The Phenomenon of Darboux Displacements

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Abstract

For a class of Schrödinger Hamiltonians the supersymmetry transformations can degenerate to simple coordinate displacements. We examine this phenomenon and show that it distinguishes the Weierstrass potentials including the one-soliton wells and periodic Lamé functions. A supersymmetric sense of the addition formula for the Weierstrass functions is elucidated.

Key-Words: Supersymmetry, factorization, addition laws

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1 Introduction

The Darboux (supersymmetry) transformations [1] are an efficient tool to extend the class of exactly solvable spectral problems for the Schrödinger’s Hamiltonians in one space dimension [2,3,4,5,6,7,8], permitting to see that new classes of exact solutions are not an exclusive achievement of General Relativity [9]. The use of the Darboux method turns of special interest in particle and statistical physics, classical mechanics, mathematical physics, biological systems and other areas [10,11,12,13] (see also the monographs [14,15,16,17]). If the initial Schrödinger’s Hamiltonian in \( L^2(\mathbb{R}) \) is:

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x) \tag{1}
\]

the original (1-st order) Darboux algorithm [1,2,3] is equivalent to the operation of ‘transporting’ through \( H \) a 1-st order differential operator \( A \) or its adjoint \( A^\dagger \):

\[
A = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + \alpha(x) \right], \quad A^\dagger = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \alpha(x) \right] \tag{2}
\]

leading to a new Hamiltonian \( \tilde{H} \)

\[
AH = \tilde{H}A \Leftrightarrow HA^\dagger = A^\dagger \tilde{H} \tag{3}
\]

again of Schrödinger’s type

\[
\tilde{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \tilde{V}(x) \tag{4}
\]

provided that the function \( \alpha(x) \) (the superpotential) fulfills the integrability condition in form of the Riccati equation:

\[
-\alpha'(x) + \alpha^2(x) = 2[V(x) - \epsilon], \tag{5}
\]

where \( \epsilon \) is a factorization constant measured in energy units though not necessarily belonging to the spectrum of \( H \) [18]. If (3-5) hold, the new potential \( \tilde{V} \) is given by

\[
\alpha'(x) + \alpha^2(x) = 2[\tilde{V}(x) - \epsilon] \quad \Rightarrow \quad \tilde{V}(x) = V(x) + \alpha'(x)
\]

The links of (3) with the traditional factorization method [2,3] are due to the identities

\[
H - \epsilon = A^\dagger A, \quad \tilde{H} - \epsilon = AA^\dagger \tag{6}
\]

(see, e.g., Andrianov [5], Nieto [4], Sukumar [19]). The Darboux transformation [3], in general, leads to a new class of spectral problems of form (4) with \( \tilde{V}(x) \) essentially different from \( V(x) \). Yet, the method allows some intriguing exceptions.

As recently discussed [20,21,22], some special periodic potentials \( V(x) \equiv V(x + T) \) admit the Darboux transformations which consist simply in coordinate displacements, \( \tilde{V}(x) = V(x + \delta) \), where \( \delta = T/2 \). As subsequently shown, the half period is
not the only displacement available. For a subclass of periodic Lamé potentials \[23\], we have detected the supersymmetric displacements \(\delta\) varying continuously in the interval \((0, T)\) \[24\]. We propose to call this phenomenon the \textit{translational invariance with respect to Darboux transformations}, or simply the \textit{Darboux invariance} \[24\]. Since very little is changed in \(V(x)\) by just shifting the argument, the Darboux displacements might look as ‘frustrated cases’ of the Darboux method (at least as far as the quest for new exact solutions in quantum mechanics is concerned). Yet, we shall show that the effect is mathematically nontrivial and of tentative physical interest. We shall formulate the necessary and sufficient conditions for a potential to be Darboux invariant and find an intimate relation between the Darboux invariance and elliptic functions. Moreover, it turns out that the Schrödinger equation with a Darboux invariant potential may be integrated by quadratures for any value of \(E\). This gives a convenient tool to construct essentially new solvable potentials by applying the finite difference Bäcklund algorithm \[25\] \[26\] \[27\].

2 The Weierstrass condition

Following our former observations \[24\], we shall look for some deeper reasons of the Darboux invariance. We shall thus consider a hypothetical potential \(V(x)\) (periodical or not) which admits Darboux transformations consisting in pure \(x\)-displacements. Let \(A\) and \(A^\dagger\) be the first order differential operators

\[
A = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + \alpha(x, \delta) \right], \quad A^\dagger = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \alpha(x, \delta) \right]
\]

producing this effect, \(i.e.,\)

\[
AH = H_\delta A
\]

where

\[
H_\delta = -\frac{1}{2} \frac{d^2}{dx^2} + V(x + \delta)
\]

Notice that if \(\delta\) is one of the admissible Darboux displacement \(\text{(7-8)}\), so is \(-\delta\) (the formal conjugation of \(\text{(7)}\) and the change of variable \(x \rightarrow x - \delta\) implying that \(\alpha'(x, -\delta) = -\alpha(x - \delta, \delta)\) is the corresponding superpotential). Note also that if the Hamiltonian \(H\) admits a Darboux displacement \(\delta\) \(\text{(7-8)}\), then any displaced version \(H_{\delta'}\) of \(H\) can be as well \(\delta\)-displaced

\[
A' H_{\delta'} = H_{\delta + \delta} A', \quad A' \equiv \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + \alpha(x + \delta', \delta) \right]
\]

The set \(D\) of all Darboux displacements \(\delta\) generated by the first order intertwiners \(\text{(2)}\) for a given Hamiltonian \(H\) is now a decisive element. Our previous study \[24\] shows that for the one soliton or Lamé potentials with \(n = 1\) the allowed displacement can be any \(\delta \in (0, T)\) (where \(T\) is either the real Lamé period or \(T = +\infty\) in the 1-soliton case). We shall see that even the existence of a finite number of Darboux displacements can be a tight structural information. Indeed, one has:
Proposition 1. If the Hamiltonian $H$ admits three Darboux displacements $\delta_1, \delta_2, \delta_3$, such that $\delta_1 + \delta_2 + \delta_3 = 0$, then up to an additive constant, the potential $V$ reduces to one of the Weierstrass functions.

Proof follows immediately from a sequence of mathematical results [28, 29, 30], concerning the invariance of the Schrödinger’s Hamiltonians under the generalized Darboux transformations, where the intertwiners $A$ in (3) can be differential operators of arbitrary order. Indeed, suppose $A_1, A_2$ and $A_3$ are three first order Darboux operators $A_i = [d/dx + \alpha_i(x)]/\sqrt{2}$ inducing the subsequent displacements

$$H \rightarrow H_{\delta_1} \rightarrow H_{\delta_1 + \delta_2} \rightarrow H_{\delta_1 + \delta_2 + \delta_3} = H.$$ (9)

Then the product $D_3 = A_3 A_2 A_1$ must commute with the Hamiltonian (1), implying that $H$ and $D_3$ form a commuting Lax pair (compare [31, 28]). Hence, there exists a constant $V_0 \in \mathbb{R}$ such that $\phi(x) = V(x) - V_0$ must fulfill the stationary KdV equation (see also [30]), leading to the 1-st order Weierstrass equation [32]

$$(\phi')^2 = 4\phi^3 - g_2\phi - g_3, \quad g_2, g_3 = \text{const}$$ (10)

The equation (10) admits two families of real solutions:

- The singular family ($S$) is given by the traditional Weierstrass functions:

$$\int_{-\infty}^{\phi} \frac{d\nu}{\sqrt{4\nu^3 - g_2\nu - g_3}} = x - a \Rightarrow \phi(x) \equiv \wp(x - a; g_2, g_3)$$ (11)

If $a, g_2, g_3$ are real, $\phi(x)$ is real too, but the family admits also an analytic continuation to complex $a$. In fact, if $\omega' = i\tau$ ($\tau \in \mathbb{R}$) is half-imaginary period of $\wp$, then one sees: $\wp(x - i\tau; g_2, g_3) = \wp(x + i\tau; g_2, g_3) = \wp(x - i\tau; g_2, g_3)$. Thus, (11) defines as well

- The regular family ($R$) is given as the ‘parallel real section’ of (11) for $a \rightarrow a + i\tau$

$$\phi(x) \equiv \wp(x - a - i\tau; g_2, g_3)$$ (12)

To obtain a geometric image of both branches, the phase portrait of (10) is relevant [33]. Take for simplicity $a = 0$. Interpreting $\phi$ and $\phi'$ respectively as the coordinate and momentum of a hypothetical point particle, with $x$ meaning the ‘time’, one can view (10) as a dynamical law defining the ‘momentum’ $p = \phi'$ as a function of the ‘position’ $\phi$, thus allowing $\phi$ to move only in the permitted areas where $P(\phi) = 4\phi^3 - g_2\phi - g_3 \geq 0$ and (10) is consistent with $(\phi')^2 \geq 0$. If all three roots $e_1, e_2, e_3$ of $P(\phi)$ are real, $e_3 < e_2 \leq e_1$, there are two allowed intervals $[R] = [e_3, e_2]$ and $[S] = [e_1, +\infty)$ where $P(\phi) = (\phi')^2$ permits the real $\phi'$. The motions in $[S]$ typically depart from and return to the infinity at a finite time $T$ (a real period of $\wp$); their repetitions paint an image of the periodic, singular
Weierstrass functions. In turn, the motions in \([R]\), in general, oscillate between two turning points \(\phi_3 = e_3, \phi_2 = e_2\), yielding the real, regular, bounded solutions of \([11]\) with a real period \(T\). If \(e_1 = (2 - m)/3, e_2 = (2m - 1)/3, e_3 = -(m + 1)/3\) and the oscillation period in \([R]\) is \(T = 2\omega\) (we adopt the notation of \([34]\)), then we obtain the Lamé function \(\varphi(x) = \text{sn}^2(x|m) - (m+1)/3\), but if \(e_1 = e_2 = 1/3 > e_3 = -2/3\) the oscillation time in \([R]\) tends to infinity and the motion reproduces the one-soliton transparent well.

As we have already observed, the regular solutions in \([R]\) can be as well obtained by an analytic continuation of the singular Weierstrass solutions in \([S]\). Thus, \(\varphi(x) = \text{sn}^2(x|m) - (m+1)/3\), while the one soliton well is the \(i\tau\)-displaced case of the singular solution \(\varphi(x)\) (see (31) in \([27]\)).

So far, \([10, 11]\) are just a necessary condition for the existence of any 3-order Darboux symmetry of the initial Hamiltonian \(H\) (and by the same for the existence of a triple Darboux displacement \([9]\) closing to identity). Quite remarkably, the condition turns also sufficient, though the proof of this last fact is less evident. We shall therefore formulate an independent criterion which is both necessary and sufficient for the existence of the Darboux displacements.

3 The supersymmetric addition law

Notice that even the existence of a single 1-st order intertwining operator producing a displacement \(\delta\) imposes strong restrictions on the corresponding potential \(V(x)\). Of course, if \(V\) is periodic, with a real period \(T\) and \(V \neq \text{const}\) then \(\delta \neq nT\) \((n \in \mathbb{Z})\). Indeed, if \(\delta = nT\), there would be a Darboux operator \([7]\) generating the identity transformation \(H_\delta = H\), \(i.e.,\) commuting with \(H\), which is impossible except if \(V(x) \equiv \text{const}\). Assume now that \([2]\) is one of operators generating a Darboux displacement \(\delta\) for the Hamiltonian \([11]\); hence

\[-\alpha'(x) + \alpha^2(x) = 2[V(x) - \epsilon], \tag{14}\]

\[\alpha'(x) + \alpha^2(x) = 2[V(x + \delta) - \epsilon] \tag{15}\]

where \(\epsilon\) is a factorization constant. Due to \([14, 15]\)

\[\alpha^2(x) = V(x) + V(x + \delta) - 2\epsilon, \tag{16}\]

\[\alpha'(x) = V(x + \delta) - V(x) \tag{17}\]

Determining \(\alpha(x)\) from \([16]\) one finds

\[\alpha(x) = \pm \sqrt{V(x) + V(x + \delta) - 2\epsilon} \tag{18}\]
Differentiating (18) and comparing with (17) one thus arrives at the following functional equation:

\[ V(x) + V(x + \delta) - \frac{1}{4} \left[ \frac{V'(x) + V'(x + \delta)}{V(x) - V(x + \delta)} \right]^2 = 2\epsilon \]  

(19)

The eq. (19) turns out a necessary condition for \( V(x) \) to admit the Darboux displacement \( V(x) \to V(x + \delta) \) with the factorization constant \( \epsilon \). Inversely, suppose \( V(x) \) fulfills (19) with certain constants \( \delta \) and \( \epsilon \). Then define \( \alpha(x) \) by (18), assuring automatically (16). Differentiating (18) one obtains:

\[ \alpha'(x) = \pm \frac{1}{2} \frac{V'(x) + V'(x + \delta)}{\sqrt{V(x) + V(x + \delta) - 2\epsilon}} \]

If the sign in (18) is “+” the superpotential \( \alpha(x) \) generates the Darboux displacement \( V(x) \to V(x + \delta) \), while the sign “−” yields the inverse displacement \( V(x + \delta) \to V(x) \). By choosing the proper sign + and by applying (19) one recovers (17). We thus arrived at

**Theorem 1.** The necessary and sufficient condition for \( V(x) \) to admit a non-trivial Darboux displacement \( \delta \) is that the left hand side of (19) is independent of \( x \). Its value defines the factorization constant \( \epsilon \) for the corresponding superpotential \( \alpha(x) \).

In order to admit a set \( D \subset \mathbb{R} \) of many Darboux displacements, the potential \( V(x) \) must satisfy a family of many simultaneous difference-differential equations of type (19). If \( D \) is non-trivial, the Proposition 1 implies that \( V(x) = \phi(x) + V_0 \) where \( \phi \) is in the Weierstrass class of functions. We shall see now that the set of conditions (19) with continuous \( \delta \) is indeed generic for the Weierstrass functions.

In fact, assume again \( a = 0 \) in (11-12). Then, examine the sense of (19) with \( V(x) \) even, for \( \delta \neq nT \) (put \( T = 0 \) if \( V \) aperiodic). The constant \( \epsilon \) has to depend on \( \delta \), \( \epsilon = \epsilon(\delta) \). Denote for simplicity \( \mathcal{E}(\delta) = -2\epsilon(\delta) \). Introducing the new variables \( u = x \) and \( v = -\delta - x \), and using the fact that \( V'(x) \) is odd, one can write (19) in the form

\[ \mathcal{E}(u + v) + V(u) + V(v) = \frac{1}{4} \left[ \frac{V'(u) - V'(v)}{V(u) - V(v)} \right]^2 \]

(20)

Notice now that this condition reduces to the well known addition formulae for the Weierstrass singular (S) and regular (R) functions. Indeed, for the singular branch (S) the traditional identity tells

\[ \wp(u + v) + \wp(u) + \wp(v) = \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 \]

(21)

for all \( u, v, u + v \) out of the singularities of (21) (see e.g. Bateman [32]). Replacing now \( u \to u - i\tau \), \( v \to v - i\tau \), \( \wp(u - i\tau) = \phi(u) \), and making use of the fact
\[ \wp(u - 2i\tau) \equiv \wp(u), \] one sees that for the regular branch (R)

\[ \mathcal{E}(u + v) + \wp(u) + \wp(v) = \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 \]  \tag{22}

where \( \mathcal{E}(\delta) \equiv \wp(\delta) \) is the Weierstrass function of the (S) branch linked with \( \wp \) by analytic continuation. We thus have

**Theorem 2.** The addition laws (20) for \( V(x) \) and (21, 22) for the Weierstrass functions are nothing else but the necessary and sufficient conditions for the existence of a continuum of the Darboux displacements.

Note, that we have thus detected a new sense of the traditional addition formulae (21, 22). Though these formulae are a part of the textbook material on the elliptic functions [32, 35], the fact that they can be so simply obtained by demanding the existence of the Darboux displacements (14-15) apparently, escaped attention. We conclude that, without calling much attention, the Darboux displacements were always present in the structure of the elliptic functions, explaining the exact form of the addition laws. Some other points may be worth making.

**Observation 1.** Though it was well established that the Weierstrass functions admit a 3-rd order symmetry (leading to the 3-rd order stationary KdV, see [28, 29, 30]), as far as we know, it was not noticed that this symmetry can be realized as a triple Darboux displacement.

**Observation 2.** Though one knows that the algebraic addition laws limit the form of the corresponding functions (permitting only rational or elliptic solutions), it has not been noticed that the composition laws (20, 21, 22) have even stronger consequencies. This is due to the fact that (20-22) are not purely algebraic, but have a form of difference-differential equations. Of course, (21, 22) can be rewritten as algebraic identities after eliminating \( \wp' \) by using (10), but this would reduce the implications to the traditional Weierstrass theorem (stating that any meromorphic function \( f \) which obeys an algebraic addition law for \( f(x) \), \( f(y) \) and \( f(x + y) \), must be an elliptic function; see e.g. Akhiezer [35], p.190). The consequencies of the difference-differential laws (20, 22) go beyond that.

**Corollary (inverse addition theorem).** If a real, differentiable, even function \( \wp(x) \) fulfills the addition law (22) for arbitrary \( u, v \), with \( \mathcal{E} \) having an isolated singularity at 0, then \( \wp(x) \) is one of Weierstrass functions.

**Proof.** Indeed, if \( \wp(x) \) is even and fulfills (22), then it also satisfies (13) for any \( \delta = u + v \) whenever \( \mathcal{E}(\delta) \) is finite. Then, due to our Theorem 1, an arbitrary \( \delta = u + v \) (out of singularities) belongs to the admissible 1-susy displacements, and in view of the Proposition 1, \( \wp(x) \) belongs either to (R) or to (S) Weierstrass classes (10-11).
Although our proof is immediate, the theorem (as far as we know) was never proved, apparently since the supersymmetry methods has not been used in the theory of the elliptic functions.

Let us also notice that the ‘supersymmetric sense’ of the addition formulae grants an explicit integrability of the Riccati equation \((5)\), making specially easy the use of the finite-difference Bäcklund algorithm \([26, 27]\) to generalize the (R) or (S) potentials. Indeed, for any \(V(x)\) obeying \((20)\) the special solutions \(\alpha(x, \delta)\) of \((5)\), generating the displacements, are explicitly given by \((18)\) without the need of solving any differential equation. Alternatively, using \((17)\) one obtains:

\[
\alpha(x, \delta) = \int [V(x + \delta) - V(x)]dx = \zeta(x) - \zeta(x + \delta) + \zeta(\delta),
\]

where \(\zeta(x)\) is the ‘non-elliptic’ Weierstrass function \([32]\) and the last (constant) term in \((23)\) was determined consistently with \((18, 20)\). This might seem a limited achievement (why to use the ‘supersymmetric machine’ just to displace the argument in \(V(x)\)?), but since \(\epsilon(\delta) = \epsilon(-\delta)\), the formula \((23)\) gives two independent solutions of the Riccati equation \((5)\) for the same factorization energy \(\epsilon\). Thus, the general solution can be easily obtained with the help of only one quadrature (see e.g. \([36]\)), \(i.e.,\) in terms of a new auxiliary function

\[
\tilde{\alpha}(x) = \Gamma e^{\int \left[\zeta(x+\delta) - \zeta(x) + 2\zeta(\delta)\right]dx} = \Gamma \frac{\sigma(x - \delta)}{\sigma(x + \delta)} e^{2\zeta(\delta)x}
\]

where \(\sigma(x)\) is another non-elliptic Weierstrass function. The general solution of \((5)\) then becomes:

\[
\alpha(x, \epsilon) = \frac{\alpha(x, \delta) - \alpha(x, -\delta)\tilde{\alpha}(x)}{1 - \tilde{\alpha}(x)}
\]

The possibility of solving generally the Riccati eq. \((5)\) for the Lame potential \((13)\) is well known, but we have never seen an argument so simple as the one based on the Darboux displacements. Moreover, by using \((23, 24)\) with varying \(|\delta|\), one has explicit expressions for the superpotentials with different factorization constants; a fact specially convenient for obtaining the transformed Lamé potentials via the purely algebraic Bäcklund algorithm \([26, 27]\)

\[
\alpha_2(x; \epsilon_1, \epsilon_2) = -\alpha_1(x; \epsilon_1) - \frac{2(\epsilon_1 - \epsilon_2)}{\alpha_1(x; \epsilon_1) - \alpha_1(x; \epsilon_2)}
\]

where \(\alpha_1\) can be either the displacement inducing solution \((23)\) or the general one \((25)\). As an example, we have used the general solution \((23, 25)\) to produce an impurity of the Lamé potential inserting a bound state into the lowest forbidden band (see Fig.1).

An atypical application of \((23)\) permits to generate as well the complex Darboux displacements. Indeed, it is known that the roots \(\epsilon_1, \epsilon_2, \epsilon_3\) of the Weierstrass polynomial \(P(\phi)\) determine the band edges of the nonsingular, periodic Weierstrass
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