Renormalized and entropy solutions for the fractional 
$p$-Laplacian parabolic equation with $L^1$ data

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1. Introduction

Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial \Omega$, $T$ is a positive number. In this paper we study the following nonlinear parabolic problem

$$
\begin{cases}
    u_t + (-\Delta)^s u = f & \text{in } \Omega_T \equiv \Omega \times (0, T), \\
    u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
    u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

(1.1)
where $0 < s < 1 < p < N$ such that $ps < N$ and $(-\Delta)_p^s$ is the fractional $p$-Laplacian operator which, up to normalization factors, is defined as

$$(-\Delta)_p^s u(x,t) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{N+ps}} \, dy$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{N+ps}} \, dy,$$ 

where $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$, P.V. is a commonly used abbreviation in the principal value sense. Moreover, we assume that $f$ and $u_0$ are nonnegative satisfying

$$f \in L^1(\Omega_T) \quad \text{and} \quad u_0 \in L^1(\Omega). \quad (1.2)$$

There have been a large number of research activities on the study of well-posedness of $p$-Laplacian type equations and general Leray-Lions problems with $L^1$ and measure data. Under these assumptions, the existence of a distributional solution, so-called SOLA (Solutions Obtained as Limit of Approximations), was proved in [13, 14, 21], but due to the lack of regularity of the solution, the distributional formulation is not strong enough to provide uniqueness. To overcome this difficulty, it is reasonable to work with renormalized solutions and entropy solutions, which need less regularity than weak solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [23] for the study of Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer to [8, 9, 10, 11, 16, 20, 31, 43, 45] for details. At the same time the notion of entropy solutions has been proposed by Bénilan et al. in [7] for the nonlinear elliptic problems. This framework was extended to related problems [4, 12, 15, 28, 34, 36, 39].

The fractional Laplacian operators and non-local operators have attracted increasing attention over the last years. This type of operators arises in a natural way in many different applications such as continuum mechanics, phase transition phenomena, population dynamics, image process, game theory and Lévy processes, see for example [5, 17, 18, 19, 33]. For this reason it is particularly important to study situations when such non-local operators are involved in equations featuring singular or irregular data. This leads to study non-local equations having $L^1$ or measure data. As far as the non-local $p$-Laplacian operator $(-\Delta)_p^s$ is concerned, the linear elliptic case $p = 2$ has been studied in [3, 24, 27]. In particular, the existence and uniqueness of
renormalized solutions for the problems of the kind
\[ \beta(u) + (-\Delta)^s u \ni f \quad \text{in } \mathbb{R}^N \]
was proved by Alibaud, Andreianov and Bendahmane in [3], where \( f \in L^1(\mathbb{R}^N) \) and \( \beta \) is a maximal monotone graph in \( \mathbb{R} \). Using a duality argument, in the sense of Stampacchia, Kenneth, Petitta and Ulusoy in [24] proved the existence and uniqueness of solutions to non-local problems like \((-\Delta)^s u = \mu \) in \( \mathbb{R}^N \) with \( \mu \) being a bounded Radon measure whose support is compactly contained in \( \mathbb{R}^N \). In [25], Kuusi, Mingione and Sire discussed the elliptic non-local case \( p \neq 2 \) with measure data and developed an existence of SOLA, regularity and Wolf potential theory. In addition, Abdellaoui et al in [1] investigated the fractional elliptic \( p \)-Laplacian equations with weight and general datum and showed that there exists a unique entropy positive solution. On the other hand, Abdellaoui et al in [2] established the results on the existence of a weak solution obtained as limit of approximations (SOLA) and the existence of nonnegative entropy solutions for the fractional \( p \)-Laplacian equations.

In this paper, we focus our attention on the well-posedness of renormalized solutions and the uniqueness of entropy solutions for the fractional \( p \)-Laplacian parabolic problem (1.1). Our results cover the case of linear parabolic non-local equations and are also new in such cases for the study of renormalized solutions. We construct an approximate solution sequence and establish some \textit{a priori} estimates. Then we draw a subsequence to obtain a limit function, and prove this function is a renormalized solution. Based on the convergence results of approximate solutions, we obtain that the renormalized solution of problem (1.1) is also an entropy solution, which leads to an inequality in the entropy formulation. By choosing suitable test functions, we prove the uniqueness of renormalized solutions and entropy solutions, and thus the equivalence of renormalized solutions and entropy solutions. Here we would like to mention that the definition of renormalized solutions is influenced by [3]. The main point is to circumvent the use of chain rules, which is not available in the non-local framework.

For the convenience of the readers, we recall some definitions and basic properties of the fractional Sobolev spaces, in which main results can be found in [22, 29, 32, 41, 42] and the references therein.

Let \( s \in (0, 1) \) and \( p > 1 \). The fractional Sobolev space
\[ W^{s,p}(\mathbb{R}^N) \equiv \left\{ u \in L^p(\mathbb{R}^N) : \int\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy < +\infty \right\} \]
is a Banach space endowed with the norm

\[ \|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}. \]

Denote \( D_\Omega = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (C\Omega \times C\Omega) \), where \( C\Omega = \mathbb{R}^N \setminus \Omega \). For every function \( u \in C^\infty_0(\Omega) \) we define \( u = 0 \) in \( C\Omega \) and then have \( u \in C^\infty_0(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \).

Now we define \( X^{s,p}_0(\Omega) \) to be the closure of \( C^\infty_0(\Omega) \) in \( W^{s,p}(\mathbb{R}^N) \). For every function \( u \in X^{s,p}_0(\Omega) \), it is clear that \( u = 0 \) a.e. in \( C\Omega \)

and

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy + 2 \int_{\Omega} |u(x)|^p \int_{C\Omega} \frac{1}{|x - y|^{N+ps}} \, dy \, dx. \]

Recall Lemma 6.1 in [22], we have

\[ \int_{C\Omega} \frac{1}{|x - y|^{N+ps}} \, dy \geq c |\Omega|^{-\frac{s}{p}}, \]

where \( c = c(N, p, s) > 0 \) and then obtain the Poincaré inequality

\[ \int_{\Omega} |u(x)|^p \, dx \leq C \int_{D\Omega} |u(x) - u(y)|^p \, d\nu, \quad \forall p \geq 1, \quad (1.3) \]

where

\[ d\nu = \frac{dx \, dy}{|x - y|^{N+ps}}. \]

Therefore, there exists a positive constant \( C = C(N, p, s, \Omega) \) such that for any \( u \in X^{s,p}_0(\Omega) \),

\[ \int_{D\Omega} |u(x) - u(y)|^p \, d\nu \leq \|u\|^p_{W^{s,p}(\mathbb{R}^N)} \leq C \int_{D\Omega} |u(x) - u(y)|^p \, d\nu. \]

Thus we can endow \( X^{s,p}_0(\Omega) \) with the equivalent norm

\[ \|u\|_{X^{s,p}_0(\Omega)} = \left( \int_{D\Omega} |u(x) - u(y)|^p \, d\nu \right)^{\frac{1}{p}}. \]
Note that $X_0^{s,p}(\Omega)$ is a uniformly convex Banach space, and hence $X_0^{s,p}(\Omega)$ is a reflexive Banach space.

For $w \in W^{s,p}(\mathbb{R}^N)$, we define the fractional $p$-Laplacian as

$$(-\Delta)_p^sw(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x - y|^{N+ps}} \, dy.$$  

It is clear that for all $w, v \in W^{s,p}(\mathbb{R}^N)$, we have

$$\langle (-\Delta)_p^sw, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y)) \, dv.$$

Now, if $w, v \in X_0^{s,p}(\Omega)$, then

$$\langle (-\Delta)_p^sw, v \rangle = \frac{1}{2} \int_{D_\Omega} |w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y)) \, dv.$$

It is easy to check that $(-\Delta)_p^s : X_0^{s,p}(\Omega) \to X_0^{s,p}(\Omega)^*$, where $X_0^{s,p}(\Omega)^*$ denotes the dual space of $X_0^{s,p}(\Omega)$. Let us define now the corresponding parabolic spaces. As in the local case, the space $L^p(0,T; X_0^{s,p}(\Omega))$ is defined as the set of function $u$ such that $u \in L^p(\Omega_T)$ with $\|u\|_{L^p(0,T; X_0^{s,p}(\Omega))} < \infty$, where

$$\|u\|_{L^p(0,T; X_0^{s,p}(\Omega))} = \left( \int_0^T \int_{D_\Omega} |u(x,t) - u(y,t)|^p \, dv \, dt \right)^{\frac{1}{p}}.$$

$L^p(0,T; X_0^{s,p}(\Omega))$ is a Banach space whose dual space is $L^{p'}(0,T; X_0^{s,p}(\Omega)^*)$.

Let $T_k$ denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k, \end{cases}$$

and its primitive $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ by

$$\Theta_k(r) = \int_0^r T_k(s) \, ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$.
We denote $u \in T_0^{s,p}(\Omega_T)$ if $u : \mathbb{R}^N \times (0,T] \to \mathbb{R}$ is measurable and $T_k(u) \in L^p(0,T; X_0^{s,p}(\Omega))$ for every $k > 0$. It is obvious that $u = 0$ a.e. in $C \Omega$. For simplicity and for any measurable function $u$, we write

$$U(x,y,t) = u(x,t) - u(y,t).$$

Next we give the following definitions of renormalized solutions and entropy solutions for problem (1.1).

**Definition 1.1.** A function $u \in T_0^{s,p}(\Omega_T) \cap C([0,T]; L^1(\Omega))$ is a renormalized solution to problem (1.1) if the following conditions are satisfied:

(i) $$\lim_{h \to \infty} \int \int \int_{\{(x,y,t) : (u(x,t),u(y,t)) \in R_h\}} |U(x,y,t)|^{p-1} d\nu dt = 0,$$

where

$$R_h = \{(u,v) \in \mathbb{R}^2 : h+1 \leq \max\{|u|,|v|\} \text{ and } (\min\{|u|,|v|\} \leq h \text{ or } uv < 0)\}.$$

(ii) For every function $\varphi \in C^1(\bar{\Omega}_T)$ with $\varphi = 0$ in $C \Omega \times (0,T)$ and $\varphi(\cdot,T) = 0$ in $\Omega$, and $S \in W^{1,\infty}(\mathbb{R})$ which is piecewise $C^1$ satisfying that $S'$ has a compact support,

$$- \int_\Omega S(u_0)\varphi(x,0) \, dx - \int_0^T \int_\Omega S(u) \frac{\partial \varphi}{\partial t} \, dx dt$$

$$+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x,y,t)|^{p-2}U(x,y,t) [(S'(u)\varphi)(x,t) - (S'(u)\varphi)(y,t)] \, d\nu dt$$

$$= \int_0^T \int_\Omega fS'(u)\varphi \, dx dt$$

holds.

**Remark 1.1.** It is not difficult to see that the symmetrization of the difference $(S'(u)\varphi)(x,t) - (S'(u)\varphi)(y,t)$ can yield the following equality:

$$\frac{1}{2} \int_0^T \int_{D_\Omega} |U(x,y,t)|^{p-2}U(x,y,t) [(S'(u)\varphi)(x,t) - (S'(u)\varphi)(y,t)] \, d\nu dt$$

$$= \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x,y,t)|^{p-2}U(x,y,t)(S'(u)(x,t) - S'(u)(y,t))$$
\[
\frac{\varphi(x,t) + \varphi(y,t)}{2} \, dv \, dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{D}_\Omega} |U(x,y,t)|^{p-2} U(x,y,t) (\varphi(x,t) - \varphi(y,t)) \\
S'(u)(x,t) + S'(u)(y,t) \, dv \, dt.
\]

**Definition 1.2.** A function \( u \in \mathcal{T}_0^{s,p}(\Omega_T) \cap C([0,T]; L^1(\Omega)) \) is an entropy solution to problem (1.1) if

\[
\int_\Omega \Theta_k(u - \phi)(T) \, dx - \int_\Omega \Theta_k(u_0 - \phi(0)) \, dx + \int_0^T \langle \phi_t, T_k(u - \phi) \rangle \, dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{D}_\Omega} |U(x,y,t)|^{p-2} U(x,y,t) \\
\cdot [T_k(u(x,t) - \phi(x,t)) - T_k(u(y,t) - \phi(y,t))] \, dv \, dt \\
\leq \int_0^T \int_{\Omega} f T_k(u - \phi) \, dx \, dt,
\]  

(1.5)

for all \( k > 0 \) and \( \phi \in C^1(\overline{\Omega}_T) \) with \( \phi = 0 \) in \( C \Omega \times (0,T) \).

Now we state our main results. The first two theorems are about the existence and uniqueness of nonnegative renormalized and entropy solutions. The third one is about the comparison principle.

**Theorem 1.1.** Assume that condition (1.2) holds. Then there exists a unique renormalized solution for problem (1.1).

**Theorem 1.2.** Assume that condition (1.2) holds. Then the renormalized solution \( u \) obtained in Theorem 1.1 is also an entropy solution for problem (1.1). And the entropy solution is unique.

**Remark 1.2.** The renormalized solution for problem (1.1) is equivalent to the entropy solution for problem (1.1).

**Theorem 1.3.** Let \( u_0, v_0 \in L^1(\Omega), f, g \in L^1(\Omega_T) \) such that \( u_0 \leq v_0 \) and \( f \leq g \). If \( u \) is the entropy solution (renormalized solution) of problem (1.1) and \( v \) is the entropy solution (renormalized solution) of problem (1.1) with \( u_0, f \) being replaced by \( v_0, g \), then \( u \leq v \) a.e. in \( \Omega_T \).
The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions to problem (1.1). We will prove the main results in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Weak solutions

In this section we will give a reasonable definition for weak solutions and prove the existence and uniqueness of weak solutions to problem (1.1).

Lemma 2.1. Assume that \( u_0 \in L^2(\Omega) \) and \( f \in L^{p'}(0, T; X_0^{s,p}(\Omega)^*) \). Then the following problem

\[
\begin{cases}
    u_t + (-\Delta)_p^s u = f & \text{in } \Omega_T, \\
    u = 0 & \text{in } C\Omega \times (0, T), \\
    u(x, 0) = u_0(x) & \text{in } \Omega
\end{cases}
\]

admits a unique weak solution \( u \in L^p(0, T; X_0^{s,p}(\Omega)) \cap C([0, T]; L^2(\Omega)) \) with \( u_t \in L^{p'}(0, T; X_0^{s,p}(\Omega)^*) \) such that for any \( \varphi \in C_0^\infty(\Omega_T) \),

\[
\int_0^T \langle u_t, \varphi \rangle \ dt + \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2} U(x, y, t)(\varphi(x, t) - \varphi(y, t)) \ d\nu dt
\]

holds.

Proof. Since the fractional \( p \)-Laplacian operator \((-\Delta)_p^s\) is monotone, the existence of weak solutions can be proved by employing the difference and variation methods. We give a sketched proof.

Let \( n \) be a positive integer. Denote \( h = T/n \). We first consider the following time-discrete problem

\[
\begin{cases}
    \frac{u_k - u_{k-1}}{h} + (-\Delta)_p^s u_k = [f]_h((k - 1)h), \\
    u_k|_{C\Omega} = 0, & \text{for } k = 1, 2, \ldots, n,
\end{cases}
\]

where \([f]_h\) denotes the Steklov average of \( f \) defined by

\[
[f]_h(x, t) = \frac{1}{h} \int_t^{t+h} f(x, \tau) \ d\tau.
\]
For \( k = 1 \), we introduce the variational problem

\[
\min \{ J(u) | u \in W \},
\]

where

\[
W = \{ u \in X_0^{s,p}(\Omega) \cap L^2(\Omega) \}
\]

and functional \( J \) is

\[
J(u) = \frac{1}{2h} \int_{\Omega} (u - u_0)^2 \, dx + \frac{1}{2p} \int_{D_\Omega} |u(x) - u(y)|^p \, d\nu - \int_{\Omega} [f]_h(0)u \, dx.
\]

By using the classical Direct Methods of the Calculus of Variations in fractional Sobolev spaces, we can prove that \( J(u) \) is lower bounded and coercive on \( W \). On the other hand, \( J(u) \) is weakly lower semicontinuous on \( W \). Therefore, there exists a function \( u_1 \in W \) such that

\[
J(u_1) = \inf_{u \in W} J(u).
\]

Thus the function \( u_1 \) is a weak solution of the corresponding Euler-Lagrange equation of \( J(u) \), which is (2.1) in the case \( k = 1 \). And it is unique since \( J(u) \) is strictly convex.

Following the same procedures, we find weak solutions \( u_k \) of (2.1) for \( k = 2, \ldots, n \). It follows that, for every \( \varphi \in W \),

\[
\int_{\Omega} \frac{u_k - u_{k-1}}{h} \varphi \, dx + \int_{\Omega} (-\Delta)^s_p u_k \varphi \, dx = \int_{\Omega} [f]_h((k-1)h) \varphi \, dx. \tag{2.2}
\]

For every \( h = T/n \), we define the approximate solutions

\[
u_h(x, t) = \begin{cases} \ u_0(x), & t = 0, \\ u_1(x), & 0 < t \leq h, \\ \ldots, & \ldots, \\ u_j(x), & (j - 1)h < t \leq jh, \\ \ldots, & \ldots, \\ u_n(x), & (n - 1)h < t \leq nh = T. \end{cases}
\]

Taking \( \varphi = u_k \) in (2.2), we can obtain an a priori estimate

\[
\int_{\Omega} u_h^2(x, t) \, dx + \frac{1}{2} \int_0^T \int_{D_\Omega} |u_h(x, t) - u_h(y, t)|^p \, d\nu dt \leq C,
\]
which implies that
\[ \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|u_h\|_{L^p(0,T;X_{0}^{s,p}(\Omega))} \leq C. \]

Thus we may choose a subsequence (we also denote it by the original sequence for simplicity) such that
\[ u_h \rightharpoonup u, \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)), \]
\[ u_h \rightharpoonup u, \quad \text{weakly in } L^p(0,T;X_{0}^{s,p}(\Omega)). \]

Recalling the fact that \( u \in L^p(0,T;X_{0}^{s,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \) and \( u_t \in L^{p'}(0,T;X_{0}^{s,p}(\Omega)^*) \), we conclude that \( u \) belongs to \( C([0,T];L^2(\Omega)) \). Therefore, we obtain the existence of weak solutions.

For uniqueness, suppose there exist two weak solutions \( u \) and \( v \) of problem (1.1). Then \( w = u - v \) satisfies the following problem
\[
\begin{cases}
w_t + [(-\Delta)^s u - (-\Delta)^s v] = 0 & \text{in } \Omega_T, \\
w = 0 & \text{in } \partial \Omega \times (0,T), \\
w(x,0) = 0 & \text{in } \Omega.
\end{cases}
\]

Choosing \( w \) as a test function in the above problem, we have, for almost every \( t \in (0,T) \),
\[
\int_{\Omega} w^2(t) \, dx + \int_0^t \int_{\Omega} \left[ |U(x,y,\tau)|^{p-2}U(x,y,\tau) - |V(x,y,\tau)|^{p-2}V(x,y,\tau) \right] \\
\cdot (U(x,y,\tau) - V(x,y,\tau)) \, dv \, d\tau = 0,
\]
where \( V(x,y,\tau) = v(x,\tau) - v(y,\tau) \). Since the two terms on the left-hand side are nonnegative, then we have \( u = v \) a.e. in \( \Omega_T \). This finishes the proof.

3. The proof of main results

Now we are ready to prove the main results. Some of the reasoning is based on the ideas developed in [2, 35, 36, 43]. First we prove the existence and uniqueness of renormalized solutions for problem (1.1).

Proof of Theorem 1.1. (1) Existence of renormalized solutions.
We first introduce the approximate problems. Define $f_n = T_n(f)$ and $u_{0n} = T_n(u_0)$, then we know that $f_n, u_{0n}$ are nonnegative, $(f_n, u_{0n}) \in L^\infty(\Omega_T) \times L^\infty(\Omega)$ and $(f_n, u_{0n}) \not\rightharpoonup (f, u_0)$ strongly in $L^1(\Omega_T) \times L^1(\Omega)$ such that
\[
\|f_n\|_{L^1(\Omega_T)} \leq \|f\|_{L^1(\Omega_T)}, \quad \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \tag{3.1}
\]

Then we consider the approximate problem of (1.1)
\[
\begin{aligned}
&\left\{
\begin{aligned}
(u_n)_t + (-\Delta)^s_p u_n = f_n & \quad \text{in } \Omega_T, \\
u_n = 0 & \quad \text{in } C\Omega \times (0, T), \\
u_n(x, 0) = u_{0n} & \quad \text{in } \Omega.
\end{aligned}
\right.
\end{aligned}
\tag{3.2}
\]

By Lemma 2.1 and comparison principle, we can find a unique nonnegative weak solution $u_n \in L^p(0, T; X_0^{s,p}(\Omega))$ for problem (3.2). Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges increasingly to a measurable function $u$, which is a renormalized solution of problem (1.1). We will divide the proof into several steps. We present a self-contained proof for the sake of clarity and readability.

**Step 1.** Prove the convergence of $\{u_n\}$ in $C([0, T]; L^1(\Omega))$ and find its subsequence which is almost everywhere convergent in $\Omega_T$.

Let $m$ and $n$ be two integers, then from (3.2) we can write the weak form as
\[
\int_0^T \langle (u_n - u_m)_t, \phi \rangle \, dt + \int_0^T \langle (-\Delta)^s_p u_n - (-\Delta)^s_p u_m, \phi \rangle \, dt
\]
\[
= \int_0^T \int_\Omega (f_n - f_m) \phi \, dx \, dt,
\]
for all $\phi \in L^p(0, T; X_0^{s,p}(\Omega)) \cap L^\infty(\Omega_T)$. Choosing $\phi = T_1(u_n - u_m)\chi_{(0,t)}$ with $t \leq T$, we have
\[
\int_0^t \langle (u_n - u_m)_t, T_1(u_n - u_m) \rangle \, d\tau
\]
\[
+ \int_0^t \langle (-\Delta)^s_p u_n - (-\Delta)^s_p u_m, T_1(u_n - u_m) \rangle \, d\tau
\]
\[
= \int_0^T \int_\Omega (f_n - f_m) T_1(u_n - u_m) \chi_{(0,t)} \, dx \, dt.
\]
Observe that
\[ \int_0^t \langle (-\Delta)_p^s u_n - (-\Delta)_p^s u_m, T_1(u_n - u_m) \rangle d\tau \]
\[ = \frac{1}{2} \int_0^t \int_{D_\Omega} \left( |U_n(x, y, \tau)|^{p-2} U_n(x, y, \tau) - |U_m(x, y, \tau)|^{p-2} U_m(x, y, \tau) \right) \]
\[ \cdot \left[ T_1(u_n(x, \tau) - u_m(x, \tau)) - T_1(u_n(y, \tau) - u_m(y, \tau)) \right] d\nu d\tau. \]

Since
\[ T_1(u_n(x, \tau) - u_m(x, \tau)) - T_1(u_n(y, \tau) - u_m(y, \tau)) \]
\[ = T'_1(\xi_{nm})(U_n(x, y, \tau) - U_m(x, y, \tau)) \]
due to the mean value theorem, where \( T'_1 \geq 0 \), we know that
\[ \int_0^t \langle (-\Delta)_p^s u_n - (-\Delta)_p^s u_m, T_1(u_n - u_m) \rangle d\tau \geq 0. \]

Then we get
\[ \int_\Omega \Theta_1(u_n - u_m)(t) dx \leq \int_\Omega \Theta_1(u_{0n} - u_{0m}) dx + \| f_n - f_m \|_{L^1(\Omega)} \]
\[ \leq \| u_{0n} - u_{0m} \|_{L^1(\Omega)} + \| f_n - f_m \|_{L^1(\Omega)} : = a_{n,m}. \]

Therefore, we conclude that
\[ \int_\Omega \frac{|u_n - u_m|^2(t)}{2} dx + \int_\Omega \frac{|u_n - u_m|(t)}{2} dx \]
\[ \leq \int_\Omega |\Theta_1(u_n - u_m)|(t) dx \leq a_{n,m}. \]

It follows that
\[ \int_\Omega |u_n - u_m|(t) dx = \int_\{ |u_n - u_m| < 1 \} |u_n - u_m|(t) dx \]
\[ + \int_\{ |u_n - u_m| \geq 1 \} |u_n - u_m|(t) dx \]
\[ \leq \left( \int_\{ |u_n - u_m| < 1 \} |u_n - u_m|^2(t) dx \right)^{\frac{1}{2}} \text{meas}(\Omega)^{\frac{1}{2}} + 2a_{n,m} \]
\[ \leq (2\text{meas}(\Omega))^{\frac{1}{2}} a_{n,m} + 2a_{n,m}. \]
Since \( \{f_n\} \) and \( \{u_{0n}\} \) are convergent in \( L^1 \), we have \( a_{n,m} \to 0 \) for \( n, m \to +\infty \). Thus \( \{u_n\} \) is a Cauchy sequence in \( C([0, T]; L^1(\Omega)) \) and \( u_n \) converges to \( u \) in \( C([0, T]; L^1(\Omega)) \). Then we find an a.e. convergent subsequence (still denoted by \( \{u_n\} \)) in \( \Omega_T \) such that
\[
    u_n \rightharpoonup u \quad \text{a.e. in } \Omega_T.
\] (3.3)

**Step 2.** Prove \( T_k(u_n) \) strongly converges to \( T_k(u) \) in \( L^p(0, T; X_0^{s,p}(\Omega)) \), for every \( k > 0 \).

Choosing \( T_k(u_n) \) as a test function in (3.2), we have
\[
    \int_\Omega \Theta_k(u_n)(T) \, dx - \int_\Omega \Theta_k(u_{0n}) \, dx \\
    + \int_0^T \langle (-\Delta)^s_p u_n, T_k(u_n) \rangle \, dt = \int_0^T \int_\Omega f_n T_k(u_n) \, dx dt.
\]
It follows from the definition of \( \Theta_k(r) \), \( 0 \leq T_k' \leq 1 \) and (3.1) that
\[
    \frac{1}{2} \int_0^T \int_{D_{T_k}} |T_k(u_n(x, t)) - T_k(u_n(y, t))|^p \, dv dt \\
    \leq \frac{1}{2} \int_0^T \int_{D_{T_k}} |U_n(x, y, t)|^{p-2} U_n(x, y, t) \langle T_k(u_n(x, t)) - T_k(u_n(y, t)) \rangle \, dv dt \\
    \leq k(\|f_n\|_{L^1(\Omega_T)} + \|u_{0n}\|_{L^1(\Omega)}) \\
    \leq k(\|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)}).
\]
Then, up to a subsequence, we deduce that
\[ T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^p(0, T; X_0^{s,p}(\Omega)). \]

In order to deal with the time derivative of truncations, we will use the regularization method of Landes [26] and use the sequence \( (T_k(u))_\mu \) as approximation of \( T_k(u) \). For \( \mu > 0 \), we define the regularization in time of the function \( T_k(u) \) given by
\[
    (T_k(u))_\mu(x, t) := \mu \int_{-\infty}^t e^{\mu(s-t)} T_k(u(x, s)) \, ds,
\]
extending \( T_k(u) \) by 0 for \( s < 0 \). Observe that \( (T_k(u))_\mu \in L^p(0, T; X_0^{s,p}(\Omega)) \cap L^\infty(\Omega_T) \), it is differentiable for a.e. \( t \in (0, T) \) with
\[
    |(T_k(u))_\mu(x, t)| \leq k(1 - e^{-\mu t}) < k \quad \text{a.e. in } \Omega_T,
\]
\[
    \frac{\partial (T_k(u))_\mu}{\partial t} = \mu (T_k(u) - (T_k(u))_\mu).
\]
After computation, we can get
\[(T_k(u))_\mu \to T_k(u) \text{ strongly in } L^p(0, T; X_0^{s,p}(\Omega)).\]

Let us take now a sequence \(\{\psi_j\}\) of \(C^\infty(\Omega)\) functions that strongly converge to \(u_0\) in \(L^1(\Omega)\), and set
\[\eta_{\mu,j}(u) \equiv (T_k(u))_\mu + e^{-\mu t}T_k(\psi_j).\]

The definition of \(\eta_{\mu,j}\), which is a smooth approximation of \(T_k(u)\), is needed to deal with a nonzero initial datum (see also [35]). Note that this function has the following properties:

\[
\begin{align*}
(\eta_{\mu,j}(u))_t &= \mu(T_k(u) - \eta_{\mu,j}(u)), \\
\eta_{\mu,j}(u)(0) &= T_k(\psi_j), \\
|\eta_{\mu,j}(u)| &\leq k, \\
\eta_{\mu,j}(u) &\to T_k(u) \text{ strongly in } L^p(0, T; X_0^{s,p}(\Omega)), \text{ as } \mu \to +\infty.
\end{align*}
\]

Fix a positive number \(k\). Let \(h > k\). We choose \(w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u))\) as a test function in (3.2). Combining the arguments in Step 2 of the proof of Theorem 1.1 in [44] and the Lemma 3.6 in [3] together with the nonnegativity and monotonicity of the sequence \(\{u_n\}\), we can conclude that

\[
\limsup_{n \to \infty} \int_0^T \int_{D_0} |T_k(u_n(x, t)) - T_k(u_n(y, t))|^p \, d\nu dt \\
\leq \int_0^T \int_{D_h} |T_k(u(x, t)) - T_k(u(y, t))|^p \, d\nu dt.
\]

It follows from
\[T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0, T; X_0^{s,p}(\Omega))\]
that
\[T_k(u_n) \to T_k(u) \text{ strongly in } L^p(0, T; X_0^{s,p}(\Omega)).\]  

\textbf{Step 3}. Show that \(u\) is a renormalized solution.
Define the function \( G_k(s) = s - T_k(s) \). For given \( h > 0 \), using \( T_1(G_h(u_n)) \) as a test function in (3.2), we find

\[
\int_{\{|u_n|>h\}} \Theta_1(u_n \mp h)(T) \, dx - \int_{\{|u_0n|>h\}} \Theta_1(u_0n \mp h) \, dx \\
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t) \\
\cdot [T_1(G_h(u_n))(x, t) - T_1(G_h(u_n))(y, t)] \, d
\nu dt \\
\leq \int_\Omega f_n T_1(G_h(u_n)) \, dx dt,
\]

which yields that

\[
\frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t) \\
\cdot [T_1(G_h(u_n))(x, t) - T_1(G_h(u_n))(y, t)] \, d
\nu dt \\
\leq \int_{\{|u_n|>h\}} |f_n| \, dx dt + \int_{\{|u_0n|>h\}} |u_0n| \, dx.
\]

It is not difficult to see that

\[
|U_n(x, y, t)|^{p-2} U_n(x, y, t) [T_1(G_h(u_n))(x, t) - T_1(G_h(u_n))(y, t)] \\
= T'_1(G_h(\xi_n)) G'_h(\xi_n) |U_n(x, y, t)|^p \geq 0.
\]

Recalling the convergence of \( \{u_n\} \) in \( C([0, T]; L^1(\Omega)) \), we have

\[
\lim_{h \to +\infty} \text{meas}\{(x, t) \in \Omega \cap : |u_n| > h\} = 0 \quad \text{uniformly with respect to } n.
\]

Since for all \((u_n(x, t), u_n(y, t)) \in R_h\),

\[
|U_n(x, y, t)|^{p-2} U_n(x, y, t) [T_1(G_h(u_n))(x, t) - T_1(G_h(u_n))(y, t)] \\
\geq |U_n(x, y, t)|^{p-1},
\]

by using Fatou’s lemma and passing to the limit first in \( n \) then in \( h \), we obtain the renormalized condition

\[
\lim_{h \to +\infty} \int \int_{\{(u(x,t), u(y,t)) \in R_h\}} |U(x, y, t)|^{p-1} \, d \nu dt = 0. \quad (3.6)
\]
Let \( S \in W^{1,\infty}(\mathbb{R}) \) be such that \( \text{supp} \ S' \subset [-M, M] \) for some \( M > 0 \). For every \( \varphi \in C^1(\bar{\Omega}_T) \) with \( \varphi = 0 \) in \( C\Omega \times (0, T) \) and \( \varphi(\cdot, T) = 0 \) in \( \Omega \), \( S'(u_n)\varphi \) is a test function in (3.2). It yields
\[
\int_0^T \int_{\Omega} \frac{\partial S(u_n)}{\partial t} \varphi \, dx \, dt
\]
\[
+ \frac{1}{2} \int_0^T \int_{D\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\varphi(x, t) - \varphi(y, t)) \cdot \frac{S'(u_n)(x, t) + S'(u_n)(y, t)}{2} \, dv \, dt
\]
\[
+ \frac{1}{2} \int_0^T \int_{D\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(S'(u_n)(x, t) - S'(u_n)(y, t)) \cdot \frac{\varphi(x, t) + \varphi(y, t)}{2} \, dv \, dt
\]
\[
= \int_0^T \int_{\Omega} f_n S'(u_n)\varphi \, dx \, dt. \tag{3.7}
\]

First we consider the first term on the left-hand side of (3.7). Since \( S \) is bounded and continuous, (3.3) implies that \( S(u_n) \) converges to \( S(u) \) a.e. in \( \Omega_T \) and weakly-* in \( L^\infty(\Omega_T) \). Then \( \frac{\partial S(u_n)}{\partial t} \) converges to \( \frac{\partial S(u)}{\partial t} \) in \( D'(\Omega_T) \) as \( n \to +\infty \), that is
\[
\int_0^T \int_{\Omega} S(u_n) \frac{\partial \varphi}{\partial t} \, dx \, dt \to \int_0^T \int_{\Omega} S(u) \frac{\partial \varphi}{\partial t} \, dx \, dt.
\]
For the right-hand side of (3.7), thanks to the strong convergence of \( f_n \), it is easy to pass to the limit:
\[
\int_0^T \int_{\Omega} f_n S'(u_n)\varphi \, dx \, dt \to \int_0^T \int_{\Omega} f S'(u)\varphi \, dx \, dt, \quad \text{as} \ n \to +\infty.
\]

For the other terms on the left-hand side of (3.7), we claim that
\[
I_1 = \int_0^T \int_{D\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\varphi(x, t) - \varphi(y, t)) \cdot \frac{S'(u_n)(x, t) + S'(u_n)(y, t)}{2} \, dv \, dt
\]
\[
\to \int_0^T \int_{D\Omega} |U(x, y, t)|^{p-2} U(x, y, t)(\varphi(x, t) - \varphi(y, t)) \cdot \frac{S'(u)(x, t) + S'(u)(y, t)}{2} \, dv \, dt, \quad \text{as} \ n \to +\infty.
\]
Assume that \( \text{supp} \, S' \subset [-M, M] \). Set

\[
D_1 = \{ (x, y, t) \in D_\Omega \times (0, T) : u_n(x, t) \geq M, u_n(y, t) \geq M \}, \\
D_2 = \{ (x, y, t) \in D_\Omega \times (0, T) : u_n(x, t) \leq M, u_n(y, t) \leq M \}, \\
D_3 = \{ (x, y, t) \in D_\Omega \times (0, T) : u_n(x, t) \geq M, u_n(y, t) \leq M \}, \\
D_4 = \{ (x, y, t) \in D_\Omega \times (0, T) : u_n(x, t) \leq M, u_n(y, t) \geq M \}.
\]

Then

\[
D_\Omega \times (0, T) = D_1 \cup D_2 \cup D_3 \cup D_4.
\]

In \( D_1 \) we have

\[
S'(u_n)(x, t) = S'(u_n)(y, t) = 0,
\]
then \( I_1 = 0 \).

In \( D_2 \) we have

\[
u_n(x, t) = T_M(u_n(x, t)), \quad u_n(y, t) = T_M(u_n(y, t)).
\]

From the strong convergence (3.5), we know that

\[
|T_M(u_n(x, t)) - T_M(u_n(y, t))|^{p-2}(T_M(u_n(x, t)) - T_M(u_n(y, t))) \\
\to |T_M(u(x, t)) - T_M(u(y, t))|^{p-2}(T_M(u(x, t)) - T_M(u(y, t)))
\]

strongly in \( L^\frac{p}{p-1}(D_\Omega \times (0, T)) \).

Moreover, \( u_n \to u \) a.e. in \( \Omega_T, S \in W^{1,\infty}(\mathbb{R}) \) and \( \varphi \in C^1(\overline{\Omega}_T) \) with \( \varphi = 0 \) in \( C\Omega \times (0, T) \) imply that

\[
\frac{\varphi(x, t) - \varphi(y, t)}{|x - y|^\frac{N+ps}{p}} \in L^p(D_\Omega \times (0, T))
\]

and

\[
\frac{\varphi(x, t) - \varphi(y, t)}{|x - y|^\frac{N+ps}{p}} S'(u_n)(x, t) + S'(u_n)(y, t) \\
\to \frac{\varphi(x, t) - \varphi(y, t)}{|x - y|^\frac{N+ps}{p}} S'(u)(x, t) + S'(u)(y, t)
\]

weakly in \( L^p(D_\Omega \times (0, T)) \).
Thus we have
\[
\int \int_{D_2} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\varphi(x, t) - \varphi(y, t))
· \frac{S'(u_n)(x, t) + S'(u_n)(y, t)}{2} \, d\nu dt
\rightarrow \int \int_{\{u(x, t) \leq M, u(y, t) \leq M\}} |U(x, y, t)|^{p-2} U_n(x, y, t)(\varphi(x, t) - \varphi(y, t))
· \frac{S'(u)(x, t) + S'(u)(y, t)}{2} \, d\nu dt, \quad \text{as } n \to \infty.
\]

In $D_3$, if $u_n(x, t) \leq M + 1$, then it can be done similarly to the estimates in $D_2$. On the other hand, if $u_n(x, t) \geq M + 1$, then
\[
\max\{u_n(x, t), u_n(y, t)\} \geq M + 1 \quad \text{and} \quad \min\{u_n(x, t), u_n(y, t)\} \leq M.
\]
It follows from (3.6) that
\[
\lim_{M \to \infty} \lim_{n \to \infty} \int \int_{\{u_n(x, y, t) \in R_M\}} |U_n(x, y, t)|^{p-1} \, d\nu dt = 0.
\]

Then we observe
\[
\lim_{M \to \infty} \lim_{n \to \infty} \int \int_{\{u_n(x, t) \geq M + 1, u_n(y, t) \leq M\}} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\varphi(x, t) - \varphi(y, t))
· \frac{S'(u_n)(x, t) + S'(u_n)(y, t)}{2} \, d\nu dt = 0.
\]

The estimates in $D_4$ can be done similarly.

Therefore, we have
\[
\lim_{M \to \infty} \lim_{n \to \infty} I_1 = \lim_{M \to \infty} \int \int_{\{u(x, y, t) \leq M, u(y, t) \leq M\}} |U(x, y, t)|^{p-2} U(x, y, t)(\varphi(x, t) - \varphi(y, t))
· \frac{S'(u)(x, t) + S'(u)(y, t)}{2} \, d\nu dt
\]
\[
+ 2 \lim_{M \to \infty} \int \int_{\{u(x, t) \leq M + 1, u(y, t) \leq M\}} |U(x, y, t)|^{p-2} U(x, y, t)(\varphi(x, t) - \varphi(y, t))
· \frac{S'(u)(x, t) + S'(u)(y, t)}{2} \, d\nu dt
\]

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\[
\int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2}U(x, y, t)(\varphi(x, t) - \varphi(y, t)) \\
\cdot S'(u)(x, t) + \frac{S'(u)(y, t)}{2} \, dvdt.
\]

The third term on the left-hand side of (3.7) can be argued similarly. Therefore, we obtain
\[
\int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2}U(x, y, t)(\varphi(x, t) - \varphi(y, t)) \\
\cdot S'(u)(x, t) + \frac{S'(u)(y, t)}{2} \, dvdt
\]
\[
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2}U(x, y, t)(\varphi(x, t) - \varphi(y, t)) \\
\cdot \frac{S'(u)(x, t) + S'(u)(y, t)}{2} \, dvdt
\]
\[
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2}U(x, y, t)[(S'(u)\varphi)(x, t) - (S'(u)\varphi)(y, t)] \, dvdt
\]
\[
= \int_0^T \int_{\Omega} fS'(u)\varphi \, dxdt,
\]
that is
\[
- \int_{\Omega} S(u_0)\varphi(x, 0) \, dx - \int_0^T \int_{\Omega} S(u) \frac{\partial \varphi}{\partial t} \, dxdt
\]
\[
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2}U(x, y, t)[(S'(u)\varphi)(x, t) - (S'(u)\varphi)(y, t)] \, dvdt
\]
\[
= \int_0^T \int_{\Omega} fS'(u)\varphi \, dxdt
\]
for any \( \varphi \in C^1(\overline{\Omega_T}) \) with \( \varphi = 0 \) in \( C\Omega \times (0, T) \) and \( \varphi(\cdot, T) = 0 \) in \( \Omega \). This completes the proof of the existence of renormalized solutions.

(2) Uniqueness of renormalized solutions.

Now we prove the uniqueness of renormalized solutions for problem (1.1) by choosing an appropriate test function motivated by [9] and [6]. Let \( u \) and \( v \) be two renormalized solutions for problem (1.1). For \( \sigma > 0 \), let \( S_\sigma \) be the
function defined by
\[
S_{\sigma}(r) = \begin{cases} 
  r & \text{if } |r| < \sigma, \\
  (\sigma + \frac{1}{2}) \mp \frac{1}{2}(r \mp (\sigma + 1))^2 & \text{if } \sigma \leq \pm r \leq \sigma + 1, \\
  \pm (\sigma + \frac{1}{2}) & \text{if } \pm r > \sigma + 1.
\end{cases}
\] (3.8)

It is obvious that
\[
S'_{\sigma}(r) = \begin{cases} 
  1 & \text{if } |r| < \sigma, \\
  \sigma + 1 - |r| & \text{if } \sigma \leq |r| \leq \sigma + 1, \\
  0 & \text{if } |r| > \sigma + 1.
\end{cases}
\]

It is easy to check \( S_{\sigma} \in W^{1,\infty}(\mathbb{R}) \) with \( \text{supp } S'_{\sigma} \subset [-\sigma - 1, \sigma + 1] \). Therefore, we may take \( S = S_{\sigma} \) in (1.4) to have
\[
\int_0^T \int_{\Omega} \partial S_{\sigma}(u) \frac{\partial \varphi}{\partial t} dx dt \\
+ \frac{1}{2} \int_0^T \int_{D_0} |U(x,y,t)|^{p-2}U(x,y,t)\varphi(x,t) - \varphi(y,t) \\
\cdot \frac{S'_{\sigma}(u)(x,t) + S'_{\sigma}(u)(y,t)}{2} dv dt \\
+ \frac{1}{2} \int_0^T \int_{D_0} |U(x,y,t)|^{p-2}U(x,y,t)\{S'_{\sigma}(u)(x,t) - S'_{\sigma}(u)(y,t)\} \\
\cdot \frac{\varphi(x,t) + \varphi(y,t)}{2} dv dt \\
= \int_0^T \int_{\Omega} f S'_{\sigma}(u) \varphi dx dt
\]

and
\[
\int_0^T \int_{\Omega} \partial S_{\sigma}(v) \frac{\partial \varphi}{\partial t} dx dt \\
+ \frac{1}{2} \int_0^T \int_{D_0} |V(x,y,t)|^{p-2}V(x,y,t)\varphi(x,t) - \varphi(y,t) \\
\cdot \frac{S'_{\sigma}(v)(x,t) + S'_{\sigma}(v)(y,t)}{2} dv dt
\]
\[ + \frac{1}{2} \int_0^T \int_{D \Omega} |V(x, y, t)|^{p-2} V(x, y, t) (S'_\sigma(v)(x, t) - S'_\sigma(v)(y, t)) \cdot \varphi(x, t) + \varphi(y, t) \, dv dt \]
\[ = \int_0^T \int_{D \Omega} f S'_\sigma(v) \varphi \, dx dt. \]

For every fixed \( k > 0 \), we plug \( \varphi = T_k(S_\sigma(u) - S_\sigma(v)) \) as a test function in the above equalities and subtract them to obtain that

\[ J_0 + J_1 + J_2 = J_3, \tag{3.9} \]

where

\[ J_0 = \int_0^T \left\langle \frac{\partial(S_\sigma(u) - S_\sigma(v))}{\partial t}, T_k(S_\sigma(u) - S_\sigma(v)) \right\rangle dt, \]
\[ J_1 = \frac{1}{2} \int_0^T \int_{D \Omega} \left[ \frac{S'_\sigma(u)(x, t) + S'_\sigma(u)(y, t)}{2} |U(x, y, t)|^{p-2} U(x, y, t) \right. \]
\[ - \frac{S'_\sigma(v)(x, t) + S'_\sigma(v)(y, t)}{2} |V(x, y, t)|^{p-2} V(x, y, t) \]
\[ \cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x, t) - T_k(S_\sigma(u) - S_\sigma(v))(y, t) \right] dv dt, \]
\[ J_2 = \frac{1}{2} \int_0^T \int_{D \Omega} \left[ |U(x, y, t)|^{p-2} U(x, y, t) \cdot (S'_\sigma(u)(x, t) - S'_\sigma(u)(y, t)) \right. \]
\[ - |V(x, y, t)|^{p-2} V(x, y, t) \cdot (S'_\sigma(v)(x, t) - S'_\sigma(v)(y, t)) \]
\[ \cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x, t) + T_k(S_\sigma(u) - S_\sigma(v))(y, t) \right] dv dt, \]
\[ J_3 = \int_0^T \int_{D \Omega} f(S'_\sigma(u) - S'_\sigma(v))T_k(S_\sigma(u) - S_\sigma(v)) \, dx dt. \]

We estimate \( J_0, J_1, J_2 \) and \( J_3 \) one by one. Recalling the definition of \( \Theta_k(r) \), \( J_0 \) can be written as

\[ J_0 = \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(T) \, dx - \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(0) \, dx. \]

Due to the same initial condition for \( u \) and \( v \), and the properties of \( \Theta_k \), we get

\[ J_0 = \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(T) \, dx \geq 0. \]
Writing
\[ J_1 = \frac{1}{2} \int_0^T \int_{D_0} (|U(x, y, t)|^{p-2}U(x, y, t) - |V(x, y, t)|^{p-2}V(x, y, t)) \cdot [T_\sigma(S_\sigma(u) - S_\sigma(v))(x, t) - T_\sigma(S_\sigma(u) - S_\sigma(v))(y, t)] \, dv dt 
+ \frac{1}{2} \int_0^T \int_{D_\alpha} \left[ 1 - \frac{S'_\sigma(u)(x, t) + S'_\sigma(v)(y, t)}{2} \right] |U(x, y, t)|^{p-2}U(x, y, t) \cdot [T_\sigma(S_\sigma(u) - S_\sigma(v))(x, t) - T_\sigma(S_\sigma(u) - S_\sigma(v))(y, t)] \, dv dt 
+ \frac{1}{2} \int_0^T \int_{D_\alpha} \left[ \frac{S'_\sigma(v)(x, t) + S'_\sigma(v)(y, t)}{2} - 1 \right] |V(x, y, t)|^{p-2}V(x, y, t) \cdot [T_\sigma(S_\sigma(u) - S_\sigma(v))(x, t) - T_\sigma(S_\sigma(u) - S_\sigma(v))(y, t)] \, dv dt \]
\[ := J_1^1 + J_1^2 + J_1^3, \]
and setting \( \sigma \geq k \), we have
\[ J_1^1 \geq \frac{1}{2} \int \int \int \left\{ |u - v| \leq k \right\} \left( |U(x, y, t)|^{p-2}U(x, y, t) - |V(x, y, t)|^{p-2}V(x, y, t) \right) \cdot [U(x, y, t) - V(x, y, t)] \, dv dt. \] (3.10)

By the Lebesgue dominated convergence theorem, we conclude that
\[ J_1^1, J_1^3 \to 0, \quad \text{as} \quad \sigma \to +\infty. \]

Furthermore, we have
\[ |J_2| \leq C \left( \int \int \int \left\{ (u(x, t), u(y, t)) \in R_\sigma \right\} |U(x, y, t)|^{p-1} \, dv dt 
+ \int \int \int \left\{ (v(x, t), v(y, t)) \in R_\sigma \right\} |V(x, y, t)|^{p-1} \, dv dt. \]

From the above estimates and (i) in Definition 1.1, we obtain
\[ \lim_{\sigma \to +\infty} (|J_1^2| + |J_1^3| + |J_2|) = 0. \]

Observing
\[ f(S'_\sigma(u) - S'_\sigma(v)) \to 0 \quad \text{strongly in} \quad L^1(\Omega_T) \]
as \( \sigma \to +\infty \) and using the Lebesgue dominated convergence theorem, we deduce that
\[
\lim_{\sigma \to +\infty} |J_3| = 0.
\]

Therefore, sending \( \sigma \to +\infty \) in (3.9) and recalling (3.10), we have
\[
\int \int \int_{\{ |u| \leq \frac{k}{2}, |v| \leq \frac{k}{2} \}} (|U(x, y, t)|^{p-2}U(x, y, t) - |V(x, y, t)|^{p-2}V(x, y, t))
\cdot (U(x, y, t) - V(x, y, t)) \, d\nu \, dt = 0,
\]
which implies \( U = V \) a.e. on the set \( \{ |u| \leq \frac{k}{2}, |v| \leq \frac{k}{2} \} \). Since \( k \) is arbitrary, we conclude that
\[
U = V \quad \text{for a.e. } x, y \in \mathbb{R}^N, t \in [0, T].
\]

It follows from the Poincaré inequality with \( p = 1 \) in (1.3) that
\[
\int_0^T \int_\Omega |u(x, t) - v(x, t)| \, dx \, dt
\leq C \int_0^T \int_{D_\Omega} |U(x, y, t) - V(x, y, t)| \, d\nu_1 \, dt = 0,
\]
where
\[
d\nu_1 = \frac{dx \, dy}{|x - y|^{N+s}}.
\]

Thus we have \( u = v \) a.e. in \( \Omega_T \). This completes the proof of Theorem 1.1. \( \square \)

Next, we prove that the renormalized solution \( u \) is also an entropy solution of problem (1.1) and the entropy solution of problem (1.1) is unique.

**Proof of Theorem 1.2.** (1) The renormalized solution is an entropy solution.

Now we choose \( v_n = T_k(u_n - \phi) \) as a test function in (3.2) for \( k > 0 \) and \( \phi \in C^1(\Omega_T) \) with \( \phi = 0 \) in \( C\Omega \times (0, T) \). Following the arguments in \( \text{[2]} \), we can prove the existence of entropy solutions.

(2) Uniqueness of entropy solutions.
Suppose that $u$ and $v$ are two entropy solutions of problem (1.1). Let $\{u_n\}$ be a sequence constructed in (3.2). Choosing $S_\sigma(u_n)$ as a test function in (1.5) for entropy solution $v$, we have

\[
\int_\Omega \Theta_k(v - S_\sigma(u_n))(T) \, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_{0n})) \, dx \\
+ \int_0^T \langle (u_n)_t, S'_\sigma(u_n)T_k(v - S_\sigma(u_n)) \rangle \, dt \\
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U(x, y, t)|^{p-2} U(x, y, t) [T_k(v(x, t) - S_\sigma(u_n)(x, t)) \\
- T_k(v(y, t) - S_\sigma(u_n)(y, t))] \, d\nu dt \\
\leq \int_0^T \int_\Omega fT_k(v - S_\sigma(u_n)) \, dx dt.
\] (3.11)

In order to deal with the third term on the left-hand side of (3.11), we take $S'_\sigma(u_n)\Psi$ with $\Psi = T_k(v - S_\sigma(u_n))$ as a test function for problem (3.2) to obtain

\[
\int_0^T \langle (u_n)_t, S'_\sigma(u_n)\Psi \rangle \, dt \\
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t) (S'_\sigma(u_n)(x, t) - S'_\sigma(u_n)(y, t)) \cdot \frac{\Psi(x, t) + \Psi(y, t)}{2} \, d\nu dt \\
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t) (\Psi(x, t) - \Psi(y, t)) \cdot \frac{S'_\sigma(u_n)(x, t) + S'_\sigma(u_n)(y, t)}{2} \, d\nu dt \\
= \int_0^T \int_\Omega f_n S'_\sigma(u_n)\Psi \, dx dt.
\] (3.12)

Thus we deduce from (3.11) and (3.12) that

\[
\int_\Omega \Theta_k(v - S_\sigma(u_n))(T) \, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_{0n})) \, dx \\
- \frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t) (S'_\sigma(u_n)(x, t) - S'_\sigma(u_n)(y, t))
\]
\[
\frac{\Psi(x, t) + \Psi(y, t)}{2} \, dvdt
\]

\[
- \frac{1}{2} \int_0^T \int_{D_\Omega} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\Psi(x, t) - \Psi(y, t))
\]

\[
\cdot \frac{S'_\sigma(u_n)(x, t) + S'_\sigma(u_n)(y, t)}{2} \, dvdt
\]

\[
+ \frac{1}{2} \int_0^T \int_{D_\Omega} |V(x, y, t)|^{p-2} V(x, y, t)(\Psi(x, t) - \Psi(y, t)) \, dvdt
\]

\[
\leq \int_0^T \int_\Omega f_T k(v - S_\sigma(u_n)) \, dx dt - \int_0^T \int_\Omega f_n S'_\sigma(u_n) T_k(v - S_\sigma(u_n)) \, dx dt.
\]

We will pass to the limit as \( n \to +\infty \) and \( \sigma \to +\infty \) successively. Let us denote \( A_3 \) for the third term on the left-hand side of the above equality for simplicity. Recalling the definition of \( S'_\sigma \), we have

\[
|A_3| \leq k \int \int \int \left\{ (u_n(x, t), u_n(y, t)) \in R_\sigma \right\} |U_n(x, y, t)|^{p-1} \, dvdt.
\]

Observe that

\[
\int_0^T \int_{D_0} |V(x, y, t)|^{p-2} V(x, y, t)(\Psi(x, t) - \Psi(y, t)) \, dvdt
\]

\[
- \int_0^T \int_{D_0} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\Psi(x, t) - \Psi(y, t))
\]

\[
\cdot \frac{S'_\sigma(u_n)(x, t) + S'_\sigma(u_n)(y, t)}{2} \, dvdt
\]

\[
= \int_0^T \int_{D_0} \left( |V(x, y, t)|^{p-2} V(x, y, t) - |U_n(x, y, t)|^{p-2} U_n(x, y, t) \right)
\]

\[
\cdot (\Psi(x, t) - \Psi(y, t)) \, dvdt
\]

\[
+ \int_0^T \int_{D_0} |U_n(x, y, t)|^{p-2} U_n(x, y, t)(\Psi(x, t) - \Psi(y, t))
\]

\[
\cdot \left( 1 - \frac{S'_\sigma(u_n)(x, t) + S'_\sigma(u_n)(y, t)}{2} \right) \, dvdt.
\]

Using the similar arguments as in Theorem 1.1 and the Lebesgue dominated convergence theorem, letting \( n \to +\infty \), we obtain

\[
\int_\Omega \Theta_k(v - S_\sigma(u))(T) \, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_0)) \, dx
\]

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\[ + \frac{1}{2} \int_0^T \int_{D_0} \left( |V(x, y, t)|^{p-2} V(x, y, t) - |U(x, y, t)|^{p-2} U(x, y, t) \right) \cdot \left[ T_k(v - S_\sigma(u))(x, t) - T_k(v - S_\sigma(u))(y, t) \right] \, dvdt \]
\[
\leq \int_0^T \int_\Omega f(1 - S'_\sigma(u)) T_k(v - S_\sigma(u)) \, dxdt \\
+ k \int \int \int \left\{ (u(x, t), u(y, t)) \in R_\sigma \right\} |U(x, y, t)|^{p-1} \, dvdt \\
+ \frac{1}{2} \int_0^T \int_{D_0} |U(x, y, t)|^{p-2} U(x, y, t) \left[ T_k(v - S_\sigma(u))(x, t) - T_k(v - S_\sigma(u))(y, t) \right] \\
\quad \cdot \left( \frac{S'_\sigma(u)(x, t) + S'_\sigma(u)(y, t)}{2} - 1 \right) \, dvdt. \] (3.14)

Now we let \( \sigma \to +\infty \). Since
\[ |\Theta_k(v - S_\sigma(u))(T)| \leq k(|v(T)| + |u(T)|), \quad |\Theta_k(u_0 - S_\sigma(u_0))| \leq k|u_0|, \]
by the Lebesgue dominated convergence theorem, we have
\[ \int_\Omega \Theta_k(u_0 - S_\sigma(u_0)) \, dx \to 0, \quad \int_\Omega \Theta_k(v - S_\sigma(u))(T) \, dx \to \int_\Omega \Theta_k(v - u)(T) \, dx. \]

According to the fact that
\[ \lim_{\sigma \to +\infty} \int \int \int \left\{ (u(x, t), u(y, t)) \in R_\sigma \right\} |U(x, y, t)|^{p-1} \, dvdt = 0 \]
and Fatou’s lemma, we deduce from (3.13) that
\[ \int_\Omega \Theta_k(v - u)(T) \, dx \\
+ \frac{1}{2} \int_0^T \int \left\{ |u| \leq \frac{1}{2}, |v| \leq \frac{1}{2} \right\} \left( |V(x, y, t)|^{p-2} V(x, y, t) - |U(x, y, t)|^{p-2} U(x, y, t) \right) \cdot \left[ V(x, y, t) - U(x, y, t) \right] \, dvdt \leq 0. \]

Using the positivity of \( \Theta_k \), we conclude that \( u = v \) a.e. in \( \Omega_T \). Therefore we obtain the uniqueness of entropy solutions. This completes the proof of Theorem 1.2. \( \Box \)
Proof of Theorem 1.3. First, we suppose that \( u_0, v_0 \in L^2(\Omega) \) and \( f, g \in L^{p'}(0,T;X_0^{s,p}(\Omega)^*) \). Then by an approximation argument, we can obtain two weak solutions \( u \) and \( v \) for problems (1.1) and
\[
\begin{cases}
v_t + (-\Delta)^s_p v = f & \text{in } \Omega_T, \\
v = 0 & \text{in } C\Omega \times (0,T), \\
v(x,0) = v_0(x) & \text{in } \Omega.
\end{cases}
\tag{3.15}
\]

Making use of the approximation argument, we choose \((u-\v)^+\chi_{(0,t)}\) as a test function and subtract the resulting equalities to get
\[
\int_0^t \int_\Omega (u-v)_t (u-v)^+ \, dx \, dt + \int_0^t \langle (-\Delta)^s_p u - (-\Delta)^s_p v, (u-v)^+ \rangle \, d\tau = \int_0^t \int_\Omega (f-g)(u-v)^+ \, dx \, dt \leq 0.
\]

Moreover, from the nonnegativity of the second term in the equality above, we have
\[
\frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt}[(u-v)^+]^2 \, dx \, dt = \frac{1}{2} \int_\Omega [(u-v)^+]^2(t) \, dx - \frac{1}{2} \int_\Omega [(u_0-v_0)^+]^2 \, dx \leq 0.
\]

Recalling \( u_0 \leq v_0 \), we conclude that
\[
(u-v)^+ = 0 \quad \text{a.e. in } \Omega_T.
\]

Thus we obtain \( u \leq v \) a.e. in \( \Omega_T \).

Now we consider \( u \) and \( v \) as the entropy solution (renormalized solution) of problems (1.1) and (3.15) with \( L^1 \) data. Find four sequences of functions \( \{f_n\}, \{g_n\} \subset C_0^\infty(\Omega_T) \) and \( \{u_{0n}\}, \{v_{0n}\} \subset C_0^\infty(\Omega) \) strongly converging respectively to \( f, g \) in \( L^1(\Omega_T) \) and to \( u_0, v_0 \) in \( L^1(\Omega) \) such that
\[
\begin{align*}
f_n & \leq g_n, \\
\|f_n\|_{L^1(\Omega_T)} & \leq \|f\|_{L^1(\Omega_T)}, \\
\|g_n\|_{L^1(\Omega_T)} & \leq \|g\|_{L^1(\Omega_T)}, \\
\|u_{0n}\|_{L^1(\Omega)} & \leq \|u_0\|_{L^1(\Omega)}, \\
\|v_{0n}\|_{L^1(\Omega)} & \leq \|v_0\|_{L^1(\Omega)}.
\end{align*}
\]

Thus we use Theorem 1.1 (Theorem 1.2) to construct two approximation sequences \( \{u_n\} \) and \( \{v_n\} \) of entropy solutions (renormalized solutions) \( u \) and
v, and apply the comparison result above to obtain \( u_n \leq v_n \) a.e. in \( \Omega_T \). Moreover, by the uniqueness of entropy solutions (renormalized solutions), we know \( u_n \to u \) and \( v_n \to v \) a.e. in \( \Omega_T \). Therefore, we conclude that \( u \leq v \) a.e. in \( \Omega_T \). This completes the proof of Theorem 1.3. \( \square \)

4. Extensions

In order to fix the ideas and to avoid unessential technicalities, we limited ourselves to the equations of principal type as the one considered in (1.1). Indeed, inspired by [10], the existence and uniqueness result of non-negative renormalized solutions obtained in Theorem 1.1 still holds for the following more general nonlinear parabolic equations

\[
\begin{cases}
\frac{\partial b(u)}{\partial t} - L_p u = f & \text{in } \Omega_T, \\
u = 0 & \text{in } \mathcal{C} \Omega \times (0, T), \\
b(u)(x, 0) = b(u_0)(x) & \text{in } \Omega,
\end{cases}
\]

where \( u_0 \) is a nonnegative measurable function such that \( b(u_0) \in L^1(\Omega) \), \( 0 \leq f \in L^1(\Omega_T) \), \( -L_p \) is a non-local operator defined by

\[
-L_p u(x, t) := \text{P.V.} \int_{\mathbb{R}^N} |u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t))K(x, y) \, dy,
\]

where \( (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ \), and \( b : \mathbb{R} \to \mathbb{R} \) is a strictly increasing \( C^1 \)-function satisfying that

\[
0 < b_0 \leq b'(s) \leq b_1, \quad b(0) = 0.
\]

Finally, the kernel \( K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is assumed to be measurable, and satisfies the following ellipticity/coercivity properties:

\[
\frac{1}{\Lambda |x - y|^{N+sp}} \leq K(x, y) \leq \frac{\Lambda}{|x - y|^{N+sp}}, \quad \forall x, y \in \mathbb{R}^N, x \neq y, \Lambda \geq 1,
\]

where \( 0 < s < 1 < p < N \) such that \( ps < N \).

The definition of renormalized solutions for problem (4.1) is as follows.

**Definition 4.1.** A function \( u \) defined on \( \mathbb{R}^N \times (0, T) \) is a renormalized solution to problem (4.1) if \( b(u) \in C([0, T]; L^1(\Omega)) \), \( T_k(b(u)) \in L^p(0, T; X_0^{s,p}(\Omega)) \) for any \( k \geq 0 \), and the following conditions are satisfied:
\[
\lim_{h \to \infty} \int \int \int \{ (x, y, t) : (b(u)(x, t), b(u)(y, t)) \in R_h \} |U(x, y, t)|^{p-1} \, d\nu \, dt = 0,
\]

where
\[
R_h = \left\{ (u, v) \in \mathbb{R}^2 : h+1 \leq \max\{|u|, |v|\} \text{ and } (\min\{|u|, |v|\} \leq h \text{ or } uv < 0) \right\}.
\]

(ii) For every function \( \varphi \in C^1(\bar{\Omega} \times (0, T)) \) with \( \varphi = 0 \) in \( C\Omega \times (0, T) \) and \( \varphi(\cdot, T) = 0 \) in \( \Omega \), and \( S \in W^{1,\infty}(\mathbb{R}) \) which is piecewise \( C^1 \) satisfying that \( S' \) has a compact support,

\[
- \int_{\Omega} S(b(u_0))\varphi(x, 0) \, dx - \int_0^T \int_{\Omega} S(b(u)) \frac{\partial \varphi}{\partial t} \, dx \, dt
+ \frac{1}{2} \int_0^T \int_{D\Omega} \tilde{U}(x, y, t)|^{p-2}\tilde{U}(x, y, t)[(S'(b(u))\varphi)(x, t) - (S'(b(u))\varphi)(y, t)] \, d\nu \, dt
= \int_0^T \int_{\Omega} f S'(b(u))\varphi \, dx \, dt
\]
holds, where
\[
\tilde{U}(x, y, t) = b(u)(x, t) - b(u)(y, t).
\]

To the best of our knowledge, it is an open problem to show the well-posedness of entropy solutions and the equivalence between renormalized and entropy solutions to the general problem (4.1).

**Acknowledgements**

The authors wish to thank Dr. Xia Zhang for careful reading an early version of this paper, pointing out a mistake in the proof of Theorem 1.1 and helping them correct the mistake.

K. Teng was supported by the NSFC (No. 11501403) and the Shanxi Province Science Foundation for Youths (No. 2013021001-3). C. Zhang was supported by the NSFC (No. 11671111) and Heilongjiang Province Postdoctoral Startup Foundation (LBH-Q16082). S. Zhou was supported by the NSFC (No. 11571020).
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