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Conformally flat submanifolds with flat normal bundle

In memory of Manfredo do Carmo

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Abstract. We prove that any conformally flat submanifold with flat normal bundle in a conformally flat Riemannian manifold is locally holonomic, that is, admits a principal coordinate system. As one of the consequences of this fact, it is shown that the Ribaucour transformation can be used to construct an associated large family of immersions with induced conformally flat metrics holonomic with respect to the same coordinate system.

A main task in conformal geometry is the study of submanifolds of conformally flat Riemannian manifolds with induced conformally flat metrics. A Riemannian manifold $M^n$ is said to be conformally flat if each point lies in an open neighborhood conformal to an open subset of Euclidean space $\mathbb{R}^n$. This is always the case for manifolds endowed with metrics of constant sectional curvature.

Even if they belong to the realm of conformal geometry, for reason of simplicity most of the results in this paper are stated for submanifolds of Euclidean space. Nevertheless, they hold true when the ambient space is just a conformally flat manifold.

E. Cartan [1] proved that a hypersurface $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, is conformally flat if and only if at each point there is a principal curvature of multiplicity at least $n-1$. If $M^n$ is free of flat points, then $f(M)$ is locally foliated by $(n-1)$-dimensional umbilical submanifolds of $\mathbb{R}^{n+1}$, or equivalently, we have that $f(M)$ is enveloped by a one-parameter family of umbilical hypersurfaces of the ambient space.

Moore [14] extended Cartan’s result to submanifolds of higher codimension. He showed that an isometric immersion of a conformally flat manifold $f : M^n \to \mathbb{R}^{n+p}$ of dimension $n \geq 4$ and codimension $p \leq n-3$ has a principal normal vector of multiplicity at least $n-p \geq 3$ at each point. Recall that a normal vector $\eta \in N_f M(x)$ is called a principal normal of $f$ at $x \in M^n$ with multiplicity $s$ if the tangent subspace defined as

$$E_\eta(x) = \{ X \in T_x M : \alpha_f(X, Y) = \langle X, Y \rangle \eta \text{ for all } Y \in T_x M \}$$
in terms of the second fundamental form $\alpha_f: TM \times TM \to N_fM$ of the immersion, satisfies $\dim E_\eta(x) = s > 0$. Clearly, principal normals are a natural generalization to submanifolds of higher codimension of principal curvatures of hypersurfaces.

A smooth normal vector field $\eta \in N_fM$ to an isometric immersion $f: M^n \to \mathbb{R}^N$ is called a principal normal vector field with multiplicity $s$ if $\dim E_\eta(x) = s > 0$ is constant, in which case the distribution $x \in M^n \mapsto E_\eta(x)$ is smooth. A principal normal vector field $\eta$ is called Dupin if it is parallel along $E_\eta$ in the normal connection, which is always the case if $s \geq 2$. The principal normal being Dupin implies that $f$ maps each leaf of the spherical distribution $x \mapsto E_\eta(x)$ into an umbilical submanifold of $\mathbb{R}^{n+p}$.

Let $f: M^n \to \mathbb{R}^N$ be an isometric immersion with flat normal bundle, that is, at any point the curvature tensor of the metric induced from the ambient space on the normal bundle of the submanifold vanishes. Submanifolds with flat normal bundle have captured the attention because they “behave like hypersurfaces”. For instance, from [17] we have that at any point $x \in M^n$ there is a unique set of pairwise distinct principal normal vectors $\eta_i \in N_fM(x), 1 \leq i \leq s(x)$, and an associate orthogonal splitting of the tangent space as $T_xM = E_{\eta_1}(x) \oplus \cdots \oplus E_{\eta_s}(x)$.

Related to Moore’s result in [14] it was proved in [12] that if at some point a conformally flat submanifold $f: M^n \to \mathbb{R}^{2n-2}, n \geq 4$, has no principal normal of multiplicity larger than one, then the normal bundle at that point has to be flat.

An isometric immersion $f: M^n \to \mathbb{R}^N$ with flat normal bundle is called holonomic if $M^n$ carries a global orthogonal principal coordinate system. That the coordinates are principal means that the corresponding coordinate vector fields diagonalize the second fundamental form of the immersion at any point. It is a classical fact that the Gauss-Codazzi equations for a holonomic submanifold can be nicely written as a completely integrable system of first order PDE’s.

Cartan [2,3] proved that if $f: M^n_\tilde{c} \to \mathbb{Q}^{n+p}_c$ is an isometric immersion of a manifold with constant sectional curvature $\tilde{c}$ into a space form of sectional curvature $c$, then the submanifold is locally holonomic if $\tilde{c} < c$ and the codimension is $p = n - 1$, which in this case is the least possible. If $\tilde{c} > c$ the same conclusion for the same codimension was obtained by Moore [15] under the additional assumption that the submanifold is free of weak-umbilic points. To prove these results, one first has to argue that the submanifold must have flat normal bundle, and then that the image of its second fundamental form spans at any point the full normal space of the immersion. An elementary argument then yields local holonomicity; for instance see Proposition 1 in [10].

Holonomic isometric immersions $f: M^n_\tilde{c} \to \mathbb{Q}^N_c$ are of particular interest because the associated Gauss–Codazzi system of equations is, in this case, a natural generalization of the sinh-Gordon, sine-Gordon, Laplace or wave equation, according to the values of the sectional curvatures. These equations are classically known to be in correspondence to constant curvature surfaces; cf. [9] and [10].

Related to the above, we proved in [8] that any proper isometric immersion of an Einstein manifold $f: M^n \to \mathbb{R}^N$ with flat normal bundle is locally holonomic.
Throughout this paper that a submanifold with flat normal bundle is proper means that it has a constant number of principal normals. Note that the only Riemannian manifolds that are simultaneously Einstein and conformally flat are the ones of constant sectional curvature.

The following is the main result of this paper.

**Theorem 1.** Let \( f : M^n \rightarrow \mathbb{R}^N \), \( n \geq 4 \), be a proper isometric immersion with flat normal bundle of a conformally flat manifold. Then \( f \) is locally holonomic with at most one principal normal vector field of multiplicity larger than one.

In Example 9 given below a large family of nonflat conformally flat submanifolds of codimension two with flat normal bundle is constructed. They possess three principal normal vector fields and the holonomic coordinates are provided by the construction.

In view of the above result, it is quite natural to consider that the conformally flat proper submanifolds with flat normal bundle belong to one of two classes according to whether they carry a principal normal vector field of multiplicity larger than one or not. Notice that in low codimension, the former situation is always the case due to the aforementioned result of Moore.

An isometric immersion \( f : M^n \rightarrow \mathbb{R}^{n+p} \) is said to be *quasiumbilical* if at any point of \( M^n \) there exists an orthonormal normal base \( \xi_1, \ldots, \xi_p \) such that each shape operator \( A_{\xi_j} \), \( 1 \leq j \leq p \), has an eigenvalue of multiplicity at least \( n - 1 \). The property of a submanifold being quasiumbilical is conformally invariant.

In view of Cartan’s result that isometric immersions with codimension one between conformally flat manifolds \( f : M^n \rightarrow \tilde{M}^{n+1}, n \geq 4 \), are quasiumbilical, it is clear that composing hypersurfaces of this type yields quasiumbilical submanifolds with higher codimension.

If \( f : M^n \rightarrow \mathbb{R}^{n+p}, n \geq 4 \), is quasiumbilical, it is easy to see that the Weyl tensor of \( M^n \) vanishes, hence \( M^n \) is conformally flat. On the other hand, Chen and Verstraelen [4] proved that if \( M^n, n \geq 4 \), is conformally flat and \( f : M^n \rightarrow \mathbb{R}^{n+p} \) has flat normal bundle with codimension \( p \leq n - 3 \), then the submanifold is quasiumbilical. Moore and Morvan [16] reached the same conclusion for codimension \( p \leq 4 \) without the assumption of flatness of the normal bundle.

To conclude that a conformally flat submanifold with flat normal bundle is quasiumbilical, the presence for a principal normal of multiplicity at least two suffices regardless of the codimension.

**Theorem 2.** Let \( f : M^n \rightarrow \mathbb{R}^{n+p}, n \geq 4 \), be a proper isometric immersion with flat normal bundle of a conformally flat manifold. If \( f \) carries a principal normal vector field of multiplicity \( m \geq 2 \) then \( p \geq n - m \) and \( f \) is quasiumbilical.

If a principal normal is trivial, say \( \eta_i = 0 \), then

\[
E_{\eta_i}(x) = \Delta(x) = \{ X \in T_x M : \alpha_f (X, Y) = 0 \text{ for all } Y \in T_x M \}
\]

is called the *relative nullity* subspace of \( f \) at \( x \in M^n \) and \( \nu(x) = \dim \Delta(x) \) the *index of relative nullity* of \( f \) at \( x \in M^n \).

That a conformally flat submanifold with flat normal bundle has index \( \nu \geq 1 \) turns out to be quite restrictive. It is convenient to state the following result for
ambient space forms $\mathbb{Q}_c^N$ and leave the definitions of generalized cone and cylinder in these spaces for later.

**Theorem 3.** Let $f : M^n \to \mathbb{Q}_c^N$, $n \geq 4$, be a proper isometric immersion with flat normal bundle of a conformally flat manifold. If $f$ has index of relative nullity $\nu \geq 1$, then one of the following holds:

(i) $M^n$ has constant sectional curvature $c$ and $f$ is locally a $\nu$-generalized cylinder over a holonomic submanifold $g : L^{n-\nu} \to \mathbb{Q}_c^N$.

(ii) $M^n$ has sectional curvature different from $c$ and $f$ is locally a $1$-generalized cone over a holonomic submanifold $g : L^{n-1} \to \mathbb{Q}_c^{N-1} \subset \mathbb{Q}_c^N$, $\tilde{c} \geq c$, with constant sectional curvature different from $\tilde{c}$.

Holonomic submanifolds are the natural object of application of the Ribaucour transformation introduced in [10]. This fact is instrumental to obtain the following result.

**Theorem 4.** Let $f : M^n \to \mathbb{R}^N$, $n \geq 4$, be a proper isometric immersion with flat normal bundle of a conformally flat manifold. Then locally there exists an $N$-parameter family of immersions $\tilde{f} : M^n \to \mathbb{R}^N$ with induced conformally flat metrics that are holonomic with respect to the same coordinate system as $f$.

We observe that some of the results in this paper have been obtained by Donaldson and Terng [18] under strong additional assumptions.

1. Preliminaries

In this section we show that the statements in this paper are conformally invariant.

Let $f : M^n \to \mathbb{R}^N$ be an isometric immersion with flat normal bundle and let $\eta_i \in N_f M(x)$, $1 \leq i \leq s(x)$, be the set of pairwise distinct principal normals at $x \in M^n$. Then, the second fundamental form $\alpha = \alpha_f$ of $f$ acquires the form

$$\alpha(X, Y)(x) = \sum_{i=1}^{s} \langle X^i, Y^i \rangle \eta_i$$

where $X \mapsto X^i$ denotes the orthogonal projection from $T_x M$ onto $E_i(x) = E_{\eta_i}(x)$. Equivalently, in terms of the shape operators of $f$ we have

$$A_{\xi} X = \sum_{i=1}^{s} \langle \xi, \eta_i \rangle X^i \quad (1)$$

for any $\xi \in N_f M$.

A submanifold $f : M^n \to \mathbb{R}^N$ with flat normal bundle is called proper if $s(x) = k$ is constant on $M^n$. In this situation, we have from [17] that the principal normal vector fields $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth. Moreover, the distributions $x \in M^n \mapsto E_i(x)$, $1 \leq i \leq k$, have constant dimension and are also smooth.
Let $\tilde{M}^N$ be endowed with conformal metrics $g_1$ and $g_2$, that is, $g_2 = \lambda^2 g_1$ where $\lambda \in C^\infty(\tilde{M})$ is positive. Given an immersion $f : M^n \to \tilde{M}^N$, we thus have the two isometric immersions with the induced metrics

$$f_j = f : (M^n, f^* g_j) \to (\tilde{M}^N, g_j), \quad 1 \leq j \leq 2.$$  

At any $x \in M^n$ the second fundamental forms of $f_1$ and $f_2$ are related by

$$\alpha_{f_2}(X, Y) = \alpha_{f_1}(X, Y) - \frac{1}{\lambda} g_1(X, Y) (\text{grad}_1 \lambda) \perp$$

and the normal curvature tensors by

$$R^\perp_2(X, Y) \xi = R^\perp_1(X, Y) \xi$$

for any $X, Y \in T_x M$ and $\xi \in N_f M(x)$. In particular, if $\eta$ is a principal normal vector of $f_1$ at $x \in M^n$ then

$$\eta - \frac{1}{\lambda} (\text{grad}_1 \lambda) \perp$$

is a principal normal vector of $f_2$ at $x \in M^n$.

We thus have the following fact.

**Proposition 5.** Let $f : M^n \to M^N$ be a proper isometric immersion with flat normal bundle and let $\tau : M^N \to \tilde{M}^N$ be a conformal diffeomorphism. Then the conformal immersion $\tilde{f} = \tau \circ f : M^n \to \tilde{M}^N$ also has flat normal bundle and is proper.

### 2. Proof of Theorem 1

The proof of Theorem 1 will follow from the two lemmas given in the sequel.

**Lemma 6.** Let $f : M^n \to \mathbb{R}^N$, $n \geq 4$, be an isometric immersion with flat normal bundle of a conformally flat manifold. Then at any point of $M^n$ there exists at most one principal normal of multiplicity at least two.

**Proof.** It is well-known that the curvature tensor of $M^n$ has the form

$$R(X, Y, Z, W) = L(X, W) \langle Y, Z \rangle - L(X, Z) \langle Y, W \rangle + L(Y, Z) \langle X, W \rangle - L(Y, W) \langle X, Z \rangle$$

in terms of the Schouten tensor given by

$$L(X, Y) = \frac{1}{n-2} \left( \text{Ric}(X, Y) - \frac{\tau}{2(n-1)} \langle X, Y \rangle \right)$$

where $\tau$ denotes the scalar curvature. In particular, the sectional curvature is given by

$$K(X, Y) = L(X, X) + L(Y, Y)$$

(3)

where $X, Y \in T M$ are orthonormal vectors.
A straightforward computation of the Ricci tensor using the Gauss equation

\[ R(X, Y, Z, W) = \langle \alpha(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle \]  \hspace{1cm} (4)

yields

\[ \text{Ric}(X, Y) = n \langle \alpha(X, Y), H \rangle - \sum_{j=1}^{n} \langle \alpha(X, X_j), \alpha(Y, X_j) \rangle \]  \hspace{1cm} (5)

where \( H \) is the mean curvature vector and \( X_1, \ldots, X_n \) an orthonormal tangent basis.

We obtain from (3) and (4) that

\[ L(X, X) + L(Y, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \| \alpha(X, Y) \|^2 \]  \hspace{1cm} (6)

for any pair \( X, Y \in TM \) of orthonormal vectors. From (5) we have

\[ \text{Ric}(X, X) = n \langle \eta_i, H \rangle - \| \eta_i \|^2 \]  \hspace{1cm} for any unit vector \( X \in E_i \).

Thus

\[ (n - 2)L(X, X) = (n - 1)\| H \|^2 + (n - 2)\langle \hat{\eta}_i, H \rangle - \| \hat{\eta}_i \|^2 - \frac{\tau}{2(n - 1)} \]  \hspace{1cm} (7)

for any unit vector \( X \in E_i \). Denoting

\[ \hat{\eta}_i = \eta_i - H, \hspace{0.5cm} 1 \leq i \leq k, \]

we obtain from (7) that

\[ (n - 2)L(X, X) = (n - 1)\| H \|^2 + (n - 2)\langle \hat{\eta}_i, H \rangle - \| \hat{\eta}_i \|^2 - \frac{\tau}{2(n - 1)} \]  \hspace{1cm} (8)

for any unit vector \( X \in E_i \).

Assume that \( \eta_1 \) has multiplicity at least two. Then (7) yields

\[ L(X, X) = L(Y, Y) \]

for any unit vectors \( X, Y \in E_1 \). Hence (6) gives that

\[ 2L(X, X) = \| \hat{\eta}_1 \|^2 + 2\langle \hat{\eta}_1, H \rangle + \| H \|^2 \]  \hspace{1cm} (9)

for any unit vector \( X \in E_1 \). It follows from (8) and (9) that

\[ \| \hat{\eta}_1 \|^2 = \| H \|^2 - \frac{\tau}{n(n - 1)}. \]  \hspace{1cm} (10)

We obtain from (9) and (10) that

\[ L(X, X) = \| H \|^2 + \langle \hat{\eta}_1, H \rangle - \frac{\tau}{2n(n - 1)} \]  \hspace{1cm} (11)

for any unit vector \( X \in E_1 \).

Given principal normals \( \eta_i \neq \eta_j \), we have from (6) that

\[ L(X, X) + L(Y, Y) = \langle \hat{\eta}_i, \hat{\eta}_j \rangle + \langle \hat{\eta}_i + \hat{\eta}_j, H \rangle + \| H \|^2 \]  \hspace{1cm} (12)
where $X \in E_i$ and $Y \in E_j$ are unit vectors. Suppose that $\eta_i$ and $\eta_j$ have both multiplicity at least two. It follows from (10) that

$$
\|\hat{\eta}_i\|^2 = \|H\|^2 - \frac{\tau}{n(n-1)} = \|\hat{\eta}_j\|^2. \tag{13}
$$

On the other hand, we obtain from (11) and (12) that

$$
\langle\hat{\eta}_i, \hat{\eta}_j\rangle = \|H\|^2 - \frac{\tau}{n(n-1)} \tag{14}
$$

We conclude from (13) and (14) that $\eta_i = \eta_j$, and this is a contradiction.

**Example 7.** A rather simple example in high codimension of a conformally flat submanifold with flat normal bundle carrying a principal normal of multiplicity at least two is as follows: Let $M^{2n}$ be the Riemannian product $S^n_1 \times U$ where $S^n_1 \subset \mathbb{R}^{n+1}$ is a round sphere and $U$ an open subset of the hyperbolic space $\mathbb{H}^{n-1}$ isometrically immersed in $\mathbb{R}^{2n-1}$. Then $M^{2n}$ is conformally flat and the product isometric immersion of $M^{2n}$ into $\mathbb{R}^{3n}$ has a principal normal of multiplicity $n$.

**Lemma 8.** Let $f : M^n \to \mathbb{R}^N$, $n \geq 4$, be an isometric immersion with flat normal bundle of a conformally flat manifold. If at some point of $M^n$ we have $k \geq 3$, then the vectors $\eta_j - \eta_m$ and $\eta_j - \eta_\ell$ are linearly independent for $1 \leq m \neq j \neq \ell \neq m \leq k$.

**Proof.** We argue by contradiction. In the sequel suppose that

$$
\eta_j - \eta_m = \mu(\eta_j - \eta_\ell) \tag{15}
$$

where $\mu \neq 0$ and $1 \leq m \neq j \neq \ell \neq m \leq k$.

If $\eta_1$ is a principal normal of multiplicity at least two, it follows from (8), (10), (11) and (12) that

$$(n - 2)\langle\hat{\eta}_1, \hat{\eta}_j\rangle + \|\hat{\eta}_j\|^2 - (n - 1)\|\hat{\eta}_1\|^2 = 0, \quad 2 \leq j \leq k,$$

which is equivalent to

$$
\|2\hat{\eta}_j + (n - 2)\hat{\eta}_1\| = n\|\hat{\eta}_1\|. \tag{16}
$$

If $\eta_i \neq \eta_j$ are principal normals of multiplicity one, we have from (8) and (12) that

$$
\|\hat{\eta}_i\|^2 + (n - 2)\langle\hat{\eta}_i, \hat{\eta}_j\rangle + \|\hat{\eta}_j\|^2 = n\|H\|^2 - \frac{\tau}{n-1}. \tag{17}
$$

By Lemma 6 there is at most one principal normal $\eta_1$ of multiplicity at least two. Suppose first that this is the case. Due to (16) the vectors

$$
\beta_j = 2\hat{\eta}_j + (n - 2)\hat{\eta}_1, \quad 2 \leq j \leq k,
$$

satisfy

$$
\|\beta_j\| = n\|\hat{\eta}_1\|, \quad 2 \leq j \leq k. \tag{18}
$$

Case (i): If $j, m, \ell \geq 2$, we have from (15) that

$$(1 - \mu)\beta_j = \beta_m - \mu\beta_\ell.$$
if \( j \neq m \neq \ell \neq j \). Hence
\[
\|\beta_j\|^2 - 2\mu\|\beta_j\|^2 + \mu^2\|\beta_j\|^2 = \|\beta_m\|^2 - 2\mu\langle \beta_m, \beta_\ell \rangle + \mu^2\|\beta_\ell\|^2
\]
which gives
\[
\|\beta_j\|^2 = \langle \beta_m, \beta_\ell \rangle.
\]
It follows from (18) that \( \eta_m = \eta_\ell \), and this is a contradiction.

Case (ii): If \( j = 1 \) and \( m \neq \ell \geq 2 \), we have from (15) that
\[
\beta_m - \mu \beta_\ell = n(1 - \mu)\hat{\eta}_1.
\]
Hence
\[
\|\beta_m\|^2 - 2\mu\langle \beta_m, \beta_\ell \rangle + \mu^2\|\beta_\ell\|^2 = n^2(1 - \mu)^2\|\hat{\eta}_1\|^2.
\]
Using (18) we obtain \( \eta_m = \eta_\ell \), and this is a contradiction.

Case (iii): If \( m = 1 \) and \( j \neq \ell \geq 2 \), we have from (15) that
\[
(1 - \mu)\beta_j + \mu \beta_\ell = n\hat{\eta}_1.
\]
We may assume that \( \mu \neq 1 \) since, otherwise, we already have a contradiction. Hence
\[
(1 - \mu)^2\|\beta_j\|^2 + 2\mu(1 - \mu)\langle \beta_j, \beta_\ell \rangle + \mu^2\|\beta_\ell\|^2 = n^2\|\hat{\eta}_1\|^2.
\]
Using (18) and \( \mu \neq 1 \) we have \( \eta_j = \eta_\ell \), and this is a contradiction.

Next assume that all principal normals have multiplicity one. From (17) we have
\[
\|\hat{\eta}_i\|^2 + (n - 2)\langle \hat{\eta}_i, \hat{\eta}_j \rangle + \|\hat{\eta}_j\|^2 = nb
\]
where \( i \neq j \) and
\[
b = \|H\|^2 - \frac{\tau}{n(n - 1)}.
\]
This is equivalent to
\[
\|\beta^i_j\|^2 = 4nb + n(n - 4)\|\hat{\eta}_j\|^2
\]
where
\[
\beta^i_j = 2\hat{\eta}_i + (n - 2)\hat{\eta}_j.
\]
We have from (15) that
\[
(1 - \mu)\beta^i_j = \beta^i_m - \mu \beta^i_\ell
\]
if \( i \neq m \neq j \neq \ell \neq i \). Hence
\[
\|\beta^i_j\|^2 - 2\mu\|\beta^i_j\|^2 + \mu^2\|\beta^i_j\|^2 = \|\beta^i_m\|^2 - 2\mu\langle \beta^i_m, \beta^i_\ell \rangle + \mu^2\|\beta^i_\ell\|^2.
\]
We obtain using (19) and \( i \neq j \) that
\[
\|\beta^i_j\|^2 = \langle \beta^i_m, \beta^i_\ell \rangle.
\]
It follows that \( \eta_m = \eta_\ell \), and this is a contradiction. \( \square \)
Proof of Theorem 1. The case \( k = 1 \) is trivial. In order to conclude holonomicity it is a standard fact that it suffices to show that the distributions \( E^\perp_j = \bigoplus_{i=1, i\neq j}^k E_i \) are integrable for \( 1 \leq j \leq k \) (see [12]). The Codazzi equation is easily seen to yield
\[
\langle X, Y \rangle \nabla^\perp_Z \eta_i = \langle \nabla_X Y, Z \rangle (\eta_i - \eta_j)
\] (20)
and
\[
\langle \nabla_X V, Z \rangle (\eta_j - \eta_\ell) = \langle \nabla_Y X, Z \rangle (\eta_\ell - \eta_i)
\] (21)
for any \( X, Y \in E_i, Z \in E_j \) and \( V \in E_\ell \) where \( 1 \leq i \neq j \neq \ell \neq i \leq k \).

It follows from (20) that the \( E_i \)'s are integrable. Thus, it is sufficient to argue for the case \( k \geq 3 \). In fact, it suffices to show that if \( X \in E_i \) and \( Y \in E_j \) then \( [X, Y] \in E_\ell^\perp \) if \( i \neq j \neq \ell \neq i \). We have from (21) that
\[
\langle \nabla_X Y, Z \rangle (\eta_\ell - \eta_j) = \langle \nabla_Y X, Z \rangle (\eta_j - \eta_i)
\]
for any \( Z \in E_\ell \). Then we obtain from Lemma 8 that
\[
\langle \nabla_X Y, Z \rangle = \langle \nabla_Y X, Z \rangle = 0
\]
which completes the proof of holonomicity. Then Lemma 6 completes the proof. \( \square \)

Example 9. A large family of nontrivial examples of conformally flat \( n \)-dimensional submanifolds in \( \mathbb{R}^{n+2} \) was constructed in [5]. This construction goes as follows. Start with two smooth spherical curves parametrized by arc-length
\[
\gamma_i : I_i \subset \mathbb{R} \to S^{m_i}_{r_i} \subset \mathbb{R}^{m_i+1}, 1 \leq i \leq 2,
\]
where \( m_1 + m_2 = n \) and \( r_1^2 + r_2^2 = 1 \). Consider the spherical surface parametrized by the isometric immersion \( h : L^2 = I_1 \times I_2 \to S^{n+1}_1 \subset \mathbb{R}^{n+2} \) defined by
\[
h(u, v) = (\gamma_1(u), \gamma_2(v)).
\]
Then, the \( n \)-dimensional submanifold of \( \mathbb{R}^{n+2} \) parametrized on the unit normal bundle \( UN_h L \) of \( h \) in \( S^{n+1}_1 \) by the map
\[
\phi(w) = h(u, v) + i_w w,
\]
where \( i : S^{n+1}_1 \rightarrow \mathbb{R}^{n+2} \) denotes the inclusion, is conformally flat. A straightforward computation shows that the submanifold has flat normal bundle.
3. Proof of Theorem 2

We first recall from [6] or [12] the following facts which can easily be proved using (1).

Lemma 10. Let \( f : M^n \to \mathbb{R}^N \) be an isometric immersion with flat normal bundle and principal normal vectors \( \eta_1, \ldots, \eta_\ell \) at \( x \in M^n \). Denote

\[
d = \dim \text{span} \{ \eta_i : 1 \leq i \leq \ell \}
\]

and

\[
S_f = \text{span} \{ \eta_i - \eta_j : 1 \leq i, j \leq \ell \}.
\]

(i) \( \dim S_f \leq \ell - 1 \) and \( d - 1 \leq \dim S_f \leq d \).
(ii) If \( \dim S_f = d - 1 \) then the unit vector \( \delta \in \text{span} \{ \eta_i, 1 \leq i \leq \ell \} \) orthogonal to \( S_f \) is umbilical, that is, \( A_\delta = aI \).

Proof of Theorem 2. It is well-known that \( M^n, n \geq 4 \), is conformally flat if and only if at any \( x \in M^n \) the following holds:

\[
K(X_1, X_2) + K(X_3, X_4) = K(X_1, X_3) + K(X_2, X_4)
\]

(22)

for every quadruple of orthogonal vectors \( X_1, X_2, X_3, X_4 \in T_x M \).

Let \( \eta_1, \ldots, \eta_{n-m+1} \) be the principal normals of \( f \) with \( \eta_1 \) the one of multiplicity \( m \geq 2 \). Choosing \( X_1, X_2 \in E_1, X_3 \in E_i \) and \( X_4 \in E_j \), we obtain from (22) that

\[
\langle \xi, \xi_j \rangle = 0, \quad \text{for all } 2 \leq i \neq j \leq n - m + 1,
\]

(23)

where

\[
\xi_i = (\eta_1 - \eta_i)/\|\eta_1 - \eta_i\|, \quad 2 \leq i \leq n - m + 1.
\]

It follows from Lemma 10 that \( \dim S_f = n - m \). In particular \( p \geq n - m \).

Observe that (23) is equivalent to

\[
\langle \xi, \xi_j \rangle = \langle \xi, \eta_1 \rangle \quad \text{for any } 2 \leq j \neq i \leq n - m + 1.
\]

(24)

According to Lemma 10 we have to distinguish the following cases.

If \( \dim S_f = d - 1 \) we have from part (ii) that

\[
\text{span} \{ \eta_i, 1 \leq i \leq n - m + 1 \} = \text{span} \{ \delta, \xi_2, \ldots, \xi_{n-m+1} \}.
\]

From (24) each \( A_{\xi_i}, 2 \leq i \leq n - m + 1 \), has an eigenvalue of multiplicity at least \( n - 1 \).

If \( \dim S_f = d \), then

\[
\text{span} \{ \eta_i, 1 \leq i \leq n - m + 1 \} = \text{span} \{ \xi_2, \ldots, \xi_{n-m+1} \},
\]

and the proof follows similarly.
4. Proof of Theorem 3

We first define generalized cylinders in space forms and subsequently generalized cones. Notice that the latter submanifolds are also generalized cylinders.

Let \( g : L^{n-s} \to \mathbb{Q}_c^N \), \( 1 \leq s \leq n-1 \), be an isometric immersion carrying a parallel flat normal subbundle \( \pi : \mathcal{L} \subset N_g L \to L^{n-s} \) of rank \( s \). The \( s \)-generalized cylinder over \( g \) determined by \( \mathcal{L} \) is the submanifold parametrized (at the open subset of regular points) by the map \( f : \mathcal{L} \to \mathbb{Q}_c^N \) given by

\[
f(x, v) = \exp_g(x) v
\]

where \( \exp \) is the exponential map of \( \mathbb{Q}_c^N \).

The following result can be found in [6] or [12].

**Proposition 11.** Let \( f : M^n \to \mathbb{Q}_c^N \) be an isometric immersion with constant index of relative nullity \( v > 0 \) such that the conullity distribution \( x \in M^n \mapsto \Delta^\perp(x) \) is integrable. Then \( f \) is locally a \( v \)-generalized cylinder over a leaf \( g : L^{n-v} \to \mathbb{Q}_c^N \) of conullity.

Let \( g : L^n \to \mathbb{Q}_\tilde{c}^m \) be an isometric immersion, and let \( i : \mathbb{Q}_\tilde{c}^m \to \mathbb{Q}_c^N, \tilde{c} \geq c \), be an umbilical inclusion. Since the normal bundle of \( \tilde{g} = i \circ g \) splits as \( N_{\tilde{g}} L = i_*(N_g L) \oplus i_* Q_{\tilde{c}}^m \), we regard \( \mathcal{L} = N_i Q_{\tilde{c}}^m \) as a subbundle of \( N_{\tilde{g}} L \). The \((N-m)\)-generalized cone over \( g \) is the submanifold parametrized (at regular points) by the map \( f : \mathcal{L} \to \mathbb{Q}_c^N \) given by

\[
f(x, v) = \exp_g(x) v
\]

where \( \exp \) is the exponential map of \( \mathbb{Q}_c^N \).

The following result can be found in [7] or [12].

**Proposition 12.** Let \( f : M^n \to \mathbb{Q}_c^N \) be an isometric immersion with constant index of relative nullity \( v > 0 \). Assume that the conullity distribution is umbilical. Then \( f \) is locally a \( v \)-generalized cone over a leaf \( g : L^{n-v} \to \mathbb{Q}_c^N \) contained in an umbilical submanifold \( \mathbb{Q}_{\tilde{c}}^{N-v} \) of \( \mathbb{Q}_c^N \) with \( \tilde{c} \geq c \).

**Proof of Theorem 3.** By Theorem 1 the conullity distribution is integrable. Proposition 11 asserts that \( f \) is locally an open neighborhood of a \( v \)-generalized cylinder over a leaf \( g = f \circ h : L^{n-v} \to \mathbb{Q}_c^N \) of the conullity distribution, where we denote by \( h : L^{n-v} \to M^n \) the inclusion map.

Let \( u_1, \ldots, u_n \) be principal coordinates for \( f \) with corresponding coordinate vector fields \( \partial_1, \ldots, \partial_n \) such that \( \Delta = \text{span} \{ \partial_1, \ldots, \partial_v \} \). Since the Levi-Civita connection satisfies (see [12, Proposition 1.12])

\[
\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = 0, \quad 1 \leq i \neq j \neq k \neq i \leq n,
\]

we have

\[
\langle \nabla_{\partial_i} \partial_j, Z \rangle = 0
\]
for $\nu + 1 \leq i \neq j \leq n$ and $Z \in \Delta$. Thus
\[ \alpha_h(\tilde{\partial}_i, \tilde{\partial}_j) = 0, \quad \nu + 1 \leq i \neq j \leq n, \]
where $h_\ast \tilde{\partial}_i = \partial_i, \quad \nu + 1 \leq i \leq n$. Recalling that $f$ is holonomic by Theorem 1, we obtain
\[ \alpha_g(\tilde{\partial}_i, \tilde{\partial}_j) = f_\ast(\alpha_h(\tilde{\partial}_i, \tilde{\partial}_j)) + \alpha_f(\partial_i, \partial_j) = 0, \quad \nu + 1 \leq i \neq j \leq n, \]
which proves that $g$ is holonomic.

We may assume $\nu = 1$ since for $\nu \geq 2$ we have from (22) choosing $X_3, X_4 \in \Delta$ and the Gauss equation that $M^n$ has constant sectional curvature $c$. Let $0 = \eta_1, \eta_2, \ldots, \eta_k$ be the distinct principal normals. From (22) we have
\[ \langle \eta_i, \eta_j \rangle = \lambda, \quad 2 \leq i \neq j \leq k. \tag{26} \]
Moreover, if $k < n$ and $\eta_r$ is the principal normal vector field with multiplicity higher than one, then
\[ \|\eta_r\|^2 = \lambda. \tag{27} \]
In fact, the above follows from (22) by choosing the quadruple of orthogonal vectors so that $X_4 \in \Delta, X_1, X_2 \in E_r$ and $X_3 \in E_j$, or $X_1 \in E_i, X_2 \in E_j$ and $X_3 \in E_r$.

We may assume $\lambda \neq 0$ since for $\lambda = 0$ we have from (26) and the Gauss equation that $M^n$ has constant sectional curvature $c$. We claim that the distribution $\Delta^\perp$ is umbilical. Since the claim is trivial if $k = 2$, we assume $k \geq 3$. From (20) we have
\[ \nabla^\perp_Z \eta_i = (\nabla_{Y_i} Y_i, Z)\eta_i, \quad 2 \leq i \leq k, \]
where $Y_i \in E_i$ is of unit length and $Z \in \Delta$. Then (26) yields
\[ Z(\lambda) = \lambda((\nabla_{Y_i} Y_i, Z) + (\nabla_{Y_j} Y_j, Z)), \quad 2 \leq i \neq j \leq k, \]
whereas (27) gives
\[ Z(\lambda) = 2\lambda(\nabla_{Y_r} Y_r, Z). \]
Since $n \geq 4$, we obtain
\[ \langle \nabla_{Y_i} Y_i, Z \rangle = \frac{Z(\lambda)}{2\lambda}, \quad 2 \leq i \leq k. \]
On the other hand, we have from (21) that
\[ \langle \nabla_{Y_i} Y_j, Z \rangle = 0, \quad 2 \leq i \neq j \leq k, \]
and the claim follows.

Proposition 12 now gives that $f$ coincides locally with the 1-generalized cone over a leaf of conullity $g : L^{n-1} \to Q^N_{\tilde{c}}$ into an umbilical submanifold $i : Q^N_{\tilde{c}} \to Q^N_c, \tilde{c} \geq c$. It is easy to see that
\[ \alpha_f(X, Y) = i_\ast \alpha_g(X, Y) \]
for any $X, Y \in TL$. Hence $L^{n-1}$ has sectional curvature $\tilde{c} + \lambda|_L$.

It remains to show that $\lambda$ is constant along the conullity leaves. From (20) we have

$$\nabla^\perp_{X_m}\eta_i = \langle \nabla_{Y_i}Y_i, X_m \rangle (\eta_i - \eta_m)$$

where $Y_i \in E_i$ is of unit length and $X_m \in E_m$ with $2 \leq m \neq i \leq k$. If $k > 3$ then (26) gives

$$X_m(\lambda) = 0, \ 2 \leq i \neq j \neq m \neq i \leq k.$$ 

If $k = 3$ we have from (26) and (27) that

$$X_i(\lambda) = 2\langle \nabla^\perp_{X_i}\eta_r, \eta_r \rangle = 0, \ 2 \leq i \neq r \leq 3,$$

where $X_i \in E_i$. Since $\eta_r$ is Dupin the proof is complete. $\square$

**Example 13.** Take any isometric immersion with flat normal bundle of an open subset of $Q_n^c$ into $Q_n^c + p$ with $p < n$. Then it is well-known that $v \geq n - p$ (see Example 1 and Corollary 1 in [15]). Hence any component of the open dense subset where the immersion is proper is an example of a generalized cylinder with constant sectional curvature.

5. Proof of Theorem 4

To prove the theorem, we first establish a one-to-one correspondence of local nature between globally conformally flat holonomic submanifolds $f : M^n \to \mathbb{R}^N$, $n \geq 4$, and flat holonomic submanifolds $F : M^n_0 \to \mathbb{L}^{N+2}$ that lie inside the light-cone $\mathbb{V}^{N+1}$ of the standard flat Lorentzian space form $\mathbb{L}^{N+2}$. Here $M^n_0$ denotes $M^n$ endowed with the flat metric conformal to the one of $M^n$. On the other hand, we have from the results in [10] and [11] that any flat submanifold $M^n_0$ in $\mathbb{L}^{N+2}$ admits locally an abundance of Ribaucour transformations with induced flat metric. Then, after restricting to the transforms that preserve lying in the light-cone, we obtain by means of the correspondence an $N$-parameter family of new conformally flat holonomic submanifolds in $\mathbb{R}^N$.

Let $\langle , \rangle_*$ be a metric conformal to the one of the Riemannian manifold $(M^n, \langle , \rangle)$ with conformal factor $e^{2\omega}$, that is,

$$\langle , \rangle_* = e^{2\omega} \langle , \rangle.$$ 

The corresponding Levi-Civita connections $\nabla^*$ and $\nabla$ are related by

$$\nabla^*_X Y = \nabla_X Y + Y(\omega)X + X(\omega)Y - \langle X, Y \rangle \text{grad} \omega$$

(28)

where the gradient is computed with respect to the metric $\langle , \rangle$. From [13] the relation between the curvature tensors $R_*$ and $R$ is given by

$$R_*(X, Y)Z = R(X, Y)Z - T(X, Y)Z$$

(29)
where

\[
T(X, Y)Z = \left( Q(Y, Z) + \langle Y, Z \rangle \| \text{grad} \, \omega \|^2 \right) X \\
- \left( Q(X, Z) + \langle X, Z \rangle \| \text{grad} \, \omega \|^2 \right) Y \\
+ \langle Y, Z \rangle Q_0 X - \langle X, Z \rangle Q_0 Y,
\]

\[
Q(X, Y) = \text{Hess} \, \omega(X, Y) - X(\omega)Y(\omega), \quad Q_0 X = \nabla_X \text{grad} \, \omega - X(\omega) \text{grad} \, \omega
\]

and everything is computed with respect to the metric \( \langle \,, \rangle \).

In the sequel \( (M^n, \langle \,, \rangle) \) stands for a globally conformally flat manifold, that is, we have globally that

\[
\langle \,, \rangle = e^{2\omega} \langle \,, \rangle_0
\]

where \( \omega \in C^\infty(M) \) and \( \langle \,, \rangle_0 \) is a flat metric on \( M^n \).

**Lemma 14.** Let \( f : M^n \to \mathbb{R}^N \), \( n \geq 3 \), be a proper isometric immersion with flat normal bundle. Then

\[
Q(X, Z) = 0
\]

for \( X \in E_i \) and any \( Z \in TM \) such that \( Z \perp X \).

**Proof.** Let \( R^f \) denote the curvature tensor with respect to the metric \( \langle \,, \rangle \). Since \( \langle \,, \rangle_0 \) is flat, we have from (29) that

\[
R^f(X, Y)Z = -T(X, Y)Z
\]

for any \( X, Y, Z \in TM \).

Assume that either \( Z \in E_i \) or that \( Z \in E_j \) with \( j \neq i \). In either case we have \( \alpha_f(X, Z) = \alpha_f(Y, Z) = 0 \) for \( Y \perp \text{span} \{X, Z\} \). The Gauss equation gives

\[
\langle R^f(X, Y)Z, W \rangle = \langle \alpha_f(X, W), \alpha_f(Y, Z) \rangle - \langle \alpha_f(X, Z), \alpha_f(Y, W) \rangle = 0
\]

for any \( W \in TM \). Using \( Z \perp X \) we have from (31) that

\[
0 = T(X, Y)Z = Q(Y, Z)X - Q(X, Z)Y,
\]

and (30) follows. \( \Box \)

**Lemma 15.** Let \( f : M^n \to \mathbb{R}^N \) be a proper isometric immersion with flat normal bundle. If \( \dim E_1 \geq 2 \), then

\[
Q(Z, Z) = -\frac{1}{2} \left( \| \eta_1 \|^2 + e^{-4\omega} \| \text{grad} \, \omega \|^2 \right)
\]

for any \( Z \in E_1 \) with \( \| Z \| = 1 \), where the gradient is taken with respect to the flat metric.
Proof. We have from (31) that
\[
\langle R^f(X, Y)Y, X \rangle = -\langle T(X, Y)Y, X \rangle
\]
where \(X, Y \in E_1\) satisfy \(X \perp Y\) and \(\|X\| = \|Y\| = 1\). The Gauss equation yields
\[
\langle R^f(X, Y)Y, X \rangle = \|\eta_1\|^2.
\]
Since
\[
\langle T(X, Y)Y, X \rangle = Q(X, X) + Q(Y, Y) + e^{-4\omega}\|\text{grad}\omega\|^2,
\]
we obtain
\[
Q(X, X) + Q(Y, Y) = -\|\eta_1\|^2 - e^{-4\omega}\|\text{grad}\omega\|^2.
\] (32)
Setting
\[
Z = \frac{1}{\sqrt{2}}(X + Y)
\]
we have that
\[
Q(Z, Z) = \frac{1}{2}(Q(X, X) + 2Q(X, Y) + Q(Y, Y)),
\]
and the proof follows from (30) and (32). □

Let \((\mathbb{L}^{N+2}, \langle \cdot, \cdot \rangle)\) denote the standard flat Lorentzian space form. The light-cone \(\mathbb{V}^{N+1}\) of \(\mathbb{L}^{N+2}\) is one of the connected components of the set of vectors
\[
\{v \in \mathbb{L}^{N+2} \setminus \{0\} : \langle v, v \rangle = 0\}
\]
endowed with a degenerate metric induced from \(\mathbb{L}^{N+2}\).

Let \(F : M^n \to \mathbb{L}^{N+2}\) be an isometric immersion of a Riemannian manifold \((M^n, \langle \cdot, \cdot \rangle)\) that lies inside \(\mathbb{V}^{N+1}\). Taking derivatives of \(\langle F, F \rangle = 0\) yields that the position vector field \(F\) is a parallel normal vector field to \(F\) such that the second fundamental form satisfies
\[
\langle \alpha_F(X, Y), F \rangle = -\langle X, Y \rangle
\] (33)
for all \(X, Y \in TM\).

Fix \(w \in \mathbb{V}^{N+1}\). We have from [19] that the subset of the light-cone
\[
\mathbb{E}^N = \mathbb{E}^N_w = \{v \in \mathbb{V}^{N+1} : \langle v, w \rangle = 1\}
\]
is a model for \(\mathbb{R}^N\). In fact, fix \(v \in \mathbb{E}^N\) and let \(A : \mathbb{R}^N \to (\text{span}\{v, w\})^\perp \subset \mathbb{L}^{N+2}\) be a linear isometry. Then the map \(\Psi = \Psi_{v,w,A} : \mathbb{R}^N \to \mathbb{L}^{N+2}\) given by
\[
\Psi(x) = v + Ax - \frac{1}{2}\|x\|_w^2
\]
is an isometric embedding such that \(\Psi(\mathbb{R}^N) = \mathbb{E}^N\). Moreover, the normal bundle is \(N_\Psi\mathbb{R}^N = \text{span}\{\Psi, w\}\) and the second fundamental form is given by
\[ \alpha_{\Psi}(X, Y) = -\langle X, Y \rangle w \]

for all \( X, Y \in T\mathbb{R}^N \).

The map \( F : M^n_0 \to \mathbb{L}^{N+2} \) laying inside \( \mathbb{V}^{N+1} \) given by

\[ F = e^{-\omega} \Psi \circ f \]

is an isometric immersion of \( M^n_0 = (M^n, \langle , \rangle_0) \) that was called in [18] the flat lift of \( f \) (see Section 9.4 in [12]). Clearly \( F \) has flat normal bundle.

**Lemma 16.** The second fundamental form of the flat lift \( F \) of \( f \) is given by

\[ \alpha_F(X, Y) = -Q(X, Y) F + e^{-\omega} \Psi \ast (\alpha_f(X, Y) - \langle X, Y \rangle_0 f_\ast \text{grad} \omega - e^{\omega}(X, Y)_0 w) \]

where everything is computed with respect to the flat metric.

**Proof.** Let \( \tilde{\nabla}, \tilde{\nabla} \) and \( \nabla \) denote the Levi-Civita connections of the metrics \( \langle \cdot, \cdot \rangle \), \( \langle , \rangle \) and \( \langle , \rangle_0 \), respectively. Then

\[ \alpha_F(X, Y) = \tilde{\nabla}_X F_\ast Y - F_\ast \nabla_X Y. \]

Since

\[ F_\ast Y = \tilde{\nabla}_Y F = -e^{-\omega} Y(\omega) \Psi \circ f + e^{-\omega} (\Psi \circ f)_\ast Y, \]

we have

\[ \tilde{\nabla}_X F_\ast Y = (X(\omega) Y(\omega) - XY(\omega)) F - e^{-\omega} (\Psi \circ f)_\ast (Y(\omega) X + X(\omega) Y) + e^{-\omega} \tilde{\nabla}_X (\Psi \circ f)_\ast Y. \]

On the other hand,

\[ \tilde{\nabla}_X (\Psi \circ f)_\ast Y = (\nabla^0_X f_\ast Y + \alpha_{\Psi}(f_\ast X, f_\ast Y)) = (\Psi \circ f)_\ast \tilde{\nabla}_X Y + \Psi_\ast \alpha_f(X, Y) - \langle X, Y \rangle w \]

where \( \nabla^0 \) is the Levi-Civita connection in \( \mathbb{R}^N \). Using (28) we obtain

\[ \tilde{\nabla}_X F_\ast Y = (X(\omega) Y(\omega) - XY(\omega)) F - e^{-\omega} (\Psi \circ f)_\ast (Y(\omega) X + X(\omega) Y - \tilde{\nabla}_X Y) + e^{-\omega} (\Psi_\ast \alpha_f(X, Y) - \langle X, Y \rangle w) = (X(\omega) Y(\omega) - XY(\omega)) F - e^{-\omega} (\Psi \circ f)_\ast (\langle X, Y \rangle_0 \text{grad} \omega - \nabla_X Y) + e^{-\omega} (\Psi_\ast \alpha_f(X, Y) - \langle X, Y \rangle w). \]

Since

\[ F_\ast (\nabla_X Y) = -\nabla_X Y(\omega) F + e^{-\omega} (\Psi \circ f)_\ast \nabla_X Y, \]
we obtain
\[
\alpha_F(X, Y) = \tilde{\nabla}_X F_* Y - F_* \nabla_X Y \\
= (X(\omega) Y(\omega) - \text{Hess} \omega(X, Y)) F - e^{-\omega} \Psi_* (X, Y)_0 f_* \text{grad} \omega \\
+ e^{-\omega} (\Psi_* \alpha_f (X, Y) - \langle X, Y \rangle w) \\
= -Q(X, Y) F + e^{-\omega} \Psi_* (\alpha_f (X, Y) \\
- \langle X, Y \rangle_0 f_* \text{grad} \omega) - e^\omega \langle X, Y \rangle_0 w,
\]
and this concludes the proof. \(\Box\)

**Proposition 17.** Let \(M^n, n \geq 4\), be a globally conformally flat Riemannian manifold and let \(f : M^n \to \mathbb{R}^N\) be a proper isometric immersion with flat normal bundle. Then the flat lift \(F : M_0^n \to \mathbb{L}^{N+2}\) of \(f\) is locally holonomic and proper with respect to the same principal coordinates.

Let \(F : M_0^n \to \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2}, n \geq 4\), be a proper isometric immersion with flat normal bundle of a flat Riemannian manifold. Then \(F\) is the flat lift of an isometric immersion \(f : M^n \to \mathbb{R}^N\) of a globally conformally flat Riemannian manifold that is locally holonomic and proper.

**Proof.** By Theorem 1, there is a local coordinate system \((u_1, \ldots, u_n)\) such that
\[
\alpha_f (X_i, X_j) = 0, \quad 1 \leq i \neq j \leq n,
\]
where \(X_i = \partial_i / \| \partial_i \|\) for \(1 \leq i \leq n\). From Lemma 14 and 16 we obtain
\[
\alpha_F(X_i, X_j) = 0, \quad 1 \leq i \neq j \leq n,
\]
hence \(F\) is holonomic with respect to the same coordinate system.

We have from Lemma 16 that
\[
\alpha_F(X_i, X_i) = -Q(X_i, X_i) F + e^{-\omega} \Psi_* (\alpha_f (X_i, X_i) - e^{-2\omega} f_* \text{grad} \omega) \\
- e^{-\omega} w, \quad 1 \leq i \leq n.
\]
Since by Theorem 1 at most one of the principal normals of \(f\) has multiplicity larger than one, it now follows from Lemma 15 that also \(F\) is proper.

We now prove the second statement. The map \(f : M^n \to \mathbb{R}^N\) given by
\[
\Psi \circ f = \frac{1}{\langle F, w \rangle} F
\]
(for appropriate \(w\)) is an isometric immersion with respect to the conformally flat metric
\[
\langle \cdot, \cdot \rangle = \frac{1}{\langle F, w \rangle^2} \langle \cdot, \cdot \rangle_0.
\]
Let \(\{Y_1, \ldots, Y_n\}\) be an orthonormal tangent base at \(x \in M^n_0\) such that
\[
\alpha_F(Y_i, Y_j) = 0, \quad i \neq j.
\]
It follows from Lemma 16 that

\[ \Psi_\ast \alpha_f(Y_i, Y_j) = \frac{1}{|\langle\langle F, w \rangle\rangle|} Q(Y_i, Y_j) F \]

and

\[
\alpha_F(Y_i, Y_i) = -Q(Y_i, Y_i) F + |\langle\langle F, w \rangle\rangle| \Psi_\ast \left( \alpha_f(Y_i, Y_i) + f_\ast \text{grad} \log |\langle\langle F, w \rangle\rangle| \right) - \frac{w}{|\langle\langle F, w \rangle\rangle|}
\]

for \(1 \leq i \neq j \leq n\). Taking norms in the first equation yields

\[ \alpha_f(Y_i, Y_j) = 0, \quad 1 \leq i \neq j \leq n. \]

Hence \( f \) has flat normal bundle. Moreover, \( f \) is proper since by the second equation and Lemma 15 there is a one to one correspondence between the principal normals of \( f \) and the ones of \( F \). Finally, we have from Theorem 1 that \( f \) is holonomic. \( \square \)

Let \( F : M^n \to \mathbb{L}^{N+2} \) be an isometric immersion of a Riemannian manifold. According to [11, Theorem 17] any Ribaucour transformation \( \tilde{F} : M^n \to \mathbb{L}^{N+2} \) of \( F \) is of the form

\[ \tilde{F} = F - 2\nu \varphi \mathcal{F} \]

where \( \mathcal{F} = F_\ast \text{grad} \varphi + \beta \) and \( \nu^{-1} = \langle\langle \mathcal{F}, \mathcal{F} \rangle\rangle \). Moreover, the function \( \varphi \in C^\infty(M) \) and the vector field \( \beta \in N_F M \) satisfy the condition

\[ \alpha_F(\text{grad} \varphi, X) + \nabla_X^\perp \beta = 0 \quad (34) \]

for any \( X \in TM \). Notice that \((\varphi + c, \beta)\) also satisfies (34) for any \( c \in \mathbb{R} \).

Now assume that \( F(M) \subset \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2} \). Then

\[ \langle\langle F, \beta \rangle\rangle = \varphi + c \quad \text{where} \quad c \in \mathbb{R}. \quad (35) \]

In fact, using (33) and (34) it follows that

\[ X \langle\langle F, \beta \rangle\rangle = \langle\langle F, \nabla_X^\perp \beta \rangle\rangle = -\langle\langle F, \alpha_F(\text{grad} \varphi, X) \rangle\rangle = X(\varphi) \]

for any \( X \in TM \).

**Proposition 18.** Assume that \( F(M) \) lies inside the light-cone \( \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2} \). Then the same holds for any Ribaucour transformation \( \tilde{F} \) of \( F \) for which \( c = 0 \) in (35).

**Proof.** We have that

\[ \langle\langle \tilde{F}, \tilde{F} \rangle\rangle = -4\nu \varphi \langle\langle F, \mathcal{F} \rangle\rangle + 4\nu^2 \varphi^2 \langle\langle \mathcal{F}, \mathcal{F} \rangle\rangle = 4\nu \varphi \left( \varphi - \langle\langle F, \beta \rangle\rangle \right) = 0, \]

and the proof follows. \( \square \)
Now assume that $M^n = M^n_0$ is flat and that $F: M^n_0 \to \mathbb{L}^{N+2}$ is holonomic. It follows from [10, Theorem 13] that the set of all Ribaucour transformations of $F$ that preserve flatness and are holonomic with respect to the same principal coordinates depends on $N + 1$ arbitrary constants. Thus, if $F(M) \subset \mathbb{V}^{N+1}$ it follows from the above that the family of Ribaucour transformations that, in addition, remain in the light-cone depends on $N$ parameters.

**Proof of Theorem 4.** The proof now follows easily using Proposition 17.

**Remark 19.** At least generically, the conformally flat metric of an element in the above family is not conformal to the original metric of $M^n$; see [19, Theorem 20].

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