Stability of Generalised Static Black Holes in Higher Dimensions

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Abstract

We discuss the stability of (charged) static black holes in higher-dimensional spacetimes with and without cosmological constant by using gauge-invariant master equations of the Schrödinger equation type for black hole perturbations derived by the authors recently. In particular, we show that the stability of higher-dimensional Schwarzschild black holes can be proved with the help of a technique called \textit{S-deformation} of the master equations. We also point out that higher-dimensional static black holes might be unstable only against scalar-type perturbations in the neutral case and in the charged case with spherically symmetric or flat horizons.

1 Introduction

The perturbation analysis of 4-dimensional black holes has a long history and has provided valuable information on various problems such as the fate of gravitational collapse, the stability and uniqueness of black holes, cosmic censorship, and gravitational wave emissions. From a technical point of view, the key point in this perturbation analysis was the fact that the perturbation equations can be reduced to a single 2nd-order ordinary differential equation (ODE) for a master variable $\Phi$ of the form

$$-\frac{d^2\Phi}{dr^2} + [V(r) - \omega^2] \Phi = S. \quad (1)$$

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Recently, starting from a gauge-invariant formalism for perturbations of higher-dimensional spacetimes with spatial symmetry\(^1\), we showed that similar reduction to single master equations is also possible for perturbations of generalised static black holes with or without charge in higher dimensions\(^2, 3\). Further, using these master equations, we analysed the stability of such black holes\(^4, 3\).

In this talk, we explain the basic idea of this formulation and describe the results of the stability analysis of (charged) static black holes in higher dimensions.

2 Tensorial Decomposition of Perturbations

2.1 Unperturbed Background

We assume that the unperturbed background spacetime is locally written \(\mathcal{M}^D \approx N^2 \times K^n\) and its metric has the form

\[
\begin{align*}
    ds^2 &= g_{ab}(y)dy^a dy^b + r^2(y)d\sigma^2_n, \\
    g_{ab} &= \delta g_{ab},
\end{align*}
\]

where \(d\sigma^2_n = \gamma_{ij}(z)dz^idz^j\) is a metric of an \(n\)-dimensional complete Einstein space \(K^n\) with \(\hat{R}_{ij} = (n-1)\hat{K}\gamma_{ij}\) (\(K = 0, \pm 1\)), and \(g_{ab}\) is a static metric of the 2-dimensional space \(N^2\) expressed as

\[
    g_{ab}dy^a dy^b = -f(r)dt^2 + \frac{dr^2}{f(r)}. 
\]

In this and the next section, we assume that this metric is a solution to the vacuum Einstein equations with cosmological constant \(\Lambda = \frac{n(n+1)}{2}\). Hence, \(f(r)\) is expressed as

\[
    f(r) = K - \frac{2M}{r^{n-1}} - \lambda r^2. 
\]

When the spacetime contains a regular black hole, the space \(K^n\) represents a spatial section of the horizon. For \(K = 1\), \(K^n\) is always compact from Meyers’ theorem and for \(n \leq 3\), \(K^n\) is locally isomorphic to \(S^n\). The spacetime contains a regular black hole if \(\lambda M^2/(n-1) < (n-1)/(n+1)(n+1)/(n-1)\). In contrast, for \(K = 0\) or \(-1\), \(K^n\) may not be compact, and the spacetime contains a regular black hole only for \(\lambda < 0\).

2.2 Perturbations

In terms of the perturbation variable \(\psi_{\mu\nu} = h_{\mu\nu} - hg_{\mu\nu}/2\) with \(h_{\mu\nu} = \delta g_{\mu\nu}\), the perturbed vacuum Einstein equations are written as

\[
-\nabla^2 \psi_{\mu\nu} - 2R_{\mu\alpha\nu\beta}\psi^{\alpha\beta} + 2\nabla(\mu \nabla^{\alpha}\psi_{\nu})_{\beta} - \nabla^{\alpha}\nabla^{\beta}\psi_{\alpha\beta}g_{\mu\nu} = 0. 
\]

In order to analyse the behaviour of perturbations using this equation, we must solve two problems. First, this equation is invariant under the gauge transformation

\[
x^\mu \to x^\mu + \xi^\mu \Rightarrow \delta h_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. 
\]
In order to extract the dynamics of the physical degrees of freedom, this gauge freedom should be eliminated. Second, 5 is actually a set of coupled equations with $(n+2)(n+3)/2$ entries and is quite hard to analyse in general.

For the special background 2, the second problem can be made tractable with the help of the following tensorial decomposition of $h_{\mu\nu}$. First, according to the tensorial behaviour on $K^n$, $h_{\mu\nu}$ can be divided into scalars $h_{ab}$, vectors $h_{ai}$ and a tensor $h_{ij}$. The vector and tensor components can be further decomposed as

$$h_{ai} = \hat{D}_i h_a + h_{ai}^{(1)}; \quad \hat{D}^j h_{ai}^{(1)} = 0,$$

$$h_{ij} = h_L \gamma_{ij} + h_{ij}^{(0)}; \quad h_{ij}^{(0)} = 0,$$

$$h_{T ij} = \left( \hat{D}_i \hat{D}_j - \frac{1}{n} \gamma_{ij} \hat{\Delta} \right) h_{T}^{(0)} + 2 \hat{D}_i h_{T j}^{(1)} + h_{T ij}^{(2)}; \quad \hat{D}^j h_{T j}^{(1)} = 0, \quad \hat{D}^j h_{T ij}^{(2)} = 0,$$

where $\hat{D}_i$ is the covariant derivative with respect to the metric $\gamma_{ij}$ on $K^n$ and $\hat{\Delta} = \hat{D} \cdot \hat{D}$. By this decomposition, we obtain the following three groups of variables:

- Scalar-type variables: $h_{ab}$, $h_a$, $h_L$, $h_T^{(0)}$.
- Vector-type variables: $h_{ai}^{(1)}$, $h_{T i}^{(1)}$.
- Tensor-type variable: $h_{T ij}^{(2)}$.

The Einstein equations written in terms of these variables are divided into three subsets each of which contains only variables belonging to one of the above three sets of variables. Further, the tensorial indices for vector-type and tensor-type variables can be eliminated if we expand these variables in terms of harmonic tensors on $K^n$, and gauge-invariant variables can be easily constructed from the harmonic expansion coefficients, as we will show below. In this way, the Einstein equations for perturbations of each type can be reduced to a set of gauge-invariant equations with a small and $n$-independent number of entries 4. In particular, for tensor perturbations, this procedure gives a single wave equation for a single variable.

3 Stability Analysis

3.1 4D Schwarzschild black hole

In a four-dimensional background spacetime with a Schwarzschild black hole, it was shown by Regge and Wheeler 7 and Zerilli 8 that by the harmonic expansion on $K^2 = S^2$ and the Fourier transformation with respect to time, $h_{ab} = f_{ab}(r)e^{-i\omega t}Y_{lm}(\theta, \phi), \ldots$, the perturbation equations for any type of perturbations can be eventually reduced to a single Schrödinger-type second-order ODE

$$- \frac{d^2 \Phi}{dr_*^2} + V(r) \Phi = \omega^2 \Phi,$$

where $r_* = \int^r dr/f(r)$ and the master variable $\Phi$ is a combination of the harmonic expansion coefficients $h_{ab}, \ldots$. The effective potential $V = \cdot \cdot \cdot$
V for vector perturbations (axial modes) is given by the Regge-Wheeler potential
\[ V_V = \frac{f}{r^2} \left( m + 2 - \frac{6M}{r} \right), \]
and the effective potential \( V = V_S \) for scalar perturbations (polar modes) is given by the Zerilli potential
\[ V_S = \frac{f}{r^2 H^2} \left( \frac{m^2 (m + 2)}{r^2} + \frac{6m M^2}{r^2} + \frac{72M^3}{r^3} \right), \]
where \( m = (l - 1)(l + 2)(l = 2, 3, \cdots) \) and \( H = m + 6M/r \). A tensor-type perturbation does not exist for \( n = 2 \).

Since the effective potentials \( V_V \) and \( V_S \) are positive outside the horizon (\( r > 2M \)), the master equations for \( \Phi \) have no regular and normalizable solution for \( \omega^2 < 0 \). Hence, the 4D Schwarzschild black hole is perturbatively stable[9].

Similar master equations for 4D black holes with \( \Lambda \neq 0 \) were derived by Cardoso and Lemos[10, 11]. With the help of these equations, the stability of these black holes were proved by us in Ref.[2, 4].

### 3.2 Vector perturbation in higher dimensions

Now, we show that even in higher dimensions, we can find a gauge-invariant master variable in terms of which the perturbation equations of each type can be reduced to a single second-order ODE of the Schrödinger type. We first treat vector perturbations.

Vector perturbations can be expanded in terms of vector harmonics on \( K^n \) satisfying
\[ (\hat{D} \cdot \hat{D} + k_V^2)V_i = 0; \quad \hat{D}^i V_i = 0, \]
and harmonic tensors derived from them,
\[ V_{ij} = -\frac{1}{2k_V} \left( \hat{D}_i V_j + \hat{D}_j V_i \right), \]
as
\[ h_{ab} = 0, \quad h_{ai} = rf_a(y) V_i, \quad h_{ij} = 2r^2 H_T(y) V_{ij}. \]
Here, in order to guarantee the completeness of the vector harmonics, we have assumed that \(-\hat{D} \cdot \hat{D}\) is extended to a self-adjoint operator in the \( L^2 \)-space of divergence-free vector fields on \( K^n \) by the Friedrichs extension. Note that \( k_V^2 \geq 0 \) in general, and for \( K = 1, k_V^2 \geq n - 1 \)[12, 13].

For generic modes with \( k_V^2 \neq (n - 1)K \), we can show that the combinations
\[ F_a := f_a + \frac{r}{k_V} D_a H_T \]
give a basis for gauge-invariant variables, where \( D_a \) is the covariant derivative with respect to the metric \( g_{ab} \) on the two-space \( N^2 \). In contrast, for the exceptional modes with \( k_V^2 = (n - 1)K \) (\( K = 0, 1 \)), \( V_{ij} \) vanishes identically and \( H_T \) is not defined. Hence, we obtain a smaller number of independent gauge-invariant variables. Since we can show that these
exceptional modes correspond to adding rotation with some angular momentum and are not dynamical\cite{2}, we do not discuss them in this article.

The vacuum Einstein equations for a vector perturbation are expressed in terms of $F_a$ as

\begin{equation}
D_a \left( r^{n+2} \epsilon^{bc} D_b (F_c/r) \right) - m_V r^{n-1} \epsilon_{ab} F^b = 0, \tag{17}
\end{equation}

\begin{equation}
k_V D_a (r^{-1} F^a) = 0, \tag{18}
\end{equation}

where $m_V = k_V^2 - (n - 1) K$. It is easy to see that these equations are equivalent to

\begin{equation}
r^{n-1} F^a = \epsilon^{ab} D_b \left( r^{n/2} \Phi \right), \tag{19}
\end{equation}

\begin{equation}
d^2 \Phi + V_V(r) \Phi = \omega^2 \Phi, \tag{20}
\end{equation}

where

\begin{equation}
V_V = \frac{f}{r^2} \left[ m_V + \frac{n(n+2)K}{4} - \frac{n(n-2)}{4} \lambda r^2 - \frac{3n^2 M}{2 r^{n-1}} \right]. \tag{21}
\end{equation}

Although the effective potential $V_V$ is not positive definite for large $n$, as is shown in Fig.\ref{fig1}, we can show that there exists no unstable mode, i.e., no $L^2$-normalizable eigenfunction $\Phi$ with $\omega^2 < 0$ by the following argument, which we call the $S$-deformation.

Let $I$ be the range of $r_*$ corresponding to the regular region outside of the horizon, $-\infty < r_* < r_*\infty$. Here, $r_*\infty$ is $+\infty$ for $\lambda \geq 0$, but is finite for $\lambda < 0$. Then, in the space $C_0^\infty (I)$ of smooth functions with compact support, the operator

\begin{equation}
A := -\frac{d^2}{dr_*^2} + V(r) \tag{22}
\end{equation}

Figure 1: Examples of $V_V$ for $\lambda = 0$ and $\lambda < 0$. 
is symmetric. We assume that $A$ is extended to a self-adjoint operator in $L^2(I)$ by the Friedrichs extension. Then, the lower bound for the spectrum of $A$ coincides with the lower bound of

$$\omega^2 = (\Phi, A\Phi)/(\Phi, \Phi), \quad \Phi \in C_0^\infty(I).$$

(23)

Here, for any regular function $S(r)$ on $I$ and $\Phi \in C_0^\infty(I)$, a partial integration yields

$$(\Phi, A\Phi) = \int dr_* \left[ |\tilde{D}\Phi|^2 + \tilde{V}|\Phi|^2 \right]$$

(24)

where

$$\tilde{D} = \frac{d}{dr_*} + S(r),$$

(25)

$$\tilde{V} = V + f \frac{dS}{dr} - S^2.$$  

(26)

Hence, if we can show that $\tilde{V}$ is non-negative for an appropriate $S$, we can conclude that $\omega^2 \geq 0$.

If we apply this method to a vector perturbation using $S = nf/(2r)$, we obtain

$$\tilde{V}_V = m_V f \frac{r}{r^2}.$$  

(27)

Since $m_V = k_V^2 - (n-1)K$ is always non-negative, this implies that the black hole is stable against the vector perturbation in any spacetime dimension $D = n + 2 \geq 4$, irrespective of the values of $K$ and $\lambda$.

### 3.3 Scalar perturbation in higher dimensions

Next, we consider scalar perturbations. Scalar perturbations can be expanded in terms of scalar harmonics on $\mathcal{K}^n$ satisfying

$$\left( \hat{\Delta} + k^2 \right) S = 0,$$

(28)

and harmonic vectors and tensors derived from them,

$$S_i := -\frac{1}{k} \hat{D}_i S,$$

(29)

$$S_{ij} := \frac{1}{k^2} \hat{D}_i \hat{D}_j S + \frac{1}{n} \gamma_{ij} S; \quad S_i^i = 0,$$

(30)

as

$$h_{ab} = f_{ab}^\mathcal{S}, \quad h_{ai} = r f_a S_i, \quad h_{ij} = 2r^2 (H_L \gamma_{ij} S + H_T S_{ij}).$$

(31)

Some comments are in order. First, since we are assuming that $\mathcal{K}^n$ is complete, $-\Delta = -\hat{D} \cdot \hat{D}$ has a unique self-adjoint extension in $L^2(\mathcal{K}^n)$, which coincides with the Friedrichs extension and is non-negative. Second, we do not consider the zero modes ($k^2 = 0$), by assuming that such a mode corresponds to a variation of the parameters of the background solution. This assumption is satisfied when $\mathcal{K}^n$ is compact and closed. Third, for $K = 1$, the second smallest eigenvalue is $k^2 = n$. For this eigenvalue, $S_{ij}$
vanishes. Since we can show that such modes are pure gauge\cite{2}, we do not consider them in this article.

From gauge transformation laws of the perturbation variables, we find that the following combinations can be used as a basis for gauge-invariant variables\cite{1}:

\[
F := H_L + \frac{1}{n} H_T + \frac{D_a r}{r} X^a, \quad F_{ab} := f_{ab} + D_a X_b + D_b X_a, \tag{32}
\]

where

\[
X_a := \frac{r}{k} \left( f_a + \frac{r}{k} D_a H_T \right). \tag{33}
\]

In terms of the gauge-invariant variable Φ constructed from these,

\[
\Phi = \frac{m^{n/2}}{H} \left( 2F + \frac{F'}{\omega r} \right), \tag{34}
\]

\[
H = m + \frac{n(n+1)M}{r^{n-1}}, \quad m = k^2 - nK, \tag{35}
\]

the Einstein equations are reduced to the single equation\cite{2}

\[
-\frac{d^2 \Phi}{dr^*} + V_S(r) \Phi = \omega^2 \Phi. \tag{36}
\]

The effective potential \(V_S\) is given by

\[
V_S(r) = \frac{fU(r)}{16r^2H^2} \tag{37}
\]

with

\[
U(r) = - \left[ n^3(n+2)(n+1)^2x + 12n^2(n+1)(n-2)mx \right.
+4(n-2)(n-4)m^2 \right] x^2 + n^4(n+1)^2x^4
+n(n+1) \left[ 4(2n^2-3n+4)m + n(n-2)(n-4)(n+1)K \right] x^2
-12n \left[ (n-4)m + n(n+1)(n-2)K \right] mx
+16m^3 + 4Kn(n+2)m^2, \tag{38}
\]

where \(x = 2M/r^{n-1}\).

We can easily see that for \(n = 2\) and \(K = 1\), \(V_S\) is positive definite. Hence, 4D spherically symmetric neutral black holes are perturbatively stable, irrespective of the value of the cosmological constant. In contrast, \(V_S\) is not positive definite for larger \(n\) in general, as illustrated in Fig.2. Nevertheless, for \(K = 1\) and \(\lambda = 0\), neutral black holes are perturbatively stable in any spacetime dimensions, because the S-deformation of \(V_S\) by

\[
S = \frac{f}{h} \frac{dh}{dr}, \quad h = r^{n/2+l-1}H \tag{39}
\]

yields

\[
\tilde{V}_S = \frac{frf}{2r^2H} \left[ ln(n+1)x + 2(l-1) \left\{ n^2 + n(3l-2) + (l-1)^2 \right\} \right] > 0. \tag{40}
\]
3.4 Tensor perturbation

In the case in which $\mathcal{K}^n$ is maximally symmetric, the only non-trivial tensor on $\mathcal{K}^n$ is the metric $\gamma_{ij}$. Hence, only the Laplacian appears as the derivative operator in the perturbation equation, and by the harmonic expansion with respect to $\hat{D} \cdot \hat{D}$, the Einstein equations reduce to equations on $N^2$ for any type of perturbation. In contrast, when $\mathcal{K}^n$ is a generic Einstein space, the Weyl curvature couples tensor perturbations, and such a reduction cannot be achieved by the harmonic expansion. However, since the covariant derivatives and the Weyl curvature appear in the Einstein equations for tensor perturbations as the combination called the Lichnerowicz operator

$$\hat{\triangle} L h_{ij} := -\hat{D} \cdot \hat{D} h_{ij} - 2 \hat{R}_{ijkl} h^{kl} + 2(n - 1) K h_{ij},$$

we can utilise the eigentensors of $\hat{\triangle} L$,

$$\hat{\triangle} L T_{ij} = \lambda L T_{ij}; \quad T^i_i = 0, \quad \hat{D}^l T_{ij} = 0.$$  

(42)

Tensor perturbations can be expanded in terms of these eigentensors as

$$h_{ab} = 0, \quad h_{ai} = 0, \quad h_{ij} = 2 r^2 H_T(y) T_{ij}.$$  

(43)

$H_T$ is gauge-invariant by itself, and the Einstein equations are reduced to

$$\Box H_T + \frac{n}{r} DH_T - \frac{\lambda L}{r^2} H_T + \frac{2(n - 1) K}{r^2} H_T = 0.$$  

(44)

In terms of

$$\Phi = r^{n/2} H_T,$$  

(45)

this equation is rewritten in the canonical form

$$-\frac{d^2 \Phi}{dr^2} + V_T(r) \Phi = \omega^2 \Phi,$$  

(46)
where
\[ V_T = \frac{f}{r^2} \left[ \lambda_L - 2(n-1)K + \frac{nrf'}{2} + \frac{n(n-2)f}{4} \right]. \tag{47} \]

Note that this equation holds for any \( f(r) \), if the background metric takes the form (2).

For \( f(r) \) given by (4), (47) can be written as
\[ V_T = \frac{f}{r^2} \left[ \frac{n(n+2)}{4}f + \frac{n(n+1)M}{r^{n-1}} + \lambda_L - (3n-2)K \right]. \tag{48} \]

When \( K^n \) is maximally symmetric, \( \lambda_L \) is related to the eigenvalue \( k^2 \) of \( -\hat{D} \cdot \hat{D} \) by \( \lambda_L = k^2 + 2nK \), and \( \lambda_L - (3n-2)K = (l-1)(l+n) > 0 \) for \( K = 1 \). Hence, \( V_T > 0 \) and the black hole is stable against the tensor perturbation.

In contrast, when \( K^n \) is a generic Einstein space, no general lower bound for \( \lambda_L \) is know, and generalised static black holes can be unstable against a tensor perturbation\cite{12}.

4 Extension to Charged Black Holes

The extension of the formulations by Regge-Wheeler and Zerilli to Reissner-Nordström black holes was done by Moncrief, Zerilli and Chandrasekhar\cite{13,14,15}, and used to prove their stability. In the charged black hole case, we have to take into account perturbations of electromagnetic fields, which couple metric perturbations non-trivially. However, by taking appropriate linear combinations of variables describing these perturbations, the perturbation equations for the Einstein-Maxwell system can be reduced to decoupled master equations, each of which is a single second-order ODE of the Schrödinger equation type. Now, we show that such a reduction is also possible in the case of higher-dimensional static black holes with charge, and discuss their stability\cite{3}.

We assume that an background spacetime is again locally written as \( \mathcal{M}^{n+2} \approx N^2 \times K^n \), and the metric is given by (2) with (3). We further assume that the electromagnetic tensor \( F_{\mu\nu} \) for the background EM field takes the form
\[ F = \frac{1}{2} E_0 \epsilon_{\alpha\beta} dy^\alpha \wedge dy^\beta. \tag{49} \]

Then, the Maxwell equations and the Einstein equations determine \( f(r) \) and \( E_0(y) \) as
\[ f(r) = K - \lambda y^2 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}}, \tag{50} \]
\[ E_0 = \frac{q}{r^n}; \quad Q^2 = \frac{\kappa^2 q^2}{n(n-1)}. \tag{51} \]

4.1 Tensor perturbation

Since the EM field does not couple tensor perturbations of the metric, the master equation for the charged case is again given by (46) with (47). If
we apply the $S$-deformation with $S = -nf/(2r)$ to $V_T$, we obtain

$$\tilde{V}_T = \frac{f}{r^2} [\lambda_L - 2(n - 1)K].$$  \hspace{1cm} (52)

Hence, a generalised static black hole with charge is still stable against tensor perturbations, if $\lambda_L \geq 2(n - 1)K$ or $k_v^2 \geq -2K$ when $K^n$ is maximally symmetric. This condition is satisfied by black holes with flat or spherical horizons. However, the case $K = -1$ ($\lambda < 0$) may be unstable even if $K^n$ is maximally symmetric.

### 4.2 Vector perturbation

From the Maxwell equation $\nabla_\nu \delta F_{\nu\lambda} = 0$, a vector perturbation of the EM field is expanded in terms of the vector harmonics as

$$\delta F_{ab} = 0, \quad \delta F_{ai} = D_a A^i, \quad \delta F_{ij} = A^i \left(\hat{D}_i V_j - \hat{D}_j V_i\right).$$  \hspace{1cm} (53)

Hence, a vector perturbation of the EM field is described by the single variable $A(y)$, which is gauge-invariant by itself. For generic modes $m_V \equiv k_v^2 - (n-1)K \neq 0$, in terms of $A$ and $F_a$, the Einstein equations are written as

$$r^{n-1} F^a = \epsilon^a_b D_b \Omega,$$  \hspace{1cm} (54)

$$r^n D_a \left(\frac{1}{r^n} D^n \Omega\right) - \frac{m_V}{r^2} \Omega = \frac{2n^2 q}{r} A,$$  \hspace{1cm} (55)

and the Maxwell equation $\delta (\nabla_\nu F^{\mu\nu}) = 0$ are written as

$$\frac{1}{r^{n-2}} D_a (r^{n-2} D^n A) - \frac{1}{r^2} \left(k_v^2 + (n - 1)K + \frac{2n(n - 1)Q^2}{r^{2n-2}}\right) A = \frac{q m_V}{r} \Omega.$$  \hspace{1cm} (56)

As shown in [3], by taking linear combinations $\Phi_{\pm} = a_{\pm} r^{-n/2} \Omega + b_{\pm} r^{n/2 - 1} A$ with some constants $a_{\pm}$ and $b_{\pm}$, these perturbation equations for the Einstein-Maxwell system are transformed to the two decoupled equations

$$\frac{d^2 \Phi_+}{dr^2} + V_{V_+}(r) \Phi_+ = \omega^2 \Phi_+.$$  \hspace{1cm} (57)

The effective potentials $V_{V_{\pm}}$ are given by

$$V_{V_{\pm}} = \frac{f}{r^2} \left[k_v^2 + \frac{(n^2 - 2n + 4)K}{4} - \frac{n(n - 2)}{4} \lambda r^2 + \frac{n(5n - 2)Q^2}{4r^{2n-2}} + \frac{\mu_{\pm}}{r^{n-1}}\right],$$  \hspace{1cm} (58)

where

$$\mu_{\pm} := -\frac{n^2 + 2}{2} M \pm \Delta; \quad \Delta := \left[\left(n^2 - 1\right) M^2 + 2n(n - 1) m_V Q^2\right]^{1/2}.$$  \hspace{1cm} (59)

In the limit $Q = 0$, $\Phi_+$ and $\Phi_-$ coincide with $A$ and $\Omega$, respectively. Hence, $\Phi_+$ and $\Phi_-$ represent the EM mode and the gravitational mode of the perturbation, respectively.
By the $S$-deformation with $S = \frac{2N}{r}$, $V_{V \pm}$ are transformed to

$$\tilde{V}_{V \pm} = \frac{f}{r^2} \left[ m_V + \frac{(n^2 - 1)M \pm \Delta}{r^{n-1}} \right].$$  \hspace{1cm} (60)

Hence, $\tilde{V}_{V \pm}$ is always positive. However, since $\Delta > (n^2 - 1)M$ for $m_V > 0$, $\tilde{V}_{V -}$ can become negative.

By examining the behaviour of $\tilde{V}_{V -}$, we find that it is positive outside the horizon for $K = 0$ and $K = 1$, while it becomes negative near the horizon for $K = -1$ if $\lambda$ is sufficiently close to $\lambda_c -$, where $\lambda_c -$ is the lower bound on $\lambda$ for a fixed value of $Q$ when the spacetime contains a regular black hole. Hence, we can conclude that a charged black hole with a flat or spherical horizon is stable against vector perturbations. However, we cannot prove the stability of a charged black hole with hyperbolic horizon by this simple method.

### 4.3 Scalar perturbation

A scalar perturbation of the EM field is expressed as

$$\delta F_{ab} + D_c (E_0 X^c) \epsilon_{ab} S = \epsilon_{ab} \left( -k^2 A + \frac{q}{2} (F_c^c - 2nF) \right) S,$$  \hspace{1cm} (61)

$$\delta F_{ai} - kE_0 \epsilon_{ab} X^b S^i = \frac{k}{r^{n-2}} \epsilon_{ab} D^a A S^i,$$  \hspace{1cm} (62)

$$\delta F_{ij} = 0.$$  \hspace{1cm} (63)

The expansion coefficient $A$ is gauge invariant and obeys the equation

$$r^{n-2}D_a \left( D^a A \right) - \frac{k^2}{r^2} A = -\frac{q}{2n^2} (F_c^c - 2nF).$$  \hspace{1cm} (64)

The Einstein equations for the scalar perturbation are equivalent to

$$f D_a D^a \Phi - V_{S\pm}(r) \Phi = \frac{f P_{S1}(r)}{r^{n/2} H^2} \kappa^2 E_0 A,$$  \hspace{1cm} (65)

where $P_{S1}(r)$ is a function of $r$, and

$$\Phi = \frac{nr^{n/2}}{H} \left( \frac{F_t}{\omega F} + 2F \right),$$  \hspace{1cm} (66)

$$H = m + \frac{n(n + 1)M}{r^{n-1}} - \frac{n^2 Q^2}{r^{2n-2}}; \hspace{0.5cm} m = k^2 - nK.$$  \hspace{1cm} (67)

In terms of the variables $\Phi_\pm$ written as linear combinations of $\Phi$ and $A$, these perturbation equations can be rewritten as the two decoupled equations

$$\frac{d^2 \Phi_\pm}{dr^2} + V_{S\pm}(r) \Phi_\pm = \omega^2 \Phi_\pm.$$  \hspace{1cm} (68)

Here, the effective potentials $V_{S\pm}$ are given by

$$V_{S\pm} = -\frac{f U_\pm}{64r^2 H^2}.$$  \hspace{1cm} (69)
where in terms of the positive constant $\delta$ defined by

$$Q^2 = (n + 1)^2 M^2 \delta (1 + m \delta),$$

(70)

$H_\pm$ are expressed as

$$H_+ = 1 - \frac{n(n + 1) \delta M}{r^{n-1}}, \quad H_- = m + \frac{n(n + 1)(1 + m \delta) M}{r^{n-1}},$$

(71)

and $U_\pm$ are expressed in terms of $x = \frac{2M}{r}$ as

$$U_+ = \left[-4n^3(n + 2)(n + 1)^2 \delta^2 x^2 - 48n^2(n + 1)(n - 2) \delta x
- 16(n - 2)(n - 4) \lambda r^2
- \delta^3 n^3 (3n - 2)(n + 1)^3 (1 + m \delta) x^4
+ 4\delta^2 n^2 (n + 1)^2 \{ (n + 1)(3n - 2)m \delta + 4n^2 + n - 2 \} x^3
+ 4\delta (n + 1) \{ (n - 2)(n - 4)(n + 1)(m + n^2 K) \delta
- 7n^3 + 7n^2 - 14n + 8 \} x^2
+ \{ 16(n + 1)(-4m + 3n^2 (n - 2) K) \delta - 16(3n - 2)(n - 2) \} x
+ 64m + 16n(n + 2) K. \right]$$

(72)

$$U_- = \left[-4n^3(n + 2)(n + 1)^2 (1 + m \delta)^2 x^2
+ 48n^2(n + 1)(n - 2)m(1 + m \delta)x - 16(n - 2)(n - 4)m^2 \right] \lambda r^2
- n^3(3n - 2)(n + 1)^3 \delta (1 + m \delta)^3 x^4
- 4n^2(n + 1)^2 (1 + m \delta)^2 \{ (n + 1)(3n - 2)m \delta - n^2 \} x^3
+ 4(n + 1)(1 + m \delta) \{ m(n - 2)(n - 4)(n + 1)(m + n^2 K) \delta
+ 4n(2n^2 - 3n + 4)m + n^2(n - 2)(n - 4)(n + 1) K \} x^2
- 16m \{ (n + 1)m (-4m + 3n^2 (n - 2) K) \delta
+ 3n(n - 4)m + 3n^2 (n + 1)(n - 2) K \} x
+ 64m^3 + 16n(n + 2)m^2 K. \right]$$

(73)

By the $S$-deformation with

$$S = \frac{f}{h} \frac{dh}{dr}; \quad h = r^{n/2 - 1} H_+,$$

(74)

the effective potential $V_{S+}$ for $\Phi_+$ is transformed to

$$\tilde{V}_{S+} = \frac{k^2 f}{2r^2 H_+} [(n - 2)(n + 1) \delta x + 2].$$

(75)

Since this is positive definite, the EM mode is always stable.

By the similar $S$-deformation

$$S = \frac{f}{h} \frac{dh}{dr}; \quad h = r^{n/2 - 1} H_-,$$

(76)

the effective potential $V_{S-}$ for $\Phi_-$ is transformed to

$$\tilde{V}_{S-} = \frac{k^2 f}{2r^2 H_-} [2m - (n + 1)(n - 2)(1 + m \delta) x].$$

(77)
Figure 3: Examples of $V_{S-}$ for $\lambda = 0$ and $\lambda < 0$.

Table 1: Stability of static black holes.

| $K = 1$ | $\lambda = 0$ | OK | OK | OK | OK | $Q \neq 0$ | OK | OK | OK | OK | $D \geq 6$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $\lambda > 0$ | OK | OK | OK | OK | $D \leq 6$ OK | $D \geq 4$ OK | $D > 6$ |
| $\lambda < 0$ | OK | OK | OK | OK | $D \leq 4$ OK | $D \geq 6$ ? | $D > 6$ ? |
| $K = 0$ | $\lambda < 0$ | OK | OK | OK | OK | $D \leq 4$ OK | $D \geq 4$ OK | $D > 5$ ? |
| $K = -1$ | $\lambda < 0$ | OK | ? | OK | ? | $D \leq 4$ OK | $D \geq 4$ OK | $D > 5$ ? |

Hence, the gravitational mode is also stable for $n = 2$.

However, for $n > 2$, $V_{S-}$ becomes negative near the horizon in general (See Fig.3). The $S$-deformation used in the neutral case does not work either, because it cannot remove a negative ditch of $V_{S-}$ produced by the charge near the horizon. We have not been able to prove the stability for $n > 2$ by this method, except for some special cases.

5 Summary

In this article, we have discussed the stability of generalised static black holes with and without charge in higher dimensions. The results of our analysis for maximally symmetric black holes are summarised in Table 1.

In particular for the case of $\Lambda = 0$, we have established the stability of a spherically symmetric black hole for arbitrary dimensions, and that of a spherically symmetric black hole with charge for $D = 4$ and $D = 5$. We have also proved the stability of 4D and 5D dS-Schwarzschild and 4D AdS-Schwarzschild black holes with charge. Although we have not

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1In this table, the results for vector and scalar perturbations are also valid for generalised static black holes, except for the case $K = 1, \lambda > 0$ and $Q = 0$, for which the stability has been proved for $D \leq 5$ in spherically symmetric case.
succeeded in proving the stability of black holes for the other cases, there is no strong reason to suspect the instability of such black holes because the negative ditch of the effective potential is not so deep.

Finally, we would like to point out that the master equations used in this article for the stability analysis can be also used in other problems related to black holes, such as perturbative analyses of the uniqueness of asymptotically de Sitter or anti-de Sitter static black holes in four and higher dimensions, the precise determination of the frequencies of quasinormal modes and the greybody factor of the Hawking radiation, and the estimation of the gravitational wave emission rate from mini black holes, which might be produced in colliders and cosmic ray events\(^2\).

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\(^2\)Explicit expressions for the source terms of the master equations are given in Ref. [3].