ON THE BEST CONSTANT IN FRACTIONAL $p$-POINCARÉ INEQUALITIES ON CYLINDRICAL DOMAINS

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Abstract. We investigate the best constants for the regional fractional $p$-Poincaré inequality and the fractional $p$-Poincaré inequality in cylindrical domains. For the special case $p = 2$, the result was already known due to Chowdhury-Csató-Roy-Sk [Study of fractional Poincaré inequalities on unbounded domains, Discrete Contin. Dyn. Syst., 41(6), 2021]. We addressed the asymptotic behaviour of the first eigenvalue of the nonlocal Dirichlet $p$-Laplacian eigenvalue problem when the domain is becoming unbounded in several directions.

1. Introduction

In the theory of partial differential equations, Poincaré inequality has always played an important role. In recent years, the study of various nonlocal analogues of Poincaré inequality has seen a steep surge. For the particular case $p = 2$, Chowdhury-Csató-Roy-Sk [CCRS21] have found the best constants for fractional Poincaré inequalities in certain unbounded domains. The aim of this article is to generalise [CCRS21] for any $p \in (1, \infty)$.

For any open set $\Omega \subseteq \mathbb{R}^n$, $0 < s < 1$, $1 \leq p < \infty$, we define the fractional Sobolev space

$$W^{s,p}(\Omega) := \{ u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty \},$$

where

$$[u]_{s,p,\Omega} := \left( \frac{C_{n,s,p}}{2} \int \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}, \quad (1.1)$$

is the so called Gagliardo seminorm. The constant $C_{n,s,p}$ [AW19] is given by

$$C_{n,s,p} = \frac{sp}{2^{2s-1}} \frac{\Gamma \left( \frac{n+sp}{2} \right)}{\Gamma(1-s)\Gamma \left( \frac{n+sp}{2} - \frac{n}{2} \right)}. \quad (1.2)$$

We endow this space with the so-called fractional Sobolev norm, given by

$$\|u\|_{s,p,\Omega} := \left( \|u\|^p_{L^p(\Omega)} + [u]_{s,p,\Omega}^p \right)^{\frac{1}{p}}.$$

At this point, we would like to introduce two more Banach spaces, directly related to the fractional Sobolev spaces $W^{s,p}(\Omega)$ defined above, which will be useful in framing the problem dealt in this article. The spaces $W^{s,p}_\Omega(\mathbb{R}^n)$ and $W^{s,p}_0(\Omega)$ denote the closures of $C^\infty_c(\Omega)$ with respect to the norms

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2020 Mathematics Subject Classification. 26D10; 35R09; 46E35; 49J40.
\begin{align}
\left(\|u\|_{L_p(\Omega)}^p + \|\partial_x u\|_{L_p(\Omega)}^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|\cdot\|_{s,p,\Omega} \text{ respectively. The Gagliardo seminorm and the fractional Sobolev norm are also important tools for studying the fractional p-Laplacian operator, defined by }
\end{align}
\begin{align}
(-\Delta_{n,p})^s u(x) := C_{n,s,p} \lim_{\ell \to 0} \int_{\mathbb{R}^n \setminus B_\ell(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+p}} \, dy, \quad x \in \mathbb{R}^n.
\end{align}

Note that while defining the Gagliardo seminorm and the fractional Sobolev norm are also important tools for studying the fractional p-Laplacian operator, in the existing literature, the constant $C_{n,s,p}$ is often ignored. However, we shall take this constant into account as this will be convenient while studying the best constants in fractional Poincaré inequalities (see remark 1.3 below). We refer the reader to [AF03, BS19, DNPV12, FSV15, FP14, LL14, DPFBLR18] for basic results regarding the fractional Sobolev spaces and the fractional p-Laplacian operator.

The regional fractional Poincaré constant $P_{n,s,p}^1(\Omega)$, and fractional Poincaré constant $P_{n,s,p}^2(\Omega)$ are defined as follows:
\begin{align}
P_{n,s,p}^1(\Omega) := \inf_{u \in W_{0,p}^s(\Omega)} \frac{\|u\|_{s,p,\Omega}^p}{\|u\|_{L_p(\Omega)}^p}, \quad \text{and} \quad P_{n,s,p}^2(\Omega) := \inf_{u \in W_{0,p}^s(\mathbb{R}^n)} \frac{\|u\|_{s,p,\mathbb{R}^n}^p}{\|u\|_{L_p(\Omega)}^p}.
\end{align}

**Definition 1.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$. We say that
\begin{itemize}
\item the *regional fractional Poincaré inequality* (RFPI) holds if $P_{n,s,p}^1(\Omega) > 0$.
\item the *fractional Poincaré inequality* (FPI) holds if $P_{n,s,p}^2(\Omega) > 0$.
\end{itemize}

For $\ell > 0$, set $\Omega_\ell := \ell \omega_1 \times \omega$, where $\omega_1$ and $\omega$ are bounded open subsets of $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively, and consider the nonlinear Dirichlet $p$-Laplacian eigenvalue problem on $\Omega_\ell$:
\begin{align}
\begin{cases}
(-\Delta_{n,p})^s u_\ell = P_{n,s,p}^2(\Omega_\ell)|u_\ell|^{p-2}u_\ell & \text{in } \Omega_\ell, \\
u_\ell = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\ell,
\end{cases}
\tag{1.3}
\end{align}
and the corresponding cross section eigenvalue problem of eq. (1.3)
\begin{align}
\begin{cases}
(-\Delta_{n-m,p})^s u = P_{n-m,s,p}^2(\omega)|u|^{p-2}u & \text{in } \omega, \\
u = 0 & \text{in } \mathbb{R}^{n-m} \setminus \omega.
\end{cases}
\tag{1.4}
\end{align}

In the existing literature, the RFPI and FPI, in unbounded domains, are not much explored yet. However, it is well-known that the FPI holds true that is $P_{n,s,p}^2(\Omega) > 0$, when $\Omega$ is a bounded domain [BLP14]. In [BC18], the authors have shown that the FPI holds true for any domain which is bounded in one direction, although the question of best fractional Poincaré constant remained unattended (see also [Yer14]). In the special case $p = 2$, it is known (see [CCRS21]) that the best fractional Poincaré constants $P_{n,s,2}^1(\mathbb{R}^{m-1} \times (-1,1))$ and $P_{n,s,2}^2(\mathbb{R}^m \times \omega)$ are equal to that of the cross sections, that is to $P_{n-s,2}^1((-1,1))$ and $P_{n-m,s,2}^2(\omega)$ respectively, where $\omega$ is a bounded domain in $\mathbb{R}^{n-m}$. The first work in this direction, to the best of our knowledge, was done in [CCRS21, AFM20]. Later, some of the results of [CCRS21] were generalized in the Orlicz fractional Sobolev setup in [BMRS20]. Regarding the RFPI, it is known that when $\Omega$ is a bounded domain with Lipschitz boundary, RFPI does not hold, that is $P_{n,s,p}^1(\Omega) = 0$ if $0 < s \leq \frac{1}{p}$ [AW19, War15]. The RFPI, however, holds true, that is $P_{n,s,p}^1(\Omega) > 0$ if $\frac{1}{p} < s < 1$, when $\Omega$ is any bounded domain in $\mathbb{R}^n$. We refer the reader to [AW19, War15] for other related results regarding regional fractional $p$-Laplacian operator. Regarding the asymptotic behaviour of the first eigenvalue $P_{n,s,p}^2(\Omega_\ell)$ of eq. (1.3), when $\ell \to \infty$, in the linear case, that is for $p = 2$, Chowdury-Roy [CR17] proved that the first eigenvalue $P_{n-s,2}^2(\Omega_\ell)$ of eq. (1.3) converges to the first eigenvalue $P_{n-m,s,2}^2(\omega)$ of eq. (1.4), when $\ell \to \infty$. Different kind of problems were studied regarding the asymptotic behaviour of $P_{n-m,s,2}^2(\Omega_\ell)$ as $\ell \to \infty$; we refer the readers
to [CRS13, ERS21, CR17, Yer14] and the references therein for more relevant information in this direction.

Our first result depicts, for a cylindrical domain, that the best constant for FPI of the domain and that of the cross-section are the same. Indeed, we have the following result:

**Theorem 1.2.** Let \( 0 < s < 1, 1 < p < \infty \) and \( \Omega_\infty = \mathbb{R}^m \times \omega \) in \( \mathbb{R}^n \) with \( 1 \leq m < n \), where \( \omega \) is a bounded open subset of \( \mathbb{R}^{n-m} \). Then we have

\[
P^2_{n,s,p}(\Omega_\infty) = P^2_{n-m,s,p}(\omega).
\] (1.5)

Furthermore, the best fractional Poincaré constant \( P^2_{n,s,p}(\Omega_\infty) \), is never achieved.

**Remark 1.3.** Recall that we took into account, the constant \( \frac{C_{n,s,p}}{2} \) (defined in eq. (1.2)) while defining the seminorm in eq. (1.1). If one ignores this constant in the definition of the seminorm, then one would get an additional multiplicative constant in eq. (1.5) in place of plain equality in theorem 1.2. The strategy for the proof of theorem 1.2, done in section 3, is somewhat analogous to the local case. It is as follows: the constant \( P^2_{n,s,p}(\Omega_\infty) \) is bounded above by the constant \( P^2_{n,s,p}(\omega) \) (see (2) of proposition 3.2). For the special case \( p = 2 \), the regularity of the first eigenfunction of eq. (1.4) is extensively used for the bounded below case. However, for general \( p > 1 \), we do not have such regularity theory of the first eigenfunction of eq. (1.4). We use simple approximation argument and discrete Picone inequality (see, lemma 2.5) to prove this part. The last part of the theorem is proven via contradiction, where the geometry of the domain has played an important role. Note that the use of discrete Picone inequality forces us to exclude the case \( p = 1 \) from the statement of the theorem 1.2.

Next, we deal with the case of RFPI. As above, we show that the best constant for RFPI for a strip is equal to that of its cross-section.

**Theorem 1.4.** Let \( 0 < s < 1, 1 \leq p < \infty \) and \( \Omega_\infty = (-1,1) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n \), then we have the following:

1. \( P^1_{n,s,p}(\Omega_\infty) = P^1_{1,s,p}((-1,1)) = 0 \), if \( 0 < s \leq 1 \).
2. \( P^1_{n,s,p}(\Omega_\infty) = P^1_{1,s,p}((-1,1)). \) Consequently, \( P^1_{n,s,p}(\Omega_\infty) > 0 \), if \( \frac{1}{p} < s < 1 \).

The proof of theorem 1.4 goes along the same line as it is done in [CCRS21] for \( p = 2 \) but with necessary modifications. However, the method of the proof differs significantly from that of theorem 1.2 and hence, in this case, we must stick to the case \( m = 1, \omega = (-1,1), \).

Finally, we come to our last main result, which shows the asymptotic behaviour of the first eigenvalue of eq. (1.3).

**Theorem 1.5.** Let \( 0 < s < 1, 1 < p < \infty, \ell > 0 \) and \( \Omega_\ell = \ell \omega_1 \times \omega \) in \( \mathbb{R}^n \) with \( 1 \leq m < n \), where \( \omega_1, \omega \) are bounded open subsets of \( \mathbb{R}^m \) and \( \mathbb{R}^{n-m} \) respectively. We then have

\[
P^2_{n-m,s,p}(\omega) \leq P^2_{n,s,p}(\Omega_\ell) \leq P^2_{n-m,s,p}(\omega) + \frac{C_1}{\ell^s} + \frac{C_2}{\ell^{sp}},
\]

where \( C_1, C_2 > 0 \) are constants independent of \( \ell \). Furthermore, if \( \Omega_\infty = \bigcup_{t>0} \Omega_\ell \)

\[
\lim_{\ell \to \infty} P^2_{n,s,p}(\Omega_\ell) = P^2_{n-m,s,p}(\omega) = P^2_{n,s,p}(\Omega_\infty).
\]

This article is organized in the following way: In section 2 we recall some results, already known in the literature. In section 3 we give proofs of theorems 1.2, 1.4 and 1.5.
2. SOME KNOWN RESULTS AND CONVENTIONS

Here we briefly discuss the notations that we shall use throughout the paper.

- $s$ will always be understood to be in $(0, 1)$.
- For any positive integer $n$ and a measurable set $\Omega \subset \mathbb{R}^n$ we write $\mathcal{L}^n(\Omega)$ to denote the Lebesgue measure of $\Omega$, or shortly $|\Omega|$ if $n$ is understood from the context.
- $B_R(x)$ denotes a ball of radius $R$ centered at $x$. We shall also write $B_R$ for $B_R(0)$.
- $\mathbb{S}^{m-1}$ is the unit sphere in the Euclidean space $\mathbb{R}^m$.
- $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure, so that

$$
\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \quad \text{where } \Gamma \text{ is the standard gamma function. (2.1)}
$$

- The beta function, for $x, y > 0$, is defined by

$$
B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt = B(x, y) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. (2.2)
$$

We now state some definitions and results, already known in literature, which we shall be using in the subsequent sections in this article. But before defining these, we would like to recall a result which follows from [DPFBLR18, Lemma 2.7].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $p > 1$. Then

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\psi(x)}{|x-y|^{n+sp}} dxdy = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\psi(x) - \psi(y))}{|x-y|^{n+sp}} dxdy,
$$

whenever the integral in the left hand side is finite or $[u]_{s,p,\mathbb{R}^n}, \|\psi\|_{L^p(\Omega)} < \infty$. Here the integral in the LHS is to be understood in the principle value (P.V.) sense.

**Proof.** Let $u, \psi \in W^{s,p}_\Omega(\mathbb{R}^n)$. Set

$$
I := \text{P.V.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\psi(x)}{|x-y|^{n+sp}} dxdy
$$

$$
:= \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\psi(x)}{|x-y|^{n+sp}} dxdy = \lim_{\epsilon \to 0} I_\epsilon.
$$

If, $I_\epsilon$ is finite, we can write, $I = \frac{1}{2} \lim_{\epsilon \to 0} (I_\epsilon + I_\epsilon)$, then we make a change of variable, by interchanging $x$ and $y$, in the second $I_\epsilon$ in the RHS and then we add the two terms in the RHS to conclude the equality.

It remains to show that $[u]_{s,p,\mathbb{R}^n}, \|\psi\|_{L^p(\Omega)} < \infty$ implies finiteness of $I_\epsilon$. The following calculation is done in [BC18, lemma 2.3]. However, we include it here for the sake of completeness. For a fixed $\epsilon > 0$, we have

$$
\left| \int_{|x-y| \geq \epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\psi(x)}{|x-y|^{n+sp}} dxdy \right|
$$

$$
\leq \int_{|x-y| \geq \epsilon} \frac{|u(x) - u(y)|^{p-1}|\psi(x)|}{|x-y|^{n+sp}} dxdy
$$
Using this lemma, we define

**Definition 2.2.** Let \( \omega \subset \mathbb{R}^{n-m} \) be a bounded open set. A function \( u \in W^{s,P}_\omega(\mathbb{R}^{n-m}) \) is said to be a weak solution of eq. (1.4) if \( u \) satisfies

\[
\frac{C_{n-m,s,p}}{2} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+m+sp}} dy \, dx = C_{n-m,s,p} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+m+sp}} dy \, dx
\]

\[
= \frac{P^2_{n-m,s,p}(\omega)}{2} \int_\omega |u(x)|^{p-2} u(x) \psi(x) dx, \quad \text{for all } \psi \in W^{s,P}_\omega(\mathbb{R}^{n-m}).
\]

Any such \( u \), not identically zero, is also called an eigenfunction of eq. (1.4), corresponding to the eigenvalue \( P^2_{n-m,s,p}(\omega) \).

**Lemma 2.3 (see [BP16]).** The constant \( P^2_{n,s,p}(\omega) \) is the first eigenvalue of the problem eq. (1.4), and the corresponding eigenfunction is strictly positive in the domain. Moreover, the corresponding eigenspace is of dimension one.

**Lemma 2.4 (see [LS10, Lemma 2.4]).** Let \( p > 0, 0 < s < 1 \) and \( \Omega \subset \mathbb{R}^n \) be a measurable set. Then for any \( u \in C^{\infty}_0(\Omega) \)

\[
2 \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy \, dx
\]

\[
= \int_{\mathbb{S}^{n-1}} dH^{n-1}(w) \int_{\{x : x \cdot w = 0\}} \int_{\{x : x + \ell w \in \Omega\}} \frac{|u(x + \ell w) - u(x + tw)|^p}{|\ell - t|^{1+sp}} d\ell dt.
\]

**Lemma 2.5 (Discrete Picone inequality, [BF14]).** Let \( p \in (1, \infty) \) and let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be two measurable functions with \( f \geq 0, g > 0 \), then \( L(f, g) \geq 0 \) in \( \mathbb{R}^n \times \mathbb{R}^n \), where

\[
L(f, g)(x, y) = |f(x) - f(y)|^p - |g(x) - g(y)|^{p-2} (g(x) - g(y)) \left( \frac{f(x)^p}{g(x)^{p-1}} - \frac{f(y)^p}{g(y)^{p-1}} \right).
\]

The equality holds if and only if \( f = \alpha g \) a.e. in \( \mathbb{R}^n \) for some constant \( \alpha \).

The following result is well known in literature. It follows directly from the definition of the Poincaré constant and of the Gagliardo seminorm.

**Proposition 2.6.** Let \( 0 < s < 1 \) and \( p \in [1, \infty) \) we have

1. **(Domain monotonicity)** If \( \Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n \), then \( P^2_{n,s,p}(\Omega_2) \leq P^2_{n,s,p}(\Omega_1) \).
(2) **(Dilation:)** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in W^{s,p}(\Omega)$, for $t > 0$. We define $v_t(x) = u(tx) \in W^{s,p}(\Omega)$. Then $[u]_{s,p,\Omega}^p = t^{-nsp}[v_t]_{s,p,\Omega}^p$, and furthermore

$$P_{n,s,p}^1(\Omega) = t^{-nsp}P_{n,s,p}^1(\Omega), \quad \text{and} \quad P_{n,s,p}^2(\Omega) = t^{-nsp}P_{n,s,p}^2(\Omega).$$

**Remark 2.7.** To the best of our knowledge, the domain monotonicity property for $P_{n,s,p}^1$ is not known in literature.

The proof of the following well-known result, in the case $sp < 1$ can be found in [Tri83, Theorem 3.4.3] for bounded $C^\infty$-domains. For bounded Lipschitz domains, in the case $sp \leq 1$, this can be found in [Dyducko04, Section 2], [AW19, Theorem 2.1], [War15, Example 4.11]. For various related results, we refer the reader to [DK21].

**Lemma 2.8.** Let $p \in [1, \infty)$, and $\Omega$ be an open bounded set in $\mathbb{R}^n$ with Lipschitz boundary. Then $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ if $0 < s < \frac{1}{p}$. In particular, in this case, we have $P_{n,s,p}^1(\Omega) = 0$.

**Lemma 2.9.** Let $p \in [1, \infty)$, and $\Omega \subset \mathbb{R}^n$ be an open set. Then $C^1_c(\Omega) \subset W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$.

**Proof.** To show the first inclusion, let us take an arbitrary $v \in C^1_c(\Omega)$. Then we apply [BM19, Lemma 8] to get a bounded open set $\Omega_1$ with smooth boundary such that supp$(v) \subset \Omega_1 \subset \Omega$. Clearly $v \in W^{1,p}(\Omega_1)$. Then we can say, from the well known trace theorem for Sobolev spaces, that $v \in W_0^{1,p}(\Omega_1) \subset W_0^{1,p}(\Omega)$.

The last inclusion follows from [DNPV12, Proposition 2.2], which implies that for any $u \in C^\infty_c(\Omega)$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq C(n, s, p) \left( \int_{\Omega} |u(x)|^p \, dx + \int_{\Omega} |\nabla u(x)|^p \, dx \right)$$

as for any $v \in W_0^{1,p}(\Omega)$, there exists a sequence of functions $v_n \in C^\infty_c(\Omega)$, converging to $v$ in $W_0^{1,p}(\Omega)$. The above inequality then suggests that the same sequence will converge to $v$ in $W^{1,p}(\Omega)$ as well. \(\square\)

**Hyper-spherical Coordinates.** Before moving further, let us recall the hyper-spherical coordinates and derive an equality, which will be used in the forthcoming section.

Let

$$A_{n-1}(\sigma) = (0, \pi)^{n-2} \times (0, 2\pi) \subset \mathbb{R}^{n-1}.$$  

The hyper spherical coordinates $H = (H_1, \ldots, H_n) : A_{n-1} \to \mathbb{S}^{n-1}$ are defined as follows: for $k = 1, \ldots, n$ and $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$

$$H_k(\sigma) = \cos \sigma_k \prod_{l=0}^{k-1} \sin \sigma_l \quad \text{with the convention} \quad \sigma_0 = \frac{\pi}{2} \quad \text{and} \quad \sigma_n = 0.$$

An elementary calculation shows $d_k(\sigma) := \left\langle \frac{\partial H_i}{\partial \sigma_k}, \frac{\partial H_i}{\partial \sigma_k} \right\rangle = \prod_{l=0}^{k-1} \sin^2 \sigma_l > 0$. One can easily verify that the metric tensor, in these coordinates, is diagonal, that is $g_{ij}(\sigma) = \left\langle \frac{\partial H_i}{\partial \sigma_l}, \frac{\partial H_j}{\partial \sigma_l} \right\rangle = \delta_{ij} d_k(\sigma)$, (Here $\delta_{ij}$ denotes the usual ‘Kronecker delta’) and hence the surface element $g_{n-1}$ is given by

$$g_{n-1}(\sigma) = \sqrt{\det g_{ij}(\sigma)} = \sqrt{\prod_{k=1}^{n-1} d_k(\sigma) } = \prod_{k=1}^{n-1} \prod_{l=0}^{k-1} \sin \sigma_l = \prod_{k=1}^{n-2} (\sin \sigma_k)^{n-k-1}. $$
3. Proof of Main Results

**Lemma 3.1.** Let $0 < s < 1$, $1 \leq p < \infty$, and for any $m, n \in \mathbb{N}$ with $1 \leq m < n$. Let $C_{n,s,p}$ be the constant as in eq. (1.2). Then we have the following:

(i) $C_{n,s,p} \Omega_{m,n,p} = C_{n-m,s,p}$, where $\Theta_{m,n,p} = \mathcal{H}^{m-1}(\mathbb{S}^{m-1}) \int_0^{\infty} \frac{t^{m-1}}{(1+t^2)^{n/2}} dt$

(ii) If $a > 0$ and $z \in \mathbb{R}^m$ then

\[
\int_{\mathbb{R}^m} \frac{dx}{1 + \left| x - z \right|^2} = a^n \Theta_{m,n,p}.
\]

**Proof.** (i) Applying the change of variable $t = \tan \theta$ in the expression of $\Theta_{m,n,p}$, followed by eqs. (2.1) and (2.2), we obtain

\[
\Theta_{m,n,p} = \mathcal{H}^{m-1}(\mathbb{S}^{m-1}) \int_0^\pi (\sin \theta)^{m-1}(\cos \theta)^{n-m+s-1} d\theta
\]

\[
= \frac{1}{2} B\left(\frac{m}{2}, \frac{n-m+s}{2}\right) 2^{\frac{n}{2}} \pi^{\frac{n}{2}} = \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n-m+s}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}
\]

From eq. (1.2) we get the desired result.

(ii) Taking the change of variable $y = \frac{x - z}{a}$, the identity follows immediately. \(\square\)

**Proposition 3.2.** Let $m, n \in \mathbb{N}$ with $m < n$, $0 < s < 1$, $1 \leq p < \infty$ and $\Omega_\infty = \mathbb{R}^m \times \omega$, where $\omega \subset \mathbb{R}^{n-m}$ is a bounded open set. Then we have

1. $P_{n,s,p}(\Omega_\infty) \leq P_{n-m,s,p}(\omega)$.

2. $P_{n,s,p}(\Omega_\infty) \leq P_{n-m,s,p}(\omega)$.

**Proof.** First, we prove (1). Note that if we can show, for any $W \in C_c^\infty(\omega)$ and $\epsilon > 0$, that there exists $u \in C_c^\infty(\Omega_\infty)$ such that

\[
\frac{\|u\|_{L^p(\Omega_\infty)}^p}{\|u\|_{L^p(\omega)}^p} \leq \frac{\|W\|_{L^p(\omega)}^p}{\|W\|_{L^p(\omega)}^p} + \epsilon,
\]

then we are done.

Therefore, we start by choosing, arbitrarily, $W \in C_c^\infty(\omega)$ and $v \in C_c^\infty(\mathbb{R}^m)$ which satisfies $\int_{\mathbb{R}^m} |v|^p = 1$. We define, for $\ell > 0$, $v_\ell(x) = \ell^{-\frac{m}{p}} v\left(\frac{x}{\ell}\right)$. Clearly $v_\ell \in C_c^\infty(\mathbb{R}^m)$ and

\[
\int_{\mathbb{R}^m} |v_\ell|^p = 1 \quad \text{for all } \ell > 0.
\]

We denote the point $x \in \mathbb{R}^n$ by $x = (X_1, X_2)$, where $X_1 \in \mathbb{R}^m$ and $X_2 \in \mathbb{R}^{n-m}$. Now we define

\[
u_\ell(X_1, X_2) = v_\ell(X_1) W(X_2).
\]

Note that we can always assume $\|W\|_{L^p(\omega)} = 1$ by normalizing $W$ appropriately. Using eq. (3.1) we get

\[
\|u_\ell\|_{L^p(\Omega_\infty)}^p = \int_{\mathbb{R}^m} \int_\omega |v_\ell(X_1)|^p |W(X_2)|^p dX_2 dX_1 = \|W\|_{L^p(\omega)}^p = 1 \quad \text{for all } \ell > 0.
\]

Therefore it only remains to show that, for sufficiently large $\ell$,

\[
[u_\ell]_{L^p, \Omega_\infty} \leq \|W\|_{L^p, \omega} + \epsilon(\ell),
\]
where \( \lim_{\ell \to \infty} \epsilon(\ell) = 0 \); again, this will follow immediately, if we can show, after redefining \( \epsilon(\ell) \) appropriately,

\[
[\epsilon]_{s,p,\Omega} \leq [W]_{s,p,\omega} + \epsilon(\ell).
\]

Using the triangle inequality of \( L^p(\Omega_\infty \times \Omega_\infty) \)-norm, we obtain

\[
[u]_{s,p,\Omega_\infty} = \left( \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}
\]

\[
= \left( \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|v_\ell(X_1)W(X_2) - v_\ell(Y_1)W(Y_2)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}
\]

\[
= \left( \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|v_\ell(Y_1)(W(X_2) - W(Y_2)) + W(X_2)(v_\ell(X_1) - v_\ell(Y_1))|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}
\]

\[
\leq I_1 + I_2,
\]

where

\[
I_1 = \left( \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|v_\ell(Y_1)|^p|W(X_2) - W(Y_2)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}
\]

and

\[
I_2 = \left( \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|W(X_2)|^p|v_\ell(X_1) - v_\ell(Y_1)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}.
\]

We shall now estimate the integrals \( I_1 \) and \( I_2 \).

**Estimate for \( I_1 \):** For \( X_2 \neq Y_2 \) and by (ii) of lemma 3.1, we get

\[
\int_{\mathbb{R}^m} \frac{dX_1}{(1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2})^{\frac{n+sp}{2}}} = |X_2 - Y_2|^m \Theta_{m,n,p} \quad \text{for any } Y_1 \in \mathbb{R}^m.
\]

Applying this identity to the definition of \( I_1 \), together with (i) of lemma 3.1 and eq. (3.1), we get

\[
I_1^p = \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|v_\ell(Y_1)(W(X_2) - W(Y_2))|^p}{|X_2 - Y_2|^{n+sp}} \left( 1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2} \right)^{\frac{n+sp}{2}} \, dx \, dy
\]

\[
= \frac{C_{n,s,p}}{2} \int_\omega \int_\omega \frac{|W(X_2) - W(Y_2)|^p}{|X_2 - Y_2|^{n+sp}} \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \frac{dX_1}{(1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2})^{\frac{n+sp}{2}}} \right) |v_\ell(Y_1)|^p \, dy_1 \, dX_2 \, dy_2
\]

\[
= \frac{C_{n,s,p}}{2} \Theta_{m,n,p} \int_\omega \int_\omega |W(X_2) - W(Y_2)|^p |X_2 - Y_2|^{-n-m+sp} \, dX_2 \, dy_2 \int_{\mathbb{R}^m} |v_\ell(Y_1)|^p \, dy_1
\]

\[
= \frac{C_{n-m,s,p}}{2} \int_\omega \int_\omega |W(X_2) - W(Y_2)|^p |X_2 - Y_2|^{-n-m+sp} \, dX_2 \, dy_2 = [W]_{s,p,\omega}^p.
\]

**Estimate for \( I_2 \):** We can write \( I_2 \) as

\[
I_2^p = \frac{C_{n,s,p}}{2} \int_{\Omega_\infty} \int_{\Omega_\infty} \frac{|(v_\ell(X_1) - v_\ell(Y_1))W(X_2)|^p}{|X_1 - Y_1|^{n+sp}} \left( 1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2} \right)^{\frac{n+sp}{2}} \, dx \, dy.
\]
Using lemma 3.1 (ii) we get
\[ \int_{\omega} \frac{dY_2}{\left(1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2}\right)^{\frac{n+m}{2}}} \leq \int_{\mathbb{R}^{n-m}} \frac{dY_2}{\left(1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2}\right)^{\frac{n+m}{2}}} = |X_1 - Y_1|^{n-m}\Theta_{n-m,n,p}. \]
Applying this to the definition of \( I_2 \) and using the fact that \( \|W\|_{L^p(\omega)} = 1 \), we obtain
\[ I_2^p \leq \frac{C_{n,s,p}}{2} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n}} \frac{|v_t(X_1) - v_t(Y_1)|^p}{|X_1 - Y_1|^{n+m+p}} dX_1dY_1 = \frac{[v_t]^p}{\ell^s}. \]
By a change of variables in the definition of \( v_t \), we get
\[ [v_t]_{s,p,\mathbb{R}^m} = \frac{\ell^{m-sp}}{\ell^m} [v]_{s,p,\mathbb{R}^m} = \frac{1}{\ell^s} [v]_{s,p,\mathbb{R}^m} \Rightarrow I_2 \leq \frac{[v]_{s,p,\mathbb{R}^m}}{\ell^s}. \]
Now plugging the above finer estimates of \( I_1 \) and \( I_2 \) into eq. (3.2), we obtain
\[ [u_t]_{s,p,\Omega_{\infty}} \leq [W]_{s,p,\omega} + \frac{[v]_{s,p,\mathbb{R}^m}}{\ell^s}. \]
This finishes the proof of (1). Proof of (2) is similar and hence omitted. □

In the next result we shall use the concept of weak formulation (see definition 2.2).

**Lemma 3.3.** Let \( x = (X_1, X_2) \in \Omega_{\infty} \) and define \( u^*(x) := W(X_2) \), where \( W \) is a weak solution of eq. (1.4). Then
\[ C_{n,s,p} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n}} \frac{|u^*(x) - u^*(y)|^p}{|x - y|^{n+m+p}} dX_2 dY_2 = P_{n-m,s,p}^2(\omega) \int_{\omega} |W(X_2)|^{p-2} W(X_2) \psi(X_2) dX_2, \text{ for all } \psi \in W_{\omega,s,p}^s(\mathbb{R}^{n-m}). \]

**Proof.** Let \( \psi \in W_{\omega,s,p}^s(\mathbb{R}^{n-m}) \). In the following calculation, we use the fact that \( W \) is a weak solution of eq. (1.4); also, we use (i) of lemma 3.1 in the second equality, and (ii) of the same lemma, with choices \( a = |X_2 - Y_2| \) and \( z = X_1 \), in the third equality. We, then, have:
\[ P_{n-m,s,p}^2(\omega) \int_{\omega} |W(X_2)|^{p-2} W(X_2) \psi(X_2) dX_2 \]
\[ = C_{n-m,s,p} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{|W(X_2) - W(Y_2)|^{p-2}(W(X_2) - W(Y_2))\psi(X_2)}{|X_2 - Y_2|^{n+m+p}} dY_2 dX_2 \]
\[ = C_{n,s,p} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{|W(X_2) - W(Y_2)|^{p-2}(W(X_2) - W(Y_2))\psi(X_2)}{|X_2 - Y_2|^{n+m+p}} \int_{\mathbb{R}^{n}} \frac{dY_1}{\left(1 + \frac{|X_2 - Y_2|^2}{|X_2 - Y_2|^2}\right)^{\frac{n+m}{2}}} dY_2 dX_2 \]
\[ = C_{n,s,p} \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^{n-m}} \frac{|W(X_2) - W(Y_2)|^{p-2}(W(X_2) - W(Y_2))\psi(X_2)}{|X_2 - Y_2|^2 + |X_2 - Y_2|^2} dY_2 dY_1 dX_2. \]
\begin{align*}
= C_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))\psi(X_2)}{|x - y|^{n+sp}} dydX_2.
\end{align*}

Since \( \psi \) is arbitrary, the lemma follows. \( \square \)

**Proof of theorem 1.2.** Suppose \( W \) is the first eigenfunction corresponding to the first eigenvalue, \( P_{n-m,s,p}^2(\omega) \) of the problem eq. (1.4), which is strictly positive in \( \omega \) (by lemma 2.3). Fix any \( v \in C_c^\infty(\Omega_\infty) \) arbitrarily. Let \( \{\rho_k\} \) be the standard mollifiers in \( \mathbb{R}^{n-m} \). Now Define \( \phi(X_1, X_2) := \frac{|v|^p}{W(X_2)^p} \), and \( \phi_k(X_1, X_2) := \frac{|v|^p}{W_k(X_2)^p} \), where \( W_k := W * \rho_k \). At this point, we fix any \( X_1 \in \mathbb{R}^m \) so that \( v(x_1, \cdot) \in C_c^\infty(\omega) \), and \( \phi_k(X_1, \cdot) \in C_c^1(\omega) \subset W_0^{q,p}(\mathbb{R}^{n-m}) \) (by lemma 2.9). Then, as \( W > 0 \) in \( \omega \), \( W_k \) are strictly positive and smooth in \( \omega \). Note that there exists \( \alpha > 0 \) such that \( W, W_k > \alpha \) in \( \text{Supp } v(x_1, \cdot) \) for any \( k \). Therefore, for any \( X_2 \in \text{Supp } v(x_1, \cdot) \),

\[
|\phi_k(X_1, X_2) - \phi_k(X_1, Y_2)| = \left| \frac{|v(X_1, X_2)|^p}{W_k(X_2)^{p-1}} - \frac{|v(X_1, Y_2)|^p}{W_k(Y_2)^{p-1}} \right| = \left| \frac{|v(X_1, X_2)|^p - |v(X_1, Y_2)|^p}{W_k^{p-1}(X_2)} + \frac{W_k^{p-2}(X_2) - W_k^{p-1}(Y_2)}{W_k^{p-1}(X_2)W_k^{p-1}(Y_2)} \right| \leq \alpha^{-1} \left( |v(X_1, X_2)|^p - |v(X_1, Y_2)|^p \right) + ||v||_\infty^p \frac{|W_k^{p-1}(Y_2) - W_k^{p-1}(X_2)|}{W_k^{p-1}(X_2)W_k^{p-1}(Y_2)} \leq \alpha^{-1} \left( |v(X_1, X_2)|^p - |v(X_1, Y_2)|^p \right) + (p - 1) ||v||_\infty^p \left( \frac{W_k^{p-2}(Y_2) + W_k^{p-2}(X_2)}{W_k^{p-1}(X_2)W_k^{p-1}(Y_2)} \right) \leq C(p, \alpha, ||v||_\infty) (|v(X_1, X_2)|^p - |v(X_1, Y_2)|^p + |W_k(X_2) - W_k(Y_2)|).
\]

This shows that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi_k(X_1, X_2) - \phi_k(X_1, Y_2)|^p}{|X_2 - Y_2|^{n-m+sp}} dX_2 dY_2 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} C(p, W, v) \left| \frac{W_k(X_2) - W_k(Y_2)}{|X_2 - Y_2|^{n-m+sp}} \right|^p dX_2 dY_2 + C(p, W, v)||v||_{s,p,\mathbb{R}^{n-m}}^p.
\]

Now one can easily check that \( W_k \) converges to \( W \) in \( W_0^{q,p}(\mathbb{R}^{n-m}) \) (see [FSV15, Lemma 11]) and also pointwise. We can apply generalised dominated convergence theorem (see [RF10, Theorem 19, Section 4.4]) to conclude that \( \phi \in W_0^{q,p}(\mathbb{R}^{n-m}) \). Define \( u(X_1, X_2) := W(X_2) \) and apply discrete Picone inequality lemma 2.5 on \( u \) and \( |v| \) to obtain

\[
|u(X_1, X_2) - u(Y_1, Y_2)|^{p-2}(u(X_1, X_2) - u(Y_1, Y_2)) = |v(X_1, X_2) - v(Y_1, Y_2)|^p \leq |v(X_1, X_2) - v(Y_1, Y_2)|^p.
\]

This gives

\[
\frac{C_{n,s,p}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|W(X_2) - W(Y_2)|^{p-2}(W(X_2) - W(Y_2))(\phi(x) - \phi(y))}{|x - y|^{n+sp}} dxdy \leq \frac{C_{n,s,p}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dxdy. \quad (3.3)
\]

Now, observe the following calculation, where we have used Fubini’s theorem, lemma 3.3 and positivity of \( W \):
Thus, we get
\[ u \equiv v \text{ in } \Omega. \]

Define \( P \) as this is true for any \( \alpha \).\( ^{[81x224]} \)

In other words, \( u \) is an eigenfunction corresponding to the first eigenvalue \( P_{n,s,p}^{2} (\Omega_{\infty}) \). Let \( h \in \mathbb{R}^{m} \), define \( v_{h}(x) = u(X_{1} + h, X_{2}). \) By change of variable, we also have \( P_{n,s,p}^{2}(\Omega_{\infty}) \leq P_{n,s,p}^{2}(\Omega_{\infty}) \), by density of \( C_{c}^{\infty}(\Omega_{\infty}) \) in \( W_{l,\infty}^{s,p}(\Omega) \).

Now for the last part of the theorem, suppose that there exist a function \( u \) such that \( P_{n,s,p}^{2}(\Omega_{\infty}) = \frac{\|u\|_{L^{p}(\Omega_{\infty})}}{\int_{\Omega_{\infty}} |u(x)|^{p} dx}. \) Then \( u \) is a weak solution of the problem

\[
\begin{align*}
(-\Delta_{n,p})^{\alpha} u &= P_{n,s,p}^{2}(\Omega_{\infty})|u|^{p-2} u \text{ in } \Omega_{\infty}, \\
u &= 0 \text{ in } \mathbb{R}^{n} \setminus \Omega_{\infty}.
\end{align*}

\]

(3.4)

In other words, \( u \) is an eigenfunction corresponding to the first eigenvalue \( P_{n,s,p}^{2}(\Omega_{\infty}) \). Let \( h \in \mathbb{R}^{m} \), define \( v_{h}(x) = u(X_{1} + h, X_{2}). \) By change of variable, we also have \( v_{h} \) is an eigenfunction of eq. (3.4) associated to the eigenvalue \( P_{n,s,p}^{2}(\Omega_{\infty}) \) for any \( h \). Since \( P_{n,s,p}^{2}(\Omega_{\infty}) \) is simple (see lemma 2.3), \( u = \alpha_{h} v_{h} \) for some constant \( \alpha_{h} \). Therefore, by a change of variable, we have

\[
\int_{\Omega_{\infty}} |u|^{p} dx = \int_{\mathbb{R}^{m}} \int_{\omega} |h(X_{1}, X_{2})|^{p} dX_{2} dx = |\alpha_{h}|^{p} \int_{\mathbb{R}^{m}} \int_{\omega} |u(X_{1}, X_{2})|^{p} dX_{2} dx = |\alpha_{h}|^{p} \int_{\Omega_{\infty}} |u|^{p} dx
\]

Thus, we get \( |\alpha_{h}|^{p} = 1 \) and this imply that \( \alpha_{h} = 1 \), because \( u \) has constant sign in \( \Omega_{\infty} \). Therefore, we get \( u(X_{1}, X_{2}) = v_{h}(X_{1}, X_{2}) \) for any \( h \in \mathbb{R}^{m} \). Hence \( u \) is independent of \( X_{1} \) variable. In particular, \( \|u\|_{L^{p}(\Omega_{\infty})} \) is infinite, which gives a contradiction. This completes the proof of theorem 1.2. \( \Box \)

**Corollary 3.4.** Let \( \{\Omega_{\ell}\} \) be an increasing sequence of bounded open sets in \( \mathbb{R}^{n} \) that is \( \Omega_{\ell} \subseteq \Omega_{\ell_{1}} \) for any \( 0 < \ell < \ell_{1} \). If \( \Omega = \bigcup_{\ell>0} \Omega_{\ell} \). Then we have

\[
P_{n,s,p}^{2}(\Omega) = \inf_{\ell>0} P_{n,s,p}^{2}(\Omega_{\ell}).
\]
Proof. By domain monotonicity property (2) of proposition 2.6, we have \( \inf_{\ell > 0} P_{n,s,p}^2(\Omega_\ell) \geq P_{n,s,p}^2(\Omega) \). So, to establish the result, we only need to show \( \inf_{\ell > 0} P_{n,s,p}^2(\Omega_\ell) \leq P_{n,s,p}^2(\Omega) \). Now, for any \( v \in C_c^\infty(\Omega) \), there exists an \( \ell > 0 \), big enough, such that \( \text{supp}(v) \subset \Omega_\ell \). Then we have \( \|v\|_{L^p(\Omega_\ell)} = \|v\|_{L^p(\Omega)} \) and \( [v|_{\Omega_\ell}]_{s,p,\mathbb{R}^n} = [v]_{s,p,\mathbb{R}^n}. \) So \( \inf_{\ell > 0} P_{n,s,p}^2(\Omega_\ell) \leq P_{n,s,p}^2(\Omega) \leq \left[ \frac{\|v\|_{s,p,\mathbb{R}^n}}{\|v\|_{L^p(\Omega)}} \right]^p \). Since this holds for any \( v \in C_c^\infty(\Omega) \), we conclude \( \inf_{\ell > 0} P_{n,s,p}^2(\Omega_\ell) \leq P_{n,s,p}^2(\Omega) \).

□

Lemma 3.5. Let \( \frac{1}{p} < s < 1, \Omega \subset \mathbb{R}^n \) be a measurable set, and \( f : \mathbb{S}^{n-1} \to [0, \infty) \) be an \( \mathcal{H}^{n-1} \)-measurable function satisfying

\[
P_{1,s,p}^1(\{t \in \mathbb{R} : x + tw \in \Omega\}) \geq f(w)
\]

for a.e. \( w \in \mathbb{S}^{n-1} \) and a.e. \( x \in \{y \in \mathbb{R}^n : y \cdot w = 0\} \). Then

\[
P_{n,s,p}^1(\Omega) \geq \frac{C_{n,s,p}}{2C_{1,s,p}} \int_{\mathbb{S}^{n-1}} f(w) d\mathcal{H}^{n-1}(w).
\]

Proof. Let us choose \( w \in \mathbb{S}^{n-1} \) and \( x \in L_w := \{y \in \mathbb{R}^n : y \cdot w = 0\} \) arbitrarily. Denote \( \Omega_{w,x} := \{t \in \mathbb{R} : x + tw \in \Omega\} \). Then from the hypotheses, we have

\[
\frac{C_{1,s,p}}{2} \int_{\Omega_{w,x}} \int_{\{t : x + tw \in \Omega\}} \frac{|u(x + tw) - u(x + tw)|^p}{|t - t'|^{1+sp}} dt \, dt \leq \int_{\Omega_{w,x}} |u(x + tw)|^p \, dt \geq f(w) \int_{\Omega_{w,x}} |u(x + tw)|^p \, dt.
\]

We apply Fubini’s theorem to get, for any \( w \in \mathbb{S}^{n-1} \),

\[
\int_{L_w} d\mathcal{H}^{n-1}(x) \int_{\Omega_{w,x}} |u(x + tw)|^p \, dt = \int_{\Omega} |u|^p,
\]

which, along with lemma 2.4, gives

\[
[u]_{s,p,\Omega}^p \geq \frac{C_{n,s,p}}{2C_{1,s,p}} \left( \int_{\mathbb{S}^{n-1}} f(w) d\mathcal{H}^{n-1}(w) \right) \int_{\Omega} |u|^p.
\]

This proves the lemma. □
In particular, using eqs. (2.1) and (2.2), for \( f(\sigma) = |\cos \sigma_1|^p \) we obtain
\[
\int_{A_{n-1}} |\cos \sigma_1|^p g_{n-1}(\sigma) d\sigma = 2\mathcal{H}^{n-2}(S^{n-2}) \int_{0}^{\pi} (\cos \sigma_1)^p (\sin \sigma_1)^{n-2} d\sigma_1 = \frac{2n-1}{\Gamma \left( \frac{n-1}{2} \right)} B \left( \frac{n-1}{2}, \frac{sp+1}{2} \right).
\] (3.5)

Now we are ready to prove theorem 1.4.

**Proof of theorem 1.4.** Part (1): Assume \( s \in (0, \frac{1}{p}] \). Note that the case \( p = 1 \) is covered here, with \( sp < 1 \). We apply proposition 3.2 with \( m = n - 1 \) and \( \omega = (-1,1) \subset \mathbb{R} \) to deduce \( P_{n,s,p}^1(\Omega_\infty) \leq P_{1,s,p}^1((-1,1)) \). Now applying lemma 2.8 on \( P_{1,s,p}^1((-1,1)) = 0 \) we get the result.

Part (2): Let us assume \( s \in \left( \frac{1}{p}, 1 \right) \). We know, from proposition 3.2, that \( P_{n,s,p}^1(\Omega_\infty) \leq P_{1,s,p}^1((-1,1)) \).

So, it is enough to prove that
\[
P_{n,s,p}^1(\Omega_\infty) \geq P_{1,s,p}^1((-1,1)).
\] (3.6)

We shall show this using lemma 3.5. Choose \( w = (w_1, \ldots, w_n) \in S^{n-1} \) and \( x \in \mathbb{R}^n \) such that \( w_1 \neq 0 \) and \( x \cdot w = 0 \). Notice that \( L^1(\{t \in \mathbb{R} : x + tw \in \Omega_\infty\}) \), i.e. the length of the intersection \( \Omega_\infty \cap \{x + tw : t \in \mathbb{R}\} \), is independent of \( x \in \omega^+ \). So, we have
\[
L^1(\{t \in \mathbb{R} : x + tw \in \Omega_\infty\}) = H^1(\Omega_\infty \cap \{x + tw : t \in \mathbb{R}\}) = |t_0(w)|,
\]
where \(-1 + t_0(w)w_1 = 1 \) i.e. \( t_0(w) = \frac{2}{w_1} \). From (ii) of proposition 2.6 we see that
\[
P_{1,s,p}^1(\{t \in \mathbb{R} : x + tw \in \Omega_\infty\}) = \left( \frac{|w_1|}{2} \right)^sp P_{1,s,p}^1((0,1)) = |w|^sp P_{1,s,p}^1((-1,1)).
\]

The above equality enables us to apply lemma 3.5, with the choice \( f(w) = P_{1,s,p}^1((-1,1))|w|^p \). We get
\[
P_{n,s,p}^1(\Omega_\infty) \geq P_{1,s,p}^1((-1,1)) \frac{C_{n,s,p}}{2C_{1,s,p}} \int_{S^{n-1}} |w|^p d\mathcal{H}^{n-1}.
\]

Again, using Hyper-spherical Coordinates, in the RHS of the above inequality, we get
\[
P_{n,s,p}^1(\Omega_\infty) \geq P_{1,s,p}^1((-1,1)) \frac{C_{n,s,p}}{2C_{1,s,p}} \int_{A_{n-1}} |\cos \sigma_1|^p g_{n-1}(\sigma) d\sigma.
\]

Now, eq. (3.5) gives
\[
P_{n,s,p}^1(\Omega_\infty) \geq P_{1,s,p}^1((-1,1)) \frac{C_{n,s,p}}{2C_{1,s,p}} 2\pi^{\frac{n-1}{2}} B \left( \frac{n-1}{2}, \frac{sp+1}{2} \right).
\]

Again, using eqs. (1.2) and (2.2), we find that
\[
\frac{C_{n,s,p}}{C_{1,s,p}} \pi^{\frac{n-1}{2}} B \left( \frac{n-1}{2}, \frac{sp+1}{2} \right) = 1,
\]
consequently \( P_{n,s,p}^1(\Omega_\infty) \geq P_{1,s,p}^1((-1,1)) \). This concludes the proof of eq. (3.6) and hence the theorem follows. \( \square \)
Proof of theorem 1.5. The domain monotonicity property ((i) of proposition 2.6) and theorem 1.2, implies $P^2_{n-m,s,p}(\omega) \leq P^2_{n-s,p}(\Omega)$. For the reverse inequality, following the same proof as in (1) of proposition 3.2, where the domain of integration $\Omega_\infty$ is replaced by $\Omega_\ell$, we obtain

$$P^2_{n-m,s,p}(\Omega_\ell) \leq \left( P^2_{n-m,s,p}(\omega) \right)^{\frac{1}{p}} + \left( \frac{[v]_{s,p,\mathbb{R}^m}}{\ell s} \right)^{\frac{p}{p'}} \leq P^2_{n-m,s,p}(\omega) + p^{p-1}$$

where we used the following elementary inequality: $(a+b)^q \leq a^q + q^{q-1}(a^{q-1}b + b^q)$ for $a, b \geq 0$ and $q \geq 1$. Combining these two estimates of $P^2_{n-s,p}(\Omega_\ell)$, the first part of the theorem follows. Now letting $\ell \to \infty$ and applying theorem 1.2 we conclude the last equality. This finishes the proof of theorem 1.5. \hfill \Box

Acknowledgment: The authors would like to thanks Prof. Prosenjit Roy, Prof. Gyula Csató and Dr. Indranil Chowdhury for fruitful discussions on this subject.

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