A SELF ADAPTIVE INERTIAL ALGORITHM FOR SOLVING
SPLIT VARIATIONAL INCLUSION AND FIXED POINT
PROBLEMS WITH APPLICATIONS

TIMILEHIN OPEYEMI ALAKOYA
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa

LATEEF OLAKUNLE JOLAOSO AND OLUWATOSIN TEMITOPE MEWOMO*
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa

(Communicated by Nobuo Yamashita)

Abstract. We propose a general iterative scheme with inertial term and self-adaptive stepsize for approximating a common solution of Split Variational Inclusion Problem (SVIP) and Fixed Point Problem (FPP) for a quasi-nonexpansive mapping in real Hilbert spaces. We prove that our iterative scheme converges strongly to a common solution of SVIP and FPP for a quasi-nonexpansive mapping, which is also a solution of a certain optimization problem related to a strongly positive bounded linear operator. We apply our proposed algorithm to the problem of finding an equilibrium point with minimal cost of production for a model in industrial electricity production. Numerical results are presented to demonstrate the efficiency of our algorithm in comparison with some other existing algorithms in the literature.

1. Introduction. In 2011, Moudafi [30] introduced the Split Variational Inclusion Problem (SVIP) defined as follows: find \( x^* \in H_1 \) such that
\[
0 \in B_1(x^*),
\]
and
\[
y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*),
\]
where 0 is the zero vector, \( H_1 \) and \( H_2 \) are real Hilbert spaces, \( B_1 : H_1 \to 2^{H_1} \) and \( B_2 : H_2 \to 2^{H_2} \) are multi-valued maximal monotone mappings, \( A : H_1 \to H_2 \) is a bounded linear operator.

The SVIP (1)-(2) constitutes a pair of variational inclusion problems, which have to be solved so that the image \( y^* = Ax^* \) under a given bounded linear operator \( A \), of the solution \( x^* \) of SVIP (1) in \( H_1 \) is the solution of another SVIP (2) in another space \( H_2 \). We denote the solution set of SVIP (1) by \( \text{SOLVIP}(B_1) \) while the solution

2020 Mathematics Subject Classification. Primary: 47H10, 47J25; Secondary: 65J15.
Key words and phrases. Split variational inclusion problems, inertia, quasi-nonexpansive mappings, k-demicontractive mappings, strong convergence.
* Corresponding author: Oluwatosin Temitope Mewomo.
set of SVIP (2) is denoted by SOLVIP(B₂). Hence, the solution set of the SVIP (1)-(2) is denoted by

$$\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}. \quad (3)$$

In recent years, SVIP has received great attention by many authors, who improved it in various ways, see, e.g. [22, 35, 37, 43] and references therein, and it has been applied in different real-world problems such as intensity-modulated radiation therapy (IMRT), in sensor networks and in computerized tomography and data compression.

Byrne et al. [9] studied the weak and strong convergence of the following iterative method for solving SVIP: Given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the following algorithm:

$$x_{n+1} = J^{B_1}_\lambda (x_n + \gamma A^*(J^{B_2}_\lambda - I)Ax_n), \quad (4)$$

for $\lambda > 0$ and $A^*$ is the conjugate transpose of $A$, $L = \|A^*A\|$, $\gamma \in (0, \frac{2}{L})$, and $J^{B_i}_\lambda := (I + \lambda B_i)^{-1}$, for $i = 1, 2$ are the resolvent operators of $B_i$. Under certain conditions it was proved that the sequence $\{x_n\}$ generated by (4) converges strongly to $x^*$, which is the solution of the SVIP. Note that $J^{B_i}_\lambda$, for $i = 1, 2$ are nonexpansive and firmly nonexpansive.

Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonlinear mapping, a point $x \in C$ is called a fixed point of $S$ if $Sx = x$. We denote by $F(S)$, the set of all fixed points of $S$. Iterative methods for finding fixed point of nonlinear mappings have recently been applied to solve convex minimization and related optimization problems, see e.g. [3, 18, 20, 21, 23, 34, 36, 42, 45, 49, 50] and the references therein. Convex minimization problems have greatly influenced the development of nearly all branches of pure and applied sciences. A typical convex minimization problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping on a real Hilbert space $H$:

$$\theta(x) = \min_{x \in C} \frac{1}{2} \langle Gx, x \rangle - \langle x, b \rangle, \quad (5)$$

where $G$ is a bounded linear operator, $C$ is the fixed point set of a nonexpansive mapping $T$ on $H$ and $b$ is a given point in $H$.

In [29], Marino and Xu introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [31]:

$$x_{n+1} = (I - \alpha_n G)Tx_n + \alpha_n\beta f(x_n), \quad n \geq 0, \quad (6)$$

where $f$ is a contraction on $H$ with contraction coefficient $\alpha \in (0, 1)$, $G$ is a strongly positive bounded linear operator on $H$ with constant $\mu$ and $0 < \beta < \frac{\mu}{\gamma}$, $T$ is a nonexpansive mapping and $\{\alpha_n\}$ is a sequence in $(0, 1)$. It was proved that if the sequence $\{\alpha_n\}$ of parameters satisfies certain conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of the variational inequality

$$\langle (G - \beta f)x^*, x - x^* \rangle \geq 0, \quad x \in C, \quad (7)$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Gx, x \rangle - h(x),$$

where $h$ is a potential function for $\beta f$ (i.e., $h'(x) = \beta f(x)$ for $x \in H$).
In this paper, we are interested in studying the problem of finding a common solution of both the SVIP (1)-(2) and FPP, that is, finding an element

\[ x \in \mathcal{F} = \Gamma \cap F(S) \neq \emptyset, \]  

(8)

where \( S \) is a quasi-nonexpansive mapping.

Recently, Wangkeeree et al. [48] introduced the following general iterative scheme for approximating a common solution of SVIP and FPP for a nonexpansive mapping in real Hilbert space setting.

\[ x_0 \in H_1, \]

\[ u_n = J_{B_1}^{\lambda}(x_n + \gamma A^* (J_{B_2}^{\lambda} - I)Ax_n), \]

\[ x_{n+1} = \alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n, \]  

(9)

where \( f : H_1 \rightarrow H_1 \) is a contraction with constant \( \alpha \in (0, 1) \), \( S : H_1 \rightarrow H_1 \) is a nonexpansive mapping, \( G : H_1 \rightarrow H_1 \) is a strongly positive bounded linear operator with constant \( \mu \) and \( 0 < \beta < \frac{\mu}{2}, \lambda > 0, \gamma \in (0, \frac{1}{2}) \), where \( L \) is the spectral radius of the operator \( A^* A \), \( \{\alpha_n\} \subset (0, 1) \) and \( B_1 : H_1 \rightarrow 2^{H_1}, B_2 : H_2 \rightarrow 2^{H_2} \) are two multi-valued mappings on \( H_1 \) and \( H_2 \) respectively. Under certain conditions, the sequence generated by the proposed Algorithm 1.1 was proved to converge strongly to a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping.

We point out here that the stepsize \( \gamma \) of Algorithm 1.1 plays an essential role in the convergence properties of the iterative method. Many of the existing iterative schemes for solving SVIP involve step-sizes that depend on the norm of the bounded linear operator \( A \). Such algorithms are usually not easy to implement because they require computation of the operator norm which oftentimes is difficult to compute.

Very recently, Tang [46] propose the following self-adaptive step-size algorithm for solving SVIP in Hilbert spaces, which does not require a prior knowledge of the operator norm.

\[ \tau_n = \frac{\rho_n f(x_n)}{||F(x_n)||^2 + ||H(x_n)||^2}, \]

and calculate the next iterate as

\[ x_{n+1} = J_{B_1}^{\lambda}(I - \tau_n A^*(I - J_{B_2}^{\lambda})Ax_n). \]

Stop Criterion: If \( x_{n+1} = x_n \), then stop. Otherwise, set \( n := n + 1 \) and return to Iterative Step,

where \( f(x) = \frac{1}{2}||(I - J_{B_2}^{\lambda})Ax||^2, F(x) = A^*(I - J_{B_2}^{\lambda})Ax \) and \( H(x) = (I - J_{B_1}^{\lambda})x \). Under mild conditions, the weak convergence of the proposed algorithm was obtained.

Based on the heavy ball methods of a two-order time dynamical system, Polyak [39] first proposed an inertial extrapolation as an acceleration process to solve the
smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several authors have constructed some fast iterative algorithms by using inertial extrapolation which includes inertial proximal methods [4, 32], inertial forward-backward methods [26], inertial split equilibrium methods [22], inertial proximal ADMM [15] and the fast iterative shrinkage thresholding algorithms FISTA [6, 14].

Very recently, Long et al. [25] introduced a new algorithm which combines the inertial technique and the viscosity method for solving SVIP in Hilbert spaces. The proposed algorithm is as follows:

Algorithm 1.3.

\[
\begin{align*}
x_0, x_1 & \in H_1, \\
w_n &= x_n + \theta_n(x_n - x_{n-1}), \\
y_n &= J_{\lambda_n}^B((I - \lambda_n A^*(I - J_{\lambda_n}^B)A)w_n), \\
x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n,
\end{align*}
\]

where \( \lambda > 0, \{\lambda_n\} \) is a sequence of real numbers such that \( 0 < a \leq \lambda_n \leq b < \left(\frac{1}{L}\right), \) \( L := \|A\|^2, f : H_1 \to H_1 \) is a contraction mapping with contraction parameter \( k \in [0, 1), \{\alpha_n\} \subset (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0, \)

\[
\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \{\theta_n\} \text{ is a sequence in } [0, \theta] \text{ for some } \theta > 0, \text{ where } \{\theta_n\} \text{ is}\]

chosen such that \( \lim_{n \to \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0. \) It was proved that the sequence \( \{x_n\} \) generated by the algorithm converges strongly to an element in the solution set of the SVIP. Note that the term \( \theta_n(x_n - x_{n-1}) \) is referred to as the inertial term and it can be regarded as a procedure for speeding up the convergence properties of an iterative scheme.

Motivated by the work of Moudafi [30], Marino and Xu [29], Wangkeeree et al., [48], Tang [46] and Long et al. [25] and by the ongoing research in this direction, in this paper we propose a general iterative scheme with inertial technique and self-adaptive stepsize for approximating a common solution of split variational inclusion problem (SVIP) and fixed point problem (FPP) for quasi-nonexpansive mapping in real Hilbert spaces. Our iterative scheme was proved to converge strongly to a common solution of SVIP and FPP for a quasi-nonexpansive mapping, which is a solution of a certain optimization problem related to a strongly positive bounded linear operator. The self-adaptive step-size ensures that no prior knowledge or estimate of the operator norm is required. The proposed algorithm was used to find an equilibrium point with minimal cost of production for a model in industrial electricity production. Numerical results for the model are presented to demonstrate the efficiency of our algorithm in comparison with some other existing algorithms in the literature. Results obtained are also applied to solve split feasibility problems and split minimization problems. Our results improve and generalize corresponding results of Moudafi [30], Marino and Xu [29], Wangkeeree et al., [48], Tang [46] and Long et al. [25] and many others in the literature.

2. Preliminaries. In this section, we recall some concepts and results which are needed in the sequel. Throughout this paper unless otherwise stated, we assume that \( H_1 \) and \( H_2 \) are two real Hilbert spaces each with inner product \( \langle \cdot , \cdot \rangle \), induced norm \( \| \cdot \| \) and \( I \) the identity operator. Let \( C \) and \( D \) be nonempty, closed and convex subsets of \( H_1 \) and \( H_2 \) respectively, let \( A : H_1 \to H_2 \) be a bounded linear operator.
and $A^* : H_2 \rightarrow H_1$ be the conjugate transpose of $A$ (in finite dimensional spaces, $A^* = A^T$). The symbols “→” and “⇒” denote the strong and weak convergence respectively. For a given sequence $\{x_n\} \subset H$, $w_\omega(x_n)$ denotes set of weak limits of $\{x_n\}$, that is, $w_\omega(x_n) := \{x \in H : x_{n_j} \rightharpoonup x\}$ for some subsequence $\{n_j\}$ of $\{n\}$.

The metric projection $P_C : H \rightarrow C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = \inf\{||x - z|| : z \in C\}.$$  

It is known that $P_C$ is nonexpansive and has the following properties:

(i) $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in C$,

(ii) for any $x \in H$ and $z \in C, z = P_C x$ if and only if \( \langle x - z, z - y \rangle \geq 0 \), for all $y \in C$,

(iii) for any $x \in H$ and $y \in C$,

$$||P_C x - y||^2 + ||P_C x - y||^2 \leq ||y - x||^2.$$  

**Definition 2.1.** A mapping $T : H \rightarrow H$ is called

(i) Lipschitzian or Lipschitz continuous on $H$, if there exists $L > 0$ such that $||Tx - Ty|| \leq L||x - y||$ for all $x, y \in H$.

If $L \in [0, 1)$, then $T$ is called a contraction mapping;

(ii) nonexpansive if $T$ is 1–Lipschitzian;

(iii) firmly nonexpansive if

$$||T(x) - T(y)||^2 \leq \langle T(x) - T(y), x - y \rangle, \text{ for all } x, y \in H;$$

or equivalently

$$||Tx - Ty||^2 \leq ||x - y||^2 - \langle (I - T)x - (I - T)y, ||x - y|| \rangle, \text{ for all } x, y \in H;$$

(iv) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - p|| \leq ||x - p|| \text{ for all } x \in H, p \in F(T);$$

(v) $k$-demicontractive if there exists a constant $k \in [0, 1)$ such that

$$||Tx - p||^2 \leq ||x - p||^2 + k||x - Tx||^2 \text{ for all } x \in H, p \in F(T).$$

Clearly, the class of $k$–demicontractive mappings properly includes the class of quasi-nonexpansive mappings, and the class of quasi-nonexpansive mappings properly contains the class of nonexpansive mappings while the class of nonexpansive mappings properly contains the class of firmly nonexpansive mappings.

**Definition 2.2.** A mapping $T : H \rightarrow H$ is called

(i) monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0 \text{ for all } x, y \in H;$$

(ii) $\alpha$-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha||x - y||^2 \text{ for all } x, y \in H;$$
(iii) $\beta$-inverse strongly monotone, if there exists a constant $\beta > 0$ such that
$$\langle Tx - Ty, x - y \rangle \geq \beta \|Tx - Ty\|^2$$
for all $x, y \in H$;

(iv) strongly positive if there exists a constant $\gamma > 0$ such that
$$\langle Tx, x \rangle \geq \gamma \|x\|^2$$
for all $x \in H$;

(v) averaged if there exists a nonexpansive mapping $S : H \to H$ and a number $c \in (0,1)$ such that
$$T = (1-c)I + cS.$$

Recall that the mapping $I - T$ is demiclosed at zero [7] if and only if $x_n \to x$ and $x_n - Tx_n \to 0$. It is well known that if $T : H \to H$ is nonexpansive, then $I - T$ is demiclosed at zero.

**Definition 2.3.** Let $B : H_1 \to 2^{H_1}$ be a multi-valued mapping and let $\lambda > 0$.

- $B$ is called monotone if
  $$\langle u - v, x - y \rangle \geq 0 \text{ for all } u \in B(x), v \in B(y),$$
  and $B$ is called maximal monotone mapping if the graph $G(B)$ of $B$
  $$G(B) := \{(x,u) \in H_1 \times H_1 | u \in B(x)\}$$
  is not properly contained in the graph of any other monotone mapping.
- The *resolvent* of $B$ with parameter $\lambda > 0$ denoted by $J^B_\lambda$ is given by
  $$J^B_\lambda := (I + \lambda B)^{-1}.$$

**Remark 1.** For $\lambda > 0$, the following results hold [46]:

1. $B$ is maximal monotone if and only if $J^B_\lambda$ is single-valued, firmly nonexpansive and $dom(J^B_\lambda) = H_1$, where $dom(B) := \{x \in H_1 | B(x) \neq \emptyset\}$.
2. The point $x^* \in B^{-1}(0)$ if and only if $x^* = J^B_\lambda x^*$.
3. The solution set $\Gamma$ of the SVIP (1)-(2) is equivalent to finding $x^* \in H_1$ with $x^* = J^B_\lambda x^*$ such that
   $$y^* = Ax^* \in H_2 \text{ and } y^* = J^{B_2} y^*.$$

**Lemma 2.4.** Let $\delta \in (0, 1)$, for $x, y \in H_1$, we have the following statements [33, 47]:

1. $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$;
2. $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2$;
3. $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

**Lemma 2.5.** (Lemma 2.6 of [40]) Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that
$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n b_n, \text{ for all } n \geq 1,$$
if $\limsup_{k \to \infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying
$$\liminf_{k \to \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0,$$ then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.6.** (Lemma 3.1 of [27]) Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that
$$a_{n+1} \leq (1 - \sigma_n) a_n + b_n + c_n \text{ for all } n \geq 0.$$
If \( b_n \leq \beta \sigma_n \) for some \( \beta \geq 0 \), then \( \{a_n\} \) is a bounded sequence.

If we have
\[
\sum_{n=0}^{\infty} \sigma_n = \infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{b_n}{\sigma_n} \leq 0,
\]
then \( \lim_{n \to \infty} a_n = 0 \).

3. Main result. In this section, we present our algorithm and discuss its convergence analysis. First, let us define the following functions:
\[
g(x) = \frac{1}{2} \|(I - J_{\lambda_n}^{B_2})Ax\|^2, \quad h(x) = \frac{1}{2} \|(I - J_{\lambda_n}^{B_1})x\|^2,
\]
and
\[
F(x) = A^*(I - J_{\lambda_n}^{B_2})Ax, \quad H(x) = (I - J_{\lambda_n}^{B_1})x.
\]
From Aubin [5], one can verify that \( g \) and \( h \) are weak lower semi-continuous and convex differentiable. We establish the convergence of the algorithm under the following conditions:

**Condition A:**

(A1) The solution set denoted by \( F = \Gamma \cap F(S) \) is nonempty;

(A2) \( A : H_1 \to H_2 \) is a bounded linear operator and \( A^* : H_2 \to H_1 \) is the conjugate transpose of \( A \);

(A3) \( B_1 : H_1 \to 2^{H_1} \) and \( B_2 : H_2 \to 2^{H_2} \) are maximal monotone mappings;

(A4) \( S : H_1 \to H_2 \) is a quasi-nonexpansive mapping such that \( I - S \) is demiclosed at zero;

(A5) \( G : H_1 \to H_1 \) is a strongly positive, bounded linear operator with coefficient \( \mu \) such that \( \|G\| = 1 \);

(A6) \( f : H_1 \to H_1 \) is a contraction mapping with coefficient \( k \in (0, 1) \) and \( 0 < \beta < \frac{k}{2} \).

**Condition B:**

(B1) \( \{\rho_n\} \) is a positive sequence satisfying \( 0 < \rho_n < 4 \) and \( \inf_{n \to \infty} (4 - \rho_n) > 0 \);

(B2) \( 0 < a \leq \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n \leq b < 1 \);

(B3) \( \{\alpha_n\} \subset (0, 1), \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);

(B4) \( \{\theta_n\} \subset [0, \theta) \) for some \( \theta > 0 \) such that
\[
\lim_{n \to \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.
\]

Now, the algorithm is presented as follows:

**Algorithm 3.1.**

**Step 0.** Select \( x_0, x_1 \in H_1 \), the parameters \( \beta, \mu, k \) and the sequences \( \{\rho_n\}, \{\beta_n\}, \{\alpha_n\} \) and \( \{\theta_n\} \) such that Condition A and Condition B above are satisfied.

**Step 1.** Set
\[
w_n = x_n + \theta_n(x_n - x_{n-1}).
\]

**Step 2.** Compute
\[
u_n = J_{\lambda_n}^{B_1}(w_n - \tau_n A^*(I - J_{\lambda_n}^{B_2})Aw_n),
\]
where
\[
\tau_n = \begin{cases} \frac{\rho_n g(w_n)}{\|F(w_n)\|^2 + \|H(w_n)\|^2}, & \|F(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]
Step 3. Compute

\[ x_{n+1} = \alpha_n \beta f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)S u_n. \]

Set \( n := n + 1 \) and return to Step 1.

First, we prove the boundedness of the sequence \( \{x_n\} \) generated by our algorithm.

Lemma 3.2. Let \( \{x_n\} \) be a sequence generated by Algorithm 3.1, then \( \{x_n\} \) is bounded.

Proof. Let \( p \in \mathcal{F} \), then we have that \( Sp = p \), \( p = J_{\lambda_1}^B p \), and \( Ap = J_{\lambda_2}^B (Ap) \). Also, since \( F(x_n) = A^*(I - J_{\lambda_1}^B)Ax_n \) and \( I - J_{\lambda_2}^B \) is firmly nonexpansive, then we obtain

\[
\langle F(w_n), w_n - p \rangle = \langle A^*(I - J_{\lambda_1}^B)Aw_n, w_n - p \rangle \\
= \langle (I - J_{\lambda_1}^B)Aw_n, Aw_n - Ap \rangle \\
= \langle (I - J_{\lambda_1}^B)Aw_n - (I - J_{\lambda_2}^B)Ap, Aw_n - Ap \rangle \\
\geq ||(I - J_{\lambda_2}^B)Ap||^2 \\
= 2g(w_n).
\]

Hence, it follows that

\[
||u_n - p||^2 = ||J_{\lambda_1}^B (w_n - \tau_n A^*(I - J_{\lambda_1}^B)Aw_n) - p||^2 \\
= ||J_{\lambda_1}^B (w_n - \tau_n A^*(I - J_{\lambda_1}^B)Aw_n) - J_{\lambda_1}^B p||^2 \\
\leq ||w_n - \tau_n A^*(I - J_{\lambda_1}^B)Aw_n - p||^2 \\
= ||w_n - p - \tau_n F(w_n)||^2 \\
\leq ||w_n - p||^2 + \tau_n^2 ||F(w_n)||^2 - 2\tau_n (F(w_n), w_n - p) \\
\leq ||w_n - p||^2 + \tau_n^2 ||F(w_n)||^2 - 4\tau_n g(w_n) \\
\leq ||w_n - p||^2 - \rho_n (4 - \rho_n) \frac{g^2(w_n)}{||F(w_n)||^2 + ||H(w_n)||^2}. \quad (11)
\]

Since \( 0 < \rho_n < 4 \), then we obtain

\[
||u_n - p|| \leq ||w_n - p||. \quad (12)
\]

Applying the triangle inequality, we get

\[
||w_n - p|| = ||x_n + \theta_n (x_n - x_{n-1}) - p|| \\
\leq ||x_n - p|| + \theta_n ||x_n - x_{n-1}|| \\
= ||x_n - p|| + \alpha_n \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}||. \quad (13)
\]

By the assumption that \( \lim_{n \to \infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0 \), it follows that there exists a constant \( M_1 > 0 \) such that \( \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| \leq M_1 \), for all \( n \geq 1 \). Hence from (13), we obtain

\[
||w_n - p|| \leq ||x_n - p|| + \alpha_n M_1. \quad (14)
\]
By applying (12) and (14), we get

\[
||x_{n+1} - p|| = ||\alpha_n f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)Su_n - p|| \\
= ||\alpha_n f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)Su_n \\
\quad - (\alpha_n G - \beta_n G + (\beta_n - \beta_n + 1)I)p|| \\
\leq \alpha_n ||f(x_n) - Gp|| + \beta_n ||x_n - p|| \\
\quad + ||((1 - \beta_n)I - \alpha_n G)||||Su_n - p|| \\
\leq \alpha_n ||f(x_n) - Gp|| + \beta_n ||x_n - p|| + (1 - \beta_n - \alpha_n \mu)||Su_n - p|| \\
\leq \alpha_n ||f(x_n) - Gp|| + \beta_n ||x_n - p|| + (1 - \beta_n - \alpha_n \mu)||u_n - p|| \\
\leq \alpha_n ||f(x_n) - Gp|| + \beta_n ||x_n - p|| + (1 - \beta_n - \alpha_n \mu)||x_n - p|| \\
\quad + (1 - \beta_n - \alpha_n \mu)||u_n - p|| \\
\leq \alpha_n \beta k ||x_n - p|| + \alpha_n ||f(p) - Gp|| \\
\quad + \beta_n ||x_n - p|| + (1 - \beta_n - \alpha_n \mu)||u_n - p|| \\
\leq \alpha_n \beta k ||x_n - p|| + \alpha_n ||f(p) - Gp|| \\
\quad + \beta_n ||x_n - p|| + (1 - \beta_n - \alpha_n \mu)(||x_n - p|| + \alpha_n M_1) \\
= (1 - \alpha_n(\mu - \beta k)) ||x_n - p|| + \alpha_n ||f(p) - Gp|| \\
\quad + (1 - \beta_n - \alpha_n \mu)\alpha_n M_1. \tag{15}
\]

Setting \(a_n := ||x_n - p||, b_n := \alpha_n ||f(p) - Gp||, c_n := (1 - \beta_n - \alpha_n \mu)\alpha_n M_1\) and \(\sigma_n := \alpha_n(\mu - \beta k)\). By Lemma 2.6(1) and using our assumptions, it follows that \(\{x_n\}\) is bounded. Consequently, \(\{w_n\}\) and \(\{u_n\}\) are also bounded. \(\square\)

Next, we prove some lemmas which will be employed in establishing the strong convergence of our algorithm.

**Lemma 3.3.** The following inequality holds for all \(p \in \mathcal{F}\) and \(n \in \mathbb{N}\):

\[
||x_{n+1} - p||^2 \leq (1 - \frac{2\alpha_n(\mu - \beta k)}{1 - \alpha_n \beta k}) ||x_n - p||^2 \\
\quad + \frac{2\alpha_n(\mu - \beta k)}{1 - \alpha_n \beta k} \left( 3M_\gamma_n \theta_n ||x_n - x_{n-1}|| \right. \\
\quad + \frac{\alpha_n \mu^2}{2(\mu - \beta k)} M_2 + \frac{1}{(\mu - \beta k)^2} (\beta f(p) - Gp, x_{n+1} - p) \left. \right) \\
\quad - \frac{\rho_n(4 - \rho_n)}{1 - \alpha_n \beta k} \frac{g^2(w_n)}{||F(w_n)||^2 + ||H(w_n)||^2}.
\]
Proof. By applying Lemma 2.4(ii), we have

\[
|x_{n+1} - p|^2 = |\alpha_n \beta f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G) Su_n - p|^2 \\
= |\alpha_n (\beta f(x_n) - Gp) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n G)(Su_n - p)|^2 \\
\leq ||\beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n G)(Su_n - p)||^2 \\
+ 2\alpha_n (\beta f(x_n) - Gp, x_{n+1} - p) \\
\leq \beta_n^2 ||x_n - p||^2 + (1 - \beta_n - \alpha_n \mu)^2 ||Su_n - p||^2 \\
+ 2\beta_n (1 - \beta_n - \alpha_n \mu)||x_n - p||||Su_n - p| \\
+ 2\alpha_n (\beta f(x_n) - Gp, x_{n+1} - p) \\
\leq \beta_n^2 ||x_n - p||^2 + (1 - \beta_n - \alpha_n \mu)^2 ||Su_n - p||^2 \\
+ \beta_n (1 - \beta_n - \alpha_n \mu)(||x_n - p||^2 + ||Su_n - p||^2) \\
+ 2\alpha_n (\beta f(x_n) - Gp, x_{n+1} - p) \\
= \beta_n (1 - \alpha_n \mu)||x_n - p||^2 + (1 - \beta_n - \alpha_n \mu)(1 - \alpha_n \mu)||Su_n - p||^2 \\
+ 2\alpha_n (\beta f(x_n) - Gp, x_{n+1} - p) \\
\leq \beta_n (1 - \alpha_n \mu)||x_n - p||^2 + (1 - \beta_n - \alpha_n \mu)(1 - \alpha_n \mu)||Su_n - p||^2 \\
+ 2\alpha_n (\beta f(x_n) - f(p), x_{n+1} - p) + 2\alpha_n (\beta f(p) - Gp, x_{n+1} - p). \tag{16}
\]

Furthermore, by applying the Cauchy-Schwartz inequality and Lemma 2.4(iii), we have that

\[
||w_n - p||^2 = ||x_n + \theta_n (x_n - x_{n-1}) - p||^2 \\
= ||x_n - p||^2 + \theta_n^2 ||x_n - x_{n-1}||^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle \\
\leq ||x_n - p||^2 + \theta_n^2 ||x_n - x_{n-1}||^2 + 2\theta_n ||x_n - x_{n-1}|| ||x_n - p|| \\
= ||x_n - p||^2 + \theta_n ||x_n - x_{n-1}|| (\theta_n ||x_n - x_{n-1}|| + 2 ||x_n - p||) \\
\leq ||x_n - p||^2 + 3M\theta_n ||x_n - x_{n-1}|| \\
= ||x_n - p||^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}||, \tag{17}
\]

where \( M := \sup_{n \in \mathbb{N}} \{ ||x_n - p||, \theta_n ||x_n - x_{n-1}|| \} > 0. \)
Now, applying (11) and (17) to (16), we obtain
\[
\|x_{n+1} - p\|^2 \leq \beta_n (1 - \alpha_n \mu) \|x_n - p\|^2 \\
+ (1 - \beta_n - \alpha_n \mu) (1 - \alpha_n \mu) \left( \|x_n - p\|^2 + 3M\alpha_n \theta_n \|x_n - x_{n-1}\| \\
- \rho_n (4 - \rho_n) \|F(w_n)\|^2 + \|H(w_n)\|^2 \right) \\
+ 2\alpha_n \beta (f(x_n) - f(p), x_{n+1} - p) + 2\alpha_n \beta f(p) - Gp, x_{n+1} - p) \\
\leq (1 - \alpha_n \mu)^2 \|x_n - p\|^2 \\
+ (1 - \beta_n - \alpha_n \mu) (1 - \alpha_n \mu) \times 3M \alpha_n \theta_n \alpha_n \|x_n - x_{n-1}\| \\
+ 2\alpha_n \beta k \|x_n - p\| \|x_{n+1} - p\| + 2\alpha_n \beta f(p) - Gp, x_{n+1} - p) \\
- \rho_n (4 - \rho_n) \|F(w_n)\|^2 + \|H(w_n)\|^2 \\
\leq (1 - \alpha_n \mu)^2 \|x_n - p\|^2 \\
+ (1 - \beta_n - \alpha_n \mu) (1 - \alpha_n \mu) \times 3M \alpha_n \theta_n \alpha_n \|x_n - x_{n-1}\| \\
+ \alpha_n \beta k (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\alpha_n \beta f(p) - Gp, x_{n+1} - p) \\
- \rho_n (4 - \rho_n) \|F(w_n)\|^2 + \|H(w_n)\|^2 \\
= (1 - \alpha_n \mu)^2 + \alpha_n \beta k \|x_n - p\|^2 + \alpha_n \beta k \|x_{n+1} - p\|^2 \\
+ 3M\gamma_n \alpha_n \theta_n \alpha_n \|x_n - x_{n-1}\| + 2\alpha_n \beta f(p) - Gp, x_{n+1} - p) \\
- \rho_n (4 - \rho_n) \|F(w_n)\|^2 + \|H(w_n)\|^2, \\
\text{where } \gamma_n = (1 - \beta_n - \alpha_n \mu) (1 - \alpha_n \mu).
\]

This implies that
\[
\|x_{n+1} - p\|^2 = \frac{(1 - 2\alpha_n \mu + (\alpha_n \mu)^2 + \alpha_n \beta k))}{(1 - \alpha_n \beta k)} \|x_n - p\|^2 \\
+ 3M\gamma_n \alpha_n \theta_n \alpha_n \|x_n - x_{n-1}\| \\
+ \frac{2\alpha_n}{(1 - \alpha_n \beta k)} \beta f(p) - Gp, x_{n+1} - p) \\
- \rho_n (4 - \rho_n) \frac{g^2(w_n)}{\|F(w_n)\|^2 + \|H(w_n)\|^2} \\
= \frac{(1 - 2\alpha_n \mu + \alpha_n \beta k)}{(1 - \alpha_n \beta k)} \|x_n - p\|^2 \\
+ 3M\gamma_n \alpha_n \theta_n \alpha_n \|x_n - x_{n-1}\| + \frac{(\alpha_n \mu)^2}{(1 - \alpha_n \beta k)} \|x_n - p\|^2 \\
+ \frac{2\alpha_n}{(1 - \alpha_n \beta k)} \beta f(p) - Gp, x_{n+1} - p)
Lemma 3.4. Let \( \lim \inf (14) \) and (15), we obtain
\[
\rho_n (4 - \rho_n) g^2(w_n) (1 - \alpha_n \beta k) \|F(w_n)\|^2 + \|H(w_n)\|^2
\]
\[
= \left( 1 - \frac{2\alpha_n (\mu - \beta k)}{(1 - \alpha_n \beta k)} \right) \|x_n - p\|^2
\]
\[
+ \frac{2\alpha_n (\mu - \beta k)}{(1 - \alpha_n \beta k)} \left( \frac{3M\gamma_n}{2(\mu - \beta k)} \right) \alpha_n \|x_n - x_{n-1}\|
\]
\[
+ \frac{\alpha_n \mu^2}{2(\mu - \beta k)} M_2 + \frac{1}{(\mu - \beta k)} \langle \beta f(p) - Gp, x_{n+1} - p \rangle
\]
\[
- \rho_n (4 - \rho_n) g^2(w_n)
\]
\[
= \frac{(1 - \alpha_n \beta k) \|F(w_n)\|^2 + \|H(w_n)\|^2}{(1 - \alpha_n \beta k)}
\]
where \( M_2 = \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\} \).

This completes the proof of the lemma. \( \Box \)

**Lemma 3.4.** Let \( p \in F \). Suppose \( \{x_{n_k}\} \) is a subsequence of \( \{x_n\} \) such that
\[
\liminf_{k \to \infty} (|x_{n_k+1} - p| - |x_{n_k} - p|) \geq 0,
\]
then \( x_{n_k} \to x^* \in F \), i.e. \( w_n \{x_n\} \subset F \).

**Proof.** Since \( \alpha_{n_k} \to 0 \) as \( k \to \infty \), then by the hypothesis of the lemma, and applying (14) and (15), we obtain
\[
0 \leq \liminf_{k \to \infty} (|x_{n_k+1} - p| - |x_{n_k} - p|)
\]
\[
\leq \liminf_{k \to \infty} \alpha_{n_k} \beta f(x_{n_k}) - Gp + |\beta_{n_k}||x_{n_k} - p| + (1 - \beta_{n_k} - \alpha_{n_k} \mu) ||Su_{n_k} - p|| - ||x_{n_k} - p||
\]
\[
= \liminf_{k \to \infty} (\beta_{n_k} ||x_{n_k} - p|| + (1 - \beta_{n_k}) ||Su_{n_k} - p|| - ||x_{n_k} - p||)
\]
\[
= \liminf_{k \to \infty} (1 - \beta_{n_k}) (||Su_{n_k} - p|| - ||x_{n_k} - p||)
\]
\[
\leq (1 - a) \liminf_{k \to \infty} (||Su_{n_k} - p|| - ||x_{n_k} - p||)
\]
\[
\leq (1 - a) \liminf_{k \to \infty} (||x_{n_k} - p|| - ||x_{n_k} - p||)
\]
\[
\leq (1 - a) \liminf_{k \to \infty} (||w_{n_k} - p|| - ||x_{n_k} - p||)
\]
\[
\leq (1 - a) \liminf_{k \to \infty} (||x_{n_k} - p|| + \alpha_{n_k} M_1 - ||x_{n_k} - p||)
\]
\[
= (1 - a) \liminf_{k \to \infty} (\alpha_{n_k} M_1)
\]
\[
\leq (1 - a) \limsup_{k \to \infty} (\alpha_{n_k} M_1) = 0.
\]

Hence, it follows from (18), (19) and (20) that
\[
||Su_{n_k} - p|| - ||x_{n_k} - p|| \to 0, \quad ||u_{n_k} - p|| - ||x_{n_k} - p|| \to 0 \text{ and } ||w_{n_k} - p|| - ||x_{n_k} - p|| \to 0,
\]
which implies that
\[
||Su_{n_k} - x_{n_k}|| \to 0 \quad (21)
\]
\[
||u_{n_k} - x_{n_k}|| \to 0 \quad (22)
\]
\[
||w_{n_k} - x_{n_k}|| \to 0 \quad (23)
\]
It also follows that
\[
\|x_{nk+1} - x_{nk}\| = \|\alpha_{nk}\beta f(x_{nk}) + \beta_{nk}x_{nk} \\
+ ((1 - \beta_{nk})I - \alpha_{nk}G)Su_{nk} - x_{nk}\| \\
= \|\alpha_{nk}\beta f(x_{nk}) + \beta_{nk}x_{nk} \\
+ ((1 - \beta_{nk})I - \alpha_{nk}G)Su_{nk} \\
- (\alpha_{nk}G - \alpha_{nk}G + (\beta_{nk} - \beta_{nk} + 1)I)x_{nk}\| \\
\leq \alpha_{nk}\|\beta f(x_{nk}) - Gx_{nk}\| + \beta_{nk}\|x_{nk} - x_{nk}\| \\
+ \|((1 - \beta_{nk})I - \alpha_{nk}G)\|Su_{nk} - x_{nk}\| \\
\leq \alpha_{nk}\|\beta f(x_{nk}) - Gx_{nk}\| + \beta_{nk}\|x_{nk} - x_{nk}\| \\
+ (1 - \beta_{nk} - \alpha_{nk}\mu)\|Su_{nk} - x_{nk}\| \to 0, \tag{24}
\]
and
\[
\|u_{nk} - w_{nk}\| \leq \|u_{nk} - x_{nk}\| + \|x_{nk} - w_{nk}\| \to 0. \tag{25}
\]
From Lemma 3.3, we have that
\[
\rho_{nk}(4 - \rho_{nk}) g^2(w_{nk}) \frac{1}{(1 - \alpha_{nk}\beta k) \|F(w_{nk})\|^2 + \|H(w_{nk})\|^2} \\
\leq \left(1 - \frac{2\alpha_{nk}(\mu - \beta k)}{(1 - \alpha_{nk}\beta k)}\right)\|x_{nk} - p\|^2 \\
- \|x_{nk+1} - p\|^2 \\
+ \frac{2\alpha_{nk}(\mu - \beta k)}{(1 - \alpha_{nk}\beta k)} \frac{3M\gamma_{nk}}{(2(1 - \alpha_{nk}\beta k) \alpha_{nk})} \|x_{nk} - x_{nk-1}\| \\
+ \frac{\alpha_{nk}\mu^2}{2(\mu - \beta k)} M_2 \\
+ \frac{1}{(\mu - \beta k)} (\beta f(p) - Gp, x_{nk+1} - p).
\]
By the hypothesis of Lemma 3.4 and the fact that \(\lim_{n \to \infty} \alpha_{nk} = 0\), it follows that
\[
\rho_{nk}(4 - \rho_{nk}) g^2(w_{nk}) \frac{1}{(1 - \alpha_{nk}\beta k) \|F(w_{nk})\|^2 + \|H(w_{nk})\|^2} \to 0.
\]
Since \(\rho_{nk}(4 - \rho_{nk}) > 0\), then we have \(g^2(w_{nk}) \to 0\), which implies that
\[
\lim_{k \to \infty} g(w_{nk}) = \lim_{k \to \infty} \frac{1}{2} \|(I - J^{B_{2z}}_\lambda)Aw_{nk}\|^2 = 0.
\]
Hence, we get
\[
\|(I - J^{B_{2z}}_\lambda)Aw_{nk}\| \to 0. \tag{26}
\]
By (21) and (22), and the demiclosedness property of \(I - S\), we have that \(u_{nk} \to x^* \in F(S)\), where \(x^*\) is a weak cluster point of \(\{w_{nk}\}\). It then follows from (22), (23) and (25) that \(w_{\omega}\{w_{nk}\} = w_{\omega}\{u_{nk}\} = w_{\omega}\{x_{nk}\} \subset F(S)\). Also, \(u_{nk} = J^{B_{1}}_\lambda(w_{nk} - \tau_{nk}A^*(I - J^{B_{2z}}_\lambda)Aw_{nk})\) can be written as
\[
(w_{nk} - u_{nk}) - \tau_{nk}A^*(I - J^{B_{2z}}_\lambda)Aw_{nk} \in B_1u_{nk}. \tag{27}
\]
By passing limit as \(k \to \infty\) in (27), applying (25) and (26), and taking into consideration the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain \(0 \in B_1(x^*)\). Moreover, since \(\{u_{nk}\}\) and \(\{w_{nk}\}\) have the same asymptotic behaviour, then \(\{Aw_{nk}\}\) converges weakly to \(Ax^*\). Again, applying (26) and the fact that the resolvent \(J^{B_{2z}}_\lambda\) is nonexpansive and Remark 1(3), we obtain
0 ∈ B₂(Ax*). This implies that x* ∈ Γ, that is, wₓ{xₙ} ⊂ Γ. Hence, wₓ{xₙ} ∈ F as required.

Now, we prove the strong convergence of our algorithm.

**Theorem 3.5.** Let H₁ and H₂ be two real Hilbert spaces. Let A : H₁ → H₂ be a bounded linear operator. Assume that B₁ : H₁ → 2H₁ and B₂ : 2H₂ are maximal monotone mappings and S : H₁ → H₁ is a quasi-nonexpansive mapping such that I − S is demiclosed at zero. Let f : H₁ → H₁ be a contraction mapping with constant k ∈ (0, 1) and \( G : H₁ → H₁ \) be a strongly positive, bounded linear operator with constant µ such that \( \|G\| = 1 \), and \( 0 < β < \frac{βG}{α} \). Then the sequence \( \{xₙ\} \) generated by Algorithm 3.1 converges strongly to \( \hat{x} \in F \), where \( \hat{x} = P_F(I - G + βf)(\hat{x}) \).

**Proof.** Let \( \hat{x} = P_F(I - G + βf)(\hat{x}) \), then it follows from Lemma 3.3 that

\[
\|xₙ₊₁ - \hat{x}\|² \leq \left( 1 - \frac{2α_n(μ - βk)}{1 - α_nβk} \right) \|xₙ - \hat{x}\|² + \frac{2α_n(μ - βk)}{1 - α_nβk} \left( \frac{3Mγn}{2(μ - βk)} α_n \right) \|xₙ - xₙ₋₁\| + \frac{α_nμ²}{2(μ - βk)} M₂ + \frac{1}{(μ - βk)} \langle βf(\hat{x}) - Gp, xₙ₊₁ - \hat{x} \rangle. \tag{28}
\]

Now, we claim that the sequence \( \{\|xₙ - \hat{x}\|²\} \) converges to zero. In order to establish this, by Lemma 2.5, it suffices to show that \( \lim_{k→∞} \sup_{n→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ₊₁ - \hat{x} \rangle ≤ 0 \) for every subsequence \( \{\|xₙ - \hat{x}\|\} \) of \( \{\|xₙ - \hat{x}\|\} \) satisfying

\[
\liminf_{k→∞} (\|xₙ₊₁ - \hat{x}\| - \|xₙ - \hat{x}\|) ≥ 0.
\]

Now, suppose that \( \{\|xₙ - \hat{x}\|\} \) is a subsequence of \( \{\|xₙ - \hat{x}\|\} \) such that

\[
\liminf_{k→∞} (\|xₙ₊₁ - \hat{x}\| - \|xₙ - \hat{x}\|) ≥ 0.
\]

Then, by Lemma 3.4 we have that \( wₓ{xₙ} \subset F \). Also by (22), we have that \( wₓ{uₙ} \subset wₓ{xₙ} \). By the boundedness of \( \{xₙ\} \), it follows that there exists a subsequence \( \{xₙ_j\} \) of \( \{xₙ\} \) such that \( xₙ_j → x^\dagger \) and

\[
\lim_{j→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ_j - \hat{x} \rangle = \limsup_{k→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ - \hat{x} \rangle = \limsup_{k→∞} \langle βf(\hat{x}) - G(\hat{x}), uₙ_k - \hat{x} \rangle.
\]

Since \( \hat{x} = P_F(I - G + βf)(\hat{x}) \), then it follows that

\[
\limsup_{k→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ - \hat{x} \rangle = \lim_{j→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ_j - \hat{x} \rangle \tag{29}
\]

\[
= \langle βf(\hat{x}) - G(\hat{x}), x^\dagger - \hat{x} \rangle ≤ 0.
\]

Combining (24) and (29), we have

\[
\limsup_{k→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ₊₁ - \hat{x} \rangle ≤ \limsup_{k→∞} \langle βf(\hat{x}) - G(\hat{x}), xₙ - \hat{x} \rangle \tag{30}
\]

\[
= \langle βf(\hat{x}) - G(\hat{x}), x^\dagger - \hat{x} \rangle ≤ 0.
\]

Applying Lemma 2.5 to (28), and using (30) together with the fact that \( \lim_{n→∞} \frac{θ_n}{α_n} \|xₙ - xₙ₋₁\| = 0 \) and \( \lim_{n→∞} α_n = 0 \), we deduce that \( \lim_{n→∞} \|xₙ - \hat{x}\| = 0 \) as required. □
Next, we use our result to approximate the fixed point of demicontractive mapping.

Let $S$ be a $k$–demicontractive mapping with $F(S) \neq \emptyset$ and set $S_\omega := (1-\omega)I+\omega S$ for $\omega \in (0, \infty)$. Then $S_\omega$ is quasi-nonexpansive for $\omega \in [0, 1-k]$ and $F(S) = F(S_\omega)$ if $\omega \neq 0$. Hence, we obtain the following result as a consequence of our main result.

**Corollary 1.** Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings and $S : H_1 \to H_1$ is a $k$-demicontractive mapping such that $I-S$ is demiclosed at zero and $S_\omega := (1-\omega)I+\omega S$, $\omega \in [0, 1-k]$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $k \in (0, 1)$ and $G : H_1 \to H_1$ be a strongly positive, bounded linear operator with constant $\mu$ such that $\|G\| = 1$, and $0 < \beta < \frac{\mu}{\alpha}$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x} = P_{\mathcal{F}}(I-G + \beta f)\hat{x}$.

**Remark 2.** Now, we consider a general case where $G$ is any strongly positive bounded linear operator with coefficient $\mu$ and $0 < \beta < \frac{\mu}{\alpha}$. We define a bounded linear operator $\hat{G} : H_1 \to H_1$ by

$$\hat{G} = \|G\|^{-1}G.$$  

(31)

It is obvious that $\hat{G}$ is strongly positive with coefficient $\|G\|^{-1}\mu > 0$ such that $\|\hat{G}\| = 1$ and

$$0 < \|G\|^{-1}\beta < \frac{\|G\|^{-1}\mu}{\alpha}.$$

By replacing $G$ with $\hat{G}$ in Theorem 3.5, we obtain the following result.

**Theorem 3.6.** Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings and $S : H_1 \to H_1$ is a quasi-nonexpansive mapping such that $I-S$ is demiclosed at zero. Let $f : H_1 \to H_1$ be a contraction mapping with constant $k \in (0, 1)$ and $G : H_1 \to H_1$ be defined as in (31) such that $\|G\| = 1$, and $0 < \|G\|^{-1}\beta < \frac{\|G\|^{-1}\mu}{\alpha}$. Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x} = P_{\mathcal{F}}(I-\|G\|^{-1}(G + \beta f))(\hat{x})$.

**Algorithm 3.7.**

**Step 0.** Select $x_0, x_1 \in H_1$, the parameters $\beta, \mu, k$ and the sequences $\{\rho_n\}, \{\beta_n\}, \{\alpha_n\}$ and $\{\theta_n\}$ such that Condition A and Condition B above are satisfied.

**Step 1.** Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

**Step 2.** Compute

$$u_n = J_{B_1}(w_n - \tau_n A^*(I-J_{\hat{A}})Aw_n),$$

where

$$\tau_n = \begin{cases} \frac{\rho_n \beta |w_n|}{\|F(w_n)\|^2 + \|H(w_n)\|^2}, & \text{if } \|F(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

**Step 3.** Compute

$$x_{n+1} = \alpha_n \beta \|G\|^{-1}f(x_n) + \beta_n x_n + ((1 - \beta_n)(1 - \alpha_n \hat{G})Su_n.$$ 

Set $n := n + 1$ and return to **Step 1**.
Proof. From Theorem 3.5, it follows that the sequence \( \{x_n\} \) converges strongly to
\[
\hat{x} = P_F(I - ||G||^{-1}(G + \beta f)(\hat{x})),
\]
which is a unique solution of the variational inequality:
\[
||G||^{-1}((G - \beta f)\hat{x}, x - \hat{x}) \geq 0, \quad x \in H_1.
\]
(32)
It is easily seen that (32) is equivalent to (7). Hence, it follows that \( \hat{x} \) is a unique
solution of the variational inequality (7) as required. \( \square \)

4. Applications. In this section, we give applications of our results to the solution
of some other optimization problems.

4.1. Split feasibility problem.
Let \( C \) and \( D \) be nonempty closed and convex subsets of real Hilbert spaces \( H_1 \) and
\( H_2 \), respectively. The Split Feasibility Problem (SFP) is defined as:
\[
\text{Find } x^* \in C \text{ such that } Ax^* \in D,
\]
where \( A : H_1 \to H_2 \) is a bounded linear operator and we denote the solution set
of SFP by \( \Omega_{\text{SFP}} \).

In 1994, Censor and Elfving [11] first introduced the SFP in
finite-dimensional Hilbert spaces for modelling inverse problems which arise from
phase retrievals and in medical image reconstruction [8]. However, SFP also finds
application in various disciplines such as image restoration, computer tomography
and radiation therapy treatment planning [12]. Many authors have researched on
the SFP and related optimization problems, see [1, 10, 19, 25, 44] and references
therein.

Let \( \hat{g} : H \to \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous convex function, the
subdifferential \( \partial \hat{g} \) of \( \hat{g} \) is defined as
\[
\partial \hat{g}(x) = \{ z \in H : \hat{g}(x) - \hat{g}(y) \leq \langle z, x - y \rangle \text{ for all } y \in H \},
\]
for all \( x \in H \). The indicator function \( i_C \) of \( C \) is defined by \( i_C(x) = 0 \) if \( x \in C \) and
\( i_C = \infty \), otherwise. Also, the normal cone of \( C \) at \( u \in C \) denoted by \( N_C(u) \) is
defined by
\[
N_C(u) = \{ z \in H : \langle z, v - u \rangle \leq 0 \text{ for all } v \in C \}.
\]
It is well known that \( i_C \) is a proper, lower semicontinuous and convex function on
\( H \). Thus, the subdifferential \( \partial i_C \) of \( i_C \) is a maximal monotone operator. Hence, we
can define the resolvent \( J_{\lambda}^{i_C} \) of \( \partial i_C \) for each \( \lambda > 0 \), as follows:
\[
J_{\lambda}^{i_C}x = (I + \lambda \partial i_C)^{-1}x, \quad \forall x \in H.
\]
Moreover, for each \( x \in C \), we have
\[
\partial i_C(x) = \{ z \in H : i_C(x) + \langle z, y - x \rangle \leq i_C(y) \text{ for all } y \in H \}
\]
\[
= \{ z \in H : \langle z, y - x \rangle \leq 0 \text{ for all } y \in C \}
\]
\[
= N_C(x).
\]
Thus, for each \( \lambda > 0 \), we obtain
\[
y = J_{\lambda}^{i_C}x \iff x \in y + \lambda \partial i_C(y) \iff x - y \in \lambda \partial i_C(y)
\]
\[
\iff \langle x - y, z - y \rangle \leq 0 \text{ for all } z \in C
\]
\[
\iff y = P_Cx.
\]
Setting $B_1 = \partial i_C$ and $B_2 = \partial i_D$ in Theorem 3.5, we obtain the following result for approximating a common solution of SFP and fixed point of quasi-nonexpansive mapping in Hilbert spaces.

**Theorem 4.1.** Let $C$ and $D$ be nonempty, closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $f : H_1 \to H_1$ be a contraction mapping with constant $k \in (0, 1)$. Let $G : H_1 \to H_1$ be a strongly positive, bounded linear operator with constant $\mu$ such that $\|G\| = 1$ and $0 < \beta < \frac{\mu}{\|G\|}$. Suppose $\Gamma = \Omega_{SFP} \cap F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a solution $\hat{x} = P_{\Gamma}(I - G + \beta f)(\hat{x})$.

**Algorithm 4.2.**

**Step 0.** Select $x_0, x_1 \in H_1$, the parameters $\beta, \mu, k$ and the sequences $\{\rho_n\}, \{\beta_n\}, \{\alpha_n\}$ and $\{\theta_n\}$ such that Condition A and Condition B above are satisfied.

**Step 1.** Set $w_n = x_n + \rho_n(x_n - x_{n-1})$.

**Step 2.** Compute $u_n = P_C(w_n - \tau_n A^*(I - P_D)A w_n)$, where

$$
\tau_n = \begin{cases} 
\rho_n g(w_n) & \text{if } \|F(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\
0 & \text{otherwise},
\end{cases}
$$

and

$$
g(x) = \frac{1}{2} \| (I - P_D) A x \|^2, \quad F(x) = A^* (I - P_D) A x,
$$

and

$$
H(x) = (I - P_C) x.
$$

**Step 3.** Compute

$$
x_{n+1} = \alpha_n \beta f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n G) Su_n.
$$

Set $n := n + 1$ and return to **Step 1**.

**Remark 3.** [13] The SFP (33) is equivalent to solving the following variational inequality problem:

Find $x^* \in C$ such that $\langle \nabla f(x^*), y - x^* \rangle \geq 0, \forall y \in C,$

where $\nabla f = A^* (I - P_D) A$, and $A^*$ is the conjugate transpose of $A$.

4.2. **Split minimization problem.**

Let $H_1$ and $H_2$ be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Given some proper, lower semicontinuous and convex functions $\varphi_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_2 : H_2 \to \mathbb{R} \cup \{+\infty\}$, the Split Minimization Problem (SMP) is defined as

Find $\bar{x} \in H_1$ such that $\bar{x} \in arg\min \varphi_1$ and $A \bar{x} \in arg\min \varphi_2$. (34)

We denote the set of solution of the SMP (34) with $\Omega_{SMP}$. The SMP was first introduced by Moudafi and Thakur [31] and has attracted lots of attention in recent years, see for instance [2, 31, 51] and reference therein. Further, the SMP has been applied in the study of many applied science problems such as multi-resolution sparse regularization, Fourier regularization, hard-constrained inconsistent feasibility and alternating projection signal synthesis problems.
The proximity operator with respect to \( \varphi_1 \) is defined by
\[
prox_{\lambda,\varphi_1}(x) := \arg\min_{z \in H_1} \left\{ \varphi_1(z) + \frac{1}{2\lambda}||x - z||^2 \right\},
\]
for all \( x \in H_1 \) and \( \lambda > 0 \). It is well known that \( \partial \varphi_1 \) is maximal monotone and
\[
0 \in \partial \varphi_1(\bar{x}) \iff \bar{x} = prox_{\lambda,\varphi_1}(\bar{x}).
\]

By setting \( B_1 = \partial \varphi_1 \) and \( B_2 = \partial \varphi_2 \) in Theorem 3.5, we obtain the following result for approximating a common solution of SMP and fixed point of quasi-nonexpansive mapping in Hilbert spaces.

**Theorem 4.3.** Let \( H_1 \) and \( H_2 \) be real Hilbert spaces and \( A : H_1 \to H_2 \) be a bounded linear operator. Let \( \varphi_1 : H_1 \to \mathbb{R} \cup \{+\infty\} \) and \( \varphi_2 : H_2 \to \mathbb{R} \cup \{+\infty\} \) be proper convex lower semicontinuous functions, \( S : H_1 \to H_1 \) be a quasi-nonexpansive mapping such that \( I - S \) is demiclosed at zero and \( f : H_1 \to H_1 \) be a contraction mapping with constant \( k \in (0,1) \). Let \( G : H_1 \to H_1 \) be a strongly positive, bounded linear operator with constant \( \mu \) such that \( ||G|| = 1 \) and \( 0 < \beta < \frac{\mu}{\lambda} \). Suppose \( \Gamma = \Omega_{\text{SMP}} \cap F(S) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by the following algorithm converges strongly to a solution \( \bar{x} = P_\Gamma(I - G + \beta f)(\bar{x}) \).

**Algorithm 4.4.**

**Step 0.** Select \( x_0, x_1 \in H_1 \), the parameters \( \beta, \mu, k \) and the sequences \( \{\rho_n\}, \{\beta_n\}, \{\alpha_n\} \) and \( \{\theta_n\} \) such that Condition A and Condition B above are satisfied.

**Step 1.** Set
\[
w_n = x_n + \theta_n(x_n - x_{n-1}).
\]

**Step 2.** Compute
\[
u_n = prox_{\lambda,\varphi_1}(w_n - \tau_nA^*(I - prox_{\lambda,\varphi_2})Aw_n),
\]
where
\[
\tau_n = \begin{cases} \frac{\rho_n g(w_n)}{||F(w_n)||^2 + ||H(w_n)||^2}, & \text{if } ||F(w_n)||^2 + ||H(w_n)||^2 \neq 0, \\ 0, & \text{otherwise,} \end{cases}
\]
\[
g(x) = \frac{1}{2}||(I - prox_{\lambda,\varphi_2})Ax||^2, \quad F(x) = A^*(I - prox_{\lambda,\varphi_2})Ax, \quad H(x) = (I - prox_{\lambda,\varphi_1})x.
\]

**Step 3.** Compute
\[
x_{n+1} = \alpha_n\beta f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)Su_n.
\]

Set \( n := n + 1 \) and return to **Step 1**.

5. **Numerical examples.** In this section, we give some numerical examples to illustrate the performance of our Algorithm 3.1. First, we consider an electricity production model which can be reformulated as a Nash-Cournot oligopolistic equilibrium problem. We perform some computational experiment by using Algorithm 4.2 to solve this model. Furthermore, we give two other numerical examples to show that our proposed Algorithm 3.1 performs better than Algorithm 1.1 of Wangecke et al. [48] and Algorithm 1.2 of Tang [46]. All numerical computations are carried-out using MATLAB 2019(a) on a HP Desktop.
Example 5.1 (Electricity Production). We consider a Nash-Cournot oligopolistic equilibrium model in electricity markets.

In this model, we assume that there are $m$ companies, each company $i$ may possess $I_i$ generating units. Let $x$ denote the vector whose entry $x_j$ stands for the power generating by unit $j$. Suppose the price $p_i(s)$ is a decreasing affine function of $s$ where $s := \sum_{j=1}^N x_j$ where $N$ is the number of all generating units. Hence, $p_i(s) := \alpha - \beta s$. Then the profit made by company $i$ is given by

$$f_i(x_i) := p_i(s) \sum_{j \in I_i} x_j - \sum_{j \in I_i} c_j(x_j),$$

where $c_j(x_j)$ is the cost for generating $x_j$ by generating unit $j$. Suppose that $K_i$ is the strategy set of company $i$, which implies that $\sum_{j \in I_i} x_j \in K_i$ for each $i$. Then the strategy set of the model is $C := K_1 \times K_2 \times \ldots \times K_m$.

Each company actually seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point $x^* \in C = K_1 \times K_2 \times \ldots \times K_m$ is an equilibrium point if

$$f_i(x^*) \geq f_i(x^*[x_i]) \quad \text{for all } x_i \in K_i, \quad i = 1, 2, \ldots, m,$$

where the vector $x^*[x_i]$ stands for the vector obtained from $x^*$ by replacing $x_i^*$ with $x_i$. Define

$$f(x, y) := \psi(x, y) - \psi(x, x)$$

with

$$\psi(x, y) := -\sum_{i=1}^m f_i(x^*[y_i]).$$

Then the problem of finding a Nash equilibrium point of our model can be formulated as

$$x^* \in C : f(x^*, x) \geq 0 \quad \text{for all } x \in C. \quad (35)$$

Suppose for every $j$, the cost $c_j$ for production is an increasingly convex function. The convexity assumption here means that the cost for producing a unit production increases as the quantity of the production gets larger. Under this convexity assumption, it is not hard to see that Problem (35) is equivalent to

Find $x^* \in C : \langle Bx^* - \bar{a}, x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0$, for all $x \in C, \quad (36)$

where $\bar{a} := (\alpha, \alpha, \ldots, \alpha)^T$, $B := \begin{pmatrix} \beta_1 & 0 & 0 & \ldots & 0 \\ 0 & \beta_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \beta_n \end{pmatrix}$, $\bar{B} := \begin{pmatrix} 0 & \beta_1 & \beta_1 & \ldots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \ldots & \beta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{pmatrix}$,

$$\varphi(x) := x^T B x + \sum_{j=1}^N c_j(x_j).$$

Note that when $c_j$ is differentiable convex for every $j$, Problem (35) is equivalent to the variational inequality

Find $x^* \in C : \langle Bx^* - \bar{a} + \nabla \varphi(x^*), x - x^* \rangle \geq 0$, for all $x \in C. \quad (37)$

Hence from Remark 3, by setting $A^*(I - P_D)A = B + \nabla \varphi$, we can apply our Algorithm 4.2 for solving problem (35). We test Algorithm 4.2 with the cost function given by

$$c_j(x_j) = \frac{1}{2} x_j^T H x_j + d^T x_j.$$
The parameters $\beta_j$ for all $j = 1, \ldots, m$, matrix $H$ and vector $d$ were generated randomly in the interval $(0, 1], [1, 30]$ and $[1, 30]$ respectively. We take $\alpha_n = \frac{1}{n+1}, \theta_n = \frac{1}{n+1}, \beta_n = \frac{2n}{5n+3}, G = 1$ and $\beta = 1$. We use different choices of $N = 4, 10, 20$, and different initial choices $x_0 = \text{rand}(N,1), x_1 = 0.5 \times \text{rand}(N,1)$ generated randomly in the interval $[1, 30]$ and $m = 10$ with $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-5}$ as stopping criterion. The projections $P_C$ and $P_D$ in this case are carried out using optimization toolbox in matlab programming. We also suppose that each company has the same lower production bound $1$ and upper production bound $30$, that is,

$$K_i := \{x_i : 1 \leq x_i \leq 30\}, \ i = 1, \ldots, 10.$$ 

We thus plot the graphs of error against number of iteration in each case. The numerical result can be seen in Figure 1 and Table 1.

| Table 1. Numerical results for Example 5.1. |
|-----------------|-----------------|-----------------|
| $N = 50$        | 19              | 0.0289          |
| $N = 100$       | 19              | 0.0386          |
| $N = 500$       | 41              | 0.1386          |
| $N = 1000$      | 138             | 0.3523          |

Next, we give some numerical examples to demonstrate the performance of our Algorithm 3.1 in comparison with some existing algorithms in the literature.

**Example 5.2.** Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers with the inner product defined by $\langle x, y \rangle = xy \forall x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. Define the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \frac{x}{2}$, $B_1, B_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ by $B_1(x) = \{2x\} \forall x \in \mathbb{R}$ and $B_2(y) = \{3y\} \forall y \in \mathbb{R}$. Let $A : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $A(x) = \frac{-x}{4} \forall x \in \mathbb{R}$ and $S : \mathbb{R} \rightarrow \mathbb{R}$ by $\frac{3}{2}x \forall x \in \mathbb{R}$. For each $n \in \mathbb{N}$, choose $\alpha_n = \frac{1}{2n}, \beta_n = \frac{2n}{5n+9}, G = I, \lambda = 0.5$ and $\beta = 1$. The sequence $\{\theta_n\}$ is chosen such that

$$\theta_n = \begin{cases} 
\frac{\omega_n}{n+1} & \text{if } x_n \neq x_{n-1}, \\
\frac{\omega_n}{2n+4} & \text{if } x_n = x_{n-1}, 
\end{cases}$$

where $\omega_n = \frac{1}{4n^4}$. We choose different initial points and compare the numerical performance of our Algorithm 3.1 with Algorithm 1.1 and Algorithm 1.2 by plotting the graphs of error against number of iterations and comparing the time taken for the computation. The stopping criterion used for our experiment is $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-5}$. The numerical results can be seen in Figure 2 and Table 2.

**Example 5.3.** Let $H_1 = H_2 = L^2([a, b])$. Define $C \subset H_1$ by $C := \{x \in L^2([a, b]) : \langle \alpha, x \rangle \leq \beta\}$, where $0 \neq \alpha \in L^2([a, b])$ and $\beta \in \mathbb{R}$, then

$$P_C(x) = \begin{cases} 
\beta - \frac{\langle \alpha, x \rangle}{||\alpha||_{L^2}} \alpha + x, & \langle \alpha, x \rangle > \beta, \\
x, & \langle \alpha, x \rangle \leq \beta. 
\end{cases}$$

Also, let $D \subset H_2$ define by $D := \{y \in L^2([a, b]) : ||x - d||_{L^2} \leq r\}$ be a closed ball centered at $d \in L^2([a, b])$ with radius $r > 0$, then

$$P_D(x) = \begin{cases} 
d + r \frac{x - d}{||x - d||_{L^2}}, & x \notin D, \\
x, & x \in D. 
\end{cases}$$
We set $B_1 = \partial i_C$, $B_2 = \partial i_D$ in Theorem 3.5 and in particular, we define

$$C = \{x \in L^2([0, 1]) : \int_0^1 (t^2 + 1)x(t)dt \leq 1\},$$

$$D = \{x \in L^2([0, 1]) : \int_0^1 |x(t) - \cos(t)|^2 dt \leq 25\}.$$

Let $A : L_2([0, 1]) \to L_2([0, 1])$ and $S : L_2([0, 1]) \to L_2([0, 1])$ be defined by $Ax(t) = \frac{3}{5} x(t)$ and $Sx(t) = \int_0^1 \frac{x(t)}{2} dt$ respectively for all $x \in L_2([0, 1])$ and $t \in [0, 1]$. Clearly
Table 2. Numerical results for Example 5.2.

| Case   | CPU time (sec) | No of Iter. |
|--------|----------------|-------------|
| I      | 0.0021         | 8           |
| II     | 0.0021         | 9           |
| III    | 0.0044         | 10          |
| IV     | 0.0062         | 10          |

A is a bounded linear operator and $S$ is a quasi-nonexpansive mapping. Define the mappings $f : H_1 \to H_1$ by $f(x) = \frac{x}{2}$, $G : H_1 \to H_1$ by $G(x) = x$, for each $n \in \mathbb{N}$, we take $\rho_n = 0.04$, $\beta_n = \frac{3n}{5n+7}$, $\alpha_n = \frac{1}{n+1}$, $\beta = 1$, $\delta = 0.06$ and choose the sequence $\{\theta_n\}$ such that

$$\theta_n = \begin{cases} 
\min \left\{ \delta, \frac{\alpha_n^2}{||x_n - x_n - 1||} \right\} & \text{if } x_n \neq x_{n-1}, \\
\delta & \text{otherwise.} 
\end{cases}$$

We apply Theorem 4.1 and compare the performance of Algorithm 3.1 with Algorithm 1.1 for finding common solution of the SVIP and fixed point of $S$ using the following initial values:

(i) $x_0 = \exp(3t)/30$ and $x_1 = \cos(2t)$,
(ii) $x_0 = \sin(3t)$ and $x_1 = t^2 + 2t$,
(iii) $x_0 = \sin(3t)/30$ and $x_1 = \cos(2t)/50$,
(iv) $x_0 = (t^2 - 1)/3$ and $x_1 = \exp(-2t)$.

For Algorithm 1.3, we choose $\alpha_n = \frac{1}{(n+1)}$, $\theta_n = \frac{1}{(n+1)^2}$, and $\lambda = 0.25$. We plot the graphs of error against number of iteration for each algorithm and use $||x_{n+1} - x_n|| < 10^{-5}$ as stopping criterion to terminate the algorithms. The numerical result can be seen in Table 3 and Figure 3.

6. Conclusion. In this paper, we study the common solution of split variational inclusion problems and fixed point problem in real Hilbert spaces. We propose a general iterative scheme with inertia and self-adaptive stepsize for approximating a common solution of split variational inclusion problem (SVIP) and fixed point problem (FPP) for quasi-nonexpansive mapping in real Hilbert spaces. Our proposed algorithm was applied to find an equilibrium point with minimal cost of production for a model in industrial electricity production. Numerical results are presented to demonstrate the efficiency of our algorithm in comparison with some other existing algorithms in the literature. Results obtained are also applied to find solution of split feasibility problems and split minimization problems. Our results improve and generalize many known results in this direction in the literature.
Figure 2. Example 5.2, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Acknowledgments. The authors sincerely thank the anonymous reviewer for his careful reading, constructive comments and fruitful suggestions that substantially improved the manuscript. The second author acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The third author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and
Table 3. Numerical results for Example 5.3.

| Choice     | Algorithm 3.1 CPU time (sec) | No of Iter. | Algorithm 1.1 CPU time (sec) | No of Iter. |
|------------|-------------------------------|-------------|-------------------------------|-------------|
| (i)        | 1.7859                        | 11          | 5.1231                        | 23          |
| (ii)       | 1.4997                        | 13          | 13.3981                       | 27          |
| (iii)      | 2.6789                        | 7           | 9.1093                        | 12          |
| (iv)       | 6.3222                        | 11          | 24.5622                       | 24          |

conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

REFERENCES

[1] H. A. Abass, K. O. Aremu, L. O. Jolaoso and O. T. Mewomo, An inertial forward-backward splitting method for approximating solutions of certain optimization problems, *J. Nonlinear Funct. Anal.*, 2020 (2020), Art. ID 6, 20 pp.

[2] M. Abbas, M. Al Sharani, Q. H. Ansari, O. S. Iyiola and Y. Shehu, Iterative methods for solving proximal split minimization problem, *Numer. Algorithms*, 78 (2018), 193–215.

[3] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, Modified inertial subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems, *Optimization*, (2020).

[4] F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Anal.*, 9 (2001), 3–11.

[5] J. P. Aubin, Optima and Equilibria: An Introduction to Nonlinear Analysis, *Springer*, 1993.

[6] A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problem, *SIAM J. Imaging Sci.*, 2 (2009), 183–202.

[7] F. E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.*, 74 (1968), 660–665.

[8] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, *Inverse Probl.*, 18 (2002), 441–453.

[9] C. Byrne, Y. Censor, A. Gibali and S. Reich, Weak and strong convergence of algorithms for the split common null point problem, *J. Nonlinear Convex Anal.*, 13 (2012), 759–775.

[10] L. C. Ceng, Q. H. Ansari and J. C. Yao, An extragradient method for solving split feasibility and fixed point problems, *Comput. Math. Appl.*, 64 (2012), 633–642.

[11] Y. Censor and T. Elfving, A multiprojection algorithms using Bregman projection in a product space, *Numer. Algorithms*, 8 (1994), 221–239.

[12] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Probl.*, 21 (2005), 2071–2084.

[13] L. C. Ceng, Q. H. Ansari and J. C. Yao, An extragradient method for solving split feasibility and fixed point problems, *Comput. Math. Appl.*, 64 (2012), 633–642.

[14] A. Chambolle and C. Dossal, On the convergence of the iterates of the “fast iterative shrinkage/thresholding algorithm”, *J. Optim. Theory Appl.*, 166 (2015), 968–982.

[15] R. H. Chan, S. Ma and J. F. Jang, Inertial proximal ADMM for linearly constrained separable convex optimization, *SIAM J. Imaging Sci.*, 8 (2015), 2239–2267.

[16] R. Glowinski and P. Le Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, *SIAM, Philadelphia*, 9 (1989).
Figure 3. Example 5.3, Top Left: Choice (i); Top Left: Choice (ii); Bottom Left: Choice (iii); Bottom Right: Choice (iv).

[17] A. N. Iusem, On some properties of paramonotone operator, Convex Anal., 5 (1998), 269–278.
[18] C. Izuchukwu, G. C. Ugwunnadi, O. T. Mewomo, A. R. Khan and M. Abbas, Proximal-type algorithms for split minimization problem in p-uniformly convex metric space, Numer. Algorithms, 82 (2019), 909–935.
[19] L. O. Jolaoso, T. O. Alakoya, A. Taiwo and O. T. Mewomo, A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems, Rend. Circ. Mat. Palermo II, (2019).
[20] L. O. Jolaoso, T. O. Alakoya, A. Taiwo and O. T. Mewomo, Inertial extragradient method via viscosity approximation approach for solving equilibrium problem in Hilbert space, *Optimization*, (2020).

[21] L. O. Jolaoso, F. U. Ogbuisi and O. T. Mewomo, An iterative method for solving minimization, variational inequality and fixed point problems in reflexive Banach spaces, *Adv. Pure Appl. Math.*, 9 (2018), 167–184.

[22] L. O. Jolaoso, K. O. Oyewole, C. C. Okeke and O. T. Mewomo, A unified algorithm for solving split generalized mixed equilibrium problem and fixed point of nonsmooth mapping in Hilbert space, *Demonstr. Math.*, 51 (2018), 211–232.

[23] L. O. Jolaoso, A. Taiwo, T. O. Alakoya and O. T. Mewomo, A self-adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces, *Demonstr. Math.*, 52 (2019), 183–203.

[24] Y. Kimura and S. Saejung, Strong convergence for a common fixed point of two different generalizations of cutter operators, *Linear Nonlinear Anal.*, 1 (2015), 53–65.

[25] L. V. Long, D. V. Thong and V. T. Dung, New algorithms for the split variational inclusion problems and application to split feasibility problems, *Optimization*, (2019).

[26] D. Lorenz and T. Pock, An inertial forward-backward algorithm for monotone inclusions, *J. Math. Imaging Vision*, 51 (2015), 311–325.

[27] P. E. Maingé, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, 325 (2007), 469–479.

[28] P. E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, *Comput. Math. Appl.*, 59 (2010), 74–79.

[29] G. Marino and H. K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, *J. Math. Anal. Appl.*, 318 (2006), 43–52.

[30] A. Moudafi, Split monotone variational inclusions, *J. Optim. Theory Appl.*, 150 (2011), 275–283.

[31] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, 241 (2000), 46–55.

[32] A. Moudafi and M. Oliny, Convergence of a splitting inertial proximal method for monotone operators, *J. Comput. Appl. Math.*, 152 (2003), 447–454.

[33] F. U. Ogbuisi and O. T. Mewomo, Convergence analysis of an inertial accelerated iterative algorithm for solving split variational inequality problem, *Adv. Pure Appl. Math.*, 10 (2019), 339–353.

[34] F. U. Ogbuisi and O. T. Mewomo, Convergence analysis of common solution of certain nonlinear problems, *Fixed Point Theory*, 19 (2018), 335–358.

[35] F. U. Ogbuisi and O. T. Mewomo, Iterative solution of split variational inclusion problem in real Banach space, *Afr. Mat.*, 28 (2017), 295–309.

[36] G. N. Ogwo, C. Izuchukwu, K. O. Aremu and O. T. Mewomo, A viscosity iterative algorithm for a family of monotone inclusion problems in an Hadamard space, *Bull. Belg. Math. Soc. Simon Stevin*, 27 (2020), 127–152.

[37] C. C. Okeke and O. T. Mewomo, On split equilibrium problem, variational inequality problem and fixed point problem for multivalued mappings, *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, 9 (2017), 255–280.

[38] P. Phairatchatniyom, P. Kumam, Y. J. Cho, W. Jirakitpuwapat and K. Sitthithakerngkiet, The modified inertial iterative algorithm for solving split variational inclusion problem for multi-valued quasi nonexpansive mappings with some applications, *Mathematics*, 7 (2019), 560.

[39] B. T. Polyak, Some methods of speeding up the convergence of iterative methods, *Zh. Vyssh. Sht. Mat. Fiz.*, 4 (1964), 1–17.

[40] S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, *Nonlinear Anal.*, 75 (2012), 742–750.

[41] Y. Shehu and D. Agbebaku, On split inclusion problem and fixed point problem for multivalued mappings, *Comput. Appl. Math.*, 37 (2018), 1807–1824.

[42] Y. Shehu and O. T. Mewomo, Further investigation into split common fixed point problem for demi-contraction operators, *Acta Math. Sin. (Engl. Ser.)*, 32 (2016), 1357–1376.

[43] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, A modified Halpern algorithm for approximating a common solution of split equality convex minimization problem and fixed point problem in uniformly convex Banach spaces, *Comput. Appl. Math.*, 38 (2019), Art. 77.
A SELF ADAPTIVE INERTIAL ALGORITHM FOR SVIP AND FPP

[44] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, Parallel hybrid algorithm for solving pseudomonotone equilibrium and split common fixed point problems, Bull. Malays. Math. Sci. Soc., 43 (2020), 1893–1918.

[45] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, General alternative regularization method for solving split equality common fixed point problem for quasi-pseudocontractive mappings in Hilbert spaces, Ric. Mat., 69 (2020), 235–259.

[46] Y. Tang, Convergence analysis of a new iterative algorithm for solving split variational inclusion problems, J. Indus. Mgt Opt., 16 (2020), 945–964.

[47] D. Van Hieu, Strong convergence of a new hybrid algorithm for fixed point problems and equilibrium problems, Math. Model. Anal., 24 (2019), 1–19.

[48] R. Wangkeeree, K. Rattanaseeha and R. Wangkeeree, The general iterative methods for split variational inclusion problem and fixed point problem in Hilbert spaces, J. Comp. Anal. Appl., 25 (2018), 19–31.

[49] H. K. Xu, An iterative approach to quadratic optimization, J. Opt. Theory Appl., 116 (2003), 659–678.

[50] H. K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc., 66 (2002), 240–256.

[51] Y. Yao, M. Postolache, X. Qin and J.-C. Yao, Iterative algorithm for proximal split feasibility problem, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 80 (2018), 37–44.

Received November 2019; revised March 2020.

E-mail address: 218086823@stu.ukzn.ac.za
E-mail address: 216074984@stu.ukzn.ac.za
E-mail address: mewomoo@ukzn.ac.za