Hyperbolicity of General Relativity in Bondi-like gauges

Thanasis Giannakopoulos, David Hilditch, and Miguel Zilhão
Centro de Astrofísica e Gravitação – CENTRA, Departamento de Física, Instituto Superior Técnico – IST, Universidade de Lisboa – UL, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

Bondi-like (single-null) characteristic formulations of general relativity are used for numerical work in both asymptotically flat and anti-de Sitter spacetimes. Well-posedness of the resulting systems of partial differential equations, however, remains an open question. The answer to this question affects accuracy, and potentially the reliability of conclusions drawn from numerical studies based on such formulations. A numerical approximation can converge to the continuum limit only for well-posed systems; for the initial value problem in the $L^2$ norm this is characterized by strong hyperbolicity. We find that, due to a shared pathologial structure, the systems arising from the aforementioned formulations are however only weakly hyperbolic. We present numerical tests for toy models that demonstrate the consequence of this shortcoming in practice for the characteristic initial boundary value problem. Working with alternative norms in which our model problems may be well-posed we show that convergence may be recovered. Finally we examine well-posedness of a model for Cauchy-Characteristic-Matching in which model symmetric and weakly hyperbolic systems communicate through an interface, with the latter playing the role of GR in Bondi gauge on characteristic slices. We find that, due to the incompatibility of the norms associated with the two systems, the composite problem does not naturally admit energy estimates.

I. INTRODUCTION

Characteristic formulations of General Relativity (GR) have advantages over more standard spacelike foliations in a number of situations. For instance, in the asymptotically flat setting, the Bondi-Sachs formalism [1, 2], crucial to the modern understanding of gravitational waves, underpins codes that aim to produce waveforms of high accuracy. This approach exploits the fact that null hypersurfaces reach future null infinity and hence allows the avoidance of systematic errors from extrapolation techniques. The general setup in these approaches is to construct a standard Cauchy problem in a finite region of the spacetime, where the main action, such as the collision of two black holes, takes place. The data on the worldtube of this finite region serve as boundary data for the characteristic initial boundary value problem (CIBVP). Solving this CIBVP one can compute quantities such as the gravitational wave news function at future null infinity. This method is often called Cauchy-characteristic extraction (CCE) [3, 4]. If the Cauchy and the CIBVP are solved simultaneously and one attempts to match the worldtube data from both the Cauchy problem and the CIBVP, then the method is called Cauchy-characteristic matching (CCM), see [5, 6] for a thorough review. In Fig. 1 an illustration of the geometric setup is given. Concerning asymptotically anti-de Sitter (AdS) spacetimes, characteristic formulations of GR are widely used in the field of numerical holography, which provides insights into the behavior of strongly coupled matter [11, 12]. We refer to the aforementioned characteristic formulations as Bondi-like or single-null.

A practical advantage of Bondi-like gauges is that the field equations can then be written as a set of nested differential equations which can be efficiently solved. For the resulting CIBVP one provides data on a timelike boundary and initial data on either an outgoing or ingoing null hypersurface depending on the physical setup. There are many examples of numerical codes making successful use of this formalism. Since these codes have successfully passed a multitude of convergence tests, and in various physical contexts, one might say that there is numerical evidence that the PDE problem solved is well-posed. To the best of our knowledge however a proof of this result is missing. By well-posedness we mean the usual notion that the problem admits unique solutions to the continuous one only for well-posed PDE problems. The answer to this question is missing. By well-posedness we mean the usual notion that the problem admits unique solutions to the continuous one only for well-posed PDE problems. The PDE systems that interest us here are of the hyperbolic class. A necessary condition for well-posedness of these systems in $L^2$, or in fact suitable Sobolev norms, is that they are strongly hyperbolic [13, 14]. Specifically, we consider PDEs in the generic form

$$
\mathcal{A}^\mu(u, x^\mu) \partial_\mu u + \mathcal{A}^\mu(u, x^\mu) \partial_\mu u + \mathcal{S}(u, x^\mu) = 0, \quad (1)
$$

FIG. 1. The CIBVP for the wave-zone of an asymptotically flat spacetime. Boundary data are given on the timelike inner boundary $T$, the worldtube $r = r_0$, and initial data on the null hypersurface $N_0$ of constant retarded time $u_0$. 
where \( \mathbf{u} = (u_1, u_2, \ldots, u_q)^T \), is the state vector of the system and
\[
\mathbf{A}^p = \begin{pmatrix} a^p_{11} & \cdots & a^p_{1q} \\ \vdots & \ddots & \vdots \\ a^p_{q1} & \cdots & a^p_{qq} \end{pmatrix}
\]
denotes the principal part matrices, with \( \det(\mathbf{A}^p) \neq 0 \).

To classify locally the character of the PDE we linearize about a background solution and then work pointwise in the frozen coefficient approximation, henceforth suppressing the explicit dependencies of the principal part matrices and source vector, and requesting the following definitions everywhere. We can construct the principal symbol
\[
\mathbf{P}^s = (\mathbf{A}^s)^{-1} \mathbf{A}^p s_1,
\]
where \( s_i \) is an arbitrary unit spatial vector. If \( \mathbf{P}^s \) has real eigenvalues for all \( s_i \), then the PDE system is called weakly hyperbolic (WH), whereas if in addition \( \mathbf{P}^s \) is diagonalizable for all \( s_i \), and there exists a constant \( K \) independent of \( s_i \) such that
\[
|\mathbf{T}_s| + |\mathbf{T}_s^{-1}| \leq K,
\]
with \( \mathbf{T}_s \) the similarity matrix that diagonalizes \( \mathbf{P}^s \), it is called strongly hyperbolic (SH).

Presently we analyze the character of the PDE systems that arise in two specific Bondi-like formulations of GR. The original systems involve second order derivatives, so we perform reductions to first order to conveniently build the principal parts. We find that, due to a degeneracy in the angular/transverse principal parts, these formulations are only WH. Consequently, they give rise to PDE problems that are ill-posed in \( L^2 \) even in the linear, frozen coefficient approximation, which prohibits well-posedness of the full system in associated Sobolev norms. We argue furthermore that this result holds true for every possible first order reduction.

Subsequently we perform careful numerical experiments that demonstrate the consequence of this shortcoming in practice. We work with two toy models, one of which is SH and the other only WH. We perform robust-stability-like [15,16] tests, suitably modified for the characteristic setting, and find that convergence in a discrete approximation to \( L^2 \) is prohibited in the WH model. Convergence with the latter model can be achieved by using a discrete approximation to a modified norm that involves a subset of derivatives of the state vector fields and adjusting the initial data for the test.

The structure of the paper is as follows. In Sec. [I] we give an overview of popular Bondi-like formulations of GR in both the asymptotically flat and AdS contexts, and present our hyperbolicity analysis of each. Afterwards, in Sec. [II] we present our toy models, then in Sec. [IV] we present numerical experiments demonstrating the effect of our analytic results in practice. Finally we conclude in Sec. [V] Geometric units are used throughout.

## II. CHARACTERISTIC FORMULATIONS

In this section we present two characteristic formulations of GR in Bondi-like gauges that are widely used in numerical work. The first, the Bondi-Sachs formulation proper, is popular in the asymptotically flat setting, whereas the second, known as the affine-null system, is used most often in numerical holography. We demonstrate that each is only weakly hyperbolic.

### A. Bondi-Sachs Gauge

In Bondi-Sachs gauge [1,2] a generic 4-dimensional axially symmetric metric can be written as
\[
ds^2 = \left( \frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right) du^2 + 2e^{2\beta} du dr
+ 2Ur^2 e^{\gamma} du d\theta - r^2 \left( e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2 \right).
\]

Here \( u \) is a null coordinate, called retarded time, \( r \) is the areal radius, and \( \theta, \phi \) give coordinates on the two-sphere in the standard way. All metric functions are functions of \((u, r, \theta)\). To make contact with [9] we adopt the signature convention \((+, -, -, -)\). In this formulation Einstein’s equations exhibit a nested structure. For axially symmetric spacetimes the PDE system consists of three equations intrinsic to the hypersurfaces of constant time,

\[
\beta_r = \frac{1}{2} \left( \gamma_r \right)^2,
\]
\[
\left[ r^4 e^{2(\gamma - \beta)} U \right]_r = 2r^2 \left[ r^2 \left( \frac{\beta_r}{r^2} \right)_r + \left( \frac{\sin^2 \theta \gamma}{\sin^2 \theta} \right)_r + 2\gamma_r \gamma_r \right],
\]
\[
V_r = -\frac{1}{4} e^{2(\gamma - \beta)} (U_r)^2 + \frac{\left( r^4 \sin \theta U \right)_r}{2r^2 \sin \theta} + e^{2(\beta - \gamma)} \left[ 1 - \left( \frac{\sin \theta \beta}{\sin \theta} \right)_r + \gamma_r + 3 \cot \theta \gamma_r \right.
- \left. \left( \beta_r \right)^2 - 2\gamma_r \left( \beta_r - \beta_r \right) \right],
\]

and one equation that involves extrinsic derivatives,
\[
4r (r\gamma)_{ur} = \left\{ 2r \gamma_r V - r^2 \left[ 2\gamma_r U + \sin \theta \left( \frac{U}{\sin \theta} \right)_r \right] \right\}_r
- 2r^2 \left( \gamma_r U \sin \theta \right)_r + \frac{1}{2} r^4 e^{2(\gamma - \beta)} (U_r)^2 + 2e^{2(\beta - \gamma)} \left( \beta_r \right)^2 + \sin \theta \left( \frac{\beta_r}{\sin \theta} \right)_r.
\]
1. First order reduction & Linearization

In [17] and [18] the authors studied existence and uniqueness of the CIBVP for the formulation given in the previous subsection. They considered the linearized and quasilinear systems, but did not study continuous dependence on given data, which will be our main focus. To treat the system in the original higher-order derivative form, we could follow [19, 20]. But for convenience in building the principal parts we instead perform an explicit first order reduction. Since this PDE is built as a reduction, there is the subtlety of the associated constraints and the specific choice of reduction, which we discuss in detail later. The minimal set of reduction variables are given by

\[ U_r = \partial_r U, \gamma_r = \partial_r \gamma, \gamma_\theta = \partial_\theta \gamma, \beta_\theta = \partial_\theta \beta. \]

We linearize the resulting equations about a fixed background. In [21] one can find the complete analysis for both Minkowski and arbitrary backgrounds. The resulting level of hyperbolicity of the system is the same regardless, and so we present the former for brevity. After this procedure the system reads

\[
\begin{align*}
\partial_r \beta &= 0, \\
\partial_r U_r - \frac{2}{r^2} \partial_r \beta_\theta + \frac{2}{r^2} \partial_r \gamma_\theta + S_2 &= 0, \\
\partial_r V + \partial_\theta \beta_\theta - \partial_\theta \gamma_\theta - 2r \partial_r U - \frac{r^2}{2} \partial_\theta U_r + S_3 &= 0, \\
4r^2 \partial_r \gamma_r + 4r \partial_r \gamma - 2r^2 \partial_r \gamma_r + 2r \partial_\theta U + r^2 \partial_\theta U_r - 2\partial_\theta \beta_\theta + S_4 &= 0, \\
\partial_r U + S_5 &= 0, \\
\partial_r \gamma + S_6 &= 0, \\
\partial_r \gamma_\theta - \partial_\theta \gamma_r &= 0, \\
\partial_r \beta_\theta &= 0,
\end{align*}
\]

where \( S_i \) denotes the various source terms and we work in the frozen coefficient approximation, so that \( r \) and so forth must be treated as constants. The variables can be collected in the state vector

\[ u = (\beta, \gamma, U, V, \gamma_r, U_r, \gamma_\theta, \gamma_\theta) \]

and the system can be written in the form \([1]\) with

\[ \mathcal{A} u \partial_r u + \mathcal{A}^I \partial_r \beta_\theta u + \mathcal{A}^0 \partial_\theta u + \mathbf{S} = 0. \]

The principal part matrix \( \mathcal{A}^u \) associated with retarded time \( u \) is not invertible (see [21] for the full calculation). In order to use the standard definitions given in the introduction we need a principal part associated to time derivatives that is invertible. We achieve this by performing a coordinate transformation to a frame that involves one timelike and three spacelike directions.

2. Coordinate transformation

We wish to bring the system \([7]\) to the form \([1]\), which has a trivial time principal part matrix

\[ \partial_t u + \mathcal{A}^\rho \partial_\rho u + \mathbf{S} = 0, \]

where \( \partial_\rho \) denotes spatial derivatives, \( u \) denotes the state vector. We therefore perform the following concrete coordinate transformation

\[ u = t - \rho, \quad r = \rho, \]

with the angular coordinates unchanged, which yields the following relation between the old and new basis vectors,

\[ \partial_u = \partial_t, \quad \partial_r = \partial_t + \partial_\rho, \]

with the remaining vectors unaltered. A schematic of the auxiliary setup is given in Fig. \( 2 \). Applying the transformation yields

\[ \mathcal{A}^I \partial_t u + \mathcal{A}^t \partial_\rho u + \mathcal{A}^0 \partial_\theta u + \mathbf{S} = 0, \]

with \( \mathcal{A}^I = \mathcal{A}^\rho + \mathcal{A}^t \) invertible. After multiplying on the left with the inverse of \( \mathcal{A}^t \) we bring the system to the desired form,

\[ \partial_t u + \mathcal{B}^\rho \partial_\rho u + \mathcal{B}^\theta \partial_\theta u + \mathbf{S} = 0, \]

where \( \mathcal{B}^\rho = (\mathcal{A}^I)^{-1} \mathcal{A}^\rho \) and \( \mathcal{B}^\theta = \rho (\mathcal{A}^I)^{-1} \mathcal{A}^\theta \) with \( \partial_\theta \equiv 1/\rho \partial_\rho \), and \( \mathbf{S} \) was redefined in the obvious manner. The solution space in this frame is equivalent to that of the original one, so in this sense the character of the PDE is invariant. For our system, the principal part matrix \( \mathcal{B}^\rho \) is diagonalizable with real eigenvalues. Although \( \mathcal{B}^\theta \) has real eigenvalues, it does not have a complete set of eigenvectors, and hence is not diagonalizable. Therefore the system resulting from the specific first order reduction we made is only WH. In [22] a subsystem of a similar first-order reduction was shown to be symmetric hyperbolic. Here, however, we are concerned with the best estimates that can be made for the full system. In
Sec. III this is written up explicitly for our homogeneous WH model.

So far we have not ruled out the existence of an alternative first order reduction that is SH however. To examine this possibility we have to understand if any potential addition of reduction constraints can make the system SH. The reduction constraints are

$$\partial_\theta \beta - \beta_\theta = 0, \quad \partial_\theta \gamma - \gamma_\theta = 0,$$  \hspace{1cm} (10)

The definitions of the variables $\gamma_r$ and $U_r$ are solved explicitly as time evolution equations within the system $[9]$ and therefore do not have an associated constraint. This subtlety, along with an examination of the form of the degeneracy follows in the next section.

3. Generalized characteristic variables

To understand the nature of the degeneracy of $B^\theta$ physically it is useful to consider the generalized eigenvalue problem,

$$l_\lambda \left( B^\theta - \lambda I \right)^m = 0,$$

with $\lambda_i$ standing for the various eigenvalues, and $l_\lambda$, representing either a true eigenvector when $m = 1$ or else a generalized eigenvector when $m > 1$. The eigenvalues of $B^\theta$ are $\lambda = \pm 1$, each with algebraic multiplicity one and $\lambda = 0$ with algebraic multiplicity six. The geometric multiplicity of each of $\lambda = \pm 1/\rho$ is also one, but $\lambda = 0$ has geometric multiplicity five. In other words one associated eigenvector is missing and we obtain one nontrivial generalized eigenvector with $m = 2$ for $\lambda = 0$. Defining the invertible matrix $T^\theta_\theta$ with the vectors $l_\lambda$, as rows, we obtain the Jordan normal form of the principal symbol in the $\theta$ direction by the similarity transformation

$$J^\theta \equiv T^\theta_\theta^{-1} B^\theta T^\theta_\theta.$$  

The same matrix can be used to construct the generalized characteristic variables of the system in the $\theta$ direction, namely the components of $v \equiv T^\theta_\theta^{-1} u$. These are of course nothing more than the left generalized eigenvectors contracted with the state vector. Working as before in the frozen coefficient approximation, focusing on the $t, \rho$ parts of $[9]$, and multiplying on the left with $T^{-1}_\theta$ we get

$$\partial_\theta v + J^\theta \partial_\theta v \simeq 0,$$  \hspace{1cm} (11)

with $\simeq$ denoting here equality up to non-principal terms and spatial derivatives transverse to $\partial_\theta$. The generalized characteristic variables with speed (eigenvalue) zero are

$$\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta, \quad \beta_\theta, \quad V,$$

$$\rho \left( -2\rho U + \frac{\rho^2}{2} U_r + \beta_\theta - \gamma_\theta \right), \quad \gamma, \quad \beta,$$

of which the third and fourth are associated with the non-trivial $2 \times 2$ Jordan block within $J^\theta$. Likewise we have

$$-\frac{\rho}{2} U + \frac{\rho^2}{4} U_r - \frac{\rho^2}{4} U_r + \frac{1}{2} \beta_\theta,$$

$$-\frac{\rho}{2} U - \frac{\rho^2}{4} U_r - \frac{\rho^2}{4} U_r + \frac{1}{2} \beta_\theta,$$

with speeds $\pm 1$ respectively. The structure of $J^\theta$ thus yields

$$-\partial_t \left( 2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0,$$

$$\partial_\rho V - \rho \partial_\theta \left( 2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0.$$  \hspace{1cm} (12)

Strongly hyperbolic systems admit a complete set of characteristic variables in each direction. In other words, if our system were strongly hyperbolic then up to non-principal and transverse derivative terms each component of $v$ would satisfy an advection equation. Presently the best we can achieve for $V$ however is $[12]$. Physically we may therefore understand weak hyperbolicity as the failure of $V$, a generalized characteristic variable, to satisfy such an advection equation. As mentioned in the previous section, we could try and cure the equations by using a different first order reduction. Observe that the choice of different reductions corresponds to the freedom to add (derivatives of) the reduction constraints to $[12]$ without introducing second derivatives. As $V$ appears at most once differentiated in the original equations there is no associated constraint, so we must hope to eradicate the $\partial_\theta$ term from $[12]$ using $[10]$ without introducing second derivatives. Even if the variable $U_\theta = \partial_\theta U$ were introduced in the reduction however, the $\partial_\theta \beta_\theta$ and $\partial_\theta \gamma_\theta$ terms would obviously persist. Thus one non-trivial generalized characteristic variable always survives and prevents the existence of a complete set of characteristic variables. Hence within the coordinate basis built from $(t, \rho, \theta)$, the field equations are at best only weakly hyperbolic regardless of the specific reduction.

B. Affine-null gauge

Although sometimes used in the asymptotically flat setting, $[23, 24]$, the affine-null gauge is particularly popular for evolutions in asymptotically AdS spacetimes $[25]$. For concreteness we will treat the specific system that occurs in the case of asymptotically AdS$_5$ spacetimes with planar symmetry, but we expect similar results in other contexts with analogous gauges. The metric is written as

$$ds^2 = -Adt^2 + \Sigma^2 \left[ e^B dx_1^2 + e^{-2B} dz^2 \right] + 2dR \left( dv + 2Fdvdz \right).$$  \hspace{1cm} (13)

Here $v$ denotes a null coordinate, called advanced time, $R$ is called the holographic coordinate, and increases from
the bulk of the spacetime towards the boundary. All metric components are functions of \((v, R, z)\). We also denote by \(dx^\perp\) the flat metric in the plane spanned by \(x^\perp\), the two coordinates associated with the symmetry. Using the convenient definitions
\[
d_x \equiv \partial_x - F \partial_R, \\
d_+ \equiv \partial_+ + \frac{2}{r} \partial_R,
\]
the field equations can be succinctly stated, and are
\[
\partial^2_x \Sigma = -\frac{1}{2} (\partial_R B)^2 \Sigma, \\
\Sigma^2 \partial^2_R F = \Sigma (6 d_x \Sigma \partial_R B + 4 \partial_R d_x \Sigma + 3 \partial_R F \partial_R \Sigma) \\
+ \Sigma^2 (3 d_+ B \partial_R B + 2 \partial_+ B d_x) - 4 d_+ \Sigma \partial_R \Sigma, \\
12 \Sigma^3 \partial_R d_+ \Sigma = -8 \Sigma^2 (-3 \Sigma^2 + 3 d_+ \Sigma \partial_R \Sigma) \\
+ e^{2B} \left\{ \Sigma^2 \left[ 4 d_x B \partial_R F - 4 d_x^2 B - 7 (d_+ B)^2 \\ \\
+ 2 \partial_R d_x F - (\partial_R B)^2 \right] \right\} + 4 (d_+ \Sigma)^2 \\
+ 2 \Sigma \left[ d_+ \Sigma (\partial_R F - 8 d_x B) - 4 d_x^2 \Sigma \right] \right\}, \\
6 \Sigma^4 \partial_R d_+ B = -9 \Sigma^3 (\partial_R \Sigma d_+ B + \partial_R B d_+ \Sigma) \\
+ e^{2B} \left\{ \Sigma^2 \left[ (d_x B)^2 - d_x B \partial_R F + d_x^2 B \\ \\
- 2 \partial_R d_x F - (\partial_R B)^2 \right] - 4 (d_+ \Sigma)^2 \right\} \\
+ \Sigma \left[ d_+ \Sigma (d_x B + 4 \partial_R F) + 2 d_x^2 \Sigma \right] \right\}, \\
6 \Sigma^4 \partial_R d_+ B = 72 \Sigma^2 d_+ \Sigma \partial_R \Sigma - 2 \Sigma^4 (9 \partial_R B d_+ B + 12) \\
+ 3 e^{2B} \left\{ \Sigma^2 \left[ 4 d_x^2 B + 7 (d_+ B)^2 - (\partial_R F)^2 \right] \\ \\
+ 8 \Sigma \left( 2 d_x B d_+ \Sigma + d_x^2 \Sigma \right) - 4 (d_+ \Sigma)^2 \right\},
\]
and finally
\[
\partial_t B = d_+ B - \frac{2}{r} \partial_R B.
\]
As in the previous section, there are also two additional equations that are not explicitly solved. The vector \(d_+\) points to the direction of the outgoing null rays and hence equations \([15]\) do involve derivatives extrinsic to the hypersurfaces of constant time. However, if one considers \(d_+B\) and \(d_+\Sigma\) as independent variables of the system, then equations \([15]\) are intrinsic to the ingoing null hypersurfaces and possess a nested structure just as in Bondi-gauge. Hence the only equation that involves derivatives extrinsic to the hypersurfaces of constant retarded time is \([16]\). To analyze the hyperbolicity of the resulting PDE system we follow exactly the same steps as in the previous setup.

1. First order reduction & Linearization

The definition \([14]\) was used earlier to write the field equations in a more compact form, but for the rest of the analysis we expand out the definition of \(d_+\). Before performing the first order reduction, we apply the coordinate transformation \(r = 1/R\), drawing the boundary to \(r = 0\). The metric components however still exhibit singular behavior there, so as elsewhere in the literature, we apply appropriate field redefinitions to obtain regular fields on the boundary, namely
\[
A(v, r, z) \rightarrow \frac{1}{r^2} + r^2 A(v, r, z), \\
B(v, r, z) \rightarrow r^4 B(v, r, z), \\
\Sigma(v, r, z) \rightarrow \frac{1}{r} + r^3 \Sigma(v, r, z), \\
F(v, r, z) \rightarrow r^2 F(v, r, z),
\]
and similarly for derivatives of the above fields. To simplify the presentation we linearize here about vacuum AdS. Our conclusions are however unaltered if we work about an arbitrary background. Full expressions in the general case can be found in \([21]\). We define reduction variables according to
\[
A_r = \partial_r A, B_r = \partial_r B, F_r = \partial_r F, \Sigma_r = \partial_r \Sigma, \\
A_z = \partial_z A, B_z = \partial_z B, F_z = \partial_z F, \Sigma_z = \partial_z \Sigma, \\
B_+ = \partial_+ B, \Sigma_+ = \partial_+ \Sigma.
\]
The complete first order system, is then
\[
r^4 \partial_r B = -S_1, \\
r^4 \partial_r B_r = \frac{r^4}{2} \partial_r B_r + r^3 \partial_r B_+ - S_2, \\
-6r \partial_r B_+ = 2r^2 \partial_r F_z + r^2 \partial_z B_z + 2r^2 \partial_z \Sigma_z - S_3, \\
\partial_r B_z = \partial_z B_r, \\
\partial_r \Sigma = -S_5, \\
r^7 \partial_r \Sigma_+ = -S_6, \\
12r \partial_r \Sigma_+ = 2r^2 \partial_r F_z + 4r^2 \partial_z B_z + 8r^2 \partial_z \Sigma_z - S_7, \\
\partial_r \Sigma_z = \partial_z \Sigma_r, \\
\partial_r F = -S_9, \\
r^4 \partial_r F_r = -4r^4 \partial_r \Sigma_z - 2r^4 \partial_r B_z - S_{10}, \\
\partial_r F_z = \partial_z F_r, \\
\partial_r A = -S_{12}, \\
6r^2 \partial_r A_r = 12r^2 \partial_r B_z + 24r^2 \partial_z \Sigma_z - S_{13}, \\
\partial_r A_z = \partial_z A_r,
\]
which can be written as
\[
\mathcal{A}^0 \partial_0 \mathbf{u} + \mathcal{A}^r \partial_r \mathbf{u} + \mathcal{A}^z \partial_z \mathbf{u} + \mathbf{S} = 0,
\]
with state vector
\[
\mathbf{u} = (A_r, B_+, \Sigma_+, \Sigma_r, F_r, B_z, \Sigma_z, B_z, A_z, F_z, A, F, B, \Sigma)^T .
\]
The principal part matrix associated with the retarded advanced time \(\mathcal{A}^r\) is again not invertible and hence we proceed with a transformation to an appropriate auxiliary frame.
2. Coordinate transformation

To obtain a suitable coordinate frame we transform from \((v, r, z)\) to \((t, \rho, z)\) with
\[
v = t - \rho, \quad r = \rho,
\]
and the remaining coordinates unaltered, which gives
\[
\partial_r = \partial_t, \quad \partial_r = \partial_t + \partial_\rho,
\]
with \(\partial_z\) unaffected. Applying the transformation yields
\[
\mathbf{A}' \partial_t \mathbf{u} + \mathbf{A}^\ast \partial_\rho \mathbf{u} + \mathbf{A}^\ast \partial_\rho \mathbf{u} + \mathbf{S} = 0,
\]
where now \(\mathbf{A}' = \mathbf{A}^\ast + \mathbf{A}^\ast\) is invertible. After multiplying from the left with the inverse of \(\mathbf{A}'\) we again bring the system to the form
\[
\partial_t \mathbf{u} + \mathbf{B}^\rho \partial_\rho \mathbf{u} + \mathbf{B}^\rho \partial_\rho \mathbf{u} + \mathbf{S} = 0, \quad (19)
\]
with \(\mathbf{B}^\rho = (\mathbf{A}')^{-1} \mathbf{A}'\) and \(\mathbf{B}^\rho = (\mathbf{A}')^{-1} \mathbf{A}'\). The principal part \(\mathbf{B}^\rho\) is diagonalizable with real eigenvalues 0 and \(\pm 1\). The principal part \(\mathbf{B}^\rho\) has the same real eigenvalues but it does not have a complete set of eigenvectors, so it is not diagonalizable. The system resulting from this specific first order reduction is thus only WH. Next, by again constructing generalized characteristic variables in the \(z\) direction we will examine whether or not an appropriate addition of the reduction constraints can render the reduction strongly hyperbolic. The reduction constraints are
\[
\partial_z A - A_z = 0, \quad \partial_z B - B_z = 0, \quad (20)
\]
\[
\partial_z \Sigma - \Sigma_z = 0, \quad \partial_z F - F_z = 0,
\]
\[
\partial_z \rho - \rho_z B_z = \frac{1}{2} \partial_z B_z - \partial_\rho B_z = 0, \quad \partial_z \Sigma - \rho_z B_z = 0,
\]
\[
\partial_z \Sigma - \rho_z \Sigma_z = \frac{1}{2} \partial_\rho \Sigma_z - \rho \Sigma_\rho - \rho \Sigma_\rho = 0.
\]

3. Generalized characteristic variables

The eigenvalues of \(\mathbf{B}^\rho\) are \(\lambda = \pm 1\) with algebraic multiplicity one and \(\lambda = 0\) with algebraic multiplicity twelve. There is one eigenvector for \(\lambda = 1\), one for \(\lambda = -1\) and nine for \(\lambda = 0\). Since the algebraic and geometric multiplicity of \(\lambda = 0\) differ by three, the Jordan normal form,
\[
\mathbf{J}^\rho \equiv \mathbf{T}_z^{-1} \mathbf{B}^\rho \mathbf{T}_z, \quad (21)
\]
must have some non-trivial block. Let us consider the \(t, z\) part of [19] and, as earlier in [11], use \(\mathbf{T}_z^{-1}\) to construct the generalized characteristic variables in the \(z\) direction,
\[
\mathbf{v} = \mathbf{T}_z^{-1} \mathbf{u}, \quad (22)
\]
with \(\simeq\) here denoting equality up to transverse derivatives and non-principal terms. The components of \(\mathbf{v}\) begin,
\[
- B_r + \frac{1}{3} B_3 z + \frac{2}{3} F_r - 2 \Sigma_r = \frac{2}{3} \Sigma_z, \quad (23)
\]
with speeds \(\mp 1\) respectively. Next we have those with vanishing speeds, which are most naturally presented in three blocks. The first of these consists of the set of true characteristic variables,
\[
B_+ - \frac{9}{7} B_r - \rho \Sigma_r, \quad \Sigma_+ - \frac{2}{3} A_+ - \frac{1}{4} B_+ + \frac{2}{3} \Sigma_+, \quad (24)
\]
and finally a coupled triplet of two generalized characteristic variables and one characteristic variable, respectively,
\[
\frac{1}{4} A_+ + \frac{3}{2} B_r + F_r + 3 \Sigma_r, \quad A, \quad F, \quad B, \quad \Sigma, \quad (25)
\]
where \(\nu_i\) referring to the field and \(\nu_{i+1}\) the next element of the pair or triplet. The question is whether or not there exists an appropriate addition of the reduction constraints \(\simeq 0\) such that equations of the form \(\simeq 0\) turn into equations of the form
\[
\partial_\nu + \lambda_\nu \partial_\nu \simeq 0, \quad (26)
\]
where we are allowing different first order reductions to adjust also characteristic speeds. This is a necessary condition for building an alternative reduction that is SH. This would mean that the generalized characteristic variable \(\nu_i\) that is originally coupled with \(\nu_{i+1}\) could be decoupled, and the respective generalized eigenvector replaced by a simple eigenvector. We examine this for the second two elements of the triplet \([24]\) and show by contradiction that this necessary condition can not be fulfilled. With our original, specific reduction we have
\[
\partial_t \left( \frac{2}{3} B_3 + \frac{1}{3} F_r + \frac{2}{3} \Sigma_z \right) \simeq 0, \quad (27)
\]
\[
\partial_t \left( - \frac{1}{4} A_+ + \frac{1}{2} B_+ + \Sigma_+ \right) + \partial_z \left( \frac{2}{3} B_3 + \frac{1}{3} F_r + \frac{2}{3} \Sigma_z \right) \simeq 0.
\]
Observe, first of all, that neither of these two equations, nor the two large terms grouped separately in the second, can be written as a linear combination (equality taken here in the sense of $\approx$) of the reduction constraints $\partial_v$. The choice of reduction lies in the freedom to add multiples of the six reduction constraints $\partial_v$ to the evolution equations. Suppose that some choice of addition of these constraints did result in a SH first order reduction. Starting with the first equation of (27), for our alternative reduction we have

$$\partial_t \left( \frac{2}{3} B_z + \frac{1}{3} F_r + \frac{4}{3} \Sigma_z \right) \approx \sum_\alpha c_\alpha C_\alpha ,$$

with the terms on the right-hand-side a linear combination of the reduction constraints $C_\alpha$. Since this alternative reduction is SH we have,

$$\sum_\alpha c_\alpha C_\alpha \approx \sum_\alpha a_\alpha^0 \partial_z v_\alpha^0 + \sum_\alpha a_\alpha^+ \partial_\Sigma v_\alpha^+ ,$$

with $v_\alpha^0$ denoting the set of 0-speed characteristic variables and $v_\alpha^\pm$ denoting the remaining characteristic variables. Using $\partial_t v_\alpha^\pm \approx \lambda_\alpha \partial_\Sigma v_\alpha^\pm$, we may therefore rewrite $\partial_t$ as

$$\partial_t \left( \frac{2}{3} B_z + \frac{1}{3} F_r + \frac{4}{3} \Sigma_z - \sum_\alpha a_\alpha^+ \lambda_\alpha^{-1} v_\alpha^+ \right) \approx \sum_\alpha a_\alpha^0 \partial_\Sigma v_\alpha^0 .$$

Now, by our observation directly after (27), the term inside the large bracket can not vanish identically. Therefore we must have $a_\alpha^0 = 0$ or we have found, on the left-hand-side, a non-trivial generalized characteristic variable, in contradiction to the assumption that our reduction is SH. Moving on to the second equation of (27), we can write the equivalent expression for the alternative first order reduction as

$$\partial_t \left( -\frac{1}{4} A_r + \frac{1}{2} B_r + \Sigma_r \right) + \partial_z \left( \frac{2}{3} B_z + \frac{1}{3} F_r + \frac{4}{3} \Sigma_z \right) \approx \sum_\alpha c_\alpha C_\alpha ,$$

again with the right-hand-side a linear combination of the reduction constraints. From here a simple calculation shows that

$$-\frac{1}{4} A_r + \frac{1}{2} B_r + \Sigma_r + \sum_\alpha a_\alpha^+ \lambda_\alpha^{-1} v_\alpha^+ ,$$

is nevertheless still a non-trivial generalized characteristic variable for a suitable choice of $a_\alpha^+$. By contradiction we have therefore shown that there is no first order reduction that gives a SH first order PDE system in the $(t, \rho, z)$ frame used here.

C. Frame independence

In the previous subsections we presented a hyperbolicity analysis of two widely used Bondi-like formulations of GR. We worked with a particular auxiliary frame with one timelike element and the remainder spacelike. The auxiliary basis was used to express the original PDEs, which were then shown to be only WH. In this subsection we argue that this result persists for other auxiliary frames. Our argument is based on the dual foliation (DF) approach of [26] and follows closely Sec. II.D of [27]. In this subsection, Latin letters $a...e$ are used as abstract indices, Greek letters run from 0 to $d=1$ for a 1+1-dimensional spacetime and a given basis and Latin indices $i, j, k$ denote only the spatial components of this basis. We also use $p$ as an abstract index for the spatial derivatives appearing on the right hand side of a first order PDE. The symbol $\partial_\alpha$ stands for the flat covariant derivative naturally defined by $x^\alpha$.

The idea of the DF approach is to express a region of spacetime in terms of two different frames, which we call uppercase and lowercase. Considering a $d+1$ split of the spacetime, let us denote as $n^a$ and $N^a$ the normal vectors on the hypersurfaces of constant time for the lower and uppercase frames, respectively. We call $v^a$ and $V^a$ the boost vectors for each frame, which are spatial with respect to the corresponding normal vector. The Lorentz factor is $W = (1 - v^a v_a)^{-1/2} = (1 - V^a V_a)^{-1/2}$ and we denote as $\gamma_{ab}$ and $(N^a)_b$ the lower and uppercase spatial metrics. The following useful relations hold

$$\delta^a_b = \gamma^a_b - n^a n_b = (N^a)_b - N^a N_b ,$$

$$n^a = W (N^a + V^a) , \quad N^a = W (n^a + v^a) .$$

Let us consider a first order PDE in the compact form

$$\mathcal{A}^a \delta^a_b \partial_\alpha u + \mathcal{S} = 0 ,$$

and $d+1$ split using the lower and uppercase frames, replacing $\delta^a_b$ by means of (29), giving

$$\mathcal{A}^a \partial_\alpha u \simeq \mathcal{A}^a \gamma^a_b \partial_\alpha u , \quad \mathcal{A}^N \partial_N u \simeq \mathcal{A}^b (N^a)_b \partial_\alpha u .$$

We obtain two evolution systems for the variables of $u$, with

$$\mathcal{A}^a n_a \equiv \mathcal{A}^n , \quad n^a \partial_\alpha n_a \equiv \partial_\alpha n ,$$

$$\mathcal{A}^N n_a \equiv \mathcal{A}^N , \quad N^a \partial_\alpha n_a \equiv \partial_\alpha N .$$

Without loss of generality we choose to identify the uppercase frame with the auxiliary frames used in subsections II.A and II.D. The definitions

$$\mathcal{A}^n \equiv A^n , \quad \mathcal{A}^a \gamma^a_b \equiv A^b ,$$

$$\mathcal{A}^N \equiv B^N , \quad \mathcal{A}^b (N^a)_b \equiv B^a ,$$

imply $B^b N_b = 0$, $A^b n_b = 0$ and lead to the following upper and lowercase first order PDE forms

$$\partial_N u = B^p \partial_p u - S , \quad A^n \partial_\alpha u = A^p \partial_p u - S ,$$

where $B^N = 1$ by assumption. The former is the same form as in equations [9] and [19]. In this form we found
the PDE systems only WH due the $2 \times 2$ Jordan blocks of the angular principal parts. This can be represented in a generalized eigenvalue problem of the form
\[ I_{\lambda_N}^N \left( P^S - 1\lambda_N \right)^M = 0, \tag{33} \]
where $S^a$ is a unit spatial vector, $P^S \equiv B^a S_a$ the principal symbol and $M$ is the rank of the generalized left eigenvector $I_N^\lambda$ with eigenvalue $\lambda_N$, with $M = 2$ for the generalized eigenvectors that correspond to the aforementioned Jordan blocks. We wish to examine if generalized eigenvalue problems of this form exist also in the lowercase frame. Hence we need to relate the two equations of (32), obtaining
\[
\begin{align*}
A^n &= W(1 + B^V), \\
A^p &= B^a(\gamma^p_a + WV_a v^p) - W(1 + B^V)v^p,
\end{align*}
\tag{34}
\]
and
\[
\begin{align*}
B^N &= 1 - W(A^n + A^v), \\
B^p &= A^a (N\gamma^p_a) - W A^n V v^p,
\end{align*}
\tag{35}
\]
where we write $B^a V_a \equiv B^v$. Let us examine $1 + B^v$. In [27] invertibility of this matrix was guaranteed by strong hyperbolicity. Here we want to analyze PDEs that are only WH and so may not assume that $B^v$ is diagonalizable. Hence, let us denote as
\[
J^{S v} = T_{S v}^{-1} B^{S v} T_{S v},
\]
the Jordan normal form of $B^{S v} = B^a (S_v)_a$, where $V^a = |V| S^a_v$ is the uppercase boost vector with norm $|V|$ pointing in the direction of $S^a_v$. One can write each block $j$ of the Jordan form $J$ with only the eigenvalue $\lambda_i$ on the diagonal as
\[
\begin{align*}
j &= \lambda_i 1 + N,
\end{align*}
\]
where $N$ is a nilpotent matrix of the size of $j$ with $N^2 = 0$. Consequently
\[
T_{S v}^{-1} (1 + B^v) T_{S v} = 1 + J^{S v} |V|,
\]
and for each block $j^{S v}$,
\[
\begin{align*}
1 + j^{S v} = \tilde{\lambda}_i^{S v} \left( 1 + |V| \frac{N^{S v}}{\tilde{\lambda}_i^{S v}} \right),
\end{align*}
\]
assuming that
\[
\begin{align*}
\tilde{\lambda}_i^{S v} = 1 + |V| \frac{N^{S v}}{\tilde{\lambda}_i^{S v}} \neq 0. \tag{36}
\end{align*}
\]
The inverse of this block is then
\[
\begin{align*}
\frac{1}{\tilde{\lambda}_i^{S v}} \left[ 1 + \sum_{j=1}^{q-1} \left( -\frac{|V|}{\tilde{\lambda}_i^{S v}} \right)^j (N^{S v})^j \right],
\end{align*}
\]
and hence $1 + B^v$ is invertible as long as condition (36) is satisfied for each $\lambda_i$. Note that in our normalization light-speed corresponds to $\lambda = 1$. Since $|V| < 1$, inequality (36) is always satisfied for physical propagation speeds, although could be violated when supraluminal gauge speeds are present. If one considers for instance the analysis of subsections II A and II B on top of Minkowski and vacuum AdS background respectively, then this condition is satisfied. We wish to find the equivalent of the upper case generalized eigenvalue problem (33) in the lower case frame. Thus, using the second equation of (35) and $S_a = s_a - WV^S_n a \equiv 27$ we express the principal symbol in the lowercase frame, namely
\[
\begin{align*}
P^S \equiv B^a S_a = A^n s_a - A^n W V^S.
\end{align*}
\]
Hence, the equivalent of (33) in the lowercase frame is
\[
\begin{align*}
I_{\lambda_N}^N \left[ (A^{s - \lambda_N W v}) - W (A^n + V^S) A^n \right]^M = 0. \tag{37}
\end{align*}
\]
Thus if in the uppercase frame the eigenproblem (33) with $M = 1$ fails to admit a complete set of left eigenvectors then so does the lowercase frame, and so both setups would be at best weakly hyperbolic. To see this we need only set $M = 1$ in (37) and note that the lowercase principal symbol in the $s_a + \lambda_N W v_a$ direction is proportional to
\[
\begin{align*}
(A^n)^{-1} (A^{s - \lambda_N W v}).
\end{align*}
\]
and so deficiency of the lower case principal symbol in this direction is equivalent to that of the upper case principal symbol stated before. Unfortunately the relationship between the upper and lowercase generalized left eigenvectors is more subtle. Returning to our specific systems and identifying the uppercase unit spatial vector $S^a$ with the unit spatial vectors in the $\partial_t$ and $\partial_z$ directions of subsection II, we conclude that weak hyperbolicity of those PDEs persists in other frames.

### III. TOY MODELS

In this section we introduce two toy models, one SH and one WH, which capture the core structure of the systems analyzed in the previous section. Our aim is to examine the consequence of the algebraic properties determined earlier on local well-posedness in the context of the CIBVP. The principal parts of the two models differ only in the angular direction $z$, with the WH model possessing a non-diagonalizable principal symbol.
A. The PDEs

The equations of motion for the WH model are,
\[
\begin{align*}
\partial_u \phi &= -S_\phi, \\
\partial_u \psi_t - \partial_z \phi &= -S_\psi, \\
\partial_u \psi - \left(1 - x^2\right)^{3/2} \partial_z \psi - \partial_z \psi &= -S_\phi,
\end{align*}
\]  
(38)

with \(x \in [0, 1], \ z \in [0, 2\pi]\) with periodic boundary conditions, \(u \geq u_0\) for some initial time \(u_0\) and \(c_x\) a constant. This PDE can be written in the form
\[
A^u \partial_u u + A^x \partial_u u + A^z \partial_z u + S = 0,
\]
(39)

where \(u = (\phi, \psi_t, \psi)^T\) is the state vector, and the principal matrices are given by
\[
A^u = \text{diag}(0, 0, 1)
\]
\[
A^x = \text{diag}(1, 1, \frac{1}{2c_x}(1 - x^2)^{3/2})
\]
and
\[
A^z = \begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

The source terms are denoted by \(S_\phi, S_\psi_t, \) and \(S_\psi\). The first two Eqs. of (38) are intrinsic to a hypersurface of constant \(u\), whereas the last is the “evolution equation” of the system. The angular principal part \(A^z\) is not diagonalizable since it has a \(2 \times 2\) Jordan block for the intrinsic equations, mimicking the core structure of the previously analyzed single-null PDEs. One may think of this model as a simplified analog of these systems with a compactified radial coordinate, similar to the way that the Bondi-Sachs formulation is used for characteristic extraction. This role can be played by the coordinate \(x\) with \(c_x\) a constant involved in the compactification. More specifically
\[
x = \sqrt{r - r_{\text{min}}} \sqrt{e^{c_\ast t} + (r - r_{\text{min}})^2},
\]
where \(r_{\text{min}}\) is the minimum physical radius that we consider and the factor \(c_x\) controls the density of points towards \(r \to \infty\), if we were to map the compactified grid \(x\) to the physical radius grid \(r\).

By removing the angular derivative from the second intrinsic equation \(38\) we obtain our SH toy model
\[
\begin{align*}
\partial_u \phi &= -S_\phi, \\
\partial_u \psi_t &= -S_\psi, \\
\partial_u \psi - \left(1 - x^2\right)^{3/2} \partial_z \psi - \partial_z \psi &= -S_\phi,
\end{align*}
\]  
(40)

which has the same principal part matrices \(A^u\) and \(A^z\) as before, but has diagonal \(A^x\). We employ this model for comparison between numerical results with SH and WH systems. The PDE problem for both systems \(38\) and \(40\) has as domain
\[
x \in [0, 1], \ z \in [0, 2\pi], \ u \in [u_0, u_f],
\]
for some initial and final times \(u_0\) and \(u_f\) respectively. We apply periodic boundary conditions in the \(z\) direction for simplicity. The initial and boundary data are
\[
\psi_t \equiv \psi(u_0, x, z),
\]
(41)
and
\[
\dot{\psi} \equiv \psi(u, 0, z), \quad \ddot{\psi} \equiv \psi(u, 0, z),
\]
(42)
respectively and are freely specifiable.

B. Algebraic determination of well-posedness

So far we have discussed the degree of hyperbolicity of GR in two gauges and constructed models that capture the basic structure we unearthed. As mentioned in the Introduction the reason we care about this algebraic characterization is that, in the linear constant coefficient approximation, it determines well-posedness of the initial value problem \([13, 29]\). In this subsection we present our well-posedness analysis, focusing on the WH toy model. The interested reader can find the complete analysis of both our models in \([21]\). In this analysis we work in the constant-coefficient approximation, following closely the philosophy and notation of \([29]\). We start with the IVP and adjust our results to the CIBVP at the end. Specifically, we wish to understand what inequalities, with what norms, can be used to bound solutions in terms of their given data, and how lower order perturbations affect such estimates.

Consider the Cauchy problem for the linear, constant coefficient system,
\[
\partial_t u = B^p \partial_u u + S \equiv B^p \partial_u u + Bu.
\]
(43)

To be well-posed in the \(L^2\)-norm we must have real constants \(K \geq 1\) and \(\alpha \in \mathbb{R}\) such that
\[
|e^{P(i\omega t)}| \leq Ke^{\alpha t},
\]
(44)

for all \(t \geq 0\) and all \(\omega \in \mathbb{R}^n\). Here
\[
P(i\omega) = i\omega_0 B^p + B
\]
(45)
is the constant-coefficient symbol of the PDE after Fourier transforming in space, with \(i\omega_0 B^p\) the principal symbol and \(Bu = -S\) the lower order term related to sources. Essentially, inequality \([14]\) states that the solution of the PDE has to be bounded at each time by an exponential that is independent of the Fourier mode \(\omega\). In this manner we obtain an estimate of the solution \(u\) at all times by the initial data
\[
||u(\cdot, t)||_{L^2} = ||e^{P(i\omega t)} \tilde{f}(\omega)||_{L^2} \leq Ke^{\alpha t} ||\tilde{f}||_{L^2} = Ke^{\alpha t} ||f||_{L^2}.
\]
In the terminology of [29], if a Cauchy problem instead satisfies
\[ |e^{P_i(\omega)t}| \leq K_1 e^{\alpha t} (1 + |\omega|^q), \]  
(46)
with \( q \) some natural number, it is called weakly well-posed. If, rather than insisting on \( L^2 \) we allow also some specific derivative, determined by the system, within the norm, we can nevertheless obtain the estimate
\[ ||u(\cdot , t)||_q \leq K_2 e^{\alpha t} ||f||_q, \]
for the solution \( u \). This would not be terrible, except that if the PDE is only weakly well-posed, then perturbations to the system by generic lower order terms will lead to frequency dependent exponential growth of the solution, and the resulting perturbed problem is ill-posed in any sense. The latter is not true for well-posed problems, which remain well-posed in the presence of lower order perturbations [29, 30].

To apply the above results directly the system needs to be written in a form where the time principal part is the derivative, determined by the system, within the specific domain and an appropriately chosen \( \rho \), with non-zero denominator for our \( \rho \) domain and an appropriately chosen \( c_e \). In this frame the principal parts are \( B^t = 1 \) and
\[ B^o = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & F \end{pmatrix}, \quad B^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G \end{pmatrix}. \]

This is the auxiliary Cauchy-type setup for the WH model, similarly to the PDEs in section [11] after the coordinate transformation. After applying a Fourier transformation, the principal symbol for the WH model is
\[ i\omega \beta B^p = i\omega \beta A^p + i\omega z A^z. \]

1. Homogeneous WH model

Focusing first on the homogeneous WH model where \( S_\phi = S_\psi = S = 0 \), we obtain
\[ e^{(i\omega \beta B^p)\omega t} = \begin{pmatrix} e^{-i|\omega|\beta t} & 0 & 0 \\ e^{-i|\omega|\beta t} & e^{-i|\omega|\beta t} & 0 \\ 0 & e^{-i|\omega|(F\beta + G\omega_z) t} \end{pmatrix}, \]
(47)
where we express the wavevector as
\[ \omega_p = |\omega|\hat{\omega}_p, \]
with \( |\omega| \) its magnitude so that \( \hat{\omega}_p^2 + \omega_z^2 = 1 \). The norm of (47) is (see chapter 2 of [29] for useful definitions)
\[ \left| e^{(i\omega_p B^o)\omega t} \right|^2 = 1 + \frac{|\omega|^2 \hat{\omega}_p^2}{2} + \left( 1 + \frac{|\omega|^2 \hat{\omega}_p^2}{2} \right)^2 - 1 \right)^{1/2}. \]
(48)
This norm behaves as \( |\omega|t \) for large \( |\omega| \) and so the homogeneous WH model obeys an inequality of the form (46), with \( q = 1 \). Hence, this PDE is only weakly well-posed, and so satisfies an estimate in some \( || \cdot ||_r \)-norm. This norm is specified for our system in Sec. [11B3]. If one would discard from the previous analysis the equation for \( \psi \) of the homogeneous WH model (38) since it is decoupled, the remaining subsystem would be symmetric hyperbolic and one might expect well-posedness of the full system in the \( L^2 \)-norm. However, as shown in Fig. 1, this expectation is not true.

2. Inhomogeneous WH model

For the homogeneous WH model we computed the norm of \( e^{(i\omega \beta B^p)\omega t} \) to estimate the behavior of solutions. However, we could also examine the form of the eigenvalues of the full symbol \( P(\omega) \) for large \( |\omega| \) to understand if the solutions exhibit exponential growth in \( \omega_p \) (see lemma 2.3.1 of [29]). If there is any eigenvalue \( \lambda \) of \( P(\omega) \) such that
\[ \Re[\lambda] \sim |\omega|^s > 0 \text{ with } s > 0, \]
for large \( |\omega| \), then solutions of the PDE may exhibit frequency dependent exponential growth, and the PDE problem is ill-posed in any sense. For the inhomogeneous WH model we consider the following possible lower order source terms
\[ B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
where \(-S = B u\). The choice \( B_1 \) is motivated by analogy with the linearized Bondi-Sachs system with \( \phi \sim \beta, \psi \sim V \) and \( \psi \sim \gamma \). In \( B_2 \) we include all possible source terms that do not break the nested structure of the intrinsic equations and finally in choice \( B_3 \), we introduce source terms that violate the nested structure, thus rendering the intrinsic system a coupled PDE. For both \( B_1 \) and \( B_2 \) the eigenvalues of \( P(\omega) \) are
\[ \lambda_1 = \lambda_2 = -i|\omega| \hat{\omega}_p, \quad \lambda_3 = i|\omega| (F\hat{\omega}_p + G\hat{\omega}_z), \]
FIG. 3. The IBVP (left) and the CIBVP (right) setups. For CCE outgoing data from the IBVP serve as boundary data on $T_0$ for the CIBVP, which can be viewed as an independent PDE problem. In this case the IBVP’s spatial domain is more extended such that data on $T_0$ are unaffected by the boundary conditions chosen for the problem. For CCM the IBVP and CIBVP are solved simultaneously and out/ingoing data are communicated from one to the other via $T_0$. Effectively, the two problems are viewed as one.

as $|\omega| \to \infty$, with the next terms appearing at order $|\omega|^0$. For these choices of lower order source terms the inhomogeneous WH model remains well-posed in the lopsided norm. On the other hand if $B = B_3$ the eigenvalues of the symbol are

$$
\lambda_1 = -i|\omega|\omega_\rho - (1)^{1/4} \sqrt{|\omega|\omega_\rho} + O(|\omega|^0) \\
\lambda_2 = -i|\omega|\omega_\rho + (1)^{1/4} \sqrt{|\omega|\omega_\rho} + O(|\omega|^0) \\
\lambda_3 = i|\omega|(F\omega_\rho + G\omega_\rho) + O(|\omega|^0)
$$

for large $|\omega|$. Since $\Re[\lambda] \sim |\omega|^{1/2}$, we conclude that when the nested structure of the intrinsic equations is broken, the solution of the inhomogeneous WH exhibits frequency dependent exponential growth. Consequently, the IVP with this system is no longer weakly well-posed but ill-posed. Note, in contrast, that for the homogeneous SH model we have

$$|e^{P(i\omega)t)}| = 1.$$ 

Hence for this model, the IVP is well-posed already in the $L^2$ norm. Unlike the WH model, well-posedness for this model is not affected by source terms.

3. The CIBVP, CCE and CCM

The previous analysis was performed in Fourier space and yielded that an IVP based on the homogeneous WH model may be well-posed in an appropriate lopsided norm, whereas on the SH model is (strongly) well-posed in the $L^2$-norm. We now present our energy estimates for solutions to the IBVP and CIBVP by working in position space. For concreteness and simplicity the PDE system for the IBVP is a homogeneous SH model (which is furthermore symmetric hyperbolic)

$$
\partial_\rho \phi + \partial_\phi \bar{\phi} + \partial_\psi \bar{\psi} = 0,
\partial_\rho \psi + \partial_\phi \bar{\psi} + \partial_\phi \bar{\psi} = 0,
\partial_\rho \bar{\psi} - \frac{1}{2} \partial_\phi \bar{\psi} - \partial_\phi \bar{\psi} = 0,
$$

with initial data $\bar{\phi}_t$, $\bar{\psi}_t$, $\bar{\psi}_\rho$ on $\Sigma_0$, boundary data $\bar{\psi}$ on $\mathcal{T}_0$ and domain $t \in [0, t_f], \rho \in (-\infty, 0]$ and the compact $z \in [0, 2\pi]$, and for the CIBVP the homogeneous WH model

$$
\partial_\rho \phi = 0,
\partial_\rho \psi_t - \partial_\rho \phi = 0,
\partial_\rho \psi - \frac{1}{2} \partial_\rho \bar{\psi} - \partial_\rho \bar{\psi} = 0,
$$

with initial data $\psi$ on $\mathcal{N}_0$, boundary data $\bar{\phi}$ and $\bar{\psi}$ on $\mathcal{T}_0$ and domain $u \in [0, u_f], x \in [0, x_f]$ and the aforementioned $z$. The domains of the two problems are illustrated in Fig. 3. We view the IBVP as a simplified analog of GR in strongly (here even symmetric) hyperbolic formulations widely used in Cauchy-type problems, with the CIBVP standing for the Bondi-Sachs gauge used in characteristic evolutions. We wish to understand whether or not problems with these features can be successfully used for CCE or CCM in principle.

For the IBVP estimate our starting point is

$$
\partial_t \|\bar{u}\|^2_{L^2(\Sigma_t)} = \partial_t \int_{\Sigma_t} \bar{u}^T \bar{u} = \partial_t \int_{\Sigma_t} \left(\phi^2 + \psi_t^2 + \bar{\psi}_z^2\right),
$$

which after using (49), the divergence theorem assuming $\bar{u} \to 0$ as $\rho \to -\infty$ and integrating in the $t$ domain, yields

$$
\|\bar{u}\|^2_{L^2(\Sigma_t)} + \|\bar{u}\|^2_{L^2(\mathcal{T}_0)} = \|\bar{u}\|^2_{L^2(\Sigma_0)} + \|\bar{u}\|^2_{L^2(\mathcal{T}_0)},
$$

where $\|\bar{u}\|^2_{L^2(\mathcal{T}_0)}$ denotes integral over $\mathcal{T}_0$ that contains only the outgoing fields $\bar{\phi}$, $\bar{\psi}$, and similarly for the ingoing. The estimate (51) states that the energy of the solution equals the energy of its given data, so that the solution is controlled by the given data.

In a Cauchy-type setup we specify all fields on the initial spacelike hypersurface and, by solving the system we obtain all of them on spacelike hypersurfaces to the future. On the contrary, in a single-null characteristic setup, fields with “evolution” equations are chosen on the initial null hypersurface and those that satisfy equations intrinsic to the null hypersurfaces are specified as boundary data. As we will see in the following, this has a natural consequence on the type of estimates that we can hope to demonstrate, both in terms of the domain on which we integrate and the particular fields that appear. This is due to the geometry of the setup.

Motivated from the IVP estimates in Fourier space of subsection 11.3.4.1 and 11.3.4.2 we might naively first con-
consider for the CIBVP the lopsided norm

\[ ||u||^2_{q\in(D)} = \int_D \left( \phi^2 + \psi^2 + \varphi^2 + (\partial_x \phi)^2 \right), \]

in some domain \( D \), where only \( \partial_x \phi \) is added to the integrand of the \( L^2 \)-norm, because precisely this term causes the pathological structure in the angular principal part of the WH model. Following our previous discussion however, it is more appropriate to split the integrand into separate pieces for the ingoing and outgoing variables. The domain \( D \) becomes \( \mathcal{N}_u \) and \( \mathcal{T}_x \) respectively for each.

For the ingoing variables we start from

\[ \partial_u ||u||^2_{q\in(\mathcal{N}_u)} = \partial_u \int_{\mathcal{N}_u} \psi^2, \]

since there are no \( \partial_x \) equations for the outgoing ones. We assume that \( \psi \to 0 \) as \( x \to x_f \) in the given data, which is the analog in our model to requiring no incoming gravitational waves from future null infinity, working on a compactified radial domain. After using (50c), the divergence theorem and integrating in the \( u \) domain we obtain

\[ 2||u||^2_{q\in(\mathcal{N}_u)} + ||u||^2_{q\in(\mathcal{T}_x)} = 2||u||^2_{q\in(\mathcal{N}_0)} \quad (52) \]

For the outgoing variables the starting point is

\[ \partial_x ||u||^2_{q\in(\mathcal{T}_x)} = \partial_x \int_{\mathcal{T}_x} \left( \phi^2 + \psi^2 + (\partial_x \phi)^2 \right), \]

and by using (50a) and (50b), the divergence theorem and integrating in the \( x \) domain up to some arbitrary \( x' \) we obtain

\[ ||u||^2_{q\in(\mathcal{T}_x)} = ||u||^2_{q\in(\mathcal{T}_0)} + \int_0^{x'} \left( \int_{\mathcal{T}_x} 2\psi \partial_x \phi \right) dx, \]

(53)

where the last term is due to the hyperbolicity of the system and would not appear for our SH example. Using \( 2\psi \partial_x \phi \leq \phi^2 + \psi^2 + (\partial_x \phi)^2 \) the latter reads

\[ ||u||^2_{q\in(\mathcal{T}_x)} \leq ||u||^2_{q\in(\mathcal{T}_0)} + \int_0^{x'} ||u||^2_{q\in(\mathcal{T}_x)} dx, \]

and by applying Grönwall’s inequality we obtain

\[ ||u||^2_{q\in(\mathcal{T}_x)} \leq e^{x'} ||u||^2_{q\in(\mathcal{T}_0)}. \]

(54)

Hence, the energy of the outgoing fields at each arbitrary timelike hypersurface \( \mathcal{T}_x \) in the characteristic domain is bounded. The sum of (52) and (54) is the complete energy estimate for the CIBVP and yields

\[ 2||u||^2_{q\in(\mathcal{N}_u)} + ||u||^2_{q\in(\mathcal{N}_u)} + \sup_{x'} ||u||^2_{q\in(\mathcal{T}_x)} \leq 2||u||^2_{q\in(\mathcal{N}_0)} + e^{x'} ||u||^2_{q\in(\mathcal{T}_0)} \quad (55) \]

where we used that \( e^{x'} \leq e^{x_f} \) for \( x' \in [0, x_f] \) and chose the supremum of \( ||u||^2_{q\in(\mathcal{T}_x)} \) to obtain the largest possible bounded left hand side, since the outgoing lopsided norm is not necessarily monotonically increasing with \( x \). Thus, the energy of the solution to the CIBVP is controlled by the given data on \( \mathcal{N}_0 \) and \( \mathcal{T}_0 \).

We first interpret these estimates in the framework of CCE. Choosing suitable data, our estimate for the IBVP shows that one obtains a smooth solution in the domain of the Cauchy-type setup. One can then use this solution to provide boundary data on \( \mathcal{T}_0 \) for the CIBVP that are finite also in the lopsided norm, and the solution to this characteristic problem has a good energy estimate as shown earlier too. Hence the CCE process is perfectly valid for our model, and provided analogous estimates for GR in the Bondi-like gauges used, would be in that context too. One question that arises for GR, but which for now we have no insight, is whether or not this procedure excludes any data of interest. For CCM the discussion is rather different, since IBVP and CIBVP are solved simultaneously and data are communicated between domains. Effectively, one joins the PDE problems and they may be viewed as one. Hence, let us try to obtain an energy estimate for the joint PDE problem, by adding (51) and (55):

\[ ||u||^2_{L^2(\mathcal{N}_u)} + ||u||^2_{L^2(\mathcal{T}_0)} + 2||u||^2_{q\in(\mathcal{N}_u)} + \sup_{x'} ||u||^2_{q\in(\mathcal{T}_x)} \leq ||u||^2_{L^2(\mathcal{N}_0)} + ||u||^2_{L^2(\mathcal{T}_0)} + 2||u||^2_{q\in(\mathcal{N}_0)} + e^{x'} ||u||^2_{q\in(\mathcal{T}_0)}, \]

(56)

where now \( \bar{u} = u \). For the joint problem there is ‘effectively’ no boundary \( \mathcal{T}_0 \) at which we are free to choose data, and hence any estimate should not involve integrals over this domain. The relevant terms can however cancel each other only if the two norms that appear coincide. This requires either that the CIBVP relies on a symmetric hyperbolic PDE system and hence is well-posed in the \( L^2 \)-norm (see for instance (31–33)), or that the IBVP relies on a system that is well-posed in the same lopsided norm as the CIBVP. But this requires special structure, above and beyond symmetric hyperbolicity, on the equations used in the IBVP. Regarding GR, the first option would translate into developing a SH (hopefully also symmetric hyperbolic) single-null formulation and the second to building a formulation that is well-posed in the same lopsided norm that Bondi-like gauges (perhaps) are. Given the long search for formulations that work for practical evolution however, such an artisanal construction seems poorly motivated. In summary; unless special structure is present in the field equations solved for the IBVP, the solution to the weakly hyperbolic CIBVP cannot be combined with that of an IBVP of a symmetric hyperbolic system in such a way as to provide a solution to the whole problem which has an energy bounded by that of the given data.
IV. NUMERICAL EXPERIMENTS

We now use the toy models introduced in Sec. III to diagnose the effects of weak hyperbolicity at the numerical level. We perform convergence tests in the single-null setup for both the WH and SH models in a discrete approximation to the $L^2$-norm, for smooth and noisy given data. We also perform convergence tests with noisy given data in the lopsided norm, for the different versions of the WH model analyzed in the previous section.

A. Implementation

As in other schemes to solve the CIBVP, several different ingredients are needed in the algorithm. These can be summarized for our models (38) and (40) as follows:

1. The domain of the PDE problem is $x \in [0, 1]$, $z \in [0, 2\pi]$ with periodic boundary conditions and $u \in [u_0, u_f]$, with $u_0$ and $u_f$ the initial and final times respectively. We always include the point $x = 1$ in the computational domain so that we do not need to impose boundary conditions at the outer boundary, since there are no incoming characteristic variables there.

2. For the initial time $u_0$ provide initial data $\psi(u_0, x, z)$ on the surface $u = u_0$ and boundary data $\phi(u_0, 0, z)$ and $\psi_b(u_0, 0, z)$.

3. Integrate the intrinsic equations of each model to obtain $\phi(u_0, x, z)$ and $\psi_b(u_0, x, z)$. We perform this integration using the two-stage, second-order strong stability preserving method of Shu and Osher (SSPRK22) [43].

4. Integrate the evolution equation of each model to obtain $\psi(u_1, x, z)$ at the surface $u = u_1 = u_0 + \Delta u$. We choose $\Delta u = 0.25\Delta x$ to satisfy the Courant-Friedrichs-Lewy (CFL) condition and the numerical integration is performed using the fourth order Runge-Kutta (RK4) method.

5. Any derivative appearing in the right-hand-sides of these integrations is approximated using second order accurate centered finite difference operators, except at the boundaries, where second order accurate forward and backward difference operators are used respectively.

6. Providing boundary data $\phi(u, 0, z)$ and $\psi_b(u, 0, z)$ as in the PDE specification (12), we repeat steps 2 and 3 to obtain $\phi(u, x, z)$, $\psi_b(u, x, z)$ and $\psi(u, x, z)$ until the final time $u_f$. This is the solution of the PDE.

No artificial dissipation is introduced. The implementation was made using the Julia language [35] with the DifferentialEquations.jl package [36] to integrate the equations. Our code is freely available [21]. We apply convergence tests to our numerical scheme for both toy models. The tests are performed for smooth, as well as for noisy given data. The latter are often called robust stability tests. They form part of the Mexico-city testbed for numerical relativity [37]. These tests have been performed widely in the literature [38–43], often, as in our case, with adaptations for the setup under consideration.

B. Convergence tests

By convergence we mean the requirement that the difference between the numerical approximation provided by a finite difference scheme and the exact solution of the continuum PDE system tends to zero as the grid spacing is increased. The finite difference scheme is called consistent when it approximates the correct PDE system and the degree to which this is achieved is its accuracy. The scheme is called stable if it satisfies a discretized version of (44) or (46). In this context versions of each continuum norm is replaced by a suitable discrete analog. Here we replace the $L^2$-norm for the single-null setup with

$$\|u\|_{h,u,h_z}^2 = \sum_{x,z} \psi^2 h_x h_z + \max_{x,z} \sum_{u,z} (\phi^2 + \psi_v^2) h_u h_z,$$

(57)

with the first sum taken over all points on the grid, with $h_x$ and $h_z$ the grid-spacing in the $x$ and $z$ directions respectively, and the second sum over all points in the $z$ and $u$ directions ($h_u = 0.25h_x$ for our setup), for all $x$ grid points and keeping the maximum in the $x$ direction. The first sum involves only ingoing and the second only outgoing variables. When, as will be the case in what follows, we have $h_u = h_z = h$ we label the norm simply with $h$. Our discrete approximation to the lopsided norm is,

$$\|u\|_{q(h_u,h_z,h)}^2 = \sum_{x,z} \psi^2 h_x h_z + \max_{x,z} \sum_{u,z} (\phi^2 + \psi_v^2 + (D_z \phi)^2) h_u h_z,$$

(58)

where $D_z$ is the second order accurate, centered, finite difference operator that replaces the continuum operator $\partial_z$, by

$$D_z f_h(x_i) = \frac{f_h(x_{i+1}) - f_h(x_{i-1})}{2h_z},$$

(59)

for a grid function $f_h$ on a grid with spacing $h_z$. When the two grid spacings are equal we again label the norm simply with $h$. This approximation to the continuum lopsided norm is not unique. If we were attempting to prove that a particular discretization converged, it might be necessary to take another. Denoting by $f$ the solution to the continuum system and as $f_h$ the numerical
approximation at resolution $h$ provided by a convergent finite difference scheme of accuracy $n$, then
\[ f = f_h + O(h^n), \tag{60} \]
and hence
\[ \|f - f_h\| = O(h^n), \tag{61} \]
in some appropriate norm $\| \cdot \|$ on the grid, with the understanding that the exact solution should be evaluated on said grid. Full definitions of the notions of consistency, stability and convergence for the IVP can be found, for example, in [13, 39, 44].

We use a second order accurate numerical approximation, so that $n = 2$. Considering numerical evolutions with coarse, medium and fine grid spacings $h_c, h_m$ and $h_f$ respectively, we can construct a useful quantity for these tests
\[ Q = \frac{h_c^n - h_m^n}{h_m^n - h_f^n}, \tag{62} \]
which we call convergence factor. In our convergence tests we solve the same discretized PDE problem for different resolutions and every time we want to increase resolution we halve the grid-spacing in all directions i.e.
\[ h_m = h_c/2, \quad h_f = h_c/4. \]
Following this approach the convergence factor is $Q = 4$. Combining (60) and (62) one can obtain the relation
\[ f_{h_c} - f_{h_c/2} = Q \left( f_{h_c/2} - f_{h_c/4} \right), \tag{63} \]
understood on shared grid-points in the obvious way, which is used to investigate pointwise convergence. In what follows the different resolutions are denoted as
\[ h_q = h_0/2^q. \]
The lowest resolution $h_0$ has $N_x = 17$ points in the $x$-grid and $N_z = 16$ in the $z$-grid. We work in units of the code in the entire section.

1. Smooth data

For the simulations with smooth given data the initial and final times are $u_0 = 0$ and $u_f = 1$ respectively. For both toy models we provide as initial data
\[ \psi(0, x, z) = e^{-100(x-1/2)^2} \sin(z), \]
and as boundary data
\[ \phi(u, 0, z) = 3e^{-100(u-1/2)^2} \sin(z), \]
and
\[ \psi_h(u, 0, z) = e^{-100(u-1/2)^2} \sin(z). \]

![FIG. 4. The fields $\phi$, $\psi$, and $\psi$ at final evolution time $u = 1$, for the SH model (left) and the homogeneous WH model (right), with the same smooth given data. Observe that the fields $\phi$ and $\psi$ in the WH case are still of the same magnitude $\sim 10^{-11}$ as the boundary data at the retarded time $u = 1$. This is not true once generic source terms are taken.](image)

For the SH model we choose the following source terms
\[ -S_\psi = \psi, \quad -S_\psi = \phi + \psi, \quad -S_\psi = \phi, \tag{64} \]
and for the WH model we choose the homogeneous case. As discussed in Sec. III B well-posedness of the SH model is unaffected by lower order source terms, so the specific choice of source terms here is not vital. However, we choose to work with the homogeneous WH model, because weakly well-posed problems are sensitive to lower order perturbations.

Runs with resolutions $h_0$, $h_1$, $h_2$, $h_3$, $h_4$ and $h_5$ were performed. In Fig. 4 the basic dynamics are plotted with each model. To first verify that the numerical scheme is implemented successfully we performed pointwise convergence tests for both models. We focus specifically here on the highest three resolutions. The algorithm is the following:

1. Consider $h_3$, $h_4$ and $h_5$ as coarse, medium and fine resolutions, respectively.
We obtain numerical solutions for the same smooth given data for both models at the various resolutions mentioned before. For triple of resolution, double resolution and quadruple resolution, we project all gridfunctions onto the coarse grid, and compute $C_{\text{self}}$ at its timesteps. In the left panel of Fig. 6 we collect the results of these norm convergence tests. Both models show similar behavior. At low resolutions curve drifts from the desired rate at early times, but the situation improves as we increase resolution, with $C_{\text{self}}$ approaching the expected value. The trend with increasing resolution is the essential behavior we are looking at in these tests. By limiting ourselves to convergence tests with smooth given data we could be misled that the WH toy model provides a well-posed CIBVP in the $L^2$-norm, since the numerical solutions appear to converge in this norm during our simulations. In other words, were we ignorant of the hyperbolicity of the system, it would be impossible to distinguish strongly and weakly hyperbolic PDEs with this test.

2. Noisy data

One can also perform norm convergence tests with random noise as given data, which is a strategy to simulate numerical error and exaggerate is expected that numerical error decreases as resolution increases, when performing simulations for these tests one must scale appropriately the amplitude of the noise as resolution improves. This scaling is important to construct a sequence of initial data that converges in a suitable norm to initial data appropriate for the continuum system. The choice of norm here is essential, and should be one which, if possible, provides a bound for the solution of (a weakly) well-posed PDE problem, in the sense of (44) and (46).

For these tests we perform simulations where the smooth part of the given data is trivial (zero), and hence the exact solution for every PDE problem based on our models vanishes identically. Knowing the exact solution, in addition to the self convergence rate (65), we can also construct the exact convergence ratio

$$C_{\text{exact}} = \log_d \frac{\|u_{h_3} - u_{\text{exact}}\|_{h_3}}{\|u_{h_2} - u_{\text{exact}}\|_{h_2}},$$

where we decrease grid spacing by a factor of $d$ when increasing resolution. $C_{\text{exact}}$ is cheaper numerically than $C_{\text{self}}$ since only two different resolutions are required to build it, and again the exact solution is understood to be evaluated on the grid itself. It is possible for a scheme to be self-convergent but fail to be convergent, for example if one were to implement the wrong field equations in error. Therefore one would like to compare the numerical solution to an exact solution wherever (rarely) possible. To calculate $C_{\text{exact}}$, we compute the discretized approximation to a suitable continuum norm at two resolutions, one twice the other. Each are computed on the

![Graph](image-url)

**Fig. 5.** Here we plot simultaneously $\psi_{h_3} - \psi_{h_4}$ and $Q(\psi_{h_4} - \psi_{h_3})$, for the SH (top) and the WH (bottom) toy models. We fix $x = 0.5$. Since our scheme is second order and we are doubling resolution we fix $Q = 4$. The results for fixed $z$ are similar. The plot is compatible with perfect second order pointwise convergence.

2. Calculate $\psi_{h_3} - \psi_{h_4}$ and $\psi_{h_4} - \psi_{h_5}$ for the gridpoints of $h_3$, for the final timestep of the evolution.

3. Plot simultaneously $\psi_{h_3} - \psi_{h_4}$ and $Q(\psi_{h_4} - \psi_{h_3})$.

As indicated from (63), for a convergent numerical scheme the two quantities should overlap, when multiplying the latter with the appropriate convergence factor.

In Fig. 5, we illustrate the results of this test for the aforementioned smooth given data for both models. At this resolution one clearly observes perfect pointwise convergence in both cases.

We also wish to examine convergence of our numerical solutions in discrete approximations of the aforementioned norms. Given that the exact solution to the PDE problem is unknown and that each time we increase resolution we decrease the grid spacing in all directions by a factor of $d$, we can build the following useful quantity

$$C_{\text{self}} = \log_d \frac{\|u_{h_3} - \frac{1}{h_c/d} u_{h_4/d}\|_{h_3}}{\|\frac{1}{h_c/d} u_{h_4/d} - \frac{1}{h_c/d^2} u_{h_5/d^2}\|_{h_3}},$$

which we call self-convergence ratio, with $u = (\phi, \psi, \psi)^T$ the state vector of the PDE system and $\phi, \psi, \psi$ grid functions. Here $\frac{1}{h_c/d}$ denotes the projection (in our setup injection) operator from the $h_c/d$ grid onto the $h_c$ grid. We calculate $C_{\text{self}}$ for a discrete analog of the $L^2$-norm. However, if one wishes to examine convergence in a different norm, $L^2$ can be replaced with that. The theoretical value of $C_{\text{self}}$ equals the accuracy $n$ of the numerical scheme, and in our specific setup

$$C_{\text{self}} = \log_d \frac{\|u_{h_3} - \frac{1}{h_c} u_{h_4/2}\|_{h_3}}{\|\frac{1}{h_c} u_{h_4/2} - \frac{1}{h_c/2} u_{h_5/4}\|_{h_3}} = 2.$$
naturally associated grid. We then take the ratio of the two at shared timesteps, corresponding to those of the coarse grid \( h_c \). In our setup \( u_{\text{exact}} = 0 \) and \( d = 2 \), hence

\[
C_{\text{exact}} = \log_2 \frac{||u_{h_c}||_{h_c}}{||u_{h_c/2}||_{h_c/2}},
\]

which again equals two for perfect convergence. As previously mentioned appropriate scaling of the random noise amplitude is crucial and is determined by the norm in which we wish to test convergence. To realize the proper amplitude is crucial and is determined by the norm in the lopsided norm (58). By replacing the \( D \) to the \( \phi \) in the lopsided norm the scaling factor is different, due to the \( D_x \phi \) term that appears in the discretized version of the lopsided norm (55). By replacing the \( L^2 \) with the lopsided norm in (68), we get

\[
C_{\text{exact}} = \log_2 \frac{||u_{h_c}||_{q(h_c)}}{||u_{h_c/2}||_{q(h_c/2)}},
\]

where now the norm estimate is dominated by the \( D_x \phi \) term. Hence, to construct noisy data that converge in the lopsided norm for our second order accurate numerical scheme, we need to multiply the amplitude of the random noise with a factor of one eighth every time we double resolution. This discussion would be more complicated if we were using either pseudospectral approximation or some hybrid scheme, which is why we focus exclusively on a straightforward finite differencing setup.

The results for norm convergence tests with appropriately scaled noisy data for the \( L^2 \)-norm, for both SH and WH models, are collected in the right column of Fig. 6. As illustrated there, the inhomogeneous SH model still exhibits convergence since with increasing resolution the exact convergence ratio tends closer to the desired value of two at all times of the evolution. On the contrary, the homogeneous WH model does not converge, and it becomes clear that with increasing resolution the exact convergence ratio of this model moves further away from two at all times.

To appreciate intuitively why noisy data allow us to diagnose a lack of strong hyperbolicity, consider the systems in frequency space as in subsection III B, which we may think of as momentum space. In practical terms, Eqn. (48) states that the homogeneous WH model does not satisfy condition (44), and so high frequency modes can grow arbitrarily fast. Considering smooth data however, predominantly low frequency modes are excited, and so using our discretized approximation the violation of inequality (44) is not visible at the limited resolutions we employ. Noisy data on the contrary excite substantially both high and low frequency modes, with the former crucial to illustrate the violation.

We also perform convergence tests in the lopsided norm (55) to examine the behavior of the different WH models. As in the previous setup, in these tests we monitor the exact convergence ratio as a function of the simulation time. As illustrated in Fig. 7, our expectations from subsection III B for the homogeneous model are verified. The homogeneous WH model converges at all times.
in the lopsided norm, provided of course that the given data are restricted to converge at second order to the trivial solution in the same norm. As also expected, the inhomogeneous case with \( B_3 \) fails to converge whatsoever during the evolution, exhibiting behavior similar to the homogeneous WH model in the \( L^2 \)-norm tests. In fact, in this test the exact convergence ratio diverges further from two with increasing resolution and at earlier times. The discussion for the inhomogeneous WH models with sources \( B_1 \) and \( B_2 \) is more subtle. Both cases initially exhibit convergence, with the \( B_1 \) case maintaining this behavior for longer. The difference lies in their late time behavior and their trend with increasing resolution. In particular, the \( B_1 \) case converges for longer with increasing resolution whereas \( B_2 \) does the opposite. At late times in the \( B_1 \) case \( C_{\text{exact}} \) reaches a plateau that converges to two with increasing resolution, which is not true with sources \( B_2 \). Thus our numerical evidence seems to indicate that the \( B_1 \) inhomogeneous WH model converges in the lopsided norm, but to disagree with the theoretical expectation at the continuum that the \( B_2 \) case does so too. This is not in contradiction with our earlier calculations however, because, as a careful examination of the approximation could reveal, purely algorithmic shortcomings may render a scheme nonconvergent.

V. CONCLUSIONS

Single-null formulations of GR are popular for applications in numerical relativity in various settings. In asymptotically flat spacetimes they are used with compactified coordinates to compute gravitational waveforms at future-null infinity. In asymptotically AdS spacetimes they are used to compute in from the timelike conformal boundary. But relatively little attention has been paid to well-posedness of the resulting PDE problems, which serves as an obstacle to the construction of rigorous error-estimates from computational work. Presently, therefore, we have examined two popular formulations, the Bondi-Sachs and affine-null systems, and performed numerical tests for toy models that illustrate the relevance of our findings. We found in a free-evolution analysis that, due to the non-diagonalizability of their angular principal part matrices, both are only weakly hyperbolic.

Our analysis employed a first order reduction, but was sufficiently general to rule out the existence of any other reduction (at least within a large class) that is strongly hyperbolic. We showed also that the degeneracy can not be avoided by a change of frame. Text book results on these systems then show that they are ill-posed in the \( L^2 \)-norm or its obvious derivatives. Considering model problems of a similar structure we saw that the same result naturally carries over to the CIBVP. In the latter case care is needed not to confuse the usual degeneracy of the norms that appear naturally in characteristic problems with high-frequency blow-up of solutions. It follows that a numerical approximation cannot converge to the exact solution of these PDE problems in any discrete approximation to \( L^2 \). We demonstrated this shortcoming numerically using our models and adapting the well-known robust-stability test. Spotting this shortcoming in practice is subtle because smooth data may, and often do, give misleading results.

Although our weakly hyperbolic toy model is ill-posed in \( L^2 \), it may be well-posed in a lopsided norm in which
the angular derivative of some specific components of the state vector are included. Thus in such a case one must be able to control the size of not only the elements of the state vector in the given data, but also some of their derivatives. This weaker notion of well-posedness is sensitive to the presence lower order source terms. For example, our weakly hyperbolic model is well-posed in a (specific) lopsided norm if it is homogeneous, or inhomogeneous with sources that respect the nested structure of the equations intrinsic to the characteristic hypersurfaces. If this nested structure is broken by the source terms, it becomes ill-posed in any sense. Again using random noise for initial data, our numerical experiments are consistent with this analytic result. There is one case in which convergence is not apparent in our approximation, despite the well-posedness of the continuum equations in the lopsided norm. This is our only example of a pure numerical instability, and it is important as it highlights the fact that for weakly hyperbolic systems numerical methods are not well-developed, and are not guaranteed to converge, even when using lopsided norms.

Bringing our attention back to the characteristic initial boundary value problem for GR, which covers both CCE and applications in numerical holography, it is clear that the two formulations we considered will be ill-posed in $L^2$. It is not clear however, in general, if they will admit estimates in suitable lopsided norms. But since the field equations do have a nested-structure, and our weakly hyperbolic model problem turned out to admit estimates in lopsided norms whenever this structure was present, there is reason to be hopeful. On the other hand, given this uncertainty, and the fact that numerical approximation to weakly hyperbolic systems (using lopsided norms) is poorly understood, it is desirable to obtain and adopt strongly or ideally symmetric hyperbolic alternatives. These could be sought out by changing gauge directly, or by the use of a dual-foliation formulation as suggested in [26]. Perhaps a simpler option would be to pay the price of evolving curvature quantities as variables. Several such formulations are known to be symmetric hyperbolic in a double-null gauge [15][47] and could be adjusted appropriately.

A true principle solution to wave-extraction would be a robust scheme for CCM, the other main alternative being the use of compactified hyperboloidal slices, a topic also under active research for full GR [18][57]. To understand the consequences of our findings for CCM we considered a model in which the IBVP is solved for a symmetric hyperbolic system, and the solutions are then glued through boundary conditions to those of a weakly hyperbolic system accepting estimates in lopsided norms. The former of these two sets of equations is viewed as a model for the formulation used in the strong-field region, the latter for a single-null formulation used on the outer characteristic domain. With this setup, we found that the fundamental incompatibility of the norms naturally associated with the two domains prohibits their combined use in building estimates. But if the weakly hyperbolic system were made symmetric hyperbolic progress could be made. A less appealing possibility would be to demonstrate that the formulation in the Cauchy domain, or some suitable replacement, admits estimates in a lopsided norm compatible with that of the characteristic region. Since this relies on very special structure in the field equations, the outlook for a complete proof of well-posedness of CCM using existing Bondi-like gauges is, unfortunately, not rosy.

Our results signpost a number of paths to follow. First and foremost, we need to recover our numerical results for toy models for full GR. Beyond that, we seek a well-posed setup for the CIBVP that can be used in numerical applications with minimum change to existing code. Work in both directions is ongoing.

ACKNOWLEDGMENTS

We are grateful to Thomas Baumgarte, Nigel Bishop, Carsten Gundlach, Luis Lehner and Denis Pollney for helpful discussions and/or comments on the manuscript. We also thank Mikel Sánchez for feedback on our Julia scripts. The work was partially supported by the FCT (Portugal) IF Program IF/00577/2015, IF/00729/2015, PTDC/MAT-APL/30043/2017 and Project No. UIDB/00099/2020. TG acknowledges financial support provided by FCT/Portugal Grant No. PD/BD/135425/2017 in the framework of the Doctoral Programme IDPASC-Portugal.

[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. A 269, 21 (1962).
[2] R. K. Sachs, Proc. Roy. Soc. A 270, 103 (1962).
[3] N. T. Bishop, R. Gómez, L. Lehner, M. Maharaj, and J. Winicour, Phys. Rev. D 56, 6298 (1997), [gr-qc/9708065].
[4] N. T. Bishop, R. Gómez, L. Lehner, and J. Winicour, Phys. Rev. D 52 (1997).
[5] Y. Zlochower, R. Gómez, S. Husa, L. Lehner, and J. Winicour, Phys. Rev. D 68, 084014 (2003).
[6] C. J. Handmer and B. Szilágyi, Class. Quant. Grav. 32, 025008 (2015) [arXiv:1406.7029 [gr-qc]].
[7] K. Barkett, J. Moxon, M. A. Scheel, and B. Szilágyi, (2019), arXiv:1910.09677 [gr-qc].
[8] J. Moxon, M. A. Scheel, and S. A. Teukolsky, (2020), arXiv:2007.01339 [gr-qc].
[9] J. Winicour, Living Rev. Relativity 15, 2 (2012) [Online article].
[10] B. Szilágyi, Cauchy-Characteristic Matching In General Relativity, Ph.D. thesis, University of Pittsburgh (2000).
[1] P. M. Chesler and L. G. Yaffe, Phys. Rev. Lett. 106, 021101 (2011), arXiv:1011.3562 [hep-th].
[2] M. Attems, J. Casalderrey-Solana, D. Mateos, D. Santos-Oliván, C. F. Sopuerta, M. Triana, and M. Zilhão, JHEP 06, 154 (2017), arXiv:1703.09681 [hep-th].
[3] B. Gustafsson, H.-O. Kreiss, and J. Oliger, Time dependent problems and difference methods (Wiley, New York, 1995).
[4] D. Hilditch, Int. J. Mod. Phys. A28, 1340015 (2013), arXiv:1309.2012 [gr-qc].
[5] M. C. Babiuc, S. Husa, D. Alic, I. Hinder, C. Lechner, E. Schnetter, B. Szilágyi, Y. Zlochower, N. Dorband, D. Pollney, and J. Winicour, Class. Quant. Grav. 25, 125012 (2008), arXiv:0709.3559 [gr-qc].
[6] apples with apples; numerical relativity comparisons and tests (1999).
[7] S. Frittelli and L. Lehner, Phys. Rev. D 59, 084012 (2004), arXiv:gr-qc/0303104.
[8] C. Gundlach and J. M. Martín-García, Class. Quantum Grav. 23, S387 (2006), gr-qc/0506037.
[9] D. Hilditch and R. Richter, Phys. Rev. D94, 044028 (2016), arXiv:1303.4783 [gr-qc].
[10] T. Giannakopoulos, D. Hilditch, and M. Zilhão, “Hyberolicity of General Relativity in Bondi-like gauges,” (2020).
[11] S. Frittelli, Phys. Rev. D 71, 024021 (2005) arXiv:gr-qc/0408035.
[12] J. Winicour, Phys. Rev. D87, 124027 (2013), arXiv:1303.6969 [gr-qc].
[13] J. A. Crespo, H. P. de Oliveira, and J. Winicour, Phys. Rev. D100, 104017 (2019), arXiv:1910.03439 [gr-qc].
[14] P. M. Chesler and L. G. Yaffe, JHEP 07, 086 (2014), arXiv:1309.1439 [hep-th].
[15] D. Hilditch, (2015), arXiv:1509.02071 [gr-qc].
[16] A. Schoepe, D. Hilditch, and M. Bugner, Phys. Rev. D97, 123009 (2018), arXiv:1712.09837 [gr-qc].
[17] D. Hilditch and A. Schoepe, Phys. Rev. D99, 104034 (2019), arXiv:1812.03485 [gr-qc].
[18] H.-O. Kreiss and J. Lorenz, Initial-boundary value problems and the Navier-Stokes equations (Academic Press, New York, 1989).
[19] O. Sarbach and M. Tiglio, Living Reviews in Relativity 15 (2012) arXiv:1203.6443 [gr-qc].
[20] N. T. Bishop, R. Gómez, P. R. Holvorcem, R. R. Matzner, P. Papadopoulos, and J. Winicour, Phys. Rev. Lett. 76, 4303 (1996).
[21] N. T. Bishop, R. Gómez, P. R. Holvorcem, R. R. Matzner, P. Papadopoulos, and J. Winicour, J. Comput. Phys. 136, 236 (1997).
[22] G. Calabrese, Class. Quant. Grav. 23, 5439 (2006), arXiv:gr-qc/0604034.
[23] C. W. Shu and S. J. Osher, J. Comput. Phys. 77, 439 (1987).
[24] Bezanson, Jeff and Edelman, Alan and Karpinski, Stefan and Shah, Viral B, SIAM Review 59, 65 (2017).
[25] C. Rackauckas and Q. Nie, The Journal of Open Research Software 5 (2017), 10.5334/jors.151.
[26] M. Alcubierre, G. Allen, T. W. Baumgarte, C. Bona, D. Fiske, T. Goodale, F. S. Guzmán, I. Hawke, S. Hawley, S. Husa, M. Koppitz, C. Lechner, L. Lindblom, D. Pollney, D. Rideout, M. Salgado, E. Schnetter, E. Seidel, H. aki Shinkai, D. Shoemaker, B. Szilágyi, R. Takahashi, and J. Winicour, Class. Quantum Grav. 21, 589 (2004), gr-qc/0305023.
[27] G. Calabrese, I. Hinder, and S. Husa, J. Comp. Phys. 218, 607 (2005), gr-qc/0503056.
[28] I. Hinder, Well-posed formulations and stable finite differencing schemes for numerical relativity, Ph.D. thesis, School of Mathematics, University of Southampton (2005).
[29] M. Boyle, L. Lindblom, H. Pfeiffer, M. Scheel, and L. E. Kidder, Phys. Rev. D75, 024006 (2007) arXiv:gr-qc/0609047.
[30] M. C. Babiuc et al., Class. Quant. Grav. 25, 125012 (2008) arXiv:0709.3559 [gr-qc].
[31] H. Witek, D. Hilditch, and U. Sperhake, Phys. Rev. D83, 104041 (2011) arXiv:1011.4407 [gr-qc].
[32] Z. Cao and D. Hilditch, Phys. Rev. D 85, 124032 (2012), arXiv:1111.2177 [gr-qc].
[33] J. Thomas, Numerical Partial Differential Equations: Finite Difference Methods, Texts in Applied Mathematics (Springer New York, 1998).
[34] A. Cabet, P. T. Chruściel, and R. T. Wafai, (2014), arXiv:1406.3009 [gr-qc].
[35] D. Hilditch, J. A. V. Kroon, and P. Zhao, (2019), arXiv:1911.00047 [gr-qc].
[36] D. Hilditch, J. A. V. Kroon, and P. Zhao, (2020), arXiv:2006.13757 [gr-qc].
[37] J. M. Bardeen, O. Sarbach, and L. T. Buchman, Phys. Rev. D83, 104045 (2011) arXiv:1011.5479 [gr-qc].
[38] A. Zenginooglu, J. Comput. Phys. 230, 2286 (2011), arXiv:1008.3809 [math.NA].
[39] A. Vaníčka-Viňuela, S. Husa, and D. Hilditch, Class. Quant. Grav. 36, 125016 (2019) arXiv:1903.06550 [gr-qc].
[40] A. Vaníčka-Viňuela, Free evolution of the hyperboloidal initial value problem in spherical symmetry, Ph.D. thesis, Iles Balears, Palma (2015), arXiv:1512.00776 [gr-qc].
[41] G. Doullis and J. Frauendiener, (2016), arXiv:1609.03584 [gr-qc].
[42] D. Hilditch, E. Harms, M. Bugner, H. Rüter, and B. Brügmann, Class. Quant. Grav. 35, 055003 (2018) arXiv:1609.08949 [gr-qc].
[43] A. Vaníčka-Viňuela and S. Husa, Class. Quant. Grav. 35, 045014 (2018) arXiv:1705.06298 [gr-qc].
[44] E. Gasperin and D. Hilditch, Class. Quant. Grav. 36, 195016 (2019) arXiv:1812.06550 [gr-qc].
[45] E. Gasperin, S. Gautam, D. Hilditch, and A. Vaníčka-Viňuela, (2019), arXiv:1909.11749 [gr-qc].
[46] B. Brügmann, Class. Quant. Grav. 35, 045014 (2018) arXiv:1705.06298 [gr-qc].
[47] E. Gasperin, and D. Hilditch, Class. Quant. Grav. 36, 195016 (2019) arXiv:1812.06550 [gr-qc].
[48] F. Beyer, J. Frauendiener, and J. Hennig, (2020), arXiv:2005.11936 [gr-qc].