Closed form of local quantum uncertainty and a sudden change of quantum correlations

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Quantum correlations play vital roles in the quantum features in quantum information processing tasks, and it is of fundamental importance in studying quantum phenomena. Among the measures of quantum correlations, recently, the local quantum uncertainty (LQU) for bipartite quantum systems is proposed [Phys. Rev. Lett. 110, 240402 (2013)]. In this paper, we have derived the closed form of the LQU for a large class of arbitrary-dimensional bipartite quantum states. We revisit the problem of sudden change of quantum correlations in the transition from separable to bound and free entangled states [Phys. Rev. Lett. 82, 1056 (1999)]. By using the LQU, we have demonstrated that the sudden change located exactly near the transition point from separable to bound entangled states.

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I. INTRODUCTION

Over the past decades, entanglement was considered to be the only ingredient of quantum properties and the main resource of the speed-up in quantum computation [1–3]. However, it has been shown that some states with subtracted entanglement can still reveal their power in quantum speed-up [4–6]. Nowadays, it is widely believed that the non-classical correlations, namely, quantum correlations, play vital roles in the quantum features in quantum information processing, and it is of fundamental importance in quantum phenomenon.

As a result, quantum correlations become the subject of intensive studies in the last two decades [7]. Among the various researches, it is of great significance to measure quantum correlations quantitatively. There are much attention put on the quantification of bipartite quantum correlations, including quantum discord [4, 6], geometric discord [8, 9], quantum deficit [10], measurement-induced disturbance [11], etc.

Recently, a measure of quantum correlations for bipartite quantum systems named the local quantum uncertainty (LQU) is proposed [12]. The LQU is defined as

$$\mathcal{U}_A = \min_K I(\rho_{AB}, K^A),$$

where we have denoted the two particles as A and B, the minimum is optimized over all the non-degenerate local projective operators on A: $K^A = A^A \otimes I_B$, and

$$I(\rho, K) = \frac{1}{2} \text{Tr}[\sqrt{\rho}(K^{d^2} + K^d)^2]$$

is the skew information introduced in Ref. [13]. It has been shown that for bipartite quantum systems, the LQU is invariant under local unitary operations, and for pure states, it is an entanglement monotone. The closed form of the LQU for 2×d quantum systems [12] is pointed out to be

$$\mathcal{U}_A = 1 - \lambda_{\text{max}}(\mathcal{W}),$$

where $\lambda_{\text{max}}$ is the maximum eigenvalue of the 3×3 matrix $\mathcal{W}$ with elements $\mathcal{W}_{ij} = \text{Tr}(\sqrt{\rho_1} \sigma_i \otimes \sqrt{\rho_2} \sigma_j)$ and $\sigma_i$ ($i = 1, 2, 3$) represent the Pauli matrices, which are the generators of SU(2). The interesting coincidence arises that for 2×d quantum systems, the LQU reduces to the linear entropy (i.e., the concurrence) for pure states. It has also been pointed out [14] that the closed form of the LQU can be achieved for d×d quantum states satisfying $\text{Tr}(\lambda_i \otimes I_d) = 0$, namely,

$$\mathcal{U}_A = \frac{2}{d} - \lambda_{\text{max}}(\mathcal{W}),$$

where $\mathcal{W}$ is a $(d^2 - 1) \times (d^2 - 1)$ matrix with elements $\mathcal{W}_{ij} = \text{Tr}(\sqrt{\rho_1} \lambda_i \otimes I_d \sqrt{\rho_2} \lambda_j \otimes I_d)$, and $\lambda_i$ represents all the generators of SU(d).

There are $d^2 - 1$ generators $\lambda_j$ (j = 1, ..., $d^2 - 1$) of SU(d) and they satisfy

$$\lambda_i \lambda_j = i \sum_k f_{ijk} \lambda_k + \sum_k g_{ijk} \lambda_k + \frac{2}{d^2 - 1} \delta_{ij},$$

where

$$f_{ijk} = \frac{1}{4i} \text{Tr}(\lambda_i \lambda_j \lambda_k), g_{ijk} = \frac{1}{4} \text{Tr}(\lambda_i \lambda_j \lambda_k).$$

The remainder of this paper is organized as follows: In Section II, we propose the closed form of the LQU for a large class of arbitrary-dimensional quantum states. In Section III, we give several examples to show the power of our approach. In Section IV, we investigate the sudden change of quantum correlations in the transition from separable to bound and free entangled states by using the LQU. In Section V, we give a short summary and prospect.

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II. CLOSED FORM OF LQU

Our goal here is to derive the closed form of LQU when the LQU is obtained by a non-degenerate $\Lambda^4$. For $d_1 \times d_2$ quantum states, any projective operator on $A$ can be expressed as $\Lambda^4 = \vec{s} \cdot \lambda$, where $\vec{s} = (s_1, s_2, \ldots, s_{d_1^2-1})$ and $|s| = 1$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d_2^2-1})^T$. Following the definition of the LQU and ignoring the non-degeneracy issue, we obtain

$$\mathcal{U}_A = \min (I(p, K^A)) = \min \{\text{Tr}(\rho(K^A)^2) - \text{Tr} (\sqrt{\text{Tr}K^A} \sqrt{\text{Tr}K^A})\}$$

$$\min \{\text{Tr}(\rho(\vec{s} \cdot \lambda \otimes I_{d_2})^2) - \text{Tr} \sqrt{\text{Tr}(\vec{s} \cdot \lambda \otimes I_{d_2})} \sqrt{\text{Tr}(\vec{s} \cdot \lambda \otimes I_{d_2})}\}. \quad (8)$$

By using Eq. (6), we can get

$$\text{Tr}(\rho(\vec{s} \cdot \lambda \otimes I_{d_2})^2) = \sum_{i,j,k} s_is_j [(iF_{ij} + G_{ij}) \text{Tr}(\rho \lambda_k \otimes I_{d_2})] + \frac{2}{d_1}. \quad (9)$$

We define

$$F_{ij} = (f_{ij}, \ldots, f_{i d_{d_2^2-1}}),$$

$$G_{ij} = (g_{ij}, \ldots, g_{i d_{d_2^2-1}}),$$

$$L = (\text{Tr}(\rho \lambda_1 \otimes I_{d_2}), \ldots, \text{Tr}(\rho \lambda_{d_2^2-1} \otimes I_{d_2}), \ldots, \text{Tr}(\rho \lambda_{d_2^2-1} \otimes I_{d_2}))^T. \quad (10)$$

Then

$$\text{Tr}(\rho(\vec{s} \cdot \lambda \otimes I_{d_2})^2) = \sum_{i,j} s_is_j [(iF_{ij} + G_{ij})L] + \frac{2}{d_1}. \quad (11)$$

Therefore

$$\mathcal{U}_A = \frac{2}{d_1} + \min \sum_{i,j} s_is_j [(iF_{ij} + G_{ij})L - \text{Tr} \sqrt{\text{Tr}(\lambda_1 \otimes I_{d_2})} \sqrt{\text{Tr}(\lambda_j \otimes I_{d_2})}]]. \quad (12)$$

It can be seen that $F_{ij}$ is antisymmetric under the transpose of the subscripts, namely, $F_{ji} = -F_{ij}$, therefore $\sum_{i,j} s_is_j F_{ij} = 0$, then we get

$$\mathcal{U}_A = \frac{2}{d_1} + \min \sum_{i,j} s_is_j [G_{ij}L - \text{Tr} \sqrt{\text{Tr}(\lambda_1 \otimes I_{d_2})} \sqrt{\text{Tr}(\lambda_j \otimes I_{d_2})}]]$$

$$= \frac{2}{d_1} - \lambda_{\text{max}}(\mathcal{W}), \quad (13)$$

where we use $\lambda_{\text{max}}$ to represent the maximum eigenvalue and $\mathcal{W}$ is a $(d_1^2 - 1) \times (d_1^2 - 1)$ matrix with elements

$$\mathcal{W}_{ij} = \text{Tr} \sqrt{\text{Tr}(\lambda_1 \otimes I_{d_2})} \sqrt{\text{Tr}(\lambda_j \otimes I_{d_2})} - G_{ij}L. \quad (14)$$

Two special cases are worthwhile to discuss. In the case where $d_1 = 2$, we have $g_{ij} = 0$, and thus $G_{ij}$ is a zero vector. It is easy to recover the result in Ref. [12]. When $\text{Tr}(\rho \lambda_1 \otimes I_{d_2}) = 0$, namely, $L = 0$, the conclusion in Ref. [14] is regained.

The procedure of our approach is straightforward. Firstly, we suppose the value of the LQU is obtained by a non-degenerate $\Lambda^4$. Then we can use the closed form derived above to calculate the LQU. As the final and most important step, we should check whether the LQU is obtained by a non-degenerate $\Lambda^4$, if not, the results we have obtained should be discarded. Therefore our approach is valid only when the LQU is obtained by a non-degenerate $\Lambda^4$.

In order to determine whether $\Lambda^4$ is degenerate when the LQU is obtained, we should get the vector $\vec{s}$ when $\mathcal{U}_A$ is minimized. Following the closed form of the LQU, $\mathcal{W}$ is the key to achieving our purpose.

The eigenvalue decomposition of $\mathcal{W}$ reads

$$\mathcal{W} = U \begin{pmatrix} w_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} U^\dagger. \quad (15)$$

A simple transformation gives

$$\mathcal{W} = U' DU'^\dagger = U' \begin{pmatrix} w_{\text{max}} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} U'^\dagger. \quad (16)$$

According to Eq. (13), the maximum of $\vec{s} \cdot \mathcal{W} \cdot \vec{s} = \vec{s} \cdot U' DU'^\dagger \cdot \vec{s}$ is achieved when $\vec{s} \cdot U' = (1, 0, \cdots, 0)$. Therefore, $\vec{s} = (1, 0, \cdots, 0) \cdot U'^\dagger$, and then $\vec{s}$ is the first row of $U'^\dagger$. By substituting $\vec{s}$ into $\Lambda^4 = \vec{s} \cdot \lambda$, one can easily check whether $\Lambda^4$ is degenerate.

It is worth noting that most bipartite quantum states do not confront with the non-degeneracy issue. Therefore, our approach is valid for a large class of arbitrary-dimensional bipartite quantum states.

III. EXAMPLES

We first consider the Werner state as an example. The three-dimensional Werner state is defined as [15]

$$\rho_w = \frac{(1-p)}{9} I + p |\psi\rangle \langle \psi|, \quad (17)$$

where $|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{2} |i\rangle |i\rangle_2$ and $0 \leq p \leq 1$. It can be verified that $\text{Tr}(\rho_w \lambda_i \otimes I_{d_2}) = 0$, namely, $L = 0$. The quantum discord [16, 17] and the LQU of the three-dimensional Werner state is illustrated in Fig. 1, which clearly demonstrates that the results obtained by the quantum discord and the LQU are consistent.
our approach reveals its power. The result is given in Fig. 1. (Color online) The quantum discord (top) and the LQU (bottom) of three-dimensional Werner state.

Di different from the three-dimensional Werner state, the $3 \times 3$ Horodecki state does not satisfy $\text{Tr}(\rho_{h} \lambda_{i} \otimes \lambda_{d}) = 0$. In this case, our approach reveals its power. The result is given in Fig. 2. (Color online) The quantum discord (top) and the LQU (bottom) of the $3 \times 3$ Horodecki state.

Then we investigated the $3 \times 3$ Horodecki state [18] expressed as [18]

$\rho_{h} = \frac{1}{8h + 1} \begin{pmatrix} h & 0 & 0 & 0 & h & 0 & 0 & 0 & h \\ 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 \end{pmatrix}$. \hspace{1cm} (18)

It is a bound entangled state and does not satisfy $\text{Tr}(\rho_{h} \lambda_{i} \otimes \lambda_{d}) = 0$ as well. The LQU is given in Fig. 3, and the results are also checked to be valid.

It can be seen that for the $4 \times 2$ Horodecki state, there exists a transition near $h = 0.22$. It needs to be noted that the algebraic lower bound of the concurrence, namely, the quantum entanglement declines near $h = 0.22$, and it decreases to zero when $h = 1$ [19]. The difference between the concurrence and LQU is that the LQU bounced back at $h = 0.55$, and it is non-zero when $h = 1$.

In the last, we consider the case where the dimension of the two particles are different. The $4 \times 2$ Horodecki state can be expressed as [18]

$\rho_{h} = \frac{1}{7h + 1} \begin{pmatrix} h & 0 & 0 & 0 & 0 & h & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & h & 0 \\ 0 & 0 & h & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & h & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & h & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 & h \end{pmatrix}$. \hspace{1cm} (19)
IV. A SUDDEN CHANGE OF QUANTUM CORRELATIONS

The sudden change of quantum correlations is an interesting and important phenomenon with a variety of researches [19, 21]. With the closed form of LQU, we revisit the sudden change of quantum correlations in a simple quantum state

$$\rho_\alpha = \frac{2}{7} |\psi_+\rangle \langle \psi_+| + \alpha \frac{5}{7} \sigma_+ + \frac{5 - \alpha}{7} \sigma_-,$$  \hspace{1cm} (20)

where

$$\psi_+ = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle),$$

$$\sigma_+ = \frac{1}{3} (|01\rangle \langle 01| + |12\rangle \langle 12| + |20\rangle \langle 20|),$$

$$\sigma_- = \frac{1}{3} (|10\rangle \langle 10| + |21\rangle \langle 21| + |02\rangle \langle 02|),$$  \hspace{1cm} (21)

in transition from separable to bound and free entanglement.

It has been shown that $\rho_\alpha$ transforms from separable state to bound entangled state, and to free entangled state with the increase of $\alpha$, namely, it is separable when $2 \leq \alpha \leq 3$, bound entangled (BE) when $3 \leq \alpha \leq 4$, and free entangled (FE) when $4 \leq \alpha \leq 5$ [21]. It is pointed out that the sudden change in quantum discord takes place near the transition point from bound to free entanglement (see Fig. 4) [17], which is strange and beyond our physical intuition [22].

We consider $\rho_\alpha$ with the LQU, the result is shown in Fig. 4, which obviously indicates that the sudden change of quantum correlations is exactly accompanied by the transition from separable state to bound entangled state. The result is close to our physical intuition.

V. CONCLUSION

We have derived the closed form of LQU for arbitrary-dimensional bipartite quantum states when the LQU is obtained by a non-degenerate $\Lambda_A$. The approach is valid for a large class of quantum states. Examples are investigated to show that our approach is capable of characterizing the quantum correlations of arbitrary-dimensional quantum states.

By using the LQU, we have revisited the sudden change of quantum correlations in the process of the transition from separable to bound and free entangled state. The result has approved our physical intuition that separable states are distance away from bound and free entangled states. On the other hand, the result has affirmed the advantage of the LQU in characterizing quantum correlations.

Further research still needs to be done on both the properties of the LQU for high-dimensional and multipartite quantum states and its applications. We expect that our result could come up with further theoretical and experimental results.

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