THE $L^p$ DIRICHLET BOUNDARY PROBLEM FOR SECOND ORDER ELLIPTIC SYSTEMS WITH ROUGH COEFFICIENTS

MARTIN DINDOŠ, SUJKJUNG HWANG, AND MARIUS MITREA

ABSTRACT. We establish solvability methods for strongly elliptic second order systems in divergence form with lower order (drift) terms on a domain above a Lipschitz graph, satisfying $L^p$-boundary data for $p$ near 2. The main novel aspect of our result is that the coefficients of the operator do not have to be constant or have very high regularity, instead they will satisfy a natural Carleson condition that has appeared first in the scalar case. A particular example of a system where this result can be applied is the Lamé operator for isotropic inhomogeneous materials.

The systems case poses substantial new challenges not present in the scalar case. In particular, there is no maximum principle for general elliptic systems and De Giorgi - Nash - Moser theory is also not available. Despite this we are able to establish estimates for the square and nontangential maximal functions for the solution of the elliptic system and use these estimates to establish $L^p$ solvability for $p$ near 2.

1. INTRODUCTION

This paper is motivated by the known results concerning boundary value problems for second order elliptic equations in divergence form, when the coefficients satisfying a certain natural, minimal smoothness condition (refer [13], [15], [23]).

Let $\Omega \subset \mathbb{R}^n$ be a domain defined by a Lipschitz function $\phi$, that is

$$\Omega = \{(x_0, x') : x_0 > \phi(x')\}. \quad (1.1)$$

Consider a second order elliptic system in divergence form given by

$$Lu = \left[ \partial_i \left( A_{ij}^{\alpha\beta}(x) \partial_j u_{\beta} \right) + B_i^{\alpha\beta}(x) \partial_\alpha u_{\beta} \right]_{\alpha} \quad (1.2)$$

for $i, j \in \{0, \ldots, n - 1\}$ and $\alpha, \beta \in \{1, \ldots, N\}$. Here the solution $u : \Omega \to \mathbb{R}^N$ is a vector-valued function. When $N = 1$ the equation is scalar, see [13] for detailed treatment of this case.

There are many differences between second order elliptic equations and elliptic systems. In general, there is no maximum principle for elliptic systems and the DeGiorgi - Nash - Moser theory that shows interior $C^\alpha$ regularity for scalar elliptic PDE might no longer hold.

This causes substantial difficulties as it forces us to consider for example a weaker version of the nontangential maximal function (defined using the $L^2$ averages). The lack of maximum principle removes the natural $L^\infty$ end-point for $L^p$ solvability results and prevents us from interpolating between the $L^2$ and $L^\infty$ solvabilities. This means that $L^p$ solvability results for $p \neq 2$ have to be obtained using different methods.
We shall say that the bounded and measurable coefficients \( A = [A_{ij}^{\alpha \beta}] \) are strongly elliptic (the condition (1.3) is usually called the Legendre condition) if there exist constants \( 0 < \lambda \leq \Lambda < \infty \) such that
\[
\lambda |\eta|^2 \leq \sum_{\alpha, \beta=1}^{N} \sum_{i,j=0}^{n-1} A_{ij}^{\alpha \beta}(x) \eta_i^\alpha \eta_j^\beta
\] 
(1.3)
for all nonzero \( \eta \in \mathbb{R}^{nN} \) and a.e. \( x \in \Omega \). We shall denote by \( \Lambda = \|A\|_{L^\infty(\Omega)} \).

In the second half of our paper it will suffice to assume a weaker condition called Legendre-Hadamard condition
\[
\lambda |p|^2|q|^2 \leq \sum_{\alpha, \beta=1}^{N} \sum_{i,j=0}^{n-1} A_{ij}^{\alpha \beta}(x) p^\alpha q^\beta q_j.
\] 
(1.4)

The main results of this paper establishes solvability of (1.2) with \( \lambda > 0 \) and \( \mu > 0 \) so that methods like layer potentials can be used. For solvability the inhomogeneous materials, provided we also have strong ellipticity. It not difficult and
\[
A_{ij}^{\alpha \beta}(x) \text{ notation (c.f. [29])}
\]
will also assume impose certain structural assumptions on the tensors \( A \) that can be achieved by rewriting (1.2) into a more convenient form.

Example. Consider the Lamé operator \( \mathcal{L} \) for isotropic inhomogeneous materials in a domain \( \Omega \) with Lamé coefficients \( \lambda(x) \) and \( \mu(x) \). Then for \( u : \Omega \to \mathbb{R}^N \) in vector notation (c.f. [29]) \( \mathcal{L} \) has the form
\[
\mathcal{L} u = \nabla \cdot (\lambda(x)(\nabla \cdot u) I + \mu(x)(\nabla u + (\nabla u)^T)) .
\] 
(1.5)

It follows that with \( B_{ij}^{\alpha \beta} = 0 \) the coefficients of the second order term are
\[
A_{ij}^{\alpha \beta}(x) = \mu(x) \delta_{ij} \delta_{\alpha \beta} + \lambda(x) \delta_{ia} \delta_{j\beta} + \mu(x) \delta_{i\beta} \delta_{j\alpha} .
\]

Observe that since
\[
\partial_i(A_{ij}^{\alpha \beta} \partial_j u_\beta) = \partial_j(A_{ij}^{\alpha \beta} \partial_i u_\beta) - \partial_j(A_{ij}^{\alpha \beta} \partial_i u_\beta) + \partial_i(A_{ij}^{\alpha \beta} \partial_j u_\beta)
\] 
(1.6)
we can rewrite the operator \( \mathcal{L} \) as
\[
\mathcal{L} u = \left[ \partial_i \left( T_{ij}^{\alpha \beta}(x) \partial_j u_\beta \right) + B_{ij}^{\alpha \beta}(x) \partial_i u_\beta \right]_\alpha ,
\] 
(1.7)
where
\[
T_{ij}^{\alpha \beta}(x) = \mu(x) \delta_{ij} \delta_{\alpha \beta} + (\lambda(x) + r(x)) \delta_{ia} \delta_{j\beta} + (\mu(x) - r(x)) \delta_{i\beta} \delta_{j\alpha}
\] 
(1.8)
and \( B_{ij}^{\alpha \beta}(x) = r(x) \partial_j(A_{ij}^{\alpha \beta}(x) - A_{ji}^{\alpha \beta}(x)) \) for any \( r(x) \in L^\infty \). This is a usual trick used in treatment of elliptic systems, when a particular choice of the function \( r \) might be more convenient than the others. In our case we will choose \( r \) so that \( T_{ij}^{\alpha \beta} = \overline{T}_{ij}^{\alpha \beta} \).

It will follow that we can apply our results to the Lamé operator for isotropic inhomogeneous materials, provided we also have strong ellipticity. It not difficult to see that (1.5) satisfies the weaker Legendre-Hadamard condition if \( \mu > 0 \) and \( \lambda + 2\mu > 0 \). Requiring strong ellipticity for our choice of \( r \) imposes a further condition \( \lambda < \mu \) (c.f. [7] for more detailed discussion).

The literature on solvability of boundary value problems for elliptic systems in \( \mathbb{R}^n \) is limited except when the tensor \( A \) has constant coefficients, or at least smooth so that methods like layer potentials can be used. For solvability the \( L^p \) Dirichlet problem of constant coefficients second order elliptic systems in the range
$2 - \varepsilon < p < 2 + \varepsilon$ see [7, 10, 16–18] and [22]. A shown in [27, 28] this range in the constant coefficient case can be extended to the interval $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ by exploring the solvability of the Regularity problem. See also [26] for more recent developments.

Of notable interest is also paper [11] where the Stationary Navier-Stokes system in nonsmooth manifolds was studied. The authors has established results for $L^p$ solvability of the linearized Stokes operator with variable coefficients via the method of layer potentials. Because of the method used, at least H"{o}lder continuity of the underlying metric tensor had to be assumed.

Another special case is when $A$ is of the block form. For block matrices $A$ in $L = \text{div}(A(x)\nabla \cdot)$, there are numerous results on on $L^p$-solvability of the Dirichlet, regularity and Neumann problems, starting with the solution of the Kato problem, where the coefficients of the block matrix are also assumed to be independent of the transverse variable (this assumption is usually referred in literature as “$t$-independent”, in our notation it is the $x_0$ variable). See [3], [20] as well as a series of papers by Auscher, Rosen(Axelsson), and McIntosh for second order elliptic systems (refer [2, 4, 6]).

There are solvability results in various special cases assuming that the solutions satisfy De Giorgi - Nash - Moser estimates. See [1] and [19] for example. The latter paper is also concerned with operators that are $t$-independent. Finally, there are perturbation results in a variety of special cases, such as [9] and [5]; the first paper shows that solvability in $L^2$ implies solvability in $L^p$ for $p$ near 2, and the second paper has $L^2$-solvability results for small $L^\infty$ perturbations of real elliptic operators when the complex matrix is $t$-independent.

In our solvability result for elliptic we do not assume “$t$-independence”. Instead, we assume the coefficients $A$ and $B$ satisfy a natural Carleson condition that has appeared in the literature so far only for real scalar elliptic PDEs ([25], [13], and [15]). The Carleson condition on $A$, (1.9) below, holds uniformly on Lipschitz subdomains, and is thus a natural condition in the context of chord-arc domains as well. However, in this work we do not go beyond the Lipschitz class of domain. Our main result is as follows.

**Theorem 1.1.** Let $\Omega$ be the Lipschitz domain $\{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x')\}$ with Lipschitz constant $L = \|\nabla \phi\|_{L^\infty}$. Assume that the coefficient tensor $A$ of the operator (1.2) is strongly elliptic with constants $\lambda, \Lambda$ (c.f. (1.3)). In addition assume that the following holds:

(i) $A^{\alpha \beta}_{\nu \nu} = \delta_{\alpha \beta} \delta_{\nu \nu}$.

(ii)  

\[
|d\mu(x) = \left[ \sup_{B_j(x)/2(x)} |\nabla A| \right]^2 + \left[ \sup_{B_j(x)/2(x)} |B| \right]^2 \delta(x) \, dx \quad (1.9)
\]

is a Carleson measure in $\Omega$.

There exists $K = K(\lambda, \Lambda, n) > 0$ and $C(\lambda, \Lambda, n, \Omega) > 0$ such that if

\[
\max \{L, \|\mu\|_{\mathcal{C}}\} \leq K \quad (1.10)
\]

then $L^p$-Dirichlet problem for the system (3.1) is solvable for all $2 - \varepsilon_0 < p < 2 + \varepsilon_0$ and the estimate

\[
\|\tilde{N}_a u\|_{L^p(\partial \Omega)} \leq C\|f\|_{L^p(\partial \Omega; \mathbb{R}^N)} \quad (1.11)
\]
holds for all energy solutions \( Lu = 0 \) with datum \( f \). Here \( \varepsilon_0 = \varepsilon_0(\lambda, \Lambda, n, K) > 0 \).

**Remark.** We will outline in section 2 how any PDE of the form (1.2) can be rewritten so that the condition (i) holds. In particular, it will follow that Theorem 1.1 applies to the operator (1.5).

**Remark 2.** It is of considerable interest to replace the condition (1.9) by another (weaker) Carleson condition

\[
d\mu(x) = \left( \frac{\text{osc}_B A}{B_{\delta(x)/2}(x)} \right)^2 \delta^{-1}(x) + \left( \sup_{B_{\delta(x)/2}(x)} |B| \right)^2 \delta(x), \quad dx, \tag{1.12}
\]

where \( \text{osc}_B A = \max_{i,j,\alpha,\beta} \left[ \sup_B A_{ij}^{\alpha\beta} - \inf_B A_{ij}^{\alpha\beta} \right] \). In the scalar case this follows from the Carleson condition (1.10) and Dahlberg-Kenig perturbation result for real and scalar elliptic PDEs. In the case of systems a similar perturbation result is not available yet. We plan to pursue this direction in the future.

**Corollary 1.2.** Let \( \Omega \) be the Lipschitz domain \( \{ (x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x') \} \) with Lipschitz constant \( L = \| \nabla \phi \|_{L^\infty} \). Assume that the Lamé coefficients \( \lambda, \mu \in L^\infty(\Omega) \) satisfy the following:

(i) There exists \( \mu_0 > 0 \) such that \( \min \{ \mu(x), \lambda(x) + 2\mu(x), \mu(x) - \lambda(x) \} \geq \mu_0 \) for a.e. \( x \in \Omega \),

(ii) \[
d\nu(x) = \sup_{B_{\delta(x)/2}(x)} (|\nabla \lambda| + |\nabla \mu|)^2 \delta(x) \, dx \tag{1.13}
\]
is a Carleson measure in \( \Omega \).

Then exists \( K = K(\mu_0, \| \lambda \|_{L^\infty}, \| \mu \|_{L^\infty}) > 0 \) and \( C(\mu_0, \| \lambda \|_{L^\infty}, \| \mu \|_{L^\infty}) > 0 \) such that if

\[
\max \{ L, \| \nu \|_{C} \} \leq K \tag{1.14}
\]

then \( L^p \)-Dirichlet problem for the Lamé system is solvable for all \( 2 \leq p < 2 + \varepsilon_0 \)

\[
\begin{align*}
Lu &= \nabla \cdot (\lambda(x)(\nabla \cdot u)I + \mu(x)(\nabla u + (\nabla u)^T)) = 0 \quad \text{in } \Omega, \\
u(x) &= f(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial \Omega, \tag{1.15}
\end{align*}
\]

and the estimate

\[
\| \tilde{N}_a u \|_{L^p(\partial \Omega)} \leq C \| f \|_{L^p(\partial \Omega; \mathbb{R}^n)} \tag{1.16}
\]

holds for all energy solutions \( u : \Omega \to \mathbb{R}^n \) with datum \( f \). Here \( \varepsilon_0 = \varepsilon_0(\mu_0, \| \lambda \|_{L^\infty}, \| \mu \|_{L^\infty}) > 0 \).

We shall also establish the following large Carleson norm result showing equivalence between the square and nontangential maximal functions.

**Theorem 1.3.** Under the assumptions of Theorem 1.1, if \( \mu \) defined by (1.9) is a Carleson measure (hence, \( \| \mu \|_{C} \) is finite though not necessarily small) then for all \( p > 0 \) any energy solution \( u \) of the problem \( Lu = 0 \) satisfies

\[
\| \tilde{N}_a u \|_{L^p(\partial \Omega)} \approx \| S_a(u) \|_{L^p(\partial \Omega)}, \tag{1.17}
\]

where the implied constants only depend on \( n, N, p, \lambda, \Lambda, a \) and \( \| \mu \|_{C} \).

In fact, the inequality \( \tilde{N} \lesssim S \) holds under a weaker hypothesis. Assume that the coefficient tensor \( A \) of the operator (1.2) satisfies the Legendre-Hadamard condition
(1.4) with constants $\lambda, \Lambda$ and $\mu$ defined by (1.9) satisfies $\|\mu\|_C < \infty$. Then for all $p > 0$ any energy solution $u$ of the problem $Lu = 0$ satisfies
\begin{equation}
\|\tilde{N}_a(u)\|_{L^p(\partial\Omega)} \lesssim \|S_a(u)\|_{L^p(\partial\Omega)},
\end{equation}
where the implied constant again only depends on $n, N, p, \lambda, \Lambda, a$ and $\|\mu\|_C$.

**Proof.** This follows from Corollary 4.5 and Proposition 5.8. □

The paper is organised as follows. In Section 2, we introduce important notions and definitions needed later. Section 3 discusses the $L^2$ Dirichlet problem and we also give there the proof of our main result. In section 4 we establish important estimates for the square function and in section 5 similar estimates for the non-tangential maximal function. Finally, the section 6 deals with the $L^p$ solvability for $p$ near 2 using extrapolation arguments.

## 2. Definitions and background results

For a vector valued function $u = [u_\alpha]_{\alpha = 1}^N : \Omega \to \mathbb{R}^N$, we use $\nabla u$ to denote the Jacobian matrix of $u$ defined by
\begin{equation}
(\nabla u)^\alpha_i = \partial_i u^\alpha = \frac{\partial u^\alpha}{\partial x_i}
\end{equation}
for $i \in \{0, \ldots, n - 1\}$ and $\alpha \in \{1, \ldots, N\}$.

For $0 \leq k \leq \infty$, we use $C^k(\Omega; \mathbb{R}^N)$ to denote the space of all smooth vector valued functions with continuous partial derivatives up to the order $k$ and let $C^k_0(\Omega; \mathbb{R}^N)$ be its subspace consisting those maps that are compactly supported in $\Omega$. For $1 \leq p < \infty$, let $W^{1,p}(\Omega; \mathbb{R}^N)$ be the Sobolev space which is the completion of $C^\infty(\Omega; \mathbb{R}^N)$ with respect to the norm
\begin{equation}
\|u\|_{W^{1,p}(\Omega)} := \left[ \int_\Omega |u(x)|^p + |(\nabla u)(x)|^p \, dx \right]^{1/p}.
\end{equation}
Also, $W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ stands for the local version of $W^{1,p}(\Omega; \mathbb{R}^N)$. Similarly, we denote by $W^{1,p}(\Omega; \mathbb{R}^N)$ the homogeneous version of the space which is the completion of $C^\infty(\Omega; \mathbb{R}^N)$ with respect to the semi-norm
\begin{equation}
\|u\|_{W^{1,p}(\Omega)} := \left[ \int_\Omega |(\nabla u)(x)|^p \, dx \right]^{1/p}.
\end{equation}

Throughout this paper, a weak solution of (1.2) means a function $u$ in $W^{1,2}_{loc}(\Omega; \mathbb{R}^N)$ satisfying $Lu = 0$ in a weak sense in $\Omega$.

### 2.1. Non-tangential maximal and square functions

On a domain of the form
\begin{equation}
\Omega = \{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x')\},
\end{equation}
where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant given by $L := \|\nabla \phi\|_{L^\infty(\mathbb{R}^{n-1})}$, define for each point $x = (x_0, x') \in \Omega$
\begin{equation}
\delta(x) := x_0 - \phi(x') \approx \text{dist}(x, \partial\Omega).
\end{equation}
In other words, $\delta(x)$ is comparable to the distance of the point $x$ from the boundary of $\Omega$. 

Definition 2.1. A cone of aperture $a > 0$ is a non-tangential approach region
to the point $Q = (x_0, x') \in \partial \Omega$ defined as
\[
\Gamma_a(Q) = \{ y = (y_0, y') \in \Omega : a |y_0 - x_0| > |x' - y'| \}. \tag{2.6}
\]

We require $a < 1/L$, otherwise the aperture of the cone is too large and might
not lie inside $\Omega$. But when $\Omega = \mathbb{R}^n_+$ all parameters $a > 0$ may be considered.
Sometimes it is necessary to truncate $\Gamma(Q)$ at height $h$, in which scenario we write
\[
\Gamma_a^h(Q) := \Gamma_a(Q) \cap \{ x \in \Omega : \delta(x) \leq h \}. \tag{2.7}
\]

Definition 2.2. For $\Omega \subset \mathbb{R}^n$ as above, the square function of some $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^N)$
at $Q \in \partial \Omega$ relative to the cone $\Gamma_a(Q)$ is defined by
\[
S_a(u)(Q) := \left( \int_{\Gamma_a(Q)} |(\nabla u(x)|^2 \delta(x)^{2-n} \, dx \right)^{1/2} \tag{2.8}
\]
and, for each $h > 0$, its truncated version is given by
\[
S_a^h(u)(Q) := \left( \int_{\Gamma_a^h(Q)} |(\nabla u(x)|^2 \delta(x)^{2-n} \right)^{1/2}. \tag{2.9}
\]

A simple application of Fubini’s theorem gives
\[
\|S_a(u)\|_{L^2(\partial \Omega)}^2 \approx \int_\Omega |(\nabla u(x)|^2 \delta(x) \, dx. \tag{2.10}
\]

Definition 2.3. For $\Omega \subset \mathbb{R}^n$ as above the nontangential maximal function of some
$u \in C^0(\Omega; \mathbb{R}^N)$ at $Q \in \partial \Omega$ relative to the cone $\Gamma_a(Q)$ and its truncated version at
height $h$ are defined by
\[
N_a(u)(Q) := \sup_{x \in \Gamma_a(Q)} |u(x)| \quad \text{and} \quad N_a^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} |u(x)|. \tag{2.11}
\]

Moreover, we shall also consider a related version of the above nontangential maximal
function. This is denoted by $\tilde{N}_a$ and is defined using $L^2$ averages over balls in
the domain $\Omega$. Specifically, given $u \in L^2_{loc}(\Omega; \mathbb{R}^N)$ we set
\[
\tilde{N}_a(u)(Q) := \sup_{x \in \Gamma_a(Q)} w(x) \quad \text{and} \quad \tilde{N}_a^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} w(x) \tag{2.12}
\]
for each $Q \in \partial \Omega$ and $h > 0$ where, at each $x \in \Omega$,
\[
w(x) := \left( \int_{B_{\delta(x)/2}(x)} |u|^2(z) \, dz \right)^{1/2}. \tag{2.13}
\]

Above and elsewhere, a barred integral indicates an averaging operation. We note
that, given $u \in L^2_{loc}(\Omega; \mathbb{R}^N)$, the function $w$ associated with $u$ as in (2.13) is
continuous and $\tilde{N}_a(u) = N_a(w)$ everywhere on $\partial \Omega$. For systems, the best regularity
we can expect from a weak solution of (1.2) is $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^N)$. In particular, $u$
might not be pointwise well-defined. In the scalar case $N = 1$ by the DeGiorgi-Nash-Moser
estimates the situation is different as the solutions are in $C^{1,1}_{loc}(\Omega)$. Hence,
while in the scalar case considering $N_a$ typically suffices, in the case of systems the
consideration of $\tilde{N}_a$ becomes necessary.
2.2. The Carleson measure condition. We begin by recalling the definition of a Carleson measure in a domain $\Omega$ as in (2.4). For $P \in \mathbb{R}^n$, define the ball centered at $P$ with the radius $r > 0$ as

$$B_r(P) := \{x \in \mathbb{R}^n : |x - P| < r\}. \quad (2.14)$$

Next, given $Q \in \partial \Omega$, by $\Delta = \Delta_r(Q)$ we denote the surface ball $\partial \Omega \cap B_r(Q)$. The Carleson region $T(\Delta_r)$ is then defined by

$$T(\Delta_r) := \Omega \cap B_r(Q). \quad (2.15)$$

**Definition 2.4.** A Borel measure $\mu$ in $\Omega$ is said to be Carleson if there exists a constant $C \in (0, \infty)$ such that for all $Q \in \partial \Omega$ and $r > 0$

$$\mu(T(\Delta_r)) \leq C \sigma(\Delta_r), \quad (2.16)$$

where $\sigma$ is the surface measure on $\partial \Omega$. The best possible constant $C$ in the above estimate is called the Carleson norm and is denoted by $\|\mu\|_C$.

As regards the elliptic operator introduced in (1.2), in all that follows we shall assume that the coefficients $A$ and $B$ satisfies the following natural conditions. First, we assume that the entries $A_{ij}^{\alpha\beta}$ of $A$ are in $\text{Lip}_{\text{loc}}(\Omega)$ while the entries $B_i^{\alpha\beta}$ are in $L^\infty_{\text{loc}}(\Omega)$. Second, we assume that

$$d\mu(x) = \left[ \left( \sup_{B_{\delta(x)/2}(x)} |\nabla A| \right)^2 + \left( \sup_{B_{\delta(x)/2}(x)} |B| \right)^2 \right] \delta(x) \, dx \quad (2.17)$$

is a Carleson measure in $\Omega$. Occasionally (but not everywhere) we will additionally assume that its Carleson norm $\|\mu\|_C$ is sufficiently small. Crucially we have the following result.

**Theorem 2.5.** Suppose that $dv = f \, dx$ and $d\mu(x) = \left[ \sup_{B_{\delta(x)/2}(x)} |f| \right] \, dx$. Assume that $\mu$ is a Carleson measure. Then there exists a finite constant $C = C(L, a) > 0$ such that for every $u \in L^p_{\text{loc}}(\Omega; \mathbb{C})$ one has

$$\int_{\Omega} |u(x)|^2 \, dv(x) \leq C \|\mu\|_C \int_{\partial \Omega} (\tilde{N}_a(u))^2 \, d\sigma. \quad (2.18)$$

**Proof.** Let

$$\Omega = \bigcup_i O_i$$

be a Whitney decomposition of $\Omega$ and assume that the Whitney sets $O_i$ are such that for any $x \in O_i$ we have $O_i \subset B_{\delta(x)/2}(x)$. Also, $|O_i| \approx |B_{\delta(x)/2}(x)|$. It follows that on each $O_i$ we have

$$\int_{O_i} |u(x)|^2 \, dv(x) \leq \left[ \sup_{O_i} |f| \right] \int_{O_i} |u(x)|^2 \, dx.$$ 

By the definition (2.13) for $w$ it follows that for any $y \in O_i$ we have

$$\int_{O_i} |u(x)|^2 \, dv(x) \lesssim \left[ \sup_{B_{\delta(y)/2}(y)} |f| \right] w(y)^2 |O_i|.$$ 

From this

$$\int_{O_i} |u(x)|^2 \, dv(x) \lesssim \int_{O_i} w(y)^2 \, d\mu(y).$$
Summing over all \( i \) we get
\[
\int_{\Omega} |u(x)|^2 \, d\nu(x) \lesssim \int_{\Omega} w(y)^p \, d\mu(y) \lesssim \|\mu\|_c \int_{\partial \Omega} N_\alpha(w)^2 \, d\sigma,
\]
where the last inequality follows from the usual inequality for the Carleson measure. Since \( \hat{N}_\alpha(u) = N_\alpha(w) \) the claim follows. \( \square \)

Moreover, the aforementioned assumption on coefficients of the system (1.2) is compatible with the useful change of variables described in the next two subsections.

### 2.3. Reformulations of (1.2) and ellipticity

In this section, we rewrite the elliptic system (1.2) in a more convenient form.

Let \( A_{ij} = [A_{ij}]_{\alpha,\beta} \) for \( i, j \in \{0, 1, \ldots, n-1\} \). Hence, each \( A_{ij} \) is an \( N \times N \) matrix. It is natural to assume that \( A_{00} \) (the principle minor of \( A \)) is invertible. For example, this is guaranteed whenever (1.3) or (1.4) holds. To proceed, consider the tensors \( \hat{A} = [\hat{A}_{ij}]_{i,j,\alpha,\beta} \) and \( \hat{B} = [\hat{B}_{ij}]_{i,\alpha,\beta} \) defined by
\[
\hat{A}_{ij} := \sum_{\gamma=1}^N \left( A_{00}^{-1}\right)^{\alpha\gamma} A_{ij}^{\gamma\beta}, \quad \hat{B}_{ij} := \sum_{\gamma=1}^N \left( A_{00}^{-1}\right)^{\alpha\gamma} \left( A_{ij}^{\gamma\beta} - \sum_{\gamma=0}^{n-1} \partial_\gamma \left( \left( A_{00}^{-1}\right)^{\alpha\gamma} A_{ij}^{\gamma\beta} \right) \right).
\]

Observe that \( \hat{A} \) is diagonalized in the \( x_0 \) variable, namely \( \hat{A}_{00} = I_{N \times N} \). Let
\[
\hat{\mathcal{L}} u := \left[ \partial_i \left( \hat{A}_{ij}^{\alpha\beta}(x) \partial_j u_{\beta} \right) + \hat{B}_{ij}^{\alpha\beta}(x) \partial_i u_{\beta} \right]_\alpha.
\]

If \( u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^N) \) is a solution of the PDE system \( \mathcal{L} u = 0 \) in \( \Omega \), then
\[
\hat{\mathcal{L}} u = \sum_{\gamma=1}^N \left( A_{00}^{-1}\right)^{\alpha\gamma} \left[ \partial_i \left( A_{ij}^{\gamma\beta} \partial_j u_{\beta} \right) + B_{ij}^{\gamma\beta} \partial_i u_{\beta} \right]
+ \sum_{\gamma=1}^N \partial_i \left( \left( A_{00}^{-1}\right)^{\alpha\gamma} A_{ij}^{\gamma\beta} \partial_j u_{\beta} \right) - \sum_{\gamma=1}^N \partial_i \left( \left( A_{00}^{-1}\right)^{\alpha\gamma} A_{ij}^{\gamma\beta} \partial_j u_{\beta} \right) = 0.
\]

Here we have used the equation \( \mathcal{L} u = 0 \) for the first two terms of (2.22). The two terms in the second line cancel out (this can be seen by permuting indices \( i \to j \to k \to i \) in the last term).

For technical reasons we will also require that \( \hat{A}_{0j} = 0_{N \times N} \) for \( j > 0 \). This can be achieved as follows. Since
\[
\partial_0 \left( \hat{A}_{0j}^{\alpha\beta} \partial_j u_{\beta} \right) = \partial_j \left( \hat{A}_{0j}^{\alpha\beta} \partial_0 u_{\beta} \right) - \partial_j \left( \hat{A}_{0j}^{\alpha\beta} \partial_0 u_{\beta} \right) + \partial_0 \left( \hat{A}_{0j}^{\alpha\beta} \right) \partial_j u_{\beta}.
\]

If follows that if we define a new coefficient tensor by taking
\[
\overline{A}_{ij}^{\alpha\beta} := \begin{cases}
\hat{A}_{ij}^{\alpha\beta}, & \text{if } i, j > 0 \text{ or } i = j = 0, \\
\hat{A}_{ij}^{\alpha\beta} + \hat{A}_{ji}^{\alpha\beta}, & \text{if } i > 0 \text{ and } j = 0, \\
0, & \text{if } i = 0 \text{ and } j > 0,
\end{cases}
\]

For technical reasons we will also require that \( \hat{A}_{0j} = 0_{N \times N} \) for \( j > 0 \). This can be achieved as follows. Since
then the difference \( \hat{L}u - \partial_i \left( \overline{A_{ij}^\alpha} \partial_j u^\beta \right) \) consists of just first order terms. As such, \( u \) may be regarded as a solution of a system similar to (2.22) with \( \overline{A_{ij}^\alpha} \) replaced by \( \hat{A}_{ij}^\alpha \) and with some slightly modified matrix coefficients for the first order terms.

Observe that the coefficients \( \overline{A}, \overline{B} \) of the new system will still satisfy a Carleson condition with norm comparable with the original \( ||\mu||_C \). That is if the original tensors \( A \) and \( B \) are such that \( (2.17) \) is a Carleson measure then

\[
d\overline{\mu}(x) = \left[ \left( \sup_{B(x/2)} |\nabla \overline{A}(x)| \right)^2 + \left( \sup_{B(x/2)} |\overline{B}(x)| \right)^2 \right] \delta(x) \, dx
\]

is also a Carleson measure in \( \Omega \) whose Carleson norm \( ||\mu||_C \) can be estimated in terms of the original norm \( ||\mu||_C \) and the Lipschitz norm of the function \( \phi \) in the definition of \( \Omega (1.1) \). In particular, if both \( ||\mu||_C \) and \( ||\nabla \phi||_L^\infty \) are small then \( ||\overline{\mu}||_C \) is correspondingly small.

Let us discuss briefly how such coefficients changes affect strong ellipticity. In general if (1.3) holds for \( A \) it might not hold anymore for (2.19) or (2.24) and for this reason we will assume strong ellipticity for (2.24). In some situations however, the strong ellipticities for \( A, \hat{A} \) or \( \overline{A} \) are equivalent. This is always true when \( N = 1 \), i.e., if the PDE (1.2) is scalar. Another important case is the one of Lamé system discussed previously. For coefficients given by (1.3) the matrix \( A_{00} = (\lambda + 2\mu)I \) and in particular the matrix multiplication in (2.19) is commutative. It is not hard to see that because of this the strong ellipticities of \( A \) and \( \hat{A} \) are equivalent.

A similar observation can be made for strong ellipticities of \( \overline{A} \) and \( \overline{\mu} \) when \( \hat{A} \) has the following symmetry

\[
\hat{A}_{ij}^\alpha = \hat{A}_{ij}^{\beta\alpha}.
\]

In the case of Lamé system this happens when we choose \( r = (\mu - \lambda)/2 \) in (1.8). For this choice of \( r \) \( A \) is strongly elliptic if and only if \( \hat{A} \) and \( \overline{\mu} \) are.

2.4. Pullback Transformation. For a domain \( \Omega \) as in (2.24), consider a mapping \( \rho : \mathbb{R}^n_+ \to \Omega \) appearing in works of Dahlberg, Nečas, Kenig-Stein and others defined by

\[
\rho(x_0, x') := (x_0 + P_\gamma x_0 \ast \phi(x'), x'), \quad \forall (x_0, x') \in \mathbb{R}^n_+,
\]

for some positive constant \( \gamma \). Here \( P \) is a nonnegative function \( P \in C_0^\infty(\mathbb{R}^{n-1}) \) and, for each \( \lambda > 0 \),

\[
P_\lambda(x') := \lambda^{n-1} P(x'/\lambda), \quad \forall x' \in \mathbb{R}^{n-1}.
\]

Finally, \( P_\lambda \ast \phi(x') \) is the convolution

\[
P_\lambda \ast \phi(x') := \int_{\mathbb{R}^{n-1}} P_\lambda(x' - y') \phi(y') \, dy'.
\]

Observe that \( \rho \) extends up to the boundary of \( \mathbb{R}^n_+ \) and maps one-to-one from \( \partial \mathbb{R}^n_+ \) onto \( \partial \Omega \). Also for sufficiently small \( \gamma \lesssim \lambda \) the map \( \rho \) is a bijection from \( \mathbb{R}^n_+ \) onto \( \Omega \) and, hence, invertible.

For \( u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^N) \) that solves \( Lu = 0 \) in \( \Omega \) with Dirichlet datum \( f \) consider \( v := u \circ \rho \) and \( \hat{f} := f \circ \rho \). The change of variables via the map \( \rho \) just described implies that \( v \in W^{1,2}_{loc}(\mathbb{R}^n_+; \mathbb{R}^N) \) solves a new PDE system of the form

\[
0 = \text{div} \left( \hat{A}^\alpha(x) \nabla u \right) + \hat{B}^\alpha(x) \cdot \nabla u, \quad \text{for } \alpha \in \{1, 2, \ldots, N\},
\]

for each \( \alpha \in \{1, 2, \ldots, N\} \).
with boundary datum \( \tilde{f} \) on \( \partial R^n_+ \). Hence, solving a boundary value problem for \( u \) in \( \Omega \) is equivalent to solving a related boundary value problem for \( v \) in \( R^n_+ \). Crucially, if the coefficients of the original system are such that (2.17) is a Carleson measure, then the coefficients of \( \tilde{A} \) and \( \tilde{B} \) satisfy an analogous Carleson condition in the upper-half space. If, in addition, the Carleson norm of (2.17) is small and \( \| \nabla \phi \|_{L^\infty} \) is also small, then the Carleson norm for the new coefficients \( \tilde{A} \) and \( \tilde{B} \) will be correspondingly small. It is also not hard to see that the strong ellipticity is preserved under this change of variables.

We need to discuss the condition (i) of Theorem 1.1 in relation to the pull-back transformation \( \rho \). Clearly, if the original tensor satisfies \( A_{i_j}^{\alpha \beta} = \delta_{i_j} \delta_{\alpha \beta} \), the new tensor \( \hat{A} \) in (2.29) after applying \( \rho \) might fail to do so and hence we might have to re-apply the change of coefficients we have just discussed in subsection 2.3. In the case we are mostly interested in the function \( \phi \) in (2.26) has a small Lipschitz norm and therefore the Jacobian of the map \( \rho \) is very close to the identity (it is as small \( L^\infty \) perturbation of \( I \) with size of the perturbation depending on the \( \| \nabla \phi \|_{L^\infty} \)). Because of this the change of coefficients from subsection 2.3 will preserve the strong ellipticity.

It follows that the map \( \rho \) allows us to reduce the problem of solving (1.2) to the special case when the underlying domain is \( \Omega = R^n_+ \) and both conditions (i) and (ii) of Theorem 1.1 holds on \( R^n_+ \).

2.5. Inequalities. Here we recall some of the basic the inequalities that hold for weak solutions of the operator \( L \).

**Proposition 2.6. (Poincaré inequality)** There exists a finite dimensional constant \( C = C(n) > 0 \) such that, for all balls \( B_R \subset R^n \) and all \( u \in W^{1,2}(B_R; R^N) \),

\[
\int_{B_R} |u - u_{B_R}|^2 \, dx \leq CR^2 \int_{B_R} |\nabla u|^2 \, dx,
\]

where \( u_{B_R} := \frac{1}{|B_R|} \int_{B_R} u(x) \, dx \) (2.30)

**Proposition 2.7. (Cacciopoli inequality)** If \( Lu = 0 \) in \( B_{2R} \), where \( L \) is as in (1.2), \( A \) is measurable and bounded and satisfies the Legendre-Hadamard condition and \( B \leq K/R \) then here exists a finite constant \( C = C(n, \lambda, \Lambda, K) > 0 \) such that, for all balls \( B_R \subset R^n \) and all \( u \in W^{1,2}(B_R; R^N) \),

\[
\int_{B_R} |\nabla u|^2 \, dx \leq CR^{-2} \int_{B_{2R}} |u|^2 \, dx.
\]

3. The \( L^2 \)-Dirichlet problem

We are ready to define the \( L^p \)-Dirichlet problem. We first recall the classical solvability via the Lax-Milgram lemma in a domain \( \Omega \) as in (2.4). Recall, that under assumptions of strong ellipticity it can be shown via standard arguments that given any \( f \in B^{2,2}_{1/2}(\partial \Omega; R^N) \) (this is the space of traces of functions in \( W^{1,2}(\Omega; R^N) \)) there exists a unique \( u \in W^{1,2}(\Omega; R^N) \) such that \( Lu = 0 \) in \( \Omega \) for \( L \) given by (1.2) and \( \text{Tr} u = f \) on \( \partial \Omega \). We will call such \( u \in W^{1,2}(\Omega; R^N) \) the energy solution of the elliptic system \( L \) in \( \Omega \). With this in hand, we can now define the notion of \( L^p \) solvability.
Definition 3.1. Let $\Omega$ be the Lipschitz domain introduced in [24] and fix an integrability exponent $p \in (1, \infty)$. Also, fix a background parameter $a > 0$. Consider the following Dirichlet problem for a vector valued function $u : \Omega \to \mathbb{R}^N$:

$$
\begin{cases}
0 = \partial_i \left( A_{ij}^{\alpha \beta}(x) \partial_j u_{\beta} \right) + B_i^{\alpha \beta}(x) \partial_i u_{\beta} & \text{in } \Omega, \quad \alpha \in \{1, 2, \ldots, N\} \\
\tilde{N}_a(u) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega,
\end{cases}
$$

(3.1)

where the usual Einstein summation convention over repeated indices ($i, j$ and $\beta$ in this case) is employed. We say the Dirichlet problem (3.1) is solvable for a given $p \in (1, \infty)$ if there exists $C = C(\Lambda, \Lambda, n, p, \Omega) > 0$ such that the unique energy solution $u \in \dot{W}^{1,2}(\Omega; \mathbb{R}^N)$, provided by the Lax-Milgram lemma, corresponding to a boundary datum $f \in L^p(\partial\Omega; \mathbb{R}^N) \cap B^{2,2}_{1/2}(\partial\Omega; \mathbb{R}^N)$ satisfies the estimate

$$
\|\tilde{N}_a u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{R}^N)}.
$$

(3.2)

Remark. By Lax-Milgram lemma the solution of (3.1) in the space $\dot{W}^{1,2}(\Omega; \mathbb{R}^N)$ is unique modulo adding an arbitrary constant to each component vector $u_\alpha$. Our additional assumption that $u = f \in L^p(\Omega; \mathbb{R}^N)$ eliminates the constant solutions and hence guarantees uniqueness. Since the space $B^{2,2}_{1/2}(\partial\Omega; \mathbb{R}^N) \cap L^p(\partial\Omega; \mathbb{R}^N)$ is dense in $L^p(\partial\Omega; \mathbb{R}^N)$ for each $p \in (1, \infty)$, it follows that there exists a unique continuous extension of the solution operator

$$
f \mapsto u
$$

(3.3)
to the whole space $L^p(\partial\Omega; \mathbb{R}^N)$, with $u$ such that $\tilde{N}_a u \in L^p(\partial\Omega)$ and the accompanying estimate $\|\tilde{N}_a u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{R}^N)}$ being valid. It is a legitimate question to consider in what sense we have a convergence of $u$ given by the solution operator (3.3) to its boundary datum $f \in L^p(\partial\Omega; \mathbb{R}^N)$. The answer can be found in the appendix of paper [13] (the proof is given for complex valued elliptic PDEs but adapts in a straightforward way to our situation). Consider the average $u_{av} : \Omega \to \mathbb{R}^N$ defined by

$$
u_{av}(x) = \frac{1}{B_{\delta(x/2)}(x)} \int_{B_{\delta(x/2)}(x)} u(y) \, dy, \quad \forall x \in \Omega.
$$

Then

$$
f(Q) = \lim_{x \to Q, x \in \Gamma(Q)} u_{av}(x), \quad \text{for } \sigma\text{-a.e. } Q \in \partial\Omega,
$$

(3.4)

where the a.e. convergence is taken with respect to the $\mathcal{H}^{n-1}$ Hausdorff measure on $\partial\Omega$.

We are now ready to establish the main result Theorem 1.1. The solutions to the Dirichlet problem in the infinite domain $\Omega = \mathbb{R}^n_+$ will be obtained as a limit of solutions in infinite strips $\Omega^h = \{x = (x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x_0 < h\}$. We define them as follows.

Definition 3.2. Let $\Omega = \mathbb{R}^n_+$, and let $\Omega^h$ be the infinite strip

$$
\Omega^h = \{x = (x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x_0 < h\},
$$

and let $p \in (1, \infty)$. Also, fix an aperture parameter $a > 0$. Let $u$ be a vector valued function $u : \Omega \to \mathbb{R}^N$ such that
From this it follows that Theorem 1.3 we have

\[ 0 = \partial_i \left( A_{ij}^{\alpha \beta}(x) \partial_j u_{\beta} \right) + B_i^{\alpha \beta}(x) \partial_i u_{\beta} \quad \text{in } \Omega^h, \quad \alpha \in \{1, 2, \ldots, N\} \]

\[ u(x, x') = 0, \quad \text{for all } x_0 \geq h, \]

\[ u(x) = f(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial \Omega, \]

\[ \tilde{N}_a(u) \in L^p(\partial \Omega), \tag{3.5} \]

where the usual Einstein summation convention over repeated indices is employed.

We say the Dirichlet problem \((3.5)\) is solvable for a given \(p \in (1, \infty)\) if there exists a \(C = C(p, \Omega) > 0\) such that for all boundary data \(f \in L^p(\partial \Omega; \mathbb{R}^N) \cap B_{1/2}^{3/2}(\partial \Omega; \mathbb{R}^N)\) we have that \(u|_{\Omega^h}\) is the unique “energy solution” to

\[ 0 = \partial_i \left( A_{ij}^{\alpha \beta}(x) \partial_j u_{\beta} \right) + B_i^{\alpha \beta}(x) \partial_i u_{\beta} \quad \text{in } \Omega^h, \quad \alpha \in \{1, 2, \ldots, N\} \]

\[ u(x, x') = 0, \quad \text{for } x_0 = h \]

\[ u(x) = f(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial \Omega, \tag{3.6} \]

and satisfies the estimate

\[ \|\tilde{N}_a(u)\|_{L^p(\partial \Omega)} \leq C\|f\|_{L^p(\partial \Omega; \mathbb{R})}. \tag{3.7} \]

Proof of Theorem 1.1. As indicated in the previous section there is no loss of generality to assume that \(\Omega = \mathbb{R}_+^n\) and the matrix \(A_{ij}^{\alpha \beta} = 0\) for \(j > 0\) (via the pullback transformation \(\rho\) from section 2.4 and the change of variables \((2.19)\) and \((2.20)\)). The new system will be strongly elliptic if the original system was thanks to the smallness of Lipschitz constant of the function \(\phi\).

We will establish the solvability of the Dirichlet problem \((3.5)\), applying the results of sections 4 and 5. The constants will not depend on the width of the strip. Then, a limiting argument (taking width of the domain to infinity) proves Theorem 1.1.

Let \(u_h\) be the energy solution in \(\Omega^h\) as in Definition 3.2. By Corollary 4.3 for some \(C = C(n, N, \lambda, \Lambda) > 0\) we have

\[ \lambda \int_{\mathbb{R}_+^n} |\nabla u_h|^2 x_0 \, dx' \, dx_0 \leq \int_{\mathbb{R}_+^{n-1}} |f(x')|^2 \, dx' + C\|\mu\|_C \int_{\mathbb{R}_+^{n-1}} \left[ \tilde{N}_a(u_h) \right]^2 \, dx'. \tag{3.8} \]

Important here is that thanks to our assumption that we work on \(\Omega_h\) and the fact that \(u_h\) is an energy solution we have

\[ \|S_a(u_h)\|_{L^2(\mathbb{R}_+^{n-1})}^2 \lesssim \int_{\mathbb{R}_+^h} |\nabla u_h|^2 \, dx < \infty, \]

where the implied constant in this estimate depends on \(h\), but the estimate itself guarantees that the \(L^2\) norm of the square function of \(u_h\) is finite. Hence by Theorem 1.3 we have

\[ \lambda \int_{\mathbb{R}_+^n} |\nabla u_h|^2 x_0 \, dx' \, dx_0 = C_a \int_{\mathbb{R}_+^{n-1}} |S_a(u_h)|^2 \, dx' \approx \int_{\mathbb{R}_+^{n-1}} \left[ \tilde{N}_a(u_h) \right]^2 \, dx'. \tag{3.9} \]

From this it follows that

\[ \int_{\mathbb{R}_+^{n-1}} \left[ \tilde{N}_a(u_h) \right]^2 \, dx' \leq C_0 \int_{\mathbb{R}_+^{n-1}} |f(x')|^2 \, dx' + C\|\mu\|_C \int_{\mathbb{R}_+^{n-1}} \left[ \tilde{N}_a(u_h) \right]^2 \, dx'. \tag{3.10} \]
The constants in this estimate are independent of \( h \). Choose \( K \) in the Theorem 1.1 such that \( C\|\mu\|_c < 1/2 \). Such a choice immediately entails

\[
\int_{\mathbb{R}^{n-1}} \left[ \tilde{N}_k(u_h) \right]^2 \, dx' \leq 2C_0 \int_{\mathbb{R}^{n-1}} |f(x')|^2 \, dx',
\]

(3.11)

for all energy solutions \( u_h \) of the system (3.5).

We now consider the limit of \( u_h \), as \( h \to \infty \). The uniform Lax-Milgram estimate on \( \|\nabla u_h\|_{L^2(\mathbb{R}^n)} \) by \( \|f\|_{B^1_{1/2}} \), and the fact that \( \text{Tr}(u_h) = f \), gives a weakly convergent subsequence to some \( u \) with \( \|\nabla u\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{B^1_{1/2}} \) and \( \text{Tr}(u) = f \). This subsequence is therefore strongly convergent to \( u \) in \( L^2_{\text{loc}}(\mathbb{R}^n) \). It follows that the \( L^2 \) averages \( w_h \) of \( u_h \) converge locally and uniformly to \( w \), the \( L^2 \) averages of \( u \) in \( C_{\text{loc}}(\mathbb{R}^n) \).

Let \( \Gamma_k(x') \) be the doubly truncated cone \( \Gamma(x') \cap \{1/k < x_0 < k\} \). Define

\[
\tilde{N}_k(u)(x') = \sup_{y \in \Gamma_k(x')} |w(y)|,
\]

and with \( \tilde{N}_k(u_h)(x') \) defined analogously. Then we have

\[
\tilde{N}_k(u_h)(x') \to \tilde{N}_k(u)(x') \quad \text{uniformly on compact subsets } K \subset \mathbb{R}^{n-1}.
\]

Finally, using (3.11), this give on each such set \( K \),

\[
\|\tilde{N}_k(u_h)\|_{L^p(K)} = \lim_{h \to \infty} \|\tilde{N}_k(u_h)\|_{L^p(K)} \leq C\|f\|_{L^p(\mathbb{R}^{n-1})}.
\]

The constant \( C \) in the estimate above is independent of \( K \) and \( k \), so that taking the supremum in each of \( k \) and \( K \) gives the desired estimate for \( u \) on \( \Omega = \mathbb{R}^n \).

The \( L^p \) solvability in the interval \( 2 < p < 2 + \varepsilon \) is established later in section 6 (c.f. (6.15)).

4. Estimates for the square function \( S(u) \) of a solution

In this section we establish a one sided estimate of the square function in terms of boundary data and the nontangential maximal function.

We fix an \( h > 1 \), and an infinite strip \( \Omega^h \) defined above, and let \( u \) be an energy solution to (3.5), extended to be zero above height \( h \). Due to the reductions we have made it suffices to work with a coefficient tensor satisfying \( A_{00} = I_{N \times N} \).

**Lemma 4.1.** Let \( u: \Omega \to \mathbb{R}^N \) be as above with the Dirichlet boundary datum \( f \in L^2(\partial \Omega; \mathbb{R}^N) \). Assume that \( A \) is strongly elliptic, satisfies \( A_{ij}^{\alpha \beta} = \delta_{ij} \delta_{\alpha \beta} \), and the measure \( \mu \) defined as in (1.2) is Carleson.

Then there exists a constant \( C = C(n, N, \lambda, \Lambda) \) such that for all \( r > 0 \)

\[
\lambda \int_{[0, r/2] \times \partial \Omega} |\nabla u|^2 x_0 \, dx' \, dx_0 + \frac{2}{r} \int_{[0, r] \times \partial \Omega} |u(x_0, x')|^2 \, dx' \, dx_0 \\
\leq \int_{\partial \Omega} |u(0, x')|^2 \, dx' + \int_{\partial \Omega} |u(r, x')|^2 \, dx' + C\|\mu\|_c \int_{\partial \Omega} \left[ \tilde{N}_r(u) \right]^2 \, dx'.
\]

(4.1)

**Proof.** Fix an arbitrary \( y' \in \partial \Omega \equiv \mathbb{R}^{n-1} \) and consider first \( r \leq h \). Pick a smooth cutoff function \( \zeta \) which is \( x_0 \)-independent and satisfies

\[
\zeta = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{outside } B_{2r}(y'). \end{cases}
\]

(4.2)
Moreover, assume that \( r |\nabla \zeta| \leq c \) for some positive constant \( c \) independent of \( y' \).

We begin by considering the integral quantity

\[
I := \iint_{[0,r] \times B_{2r}(y')} \alpha_{ij} \beta_{ij} u_\alpha x_\beta \partial_j u_\beta \partial_i u_\alpha x_\beta \zeta \, dx' \, dx_0
\]

(4.3)

with the usual summation convention understood. In relation to this we note that the uniform ellipticity (1.3) gives

\[
I \geq \lambda \iint_{[0,r] \times B_{2r}} |\nabla u_\alpha|^2 x_\alpha \zeta \, dx' \, dx_0 = \lambda \iint_{[0,r] \times B_{2r}} |\nabla u|^2 x_0 \zeta \, dx' \, dx_0,
\]

(4.4)

where we agree henceforth to abbreviate \( B_{2r} := B_{2r}(y') \) whenever convenient. The idea now is to integrate by parts the formula for \( I \) in order to relocate the \( \partial_i \) derivative. This gives

\[
I = \frac{\partial}{\partial(0,r) \times B_{2r}} A_\alpha \beta \partial_j u_\alpha x_\beta \nu_x \sigma \, \partial_i u_\beta \partial_j u_\alpha \zeta \, dx' \, dx_0
\]

\[
- \iint_{[0,r] \times B_{2r}} \partial_i \left( A_\alpha \beta \partial_j u_\beta \right) u_\alpha x_\beta \zeta \, dx' \, dx_0
\]

\[
- \iint_{[0,r] \times B_{2r}} A_\alpha \beta \partial_j u_\beta u_\alpha \partial_i x_\beta \zeta \, dx' \, dx_0
\]

\[
- \iint_{[0,r] \times B_{2r}} A_\alpha \beta \partial_j u_\beta u_\alpha \partial_i x_\beta \zeta \, dx' \, dx_0
\]

\[
=: I + II + III + IV
\]

(4.5)

where \( \nu \) is the outer unit normal vector to \((0,r) \times B_{2r}(y')\). Bearing in mind \( A_\alpha \beta = 0 \) for \( j > 0 \) and upon recalling that we are assuming \( A_{00} = I_{N \times N} \), the boundary term \( I \) simply becomes

\[
I = \iint_{B_{2r}} \partial_0 u_\beta (r, x') u_\beta (r, x') r \zeta \, dx'.
\]

(4.6)

As \( u \) is a weak solution of \( Lu = 0 \) in \( \Omega \), we use this PDE to transform \( II \) into

\[
II = \iint_{[0,r] \times B_{2r}} B_\alpha \beta (\partial_i u_\beta) u_\alpha x_\beta \zeta \, dx' \, dx_0.
\]

(4.7)

To further estimate this term we use Cauchy-Schwarz inequality, the Carleson condition for \( B \) and Theorem 2.5 in order to write

\[
II \leq \left( \iint_{[0,r] \times B_{2r}} B_\alpha \beta^2 \, |u_\alpha|^2 x_\beta \, dx' \, dx_0 \right)^{1/2} \cdot \left( \iint_{[0,r] \times B_{2r}} |\partial_j u_\beta|^2 x_\beta \, dx' \, dx_0 \right)^{1/2}
\]

\[
\leq C\lambda, \Lambda, N \left( ||\mu||c \int_{B_{2r}} \left[ N^r_\alpha (u) \right]^2 \, dx' \right)^{1/2} \cdot \Omega^{1/2}.
\]

(4.8)

As \( \partial_i x_0 = 0 \) for \( i > 0 \) the term \( III \) is non-vanishing only for \( i = 0 \). We further split this term by considering the cases when \( j = 0 \) and \( j > 0 \). When \( j = 0 \), we use
that $A_{00}^{\alpha \beta} = I_{N \times N}$. This yields

$$III_{j=0} = -\frac{1}{2} \int_{[0,r] \times B_{2r}} \sum_{\beta} \partial_0 \left( u_\beta^2 \zeta \right) dx' dx_0$$

$$= -\frac{1}{2} \int_{B_{2r}} \sum_{\beta} u_\beta(r, x')^2 \zeta dx' + \frac{1}{2} \int_{B_{2r}} \sum_{\beta} u_\beta(0, x')^2 \zeta dx'.$$

(4.9)

Corresponding to $j > 0$ we simply recall that $A_{0j}^{\alpha \beta} = 0$ for $j > 0$ to conclude that $III_{j=0} = 0$.

We add up all terms we have so far to obtain

$$I \leq \int_{B_{2r}} \partial_0 u_\beta(r, x') u_\beta(r, x') r \zeta dx'$$

$$- \frac{1}{2} \int_{B_{2r}} \sum_{\beta} u_\beta(r, x')^2 \zeta dx' + \frac{1}{2} \int_{B_{2r}} \sum_{\beta} u_\beta(0, x')^2 \zeta dx'$$

$$+ C(\lambda, \Lambda, n, N) \|\mu\| \int_{B_{2r}} \left[ \tilde{N}_a^r(u) \right]^2 dx' + \frac{1}{2} I + IV,$$

(4.10)

where we have used the arithmetic-geometric inequality for expression bounding the term $II$ in (4.3).

To obtain a global version of (4.10), consider a sequence of disjoint boundary balls $(B_r(y_k'))_{k \in \mathbb{N}}$ such that $\bigcup_k B_{2r}(y_k')$ covers $\partial \Omega = \mathbb{R}^{n-1}$ and consider a partition of unity $(\zeta_k)_{k \in \mathbb{N}}$ subordinate to this cover. That is, assume $\sum_k \zeta_k = 1$ on $\mathbb{R}^{n-1}$ and each $\zeta_k$ is supported in $B_{2r}(y_k')$. Write $IV_k$ for each term as the last expression in (4.5) corresponding to $B_{2r} = B_{2r}(y_k')$. Given that $\sum_k \partial_i \zeta_k = 0$ for each $i$, by summing (4.10) over all $k$’s gives $\sum_k IV_k = 0$. It follows that

$$\frac{\lambda}{2} \int_{[0,r] \times \mathbb{R}^{n-1}} |\nabla u|^2 x_0 dx' dx_0$$

$$\leq \int_{\mathbb{R}^{n-1}} \partial_0 u_\beta(r, x') u_\beta(r, x') r dx'$$

$$- \frac{1}{2} \int_{\mathbb{R}^{n-1}} \sum_{\beta} u_\beta(r, x')^2 dx' + \frac{1}{2} \int_{\mathbb{R}^{n-1}} \sum_{\beta} u_\beta(0, x')^2 dx'$$

$$+ C(\|\mu\|) \int_{\mathbb{R}^{n-1}} \left[ \tilde{N}_a^r(u) \right]^2 dx'.$$

(4.11)

We have established (4.11) for $r \leq h$, but we now observe that (4.11) holds also for $r > h$, as $u = 0$ when $r > h$. From this (4.11) follows by integrating (4.11) in $r$ over $[0, r']$ and then dividing by $r'$.

Lemma 4.1 has three important corollaries.

**Corollary 4.2.** Retain the assumptions of Lemma 4.1. Then, given a weak solution $u$ of (3.3), for any $r > 0$ we have

$$\lambda \int_{[0, r/2] \times \partial \Omega} |\nabla u|^2 x_0 dx' dx_0 \leq C(1 + \|\mu\|) \int_{\partial \Omega} \left[ \tilde{N}_a^r(u) \right]^2 dx'.$$

(4.12)
That is, \( \|S_a(u)\|_{L^2(\partial\Omega)} \leq C\|\tilde{N}_a(u)\|_{L^2(\partial\Omega)} \) with the intervening constant depending only on \( \lambda, \Lambda, n, N, a, \) and \( \|\mu\|_C .\) In particular, letting \( r \to \infty \) yields a version for the global square and nontangential maximal functions, namely
\[
\|S_a(u)\|_{L^2(\partial\Omega)} \leq C\|\tilde{N}_a(u)\|_{L^2(\partial\Omega)},
\] (4.13)
for all energy solutions \( u \) of (3.5).

**Corollary 4.3.** Under the assumptions of Lemma 4.1 for any energy solution \( u \) of (3.5) we have
\[
\lambda \int_{\mathbb{R}^n_+} |\nabla u|^2 x_0 \, dx \, dx_0 \leq \int_{\mathbb{R}^{n-1}} |u(0, x')|^2 \, dx' + C\|\mu\|_C \int_{\mathbb{R}^{n-1}} \left[ \tilde{N}_a(u) \right]^2 \, dx'.
\] (4.14)

**Corollary 4.4.** Under the assumptions of Lemma 4.1 for any energy solution \( u \) of (3.5) we have for any \( x' \in \mathbb{R}^{n-1} \) and \( r > 0 \)
\[
\int_{[0, r/2] \times B_r} |\nabla u|^2 x_0 \, dx \, dx_0 \leq C \left[ \int_{B_{2r}} |u(0, x')|^2 \, dx' + \int_{B_{2r}} |u(r, x')|^2 \, dx' + \|\mu\|_C \int_{B_{2r}} \left[ \tilde{N}^{r}_a(u) \right]^2 \, dx' \right]
\leq C(2 + \|\mu\|_C) \int_{B_{2r}} \left[ \tilde{N}^{2r}_a(u) \right]^2 \, dx'.
\] (4.15)

This is a local version of the Corollary. To see this one proceeds exactly as in the proof above until (4.10). Then instead of summing over different balls \( B_r \) covering \( \mathbb{R}^{n-1} \) we estimate the terms \( IV \). Both of these terms are of the same type and can be bounded (up to a constant) by
\[
\int_{[0, r] \times B_{2r}} |\nabla u||u||x_0|\partial_T \zeta| \, dx' \, dx_0,
\] (4.16)
where \( \partial_T \zeta \) denotes any of the derivatives in the direction parallel to the boundary. Recall that \( \zeta \) is a smooth cutoff function equal to 1 on \( B_r \) and 0 outside \( B_{2r} \). In particular, we may assume \( \zeta \) to be of the form \( \zeta = \eta^2 \) for another smooth function \( \eta \) such that \( |\nabla \eta| \leq C/r \). By Cauchy-Schwartz (4.10) can be further estimated by
\[
\left( \int_{[0, r] \times B_{2r}} |\nabla u|^2 x_0(\eta) \, dx' \, dx_0 \right)^{1/2} \left( \int_{[0, r] \times B_{2r}} |u|^2 x_0 |\nabla \eta|^2 \, dx' \, dx_0 \right)^{1/2}
\leq T^{1/2} \left( \frac{1}{r} \int_{[0, r] \times B_{2r}} |u|^2 x_0 \, dx' \, dx_0 \right)^{1/2} \leq \varepsilon T + C_{\varepsilon} \int_{B_{2r}} \left[ \tilde{N}_a(u) \right]^2 \, dx'.
\] (4.17)

In the last step we have used AG-inequality and a trivial estimate of the solid integral \(|u|^2 \) by the averaged non-tangential maximal function. Substituting (4.17) into (4.15) the estimate (4.14) follows by integrating in \( r \) over \([0, r']\) and dividing by \( r' \) exactly as done above.

**Corollary 4.5.** Under the assumptions of Lemma 4.1 for any energy solution \( u \) of (3.5) we have for any \( p > 0 \) and \( a > 0 \) the following. There exists a finite constant \( C = C(n, N, \Lambda, \lambda, p, a, \|\mu\|_C) > 0 \) such that
\[
\|S_a(u)\|_{L^p(\mathbb{R}^{n-1})} \leq C\|\tilde{N}_a(u)\|_{L^p(\mathbb{R}^{n-1})}, \quad (4.18)
\]
This is a consequence of Corollary 4.4 following [21], see a more detailed discussion in the proof of Proposition 5.8.
5. Bounds for the nontangential maximal function by the square function

As before, we shall work under the assumption that $\Omega = \mathbb{R}^n_+$. We will only assume the Legendre-Hadamard condition \((1.4)\) and large Carleson condition on the coefficients. Via the pullback map $\rho$ the problem reduces to the domain $\mathbb{R}^n_+$. Our aim in this section is to establish a reverse version of the inequality in Corollary \(4.2\). The approach necessarily differs from the usual argument in the scalar elliptic case due to the fact that certain estimates, such as interior H"older regularity of a weak solution, are unavailable for the class of systems presently considered. Hence, alternative arguments bypassing such difficulties must be devised.

The major innovation is the use of an entire family of Lipschitz graphs on which the nontangential maximal function is large in lieu of a single graph constructed via a stopping time argument. This is necessary as we are using $L^2$ averages of solutions to define the nontangential maximal function and hence the knowledge of certain bounds for a solution on a single graph provides no information about the $L^2$ averages over interior balls.

The energy solutions $u_h$ constructed using Lax-Milgram lemma on $\Omega_h$ and extended by zero on $\{(x_0, x') : x_0 > h\}$ a priori belong to the space $\dot{W}^{1,2}(\mathbb{R}^n_+; \mathbb{R}^N)$. Since $u(h, \cdot) = 0$ this implies $\dot{W}^{1,2}(\mathbb{R}^n_+; \mathbb{R}^N)$ (with norm depending of $h$). We drop dependence on $h$ for now and consider $u = u_h$. For the function $w$ defined in $\Omega$ as in \((2.13)\), and a constant $\nu > 0$, define the set

$$E_{\nu,a} := \{x' \in \partial \Omega : N_a(w)(x') > \nu\} \quad (5.1)$$

(\text{where, as usual, $a > 0$ is a fixed background parameter}), and consider the map $h : \partial \Omega \to \mathbb{R}$ given at each $x' \in \partial \Omega$ by

$$h_{\nu,a}(w)(x') := \inf \left\{ x_0 > 0 : \sup_{z \in \Gamma_a(x_0, x')} w(z) < \nu \right\} \quad (5.2)$$

with the convention that $\inf \emptyset = \infty$. We remark $h$ is differs from the function $\tilde{h} : \partial \Omega \to \mathbb{R}$ defined at each $x' \in \partial \Omega$ as

$$\tilde{h}_{\nu,a}(w)(x') := \sup \left\{ x_0 > 0 : \sup_{z \in \Gamma_a(x_0, x')} w(z) > \nu \right\}. \quad (5.3)$$

The function $\tilde{h}$ has been used in the argument for scalar equations (cf. \cite{25}, pp. 212 and \cite{24}). While there are clear similarities in the manner in which the functions $h$ and $\tilde{h}$ are defined, throughout this paper we prefer to use $h$ as it works better for elliptic systems.

At this point we observer that $h_{\nu,a}(w, x') < \infty$ for all points $x' \in \partial \Omega$. This is due to the fact that the function $u$ vanishes above height $h$ and hence the averages $w$ vanish above the height $2h$. It follows that $h_{\nu,a}(w)(x') < \infty$, in fact $h_{\nu,a}(w)(x') < 2h$.

Lemma 5.1. Let $u$ be an energy solution of \((3.5)\), and associated with it the function $w$ as in \((2.13)\). Also, fix two positive numbers $\nu, a$. Then the following properties hold.

(i) The function $h_{\nu,a}(w)$ is Lipschitz, with a Lipschitz constant $1/a$. That is,

$$|h_{\nu,a}(w)(x') - h_{\nu,a}(w)(y')| \leq a^{-1}|x' - y'| \quad (5.4)$$
for all $x', y' \in \partial \Omega$.

(ii) Given an arbitrary $x' \in E_{\nu,a}$, let $x_0 := h_{\nu,a}(w)(x')$. Then there exists a point $y = (y_0, y') \in \partial \Gamma_a(x_0, x')$ such that $w(y) = \nu$ and $h_{\nu,a}(w)(y') = y_0$.

Proof. To prove the claim formulated in part (i), pick a pair of arbitrary points $x', y' \in \partial \Omega$ and set $y_0 := h_{\nu,a}(w)(y')$, $x_0 := h_{\nu,a}(w)(x')$. Without loss of generality it may be assumed that $y_0 < x_0$. In particular, this forces $x_0 \in (0, \infty)$. Seeking a contradiction, suppose

$$|x' - y'| < a(x_0 - y_0). \quad (5.5)$$

Then simple geometric considerations give

$$\Gamma_a(x_0, x') \subset \Gamma_a(y_0, y'). \quad (5.6)$$

In particular, there exists $\varepsilon \in (0, 2x_0)$ with the property that

$$\Gamma_a(x_0, x') \subset \Gamma_a(y_0 + \varepsilon, y'). \quad (5.7)$$

Hence,

$$\Gamma_a(x_0 - \varepsilon/2, x') \subset \Gamma_a(y_0 + \varepsilon/2, y'). \quad (5.8)$$

It follows that

$$\sup_{\partial \Gamma_a(x_0 - \varepsilon/2, x')} w \leq \sup_{\partial \Gamma_a(y_0 + \varepsilon/2, y')} w < \nu, \quad (5.9)$$

the last inequality being true by the definition of $y_0 = h_{\nu,a}(w)(y')$ in (5.2). This however implies that

$$x_0 - \varepsilon/2 \geq h_{\nu,a}(x') = x_0, \quad (5.10)$$

which is the desired contradiction. Therefore the assumption made in (5.5) is false which then entails $0 < a(x_0 - y_0) \leq |x' - y'|$. From this the claim in part (i) follows.

To justify the claim recorded in part (ii), fix some $x' \in E_{\nu,a}$ and note that this implies $x_0 = h_{\nu,a}(w)(x') > 0$. To show that there exists a point $y = (y_0, y') \in \partial \Gamma_a(x_0, x')$ such that $w(y) = \nu$ we employ a compactness argument. Due to the decay of $w$ at infinity it follows that for a sufficiently large $r$ (depending on $u$) we have

$$\sup_{\{z \in \mathbb{R}^3 : z_0 \geq r\}} w(z) \leq \nu/2. \quad (5.11)$$

If it were true that $w(z) < \nu$ for all $z \in \partial \Gamma_a(x_0, x') \cap \{z_0 \leq r\}$ then, as $w$ is continuous, each such point $z$ would posses a neighborhood $\mathcal{O}_z$ where $w < \nu$. The family $\{\mathcal{O}_z\}_z$ then constitutes an open cover of the compact set $\partial \Gamma_a(x_0, x') \cap \{z_0 \leq r\}$ and may therefore be refined to a finite subcover, say $\{\mathcal{O}_{z_i}\}_{1 \leq i \leq k}$. Upon introducing

$$\mathcal{S} := \left( \bigcup_{i=1}^{k} \mathcal{O}_{z_i} \right) \cup \Gamma_a(x_0, x') \cup \{(z_0, z') : z_0 \geq r\} \quad (5.12)$$

it follows that

$$w(z) < \nu, \quad \forall z \in \mathcal{S}. \quad (5.13)$$

However, for some small $\varepsilon \in (0, x_0)$

$$\Gamma_a(x_0 - \varepsilon, x') \subset \mathcal{S}, \quad (5.14)$$

and the compactness of the set $\mathcal{S} \cap \{z_0 \leq r\}$ together with decay of $w$ above height $r$ entail

$$\sup_{z \in \Gamma_a(x_0 - \varepsilon, x')} w(z) < \nu. \quad (5.15)$$
This contradicts the definition of \( x_0 = h_{\nu,a}(w)(x') \) in (5.2). Bearing in mind the definition of \( h_{\nu,a}(w)(x') \) and the continuity of \( w \), we conclude that for some point \( y = (y_0, y') \in \partial \Gamma_a(x_0, x') \) we must have \( w(y) = \nu \). In turn, this forces \( h_{\nu,a}(w)(y') \geq y_0 \). On the other hand, since \( \Gamma_a(y_0, y') \subseteq \Gamma_a(x_0, x') \), we have

\[
\Gamma_a(y_0 + \varepsilon, y') \subseteq \Gamma_a(x_0 + \varepsilon, x') \quad \text{for every} \quad \varepsilon > 0
\]

which implies

\[
\sup_{\Gamma_a(y_0 + \varepsilon, y')} w \leq \sup_{\Gamma_a(x_0 + \varepsilon, x')} w < \nu \quad \text{for every} \quad \varepsilon > 0.
\]

In turn, this allows us to conclude that \( h_{\nu,a}(w)(y') \leq y_0 + \varepsilon \) for every \( \varepsilon > 0 \). Hence, ultimately it follows that \( h_{\nu,a}(w)(y') = y_0 \), as claimed. \( \square \)

**Lemma 5.2.** Assume as before that \( u \) is an energy solution of the system \((3.5)\) in \( \Omega = \mathbb{R}_+^n \). For any \( a > 0 \) there exists \( b = b(a) > a \) and \( \gamma = \gamma(a) > 0 \) such that the following holds. Having fixed an arbitrary \( \nu > 0 \), for each point \( x' \) from the set

\[
\{ x' : N_a(w)(x') > \nu \quad \text{and} \quad S_b(u)(x') \leq \gamma \nu \}
\]

there exists a boundary ball \( R \) with \( x' \in 2R \) and such that

\[
|w(h_{\nu,a}(w)(z'), z')| > \nu/2 \quad \text{for all} \quad z' \in R.
\]

**Proof.** Let \( x' \in \partial \Omega \) be such that \( N_a(w)(x') > \nu \) and \( S_b(u)(x') \leq \gamma \nu \). As before, set \( x_0 := h_{\nu,a}(w, x') \). From part (ii) in Lemma 5.1 we know that there exists a point \( y = (y_0, y') \in \partial \Gamma_a(x_0, x') \) such that \( w(y) = \nu \). Let \( d := |x' - y'| \) and define \( R = \{ z' \in \partial \Omega : |z' - y'| < 3a y_0/2 \} \). Then \( x' \in 2R \) since \( d \leq ay_0 \). This choice also guarantees that \( h_{\nu,a}(w, z') \in [y_0/3, 5y_0/3] \) by (i) in Lemma 5.1.

To proceed, consider the set

\[
\mathcal{O} := \{ z = (z_0, z') \in \Omega : z' \in R \quad \text{and} \quad z_0 \in [y_0/3, 5y_0/3] \}.
\]

In particular, \( y \in \mathcal{O} \). Then all claims in the current lemma are justified as soon as we establish that

\[
w(z) > \nu/2 \quad \text{for all} \quad z \in \mathcal{O}.
\]

With this goal in mind, consider \( \bigcup_{z \in \mathcal{O}} B_{z_0/2}(z) \). All points of this set are at least \( y_0/6 \) away from the boundary of \( \Omega \) and the diameter of this set is comparable to \( y_0 \). Select the number \( b > a \) so that

\[
B := \bigcup_{z \in \mathcal{O}} B_{z_0/2}(z) \subseteq \Gamma_b(0, x').
\]

A simple geometrical argument shows that \( b \) can be chosen independently of the location of points \( x', y' \), and only depends on the size of \( a \). Our goal is to estimate the difference \( |w(z) - w(y)| \) for all \( z \in \mathcal{O} \). To this end, fix some \( z \in \mathcal{O} \). Abbreviating \( B := B_{1/2}(0) \) then permits us to express

\[
w(z) = \left( \int_B |u(z + z_0 \xi)|^2 \, d\xi \right)^{1/2}, \quad w(y) = \left( \int_B |u(y + y_0 \xi)|^2 \, d\xi \right)^{1/2}.
\]
It follows that
\[ w(z) = \left( \int_B |u(y + y_0 \xi) + [u(z + z_0 \xi) - u(y + y_0 \xi)]|^2 \, d\xi \right)^{1/2} \]
\[ \leq \left( \int_B |u(y + y_0 \xi)|^2 \, d\xi \right)^{1/2} + \left( \int_B |u(z + z_0 \xi) - u(y + y_0 \xi)|^2 \, d\xi \right)^{1/2} \]
\[ = w(y) + \left( \int_B |u(z + z_0 \xi) - u(y + y_0 \xi)|^2 \, d\xi \right)^{1/2}. \]  
(5.24)

Since a similar estimate holds when the roles of \( y \) and \( z \) are interchanged, we eventually conclude that
\[ |w(z) - w(y)|^2 \leq \int_B |u(z + z_0 \xi) - u(y + y_0 \xi)|^2 \, d\xi. \]  
(5.25)

Going further, the Fundamental Theorem of Calculus gives that for any two points \( z_1, z_2 \in B \) we have
\[ |u(z_1) - u(z_2)|^2 \leq \left| \int_0^1 (\nabla u)(z_1 + (z_2 - z_1)\tau) \cdot (z_1 - z_2) \, d\tau \right|^2 \]
\[ \leq |z_1 - z_2|^2 \int_0^1 |(\nabla u)(z_1 + (z_2 - z_1)\tau)|^2 \, d\tau \]
\[ = y_0^{n-2}|z_1 - z_2|^2 \int_0^1 |(\nabla u)(z_1 + (z_2 - z_1)\tau)|^2 y_0^2 \, d\tau \]
\[ \leq C y_0^{n-1} \int_{[z_1, z_2]} |(\nabla u)(q)|^2 q_0^{2-n} \, ds(q), \]  
(5.26)

where the last integral is understood as a line integral over the segment joining \( z_1 \) and \( z_2 \). We have also used the fact that \( |z_1 - z_2| \leq C y_0 \) for all \( z_1, z_2 \in B \). We apply this formula to generic pairs of points of the form \( z + z_0 \xi, y + y_0 \xi \) for \( z \in \mathcal{O} \) and \( \xi \in B \) (which, by design, are in \( B \)) and then integrate in \( \xi \). Notice that, for various points \( \xi \), the lines joining \( z + z_0 \xi \) with \( y + y_0 \xi \) are almost parallel; in fact they are genuinely parallel when \( z_0 = y_0 \). When integrating in \( \xi \) over \( B \) a typical point \( q \) in the very last expression in (5.26) considered with \( z_1 := z + z_0 \xi \) and \( z_2 := y + y_0 \xi \) will belong to certain line segments joining these points with \( \xi \) belonging to a certain subset of \( B \) of 1-dimensional Hausdorff measure, having size \( O(1) \) relative to this measure. Hence,
\[ \frac{1}{|B|} \int_B |u(z + z_0 \xi) - u(y + y_0 \xi)|^2 \, d\xi \leq C \int_{\mathcal{H}} |(\nabla u)(q)|^2 q_0^{2-n} \, dq, \]  
(5.27)

where \( \mathcal{H} \) denotes the convex hull of the set \( B_{z_0/2}(z) \cup B_{y_0/2}(y) \subset \Gamma_b(0, x') \), which is a set of diameter comparable to \( y_0 \). The factor \( y_0^{n-1} \) in (5.26) disappears after integrating in \( \xi \) due to the natural change of variables which takes \( ds(q) d\xi \) into \( dq \) in (5.27), the natural Lebesgue measure on \( \mathcal{H} \). Because \( \mathcal{H} \) is contained in \( \Gamma_b(0, x') \), the right-hand side of (5.27) may be further estimated by \( S^2_{\xi}(u)(x') \leq \gamma^2 \nu^2 \). Hence, by combining (5.25)-(5.27) we obtain
\[ |w(z) - w(y)|^2 \leq C(a, n, N)(\gamma \nu)^2 \leq \frac{\mu^2}{4}, \]  
(5.28)
if $\gamma$ is chosen so that $C(a, n, N)\gamma^2 < 1/4$. It follows that for any $z \in \mathcal{O}$ we have

$$w(z) \geq w(y) - |w(y) - w(z)| \geq \nu - \frac{\nu}{2} = \frac{\nu}{2}. \quad (5.29)$$

Hence the claim in (5.21) follows, finishing the proof of the lemma.

Given a Lipschitz function $h : \mathbb{R}^{n-1} \to \mathbb{R}$, denote by $M_h$ the Hardy-Littlewood maximal function considered on the graph of $h$. That is, given any locally integrable function $f$ on the Lipschitz surface $\Lambda_h = \{(h(z'), z') : z' \in \mathbb{R}^{n-1}\}$, define $(M_hf)(x) := \sup_{r > 0} \int_{\Lambda_h \cap B_r(x)} |f| \, d\sigma$ for each $x \in \Lambda_h$.

**Corollary 5.3.** Let $u$ is an energy solution of the system (3.3) in $\Omega = \mathbb{R}^n_+$ and fix $a > 0$. Associated with these, let $b, \gamma$ be as in Lemma 5.2. Then there exists a finite constant $C = C(n) > 0$ with the property that for any $\nu > 0$ and any point $x' \in E_{\nu, a}$ such that $S_b(u)(x') \leq \gamma \nu$ one has

$$(M_{\nu, a}(w)(h_{\nu, a}(x'), x') \geq C \nu. \quad (5.30)$$

**Proof.** Fix a point $x' \in E_{\nu, a}$ where $S_b(u)(x') \leq \gamma \nu$. Lemma 5.2 then guarantees the existence of a boundary ball $R$ with the property that $w(h_{\nu, a}(w)(z'), z') > \nu/2$ for all $z' \in R$ and $x' \in 2R$. Granted this, it follows that

$$(M_{\nu, a}(w)(h_{\nu, a}(w)(x'), x') \geq \frac{1}{|2R|} \int_R w(h_{\nu, a}(w)(z'), z') \, dz' \geq \frac{|R|}{|2R|} \frac{\nu}{2}, \quad (5.31)$$

as desired. \hfill \square

**Lemma 5.4.** Consider the system (3.1) with coefficients satisfying Carleson condition and the condition (1.1). Then there exists a $\theta > 0$ with the following significance. Suppose $u$ is a weak solution of (3.3) in $\Omega = \mathbb{R}^{n+}_+$. Select $\theta \in [1/6, 6]$ and, having picked $\nu > 0$ arbitrary, let $h_{\nu, a}(w)$ be as in (5.2). Also, consider the domain $\mathcal{O} = \{x_0, x' \in \Omega : x_0 > \theta h_{\nu, a}(x')\}$ with boundary $\partial \mathcal{O} = \{(x_0, x') \in \Omega : x_0 = \theta h_{\nu, a}(x')\}$. In this context, for any surface ball $\Delta_r := B_r(Q) \cap \partial \mathcal{O}$, with $Q \in \partial \mathcal{O}$ and $r > 0$ chosen such that $h_{\nu, a}(w) \leq 2r$ pointwise on $\Delta_{2r}$, one has

$$\int_{\Delta_r} |u(x)|^2 \, dx' \leq C(1 + ||\mu||_{C^1}^{1/2}) ||S_b(u)||_{L^2(\Delta_{2r})} ||\tilde{N}_a(u)||_{L^2(\Delta_{2r})}$$

$$+ C ||S_b(u)||_{L^2(\Delta_{2r})} + \frac{C}{r} \int_{\mathcal{K}} |u|^2 \, dX. \quad (5.32)$$

Here $C = C(\lambda, \Lambda, n, N) \in (0, \infty)$ and $\mathcal{K}$ is a region inside $\mathcal{O}$ of diameter, distance to the boundary $\partial \mathcal{O}$, and distance to $Q$, are all comparable to $r$. Also, the parameter $b > a$ is as in Lemma 5.2 and the cones used to define the square and nontangential maximal functions in this lemma have vertices on $\partial \mathcal{O}$.

Moreover, the term $\int_{\mathcal{K}} |u|^2 \, dX$ appearing in (5.32) may be replaced by the quantity

$$Cr^{n-1} |u_{av}(A_r)|^2 + C \int_{\Delta_{2r}} S^2_b(u) \, d\sigma, \quad (5.33)$$

where $A_r$ is any point inside $\mathcal{K}$ (usually called a corkscrew point of $\Delta_r$) and

$$u_{av}(X) := \int_{B_h(X) \cap \partial(X)} u(Z) \, dZ. \quad (5.34)$$
Proof. Fix \( \theta \in [1/6, 6] \). We first consider the case when \( r \) is small, that is \( 2r \leq h \). This implies that \( u = u_h \) solves the PDE system \( Lu = 0 \) on the set we shall integrate over.

Consider the pullback transformation \( \rho : \mathbb{R}^n_+ \to \mathcal{O} \) defined as in section 2.3 relative to the Lipschitz function \( \theta h_{v,a}(w) \). Let \( v = (v_\beta)_{1 \leq \beta \leq N} \) be given by \( v := u \circ \rho \) in \( \mathbb{R}^n_+ \). Thanks to the assumptions made on the system (3.1), the vector-valued function \( v : \mathbb{R}^n_+ \to \mathbb{R}^N \) will satisfy a PDE similar to that of \( u \). Specifically, we have

\[
\left[ \partial_i \left( A^{\alpha \beta}_\nu(x) \partial_j v_\beta \right) + B^{\alpha \beta}_\nu(x) \partial_i v_\beta \right] = 0,
\]

where \( \bar{A} \) is uniformly elliptic and the coefficients \( \bar{A} \) and \( \bar{B} \) are such that

\[
d\bar{\mu}(x) = \left[ \left( \sup_{B_{\delta/2}(x)} |\nabla \bar{A}(x)| \right)^2 + \left( \sup_{B_{\delta/2}(x)} |\bar{B}(x)| \right)^2 \right] \delta(x) \, dx
\]

is a Carleson measure in \( \mathbb{R}^n_+ \). Moreover, the Carleson norm \( ||\bar{\mu}||_c \) only depends on the Carleson norm of the original coefficients and the Lipschitz norm of the function \( h_{v,a} \). When the Lipschitz norm of this function goes to zero we have

\[
\lim \sup ||\bar{\mu}||_c \leq ||\mu||_c
\]

and hence the parameter \( a > 0 \) may be chosen large enough so that the Lipschitz norm of the function \( \theta h_{v,a} \) is sufficiently small (at most \( 6/a \)) such that \( ||\bar{\mu}||_c \leq 2 ||\mu||_c \). As we have observed before for the original equation we may arrange (by change of variables) that \( A_{00} = I_{N \times N} \). This is true even if we only assume (1.4) as the condition implies invertibility of the matrix \( A_{00} \). Hence we can use (2.19)–(2.20).

Having fixed a scale \( r > 0 \), we localize to a ball \( B_r(y') \) in \( \mathbb{R}^{n-1} \). Let \( \zeta \) be a smooth cutoff function of the form \( \zeta(x_0, x') = \zeta_0(x_0)\zeta_1(x') \) where

\[
\zeta_0 = \begin{cases} 1 & \text{in } [0, r], \\ 0 & \text{in } [2r, \infty), \end{cases} \quad \zeta_1 = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{in } \mathbb{R}^n \setminus B_{2r}(y') \end{cases}
\]

and

\[
r|\partial_0 \zeta_0| + r|\nabla x_\alpha \zeta_1| \leq c
\]

for some constant \( c \in (0, \infty) \) independent of \( r \). Our goal is to control the \( L^2 \) norm of \( u(\theta h_{v,a}(w), \cdot) \). Since after the pullback under the mapping \( \rho \) the latter is comparable with the \( L^2 \) norm of \( v(0, \cdot) \), we fix \( \alpha \in \{1, \ldots, N\} \) and proceed to estimate

\[
\int_{B_{2r}(y')} v_\alpha^2(0, x')\zeta(0, x') \, dx'
\]

\[
= - \int_{[0, 2r] \times B_{2r}(y')} \partial_0 \left[ v_\alpha^2(x_0, x')\zeta(x_0, x') \right] \, dx_0 \, dx'
\]

\[
= -2 \int_{[0, 2r] \times B_{2r}(y')} v_\alpha \partial_0 v_\alpha \zeta \, dx_0 \, dx'
\]

\[
- \int_{[0, 2r] \times B_{2r}(y')} v_\alpha^2(0, x') \partial_0 \zeta \, dx_0 \, dx'
\]

\[
=: \mathcal{A} + IV.
\]
We further expand the term $A$ as a sum of three terms obtained via integration by parts with respect to $x_0$ as follows:

$$A = -2 \int_{[0,2r] \times B_{2r}(y')} v_\alpha \partial_0 v_\alpha (\partial_0 x_0) \zeta \, dx_0 \, dx'$$

$$= 2 \int_{[0,2r] \times B_{2r}(y')} |\partial_0 v_\alpha|^2 x_0 \zeta \, dx_0 \, dx'$$

$$+ 2 \int_{[0,2r] \times B_{2r}(y')} v_\alpha \partial_{\partial_0 0}^2 v_\alpha x_0 \zeta \, dx_0 \, dx'$$

$$+ 2 \int_{[0,2r] \times B_{2r}(y')} v_\alpha \partial_0 v_\alpha \partial_0 \zeta \, dx_0 \, dx'$$

$$= I + II + III.$$  \hspace{1cm} (5.40)

We start by analyzing the term $II$. In view of the fact that $\bar{A}_{00} = I_{N \times N}$, the PDE recorded in (5.35) allows us to write

$$\partial_{\partial_0 0}^2 v_\alpha = -\sum_{(i,j) \neq (0,0)} \partial_i \left( \bar{A}_{ij}^\alpha \partial_j v_\beta \right) - B_{i}^{\alpha \beta} \partial_i v_\beta.$$  \hspace{1cm} (5.41)

In turn, this permits us to express

$$II = -2 \sum_{(i,j) \neq (0,0)} \int_{[0,2r] \times B_{2r}} \partial_i \left( \bar{A}_{ij}^\alpha \right) v_\alpha \partial_j v_\beta x_0 \zeta \, dx_0 \, dx'$$

$$- 2 \int_{[0,2r] \times B_{2r}} B_{i}^{\alpha \beta} v_\alpha \partial_0 v_\beta x_0 \zeta \, dx_0 \, dx'$$

$$- 2 \sum_{(i,j) \neq (0,0)} \int_{[0,2r] \times B_{2r}} \bar{A}_{ij}^\alpha v_\alpha \partial_{\partial_0 0}^2 v_\beta x_0 \zeta \, dx_0 \, dx'$$

$$=: II_1 + II_2 + II_3.$$  \hspace{1cm} (5.42)

The last term above requires some further work. Let us temporarily fix $i, j$ and denote by $II_3^{ij}$ the corresponding term in $II_3$. Since in the present context we have $(i,j) \neq (0,0)$, at least one of the two indices involved is not zero, say $i > 0$. Integrating by parts with respect to the variable $x_i$ then yields (in what follows we do not sum over indices $i$ and $j$)

$$II_3^{ij} = 2 \int_{[0,2r] \times B_{2r}} \partial_i \left( \bar{A}_{ij}^\alpha \right) v_\alpha \partial_j v_\beta x_0 \zeta \, dx_0 \, dx'$$

$$+ 2 \int_{[0,2r] \times B_{2r}} \bar{A}_{ij}^\alpha v_\alpha \partial_0 v_\beta x_0 \zeta \, dx_0 \, dx'$$

$$+ 2 \int_{[0,2r] \times B_{2r}} \bar{A}_{ij}^\alpha v_\alpha \partial_j v_\beta x_0 \partial_0 \zeta \, dx_0 \, dx'$$

$$=: J_1^{ij} + J_2^{ij} + J_3^{ij}.$$  \hspace{1cm} (5.43)
The treatment of $II_i^J$ in the case when $i = 0$ proceeds along the same lines, except that we now integrate in the variable $x_j$. Since the resulting terms are of a similar nature as above, we omit writing them explicitly.

We now group together terms that are of the same type. Firstly, we have

$$I + J_2 \leq C(\lambda, \Lambda, n, N)\|S_b(u)\|_{L^2(B_{2r})}.$$  

(5.44)

Here, the estimate would be true even with $\|S_b^\nu(v)\|_{L^2(B_{2r})}$ which is at every point dominated by $\|S_b(u)\|_{L^2(B_{2r})}$. Secondly, the Carleson condition (5.36) and the Cauchy-Schwarz inequality imply

$$II_1 + II_2 + J_1 \leq C(n, N)\|\mu\|_{C^1}^{1/2}\|S_b(u)\|_{L^2(B_{2r})}\|\tilde{N}_a(u)\|_{L^2(B_{2r})}.$$  

(5.45)

Next, corresponding to the case when the derivative falls on the cutoff function $\zeta$ we have

$$J_3 + III \leq C(\lambda, \Lambda, n, N)\int_{[0,2r] \times B_{2r}} |\nabla v| |v| \frac{\chi_0}{r} \, dx_0 \, dx'$$

$$\leq C(\lambda, \Lambda, n, N) \left(\int_{[0,2r] \times B_{2r}} |v| \frac{\chi_0}{r} \, dx_0 \, dx'\right)^{1/2} \|S_b^\nu(v)\|_{L^2(B_{2r})}$$

$$\leq C(\lambda, \Lambda, n, N)\|S_b(u)\|_{L^2(B_{2r})}\|\tilde{N}_a(u)\|_{L^2(B_{2r})}.$$  

(5.46)

Finally, the interior term $IV$, which arises from the fact that $\partial_0 \zeta$ vanishes on the set $(0, r) \cup (2r, \infty)$ may be estimated as follows:

$$IV \leq \frac{c}{r} \int_{[r,2r] \times B_{2r}} |v|^2 \, dx_0 \, dx'.$$  

(5.47)

Summing up all terms, the above analysis ultimately yields

$$\int_{B_r(y')} |v(0, x')|^2 \, dx'$$

$$\leq C(\lambda, \Lambda, n, N)(1 + |\mu|_{C^1}^{1/2})\|S_b(u)\|_{L^2(B_{2r})}\|\tilde{N}_a(u)\|_{L^2(B_{2r})}$$

$$+ C(\lambda, \Lambda, n, N)\|S_b(u)\|_{L^2(B_{2r})}^2 + \frac{c}{r} \int_{[r,2r] \times B_{2r}} |v|^2 \, dx_0 \, dx'.$$  

(5.48)

With this in hand, the estimate in (5.32) follows (by passing from $v$ back to $u$ via the map $\rho$).

The case $r >> h$ requires some extra care. However we can observe that for $\theta h_{\nu,a}(w)(x') \geq h$ we have $u(\theta h_{\nu,a}(w)(x'), x') = 0$ and hence for such points the lefthand side of (5.32) vanishes. It follows that without loss of generality we may modify our function $h_{\nu,a}$ assume that $\theta h_{\nu,a}(w) \leq h$ in $\Delta_r$ without changing the value of the lefthand side of (5.32). What this implies is that the estimate (5.32) for $\Delta_r$ can be deduced from adding up estimates (5.32) for smaller balls $\Delta_{r'} \subset \Delta_r$ where $r' \approx h$ and hence we still have $h_{\nu,a} \leq 2r'$. However, the estimate for such small balls was established above and hence we can conclude that (5.32) holds for balls of all sizes.

Finally, the last claim in the statement of the lemma can be seen as follows. If $K = B_{3(X)/2}(X)$ and $A_r = X$ then the claim in question becomes a direct
consequence of Poicaré’s inequality (cf. Lemma 2.0). For more general $\mathcal{K}$, there is a finite covering of $\mathcal{K}$ by balls of the form $B_i = B_{\delta(X_i)/2}(X_i)$. Then

$$\int_{\mathcal{K}} |u|^2 dX \leq \sum_i \int_{B_i} |u|^2 dZ \leq C \sum_i r_i^{n-1} |u_{av}(X_i)|^2 + \int_{\Delta_2r} S_b(u) d\sigma, \quad (5.49)$$

by Poincaré’s inequality. Furthermore, for each $i$ we have (abbreviating $r_i := \delta(X_i)$, $\tilde{r} := \delta(A_r)$, and $B := B_{1/2}(0)$):

$$|u_{av}(X_i)|^2 \leq 2 |u_{av}(A_r)|^2 + 2 |u_{av}(X_i) - u_{av}(A_r)|^2$$

$$\leq 2 |u_{av}(A_r)|^2 + 2 \left( \int_B |u(X_i + r_i \xi) - u(A_r + \tilde{r}\xi)| d\xi \right)^2$$

$$\leq 2 |u_{av}(A_r)|^2 + 2 \int_B |u(X_i + r_i \xi) - u(A_r + \tilde{r}\xi)|^2 d\xi. \quad (5.50)$$

Note that the last term above is of the same type as the right-hand side of (5.29). As in the past, the term in question may once again be estimated as in (5.27). Hence, ultimately, this is $\leq C(S_b(u)(Q))^2$ for all $Q \in \Delta_2r$. The desired conclusion now readily follows from this.

We now make use of Lemma 5.3 involving the stopping time Lipschitz functions $\theta_{h_{\nu,a}}(w)$, in order to obtain the good-\(\lambda\) inequality stated in the next lemma. As a preamble, we agree to let $Mf(x') := \sup_{r>0} \int_{x'-r < z'} |f(z')| dz'$, for $x' \in \mathbb{R}^{n-1}$, denote the standard Hardy-Littlewood maximal function on $\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$.}

**Lemma 5.5.** Consider the system (5.1) with coefficients satisfying the Carleson condition and (1.4) in $\mathbb{R}^n_+$. Then for each $\gamma \in (0,1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma) \rightarrow 0$ as $\gamma \rightarrow 0$ and with the property that for each $\nu > 0$ and each energy solution $u$ of (5.1) there holds

$$\left| \left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u)(x') > \nu/32 \right\} \right| \leq C(\gamma) \left| \left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u)(x') > \nu / 2 \right\} \right|. \quad (5.51)$$

**Proof.** For starters, observe that $\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u)(x') > \nu / 32 \}$ is an open subset of $\mathbb{R}^{n-1}$. When this set is empty or the entire Euclidean ambient, estimate (5.51) is trivial, so we focus on the case when the set in question is both nonempty and proper. Granted this, we may consider a Whitney decomposition $(\Delta_i)_{i \in I}$ of it, consisting of open cubes in $\mathbb{R}^{n-1}$. Let $F^i_{\nu}$ be the set appearing on the left-hand side of (5.51) intersected with $\Delta_i$. We may streamline the index set $I$ by retaining only those $i$’s for which $F^i_{\nu} \neq \emptyset$. Let $B_i$ be a ball of radius $r_i$ in $\mathbb{R}^n$ such that $\Delta_i \subset B_i \cap \{ x_0 = 0 \}$ and there exists a point $p' \in 2B_i \cap \partial \mathbb{R}^n_+$ with $\tilde{N}_a(u)(p') = N_a(w)(p') \leq \nu / 32$. The existence of such point $p'$ is guaranteed by the very nature of the Whitney decomposition. Indeed, there exists a point near $\Delta_i$ not contained in the set $\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u)(x') > \nu / 32 \}$.

This clearly implies that $w(z) \leq \nu / 32$ for all $z \in \Gamma_a(p')$. In particular, for all $x' \in \Delta_i$ we have $w(z) \leq \nu / 32$ for all $z \in \Gamma_a(x') \cap \Gamma_a(p')$, so we focus on estimating the size of $w(z)$ for $z \in \Gamma_a(x') \setminus \Gamma_a(p')$ with $z_0 \geq 2r$. Since we also assume that for at least one $x' \in \Delta_i$ we have $M(S^2_b(u))(x') \leq (\nu \gamma)^2$, by the same type of estimates established in the proof of Lemma 5.2 (cf. (5.28) in particular) we may conclude...
that for sufficiently small $\gamma > 0$ we have that for any $z \in \Gamma_a(x')$ with $z_0 \geq 2r$ there is a point $\tilde{z} \in \Gamma_a(p')$ with
\[
|z - \tilde{z}| \leq C r_i \quad \text{and} \quad |w(z) - w(\tilde{z})| \leq \nu/32. \tag{5.52}
\]
It follows that for all such $z$ we have $w(z) \leq \nu/16$. Hence for all $x' \in \Delta_i$ we have
\[
\nu < \tilde{N}_a(u)(x') = N_a(w)(x') = N_a^{2r}(w)(x'), \tag{5.53}
\]
where $N_a^{2r}$ is the truncated nontangential maximal function at height $2r$. In particular this also implies
\[
h_{\nu, a}(w) \leq 2r_i \quad \text{pointwise on} \quad \Delta_i. \tag{5.54}
\]

Let us also note that we can find a point $q$ (specifically, a corkscrew point for $12\Delta_i$) with distance to $\Delta_i$ and the boundary equal to $12r_i$ such that $w(q) \leq \nu/16$. When $h \lesssim r_i$ since $u$ vanishes above height $h$ and we might actually take $q$ such that $w(q) = 0$.

As $w$ is the $L^2$ average of $|u|$, in terms of $u_{av}(q) = \int_{B_i(q)/2(q)} u(z) dz$ the latter estimate gives
\[
|u_{av}(q)| \leq w(q) \leq \nu/16. \tag{5.55}
\]
Next, consider $\tilde{u} := u - u_{av}(q)$. (For $h \lesssim r_i$ this is just $u$ as $u_{av}(q) = 0$). Then $L\tilde{u} = 0$, hence $\tilde{u}$ still solves the system (5.3) and $\tilde{u}_{av}(q) = 0$. Denote by $\tilde{w}$ the $L^2$ averages of $|\tilde{u}|$. For all $x' \in F_\nu'$ we have
\[
N_a^{2r}(\tilde{w})(x') \geq N_a^{2r}(w)(x') - |u(q)| \geq \nu - \nu/16 > \nu/2. \tag{5.56}
\]
With $\tilde{h} := h_{\nu, a}(w)$ and for $M_{\tilde{h}}$ defined on the graph of $\tilde{h}$ in Corollary 5.3 we see that Corollary 5.3 applied to $\tilde{u}$ implies
\[
M_{\tilde{h}}(\tilde{w}\chi_{4B_i})(h(x'), x') \geq C(n)\nu. \tag{5.57}
\]
Here we are allowed to apply the cutoff function $\chi_{4B_i}$ since values of $\tilde{w}$ are small above the height $2r_i$ and hence this put a limit on the distance and the diameter of the boundary ball $R$ constructed in Corollary 5.3 from the point $x'$ (both are bounded by $\lesssim r_i$). Thus by the maximal function theorem
\[
|F_{\nu}'| \leq \frac{C}{\nu^2} \int_{4\Delta_i} \left( M_{\tilde{h}}(\tilde{w}\chi_{4B_i}) \right)^2(h(x'), x') dx' \leq \frac{C}{\nu^2} \int_{4\Delta_i} \tilde{w}^2(h(x'), x') dx'. \tag{5.58}
\]

At this stage, we bring in the following lemma.

**Lemma 5.6.** For any surface ball $\Delta$ if $a > 0$ we have for $h = h_{\nu, a}(w)$
\[
\int_{\Delta} \tilde{w}^2(h(x'), x') dx' \leq C \int_{1/6}^6 \int_{4\Delta} |\tilde{u}(\theta h(x'), x')|^2 dx' d\theta. \tag{5.59}
\]

1 Technically $\tilde{u} \in W^{1,2}_{loc}(\Omega)$ is not an energy solution, but in the proof the smallness of the solution is only need above a certain distance from the boundary. In our case we obviously have $\tilde{w}(z) \leq w(z) + |w(q)| \leq \nu/8$ for points $z$ whose distance to the boundary exceeds $2r_i$ which suffices for our purposes.
Accepting for the moment this lemma, whose proof we postpone for a later occasion, we have (taking $a > 0$ as in Lemma 5.4)

$$|F_\nu^i| \leq \frac{C}{\nu^2} \int_{1/6}^6 \int_{12\Delta_i} \left| \hat{u}(\theta h(x'), x') \right|^2 dx' d\theta. \quad (5.60)$$

For each $\theta$, we apply the conclusion in Lemma 5.4 (in the version recorded in the very last part of its statement) to the solution $\hat{u}$. This gives

$$\int_{12\Delta_i} \left| \hat{u}(\theta h(x'), x') \right|^2 dx'$$

$$\leq C(1 + \|\mu\|_{L^2}^{1/2}) \|S_b(u)\|_{L^2(24\Delta_i)} \|N_a(\hat{v})\|_{L^2(24\Delta_i)}$$

$$+ C\|S_b(u)\|^2_{L^2(24\Delta_i)} + C\|\nu\|_{L^2(24\Delta_i)}^2$$

$$\leq C(1 + \|\mu\|_{L^2}^{1/2}) \|S_b(u)\|_{L^2(24\Delta_i)} \|N_a(w + w(q))\|_{L^2(24\Delta_i)}$$

$$+ C\|S_b(u)\|^2_{L^2(24\Delta_i)}. \quad (5.61)$$

Observe that we have dropped the term $C\|\nu\|_{L^2(24\Delta_i)}^2$ as we have arranged previously that $\hat{u}(\theta h(x')) = 0$ since $F_\nu^i \neq \emptyset$ and $|w(q)| \leq \nu/16$ the term in the penultimate line of (5.61) may be bounded by

$$C|24\Delta_i| \left( \int_{24\Delta_i} S_b^2(u) dx' \right)^{1/2} \left[ \int_{24\Delta_i} N_a^2(u) dx' \right]^{1/2} + \frac{\nu}{16} \int_{24\Delta_i} S_b^2(u) dx'$$

$$\leq C|24\Delta_i| \left( M(S_b^2(u))(x')M(N_a^2(u))(x') \right)^{1/2} + \frac{\nu}{16} M(S_b^2(u))(x')^{1/2}$$

$$\leq C|24\Delta_i|(\gamma^2 + \gamma/16)\nu^2 = C(\gamma)|\Delta_i|\nu^2. \quad (5.62)$$

Here $x' \in F_\nu^i$ is a point where we use the assumptions for the set on the left-hand side of (5.51). Also, we have used that $|24\Delta_i| \lesssim |\Delta_i|$ by the doubling property of the Lebesgue measure. The estimate for the very last term of (5.61) is analogous. By design, we have $C(\gamma) \to 0$ as $\gamma \to 0$. Using this back in (5.60) we obtain

$$|F_\nu^i| \leq C'(\gamma)|\Delta_i|. \quad (5.63)$$

Summing over all $i$ we obtain (5.58), as desired. \qed

At this stage, it remains to prove Lemma 5.6.

**Proof.** Write $\mathbb{R}^{n-1} = \bigcup_{i \in \mathbb{Z}} \Delta_i$ where, for each $i$,

$$\Delta_i := \{ x' \in \mathbb{R}^{n-1} : 2^{i-1} \leq h(x') < 2^i \}. \quad (5.64)$$

Consider $y = (y_0, y') \in B_{h(x')/2}(h(x'), x')$ for $x' \in \Delta_i \cap \Delta$. Then

$$|y_0 - h(x')| \leq h(x')/2 \quad \text{and} \quad |x' - y'| \leq h(x')/2. \quad (5.65)$$

The goal is to estimate $h(y')$. Since $h = h_{v, a}(w)$ is a Lipschitz function with Lipschitz constant $1/a < 1$ (cf. Lemma 5.1) we have

$$h(y') \geq h(x') - |h(x') - h(y')| \geq h(x') - |x' - y'| > \frac{h(x')}{2} \quad (5.66)$$

and

$$h(y') \leq h(x') + |h(x') - h(y')| \leq h(x') + |x' - y'| < 3h(x')/2. \quad (5.67)$$
It follows that if \( O := \bigcup_{x' \in \Delta \cap \Delta} B_{h(x')/2}(h(x'), x') \) then
\[
(y_0, y') \in O \implies \begin{cases}
y' \in \widetilde{\Delta} := \Delta^{i-1} \cup \Delta^i \cup \Delta^{i+1}, \\
y' \in 3\Delta, \\
y_0 \in \{2^{i-2}, 3 \cdot 2^{i-1}\}.
\end{cases}
\tag{5.68}
\]
The fact that \( y' \in 3\Delta \) follows from (5.54). Hence we have
\[
\int_{\Delta \cap \Delta} \tilde{u}^2(h(x'), x') \, dx' = \int_{x' \in \Delta \cap \Delta} \int_{B_{h(x')/2}(h(x'), x')} |\tilde{u}(z)|^2 \, dz \, dx' \leq C 2^{-i} \int \int_{O} |\tilde{u}(z)|^2 \, dz \, dx',
\]
where in the last step we have interchanged the order of integration. For a fixed \( z \in O \) we have
\[
\left| \left\{ x' \in \Delta^i \cap \Delta : z \in B_{h(x')/2}(h(x'), x') \right\} \right| \leq \left| \left\{ x' \in \mathbb{R}^{n-1} : z \in B_{3h(x')/2}(h(x'), x') \right\} \right|.
\tag{5.70}
\]
Since for such \( z = (z_0, z') \) we have \( z_0 \in [2^{i-2}, 3 \cdot 2^{i-1}] \) and
\[
\frac{h(x')}{2} < z_0 < \frac{3h(x')}{2} \implies h(x') \in (2z_0/3, 2z_0) \subset (2^{i-1}/3, 3 \cdot 2^i).
\tag{5.71}
\]
From this we then conclude
\[
\left\{ x' \in \mathbb{R}^{n-1} : z \in B_{h(x')/2}(h(x'), x') \right\} \subset \left\{ x' \in \mathbb{R}^{n-1} : |x' - z'| < 2^{i+2} \right\}
\tag{5.72}
\]
hence, further,
\[
\left| \left\{ x' \in \Delta^i \cap \Delta : z \in B_{h(x')/2}(h(x'), x') \right\} \right| \leq C 2^{i(n-1)}.
\tag{5.73}
\]
Using this back in (5.69) then yields
\[
\int_{\Delta \cap \Delta} \tilde{w}^2(h(x'), x') \, dx' \leq C 2^{-i} \int \int_{O} |\tilde{u}(z)|^2 \, dz \, dx' \leq C 2^{-i} \int_{O} \int_{z_0(2^{-2}, 3 \cdot 2^{i-1})} |\tilde{u}(z)|^2 \, dz_0 \, dz',
\tag{5.74}
\]
where (with \( \widetilde{\Delta} \))
\[
\mathcal{P}(O) := \left\{ z' \in \mathbb{R}^{n-1} : z_0 \text{ such that } (z_0, z') \in O \right\} \subset \widetilde{\Delta} \cap 3\Delta.
\tag{5.75}
\]
Clearly since \( z' \in \mathcal{P}(O) \) we have \( h(z') \in [2^{i-2}, 3 \cdot 2^{i-1}] \) and, therefore,
\[
(2^{i-2}, 3 \cdot 2^{i-1}) \subset (h(z')/6, 6h(z')).
\tag{5.76}
\]
Hence (5.74) may be also written as
\[
\int_{\Delta \cap \Delta} \tilde{w}^2(h(x'), x') \, dx' \leq C \int_{\widetilde{\Delta} \cap 3\Delta} \int_{1/6}^{6} |\tilde{u}(\theta(h(z'), z'))|^2 \, d\theta \, dz'.
\tag{5.77}
\]
By interchanging the order of integration and then summing over all $i \in \mathbb{Z}$ we arrive at

$$
\int_{\Delta} \tilde{u}^2(h(x'), x') \, dx' \\
\leq C \int_{1/6}^{1} \sum_{i} \int_{\Delta \cap 3\Delta} |\tilde{u}(\theta h(z'), z')|^2 \, dz' \, d\theta \\
= C \int_{1/6}^{1} \sum_{i} \left( \int_{\Delta \cap 3\Delta} + \int_{\Delta \cap 3\Delta} + \int_{\Delta \cap 3\Delta} \right) |\tilde{u}(\theta h(z'), z')|^2 \, dz' \, d\theta \\
= 3C \int_{1/6}^{1} \sum_{i} \int_{\Delta \cap 3\Delta} |\tilde{u}(\theta h(z'), z')|^2 \, dz' \, d\theta \\
= 3C \int_{1/6}^{1} \int_{\Delta \cap 3\Delta} |\tilde{u}(\theta h(z'), z')|^2 \, dz' \, d\theta,
$$

as wanted. This finishes the proof of Lemma 5.6 and completes the proof of Lemma 5.7. \qed

Lemma 5.7 has a localized version on any boundary ball $\Delta_d \subset \mathbb{R}^{n-1}$.

**Lemma 5.7.** Consider the system (3.1) with coefficients satisfying the Carleson condition and (1.4) in $\mathbb{R}^{n_1}_{+}$. Consider any boundary ball $\Delta_d = \Delta_d(Q) \subset \mathbb{R}^{n-1}$, let $A_d = (d/2, Q)$ be its corkscrew point and let

$$
\nu_0 = \left( \frac{\int_{B_{d/4}(A_d)} |u(z)|^2 \, dz}{2} \right)^{1/2}.
$$

Then for each $\gamma \in (0, 1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma) \to 0$ as $\gamma \to 0$ and with the property that for each $\nu > 2\nu_0$ and each energy solution $u$ of (3.1) there holds

$$
\left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u\chi_{T(\Delta_d)}) > \nu, (M(S^2_\theta(u)))^{1/2} \leq \gamma \nu, (M(S^2_\theta(u))(\tilde{N}_a(u\chi_{T(\Delta_d)})))^{1/4} \leq \gamma \nu \right\} \\
\leq C(\gamma) \left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u\chi_{T(\Delta_d)})(x') > \nu/32 \right\}.
$$

Here $\chi_{T(\Delta_d)}$ is the indicator function of the Carleson region $T(\Delta_d)$ and the square function $S_\theta$ in (5.80) is truncated at the height $2d$. Similarly, the Hardy-Littlewood maximal operator $M$ is only considered over all balls $\Delta' \subset \Delta_{md}$ for some enlargement constant $m = m(a) \geq 2$.

**Proof.** The proof is similar to Lemma 5.5 and hence we only point out the main differences introduced by considering $N$ of $u\chi_{T(\Delta_d)}$ instead of $u$. Let $w_1$ be the $L^2$ averages of $u\chi_{T(\Delta_d)}$ instead of $u$. We consider $h_{\nu,a}(w_1)$ as in (5.2), the Lemma 5.1 holds for $h_{\nu,a}(w_1)$ as before.

Because $u\chi_{T(\Delta_d)} = 0$ outside $T(\Delta_d)$ we see that $w_1 = 0$ outside $T(\Delta_{2d})$ and it follows that there exists $m = m(a) > 1$ such that

$$
N_a(w_1) = 0 \text{ on } \mathbb{R}^{n-1} \setminus \Delta_{md} \text{ and } \sup_{\Delta_d} h_{\nu,a}(w_1) = \sup_{\Delta_d} h_{\nu,a}(w_1).
$$
We only consider $\nu > 2\nu_0$. We claim that for such choice of $\nu$ Lemma 5.2 and thus Corollary 5.3 remain valid and only require minor changes we outline below. We choose $b = b(a) > 0$ such that whenever $x' \in \Delta_{md}$ then
\[ [d/48, d] \times \Delta_d \subset \Gamma_b(x'). \]
With this at hand we will have thanks to (5.28) the following estimate for all $(y_0, y') \in [d/24, 2d] \times \Delta_d$
\[ |w(A_d) - w(y_0, y')|^2 \lesssim C\gamma^2 \nu^2. \]

Here $w$ as before denotes the $L^2$ averages of un-truncated function $u$. We choose $\gamma \leq \gamma_0$ where $C\gamma_0^2 = 1/4$. As $w(A_d) = w_1(A_d)$ we therefore obtain a one-sided estimate
\[ |w_1(y_0, y')| \leq |w(y_0, y')| \leq |w(A_d)| + |w(y_0, y') - w(A_d)| < \nu/2 + \nu/2 = \nu. \]
In particular, this implies that on $\Delta_d$ we have $h_{\nu,a}(w_1) \leq d/24$ and hence thanks to (5.31) $h_{\nu,a}(w_1) \leq d/24$ everywhere.

With this at our disposal the proof of Lemma 5.2 only requires a minor modification. We again find a point $y = (y_0, y') \in \partial \Gamma_a(x_0, x')$ such that $w_1(y) = \nu$ and define $R$ as before.

Consider a subregion $R'$ of $R$ defined as follows
\[ R' = \{ z' \in R : |B_{z_0/2}(z_0, z') \cap T(\Delta_d)| \geq |B_{y_0/2}(y_0, y) \cap T(\Delta_d)|/2 \text{ for all } (z_0, z') \in \mathcal{O} \}. \]
By simple geometric consideration we will have $R \subset 4R'$. Now repeating the calculation (5.26) for any pair of points $z_1 \in B_{z_0/2}(z_0, z') \cap T(\Delta_d)$ and $z_2 \in B_{y_0/2}(y_0, y) \cap T(\Delta_d)$ we obtain a bound from below on the size of $w_1(z_0, z)$ it terms of $w_1(y_0, y')$ (it is a calculation similar to (5.27) but trickier as the sets $B_{z_0/2}(z_0, z) \cap T(\Delta_d)$ and $B_{y_0/2}(y_0, y) \cap T(\Delta_d)$ are not necessary balls any more). We obtain
\[ w_1(z_0, z) \geq w_1(y_0, y')/2 - C(a, n, N)\gamma \nu > \nu/2 - \nu/4 = \nu/4, \]
for $\gamma$ chosen such that $C(a, n, N)\gamma < 1/4$.

It follows that Lemma 5.2 holds for $w_1$ with $R'$ replacing $R$ and slightly weaker claims $x' \in 8R'$ and
\[ |w_1(h_{\nu,a}(w_1)(z'), z')| > \nu/4. \]
for all $z' \in R'$, instead of (5.19). However, this is still sufficient to conclude that Corollary 5.3 holds for $w_1$ as well.

We now look at Lemma 5.4 and in particular the place it is used in the good-$\lambda$ Lemma 5.5. Recall that we apply this lemma in one place only, namely the estimate (5.60), where $\Delta_i$ are Whitney cubes. Hence we might as well arrange that the balls $\Delta_r$ we consider in Lemma 5.4 are from a dyadic grid on $\mathbb{R}^{n-1}$. Similarly, in the claim of Lemma 5.4 it suffices to consider $\Delta_d$ dyadic.

Hence, whenever $\Delta_r \cap \Delta_d \neq \emptyset$ then either $\Delta_r \subset \Delta_d$ or $\Delta_d \subset \Delta_r$. If $\Delta_d \subset \Delta_r$ then clearly if we prove the claim of Lemma 5.4 for $\Delta_d$ and function $u \chi_T(\Delta_d)$ it will also hold for the larger ball $\Delta_r$ as the lefthand side of (5.32) vanishes outside $\Delta_d$. The terms on the righthand side will be bigger or comparable if we replace $\Delta_d$ by $\Delta_r$ there. This is also true for the last term of (5.32) because although we have $\Delta_d \subset \Delta_r$ we must have $r \approx d$. This is due to the fact that the set $\Delta_r$ comes from Whitney decomposition of the set $\{ \tilde{N}(w_1) > \nu/32 \} \subset \Delta_{md}$ implying the inequality $r \lesssim d$. 
Hence it suffices to consider $\Delta_r \subset \Delta_d$ or $\Delta_r \cap \Delta_d = \emptyset$ in Lemma 5.4. We consider these two cases.

- If $\Delta_r \cap \Delta_d = \emptyset$ then Lemma 5.4 hold trivially for $u\chi_{T(\Delta_d)}$ as the function vanishes on $\Delta_r$.
- If $\Delta_r \subset \Delta_d$. We have already established above that $h_{a,\nu} \leq d/24$ and therefore $\theta h_{a,\nu} \leq d/4$. It follows that all terms in (5.32) are either the same or comparable when $u$ is replaced by $u\chi_{T(\Delta_d)}$ on the lefthand side of (5.32) and in the term $\tilde{N}_a$ as the functions $u$ and $u\chi_{T(\Delta_d)}$ coincide in $\Delta_r \times (0, d)$.

We also clearly the estimate (5.32) only requires truncated versions of $S_6$ and $\tilde{N}$.

Therefore we can use Lemma 5.4 to prove Lemma 5.7 the same way as we did Lemma 5.5. This shows that the local good-$\lambda$ inequality (5.80) holds. □

Finally we have the following.

**Proposition 5.8.** Let $u$ be an arbitrary energy solution of (3.1) in $\Omega = \mathbb{R}^n$ and the measure $\mu$ defined as in (1.3) is Carleson with norm $\|\mu\|_{C} < \infty$. Assume that the coefficients satisfy the Legendre-Hadamard condition (1.4). Then for any $p > 0$ and $a > 0$ there exists an integer $m = m(a) \geq 2$ and a finite constant $C = C(n, N, \lambda, a, \|\mu\|_{C}) > 0$ such that for all balls $\Delta_d \subset \mathbb{R}^{n-1}$ we have

$$\|\tilde{N}_a(u)\|_{L^p(\Delta_d)} \leq C\|S_{a}^{2r}(u)\|_{L^p(\Delta_{md})} + C d^{(n-1)/p}|u_{av}(A_d)|,$$  

(5.82)

where $A_d$ denotes the corkscrew point of the ball $\Delta_d$ and $u_{av}$ is as in (5.34).

We also have a global estimate for any $p > 0$ and $a > 0$. There exists a finite constant $C = C(n, N, \lambda, a, \|\mu\|_{C}) > 0$ such that

$$\|\tilde{N}_a(u)\|_{L^p(\mathbb{R}^{n-1})} \leq C\|S_{a}(u)\|_{L^p(\mathbb{R}^{n-1})}.$$  

(5.83)

**Proof.** When $p > 2$ (5.82) follows immediately by a standard argument (multiplying the good-$\lambda$ inequality (5.80) by $\nu^{p-1}$ and integrating w.r.t. over the interval $(2\nu_0, \infty)$). Note that the fact that the square function $S_{a}^{2r}$ is only integrated over some enlargement of $\Delta_d$ instead of the whole $\mathbb{R}^{n-1}$ follows from the fact that the set $\{x \in \mathbb{R}^{n-1} : \tilde{N}_a(u\chi_{T(\Delta_d)})(x') > \nu/32\}$ on the lefthand side of (5.81) vanishes outside a ball of diameter comparable to $\Delta_d$. For this reason the maximal operators $M$ in (5.80) can be restricted to such enlarged ball $\Delta_{md}$.

We do not quite get (5.82), instead we get on the righthand side

$$\|S_{a}^{2r}(u)\|_{L^p(\Delta_{md})} + d^{(n-1)/p}\left(\int_{B_{d/4}(A_d)} |u(z)|^2 dz\right)^{1/2},$$  

(5.84)

but then using Poincaré as in (5.49) we have for the second term an estimate

$$d^{(n-1)/p}\left(\int_{B_{d/4}(A_d)} |u(z)|^2 dz\right)^{1/2} \lesssim d^{(n-1)/p}|u_{av}(A_d)| + \left(\int_{\Delta_{2d}}|S_{b}^{2r}(u)(Q)|^2 dQ\right)^{1/2}.$$  

The argument that shows (5.83) for all $p > 0$ can be found in [21]. The local estimate (5.82) for $p > 2$ is the necessary ingredient for what is otherwise a purely real variable argument. Further details can be found in [21]. □
6. $L^p$ Dirichlet problem for $p$ near 2.

Following [8] we explore the extrapolation of solvability from $L^2$ to $L^p$ values of $p$ near 2. Consider $2 < p < 2 + \varepsilon$. Because our main estimate (following [8]) will require a local estimate on solvability of the $L^2$ Dirichlet problem on graph domains we record here statements that are sufficient for our purposes.

The domains we shall consider will all be of the following the form. Let $\Delta_d \subset \mathbb{R}^{n-1}$ be a boundary ball or a cube or diameter $d$. We denote by $\mathcal{O}_{\Delta_d,a}$

$$\mathcal{O}_{\Delta_d,a} = \bigcup_{Q \in \Delta_d} \Gamma_a(Q). \quad (6.1)$$

Here as before $\Gamma_a(Q)$ denotes the nontangential region with aperture $a$ at a point $Q$ (c.f. Definition 2.1).

Clearly, the $L^2$ solvability result from Theorem 1.1 applies to domains like (6.1) as these are domains with Lipschitz constant 1/a. It follows if $\mathcal{L}$ satisfies assumptions of this theorem on $\mathbb{R}^n$ it also satisfies it on any domain $\mathcal{O}_{\Delta_d,a}$, provided 1/a is sufficiently small. We fix $a > 0$ for which we have such solvability. Theorem 1.1 then implies the estimate

$$\|\tilde{N}_{a/2}u\|_{L^2(\partial \mathcal{O}_{\Delta_d,a})} \leq C\|u\|_{L^2(\partial \mathcal{O}_{\Delta_d,a};\mathbb{R}^N)}, \quad (6.2)$$

for all energy solutions $u$ of $\mathcal{L}u = 0$. The constant $C > 0$ in the estimate above is independent of $\Delta_d$. Here the nontangential maximal function $\tilde{N}$ must be take with respect to nontangential approach regions that are contained inside $\mathcal{O}_{\Delta_d,a}$, that is we need to take regions $\Gamma_b(\cdot)$ for any $b < a$. Without loss of generality we choose $b = a/2$ and fix it for the remaining part of this section.

We require a local version of the estimate (6.2). For ease of notation we drop the dependence of the domain $\mathcal{O}_{\Delta_d,a}$ on $\Delta_d$ and $a$ and use $\mathcal{O} = \mathcal{O}_{\Delta_d,a}$.

Applying the local results on parts of boundary such as Corollary 4.4 and (5.82) and putting pieces together we have the following result.

**Lemma 6.1.** Let $\mathcal{L}$ be as in Theorem 1.1 on the domain $\mathbb{R}^n_+$. Let $\mathcal{O}$ be a Lipschitz domain as above and assume $u$ is an arbitrary energy solution of (3.1) in $\mathbb{R}^n$ with the Dirichlet boundary datum $f \in L^2(\partial \mathcal{O};\mathbb{R}^N)$. Then the following estimate holds:

$$\|\tilde{N}_{a/2}u\|_{L^2(\Delta_d)} \leq C\|f\|_{L^2(\partial \mathcal{O} \cap \overline{T(\Delta_{md})}\mathbb{R}^N)} + Cd^{(n-1)/2} \sup_{x \in \mathcal{O} \setminus \{\delta(x) > d\}} w(x), \quad (6.3)$$

where $\delta(x) = \text{dist}(x, \partial \mathbb{R}^n_+)$, $w(x) = \left(\int_{B_{\delta(x)/2}(x)} |u(y)|^2 dy\right)^{1/2}$ and $m = m(a) > 1$ is sufficiently large.

**Proof.** In last term of (6.3) because of the way $\mathcal{O}$ is defined we clearly have

$$\{(x_0, x') \in \mathcal{O} : x' \notin \Delta_{(1+a)d}\} \subset \mathcal{O} \cap \{\delta(x) > d\}. \quad (6.4)$$

If follows that by considering the map $\rho : \mathbb{R}^n_+ \to \mathcal{O}$ defined in (2.26) proving (6.3) is equivalent to establishing

$$\|\tilde{N}u\|_{L^2(\Delta_d)} \leq C\|f\|_{L^2(\Delta_{md};\mathbb{R}^N)} + Cd^{(n-1)/2} \sup_{x \in \mathbb{R}^n_+ \setminus T(\Delta_{(1+a)d})} w(x), \quad (6.5)$$

where we now work on the domain $\mathbb{R}^n_+$ with $u$ solving $\mathcal{L}u = 0$ in $\mathbb{R}^n_+$ for $\mathcal{L}$ as in Theorem 1.1. We start with the term on the lefthand side of (6.5). If follows from
gives estimate the first term on the righthand side of (6.6) we use Corollary 4.4. This gives
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq C \int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx + C d^{n-1} |u_{av}(A_d)|^2. \tag{6.6}
\]

The last term above has a trivial bound by \(C d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2\). To estimate the first term on the righthand side of (6.6) we use Corollary 4.4. This gives
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq \int_{\Delta_{md}} |u(0, x')|^2 \, dx' + \int_{\Delta_{md}} |u(2md, x')|^2 \, dx' + \|\mu\|c \int_{\Delta_{md}} \big[ \tilde{N}^{2md}(u) \big]^2 \, dx'. \tag{6.7}
\]

The second term in the last line can be estimated by \(C d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2\) using the averaging procedure. By varying \(d\) in (6.7) between say \(d_0\) to \(2d_0\) the second term turns into a solid integral over a set that is contained in \(\mathbb{R}^n \setminus T(\Delta_{(1+a)d})\) and hence the estimate holds. This gives
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq \int_{\Delta_{md}} |f(x')|^2 \, dx' + \|\mu\|c \int_{\Delta_{md}} |\nabla u|^2 \delta(x) \, dx + d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2. \tag{6.8}
\]

Finally for the second term in the last line we again use (5.82). We get
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq \int_{\Delta_{md}} |f(x')|^2 \, dx' + \|\mu\|c \int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx + d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2. \tag{6.9}
\]

For sufficiently small \(\|\mu\|c\) we can hide part of the second term in the last line on the righthand side of (6.9). Hence
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq \int_{\Delta_{md}} |f(x')|^2 \, dx' \tag{6.10}
\]
\[
+ \|\mu\|c \int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx + d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2. \tag{6.11}
\]

We claim that by the Caccioppoli inequality we have
\[
\int_{T(\Delta_{md})} |\nabla u|^2 \delta(x) \, dx \leq d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2. \tag{6.11}
\]

This is obvious on the set \(T(\Delta_{md}) \cap \{ \delta(x) \geq d \}\) which is clearly in the interior of \(\mathbb{R}^n_+\). However, let us recall (6.4). It follows that all points of \(T(\Delta_{md}) \setminus T(\Delta_{md})\) are in the interior of the original domain \(\Omega\) and hence we can use Caccioppoli inequality in the original domain.

Finally, by combining (6.6), (6.10) and (6.11) we see that (6.5) holds. We can remove the truncation of \(N\) at height \((1+a)d\) in (6.6) as for points above this height the term \(d^{n-1} \sup_{x \in \mathbb{R}^n \setminus T(\Delta_{(1+a)d})} |w(x)|^2\) controls the nontangential maximal function. \qed
We now establish an analogue of (2.15) from [8]. As before it suffices to work on \( \mathbb{R}^n_\tau \). Let

\[ E_\nu = \{ x' \in \mathbb{R}^{n-1} : \tilde{N}_\alpha(u)(x') > \nu \}. \]

Here, \( \alpha > 0 \) will be determined later. Denote by \( g \)

\[ g(x') = \sup_{B \ni x'} \left( \int_B |f(y')|^2 dy' \right)^{1/2}, \]

for all \( x' \in \mathbb{R}^{n-1} \) where the supremum is taken over all boundary balls \( B \) containing \( x \).

Let \( (\Delta^i) \) be the Whitney decomposition of \( E_\nu \) with the property that \( 2m\Delta^i \subset E_\nu \) and \( 2m\Delta^i \) have finite overlaps. Here \( m \) is chosen as in Lemma 6.1. We look at those Whitney cubes such that

\[ F^i = \Delta^i \cap \{ x' : g(x') \leq \nu \} \neq \emptyset. \]

Since \( \Delta^i \) is the Whitney cube there exists a point \( x_i \in \mathbb{R}^{n-1} \setminus E_\nu \) with

\[ \text{dist}(x_i, \Delta^i) \leq C_\alpha \text{diam}(\Delta^i). \]

For \( 1 < \tau < 2 \) consider the Lipschitz domains

\[ \Omega_\tau = \mathcal{O}_{\tau \Delta^i,a} \]

where \( \tau\Delta^i \) is an enlargement of \( \Delta^i \) by factor of \( \tau \) and \( a \) was chosen earlier (so that the solvability of \( \Omega_\tau \) holds). Set \( A_\tau = \partial \Omega_\tau \cap \Gamma_\alpha(x_i) \), \( B_\tau = (\partial \Omega_\tau \cap \mathbb{R}^n) \setminus \Gamma_\alpha(x_i) \).

Because of the choices we have made for \( \tau \in (1, 2) \) the height of \( B_\tau \) is bounded, namely we have

\[ h := \sup\{ y_0 : (y_0, y') \in B_\tau \} \leq C_\alpha \alpha^{-1} \text{diam}(\Delta^i). \tag{6.12} \]

Since \( F^i \neq \emptyset \) we have

\[ \int_{2m\Delta^i} |f(x')|^2 dx' \lesssim \int_{2m\Delta^i} |g(x')|^2 dx' \lesssim \nu^2 |\Delta^i|. \tag{6.13} \]

It follows by Lemma 6.1 for each \( \Omega_\tau \) we have by (6.13)

\[ ||\tilde{N}u||^2_{L^2(\Delta^i)} \leq C ||f||^2_{L^2(\partial \Omega_\tau \cap \Gamma_\alpha(x_i) \cap \mathbb{R}^n)} + C d^{n-1} \sup_{x \in \Omega_\tau \cap \{\delta(x) > d\}} |w(x)|^2. \tag{6.14} \]

Here \( d = \text{diam}(\Delta^i) \) and \( \tilde{N} \) is defined using cones \( \Gamma_k \) (see above). We deal with the terms on the righthand side. Firstly, for sufficiently large \( \alpha > 0 \) we have \( \Omega_\tau \cap \{\delta(x) > d\} \subset \Gamma_\alpha(x_i) \) and hence

\[ d^{-n-1} \sup_{x \in \Omega_\tau \cap \{\delta(x) > d\}} |w(x)|^2 \lesssim \nu^2 |\Delta^i|. \]

The boundary \( \partial \Omega_\tau \) consists of three pieces, \( A_\tau, B_\tau \) and \( \partial \Omega_\tau \cap \mathbb{R}^{n-1} \subset 2m\Delta^i \), for the last piece we already have the estimate (6.13). Hence by (6.14)

\[ ||\tilde{N}u||^2_{L^2(\Delta^i)} \leq C ||u||^2_{L^2(A_\tau \cap \mathbb{R}^{n-1})} + C ||u||^2_{L^2(B_\tau)} + C \nu^2 |\Delta^i|. \tag{6.15} \]

We integrate (6.15) in \( \tau \) over the interval \( (1, 2) \) in \( \tau \). Since \( A_\tau \subset \Gamma_\alpha(x_i) \) integrating in \( \tau \) turns this into a solid integral which has the following estimate

\[ \int_1^2 ||u||^2_{L^2(A_\tau)} d\tau \lesssim d^{-1} \int \int_{A_\tau} |u(x)|^2 dx \lesssim d^{-1} \int \int_{\Gamma_\alpha(x_i) \cap \mathbb{R}^{n-1}} |u(x)|^2 dx \lesssim \nu^2 |\Delta^i|. \]
We have a similar estimate for $B_r$. 
\[
\int_1^2 \|u\|^2_{L^2(B_r)} dx \lesssim d^{-1} \int_{\bigcup_j B_r} \|u(x)\|^2 dx \lesssim d^{-1} \int_{T(2m\Delta^i) \cap \{x \leq h\}} |u(x)|^2 dx.
\]

However thanks to (6.12) we conclude
\[
d^{-1} \int_{T(2m\Delta^i) \cap \{x \leq h\}} |u(x)|^2 dx \lesssim d^{-1} \alpha^{-1} d \int_{2m\Delta^i} |\tilde{N}u(x')|^2 dx'.
\]

Putting all terms together yields
\[
\|\tilde{N}u\|_{L^2(\Delta^i)}^2 \leq C\nu^2|\Delta^i| + C\alpha^{-1}\|\tilde{N}u\|_{L^2(2m\Delta^i)}^2.
\]

(6.16)

Summing over all indices $i$ (using finite overlap of the Whitney cubes $(2m\Delta^i)$) finally yields
\[
\int_{E_{\nu} \cap \{x \leq \nu\}} \left[\tilde{N}u(x')\right]^2 dx' \leq C\nu^2|E_{\nu}| + C\alpha^{-1} \int_{E_{\nu}} \left[\tilde{N}u(x')\right]^2 dx'.
\]

(6.17)

This is the analogue of (2.15) from [8]. From this as on p. 449 of [8] we conclude (by purely real variable argument) that there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ there is $C(\delta) > 0$ such that
\[
\int_{\mathbb{R}^{n-1}} \left[\tilde{N}u(x')\right]^{2+\delta} dx' \leq C \int_{\mathbb{R}^{n-1}} |f(x')|^{2+\delta} dx'.
\]

(6.18)

From this $L^{2+\delta}$ solvability of the Dirichlet problem in Theorem 1.1 follows.

We now turn to the case $2 - \varepsilon < p < 2$. Following the real variable argument of [9] we work with two family of cones $\Gamma_b(\cdot)$ and $\Gamma_a(\cdot)$ with $b < a$ so that the cones $\Gamma_a(Q)$ contain $\overline{\Gamma_b(Q)} \setminus Q$. Set for ease of notation
\[
m(x') = (\tilde{N}_b u)(x'), \quad \overline{m}(x') = (\tilde{N}_a u)(x'),
\]

and let for $\nu > 0$
\[
F_{\nu} = \{x' \in \mathbb{R}^{n-1} : \overline{m}(x') \leq \nu\}.
\]

Finally, let
\[
\overline{F}_{\nu} = \mathcal{O}_{F_{\nu},a} = \bigcup_{Q \in F_{\nu}} \Gamma_a(Q).
\]

By (6.12) we can conclude that
\[
\int_{F_{\lambda}} m^2(x') dx' \leq C \int_{F_{\lambda}} |f|^2 dx' + C \int_{\partial F_{\lambda} \setminus F_{\nu}} u^2 d\sigma
\]

(6.19)

where the second term can be estimated by $C\nu^2\sigma(\mathbb{R}^{n-1} \setminus F_{\nu})$ by averaging and using the definition of the set $\overline{F}_{\nu}$. Hence we have
\[
\int_{F_{\lambda}} m^2(x') dx' \leq C \int_{F_{\lambda}} f^2 dx' + C\nu^2\sigma(\mathbb{R}^{n-1} \setminus F_{\nu}).
\]

(6.20)

Now as in [9] we have
\[
\int_{\mathbb{R}^{n-1}} m^{2-\varepsilon}(x') dx' \leq C \int_{\mathbb{R}^{n-1}} m^2(x')\overline{m}^{-\varepsilon}(x') dx',
\]

(6.21)

which follows from the fact that for any $0 < \varepsilon < 1$, $M(m)^{\varepsilon}$ is a Muckenhoupt weight of class $A_1$. In particular it is an $A_2$ weight and hence so is $M(m)^{-\varepsilon}$. Hence
by the Muckenhoupt theorem it follows that the maximal operator is bounded on $L^2(\mathbb{R}^{n-1}, M(m)^{-\varepsilon} \, dx')$ and hence

$$
\int_{\mathbb{R}^{n-1}} M(m)^2(x')M(m)^{-\varepsilon}(x') \, dx' \leq \int_{\mathbb{R}^{n-1}} m^2(x')M(m)^{-\varepsilon}(x') \, dx' \\
\leq \int_{\mathbb{R}^{n-1}} m^2(x')M^{-\varepsilon}(x') \, dx',
$$

(6.22)

where the last estimates uses the pointwise bound $m(x') \leq CM(m)(x')$.

It follows by (6.21) that

$$
\int_{\mathbb{R}^{n-1}} m^2(x') \, dx' \leq \int_{\mathbb{R}^{n-1}} m^2(x')M^{-\varepsilon}(x') \, dx' = \varepsilon \int_{\mathbb{R}^{n-1}} \nu^{-1-\varepsilon} \left( \int_{\{x' : m(x') \leq \nu\}} m^2(y) \, dy' \right) \, d\nu.
$$

(6.23)

Hence by (6.20) this further estimates as

$$
\int_{\mathbb{R}^{n-1}} m^2(x') \, dx' \leq C\varepsilon \int_{\mathbb{R}^{n-1}} |f|^2 \mu^{-\varepsilon} \, dx' + \varepsilon \int_{\mathbb{R}^{n-1}} m^2 \, dx' + C\varepsilon \int_{\mathbb{R}^{n-1}} \mu^{-\varepsilon} \sigma_{\{x' : m(x') > \nu\}} \, d\nu
$$

\leq C \int_{\mathbb{R}^{n-1}} |f|^2 \mu^{-\varepsilon} \, dx' + \varepsilon \int_{\mathbb{R}^{n-1}} m^2 \, dx'.
$$

(6.24)

By classical arguments [21]

$$
\int_{\mathbb{R}^{n-1}} m^2 \, dx' \lesssim \int_{\mathbb{R}^{n-1}} m^2 \, dx',
$$

and hence for sufficiently small $\varepsilon > 0$ this yields

$$
\int_{\mathbb{R}^{n-1}} m^2(x') \, dx' \leq C \int_{\mathbb{R}^{n-1}} |f|^2 \mu^{-\varepsilon} \, dx'.
$$

(6.25)

Since for almost every $x'$ we have $|f(x')| \leq \mu(x')$ this then gives us the desired estimate

$$
\int_{\mathbb{R}^{n-1}} m^2(x') \, dx' \leq C \int_{\mathbb{R}^{n-1}} |f(x')|^2 \mu^{-\varepsilon} \, dx',
$$

proving solvability for $p < 2$ close to 2.

Compliance with ethical standards: The authors are not aware of any conflict of interest connected with research contained in this paper. The paper does not contain any research involving Human Participants and/or Animals and hence no Informed consent was sought.

References

[1] M. Angeles Alfonseca, Pascal Auscher, Andreas Axelsson, Steve Hofmann, and Seick Kim, Analyticity of layer potentials and $L^2$ solvability of boundary value problems for divergence form elliptic equations with complex $L^\infty$ coefficients., Adv. Math 226 (2011), no. 5, 4533–4606.

[2] Pascal Auscher and Andreas Axelsson, Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I, Invent. Math. 184 (2011), no. 1, 47–115.
[3] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Philippe Tchamitchian, The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^p$, Ann. Mat. 156 (2001), no. 2, 633–654.

[4] Pascal Auscher and Andreas Rosén, Weighted maximal regularity estimates and solvability of nonsmooth elliptic systems, II, Anal. PDE 5 (2012), no. 5, 983–1061.

[5] Pascal Auscher, Andreas Axelsson, and Steve Hofmann, Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems, J. Func. Anal 255 (2008), no. 2, 374–448.

[6] Pascal Auscher, Andreas Axelsson, and Alan McIntosh, Solvability of elliptic systems with square integrable boundary data, Ark. Mat. 48 (2010), no. 2, 253–287.

[7] Russell M. Brown and Irina Mitrea, The mixed problem for the Lam system in a class of Lipschitz domains, J. Diff. Eq. 246 (2009), 2577–2589.

[8] Björn E. J. Dahlberg and Carlos E. Kenig, Hardy spaces and the Neumann problem in $L^p$ for Laplace’s equation in Lipschitz domains, Ann. of Math. (2) 125 (1987), no. 3, 437–465.

[9] Björn E. J. Dahlberg, Carlos E. Kenig, and Gregory Verchota, The Dirichlet problem for the biharmonic equation in Lipschitz domain, Ann. Inst. Fourier 36 (1986), no. 3, 109–136.

[10] ———, Boundary value problems for the systems of elastostatics in Lipschitz domains, Duke Math. J. 57 (1988), 795–818.

[11] Martin Dindoš and Marius Mitrea, The stationary Navier-Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and $C^1$ domains, Arch. Ration. Mech. Anal. 174 (2004), no. 1, 1–47.

[12] Martin Dindoš and Sukjung Hwang, The Dirichlet boundary problem for second order parabolic operators satisfying Carleson condition, Rev. Math. Iber. 34 (2018), no. 2, 767–810.

[13] Martin Dindoš, Stefanie Petermichl, and Jill Pipher, The $L^p$ Dirichlet problem for second order elliptic operators and a $p$-adapted square function, J. Funct. Anal. 249 (2007), no. 2, 372–392.

[14] Martin Dindoš and Jill Pipher, Regularity theory for solutions to second order elliptic operators with complex coefficients and the $L^p$ Dirichlet problem, arXiv:1612.01568.

[15] Martin Dindoš, Jill Pipher, and David Rule, The boundary value problems for second order elliptic operators satisfying a Carleson condition, Com. Pure Appl. Math. 70 (2017), no. 2, 1316–1365.

[16] Eugene Fabes, Layer potential methods for boundary value problems on Lipschitz domains, In: Lecture Notes in Mathematics 1344 (1988), 55–80.

[17] Eugene Fabes, Carlos E. Kenig, and Gregory Verchota, Boundary value problems for the Stokes system on Lipschitz domains, Duke Math. J. 57 (1988), 769–793.

[18] W. Gao, Boundary value problems on Lipschitz domains for general elliptic systems, J. Funct. Anal. 95 (1991), 377–399.

[19] Steve Hofmann, Carlos Kenig, Svitlana Mayboroda, and Jill Pipher, The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients, Math. Ann. 361 (2015), 863–907.

[20] Steve Hofmann and Jose Martell, $L^p$ bounds for Riesz transforms and square roots associated to second order elliptic operators, Pub. Mat. 47 (2003), 497–515.

[21] Charles Fefferman and Elias Stein, $H^p$ spaces of several variables, Acta Mat. 129 (1972), 137–193.

[22] Carlos E. Kenig, Elliptic boundary value problems on Lipschitz domains, In: Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud. 112 (1986), 131–183.

[23] Carlos E. Kenig and Jill Pipher, The Dirichlet problem for elliptic equations with drift terms, Publ. Math. 45 (2001), no. 1, 199–217.

[24] Carlos E. Kenig, Herbert Koch, Jill Pipher, and Tatiana Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations, Adv. Math. 153 (2000), no. 2, 231–298.

[25] Carlos E. Kenig and Jill Pipher, The Dirichlet problem for elliptic equations with drift terms, Publ. Math. 45 (2001), no. 1, 199–217.

[26] José M. Martell, Dorina Mitrea, Irina Mitrea, and Marius Mitrea, The Dirichlet problem for elliptic systems with data in Köthe function spaces, Rev. Mat. Iberoam. 32 (2016), no. 3, 913–970.

[27] Zhongwei Shen, Necessary and sufficient conditions for the solvability of the $L^p$ Dirichlet problem on Lipschitz domains, Math. Ann. 336 (2006), 697–725.
[28] ______, The $L^p$ Dirichlet problem for elliptic systems on Lipschitz domains, Math. Res. Lett. 13 (2006), 143–159.

[29] Gunther Uhlmann and Jenn-Nan Wang, Complex spherical waves for the elasticity system and probing of inclusions, SIAM J. Math. Anal. 38 (2007), 1967–1980.

School of Mathematics, The University of Edinburgh and Maxwell Institute of Mathematical Sciences, UK
E-mail address: M.Dindos@ed.ac.uk

Department of Mathematics, Yonsei University, Korea
E-mail address: S.Hwang@ed.ac.uk

Department of Mathematics, The University of Missouri at Columbia, USA
E-mail address: mitream@math.missouri.edu