Transformation à la Foata pour des Vents Spéciaux de Descents et d’Excedances

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1. Introduction and notations

The distribution of the number of descents has been widely studied on several classes of combinatorial objects such as permutations [14], cycles [7, 8], and words [3, 10]. Many interpretations of this statistic appear in several fields as Coxeter groups [4, 11] or lattice path theory [12]. One of the most famous result involves the Foata fundamental transformation [9] to establish a one-to-one correspondence between descents and excedances on permutations. This bijection provides a more straightforward proof than those of MacMahon [14] for the equidistribution of these two Eulerian statistics.

In this paper, we present a bijection à la Foata on the symmetric group that exchanges pure descents and pure excedances of a special kind. As a byproduct, we prove that the popularity of pure excedances equals those of pure descents on permutations, while their distributions are different.

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patterns \( p_i, i \in [0, 2] \). We refer to Figure 1 for a graphical illustration. On the other hand, we define a pure excedance as an occurrence of an excedance, i.e., \( \pi_i > i \), with the additional restriction that there is no point \((j, \pi_j)\) such that \(1 \leq j \leq i - 1\) with \(i \leq \pi_j < \pi_i\). Although such a pattern (called \( p_{ex} \)) is not a mesh pattern, we can represent it graphically as shown in Figure 1.

\[
p_0 = \quad p_1 = \quad p_2 = \quad p_{ex} =
\]

Figure 1: Illustration of the mesh patterns \( p_0 \), \( p_1 \), \( p_2 \) and \( p_{ex} \); \( p_1 \) and \( p_{ex} \) correspond respectively to a pure descent and a pure excedance.

A statistic is an integer-valued function from a set \( A \) of \( n \)-length permutations (we use the boldface to denote statistics). For a pattern \( p \), we define the pattern statistic \( \mathbf{p} : A \to \mathbb{N} \) where the image \( \mathbf{p} \pi \) of \( \pi \in A \) by \( \mathbf{p} \) is the number of occurrences of \( p \) in \( \pi \). The popularity of \( p \) in \( A \) is the total number of occurrences of \( p \) over all objects of \( A \), that is \( \sum_{\pi \in A} \mathbf{p} \pi \) (see [5] for instance). Below, we present statistics that we use throughout the paper:

\[
\begin{align*}
\text{exc } \pi & = \text{number of excedances in } \pi, \\
\text{pex } \pi & = \text{number of pure excedances in } \pi, \\
\text{des } \pi & = \text{number of descents in } \pi, \\
\text{des}_i \pi & = \text{number of patterns } p_i \text{ in } \pi, \quad 0 \leq i \leq 2, \\
\text{fix } \pi & = \text{number of fixed points in } \pi, \\
\text{cyc } \pi & = \text{number of cycles in the decomposition of } \pi, \\
\text{pcyc } \pi & = \text{number of pure cycles (i.e. cycles of length at least two)} \text{ in } \pi, \\
& = \text{cyc } \pi - \text{fix } \pi
\end{align*}
\]

We organize the paper as follows. In Section 2, we focus on patterns \( p_i, 0 \leq i \leq 2 \). We prove that the statistics \( \text{des}_0 \) and \( \text{des}_1 \) are equidistributed by giving algebraic and bijective proofs. Next, we provide the bivariate exponential generating function for the distribution of \( p_2 \), and we deduce that \( p_2 \) has the same popularity as \( p_0 \) and \( p_1 \), without having the same distribution. In Section 3, we present a bijection on \( S_n \) that transports pure excedances into patterns \( p_2 \). Notice that the Foata’s first transformation [9] is not a candidate for such a bijection. As a consequence, pure descents and pure excedances are equipopular on \( S_n \), but they do not have the same distribution. Combining all these results, we deduce that patterns \( p_i, 0 \leq i \leq 2 \), and \( p_{ex} \) are equipopular on the symmetric group \( S_n \). Finally we present two conjectures about the equidistribution of \( (\text{cyc}, \text{des}_2) \) and \( (\text{cyc}, \text{pex}) \), and that of \( (\text{des}, \text{des}_2) \) and \( (\text{exc}, \text{pex}) \).

2. The statistics \( \text{des}_i, \ 0 \leq i \leq 2 \)

For \( 0 \leq i \leq 2 \), let \( A_{n,k}^i \) be the set of \( n \)-length permutations having \( k \) occurrences of \( p_i \), and denote by \( a_{n,k}^i \) its cardinality. Let \( A^i(x, y) \) be the bivariate exponential generating function \( \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} a_{n,k}^i x^n y^k \). In [2,13], it is proved that \( a_{n,k}^i \) equals the signless Stirling numbers of the first kind \( c(n, k + 1) \) (see Sequence A132393 in [15]). Indeed, a permutation \( \pi \in A_{n,k}^i \) can be uniquely obtained from an \( (n-1) \)-length permutation \( \pi' \) by one of the two following constructions:

(i) if \( \pi \in A_{n-1,k-1}^i \), then we increase by one all values of \( \pi' \) greater than or equal to \( \pi_{n-1} \), and we add \( \pi_{n-1} \) at the end;

(ii) if \( \pi \in A_{n-1,k}^i \), then we increase by one all values of \( \pi' \) greater than or equal to a given value \( x \leq n \), \( x \neq \pi_{n-1} \) and we add \( x \) at the end.

Then, we deduce the recurrence relation \( a_{n,k}^i = a_{n-1,k-1}^i + (n-1)a_{n-1,k}^i \) with \( a_{n,0}^i = (n-1)! \) for \( n \geq 1 \), \( a_{0,0}^i = 1 \) and the bivariate exponential generating function is

\[
A^i(x, y) = \frac{1}{y(1-x)^x} = \frac{1}{y} + 1
\]

which proves that \( a_{n,k}^i = c(n, k + 1) \).

Below, we prove that \( a_{n,k}^i \) also counts \( n \)-length permutations having \( k \) occurrences of the pattern \( p_0 \).
Theorem 2.1. The number $a_{n,k}^{0}$ of $n$-length permutations having $k$ occurrences of pattern $p_0$ equals $a_{n,k}^{1} = e(n, k + 1)$.

Proof. An $n$-length permutation $\sigma \in A_{n,k}^{0}$ can be uniquely obtained from an $(n - 1)$-length permutation $\pi$ by one of the two following constructions:

(i) if $\pi \in A_{n-1,k-1}^{0}$, then we increase by one all values of $\pi$ and we add 1 at the end;

(ii) if $\pi \in A_{n-1,k}^{0}$, then we increase by one all values of $\pi$ greater than or equal to a given value $x$, $1 < x \leq n$, and we add $x$ at the end.

We deduce the recurrence relation $a_{n,k}^{0} = a_{n-1,k-1}^{0} + (n - 1)a_{n-1,k}^{0}$ with the initial condition $a_{n,0}^{0} = (n - 1)!$, and then $a_{n,k}^{0} = a_{n,k}^{1} = c(n, k + 1)$. □

Now, we focus on the distribution of the pattern $p_2$. Table 1 provides exact values for small sizes.

Theorem 2.2. We have

$$A^{2}(x, y) = \frac{e^{(x-1)y}}{(1 - x)^{y}},$$

and the general term $a_{n,k}^{2}$ satisfies for $n \geq 2$ and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$a_{n,k}^{2} = na_{n-1,k}^{2} + (n - 1)a_{n-2,k-1}^{2} - (n - 1)a_{n-2,k}^{2}$$

with the initial conditions $a_{n,0}^{2} = 1$ and $a_{n,k}^{2} = 0$ for $n \geq 0$ and $k > \left\lfloor \frac{n}{2} \right\rfloor$ (see Table 1 and Sequence A136394 in [15]).

Proof. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ denote a permutation of length $n$ having $k$ occurrences of pattern $p_2$. Let $u_{n,k}$ (resp. $v_{n,k}$) be the number of such permutations satisfying $\sigma_n = n$ (resp. $\sigma_n < n$). Obviously, we have

$$a_{n,k}^{2} = u_{n,k} + v_{n,k}.$$

A permutation $\sigma$ with $\sigma_n = n$ can be uniquely constructed from an $(n - 1)$-length permutation $\pi$ as $\sigma = \pi_1\pi_2 \ldots \pi_{n-1}n$. No new occurrences of $p_2$ are created, and we obtain

$$u_{n,k} = a_{n-1,k}^{2}.$$

A permutation $\sigma$ satisfying $\sigma_n < n$ can be uniquely obtained from an $(n - 1)$-length permutation $\pi$ by adding a value $x < n$ on the right side of its one-line notation, after increasing by one all the values greater than or equal to $x$. This construction creates a new pattern $p_2$ if and only if $\pi$ ends with $n - 1$. Thus, we deduce

$$v_{n,k} = (n - 1)u_{n-1,k-1} + (n - 1)v_{n-1,k}.$$

Combining the equations, we obtain for $n \geq 2$ and $k \geq 1$

$$a_{n,k}^{2} = na_{n-1,k}^{2} + (n - 1)a_{n-2,k-1}^{2} - (n - 1)a_{n-2,k}^{2},$$

which implies the following differential equation

$$\frac{\partial A^{2}(x, y)}{\partial x} = (y - 1)x A^{2}(x, y) + \frac{\partial (x A^{2}(x, y))}{\partial x},$$

where $A^{2}(x, 0) = 1$.

A simple calculation provides the claimed closed form for the generating function $A^{2}(x, y)$. □

Corollary 2.1. For $0 \leq i \leq 2$, the patterns $p_i$ are equipopular on $S_n$. Their popularity is given by the generalized Stirling number $n! \cdot (H_n - 1)$ (see Sequence A001705 in [15]) where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the $n$th harmonic number.

Proof. The generating function of the popularity is directly deduced from the bivariate generating function of pattern distribution by calculating

$$\frac{\partial A^{1}(x, y)}{\partial y} \bigg|_{y=1} = \frac{\partial A^{2}(x, y)}{\partial y} \bigg|_{y=1}.$$

□

The statistic $\text{des}_2$ has a different distribution from $\text{des}_0$ and $\text{des}_1$, but the three patterns $p_0, p_1, p_2$ have the same popularity. Below we present a bijection on $S_n$ that transports the statistic $\text{des}_2$ to the statistics $\text{pcyc} = \text{cyc} - \text{fix}$. 
There is a one-to-one correspondence \( \phi \) on \( S_n \) such that for any \( \pi \in S_n \), we have

\[
\text{des}_2 \pi = \text{pcyc} \phi(\pi).
\]

Proof. Let \( \pi \) be a permutation of length \( n \) having \( k \) occurrences of \( p_2 \). We decompose

\[
\pi = B_0 \pi_{i_1} A_1 B_1 \pi_{i_2} A_2 B_2 \pi_{i_3} \cdots \pi_{i_k} A_k B_k,
\]

where

- \( \pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k} \) are the tops of the occurrences of \( p_2 \), i.e. values \( \pi_{i_j} > \pi_{i_{j+1}} \) such that there does not exist \( \ell < i_j \) such that \( \pi_{i_\ell} > \pi_{i_j} \),
- \( A_j \) is a maximal sequence such that all its values are lower than \( \pi_{i_j} \),
- for \( 0 \leq j \leq k \), \( B_j \) is an increasing sequence such that \( \pi_{i_j} < \min B_j \) and \( \max B_j < \pi_{i_{j+1}} \).

Now we construct an \( n \)-length permutation \( \phi(\pi) \) with \( k \) pure cycles as follows:

\[
\phi(\pi) = (\pi_{i_1} A_1) \cdot (\pi_{i_2} A_2) \cdot \cdots \cdot (\pi_{i_k} A_k).
\]

For instance, if \( \pi = 125346879 \) then \( \phi(\pi) = (5,3,4) \cdot (8,7) \). The map \( \phi \) is clearly a bijection on \( S_n \) such that \( \text{des}_2 \pi \) equals the number of pure cycles in \( \phi(\pi) \).

Note that \( \phi^{-1} \) is closely related to the Foata fundamental transformation [9].

3. The statistic \( \text{pex} \) of pure excedances

In order to prove the equidistribution of \( \text{pex} \) and \( \text{des}_2 \), regarding Theorem 2.3, it suffices to construct a bijection on \( S_n \) that transports pure excedances to pure cycles. Here, we first exhibit a bijection on the set \( D_n \) of \( n \)-length derangements (permutations without fixed points), then we extend it to the set of all permutations \( S_n \).

Any permutation \( \pi \in S_n \) is uniquely decomposed as a product of transpositions of the following form:

\[
\pi = \langle t_1,1 \rangle \cdot \langle t_2,2 \rangle \cdots \langle t_n,n \rangle
\]

where \( t_i \) are integers such that \( 1 \leq t_i \leq i \). The transposition array of \( \pi \) is defined by \( T(\pi) = t_1 t_2 \cdots t_n \), which induces a bijection \( \pi \mapsto T(\pi) \) from \( S_n \) to the product set \( T_n = [1] \times [2] \times \cdots \times [n] \). By Lemma 1 from [1], the number of cycles of a permutation \( \pi \) is given by the number of fixed points in \( T(\pi) \). Moreover, it is straightforward to check the two following properties:

- if \( t_i = i \), then \( \pi_i = i \) if and only if there is no number \( j > i \) such that \( t_j = t_i = i \);
- if \( t_i = i \) and \( \pi_i = i \), then \( i \) is the minimal element of a cycle of length at least two in \( \pi \).

So, we deduce the following lemma.

Lemma 3.1. The transposition array \( t_1 t_2 \cdots t_n \in T_n \) corresponds to a derangement if and only if: \( t_i = i \Rightarrow \) there is a \( j > i \) such that \( t_j = i \).

Given a derangement \( \pi = \pi_1 \pi_2 \cdots \pi_n \in D_n \) and its graphical representation \( \{ (i, \pi_i), i \in [n] \} \). We say that the square \( (i,j) \in [n] \times [n] \) is free if all following conditions hold:

(i) Neither \( \pi_i \) nor \( i \) is a position of a pure excedance;
(ii) \( (i,j) \) is not on the first diagonal, i.e. \( j \neq i \);
(iii) there does not exist \( k > i \) such that \( \pi_k = j \);
(iv) $j$ is not a pure excedance such that $j < i$ and $\pi^{-1}(j) < i$;

(v) there does not exist $k < i$, with $\pi_k = j > i$ such that all values of the interval $[i, j - 1]$ appear on the right of $\pi_i$ in $\pi$.

Whenever at least one of the statements above is not satisfied, we say that the square $(i, j)$ is unfree. Notice that if $i$ and $\pi_i$ are not the positions of a pure excedance, then the square $(i, \pi_i)$ is always free. So, for a column $i$ of the graphical representation of $\pi$ such that $i$ and $\pi_i$ are not the positions of a pure excedance, we label by $j$ the $j$th free square from the bottom to the top. We refer to Figure 2 for an example of this labeling.

Now we define the map $\lambda$ from $D_n$ to the set $T_n^*$ of transposition arrays of length $n$ satisfying the property of Lemma 3.1.

For a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n \in D_n$, we label its graphical representation as defined above, and $\lambda(\pi) = \lambda_1 \lambda_2 \ldots \lambda_n$ is obtained as follows:

- if $i$ is a pure excedance in $\pi$, then we set $\lambda_i = i$ and $\lambda_{\pi^{-1}(i)} = i$;
- otherwise, $\lambda_j$ is the sum of the label of the free square $(i, \pi_i)$ with the number of pure excedances $k < i$ such that $\pi^{-1}(k) < i$.

For instance, if $\pi = 6 8 1 2 5 4 7 3 2 1 1 9 10$ then we obtain $\lambda(\pi) = 1 1 2 4 4 2 1 1 9 1 9 10$ (see Figure 2).

Figure 2: Illustration of the bijection $\lambda$ for $\pi = 6 8 1 2 5 4 7 3 2 1 1 9 10$ and $\lambda(\pi) = 1 1 2 4 4 2 1 1 9 1 9 10$.

**Theorem 3.1.** The map $\lambda$ from $D_n$ to $T_n^*$ is a bijection such that

\[ \text{pex } \pi = \text{fix } \lambda(\pi). \]

**Proof.** Since the cardinality of $T_n^*$ equals that of $D_n$, and the image of $D_n$ by $\lambda$ is contained in $T_n^*$, it suffices to prove the injectivity.

Let $\pi$ and $\sigma$, $\pi \neq \sigma$, be two derangements in $D_n$. If $\pi$ and $\sigma$ do not have the same pure excedances, then, by construction, $\lambda(\pi)$ and $\lambda(\sigma)$ do not have the same fixed points, and thus $\lambda(\pi) \neq \lambda(\sigma)$.

Now, let us assume that $\pi$ and $\sigma$ have the same pure excedances. If there is a pure excedance $i$ such that $\pi^{-1}(i) \neq \sigma^{-1}(i)$ then the definition implies $\lambda(\pi) \neq \lambda(\sigma)$. Otherwise the two permutations have the same pure excedances $i$, and for each of them we have $\pi^{-1}(i) = \sigma^{-1}(i)$. Let $j$ be the greatest integer such that $\pi_j \neq \sigma_j$ (without loss of generality, we assume $\pi_j < \sigma_j$). In this case, $j$ is not a pure excedance for the two permutations. Thus, $\lambda(\pi)_j$ (resp. $\lambda(\sigma)_j$) is the sum of the label of $(j, \pi_j)$ (resp. $(j, \sigma_j)$) with the number of pure excedances
k < j such that \( \pi^{-1}(k) < j \) (resp. \( \sigma^{-1}(k) < j \)). Since we have \( \pi_j < \sigma_j \), the label of \((j, \pi_j)\) is less than the label of \((j, \sigma_j)\), and the number of pure excedances \( k < j \) such that \( \pi^{-1}(k) < j \) is less than or equal to the number of pure excedances \( k < j \) such that \( \sigma^{-1}(k) < j \). Then we have \( \lambda(\pi_j) < \lambda(\sigma_j) \). Then \( \lambda \) is an injective map, and thus a bijection.

**Theorem 3.2.** There is a one-to-one correspondence \( \psi \) on the set \( D_n \) of \( n \)-length derangements such that for any \( \pi \in D_n \),

\[
\text{pex} \ \pi = \text{cyc} \ \psi(\pi).
\]

**Proof.** Considering Theorem 2.3 and Theorem 3.1, we define for any \( \pi \in D_n \), \( \psi(\pi) = \phi(\sigma) \) where \( \sigma \) is the permutation having \( \lambda(\pi) \) as transposition array.

**Theorem 3.3.** The two bistatistics \( (\text{pex, fix}) \) and \( (\text{pcyc, fix}) \) are equidistributed on \( S_n \).

**Proof.** Considering Theorem 3.2, we define the map \( \bar{\psi} \) on \( S_n \). Let \( \bar{\pi} \) be the permutation obtained from \( \pi \) by deleting all fixed points and after rescaling as a permutation. Let \( I = \{i_1, i_2, \ldots, i_k\} \) be the set of fixed points of \( \pi \). Then, we set \( \pi'' = \bar{\psi}(\bar{\pi}) \). So, \( \sigma = \psi(\pi) \) is obtained from \( \pi'' \) by inserting fixed points \( i \in I \) after a shift of all other entries in order to produce a permutation in \( S_n \). By construction, we have \( \text{pex} \ \pi = \text{pcyc} \ \sigma \) and \( \text{fix} \ \pi = \text{fix} \ \sigma \) which completes the proof.

A byproduct of this theorem is

**Corollary 3.1.** The statistics \( \text{cyc} \) and \( \text{pex} + \text{fix} \) are equidistributed on \( S_n \).

Also, a direct consequence of Theorems 2.3 and 3.3 is

**Theorem 3.4.** The two statistics \( \text{pex} \) and \( \text{des}_2 \) are equidistributed on \( S_n \).

Notice that Foata’s first transformation is not a candidate for proving the equidistribution of \( \text{pex} \) and \( \text{des}_2 \), while it transports \( \text{exc} \) to \( \text{des} \). Combining Theorem 3.4 and Corollary 2.1 we have the following.

**Corollary 3.2.** For \( 0 \leq i \leq 2 \), the patterns \( p_i \) and \( \text{pex} \) are equipopular on \( S_n \) (see Sequence A001705 in [15]).

Finally, we present two conjectures for future works.

**Conjecture 3.1.** The two bistatistics \( (\text{des}_2, \text{cyc}) \) and \( (\text{pex, cyc}) \) are equidistributed on \( S_n \).

**Conjecture 3.2.** The two bistatistics \( (\text{des}_2, \text{des}) \) and \( (\text{pex, exc}) \) are equidistributed on \( S_n \).

It is interesting to remark that \( (\text{des}, \text{cyc}) \) and \( (\text{exc, cyc}) \) are not equidistributed. Indeed, there are 3 permutations in \( S_3 \) having \( \text{exc} = 1 \) and \( \text{cyc} = 2 \), namely 132, 213, 321, but only 2 permutations with \( \text{des} = 1 \) and \( \text{cyc} = 2 \), videlicet 132 and 213. So, if the Conjectures 3.1 and 3.2 are true then their proofs are probably independent.

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