KERNELS FOR PRODUCTS OF HILBERT L-FUNCTIONS

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Abstract. We study kernel functions of \( L \)-functions and products of \( L \)-functions of Hilbert cusp forms over real quadratic fields. This extends the results on elliptic modular forms in [4] [5].

1. Introduction

One of the central problems in number theory is to explore the nature of special values of various Dirichlet series such as Riemann zeta function, modular \( L \)-functions, automorphic \( L \)-functions, etc. The known main idea to study arithmetic properties of the special values of modular \( L \)-functions is to compare such values with certain inner product of modular forms.

Such an idea was first introduced by Rankin [13], expressing the product of two critical \( L \)-values of an elliptic Hecke eigenform in terms of the Petersson scalar product of an elliptic Hecke eigenform with a product of Eisenstein series. Much later Zagier ([16], p 149) extended Rankin’s result to express the product of any two critical \( L \)-values of an elliptic Hecke eigenform in terms of the Petersson scalar product of the Hecke eigenform with the Rankin-Cohen brackets of two Eisenstein series. Shimura [14] and Manin [11] developed theories to study arithmetic properties of modular \( L \)-values on the critical strip. Kohnen-Zagier and Choie-Park-Zagier [10] [2] further studied the space of modular forms whose \( L \)-values on the critical strip are rational and showed that such a space can be spanned by Cohen kernel introduced by Cohen [3]. Recently double Eisenstein series has been introduced by Diamond and O’Sullivan [4] [5] as a kernel yielding products of two \( L \)-values of elliptic Hecke eigenforms. It turns out that Rankin-Cohen brackets [17] of two

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Eisenstein series can be realized as a double Eisenstein series [5]. Generalizing Cohen kernel, the arithmetic results of $L$-values by Manin [11] and Shimura [14] could be recovered [4, 5].

The purpose of this paper to state above results to the space of Hilbert modular forms by extending kernel functions introduced in [4, 5]. More precisely, a double Hilbert Eisenstein series is a kernel function of two $L$-values of a primitive form in terms of the Petersson scalar product. Also one can recover the arithmetic results [14] of $L$-values of Hilbert cusp forms by studying Cohen kernel over real quadratic fields. Furthermore it turns out that the Rankin-Cohen bracket of two Hilbert Eisenstein series is the special value of a double Hilbert Eisenstein series.

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2. Notations and Main Theorems

Throughout of this paper, for simplicity, we only consider the space of Hilbert modular forms over real quadratic fields $F$ with narrow class number one on the full Hilbert modular group $\Gamma = \text{SL}_2(\mathcal{O})$.

2.1. Notations. Let $F$ be a real quadratic field with narrow class number equal to 1. Let $D$, $\mathcal{O}$ and $\mathfrak{d}$ be the fundamental discriminant, the ring of integers and the different of $F$ respectively. Let $N$ and $\text{Tr}$ be the norm and the trace on $F$ defined by $N(a) = aa'$, $\text{Tr}(a) = a + a'$ with $a'$ the algebraic conjugate of $a \in F$. We denote $a \gg 0$ for $a \in F$ if $a$ is totally positive, that is $a > 0$ and $a' > 0$. For $B \subset F$, let $B_+$ denote the subset of totally positive elements in $B$. So $\mathcal{O}_+$ and $\mathcal{O}_+^\times$ denote the set of totally positive integers and the set of totally positive units respectively.
For a $2 \times 2$ matrix $\gamma$ in $GL_2^+(F)$, we usually denote its entries by $\gamma = \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a'_{\gamma} & b'_{\gamma} \\ c'_{\gamma} & d'_{\gamma} \end{pmatrix}$. The group $GL_2^+(F)$ acts on two copies of the complex upper half plane $\mathbb{H}^2$ by \( \gamma z := (\gamma z_1, \gamma' z_2) = \left( \frac{a_{\gamma} z_1 + b_{\gamma}}{c_{\gamma} z_2 + d_{\gamma}}, \frac{a'_{\gamma} z_2 + b'_{\gamma}}{c'_{\gamma} z_2 + d'_{\gamma}} \right) \) as linear fractional transformations for all $\gamma \in GL_2^+(F)$ and $z = (z_1, z_2) \in \mathbb{H}^2$.

Let $\Gamma = SL_2(\mathcal{O})$ be the modular group of $2 \times 2$ matrices with determinant equal to one over $\mathcal{O}$. Denote $\Gamma_\infty$ the subgroup of upper-triangular elements and $\Gamma_\infty^+$ the subgroup of elements with totally positive diagonal entries in $\Gamma_\infty$. Let $A$ denote the subgroup of diagonal elements in $\Gamma_\infty^+$, so $A = \{ \text{diag}(\varepsilon, \varepsilon^{-1}) : \varepsilon \in \mathcal{O}_\infty^+ \}$. Throughout the note, we employ the standard multi-index notation. In particular, for $\gamma \in GL_2^+(F)$, $z = (z_1, z_2) \in \mathbb{H}^2$ and $k \in \mathbb{Z}$, we denote $1 = (1, 1)$, $f(\gamma, z)^k = N(\gamma z)^k = (\gamma z_1)(\gamma' z_2)$, $|z| = (|z_1|, |z_2|)$, $|z|^k = |z_1|^k|z_2|^k$ and the automorphic factor by
\[
j(\gamma, z)^k = N(j(\gamma, z))^k = j(\gamma, z_1)^k j(\gamma', z_2)^k = (c_{\gamma} z_1 + d_{\gamma})^k (c'_{\gamma} z_2 + d'_{\gamma})^k.
\]

For any function $f$ on $\mathbb{H}^2$ and $\gamma \in GL_2^+(F)$, define the slash operator by
\[
(f|_k \gamma)(z) = N(\det(\gamma))^{\frac{k}{2}} N(j(\gamma, z))^{-k} f(\gamma z).
\]

A Hilbert modular form of (parallel) weight $k$ for $\Gamma$ is a holomorphic function $f$ on $\mathbb{H}^2$ such that $f|_k \gamma = f$ for any $\gamma \in \Gamma$. Then $f$ has the following Fourier expansion
\[
f(z) = a_f(0) + \sum_{\alpha \in \mathcal{O}_\infty^{-1}} a_f(\alpha) e^{2\pi i \alpha z}.
\]

If $a_f(0) = 0$, we call $f$ a Hilbert cusp form. For a Hilbert cusp form $f$ and a Hilbert modular form $g$ of weight $k$ on $\Gamma$, their Petersson scalar product is defined by
\[
\langle f, g \rangle := \int_{\mathcal{F}} f(z) \overline{g(z)} \, d\mu = \int_{\mathcal{F}} f(z) g(z) \, d\mu,
\]
where $\mathcal{F}$ is a fundamental domain of $\Gamma$ on $\mathbb{H}^2$ and
\[
d\mu = (y_1 y_2)^{-2} dx_1 dx_2 dy_1 dy_2 = N(y)^{-2} N(dx) N(dy).
\]
Here $z = x + iy$, $Re(z) = x = (x_1, x_2)$ and $Im(z) = y = (y_1, y_2)$. 
Note that this “unnormalized” Petersson inner product is different from Shimura’s \[14\]. For a Hilbert cusp form \(f\) of weight \(k\) for \(\Gamma\), define the associated \(L\)-function by

\[
L(f, s) = \sum_{\alpha \in \mathcal{O}\times} a_f(\alpha) N(\alpha) - s = \sum_{\alpha} a_f(\alpha) N(\alpha)^{-s},
\]

where \(a_f(\alpha) := a_f(\alpha)\) for \(\alpha \mathfrak{d} = \alpha\). It is known \[7\] that the complete \(L\)-function satisfies

\[
\Lambda(f, s) := D(2\pi)^{-2s} \Gamma(s)^2 L(f, s) = (-1)^k \Lambda(f, k - s)
\]

and has an analytic continuation to the entire \(\mathbb{C}\).

Next we recall the theory of Hecke operators on spaces of Hilbert modular forms. For each nonzero integral ideal \(\mathfrak{n}\) of \(\mathcal{O}\), let \(M_\mathfrak{n}\) be the set of \(2 \times 2\) matrices \(\gamma\) over \(\mathcal{O}\) such that \(\text{det}(\gamma) \gg 0\) and \((\text{det}(\gamma)) = \mathfrak{n}\). Moreover, let \(Z \cong \mathcal{O}\times\) denote the \(2 \times 2\) scalar matrices with diagonal entries in \(\mathcal{O}\times\). The \(\mathfrak{n}\)-th Hecke operator \(T_\mathfrak{n}\) on \(S_k(\Gamma)\), the space of cusp forms for \(\Gamma\) of parallel weight-\(k\), is defined as

\[
T_\mathfrak{n}(f(z)) = N(\mathfrak{n})^{k/2 - 1} \sum_{\gamma \in Z \setminus M_\mathfrak{n}} f|\gamma(z).
\]

The operators \(T_\mathfrak{n}\) are self-adjoint with respect to the Petersson inner product and generate a commutative algebra. It follows that there exists a basis \(H_k\), consisting of normalized cuspidal Hecke eigenforms, of \(S_k(\Gamma)\). We call elements in \(H_k\) “primitive forms”. Here \(f\) is normalized if the Fourier coefficient \(a_f(\mathcal{O}) = 1\) or equivalently if \(\mathfrak{d}^{-1} = (\alpha)\) with \(\alpha \gg 0\), then \(a_f(\alpha) = 1\). Therefore, for \(f \in H_k\), \(T_\mathfrak{n}f = a_f(\mathfrak{n})f\), so \(a_f(\mathfrak{n})\) is real. For details see Section 1.15 of \[7\].

### 2.2. Main Theorems.

Fix \(k \in \mathbb{Z}\). We define the Cohen kernel \(C_k^{Hil}(z; s)\) on \(\mathbb{H}^2 \times \mathbb{C}\) by

\[
C_k^{Hil}(z; s) = \frac{1}{2} c_{k, s, D}^{-2} \sum_{\gamma \in A \setminus \Gamma} (\gamma z)^{-s} j(\gamma, z)^{-k},
\]

with

\[
c_{k, s, D} = \frac{D^{k - 1} 2^{2-k} \pi \Gamma(k - 1)}{e^{\pi i} \Gamma(s) \Gamma(k - s)}
\]
and $A = \{ \text{diag}(\varepsilon, \varepsilon^{-1}) : \varepsilon \in \mathcal{O}_k^* \}$. Note that if $k$ is odd, this definition gives zero function.

**Theorem 2.1. (Cohen kernel)** Let $k \geq 4$ be even. Then the following hold:

1. $C_k^{\text{Hil}}(z; s)$ converges absolutely and uniformly on all compact subsets in the region given by

\[ 1 < \text{Re}(s) < k - 1, \quad z \in \mathbb{H}^2. \]

2. For each $s \in \mathbb{C}$,

\[ C_k^{\text{Hil}}(z; s) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f, k - s)}{\langle f, f \rangle} f(z), \]

where $\mathcal{H}_k$ is the set of primitive forms $s$ of weight $k$ on $\Gamma$.

3. $C_k^{\text{Hil}}(z; s)$ can be analytically continued to the whole $s$-plane and for each $s \in \mathbb{C}$, $C_k^{\text{Hil}}(z; s)$ is a cusp form for $\Gamma$ of weight $k$ in $z$.

Next we define the **double Eisenstein series** as follows: for $s, w \in \mathbb{C}, z \in \mathbb{H}^2$ and even integer $k \geq 6$,

\[ E_{s,k-s}^{\text{Hil}}(z; w) = \sum_{\gamma, \delta \in \Gamma \backslash \mathbb{H}^2} (c_{\gamma \delta^{-1}})^{(w-1)1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s1} j(\delta, z)^{-k1}, \]

and a completed double Eisenstein series by

\[ E_{s,k-s}^{*,\text{Hil}}(z; w) = 2\alpha_{k,s,w,D} \cdot E_{s,k-s}^{\text{Hil}}(z; w) \]

with

\[ \alpha_{k,s,w,D} := D^{k-w} \zeta_F(1 - w + s) \zeta_F(1 - w + k - s) \times \left( e^{\frac{\pi i}{2}} (2\pi)^{-k-2} \frac{\Gamma(s) \Gamma(k - s) \Gamma(k - w)}{\Gamma(k - 1)} \right)^2. \]

Then we have the following:

**Theorem 2.2. (double Eisenstein series)** Let $k \geq 6$ be even.
(1) $E_{s,k-s}^{\text{Hil}}(z;w)$ converges absolutely and uniformly on compact subsets in the region $\mathcal{R}$ of points $(z, (s, w))$ in $\mathbb{H}^2 \times \mathbb{C}^2$ subject to 
$$2 < \text{Re}(s) < k - 2, \text{Re}(w) < \min\{\text{Re}(s) - 1, k - 1 - \text{Re}(s)\}.$$ 

(2) $E_{s,k-s}^{*,\text{Hil}}(z;w)$ has an analytic continuation to all $s, w \in \mathbb{C}$ and is a Hilbert cusp form of weight $k$ on $\Gamma$ as a function in $z$.

(3) 
$$E_{s,k-s}^{*,\text{Hil}}(z;w) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f, s) \Lambda(f, w)}{\langle f, f \rangle} f,$$

where $\mathcal{H}_k$ is the set of primitive forms of weight $k$.

(4) For $f \in \mathcal{H}_k$, $\langle E_{s,k-s}^{*,\text{Hil}}(z;w), f \rangle = \Lambda(f, s) \Lambda(f, w)$, for all $s, w \in \mathbb{C}$.

(5) $E_{s,k-s}^{*,\text{Hil}}(z;w)$ satisfies functional equations: 
$$E_{s,k-s}^{*,\text{Hil}}(z;w) = E_{w,k-w}^{*,\text{Hil}}(z;s), \quad E_{k-s,s}^{*,\text{Hil}}(z;w) = E_{s,k-s}^{*,\text{Hil}}(z;w).$$

The following gives a relation between Rankin-Cohen brackets and a double Eisenstein series. Rankin-Cohen brackets on spaces of Hilbert modular forms have been studied in [H]. Let us recall the definition of Rankin-Cohen brackets: for each $j = 1, 2$, let $f_j : \mathbb{H}^2 \to \mathbb{C}$ be holomorphic, $k_j \in \mathbb{N}$ and $\ell = (\ell_1, \ell_2), \nu = (\nu_1, \nu_2) \in \mathbb{Z}_2 \geq 0$. Define the $\nu$-th Rankin-Cohen bracket

$$[f_1, f_2]_{\nu}^{\text{Hil}} = \sum_{0 \leq \ell_j \leq \nu_j, j = 1, 2} (-1)^{\ell_1+\ell_2} \left( \frac{k_1+\nu_1-1}{\nu_1-\ell} \right) \left( \frac{k_2+\nu_2-1}{\nu_2-\ell} \right) f_1^{(\ell)} f_2^{(\nu-\ell)}.$$

Here $f_1^{(\ell)}(z) = \left( \frac{\partial^{\ell_1+\ell_2}}{\partial z_1^{\ell_1} \partial z_2^{\ell_2}} f \right)(z)$ and $\left( \frac{k_1+\nu_1-1}{\nu_1-\ell} \right) = \left( \frac{k+\nu_1-1}{\nu_1-\ell} \right)$.

In the following, we only need parallel $\nu$, that is $\nu_1 = \nu_2$.

**Theorem 2.3. (Rankin-Cohen brackets and a double Eisenstein series)** For $\nu \in \mathbb{Z}_{\geq 0}$ and $k_j \in 2\mathbb{N}, j = 1, 2$, we have

$$\left( \frac{\Gamma(k_1) \Gamma(\nu+1)}{\Gamma(k_1+\nu)} \right)^2 [E_{k_1}, E_{k_2}]_{(\nu,\nu)}^{\text{Hil}} = 4 \left( \frac{\Gamma(k_2+\nu)}{\Gamma(k_2)} \right)^2 E_{k_1+\nu,k_2+\nu}^{\text{Hil}}(z;\nu+1),$$

where $E_k(z)$ is the usual Hilbert Eisenstein series of weight $k$ on $\Gamma$ defined by

$$E_k(z) := \sum_{\gamma \in \mathbb{H}\backslash \Gamma} j(\gamma, z)^{-k1}.$$
Remark 2.4. (1) Cohen kernel (see [3] and [10]) is an elliptic cusp form $R_n$ of weight $2k$ on $\text{SL}_2(\mathbb{Z})$ characterized by, for each $0 \leq n \leq 2k - 2$,

$$\langle f, R_n \rangle = n!(2\pi)^{-n-1}L(f, n + 1),$$

for all $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$.

Diamantis and O’Sullivan in [4] generalized Cohen kernel $C_{\text{ell}}(\tau, s)$ to get

$$\langle f, C_{\text{ell}}(\tau, s) \rangle = \Gamma(s)(2\pi)^s L(f, s), s \in \mathbb{C}.$$  

(2) Double Eisenstein series was introduced and studied in [4, 5] as a kernel yielding products of the periods of an elliptic Hecke eigenform at critical values as well as producing products of $L$-functions for Maass cusp forms.

In the following theorem, we recover Shimura’s result on the algebraicity of critical values of $L(f, s)$ (Theorem 4.3 of [14]). For a primitive form $f$ of even weight $k$, let $\mathbb{Q}(f)$ denote the number field generated by the Fourier coefficients of $f$ over $\mathbb{Q}$.

**Theorem 2.5. (rationality)** Let $f$ be a primitive form of even weight $k \geq 6$ for $\Gamma$. Then there exist complex numbers $\omega_\pm(f)$ with $\langle f, f \rangle = \omega_+(f)\omega_-(f)$ such that for even $m$ and odd $\ell$ with $1 \leq m, \ell \leq k - 1$,

1. $$\frac{\Lambda(f, m)}{\omega_+(f)}, \frac{\Lambda(f, \ell)}{\omega_-(f)} \in \mathbb{Q}(f),$$

2. for each $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\left( \frac{\Lambda(f, m)}{\omega_+(f)} \right)^\sigma = \frac{\Lambda(f^\sigma, m)}{\omega_+(f^\sigma)}, \quad \left( \frac{\Lambda(f, \ell)}{\omega_-(f)} \right)^\sigma = \frac{\Lambda(f^\sigma, \ell)}{\omega_-(f^\sigma)}.$$ 

Remark 2.6. (1) The above theorem is an analogous result of that for elliptic modular forms proved in [10] (Theorem in page 202). We can also extend the rationality easily to arbitrary $L$-values as did in Theorem 8.3 of [5].

(2) The above theorem is a special case of Shimura theorem (Theorem 4.3 in [14]) by taking $n = 2, \psi = 1$, and $k_1 = k_2 = k$. 


3. Proofs

We need the following multi variable Lipschitz summation formula.

**Lemma 3.1. (multi-variable Lipschitz summation formula)** Assume that $\text{Im}(s) > 2$. For $z \in \mathbb{H}^2$,

$$
\sum_{x \in \mathcal{O}} (z + x)^{-s} = \frac{(2\pi)^{2s}}{e^{\pi i \Gamma(s)^2 D^{1/2}}} \sum_{\xi \in \mathfrak{d}^{-1}} N(\xi)^{s-1} \exp(2\pi i \text{Tr}(\xi z)),
$$

**Proof.** By the multi-index notation,

$$
\sum_{x \in \mathcal{O}} (z + x)^{-s} = \sum_{x \in \mathcal{O}} N(z + x)^{-s} = \sum_{x \in \mathcal{O}} (z_1 + x)^{-s}(z_2 + x')^{-s}.
$$

Following [9], define

$$
f(x) = N(x)^{s-1} \exp(2\pi i \text{Tr}(xz))
$$

for $x = (x_1, x_2) \gg 0$ and 0 otherwise, so for $\text{Im}(s) > 2$ and $z \in \mathbb{H}^2$, $f$ is clearly $L^1$ on the quadratic space $V = \mathbb{R}^2$ with the trace form. The computation of [9 Theorem 1] shows that the Fourier transform $\hat{f}(w)$ is given by

$$
\hat{f}(w) = \frac{\Gamma(s)^2}{(-2\pi i)^{2s}} (z + w)^{-s}, \quad w \in \mathbb{R}^2.
$$

It is clear that for $x \in \mathbb{R}^2$,

$$
|f(x)| + |\hat{f}(x)| \ll (1 + ||x||)^{-2-\delta}
$$

for any positive $\delta$, where $|| \cdot ||$ is the Euclidean norm. Therefore, we may apply the Poisson summation formula (see page 252 of [15]), and for a general lattice $M$ in $V$ with integral dual lattice $M^\vee$, the Poisson summation formula reads

$$
\sum_{\alpha \in M} f(\alpha) = \sqrt{|M/M^\vee|} \sum_{\alpha \in M^\vee} \hat{f}(\alpha).
$$

Now set $M = \mathfrak{d}^{-1}$, then $M^\vee = \mathcal{O}$, $|M/M^\vee| = D$ and the Lipschitz summation formula follows easily. \qed

Now we prove Theorem 2.1 about Cohen kernel.

**Proof of Theorem 2.1:** To show the convergence, we follow the treatment of Section 1.15 in [7]. Firstly, we prove the uniform absolute convergence on compact subsets, using the fact that $L^1$-convergence implies uniform convergence.
on compact subset for series of holomorphic functions (See Lemma on Page 52 of \cite{7}). It suffices to treat the case for \( z \) in a small neighborhood \( U \) such that \( \mathcal{U} \) is compact, \( N(\text{Im}z) > X^{-1} \) and \( N(\text{Im}\gamma z) < X \) for any \( \gamma \in \Gamma \) and \( z \in U \) for fixed big \( X > 0 \). Note that this essentially picks a Siegel set where \( \mathcal{U} \) lives. In this case, we only have to prove that

\[
\int_{\Gamma \backslash \mathbb{H}^2_X} \sum_{\gamma \in A \backslash \Gamma} |N(j(\gamma, z))|^{-k} |\gamma z|^{-\sigma_1} d\mu(z) < \infty
\]

where \( \sigma = \text{Re}(s) \) and \( \mathbb{H}^2_X \) is the subset of \( z \) with \( N(\text{Im}z) < X \) in \( \mathbb{H}^2 \). Here we denote \( |z| = (|z_1|, |z_2|) \) and employ the multi-index notation. The left-hand side is bounded by

\[
\leq X^{\frac{k}{2}} \int_{\Gamma \backslash \mathbb{H}^2_X} \sum_{\gamma \in A \backslash \Gamma} (\text{Im} \gamma z)^{\frac{k}{2}} |\gamma z|^{-\sigma_1} d\mu(z)
\]

\[
\ll \sum_{\gamma \in A \backslash \Gamma} \int_{\Gamma \backslash \mathbb{H}^2_X} (\text{Im} \gamma z)^{\frac{k}{2}} |\gamma z|^{-\sigma_1} d\mu(z)
\]

\[
= \int_{A \backslash \mathbb{H}^2_X} (\text{Im} z)^{\frac{k}{2}} |z|^{-\sigma_1} d\mu(z).
\]

The space \( A \backslash \mathbb{H}^2_X \) can be viewed as a subspace of

\[
\{(z_1, z_2) : N(\text{Im}z) < X, Y^{-1} \leq y_1/y_2 \leq Y\}
\]

for some positive \( Y \) (\( Y \) can be chosen as the smallest totally positive unit bigger than 1). Moreover, that \( N(\text{Im}(-1/z)) < X \) implies \( N(|z|^2) > X^{-1} N(\text{Im}z) \). For \( 1 < r < \sigma < k - 1 \), the last quantity is equal to

\[
\int_{A \backslash \mathbb{H}^2_X} (N(\text{Im}z))^{\frac{k}{2}} |Nz|^{-r-(\sigma-r)} d\mu(z)
\]

\[
\ll \int_{A \backslash \mathbb{H}^2_X} (N(\text{Im}z))^{\frac{k-\sigma+r}{2}} |Nz|^{-r} d\mu(z)
\]

\[
\ll \int_{y_1 y_2 < X, Y^{-1} < y_1/y_2 < Y} (N(\text{Im}z))^{\frac{k-\sigma+r}{2}} (N(\text{Im}z))^{1-r} \frac{dy_1 dy_2}{(y_1 y_2)^2}
\]

\[
= \int_{y_1 y_2 < X, Y^{-1} < y_1/y_2 < Y} (N(\text{Im}z))^{\frac{k-\sigma+r}{2}} dy_1 dy_2 < \infty
\]

where in the third line we applied Equation (5.8) of \cite{4} for the integration on \( x \). This is part (1).
For part (2), first note that the absolutely uniformly convergence implies that $C_{k}^{Hil}(z; s)$ converges to a Hilbert modular form in the strip $2 < \sigma < k - 1$ since $C_{k}^{Hil}(z; s)$ is $\Gamma$-invariant with a proper automorphic factor. Secondly, we write

$$2c_{k,s,D}^{2} \cdot C_{k}^{Hil}(z; s) = \sum_{\alpha \in A \setminus \Gamma_{\infty}^{+} \setminus \Gamma} \sum_{\gamma \in \Gamma_{\infty}^{+} \setminus \Gamma} j(\alpha \gamma, z)^{-k1}(\alpha \gamma z)^{-s1}$$

$$= \sum_{\gamma \in \Gamma_{\infty}^{+} \setminus \Gamma} j(\gamma, z)^{-k1} \sum_{\alpha \in A \setminus \Gamma_{\infty}^{+}} (\alpha \gamma z)^{-s1} = \sum_{\gamma \in \Gamma_{\infty}^{+} \setminus \Gamma} j(\gamma, z)^{-k1} \sum_{x \in O} (\gamma z + x)^{-s1}.$$

Applying the Lipschitz summation formula in Lemma 3.1 with $2 < \sigma < k - 1$, we have

$$2c_{k,s,D}^{2} \cdot C_{k}^{Hil}(z; s) = \frac{(2\pi)^{2s}}{e^{\pi i s}(s)2D^{1/2}} \sum_{\gamma \in \Gamma_{\infty}^{+} \setminus \Gamma} j(\gamma, z)^{-k1} \sum_{\xi \in \Theta_{+}^{-1}} (N\xi)^{s-1} \exp(2\pi i Tr(\xi \gamma z))$$

$$= \frac{(2\pi)^{2s}}{e^{\pi i s}(s)2D^{1/2}} \sum_{\xi \in \Theta_{+}^{-1}} (N\xi)^{s-1} \sum_{\gamma \in \Gamma_{\infty}^{+} \setminus \Gamma} j(\gamma, z)^{-k1} \exp(2\pi i Tr(\xi \gamma z))$$

$$= \frac{(2\pi)^{2s}}{e^{\pi i s}(s)2D^{1/2}} \sum_{\xi \in \Theta_{+}^{-1}/(G_{\infty}^{+})^{2}} (N\xi)^{s-1} \sum_{\gamma \in U \setminus \Gamma} j(\gamma, z)^{-k1} \exp(2\pi i Tr(\xi \gamma z))$$

$$= \frac{(2\pi)^{2s}}{e^{\pi i s}(s)2D^{1/2}} \sum_{\xi \in \Theta_{+}^{-1}/(G_{+}^{2})^{2}} (N\xi)^{s-1} \sum_{\gamma \in U \setminus \Gamma} j(\gamma, z)^{-k1} \exp(2\pi i Tr(\xi \gamma z)),$$

where $U$ is the subgroup of elements of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in $\Gamma$. On the other hand, recall the $\xi$-th Poincaré series [7]

$$P_{k}(z; \xi) = \sum_{\gamma \in U \setminus \Gamma} j(\gamma, z)^{-k1} \exp(2\pi i Tr(\xi \gamma z))$$

and that it is a cusp form with

$$P_{k}(z; \xi) = \frac{\Gamma(k - 1)^{2} D^{1/2}}{(4\pi)^{2k-2} N(\xi)^{k-1}} \sum_{f \in H_{k}} a_{f}(\xi) \frac{f}{(f, f)}.$$
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We see that up to a constant factor (depending on $s$) $C^H_{k}(z; s)$ is equal to

$$
\sum_{\xi \in \mathcal{O}_+^{-1}/(\mathcal{O}_+^\times)^2} (N\xi)^{s-1}(N\xi)^{1-k} \sum_{f \in \mathcal{H}_k} \bar{a}_f(\xi)f(z) \langle f, f \rangle
$$

$$
= \sum_{\xi \in \mathcal{O}_+^{-1}/(\mathcal{O}_+^\times)^2} (N\xi)^{s-k} \sum_{f \in \mathcal{H}_k} \bar{a}_f(\xi)f(z) \langle f, f \rangle
$$

$$
= 2 \sum_{\xi \in \mathcal{O}_+^{-1}/(\mathcal{O}_+^\times)} (N\xi)^{s-k} \sum_{f \in \mathcal{H}_k} \bar{a}_f(\xi)f(z) \langle f, f \rangle
$$

$$
= 2D^{k-s} \sum_{f \in \mathcal{H}_k} \frac{f(z)}{\langle f, f \rangle} \sum_{\xi \in \mathcal{O}_+^{-1}/(\mathcal{O}_+^\times)} (N\xi)^{s-k} \bar{a}_f(\xi)
$$

$$
= 2D^{k-s} \sum_{f \in \mathcal{H}_k} \frac{f(z)\Lambda(f,k-s)}{\langle f, f \rangle},
$$

where we used the fact that $a_f(\xi)$ is real. Putting everything together, we see that

$$
2c^2_{k,s,D} \cdot C^H_{k}(z; s) = \frac{2^{5-2k}\pi^2\Gamma(k-1)^2}{\pi is\Gamma(s)^2\Gamma(k-s)^2} \sum_{f \in \mathcal{H}_k} \Lambda(f,k-s)f(z) \langle f, f \rangle
$$

It follows that $C^H_{k}(z; s)$ is cuspidal on the region $2 < \sigma < k - 1$, and that

$$
C^H_{k}(z; s) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f,k-s)f(z)}{\langle f, f \rangle}.
$$

For part (3): The expression of $C^H_{k}(z; s)$ in part (2) gives the analytic continuation to $s \in \mathbb{C}$ and that for each $s \in \mathbb{C}$, $C^H_{k}(z; s)$ is a cusp form. This completes the proof. □

Next, to prove Theorem 2.2 we first need to show a connection between Cohen kernel and double Eisenstein series, which is obtained in the following lemma:

**Lemma 3.2.** On the region $\mathcal{R}$, we have

$$
\zeta_F(1-w+s)\zeta_F(1-w+k-s)E^H_{s,k-s}(z; w) = 2c^2_{k,s,D} \sum_{n} N(n)^{w-k}T_n(c^H_{k}(z; s)),
$$

with $T_n$ the $n$-th Hecke operator and $\zeta_F(s)$ the Dedekind zeta function for $F$ defined as

$$\zeta_F(s) = \sum_a N(a)^{-s} = \sum_{a \in \mathcal{O}_F / \mathcal{O}_F^\times} N(a)^{-s},$$

where $a$ runs through all integral nonzero ideals.

**Proof.** On $\mathcal{R}$, the series expansions of the two $\zeta_F$-factors converge absolutely. Therefore, on $\mathcal{R}$, by sending $\gamma$ to $(c, d)$, the left-hand side is equal to

$$\zeta_F(1 - w + s) \zeta_F(1 - w + k - s) E_{s,k-s}(z; w)$$

$$= \sum_{u, \tilde{u}} N(u)^{w-1-s} N(\tilde{u})^{w+s-1-k} \sum_{(c,d),(\tilde{c},\tilde{d})} (c\tilde{d} - d\tilde{c})^{(w-1)1} \left( \frac{cz + d}{\tilde{c}z + \tilde{d}} \right)^{-s1} (\tilde{c}z + \tilde{d})^{-k1},$$

where $u, \tilde{u} \in \mathcal{O}_F \setminus \mathcal{O}_F$ and $(c, d), (\tilde{c}, \tilde{d}) \in \mathcal{O}_F \setminus \mathcal{O}^2$ such that $\mathcal{O}c + \mathcal{O}d = \mathcal{O}\tilde{c} + \mathcal{O}\tilde{d} = \mathcal{O}$ and $c\tilde{d} - d\tilde{c} \gg 0$. Combining the two summations, we have

$$\sum_a \sum_{(c,d),(\tilde{c},\tilde{d})} (c\tilde{d} - d\tilde{c})^{(w-1)1} \left( \frac{cz + d}{\tilde{c}z + \tilde{d}} \right)^{-s1} (\tilde{c}z + \tilde{d})^{-k1},$$

where this time $a, \tilde{a}$ are over all nonzero integral ideals and the inner summation is over $(c, d), (\tilde{c}, \tilde{d}) \in \mathcal{O}_F \setminus \mathcal{O}^2$ such that $\mathcal{O}c + \mathcal{O}d = a, \mathcal{O}\tilde{c} + \mathcal{O}\tilde{d} = \tilde{a}$ and $c\tilde{d} - d\tilde{c} \gg 0$. Then we can remove the summation over $a, \tilde{a}$ and it equals to

$$\sum_{(c,d),(\tilde{c},\tilde{d})} (c\tilde{d} - d\tilde{c})^{(w-1)1} \left( \frac{cz + d}{\tilde{c}z + \tilde{d}} \right)^{-s1} (\tilde{c}z + \tilde{d})^{-k1}$$

$$= \sum_n \sum_{(c,d),(\tilde{c},\tilde{d})} (c\tilde{d} - d\tilde{c})^{(w-1)1} \left( \frac{cz + d}{\tilde{c}z + \tilde{d}} \right)^{-s1} (\tilde{c}z + \tilde{d})^{-k1},$$

where $n$ is over all nonzero integral ideals and the inner summation is over $(c, d), (\tilde{c}, \tilde{d}) \in \mathcal{O}_F \setminus \mathcal{O}^2$ such that $c\tilde{d} - d\tilde{c} \gg 0$ and $(c\tilde{d} - d\tilde{c}) = n$. Note that the two summations over $(c, d), (\tilde{c}, \tilde{d})$ in the preceding equation have different ranges.

Let $\mathbf{A}$ denote the group of diagonal $2 \times 2$ matrices with entries in $\mathcal{O}_F^\times$, so clearly $\mathbf{A} \subset Z\Gamma$ and $\mathbf{A} \setminus Z\Gamma \cong A \setminus \Gamma$. Note that the inner summation set is
mapped bijectively to \( \tilde{A} \setminus M_{n} \) via

\[
((c, d), (\tilde{c}, \tilde{d})) \mapsto \begin{pmatrix} c & d \\ \tilde{c} & \tilde{d} \end{pmatrix}.
\]

Therefore, above expression is equal to

\[
\sum_{n} \sum_{\gamma \in \tilde{A} \setminus M_{n}} (\det(\gamma))^{(w-1)1} (\gamma z)^{-s} j(\gamma, z)^{-k1}
\]

\[
= \sum_{n} \sum_{\gamma \in \tilde{A} \setminus M_{n}} (\det(\beta\gamma))^{(w-1)1} (\beta\gamma z)^{-s} j(\beta\gamma, z)^{-k1}
\]

\[
= \sum_{n} \sum_{\gamma \in \tilde{A} \setminus M_{n}} (\det(\gamma))^{(w-1)1} (\beta\gamma z)^{-s} j(\beta\gamma, z)^{-k1}
\]

\[
= 2c_{k,s,D}^{2} \cdot \sum_{n} N(n)^{-\frac{k}{2}+w-1} \sum_{\gamma \in \tilde{A} \setminus M_{n}} C_{k}^{Hil}(z; s) |k\gamma|
\]

\[
= 2c_{k,s,D}^{2} \cdot \sum_{n} N(n)^{w-k} T_{n}(C_{k}^{Hil}(z; s)),
\]

which is the right-hand side. \( \square \)

Using the preceding lemma we prove the following main theorem:

**Proof of Theorem 2.2**: For part (1), apply the proof of Lemma 4.1 in [5] for each component and we have

\[
N(c_{\gamma\delta}^{-1}) \leq N(\Im(\gamma z))^{-1/2} N(\Im(\delta z))^{-1/2},
\]

for any \( \gamma, \delta \in \Gamma \) with \( c_{\gamma\delta}^{-1} \gg 0 \). Let \( r = \max\{\Re(w), 1\} \). Since \( [\Gamma_{\infty} : \Gamma_{\infty}^{\circ}] \) is finite, \( E_{s,k-s}(z; w) \) is absolutely bounded up to a constant by

\[
\sum_{a, \delta \in \Gamma_{\infty}^{+} \setminus \Gamma, c_{\gamma\delta}^{-1} \gtrless 0} (Nc_{\gamma\delta}^{-1})^{\Re(w)-1} |Nj(\gamma, z)|^{-\Re(s)} |Nj(\delta, z)|^{\Re(s)-k}
\]

\[
\leq \sum_{a, \delta \in \Gamma_{\infty}^{+} \setminus \Gamma, c_{\gamma\delta}^{-1} \gtrless 0} N(\Im(\gamma z))^{\frac{1-r}{2}} N(\Im(\delta z))^{\frac{1-r}{2}} |Nj(\gamma, z)|^{-\Re(s)} |Nj(\delta, z)|^{\Re(s)-k}
\]

\[
\leq \sum_{a, \delta \in \Gamma_{\infty}^{+} \setminus \Gamma, c_{\gamma\delta}^{-1} \neq 0} N(\Im(\gamma z))^{\frac{1-r}{2}} N(\Im(\delta z))^{\frac{1-r}{2}} |Nj(\gamma, z)|^{-\Re(s)} |Nj(\delta, z)|^{\Re(s)-k}
\]

\[
\ll \sum_{a, \delta \in \Gamma_{\infty} \setminus \Gamma, c_{\gamma\delta}^{-1} \neq 0} N(\Im(\gamma z))^{\frac{1-r}{2}} N(\Im(\delta z))^{\frac{1-r}{2}} |Nj(\gamma, z)|^{-\Re(s)} |Nj(\delta, z)|^{\Re(s)-k}
\]
which is the product of two Eisenstein series whose absolute convergence is well-known (see, for example, 5.7 Lemma of Chapter I in [6]). So absolute convergence follows if we have

$$\frac{\text{Re}(s) - r + 1}{2} > 1 \quad \text{and} \quad \frac{k - \text{Re}(s) - r + 1}{2} > 1.$$ 

One sees easily that \(E_{s,k-s}(z; w)\) transforms correctly under \(\Gamma\). In the above estimate

$$\sum_{\gamma, \delta \in \Gamma_{\infty} \setminus \Gamma; c_{\gamma, \delta} - 1 \neq 0} \frac{\text{N}(\text{Im}(\gamma z))}{2} \frac{\text{N}(\text{Im}(\delta z))}{2} \frac{\text{N}(\text{Im}(\gamma z))}{2} \frac{\text{N}(\text{Im}(\delta z))}{2},$$

where \(E(z, s) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} N(\text{Im}(\gamma z))^{-s} = N(y)^s + A(s) N(y)^{1-s} + o(1)\). By removing the highest terms \(N(y)^{\frac{k}{2}-r+1}\) from the difference, the rest are all \(o(N(y)^\frac{k}{2})\). This shows that \(E_{s,k-s}(z; w) \to 0\) as \(N(y) \to \infty\), and hence proves part (2) that \(E_{s,k-s}(z; w)\) is a cuspform since only one cusp exists.

For part (3), by Theorem 2.1 the Cohen kernel are cuspforms. By Lemma 3.2

$$\zeta_F(1 - w + s)\zeta_F(1 - w + k - s)E^{Hil}_{s,k-s}(z; w)$$

$$=2c_{k,s,D}^2 \sum_n N(n)^{w-k} T_n \left( C^H_{k-s}(z; s) \right)$$

$$=2c_{k,s,D}^2 \sum_n N(n)^{w-k} \sum_{f \in \mathcal{H}_k} \frac{\langle T_n C^H_{k-s}(z; s), f \rangle}{\langle f, f \rangle} f(z)$$

$$=2c_{k,s,D}^2 \sum_n N(n)^{w-k} \sum_{f \in \mathcal{H}_k} \frac{\langle C^H_{k-s}(z; s), T_n f \rangle}{\langle f, f \rangle} f(z)$$

$$=2c_{k,s,D}^2 \sum_n N(n)^{w-k} \sum_{f \in \mathcal{H}_k} a_f(n) \frac{\langle C^H_{k-s}(z; s), f \rangle}{\langle f, f \rangle} f(z),$$
since \( \alpha_f(n) = n_f(n) \). We have shown in Theorem 2.1 that
\[
2c_{k,s,D}^2\left< \mathcal{O}_k^{Hil}(z,s), f \right> = \frac{2^{5-2k}\pi^2\Gamma(k-1)^2}{e^{i\pi s}\Gamma(s)^2\Gamma(k-s)^2}\Lambda(f,k-s), \text{ for } f \in \mathcal{H}_k.
\]

By defining
\[
E_{s,k-s}^{*,Hil}(z,w) = 2\alpha_{k,s,w,D}E_{s,k-s}^{Hil}(z,w)
\]
with \( \alpha_{k,s,w,D} \) in (2.2) and using the result of Theorem 2.1, we obtain
\[
E_{s,k-s}^{*,Hil}(z,w) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f,k-w)\Lambda(f,k-s)}{\langle f,f \rangle}f(z) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f,w)\Lambda(f,s)}{\langle f,f \rangle}f(z).
\]

Part (4) follows easily from part (3) since \( f \) is a primitive form. Finally, by part (3), \( E_{s,k-s}^{*,Hil}(z,w) \) has meromorphic continuation to all of \( s,w \in \mathbb{C} \), and reflected from properties of \( \Lambda(f,s) \), it satisfies functional equations
\[
E_{s,k-s}^{*,Hil}(z,w) = E_{w,k-s}^{*,Hil}(z,s), \quad E_{k,-s,s}^{*,Hil}(z,w) = E_{s,k-s}^{*,Hil}(z,w),
\]
proving part (5) and hence the whole theorem. \( \square \)

Using the result about Rankin-Cohen brackets studied in [1], we prove Theorem 2.3:

**Proof of Theorem 2.3** One checks (from Proposition 1 in [1])
\[
\left( \frac{(k_1-1)!\nu!}{(k_1+\nu-1)!} \right)^2 [E_{k_1}, E_{k_2}]_{Hil}^{\nu,\nu} = \sum_{\delta \in \Gamma_\infty \setminus \Gamma} j(\delta, z)^{-k_1}E_{k_2}^{(\nu)}|_{k_2+2\nu} \delta.
\]

Since
\[
E_{k_2}^{(\nu)} = \left( \frac{(k_2-1+\nu)!}{(k_2-1)!} \right)^2 \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} N(c_\gamma)^\nu j(\gamma, z)^{-(k_2+\nu)}1,
\]
by Lemma 1 in [1], this in turn is equal to
\[
\left( \frac{(k_2-1+\nu)!}{(k_2-1)!} \right)^2 \sum_{\delta,\gamma \in \Gamma_\infty \setminus \Gamma} j(\delta, z)^{-k_1}1N(c_\gamma)^\nu j(\gamma, \delta(\gamma))^{-(k_2+\nu)}1 j(\delta, z)^{-(k_2+2\nu)}1
\]
\[
= \left( \frac{(k_2-1+\nu)!}{(k_2-1)!} \right)^2 \sum_{\delta,\gamma \in \Gamma_\infty \setminus \Gamma} N(c_{\gamma\delta-1})^\nu j(\delta, z)^{-(k_1+\nu)}1 j(\gamma, z)^{-(k_2+\nu)}1.
\]

In such a particular situation, we see easily that the summand is actually well-defined on \( \Gamma_\infty \setminus \Gamma \). Denote \( S \) the subset of \( (\delta, \gamma) \in (\Gamma_\infty^+ \setminus \Gamma)^2 \) with \( c_{\gamma\delta-1} \neq 0 \) and
$S_{\pm, \pm} \subset S$ consists of elements whose $c_{\gamma \delta-1}$ has the prescribed sign vector. In particular, $S_{+, +}$ consists of elements with $c_{\gamma \delta-1} \gg 0$. It is obvious that the sums over these four subsets are all equal, since we may multiply on left by ±1 and ±diag$(\varepsilon_0, \varepsilon_0^{-1}) \in \Gamma_\infty$ to adjust the signs; here $\varepsilon_0$ is the fundamental unit. That said, we have

$$\left( \frac{(k_1-1)!\nu!}{(k_1+\nu-1)!} \right)^2 [E_{k_1}, E_{k_2}]_{(\nu, \nu)}^{Hil} = 4 \left( \frac{(k_2-1+\nu)!}{(k_2-1)!} \right)^2 E_{k_1+\nu, k_2+\nu}^{Hil}(z, \nu + 1),$$

and it finishes the proof.

**Proof of Theorem 2.5** We follow the lines in Section 8A of [5] and first prove that for even $m$ and odd $\ell$ with $1 \leq m, \ell \leq k-1$, both of $E_{m, k-m}^{\ast, \ast}(z; k-1)$ and $E_{k-2, 2}^{\ast, \ast}(z; \ell)$ have rational Fourier coefficients. By the functional equations in Theorem 2.2,

$$E_{m, k-m}^{\ast, \ast}(z; k-1) = E_{m, k-m}^{\ast, \ast}(z; 1),$$

and it suffices to prove that the Fourier coefficients of $E_{m, k-m}^{\ast, \ast}(z; \ell)$ are rational for even $m$ and odd $\ell$ with $1 \leq \ell < m \leq k/2$. By Theorem 2.3, $E_{m, k-m}^{\ast, \ast}(z; \ell) = C[E_{m+1-\ell, k+1-m-\ell}^{\ast, \ast}]_{\ell-1}$, where $C$ is a rational multiple of $\pi^{2-2\ell}$ by Theorem 9.8 on page 515 of [12]. It follows that the Fourier coefficients of $E_{m, k-m}^{\ast, \ast}(z; \ell)$ belong to $\mathbb{Q}$.

Next, for primitive $f \in \mathcal{H}_k$, by Proposition 4.15 of [14] and Theorem 2.2, we have $(f, E_{k-1, 2}^{\ast, \ast}(z; k-1)) = \alpha_f(f, f) = \Lambda(f, k-1)\Lambda(f, k-2)$, for certain $\alpha_f \in \mathbb{Q}(f)$. Again by Proposition 4.15, since $E_{k-1, 2}^{\ast, \ast}(z; k-1)$ has rational Fourier coefficients, $\alpha_f^\sigma = \alpha_f$ for each $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$. Also note $\alpha_f \neq 0$ because of the convergence of the Euler product of $\Lambda(f, s)$ for Re$(s) \geq k/2+1$ (see Kim-Sarnak’s bound in [8]). Define

$$\omega_+(f) = \frac{\alpha_f(f, f)}{\Lambda(f, k-1)}, \quad \omega_-(f) = \frac{\langle f, f \rangle}{\Lambda(f, k-2)}.$$

Then for even $m$, odd $\ell$ with $1 \leq m, \ell < k-1$,

$$\frac{\Lambda(f, m)}{\omega_+(f)} = \frac{\langle f, E_{m, k-m}^{\ast, \ast}(z; k-1) \rangle}{\alpha_f(f, f)} \in \mathbb{Q}(f)$$

again by Proposition 4.15 of [14] and similarly $\frac{\Lambda(f, \ell)}{\omega_-(f)} \in \mathbb{Q}(f)$. It is clear that $\omega_+(f)\omega_-(f) = \langle f, f \rangle$. Finally, the assertion (4.16) of [14] and that $\alpha_f^\sigma = \alpha_f$. 


for each \( \sigma \in \text{Gal}(\overline{Q}/Q) \) implies that
\[
\left( \frac{\Lambda(f, m)}{\omega_+(f)} \right)^\sigma = \frac{\Lambda(f^\sigma, m)}{\omega_+(f^\sigma)}, \quad \left( \frac{\Lambda(f, \ell)}{\omega_-(f)} \right)^\sigma = \frac{\Lambda(f^\sigma, \ell)}{\omega_-(f^\sigma)},
\]
finishing the proof. \( \square \)

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