The 4D geometric quantities versus the usual 3D quantities. 

The resolution of Jackson’s paradox

In this paper we present definitions of different four-dimensional (4D) geometric quantities (Clifford multivectors). New decompositions of the torque $N$ and the angular momentum $M$ (bivectors) into 1-vectors $N_x$, $N_y$, $N_z$ and $M_x$, $M_y$, $M_z$ respectively are given. The torques $N_x$, $N_y$, $N_z$ (the angular momentums $M_x$, $M_y$, $M_z$), taken together, contain the same physical information as the bivector $N$ (the bivector $M$). The usual approaches that deal with the 3D quantities $E$, $B$, $F$, $L$, $N$, etc. and their transformations are objected from the viewpoint of the invariant special relativity (ISR).

In the ISR it is considered that 4D geometric quantities are well-defined both theoretically and experimentally in the 4D spacetime. This is not the case with the usual 3D quantities. It is shown that there is no apparent electrodynamic paradox with the torque, and that the principle of relativity is naturally satisfied, when the 4D geometric quantities are used instead of the 3D quantities.

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I. INTRODUCTION

It is almost generally accepted that the covariant quantities, e.g., the covariant angular momentum four-tensor $M^{\mu\nu}$, the torque four-tensor $N^{\mu\nu}$, the electromagnetic field strength tensor $F^{\alpha\beta}$, etc. are only auxiliary mathematical quantities from which “physical” three-dimensional (3D) quantities, the angular momentum $L$, the torque $N$, the electric and magnetic fields $E$ and $B$, etc., are deduced. (The vectors in the 3D space will be designated in bold-face.) The transformations of the 3D quantities are derived from the Lorentz transformations (LT) of the corresponding covariant quantities. (For such approaches see, e.g., [1-4].) However a geometric approach to special relativity (SR) is recently developed, which exclusively deals with 4D geometric quantities; it is called the invariant special relativity (ISR). In the ISR one considers that the 4D geometric quantities are well-defined both theoretically and experimentally in the 4D spacetime, and not, as usual, the 3D quantities. This geometric approach is presented in [5-8] (tensor formalism, with tensors as geometric quantities) and [9-13] (geometric algebra formalism). (See also [14] in which the covariant 4-momentum of the electromagnetic field is expressed in terms of 4-vectors of the electric and magnetic fields.) It is shown in the mentioned references that such geometric approach is in a complete agreement with the principle of relativity and, what is the most important, with experiments, see [7] (tensor formalism) and [9-13] (geometric algebra formalism).

In this paper the investigation with 4D geometric quantities will be done in the geometric algebra formalism, see, e.g., [15, 16]. Physical quantities will be represented by 4D geometric quantities, multivectors, that are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, they are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis.

In Sec. II we present the expressions for different 4D AQs and CBGQs: the Lorentz force $K_L$ (1-vector) expressed in terms of the electromagnetic field $F(x)$ (bivector), Eq. (1), or in terms of the electric and magnetic fields, $E$ and $B$ (1-vectors), Eq. (5), the angular momentum $M$ (bivector) and the torque $N$ (bivector), Eq. (11). The decomposition of $F(x)$ into 1-vectors $E$ and $B$ is given in (4). The new decomposition of the torque $N$ into two 1-vectors, the “space-space” torque $N_x$ and the “time-space” torque $N_t$ is given in (13); they together contain the same physical information as the bivector $N$. The similar decomposition of the angular momentum $M$ into two 1-vectors $M_x$ and $M_t$ is presented in (17). $F(x)$ for a charge $Q$ moving with constant velocity $u_Q$ (1-vector) is given in (2). The new expressions for the 1-vectors $E$ and $B$ for the same case are given in (5).

In Sec. III we first discuss Jackson’s [2] paradox with the 3D torque $N$. The paradox consists in the fact that there is a 3D torque $N$ and so a time rate of change of 3D angular momentum $(N = dL/dt)$ in one inertial frame, but no 3D angular momentum $L'$ and no 3D torque $N'$ in another relatively moving inertial frame. Then it is shown that, contrary to the general opinion, the transformations of the components of the 3D quantities (e.g., Eq. (22) for the components of the 3D angular momentum $L$) drastically differ from the LT of the corresponding 4D quantities (e.g., Eq. (24) for the components of the 4D angular momentum $M$). Furthermore, a 4D geometric quantity, for example, 1-vector $M_x$, is an invariant quantity under the LT, as can be seen from Eq. (25): it is the same quantity for relatively moving inertial observers, which can use different systems of coordinates. On the other hand the corresponding 3D vector $L$ in the inertial frame $S$ is completely different than $L'$ in the relatively moving $S'$ frame, as seen from Eq.
The same fundamental difference between the 3D quantities and their transformations and the corresponding 4D geometric quantities and their LT is discussed for some other quantities and equations with them.

In Secs. IV - IV C the 4D geometric quantities from Sec. II are used to resolve Jackson’s paradox. First, in Sec. IV A, we considered the whole problem using the bivector \( N \) as an AQ and a CBGQ. It is shown that the paradox with the 3D torque arises since all space-space components of \( N \) as a CBGQ in the \( S' \) frame, \( N'^{ij} \), are zero but, as shown in (13), \( N^{12} \) is different from zero in the \( S \) frame. Since the components of the 3D torque are associated with the space-space components of \( N \) this means that \( N' = 0 \) but \( N \neq 0 \). From the point of view of the ISR the fact that \( N' \neq N \) means that \( N \) is not obtained by the LT from \( N' \) and thus it is not the same 4D quantity for observers in the \( S' \) and \( S \) frames. On the other hand when the 4D torque \( N \) is used then it is shown that \( N' \) as a CBGQ in \( S \) (see (14)) is obtained by the LT from \( N \) as a CBGQ in \( S' \) (see (12)); they represent the same 4D quantity in two relatively moving inertial frames, \( N \) (12) = \( N' \) (13). Hence in the approach with the 4D torque \( N \) the principle of relativity is naturally satisfied and there is no paradox. In Secs. IV B and IV C we have considered the same problem using the decomposition of \( N \) into the “space-space” torque \( N_s \) and the “time-space” torque \( N_t \). Again the same result that there is no paradox is achieved. These solutions with 4D geometric quantities can be simply applied to the explanation of the Trouton-Noble experiment as shown in [12] and [13].

In Sec. V the conclusions are presented.

II. DEFINITIONS OF DIFFERENT 4D ABSOLUTE QUANTITIES

In this section, as already mentioned in the Introduction, we shall examine different AQs and CBGQs. For simplicity and for easier understanding only the standard basis \( \{\gamma_\mu; 0, 1, 2, 3\} \) of orthonormal 1-vectors, with timelike vector \( \gamma_0 \) in the forward light cone, will be used for CBGQs. It is worth noting that the standard basis corresponds, in fact, to the specific system of coordinates that we call Einstein’s system of coordinates. In Einstein’s system of coordinates the standard basis corresponds, in fact, to the explanation of the Trouton-Noble experiment as shown in [12] and [13].

In order to treat different systems of coordinates on an equal footing we exposed and exploited throughout the paper. In order to treat different systems of coordinates on an equal footing we have considered the whole problem using the bivector field \( F \) (i.e., the electromagnetic field \( F(x) \)) for a charge \( Q \) moving with constant velocity \( u = dx/d\tau \). The Lorentz force as a 4D AQ (1-vector) is

\[
K_L = (q/c)F \cdot u,
\]

where \( u \) is the velocity 1-vector of a charge \( q \) (it is defined to be the tangent to its world line). The bivector field \( F(x) \) (i.e., the electromagnetic field \( F(x) \)) for a charge \( Q \) moving with constant velocity \( u_Q \) (1-vector) is

\[
F(x) = kQ(x \wedge (u_Q/c))/|x \wedge (u_Q/c)|^3,
\]

where \( k = 1/4\pi \varepsilon_0 \), see [13] and references therein. (For the charge \( Q \) at rest, \( u_Q/c = \gamma_0 \).)

All AQs in Eq. (2) can be written as CBGQs in some basis. We shall write them in the standard basis \( \{\gamma_\mu\} \). In the \( \{\gamma_\mu\} \) basis \( x = x^\mu \gamma_\mu \), \( u_Q = u_Q^\mu \gamma_\mu \), \( F = (1/2)F^{\alpha \beta} \gamma_\alpha \wedge \gamma_\beta \); the basis components \( F^{\alpha \beta} \) are determined as \( F^{\alpha \beta} = \gamma^\beta \cdot (\gamma^\alpha \cdot F) = (\gamma^\beta \cdot \gamma^\alpha) \cdot F \). Every 4D CBGQ is invariant under the passive Lorentz transformations (LT); the components transform by the LT and the basis by the inverse LT leaving the whole CBGQ unchanged. (This is the reason for the name ISR.) The invariance of some 4D CBGQ under the passive LT reflects the fact that such mathematical, invariant, 4D geometric quantity represents the same physical quantity for relatively moving inertial observers. Due to the invariance of any 4D CBGQ under the passive LT it will hold that, e.g., \( F = (1/2)F^{\mu \nu} \gamma_\mu \wedge \gamma_\nu = (1/2)F^{\mu \nu} \gamma_\mu \wedge \gamma_\nu \), where \( F^{\mu \nu} \) are components and \( \gamma_\mu \) and \( \gamma_\nu \) are the basis 1-vectors in two relatively moving inertial frames \( S \) and \( S' \) respectively. (Of course one could also use another basis, e.g., the basis \( \{r_\mu\} \) with “\( r \)” synchronization, in which \( F \) will be represented as \( F = (1/2)F^{-} r_\mu \wedge r_\nu = (1/2)F^+ r_\mu \wedge r_\nu \), where the primed quantities are the Lorentz transforms of the unprimed ones.) The use of CBGQs enables us to have clearly and correctly defined concept of sameness of a physical system for different observers. In the ISR only quantities that do
not change upon the passive LT have an independent physical reality, both theoretically and experimentally. When the physical laws are written with such Lorentz invariant quantities as in SR then the principle of relativity is automatically satisfied and there is no need to postulate it as in Einstein’s SR. It is worth noting that Einstein’s [17] formulation of SR deals with Lorentz contraction, dilatation of time and the usual transformations of the 3D vectors \(\mathbf{E}\) and \(\mathbf{B}\) (see, e.g., Eqs. (11.148) and (11.149) in [1]). However, e.g., the rest length and the Lorentz contracted length are not the same 4D quantity for relatively moving observers, since the transformed length \(L' = (1 - \beta^2)^{1/2} L\) is different than the rest length \(L\). Rohrlich [18] named the Lorentz contraction and other transformations which do not refer to the same 4D quantity as the “apparent” transformations (AT). Similar ideas are expressed by Gamba [19].

Both Rohrlich [18] and Gamba [19] considered that the same quantity for relatively moving frames is a covariantly defined quantity (components of tensors) that retain the same form under the LT. But any covariant quantity, e.g., the electromagnetic field strength tensor \(F^{\alpha\beta}\), consists of components (numbers) that are taken (implicitly) in some basis. It is true that these components refer to the same tensor quantity, but they cannot be equal since the bases are not included, e.g., \(F^{\mu\nu} \neq F'^{\mu\nu}\). \(F^{\mu\nu}\) are Lorentz transformed components. When the 4D geometric quantities are used, as in the ISR, the concept of sameness becomes very clear since, as mentioned above, every 4D CGBQ is invariant under the passive LT. Using 4D geometric quantities and the concept of sameness we have shown in [6] and [7] that not only the Lorentz contraction but the dilatation of time as well are the AT. Recently a fundamental result is achieved in [8, 10] and [11] (both in the tensor formalism and in the Clifford algebra formalism). There it is proved that the usual transformations of the 3D \(\mathbf{E}\) and \(\mathbf{B}\) are also the AT; they markedly differ from the LT of the 4D geometric quantities that represent the electric and magnetic fields. This result indicates that, contrary to the general belief, which prevails from Einstein’s fundamental paper [17], the usual transformations of the 3D \(\mathbf{E}\) and \(\mathbf{B}\) are not relativistically correct. Comparison with experiments in [7] and [10-12] clearly showed that the approach to SR with 4D geometric quantities, i.e., the ISR, is in a true agreement with all considered experiments. That agreement is independent of the chosen frame and of the chosen system of coordinates in it.

Now let us see how the bivector field \(\mathbf{F}\) can be decomposed. Usually the spacetime split is used for the decomposition of \(\mathbf{F}\) into the electric and magnetic fields that are represented by bivectors, see Eqs. (58)-(60) in Hestenes’ paper [15]. This means that Hestenes’ decomposition is an observer dependent decomposition; an observer independent quantity \(\mathbf{F}\) is decomposed into observer dependent bivectors of the electric and magnetic fields. Instead of using the observer dependent decomposition from [15,16] we shall make an analogy with the tensor formalism [20] (see also [5, 6] and [8]) and represent the electric and magnetic fields by 1-vectors \(\mathbf{E}\) and \(\mathbf{B}\) [9] that are defined without reference frames, i.e., as AQs

\[
\begin{align*}
F &= (1/c)\mathbf{E} \wedge \mathbf{v} + (\mathbf{IB}) \cdot \mathbf{v}, \\
E &= (1/c)F \cdot \mathbf{v}, \quad B = -(1/c^2)I(F \wedge \mathbf{v}),
\end{align*}
\]

where \(I\) is the unit pseudoscalar. \((I\) is defined algebraically without introducing any reference frame, as in [21], Sec. 1.2.) The velocity \(\mathbf{v}\) and all other quantities entering into the relations (3) are AQs. That velocity \(\mathbf{v}\) characterizes some general observer. We can say, as in tensor formalism [20,6,8] that \(\mathbf{v}\) is the velocity (1-vector) of a family of observers who measures \(\mathbf{E}\) and \(\mathbf{B}\) fields. Of course the relations for \(\mathbf{E}\) and \(\mathbf{B}\), Eq. (3), hold for any observer; they are manifestly Lorentz invariant equations. Note that

\[
E \cdot \mathbf{v} = B \cdot \mathbf{v} = 0,
\]

which yields that only three components of \(\mathbf{E}\) and three components of \(\mathbf{B}\) are independent quantities. The relations (3) and (4) that connect \(\mathbf{F}\) and 1-vectors \(\mathbf{E}\) and \(\mathbf{B}\) are explained in more detail in [9-11]. We also remark that a complete and consistent formulation of classical electromagnetism with the bivector field \(\mathbf{F}\) as the primary quantity is presented in [12].

(It is shown in [10,11] that one can use another equivalent decomposition of \(\mathbf{F}\) into bivectors \(E_{Hv}\) and \(B_{Hv}\)

\[
\begin{align*}
F &= E_{Hv} + cIB_{Hv}, \quad E_{Hv} = (1/c^2)(F \cdot \mathbf{v}) \wedge \mathbf{v}, \\
B_{Hv} &= -(1/c^3)I[(F \wedge \mathbf{v}) \cdot \mathbf{v}], \quad IB_{Hv} = (1/c^3)(F \wedge \mathbf{v}) \cdot \mathbf{v}.
\end{align*}
\]

However we shall use the decomposition (4) into 1-vectors \(\mathbf{E}\) and \(\mathbf{B}\) since it is much simpler and closer to the usual formulation with the 3D \(\mathbf{E}\) and \(\mathbf{B}\).

The 1-vectors \(\mathbf{E}\) and \(\mathbf{B}\) for a charge \(Q\) moving with constant velocity \(u_Q\) can be determined from (3) and the expression for the bivector field \(\mathbf{F}\) (2). They are

\[
\begin{align*}
E &= (D/c^2)[(x \wedge u_Q) \cdot \mathbf{v}] \\
B &= -(D/c^3)I(x \wedge u_Q \wedge \mathbf{v}),
\end{align*}
\]
where \( D = kQ/|x \wedge (uQ/c)|^3 \). Note that \( B \) in (3) can be expressed in terms of \( E \) as
\[
B = (1/c^3)I(uQ \wedge E \wedge v).
\] (7)

When the world lines of the observer and the charge \( Q \) coincide, \( uQ = v \), then (3) yields that \( B = 0 \) and only an electric field (Coulomb field) remains.

The Lorentz force can be written in terms of 4D AQs, 1-vectors \( E \) and \( B \), as [9,10]
\[
K_L = (q/c) [(1/c)E \wedge v + (IB) \cdot v] \cdot u.
\] (8)

Particularly from the definition of the Lorentz force \( K_L \) and the relation \( E = (1/c)F \cdot v \) (from (8)) it follows that the Lorentz force ascribed by an observer comoving with a charge, \( u = v \), is purely electric \( K_L = qE \). When \( K_L \) is written as a CBGQ in \( S \) and in the \( \{ \gamma_\mu \} \) basis it is given as
\[
K_L = (q/c^2)[v_\nu u_\mu E^\mu + \varepsilon_{\mu\nu\rho\sigma} CB_{\rho\sigma} - (E^\nu u_\nu) v^\mu]\gamma_\mu,
\] (9)

where \( \varepsilon_{\mu\nu\rho\sigma} \equiv \varepsilon_{\lambda\mu\nu\rho} v^\lambda \) is the totally skew-symmetric Levi-Civita pseudotensor induced on the hypersurface orthogonal to \( v \).

When the force, e.g., the Lorentz force \( K_L \), is known we can solve the equation of motion, Newton’s second law, written as
\[
K = dp/d\tau, \quad p = mu.
\] (10)

where \( p \) is the proper momentum (1-vector); \( p \) as a CBGQ in \( S \) and in the \( \{ \gamma_\mu \} \) basis is \( p = p^\nu \gamma_\nu \), \( p^\nu = (\gamma_\nu m c, \gamma_\nu p_2, \gamma_\nu p_3, \gamma_\nu p_4) \), where \( p_{x,y,z} \) are the components of the 3D momentum \( p = m u \), \( \gamma_\nu = (1-\beta^2)^{-1/2} \), \( \beta_u = |u|/c \).

Furthermore the angular momentum \( M \) (bivector), the torque \( N \) (bivector) for the force \( K \) and manifestly Lorentz invariant equation connecting \( M \) and \( N \) are defined as
\[
M = x \wedge p, \quad N = x \wedge K;
\]
\[
N = dM/d\tau.
\] (11)

When \( M \) and \( N \) are written as CBGQs in the \( \{ \gamma_\mu \} \) basis they become
\[
M = (1/2)M^{\mu\nu} \gamma_\nu \wedge \gamma_\nu, \quad M^{\mu\nu} = m(x^\mu u^\nu - x^\nu u^\mu),
\]
\[
N = (1/2)N^{\mu\nu} \gamma_\nu \wedge \gamma_\nu, \quad N^{\mu\nu} = x^\mu K^\nu - x^\nu K^\mu.
\] (12)

We see that the components \( M^{\mu\nu} \) from (12) are identical to the covariant angular momentum four-tensor given by Eq. (A3) in Jackson’s paper [2]. However \( M \) and \( N \) from (11) are 4D geometric quantities, the 4D AQs, which are independent of the chosen reference frame and of the chosen system of coordinates in it, whereas the components \( M^{\mu\nu} \) and \( N^{\mu\nu} \) that are used in the usual covariant approach, e.g., Eq. (A3) in [2], are coordinate quantities, the numbers obtained in the specific system of coordinates, i.e., in the \( \{ \gamma_\mu \} \) basis. Notice that, in contrast to the usual covariant approach, \( M \) and \( N \) from (12) are also 4D geometric quantities, the 4D CBGQs, which contain both components and a basis, here bivector basis \( \gamma_\mu \wedge \gamma_\nu \).

In the same way as \( F \) is decomposed in (8) into 1-vectors \( E \) and \( B \) and the unit time-like 1-vector \( v/c \) we can decompose the bivector \( N \) defined either as 4D AQs (by the relation (11)) or as 4D CBGQ (by equation (12)) into two 1-vectors \( N_s \) and \( N_t \)
\[
N = (v/c) \wedge N_t + (v/c) \cdot (N_s I),
\]
\[
N_t = (v/c) \cdot N, \quad N_s = I(N \wedge v/c),
\] (13)

with the condition
\[
N_s \cdot v = N_t \cdot v = 0;
\] (14)

only three components of \( N_s \) and three components of \( N_t \) are independent since \( N \) is antisymmetric. Here, as in (8), \( v \) is the velocity (1-vector) of a family of observers who measures \( N \). All quantities in (13) are 4D AQs. It is worth noting that the introduction of \( N_s \) and \( N_t \) and the decomposition of the bivector \( N \) into 1-vectors \( N_s \) and \( N_t \), equations (13) and (14), are not earlier mentioned in the literature, as I am aware. When \( N_s \) and \( N_t \) are written as CBGQs in the \( \{ \gamma_\mu \} \) basis they become
\[
N_s = (1/2c)\varepsilon^{\alpha\beta\mu\nu} N_{\alpha\beta} v_\nu \gamma_\nu, \quad N_t = (1/c)N^{\mu\nu} v_\mu \gamma_\nu.
\] (15)
It is seen from (15) that in the frame of “fiducial” observers, in which the observers who measure \( N_s \) and \( N_t \) are at rest, and in the \( \{\gamma_\mu\} \) basis, \( v^\mu = (c, 0, 0, 0) \), \( N_s^0 = N_t^0 = 0 \) and only the spatial components \( N_s^i \) and \( N_t^i \) remain

\[
\begin{align*}
N_s^0 &= 0, \quad N_s^i = (1/2)\varepsilon^{ijk} N_{jk}, \quad N_t^0 = 0, \quad N_t^i = N_t^0, \\
N_s^1 &= N^{23} = x^2 K_L^2 - x^3 K_S^2, \quad N_s^2 = N^{31}, \quad N_s^3 = N^{12}. 
\end{align*}
\] (16)

Thus in our approach the torque in the 4D spacetime is the bivector \( N \) defined either as 4D AQs (by the relation 11) or as 4D CBGQ (by equation 12). From the bivector \( N \) we have constructed two 1-vectors, the “space-space” torque \( N_s \) and the “time-space” torque \( N_t \) (the relation 13 with the condition 14), which together contain the same physical information as the bivector \( N \). Hence both \( N_s \) and \( N_t \) are 4D torques which taken together are equivalent to the 4D torque, the bivector \( N \).

The whole discussion with the torque can be completely repeated for the angular momentum replacing \( N \), \( N_s \) and \( N_t \) by \( M \), \( M_s \) and \( M_t \). Thus we have

\[
\begin{align*}
M &= (v/c) \wedge M_t + (v/c) \cdot (M_s I), \\
M_t &= (v/c) \cdot M, \quad M_s = I(M \wedge v/c), 
\end{align*}
\]

(17) with the condition

\[ M_s \cdot v = M_t \cdot v = 0. \] (18)

1-vectors \( M_s \) and \( M_t \) correspond to \( L \) and \( K \) from [2] respectively in the usual 3D picture. It has to be remarked that, according to my knowledge, the relations 17 and 15 were not earlier mentioned in the literature. It is usually considered that only \( M_s \) is angular momentum, see, e.g., Ludvigsen’s book, [20] Sec. 8.3 (with tensors as geometric quantities). In our approach both \( M_s \) and \( M_t \) are angular momentums, which have to be treated on an equal footing. They contain the same physical information as the bivector \( M \) only when they are taken together.

When \( M_s \) and \( M_t \) are written as CBGQs in the \( \{\gamma_\mu\} \) basis, they become

\[ M_s = (1/2c)\varepsilon^{\alpha\beta\mu\nu} M_{\alpha\beta} v_\mu \gamma_\nu, \quad M_t = (1/c)M^{\mu\nu} v_\mu \gamma_\nu. \] (19)

The representation of the angular momentum with two 1-vectors \( M_s \) and \( M_t \) will surely have important consequences in the quantum theory and in the quantum field theory.

Instead of using decompositions of \( N \) and \( M \) into 1-vectors \( N_s \), \( N_t \) and \( M_s \), \( M_t \) we could decompose them into bivectors in a complete analogy with the decomposition of \( F \) into the bivectors \( E_{H\nu} \) and \( B_{H\nu} \) 5, but it will not be done here.

**III. JACKSON’S PARADOX. THE 3D QUANTITIES AND THEIR APPARENT TRANSFORMATIONS**

Having discussed different 4D geometric quantities we now examine the corresponding 3D quantities and their transformations. A nice example from the recent literature will help to better understand the essential difference between the approach with 4D geometric quantities and the usual approach with 3D quantities.

In a recent paper Jackson [2] discussed the apparent paradox of different mechanical equations for force and torque governing the motion of a charged particle in different inertial frames. Two inertial frames \( S \) (the laboratory frame) and \( S' \) (the moving frame) are considered (they are \( K \) and \( K' \) respectively in Jackson’s notation). In \( S' \) a particle of charge \( q \) and mass \( m \) experiences only the radially directed electric force caused by a point charge \( Q \) fixed permanently at the origin. Consequently both the angular momentum \( L' \) and the torque \( N' \) are zero in \( S' \), see Fig. 1(a) in [2]. In \( S \) the charge \( Q \) is in uniform motion and it produces both an electric field \( E \) and a magnetic field \( B \). The existence of \( B \) in \( S \) is responsible for the existence of the 3D magnetic force \( F = q u \times B \) and this force provides a 3D torque \( N = (x \times F) \) on the charged particle, see Fig. 1(b) in [2]. Consequently a nonvanishing 3D angular momentum of the charged particle changes in time in \( S \), \( N = dL/dt \). Thus there is a 3D torque and so a time rate of change of 3D angular momentum in one inertial frame, but no 3D angular momentum and no 3D torque in another. Jackson [2] considers that there is no paradox and that such result is relativistically correct result. (It has to be mentioned that exactly the same paradox appears in the Trouton-Noble experiment, see, e.g., [4] and references therein.)

In Sec. III in [2] Jackson discusses “Lorentz transformations of the angular momentum between frames.” He starts with the usual covariant definition of the angular momentum tensor \( M^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} \), Eq. (8) in [2]. Notice that the standard basis \( \{\gamma_\mu\} \), i.e., Einstein’s system of coordinates, is implicit in that definition, actually, the implicit basis
is bivector basis as in (12). Then the components \( L_i \) of the 3D vector \( \mathbf{L} \) (which is called the angular momentum) are identified with the space-space components of \( M^\mu \nu \) and the components \( L_{t,i} \) of the 3D vector \( \mathbf{L}_t \) (for which a physical interpretation is not given) are identified with the three time-space components of \( M^\mu \nu \) (we denote Jackson’s \( K_i \) with \( L_{t,i} \), \( \mathbf{K} \) with \( \mathbf{L}_t \)). (Note that instead of the 3D \( \mathbf{L} \) and \( \mathbf{L}_t \) we are dealing with 4D geometric quantities, 1-vectors \( M_s \) and \( M_{t,i} \), defined by (17) and (18).)

This is in a complete analogy with the way in which (see [1] Sec. 11.9) the components of 3D vectors \( \mathbf{B} \) and \( \mathbf{E} \) are identified with the space-space and the time-space components respectively of \( F^{\mu \nu} \)

\[
B_i = (1/2c)\varepsilon_{ikl}F^{kl}, \quad E_i = F^{i0}.
\] (20)

(It is worth noting that Einstein’s fundamental work [22] is the earliest reference on covariant electrodynamics and on the identification of components of \( F^{\alpha \beta} \) with the components of the 3D \( \mathbf{E} \) and \( \mathbf{B} \).

The mentioned identification for \( L_t \) and \( L_{t,i} \) is

\[
L_i = (1/2)\varepsilon_{ikl}M^{kl}, \quad L_{t,i} = M^{t0}.
\] (21)

The relations (20) and (21) show that the components \( L_i \) correspond to \(-B_i\) and \( L_{t,i}\) to \(-E_i\). In (20) and (21) the components of the 3D vectors \( \mathbf{B}, \mathbf{E} \) and \( \mathbf{L}, \mathbf{L}_t \) respectively are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric \( \varepsilon \) tensor too. The super- and subscripts are used only on the components of the 4D quantities. The 3D vectors \( \mathbf{L} \) and \( \mathbf{L}_t \), as geometric quantities in the 3D space, are constructed multiplying the components \( M^{\mu \nu} \) of a 4D geometric quantity \( M \), by the unit 3D vectors \( i, j, k \), e.g., \( \mathbf{L} = M^{23}i + M^{31}j + M^{12}k \). Such procedure clearly shows that in the approach from [2] the physical reality is attributed to the 3D vector \( \mathbf{L} \) (but what is with a physical interpretation for \( \mathbf{L}_t \) and not to the whole set of components \( M^{\mu \nu} \), i.e., the 4D geometric quantity \( M \)). Note that exactly the same procedure is applied to construct geometric quantities in the 3D space \( \mathbf{B} \) and \( \mathbf{E} \) from the components \( F^{\mu \nu} \) and the unit 3D vectors \( i, j, k \). The objections to such procedure for the construction of \( \mathbf{B} \) and \( \mathbf{E} \) are considered in detail in, e.g., [8, 10] and [11], and they apply in the same measure to the construction of \( \mathbf{L} \) and \( \mathbf{L}_t \). Some of them are the following.

(i) The whole procedure is made in an inertial frame of reference with the Einstein system of coordinates, i.e., the standard basis \( \{\gamma_\mu\} \). In another system of coordinates that is different than the Einstein system of coordinates, e.g., differing in the chosen synchronization (as it is the ‘r’ synchronization considered in [5-7]), the identification of \( E_i \) with \( F^{i0} \), as in (20) (and also for \( B_i \), or \( L_{t,i} \) with \( M^{t0} \) in (21), is impossible and meaningless.

(ii) Furthermore the components \( E_i, B_i, \) and \( L_{t,i}, L_i \), of the 3D vectors \( \mathbf{E}, \mathbf{B} \) and \( \mathbf{L}, \mathbf{L}_t \) respectively are determined from 4D quantities written in the standard basis \( \{\gamma_\mu\} \). Hence when forming the geometric quantities the components would need to be multiplied with the unit 1-vectors \( \gamma_i \) and not with the unit 3D-vectors.

It is considered in [2] that the relations (21) hold both in \( S' \), the rest frame of the charges \( q \) and \( Q \), and in \( S \), the laboratory frame, which then leads to the usual transformations of the components of the 3D vector \( \mathbf{L} \) that are given by Eq. 11 in [2]. We write them as

\[
L_1 = L'_1, \quad L_2 = \gamma(L'_2 - \beta L'_{t,3}), \quad L_3 = \gamma(L'_3 + \beta L'_{t,2}).
\] (22)

Note that the components \( L_i \) in \( S \) are expressed by the mixture of components \( L'_i \) and \( L'_{t,i} \) from \( S' \). It is clear from the usual transformations (22) that the components of the 3D angular momentum do not vanish in the laboratory frame \( S \), even if they do in \( S' \). In this case \( L_3 \) is different from zero due to contribution from \( L'_{t,2} \). Then Jackson [2] calculates \( dL_3/dt \), where \( L_3 \) is obtained “via a Lorentz transformation,” i.e., via the transformations (22) (note that in [2] the derivative of \( L_3 \) is relative to the coordinate time \( t \) and not, as in (11), relative to the proper time \( \tau \)). It is shown in [2] that \( dL_3/dt = N_3 \), where the torque \( N_3 \) (\( N_z \) in Eq. (7) in [2]) is “directly obtained from the force equation in the laboratory.” \( N_z \) in Eq. (7) in [2] is obtained using the force equation (Eq. (4) in [2]) with the 3D vectors \( \mathbf{p}, \mathbf{F}, \mathbf{E} \) and \( \mathbf{B} \), but again the derivative is relative to the coordinate time \( t \). Jackson [2] finds the consistency in both calculations and states: “The time rate of change of the particle’s angular momentum obtained via a Lorentz transformation is equal to the torque directly obtained from the force equation in the laboratory, as it must.” In our opinion what is found in [2] is that when using the 3D vectors and their transformations (like Eq. (22)) the paradox is always obtained and, actually, the principle of relativity is violated.

Let us examine in more detail the transformations of components of the 3D quantities and also of the 3D vectors. As already said both \( \mathbf{L}' \) and \( \mathbf{L} \), as geometric quantities in the 3D space, are constructed multiplying the components \( L'_i \) and \( L_{t,i} \) (given by (22)) by the unit 3D vectors \( i', j', k' \) and \( i, j, k \) respectively. This gives another objection to the usual construction of \( \mathbf{L}' \) and \( \mathbf{L} \).

(iii) The components \( L_{t,i} \) are determined by the transformations (22), but there is no transformation which transforms the unit 3D vectors \( i', j', k' \) into the unit 3D vectors \( i, j, k \). Hence it is not true that the 3D vector \( \mathbf{L} = L_1i + L_2j + L_3k \)
relatively moving inertial observers, \( \mathbf{L} \neq \mathbf{L}' \)

\[
L_1 i + L_2 j + L_3 k \neq L_1' i' + L_2' j' + L_3' k'.
\]

(23)

Thus contrary to the general opinion, the transformations (22) are not the LT but the AT of the 3D \( \mathbf{L} \). The same situation happens with the transformations of the 3D \( \mathbf{B} \) and \( \mathbf{E} \) as explained in detail in [8, 10] and [11].

On the other hand, as already mentioned in Sec. II, every 4D CBGQ is invariant under the passive LT, which means that such 4D geometric quantity represents the same physical quantity for relatively moving inertial observers. Hence, it holds, e.g., \( M_1 = (1/c) M^{\mu \nu} u_\nu \gamma_\nu = (1/c) M^{\mu \nu} u'_\nu \gamma'_\nu \), where all primed quantities are obtained by the LT from the unprimed ones. \( M_2 \) and \( M_4 \), written in 4D CBGQs, transform under the LT as every 1-vector (as 4D CBGQ) transforms, which means that the components \( M^\mu_2 \) transform again to \( M'^\mu_2 \) and similarly \( M^\mu_4 \) transform to \( M'^\mu_4 \); there is no mixing of components. Thus the equation corresponding to Eq. (11) in [2], i.e., to Eq. (22), will be

\[
M^0_2 = \gamma(M^0_2 + \beta M^1_2), \quad M^1_2 = \gamma(M^1_2 + \beta M^0_2), \quad M^2,3_2 = M^2,3_2, \quad M^0_2 = \gamma(M^0_2 + \beta M^1_2), \quad M^1_2 = \gamma(M^1_2 + \beta M^0_2), \quad M^2,3_2 = M^2,3_2,
\]

(24)

and the same for \( M^\mu_4 \). This is in a sharp contrast to the AT (22) in which the transformed components of the 3D force \( F \) is constructed from the components \( M^\mu_2 \) and \( M^\mu_4 \) are multiplied by the unit 1-vectors \( \gamma_\mu \), while, as we mentioned, the 3D angular momentum \( L \) is formed multiplying the components \( M^{\mu \nu} \) (i.e., \( L_i \)) of a 4D geometric quantity \( M \), by the unit 3D vectors \( i, j, k \). Of course, it holds that \( M_2 \) is the same quantity for observers in \( S \) and \( S' \), which can use different basis, e.g., \( \{\gamma_\mu\}, \{\nu\} \), and so on. Thus

\[
M_2 = M^\mu_2 \gamma_\mu = M'^\mu_2 \gamma'_\mu = M^\mu_2 r^\mu r_\mu = M'^\mu_2 r'^\mu r'_\mu = \ldots,
\]

(25)

where the primed quantities are the Lorentz transforms of the unprimed ones; see the discussion at the beginning of Sec. II.

This is in a complete analogy with the fundamental difference between the AT of the 3D \( \mathbf{E} \) and \( \mathbf{B} \) and the LT of 1-vectors \( E \) and \( B \) that are defined by (3), see, e.g., [10, 11], in which this fundamental difference is exactly proved. There, it is also shown that the LT of 1-vectors \( E \) and \( B \) are in a complete agreement with experiments on motional emf and Faraday disk, while it is not the case with the AT of the 3D \( \mathbf{E} \) and \( \mathbf{B} \). (Regarding the mentioned analogy, e.g., the AT for the components of the 3D \( \mathbf{B} \) are the same as the AT for \( L_2 \) that are given by (22); \( L_i, L'_i \) have to be replaced by \( B_i, B'_i \) (components of the 3D \( \mathbf{B} \)). On the other hand the LT for the components \( B^\mu \) of the 1-vector \( B \) are the same as the LT (24), but \( M^\mu_2, M^\mu_4 \) have to be replaced by \( B^\mu, B'^\mu \).)

The above consideration suggests that the transformations of other 3D quantities are the AT as well. For example, the AT of the 3D \( \mathbf{N} \), which are the same as (22), are found, e.g., in Jefimenko’s book [3] and given in [4] Eqs. (1)-(3). In Sec. 8 of [3], under the title: “From relativistic electromagnetism to relativistic mechanics,” the AT of different 3D quantities are presented. Notice that the AT of the 3D \( \mathbf{E}, \mathbf{B}, \mathbf{L} \) and \( \mathbf{N} \) all may be obtained in the same way by the identification of the components of the 3D vectors with the components of second-rank 4D tensors, i.e., bivectors in the 4D spacetime.

Furthermore the transformations of the 3D force \( \mathbf{F} \) are also the AT; they are given, e.g., by Eqs. (8-5.4)-(8-5.6) in [3] (or by Eqs. (1.53)-(1.55) in [23])

\[
F'_x = [F_x - (\beta_u/u)(\mathbf{F}u)]/(1 - (\beta_u u_x/c)), \quad F'_y, z = F_y, z/\gamma(1 - (\beta_u u_x/c)),
\]

(26)

where \( u \) is the 3D velocity of a particle. All previously mentioned objections, (i) - (iii), regarding the construction of the 3D vectors, are also at place here. The 3D forces \( \mathbf{F}, \mathbf{F}' \) are constructed from the components \( F_{x,y,z}, F'_{x,y,z} \) (determined by (26)) and the unit 3D vectors \( i, j, k \), and \( i', j', k' \) respectively and \( i', j', k' \) are not obtained by any transformations from \( i, j, k \). Particularly it is visible from (26) that \( \mathbf{F}' \neq \mathbf{F} \); they do not refer to the same quantity in the 4D spacetime and the transformations (26) are not the LT but the AT. The same holds for the well-defined transformations of the 3D velocity \( u \) that are given, e.g., by equations (11.31) in [1], or by equations (7-2.5)-(7-2.7) in [3].

From the ISR viewpoint, the correctly defined quantities in the 4D spacetime, both theoretically and experimentally, are 1-vectors \( K = K' r_\mu \gamma_\mu \) and \( u = u' r_\mu \gamma_\mu \), or in another basis \( \{r_\mu\} \) these 1-vectors are \( K = K' r_\mu \gamma_\mu \) and \( u = u' r_\mu \gamma_\mu \). In contrast to awkward transformations of the components of the 3D force \( \mathbf{F} \) (26) the LT of the components \( K' \) of the 1-vector \( K \) are very simple (the LT (26) but with \( K' \), \( K'' \) replacing \( M^\mu_2, M^\mu_4 \)). When the components of the 4-force \( K \) and of the 4-velocity \( u \) are determined in the standard basis \( \{\gamma_\mu\} \) then they can be expressed in terms of components of the 3D force \( \mathbf{F} \), \( (F_x, F_y, F_z) \), and of the 3D velocity \( u \), \( (u_x, u_y, u_z) \). They are \( K = (\gamma_\mu F_x/c, \gamma_\mu F_y/c, \gamma_\mu F_z/c, \gamma_\mu F_\nu) \) and \( u' = (\gamma_\mu c, \gamma_\mu u_x/c, \gamma_\mu u_y/c, \gamma_\mu u_z/c) \), where \( \gamma_\mu = (1 - |u|^2/c^2)^{-1/2} \). We see that, in general, the spatial components \( K^i, u^i \)
differ from the components of the 3D quantities $\mathbf{F}$, $\mathbf{u}$. Only in the case when the considered particle is at rest, i.e., $u_x, u_y, u_z = 0$, $\gamma_u = 1$ and consequently $u' = (c, 0, 0, 0)$, then $K'$ will be exclusively determined with the components $F_{x', y', z'}$, i.e., $K' = (0, F_x, F_y, F_z)$. However even in that case $u'$ and $K'$ are the components of the 4D geometric quantities $u = u' \gamma_u$ and $K = K' \gamma_u$ in the $\{\gamma_u\}$ basis and not the components of some 3D geometric quantities $\mathbf{u}$ and $\mathbf{F}$, see also the discussion in [10] Sec. 3.2. This discussion additionally shows that the transformations (20) have nothing in common with the LT of the 1-vector $K$.

It is generally accepted, e.g., [1, 3, 23], that the “relativistic” equations of motion have the same form in two relatively moving inertial frames $S$ and $S'$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad \mathbf{p} = m\gamma_u \mathbf{u}$$

$$\mathbf{F}' = \frac{d\mathbf{p}'}{dt'}, \quad (27)$$

see, for example, Eqs. (1.39) and (1.40) in [23], or Sec. 12.2 and 12.4 (with the Lorentz force) in [1].

In Einstein’s formulation [17] of SR the principle of relativity is a fundamental postulate that is supposed to hold for all physical laws including those expressed by 3D quantities, e.g., the Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$; Einstein [17] used that postulate to derive the transformations of the 3D $\mathbf{E}$ and $\mathbf{B}$. It is proved in [11], both in the geometric algebra and tensor formalisms, that the usual Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$ change their form under the LT and thus that they are not covariant under the LT. This result explicitly shows that the principle of relativity does not hold for physical laws expressed by 3D quantities (a fundamental achievement). The results from this section also reveal that a 3D quantity cannot correctly transform under the LT, which means that it does not have an independent physical reality in the 4D spacetime; it is not the same quantity for relatively moving observers in the 4D spacetime. Hence it is not true that Eqs. (21) are the relativistic equations of motion since the primed 3D quantities are not obtained by the LT from the unprimed ones, but they are obtained in terms of the AT for the 3D force $\mathbf{F}$ (20) and the 3D momentum $\mathbf{p}$, i.e., the 3D velocity $\mathbf{u}$, Eq. (11.31) in [1]. Instead of Eqs. (27) one has to use equation of motion with 4D geometric quantities (10).

Similarly it is generally accepted in usual approaches that the Lorentz force law remains of the same form in relatively moving inertial frames $S$ and $S'$

$$\mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}, \quad S'; \quad \mathbf{F}' = q\mathbf{E}' + q\mathbf{u}' \times \mathbf{B}', \quad (28)$$

where all primed quantities are considered to be obtained by the LT from the unprimed ones. For example, it is argued, e.g., in [3] Sec. 8: “This law does not depend on the inertial reference frame in which $q$, $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{B}$ are measured.” The same assertions about the form of the Lorentz force law in two inertial frames can be found in [23], Eqs. (6.42) and (6.43). There, this form invariance of the Lorentz force, together with the AT of the 3D force $\mathbf{F}$ (20) and the AT of the 3D velocity $\mathbf{u}$, Eqs. (1.26)-(1.28) in [23], are used to derive the AT for the 3D $\mathbf{E}$ and $\mathbf{B}$. However the above discussion clearly shows that the 3D quantities in $S'$ are not obtained by the LT from the corresponding 3D quantities in $S$ than by the use of the AT. Therefore the Lorentz force law has to be written by means of 4D geometric quantities, e.g., Eq. (11) with bivector field $F$, or Eq. (5) with 1-vectors $E$ and $B$ and, of course, so defined $K_L$ is an invariant quantity under the LT.

IV. THE RESOLUTION OF JACKSON’S PARADOX USING 4D TORQUES

Instead of dealing with 3D quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{L}$ and $\mathbf{N}$ and their AT as in [2] we shall examine Jackson’s paradox using the expressions for the 4D geometric quantities from Sec. II. First we write $N$ from (11) using the expression (1) for $K_L$ and (2) for $F$. Then $N$ becomes

$$N = (Dq/c^2)(u \cdot x)(u_Q \wedge x), \quad (29)$$

where $D$ is already defined in (9), $D = kQ/|x \wedge (u_Q/c)|^3$. This is the most general expression for the considered torque $N$ written as an AQ. Then $N_s$ and $N_t$ are determined from (18) and (29) as

$$N_s = (Dq/c^3)(u \cdot x)I(x \wedge v \wedge u_Q),$$

$$N_t = (Dq/c^3)(u \cdot x)[(x \wedge u_Q) \cdot v]. \quad (30)$$

Comparison with (9) shows that $N_s$ and $N_t$ can be expressed in terms of $B$ and $E$ as

$$N_s = q(u \cdot x)B,$$

$$N_t = (q/c)(u \cdot x)E. \quad (31)$$
As already said, in connection with (11), when \( u_Q = \nu \) then \( B = 0 \). The relations (10) and (11) reveal that in that case \( N_s = 0 \) as well.

A. \( N \) as a CBGQ in \( S' \) and \( S \) frames

Let us now write all AQs from (20) as CBGQs in \( S' \), the rest frame of the charge \( Q \), in which \( u_Q = c\gamma_0' \). Then \( N = (Dq/c)(u \cdot x)(\gamma_0' \wedge x) \), and in the \( \{ \gamma_\mu' \} \) basis it is explicitly given as

\[
N = (1/2)N^{\mu\nu}\gamma_{\mu'} \wedge \gamma_{\nu'} = N^{01}(\gamma_0' \wedge \gamma_1') + N^{02}(\gamma_0' \wedge \gamma_2'),
\]

\[
N^{01} = (Dq/c)(u^\mu x_\mu')x^0, \quad N^{02} = (Dq/c)(u^\mu x_\mu')x^2.
\]

(32)

The components \( x^\mu' \) are \( x^\mu' = (x^{0}, c t', x^1, x^2, 0) \) where \( x^0 = r' \cos \theta', x^1 = r' \sin \theta' \). In \( S' \) the velocity 1-vector of the charge \( q \) \( (u = dx/d\tau) \) at any \( t' \) is \( u = u^\mu' \gamma_{\mu'} \), where \( u^\mu = dx^\mu / d\tau = (u_0, u^1, u^2, 0) \). The components \( N^{\mu\nu} \) \( \left( N^{\mu\nu} = x^\mu K_L^\nu - x^\nu K_L^\mu \right) \) that are different from zero are only \( N^{01} \) and \( N^{02} \).

It can be easily seen that all \( N^{\alpha\beta} \) are zero in the \( S' \) frame when it is supposed that at \( t' = 0 \) the charge \( q \) is still at rest, i.e., \( u^\mu = (c, 0, 0, 0) \). From the invariance of any 4D CBGQ under the passive LT it follows that at \( t' = 0 \) the whole \( N \) is zero not only in \( S' \) but in the laboratory frame \( S \) as well. This case explicitly refers to the Truout-Noble paradox as discussed in [12, 13].

Similarly in order to find the torque \( N \) in the \( S \) frame we write all AQs from (20) as CBGQs in \( S \) and in the \( \{ \gamma_\mu \} \) basis. In \( S \) the charge \( Q \) is moving with velocity \( u_Q = \gamma_Q c_0 + \gamma_Q \beta_Q c_\gamma_1 \), where \( \beta_Q = |u_Q|/c \) and \( \gamma_Q = (1 - \beta_Q^2)^{-1/2} \). Another way to find \( N \) in \( S \) is to make the LT of \( N \) in \( S' \) (22). The result is

\[
N = (1/2)N^{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu} = N^{01}(\gamma_0 \wedge \gamma_1) + N^{02}(\gamma_0 \wedge \gamma_2) + N^{12}\gamma_1 \wedge \gamma_2,
\]

(33)

where \( N^{01} = N^{01}, N^{02} = \gamma_Q N^{02} \) and \( N^{12} = \gamma_Q \beta_Q N^{02} \). When \( N \) is written explicitly in terms of quantities in \( S \) it becomes

\[
N = (Dq/c)(u^\mu x_\mu)|\gamma_Q(x^1 - \beta_Q x^0)(\gamma_0 \wedge \gamma_1)
+ \gamma_Q x^2(\gamma_0 \wedge \gamma_2) + \beta_Q \gamma_Q x^2(\gamma_1 \wedge \gamma_2)|.
\]

(34)

We see that in the laboratory frame, where the charge \( Q \) is moving, the components \( N^{\mu\nu} \) that are different from zero are not only \( N^{01} \) and \( N^{02} \) but also \( N^{12} = \beta_Q N^{02} \).

Of course, we could start with the 4D angular momentum \( M = x \wedge p \) and calculate it in both frames \( S' \) and \( S \). Then using the relation \( N = dM / d\tau \) one can again find the same expressions (22) and (34) for \( N \).

The component \( N^{12} \) can be written in the form similar to the expression for \( N_z \), Eq. (7) in [2]. Thus

\[
N^{12} = \beta_Q ct K_L^2 - \beta_Q y K_L^0 = \beta_Q ct K_L^2 - (q/c)\beta_Q y (F^{01} u_1).
\]

(35)

\( K_L^2 \) can be determined from (11) and it is \( K_L^2 = (q/c)(F^{20} u_0 + F^{21} u_1) \). Comparing \( N_z \) from Eq. (7) in [2] and our \( N^{12} \) from (35) we see that instead of the components of the 3D Lorentz force \( F \), we have the components \( K_L^2 \) and \( K_L^0 \) of the 4D Lorentz force \( K_L \), which is defined by (11).

This form for \( N^{12} \) clearly shows the essential difference between our approach and the usual approach, e.g., [2]. The paradox with the 3D torque arises since all \( N^{\mu\nu} \) are zero but, according to (35), \( N^{12} \) is different from zero in the \( S \) frame, which means that \( N' = 0 \) but \( N \neq 0 \). From the point of view of the ISR the fact that \( N' \neq N \) means that \( N \) is not obtained by the LT from \( N' \) and thus it is not the same 4D quantity for observers in the \( S' \) and \( S \) frames. In contrast to the usual approach [2] with 3D quantities, \( N \) determined by (35) is obtained by the LT from \( N \) given by (22), they represent the same 4D quantity in two relatively moving inertial frames, \( N (22) = N (35) \). Hence the principle of relativity is naturally satisfied in our geometric approach and there is not any paradox.

This consideration indicates that in order to check the validity of the above relations with 4D quantities, and thus the validity of the principle of relativity and, more generally, of the SR, the measurements must be changed relative to the usual measurements of the 3D quantities. For the 4D quantities the experimentalists have to measure all components of \( N \) and \( M \) in both frames \( S' \) and \( S \). The observers in \( S' \) and \( S \) are able to compare only such complete set of data which corresponds to the same 4D geometric quantity. Such point of view is illustrated in much more detail in [7].

B. \( N_s \) and \( N_i \) as CBGQs in \( S' \) and \( S \) frames. \( S' \) is the frame of “fiducial” observers
Let us now make the same consideration as in Sec. IV A, but with \(N_s\) and \(N_t\) as CBGQs. From the relations (13) and (14) we see that 1-vectors \(N_s\) and \(N_t\) are not uniquely determined by \(N\), but their explicit values depend also on \(v\). This means that it is important to know which frame is chosen to be the frame of “fiducial” observers, in which the observers who measure \(N_s\) and \(N_t\) are at rest. As seen from (31), (4) and [13], [14] the same conclusions refer also to the determination of \(E\), \(B\) and \(M\), \(M_t\) from \(F\) and \(M\) respectively.

First, it will be assumed that \(S'\), the rest frame of the charge \(Q\) is the \(\gamma_0\)-system, i.e., the frame of “fiducial” observers. Hence, in \(S'\) \((\gamma_1\}\) basis), \(u_Q = c = v\), and the velocity 1-vector of the charge \(q\) at any \(t'\) is \(u = u^{\mu}_s\gamma_\mu\), where \(u^{\mu} = (u^0, u^1, u^2, 0)\). The results for 1-vectors \(N_s\) and \(N_t\) can be simply obtained using (30) which yields that the “space-space” torque \(N_s\) as a CBGQ in \(S'\) is

\[
N_s = N^{\mu}_s\gamma_\mu = 0, \tag{36}
\]

and the “time-space” torque \(N_t\) as a CBGQ in \(S'\) is

\[
N_t = N^{t1}_s\gamma_1 + N^{t2}_s\gamma_2 = N^{01}_s\gamma_1 + N^{02}_s\gamma_2, \tag{37}
\]

where \(D = kQ/r^3\) and \(N^{a0}_s\), \(N^{02}_s\) are given by (32).

The same results for \(N_s\) and \(N_t\) in \(S'\) can be obtained using Eqs. (31) and (6), but written in terms of CBGQs. In that case \(B\) and \(E\) from (6) are

\[
B = B^{\mu}_s\gamma_\mu = 0, \quad E = E^{\mu}_s\gamma_\mu = D(x^1\gamma_1^0 + x^2\gamma_2^0). \tag{38}
\]

Note that the spatial components of \(E\) are the same as the components of the 3D \(E'\) as it must.

At \(t' = 0\) and when \(u^{\mu} = (0, 0, 0, 0)\) all \(N^{a\beta}_s\) are zero and consequently both \(N_s\) and \(N_t\) are zero not only in \(S'\) but in the laboratory frame \(S\) as well.

Let us now determine 1-vectors \(N_s\) and \(N_t\) as CBGQs in \(S\) \((\{\gamma_1\}\) basis). Relative to the \(S\) frame both the charge \(Q\) and the “fiducial” observers are moving with velocity \(u_Q = v = \gamma QC_0 + \gamma Q\beta QC_1\). Then \(N_s\) and \(N_t\) in \(S\) can be obtained either directly from (30), or by means of the LT of the 1-vectors \(N_s\) and \(N_t\). Due to invariance of any 4D CBGQ under the passive LT the “space-space” torque \(N_s\) is zero in the laboratory frame \(S\) too

\[
N_s = N^{\mu}_s\gamma_\mu = 0. \tag{39}
\]

The “time-space” torque \(N_t\) as a CBGQ in \(S\) is

\[
N_t = \gamma_Q\beta_QN^{01}_s\gamma_1 + \gamma_QN^{01}_s\gamma_1 + (1/\gamma_Q)N^{02}_s\gamma_2, \tag{40}
\]

where \(N^{01}_s\) and \(N^{02}_s\) are determined by (32).

The same \(N_s\) and \(N_t\) in \(S\) can be found using (31) and (6) and writing all AQs as CBGQs in the \(S\) frame. \(E\) and \(B\) in \(S\) are determined as the LT of the 1-vectors \(E\) and \(B\) given by (31) (for the LT of \(E\) and \(B\) see [10]). This yields

\[
B = B^{\mu}_s\gamma_\mu = 0, \quad E = E^{\mu}_s\gamma_\mu, \\
E^0 = D\beta_Q\gamma_\mu^2(x^1 - \beta_Qx^0), \quad E^1 = E^0/\beta_Q, \quad E^2 = Dx^2, \quad E^3 = 0. \tag{41}
\]

Hence \(N_s = N^{\mu}_s\gamma_\mu = 0\) and the same \(N_t\) in \(S\) is obtained as in (40).

Again, we could start with the 4D angular momentum \(M_s\) and \(M_t\) defined by (17) and (18) and calculate them in both frames \(S'\) and \(S\). Then using the relations \(N_{s,t} = dM_{s,t}/dt\) one can again find the same expressions (30), (37) and (39), (40) for \(N_s\), \(N_t\) in \(S'\) and \(S\) respectively.

It can be easily checked that \(N_t\) given by (37) in \(S'\) is the same 4D CBGQ as \(N_t\) given by (10) in \(S\), i.e., that \(N^{t1}_s\gamma_1 + N^{t2}_s\gamma_2 = N^{01}_s\gamma_1 + N^{02}_s\gamma_2\), and it is seen from (31) and (32) that \(N_s\) is the same 4D CBGQ for observers in \(S'\) and \(S\). This again shows that the principle of relativity is naturally satisfied in our ISR and that there is not any paradox.

Inserting \(N_s\) and \(N_t\) from (39), (37) and (39), (40) into the relation (13) (written with CBGQs), we can directly check the validity of these relations.

According to our result (39) the 1-vector \(N_s\) is zero both in \(S'\) and \(S\) at any \(t'\) when the “fiducial” frame is the \(S'\) frame. However \(N_t\) is different from zero in both frames. As already said only \(N_s\) and \(N_t\) taken together are equivalent to the bivector \(N\), which means that validity of the above relations can be checked measuring all six independent components of \(N_s\) and \(N_t\) in both frames. It has to be remarked that the usual 3D \(N\) is connected with the three spatial components of \(N_s\).

Note that the usual 3D rotation requires measurement of only three independent variables. Therefore in order to test SR, e.g., by means of the Trouton-Noble type experiments, it is not enough, as usually done, to measure three
independent parameters of the 3D rotation (i.e., three independent components of \( N_s \), or \( M_s \)), but also one has to measure the other three relevant variables (i.e., three independent components of \( N_t \), or \( M_t \)).

**C. \( N_s \) and \( N_t \) as CBGQs in \( S' \) and \( S \) frames. \( S \) is the frame of “fiducial” observers**

Let us now assume that the laboratory frame \( S \) is the \( \gamma_0 \)-system, i.e., the frame of “fiducial” observers, in which the observers who measure \( N_s \) and \( N_t \) are at rest, that is, \( v = v^\mu \gamma_\mu = c \gamma_0, \quad u^\mu = (c, 0, 0, 0) \). Then from \( \text{[15]} \), i.e., \( \text{[16]} \), it follows that in \( S \) the temporal components of the 1-vectors \( N_s \) and \( N_t \) are zero and only their spatial components remain. In the laboratory frame \( S \) the charge \( Q \) is moving and the components of the CBGQ \( u_0^\mu \gamma_\mu \) are given as \( u_0^\mu = (\gamma_0 Q, \gamma_0 \beta Q c, 0, 0) \).

The 1-vectors \( N_s \) and \( N_t \) will be determined either directly from \( \text{[30]} \) and the above expressions for \( v \) and \( u_Q \), or by the use of the already known expression \( \text{[34]} \) for the bivector \( N \) in \( S \) and the relation \( \text{[14]}, \) i.e., \( \text{[16]} \). This yields that

\[
N_s = N_s^\mu \gamma_\mu = N^{12} \gamma_3, \quad N_t = N_t^1 \gamma_1 + N_t^2 \gamma_2 = N^{01} \gamma_1 + N^{02} \gamma_2,
\]

where \( N^{12} \) is from \( \text{[33]} \) or \( \text{[34]} \) and \( N^{01}, N^{02} \) are from \( \text{[34]} \). It is visible from \( \text{[12]} \) that in the case when \( S \) is the frame of “fiducial” observers the “space-space” torque \( N_s \) is different from zero.

In the same way as in Sec. IV B we find \( N_s \) and \( N_t \) using \( \text{[31]} \) and \( \text{[0]} \). Now the charge \( Q \) moves in the frame of “fiducial” observers, the \( S \) frame, which yields that both \( E \) and the magnetic field \( B \) are different from zero. Then

\[
E = E^\mu \gamma_\mu, \quad E^0 = E^3 = 0, \quad E^1 = D \gamma_Q (x^1 - \beta_Q x^0), \quad E^2 = D \gamma_Q x^2,
\]

and the magnetic field is

\[
B = B^\mu \gamma_\mu, \quad B^0 = B^1 = B^2 = 0, \quad B^3 = (D/c) \gamma_Q \beta_Q x^3 = \beta_Q E^2/c.
\]

The spatial components \( E^i \) and \( B^j \) from \( \text{[13]} \) and \( \text{[14]} \) are the same as the usual expressions for the components of the 3D vectors \( E \) and \( B \) for an uniformly moving charge. Inserting \( \text{[13]} \) and \( \text{[14]} \) into \( \text{[31]} \) we again find \( N_s \) and \( N_t \) as in \( \text{[32]} \). \( N_s \) is different from zero since \( B \) given by \( \text{[14]} \) is different from zero.

Instead of expressing components of \( K_L \) in \( \text{[35]} \) in terms of components of the electromagnetic field \( F \) we shall now write \( N_s \) from \( \text{[32]} \) in terms of 1-vectors \( E \) and \( B \), which are explicitly given by Eqs. \( \text{[13]} \) and \( \text{[14]} \). Then \( N_s \) becomes

\[
N_s = N^3 \gamma_3 = N^{12} \gamma_3 = (\beta_Q c t K^2_L + (q/c) \beta_Q y (E^\mu u_\mu)) \gamma_3.
\]

Remember that \( E^0 = B^0 = 0 \), Eqs. \( \text{[33]} \) and \( \text{[34]} \), when \( S \) is the frame of “fiducial” observers. In the usual approach, e.g., [2], it is considered that in the \( S \) frame the whole physical torque is the 3D \( N \), i.e., \( N_s \), given by Eq. (7) in [2]. We see that in the 4D spacetime the physical torque is the bivector \( N \) that is given by relation \( \text{[33]} \) as a CBGQ in \( S \) and in the \( \{\gamma_\mu\} \) basis. When that \( S \) is chosen to be the frame of “fiducial” observers then \( N \) can be represented by two 1-vectors \( N_s \) and \( N_t \) given by \( \text{[32]} \) and \( \text{[35]} \), which are both physical and have to be determined theoretically and experimentally. Only when the laboratory frame \( S \) is the frame of “fiducial” observers the spatial components of \( N_s \) have some resemblance with the components of the 3D \( N \). However note that in \( \text{[15]} \) all components are the components of the 4D quantities, 1-vectors \( x, u, K_L, E \) and \( B \), while in Eq. (7) in [2] only the corresponding 3D vectors are involved.

Let us now determine 1-vectors \( N_s \) and \( N_t \) as CBGQs in \( S' \). Relative to the \( S' \) frame the charge \( Q \) is at rest \( u_Q = c \gamma_0', \) but the “fiducial” observers are moving with velocity \( v = \gamma_Q c \gamma_0' - \gamma_Q \beta_Q c \gamma_1' \). Then \( N_s \) and \( N_t \) in \( S' \) can be obtained either directly from \( \text{[30]} \) or by means of the LT of 1-vectors \( N_s \) and \( N_t \) as CBGQs, which are given by \( \text{[12]} \). We find that \( N_s \) is different from zero not only in \( S \) but in the \( S' \) frame as well

\[
N_s = N_s^\mu \gamma_\mu = N_s^3 \gamma_3, \quad N_s^3 = \gamma_Q \beta_Q N^{02}.
\]

For \( N_t \) one gets

\[
N_t = N_t^\mu \gamma_\mu, \quad N_t^0 = -\beta_Q \gamma_Q N^{01}, \quad N_t^{1,2} = \gamma_Q N^{01,2}, \quad N_t^3 = 0.
\]

The same results for \( N_s \) and \( N_t \) in \( S' \) can be obtained using \( \text{[31]} \) and \( \text{[0]} \) and writing all AQs as CBGQs in the \( S' \) frame. \( E \) and \( B \) in \( S' \) are determined by the LT of 1-vectors \( E \) and \( B \) given by \( \text{[13]} \) and \( \text{[14]} \) respectively. They are

\[
E = E^\mu \gamma_\mu, \quad E^0 = -\beta_Q \gamma_Q x^1, \quad E^1 = D \gamma_Q x^1, \quad E^2 = D \gamma_Q x^2, \quad E^3 = 0, \quad B = B^\mu \gamma_\mu, \quad B^0 = B^1 = B^2 = 0, \quad B^3 = (D/c) \gamma_Q \beta_Q x^3 = \beta_Q E^2/c.
\]
Inserting (48) into (31) the same $N_s$ and $N_t$ are found as in (46) and (17). $N_s$ in $S'$ is different from zero since $B$ given by (48) is different from zero.

Of course it can be again easily seen that $N_s$ ($N_t$) from (42) is equal to $N_s$ ($N_t$) from (46) ((47)); it is the same 4D CBGQ for observers in $S$ and $S'$.

Inserting $N_s$ and $N_t$ from (42) and also from (46) and (47) into the relation (13), which connects $N$ with $N_s$ and $N_t$, we find the same expressions (34) and (32) for $N_s$ in $S'$ and $S$, and we already know that they are equal, $N (32) = N (34)$.

The above discussion shows that the explicit expressions for $N_s$ and $N_t$ depend on the choice for the frame of “fiducial” observers. For example, in the $S$ frame $N_s$ and $N_t$ are given by the relations (39) and (40), or (42), when $S'$, or $S$, are chosen for the frame of “fiducial” observers. However when they are inserted into (13) they will always give the same $N$.

The paradox does not appear in the considered representations for the torques since the principle of relativity is automatically satisfied in such an approach to SR which exclusively deals with 4D geometric quantities, i.e., AQs or CBGQs. In the standard approach to SR [17] the principle of relativity is postulated outside the framework of a mathematical formulation of the theory, and, as we already discussed, it is considered that the principle of relativity holds for the equations written with the 3D quantities.

V. CONCLUSIONS

The whole consideration exposed in previous sections strongly suggests that the violation of the principle of relativity and the existence of the electrodynamic paradox come from the use of the 3D quantities, as physical quantities in the 4D spacetime, and from using their apparent transformations. We have shown that, in the 4D spacetime, the well-defined angular momentum with relativistically correct transformation properties is not the 3D vector $L$, with its apparent transformations (22), but the Lorentz invariant 4D geometric quantities, the bivector $M$, or 1-vectors $M_s$ and $M_t$, which are derived from $M$ and which together contain the same physical information as $M$. The same result refers to the 3D torque $N$ and the bivector $N$, or the torques $N_s$ and $N_t$ that together correspond to the torque $N$. It is already proved in, e.g., [8 - 12], that the same situation exists with the 3D vectors $E$ and $B$ and the 4D geometric quantities, the bivector $F$, or, derived from it, the 1-vectors $E$ and $B$.

We hope that the results obtained in this paper will have important consequences for all branches of physics in which the relativistic effects have to be taken into account, particularly for classical and quantum relativistic electrodynamics.

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