Anisotropic Weyl symmetry and cosmology

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November 10, 2010

Abstract

We construct an anisotropic Weyl invariant theory in the ADM formalism and discuss its cosmological consequences. It extends the original anisotropic Weyl invariance of Hořava-Lifshitz gravity using an extra scalar field. The action is invariant under the anisotropic transformations of the space and time metric components with an arbitrary value of the critical exponent \( z \). One of the interesting features is that the cosmological constant term maintains the anisotropic symmetry for \( z = -3 \). We also include the cosmological fluid and show that it can preserve the anisotropic Weyl invariance if the equation of state satisfies \( P = z\rho/3 \). Then, we study cosmology of the Einstein-Hilbert-anisotropic Weyl (EHaW) action including the cosmological fluid, both with or without anisotropic Weyl invariance. The correlation of the critical exponent \( z \) and the equation of state parameter \( \omega \) provides a new perspective of the cosmology. It is also shown that the EHaW action admits a late time accelerating universe for an arbitrary value of \( z \) when the anisotropic conformal invariance is broken, and the anisotropic conformal scalar field is interpreted as a possible source of dark energy.

1 Introduction

Gravity theory with a local Weyl invariance was proposed as an alternative theory of gravity [1] and various aspects have been investigated for a long time. Among them, there are two main avenues. The first one is conformal gravity where the theory is built on the local conformal invariance and the general covariance. In this theory, the conformal invariant action can be realized by introducing the quadratic Weyl curvature tensor. This theory has the dimensionless gravitational coupling constant and the property that it is renormalizable.

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asymptotically free and could be potentially unitary. The other way of achieving conformal invariance is to introduce an extra gauge scalar field to compensate the non-invariance of the Einstein-Hilbert action. Ever since, many attempts of incorporating conformal invariance in the theory of general relativity have been carried out in the diverse areas of theoretical physics.

One application of conformal symmetry in the latter case is to consider the conformal scalar field which is non-minimally coupled to the curvature scalar with a coupling constant. In such a theory, the gravity sector is described by the Einstein-Hilbert action, and the conformal scalar field is treated as conformal matter. The exact conformal symmetry is imposed with a specific choice of the non-minimal coupling constant. In particular, this approach found wide applications in cosmological models in which it was first proposed to describe decaying cosmological constant.

The original conformal invariance is isotropic in the sense that the time and space components of the metric transform in the same manner. On the other hand, the canonical ADM formalism decomposes space-time into space and time. In this background, one could envisage an anisotropic Weyl transformation in which the space and time components of the metric transform differently, but still leaves the “action” invariant. Especially, such an attempt is well motivated by the recent upsurge of interest in the Hořava-Lifshitz gravity which has the feature of anisotropy between the space and time. In fact, in it is shown that classical action can have an anisotropic (local) Weyl invariance for specific values of the critical exponent $z = 3$ and the free parameter of the metric on the space of metrics. In this case, each component of the metric transforms as $g_{00} \rightarrow e^{6\omega(t,x)} g_{00}$, $g_{0i} \rightarrow e^{2\omega(t,x)} g_{0i}$, $g_{ij} \rightarrow e^{2\omega(t,x)} g_{ij}$.

As was mentioned before, however, one can also construct a conformally invariant gravity with curvature scalar by introducing an extra scalar field. Therefore, it seems natural to attempt to extend the anisotropic Weyl invariance of Hořava-Lifshitz gravity in this way. The purpose of this work is to show that this type of extension is indeed possible. We first review the anisotropic Weyl invariance of Hořava-Lifshitz gravity in the ADM formalism. Then, we extend the analysis to anisotropic Weyl invariant theory by introducing a scalar field with a suitable conformal weight which is given by the critical exponent. It turns out that the conformal invariance can be preserved for any value of the critical exponent, especially under the transformation $g_{00} \rightarrow e^{2z\omega(t,x)} g_{00}$. We will treat the resulting anisotropic conformal scalar field as describing the conformal matter and couple to the Einstein-Hilbert action. For $z = 1$, the action reduces to the conformal matter theory of Ref. Otherwise, the local Lorentz invariance is explicitly broken.

One of the motivations for considering an arbitrary value of $z$ is that we look for the role of the critical exponent in the cosmology and interpret the anisotropic conformal scalar field as a possible source of dark energy. In order to describe the cosmology in this context, we first break the conformal invariance explicitly by considering an arbitrary potential term for the scalar field. Then, we search for cosmological solutions paying attention

\footnote{In this paper, we only pay attention to the anisotropic Weyl invariance at the lowest curvature level and do not include the higher derivative terms such as the Cotton tensor, $R^2$ term, etc. in the action.}
to the possible role of $z$ in the evolution of the universe. For example, we compare with the cosmological model with $z = 1$, and show that accelerating phase can exist in vacuum with an arbitrary value of $z$. We also introduce a cosmological fluid with equation of state $\bar{\omega}$, and study the cosmology. If the conformal invariance is imposed on the fluid, we find that the critical exponent and the equation of state must be correlated. When the conformal invariance is broken, there is no correlation and the critical exponent $z$ remains as a free parameter. We will present the conformal preserving case also, because it seems that the cosmology in this case also shows some interesting feature. For example, the cosmological constant term enjoys anisotropic Weyl invariance with $z = -3$, in which case a cosmological solution which extrapolates between the matter dominated epoch and the accelerating phase exists.

This paper is organized as follows. In Sec.2, we briefly review the anisotropic Weyl invariance in the Hořava-Lifshitz gravity. In Sec.3, we consider the Einstein-Hilbert action with an (isotropic) conformal symmetry in the ADM formalism and extend to the action with an anisotropic Weyl invariance by introducing a scalar field. In particular, we show that cosmological fluid can preserve the anisotropic Weyl invariance if the equation of state satisfies $P = z\rho/3$. In Sec.4, we study cosmology with the FRW metric and apply it to the EHAW action with an arbitrary potential including also the cosmological fluid with or without anisotropic Weyl invariance. We briefly summarize the results and discuss them in Sec.5.

2 Anisotropic Weyl invariance in the Hořava-Lifshitz gravity

Let us consider an action of $z = 3$ gravity theory in $1 + 3$ dimensions [20]:

$$S_{aH} = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} \right\},$$

(2.1)

where $\kappa, w$ are dimensionless constant parameters, $K_{ij}$ is the extrinsic curvature which is defined by

$$K_{ij} = -\frac{1}{2N} (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i)$$

(2.2)

and

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R^j_{\ell} - \frac{1}{4} R \delta^j_{\ell} \right)$$

(2.3)

is the Cotton tensor. Here, $R$ is the curvature scalar in 3 space dimensions. Under anisotropic Weyl transformation,

$$N \to e^{3\omega(t,x)} N, \quad N_i \to e^{2\omega(t,x)} N_i, \quad g_{ij} \to e^{2\omega(t,x)} g_{ij},$$

(2.4)
the action (2.1) transforms to

\[
S_{aH} \rightarrow S_{aH} = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} \left[ K_{ij} K^{ij} - \lambda K^2 - 2(1 - 3\lambda)K \left( \frac{\hat{\omega} - \nabla_i \omega \nabla^i N}{N} \right) \right] \right. \\
+ \left. 3(1 - 3\lambda) \left( \frac{\hat{\omega} - \nabla_i \omega \nabla^i N}{N} \right)^2 \right\} - \frac{\kappa^2}{2\omega^4} C_{ij} C^{ij},
\] (2.5)

In the above equation (2.5), the third and fourth terms vanish with \( \lambda = 1/3 \), and the action (2.1) has an anisotropic Weyl invariance. As was shown in [20], this symmetry is an additional gauge symmetry supplementing the foliated diffeomorphisms.

It is interesting to note that under general anisotropic Weyl transformation,

\[
N \rightarrow e^{z\omega(t,x)} N, \quad N_i \rightarrow e^{2\omega(t,x)} N_i, \quad g_{ij} \rightarrow e^{2\omega(t,x)} g_{ij},
\] (2.6)

the extrinsic curvature terms become

\[
K_{ij} K^{ij} - \frac{1}{3} K^2 \rightarrow e^{-2\omega}(K_{ij} K^{ij} - \frac{1}{3} K^2).
\] (2.7)

Since the volume element transforms as \( \sqrt{g} N \rightarrow e^{(3+z)\omega} \sqrt{g} N \) under the transformation (2.6), it has no anisotropic Weyl invariance unless \( z = 3 \). One may, however, write an anisotropic Weyl invariant action by introducing some scalar field which can compensate the conformal weight \( 3 - z \). With a proper power of this scalar field, the conformal weight \( z - 3 \) coming from the volume element and the Cotton tensor term which transforms as \( C_{ij} C^{ij} \rightarrow e^{-6\omega} C_{ij} C^{ij} \), can also be compensated. In the next section, we will explicitly construct anisotropic Weyl action which is invariant for an arbitrary \( z \) including curvature scalar. As was mentioned before, we will drop the Cotton tensor term and include only terms up to second derivatives.

### 3 Anisotropic Weyl invariant gravity for general \( z \)

Let us first consider a conformally invariant action in four dimensions. This is given by

\[
S_C = \int d^4x \sqrt{-g} \varphi^2 \left( R_{(4)} - 6 \nabla_i \nabla^i \varphi \right).
\] (3.1)

The above action is invariant under \([5,10,25]\)

\[
g_{\mu\nu} \rightarrow e^{2\omega} g_{\mu\nu}, \quad \varphi \rightarrow e^{-\omega} \varphi,
\] (3.2)

where \( \omega = \omega(t,x) \). Considering the ADM decomposition and rearranging terms, it can be rewritten as \([10]\)

\[
S_C = \int dt d^3x N \sqrt{g} \varphi^4 \left( R - 8 \nabla_i \nabla^i \varphi + B_{ij} B^{ij} - B^2 \right),
\] (3.3)
where \( \varphi^2 = \phi \) and

\[
B_{ij} = K_{ij} - \frac{2}{N\varphi} g_{ij}(\varphi - \nabla_i \varphi N^i) \tag{3.4}
\]

\[
\equiv K_{ij} + \frac{\theta}{2N} g_{ij}, \tag{3.5}
\]

with \( \theta = -4(\dot{\varphi} - \nabla_i \varphi N^i)/\varphi \). The above action \( \mathcal{S}_3 \) is invariant under

\[
N \rightarrow e^\omega N, \quad N_i \rightarrow e^{2\omega} N_i, \\
g_{ij} \rightarrow e^{2\omega} g_{ij}, \quad \varphi \rightarrow e^{-\frac{\omega}{2}} \varphi. \tag{3.6}
\]

We can extend the above procedure to an anisotropic Weyl invariance. One can show that the following action

\[
\mathcal{S}_\varphi = \int dt d^3x N \sqrt{g} \left[ \varphi^{2z+2} \left( R - \frac{8 \nabla_i \nabla^i \varphi}{\varphi} \right) + \varphi^{-2z+6} B_{ij} B^{ij} - \varphi^{-2z+6} B^2 \right], \tag{3.7}
\]

is invariant with respect to

\[
N \rightarrow e^{\omega} N, \quad N_i \rightarrow e^{2\omega} N_i, \\
g_{ij} \rightarrow e^{2\omega} g_{ij}, \quad \varphi \rightarrow e^{-\frac{\omega}{2}} \varphi. \tag{3.8}
\]

Note that we have a factor \( z \) in the above equations which turns out to be coincident with the critical exponent. That is, beside the local transformations \( \text{(3.8)} \), the above action \( \text{(3.7)} \) is also invariant with respect to a global transformation

\[
t \rightarrow b^z t, \quad x \rightarrow bx, \quad N_i \rightarrow b^{1-z} N_i, \quad \varphi \rightarrow b^{-\frac{1}{z}} \varphi, \tag{3.9}
\]

with \( N \) and \( g_{ij} \) being unchanged. When \( z = 1 \), the above action \( \text{(3.7)} \) reduces to the ADM decomposition of the conformally invariant action \( \text{(3.1)} \). On the other hand, in the same manner, considering the transformation law \( \text{(2.7)} \) in the previous section, the extrinsic curvature terms with an anisotropic Weyl invariance can be written as

\[
S_{aK} = \int dt d^3x \sqrt{g} N \left\{ \varphi^{-2z+6} \left( K_{ij} K^{ij} - \frac{1}{3} K^2 \right) \right\}. \tag{3.10}
\]

From the action \( \text{(3.7)}, \text{(3.10)} \), one can construct the general action with anisotropic Weyl invariance as follows

\[
S_{aW} = \int dt d^3x N \sqrt{g} \left[ \varphi^{2z+2} \eta \left( R - \frac{8 \nabla_i \nabla^i \varphi}{\varphi} \right) + \varphi^{-2z+6}(\eta + \xi) B_{ij} B^{ij} - \varphi^{-2z+6}(\eta + \xi B^2 - V(\varphi)) \right], \tag{3.11}
\]

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where η, ξ are some constants and for the anisotropic Weyl invariance, \( V(\phi) = \alpha \phi^{2(z+3)} \) with some constant \( \alpha \). Note that in the case of \( z = 1 \) the transformation law of \( N \) becomes isotropic Weyl transformation, i.e., ordinary conformal transformation. As expected, for \( \eta = 1, \xi = \alpha = 0 \) and \( \eta = \alpha = 0, \xi = 1 \) the action (3.11) becomes (3.7) and (3.10) respectively. We also note that in the case of \( z = -3 \), the potential term becomes a cosmological constant.

The most general action which includes the cosmological fluid will be given by

\[
S_{awf} = S_{aw} + S_f, \quad (3.12)
\]

where \( S_f \) is the action which has anisotropic Weyl invariance without \( \phi \) term. We show in the following lines that the equation of the state for this cosmological fluid satisfies

\[
P = \frac{z}{3} \rho. \quad (3.13)
\]

In order to see this, we first vary for \( N, N^i, g^{ij}, \phi \), and obtain the equations of motion:

\[
\begin{align*}
\delta_N S_{awf}: & \quad \varphi^{2z+2} \{ \eta \left( R - 8 \frac{\nabla_i \nabla^i \phi}{\phi} \right) - \varphi^{-4z+4}(\eta + \xi) B_{ij} B^{ij} + \varphi^{-4z+4}(\eta + \frac{\xi}{3}) B^2 \} - \alpha \varphi^2 (z+3) - \rho = 0, \quad (3.14) \\
\delta_N S_{awf}: & \quad 2 \varphi^{-2z+6} \{ - (\eta + \xi) \nabla_j B^j_i + 2(z - 3)(\eta + \xi) \frac{\nabla_j \phi}{\phi} B^j_i \} + 2 \varphi^{-2z+6} \{ (2(z - 3)) \eta + \frac{2}{3} (3 - z) \xi \} \nabla_i \phi B + (\eta + \frac{\xi}{3}) \nabla_i B \} = 0, \quad (3.15) \\
\delta g^{ij} S_{awf}: & \quad N \varphi^{2z+2} \{ \eta A^{(1)}_{ij} + \varphi^{-4z+4}(\eta + \xi) A^{(2)}_{ij} - \varphi^{-4z+4}(\eta + \frac{\xi}{3}) A^{(3)}_{ij} \} + \frac{N}{2} \alpha \varphi^2 (z+3) g_{ij} - \frac{N}{2} g_{ij} P = 0, \quad (3.16)
\end{align*}
\]

where \( \rho = - \frac{1}{\sqrt{g}} \frac{\delta S_{aw}}{\delta g} \), \( P_{g^{ij}} = - \frac{2}{N \sqrt{g^2 g^{ij}}} \),

\[
A^{(1)}_{ij} = R_{ij} - \frac{1}{2} g_{ij} R - \frac{\nabla_i \nabla_j N}{N} - 4(z - 1) \frac{\nabla_i N \nabla_j \phi}{\phi} + 4 \frac{\nabla_i N \nabla_j \phi}{\phi} g_{ij}
- 2(z + 1)(z - 3) \frac{\nabla_i \phi \nabla_j \phi}{\phi^2} + \frac{\nabla_k \nabla^k N}{N} g_{ij} + 2(z - 1)(z + 1) \frac{\nabla_k \phi \nabla^k \phi}{\phi^2} g_{ij} - 2(z + 1) \frac{\nabla_i \phi \nabla_j \phi}{\phi} + 2(z + 1) \frac{\nabla_k \phi \nabla^k \phi}{\phi} g_{ij},
\]

\[
A^{(2)}_{ij} = - \frac{N_i \nabla_k B^k_j}{N} - \frac{N_j \nabla_k B^k_i}{N} + \frac{\nabla_i N_k B^k_j}{N} + \frac{\nabla_j N_k B^k_i}{N} + \frac{N_k \nabla^k B_{ij}}{N} - 2 B^k_i B_{jk} - \frac{1}{2} B_{kl} B^{kl} g_{ij} + B B_{ij} - \frac{\dot{B}_{ij}}{N} + (1 - \frac{z}{2}) \frac{\theta B_{ij}}{N} + 4(z - 3) \frac{\nabla_k \phi N_i B^k_j}{\phi} + 4 \frac{\nabla_i \phi N_j B}{\phi} g_{ij},
\]

\[
A^{(3)}_{ij} = \frac{B^2}{2} g_{ij} - \frac{B \nabla_i B N_j}{N} - \frac{\nabla_i B N_j}{N} + 2 z \frac{\nabla_i \phi N_i B}{\phi} + 2 z \frac{\nabla_i \phi N_j B}{\phi} - \frac{z \theta}{2 N} g_{ij} - \frac{\nabla_i B N_k g_{ij}}{N} - \frac{\dot{B} g_{ij}}{N},
\]
and
\[
\delta \varphi S_{aWf}; \quad N\varphi^{2z+1}[(2z + 2)\eta R - 16(2z + 1)\eta \frac{\nabla_i \nabla^i \varphi}{\varphi} - 16z(2z + 1)\eta \frac{\nabla_i \varphi \nabla^i \varphi}{\varphi^2} + \\
\{2(\eta - \xi) + 2z(\eta + \frac{\xi}{3})\} B^2 \varphi^{-4z+4} - 4z\eta N B \theta \varphi^{-4z+4} + 8\eta \nabla_i B N^i \varphi^{-4z+4} + \\
(-2z + 6)(\eta + \xi) B_{ij} B^{ij} \varphi^{-4z+4} - 8\eta \nabla_i \nabla^i N - 16(2z + 1)\eta \nabla_i N \frac{\nabla^i \varphi}{\varphi} - 8\eta \dot{B} \varphi^{-4z+4}]
\]
\[-2(z + 3)N\alpha \varphi^{2z+5} = 0. \quad (3.17)
\]

In the Appendix, we show that Eq. (3.13) must be satisfied in order to be consistent with Eqs. (3.14)~(3.17). Note that this condition is only for cosmological fluid with anisotropic Weyl invariance\footnote{When $z = 1$, this condition corresponds to $T_{\mu}^\mu = 0$ which represents the condition of isotropic conformally invariant fluid. That is, the above condition (3.13) is equivalent to $P = \frac{1}{3}\rho$. It is pointed out that the above condition (3.13) also can be obtained by dimensional analysis as shown in [26].}. It can be violated, if we do not insist on the symmetry. In the next section, we will consider cosmological consequences of both unbroken and broken cases. In the broken case, we consider an arbitrary potential $V(\varphi)$ breaking anisotropic Weyl invariance in Eq. (3.11).

\section{4 Cosmological solutions}

In order to investigate cosmological consequences, we first recall that the Einstein-Hilbert action in the ADM formalism is given by

\[
S_{EH} = \int dt d^3x N \sqrt{g} \frac{1}{2\kappa^2} (R + K_{ij}K^{ij} - K^2), \quad (4.1)
\]

and consider

\[
S_{EHaw} = S_{EH} + S_{aWf}, \quad (4.2)
\]

where $S_{aWf}$ is the anisotropic Weyl action given in Eq. (3.12) with an arbitrary potential $V(\varphi)$. Let us introduce the Friedmann-Robertson-Walker metric via

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \bar{\kappa}r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4.3)
\]

where $\bar{\kappa} = -1, 0, 1$ corresponding to an open, flat, closed universe respectively. Note that for $z = 1, \eta = 1$, and $\xi = 0$, the action (4.2) is reduced to the conformal quintessence [14–16]. In this background, one finds the following results,

\[
K_{ij} = -H g_{ij}, \quad K = -3H, \quad (4.4)
\]

\[
R_{ij} = \frac{2\bar{\kappa}}{a^2} g_{ij}, \quad R = \frac{6\bar{\kappa}}{a^2}. \quad (4.5)
\]
For the above action (4.2), the equations of motion become
\[
\begin{align*}
\delta_N S_{EHaW}: & \quad \frac{1}{2\kappa^2} (R - K_{ij} K^{ij} + K^2) + \delta_N S_{awf}, \\
\delta_{N^i} S_{EHaW}: & \quad \frac{1}{2\kappa^2} (-2 \nabla_j K^i_j + 2 \nabla_i K) + \delta_{N^i} S_{awf}, \\
\delta_{g^{ij}} S_{EHaW}: & \quad \frac{N}{2\kappa^2} A^{(0)}_{ij} + \delta_{g^{ij}} S_{awf},
\end{align*}
\]  
(4.6)  
(4.7)  
(4.8)

where
\[
A^{(0)}_{ij} = R_{ij} - \frac{1}{2} g_{ij} R - \frac{\nabla_i \nabla_j N}{N} + \frac{\nabla_k \nabla^k N}{N} g_{ij} - \frac{N_i \nabla_k K^k_j}{N} - \frac{N_j \nabla_k K^k_i}{N} + \frac{\nabla_i N^k K_{jk}}{N} + \frac{\nabla_j N^k K_{ik}}{N} - 2 K^k_i K^k_j - \frac{1}{2} K_{kl} K^{kl} g_{ij} + K K_{ij} - \frac{K_{ij}}{N}.
\]  
(4.9)

Assuming a homogeneous scalar field \( \varphi(x) = \varphi(t) \), one obtains the following equations from Eqs. (4.6) ~ (4.8) and (3.17):
\[
\frac{1}{2\kappa^2} \varphi^{2z-6} + \eta \dot{H} = \eta H \theta - \frac{\theta^2}{4} + \frac{1}{6} \varphi^{2z-6} V + \frac{\rho}{6} \varphi^{2z-6} - \frac{\bar{\kappa}}{a^2} \left( \frac{\varphi^{2z-6}}{2\kappa^2} + \eta \varphi^{4z-4} \right),
\]  
(4.10)
\[
\frac{1}{2\kappa^2} \varphi^{2z-6} + \eta \dot{H} = -\frac{\varphi^{2z-6}}{4} (\rho + P) + \frac{\dot{\varphi}}{2} + \frac{\eta z}{2} (-H \theta + \dot{\theta}^2 + 2 \bar{\kappa} \left( \frac{\varphi^{2z-6}}{2\kappa^2} + \eta \varphi^{4z-4} \right)),
\]  
(4.11)

and
\[
\eta(z + 1) \frac{\bar{\kappa}}{a^2} \varphi^{4z-4} + \eta(z + 3) H^2 - 3 \eta H \theta - \frac{\eta(z - 3)}{4} \theta^2 + 2 \eta \dot{H} - \eta \dot{\theta} - \frac{1}{12} \varphi^{2z-5} V' = 0,
\]  
(4.12)

where \( \dot{} \) denotes differentiation with respect to \( \varphi \). Note that Eq. (4.12) is satisfied trivially. When \( \eta = V = 0 \), one can check that Eqs. (4.10), (4.11) are equivalent to the ordinary Friedmann equations. We assume flat universe with \( \bar{\kappa} = 0 \) from here on. Then, Eqs. (4.10) ~ (4.12) can combine into the following equation,
\[
(z + 3) H^2 + 2 \dot{H} = \kappa^2 \left( \frac{z}{3} \rho - P \right) - \frac{\kappa^2}{3} \left( \frac{\varphi}{2} V'(\varphi) - (z + 3) V(\varphi) \right).
\]  
(4.13)

In the next subsections, we discuss possible solutions of these equations for both with or without anisotropic Weyl invariance.

### 4.1 Case with anisotropic invariance

In the case of cosmological fluid with anisotropic invariance, introducing \( P = \bar{\omega} \rho \) then \( z = 3 \bar{\omega} \). Here \( \bar{\omega} \) is the equation of state parameter. Since the right hand side of Eq. (4.13) vanishes in this case, we have
\[
(z + 3) H^2 + 2 \dot{H} = 0.
\]  
(4.14)
In the above equation, one can check easily that in the case of \( z \neq -3 \), \( H \) behaves as \( 1/t \) which correspond the power law solutions and for \( z = -3 \), \( H \) is a constant value which represents the exponentially accelerating solution.

To see these behaviors in more detail, we find explicit solutions of Eqs. (4.10) \( \sim \) (4.12). For power law solutions, we have

\[
\rho = \rho_0 a^{-3(1+\bar{\omega})}, \quad a = a_0 t^{\frac{2}{3(1+\bar{\omega})}}, \quad \varphi = \varphi_0 t^{-\frac{1}{3(1+\bar{\omega})}}, \quad \alpha = 0,
\]

where \( \rho_0, a_0, \varphi_0 \) are positive constants which satisfy the relations \( \rho_0 a_0^{-3(1+\bar{\omega})} = \frac{4}{3\kappa^2(1+\bar{\omega})^2} \).

This solution preserves the standard cosmology. Most of all, in this case, it should be remarked that \( z = 1, 0 \) correspond to \( \bar{\omega} = 1/3 \) (radiation dominance), \( 0 \) (matter dominance) respectively. In the region \( -3 < z < -1 \), it has an accelerating phase, whereas decelerating phase in \( z > -1 \). In particular, for \( z = -3 \) the solution is

\[
\rho = P = 0, \quad a = a_0 e^{Ht}, \quad \varphi = \varphi_0 e^{-\frac{H}{3}t},
\]

where \( H^2 = \kappa^2\alpha/3 \).

Note that with the anisotropic Weyl invariant fluid with \( P = z\rho/3 \), the universe is described by different values of the critical exponent in radiation-dominated, matter-dominated, and vacuum-dominated epochs, but still can maintain the anisotropic Weyl invariance.

However, if we break the anisotropic Weyl invariance of the cosmological fluid, while preserving it for the potential part, we can describe the extrapolation from matter dominance to vacuum dominance of the universe with a single value of the critical exponent. In this case, with \( P = \bar{\omega} \rho \), Eq. (4.13) becomes

\[
(z + 3)H^2 + 2\dot{H} = \frac{\kappa^2}{3} (z - 3\bar{\omega})\rho.
\]

In particular for \( z = -3 \),

\[
\dot{H} = -\frac{\kappa^2}{2} (1 + \bar{\omega})\rho.
\]

Then, from Eqs. (4.10) \( \sim \) (4.12) we find the following solution

\[
H = \sqrt{\frac{\kappa^2\alpha}{3} \coth[At]}, \quad \varphi = B (\sinh[At])^{-\frac{1}{3(1+\bar{\omega})}}, \quad \rho = \alpha (\sinh[At])^{-2},
\]

Note that the critical exponent decreases as the universe evolves. One possible interpretation of the result might be that this running behavior of the parameter \( z \) in the Lagrangian is due to the renormalization properties of the scalar field theory and could be viewed as being reasonable in the sense of the renormalization group. However, a demonstration of this running behavior is beyond the scope of the present paper, and since the macroscopic equations of state in each epoch is involved with different cosmological fluids, it may be hard that it could actually be realized at the microscopic level. Another interpretation is that there are several anisotropic conformal invariant sectors described by \( z \) and each epoch corresponds to one of these sectors via cosmological fluid contents at that epoch.
where \( A = (1 + \bar{\omega})\sqrt{3\kappa^2\alpha}/2 \), \( B \) is a constant and \( a(t) = a_0(\sinh(At))^{2/3(1+\bar{\omega})} \). For early universe with small \( t \), \( a(t) \sim t^{2/3(1+\bar{\omega})} \), and it describes the standard power law expansion with equation of state parameter with \( \bar{\omega} \). For late time with large \( t \), \( a(t) \sim e^{\sqrt{\Lambda t}} \) with \( \Lambda = \kappa^2\alpha \). This solution describes an extrapolation from matter-dominance into vacuum dominance.

### 4.2 Case without anisotropic invariance

In this case, \( P = \bar{\omega}\rho \) and \( V(\varphi) \) is different from the anisotropic invariant potential \( \alpha\varphi^{2(z+3)} \). Eq. (4.13) can be solved in three cases.

In the case \( \rho = 0 \), there are two potential forms that are interesting from the cosmological point of view.

i) \( V(\varphi) = A_1\varphi^{-2z+6} + A_2\varphi^{-2z+10} \), where \( A_1, A_2 \) are some constants. Then Eqs. (4.10)~(4.12) and Eq. (4.13) yields the following solution:

\[
H = B_1 \tanh[B_2t], \quad \varphi = \left(-2\eta\kappa^2\right)^{1/2-z-6},
\]

where \( B_1 = \sqrt{A_2/(3\eta(z+3))(-2\eta\kappa^2)^{1/2-z-3}} \), \( B_2 = \sqrt{A_2(z+3)/(12\eta)(-2\eta\kappa^2)^{1/2-z-3}} \) and \( A_1 = (-2\eta\kappa^2)^{2/2-z-3}A_2 \). Here \( \eta < 0 \), \( z (\neq 3) > -3 \) and \( A_2 < 0 \). It is interesting to note that in this case, the effective equation of state is given by

\[
\omega(t) = -1 - \frac{z + 4}{3} \text{csch}^2[B_2t] < -1
\]

(4.21)
corresponding to the phantom model [21].

ii) \( V(\varphi) = \left(\frac{\varphi^{2z-6}}{1 + 2\eta\kappa^2\varphi^{2z-6}}\right)^{\frac{1}{3}} \). With \( \varphi = \varphi_0 = \text{const} \), we find the de Sitter solution, \( a \sim e^{Ht} \), where

\[
H = \sqrt{\frac{1}{6\eta} \left(\frac{\varphi_0^{2z-6}}{1 + 2\eta\kappa^2\varphi_0^{2z-6}}\right)^{\frac{1}{2-z-3}}},
\]

(4.22)

In the case of \( z = 3 \) the above solution diverges and is replaced with

\[
V(\varphi) = \varphi^{12/2\eta\kappa^2+1}, \quad H = \frac{1}{6\eta(1 + 1/2\eta\kappa^2)}\varphi_0^{12/2\eta\kappa^2+1}.
\]

When \( \rho \neq 0 \), the available solution is given when \( V = V_0 = \text{const} \) and \( z \neq -3 \):

\[
H = \sqrt{\frac{\kappa^2V_0}{3}} \coth[Ct], \quad \varphi = D(\sinh[Ct])^{-\frac{1}{2(1+\bar{\omega})}}, \quad \rho = V_0(\sinh[Ct])^{-2},
\]

(4.23)

where \( C = (1 + \bar{\omega})\sqrt{3\kappa^2V_0}/2 \), \( D \) is a constant and \( a(t) = a_0(\sinh[Ct])^{2/3(1+\bar{\omega})} \). This is also an extrapolating solution from matter dominance to vacuum dominance.
5 Conclusion and discussion

In this paper, we were able to construct an anisotropic Weyl invariant action in the ADM formalism with the help of extra scalar field, generalizing the original work of Hořava. Among possible applications of the result, we treated this action as an anisotropic Weyl matter field and considered the EHaW action adding the cosmological fluid sector, and studied cosmological consequences. It is found that if the anisotropic Weyl invariance is imposed, there must be a correlation of the critical exponent $z$ and the equation of state parameter $\tilde{\omega}$ satisfying $\tilde{\omega} = z/3$. According to this condition, it is possible to reinterpret cosmological evolution not by $\tilde{\omega}$ but by $z$. In the early universe, radiation and matter dominance correspond to $z = 1$ and $z = 0$ anisotropic Weyl invariance respectively. At late times, it has $z = -3$ which is de-Sitter phase, i.e., $\tilde{\omega} = -1$. It is also shown that for particular value of $z = -3$, the potential term for the scalar field becomes cosmological constant and there exists an extrapolating solution from matter dominated epoch into a late time accelerating universe in the broken cosmological fluid case. We also found de Sitter solution in the case where the anisotropic conformal invariance is broken by the potential term, and especially in the polynomial potential (case i) of Sec. 4.2), the effective equation of state parameter is less than -1. The compatibility of the cosmology considered in this work with the observed CMB anisotropies and structure formation remains to be seen.

We recall that in the standard cosmology, the cosmological fluid sector breaks conformal invariance, unless $\tilde{\omega} = 1/3$, i.e., radiation dominated. In the present anisotropic Weyl invariance, cosmological fluid sector can maintain the invariance for an arbitrary value of $\tilde{\omega}$ due to the constraint $\tilde{\omega} = z/3$. However, for realistic cosmology, the anisotropic conformal invariance has to be broken, and the parameter $z$ is free. The physical significance of this parameter, in general including the cosmological case is yet to be explored. One example of anisotropic local conformal invariance appears in the condensed matter system [27], and considering the AdS/CMT correspondence might shed some light on this.

We conclude with a couple of remarks on the issues related to Hořava-Lifshitz gravity in the case of the anisotropic Weyl invariant action. The first one is the question on the possible existence of the scalar graviton which shows pathological behavior. In the original Hořava-Lifshitz gravity, this was pointed out to cause serious problems, but, subsequently it was shown that this could be cured via a natural extension of the Hořava-Lifshitz gravity by abandoning the projectability and adding suitable space dependent lapse functions [28]. Since local Lorentz invariance is broken when $z \neq 1$, the action (3.11) confronts the same problem and the scalar graviton persists in the anisotropic Weyl action (3.11). To see this more closely, we first fix the gauge by choosing a constant value for the field $\varphi$. Then, one can show that the $\eta$ term in the action (3.11) does not produce any scalar graviton mode since the ratio of $B_{ij}B^{ij}$ to $B^2$ is equal to 1. Also, for the $\xi$ term with Weyl invariance of the original Hořava-Lifshitz gravity, it has been shown explicitly that this $\xi$ term also does not produce graviton mode [29]. Even though these terms do not propagate the scalar graviton separately, their sum do propagate the extra mode and the conformal action (3.11) turns out to coincide with the low-energy limit of the non-projectable Hořava-Lifshitz gravity.
which was shown to propagate extra graviton mode \[30\]. Unfortunately, this scalar graviton problem is not cured for the full Einstein-Hilbert anisotropic Weyl action (4.2). In this case, conformal symmetry is not present and we cannot gauge fix \(\varphi\) away. Assuming that the action admits a flat background with a constant \(\varphi = \varphi_0\) which would require the potential to have a minimum with \(V(\varphi_0) = 0\), we can consider a perturbation around \(\varphi_0, \varphi = \varphi_0 + \tilde{\varphi}\). It turns out that the Einstein-Hilbert action has the effect of changing the coefficient of the scalar graviton mode, but the perturbed action reduces to the theory where the \(\tilde{\varphi}\) is coupled with the low-energy limit of the non-projectable Hořava-Lifshitz gravity previously mentioned. To investigate further the nature of this scalar-tensor-type theory and especially to check whether the scalar graviton problem can be cured along the line of Ref. [28] are left as open problems. The other is to check whether the anisotropic Weyl action (3.11) could be derived using the detailed balance condition\[20\] of Ref. [20]. This condition may also restrict the diverse terms which will appear when the anisotropic Weyl invariance is exploited to construct the higher curvature terms not considered in this work. This remains as a future study.

Appendix

In the action (3.12), \(S_{awf} = S_{aw} + S_f\) is invariant with respect to

\[
N \rightarrow e^{z\omega}N, \quad N_i \rightarrow e^{2\omega}N_i, \\
g_{ij} \rightarrow e^{2\omega}g_{ij}, \quad \varphi \rightarrow e^{-z\omega} \varphi.
\]

For the infinitesimal \(\omega\), one can find the followings,

\[
\delta N = z\omega N, \quad \delta N_i = 0, \\
\delta g^{ij} = -2\omega g^{ij}, \quad \delta \varphi = -\frac{\omega}{2} \varphi.
\]

And for the above transformation, \(\delta S_{awf}\) is

\[
\delta S_{awf}(N, g^{ij}, N^i, \varphi) = 0 = \frac{\delta S_{awf}}{\delta N} \delta N + \frac{\delta S_{awf}}{\delta g^{ij}} \delta g^{ij} + \frac{\delta S_{awf}}{\delta N^i} \delta N^i + \frac{\delta S_{awf}}{\delta \varphi} \delta \varphi
\]

\[
= \omega \left( zN \frac{\delta S_{awf}}{\delta N} - 2g^{ij} \frac{\delta S_{awf}}{\delta g^{ij}} - \frac{\varphi}{2} \frac{\delta S_{awf}}{\delta \varphi} \right). \tag{5.1}
\]

Substituting (3.14) \sim (3.17) into (5.1) and after some tedious calculations one can find the following condition,

\[
2zN(-\rho) + 6NP = 0
\]

\[
\rightarrow P = \frac{z}{3} \rho.
\]

\[\text{In Ref. [29] it was argued that the anisotropic Weyl invariant action of Hořava-Lifshitz gravity might be derived from the detailed balance condition.}\]
Acknowledgments

We thank the anonymous referee for valuable suggestions. We also like to thank Seyen Kouwn, Joohan Lee, Tae Hoon Lee, and Won-Il Myeong for useful comments. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number 2005-0049409.

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