Structure constants for the osp(1|2) current algebra

I.P. Ennes⋆, A.V. Ramallo† and J. M. Sanchez de Santos‡

Departamento de Física de Partículas,
Universidad de Santiago,
E-15706 Santiago de Compostela, Spain.

ABSTRACT

We study the free field realization of the two-dimensional osp(1|2) current algebra. We consider the case in which the level of the affine osp(1|2) symmetry is a positive integer. Using the Coulomb gas technique we obtain integral representations for the conformal blocks of the model. In particular, from the behaviour of the four-point function, we extract the structure constants for the product of two arbitrary primary operators of the theory. From this result we derive the fusion rules of the osp(1|2) conformal field theory and we explore the connections between the osp(1|2) affine symmetry and the $N = 1$ superconformal field theories.

⋆ E-mail: ENNES@GAES.USC.ES
† E-mail: ALFONSO@GAES.USC.ES
‡ E-mail: SANTOS@GAES.USC.ES
1. Introduction

Current algebras play a prominent role in two-dimensional Conformal Field Theory (CFT) [1]. In fact, the theories endowed with an affine Kac-Moody symmetry can be considered as the building blocks out of which most known CFT’s can be constructed. The basic procedures to generate new CFT’s from models possessing a current algebra symmetry are the coset construction [2] and the hamiltonian reduction [3]. Moreover, an $sl(N)$ affine algebra shows up when quantum two-dimensional $W_N$ gravity is analyzed in the light-cone gauge [4] and, as was shown in refs. [5, 6], the topological $sl(N)/sl(N)$ coset theories can be used to describe non-critical $W_N$-strings.

One of the simplest Lie superalgebras is $osp(1|2)$. In CFT the affine version of this superalgebra is frequently encountered when one studies models with $N = 1$ supersymmetry. Indeed, the $osp(1|2)$ current algebra is the starting point in the construction of the $N = 1$ superconformal minimal models by means of the hamiltonian reduction procedure [7]. It also appears in the quantization of two-dimensional supergravity in the light-cone gauge [8] and its topological version, i.e. the $osp(1|2)/osp(1|2)$ coset model, is related to the Ramond-Neveu-Schwarz non-critical superstrings [9, 10]. It is thus interesting to have a clear understanding of this current algebra symmetry in order to improve our knowledge of the two-dimensional superconformal symmetry and, in particular, to unravel the connection of the latter with the Lie superalgebra theory.

In this paper we shall determine the operator product algebra of the primary fields of the $osp(1|2)$ current algebra. With this purpose in mind, we shall develop a free field representation of the $osp(1|2)$ theory, which was previously introduced in ref. [7] to study its hamiltonian reduction. These free field constructions have become a powerful tool in CFT. Indeed, the Feigin-Fuchs formalism [11], as was spelt out by Dotsenko and Fateev in ref. [12], has allowed to represent the conformal blocks of the minimal Virasoro models and to obtain their operator algebra. A similar analysis has been performed in refs. [13, 14, 15] for the minimal
superconformal models.

The free field representation of the bosonic current algebras, the so-called Wakimoto representation, has been introduced in refs. [16, 17, 18]. This representation has been used by Dotsenko in ref. [19] to evaluate the correlation functions and the structure constants of the $sl(2)$ current algebra. In the $osp(1|2)$ case we shall follow the methodology of refs. [12, 19]. We shall restrict ourselves to the case in which the level of the affine $osp(1|2)$ algebra is a positive integer. This is equivalent in the $sl(2)$ case to consider integrable representations. The structure constants and fusion rules we have obtained have been reported, without proof, in our previous paper [20]. We present here a full account of our results.

The organization of the paper is as follows. In section 2 we introduce the basic ingredients of our Coulomb gas formalism. After recalling the free field representation of the currents, which we take from ref. [7], we consider the realization of the primary fields. As always happens in this kind of free field realizations, we have two representations for the same primary operator. In one of these representations, which we shall refer to as the conjugate representation, the unity is not realized as the trivial operator. An analysis of the form of this conjugate unit serves to establish the basic rules to compute vacuum expectation values in the Fock space of our free field representation. The last ingredient needed to compute correlation functions is the screening charge. A local operator, satisfying the requirements demanded to one of such a charge, has been found in ref. [7]. Once the basic set-up of the formalism is in place, we can start to compute the correlation functions of the theory. An important consistency check of our approach is the verification that our expectation values satisfy the selection rules dictated by the $osp(1|2)$ representation theory, a brief review of which is given in appendix A. At the end of section 2 we analyze the two- and three-point functions of the model and we show that, indeed, our free field construction incorporates the selection rules expected for the coupling of $osp(1|2)$ representations.

Section 3 is devoted to the study of the four-point functions. By looking at
the local behaviour of the conformal blocks, we determine the possible s-channel intermediate states and check that they correspond precisely to the non-vanishing couplings found in the analysis of the three-point functions. In the computation of the structure constants we shall need the value of the normalization integrals of the conformal blocks. The evaluation of these integrals is highly non-trivial and it is done in appendix B.

The determination of the operator algebra of the model is the objective of section 4. We first construct the monodromy invariant four-point correlators. Following the standard procedure introduced in ref. [12] for the minimal models, the structure constants can be obtained from the coefficients appearing in the power expansion of the physical four-point correlator. Actually, a suitable normalization must be performed in order to properly identify the structure constants. After this is done, one is left with a long and uninspiring expression for these constants. It turns out, however, that one can convert this expression into a symmetric and rather transparent equation which is, actually, very similar to the one found in refs. [21, 19] for the $sl(2)$ current algebra.

The fusion rules that follow from our structure constants are given in section 5. As we shall argue in this section, these fusion rules provide some new insights on the relation between the osp(1|2) theory and the minimal superconformal models, a fact which was, actually, one of the main motivations for our work. We end this section with some comments on how to extend our analysis to the case in which $k$ is not integer. Finally, in section 6 we draw some conclusions from our results and indicate some possible lines of future research.
2. Free field representation of the osp(1|2) current algebra

The osp(1|2) Lie superalgebra has three even (bosonic) generators and two odd (fermionic) ones. Its affine version, \( i.e. \) the osp(1|2) current algebra, is generated by three bosonic and two fermionic currents. We shall denote the former by \( J_\pm \) and \( H \) whereas for the latter we shall use the symbols \( j_\pm \). These currents can be realized in terms of a scalar field \( \phi \), a pair of two conjugate bosonic fields \( (w, \chi) \) and two fermionic fields \( \psi \) and \( \bar{\psi} \). The non-vanishing operator product expansions (OPE’s) among them will be taken as:

\[
\begin{align*}
  w(z_1) \chi(z_2) &= \psi(z_1) \bar{\psi}(z_2) = \frac{1}{z_1 - z_2}, \\
  \phi(z_1) \phi(z_2) &= -\log (z_1 - z_2). \quad (2.1)
\end{align*}
\]

The fields \( \bar{\psi} \) and \( \psi \) (\( w \) and \( \chi \)) constitute a \( bc \) (\( \beta\gamma \)) system with conformal dimensions \( \Delta(\bar{\psi}) = 1 \) and \( \Delta(\psi) = 0 \) (\( \Delta(w) = 1 \) and \( \Delta(\chi) = 0 \) respectively). The explicit form of the osp(1|2) currents is [7]:

\[
\begin{align*}
  J_+ &= w \\
  J_- &= -w\chi^2 + i\sqrt{2k+3} \chi \partial \phi - \chi \psi \bar{\psi} + k \partial \chi + (k+1) \psi \partial \psi \\
  H &= -w\chi + \frac{i}{2} \sqrt{2k+3} \partial \phi - \frac{1}{2} \psi \bar{\psi} \\
  j_+ &= \bar{\psi} + w\psi \\
  j_- &= -\chi(\bar{\psi} + w\psi) + i\sqrt{2k+3} \psi \partial \phi + (2k+1) \partial \psi,
\end{align*}
\]

where the c-number \( k \) is the level of the affine osp(1|2) superalgebra. In what follows we shall restrict ourselves to the case in which \( k \) is a positive integer. It may be easily verified using (2.1) that the currents \( J_\pm \) and \( H \) close an \( sl(2) \) algebra. The fermionic operators \( j_\pm \) extend this \( sl(2) \) algebra to the full osp(1|2) symmetry. In order to construct a CFT in which the currents (2.2) are dimension-one primary fields, we must first define the energy-momentum tensor \( T \) of the model. For current algebras the Sugawara construction provides a method to get \( T \) as an
expression quadratic in the currents. In our case this construction yields:

$$T^J = \frac{1}{2k+3} \left[ J_+ J_- + J_- J_+ + 2H^2 - \frac{1}{2} j_+ j_- + \frac{1}{2} j_- j_+ \right], \quad (2.3)$$

where normal-ordering is understood. Using standard methods one can obtain $T$ as a function of our basic set of free fields:

$$T = w \partial \chi - \bar{\psi} \partial \psi - \frac{1}{2} (\partial \phi)^2 + \frac{i}{2} \alpha_0 \partial^2 \phi. \quad (2.4)$$

In eq. (2.4) $\alpha_0$ is a background charge for the field $\phi$ which, in terms of the level $k$, can be written as:

$$\alpha_0 = -\frac{1}{\sqrt{2k+3}}. \quad (2.5)$$

It is straightforward to prove that the operator $T$ given in eqs. (2.3) and (2.4) satisfies the Virasoro algebra with central charge given by:

$$c = \frac{2k}{2k+3}. \quad (2.6)$$

Let us construct now the primary fields of the theory. One should have a multiplet of such fields associated to each irreducible representation of the superalgebra. The representation theory of the $osp(1|2)$ Lie superalgebra has been studied in ref. [22](see also ref. [23] for an account of the general theory of Lie superalgebras). Let us recall some of its basic features. A more detailed review is given in appendix A. The $osp(1|2)$ finite dimensional irreducible representations closely resemble those of the $sl(2)$ algebra. They are characterized by an integer or half-integer number $j$ which we shall refer to as the isospin of the representation. A general state of the isospin $j$ representation will be denoted by $|j, m>$, where $m$ is the eigenvalue of the Cartan generator $H$ ($m = -j, -j + \frac{1}{2}, \cdots, j - \frac{1}{2}, j$). Acting with the odd generators $j_{\pm}$, the value of $m$ is shifted by $\pm 1/2$, while the bosonic currents $J_{\pm}$ produce a change in the eigenvalue $m$ of one unit. The highest weight state of the
isospin $j$ representation, whose dimensionality is $4j+1$, is $|j, j>$. To completely characterize the representation we must specify, in addition, the statistics of its highest weight state. If $|j, j>$ is bosonic (fermionic) we will say that the representation is even (odd). It is important to point out that when $j - m$ is integer (half-integer) the states $|j, m>$ and $|j, j>$ have the same (opposite) statistics.

The primary field associated to the $|j, m>$ state will be denoted by $\Phi^{j}_{m}$. The conformal dimensions $\Delta_j$ of these operators can be written in terms of the quadratic Casimir invariant $C_2$ of the osp(1|2) superalgebra. The expression of $C_2$ is given in appendix A. By comparing this expression with the bilinear form appearing in $T$ (see eq. (2.3)) one easily concludes that:

$$\Delta_j = \frac{2C_2}{2k + 3}.$$  \hfill (2.7)

Taking into account that for an isospin $j$ representation the value of $C_2$ is $j(j + \frac{1}{2})$ (see eq. (A5)), one can rewrite eq. (2.7) as:

$$\Delta_j = \frac{j(2j + 1)}{2k + 3}.$$  \hfill (2.8)

The degenerate admissible representations of osp(1|2) have been studied in refs. [24, 25] by considering the coset decomposition of osp(1|2) into the $sl(2)$ algebra generated by its bosonic currents and an osp(1|2)/$sl(2)$ theory. It was shown in refs. [24, 25] that when the osp(1|2) level $k$ is admissible with respect to the even $sl(2)$ algebra, the osp(1|2)/$sl(2)$ theory can be identified with one of the models of the Virasoro minimal series. The osp(1|2) levels $k$ for which degenerate representations appear are parametrized by two integers $p$ and $q$, by means of the relation $2k + 3 = \frac{q}{p}$, where $q + p$ is even and $p$ and $\frac{p+q}{2}$ are relatively prime. The isospins $j$ corresponding to the admissible representations are determined by the equation $4j + 1 = r - s\frac{q}{p}$ with $r + s$ odd and $r, s$ taking values in the ranges $r = 1, \cdots, q - 1$ and $s = 0, \cdots, p - 1$. In our case, i.e. when $k$ is a positive integer, $p = 1$ and thus $s = 0$. Therefore, since $r$ must be odd, the highest value it can take
is $2k + 1$ and, thus, we conclude that the admissible representations have integer and half-integer isospins $j$ that satisfy $j \leq k/2$. Notice that this corresponds to taking integral representations for the even $\mathfrak{sl}(2)$ algebra. It will be understood from now on that this constraint is satisfied by all the primary fields $\Phi^j_m$ we shall consider.

The actual form of the operators $\Phi^j_m$ in our free field realization can be determined as follows. First of all, it is clear that one can obtain the fields $\Phi^j_m$ with $m < j$ by acting with the lowering operators $J_-$ and $j_-$ on the highest weight field $\Phi^j_j$. Moreover, the OPE of the raising currents $J_+$ and $j_+$ with $\Phi^j_j$ must vanish. By inspecting the realization of $J_+$ and $j_+$ in eq. (2.2), one immediately reaches the conclusion that in the expression of $\Phi^j_j$ only the fields $w$ and $\phi$ can appear. Let us suppose that we adopt an ansatz for $\Phi^j_j$ in which the field $w$ is not present. We therefore shall assume that $\Phi^j_j$ can be written as a vertex operator of the form:

$$
\Phi^j_j = e^{i\alpha_j \alpha_0 \phi},
$$

(2.9)

where $\alpha_j$ is a constant whose exact dependence on $j$ has to be determined. The easiest way to fix the value of $\alpha_j$ is by requiring that the current $H$ acts diagonally on $\Phi^j_j$ with eigenvalue equal to $j$. The $H$-charge of the operator (2.9) is $-\alpha_j/2$, as an straightforward calculation using eqs. (2.2) and (2.5) shows and, thus, we must take $\alpha_j = -2j$. It is also simple to verify that for this value of $\alpha_j$ the conformal dimension of $\Phi^j_j$ is given precisely by eq. (2.8). As we have pointed out above, the other members $\Phi^j_m$ of the field multiplet can be obtained by successive application of the operators $J_-$ and $j_-$. The result one arrives at is the following:

$$
\Phi^j_m = \begin{cases} 
\chi^{j-m} e^{-2ij\alpha_0 \phi} & \text{if } j - m \in \mathbb{Z} \\
\chi^{j-m-\frac{1}{2}} \psi e^{-2ij\alpha_0 \phi} & \text{if } j - m \in \mathbb{Z} + \frac{1}{2}.
\end{cases}
$$

(2.10)

Notice that the conformal dimensions of the operators (2.10) are $m$-independent and given by eq. (2.8)( the fields $\chi$ and $\psi$ have vanishing conformal weight). It is
also easy to obtain the action of the currents on the operators (2.9). The OPE of the Cartan current $H$ and the fields (2.10) is:

$$H(z_1) \Phi^j_m(z_2) = m \frac{\Phi^j_m(z_2)}{z_1 - z_2},$$  \hspace{0.5cm} (2.11)

which confirms that the integer $m$ in eq. (2.10) is the $H$-charge. Moreover, the operators $J_\pm$ connect two fields whose value of $m$ differs in one unit:

$$J_\pm(z_1) \Phi^j_m(z_2) = \begin{cases} 
(j \mp m) \frac{\Phi^j_{m\pm1}(z_2)}{z_1 - z_2} & \text{if } j - m \in \mathbb{Z} \\
(j \mp m - \frac{1}{2}) \frac{\Phi^j_{m\pm1}(z_2)}{z_1 - z_2} & \text{if } j - m \in \mathbb{Z} + \frac{1}{2}.
\end{cases}$$  \hspace{0.5cm} (2.12)

Finally, as they should, the fermionic currents $j_\pm$ change the value of $m$ by one-half unit:

$$j_\pm(z_1) \Phi^j_m(z_2) = \begin{cases} 
(j \mp m) \frac{\Phi^j_{m\pm1/2}(z_2)}{z_1 - z_2} & \text{if } j - m \in \mathbb{Z} \\
\pm \frac{\Phi^j_{m\pm1/2}(z_2)}{z_1 - z_2} & \text{if } j - m \in \mathbb{Z} + \frac{1}{2}.
\end{cases}$$  \hspace{0.5cm} (2.13)

The representation (2.10) of the primary fields is not unique. Indeed, as we will show below, one can find a representation for these fields which is conjugate to the one written in (2.10). Recall that in (2.9) we have discarded the possibility of having a power of the field $w$. Let us now include this type of term in our ansatz for the highest weight operator. For reasons that soon will become apparent, it is convenient to find first the conjugate of the isospin zero operator. Let us call $I$ the operator (2.10) for $j = m = 0$. It is obvious from (2.10) that $I = 1$, so we are trying to get a new representation for the unit operator. This conjugate unit will
be denoted by $\tilde{I}$. Its expression will be of the form:

$$
\tilde{I} = w^A e^{iB \alpha_0 \phi},
$$

(2.14)

where $A$ and $B$ are constants. As we have already discussed, the highest weight conditions:

$$
j_+(z_1) \tilde{I}(z_2) = J_+(z_1) \tilde{I}(z_2) = 0,
$$

(2.15)

are satisfied for any value of $A$ and $B$. Moreover, the singular terms in the product of the current $H$ and $\tilde{I}$ are given by:

$$
H(z_1) \tilde{I}(z_2) = (A - \frac{1}{2} B) \frac{\tilde{I}(z_2)}{z_1 - z_2}.
$$

(2.16)

Therefore, if we require neutrality of $\tilde{I}$ with respect to $H$ (recall that $\tilde{I}$ corresponds to $m = 0$), the following condition must be imposed

$$
B = 2A.
$$

(2.17)

Eq. (2.17) fixes $B$ in terms of $A$. This latter constant can be determined by looking at the current algebra descendants of the conjugate unit $\tilde{I}$. In fact, acting with $j_-$ and $J_-$ on $\tilde{I}$, one gets:

$$
j_-(z_1) \tilde{I}(z_2) = \frac{\xi(z_2)}{z_1 - z_2},
$$

$$
J_-(z_1) \tilde{I}(z_2) = A(k + A + 1) \frac{w(z_2)^{A-1} e^{2iA \alpha_0 \phi(z_2)}}{(z_1 - z_2)^2} + \frac{\Xi(z_2)}{z_1 - z_2},
$$

(2.18)

where $\xi(z)$ and $\Xi(z)$ are operators whose precise form will not be needed. One should not have double pole singularities in the OPE’s of $j_-$ and $J_-$ with a highest weight operator. These double pole singularities do not appear in (2.18) when $A = -k - 1$. Moreover, one can easily verify that, apart from the trivial solution
\( A = 0 \), only for this value of \( A \) the conformal dimension of \( \tilde{I} \) is zero. We are thus led to adopt the following expression for \( \tilde{I} \):

\[
\tilde{I} = w^s e^{2i\alpha_0 \phi},
\]

where

\[
s = -k - 1.
\]

Similarly to what happens in the \( sl(2) \) case [19], one can show that the descendant fields \( \xi \) and \( \Xi \) generate null vectors in the module of the trivial \( j = 0 \) representation. Indeed, one can check that \( \xi \) and \( \Xi \) satisfy the highest weight conditions for the \( osp(1|2) \) current algebra and, thus, they are going to decouple (i.e. to vanish) in the conformal blocks of the model. Moreover, the form (2.19) of the conjugate identity fixes the charge asymmetry of the Fock space metric. This charge asymmetry is present in other Coulomb gas representations of CFT’s [12, 13]. The best way to determine it is by requiring the vacuum expectation value of the conjugate identity to be non-vanishing. By inspecting (2.19), one can easily obtain a series of selection rules that the non-vanishing correlators must satisfy.

Let us imagine that we are computing the expectation value \(< \prod_i O_i >\), where \( O_i \) are general operators of the form \( O_i = w^{n_i} \chi^{m_i} e^{i\alpha_i \phi} \). Calling \( N(w) = \sum_i n_i \) and \( N(\chi) = \sum_i m_i \), one gets the following conditions:

\[
N(w) - N(\chi) = s
\]

\[
\sum_i \alpha_i = 2\alpha_0 s.
\]

A possible way to implement the conditions (2.21) is by defining the \( osp(1|2) \) correlators with some fields inserted at the point at infinity. We will not need, however, to be very explicit about this point as far as the conditions (2.21) are satisfied. Notice that when \( k \) is a positive integer, \( s \) is negative. Therefore, in some of the expressions of the conjugate fields (such as the one of \( \tilde{I} \) in (2.19)) we
are going to have negative powers of the \( w \) field. It will be understood in what follows that these negative powers have been properly defined (the situation is similar to the \( sl(2) \) case, see [19]).

For \( j > 0 \) the form of the conjugate operators can be obtained in a similar way. In general, the highest weight operator \( \tilde{\Phi}^j \) will be given by an expression of the type (2.14). Requiring now the \( H \)-eigenvalue to be \( j \), one gets the constraint \( A - \frac{B}{2} = j \), whereas by fixing the conformal dimension of \( \tilde{\Phi}_j \) to the value (2.8) one obtains \( A = 2j + s \) and \( B = 2j + 2s \). Therefore \( \tilde{\Phi}^j \) is given by:

\[
\tilde{\Phi}_j = w^{2j+s} e^{2i(j+s)\alpha_0 \phi}.
\]

The remaining operators \( \tilde{\Phi}_m \) of the conjugate multiplet can be obtained by acting with the currents \( j_- \) and \( J_- \) on the highest weight \( \tilde{\Phi}_j \). After a simple calculation one gets:

\[
j_-(z_1) \tilde{\Phi}_j (z_2) = \frac{\tilde{\Phi}_{j-1/2} (z_2)}{z_1 - z_2}, \tag{2.23}
\]

\[
J_-(z_1) \tilde{\Phi}_j (z_2) = \frac{\tilde{\Phi}_{j-1} (z_2)}{z_1 - z_2},
\]

where \( \tilde{\Phi}_{j-1/2} \) is given by:

\[
\tilde{\Phi}_{j-1/2} = \frac{1}{2j} \left[ (2j + s) \bar{\psi} w^{2j+s-1} - s w^{2j+s} \psi \right] e^{2i(j+s)\alpha_0\phi}, \tag{2.24}
\]

and \( \tilde{\Phi}_{j-1} \) can be written as:

\[
\tilde{\Phi}_{j-1} = \left[ \chi \omega^{2j+s} - \frac{(2j + s)(2j + s - 1)}{2j} \partial \omega \omega^{2j+s-2} + \frac{2j + s}{2j} \left[ \bar{\psi}\psi - i\sqrt{2k + 3} \partial\phi \right] \omega^{2j+s-1} \right] e^{2i(j+s)\alpha_0\phi}. \tag{2.25}
\]

Notice that the OPE’s (2.23) are the same as those of \( j_- \) and \( J_- \) with the fields (2.10)(see eqs. (2.12) and (2.13)). This fact can be confirmed by computing other
singular product expansions such as:

\[ j_\pm(z_1) \tilde{\Phi}^j_{j-1/2}(z_2) = \pm 2j \frac{\tilde{\Phi}^j_{j-1/2\pm 1/2}(z_2)}{z_1 - z_2} \]

\[ j_+(z_1) \tilde{\Phi}^j_{j-1}(z_2) = \frac{\tilde{\Phi}^j_{j-1/2}(z_2)}{z_1 - z_2}. \]

By successive application of the currents \( j_- \) and \( J_- \), one can generate other components of the conjugate multiplet of primary fields. In general, the expressions of \( \tilde{\Phi}^j_m \) found in this way are increasingly complicated as \( m \) is decreased. Fortunately for our purposes the knowledge of the highest weight conjugate field \( \tilde{\Phi}^j_j \) will be enough.

Following the standard procedure to compute correlators in the Coulomb gas realizations of CFT’s, one must represent the conformal blocks of the theory as expectation values of products of the fields \( \Phi^j_m \) and their conjugates \( \tilde{\Phi}^j_m \). However, in order to get a non-vanishing result, the conditions (2.21) must be fulfilled. This can only be achieved if the fields are screened by means of the insertion of a new operator \( Q \) (the screening charge), which must be invariant under the action of the \( \text{osp}(1|2) \) currents and must have zero conformal dimension. We shall represent \( Q \) as an integral of a local operator \( S(z) \) over a closed contour:

\[ Q = \oint dz \, S(z), \tag{2.27} \]

where \( S(z) \) has conformal weight equal to one and is such that its OPE’s with the \( \text{osp}(1|2) \) currents have only total derivatives. This last condition guarantees the (anti)commutation of \( Q \) with \( J_\pm, H \) and \( j_\pm \). As it has been checked in ref. [7], \( S \) can be taken as:

\[ S = (\bar{\psi} - w\psi) e^{i\alpha_0 \phi}. \tag{2.28} \]

In the remaining of this section we shall study the two- and three-point functions of the model. We shall verify that the formalism we have introduced correctly
reproduces the features of these correlators which are to be expected from the conformal invariance and the osp(1|2) representation theory (see appendix A).

The two-point function can be represented as an expectation value of the product of a field (2.10) and its conjugate. It can be easily seen that the conditions (2.21) can be satisfied without the insertion of the screening charge $Q$. Let us check this fact in the simplest case in which the highest weight conjugate operator (2.22) is used. The expectation value to be computed is:

$$\langle \Phi_j(z_1) \tilde{\Phi}_j(z_2) \rangle = \langle \chi(z_1)^{2j} e^{-2i j \alpha \phi(z_1)} [w(z_2)]^{2j+s} e^{2i(j+s)\alpha \phi(z_2)} \rangle .$$

(2.29)

A simple counting shows that, indeed, the conditions (2.21) are satisfied. Moreover, an elementary application of Wick’s theorem allows to write the right-hand side of eq. (2.29) as:

$$\langle \Phi_j(z_1) \tilde{\Phi}_j(z_2) \rangle = C \frac{1}{(z_1 - z_2)^{2\Delta_j}} ,$$

(2.30)

where $\Delta_j$ is given in (2.8) and $C$ is a constant proportional to the expectation value of $w^s$ in the osp(1|2) Fock space. One can also verify that for other $H$-eigenvalues of the fields, such as in $\langle \Phi_{j+\frac{1}{2}}(z_1) \tilde{\Phi}_{j-\frac{1}{2}}(z_2) \rangle$, the correlator is also given by the right-hand side of (2.30) (with the same value of the constant $C$). This is precisely the behaviour expected for an osp(1|2) CFT.

The three-point conformal blocks can be represented as correlators of two fields (2.10) and one conjugate operator $\tilde{\Phi}_m$. In this case screening charges must be inserted in order to satisfy the osp(1|2) charge asymmetry conditions. Therefore we must consider an expectation value of the form:

$$\langle \Phi_{m_1}^{j_1}(z_1) \Phi_{m_2}^{j_2}(z_2) \tilde{\Phi}_{m_3}^{j_3}(z_3) Q^n \rangle .$$

(2.31)

It is a simple exercise to determine the number $n$ of screening charges needed in order to have a non-zero result. Although we have not obtained the explicit expression of $\tilde{\Phi}_m$ for arbitrary $m$, we do know that the coefficient of the $\phi$ field
in the exponential is not changed by the action of the \( j_- \) and \( J_- \) currents and, therefore, for all values of \( m \) this coefficient is the same as in eq. (2.22). Thus we can write the value of the left-hand side of the second equation in (2.21) for the correlator (2.31) as:

\[
\sum \alpha_i = 2\alpha_0 \left( s + j_3 - j_1 - j_2 + \frac{n}{2} \right). \tag{2.32}
\]

Therefore from (2.21) we obtain that \( n \) is related to \( j_1, j_2 \) and \( j_3 \) through the expression:

\[
0 \leq j_1 + j_2 - j_3 = \frac{n}{2}, \tag{2.33}
\]

where the inequality is an obvious consequence of the fact that \( n \) is a positive integer. Eq. (2.33) implies, in particular, that \( j_3 \leq j_1 + j_2 \). Moreover, the three-point function we are considering could equally be represented by taking the conjugate operator to be the one of isospin \( j_1 \) or that of isospin \( j_2 \). In these two cases from the screening condition we get two inequalities similar to (2.33), namely:

\[
\begin{align*}
& j_1 + j_3 - j_2 \geq 0 \\
& j_2 + j_3 - j_1 \geq 0.
\end{align*} \tag{2.34}
\]

Notice that, combining the two inequalities in (2.34), we get that \( j_3 \geq |j_1 - j_2| \). Taking (2.33) into account, we get that the values of \( j_3 \) that can have a non-vanishing coupling to the isospins \( j_1 \) and \( j_2 \) are contained in the interval:

\[
|j_1 - j_2| \leq j_3 \leq j_1 + j_2. \tag{2.35}
\]

The similarity of eq. (2.35) with what happens in \( sl(2) \) is manifest. It is important to point out, however, that in the \( osp(1|2) \) case \( j_1 + j_2 - j_3 \) is in general half-integer (see eq. (2.33)). These are well-known results of the \( osp(1|2) \) representation theory. The fact that we were able to reproduce them within our Coulomb gas formalism is a confirmation of the correctness of our approach. As a further verification, let us...
check that the expectation value (2.31) is non-vanishing only when \( m_1 + m_2 + m_3 = 0 \). To prove this statement the first of our selection rules (2.21) will become crucial. Actually, we are only going to consider the simplest case in which \( m_3 = j_3 \). Thus, we will be dealing with the correlator:

\[
< \Phi_{m_1}^{j_1}(z_1) \Phi_{m_2}^{j_2}(z_2) \tilde{\Phi}_{j_3}^{j_3}(z_3) Q^n >.
\]

(2.36)

The screening charge \( Q \) is the sum of two terms, and only in one of them the \( w \) field is present. Therefore in \( Q^n \) there will be contributions with different powers \( w^l \) with \( 0 \leq l \leq n \). Only one of these terms, \( i.e. \) the one that satisfies (2.21), contributes to the correlator (2.36). Therefore, inside this expectation value we can substitute:

\[
Q^n \sim \oint w^l \psi^l \bar{\psi}^{n-l},
\]

(2.37)

where a symbolic notation for the multiple contour integral has been adopted. Let us now determine the value of \( l \) in (2.37). Our first observation is that the number of \( \chi \) fields in the operator \( \Phi_m^j \) is equal to the integer part of \( j - m \), which we shall denote by \( [j - m] \) (see eq. (2.10)). Therefore for the correlator (2.36) one has:

\[
N(w) - N(\chi) = 2j_3 + s + l - [j_1 - m_1] - [j_2 - m_2].
\]

(2.38)

After taking into account (2.21), we conclude that \( l \) is given by:

\[
l = [j_1 - m_1] + [j_2 - m_2] - 2j_3.
\]

(2.39)

The number of \( \psi \)'s and \( \bar{\psi} \)'s inside a correlator must be equal if we want to have a chance of getting a non-vanishing result. It is straightforward to prove that the number of \( \psi \) fields in the operator \( \Phi_m^j \) is \( 2( j - m - [j - m] ) \). Since \( \tilde{\Phi}_j^j \) does not contain any \( \psi \) field in its expression (see eq. (2.22)), the number of \( \psi \)'s in the
non-vanishing terms of (2.36) is:

\[ l + 2(j_1 - m_1 + j_2 - m_2 - [j_1 - m_1] - [j_2 - m_2]) \].

(2.40)

The only source of $\psi$’s in (2.36) is the screening charge $Q$. After inspecting eq. (2.37), we conclude that their number in the correlator is $n - l$. Taking into account the values of $n$ and $l$ given in (2.33) and (2.39), it is easy to prove that (2.40) is equal to $n - l$ only when $m_1 + m_2 + j_3 = 0$. This is the result we wanted to demonstrate.

3. The four-point functions

In this section we shall apply the free field representation studied in section 3 to the computation of the four-point conformal blocks of the model. As in the case of the two- and three-point functions, we shall represent these blocks as expectation values of primary fields in the $\text{osp}(1|2)$ Fock space. In order to satisfy the charge asymmetry conditions of the latter, one of the four primary fields will be taken in the conjugate representation. Therefore we shall consider a correlator of the form $< \Phi_{j_1 m_1}(z_1) \Phi_{j_2 m_2}(z_2) \Phi_{j_3 m_3}(z_3) \tilde{\Phi}_{j_4 m_4}(z_4) Q^n >$. The number $n$ of screening charges can be easily determined from the second condition in (2.21). Indeed, one can immediately demonstrate that only when $n = 2(j_1 + j_2 + j_3 - j_4)$ this correlator is non-vanishing. By using the $sl(2)$ projective invariance of the Virasoro algebra, we can fix the positions of the four fields to the values $z_1 = 0$, $z_2 = z$, $z_3 = 1$ and $z_4 = \infty$. After this fixing, the correlator is a function of the variable $z$. We want to investigate the analytical structure of these blocks and, as a result of this study, we would like to determine the operator algebra of the model. We shall follow the method developed in ref. [12] for the minimal models and extended to $sl(2)$ current algebras in ref. [19]. As it happened in this latter case, the study of the correlator for some particular values of the isospins and $H$-charges is enough to determine the structure constants of the model. Therefore, as in ref. [19], we shall restrict ourselves to the situation in which $j_3 = j_2$ and $j_4 = j_1$ with $j_1 \geq j_2$. Notice
that in this case \( n = 4j_2 \). This implies that the number of screening operators must be even. In addition, the \( H \)-charges of the four primary fields will be taken to be \( m_1 = -j_1, m_2 = j_2, m_3 = -j_2 \) and \( m_4 = j_1 \). Therefore we will center our efforts in the analysis of the quantity:

\[
I(z) \equiv < \Phi^{j_1}_{-j_1}(0) \Phi^{j_2}_{j_2}(z) \Phi^{j_2}_{-j_2}(1) \tilde{\Phi}^{j_1}_{j_1}(\infty) Q^{4j_2} > . \tag{3.1}
\]

We shall suppose, finally, that the four representations involved in the correlator (3.1) are even, \( i.e. \) with bosonic highest weights. As is reviewed in appendix A, for these even representations one can choose a metric such that all the states of the corresponding multiplet have positive norm.

Using the expressions of the primary fields and the screening charge, it is immediate to get the explicit representation of \( I(z) \). One has:

\[
I(z) = \prod_{i=1}^{n} \oint_{C_i} d\tau_i \lambda(z, \{\tau_i\}) \eta(\{\tau_i\}) . \tag{3.2}
\]

In eq. (3.2) \( \tau_i \) are the integration variables that appear in the screening charges, the integration contours \( C_i \) will be specified below and the function \( \lambda(z, \{\tau_i\}) \) is the part of the correlator that corresponds to the field \( \phi \), namely:

\[
\lambda(z, \{\tau_i\}) = < e^{-2ij_1a_0\phi(0)} e^{-2ij_2a_0\phi(z)} e^{-2ij_2a_0\phi(1)} e^{2i(s+j_1)a_0\phi(\infty)} \times e^{i\alpha_0 \phi(\tau_1)} \cdots e^{i\alpha_0 \phi(\tau_n)} > . \tag{3.3}
\]

The function \( \eta(\{\tau_i\}) \) contains the contribution of the fields \( w, \chi, \psi \) and \( \bar{\psi} \) to the vacuum expectation value (3.1). Notice that one gets many terms in \( \eta(\{\tau_i\}) \) when \( Q^{4j_2} \) is expanded as in eq. (2.37)(with \( n = 4j_2 \)). Actually, only one type of these terms is non-vanishing. Indeed, as in the primary fields involved in (3.1), the fields \( \psi \) and \( \bar{\psi} \) do not appear, only those pieces of \( Q^{4j_2} \) with equal number of \( \psi \)'s and \( \bar{\psi} \)'s survive in the expectation value (3.1). Inspecting eq. (2.37) we conclude that
those contributions appear when \( l = n/2 = 2j_2 \). Therefore we can write:

\[
\eta(\{\tau_i\}) = (-1)^{2j_2} < (\chi(0))^{2j_1} (\chi(1))^{2j_2} (w(\infty))^{2j_1+s} w(\tau_1) \cdots w(\tau_{2j_2}) > \times
\]

\[\times < \psi(\tau_1) \cdots \psi(\tau_{2j_2}) \bar{\psi}(\tau_{2j_2+1}) \cdots \bar{\psi}(\tau_{4j_2}) > + \text{permutations.} \tag{3.4}\]

In eq. (3.4) the sum over permutations has its origin in all the possible terms of the type (2.37) with \( l = 2j_2 \). Notice that in eq. (3.4) \( N(w) = 2j_1 + 2j_2 + s \) and \( N(\chi) = 2j_1 + 2j_2 \) and thus the selection rule (2.21) is satisfied.

We will use in eq. (3.2) the canonical set of contours that give rise to the \( s \)-channel conformal blocks [12, 19]. Notice that our correlators (3.1) have \( z_1 = 0, z_2 = z, z_3 = 1 \) and \( z_4 = \infty \) as singular points. The \( C_i \)'s will be contours connecting these points as follows. We shall take the first \( n - p + 1 \) integrals along a path lying on the real axis and joining the points \( \tau = 1 \) and \( \tau = \infty \). The remaining \( p - 1 \) integrals, \( i.e. \) those involving the variables \( \tau_{n-p+1+i} \) for \( i = 1, \cdots, p-1 \), will be taken along the segment \( (0, z) \). Obviously the range of values of \( p \) is \( 1 \leq p \leq 4j_2+1 \).

We have thus divided our integrations in two sets: \( n - p + 1 \) of them are performed in the interval \( (1, \infty) \) while for the other \( p - 1 \) the domain of integration is the segment that goes from \( \tau = 0 \) to \( \tau = z \). Within each of these two intervals the integration variables will be taken as ordered. Thus, if we relabel the \( \tau_i \)'s as \( u_i = \tau_i \) for \( i = 1, \cdots, n - p + 1 \) and \( v_i = \tau_{n-p+1+i} \) for \( i = 1, \cdots, p-1 \), the conformal block \( I_p(z) \) can be written as:

\[
I_p(z) = \int_{1}^{\infty} du_1 \cdots \int_{1}^{u_{n-p}} du_{n-p+1} \int_{0}^{z} dv_1 \cdots \int_{0}^{v_{p-2}} dv_{p-1} \lambda_p(z, \{u_i\}, \{v_i\}) \eta_p(\{u_i\}, \{v_i\}). \tag{3.5}\]

In eq. (3.5) the quantities \( \lambda_p(z, \{u_i\}, \{v_i\}) \) and \( \eta_p(\{u_i\}, \{v_i\}) \) are, respectively, the functions \( \lambda(z, \{\tau_i\}) \) and \( \eta(\{\tau_i\}) \) after the relabelling of variables introduced above. By applying Wick’s theorem to the vacuum expectation value (3.3), one can readily
prove that \( \lambda_p(z, \{u_i\}, \{v_i\}) \) is given by:

\[
\lambda_p(z, \{u_i\}, \{v_i\}) = z^{8j_1j_2\rho} (1 - z)^{8j_2\rho} \prod_{i=1}^{n-p+1} u_i^a (u_i - z)^b (u_i - 1)^b \prod_{i<j} (u_i - u_j)^{2\rho} \times \\
\times \prod_{i=1}^{p-1} v_i^a (z - v_i)^b (1 - v_i)^b \prod_{i<j} (v_i - v_j)^{2\rho} \prod_{i=1}^{n-p+1} \prod_{j=1}^{p-1} (u_i - v_j)^{2\rho},
\]

(3.6)

where

\[
\rho = \frac{\alpha_0^2}{2} = \frac{1}{2(2k + 3)},
\]

(3.7)

and the constants \( a \) and \( b \) are defined as:

\[
a = -2j_1\alpha_0^2, \quad b = -2j_2\alpha_0^2.
\]

(3.8)

The representation (3.5) can be used to obtain the non-analytic behaviour of the blocks around the point \( z = 0 \). In general, one expects that, as \( z \to 0 \)

\[
I_p(z) \sim N_p z^{\gamma_p},
\]

(3.9)

where \( N_p \) and \( \gamma_p \) are constants and only the leading term of the expansion has been written down. Eq. (3.9) corresponds to a well-defined s-channel exchange. The exponents \( \gamma_p \) are related to the conformal weights of the s-channel intermediate states, whereas the \( N_p \)'s measure the coupling constants of the intermediate channel and are related to the structure constants of the operator algebra of the model (see below). In order to make the \( z \sim 0 \) behaviour more explicit let us rescale the \( u_i \) integration variables as \( v_i = z t_i \). These new variables \( t_i \) are integrated over the interval \( (0, 1) \). It is an easy exercise to obtain the leading term of the \( z \to 0 \)
expansion of \( \lambda_p(z, \{u_i\}, \{zt_i\}) \). One has:

\[
\lambda_p(z, \{u_i\}, \{zt_i\}) \sim z^{8j_1j_2\rho+(p-1)[a+b+(p-2)\rho]} \prod_{i=1}^{n-p+1} u_i^{a+b+2\rho(p-1)} (u_i - 1)^b \prod_{i<j} (u_i - u_j)^{2\rho} \times \\
\times \prod_{i=1}^{p-1} t_i^{a}(1 - t_i)^b \prod_{i<j} (t_i - t_j)^{2\rho}.
\]

(3.10)

Notice that in (3.10) the coefficients multiplying the leading power of \( z \) factorize in the variables \( \{u_i\} \) and \( \{t_i\} \).

In order to get the values of \( N_p \) and \( \gamma_p \), let us study in detail the \( z \to 0 \) expansion of the function \( \eta_p(\{u_i\}, \{zt_i\}) \). The key point in this analysis is the fact that the rescaling \( v_i = zt_i \) introduces in \( \eta_p \) a \( z \)-dependence, whose leading term we want to determine. Due to the presence of the fermionic correlator in (3.4), one must distinguish two cases, depending on the even or odd character of the number \( p - 1 \) of the \( v_i \) variables. Let us start with the case in which \( p - 1 \) is even. The first relevant observation we must make is that, when the field \( w(zt_i) \) is contracted with \( \chi(0) \), a \( z^{-1} \) factor is generated. Therefore, in the leading term, all the \( w(v_i) \) fields must be contracted to \( \chi(0) \). Moreover, it is clear that the dominant power of \( z \) coming from the fermionic correlator is generated when the maximum number of \( \psi(v_i) \) and \( \bar{\psi}(v_i) \) are contracted among themselves.

Our previous discussion shows that we have two sources of powers of \( z \) which, actually, are not independent. Indeed, it is clear from (3.4) that, for each \( \psi \) field in the fermionic correlator, we must have a \( w \) field evaluated at the same point. One might wonder if it is possible to have an excess of \( \psi(v_i) \)'s with respect to the \( \bar{\psi}(v_i) \)'s, since the power of \( z \) lost in the fermionic correlator could be compensated by the contraction of the extra \( w(v_i) \)'s to \( \chi(0) \). It turns out, however, that these terms, which by a simple power counting could be present, do not contribute because the corresponding fermionic correlators are zero at leading order. Let us illustrate this point with an example. Suppose that the number of \( \psi(v_i) \)'s is greater in two units than those of the \( \bar{\psi}(v_i) \)'s. This means that two \( \psi(v_i) \)'s must be contracted with
two \( \tilde{\psi}(u_i) \)'s. The fermionic correlator of these contributions must contain pieces of the type \(< \tilde{\psi}(u_k) \tilde{\psi}(u_l) \psi(v_i) \psi(v_j) >\). After substituting \( v_i = z t_i \) and taking the limit \( z \to 0 \) one gets:

\[
\lim_{z \to 0} < \tilde{\psi}(u_k) \tilde{\psi}(u_l) \psi(z t_i) \psi(z t_j) > = 0, \quad (3.11)
\]

which can be regarded as a consequence of the antisymmetric character of the fermionic correlators. It is thus clear that only those permutations in (3.4) having \( \frac{p-1}{2} \) \( \psi(v_i) \) and \( \tilde{\psi}(v_i) \) fields in the fermionic expectation value contribute to the leading term when \( z \to 0 \) (recall that we are considering the case in which \( p - 1 \) is even). It is not difficult to convince oneself that, in the leading term, the fermionic correlator factorizes into the product of two vacuum expectation values, each of which involving fields that take values either in the \((1, \infty)\) or \((0, 1)\) intervals of the real line. The expression that one arrives at is:

\[
\eta_p(\{u_i\}, \{zt_i\}) \sim (-1)^n z^{1-p} \frac{(2j_1)!}{(2j_1 - \frac{p-1}{2})!} \times 
\]

\[
\times \left\{ < (\chi(0))^{2j_1} - \frac{p-1}{2} (\chi(1))^{2j_2} (w(\infty))^{2j_3 + s} w(u_1) \cdots w(u_{n-p+1}) > \times 
\right.
\]

\[
\times < \tilde{\psi}(u_1) \cdots \tilde{\psi}(u_{n-p+1}) \tilde{\psi}(u_{n-p+3}) \cdots \tilde{\psi}(u_{n-p+1}) > + \text{permutations} \} \times (3.12)
\]

\[
\times \left\{ \prod_{i=1}^{\frac{p-1}{2}} t_i^{-1} < \psi(t_1) \cdots \psi(t_{\frac{p-1}{2}}) \psi(t_{\frac{p+1}{2}}) \cdots \psi(t_{p-1}) > + \text{permutations} \right. 
\]

The origin of the different terms in eq. (3.12) is clear. The combinatorial and \( t_i^{-1} \) factors come from the contractions between \( w(z t_i) \) and \( \chi(0) \). The power of \( z \) displayed in (3.12) has a double origin: a factor \( z^{1-p} \) comes from the contractions of the \( w(z t_i) \)'s with \( \chi(0) \), whereas another \( z^{1-p} \) contribution is generated in the fermionic correlator.
The situation when $p - 1$ is odd is slightly different. In this case, the number of fermionic operators taking values in the $(0, z)$ interval is odd. According to the general arguments given above, the leading term is generated when the number of $\psi$ fields in the $(0, z)$ segment exceeds in one unit to that of the $\bar{\psi}$'s. Notice that now the antisymmetry argument employed in (3.11) does not work. Moreover, the fermionic correlator multiplying the leading term does not factorize in a naive way. There is, however, a form of writing a factorized expression for this quantity. It consists in the introduction of two spectator fermions, located at the points $z = 0$ and $z = \infty$ of the complex plane, that are inserted in each of the two correlators in which the initial vacuum expectation value splits. In fact one can prove that, for any pair of two positive integers $N$ and $M$, one has:

$$< \psi(u_1) \cdots \psi(u_N) \bar{\psi}(u_{N+1}) \cdots \bar{\psi}(u_{2N+1}) \psi(v_1) \cdots \psi(v_M) \bar{\psi}(v_{M+1}) \cdots \bar{\psi}(v_{2M-1}) > \sim$$

$$\sim z^{1-M} \lim_{R \to 0} < \psi(R) \psi(u_1) \cdots \psi(u_N) \bar{\psi}(u_{N+1}) \cdots \bar{\psi}(u_{2N+1}) > \times$$

$$\times \lim_{R \to \infty} R < \psi(t_1) \cdots \psi(t_M) \bar{\psi}(t_{M+1}) \cdots \bar{\psi}(t_{2M-1}) \bar{\psi}(R) >,$$

(3.13)

where we have only kept the leading order term in $z$ and the variables $v_i$ and $t_i$ are related as above (i.e. $v_i = z t_i$). The proof of (3.13) is straightforward and can be performed, for example, by means of the Cauchy determinant formula (see eq. (B16)). We shall need (3.13) for $N = \frac{n-p}{2}$ and $M = \frac{p}{2}$. Thus, for $p - 1$ odd, we obtain:
\[ \eta_p(\{u_i\}, \{zt_i\}) \sim (-1)^{\frac{p}{2}} z^{1-p} \frac{(2j_1)!}{(2j_1-p)!} \times \]

\[ \times \left\{ \langle \chi(0)^{2j_1-\frac{p}{2}} (\chi(1))^{2j_2} (w(\infty))^{2j_1+s} w(u_1) \cdots w(u_{n-p}) > \times \right. \]

\[ \times \lim_{R \to 0} < \psi(R)\psi(u_1) \cdots \psi(u_{n-p}) \bar{\psi}(u_{n-p+1}) \bar{\psi}(u_{n-p+1}) > + \text{permutations} \times \]

\[ \times \left\{ \lim_{R \to \infty} R \prod_{i=1}^{\frac{n}{2}} t_i^{-1} < \psi(t_1) \cdots \psi(t_n) \bar{\psi}(t_{n+1}) \cdots \bar{\psi}(t_{p-1}) \bar{\psi}(R) > + \text{permutations} \right\}. \]

(3.14)

It is interesting to point out that the power of \( z \) appearing in (3.12) and (3.14) is the same, \textit{i.e.} \( \eta_p \sim z^{1-p} \) for any value of \( p \). Moreover, in the change of variables \( v_i \to zt_i \) a Jacobian factor is introduced. Indeed one has:

\[ \int_0^z dv_1 \cdots \int_0^{v_{p-2}} dv_{p-1} \cdots = z^{p-1} \int_0^1 dt_1 \cdots \int_0^{t_{p-2}} dt_{p-1} \cdots. \]  

(3.15)

The power of \( z \) in eq. (3.15) just cancels the one in \( \eta_p \). Therefore, \( \gamma_p \) can be read from the expression of the leading term of \( \lambda_p \) (eq. (3.10)):

\[ \gamma_p = 8j_1j_2\rho + (p-1) [a + b + (p-2)\rho]. \]  

(3.16)

After some elementary algebraic manipulations, the exponents \( \gamma_p \) can be written as differences of osp(1|2) conformal weights:

\[ \gamma_p = \Delta_{j_3} - \Delta_{j_1} - \Delta_{j_2}, \]  

(3.17)

where the quantities \( \Delta_{j_i} \) are given in (2.8) and \( j_3 \), as a function of \( p \), is:

\[ j_3 = j_1 + j_2 + \frac{1 - p}{2}. \]  

(3.18)

Eq. (3.17) allows to interpret \( j_3 \) as the isospin of the s-channel intermediate state. Notice that as \( p = 1, \cdots, 4j_2 + 1 \) the values taken by \( j_3 \) are \( j_1 - j_2, j_1 - j_2 + \)
\begin{align*}
\frac{1}{2}, \cdots, j_1 + j_2. \text{ Remarkably, these are the values of the isospin that appear in the Clebsch-Gordan decomposition of the tensor product of two } \text{osp}(1|2) \text{ representations of isospins } j_1 \text{ and } j_2. \text{ This result confirms our analysis of the three-point functions and encourages us to proceed with the study of the four-point function.}

The coefficients } N_p \text{ will be given by multiple integrals, whose explicit expressions can be obtained by gathering the different contributions coming from our previous equations. Let us consider first the case in which } p - 1 \text{ is even. In general, } N_p \text{ will depend on the product of two integrals: one over the variables } u_i \text{ and the other involving the } t_i's. \text{ The expression of the latter can be obtained by collecting the factors depending on } t_i \text{ of eqs. (3.10) and (3.12). The result is:}

\begin{align*}
S_p & \equiv \int_0^1 dt_1 \cdots \int_0^{t_{p-2}} dt_{p-1} \times \\
& \times \left\{ \prod_{i=1}^{p-1} t_i^{a-1} \prod_{i=m+1}^{p-1} t_i \cdot \psi(t_1) \cdots \psi(t_{m+1}) \bar{\psi}(t_{m+2}) \cdots \bar{\psi}(t_{p-1}) > + \text{ permutations} \right\} \times \\
& \times \prod_{i=1}^{p-1} (1 - t_i)^b \prod_{i<j} (t_i - t_j)^{2\rho}.
\end{align*}

(3.19)

\(S_p\) is given by an integral of the type studied by Selberg [26]. In appendix B we have studied the integrals of this kind that we shall need in our calculation. Actually, } S_p \text{ belongs to the family of even-dimensional integrals } J_{2r}^m \text{ defined in eq. (B11). Indeed, comparing the right-hand sides of eqs. (3.19) and (B11), one finds:}

\begin{align*}
S_p = J_{p-1}^{m+1} (a - 1, b, \rho).
\end{align*}

(3.20)

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\begin{align*}
S_p = J_{p-1}^{m+1} (a - 1, b, \rho).
\end{align*}

(3.20)

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\begin{align*}
S_p = J_{p-1}^{m+1} (a - 1, b, \rho).
\end{align*}

(3.20)
in eqs. (3.12) and (3.14). Apart from an irrelevant constant, this quantity can be written as:

\[
< (\chi(0))^{2j_1 - \lfloor \frac{p}{2} \rfloor} (\chi(1))^{2j_2} (w(\infty))^{2j_1 + s} w(u_1) \cdots w(u_{n-p+1}) > = \\
\sum_{m=0}^{\lfloor \frac{n-p+1}{2} \rfloor} B_m \left\{ \prod_{i=1}^{m} u_i^{-1} \prod_{i=m+1}^{\lfloor \frac{n-p+1}{2} \rfloor} (u_i - 1)^{-1} + \text{permutations} \right\},
\]

(3.21)

where the combinatorial constants \( B_m \) are given by:

\[
B_m = \frac{(2j_1 - \lfloor \frac{p}{2} \rfloor)!(2j_2)!(2j_2 - \lfloor \frac{n-p+1}{2} \rfloor + m)!}{(2j_1 - \lfloor \frac{p}{2} \rfloor - m)!(2j_2)!}. \quad (3.22)
\]

Notice that (3.21) is the \( w\chi \) correlator needed for an arbitrary value of \( p \) (i.e. eq. (3.22) can be used both in (3.12) and (3.14)). Coming back to the case in which \( p-1 \) is even, after eq. (3.21) is substituted in (3.12), one realizes that the integrals to be computed are:

\[
S_m^p \equiv \int_1^{\infty} du_1 \cdots \int_1^{u_{n-p}} \left\{ \prod_{i=1}^{n-p+1} u_i^{a+b+2p-2} (u_i - 1)^b \times \right. \\
\times \left[ \prod_{i=1}^{m} u_i^{-1} \prod_{i=m+1}^{\lfloor \frac{n-p+1}{2} \rfloor} (u_i - 1)^{-1} + \text{permutations} \right] \times \\
\times < \psi(u_1) \cdots \psi(u_{n-p+1}) \bar{\psi}(u_{n-p+1}) \cdots \bar{\psi}(u_{n-p+1}) > + \text{permutations} \right\} \times \\
\times \prod_{i<j} (u_i - u_j)^{2p}. 
\]

(3.23)

We would like to recast \( S_m^p \) as a multiple ordered integral in the (0,1) interval. This can be achieved after a change of variables that involves an inversion and reordering of the \( u_i \)'s. Let us introduce new variables \( \xi_i, i = 1, \ldots, n-p+1, \) by means of the equation:

\[
\xi_i = \left[ u_{n-p+2-i} \right]^{-1}. \quad (3.24)
\]

Notice that, in the integration domain appearing in the definition of \( S_m^p \), the variables \( \xi_i \) satisfy the inequalities \( 1 \geq \xi_1 \geq \cdots \geq \xi_{n-p+1} \geq 0 \). After an straightfor-
ward calculation one can rewrite eq. (3.23) as:

\[
\mathcal{S}_m^p = \int_0^1 d\xi_1 \cdots d\xi_{n-p} \left\{ \prod_{i=1}^{n-p+1} \xi_i \bar{a}_{n-p+1+i} (1 - \xi_i)^b \times \right.
\]

\[
\times \left[ \prod_{i=1}^{m} \left( 1 - \xi_{n-p+1+i} \right)^b \prod_{i=m+1}^{n-p+1} (1 - \xi_{n-p+1+i})^{b-1} + \text{permutations} \right] \times \right.
\]

\[
\times < \psi(\xi_1) \cdots \psi(\xi_{n-p+1}) \bar{\psi}(\xi_{n-p+1}) \cdots \bar{\psi}(\xi_{n-p+1}) > + \text{permutations} \right\} \times \right.
\]

\[
\times \prod_{i<j} (\xi_i - \xi_j)^{2\rho},
\]

where \( \bar{a} \) is defined as:

\[
\bar{a} = -1 - a - 2b - 2\rho(n - 1).
\]

Eq. (3.25) allows to identify \( \mathcal{S}_m^p \) with an integral of those computed in appendix B. In fact, one has:

\[
\mathcal{S}_m^p = \mathcal{J}_{n-p+1}^m (\bar{a}, b, \rho).
\]

The functions \( \mathcal{J}_{2r+1}^m (a, b, \rho) \) have been defined in eq. (B36). Their values, again given in terms of \( \Gamma \)-functions, have been written down in eq. (B37).

For \( p - 1 \) odd one can proceed similarly. The odd-dimensional integrals needed to compute \( N_p \) match the definitions of the functions \( \mathcal{J}_{2r+1}^m \) and \( \mathcal{J}_{2r+1}^m \) adopted in appendix B (eqs. (B41) and (B43) respectively). The only subtle point in the calculation of \( N_p \) appears when one studies the behaviour of the fermionic correlator under the change of variables (3.24). Actually, one can prove an equation relating the correlator in the variables \( u_i \), with an additional fermionic spectator at the origin, to a correlator of the fields \( \psi(\xi_i) \) and \( \bar{\psi}(\xi_i) \), which has a fermionic insertion
at infinity. This equation is:

$$\lim_{R \to 0} < \psi(R) \psi(u_1) \cdots \psi(u_{n-p}) \bar{\psi}(u_{n-p+1}) \cdots \bar{\psi}(u_{n-p+1}) > =$$

$$= \left( \prod_{i=1}^{n-p+1} \xi_i \right) \lim_{R \to \infty} R < \psi(\xi_1) \cdots \psi(\xi_{n-p+1}) \bar{\psi}(\xi_{n-p+2}) \cdots \bar{\psi}(\xi_{n-p+1}) \bar{\psi}(R) > .$$

(3.28)

It is a highly non-trivial fact (see appendix B) that there exist closed expressions for the functions \( J^m_N \) and \( J^m_N \) for even and odd values of \( N \). These expressions are written in eqs. (B42) and (B44). Moreover, one can put \( N_p \) for arbitrary \( p \) in terms of these functions:

$$N_p = (-1)^{\frac{p}{2}} \frac{(2j_1)!}{(2j_1 - \frac{2p}{\rho})!} \sum_{m=0}^{\frac{n-p+1}{2}} B_m J^m_{n-p+1} (a, b, \rho).$$

(3.29)

In the next section we shall use the value of \( N_p \) written in eq. (3.29) to obtain the operator algebra of the model.

4. The operator product algebra

So far in our study of the osp(1|2) correlators we have only considered the holomorphic sector of the theory. In order to compute the physical correlation functions \( G(z, \bar{z}) \), one must combine the holomorphic and antiholomorphic blocks in such a way that \( G(z, \bar{z}) \) becomes monodromy invariant. We shall restrict ourselves to operators whose \( z \) and \( \bar{z} \) quantum numbers are the same. This, in particular, means that they will have equal holomorphic and antiholomorphic conformal dimensions, i.e. the operators we shall consider will have zero conformal spin. According to the general arguments valid for the Coulomb gas representations, one constructs \( G(z, \bar{z}) \) as a quadratic expression of the functions \( I_p(z) \) and their complex conjugates:

$$G(z, \bar{z}) = \sum_p X_p \left| I_p(z) \right|^2,$$

(4.1)

where the coefficients \( X_p \) are such that the right-hand side of eq. (4.1) is mon-
odromy invariant. In ref. [12] a technique to compute these coefficients $X_p$ has been developed. This method is based on the representation of the functions $I_p(z)$ as contour integrals in the complex plane, and allows to obtain the $X_p$'s up to a global (i.e. $p$-independent) constant. It is straightforward to adapt the results of [12] to our case. One gets:

$$X_p = \prod_{i=1}^{p-1} s(i\rho - \frac{1}{2}) \prod_{i=0}^{p-2} \frac{s(a + i\rho - \frac{1}{2}) s(1 + b + i\rho - \frac{1}{2})}{s(1 + a + b + (p - 2 + i)(\rho - \frac{1}{2}))} \times$$

$$\times \prod_{i=1}^{n-p+1} s(i\rho - \frac{1}{2}) \prod_{i=0}^{n-p} \frac{s(1 - a - 2b - (\rho - \frac{1}{2})(2(n - 1) - i)) s(b + i\rho - \frac{1}{2})}{s(1 - a - b - (\rho - \frac{1}{2})(2(p - 1) + i))},$$

(4.2)

where we have introduced the notation

$$s(x) \equiv \sin(\pi x).$$

(4.3)

It is now easy to get the leading $|z| \to 0$ contributions to $G(z, \bar{z})$ of the different channels. Indeed, after combining eqs. (3.9), (3.17) and (4.1), one obtains:

$$G(z, \bar{z}) \sim \sum_p \left[ \frac{S_p}{|z|^{2(\Delta_{j1} + \Delta_{j2} - \Delta_{j3})}} + O(z) \right].$$

(4.4)

In eq. (4.4) $j_3$ and $p$ are related as in eq. (3.18) and the coefficients $S_p$ are:

$$S_p = X_p (N_p)^2.$$ 

(4.5)

The operator product algebra of the model can be determined by comparing the expansion of eq. (4.4) with the one obtained by performing some appropriate OPE's of primary fields inside the correlator $G(z, \bar{z})$. In general, the operator product algebra of the theory will have the form:

$$\Phi_{m1}^{j_1}(z_1, \bar{z}_1) \Phi_{m2}^{j_2}(z_2, \bar{z}_2) = \sum_{j_3, m_3} D_{j_1, m_1; j_2, m_2}^{j_3, m_3} \left[ \frac{\Phi_{m_3}^{j_3}(z_2, \bar{z}_2)}{|z_1 - z_2|^{2(\Delta_{j1} + \Delta_{j2} - \Delta_{j3})}} + O(z_1 - z_2) \right].$$

(4.6)

The coefficients $D_{j_1, m_1; j_2, m_2}^{j_3, m_3}$ in eq. (4.6) are the so-called structure constants of
the model. Their determination from the quantities $S_p$ is the main objective of the present section. The value of the structure constants depends on the normalization chosen for the two-point correlator $\langle \Phi^{j_1}_{m_1}(z_1, \bar{z}_1) \Phi^{j_2}_{m_2}(z_2, \bar{z}_2) \rangle$. The standard choice for the normalization of these functions is:

$$
\langle \Phi^{j_1}_{m_1}(z_1, \bar{z}_1) \Phi^{j_2}_{m_2}(z_2, \bar{z}_2) \rangle = \delta_{j_1,j_2} \delta_{m_1,-m_2} |z_1 - z_2|^{4\Delta_{j_1}},
$$

which implies the following constraint for the structure constants:

$$
D_{j_1,m_1;j_1,-m_1}^{0,0} = 1.
$$

For primary fields $\Phi^j_m$ that correspond to states $|j, m>$ with negative norm we shall include a minus sign in the right-hand side of (4.7). As is explained in appendix A, these negative norm states appear in the odd representations of osp(1|2). Although the representations involved in our correlator $G(z, \bar{z})$ are even, odd representations do appear in the intermediate states $|j_3, m_3>$ and, therefore, we must be careful with this sign (see below). On the other hand, we can reobtain the power behaviour (4.4) by substituting the OPE’s $\Phi^{j_1}_{-j_1}(z_1, \bar{z}_1) \Phi^{j_2}_{j_2}(z_2, \bar{z}_2)$ and $\Phi^{j_3}_{-j_3}(z_3, \bar{z}_3) \Phi^{j_1}_{j_1}(z_4, \bar{z}_4)$ for $z_1 = 0$, $z_2 = z$, $z_3 = 1$ and $z_4 = \infty$ in the correlator $G(z, \bar{z})$. The equation we arrive at is:

$$
G(z, \bar{z}) \sim \sum_{j_3, m_3} (-1)^{\sigma(j_3, m_3)} \left[ \frac{[D_{j_1,j_1;j_2,-j_2}^{j_3,m_3}]^2}{|z|^{2(\Delta_{j_1}+\Delta_{j_2}-\Delta_{j_3})}} + O(z) \right],
$$

where $\sigma(j_3, m_3)$ is 0(1) if the state $|j_3, m_3>$ has positive(negative) norm. It is clear by comparing eqs. (4.4) and (4.9) that one has the identification:

$$
(-1)^{\sigma(j_3, m_3)} \left[ D_{j_1,j_1;j_2,-j_2}^{j_3,m_3} \right]^2 \sim S_p,
$$

which allows to get the structure constants in terms of the quantities $S_p$. Notice, however, that the global factor ambiguity in the determination of the $X_p$’s is inherited in the constants $S_p$. As will be discussed below, this ambiguity can be eliminated by imposing the normalization condition (4.8).
Before plunging into the calculation of the coefficients $S_p$ and the structure
constants, let us obtain a remarkable simplification of the expression of the $N_p$’s
given in (3.29). First of all, let us rewrite eq. (3.29) using the explicit expression
of the coefficients $B_m$ written in eq. (3.22):

$$N_p = (-1)^{\frac{n}{2}} J_{p-1}^{\frac{n-p+1}{2}} (a - 1, b, \rho) \times$$

$$\times \sum_{m=0}^{\left[\frac{n-p+1}{2}\right]} \frac{(2j_1)!}{(2j_1 - \left[\frac{n-p+1}{2}\right] - m)!} \frac{(2j_2)!}{(2j_2 - \left[\frac{n-p+1}{2}\right] + m)!} J_{n-p+1}^m (\bar{a}, b, \rho).$$

(4.11)

The value of the integrals $J_{n-p+1}^m (\bar{a}, b, \rho)$ is given in eq. (B44). From this equation
it is easy to relate these integrals for arbitrary $m$ to the same functions with $m = 0$.
Indeed, using in (B44) the elementary properties of the $\Gamma$-function, one gets:

$$J_{n-p+1}^m (\bar{a}, b, \rho) = \left(\frac{n-p+1}{2}\right)^{[n-p+1]} \prod_{i=0}^{[n-p+1]-m-1} \frac{1 + \bar{a} + b + 2\rho\left(\frac{n-p+1}{2}\right) + i}{b + 2\rho i} J_{n-p+1}^0 (\bar{a}, b, \rho).$$

(4.12)

Employing eq. (4.12) we will be able to perform the summation in $m$ in eq. (4.11).
Our first step will consist in writing the product (4.12) in terms of factorials. Using
the expressions of $\bar{a}$ (eq. (3.26)), $b$ (eq. (3.8)) and $\rho$ (eq. (3.7)) in terms of $j_1$ and $j_2$
one gets:

$$\prod_{i=0}^{[n-p+1]-m-1} \frac{1 + \bar{a} + b + 2\rho\left(\frac{n-p+1}{2}\right) + i}{b + 2\rho i} =$$

$$= (-1)^{[n-p+1]-m} \frac{(2j_1 - \left[\frac{n-p+1}{2}\right] + m)!}{(2j_1 - \left[\frac{n-p+1}{2}\right])!} \frac{(2j_2 - \left[\frac{n-p+1}{2}\right] + m)!}{(2j_2)!}.$$

(4.13)

Substituting the right-hand side of eq. (4.13) in eq. (4.11) one realizes that the
factors depending on $j_2$ disappear. Moreover, the sum to be computed can be

30
written as:

\[
\sum_{m=0}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor} (-1)^m \left( \left\lfloor \frac{n-p+1}{2} \right\rfloor \right) \frac{(2j_1 - \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{n-p+1}{2} \right\rfloor - m)!}{(2j_1 - \left\lfloor \frac{p}{2} \right\rfloor - m)!} = \\
= \sum_{m=0}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor} \left( \left\lfloor \frac{n-p+1}{2} \right\rfloor \right) \prod_{i=0}^{m-1} (-2j_1 + \left\lfloor \frac{p}{2} \right\rfloor + i) \prod_{i=0}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor - m-1} (2j_1 - \left\lfloor \frac{p}{2} \right\rfloor + 1 + i).
\]

(4.14)

In order to evaluate the right-hand side of (4.14), we shall use the identity (which can be easily demonstrated by induction):

\[
\sum_{m=0}^{N} \binom{N}{m} \prod_{i=0}^{m-1} (A + i\rho) \prod_{i=0}^{N-m-1} (B + i\rho) = \prod_{i=0}^{N-1} (A + B + i\rho).
\]

(4.15)

Putting \(A = -B + 1 = -2j_1 + \left\lfloor \frac{p}{2} \right\rfloor\), \(N = \left\lfloor \frac{n-p+1}{2} \right\rfloor\) and \(\rho = 1\) in (4.15), one can get the value of the sum (4.14), namely:

\[
\sum_{m=0}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor} (-1)^m \left( \left\lfloor \frac{n-p+1}{2} \right\rfloor \right) \frac{(2j_1 - \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{n-p+1}{2} \right\rfloor - m)!}{(2j_1 - \left\lfloor \frac{p}{2} \right\rfloor - m)!} = \left( \left\lfloor \frac{n-p+1}{2} \right\rfloor \right)!. 
\]

(4.16)

Using this result in (4.11), and taking into account that \(J_0^{0}_{n-p+1}(a, b, \rho) = J_{n-p+1}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor}(\bar{a}, b, \rho)\) (compare the definitions of both integrals in appendix B), one can obtain the simplified expression of \(N_p\) we were looking for, i.e.:

\[
N_p = (-1)^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{(2j_1)! \left( \left\lfloor \frac{p}{2} \right\rfloor \right)!}{(2j_1 - \left\lfloor \frac{p}{2} \right\rfloor)!} J_{p-1}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor}(a - 1, b, \rho) J_{n-p+1}^{\left\lfloor \frac{n-p+1}{2} \right\rfloor}(\bar{a}, b, \rho).
\]

(4.17)

We can now use the value of the integrals \(J_N^m\), which has been written in eq. (B42),
to get the following representation of $N_p$:

$$
N_p = (-1)^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{(2j_1)! \left( \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right)!}{(2j_1 - \left\lfloor \frac{n}{2} \right\rfloor)!} \mu_{p-1}(\rho) \mu_{n-p+1}(\rho) \times
$$

$$
\times \prod_{i=0}^{p-2} \frac{\Gamma(a + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}]) \Gamma(1 + b + i(\rho - \frac{1}{2}) + [\frac{i}{2}])}{\Gamma(a + b + (\rho - \frac{1}{2})(p - 2 + i) + [\frac{i+2}{2}])} \times (4.18)
$$

$$
\times \prod_{i=0}^{n-p} \frac{\Gamma(1 + \alpha + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}]) \Gamma(1 + b + i(\rho - \frac{1}{2}) + [\frac{i}{2}])}{\Gamma(1 + \alpha + b + (\rho - \frac{1}{2})(n - p + i) + [\frac{i+n-p+2}{2}])}.
$$

The function $\mu_N(\rho)$ appearing in eq. (4.18) has been defined in eq. (B13). Having the expression (4.18) at our disposal, we can resume our calculation of the constants $S_p$. We must substitute $X_p$ and $N_p$, as given in eqs. (4.2) and (4.18), in the right-hand side of eq. (4.5). In this calculation we shall make use of the relation $s(x) \Gamma(x)^2 = \pi \Gamma(x)/\Gamma(1 - x)$. The final result can be compactly written in terms of the functions:

$$
\Pi_N(a, b, \rho) \equiv \prod_{i=0}^{N} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}]) \Gamma(1 + b + i(\rho - \frac{1}{2}) + [\frac{i}{2}])}{\Gamma(-a - i(\rho - \frac{1}{2}) - [\frac{i+1}{2}]) \Gamma(-b - i(\rho - \frac{1}{2}) - [\frac{i}{2}])} \times
$$

$$
\times \prod_{i=0}^{N} \frac{\Gamma(-a - b - (\rho - \frac{1}{2})(N + i) - [\frac{i+N+2}{2}])}{\Gamma(1 + a + b + (\rho - \frac{1}{2})(N + i) + [\frac{i+N+2}{2}])}
$$

$$
\hat{\mu}_N(\rho) \equiv \prod_{i=1}^{N} \frac{\Gamma(i(\rho + \frac{1}{2}) - [\frac{i}{2}])}{\Gamma(1 - i(\rho + \frac{1}{2}) + [\frac{i}{2}])}. \quad (4.19)
$$

With this definition, the constants $S_p$ are given by:

$$
\left( \frac{\Gamma(\rho + \frac{1}{2})}{\pi} \right)^{2n} S_p = \left( \frac{(2j_1)! \left( \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right)!}{(2j_1 - \left\lfloor \frac{n}{2} \right\rfloor)!} \right)^2 \hat{\mu}_{p-1}(\rho) \hat{\mu}_{n-p+1}(\rho) \Pi_{p-2}(a-1, b, \rho) \Pi_{n-p}(\alpha, b, \rho),
$$

(4.20)

where we have multiplied $S_p$ by a $p$-independent constant, which disappears when these quantities are properly normalized. We shall need the values of $S_p$ in terms
of \( j_1, j_2 \) and \( j_3 \). Substituting the values of \( a, \bar{a}, b, p \) and \( \rho \) as functions of the isospins and recalling that \( n = 4j_2 \), one can rewrite eq. (4.20) as:

\[
\left( \frac{\Gamma(\rho + \frac{1}{2})}{\pi} \right)^{2n} S(j_1, j_2; j_2, j_1|j_3) = \]

\[
= \left( \frac{(2j_1)!([j_2 + j_3 - j_1]!)}{([j_1 + j_3 - j_2]!)} \right)^2 \hat{\mu}_{2j_1+2j_2-2j_3}(\rho) \hat{\mu}_{2j_2+2j_3-2j_1}(\rho) \times \]

\[
\times \prod_{i=0}^{2j_1+2j_2-2j_3-1} \left[ \frac{\Gamma(\rho(i - 4j_1) - \frac{i}{2} + \left[ \frac{i+1}{2} \right])}{\Gamma(1 - \rho(i - 4j_1) + \frac{1}{2} - \left[ \frac{i+1}{2} \right])} \frac{\Gamma(1 + \rho(i - 4j_2) - \frac{i}{2} + \left[ \frac{i}{2} \right])}{\Gamma(-\rho(i - 4j_2) + \frac{1}{2} - \left[ \frac{i}{2} \right])} \right] \times \]

\[
\times \prod_{i=0}^{2j_2+2j_3-2j_1-1} \left[ \frac{\Gamma(\rho(4j_2 + 2 + i) - \frac{i}{2} + \left[ \frac{i+1}{2} \right])}{\Gamma(1 - \rho(4j_3 + 2 + i) + \frac{1}{2} - \left[ \frac{i+1}{2} \right])} \frac{\Gamma(1 + \rho(4j_1 - 2 + i) - \frac{i}{2} + \left[ \frac{i}{2} \right])}{\Gamma(-\rho(4j_1 - 2 + i) + \frac{1}{2} - \left[ \frac{i}{2} \right])} \right] \times \]

\[
\times \prod_{i=0}^{2j_3} \left[ \frac{\Gamma(\rho(i - 4j_3) - \frac{i}{2} + \left[ \frac{i+1}{2} \right])}{\Gamma(1 - \rho(i - 4j_3) + \frac{1}{2} - \left[ \frac{i+1}{2} \right])} \frac{\Gamma(1 + \rho(i - 4j_2) - \frac{i}{2} + \left[ \frac{i}{2} \right])}{\Gamma(-\rho(i - 4j_2) + \frac{1}{2} - \left[ \frac{i}{2} \right])} \right],
\]

(4.21)

where we have called \( S(j_1, j_2; j_2, j_1|j_3) \) to what we were previously denoting by \( S_p \).

Let us now consider the question of the normalization of the \( S_p \)'s needed to convert the identification (4.10) in a true equality. As we have previously mentioned, this normalization can be fixed by requiring the fulfillment of eq. (4.8). Following the analysis of refs. [12, 19], we shall achieve this by dividing the constants (4.21) by \( S(j_2, j_2; j_2, j_2|0) \), which correspond to the conformal blocks with \( j_1 = j_2 \) and the trivial representation, \( i.e. \) that with isospin \( j_3 = 0 \), in the intermediate state. Moreover, let us notice that the state exchanged in the s-channel is \( |j_3, j_1 - j_2 > \). In order to fix the sign in the left-hand side of eq. (4.10), we must determine under which conditions \( |j_3, j_1 - j_2 > \) has negative norm. This state appears in the tensor product of \( |j_1, j_1 > \) and \( |j_2, -j_2 > \), which are both bosonic since we are assuming
that they belong to even representations of osp(1|2). Thus the state $|j_3, j_1 - j_2 >$ is bosonic. Recall that, in general, $|j, m >$ is bosonic when $j - m$ is integer (half-integer) for an even (odd) representation. Moreover, as is explained in appendix A, one can arrange the normalization conventions in such a way that only those states $|j, m >$ belonging to an odd representation and with $j - m$ half-integer have negative norm. These results imply in our case that $|j_3, j_1 - j_2 >$ has negative norm only when $j_3 - j_1 + j_2$ is half-integer and, therefore, in eq. (4.10) we must take:

$$(-1)^{\sigma(j_3,m_3)} = (-1)^{2j_1+2j_2-2j_3}.$$  \hfill (4.22)

All these considerations lead us to write:

$$\left[D_{j_3,j_1:j_2,-j_2}^{j_1,j_2} \right]^2 = (-1)^{2j_1+2j_2-2j_3} \frac{S(j_1,j_2; j_2,j_1|j_3)}{S(j_2,j_2; j_2,j_2|0)}.$$ \hfill (4.23)

In order to compute the right-hand side of eq. (4.23), it is convenient to introduce the quantities $c_j$, defined as:

$$c_j \equiv \prod_{i=1}^{4j} \frac{\Gamma(1 - [i/2] + i\rho + [i/2]) \Gamma(1+i/2 - (i+1)\rho - [i/2])}{\Gamma(1+i/2 - i\rho - [i/2]) \Gamma(1+i/2 + (i+1)\rho + [i/2])}.$$ \hfill (4.24)

After some rearrangements of the products in (4.21), one can put the normalization factor in (4.23) in the form:

$$\left(\frac{\Gamma(\rho + 1/2)}{\pi}\right)^{2n} S(j_2,j_2; j_2,j_2|0) = \frac{(2j_2)!}{c_{j_2}}.$$ \hfill (4.25)

One can perform similar manipulations to the numerator of the right-hand side of eq. (4.23). In fact, it is possible to arrive at an expression in which most of the
The total combinatorial factor multiplying the product of $\lambda$ displayed in (4.26) can be put as:

$$\frac{S(j_1, j_2; j_3, j_3)}{S(j_2, j_2; j_2, j_2|0)} = \left( \frac{(2j_1)!(j_2 + j_3 - j_1)!}{(j_1 + j_3 - j_2)! (2j_2)!} \right)^2 \frac{c_{j_1} c_{j_2}}{c_{j_3}} \prod_{i=1}^{2j_1+2j_2-2j_3-1} \frac{\Gamma(\rho(i - 4j_1) - \frac{i}{2} + \left\lfloor \frac{i+1}{2} \right\rfloor)}{\Gamma(1 - \rho(i - 4j_1) + \frac{i}{2} - \left\lfloor \frac{i+1}{2} \right\rfloor)} \Gamma(1 + \rho(i - 4j_2) - \frac{i}{2} + \left\lfloor \frac{i+1}{2} \right\rfloor) \times \Gamma(1 - \rho(i - 4j_2) + \frac{i}{2} - \left\lfloor \frac{i+1}{2} \right\rfloor) \times \frac{\Gamma(\rho(4j_3 + 2 + i) - \frac{i}{2} + \left\lfloor \frac{i+1}{2} \right\rfloor)}{\Gamma(1 - \rho(4j_3 + 2 + i) + \frac{i}{2} - \left\lfloor \frac{i+1}{2} \right\rfloor)} \right)^2 \times \prod_{i=1}^{j} \frac{\Gamma(i + j - \frac{i}{2})}{\Gamma(i - j - \frac{i}{2})} \times \left( 2j_1 + 2j_2 - 2j_3 \right) \left( 2\rho \right)^{j_1 + j_2 - 4j_3} \left( \frac{(2j_1)!(2j_2)!}{(2j_3)!} \right)^2 \lambda(1) \frac{\lambda(4j_3 + 1)}{\lambda(4j_1 + 1)\lambda(4j_2 + 1)}.$$ (4.26)

We would like to convert (4.26) into an equation symmetric in the three isospins $j_1$, $j_2$ and $j_3$. In order to attain this purpose let us introduce the functions $\lambda(j)$ and $P(j)$. The former is defined as:

$$\lambda(j) = \frac{\Gamma(\frac{j}{2} + j\rho - \left\lfloor \frac{j}{2} \right\rfloor)}{\Gamma(\frac{j}{2} - j\rho - \left\lfloor \frac{j}{2} \right\rfloor)},$$ (4.27)

while $P(j)$ is given by:

$$P(j) = \prod_{i=1}^{j} \lambda(i) = \prod_{i=1}^{j} \frac{\Gamma(\frac{i}{2} + j\rho - \left\lfloor \frac{i}{2} \right\rfloor)}{\Gamma(\frac{i}{2} - j\rho - \left\lfloor \frac{i}{2} \right\rfloor)}.$$ (4.28)

It turns out that the different contributions in (4.26) can be written in terms of these two functions and some combinatorial factors. For example, the ratio of $c_j$’s displayed in (4.26) can be put as:

$$\frac{c_{j_1} c_{j_2}}{c_{j_3}} = (-1)^{2j_1+2j_2-2j_3} (2\rho)^{j_1+j_2-4j_3} \left( \frac{(2j_1)!(2j_2)!}{(2j_3)!} \right)^2 \lambda(1) \frac{\lambda(4j_3 + 1)}{\lambda(4j_1 + 1)\lambda(4j_2 + 1)}.$$ (4.29)

The total combinatorial factor multiplying the product of $\lambda$ and $P$ functions is:

$$\left[ \frac{C_{j_1, j_1, j_1; j_2, j_2, -j_2}}{C_{j_1, j_1, j_1; j_2, j_2, -j_2}} \right]^2 = \frac{(2j_1)!(2j_2)!}{(\left[j_1 + j_2 + j_3 + \frac{1}{2} \right]! (\left[j_1 + j_2 - j_3 \right]!)}.$$ (4.30)

Remarkably, the quantity $C_{j_1, j_1, j_1; j_2, j_2, -j_2}$ in eq. (4.30) is the osp(1|2) Clebsch-Gordan coefficient for the coupling of the three states $|j_1, j_1 >, |j_2, -j_2 >$ and $|j_3, j_1 - j_2 >$.
(see appendix A). With this identification, we can write the structure constants for arbitrary values of the isospins and Cartan components as:

\[
\left[ D_{j_1,m_1;j_2,m_2}^{j_3,m_3} \right]^2 = \left[ C_{j_1,m_1;j_2,m_2}^{j_3,m_3} \right]^4 \lambda(1) \mathcal{P}^2(2j_1 + 2j_2 + 2j_3 + 1) \times \\
\times \prod_{i=1}^{3} \frac{\lambda(4j_i + 1) \mathcal{P}^2(2j_1 + 2j_2 + 2j_3 - 4j_i)}{\mathcal{P}^2(4j_i + 1)}.
\] (4.31)

The result (4.31) constitutes the culmination of our efforts. Its consequences and implications will be analyzed in the next section. Before proceeding with this analysis several remarks are in order. First of all, let us notice that the structure constants \( D_{j_1,m_1;j_2,m_2}^{j_3,m_3} \) depend on the \( m_i \)'s only through the corresponding Clebsch-Gordan coefficients. This fact, which was to be expected on general grounds, can be checked by studying some other correlators, different from the one displayed in eq. (3.1). In this calculation the representations (2.24) and (2.25) of the components of the multiplet of conjugate primary fields must be used. We have verified in some of these cases that, indeed, the result for the structure constants is the one appearing in eq. (4.31). Moreover, it is interesting to point out that the \( m_i \)-independent factor in the right-hand side of eq. (4.31) is symmetric under the permutation of \( j_1 \), \( j_2 \) and \( j_3 \). Finally, let us notice the close similarity of the result in eq. (4.31) and the value of the structure constants for the \( sl(2) \) current algebra (see refs. [21, 19]). In the next section this similarity between the \( sl(2) \) and \( osp(1|2) \) cases will become more manifest.
5. Fusion rules and connection with the superconformal minimal models

With eq. (4.31) at our disposal we can obtain the selection rules of the operator algebra, i.e. the fusion rules, for the osp(1|2) current algebras. In order to determine these rules we must characterize the values of $j_1$, $j_2$ and $j_3$ for which $D^{j_3,m_3}_{j_1,m_1;j_2,m_2}$ is non-vanishing. Let us first of all recall (see section 2) that, when $k$ is a positive integer, only those representations of the affine osp(1|2) superalgebra with isospins

$$j \leq \frac{k}{2},$$  

are allowed. Therefore, we shall assume that the constraint (5.1) is satisfied by $j_1$, $j_2$ and $j_3$. It is important to point out that when this happens a possible divergence in (4.31) due to the terms $1/\mathcal{P}^2(4j_i + 1)$ never takes place. We can therefore concentrate ourselves on the zeros coming from the other terms in (4.31). Generally these zeros are generated when a $\Gamma$-function evaluated at zero or a negative integer appears in the denominator of eq. (4.31). So, for example, a detailed analysis of the factor $\mathcal{P}^2(2j_1 + 2j_2 + 2j_3 + 1)$ using the definition (4.28) shows that it is non-vanishing only when $j_3$ satisfies the condition:

$$j_3 \leq k + \frac{1}{2} - j_1 - j_2. \tag{5.2}$$

Moreover, the term $\mathcal{P}^2(2j_1 + 2j_2 - 2j_3)$ is different from zero if

$$j_3 \leq j_1 + j_2, \tag{5.3}$$

whereas, when (5.2) and (5.3) are satisfied, the requirement that the remaining factors $\mathcal{P}^2(2j_2 + 2j_3 - 2j_1)$ and $\mathcal{P}^2(2j_1 + 2j_3 - 2j_2)$ never vanish yields the condition:

$$j_3 \geq |j_1 - j_2|. \tag{5.4}$$

Notice that the inequalities (5.3) and (5.4) do not depend on the level $k$ and, in fact, are the same that appear in the non-affine osp(1|2) representation theory. As
was mentioned before, they are automatically incorporated in our approach when $j_3$ is given by eq. (3.18). On the other hand, eq. (5.2) is a non-trivial constraint on the maximum value of $j_3$ which does depend on the level $k$. It is elementary to verify the compatibility of (5.2) with our general condition (5.1). Indeed, by adding (5.2) and (5.3) one gets $j_3 \leq k + \frac{1}{4}$ which, as $k$ is integer and $j_3$ can only take integer or half-integer values, reduces to $j_3 \leq \frac{k}{2}$. Gathering all the conditions we have found, we can write the osp(1$|$2) fusion rules as:

$$[j_1] \times [j_2] = \min(j_1 + j_2, k + \frac{1}{2} - j_1 - j_2) \sum_{j_3 = |j_1 - j_2|}^{2(j_1 - j_1 - j_2) \in \mathbb{Z}} [j_3]. \quad (5.5)$$

Interestingly, the fusion rules (5.5) coincide, for this $0 < k \in \mathbb{Z}$ case, with the ones found in ref. [25] from the modular properties of the osp(1$|$2) characters. This coincidence is an argument in favor of the validity in this case of the Verlinde formula which, with a suitable reinterpretation, was used in [25] to arrive at eq. (5.5).

Let us now compare the fusion rules (5.5) with the ones corresponding to the $N = 1$ superconformal minimal models in the Neveu-Schwarz (NS) sector. The $(p', p)$ superconformal model has central charge:

$$c_{p', p} = \frac{3}{2} \left[1 - \frac{2(p' - p)^2}{p'p} \right]. \quad (5.6)$$

The primary operators in the NS sector are labelled by two integers $m'$ and $m$ which must be such that $1 \leq m' \leq p' - 1$, $1 \leq m \leq p - 1$ and $m' - m \in 2\mathbb{Z}$. The fusion rules for these operators can be written as [13]:

$$[m'_1, m_1] \times [m'_2, m_2] = \min(m'_1 + m'_2 - 1, 2p' - m'_1 - m'_2 - 1) \sum_{m'_3 = |m'_1 - m'_2| + 1}^{m'_1 + m'_2 - 1} \sum_{m_3 = |m_1 - m_2| + 1}^{m_1 + m_2 - 1} [m'_3, m_3]. \quad (5.7)$$

By inspecting (5.7), one concludes that these fusion rules are equivalent to those
of two independent osp(1|2) algebras, when the isospins \((j', j)\) and levels \((k', k)\) of the latter are properly identified with the quantum numbers \((m', m)\) and the \((p', p)\) parameters of the \(N = 1\) superconformal field theory. This identification is:

\[
m' = 4j' + 1 \quad m = 4j + 1 \\
p' = 2k' + 3 \quad p = 2k + 3 .
\] (5.8)

One can check, using (5.8), that the ranges of allowed values of \(m\) and \(m'\) (i.e. \(1 \leq m' \leq p' - 1, 1 \leq m \leq p - 1\) and \(m' - m \in 2\mathbb{Z}\)) correspond precisely to the ranges \(j' \leq k'/2\) and \(j \leq k/2\) for the osp(1|2) isospins.

The relation between the osp(1|2) current algebras and the \(N = 1\) superconformal theories goes beyond the similarity of their fusion rules. Actually, the structure constants for the products of thermal operators \(\phi_{1,m}\) of the \(N = 1\) SCFT and those of the osp(1|2) current algebras are closely related. Let us denote by \(D_{m_1,m_2}^{m_3}\) the structure constants appearing in the correlator \(<\phi_{1,m_1}(z_1)\phi_{1,m_2}(z_2)\phi_{1,m_3}(z_3)\>\) and, for the \((p', p)\) N=1 minimal model, let \(\tilde{\rho}\) be defined as:

\[
\tilde{\rho} = \frac{p'}{2p} .
\] (5.9)

In order to express the constants \(D_{m_1,m_2}^{m_3}\) in a compact fashion, let us introduce the quantities

\[
s_m \equiv 1 + \sum_{i=1}^{3} \frac{m_i}{4} , \\
Q(N) \equiv \prod_{i=1}^{N} \frac{1}{\left(\frac{1}{2} - (2i+1)\tilde{\rho}\right)^2} ,
\] (5.10)

in terms of which the function \(\Lambda(m_1, m_2, m_3)\) is defined as:

\[
\Lambda(m_1, m_2, m_3) = 2^4(s_m - \lfloor s_m \rfloor) (-1)^{2s_m} \frac{Q^2(\lfloor s_m \rfloor)}{(s_m - 1/2)!} \times \\
\times \prod_{i=1}^{3} \frac{(m_i - 1)!}{\lfloor m_1 + m_2 + m_3 - 2m_i - 1 \rfloor!} \frac{Q(m_i - 1/4)}{Q(m_i - 1)} .
\] (5.11)
The value of the \( D_{m_1,m_2}^{m_3} \) constants has been obtained in refs. [14, 15]. With our notation the result of these references can be written as:

\[
\left[ D_{m_1,m_2}^{m_3} \right]^2 = \Lambda(m_1, m_2, m_3) \tilde{\lambda}(1) \tilde{\mathcal{P}}^2 \left( \frac{m_1 + m_2 + m_3 - 1}{2} \right) \times
\]

\[
\times \prod_{i=1}^3 \frac{\tilde{\lambda}(m_i)}{\tilde{\mathcal{P}}^2(m_i)} \tilde{\mathcal{P}}^2 \left( \frac{m_1 + m_2 + m_3 - 2m_i - 1}{2} \right),
\]

where \( \tilde{\lambda}(j) \) and \( \tilde{\mathcal{P}}(j) \) are given by eqs. (4.27) and (4.28) with \( \rho \) substituted by \( \tilde{\rho} \).

Notice that \( \Lambda(m_1, m_2, m_3) \) never vanishes and it is totally symmetric in \( m_1, m_2 \) and \( m_3 \). Moreover, comparing the \( \Gamma \)-function terms in (4.31) and (5.12) one easily concludes that they are the same if one identifies \( \rho \) and \( \tilde{\rho} \) and, as was done in eq. (5.8), if we take \( m_i = 4j_i + 1 \). This identification \( \rho = \tilde{\rho} \) corresponds to taking \( 2k + 3 = p/p' \), which is precisely the fractional level required to construct the \( N = 1 (p', p) \) minimal supersymmetric models from the Hamiltonian reduction of the \( \text{osp}(1|2) \) current algebra. It is interesting at this point to recall that, between the structure constants of the \( \text{sl}(2) \) CFT and those of the thermal operators of the minimal non-supersymmetric models, there exists a relation similar to the one we have found here between the two sets of structure constants of eqs. (4.31) and (5.12).

As was pointed out above, the identification \( \rho \equiv \tilde{\rho} \) requires to consider a value of \( k \) which is, in general, rational. For these rational values of \( k \) a larger class of \( \text{osp}(1|2) \) admissible representations must be considered (see sect. 2). Indeed, in this case, the isospin \( j \) can also take rational values. Comparing with a similar situation in the \( \text{sl}(2) \) algebra, it is plausible to think that, in order to deal with this more general case, one must consider a formalism in which new variables are introduced to represent the action of the superalgebra. For the \( \text{sl}(2) \) Lie algebra the primary fields in this new formalism depend on the space-time coordinate \( z \) and on a new bosonic coordinate \( x \). In the \( \text{osp}(1|2) \) case it is natural to introduce, in addition, a new Grassmann variable \( \theta \) in such a way that the generalized primary
fields are:

\[ \varphi(z, x, \theta) = (1 + x\chi + \theta\psi)^{2j} e^{-2i\alpha_0 \phi} . \]  

(5.13)

Notice that, for integer or half-integer \( j \), \( \varphi(z, x, \theta) \) can be expanded in a finite sum involving different powers of \( x \) and \( \theta \). The coefficients in this power sum are precisely the fields \( \Phi^m_{j} \) defined in (2.10). In fact, the terms that do not contain the Grassmann variable \( \theta \) correspond to \( \Phi^m_{j} \) with \( j - m \in \mathbb{Z} \), whereas those that contain \( \theta \), and therefore also the field \( \psi \), can be identified with the \( \Phi^m_{j} \) operators with \( j - m \in \mathbb{Z} + \frac{1}{2} \). For general \( j \), the OPE’s of the osp(1|2) currents with the field \( \varphi(z, x, \theta) \) can be written as:

\[
H(z) \varphi(w, x, \theta) = \frac{D^j_3 \varphi(w, x, \theta)}{z - w}, \\
J^\pm(z) \varphi(w, x, \theta) = \frac{D^j_\pm \varphi(w, x, \theta)}{z - w}, \\
j^\pm(z) \varphi(w, x, \theta) = \frac{d^j_\pm \varphi(w, x, \theta)}{z - w},
\]

(5.14)

where \( D^j_3, D^j_\pm \) and \( d^j_\pm \) are the following differential operators on the variables \( x \) and \( \theta \):

\[
D^j_3 = -x \partial_x - \frac{1}{2} \theta \partial_\theta + j \\
D^j_\pm = -x^2 \partial_x + 2jx - \theta x \partial_\theta \\
D^j_- = \partial_x \\
d^j_+ = x \partial_\theta + \theta x \partial_x - 2j\theta \\
d^j_- = \partial_\theta + \theta \partial_x .
\]

(5.15)

It is straightforward to verify that the operators (5.15) can be used to represent the osp(1|2) algebra (this is the so-called isotopic representation).

Let us finally point out that, when the isospin \( j \) is not integer or half-integer, the charge \( Q \) given in eqs. (2.27) and (2.28) is unable to screen a general correlator. We must thus find, as it happens in the \( sl(2) \) CFT, a second screening operator.
This is not difficult and, in fact, one can prove that the OPE’s of the osp(1|2) currents with the field

\[
\hat{S} = w^{-(k+2)} (\bar{\psi} - w\psi) e^{-\frac{1}{\alpha_0} \phi},
\]  

(5.16)

have only total derivatives. Therefore \( \oint dz \hat{S}(z) \) can be taken as a screening charge. Notice that \( \hat{S}(z) \) is non-local (this also happens for \( sl(2) \)) and, thus, one must give a prescription to compute correlators involving it. Presumably one could apply, for this purpose, the fractional calculus technique of ref. [27].

6. Concluding remarks

Let us recapitulate our main results. We have studied the free field realization of the osp(1|2) current algebra. We have given a representation of the primary fields of the theory which allows to compute the conformal blocks of the model. We have been able to define a set of selection rules for the computation of the Fock space expectation values, which are such that incorporate the basic features of the osp(1|2) representation theory. We have performed in detail the analysis of the four point-functions. Our goal in this study was to obtain the structure constants appearing in the operator algebra of the theory.

The technique we have used has been successfully employed previously for the minimal models [12, 14, 15] and for the \( sl(2) \) current algebras[19]. The computation of the structure constants requires the evaluation of the normalization integrals of the blocks. This is, in fact, the greatest technical difficulty of this approach. In our case, the integrals needed can be computed in some cases by reducing them to known results, and by imposing some consistency conditions in other situations. Although the expression of these integrals might appear cumbersome, the final result (4.31) for the structure constants is rather simple and, actually, very similar to the \( sl(2) \) case. This is so because some remarkable simplifications occur in the intermediate steps of the calculation.
Once the structure constants are known one can perform an analysis to determine when they vanish. The outcome of this study are the fusion rules (5.5) of the model. The comparison of these rules with the ones corresponding to the minimal superconformal theories can help to shed light on the relation between them. For example, the interpretation of the fusion rules for the $N = 1$ minimal superconformal models as given by two independent osp(1|2) algebras is reminiscent of a similar relationship between the minimal Virasoro models and the $sl(2)$ current algebras. This latter relation can be easily understood when one constructs the minimal Virasoro models as sl(2) coset theories. Our result suggests the existence of a similar construction relating the osp(1|2) superalgebras and the $N = 1$ minimal superconformal models.

Obviously, many aspects of our free field construction remain to be explored. For example, one should be able to obtain the full duality structure of the osp(1|2) theory by applying the ideas of ref. [28] to our approach. One expects that a quantum deformation of osp(1|2) [29] will show up as the result of this analysis. It is also likely that the fusion rules (5.5) could be obtained directly by using the cohomological methods of ref. [30].

One should be able to incorporate to our formalism the fractional isospins that appear when the level $k$ is not integer. The understanding of the conformal blocks for these representations is essential if one wants to implement a hamiltonian reduction procedure, relating the osp(1|2) theory with the supersymmetric theories, and, eventually, with the two-dimensional supergravity and the non-critical superstrings. In this respect, the isotopic approach introduced at the end of section 5 should be relevant. Within this approach, it is natural to assemble the two internal coordinates $x$ and $\theta$ into an isotopic superspace coordinate $X = (x, \theta)$. On the other hand, it was argued in ref. [31] that the hamiltonian reduction of the $sl(2)$ theory can be performed by identifying the isotopic and the space-time coordinates, i.e. by putting $x = z$. Trying to generalize this result to the osp(1|2) superalgebra, one is led to think that one should supplement the osp(1|2) currents with some supersymmetric partners. The resulting model would be a Kac-Todorov
system for the osp(1|2) Lie superalgebra, for which a natural superspace coordinate \( Z = (z, \eta) \) can be defined. It would be very interesting to investigate if the identification of \( Z \) and \( X \) could be used to generate the conformal blocks for the \( N = 1 \) superconformal models from those of the osp(1|2) system. The addition of extra fields to the osp(1|2) current algebra has been considered in other approaches to hamiltonian reduction [33].

Let us finally point out that the methods employed here can be applied, in principle, to other Lie superalgebras. The simplest of these superalgebras is osp(2|2), which is related to the \( N = 2 \) superconformal symmetry [7]. The osp(2|2) representation theory is well established [22] and the free field representation of the corresponding current algebra has been given in ref. [7].

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**APPENDIX A**

The osp(1|2) Lie superalgebra contains three bosonic generators \( T_{\pm} \) and \( T_3 \), which form the Lie algebra \( sl(2) \), together with two fermionic generators \( F_{\pm} \). The (anti)commutators that define the osp(1|2) superalgebra are:

\[
\begin{align*}
[T_3, T_{\pm}] &= \pm T_{\pm} \\
[T_3, F_\pm] &= \pm \frac{1}{2} F_\pm \\
\{F_{\pm}, F_\mp\} &= 2T_3 \\
\{F_{\pm}, F_{\mp}\} &= \pm 2T_3 \\
[T_{\pm}, F_\mp] &= F_\pm \\
[T_{\pm}, F_\mp] &= 0
\end{align*}
\] (A.1)

Notice that the bosonic generators \( T_{\pm} \) and \( T_3 \) correspond to the currents \( J_{\pm}(z) \) and \( H(z) \) respectively, whereas the fermionic operators \( F_{\pm} \) correspond to the currents
$j_{\pm}(z)$. It is not difficult to prove, using the relations (A.1), that the operator

$$ C_2 = T_3^2 + \frac{1}{2} \left[ T_- T_+ + T_+ T_- \right] + \frac{1}{4} \left[ F_- F_+ - F_+ F_- \right], \tag{A.2} $$

commutes with all the osp(1|2) generators. Actually, $C_2$ is the quadratic Casimir operator of the osp(1|2) superalgebra. Using the (anti)commutators (A.1) we can reexpress $C_2$ as:

$$ C_2 = T_3^2 + \frac{1}{2} T_3 + T_- T_+ + \frac{1}{2} F_- F_+ . \tag{A.3} $$

Let us now study [22] the matrix representations of the algebra (A.1). The representation theory of osp(1|2) has many features that are similar to those of the $sl(2)$ Lie algebra. As in this latter case, we shall represent the Cartan generator $T_3$ by a diagonal operator and, thus, we can label the states of our vector space basis by their $T_3$ eigenvalues, which are the weights of the representation. It is a simple exercise to check from (A.1) that the $T_+$ ($T_-$) operator raises (lowers) the eigenvalue of the $T_3$ eigenstates in one unit without changing its Grassmann parity, whereas $F_+$ ($F_-$) increases (decreases) the $T_3$ eigenvalue of these states in one-half unit and changes their statistics. The finite dimensional irreducible representations $\mathcal{R}_j$ of osp(1|2) are characterized by the value $j$ of their highest weight, which can be integer or half-integer. In close parallel with the $sl(2)$ case, we shall call $j$ the isospin of the representation. If we denote the highest weight vector by $|j, j >$, it is evident that it must satisfy:

$$ T_+ |j, j > = F_+ |j, j > = 0. \tag{A.4} $$

A general basis state for the representation $\mathcal{R}_j$ will be denoted by $|j, m >$, $m$ being the $T_3$ eigenvalue. The quadratic Casimir operator $C_2$ acts on these states as a multiple of the unit operator. It is easy to obtain its value by computing $C_2 |j, j >$ using (A.4). One immediately gets:

$$ C_2 |j, m > = j \left( j + \frac{1}{2} \right) |j, m >. \tag{A.5} $$

Using eqs. (A.5), (A.2) and the defining relations of the algebra (eq. (A.1)),
it is not difficult [22] to find the matrix elements of the different generators in the representation $\mathcal{R}_j$. The result that one finds for the bosonic operators is:

$$T_3 |j, m > = m |j, m >$$  \hspace{1cm} (A.6)

$$T_\pm |j, m > = \sqrt{[j \mp m] [j \pm m + 1]} |j, m \pm 1 > ,$$

where, as in the main text, $[x]$ represents the integer part of a non-negative integer or half-integer $x$. The matrix elements of the odd operators $F_\pm$ are:

$$F_\pm |j, m > = \begin{cases} 
-\sqrt{j \mp m} |j, m \pm \frac{1}{2} > & \text{if } j - m \in \mathbb{Z} \\
\mp \sqrt{j \pm m + \frac{1}{2}} |j, m \pm \frac{1}{2} > & \text{if } j - m \in \mathbb{Z} + \frac{1}{2}.
\end{cases}$$  \hspace{1cm} (A.7)

From (A.6) and (A.7) one easily concludes that, indeed, when $j$ is integer or half-integer $\mathcal{R}_j$ is finite dimensional. In fact, in this case, only the states $|j, m >$ with $m = -j, -j + \frac{1}{2}, \cdots, j - \frac{1}{2}, j$ are connected by the action of the $\text{osp}(1|2)$ generators. The dimension of $\mathcal{R}_j$ is thus:

$$\dim (\mathcal{R}_j) = 4j + 1.$$  \hspace{1cm} (A.8)

Moreover, eq. (A.6) shows that for $j \neq 0 \mathcal{R}_j$ decomposes under the even part of the superalgebra into two $sl(2)$ multiplets with isospins $j$ and $j - \frac{1}{2}$. The members of these multiplets are the states $|j, m >$ with $j - m \in \mathbb{Z}$ and $j - m \in \mathbb{Z} + \frac{1}{2}$ respectively. These two $sl(2)$ multiplets are connected by the action of $F_\pm$ and thus they have opposite statistics. Actually, an $\text{osp}(1|2)$ representation $\mathcal{R}_j$ is completely characterized if, together with its isospin $j$, we also give the statistics of its highest weight state $|j, j >$. The Grassmann parity of $|j, j >$ will be denoted by $p(j)$ ($p(j) = 0, 1$). We will say that the representation is even (odd) when $|j, j >$ is bosonic(fermionic), i.e. when $p(j) = 0$ ($p(j) = 1$).
For Lie superalgebras it is possible to define a generalized adjoint operation [22], denoted by $\dagger$, such that for any operator $A$ and any two states $\alpha$ and $\beta$ one has:

$$<A^\dagger \alpha | \beta> = (-1)^{p(A)p(\alpha)} < \alpha | A \beta>,$$  \hspace{1cm} (A.9)

where $p(A)$ and $p(\alpha)$ denote respectively the Grassmann parities of the operator $A$ and the state $\alpha$. We will say that $A^\dagger$ is the superadjoint of $A$. From the property (A.9), one can verify that:

$$(AB)^\dagger = (-1)^{p(A)p(B)} B^\dagger A^\dagger.$$ \hspace{1cm} (A.10)

It is not difficult to obtain the explicit form of the superadjoint operation for the $\text{osp}(1|2)$ generators. In fact, by requiring compatibility of this operation with the (anti)commutators (A.1), one can easily establish that $T^\dagger_\pm = T_\mp$ and $T^\dagger_3 = T_3$, as expected, while the rule for the fermionic generators is:

$$F^\dagger_+ = \eta F_- \hspace{1cm} F^\dagger_- = -\eta F_+,$$ \hspace{1cm} (A.11)

where $\eta$ can take the values $\pm 1$. Notice that, independently of $\eta$, $((F_\pm)^\dagger)^\dagger = -F_\pm$.

The actual value of $\eta$ can be determined, as a consequence of eq. (A.9), from the norm of the basis states and the parity of the representation. Let us suppose that $<j, m| j, m> = \epsilon(\epsilon')$ if $j - m$ is integer (half-integer), where $\epsilon$ and $\epsilon'$ can take the values $\pm 1$. Putting in eq. (A.9) $\alpha = |j, j>$, $\beta = |j, j - \frac{1}{2}>$ and $A = F_+$, and taking eq. (A.7) into account, one gets:

$$\eta = (-1)^{p(j)} \epsilon \epsilon'.$$ \hspace{1cm} (A.12)

We shall conventionally choose $\eta = 1$, which means that $F^\dagger_+ = F_-$ and $F^\dagger_- = -F_+$. For even representations this election implies that $\epsilon \epsilon' = +1$, whereas, on the contrary, $\epsilon \epsilon'$ must be negative for odd representations (see eq. (A.12)). According to this result we shall take $\epsilon = \epsilon' = +1$ ($\epsilon = -\epsilon' = +1$) for even(odd) representations and, thus, only the states $|j, m>$ with $p(j) = 1$ and $j - m \in \mathbb{Z} + \frac{1}{2}$ will have negative norm.
Let us now consider the tensor product of two representations. By using the well-known methods of angular momentum theory, one can easily convince oneself that the coupling of isospins \( j_1 \) and \( j_2 \) gives rise to isospins \( j_3 = |j_1 - j_2|, |j_1 - \frac{1}{2}, \cdots, j_1 + j_2. \) Actually, one can write the tensor product decomposition of \( \mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} \) as:

\[
\mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} = \bigoplus_{j_3=|j_1 - j_2|, 2(j_3 - j_1 - j_2) \in \mathbb{Z}} j_3
\]

Furthermore, the parity of the representations \( \mathcal{R}_{j_3} \) in the right-hand of eq. (A.13) is:

\[
p(j_3) = p(j_1) + p(j_2) + 2(j_1 + j_2 - j_3) \mod (2)
\]

As usual, the states \( |j_3, m_3> \) can be obtained from those of \( \mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} \) by means of the \( \text{osp}(1|2) \) Clebsch-Gordan coefficients \( C_{j_1,m_1; j_2,m_2}^{j_3,m_3} \):

\[
|j_3, m_3> = \sum_{m_1,m_2} C_{j_1,m_1; j_2,m_2}^{j_3,m_3} |j_1, m_1> \otimes |j_2, m_2>
\]

In ref. [22], the \( C_{j_1,m_1; j_2,m_2}^{j_3,m_3} \) have been computed in terms of the \( \text{sl}(2) \) Clebsch-Gordan coefficients. Let us denote the latter by \( \widehat{C}_{j_1,m_1; j_2,m_2}^{j_3,m_3} \). In our calculation of the structure constants of the \( \text{osp}(1|2) \) current algebra, we shall only need the value of the \( C_{j_1,m_1; j_2,m_2}^{j_3,m_3} \) for \( j_1 - m_1 \in \mathbb{Z} \) and \( j_2 - m_2 \in \mathbb{Z}. \) In this case the result given in ref. [22] can be written as:

\[
C_{j_1,m_1; j_2,m_2}^{j_3,m_3} = \begin{cases} 
\sqrt{\frac{j_1+j_2+j_3+1}{2j_3+1}} \widehat{C}_{j_1,m_1; j_2,m_2}^{j_3,m_3} & \text{if } j_3 - m_3 \in \mathbb{Z} \\
(-1)^{p(j_1)+1} \sqrt{\frac{j_1+j_2-j_3+\frac{1}{2}}{2j_3}} \widehat{C}_{j_1,m_1; j_2,m_2}^{j_3-\frac{1}{2},m_3} & \text{if } j_3 - m_3 \in \mathbb{Z} + \frac{1}{2}.
\end{cases}
\]

Let us particularize eq. (A.16) to the situation in which \( m_1 = j_1 \) and \( m_2 = -j_2. \)
The $sl(2)$ Clebsch-Gordan coefficients are:

\[ \hat{C}_{j_1, j_1; j_2, -j_2} = \sqrt{2j+1} \left[ \frac{(2j_1)!(2j_2)!}{(j_1+j_2+j+1)!(j_1+j_2-j)!} \right]^{1/2}. \quad (A.17) \]

Substituting this result for $j = j_3$ and $j = j_3 - \frac{1}{2}$ in eq. (A.16), the following expression for $C_{j_3, j_1; j_2, -j_2}$ is obtained:

\[ C_{j_3, j_1; j_2, -j_2} = (-1)^{2p(j_1)+1}(j_1+j_2-j_3) \left[ \frac{(2j_1)!(2j_2)!}{([j_1+j_2+j_3+\frac{1}{2}]!([j_1+j_2-j_3])!)} \right]^{1/2}, \quad (A.18) \]

which is precisely the result needed in section 4 (see eq. (4.30)).

APPENDIX B

In this appendix we will evaluate the multiple integrals needed to obtain the expression of the osp(1|2) structure constants. In general, the integrals to compute will involve functions of $N$ variables $t_1, t_2, \ldots, t_N$. The domain of integration will be the subset of the $N$-dimensional unit hypercube defined by $1 \geq t_1 \geq \cdots \geq t_{N-1} \geq t_N \geq 0$. If $f(\{t_i\})$ is the function to integrate, we shall adopt the following notation for these ordered integrations:

\[ \oint f(\{t_i\}) \equiv \int_0^1 dt_1 \cdots \int_0^{t_{N-1}} dt_N \ f(\{t_i\}). \quad (B.1) \]

First of all, let us consider the class of integrals defined by:

\[ I_n^m (a, b, \rho) \equiv \oint \left[ (t_1 \cdots t_m)^{a+1} (t_{m+1} \cdots t_n)^a + \text{permutations} \right] \times \]

\[ \times \prod_{i=1}^n (1-t_i)^b \prod_{i<j}^n (t_i - t_j)^{2\rho}, \quad (B.2) \]

where $0 \leq m \leq n$. In eq. (B.2) the dimensionality of the integration domain is $N = n$ and, in the integrand, $m$ of the integration variables are raised to the
power \( a + 1 \) whereas the exponent of \( n - m \) of them is \( a \). A sum over all the possible elections of these \( n \) and \( n - m \) variables is performed in order to make the integrand in (B.2) completely symmetric in its arguments. The result of the integrals \( I_n^0(a,b,\rho) \) and \( I_n^a(a,b,\rho) \) has been given by Dotsenko and Fateev [12] as a product of Euler \( \Gamma \)-functions. It is easy to get an expression interpolating between these two extreme values of \( m \). Actually we are going to argue that the integrals \( I_n^m(a,b,\rho) \) are given by:

\[
I_n^m(a,b,\rho) = \binom{n}{m} \lambda_n(\rho) \prod_{i=0}^{n-m-1} \frac{\Gamma(1 + a + i\rho) \Gamma(1 + b + i\rho)}{\Gamma(2 + a + b + (n-1+i)\rho)} \times \\
\times \prod_{i=n-m}^{n-1} \frac{\Gamma(2 + a + i\rho) \Gamma(1 + b + i\rho)}{\Gamma(3 + a + b + (n-1+i)\rho)},
\]

where the function \( \lambda_n(\rho) \) is:

\[
\lambda_n(\rho) = \prod_{i=1}^{n} \frac{\Gamma(i\rho)}{\Gamma(\rho)}.
\]

Notice that, in the last \( m \) factors of the right-hand side of (B.3), \( a \) is shifted in one unit with respect to the first \( n - m \) ones. As a first check of eq. (B.3), let us consider the case \( \rho = 0 \). When \( \rho \) vanishes the multiple integral in eq. (B.2) decouples into \( n \) one-dimensional integrals whose expression is given in terms of the Euler beta function. It is straightforward to verify that the value of \( I_n^m(a,b,0) \) so obtained coincides with the one dictated by (B.3) when \( \rho = 0 \). As a more restrictive test of (B.3), we are going to get a functional relation that the integrals \( I_n^m(a,b,\rho) \) must satisfy. This relation can be obtained directly from the definition (B.2). Suppose that we use in this equation the identity \( t_i^{a+1} = t_i^{a} - t_i^{a}(1-t_i) \) in all the variables whose exponent is \( a+1 \). After changing variables as \( t_i \rightarrow 1-t_{n+1-i} \) for \( i = 1, \cdots, n \), we arrive at the following relation:

\[
I_n^m(a,b,\rho) = \sum_{p=0}^{m} (-1)^{m-p} \binom{p+n-m}{p} I_n^{m-p}(b,a,\rho).
\]

For \( m = 0 \) eq. (B.5) simply states that \( I_n^0(a,b,\rho) \) is symmetric under the inter-
change of $a$ and $b$. Notice that this property is crucial in the Dotsenko-Fateev derivation[12]. For $m > 0$ eq. (B.5) relates the integrals (B.2) to functions of the same kind with lower values of their upper index $m$ and with $a$ and $b$ exchanged. Using the elementary properties of the $\Gamma$-function, it is not difficult to verify that our result (B.3) satisfies eq. (B.5). In order to get a general proof of this statement, the binomial identity (4.15) is very useful.

As a final check of eq. (B.3), let us compute the integrals (B.2) for $m = n - 1$ by means of the following trick. First of all, we define the functions $K_n(a, b, \rho, x)$ depending on an additional variable $x$:

$$K_n(a, b, \rho, x) \equiv \int_0^x dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i=1}^n t_i^a (x - t_i)^{b+1} \prod_{i<j}^n (t_i - t_j)^{2\rho}. \quad (B.6)$$

It is evident from their definition that the integrals $K_n(a, b, \rho, x)$ reduce to the functions written in eq. (B.2) when the variable $x$ is equal to one. Namely, one has:

$$K_n(a, b, \rho, 1) = I_0^n(a, b + 1, \rho). \quad (B.7)$$

Moreover, the dependence of $K_n(a, b, \rho, x)$ on $x$ can be easily extracted by performing a rescaling $t_i \to xt_i$ of the integration variables in eq. (B.6). One easily arrives at:

$$K_n(a, b, \rho, x) = x^{2n+na+nb+n(n-1)\rho} I_0^n(a, b + 1, \rho). \quad (B.8)$$

Let us now calculate the derivative of $K_n(a, b, \rho, x)$ with respect to $x$ at the point $x = 1$. Computing it directly from the definition (B.6), one gets:

$$\frac{\partial K_n(a, b, \rho, x)}{\partial x}|_{x=1} = (b + 1) I_{n-1}^n(b, a, \rho). \quad (B.9)$$

The left-hand side of eq. (B.9) can also be obtained from the $x$ dependence displayed in eq. (B.7). Comparing both ways of computing this derivative, one arrives
at the result:

\[ I_{n-1}^{n-1}(a, b, \rho) = n \frac{2 + a + b + (n-1)\rho}{1 + a} I_n^0(b, a + 1, \rho), \quad \text{(B.10)} \]

which gives the integrals (B.2) for \( m = n - 1 \) in terms of those with \( m = 0 \). Using the Dotsenko-Fateev result for \( I_n^0(a, b, \rho) \), one easily proves that the values of \( I_{n-1}^{n-1}(a, b, \rho) \) given in eqs. (B.10) and (B.3) coincide.

We are now going to study a family of integrals that appear directly in our evaluation of the osp(1\mid2) operator algebra. They are the 2\( n \)-dimensional integrals \( J_{2n}^m(a, b, \rho) \), with \( 0 \leq m \leq n \), defined as:

\[
J_{2n}^m(a, b, \rho) \equiv \oint \{ (t_1 \cdots t_n)^a [ (t_{n+1} \cdots t_{n+m})^{a+1} (t_{n+m+1} \cdots t_{2n})^a + \text{permutations}] \times \]
\[
\times \langle \psi(t_1) \cdots \psi(t_n) \bar{\psi}(t_{n+1}) \cdots \bar{\psi}(t_{2n}) \rangle + \text{permutations} \}
\times \prod_{i=1}^{2n} (1 - t_i)^b \prod_{i<j} (t_i - t_j)^{2\rho}.
\]

\[ \text{(B.11)} \]

Notice the close similarity between the definitions (B.2) and (B.11). The main difference between \( I_n^m \) and \( J_{2n}^m \) is the presence in the latter of a fermionic correlator involving the Dirac fields \( \psi \) and \( \bar{\psi} \). Our conventions for the normalization of these fields are the same as those used in section 2 (see eq. (2.1)). In eq. (B.11) \( m \) of the arguments of the \( \bar{\psi} \) fields appear raised to the power \( a+1 \). A double symmetrization of the integrand of (B.11) is performed. First of all, one must sum over all the possible elections of the \( m \) variables among the \( n \) arguments of the \( \bar{\psi} \) fields inserted in the correlator. Secondly, one must sum over all the possible locations of the fields \( \psi \) and \( \bar{\psi} \) inside the vacuum expectation value. It is important to point out that the arguments of the fields appearing in these correlators are ordered as the integration limits, i.e. when \( i < j \), the field with argument \( t_i \) is always to the left of those with arguments \( t_j \).

The value of the integrals \( J_{2n}^m \) can be obtained following the same steps that
led us to (B.3). The expression that one arrives at is:

\[
J_{2n}^m(a, b, \rho) = \binom{n}{m} \mu_{2n}(\rho) \prod_{i=0}^{2n-2m-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \lceil \frac{i}{2} \rceil) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \lceil \frac{i}{2} \rceil)}{\Gamma(1 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + \lceil \frac{i}{2} \rceil)} \times \\
\times \prod_{i=2n-2m}^{2n-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \lceil \frac{i+1}{2} \rceil) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \lceil \frac{i}{2} \rceil)}{\Gamma(1 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + \lceil \frac{i+1}{2} \rceil)},
\]

\[(B.12)\]

where the function \( \mu_N(\rho) \) is given by:

\[
\mu_N(\rho) = \prod_{i=1}^{N} \frac{\Gamma(i(\rho + \frac{1}{2}) - \lceil \frac{i}{2} \rceil)}{\Gamma(\rho + \frac{1}{2})}.
\]

\[(B.13)\]

In eqs. (B.12) and (B.13) \( \lceil \frac{i}{2} \rceil \) represents the integer part of \( \frac{i}{2} \) for any positive integer \( i \). In order to verify the correctness of eqs. (B.12) and (B.13), let us consider some particular cases. Let us first take \( m = 0 \) in eq. (B.12). For this value of \( m \) the fermionic correlators appearing in the definition of \( J_{2n}^m \) are all multiplied by the same factor. It turns out that the combination of vacuum expectation values of products of \( \psi \) and \( \bar{\psi} \) appearing in \( J_{2n}^0 \) can be put as a single correlator of a new fermionic field. Indeed, let \( \lambda(t) \) be a fermionic Majorana field normalized in such a way that its basic OPE is \( \lambda(t_1)\lambda(t_2) = (t_1 - t_2)^{-1} \). It can be easily proved that:

\[
< \psi(t_1) \cdots \psi(t_n)\bar{\psi}(t_{n+1}) \cdots \bar{\psi}(t_{2n}) > + \text{permutations} = \\
= 2^n < \lambda(t_1) \cdots \lambda(t_n)\lambda(t_{n+1}) \cdots \lambda(t_{2n}) > .
\]

\[(B.14)\]

Integrals of the type studied above, i.e. with a correlator of Majorana fields in the integrand, appear in the Feigin-Fuchs calculation of the structure constants of the minimal supersymmetric models[14, 15]. In fact, a general expression for these integrals has been given in ref. [14]. After taking eq. (B.14) into account, it can be seen that our expression (B.12) for \( m = 0 \) is in agreement with the result of ref. [14].

53
It would be interesting to find a particular value of $\rho$ for which the $2n$-dimensional integral (B.11) decouples. Notice that, due to the presence of the fermionic correlator, this decoupling does not occur now at $\rho = 0$. In order to find the new decoupling point, let us study for a while the fermionic correlator. Using the two-point function for the $\psi$ and $\bar{\psi}$ fields (see eq. (2.1)) and taking the anticommutative character of these fields into account, one has:

$$<\psi(u_1)\cdots\psi(u_n)\bar{\psi}(v_1)\cdots\bar{\psi}(v_n)> = (-1)^{\frac{n(n-1)}{2}} \det[\frac{1}{u_i - v_j}]. \quad (B.15)$$

By means of the so-called Cauchy determinant formula,

$$\det[\frac{1}{u_i - v_j}] = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{i<j} (u_i - u_j)(v_i - v_j)}{\prod_{i,j} (u_i - v_j)}, \quad (B.16)$$

the correlator appearing in (B.11) can be written as:

$$<\psi(t_1)\cdots\psi(t_n)\bar{\psi}(t_{n+1})\cdots\bar{\psi}(t_{2n})> = \frac{\prod_{i<j}^n (t_i - t_j)^2(t_{n+i} - t_{n+j})^2}{\prod_{i<j}^{2n} (t_i - t_j)}. \quad (B.17)$$

Inserting the value given in eq. (B.17) for the fermionic vacuum expectation value into the definition of $J_{2n}^m(a,b,\rho)$, it is easy to realize that for the particular value $\rho = \frac{1}{2}$ the $2n$-dimensional integrals (B.11) reduce to the product of two $n$-dimensional integrals of the type (B.2). Actually, one has:

$$J_{2n}^m(a,b,\frac{1}{2}) = I_n^0(a,b,1) I_n^m(a,b,1). \quad (B.18)$$

It can be easily proved that our result (B.12) satisfies (B.18). In fact what happens is that for $\rho = \frac{1}{2}$ the factors in (B.12) with even (odd) product index $i$ give rise to the function $I_n^0(a,b,1)$ $(I_n^m(a,b,1))$ respectively. On the other hand the integrals $J_{2n}^m(a,b,\rho)$ satisfy a recursion relation similar to the one satisfied by the functions $I_n^m(a,b,\rho)$. Proceeding as in the derivation of eq. (B.5), we get:

$$J_{2n}^m(a,b,\rho) = \sum_{p=0}^{m} (-1)^{m-p} \binom{p + n - m}{p} J_{2n}^{m-p}(b,a,\rho). \quad (B.19)$$

It is not difficult to prove that our ansatz (B.12) satisfies the relation (B.19). As happened with eq. (B.3), this fact is a highly non-trivial check of eq. (B.12).
Closely related to the functions $J_m^{2n}(a, b, \rho)$ are the integrals:

$$\tilde{J}_m^{2n}(a, b, \rho) \equiv \oint \{ [(t_1 \cdots t_m)^{a+1}(t_{m+1} \cdots t_n)^a + \text{permutations}] (t_{n+1} \cdots t_{2n})^{a+1} \times$$

$$\times <\psi(t_1) \cdots \psi(t_n) \bar{\psi}(t_{n+1}) \cdots \bar{\psi}(t_{2n}) > + \text{permutations} \} \times$$

$$\times \frac{2n}{\prod_{i=1}^{2n}(1-t_i)^b \prod_{i<j}(t_i - t_j)^{2\rho}},$$

where again $0 \leq m \leq n$. In $\tilde{J}_m^{2n}(a, b, \rho)$, $n - m$ variables chosen among the $n$ arguments of the fields $\psi$ have an exponent which is one unit lower than the exponents of the remaining variables. One can also write down a closed expression for these multiple integrals:

$$\tilde{J}_m^{2n}(a, b, \rho) = \binom{n}{m} \mu_{2n}(\rho) \prod_{i=0}^{2n-2m-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \frac{i+1}{2}) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \frac{i+1}{2})}{\Gamma(1 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + \frac{i+1}{2})} \times$$

$$\times \prod_{i=2n-2m}^{2n-1} \frac{\Gamma(2 + a + i(\rho - \frac{1}{2}) + \frac{i}{2}) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \frac{i}{2})}{\Gamma(2 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + \frac{i}{2})}.$$  

Let us present the arguments in support of the result (B.21). First of all, for the extreme values of $m$ (i.e. for $m = 0$ and $m = n$), $\tilde{J}_m^{2n}(a, b, \rho)$ reduce to the previously studied functions $J_m^{2n}$. In fact, by inspecting the definitions of these two types of integrals (eqs. (B.11) and (B.20)), one easily concludes that:

$$\tilde{J}_0^{2n}(a, b, \rho) = J_0^n(a, b, \rho)$$

$$\tilde{J}_n^{2n}(a, b, \rho) = J_0^n(a + 1, b, \rho).$$

Secondly, for $\rho = \frac{1}{2}$, the integrals (B.20) can be put in terms of the functions $I_m^n$ at $\rho = 1$:

$$\tilde{J}_m^{2n}(a, b, \frac{1}{2}) = I_m^n(a, b, 1) I_m^n(a, b, 1).$$

Moreover, for $m = n - 1$ the value of the right-hand side of (B.20) can be given in
terms of the known function $\tilde{J}_{2n}^n$:

$$
\tilde{J}_{2n}^{n-1}(a, b, \rho) = n \frac{3 + a + b + (2n - 1)\rho}{1 + a} \tilde{J}_{2n}^n(b - 1, a + 1, \rho).
$$

(B.24)

The result (B.24) can be obtained by employing the same method used to derive (B.10). In can be easily verified that the expression (B.21) satisfies eqs. (B.22)-(B.24). It is important to point out that, although the integrals $\tilde{J}_{2n}^m$ do not appear directly in our calculation of the osp(1|2) structure constants, they are needed in some intermediate steps (see below).

We shall also need integrals where the powers of some of the factors $1 - t_i$ are lowered in one unit. For illustrative purposes, let us first consider the case in which there is no fermionic correlator in the integrand. We define the integrals $\mathcal{I}_{2n}^m(a, b, \rho)$ with $0 \leq m \leq n$ by means of the expression:

$$
\mathcal{I}_{2n}^m(a, b, \rho) = \oint \left\{ \prod_{i=1}^{n} t_i^a t_{n+i}^{a+1} (1 - t_{n+i})^b \prod_{i=m+1}^{n} (1 - t_{n+i})^{b-1} + \text{permutations} \right\} \times
$$

$$
\times \prod_{i=1}^{n} (1 - t_i)^b + \text{permutations} \prod_{i<j}^{2n} (t_i - t_j)^{2\rho}.
$$

(B.25)

For low values of $n$, the $\mathcal{I}_{2n}^m$’s can be given in terms of our previous results. From these particular cases one can easily guess the general form of these functions. One expects now to have products of $\Gamma$-functions similar to the ones in (B.3), where now, in addition, $b$ is shifted in some of the arguments of the $\Gamma$’s. In fact, the general expression of $\mathcal{I}_{2n}^m(a, b, \rho)$ is given by:

$$
\mathcal{I}_{2n}^m(a, b, \rho) = \binom{2n}{n} \binom{n}{m} \lambda_{2n}(\rho) \prod_{i=0}^{n-m-1} \frac{\Gamma(1 + a + i\rho) \Gamma(b + i\rho)}{\Gamma(2 + a + b + (2n - 1 + i)\rho)} \times
$$

$$
\times \prod_{i=n-m}^{n-1} \frac{\Gamma(1 + a + i\rho) \Gamma(1 + b + i\rho)}{\Gamma(2 + a + b + (2n - 1 + i)\rho)} \prod_{i=n}^{2n-m-1} \frac{\Gamma(2 + a + i\rho) \Gamma(1 + b + i\rho)}{\Gamma(2 + a + b + (2n - 1 + i)\rho)} \times
$$

$$
\times \prod_{i=2n-m}^{2n-1} \frac{\Gamma(2 + a + i\rho) \Gamma(1 + b + i\rho)}{\Gamma(3 + a + b + (2n - 1 + i)\rho)}.
$$

(B.26)
It is an easy exercise to check that eq. (B.26) gives the correct result in the decoupling point $\rho = 0$. Moreover, one can check eq. (B.26) for the extreme values of $m$. Indeed, for $m = n$ one must have (compare the definitions (B.2) and (B.25)):

$$I_n^{2n}(a, b, \rho) = I_n^{2n}(a, b, \rho). \quad \text{(B.27)}$$

It is straightforward to verify that our solution (B.26) satisfies eq. (B.27) when $I_n^{2n}(a, b, \rho)$ is given by (B.3). When $m = 0$, it is also possible to derive the form of the integrals (B.25) for arbitrary $n$. Suppose that in the definition of $I_0^{2n}(a, b, \rho)$,

$$I_0^{2n}(a, b, \rho) = \int \left\{ \prod_{i=1}^{n} t_i^a t_{n+i}^{a+1} (1 - t_i)^b (1 - t_{n+i})^{b-1} + \text{permutations} \right\} \times \prod_{i<j}^{2n} (t_i - t_j)^{2\rho}, \quad \text{(B.28)}$$

we substitute the identity $(1 - t_i)^b = (1 - t_i)^{b-1} (1 - t_i)$ in all the $1 - t_i$ factors raised to the power $b$. The resulting integrals are of the form (B.2) and, actually, one has:

$$I_0^{2n}(a, b, \rho) = \sum_{l=0}^{n} (-1)^l \binom{n+l}{n} I_0^{n+l}(a, b - 1, \rho), \quad \text{(B.29)}$$

which gives $I_0^{2n}(a, b, \rho)$ in terms of known quantities. Let us prove that the value of $I_0^{2n}(a, b, \rho)$ obtained from the right-hand side of (B.29) is equal to the one displayed in eq. (B.26). The explicit value of $I_0^{n+l}(a, b - 1, \rho)$ is (see eq. (B.3)):

$$I_0^{n+l}(a, b - 1, \rho) = \binom{2n}{n+l} \lambda_{2n}(\rho) \prod_{i=0}^{n-l-1} \frac{\Gamma(1 + a + i\rho) \Gamma(b + i\rho)}{\Gamma(1 + a + b + (2n - 1 + i)\rho)} \times \prod_{i=n-l}^{2n-1} \frac{\Gamma(2 + a + i\rho) \Gamma(b + i\rho)}{\Gamma(2 + a + b + (2n - 1 + i)\rho)}. \quad \text{(B.30)}$$

Using the property $\Gamma(1 + x) = x\Gamma(x)$ in eq. (B.30), one can extract the $l$-dependent
part of $I_{2n}^{n+l}(a, b−1, ρ)$ as follows:

$$I_{2n}^{n+l}(a, b−1, ρ) = \binom{2n}{n+l} C_n^l(a, b, ρ) \Omega_{2n}(a, b, ρ). \quad (B.31)$$

In eq. (B.31) $C_n^l(a, b, ρ)$ is given by:

$$C_n^l(a, b, ρ) = \prod_{i=0}^{n-l-1} \left[1 + a + b + (2n − 1 + i)ρ\right] \prod_{i=n-l}^{n-1} (1 + a + iρ), \quad (B.32)$$

while $\Omega_{2n}(a, b, ρ)$ denotes the quantity:

$$\Omega_{2n}(a, b, ρ) = \lambda_{2n}(ρ) \prod_{i=0}^{n-1} \frac{\Gamma(1 + a + iρ) \Gamma(b + iρ)}{\Gamma(2 + a + b + (2n − 1 + i)ρ)} \times \prod_{i=n}^{2n-1} \frac{\Gamma(2 + a + iρ) \Gamma(b + iρ)}{\Gamma(2 + a + b + (2n − 1 + i)ρ)}. \quad (B.33)$$

Amazingly, the sum in (B.29) can be done explicitly by means of the binomial identity (4.15). One has:

$$\sum_{l=0}^{n} (-1)^l \binom{n+l}{n} \binom{2n}{n+l} C_n^l(a, b, ρ) = \binom{2n}{n} \prod_{i=n}^{2n-1} (b + iρ). \quad (B.34)$$

Therefore the expression of $\mathcal{I}^0_{2n}(a, b, ρ)$ that we get is:

$$\mathcal{I}^0_{2n}(a, b, ρ) = \binom{2n}{n} \left( \prod_{i=n}^{2n-1} (b + iρ) \right) \Omega_{2n}(a, b, ρ). \quad (B.35)$$

Using again $\Gamma(1 + x) = x\Gamma(x)$, one easily proves that the right-hand side of eq. (B.35) equals the value given by eq. (B.26) for $m = 0$. 

58
The same methodology that we have applied to obtain eq. (B.26) can be used to get the values of the integrals:

\[
\mathcal{J}_m^{2n}(a,b,\rho) \equiv \oint \left\{ \prod_{i=1}^{n} t_i^{a+1} (1-t_i)^b \prod_{i=m+1}^{n} (1-t_{n+i})^{b-1} \text{ + permutations} \right\} \times \langle \psi(t_1) \cdots \psi(t_n) \bar{\psi}(t_{n+1}) \cdots \bar{\psi}(t_{2n}) \rangle + \text{permutations} \times \prod_{i<j} (t_i - t_j)^{2\rho},
\]

where \(0 \leq m \leq n\). Notice that the only difference between the definitions (B.25) and (B.36) is the presence in the integrand of the latter of the fermionic correlator. As we are now going to argue, the \(\mathcal{J}_m^{2n}\)'s are given by:

\[
\mathcal{J}_m^{2n}(a,b,\rho) = \left( \begin{array}{c} n \\ m \end{array} \right) \mu_{2n}(\rho) \prod_{i=0}^{2n-2m-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}]) \Gamma(1 + b + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}])} {\Gamma(1 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + [\frac{i}{2}])} \times \prod_{i=2n-2m}^{2n-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + [\frac{i+1}{2}]) \Gamma(1 + b + i(\rho - \frac{1}{2}) + [\frac{i}{2}])} {\Gamma(1 + a + b + n + (\rho - \frac{1}{2})(2n - 1 + i) + [\frac{i+1}{2}])}.
\]

Indeed, eq. (B.37), when \(\rho = \frac{1}{2}\), can be written as:

\[
\mathcal{J}_m^{2n}(a,b,\frac{1}{2}) = I_n^0(a,b,1) I_n^m(b-1,a+1,1),
\]

which is the result expected from the definition (B.36). Moreover, for \(m = n\) one should have:

\[
\mathcal{J}_n^{2n}(a,b,\rho) = \tilde{J}_n^0(a,b,\rho) = J_n^{2n}(a,b,\rho),
\]

and a simple inspection of eqs. (B.37) and (B.21) shows that this is indeed the case. Finally, for \(m = 0\) the analogue of eq. (B.29) is:

\[
\mathcal{J}_0^{2n}(a,b,\rho) = \sum_{l=0}^{n} (-1)^l \tilde{J}_l^{2n}(a,b-1,\rho).
\]

The sum of eq. (B.40) can also be performed and the result, similar to (B.35),
matches perfectly with our ansatz (B.37).

Let us now try to generalize the definitions (B.11) and (B.36) to include the case in which the integrals are performed in an odd-dimensional region. The obvious difficulty to overcome in this case is the fact that the correlator of an odd number of fermionic fields vanishes. To solve this problem, we include in the correlator an extra field $\bar{\psi}$ with its argument placed at infinity. In fact, the definition of $J_{2n+1}^m(a,b,\rho)$ that we shall adopt is:

$$J_{2n+1}^m(a,b,\rho) \equiv - \oint \{ (t_1 \cdots t_{n+1})^a [(t_{n+2} \cdots t_{n+m+1})^{a+1} (t_{n+m+2} \cdots t_{2n+1})^a + \text{permutations}] \times \lim_{R \to \infty} R < \psi(t_1) \cdots \psi(t_{n+1}) \bar{\psi}(t_{n+2}) \cdots \bar{\psi}(t_{2n+1}) \bar{\psi}(R) > + \text{permutations} \} \times \prod_{i=1}^{2n+1} (1-t_i)^b \prod_{i<j}^{2n+1} (t_i - t_j)^{2\rho},$$  \hspace{1cm} \text{(B.41)}

where we have multiplied the correlator by $R$ in order to get a non-vanishing result in the limit $R \to \infty$. The minus sign in (B.41) appears because, for all the permutations, $\bar{\psi}(R)$ is inserted to the right of all the other fields in the correlator (i.e. the field $\bar{\psi}(R)$ behaves as an spectator). The final justification to define $J_{2n+1}^m(a,b,\rho)$ as in eq. (B.41) is the fact that, in our calculation of the osp(1|2) structure constants, these integrals appear precisely in this form. Moreover, there is a nice generalization of the eq. (B.12) that gives the value of $J_N^m(a,b,\rho)$ for any integer $N$ and $m \leq \lfloor N/2 \rfloor$. This expression is:

$$J_N^m(a,b,\rho) = \left( \frac{\lfloor N/2 \rfloor}{m} \right) \mu_N(\rho) \prod_{i=0}^{2\lfloor N/2 \rfloor - 2m - 1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \lfloor \frac{i}{2} \rfloor) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \lfloor \frac{i}{2} \rfloor)}{\Gamma(1 + a + b + (\rho - \frac{1}{2})(N - 1 + i) + \lfloor \frac{i + N}{2} \rfloor)} \times \prod_{i=2\lfloor N/2 \rfloor - 2m}^{N-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \lfloor \frac{i + 1}{2} \rfloor) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \lfloor \frac{i}{2} \rfloor)}{\Gamma(1 + a + b + (\rho - \frac{1}{2})(N - 1 + i) + \lfloor \frac{i + N + 1}{2} \rfloor)}.$$  \hspace{1cm} \text{(B.42)}

Notice that the different behaviour for even and odd $N$ is reproduced by putting $N$
inside the integer part symbol in eq. (B.42). Similarly, we can define $J^m_N(a, b, \rho)$ for odd $N$ as follows:

$$J^m_{2n+1}(a, b, \rho) = - \oint \left\{ \prod_{i=1}^{n+1} a^{n_i} (1-t_i)^b \prod_{i=1}^{n} a^{n_{i+1}} \times ight.$$ 

$$\times \left[ \prod_{i=1}^{m} (1-t_{n+i+1})^b \prod_{i=m+1}^{n} (1-t_{n+i+1})^{b-1} + \text{permutations} \right] \times$$ 

$$\times \lim_{R \to \infty} R < \psi(t_1) \cdots \psi(t_{n+1}) \tilde{\psi}(t_{n+2}) \cdots \tilde{\psi}(t_{2n+1}) \tilde{\psi}(R) > + \text{permutations} \right\} \times$$ 

$$\times \prod_{i<j}^ {2n+1} (t_i - t_j)^{2\rho},$$

(B.43)

and the same rule used to pass from eq. (B.12) to (B.42) serves to generalize (B.37). If $m \leq \lfloor \frac{N}{2} \rfloor$ one has:

$$J^m_N(a, b, \rho) =$$

$$= \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^m \mu_N(\rho) \prod_{i=0}^{2\left\lfloor \frac{N}{2} \right\rfloor - 2m - 1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \left\lfloor \frac{i+1}{2} \right\rfloor) \Gamma(b + i(\rho - \frac{1}{2}) + \left\lfloor \frac{i+1}{2} \right\rfloor)}{\Gamma(1 + a + b + (\rho - \frac{1}{2})(N - 1 + i) + \left\lfloor \frac{i+1}{2} \right\rfloor)} \times$$ 

$$\times \prod_{i=2\left\lfloor \frac{N}{2} \right\rfloor - 2m}^{N-1} \frac{\Gamma(1 + a + i(\rho - \frac{1}{2}) + \left\lfloor \frac{i+1}{2} \right\rfloor) \Gamma(1 + b + i(\rho - \frac{1}{2}) + \left\lfloor \frac{i}{2} \right\rfloor)}{\Gamma(1 + a + b + (\rho - \frac{1}{2})(N - 1 + i) + \left\lfloor \frac{i+N+1}{2} \right\rfloor)}.$$

(B.44)

Let us finally point out that we have numerically checked, for low dimensions, the values of the integrals given in this Appendix.
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