REPRESENTATIONS OF THE QUANTUM ALGEBRA $U_q(\mathfrak{u}_{n,1})$

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Abstract

Infinite dimensional representations of the real form $U_q(\mathfrak{u}_{n,1})$ of the Drinfeld–Jimbo algebra $U_q(\mathfrak{gl}_{n+1})$ are defined. The principal series of representations of $U_q(\mathfrak{u}_{n,1})$ is studied. Intertwining operators for pairs of the principal series representations are calculated in an explicit form. The structure of reducible representations of the principal series is determined. Irreducible representations of $U_q(\mathfrak{u}_{n,1})$, obtained from irreducible and reducible principal series representations, are classified. All $*$-representations in this set of irreducible representations are separated. Unlike the classical case, the algebra $U_q(\mathfrak{u}_{n,1})$ has finite dimensional irreducible $*$-representations.

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1. Introduction

Under quantum algebras we mean quantized universal enveloping algebras defined by Drinfeld [1] and Jimbo [2]. We denote them by $U_q(g)$, where $q$ is a deformation parameter and $g$ is a corresponding semisimple Lie algebra. Finite dimensional irreducible representations of $U_q(g)$ were classified by Rosso [3] and Lusztig [4] (see also [5] and [6]). It was shown that every irreducible finite dimensional representation of a simple Lie algebra $g$ can be deformed to be an irreducible representation of the corresponding quantum algebra $U_q(g)$. Moreover, if $q$ is not a root of unity, then every finite dimensional irreducible representation of $U_q(g)$ is essentially obtained in this way (after possibly tensoring by a one-dimensional representation).

Along with finite dimensional representations of quantum algebras their infinite dimensional representations are also important. For example, infinite dimensional representations of the algebra $U_q(su_{1,1})$ are closely related to the $q$-oscillator algebra. They were used in the theory of $q$-special functions. Our aim in this paper is to construct infinite dimensional representations of the real form $U_q(u_{n,1})$ of the quantum algebra $U_q(gl_{n+1})$.

Simplest infinite dimensional irreducible representations of quantum algebras are determined by Verma modules. The theory of Verma modules of $U_q(g)$ when $q$ is not a root of unity is similar to that for the corresponding universal enveloping algebras $U(g)$ (the details of this construction see in [4] and [6]) and hence we do not concern Verma modules in this paper.

It is possible to give different definitions of infinite dimensional irreducible representations of $U_q(u_{n,1})$. We use the definition similar to that for the classical case. As in the classical case, we demand that the restriction of a representation $T$ of $U_q(u_{n,1})$ to a maximal compact subalgebra $U_q(u_n \oplus u_1)$ decomposes into a direct sum of finite dimensional irreducible representations of this subalgebra.

In the case of infinite dimensional representations of real semisimple Lie algebras, there is a one-to-one correspondence between irreducible infinite dimensional representations of a connected simply connected real semisimple Lie group $G$ and those of its Lie algebra $g$ (we consider only those representations of $G$ and $g$ which correspond to Harish-Chandra $g$-modules). Let $P = MAN$ be a minimal parabolic subgroup in $G$ [7], and let $\omega$ be an irreducible finite dimensional representation of $P$. We induce by $\omega$ the representation $T_\omega$ of $G$. The representations $T_\omega$ constitute the principal nonunitary series of $G$. Using the method described in [8] we can construct the corresponding representations $T_\omega$ of $g$ induced by representations $\omega$ of the subalgebra $p = m + a + n$ (the Lie algebra of $P$). By Harish-Chandra’s theorem, every irreducible representation of $g$ (or of $G$) is equivalent to some irreducible representation $T_\omega$ or to a subrepresentation in a quotient representation of some reducible representation $T_\omega$. By means of this theorem a classification of all irreducible representations of $g$ and of $G$ has been obtained.

An essential item in construction of these representations is the Iwasawa decomposition of $G$ (and of $g$) and parabolic subgroups. This method cannot be extended to
the quantum case since we have no Iwasawa decomposition and parabolic subalgebras for real forms of quantum algebras.

In order to construct infinite dimensional representations of $U_q(u_{n,1})$ we write down explicit formulas for representation operators corresponding to the generating elements of $U_q(u_{n,1})$. Then we prove that they indeed determine representations of $U_q(u_{n,1})$. In this way we obtain for $U_q(u_{n,1})$ a $q$-analogue of the principal nonunitary series. Then we succeed as in the classical case, that is, we separate in this set of representations irreducible ones and find all possible irreducible components of reducible representations. Thus, we receive a set $\Omega$ of irreducible representations of $U_q(u_{n,1})$. It will be shown in the separate paper that the set $\Omega$ exhausts all irreducible representations of $U_q(u_{n,1})$. By the standard method, irreducible $^\ast$-representations are separated in the set $\Omega$. Note that the results of this paper were announced (without proofs) in the preprint [9].

Since in the set $\Omega$ there are irreducible finite dimensional representations of $U_q(g_{n+1})$, then we obtain a new proof of the Gel'fand–Tsetlin formulas for finite dimensional representations of the quantum algebra $U_q(gl_{n+1})$. Remark that these formulas for $U_q(gl_{n+1})$ were stated without proof by Jimbo [10]. The proof was given in [11]. However, this proof is rather very complicated. Our proof is much more simple and natural.

2. The quantum algebras $U_q(u_n(\varepsilon))$

The quantum algebra $U_q(gl_n)$ is a $q$-deformation of the universal enveloping algebra $U(gl(n, \mathbb{C}))$ and is generated by the elements

$k_i, k_i^{-1}, e_j, f_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n - 1,$

satisfying the relations

1. $k_i k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$
2. $k_i e_j k_i^{-1} = q^{\delta_{ij} - \delta_{i+1,j+1}} e_j, \quad k_i f_j k_i^{-1} = q^{-\delta_{ij} + \delta_{i+1,j+1}} f_j,$
3. $[e_i, f_j] = \delta_{ij} \frac{k_i k_{i+1}^{-1} - k_{i+1}^{-1} k_i}{q - q^{-1}},$
4. $[e_i, e_j] = [f_i, f_j] = 0, \quad |i - j| > 1,$
5. $e_i^2 e_{i \pm 1} - (q + q^{-1}) e_i e_{i \pm 1} e_i + e_{i \pm 1} e_i^2 = 0,$
6. $f_i^2 f_{i \pm 1} - (q + q^{-1}) f_i f_{i \pm 1} f_i + f_{i \pm 1} f_i^2 = 0.$

The last two relations are called the $q$-Serre relations. A structure of a Hopf algebra is introduced into $U_q(sl_n)$ (see, for example [5]). We do not need this Hopf algebra structure in this paper and shall use only the algebraic structure of $U_q(gl_n)$.

We suppose throughout the paper that $q$ is a positive real number and equip $U_q(gl_n)$ with $^\ast$-structures. Different $^\ast$-structures define real forms of $U_q(gl_n)$. Let
\[ \epsilon := (\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}), \ \epsilon_i = \pm1. \] The \(*\)-structure of \( U_q(\mathfrak{g}_n) \) associated with \( \epsilon \) is determined by

\[ (k_i^{\pm1})^* = k_i^{\pm1}, \ \ e_i^* = \epsilon_i-1 \epsilon_i f_i, \ \ f_i^* = \epsilon_i-1 \epsilon_i e_i \]

and is denoted by \( U_q(\mathfrak{u}_n(\epsilon)) \). In particular, the \(*\)-structure determined by

\[ (k_i^{\pm1})^* = k_i^{\pm1}, \ \ e_p^* = -f_p, \ \ f_p^* = -e_p, \] \hspace{1cm} (7)

\[ e_i^* = f_i, \ \ f_i^* = e_i, \ \ i \neq p, \] \hspace{1cm} (8)

defines the real form denoted by \( U_q(\mathfrak{u}_{p,n-p}) \). The real form \( U_q(\mathfrak{u}_n(\epsilon)) \) with \( \epsilon = (1,1,\ldots,1) \) is called the compact real form of \( U_q(\mathfrak{g}_n) \) and is denoted by \( U_q(\mathfrak{u}_n) \).

Note that the quantum algebras \( U_q(\mathfrak{u}_n(\epsilon)) \) are of a considerable interest since they are closely related to the quantum hyperboloids. The quantum hyperboloid \( M^{2n-1}_q \) is the associative algebra generated by the elements \( z_i, \hat{z}_i, i = 0,1,\ldots,n-1, \) and \( \kappa \) with the defining relations

\[ z_i z_j = q^{-1} z_j z_i, \ \ \hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \ \ i < j, \]

\[ z_i \hat{z}_j = q^{-1} \hat{z}_j z_i, \ \ \hat{z}_i \hat{z}_j \neq j, \ \ z_i \hat{z}_i - \hat{z}_i z_i = (q^2 - 1) \left( \sum_{k \geq i} z_k \hat{z}_k + \kappa \right), \]

\[ z_i \kappa = q^{-2} \kappa z_i, \ \ \hat{z}_i \kappa = q^2 \kappa \hat{z}_i \]

and equipped with the \(*\)-structure determined by the formulas

\[ z_i^* = \epsilon_i \epsilon_{n-1} \hat{z}_i, \ \ \kappa^* = \kappa. \]

The algebra \( U_q(\mathfrak{u}_n(\epsilon)) \) acts on \( M^{2n-1}_q \) as

\[ e_i : \ z_j \rightarrow \delta_{ij} q^{1/2} z_{j-1}, \ \ \hat{z}_j \rightarrow -\delta_{i-1,j} q^{3/2} \hat{z}_{j+1}, \ \ \kappa \rightarrow 0, \]

\[ f_i : \ z_j \rightarrow \delta_{i-1,j} q^{-1/2} z_{j+1}, \ \ \hat{z}_j \rightarrow -\delta_{ij} q^{-3/2} \hat{z}_{j-1}, \ \ \kappa \rightarrow 0, \]

\[ k_i : \ z_{i-1} \rightarrow q^{1/2} z_{i-1}, \ \ \hat{z}_{i-1} \rightarrow q^{-1/2} \hat{z}_{i-1}, \ \ \kappa \rightarrow \kappa, \]

\[ k_i : \ z_i \rightarrow q^{-1/2} z_i, \ \ \hat{z}_i \rightarrow q^{1/2} \hat{z}_i, \ \ z_j \rightarrow z_j, \ \ \hat{z}_j \rightarrow \hat{z}_j, \ \ j \neq i-1,i. \]

This action turns \( M^{2n-1}_q \) into \( U_q(\mathfrak{u}_{n}(\epsilon))-\)module algebra. Harmonic analysis on the hyperboloid \( M^{2n-1}_q \) demands to have irreducible \(*\)-representations of \( U_q(\mathfrak{u}_n(\epsilon)) \). The harmonic analysis on the quantum hyperboloid \( M^3_q \), related to the quantum algebra \( U_q(\mathfrak{su}_{1,1}) \), see in [12].

In order to describe representations of the algebra \( U_q(\mathfrak{u}_{n,1}) \) we need finite dimensional representations of the subalgebra \( U_q(\mathfrak{u}_n) \).

3. Irreducible representations of \( U_q(\mathfrak{u}_n) \)

The irreducible finite dimensional representations of the algebra \( U_q(\mathfrak{u}_n) \) are given by \( n \) integers \( \mathbf{m} = (m_1, m_2, \ldots, m_n) \) such that

\[ m_1 \geq m_2 \geq \cdots \geq m_n. \]
These representations will be denoted by $T_m$. The set of numbers $m = (m_1, m_2, \ldots, m_n)$ is called the highest weight of the representation $T_m$. The representations $T_m$ and $T_{m'}$ are not equivalent if $m \neq m'$.

The Gel'fand–Tsetlin bases of carrier spaces of irreducible representations $T_m$ are formed by successive restrictions of the representations to the subalgebras $U_q(u_{n-1})$, $U_q(u_{n-2})$, $\ldots$, $U_q(u_1) \equiv U(u_1)$. The decomposition of the representation $T_m$ of $U_q(u_n)$ into irreducible representations of $U_q(u_{n-1})$ is the same as for the corresponding representation $T_m$ of $gl(n, \mathbb{C})$. Hence the restriction of $T_m$, $m = (m_1, \ldots, m_n)$, to $U_q(u_{n-1})$ decomposes into the irreducible representations $T_{m_{n-1}}$, $m_{n-1} = (m_{1,n-1}, \ldots, m_{n-1,n-1})$, such that

$$m_1 \geq m_{1,n-1} \geq m_2 \geq m_{2,n-1} \geq \cdots \geq m_{n-1,n-1} \geq m_n$$

and each of these representations enters into the decomposition exactly once. Since the irreducible representations of $U(u_1)$ are one-dimensional, we obtain a basis of the carrier space $V_m$ of the representation $T_m$ of $U_q(u_n)$ labeled by the Gel'fand–Tsetlin tableaux

$$M = \begin{pmatrix}
m_{1,n} & m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & m_{n,n} \\
m_{1,n-1} & m_{2,n} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
m_{11} & & & & m_{n,n}
\end{pmatrix},$$

where $m_{i,n} \equiv m_i$. The entries in (10) are integers satisfying the betweenness conditions

$$m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1}, \quad i = 1, 2, \ldots, \hat{j}, \quad j = 1, 2, \ldots, n - 1.$$  

The set of all tableaux (10), satisfying these conditions, labels the basis elements of the carrier space of $T_m$. If $M$ is a tableau (10), then the corresponding basis element will be denoted by $|M\rangle$.

It was stated by Jimbo [10] that the generators of $U_q(u_n)$ act on the Gel'fand–Tsetlin basis of the representation by the formulas

$$T_m(k_r)|M\rangle = q^{a_r}|M\rangle, \quad a_r = \sum_{i=1}^r m_{ir} - \sum_{i=1}^{r-1} m_{i,r-1}, \quad 1 \leq r \leq n,$$  

$$T_m(e_r)|M\rangle = \sum_{j=1}^r A^+_j(M)|M^{+\rangle_j}\rangle, \quad T_m(f_r)|M\rangle = \sum_{j=1}^r A^-_j(M^{-\langle j}|M^{-j}\rangle\rangle, \quad 1 \leq r \leq n - 1.$$  

Here $M^{+\langle j}$ is the Gel'fand–Tsetlin tableau obtained from the tableau (10) if $m_{jr}$ is replaced by $m_{jr} \pm 1$, and $A^+_j(M)$ is the expression

$$A^+_j(M) = \left( -\frac{\prod_{i=1}^{r+1} |l_{i,r+1} - l_{jr}| \prod_{i=1}^{r-1} |l_{i,r-1} - l_{jr} - 1|}{\prod_{i \neq j} |l_{ir} - l_{jr}| |l_{ir} - l_{jr} - 1|} \right)^{1/2},$$

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where $l_i = m_i s - i$, the positive value of the square root is taken and a number in square brackets denotes a $q$-number defined by

\[ [m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \]

4. Definition of representations of $U_q(u_{n,1})$

In this paper we are interested in representations of the quantum algebra $U_q(u_{n,1})$. We distinguish for this algebra the notions of representations and $*$-representations. Roughly speaking, the notion of a representation does not take into account the $*$-structure of $U_q(u_{n,1})$. A $*$-representation is a representation conserving the $*$-structure. The strict definitions are as follows.

A representation $T$ of the algebra $U_q(u_{n,1})$ is an algebraic homomorphism from $U_q(u_{n,1})$ to an algebra of linear (bounded or unbounded) operators on a Hilbert space $H$ for which the following conditions are fulfilled:

(a) the restriction of $T$ onto the maximal compact subalgebra $U_q(u_n + u_1)$ decomposes into a direct sum of its finite dimensional irreducible representations;

(b) operators of a representation $T$ are defined on everywhere dense subspace $V$ of $H$, containing all subspaces which are carrier spaces of irreducible finite dimensional subrepresentations of $U_q(u_n + u_1)$ from the restriction of $T$ onto this subalgebra;

(c) the subspace $V$ is invariant with respect to all operators of the representation $T$.

To determine a representation $T$ of $U_q(u_{n,1})$ it is sufficient to give the operators $T(k_i^{\pm 1})$, $T(e_j)$, $T(f_j)$, $i = 1, 2, \cdots, n + 1$, $j = 1, 2, \cdots, n$, satisfying the relations (1)–(6) written down for the algebra $U_q(gl_{n+1})$.

A representation $T$ of $U_q(u_{n,1})$ is called $U_q(u_n + u_1)$-finite if each irreducible representation of $U_q(u_n + u_1)$ is contained in the restriction of $T$ to $U_q(u_n + u_1)$ with finite multiplicity. In other words, $U_q(u_n + u_1)$-finite representations of $U_q(u_{n,1})$ are Harish-Chandra modules of $U_q(u_{n,1})$ with respect to $U_q(u_n + u_1)$. A linear span of irreducible subspaces of $H$ with respect to the subalgebra $U_q(u_n + u_1)$ will be denoted by $D$. We shall see that $D$ is invariant with respect to $U_q(u_n + u_1)$-finite representations of $U_q(u_{n,1})$.

Below we consider only $U_q(u_n + u_1)$-finite representation of $U_q(u_{n,1})$ and shall not emphasize this every time. So, everywhere below under a representation of $U_q(u_{n,1})$ we understand its $U_q(u_n + u_1)$-finite representation.

A representation $T$ of $U_q(u_{n,1})$ on $H$ is called irreducible if the subspace $D$ has no non-trivial invariant subspaces (that is, $H$ has no non-trivial invariant subspaces with closure coinciding with $H$).

If operators of a representation $T$ satisfy relations (7) and (8) on the common domain of definition $V$, then $T$ is called a $*$-representation.

Two representations $T$ and $T'$ of $U_q(u_{n,1})$ on Hilbert spaces $H$ and $H'$ are called (algebraically) equivalent if there exists a one-to-one operator $A : D \to D'$ such that $T'(a)Av = AT(a)v$ for all $a \in U_q(u_{n,1})$ and $v \in D$. 

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Lemma 1. Let $T$ be a representation of $U_q(u_{n,1})$ on a Hilbert space $\mathcal{H}$ such that the restriction of $T$ to the subalgebra $U_q(u_n)$ decomposes into a direct sum of its irreducible finite dimensional representations $T_{m_n}$ and each of them is contained in the decomposition with the unit multiplicity. Let

$$\mathcal{H} = \oplus_{m_n} \mathcal{V}_{m_n}$$ \hspace{1cm} (15)$$

be the corresponding decomposition of $\mathcal{H}$ into irreducible $U_q(u_n)$-invariant subspaces. If $\mathcal{H}'$ is a $U_q(u_{n,1})$-invariant subspace of $\mathcal{H}$, then $\mathcal{H}'$ is a direct sum of some subspaces $\mathcal{V}_{m_n}$ of the decomposition (15).

Proof. Let $\mathcal{H}'$ be a $U_q(u_{n,1})$-invariant subspace of $\mathcal{H}$ and let $v = v_{m_n} + v_{m'_n} \in \mathcal{H}'$, where $v_{m_n} \in \mathcal{V}_{m_n}$ and $v_{m'_n} \in \mathcal{V}_{m'_n}$ (for simplicity, we consider the case of two summands; the case of more summands is considered similarly). By the results of [13] (see also Proposition 7.9 in [5]), there exists an element $Z$ of the center of $U_q(u_n)$ such that $\lambda_{m_n} \equiv T_{m_n}(Z) \neq \lambda_{m'_n} \equiv T_{m'_n}(Z)$. Since

$$T(Z)v = T_{m_n}(Z)v_{m_n} + T_{m'_n}(Z)v_{m'_n} = \lambda_{m_n}v_{m_n} + \lambda_{m'_n}v_{m'_n} \in \mathcal{H},$$

then $T(Z)v - \lambda_{m_n}v = (\lambda_{m'_n} - \lambda_{m_n})v_{m'_n} \in \mathcal{H}'$. Thus, $v_{m_n} \in \mathcal{H}'$ and $v_{m'_n} \in \mathcal{H}'$. This means that $T(U_q(u_n))v_{m_n} = \mathcal{V}_{m_n} \in \mathcal{H}'$ and $T(U_q(u_n))v_{m'_n} = \mathcal{V}_{m'_n} \in \mathcal{H}'$. Lemma is proved.

5. The principal series representations of $U_q(u_{n,1})$

In this section we construct a series of representations of $U_q(u_{n,1})$ which is a $q$-analogue of the principal nonunitary series of the real Lie algebra $u_{n,1}$. We first describe these representations and then prove that they indeed are representations of $U_q(u_{n,1})$.

Let $c_1$ and $c_2$ be complex numbers such that $c_1 + c_2 = m_0$ is an integer, and let $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$ be a set of integers such that $m_1 \geq m_2 \geq \cdots \geq m_{n-1}$. The number $m_0$ determines a one-dimensional representation of the subalgebra $U_q(u_1)$ and $\mathbf{m}$ determines an irreducible finite dimensional representation of the subalgebra $U_q(u_{n-1})$. The numbers $\mathbf{m}, c_1, c_2$ determine the representation $T_{\mathbf{m},c_1,c_2}$ of the quantum algebra $U_q(u_{n,1})$ which is defined as follows. The restriction of $T_{\mathbf{m},c_1,c_2}$ onto $U_q(u_n)$ decomposes into the direct sum of all irreducible representations $T_{m_n}$ of this subalgebra with highest weights $\mathbf{m}_n = (m_{1n}, \ldots, m_{nn})$ such that

$$m_{1n} \geq m_1 \geq m_{2n} \geq m_2 \geq \cdots \geq m_{n-1} \geq m_{nn}. \hspace{1cm} (16)$$

Every of these representations of $U_q(u_n)$ is contained in $T_{\mathbf{m},c_1,c_2}$ exactly once. This restriction determines the Hilbert space $\mathcal{H}_\mathbf{m}$ (independent of $c_1$ and $c_2$) on which $T_{\mathbf{m},c_1,c_2}$ acts. This space is described by the decomposition $\mathcal{H}_\mathbf{m} = \oplus_{\mathbf{m}_n} \mathcal{V}_{\mathbf{m}_n}$, where $\mathcal{V}_{\mathbf{m}_n}$ is the subspace on which the irreducible representation of $U_q(u_n)$ with highest weight $\mathbf{m}_n$ is realized and the sum is over all highest weights satisfying (16). We choose in $\mathcal{H}_\mathbf{m}$ the orthonormal basis consisting of the Gel’fand-Tsetlin bases of the subspaces $\mathcal{V}_{\mathbf{m}_n}$. The basis elements are denoted by $|M \rangle \equiv |\mathbf{m}_n, \alpha\rangle$, where $M$ is a Gel’fand–Tsetlin tableau.
(10) and \( \alpha \) is the tableau \( M \) without the first row \( m_n \). The operators \( T_{m, e_1, e_2}(e_n) \) and \( T_{m, e_1, e_2}(f_n) \) of the representation \( T_{m, e_1, e_2} \) act upon \( |m_n, \alpha \rangle \) by the formulas

\[
T_{m, e_1, e_2}(e_n)|m_n, \alpha \rangle = \sum_{s=1}^{n}[l_{sn} - c_1]\omega_s(m, m_n, \alpha)|m_n^+, \alpha \rangle,
\]

\[
T_{m, e_1, e_2}(f_n)|m_n, \alpha \rangle = \sum_{s=1}^{n}[-l_{sn} + c_2 + 1]\omega_s(m, m_n^-, \alpha)|m_n^-, \alpha \rangle,
\]

where

\[
\omega_s(m, m_n, \alpha) = \left( \frac{\prod_{j=s}^{n-1}[l_{jn, n-1} - l_{sn} - 1][l_j - l_{sn}]}{\prod_{r\neq s}[l_{sn} - l_{rn} + 1][l_{sn} - l_{rn}]} \right)^{1/2},
\]

\[
l_j = m_j - j - 1, \quad j = 1, 2, \ldots, n - 1; \quad l_{sk} = m_{sk} - s, \quad s = 1, 2, \ldots, k,
\]

and \( m_n^+ = (m_1n, \cdots, m_{s-1}n, m_{sn} \pm 1, m_{s+1}n, \cdots, m_{nn}) \) if \( m_n = (m_1n, \cdots, m_{nn}) \). The operator \( T_{m, e_1, e_2}(k_{n+1}) \) is given by the formula

\[
T_{m, e_1, e_2}(k_{n+1})|m_n, \alpha \rangle = q^a|m_n, \alpha \rangle, \quad a = (c_1 + c_2 + n + 2 + \sum_{j=1}^{n-1}m_j - \sum_{j=1}^{n}m_{jn}).
\]

The other generators \( e_i, f_j, h_k \) belong to the subalgebra \( U_q(u_{n,1}) \) and the operators \( T_{m, e_1, e_2}(e_i), T_{m, e_1, e_2}(f_j), T_{m, e_1, e_2}(h_k) \) act upon the basis elements \( |m_n, \alpha \rangle \) by the corresponding formulas (12)–(14).

**Theorem 1.** The mapping \( a \to T_{m, e_1, e_2}(a), a \in U_q(u_{n,1}) \), described above, indeed determines a representation of the algebra \( U_q(u_{n,1}) \).

**Proof.** For simplicity, in this proof we denote the operators \( T_{m, e_1, e_2}(e_i), T_{m, e_1, e_2}(f_j) \) and \( T_{m, e_1, e_2}(h_r) \) by \( e_i, f_j \) and \( h_r \), respectively. We have to prove the following relations

\[
1 \leq i \leq n, \quad 1 \leq j \leq n - 2:
\]

\[
k_{n+1}k_{n+1}^{-1} = k_{n+1}^{-1}k_{n+1} = 1, \quad k_{n+1}k_i = k_ik_{n+1},
\]

\[
k_{n+1}e_i k_{n+1}^{-1} = q^{-\delta_{i,n}} e_i, \quad k_{n+1}f_i k_{n+1}^{-1} = q^{\delta_{i,n}} f_i, \quad k_i e_i k_i^{-1} = q^{\delta_{i,n}} e_i, \quad k_i f_i k_i^{-1} = q^{-\delta_{i,n}} f_i,
\]

\[
[e_n, f_{n-1}] = [e_n, f_n] = [e_n, e_j] = [f_n, f_j] = [e_n, f_n] = [e_j, f_n] = 0,
\]

\[
e_n^2 + e_{n-1}e_n^2 - 2e_ne_{n-1}e_n = 0, \quad e_{n-1}^2 + e_ne_n^2 - 2e_{n-1}e_ne_{n-1} = 0,
\]

\[
f_{n-1}^2 + f_{n-1}f_n^2 - 2f_nf_{n-1}f_n = 0, \quad e_{n-1}f_n + f_ne_{n-1} = 2f_{n-1}f_n f_{n-1} = 0,
\]

\[
[e_n, f_n] = \frac{k_{n+1}k_{n+1}^{-1} - k_{n+1}^{-1}k_{n+1}}{q - q^{-1}}.
\]

The relations (21) are trivial. The relations (22) are easily verified by means of formulas (12)–(14) and (17)–(20). The relations (23) are also easily verified. As a sample, we check the relation \( [e_n, f_{n-1}] = 0 \). Acting by the left hand side upon the basis vector
$|M\rangle := |\mathbf{m}_n, \alpha\rangle$ and then collecting all coefficients at the basis vector $|(M_{n+j}^{+j'})_{n-1}\rangle$ we obtain the expression

$$[l_{j',n} - c_1] \omega_{j'}(M_{n+1}^{j'})(M_{n-1}^{j'}) - A_{n-1}^j((M_{n+j}^{+j'})_{n-1}(M_{n+1}^{j'})) [l_{j',n} - c_1] \omega_{j'}(M).$$

Substituting here the expressions for $\omega_{j'}$ and $A_{n-1}^j$ we reduce it to the expression

$$D([l_{j,n-1} - l_{j',n} - 2][l_{j',n} - l_{j,n-1} + 1] - [l_{j,n-1} - l_{j',n} - 1][l_{j',n} - l_{j,n-1} + 2]),$$

where $D$ stands for a certain expression, explicit form of which is not important for us. The expression in the brackets trivially vanishes.

All relations in (24) and (25) are verified in the same manner. As a sample, we check the relation

$$e_n^2 e_{n-1} + e_{n-1} e_n^2 - [2] e_{n-1} e_n = 0. \tag{27}$$

We act by the left hand side upon the basis vector $|M\rangle$. As a result we obtain a linear combinations of basis vectors of the types $|((M_{n+j}^{+j'})_{n+2}^{+j'})_{n-1}\rangle$ and $|((M_{n+j}^{+j'})_{n+2}^{+j'})_{n-1}\rangle$, $j' \neq j''$. Collecting the coefficients at the vector $|((M_{n+j}^{+j'})_{n+2}^{+j'})_{n-1}\rangle$ we obtain the expression

$$[(l_{j',n} + 1) - c_1] \omega_{j'}((M_{n+1}^{j'})_{n+2}^{+j'}) [l_{j',n} - c_1] \omega_{j'}(M_{n+1}^{j'}) + A_{n-1}^j((M_{n+j}^{+j'})_{n+2}^{+j'}) [l_{j',n} - c_1] \omega_{j'}(M) -$$

$$-2[(l_{j',n} + 1) - c_1] \omega_{j'}((M_{n+j}^{+j'})_{n+2}^{+j'}) A_{n-1}(M_{n+1}^{j'}) [l_{j',n} - c_1] \omega_{j'}(M). \tag{28}$$

Substituting the expressions for $\omega_{j'}$ and $A_{n-1}^j$ we reduce it (up to some common coefficient) to the expression

$$(-[l_{j,n-1} - l_{j',n} - 1][l_{j,n-1} + l_{j',n} - l_{j,n-1}])^{1/2} +$$

$$+(-[l_{j,n-1} - l_{j',n} - 2][l_{j,n-1} - l_{j',n} - 1][l_{j',n} - l_{j,n-1} + 2])^{1/2} -$$

$$-[2][-l_{j,n-1} - l_{j',n} - 1][l_{j,n-1} - l_{j',n} - 1][l_{j',n} - l_{j,n-1} + 1])^{1/2}.$$
where \( \omega_j'(M_n^{+j''}) \) means the expression in the first big round brackets with \( j' \) replaced by \( j'' \) and \( j'' \) by \( j' \). Taking the explicit forms for \( \omega_j \) and \( A_n^{j,1} \), and then using the identity (29) and the equalities

\[
[x][y] = \frac{[(x + y)/2]^2 - [(x - y)/2]^2}{(x - y)}
\]

we reduce the expression in the first big round brackets to

\[
- [l_{j',n} - l_{j''}, n - 1]([l_{j,n-1} - l_{j',n}][l_{j,n-1} - l_{j''}, n][l_{j,n-1} - l_{j',n} - 1][l_{j,n-1} - l_{j''}, n - 1])^{-1/2}.
\]

The expression in the second big round brackets can be obtained from (31) by replacement \( j' \leftrightarrow j'' \). Substituting these expressions and the explicit forms for \( \omega_j \) and \( \omega_j' \) into (30) we represent (30) in the form

\[
N \left( \frac{[l_{j',n} - l_{j''}, n - 1]}{([l_{j',n} - l_{j''}, n][l_{j',n} - l_{j''}, n - 1][l_{j',n} + 1][l_{j',n} - l_{j''}, n])^{1/2}} + \frac{[l_{j',n} - l_{j''}, n - 1]}{([l_{j',n} - l_{j''}, n][l_{j',n} - l_{j''}, n - 1][l_{j',n} - l_{j''}, n + 1][l_{j',n} - l_{j''}, n])^{1/2}} \right),
\]

where \( N \) stands for a certain expression, explicit form of which is not important for us. It is easily checked that the expression in the brackets (at the coefficient \( N \)) vanishes. Therefore the expression (30) vanishes and therefore the relation (27) is proved.

Now let us prove the relation (26). Acting by the left hand side upon the basis vector \(|M\rangle\) we obtain a linear combination of the basis vectors \(|(M_n^{+j'})^{-j''}\rangle\), \( j' \neq j'' \), and \(|M\rangle\). Acting by the right hand side of (26) upon the basis vector \(|M\rangle\) we obtain the same basis vector with a coefficient.

Acting by both sides of (26) upon \(|M\rangle\) and collecting coefficients at the basis vector \(|(M_n^{+j'})^{-j''}\rangle\), \( j' \neq j'' \) we obtain the relation

\[
\omega_{j'}(M_n^{-j''})\omega_{j''}(M_n^{-j''}) - \omega_{j''}(M_n^{+j'})\omega_{j'}(M) = 0.
\]

Using the explicit form of \( \omega_{j'} \) and \( \omega_{j''} \) it reduces to the relation

\[
-[(l_{j',n} - 1) - l_{j',n}][(l_{j'',n} - 1) - l_{j'',n} - 1][(l_{j',n} - (l_{j'',n} - 1))][l_{j',n} - (l_{j'',n} - 1)]^{-1/2} - \frac{[(l_{j',n} + 1) - (l_{j'',n} - 1)][(l_{j',n} + 1) - (l_{j'',n} - 1)][l_{j',n} - l_{j'',n}][l_{j',n} - l_{j'',n} - 1)]^{-1/2} = 0,
\]

which is obviously fulfilled.

Equating diagonal matrix elements of both sides of (26) we obtain the relation

\[
\Phi\{l_{1,n+1}, l_{1,n}, l_{1,n-1}; l_{2,n+1}, l_{2,n}, l_{2,n-1}; \cdots; l_{n,n+1}, l_{n,n}; l_{n+1,n+1}\} := \sum_{j=1}^{n} \left( (A_n^{j,1}(M_n^{-j}))^2 - (A_n^{j,1}(M))^2 \right) = \left[ 2 \sum_{j=1}^{n} l_{j,n} - \sum_{j=1}^{n+1} l_{j,n+1} - \sum_{j=1}^{n-1} l_{j,n-1} - 1 \right],
\]

(33)
where the notations \( l_{1,n+1} := c_1 \), \( l_{n+1,n+1} := c_2 \) and \( l_{j+1,n+1} := l_j \) \((1 \leq j \leq n - 1)\) were used and \((A^2_n(M))^2\) is the square of the expression (14) taken for \( r = n \). We consider the relation (33) for arbitrary complex \( l_{1,n+1} \) and \( l_{n+1,n+1} \), with their sum not necessary to be integer.

We first suppose that the relation (33) is valid for \( l_{1,n+1} = l_{1,n} \) and show that then (33) is true for any value of \( l_{1,n+1} \). For this, we derive some properties of the function \( \Phi \).

Using the identity \([a][b] - [a + x][b - x] = [x][a - b + x]\), which is valid for arbitrary complex \( a, b, x \), it is directly verified the following identity for function \( \Phi \):

\[
\Phi(\{l_{1,n+1}, l_{1,n}, l_{1,n-1}\}; \ldots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1}\}) = -\Phi(\{l_{1,n+1} + x, l_{1,n}, l_{1,n-1}\}; \ldots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1} - x\}) = [x][l_{1,n+1} - l_{n+1,n+1} + x] \Phi(\{\ldots; \{l_{n,n+1}, l_{n,n}\}; \{\ldots\}, (34)
\]

where the dot "·" on the places of \( l_{1,n+1} \) and \( l_{n+1,n+1} \) in the last \( \Phi \) means that all the multipliers depending on \( l_{1,n+1} \) and \( l_{n+1,n+1} \) in the expression for \( \Phi \) must be omitted.

Now let us prove the identity

\[
\Phi(\{l_{1,n} + 1, l_{1,n}, l_{1,n-1}\}; \ldots) = \Phi(\{l_{1,n} + 1, l_{1,n} + 1, l_{1,n-1}\}; \ldots)
\]

(35)

For summands in \( \Phi(\{l_{1,n} - 1, l_{1,n}, l_{1,n-1}\}; \ldots) \) we have

\[
(A^n_1(\{l_{1,n} - 1, l_{1,n} - 1, l_{1,n-1}\}; \ldots))^2 = 0,
\]

\[
(A^n_2(\{l_{1,n} - 1, l_{1,n}, l_{1,n-1}\}; \ldots))^2 = \frac{[(l_{1,n} - 1) - l_{1,n}] \prod_{s=2}^{n+1}[l_{s,n+1} - l_{1,n}] \prod_{s=1}^{n-1}[l_{s,n-1} - l_{1,n} - 1]}{\prod_{s=2}^{n}[l_{s,n} - l_{1,n}] [l_{s,n} - l_{1,n} - 1]},
\]

\[
(A^n_2(\{l_{1,n} - 1, l_{1,n}, l_{1,n-1}\}; \ldots))^2 = \frac{[(l_{1,n} - 1) - l_{j,n}] \prod_{s=2}^{n+1}[l_{s,n+1} - l_{j,n}] \prod_{s=1}^{n-1}[l_{s,n-1} - l_{j,n} - 1]}{[l_{1,n} - l_{j,n}][l_{1,n} - l_{j,n} - 1] \prod_{s=2}^{n+1}[l_{s,n} - l_{j,n}][l_{s,n} - l_{j,n} - 1]}, \quad 2 \geq j \geq n,
\]

and for summands in \( \Phi(\{l_{1,n} + 1, l_{1,n} + 1, l_{1,n-1}\}; \ldots) \) we obtain

\[
(A^n_1(\{l_{1,n} + 1, l_{1,n} + 1, l_{1,n-1}\}; \ldots))^2 = \frac{\prod_{s=2}^{n+1}[l_{s,n+1} - l_{1,n}] \prod_{s=1}^{n-1}[l_{s,n-1} - l_{1,n} - 1]}{[l_{1,n} - l_{j,n}] \prod_{s=2}^{n+1}[l_{s,n} - l_{j,n}][l_{s,n} - l_{j,n} - 1] [l_{s,n} - l_{j,n} - 1] [l_{s,n} - l_{j,n} - 1]}, \quad 2 \geq j \geq n.
\]
We easily see that the above expressions for the summands in $\Phi(\{l_{1,n-1}, l_{1,n}, l_{1,n-1}\}; \cdots)$ coincide with those for the summands in $\Phi(\{l_{1,n} + 1, l_{1,n} + 1, l_{1,n-1}\}; \cdots)$. Thus, the relation (35) is proved.

From the identity (35) with $l_{n+1,n+1}$ replaced by $l_{n+1,n+1} + 1$ and from the assumption that (33) is true for $l_{n+1} = l_{1,n}$ we derive that

$$
\Phi(\{l_{1,n} - 1, l_{1,n}, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1} + 1\}) = \\
\Phi(\{l_{1,n} + 1, l_{1,n} + 1, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1} + 1\}) = \\
\left[ l_{1,n} + 2 \sum_{j=2}^{n} l_{j,n} - \sum_{j=2}^{n+1} l_{j,n+1} - \sum_{j=1}^{n-1} l_{j,n-1} - 1 \right] = \\
\Phi(\{l_{1,n}, l_{1,n}, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1}\}).
$$

It follows from here that at $l_{1,n+1} = l_{1,n}$ and $x = -1$ ($l_{n+1,n+1}$ is an arbitrary complex number) the left hand side of (34) identically vanishes. Hence the right hand side vanishes. Since $l_{n+1,n+1}$ is an arbitrary complex number, we obtain that $\Phi$ from the right hand side vanishes:

$$
\Phi(\{\cdot, l_{1,n}, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{\cdot\}) = 0.
$$

Using again the identity (34) with arbitrary $l_{1,n+1}$ and with $x = l_{1,n} - l_{1,n+1}$ and then the relation (33) at $l_{1,n+1} = l_{1,n}$ we have

$$
\Phi(\{l_{1,n+1}, l_{1,n}, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1}\}) = \\
\Phi(\{l_{1,n}, l_{1,n}, l_{1,n-1}\}; \cdots; \{l_{n,n+1}, l_{n,n}\}; \{l_{n+1,n+1} + (l_{1,n+1} - l_{1,n})\}) = \\
\left[ 2 \sum_{j=1}^{n} l_{j,n} - \sum_{j=1}^{n+1} l_{j,n+1} - \sum_{j=1}^{n-1} l_{j,n-1} - 1 \right].
$$

This gives the relation (33) for arbitrary $l_{1,n+1}$.

It is shown similarly that the relation (33) with the admissible values of $l_{1,n-1}$ follows from (33) with $l_{1,n-1} = l_{1,n}$.

Thus, in order to prove the relation (33) with arbitrary $l_{1,n+1}$, $l_{1,n}$, $l_{1,n-1}$ it is enough to prove the special case of (33) when $l_{1,n+1} = l_{1,n} = l_{1,n-1}$. In order to prove this special case we note that

$$
\Phi(\{l_{1,n}, l_{1,n}, l_{1,n}\}; \{l_{2,n+1}, l_{2,n}, l_{2,n-1}\}; \cdots) = \Phi(\{l_{2,n+1}, l_{2,n}, l_{2,n-1}\}; \cdots), \quad (36)
$$

where on the right hand side all the multipliers depending on $l_{1,n+1}$, $l_{1,n}$, $l_{1,n-1}$ are omitted. Indeed, we have the equalities

$$
(A_n^1(\{l_{1,n}, l_{1,n} - 1, l_{1,n}\}; \cdots))^2 = 0, \quad (A_n^1(\{l_{1,n}, l_{1,n}, l_{1,n}\}; \cdots))^2 = 0,
$$

$$
(A_n^2(\{l_{1,n}, l_{1,n}, l_{1,n}\}; \{l_{2,n+1}, l_{2,n}, l_{2,n-1}\}; \cdots))^2 = (A_n^2(\{l_{2,n+1}, l_{2,n}, l_{2,n-1}\}; \cdots))^2.
$$

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with \(2 \leq j \leq n\), which prove (36).

The right hand side of (33) at \(l_{1,n+1} = l_{1,n} = l_{1,n-1}\) coincides with \([2 \sum_{j=2}^{n} l_{j,n} - \sum_{j=2}^{n+1} l_{j,n+1} - \sum_{j=2}^{n-1} l_{j,n-1} - 1]\). This means that the relation (33) at \(l_{1,n+1} = l_{1,n} = l_{1,n-1}\) coincides with the same relation written down for the algebra \(U_q(u_{n-1,1})\). Hence, the validity of the relation (33), which correspond to \(U_q(u_{n,1})\), follows from the validity of (33) for \(U_q(u_{n-1,1})\). If we continue the described procedure we reduce the proof of (33) to the obvious identity

\[
\Phi(l_{n,n}, l_{n,n}; l_{n+1,n+1}) = -[l_{n,n} - l_{n,n} + 1][l_{n+1,n+1} - l_{n,n} + 1] = [2l_{n,n} - l_{n,n} - l_{n+1,n+1} - 1].
\]

Theorem is proved.

There exist equivalence relations in the set of the representations \(T_{m,c_1,c_2}\). A part of these equivalences is related to the periodicity of analytical function

\[
w(z) = [z] \equiv (q^z - q^{-z})/(q - q^{-1}).
\]

We set \(q = \exp h\). Then \(w(z)\) is a periodic function with period \(2\pi i/h\). Besides, we have \(w(z) = -w(z + \pi i/h)\). This means that the representations

\[
T_{m,c_1,c_2} \text{ and } T_{m,c_1+2\pi i/k,h,c_2-2\pi i/k,h}, \quad k \in \mathbb{Z},
\]

coincide and the representations

\[
T_{m,c_1,c_2} \text{ and } T_{m,c_1+\pi i/k,h,c_2-\pi i/k,h}, \quad k \in \mathbb{Z},
\]

(37)

are equivalent. Therefore, we may restrict ourselves by consideration of the representations \(T_{m,c_1,c_2}\) with \(-\pi/2h < \text{Im } c_1 \leq \pi/2h\). However, it will be often convenient to assume that \(\text{Im } c_1\) is defined modulo \(\pi/h\) and \(c_2 = m_0 - c_1\).

In order to study irreducibility of the representations \(T_{m,c_1,c_2}\) we prove the following assertion:

**Proposition 1.** The representation \(T_{m,c_1,c_2}\) with \(-\pi/2h < \text{Im } c_1 \leq \pi/2h\) is irreducible if for each irreducible representation \(T_{m_n}\) of \(U_q(u_n)\), contained in the restriction of \(T_{m,c_1,c_2}\) to \(U_q(u_n)\), no of the numbers \(l_{s,n} - c_1\) and \(l_{s,n} - c_2 - 1\), \(s = 1, 2, \ldots, n\), vanishes. If \(l_{s,n} - c_1 = 0\) for some representation \(T_{m_n}\) such that \(T_{m_n}^s\) is not contained in the restriction \(T_{m,c_1,c_2}\) of \(U_q(u_n)\) or \(l_{s,n} - c_2 - 1 = 0\) for some representation \(T_{m_n}\) such that \(T_{m_n}^s\) is not contained in the restriction \(T_{m,c_1,c_2}\) of \(U_q(u_n)\), then \(T_{m,c_1,c_2}\) is also irreducible.

**Proof.** Since \(q\) is a positive number, the number \(l_{s,n} - c_1\) (resp. the number \(l_{s,n} - c_2 - 1\)) vanishes if and only if the \(q\)-number \([l_{s,n} - c_1]\) (resp. the \(q\)-number \([l_{s,n} - c_2 - 1]\)) vanishes. These \(q\)-numbers are parts of the coefficients at the vectors on the right hand sides of (17) and (18). Multipliers \(\omega_s(m, m_n, \alpha)\) and \(\omega_s(m, m_n^\alpha, \alpha)\) in these coefficients do not vanish. Thus, under the conditions of our Proposition the coefficients at \([m_n^\alpha, \alpha]\) and \([m_n^\alpha, \alpha]\) in formulas (17) and (18) do not vanish.
Let $\mathcal{H}'$ be a $U_q(u_{n,1})$-invariant subspace of the space $\mathcal{H}_m$ of the representation $T_{m,c_1,c_2}$. By Lemma 1, $\mathcal{H}' = \oplus_{m_0} V_{m_0}$. We take a fixed vector $|m',\alpha\rangle$ from this subspace. It follows from (17) and (18) that the operators $T_{m,c_1,c_2}(e_j)$, acting upon the vectors $|m_n,\alpha\rangle$, increase the numbers $m_{ij}$ and the operators $T_{m,c_1,c_2}(f_j)$ decrease these numbers. Since the coefficients in (17) and (18) do not vanish, then acting successively upon $|m',\alpha\rangle$ by the operators $T_{m,c_1,c_2}(e_i)$ and $T_{m,c_1,c_2}(f_i)$, $i = 1, 2, \ldots, n$, we can obtain a vector $v = \sum a_{m_n,\alpha'}|m_n,\alpha'\rangle$ with non-vanishing coefficient at $|m_n,\alpha'\rangle$, where $m_n$ is any fixed highest weight satisfying the condition (16). Since $\mathcal{H}'$ is $U_q(u_{n,1})$-invariant, we have $v \in \mathcal{H}'$ and therefore $|m_n,\alpha'\rangle \in \mathcal{H}'$ by Lemma 1. Hence $\mathcal{H}' = \mathcal{H}$. Proposition is proved.

**Corollary 1.** The representation $T_{m,c_1,c_2}$ with $-\pi/2h < \text{Im} c_1 \leq \pi/2h$ is irreducible if $c_1$ and $c_2$ are not integers or if both $c_1$ and $c_2$ are integers coinciding with some of the numbers $l_1, l_2, \ldots, l_{n-1}$, $l_i = m_i - i - 1$.

**Proof.** For such values of $c_1$ and $c_2$ the numbers from Proposition 1 do not vanish. This proves our Corollary.

Below we show that the conditions of Corollary 1 are also sufficient for irreducibility of $T_{m,c_1,c_2}$.

Note that the algebra $U_q(u_{n,1})$ has one-dimensional representations which are in fact representations of $U_q(gl_{n+1})$. As in the case of finite dimensional representations, new infinite dimensional representations of $U_q(u_{n,1})$ may be obtained by taking tensor products of the representations $T_{m,c_1,c_2}$ with one-dimensional ones. But these new representations are not essential and we do not consider them below.

Below we shall need the following

**Lemma 2.** Let $T$ be a representation of $U_q(u_{n,1})$ such that $T \downarrow U_q(u_n)$ contains irreducible representations of $U_q(u_n)$ not more than once. Let the operator $T(k_{n+1})$ be diagonal in the Gel'fand–Tsetlin basis $\{|m_n,\alpha\rangle\}$ of the representation space, and let $T(e_n)$ and $T(f_n)$ are given by

$$T(e_n)|m_n,\alpha\rangle = \sum_{s=1}^{n} C_s(m_n,\alpha)|m_n^{+s},\alpha\rangle, \quad T(f_n)|m_n,\alpha\rangle = \sum_{s=1}^{n} B_s(m_n,\alpha)|m_n^{-s},\alpha\rangle.$$

If the coefficients $C_s(m_n,\alpha)$ and $B_s(m_n,\alpha)$ vanish only if $m_n^{+s}$ and $m_n^{-s}$ are not contained in $T \downarrow U_q(u_n)$, then the representation $T$ is irreducible.

**Proof** is the same as that of Proposition 1 and we omit it.

**6. Intertwining operators**

In section 5 we found that the representations $T_{m,c_1,c_2}$ and $T_{m,c_1+c_2,\pi k/h,c_2,\pi k/h}$, $k \in \mathbb{Z}$, are equivalent. Let us find other equivalence relations in the set of the irreducible representations $T_{m,c_1,c_2}$.

Assume that the representations $T_{m,c_1,c_2}$ and $T_{m',c_1',c_2'}$ are equivalent. Then the restrictions $T_{m,c_1,c_2} \downarrow U_q(u_n)$ and $T_{m',c_1',c_2'} \downarrow U_q(u_n)$ are equivalent representations of $U_q(u_n)$.
This means that these restrictions consist of the same irreducible representations of $U_q(u_n)$. It follows from (16) that it is possible if and only if $m = m'$.

Equivalence of the irreducible representations $T_{m,c_1,c_2}$ and $T_{m',c'_1,c'_2}$ means that there exists an invertible operator $A : \mathcal{D} \to \mathcal{D}$ such that

$$AT_{m,c_1,c_2}(a) = T_{m',c'_1,c'_2}(a)A, \quad a \in U_q(u_{n,1}),$$

(38)
on $\mathcal{D}$. Since $T_{m,c_1,c_2} | U_q(u_n) = T_{m',c'_1,c'_2} | U_q(u_n)$, then it follows from Schur’s lemma that $A$ is a constant operator on each of the subspaces $\mathcal{V}_{m_n}$, on which the irreducible representations $T_{m_n}$ of $U_q(u_n)$ are realized. Thus, the operator $A$ is diagonal in the basis $\{ |m_n, \alpha\} \}$ of the representation space $\mathcal{H}_m$ and

$$\langle m_n, \alpha | A | m_n, \alpha \rangle = b_{m_n},$$

where $b_{m_n}$ does not depend on $\alpha$. Representing the relation (38) in the matrix form and setting $a = c_n$ and then $a = f_n$, we obtain the set of the equalities

$$b_{m_n}[l_{sn} - c_1] = b_{m_n}[l_{sn} - c'_1], \quad b_{m_n}[-l_{sn} + c_2] = b_{m_n}[-l_{sn} + c'_2]. \quad (39)$$

It follows from here that for $s = 1, 2, \cdots, n$ and for all admissible values of $l_{sn}$ we have

$$\frac{[l_{sn} - c_1]}{[l_{sn} - c'_1]} = \frac{[l_{sn} - c_2]}{[l_{sn} - c'_2]}.$$ 

These equalities are possible only if

$$c'_1 = c_1 + i\pi k/h, \quad c'_2 = c_2 - i\pi k/h \quad \text{or if} \quad c'_1 = c_2 + i\pi k/h, \quad c'_2 = c_1 - i\pi k/h, \quad (40)$$

where $k \in \mathbb{Z}$. Thus, we proved the following

**Proposition 2.** The irreducible representations $T_{m,c_1,c_2}$ and $T_{m',c'_1,c'_2}$ are equivalent if and only if $m = m'$ and one of the conditions (40) is satisfied.

The recurrence equations (39) allow us to calculate the matrix elements $b_{m_n} \equiv b_{m_n}(m, c_1, c_2)$ of the intertwining operator $A$ of the representations $T_{m,c_1,c_2}$ and $T_{m,c_2,c_1}$.

A direct calculation shows that they are given by

$$b_{m_n}(m, c_1, c_2) = b \frac{a_{m_n}(m, c_1, c_2)}{a_{m_n}(m, c_2, c_1)},$$

where $b$ is an arbitrary fixed complex number and $a_{m_n}(m, c_1, c_2)$ is determined by one of the expressions

$$a_{m_n}(m, c_1, c_2) = \prod_{\tau=1}^{k} \prod_{\sigma=l_n+1}^{l_{\tau-1}} [\sigma - c_2] \prod_{s=k+1}^{n} \prod_{\tau=l_{\tau+1}}^{l_{\tau-1}} [\tau - c_1], \quad k = 1, 2, \cdots, n - 1.$$ 

All $n - 1$ expressions for $a_{m_n}(m, c_1, c_2)$ lead (up to a constant) to the same expression for $b_{m_n}(m, c_1, c_2)$. Everywhere below we assume that $a_{m_n}(m, c_1, c_2)$ is determined by $k = n - 1$. 

7. Reducible representations $T_{m,c_1,c_2}$

Now we consider the representations $T_{m,c_1,c_2}$ for which the conditions of Corollary 1 are not satisfied. Thus, in this section the numbers $c_1$ and $c_2$ in the representations $T_{m,c_1,c_2}$ are integers.

Let $l_0, l_1, l_2, \cdots l_{n-1}, l_n$ be the numbers defined by $l_0 = \infty$, $l_i = m_i - i - 1$ and $l_n = -\infty$. Let $c$ be an integer such that $l_{i-1} > c > l_i$ for some $i$ ($i = 1, 2, \cdots, n$). We denote by $\Gamma^-$ and $\Gamma^+$ the set of the subspaces $V_{m, c}$ of the representation space $\mathcal{H}_m$ for which the highest weights $m$ satisfy the conditions (16) and the condition $l_{in} \equiv m_{in} - i \leq c$ and $l_{in} > c$, respectively. Let $E_c^-$ and $E_c^+$ be the projectors mapping $\mathcal{H}_m$ onto the closure of the span of all subspaces from $\Gamma^-$ and $\Gamma^+$, respectively.

Under studying reducible representations $T_{m,c_1,c_2}$ we distinguish four cases depending on the placement of $c_1$ and $c_2$ between the numbers $l_0, l_1, \cdots, l_{n-1}, l_n$.

Case 1: $l_{r-1} > c_1 > l_r$, $l_{s-1} > c_2 > l_s$, $1 \leq r < s \leq n$. In this case in formula (17) one coefficient $[l_{mn} - c_1] \omega_r(m, m_n, \alpha)$ vanishes at $l_{mn} = c_1$ and in formula (18) one coefficient $[-l_{mn} + c_2 + 1] \omega_r(m, m_n^+, \alpha)$ vanishes at $l_{mn} = c_2 + 1$. It is seen from (17) and (18) that the operator $T_{m,c_1,c_2}(e_n)$ only increases values of the numbers $m_{in}$ and the operator $T_{m,c_1,c_2}(f_n)$ only decreases these values. Therefore, vanishing of the coefficients in (17) and (18) means appearance of invariant subspaces. There exist three invariant subspaces

$$\mathcal{H}^- := E_{c_1}^{-1} E_{c_1}^{+} \mathcal{H}_m, \quad \mathcal{H}^- := E_{c_1}^{-1} \mathcal{H}_m, \quad \mathcal{H}^+ := E_{c_2}^{+} \mathcal{H}_m$$

in the representation space $\mathcal{H}_m$. Irreducible representations of $U_q(u_{n,1})$ are realized on the subspace $\mathcal{H}^+$ and on the quotient spaces $\mathcal{H}^-/\mathcal{H}^+$, $\mathcal{H}^+/\mathcal{H}^-$ and $\mathcal{H}_m/(\mathcal{H}^+ + \mathcal{H}^-)$ (their irreducibility follows from Lemma 2). We denote these representations by

$$\hat{R}^{rs}(m, c_1, c_2), \quad R^{rs}_-(m, c_1, c_2), \quad R^{rs}_+(m, c_1, c_2), \quad \hat{R}^{rs}_-(m, c_1, c_2),$$

respectively.

Case 2: $l_{r-1} > c_1 > c_2 > l_r$, $1 \leq r < n$. Again, vanishing of the coefficients in (17) and (18) in this case means the representation space $\mathcal{H}_m$ has three invariant subspaces

$$\mathcal{H}^+ := E_{c_1}^{-1} E_{c_1}^{+} \mathcal{H}_m, \quad \mathcal{H}^- := E_{c_1}^{-1} \mathcal{H}_m, \quad \mathcal{H}^+ := E_{c_2}^{+} \mathcal{H}_m.$$

Irreducible representations of $U_q(u_{n,1})$ are realized on the subspace $\mathcal{H}^+$ and on the quotient spaces $\mathcal{H}^-/\mathcal{H}^+$ and $\mathcal{H}^+/\mathcal{H}^+$ (their irreducibility follows from Lemma 2). We denote these representations respectively by

$$\hat{R}^{rr}(m, c_1, c_2), \quad R^{rr}_-(m, c_1, c_2), \quad R^{rr}_+(m, c_1, c_2).$$

Case 3: $c_1 = l_r$, $r = 1, 2, \cdots, n - 1$, and $l_{s-1} > c_2 > l_s$, $1 \leq s \leq n$. In this case the representation space has only one invariant subspace $\mathcal{H}^+ := E_{c_2}^{+} \mathcal{H}_m$. On this subspace and on the quotient space $\mathcal{H}_m/\mathcal{H}^+$, irreducible representations of $U_q(u_{n,1})$ are realized. They will be denoted respectively by

$$\hat{R}^{rs}_+(m, c_1, c_2), \quad \hat{R}^{rs}_-(m, c_1, c_2).$$
Case 4: $c_1 = c_2 = c$ and $l_{r-1} > c > l_r$, $1 \leq r \leq n$. In this case the representation $\hat{T}_{c_1,c_2}$ is a direct sum of two irreducible representations $R^+_{c_1}(m,c_1,c_2)$ and $R^-_{c_2}(m,c_1,c_2)$ of $U_q(u_{n,1})$ which are realized on the subspaces $E^+_cH_m$ and $E^-_cH_m$, respectively.

There are other two cases for consideration: the case when $l_{r-1} > c_2 > l_r$, $l_{s-1} > c_1 > l_s$, $1 \leq r < s \leq n$ and the case when $l_{r-1} > c_2 > c_1 > l_r$, $1 \leq r \leq n$. However, direct calculation shows that the reducible representation $\hat{T}_{c_1,c_2}$ with $l_{r-1} > c_2 > l_r$, $l_{s-1} > c_1 > l_s$, $1 \leq r < s \leq n$ contains the same irreducible constituents as the representation $\hat{T}_{c_2,c_1}$ which belongs to Case 1 and the reducible representation $\hat{T}_{c_1,c_2}$ with $l_{r-1} > c_2 > c_1 > l_r$, $1 \leq r \leq n$ contains the same irreducible constituents as the representation $\hat{T}_{c_2,c_1}$ which belongs to Case 2. The equivalence (intertwining) operators for these irreducible constituents are constructed in Proposition 5 below.

The results of this section show that each representation $\hat{T}_{c_1,c_2}$ of $U_q(u_{n,1})$, which is not included into Corollary 1, is reducible, that is, we have the following

**Proposition 3.** The representation $\hat{T}_{c_1,c_2}$ with $-\pi/2h < \text{Im} c_1 \leq \pi/2h$ is irreducible if and only if $c_1$ and $c_2$ are not integers or if both $c_1$ and $c_2$ coincide with some of the numbers $l_1, l_2, \ldots, l_{n-1}$.

There are equivalence relations between irreducible constituents of reducible representations of Case 1 and of Case 2. In order to find these equivalence relations we note that equivalent irreducible constituents must contain the same irreducible representations of the subalgebra $U_q(u_n)$. Analysing the sets of irreducible representations of $U_q(u_n)$ contained in irreducible constituents of reducible representations $\hat{T}_{c_1,c_2}$ we find pairs of irreducible constituents which are possibly equivalent. Using the method of section 6, we try to construct equivalence operators for these pairs. In this way, we find that in Case 1 the irreducible representation $\hat{R}^s_{c_1,c_2}$ is equivalent to the representation $R^+_{c_1,c_2}$ if $s < n$, where the set of numbers $(l_1', l_2', \ldots, l_{n-1}', c_1', c_2')$, $l_j' = m_j' - j - 1$, is obtained from the set of numbers $(l_1, l_2, \ldots, l_{n-1}, c_1, c_2)$, $l_j = m_j - j - 1$, by permutation of the $s$-th and $(n+1)$-th numbers, and to the representation $R^+_{c_1,c_2}$ if $s > 1$, where the set of numbers $(l_1', l_2', \ldots, l_{n-1}', c_1', c_2')$, $l_j' = m_j' - j - 1$, is obtained from the set of numbers $(l_1, l_2, \ldots, l_{n-1}, c_1, c_2)$, $l_j = m_j - j - 1$, by permutation of the $(r-1)$-th and $n$-th numbers.

It is similarly shown that in Case 1 the irreducible representation $\hat{R}^s_{c_1,c_2}$ is equivalent to the representation $R^+_{c_1,c_2}$, where the set of numbers $(l_1', l_2', \ldots, l_{n-1}', c_1', c_2')$, $l_j = m_j - j - 1$, by permutation of the $r$-th and $n$-th numbers.

The representation $\hat{R}^r_{c_1,c_2}$ from Case 2 is equivalent to the representation $R^+_{c_1,c_2}$ from Case 1 if $r < n$, where the set of numbers $(l_1', l_2', \ldots, l_{n-1}', c_1', c_2')$, $l_j' = m_j' - j - 1$, is obtained from the set of numbers $(l_1, l_2, \ldots, l_{n-1}, c_1, c_2)$, $l_j = m_j - j - 1$, by permutation of the $r$-th and $(n+1)$-th numbers and to the representation $R^+_{c_1,c_2}$ from Case 1 if $r > 1$, where the set of numbers $(l_1', l_2', \ldots, l_{n-1}', c_1', c_2')$, $l_j' = m_j' - j - 1$, is obtained from the set of numbers $(l_1, l_2, \ldots, l_{n-1}, c_1, c_2)$, $l_j = m_j - j - 1$, by permutation of the $(r-1)$-th and $n$-th numbers.
Other equivalence relations between irreducible representations from Cases 1–4 will be described in Theorem 2.

We also note that the irreducible representation \( \hat{\mathbf{R}}_{1,n}^{i}(\mathbf{m}, c_1, c_2) \) from Case 1 is equivalent to the irreducible finite dimensional representation \( T_{\mathbf{m}_{n+1}} \) of the algebra \( U_q(\mathfrak{u}_{n,1}) \) with highest weight \( \mathbf{m}_{n+1} \) such that

\[
m_{1,n+1} = c_1 + 1, \quad m_{i,n+1} = l_{i-1} + i, \quad i = 2, 3, \ldots, n, \quad m_{n+1,n+1} = c_2 + n + 1.
\]

Now we can describe a structure of reducible representations \( T_{\mathbf{m},c_1,c_2} \). Irreducible constituents are contained in these representations in the form of semidirect sum (that is, there are irreducible constituents realized on quotient spaces and there are irreducible constituents realized on invariant subspaces). Below we show decompositions of reducible representations \( T_{\mathbf{m},c_1,c_2} \) into irreducible constituents. In these decompositions, two constituents \( R \) and \( R' \) are connected by an arrow: \( R \to R' \) if \( R' \) is a subrepresentation and \( R \) is realized on a quotient space. If constituents \( R \) and \( R' \) are connected as \( R \oplus R' \), then they are contained in the decomposition as a direct sum of representations, that is, \( R \) and \( R' \) are realized on invariant subspaces.

For convenience, below we denote the representations \( R(\mathbf{m}, c_1, c_2) \) (with \( R \) equipped with symbols) also by \( R(L) \) (with \( R \) equipped with the same symbols), where \( L = (l_1, l_2, \cdots, l_{n-1}, c_1, c_2), l_i = m_i - i - 1 \). Then \( s_{ik}L \) means \( L \) with permuted \( i \)-th and \( k \)-th numbers.

Let \( c_1 \) and \( c_2 \) be integers. If \( l_{r-1} > c_1 > c_2 > l_r, 1 \leq r \leq n \), then

\[
T_{\mathbf{m},c_1,c_2} = \{ R^{r,r}(L) \oplus R^{r,r}(L) \} \to R^{r+1,r}(s_{r,n+1}L) \quad \text{if} \quad r < n,
\]

\[
T_{\mathbf{m},c_1,c_2} = \{ R^{r,r}(L) \oplus R^{r,r}(L) \} \to R^{-1,r}(s_{r-1,n}L) \quad \text{if} \quad r > 1.
\]

For \( T_{\mathbf{m},c_2,c_1} \), the decomposition coincides with the decomposition for \( T_{\mathbf{m},c_1,c_2} \) if to reverse arrows to the opposite sides.

If \( l_{r-1} > c_1 > l_r, l_{s-1} > c_2 > l_s, 1 \leq r < s \leq n \), then

\[
T_{\mathbf{m},c_1,c_2} = R^{r+1,s}(s_{r,n}L) \to \{ R^{r,s}(L) \oplus R^{r,s}(L) \} \to R^{r+1,s}(s_{s,n+1}L) \quad \text{if} \quad s < n,
\]

\[
T_{\mathbf{m},c_1,c_2} = R^{r+1,s}(s_{r,n}L) \to \{ R^{r,s}(L) \oplus R^{r,s}(L) \} \to R^{r-1,s}(s_{r-1,n}L) \quad \text{if} \quad r > 1,
\]

\[
T_{\mathbf{m},c_1,c_2} = R^{r+1,s}(s_{r,n}L) \to \{ R^{r,s}(L) \oplus R^{r,s}(L) \} \to T_{\mathbf{m}_{n+1}} \quad \text{if} \quad r = 1, s = n,
\]

where \( T_{\mathbf{m}_{n+1}} \) is the irreducible finite dimensional representation of \( U_q(\mathfrak{u}_{n,1}) \) with highest weight \( \mathbf{m}_{n+1} \) described above. For \( T_{\mathbf{m},c_2,c_1} \) the decomposition coincides with the decomposition for \( T_{\mathbf{m},c_1,c_2} \) if to reverse arrows to the opposite sides.

If \( c_1 = l_r, r = 1, 2, \cdots, n - 1, \) and \( l_{s-1} > c_2 > l_s, 1 \leq s \leq n \), then

\[
T_{\mathbf{m},c_1,c_2} = \hat{R}^{rs}(L) \to \tilde{R}^{rs}(L).
\]

If \( c_1 = c_2 = c \) and \( l_{r-1} > c > l_r, 1 \leq r \leq n \), then

\[
T_{\mathbf{m},c_1,c_2} = R^c_{-}(L) \ominus R^c_{+}(L).
\]
8. Intertwining operators for reducible representations

In section 6, we constructed intertwining operators for pairs of the irreducible representations \( T_{m,c_1,c_2} \). Intertwining operators exist also for pair of the reducible representations. Before to construct them we first prove the following

**Proposition 4.** Representations \( T \) and \( T' \) have a nonzero intertwining operator if and only if there exist subrepresentations \( R \) and \( R' \) of \( T \) and \( T' \), respectively, such that the quotient representation \( T/R \) and the subrepresentation \( R' \) are equivalent.

**Proof.** Let \( A \) be an intertwining operator for the representations \( T \) and \( T' \), that is, \( AT = T'A \). Then \( T \) induces a subrepresentation \( R \) on \( \text{Ker} \, A \) and \( T' \) determines a subrepresentation \( R' \) on \( \text{Im} \, A \). The operator \( A \) induces the operator \( A' \colon \mathcal{H}/\text{Ker} \, A \to \text{Im} \, A \) (\( \mathcal{H} \) is a representation space for \( T \)) which is an equivalence operator for the representations \( T/R \) and \( R' \). Conversely, let \( R \) and \( R' \) be subrepresentations of \( T \) and \( T' \), respectively, such that the quotient representation \( T/R \) and the subrepresentation \( R' \) are equivalent and let \( B \) be an equivalence operator for \( T/R \) and \( R' \). We represent the representations \( T \) and \( T' \) in the form of semidirect sums \( T \sim R \to T' \sim R' \to R' \), where arrows mean the same as in section 7. According to these decompositions we have the operator from the representation space of \( T \) to the representation space of \( T' \) representable in the block form as \( \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \). It is an intertwining operator for the representations \( T \) and \( T' \). Proposition is proved.

In section 6, we constructed intertwining operators \( A(m,c_1,c_2) \) for pairs of the irreducible representations \( T_{m,c_1,c_2} \) and \( T_{m,c_2,c_1} \) which are diagonal in the basis \( \{m_n, \alpha\} \). Let us consider \( A(m,c_1,c_2) \) as an operator function of \( c_1 \). Then for every fixed \( m \) and \( m_0 = c_1 + c_2 \), \( A(m,c_1,c_2) \) is an analytical function of the complex variable \( c_1 \) in all points \( c_1 \) for which the representations \( T_{m,c_1,c_2} \) are irreducible. This means that its diagonal matrix elements \( b_{m_n}(m,c_1,c_2) \) are analytical functions of \( c_1 \). Let us take the meromorphic continuation of \( A(m,c_1,c_2) \) to the points \( c_1 \) in which the representations \( T_{m,c_1,c_2} \) are reducible, that is, to the whole complex plane. We denote the analytically continued operator function also by \( A(m,c_1,c_2) \). We have explicit expressions for matrix elements of these operator functions in the basis \( \{m_n, \alpha\} \). Analysing analytical properties of the matrix elements \( b_{m_n}(m,c_1,c_2) \) we obtain the following

**Proposition 5.** For fixed \( m \) and \( m_0 = c_1 + c_2 \), the operator function \( A(m,c_1,c_2) \) of the complex variable \( c_1 \) is regular in all points of the strip \(-i\pi/2h \leq \text{Im} \, c_1 \leq i\pi/2h \) except for the integral points \( c_1 \) for which one of the following conditions is fulfilled:

(a) \( c_1 \) and \( c_2 \) lie in different intervals \((l_{i-1}, l_i)\) and \((l_{j-1}, l_j)\), \( i = 1, 2, \ldots, n - 1, j = 1, 2, \ldots, n \), and do not coincide with their ends;

(b) \( l_{i-1} > c_2 > c_1 > l_i, \ i = 1, 2, \ldots, n; \)

(c) \( c_1 = l_i, \ i = 1, 2, \ldots, n - 1, l_{n-1} > c_2; \)

(d) \( c_2 = l_j, \ i = 1, 2, \ldots, n - 1, l_{j-1} > c_1 > l_j, \ j = 1, 2, \ldots, n - 1. \)
In the points (a)–(d) (except for the points \(l_{j-1} > c_1 > l_j, j = 1, 2, \ldots, n-1\), \(c_2 < l_{n-1}\) for which \(A(m, c_1, c_2)\) has second order poles) the operator function \(A(m, c_1, c_2)\) has first order poles.

In all regularity points except for the integral points \(c_1\) for which one of the following conditions is fulfilled:

1. \(l_{i-1} > c_1 > c_2 > l_i, i = 1, 2, \ldots, n;\)
2. \(c_1 = l_i, i = 1, 2, \ldots, n-1, l_{j-1} > c_2 > l_j, j = 1, 2, \ldots, n-1;\)
3. \(c_2 = l_i, i = 1, 2, \ldots, n-1, c_1 < l_{n-1};\)
4. \(c_1 < l_{n-1}, l_{i-1} > c_2 > l_i, i = 1, 2, \ldots, n-1,\)

the kernel of the operator \(A(m, c_1, c_2)\) consists only of the zero element. In the points (1) – (4) this kernel coincide

with \(E_{c_2}^+ E_{c_1}^- D_{m, c_1, c_2}\) in the case (1);
with \(E_{c_2}^+ D_{m, c_1, c_2}\) in the case (2);
with \(E_{c_1}^- D_{m, c_1, c_2}\) in the case (3);
with \((1 - E_{c_2}^+ E_{c_1}^-) D_{m, c_1, c_2}\) in the case (4),

where \(D_{m, c_1, c_2}\) is the span of the subspaces of irreducible representations of \(U_q(u_n)\).

In every point \(c_1 = c_0,\) in which \(A(m, c_1, c_2)\) has a first order pole, the operator

\[
B(m, c_0, -c_0 + m_0) = \text{Res}_{c_1 = c_0} A(m, c_1, c_2)
\]

is an intertwining operator for the representations \(T_{m, c_0, -c_0 + m_0}\) and \(T_{m, -c_0 + m_0, c_0}\). If \(c_1\) satisfies the condition \(l_{i-1} > c_1 \geq c_2 > l_i, i = 1, 2, \ldots, n\), then the operator \(A(m, c_1, c_2)\) is a direct sum of two intertwining operators \(A(m, c_1, c_2) E_{c_1}^+\) and \(A(m, c_1, c_2) E_{c_2}^-\). In every point \(c_1,\) in which \(A(m, c_1, c_2)\) has a second order pole, the operator \(B(m, c_0, -c_0 + m_0)\) which is a second order residue of the operator function \(A(m, c_1, c_2)\) in the point \(c_1 = c_0\) is an intertwining operator for the representations \(T_{m, c_0, -c_0 + m_0}\) and \(T_{m, -c_0 + m_0, c_0}\). The kernel of the operator \(B(m, c_0, -c_0 + m_0)\) coincides

with \((1 - E_{c_2}^+ E_{c_1}^-) D_{m, c_1, c_2}\) if \(c_1 = c_0\) satisfies the condition (a);
with \((E_{c_1}^- + E_{c_2}^+ D_{m, c_1, c_2}\) if \(c_1 = c_0\) satisfies the condition (b);
with \(E_{c_2}^+ D_{m, c_1, c_2}\) or \(E_{c_1}^- D_{m, c_1, c_2}\) if \(c_1 = c_0\) satisfies the condition (c) or (d), respectively.

Remark that Proposition 5 gives the equivalence operators for the irreducible constituents of the reducible representations \(T_{m, c_1, c_2}\) and \(T_{m, c_2, c_1}\) mentioned before Proposition 3.

9. Irreducible representations of \(U_q(u_{n, 1})\)

Now we select the set of all irreducible representations which can be obtained from the representations \(T_{m, c_1, c_2}\). We obtain the following theorem in which \(\text{Im} c_1\) is considered modulo \(\pi/h\) and \(c_2 = m_0 - c_1\).

**Theorem 2.** The set of irreducible representations of the algebra \(U_q(u_{n, 1})\) consisting of all irreducible representations \(T_{m, c_1, c_2}\) and all irreducible constituents of reducible representations \(T_{m, c_1, c_2}\) give the following classes of representations:
(a) The representations \( T_{m,c_1,c_2} \) for which \( c_1 \) and \( c_2 \) are not integers or for which \( c_1 \) and \( c_2 \) coincide with some of the numbers \( l_1, l_2, \ldots, l_{n-1} \).

(b) The representations \( R^{r,s}_-(m, c_1, c_2) \) and \( R^{r,s}_+(m, c_1, c_2) \), \( 1 \leq r \leq s \leq n, c_1 > c_2 \), where the numbers \( c_1 \) and \( c_2 \) are integral no of which coincides with any of the numbers \( l_1, l_2, \ldots, l_{n-1} \).

(c) The representations \( \tilde{R}^{r,s}_-(m, c_1, c_2) \) and \( \tilde{R}^{r,s}_+(m, c_1, c_2) \), \( 1 \leq r \leq n-1, 1 \leq s \leq n \), where \( c_1 = l_r \) and \( l_{s-1} > c_2 > l_s \).

(d) The representations \( R^r_- (m,c_1,c_2) \) and \( R^r_+ (m,c_1,c_2), 1 \leq r \leq n, \) where \( c_1 = c_2 = c \) and \( l_{r-1} > c > l_r \).

(e) All irreducible finite dimensional representations \( T_{m,c_1,c_2} \).

Between the irreducible representations \( T_{m,c_1,c_2} \) there exist the following equivalence relations: \( T_{m,c_1,c_2} \sim T_{m,c_2,c_1} \). In the set of the representations of classes (b)–(e) there exist the following equivalence relations:

\[
R^{r,s}_-(L) \sim R^{r-1,s-1}_-(s_{r-1,n}s_{s-1,n+1},L), \quad r \neq s, \quad r \neq 1, \quad (41)
\]
\[
R^{r,r}_-(L) \sim R^{r-1,r-1}_-(s_{r-1,n}s_{n,n+1},L), \quad r \neq 1, \quad (42)
\]
\[
\tilde{R}^{r,s}_-(L) \sim \tilde{R}^{r,r-1}_-(s_{s-1,n+1},L), \quad s \neq 1, \quad (43)
\]
\[
R^r_- (L) \sim \tilde{R}^{r-1,r-1}_+(s_{r-1,n+1},L), \quad r \neq 1, \quad R^r_- (L) \sim \tilde{R}^{-r,r+1}_-(s_{r,n+1},L), \quad r \neq n. \quad (44)
\]

Any another equivalence relation between the representations of classes (a)–(e) is obtained from the above ones by means of superpositions.

Proof. The first part of the theorem follows from the above reasoning. Existence of the equivalence relations stated in the theorem is proved by constructing the intertwining operators for these pairs of representations (they are diagonal in the basis \( \{m_n, \alpha\} \) and can be easily found as in section 6).

All equivalence relations in the set of the irreducible representations \( T_{m,c_1,c_2} \) were described above. The reducible representation \( T_{m,c_1,c_2} \) cannot be equivalent to any of the representations \( R \) from classes (b)–(e) since the restrictions \( T_{m,c_1,c_2} \downarrow U_q(u_n) \) and \( R \downarrow U_q(u_n) \) do not coincide. From other side, direct verification shows that if two representations from classes (b)–(d) are not related by an equivalence relation from (41)–(44) or by their superposition, then they have not coinciding restrictions to the subalgebra \( U_q(u_n) \). This proves the second part of the theorem. Theorem is proved.

It will be proved in a separate paper that the representations of Theorem 2 exhaust (up to tensoring by one-dimensional representations) all irreducible representations of the algebra \( U_q(u_{n,1}) \).

By results of section 7, every reducible representation \( T_{m,c_1,c_2} \) has one of the forms

\[
T_{m,c_1,c_2} \sim R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_4, \quad T_{m,c_1,c_2} \sim R_1 \rightarrow R_2 \rightarrow R_3, \quad T_{m,c_1,c_2} \sim R_1 \rightarrow R_2,
\]

where \( R_i \) are irreducible constituents and in some cases an arrow must be replaced by the symbol of a direct sum. Then the representation \( T_{m,c_2,c_1} \) is of the form

\[
T_{m,c_2,c_1} \sim R'_4 \rightarrow R'_3 \rightarrow R'_2 \rightarrow R'_1, \quad T_{m,c_2,c_1} \sim R'_3 \rightarrow R'_2 \rightarrow R'_1, \quad T_{m,c_1,c_2} \sim R'_2 \rightarrow R'_1,
\]
respectively, where $R_i$ is equivalent to $R'_i$. Moreover, it follows from the results of section 7 that every irreducible representation $R$ from Theorem 2 can be realized as the constituent $R_1 \sim R'_1$ in the decompositions

$$T_{m,c_1,c_2} \sim R_1 \rightarrow R_2 \rightarrow \cdots \quad \text{and} \quad T_{m,c_2,c_1} \sim \cdots \rightarrow R'_2 \rightarrow R'_1.$$ 

From other side, Proposition 5 shows that there is an intertwining operator for $T_{m,c_1,c_2}$ and $T_{m,c_2,c_1}$ which is an equivalence operator for the representations $R_1$ and $R'_1$, and turns into zero on other parts of the representation space. This fact will be used in the following section for obtaining irreducible $*$-representations of $U_q(u_{n,1})$.

### 10. Irreducible $*$-representations of $U_q(u_{n,1})$

The aim of this section is to separate in Theorem 2 all representations which are equivalent to $*$-representations. For this, we introduce the notion of Hermitian-adjoint representations of $U_q(u_{n,1})$. Two representations $T := T_{m,c_1,c_2}$ and $T' := T_{m,c'_1,c'_2}$ on a Hilbert space $H_m$ are called Hermitian-adjoint if for any $a \in U_q(u_{n,1})$ we have

$$(T(a^*)v, v') = (v', T'(a)v) \quad \text{for any} \quad a \in U_q(u_{n,1}).$$

Setting here $v = v' = |m_n, \alpha \rangle$ and $a = k_n$, we find that $c_1 + c_2 = c'_1 + c'_2$. Then setting $v' = |m_n, \alpha \rangle$, $v = |m_n^{+\ast}, \alpha \rangle$ and $a = c_n$, we derive that $[l_{sn} - c_2] = [l_{sn} - c'_1]$, that is, $c'_1 = \overline{c_2} + 2\pi k/h$ and $c'_2 = \overline{c_1} - 2\pi k/h$, where $k \in \mathbb{Z}$. Direct calculation shows that the representations $T := T_{m,c_1,c_2}$ and $T' := T_{m,c'_1,c'_2}$ are indeed Hermitian-adjoint. Thus, if $-\pi/2h < \text{Im} c_1, \text{Im} c'_1 \leq \pi/2h$, then the representations $T := T_{m,c_1,c_2}$ and $T' := T_{m,c'_1,c'_2}$ are Hermitian-adjoint if and only if $c'_1 = \overline{c_2}$ and $c'_2 = \overline{c_1}$.

Now we shall find which of the representations of Theorem 2 are equivalent to $*$-representations. The representation $T := T_{m,c_1,c_2}$ on the Hilbert space $H_m$ is equivalent to such a representation if there exists a scalar product (that is, a strictly positive Hermitian form) $H(\cdot, \cdot)$ of $D_m$ such that

$$H(T(a^*)v, v') = H(v, T(a)v'), \quad a \in U_q(u_{n,1}).$$

(45)

Any Hermitian form $H(\cdot, \cdot)$ on $D_m$ can be represented as

$$H(v, v') = (v, Qv'),$$

where $(\cdot, \cdot)$ is the scalar product on $H_m$ and $Q$ is a Hermitian operator on $D_m$. It follows from (45) that

$$H(v, T(a)v') = (v, QT(a)v') = H(T(a^*)v, v') = (T(a^*)v, Qv') = (v, T'(a)Qv'),$$

where $T' := T_{m,\overline{c_2},\overline{c_1}}$. Hence, $QT(a) = T'(a)Q$, that is, $Q$ is an intertwining operator for the representations $T = T_{m,c_1,c_2}$ and $T' = T_{m,\overline{c_2},\overline{c_1}}$. As we have seen in section 6, if the representation $T$ is irreducible, then such an operator exists if $\overline{c_2} = c_1$ or $\overline{c_2} = c_2$, $c_1 = c_2$ or $\overline{c_2} = c_1 + i\pi/h$, $\overline{c_1} = c_2 - i\pi/h$. In the first case the representation $T = T_{m,c_1,c_2}$ coincides with $T' = T_{m,\overline{c_2},\overline{c_1}}$ and the operator $Q$ is multiple to the identity operator. In
the second case, the numbers $c_1$ and $c_2$ are real, and the operator $Q$ is the intertwining operator for $T = T_{m,c_1,c_2}$ and $T' = T_{m,c_2,c_1}$ constructed in section 6. In the third case $\text{Im } c_1 = -\text{Im } c_2 = \pi/2h$ and the intertwining operator is from section 6.

In the first case, the Hermitian form $H(\cdot, \cdot)$ coincides with the scalar product and, consequently, the representation $T_{m,c_1,c_2}$ with $c_2 = c_1$ is a $*$-representation with respect to the scalar product $(\cdot, \cdot)$. In the second and third cases, the Hermitian form has the form $H(v, v') = (v, Qv')$, where $Q$ is a non-trivial intertwining operator. We have to find when this form is positive. Since the matrices of all constructed intertwining operators are diagonal with respect to the basis $\{|m_n, \alpha\rangle\}$, then we must find when matrix elements of the intertwining operator are all positive or all negative. Making these calculations explicitly we obtain all irreducible representations $T_{m,c_1,c_2}$ which are equivalent to $*$-representations.

The same method is appropriate for the irreducible representations from classes (b)–(e) of Theorem 2. For every representation $R$ from this set there exist (according to the remark at the end of section 9) the representations $T_{m,c_1,c_2}$ and $T_{m,c_2,c_1}$ such that $T_{m,c_1,c_2} \sim R \sim \cdots$ and $T_{m,c_2,c_1} \sim \cdots \sim R'$, $R \sim R'$. Now we continue as above. The difference is that now the Hermitian form $H(v, v') = (v, Qv')$ must be only positive (not strictly positive). As above, we derive that $Q T_{m,c_1,c_2}(a) = T_{m,c_2,c_1}(a)Q$, $a \in U_q(u_{n,1})$, and $Q$ is an equivalence operator for $R$ and $R'$. On the carrier space $\mathcal{H}$ of $R'$ the Hermitian form $H$ is strictly positive if $R$ is equivalent to a $*$-representation. The form $H$ is strictly positive on $\mathcal{H}$ if $Q$ is a positive (or negative) Hermitian operator, that is, if his non-vanishing diagonal matrix elements are all positive or all negative. Making these calculations explicitly, we obtain the theorem formulated below. In the formulation of this theorem, a series of numbers $a_1, a_2, \cdots, a_k$ is called dense if $a_i = a_{i-1} - 1$, $i = 2, 3, \cdots, k$.

**Theorem 3.** The following irreducible representations of $U_q(u_{n,1})$ from Theorem 2 are equivalent to $*$-representations:

(a) the representations $T_{m,c_1,c_2}$, $c_1 = \overline{c_2}$ (principal series of $*$-representations);
(b) the representations $T_{m,c_1,c_2}$, $\text{Im } c_1 = -\text{Im } c_2 = \pi/2h$ (the strange series);
(c) the representations $T_{m,c_1,c_2}$, where $c_1$ and $c_2$ are real numbers for which there exist numbers $l_r = m_r - r - 1$, $l_s = m_s - s - 1$, $r, s = 1, 2, \cdots, n-1$, such that $|l_r - c_1| < 1$, $|l_s - c_2| < 1$ and the series $l_r, l_{r+1}, \cdots, l_s$ (for $c_1 > c_2$) or the series $l_s, l_{s+1}, \cdots, l_r$ (for $c_1 < c_2$) is dense (the supplementary series);
(d) the representations $R^i_{ij}(m, c_1, c_2)$, $c_1 > c_2$, if $i = j$ or if the series $l_1, l_1, l_{i+1}, \cdots, l_{j-1}$ is dense;
(e) the representations $R^i_{ij}(m, c_1, c_2)$, $c_1 > c_2$, if $i = j$ or if the series $l_i, l_i, l_{i+1}, \cdots, l_{j-1}, c_2$ is dense;
(f) the representations $\tilde{R}^i_{ij}(m, c_1, c_2)$, if $i < j$ and the series $l_i, l_i, l_{i+1}, \cdots, l_{j-1}, c_2$ is dense or if $i \geq j$ and the series $l_l, l_{j+1}, \cdots, l_i$ is dense;
(g) the representations $\tilde{R}^i_{ij}(m, c_1, c_2)$, if $i < j$ and the series $c_2, l_j, l_{j+1}, \cdots, l_i$ is dense.
(h) all the representations $R^+_+(m, c, c)$ and $R^-_+(m, c, c)$.

Note that there is a one-to-one correspondence between nonequivalent irreducible $*$-representations of $U_q(u_{n,1})$ which are irreducible components of reducible representations $T(m, c_1, c_2)$ and nonequivalent unitary irreducible representations of the same type of the group $U(n, 1)$. The list of the last representations can be found in [14].

Remark that the strange series of representations disappears when $q \to 1$. This means that this series is absent for the classical case.

Theorem 3 classifies irreducible $*$-representations of the algebra $U_q(u_{n,1})$ in the set of representations of Theorem 2. However, the algebra $U_q(u_{n,1})$ has also irreducible $*$-representations belonging to the set of representations obtained from the representations of Theorem 2 by tensoring by one-dimensional representations. Below, we give a list of such representations.

**Proposition 6.** (a) If $T$ is a $*$-representation from Theorem 3, then the representation $T'$ determined by the operators $T'(k_j) = -T(k_j)$, $T'(e_r) = T(e_r)$, $T'(f_r) = T(f_r)$, $j = 1, 2, \cdots, n + 1$, $r = 1, 2, \cdots, n$, is a $*$-representation of $U_q(u_{n,1})$.

(b) Let $T_{m_{n+1}}$ be a finite dimensional representation of $U_q(u_{n,1})$ defined by formulas (12)–(14), written down for $U_q(gl_{n+1})$. Then the representation $T'_{m_{n+1}}$, determined by the formulas $T'_{m_{n+1}}(k_{n+1}) = -T_{m_{n+1}}(k_{n+1})$, $T'_{m_{n+1}}(e_n) = T_{m_{n+1}}(e_n)$, $T'_{m_{n+1}}(f_n) = -T_{m_{n+1}}(f_n)$ and $T'_m(a) = T_m(a)$ for all other generating elements of $U_q(gl_{n+1})$, is a $*$-representation of $U_q(u_{n,1})$.

**Proof** is given by a direct verification.

By Proposition 6, $U_q(u_{n,1})$ has finite dimensional irreducible $*$-representations. It is not a case for the classical case.

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