On the universality of fluctuations for the cover time

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March 22, 2023

Abstract

We consider random walks on finite vertex-transitive graphs Γ of bounded degree. We find a simple geometric condition which characterises the cover time fluctuations: the renormalised cover time \( \frac{\tau_{\text{cov}} - \log |\Gamma|}{n} \) converges to a standard Gumbel variable if and only if \( \text{Diam}(\Gamma)^2 = o\left(n/\log n\right) \), where \( n = |\Gamma| \). We prove that this condition is furthermore equivalent to the decorrelation of the uncovered set. The arguments rely on recent breakthroughs by Tessera and Tointon on finitary versions of Gromov’s theorem on groups of polynomial growth, which we leverage into strong heat kernel bounds, and refined quantitative estimates on Aldous and Brown’s exponential approximation of hitting times, which are of independent interest.

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1 Introduction

1.1 Context

Let $\Gamma$ be a finite (connected) vertex-transitive graph. Let $(X_t)_{t \geq 0}$ denote the simple random walk in continuous time, which at constant rate 1 jumps to a randomly chosen neighbour, starting from a designated vertex called the root and denoted by $o$. Let us denote by $T_x = \inf \{ t \geq 0 : X_t = x \}$ the hitting time of the vertex $x$. The cover time variable of $\Gamma$ by $X$ is the first time that the walk has visited every vertex:

$$\tau_{\text{cov}} = \max_{x \in \Gamma} T_x,$$

where with a slight abuse of notation we have used $\Gamma$ to also denote the vertex set of the graph $\Gamma$. In this article we are concerned with obtaining general fluctuation results for the cover time $\tau_{\text{cov}}$ of random walks on vertex-transitive graphs, as the vertex set size $|\Gamma|$ tends to infinity. As we will see this question is deeply intertwined with the study of the structure of the uncovered set $U(t) = \{ x \in \Gamma : T_x > t \}$, for $t$ close to the (expected) cover time.

Obtaining quantitative estimates on the cover time is a natural problem which parallels the intensively studied question of mixing time and cutoff for random walks on graphs (i.e. understanding how far the law of $X_t$ is from the stationary distribution, see Section 2.2 for some definitions, and [LP17] and the references therein for an introduction). In common with much of the literature on the subject, we will focus in this paper on vertex-transitive graphs. The restriction to this class of graphs is natural in order to avoid pathological examples, which can be arbitrarily badly behaved. At the same time it allows for a very rich range of behaviours, as we are about to discuss.

Results in this direction go back at least to the seminal work of Aldous [Ald83] who proved that for random walks on finite groups and under mild geometric conditions, the cover time $\tau_{\text{cov}}$ is concentrated around its mean $t_{\text{cov}} = E(\tau_{\text{cov}})$, i.e. that $\frac{\tau_{\text{cov}}}{t_{\text{cov}}} \to 1$ in probability, and obtained the leading order behaviour of $t_{\text{cov}}$. This was complemented a few years later by Matthews [Mat88] who obtained a general upper bound (valid for any graph) on the expected cover time. The problem of the correlation structure of the uncovered set was raised in the physics literature through the work of Brummelhuis and Hilhorst [BH91, BH92].
It is worth noting that even for basic graphs such as the $d$-dimensional torus of side-length $n$, i.e. when $\Gamma = (\mathbb{Z}/n\mathbb{Z})^d$, the problem is highly nontrivial. It was only in 2004 that the first order of the cover time for the two-dimensional torus $(\mathbb{Z}/n\mathbb{Z})^2$ was found by Dembo, Peres, Rosen and Zeitouni in [DPRZ04]. This result was later refined, see [BK17, Abe21, BRZ20], but obtaining a convergence in distribution for the fluctuations of the cover time in two dimensions remains an open problem to this day (it is widely believed however that the fluctuations will be in any case different from the higher dimensional regime described below).

In dimensions $d \geq 3$, it proved by Belius [Bel13] in 2013 (solving an open question of Aldous and Fill [AF]), building on Sznitman’s random interlacement model [Szn10] (who was in fact motivated by the work of Brummelhuist and Hilhorst [BH91, BH92]), that the fluctuations of the cover time are asymptotically distributed according to a standard Gumbel law:

$$
\mathbb{P}\left( \frac{\tau_{\text{cov}}}{t_{\text{hit}}} - \log |\Gamma| \leq s \right) \to e^{-e^{-s}},
$$

where $t_{\text{hit}} := \max_{x,y \in \Gamma} \mathbb{E}_x[T_y]$ is the maximal expected hitting time. (See also the results in Prata’s thesis [PdP12] for partial results valid for more general graphs but under some restrictive assumptions.)

The Gumbel law is significant because it describes the asymptotic maximum of i.i.d. random variables, subject to some conditions on their common distribution. The Gumbel fluctuations in the result above therefore suggest that the uncovered set at time $t_s := t_{\text{hit}}(\log(|\Gamma|) + s)$ might asymptotically be close to a product measure, where each vertex is uncovered with probability $e^{-s}/|\Gamma|$ independently of other vertices. More formally, if $\mu_s$ is a Bernoulli variable of parameter $e^{-s}/|\Gamma|$, we might expect that as $|\Gamma| \to \infty$,

$$
d_{\text{uncov}}(t_s) := d_{\text{TV}}(\mathcal{L}(U(t_s)), \mu_s^{\otimes \Gamma}) \to 0, \quad (1.1)
$$

where $\mathcal{L}(U(t_s))$ denotes the law of the uncovered set at time $t_s$, and the total variation distance between two probability measures $\mu$ and $\nu$ on a finite space $S$ is given by

$$
d_{\text{TV}}(\mu, \nu) = \max_{A \subseteq S} |\mu(A) - \nu(A)|.
$$

Part of the goal of this paper is to study the uncovered set and in particular prove (1.1) in a general framework.

We note that questions concerning the geometry of the uncovered set (even when the cover time is well understood) have recently become prominent. Even on the $d$-dimensional torus, a number of basic problems remain open. For instance, it was proved in [MS17] and [OTS20] that the uncovered set at time $t_{\text{cov}}$ is decorrelated in the above sense if $a > 7/8$ and correlated (in the sense that (1.1) does not hold) if $a < \frac{1+\sqrt{5}}{2}$, where $p_d = \mathbb{P}_o(T_o^+ < \infty)$ for the simple random walk on $\mathbb{Z}^d$. It nevertheless still remains an open question whether there actually is a phase transition, i.e. whether there exists a critical value $a^*$ such that for $a \neq a^*$, $d_{\text{uncov}}(a t_{\text{cov}}) \to 1_{a a^*}$. More generally, the study of the uncovered set fits into the theme of exceptional points for random walks. In two dimensions the structure of those exceptional points has recently been proved to be linked with Liouville quantum gravity, see [AB22], and, away from the cover time, to an even more singular and in some way intriguing object called Brownian multiplicative chaos, see [Jeg20].

Finally, we note that the expected cover time $t_{\text{cov}}$ of a graph is also closely related to the typical value of the maximum of the associated Gaussian free field, see for instance [DLP12], [Din14] and [Zha18].
1.2 Main results

Let $\Gamma$ be a finite (connected) bounded degree vertex-transitive graph. Write $\Gamma$ also for the vertex set of $\Gamma$ as above. We denote by $d(x, y)$ the graph distance between the vertices $x$ and $y$, by $D := \max_{x, y \in \Gamma} d(x, y)$ the diameter of $\Gamma$, and by $\pi$ its stationary distribution.

Our first main results shows that Gumbel fluctuations are universal. Perhaps even more surprisingly we obtain a sharp (necessary and sufficient) geometric condition for this universality.

**Theorem 1.1.** Let $(\Gamma)$ be a collection of finite (connected) vertex-transitive of fixed degree $d$, and let $\chi$ be a standard Gumbel variable, i.e. $\mathbb{P}(\chi \leq s) = e^{-e^{-s}}$ for $s \in \mathbb{R}$. Then

$$\frac{\tau_{\text{cov}}}{\text{hit}} - \log |\Gamma| \xrightarrow{(d) |\Gamma| \to \infty} \chi$$

if and only if

$$\frac{D^2 \log |\Gamma|}{|\Gamma|} \xrightarrow{|\Gamma| \to \infty} 0.$$  \hspace{1cm} (DC)

For future reference we note that, trivially, if we let $n = |\Gamma|$ denote the number of vertices of $\Gamma$ (which tends to infinity by assumption), then (DC) is equivalent to $D^2 = o(n/\log n)$, and (1.2) is equivalent to the condition that for all $s \in \mathbb{R}$, as $n \to \infty$,

$$\mathbb{P}(\tau_{\text{cov}} \leq \text{hit}(\log n + s)) \to \exp(-e^{-s}).$$

We complement this result with a refined statement on the structure of the uncovered set.

**Theorem 1.2.** Let $(\Gamma)$ be a collection of finite (connected) vertex-transitive graphs of fixed degree $d$, and $s \in \mathbb{R}$. For $s \in \mathbb{R}$, let $t_s := \text{hit}(\log(|\Gamma|) + s)$ and let $\mu_s^{\otimes \Gamma}$ denote the product over all the vertices of the graph of the Bernoulli law $\mu_s$ with parameter $e^{-s}/|\Gamma|$. Then

$$d_{TV}(\mathcal{L}(U(t_s)), \mu_s^{\otimes \Gamma}) \xrightarrow{|\Gamma| \to \infty} 0.$$  \hspace{1cm} (1.3)

for every $s \in \mathbb{R}$ if and only if the diameter condition (DC) holds.

This result will be strengthened in various ways under the assumption (DC) in Section 4. For instance, in Theorem 4.5 we discuss the law of the uncovered set at the first time that exactly $k$ points remain to be covered.

In particular, we recover that (1.1) holds for the tori $\mathbb{Z}^d$, for $d \geq 3$ (this is in fact already mentioned in the thesis of Prata [PdP12], although the result was never published).

To get a feel for the condition (DC), the reader might want to keep in mind a “thin” torus where $\Gamma = (\mathbb{Z}/m\mathbb{Z})^2 \times (\mathbb{Z}/h\mathbb{Z})$. This graph is nothing but a box (with periodic boundary conditions) of sidelength $m$ and height $h = h_m$, where we assume that $1 \leq h_m \leq m$. (Thus the extreme cases $h_m = 1$ and $h_m = m$ correspond to the familiar two- and three-dimensional situations respectively, while very little is known in general about intermediate situations). Then the diameter $D = m$, and the volume is $n = m^2 h_m$, so the condition (DC) holds if and only if $h_m \gg \log m$ (we will write $a_n \gg b_n$ when $a_n/b_n \to \infty$ as $n \to \infty$).

The reader might find it surprising that the condition (DC) determines the behaviour (1.2) (and (1.3)). It would indeed be natural to expect that the local structure of the graphs plays an important role in the decorrelation of the uncovered set. Theorem 1.2 tells us that it is not the case: decorrelation in the uncovered set depends only on the relation of the diameter of the graph to its size, a completely global geometric condition.
We also point out that despite considerable effort over the last 50 years, in the parallel problem of the mixing time mentioned above, a simple characterisation of the cutoff phenomenon (and even more so of the limiting profile) is currently completely out of reach.

In the proof that (DC) is necessary for (1.3), we will see that even the first moment of \( U(t_s) \) is different from that of \( \mu_s^\otimes \Gamma \). In that sense the theorem above does not capture the correlations of the uncovered set close to the cover time. However, there are other natural ways to adjust the time scaling, for instance by considering (for any \( s \in \mathbb{R} \)) the uncovered set at the time \( t_s^* \) such that \( |U(t_s^*)| = e^{-s} \). Note that such changes of normalisation would not affect the above results under (DC). That is, under (DC), both (1.2) and (1.3) hold with \( t_s \) replaced by \( t_s^* \). Indeed, we will prove in Corollary 2.12 and Proposition 2.13 that under the assumption (DC), we have \( t_s = t_s^* + o(n) \), where \( n = |\Gamma| \) is the number of vertices of \( \Gamma \). It is therefore natural to ask whether \( U(t_s^*) \) is close to a product measure. For this as well, we will show that the diameter condition (DC) is the sharp criterion for decorrelation. However, in order to state this stronger result we will need to make an additional geometric assumption.

**Theorem 1.3.** Let \( (\Gamma) \) be a collection of finite (connected) vertex-transitive graphs of fixed degree \( d \), and let \( n = |\Gamma| \). Suppose that \( D^2/n \to 0 \). For \( s \in \mathbb{R} \), let \( \mu_s^\otimes \Gamma \) denote the product over all the vertices of the graph of the Bernoulli law \( \mu_s \) with parameter \( e^{-s}/|\Gamma| \). Fix \( s \in \mathbb{R} \). The following are equivalent:

(i) \( d_{TV}(\mathcal{L}(U(t_s^*)), \mu_s^\otimes \Gamma) \xrightarrow{|\Gamma| \to \infty} 0 \),

(ii) The diameter condition (DC) holds.

**Remark 1.4.** In fact, we will prove an even stronger result: namely, the conclusion is valid as soon as \( t_{rel} = o(t_{hit}) \), where \( t_{rel} \) is the relaxation time, or inverse spectral gap. We will also show that these conditions are equivalent to a third one, where instead of \( t_s^* \) we use the time \( t(s) := t_{hit}(\log n + s) \), where \( t_{hit} := \mathbb{E}_x T_0 \); this is in fact an integral part of our proof. The same statement also holds with \( t_s \) instead of \( t_s^* \). In other words, if \( t_{rel} = o(t_{hit}) \) holds, we can strengthen Theorem 1.2, replacing “for every \( s \in \mathbb{R} \)” by “for some \( s \in \mathbb{R} \).” The theorem will be proved in these forms in Section 6.3, where we will also briefly explain why the assumption \( D^2/n \to 0 \) implies \( t_{rel} = o(t_{hit}) \).

We conjecture in fact that the theorem is valid without any assumption on \( t_{rel} \) or \( t_{hit} \).

Central to the arguments of this paper will be exponential approximations for hitting times of arbitrary sets of vertices (irrespective of the relative geometry of the points). Such exponential approximations go back to the work of Aldous and Brown [AB92], who obtained quantitative error bounds which we recall below. However, our proofs that the diameter condition (DC) is sharp require us crucially to strengthen the quantitative bounds in these approximations, as we now detail.

Let us recall that \( t_{rel} \) is the relaxation time of the chain and define, for \( A \subset \Gamma \), the \textbf{quasistationary distribution} by, for \( x \in \Gamma \),

\[
\alpha_A(x) = \lim_{t \to \infty} \mathbb{P}(X_t = x \mid T_A > t).
\]  

(See, e.g., (3.86) in [AF] for a proof of the existence of this limit). It is not hard to see that, starting from the quasi-stationary distribution \( \alpha_A \), the hitting time of \( A \) is exactly an exponential random variable. A seminal result of Aldous and Brown in this direction is the following.

**Theorem 1.5 ([AB92], Theorem 3 and Equation (1)).** Let \( (X_t)_{t \geq 0} \) be an irreducible reversible Markov chain on a finite state space \( V \), and \( \pi \) be its stationary distribution. Let \( A \subset V \), and
denote by $T_A$ the first hitting time of $A$. Then for all $t > 0$,
\[
0 \leq 1 - \frac{\mathbb{P}_\pi(T_A > t)}{\mathbb{P}_{\alpha_A}(T_A > t)} = O \left( \frac{t_{\text{rel}}}{\mathbb{E}_{\alpha_A} T_A} \right),
\]
and
\[
0 \leq 1 - \frac{\mathbb{E}_n T_A}{\mathbb{E}_{\alpha_A} T_A} = O \left( \frac{t_{\text{rel}}}{\mathbb{E}_{\alpha_A} T_A} \right).
\]

Among other things, these approximations are very useful to estimate the capacity $q_A := \frac{\mathbb{E}_n T_A}{\mathbb{E}_{\alpha_A} T_A}$ of finite sets $A$, which play a crucial role in our analysis. Indeed, when the diameter condition (DC) holds, the error term above is $o(1/\log n)$, which will turn out to be sufficient. However, as soon as (DC) fails, this error term becomes too large (even to estimate the expected number of uncovered points). To prove our results when (DC) does not hold, we will instead rely on the following theorem, which is a refinement of Theorem 1.5, and which we view as the main technical innovation of our paper. We expect the error bounds be sharp up to a constant factor for any family of vertex-transitive graphs of fixed degree (without further assumptions).

**Theorem 1.6.** Let $(\Gamma)$ be a collection of finite vertex-transitive graphs of fixed degree $d$, and let $k \geq 1$. Then, uniformly over all sets $A \subset \Gamma$ of size $k$, and $t \geq 0$, we have as $n = |\Gamma| \to \infty$
\[
0 \leq 1 - \frac{\mathbb{P}_\pi(T_A > t)}{\mathbb{P}_{\alpha_A}(T_A > t)} = O_{d,k} \left( \frac{D^4}{(\mathbb{E}_{\pi T_0})^4} \left( 1 + \frac{n}{D^4} \int_0^D \frac{s^3 ds}{V(s)} \right) \right),
\]
and
\[
0 \leq 1 - \frac{\mathbb{E}_n T_A}{\mathbb{E}_{\alpha_A} T_A} = O_{d,k} \left( \frac{D^4}{(\mathbb{E}_{\pi T_0})^4} \left( 1 + \frac{n}{D^4} \int_0^D \frac{s^3 ds}{V(s)} \right) \right),
\]
where $V(s)$ denotes the volume of the ball of radius $|s| \geq 0$. Moreover, the right-hand sides of (1.7) and (1.8) are $O_{d,k} (t_{\text{rel}} / t_{\text{hit}})^2$ if $D \gtrsim n^{1/4} \log n$ and $O_{d,k}(1/n)$ if $D \lesssim n^{1/4} \log n$.

The full version of this result will be stated in Theorem 5.2 and Theorem 5.5. See also the other results in that section for complements.

We believe that those technical improvements will prove useful also for other problems, for instance in order to establish the transition phase of the uncovered set at time $a_{\text{cov}}$ for $0 < a < 1$ on tori of dimension $d \geq 3$, which would answer the questions raised by Miller and Sousi in [MS17] (see also [OTS20]).

This type of improvement is fundamental to handle borderline cases. It is in fact even significant for 2-dimensional tori: if $\Gamma = (\mathbb{Z}/m\mathbb{Z})^2$, then this error term is $O(1/(\log n)^2) = o(1/\log n)$, compared to $O(1/\log n)$ in (1.5) and (1.6).

The setup of our improvements is actually fairly general: not all statements require the graphs $\Gamma$ to be vertex-transitive, and the size of the sets $A$ can be allowed to diverge. The interested reader can find more precise statements in Section 5.

### 1.3 Diameter condition and local transience

We have already mentioned that the sharpness of the diameter condition (DC) is a little surprising. Initially (and this was in fact our own belief when we began this work), one might have suspected that Gumbel fluctuations are perhaps more naturally linked with the following notions of local transience which we now define. For this it will be useful to recall the definition of the mixing time $t_{\text{mix}}(\varepsilon)$ at level $0 < \varepsilon < 1$ for a positive recurrent Markov chain $(X_t)_{t \in \mathbb{R}_+}$ on some state space $\Gamma$ with invariant distribution $\pi$:
\[
t_{\text{mix}}(\varepsilon) = \inf \{ t \geq 0 : \max_{x \in \Gamma} d_{\text{TV}}(P_t(x, .); \pi(.)) \leq \varepsilon \},
\]
where $P_t(x, .)$ denotes the law of the Markov chain at time $t$ starting from $x \in \Gamma$. 

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**Definition 1.7.** We say that a sequence of finite Markov chains with state spaces \( \Gamma_n \), transition matrices \( P = P(n) \) and stationary distributions \( \pi = \pi_n \) satisfying \( \lim_{n \to \infty} \max_{x \in \Gamma_n} \pi(x) = 0 \) is **weakly uniformly transient (WUT)**, or **uniformly locally transient**, if

\[
\max_{o \in \Gamma_n} \mathbb{E}_o(L_o(t)) = O(1),
\]

where \( t = t_{\text{mix}}(1/4) \), and \( L_x(t) := \int_0^t 1_{\{X_s = x\}} \, ds \) is the *local time* of the walk at \( x \) up to time \( t \). Again, writing \( t = t_{\text{mix}}(1/4) \), we say that the sequence is (SUT) **strongly uniformly transient**, or **uniformly globally transient**, if

\[
\lim_{s \to \infty} \limsup_{n \to \infty} \max_{o \in \Gamma_n} \mathbb{E}_o(L_o(t) - L_o(s)) = 0.
\]

We say that a sequence of graphs \( \Gamma_n := (V_n, E_n) \) is WUT (resp. SUT) if the sequence of simple random walks on \( \Gamma_n \) is WUT (resp. SUT).

Clearly, (1.11) implies (1.10). The reason why one might suspect that this notion might be related to Gumbel fluctuations is that uniform transience, especially strong uniform transience, should prevent clusterisation and hence lead to decorrelation in the uncovered set.

For example, a torus in dimension \( d \geq 3 \) is strongly uniformly transient, but a torus of dimension \( d = 1, 2 \) is not even weakly uniformly transient. More generally, let us return to the example of the thin torus \( \Gamma = (\mathbb{Z}/m\mathbb{Z})^2 \times (\mathbb{Z}/h\mathbb{Z}) \) discussed above, where \( h = h_m \). For which values of \( h \) is this weakly/strongly uniformly transient? Since a two-dimensional random walk returns to the origin roughly \( \log m \) times by time \( t = t_{\text{mix}}(1/4) \propto m^2 \), it is not hard to see that the thin torus is strongly uniformly transient if and only if \( h_m \gg \log m \) (and weakly uniformly transient if and only if \( h_m \gtrsim \log m \)). This condition coincides with (DC), as already observed.

This immediately raises the following question: could it be the case that the diameter condition is equivalent to strong (or weak) uniform transience? We will see as part of our analysis that any sequence of vertex-transitive graphs \( \Gamma_n = (V_n, E_n) \) satisfying the diameter condition (DC) satisfies that

\[
\lim_{r \to \infty} \limsup_{n \to \infty} \max_{y \in V_n, d(o,y) \geq r} \mathbb{E}_y(L_o(2t_{\text{mix}}(1/4))) = 0,
\]

where \( d(o,y) \) is the graph distance between \( y \) and \( o \) (see, e.g., Lemma 2.17). Strong uniform transience can be deduced from this relatively easily (in Proposition 2.18 we give a more direct argument). Thus

\[
(\text{DC}) \implies (\text{SUT}).
\]

However the converse is, perhaps surprisingly again, not true. In Section 7 we will construct a sequence of finite vertex-transitive graphs \( \Gamma_n \) of uniformly bounded degree which satisfy SUT but not (DC). In particular, as a consequence of Theorem 1.2, despite being locally transient in this strong sense, the uncovered set will display nontrivial correlations. This example will be constructed by considering the product of a Ramanujan graph and a cycle of suitable sizes. (Essentially, in this product, one direction is very recurrent, but the other is very transient). See Section 7 for the precise definition.

The implication (1.12) is closely related to a conjecture of Benjamini and Kozma [BK05]. This conjecture states that for vertex-transitive graphs of bounded degree, if we assume \( D^2 \lesssim n/\log n \) then the effective resistance \( R_{x+y} \) between vertices of the graph is uniformly bounded above by some constant. This conjecture was recently solved by Tessera and Tointon [TT21]. Furthermore, using the tools developed in their paper, it is not hard to show that the effective resistance is
uniformly bounded if and only if (WUT) holds. The implication (1.12) is therefore the direct “strong” analogue of the Benjamini–Kozma conjecture (the left hand side replaces the assumption $D^2 \lesssim n/\log n$ by $D^2 \ll n/\log n$, and the weak uniform transience in the right hand side by the strong uniform transience).

We end this discussion with an instructive comparison with the results in a recent paper of Dembo, Ding, Miller and Peres [DDMP19]. This describes the first order (cutoff) behaviour of the mixing time of the lamplighter walk on the thin torus $\Gamma_m = (\mathbb{Z}/m\mathbb{Z})^2 \times (\mathbb{Z}/h\mathbb{Z})$ with $h = a \log m$. The (total variation) mixing time of the lamplighter group on the base graph $\Gamma_m$ is known to be closely related to the cover time of the simple random walk on that base graph $\Gamma_m$. This led the authors of [DDMP19] to use the ratio between the mixing time of the lamplighter on $\Gamma_m$ and the cover time on $\Gamma_m$ as an indicator of low- vs. high-dimensional behaviour. Indeed, this ratio is asymptotically equal to 1 for a completely flat, two-dimensional torus by results of [DPRZ04] and [PR04], whereas it is asymptotically 1/2 for a three-dimensional torus, by results of Miller and Peres [MP12] (as well as on the complete graph, where the lamplighter walk reduces to the well known walk on the hypercube). Surprisingly, the authors in [DDMP19] show that on a thin torus, this ratio is strictly contained in the interval $(1/2, 1)$ for $0 < a < a_*$ and becomes asymptotically equal to 1/2 (as in the three-dimensional case) for $a \geq a_*$, for some explicit $a_*$. They interpret this as a phase transition between low-dimensional (“recurrent”) and high-dimensional (“transient”) behaviour; see the discussion after Theorem 1.3 in that paper. By contrast, our results show that, at the level of fluctuations, high-dimensional behaviour only kicks in when $a = a_m \to \infty$, arbitrarily slowly, rather than for $a \geq a_*$. 

1.4 Discussion of proof ideas and organisation of paper

Our starting point for this paper is the remarkable series of papers by Tessera and Tointon [TT21], [TT20] which gives a quantitative form of Gromov’s theorem on groups of polynomial growth. Recall that, since the graph is vertex-transitive, by results of Trofimov [Tro84] it is roughly isometric to the Cayley graph of a finite group. Recall also that, for infinite groups, Gromov’s theorem [Gro81] shows that if the volume of balls of radius $r$ grow polynomially in the radius $r$ then the growth exponent $\alpha$ must be integer and the group is then (in the Gromov–Hausdorff sense) close to a $d$-dimensional Euclidean lattice.

The diameter condition (DC), which says that the graph is slightly-more than two-dimensional, combined with the results of Tessera and Tointon, therefore implies that at least for relatively moderate distances the volume growth is at least three-dimensional. In combination with isoperimetric profile bounds (coming for instance from the theory of evolving sets of Morris and Peres [MP05]), this translates into very good decay for the heat kernel at small times and so gives excellent control on the number of returns by random walk to its starting point in this time. Later visits to this point are controlled using two-dimensional estimates. As we will see now, it turns out this is at the root of the decorrelation in the uncovered set.

These heat kernel bounds are used as follows. Suppose that (DC) holds and we wish to prove (1.2). Let us compute the $k$th cumulant of the size of the uncovered set $Z_s$ at time $t_{(s)} = t_{hit}(\log n + s)$. It suffices to show that this cumulant converges to the $k$th cumulant of a Poisson random variable with parameter $e^{-s}$. Now, this cumulant can be expressed as a sum over sets $A$ of size $k$ of the probability that $A$ was not visited by time $t_{(s)}$, and we can assume without loss of generality that the starting distribution is the uniform distribution $\pi$, namely $\pi(A > t_{(s)}).$ The Aldous–Brown approximation [AB92] (recall also Theorem 1.5) shows that this hitting time is approximately an exponential random variable.

It remains to get good approximation for the quasistationary exponential rate. We show (using again the Aldous–Brown approximation) that this quasistationary exponential rate is
close to the ratio $q_A := \frac{E_\pi(T_O)}{E_\pi(T_A)}$ of expectations of hitting time of $A$ compared to that of a single point; see Proposition 2.24. For sets $A$ such that the points in $A$ are well separated, we typically expect $q_A \approx k$, because the hitting time of $A$ is close to the minimum of $k$ independent exponential random variables. The challenge is to quantify this approximation and show that the contribution coming from sets where the points are not so well separated is negligible. It is here that the heat kernel bounds are very useful: indeed, the on-diagonal decay of $p_t(x, y)$ translates into a strong off-diagonal decay (using a subgaussian estimate, namely a recent variant due to Foll [Foll11]) and implies good quantitative bounds of the desired form for $q_A$.

In the most delicate case where $n$ is barely larger than $D^2 \log n$, the analysis uses a somewhat elaborate induction over scales which requires strong quantitative bounds.

In the opposite direction, when (DC) fails, the key task (both for Theorems 1.1 and 1.2) is to show that the expected number of uncovered points at time $t_{\langle \rangle}$ is strictly smaller (by a factor $c$ bounded away from 1) than $e^{-s}$. Indeed this immediately implies Theorem 1.2, and implies Theorem 1.1 by considering the tail at $+\infty$ of $\tau_{\text{cov}}$ and a simple union bound.

The proof of Theorem 1.3 is more complicated. It might initially be tempting to show that the uncovered set is “too” clustered, i.e., the probability that two relatively nearby points are uncovered is larger than it should be in an independent scenario. However, this turns out to be very difficult to control (moments of order at least two of the size of the uncovered set can explode when (DC) does not hold, precisely because of the contribution coming from nearby points). Instead we show that the uncovered set is negatively correlated at large distances. This requires controlling the macroscopic variations of the Green function, a task which occupies a good part of Section 6.

**Organisation of the paper.** We start in Section 2 with the preliminaries in which we setup the notations and obtain the heat kernel bounds (Corollary 2.5 for the on-diagonal term, and Section 2.3 for the off-diagonal terms). We also discuss the original Aldous–Brown approximation and some elementary consequences in Section 2.4.

Section 3 is the proof of the main theorem (Theorem 1.1) under the assumption (DC), and discusses the cumulants of the uncovered set. A detailed strategy is provided in Section 3.1.

Section 4 contains the proofs of Theorem 1.2 under the assumption (DC) and its refinements on the structure of the uncovered set.

Section 5 brings new insights to understand quasistationary distributions and proves Theorem 1.6 improving the Aldous–Brown approximations. Section 5.3 provides a somewhat shorter (at least given the results of that section) proof of Theorem 1.1 under the condition (DC).

Section 6 studies the cover time and uncovered set of graphs which do not satisfy (DC) and completes the proof of the theorems 1.1, 1.2, and 1.3.

Section 7 contains the construction of an explicit example of graphs that are (SUT) but do not satisfy (DC).

**Acknowledgements.** We thank Roberto Imbuzeiro Oliveira and Matt Tointon for some useful discussions. In particular we thank Matt Tointon for asking us about the relation between Definition 1.7 and the definition in [TT20]. This work started when the first two authors were at the University of Cambridge, during which time they were supported by EPSRC grant EP/L018896/1. The paper was finished while the first author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2022 semester on *Analysis and Geometry of Random Spaces*, which was supported by the National Science Foundation under Grant No. DMS-1928930. J.H.’s research is supported by NSERC grants.
2 Preliminaries

2.1 A priori bounds on the heat kernel

From now on and for the rest of the article to ease notations, we will write with a small abuse of notation $\Gamma$ for the graph and will denote the size of the vertex set of $\Gamma$ by $n$. Our constants will implicitly be allowed to depend on the degree $d$ (and also on the size $k \geq 1$ of a set when we apply the moment method to compute the size of the uncovered set). We will not always recall this. This also applies to notations such as $a_n = o(b_n)$, $a_n = O(b_n)$. We will also sometimes use $a_n \ll b_n$ (which is by definition equivalent to $a_n = o(b_n)$, i.e., $a_n/b_n \to 0$) and $a_n \lesssim b_n$ (which by definition means $a_n = O(b_n)$, i.e., $a_n/b_n$ is bounded).

It will be at times convenient to allow the graph $\Gamma$ to not be vertex-transitive. Let $P(\cdot, \cdot)$ be the transition kernel of our walk. Define the conductance bounds on the heat kernel given a conductance profile. Let $W e first recall here a crucial consequence of the theory of evolving sets which allows us to get

\[ \Phi(S) = \frac{Q(S, S^c)}{\pi(S)}, \]

where for $A, B \subset \Gamma$,

\[ Q(A, B) = \sum_{a \in A, b \in B} \pi(a) P(a, b). \]

We call conductance profile the function $\phi$ defined, for $\min_{x \in \Gamma} \pi(x) \leq u \leq 1/2$, by

\[ \phi(u) = \inf \{ \Phi(S) : \pi(S) \leq u \}. \]

We first recall here a crucial consequence of the theory of evolving sets which allows us to get bounds on the heat kernel given a conductance profile. Let $\varepsilon > 0$. Then we know by [MP05, Theorem 13], considering only diagonal transitions, that for $\varepsilon$ such that $4/\varepsilon \leq 1/2$ and every $x \in \Gamma$,

\[ t \geq \int_{4\pi(x)}^{4/\varepsilon} \frac{8du}{u\phi(u)^2} \implies p_t(x, x) \leq (1 + \varepsilon)\pi(x). \]

Consider now the vertex-transitive case where $\pi(x) = 1/n$ for every $n$. When applying this inequality it is essential to note that the condition on $\varepsilon$ is such that $8 \leq \varepsilon \leq n$. It may be slightly perverse to name $\varepsilon$ a quantity which by assumption is greater or equal to 8, but these notations are by now relatively well established so we do not tamper with them. ($w = n/\varepsilon$ plays the role of the relevant volume in the graph, namely we get $p_t(o, o) \asymp 1/w$. Therefore $\varepsilon$ will typically be small compared to the volume $n$.)

Our first task will be to transform the condition on the conductance profile into a condition on the volume growth, or equivalently, on isoperimetry. (Combined with the work of Tessera and Tointon on a quantitative form of Gromov’s theorem giving strong control on the volume growth, this will give us excellent control on the return probabilities.)

Proposition 2.1. Recall that $n = |\Gamma|$ and assume that $n \geq D^q$. Then, we have uniformly over all $t \leq D^2$, writing $m = [q]$ and $R = D^{[q]}$,

\[ p_t(o, o) \lesssim \max\left(\frac{1}{t^{(m+1)/2}}, \frac{1}{R^{tm/2}}\right), \]

where the implicit constant depends on $m$ and $d$.

Remark 2.2. For a given $m = [q]$ this upper bound is sharp, as is easily shown by considering “flat” tori of the form $\Gamma = (\mathbb{Z}/L\mathbb{Z})^m \times (\mathbb{Z}/R\mathbb{Z})$ with $1 \ll R \leq L$. In this case, the heat kernel initially decays like in dimension $(m+1)$; but eventually (for $t \gg R^2$) the decay becomes only $m$-dimensional.
Remark 2.3. We will only apply this result with \( m \leq 5 \), so we don’t need to track the dependence in \( m \).

Remark 2.4. By using [MP05, Theorem 1] instead of [MP05, Theorem 13], the same estimate holds for the discrete time transition probability \( p_t^F(o, o) \), a fact which will also be useful in some of the steps of the proof of Lemma 3.18.

Proof. The key is to observe that we have the following bound on the isoperimetric profile: for every nonempty \( S \subset \Gamma \) such that \( \pi(S) \leq 1/2 \), then

\[
\Phi(S) \geq \min \left( |S|^{-1/(m+1)}, R^{1/m} |S|^{-1/m} \right). \tag{2.6}
\]

This is essentially [TT20, Theorem 6.1], which is in itself a simple consequence of their bound on the volume growth [TT20, Proposition 5.1] and an isoperimetric inequality [TT20, Proposition 4.1] for vertex-transitive graphs. (In fact such an inequality can also be found in [LP16, Lemma 10.46], and for the infinite case in [LMS08, Lemma 7.2].) For \( v \leq n/2 \), considering sets \( S \) such that \( v/n \leq \pi(S) \leq 1/2 \), we get immediately that

\[
\phi(v/n) \geq \min \left( v^{-1/(m+1)}, R^{1/m} v^{-1/m} \right). \tag{2.7}
\]

We deduce, making the change of variables \( v = nu \) and setting \( w = 8n/\varepsilon \) with \( 8 \leq \varepsilon < n \),

\[
\int_{4/n}^{4/\varepsilon} \frac{8du}{u \phi(u)^2} = \int_{4}^{4n/\varepsilon} \frac{8dv}{v \phi(v/n)^2} \leq \int_{4}^{4n/\varepsilon} \max \left( v^{2/(m+1)}, \frac{2(m+1)}{R^2/m} \right) dv \leq \max \left( w^{2/(m+1)}, \frac{2(m+1)}{R^2/m} \right).
\]

Consequently, there exists \( C \geq 1 \) such that if \( t = C \max \left( w^{2/(m+1)}, \frac{2(m+1)}{R^2/m} \right) \), we may apply (2.4) and obtain (recall that \( w = 8n/\varepsilon \)),

\[
p_t(o, o) \leq \frac{1 + \varepsilon}{n} = \frac{1}{n} + \frac{8}{w} \leq \frac{9}{w}. \tag{2.9}
\]

As \( t = C \max \left( w^{2/(m+1)}, \frac{2(m+1)}{R^2/m} \right) \), we have in particular \( w \geq \min \left( t^{(m+1)/2}, R t^{m/2} \right) \), and the bound on \( p_t(o, o) \) can be rewritten as

\[
p_t(o, o) \leq \max \left( \frac{1}{t^{(m+1)/2}}, \frac{1}{R t^{m/2}} \right). \tag{2.8}
\]

Finally, as we defined first \( 1 < w \leq n \), and then \( t = C \max \left( w^{2/(m+1)}, \frac{2(m+1)}{R^2/m} \right) \), (2.8) holds for all \( t \) such that \( C < t \leq CD^2 \), and hence, as we took \( C \geq 1 \) and up to changing the implicit constant because of the values of \( t \) smaller than \( C \), for all \( t \leq D^2 \), as desired.

To avoid separating the proof into too many cases and cutting the integrals into several pieces, the following Corollary will be helpful. It provides upper bounds on the diagonal return probabilities which are true for all \( t \).

Corollary 2.5. Recall that \( n = D^2 f(n) \).

(a) Uniformly over all finite vertex-transitive graphs of degree less or equal to \( d \) such that \( n \geq D^2 \), and \( t \geq 0 \), we have

\[
p_t(o, o) \leq \frac{1}{t^{3/2}} \max \left( \frac{1}{f(n)t}, \frac{1}{f(n) t} \right), \quad 1 \leq t \leq D^2, \tag{2.9}
\]
and
\[ p_t(o, o) \lesssim \frac{1}{D^3} + \frac{1}{n}, \quad t \geq D^2. \tag{2.10} \]

(b) Uniformly over all finite vertex-transitive graphs of degree less or equal to \( d \) such that \( n \geq D^5 \), and \( t \geq 0 \), we have
\[ p_t(o, o) \lesssim \frac{1}{t^{5/2}} + \frac{1}{D^5}. \tag{2.11} \]

Moreover, the implicit constants depend only on \( d \).

**Remark 2.6.** The bounds we give are not necessarily optimal if we fix the value of \( m = \lfloor q \rfloor \), but have the advantage that they do not depend on \( m \) and so can be used regardless. This leads to fewer cases to treat separately further down the proof and so makes the argument more unified.

**Proof.** (a) First assume that \( f(n) < D \). Setting \( q \) such that \( n = D^q \), we have from Proposition 2.1, noting that in this case \( R = f(n) \), that for \( 1 \leq t \leq D^2 \),
\[ p_t(o, o) \lesssim \frac{1}{t^{3/2}} + \frac{1}{f(n)t}. \tag{2.12} \]

On the other hand, if \( f(n) \geq D \), i.e. \( n \geq 3 \), we have, applying Proposition 2.1 with \( q = 3 \) that for all \( 1 \leq t \leq D^2 \), \( p_t(o, o) \lesssim \frac{1}{t^{3/2}} \). Observing moreover that as \( t \leq D^2 \), \( \frac{1}{t^{3/2}} \leq \frac{1}{D^2} \), we get that uniformly over all vertex-transitive graphs (of degree less or equal to \( d \)) such that \( n \geq D^2 \), and \( t \leq D^2 \), we have
\[ p_t(o, o) \lesssim \frac{1}{t^{3/2}} + \frac{1}{f(n)t}. \tag{2.13} \]

For \( t \geq D^2 \), as \( t \mapsto p_t(o, o) \) is decreasing, we have, using the previous bound at time \( D^2 \),
\[ p_t(o, o) \leq p_{D^2}(o, o) \lesssim \frac{1}{D^3} + \frac{1}{n}. \tag{2.14} \]

This concludes the proof of (a). (b) This follows by Proposition 2.1 with \( q = 5 \), proceeding exactly as in the proof of (a).

### 2.2 Relaxation, mixing, and hitting times

Recall that the relaxation time of the chain is the inverse of the spectral gap:
\[ t_{rel} = \frac{1}{1 - \lambda_2}, \tag{2.15} \]

where \( \lambda_2 \) is the second largest eigenvalue of the chain. By a classical argument (see e.g. [Chu97, Theorem 7.6]) the following bound on the relaxation time is valid on every finite (connected) vertex-transitive graph:
\[ t_{rel} \leq dD^2. \tag{2.16} \]

We now show that for graphs of polynomial growth, \( t_{rel} \propto D^2 \).

**Lemma 2.7.** There exists a constant \( C \) such that for every finite (connected) vertex-transitive graphs of degree \( d \) such that \( n = |\Gamma| \leq D^5 \),
\[ CD^2 \leq t_{rel} \leq dD^2. \tag{2.17} \]
Proof. We follow arguments from the proof of the lower bound of [DSC93, Theorem 3.1].

By the minimax characterisation of the spectral gap (see for instance [LP17, Lemma 13.7 and Remark 13.8]) we have, setting \( g = d(o, \cdot) \),

\[
1 - \lambda_2 \leq \frac{\mathcal{E}(g)}{\text{Var}_\pi(g)}.
\]  

(2.18)

Since \( g \) is 1-Lipschitz, \( \mathcal{E}(g) \leq 1/2 \). We hence have

\[
t_{\text{rel}} \geq 2 \text{Var}_\pi(g) = \frac{1}{n^2} \sum_{x,y \in \Gamma} (g(x) - g(y))^2.
\]

(2.19)

Let \( z \in \Gamma \) such that \( d(o, z) = D \). We deduce that

\[
t_{\text{rel}} \geq \frac{1}{n^2} \sum_{x \in B(o, D/4), y \in B(z, D/4)} (D/2)^2 = \frac{D^2}{4} \left( \frac{V(D/4)}{n} \right)^2.
\]

(2.20)

Moreover, by [TT20, Proposition 5.1], there exists a (universal) constant \( c \) such that for all finite (connected) vertex-transitive graphs satisfying \( n \leq D^5 \),

\[
V(D/4) \geq cn.
\]

(2.21)

For those graphs, we hence have, setting \( C = c^2/4 \)

\[
CD^2 \leq t_{\text{rel}} \leq dD^2.
\]

(2.22)

\( \square \)

Let us now define mixing times and the distance to stationarity. We restrict here to our framework of simple random walks on vertex-transitive graphs, so the walk is in particular transitive and the stationary distribution is the uniform distribution \( \pi \). See [LP17, Chapter 4] for more general definitions. For \( 1 \leq p < \infty \), we define the \( L^p \) distance to stationarity of the chain \( (X_t)_{t \geq 0} \) at time \( t \) by

\[
d^{(p)}(t) = \left( \frac{1}{n} \sum_{x \in \Gamma} |np_t(o, x) - 1|^p \right)^{1/p}.
\]

(2.23)

We also define the \( L^\infty \) distance to stationarity as (see [LP17, Proposition 4.15]).

\[
d^{(\infty)}(t) = np_t(o, o) - 1.
\]

(2.24)

We define the \( L^p \) mixing time at level \( \varepsilon \), for \( \varepsilon < 0 \) and \( 1 \leq p \leq \infty \), by

\[
t^{(p)}_{\text{mix}}(\varepsilon) = \inf \left\{ t \geq 0 : d^{(p)}(t) \leq \varepsilon \right\}.
\]

(2.25)

Note that the total variation distance is just half of the \( L^1 \) distance. For all \( x, y \in \Gamma \) and \( t \geq 0 \), we set

\[
h_t(x, y) = p_t(x, y) - \frac{1}{n}.
\]

(2.26)

Proposition 2.8. For \( 0 < \varepsilon < 1 \), we have, uniformly over all finite (connected) vertex-transitive graphs of degree \( d \) such that \( n = |\Gamma| \leq D^5 \),

\[
t^{(\infty)}_{\text{mix}}(\varepsilon) \asymp_{d, \varepsilon} t^{(1)}_{\text{mix}}(\varepsilon) \asymp_{d, \varepsilon} t_{\text{rel}} \asymp_d D^2.
\]

(2.27)
Proof. We already proved in Lemma 2.7 that \( t_{\text{rel}} \simeq_d D^2 \). By convexity, if \( 1 \leq p \leq q \leq \infty \), and \( t \geq 0, d^{(p)}(t) \leq d^{(q)}(t) \), and hence for \( 0 < \varepsilon < 1 \), we have \( \epsilon_{\text{mix}}^{(p)}(\varepsilon) \leq \epsilon_{\text{mix}}^{(q)}(\varepsilon) \). By [LP17, Lemma 20.11], we hence have \( d^{(\infty)}(t) \geq d^{(1)}(t) \geq e^{-t/t_{\text{rel}}} \) (since the total variation distance is exactly the half of the \( L^1 \) distance) and thus for \( 0 < \varepsilon < 1 \),

\[
\epsilon_{\text{mix}}^{(\infty)}(\varepsilon) \geq \epsilon_{\text{mix}}^{(1)}(\varepsilon) \geq \log(1/\varepsilon) t_{\text{rel}}.
\] (2.28)

Let us now prove that \( \epsilon_{\text{mix}}^{(\infty)}(\varepsilon) \lesssim t_{\text{rel}} \). By spectral estimates, we have for all \( t, s \geq 0 \),

\[
h_{t+s}(o, o) \leq e^{-s/t_{\text{rel}}} h_t(o, o).
\] (2.29)

Moreover, by Proposition 2.1, there is a constant \( C(d) \) such that uniformly over all finite connected vertex-transitive graphs of degree \( d \) such that \( n \leq D^5 \), \( h_{D^2}(o, o) \leq C(d) \). Hence, at time \( D^2 + i t_{\text{rel}} \), where \( i = \lfloor \log(C(d)/\varepsilon) \rfloor \),

\[
h_{D^2+i t_{\text{rel}}}(o, o) \leq e^{-i} h_{D^2}(o, o) \leq \varepsilon,
\] (2.30)

so, recalling that \( t_{\text{rel}} \leq D^2 \),

\[
\epsilon_{\text{mix}}^{(\infty)}(\varepsilon) \leq D^2 + \lfloor \log(C(d)/\varepsilon) \rfloor t_{\text{rel}} \leq D^2 (1 + d \lfloor \log(C(d)/\varepsilon) \rfloor),
\] (2.31)

concluding the proof.

Remark 2.9. Proposition 2.8 shows that when \( n \leq D^5 \), the relaxation time and the mixing time are both of order \( D^2 \). The explicit bounds (2.28) and (2.31) show in particular that there is no cutoff, i.e. that for some \( 0 < \varepsilon < 1 \), \( \frac{\epsilon_{\text{mix}}^{(\infty)}(\varepsilon)}{\epsilon_{\text{mix}}^{(1)}(\varepsilon)} \) does not converge to 1 as \( n = |\Gamma| \to \infty \). See [LP17, Chapter 18] more details on the cutoff phenomenon.

The following proposition is a classical result which can be traced back to [Ald82].

Proposition 2.10. Uniformly over all finite (connected) vertex-transitive graphs of degree \( d \) such that \( n = |\Gamma| \leq D^5 \), we have

\[
t_{\text{hit}} - t_{(\text{hit})} \simeq_d D^2.
\] (2.32)

Proof. Let \( \tau_i \) for \( 1 \leq i \leq 4 \) be as in [Ald82], and set \( \tau_5 := t_{\text{hit}} - t_{(\text{hit})} \). From [Ald82, Equation (17)], we have \( \tau_5 \leq \tau_2 \). On the other hand, we have

\[
\tau_3 = \max_{x,y \in \Gamma} \sum_{z \in \Gamma} \pi(z) |E_x T_z - E_y T_z| = 2 \max_{x,y \in \Gamma} \sum_{z \in \Gamma : E_x T_z \geq E_y T_z} \pi(z) (E_x T_z - E_y T_z).
\] (2.33)

Moreover, for any \( x, y, z \in \Gamma \), we have \( E_x T_z - E_y T_z \leq t_{\text{hit}} - E_y T_z \), and (by definition of \( t_{\text{hit}} \)) \( t_{\text{hit}} - E_y T_z \geq 0 \). We deduce from this that

\[
\tau_3 \leq 2 \max_{x,y \in \Gamma} \sum_{z \in \Gamma} \pi(z) (t_{\text{hit}} - E_y T_z) = 2\tau_5,
\] (2.34)

so we have proved that \( \tau_3/2 \leq \tau_5 \leq \tau_2 \). It finally follows from [Ald82, Theorem 5] and Proposition 2.8 that

\[
\tau_5 \simeq \tau_1 = \frac{1}{2} \epsilon_{\text{mix}}^{(1)} ((2\varepsilon)^{-1}) \simeq_d D^2.
\] (2.35)
Remark 2.11. Proposition 2.10 also holds if we replace \( t_{\text{hit}} = \mathbb{E}_x T_o \) by \( \mathbb{E}_{\alpha_o} T_o \), where \( \alpha_o \) is the quasi-stationary distribution (defined in (1.4)) associated to \( A = \{o\} \). This will be shown in Proposition 6.11.

Corollary 2.12. Under the assumptions of Theorem 1.2, we have when (DC) holds,

\[
t_{\text{hit}}((\log n) + s) = t_{\text{hit}}((\log n) + s + o(1)).
\]

Proof. From Proposition 2.10, we have

\[
t_{\text{hit}} = t_{\text{hit}} \left( 1 + O \left( \frac{D^2}{t_{\text{hit}}} \right) \right).
\]

Moreover, by [LP17, Proposition 1.19],

\[
t_{\text{hit}} = \max_{x,y} \mathbb{E}_x T_y \geq \max_x \mathbb{E}_x T_o^+ - 1 = \mathbb{E}_{\alpha_o} T_o^+ - 1 = n - 1.
\]

We finally deduce (using that (DC) holds for the last equality), that

\[
t_{\text{hit}}((\log n) + s) = t_{\text{hit}} \left( (\log n) + s + O \left( \frac{D^2 \log n}{n} \right) \right) = t_{\text{hit}} \left( (\log n) + s + o(1) \right).
\]

\[\Box\]

Proposition 2.13. Recall that for \( s \in \mathbb{R} \), \( t_s = t_{\text{hit}}((\log n) + s) \), and \( t_s^* \) is such that \( \mathbb{E}|U(t_s^*)| = e^{-s} \). Then under (DC), we have

\[
t_s^* = t_s + o(n).
\]

Proof. First observe that we have for every \( t \geq 0 \), setting \( \alpha = \alpha_o \) the quasi-stationary distribution associated to \( A = \{o\} \),

\[
\mathbb{E}|U(t)| = n \mathbb{P}_{\pi}(T_o > t) = n \frac{\mathbb{P}_{\pi}(T_o > t)}{\mathbb{P}_{\pi}(T_o > t)} \mathbb{P}_{\alpha}(T_o > t) = n \frac{\mathbb{P}_{\alpha}(T_o > t)}{\mathbb{P}_{\alpha}(T_o > t)} e^{-t/\mathbb{E}_{\alpha} T_o}.
\]

It follows, since \( t \mapsto \mathbb{E}|U(t)| \) is decreasing, that \( t_s^* \) is unique and satisfies

\[
t_s^* = \mathbb{E}_{\alpha} T_o \left( (\log n) + s - \log \left( \frac{\mathbb{P}_{\alpha}(T_o > t_s^*)}{\mathbb{P}_{\pi}(T_o > t_s^*)} \right) \right).
\]

Since (DC) holds, \( t_{\text{rel}} / t_{\text{hit}} \lesssim D^2/n = o(1/\log n) \) (by (2.16) and (2.38)). We deduce from the Aldous–Brown approximation (Theorem 1.5) that \( \mathbb{E}_{\alpha} T_o = \mathbb{E}_{\pi} T_o (1 + o(1/\log n)) \), and (using also that \( \log(1+x) \leq x \) for \( x > -1 \)) that \( \log \left( \frac{\mathbb{P}_{\pi}(T_o > t_s^*)}{\mathbb{P}_{\pi}(T_o > t_s^*)} \right) = o(1/\log n) = o(1) \). This implies

\[
t_s^* = t_s + o(t_{\text{hit}}).
\]

To conclude it suffices to show that \( t_{\text{hit}} = O(n) \). To see this, note that by using reversibility and the commute-time identity (see [LP17, Proposition 10.7]), we have for all \( x, y \in \Gamma \)

\[
\mathbb{E}_x(T_g) + \mathbb{E}_y(T_x) = d \mathcal{R}(x \leftrightarrow y)n,
\]

so it suffices to show that the effective resistance between vertices is uniformly bounded. However, writing \( \mathcal{R}_{\Gamma,2} := \max_{x,y \in \Gamma} \mathcal{R}(x \leftrightarrow y) \), we have from [TT20, Theorem 1.12], that

\[
\mathcal{R}_{\Gamma,2} \lesssim \frac{1}{d} + \frac{D^2 \log n}{n} = \frac{1}{d} + o(1) = O(1).
\]

This concludes the proof. \[\Box\]
A crucial part of the argument will be to obtain good bounds on the hitting time of a finite arbitrary set $A$ of fixed cardinality $k \geq 1$. Problems usually arise when some points in $A$ are relatively close to one another. The following proposition shows that under (DC), the expected hitting time of $A$ is always of order $n$.

**Proposition 2.14.** Let $A$ be a finite set of vertices of cardinality $k$ of $\Gamma$. Then, as $n \to \infty$,

$$\mathbb{E}_\pi T_A \gtrsim n,$$  \hfill (2.44)

and besides assuming (DC) we have

$$\mathbb{E}_\pi T_A \asymp n,$$  \hfill (2.45)

where the implicit constants depend on $k$ and $d$.

**Proof.** Averaging (2.49) with respect to $x$, and taking $y = o$, we have for every $t \geq 0$

$$\mathbb{P}_\pi(T_o \leq t) \leq e\mathbb{E}_\pi L_o(t + 1) = e(t + 1)/n.$$  \hfill (2.46)

It follows that for every $t \geq 1$, as $t + 1 \leq 2t$, and taking $t = n/(4ek)$,

$$\mathbb{P}_\pi(T_A \leq t) \leq k\mathbb{P}_\pi(T_o \leq t) \leq 2ekt/n \leq 1/2.$$  

By Markov’s inequality, we deduce

$$\mathbb{E}_\pi T_A \geq t\mathbb{P}_\pi(T_A > t) = t(1 - \mathbb{P}_\pi(T_A \leq t)) \geq t/2 \gtrsim n,$$

which concludes the proof of the lower bound.

The upper bound is straightforward, since $\mathbb{E}_\pi T_A \leq \mathbb{E}_\pi T_o \leq t_{\text{hit}}$, and we have already observed in the proof of Proposition 2.13 that under (DC), $t_{\text{hit}} = O(n)$.

2.3 Off-diagonal heat kernel bounds

Our first task will be to translate the on-diagonal bounds described in the previous section into off-diagonal bounds. A general upper bound can always be obtained with the Carne–Varopoulos inequality, but we will need a sharper recent result due Folz, see [Fol11, Corollary 1.2]. This general result takes as an input a time-dependent bound on the on-diagonal heat kernel and deduces from this a general off-diagonal bound. The result is actually quite general but we will only use it in the most simple case where the Lipschitz function is taken to be the graph distance, and the “volume” is measured with respect to cardinality.

Set $1/g(t)$ to be the function on the right hand side of (2.13), so that $1/g(t)$ is a bound on $p_t(o, o)$; that is,

$$1/g(t) = \begin{cases} \frac{C}{t^{1/2}} & \text{for } 1 \leq t \leq f(n)^2 \\ \frac{C}{t^{1/2}} & \text{for } f(n)^2 \leq t \leq D^2. \end{cases}$$  \hfill (2.47)

(Recall that if $m = 2$ then $R = f(n)$.) A consequence of his result is the following inequality. Assume that $f(n) < D$. Then there exists a constant $c > 0$ such that for every $x \neq y \in \Gamma$ and $d(x, y) \leq t \leq D^2$,

$$p_t(x, y) \lesssim \frac{1}{g(t)} \exp \left( -c \frac{d(x, y)^2}{t} \right),$$  \hfill (2.48)

where the implicit constant in $\lesssim$ depends only on the degree bound. For smaller values of $t$, we will simply bound the heat kernel via the Carne–Varopoulos bound,

$$p_t(x, y) \lesssim \exp(-cd(x, y)^2/t) \leq \exp(-cd(x, y))$$
when \( t \leq d(x, y) \).

We will use (2.48) to get upper bounds on the off-diagonal heat kernel, and therefore by integrating, on the expected local time at a given vertex. In turn, this can be used to upper bound the probability to visit a vertex \( y \) far away from \( x \) in a relatively short time, via the following elementary lemma.

**Lemma 2.15.**

\[
\mathbb{E}_x L_y (t+1) \geq \frac{1}{e} \mathbb{P}_x (T_y \leq t). \tag{2.49}
\]

**Proof.** Let us define the event \( E \) by

\[
E := \{ \text{the walk stays for time at least 1 at } y \text{ just after } T_y \}. \tag{2.50}
\]

Then we have

\[
\mathbb{E}_x L_y (t+1) \geq \mathbb{E}_x (L_y (t+1) | T_y \leq t, E) \mathbb{P}_x (T_y \leq t, E) \geq \mathbb{P}_x (T_y \leq t) \mathbb{P} (E).
\]

This proves the lemma. \( \square \)

We now combine the above ideas to get the following bounds:

**Proposition 2.16.** Let \( 1 \leq \delta \leq D/2 \).

(a) Uniformly over all \( x, y \in \Gamma \) such that \( d(x, y) \geq \delta \), and \( t \geq 1 \),

\[
\mathbb{E}_x L_y (t) \lesssim \frac{1}{\delta^2} + \frac{\log(D/\delta)}{f(n)} + \frac{t}{\delta D^2} + t\frac{1}{\delta D^2}. \tag{2.51}
\]

(b) If we moreover assume that \( D^5 \leq n \), the bound becomes

\[
\mathbb{E}_x L_y (t) \lesssim \frac{1}{\delta^3} + \frac{t}{D^3}. \tag{2.52}
\]

Note that by Lemma 2.15, the same bounds hold also for \( \mathbb{P}_x (T_y \leq t) \).

**Proof.** Let \( x, y \in \Gamma \). Since for all \( s \), \( p_s (x, y) \leq p_s (o, o) \) (see [AF, Lemma 20, in particular Equation (3.60)])

\[
\mathbb{E}_x L_y (t) = \int_0^t p_s (x, y) ds \leq \int_0^{d(x,y)} p_s (x, y) ds + \int_{d(x,y)}^{d(x,y)^2} p_s (x, y) ds + 1_{t \geq d(x,y)^2} \int_{d(x,y)^2}^t p_s (o, o) ds.
\]

The first integral is by the Carne–Varopoulos bound smaller or equal to \( \delta \exp(-c\delta) \lesssim 1/\delta \). Let us consider the second integral. By (2.48), we have

\[
\int_{d(x,y)^2}^{d(x,y)^2} p_s (x, y) ds \lesssim \int_0^{d(x,y)^2} \left( s^{-3/2} + \frac{1}{f(n)s} \right) \exp(-c\delta^2/s) ds.
\]

Now, studying the function \( t \mapsto t^{-b} \exp(-A/t) \) we see that this is maximised at \( t = A/b \) and so is always \( < (A/b)^{-b} \). Applying this with \( b = 1 \) and \( b = 3/2 \) as well as \( A \propto \delta^2 \), we get

\[
\int_0^{d(x,y)^2} p_s (x, y) ds \lesssim \delta^2 \frac{1}{\delta^3} + \delta^2 \frac{1}{\delta^2 f(n)} \leq \frac{1}{\delta} + \frac{1}{f(n)}.
\]
For the third integral, using Corollary 2.5, (a), we immediately get
\[ \int_t^1 p_s(o,o)ds \lesssim \int_0^{D^2} \left( \frac{1}{s^{3/2}} + \frac{1}{f(n)s} \right) ds + 1_{t \geq D^2} \int_{D^2}^t \left( \frac{1}{D^3} + \frac{1}{n} \right) ds \]
\[ \lesssim \frac{1}{\delta} + \frac{\log(D/\delta)}{f(n)} + \frac{t}{n} + \frac{t}{D^3}. \]

We finally have, as \( 1 \leq \delta \leq D/2 \),
\[ \mathbb{E}_x L_y(t) \lesssim \frac{1}{\delta} + \frac{\log(D/\delta)}{f(n)} + \frac{t}{n} + \frac{t}{D^3} + \frac{1}{D^3} \]
\[ \lesssim \frac{1}{\delta} + \frac{\log(D/\delta)}{f(n)} + \frac{t}{n} + \frac{t}{D^3}, \quad (2.54) \]
which proves (a). The second bound is proved the same way, using part (b) of Corollary 2.5 and taking \( 1/g(t) = C/t^{5/2} \) for \( 1 \leq t \leq D^2 \).

**Corollary 2.17.** Let \( t = t(n) = o \left( \min(n,D^3) \right) \) and fix a sequence \( \omega_n \to \infty \). Then, uniformly over \( x,y \) with \( d(x,y) \geq \omega_n \),
\[ \mathbb{E}_x L_y(t) \to 0. \quad (2.55) \]
In particular,
\[ \mathbb{P}_x(T_y \leq t) \to 0. \quad (2.56) \]

**Proof.** It follows immediately from part (a) of Proposition 2.16, Lemma 2.15, and the assumption \( \log n \ll f(n) \).

When we consider the diagonal case we get a similar bound, but this time of order 1; in this case we can therefore afford to consider times that are as big as the volume. Such a bound is also a signature of our local transience condition. As mentioned in the introduction, Tessera and Tointon [TT20] proved (2.57). We present its proof for the sake of completeness.

**Proposition 2.18.** Let \( \Gamma_n = (|V_n|, E_n) \) be a sequence of finite vertex-transitive graphs of uniformly bounded degree, and diameters \( D = D_n \) satisfying \( D^2 \asymp |V_n|/f(n) = n/f(n) \). If \( f(n) \gtrsim \log n \) then
\[ \mathbb{E}_o L_o(n) = O(1). \quad (2.57) \]
If \( f(n) \gg \log n \) then
\[ \lim_{s \to \infty} \lim_{n \to \infty} \max_{o \in V_n} \mathbb{E}_o(L_o(t_{\text{mix}}(1/4)) - L_o(s)) = 0. \quad (2.58) \]

**Proof.** We first prove (2.57). We start as in the proof of Proposition 2.16, cutting the integral at time 1 instead of time \( d(x,y)^2 \). At time \( t = D^2 \), we have
\[ \mathbb{E}_o L_o(D^2) = \int_0^1 p_s(o,o)ds + \int_{D^2}^1 p_s(o,o)ds \lesssim 1 + \left( 1 + \frac{\log D}{f(n)} + \frac{t}{n} + \frac{t}{D^3} \right) = O(1). \quad (2.59) \]
Recall from (2.16) that \( t_{\text{rel}} \leq dD^2 \). Furthermore, we have, for every \( t \geq D^2 \), (see for instance [LP17, Lemma 4.18 and Equation 20.18])
\[ p_t(o,o) - \frac{1}{n} \leq \exp \left( -\frac{t - D^2}{dD^2} \right) p_{D^2}(o,o). \quad (2.60) \]
Hence,
\[ \int_{D^2}^n p_s(o,o) ds = \int_{D^2}^n \left( p_s(o,o) - \frac{1}{n} \right) ds + \int_{D^2}^n \frac{1}{n} ds \]
\[ \lesssim \frac{1}{n} \int_{D^2}^\infty \exp \left( -\frac{t}{dD^2} \right) ds + 1 \]
\[ = O(1), \]
which concludes the proof. We now prove (2.58). By Corollary 2.5 we have that \( t_{\text{mix}}(1/4) \ll n \).
The remainder of the proof is analogous to that of (2.57) and hence omitted. \( \square \)

With this, it is easy to deduce that any point (distinct from the starting point) can be avoided with positive probability up to a time slightly smaller than the volume.

**Proposition 2.19.** There exists a constant \( \beta > 0 \) such that for \( t = t(n) \ll n \) and uniformly over \( x, y \) distinct points of \( \Gamma \), we have
\[ P_x(T_y > t) \geq \beta + o(1). \] (2.61)

**Proof.** This can be proved using [TT20, Theorem 1.8], but we give a simple, self-contained argument. We argue by contradiction and assume (without loss of generality, perhaps taking a subsequence if needed) that there exists \( x, y \) such that \( P_x(T_y \leq t) = 1 - o(1) \). As the law of \( T_x \) starting from \( y \) is the same as the law of \( T_y \) starting from \( x \) by symmetry (see the proof of [Ald89, Proposition 2]), this implies, using the Markov property, that we also have
\[ P_x(T_x^+ \leq 2t) \geq P_x(T_y \leq t)P_y(T_x \leq t) = 1 - o(1). \] (2.62)

By the same argument, for every integer \( m \geq 1 \),
\[ P_x(\text{the walk returns to } x \text{ at least } m \text{ times before time } 2mt) \geq P_x(T_x^+ \leq 2t)^m = 1 - o(1). \] (2.63)
Since \( t = t(n) = o(n) \), we also have \( 2mt = o(n) \) and hence we deduce that \( E_0(L_0(n)) \geq m(1 - o(1)) \) and so is unbounded. This contradicts Proposition 2.18. \( \square \)

**2.4 Exponential approximation of the hitting time**

The aim of this subsection is to approximate the hitting time of a fixed set \( A \) of cardinality \( k \geq 1 \) (starting from stationarity) by an exponential random variable with mean \( E_x T_A \). The key tool to do this will be to consider the quasi-stationary distribution \( \alpha_A \), for which the hitting distribution is exactly exponential. Such an idea goes back to the work of Aldous and Brown [AB92].

If \( \emptyset \neq A \subseteq \Gamma \), we will denote by \( \alpha = \alpha_A \) its quasi-stationary distribution. This is a distribution whose support is contained in \( B := A^c \), and is unique and of support \( B \) when \( P_a[T_b < T_A] \) for all \( a, b \in B \). In fact, it is in principle possible for the set \( A \) to disconnect \( \Gamma \setminus A \) into several connected components, in which case there are multiple quasi-stationary distribution. However, under our diameter assumption (in fact, as soon as \( n \geq D^2 \) say), only one of these components may be of macroscopic size as \( n \to \infty \) while \( k \geq 1 \) is fixed (for a fixed \( k \), under our diameter condition, for all sufficiently large \( n \) the rest of the components are of bounded sizes, where the bound on their sizes depends only on \( k \) and the degree). In such a case we therefore consider \( \alpha_A \) to be the quasi-stationary distribution associated with that component, and we do so without further commenting on this case. We start with an elementary lemma to bound the error term in the Aldous–Brown approximation.
**Lemma 2.20.** Uniformly over all sets $A \subset \Gamma$ of fixed size $k \geq 1$, we have
\[
\frac{t_{\text{rel}}}{E_\alpha T_A} \lesssim \frac{D^2}{n},
\]
(2.64)
where $\alpha = \alpha_A$ denotes the quasistationary distribution associated to $A$.

**Proof.** From (2.16), we have $t_{\text{rel}} \lesssim D^2$. Moreover, from Theorem 1.5, $E_\alpha T_A \geq E_\pi T_A$. Finally, from Proposition 2.14, $E_\pi T_A \gtrsim n$. This concludes the proof. \qed

The following lemma shows that the quantities $E_\pi T_A$ and $E_\alpha T_A$ are very similar.

**Lemma 2.21.** Let $k \in \mathbb{N}^*$, and $A$ a finite subset of $\Gamma$ of cardinality $k$. Then
\[
E_\pi T_A = E_\alpha T_A \left(1 + O \left(\frac{1}{f(n)}\right)\right).
\]
(2.65)

**Proof.** This follows immediately from the Aldous–Brown approximation (Theorem 1.5) and Lemma 2.20. \qed

We now observe that $P_\pi (T_A > t)$ and $P_\alpha (T_A > t)$ are also very similar.

**Lemma 2.22.** Let $k \in \mathbb{N}^*$ and $A$ be a finite subset of $\Gamma$ of cardinality $k$. Then for every $t \in \mathbb{R}^+$,
\[
P_\alpha (T_A > t) = P_\pi (T_A > t) \left(1 + O \left(\frac{1}{f(n)}\right)\right),
\]
(2.66)
where the implicit constants do not depend on $t$, nor on $A$ (among sets of same cardinality $k$).

**Proof.** This follows again immediately from the Aldous–Brown approximation (Theorem 1.5) and Lemma 2.20. \qed

**Remark 2.23.** The most precise approximation stated in [AB92] (as Equation (1) and Theorem 3) is the following. For every $t \in \mathbb{R}^+$,
\[
P_\alpha (T_A > t) \left(1 - \frac{t_{\text{rel}}}{E_\alpha T_A}\right) \leq P_\pi (T_A > t) \leq P_\alpha (T_A > t) (1 - \pi(A)).
\]
(2.67)
Combining those two lemmas leads to an exponential approximation of $P_\pi (T_A > t_{(s)})$, which will later be useful to rewrite the moments of the uncovered set at time $t_{(s)}$, which we recall from the introduction is given by, for $s \in \mathbb{R}$ fixed,
\[
t_{(s)} = E_\pi T_o (\log(n) + s).
\]

**Proposition 2.24.** Let $A$ be a finite subset of $\Gamma$ of cardinality $k \geq 1$. Then
\[
P_\pi (T_A > t_{(s)}) = \exp \left(-\frac{E_\pi T_o}{E_\pi T_A} (\log(n) + s)\right) (1 + o(1)),
\]
(2.68)
where the $o(1)$ is uniform over all the $A$ of cardinality $k$. 20
Proof. From Proposition 2.22,
\[ P_\pi (T_A > t(\langle s \rangle)) = P_\alpha (T_A > t(\langle s \rangle)) (1 + o(1)). \] (2.69)

Now, by the definition of quasi-stationarity and Proposition 2.21,
\[ P_\alpha (T_A > t(\langle s \rangle)) = \exp \left(- \frac{\langle s \rangle}{E_\alpha T_A} \right) \left(1 + O \left( \frac{1}{f(n)} \right) \right). \] (2.70)

Noting that \( \frac{E_\pi T_o}{E_\pi T_A} = \Theta(1) \) and recalling that \( t(\langle s \rangle) = E_\pi T_o (\log(n) + s) \), this leads to
\[ P_\alpha (T_A > t(\langle s \rangle)) = \exp \left(- \frac{E_\pi T_o}{E_\pi T_A} (\log(n) + s) + O (\log(n) + s) \right). \] (2.71)

As \( f(n) \gg \log n \), this concludes the proof. \( \square \)

Since exponential functions are approximately linear for small times, we obtain a relation between \( P_\pi (T_A > t) \) and \( E_\pi T_A \) for not too large values of \( t \).

Proposition 2.25. Let \( k \in \mathbb{N}^* \), let \( A \) be a finite subset of vertices of cardinality \( k \), and \( t \ll n \). Then
\[ E_\pi T_A = t \frac{P_\pi (T_A \leq t)}{P_\pi (T_A \leq t)} \left(1 + O \left( \frac{n}{tf(n)} \right) + O \left( \frac{t}{f(n)} \right) \right). \] (2.72)

Moreover, the \( O(\cdot) \) are uniform over such \( A \) and \( t \).

Proof. Let \( t \geq 0 \). From Lemma 2.22, which is true uniformly for all \( t \geq 0 \),
\[ P_\pi (T_A > t) = \exp \left(- \frac{t}{E_\pi T_A} \right) \left(1 + O \left( \frac{1}{f(n)} \right) \right). \]

Suppose that \( t \ll n \), so that \( t/E_\alpha(T_A) = o(1) \). Then by Taylor expansion of the exponential,
\[ P_\pi (T_A > t) = \left(1 - \frac{t}{E_\alpha(T_A)} + O \left( \frac{t}{E_\alpha(T_A)} \right)^2 \right) \left(1 + O \left( \frac{1}{f(n)} \right) \right) \]
\[ = 1 - \frac{t}{E_\alpha(T_A)} + O \left( \frac{t}{E_\alpha(T_A)} \right)^2 + O \left( \frac{1}{f(n)} \right). \]

Hence, recalling that \( E_\alpha(T_A) \asymp n \), we get
\[ P_\pi(T_A \leq t) = \frac{t}{E_\alpha(T_A)} \left(1 + O \left( \frac{t}{f(n)} \right) \right). \] (2.73)

The following corollary will be particularly useful in what follows.

Corollary 2.26. Uniformly over \( A \subset \Gamma \) of size \( k \) and \( t \ll n \), we have
\[ \frac{E_\pi T_o}{E_\pi T_A} = \frac{P_\pi (T_A \leq t)}{P_\pi (T_o \leq t)} \left(1 + O \left( \frac{n}{tf(n)} \right) + O \left( \frac{t}{f(n)} \right) \right). \] (2.74)
Remark 2.27. The ratio $q_A := \frac{E_\pi T_o}{E_\pi T_A}$ will play a crucial role in our analysis. In what follows we will want to apply Corollary 2.26 with $t = D^2 \sqrt{f(n)} = n/\sqrt{f(n)}$. Hence for this choice of $t$ we obtain:

$$\frac{E_\pi T_o}{E_\pi T_A} = \frac{\mathbb{P}_\pi(T_A \leq t)}{\mathbb{P}_\pi(T_o \leq t)} \left(1 + O\left(\frac{1}{\sqrt{f(n)}}\right)\right).$$

(2.75)

As we will see, this approximation is very useful, as the hitting probabilities $\mathbb{P}_\pi(T_A \leq t)$ are easier to estimate than the expectations $E_\pi T_A$. This approximation is particularly good when $n \gg D^2 (\log n)^2$, i.e. $f(n) \gg (\log n)^2$, as in this case error term will be small:

$$\frac{E_\pi T_o}{E_\pi T_A} = \frac{\mathbb{P}_\pi(T_A \leq t)}{\mathbb{P}_\pi(T_o \leq t)} \left(1 + o\left(\frac{1}{\log n}\right)\right).$$

(2.76)

As we will soon see (see (3.7)), an error term of $o(1/\log n)$ is indeed sufficient for our purpose. However, when we merely have $f(n) \gg \log n$, this approximation is not precise enough anymore, and more arguments will be required. We will show two related but distinct ways to deal with this difficulty in the general case. The first approach will be to work directly with the expectations $E_\pi T_A$, and through a bootstrap argument. It is technical but rather elementary. This will be done in Section 3.7. The second will be to improve on the results of Aldous and Brown [AB92], so that the approximation (2.76) still holds, even when we merely have $f(n) \gg \log n$. This simplifies some aspects of the proof (essentially, the base case of bootstrap argument is simpler, although the bootstrap itself remains needed). This second approach will be carried out in Section 5.

3 Convergence of the uncovered set

We now start the proof of the main result of this paper, and so recall our standing assumptions where we fix $s \in \mathbb{R}$, and assume that $f(n) \gg \log n$. We also recall that $Z = Z_s$ denotes the size of the uncovered set at time $t(s)$, and we want to prove that for any fixed $k \geq 1$,

$$\mathbb{E}_\pi \left[Z_{\downarrow k}\right] \to e^{-ks},$$

(3.1)

as $n \to \infty$, where we recall that $z_{\downarrow k} = z(z-1)\cdots(z-k+1)$. Indeed, once the convergence (3.1) is known, a standard application of the method of moments shows that $Z_s$ converges in distribution to a Poisson random variable with parameter $e^{-s}$: that is,

$$\mathbb{P}(Z_s = k) \to \exp(-e^{-s}) \frac{e^{-ks}}{k!}.$$  

(3.2)

Consequently,

$$\mathbb{P}(\tau_{cov} \leq t(s)) = \mathbb{P}(Z_s = 0) \to \exp(-e^{-s}),$$

as desired.

Our first task will be to rewrite these factorial moments in terms of the ratio

$$q_A = \frac{E_\pi T_o}{E_\pi T_A}$$

(3.3)

mentioned in the previous section, and which will play a crucial role in all that follows.
3.1 Strategy

We recall that \( t_{(i)} = E \pi T_{o} (\log(n) + s) \). As \( Z = \sum_{x \in \Gamma} I_{T_{x} > t_{(i)}} \), we can see that

\[
Z^{ik} = \sum_{A \in \mathcal{A}} I_{T_{A} > t_{(i)}},
\]

where \( \mathcal{A} \) denotes the set of all (ordered) \( k \)-tuples of pairwise distinct elements of \( \Gamma \). Note that \( \mathcal{A} \) depends on \( k \) but we do not indicate this in the notation for ease of readability, and since \( k \geq 2 \) is completely fixed throughout the proof. (We made here another small abuse of notation, where we identify the sets \( A \in \mathcal{A} \) with a subset of \( \Gamma \).)

We can hence rewrite the expectation as

\[
E_{\pi} \left[ Z_{\downarrow k} \right] = \sum_{A \in \mathcal{A}} P_{\pi} \left( T_{A} > t_{(i)} \right).
\]

Now, let us consider the quantity \( q_{A} \) defined in (3.3) for \( A \in \mathcal{A} \). Applying Proposition 2.24, we get the approximation

\[
E_{\pi} \left[ Z^{ik} \right] (1 + o(1)) = \sum_{A \in \mathcal{A}} \exp \left( -E_{\pi} T_{o} (\log(n) + s) \right) = \sum_{A \in \mathcal{A}} n^{-q_{A}} e^{-q_{A}s}.
\]

Let us first give some intuition on the rest of the proof. Our aim will be to get a good approximation of \( q_{A} \). If all the points of \( A \) are “far away” from each other, then we expect that hitting \( A \) occurs on average \( k \) times faster as hitting a single point. This suggests that \( q_{A} \) is approximately equal to \( k \).

Suppose for a moment that we were given the following uniform bound: for all \( A \in \mathcal{A} \),

\[
q_{A} = k + o \left( \frac{1}{\log n} \right).
\]

Then

\[
n^{-q_{A}} = n^{-k}(1 + o(1))
\]

and since \( |\mathcal{A}| = n^{k}(1 + o(1)) \), we would immediately get

\[
E_{\pi} \left[ Z^{ik} \right] = (1 + o(1)) \sum_{A \in \mathcal{A}} n^{-q_{A}} e^{-q_{A}s} = (1 + o(1)) n^{k} n^{-k} e^{-ks} = (1 + o(1)) e^{-ks},
\]

which is exactly what we want. Therefore, for the proof of the main theorem it would suffice to prove (3.7). Of course, such an approximation is wrong if some of the points of \( A \) are too close to one another. The rest of the proof will develop arguments to show that the sets \( A \) for which (3.7) fails are not too numerous (and of course we will also need to control \( q_{A} \) in those cases). For now, it suffices to point out that as soon as (3.7) holds, we do not need to study further the properties of \( A \).

Remark 3.1. Note that for \( k = 1 \), \( q_{A} = 1 \) for every singleton \( A \subset \Gamma \), which proves that

\[
E_{\pi} Z \xrightarrow{n \to \infty} e^{-s}.
\]

In what follows we will therefore assume that \( k \geq 2 \).

Initially it is not clear whether one needs to worry about the whole spectrum of possibilities for the mutual distances between points of \( A \) or if a cruder bound on the minimum distance between points of \( A \) is sufficient to distinguish between the good and the bad cases. As it turns
out, we are rather lucky that this cruder strategy is sufficient. We will therefore introduce the following quantity:

$$\text{mindist}(A) := \min_{x, y \in A, x \neq y} d(x, y).$$  \hfill (3.9)

It will also be convenient to introduce for $1 \leq \delta \leq D$ the notation:

$$A_\delta = \{ A \in \mathcal{A} : \text{mindist}(A) \leq \delta \},$$  \hfill (3.10)

as well as the partial sums:

$$S(\delta) = \sum_{A \in A_\delta} n^{-q_A} e^{-q_A s}.$$  \hfill (3.11)

With these notations, our aim is to show that

$$S(D) \to e^{-ks}.$$  

We record here a simple argument to say that if we assume (3.7) for a class of sets $A$ with $\text{mindist}(A) \ll D$ and that the remaining contribution is negligible, then we have the desired conclusion. Although intuitively clear, this argument will appear several times in the next subsections, so we prefer to state it here once and for all.

**Lemma 3.2.** Fix a sequence $\delta = \delta(n)$ such that $1 \leq \delta \ll D$. Suppose that

$$S(\delta) = o(1)$$  \hfill (3.12)

and that (3.7) is fulfilled uniformly for all $A \notin A_\delta$. Then

$$\mathbb{E}_\pi [Z^{\delta}] = e^{-ks}(1 + o(1)).$$  \hfill (3.13)

**Proof.** Let $V(r)$ denote the volume of a ball of radius $r$. Observe that for $r \geq 1$,

$$V(r) \leq \frac{3r}{D} n$$  \hfill (3.14)

because we can fit $\lfloor D/(2r+1) \rfloor$ disjoint balls of radius $r$ in $\Gamma$. Consequently, since $\delta = o(n)$, it follows that $V(\delta) = o(n)$ and therefore

$$|A_\delta| = o(n^k).$$  \hfill (3.15)

We therefore have

$$S(D) = S(\delta) + \sum_{A \in A \setminus A_\delta} n^{-q_A} e^{-q_A s}$$

$$= o(1) + n^k (1 - o(1)) n^{-k} e^{-ks} (1 + o(1))$$

$$= e^{-ks} (1 + o(1)),$$

and (3.6) allows to conclude.

We also remark that we always trivially have

$$q_A \leq k + o(1),$$  \hfill (3.16)

by using (2.75) and a union bound $\mathbb{P}_\pi(T_A < t^*) \leq k \mathbb{P}_\pi(T_A < t^*)$ on the numerator. Such an upper bound is useful since $q_A$ appears not only in the term $n^{-q_A}$ but also in the term $e^{-q_A s}$, while $s \in \mathbb{R}$ can be negative.
3.2 Organisation of the section

The rest of the section is organised as follows.

- In Section 3.3 we simplify a set $A$ into a modified set in which points are sufficiently far away from each other, controlled by a threshold $\delta$ which we can choose arbitrarily. We call this the $\delta$-skeleton of the set $A$. We then derive estimates for the probabilities to hit the skeleton within a suitably chosen timescale (and thus on $q_A$) which will be useful throughout.

- In Section 3.4 we show that if $\text{mindist}(A) \leq (\log n)^{1/2}$ (the “microscopic case”) then $q_A \geq k - 1 + \beta \gamma$ for some $\beta > 0$. Such a bound is sufficient to neglect the overall contribution of such sets to (3.6).

- In Section 3.5, we make the strong assumption that $n \geq D^5$ (roughly speaking, $\Gamma$ is at least five-dimensional, so that we are in the “high-dimensional case”). In that case, we complement the results of the previous section by sharper estimates on $q_A$ when $\text{mindist}(A) \geq (\log n)^{1/2}$, and show directly that (3.7) holds. This concludes the proof of the main theorem in the high-dimensional case.

- In Section 3.6, we suppose $(\log n)^2 \ll f(n)$ and $D^5 \leq n$. This is the “intermediate regime” where $\Gamma$ is no more than five-dimensional but we assume slightly more than the optimal condition $f(n) \gg \log n$. In that case, we introduce an intermediary scale, the “mesoscopic scale”, where $(\log n)^{1/2} \leq \text{mindist}(A) \leq D^{1/2}$, and show in Proposition 3.10 that $q_A \geq k + o(1)$ and therefore the overall contribution to (3.6) of mesoscopic sets is negligible. The remaining (macroscopic) scales are handled in Proposition 3.11.

- Finally, we handle the most delicate case where $f(n)$ is only assumed to be $\gg \log n$ in Sections 3.7 and 3.8. This is done using an elaborate bootstrap argument which will be detailed later.

3.3 Skeleton of a set

We will need to group the points of $A \in \mathcal{A}_\delta$ into subsets of points which are close to one another. Let us define an equivalence relation which will allow us to realize such partitions in a convenient way.

**Definition 3.3.** Let $A \subset \Gamma$ and $1 \leq \delta \leq D$. We say that two points $x, y \in A$ are $(A, \delta)$-linked if there exist an integer $r \geq 2$ and a sequence $x = x_1, \ldots, x_r = y$ of points in $A$ such that for all $1 \leq i \leq r - 1$, $d(x_i, x_{i+1}) \leq \delta$. We will write $A$ under the form

$$A = A_1 \sqcup \ldots \sqcup A_{\ell_\delta},$$

where the $A_i$ are the $(A, \delta)$-connected components of $A$, and $\ell_\delta(A)$ is the number of such components.

**Remark 3.4.** It is worth noting that some points can be closer to points in different connected components than to some points in their own components. For example on $\mathbb{Z}$, take $\delta = 2$ and $A = \{0, 2, 4, 7\}$. The partition is then $\{\{0, 2, 4\}, \{7\}\}$, though 4 is closer to 7 than to 0.

Let $A \in \mathcal{A}_\delta$. Then, since at least two points of $A$ are in the same component (we assume without loss of generality that $|A_1| \geq 2$), we have $\ell_\delta(A) \leq k - 1$. Let now $t^* = t^*(n)$ be (a sequence of times) such that $D^2 \ll t^* \ll n$. From Corollary 2.26 we have in particular

$$q_A = \frac{P_x(T_A \leq t^*)}{P_x(T_o \leq t^*)} \left(1 + o(1)\right),$$

(3.18)
where the $o(1)$ is explicit and is of the form $O(n/(t^* f(n))) + O(t^*/n)$.

Let $\gamma$ be the set of points visited by a random walk starting from the uniform distribution $\pi$ and run for some time $t^*$, which is independent of our underlying walk. With this notation and Fubini’s theorem we can rewrite $\mathbb{P}(T_o \leq t^*) = \mathbb{P}(o \in \gamma)$ and $\mathbb{P}(T_A \leq t^*) = \mathbb{P}(A \cap \gamma \neq \emptyset) = \mathbb{P}(\bigcup_{i=1}^{f(A)} \{A_i \cap \gamma \neq \emptyset\})$. It is then natural to define and study, for $1 \leq i \leq \ell(A)$, the events

$$E_i = \{A_i \cap \gamma \neq \emptyset\}.$$  

(3.19)

What we want to show is that as the sets $A_i$ are quite far from one another, the event $E_i$ are essentially independent.

**Proposition 3.5.** With the same notations as above, we have

$$\frac{\mathbb{P}(E_i \cap E_j)}{\mathbb{P}(T_o < t^*)} \lesssim \max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*).$$  

(3.20)

**Proof.** First observe that for all $x, y \in \Gamma$, if $T_x$ is the hitting of $x$ by $\gamma$, then

$$\mathbb{P}\{x, y \subset \gamma, T_x < T_y\} = \mathbb{P}(x \in \gamma) \mathbb{P}(T_x < T_y | T_x < t^*) \leq \mathbb{P}(x \in \gamma) \mathbb{P}_x(T_y < t^*)$$

by the Markov property. On the other hand, $\mathbb{P}_x(T_y < t^*) = \mathbb{P}_y(T_x < t^*)$ by symmetry, so altogether,

$$\mathbb{P}\{x, y \subset \gamma\} \leq 2\mathbb{P}(o \in \gamma) \mathbb{P}_x(T_y < t^*).$$

Therefore,

$$\mathbb{P}(E_i \cap E_j) = \mathbb{P}\left(\bigcup_{x, y \in A_i \cap A_j} \{x, y \subset \gamma\}\right)$$

$$\leq |A_i| |A_j| \max_{x,y \in \Gamma} \mathbb{P}\{x, y \subset \gamma\}$$

(3.21)

$$\leq 2k^2 \mathbb{P}(o \in \gamma) \max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*).$$

(3.22)

The proof is concluded by noting that $\max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*) = \max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*)$.  

(3.23)

Let us conclude this subsection with a useful lower bound on $q_A$.

**Proposition 3.6.**

$$q_A - \frac{\ell(A)}{\sum_{i=1}^{f(A)} \mathbb{P}(E_i)} \geq - \left(\max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*) + \frac{n}{t^* f(n)} + \frac{t^*}{n}\right).$$

(3.24)

In particular, if $\delta < \text{mindist}(A)$ and $D^2 \log n \ll t^* \ll n / \log n$ we have

$$q_A - k \geq - \max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t^*) + o\left(\frac{1}{\log n}\right).$$

(3.25)
Proof. From the Bonferroni inequality, we have

\[
\mathbb{P}_\pi(T_A < t^*) = \mathbb{P} \left( \bigcup_{i=1}^{\ell_\delta(A)} E_i \right) \geq \sum_{i=1}^{\ell_\delta(A)} \mathbb{P}(E_i) - \sum_{1 \leq i < j \leq \ell_\delta(A)} \mathbb{P}(E_i \cap E_j). \tag{3.26}
\]

Dividing on both sides by \(\mathbb{P}_\pi(T_o < t^*)\), we obtain

\[
\frac{\mathbb{P}_\pi(T_A < t^*)}{\mathbb{P}_\pi(T_o < t^*)} - \sum_{i=1}^{\ell_\delta(A)} \frac{\mathbb{P}(E_i)}{\mathbb{P}(T_o < t^*)} \geq - \sum_{1 \leq i < j \leq \ell_\delta(A)} \frac{\mathbb{P}(E_i \cap E_j)}{\mathbb{P}(T_o < t^*)}. \tag{3.27}
\]

The result then follows from Corollary 2.26 and Proposition 3.5.

3.4 Microscopic scale

The aim of this subsection is to show that the contribution of the \(A \in \mathcal{A}\) which have two very close points is negligible. More precisely, we set \(\delta_{\text{micro}} = (\log n)^{1/2}\), and we want to show that

\[
S(\delta_{\text{micro}}) = o(1). \tag{3.28}
\]

This will be proved in Proposition 3.8. First, we combine Proposition 3.6 with Corollary 2.17 and Proposition 2.19 to get a lower bound on \(q_A\).

**Proposition 3.7.** There exists \(\beta' > 0\) independent of \(n\) such that for all \(n\) sufficiently large, we have, uniformly over all \(A \in \mathcal{A}_{\delta_{\text{micro}}},\)

\[
q_A \geq \ell_{\delta_{\text{micro}}}(A) + \beta'. \tag{3.29}
\]

**Proof.** To lighten notations, we will write \(\delta\) for \(\delta_{\text{micro}}\) in this proof. Let \(t^* = t^*(n)\) such that \(D^2 \ll t^* \ll \min(n, D^3)\). Applying the lower bound on \(q_A\) proven in Proposition 3.6, together with Corollary 2.17, we have,

\[
q_A - \sum_{i=1}^{\ell_\delta(A)} \frac{\mathbb{P}(E_i)}{\mathbb{P}(T_o < t^*)} \geq o(1). \tag{3.30}
\]

Let \(x, y\) be distinct points of \(A_1\) such that \(d(x, y) \leq \delta\) and \(E'_1\) be the event \(\{x \in \gamma\} \cup \{y \in \gamma\}\). Then we have by symmetry,

\[
\mathbb{P}(E'_1) = \mathbb{P}(x \in \gamma) + \mathbb{P}(y \in \gamma) - \mathbb{P}(\{x, y\} \subset \gamma) = 2\mathbb{P}(o \in \gamma) - \mathbb{P}_\pi(T_x < t^*, T_y < t^*) \tag{3.31}
\]

Now observe that

\[
\mathbb{P}_\pi(T_x < t^*, T_y < t^*) \leq \mathbb{P}_\pi(T_{(x,y)} < t^*) (1 - \beta) \tag{3.32}
\]

where \(\beta\) is the constant appearing in Proposition 2.19, \(T_{(x,y)}\) is the hitting time of the set \(\{x, y\}\) and we have used the strong Markov property at this time (together with Proposition 2.19). Combining (3.31) and (3.32) together we get

\[
\mathbb{P}(E'_1)(2 - \beta) \geq 2\mathbb{P}(o \in \gamma),
\]
so that
\[ \mathbb{P}(E_1) \geq \mathbb{P}(E'_1) \geq \frac{2}{2-\beta} \mathbb{P}_\pi(T_o < t^*). \]

Therefore we have proved that
\[ \frac{\mathbb{P}(E_1)}{\mathbb{P}_\pi(T_o < t^*)} \geq \frac{2}{2-\beta}. \] (3.33)

For the \( \ell_\delta(A) - 1 \) other \( E_i \)'s, a very crude bound is sufficient. As the sets \( A_i \) are non-empty, we have
\[ \frac{\mathbb{P}(E_i)}{\mathbb{P}_\pi(T_o < t^*)} \geq 1, \] (3.34)
and we finally get, putting everything together,
\[ q_A \geq \frac{2}{2-\beta} + (\ell_\delta(A) - 1) + o(1) = \ell_\delta(A) + \beta' + o(1). \] (3.35)
for some \( \beta' > 0 \) independent of \( n \).

We are now able to conclude this subsection with the aforementioned result.

**Proposition 3.8.**
\[ S(\delta_{\text{micro}}) = o(1). \] (3.36)

**Proof.** Let \( 1 \leq \ell \leq k \). As points in the same component are at distance at most \( k\delta_{\text{micro}} \) from one another, the number of sets \( A \in \mathcal{A}_{\delta_{\text{micro}}} \) with \( \ell \) connected components is
\[ N(\ell) := | \{ A \in \mathcal{A}_{\delta_{\text{micro}}}: \ell_\delta(A) = \ell \} | \lesssim n^\ell V_k(\ell \delta_{\text{micro}})^k \leq n^\ell (k \delta_{\text{micro}})^k. \] (3.37)

As \( \delta_{\text{micro}} = o(\log n) \), we deduce that
\[ N(\ell) \lesssim n^\ell(d+1)^{k\delta_{\text{micro}}} = n^{\ell + o(1)}. \] (3.38)

Combining this with Proposition 3.7 gives finally (recalling from (3.16) that \( 1 \leq q_A \leq k + o(1) \))
\[ S(\delta_{\text{micro}}) \leq \sum_{\ell=1}^{k-1} N(\ell)n^{-(\ell+\beta')}e^{q_A|s|} \lesssim n^{-\beta'} = o(1), \] (3.39)
which concludes the proof. \( \square \)

### 3.5 Convergence in the high-dimensional case

In this subsection, we treat the case \( n \geq D^5 \) where the diameter is very small compared to the volume. As the growth is faster, the walk is “more transient”, and we can use cruder bounds to conclude quickly. The point of including this calculation is that it is quick and illustrates the ideas developed so far usefully, while preparing the ground for the more sophisticated arguments below. Getting the high-dimensional case out of the way from the start is also useful to simplify some arguments later on.

We will show that if \( \text{mindist}(A) > \delta_{\text{micro}} \), then (3.7) is satisfied under the high-dimensional assumption \( n \geq D^5 \).

**Proposition 3.9.** Assume that \( n \geq D^5 \). Then we have
\[ \mathbb{E}_\pi \left[ Z^k \right] \to e^{-ks}. \] (3.40)
Proof. Let $t^* = D^2(\log n)^{1.1}$. Then, from Proposition 2.16, (b), and as $D \gtrsim \log n$, we have uniformly over all $x, y \in \Gamma$ such that $d(x, y) \geq \delta_{\text{micro}} = (\log n)^{1/2}$,

$$P_x(T_y < t^*) \lesssim \frac{1}{\delta_{\text{micro}}^3} + \frac{t^*}{D^5} = o\left(\frac{1}{\log n}\right).$$

(3.41)

Combining this with Proposition 3.6, and noting that $D^2 \log n \ll t^* \ll n/\log n$, we have, uniformly over all $A \in A \setminus A_{\text{micro}}$,

$$q_A \geq k + o\left(\frac{1}{\log n}\right).$$

(3.42)

Then, since from Proposition 3.8 we know that $S(\delta_{\text{micro}}) = o(1)$ the result follows from Proposition 3.2. \hfill \Box

3.6 Convergence in the case of moderate polynomial growth

In this section, we consider the case

$$D^2(\log n)^2 \ll n \leq D^5.$$  

(3.43)

This time, Condition (3.7) does not necessarily hold over all $A$ such that $\text{mindist}(A) > \delta_{\text{micro}}$, and we need to consider another scale, which we call mesoscopic. To this purpose we set

$$\delta_{\text{meso}} := D^{1/2}. $$

(3.44)

We will prove that Condition (3.7) is satisfied if $\text{mindist}(A) \geq \delta_{\text{meso}}$, but first we need to reach this scale and prove that $S(\delta_{\text{meso}}) = o(1)$. This will be our first task. In fact, this estimate will also be needed to start the bootstrap argument in Section 3.8 for the proof of Theorem 1.1 in its full generality. Therefore we state it under the condition $D^2 \log n \ll n \leq n^5$, so it can be used there as well.

**Proposition 3.10.** Assume that $D^2 \log n \ll n \leq D^5$. Then

$$S(\delta_{\text{meso}}) = o(1). $$

(3.45)

**Proof.** We already know from Proposition 3.8 that $S(\delta_{\text{micro}}) = o(1)$. Therefore, we only have to study the contribution of the sets $A$ such that $\delta_{\text{micro}} < \text{mindist}(A) \leq \delta_{\text{meso}}$. We know from Proposition 3.6 (see in particular (3.25)), Corollary 2.17, and (3.16), that for such sets we have $q_A = k + o(1)$. Moreover, from the volume bound $V(r) \leq \frac{3r}{D^2} n$ (see (3.14)), and as $n \leq D^5$,

$$V(\delta_{\text{meso}}) \lesssim \frac{n}{D^{1/2}} \leq n^{9/10}. $$

(3.46)

Setting

$$A_{\text{meso}} = \{ A \in A : \delta_{\text{micro}} < \text{mindist}(A) \leq \delta_{\text{meso}} \}, $$

(3.47)

we have

$$|A_{\text{meso}}| \lesssim n^{k-1} V(\delta_{\text{meso}}) \lesssim n^{k-1/10}, $$

(3.48)

and we finally deduce that

$$\sum_{A \in A_{\text{meso}}} n^{-q_A} e^{-q_A} s \lesssim |A_{\text{meso}}| n^{-k+o(1)} \leq n^{-1/10+o(1)} = o(1). $$

(3.49)

Since $S(\delta_{\text{micro}}) = o(1)$, this concludes the proof of Proposition 3.10. \hfill \Box

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Proposition 3.11. Assume that $D^2(\log n)^2 \ll n \leq D^5$. Then we have
\begin{equation}
\mathbb{E}_\pi \left[ Z^{ik} \right] \to e^{-ks}.
\end{equation}

Proof. Let $t^* = \min \left( D^2 \sqrt{f(n)}, D^2(\log n)^{1.1} \right)$, so that $D^2 \log n \ll t^* \ll n / \log n$. Then, from Proposition 2.16, (a), we have uniformly over all $x, y \in \Gamma$ such that $d(x, y) \geq \delta_{\text{meso}} = D^{1/2}$, recalling that $D \geq n^{1/5}$ and $f(n) \gg (\log n)^2$,
\begin{equation}
\mathbb{P}_x(T_y < t^*) \ll \frac{1}{\delta_{\text{meso}}} + \frac{\log D}{f(n)} + \frac{t^*}{n} + \frac{t^*}{D^3} = o\left( \frac{1}{\log n} \right).
\end{equation}
Combining this with Proposition 3.6 (once again specifically (3.25)), and noting that $D^2 \log n \ll t^* \ll n / \log n$, we have, uniformly over all $A \in A \setminus A_{\delta_{\text{meso}}}$,
\begin{equation}
q_A \geq k + o\left( \frac{1}{\log n} \right).
\end{equation}
Furthermore, from Proposition 3.10 we know that $S(\delta_{\text{meso}}) = o(1)$. The result thus follows from Proposition 3.2. \qed

### 3.7 Bootstrap argument

The basis of the bootstrap argument is the following estimate. We first postpone its proof and explain how this implies the desired estimate. Recall that $\delta_{\text{meso}} = D^{1/2}$.

Proposition 3.12. There exists a constant $K$ such that for all $\delta_{\text{meso}} \leq \delta \leq D/2$ and $A \in \mathcal{A}$ such that $\delta \leq \text{mindist}(A)$, we have
\begin{equation}
q_A \geq k - K \frac{\log(D/\delta)}{f(n)}.
\end{equation}
Moreover, the constant $K$ depends only on $k$ and the degree bound $d$.

Remark 3.13. In reality this proposition will only be used with $\delta = o(D)$, so the restriction $\delta \leq D/2$ is not important.

Proof. We postpone the proof until Section 3.8. \qed

We now explain how Proposition 3.12 implies the desired estimate. Recall that
\begin{equation}
\mathbb{E}_\pi \left[ Z^{ik} \right] (1 + o(1)) = \sum_{A \in \mathcal{A}} n^{-q_A} e^{-q_A s} = S(D),
\end{equation}
and recall that by (3.16) $q_A \leq k + o(1)$, uniformly over all $A \in \mathcal{A}$, and by (3.52) $q_A \geq k + o(1)$, uniformly over all $A \in \mathcal{A}$, such that $\text{mindist}(A) \geq D^{1/2}$. (The upper bound comes from , and see e.g. (3.52) for the lower bound). As $s \in \mathbb{R}$ is fixed, it follows that $e^{-q_A s} \leq e^{(k+1)|s|} = O(1)$, so the only term we have to study is $n^{-q_A}$. Let us also write
\begin{equation}
b(n) := f(n) / \log n,
\end{equation}
which tends to infinity by the hypothesis $n/(D^2 \log n) \gg 1$. 30
Lemma 3.14. For all \( \delta_{\text{meso}} \leq \delta \leq \delta' \leq D/2 \), we have (with \( K \) the constant from Proposition 3.12),

\[
S(\delta') - S(\delta) \lesssim \frac{\delta'}{D} \left( \frac{D}{\delta} \right)^{K/b(n)},
\]

where the implicit constant depends only on \( k, s \).

Proof. From Proposition 3.12, we have for \( A \in \mathcal{A}_{\delta'} \backslash \mathcal{A}_{\delta} \),

\[
n^{-q_A} \leq n^{-k} n^{K\log(D/\delta)/f(n)} = n^{-k} (D/\delta)^{K\log n}/f(n) = n^{-k} (D/\delta)^{K/b(n)}.
\]

Moreover, from the volume bound \( V(r) \leq \frac{3\delta'}{D} n \) for \( r \geq 1 \),

\[
|\mathcal{A}_{\delta'} \backslash \mathcal{A}_{\delta}| \leq |\mathcal{A}_{\delta'}| \leq \left( \frac{k}{2} \right) V(\delta') n^{k-1} \leq \left( \frac{k}{2} \right) \frac{3\delta'}{D} n n^{k-1} = \left( \frac{k}{2} \right) \frac{3\delta'}{D} n^k
\]

For \( \delta \) sufficiently large, since \( e^{-q_A s} \leq e^{(k+1)|s|} \) (since \( q_A = k + o(1) \)), we finally obtain

\[
S(\delta') - S(\delta) = \sum_{A \in \mathcal{A}_{\delta'} \backslash \mathcal{A}_{\delta}} n^{-q_A} e^{-q_A s} \leq \left( 3e^{(k+1)|s|} \right) \left( \frac{k}{2} \right) \frac{3\delta'}{D} D^{K/b(n)},
\]

as desired. \( \square \)

Lemma 3.14 allows us to increase the value of \( \delta \) iteratively in such a way that the contribution to (3.6) is negligible (except for macroscopic scales). More precisely, set

\[
J = J(n) := \left\lfloor \frac{4 \log \log n}{\log(b(n))} - 1 \right\rfloor.
\]

In particular, we have

\[
4 \frac{\log \log n}{\log(b(n))} - 2 \leq J \leq 4 \frac{\log \log n}{\log(b(n))} - 1.
\]

Furthermore, we define a sequence of scales \( (\delta_j)_{1 \leq j \leq J(n)} \) by setting for all \( 1 \leq j \leq J(n) \),

\[
\delta_j := D \exp \left( - \frac{\log n}{b(n)j/4} \right) = D n^{-1/b(n)j/4}.
\]

Note that \( \delta_1 \geq \delta_{\text{meso}} \), and that from (3.60), we have

\[
b(n)^{J/4} \geq (b(n)^{\log \log n}/\log(b(n))^{-1/2} = \frac{\log n}{\sqrt{b(n)}},
\]

so

\[
\delta_j \geq D n^{-\sqrt{b(n)/\log n}} = De^{-\sqrt{b(n)}}.
\]

Plugging this into the estimate from Proposition 3.12, we see that \( q_A = k+ \) error, where the error (if \( \text{mindist}(A) \geq \delta_j \)) is at most

\[
K \frac{\log(D/\delta_j)}{f(n)} \leq K \frac{\sqrt{b(n)}}{f(n)} = \frac{K}{(\log n) \sqrt{b(n)}} = o\left( \frac{1}{\log n} \right).
\]

We already know that such an estimate suffices to imply the desired result (see (3.7)). Therefore, it will suffice for our purposes to prove the following proposition:
Proposition 3.15. We have $S(\delta_j) = o(1)$.

Proof. We already know from Proposition 3.10 that $S(\delta_{\text{meso}})$ tends to zero. Furthermore, using Lemma 3.14, we see that we also have $S(\delta_1) - \delta_{\text{meso}} \to 0$.

Now observe that for $1 \leq j \leq J - 1$, from Lemma 3.14 and some elementary computations,

$$S(\delta_{j+1}) - S(\delta_j) \lesssim n^{-\phi},$$

where as before the implicit constant is uniform on $j$, depending only on $k, s, d$, and

$$\phi := \frac{1}{b(n)^{\frac{j+1}{4}}} - \frac{K}{b(n)^{\frac{j+1}{4} + 1}}.$$

Hence, for $n$ large enough (which we assume in the following), we have $\phi \geq \frac{1}{2b(n)^{(i+1)/4}}.$ Then,

$$S(\delta_j) = o(1) + \sum_{j=1}^{J-1} (S(\delta_{j+1}) - S(\delta_j)) \lesssim o(1) + \sum_{j=1}^{J-1} n^{-1/(2b(n)^{(i+1)/4})}.$$

Note that from 3.60, we have

$$b(n)^{J/4} \leq (\log n)/b(n)^{1/4}. \quad (3.66)$$

Making the change of variables $i = J - j$, we therefore have

$$\sum_{j=1}^{J-1} n^{-1/(2b(n)^{(i+1)/4})} = \sum_{i=1}^{J-1} n^{-1/(2b(n)^{(i-1)/4})} \leq \sum_{i=1}^{J-1} e^{-b(n)^{(i-1)/4}}/2 = \sum_{i=1}^{J-1} u_i, \quad (3.67)$$

with $u_i := \exp(-b(n)^{(i-1)/4}/2).$ Note also that for $n$ sufficiently large, we have for all $1 \leq i \leq J - 1$ that

$$\frac{u_{i+1}}{u_i} \leq 1/e. \quad (3.68)$$

The sum $\sum_{i=1}^{J(n)-1} u_i$ being subgeometric, it is of the same order of magnitude as its first term, $u_1$, which tends to 0 since $b(n) \to \infty$. This concludes the proof of Proposition 3.15.

\[\square\]

3.8 Proof of Proposition 3.12

Let $\delta_{\text{meso}} \leq \delta \leq D/2$ and $A \in \mathcal{A}$ such that $\delta < \text{mindist}(A)$. To estimate $q_A$ we start from the definition of $q_A$ as

$$q_A = \frac{\mathbb{E}_\pi(T_o)}{\mathbb{E}_\pi(T_A)}.$$

We will rely on the following well-known identity relating the expected hitting time of a vertex to the “fundamental matrix” of the chain. For this it is useful to introduce the discrete time lazy random walk $(X^\#_t, t = 0, 1, \ldots)$ on $\Gamma$, which is the Markov chain with transition matrix $K = (I + P)/2$ where $P$ is the transition matrix of simple random walk on $\Gamma$. We note that if $T^\#_o$ and $T^\#_A$ denote its hitting times of $o$ and $A$ respectively, then by conditioning on the jump chain we easily get

$$\mathbb{E}_\pi(T_o) = (1/2)\mathbb{E}_\pi(T^\#_o) \quad \text{and} \quad \mathbb{E}_\pi(T_A) = (1/2)\mathbb{E}_\pi(T^\#_A),$$

so that $q_A = \mathbb{E}_\pi(T^\#_o)/\mathbb{E}_\pi(T^\#_A)$. These expected hitting times can then be evaluated through the following identity (see [LP17, Proposition 10.26] and [AF, Lemma 2.11]).
**Lemma 3.16.** Let $X$ be a discrete-time irreducible and aperiodic Markov chain on a finite state space $S$ with invariant distribution $\pi = (\pi_i)_{i \in S}$. For $j \in S$, let $T_j$ denote the hitting time of $j$ by $X$ and let $p_m(x,y)$ denote its $m$-step transition probabilities. Then

$$
\mathbb{E}_\pi(T_j) = \frac{1}{\pi(j)} \sum_{m=0}^{\infty} [p_m(j,j) - \pi(j)].
$$

While this is stated for the hitting time of a single state, we can also use this lemma to compute $\mathbb{E}_\pi(T_A^\#)$. One should however be a little careful. Indeed, as highlighted in [AF] (see Section 2.3.3) one cannot naively hope that the last formula holds with replacing $p_m(j,j)$ by $\mathbb{P}_{\pi_A}[X_m \in A]$ in Lemma 3.16, where $\pi_A$ denotes the uniform distribution on $A$. Instead, we will apply this lemma to the lazy random walk on the following **weighted collapsed graph** $\Gamma_A$ defined as follows. The vertices of $\Gamma_A$ are obtained by replacing all the vertices in $A$ by a single vertex, which we denote (with a small abuse of notation) by $A$. The edges of $\Gamma_A$ are those induced by the edges of $\Gamma$ and this wiring. Note that $\Gamma_A$ may be a multigraph (there can be several edges between $A$ and a vertex $x$ if $x$ had several neighbours in $A$ in the original graph $\Gamma$) and may contain loops if $A$ contained adjacent vertices in the original graph $\Gamma$. To the edges of this (multi)graph $\Gamma_A$ we associate weights $w$ as follows: let $e$ be an edge of $\Gamma_A$. Then

- if at least one endpoint of $e$ is distinct from $A$ we set $w(e) = 1$
- if instead $e$ is a loop from $A$ to $A$ then we set $w(e) = 2$.

Let $\tilde{p}_m^\#(x,y)$ denote the $m$-step transition probabilities of the random walk in discrete time $(\tilde{X}_m^\#, m = 0, 1, \ldots)$ on this graph, with laziness parameter $1/2$ at each vertex (including at $A$). In other words, this is the $m$-step transition probability of the Markov chain $\tilde{X}^\#$ whose transition matrix is by definition $\tilde{K} = (I + \tilde{P})/2$, where $\tilde{P}(x,y) = n(x,y)w(x,y)/\sum_z n(x,z)w(x,z)$, and $n(x,y)$ is the number of edges in the multigraph $\Gamma_A$ that lead from $x$ to $y$. An elementary computation shows that the Markov chain $\tilde{X}^\#$ is reversible w.r.t. the distribution $\tilde{\pi}$ given by $\tilde{\pi}(x) = 1/n$ for $x \in \Gamma_A$ with $x \neq A$, and $\tilde{\pi}(A) = k/n$. In particular, $\tilde{\pi}$ is the invariant distribution of $\tilde{X}^\#$. In other words, $\tilde{\pi} = \pi$, with a small abuse of notation (this is one of the reasons for giving weight 2 to edges internal to $A$).

Note that until hitting $A$, the Markov chain $\tilde{X}^\#$ on $\Gamma_A$ coincides with the original lazy random walk $X^\#$ on $\Gamma$, so that $\mathbb{E}_\pi(T_A^\#) = \mathbb{E}_\pi(\tilde{T}_A^\#)$, where $\tilde{T}_A^\#$ is the hitting time of (the vertex) $A$ by $X^\#$.

Therefore, by Lemma 3.16,

$$
\mathbb{E}_\pi(T_A^\#) = \frac{1}{\pi(A)} \sum_{m=0}^{\infty} [\tilde{p}_m^\#(A,A) - \pi(A)].
$$

Since $\pi(A)/\pi(o) = k$, we can rewrite $q_A$ under the discrete form

$$
q_A = k \frac{\sum_{m=0}^{\infty} [\tilde{p}_m^\#(a,o) - \pi(o)]}{\sum_{m=0}^{\infty} [\tilde{p}_m^\#(A,A) - \pi(A)]}
$$

and in order to prove Proposition 3.12 it suffices to show that
\[
\sum_{m=0}^{\infty} \frac{[p^\#_{m}(o,o) - \pi(o)]}{[\tilde{p}^\#_{m}(A,A) - \pi(A)]} \geq 1 - C \frac{\log(D/\delta)}{f(n)} \tag{3.70}
\]

for some constant \(C\) depending only on \(k, s, d\).

We start with the numerator, and observe that since the chain \(X^\#\) is lazy (hence all its eigenvalues are nonnegative), \(p^\#_{m}(o,o) \geq \pi(o)\) for all \(m\), therefore:

\[
\sum_{m=0}^{\infty} [p^\#_{m}(o,o) - \pi(o)] \geq \sum_{m=0}^{\delta^2} [p^\#_{m}(o,o) - \frac{1}{n}] \geq \left( \sum_{m=0}^{\delta^2} p^\#_{m}(o,o) \right) - \frac{1}{f(n)},
\]

where in the last inequality we used \(\delta^2 \leq D^2 = n/f(n)\). The bulk of the proof of (3.70) is therefore to give an upper bound to the denominator. We divide the proof into the following two steps.

**Lemma 3.17.** We have, uniformly over sets \(A\) of size \(k\) such that \(\text{mindist}(A) \geq \delta \geq \delta_{\text{meso}}\),

\[
\sum_{m=0}^{\delta^2} \tilde{p}^\#_{m}(A,A) \leq \sum_{m=0}^{\delta^2} p^\#_{m}(o,o) + O(1/f(n)).
\]

**Lemma 3.18.** We have, uniformly over sets \(A\) of size \(k\) such that \(\text{mindist}(A) \geq \delta \geq \delta_{\text{meso}}\),

\[
\sum_{m=0}^{2D^2} \tilde{p}^\#_{m}(A,A) \lesssim \frac{\log(D/\delta)}{f(n)}.
\]

Before proving these lemmas, we first explain how they imply (3.70). By Corollary 3.27 in [AF] we observe that the spectral gap of the collapsed chain is greater than that of the original chain and therefore (as already remarked in (2.16)) the relaxation time of the lazy version of the collapsed chain is at most \(2dD^2\). From Lemma 3.18 and the Poincaré inequality it therefore follows that

\[
\sum_{m=0}^{\infty} [p^\#_{m}(A,A) - \pi(A)] \lesssim \frac{\log(D/\delta)}{f(n)}, \tag{3.71}
\]

Indeed the Poincaré inequality (for the collapsed chain) implies that the summand decays geometrically fast for \(t \gtrsim D^2\) since the relaxation time is \(\lesssim D^2\). Hence the integral from \(D^2\) to infinity is bounded by a geometric sum which is itself bounded by its first term up to a constant. This first term is at most the right hand side of Lemma 3.18.

Let us explain this argument in more detail. First recall that by spectral decomposition (since \(\tilde{K}\) is reversible, see for instance [LP17, Lemma 12.2]) for every \(m = 0, 1, \ldots\) and

\[
\frac{\tilde{p}^\#_{m}(x,y)}{\tilde{\pi}(y)} = 1 + \sum_{j=2}^{n} f_{j}(x)f_{j}(y)\lambda_{j}^{m}, \tag{3.72}
\]

where \(0 \leq \lambda_{n} \leq \ldots \leq \lambda_{2} < \lambda_{1} = 1\) are the eigenvalues of \(\tilde{K}\) (which are nonnegative since the chain is lazy) and \(f_{1}, \ldots, f_{n}\) are the associated eigenfunctions. We immediately deduce, that for all \(t \geq 0\) and a vertex \(x\),

\[
\frac{\tilde{p}^\#_{t+s}(x,x)}{\tilde{\pi}(x)} - 1 = \sum_{j=2}^{n} f_{j}(x)^{2} \lambda_{j}^{t+s} \leq \left( \frac{\tilde{p}^\#_{t}(x,x)}{\tilde{\pi}(x)} - 1 \right) \lambda_{2}^{s} \leq \left( \frac{\tilde{p}^\#_{t}(x,x)}{\tilde{\pi}(x)} - 1 \right) e^{-s/\tilde{\tau}_{\text{meso}}},
\]

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where \( \tau_{\text{rel}} = 1/(1 - \lambda_2) \). In other words,
\[
\tilde{p}_t^\#(x, x) - \tilde{\pi}(x) \leq e^{-s/\tilde{\tau}_{\text{rel}}}(\tilde{p}_t^\#(x, x) - \tilde{\pi}(x)).
\] (3.73)
We can sum this inequality over intervals of length roughly \( \tilde{\tau}_{\text{rel}} \) to deduce that the sum from \( m\tilde{\tau}_{\text{rel}} \) to \( (m + 1)\tilde{\tau}_{\text{rel}} \) of \( \tilde{p}_t^\#(x, x) - \tilde{\pi}(x) \) decays exponentially in \( m \). That is, taking \( x = A \),
\[
\sum_{t = D^2}^\infty \sum_{m = 1}^{(m + 1)D^2 - 1} [\tilde{p}_t^\#(A, A) - \tilde{\pi}(A)] 
\leq \sum_{m = 1}^{\infty} \sum_{t = mD^2}^{2D^2} e^{-(m-1)D^2/\tilde{\tau}_{\text{rel}}} [\tilde{p}_t^\#(A, A) - \tilde{\pi}(A)]
\leq \sum_{t = D^2}^{2D^2} [\tilde{p}_t^\#(A, A) - \tilde{\pi}(A)],
\]
where in the last line we used that \( 2D^2/\tilde{\tau}_{\text{rel}} \geq 1/d \), as it is easy to check that \( \tilde{\tau}_{\text{rel}} \leq 2 \tilde{\tau}_{\text{rel}} \leq 2dD^2 \) (the inequality \( \tilde{\tau}_{\text{rel}} \leq 2 \tilde{\tau}_{\text{rel}} \) follows easily from comparing the Dirichlet forms associated to \( \tilde{K} \) and the original matrix \( K \), see, e.g., Corollary 4.1 in [Ber16], or [LP17, Lemma 13.7]).

Therefore, Lemmas 3.17 and 3.18 together imply (3.70).

It remains to prove Lemmas 3.17 and 3.18.

**Proof of Lemma 3.17.** A key idea for this lemma will be a probabilistic construction of \( \tilde{X}^\# \) in terms of excursions away from \( o \) of the original lazy walk \( X^\# \) on the vertex-transitive graph \( \Gamma \). We explain this construction first. Let \( (e_i, i \geq 1) \) be a sequence of i.i.d. *excursions* of the walk \( X^\# \) (on \( \Gamma \)) from \( o \) to \( o \): that is, each \( e_i \) is a lazy random walk on \( \Gamma \) starting at \( o \), and killed upon returning to \( o \) for the first time (let \( \zeta_i \) be this time). Consider also an independent and i.i.d. sequence \( (a_i)_{i \geq 1} \) such that \( a_i \) is uniformly distributed on \( A \) for each \( i \). Roughly speaking, we will use \( a_i \) to translate the excursion \( e_i \): that is, for a vertex \( v \) of \( \Gamma \), fix \( \phi_v \) a graph isomorphism such that \( \phi_v(o) = v \). We then set
\[
\tilde{e}_i := \phi_{a_i}(e_i),
\]
which defines an independent sequence of excursions based at \( a_i \) (instead of \( o \)) respectively. The idea is then to concatenate the excursions \( \tilde{e}_i \) to form the Markov chain \( \tilde{X}^\# \) on the collapsed graph \( \Gamma_A \). However, we need to pay attention to the fact that during the excursion \( \tilde{e}_i \) it is also possible to hit \( A \) through a different point than merely \( a_i \). Let us define
\[
\tau_i := \inf\{t = 1, 2, \ldots : \tilde{e}_i(t) \in A \setminus \{a_i\}\}
\]
with the usual convention that \( \inf\emptyset = \infty \). To obtain the Markov chain \( \tilde{X}^\# \) on \( \Gamma_A \), we simply concatenate the truncated excursions \( \eta_i := (\tilde{e}_i(t), 0 \leq t \leq \zeta_i \wedge \tau_i - 1) \). We leave it to the reader to check that the Markov property the following claim:

**Claim:** the concatenation of \( (\eta_i)_{i \geq 1} \) is a realisation of \( (\tilde{X}^\#_i, t = 0, 1, \ldots) \) on \( \Gamma_A \).

When the stopping time \( \tau_i \) is infinite, this corresponds to the chain \( \tilde{X}^\# \) returning to \( A \) in a way which we think of as “simple”, corresponding to an excursion from a point \( a_i \in A \) to itself, when the walk is seen in \( \Gamma \); as we will see below the expected number of returns to \( A \) in this fashion will be simply controlled by the heat kernel \( p_t^\#(o, o) \) on the base graph, for which we have good estimates. We will therefore be especially interested in controlling returns to \( A \) that occur in the opposite case when \( \tau_i < \zeta_i \), i.e., when the excursion seen in \( \Gamma \) corresponds to a path.
from a point \(a_i \in A\) to a distinct point \(b_i \in A\). We will refer to such an event as a **thread transition** (from \(A\) to \(A\)).

Let \(\tau\) denote the first time that there is such a thread transition. The main argument will be to show that

\[
P(\tau \leq \delta^2) = O(1/f(n)). \tag{3.74}
\]

Let us first explain why (3.74) implies the conclusion of the lemma. Let \(\tilde{L}_A^\#(m) = \sum_{i=0}^{m-1} 1\{X_i^\# = A\}\) and \(L_0^\#(m) = \sum_{i=0}^{m-1} 1\{X_i^\# = o\}\). We observe that

\[
E_A(\tilde{L}_A^\#(\tau \land \delta^2)) = E_o(L_0^\#(\tau \land \delta^2)) \leq E_o(L_0^\#(\delta^2)),
\]

since before \(\tau\) the only returns to \(A\) by \(\tilde{X}^\#\) correspond to the lifetimes \(\zeta_i\) of the excursions \(\tilde{e}_i\) or, equivalently, of \(e_i\). When we concatenate the excursions \(e_i\) we get a random walk \(X^\#\) on the original graph \(\Gamma\), and so these returns correspond to \(X^\#\) hitting \(o\). Therefore, applying the strong Markov property of \(X^\#\) at time \(\tau\), we get

\[
E_A(\tilde{L}_A^\#(\delta^2)) = E_A(\tilde{L}_A^\#(\tau \land \delta^2)) + E_A((\tilde{L}_A^\#(\delta^2) - \tilde{L}_A^\#(\tau \land \delta^2)))
\]

\[
\leq E_o(L_0^\#(\delta^2)) + P(\tau < \delta^2)E_A(\tilde{L}_A^\#(\delta^2))
\]

\[
\leq E_o(L_0^\#(\delta^2)) + O(1/f(n))E_A(\tilde{L}_A^\#(\delta^2))
\]

where the second inequality follows by (3.74), and the first by averaging over \(\tau\) and noting that \(\tilde{L}_A^\#(\cdot)\) is non-decreasing. Therefore, putting the last term on the right hand side in the left hand side, we obtain

\[
E_A(\tilde{L}_A^\#(\delta^2)) \leq (1 + O(1/f(n)))E_o(L_0^\#(\delta^2)) = E_o(L_0^\#(\delta^2)) + O(1/f(n)),
\]

recalling that \(E_o(L_0^\#(\delta^2)) = O(1)\), using continuous time estimates (see, e.g., Proposition 2.18) and using basic properties of Poisson processes. This is the required bound for Lemma 3.17.

It remains to prove (3.74). For this we will again use the construction above in terms of excursions and use the following crude bound.

Consider the finite set

\[
B := \bigcup_{a \in A} \phi_a^{-1}(A \setminus \{a\}). \tag{3.76}
\]

In words, \(B\) represents “all the ways to see the rest of \(A\), viewed from an arbitrary point in \(A\).” Note that \(B\) is a set of cardinality at most \(k(k - 1) \leq k^2\).

Note also that \(d(o, b) \geq \delta\) for all \(b \in B\) (since \(\text{mindist}(A) \geq \delta\)). Furthermore, by Markov’s inequality,

\[
P(\tau < \delta^2) \leq P_o(T_B^\# < \delta^2) \leq E_o(L_B^\#(\delta^2))
\]

\[
\leq \sum_{y \in B} \sum_{t=0}^{\delta^2} p_t^\#(o, y).
\]

We bound the expected discrete local time by the expected continuous local time in order to use our off-diagonal estimates on the heat kernel of Proposition 2.16. More precisely, note that if \(N\) is a Poisson random variable with mean \(10\delta^2\) (corresponding to the number of jumps in continuous time which occurred by time \(t = 10\delta^2\)), then for some absolute constant \(c > 0\)

\[
\sum_{m=0}^{\delta^2} p_m^\#(o, y) \leq \int_0^{10\delta^2} p_t(o, y) dt + \delta^2 P(N \leq 2\delta^2)
\]

\[
\leq O(1/f(n)) + \delta^2 \exp(-c\delta^2),
\]

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where for the first we used Proposition 2.16 and the fact that \( d(o, y) \geq \delta \), and for the second term, we used standard Chernoff bounds for Poisson random variables. Since \( \delta \geq \delta_{\text{meso}} \geq \sqrt{D} \), we clearly have that the second term is negligible compared to the first.

This concludes the proof of (3.74) and therefore also of Lemma 3.17.

Proof of Lemma 3.18. We wish to argue as in Proposition 2.1, but we cannot directly apply this result since the graph \( \Gamma_A \) of the collapsed chain is not vertex-transitive. Nevertheless, it is easy to see that the same bounds apply to this chain.

First observe that the conductance profile \( \phi_A \) of the collapsed chain is larger than that of the original chain: that is, \( \phi(u) \leq \phi_A(u) \).

Indeed, to every set \( \tilde{S} \subset \Gamma_A \) corresponds naturally an uncollapsed set \( S \) of vertices in \( \Gamma \) (this set \( S \) is nothing but \( \tilde{S} \setminus \{\{A\}\} \cup A \)), and since \( A \) is sparse (recall that \( \text{mindist}(A) \geq \delta_{\text{meso}} \)), those sets satisfy \( Q_A(\tilde{S}, \tilde{S}^c) = Q(S, S^c) \). Conversely, given any set \( S \) of vertices in \( \Gamma \) such that \( S \supset A \) or \( S \cap A = \emptyset \) (i.e., \( S \) does not contain just a portion of \( A \)) then we see that \( S \) corresponds to a set \( \tilde{S} \) of vertices in \( \Gamma_A \) such that \( Q_A(\tilde{S}, \tilde{S}^c) = Q(S, S^c) \) and thus \( \Phi(S) = \Phi_A(\tilde{S}) \).

We deduce that for \( 1/n \leq u \leq 1/2 \),

\[
\phi(u) = \inf \{ \Phi(S) : \pi(S) \leq u \} \\
\leq \inf \{ \Phi(S) : \pi(S) \leq u \text{ and } [S \supset A \text{ or } S \cap A = \emptyset] \} \\
= \inf \{ \Phi_A(\tilde{S}) : \pi(\tilde{S}) \leq u \} \\
= \phi_A(u).
\]

Combining this inequality and (2.7), we see that Proposition 2.1 still applies here, even without vertex-transitivity, and that only the constants differ (there is for instance an extra factor \( k \) in (2.4), as \( \pi(A) = k\pi(o) \)). This also holds in discrete-time, as noted in Remark 2.4.

Therefore, reasoning as in Corollary 2.5, we also have the following bounds: If \( f(n) < D \), then for every \( x \in \Gamma_A \), for \( 1 \leq t \leq f(n)^2 \), \( \tilde{p}_t^x(x, x) \lesssim t^{-3/2} \), and for every \( f(n)^2 \leq t \leq D^2 \),

\[
\tilde{p}_t^x(x, x) \lesssim \frac{1}{tf(n)}.
\]  
(3.77)

As mentioned above, the only difference is now the implicit constants depend also on \( k \). Using this, we immediately deduce (recalling that \( \delta^2 \geq f(n)^2 \)) that

\[
\sum_{t=\delta^2}^{D^2} \tilde{p}_t^x(A, A) \lesssim \sum_{t=\delta^2}^{D^2} \frac{1}{tf(n)} \lesssim \frac{\log(D/\delta)}{f(n)}.
\]  
(3.78)

This completes the proof of Lemma 3.18 and thus also of (3.70). In turn, the proof of Theorem 1.1 is complete.

\section{Law of the uncovered set}

The goal of this section is to show that the information on the law of the cover time can be supplemented by a precise description of the law of the uncovered sets before the cover time. From the results in Section 3 and in particular from (3.1), we know that at a time \( t(o) = \mathbb{E}_x T_o(s + \log n) \), the size of the uncovered set converges to a Poisson random variable with parameter \( e^{-s} \). We will then turn to describe the geometry of this uncovered set: roughly, we aim to show that the uncovered points are approximately uniformly chosen from the vertex set of the graph.
There are two different ways to express this idea. The first one is to consider the fixed time \( t_{(s)} \) and condition on the size \( Z_s = |U(t_{(s)})| = k \) of the uncovered set at this time, and show that the law of the set \( U(t_{(s)}) \) itself is close (in the total variation sense) to a uniformly chosen set of size \( k \). This is carried out in Section 4.1. Another way is to consider the stopping time \( \tau_k \) which is the first time at which the size of uncovered set is equal to \( k \) (so \( \tau_0 = \tau_{cov} \)), and prove the same approximate uniformity. This is carried out in Section 4.2.

As we will see, a stronger form of convergence (namely, convergence in total variation) for the cover time itself will follow relatively quickly from these results. This will be explained in Section 4.3.

### 4.1 Convergence to a product measure

In this subsection, we prove the first form of uniformity for the uncovered set mentioned above. The statement is as follows. We recall that from the proof in Section 3, in all growth cases, there exists \( \delta_s = o(D) \) such that \( q_A \geq k - o \left( \frac{1}{\log n} \right) \) uniformly over sets \( A \) of size \( k \) such that \( \text{mindist}(A) > \delta_s \). (Indeed, \( \delta_s \) can be taken to be \( \delta_{micro} \) in the high dimensional case, \( \delta_{meso} \) in the intermediate regime, and \( \delta_{macro} \) in the most difficult regime, the low-dimensional case).

For convenience, let \( A_s \) denote the subsets \( A \subset \Gamma \) (i.e., \( A \) is a set of vertices of \( \Gamma \)) such that \( \text{mindist}(A) > \delta_s \).

**Theorem 4.1.** Let \( k \geq 1 \) and \( s \in \mathbb{R} \). Uniformly over sets \( A \in A_s \) of size \( k \),

\[
\mathbb{P}(U(t_{(s)}) = A | Z_s = k) = \frac{k!}{n^k}(1 + o(1)).
\]

In particular, if \( \text{Unif}_k \) denotes the uniform law on subsets of size \( k \), and if \( U(t_{(s)} | k) \) denotes the law of \( U(t_{(s)}) \) conditionally given \( Z_s = k \), then

\[
\text{d}_{TV}(U(t_{(s)} | k); \text{Unif}_k) \to 0
\]
as \( n \to \infty \).

**Remark 4.2.** It is not hard to see (using the thinning property of Poisson random measures) that an equivalent formulation of Theorem 4.1 is as follows: without any conditioning, the law of \( U(t_{(s)}) \) is close in the total variation sense to the law of the set obtained by tossing an independent coin for each vertex, with probability of heads given by \( e^{-s}/n \). In other words, the law of \( U(t_{(s)}) \) is close to that of a product measure \( \mu_s \otimes \Gamma \) where the product is over all vertices of the graph, and the law \( \mu_s \) is Bernoulli with parameter \( e^{-s}/n \). Thus,

\[
\text{d}_{TV}(U(t_{(s)}), \mu_s \otimes \Gamma) \to 0,
\]

where, as is standard, if \( X \) is a random variable and \( \mu \) a law, \( \text{d}_{TV}(X, \mu) \) denotes the total variation between the law of \( X \) and \( \mu \).

We start the proof of Theorem 4.1 with the following simple lemma.

**Lemma 4.3.** Let \( k \geq 1 \) and \( s \in \mathbb{R} \). Then we can neglect sets that are not sufficiently well separated:

\[
\sum_{A \in A_s, |A| = k} \mathbb{P}(U(t_{(s)}) = A) = o(1).
\]

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Proof. This follows from the trivial bound
\[ \mathbb{P}(U(t_\phi) = A) \leq \mathbb{P}(T_A > t_\phi) \] (4.3)
and bounds obtained previously. Indeed, by Proposition 2.24 we have
\[ \mathbb{P}(T_A > t_\phi) = n^{-q_A}e^{-q_A s}(1 + o(1)), \] (4.4)
and by choice of \( \delta_* \), as shown in Section 3, for all growth cases, \( S(\delta_*) = o(1) \). This implies that
\[ \sum_{A \in \mathcal{A} \mid |A| = k} \mathbb{P}(U(t_\phi) = A) \leq S(\delta_*)(1 + o(1)) = o(1), \] (4.5)
as desired. \( \square \)

The proof of Theorem 4.1 will therefore follow from the following lemma and Lemma 4.3.

Lemma 4.4. Let \( k \geq 1 \) and \( s \in \mathbb{R} \). Then, uniformly over sets \( A \in \mathcal{A}_* \) of size \( k \),
\[ \mathbb{P}(U(t_\phi) = A) = \frac{e^{-e^{-s}e^{-ks}}}{n^k}(1 + o(1)). \] (4.6)

Proof. We cannot directly apply here the moment method (or factorial moment method) as we did in Section 3 but we note that there is a relatively simple way to use the work done in this section nevertheless, by exploiting instead the Bonferroni inequalities.

Let us fix a set \( A \) as in the lemma, and observe that we can rewrite
\[ \mathbb{P}(U(t_\phi) = A) = \mathbb{P}(A \text{ is not touched but all the points in } \Gamma \setminus A \text{ are}) \] (4.7)
\[ = \mathbb{P}\left( \bigcap_{x \in \Gamma \setminus A} \{ T_x \leq t_\phi \} \right). \] (4.8)

For \( X \subset \Gamma \setminus A \), set \( E_X := \{ T_{A \cup X} \leq t_\phi \} \), with the standard abuse of notations when \( X \) is a singleton. Then
\[ \mathbb{P}(U(t_\phi) \neq A) = \mathbb{P}\left( \bigcup_{x \in \Gamma \setminus A} \{ T_x \leq t_\phi \} \right) \] (4.9)
\[ = \mathbb{P}(T_A \leq t_\phi) + \mathbb{P}\left( \bigcup_{x \in \Gamma \setminus A} \{ T_x \leq t_\phi \} \right) \] (4.10)
\[ = \mathbb{P}(T_A \leq t_\phi) + \mathbb{P}\left( \bigcup_{x \in \Gamma \setminus A} E_x \right). \] (4.11)

By Proposition 2.24, uniformly over sets \( A \in \mathcal{A}^* \) of size \( k \),
\[ \mathbb{P}(T_A > t_\phi) = (1 + o(1)) \frac{e^{-ks}}{n^k}. \] (4.12)
Consequently, we have
\[ \mathbb{P}(U(t_\phi) = A) = (1 + o(1)) \frac{e^{-ks}}{n^k} - \mathbb{P}\left( \bigcup_{x \in \Gamma \setminus A} E_x \right). \] (4.13)
Observe that $E_X \cap E_Y = E_{X\cup Y}$. Note also that if $\text{mindist}(A \cup X) \geq \delta_*$ and $|X| = j$, then $q_{A\cup X} \geq k + j - o(1/\log n)$. Therefore,

$$
n^k \sum_{X \subseteq \Gamma \setminus A, |X| = j, A \cup X \in \mathcal{A}_*} \mathbb{P}(E_X) \to \frac{e^{-(k+j)s}}{j!}.
$$

(4.14)

Choose $J = J(k, \varepsilon, s)$ such that

$$
\sum_{j=J+1}^{\infty} \frac{e^{(k+j)s}}{(k+j)!} \leq \varepsilon.
$$

Having chosen $J$, proceeding as in the proof of Lemma 4.3, we can ignore sets $A \cup X \in \mathcal{A}_*^{\setminus A}$ and write for every $1 \leq j \leq J + 1$,

$$
n^k \sum_{X \subseteq \Gamma \setminus A, |X| = j, A \cup X \in \mathcal{A}_*^{\setminus A}} \mathbb{P}(E_X) \leq \frac{\varepsilon}{J}.
$$

(4.15)

We deduce from the Bonferroni inequalities that for $n$ large enough, and separating according to whether $A \cup X \in \mathcal{A}_*$ or not,

$$
n^k \left| \mathbb{P} \left( \bigcup_{x \in \Gamma \setminus A} E_x \right) - \sum_{j=1}^{J} (-1)^{j+1} \sum_{X \subseteq \Gamma \setminus A, |X| = j} \mathbb{P}(E_X) \right| \leq \sum_{X \subseteq \Gamma \setminus A, |X| = J+1} \mathbb{P}(E_X)
$$

$$
\leq \frac{\varepsilon}{J} + \frac{e^{(k+J+1)s}}{(k+J+1)!} + \varepsilon
$$

(4.17)

$$
\leq 3\varepsilon.
$$

(4.18)

Consequently,

$$
n^k \left| \mathbb{P} \left( \bigcup_{x \in \Gamma \setminus A} E_x \right) - \sum_{j=1}^{J} (-1)^{j+1} \sum_{X \subseteq \Gamma \setminus A, |X| = j, A \cup X \in \mathcal{A}_*} \mathbb{P}(E_X) \right| \leq 3\varepsilon + J\varepsilon/J = 4\varepsilon.
$$

(4.19)

By (4.14), we may assume that $n$ is large enough that for $1 \leq j \leq J$,

$$
n^k \left( \sum_{X \subseteq \Gamma \setminus A, |X| = j, A \cup X \in \mathcal{A}_*} \mathbb{P}(E_X) \right) - \frac{e^{-(k+j)s}}{j!} \leq \frac{\varepsilon}{J}.
$$

(4.20)

Combining this with the definition of $J$ and by another triangle inequality, we obtain, as

$$
e^{-ks} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{e^{-(k+j)s}}{j!} = e^{-ks} e^{-e^{-s}},
$$

(4.21)

that for $n$ large enough, we have

$$
\left| n^k \mathbb{P}(U(t_0) = A) - e^{-ks} e^{-e^{-s}} \right| \leq 6\varepsilon,
$$

(4.22)

which concludes the proof. \qed
4.2 Convergence of the last $k$ points

In this section we are interested in the first time at which the uncovered set has size $k$, that is:

$$
\tau_k := \inf\{t \geq 0 : |\{X_u, u \leq t\}^c| = k\}.
$$

We want to show that the distribution of $U(\tau_k)$ is close to the uniform distribution $\text{Unif}_k$ over sets of size $k$.

**Theorem 4.5.** As $n \to \infty$, we have

$$
d_{TV}(U(\tau_k), \text{Unif}_k) \to 0. \quad (4.23)
$$

We will need two ingredients.

Our first lemma improves on Theorem 4.1 by showing that the position of the walk at time $t_{(s)}$ is approximately independent of $U(t_{(s)})$ and is distributed (approximately again) according to $\pi$.

**Lemma 4.6.** Let $s \in \mathbb{R}$. Recall the product law $\mu_s^{\otimes \Gamma}$ from Remark 4.2. Then

$$
d_{TV}([U(t_{(s)}), X_{t_{(s)}}]; (\mu_s^{\otimes \Gamma} \otimes \pi)] \to 0.
$$

**Proof.** Recall that $Z_s = |U(t_{(s)})|$. Fix $s' < s$ in such a way that

$$
D^2 \ll t_{(s)} - t_{(s')} \ll n = D^2 f(n). \quad (4.24)
$$

The upper bound of (4.24) implies that $s - s' = o(1)$ and so that $E(Z_{s'} - Z_s) \to 0$. As moreover $Z_s \leq Z_{s'}$, we deduce that

$$
P(Z_s \neq Z_{s'}) = P(Z_{s'} - Z_s \geq 1) \leq E(Z_{s'} - Z_s) \to 0. \quad (4.25)
$$

Furthermore, we already know from Theorem 4.1 that $d_{TV}(U(t_{(s')}); \mu_s^{\otimes \Gamma}) \to 0$. But since $|s - s'| \to 0$, we also have $d_{TV}(U(t_{(s')}), \mu_s^{\otimes \Gamma}) \to 0$.

To conclude, let us compare $P(X_{t_{(s)}} = x, U(t_{(s)}) = A)$ with our target $\pi(x)\lambda(A)$, where we set $\lambda(A) = e^{-e^{-s}e^{-|A|s}/n|A|}$, i.e., $\lambda = \mu_s^{\otimes \Gamma}$. Then

$$
\left|P(X_{t_{(s)}} = x, U(t_{(s)}) = A) - \pi(x)\lambda(A)\right| \leq \left|P(X_{t_{(s)}} = x, U(t_{(s)}) = A) - P(X_{t_{(s)}} = x, U(t_{(s')} = A)\right|
$$

$$
+ \left|P(X_{t_{(s)}} = x, U(t_{(s')} = A) - \pi(x)\lambda(A)\right|
$$

$$
\leq P(X_{t_{(s)}} = x, U(t_{(s')} = A; Z_s \neq Z_{s'})
$$

$$
+ \left|P(U(t_{(s')}) = A)P(X_{t_{(s)}} = x|U(t_{(s')} = A) - \pi(x)\lambda(A)\right|
$$

Summing over all $x, A$, for the first term in the right hand side above, we get

$$
\sum_x \sum_A P(X_{t_{(s)}} = x, U(t_{(s')} = A; Z_s \neq Z_{s'}) = P(Z_s \neq Z_{s'}),
$$

which converges to zero by (4.25). The second term, on the other hand, can be written by Theorem 4.1 (or more precisely Remark 4.2) as

$$
\left|\lambda(A)(1 + o(1))P(X_{t_{(s)}} = x|U(t_{(s')} = A) - \pi(x)\lambda(A)\right|
$$

41
But observe that by choice of \( s' \) compared to \( s \), in the interval from \( t_{(s')} \) to \( t_{(s)} \) the walk has had time to mix. More specifically, the lower bound of \( (4.24) \), \( D^2 \ll t_{(s)} - t_{(s')} =: t \), implies, through \([LP17, \text{Proposition 4.15}], \) \((2.60)\), and Corollary 2.5, that uniformly over \( x, y \in \Gamma \),

\[
|p_t(x, y) - 1/n| \leq p_t(o, o) - 1/n \leq \exp\left(-\frac{t - D^2}{dD^2}\right) p_{D^2}(o, o) = o\left(\frac{1}{n}\right), \tag{4.26}
\]

i.e. that \( p_t(x, y) = \pi(y)(1 + o(1)) \).

Thus, by averaging over \( X_{t_{(s')}} \), we have \( \mathbb{P}(X_{t_{(s)}} = x|U(t_{(s')})) = A) = \pi(x)(1 + o(1)) \), and we conclude that this second term is equal to \( \lambda(A)\pi(x)o(1) \). Summing over all \( x \) and \( A \), we see that the sum is \( o(1) \) and we deduce that

\[
\sum_x \sum_A |\mathbb{P}(X_{t_{(s)}} = x, U(t_{(s)}) = A) - \pi(x)\lambda(A)| \to 0
\]

which is the desired result.

\[\square\]

**Lemma 4.7.** Let \( k \geq 2 \) and \( A \in \mathcal{A}_* \) of size \( k \), and let \( x \in A \). Then the probability that \( x \) is the first point of \( A \) to be touched by the walk starting from uniformity is \( 1/k + o(1) \):

\[
\mathbb{P}_\pi(T_A = T_x) = \frac{1}{k} + o(1). \tag{4.27}
\]

**Proof.** By going in discrete time it would be possible to prove this via a reversibility argument and some simple estimates. We opt however for the following idea using the skeleton of Section 3.3. Let \( \alpha \) be the quasi-stationary distribution associated to \( A \), and let

\[
p := \mathbb{P}_\alpha(T_A = T_x). \tag{4.28}
\]

Remark that as \( d_{TV}(\pi, \alpha) = o(1) \) (either by Aldous–Brown, or more precisely by Theorem 5.4), we have \( \mathbb{P}_\pi(T_A = T_x) = p + o(1) \).

Moreover, we have by the memoryless property of the hitting time of \( A \) under the quasi-stationary distribution, that for every \( t > 0 \),

\[
p = \mathbb{P}_\alpha(T_A = T_x|T_A \leq t)\mathbb{P}(T_A \leq t) + \mathbb{P}_\alpha(T_A = T_x|T_A > t)\mathbb{P}(T_A > t)
\]

\[= \mathbb{P}_\alpha(T_A = T_x|T_A \leq t)\mathbb{P}(T_A \leq t) + p\mathbb{P}(T_A > t),
\]

so simplifying this identity and doing the cancellations,

\[
p = \mathbb{P}_\alpha(T_A = T_x|T_A \leq t) = \frac{\mathbb{P}_\alpha(T_A = T_x, T_A \leq t)}{\mathbb{P}_\alpha(T_A \leq t)}. \tag{4.29}
\]

From now on, fix \( t \) such that \( D^2 \ll t \ll \min(n, D^1) \).

For the numerator, let us use again the uniform distribution instead of the quasistationary distribution. From the Cauchy-Schwarz inequality and Theorem 5.4 we have:

\[
|\mathbb{P}_\alpha(T_A = T_x, T_A \leq t) - \mathbb{P}_\pi(T_A = T_x, T_A \leq t)| \leq d_{TV}(\pi, \alpha) = O(1/n). \tag{4.30}
\]

To estimate \( \mathbb{P}_\pi(T_A = T_x, T_A \leq t) \) we make use of our results in Subsection 3.3. Recall the notion of skeleton, and note that under the assumption \( A \in \mathcal{A}_* \), the set \( A \) has \( k \) \( \delta_s \)-components, which are all singletons. We can write for instance \( A = \{a_1, \ldots, a_k\} \), and we have \( E_i = \{a_i \in \gamma\} \), where \( \gamma \) is a path starting from uniformity and of time-length \( t \). In particular, assuming without loss of generality that \( x = a_1 \), we have a trivial upper bound

\[
\mathbb{P}_\pi(T_A = T_{a_1}, T_A \leq t) \leq \mathbb{P}_\pi(T_{a_1} \leq t). \tag{4.31}
\]
For the lower bound, observe that
\[
\{T_A = T_{a_1}, T_A \leq t\} \supset \{T_{a_1} \leq t \text{ and } T_{a_i} > t \text{ for all } 2 \leq i \leq k\} = \{T_{a_1} \leq t\} \setminus \bigcup_{2 \leq i \leq k} \{T_{a_i} \leq t \text{ and } T_{a_i} \leq t\}.
\]
Hence, as for each $2 \leq i \leq k$ we have from Proposition 3.5 and Corollary 2.17 that
\[
\mathbb{P}_\pi(T_{a_1} \leq t \text{ and } T_{a_i} \leq t) = \mathbb{P}(E_1 \cap E_i) \leq \mathbb{P}_\pi(T_o \leq t) \max_{x,y \in \Gamma} \mathbb{P}_x(T_y < t) = \mathbb{P}_\pi(T_o \leq t) \cdot o(1).
\]
which finally proves that
\[
\mathbb{P}_\pi(T_A = T_{a_1}, T_A \leq t) = \mathbb{P}_\pi(T_o \leq t)(1 + o(1)).
\]
(4.32)
As $\mathbb{P}_\pi(T_o \leq t) \gtrsim t/n \gg 1/n$ (for instance by considering again the quasi-stationary distribution of $o$), we have also using (4.30)
\[
\mathbb{P}_\alpha(T_A = T_x, T_A \leq t) = \mathbb{P}_\pi(T_o \leq t)(1 + o(1)).
\]
(4.34)
We deduce that
\[
p = \frac{\mathbb{P}_\pi(T_o \leq t)}{\mathbb{P}_\alpha(T_A \leq t)}(1 + o(1)).
\]
(4.35)
Note that this obviously does not depend on the choice of $x$ in $A$, and so must be $1/k + o(1)$. Alternatively, by Corollary 2.26 and the fact that $\pi$ and $\alpha$ are close, the quotient on the right hand side is $(1 + o(1))/q_A$, so it is indeed equal to $1/k + o(1)$.

**Lemma 4.8.** Let $A \in \mathcal{A}_s$ of size $k$, and let $y$ be such that $d(y, A) \geq \delta_s$. Let $x \in A$. Then starting from $y$, the probability that $x$ is the first point of $A$ to be touched by the walk is $1/k + o(1)$:
\[
\mathbb{P}_y(T_A = T_x) = \frac{1}{k} + o(1).
\]
(4.36)
**Proof.** This follows simply from the previous lemma (Lemma 4.7) if we allow a burn-in period during which the walk can mix but is unlikely to touch $A$. Let $t^*$ be such that $D^2 \ll t^* \ll \min(n, D^3)$. Note that by Proposition 2.17 (see in particular (2.56)), $\mathbb{P}_y(T_A \leq t^*) \to 0$. Furthermore, since $t_{\text{mix}} \lesssim D^2 \ll t^*$, the distribution of the walk at time $t^*$ is $o(1)$ away (in the total variation sense) from $\pi$. Hence
\[
\mathbb{P}_y(T_A = T_x) = \mathbb{P}_y(T_A = T_x; T_A \leq t^*) + \mathbb{P}_y(T_A = T_x; T_A > t^*) = o(1) + \mathbb{P}_\pi(T_A = T_x) + o(1) = 1/k + o(1),
\]
as desired.

We can now prove the main result of this subsection.

**Proof of Theorem 4.5.** Let $k \geq 2$ and $\varepsilon > 0$. Let $s = s(k, \varepsilon) \in \mathbb{R}$ be such that for $n$ large enough,
\[
\mathbb{P}(\tau_k > t_{(s)}) \geq 1 - \varepsilon.
\]
(4.37)
Let also $K = K(s, k, \varepsilon) \geq k + 1$ be such that for all sufficiently large $n$ we have that,
\[
\mathbb{P}(\tau_K \leq t_{(s)}) \geq 1 - \varepsilon.
\]
(4.38)
In particular, with probability at least $1 - 2\varepsilon$, there are between $k + 1$ and $K$ uncovered points at time $t_{(j)}$:

$$\mathbb{P}(k + 1 \leq Z_s \leq K) = \mathbb{P}(\tau_K \leq t_{(j)} < \tau_k) \geq 1 - 2\varepsilon. \quad (4.39)$$

Fix $j$ such that $k + 1 \leq j \leq K$, and condition on the event $Z_s = j$. Let us now describe the evolution of $U(t_{(j)})$, up to events of probability $o(1)$, in order to get an approximation of $U(\tau_k)$ in the total variation sense. Conditionally on $\{Z_s = j\}$, we know by Theorem 4.1 that $U(t_{(j)})$ is a uniformly chosen set of size $j$, Unif$_j$, which we may assume is in $A_*$. Furthermore, by Lemma 4.6, the position of the walk at time $t_{(j)}$ is uniformly distributed on $\Gamma$, independently from $U(t_{(j)})$. (Again these descriptions refer in reality to approximation in the total variation sense.) By Lemma 4.7, the next point that is removed from $U(t_{(j)})$ is therefore uniformly chosen among $U(t_{(j)})$. Applying next Lemma 4.8 $j - 1 - k$ times successively, from this point onwards, at each successive stage until time $\tau_k$, points are removed uniformly at random. Since $U(t_{(j)})$ was uniformly distributed among sets of size $j$ initially, it therefore follows that (still under the conditional law given $\{Z_s = j\}$) the law of $U(\tau_k)$ is (close to, in the total variation sense) Unif$_K$. Since this is true for every $k + 1 \leq j \leq K$, we deduce

$$d_{TV}(U(\tau_k), \text{Unif}_K) \leq 2\varepsilon + o(1).$$

The result follows. \hfill \square

### 4.3 Convergence in total variation of cover time

We illustrate the results above by strengthening the mode of convergence for the rescaled cover time: namely the distribution $\nu_n$ of the random variable $Y_n = \frac{Z_n}{\log n} - \log n$, converges in total variation to a standard Gumbel distribution $\nu$:

**Theorem 4.9.** As $n$ tends to infinity, we have

$$d_{TV}(\nu_n, \nu) \to 0. \quad (4.40)$$

**Proof.** We need several ingredients. First, we saw in Theorem 4.5 that for each $k \geq 1$, $U(\tau_k)$ is approximately uniform, in particular, at time $\tau_1$, the walk is with probability $1 - o(1)$ macroscopically far from the last point $x_0$ to be visited, and hence, with the same arguments as for the convergence of the $k$ last points, the walk with probability $1 - o(1)$ get mixed again before touching $x_0$. Recalling moreover that the distance in total variation between the uniform distribution and the quasi-stationary distribution associated to $x_0$ is $o(1)$, we have that

$$d_{TV}(\frac{\tau_0 - \tau_1}{\mathbb{E}_\pi(T_0)}. \text{Exp}(1)) \to 0.$$

Let $\nu_{\text{cont}}$ denote the law of $\frac{\tau_0}{\mathbb{E}_\pi(T_0)} + X$, where $X$ is an independent exponential random variable with mean 1. Note that $\nu_{\text{cont}}$ is obtained by convolution with an exponential law and so has a 1-Lipschitz density with respect to Lebesgue measure. Furthermore, since $\tau_{\text{cov}} = \tau_0 = \tau_1 + (\tau_0 - \tau_1)$,

$$d_{TV}(\frac{\tau_{\text{cov}}}{\mathbb{E}_\pi(T_0)}, \nu_{\text{cont}}) \to 0. \quad (4.41)$$

Second, we have already shown that for each fixed $s \in \mathbb{R}$,

$$F_n(s) := \mathbb{P}(Y_n \leq s) = \mathbb{P}(Z_s = 0) \to e^{-e^{-s}} =: F(s). \quad (4.42)$$
Let now $\varepsilon > 0$ and let us fix $S = S(\varepsilon)$ large enough such that for $n$ large enough, we have $F_n(S) \geq 1 - \varepsilon$ and $F_n(-S) \leq \varepsilon$. In particular, for such $n$’s, we have

$$\nu_n([-S, S]) = \mathbb{P}(Y_n \in [-S, S]) \geq 1 - 2\varepsilon. \quad (4.43)$$

Let $f_Y$ be the density of $Y$ (which is just the density of the Gumbel law) and let $f_n$ denote the density of $\nu_{\text{cont}}$ after translating by $\log n$. Since $d_{\text{TV}}(\nu_n, f_n) \to 0$ it suffices to show that

$$d_{\text{TV}}(f_n, f_Y) = (1/2) \int_{s \in \mathbb{R}} |f_n(s) - f_Y(s)| \, ds \to 0. \quad (4.44)$$

Observe that $f_n$ is 1-Lipschitz over $[-S, S]$, since it is a convolution of some given law with an exponential law. It is also pointwise bounded at, say $s = 0$ (indeed, since $f_n$ is Lipschitz, it cannot be large at any point without its integral being large, which is not possible by (4.41)). It is therefore uniformly equicontinuous, and by the Ascoli–Arzela theorem has subsequential uniform limits. However, the limit can only be $f_Y$, again by (4.41). Thus $f_n$ converges to $f_Y$ uniformly over $[-S, S]$, and hence also in the $L^1$ sense over $[-S, S]$. This proves (4.44). \qed

## 5 Refining the Aldous–Brown approximation

In this section we provide a refinement of Aldous and Brown’s result about hitting time from stationarity ([AB92], our equation (2.67)). We believe this refinement is of interest in its own right. In recent years there has been much interest in understanding the quasi-stationary distribution, and the rate of convergence to it for the chain conditioned on not hitting the corresponding set, e.g., [DM09, DM15, DHESC21]. The results below should be useful in that context too.

We first explain why such a refinement is needed here. In the examples studied in §7 we have that $t_{\text{ref}} \log n \asymp \mathbb{E}_{\pi}[T_A]$. As always, denote the quasi-stationary distribution of $A$ by $\alpha$ (suppressing the dependence of $\alpha$ on $A$ in the notation). Then by (2.67) we get that

$$\mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A] = O(\mathbb{E}_\pi[T_A]/\log n).$$

Since we consider $\mathbb{P}_\pi[T_A > t]$ for times $t \asymp \mathbb{E}_\pi[T_A] \log n$, in order to replace $\mathbb{E}_\alpha[T_A]$ with $\mathbb{E}_\pi[T_A]$ in the term $\exp(-t/\mathbb{E}_\alpha[T_A])$ in (2.67), it is crucial for us that $\mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A] = o(\mathbb{E}_\pi[T_A]/\log n)$. Indeed, for $t = (1 \pm o(1))\mathbb{E}_\alpha[T_A] \log n$, if $\mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A] \gtrsim \mathbb{E}_\alpha[T_A]/\log n$, then

$$\frac{\exp(-t/\mathbb{E}_\alpha[T_A])}{\exp(-t/\mathbb{E}_\pi[T_A])} = \exp\left(\frac{t (\mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A])}{\mathbb{E}_\alpha[T_A] \mathbb{E}_\pi[T_A]}\right) \gtrsim 1,$$

whereas we need this ratio to be $1 + o(1)$ for our purposes. However, when $t_{\alpha} \log n \asymp \mathbb{E}_\pi[T_A]$ (2.67) only implies that $\mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A] = O(\mathbb{E}_\pi[T_A]/\log n)$. A refinement of (2.67) is therefore required.

Consider a reversible Markov chain with a finite state space $V$. Denote the transition matrix of the chain by $P$ and its stationary distribution by $\pi$. Let $\emptyset \neq A \subsetneq V$. Throughout this section we assume without repeating this assumption that for all $a, b \in A^c$ we have that $\mathbb{P}_a[T_b < T_A] > 0$. (The next remark explains how can this condition be relaxed.)

For any non-empty $B \subset V$, we write $\pi_B$ for the distribution of $\pi$ conditioned on $B$. That is, $\pi_B(\cdot) := \frac{\pi(\cdot) 1_{x \in B}}{\pi(B)}$. 45
Remark 5.1. If the above condition fails, one can consider equivalent classes \(B_1, \ldots, B_\ell\) of the equivalence relation (on \(B\)) \(a \sim b\) iff \(\mathbb{P}_a[T_B < T_A] > 0\). By reversibility this relation is indeed symmetric, whereas transitivity follows by the strong Markov property. In this case

\[
\mathbb{P}_{\pi_A}[T_A > t] = \sum_{i=1}^{\ell} \frac{\pi(B_i)}{\pi(A^c)} \mathbb{P}_{\pi_{B_i}}[T_{B_i} > t].
\]

The theory we develop below can be applied to \(\mathbb{P}_{\pi_{B_i}}[T_{B_i} > t]\) for all \(i\).

Before stating our refinement of Aldous-Brown we need to introduce some additional notation. We denote by \(\alpha/\pi : V \to \mathbb{R}\) the Radon-Nikodym derivative of \(\alpha\) w.r.t. \(\pi\). That is 

\[
(\alpha/\pi)(a) := \alpha(a)/\pi(a)
\]

for all \(a \in V\) (with the convention that \(\alpha(a) = 0\) if \(a \notin A\)). For \(f, g : V \to \mathbb{R}\) we define the inner-product \((f, g)_\pi := \mathbb{E}_\pi[fg]\) (where \(\mathbb{E}_\pi[h] := \sum_{x \in V} \pi(x)h(x)\)), \(L^2\) norm \(\|f\|_2 := \sqrt{(f, f)_\pi}\) and variance w.r.t. \(\pi\), \(\text{Var}_\pi f := \|f - \mathbb{E}_\pi[f]\|_2^2\). Lastly, we denote the \(\varepsilon\) total variation mixing time of this chain by \(\text{t}_{\text{mix}}(\varepsilon)\) and its relaxation-time by \(t_{\text{rel}}\). Below we consider the continuous-time setup. We first state all of our results for this section.

**Theorem 5.2** (Refinement of Aldous-Brown). In the above setup and notation we have that for all \(t \geq 0\)

\[
0 \leq \mathbb{P}_\pi[T_A > t] - \frac{1}{\|\alpha/\pi\|_2^2} \exp \left(-\frac{t}{\mathbb{E}_\alpha[T_A]}\right) \leq \left(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A)\right) \exp \left(-\frac{t}{t_{\text{rel}}}\right).
\]

Consequently, integrating over \(t > 0\),

\[
0 \leq \mathbb{E}_\pi[T_A] - \frac{1}{\|\alpha/\pi\|_2^2} \mathbb{E}_\alpha[T_A] \leq \left(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A)\right) t_{\text{rel}},
\]

and in particular

\[
0 \leq \mathbb{E}_\alpha[T_A] - \mathbb{E}_\pi[T_A] \leq \left(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} \right) \mathbb{E}_\alpha[T_A].
\]

The \(L^2\) norm (with respect to \(\pi\)) of a signed measure \(\rho\) is defined as \(\|\rho\|_{2,\pi} := \|\rho/\pi\|_2\). Observe that \(\|\alpha/\pi\|_2^2 - 1 = \|\alpha/\pi - 1\|_2^2\) is exactly \(\|\alpha - \pi\|_{2,\pi}^2\). We also observe that by Cauchy–Schwarz

\[
\|\alpha/\pi\|_2^2 \pi(B) \geq \mathbb{E}_\pi[(\alpha/\pi)1_B]^2 = \|\alpha/\pi\|_2^2 = 1.
\]

Hence \(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) = \pi(B) - \frac{1}{\|\alpha/\pi\|_2^2} \geq 0\).

**Remark 5.3.** If instead of \(\mathbb{E}_\pi[T_A]\) or \(\mathbb{P}_\pi[T_A > t]\) we consider \(\mathbb{E}_{\pi_B}[T_A]\) or \(\mathbb{P}_{\pi_B}[T_A > t]\), respectively, then the term \(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A)\) in (5.1) and (5.2) is replaced by \(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2}\).

To make use of Theorem 5.2 we require an upper bound on \(\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A)\). The next theorem gives such a bound whenever \(t_{\text{rel}}/\mathbb{E}_\alpha[T_A]\) is small.

Denote by \(P_B\) the transition matrix of the chain killed upon hitting \(A\) (that is, \(P_B(x, y) = P(x, y)1\{x, y \in B\}\)). Observe that by the reversibility of \(P\) we have that \(P_B\) is self-adjoint with respect to the inner-product \((f, g)_\pi := \mathbb{E}_\pi[f g]\) on the space

\[
C_0(B) := \{f \in \mathbb{R}^V : f(x) = 0 \text{ for all } a \in A\}.
\]

Hence \(P_B\) has \(m := |B|\) real eigenvalues \(1 - \lambda = \gamma_1 > \gamma_2 \geq \cdots \geq \gamma_m \geq -|\gamma_1|\), where the last inequality follows by the Perron-Frobenius Theorem, which also asserts that \(\gamma_m = -|\gamma_1|\) if and
only if the restriction of $P_B$ to $B$ has period 2, and that $\gamma_2 < \gamma_1$, due to our assumption that $\min_{a,b \in B} \mathbb{P}_a[T_b < T_A] > 0$, which is equivalent to the restriction of $P_B$ to $B$ being irreducible. A fairly standard application of the interlacing eigenvalues theorem (see Lemma 5.7) yields that

$$1 - \gamma_2 \geq 1/ t_{\text{rel}}.$$  \hspace{1cm} (5.4)

Denote $c_i := \mathbb{E}_\pi[f_i] = \sum_x \pi(x)f_i(x)$ and $\lambda_i := 1 - \gamma_i$.

We shall show later in this section that the law of $T_A$ under $\mathbb{P}_\pi$ can be written as a mixture $\pi(A)\delta_0 + \sum_{i=1}^m c_i^2 \nu_i$, where for all $i,j \in [m]$ we have that $\nu_i$ is the Exponential distribution with parameter $\lambda_i$, where for some $f_1, \ldots, f_m : V \to \mathbb{R}$ in $C_0(B)$ we have for all $i,j \in [m]$ that $c_i := \mathbb{E}_\pi[f_i]$, $\mathbb{E}_\pi[f_if_j] = 1 \{ i = j \}$, where $\mathbb{E}_\pi[g] := \sum_{x \in V} \pi(x)g(x)$, and that $P_B f_i = (1 - \lambda_i) f_i$. Moreover, we may take $f_1 := \|\alpha/\pi\|^2$, where $\alpha$ is the quasi-stationary distribution on $B$, which satisfies $\alpha P_B = (1 - \lambda_1) \alpha$. In particular,

$$\sum_{i=1}^m c_i^2 = \mathbb{P}_\pi[T_A > 0] = \pi(B) \quad \text{and} \quad c_i^2/\lambda_1 \leq \mathbb{E}_\pi[T_A] = \sum_{i=1}^m c_i^2/\lambda_i \leq \pi(B)/\lambda_1,$$

$$c_1^2 = 1 - \frac{\|\alpha/\pi\|^2}{\|\alpha/\pi\|^2} - \pi(A) \quad \text{and} \quad p := \sum_{j=2}^m c_j^2 = \pi(B) - c_1^2 - \frac{\|\alpha/\pi\|^2}{\|\alpha/\pi\|^2} - \pi(A).$$  \hspace{1cm} (5.5)

Finally, let us define $t_{\text{med}}$ by

$$\sum_{i=2}^m c_i^2 (1 - e^{-\lambda_i t_{\text{med}}})^2 = \frac{1}{2} \sum_{i=2}^m c_i^2.$$  

By (5.4) we have that $t_{\text{med}} \leq \log(1 + \frac{1}{\sqrt{2}})/\lambda_2 \leq \log(1 + \frac{1}{\sqrt{2}}) t_{\text{rel}} \leq \frac{1}{\sqrt{2}} t_{\text{rel}}$.

**Theorem 5.4** (Bounding $\|\alpha/\pi\|^2 - 1$). For all $\emptyset \neq A \subset V$ we have that

$$\frac{\|\alpha/\pi\|^2}{\|\alpha/\pi\|^2} - \pi(A) = 2 \sum_x \pi(x) (\mathbb{P}_\pi[T_A \leq t_{\text{med}}])^2 - 2 \frac{1}{\|\alpha/\pi\|^2} (1 - e^{-\lambda_1 t_{\text{med}}})^2.$$  \hspace{1cm} (5.6)

Note that when $\pi(A)$ is small (5.6) offers an improvement over the estimate $\frac{\|\alpha/\pi\|^2 - 1}{\|\alpha/\pi\|^2} \leq t_{\text{rel}}/\mathbb{E}_\alpha[T_A]$ of Aldous and Brown. Indeed, if $t_{\text{rel}} = \varepsilon \mathbb{E}_\alpha[T_A]$ for $\varepsilon \leq 1/2$ then by (5.1)

$$\sum_x \pi(x) (\mathbb{P}_\pi[T_A \leq t_{\text{rel}}])^2 \leq \mathbb{P}_\pi[T_A \leq t_{\text{rel}}] \leq 2 t_{\text{rel}}/\mathbb{E}_\alpha[T_A] = 2 \varepsilon.$$

Of course the first inequality can be very wasteful. As we shall see in the next theorem, in many cases we have that $\sum_x \pi(x) (\mathbb{P}_\pi[T_A \leq t_{\text{rel}}])^2 \leq (t_{\text{rel}}/\mathbb{E}_\alpha[T_A])^2$. We now specialize Theorems 5.2 and 5.4 to the case of vertex-transitive graphs. For the applications in this paper we are mostly interested in the “two-dimensional case” $m = 2$ (in the next theorem $m$ no longer stands for the cardinality of $B$). In that case the main term of the right hand side is of order $\lesssim 1/R^2$ which corresponds to $\lesssim 1/f(n)^2$ in the notations of Theorem 1.1.

**Theorem 5.5.** Let $G = (V, E)$ be a connected vertex-transitive graph of degree $d$ with $n$ vertices, $o \in V$, and $\alpha = \alpha_o$. Then

$$\frac{\|\alpha/\pi\|^2}{\|\alpha/\pi\|^2} - \pi(A) \lesssim_{d,k} \frac{D^4}{(\mathbb{E}_\pi T_0)^2} (1 + \frac{n}{D^4} \int_0^D s^3 ds \mathbb{V}(s)).$$  \hspace{1cm} (5.7)
In particular, if $D^mR = n$ where $1 \leq R < D$ and $2 \leq m \in \mathbb{N}$, we have

$$\|\alpha/\pi\|_2^2 - 1 - \pi(A) \lesssim_{d,k} \begin{cases} \frac{P^i_A}{(E_{\alpha}(\pi_i))^2} & m = 1, 2 \\ \frac{P^i_A}{(E_{\alpha}(\pi_i))^2} \left( 1 + \frac{R \log R}{D} \right) & m = 3 \\ \frac{P^i_A}{(E_{\alpha}(\pi_i))^2} (R + \log(\frac{D}{R})) & m = 4 \\ \frac{1}{1/n} & m \geq 5 \end{cases}.$$  

(5.8)

### 5.1 Preparation towards the proofs

We now present some background on quasi-stationary distributions and their relation to hitting times. Let $\emptyset \neq A \subseteq V$. Denote $B := A^c$. We begin the analysis by recalling (e.g. [AF, Ch. 4]) that the quasi-stationary distribution $\alpha = \alpha_B$ corresponding to the set $B$ satisfies that the first hitting time of $A$ under $P_\alpha$ has in continuous-time an Exponential distribution whose parameter $\lambda$ is the smallest eigenvalue of $I_B - P_B$, where $P_B(x,y) := P(x,y)1\{x,y \in B\}$ and $I_B(a,b) := 1\{a = b \in B\}$. Note that $\lambda > 0$ since $P_B$ is sub-stochastic. The matrix $P_B$ is the transition matrix of the chain killed upon hitting $A$, and $P_B - I_B$ is the Markov generator corresponding to the rate one continuous-time version of this killed chain. As we shall work in continuous-time, we introduce the notation $P_{t,B} := e^{-t(I_B-P_B)} = e^{-t \sum_{i=0}^{\infty} (P_B)^\ell}$.

We now prove the above claim. It is an immediate consequence of the Perron-Frobenius Theorem that there exists a distribution $\alpha$ such that $\alpha P_B = (1 - \lambda)\alpha$ from which it follows that for all $t \geq 0$ we have that

$$\alpha P_{t,B} = e^{-t(I_B-P_B)} = e^{-t \sum_{\ell=0}^{\infty} \frac{[(1 - \lambda)t]^{\ell}}{\ell!}}\alpha = e^{-\lambda t}\alpha,$$

and so $P_\alpha[T_A > t] = \alpha P_{t,B}(B) = e^{-\lambda t}$, where we used the fact that for all $x$, $\alpha P_{t,B}(x) := \sum_b \alpha(b)P_{t,B}(b,x) = P_\alpha[X_t = x, T_A > t]$. It follows that starting from $\alpha$ the law of $T_A$ is exponential with mean $1/\lambda$, and so $1/\lambda = E_\alpha[T_A]$.

The identity $\alpha P_{t,B} = e^{-\lambda t}\alpha$ also implies that for all $x \in B$ and all $t \geq 0$

$$P_\alpha[X_t = x \mid T_A > t] = e^{\lambda t}P_\alpha[X_t = x, T_A > t] = e^{\lambda t}\alpha P_{t,B}(x) = \alpha(x).$$

The last equation justifies the name “quasi-stationary distribution”.

We now present the derivation of the well-known fact that for continuous-time reversible chains, the hitting time of a set $A$ starting from the stationary distribution $\pi$ is a mixture of exponentials. In other words, this law is completely monotone. The proof we present for this well-known fact shall play a role in our refinement of Aldous-Brown.

It is natural to identify $P_{t,B}$ as an operator with its extension to $C_0(B)$ defined as follows: for $f : V \to \mathbb{R}$ supported on $B$

$$(P_{t,B}f)(a) := \sum_b P_{t,B}(a,b)f(b) = E_\alpha[f(X_t)1\{T_A > t\}].$$

Denote $m = |B|$. Let $f_1, \ldots, f_m$ be an orthonormal (w.r.t. the inner-product $\langle \cdot, \cdot \rangle_\pi$) basis of $C_0(B)$, such that $P_B f_i = \gamma_i f_i$ for all $i$. By reversibility we can take $f_1 = \frac{\alpha/\pi}{\|\alpha/\pi\|_2}$. Then $1_B = \sum_{i=1}^m c_i f_i$, where $c_i := E_\pi[f_i]$ and $P_{t,B}1_B = \sum_{i=1}^m c_i e^{-t(1-\gamma_i)} f_i$. Hence, by orthonormality, we have that

$$P_\pi[T_A > t] = \langle P_{t,B}1_B, 1_B \rangle_\pi = \sum_{i=1}^m c_i^2 e^{-t(1-\gamma_i)} = \frac{1}{\|\alpha/\pi\|_2^2} e^{-t/E_\alpha[T_A]} + \sum_{i=2}^m c_i^2 e^{-t(1-\gamma_i)}$$  

(5.9)
(similarly, in discrete time we have that $\mathbb{P}_\pi[T_A > t] = \langle P_B^t 1_B, 1_B \rangle_\pi = \sum_{i=1}^m c_i^2 \gamma_i^t$). Taking $t = 0$ we see that $\sum_{i=1}^m c_i^2 = \pi(B)$. Hence $\sum_{i=2}^m c_i^2 = \|\alpha/\pi\|_2^2 - 1 - \pi(A)$, and so

$$\frac{1}{\|\alpha/\pi\|_2^2} e^{-t/\mathbb{E}_\alpha[T_A]} \leq \mathbb{P}_\pi[T_A > t] \leq \frac{1}{\|\alpha/\pi\|_2^2} e^{-t/\mathbb{E}_\alpha[T_A]} + \left( \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \right) e^{-t(1-\gamma_2)}. \quad (5.10)$$

Since $\mathbb{P}_\alpha[T_A > t] = e^{-t(1-\gamma_1)} = e^{-t/\mathbb{E}_\alpha[T_A]}$, the last inequality can also be written as

$$0 \leq \mathbb{P}_\pi[T_A > t] - \frac{1}{\|\alpha/\pi\|_2^2} \mathbb{P}_\alpha[T_A > t] \leq \left( \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \right) e^{-t(1-\gamma_2)}. \quad (5.11)$$

For the sake of completeness, we recall the following lemma of Aldous and Brown [AB92, Lemma 10], which in conjunction with Lemma 5.7 and (5.10) immediately implies that for all $t \geq 0$, $\mathbb{P}_\pi[T_A > t] \geq (1 - \frac{1}{\mathbb{E}_\alpha[T_A]}) e^{-t/\mathbb{E}_\alpha[T_A]}$, which is the non-trivial direction of (2.67). We note that while the next lemma is taken from [AB92], the derivation of the (2.67) in [AB92] is considerably more complicated than the one presented here.

**Lemma 5.6.** In the above setup and notation we have that

$$\frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} \leq \frac{t_{\text{rel}}}{\mathbb{E}_\alpha[T_A]}. \quad (5.12)$$

**Proof.** Since $f_1 = \frac{\alpha/\pi}{\|\alpha/\pi\|_2}$ is supported on $B$ we get that $\langle (I - P)f_1(x) \rangle f_1(x) = \lambda f_1(x)^2$ for all $x$. This, together with the fact that $\|f_1\|_2 = 1$ gives that

$$\frac{1}{\mathbb{E}_\alpha[T_A]} = \lambda = \frac{\langle (I - P)f_1(x) \rangle}{\|f_1\|_2^2} = \frac{\text{Var}_\pi f_1}{\text{Var}_\pi f_1}. \quad (5.12)$$

Using the extremal characterization of the relaxation-time (see, e.g., Theorem 3.1 in [Ber16]), this implies that $\frac{\langle (I - P)f_1(x) \rangle}{\text{Var}_\pi f_1} \geq 1/t_{\text{rel}}$ and observing that $\text{Var}_\pi f_1 = \frac{\|\alpha/\pi - 1\|^2_2}{\|\alpha/\pi\|^2_2} = \frac{\|\alpha/\pi\|^2_2 - 1}{\|\alpha/\pi\|^2_2}$ conclude the proof. \qed

We recall the construction of the auxiliary Markov chain in which $A$ is collapsed into a single point. Its state space is $B \cup \{A\}$, where $B = A^c$. Its transitions are given by $K(x,y) = P(x,y)$, $K(x,\{A\}) = P(x,A)$, $K(\{A\},x) = \sum_{a \in A} \pi_A(a)P(a,x)$ for $x,y \in B$ and $K(\{A\},\{A\}) = \sum_{a \in A} \pi_A(a)P(a,A)$. This is a reversible chain w.r.t. the distribution $\pi$ given by $\hat{\pi}(\{A\}) = \pi(A)$ and $\hat{\pi}(x) = \pi(x)$ for all $x \in B$. Denote the relaxation time of $K$ by $t_{\text{rel}}(K)$.

**Lemma 5.7 (Interlacing eigenvalues theorem).** We have that

$$\frac{1}{1 - \gamma_2} \leq t_{\text{rel}}(K) \leq t_{\text{rel}}. \quad (5.13)$$

**Proof.** Denote the second largest eigenvalue of $K$ by $\hat{\lambda}$. By the extremal characterization of the second largest eigenvalue (e.g. [LP17, Remark 13.8]) we get that

$$\frac{1}{t_{\text{rel}}} = \frac{\min_{f: \text{Var}_\pi f \neq 0} \langle (I - P)f, f \rangle}{\text{Var}_\pi f} \leq \frac{\min_{f: \text{Var}_\pi f \neq 0, f(a) = f(b) \text{ for all } a,b \in A} \langle (I - P)f, f \rangle}{\text{Var}_\pi f} \leq \frac{\langle (I - K)f, f \rangle}{\text{Var}_\pi f} = \frac{\langle (I - K)f, f \rangle}{\text{Var}_\pi f},$$

where $\hat{f}(x) = f(x)$ for all $x \in B$ and $\hat{f}(\{A\}) = f(a)$ for $a \in A$. Hence the r.h.s. of the last display equals $1 - \lambda = 1/t_{\text{rel}}(K)$. To conclude the proof it remains to show that $\hat{\lambda} \geq \gamma_2$. Observe that $P_B$ is obtained from $K$ by deleting the row and column corresponding to $\{A\}$. It follows from the interlacing eigenvalues theorem that indeed $\hat{\lambda} \geq \gamma_2$, as desired. \qed
5.2 Proofs

Proof of Theorem 5.2. This follows at once by combining (5.10) and Lemma 5.7.

Proof of Theorem 5.4. Denote \( t : = t_{\text{med}} \). Let \( f_1 = \frac{\alpha/\pi}{\|\alpha/\pi\|_2^2}, f_2, \ldots, f_m \) be an orthonormal basis of \( C_0(B) \) such that \( P_{B,\pi} f_i = e^{-\lambda_i t} f_i \) for all \( i \in [m] \) and \( s \geq 0 \). We see that for all \( x, y \in B \)

\[
\mathbb{P}_x[X_t = y, T_A > t] = \langle P_{t,B} 1_y, \frac{1}{\pi(x)} \rangle_\pi = \sum_{i=1}^m \pi(y) f_i(y) f_i(x) e^{-t(1-\gamma_i)} = \alpha(y) \frac{\pi(x)}{\|\alpha/\pi\|_2^2} e^{-t(1-\gamma)} + \sum_{i=2}^m \pi(y) f_i(y) f_i(x) e^{-t(1-\gamma_i)}.
\]

Summing over \( y \) (and recalling that \( c_i : = \mathbb{E}_{\pi} f_i \)) gives

\[
\mathbb{P}_x[T_A > t] = \alpha(x) \frac{\pi(x)}{\|\alpha/\pi\|_2^2} e^{-t(1-\gamma)} + \sum_{i=2}^m c_i f_i(x) e^{-t(1-\gamma_i)}. \tag{5.14}
\]

Let \( h(x) : = \mathbb{P}_x[T_A \leq t] - c_1 f_1(x)(1-e^{-\lambda_1 t}) \). By (5.14) we have that \( h(x) = \sum_{i=2}^m c_i f_i(x)(1-e^{-\lambda_i t}) \) and \( \mathbb{P}_x[T_A \leq t] = \sum_{i=1}^m c_i f_i(x)(1-e^{-\lambda_i t}) \), where \( c_i : = \mathbb{E}_{\pi} f_i \). Hence, by orthonormality of \( f_1, \ldots, f_m \), the definition of \( t_{\text{med}} \) and the fact that \( \sum_{i=2}^m c_i^2 = \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \) (as explained above (5.10)), we get that

\[
\sum_x \pi(x) h^2(x) = \sum_{i=2}^m c_i^2 (1-e^{-\lambda_i t})^2 = \frac{1}{2} \sum_{i=2}^m c_i^2 = \frac{1}{2} \left( \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \right),
\]

\[
\sum_x \pi(x) (\mathbb{P}_x[T_A \leq t])^2 = \sum_{i=1}^m c_i^2 (1-e^{-\lambda_i t})^2.
\]

Hence,

\[
\sum_x \pi(x) (\mathbb{P}_x[T_A \leq t])^2 = \sum_x \pi(x) h^2(x) + c_1^2 (1-e^{-\lambda_1 t})^2
\]

\[
= \frac{1}{2} \left( \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \right) + \frac{1}{2} \left( \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \right). \tag{5.15}
\]

Proof of Theorem 5.5. We first prove (5.8). Let \( A \) be of size \( k \) and \( o \in V \). Denote \( t : = t_{\text{mid}} \leq 1/\sqrt{2} t_{\text{rel}} \). By Theorem 5.4 our goal is to show that \( \frac{\sum_x \pi(x) (\mathbb{P}_x[T_A \leq t])^2}{C(m)(dk)^2} \) is upper bounded by the r.h.s. of (5.8). Let \( \tau \sim \text{Exp}(\frac{1}{t_1}) \) be independent of the random walk, where \( t_1 : = 2 t_{\text{rel}} \). Then \( \mathbb{P}(\tau \geq t)^2 \geq \exp(-\frac{1}{\sqrt{2}}) \geq \frac{2}{3} \) and so by transitivity (used to argue that \( \mathbb{P}_\pi[T_A \leq \tau] \) is independent of \( A \)),

\[
\frac{2}{3} \sum_x \pi(x) (\mathbb{P}_x[T_A \leq t])^2 \leq \sum_x \pi(x) (\mathbb{P}_x[T_A \leq t])^2 \leq \sum_x \pi(x) \left( \sum_{a \in A} \mathbb{P}_x[T_A \leq t] \right)^2 \leq k^2 \sum_x \pi(x) (\mathbb{P}_x[T_{\text{rel}} \leq \tau])^2, \tag{5.16}
\]

by the Cauchy–Schwarz inequality. Let \( N = L_o(\tau) \) be the time the chain spent at \( o \) by time \( \tau \). Note that \( \mathbb{P}_x[X_s = x, \tau > s] = P_s(x,y) e^{-\frac{\sigma}{\tau}} \). Hence \( \mathbb{E}_x[N] = \int_0^\infty P_s(x,o) e^{-\frac{\sigma}{\tau}} \mathrm{d}s \). By the memory
property of the exponential distribution $\kappa := \mathbb{E}_x[\mathbb{N} \mid \mathbb{N} > 0] = \mathbb{E}_o[\mathbb{N}] = \int_0^\infty p_s(o,o)e^{-\frac{t}{\kappa}}ds$.

Using

$$
P_x[T_o \leq \tau] = \frac{\mathbb{E}_x[\mathbb{N}]}{\mathbb{E}_x[\mathbb{N} \mid \mathbb{N} > 0]},
$$

and reversibility, and the change of variables $s + r = u$, we get that

$$
\sum_x \pi(x)(\mathbb{P}_x[T_o \leq \tau])^2 = \kappa^{-2} \sum_x \pi(x) \left( \int_0^\infty p_s(x,o)e^{-\frac{t}{\kappa}}ds \right)^2
\begin{align*}
&= \kappa^{-2} \sum_x \pi(o) \left( \int_0^\infty p_s(o,x)e^{-\frac{t}{\kappa}}ds \right) \left( \int_0^\infty p_r(x,o)e^{-\frac{r}{\kappa}}dr \right) \\
&= (\kappa)^{-2} \int_0^\infty nuP_u(o,o)e^{-\frac{u}{\kappa}}du \\
&=: J.
\end{align*}
$$

Let $f_1, \ldots, f_n$ be an orthonormal basis of $\mathbb{R}^V$ with respect to $\langle \cdot, \cdot \rangle_\pi$, such that $P_s f_i = e^{-\beta_i s} f_i$ for all $i$. We now argue that

$$
\int_0^\infty ns p_s(o,o)e^{-\frac{s}{\kappa}}ds \leq t_1^2 + \sum_{i=2}^n \beta_i^{-2}.
$$

Indeed, $\int_0^\infty ns p_s(o,o)e^{-\frac{s}{\kappa}}ds = t_1^2 + \int_0^\infty ns (p_s(o,o) - \pi(o)) e^{-\frac{s}{\kappa}}ds$ and so by transitivity, we get that

$$
\int_0^\infty ns p_s(o,o)e^{-\frac{s}{\kappa}}ds - t_1^2 = \int_0^\infty \sum_x s (p_s(x,x) - \pi(x)) e^{-\frac{s}{\kappa}}ds = \sum_{i=2}^n \int_0^\infty se^{-\beta_i s - \frac{s}{\kappa}}ds
\begin{align*}
&= \sum_{i=2}^n \frac{1}{\beta_i + 1/t_1} \leq \sum_{i=2}^n \beta_i^{-2}.
\end{align*}
$$

Similarly, using the fact that $\beta_i + \frac{1}{\kappa t_1} \leq \frac{3}{2}\beta_i$ for $i \geq 2$ (by the choice of $t_1$)

$$
n\kappa - t_1 = \int_0^\infty \sum_x (p_s(x,x) - \pi(x)) e^{-\frac{s}{\kappa}}ds
\begin{align*}
&= \sum_{i=1}^n \int_0^\infty e^{-s(\beta_i + 1/t_1)}ds \\
&\geq \sum_{i=2}^n \frac{1}{\beta_i + 1/t_1} \geq \frac{2}{3} \sum_{i=2}^n \frac{1}{\beta_i} = \frac{2}{3} \mathbb{E}_\pi[T_o],
\end{align*}
$$

where we used the eigentime identity $\sum_{i=2}^n \beta_i^{-1} = \sum_x \pi(x)\mathbb{E}_\pi[T_x] = \mathbb{E}_\pi[T_o]$ (see e.g. [AF, p. 117]), where the last equality holds by transitivity.

We now bound $\frac{1}{n} \sum_{i=2}^n \beta_i^{-2}$ from above. Let $\mu = (1/n) \sum_{i=1}^n \delta_{\beta_i}$ be the spectral measure and recall that $\beta_1 = 0$. It will be convenient to also consider $\nu = \mu - \delta_{\{0\}}/n$. Let $X \sim \mu$. Then, since $\mu((0,r]) = \mu([0,r]) - 1/n = \nu((0,r])$,

$$
\frac{1}{n} \sum_{i=2}^n \beta_i^{-2} = \mathbb{E}[X^{-2} \{X > 0\}] = \int_0^{\beta_2^{-2}} \mathbb{P}[s \leq X^{-2} < \infty]ds \leq 1/4 + \int_{1/4}^{\beta_2^{-2}} \mathbb{P}[s \leq X^{-2} < \infty]ds
\begin{align*}
&= \frac{1}{4} + \int_{\beta_2^{-2}}^{2^{1/4}} \frac{2}{\sqrt{s}} \mathbb{P}[0 < X \leq r]dr \quad \text{(change of variables } r = \frac{1}{\sqrt{s}}).\end{align*}
$$

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when

\[ \int_{\log 2}^{\log 8} \frac{2}{r^4} \nu([0, r]) dr \leq \frac{1}{4} + \int_{\log 2}^{\log 8} \frac{2}{r^4} \nu([0, r]) dr, \]

where we used the fact that by transitivity \( \beta_2 \geq \frac{1}{D^2} \) (see e.g. [LOG18, Theorem 1.7]).

Denote the number of vertices in a ball of radius \( |\rho| \) by \( V(\rho) \). Theorem 1.7 in [LOG18] (with their parameter \( \alpha \) set to \( \alpha = 1/2 \)) asserts that

\[ \nu([0, r]) \leq \frac{4}{V(\sqrt{(2dr)^{-1}})}. \]

Substituting this bound, and using the change of variables \( s = \sqrt{(2dr)^{-1}} \) yields

\[ \int_{\log 2}^{\log 8} \frac{2}{r^4} \nu([0, r]) dr \leq 64d^2 \int_{1/\sqrt{2}}^{D/\sqrt{2}} s^3 ds / V(s). \]

Putting everything together we get that

\[ J \leq \left( t_1 + \frac{4}{V(T_0)} \right)^2. \]

Recall that from (5.15), (5.16), and (5.17), \( \|\alpha/\pi\|_2^2 - \pi(A) \leq 5k^2 J \). Since \( t_1 \leq 2dD^2 \) (and \( D \geq 1 \)), we deduce that

\[ \frac{\|\alpha/\pi\|_2^2 - 1}{\|\alpha/\pi\|_2^2} - \pi(A) \leq \frac{5k^2}{4} \frac{25D^4 + 64d^2 n \int_{0}^{D} s^3 ds / V(s)}{(E_T T_0)^2} \]

\[ \leq 720 (kd)^2 \frac{D^4 + n \int_{0}^{D} s^3 ds / V(s)}{(E_T T_0)^2} \]

\[ = 720 (kd)^2 \frac{D^4}{(E_T T_0)^2} \left( 1 + \frac{n}{D^4} \int_{0}^{D} s^3 ds / V(s) \right). \]

The proof of (5.8) is concluded by substituting the bounds on \( V(s) \) of Tessera and Tointon [TT21, Corollary 1.5]

\[ V(s) \geq \begin{cases} 
    c(m)s^{m+1} & s \leq R \\
    c(m)Rs^m & s > R 
\end{cases} \]

(5.20)

and using for \( m \geq 5 \) the fact that \( E(T_0) = \sum_{n=2}^{\infty} \frac{1}{n} \geq \frac{1}{2}(n - 1) \geq \frac{1}{4} \) Note that replacing \( c(m) \) by \( \min_{1 \leq m \leq 5} c(m) \) removes the dependence in \( m \) of the constant.

5.3 Alternative proof for the low dimensional case

In Section 3, we proved that if \( f(n) \gg \log n \), then the factorial moments \( E[Z_i^k] \) converge to \( e^{-ks} \). A difficulty was that the approximation (2.76) of \( q_A \) did not hold if \( f(n) \lesssim (\log n)^2 \). To prove that the result holds under the sharp diameter condition \( n \gg D^2 \log n \), we had to approximate directly the expected hitting times \( E_T T_0 \) using auxiliary chains. The reason for this is that the approximation appearing in the results of Aldous–Brown, which we recalled in (2.67), is not precise enough. We will show as a consequence of Theorem 5.5 that (2.76) holds even in the sharp case \( n \gg D^2 \log n \), and we will give a simpler proof of Proposition 3.12. We will first prove an approximation under the more general assumption \( n \gtrsim D^2 \), which will be useful in Section 7 where we will have \( n \asymp D^2 \log n \), and then specify to the case \( n \gg D^2 \log n \). The first proposition we prove refines Lemma 2.22 when \( t \) is not too large.
**Proposition 5.8.** Assume that $1 \leq f(n) < D$. Then we have uniformly over $A \subset \Gamma$ of size $k$ and $t \geq 0$ that

$$0 \leq \mathbb{P}_\alpha[T_A > t] - \mathbb{P}_\pi[T_A > t] = O\left(\frac{1}{f(n)^2}\right),$$

where the implicit constant depends on $d$ and $k$.

**Proof.** We have, from Theorem 5.5, noting that $f(n) = R$ and that we are in the case $m = 2$,

$$\frac{\|\alpha/\pi\|^2 - 1}{\|\alpha/\pi\|^2} - \pi(A) \geq \frac{d^2D^4 + n(D^2 - f(n)^2)}{f(n)(E_\pi[T_o])^2} \leq \frac{nD^2(1 + d^2/f(n))}{f(n)(E_\pi[T_o])^2} \leq \frac{(1 + d^2)n^2}{f(n)^2(E_\pi[T_o])^2}.$$  \hspace{1cm} (5.22)

We deduce from Theorem 5.2, that (uniformly) for every $t \geq 0$,

$$0 \leq \mathbb{P}_\pi[T_A > t] - \mathbb{P}_\alpha[T_A > t] \lesssim C(d,k)\left(\frac{n}{f(n)E_\pi[T_o]}\right)^2 \exp\left(-\frac{t}{t_{rel}}\right).$$  \hspace{1cm} (5.23)

Note also that from (5.22) (for the right hand side), we have

$$0 \leq \frac{\|\alpha/\pi\|^2 - 1}{\|\alpha/\pi\|^2} \lesssim \left(\frac{n}{f(n)E_\pi[T_o]}\right)^2,$$

so

$$\frac{1}{\|\alpha/\pi\|^2} = 1 - O\left(\frac{n}{f(n)E_\pi[T_o]}\right)^2.$$ \hspace{1cm} (5.25)

We deduce finally, as $e^{-t/t_{rel}} \leq 1$ and using that $E_\pi[T_o] \geq n/(8e)$ (see the proof of the lower bound of Proposition 2.14) that uniformly over every $t \geq 0$,

$$0 \leq \mathbb{P}_\alpha[T_A > t] - \mathbb{P}_\pi[T_A > t] \lesssim \left(\frac{n}{f(n)E_\pi[T_o]}\right)^2 \lesssim \frac{1}{f(n)^2}.$$ \hspace{1cm} (5.26)

where the implicit constant depends on $d$ and $k$. \hfill \square

**Corollary 5.9.** Assume that $1 \leq f(n) < D$. Then we have at time $t = E_\alpha T_A/f(n)$, uniformly over $A \subset \Gamma$ of size $k$, that

$$q_A = \frac{\mathbb{P}_\pi[T_A \leq t]}{\mathbb{P}_\pi[T_o \leq t]} \left(1 + O\left(\frac{1}{f(n)}\right)\right).$$ \hspace{1cm} (5.27)

In particular, if $f(n) \gg \log n$, we have at time $t = D^2$, that

$$q_A = \frac{\mathbb{P}_\pi(T_A \leq t)}{\mathbb{P}_\pi(T_o \leq t)} \left(1 + o\left(\frac{1}{\log n}\right)\right).$$ \hspace{1cm} (5.28)

Moreover, the last estimation still holds if we take $t = n/\sqrt{f(n)\log n}$ instead of $t = D^2$.

**Proof.** We know from Proposition 5.8 that $E_\alpha T_A = E_\pi T_A \left(1 + O(1/f(n)^2)\right)$, and from the proof of Proposition 2.14 that $n/(8ek) \leq E_\pi T_A$, so $E_\pi T_A \gtrsim n$. Consequently, we have for $t \leq n$ that

$$\mathbb{P}_\alpha[T_A > t] = \exp(-t/E_\alpha T_A) \gtrsim 1.$$ \hspace{1cm} (5.29)

We deduce from Proposition 5.8 that for $t \leq n$,

$$\mathbb{P}_\pi(T_A > t) = \exp\left(-\frac{t}{E_\alpha T_A}\right) \left(1 + O\left(\frac{1}{f(n)^2}\right)\right).$$

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Assume now moreover that \( t \ll n \), then we can follow the proof of Proposition 2.25. The only difference is that we do not upper bound \( \mathbb{E}_n T_A \) by \( n \) as they are not necessarily of the same order of magnitude. First, we have

\[
\mathbb{E}_n T_A = \frac{t}{\mathbb{P}_\pi(T_A \leq t)} \left( 1 + O \left( \frac{\mathbb{E}_n T_A}{tf(n)^2} \right) + O \left( \frac{t}{\mathbb{E}_n(T_A)} \right) \right),
\]

Taking quotients, we get

\[
q_A = \frac{\mathbb{P}_\pi(T_A \leq t)}{\mathbb{P}_\pi(T_o \leq t)} \left( 1 + O \left( \frac{\mathbb{E}_n T_A}{tf(n)^2} \right) + O \left( \frac{t}{\mathbb{E}_n(T_A)} \right) \right),
\]

and we finally have at time \( t = \mathbb{E}_n T_A/f(n) \),

\[
q_A = \frac{\mathbb{P}_\pi(T_A \leq t)}{\mathbb{P}_\pi(T_o \leq t)} \left( 1 + O \left( \frac{1}{f(n)} \right) \right).
\]

For the last point, if \( f(n) \gg \log n \), we know from Proposition 2.14 (and Proposition 2.21) that \( \mathbb{E}_n T_A \asymp n \). At time \( t = n/\sqrt{f(n) \log n} \), we have both \( n/(tf(n)^2) = o(1/\log n) \) and \( t/n = o(1/\log n) \) so the error in (5.30) is \( o(1/\log n) \). Taking quotients, we get the desired result.

This allows us to strengthen the results of Section 3.3. In particular, the second part of Proposition 3.6 becomes:

**Proposition 5.10.** Assume that \( n \gg D^2 \log n \) and \( \delta < \text{mindist}(A) \). Then at time \( t = n/\sqrt{f(n) \log n} \), we have

\[
q_A - k \geq - \max_{x,y \in \Gamma, d(x,y) = \delta} \mathbb{P}_x(T_y < t) + o \left( \frac{1}{\log n} \right).
\]

We can now give a simpler proof of Proposition 3.12.

**Alternative proof of Proposition 3.12:** Let \( \delta \) such that \( \delta_{\text{meso}} \leq \delta \leq D/2 \) and \( A \subset \Gamma \) such that \( |A| = k \), and \( \text{mindist}(A) \geq \delta \). Then from the intermediate step of (2.54), and Lemma 2.15 we have immediately, at \( t = n/\sqrt{f(n) \log n} \),

\[
\max_{x,y \in \Gamma, \delta_{d(x,y)} = \delta} \mathbb{P}_x(T_y < t) \lesssim \mathbb{E}_x L_y(t) \lesssim \frac{1}{\delta} + \frac{\log(D/\delta)}{f(n)} + \frac{t}{D^3} + \frac{1}{f(n)} = \frac{\log(D/\delta)}{f(n)} + o \left( \frac{1}{\log n} \right).
\]

### 6 When the diameter condition fails

#### 6.1 Green function estimates

In this section \( \Gamma = (V,E) \) is a vertex-transitive graph of degree \( d \). Denote the transition matrix of simple random walk on \( \Gamma \) by \( P \) and the time \( t \) transition probabilities for the rate 1 continuous-time simple random walk on \( \Gamma \) by \( P_t = e^{-t(I-P)} \) (so that we have \( P_t(x,y) = p_t(x,y) \)). Recall that for \( x, y \in V \) and \( t \geq 0 \),

\[
h_t(x, y) = p_t(x, y) - \frac{1}{n}.
\]

We start by adapting a gradient inequality, due to Diaconis and Saloff-Coste [DSC94] for Cayley graphs, to vertex-transitive graphs. The main difference is that we need to use the mass transport principle.
Proposition 6.1. Let \( \Gamma = (V, E) \) be a finite vertex-transitive graph of degree \( d \). Then for all \( s, t \geq 0 \) and \( o, y, z \in V \) we have that

\[
|h_{t+s}(o, y) - h_{t+s}(o, z)| \leq d(y, z) \sqrt{\frac{d}{2es}} h_t(o, o). \tag{6.2}
\]

Proof. By the triangle inequality it suffices to prove the inequality when \( y \) and \( z \) are neighbours. Let then \( y, z \in V \) such that \( y \sim z \). First observe that

\[
h_{t+s}(o, y) - h_{t+s}(o, z) = p_{t+s}(o, y) - p_{t+s}(o, z)
= \sum_{w \in V} p_{t/2}(o, w)(p_{t/2+s}(w, y) - p_{t/2+s}(w, z))
= \sum_{w \in V} h_{t/2}(o, w)(h_{t/2+s}(w, y) - h_{t/2+s}(w, z))
\]

because, by reversibility, \( \sum_{w \in V}(h_{t/2+s}(w, y) - h_{t/2+s}(w, z)) = 0 \). By reversibility again, we have

\[
\sum_{w \in V} h_{t/2}(o, w)^2 = \left( \sum_{w \in V} p_{t/2}(o, w)p_{t/2}(w, o) \right) - \frac{1}{n} = p_t(o, o) - \frac{1}{n} = h_t(o, o). \tag{6.3}
\]

It follows from the triangle inequality and the Cauchy–Schwarz inequality that

\[
|h_{t+s}(o, y) - h_{t+s}(o, z)|^2 \leq \left( \sum_{w} h_{t/2}(o, w)^2 \right) \left( \sum_{w} |h_{t/2+s}(w, y) - h_{t/2+s}(w, z)|^2 \right)
= h_t(o, o) \sum_{w} |p_{t/2+s}(y, w) - p_{t/2+s}(z, w)|^2
\leq h_t(o, o) \sum_{w} \sum_{y', y''} |p_{t/2+s}(y, w) - p_{t/2+s}(y', w)|^2
= h_t(o, o) \sum_{w} F(y, w),
\]

where

\[
F : (a, b) \mapsto \sum_{x : x \sim a} |p_{t/2+s}(a, b) - p_{t/2+s}(x, b)|^2.
\]

Observe that for any \( \gamma \in \text{Aut}(\Gamma) \), and \( a, b \in V \), we have \( F(\gamma(a), \gamma(b)) = F(a, b) \). Moreover \( \text{Aut}(\Gamma) \) is a discrete group of automorphisms, so by \([LP16][Corollary 8.9]\) is unimodular. Since \( \Gamma \) is vertex-transitive, \( \text{Aut}(\Gamma) \) is (by definition) transitive, so we can apply the mass transport principle. By \([LP16][Equation (8.4)]\), we hence have, for all \( y \in V \), that

\[
\sum_{w \in V} F(y, w) = \sum_{w \in V} F(w, y) = \sum_{w \in V} \sum_{w' : w' \sim w} |p_{t/2+s}(w, y) - p_{t/2+s}(w', y)|^2. \tag{6.5}
\]

Denote \( P_L := \frac{1}{2}(I + P) \). For \( f, g : V \to \mathbb{R} \) denote \( \langle f, g \rangle = \pi(o) \sum_{v \in V} f(v)g(v) \) and \( \|f\|^2 := \langle f, f \rangle \). Since \( I - P_L \) is self-adjoint and \( \langle (I - P_L)f, f \rangle \geq 0 \) for all \( f : V \to \mathbb{R} \), we may consider \( K := \sqrt{I - P_L} \) which is a self-adjoint operator satisfying \( K^2 = I - P_L \), and \( \langle (I - P_L)f, g \rangle = \langle Kf, Kg \rangle \) for all \( f, g : V \to \mathbb{R} \). Noting that \( P_s = e^{-2s(I - P_L)} \) we have that \( KP_s = q(I - P_L) \), where \( q : [0, 1] \to [0, 1] \) is defined by \( q(u) = \sqrt{1 - 2su} \), notation which we also extend (slightly abusing notation) to matrices. Since the spectral measure of \( I - P_L \) is supported on \([0, 1]\), it is standard that \( \|KP_s\|^2 = \|q(I - P_L)\|^2 \leq \max_{u \in [0, 1]} q(u)^2 = \frac{1}{4us} \), where \( \|KP_s\|_2 := \sup_{f \in \ell^2} \|f\|_2 = 1 \) \( \|KP_sf\|_2 \) is the operator norm of \( KP_s \).
For $r \geq 0$ and $w \in V$, denote $f_r(w) = h_r(w, y)$. By reversibility, the sum on the right hand side of (6.5) equals $2d$ times (see e.g., [LP17, Lemma 13.6])

$$\frac{1}{\pi(o)}((I - P_L) f_{t/2+s}, f_{t/2+s}) = \frac{1}{\pi(o)}(K^2 P_s f_{t/2}, P_s f_{t/2}) \leq \frac{1}{\pi(o)}\|KP_s\|_2^2 \|f_{t/2}\|_2^2 \leq \frac{h_t(o, o)}{4es},$$

where in the last inequality we used that $\|KP_s\|_2^2 \leq \frac{1}{4es}$ and (6.3).

It follows that the right hand side of (6.5) equals $\frac{dh_t(o, o)}{2es}$, which in conjunction with (6.4) concludes the proof. $\square$

We now prove that at times $t$ proportional to $D^2$ the distribution of the walk is still far from being uniform.

**Proposition 6.2.** Assume that $n \leq D^3$. There exists a constant $0 < a = a(d) < 1$ such that for all $t \leq aD^2$,

$$nh_t(o, o) \geq \frac{1}{2}. \quad (6.6)$$

**Proof.** Recall the off-diagonal heat kernel bound (2.48). Note that the constant $C$ in (2.47), as well as the constant $c$ and the implicit constant in (2.48) (call it $C'$ for this proof), depend only on the graph degree, and the growth of the function $g$ defined in (2.47). We can hence take the same constants for all (connected) finite vertex-transitive graphs of degree $d$ such that $D^3 \geq n$.

First assume that $D^2 \leq n < D^3$, and let $x$ such that $d(o, x) = D$. We have for every $D \leq t \leq D^2$,

$$p_t(o, x) \leq C' C \min(f(n)t, t^{3/2}) \exp\left(-c \frac{D^2}{t}\right). \quad (6.7)$$

Let $0 < a < 1$. At time $t = aD^2$ we have

$$\min(f(n)t, t^{3/2}) = \min(an, a^{3/2}D^3) \geq a^{3/2}n. \quad (6.8)$$

It follows that

$$p_t(o, x) \leq \frac{CC'}{a^{3/2}} \exp\left(-c \frac{1}{a}\right). \quad (6.9)$$

Fixing $a = a(c, C, C')$ small enough, we hence have at time $t = aD^2$ that $np_t(o, x) \leq 1/2$, i.e.

$$nh_t(o, x) \leq -\frac{1}{2}. \quad (6.10)$$

Finally, we have $h_t(o, o) = \max_{z, w \in \Gamma} |h_t(z, w)|$ by [LP17, Proposition 4.15] and vertex-transitivity. We conclude that

$$nh_t(o, o) \geq \frac{1}{2}. \quad (6.11)$$

The proof for $D \leq n < D^2$ is similar. $\square$

We define the Green function between two points $x, y \in V$ by

$$G(x, y) = \int_0^\infty h_t(x, y)dt. \quad (6.12)$$

The following proposition collects some useful identities involving Green functions.
Proposition 6.3. Let $\Gamma$ be any finite connected vertex-transitive graph. We have the following identities.

1. Fix $o \in \Gamma$. We have
   \begin{equation}
   \mathbb{E}_\pi T_o = nG(o, o) \geq n - (1 + \log n). \tag{6.13}
   \end{equation}

2. For every $x, y \in \Gamma$,
   \begin{equation}
   \mathbb{E}_x T_y = n \left( G(o, o) - G(x, y) \right). \tag{6.14}
   \end{equation}

3. For any subset set $A = \{x, y\}$ of size 2 of $\Gamma$,
   \begin{equation}
   q_A = \frac{2}{1 + \frac{G(x, y)}{G(o, o)}}. \tag{6.15}
   \end{equation}

Proof. The equalities in the first two points are stated in [AF] in a more general framework, as Lemma 2.11 and Lemma 2.12, respectively, and Section 2.2.3 explains why they also hold in continuous time. We moreover have, lower bounding the probability to be at $o$ by the probability to have never jumped:
   \begin{equation}
   G(o, o) \geq \int_0^{\log n} \left[ e^{-t} - \frac{1}{n} \right] dt = 1 - \frac{1 + \log n}{n}, \tag{6.16}
   \end{equation}
proving the inequality from the first point.

For the third point, let us denote with tildes the collapsed chain where $A$ is reduced to a point which we denote $\tilde{A}$ (which, if $x$ and $y$ are neighbours, has a loop at $\tilde{A}$), and jumps at rate 1. Then by [AF], Lemma 2.11, and transitivity of the original graph $\Gamma$, we have
   \begin{equation}
   \mathbb{E}_\pi T_A = \mathbb{E}_\pi T_{\tilde{A}} = \frac{1}{\pi(\tilde{A})} \tilde{G}(\tilde{A}, \tilde{A}) = \frac{2}{n} \left( G(o, o) + G(x, y) \right), \tag{6.17}
   \end{equation}
using the same idea as in the proof of Lemma 3.17 (the difference being that the set $A$ here has size two, so the argument is much simpler, and we are in continuous time with the walk jumping out of every vertex – including $\tilde{A}$ – at rate 1).

Therefore,
   \begin{equation}
   q_A = \frac{\mathbb{E}_\pi T_o}{\mathbb{E}_\pi T_A} = \frac{nG(o, o)}{G(o, o) + G(x, y)} = \frac{2}{1 + \frac{G(x, y)}{G(o, o)}}, \tag{6.18}
   \end{equation}
as desired.

We first show that the tail of the integral defining the Green function above decreases exponentially fast (this is similar to arguments already used in Section 3.7 although this was written in discrete time).

Lemma 6.4. There exist constants $c_1, c_2$ depending on $d$ such that uniformly over all real $b \geq 1$, we have, uniformly over all finite (connected) vertex-transitive graphs of degree $d$ such that $n \leq D^5$,
   \begin{equation}
   \int_{bD^2}^{\infty} h_t(o, o) dt \leq c_1 e^{-c_2 b D^2 / n}. \tag{6.19}
   \end{equation}

Proof. By spectral estimates, we have for all $t, s \geq 0$.
   \begin{equation}
   h_{t+s}(o, o) \leq e^{-s/t} h_t(o, o). \tag{6.20}
   \end{equation}
We deduce, splitting the integral into parts of length $D^2$, that
\[ \sum_{j=0}^{\infty} \int_{D^2}^{(j+1)D^2} h_t(o,o)dt \leq \left( \int_{D^2}^{2D^2} h_t(o,o)dt \right) \sum_{j=0}^{\infty} e^{-jD^2/t_{\text{rel}}}. \] (6.21)
By (2.8) and since $t \mapsto h_t(o,o)$ is decreasing, there exists a constant $c_0 = c_0(d)$ such that
\[ \int_{D^2}^{2D^2} h_t(o,o)dt \leq h_{D^2}(o,o) \cdot D^2 \leq c_0 \frac{D^2}{n}. \] (6.22)
Finally, by Lemma 2.7, $t_{\text{rel}} \leq dD^2$, hence $e^{-D^2/t_{\text{rel}}} \leq e^{-1/d}$. We deduce
\[ \sum_{j=0}^{\infty} e^{-jD^2/t_{\text{rel}}} \leq \frac{1}{1 - e^{-1/d}}. \] (6.23)
We have proved the claim for integers $b$. The claim for real $b$ follows immediately, since $b \mapsto \int_{bD^2}^{\infty} h_t(o,o)dt$ is decreasing. \(\square\)

**Proposition 6.5.** Let $(\Gamma)$ be a sequence of finite (connected) vertex-transitive graphs of fixed degree $d$, such that $|\Gamma| = n \leq D^3$. There exist positive constants $\eta = \eta(d), C = C(d)$ such that for all $x \in B(o, \eta D)$,
\[ G(o,x) \geq C \frac{D^2}{n}. \] (6.24)

**Proof.** Let $\varepsilon > 0$, to be chosen later. By Lemma 6.4, we can chose $b = b(d, \varepsilon) \geq 1$ large enough such that
\[ \int_{bD^2}^{\infty} h_t(o,o)dt \leq \varepsilon \frac{D^2}{n}. \] (6.25)
We hence have
\[ G(o,x) = \int_{0}^{\varepsilon D^2} h_t(o,x)dt + \int_{\varepsilon D^2}^{bD^2} h_t(o,x)dt + \int_{bD^2}^{\infty} h_t(o,x)dt. \]
Since for every $t$, $h_t(o,x) \geq -1/n$, and $h_t(o,x) \geq -h_t(o,o)$, we deduce that
\[ G(o,x) \geq \int_{\varepsilon D^2}^{bD^2} h_t(o,x)dt - 2\varepsilon \frac{D^2}{n}. \]
By Proposition 6.1, we have
\[ \int_{\varepsilon D^2}^{bD^2} h_t(o,x)dt \geq \int_{\varepsilon D^2}^{bD^2} h_t(o,o)dt - d(o,x) \int_{\varepsilon D^2}^{bD^2} \sqrt{\frac{d}{d t}} h_{t/2}(o,o)dt. \] (6.26)
By Proposition 6.2, we have, assuming without loss of generality that $\varepsilon \leq a/2$, that
\[ \int_{\varepsilon D^2}^{bD^2} h_t(o,o)dt \geq \int_{aD^2/2}^{aD^2} h_t(o,o)dt \geq \frac{a D^2}{4 n}. \] (6.27)
Finally, from (2.13) there exists a constant $C = C(d, \varepsilon)$ such that $h_{\varepsilon D^2/2} \leq C(d, \varepsilon)/n$, so we have, setting $C' = Cb \sqrt{\frac{d}{\varepsilon D^2}}$, for every $x \in B(o, \varepsilon D)$, that
\[ d(o,x) \int_{\varepsilon D^2}^{bD^2} \sqrt{\frac{d}{d t}} h_{t/2}(o,o)dt \leq d(o,x)(b - \varepsilon)D^2 \sqrt{\frac{d}{\varepsilon D^2}} h_{\varepsilon D^2/2}(o,o) \leq C' \frac{d(o,x) D^2}{D} \leq \frac{\varepsilon D^2}{n}. \] (6.28)
The arguments above show that taking $\varepsilon = a/24$, we have, for some $\eta > 0$ (depending only on $d$), that for every $x \in B(o, \eta D)$,

$$G(o, x) \geq \left(\frac{a}{4} - 3\varepsilon\right) \frac{D^2}{n} \geq \frac{a}{8} \frac{D^2}{n},$$

which concludes the proof. \qed

**Lemma 6.6.** Let $(\Gamma)$ be a sequence of finite (connected) vertex-transitive graphs of fixed degree $d$, such that $|\Gamma| = n \leq D^3$. There exists a constant $C = C(d) > 0$ such that for all $x \in \Gamma$,

$$G(o, x) \geq -C \frac{D^2}{n}.$$  \hspace{1cm} (6.30)

**Proof.** We have

$$G(o, y) = \int_0^\infty h_t(o, x) dt \geq \int_0^{D^2} \left(-\frac{1}{n}\right) dt - \int_{D^2}^\infty h_t(o, o) dt.$$  \hspace{1cm} (6.31)

Applying Lemma 6.4 to the second integral on the right hand side concludes the proof. \qed

Let us set, for every $x \in \Gamma$ and $c > 0$,

$$S_{o,c} := \left\{ z \in \Gamma : G(o, z) \leq -c \frac{D^2}{n} \right\}.$$  \hspace{1cm} (6.32)

**Corollary 6.7.** There exist constants $c = c(d) > 0$ and $c' = c'(d) > 0$ such that for all finite (connected) vertex-transitive graphs of degree $d$ such that $n = |\Gamma| \leq D^3$, we have

$$|S_{o,c}| \geq c'n.$$

**Proof.** Let us call for this proof $C_1$ the constant from Proposition 6.5 and $C_2$ the constant from Lemma 6.6. Also, by [TT20][Proposition 5.6], there is a constant $C_3 = C_3(d, \eta)$ such that $V(\eta D) \geq C_3 n$. We hence have

$$0 = \sum_{x \in \Gamma} G(o, x) = \sum_{x \in \Gamma} G(o, x) + \sum_{x \in \Gamma} G(o, x) + \sum_{x \in \Gamma} G(o, x)$$

$$= A_1 + A_2 + A_3.$$

The first term can be lower bounded using Proposition 6.5:

$$A_1 \geq \sum_{x \in B(o, \eta D)} G(o, x) \geq C_1 \frac{D^2}{n} V(\eta D) \geq C_1 C_3 D^2.$$  \hspace{1cm} (6.34)

Since the second sum is over at most $n$ terms, we have the raw bound

$$A_2 \geq -C_2 \frac{D^2}{n}.$$  \hspace{1cm} (6.35)

Finally, since by Lemma 6.6 for every $x$, $G(o, x) \geq -C_2 \frac{D^2}{n}$,

$$A_3 \geq -|S_{o,c}| C_2 \frac{D^2}{n}.$$  \hspace{1cm} (6.36)
where \( c = C_1 C_3 / 2 \).

All in all, dividing by \( D^2 \), we have proved that
\[
0 \geq C_1 C_3 - \frac{C_1 C_3}{2} - \frac{C_2}{n} |S_{o,c}| = c - \frac{C_2}{n} |S_{o,c}|,
\]
i.e., setting \( c := \frac{C_1 C_3}{2} \) and \( c' := c / C_2 = \frac{C_1 C_3}{2 C_2} \), that
\[
|S_{o,c}| \geq c'n,
\]as desired.

Roughly speaking, one should think of the set \( S_{o,c} \) as the set of points which are “far away” from \( o \). However, it turns out that the variations of the Green function are \textit{macroscopically continuous}, and hence that the set \( S_{o,c} \) contains a ball of size \( \asymp n \). This is the content of Proposition 6.8 and Proposition 6.9.

**Proposition 6.8.** Let \((\Gamma)\) be a sequence of finite (connected) vertex-transitive graphs of fixed degree \( d \), such that \(|\Gamma| = n \leq D^3\). Let \( 0 < \rho < 1 \). For every \( \varepsilon > 0 \), there exists a constant \( \delta = \delta(\rho, \varepsilon, d) > 0 \) such that for every \( x, y, z \in \Gamma \) such that \( d(o, x) \geq \rho D, d(x, y) \leq \delta D, \) and \( d(o, z) \leq \delta D, \) we have
\[
|G(o, x) - G(z, y)| \leq \varepsilon \frac{D^2}{n}.
\]

**Proof.** Let \( \varepsilon > 0 \). We first assume that \( z = o \) and \( D^2 \leq n \leq D^3 \). Let \( a, b > 0 \) to be fixed, and let \( x, y \in B(o, \rho D) \). Denote for this proof, for \( 0 < t_1 < t_2 \leq \infty, I(t_1, t_2) := \int_{t_1}^{t_2} |h_t(o, x) - h_t(o, y)| dt \). By the triangle inequality,
\[
|G(o, x) - G(o, y)| \leq I(o, D) + I(D, aD^2) + I(aD^2, bD^2) + I(bD^2, \infty).
\]
By the Carne-Varopoulos inequality, we have \( I(o, D) = o(D^2 / n) \). Now observe that for all \( t, |h_t(o, x) - h_t(o, y)| = |p_t(o, x) - p_t(o, y)| \leq \max(p_t(o, x), p_t(o, y)) \). To bound \( p_t(o, x) \) (and \( p_t(o, y) \)) we proceed as in the proof of Proposition 6.2 to get (6.9). (Now \( d(o, x) \geq \rho D \) instead of having \( d(o, x) = D \).) Let \( c \) as in (6.9). Observing that the bound on \( p_t(o, x) \) in the integral is maximised at \( t = aD^2 \), we have
\[
I(D, aD^2) \lesssim \int_D^{aD^2} \frac{1}{a^{3/2} n} \exp \left( -\frac{\rho^2}{a} \right) dt \leq \frac{e^{\rho^2 / a} D^2}{a^{1/2} n},
\]
which is less than \( \varepsilon D^2 / n \) if we fix \( a > 0 \) small enough. Let us also fix \( b > 0 \) large enough so that (6.25) holds. Then by the triangle inequality (and since for all \( t, h_t(o, o) = \max_x |h_t(o, x)| \)), we have \( I(bD^2, \infty) \leq 2 \varepsilon D^2 / n \). Finally, by Proposition 6.1 with \( s = aD^2 / 2 \) and making a change of variables, we have
\[
I(aD^2, bD^2) \leq \sqrt{d/e} \frac{d(x, y)}{D} \int_{aD^2 / 2}^{(b-a/2)D^2} h_1(o, o) dt \lesssim_{a,d} \frac{d(x, y) D^2}{D n},
\]
so for \( \delta > 0 \) fixed small enough, if \( d(x, y) \leq \delta D, \) we also have \( I(aD^2, bD^2) \leq \varepsilon D^2 / n \). Putting everything together, we have proved that for \( a, \delta > 0 \) fixed as above and \( x, y \) such that \( d(x, y) \leq \delta D, \) we have \( |G(o, x) - G(o, y)| \leq (4\varepsilon + o(1)) \frac{D^2}{n} \). We can bound similarly \( |G(o, y) - G(z, y)| \), and hence by the triangle inequality we have \( |G(o, x) - G(z, y)| \leq (8\varepsilon + o(1)) \frac{D^2}{n} \), concluding the proof when \( D^2 \leq n \leq D^3 \). The proof for \( D \leq n \leq D^2 \) is analog. \( \square \)
Proposition 6.9. There exist constants \(c, \rho > 0\) (depending on \(d\)) such that for all finite (connected) vertex-transitive graphs of degree \(d\) such that \(n = |\Gamma| \leq D^5\), there exist two (disjoint) balls \(S_1\) and \(S_2\) of radius \(\rho D\) such that for every \(x \in S_1\) and \(y \in S_2\),

\[
G(x, y) \leq -c \frac{D^2}{n}.
\]  

(6.43)

In particular, for such \(c, \rho\), \(S_{o,c}\) contains a ball of radius \(\rho D\).

Proof. By Corollary 6.7, there exists \(c > 0\) and \(x \in \Gamma\) such that \(G(o, x) \leq -cD^2/n\). Therefore, \(d(o, x) \geq \eta D\) where \(\eta\) is as in Proposition 6.5. Therefore, by Proposition 6.8 (where the role of \(\rho\) there is played by \(\eta\) here) we can find \(\delta > 0\) such that for every \(y \in B(x, \delta D)\), and \(z \in B(o, \delta D)\)

\[
|G(z, y) - G(o, x)| \leq c \frac{D^2}{n}.
\]  

(6.44)

It follows that for such \(y, z\), we have \(G(z, y) \leq -c \frac{D^2}{n}\), concluding the proof. \(\square\)

We will also need the following bound on \(t_{\text{hit}}\).

Proposition 6.10. Let \(C > 0\). Let \((\Gamma)\) be a collection of finite (connected) vertex-transitive of fixed degree \(d\), such that \(n = |\Gamma| \leq CD^2 \log n\). Then

\[
t_{\text{hit}} \lesssim_{d,C} D^2 \log n.
\]  

(6.45)

Proof. Recall that we have \(t_{\text{hit}} = nG(o, o)\) from Lemma 2.11 in [AF]. We split the proof into two cases, depending on the diameter of \(\Gamma\).

First, if \(D^2 \leq n \leq CD^2 \log n\), we write \(n = D^2 R\). Integrating the bound on \(p_t(o, o)\) from Proposition 2.1, and bounding the tail of the integral with Proposition 6.4, we have

\[
G(o, o) = \int_0^\infty h_t(o, o) dt \lesssim_d 1 + \frac{1}{R} + \frac{\log(D/R)}{R} + \frac{D^2}{n} \lesssim_{d,C} \frac{\log n}{R} = \frac{1}{n} D^2 \log n,
\]  

(6.46)

as desired.

Suppose now that \(D \leq n \leq D^2\), and let \(H\) such that \(n = DH\). Proceeding as above, we get

\[
G(o, o) \lesssim_d 1 + \log H + \frac{D}{H} + \frac{D^2}{n} \lesssim \log n + \frac{D}{H}.
\]  

(6.47)

We hence have

\[
nG(o, o) \lesssim DH \log n + D^2 \lesssim D^2 \log n,
\]  

(6.48)

which concludes the proof. \(\square\)

6.2 Proof that the diameter condition is necessary in Theorem 1.1 and Theorem 1.2

In this section, we prove that when the diameter condition (DC) does not hold, then the renormalised cover time \(\frac{t_{\text{hit}}}{n} - \log |\Gamma|\) does not have asymptotic Gumbel fluctuations. We first prove that Proposition 2.10 still holds when the average hitting time is replaced by the quasistationary hitting time.

Proposition 6.11. Uniformly over all finite (connected) vertex-transitive graphs of degree \(d\) such that \(n = |\Gamma| \leq D^5\), we have

\[
t_{\text{hit}} - \mathbb{E}_\alpha T_o \asymp_d D^2,
\]  

(6.49)

where \(\alpha\) is the quasistationary distribution associated to \(o\).
To prove Proposition 6.11 we will need a few lemmas. For \(1 \leq r \leq D\), set \(B_r := B(o, r)\), and \(\beta(r) = 1/\mathbb{E}_{o,W} T_{B_r^c}\).

**Lemma 6.12.** For every \(2 \leq \ell < D\) and \(t \geq 0\), we have, writing \(r := \lfloor \ell/2 \rfloor\), that
\[
\mathbb{P}_o[T_{B(o,\ell)^c} > t] \geq \max_{v \in B(o, r)} \mathbb{P}_v[T_{B(o,\ell)^c} > t] \geq e^{-t\beta(r)},
\]
(6.50)
\[
P_t(o,o) \geq \frac{\mathbb{P}_o[X_t \in B(o, \ell)]}{V(\ell)} \geq \frac{\mathbb{P}_o[T_{B(o,\ell)^c} > t]}{V(\ell)} \geq e^{-t\beta(r)},
\]
(6.51)
\[
\beta(\ell) \leq 4V(\ell) / (r^2).
\]
(6.52)

**Proof.** Let \(v \in B(o, r)\). The first inequality in (6.50) follows by noting that if \(X_0 = v\) then \(d(X_t, X_0) \leq 2r \leq \ell\) for all \(t < T_{B(o,\ell)^c}\) and so \(T_{B(o,\ell)^c} \leq T_{B(v,\ell)^c}\). It follows by transitivity that \(\mathbb{P}_v[T_{B(o,\ell)^c} > t] \leq \mathbb{P}_v[T_{B(v,\ell)^c} > t] = \mathbb{P}_o[T_{B(o,\ell)^c} > t]\) for all \(t \geq 0\), as desired.

Let \(A = B(o, \ell)^c\), and \(\alpha = \alpha_A\) be the quasi-stationary distribution associated to \(A\). The second inequality in (6.50) follows from the fact that
\[
\max_{v \in B(o, r)} \mathbb{P}_v[T_{B(o,\ell)^c} > t] \geq \mathbb{P}_\alpha[T_{B(o,\ell)^c} > t] = e^{-t\beta(r)}.
\]
The first inequality in (6.51) follows from the fact that by transitivity \(p_t(o,o) = \max_{x \in \Gamma} p_t(o,x)\), while the second is trivial and the third is exactly (6.50).

We now prove (6.52). For a distribution \(\nu\) on \(V\) and \(g, g' : V \to \mathbb{R}\) we write \(\langle g, g'\rangle_{\nu} := \mathbb{E}_\nu[gg'] = \sum_{x \in V} \nu(x)g(x)g'(x)\). Let \(C_\ell\) be the collection of all \(g : V \to \mathbb{R}\) whose support \(\{x \in V : g(x) \neq 0\}\) is non-empty and contained in \(B(o, \ell)\), and denote by \(\pi_\ell\) the uniform distribution on \(B(o, \ell)\). Observe that for all \(g \in C_\ell\) we have that
\[
\langle (I - P_{B(o,\ell)})g, g\rangle_{\pi_\ell} = \langle (I - P)g, g\rangle_{\pi} = \langle (I - P)g, g\rangle_{\pi} / \pi (B(o, \ell))
\]
\[
= \frac{1}{2\pi(B(o,\ell))} \mathbb{E}_\pi \left[ (g(X_0) - g(X_1))^2 \right]
\]
\[
= \mathbb{E}_{\pi_\ell} \left[ (g(X_0) - g(X_1))^2 \frac{1 + 1 \{ X_1 \notin B(o, \ell) \} }{2} \right]
\]
(6.53)
(c.f. [LP17, Lemma 13.6] for the third equality). Since \(\beta(\ell)\) is the spectral gap of the chain killed at \(B(o, \ell)^c\), we have by the Courant-Fischer characterization of \(\beta(\ell)\) that
\[
\beta(\ell) = \min_{g \in C_\ell} \left\{ \frac{\langle (I - P_{B(o,\ell)})g, g\rangle_{\pi_\ell}}{\mathbb{E}_{\pi_\ell}[g^2]} \right\}.
\]
(6.54)

Using (6.53), and since the test function \(f : x \mapsto d(x, B(o, \ell)^c)\) is 1-Lipschitz we get that \(\langle (I - P_{B(o,\ell)})f, f\rangle_{\pi_\ell} \leq 1\). Since moreover \(f \in C_\ell\), we have
\[
\frac{1}{\beta(\ell)} \geq \frac{\mathbb{E}_{\pi_\ell}[f^2]}{\langle (I - P_{B(o,\ell)})f, f\rangle_{\pi_\ell}} \geq \mathbb{E}_{\pi_\ell}[f^2] \geq (\ell - r)^2 \pi_\ell(B(o, r)) \geq \frac{\ell^2 V(r)}{4V(\ell)},
\]
where we used the fact that \(f(x) \geq \ell - r\) for all \(x \in B(o, r)\). This concludes the proof. \(\Box\)

**Lemma 6.13.** Let \((\Gamma)\) be a sequence of finite (connected) vertex-transitive graphs of degree \(d\) such that \(n = |\Gamma| \leq D^5\), and let \(a, b > 0\). There exists \(C = C(a, b, d) > 0\) such that
\[
\mathbb{P}_o(T_{B_{aD}^c} > bD^2) \geq C.
\]
(6.55)
Proof. Let $\ell := 2\,|aD/2|$, $r = \ell/2$, and $t = bD^2$, so that $\ell$ is even and $\mathbb{P}_o(T_{B_{\ell}D} > bD^2) \geq \mathbb{P}_o(T_{B_{\ell}D} > t)$. By [TT21, Corollary 1.5] (which we recalled in (5.20)), we have $V(r) \gtrsim n$. Plugging this into (6.52), we deduce that $\beta(r) \lesssim r^{-2}$, and therefore that $e^{-\beta(r)t} \gtrsim 1$. Plugging this into (6.50) concludes the proof.

Lemma 6.14. Let $(\Gamma)$ be a sequence of finite (connected) vertex-transitive graphs of degree $d$ such that $n = |\Gamma| \leq D^5$, and let $\varepsilon > 0$. There exists $c > 0$ such that
\begin{equation}
\alpha_o(B_{\varepsilon D}) \geq c(\varepsilon).
\end{equation}

Proof. The intuition of this proof is that if a walk starts more than $\varepsilon D$ away from $o$, there should be a positive probability that, after time of order $D^2$, it finds itself in the ball $B_{\varepsilon D}$ (indeed, mixing times are hitting times of large sets, see [PS15]) but has not touched $o$ yet (since starting from the quasistationary distribution $\alpha_o$ and conditioning on hitting $o$, the distribution at any given later time remains $\alpha_o$).

Suppose $x \in B_{\varepsilon D}$. Then
\begin{equation}
\mathbb{P}_x(T_{B_{\varepsilon D}/2} \leq \ell_{\text{mix}}^{(1/2)}) \geq \mathbb{P}_x(X_{\ell_{\text{mix}}^{(1/2)}} \in B_{\varepsilon D/2}) \geq \frac{1}{2} \frac{V(\varepsilon D/2)}{n} \gtrsim 1.
\end{equation}

Furthermore, by Lemma 6.13 (recalling that $\ell_{\text{mix}}^{(1/2)} \asymp D^2$ by Proposition 2.8), given that the walk enters $B_{\varepsilon D/2}$, the conditional probability that it remains in the annulus $B_{\varepsilon D/4} \setminus B_{\varepsilon D/4}$ until time $\ell_{\text{mix}}^{(1/2)}$ is at least $c$ for some constant $c$. Therefore, setting $t = \ell_{\text{mix}}^{(1/2)}$,
\begin{align*}
\alpha(B_{\varepsilon D}) &= \mathbb{P}_\alpha(X_t \in B_{\varepsilon D}|T_o \geq t) \\
&\geq \alpha(B_{\varepsilon D}^c) \mathbb{P}_\alpha|B_{\varepsilon D}(X_t \in B_{\varepsilon D}|T_o \geq t) \\
&\geq \alpha(B_{\varepsilon D}^c) \inf_{x \in B_{\varepsilon D}^c} \mathbb{P}_x(X_t \in B_{\varepsilon D}; T_o \geq t) \\
&\gtrsim \alpha(B_{\varepsilon D}^c).
\end{align*}

Thus $\alpha(B_{\varepsilon D})$ is bounded away from zero, as desired.

Proof of Proposition 6.11. By Proposition 6.3, for $x, y \in \Gamma$, we have $E_xT_y = E_xT_o - nG(x, y)$, so $E_xT_y \leq E_xT_o$ if and only if $G(x, y) \geq 0$. Combining this with Proposition 6.5, there exists $\eta > 0$ such that for every $x \in B(o, \eta D) =: B_\eta$, 
\begin{equation}
E_xT_o \leq E_\eta T_o.
\end{equation}

Therefore, since $\alpha(B_{\eta}) \gtrsim 1$ by Lemma 6.14, and from Proposition 2.10, we conclude that
\begin{equation}
t_{\text{hit}} - E_\alpha T_o \geq \sum_{x \in B_\eta} \alpha(x) (t_{\text{hit}} - E_\pi T_o) = \alpha(B_{\eta D}) (t_{\text{hit}} - E_\pi T_o) \gtrsim D^2,
\end{equation}

as desired.

Proposition 6.15. Let $(\Gamma)$ be a collection of finite (connected) vertex-transitive of fixed degree $d$, and assume that as $n = |\Gamma| \to \infty$,
\begin{equation}
D^2 \log n \gtrsim n.
\end{equation}

Then there exists a constant $\kappa < 1$ such that for $s \in \mathbb{R}$ large enough, we have at time $t_s = t_{\text{hit}}((\log n) + s)$
\begin{equation}
\mathbb{P}(\tau_{\text{cov}} > t_s) \leq \kappa \left(1 - e^{-e^{-s}}\right).
\end{equation}

In particular $\tau_{\text{hit}} - \log |\Gamma|$ asymptotically does not have Gumbel fluctuations, and for $s$ large enough, $d_{TV}(\mathcal{L}(U(t_s)), \mu_\gamma^{\otimes \Gamma})$ does not converge to 0.
Proof. From Proposition 6.11, there exists a constant \( c > 0 \) such that
\[
\frac{t_{\text{hit}}}{E_{\alpha}T_o} \geq 1 + c\frac{D^2}{E_{\alpha}T_o}.
\] (6.62)
It follows from Proposition 6.10 that there exists a constant \( C > 0 \) (depending on \( d \), on \( c \) and the implicit constant in (6.60)), such that
\[
\frac{t_{\text{hit}}}{E_{\alpha}T_o} \geq 1 + \frac{C}{\log n}.
\] (6.63)
From a simple union bound and (6.63), we deduce that
\[
\mathbb{P}(\tau_{\text{cov}} > t_s) \leq n\mathbb{P}\mathcal{P}(T_o > t_s) \leq n\mathbb{P}_\alpha(T_o > t_s) = n \exp\left(-\frac{t_{\text{hit}}}{E_{\alpha}T_o}((\log n) + s)\right) \leq e^{-s}e^{-C}. \] (6.64)
Hence, for \( s \) larger than some \( s_0 = s_0(C) \), we have
\[
\mathbb{P}\left(\tau_{\text{cov}} > t_s, \log |\Gamma| > s\right) = \mathbb{P}(\tau_{\text{cov}} > t_s) \leq e^{-C}\mathbb{P}(\chi > s) \leq \kappa \left(1 - e^{-\kappa}\right),
\] (6.65)
where \( \kappa := \frac{4\log C}{2} < 1 \). This proves that \( \tau_{\text{cov}} - \log |\Gamma| \) does not have Gumbel fluctuations. Since \( \mathbb{P}(\tau_{\text{cov}} > t_s) = \mathbb{P}(U(t_s) \neq \emptyset) \), it also shows that for \( s \geq s_0 \), we have
\[
d_{\text{TV}}(\mathcal{L}(U(t_s)), \mu_s^{\otimes \Gamma}) \geq (1 - \kappa) \left(1 - e^{-\kappa}\right). \] (6.66)
\[ \square \]

6.3 Proof that the diameter condition is necessary in Theorem 1.3

In this section, we assume that \( t_{\text{rel}} = o(t_{\text{hit}}) \). We want to show that the law of the uncovered set at time \( t_s^* \) is far, in the total variation sense, from the product measure \( \mu_s^{\otimes \Gamma} \). Note that this immediately implies Theorem 1.3: indeed, since \( t_{\text{rel}} \leq dD^2 \) by (2.7) and \( t_{\text{hit}} \geq t_{\text{hit}} = nG(o, o) \geq n(1 - o(1/\log n)) \) by Proposition 6.3, the assumption \( dD^2 \ll n \) implies that \( t_{\text{rel}} \ll t_{\text{hit}} \). Therefore, it suffices to prove the result under the sole assumption \( t_{\text{rel}} = o(t_{\text{hit}}) \).

To prove that the law of the uncovered set is far from a product measure when the diameter condition fails, we might initially be tempted to show that the uncovered set is “too” clustered, i.e., the probability that two relatively nearby points are uncovered is larger than it should be under the independent scenario. However, this turns out to be very difficult to control as the contribution to the moments of order \( k \) of the size of the uncovered set coming from nearby points start exploding when the diameter condition fails. We cannot translate this into estimates about events of positive probability for the uncovered set; roughly speaking, either the Bonferroni inequality goes in the wrong direction, or one would need to keep track of moments of higher order and compare how they blow up.

Instead, we show that points that are sufficiently far apart are negatively correlated. It turns out that this enables us to use the Bonferroni inequality (i.e., union bound) as this is an upper bound.

Proposition 6.16. Let \( \gamma > 0 \), and \( (\Gamma) \) be a collection of finite (connected) vertex-transitive graphs of fixed degree \( d \), such that \( n = |\Gamma| \leq \gamma D^2 \log n \) and \( t_{\text{rel}} = o(t_{\text{hit}}) \), and let (for every \( s \in \mathbb{R} \)) \( \mu_s^{\otimes \Gamma} \) denote the product over all the vertices of the graph of the Bernoulli law \( \mu_s \) with parameter \( e^{-s}/|\Gamma| \). Then for every fixed \( s \in \mathbb{R} \), there exists a constant \( c^* = c^*(s,d,\gamma) \) such that as \( |\Gamma| \to \infty \), for \( t \in \{t_{(s)}, t_s^*, t_s\} \),
\[
d_{\text{TV}}(\mathcal{L}(U(t)), \mu_s^{\otimes \Gamma}) \geq c^* + o(1).
\]
We start by comparing $t_{(s)}$ and $t_s^*$. Recall that $t_{(s)} = t_{(\text{hit})}((\log n) + s)$ and $t_s^*$ is such that $\mathbb{E}[U(t_s^*)] = e^{-s}$.

**Lemma 6.17.** Let $s \in \mathbb{R}$. For $n$ large enough, we have $\mathbb{E}(U(t_{(s)})) \geq e^{-s}$, or in other words $t_{(s)} \leq t_s^*$.

**Proof.** Set $\alpha = \alpha_o$, and let $s \in \mathbb{R}$. We have

$$\mathbb{E}[U(t_{(s)})] = n \mathbb{P}_x(T_o > t_{(s)}) = n \frac{\mathbb{P}_x(T_o > t_{(s)})}{\mathbb{P}_\alpha(T_o > t_{(s)})} e^{-t_{(s)}/E_n T_o} = n \frac{\mathbb{P}_x(T_o > t_{(s)})}{\mathbb{P}_\alpha(T_o > t_{(s)})} e^{-\mathbb{E}_n T_o (\log n) + s}. \tag{6.67}$$

By Theorem 5.2, setting $\theta := 1 - \frac{1}{\|\alpha/\|_2^2}$, we have

$$\frac{\mathbb{P}_x(T_o > t_{(s)})}{\mathbb{P}_\alpha(T_o > t_{(s)})} \geq 1 - \theta, \tag{6.68}$$

and,

$$\frac{\mathbb{E}_n T_o}{\mathbb{E}_\alpha T_o} \leq 1 - \theta + \theta \frac{t_{rel}}{\mathbb{E}_\alpha T_o}. \tag{6.69}$$

Since $t_{rel} = o(t_{\text{hit}})$ by hypothesis, we have $\frac{t_{rel}}{\mathbb{E}_\alpha T_o}$ and $\theta \to 0$. Therefore, using the inequality $e^x \geq 1 + x$, valid for all $x \in \mathbb{R}$, we have

$$\mathbb{E}[U(t_{(s)})] \geq e^{-s} (1 - \theta)(1 + \theta \log n + o(\theta \log n)) = e^{-s} (1 + \theta \log n + o(\theta \log n)), \tag{6.70}$$

which concludes the proof. \hfill \square

We now prove a technical lemma.

**Lemma 6.18.** Let $(\Gamma)$ as in Proposition 6.16, and $c, S_1, S_2$ as in Proposition 6.9. There exists a constant $C > 0$ such that for every $(x, y) \in S_1 \times S_2$ and $s \in \mathbb{R}$, we have (as $|\Gamma| \to \infty$), setting $A = \{x, y\}$,

$$\mathbb{P}_\pi(T_A > t_{(s)}) \leq \frac{e^{-2s}}{n^2} e^{-C + o(1)}. \tag{6.71}$$

**Proof.** Let us fix $s \in \mathbb{R}$. For every $(x, y) \in S_1 \times S_2$, we have by (6.15)

$$q_A = \frac{2}{1 + \frac{G(x, y)}{G(o, o)}} \geq \frac{2}{1 - c \frac{D^2}{nG(o, o)}} \geq 2 \left(1 + c \frac{D^2}{nG(o, o)}\right). \tag{6.72}$$

Moreover, using Equation (1) in [AB92] (which we recalled in Theorem 1.5) for the first inequality, and by Theorem 1.6, we have uniformly over all $A \subset \Gamma$ of size 2,

$$\mathbb{P}_\pi(T_A > t_{(s)}) \leq \mathbb{P}_\alpha(T_A > t_{(s)}) = \exp \left(-q_A t_{(s)} \left(1 + O\left(\frac{t_{rel}}{t_{\text{hit}}^2}\right)\right)\right). \tag{6.73}$$

It follows, recalling that $\frac{D^2}{nG(o, o)} \simeq \frac{t_{\text{hit}}}{t_{\text{hit}}}$, that there exists a constant $c' > 0$ such that for every $(x, y) \in S_1 \times S_2$,

$$\mathbb{P}_\pi(T_A > t_{(s)}) \leq \frac{e^{-2s}}{n^2} \exp \left(-2c' \frac{t_{rel}}{t_{\text{hit}}} (\log n) \left(1 + \frac{s}{\log n} + O\left(\frac{t_{rel}}{t_{\text{hit}}}\right)\right)\right). \tag{6.74}$$
From Proposition 6.10, there exists a constant $C$ such that (recalling that $t_{\text{rel}} \asymp D^2$)

$$2e^{t_{\text{rel}} \log n} \geq C,$$  \hspace{1cm} (6.75)

which allows us to conclude, using that $t_{\text{rel}} \ll t_{\text{hit}}$, that

$$\mathbb{P}_\pi (T_A > t_{(s)}) \leq \frac{e^{-2s}}{n^2} e^{-C + o(1)},$$  \hspace{1cm} (6.76)

as desired. \hfill \qed

**Proof of Proposition 6.16.** Let $s \in \mathbb{R}$. By definition of the total variation distance, it is enough to find a subset $B$ of $\mathcal{P}(\Gamma)$ and a constant $c^*$ such that for $t \in \{ t_{(s)}, t_s, t_{*s} \}$,

$$\mu_s^\oplus \Gamma (B) - \mathbb{P} (U(t) \in B) > c^*.$$  \hspace{1cm} (6.77)

Let $c, S_1, S_2$ as in Proposition 6.9, and $C > 0$ from Lemma 6.18. Set $b = b(s) = e^{-s}$ and let $a > 0$ be small enough such that $|S_1| > an$ (for all $\Gamma$ as in the statement of the proposition), and $1 - 2e^{-ab} e^{-2ab} \geq (ab)^2 (1 + e^{-C})/2$. (Such a choice for $a$ is possible since as $a \to 0$, we have $1 - 2e^{-ab} e^{-2ab} = (ab)^2 + O(a^3)$.) Let $S_1'$ and $S_2'$ be sub-balls of (respectively) $S_1$ and $S_2$ such that $|S_1'| = |S_2'| = (a + o(1))n$, and set

$$B := \{ A \subset \Gamma : A \cap S_1' \neq \emptyset \text{ and } A \cap S_2' \neq \emptyset \}.$$  \hspace{1cm} (6.78)

By a union bound and Lemma 6.18, we get the following upper bound on $\mathbb{P} (U(t_{(s)}) \in B)$

$$\mathbb{P} (U(t_{(s)}) \in B) \leq \sum_{(x,y) \in S_1' \times S_2'} \mathbb{P}_\pi (\{ x, y \} \subset U(t_{(s)})) \leq \frac{|S_1'| |S_2'|}{n^2} e^{-2s} (e^{-C} + o(1))$$  \hspace{1cm} (6.79)

$$= (ab)^2 e^{-C} + o(1).$$

Let us now lower bound $\mu_s^\oplus \Gamma (B)$. Let $B_i = \{ A \subset \Gamma : A \cap S_i' \neq \emptyset \}$ for $i \in \{ 1, 2 \}$. Since $B = B_1 \cap B_2$, we have

$$\mu_s^\oplus \Gamma (B) = 1 - \mu_s^\oplus \Gamma (B_1^c \cup B_2^c) = 1 - (\mu_s^\oplus \Gamma (B_1^c) + \mu_s^\oplus \Gamma (B_2^c) - \mu_s^\oplus \Gamma (B_1^c \cap B_2^c)).$$  \hspace{1cm} (6.80)

Moreover,

$$\mu_s^\oplus \Gamma (B_1^c) = \mu_s^\oplus \Gamma (B_2^c) = \left( 1 - \frac{b}{n} \right)^{(a + o(1))n} = e^{-ab + o(1)},$$  \hspace{1cm} (6.81)

and we have similarly $\mu_s^\oplus \Gamma (B_1^c \cap B_2^c) = e^{-2ab + o(1)}$. If follows that

$$\mu_s^\oplus \Gamma (B) \geq 1 - 2e^{-ab + o(1)} + e^{-2ab + o(1)},$$  \hspace{1cm} (6.82)

and hence, by definition of $a$, that

$$\mu_s^\oplus \Gamma (B) - \mathbb{P} (U(t_{(s)}) \in B) \geq (ab)^2 \frac{1 + e^{-C}}{2} - (ab)^2 e^{-C} + o(1) = (ab)^2 \frac{1 - e^{-C}}{2} + o(1).$$  \hspace{1cm} (6.83)

Recall that $t_s' \geq t_{(s)}$ (for $n$ large enough) and $t_s \geq t_{(s)}$ (for all $n$), so since $t \mapsto \mathbb{P}_\pi (T_A > t)$ is decreasing for every $A$, (6.79) also holds with $\mathbb{P} (U(t_s') \in B)$ or $\mathbb{P} (U(t_s) \in B)$ on the left hand side. Therefore, (6.83) also holds with $t_s'$ or $t_s$ instead of $t_{(s)}$, and the proof is complete. \hfill \qed
7 A strongly uniformly transient graph without Gumbel fluctuations

In this section, we construct an example of a sequence of vertex-transitive graphs which satisfy the strong uniform transience (SUT) of (1.11), but not the diameter condition (DC).

The idea is to consider the following graph: for \( m \geq 2 \) even, let \( \Gamma = C_m \times G \) be the Cartesian product of a cycle \( C_m \) of length \( m \), together with an expander Cayley graph \( G \) (in fact it will be convenient to take a Ramanujan graph of degree greater than three, see Morgenstern [Mor94] for an explicit construction generalising the famous construction of Lubotzky, Phillips and Sarnak [LPS88]) of size \( mh \), where \( h = h_m \geq 1 \), and say \( h_m \leq m \), say. (The interesting examples will be those for which \( h \lesssim \log m \).)

Thus the total size (number of vertices) of \( \Gamma \) is \( n = m^2h \) and the diameter is \( \asymp m \) and for \( h \lesssim \log m \), \( \Gamma \) does not satisfy the diameter condition (DC).

**Proposition 7.1.** The graph \( \Gamma \) above satisfies the strong uniform transience (SUT) condition whenever \( h_m \to \infty \), no matter how slowly.

**Proof.** For a vertex \( x \in \Gamma \), let us write \( x = (\hat{x}, \hat{\lambda}) \) with \( \hat{x} \in C_m \) and \( \hat{\lambda} \in G \) so that \( \hat{x} \) denote the cycle coordinate and \( \hat{\lambda} \) the Ramanujan coordinate. Since both \( C_m \) and \( G \) are vertex-transitive, for a continuous-time random walk \( X_t = (X_t, \hat{X}_t) \in \Gamma \), the coordinates \( \hat{X} \) and \( \hat{X} \) are in fact independent continuous time random walks with rates \( 2/d \) and \( 1 - 2/d \) respectively (where \( d = \deg(x) \) is the total degree on \( \Gamma \)).

Since the coordinates are independent, we have that for every \( t \geq 0 \),

\[
p_t(x, y) = \hat{p}_t(\hat{x}, \hat{y}) \hat{p}_t(\hat{x}, \hat{y}),
\]

where \( \hat{p}_t \) denote the transition probabilities of the random walk on the cycle \( C_m \) (with rate \( 2/d \)), and \( \hat{p}_t \) denote those on the Ramanujan graph \( G \) (with rate \( 1 - 2/d \)).

Furthermore, by Proposition 2.8, \( t_{\text{mix}} = t_{\text{mix}}(1/4) \lesssim m^2 \). It is therefore easy to estimate both \( \hat{p}_t \) and \( \hat{p}_t \) for \( t \leq m^2 \): namely, on the cycle we know (for instance from Proposition 2.1) that

\[
\hat{p}_t(\hat{\lambda}, \hat{\lambda}) \lesssim \frac{1}{\sqrt{t+1}};
\]

and on the Ramanujan component, by (2.29),

\[
|\hat{p}_t(\hat{\lambda}, \hat{\lambda}) - \hat{\pi}(\hat{\lambda})| \leq e^{-\lambda t}
\]

where \( \lambda > 0 \) is the spectral gap on the Ramanujan component (which, by definition, is bounded away from zero), and \( \hat{\pi} \) is the uniform distribution on \( G \). Since \( \hat{\pi}(\hat{\lambda}) = 1/(mh) \), we deduce that

\[
\hat{p}_t(\hat{\lambda}, \hat{\lambda}) \leq 1/(mh) + e^{-\lambda t}.
\]

For each fixed \( s > 0 \)

\[
\mathbb{E}_o[L_o(t_{\text{mix}}) - L_o(s)] = \int_s^{t_{\text{mix}}} p_t(o, o) dt
\]

\[
\lesssim \int_s^{t_{\text{mix}}} \frac{1}{\sqrt{t+1}} (\frac{1}{mh} + e^{-\lambda t}) dt
\]

\[
\lesssim \frac{1}{mh} \int_s^{t_{\text{mix}}} \frac{1}{\sqrt{t+1}} dt + \int_s^{\infty} e^{-\lambda t} \frac{1}{\sqrt{t+1}} dt
\]

Since \( t_{\text{mix}} \lesssim m^2 \) and \( h = h_m \to \infty \), the first term tends to zero. The second term does not depend on \( m \), and is the integral of a function which is clearly integrable. Consequently, the
\[
\limsup_{s \to \infty} \limsup_{n \to \infty} \mathbb{E}_o(L_0(t_{\text{mix}}) - L(s)) = 0,
\]
which shows (1.11).

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