ON THE WARING-GOLDBACH PROBLEM FOR TENTH POWERS

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1. Introduction

Define $H(k)$ to be the least $s$ such that all sufficiently large $n$ satisfying some congruence condition are the sum of $s k^{th}$ powers of primes.

Work done by Vinogradov [17], Hua [3, 4], and Davenport [1], and later work by Thanigasalam [10, 11, 12] and Vaughan [15] resulted in the knowledge that

\[ H(4) \leq 15, \quad H(5) \leq 23, \quad H(6) \leq 33, \quad H(7) \leq 47, \quad H(8) \leq 63, \quad H(9) \leq 83, \quad H(10) \leq 107. \]

Recently, Kawada and Wooley [5] and Kumchev [6] showed that

\[ H(4) \leq 14, \quad H(5) \leq 21, \quad H(7) \leq 46, \]

and even more recently, Zhao [18] has shown that

\[ H(4) \leq 13, \quad H(6) \leq 32. \]

The purpose of this paper is to establish similar results for tenth powers.

In particular, we shall obtain the following:

**Theorem 1.**

$H(10) \leq 105$. In particular, every sufficiently large integer congruent to $6$ modulo $33$ is the sum of $105$ tenth powers of primes.

The above result will be established with the Hardy-Littlewood method. The results which allow us to improve the previous bound of $107$ are the new Weyl sum estimates obtained by Kumchev [7] and the mean value estimates in [13].

2. Preliminaries

2.1. Notation. In all cases, unless otherwise specified, $p$ will refer to a prime.

For $x \in \mathbb{R}$, let $e(x) = e^{2\pi i x}$.

As usual, we will write $O(f)$ to denote some quantity bounded above by $C|f|$ for some $C$, and $f \ll g$ means $|f| \leq Cg$ for some $C$, and $\epsilon$ will refer to a sufficiently small positive real number. All sums will be over the natural numbers unless otherwise specified.

We write

\[ \mathfrak{M}(g, a; Q, P) = \left\{ \alpha \in [0, 1] : |g\alpha - a| \leq \frac{Q}{P} \right\}. \]

We define the primorial $Q#$ of $Q > 0$ to be the product of all primes less than or equal to $Q$. Also, $c$ will refer to some constant, and will not necessarily be the same each time it is mentioned in the paper.

Call a set $\{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{R}^+$ admissible if the number of solutions $S$ to

\[ \sum_{i \leq k} (x_i^k - y_i^k) = 0 \]

satisfying $P^{\lambda_i} < x_i, y_i \leq 2P^{\lambda_i}$ satisfies $S \ll P^{\lambda_1 + \cdots + \lambda_k + \epsilon}$.

\[ ^{1}\text{Since the writing of this paper, the bound } H(10) \leq 89 \text{ has been achieved in [9].} \]
2.2. Exponential Sum Estimates.

Lemma 2. For all \( \alpha \) for which for all coprime \( 0 \leq a \leq q \leq P^{1/4} \) s.t. \(|q\alpha - a| > P^{1/4}P^{-10}\),

\[
\sum_{p < p \leq 2P} e(\alpha p^{10}) \ll P^{1-1/480+\varepsilon}
\]

Proof. We have that by Theorem 2 in Kumchev [7],

\[
\sum_{n \leq P} \Lambda(n)e(\alpha n^{10}) = \sum_{p \leq P} \log pe(\alpha p^{10}) + O(\sqrt{P}) \ll P^{1-1/480+\varepsilon}.
\]

Then, by partial summation, we have that

\[
\sum_{p \leq P} e(\alpha p^{10}) = \frac{1}{\log P}\sum_{p \leq P} \log pe(\alpha p^{10}) - \int_{2}^{P} \left( -\frac{1}{t \log^2 t} \right) \sum_{p \leq t} \log pe(\alpha p^{10}) dt.
\]

Note that

\[
\int_{2}^{P} \left( -\frac{1}{t \log^2 t} \right) \sum_{p \leq t} \log pe(\alpha p^{10}) dt \leq \int_{2}^{P} \frac{t dt}{t^{1/480} \log^2 t} \ll P^{1-1/480+\varepsilon}.
\]

The desired result follows. \( \square \)

Lemma 3. For some \( a, q, \alpha \) satisfying \( (a, q) = 1, |q\alpha - a| \leq QP^{-10} \) for some \( q \leq Q \leq P \),

\[
\sum_{p < p \leq 2P} e(\alpha p^{10}) \ll q(q^6 + P^{11/20} + P^{5/2} |q\alpha - a|^{-1/2}).
\]

Proof. This is just Lemma 5.6 in Kumchev [6] with \( M = 1/2, z = \sqrt{2P} \). \( \square \)

3. Mean-Value Estimates

Lemma 4. There exist admissible exponents \( 1 = \lambda_1, \ldots, \lambda_{51} \) satisfying

\[
\alpha_{51} = \frac{\lambda_1 + \cdots + \lambda_{51}}{10} > 0.999553 > 1 - \frac{1}{2230}.
\]

Proof. This follows from Lemma 18 in Thanigasalam [14] and (10.5) in Thanigasalam [12]. \( \square \)

For \( 1 \leq j \leq 51 \), write \( P_j = P^{\lambda_j} \) where the \( \lambda_i \) are as in Lemma 4, and let

\[
f_j(\alpha) = \sum_{P_j < n \leq 2P_j} e(\alpha n^{10})
\]

Then, since the \( \lambda_j \) are admissible, we have that the following, which is the main result of this section holds:

Lemma 5. We have that

\[
\int_{0}^{1} |f_1(\alpha) \cdots f_{51}(\alpha)|^2 d\alpha \ll P^{10\alpha_{51}+\varepsilon},
\]

where \( \alpha_{51} \) is the constant mentioned in Lemma 4.
4. Proof of the main theorem

Let $N$ be some large integer congruent to 6 (mod 33), let $B$ be a sufficiently large real number, and set

$$P = \frac{1}{2} N^{1/10}, \quad X = P^5 P_2 \ldots P_{51} N^{-1}, \quad L = \log^B P.$$  

Let

$$\mathfrak{N}(q, a) = \mathfrak{M}(q, a; L, P^{10}), \quad \mathfrak{N} = \bigcup_{0 \leq a \leq q \leq \log^B P}^\text{(a.q=1)} \mathfrak{N}(q, a)$$

$$\mathfrak{M} = \bigcup_{0 \leq a \leq q \leq P^{1/4}}^\text{(a.q=1)} \mathfrak{M}(q, a; P^{1/4}, P^{10})$$

and let $m = [0, 1) \setminus \mathfrak{M}$, $n = [0, 1) \setminus \mathfrak{N}$

For $1 \leq i \leq 51$, define

$$g_i(\alpha) = \sum_{P < p \leq 2P} e(\alpha p^{10}).$$

For some measurable $\mathfrak{B} \subseteq [0, 1)$, write

$$R(N; \mathfrak{B}) = \int_0^1 g_1(\alpha)^5 g_2(\alpha)^2 \ldots g_{51}(\alpha)^2 d\alpha.$$  

Let

$$R(N) = |\{(p_1, \ldots, p_{105}) : p_{10}^1 + \cdots + p_{105}^{10} = N\}|,$$

for primes $p_1, \ldots, p_{105}$ satisfying

$$p_1 < p_2, p_3, p_4, p_5 \leq 2P_1, P_i < p_{2i+2}, p_{2i+3} \leq 2P_i \text{ for } 2 \leq i \leq 51.$$

Then, by orthogonality, we have that $R(N) = R(N; [0, 1))$.

Note that in order to prove Theorem 1, it is sufficient to show that for all sufficiently large $N$, $R(N) > 0$.

4.1. The major arcs. In this section, we shall consider the contribution to $R(N)$ from the major arcs $\mathfrak{N}$. Write

$$S(q, a) = \sum_{1 \leq k \leq q}^\text{(k.q=1)} e\left(\frac{ak^{10}}{q}\right),$$

$$v_1(\beta) = \int_{P_1}^{2P_1} \frac{e(\beta t^{10})}{\log t} dt,$$

$$B(N, q) = \frac{1}{\phi(q)^{105}} \sum_{1 \leq a \leq q}^\text{(a.q=1)} S(q, a) e\left(-\frac{aN}{q}\right),$$

$$\mathcal{S}(N) = \sum_q B(n, q),$$

$$J(N; \xi) = \int_{-\xi}^\xi v_1(\beta)^5 v_2(\beta)^2 \ldots v_{51}(\beta)^2 e(-N\alpha) d\beta,$$

and let $J(N) = J(N; \infty)$.

Note that by Theorem 12 in [4], $\mathcal{S}(N) \asymp 1$. We also have that $J(N) \asymp X \log^{-105} P$.

We then have that by partial summation and the Siegel-Walfisz Theorem that for $\alpha \in \mathfrak{N}_0(q, a)$

$$g_i(\alpha) = \phi(q)^{-1} S(q, a) v(\alpha - a/q) + O(P_i L^{-3})$$

Therefore, since the measure of $\mathfrak{N}$ is $O(L^2 n^{-1})$

$$R(N; \mathfrak{N}) = \mathcal{S}(N) J(N) + O(X L^{-1}) \gg X \log^{-105} P$$
4.2. The minor arcs. In this section we shall bound the contribution from \( \mathfrak{R} = \mathfrak{M} \cap \mathfrak{n} \) and \( \mathfrak{m} \)

Lemma 6. There exists \( \eta > 0 \) s.t.

\[
R(N; \mathfrak{m}) \ll XP^{-\eta + \epsilon}
\]

Proof. In fact, we shall prove that this is the case for all \( \eta = 1/160 - 1/223 \). We have that by (2.1),

\[
\sup_{\alpha \in \mathfrak{m}} |g_1(\alpha)| \ll P^{1-1/480+\epsilon}.
\]

It then follows from (3.1) and (3.2) by considering the underlying diophantine equation that

\[
R(N; \mathfrak{m}) = \int_{\mathfrak{m}} g_1(\alpha)^5 g_2(\alpha)^2 \cdots g_{51}(\alpha)^2 d\alpha
\]

\[
\ll \left( \sup_{\alpha \in \mathfrak{m}} |g_1(\alpha)| \right)^3 \int_0^1 |f_1(\alpha) \cdots f_{51}(\alpha)|^2 d\alpha \ll P^{3-1/160}(P_1 \cdots P_{51})^2 P^{10\alpha_{51}+\epsilon}
\]

\[
\ll P^{1/223-1/160+\epsilon} P^{-10}(P_1 \cdots P_{51})^2 \ll XP^{-\eta + \epsilon}
\]

as desired.

□

Lemma 7. We have that

\[
R(N; \mathfrak{R}) \ll XL^{-1} \log^c P.
\]

Proof. Note that \( \mathfrak{R} \) is the disjoint union of \( \mathfrak{R}(q, a) \) for coprime \( a, q \) satisfying \( 0 \leq a \leq q \leq P^{1/4} \), where \( \mathfrak{R}(q, a) = \mathfrak{M}(q, a) \setminus \mathfrak{N}(q, a) \) for \( q \leq L \) and \( \mathfrak{R}(q, a) = \mathfrak{M}(q, a) \) otherwise.

Then, it follows by applying Lemma 4 that

\[
\int_{\mathfrak{R}} g_1(\alpha)^5 g_2(\alpha)^2 \cdots g_{51}(\alpha)^2 d\alpha
\]

\[
\ll \int_{\mathfrak{R}} |g_1(\alpha)|^5 |g_2(\alpha)|^2 |g_3(\alpha)||g_4(\alpha)\cdots g_{51}(\alpha)|^2 d\alpha
\]

\[
\ll Xn \sum_{q \leq P^{1/4}} \sum_{1 \leq a \leq q} \frac{\log P d\alpha}{q^4(1 + n|\alpha - a/q|)^2}.
\]

The desired result follows.

□

Now, it follows from this and (4.1), by making \( B \) sufficiently large, that

\[
R(N) = R(N; \mathfrak{N}) + R(N; \mathfrak{R}) + R(N; \mathfrak{m}) \gg X \log^{-105} P,
\]

so Theorem 1 holds.

5. Acknowledgements

The author is thankful to T. D. Wooley and D. Goldston for providing corrections and suggestions.
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