NOTES ON DIFFEOMORPHISM CLASSES OF THE DOUBLING
CALABI-YAU THREEFOLDS

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Abstract. Previously we constructed Calabi-Yau threefolds by a differential-geometric gluing method using Fano threefolds with their smooth anticanonical $K3$ divisors [DY14]. In this article we further consider the diffeomorphic types of the resulting Calabi-Yau threefolds starting from different pairs of Fano threefolds of Picard number one.

1. Introduction

The purpose of this paper is to consider the diffeomorphism classes of certain Calabi-Yau $3$-folds constructed in our gluing construction [DY14 Table 6.1]. Calabi-Yau manifolds play the crucial role both in algebraic geometry and differential geometry. In particular, they form one of the building blocks in the classification of algebraic varieties up to birational isomorphism. Recall that a Calabi-Yau $n$-fold is an $n$-dimensional compact Kähler manifold $X$ whose canonical bundle is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. Then $X$ admits a Ricci-flat Kähler metric with holonomy $SU(n)$. In the case of $n = 1$, the only examples are genus 1 curves and they are all diffeomorphic each other. In the case of $n = 2$, Calabi-Yau surfaces are so-called $K3$ surfaces. Any two $K3$ surfaces are diffeomorphic as smooth $4$-manifolds (see Theorem 7.1.1 in [Huy16]).

In higher dimensions the classification of Calabi-Yau manifolds is hard to deal with and many problems are still open. For example, it is an open problem whether or not the number of topological types of Calabi-Yau $3$-folds is bounded, while many researchers have made substantial progress [GHJ03].

From a differential-geometric point of view, it is natural to classify Calabi-Yau $3$-folds up to diffeomorphism classes. For this purpose we use the classification of closed, oriented simply-connected $6$-manifolds by Wall, Jupp and Zhubr (see Section 3 and the website of the Manifold Atlas Project, $6$-manifolds: $1$-connected [MAP] for a good overview which includes further references). In a word, two closed simply-connected $6$-manifolds that are homeomorphic are necessarily diffeomorphic. Classification of simply-connected $6$-manifolds with torsion free homology up to diffeomorphism classes is determined by the basic invariants in [OkV95 p.300]. The essential invariants here are the cubic intersection forms on $H^2(X, \mathbb{Z})$ and the Chern classes.

We recall that two compact complex manifolds $X_1$ and $X_2$ are said to be deformation equivalent if there is a smooth proper holomorphic map $\varphi : \mathcal{X} \to B$ satisfying the following conditions:

(i) The total space $\mathcal{X}$ and the base space $B$ are connected.

(ii) There exist two points $t_1, t_2 \in B$ such that $\varphi^{-1}(t_i) \cong X_i (i = 1, 2)$.

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The reader should bear in mind that deformation equivalent Calabi-Yau 3-folds are in particular diffeomorphic although the converse is not true in general. In fact, there exist diffeomorphic Calabi-Yau 3-folds which are not deformation equivalent. See [Gr97], [Th99, p. 420].

There are several ways to construct Calabi-Yau 3-folds in the study of algebraic and complex geometry. The large amount of examples of Calabi-Yau 3-folds are brought by Batyrev construction in the context of mirror symmetry [Bat94]. Starting from a 4-dimensional reflexive polytope $\Delta$, we consider the associated Gorenstein toric Fano variety $P_\Delta$ and take a generic anti-canonical section $X$ in $P_\Delta$. By taking a crepant resolution of $X$, we obtain a Calabi-Yau 3-fold whose Hodge numbers are explicitly calculated by combinatorial data of $\Delta$.

Another construction was developed by Kawamata and Namikawa [KN94]. They investigated log deformation theory of normal crossing varieties and constructed examples of Calabi-Yau 3-folds by smoothing simple normal crossing varieties. Later on Lee investigated log deformations of simple normal crossing varieties of two irreducible components which are so-called Tyurin degenerations (see [Lee, Definition V.1]). In particular he gave an explicit description of the Picard groups, Hodge numbers and the second Chern classes of Tyurin degenerations [Lee10].

Differential-geometric counterpart of Lee’s construction was studied by Doi and the author in [DY14]. They gave a differential-geometric construction (doubling construction) of Calabi-Yau 3-folds by gluing two asymptotically cylindrical Ricci-flat Kähler manifolds along their cylindrical ends in appropriate conditions. Also they obtained the set of all Hodge numbers $(h^{1,1}, h^{2,1})$ of Calabi-Yau 3-folds arising from this construction. See Figure 1 and Section 2.2 for details. Consequently we found 61 topologically distinct types of Calabi-Yau 3-folds in our construction (Proposition 2.1). As ingredients, we used admissible pairs $(Y, D)$ consists of a three dimensional compact Kähler manifold $Y$ and a smooth anticanonical divisor $D$ on $Y$. One can construct admissible pairs from Fano 3-folds as follows. Beginning with a Fano 3-fold $V$ with a smooth anticanonical $K3$ divisor $D$, one can show that if we blow-up $V$ along a curve representing $D \cdot D$ to obtain $Y$, then $Y$ has an anticanonical divisor isomorphic to $D$ (denoted by $D$ again) with the holomorphic normal bundle $N_{Y/D}$ trivial. In particular, $(Y, D)$ is an admissible pair (Theorem 2.3). Note that $Y$ itself is not a Fano 3-fold in this construction. According to the classification of smooth Fano 3-folds [Is77, MM81, MM03], there are 105 algebraic families with Picard numbers $1 \leq \rho(V) \leq 10$. It is remarkable that 61 (= the number of topological types of the resulting Calabi-Yau 3-folds) is much less than 105 families of Fano 3-folds. This means that many of the resulting Calabi-Yau 3-folds in our construction have the same Hodge numbers $(h^{1,1}, h^{2,1})$ even though we use distinct Fano 3-folds as ingredients. Hence it is natural to ask whether they are further diffeomorphic or not.

**Question 1.** Let $V_1$ and $V_2$ be distinct Fano 3-folds among 105 families. Let $M_1$ and $M_2$ be the corresponding Calabi-Yau 3-folds obtained by the doubling construction (see Theorems 2.3 and 2.4). If $M_1$ and $M_2$ have the same Hodge numbers $(h^{1,1}, h^{2,1})$, are they also diffeomorphic?

However the answer to Question 1 is negative by the following.
Theorem 1.1. Let $V_1 = \mathbb{C}P^3$ and $V_2$ be a smooth quartic hypersurface in $\mathbb{C}P^4$. Then the corresponding doubling Calabi-Yau 3-folds have the same Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 86)$ as in [DY14, Table 6.1]. However they are not diffeomorphic.

See Section 4 for more precise description.

2. Calabi-Yau constructions

2.1. Algebro-geometric approach. The smoothing problem of normal crossing variety was pioneered by R. Friedman in the early 1980’s [Fr83]. He proved a smoothing theorem for normal crossing complex surfaces and showed that any $d$-semistable $K3$ surface $X$ has a family of smoothings $\varphi : \mathcal{X} \to \Delta \subset \mathbb{C}$ with $K_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$, where $\mathcal{X}$ is a 3-dimensional complex manifold and $\varphi$ is a proper map between $\mathcal{X}$ and a domain $\Delta$ in $\mathbb{C}$. This result was generalized by Kawamata and Namikawa to higher dimensional case [KN94]. They used $T^1$-lifting property for proving the unobstructedness of the deformation space of smooth Calabi-Yau $n$-folds [Ra92, Kaw92]. In particular we have to require the cohomological condition $H^{n-1}(X, \mathcal{O}_X) = 0$ in order to carry out $T^1$-lifting property. Recall that a complex simple normal crossing (SNC for short) variety $X = \bigcup_{i=1}^N Y_i$ is
$d$-semistable if

$$
\left( \bigoplus_{i=1}^{N} \mathcal{I}_{Y_{i}}/\mathcal{I}_{Y_{i}} \right)^* \simeq \mathcal{O}_{D}
$$

for $D = \text{Sing} X$, where $\mathcal{I}_{Y_{i}}$ and $\mathcal{I}_{D}$ are the ideal sheaves of $X_{i}$ and $D$ in $X$ respectively. If $N = 2$ (i.e. $X = Y_{1} \cup Y_{2}$), then $X$ is $d$-semistable if and only if

(2.1) $$
\mathcal{N}_{D/Y_{1}} \otimes \mathcal{N}_{D/Y_{2}} \simeq \mathcal{O}_{D}
$$

where $\mathcal{N}_{D/Y_{i}}$ is the normal bundle of $D$ in $Y_{i}$. A degeneration $\varpi : \mathcal{X} \to \Delta$ of $X$ is semistable if the total space $\mathcal{X}$ is smooth and the central fiber $X_{0} := \varpi^{-1}(0) \cong X$ is Kähler. It is known that a SNC variety $X$ is $d$-semistable if $X$ is the central fiber in a semistable degeneration [Fr83, Corollary 1.12]. Although the converse is not true in general, in the case of Calabi-Yau manifolds, the theorem of Kawamata and Namikawa states the following.

**Theorem 2.1** (Kawamata-Namikawa). Let $X = \bigcup_{i=1}^{N} Y_{i}$ be a proper simple normal crossing (SNC) variety with $\dim_{\mathbb{C}} X = n \geq 3$ satisfying the following conditions:

1. The dualizing sheaf is trivial, that is, $\omega_{X} \cong \mathcal{O}_{X}$.
2. $H^{n-1}(X, \mathcal{O}_{X}) = 0$ and $H^{n-2}(Y_{i}, \mathcal{O}_{Y_{i}}) = 0$ for any $i$.
3. $X$ is $d$-semistable.

Then $X$ has a family of smoothings $\varpi : \mathcal{X} \to \Delta$ with the smooth total space.

**Remark 1.** Originally it is assumed that $X$ is Kähler in [KN94, Theorem 4.2]. However we only need to assume that $X$ is a proper SNC variety. See Remark 2.9 in [HS19] for more details. In particular, Hashimoto and Sano constructed infinitely many topological types (and hence diffeomorphic classes) of non-Kähler compact complex 3-fold with trivial canonical bundle [HS19, Theorem 1.2].

Later on Lee investigated log deformations of SNC varieties consisting of two irreducible components which occur quite often. Moreover he described the Picard group of a Calabi-Yau $n$-fold with $n \geq 3$ obtained by Theorem 2.1.

For our purpose, we focus on complex three dimensional case and the case where the central fiber has only two components, that is, $X = Y_{1} \cup Y_{2}$. Let $D := Y_{1} \cap Y_{2}$. Then $X$ is projective (and so is Kähler) if and only if there are ample divisors $H_{1}$ on $Y_{1}$ and $H_{2}$ on $Y_{2}$ such that $H_{1}|_{D}$ is linearly equivalent to $H_{2}|_{D}$. In particular, the above conditions (1)–(3) in Theorem 2.1 are equivalent to the following conditions:

1. $D$ is an anticanonical divisor on each $Y_{i}$: $D \in | - K_{Y_{i}} |$.
2. $H^{1}(Y_{i}, \mathcal{O}_{Y_{i}}) = H^{1}(D, \mathcal{O}_{D}) = 0$.

By condition (1) and the adjunction formula, we have

$$
K_{D} = (K_{X} \otimes [D])|_{D} = [-D]|_{D} \otimes [D]|_{D} \simeq \mathcal{O}_{D}.
$$

Thus condition (2)’ yields that $D$ is a $K3$ surface. As we have seen in (2.1), we require

(3)’ $$
\mathcal{N}_{D/Y_{1}} \otimes \mathcal{N}_{D/Y_{2}} \simeq \mathcal{O}_{D}
$$

for $X$ to be $d$-semistable. Then by Theorem 2.1, $X$ is smoothable to a complex 3-fold $M_{t} := \varpi^{-1}(t)$ with trivial canonical bundle and $H^{i}(M_{t}, \mathcal{O}_{M_{t}}) = H^{0}(M_{t}, \Omega^{i}_{M_{t}}) = 0$ for $i = 1, 2$. Especially $M_{t}$ is a Calabi-Yau 3-fold. Lee gave the following description of $H^{2}(M_{t}, \mathbb{Z})$ in terms of the normal crossing central fiber. 


Theorem 2.2 (Corollary 8.2 in [Lee10]). Let $M_t$ be the Calabi-Yau 3-fold obtained in the above and $k = \text{rank }(H^2(M_t, \mathbb{Z}))$. Assume that there are some elements $\alpha_1, \ldots, \alpha_k$ in $G := \{ (\ell, \ell') \in H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z}) \mid i^*_1(\ell) = i^*_2(\ell') \}$ and $\beta_1, \ldots, \beta_k$ in $\{ (\tilde{\ell}, \tilde{\ell}') \in H^4(Y_1, \mathbb{Z}) \times H^4(Y_2, \mathbb{Z}) \mid i^*_1(\tilde{\ell}) = i^*_2(\tilde{\ell}') \}$ such that the $k \times k$ matrix $(\alpha_i \cdot \beta_j)$ is unimodular, where $i_a : D \to Y_a$ ($a = 1, 2$) is the inclusion map. Then
\begin{equation}
H^2(M_t, \mathbb{Z}) \cong G / \langle D, -D \rangle
\end{equation}
up to torsion with the cup products being preserved.

2.2. Differential-geometric approach. In [DY14], we gave a differential-geometric construction of Calabi-Yau 3-folds building upon the work of Kovalev [Kov03] and Kovalev-Lee [KL11]. Ingredients in our construction are admissible pairs. Here an admissible pair $(Y, D)$ consists of a 3-dimensional compact Kähler manifold $Y$ and a smooth anticanonical $K3$ divisor $D$. More precisely $(Y, D)$ is said to be an admissible pair if the following conditions hold:

(a) $Y$ is a 3-dimensional compact Kähler manifold.
(b) $D$ is a smooth anticanonical divisor on $Y$.
(c) The normal bundle $N_{Y/D}$ is trivial.
(d) $Y$ and $Y \setminus D$ are simply-connected.

Moreover one can construct admissible pairs (which are called of Fano type) using a Fano 3-fold with a smooth anticanonical $K3$ divisor.

Theorem 2.3 (Proposition 6.42 in [Kov03]). Let $V$ be a Fano 3-fold, $D \in |-K_V|$ a $K3$ surface, and let $C$ be a smooth curve in $D$ representing the self-intersection class of $D \cdot D$. Let $\varphi : Y = \text{Bl}_C(V) \dashrightarrow V$ be the blow-up of $V$ along the curve $C$. Taking the proper transform of $D$ under the blow-up $\varphi$, we still denote it by $D$. Then $(Y, D)$ is an admissible pair.

Let $(Y, D)$ be an admissible pair. Let $N = N_{D/Y}$ denote the normal bundle. Then there exists a Calabi-Yau structure on $Y \setminus D$ asymptotic to a cylindrical Calabi-Yau structure $N \setminus D$ due to the work of [HHN15]. Hence one can obtain a compact manifold $M$ by gluing two copies of $Y \setminus D$ along their cylindrical ends. Furthermore we have the following.

Theorem 2.4 (Corollary 3.12 in [DY14]). Let $(Y, D)$ be an admissible pair of Fano type with $\dim_C Y = 3$. Then we can construct the Calabi-Yau 3-fold by gluing two copies of $Y \setminus D$ along their cylindrical ends.

We call the resulting manifold $M$ the doubling Calabi-Yau 3-fold. Combining Theorems 2.3 and 2.4 we can construct Calabi-Yau 3-folds systematically starting from Fano 3-folds.

Proposition 2.1. Let $M$ be the doubling Calabi-Yau 3-fold obtained by an admissible pair of Fano type. Then the number of distinct topological types of $(h^{1,1}(M), h^{2,1}(M))$ is 61.

Proof. See Figure 1 and Tables 6.1–6.5 in [DY14].
3. DIFFEOMORPHISM CLASSES OF THREE DIMENSIONAL CALABI-YAU MANIFOLDS

In the rest of this paper, we shall consider the diffeomorphism classes of 3-dimensional Calabi-Yau manifolds. Hence our main concern is to distinguish given Calabi-Yau 3-folds by their diffeomorphism type. A good way to achieve this is to use the classification of closed simply-connected 6-manifolds which is very complete. Remark that if $M$ and $N$ are homotopy equivalent through a map which preserves the characteristic classes of the tangent bundle, then they are diffeomorphic. This technique also works for 5-manifolds, but it is false in real dimension 4 and 7. In a word, two closed simply-connected smooth 6-manifolds that are homeomorphic are necessarily diffeomorphic. The spin case is due to C.T.C Wall [Wa66] where he described the invariants that determine the diffeomorphism classes of simply-connected, spin, oriented, closed 6-manifold with torsion free cohomology. The following characterization is due to Jupp [J73], that is, we can classify any compact simply-connected complex 3-folds $M$ up to diffeomorphism by

(i) a symmetric trilinear form (cubic form)
\[
\mu : H^2(M, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) \longrightarrow H^6(M, \mathbb{Z}) \cong \mathbb{Z}
\]
(3.1)
where $\cup$ denotes the cup product of differential forms, and

(ii) the Chern classes of $M$.

See [OkVa95, KW14, BI15] for further details. In Section 4 we pick up specific two Calabi-Yau 3-folds $M$ and $M'$ from the list in [DY14] and prove that they are not diffeomorphic. Note that we still can not conclude that the cubic forms of $M$ and $M'$ are non-isomorphic even though the values of (3.1) for each are different. This is because the values of cubic forms may change if we use different basis of $H^2(M, \mathbb{Z})$ (resp. $H^2(M', \mathbb{Z})$). Therefore we still need to show that the cubic forms for each $M$ and $M'$ are not isomorphic after calculating them for some specific basis. In order to solve this problem, we use the following $\lambda$-invariant defined in [Lee20] to distinguish $M$ and $M'$ up to diffeomorphism.

Let $M$ be a compact complex 3-fold with $h^2(M) := \text{rank}(H^2(M, \mathbb{Z})) = 2$ whose second Chern class $c_2(M)$ is not zero in
\[
H^4(M, \mathbb{Z})_f = H^4(M, \mathbb{Z})/H^4(M, \mathbb{Z})_t
\]
where $H^4(M, \mathbb{Z})_t$ denotes the torsion part of $H^4(M, \mathbb{Z})$. Then the subgroup
\[
\{ \ell \in H^2(M, \mathbb{Z})_f \mid c_2(M) \cdot \ell = 0 \}
\]
of $H^2(M, \mathbb{Z})_f$ is generated by a single element and we denote it by $m$. In [Lee20], Lee defined the $\lambda$-invariant of $M$ by
\[
\lambda(M) := |m^3|
\]
in order to distinguish whether given two compact complex 3-folds $M$ and $M'$ are homeomorphic (i.e. diffeomorphic) or not. Namely we deduce the following.

**Proposition 3.1.** Let $M$ and $M'$ be two Calabi-Yau 3-folds with $h^2(M) = h^2(M') = 2$. We assume that their second Chern classes are not zero. If the values of the $\lambda$-invariants for $M$ and $M'$ are different, then the cubic forms of $M$ and $M'$ are not isomorphic. In particular, $M$ and $M'$ are not diffeomorphic (i.e. non-homeomorphic).

We will give explicit computations on these invariants in the following section. In order to describe two dimensional cohomology, we apply Theorem 2.2.
4. Computation of Cubic Forms and the $\lambda$-invariants

Let $M$ and $M'$ be two simply-connected doubling Calabi-Yau 3-folds with the same Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 86)$. More precisely we consider the following two cases:

Case I. Starting from $\mathbb{CP}^3$ with a smooth an canonical $K3$ divisor $D \in |\mathcal{O}_{\mathbb{CP}^3}(4)|$, we take a smooth curve $C$ in $D$ representing the self-intersection class of $D \cdot D$. Let $\varphi : Y \dasharrow \mathbb{CP}^3$ be the blow-up of $\mathbb{CP}^3$ along the curve $C$. Taking the proper transform of $D$ under the blow-up $\varphi$, we still denote it by $D$. By applying Theorem \ref{thm:main}, we obtain the doubling Calabi-Yau 3-fold $M$ with $(h^{1,1}, h^{2,1}) = (2, 86)$ which is labeled as No.2 in \cite[Table 6.1]{DY14}.

Case II. Let $V := V(4) \subset \mathbb{CP}^4$ be a smooth quartic hypersurface, $D \in |- K_V|$ a smooth an canonical $K3$ divisor. By repeating the procedure in Case I, we obtain the Calabi-Yau 3-fold $M'$ having the same Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 86)$ which is labeled No.17 in \cite[Table 6.1]{DY14}.

Hence it is natural to ask whether they are diffeomorphic to each other. The main result of this paper is to show the following statement.

**Theorem 4.1.** Let $M$ and $M'$ be as above. Let $e_i$ and $f_i$ $(i = 1, 2)$ be the generators of $H^2(M, \mathbb{Z})$ and $H^2(M', \mathbb{Z})$ respectively. Then the cubic products of each doubling Calabi-Yau 3-fold are:

\[
\begin{align*}
(e_1^3 &= 2, \quad e_1^2e_2 = 4, \quad e_1e_2^2 = 0, \quad e_2^3 = 0, \quad \lambda(M) = 4320 \quad \text{and} \\
f_1^3 &= 8, \quad f_1^2f_2 = 4, \quad f_1f_2^2 = 0, \quad f_2^3 = 0, \quad \lambda(M') = 540.
\end{align*}
\]

In particular, $M$ and $M'$ are not diffeomorphic.

**Proof.** According to Proposition \ref{prop:invariants}, we deduce the second statement from (4.1). We shall find the values of invariants in (4.1) by explicit computations.

Case I. Let $V$ be the 3-dimensional projective space $\mathbb{CP}^3$. Let $D \in |\mathcal{O}_V(4)|$ be a smooth quartic $K3$ divisor and $C$ a smooth curve in $D$ representing $D \cdot D$. Taking $Y_1$ to be the blow-up $\text{Bl}_C(V)$ of $V$ along $C$, we denote the exceptional divisors $E_i := \pi_i^{-1}(C)$ for $i = 1, 2$ where $\pi_i : Y_i \dasharrow V$. Then the cohomology ring $H^*(Y_i) = H^*(Y_i, \mathbb{C})$ can be computed as

\[
H^2(Y_i) = \pi_i^*H^2(V) \oplus \mathbb{C}\langle E_i \rangle = \mathbb{C}\langle H_i, E_i \rangle
\]

where $H_i = \pi_i^*(H) \subset Y_i$ for the ample generator $H \in H^2(Y, \mathbb{Z})$ (see \cite[p.621]{GH78}). A straightforward computation shows that the proper transform $\tilde{D}_i$ of $D_i$ in $Y_i$ is $4H_i - E_i$ for each $i$. Let $\delta = \langle -\tilde{D}_1, \tilde{D}_2 \rangle = \langle E_1 - 4H_1, 4H_2 - E_2 \rangle$. Here and hereafter we denote the proper transforms $\tilde{D}_i$ by $D$ which can be regarded as a divisor in $Y = Y_1 \cup Y_2$. Since $H^2(Y_i, \mathbb{Z}) = \langle H_i, E_i \rangle$, any element $\ell_i \in H^2(Y_i, \mathbb{Z})$ for $i = 1, 2$ is written as

\[
\ell_1 = aH_1 + bE_1, \quad \text{and} \quad \ell_2 = cE_2 + dH_2 \quad (a, b, c, d \in \mathbb{Z}).
\]

Thus the condition $i_1^*(\ell_1) = i_2^*(\ell_2)$ for the inclusion map $i_\alpha : D \to Y_\alpha$ $(\alpha = 1, 2)$ implies that $a + 4b = 4c + d$ which means $d = a + 4b - 4c$. Then we see that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ can be expressed as

\[
(aH_1 + bE_1, cE_2 + (a + 4b - 4c)H_2) = (a + 4b)(H_1, H_2) - (b + c)(4H_1 - E_1, 0) - c\delta.
\]
Recall that the cup product is given by the rule
\[
H^2(Y, \mathbb{Z}) \times H^2(Y, \mathbb{Z}) \times H^2(Y, \mathbb{Z}) \to \mathbb{Z}
\]
\[
((l_1, l_2), (m_1, m_2), (n_1, n_2)) \mapsto l_1m_1n_1 + l_2m_2n_2.
\]
We remark that the condition in Theorem 2.2 has been checked in [Lee20] p.215. Hence (2.2) yields that
\[
H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (4H_1 - E_1, 0) \rangle
\]
up to torsion, where \(M\) is the doubling Calabi-Yau 3-fold in [DY14, Table 6.1, No.17]. Thus we can take \(e_1 = (H_1, H_2)\) and \(e_2 = (4H_1 - E_1, 0)\) as generators of \(H^2(M, \mathbb{Z})\).

Next we shall calculate the cubic products in \(H^{2*}(M, \mathbb{Z})\). Let \(L\) be a fiber over a point on \(C\) under the blow-up \(\pi_1\). Then
\[
H_1^3 = 1, \quad H_1L = 0, \quad E_1L = -1 \quad \text{and} \quad H^2E_1 = 0
\]
by [GH78] p.620. Since a hyperplane in \(\mathbb{CP}^3\) meets \(C\) in 16 points, we find \(H_1E_1 = 16L\). Moreover
\[
E_1^2 = -16H_1^2 + (4 \cdot 16 + (-K_{\mathbb{CP}^3}^3))L = -16H_1^2 + 128L
\]
by the table in [GH78] p.623. Then (4.2) and (4.3) imply that
\[
H_1E_1^2 = -16H_1^3 + 128H_1L = -16 \quad \text{and} \quad E_1^3 = -16H_1^2E_1 + 128E_1L = -16 \cdot 0 + 128 \cdot (-1) = -128.
\]
Consequently we see that
\[
e_1^3 = (H_1, H_2)^3 = H_1^3 + H_2^3 = 1 + 1 = 2,
\]
\[
e_1^2e_2 = (H_1, H_2)^2(4H_1 - E_1, 0) = H_1^2(4H_1 - E_1) = 4H_1^3 - H_2E_1 = 4
\]
\[
e_1e_2^2 = (H_1, H_2)(4H_1 - E_1, 0)^2 = H_1^2(4H_1 - E_1)^2 = H_1(16H_1^2 - 8H_1E_1 + E_1^2)
\]
\[
= 16H_1^3 + H_1E_1^2 = 16 - 16 = 0,
\]
\[
e_2^3 = (4H_1 - E_1)^3 = 64H_1^3 - 48H_1^2E_1 + 12H_1E_1^2 - E_1^3
\]
\[
= 64 - 0 + 12 \cdot (-16) + 128 = 0.
\]

Now we compute the \(\lambda\)-invariant. For this we first need to find the Chern classes as follows. According to [Lee10] Section 7, let us denote \(c_2(M) = (c_2(Y_1), c_2(Y_2))\). By the blow-up formula of Chern classes [GH78] p.610, we see that
\[
c_2(Y_i) = \pi_i^*(c_2(\mathbb{CP}^3) + \eta_C) - \pi_i^*c_1(\mathbb{CP}^3) \cdot E_i
\]
\[
= (6H_1^2 + 16H_1^2 - 4H_1E_i = 22H_1^2 - 4H_1E_i
\]
for \(i = 1, 2\), where \(\eta_C \in H^4(Y_i, \mathbb{Z})\) is the class of the blow-up center \(C\). Then the products of \(c_2(M)\) and \(e_i\) \((i = 1, 2)\) are
\[
e_1 \cdot c_2(M) = H_1 \cdot c_2(Y_1) + H_2 \cdot c_2(Y_2)
\]
\[
= 22H_1^3 - 4H_1^2E_1 + 22H_2^3 - 4H_2^2E_2 = 44 \cdot 4 \cdot 11
\]
and
\[
e_2 \cdot c_2(M) = (4H_1 - E_1) \cdot c_2(Y_1) + 0 = 88H_1^3 + 4H_1E_1^2 = 24 \cdot 4 \cdot 6.
\]
Hence the subgroup (3.2) is generated by a unique element:
\[
\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \} = \langle 6e_1 - 11e_2 \rangle.
\]
Consequently we find that
\[ \lambda(M) = |(6e_1 - 11e_2)^3| = |216e_1^3 - 1188e_2^2e_1| = 4320 \]
by the results in (4.4).

**Case II.** Let \( V \) be a quartic hypersurface in \( \mathbb{C}P^4 \). Note that \( V \) is the Fano 3-fold with the Picard number one and \( -K_V^3 = 4 \) (see [IsPr99, p.215]). By Lefschetz Hyperplane Theorem, we have more specific description of \( V \) such as

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & & \\
\end{array}
\]

\[
h^{p,q}(V) = 0 \quad 30 \quad 30 \quad 0 \\
g = g(V) = \frac{H^3}{2} + 1 = \frac{-K_V^3}{2} + 1 = 3
\]

where \( g \) denotes the genus of Fano variety. In particular, \( H^3 = 4 \) for the ample generator \( H \in H^2(V, \mathbb{Z}) \). Let \( D \in | - K_V | \) a smooth divisor, \( C \in |O_D(1)| \) a smooth curve in \( D \). Then the degree of \( C \) is \( 2g - 2 \) and this is the reason why \( g = \frac{-K_V^3}{2} + 1 \) is called the genus of a Fano 3-fold [IsPr99, p.32]. Taking \( Y_i \) to be the blow-up \( \text{Bl}_C(V) \) of \( V \) along \( C \), we again denote the exceptional divisors by \( E_i \) for \( i = 1, 2 \). The cohomology ring is

\[ H^2(Y_i) = \mathbb{C}\langle \pi_i^*(H), E_i \rangle = \mathbb{C}\langle H_i, E_i \rangle \]

and the proper transforms \( D_i \) in \( Y_i \) are \( H_i - E_i \). Thus \( \delta = \langle -D_1, D_2 \rangle = \langle E_1 - H_1, H_2 - E_2 \rangle \). In the same way as Case I, we see that any element in \( H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z}) \) is written as

\[ (aH_1 + bE_1, cE_2 + (a + b - c)H_2) = (a + b)(H_1, H_2) - (b + c)(H_1 - E_1, 0) - c\delta. \]

Moreover one can check that the condition in Theorem 2.2 almost in the same way as in [Lee20, p.215]. Thus we conclude that

\[ H^2(M', \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0) \rangle \]

up to torsion. Hence in this case, we take \( f_1 = (H_1, H_2) \), \( f_2 = (H_1 - E_1, 0) \) as generators of \( H^2(M', \mathbb{Z}) \).

Let \( L \) be a fiber over a point on \( C \) under the blow-up \( \pi_1 \). Since the intersection number is preserved by the total transform, we see that \( H_1^3 = (\pi_1^*H)^3 = H^3 = 4 \). Moreover, \( H_1L = 0 \) and \( E_1L = -1 \) in the same way as Case I. Let \( d \) be the degree of \( C \). Since a hyperplane in \( V \) will intersect \( C \) in \( d \) points, its inverse image \( H_1 \) in \( Y_1 \) will meet the exceptional divisor \( E_1 \) in \( d \) fibers. Thus

\[ H_1E_1 = dL = (2g - 2)L = 4L \quad \text{and} \quad E_1^2 = -4H_1^2 + 8L. \]

Then we see that

\[ H_1^2E_1 = 4H_1L = 0, \quad H_1E_1^2 = 4E_1L = -4 \quad \text{and} \quad E_1^3 = -4H_1^2E_1 + 8LE_1 = -8. \]

In sum, we find the following table of the multiplication of the intersection form on \( H^{2*}(Y_1, \mathbb{Z}) \):
Plugging these values into the products, we find that
\[ f_1^3 = (H_1, H_2)^3 = H_1^3 + H_2^3 = 8, \]
\[ f_1^2 f_2 = (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 4, \]
\[ f_1 f_2^2 = (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = 4 - 4 = 0, \]
\[ f_2^3 = (H_1 - E_1)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = 4 - 0 + 3 \cdot (-4) - (-8) = 0. \]

Next we calculate the \( \lambda \)-invariant. Since \( V \) is a degree 4 smooth hypersurface in \( \mathbb{C}P^4 \), the total Chern classes of \( V \) are given by the formula
\[ (1 + 5H + 10H^2)(1 - 4H + 16H^2) = 1 + H + 6H^2 + 40H^3. \]
Hence we find that
\[ c_2(Y_i) = \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i = 7H_i^2 - H_i E_i \]
by [GH78, p.610], where \( \eta_C \) denotes the class of the blow-up center \( C \in |O_D(1)|. \) Then the products of \( c_2(M') \) and \( f_i \) (\( i = 1, 2 \)) are
\[ f_1 \cdot c_2(M') = 7H_1^3 - H_1^2 E_1 + 7H_2^2 - H_2^2 E_2 = 56 = 8 \cdot 7, \]
\[ f_2 \cdot c_2(M') = (7H_1^2 - H_1 E_1)(H_1 - E_1) \]
\[ = 7H_1^3 - H_1^2 E_1 - 7H_1^2 E_1 + H_1 E_1^2 \]
\[ = 7 \cdot 4 - 4 = 24 = 8 \cdot 3. \]

Since the subgroup \( \{ f \in \langle f_1, f_2 \rangle \ | \ f \cdot c_2(M') = 0 \} \) of \( H^2(M', \mathbb{Z}) \) is generated by a single element \( 3f_1 - 7f_2 \), the \( \lambda \)-invariant of \( M' \) is
\[ \lambda(M') = |(3f_1 - 7f_2)|^3 = |27f_1^3 - 189f_1^2 f_2 + 441f_1 f_2^2 - 343f_2^3| \]
\[ = |27 \cdot 8 - 189 \cdot 4| = 540. \]

The assertion then follows from Proposition 3.1 \( \square \)

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