im } Z)_1 = 2 \text{Im}(\omega) \text{Im}(\tau) - (\text{Im}(3), \text{Im}(3)) > 0.
\]

We fix the isomorphism \([\text{pr}] : \mathcal{H}(L) \to D(L_2)\) defined by the 1-dimensional cusp \( F \) fixed above

\[
Z = \begin{pmatrix} \omega \\ 3 \\ \tau \end{pmatrix} \mapsto \text{pr}(Z) = \begin{pmatrix} \frac{1}{2}(Z, Z) \\ \omega \\ 3 \\ \tau \\ 1 \end{pmatrix} \mapsto [\text{pr}(Z)]. \tag{7}
\]

The map \( \text{pr} \) gives us the embedding of \( \mathcal{H}(L) \) into the affine cone \( D^*(L_2) \) over \( D(L_2) \). Using the map \([\text{pr}]\), we can define a linear-fractional action of \( M \in O^+(L_2 \otimes \mathbb{R}) \) on the tube domain

\[
M \cdot \text{pr}(Z) = J(M, Z) \cdot \text{pr}(M(Z))
\]

where the automorphic factor \( J(M, Z) \) is the last (non-zero) element of the column vector \( M \cdot \text{pr}(Z) \in D^*(L_2) \). In particular, for the standard elements of the Jacobi group, we have the following action

\[
[A] \langle Z \rangle = \begin{pmatrix} \omega - \frac{c(3, 3)}{2(c\tau + d)} & i\frac{3}{c\tau + d} & \frac{a\tau + b}{c\tau + d} \\ a & c & d \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R});
\]

\[
[x, y; r] \langle Z \rangle = \begin{pmatrix} \omega + \frac{1}{2}(x, x)\tau + (x, 3) + \frac{1}{2}(y, y) + r, i(3 + x\tau + y), \tau \end{pmatrix}
\]

\( x, y \in L \otimes \mathbb{R} \) and \( r \in \mathbb{R} \). We note that \( J([A], Z) = c\tau + d \) and \( J([x, y; r], Z) = 1 \). For a function \( \psi : \mathcal{H}(L) \to \mathbb{C} \), we define as usual

\[
(\psi|_k M)(Z) := J(M, Z)^{-k} \psi(M(Z)), \quad M \in O^+(L_2 \otimes \mathbb{R}).
\]

In the next definition, Jacobi forms are considered as modular forms with respect to the parabolic subgroup \( \Gamma^J(L) \) of \( O^+(L_2) \).

**Definition 2.2** Let \( \chi \) be a character (or a multiplier system) of finite order of \( \Gamma^J(L) \), \( k \) be integral or half-integral and \( t \) be a (positive) rational number. A holomorphic function
$\varphi : \mathbb{H}_1 \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a holomorphic *Jacobi form* of weight $k$ and index $t$ with a character (or a multiplier system) $\chi$ if the function

$$
\tilde{\varphi}(Z) = \varphi(\tau, Z)e^{2\pi i t \omega}, \quad Z = \begin{pmatrix} \omega \\ \tau \end{pmatrix} \in \mathcal{H}(L)
$$

satisfies the functional equation

$$
\tilde{\varphi}|_k M = \chi(M) \tilde{\varphi} \quad \text{for any } M \in \Gamma^J(L)
$$

and is holomorphic at “infinity” (see the condition (11) below).

**Remarks**
(1) We show below that for any non zero Jacobi form of rational index $t$ we have $t \cdot s(L) \in \mathbb{Z}$ where $s(L)$ is the scale of the lattice (see Proposition 2.1).

(2) One can reduce this definition to the only two cases $t = 1$ and $t = 1/2$ (see Proposition 2.3).

(3) One can give another definition of Jacobi forms in more intrinsic terms (see (9), (10) and Definition 2.2').

In order to precise the condition “to be holomorphic at infinity”, we analyze the character $\chi$, the functional equation and the Fourier expansion of Jacobi forms. We decompose the character into two parts

$$
\chi = \chi|_{SL_2(\mathbb{Z})} \times \chi|_{H(L)} = \chi_1 \times \nu,
$$

where $\chi_1 = v_D$ (see (5)) and $\nu$ satisfies the condition of Proposition 2.1. For a central element $[0, 0; (x, y)] \in H_1(L)$ $(x, y \in L)$, the Eq. (8) gives $\nu([0, 0; (x, y)]) = e^{2\pi it(x,y)} = 1$. Therefore

$$
t \cdot s(L) \in \mathbb{Z} \quad \text{if } \varphi \text{ is not identically zero}
$$

and

$$
\nu([x, y; r]) = e^{\pi i t ((x,x) + (y,y) - (x,y) + 2r)}, \quad [x, y; r] \in H(L)
$$

as in Proposition 2.1.

The formulae above for the action of the generators of the Jacobi group on the tube domain define also an action, denoted by $M(\tau, Z)$, of the real Jacobi group $\Gamma^J(L \otimes \mathbb{R})$ on the domain $\mathbb{H}_1 \times (L \otimes \mathbb{C})$. We can always add to any $(\tau, Z) \in \mathbb{H}_1 \times (L \otimes \mathbb{C})$ a complex number $\omega \in \mathbb{H}_1$ such that $Z = \begin{pmatrix} \omega \\ \tau \end{pmatrix}$ belongs to $\mathcal{H}(L)$. If we denote the first component of $M(Z)$ (that is the component along the vector $e_1$ of our basis) by $\omega[M(Z)]$ for $M \in \Gamma^J(L \otimes \mathbb{R})$ then

$$
J_{k,t}(M; \tau, Z) = J(M, Z)^k e^{-2\pi i t |M(Z)|} e^{2\pi i t \omega}
$$

defines an automorphic factor of weight $k$ and index $t$ for the Jacobi group. For the generators of the Jacobi group, we get

$$
J_{k,t}([A]; \tau, Z) = (c\tau + d)^k e^{i\pi t \frac{c(3)}{c + d}}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})
$$

and

$$
J_{k,t}([x, y; r]; \tau, Z) = e^{-2\pi i t \frac{1}{2}(x,x)\tau + (x,\bar{y}) + \frac{1}{2}(x,y) + r}, \quad x, y \in L \otimes \mathbb{R}, \ r \in \mathbb{R}.
$$
We also get an action of the Jacobi group on the space of functions defined on $\mathbb{H} \times (L \otimes \mathbb{C})$:

$$(\varphi|_{k,t} M)(\tau, \mathfrak{z}) := J_{k,t}^{-1}(M; \tau, \mathfrak{z}) \varphi(M\langle \tau, \mathfrak{z} \rangle).$$

Then the Eq. (8) in the definition of Jacobi forms is equivalent to

$$(\varphi|_{k,t} M)(\tau, \mathfrak{z}) = \chi(M) \varphi(\tau, \mathfrak{z}), \quad M \in \Gamma^J(L).$$

For the generators of the Jacobi group we obtain

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{3}{c\tau + d}\right) = \chi(A)(c\tau + d)^k e^{\frac{i\pi t}{(3,3)}} \varphi(\tau, \mathfrak{z})$$

(9)

for all $A = (\begin{array}{cc} a & b \\ c & d \end{array}) \in \text{SL}_2(\mathbb{Z})$ and

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \chi \left( \left[ x, y; \frac{1}{2} (x, y) \right] \right) e^{-i\pi t((x,x)\tau + 2(x,\mathfrak{z}))} \varphi(\tau, \mathfrak{z})$$

(10)

for all $x, y \in L$.

(Note that $t(x,y) \in \mathbb{Z}$ for any $x, y$ in $L$ if $\varphi \not\equiv 0$.) The variables $\tau$ and $\mathfrak{z}$ are called modular and abelian variables. To clarify the last condition of Definition 2.2, we consider the Fourier expansion of a Jacobi form $\varphi$.

We see that the function $\varphi$ have the following periodic properties

$$\varphi(\tau + 1, \mathfrak{z}) = e^{2\pi i \frac{B}{2}} \varphi(\tau, \mathfrak{z}) \quad \text{and} \quad \varphi(\tau, \mathfrak{z} + 2y) = \nu(\{0, 2y; 0\}) \varphi(\tau, \mathfrak{z}) = \varphi(\tau, \mathfrak{z}).$$

The function $\varphi$ is called holomorphic at infinity if it has the Fourier expansion of the following type

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Q}_{\geq 0}, \frac{n \equiv D}{2n(l,l) \geq 0} \mod \mathbb{Z}, l \in \frac{1}{2} L^\vee} f(n, l)e^{2\pi i (nt + (l, \mathfrak{z}))}$$

(11)

where $L^\vee$ is the dual lattice of the even positive definite lattice $L$. This condition is equivalent to the fact that the function $\tilde{\varphi}$ is holomorphic at the zero-dimensional cusp defined by the isotropic vector $f$ in the first copy $U$ in the lattice $L_2 = U \oplus U_1 \oplus L(-1)$.

The Definition 2.2 suits well for the applications considered in this paper but we can give another definition which does not depend on the orthogonal realization of the Jacobi group $\Gamma^J_s(L) \simeq \text{SL}_2(\mathbb{Z}) \rtimes H_s(L)$.

**Definition 2.2** A holomorphic function $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a holomorphic Jacobi form of weight $k \in \frac{1}{2} \mathbb{Z}$ and index $t \in \mathbb{Q}$ with a character (or a multiplier system) of finite order $\chi : \Gamma^J_s(L) \to \mathbb{C}^*$ if $\varphi$ satisfies the functional equations (9) and (10) and has a Fourier expansion of type (11).

**Remarks** (1) The classical Jacobi forms of Eichler and Zagier. Note that for $n_0 = 1$, the tube domain $\mathcal{T}(L)$ is isomorphic to the classical Siegel upper half-space of genus 2. If $L \simeq \mathbb{Z}(2)$ is the lattice $\mathbb{Z}$ with quadratic form $2x^2$ and $\chi = \text{id}$ then the definition above is identical to the definition of Jacobi forms of integral weight $k$ and index $t$ given in the book [8].
The difference between the Definitions 2.2 and 2.2′ is the character of the center of the “orthogonal” Heisenberg group $H(L)$. It is more natural to consider a Jacobi forms $\varphi(\tau, Z)$ as a modular form with respect to the minimal Jacobi group $\Gamma_J(L)$ and the extended Jacobi form $\tilde{\varphi}(Z) = \varphi(\tau, Z)e^{2\pi it\omega}$ with respect to $\Gamma^J(L)$.

We denote the vector space of Jacobi forms from Definition 2.2′ by $J_{k,L}^{1/2}(\chi)$ where $\chi = v_n^D \times v$ is defined by a character (or a multiplier system) $v_n^D$ of $SL_2(\mathbb{Z})$ and a binary (or trivial) character $v$ of $H_s(L)$. The space of Jacobi forms from Definition 2.2 is denoted by $\tilde{J}_{k,L}^{1/2}(v_n^D \times \tilde{\nu})$ with evident modification for the central part of $\tilde{\nu}$. The character $\chi$ of $\Gamma_J(L)$ and its restriction $\tilde{\chi} = \chi|_{\Gamma^J(L)}$ determine each other uniquely and we denote both of them by $\chi$. We denote the vector space of Jacobi forms from Definition 2.2 by $J_{k,L}(\chi)$.

A function $\varphi(\tau, Z) = \sum_{n,l} f(n,l)e^{2\pi i(n\tau + (l, Z))} \in J_{k,L}(\chi)$ is called a Jacobi cusp form if $f(n,l) \neq 0$ only if the hyperbolic norm of its index is positive: $2nt - (l,l) > 0$. We denote this vector space by $J_{k,L}^{cusp}(\chi)$. We define the order of $\varphi$ as follows

$$\text{Ord}(\varphi) = \min_{f(n,l) \neq 0} \{2nt - (l,l)\}.$$ (12)

When the character (or the multiplier system) is trivial, we omit it in the notation of these spaces. If $\chi = v_n^D \times \text{id}$, we omit the trivial part. We see from the definition that $\varphi \equiv 0$ if $t < 0$. If $t = 0$ then $J_{k,L,0}(\chi) = M_k(SL_2(\mathbb{Z}), \chi|_{SL_2})$. In fact, a Jacobi form corresponds to a vector-valued modular form of integral or half-integral weight related to the Weil representation of the lattice $L(t)$ and $J_{k,L}(\chi)$ is finite dimensional (see Sect. 3).

The notation $L(t)$ stands for the lattice $L$ equipped with bilinear for $t(\cdot, \cdot)$. We proved above that if $J_{k,L,0}(\chi) \neq \{0\}$ then the lattice $L(t)$ is integral. Any Jacobi form with trivial character can be considered as Jacobi form of index 1 (see [11, Lemma 4.6]). In general, we have the following

**Proposition 2.3** 1. If $L(t)$ is an even lattice then

$$J_{k,L(t)}(\chi) = J_{k,L(t),1}(\chi).$$

If this space is non-trivial then the Heisenberg part $v = \chi|_{H_s(L)}$ of the character is trivial.

2. If $L(t)$ is integral odd (non-even) lattice then

$$J_{k,L(t)}(\chi) = J_{k,L(t),1/2}(\chi).$$

In this case, the character $v = \chi|_{H_s(L)}$ is of order 2.

3. If $\varphi(\tau, Z) \in J_{k,L(t)}(v_n^D \times v)$ then $\varphi(\tau, 2Z) \in J_{k,L,t}(v_n^D \times \text{id})$.

**Remark** This proposition shows that we have to distinguish in fact only between index 1 and $1/2$. In what follows, we denote by $J_{k,L}(\chi)$ the space of Jacobi forms of index 1.

**Proof** If $L(t)$ is even then $t(x, x) \in 2\mathbb{Z}$ for any $x \in L$. Therefore $\chi([x, 0; 0]) = e^{\pi i t(x,x)} = 1$ and the Heisenberg part of $\chi$ is trivial. If $L(t)$ is odd then there exists $x \in L$ such that $t(x, x)$ is odd. Therefore the Heisenberg part of $\chi$ is non-trivial.
We prove the proposition about the indexes using a map that we will need in Sect. 2. We define an application for $N \in \mathbb{Q}_{>0}$

$$
\pi_N : \mathcal{H}(L(N)) \to \mathcal{H}(L), \quad \pi_N : \left( \frac{\omega}{\tau} \right) \mapsto \left( \frac{\omega/N}{\tau} \right).
$$

(13)

This map corresponds to the multiplication $I_N \cdot \text{pr}(Z)$ with $Z \in \mathcal{H}(L(N))$ and $\text{pr}(Z) \in \mathcal{D}^*(U \oplus U \oplus L(-N))$ where $I_N = \text{diag}(N^{-1} \mathbf{1}_2, \mathbf{1}_{n_0}, \mathbf{1}_2)$. We prove the second claim. (The proof of the first one is similar.)

If $\varphi \in J_{k,L;1}(\chi)$ then

$$
\tilde{\varphi}_{1/2}(Z) = \varphi(\tau, \mathbf{z}) e^{\pi \text{i} \omega} = \tilde{\varphi} \circ \pi_{2t}(Z), \quad Z \in \mathcal{H}(L(2t))
$$

is a holomorphic function on $\mathcal{H}(L(2t))$. We add index $2t$ to $\{A\}$ and $h$ in order to indicate the elements of the Jacobi groups $\Gamma^J(L(2t))$. First, we see that $I_{2t} \{A\} I_{2t}^{-1} = \{A\}$. Therefore

$$
\tilde{\varphi}_{1/2}|_{k \{A\} 2t}(Z) = \chi(A) \tilde{\varphi}_{1/2}(Z).
$$

Secondly, we have

$$
I_{2t}[x, y; r]_{2t} I_{2t}^{-1} = \left[ x, y; \frac{r}{2t} \right] = \left[ x, y; \frac{1}{2}(x, y) \right] \cdot \left[ 0, 0; \frac{r-t(x, y)}{2t} \right] \quad \text{for } x, y \in L.
$$

Therefore

$$
\tilde{\varphi}_{1/2}|_{k \{x, y; r\} 2t}(Z) = \tilde{\varphi}|_{k} \left( \left[ x, y; \frac{1}{2}(x, y) \right] \cdot \left[ 0, 0; \frac{r}{2t} - \frac{1}{2}(x, y) \right] \right)(\pi_{2t}(Z))
$$

$$
= \chi \left( \left[ x, y; \frac{1}{2}(x, y) \right] \right) e^{\pi \text{i}((r-t(x, y))}|_{\tilde{\varphi}_{1/2}(Z)} \quad \text{where } r - t(x, y) \in \mathbb{Z}.
$$

It means that $\tilde{\varphi}_{1/2}$ is an (extended) Jacobi form of index $\frac{1}{2}$ with the same $\text{SL}_2(\mathbb{Z})$- and Heisenberg characters with respect to the even lattice $L(2t)$. \hfill \Box

In the Definitions 2.2 and 2.2' and in the proof of the last proposition, we used two interpretations of Jacobi forms as a function on $\mathbb{H}_1 \times (L \otimes \mathbb{C})$ and on the tube domain $\mathcal{H}(L)$. For any $\tau = u + iv \in \mathbb{H}_1$ and $\mathbf{3} \in L \otimes \mathbb{C}$, we can find $\omega = u_1 + i v_1 \in \mathbb{H}_1$ such that $2v_1 v - (\text{Im}(\mathbf{3}), \text{Im}(\mathbf{3})) > 0$ or, equivalently, such that $(\omega, \mathbf{3}, \tau) \in \mathcal{H}(L)$. In the next lemma, we fix an independent part of this parameter $\omega$.

**Lemma 2.4** Let $Z = \mathbf{1}(\omega, \mathbf{3}, \tau) \in \mathcal{H}(L)$. Then the quantity

$$
\tilde{v}(Z) = v_1 - \frac{(\text{Im}(\mathbf{3}), \text{Im}(\mathbf{3}))}{2v} > 0
$$

is invariant with respect to the action of the real Jacobi group $\Gamma^J(L \otimes \mathbb{R})$.

**Proof** For any $Z = X + iY \in \mathcal{H}(L)$ we consider its image $[Z] = [X + iY] = [\text{pr}(Z)] = [\mathbf{1}(\omega - \mathbf{3}/2, \mathbf{Z}, 1)]$ in the projective homogeneous domain $\mathcal{D}(L_2)$.

For any $M \in \text{O}^+(L_2 \otimes \mathbb{R})$, we have

$$
(M \mathbf{Z}, M \mathbf{Z}) = (\mathbf{Z}, \mathbf{Z}) = 2(Y, Y)_1 = 2(2v_1 \cdot v - (\text{Im}(\mathbf{3}), \text{Im}(\mathbf{3}))) = 4v \cdot \tilde{v}(Z).
$$
By the definition of the action of the group $O^+ (L_2 \otimes \mathbb{R})$ on the tube domain, we have

$$4v \cdot \tilde{v}(Z) = 2(Y, Y)_1 = (MZ, MZ) = J(M, Z) \cdot \overline{J(M, Z)}(MZ, MZ)$$

$$= 2|J(M, Z)|^2 (Y(MZ)), Y(MZ))_1 = 4|J(M, Z)|^2 v(MZ) \cdot \tilde{v}(MZ)$$

$$= 4v \cdot \tilde{v}(MZ).$$

**Remark** The quantity $\tilde{v}(Z) \in \mathbb{R}_{>0}$ is a free part of the variable $Z$ in the extended Jacobi form $\tilde{\varphi}(Z)$:

$$(\tau, Z) \mapsto \iota \left( x_1 + i \left( \tilde{v} + \frac{(\text{Im}(3), \text{Im}(3))}{2v} \right), \cdot, \tau \right) \in \mathcal{H}(L).$$

The Jacobi forms with respect to a lattice $L$ form a bigraded ring $J_{s, L; t}$ with respect to weights and indexes. In the next proposition, we define a direct (or tensor) product of two Jacobi forms. Its proof follows directly from the definition.

**Proposition 2.5** Let $\varphi_1 \in J_{k_1, L_1; t} (\chi_1 \times v_1)$ and $\varphi_2 \in J_{k_2, L_2; t} (\chi_2 \times v_2)$ two Jacobi forms of the same index $t$. Then

$$\varphi_1 \otimes \varphi_2 (\tau, Z_1, Z_2) := \varphi_1 (\tau, Z_1) \cdot \varphi_2 (\tau, Z_2) \in J_{k_1 + k_2, L_1 \oplus L_2; t} (\chi_1 \chi_2 \times v_1 v_2)$$

where $v_1 v_2$ is a character of the group $H (L_1 \oplus L_2)$ defined by

$$(v_1 v_2)([x, y; r]) = v_1 \left( \left[ x_1, y_1; \frac{1}{2} (x_1, y_1) \right] \right) v_2 \left( \left[ x_2, y_2; \frac{1}{2} (x_2, y_2) \right] \right) e^{i \tau (x_1 y_1 + x_2 y_2 + 2r)}$$

for $[x, y; r] = [x_1 \oplus x_2, y_1 \oplus y_2; r] \in H (L_1 \oplus L_2)$. The tensor product of two Jacobi forms is a cusp form if at least one of them is a cusp form.

It is known (see [10]) that the space $J_{k, L; t} (\chi)$ is trivial if $k < \frac{n_0}{2}$ where rank $L = n_0$. The minimal possible weight $k = \frac{n_0}{2}$ is called singular weight. For any Jacobi form $\varphi$ of singular weight the hyperbolic norm of the index of a non-zero Fourier coefficient $f(n, l)$ (see (11)) is equal to zero: $2nt - (l, l) = 0$, i.e. $\text{Ord}(\varphi) = 0$.

**Example 2.6** (The Jacobi theta-series) The Jacobi theta-series of characteristic $(\frac{1}{2}, \frac{1}{2})$ (see [29]) is defined by

$$\vartheta (\tau, z) = \sum_{n \in \mathbb{Z}} \left( -\frac{4}{n} \right) q^{\frac{1}{2} n^2} r^2 = -q^{1/8 r^{-1/2}} \prod_{n \geq 1} (1 - q^n r^{-1})(1 - q^n r^{-1})(1 - q^n)^2.$$ (14)

where $q = e^{2 \pi i \tau}, \tau \in \mathbb{H}_1$ and $r = e^{2 \pi i z}, z \in \mathbb{C}$. This is the simplest example of Jacobi form of half-integral index. The theta-series $\vartheta$ satisfies two functional equations

$$\vartheta (\tau, z + x \tau + y) = (-1)^{x+y} e^{-\pi i (x^2 + y^2)} \vartheta (\tau, z), \quad (x, y) \in \mathbb{Z}^2$$

and

$$\vartheta \left( \frac{z}{c \tau + d} \right) = \vartheta (A \tau, \frac{z}{c \tau + d}) = v_3^b (A) (c \tau + d)^{\frac{1}{2}} e^{\pi i \frac{cz^2}{c \tau + d}} \vartheta (\tau, z), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2 (\mathbb{Z})$$
where \( v_\eta \) is the multiplier system of the Dedekind \( \eta \)-function. Using our notations, \( A_1 = (2) \), we have

\[
\vartheta \in J_{1/2, A_1; 1/2} \left( v_\eta^3 \times v_H \right)
\]

where, for short, \( v_{H(A_1)} = v_H \) is defined by:

\[
v_H ([x, y; r]) = (-1)^{x+y+x+y+r}, \quad [x, y; r] \in H(A_1) = H(\mathbb{Z}).
\]

The Jacobi theta-series \( \vartheta \) is the Jacobi form of singular weight \( 1/2 \) with a non-trivial character of the Heisenberg group. This Jacobi form was not mentioned in [8] but it plays an important role in the construction of the basic Jacobi forms and reflective modular forms (see [13, 14, 19]). We remind that

\[
\text{div } \vartheta = \{ h(z = 0) \mid h \in H(\mathbb{Z}) \} = \{ z = x\tau + y \mid x, y \in \mathbb{Z} \}.
\]

The Jacobi theta-series \( \vartheta \) having the triple product formula (14) will be the first main function in our construction of Jacobi forms for orthogonal lattices.

**Example 2.7 (Jacobi forms of singular weight for \( mA_1 \))** Using the Jacobi theta-series, we can construct Jacobi forms of singular weight for \( mA_1 = (2) \oplus \cdots \oplus (2) \). The tensor product of \( m \) Jacobi theta-series is a Jacobi form of singular weight and index \( 1/2 \) for \( mA_1 \):

\[
\vartheta_{mA_1} (\tau, z_1, \ldots, z_m) = \prod_{1 \leq j \leq m} \vartheta (\tau, z_j) \in J_{m, mA_1; 1/2} \left( v_{mA_1}^3 \times v_H^m \right)
\]

where

\[
v_{H}^m ([x, y; r]) = v_{H(mA_1)} ([x, y; r]) = (-1)^{r + \sum_{j=1}^{m} x_j + y_j + x_j y_j}
\]

for any \( x_j, y_j \) and \( r \) in \( \mathbb{Z} \). For even \( m \), we can construct Jacobi forms of singular weight and index 1 because

\[
\vartheta_{mA_1}^{(1)} (\tau, z_1, z_2) = \vartheta (\tau, z_1 + z_2) \cdot \vartheta (\tau, z_1 - z_2) \in J_{1, 2A_1} (v_\eta^6).
\]

Taking different orthogonal decompositions of the lattice \( 8A_1 \), we obtain 105 Jacobi forms of weight 4 and index 1 with trivial character.

**Example 2.8 (Jacobi forms of singular weight for \( D_m \))** We recall the definition of the even quadratic lattice \( D_m \). (We denote by \( A_m, D_m, E_m \) the lattices generated by the corresponding root systems.) We use the standard Euclidian basis \( \langle e_i \rangle_{i=1}^{m} \) \((e_i, e_j) = \delta_{ij}) in \( \mathbb{Z}^m \). Then

\[
D_m = \left\{ (x_1, \ldots, x_m) \in \mathbb{Z}^m \mid x_1 + \cdots + x_m \in 2\mathbb{Z} \right\} \quad (m \geq 1)
\]

is the maximal even sublattice in \( \mathbb{Z}^m \). The theta-product (15) is a Jacobi form of index 1 for \( D_m \) with trivial Heisenberg character because the quadratic form \( x_1^2 + \cdots + x_m^2 \) is even on \( D_m \)

\[
\vartheta_{D_m} (\tau, z_m) = \vartheta (\tau, z_1) \cdots \vartheta (\tau, z_m) \in J_{2, D_m} (v_\eta^m).
\]

We note that \( D_2 \cong 2A_1 \) and \( \vartheta_{D_2} = \vartheta_{2A_1}^{(1)} \). For the lattice \( D_4 \) we can give two more examples:

\[
\vartheta_{D_4}^{(2)} (\tau, z_4) = \vartheta \left( \tau, \frac{-z_1 + z_2 + z_3 + z_4}{2} \right) \vartheta \left( \tau, \frac{z_1 - z_2 + z_3 + z_4}{2} \right).
\]
\[
\times \vartheta\left(\tau, \frac{z_1 + z_2 - z_3 + z_4}{2}\right)\vartheta\left(\tau, \frac{z_1 + z_2 + z_3 - z_4}{2}\right) \in J_{2, D_4}(v^{12}_\eta) \quad (18)
\]

and
\[
\vartheta^{(3)}_{D_4}(\tau, 3_4) = \vartheta\left(\tau, \frac{z_1 + z_2 + z_3 + z_4}{2}\right)\vartheta\left(\tau, \frac{z_1 + z_2 - z_3 - z_4}{2}\right)
\times \vartheta\left(\tau, \frac{z_1 - z_2 - z_3 + z_4}{2}\right)\vartheta\left(\tau, \frac{z_1 - z_2 + z_3 - z_4}{2}\right) \in J_{2, D_4}(v^{12}_\eta). \quad (19)
\]

Analyzing the divisors of the Jacobi forms, we obtain the relation
\[
\vartheta_{D_4}(\tau, 3_4) = \vartheta^{(2)}_{D_4}(\tau, 3_4) + \vartheta^{(3)}_{D_4}(\tau, 3_4).
\]

**Jacobi forms and Weil representation** The Jacobi forms can be considered as vector-valued \(SL_2(\mathbb{Z})\)-modular forms (see [3, 11, 26, 30, 32]) related to the Weil representation. To compare the examples considered above with vector-valued modular forms, we recall the definitions from [11] for Jacobi forms of index one.

Let \(L\) an even positive definite lattice of rank \(n_0\) and
\[
\varphi(\tau, 3) = \sum_{n \in \mathbb{Z}, \ l \in L^\vee} f(n, l)e^{2i\pi(n\tau + \langle l, 3 \rangle)} \in J_{k,L}
\]
a Jacobi form of weight \(k\) and index one. By \(q = q(L)\) we denote the level of the lattice \(L\), i.e. the smallest integer such that \(L^\vee(q)\) is an even lattice. Then we have the following representation (see [11, Lemma 2.3] with \(m = 1\))
\[
\varphi(\tau, 3) = \sum_{\mu \in D(L)} \phi_\mu(\tau)\theta^L_\mu(\tau, 3)
\]
where \(D(L) = L^\vee/L\) is the discriminant group of \(L\),
\[
\phi_\mu(\tau) = \sum_{r \geq 0} f_h(r) \exp\left(\frac{2\pi i r}{q} \frac{r}{\tau}\right), \quad f_\mu(r) = f\left(\frac{2r + \langle \mu, \mu \rangle - 2}{q}, \mu\right)
\]
and
\[
\theta^L_\mu(\tau, 3) = \sum_{l \in \mu + L} e^{i\pi\langle (l, l) + 2\langle l, 3 \rangle \rangle}
\]
is the theta-series with characteristic \(\mu\). For any matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\), the theta-vector \(\Theta_L(\tau, 3) = (\theta^L_\mu(\tau, 3))_{\mu \in D(L)}\) has the following transformation property
\[
\Theta_L\left(\frac{a\tau + b}{c\tau + d}, \frac{3}{c\tau + d}\right) = (c\tau + d)\eta_0^q U(M) \exp\left(\frac{\pi ic(3, 3)}{c\tau + d}\right) \Theta_L(\tau, 3)
\]
where \(U(M)\) is a unitary matrix. In particular, for \(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) and \(S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) we have
\[
U(T) = \text{diag}(e^{i\pi\langle \mu, \mu \rangle})_{\mu \in D(L)}, \quad U(S) = (-i)^{\eta_0} \left(\sqrt{|D(L)|}\right)^{-\frac{1}{2}} (e^{-2i\pi\langle \mu, \nu \rangle})_{\mu, \nu \in D(L)}.
\]
Therefore $\Phi(\tau) = (\phi_\mu(\tau))_{\mu \in D(L)}$ is a holomorphic vector-valued modular form of weight $k - \frac{n_0}{2}$ for the conjugated representation $\overline{U(M)}$ of $SL_2(\mathbb{Z})$. In particular, the weight of any holomorphic Jacobi form is greater or equal to $\frac{n_0}{2}$ (see [10]). We note also that the theta-series $\theta_\mu^L$ are linearly independent and $\Theta_L$ is invariant with respect to the action of the stable orthogonal group $\tilde{O}(L)$ (see Theorem 3.2 below) and, in particular, with respect to the Weyl group of the lattice $L$, $W_2(L)$, which is a subgroup of $\tilde{O}(L)$ generated by 2-reflections in the lattice $L$.

We get the simplest example for an even unimodular lattice $N$ of rank $n_0$ (see [10] and [11, Lemma 4.1])

$$\Theta_N(\tau, \mathcal{Z}) = \sum_{l \in N} e^{\pi i (l, l) \tau + \pi i (l, \mathcal{Z})} \in J_{n_0/2, N}.$$ (20)

Moreover we have that two linear spaces of Jacobi forms are isomorphic

$$J_{k_1, L_1} \cong J_{k_1 + \frac{n_2 - n_1}{2}, L_2}$$

if $L_1$ (rank $L_1 = n_1$) and $L_2$ (rank $L_2 = n_2$) are two lattices with isomorphic discriminant forms (see [11, Lemma 2.4]).

**Example 2.9 (Weil representation for $D_m$)** Recall that $|D_m^\vee / D_m| = 4$ and

$$D_m^\vee / D_m = \{ \mu_i, i \mod 4 \} = \left\{ 0, \frac{1}{2} (e_1 + \cdots + e_m), e_1, \frac{1}{2} (e_1 + \cdots + e_{m-1} - e_m) \mod D_m \right\}$$

is the cyclic group of order 4 generated by $\mu_1 = \frac{1}{2} (e_1 + \cdots + e_m) \mod D_m$, if $m$ is odd, and the product of two groups of order 2, if $m$ is even. We have the following matrix of inner products in the discriminant group of $D_m$ of the non-trivial classes modulo $D_m$

$$((\mu_i, \mu_j))_{i,j \neq 0} = \begin{pmatrix} m/4 & 1/2 & m/4 - 2/4 \\ 1/2 & 1 & 1/2 \\ m/4 - 2/4 & 1/2 & m/4 \end{pmatrix} (\mu_i \in D_m^\vee / D_m)$$

where the diagonal elements are taken modulo 2$\mathbb{Z}$ and the non-diagonal elements are taken modulo $\mathbb{Z}$. We note that the discriminant group of $D_m$ depends only on $m \mod 8$. This gives the formulae for $U(T)$ and $U(S)$.

(1) For $m \equiv 4 \mod 8$, we have

$$U(T) = \text{diag}(1, -1, -1, -1), \quad U(S) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$  

We put $\theta_{D_m}^i(\tau, \mathcal{Z}_m) := \theta_{D_m}^i(\tau, \mathcal{Z}_m)$ for $i \mod 4$. The matrices $U(T)$ and $U(S)$ have the three common eigenvectors:

$$\theta_1^{D_m} - \theta_3^{D_m}, \quad \theta_1^{D_m} - \theta_2^{D_m}, \quad \theta_2^{D_m} - \theta_3^{D_m} \quad (m \equiv 4 \mod 8).$$

If $m = 4$ we get the Jacobi forms $\vartheta_{D_4}$, $\vartheta_{D_4}^{(1)}$ and $\vartheta_{D_4}^{(2)}$ obtained above as theta-products.
(2) For \( m \equiv 0 \mod 8 \), we have

\[
U(T) = \text{diag}(1, 1, -1, 1), \quad U(S) = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

These lattices have again two linearly independent common eigenvectors. The first one is the theta-product \( \vartheta_{D_m} = \theta_{D_m} - \theta_{D_m}^3 \). The second eigenvector \( \vartheta_{D_m}^+ = \theta_{D_m}^0 + \theta_{D_m}^3 \) is equal to the Jacobi theta-series of the unimodular lattice \( D_m^+ = \langle D_m, \mu_1 \rangle \). In particular, \( D_8^+ = E_8 \) and \( \vartheta_{D_8} = \Theta_{E_8} \) (see (20)). To understand better the role of the Jacobi theta-series \( \vartheta \), we consider one more case.

(3) For \( m \equiv 1 \mod 8 \), we have

\[
U(T) = \text{diag}\left(1, e^{i \pi/4}, -1, e^{i \pi/4}\right), \quad U(S) = \frac{1}{2} e^{-i \pi/4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
i & i & -1 & -i
\end{pmatrix}
\]

and \( \vartheta_{D_m} = \theta_{D_m}^0 - \theta_{D_m}^3 \) is the only Jacobi form of singular weight. Moreover, for \( m = 1 \) we get

\[
\vartheta_{D_1}(\tau, z) \in J_{\frac{1}{2}}(v_3^3) = J_{\frac{1}{2}}(v_3^3) = J_{\frac{1}{2}}(v_3^3).
\]

The last space is the space of classical Jacobi forms of weight \( \frac{1}{2} \), index 2 with the multiplier system \( v_3^3 \). It is easy to check that \( \vartheta_{D_1}(\tau, 0) = \vartheta_{D_1}(\tau, \frac{1}{2}) = 0 \). Therefore

\[
\vartheta_{D_1} = \vartheta(\tau, 2z).
\]

(4) Analyzing \( U(T) \) and \( U(S) \) for all other \( m \) modulo 8, we get only one common eigenvector corresponding to the theta-product \( \vartheta_{D_m} = \theta_{D_m}^0 - \theta_{D_m}^3 \). Therefore Example 1.8 contains all possible Jacobi forms of singular weight (and index one) for \( D_m \).

**Example 2.10** (The lattice \( E_6 \)) Let \( E_6^\vee \) be the dual lattice of \( E_6 \) and \( D(E_6) \) its discriminant group. We have

\[
D(E_6) = E_6^\vee/E_6 \simeq \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad q_{D(E_6)} = -q_{D(A_2)}.
\]

The discriminant group has the following system of representatives (see [4], Planche V): \( D(E_6) = \{0, \mu, 2\mu\} \) where \( \mu^2 = \frac{4}{3} \mod 2\mathbb{Z} \). We have

\[
U(T) = \text{diag}(1, \rho, \rho^2), \quad U(S) = \frac{i}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \rho^2 & 1 \\
1 & \rho & \rho^2
\end{pmatrix}
\]

with \( \rho = e^{2i \pi/3} \). We get

\[
\theta_{E_6}(\tau, \zeta_6) = (\theta_1 - \theta_2)(\tau, \zeta_6) \in J_{3,E_6}(v_3^{16}).
\]

This Jacobi form is invariant with respect to the Weyl group \( W(E_6) \).
The simple construction of Jacobi forms using products of Jacobi theta-series has a lot of advantages. First, we get Jacobi forms of singular weight with a very simple divisor. Second, we can easily determine the maximal group of symmetries with respect to the abelian variable. This fact is important in the next section in which we construct modular forms of singular weight with respect to orthogonal groups.

Example 2.11 (The Jacobi theta-series $\vartheta_{3/2}$) We can get more examples using the second theta-series of weight $1/2$ and index $3/2$ with respect to the full modular group $\text{SL}_2(\mathbb{Z})$. This function is related to twisted affine Lie algebras and is important in the construction of basic reflective Siegel modular forms (see [19])

$$\vartheta_{3/2}(\tau, z) = \frac{\eta(\tau) \vartheta(\tau, 2z)}{\vartheta(\tau, z)} \in J_{1/2, A_1; 3/2}(v_\eta \times v_H)$$

which is given by the quintuple product formula

$$\vartheta_{3/2}(\tau, z) = \sum_{n \in \mathbb{Z}} \left(\frac{12}{n}\right) q^{\frac{3}{2}r^2}$$

$$= q^{\frac{3}{2}r^{-1}} \prod_{n \geq 1} (1 + q^{n-1}r)(1 + q^{n}r^{-1})(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2})(1 - q^n).$$

We have

$$\vartheta_{mA_1}(3)(\tau, z_1, \ldots, z_m) = \prod_{1 \leq j \leq m} \vartheta_{3/2}(\tau, z_j) \in J_{m, (6); 1}(v_\eta \times v_H)$$

(we recall that $m \langle 6 \rangle = mA_1(3)$ denotes the orthogonal sum of $m$ copies of the lattice $\langle 6 \rangle$ of rank one). The same theta-product can be considered as a Jacobi form of index 1 for the lattice $D_m(3)$ and

$$\vartheta_{D_m(3)}(\tau, 3_m) = \vartheta_{3/2}(\tau, z_1) \cdots \vartheta_{3/2}(\tau, z_m) \in J_{m, D_m(3)}(v_\eta^m)$$

where $D_m(3)$ is the lattice $D_m$ renormalized by 3. In this simple way, we construct examples of Jacobi forms of singular weight with trivial character for even $n_0 \geq 8$: $D_8$, $D_7 \oplus D_5(3)$, $D_6 \oplus D_6(3)$, $D_5 \oplus D_9(3)$ and so on (see Proposition 4.6).

3 The lifting of Jacobi forms of half-integral index

The lifting of the Jacobi form $\vartheta_{D_8}$ (see (17)) is a reflective modular form with respect to the orthogonal group $O^+(2U \oplus D_8(-1))$ (see [14]) which is equal to the Borcherds–Enriques automorphic discriminant $\Phi_4$ of the moduli space of the Enriques surfaces introduced in [2]. The lifting of the Jacobi form determined the unique canonical differential form on the modular variety of the orthogonal group $\text{SO}_0^+(2U \oplus D_8(-1))$ having Kodaira dimension 0. In [14], there were found three such modular varieties of dimension 4, 6 and 7. The cusp form of the modular variety of dimension 4 is defined by a Jacobi form of half-integral index with a character of order 2.
Modular forms of orthogonal type and Jacobi theta-series

In this section, we give a variant of the lifting of Jacobi forms of half-integral index with a character. This theorem is a generalization of Theorem 3.1 in [11] (the case of Jacobi forms of orthogonal type with trivial character) and Theorem 1.12 in [19] (the case of Siegel modular forms with respect to a paramodular group of genus 2). All these constructions are particular cases of Borcherds additive lifting (see [3, Sect. 14]) of vector-valued modular forms. Nevertheless, our approach related to Jacobi forms gives in a natural way many new important examples of reflective modular forms for orthogonal groups. The Theorem 3.2 is a necessary tool for this purpose.

We can define a Hecke operator which multiplies the index of Jacobi forms. This operator is similar to the operator \( V_m \) of [8] or to the ‘minus’-Hecke operator introduced in [11, 12] in the case of Siegel modular forms of arbitrary genus or for the modular forms for orthogonal groups. We apply such operators to elements of \( J_{k,L,t}(v^D_\eta \times \nu) \) where \( \nu \) is a binary character of the minimal integral Heisenberg group \( H_s(L) \).

**Proposition 3.1** Let \( \phi \in J_{k,L,t}(v^D_\eta \times \nu) \) not identically zero. We assume that \( k \) is integral, \( t \) is rational and \( D \) is an even divisor of 24. If \( Q = \frac{24}{D} \) is odd, we assume that the character of the minimal integral Heisenberg group \( \nu : H_s(L) \rightarrow \{ \pm 1 \} \) is trivial. Then, for any natural \( m \) coprime to \( Q \), the function

\[
\varphi_{k,t} T^{(Q)}(m) = \sum_{a d = m, \sigma_a \equiv 0 \mod Q} a^k v^D_\eta(\sigma_a) \varphi\left( \frac{a^2 + bQ}{d}, a \right),
\]

where \( \sigma_a \in SL_2(\mathbb{Z}) \) such that \( \sigma_a \equiv \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right) \mod Q \), belongs to \( J_{k,L,m,\tau}(v^D_\eta \times \nu) \). The new \( SL_2(\mathbb{Z}) \)-character is defined as follows:

\[
v^D_{\eta,m}(A) = v^D_\eta(A_m) \quad \text{for all} \quad A \in SL_2(\mathbb{Z})
\]

with \( A_m \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & m \end{array} \right) \mod Q \). The character \( v^D_{\eta,m} \) depends only on \( m \mod Q \).

**Proof** It is known that \( \text{Ker} \ v^D_\eta \) contains the principle congruence subgroup \( \Gamma(Q) < SL_2(\mathbb{Z}) \) (see [19, Lemma 1.2]). We consider the following subgroup \( \Gamma^J(Q) \simeq \Gamma(Q) \times \text{Ker}(\nu) \) of the Jacobi group. We identify it with the corresponding parabolic subgroup in the orthogonal group \( SO^+(2U \oplus L(-1)) \). For \( (m, Q) = 1 \), let

\[
T^{(Q)}(m) = \Gamma(Q) \sum_{a d = m, \sigma_a \equiv 0 \mod Q} \sigma_a \left( \begin{array}{cc} a & bQ \\ 0 & d \end{array} \right)
\]

be the usual Hecke operator for \( \Gamma(Q) \). To the element \( T^{(Q)}(m) \), we associate the element \( T_+^{(Q)}(m) \) of the Hecke ring of the parabolic subgroup (see [11] and [19])

\[
T_+^{(Q)}(m) = \Gamma^J(Q) \sum_{a d = m, \sigma_a \equiv 0 \mod Q} \{ \sigma_a \} M_{a,b,d}
\]

where \( M_{a,b,d} = \text{diag}\left( \left( \begin{array}{cc} a & bQ \\ 0 & d \end{array} \right), 1, m^{-1} \left( \begin{array}{cc} a & bQ \\ 0 & d \end{array} \right) \right) \). This is a sum of some double cosets with respect to \( \Gamma^J(Q) \). We consider the extended Jacobi form \( \tilde{\varphi}(Z) = \varphi(\tau, Z) e^{2\pi i t_0} \) with \( Z = \)
'(ω, τ, τ) ∈ ℋ(L) which is modular with respect to the parabolic subgroup. Then we have

\[ \tilde{\psi}(Z) = (\tilde{\varphi}_k T_\omega^{(Q)}(m))(Z) = \sum_{a d = m, a > 0} \sum_{b \mod d} (\tilde{\varphi}_k \sigma_a) M_{a,b,d}(Z). \]

By definition, we have

\[ (\tilde{\varphi}_k \sigma_a) M_{a,b,d}(Z) = a^k v_d^0(\sigma_a) \varphi \left( \frac{a \tau + b Q}{d}, a \frac{\omega}{d} \right) e^{2\pi i m t \omega}. \]

Therefore, the Hecke operator of the proposition corresponds to the Hecke operator \( T_\omega^{(Q)}(m) \) of the parabolic subgroup \( \Gamma^J(Q) \) acting on the modular forms with respect to the parabolic subgroup \( \Gamma^J(Q) \). We remark that the new index of the extended function on \( \mathcal{H}(L) \) is equal to \( m t \). The case of modular transformations is similar to the theory of usual Hecke operators (see [31]). If \( A \in SL_2(\mathbb{Z}) \), then somewhat lengthy but easy calculations give us

\[ \tilde{\psi} |k[A] = \sum_{a, d = m, a > 0} \sum_{b \mod d} (\tilde{\varphi}_k \sigma_a) M_{a,b,d}(A) \tilde{\psi}. \]

This is due to the fact that the group \( \Gamma(Q) \) is normal in \( SL_2(\mathbb{Z}) \) and then

\[ \Gamma^J(Q) \{ A_m \}^{-1} \sigma_a M_{a,b,d} \{ A \} \neq \Gamma^J(Q) \{ A_m \}^{-1} \sigma_{a'} M_{a,b,d} \{ A \} \]

for distinct \( a \) and \( a' \) prime to \( Q \). Secondly, we consider the abelian transformations. Let \( h = \{ x, y; r \} \in H_s(L) \), then

\[ \tilde{\psi} |k[h] = \sum_{a d = m, a > 0} \sum_{b \mod d} v(h'_{a,b,d}) \tilde{\psi} |k[s_{a}] M_{a,b,d} \]

where \( h'_{a,b,d} = \{ \sigma_a \} (M_{a,b,d} \cdot h) \{ \sigma_a^{-1} \} = \{ x', y'; r' \} \) and

\[ \{ x', y'; r' \} = \{ (\delta d + \gamma b Q) x - a y, -(\beta d + \alpha b Q) x + a y; mr \} \in H(L) \]

with \( \sigma_a = \left( \begin{array}{cc} a & \beta \\ \gamma & d \end{array} \right) \equiv \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right) \mod Q \). We note that

\[ \{ x', y' \} = \{ m(\alpha \delta + \beta \gamma) + 2 \alpha \gamma ab Q \} \equiv m(\alpha \beta y; y) \mod 2 s(L). \]

Therefore, if \( \nu = \text{id} \) then

\[ \nu(\{ x', y'; r' \}) = e^{2\pi i t(mr - \frac{1}{2} m(x,y))} \]

because \( t \cdot s(L) \in \mathbb{Z} \). This proves the formula for odd \( Q \). If \( Q \) is even, we have \( \{ x', y'; r' \} = \{ mx + Q\tilde{x}, y + Q\tilde{y}; mr \} \). Then

\[ \left[ -Q\tilde{x}, -Q\tilde{y}; -\frac{Q^2}{2}(\tilde{x}, \tilde{y}) \right] \cdot \{ x', y'; r' \} = \left[ mx, y; mr + \frac{Q}{2}(-m(\tilde{y}, x') + m(\tilde{y}, x') - Q(\tilde{y}, \tilde{x})) \right]. \]
As $Q$ is even, we have $v([-Q\vec{x}, -Q\vec{y}; -\frac{Q^2}{2} (\vec{x}, \vec{y})]) = 1$. But, in this case, $m = 2m_0 + 1$ is odd so

$$v([x', y'; r']) = v([mx, y; mr]) = v([x, y; mr - m_0(x, y)]) = v([x, y; r]).$$

We calculate the Fourier expansion of $\varphi|_{k,l} T_\varphi(Q)(m)$ in the proof of Theorem 3.2 (see below). It shows that it is a holomorphic Jacobi form.

Let $L$ be an even lattice. The stable orthogonal group $\tilde{O}^+(L)$ is the subgroup of $O^+(L)$ whose elements induce the identity on the discriminant group $D(L) = L^\vee / L$

$$\tilde{O}^+(L) = \{ g \in O^+(L) | \forall l \in L^\vee : g(l) - l \in L \}.$$  

**Theorem 3.2** Let $\varphi \in J_{k,L; (v_n^D \times \nu)}$, $k$ be integral, $t$ be rational and $D$ be an even divisor of 24. If the conductor $Q = \frac{D}{24}$ is odd, we assume that $\nu$ is trivial. Fix $\mu \in (\mathbb{Z} / Q\mathbb{Z})^*$. Then the function

$$\text{Lift}_\mu(\varphi)(Z) = f(0, 0) E_k(\tau) + \sum_{m \equiv \mu \text{ mod } Q} m^{-1}(\varphi|_{k,L} T_\varphi(Q)(m)) \circ \pi_{Qt}(Z),$$

is a modular form of weight $k$ with respect to the stable orthogonal group $\tilde{O}^+(2U \oplus L(Qt))$ of the even lattice $L(Qt)$ with a character of order $Q$ induced by $v_n^D$, the binary Heisenberg character $\nu$ of $H_\nu(L(Qt))$ and the character $e^{2\pi i \frac{\mu}{Q}}$ of the center of $H(L(Qt))$. In the formula above, $f(0, 0)$ is the zeroth Fourier coefficient of $\varphi$, $E_k$ is the Eisenstein series of weight $k$ with respect to $SL_2(\mathbb{Z})$ and the map $\pi_{Qt}$ was defined in (13).

**Proof** The Eisenstein series $E_k$. First we note that $f(0, 0)$ could be non-zero only for the trivial character $v_n^D = \text{id}$. In this case, $\varphi(\tau, 0) = f(0, 0) + \cdots$ is a non-zero modular form of weight $k$ with respect to $SL_2(\mathbb{Z})$. Therefore $k \geq 4$ and $E_k$ is well defined. We note that $E_k$ is a Jacobi form of index 0.

The lattice $L(Qt)$. The lattice $L(t)$ is integral for a non zero Jacobi form $\varphi$. If $Q$ is odd then $L(t)$ is even because the character $\nu$ is trivial in this case (see Proposition 2.3). Therefore, for all $Q$, the lattice $L(Qt)$ is even.

The character of $\Gamma^J(L(Qt))$. According to Proposition 3.1

$$\varphi_m(\tau, 3) = (\varphi|_{k,l} T_\varphi(Q)(m))(\tau, 3) \in J_{k,L;mt}(v_n^D \times \nu).$$

We can defined an extended Jacobi form using the map $\pi_{Qt}$ (see (13)). According to Proposition 2.3

$$\varphi_m(\tau, 3)e^{2\pi i \frac{m\omega}{Q}} \in \tilde{J}_{k,L;Q}(v_n^D \times \nu)$$

is a modular form of weight $k$ with respect to the parabolic subgroup $\Gamma^J(L(Qt))$ of the orthogonal group $SO^+(2U \oplus L(-Qt))$. We note that the character $\nu$ of the minimal integral Heisenberg group $H_\nu(L(Qt))$ is extended to the center of $H(L(Qt))$ by the formula

$$v([0, 0; t]) = e^{2\pi i \frac{m}{Qt}} = e^{2\pi i \frac{m}{\nu}}.$$
If \( f(0, 0) \neq 0 \) then \( v_{\eta}^D = \text{id} \), i.e. \( D = 24 \), \( Q = 1 \) and \( \nu = \text{id} \). Therefore all terms in the sum defining the lifting \( \text{Lift}_\mu(\varphi) \) have the same character with respect to \( \Gamma^J(L(Q)t) < \tilde{SO}^+(2U \oplus L(−Q)t) \).

**Convergence.** Let \( Z = (\omega, (3, \tau)) \in \mathcal{H}(L(Q)t) \). The extended Jacobi form \( \tilde{\varphi}(Z) = \varphi(\tau, 3) \exp(2\pi i \omega) \) of index \( 1 \) is holomorphic at “infinity” \( (\text{Im} \omega \to +\infty) \). Therefore \( |\tilde{\varphi}| \) is bounded in any neighborhood of infinity (see [5] and [27]). We can rewrite this fact using the free parameter \( \tilde{\nu} = \tilde{\nu}(Z) > 0 \) from Lemma 2.4. Then we have

\[
|\varphi(\tau, 3)| \exp\left(-\frac{2\pi}{Q} \frac{(\text{Im}(3), \text{Im}(3))}{2\nu}\right) < C
\]

is bounded for \( \nu = \text{Im}(\tau) > \varepsilon \) and the exponential term does not depend on the action of \( \Gamma^J(L(Q)t) \). Using the action of \( \text{SL}_2(\mathbb{Z}) < \Gamma^J(L(Q)t) \), we obtain that

\[
|\varphi(\tau, 3)| \exp\left(-\frac{2\pi}{Q} \frac{(\text{Im}(3), \text{Im}(3))}{2\nu}\right) < C\nu^{-k}
\]

if \( \nu \leq \varepsilon \) (see [5, Sect. 2] for similar considerations). Now we can get an estimation of all terms in the sum for \( \text{Lift}_\mu(\varphi) \) for \( \nu > \varepsilon \). We have

\[
|a^k \varphi\left(\frac{a\tau + bQ}{d}, 3\right) \exp\left(-2\pi \frac{1}{Q} \left(\frac{(\text{Im}(a\delta), \text{Im}(a\delta))}{2va/d}\right)\right)| < Cd^{k\nu^{-k}}
\]

if \( \frac{a}{d} \nu \leq \varepsilon \). If \( \frac{a}{d} \nu > \varepsilon \) then we have \( < Ca^k \). In the both cases we see that the term above depending on \((a, b, d)\) is smaller than \( C_{\varepsilon} m^k \). It gives us

\[
|m^{-1}\varphi_m(\tau, 3)e^{2i\pi \frac{m}{Q}}| < C_{\varepsilon} m^k \sigma_0(m) \exp\left(-\frac{2\pi m}{Q} \tilde{\nu}\right) < C_{\varepsilon} m^{k+1} \exp\left(-\frac{2\pi m}{Q} \tilde{\nu}\right)
\]

where \( \tilde{\nu}(Z) > 0 \). Therefore the function \( \text{Lift}_\mu(\varphi) \) is well defined and it transforms like a modular form of weight \( k \) and character \( v_{\eta}^D \times \nu \times e^{2\pi i \frac{\mu}{Q} \tau} \) with respect to the parabolic subgroup \( \Gamma^J(L(Q)t) \).

**Fourier expansion of \text{Lift}_\mu(\varphi).** In the summation of the Fourier expansion of \( \varphi \in J_{k,L}(v_{\eta}^D \times \nu) \), we have \( n = \frac{D}{24} \mod \mathbb{Z} \) (see (11)). Rewriting \( n \) in terms of the conductor \( Q = \frac{24}{D} \), the Fourier expansion of the function \( \varphi \) has the following form

\[
\varphi(\tau, 3) = \sum_{n \equiv 1 \mod Q, n \geq 0} \sum_{l \in \frac{1}{2} \mathbb{L}, (l, l) \geq 0} f\left(\frac{n}{Q}, l\right) e^{2\pi i \left(\frac{n}{Q} \tau + (l, 3)\right)}.
\]

After the summation over \( b \mod d \) in the action of the Hecke operator, we get

\[
m^{-1}(\tilde{\varphi}|_k T_{\nu}(\delta)(m)) \circ \pi_{Qt}(Z)
\]

\[
= \sum_{a, d = m \mod \delta > 0} a^{k-1} v_{\eta}^D(\sigma_\alpha) \sum_{n \equiv 1 \mod Q, n \geq 0} \sum_{l \in \frac{1}{2} \mathbb{L}, (l, l) \geq 0} f\left(\frac{nd}{Q}, l\right) e^{2\pi i \left(\frac{n}{Q} \tau + a(l, 3) + \frac{ad}{Q} \omega\right)}
\]
So we have

$$\text{Lift}_\mu(\varphi)(Z) = \sum_{\substack{m \equiv \mu \mod Q \quad m \geq 1}} \sum_{\substack{a \geq 1 \quad a \mid m}} a^{-1} v_D^\mu(\sigma_a)$$

$$\times \sum_{\substack{nd \equiv 1 \mod Q \quad \ell \in \frac{1}{2}L^\lor \quad 2nd \ell - (l,l) \geq 0}} f\left(\frac{nd}{Q}, l\right) e^{2i\pi\left(\frac{na}{Q} \tau + a(l,3) + \frac{ad}{Q} \omega\right)}.$$ 

But \(nd \equiv 1 \mod Q \iff an \equiv \mu \mod Q\) because for any \((\mu, 24) = 1\) we have \(\mu^2 \equiv 1 \mod 24\). (We note that 24 is the maximal natural number with this property.) Using this property, we obtain the Fourier expansion of the lifting

$$\text{Lift}_\mu(\varphi)(Z) = \sum_{\substack{m,n \equiv \mu \mod Q \quad m,n \geq 1 \quad l \in \frac{1}{2}L^\lor \quad 2nm = (l,l) \geq 0}} \left( \sum_{\substack{a \mid (n,l,m) \quad \ell \in \frac{1}{2}L^\lor \quad 2nm \ell - (l,l) \geq 0}} f\left(\frac{nm}{Qa^2}, l\right) \right) e^{2i\pi\left(\frac{na}{Q} \tau + (al,3) + \frac{m}{Q} \omega\right)}.$$ 

We can reformulate the condition on the hyperbolic norm of the index \((n,l,m)\) of the Fourier coefficient in terms of the lattice \(L(Q^t)\):

$$2nmQ^2 - \frac{1}{Q^t} (l,l) \geq 0.$$ 

The formula for the Fourier expansion is symmetric with respect to \(\tau\) and \(\omega\). The involution \(V\) which permutes the isotropic vectors \(e_1\) and \(f_1\) in the second copy of the hyperbolic plane of the lattice \(U \oplus U_1 \oplus L(-Q^t)\) realizes the transformation \(\tau \leftrightarrow \omega\) and \(3 \leftrightarrow 3\). We see that \(V \in \tilde{O}^+(2U \oplus L(-Q^t))\), \(\det(V) = -1\), \(J(V, Z) = 1\) and

$$\text{Lift}_\mu(\varphi)|_k V = \text{Lift}_\mu(\varphi).$$

It is known (see [11, p. 1194] or [22, Proposition 3.4]) that

$$\tilde{O}^+(2U \oplus L(-Q^t)) \subseteq \langle \Gamma^J(L(Q^t)), V \rangle.$$ 

Therefore, the function \(\text{Lift}_\mu(\varphi)\) is a modular form of weight \(k\) with a character of order \(Q\) with respect to \(\tilde{O}^+(2U \oplus L(-Q^t)).\)

Remark to Theorem 3.2 If \(\mu = 1\) then \(\text{Lift}(\varphi) = \text{Lift}_1(\varphi) \neq 0\) because its first Fourier–Jacobi coefficient \(\tilde{\varphi}\) is not zero. For \(\mu \neq 1\) the function \(\text{Lift}_\mu(\varphi)\) might be identically zero. See [19, Example 1.15] for a non-zero \(\mu\)-lifting in the case of signature \((2, 3)\).

At the end of the section we give the first application of Theorem 3.2.

Example 3.3 (Modular forms of singular weight) The first example of such modular forms was given in [10]:

$$\text{Lift}(\Theta_{E_8}) = \frac{1}{240} + \sum_{\substack{n,m,l \geq 1 \quad \ell \in E_8 \quad 2nm\ell = (l,l) \quad (n,m) \neq (0,0)}} \sigma_3((n, \ell, m)) e^{2\pi i(n\tau + (\ell,3) + m\omega)} \in M_4(O^+(JI_2, 10))$$

where \(\sigma_3((n, \ell, m))\) is the sum of the cubes of all divisors of the greatest common divisor of \(n, m\) and \(\ell \in E_8\). This function is sometimes called the simplest modular form. Using
the theta-products (15)–(23), we can define modular forms of singular weight on orthogonal groups with a character induced by \(v^\mathbb{D}_n\)-character for all even divisors of 24. We give some examples below in order to illustrate different cases:

\[
\text{Lift}(\vartheta_{D_8}) \in M_4(\overline{O}^+(2U \oplus D_8(-1))), \quad \text{Lift}(\vartheta_{D_8(3)}) \in M_4(\overline{O}^+(2U \oplus D_8(-9)), \chi_3).
\]

\[
\text{Lift}(\vartheta_{4A_1}) \in M_2(\overline{O}^+(2U \oplus 4A_1(-1)), \chi_2), \quad \text{Lift}(\vartheta_{D_{24}(3)}) \in M_{12}(\overline{O}^+(2U \oplus D_{24}(-3))).
\]

\[
\text{Lift}(\vartheta_{2A_1}) \in M_4(\overline{O}^+(2U \oplus 2(-4)), \chi_4), \quad \text{Lift}(\vartheta_{D_{24}(3)}) \in M_4(\overline{O}^+(2U \oplus 2(-36)), \chi_{12})
\]

where \(\chi_n\) denotes a character of order \(n = \frac{24}{\ell}\) of the corresponding orthogonal group.

We note that in many cases the maximal modular group of the lifting is larger than the stable orthogonal group \(\overline{O}^+(2U \oplus L(-1))\). For example, the maximal modular group of \(\text{Lift}(\eta^d \vartheta_{D_m})\) for any \(d\) and \(m\) such that \(d + 3m \equiv 0 \text{ mod } 24\) is the full orthogonal group \(O^+(2U \oplus D_{m}(-1))\) if \(m \neq 4\). The form \(\text{Lift}(\eta^d \vartheta_{D_m})\) is anti-invariant with respect to the involution of the Dynkin diagram (the reflection with respect to a vector with square 4). If \(m = 4\) then

\[
O^+(2U \oplus D_4(-1)) \cap \overline{O}^+(2U \oplus D_4(-1)) \cong S_3.
\]

The liftings of \(\vartheta_{D_8}, \vartheta_{D_8}^{(2)}, \vartheta_{D_8}^{(3)}\) (see Example 1.8) are modular with respect to three different subgroups of order 3 in \(O^+(2U \oplus D_4(-1))\).

The lifting of any theta-products vanishes along the divisors of the corresponding Jacobi forms. In particular \(\text{Lift}(\vartheta_{4A_1})\) vanishes with order one along \(z_i = 0\). It is known that the full divisor of this modular form is equal to the union of all modular transformations of \(z_i = 0\), i.e. this is a singular reflective modular form with the simplest possible divisor (see [14]). The same is true for \(\text{Lift}(\vartheta_{D_8})\). The Fourier expansion of \(\text{Lift}(\vartheta_{4A_1})\) (or \(\text{Lift}(\vartheta_{D_8})\)) written in a fixed Weyl chamber of the corresponding orthogonal group will define generators and relations of Lorentzian Kac–Moody algebras (see [17–19] and a forthcoming paper of Gritsenko and Nikulin about reflective groups of rank \(\geq 4\)). Here we consider the formula for \(4A_1\) which was given without proof in [14].

**Example 3.4** (Jacobi lifting, the modular tower \(4A_1\) and modular forms of “Calabi–Yau type”) We consider the following theta-product as a Jacobi form of index \(\frac{1}{2}\)

\[
\vartheta_{4A_1}(\tau, 3_4) = \vartheta(\tau, z_1) \cdots \vartheta(\tau, z_4) \in J_{2,4A_1;\frac{1}{2}}(v^\mathbb{H}_4 \times v^\mathbb{D}_4).
\]

According to Theorem 3.2, we get

\[
\Phi_2(Z) := \text{Lift}(\vartheta_{4A_1})(\tau, 3_4, \omega) \in M_2(O^+(2U \oplus 4A_1(-1)), \chi_2)
\]

where \(\chi_2\) is a character of order 2 of the full orthogonal group. The modular form \(\Phi_2\) is reflective with the simplest possible divisor (see [14]). The Fourier expansion of this fundamental reflective form of singular weight is the following

\[
\Phi_2(Z) = \sum_{l=(l_1, \ldots, l_4), l_j = \frac{1}{2} \text{ mod } Z} \sum_{n, m \in \mathbb{Z}^2, \frac{nm}{nm - (\ell, \ell) = 0}} \sigma_1((n, \ell, m)) \left(\frac{-4}{2l_1}\right) \cdots \left(\frac{-4}{2l_4}\right) e^{\pi i(n\tau + (\ell, 3_4) + m\omega)}
\]
where $\sigma_1(n) = \sum_{d|n} d$. The quasi-pullbacks (see [21]) of $\Phi_2$ along the divisors are again reflective (see [14]). In this way we obtain the $4A_1$-tower of reflective modular forms in six, five, four and three variables with respect to $O^+(2U \oplus nA_1(-1))$ for $n = 4, 3, 2$ and $1$:

$$
\Phi_2 = \text{Lift}(\vartheta_{4A_1}), \quad \text{Lift}(\eta^3 \vartheta_{3A_1}),
$$

$$
K_4(\tau, z_1, z_2, \omega) := \text{Lift}(\eta(\tau)\vartheta(\tau, z_1)\vartheta(\tau, z_2)), \quad \Delta_5 = \text{Lift}(\eta^5(\tau)\vartheta(\tau, z))
$$

where $\Delta_5 \in S_5(Sp_2(\mathbb{Z}), \chi_2)$ is the Igusa modular form (a square root of the first Siegel cusp form of weight 10). The modular form $\Delta_5$ determines one of the most fundamental Lorentzian Kac–Moody algebras related to the second quantized elliptic genus of $K3$ surfaces (see [7, 17] and [13]). The modular form

$$
K_4 \in S_4(\tilde{SO}^+(2U \oplus 2A_1(-1)), \chi_2)
$$

is the second member of the modular $4A_1$-tower based on $\Delta_5$. This form defines an (elliptic) Lorentzian Kac–Moody algebra of signature $(1, 3)$ (see a forthcoming paper of Gritsenko and Nikulin). Moreover $K_4(Z)dZ$ is the only canonical differential form on the orthogonal modular variety

$$
M_{\chi_2}(2A_1) = \Gamma_{\chi_2} \setminus \mathcal{D}(2U \oplus 2A_1(-1))
$$

of complex dimension 4 and of Kodaira dimension 0 where $\Gamma_{\chi_2} = \ker(\chi_2)$ (see [14]). The first example of cusp forms of this type was considered in [16] where it was shown that the modular form

$$
\Delta_1 = \text{Lift}(\eta(\tau)\vartheta(\tau, z)) \in S_1(\tilde{O}^+(2U \oplus (-6)), \chi_6)
$$

determines the unique, up to a constant, canonical differential form $\Delta_1^3(Z)dZ$ on the Barth-Nieto modular Calabi–Yau three-fold. The second example of Siegel cusp forms of canonical weight with the simplest possible divisor was constructed in [5]:

$$
\nabla_3 = \text{Lift}(\eta(\tau)\eta(2\tau)^4\vartheta(\tau, z)) \in S_3(\Gamma_0^{(2)}(2), \chi_2)
$$

where $\Gamma_0^{(2)}(2) < Sp_2(\mathbb{Z})$ and $\chi_2$ is its character of order 2. A Calabi-Yau model of the Siegel modular three-fold $\Gamma_0^{(2)}(2)_{\chi_2} \setminus \mathbb{H}_2$ was found in [9]. The modular form in four variables $K_4$ is the next example of a cusp form of “Calabi–Yau type” similar to the Siegel modular forms $\Delta_1^3$ and $\nabla_3$.

Question 3.5 We can ask a question about the existence of a compact model of Calabi–Yau type of the modular variety $M_{\chi_2}(2A_1)$ of dimension 4 defined above.

4 Modular forms of singular and critical weights

The minimal possible weight (singular weight) of holomorphic Jacobi form for $L$ is $n_0/2$ where $n_0 = \text{rank} L$. The first weight for which Jacobi cusp forms might appear is equal to $n_0/2 + 1/2$. This weight is called critical. In the case of classical modular forms in one variable, the critical weight is equal to 1. The simplest possible example of modular forms of critical weight in our context is the cusp form $\Delta_1 = \text{Lift}(\eta\vartheta)$ of weight 1 with a character of order 6 for the lattice $2U \oplus (-6)$ of signature $(2, 3)$. We mentioned in Example 3.4 that this
function determines one of the basic Lorentzian Kac–Moody algebras in the Gritsenko–Nikulin classification (see [18, 19]) and it induces the unique canonical differential form on a special Calabi–Yau three-folds, the Barth–Nieto quintic. We can construct a simple example of modular form of critical weight with trivial character using Theorem 3.2. This is

\[ \text{Lift}(\eta \vartheta_D^{23}) \in M_{12}(\bar{O}^+ (2U \oplus D_{23}(-3))) \]

which is a modular form with trivial character with respect to the orthogonal group of signature \((2,25)\). In this section, we construct examples of Jacobi cusp forms of critical weight for all even ranks. For this aim, we use the pullback of Jacobi forms of singular weight such that its Fourier coefficient \( f(0,0) = 0 \). This is exactly the case of \( \vartheta_D^{23} \).

Let \( M < L \) be an even sublattice of \( L \). We can consider the Heisenberg group of \( M \) as a subgroup of \( H(L) \). Therefore if \( \text{rank}(M) = \text{rank}(L) \) then the Jacobi forms with respect to \( L \) can be considered as Jacobi forms with respect to \( M \). In the next proposition, we consider the operation of pullback.

**Proposition 4.1** Let \( M < L \) be a sublattice of \( L \) and \( \text{rank}(M) < \text{rank}(L) \)

\[ M \oplus M^\perp < L, \quad \exists = \exists_m \oplus \exists_{\perp} \in L \otimes \mathbb{C} = (M \oplus M^\perp) \otimes \mathbb{C}. \]

For any \( \varphi(\tau, \exists) \in J_{k,L}^1 (\chi \times \nu) \) its pullback is also a Jacobi form

\[ \varphi|_M := \phi(\tau, \exists_m) = \varphi(\tau, \exists)|_{(\exists_{\perp} = 0)} \in J_{k,M}^1 (\chi \times \nu|_{\Gamma_J(M)}). \]

The pullback of a Jacobi cusp form is a cusp form or 0.

**Proof** We note that the pullback of Jacobi form might be the zero-function. What is more interesting is that the pullback might be a cusp form although the original function is not.

The functional equations (9)–(10) are evidently true for \( \varphi|_M \). To calculate its Fourier expansion, we consider the embedding of the lattices

\[ M \oplus M^\perp < L < L^\vee < M^\vee \oplus (M^\perp)^\vee. \]

We have to analyze the \( M \)-projection of any vector \( l \) in \( \frac{1}{2}L^\vee = L(2)^\vee \) in the Fourier expansion (11). If the character \( \nu \) of the minimal Heisenberg group is trivial then we do not need the coefficient \( \frac{1}{2} \) before the lattices dual to \( L \) and \( M \) in the calculation below. For any \( l \in \frac{1}{2}L^\vee = L(2)^\vee \), we have the following decomposition

\[ l = l_m \oplus l_{\perp} = \text{pr}_{M(2)^\vee} (l) \oplus \text{pr}_{(M(2)^\perp)^\vee} (l) \in M(2)^\vee \oplus (M(2)^\perp)^\vee. \]

In the coordinates \( \exists = \exists_m \oplus \exists_{\perp} \), we have

\[ \varphi(\tau, \exists) = \sum_{n \geq 0, l = l_m \oplus l_{\perp}} f(n, l) e^{2\pi i (n\tau + (l_m, \exists_m) + (l_{\perp}, \exists_{\perp}))}. \]

Therefore

\[ \varphi|_M(\tau, \exists_m) = \sum_{n \geq 0, l_m \in M(2)^\vee} \left( \sum_{l_{\perp} \in (M(2)^\perp)^\vee_{l_m \oplus l_{\perp} \in L(2)^\vee}} f(n, l_m \oplus l_{\perp}) \right) e^{2\pi i (n\tau + (l_m, \exists_m))}. \]

We note that \( 2nt - (l_m, l_m) \geq (l_{\perp}, l_{\perp}) \geq 0 \). The last inequality is strict if \( \varphi \) is a cusp form. □
Using the operation of pullback, we can construct Jacobi cusp forms of critical weight starting from Jacobi forms of singular weight if the constant term $f(0,0)$ of the last one is equal to zero. The estimation on $2nt - (l_m, l_m)$ at the end of the proof of the last proposition gives us the following estimation of the order at infinity (see (12)) of the pullback.

**Corollary 4.2** In the conditions of Proposition 4.1 we have

$$\text{Ord}(\phi|_M(\tau, 3_m)) \geq \min\{(l_\perp, l_\perp) | l_\perp = \text{pr}_{(M_\perp)'}(l) \text{ such that } f(n, l) \neq 0\}.$$  

In particular, if $\text{pr}_{(M_\perp)'}(l) \neq 0$ for all $f(n, l) \neq 0$ then the pullback $\phi|_M$ is a cusp form or the zero-function.

Using the last corollary, we can construct new important examples of Jacobi forms of singular and critical weights. We recall that by $J_{k,L}$ we denote the space of Jacobi forms of index one.

We define the root lattice $A_m$ as a sublattice of $D_{m+1}$

$$A_m = \{(x_1, \ldots, x_{m+1}) \in \mathbb{Z}^{m+1} | x_1 + \cdots + x_{m+1} = 0\} < D_{m+1}.$$  

We note that $A_1 \cong \langle 2 \rangle$, $A_1 \oplus A_1 \cong D_2$ and $A_3 \cong D_3$.

**Proposition 4.3** (1) Let $v = 2(b_1, \ldots, b_m) \in \mathbb{Z}^m$ be an element in $D_m$ with at least two non-zero coordinates $b_i$ such that $(b_1 + \cdots + b_m) \equiv 1 \mod 2$ and g.c.d. $(b_1, \ldots, b_m) = 1$. Then

$$\vartheta_{D_m}|_{v_\perp} \in J_{3m}^{\text{cusp}}(v_3^m)$$

and

$$\vartheta_{D_m(3)}|_{v_\perp} \in J_{3m}^{\text{cusp}}(v_3^m)$$

is a non-zero Jacobi cusp form of critical weight such that

$$\text{Ord}(\vartheta_{D_m}|_{v_\perp}) = \frac{1}{\langle v, v \rangle} > 0 \quad \text{and} \quad \text{Ord}(\vartheta_{D_m(3)}|_{v_\perp}) = \frac{1}{3\langle v, v \rangle} > 0.$$  

(2) The theta-product

$$\vartheta_{A_m}(\tau, z_1, \ldots, z_m) = \vartheta(\tau, z_1) \cdots \vartheta(\tau, z_m) \cdot \vartheta(\tau, z_1 + \cdots + z_m) \in J_{\frac{m+1}{2},A_m}(v_3^{3m+3})$$

is a Jacobi form of critical weight. If $m$ is even then $\vartheta_{A_m}$ is a Jacobi cusp form and

$$\text{Ord}(\vartheta_{A_m}) = \frac{1}{4(m+1)} > 0.$$  

(3) For the renormalized lattice $A_m(3)$, the Jacobi form

$$\vartheta_{A_m(3)}(\tau, z_1, \ldots, z_m) = \vartheta_{3/2}(\tau, z_1) \cdots \vartheta_{3/2}(\tau, z_m) \cdot \vartheta_{3/2}(\tau, z_1 + \cdots + z_m)$$

belongs to $J_{\frac{m+1}{2},A_m(3)}(v_3^{m+1})$. For even $m$,

$$\text{Ord}(\vartheta_{A_m(3)}) = \frac{1}{12(m+1)} > 0.$$  

[ Springer]
Proof (1) If in \( v \) only one \( b_i \neq 0 \) then \( \vartheta_{D_m}|_{v^\perp} \equiv 0 \). To prove the lemma, we calculate the Fourier expansion of the pullback function. The discriminant group of \( D_m \) was given in Example 2.9. The Fourier expansion of \( \vartheta_{D_m} \) has the following form

\[
\vartheta_{D_m}(\tau, Z_m) = \sum_{n \in \mathbb{Q} > 0, \ell \in \mathbb{Z}^m} f(n, \ell) e^{2\pi i (n\tau + (\ell, Z_m))}
\]

where \( f(n, \ell) = \left( -\frac{4}{2\ell_1} \right) \cdots \left( -\frac{4}{2\ell_m} \right) \) is the product of the generalized Kronecker symbols modulo 4. In particular, all coordinates \( 2\ell_i \) are odd. If \( v \) is a vector which satisfies the condition of the proposition then \( (l, v) = 2(l_1 b_1 + \cdots + l_m b_m) \equiv 1 \mod 2 \). The lattice \( \langle v \rangle^\perp \) is generated by \( \frac{v}{(v, v)} \). We get that

\[
I_\perp = \text{pr}_{\langle v \rangle^\perp}(l) = (l, v) \frac{v}{(v, v)} \neq 0
\]

is always non trivial. Moreover, there exists a vector \( 2l = (2l_i) \) with odd coordinates such that \( (l, v) = 1 \). According to Corollary 4.2

\[
\text{Ord}(\vartheta_{D_m}|_{v^\perp}) = \left| \left( \frac{v}{(v, v)} \cdot \frac{v}{(v, v)} \right) \right| = \frac{1}{|v|} > 0
\]

and \( \vartheta_{D_m}|_{v^\perp} \) is a Jacobi cusp form. The proof for \( D_m(3) \) is quite similar.

(2) We have \( A_m = v_{D_{m+1}}^\perp \) where \( v = 2(1, \ldots, 1) \in D_{m+1} \). In particular \( z_{m+1} = -(z_1 + \cdots + z_m) \) and \( \vartheta_{A_m} = -\vartheta_{D_{m+1}}|_{v^\perp} \). If \( m \) is even then \( v \) satisfies the condition in (1) and \( \text{Ord}(\vartheta_{A_m}) = \frac{1}{4(m+1)} \). The proof of (3) is similar. \( \square \)

Example Since \( A_3 \cong D_3 \), there exist a Jacobi form of singular and two Jacobi forms (cusp and non-cusp) of critical weight for this lattice

\[
\vartheta_{A_3} \in J_{2, A_3}(v_8^9), \quad \eta \vartheta_{A_3} \in J_{2, D_3}^{\text{cusp}}(v_{10}^{10}), \quad \vartheta_{A_3} \in J_{2, A_3}(v_{12}^{12}).
\]

We can construct many Jacobi forms of singular, critical and other small weights using the equalities of the previous proposition. For any \( \varphi \in J_{k, L; t}(\chi) \), we denote by \( \varphi^{[n]} \) the direct (tensor) product of \( n \)-copies of \( \varphi \), i.e. the Jacobi form for the lattice \( nL \)

\[
\varphi^{[n]}(\tau, (Z_1, \ldots, Z_n)) = \varphi(\tau, Z_1) \cdots \varphi(\tau, Z_n) \in J_{nk, nL; t}(\chi^n).
\]

The next example is very important.

**Corollary 4.4** There exists a Jacobi form of singular weight for \( A_2 \)

\[
\sigma_{A_2}(\tau, z_1, z_2) = \frac{\vartheta(\tau, z_1) \vartheta(\tau, z_2) \vartheta(\tau, z_1 + z_2)}{\eta(\tau)} \in J_{1, A_2}(v_8^9).
\]

In particular, the Jacobi form of singular weight \( \sigma_{3 A_2} = \sigma_{A_2}^{[3]} \in J_{3, 3 A_2} \) has trivial character.
Proof We note that \( \eta(\tau)^{-1} = q^{-1/24} (1 + q(\ldots)) \). Therefore

\[
\text{Ord}\left( \frac{\varphi(\tau, \frac{3}{\eta(\tau)})}{\eta(\tau)} \right) = \text{Ord}(\varphi(\tau, 3)) - \frac{1}{12}.
\]

Thus \( \sigma_{A_2} \) is holomorphic Jacobi form of singular weight for \( A_2 \). \( \square \)

Remarks (1) The Jacobi form \( \sigma_{A_2} \) is equal to the denominator function of the affine Lie algebra \( A_2 \) (see [26] and [6]). We consider the Jacobi forms related to the denominator functions of all affine Kac–Moody Lie algebras in a forthcoming paper of V. Gritsenko and K.-I. Iohara.

(2) The lifting of \( \sigma_{A_2} \) is a reflective modular form of singular weight. The lifting of \( \eta^8 \sigma_{A_2} \) determined the unique canonical differential form on a modular variety of Kodaira dimension 0 (see [14]).

(3) The form \( \sigma_{A_2} \) is the first example of Jacobi form obtained as theta/eta-quotients. Using such Jacobi form, we can produce important classical Jacobi forms in one variable called theta-blocks, see Corollary 4.9 and [23].

Using the same principle, we obtain

**Corollary 4.5** The Jacobi forms given below are cusp forms of critical weight.

\[
\kappa_{2A_4} = \frac{\partial_{A_4} \otimes \partial_{A_4}}{\eta} \in J^\text{cusp}_{2, A_4}(v_5^5), \quad \kappa_{A_4 \oplus A_6} = \frac{\partial_{A_4} \otimes \partial_{A_6}}{\eta} \in J^\text{cusp}_{11, A_4 \oplus A_6}(v_7^{11}).
\]

Let \( v_5 = 2(2, 1, 0, \ldots, 0) \in D_m \) (\( m \geq 2 \)) and \( v_7 = 2(2, 1, 1, 0, \ldots, 0) \in D_n \) (\( n \geq 4 \)). Then

\[
\partial_{D_m\vert v_5^5} \otimes \partial_{D_n\vert v_7^7} \in J^\text{cusp}_{m+n, A_4 \oplus A_6}(v_7^{3(m+n)-1})
\]

where \( a = 5 \) or 7.

Proof According to Proposition 4.3

\[
\text{Ord}(\kappa_{2A_4}) = \frac{1}{60}, \quad \text{Ord}(\kappa_{A_4 \oplus A_6}) = \frac{1}{420}, \quad \text{Ord}(\partial_{D_m\vert v_5^5}) = \frac{1}{20},
\]

\[
\text{Ord}(\partial_{D_n\vert v_7^7}) = \frac{1}{28}.
\]

Remark In the same way, we get non-cusp Jacobi forms of weight \( \frac{n_0}{2} + 1 \) (singular weight +1):

\[
\frac{\partial_{A_4}^{[5]}}{\eta^5} \in J_{11,5, A_4}, \quad \frac{\partial_{A_6}^{[7]}}{\eta^3} \in J_{23,7, A_6}, \quad \frac{\partial_{A_8}^{[3]}}{\eta} \in J_{13,3, A_8}(v_7^8).
\]

The first two functions have trivial character. It might be that these functions are interesting Eisenstein series. We can also mention the non-cusp form

\[
\frac{\partial_{A_2(3)}^{[3]}}{\eta} \in J_{4,3, A_2(3)}(v_0^8).
\]

**Proposition 4.6** Theta-products give examples of Jacobi forms of singular weight with trivial character for some lattices of all even ranks \( \geq 6 \).
Proof The corresponding Jacobi forms are tensor products of the following Jacobi theta-products

\[ \sigma_{A_2} \quad \text{with character } v_4^8, \quad \vartheta_{D_m} \quad \text{with character } v_m^{3m}, \]

\[ \vartheta_{D_4}^{(i)} \quad \text{with character } v_m^{3m} \quad (i = 2, 3), \quad \vartheta_{2A_1}^{(i)} \quad \text{with character } v_4^6, \]

\[ \vartheta_{D_{m(3)}} \quad \text{with character } v_m^m. \]

See (24), (17), (18), (19), (16) and (23). Below, we give a list of lattices of rank smaller or equal to 24 since for larger ranks one can use the periodicity of the characters:

- \( n = 6, \quad 3A_2 \);
- \( n = 8, \quad D_8, \quad 2D_4, \quad 8A_1 \);
- \( n = 10, \quad D_7 \oplus D_3(3), \quad A_2 \oplus D_4 \oplus D_4(3) \);
- \( n = 12, \quad D_6 \oplus D_6(3), \quad 2A_2 \oplus D_8(3), \quad 6A_2 \);
- \( n = 14, \quad D_5 \oplus D_9(3) \);
- \( n = 16, \quad D_8 \oplus D_{12}(3) \);
- \( n = 18, \quad D_9 \oplus D_{15}(3) \);
- \( n = 20, \quad D_2 \oplus D_{20}(3) \);
- \( n = 22, \quad D_1 \oplus D_{21}(3) \);
- \( n = 24, \quad D_{24}, \quad 12A_2, \quad D_{24}(3) \).

We note that we consider \( 8A_1 \) as \( 4(A_1 \oplus A_1) \). The corresponding Jacobi form is the product of four functions of type \( \vartheta_{2A_1} \). Moreover, instead of any \( D_m \) in this list, we can put a direct sum \( D_{m_1} \oplus \cdots \oplus D_{m_k} \) with \( m_1 + \cdots + m_k = m \).

Jacobi forms of singular weight with respect to the full Jacobi group of a lattice \( L \) have a \( SL_2(\mathbb{Z}) \)-character of type \( v_m^{2m} \) if the rank of \( L \) is even (the singular weight is integral) or a multiplier system of type \( v_m^{2m+1} \) if the rank is odd (the singular weight is half-integral). Analyzing the examples of theta-products given above, we get the following table of possible characters \( v_m^m \)

| rank \( n \) | \( d \) : character of type \( v_\eta^d \) |
|---|---|
| 1 | 1, 3 |
| 2 | 2, 4, 6, 8 |
| 3 | 3, 5, 7, 9, 11 |
| 4 | 4, 6, 8, 10, 12, 14, 16 |
| 5 | 5, 7, 9, 11, 13, 15, 17, 19 |
| 6 | 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 |
| 7 | 7, 9, 11, 13, 15, 17, 19, 21, 23, 1, 3 |
| 8 | 8, 10, 12, 14, 16, 18, 20, 22, 24, 2, 4, 6 |

As corollary, we obtain

**Proposition 4.7** If \( n \geq 8 \) is even (respectively, \( n \geq 9 \) is odd) and \( d \equiv n \mod 2 \) then there exists a lattice \( L \) of rank \( n \) such that the space of Jacobi forms of singular weight \( J_{\mathbb{Z},L}(v_\eta^d) \) is not empty.

**Remark** For some \( n \), we can prove that this table contains all possible characters. We are planning to come to this question in another publication.
Now we would like to analyze Jacobi forms of critical weight. First, note that the multiplication by $\eta$ gives us the simplest such Jacobi form

$$\eta \vartheta_{D_{23}(3)} \in J_{12,D_{23}(3)}.$$ (25)

The tensor product of a Jacobi form of singular weight and Jacobi form of critical weight has critical weight for the corresponding lattice. In particular, there exist two simple series of Jacobi cusp forms with trivial character for the lattices $A_m \oplus D_n$ where $m$ is even and $m + n \equiv 7 \mod 8$

$$\vartheta_{A_m \oplus D_n} = \vartheta_{A_m}(\tau, 3m) \otimes \vartheta_{D_n}(\tau, 3n) \in J^{\text{cusp}}_{(m+n+1)/2, A_m \oplus D_n},$$ (26)

$$\vartheta_{A_m \oplus D_{3n}(3)} = \vartheta_{A_m}(\tau, 3m) \otimes \vartheta_{D_{3n}(3)}(\tau, 3n) \in J^{\text{cusp}}_{(m+3n+1)/2, A_m \oplus D_{3n}(3)}.$$ (27)

In particular, we get examples of Jacobi cusp forms of weight one with character in one abelian variable. The simplest examples of such forms can be found in [18] (see also [15] where many different cusp theta-products of small weights were considered):

$$\eta(\tau) \vartheta_{3/2}(\tau, 2z) \in J_{1,D_1(3)}(v^3_\eta), \quad \eta(\tau) \vartheta(\tau, 2z) \in J_{1,D_1(3)}(v^4).$$

To get more interesting examples, we take the pullback of $\sigma_{A_2}$. We consider $A_2$ as the sublattice $vD_3$ where $v = 2(1, 1, 1)$. Let $u = 2(u_1, u_2, u_3) \in D_3$ and $u_a$ be the projection of $u$ on $A_2^\perp$, i.e., $u = u_a + u_v$ where $u_a \in \langle v^\perp \rangle = A_2^\perp$ and $u_v \in \langle v \rangle = \{ v/12 \}$. We set $\sigma_{A_2|u} := \sigma_{A_2 | (u_a)}_{A_2}^\perp$.

**Proposition 4.8** Let $u = 2(u_1, u_2, u_3) \in D_3$ such that $u_i \neq u_j$ and $u_1 + u_2 + u_3 \neq 0 \mod 3$. Then $\sigma_{A_2|u}$ is a Jacobi cusp form of critical weight 1 with character $v^8_\eta$.

**Proof** We note first that if $u_i \neq u_j$ then the pullback $\sigma_{A_2|u}$ is not identically zero. According to the proof of Proposition 4.1 and Corollary 4.4, the Fourier expansion of the Jacobi form $\sigma_{A_2}$ of singular weight 1 has the following form

$$\sigma_{A_2}(\tau, 3_2) = \sum_{n>0, l_a \in A_2^\perp, l_v \in 1/2Z^3} f(n, l_a) e^{2\pi i (n\tau + (l_a, 3_2))}.$$ 

More exactly, in the last summation, we have $l' = l_a + l_v = 1/2 (l_1, l_2, l_3)$ with odd $l_i$ because the division by $\eta$ does not change the $3_2$-part of $\vartheta_{A_2}$. Let $u_a$ be the projection of $u$ on $A_2^\perp$, i.e., $u = u_a + u_v$ where $u_a \in \langle v^\perp \rangle = A_2^\perp$ and $u_v \in \langle v \rangle = \{ v/12 \}$. We have to analyze the Fourier expansion of $\sigma_{A_2|u}$. As in the proof of Proposition 4.1, we set $l_a = l_a \oplus l_\perp$ where $(l_a, u_a) = 0$. If the hyperbolic norm of the index of a Fourier coefficient $f_a(n, l_a)$ of $\sigma_{A_2|u}$ is equal to zero then $l_\perp = 0$. Therefore $(l_a, u_a) = (l_a, u_a) = (l_a, u) = 0$ and

$$\langle l', u \rangle = \pm \frac{(u, v)}{12} = \pm \frac{u_1 + u_2 + u_3}{3} \in Z.$$

The last inclusion is not possible. Thus the pullback $\sigma_{A_2|u}$ is a cusp form. $\square$

We note that the Jacobi form in Proposition 3.8 is a classical Jacobi form of type [8]. We give its more explicit form in the next corollary
**Corollary 4.9** Let \( a, b \in \mathbb{Z}_{>0} \). The following function, called theta-quark,
\[
\theta_{a,b}(\tau, z) = \frac{\vartheta(\tau, az) \vartheta(\tau, bz) \vartheta(\tau, (a+b)z)}{\eta(\tau)} \in J_{1, A_1; a^2 + ab + b^2}(v_8^\delta)
\]
is holomorphic Jacobi form of Eichler–Zagier type of weight 1, index \((a^2 + ab + b^2)\) and character \(v_8^\delta\). This is a Jacobi cusp form if \(a \not\equiv b \mod 3\).

**Proof** We can assume that \(a\) and \(b\) are coprime. We obtain this function as \(\sigma_{A_2}|_u\) for \(u = 2(b, -a, 0)\).

**Remark** The Jacobi form \(\theta_{a,b}\) was proposed by the second author many years ago in his talks on canonical differential forms on Siegel modular three-folds. The Jacobi forms of similar types, called *theta-blocks*, are studied in the paper [23] where the Fourier expansion of theta-quark \(\theta_{a,b}\) is found explicitly. The method of the proof of Proposition 4.8 can be used for other Jacobi forms when one takes a pullback on a sublattice of co-rank 2.

Propositions 4.3 and 4.8 give a method to pass from Jacobi forms of singular weight to Jacobi forms of critical weight. We have noticed that the tensor product of Jacobi forms of singular and critical weights is a form of critical weight. In some cases, we can divide some products of two forms of critical weight by \(\eta\) (see Corollary 4.5). We can control that the obtained Jacobi form is a cusp (or non-cusp) form. Analyzing the table of characters before Proposition 4.7, we obtain

**Proposition 4.10** If \(n \geq 7\) is odd (respectively, \(n \geq 8\) is even) and \(d \equiv n + 1 \mod 2\) then there exists a lattice \(L\) of rank \(n\) such that the space of Jacobi forms of critical weight \(J_{2n+1, L}(v_8^d)\) is not empty.

The analogue of Proposition 4.6 is the following

**Proposition 4.11** Theta-products give examples of Jacobi cusp forms of critical weight with trivial character for some lattices of all odd ranks \(\geq 5\).

**Proof** The corresponding Jacobi forms of critical weight are pullbacks (see Proposition 4.3) of Jacobi forms of singular weight of Proposition 4.6. One can also use \(\vartheta_{Am(3)}\) instead of \(\vartheta_{A_m}\) in theta-products.

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