ON THE LOCAL GEOMETRY OF GRAPHS IN TERMS OF THEIR SPECTRA

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Abstract. In this paper, we consider the relation between the spectrum and the number of short cycles in large graphs. Suppose $G_1, G_2, G_3, \ldots$ is a sequence of finite and connected graphs that share a common universal cover $T$ and such that the proportion of eigenvalues of $G_n$ that lie within the support of the spectrum of $T$ tends to 1 in the large $n$ limit. We prove such a sequence of graphs is asymptotically locally tree-like. This is deduced by way of an analogous spectral rigidity theorem proved for certain infinite sophic graphs. We present additional results and questions in this spirit.

1. Introduction

This paper is about how the spectrum of a big, bounded degree graph determines its local geometry around typical vertices. For a finite and connected graph $G$, let

$$\lambda_1(G) > \lambda_2(G) \geq \lambda_3(G) \geq \cdots$$

be the eigenvalues of its adjacency matrix. Let $T$ be the universal cover tree of $G$ and denote by $\rho(T)$ its spectral radius, which is the operator norm of the adjacency matrix of $T$ acting on $\ell^2(T)$. If $G$ is $d$-regular then $\lambda_1(G) = d$ and $\rho(T) = 2\sqrt{d-1}$, $T$ being the $d$-regular tree.

It is easy to see that $\lambda_1(G) \geq \rho(T)$. Various extensions of the Alon-Boppana Theorem state that a positive proportion of the eigenvalues of $G$ lie outside the interval $[-\rho(T) + \varepsilon, \rho(T) - \varepsilon]$, independently of the size of $G$ but dependent on $\varepsilon > 0$; see [5, 7, 15, 16]. What happens when the eigenvalues actually concentrate within $[-\rho(T), \rho(T)]$?

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A graph $G$ is Ramanujan if $|\lambda_i(G)| \leq \rho(T)$ for every $i \geq 2$. It is a fairly well-understood theme that large, $d$-regular Ramanujan graphs locally resemble the $d$-regular tree in that they contain few short cycles. For an illustration of such results, see [1, 4, 9, 11, 12] and references therein. This relation is not as well understood for sparse, irregular graphs. We prove the following relation between the spectral measure and the local geometry.

Suppose $G_1, G_2, G_3, \ldots$ is a sequence of finite and connected graphs. They are weakly Ramanujan if they have a common universal cover tree $T$ and if, counting with multiplicity,

$$\frac{\#\{\text{Eigenvalues of } G_n \text{ s.t. } |\lambda_i(G_n)| \leq \rho(T)\}}{|G_n|} \longrightarrow 1 \text{ as } n \to \infty.$$  

**Theorem 1.** Consider a sequence of weakly Ramanujan graphs as in (1.1). Suppose that $|G_n| \to \infty$. Then the graphs are asymptotically locally tree-like in that for every $r > 0$,

$$\frac{\#\{\text{Vertices } v \in G_n \text{ s.t. its } r\text{-neighbourhood is a tree}\}}{|G_n|} \longrightarrow 1 \text{ as } n \to \infty.$$  

The version of Theorem 1 for $d$-regular weakly Ramanujan graphs has been proved earlier in [1]. In this case many tools, such as the Green’s function and spectral measure of the $d$-regular tree, are available in precise form. This is lacking for general universal covers where even the computation of the spectral radius is difficult (although an algorithm is provided in [14] and various bounds are given in [8, 15]).

Let us make some remarks about Theorem 1. First, if the $r$-neighbourhood of a vertex $v \in G$ is a tree then it agrees with the $r$-neighbourhood of any vertex $\hat{v}$ in the universal cover of $G$ that maps to $v$ under the the cover map. So roughly speaking, large weakly Ramanujan graphs locally look like their universal covers around most vertices.

Second, it is natural in our context to assume a sequence of graphs share a common universal cover. For one thing it is a generalization of a sequence of $d$-regular graphs, $(a, b)$-biregular graphs, etc. But more so, it provides a way to compare the spectra and geometry of graphs with differing sizes on a common scale. For example, two finite graphs with the same universal cover have the same degree distribution and, hence, the same average and maximal degree. They also have the same maximal eigenvalue which follows from a theorem in [10]. The definition of Ramanujan graphs in terms of their universal covers was introduced in [7] (also stated in [9]). Theorem 1 also answers a question of the second author from [15].
Finally, the assumption in Theorem 1 that $|G_n| \to \infty$ is necessary. For example, if all the graphs are equal to a common cyclic graph then the sequence is weakly Ramanujan. (The universal cover is $\mathbb{Z}$ with $\rho(\mathbb{Z}) = 2$, and all the eigenvalues lie in the interval $[-2,2]$.) However, this is the only obstruction as being weakly Ramanujan implies $|G_n| \to \infty$ so long as the common average degree of the graphs is larger than 2. This follows from the following theorem which asserts that $\lambda_1(G_n) > \rho(T)$ when the common average degree is more than 2.

**Theorem 2.** Let $G$ be a finite and connected graph with universal cover $T$. Then $\lambda_1(G) = \rho(T)$ if and only if $G$ has at most one cycle or, equivalently, if and only if the average degree of $G$ is at most 2.

Despite being intuitive, the proof of this theorem is more delicate than one may presume. For instance, consider the bowtie graph $G$ obtained by gluing together two triangles at a common vertex. It has $\lambda_1(G) = (1+\sqrt{17})/2$ and $\rho(T) = (\sqrt{3} + \sqrt{11})/2$. The spectral gap is about 0.04 and the average degree is also smaller than $\rho(T)$. In general, the spectral gap can be arbitrarily small for graphs formed by gluing together two large cycles at a common vertex. So the proof of Theorem 2 requires some work.

Theorem 1 is proved in the following section. The idea behind the proof is to reduce the theorem to an analogous theorem about certain infinite, random rooted graphs (often called sophic graphs) by using the notion of local convergence of graphs. The key result of the paper is a proof of the analogue of Theorem 1 for these infinite graphs, which is stated as Theorem 3 below and proved in Section 3. The proof establishes a lower bound on the spectral radius of such graphs in terms of a probabilistic notion of cycle density. The aforementioned Theorem 2 is proved in Section 4. Section 5 concludes with some questions.

### 2. A Reduction of Theorem 1

We begin with a description of the notion of local convergence of graphs and its properties utilized in the proof of Theorem 1. A complete account, including proofs, may be found in [2, 3, 6].

Let $B_r(G, v)$ be the $r$-neighbourhood of a vertex $v$ in a graph $G$. A sequence of finite and connected graphs $G_1, G_2, G_3, \ldots$ converges locally if the following holds. For every $r$ and every rooted, connected graph $(H, o)$ having radius at most $r$ from the root $o$, the ratio

$$\frac{\# \{\text{Vertices } v \in G_n \text{ s.t. } B_r(G_n, v) \cong (H, o)\}}{|G_n|}$$

converges as $n \to \infty$. 
The isomorphism relation $\cong$ is for rooted graphs, i.e., the isomorphism must take the root of one to the other. Local convergence is sometimes called Benjamini-Schramm convergence as it was formulated by them.

A locally convergent sequence of graphs may be represented as a random rooted graph in the following way. Let $G$ be the set of all rooted and connected graphs whose vertex sets are subsets of the integers. Identify the graphs in $G$ up to their rooted isomorphism class. The set $G$ is a complete and separable metric space with the distance between $(H, o)$ and $(H', o')$ being $2^{-r}$, where $r$ is the maximal integer such that $B_r(H, o) \cong B_r(H', o')$. A random rooted graph is simply a Borel probability measure on $G$ or, in other words, a $G$-valued random variable $(G, o)$ that is Borel-measurable. Given a locally convergent sequence of graphs as above, there is a random rooted graph $(G, o)$ such that for every $r$ and $(H, o)$ as above,

$$\frac{\#\{\text{Vertices } v \in G_n \text{ s.t. } B_r(G_n, v) \cong (H, o)\}}{|G_n|} \to \Pr [B_r(G, o) \cong (H, o)].$$

A random rooted graph that is obtained from a locally convergent sequence of finite graphs is called sophic. A simple example is any finite and connected graph rooted at an uniformly random vertex. More examples may be found in [2, 3, 6]. Sophic graphs satisfy an important property known as the mass transport principle, as we explain.

Suppose $(G, o)$ is sophic. Consider a bounded and measurable function $F(G, u, v)$ defined over doubly rooted graphs $(G, u, v)$ such that it depends only on the double-rooted isomorphism class of $(G, u, v)$. The mass transport principle states that

$$\mathbf{E} \sum_{v \in G} F(G, o, v) = \mathbf{E} \sum_{v \in G} F(G, v, o).$$

The above is readily verified for a finite graph rooted at a uniformly random vertex, and it continues to hold in the local limit, which is why it holds for a sophic graph.

Let us describe the universal cover of a sophic graph. Recall the universal cover of a graph $G$ is the unique tree $T$ for which there is a surjective graph homomorphism $\pi : T \to G$, called the cover map, such that $\pi$ is locally bijective: for every $\hat{v} \in T$, $\pi$ provides a bijection $B_1(T, \hat{v}) \leftrightarrow B_1(G, \pi(\hat{v}))$. If $\pi(\hat{v}) = \pi(\hat{u})$ then the rooted graphs $(T, \hat{v}) \cong (T, \hat{u})$. Therefore, the universal cover of a sophic graph $(G, o)$ may be defined as its samplewise universal cover $(\hat{T}, \hat{o})$, where $\hat{o}$ is any vertex that is mapped to $o$ by the cover map.

The spectral radius of a sophic graph $(G, o)$ is defined as follows. Let $W_k(G, o)$ be the set of closed walks of length $k$ from $o$ in a graph $G$ and denote by $|W|_k(G, 0)$
its size. The spectral radius of \((G,o)\) is
\[
\rho(G) = \lim_{k \to \infty} (E|W|_{2k}(G,o))^{1/2k}.
\]
Recall that the spectral radius of a connected graph \(G\) is also the exponential growth rate of \(|W|_{2k}(G,v)\) for any vertex \(v\). The connection of \(\rho(G)\) to the adjacency matrix of \(G\) is that it equals the sup norm, \(||\rho(G,o)||_{\infty}\), of the samplewise spectral radius of \((G,o)\). It is also the largest element in the support of the “averaged” spectral measure of \((G,o)\), which is a Borel probability measure on the reals whose moments are \(E|W|_0(G,o), E|W|_1(G,o), E|W|_2(G,o)\), and so on.

Our theorem regarding the spectrum and geometry of an infinite sophic graph is the following.

**Theorem 3.** Let \((G,o)\) be an almost surely infinite sophic graph that is the limit of a sequence of finite graphs sharing a common universal cover \(T\). If \(\rho(G) = \rho(T)\) then \(G\) is isomorphic to \(T\) almost surely.

This theorem may be readily extended to an unimodular network whose universal cover is non-random and quasi-transitive.

### 2.1. Reducing Theorem 1 to sophic graphs.

The theorem may be reformulated as stating that given a sequence of weakly Ramanujan graphs sharing a common universal cover \(T\), and with their sizes asymptotically large, the sequence converges locally to \(T\). The root of \(T\) will be a random vertex whose distribution is uniquely determined by the convergent sequence. To be more precise, it will be a probability measure on a predetermined set of vertices
\[
\{ \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m \}
\]
of \(T\) with the following property. For any graph \(G\) that is covered by \(T\) and any vertex \(v \in G\) and \(\hat{v} \in T\) such that \(\pi(\hat{v}) = v\), there is a \(T\)-automorphism that takes \(\hat{v}\) to some \(\hat{v}_j\). \((T\) is quasi-transitive and \(m\) is at most the size of the smallest graph that is covered by \(T\).)

Suppose \(G_1, G_2, G_3, \ldots\) is a sequence of weakly Ramanujan graphs as in the statement of the theorem. Since they share a common universal cover, their vertex degrees are bounded by some integer \(\Delta\). By a simple diagonalization argument (there are at most \(\Delta^{r+1}\) rooted graphs of radius \(\leq r\) and maximal degree \(\leq \Delta\)), the sequence is pre-compact in the local topology. We must prove that its only limit point is \(T\) (up to having the random root).

Suppose \((G,o)\) is a limit point of the sequence. Then its universal cover \((T,o)\) is such that \(T\) is isomorphic to \(T\) almost surely. This is because the sequence has a common universal cover and the universal cover is a local function of its base.
(If $T$ is the universal cover of $G$ and $\hat{v} \in T$, then the vertices of $T$ are in one to one correspondence with the non-backtracking walks of $G$ that start at $\pi(\hat{v})$ and two such walks are adjacent in $T$ if one is an extension of the other by exactly one edge; see [13]). Theorem 1 is proved if $G$ is isomorphic to $T$ almost surely as unrooted graphs. By Lemma 2.1 below, $\rho(G) = \rho(T)$. The theorem now follows from Theorem 3.

**Lemma 2.1.** Let $G_1, G_2, G_3, \ldots$ be a locally convergent sequence of weakly Ramanujan graphs. Suppose $(G, o)$ is its limit and $T$ is the common universal cover. Then $\rho(G) = \rho(T)$.

**Proof.** By continuity in the local topology, $\mathbb{E}|W|_{2k}(G, o)$ is the limit of the averages $\frac{1}{|G_n|} \sum_{v \in G_n} |W|_{2k}(G_n, v)$. Let $p_n$ be the proportion of eigenvalues of $G_n$ that are at most $\rho(T)$ in absolute value, so then $p_n \to 1$. Let $\Delta$ be the maximal vertex degree of the graph sequence and note that all eigenvalues are bounded by $\Delta$ in absolute value. The aforementioned average is the trace of the $(2k)$-th power of the adjacency matrix of $G_n$, normalized by $|G_n|$. Thus,

$$\frac{1}{|G_n|} \sum_{v \in G_n} |W|_{2k}(G_n, v) \leq p_n \rho(T)^{2k} + (1 - p_n) \Delta^{2k}.$$ 

Upon taking limits we conclude that $\rho(G) \leq \rho(T)$.

For the inequality in the other direction, note that if $T$ is the universal cover of $G$ and $\hat{v} \in T$ is mapped to $v \in G$ by the cover map then $|W|_{2k}(G, v) \geq |W|_{2k}(T, \hat{v})$. This is because the cover map provides an injection from $W_{2k}(G, v)$ into $W_{2k}(T, \hat{v})$. Now recall the vertices $\hat{v}_1, \ldots, \hat{v}_m$ from (2.1). There is a probability measure $(p_1(G), \ldots, p_m(G))$ on these vertices such that $p_j(G)$ proportion of the vertices of $G$ have a preimage in $T$, under the cover map, which can be sent to $\hat{v}_j$ by a $T$-automorphism. Consequently,

$$\frac{1}{|G|} \sum_{v \in G} |W|_{2k}(G, v) \geq m \sum_{j=1}^m p_j(G) |W|_{2k}(T, \hat{v}_j).$$

Applying the inequality above to every $G_n$ and taking the large $n$ limit gives

$$\mathbb{E}|W|_{2k}(G, o) \geq \sum_{j=1}^m p_j |W|_{2k}(T, \hat{v}_j).$$

Here $(p_1, \ldots, p_m)$ is the limit of $(p_1(G_n), \ldots, p_m(G_n))$, which exists due to local convergence of the graph sequence. Since $\rho(T)$ is the large $k$ limit of $|W|_{2k}(T, \hat{v}_j)$ for every $\hat{v}_j$, the inequality above implies that $\rho(G) \geq \rho(T)$. $\square$
3. A spectral rigidity theorem

Theorem 3 will be proved by showing that if there is an \( \ell \) such that
\[
\Pr \left[ o \text{ lies in an } \ell \text{ -- cycle of } G \right] > 0,
\]
then \( \rho(G)/\rho(T) \geq 1+\delta \) for some positive \( \delta \). This result is built up in the subsequent sections by drawing a connection between the spectral radius of \( G \) and of \( T \) in terms of the norms of certain Markov operators associated to random walks on the fundamental group of \( G \). This connection was established in \( [1] \). The key ingredients are Lemma 3.3 and Lemma 3.4.

3.1. Walk counting using the fundamental group. Consider a connected graph \( H \) which may be countably infinite and may have multi-edges and loops around its vertices. (A loop contributes degree 2 to its vertex.) Let \( \pi(H,v) \) be its fundamental group based at vertex \( v \), which consists of homotopy classes of closed walks from \( v \) under the operation of concatenation. It is a free group. (See \( [13] \) for a comprehensive account on the fundamental group of graphs and its properties mentioned herein.)

Let \( W_k(u,v) \) be all walks in \( H \) of length \( k \) from \( u \) to \( v \). Note \( W_k(v,u) = W_k^{-1}(u,v) \), where the inverse means walking in the opposite direction. The set
\[
WW^{-1} = \{ PQ^{-1} : P, Q \in W_k(u,v) \}
\]
consists of closed walks from \( u \) of length \( 2k \) and is itself closed under taking inverses. It naturally maps into \( \pi(H,u) \), and the uniform measure on it pushes forward to a measure on the image \( WW^{-1} \subset \pi(H,u) \). Note the push forward may not be uniform measure on the image as different closed walks in \( WW^{-1} \) may be homotopy equivalent.

Consider the random walk on \( \pi(H,u) \) whose step distribution is the aforementioned pushed forward measure of \( WW^{-1} \). Since \( WW^{-1} \) is closed under taking inverses, this is a symmetric random walk on the Cayley graph of the subgroup of \( \pi(H,u) \) generated by \( WW^{-1} \) (but not necessarily the simple random walk). Denote the norm of its associated Markov operator by
\[
\|M_k\|(u,v).
\]

Now fix a vertex \( o \in H \), and also a path \( P \) from \( o \) to \( u \) and another path \( Q \) from \( o \) to \( v \). The set \( PW_k(u,v)Q^{-1} \) consists of closed walks from \( o \). Consider the random walk on \( \pi(H,o) \) whose step distribution is the push forward of the uniform measure on this set under its the natural mapping into \( \pi(H,o) \). Denote by \( \sqrt{\|M_k\|}(u,v) \) the norm of the Markov operator of this random walk. This operator may not be
symmetric since the set \( PW_k(u, v)Q^{-1} \) is not closed under taking inverses. However, 
\[
\sqrt{||M_k||(u, v)^2} = ||M_k||(u, v)
\]
because the norm in question is the square root of the norm of the Markov operator for the random walk on \( \pi(H, o) \) associated to the set 
\[
(PW_k(u, v)Q^{-1})(PW_k(u, v)Q^{-1})^{-1} = PWW^{-1}P^{-1}.
\]
The Markov operator for \( PWW^{-1}P^{-1} \) is isomorphic – as an operator on \( \ell^2(\pi(H, o)) \) – to the Markov operator for \( WW^{-1} \) on \( \ell^2(H, u) \). The isomorphism comes from the natural isomorphism of groups \( \pi(H, o) \leftrightarrow \pi(H, u) \). The norm of the Markov operator for \( WW^{-1} \) is 
\[
||M_k||(u, v).
\]

3.2. The counting argument. Let \( H \) be a graph as above. A purely backtracking walk in \( H \) is a closed walk that is homotopic to the empty walk, that is, it reduces to the identity in the fundamental group of \( H \). Purely backtracking walks in \( H \) from a base point \( o \) are in one to one correspondence with closed walks in the universal cover of \( H \) from a base point \( \hat{o} \) such that \( \pi(\hat{o}) = o \) (\( \pi \) being the cover map). This is due to the path lifting property of the universal cover map.

Choose an arbitrary vertex \( o \in H \). Let \( n \) and \( k \) be arbitrary integers with being \( nk \) even. Denote by \( W \) all closed walks from \( o \) of length \( nk \). Denote by \( N \) all purely backtracking walks from \( o \) of length \( nk \). The following is a key inequality.

\[
(3.2) \log |W| - \log |N| \geq \frac{1}{|N|} \sum_{P \in N} \sum_{j=1}^{n} -\frac{1}{2} \log \langle M_k ||(P_{j-1})k, kj \rangle.
\]

The proof is based on partitioning the set \( W \) in the following way. Two walks in \( W \) are equivalent if their locations coincide at the times \( 0, k, 2k, \ldots, nk \). Let \( W_N \) denote the set of walks in \( W \) that are equivalent to some purely backtracking walk. Observe that

\[
|W_N| = \sum_{P \in N} \frac{||P||}{||P \cap N||}.
\]

The term \( ||P||/||P \cap N|| \) is the reciprocal of the probability that a uniform random walk in \( [P] \) is purely backtracking. The probability can be interpreted in the following way. Consider the random walk on \( \pi(H, o) \) whose step distribution is the push forward of the uniform measure on \( [P] \) \( \mapsto \pi(H, o) \). The probability under consideration is the one-step return probability of this random walk. It may be expressed as \( \langle M_P \text{id}, \text{id} \rangle \), where \( M_P \) is the Markov operator of this random walk. Therefore,

\[
|W| \geq \sum_{P \in N} \langle M_P \text{id}, \text{id} \rangle^{-1}.
\]
Every \( Q \in \{P\} \) agrees with \( P \) at the times \( 0, k, \ldots, nk \). This allows us to decompose \( Q \) into petals as in Figure 1.

Figure 1. Decomposing a closed walk into petals.

Here, \( Q_j \) is the segment of \( Q \) from \( Q_{(j-1)k} = P_{(j-1)k} \) to \( Q_{jk} = P_{jk} \). \( R_j \) is a fixed path from \( o \) to \( P_{jk} \) chosen independently of \( Q \). The decomposition is that
\[
Q = (Q_1 R_1^{-1}) \cdot (R_1 Q_2 R_2^{-1}) \cdots (R_{n-1} Q_n) .
\]
Under this decomposition, a uniformly random element \( Q \in \{P\} \) becomes the product \( T_1 \cdots T_n \), where \( T_j \) is a uniformly random element of \( R_{j-1} W_k( P_{(j-1)k}, P_{jk} ) R_j^{-1} \).
This uses that the locations of \( Q \) are pinned at the times \( 0, k, \ldots, nk \).

Let \( M_j \) be the Markov operator for the random walk on \( \pi(H, o) \) with step distribution \( T_j \). Then \( M_P = M_1 \cdots M_n \), and
\[
\langle M_P \text{id}, \text{id} \rangle \leq ||M_P|| \leq \prod_j ||M_j|| .
\]
Each \( ||M_j|| \) equals \( \sqrt{||M_k||} (P_{(j-1)k}, P_{jk}) \). Therefore,
\[
|W| \geq \sum_{P \in N} \prod_{j=1}^n ||M_k||(P_{(j-1)k}, P_{jk})^{-1/2} .
\]
Dividing the above by \( |N| \), using the inequality of arithmetic-mean and geometric-mean, and then taking the logarithm gives the inequality from (3.2).
Let \((G, o)\) be an infinite sophic graph as in the statement of Theorem 3. The mass transport principle simplifies the right hand side of (3.2) for \((G, o)\) as follows.

**Lemma 3.1.** In this setting the following equation holds for \(j = 1, \ldots, n\).

\[
\frac{1}{|N|_{nk}(G, o)} \sum_{P \in N_{nk}(G, o)} \log |M_k|(P_{(j-1)k}, P_{jk}) = \frac{\sum_{P \in N_{nk}(G, o)} \log |M_k|(o, P_k)}{|N|_{nk}(P_{(n-j+1)k})}.
\]

**Proof.** Consider the function

\[
F(H, u, v) = \frac{1}{|N|_{nk}(H, v)} \sum_{P \in N_{nk}(H, u)} 1_{\{P_{(j-1)k} = u\}} \log |M_k|(u, P_k).
\]

It depends on the doubly-rooted isomorphism class of \((H, u, v)\). Now,

\[
\sum_{u \in H} F(H, u, v) = \frac{1}{|N|_{nk}(H, v)} \sum_{P \in N_{nk}(H, u)} \log |M_k|(P_{(j-1)k}, P_{jk}).
\]

On the other hand,

\[
F(H, u, v) = \frac{1}{|N|_{nk}(H, u)} \sum_{P \in N_{nk}(H, u)} 1_{\{P_{(n-j+1)k} = v\}} \log |M_k|(u, P_k)
\]

because we can also sum over the walks by starting them at \(u\) instead of \(v\). Therefore,

\[
\sum_{v \in H} F(H, u, v) = \frac{\sum_{P \in N_{nk}(H, u)} \log |M_k|(u, P_k)}{|N|_{nk}(H, P_{(n-j+1)k})}.
\]

The mass-transport principle for \((G, o)\) states that

\[
E \sum_{u \in G} F(G, u, o) = E \sum_{v \in G} F(G, o, v),
\]

which is the equation in the statement of the lemma. \(\square\)

### 3.3. Bounds.

Applying the bound from (3.2) to \((G, o)\), taking the expectation value, applying Lemma 3.1 and then dividing by \(nk\) gives

\[
\frac{E \log |W|_{nk}(G, o) - E \log |N|_{nk}(G, o)}{nk} \geq \frac{\sum_{P \in N_{nk}(G, o)} \frac{1}{n} \sum_{j=1}^{n} -(2k)^{-1} \log |M_k|(o, P_k)}{|N|_{nk}(G, P_{(n-j+1)k})}.
\]

The term \(-(2k)^{-1} \log |M_k|(P_{o}, P_k)\) is non-negative. We would thus like to replace each of the terms \(|N|_{nk}(G, P_{(n-j+1)k})\) by \(|N|_{nk}(G, o)\), after which the average over the parameter \(j\) would be replaced by unity. Recall the universal cover of \(G\) is the non-random tree \(T\) and \(W_{nk}(T, \hat{v}) = N_{nk}(G, \pi(\hat{v}))\). Therefore, the cost of replacing \(|N|_{nk}(G, P_{(n-j+1)k})\) by \(|N|_{nk}(G, o)\) while preserving the \(\geq\) inequality is
given by the multiplicative factor
\[ r_{nk} = \min_{i,j} \frac{|W_{nk}(T, \hat{v}_i)|}{|W_{nk}(T, \hat{v}_j)|}, \]
where \( \hat{v}_1, \ldots, \hat{v}_m \) are a set of orbit representatives for \( T \) as explained in [2,1]. Part 1 of the following lemma shows that \( r_{nk} \geq \Delta^{-2d} \), where \( d \) is the maximum distance between any two of the \( \hat{v}_i \)s and \( \Delta \) is the maximal degree of \( T \).

**Lemma 3.2.** Let \( H \) be a connected graph having maximum degree at most \( \Delta \). Let \( x \) and \( y \) be two of its vertices having distance \( d \) between them.

1. \( |W|_{2k}(H,y) \leq \Delta^{2d} |W|_{2k}(H,x) \).
2. \( |W|_{2k+2j}(H,x) \leq \Delta^{2j} |W|_{2k}(H,x) \).

Proof. Let \( A \) be the adjacency matrix of \( H \) acting on \( \ell^2(H) \) (\( H \) may be countably infinite).

The inequality in (1) follows from the inequality in (2) upon observing that \( |W|_{2k}(H,y) \leq |W|_{2k+2d}(H,x) \). For the proof of (2), we have \( |W|_{2k+2j}(H,x) = \langle A^{2k+2j} \delta_x, \delta_x \rangle \) and the latter equals \( \langle A^{2j}(A^k \delta_x), (A^k \delta_x) \rangle \). Thus,
\[
|W|_{2k+2j}(H,x) \leq \|A^{2j}\| \langle A^k \delta_x, A^k \delta_x \rangle = \Delta^{2j} |W|_{2k}(H,x) .
\]

**Lemma 3.3.** Let \( \Delta \) be the maximal degree of \( T \). The following inequality holds:
\[
\log \rho(G) - \log \rho(T) \geq \sup_{k \geq 1} \frac{\mathbb{E} - \log \|M_{2k}\|(o,o)}{4k \Delta^{2d+2k}}.
\]

Proof. Observe that \( G \) has maximal degree \( \Delta \) almost surely because it is covered by \( T \). By Lemma 3.2
\[
\mathbb{E} \log |W|_{nk}(G,o) - \mathbb{E} \log |N|_{nk}(G,o) \geq \Delta^{-2d} \mathbb{E} \frac{1}{|N|_{nk}(G,o)} \sum_{P \in N_{nk}(G,o)} -\log \|M_k\|(o,P_k) \cdot \frac{2k}{2k}.
\]

The expectation on the right hand side of (3.3) is an average over \( (G,o,P^n) \), where \( P^n \) is a uniformly random purely backtracking walk in \( G \) starting at \( o \) and having length \( nk \). If \( k \) is even then \( \Pr[P^n_k = o] \geq \Delta^{-k} \). This is because a purely backtracking walk from \( o \) of length \( nk \) will be at \( o \) at time \( k \) if it consists of a purely backtracking walk from \( o \) of length \( k \) followed by a purely backtracking walk from \( o \) of length \( nk - k \). Consequently,
\[
\Pr[P^n_k = o] \geq \mathbb{E} \frac{|N|_k(G,o) \cdot |N|_{nk-k}(G,o)}{|N|_{nk}(G,o)} \geq \Delta^{-k}.
\]
where the last inequality used that $|N|^k(G, o) \geq 1$ and also, by part 2 of Lemma 3.2 that $|N|_{nk-k}(G, o) \geq \Delta^{-k}|N|_{nk}(G, o)$. Since $-\log||M_k||(u, v)$ is non-negative, (3.3) implies that for every even $k$

$$\frac{E \log |W|_{nk}(G, o) - E \log |N|_{nk}(G, o)}{nk} \geq \frac{E - \log ||M_k||(o, o)}{2k \Delta^{2d} + k}.$$

We may take a large $n$ limit supremum of the left hand side of the above for every even value of $k$. In the limit as $n \to \infty$, the left hand side is at most $\log \rho(G) - \log \rho(T)$. This is because $E \log |W| \leq \log E |W|$ by concavity of log and, as argued in Lemma 2.1, $E \log |N|_{nk}(G, o)$ is the average over a finitely supported probability measure (on at most $m$ points) and each term in this average converges to $\log \rho(T)$ after division by $nk$ and letting $n$ tend to infinity. The inequality from the lemma now follows due to $k$ being an arbitrary even integer. \hfill \Box

3.4. Completion of the proof. Let $(H, v)$ be a rooted and connected graph. Given an even integer $k$ and another integer $\ell$, let us say $H$ contains a bouquet if it has two disjoint $\ell$-cycles, $C_1$ and $C_2$, such that if the distance from $v$ to $C_j$ is $r_j$, $j = 1, 2$, then $k \geq \ell + \max\{r_1, r_2\}$. The situation is pictured below.

![Figure 2](image)

**Figure 2.** A bouquet around the root.

Suppose $(H, v)$ contains a bouquet for the parameter values $k$ and $\ell$. They provide two closed walks in $W_{2k}(v, v)$, say $P_1$ and $P_2$, in the following way. The walk $P_j$ is obtained by walking from $v$ to the closest vertex on $C_j$, traversing the cycle twice, then walking back to $v$ along the reverse of the initial segment and appending some purely backtracking walk at the end to ensure $2k$ steps in total.

Recall the walks in $W_{2k}(v, v)$ naturally map to a set $\overline{W_{2k}(v, v)} \subset \pi(H, v)$ by homotopy equivalence. In this way the walks $P_1$ and $P_2$ correspond to two elements of $\pi(H, v)$ that have the form $g_1^2$ and $g_2^2$, respectively. Indeed, $g_j$ corresponds to the walk that results from going from $v$ to the closest vertex to $C_j$, traversing in once, and then walking back to $v$ along the reverse of the initial segment.
The elements $g_1$ and $g_2$ are free in $\pi(H,v)$. They can be extend to a minimal set of mutually free elements of $\pi(H,v)$, such that every element of $W_{2k}(v,v)$ can be expressed a product of these $g_j$s and their inverses. Let $\Gamma$ be the subgroup of $\pi(H,v)$ generated by $\{g_1, g_2, \ldots, g_m\}$. It is a finitely generated free group that contains $W_{2k}(v,v)$. Recall that the uniform measure on $W_{2k}(v,v)$ pushes forward to a measure $p(\cdot)$ on $W_{2k}(v,v)$, which induces a symmetric random walk on $\Gamma$ whose step distribution is $p$ and whose Markov operator is denoted $M$.

**Lemma 3.4.** Suppose $(H,v)$ contains a bouquet as in the setup described above. Then the norm of the operator $||M|| < 1$.

**Proof.** Consider the Cayley graph of $\Gamma$ generated by right multiplication by the $g_j$s and their inverses. This is a regular tree. Denote it $T$ and denote $d(\cdot, \cdot)$ its graph distance. Root $T$ at the identity. Also, denote the set $W_{2k}(v,v)$ by $S$.

The operator norm of $M$ is

\[
||M|| = \sup_{x \in \ell^2(\Gamma)} \left| \sum_{u,v \in \Gamma: uv^{-1} \in S} p(uv^{-1}) x_u x_v \right|.
\]

Fix an $x \in \ell^2(\Gamma)$ having norm 1. We estimate each term $x_u x_v$ ($u \neq v$) from the right hand side of (3.4) by

\[
|x_u x_v| \leq \frac{1}{2} \left[ \lambda(u,v)x_u^2 + \frac{1}{\lambda(u,v)x_v^2} \right],
\]

where

\[
\lambda(u,v) = \begin{cases} 
1 - \epsilon, & \text{if } d(id,u) < d(id,v); \\
\frac{1}{1-\epsilon}, & \text{if } d(id,u) > d(id,v); \\
1, & \text{if } d(id,u) = d(id,v),
\end{cases}
\]

for some $\epsilon$ to be determined.

The right hand side of (3.4) is bounded from above by $\sum_{u \in \Gamma} \sigma(u) x_u^2$, where

\[
\sigma(u) = p(id) + \sum_{s \in S \neq id} p(s) \lambda(u,us).
\]

Let $s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}$ be an enumeration of the non-identity elements of $S$, noting that no $s_i = s_i^{-1}$ because $\Gamma$ is a free group. As observed earlier, this set contains $g_1^2$ and $g_2^2$. We may assume that $s_1 = g_1^2$ and $s_2 = g_2^2$. 
Since \( p(s) = p(s^{-1}) \) for every \( s \in S \), we have that

\[
\sigma(u) = p(\text{id}) + \sum_{i=1}^{m} p(s_i) \left[ \lambda(u, us_i) + \lambda(u, us_i^{-1}) \right].
\]

Lemma 3.5 below implies that for each \( s_i \),

\[
\lambda(u, us_i) + \lambda(u, us_i^{-1}) \leq (1 - \epsilon) + \frac{1}{1 - \epsilon} = 2 + \frac{\epsilon^2}{1 - \epsilon}.
\]

Suppose the edge from \( u \) to its parent in \( T \) – the neighbour of \( u \) closer to the root – is labelled by the generator \( g_j \) or \( g_j^{-1} \). If \( g_j \) is not \( g_1 \) or \( g_2 \) then both of \( us_1 \) and \( us_1^{-1} \), as well as \( us_2 \) and \( us_2^{-1} \), are descendants of \( u \) in \( T \). On the other hand, if \( g_j = g_1 \) then \( us_2 \) and \( us_2^{-1} \) are descendants of \( u \), or the same thing with \( s_1 \) if \( g_j = g_2 \). In any case, one of the pairs \{\( us_1, us_1^{-1} \)\} or \{\( us_2, us_2^{-1} \)\} consist of descendants. Suppose it holds for \( s_1 \). Then

\[
\lambda(u, us_1) = \lambda(u, us_1^{-1}) = 1 - \epsilon.
\]

Note that \( \sum_{i=1}^{m} p(s_i) = (1 - p(\text{id}))/2 \). So \( \sigma(u) \) satisfies the following bound by using (3.5) and (3.6).

\[
\sigma(u) \leq p(\text{id}) + p(s_1)(2 - 2\epsilon) + \sum_{i=2}^{m} p(s_i) \left[ 2 + \frac{\epsilon^2}{1 - \epsilon} \right]
= 1 - \epsilon \left[ 2p(s_1) - \frac{\epsilon}{2(1 - \epsilon)}(1 - p(\text{id}) - 2p(s_1)) \right].
\]

Since \( p(s_1) > 0 \), by making \( \epsilon \) sufficiently small we can ensure that the term inside the square parentheses above is positive. Moreover, \( \epsilon \) may be chosen without dependence on the vertex \( u \) (it will depend on \( p(\text{id}), p(s_1) \) and \( p(s_2) \)). Consequently, there is a \( \delta > 0 \) such that \( \sigma(u) \leq 1 - \delta \) for every \( u \). Thus, \( ||M|| \leq 1 - \delta \). \( \Box \)

**Lemma 3.5.** Let \( T \) be the Cayley tree associated to \( \Gamma \) as above and let \( d(\cdot, \cdot) \) denote its graph distance. For \( u, v \in \Gamma \) with \( v \neq \text{id} \),

\[
\max \{ d(\text{id}, uv), d(\text{id}, uv^{-1}) \} > d(\text{id}, u).
\]

**Proof.** First, observe that \( d(\text{id}, uv) = d(u, v^{-1}) \) and \( d(\text{id}, uv^{-1}) = d(u, v) \). Therefore, it suffices to prove that

\[
\max \{ d(u, v), d(u, v^{-1}) \} > d(u, \text{id}).
\]

For all vertices \( w_1, w_2 \in T \), let \( P(w_1, w_2) \) denote the path from \( w_1 \) to \( w_2 \) in \( T \). Let \( v' \) be the last common vertex of \( P(\text{id}, v) \) and \( P(\text{id}, v') \). Because \( v \) and \( v^{-1} \) are equidistant from \( \text{id} \), \( d(v', v) = d(v', v^{-1}) \). The key observation is that

\[
d(v', \text{id}) < d(v', v) = d(v', v^{-1}).
\]
The proof of this observation is as follows. Suppose $d(v', \text{id}) \geq d(v, \text{id})$. Then, $d(v', \text{id}) \geq \frac{1}{2} d(v, \text{id})$. So when $v$ and $v^{-1}$ are written as minimal length products of generators, say

$$v = v_1 v_2 \cdots v_n \text{ and } v^{-1} = v_n^{-1} v_{n-1}^{-1} \cdots v_1^{-1},$$

they agree in the first $\left\lceil \frac{n}{2} \right\rceil$ terms. However, this implies that they agree everywhere, i.e. $v = v^{-1}$. This is impossible in a free group unless $v = \text{id}$.

Now, view $T$ as a tree rooted at $v'$. Each child of $v'$ is the root of a subtree of $T$, and the vertices $v, v^{-1}$ lie in different subtrees. If $u$ is not in the same subtree as $v$ then $d(u, v) > d(u, \text{id})$. If $u$ is in the same subtree as $v$ then $d(u, v^{-1}) > d(u, \text{id})$. □

In light of Lemma 3.4, the following lemma implies

$$\sup_{k \geq 1} \frac{E - \log ||M_{2k}||(o, o)}{4k \Delta^{2d+2k}} > 0,$$

from which the proof of Theorem 3 follows by Lemma 3.3.

**Lemma 3.6.** Let $(G, o)$ be an infinite sophic graph such that for some $\ell$,

$$\Pr[o \text{ lies in an } \ell - \text{cycle of } G] > 0.$$

Then there is a deterministic integer $k$ such that, with positive probability, $(G, o)$ contains a bouquet with respect to the parameters $k$ and $\ell$.

**Proof.** Let $N_R(v)$ be the number of distinct $\ell$-cycles of a graph $H$ within distance $R$ of a vertex $v$. We will show below that for $(G, o)$,

$$E N_R(o) \geq (R/\ell) \Pr[o \text{ lies in an } \ell - \text{cycle of } G].$$

Assuming this, we may choose an $R$ in terms of $\ell$ such that $E N_R(o) \geq \ell \Delta^\ell + 2$.

In this case, with positive probability, there are at least $\ell \Delta^\ell + 2$ different $\ell$-cycles in $G$ within distance $R$ of the root $o$. Whenever this happens there must be two disjoint $\ell$-cycles within distance $R$ of the root. We may take $k$ to be the smallest even integer that is at least $R + \ell$.

The reason for the existence of disjoint $\ell$-cycles is as follows. If a graph $H$ has maximal degree $\Delta$, and $v$ is a vertex, then there can be at most $\Delta^\ell$ different $\ell$-cycles that pass through $v$. This means that any specific $\ell$-cycle can meet at most $\ell \Delta^\ell$ other $\ell$-cycles. So when there are $\ell \Delta^\ell + 2$ different $\ell$-cycles, some two must be disjoint.

In order to get the lower estimate on $E N_R(o)$ consider the function

$$F(H, u, v) = 1 \{\text{dist}(u, v) \leq R \text{ and } u \text{ lies in an } \ell - \text{cycle of } H\}.$$
Then,
\[ \sum_{u \in H} F(H, u, v) = \# \{ \text{vertices in } \ell \text{- cycles of } H \text{ within distance } R \text{ of } v \} \leq \ell N_R(v), \]
and
\[ \sum_{v \in H} F(H, u, v) = |B_R(H, u)| \mathbf{1}\{u \text{ lies in an } \ell \text{- cycle of } H\}. \]
Since \((G, o)\) is infinite almost surely, \(|B_R(G, o)| \geq R\). The mass transport principle then provides the lower bound on \(E N_R(o)\) as displayed above. □

4. A spectral gap theorem for finite graphs

In this section we prove Theorem 2. Let \(G\) be a finite and connected graph with universal cover \(T\) and cover map \(\pi\). Since \(\lambda_1(G)\) is also the largest eigenvalue of \(G\) in absolute value, we denote it \(\rho(G)\) henceforth.

For a graph \(H\) and \(x \in \ell^2(H)\), let
\begin{equation}
(4.1) \quad f_H(x) = 2 \sum_{\{u, v\} \in H} x_u x_v,
\end{equation}
where the summation is over the edges of \(H\) counted with multiplicity as there may be multi-edges and loops (recall a loop contributes degree 2 to its vertex). Thus,
\[ \rho(T) = \sup_{x \in \ell^2(T), ||x||=1} |f_T(x)| \text{ and } \rho(G) = \sup_{x \in \ell^2(G), ||x||=1} |f_G(x)|. \]

Theorem 2 follows from the Propositions 4.1 and 4.2 given below.

4.1. Spectral radius of an unicyclic graph.

**Proposition 4.1.** Let \(G\) be a finite and connected graph with at most one cycle. Then \(\rho(G) = \rho(T)\).

**Proof.** There is nothing to prove if \(G\) is a tree, so assume that \(G\) has exactly one cycle (possibly a loop, or a 2-cycle made by a pair of multi-edges). We give an explicit description of \(T\) in terms of \(G\).

Let the unique cycle in \(G\) consist of vertices \(v_1, \ldots, v_n\), in that order. Let \(H\) be the graph obtained by deleting edge \((v_n, v_1)\) from \(G\). We construct countably infinite copies \(\ldots, H_{-1}, H_0, H_1, \ldots\) of \(H\), indexed by \(\mathbb{Z}\). For each \(k \in \mathbb{Z}\), we draw an edge between \(v_n\) in \(H_k\) and \(v_1\) in \(H_{k+1}\). The resulting graph is \(T\).
Let \( y \in \ell^2(G) \) be the maximal eigenvector of \( G \), normalized to \( ||y|| = 1 \) and with positive entries. Thus, \( f_G(y) = \rho(G) \). We will construct an \( x \in \ell^2(T) \), with \( ||x|| = 1 \), such that \( f_T(x) \) approximates \( \rho(G) \) arbitrarily closely.

Fix an arbitrary \( N \in \mathbb{N} \). For \( v' \in H_1, \ldots, H_N \), set \( x_{v'} = \frac{1}{\sqrt{N}} y_{\pi(v')} \). For all other \( v' \in T \), set \( x_{v'} = 0 \).

It is evident that \( ||x|| = ||y|| = 1 \). Moreover,

\[
f_T(x) = 2 \sum_{(u',v') \in T} \frac{1}{N} y_{\pi(u')} y_{\pi(v')}.
\]

For each edge \((u, v) \in G\) the term \( \frac{1}{N} y_u y_v \) appears \( N \) times in the above sum, except for \( \frac{1}{N} y_{v_1} y_{v_n} \), which appears \( N - 1 \) times. Therefore,

\[
f_T(x) = 2 \sum_{(u, v) \in G} y_u y_v - \frac{2}{N} y_{v_1} y_{v_n} = \rho(G) - \frac{2}{N} y_{v_1} y_{v_n}.
\]

As \( N \) was arbitrary, the error term \( \frac{2}{N} y_{v_1} y_{v_n} \) can be made arbitrarily small. \( \square \)

4.2. Spectral gap for a multi-cyclic graph.

**Proposition 4.2.** Let \( G \) be a finite and connected graph with at least two cycles. Then \( \rho(T) < \rho(G) \).

For the remainder of this section we assume \( G \) is a finite and connected graph with at least two cycles (which may intersect, may be loops, or cycles made by multi-edges).

The 2-core of \( G \) is defined by the following procedure. If \( G \) has at least one leaf, pick an arbitrary leaf and delete it. This operation may produce more leaves. Repeat the leaf removal operation until there are no leaves. The resulting subgraph of \( G \) is its 2-core.

The 2-core of a graph is non-empty if and only if it contains a cycle. Moreover, all cycles are preserved in its 2-core. Consequently, since \( G \) has two distinct cycles, so does its 2-core.

Let \( G_{\text{int}} \) denote the 2-core of \( G \). Let \( V_{G_{\text{int}}} \) denote the vertices of \( G_{\text{int}} \). Let \( E_{G_{\text{int}}} \) be the edges of \( G_{\text{int}} \) directed both ways, so that every edge \( \{u, v\} \in G_{\text{int}} \) becomes two directed edges \((u, v)\) and \((v, u)\) in \( E_{G_{\text{int}}} \).

Denote by \( V_{G_{\text{ext}}} \) the vertices of \( G \setminus G_{\text{int}} \). Let \( E_{G_{\text{ext}}} \) be the edges of \( G \setminus G_{\text{int}} \) such that they are directed away from the 2-core. This is possible because for every edge \( \{u, v\} \) in \( G \setminus G_{\text{int}} \), there is a unique shortest path from \( G_{\text{int}} \) that terminates at \( \{u, v\} \). The
orientation of \{u, v\} is then in the direction this path enters the edge. The figure below gives an illustration of these definitions.

![Diagram](image)

**Figure 3.** An example illustrating the definitions of $V^G_{\text{int}}, V^G_{\text{ext}}, E^G_{\text{int}},$ and $E^G_{\text{ext}}$. $V^G_{\text{int}}$ and $E^G_{\text{int}}$ are coloured blue. $V^G_{\text{ext}}$ and $E^G_{\text{ext}}$ are coloured red.

**Lemma 4.1.** There exists a positive-valued function $\Gamma : E^G_{\text{int}} \to [1, 2)$ such that for each directed edge $(u, v) \in E^G_{\text{int}},$

$$\sum_{u: (v, w) \in E^G_{\text{int}}, w \neq u} \Gamma(v, w) > \Gamma(u, v).$$

Note $\Gamma$ is not symmetric, i.e., $\Gamma(u, v)$ need not equal $\Gamma(v, u)$.

**Proof.** Every vertex of $G_{\text{int}}$ has degree at least 2 within this subgraph. The following property of $G_{\text{int}}$ is crucial: since $G_{\text{int}}$ has at least two cycles and is connected, every cycle of $G_{\text{int}}$ contains a vertex of degree more than 2.

For each directed edge $(u, v) \in E^G_{\text{int}}$ with $\deg u > 2$, set $\Gamma(u, v) = 1$. The remaining values of $\Gamma(u, v)$ will correspond to directed edges $(u, v)$ with $\deg u = 2$. We assign these values by the following iterative procedure.

Fix an $\epsilon > 0$ to be determined. If $\deg u = 2$, $u$ is adjacent to $v_1$ and $v_2$, and $\Gamma(v_1, u)$ has been assigned, assign $\Gamma(u, v_2) = \Gamma(v_1, u) + \epsilon$. Due to the aforementioned crucial property, this procedure assigns a value of the form $1 + m\epsilon$ to every $\Gamma(u, v)$ with $(u, v) \in E^G_{\text{int}}$. Finally, since $G$ is finite we may choose $\epsilon$ small enough such that $\Gamma$ is strictly less than 2 everywhere.

Now if $(u, v) \in E^G_{\text{int}}$ and $\deg v > 2,$

$$\sum_{u: (v, w) \in E^G_{\text{int}}, w \neq u} \Gamma(v, w) \geq 2 > \Gamma(u, v).$$
If \((u, v) \in E_{\text{int}}^G\) and \(\deg v = 2\),
\[
\sum_{w: (v, w) \in E_{\text{int}}^G, w \neq u} \Gamma(v, w) = \Gamma(u, v) + \epsilon > \Gamma(u, v).
\]

\[\square\]

**Lemma 4.2.** There exists a positive-valued function \(\Delta : E_{\text{ext}}^G \rightarrow (0, 1]\) such that for each directed edge \((u, v) \in E_{\text{ext}}^G\),
\[
\sum_{w: (v, w) \in E_{\text{ext}}^G} \Delta(v, w) < \Delta(u, v).
\]

**Proof.** The edges in \(E_{\text{ext}}^G\) form trees, rooted at vertices in \(V_{\text{int}}^G\) and directed toward the leaves.

For each edge \((u, v) \in E_{\text{ext}}^G\), where \(u \in V_{\text{int}}^G\), set \(\Delta(u, v) = 1\). Assign the remaining variables by recursing down the trees in the following way. If \(\Delta(u, v)\) has been assigned and \(v\) has \(d\) out-edges \((v, w) \in E_{\text{ext}}^G\), set \(\Delta(v, w) = \frac{1}{d+1} \Delta(u, v)\) for each out-edge \((v, w)\). Then,
\[
\sum_{w: (v, w) \in E_{\text{ext}}^G} \Delta(v, w) = \frac{d}{d+1} \Delta(u, v) < \Delta(u, v),
\]
so the desired inequality holds. \(\square\)

**Proof of Proposition 4.2.** Let \(y\) be the eigenvector of the maximal eigenvalue of \(G\) chosen such that all its entries are positive and \(||y|| = 1\). Note this identity for every vertex \(u \in G\):
\[
(4.2) \quad \sum_{v: \{u, v\} \in G} \frac{y_v}{y_u} = \rho(G).
\]

Root \(T\) at any vertex \(r\) such that \(\pi(r) \in V_{\text{int}}^G\). For the rest of this proof, when we refer to an edge \((u, v) \in T\) the first vertex \(u\) is the parent, that is, closer to the root than \(v\).

Let \(V_{\text{int}}^T\) be the vertices in \(T\) with infinitely many descendants, and \(V_{\text{ext}}^T\) be the vertices in \(T\) with finitely many descendants. Let \(E_{\text{int}}^T\) denote the edges \((u, v) \in T\) with \(v \in V_{\text{int}}^T\), and \(E_{\text{ext}}^T\) the edges \((u, v) \in T\) with \(v \in V_{\text{ext}}^T\).

Observe that \(u \in V_{\text{int}}^T\) (resp. \(V_{\text{ext}}^T\)) if and only if \(\pi(u) \in V_{\text{int}}^G\) (resp. \(V_{\text{ext}}^G\)). Similarly, \((u, v) \in E_{\text{int}}^T\) if and only if \((\pi(u), \pi(v)) \in E_{\text{int}}^G\), and \((u, v) \in E_{\text{ext}}^T\) if and only if \((\pi(u), \pi(v)) \in E_{\text{ext}}^G\). In the latter case it is crucial that \(u\) is the parent of \(v\); this requires \(v\) to be farther than \(u\) from \(V_{\text{int}}^T\), so \(\pi(v)\) is farther than \(\pi(u)\) from \(V_{\text{int}}^G\). Thus \((\pi(u), \pi(v))\) has the necessary orientation of an edge in \(E_{\text{ext}}^G\).
Consider the functions $\Gamma$ and $\Delta$ from Lemmas 4.1 and 4.2. Let $\gamma, \delta > 0$ be (small) constants to be determined later. Throughout the following argument we will use that

$$2|ab| \leq \eta a^2 + \eta^{-1}b^2$$

for $\eta > 0$.

For each edge $(u, v) \in E^T_{\text{int}}$, we have

$$2|x_u x_v| \leq \frac{y_{\pi(v)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(u), \pi(v)) \gamma}{y_{\pi(u)} y_{\pi(v)}}\right)^{-1} x_u^2 + \frac{y_{\pi(u)}}{y_{\pi(v)}} \left(1 + \frac{\Gamma(\pi(u), \pi(v)) \gamma}{y_{\pi(u)} y_{\pi(v)}}\right) x_v^2.$$  

(4.3)

Analogously, for each edge $(u, v) \in E^T_{\text{ext}}$,

$$2|x_u x_v| \leq \frac{y_{\pi(v)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(v)) \delta}{y_{\pi(u)} y_{\pi(v)}}\right)^{-1} x_u^2 + \frac{y_{\pi(u)}}{y_{\pi(v)}} \left(1 + \frac{\Delta(\pi(u), \pi(v)) \delta}{y_{\pi(u)} y_{\pi(v)}}\right) x_v^2.$$  

(4.4)

The quantity $\Delta(\pi(u), \pi(v))$ is defined because $(\pi(u), \pi(v))$ has the correct orientation of an edge in $E^G$, as noted above.

Recall $f_T$ from [4.1]. The estimates (4.3) and (4.4) imply that

$$|f_T(x)| \leq 2 \sum_{\{u,v\} \in T} |x_u x_v| \leq \sum_{u \in T} g(u) x_u^2,$$  

(4.5)

where $g(u)$ is as follows. Let $\text{pa}(u)$ denote the parent of vertex $u \in T$ and $\text{ch}(u)$ denote the set of all children of $u$. If $u \in V^T_{\text{int}}$ then

$$g(u) = \frac{y_{\text{pa}(u)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\text{pa}(u), \pi(u)) \gamma}{y_{\text{pa}(u)} y_{\pi(u)}}\right) + \sum_{c \in \text{ch}(u)} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(c), \pi(c)) \gamma}{y_{\pi(u)} y_{\pi(c)}}\right)^{-1} + \sum_{d \in \text{ch}(u) \cap V^T_{\text{ext}}} \frac{y_{\pi(d)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(d)) \delta}{y_{\pi(u)} y_{\pi(d)}}\right).$$

If $u \in V^T_{\text{ext}}$ then

$$g(u) = \frac{y_{\text{pa}(u)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\text{pa}(u), \pi(u)) \delta}{y_{\text{pa}(u)} y_{\pi(u)}}\right)^{-1} + \sum_{d \in \text{ch}(u)} \frac{y_{\pi(d)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(d)) \delta}{y_{\pi(u)} y_{\pi(d)}}\right).$$

Due to (4.5), the proposition will be proved by showing that $g(u)$ is uniformly bounded away from $\rho(G)$ over all vertices $u$. We separately consider the two cases $u \in V^T_{\text{int}}$ and $u \in V^T_{\text{ext}}$.

Suppose $u \in V^T_{\text{int}}$. Then for all sufficiently small $\gamma > 0$ we have the bound

$$\sum_{c \in \text{ch}(u) \cap V^T_{\text{int}}} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(u), \pi(c)) \gamma}{y_{\pi(u)} y_{\pi(c)}}\right)^{-1} \leq \sum_{c \in \text{ch}(u) \cap V^T_{\text{int}}} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 - \frac{\Gamma(\pi(u), \pi(c)) \gamma}{y_{\pi(u)} y_{\pi(c)}}\right) + C_u \gamma^2,$$  

(4.6)
for some constant $C_u \geq 0$ depending on $u$.

The terms in (4.6) depend only on the vertices $\pi(u)$ and $\pi(c)$ for $c \in \text{ch}(u)$. These are vertices of $G$ and, since $G$ is finite, there are only finitely many distinct values of $C_u$. Let $C$ be the maximum of the $C_u$s. In the inequality (4.6) we may replace every $C_u$ by $C$, as we do henceforth.

Inequality (4.6) implies the following bound for every $u \in V_{\text{int}}^T$ and all sufficiently small $\gamma > 0$.

$$g(u) \leq \rho(G) + \frac{\gamma}{y_{\pi(u)}} \left[ \Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) \right] + \frac{\delta}{y_{\pi(u)}} \sum_{d \in \text{ch}(u) \cap V_{\text{ext}}^T} \Delta(\pi(u), \pi(d)) + C\gamma^2. \tag{4.7}$$

This is obtained by substituting (4.6) into the definition of $g(u)$, then multiplying out the terms and simplifying the sums by using the eigenvector equation (4.2).

By Lemma 4.1,

$$\Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) < 0$$

for every $u \in V_{\text{int}}^T$. Moreover, as $u$ ranges over $V_{\text{int}}^T$ the quantities

$$u \mapsto \frac{1}{y_{\pi(u)}^2} \left[ \Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) \right]$$

are determined by the graph $G$. So they attain finitely many values and have a maximum value $C_{\text{int}} < 0$. Analogously, the quantities

$$u \mapsto \frac{1}{y_{\pi(u)}^2} \sum_{d \in \text{ch}(u) \cap V_{\text{ext}}^T} \Delta(\pi(u), \pi(d))$$

have a maximum value $D_{\text{int}} \geq 0$ as $u$ ranges over $V_{\text{int}}^T$. So we infer that for every $u \in V_{\text{int}}^T$ and all sufficiently small $\gamma > 0$,

$$g(u) \leq \rho(G) + C_{\text{int}}\gamma + C\gamma^2 + D_{\text{int}}\delta. \tag{4.9}$$

Now suppose that $u \in V_{\text{ext}}^T$. By an analogous argument as above, there exists a constant $D \geq 0$ independently of $u$ such that for all sufficiently small $\delta > 0$,

$$g(u) \leq \rho(G) + \frac{\delta}{y_{\pi(u)}} \left[ -\Delta(\pi(\text{pa}(u)), \pi(u)) + \sum_{d \in \text{ch}(u)} \Delta(\pi(u), \pi(d)) \right] + D\delta^2.$$
By Lemma 4.2
\[-\Delta(\pi(pa(u)), \pi(u)) + \sum_{d \in \text{ch}(u)} \Delta(\pi(u), \pi(d)) < 0\]
for all $u \in V_T$. Therefore, as before, there is a $D_{\text{ext}} < 0$ such that for every $u \in V_{\text{ext}}^T$ and all sufficiently small $\delta > 0$,
\[(4.10) \quad g(u) \leq \rho(G) + D_{\text{ext}} \delta + D \delta^2.\]

Finally, we select $\gamma > 0$ small enough that (4.9) holds and $C_{\text{int}} \gamma + C \gamma^2 < 0$. This is possible because $C_{\text{int}} < 0$. Then we select $\delta > 0$ small enough such that (4.10) holds while both $C_{\text{int}} \gamma + C \gamma^2 + D_{\text{int}} \delta < 0$ and $D_{\text{ext}} \delta + D \delta^2 < 0$. This is possible due to the choice of $\gamma$ and because $D_{\text{ext}} < 0$.

In light of (4.9) and (4.10), our choice of $\gamma$ and $\delta$ above imply that there is an $\epsilon > 0$ such that for every vertex $u \in T$, $g(u) \leq \rho(G) - \epsilon$. This implies $\rho(T) < \rho(G)$. \(\square\)

5. Future directions

It would be interesting to find an effective version of Theorem 1 in the following sense. Let $G_1, G_2, G_3, \ldots$ be finite, connected graphs with $|G_n| \to \infty$. Suppose they have a common universal cover $T$ and are Ramanujan in the sense that all but their largest eigenvalue are at most $\rho(T)$ in absolute value. What is the “essential girth” of $G_n$ in terms of its size, meaning, the asymptotic girth of $G_n$ after possibly having removed at order of $o(|G_n|)$ edges? For $d$-regular Ramanujan graphs it is known that the essential girth is at least of order $\log \log |G|$ while known constructions provide graphs having girth of order $\log |G|$. It seems that a lower bound of order $\log \log |G|$ for the girth is unknown even for Cayley graphs that are Ramanujan.

It would also be interesting to find an effective form of Theorem 2 in terms of the size and the maximal degree of $G$. The theorem is in some ways an analogue of Theorem 3 for finite graphs, although, its word-for-word reformulation is false for infinite graphs. In this regard it would be interesting to prove a spectral gap between $\rho(G)$ and $\rho(T)$ under natural hypotheses on an infinite graph $G$. For instance, to prove an effective spectral gap when the $R$-neighbourhood of every vertex in $G$ contains a cycle.

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