Counting statistics of charge pumping in an open system

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Electron counting statistics of a current pump in an open system has universal form in the weak pumping current regime. In the time domain, charge transmission is described by two uncorrelated Poisson processes, corresponding to electron transmission in the right and left direction. Overall noise is super-poisonian, and can be reduced to poissonian by tuning the amplitude and phase of driving signal so that current to noise ratio is maximized. Measuring noise in this regime provides a new method for determining charge quantum in an open system without any fitting parameters.

Electric current through an open electron system, such as a quantum dot well coupled to the leads, can be induced by modulating its area, shape, or other parameters \( A(t) \). The possibility to generate a DC current through a quantum dot by cycling potentials on the gates was proposed by Spivak et al. \( [2] \) and realized experimentally by Switkes et al. \( [3] \). Theory of pumping in open systems was developed by Brouwer \( [4] \) and by Zhou et al. \( [5] \). Brouwer made an interesting observation that time averaged pumped current is a purely geometric property of the path in the scattering matrix parameter space, insensitive to path parameterization (also, see Refs. \( [12,13] \)). Zhou et al. demonstrated \( [5] \) that pumping provides a new approach to a detailed understanding of mesoscopic transport. Recently, a number of issues related to incomplete agreement between theory and experiment were addressed \( [1,11] \). An interesting extension of these ideas to mesoscopic superconducting systems was proposed \( [14] \).

In this article we discuss current fluctuations in parametrically driven open systems. In the regime of interest, called “adiabatic pumping,” system parameters change slowly compared to transport time through the system. This problem is different from adiabatic transport proposed by Thouless \( [15] \), involving an open system with a gap in the excitation spectrum. Thouless pump is adiabatic in the quantum-mechanical evolution sense, provided that the driving frequency \( f \) is smaller than the energy gap in the system. Current in the Thouless pump is quantized in the units of \( e f \), which has been demonstrated in quantum dots \( [6-11] \) and also motivated proposals to detect fractional Quantum Hall charge \( [16,17] \). We demonstrate below that, although in an open system pumped current is not quantized, charge quantum can still be detected from noise measurement.

Coherent transport through an open mesoscopic system is described \( [16] \) by a unitary scattering matrix \( S \) which depends on externally driven parameters, and thus varies with time. The matrix \( S(t) \), as a function of time, defines a path in the space of all scattering matrices. For a system with \( n \) scattering channels, the matrix space is the group \( U(n) = SU(n) \times U(1) \).

In the pumping experiment one can, in principle, realize any path in the space of scattering matrices. In this article we consider the regime of a weak pumping field, when the path \( S(t) \) is a small loop. We show that in this case the distribution of charge \( q \) transmitted per cycle is fully characterized by only two parameters, average charge flow per cycle, \( I = f \langle q \rangle \), and noise, \( J = f \langle q^2 \rangle \). In the time domain charge transport is described by two uncorrelated Poisson processes for independent single electron transmission to the right and to the left. The generating function for charge distribution over \( N \) pumping cycles in this case is

\[
\chi(\lambda) = e^{iuN(e^{i\lambda}-1)}e^{ivN(e^{-i\lambda}-1)},
\]

FIG. 1. Current to noise ratio, \( I/J = e^{-1}(u-v)/(u+v) \), as a function of the driving signal parameters \( [2] \). Maximum and minimum, as a function of \( w \), are \( I/J = \pm e^{-1} \), where \( e \) is elementary charge.

Here the rates \( u \) and \( v \) of right and left transmission per cycle are given by \( u - v = I/e f \), \( u + v = J/e^2 f \), where \( e \) is elementary charge. Counting probabilities \( p_n \) can be found from Fourier decomposition, \( \chi(\lambda) = \sum_n e^{in\lambda}p_n \).

Poisson statistics, identical to that of conventional classical shot noise, makes it possible to use pumped current noise as a new method of measuring charge quantum. However, since right and left current fluctuations \( [10] \) are independent, one needs a way to separate the two Poisson
processes. This can be achieved, as discussed below, by varying amplitudes and relative phases of external driving signals. Either the right or the left transmission rate, \( u \) or \( v \), can be nulled. The parameters for which this happens give extremum (maximum or minimum) to the current-to-noise ratio \( I/J \) — see Fig. 4. Once \( \beta \) is reduced to a single Poisson process, this ratio gives charge quantum without any fitting parameters, \( J/I = e \).

So far, only noise in nearly open systems has been used to detect quasiparticle charge. In particular, shot noise measurements in Quantum Hall point contacts also use backscattering current of a conductance plateau. Theoretical discussion of ways to detect fractional charge in Quantum Hall systems and in Luttinger liquid also focuses on weak backscattering regime. Based on the present analysis, we conjecture that the requirement of ballistic transport is not necessary if current is induced by weak pumping, rather than by a DC voltage. The method discussed below allows to determine charge quantitatively as follows. The change of scattering matrix, \( \psi_t \), generally changes it. One can explain the result (2) qualitatively as follows. The change of scattering matrix, \( S(t) \rightarrow S'(t) = S(t)S_0 \), is equivalent to replacing states in the incoming scattering channels by their superpositions \( \psi'^{\sigma} = S_{0 \sigma}^{\sigma} \psi^{\sigma} \). At zero temperature, however, Fermi reservoirs are noiseless and also such are any their superpositions. Correlation between superposition states of noiseless reservoirs is negligible, because all current fluctuations arise only during time-dependent scattering. Therefore, noise statistics remain unchanged. A simple formal proof of the result (2) is given below.

I. GENERAL APPROACH

The distribution of transmitted charge is characterized by electron counting probabilities \( p_n \), usually accumulated in one generating function \( \chi(\lambda) = \sum e^{i n \lambda} p_n \). The function \( \chi(\lambda) \) is given by Keldysh partition function, describing evolution in the presence of a counting field \( \lambda \) which is an auxiliary gauge field having opposite signs on the forward and backward parts of Keldysh time contour. By a gauge transformation, the time-dependent scattering matrix becomes

\[
S_\lambda(t) = e^{i \frac{\lambda}{2} \sigma_3} S(t) e^{-i \frac{\lambda}{2} \sigma_3}
\]

for the forward time direction, and

\[
S_\lambda(t) = e^{-i \frac{\lambda}{2} \sigma_3} S(t) e^{i \frac{\lambda}{2} \sigma_3}
\]

for the backward direction, where \( \sigma_3 \) is a diagonal matrix with eigenvalues 1 and -1 for the right and left channels. Then

\[
\chi(\lambda) = \det \left( \mathbb{1} + n(t, t') \left( \hat{T}_\lambda(t) - \mathbb{1} \right) \right)
\]

(3)

\[
\hat{T}_\lambda(t) = S^\dagger_\lambda(t) S_\lambda(t), \quad n(t, t') = \frac{i}{2\pi(t - t' + i\delta)}
\]

(4)

where \( \hat{n} \) is the density matrix of reservoirs at zero temperature. The determinant of an infinite matrix requires formal proof of the result (2) is given below. For periodic \( S(t) \), one can use frequency representation in which \( \hat{n} \) is a diagonal operator, \( n(\omega) = \theta(-\omega) \). The operator \( S(t) \) has off-diagonal matrix elements \( S_{\omega, \omega} \) with frequency change being a multiple of external (pumping) frequency, \( \omega' - \omega = n\Omega = 2\pi n f \). In this method the energy axis is divided into intervals \( n\Omega < \omega < (n+1)\Omega \), and each interval is treated as a separate conduction channel. In doing so it is convenient (and some times necessary) to assign separate counting field \( \lambda \) to each frequency channel, so that the field \( \lambda \) may acquire frequency channel index in addition to the usual conduction channel dependence given by \( \sigma_3 \) in (3). This procedure brings (3) to the form of a determinant of a matrix with an infinite number of rows and columns. This matrix is then truncated at very high and low frequencies, eliminating empty states and the states deep in the Fermi sea which do not contribute to noise.

This method was used in Ref. 26 to study noise in a two channel problem described by a \( 2 \times 2 \) matrix

\[
S(\tau) = \begin{pmatrix}
  r & r' \\
  t & t'
\end{pmatrix}
= \begin{pmatrix}
  B + be^{-i\Omega \tau} & \bar{A} + \bar{a}e^{i\Omega \tau} \\
  A + ae^{-i\Omega \tau} & -B - be^{i\Omega \tau}
\end{pmatrix}
\]

(5)

which is unitary for \( |A|^2 + |B|^2 = 1 \), \( A\bar{a} + B\bar{b} = 0 \). Charge distribution for \( N \) pumping cycles is

\[
\chi(\lambda) = \left( 1 + p_1(e^{i\lambda} - 1) + p_2(e^{-i\lambda} - 1) \right)^N
\]

(6)

with \( p_1 = |a|^4/(|a|^2 + |b|^2) \) and \( p_2 = |b|^4/(|a|^2 + |b|^2) \).

Alternatively, the determinant in (3) can be analyzed in the time domain. This representation is beneficial when an orthogonal basis of functions can be found that provides a simple enough representation of (3). For dealing with periodically driven sys"
are stationary in time, so that $\chi(\lambda)$ is multiplicative, and
$\ln(\chi(\lambda)) \propto N$ at $N \gg 1$. Conveniently, the periodicity of
(3) makes multiplicative character of $\chi(\lambda)$ an exact property, true even for $N \approx 1$.

One encounters a number of interesting cases which can be handled in the time domain in the problem of noise in voltage-driven systems. An external driving voltage $V(t)$ applied across the system can be incorporated in $S$ as a time-dependent forward scattering phase. This is achieved by a gauge transformation making the scattering matrix time-dependent:

$$S(t) = e^{-\frac{i}{2} \varphi(t) \sigma_3} e^{\frac{i}{2} \varphi(t) \sigma_3}, \quad \varphi(t) = \frac{e}{b} V(t). \quad (8)$$

The formula (8) defines a circular path in $U(m)$ of radius which depends on the system conductance. Full statistics have been studied for a large family of paths of the form (8). The statistics was found to be binomial for the DC voltage case [23] as well as for the AC case [27,28] with a particular time dependence $V(t)$ obtained from the criterion of minimal noise.

An attempt to adapt these results to the problem of pumping noise was made by Andreev and Kamenev [29]. For several matrix paths $S(t) \in SU(2)$ obtained from (4) by exchanging incoming scattering channels, binomial charge distributions arise, not surprisingly, with transmission and reflection probabilities exchanged. Another matrix considered in Ref. [29] is $r = r' = \cos \Omega t$, $t = t' = \sin \Omega t$, is related to (5) by the transformation (2) with $S_0 = (\sigma_3 + \sigma_2)/\sqrt{2}$ and parameter values $a, b = 1/\sqrt{2}, A, B = 0$. The result (Eq.8 of Ref. [29]) in this case agrees with (4), in full accord with the invariance property (2). However, in the context of the pumping problem, the paths $S(t)$ considered in Ref. [29] appear to be less relevant than, say, the paths (4) in voltage-driven systems. In general, the dependence of the scattering matrix $S(t)$ on the parameters externally controlled in the pumping experiment is not known. Because of that, the results for particular paths $S$ are of less interest than the properties that hold for sufficiently general families of paths.

II. CALCULATION

For a weak pumping field we shall evaluate (3) in the time domain by expanding $\ln\det(\cdots)$ in powers of $\delta S$ and keeping non-vanishing terms of lowest order. In doing so, however, we preserve full functional dependence on $\lambda$ which gives all moments of counting statistics. We write $S(t) = e^{A(t)} S(0)$ with antihermitian $A(t)$ representing small perturbation, $\text{tr} A^2 A \ll 1$. Here $S(0)$ is the scattering matrix of the system in the absence of pumping. Substituting this into (3) one obtains

$$\hat{T}_\lambda(t) = \hat{T}_\lambda(0) + \delta T_\lambda(t) = S_{\lambda}(0) e^{-A_{-\lambda}(t)} e^{A_\lambda(t)} S_{\lambda}(0) \quad (9)$$

with $\hat{T}_\lambda(t) = \hat{S}_{\lambda}(0) + \hat{S}_{\lambda}(t)$ and $A_\lambda(t) = e^{i\frac{1}{2} \sigma_3} A(t) e^{-i\frac{1}{2} \sigma_3}$. Now, we expand (3):

$$\ln(\chi(\lambda)) = \ln(\det Q_0 + tr R - \frac{1}{2} tr R^2 + \frac{1}{3} tr R^3 - \cdots) \quad (10)$$

where $Q_0 = 1 + \hat{n}(\hat{T}_\lambda(0))$ and $R = Q_\lambda^{-1} \hat{n} \delta T_\lambda$. At zero temperature, from $\hat{n}_\lambda^2 = \hat{n}$ it follows that $\det Q_0 = 1$ and $R = S_{\lambda}(0)^{-1} \hat{n} (e^{-A_{-\lambda}(t)} e^{A_\lambda(t)} - 1) S_{\lambda}(0)$. Therefore,

$$\ln(\chi(\lambda)) = tr \hat{n} \hat{M} - \frac{1}{2} tr(\hat{n} \hat{M})^2 + \frac{1}{3} tr(\hat{n} \hat{M})^3 - \cdots \quad (11)$$

where $\hat{M} = e^{-A_{-\lambda}(t)} e^{A_\lambda(t)} - 1$. Note that at this stage there is no dependence left on the constant matrix $S(0)$, which proves invariance under the group shift (2).

We need to expand (11) in powers of the pumping field, which amounts to taking the lowest order terms of the expansion in powers of the matrix $A(t)$. One can check that the two $O(A)$ terms arising from the first term on the RHS of (11) vanish. The $O(A^2)$ terms arise from the first and second term in (11) and have the form

$$\ln(\chi(\lambda)) \equiv \frac{1}{2} tr (\hat{n} (A_{-\lambda}^2 + A_{\lambda}^2 - 2 A_{-\lambda} A_\lambda)) - \frac{1}{2} tr(\hat{n} B_\lambda)^2 \quad (12)$$

with $B_\lambda(t) = A_\lambda(t) - A_{-\lambda}(t)$. At zero temperature, by using $\hat{n}^2 = \hat{n}$, one can bring (12) to the form

$$\frac{1}{2} tr (\hat{n} [A_\lambda, A_{-\lambda}]) + \frac{1}{2} tr (\hat{n} B_\lambda)^2 \quad (13)$$

The first term of (13) has to be regularized in the Schwinger anomaly fashion, by splitting points, $t', t'' = t \pm \epsilon/2$, which gives

$$\frac{1}{2} \int n(t', t'') tr (A_{-\lambda}(t'') A_\lambda(t') - A_\lambda(t'') A_{-\lambda}(t')) dt \quad (14)$$

Averaging over small $\epsilon$ can be achieved either by inserting in (14) additional integrals over $t', t''$, or simply by replacing $A_\lambda(t) \rightarrow \frac{i}{\epsilon} (A_\lambda(t) + A_\lambda(t'))$, etc. After taking the limit $\epsilon \rightarrow 0$, Eq. (14) becomes

$$\frac{i}{8\pi} \int tr (A_{-\lambda} \partial_t A_\lambda - A_\lambda \partial_t A_{-\lambda}) dt \quad (15)$$

The second term of (13) can be written as

$$\frac{1}{4(2\pi)^2} \int \int \frac{tr(B_\lambda(t) - B_\lambda(t'))^2}{(t - t')^2} dt dt' \quad (16)$$

Now, we decompose $A = a_0 + z + z^\dagger$, so that $[\sigma_3, a_0] = 0$, $[\sigma_3, z] = -2z$, $[\sigma_3, z^\dagger] = 2z^\dagger$. Then

$$A_\lambda \equiv e^{i\frac{1}{2} \sigma_3} A e^{-i\frac{1}{2} \sigma_3} = a_0 + e^{i\frac{1}{2} z^\dagger} z + e^{-i\frac{1}{2} z} \quad (17)$$

$$B_\lambda = \left( e^{i\frac{1}{2} z^\dagger} - e^{-i\frac{1}{2} z} \right) W, \quad W \equiv z^\dagger - z \quad (18)$$
Substituting this into (15) and (16) one finds that in terms of \( W(t) \) these two expressions become

\[
\frac{\sin \lambda}{8\pi} \int \text{tr} \left( [\sigma_3, W] \partial_t W \right) dt
\]

(19)

and

\[
\frac{(1 - \cos \lambda)}{2(2\pi)^2} \int \int \text{tr} \left( W(t) - W(t') \right)^2 \frac{dt dt'}{(t - t')^2}
\]

(20)

Hence \( \ln \chi \) indeed depends on \( \lambda \) as \( u(e^{i\lambda} - 1) + v(e^{-i\lambda} - 1) \).

Eq. (19) is essentially identical to the result obtained by Brouwer for average pumped current \([4]\). The integral in (19) is invariant under reparameterization, and thus has a purely geometric character determined by the contour \( S(t) \) in \( U(m) \). Eq. (20) represents a generalization of the expression for noise induced by time-dependent external field considered in Refs. \([8, 27]\).

The parameters \( u \) and \( v \) in (1) can be expressed through \( z(t) \) and \( z(t') \) in a simple way. Let us write \( z(t) \) as \( z_+(t) + z_-(t) \), where \( z_+(t) \) and \( z_-(t) \) contain only positive or negative Fourier harmonics, respectively. Then

\[
u = \frac{i}{4\pi} \int \text{tr} \left( z_+^\dagger \partial_t z_+ - z_+ \partial_t z_+^\dagger \right) dt = \sum_{\omega > 0} \omega \text{tr} z_+\dagger \omega z_\omega,
\]

(21)

where \( V_{1,2} \) are pumping signal amplitudes, and complex parameters \( z_{1,2} \) depend on microscopic details. From (21) we find the Poisson rates

\[
u = \frac{1}{4} \left| z_1 V_1 e^{i\theta} + z_2 V_2 \right|^2, \quad \omega = \frac{1}{4} \left| z_1 V_1 e^{-i\theta} + z_2 V_2 \right|^2
\]

(24)

Note that \( u \geq 0 \) and \( v \geq 0 \). It is straightforward to show that (13) equals \( i \sin \lambda(u - v) \), whereas (20) equals \( (\cos \lambda - 1)(u + v) \), which completes the proof of (1).

Now we consider a single channel pump, \( S(t) \in U(2) \). In this case, \( z \) and \( \tilde{z} \) are complex numbers. For harmonic driving signal, without loss of generality, one can write

\[
z(t) = z_1 V_1 \cos(\Omega t + \theta) + z_2 V_2 \cos(\Omega t),
\]

(23)

where \( V_{1,2} \) are pumping signal amplitudes, and complex parameters \( z_{1,2} \) depend on microscopic details. From (21) we find the Poisson rates

\[
u = \frac{1}{4} \left| z_1 V_1 e^{i\theta} + z_2 V_2 \right|^2, \quad \omega = \frac{1}{4} \left| z_1 V_1 e^{-i\theta} + z_2 V_2 \right|^2
\]

(24)

Note that \( u \) and \( v \) vanish at particular signal amplitudes ratio \( V_1/V_2 \) and phase \( \theta \). Once the two Poisson processes (1) are reduced to one, the current to noise ratio gives elementary charge, \( I/J = \pm e^{-1} \). This happens at the extrema of \( I/J \) as a function of \( w = (V_1/V_2) e^{i\theta} \), for (24) reached at \( w = -z_2/z_1, -z_2^*/z_1 \). This behavior is illustrated in Fig. [1].

Reducing the counting statistics (1) to purely Poissonian by varying pumping signal parameters, in principle, is possible for any number of channels \( n \). However, since the number of parameters to be tuned is \( 2n^2 \), this method is practical perhaps in the single channel case only. Although the method is demonstrated for non-interacting fermions, we argue that it can be applied to interacting systems as well. Poisson distribution results from the absence of correlations of subsequently transmitted particles, which must be the case in any system at small pumping current. Using the dependence of Poisson rates \( u, v \) on the driving signal to maximize \( I/J \) allows to eliminate one of the two Poisson processes (1) and then obtain charge quantum in the standard way as \( e = J/I \).

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