Edge coloring graphs with large minimum degree

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Abstract
Let $G$ be a simple graph with maximum degree $\Delta(G)$. A subgraph $H$ of $G$ is overfull if $E(H) > [\Delta(G)] [V(H)]/2]$. Chetwynd and Hilton in 1986 conjectured that a graph $G$ with $\Delta(G) > [V(G)]/3$ has chromatic index $\Delta(G)$ if and only if $G$ contains no overfull subgraph. The best previous results supporting this conjecture have been obtained for regular graphs. For example, Perković and Reed verified the conjecture for large regular graphs $G$ with degree arbitrarily close to $[V(G)]/2$. We provide a similar result for general graphs asymptotically, showing that for any given $0 < \varepsilon < 1$, there exists a positive integer $n_0$ such that the following statement holds: if $G$ is a graph on $2n \geq n_0$ vertices with minimum degree at least $(1 + \varepsilon)n$, then $G$ has chromatic index $\Delta(G)$ if and only if $G$ contains no overfull subgraph.

KEYWORDS
1-factorization, chromatic index, overfull conjecture, overfull graph

1 | INTRODUCTION

In this paper, the terminology “graph” is used to mean a simple graph and a “multigraph” may contain parallel edges but no loops. Let $G$ be a multigraph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively, and by $e(G)$ the cardinality of $E(G)$. For $v \in V(G)$, $N_G(v)$ is the set of neighbors of $v$ in $G$, and $d_G(v)$, the degree of $v$ in $G$, is the number of edges of $G$ that are incident with $v$. When $G$ is simple, $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$, the subgraph of $G$ induced on $S$ is denoted by $G[S]$, and $G - S := G[V(G) \setminus S]$. If $F \subseteq E(G)$, then $G - F$ is obtained from $G$ by deleting all the edges of $F$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in $G$ with one end in $V_1$ and the other end in $V_2$, and...
We write \( e_G(v, V_2) := |E_G(V_1, V_2)| \). We write \( e_G(v, V_2) \) and \( e_G(v, V_2) \) if \( V_1 = \{ v \} \) is a singleton. Define \( \mu(G) = \max\{e_G(u, v) : u, v \in V(G)\} \) to be the multiplicity of \( G \). We also write \( G[V_1, V_2] \) to denote the bipartite subgraph of \( G \) with vertex set \( V_1 \cup V_2 \) and edge set \( E_G(V_1, V_2) \).

For two integers \( p, q \), let \( [p, q] = \{ i \in \mathbb{Z} : p \leq i \leq q \} \). Let \( k \geq 0 \) be an integer. An edge \( k\text{-coloring} \) of a multigraph \( G \) is a mapping \( \varphi \) from \( E(G) \) to the set of integers \( [1, k] \), called \textit{colors}, such that no two adjacent edges receive the same color with respect to \( \varphi \). The \textit{chromatic index} of \( G \), denoted \( \chi'(G) \), is defined to be the smallest integer \( k \) so that \( G \) has an edge \( k\text{-coloring} \). We denote by \( C^k(G) \) the set of all edge \( k\text{-colorings} \) of \( G \).

In the 1960s, Gupta [11] and, independently, Vizing [26] proved that for all graphs \( G \), \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \). This leads to a natural classification of graphs. Following Fiorini and Wilson [8], we say a graph \( G \) is of class 1 if \( \chi'(G) = \Delta(G) \) and of class 2 if \( \chi'(G) = \Delta(G) + 1 \). Holyer [13] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. Nevertheless, if \( |E(G)| > \Delta(G)|V(G)|/2 \), then we must use \( \Delta(G) + 1 \) colors to edge color \( G \). Such graphs are \textit{overfull}. An overfull subgraph \( H \) of \( G \) with \( \Delta(H) = \Delta(G) \) is called a \( (G)\text{-overfull subgraph} \) of \( G \). A number of long-standing conjectures listed in \textit{Twenty Pretty Edge Coloring Conjectures} in [24] lie in deciding when a graph is overfull. Chetwynd and Hilton [3,4], in 1986, proposed the following conjecture.

**Conjecture 1.1** (Overfull Conjecture). Let \( G \) be a simple graph with \( \Delta(G) > \frac{1}{3}|V(G)| \). Then \( \chi'(G) = \Delta(G) \) if and only if \( G \) contains no \( (G)\text{-overfull subgraph} \).

The 3-critical graph \( P^* \), obtained from the Petersen graph by deleting one vertex, has \( \chi'(P^*) = 4 \), satisfies \( \Delta(P^*) = \frac{1}{3}|V(P^*)| \) but contains no 3-overfull subgraph. Thus the degree condition \( \Delta(G) > \frac{1}{3}|V(G)| \) in the conjecture above is best possible. Applying Edmonds' matching polytope theorem, Seymour [22] showed that whether a graph \( G \) contains an overfull subgraph of maximum degree \( \Delta(G) \) can be determined in polynomial time. Thus if the Overfull Conjecture is true, then the NP-complete problem of determining the chromatic index becomes polynomial-time solvable for graphs \( G \) with \( \Delta(G) > \frac{1}{3}|V(G)| \). There have been some fairly strong results supporting the Overfull Conjecture in the case when \( G \) is regular. It is easy to verify that when \( G \) is regular with even order, \( G \) has no \( (G)\text{-overfull subgraphs} \) if its vertex degrees are at least \( |V(G)|/2 \). Thus the well-known 1-Factorization Conjecture stated below is a special case of the Overfull Conjecture.

**Conjecture 1.2** (1-Factorization Conjecture). Let \( G \) be a regular graph of order \( 2n \) with degree at least \( n \) if \( n \) is odd, or at least \( n - 1 \) if \( n \) is even. Then \( G \) is 1-factorable; equivalently, \( \chi'(G) = \Delta(G) \).

Hilton and Chetwynd [2] verified the 1-Factorization Conjecture if the vertex degree is at least \( 0.823|V(G)| \). Perković and Reed [19] showed in 1997 that the 1-Factorization Conjecture is true for large regular graphs with vertex degree at least \( V(G)|/2 - \epsilon \). In 2016, Csaba et al. [6] verified the conjecture for sufficiently large \( V(G) \). Much less is known about the truth of the Overfull Conjecture if we no longer require that \( G \) is regular. It was confirmed for graphs with \( \Delta(G) \geq |V(G)| - 3 \) by Chetwynd and Hilton in 1989 [4]. Plantholt [20] in 2004 verified the conjecture for graphs of even order and minimum degree at least \( 0.8819|V(G)| \). More recently, Plantholt [21] showed the conjecture is true for sufficiently large even order graphs with
minimum degree at least \(2|V(G)|/3\). We extend these results and give an asymptotic result for general graphs that is similar to the Perković-Reed result for regular graphs, by obtaining the result below.

**Theorem 1.3.** For all \(0 < \varepsilon < 1\), there exists \(n_0\) such that the following statement holds: if \(G\) is a graph on \(2n \geq n_0\) vertices with \(\delta(G) \geq (1 + \varepsilon)n\), then \(\chi'(G) = \Delta(G)\) if and only if \(G\) contains no \(\Delta(G)\)-overfull subgraph. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

Define \(V_i(G) = \{v \in V(G) : d_G(v) = i\}\), and we write \(V_i\) for \(V_{\delta(G)}\) if \(G\) is clear. Furthermore, \(V_{\delta(G)}\) and \(V_{\Delta(G)}\) are simply written as \(V_{\delta}\) and \(V_{\Delta}\), respectively. The proof of Theorem 1.3 is based on the following result.

**Theorem 1.4.** For all \(0 < \varepsilon < 1\), there exists \(n_0\) such that the following statement holds: if \(G\) is a graph on \(2n \geq n_0\) vertices satisfying one of the following three conditions:

(a) \(G\) is regular with \(\delta(G) \geq (1 + 4\varepsilon/5)n\),

(b) \(G\) has two distinct vertices \(x, y\) such that \(d(x) = d(y) \geq (1/2 + 3\varepsilon/2)n\), for all \(z \in V(G) \setminus \{x, y\}\), \(d(z) = \Delta(G) \geq (1 + \varepsilon)n\), and \(\Delta(G) - \delta(G) \leq (1/2 - \varepsilon/2)n\),

(c) \(\Delta(G) - \delta(G) \geq n^{6/7}, |V_{\delta}| \geq n^{6/7}\) and \(V_{\Delta} \geq n + 1\), and \(\delta(G) \geq (1 + \varepsilon)n\),

then \(\chi'(G) = \Delta(G)\). Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

The proof of Theorem 1.4 develops an approach to edge coloring even order large graphs \(G\) that have minimum degree arbitrarily close to \(\delta(G)\) but are not regular. The approach is based on the proof scheme of Lemma 14 from [25] by Vaughan but the scheme there is only for regular graphs. The new technique is essentially different from the main ideas used in [23] by the second author and in [21] by the first author, where the graphs can be reduced into a regular graph still with good properties by taking off edge-disjoint linear forests or matchings.

The remainder of this paper is organized as follows. In the next section, we introduce some notation and preliminary results. In Section 3, we prove Theorem 1.3 by applying Theorem 1.4. Theorem 1.4 is then proved in the last section.

## 2 | NOTATION AND PRELIMINARIES

Let \(G\) be a multigraph and \(\varphi \in \mathcal{C}^k(G)\) for some integer \(k \geq 0\). For any \(v \in V(G)\), the set of colors present at \(v\) is \(\varphi(v) = \{\varphi(e) : e \in E(G)\text{ is incident with } v\}\), and the set of colors missing at \(v\) is \(\varphi^{-1}(v) = [1, k] \setminus \varphi(v)\). For a subset \(X\) of \(V(G)\) and a color \(i \in [1, k]\), define \(\varphi_X^{-1}(i) = \{v \in X : i \in \varphi(v)\}\), and we write \(\varphi^{-1}(i)\) for \(\varphi^{-1}(i)\). An edge \(k\)-coloring of a multigraph \(G\) is said to be equalized if each color class contains either \(|E(G)|/k|\) or \(|E(G)|/k|\) edges.

For \(x \in V(G)\), the deficiency of \(x\) in \(G\) is \(d_G(x) = \Delta(G) - d_G(x)\). For \(X \subseteq V(G)\), \(d_G(X) = \sum_{x \in X} d_G(x)\). We simply write \(d_G(V(G))\) as \(d_G\). A subgraph \(H\) of \(G\) with an odd order is \(\Delta(G)\)-full if \(|E(H)| = \Delta(G)|V(H)|/2|\).
We will use the following notation: $0 < a \ll b \leq 1$. Precisely, if we say a claim is true provided that $0 < a \ll b \leq 1$, then this means that there exists a nondecreasing function $f : (0, 1] \rightarrow (0, 1]$ such that the statement holds for all $0 < a, b \leq 1$ satisfying $a \leq f(b)$.

In the 1960s, Gupta [11] and, independently, Vizing [26] provided an upper bound on the chromatic index of multigraphs, and König [15] gave an exact value of the chromatic index for bipartite multigraphs.

**Theorem 2.1** (Gupta [11] and Vizing [26]). Every multigraph $G$ satisfies $\chi'(G) \leq \Delta(G) + \mu(G)$.

**Theorem 2.2** (König [15]). Every bipartite multigraph $G$ satisfies $\chi'(G) = \Delta(G)$.

McDiarmid [16] observed the following result.

**Theorem 2.3.** Let $G$ be a multigraph with chromatic index $\chi'(G)$. Then for all $k \geq \chi'(G)$, there is an equalized edge-coloring of $G$ with $k$ colors.

Let $G$ be a multigraph, $k \geq 0$ be an integer and $\varphi \in \mathcal{C}^k(G)$. There is a polynomial time algorithm to modify $\varphi$ into an equalized edge-coloring of $G$ with $k$ colors. To see this, suppose that $\varphi$ is not equalized and so we take two colors $i, j \in [1, k]$ such that $|\varphi^{-1}(i)| - |\varphi^{-1}(j)|$ is largest. Since $\varphi$ is not equalized, $|\varphi^{-1}(i)| - |\varphi^{-1}(j)| \geq 4$. Assume by symmetry that $|\varphi^{-1}(i)| - |\varphi^{-1}(j)| \geq 4$. Consider the submultigraph of $G$ induced on the set of edges colored by $i$ or $j$, then the submultigraph must have a component that is a path $P$ starting at an edge colored by $j$ and ending at an edge colored by $j$. By swapping the colors $i$ and $j$ along this path $P$, we decreased $|\varphi^{-1}(i)| - |\varphi^{-1}(j)|$ by 4. Repeating this process, we can obtain an equalized edge-coloring of $G$ with $k$ colors after at most $k^2|V(G)|$ rounds.

Given an edge coloring of $G$ and a given color $i$, since vertices presenting $i$ are saturated by the matching consisting of all edges colored by $i$, we have the Parity Lemma below. The result had appeared in many papers, for example, see [10, lemma 2.1].

**Lemma 2.4** (Parity Lemma). Let $G$ be a multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq \Delta(G)$. Then $|\varphi^{-1}(i)| \equiv |V(G)| \pmod{2}$ for every color $i \in [1, k]$.

We need the following classic result of Hakimi [12] on multigraphic degree sequence.

**Theorem 2.5.** Let $0 \leq d_n \leq \cdots \leq d_1$ be integers. Then there exists a multigraph $G$ on vertices $x_1, \ldots, x_n$ such that $d_G(x_i) = d_i$ for all $i$ if and only if $\sum_{i=1}^n d_i$ is even and $\sum_{i=1} d_i \geq d_i$.

Though it is not explicitly stated in [12], the inductive proof yields a polynomial time algorithm which finds an appropriate multigraph if it exists.

**Theorem 2.6** (Dirac [7]). Let $G$ be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then $G$ is hamiltonian; and if $\delta(G) \geq \frac{n+1}{2}$, then $G$ is hamiltonian-connected.

Following the proof of Dirac [7], a hamiltonian cycle can be constructed in polynomial time in $n$ if $\delta(G) \geq \frac{n}{2}$. In fact, there is a polynomial time algorithm that constructs the closure of a
graph $G$ and finds a hamiltonian cycle of $G$ if its closure is a complete graph (see [1, exercise 4.2.15, p. 62]).

**Lemma 2.7.** Let $G$ be an $n$-vertex graph such that all vertices of degree less than $\Delta(G)$ are mutually adjacent in $G$. Then $V_\Delta > \frac{n}{2}$.

**Proof.** Suppose the set $A$ of maximum degree vertices has cardinality $k$, and the number of vertices of degree less than maximum degree is $kr + r$ with $r \geq 0$. Deleting $r$ vertices not in $A$, we get a new graph $H$ with $k$ vertices, $k$ of them forming $A$, and the remaining $k$ forming a set of vertices $B$ such that each vertex in $B$ has degree less than each vertex of $A$ in $H$. But $B$ induces a complete graph in $H$ so in $H$ the sum of the vertex degrees in $A$ is less than or equal to the degree sum of the vertices in $B$. Since every vertex of $V(G) \setminus V(H)$ is adjacent in $G$ to every vertex of $B$, it follows that in $G$ the sum of the vertex degrees in $A$ is less than or equal to the degree sum of the vertices in $B$. This gives a contradiction. □

The two lemmas below concern existences of overfull subgraphs in graphs.

**Lemma 2.8** (Plantholt [20]). Let $G$ be a graph of even order $n$ with $\delta(G) > \frac{n}{2}$. If $H$ is an induced proper subgraph of $G$ such that $H$ is either $\Delta(G)$-overfull or $\Delta(G)$-full, then $H = G - v$ for some vertex $v \in V_\delta$.

**Lemma 2.9.** Let $G$ be a graph of even order $n$ with $\delta(G) > \frac{n}{2}$. Then $G$ contains no $\Delta(G)$-overfull subgraph if $|V_\delta| \geq 2$.

**Proof.** Let $x, y \in V_\delta$ be distinct. Then $\sum_{v \in V(G-x)}(\Delta(G) - d_{G-x}(v)) = d_G(x) + (\Delta(G) - d_G(y)) + \text{def}_G(V(G) \setminus \{x, y\}) \geq \Delta(G)$. Thus $G - x$ is not $\Delta(G)$-overfull. By Lemma 2.8, $G$ contains no $\Delta(G)$-overfull subgraph. □

**Lemma 2.10.** Let $0 < \varepsilon < 1$, $n_0$ be a positive integer, and $G$ be a graph on $2n \geq n_0$ vertices with $\delta(G) \geq (1 + \varepsilon)n$. If $G$ contains a $\Delta(G)$-full subgraph, then $G$ contains a spanning $\delta(G)$-regular subgraph obtained from $G$ by deleting $\Delta(G) - \delta(G)$ matchings iteratively. As a consequence, $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

**Proof.** Define $g = \Delta(G) - \delta(G)$. If $G$ is regular, then we are done by Theorem 1.4. Thus $G$ is not regular and so $g \geq 1$. The graph $G$ contains a $\Delta(G)$-full subgraph, which by Lemma 2.8 must be $G - x$ for some vertex $x \in V_\delta$. Also, if $G$ contains a $\Delta(G)$-overfull subgraph, then $G - x$ must be $\Delta(G)$-overfull also by Lemma 2.8. Since $G - x$ is $\Delta(G)$-full, we conclude that $G$ contains no $\Delta(G)$-overfull subgraph and so has another vertex of degree less than $\Delta(G)$. We let $y \in V(G) \setminus \{x\}$ such that $d_G(y)$ is smallest among all vertices in $V(G) \setminus \{x\}$. Since $G - x$ is $\Delta(G)$-full, we have $\Delta(G) = \text{def}(G - x) = d_G(x) + (\Delta(G) - d_G(y)) + \text{def}_G(V(G) \setminus \{x, y\})$. As $d_G(x) = \delta(G)$, if $d_G(y) = \delta(G)$, then $\text{def}_G(V(G) \setminus \{x, y\}) = 0.$
This implies that if \( d_G(y) = \delta(G) \), then every vertex from \( V(G) \setminus \{x, y\} \) has degree \( \Delta(G) \) in \( G \); and if \( d_G(y) > \delta(G) \), then as \( y \) is chosen to have smallest degree in \( G \) among vertices from \( V(G) \setminus \{x\}, V(G) \setminus \{x, y\} \) contains no vertex of degree \( \delta(G) \) in \( G \). Since \( \delta(G - x - y) \geq n - 1 \), \( G - x - y \) has a hamiltonian cycle by Theorem 2.6. As \( n - 2 \) is even, we know that \( G - x - y \) has a perfect matching \( M_t \). Now we have \( \delta(G - M_t) = \delta(G) \) and \( \Delta(G - M_t) - \delta(G - M_t) = g - 1 < g \). Let \( G_1 = G - M_t \). Since

\[
\def(G_1 - x) = d_G(x) + (\Delta(G_1) - d_G(y)) + \def_G(V(G) \setminus \{x, y\})
= d_G(x) + \def_G(V(G) \setminus \{x\}) - 1
= \def(G - x) - 1 = \Delta(G) - 1 = \Delta(G_1),
\]

we see that \( G_1 - x \) is \( \Delta(G_1) \)-full. Thus we may repeat the procedure, and reach a \( \delta(G) \)-regular graph \( G^* \) after taking \( g \) matchings \( M_1, \ldots, M_g \).

Now by Theorem 1.4, \( \chi'(G^*) = \Delta(G^*) = \delta(G) \). Coloring each of the \( g \) matchings \( M_1, \ldots, M_g \) using a different color together with an edge \( \delta(G) \)-coloring of \( G^* \) gives an edge \( \Delta(G) \)-coloring of \( G \). Thus \( \chi'(G) = \Delta(G) \).

It is polynomial-time to find a hamiltonian cycle in graphs \( H \) with \( \delta(H) \geq \frac{1}{2}|V(H)| \) by the comments immediately after Theorem 2.6. Thus all the matchings \( M_1, \ldots, M_g \) can be found in polynomial time. As an optimal edge coloring can be found in polynomial time for graphs satisfying the conditions in Theorem 1.4, we can find an edge \( \Delta(G) \)-coloring of \( G^* \) in polynomial time. Therefore, there is a polynomial time algorithm that finds an edge \( \Delta(G) \)-coloring for \( G \).

**Lemma 2.11.** Let \( G[X, Y] \) be bipartite graph with \(|X| = |Y| = n\). Suppose \( \delta(G) = t \) for some \( t \in [1, n] \), and except at most \( t \) vertices all other vertices of \( G \) have degree at least \( n/2 \) in \( G \). Then \( G \) has a perfect matching.

**Proof.** We show that \( G[X, Y] \) satisfies Hall’s Condition. If not, we let \( S \subseteq X \) with smallest cardinality such that \(|S| > |N_G(S)|\). By this choice, \(|S| = |N_G(S)| + 1 \) and \(|N_G(S)| < |Y|\). As \(|S| > |N_G(S)|\), it follows that \(|S| \geq \delta(G) + 1 \geq t + 1\). As \( G \) has at most \( t \) vertices of degree less than \( n/2 \), it then follows that \(|S| > n/2\). Thus \(|X \setminus S| < n/2\). Since \(|N_G(S)| < |Y|\), there exists \( y \in Y \setminus N_G(S) \) such that \( N_G(y) \subseteq X \setminus S \). As \( \delta(G) \geq t \), we have \(|X \setminus S| \geq t\). As \( |Y \setminus N_G(S)| = |Y| - |S| + 1 = |X| - |S| + 1 \geq t + 1 \) and \( G \) has at most \( t \) vertices of degree less than \( n/2 \), \( Y \setminus N_G(S) \) contains a vertex of degree at least \( n/2 \) in \( G \). However \(|X \setminus S| < n/2\), we obtain a contradiction. Hence \( G \) has a perfect matching. \( \square \)

A path \( P \) connecting two vertices \( u \) and \( v \) is called a \((u, v)\)-path, and we write \( uPv \) or \( vPu \) to specify the two endvertices of \( P \). Let \( uPv \) and \( xPy \) be two disjoint paths. If \( vx \) is an edge, we write \( uPvxQy \) as the concatenation of \( P \) and \( Q \) through the edge \( vx \). If \( P \) is a path and \( x, y \in V(P) \), then \( xPy \) is the subpath of \( P \) with endvertices \( x \) and \( y \).

**Lemma 2.12.** Let \( 0 < 1/n_0 \ll \varepsilon < 1 \), and \( G \) be a graph on \( n \geq n_0 \) vertices such that \( \delta(G) \geq (1 + \varepsilon)n/2 \). Moreover, let \( M = \{a_1b_1, \ldots, a_ib_i\} \) be a matching in the complete graph on \( V(G) \) of size at most \( \varepsilon n/8 \). Then there exist vertex-disjoint path \( P_1, \ldots, P_t \) in \( G \) such that \( \cup V(P_i) = V(G) \) and \( P_i \) joins \( a_i \) to \( b_i \), and these paths can be found in polynomial time.
Proof. For $i \in [1, t-1], |N_G(a_i) \cap N_G(b_i)| \geq \varepsilon n$, so we can greedily find vertices $c_i \in N_G(a_i) \cap N_G(b_i)$ such that $c_i \neq c_j$ for distinct $i, j \in [1, t-1]$. Thus we let $P_i = a_i c_i b_i$. Let $G^* = G - \bigcup_{i=1}^{t-1} V(P_i)$. Then $\delta(G^*) \geq (1 + \varepsilon)n/2 - 3(t - 1) \geq (1 + \varepsilon/8)n/2$, and so $G^*$ is hamiltonian-connected by Theorem 2.6. Thus we can find an $(a_i, b_i)$-hamiltonian path $P_i$ in $G^*$.

It is clear that each of $P_1, ..., P_{t-1}$ can be found in polynomial time. For the path $P_t$, we construct it as follows. By the comments immediately after Theorem 2.6, we can find a hamiltonian cycle $C$ of $G^*$ in polynomial time. By taking a longer segment between $a_t$ and $b_t$ from $C$, we get an $(a_t, b_t)$-path $Q_1$ that contains at least $V(G^*)/2$ vertices. We will extend $Q_1$ into a hamiltonian $(a_t, b_t)$-path of $G^*$. Denote by $Q_2$ the remaining segment of $C$ that is disjoint from $Q_1$ and let $c$ and $d$ be the endvertices of $Q_2$. Let $|V(Q_2)| = p$. Then as $\delta(G^*) \geq (1 + \varepsilon/8)n/2$, each of $c$ and $d$ has on $Q_1$ at least $(1 + \varepsilon/8)n/2 - (p - 1) = (1 + \varepsilon/8)n/2 - p + 1$ neighbors. Since $2((1 + \varepsilon/8)n/2 - p) + |V(Q_2)| > |V(G^*)|$, it follows that one of the following two situations must happen: (a) there is a vertex $c_1 \in N_G^*(c) \cap V(Q_1)$ and a vertex $d_1 \in N_G^*(d) \cap V(Q_1)$ such that $c_1 Q_1 d_1$ contains less than $p + 2$ vertices, and (b) $c$ or $d$ has on $Q_1$ two neighbors that are consecutive on $Q_1$. When (a) happens, assume by symmetry that $c_1$ is between $a_t$ and $d_1$ on $Q_1$, then $Q^*_t = a_t Q_1 c_1 Q_2 d_1 Q_1 b_t$ is longer than $Q_1$ and the component of $G^* - V(Q^*_t)$ still contains a hamiltonian path. Similarly, we can extend $Q_1$ into a longer $(a_t, b_t)$-path such that the subgraph of $G^*$ outside the path is hamiltonian if (b) happens. Repeating this procedure at most $n/2$ times, we obtain a hamiltonian $(a_t, b_t)$-path of $G^*$. Therefore, all the paths $P_1, ..., P_t$ can be found in polynomial time. 

3 | PROOF OF THEOREM 1.3

**Theorem 1.3.** For all $0 < \varepsilon < 1$, there exists $n_0$ such that the following statement holds: if $G$ is a graph on $2n \geq n_0$ vertices with $\delta(G) \geq (1 + \varepsilon)n$, then $\chi'(G) = \Delta(G)$ if and only if $G$ contains no $\Delta(G)$-overfull subgraph. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

**Proof.** Choose positive integer $n_0$ such that $0 < 1/n_0 \ll \varepsilon$.

If $G$ is regular, then we are done by Theorem 1.4. Thus we assume that $G$ is not regular. If $G$ contains a $\Delta(G)$-overfull subgraph, then $\chi'(G) = \Delta(G) + 1$. Thus we assume that $G$ contains no $\Delta(G)$-overfull subgraph. As a consequence, $\text{def}(G) \geq \Delta(G)$. By Lemma 2.10, we may assume that $G$ contains no $\Delta(G)$-full subgraph. Therefore, if two vertices with degree less than $\Delta(G)$ are not adjacent in $G$, we may add the edge between them without creating a $\Delta(G)$-overfull subgraph, or increasing $\Delta(G)$. We iterate this edge-addition procedure. If at some point we create a $\Delta(G)$-full subgraph, the result follows by Lemma 2.10. Otherwise, we reach a point where we may now assume that in $G$ all vertices with degree less than $\Delta(G)$ are mutually adjacent, and so by Lemma 2.7, we have $|V_2| \geq n + 1$.

Define $n_1 = |V_2|$. Note that $n_1 < n$. If $n_1 \geq n^{6/7}$ and $\Delta(G) - \delta(G) \geq n^{6/7}$, then we are done by Theorem 1.4. Thus we assume $n_1 < n^{6/7}$ or $\Delta(G) - \delta(G) < n^{6/7}$, and we
consider the two cases below. We call a vertex of degree less than \( \Delta(G) \) but greater than \( \delta(G) \) a middle degree vertex.

**Case 1.** \( n_1 < n^{6/7} \).

Note that for any \( v \in V(G) \setminus V_\delta, \delta(G - v - V_\delta) \geq n \) and so both \( G - V_\delta \) and \( G - v - V_\delta \) are hamiltonian by Theorem 2.6. Thus if \( G - V_\delta \) and \( G - v - V_\delta \) have even order, then they each has a perfect matching. Hence if \( n_1 \) is even, we can decrease \( \Delta(G) - \delta(G) \) but preserve \( \delta(G) \) in deleting a perfect matching \( M \) of \( G - V_\delta \). If \( n_1 \) is odd but \( G \) has a middle degree vertex \( v \), we can decrease \( \Delta(G) - \delta(G) \) but preserve \( \delta(G) \) in deleting a perfect matching \( M \) of \( G - v - V_\delta \). Denote by \( G_1 \) the reduced graph from \( G \) by deleting \( M \) in either of these two cases. If \( |V_\delta| \geq 2 \), then as \( V_\delta \subseteq V_\delta(G_1) \), we know that \( G_1 \) still contains no \( \Delta(G_1) \)-overfull subgraph by Lemma 2.9. Thus \( |V_\delta| = 1 \). Let \( V_\delta = \{u\} \). Note that \( u \in V(G) \). Then

\[
\text{def}(G_1 - u) = d_G(u) + (\Delta(G_1) - d_G(v)) \leq d_G(u) + (\Delta(G) - 1 - d_G(v)) + \text{def}(G(V(G) \setminus \{u, v\})) = d_G(u) + (\Delta(G) - \Delta(G_1)) = \sum_{w \in V(G(v))} (\Delta(G) - d_G(w)) - 1.
\]

Since \( G \) contains no \( \Delta(G) \)-overfull subgraph, we have \( \sum_{w \in V(G(v))} (\Delta(G) - d_G(w)) \geq \Delta(G) \). Thus \( \text{def}(G_1 - u) \geq \Delta(G) - 1 = \Delta(G_1) \) and so \( G_1 \) contains no \( \Delta(G_1) \)-overfull subgraph by Lemma 2.8. Furthermore, \( \chi'(G_1) = \Delta(G_1) \) implies that \( \chi'(G) = \Delta(G) \). Thus in these two cases, we can consider \( G_1 \) in place of \( G \) and show that \( G_1 \) is a class 1 graph.

Thus we assume that \( n_1 \) is odd and \( G \) has no middle degree vertex. This in particular, implies that \( \delta(G) \) and \( \Delta(G) \) have the same parity. As \( G \) has no \( \Delta(G) \)-overfull subgraph, \( |V_\delta| \geq 3 \). Let \( x, y \in V_\delta \) be distinct. We find a perfect matching \( M_{11} \) in \( G - (V_\delta \setminus \{x\}) \) and a perfect matching \( M_{12} \) in \( G - (V_\delta \setminus \{y\}) \). The matchings exist by Theorem 2.6. Let \( G_1 = G - M_{11} - M_{12} \). We repeat this same process and find a perfect matching \( M_{13} \) in \( G_1 - (V_\delta \setminus \{x\}) \) and a perfect matching \( M_{22} \) in \( G_1 - (V_\delta \setminus \{y\}) \). For \( i \in [2, (\Delta(G) - \delta(G))/2] \), we get \( G_i = G_{i-1} - M_{11} - M_{12} \). We have

\[
d_G(x) = d_G(y) = \delta(G) - i.
\]

As \( \Delta(G) - \delta(G) \leq 2n - (1 + \varepsilon)n = (1 - \varepsilon)n \), we see that

\[
d_G(z) \geq \delta(G) \geq 1 + \varepsilon)n.
\]

Let \( x^* \) be a neighbor of \( x \) in \( G_i - (V_\delta \setminus \{x\}) \) and \( y^* \) be a neighbor of \( y \) in \( G_i - (V_\delta \setminus \{y\}) \). Then \( G_i - (V_\delta \setminus \{x, y\}) \) has a perfect matching \( M^*_{(i+1)1} \) and \( G_i - (V_\delta \setminus \{y\}) - \{x, y^*\} \) has a perfect matching \( M^*_{(i+1)2} \). Let \( M^*_{(i+1)1} = M^*_{(i+1)1} \cup \{xx^*\} \) and \( M^*_{(i+1)2} = M^*_{(i+1)2} \cup \{yy^*\} \). Thus for each \( i \in [2, (\Delta(G) - \delta(G))/2] \), we find matchings \( M_{11}, M_{12} \), respectively from \( G_{i-1} - (V_\delta \setminus \{x\}) \) and \( G_{i-1} - (V_\delta \setminus \{y\}) \).

We claim \( G^* := G_{(\Delta(G) - \delta(G))/2} \) satisfies Condition (b) of Theorem 1.4. By the analysis above, we have \( d_G(x) = d_G(y) \leq (1/2 + 3\varepsilon/2)n \). Also

\[
\Delta(G^*) - \delta(G^*) = \delta(G) - (\delta(G) - (\Delta(G) - \delta(G))/2) = \frac{1}{2}\Delta(G) \leq \frac{1}{2}(2n - (1 + \varepsilon)n) = (1/2 - \varepsilon/2)n.
\]

By Theorem 1.4, \( \chi'(G^*) = \Delta(G^*) \). Taking an edge \( \delta(G) \)-coloring of \( G^* \), coloring edges in \( M_{11} \) with color \( \delta(G) + 2i - 1 \) and coloring edges in \( M_{12} \) with color \( \delta(G) + 2i \) for each \( i \in [1, (\Delta(G) - \delta(G))/2] \), we obtain an edge \( \Delta(G) \)-coloring of \( G \).

**Case 2.** \( \Delta(G) - \delta(G) < n^{6/7} \).
Let $V(G) = \{x_1, \ldots, x_{2n}\}$ and we assume $\text{def}_G(x_i) \geq \cdots \geq \text{def}_G(x_{2n}) = 0$. Since $x_i$ has the smallest degree in $G$ and $G - x_i$ is not $\Delta(G)$-overfull by our assumption, $\sum_{i \geq 2} \text{def}_G(x_i) \leq \text{def}_G(x_i)$. Since $|V(G)| = 2n$ is even, $\sum_{i \geq 1} \text{def}_G(x_i)$ is even. Then by Theorem 2.5, there exists a multigraph $H$ on $V(G)$ such that $d_H(x) = \text{def}_G(x_i)$ for each $i \in [1, 2n]$. This multigraph $H$ will aid us to find a spanning regular subgraph of $G_k$ whose size at most $\Delta(G) - \delta(G) < n^6/7$ and $H$ contains isolated vertices.

Note that $\Delta(H) = \text{def}_G(x_i) = \Delta(G) - \delta(G) < n^6/7$ and $H$ contains isolated vertices. Thus $\chi'(H) \leq \Delta(H) + \mu(H) \leq 2\Delta(H) \leq 2n^6/7$. Hence we can greedily partition $E(H)$ into $k \leq 10n^6/7/\varepsilon$ matchings $M_1, \ldots, M_k$ each of size at most $en/5$. Now we take out linear forests from $G$ by applying Lemma 2.12 with $M_1, \ldots, M_k$. More precisely, define spanning subgraphs $G_0, \ldots, G_k$ of $G$ and edge-disjoint linear forests $F_1, \ldots, F_k$ such that

1. $G_0 := G$ and $G_i = G_{i-1} - E(F_i)$ for $i \in [1, k]$,
2. $F_i$ is a spanning linear forest (each vertex of $G_{i-1}$ has degree 1 or 2 in $F_i$) in $G_{i-1}$ whose leaves are precisely the vertices in $M_i$.

Let $G_0 = G$ and suppose that for some $i \in [1, k]$, we already defined $G_0, \ldots, G_{i-1}$ and $F_1, \ldots, F_{i-1}$. As $\Delta(F_i \cup \cdots \cup F_{i-1}) \leq 2(i - 1) \leq 20n^6/7/\varepsilon$, it follows that $\delta(G_{i-1}) \geq (1 + \varepsilon)n - 20n^6/7/\varepsilon \geq (1 + 4\varepsilon/5)n$. Since $M_i$ has size at most $en/5$, we can apply Lemma 2.12 to $G_{i-1}$ and $M_i$ and obtain a spanning linear forest $F_i$ in $G_{i-1}$ whose leaves are precisely the vertices in $M_i$. Set $G_i := G_{i-1} - E(F_i)$.

We claim that $G_k$ is regular. Consider any vertex $x \in V(G_k)$. For every $i \in [1, k]$, $d_{F_i}(u) = 1$ if $u$ is an endvertex of some edge of $M_i$ and $d_{F_i}(u) = 2$ otherwise. Since $M_1, \ldots, M_k$ partition $E(H)$, we know that $\sum_{i=1}^k d_{F_i}(u) = 2k - d_H(u) = 2k - \text{def}_G(u)$. Thus

$$d_{G_k}(u) = d_G(u) - \sum_{i=1}^k d_{F_i}(u) = d_G(u) - (2k - \text{def}_G(u)) = \Delta(G) - 2k.$$

Note that $\Delta(G) \geq (1 + \varepsilon)n - 20n^6/7/\varepsilon \geq (1 + 4\varepsilon/5)n$. Now $\chi'(G_k) = \Delta(G_k)$ by Theorem 1.4. We color the edges of $F_i$ using two distinct colors from $[\Delta(G) - 2k + 1, \Delta(G)]$ for each $i \in [1, k]$. It is clear that any edge $\Delta(G_k)$-coloring of $G_k$ together with this coloring of $\bigcup_{i=1}^k F_i$ gives an edge coloring of $G$ using $\Delta(G_k) + 2k = \Delta(G)$ colors.

We lastly check that the procedure above yields a polynomial time algorithm. Given $G$, taking a vertex $u$ of minimum degree in $G$, we first check if $G - u$ is $\Delta(G)$-overfull. If yes, then $\chi'(G) = \Delta(G) + 1$ and $G$ can be edge colored using $\Delta(G) + 1$ colors in polynomial time [17]. Thus $G$ contains no $\Delta(G)$-overfull subgraph. If $G$ contains a $\Delta(G)$-full subgraph, then an edge $\Delta(G)$-coloring of $G$ can be found in polynomial time by Lemma 2.10. Thus $G$ contains no $\Delta(G)$-full subgraph. If there exist nonadjacent $u, v \in V(G) \setminus V_A$, we add the edge $uv$ in $G$. If we reach a point where the resulting graph contains a $\Delta(G)$-full subgraph, we then find an edge $\Delta(G)$-coloring of the graph in polynomial time by Lemma 2.10, which also gives an edge $\Delta(G)$-coloring of $G$. Thus we assume that every two vertices from $V(G) \setminus V_A$ are adjacent in $G$. If $G$ is in Condition (c) of Theorem 1.4, then we find an edge $\Delta(G)$-coloring of $G$ in polynomial time by Theorem 1.4. Thus we have Case 1 or Case 2 as described in this proof. If $G$ is in Case 1, it is polynomial time to find the desired matchings (basically find hamiltonian cycles of even length in graphs with large minimum degree by the comments immediately after
Theorem 2.6) to reduce $G$ into a graph satisfying one of the conditions in Theorem 1.4. Then we find an edge $\Delta(G)$-coloring of $G$ in polynomial time by Theorem 1.4. If $G$ is in Case 2, then can construct an edge $\Delta(G)$-coloring of $G$ through the process as described in Case 2. Since Theorem 2.5, Lemma 2.12 and Theorem 1.4 give appropriate running time statements, this can be achieved in time polynomial in $n$. □

4 | PROOF OF THEOREM 1.4

We will need the following result, which was proved using Chernoff bound.

**Lemma 4.1** (Shan [23], lemma 3.2). There exists a positive integer $n_0$ such that for all $n \geq n_0$ the following holds. Let $G$ be a graph on $2n$ vertices, and $N = \{x_1, y_1, ..., x_t, y_t\} \subseteq V(G)$, where $t \in [1, n]$. Then $V(G)$ can be partitioned into two parts $A$ and $B$ satisfying the properties below:

(i) $|A| = |B|$
(ii) $|A \cap \{x_i, y_i\}| = 1$ for each $i \in [1, t]$;
(iii) $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ for each $v \in V(G)$, where $d_S(v) = |N_G(v) \cap S|$ for any $S \subseteq V(G)$.

Furthermore, one such partition can be constructed in $O(n^3 \log_2(2n^3))$-time.

**Theorem 1.4.** For all $0 < \varepsilon < 1$, there exists $n_0$ such that the following statement holds. If $G$ is a graph on $2n \geq n_0$ vertices satisfying one of the following three conditions:

(a) $G$ is regular with $\delta(G) \geq (1 + 4\varepsilon/5)n$,
(b) $G$ has two distinct vertices $x, y$ such that $d(x) = d(y) \geq (1/2 + 3\varepsilon/2)n$, for all $z \in V(G) \setminus \{x, y\}$, $d(z) = \Delta(G) \geq (1 + \varepsilon)n$, and $\Delta(G) - \delta(G) \leq (1/2 - \varepsilon/2)n$,  
(c) $\Delta(G) - \delta(G) \geq n^{6/7}$, $|V_\delta| \geq n^{6/7}$ and $|V_\delta| \geq n + 1$, and $\delta(G) \geq (1 + \varepsilon)n$,

then $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

**Proof.** Choose positive integer $n_0$ such that $0 < 1/n_0 \ll \varepsilon$.

If $G$ is in Condition (a), we let $N = \emptyset$. If $G$ is in Condition (b), we let $N = \{x_1, y_1\}$, where $x_1 = x$ and $y_1 = y$. If $G$ is in Condition (c), we take $2[(2n - |V_\delta|)/2]$ vertices from $V(G) \setminus V_\delta$ and name them as $x_1, y_1, ..., x_t, y_t$, where $t := [(2n - |V_\delta|)/2]$ and we assume that the first $|V_\delta|/2$ pairs of vertices $x_i, y_i$ are all from $V_\delta$. Let $N = \{x_1, y_1, ..., x_t, y_t\}$. Applying Lemma 4.1 on $G$ and $N$, we obtain a partition $\{A, B\}$ of $V(G)$ satisfying the following properties:

P.1 $|A| = |B|$
P.2 $|A \cap \{x_i, y_i\}| = 1$ for each $i \in [1, t]$;
P.3 $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ for each $v \in V(G)$.

Thus when $G$ is in Condition (b), we may assume $x \in A$ and $y \in B$. When $G$ is in Condition
(c), we know that $|A \cap V_{A}| \geq \frac{1}{2}(|V_{A}| - 1)$, $|B \cap V_{B}| \geq \frac{1}{2}(|V_{B}| - 1)$, $|A \cap V_{B}| \geq n/2$ and $|B \cap V_{A}| \geq n/2$. By P.3, for any $v \in V(G)$, we have

$$\frac{1}{2}(d_G(v) - n^{2/3}) \leq d_A(v), d_B(v) \leq \frac{1}{2}(d_G(v) + n^{2/3}).$$

Let

$$G_1 = G[A], G_2 = G[B], \text{ and } H = G[A,B].$$

To prove the theorem, we will construct an edge coloring of $G$ using $\Delta(G)$ colors. We provide below an overview of the steps. At the start of the process, $E(G)$ is assumed to be uncolored, and throughout the process, the partial edge coloring of $G$ is always denoted by $\varphi$, which is updating step by step.

Step 1 Define $S = \{v \in V(G) : \Delta(G) - d_G(v) \geq 7n^{2/3}\}$. Let $k = \max\{\Delta(G_A), \Delta(G_B)\} + 1$. By Theorem 2.1, we find an edge $k$-coloring $\varphi$ of $G_A \cup G_B$. If there exist distinct $u, v \in S \cap A$ or distinct $u, v \in S \cap B$ such that $\varphi(u) \cap \varphi(v) \neq \emptyset$, we add an edge joining $u$ and $v$ and color the new edge by a color in $\varphi(u) \cap \varphi(v)$. The edge coloring $\varphi$ is updated and we still call it $\varphi$. We iterate this process of adding and coloring edges and call the multigraphs resulting from $G_A$ and $G_B$, respectively, $G_A^*$ and $G_B^*$, and call $G^*$ the union of $G_A^*, G_B^*$ and $H$. We will modify the current edge coloring, which is still named $\varphi$, such that the following properties are satisfied:

1.1 When $G$ is in Conditions (a) or (b),

$$\left|\varphi_A^{-1}(i)\right| = \left|\varphi_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

When $G$ is in Condition (c), assume by symmetry that $e(G_A^*) \leq e(G_B^*)$, then

$$\left|\varphi_A^{-1}(i)\right| \geq \left|\varphi_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

S1.2

$$\sum_{u \in A} \left|\varphi(u)\right| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n,$$

$$\sum_{u \in B} \left|\varphi(u)\right| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n,$$

$$\left|\varphi_A^{-1}(i)\right| \leq 4n^{2/3} \quad \text{and} \quad \left|\varphi_B^{-1}(i)\right| \leq 4n^{2/3} \quad \text{for every } i \in [1, k].$$

Step 2 Modify the partial edge-coloring of $G^*$ obtained in Step 1 by exchanging alternating paths. When this step is finished, each of the $k$ color class will be a 1-factor of $G^*$. During the process of this step, a few edges of $H$ will be colored.
and a few edges of $G_A^*$ and $G_B^*$ will be uncolored. Denote by $R_A$ and $R_B$, respectively, the submultigraphs of $G_A^*$ and $G_B^*$ consisting of the uncolored edges. The two multigraphs $R_A$ and $R_B$ will initially be empty, but one, two or three edges will be added to at least one of them when each time we exchange an alternating path. The conditions below will be satisfied at the completion of this step:

S2.1 The number of uncolored edges in each of $G_A^*$ and $G_B^*$ is less than $12n^{5/3}$.

When $G$ is in Conditions (a) or (b), $G_A^*$ and $G_B^*$ have the same number of uncolored edges; and when $G$ is in Condition (c), the number of uncolored edges in $G_B^*$ is greater than or equal to the number of uncolored edges in $G_A^*$ (this follows from our assumption that $e(G_A^*) \leq e(G_B^*)$).

S2.2 $\Delta(R_A)$ and $\Delta(R_B)$ are less than $n^{5/6} + 1$.

S2.3 Define $S_A = \{u \in S \cap A : d_{G_A^*}(u) \leq k - 2n^{2/3}\}$ and $S_B = \{u \in S \cap B : d_{G_B^*}(u) \leq k - 2n^{2/3}\}$.

We require

S2.3.1 Every vertex in $V(G^*) \setminus (S_A \cup S_B)$ is incident in $G^*$ with fewer than $2n^{5/6}$ colored edges of $H$.

S2.3.2 When $G$ is in Condition (b), each of the vertex from $S_A \cup S_B$ is incident in $G^*$ with fewer than $(\frac{1}{4} - \frac{1}{5\varepsilon})n$ colored edges of $H$.

S2.3.3 When $G$ is in Condition (c), each of the vertex from $S_A \cup S_B$ is incident in $G^*$ with fewer than $(\frac{1}{2} - \frac{1}{3\varepsilon})n$ colored edges of $H$.

Step 3 We will edge color $R_A$ and $R_B$ and a few uncolored edges of $H$ using another $\ell$ colors, where $\ell = \lceil 2n^{5/6} \rceil$. The goal is to ensure that each of these $\ell$ new color classes obtained at the completion of Step 3 presents at all vertices from $V(G^*) \setminus V_\delta$ while preserving the $k$ 1-factors already obtained through Steps 1 and 2.

Step 4 At the start of Step 4, all of the uncolored edges of $G^*$ belong to $H$. Denote by $R$ the subgraph of $G^*$ consisting of the uncolored edges. It will be shown that $\Delta(R) = \Delta(G^*) - k - \ell$. This subgraph is bipartite, so we can color its edges using $\Delta(G^*) - k - \ell$ colors by Theorem 2.2.

When Step 4 is completed, we obtain an edge coloring of $G^*$ using exactly $\Delta(G^*)$ colors. We now give the details of each step, and for concepts that were already defined in the outline above, we will use them directly.

Step 1: Coloring $G_A$ and $G_B$

Recall $S = \{v \in V(G) : \Delta(G) - d_G(v) \geq 7n^{2/3}\}$. Note that when $G$ is in Condition (a), $S = \emptyset$; when $G$ is in Condition (b), then $S \subseteq \{x, y\}$; and when $G$ is in Condition (c), then $V_\delta \subseteq S$. Following the operations described in the outline of Step 1, for the current edge coloring $\phi$ of $G_A^* \cup G_B^*$, the following statement holds: $\phi(u) \cap \phi(v) = \emptyset$ for any two distinct $u, v \in S \cap A$ or any two distinct $u, v \in S \cap B$. Therefore,
\[
\sum_{u \in A \cap S} |\varphi(u)| = \sum_{u \in A \cap S} (k - d_{G^*}(u)) \leq k, \quad \sum_{u \in B \cap S} |\varphi(u)| = \sum_{u \in B \cap S} (k - d_{G^*}(u)) \leq k. \quad (S1.II)
\]

We will in the rest of the proof show that \(\chi'(G^*) = \Delta(G^*)\), this is because \(G\) is a subgraph of \(G^*\) and \(\Delta(G^*) = \Delta(G)\). The latter is seen as follows: for any \(u \in S \cap A\), we have

\[
d_{G^*}(u) \leq k + e_G(u, B) \leq \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 + \frac{1}{2}(\Delta(G) - 7n^{2/3} + n^{2/3}) \leq \Delta(G).
\]

Similarly, we have \(d_{G^*}(u) \leq \Delta(G)\) for any \(u \in S \cap B\). In particular, if \(u \in V_5\), as \(\Delta(G) - \delta(G) \geq n^{6/7}\), we have

\[
d_{G^*}(u) \leq \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 + \frac{1}{2}(\Delta(G) - n^{6/7} + n^{2/3}) \leq \Delta(G) - \frac{1}{3}n^{6/7}. \quad (S1.III)
\]

Let \(\varphi_A\) and \(\varphi_B\) be the restrictions of \(\varphi\) on \(G^*_A\) and \(G^*_B\), respectively. By Lemma 2.3 and the comments immediately below the lemma, we modify \(\varphi_A\) and \(\varphi_B\) into equitable edge \(k\)-colorings of \(G^*_A\) and \(G^*_B\), respectively, and still call \(\varphi\) the edge \(k\)-coloring of \(\cup G^*_A G^*_B\) consisting of the modifications of \(\varphi_A\) and \(\varphi_B\). Note that under the new colorings, it is possible that \(\varphi(u) \cap \varphi(v) \neq \emptyset\) for some distinct \(u, v \in S \cap A\) or distinct \(u, v \in S \cap B\). However the inequalities in (S1.II) still hold.

When \(G\) is in Conditions (a) or (b), we have \(|S| \leq 2\) and \(e(G_A) = e(G_B)\) by the partition \([A, B]\) of \(V(G)\). Since \(|S \cap A| = |S \cap B| \leq 1\), it follows that \(G^*_A = G_A\) and \(G^*_B = G_B\). Thus \(e(G^*_A) = e(G^*_B)\). Since \(\varphi_A\) and \(\varphi_B\) are equitable edge \(k\)-colorings of \(G^*_A\) and \(G^*_B\), by renaming some color names in \(G^*_A\) if necessary, we assume

\[
|\varphi^{-1}_A(i)| = |\varphi^{-1}_B(i)| \quad \text{for every } i \in [1, k].
\]

When \(G\) is in Condition (c), by symmetry, we assume \(e(G^*_A) \leq e(G^*_B)\). For the same reasoning as above, we assume

\[
|\varphi^{-1}_A(i)| \geq |\varphi^{-1}_B(i)| \quad \text{for every } i \in [1, k].
\]

By the Parity Lemma, \(|\varphi^{-1}_A(i)| - |\varphi^{-1}_B(i)|\) is even for every \(i \in [1, k]\). Therefore, we have the statement S1.1 as stated in the outline of Step 1.

Next, we verify that every color \(i \in [1, k]\) is missing at a small number of vertices. Property P.2 of the partition \([A, B]\) implies \(|A \cap V_6| \geq n/2\) and \(|B \cap V_6| \geq n/2\), and each vertex \(u \in V_A\) satisfies \(\varphi(u) \leq n^{2/3} + 1\), call this Fact 1. By the definition of \(S\), for every \(u \in V(G^*) \setminus S\), \(d_{G^*}(u) = d_G(u) \geq \Delta(G^*) - 7n^{2/3}\), and Property P.3 of the partition \([A, B]\) implies \(d_{G^*_A}(u) \geq \frac{1}{2}d_G(u) - n^{2/3}\) for every \(u \in A \setminus S\) and \(d_{G^*_B}(u) \geq \frac{1}{2}d_G(u) - n^{2/3}\) for every
Thus \(|\overline{\varphi}(u)| \leq k - (1/2)d_G(u) - n^{2/3}) < 6n^{2/3}\) for every \(u \in V(G^*)\setminus S\), call this Fact 2. These two facts together with the fact in (S1.II), give

\[
\sum_{u \in A} |\varphi(u)| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n.
\]

Similarly,

\[
\sum_{u \in B} |\varphi(u)| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n.
\]

Since \(\varphi_A\) and \(\varphi_B\) are equitable edge \(k\)-colorings of \(G^*_A\) and \(G^*_B\), we get

\[
|\overline{\varphi}_A^{-1}(i)| \leq 4n^{2/3} \quad \text{and} \quad |\overline{\varphi}_B^{-1}(i)| \leq 4n^{2/3}.
\]

Therefore, we have the statement S1.2 as stated in the outline of Step 1.

Step 2: Extending existing color classes into 1-factors

Each of the \(k\) color classes obtained in Step 1 will be extended into \(k\) 1-factors of \(G^*\) through exchanging of alternating paths, which consist of colored edges and uncolored edges. The colored edges and uncolored edges of these alternating paths are from \(G^*_A \cup G^*_B\) and \(H\), respectively. Thus during the procedure of Step 2, we will uncolor some of the edges of \(G^*_A\) and \(G^*_B\), and will color some of the edges of \(H\). Recall that \(R_A\) and \(R_B\) are the submultigraphs of \(G^*_A\) and \(G^*_B\) consisting of the uncolored edges, which are empty initially.

To ensure Condition S2.2 is satisfied, we say that an edge \(e = uv \in E(G^*_A \cup G^*_B)\) is good if \(e \not\in E(R_A \cup R_B)\) and the degree of \(u\) and \(v\) in both \(R_A\) and \(R_B\) is less than \(n^{5/6}\) (actually, note that when \(uv \in E(G^*_A)\), then the degree of \(u\) and \(v\) is zero in \(R_B\) and vice versa). Thus a good edge can be added to \(R_A\) or \(R_B\) without violating S2.2.

By S1.1, for each color \(i \in [1, k]\), we pair up each vertex from \(\overline{\varphi}_B^{-1}(i)\) with a vertex from \(\overline{\varphi}_A^{-1}(i)\), and then pair up the remaining unpaired vertices from \(\overline{\varphi}_A^{-1}(i)\) as \(|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)|\) is even and we assumed \(|\overline{\varphi}_A^{-1}(i)| \geq |\overline{\varphi}_B^{-1}(i)|\). Each of those pairs is called a missing-common-color pair or MCC-pair in short with respect to the color \(i\). In particular, when \(G\) is in Conditions (a) or (b), every vertex from \(\varphi_A^{-1}(i)\) is paired up with a vertex from \(\varphi_B^{-1}(i)\).

For every MCC-pair \((a, b)\) with respect to some color \(i \in [1, k]\), we will exchange an alternating path \(P\) from \(a\) to \(b\) with at most 11 edges, where, if exist, the first, third, fifth, seventh, ninth, and eleventh edges are uncolored and the second, fourth, sixth, eighth, and tenth edges are good edges colored by \(i\). After \(P\) is exchanged, \(a\) and \(b\) will be incident with edges colored by \(i\), and at most three good edges will be added to each of \(R_A\) and \(R_B\). With this information at hand, before demonstrating the existence of such paths, we show that Conditions S2.1, S2.2, and S2.3 can be guaranteed at the end of Step 2. After the completion of Step 1, by (S1.I), the total number of missing colors from vertices in \(A\) or from vertices in \(B\) is at most \(4n^{5/3} - 2n\). Thus there are at most \(4n^{5/3} - 2n\) MCC-pairs. For each MCC-pair \((a, b)\) with \(a, b \in V(G^*)\), at most three edges will be added to each of
and when we exchange an alternating path from $a$ to $b$. Thus there will always be fewer than
\[ 3(4n^{5/3} - 2n) < 12n^{5/3} \]
edges in each of $R_A$ and $R_B$. Each of the $k$ color classes is a 1-factor of $G^*$ at the end of Step 2. Thus the number of colored edges in $G^*_A$ is the same as that in $G^*_B$. Since $e(G^*_A) = e(G^*_B)$ when $G$ is in Conditions (a) or (b), and $e(G^*_B) \geq e(G^*_A)$ when $G$ is in Condition (c), we have $e(R_A) = e(R_B)$ when $G$ is in Condition (a) or (b), and $e(R_B) \geq e(R_A)$ when $G$ is in Condition (c). Thus Condition S2.1 will be satisfied at the end of Step 2. And as we only ever add good edges to $R_A$ and $R_B$, Condition S2.2 will hold automatically. We now show that Condition S2.3 will also be satisfied. Recall
\[ S_A = \{ u \in S \cap A : d_{G^*_A}(u) \leq k - 2n^{2/3} \} \quad \text{and} \quad S_B = \{ u \in S \cap B : d_{G^*_B}(u) \leq k - 2n^{2/3} \}. \]
Since $\sum_{u \in S \cap A} (k - d_{G^*_A}(u))$, $\sum_{u \in S \cap B} (k - d_{G^*_B}(u)) \leq k \leq \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 < 2n$, it follows that
\[ |S_A| < n^{1/3} \quad \text{and} \quad |S_B| < n^{1/3}. \]
Thus for every vertex $u \in S \setminus (S_A \cup S_B)$, $|\varphi(u)| < 2n^{2/3}$. For every vertex $u \in V(G^*) \setminus S$, as $d_G(u) = d_{G^*_A}(u) > \Delta(G) - 7n^{2/3}$, it follows that $|\varphi(u)| < k - (\Delta(G) - 7n^{2/3}) < 6n^{2/3}$. Thus for any $u \in V(G^*) \setminus (S_A \cup S_B)$, we have $|\varphi(u)| < 6n^{2/3}$. In the process of Step 2, the number of newly colored edges of $H$ that are incident with a vertex $u \in V(G^*) \setminus (S_A \cup S_B)$ will equal the number of alternating paths containing $u$ that have been exchanged. The number of such alternating paths of which $u$ is the first vertex will equal the number of colors that missed at $u$ at the end of Step 1, which is less than $6n^{2/3}$. The number of alternating paths in which $u$ is not the first vertex will equal the degree of $u$ in $R_A \cup R_B$, and so will be less than $n^{5/6} + 1$. Hence the number of colored edges of $H$ that are incident with $u$ will be less than
\[ 6n^{2/3} + n^{5/6} + 1 < 2n^{5/6}. \]
This applies to all vertices in $V(G^*) \setminus (S_A \cup S_B)$, and so Condition S2.3.1 will be satisfied.

When $G$ is in Condition (b), for any vertex $u \in S_A \cup S_B = \{x, y\}$, since $\Delta(G) - \delta(G) \leq \frac{1}{2}(1 - \varepsilon)n$, we have
\[ |\varphi(u)| \leq k - \frac{1}{2}(d_G(u) - n^{2/3}) \leq k - \frac{1}{2}\left(\Delta(G) - \frac{1}{2}(1 - \varepsilon)n - n^{2/3}\right) \leq \frac{1}{4}(1 - \varepsilon)n + n^{2/3} + 1. \]
Hence the number of colored edges of $H$ that are incident with $u$ will be less than
\[
\frac{1}{4} (1 - \varepsilon)n + n^{2/3} + 1 + n^{5/6} + 1 < \left( \frac{1}{4} - \frac{1}{5} \right)n.
\]

Therefore, Condition S2.3.2 will be satisfied.

When \( G \) is in Condition (c), for any vertex \( u \in S_A \cup S_B \), since \( \Delta(G) - \delta(G) \leq (1 - \varepsilon)n \), we have

\[
|\varphi(u)| \leq k - \frac{1}{2} (d_G(u) - n^{2/3}) \leq k - \frac{1}{2} (\Delta(G) - (1 - \varepsilon)n - n^{2/3})
\]

\[
\leq \frac{1}{2} (1 - \varepsilon)n + n^{2/3} + 1.
\]

Hence the number of colored edges of \( H \) that are incident with \( u \) will be less than

\[
\frac{1}{2} (1 - \varepsilon)n + n^{2/3} + 1 + n^{5/6} + 1 < \left( \frac{1}{2} - \frac{1}{3} \right)n.
\]

Therefore, Condition S2.3.3 will be satisfied.

We now show below the existence of alternating paths for MCC-pairs. For a given color \( i \in [1, k] \), and vertices \( a \in A \) and \( b \in B \), let \( N_B(a) \) be the set of vertices in \( B \) that are joined with \( a \) by an uncolored edge and are incident with a good edge colored \( i \) such that the good edge is not incident with any vertex of \( S_B \), and let \( N_A(b) \) be the set of vertices in \( A \) that are joined with \( b \) by an uncolored edge and are incident with a good edge colored \( i \) such that the good edge is not incident with any vertex of \( S_A \). To estimate the sizes of \( N_A(b) \) and \( N_B(a) \), we show that \( A \) and \( B \) contain only a few vertices that either miss the color \( i \) or are incident with a non-good edge colored \( i \). By S2.1, there are at most \( 12n^{5/3} \) edges in \( R_B \), so there are fewer than \( 24n^{5/6} \) vertices of degree at least \( n^{5/6} \) in \( R_B \). Each non-good edge is incident with one or two vertices of \( R_B \) through the color \( i \), so there are fewer than \( 48n^{5/6} \) vertices in \( B \) that are incident with a non-good edge colored \( i \). Furthermore, there are at most \( 2|S_B| \leq 2n^{1/3} \) vertices in \( B \) that are either contained in \( S_B \) or adjacent to a vertex from \( S_B \) through an edge with color \( i \). Finally, there are fewer than \( 4n^{2/3} \) vertices in \( B \) that are missed by the color \( i \). So the number of vertices in \( B \) that are not incident with a good edge colored \( i \) such that the good edge is not incident with any vertex from \( S_B \) is less than

\[
48n^{5/6} + 2n^{1/3} + 4n^{2/3} < 49n^{5/6}.
\]

By symmetry, the number of vertices in \( A \) that are not incident with a good edge colored \( i \) such that the good edge is not incident with any vertex from \( S_A \) is less than \( 49n^{5/6} \). By S2.3.1, when \( \{a, b\} \cap (S_A \cup S_B) = \emptyset \),

\[
|N_A(b)|, |N_B(a)| \geq \frac{1}{2} ((1 + 4\varepsilon/5)n - n^{2/3}) - 2n^{5/6} - 49n^{5/6} > \left( \frac{1}{2} - \frac{1}{3} \right)n. \quad (S2.1)
\]

When \( G \) is in Condition (b) and \( \{a, b\} \cap \{x, y\} \neq \emptyset \), by S2.3.2, we have
\[|N_A(b)|, |N_B(a)| \geq \frac{1}{2}((1/2 + 3\varepsilon/2)n - n^{2/3}) - \left(\frac{1}{4} - \frac{1}{3}\varepsilon\right)n - 49n^{5/6} > \frac{3}{4}\varepsilon n.\]  

(S2.II)

When \(G\) is in Condition (c) and \([a, b] \cap (S_A \cup S_B) \neq \emptyset\), by S2.3.3, we have

\[|N_A(b)|, |N_B(a)| \geq \frac{1}{2}((1 + \varepsilon)n - n^{2/3}) - \left(\frac{1}{2} - \frac{1}{3}\varepsilon\right)n - 49n^{5/6} > \frac{1}{2}\varepsilon n.\]  

(S2.III)

Let \(M_B(a)\) be the set of vertices in \(B\) that are joined with a vertex in \(N_B(a)\) by an edge of color \(i\), and let \(M_A(b)\) be the set of vertices in \(A\) that are joined with a vertex in \(N_A(b)\) by an edge of color \(i\). Note that \((S_A \cup S_B) \cap (M_A(b) \cup M_B(a)) = \emptyset\) by the choice of \(N_A(b)\) and \(N_B(a)\). Note also that \(|M_B(a)| = |N_B(a)|\) but some vertices may be in both. Similarly \(|M_A(b)| = |N_A(b)|\).

For a MCC-pair \((a, b)\), to have a unified discussion as in the case that \([a, b] \cap (S_A \cup S_B) = \emptyset\), if necessary, by exchanging an alternating path of length 2 from \(a\) to another vertex \(a^*\), and exchanging an alternating path from \(b\) to another vertex \(b^*\), we will replace the pair \((a, b)\) by \((a^*, b^*)\) such that \([a^*, b^*] \cap (S_A \cup S_B) = \emptyset\). Precisely, we will implement the following operations to vertices in \(S_A \cup S_B\). For any vertex \(a \in S_A\), and for each color \(i \in \varphi(a)\), we take an edge \(b_1b_2\) with \(b_1 \in N_B(a)\) and \(b_2 \in M_B(a)\) such that \(b_1b_2\) is colored by \(i\), where the edge \(b_1b_2\) exists by (S2.II)–(S2.III) and the fact that \(|M_B(a)| = |N_B(a)|\). Then we exchange the path \(ab_1b_2\) by coloring \(ab_1\) with \(i\) and uncoloring the edge \(b_1b_2\) (See Figure 1A). After this, the edge \(ab_1\) of \(H\) is now colored by \(i\), and the uncolored edge \(b_1b_2\) is added to \(R_B\). We then update the original MCC-pair that contains \(a\) with respect to the color \(i\) by replacing the vertex \(a\) with \(b_2\). We do this at the vertex \(a\) for every color \(i \in \varphi(a)\) and then repeat the same process for every vertex in \(S_A\). Similarly, for any vertex \(b \in S_B\), and for each color \(i \in \varphi(b)\), we take an edge \(a_1a_2\) with \(a_1 \in N_A(b)\) and \(a_2 \in M_A(b)\) such that \(a_1a_2\) is colored by \(i\), where the edge \(a_1a_2\) exists by (S2.II)–(S2.III) and the fact that \(|M_B(b)| = |N_B(b)|\). Then we exchange the path \(ba_1a_2\) by coloring \(ba_1\) with \(i\) and uncoloring the edge \(a_1a_2\). The same, we update the original MCC-pair that contains \(b\) with respect to the color \(i\) by replacing the vertex \(b\) with \(a_2\).

After the procedure above, we have now three types MCC-pair \((u, v)\): \(u, v \in A\), \(u, v \in B\), and \(A\) contains exactly one of \(u\) and \(v\) and \(B\) contains the other. However, in either case, \([u, v] \cap (S_A \cup S_B) = \emptyset\). We will exchange alternating path for each of such pairs.

We deal with each of the colors from \([1, k]\) in turn. Let \(i \in [1, k]\) be a color. We consider first an MCC-pair \((a, a^*)\) with respect to \(i\) such that \(a, a^* \in A\). By (S2.I), we have \(|M_B(a^*)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n\). We take an edge \(b_1^*b_2^*\) colored by \(i\) with \(b_1^* \in N_B(a^*)\) and \(b_2^* \in M_B(a^*)\). Then again, by (S2.I), we have \(|M_B(a)|, |M_A(a^*)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n\). Therefore, as each vertex \(c \in M_A(b_2^*)\) satisfies \(N_B(c) > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n\), we have \(|N_B(c) \cap M_B(a)| \geq \frac{2}{3}\varepsilon n\). We take \(a_2a_2^*\) colored by \(i\) with \(a_2^* \in N_A(b_2^*)\) and \(a_2 \in M_A(b_2^*)\). Then we let \(b_2 \in N_B(a_2) \cap M_B(a)\), and let \(b_1\) be the vertex in \(N_B(a)\) such that \(b_1b_2\) is colored by \(i\). Now we get the alternating path \(P = ab_1b_2a_2a_2^*b_2^*a^*\) (See Figure 1C). We exchange \(P\) by coloring \(ab_1, b_2a_2, a_2^*b_2^*\) and \(b_1^*a^*\) with color \(i\) and uncoloring the edges \(b_1b_2, b_1^*b_2^*\) and
After the exchange, the color $i$ appears on edges incident with $a$ and $a^*$, the edges $b_1b_2$ and $b_1^*b_2^*$ are added to $R_B$ and the edge $a_2a_2^*$ is added to $R_A$. We added at most one edge to each of $R_A$ and $R_B$ when we updated the original MCC-pair corresponding to $(a,a^*)$. Thus we added at most three edges to each of $R_A$ and $R_B$ when we modify $\varphi$ to have the color $i$ present at both of the vertices in the original MCC-pair corresponding to $(a,a^*)$. By symmetry, we can deal with an MCC-pair $(b,b^*)$ with respect to $i$ such that $b,b^* \in B$ similarly as above.

Thus we consider an MCC-pair $(a,b)$ with respect to $i$ such that $a \in A$ and $b \in B$. By (S2.I), we have $|M_B(a)|, |M_A(b)| \geq \left(\frac{1}{2} + \frac{1}{3}\epsilon\right)n$. We choose $a_1a_2$ with color $i$ such that $a_1 \in N_A(b)$ and $a_2 \in M_A(b)$. Now as $M_B(a), |N_B(a)| > \left(\frac{1}{2} + \frac{1}{3}\epsilon\right)n$ by (S2.I), we know that $N_B(a_2) \cap M_B(a) \neq \emptyset$. We choose $b_2 \in N_B(a_2) \cap M_B(a)$ and let $b_1 \in N_B(a)$ such that $b_1b_2$ is colored by $i$. Then $P = ab_1b_2a_2a_1b$ is an alternating path from $a$ to $b$ (See Figure 1B). We exchange $P$ by coloring $ab_1, b_2a_2$ and $a_1b$ with color $i$ and uncoloring the edges $a_1a_2$ and $b_1b_2$. After the exchange, the color $i$ appears on edges incident with $a$ and $b$, the edge $a_1a_2$ is added to $R_A$ and the edge $b_1b_2$ is added to $R_B$. We added at most one edge to each of $R_A$ and $R_B$ when we updated the original MCC-pair corresponding to $(a,b)$. Thus we added at most three edges to each of $R_A$ and $R_B$ when we modify $\varphi$ to have the color $i$ present at both of the vertices in the original MCC-pair corresponding to $(a,b)$. By finding such paths for all MCC-pairs with respect to the color $i$, we can increase the number of edges colored $i$ until the color class is a 1-factor of $G^*$. By doing this for all colors, we can make each of the $k$ color classes a 1-factor of $G^*$.

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**FIGURE 1** The alternating path $P$. Dashed lines indicate uncolored edges, and solid lines indicate edges with color $i$. (A) $P$ with 3 edges (B) $P$ with 5 edges (C) $P$ with 7 edges
Step 3: Coloring $R_A$ and $R_B$ and extending the new color classes

Each of the color classes for the colors from $[1, k]$ is now a 1-factor of $G^*$. We now consider the multigraphs $R_A$ and $R_B$ that consist of the uncolored edges of $G_A^*$ and $G_B^*$. By Condition S2.1, $R_A$ and $R_B$ each has fewer than $12n^{5/3}$ edges, and $\Delta(R_A), \Delta(R_B) < n^{5/6} + 1$. Note that $R_A$ and $R_B$ might contain parallel edges with endvertices in $S$. By Theorem 2.1 and Theorem 2.3, $R_A$ and $R_B$ each have an equalized edge-coloring with exactly $\ell := \lceil 2n^{5/6} \rceil$ colors $k + 1, ..., k + \ell$.

If $G$ is in Conditions (a) or (b), then we have $e(R_A) = e(R_B)$. Under these two conditions, by renaming some color classes of $R_A$ if necessary, we can assume that in the edge colorings of $R_A$ and $R_B$, each color appears on the same number of edges in $R_A$ as it does in $R_B$. When $G$ is in Condition (c), by our assumption that $G_B^*$ has more edges than $G_A^*$ does, we have $e(R_A) \leq e(R_B)$. In this case, we can assume that in the edge colorings of $R_A$ and $R_B$, the number of edges with a color $i \in [k + 1, k + \ell]$ in $R_B$ is at least the number of edges with a color $i \in [k + 1, k + \ell]$ in $R_A$.

There are fewer than $12n^{5/3}$ edges in each of $R_A$ and $R_B$, and $\ell > n^{5/6}$, so each of the color $i \in [k + 1, k + \ell]$ appears on fewer than $12n^{5/6} + 1$ edges in each of $R_A$ and $R_B$. We will now color some of the edges of $H$ with the $\ell$ colors from $[k + 1, k + \ell]$ so that each of these color classes present at vertices from $V(G^*) \setminus V_5$. We perform the following procedure for each of the $\ell$ colors in turn.

Given a color $i$ with $i \in [k + 1, k + \ell]$, we let $A_i$ and $B_i$ be the sets of vertices in $A$ and $B$ respectively that are incident with edges colored $i$. Note that $|A_i| \leq |B_i| < 2(12n^{5/6} + 1)$ as $R_A$ and $R_B$ each contains fewer than $12n^{5/6} + 1$ edges colored $i$. Note that if $G$ is in Conditions (a) or (b), we have $|A_i| = |B_i|$; and we might have $|B_i| \geq |A_i|$ when $G$ is in Condition (c). When $G$ is in Condition (c) and $|B_i| > |A_i|$, we let

$$A_i^* \subseteq (V_6 \cap A) \setminus A_i$$

such that $|A_i^*| + |A_i| = |B_i|$, and just let $A_i^* = \emptyset$ otherwise. Note that such $A_i^*$ exists as $|V_6 \cap A_i| \geq \frac{1}{2}n^{6/7} - 1$ and $|A_i|, |B_i| < 2(12n^{5/6} + 1)$. Let $H_i$ be the subgraph of $H$ obtained by deleting the vertex sets $A_i \cup A_i^* \cup B_i$ and removing all colored edges. We will show next that $H_i$ has a perfect matching and we will color the edges in the matching by the color $i$.

Each vertex in $V(G^*) \setminus (S_A \cup S_B)$ is incident with fewer than $2n^{5/6} + \ell \leq 5n^{5/6}$ edges of $H$ that are colored, since fewer than $2n^{5/6}$ were colored in Step 2 by S2.3.1 and at most $2n^{5/6} + 2 < 3n^{5/6}$ have been colored in Step 3. Also each vertex in $G^*$ has fewer than $2(12n^{5/6} + 1)$ edges that join it with a vertex in $A_i$ or $B_i$. So each vertex from $V(H_i) \setminus (S_A \cup S_B)$ is adjacent in $H_i$ to more than

$$\frac{1}{2}((1 + 4\varepsilon/5)n - n^{2/3}) - 5n^{5/6} - 2(12n^{5/6} + 1) > \frac{1}{2}(1 + \varepsilon/2)n$$

vertices.

When $G$ is in condition (b), each vertex in $S_A \cup S_B$ is incident with fewer than $(\frac{1}{4} - \frac{1}{5}\varepsilon)n + 3n^{5/6}$ edges of $H$ that are colored, since fewer than $(\frac{1}{4} - \frac{1}{5}\varepsilon)n$ were colored in Step 2 by S2.3.2 and at most $3n^{5/6}$ have been colored in Step 3. Also each vertex in $G^*$
has fewer than $2(12n^{5/6} + 1)$ edges that join it with a vertex in $A_i$ or $B_i$. So when $G$ is in Condition (b), each vertex from $S_A \cup S_B$ is adjacent in $H_i$ to more than

$$\frac{1}{2}((1/2 + 3\varepsilon/2)n - n^{2/3}) - \left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n + 3n^{5/6} - 2(12n^{5/6} + 1) > \frac{3}{4}\varepsilon n$$

vertices.

When $G$ is in condition (c), each vertex in $S_A \cup S_B$ is incident with fewer than $(\frac{1}{2} - \frac{1}{2}\varepsilon)n + 3n^{5/6}$ edges of $H$ that are colored, since fewer than $(\frac{1}{2} - \frac{1}{3}\varepsilon)n$ were colored in Step 2 by S2.3.3 and at most $3n^{5/6}$ have been colored in Step 3. Also each vertex in $G^*$ has fewer than $2(12n^{5/6} + 1)$ edges that join it with a vertex in $A_i \cup A_i^*$ or $B_i$. So when $G$ is in Condition (c), each vertex from $S_A \cup S_B$ is adjacent in $H_i$ to more than

$$\frac{1}{2}((1 + \varepsilon)n - n^{2/3}) - \left(\frac{1}{2} - \frac{1}{3}\varepsilon\right)n + 3n^{5/6} - 2(12n^{5/6} + 1) > \frac{1}{2}\varepsilon n$$

vertices.

Thus $\delta(H_i) \geq \frac{1}{2}\varepsilon n$ in either case and $H_i$ has at most $S_A \cup S_B \leq 2n^{1/3} < \frac{1}{2}\varepsilon n$ vertices of degree less than $\frac{1}{3}n$. So $H_i$ has a 1-factor $F$ by Lemma 2.11. If we color the edges of $F$ with the color $i$, then every vertex in $V(G^*) \setminus A_i^*$ is incident with an edge of color $i$. We repeat this procedure for each of the colors from $[k + 1, k + \ell]$. After this has been done, each of these $\ell$ colors presents at all vertices from $V(G^*) \setminus V_5$. So at the conclusion of Step 3, all of the edges in $G_A^*$ and $G_B^*$ are colored, some of the edges of $H$ are colored, each of the $k$ color classes for colors from $[1, k]$ is a 1-factor of $G^*$, and each of the $\ell$ colors from $[k + 1, k + \ell]$ presents at all vertices from $V(G^*) \setminus V_5$.

Step 4: Coloring the graph $R$

Let $R$ be the subgraph of $G^*$ consisting of the remaining uncolored edges. These edges all belong to $H$, so $R$ is a subgraph of $H$ and hence is bipartite. We claim that $\Delta(R) = \Delta(G^*) - k - \ell$. Note that every vertex from $V(G^*) \setminus V_5$ presents every color from $[1, k + \ell]$ and so those vertices have degree at most $\Delta(G^*) - k - \ell$ in $R$. For the vertices from $V_5$, they present all the colors from $[1, k]$. Thus by (S1.III), those vertices have degree at most

$$\Delta(G^*) - \frac{1}{3}n^{6/7} - k < \Delta(G^*) - k - \ell$$

in $R$. By Theorem 2.2 we can color the edges of $R$ with $\Delta(R)$ colors from $[k + \ell + 1, \Delta(G^*)]$. Thus $\chi'(G^*) \leq k + \ell + (\Delta(G^*) - k - \ell) = \Delta(G^*)$ and so $\chi'(G^*) = \Delta(G^*)$, as desired.

Lastly, we check that there is a polynomial time algorithm to obtain an edge $\Delta(G)$-coloring of $G$. By Lemma 4.1, we can obtain a desired partition $\{A, B\}$ of $V(G)$ in polynomial time. Also, it is polynomial time to edge color $G_A$ and $G_B$ by an algorithm described in [17]. Modifying $G_A$ and $G_B$ into $G_A^*$ and $G_B^*$ and the corresponding edge colorings into equalized edge-colorings can be done in polynomial time too. In Step 2, the construction of the alternating paths and swaps of the colors on the paths can be done in $O(n^3)$-time, as the total number of colors missing at vertices is $O(n^2)$ and it takes $O(n)$-time to find an alternating path for a MCC-pair. In Step 3, there is a polynomial time
algorithm (see e.g. [14]) to edge color $R_A$ and $R_B$ using at most $\ell$ colors. Then by doing Kempe changes as mentioned in the comments immediately after Theorem 2.3, these edge colorings can be modified into equalized edge-colorings in polynomial time. The last step is to edge color the bipartite graph $R$ using $\Delta(R)$ colors, which can be done in polynomial-time in $n$, for example, using an algorithm from [5]. Thus, there is a polynomial time algorithm that gives an edge coloring of $G$ using $\Delta(G)$ colors. □

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