DISTINGUISHED REPRESENTATIONS, BASE CHANGE, AND REDUCIBILITY FOR UNITARY GROUPS

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Abstract. We show the equality of the local Asai $L$-functions defined via the Rankin-Selberg method and the Langlands-Shahidi method for a square integrable representation of $GL_n(E)$. As a consequence we characterise reducibility of certain induced representations of $U(n,n)$, and the image of the base change map from $U(n)$ to $GL_n(E)$ in terms of $GL_n(F)$-distinguishedness.

1. Introduction

A representation $(\pi, V)$ of a group $G$ is said to be distinguished with respect to a character $\chi$ of a subgroup $H$, if there exists a linear form $l$ of $V$ satisfying $l(\pi(h)v) = \chi(h)l(v)$ for all $v \in V$ and $h \in H$. When the character $\chi$ is taken to be the trivial character, such representations are also called as distinguished representations of $G$ with respect to $H$.

The concept of distinguished representations can be carried over to a continuous context of representations of real and $p$-adic Lie groups, as well in a global automorphic context (where the requirement of a non-zero linear form is replaced by the non-vanishing of a period integral). The philosophy, due to Jacquet, is that representations of a group $G$ distinguished with respect to a subgroup $H$ of fixed points of an involution on $G$ are often functorial lifts from another group $G'$.

In this paper we consider $G = \text{Res}_{E/F} GL(n)$ and $H = GL(n)$ where $E$ is a quadratic extension of a non-Archimedean local field $F$ of characteristic zero. In this case, the group $G'$ is conjectured to be the quasi-split unitary group with respect to $E/F$,

$$G' = U(n) = \{ g \in GL_n(E) \mid gJ'\bar{g} = J \},$$

where $J_{ij} = (-1)^{n-i}\delta_{i,n-j+1}$ and $\bar{g}$ is the Galois conjugate of $g$. There are two base change maps from $U(n)$ to $GL(n)$ over $E$ called the stable and the unstable base change maps (see Section 4.2). We have the following conjecture due to Flicker and Rallis [4]:

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Conjecture. Let $\pi$ be an irreducible admissible representation of $GL_n(E)$. If $n$ is odd (resp. even), then $\pi$ is $GL_n(F)$-distinguished if and only if it is a stable (resp. unstable) base change from $U(n)$.

When $n = 1$ the above conjecture is just Hilbert Theorem 90. The case $n = 2$ is established by Flicker [4]. The following theorem proves the conjecture for a supercuspidal representation when $n = 3$.

**Theorem 1.** A supercuspidal representation $\pi$ of $GL_3(E)$ is distinguished with respect to $GL_3(F)$ if and only if it is a stable base change lift from $U(3)$.

Let $G$ be a reductive $p$-adic group. An irreducible tempered representation of $G$ occurs as a component of an induced representation $I(\pi)$, parabolically induced from a square integrable representation $\pi$ of the Levi component $M$ of a parabolic subgroup $P$ of $G$. Thus the tempered spectrum of $G$ is determined from a knowledge of the discrete series representations of the Levi components of different parabolics and knowing the decomposition of induced representations. The decomposition of $I(\pi)$ is governed by the theory of $R$-groups.

Let $G = U(n,n)$ be the quasi-split unitary group in $2n$ variables over a $p$-adic field $F$, defined with respect to a quadratic extension $E$ of $F$. Let $P$ be a parabolic subgroup of $G$ with a Levi component $M$ isomorphic to $GL_{n_1}(E) \times \cdots \times GL_{n_t}(E)$ for some integers $n_i \geq 1$ satisfying $\sum_{i=1}^{t} n_i = n$. Let $\pi_i$, $1 \leq i \leq t$ be discrete series representations of $GL_{n_i}(E)$. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_t$ be the discrete series representation of $M$. Let $\omega_{E/F}$ denote the quadratic character of $F^*$ associated to the quadratic extension $E/F$. The following theorem gives a description of the $R$-group $R(\pi)$ in terms of distinguishedness of the representations $\pi_i$:

**Theorem 2.** With the above notation, 

$$R(\pi) \simeq (\mathbb{Z}/2\mathbb{Z})^r,$$

where $r$ is the number of inequivalent representations $\pi_i$ which are $\omega_{E/F}$-distinguished with respect to $GL_{n_i}(F)$.

**Corollary 1.** Let $P$ be a maximal parabolic of $U(n,n)$ with Levi component isomorphic to $GL_n(E)$, and $\pi$ be a discrete series representation of $GL_n(E)$. Then $I(\pi)$ is reducible if and only if $\pi$ is $\omega_{E/F}$-distinguished with respect to $GL_n(F)$.

A particular consequence of the corollary is the following result about the Steinberg representation of $GL_n(E)$, which is part of a more general conjecture, due to D. Prasad, about the Steinberg representation of $G(E)$ where $G$ is a reductive algebraic group over $F$ [15].
Theorem 3. Let $\pi$ be the Steinberg representation of $GL_n(E)$. Then $\pi$ is distinguished with respect to a character $\chi \circ \det$, for a character $\chi$ of $F^\ast$, of $GL_n(F)$ if and only if $n$ is odd and $\chi$ is the trivial character, or $n$ is even and $\chi = \omega_{E/F}$.

Our approach to the above theorems is via the theory of Asai $L$-functions. The Asai $L$-function, also called the twisted tensor $L$-function, can be defined in three different ways: one via the local Langlands correspondence and in terms of Langlands parameters denoted by $L(s, As(\pi))$; via the theory of Rankin-Selberg integrals \cite{rankin, selberg, shahidi} denoted by $L_1(s, As(\pi))$; and the Langlands-Shahidi method (applied to a suitable unitary group) \cite{shahidi, wang} denoted by $L_2(s, As(\pi))$. It is of course expected that all the above three $L$-functions match.

The main point is that the analytical properties of the different definitions of Asai $L$-function give different insights about the representation: the Asai $L$-function defined via the Rankin-Selberg method can be related to distinguishedness with respect to $GL_n(F)$, whereas the Asai $L$-function defined via the Langlands-Shahidi method is related to the base change theory from $U(n)$, and to reducibility questions for $U(n, n)$. Thus the following theorem, proved using global methods, is a key ingredient towards a proof of the above theorems:

Theorem 4. Let $\pi$ be a square integrable representation of $GL_n(E)$. Then $L_1(s, As(\pi)) = L_2(s, As(\pi))$.

2. Asai $L$-functions

2.1. Langlands parameters. Let $F$ be a non-archimedean local field and let $E$ be a quadratic extension of $F$. The Weil-Deligne group $W'_E$ of $E$ is of index two in the Weil-Deligne group $W'_F$ of $F$. Choose $\sigma \in W'_F \setminus W'_E$ of order 2. Given a a continuous, $\Phi$-semisimple representation $\rho$ of $W'_E$ of dimension $n$, the representation $As(\rho) : W'_F \to GL_n(\mathbb{C})$ given by tensor induction of $\rho$ is defined as,

$$
As(\rho)(x) = \begin{cases} 
\rho(x) \otimes \rho(\sigma^{-1}x\sigma) & \text{if } x \in W'_E \\
\rho(\sigma x) \otimes \rho(x\sigma) & \text{if } x \notin W'_E.
\end{cases}
$$

Let $\pi$ be an irreducible, admissible representation of $GL_n(E)$ with Langlands parameter $\rho_\pi$. The Asai $L$-function $L(s, As(\pi))$ is defined to be the $L$-function $L(s, As(\rho_\pi))$.

2.2. Rankin-Selberg method.
2.2.1. Local theory. We recall the Rankin-Selberg theory of the Asai $L$-function [3], [5], and [12]. Let $F$ be a non-archimedean local field and let $E$ be either a quadratic extension of $F$ or $F \oplus F$. Let $\pi$ be an irreducible admissible generic representation of $GL_n(E)$. We take an additive character $\psi$ of $E$ which restricts trivially to $F$. There exists an additive character $\psi_0$ of $F$ such that $\psi(x) = \psi_0(\Delta(x - \bar{x}))$ where $\Delta$ is a trace zero element of $E^*$. Let $W(\pi, \psi)$ denote the Whittaker model of $\pi$ with respect to $\psi$. Let $N_n(F)$ be the unipotent radical of the Borel subgroup of $GL_n(F)$. Consider the integral (see [3])

$$\Psi(s, W, \Phi) = \int_{N_n(F)\backslash GL_n(F)} W(g)\Phi((0, 0, \ldots, 1)g) \, |\det g|_F^s \, dg,$$

where $\Phi \in S(F^n)$, the space of locally constant compactly supported functions on $F^n$, and $dg$ is a $GL_n(F)$-invariant measure on $N_n(F) \backslash GL_n(F)$.

In [5], Flicker proves that the above integral converges absolutely in some right half plane to a rational function in $X = q^{-s}$, where $q = q_F$ is the cardinality of the residue field of $F$. The space spanned by $\Psi(s, W, \Phi)$ (as $W$ and $\Phi$ vary) is a fractional ideal in $\mathbb{C}[X, X^{-1}]$ containing the constant function 1. We can choose a unique generator of this ideal of the form $P_1(X)^{-1}$, $P_1(X) \in \mathbb{C}[X]$ such that $P_1(0) = 1$. Define the Asai $L$-function $L_1(s, As(\pi))$ as

$$L_1(s, As(\pi)) = P_1(q^{-s})^{-1}.$$

This does not depend on the choice of the additive character $\psi$. Moreover $\Psi(s, W, \Phi)$ satisfies the functional equation

$$\Psi(1 - s, \tilde{W}, \tilde{\Phi}) = \gamma_1(s, As(\pi), \psi)\Psi(s, W, \Phi)$$

where $\tilde{W}(g) = W(w^t g^{-1})$, $w$ is the longest element of the Weyl group, and $\tilde{\Phi}$ is the Fourier transform of $\Phi$ with respect to $\psi_0$. The epsilon factor,

$$\epsilon_1(s, As(\pi), \psi) = \gamma_1(s, As(\pi), \psi) \frac{L_1(s, As(\pi))}{L_1(1 - s, As(\pi^\vee))}$$

is a monomial in $q_F^{-s}$.

If $E = F \oplus F$ write $\pi = \pi_1 \times \pi_2$ considered as a representation of $G_n(F) \times GL_n(F)$. Then

(1) $$L_1(s, As(\pi)) = L(s, \pi_1 \times \pi_2),$$

where the right hand side is the Rankin-Selberg $L$-factor of $\pi_1 \times \pi_2$.

We have the following proposition [3] Proposition in Section 3]:

**Proposition 5.** Suppose $E/F$ is an unramified quadratic extension. Let $\pi = Ps(\mu_1, \ldots, \mu_n)$ be an unramified unitary representation induced
from the character \((t_1, \ldots, t_n) \mapsto \prod \mu_i(t_i)\) of the diagonal torus in \(GL_n(E)\). Let \(W^0_\pi\) be the spherical Whittaker function, and \(\Phi^0_F\) be the characteristic function of \(O^n_F\). Then

\[
\Psi(s, W^0_\pi, \Phi^0_F) = \prod_{j=1}^{n} (1 - \mu_j(\varpi_F)q^{-s})^{-1} \cdot \prod_{i<j} (1 - \mu_i(\varpi_F)\mu_j(\varpi_F)q^{-2s})^{-1}
\]

where \(\varpi_F\) is a uniformizing parameter of \(F\).

The following proposition is proved in [12, Theorem 4]:

**Proposition 6.** Let \(\pi\) be a square integrable representation of \(GL_n(E)\). Then \(L_1(s, \text{As}(\pi))\) is regular in the region \(\text{Re}(s) > 0\).

We remark that for the proof of Theorem 4 all that we require is that \(L_1(s, \text{As}(\pi))\) be regular in the region \(\text{Re}(s) \geq 1/2\).

2.2.2. **Global theory.** Now let \(L/K\) be a quadratic extension of number fields. We assume that the archimedean places of \(K\) split in \(L\). Let \(\psi_0\) be a non-trivial character of \(A_K/K\), and let \(\psi = \psi_0(\Delta(x - \bar{x}))\). For a global field \(K\), let \(\Sigma_K\) denote the set of places of \(K\). Let \(\Pi = \bigotimes_{w \in \Sigma_L} \Pi_w\) be a representation of \(GL_n(A_L)\). Let \(T\) be a finite set of places of \(K\) containing the following places:

- the archimedean places of \(K\).
- the ramified places of the extension \(L/K\).
- the places \(v\) of \(K\) dividing a place \(w\) of \(L\), where either \(\psi_{0,v}\), \(\psi_{L_w}\) or \(\Pi_w\) is ramified.

Define,

\[
L_{1,v}(s, \text{As}(\Pi)) = \begin{cases} 
L_1(s, \text{As}(\Pi_w)) & w|v, \ v \in T \text{ and } v \text{ inert}, \\
\Psi_v(s, W^0_{\Pi_w}, \Phi^0_{F_v}) & v \text{ inert}, \ v \notin T, \\
L(s, \Pi_{w_1} \times \Pi_{w_2}) & v \text{ splits}, \ v = w_1w_2.
\end{cases}
\]

**Remark.** Let \(v\) be a place of \(K\) not in \(T\), inert in \(L\) and \(w\) the place of \(L\) dividing \(v\). It is not known that \(L_1(s, \text{As}(\Pi_w)) = \Psi(s, W^0_{\Pi_w}, \Phi^0_{K_v})\).

In the notation of Proposition 5, the right hand side is the \(L\)-factor associated by Langlands functoriality.

Following Kable [12], we define the Rankin-Selberg Asai \(L\)-function \(L_1(s, \text{As}(\Pi))\) as,

\[
L_1(s, \text{As}(\Pi), T) = \prod_{v \in \Sigma_K} L_{1,v}(s, \text{As}(\Pi)).
\]

We have the functional equation:
Proposition 7 (Theorem 5, [12]). Let $\Pi$ be a cuspidal automorphic representation of $GL_n(A_L)$. Then $L_1(s, \text{As}(\Pi), T)$ admits a meromorphic continuation to the entire plane and satisfies the functional equation

$$L_1(s, \text{As}(\Pi), T) = \epsilon_1(s, \text{As}(\Pi), T) L_1(1-s, \text{As}(\Pi^\vee), T)$$

where the function $\epsilon_1(s, \text{As}(\Pi), T)$ is entire and non-vanishing.

2.3. Langlands-Shahidi method.

2.3.1. Local theory. We now recall the Langlands-Shahidi approach to the Asai $L$-function [6, 18]. Let $G = U(n, n)$ be the quasi-split unitary group in $2n$ variables with respect to $E/F$. The group $M = R_{E/F}GL_n$ can be embedded as a Levi component of a maximal parabolic subgroup $P$ of $G$ with unipotent radical $N$. Let $r$ be the adjoint representation of the $L$-group of $M$ on the Lie algebra of the $L$-group of $N$. Fix an additive character $\psi_0$ of $F$. The Langlands-Shahidi gamma factor $\gamma_2(s, \pi, r, \psi_0)$ defined in [18], is a rational function of $q^{-s}$. Let $P_2(X)$ be the unique polynomial satisfying $P_2(0) = 1$ such that $P_2(q^{-s})$ is the numerator of $\gamma_2(s, \pi, r, \psi_0)$. For a tempered $\pi$, the Langlands-Shahidi Asai $L$-function is defined as

$$L_2(s, \text{As}(\pi)) = 1/P_2(q^{-s}).$$

The $L$-function is independent of the additive character. The quantity

$$\epsilon_2(s, \text{As}(\pi), \psi_0) = \gamma_2(s, \pi, r, \psi_0) \frac{L_2(s, \text{As}(\pi))}{L_2(1-s, \text{As}(\pi^\vee))}$$

is the Langlands-Shahidi epsilon factor, and is a monomial in $q^{-s}$.

The analytical properties of $L_2(s, \text{As}(\pi))$ are proved in [18, Theorem 3.5, Proposition 7.2]:

Proposition 8. Let $\pi$ be an irreducible admissible representation of $GL_n(E)$. Then the following holds:

1. If $E$ is an unramified extension of $F$, and $\pi = Ps(\mu_1, \ldots, \mu_n)$ be a unitary unramified representation of $GL_n(E)$, as in the hypothesis of Proposition 5. Then

$$L_2(s, \text{As}(\pi)) = \prod_{j=1}^{n} (1 - \mu_j(\varpi_F q_F^{-s})^{-1} \cdot \prod_{i<j} (1 - \mu_i(\varpi_F) \mu_j(\varpi_F) q_F^{-2s})^{-1}.$$

2. Let $\pi$ be a tempered representation of $GL_n(E)$. Then $L_2(s, \text{As}(\pi))$ is regular in the region $\text{Re}(s) > 0$. 
2.3.2. Global theory. Let $L/K$ be a quadratic extension of number fields, and let $\Pi = \bigotimes_w \Pi_w$ be a representation of $GL_n(\mathbb{A}_L)$. Define for a place $v$ of $K$,

\[
L_{2,v}(s, As(\Pi)) = \begin{cases} 
L_2(s, As(\Pi_w)) & w|v, \ v \text{ inert,} \\
L(s, \Pi_{w_1} \times \Pi_{w_2}) & v \text{ splits, } v = w_1w_2.
\end{cases}
\]

Define the global $L$-function

\[
L_2(s, As(\Pi)) = \prod_{v \in \Sigma_K} L_{2,v}(s, As(\Pi)).
\]

Then we have the functional equation [18]:

**Proposition 9.** Let $\Pi$ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$. Then $L_2(s, As(\Pi))$ admits a meromorphic continuation to the entire plane and satisfies the functional equation

\[
L_2(s, As(\Pi)) = \epsilon_2(s, As(\Pi))L_2(1 - s, As(\Pi^\vee))
\]

where the function $\epsilon_2(s, As(\Pi))$ is entire and non-vanishing.

3. Proof of Theorem

The proof of Theorem is via global methods. The following proposition embedding a square integrable representation $\pi$ as the local component of a cuspidal automorphic representation is well known [12, Lemma 5], [2, Lemma 6.5 of Chapter 1]:

**Proposition 10.** Let $E/F$ be a quadratic extension of non-archimedean local fields of characteristic zero and residue characteristic $p$. Let $\pi$ be a square integrable representation of $GL_n(E)$. Then the following holds:

1. There exists a number field $K$, a quadratic extension $L$ of $K$ and a place $v_0$ of $K$ inert in $L$, such that $K_{v_0} \simeq F$ and $L_{v_0} \simeq E$, where $w_0$ is the unique place of $L$ dividing $v_0$. Further, $v_0$ is the unique place of $K$ lying over the rational prime $p$, and the real places of $K$ are split in $L$.

2. There exists a cuspidal automorphic representation $\Pi$ of $GL_n(\mathbb{A}_L)$ such that $\Pi_{w_0} \simeq \pi$.

Let $\Pi$ be a cuspidal representation of $GL_n(\mathbb{A}_L)$ satisfying the properties of the above proposition. Choose a finite set $T$ of places of $K$ as in Proposition Consider the ratio

\[
F(s, \Pi) = \frac{L_2(s, As(\Pi))}{L_1(s, As(\Pi), T)}.
\]
If $v = w_1 w_2$ is a place of $K$ which splits into two places $w_1$ and $w_2$ of $L$, then

$$L'_{1,v}(s, \text{As}(\Pi)) = L_{2,v}(s, \text{As}(\Pi)) = L(s, \Pi_{w_1} \times \Pi_{w_2}).$$

By Propositions 5 and 8 if $v$ is a place of $K$ which is inert and not in $T$, then

$$L'_{1,v}(s, \text{As}(\Pi)) = L_{2,v}(s, \text{As}(\Pi)).$$

Hence

$$F(s, \Pi) = \prod_{v \in T} \frac{L_{2,v}(s, \text{As}(\Pi))}{L'_{1,v}(s, \text{As}(\Pi))}.$$ 

Write

$$F(s, \Pi) = G(s, \Pi)Q(s, \Pi)P_0(s, \Pi),$$

where

- The function $G(s, \Pi)$ is the ratio of the $L$-factors at the archimedean places; it is a ratio of products of Gamma functions of the form $\Gamma(as + b)$ for some suitable constants $a, b$.
- The function

$$Q(s, \Pi) = \prod_{i=1}^{n} \frac{(1 - \alpha_i q_{v_i}^{-s})}{\prod_{j=1}^{n} (1 - \beta_j q_{v_j}^{-s})}, \quad v_i, v_j \in T' := T \setminus \{v_0\}$$

is a ratio of the $L$-factors at the finite set of places of $T$ not equal to $v_0$; it is a ratio of products of distinct functions of the form $(1 - \beta q_v^{-s})$, $\beta \neq 0$, where $v \in T' := T \setminus \{v_0\}$, and $q_v$ is the number of elements of the residue field. By our assumption on $K$, $(p, q_{v_0}) = 1$.
- The function

$$P_0(s, \Pi) = \frac{L_{2}(s, \text{As}(\pi))}{L_{1}(s, \text{As}(\pi))}$$

is a ratio of products of functions of the form $(1 - \alpha q_{v_0}^{-s})$.

By Propositions 5 and 8 the functions $P_0(s, \Pi)$ and $P_0(s, \Pi')$ are regular and non-vanishing in the region $\text{Re}(s) \geq 1/2$.

We claim the following:

Claim. Let $\gamma_0$ be a pole (resp. zero) of $P_0(s, \Pi)$. The function $F(s, \Pi)$ has a pole (resp. zero) at all but finitely many elements of the form $\gamma_0 + 2\pi ik/\log q_{v_0}$, $k \in \mathbb{Z}$.

Proof of Claim. Suppose that the function $F(s, \Pi)$ is regular at points of the form $\gamma_0 + 2\pi il/\log q_{v_0}$ for integers $l \in C$, where $C$ is an infinite subset of the integers. Since $G(s)$ can contribute only finitely many zeros on any line with real part constant, these poles have to be cancelled by zeros of $Q(s, \Pi)$. Since $T$ is finite, and the local $L$-factors are
polynomial functions in $q_v^{-s}$, there is a $v \in T'$, $\gamma \in \mathbb{C}$ and a function $f : C \to \mathbb{Z}$ such that,

$$\gamma_0 + 2\pi i l / \log q_{v_0} = \gamma + 2\pi i f(l) / \log q_v$$

for infinitely many $l \in C$. Taking the difference of any two elements, we get $\log q_{v_0} / \log q_v \in \mathbb{Q}$. This is not possible as $q_{v_0}$ and $q_v$ are coprime integers. Hence, all but finitely many poles of the form $\gamma_0 + 2\pi ik / \log q_{v_0}$, $k \in \mathbb{Z}$ are poles of $F(s, \Pi)$. □

Since $P_0(s, \Pi)$ is regular in the region $\text{Re}(s) \geq 1/2$, we obtain $\text{Re}(\gamma_0) < 1/2$. From the global functional equations given by Propositions 7 and 9, $F(s, \Pi)$ satisfies a functional equation,

$$F(s, \Pi) = \eta(s, \Pi) F(1 - s, \Pi')$$

where $\eta(s, \Pi)$ is an entire non-vanishing function. Hence $F(s, \Pi')$ has infinitely many poles of the form $1 - \gamma_0 + 2\pi ik / \log q_{v_0}$ with $k \in \mathbb{Z}$. Since $P_0(s, \Pi')$ is regular in the region $\text{Re}(s) \geq 1/2$, these poles have to be poles of $G(s, \Pi') Q(s, \Pi')$. Arguing as in proof of the above claim, we obtain a contradiction. Arguing similarly with the zeros instead of poles, we obtain that $P_0(s, \Pi)$ is an entire non-vanishing function and hence it is a constant. Since the $L$-factors are normalised, we obtain a proof of Theorem 4.

Remark. The method of proof of Theorem 4 is a general method allowing us to establish an equality for two possibly different definitions of $L$-factors at ‘bad’ places. This requires a global functional equation, equality of the $L$-factors at all good places, and regularity in the region $\text{Re}(s) \geq 1/2$ for the ‘bad’ $L$-factors. The method is illustrated in [10] in the context of functoriality, but allowing the use of cyclic base change. It is used by Kable in [12] to prove, for a square integrable representation, that the Rankin-Selberg $L$-factor $L(s, \pi \times \pi)$ factorises as a product of $L_1(s, As(\pi))$ times $L_1(s, As(\pi \otimes \tilde{\omega}))$, where $\tilde{\omega}$ is an extension of $\omega_{E/F}$, the quadratic character corresponding to the extension $E/F$. A proof of strong multiplicity one in the Selberg class using similar arguments is given in [13].

Remark. It has been shown by Henniart [10] using similar global methods, that for any irreducible, admissible representation $\pi$ of $GL_n(E)$, the equality $L(s, As(\pi)) = L_2(s, As(\pi))$. Henniart’s proof uses cyclic base change and the inductivity of $\gamma$-factors to go from square integrable to all irreducible, admissible representations. Since we do not know inductivity of the Rankin-Selberg $\gamma$-factors $\gamma_1(s, As(\pi), \psi)$, we cannot derive a similar statement for the Rankin-Selberg $L$-factors.
Remark. Using cyclic base change as in [10] or [10], it is possible to show that the \( \epsilon \)-factors \( \epsilon_1(s, As(\pi), \psi) \) and \( \epsilon_2(s, As(\pi), \psi_0) \) are equal up to a root of unity, when \( \pi \) is square integrable.

4. Applications

4.1. Analytic characterisation of distinguished representations.

The proofs of Theorems 1 and 2 use the following proposition relating the concept of distinguishedness with the analytical properties of the (Rankin-Selberg) Asai \( L \)-function [1, Corollary 1.5]:

Proposition 11. Let \( \pi \) be a square integrable representation of \( GL_n(E) \). Then \( \pi \) is distinguished with respect to \( GL_n(F) \) if and only if \( L_1(s, As(\pi)) \) has a pole at \( s = 0 \).

4.2. Base change for \( U(3) \).

Let \( W_{E/F} \) be the relative Weil group of \( E/F \) defined as the semidirect product of \( E^* \rtimes \text{Gal}(E/F) \) for the natural action of \( \text{Gal}(E/F) \) on \( E^* \). The Langlands dual group of \( U(n) \) is given by \( L U(n) = GL_n(\mathbb{C}) \rtimes W_{E/F} \), where \( W_{E/F} \) acts via the projection to \( \text{Gal}(E/F) \), and the non-trivial element \( \sigma \in \text{Gal}(E/F) \) acts by \( \sigma(g) = J^{-1}t^g J^{-1} \) on \( GL_n(\mathbb{C}) \). The Langlands dual group of \( R_{E/F}(GL_n) \) is given by

\[
L R_{E/F}(U(n)) = [GL_n(\mathbb{C}) \times GL_n(\mathbb{C})] \rtimes W_{E/F}.
\]

Here again the action of \( W_{E/F} \) is via the projection to \( \text{Gal}(E/F) \), and \( \sigma \) acts by \( (g,h) \mapsto (J^{-1}t^h J^{-1}, J^{-1}t^g J^{-1}) \).

There are two natural mappings from the \( L \)-group of \( U(n) \) to the \( L \)-group of \( GL_n(E) \), called the stable and the unstable base change maps. At the \( L \)-group level, the stable base change map, which corresponds to the restriction of parameters from the Weil group \( W_F \) of \( F \) to the Weil group \( W_E \) of \( E \), is given by the diagonal embedding \( \psi : L U(n) \to L R_{E/F}(U(n)) \). The unstable base change map is defined by first choosing a character \( \bar{\omega} \) of \( E^* \) extending the quadratic character \( \omega_{E/F} \) of \( F^* \) associated to the quadratic extension \( E/F \). At the level of \( L \)-groups, the unstable base change corresponds to the homomorphism \( \psi' : L U(n) \to L R_{E/F}(U(n)) \) given by \( \psi'(g \times w) = (\bar{\omega}(w)g, \bar{\omega}(w)^{-1}g) \times w \) for \( w \in E^*, g \in GL_n(\mathbb{C}) \) and \( \psi'(1, \sigma) = (1, -1) \times \sigma \). The base change lift for \( n = 3 \) has been established by Rogawski [17].

Proof of Theorem 1. By [6, Corollary 4.6], a supercuspidal representation \( \pi \) of \( GL_3(E) \) is a stable base change lift from \( U(3) \) if and only if the Langlands-Shahidi Asai \( L \)-function \( L_2(s, As(\pi)) \) has a pole at \( s = 0 \). By Theorem 1, this amounts to saying that the Rankin-Selberg Asai \( L \)-function \( L_1(s, As(\pi)) \) has a pole at \( s = 0 \). Now the theorem follows by appealing to Proposition 11. □
Remark. If $\pi$ is a square integrable representation such that $\pi^\vee \cong \tilde{\pi}$, and the central character of $\pi$ has trivial restriction to $F^*$, then Kable [12] has proved that $\pi$ is distinguished or distinguished with respect to $\omega_{E/F}$, the quadratic character associated to the extension $E/F$ (see [9, 15] for earlier results in this direction). The given conditions on $\pi$ are expected to be necessary for $\pi$ to be in the image of the base change map from $U(n)$. Thus Kable’s result can be thought of as a weaker version of the conjecture stated in the introduction. On the other hand, it is expected that $U(n)$-distinguished representations of $GL_n(E)$ are base change lifts from $GL_n(F)$. This has been proved in several cases [8, 15].

4.3. Reducibility for $U(n,n)$. We now prove Theorem 2. In [6, 7], Goldberg proves that for a discrete series representation $\pi$ with $\pi^\vee \cong \tilde{\pi}$, $I(\pi)$ is irreducible if and only if $L_2(s, As(\pi))$ has a pole at $s = 0$ (see also [11]). By [7, Theorem 3.4], $R(\pi) \cong (\mathbb{Z}/2\mathbb{Z})^r$, where $r$ is the number of inequivalent representations $\pi_i$ satisfying $\pi_i^\vee \cong \tilde{\pi}_i$ and the Plancherel measure $\mu(s, \pi_i)$ has no zero at $s = 0$. By [18, Corollary 3.6], the latter condition amounts to knowing that the Asai $L$-functions $L_2(s, As(\pi_i))$ are regular at $s = 0$.

Theorem 2 follows from the following claim:

Claim. An irreducible, square integrable representation $\pi$ of $GL_n(E)$ is $\omega_{E/F}$ distinguished if and only if $\pi^\vee \cong \tilde{\pi}$ and $L_2(s, As(\pi))$ is regular at $s = 0$.

Proof of Claim. By [6, Corollary 5.7],

$$L(s, \pi \times \tilde{\pi}) = L_2(s, As(\pi))L_2(s, As(\pi \otimes \tilde{\omega})),$$

where $\tilde{\omega}$ is a character of $E^*$ which restricts to $\omega_{E/F}$ on $F^*$. Now $L(s, \pi \times \tilde{\pi})$ has a pole at $s = 0$ if and only if $\pi^\vee \cong \tilde{\pi}$. Hence $\pi^\vee \cong \tilde{\pi}$ and $L_2(s, As(\pi))$ is regular at $s = 0$ is equivalent to saying that $L_2(s, As(\pi \otimes \tilde{\omega}))$ has a pole at $s = 0$. By Theorem 4 this is the same as saying that $L_1(s, As(\tilde{\pi} \otimes \tilde{\omega}))$ has a pole at $s = 0$. By Proposition 11, the latter condition is equivalent to saying that $\pi$ is $\omega_{E/F}$ distinguished. This proves Theorem 2.

Remark. The $R$-group in this context is also computed in terms of the Langlands parameters by D. Prasad [14, Proposition 2.1]. According to this computation, $R(\pi)$ is a product of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$’s, where $r$ is the number of $\pi_i$’s such that $\pi_i^\vee \cong \tilde{\pi}_i$, and $c(\sigma_i) = -1$, where $\sigma_i$ is the Langlands parameter of $\pi_i$. Here $c(\sigma_i) \in \{\pm 1\}$ denotes the constant introduced by Rogawski [17, Lemma 15.1.1]. Also $c(\sigma_i) = (-1)^{n_i-1}$ if and only if $\sigma_i$ can be extended to a parameter for $U(n_i)$. Together
4.4. Distinguishedness of Steinberg representation of $GL(n)$. 

Now let $G = GL(n)$. For a representation $\pi$ of $GL_n(E)$, let $I(\pi)$ be the parabolically induced representation of $U(n, n)$. If $\pi$ is a discrete series representation such that $\pi^\vee \not\cong \overline{\pi}$, then $I(\pi)$ is known to be irreducible [6]. Suppose $\pi^\vee \cong \pi$. Let $a$ and $b$ be integers such that $ab = n$, such that $\pi$ is the unique square integrable constituent of the representation induced from $\pi_1 \otimes \ldots \otimes \pi_b$ where $\pi_i = \pi_0 \otimes |E|^{b_i - 2i/2}$, and $\pi_0$ a supercuspidal representation of $GL_a(E)$. Then $\pi_0^\vee \cong \overline{\pi}_0$. We have the following result of Goldberg [6, Section 7]:

**Proposition 12.** The representation $I(\pi)$ of $U(n, n)$ is irreducible if and only if $L_2(s, As(\pi_0))$ (resp. $L_2(s, As(\pi_0 \otimes \overline{\omega})$) has a pole at $s = 0$ if $b$ is odd (resp. even). Here $\overline{\omega}$ is a character of $E^*$ that restricts to $\omega_{E/F}$.

Now if $\pi$ is the Steinberg representation of $GL_n(E)$, then $a = 1$, $b = n$, and $\pi_0$ is the trivial character. Thus $I(\pi)$ is irreducible when $n$ is odd, and reducible when $n$ is even. By the corollary to Theorem 2, $\pi$ is $\omega_{E/F}$-distinguished when $n$ is even, and $\pi$ is not $\omega_{E/F}$-distinguished when $n$ is odd.

Since $\pi^\vee \cong \pi$ and $\omega_{E/F} = 1$, we know that $\pi$ is either distinguished or $\omega_{E/F}$-distinguished, but not both (see [12] Theorem 7 and [1] Corollary 1.6)). Therefore it follows that when $n$ is odd (resp. even), $\pi$ is distinguished (resp. $\omega_{E/F}$-distinguished), and that $\pi$ is not distinguished with respect to any other character.

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