Hardness results for rainbow disconnection of graphs

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Abstract

Let $G$ be a nontrivial connected, edge-colored graph. An edge-cut $S$ of $G$ is called a rainbow cut if no two edges in $S$ are colored with a same color. An edge-coloring of $G$ is a rainbow disconnection coloring if for every two distinct vertices $s$ and $t$ of $G$, there exists a rainbow cut $S$ in $G$ such that $s$ and $t$ belong to different components of $G\setminus S$. For a connected graph $G$, the rainbow disconnection number of $G$, denoted by $rd(G)$, is defined as the smallest number of colors such that $G$ has a rainbow disconnection coloring by using this number of colors. In this paper, we show that for a connected graph $G$, computing $rd(G)$ is NP-hard. In particular, it is already NP-complete to decide if $rd(G) = 3$ for a connected cubic graph. Moreover, we prove that for a given edge-colored (with an unbounded number of colors) connected graph $G$ it is NP-complete to decide whether $G$ is rainbow disconnected.

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1 Introduction

All graphs in this paper are simple, finite and undirected. We follow [1] for graph theoretical notation and terminology not described here. Let $G$ be a graph. We use $V(G)$, $E(G)$, $n(G)$, $m(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex-set, edge-set, number of vertices, number of edges, minimum degree and maximum degree of $G$, respectively. Let $c : E(G) \rightarrow [k] = \{1, 2, ..., k\}$, $k \in \mathbb{N}$ be an edge-coloring of $G$, where adjacent edges may be colored with a same color. When adjacent edges of $G$ receive different colors under $c$, the edge-coloring $c$ is called proper. The chromatic index of $G$, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper coloring of $G$. By a famous theorem of Vizing [8] we have

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph $G$. And, if $\chi'(G) = \Delta(G)$, then $G$ is of Class 1; if $\chi'(G) = \Delta(G) + 1$, then $G$ is of Class 2.

A path of an edge-colored graph $G$ is said to be a rainbow path if no two edges on the path have the same color. The graph $G$ is called rainbow connected if every pair of distinct vertices of $G$ is connected by a rainbow path in $G$. An edge-coloring of a connected graph is a rainbow connection coloring if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [5] in 2008. For a connected graph $G$, the rainbow connection number $rc(G)$ of $G$ is defined as the smallest number of colors that are needed in order to make $G$ rainbow connected. The reader who are interested in this topic can see [6, 7] for a survey.

An edge-cut of a nontrivial connected graph $G$ is a set $S$ of edges of $G$ such that $G \setminus S$ is disconnected. The minimum number of edges in an edge-cut of $G$ is defined as the edge-connectivity $\lambda(G)$ of $G$. We then have the well-known inequality $\lambda(G) \leq \delta(G)$. For two distinct vertices $s$ and $t$ of $G$, let $\lambda(s, t)$ denote the minimum number of edges in an edge-cut $S$ of $G$ such that $s$ and $t$ lie in different components of $G \setminus S$. The so-called upper edge-connectivity $\lambda^+(G)$ of $G$ is defined by

$$\lambda^+(G) = \max \{\lambda(s, t) : s, t \in V(G)\}.$$ 

$\lambda(G)$ denotes the global minimum edge-connectivity of a graph, while $\lambda^+(G)$ denotes the local maximum edge-connectivity of a graph.

An edge-cut $S$ of a nontrivial connected graph $G$ is called a rainbow cut if no two edges in $S$ are colored with a same color. A rainbow cut $S$ is said to separate two vertices $s$ and $t$ if $s$ and $t$ belong to different components of $G \setminus S$. Such a rainbow cut is called
a $s - t$ rainbow cut. An edge-colored graph $G$ is called a rainbow disconnected if for every two vertices $s$ and $t$ of $G$, there exists an $s - t$ rainbow cut in $G$. In this case, the edge-coloring $c$ is called a rainbow disconnection coloring of $G$. Similarly, we define the rainbow disconnection number of $G$, denoted by $rd(G)$, as the smallest number of colors such that $G$ has a rainbow disconnected coloring by using this number of colors. A rainbow disconnection coloring with $rd(G)$ colors is called an rd-coloring of $G$. This concept of rainbow disconnection of graphs was introduced by Chartrand et al. \cite{2} very recently in 2018.

In this paper, we show that for a connected graph $G$ computing $rd(G)$ is NP-hard. In particular, it is already NP-complete to decide if $rd(G) = 3$ for a connected cubic graph $G$. Moreover, we show that for a given edge-colored (with an unbounded number of colors) connected graph $G$ it is NP-complete to decide whether $G$ is rainbow disconnected.

\section{Hardness results}

At the very beginning, we state some fundamental results on the rainbow disconnection of graphs, which will be used in the sequel.

\begin{lemma} \cite{2} If $G$ is a nontrivial connected graph, then
\[ \lambda(G) \leq \lambda^+(G) \leq rd(G) \leq \chi'(G) \leq \Delta(G) + 1. \]
\end{lemma}

Next result is due to Holyer \cite{3}, which is on the complexity of the chromatic index of a cubic graph.

\begin{lemma} \cite{3} It is NP-complete to determine whether the chromatic index of a cubic graph is 3 or 4.
\end{lemma}

At first we should show that our problem is in NP for a fixed integer $k$.

\begin{lemma} For a fixed positive integer $k$, given a $k$-edge-colored graph $G$, deciding whether $G$ is rainbow disconnected is in $P$.
\end{lemma}

\textit{Proof.} Let $n, m$ be the number of vertices and edges of $G$ respectively. Let $s, t$ be two vertices in $G$. Since $G$ is $k$-edge-colored, any rainbow cut set $S$ contains at most $k$ edges, and so, we have no more than $\binom{k}{m}$ choices of $S$. Given a set $S$ of edges, it is easy to check whether $s$ and $t$ lie in different components of $G \setminus S$. And there are
at most \( C_2^2 \) pairs of vertices in \( G \). Then, we can deduce that deciding whether \( G \) is rainbow disconnected can be checked in polynomial-time.

Next lemma is crucial for proof of our main result.

**Lemma 2.4** Let \( G \) be a 3-edge-connected cubic graph. Then \( \chi'(G) = 3 \) if and only if \( rd(G) = 3 \).

**Proof.** Assume that \( \chi'(G) = 3 \), and let us show that \( rd(G) = 3 \). Noticing that \( G \) is 3-edge-connected, we have that \( rd(G) \geq 3 \). Since \( rd(G) \leq \chi'(G) \) by Lemma 2.1, we then have \( rd(G) = 3 \).

Assume that \( rd(G) = 3 \). Let \( S = \{u_1v_1, u_2v_2, u_3v_3\} \) be a rainbow 3-edge cut of \( G \), and \( G \setminus S \) has two non-trivial component (which means the component is not a singleton) \( C_1 \) and \( C_2 \). If all the edges of \( S \) share a common vertex, then one of \( C_1 \) of \( C_2 \) is a singleton, a contradiction. If two edges of \( S \) are adjacent, say \( u_1 = u_2 \), let \( e \) be the third edge which is adjacent to \( u_1 \), then \( S' = \{e, u_3v_3\} \) is a 2-edge cut of \( G \), a contradiction. If non edges of \( S \) are adjacent, then we employ a new vertex \( x_1 \) which is adjacent to \( u_1, u_2, u_3 \) in \( C_1 \), and \( u_i x_1 \) receive the same color as \( u_i v_i \) for \( i = 1, 2, 3 \). Similarly, we employ a new vertex \( x_2 \) which is adjacent to \( v_1, v_2, v_3 \) in \( C_2 \), and \( x_2 v_i \) receive the same color as \( u_i v_i \) for \( i = 1, 2, 3 \). Now we get 3-edge connected cubic graphs \( C'_1 \) and \( C'_2 \). Repeat the operation on \( C'_1 \) and \( C'_2 \), and finally we get a graph sequence \( T = \{T_1, T_2 \cdots T_r\} \). Let \( s, t \) be two vertices in \( T_j \in T \), and \( S_{s,t} \) be the rainbow cut, we then have that \( |S_{s,t}| = 3 \) and the three edges of \( S_{s,t} \) are incident with one of \( s, t \). Next, we deduce that every vertex of \( T_j \) is incident with three rainbow colored edges except for one vertex, say \( s_0 \). Let the number of edges with color \( i \) incident with \( s_0 \) be \( k_i \), and \( T_{12} \) be the subgraph induced by the set of edges of \( T_j \) which are colored with colors 1 or 2. Let \( s_{12} \neq s_0 \) be a vertex of \( T_{12} \), then \( d(s_{12}) = 2 \). Since the degree sum of \( T_{12} \) is an even number, we have \( k_1 + k_2 = 0 \) (mod 2), which gives \( k_1 = k_2 \) (mod 2). Similarly, \( k_2 = k_3 \) (mod 2). So, we have that \( k_1 = k_2 = k_3 = 1 \) and \( s_0 \) is incident with three rainbow colored edges. So, \( T_j \) is properly colored for \( 1 \leq j \leq r \). Now we consider the original graph \( G \), the coloring satisfies that \( rd(G) = 3 \) is also a proper edge-coloring. Then, \( \chi'(G) = 3 \). 

In the proof of the following result, we will use the graph \( G_\phi \) which contains some copies of Figures 1, 2 and 3 employed in Holyer’s paper [3].

**Theorem 2.5** It is NP-complete to determine whether the rainbow disconnection number of a cubic graph is 3 or 4.

**Proof.** Clearly, the problem is in NP by Lemma 2.3. We prove that it is NP-complete by reducing 3-SAT to it. Given a 3CNF formula \( \phi = \bigwedge_{i=1}^n C_i \) over \( n \) variables
Figure 1: The inverting component and its symbolic representation.

Figure 2: The variable-setting component made from 8 inverting components and having 4 output pairs of edges. More generally, it is made from $2n$ inverting components and has $n$ output pairs ($n \geq 2$).

$x_1, x_2, \ldots, x_n$, we use the cubic graph $G_\phi$ that was used by Holyer in Lemma 2.2, such that $rd(G_\phi) = 3$ if and only if $\phi$ is satisfiable.
Noticing that $G(\phi)$ is 3-edge-connected, we then can verify that $rd(G_\phi) = 3$ if and only if $\phi$ is satisfiable by Lemma 2.4.

Deciding whether a $k$-edge-colored graph $G$, where $k$ is a constant, is rainbow disconnected is in $P$. However, it is NP-complete to decide whether a given edge-colored (with an unbounded number of colors) graph is rainbow disconnected. The proof of the following result uses a similar technique of [5].

**Theorem 2.6** Given an edge-colored graph $G$ and two vertices $s, t$ of $G$, deciding whether there is a rainbow cut between $s$ and $t$ is NP-complete.

**Proof.** Clearly, the problem is in NP. We prove that it is NP-complete by reducing 3-SAT to it. Given a 3CNF formula $\phi = \land_{i=1}^m c_i$ over $n$ variables $x_1, x_2, \ldots, x_n$, we construct a graph $G_\phi$ with two special vertices $s, t$ and a coloring $c : E(G_\phi) \to [E(G_\phi)]$ such that there is a rainbow cut between $s$ and $t$ in $G_\phi$ if and only if $\phi$ is satisfiable.

We define $G_\phi$ as follows:

$$V(G_\phi) = \{c_i, c_i^1, c_i^2, c_i^3 : i \in [m]\} \cup \{x_i^0, x_i^1 : i \in [n]\} \cup \{s, t\}$$
The coloring $c$ is defined as follows:

- the edges $\{s, t\}$ are colored with a special color $r_0$;
- the edges $\{s, x_i^0\}, \{s, x_i^1\}$ are colored with a special color $r_i$, $i \in [n]$;
- the edge $\{x_j^0, c_i\}$ or $\{x_j^1, c_i\}$ is colored with a special color $r_i^k$ when $x_j \in c_i$ is positive or negative in $\phi$ respectively, $i \in [m], k \in [1, 2, 3]$;
- the edge $\{c_i^k, x_j^0\}$ or $\{c_i^k, x_j^1\}$ is colored with a special color $r_i^4$ when $x_j \in c_i$ is positive or negative in $\phi$ respectively, $i \in [m], k \in [1, 2, 3]$;
- the edges $\{c_i, c_j^l\}$ are colored with a special color $r_i^5$, $i \in [m]$.

Now we can verify that there is a rainbow cut between $s$ and $t$ in $G_\phi$ if and only if $\phi$ is satisfiable.

Assume that there is a rainbow cut $S$ between $s$ and $t$ in $G_\phi$ under $c$, and let us show that $\phi$ is satisfiable. At first, we consider the color $r_0$. Since $s$ and $t$ are adjacent in $G(\phi)$, then the edge $\{s, t\}$ is in $S$. Clearly, $S$ separates $s$ and the set $\{t\} \cup \{c_i : i \in [m]\}$. Next, the color $r_i$ appears twice in $G(\phi)$. Without loss of generality, we can assume that there is exactly one of $\{s, x_i^0\}$ and $\{s, x_i^1\}$ in $S$, which corresponds to the value of variable $x_i$, $i \in [n]$. At last, we consider the colors $r_i^4$ and $r_i^5$, $i \in [m]$. There are at most two edges that have color $r_i^4$ or $r_i^5$ in $S$, which means that $c_i$ (a clause in $\phi$) is satisfiable, $i \in [m]$. As a result, $\phi$ is satisfiable.

Assume that $\phi$ is satisfiable, and let us construct a rainbow cut $S$ between $s$ and $t$ in $G_\phi$ under $c$. At first, the edges $\{s, x_i^{[x_j]}\}, i \in [n]$ and $\{s, t\}$ are in $S$. If the vertex $x_j^{[x_j]}$ is adjacent to $c_i$, then we choose one edge colored with $r_i^4$ or $r_i^5$ corresponding to variable $x_j$. If the vertex $x_j^{[x_j]}$ is adjacent to $c_i^k$, then we choose the edge $\{c_i, x_j^{[x_j]}\}$ that is colored with $r_i^k$ and corresponds to variable $x_j$. Notice that $\phi$ is satisfiable and no more than two edges colored with $r_i^4$ or $r_i^5$ are chosen. Add these chosen edges to $S$, and now $S$ is a rainbow cut between $s$ and $t$ in $G_\phi$ under $c$. ■
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