On the final limit of a transition matrix

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Abstract. For a finite intensity matrix $B$ the final limit of its transition matrix $\exp(tB)$ exists. This is a well-known fact in the realm of continuous-time Markov processes where it is proven by probability theoretic means. A simple proof is presented with help of a Tauberian theorem of complex analytic functions which is used also in [3] to proof the prime number theorem.

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1. Introduction

Square matrices with finite index set $Z$ are considered. A right intensity matrix $B = (b_{ij})$ is defined by $b_{ij} \geq 0$ for all $i, j \in Z$ with $i \neq j$ and by vanishing row sums $\sum_{j \in Z} b_{ij} = 0$ for all $i \in Z$. Its transition matrix is

$$\exp(tB) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n, t \in \mathbb{R}.$$  

Theorem 1. For a right intensity matrix $B$ the limit $\lim_{t \to \infty} \exp(tB)$ exists.

Proof. A standard proof is given in [6], thm. 5.4.6. The assertion follows also by the Tauberian theorem [2], ch. III, thm. 7.1 from the complex analytic Proposition[5] which is deferred to the next section. □

By matrix transposition one obtains an analogous theorem for vanishing column sums. We still restrict to row vectors.

1The probability distribution $P$ of the corresponding Markov process $\{X_t\}_{t \geq 0}$ on the state space $Z$ with initial probability row vector $\pi$ fulfills $(P\{X_t = j\})_{j \in Z} = \pi \exp(tB), t \geq 0.$
2. Complex analytic proof of the theorem

First we recall some theory of homogenous linear differential equation systems with constant coefficients. The identity matrix is denoted by $I$.

**Lemma 2.** For a complex square matrix $A$ the matrix $Y(z) := \exp(zA)$ fulfills $Y(0) = I$, $Y' = YA = AY$ and $Y(w + z) = Y(w) + Y(z)$ for all $w, z \in \mathbb{C}$. In case the norm of $Y$ is bounded on the positive real axis $t > 0$ every entry of the Laplace transform $z \mapsto (zI - A)^{-1}$ of $Y$ is holomorphic on the right half plane $\Re(z) > 0$.

**Proof.** The formulas follow easily from the definition. Laplace transformation of $Y' = YA$ in each entry yields $z\hat{Y}(z) - Y(0) = \hat{Y}(z)A$ for the Laplace transform $\hat{Y}$ of $Y$. Because of $Y(0) = I$ we have indeed $\hat{Y}(z) = (zI - A)^{-1}$. Boundedness of the norm of $Y$ implies boundedness of each entry of $Y$. So $\hat{Y}$ is well defined on $\Re(z) > 0$. Holomorphy follows from $M^{-1} = \text{adj}M/|M|$ (s. [1], Kap. 1, (25)) with the adjoint $\text{adj}M$ of a matrix $M$ with determinant $|M| \neq 0$. \hfill \Box

Next we need some some theory of stochastic matrices. A square matrix $Q = (q_{ij})_{i,j \in \mathbb{Z}}$ is called (right) stochastic when all its entries $q_{ij} \geq 0$ are non-negative and all its row sums are equal to one. Thus the row sum norm $\|Q\|_\infty = 1$ of $Q$ equals one.

**Lemma 3.** For a right intensity matrix $B$ its transition matrix $\exp(tB)$ is stochastic, hence fulfills $\|\exp(tB)\|_\infty = 1$ for all $t \in \mathbb{R}$.

**Proof.** The product $AB$ of any matrix $A$ with $B$ has vanishing row sums. Therefore $B^n$ has vanishing row sums for all $n \in \mathbb{N}$. Hence the partial sums

$$
\sum_{n=0}^{N} \frac{t^n}{n!} B^n = E + tB + \ldots + \frac{t^N}{N!} B^N, \quad N \in \mathbb{N}, t \in \mathbb{R}
$$

have row sums one. By the limiting process with $N \to \infty$ the same assertion holds for $\exp(tB)$. For proving non-negativity we choose a real number $\mu$ such that $T := B + \mu I$ has non-negative entries (like in the proof of [4], thm. 2.7). Since $\exp(tT)$ has non-negative entries and $e^{-\mu t} > 0$ the assertion follows by $\exp(tB) = \exp(-\mu I) \exp(tT) = e^{-\mu t} \exp(tT)$. \hfill \Box

A complex square matrix $(a_{ij})$ is called reducible when there is some non-empty, proper subset $J$ of the index set $Z$ such that for all $i \in Z \setminus J$ and all $j \in J$ it holds $a_{ij} = 0$. Otherwise the matrix is called irreducible. Now we use the notation $D = \text{diag}(D_1, \ldots, D_k)$ for a matrix $D$ of block diagonal form with $k \in \mathbb{N}$ block matrices $D_1, \ldots, D_k$ (in the given order) as main diagonal submatrices of $D$ (with the index set of $D$ as a disjoint union of the index sets of the $D_i$) and with all other submatrices besides the $D_i$ blockwise equal to the zero matrix $O$ (of appropriate dimension).
Lemma 4. By permutation of the index set a stochastic matrix has either the block diagonal form $R := \text{diag}(R_1, ..., R_k)$ with irreducible stochastic matrices $R_i$ or else the form \( \begin{pmatrix} R & O \\ S & T \end{pmatrix} \) for some matrix $S \neq O$ of appropriate dimension and some square matrix $T$ such that $I - T$ is invertible. For every $i \in \{1, ..., k\}$ the polynomial $z \mapsto |zI - R_i|$ has the simple root $z = 1$.

Proof. One may choose iteratively stochastic main submatrices $R_i$ of $B$ of maximal dimension such that they are irreducible. The iteration stops until there is no further stochastic main sumatrix of $B$. In the second case it means that $T$ is not stochastic, hence $S \neq O$, and that $T$ has no stochastic submatrix. Then the assertions follow by [1], Kap. 6, Satz 7'&10. \(\square\)

Proposition 5. For an intensity matrix $B$ it exists a $\delta > 0$ such that every holomorphic entry of the matrix valued function $z \mapsto z(zI - B)^{-1}$, $\Re(z) > 0$ is analytically continuable to the half plane $\Re(z) > -\delta$.

Proof. By Lemma [2] & [3] the given function $F(z) := z(zI - B)^{-1}$ is indeed well-defined and (entry-wise) holomorphic on the half plane $\Re(z) > 0$. The polynomial $z \mapsto |zI - B|$ of $B$ has finitely many roots. So $F$ is meromorphic on the whole complex plane. According to [5], prop. 5.12 all eigenvalues of $B$ are elements of the compact disc of radius $\rho := \max\{-b_{ii} \mid i \in Z\}$ around $-\rho$. Hence the non-zero eigenvalues of $B = (b_{ij})_{i,j \in Z}$ have negative real part. So it suffices to show that $F$ is analytically continuable in $z = 0$. Therefore let $D$ be the diagonal matrix with entries $d_i := -b_{ii}, i \in Z$. Let $e_i := (i-th row of I)$ denote the $i$-th canonical unit vector. In case $d_i = 0$ set the $i$-th row of $Q$ equal to $e_i$, otherwise $q_{ii} := 0$ and $q_{ij} := b_{ij}/d_i$ for $j \neq i$. This defines a stochastic matrix $Q = (q_{ij})_{i,j \in Z}$, since in case $d_i \neq 0$ we have

$$\sum_{j \in Z} q_{ij} = \frac{1}{d_i} \sum_{j \neq i} b_{ij} = \frac{-b_{ii}}{d_i} = 1.$$  

Moreover it holds $B = DQ - D$. For $j \in Z$ we have $e_jB$ equal to zero if and only if $e_jQ = e_j$. Let $J$ be the set of such indices $j$; i.e. the index set of zero-rows of $B$. For the linear hull $U$ of $\{e_j \mid j \in J\}$ there are linear subspaces $V, W$ such that the left null space $N$ of $B$ is the direct sum of $U, V$, and the left eigenspace $E$ of $Q$ with eigenvalue one is the direct sum of $U, W$. Due to $B = DQ - D$ the linear map $x \mapsto xD$ maps $V$ isomorphically onto $W$. So $N$ and $E$ are isomorphic. By suitable permutation of $Z$ we have $Q$ in block form like in Lemma [4] Then also $B$ is in that block form since for all $i \neq j$ holds $q_{ij} = 0$ if and only if $b_{ij} = 0$. In the first case $B$ is of block diagonal form $B = \text{diag}(B_1, ..., B_k)$ with intensity matrices $B_i$. Let $D = \text{diag}(D_1, ..., D_k)$ be the corresponding block form of diagonal matrices $D_i$. Then we have $B_i = D_iR_i - D_i$ for all $i \in \{1, ..., k\}$. In the second case we have an additional diagonal matrix $D_k+1$ such that $C := D_{k+1}T - D_{k+1}$ is the last diagonal block of $B$ and $A := D_{k+1}S - D_{k+1}$ the lower left block of $B$. By the above isomorph the polynomials $z \mapsto |zI - B_i|$ (with $I$ of
appropriate dimension) have the simple root zero and holds \(|C| \neq 0\). From 
\[ \text{diag}(zI - B_1, ..., zI - B_k)^{-1} = \text{diag}((zI - B_1)^{-1}, ..., (zI - B_k)^{-1}) \] 
and 
\[
\begin{pmatrix}
zI - B & O \\
A & zI - C
\end{pmatrix}^{-1} = \begin{pmatrix}
(zI - B)^{-1} \\
-(zI - C)^{-1}A(zI - B)^{-1} & (zI - C)^{-1}
\end{pmatrix}
\]
the assertion follows now. \(\square\)

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