Classical trajectories compatible with quantum mechanics

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Consider any stationary Schroedinger wave equation (SWE) solution $\psi(x)$ for a particle. The corresponding PDF on position $x$ of the particle is $p_x(x) = |\psi(x)|^2$. There is a classical trajectory $x(t)$ for the particle that is consistent with this PDF. The trajectory is unique to within an additive constant corresponding to an initial condition $x(0)$. However the value of $x(0)$ cannot be known. As an example, a free particle in its ground state in a box of length $L$ obeys a classical trajectory $x/L - (1/2\pi)\sin(2\pi x/L) + t_0 = t$. The constant $t_0$ is an unknowable time displacement. Momentum values, however, cannot be determined by merely differentiating $d/dt$ the trajectory $x(t)$ and, instead, follow the usual quantification rules of Heisenberg’s. This permits position and momentum to remain complementary variables. Our approach is fundamentally different from that of D. Bohm.

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I. INTRODUCTION

Suppose that a non-relativistic particle is allowed to move in one dimension $x$ (for simplicity) under the influence of a potential function. There are two general perspectives for analyzing the motion of the particle: (i) that of classical physics, according to which the particle obeys Newtonian mechanics, or (ii) that of quantum mechanics, according to which the particle obeys, e.g., the Schroedinger wave equation (SWE). (We ignore for the moment the combined quantum-classical approach of D. Bohm [1].) Here we will show that there is, in fact, a third, intermediary viewpoint, whereby the particle obeys to a degree both classical mechanics and quantum mechanics.

Let the particle have mass $m$, and be moving under the influence of a stationary potential $V(x)$. Later on we treat the more general, time-dependent case. The particle is otherwise not perturbed by any exterior effect including measurement. Assuming non-relativistic speeds, the kinematics of the particle obey the Schroedinger wave equation (SWE). In the presence of our stationary potential this has as its solution a separable wave amplitude function $\Psi(x,t) = \psi(x) \exp(iEt/\hbar)$, where $\psi(x)$ is any complex amplitude function, $E$ is a definite energy value, and $\hbar$ stands for Planck’s constant (divided by $2\pi$).

The probability density function (PDF) for the separable case loses the time-dependence, $p_X(x) = |\psi(x)|^2$. This means that, from a quantum viewpoint, the coordinate $x$ is independent of the time $t$. The same PDF on position is obeyed at all times. However, from a classical viewpoint, in the presence of the potential $V(x)$ a particle has a definite trajectory $x(t)$ through time. To what extent can the quantum result and the classical viewpoint agree?

II. TRAJECTORY

Regard the particle as having a definite trajectory $x(t)$ during a time interval $(0, T)$. Consider the histogram of position values $x$ taken on by the particle during that time. The main point of this paper is that for any quantum solution $\psi(x)$ there is a classical trajectory $x(t)$ whose density of position values $x$ over the given time interval exactly coincides with $|\psi(x)|^2$. This trajectory is found next.

Central to the calculation is the following effect. Among the four coordinates of space-time, each time coordinate value is unique in occurring once and only once over a given interval $(0, T)$. That is, the time values as coordinates occur predictably in sequence one after the other. Of course, such a sequence of numbers has a density $p_T(t)$ of values that is uniform or flat over any time interval $(0, T)$,

$$p_T(t) = 1/T, \ t = (0, T). \quad (1)$$

This expression is unusual in expressing a density of deterministic events. However, whether deterministic or random, the density still must obey Jacobian transformation theory [2].

One defining property of a trajectory $x(t)$ is that to each time value $t$ in the sequence there is one and only one $x$ value. Consider the density $p_X(x)$ of all $x$ values traveled over by the particle during the given time interval. Note that like $p_T(t)$, density $p_X(x)$ is not a probability density, since the events $x$ follow a deterministic trajectory $x(t)$. Since to each $t$ value there is only one $x$ value the frequency of occurrence of each $x$ (actually, of each interval $(x, x + dx)$) equals that of its corresponding value $t$. Or, in terms of density functions [2],

$$p_X(x)dx = p_T(t)dt, \quad (2)$$

where $x = x(t)$. Combining this with Eq. (1) gives
The absolute value sign for $dx/dt$ defines two possible scenarios: either $x$ increases with $t$ during the entire time period $T$ or $x$ decreases with $t$ during the period. (Note that for reasons of continuity in $x$, the choice of sign cannot be changed during the period.) By either sign, $x$ is a monotonic function of $t$. Thus the particle makes one pass through the system in either the positive $x$ direction or in the negative $x$ direction. However, since the trajectory obeys the required PDF $p_X(x)$ via Eq. (3) for either direction (either choice of sign for $dx/dt$), the particle can alternatively obey periodic motion. This would be for any number of such periods $T$, i.e., any number of traversals to and fro through the system. At each such traversal the particle position would be a monotonic function of the time (as above) as defined by Eq. (3). In summary, the particle motion can be either single pass, or oscillatory.

To be definite, consider the monotonically increasing orbit $dx(t)/dt \geq 0$. Let the density $p_X(x)$ of deterministic events $x$ equal the density $|\psi(x)|^2$ of random events $x$ as given by the SWE. Then Eq. (3) shows that the probabilistic density $|\psi(x)|^2$ is achieved by a classical trajectory $x(t)$ obeying

$$\frac{dx}{dt} = \frac{1}{T |\psi(x)|^2}. \quad (4)$$

This shows a possible problem at positions $x$ for which $|\psi(x)|^2 \to 0$. The required particle speed $dx/dt$ becomes so large as to exceed $c$, the speed of light. However, Eq. (4) also shows an inverse dependence upon the period $T$, and $T$ has yet to be chosen. Obviously, if $|\psi(x)|^2$ is small but not zero the problem can be avoided by making $T$ large enough. Furthermore, even at $x$ for which $|\psi(x)|^2 = 0$, with $T$ chosen large enough the unphysical region $x$ within which relativity is violated can be made as narrow (as improbable) as is required.

Continuing with the analysis, Eq. (4) is straightforward to integrate, giving

$$\int dx|\psi(x)|^2 = (t - t_0)/T, \quad (5)$$

with $t_0 = \text{const.}$ The integral is an indefinite one, resulting in a well-defined function $g(x)$ corresponding to the given $\psi(x)$. Hence Eq. (5) expresses the trajectory as

$$g(x) = (t - t_0)/T, \quad (6)$$

which has to be inverted to obtain the desired trajectory

$$x = g^{-1}[(t - t_0)/T] = x(t). \quad (7)$$

This is the formal solution to the problem.

There are two distinct interpretations to this result. As it was derived, it gives the trajectory of $x$ values that, if binned deterministically at each time $t$ over the interval $(0, T)$ will give the required PDF $|\psi(x)|^2$. But alternatively, the binned trajectory must also give the required PDF if it is sampled with uniform randomness (1) in time. The analysis is blind to whether $t$ is random or deterministic, since it depends only upon the Jacobian transformation of density functions, and Jacobian theory holds irrespective of whether the densities define deterministic or random variables.

The time $t_0$ is of interest. With the function $g(x)$ fixed for a given problem, Eq. (6) shows that the value of $t_0$ is defined by knowledge of initial conditions. For example, if it is known that at $t = 0$ position $x$ has a certain value $x_0$ then $t_0$ is fixed as $-T g(x_0)$. Therefore, to know $t_0$ requires observation of the particle at a particular space-time point. But if this observation were made the particle’s state would be changed, and it would no longer have the amplitude $\psi(x)$ that has been presumed in the calculation. In fact the state would be a function of time, contrary to the stationary state solution assumed.

Therefore the time $t_0$ remains an unknown constant of the theory, and the trajectory (7) suffers an unknown time displacement. We conclude that the trajectory is deterministic only up to an unknown additive constant. Another way of saying this is that the trajectory is one of a family of curves $x(t; t_0)$ where each member of the family is defined by a particular value of $t_0$. This defines, in fact, a stochastic process [2]. We spoke at the beginning of defining a form of mechanics that is intermediary between classical and quantum mechanics. It is this stochastic process that is intermediary between the two, preserving aspects of each. Thus, the trajectory $x(t)$ is deterministic, or classical, to an additive random constant, and it is the latter that gives a quantum aspect to the trajectory.

It may be noted that any member of the ensemble of trajectories satisfies the required PDF $|\psi(x)|^2$. Eq. (4) depends upon the value of $dx/dt$ which, according to Eq. (6), is independent of the unknowable $t_0$.

III. CASE OF A PARTICLE IN A BOX

To find a particular trajectory we have to specify a particular case $\psi(x)$. Consider the case where the particle is in a box of length $L$. With the origin for coordinate $x$ at the center of the box, the wave function in the ground state obeys [3]

$$\psi(x) = \sqrt{2/L} \cos(\pi x/L), \quad -L/2 \leq x \leq L/2. \quad (8)$$

Using this in Eq. (5) gives a solution

$$x/L + (2\pi)^{-1} \sin(2\pi x/L) = (t - t_0)/T. \quad (9)$$
This defines the trajectory $x(t)$ of the particle for positions $x$ over the interval $(0, L)$. Since the left-hand side of Eq. (9) is transcendental in $x$, the equation must be inverted numerically for the desired trajectory $x(t)$. A particle that traverses the given curve $x(t)$ back and forth any integral number of times will satisfy the required PDF $p(x, t)$. As a check on the theory, the $x$-values along the curve can be ‘binned’ to form a histogram $p(x)$ of occurrences $x$. One would then ascertain that it matches the theoretical answer $(2/L) \cos^2(\pi x/L)$. The net PDF on $x$, $p_x(t)$, is defined as the probability density of the particle for positions $x$ at (conditional on) a fixed, nonrandom value $t$ of time. Hence, we constructed to obey the partition law of statistics [2], that this PDF on random values varies with the time, $t$

$$p_x(x) = \int dt p(x|t)p_T(t). \quad (11)$$

Then by Eqs. (1) and (10), Eq. (11) becomes

$$p_x(x) = \frac{1}{T} \int dt |\Psi(x, t)|^2. \quad (12)$$

This is now used in place of quantity $|\psi(x)|^2$ in Eq. (5) to get the required trajectory. If the positions $x$ along this trajectory are sampled with uniform randomness in time, the resulting histogram will obey $p_x(x)$ given by Eq. (12).

**IV. NON-STATIONARY PROBLEMS**

We may now generalize the approach. Suppose that the potential $V$ is now more generally a function of both $x$ and $t$. Then the SWE solution is of a generally nonseparable form $\Psi(x, t)$. Accordingly, the particle has a PDF on position that varies with the time,

$$|\Psi(x, t)|^2 = p(x|t). \quad (10)$$

The vertical solidus on the right-hand side emphasizes that this PDF on random values is at (conditional upon) a fixed, nonrandom value $t$ of time. May the particle still be describable by a trajectory $x(t)$?

The random variable in the amplitude function $\Psi(x, t)$ is $x$. The time $t$ is assumed known. However, in practice, a particle is tracked with uniform randomness (1) in time. The net PDF on $x$ over all such time values obeys the partition law of statistics [2],

$$p_x(x) = \int dt p(x|t)p_T(t). \quad (11)$$

After use of Eq. (1). Next, by Eq. (4)

$$\frac{dv}{dt} = \frac{d}{dt} \left( \frac{1}{|\psi(x)|^2} \right) = -\frac{2}{T|\psi(x)|^3} \frac{d|\psi|}{dx}, \quad (14)$$

after use of the chain rule of differentiation. Substitution in Eq. (13) gives

$$p_v(v) = \frac{1}{2|v|} \frac{|\psi|^3}{d|\psi|/dx}, \quad (15)$$

Although values of $\psi(x)$ are continuous, its derivatives $d\psi/dx$ can be non-continuous [3]. At such velocities a PDF $p_v(v)$ computed via Eq. (15) would be ill-defined. Hence, this approach to finding $p_v(v)$ is untenable in general. Likewise the corresponding PDF on momentum by Jacobian transformation to $\mu = mv$ would be ill-defined in general. Hence, the momentum of our particle cannot be obtained as the mass times velocity. This leaves the momentum representation of Heisenberg ($\mu \rightarrow -i \hbar \node$) as the alternative. Thus, the probability amplitude $\Phi(\mu)$ on momentum values $\mu$ is the usual Fourier transform of $\Psi(x, t)$. This Fourier relation implies that the particle obeys the Heisenberg uncertainty principle, as it must.

In summary, this approach to particle dynamics defines the particle positions by the stochastic process $x(t; t_0)$, where trajectory $x(t)$ is deterministic up to the unknown constant $t_0$, and defines momentum values by working in the usual Heisenberg representation $p_{v(t)} \rightarrow -i \hbar \node$. A general wave amplitude $\psi(x)$ is complex and hence has a finite phase function. How is phase information reflected in the trajectory $x(t)$? The answer is that it is NOT, since by Eq. (4) $dx/dt$ is blind to the phase function. Instead, the phase function follows from the Heisenberg representation for momentum, by taking in the usual way the inverse Fourier-transform of the probability amplitude $\Phi(\mu)$ on momentum.

**V. MOMENTUM AND PHASE VALUES**

We now turn to the question of the particle’s momentum values $\mu$ (note: this is not the usual notation $p$, because of previous use of $p$ to denote probabilities.) The momentum of a purely classical particle obeys $\mu = mv = m dx/dt$, suggesting that a momentum

‘trajectory’ $\mu(t)$ could be found by simply differentiating $d/dt$ the previously found trajectory $x(t)$. However, in general this gives an ill-defined answer for the corresponding PDF on momentum (or velocity $v$), as shown next. Corresponding to Eq. (2) is

$$p_v(v) = \frac{1}{T|dv/dt|}, \quad (13)$$

after use of Eq. (1). Next, by Eq. (4)

$$\frac{dv}{dt} = \frac{d}{dt} \left( \frac{1}{T|\psi(x)|^2} \right) = -\frac{2}{T|\psi(x)|^3} \frac{d|\psi|}{dx}. \quad (14)$$

This defines the trajectory $x(t)$ of the particle for positions $x$ over the interval $(0, L)$. Since the left-hand side of Eq. (9) is transcendental in $x$, the equation must be inverted numerically for the desired trajectory $x(t)$. A particle that traverses the given curve $x(t)$ back and forth any integral number of times will satisfy the required PDF $(2/L) \cos^2(\pi x/L)$. As a check on the theory, the $x$-values along the curve can be ‘binned’ to form a histogram $p(x)$ of occurrences $x$. One would then ascertain that it matches the theoretical answer $(2/L) \cos^2(\pi x/L)$. The net PDF on $x$, $p_x(t)$, is defined as the probability density of the particle for positions $x$ at (conditional on) a fixed, nonrandom value $t$ of time. Hence, we constructed to obey the partition law of statistics [2], that this PDF on random values varies with the time, $t$

$$p_x(x) = \int dt p(x|t)p_T(t). \quad (11)$$

Then by Eqs. (1) and (10), Eq. (11) becomes

$$p_x(x) = \frac{1}{T} \int dt |\Psi(x, t)|^2. \quad (12)$$

This is now used in place of quantity $|\psi(x)|^2$ in Eq. (5) to get the required trajectory. If the positions $x$ along this trajectory are sampled with uniform randomness in time, the resulting histogram will obey $p_x(x)$ given by Eq. (12).

**VI. EFFECTIVE POTENTIAL $V$**

The trajectory $x(t)$, which we constructed to obey the SWE, can also be made to obey Newton’s laws, through the effect of an appropriate potential. Newton’s second law is

$$F = -\frac{\partial V}{\partial x} = m \frac{dv}{dt}. \quad (16)$$
where $\overline{V}$ is to be an effective potential function. The acceleration $dv/dt$ borne of the SWE obeys Eq. (14). Using this in Eq. (4) forces the acceleration to be responding as well to a classical force $F$. Using as well Eq. (4) for $v$, gives
\[
-\frac{d\overline{V}}{dx} = -\frac{2m}{T^2|\psi|^5} \frac{d|\psi|}{dx}, \quad \psi = \psi(x).
\] (17)

This equation defines the effective potential function $\overline{V}(x)$ that must be present for the particle to simultaneously obey the SWE and Newton’s second law. It may be integrated directly, to give
\[
\overline{V}(x) = -\frac{m}{2T^2}|\psi(x)|^{-4}.
\] (18)

The dependence upon $\psi$ makes sense classically in that where the particle tends not to be, i.e. at $x$ for which $\psi(x)$ is low, the potential tends to be high, acting as a classical barrier. Also, the negative sign means that the effective potential is always attractive, i.e., tending to produce periodic motion.

In the more general case (Sec. IV) of a time-dependent potential $V(x,t)$, by similar steps the effective potential obeys
\[
\overline{V}(x) = -\frac{m}{2T^2}p_x(x)^{-2}.
\] (19)

**VII. DISCUSSION**

The semi-classical theory of D. Bohm [1] attempts, as here, to ascribe a classical trajectory to a particle that a priori obeys the SWE. As with our approach, the particle trajectory by the Bohm approach is only knowable to an unknown, additive constant. However, Bohm’s theory differs from ours both operationally (mathematically) and in its physical assumptions. Operationally, instead of Eq. (4), the Bohm trajectory for a particle obeys [4]
\[
\frac{dx}{dt} = \frac{j(x,t)}{|\psi(x,t)|^2},
\] (20)

where $j(x,t)$ is the quantum mechanical probability current. Comparing Eqs. (4) and (20) shows that, unless the current is unity, the two approaches give different trajectories for the particle. In general the current obeys
\[
j(x,t) = \frac{i\hbar}{2m}[\psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \psi^*}{\partial x}].
\] (21)

This rarely has the value unity. For example, if $\Psi$ is of the separable form $\Psi(x,t) = (1/L)^{-1/2} \exp(ikx) \exp(-iEt/\hbar)$, Eq. (21) gives a current $j = \frac{\hbar k}{mL}$. Or, where $\Psi$ is of the separable form $\Psi(x,t) = \psi(x) \exp(iEt/\hbar)$ with $\psi(x)$ real, Eq. (21) gives a current $j = 0$. Hence the Bohm trajectories usually differ from ours. As an example, in the particle-in-box case above, $j = 0$ so that by Eq. (21) the Bohm trajectory obeys $dx/dt = 0$, i.e., the Bohm particle does not move in the box. As with our effective potential Eq. (18), the Bohm particle moves in the field of an effective potential function that depends upon the wave amplitude function. However the Bohm effective potential, commonly called the "quantum potential", is of an entirely different form [1].

The physical underpinning of the Bohm approach consists of four assumptions. We list these in the following, along with corresponding assumptions of our particle model:

1. A SWE solution $\Psi(x,t)$ is not a probability amplitude but, rather, a “field” analogous to the electromagnetic field. Our approach preserves the standard probabilistic nature of $\Psi(x,t)$.

2. The phase $S(x,t)$ of $\Psi(x,t)$ is also a classical Hamilton-Jacobi function of mechanics. We do not assume this. Instead, the classical trajectory follows from the special nature of the time coordinate as in Eqs. (1) and (2).

3. The Bohm semi-classical particle is acted upon by both the given potential function $V(x,t)$ and an added potential function called the “quantum potential” where the latter is a function of $\Psi(x,t)$ as well. Our classical particle is acted upon in parallel by the two potential functions of the problem. These are (i) a conventional potential function $V(x)$ that defines, via the SWE, the probability field $|\psi(x)|^2$ on position $x$ that the particle is required to obey, and (ii) an effective potential function $\overline{V}(x)$ that constrains the classical particle to take the particular trajectory that satisfies the required $|\psi(x)|^2$. The effective potential function $\overline{V}(x)$ derives from the conventional potential function $V(x)$ via solution $|\psi(x)|^2$ to the SWE and Eq. (18). In this manner the SWE and the conventional potential function $V(x)$ fix both the statistics of the particle and its required trajectory (to an additive constant).

4. The Bohm trajectory obeys Eq. (20), which is based upon the assumption that the quantum mechanical current $j$ is also a classical current. As we found, this assumption in general implies trajectories that are different from ours. The assumption that the quantum mechanical current $j$ is a classical current is difficult to verify, since the quantum mechanical current $j$ is not a physical observable [3].

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