Noncompact Gepner Models with Discrete Spectra

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Abstract

We investigate a noncompact Gepner model, which is composed of a number of SL(2,\(\mathbb{R}\))/U(1) Kazama-Suzuki models and \(\mathcal{N} = 2\) minimal models. The SL(2,\(\mathbb{R}\))/U(1) Kazama-Suzuki model contains the discrete series among the SL(2,\(\mathbb{R}\)) unitary representations as well as the continuous series. We claim that the discrete series contain the vanishing cohomology and the vanishing cycles of the associated noncompact Calabi-Yau manifold. We calculate the Elliptic genus and the open string Witten indices. In the A\(_{N-1}\) ALE models, they actually agree with the vanishing cohomology and the intersection form of the vanishing cycles.

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1 Introduction

Recently, the worldsheet CFT of a noncompact singular Calabi-Yau manifold has been largely investigated in the context of the holographic description of the little string theories [1–3]. The modular invariant partition functions have been constructed [4–8], and D-branes on a noncompact singular Calabi-Yau manifold have been studied by using the CFT descriptions [9, 10].

However, the modular invariant partition function constructed in [4–8] includes only “the principal continuous series” of the SL(2,\(\mathbb{R}\)) unitary representations. If we naively treat this partition function [7],[8], the elliptic genus in the closed string theory and the open string Witten indices vanish, and we cannot obtain the vanishing cohomology of the noncompact Calabi-Yau manifold, nor the intersection form of the vanishing cycles of the manifold. Then, where are these compact cohomology and vanishing cycles in the CFT description?\(^1\)

Besides the principal continuous series, we can include the highest and lowest weight discrete series [11]. We claim that we should include this discrete series to obtain the right Calabi-Yau

\(^1\)In [10], reasonable nonzero open string Witten indices are obtained by treating the Liouville potential appropriately and using the Ramond ground states of discrete series.
sigma model; the vanishing cohomology and boundary states of the vanishing cycles belong to this sector.

The wave functions of most of the states in the discrete series (we use only these states) are $L^2$ normalizable in the SL(2, $\mathbb{R}$) group manifold. Therefore, they are localized near the deformed singularity in the noncompact Calabi-Yau manifold, and they are “bound states” in this theory. It is also reasonable that the compact cohomology belongs to this class of states.

It is difficult, however, to construct a modular invariant partition function of the SL(2, $\mathbb{R}$) discrete series [12]. We use, in this paper, the one proposed in [13–15]. It includes the new sectors obtained by “spectral flow” transformation — a series of automorphisms of the affine SL(2, $\mathbb{R}$).

We construct modular invariant partition functions of rather general “noncompact Gepner models”, which consist of a number of SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki models with integral levels and $\mathcal{N} = 2$ minimal models. We also calculate the elliptic genera in these models in order to study the topological properties and compare with the geometric ones. We also construct the boundary states and calculate the open string Witten indices between them in order to compare with the geometrical cycles and the intersection form. We obtain nonzero closed and open string Witten indices independent of the moduli parameter of the torus or annulus.

As an example, we consider the $A_{N-1}$ ALE model; this model consists of an level $N$ SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki model and an level $(N - 2) \mathcal{N} = 2$ minimal model [16]. As a result, we obtain the closed string Witten index as $(N - 1)$. This corresponds to the $(N - 1)$ dimensional $(2, 2)$ compact cohomology of the $A_{N-1}$ ALE space. We also obtain the open string Witten indices. They agree with the intersection form of vanishing cycles in the $A_{N-1}$ ALE space. This intersection form also coincides with the one proposed in [17].

2 SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki model

In this section, we construct the characters of the SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki model by summing up the new sectors [13–15]. This character is used to construct “noncompact Gepner models” in the next section.

In this paper, we use the convention in appendix A of [1].

2.1 Currents of SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki model

Let us first introduce the supersymmetric level $k$ SL(2, $\mathbb{R}$) WZW model; this consists of a set of level $\tilde{k} = k + 2$ bosonic SL(2, $\mathbb{R}$) currents $\hat{J}^{\pm,3}$ and three free fermions $\psi^{\pm,3}$. These currents satisfy the following OPE relations

\[
\psi^3(z)\psi^3(w) \sim \frac{-1}{z-w}, \quad \psi^+(z)\psi^-(w) \sim \frac{2}{z-w},
\]

\[
\hat{\tilde{J}}^3(z)\hat{\tilde{J}}^3(w) \sim \frac{-\tilde{k}/2}{(z-w)^2}, \quad \hat{\tilde{J}}^3(z)\hat{\tilde{J}}^{\pm}(w) \sim \frac{\pm\hat{J}^{\pm}(w)}{z-w}, \quad \hat{\tilde{J}}^+(z)\hat{\tilde{J}}^-(w) \sim \frac{\hat{k}}{(z-w)^2} + \frac{-2\hat{J}^3(w)}{z-w}.
\]

We want to construct the SL(2, $\mathbb{R}$)/U(1) Kazama-Suzuki model from these currents. The supersymmetric affine U(1) in the denominator is generated by the time-like boson $J^3 = \hat{J}^3 +$
These new generators represent the zero modes are infinite dimensional.

If we denote the Verma module of \( \hat{\mathcal{H}}_\ell, \mathcal{H}_s, \) and \( \mathcal{H}_m \) respectively, the Verma module of the \( \text{SL}(2, \mathbb{R})/U(1) \) Kazama-Suzuki model \( \hat{\mathcal{H}}_{\ell,s} \) is defined by the decomposition

\[
\hat{\mathcal{H}}_\ell \otimes \mathcal{H}_s = \sum_m \hat{\mathcal{H}}_{\ell,m} \otimes \mathcal{H}_m.
\]

If we know the structures of \( \hat{\mathcal{H}}_\ell, \mathcal{H}_s, \) and \( \mathcal{H}_m \), then we can determine the structure of \( \hat{\mathcal{H}}_{\ell,m} \).

The Verma modules of free fermions and \( U(1)_{-k} \) are not difficult. We analyze them in subsection 2.3. In the next subsection, we concentrate on the \( \text{SL}(2, \mathbb{R}) \) Kazama-Suzuki model especially, the character of the discrete series.

### 2.2 \( \text{SL}(2, \mathbb{R}) \) Discrete Series

Let us consider the lowest weight discrete representations of \( \text{SL}(2, \mathbb{R})_k \). Each of the Verma modules of \( L^2 \) normalizable lowest weight representations of \( \text{SL}(2, \mathbb{R})_k \) is labeled by a number \( \ell \in \mathbb{Z}, \ 1 < \ell < \hat{k} - 1 \). The character of this Verma module is known to be (for example, see [13])

\[
\chi_\ell^+(\tau, z) := \text{Tr} \left[ q^{L_0 - \frac{k}{8\pi} y^3} \right] = \frac{q^{-\frac{1}{4\pi}(\ell-1)^2 y^4} y^{\frac{1}{2}(\ell-1)}}{-i\theta_1(\tau, z)},
\]

where \( q = \exp(2\pi i \tau), \ y = \exp(2\pi i z) \). This character is divergent if we set \( z = 0 \) because the representations of the zero modes are infinite dimensional.

The modular transformation law of this character is not good. To make this good, let us introduce a series of automorphisms of the algebra, called “spectral flow”.

\[
\hat{j}_n^3(v) = \hat{j}_n^3 - \frac{k}{2} \delta_{n,0}, \quad \hat{j}_n^\pm(v) = \hat{j}_n^{\pm,v}, \quad v \in \mathbb{Z}.
\]

These new generators \( \hat{j}_n^{\pm,3(v)} \) satisfy the same algebra as \( \hat{j}_n^{\pm,3} \).

The character of the new sector obtained by this spectral flow operation can be written as

\[
\chi_\ell^{\pm,v}(\tau, z) := \text{Tr} \left[ q^{L_0(v) - \frac{k}{8\pi} y^3(v)} \right] = \frac{(-1)^v q^{-\frac{1}{4\pi}(\ell-1-kv)^2 y^{\frac{1}{2}(\ell-1-kv)}}{-i\theta_1(\tau, z)}.
\]

2 The notation \( j = \ell/2 \) is often used in other papers. The Casimir eigenvalue of a representation can be written as \( -j(j-1) \) by using this \( j \).
Now, we sum up for \( v \in 2\mathbb{Z} \) in order to make a character with good modular properties

\[
\sum_{v \in 2\mathbb{Z}} \chi_\ell^{+(v)}(\tau, z) = \frac{\Theta_{\ell-1,-k}(\tau, z)}{-i\theta_1(\tau, z)}.
\]

This character includes a negative level theta function. Therefore, this character diverges in the region \( \text{Im}\tau > 0 \). Here, we treat this character as a formal series.

If we sum up for odd integer \( v \), we obtain the same result as the case of the \( \hat{\ell} - \ell \) highest weight representation for even \( v \)

\[
\sum_{v \in 2\mathbb{Z}+1} \chi_\ell^{+(v)}(\tau, z) = \sum_{v \in 2\mathbb{Z}} \chi_\ell^{-}(v)(\tau, z) = \frac{-\Theta_{-k+\ell-1,-k}(\tau, z)}{-i\theta_1(\tau, z)}.
\]

This is because the \( v = 1 \) spectral flow maps \( \ell \) lowest weight representation to \( \hat{\ell} - \ell \) highest weight representation \([15]\). We have only to consider the lowest weight representations for this reason.

Next, we define the following character \( \hat{\chi}_\ell \), and use this character in the rest of this paper

\[
\hat{\chi}_\ell(\tau, z) := \sum_{v \in 2\mathbb{Z}} \chi_\ell^{+(v)}(\tau, z) + \sum_{v \in 2\mathbb{Z}+1} \chi_\ell^{-}(v)(\tau, z) = \frac{\Theta_{\ell-1,-k}(\tau, z) - \Theta_{-(\ell-1),-k}(\tau, z)}{-i\theta_1(\tau, z)}.
\] (2.2)

Note that each coefficient of the power of \( q \) in this character is convergent even if we set \( z = 0 \), in contrast with \( \chi_\ell^{+} \) or \( \sum_{v \in 2\mathbb{Z}} \chi_\ell^{+(v)} \). But the sum is still divergent.

### 2.3 The characters of \( \text{SL}(2, \mathbb{R})/\text{U}(1) \)

The characters of the \( U(1)_{-k} \) generated by \( J^3 = j^3 + \frac{1}{2} \psi^+ \psi^- \) is expressed as

\[
\text{Tr}_{\mathcal{H}_m}[q^{L_0 - \frac{1}{2} y^0}] = \Theta_{m,-k}(\tau, z)/\eta(\tau), \quad m \in \mathbb{Z}_{2k}.
\] (2.3)

This formula includes a theta function of negative level and is divergent because the \( J^3 \) direction is time-like. We treat this character as a formal series as the same way as \( \hat{\chi}_\ell \).

There remaining is the characters of two fermions \( \text{SO}(2)_1 \). This character can be written as

\[
\text{Tr}_{\mathcal{H}_s}[q^{L_0 - \frac{1}{2} y^0}] = \Theta_{s,2}(\tau, z)/\eta(\tau), \quad s \in \mathbb{Z}_4,
\] (2.4)

where \( J_0^{(F)} \) is the zero mode of the fermion number current \( J^{(F)} = \frac{1}{2} \psi^+ \psi^- \).

Collecting the characters \((2.2), (2.3), (2.4)\), and looking at the forms of the currents \((2.1)\), we obtain the characters of \( \text{SL}(2, \mathbb{R})/\text{U}(1) \) KS model \( \hat{\chi}_m^{\ell,s} := \text{Tr}_{\mathcal{H}_m}[q^{L_0 - \frac{1}{2} y^0}] \) through the decomposition

\[
\hat{\chi}_\ell \left( \tau, -\frac{2}{k} z_1 + z_2 \right) \Theta_{s,2} \left( \tau, -\frac{\hat{k}}{k} z_1 + z_2 \right) = \sum_{m_0 \in \mathbb{Z}_k} \hat{\chi}_m^{\ell,s}(\tau, z_1) \Theta_{m,-k}(\tau, z_2).
\] (2.5)

In this formula, \( \hat{\chi}_\ell \) is a divergent series, however, \( \Theta_{m,-k}(\tau, z_2) \) is also divergent. Therefore, the character \( \hat{\chi}_m^{\ell,s}(\tau, z) \) is possibly convergent.
Let us write the explicit form of $\tilde{\chi}_{\ell,s}^{m}(\tau,z)$. For this purpose, we define the “string function” $\tilde{c}_{m'}^{\ell}$ of SL(2, R) as

$$\tilde{\chi}(\tau,z) = \sum_{m' \in \mathbb{Z}} \tilde{c}_{m'}^{\ell}(\tau)\Theta_{m',-k}(\tau,z).$$

Using this string function, we express the character of SL(2, R)/U(1) Kazama-Suzuki model as

$$\tilde{\chi}_{\ell,s}^{m}(\tau,z) = \sum_{r \in \mathbb{Z}} \tilde{c}_{m-s+4r}^{\ell}(\tau)\Theta_{m,-k}(\tau,z).$$

Note that if all the string functions $\tilde{c}_{m'}^{\ell}(\tau)$ are convergent, then $\tilde{\chi}_{\ell,s}^{m}(\tau,z)$’s are convergent.

The explicit form of $\tilde{c}_{m'}^{\ell}(\tau)$ can be obtained by using the results of [19]. If $\ell + m'$ is an odd integer, then $\tilde{c}_{m'}^{\ell} = 0$. If $\ell + m'$ is an even integer, then

$$\tilde{c}_{m'}^{\ell} = \eta(\tau)^{-3} \sum_{r \in \mathbb{Z}} \sum_{u=0}^{\infty} (-1)^{u} \left[ q^{-k\left(\frac{\ell}{4k} + \frac{u}{2}\right)^2 + k\left(\frac{\ell'}{4k} + \frac{u}{2} + r\right)^2} + q^{-k\left(\frac{\ell}{4k} - \frac{u}{2}\right)^2 + k\left(\frac{\ell'}{4k} - \frac{u}{2} + r\right)^2} \right].$$

This function is shown to be actually convergent in [14]. Consequently, we conclude that the character $\tilde{\chi}_{\ell,s}^{m}(\tau,z)$ is convergent.

Let us consider the modular transformation properties of these characters in order to construct modular invariant partition functions. The modular transformation law can be read from (2.5) as follows.

$$\tilde{\chi}_{\ell,s}^{m}(\tau + 1,z) = e^{-\frac{\ell(\ell - 2)}{4k} + \frac{s^2}{8} + \frac{m^2}{4k} - \frac{k}{8k}} \tilde{\chi}_{\ell,s}^{m}(\tau,z),$$

$$\tilde{\chi}_{\ell,s}^{m}(-1/\tau, z/\tau) = e^{\frac{k}{2k} \frac{z^2}{\tau}} \sum_{\ell,m,s} (-1) \sqrt{2k} \pi \frac{\sin \pi (\ell - 1)(\ell' - 1)}{k} \times \frac{1}{\sqrt{8k}} e^{-\frac{ss' - mnn'}{4k}} \tilde{\chi}_{\ell',s'}^{m'}(\tau,z).$$

Note that from (2.5), we can read off the charge of the states in $\mathcal{H}_{\text{mv}}^{\ell,s}$ to be $-s/2 - m/k$ mod 2. This relation is used to perform the GSO projection.

Also, it is convenient to define a “index” $\tilde{I}_{\ell}^{m}(\tau,z) := \tilde{\chi}_{\ell}^{1}(\tau,z) - \tilde{\chi}_{\ell}^{-1}(\tau,z)$ in order to construct the elliptic genera and Witten indices. This $\tilde{I}_{\ell}^{m}$ has the following properties from (2.5).

$$\tilde{I}_{\ell}^{m}(\tau, 0) = -\delta_{m,\ell-1} + \delta_{m,-(\ell-1)}. \quad (2.6)$$

$\tilde{I}_{\ell}^{m}(\tau, 0)$ actually has no $\tau$ dependence because of worldsheet supersymmetry; there are only contributions from Ramond ground states.

3 Noncompact Gepner models

In this section, we construct the “noncompact Gepner models” constructed by a number of integral level SL(2, R)/U(1) Kazama-Suzuki models and $\mathcal{N} = 2$ minimal models (SU(2)/U(1)
Kazama-Suzuki models). The Gepner-like description of ALE [13], and Seiberg-Witten curve [17] are included in this class of models.

In this paper, we construct only the discrete part. However, there is also the continuous part for each SL(2, R)/U(1).

3.1 The closed string theory

Let us first consider the closed string theory and construct the toroidal partition function. The theory we consider here is

\[
\left( \frac{\text{SL}(2, \mathbb{R})_{N_1}}{U(1)} \times \frac{\text{SL}(2, \mathbb{R})_{N_2}}{U(1)} \times \cdots \times \frac{\text{SU}(2)_{N_{r}}}{U(1)} \right) / \mathbb{Z}_K,
\]

where \( K = \text{lcm}(N_j, N_j) \). This orbifold projection is the one onto the integer charged states. This theory can be expressed as an \( \mathcal{N} = 2 \) Landau-Ginzburg orbifold with a superpotential

\[
W = Y_1^{-N_1} + Y_2^{-N_2} + \cdots + Y_r^{-N_r} + X_1^{N_1} + \cdots + X_r^{N_r},
\]

where \( Y_1, \ldots, Y_r, X_1, \ldots, X_r \) are chiral superfields. We construct the partition function of this theory by the beta-method [20] and determine the combination of left mover and right mover.

The central charge of this theory is expressed as

\[
c/3 = \sum_{j=1}^{r} \frac{N_j + 2}{N_j} + \sum_{j=1}^{r} \frac{N_j - 2}{N_j} = r + \hat{r} + \sum_{j=1}^{r} \frac{2}{N_j} - \sum_{j=1}^{r} \frac{2}{N_j}.
\]

Since we expect that this theory is equivalent to a sigma model on a Calabi-Yau manifold, we consider the case that \( c/3 \) is an integer. We also concentrate to the case that \( \hat{r} + r - c/3 \) is an even integer for a technical reason.

A Verma module of this theory is a tensor product of ones of each sub-theory. The character of the Verma module \( f_\alpha(\tau) \) is a product of ones of each sub-theory

\[
f_\alpha(\tau, z) := \prod_{\ell, m, s} \chi^{\ell, m, s}(\tau, z) = \prod_{\ell, m, s} \chi^{\ell, m, s}(\tau, z)
\]

where \( \chi^{\ell, m, s} \) is an \( \mathcal{N} = 2 \) minimal model character [20], and the label \( \alpha \) is defined as

\[
\ell = (\ell_1, \ldots, \ell_r; \ell_1, \ldots, \ell_r), \quad m = (\bar{m}_1, \ldots, \bar{m}_r; m_1, \ldots, m_r), \quad s = (\bar{s}_1, \ldots, \bar{s}_r; s_1, \ldots, s_r),
\]

\[
\alpha = (\ell, m, s),
\]

and each \( (\ell_j, m_j, s_j) \) is a label of a Verma module of \( \text{SL}(2, \mathbb{R})_{N_j}/U(1) \), and each \( (j, m_j, s_j) \) is that of \( \text{SU}(2)_{N_j}/U(1) \). We define, for the sake of convenience, the inner products between two \( \bar{m} \)'s, and between two \( \bar{s} \)'s as follows.

\[
\bar{m} \cdot \bar{m}' := -\sum_j \frac{\bar{m}_j \bar{m}'_j}{2N_j} + \sum_j \frac{m_j m'_j}{2N_j}, \quad \bar{s} \cdot \bar{s}' := -\sum_j \frac{s_j s'_j}{4} - \sum_j \frac{s_j s'_j}{4}.
\]

Also, we introduce a special vector \( \vec{\beta} = (2, \ldots, 2; 2, \ldots, 2) \) which is the same type of \( \bar{m} \). Using this vector, we can write the charge integrality condition as \( \vec{\beta} \cdot \bar{m} \in \mathbb{Z} \).
Next, we need the modular transformation laws of \( f_a \) to construct the modular invariant partition function and determine the combination of left mover and right mover. These modular properties can be written as

\[
f_a(\tau + 1, z) = e^{\left[ -\sum_j \ell_j (\ell_j - 2) \cdot 4N_j + \sum_j \ell_j (\ell_j + 2) \cdot 4N_j - \frac{1}{2} (\vec{s} \cdot \vec{s} + \vec{m} \cdot \vec{m}) + \frac{c}{12} \right]} f_a(\tau, z),
\]

\[
f_a(-1/\tau, z/\tau) = e^{\left[ \frac{cz^2}{6\tau} \right]} \sum_{a'} S_{aa'} f_{a'}(\tau, z),
\]

\[
S_{aa'} := \left( \prod_{j=1}^r (-1) A_{\ell_j-2, \ell_j-2} \right) \left( \prod_{j=1}^r A_{\ell_j, \ell_j} \right) \frac{1}{8N} e^{[\vec{s} \cdot \vec{s}'] + \vec{m} \cdot \vec{m}'], \quad (3.1)
\]

where \( A_{\ell\ell}' \)'s are \( SU(2) \) S matrices.

With these notations, we can write the modular invariant partition function of NS sector as

\[
Z(\tau, \bar{\tau}) = \frac{1}{2^{r+\tau}} \sum_{b=0}^{K-1} \sum_{\vec{\ell}, \vec{m}, \vec{s}, \vec{s}, \vec{m} \in \mathbb{Z}} f_{\vec{\ell}, \vec{m}, \vec{s}, \vec{s}}(\tau) \tilde{f}_{\vec{\ell}, \vec{m} + b\vec{m}, \vec{m}}(\bar{\tau}),
\]

We can check that this partition function is actually invariant under the transformation (3.1).

Now, let us construct the elliptic genus in order to investigate the topological feature of this model. The elliptic genus \( I(\tau, \bar{\tau}, z) := \text{Tr}_{RR} [(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} y^{j_0}] \) becomes a sum of products

\[
I(\tau, \bar{\tau}, z) = \frac{1}{2^{r+\tau}} \sum_{b=0}^{K-1} \sum_{\vec{\ell}, \vec{m}} \tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z) \tilde{f}^{\vec{\ell}}_{\vec{m} + b\vec{m}, \vec{m}}(\bar{\tau}, 0),
\]

\[
\tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z) = \tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z) \ldots \tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z) \tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z) \ldots \tilde{I}^{\vec{\ell}}_{\vec{m}}(\tau, z), \quad (3.2)
\]

where \( I_{\vec{m}} = \chi_{\vec{m}}^{\ell_1} - \chi_{\vec{m}}^{\ell_1-1} \) and \( \tilde{I}^{\vec{\ell}}_{\vec{m}} \) is defined in section 2.3. We can check that this elliptic genus has the right modular properties [21].

The Witten index is obtained as \( I(\tau, \bar{\tau}, 0) \). This is actually independent of \( \tau \) from the relation (2.6).

### 3.2 D-branes

In this subsection, we consider the boundary states, the annulus amplitude and the open string Witten indices in the model defined in the previous subsection. We use almost the same method as [22], and we only show the results here.

There are two types of gluing conditions of the \( \mathcal{N} = 2 \) superconformal algebra: the A-type and the B-type [23]. In both cases, the boundary conditions (boundary states) are labeled by \( \alpha = (\vec{L}, \vec{M}, \vec{S}) \) which is the same label as \( a \).
In the case of the A-type gluing condition, the open string partition function becomes

$$Z^A_{\alpha \bar{\alpha}} := \frac{1}{2^{r+r}} \sum_{a'}^{NS} K^{-1} \left( \prod_{j=1}^r N_j \delta_{-M_j + \bar{M}_j + m'_j + 2b} \right) f_{a'}(\tau),$$

where, $N_j$'s are SU(2) fusion coefficients.

The A-type open string Witten index becomes

$$I^A_{\alpha \bar{\alpha}} := \frac{1}{2^{r+r}} \left( \sum_{a'}^{NS} \prod_{j=1}^r N_j \delta_{-M_j + \bar{M}_j + m'_j + 2b} \right) f_{a'}(\tau),$$

(3.3)

where $S = \sum j=1^r S_j + \sum j=1^r \bar{S}_j$, $\bar{S} = \sum_{j=1}^r \bar{S}_j + \sum_{j=1}^r \bar{S}_j$.

On the other hand, in the B-type case, open string partition function becomes

$$Z^B_{\alpha \bar{\alpha}} := \frac{1}{2^{r+r}} \sum_{a'}^{NS} \left( \prod_{j=1}^r N_j \delta_{-M_j + \bar{M}_j + m'_j + 2b} \right) f_{a'}(\tau),$$

where, $M = K\vec{\beta} \cdot \vec{M}$, $\bar{M} = K\vec{\beta} \cdot \vec{\tau}$.

The open string Witten index is written as

$$I^B_{\alpha \bar{\alpha}} := \frac{1}{2^{r+r}} \sum_{a'}^{NS} \left( \prod_{j=1}^r N_j \delta_{-M_j + \bar{M}_j + m'_j + 2b} \right) f_{a'}(\tau),$$

(3.4)

In both types of gluing conditions, the open string Witten indices are actually independent of $\tau$.

### 3.3 An Example — ALE space —

In this subsection, we consider the properties of the A$_{N-1}$ ALE model in detail; this model is composed of an SL(2,$\mathbb{R}$)$_N$/U(1) Kazama-Suzuki model and an $\mathcal{N} = 2$ level $(N - 2)$ minimal model [16].

The elliptic genus of the A$_{N-1}$ ALE model is obtained as the special case of (3.2). In particular, the closed string Witten index becomes $I(\tau, \bar{\tau}, z = 0) = N - 1$. This exactly correspond to the $(N - 1)$ dimensional (2,2) compact cohomology elements of the A$_{N-1}$ ALE space. Other compact cohomology elements of the A$_{N-1}$ ALE space are known to be 0 dimensional.

Associated open string Witten indices are also obtained as the special case of (3.3) and (3.4). In this special case, the A-type open string Witten indices are the same as the B-type ones and the result is

$$I_{\alpha \bar{\alpha}} = (-1)^{(S-\bar{S})/2} \sum_{m=0}^{2N-1} N_{L_1 \bar{L}_1}^{N - \bar{M} + m} N_{L_1 \bar{L}_1}^{m},$$

(3.5)
Figure 1: The image of the cycle associated with a boundary state of $\bar{L}_1, L_1, M = -1, S = 0$ with $\bar{L}_1 + L_1 \in 2\mathbb{Z}$. The total space is the $x$-plane when we express the ALE space as $x^N + y^2 + z^2 = \mu$ in $\mathbb{C}^3$. The dots are the root of the equation $x^N = \mu$. A line between two dots is a 2-cycle $C_{-\nu, \nu}$, $\nu = b, b+1, \ldots, a$, $a = (\bar{L}_1 + L_1 - 1)/2$, $b = |\bar{L}_1 - L_1 - 2|/2 + 1/2$. The sum of these cycles corresponds to the boundary state mentioned above.

This fact the A-type open string Witten indices coincides with the B-type ones is consistent to the fact that the $A_{N-1}$ ALE space is self mirror.

If we set $\bar{L}_1 = \tilde{L}_1 = 2$ in (3.5), this gives the same result as [17] and it is the correct intersection form of the vanishing 2-cycles of the $A_{N-1}$ ALE space. In particular, this intersection form becomes the $A_{N-1}$ extended Cartan matrix when we set $S = \tilde{S} = L_1 = \tilde{L}_1 = 0$.

In the general values of $\bar{L}_1$ and $L_1$, the associated cycles in the ALE space is as follows. Let us denote the 2-cycles in ALE space as $C_{\nu}$, $\nu \in \mathbb{Z}$, $C_{\nu+N} = C_{\nu}$, $\sum_{\nu=0}^{N-1} C_{\nu} = 0$. Here, we set the intersection form as $\langle C_{\nu}, C_{\nu}' \rangle = 2\delta_{\nu, \nu}' \mod N - \delta_{\nu, \nu+1}' - \delta_{\nu, \nu-1}'$. We also define $C_{\nu_1 \nu_2} := C_{\nu_1} + C_{\nu_1+1} + \cdots + C_{\nu_2-1}$ for the sake of convenience. For the boundary states with $\bar{L}_1 \leq (N+2)/2$, $L_1 \leq (N-2)/2$, $\bar{L}_1 + L_2 \in 2\mathbb{Z}$, $M = S = 0$, we can write the associated cycle as

$$\gamma(\bar{L}_1, L_1, M=0, S=0) = C_{-a+1,a} + C_{-a+2,a-1} + \cdots + C_{-b,b+1},$$

where, $a = \frac{1}{2}(\bar{L}_1 + L_1)$, $b = \frac{1}{2}|\bar{L}_1 - L_1 - 2|$.

On the other hand, for the boundary states with $\bar{L}_1 \leq (N+2)/2$, $L_1 \leq (N-2)/2$, $\bar{L}_1 + L_2 \in 2\mathbb{Z} + 1$, $M = -1$, $S = 0$, we can write the associated cycle as

$$\gamma(\bar{L}_1, L_1, M=-1, S=0) = C_{-a,a} + C_{-a+1,a-1} + \cdots + C_{-b,b},$$

where, $a = \frac{1}{2}(\bar{L}_1 + L_1 - 1)$, $b = \frac{1}{2}|\bar{L}_1 - L_1 - 2| + \frac{1}{2}$.

Actually, the open string Witten indices (3.5) and the intersection number between these cycles are the same. The image of this cycle is shown in Fig.4. The cycles for general values of $M$ are obtained by shifting by the $\mathbb{Z}_N$ symmetry.

4 Conclusion

In this paper, we investigate the discrete series of the noncompact Gepner model. We claim that the compact cohomology and the compact cycles belong to this sector. We check it in the case
of the $A_{N-1}$ ALE model. We treat only the A-type ALE in this paper, but D-type and E-type ALE can be also treated as the same way and the results will be the same as [24].

Other interesting noncompact models are ADE type Calabi-Yau 3-fold and 4-fold singularities. In these cases, the level of the $\text{SL}(2, \mathbb{R})/\text{U}(1)$ becomes fractional and this model does not belong to the models treated in this paper. We should consider some multiple covering of the $\text{SL}(2, \mathbb{R})$ group manifold and a fractional $\ell$ in order to treat the discrete series of the fractional level $\text{SL}(2, \mathbb{R})/\text{U}(1)$ Kazama-Suzuki models. We postpone this in a future work.

The relation between the continuous series and discrete series is also interesting. At the level of the modular invariance, these two are independent of each other. However, in the OPE level, they are interacting.

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