The local (adjacency) metric dimension of split related complete graph

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Abstract. Let $G$ be a simple graph. A set of vertices, called $V(G)$ and a set of edges, called $E(G)$ are two sets which form graph $G$. $W$ is a local adjacency resolving set of $G$ if for every two distinct vertices $x, y$ and $x$ adjacent with $y$ then $rA(x|W) \neq rA(y|W)$. A minimum local adjacency resolving set in $G$ is called local adjacency metric basis. The cardinality of vertices in the basis is a local adjacency metric dimension of $G$ ($\text{dim}_{A,l}(G)$). We present the exact value of local adjacency metric dimension of $m$-splitting complete and bipartite graphs.

1. Introduction
This research in this paper uses simple and connected graphs. A set of vertices, called $V(G)$ and a set of edges, called $E(G)$ are two sets which form graph $G$. [4], [5], [6], [7], [2] A split graph is a graph derived by adding new vertex $v'$ in every vertex $v$ such that $v'$ adjacent to $v$ in graph $G$. An $m$-splitting graph is a graph which has $m$ $v'$-vertices, denoted by $m\text{-Spl}(G)$. [3] The local adjacency metric dimension is one of graph topic. Suppose there are three neighboring vertex $a, b, c$ in path $a - c$. Path $a - c$ is called local if $a, b, c$ where each has representation: $a$ is not equals $b$ and $a$ may equals $c$. [1] Let’s say, $x, y \in G$. For an order set of vertices $W = \{w_1, w_2, \ldots, w_k\}$, the adjacency representation of $v$ with respect to $W$ is the ordered $k$-tuple $rA(x|W) = (dA(x, w_1), dA(x, w_2), \ldots, dA(x, w_k))$, where $dA(x, w)$ represents the adjacency distance $x - w$. $dA(x, w)$ defined by $0$ if $x = w_1$, $1$ if $x$ adjacent with $w$, and $2$ if $x$ does not adjacent with $w$. $W$ is a local adjacency resolving set of $G$ if for every two distinct vertices $x, y$ and $x$ adjacent with $y$ then $rA(x|W) \neq rA(y|W)$. A minimum local adjacency resolving set in $G$ is called local adjacency metric basis. The cardinality of vertices in the basis is a local adjacency metric dimension of $G$ ($\text{dim}_{A,l}(G)$).

2. Result
2.1. $m$–Splitting of Complete Graph
A $m$–splitting of complete graph ($m\text{-Spl}(K_n)$) is a graph obtained from a complete graph ($K_n$) by adding new vertex $v'$ in every vertex $v$ as $n$ such that $v'$ adjacent $v$ in $K_n$. $m$–splitting graph is graph which has the number of vertex $v'$ as $m$. Let $G = m\text{-Spl}(K_n)$ with vertex set...
Figure 1. $\text{Spl}(K_4)$ Graph

$V(G) = \{u_1, u_2, \ldots, u_i\} \cup \{u^1_1, u^1_2, \ldots, u^k_i\}$, where $u_i$ is vertex of $K_n$ and $u^k_i$ is copy of vertex $u_i$ around $K_n$ for $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. We can see at 2.1 as illustration.

**Theorem 2.1:** Let $G$ be $m$–splitting of complete graph ($m\text{Spl}(K_n)$) with $|V(G)| = 2n$. For $n \geq 4$ and $m, n \in \mathbb{N}$, then $\dim_{A,l}(G) = n - 1$

**Proof 2.1** Choose $S = \{u_1, u_2, \ldots, u_{n-1}\} \subset V(G)$. We will show that $S$ is a local adjacency resolving set of $G$. The local adjacency representations of vertices from $V(G) - S$ are as follow:

- $r_A(u_i|S) = (11 \ldots 1)$
- $r_A(u^k_i|S) = (2111 \ldots 1)$
- $r_A(u^k_2|S) = (1211 \ldots 1)$
- $r_A(u^k_3|S) = (1121 \ldots 1)$
- $\vdots$
- $r_A(u^k_i|S) = (11 \ldots 112)$

As we see that all of the adjacency representations of adjacent vertices are distinct. So, $S = \{u_1, u_2, \ldots, u_{n-1}\}$ is a local adjacency resolving set for $G$. The cardinality of $S$, $|S| = n - 1$ is minimum, because if $|S| < n - 1$ certainly there are $a \neq b \in V(G) - S$ such that $r(a|S) = r(b|S)$.

Suppose $S_1 = \{u_1, u_2, \ldots, u_{n-2}\}$, $|S| = n - 2 < n - 1$. Then, $r_A(u_i|S) = (11 \ldots 1) = r_A(u_{i-1}|S)$ and $u_i \sim u_{i-1}$. Thus, $\dim_{A,l}(G) = n - 1$. $\square$

2.2. $m$–Splitting of Complete Bipartite Graph

A $m$–splitting of complete bipartite graph ($m\text{Spl}(K_{n,t})$) is a graph obtained from a complete bipartite graph ($K_{n,t}$) by adding new vertex $v'$ in every vertex $v$ as $n+t$ such that $v'$ adjacent $v$ in $K_{n,t}$. $m$–splitting graph is graph which has the number of vertex $v'$ as $m$. Let $G = m\text{Spl}(K_{n,t})$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_i\} \cup \{u^1_1, u^1_2, \ldots, u^k_i\}$, where $u_i$ is vertex of $K_{n,t}$ and $u^k_i$ is copy of vertex $u_i$ around $K_{n,t}$ for $i \in \{1, 2, \ldots, n + t\}$ and $k \in \{1, 2, \ldots, m\}$. We can see at 2.2, 2.2, and as illustration.
Figure 2. $\text{1Spl}(K_{2,2})$ Graph

Figure 3. $\text{1Spl}(K_{2,3})$ Graph

Theorem 2.2: Let $G$ be $m$–splitting of complete bipartite graph $(m\text{Spl}(K_{n,t}))$ with $|V(G)| = n + t$. For $n, t > 1$ and $n, t, m \in \mathbb{N}$, then $\text{dim}_{A,l}(G) = 1$

Proof 2.2 We divide the proof till some cases. We prove this theorem by see the construct of the based graph, complete bipartite graph $(K_{n,t})$ for $n, t > 1$ and $n, t, m \in \mathbb{N}$.

(i) Case 1. For $n = t$. Choose $S = \{a_1\} \subseteq V(G)$. We will show that $S$ is a local adjacency
resolving set of $G$. We know that $d$ is defined by

$$d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ adjacent with } w; \\
2 & \text{if } v \text{ does not adjacent with } w. 
\end{cases}$$

Suppose we call the "inside" vertices of $G$ is the set of vertices in $K_{n,t}$ and the "outside" vertices of $G$ is the set of vertices outside $K_{n,t}$ (or in $m$–split of $K_{n,t}$). Based on the construction of $K_{n,t}$, then there are three cases to prove the theorem, such that:

(a) When the resolving vertices set are inside the $G$. Choose resolving vertice of $S$ as much as $1$. Suppose we put any vertices of $S$ inside $G$. Based on the construction of $K_{n,t}$, every vertex in $(v_b^k)$ has neighbour as $(v_1)$. Suppose we have $(v_b)$ for $b \in \{1, 2, \ldots, n\}$ and $(v_c)$ for $c \in \{1, 2, \ldots, t\}$. When we put $v_1$ in $(v_b)$ then every vertex in $(v_c)$ and $(v_b^k)$ has same $r$ such that $1$. Otherwise, every vertex in $(v_b)$ and $(v_b^k)$ has same $r$ such that $2$ except $r(a_1) = 0$. But every vertex in $(v_c)$ or $(v_b)$ is not adjacent. Then it ensures that all vertices in $S$ are distinct.

(b) When the resolving vertices set are outside the $G$. Choose resolving vertices of $S$ as much as $1$. Suppose we put any vertices of $S$ outside $G$. Without loss the generality, let $j$ be even number of $N$. Let $v_{i+1}, v_{i+2}, \ldots, v_{i+j}$ in $S$. Then there must be minimum an outside vertex $(v_{i+1})$ adjacent to inside vertex $(v_i)$ which have same $r = (2)$.

Based on two points above, we focus in the first point of case. As we see that all of the adjacency representations of adjacency vertices are distinct. So, $S = \{a_1\}$ is a local adjacency resolving set for $G$. The cardinality of $S$, $|S| = 1$ is minimum. Thus, $\dim_{A,l}(G) = 1$ for $n = t$.

(ii) Case 2. For $n$ is odd and $r$ is even and otherwise. Choose $S = \{a_1\} \subseteq V(G)$. We will show that $S$ is a local adjacency resolving set of $G$. We know that $d$ is defined by

$$d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ adjacent with } w; \\
2 & \text{if } v \text{ does not adjacent with } w. 
\end{cases}$$

Suppose we call the "inside" vertices of $G$ is the set of vertices in $K_{n,t}$ and the "outside" vertices of $G$ is the set of vertices outside $K_{n,t}$ (or in $m$–split of $K_{n,t}$). Based on the construction of $K_{n,t}$, then there are three cases to prove the theorem, such that:

(a) When the resolving vertices set are inside the $G$. Choose resolving vertice of $S$ as much as $1$. Suppose we put any vertices of $S$ inside $G$. Based on the construction of $K_{n,t}$, every vertex in $(v_b^k)$ has neighbour as $(v_1)$. Suppose we have $(v_b)$ for $b \in \{1, 2, \ldots, n\}$ and $(v_c)$ for $c \in \{1, 2, \ldots, t\}$. When we put $a_1$ in $(v_b)$ then every vertex in $(v_c)$ and $(v_b^k)$ has same $r$ such that $1$. Otherwise, every vertex in $(v_b)$ and $(v_b^k)$ has same $r$ such that $2$ except $r(a_1) = 0$. But every vertex in $(v_c)$ or $(v_b)$ is not adjacent. Then it ensures that all vertices in $S$ are distinct.

(b) When the resolving vertices set are outside the $G$. Choose resolving vertices of $S$ as much as $1$. Suppose we put any vertices of $S$ outside $G$. Without loss the generality, let $j$ be even number of $N$. Let $v_{i+1}, v_{i+2}, \ldots, v_{i+j}$ in $S$. Then there must be minimum an outside vertex $(v_{i+1})$ adjacent to inside vertex $(v_i)$ which have same $r = (2)$.

Based on two points above, we focus in the first point of case. As we see that all of the adjacency representations of adjacency vertices are distinct. So, $S = \{a_1\}$ is a local adjacency resolving set for $G$. The cardinality of $S$, $|S| = 1$ is minimum. Thus, $\dim_{A,l}(G) = 1$ for $n$ is odd and $r$ is even and otherwise.
Figure 4. $\text{Spl}(K_{2,4})$ Graph

(iii) Case 3. For $n$ and $r$ are even or for $n$ and $r$ are odd. Choose $S = \{a_1\} \subseteq V(G)$. We will show that $S$ is a local adjacency resolving set of $G$. We know that $d$ is defined by

$$d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ adjacent with } w; \\
2 & \text{if } v \text{ does not adjacent with } w.
\end{cases}$$

Suppose we call the "inside" vertices of $G$ is the set of vertices in $K_{n,t}$ and the "outside" vertices of $G$ is the set of vertices outside $K_{n,t}$ (or in $m$—split of $K_{n,t}$). Based on the construction of $K_{n,t}$, then there are three cases to prove the theorem, such that:

(a) When the resolving vertices set are inside the $G$. Choose resolving vertex of $S$ as much as 1. Suppose we put any vertices of $S$ inside $G$. Based on the construction of $K_{n,t}$, every vertex in $(v_i^j)$ has neighbour as $(v_i)$. Suppose we have $(v_b)$ for $b \in \{1, 2, \ldots, n\}$ and $(v_c)$ for $c \in \{1, 2, \ldots, t\}$. When we put $a_1$ in $(v_b)$ then every vertex in $(v_c)$ and $(v_b)$ has same $r$ such that 1. Otherwise, every vertex in $(v_b)$ and $(v_b)$ has same $r$ such that 2 except $r(a_1) = 0$. But every vertex in $(v_c)$ or $(v_b)$ is not adjacent. Then it ensures that all vertices in $S$ are distinct.

(b) When the resolving vertices set are outside the $G$. Choose resolving vertices of $S$ as much as 1. Suppose we put any vertices of $S$ outside $G$. Without loss the generality, let $j$ be even number of $N$. Let $v_{i+1}^k, v_{i+2}^k, \ldots, v_{i+j}^k$ in $S$. Then there must be minimum an outside vertex $(v_{i+1}^k)$ adjacent to inside vertex $(v_i)$ which have same $r = (2)$.

Based on two points above, we focus in the first point of case. As we see that all of the adjacency representations of adjacency vertices are distinct. So, $S = \{a_1\}$ is a local adjacency resolving set of $G$. We know that $d$ is defined by

$$d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ adjacent with } w; \\
2 & \text{if } v \text{ does not adjacent with } w.
\end{cases}$$
resolving set for $G$. The cardinality of $S$, $|S| = 1$ is minimum. Thus, $\text{dim}_{A,l}(G) = 1$ for $n$ and $r$ are even or for $n$ and $r$ are odd.

\[ \square \]

3. Concluding Remark
We have discussed about the local adjacency metric dimension of some $m$ splitting related wheel graphs for several sets of value $(n, t, m)$ in this paper. Two basic theorems are about complete graph and complete bipartite graph which has any solutions for being a basic graph of operation $m$ splitting.

Open Problem
Find local adjacency metric of $m\text{Spl}(H_n)$ graph for any $n$ and $m$ where $H$ is any graph.

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