Efficient Numerical Strategy for High Dimensional Stochastic Problems

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\begin{abstract}
Uncertainty quantification appears today as a crucial point in numerous branches of science and engineering. In the past two decades, a growing interest has been devoted to stochastic finite element method (SFEM) for the propagation of uncertainties through physical models governed by stochastic partial differential equations (SPDEs). Despite its success and applications, the SFEM is mainly limited to small-scale and low-dimensional stochastic problems due to the extreme computational cost. In this article, by developing an universal construct of stochastic solution and a general algorithm for linear/nonlinear SFE equation, we explore a new strategy for the solution of high-dimensional stochastic problems, where stochastic problems are transformed into deterministic problems and stochastic algebraic equations. Since computational cost is almost proportional to the stochastic dimensionality of the problem, our method beats the so-called Curse of Dimensionality with great success. Numerical examples, including linear, nonlinear and high-dimensional stochastic problems, are used to demonstrate the method. Results show that our algorithm provides a highly efficient and unified frame-
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work for problems involving uncertainties, and is particularly appropriate for high stochastic dimensional problems of practical interest.

**Keywords:** High stochastic dimensions, Stochastic Galerkin, Curse of dimensionality, Stochastic nonlinerity

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1. **Introduction**

Due to the significant development in computational hardware and scientific computing techniques, it is now possible to solve very high resolution models in various computational physics problems, ranging from fluid mechanics to nano-bio mechanics. In particular, the finite element method (FEM) [1, 2] and closely related approximations have become state-of-the-art. On the other hand, however, the considerable influence of inherent uncertainties on system behavior has led the scientific community to recognize the importance of uncertainty quantification (UQ) to realistic physical systems. More than ever, the goal becomes to represent and propagate uncertainties from the available data to the desired results through stochastic partial differential equations (SPDEs) [3, 4].

Over the last decade, there has been tremendous progress in posing and solving SPDEs with the methods broadly speaking be divided into intrusive and non-intrusive ones. Although various non-intrusive methods, e.g., Monte Carlo simulation, regression and projection methods [5, 6, 7], enable the use of the already existing deterministic solvers to solve the stochastic problem at carefully chosen samples, this is the method of last resort since the number of realizations required is usually quite large. To address the low convergence rate, stochastic basis expansion, for instance, in polynomial
chaos and multivariate numerical integration/interpolation, such as those based on sparse grid collocation [8, 9], have been recently developed. While proven efficient on numerous problems, computational complexity of these techniques grows exponentially fast with respect to the number of input random variables: an issue known as the *curse-of-dimensionality* [10, 11]. The reason for such a fast growth is the tensor product construction of multidimensional bases (in polynomial chaos methods) or quadrature rules (in sparse grid collocation approaches) from one dimensional bases or quadrature rules, respectively. Such tensorizations, therefore, impose an explicit dependence on the random input dimensionality [12].

Addressing some of the aforementioned concerns have led to a most popular *intrusive* method, known as Galerkin-type stochastic finite element method (SFEM) [13, 14, 15, 16, 17], or spectral method, for formulating and discretizing SPDEs. In this method, the target function is projected onto a stochastic space spanned by generalized polynomial chaos basis and Galerkin projection scheme is then used to transform the original SPDE into a system of coupled deterministic equations whose size can be up to orders of magnitude larger than that of the corresponding deterministic problems. Although various iterative solvers [17, 18] have been developed to decrease the substantial computational requisite, the solution of such augmented algebraic systems is still challenging due to the increased memory and computational resources required, especially for large-scale problems. Furthermore, the coupled nature of the resulting equations makes the solution of the stochastic problem extremely complex as the number of stochastic dimensions and/or the number of expansion terms increase, the so called *curse-of-
dimensionality. For this line of approach to be successful in practice, it is crucial to have general-purpose and highly efficient numerical schemes that can beat curse-of-dimensionality for the solution of stochastic problem.

Here we develop highly efficient numerical strategies for the explicit and high precision solution of SPDE with application to problems that involve high-dimensional uncertainties. An universal construct of solution [19] to SFE equations is firstly developed. Based on this solution construct, we further develop an unified numerical strategy for solving linear and nonlinear SFE equations, where the stochastic analysis and deterministic analysis in the solution procedure can be implemented in their individual spaces. Thus, the distinguish feature of the non-intrusive method, i.e., application of available FEM codes, is readily realized in the framework of Galerkin-type method. Another beauty of the method is that curse-of-dimensionality can be circumvent to great extent since the computational cost is proportional to stochastic dimensions of the problem. In this way, our method is computationally possible to solve very high-dimensional stochastic problems encountered in science and engineering.

2. Method Description

Given a complete probability space \((\Theta, \Sigma, P)\) with sample space \(\Theta\), \(\sigma\)-algebra \(\Sigma\) on \(\Theta\), and probability measure \(P\). The formulation of a stochastic problem by SFEM is generally to find a stochastic function \(u(\theta)\) such that the following discretized SFE equation almost surely holds,

\[
K (u (\theta) , \theta) u (\theta) - F (\theta) = 0
\]  

(1)
where $K$ is an operator representing properties of the physical model under investigation, being stochastic constant matrix or stochastic matrix function related to the solution $u(\theta)$, and $F(\theta)$ is a right-hand side associated with the source terms. The stochastic nature of the problem is expressed by variable $\theta \in \Theta$. In particular, when SPDE is defined in a high dimensional stochastic space $\Theta$, i.e., $\theta := \{\theta_1, \ldots, \theta_M\}$ with a large value of $M$, considerable challenges arise in the solution of Eq. (1) due to the curse-of-dimensionality.

2.1. Solution Construct

We firstly explore an universal solution construct of Eq. (1). Although it is very natural to represent the random solution process by means of random field expansion theories, available techniques are inactive because no knowledge about $u(\theta)$ can be used. In this case, we construct the solution, $u(\theta)$, under the form

$$u(\theta) = \sum_{i=1}^{\infty} \lambda_i(\theta) d_i \quad (2)$$

where $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ are random variables and $\{d_i\}_{i=1}^{\infty}$ are discretized deterministic basis vectors. Note that several tentative constructs of $u(\theta)$ can be found in Appendix, 1. Inspired by Karhunen-Loève (KL) expansion and Principal Component Analysis (PCA) [20, 21], the following bi-orthogonal condition is further introduced

$$d_i^T d_j = \delta_{ij}, \quad E\{\lambda_i(\theta) \lambda_j(\theta)\} = \gamma_i \delta_{ij} \quad (3)$$

to improve the accuracy and efficiency of the expansion in Eq. (2), where $E\{\cdot\}$ denotes expectation operator and $\gamma_i = E\{\lambda_i^2(\theta)\}$.

We emphasize that the construct of solution is independent of the form of Eq. (1) since operator $K$ has not been emerged in the derivation. As a
result, the developed expansion in Eq. (2) is applicable for both linear and nonlinear stochastic problems. On the other hand, since the solution space is decoupled into stochastic space and deterministic space, as shown in Eq. (2), it is possible to determine \( \{\lambda_i(\theta)\}_{i=1}^{\infty} \) and \( \{d_i\}_{i=1}^{\infty} \) in their individual space, respectively. In practice, one only requires to seek a set of deterministic orthogonal vectors \( \{d_i\}_{i=1}^{\infty} \) and corresponding uncorrelated random variables \( \{\lambda_i(\theta)\}_{i=1}^{\infty} \) such that the expanded solution in Eq. (2) satisfies Eq. (1). In this way, the difficulties in expanding unknown solution process of Eq. (1) is overcome.

### 2.2. Solving Procedure

The solution construct in Eq. (2) makes it possible to solve SPDE from new viewpoint. The idea is to determine a set of couples \( \{\lambda_i(\theta), d_i\}_{i=1}^{\infty} \) such that the truncated solution in Eq. (2) satisfies Eq. (1). At this point, neither \( \{d_i\}_{i=1}^{\infty} \) nor \( \{\lambda_i(\theta)\}_{i=1}^{\infty} \) is known a priori, we hence successively determine these unknown couples \( \{\lambda_i(\theta), d_i\} \) one after another via iterative method. Note that, even for linear problem, i.e., operator \( K \) is independent of the response \( u(\theta) \), Eq. (1) is still a ‘nonlinear’ equation due to the couple of elementary event \( \theta \). In order to determine the solution of this nonlinear stochastic equation, we introduce the following stochastic increment as

\[
u_k(\theta) = u_{k-1}(\theta) + \Delta u_k(\theta)
\]  

where \( u_{k-1}(\theta) \) denotes the approximate response obtained in the previous \( k-1 \) iterations and \( \Delta u_k(\theta) \) denotes the incremental response obtained in the \( k \)-th iteration, and then let \( u_{k-1}(\theta) = \sum_{j=1}^{k-1} \lambda_j(\theta)d_j \) and \( \Delta u_k(\theta) = \lambda_k(\theta)d_k \). From this point, we consider \( \lambda_k(\theta)d_k \) as the stochastic increment of solution.
In this sense, the summation of all $k$ increments $\{\lambda_i(\theta)d_i\}_{i=1}^k$, which is similar as the summation of all components as shown in Eq. (2), constitute the solution $u(\theta)$. Solution $u(\theta)$ can be approximated by truncating the increment after $k$ terms, and the more stochastic increment $\lambda_k(\theta)d_k$ retains, the more accurate approximation can be obtained.

Several strategies on the solution of Eq. (1) by Eq. (4) are discussed in Appendix, 2. Here we develop a highly efficient and practical solving procure. Substituting the stochastic increment introduced in Eq. (4) into Eq. (1) yields

$$K(u_{k-1}, \theta)(u_{k-1}(\theta) + \lambda_k(\theta)d_k) - F(\theta) = 0 \quad (5)$$

In the stochastic case, the iterative methods can not be readily employed to determine the stochastic increment since it is difficult to compute the ‘inverse’ of the stochastic matrix $K(\theta)$. We employ stochastic Galerkin projection to convert the stochastic increment into deterministic quantity, so that the iterative method can be used. For this purpose, suppose that the approximate solution $u_{k-1}(\theta)$ has been obtained. Then if random variable $\lambda_k(\theta)$ has been determined (or given as an initial value), $d_k$ can be determined by using stochastic Galerkin orthogonality. This corresponds to

$$E\{\lambda_k(\theta)[K(u_{k-1}, \theta)(u_{k-1}(\theta) + \lambda_k(\theta)d_k) - F(\theta)]\} = 0 \quad (6)$$

which is a linear equation about $d_k$ and the size is same to that of the original stochastic problem in Eq. (1). Details for the solution of Eq. (6) can be found in Appendix, 3.

Once $d_k$ has been determined through Eq. (6), random variable $\lambda_k(\theta)$ can be subsequently updated via the similar procedure. This requires to multiply
$d_k$ on both sides of Eq. (5) to yield

$$d_k^T [K (u_{k-1}, \theta) (u_{k-1} (\theta) + \lambda_k (\theta) d_k) - F (\theta)] = 0 \quad (7)$$

Analytic methods for the solution of Eq. (7) are to represent random variable $\lambda_k(\theta)$ in terms of a set of stochastic basis functions $\{H_m(\theta)\}$, known as polynomial chaos expansion (PCE) [13, 14]. Details for solving Eq. (7) with PCE can be found in Appendix, 3. However, curse-of-dimensionality hinders the application of such methods in low stochastic dimensional and/or small-scale stochastic problems. Here we develop a simulation-based method to determine $\lambda_k(\theta)$. For the positive definite matrix $K(u, \theta)$ and each realization of $\{\theta^{(m)}\}_{m=1}^N$, $\lambda_k(\theta^{(m)})$ can be obtained by solving Eq. (7) as

$$\lambda_k(\theta^{(m)}) = \frac{d_k^T [F(\theta^{(m)}) - K(u_{k-1}, \theta^{(m)})u_{k-1}(\theta^{(m)})]}{d_k^T K(u_{k-1}, \theta^{(m)})d_k} \quad (8)$$

It is important to note that since Eq. (8) has become a one-dimensional linear algebraic equation about $\lambda_k(\theta^{(m)})$, the total computational cost for determining $\{\lambda_k(\theta^{(m)})\}_{m=1}^N$ is low even for problems with very high stochastic dimensions. By determining random variable $\lambda_k(\theta)$ from a set of its realizations, the curse-of-dimensionality can be circumvent to great extent because the computation in Eq. (8) is insensitive to the dimensions of $\theta$. Hence, the developed method will be particularly promising for a wide range of high-dimensional stochastic problems encountered in science and engineering.

We also note that, a good initial couple of $(\lambda_1 (\theta), d_1)$ is required for the solution of Eq. (5). This can be accomplished by solving the following simplified-version of stochastic nonlinear equation

$$K (\lambda_1 (\theta) d_1, \theta) \lambda_1 (\theta) d_1 - F (\theta) = 0 \quad (9)$$
Procedures in Eq. (6) and Eq. (7) can be also used to solve Eq. (9). Further details can be found in Appendix, 4 and next section.

2.3. Numerical Algorithm

The resulting procedure for approximating the solution of Eq. (1) is summarized in Algorithm 1, which includes a double-loop iteration procedure. The inner loop, which is from step 9 to 14, is used to determine the couple of \((\lambda_k(\theta), d_k)\), while the outer loop, which is from step 4 to 16, corresponds to recursively building the set of couples and thereby the approximate solution \(u_k(\theta)\). With an initial random variable \(\lambda_k^{(0)}(\theta)\) given in step 8, \(d_k^{(j)}\) can be determined in step 10 and 11, where superscript \(j\) represents the \(j\)-th round of iteration. With the obtained \(d_k^{(j)}\), random variable \(\lambda_k^{(j)}(\theta)\) is then updated in step 12 and 13. Note that both \(d_k^{(j)}\) and \(\lambda_k^{(j)}(\theta)\) require orthogonalization such that the bi-orthogonal condition in Eq. (3) holds along the whole process, here we employ the Gram-Schmidt Orthogonalization scheme in step 11 and 13. The outer-loop iteration then generates a set of couples such that the approximate solution in step 15 satisfies Eq. (1). Note that, for nonlinear stochastic problems, additional two steps need to be performed, i.e., solving \((\lambda_1(\theta), d_1)\) in step 2 and updating \(K(u_{k-1}(\theta), \theta)\) in step 6.

3. Applications on SPDEs

In this section, our method is applied to linear and nonlinear elliptic SPDEs to show its capacity for efficiently solving high stochastic dimensional problems. Details on determination of initial couple of \((\lambda_1(\theta), d_1)\) in Eq. (9) for nonlinear stochastic problems are also demonstrated.
Algorithm 1 Algorithm for Linear, Nonlinear and High Dimensional Stochastic Finite Element Equations

1: if nonlinear problems then
2: compute \((\lambda_1(\theta), d_1)\) by solving stochastic nonlinear equation Eq. \([9]\);
3: end if
4: while \(E\{\lambda_{k-1}^2(\theta)\} / \sum_{i=1}^{k-1} E\{\lambda_i^2(\theta)\} > \varepsilon_1\) do
5: if nonlinear problems then
6: update \(K(u_{k-1}(\theta), \theta)\);
7: end if
8: initial \(\lambda_k^{(0)}(\theta)\);
9: repeat
10: compute \(d_k^{(j)}\) by solving Eq. \([6]\);
11: orthogonalization \(d_k^{(j)} \perp d_i, i = 1, \cdots, k - 1\) and unitization \(d_k^{(j)} = d_k^{(j)} / \|d_k^{(j)}\|\);
12: compute \(\lambda_k^{(j)}(\theta)\) by Eq. \([8]\);
13: orthogonalization \(\lambda_k^{(j)}(\theta) \perp \lambda_i(\theta), i = 1, \cdots, k - 1\);
14: until \(\|d_k^{(j)} - d_k^{(j-1)}\| < \varepsilon_2\)
15: \(u_k(\theta) = \sum_{i=1}^{k-1} \lambda_i(\theta) d_i + \lambda_k(\theta) d_k, k \geq 2\);
16: end while

3.1. Problem Setup

Consider an elliptic PDE

\[-\nabla (a(u, x, \theta) \nabla u(x, \theta)) + b(x, \theta) u(x, \theta) = f(x, \theta)\]  (10)
defined in domain $\Omega$ with Dirichlet boundary condition $u = 0$ on the boundary of $\Omega$. The variational form and SFE equation of Eq. (10) can be found in Appendix, 5. For linear case, $a(u,x,\theta) = a(x,\theta)$ is independent of the solution $u(x,\theta)$, and we set $a(u,x,\theta) = a(x,\theta) u(x,\theta)$ for nonlinear case. Generally, the random field $a(x,\theta)$ (and $b(x,\theta)$, $f(x,\theta)$) can be represented as

$$a(x,\theta) = \sum_{i=0}^{M} \xi_i(\theta) a_i(x)$$ (11)

where $\xi_0(\theta) \equiv 1$, $a_0(x)$ is the mean function and $M$ is the number of retained terms. Expansion in Eq. (11) can be implemented by various methods, such as Karhunen-Loève expansion and Polynomial Chaos expansion [14, 13]. Here we introduce high dimensional stochastic space $\{\xi_i(\theta)\}_{i=1}^{M}$ with large $M$. Note that Eq. (11) is not necessary condition for our method since high-dimensional stochastic space arises in many real-world problems.

3.2. Linear SPDE

In the linear case of Eq. (10), i.e., $a(u,x,\theta) = a(x,\theta)$, by virtue of Eq. (11), $K(u(\theta),\theta)$ in Eq. (1) becomes (details refer to Appendix, 5)

$$K(\theta) = \sum_{i=0}^{M} \xi_i(\theta) K_i$$ (12)

where $\xi_0(\theta) \equiv 1$. Substituting Eq. (12) into Eq. (6) yields

$$\left(\sum_{i=0}^{M} h_{ikk} K_i\right) d_k = f_k - \sum_{i=0}^{M} \sum_{j=1}^{k-1} h_{ijk} K_i d_j$$ (13)

where $h_{ijk} = E\{\xi_i(\theta) \lambda_j(\theta) \lambda_k(\theta)\}$, $f_k = E\{\lambda_k(\theta) F(\theta)\}$ are easy to compute with low computational cost. Since the size of $\sum_{i=0}^{M} h_{ikk} K_i$ is same as that
of the original stochastic matrix $K(\theta)$, the method can be applied to large-scale problems without extra computational burden. Further, by substituting Eq. (12) into Eq. (8), we have

$$
\lambda_k(\theta^{(m)}) = \frac{d_k^T F(\theta^{(m)}) - \sum_{j=1}^{k-1} \lambda_j(\theta^{(m)}) \xi(\theta^{(m)}) g_{jk}}{\xi(\theta^{(m)}) g_{kk}}
$$

where $\xi(\theta) = (\xi_0(\theta), \cdots, \xi_M(\theta))$, and $g_{jk} = d_k^T K_j d_j$. $\lambda_k(\theta)$ can thus be determined from the set of its samples in Eq. (14). As mentioned earlier, for each $\theta^{(m)}$, $g_{kk}$ is invariant and $\xi(\theta)$ is insensitive to the dimensionality $M$. Even for large value of $M$, computational cost in Eq. (14) is very low, and curse-of-dimensionality can thus be circumvent to great extent. In this sense, our method will be particularly appropriate for the solution of high-dimensional stochastic problems.

3.3. Nonlinear SPDE

In the nonlinear case of Eq. (10), we assume $a(u, x, \theta) = a(x, \theta) u(x, \theta)$, then $K(u(\theta), \theta)$ in Eq. (1) becomes (details refer to Appendix, 5)

$$
K(u_k, \theta) = \sum_{j=1}^{k} \lambda_j(\theta) \sum_{i=0}^{M_1} \xi_i(\theta) K_i(d_j) + \sum_{l=0}^{M_2} \eta_l(\theta) K_l
$$

where $\xi_0(\theta) = \eta_0(\theta) \equiv 1$. The procedure in Eq. (6) and Eq. (7) can be also used to solve Eq. (9): for fixed $\lambda_1(\theta)$, $d_1$ is determined by

$$
\left( \sum_{i=0}^{M_1} c_{1i} K_i(d_1) + \sum_{l=0}^{M_2} c_{2l} K_l \right) d_1 - f_1 = 0
$$

where $c_{1i} = E \{ \xi_i(\theta) \lambda_1^2(\theta) \}$, $c_{2l} = E \{ \eta_l(\theta) \lambda_1^2(\theta) \}$, $f_1 = E \{ \lambda_1(\theta) F(\theta) \}$. For fixed $d_1$, $\lambda_1(\theta)$ is determined by

$$
\beta_1(\theta^{(m)}) \lambda_1^2(\theta^{(m)}) + \beta_2(\theta^{(m)}) \lambda_1(\theta^{(m)}) - d_1^T F(\theta) = 0
$$
where $\beta_1(\theta) = \sum_{i=0}^{M_1} \xi_i(\theta) d_1^T K_i (d_1) d_1$, $\beta_2(\theta) = \sum_{l=0}^{M_2} \eta_l(\theta) d_1^T K_l d_1$.

For nonlinear problems, the determination of initial couple of $(\lambda_1(\theta), d_1)$, which is required in advance, can be efficiently accomplished by means of step 8 to 14 in Algorithm 1. We note that once Eq. (16) and Eq. (17) are used to determine $d_1^{(j)}$ in step 10 and $\lambda_1^{(j)}(\theta)$ in step 12, respectively, orthogonalizations in step 11 and 13 are no longer required since $(\lambda_1(\theta), d_1)$ is the first term. Various methods are available to solve deterministic nonlinear finite element equation Eq. (16) and the one-dimensional nonlinear algebraic equation can be also computed efficiently. Once $(\lambda_1(\theta), d_1)$ is obtained, then $(\lambda_k(\theta), d_k)$ can be readily computed by means of Eq. (5), which has reduced to the linear problem.

4. Numerical Applications

Implementation of our method is illustrated with aid of three numerical applications in this section. The first application involves a linear stochastic Euler Bernoulli beam, and the second one considers a nonlinear stochastic Burgers equation. These two examples are served to verify our method, and also to demonstrate the universal solution construct for different types of problems. The third example, which is employed to illustrate the capacity of our method for dealing with high-dimensional stochastic problem, considers a linear Kirchhoff-Love plate with stochastic material properties defined in high dimensional stochastic space. For all problems considered, convergence criteria in Algorithm 1 are set as $\varepsilon_1 = \varepsilon_2 = 10^{-6}$.
4.1. Stochastic Euler Bernoulli Beam

Consider a Euler Bernoulli beam shown in Fig. 1 of length $L$, clamped at one end and subjected to a deterministic transverse load $P(x)$. Here, we consider the transverse displacement of the beam. It is assumed that the bending rigidity $\omega(x, \theta) = EI(x, \theta)$ is the realization of a Gaussian random process with covariance function $C_{\omega\omega}(x_1, x_2) = \sigma^2_\omega e^{-|x_1 - x_2|}$ and mean function $\bar{\omega}(x) = 1$ indexed over the spatial domain occupied by the beam.

Parameters in this problem are $L = 1$, $P(x) = 1$, $\sigma^2_\omega = 0.3$.

Results

We use KL expansion with a five-term truncation to represent random field $\omega(x, \theta)$ to arrive at the linear SFE equation. Given the initial random variable samples $\{\lambda_k^{(0)}(\theta^{(m)})\}_{m=1}^{1 \times 10^4}$, the corresponding probability density functions (PDF) of random variables $\{\lambda_k(\theta)\}$ can be determined by mensa of the linear case of Algorithm 1 as shown in Fig. 2. In this example, the number of couples $\{\lambda_k(\theta), d_k\}$ that constitutes the stochastic response is chosen as $k = 4$. It is seen from Fig. 2 that, with increasing of the number of couples, the range of the corresponding random variable is more closely...
approaching to zero, indicating that the contribution of the higher order random variables to the approximate solution decays dramatically. Appendix, 6 compare the convergence in step 4 of Algorithm 1 and the resulting approximate PDF of point $A$ in Fig. 1 with those from standard Monte Carlo simulation. Obviously, the results of the four-term approximation is in very good accordance with that from the Monte Carlo simulation. According to our experience, further increasing the number of couples will not significantly improve the result because the first few couples has make the series in Eq. (2) converge to solution of the problem. This example demonstrates the success of our proposed solution construct and Algorithm 1 for the linear stochastic problems.

4.2. Stochastic Burgers Equation

Deterministic Burgers equation is an important PDE occurring in various areas [23], such as fluid mechanics, nonlinear acoustics and gas dynamics. To better model the randomness inherent of turbulence, the following stochastic
Burgers equation [23, 24, 25] is introduced,

\[ u(x, \theta) \frac{\partial u(x, \theta)}{\partial x} = d \frac{\partial^2 u(x, \theta)}{\partial x^2} - \gamma \frac{\partial \rho(x, \theta)}{\partial x} \]

where \( \gamma \frac{\partial \rho(x, \theta)}{\partial x} \), which can be obtained from Eq. (10), represents modeled randomness. Parameters in this problem are \( x \in [0, 1], d = 1, \gamma = 1 \) and \( \rho(x, \theta) \) is a Gaussian random process with covariance function \( C_{\rho\rho}(x_1, x_2) = \min(x_1, x_2) \) and mean function \( x^2 \).

**Results**

Similar to the previous example, we use five-term KL truncation to expand random field \( \rho(x, \theta) \) (details can be found in Appendix, 7), and then formulate the nonlinear SFE equation. \( \{\lambda_k^{(0)}(\theta^{(m)})\}_{m=1}^{1 \times 10^4} \) is given for each initial random variable samples of \( \lambda_k(\theta) \). After determining the initial couple \( (\lambda_1(\theta), d_1) \) by use of Eq. (9), the set of deterministic basis vectors \( \{d_k\} \) and corresponding PDF of random variables \( \{\lambda_k(\theta)\} \) can be determined by means of the nonlinear case of Algorithm 1. As shown in Fig. 3, there needs six re-

![Figure 3: Solution2: \( \{d_k\}_{k=1}^{6} \) (Left) and PDFs of \( \{\lambda_k(\theta)\}_{k=1}^{6} \) (Right)]
tained items \( \{d_i\}_{i=1}^6 \) and \( \{\lambda_i\}_{i=1}^6 \) to achieve target accuracy. Convergence of the procedure in Algorithm 1 and the resulting approximate PDF of point \( x = 0.2 \) are compared with those from standard Monte Carlo simulation, as shown in Appendix, 7. Similar as the linear case, Algorithm 1 leads to good convergence for nonlinear problems. Analogously, conclusions from the previous linear problem is also appropriate for the nonlinear case since they have the same solution form. The proposed algorithm and the solution construct are, therefore, once again verified by the nonlinear stochastic problem. We also note that, as a direct result from the developed solution construct, \( d_k \) and \( \lambda_k(\theta) \) can be solved in their individual spaces, and as a result, the mature deterministic nonlinear FEM codes can be readily embedded into step 10 in Algorithm 1 to determine the set of base vectors \( d_i \), or equivalently, deterministic nonlinear ODE codes can be incorporated into step 12 in Algorithm 1 to compute set of \( \lambda_k(\theta) \). Therefore, the proposed algorithm is particularly appropriate for the solution of large-scale complex stochastic problems of practical interest without recompiling numerous new programs.

4.3. Stochastic Kirchhoff-Love Plate

Consider a Kirchhoff-Love thin plate with length \( L \), width \( D \), thickness \( t \) and Poisson’s ratio \( \nu \), which is subjected to a deterministic distributed load \( q(x, y) \) and simply supported on four edges, as shown in Fig.4. For the sake of simplicity, we neglect self-weight of the plate and assume Young’s modulus \( E(x, y, \theta) \) as the realization of a Gaussian random field with mean function \( \mu_E \) and covariance function \( C_{EE}(x_1, y_1; x_2, y_2) = \sigma_E^2 e^{-|x_1-x_2|/l_x-|y_1-y_2|/l_y} \) with correlation lengths \( l_x, l_y \). Parameters are set as \( L = 4m, D = 2m, t = 0.05m, \nu = 0.3, q = -10kN/m^2, \mu_E = 210GPa, \sigma_E = 22GPa, l_x = 2m, l_y = 4m. \)
The above model definition and Fig. 4 are from [26], thereby explain.

As the previous examples, Young’s modulus \( E(x, y, \theta) \) is represented by KL expansion as

\[
E(x, y, \theta) = \mu_E + \sum_{i=1}^{M_{\text{terms}}} \xi_i(\theta) E_i(x, y)
\]

(18)

where \( M_{\text{terms}} \) is the number of truncated terms. We test our algorithm and extend this problem to high dimensional stochastic space by taking different large values of \( M_{\text{terms}} \).

Results 1

We use Kirchhoff-Love finite element theory of plates and four-node Melosh-Zienkiewicz-Cheung (MZC) element to divide the plate into 861 nodes and 800 elements. The unknown node displacement \( u(\theta) \) is introduced as \( u(\theta) = [u_\omega(\theta), u_x(\theta), u_y(\theta)]^T \), which are the vertical displacement, rotations in \( x \) and \( y \) axes, respectively, then 2583 degrees of freedom are defined, as shown in Fig. 4. For the case of low stochastic dimensions, i.e., value of \( M_{\text{terms}} \) is relatively small, further numerical details of the expansion in Eq. (18), obtained
\{\lambda_k(\theta), d_k\}, convergence of step 4 in Algorithm 1 and the resulting approximate PDF of \(u(\theta) = [u_\omega(\theta), u_x(\theta), u_y(\theta)]^T\) at the node 325, are compared with those from standard Monte Carlo simulation. (results are given in Appendix, 8). Conclusions on the previous two examples still work here and this example further supports the validation of our strategy for low-dimensional stochastic problems.

**Results 2**

Our final goal is to apply our method to the case of high stochastic dimensions. Here we introduce high dimensional space by taking large values of \(M_{terms}\) and only check the efficiency of Algorithm 1 regardless of the accuracy since a small stochastic dimension can satisfy the target accuracy. A personal laptop is used to test the stochastic dimensions \(M_{terms}\) ranging from 100 to 6000 (which may be even larger). Results are shown in Fig.5, where the blue line denotes the time spent on the solution of linear case of Eq. (1). It is clear that our algorithm is highly efficient for problems with high stochastic dimensions. As has been mentioned in Eq. (8) or Eq. (14), the method is insensitive to the dimensions, which is illustrated by the fact that computational cost increases slowly with the dramatically increasing of the stochastic dimensions. This may interpret the reason of the circumvent of the curse of dimensionality. Red line in Fig.5 denotes the cost for assembling the deterministic matrix of \(K(\theta)\) in linear case Eq. (1), or parallel, \(\{K_i\}\) in Eq. (12). It is seen that the assembling cost is linear to the dimension \(M_{terms}\), since the total assembling cost will be approximately \(M_{terms}t_0\) if the assembling cost of single \(K_i\) is \(t_0\). The black line in Fig.5 denotes the total time of solution time and assembling time. It is seen that the assembling
time dominates the total time needed, indicating that the efficiency of our algorithm is heavily dependent on the assembling cost. We also emphasize that, discretization of original problem and sample generation of \( \{\theta_i\}_{i=1}^{M_{terms}} \), which requires much computational cost in the solution of the problem, also affect the time required of our method. Therefore, efficiency of our method can be further improved through highly efficient assembly or discretization.

5. Conclusions

We develop highly efficient numerical strategies for the high precision solution of SPDE with application to problems involving uncertainties. In our method, an universal construct of solution to stochastic problems is firstly explored. Based on this solution construct, we then develop an unified nu-
merically algorithm for solving linear and nonlinear SFE equations, where the stochastic analysis and deterministic analysis in the solution procedure can be implemented in their individual spaces. One of the most challenging issue in stochastic analysis, known as *curse-of-dimensionality*, can be circumvent with great success as the computational cost of our method is almost proportional to the stochastic dimensionality of the problem. Since the available FEM and ODE codes can be readily incorporated into the computational procedure, our method is particularly appropriate for large-scale and high-dimensional stochastic problems of practical interest. It is also worthwhile pointing out that although the proposed algorithm is only applied to time-independent problems, it can be straightforwardly extended to time-dependent problems. In this sense, our algorithm is general-purpose and has great potential in the uncertainty quantification problems in science and engineering. In the follow-up research, we hopefully further improve the theoretical analysis of proposed algorithm [27] and apply the algorithm to a wider range of problems, such as complex Bayesian inference [28] and weather prediction [29].

Appendix

1. Construct of Solutions to SFE equations

Consider stochastic finite element (SFE) equation under the form Eq. (1). Since the solution \( u(\theta) \) of Eq. (1) is a random field, random field expansion theory can be readily used to represent \( u(\theta) \). Here we choose the Karhunen-Loève (KL) expansion theory [14] and Principal Components Analysis (PCA) [21] due to the optimal convergence. Note that PCA can be considered as a
kind of discrete KL expansion in some cases. It is known that an arbitrary random field $\omega(x, \theta)$ indexed on a bounded domain $D$, having mean function $f_0(x)$ and finite variance function $C(x_1, x_2)$, can be decomposed using an infinite KL series as

$$
\omega(x, \theta) = \sum_{i=0}^{\infty} \xi_i(\theta) f_i(x) \tag{19}
$$

where $\xi_0(\theta) \equiv 1$, $\{\xi_i(\theta)\}_{i=1}^{\infty}$ is a set of uncorrelated random variables, and $f_i(x)_{i=1}^{\infty} \in \mathbb{R}^d$ are the eigenfunctions of covariance $C(x_1, x_2)$ of the random field $\omega(x, \theta)$, obtained from solving the homogeneous Fredholm integral equation of the second kind

$$
\int_D C(x_1, x_2) f_i(x_1) \, dx_1 = \gamma_i f_i(x_2)
$$

The expansion in Eq. (19) has generated great interest due to its bi-orthogonal property, i.e., both the deterministic basis functions and the random coefficients are orthogonal

$$
\int_D f_i(x) f_j(x) \, dx = \delta_{ij}, \quad E\{\xi_i(\theta) \, \xi_j(\theta)\} = \gamma_i \delta_{ij}
$$

where $\delta_{ij}$ is the Kronecker delta. However, since KL expansion requires knowledge of the covariance of the random field being expanded, Eq. (19) is only of theoretically importance and can not be used for representing the unknown solution random field $u(\theta)$. In order to circumvent this problem, it is natural to express the eigenfunctions $f_i(x)$ in Eq. (19) in terms of a set of unknown completely discretized basis vectors $\{u_j\}_{j=1}^{\infty}$ as $f_i(x) = \sum_{j=1}^{\infty} c_{ij} u_j$, where $c_{ij}$ are the corresponding coefficients. By substituting the expanded $f_i(x)$ into Eq. (19), the solution $u(\theta)$ of Eq. (1) can be expanded as

$$
u(\theta) = \sum_{j=1}^{\infty} \eta_j(\theta) u_j \tag{20}$$
where \( \{ \eta_j(\theta) \}_{j=1}^{\infty} \) is the random variables to be determined. We emphasize that, unlike the KL expansion, both the deterministic complete basis \( u_j \) and the random coefficients \( \eta_j(\theta) \) in Eq. (20) do not satisfy the orthogonal conditions. Hence, the expansion in Eq. (20) is not an optimal formula when compared to KL expansion. Based on [19, 30], its accuracy and efficiency are possible to be further improved [30]. Intuitively, it is expected to seek a bi-orthogonal formula such that efficiency of the expansion is comparable to that of KL expansion or PCA, while the construct of the expansion is more suitable for the solution of SFE equations. For this purpose, we introduce the following lemma

**Lemma 1.** If the random field \( \omega(x, \theta) \) defined in Eq. (19), can be represented in terms of an arbitrary set of complete unit orthogonal functions \( \{ g_i(x) \}_{i=1}^{\infty} \) and a set of uncorrelated random variables \( \{ \lambda_i(\theta) \} \) as

\[
\omega(x, \theta) = \sum_{i=1}^{\infty} \lambda_i(\theta) g_i(x)
\]

then,

\[
\int_D C(x_1, x_2) g_i(x_1) \, dx_1 = \kappa_i g_i(x_2)
\]

where \( C(x_1, x_2) \) is the covariance of \( \omega(x, \theta) \).

**Proof 1.** It follows directly from Lemma 1 that

\[
E \{ \lambda_i(\theta) \omega(x, \theta) \} = \sum_{j=1}^{\infty} E \{ \lambda_i(\theta) \lambda_j(\theta) \} g_j(x) = \kappa_i g_i(x)
\]

On the other hand, since \( \{ g_i(x) \}_{i=1}^{\infty} \) is orthogonal, we have

\[
\lambda_i(\theta) = \int_D \omega(x, \theta) g_i(x) \, dx
\]
Thus,

\[ E \{ \lambda_i(\theta) \omega(x, \theta) \} = \int_D E \{ \omega(y, \theta) \omega(x, \theta) \} g_i(y) dy = \int_D C(y, x) g_i(y) dy \]

By comparing the above two equations, we have

\[ \int_D C(y, x) g_i(y) dy = \kappa_i g_i(x) \]

Lemma 1 illustrates that the expansion of a random field has comparable convergence as that of the KL expansion or PCA. Thus, according to Lemma 1, the solution of SFE equation in Eq. (1) can be constructed under the form

\[ u(\theta) = \sum_{i=1}^{\infty} \lambda_i(\theta) d_i \tag{21} \]

where the deterministic discretized basis vectors \( \{d_i\}_{i=1}^{\infty} \) and the random variables \( \{\lambda_i(\theta)\}_{i=1}^{\infty} \) are both orthogonal, i.e.,

\[ d_i^T d_j = \delta_{ij}, \quad E \{ \lambda_i(\theta) \lambda_j(\theta) \} = \kappa_i \delta_{ij} \tag{22} \]

It will be shown that the new expansion in Eq. (21) is of practically importance in the solution of SFE equation, as opposed to the KL expansion. In this way, the difficulties of KL expansion in expanding the unknown solution random field of Eq. (1) can be overcome. It is also noted that, although the expansion in Eq. (20) and that in Eq. (21) have the same formula, the bi-orthogonal property makes the construct of expansion as given in Eq. (21) more suitable for determining the solution of SFE equation. One only requires to seek a set of deterministic orthogonal vectors \( \{d_i\}_{i=1}^{\infty} \) and the corresponding uncorrelated random variables \( \{\lambda_i(\theta)\}_{i=1}^{\infty} \) such that the expanded solution in Eq. (21) satisfies Eq. (1).
2. Solving nonlinear SFE equation

In order to solve nonlinear SFE equations in Eq. (1), a natural idea is to substitute \( u_k(\theta) = u_{k-1}(\theta) + \lambda_k(\theta)d_k \) into Eq. (1) to yield

\[
K\left(u_{k-1}(\theta) + \lambda_k(\theta)d_k, \theta\right)(u_{k-1}(\theta) + \lambda_k(\theta)d_k) = F(\theta)
\]

which means a large number of nonlinear equations have to be solved for each couple of \((\lambda_i(\theta), d_i)\) shown in Fig. 6a in the solution of nonlinear SFE equation. This is obviously a disaster for large-scale problems. In order to circumvent this difficulty, we borrow the idea from deterministic nonlinear FEM by ignoring the effect of incremental solution in each iteration, known as 'linearization'. Thus, Eq. (23) can be simplified as

\[
K\left(u_{k-1}(\theta), \theta\right)(u_{k-1}(\theta) + \lambda_k(\theta)d_k) = F(\theta)
\]

which is a linear stochastic equation and the computation of operator \(A\) requires the knowledge of \(u_{k-1}(\theta)\). In this case, an additional couple of \((\lambda_0(\theta), d_0)\) is then required to proceed the iteration. As demonstrated in

\[\text{Figure 6: Nonlinear iterative schema}\]
Fig. 6, \((\lambda_0(\theta), d_0)\) is an initial couple specified by users. With different choices of \((\lambda_0(\theta), d_0)\), the resulting set of couples \(\{\lambda_i(\theta), d_i\}_{i=1}^{k}\) varies significantly, and may lead to the convergence issue. In general, it is difficult to choose an appropriate initial couple in advance, and thereby a practical initialization scheme needs to be considered. In fact, Eq. (24) can be slightly modified for the nonlinear SFE equation as long as one specify the initial couple from \((\lambda_1(\theta), d_1)\), instead of \((\lambda_0(\theta), d_0)\), as demonstrated in Fig. 6c. In this context, the initialization \((\lambda_1(\theta), d_1)\) can be approximately determined by solving the simplified version of Eq. (23) as

\[
K(\lambda_1(\theta), \theta) \lambda_1(\theta) d_1 = F(\theta)
\]

3. Solving the couple \((\lambda_k(\theta), d_k)\)

\(d_k\) can be determined from solving

\[
E\{\lambda_k(\theta)[K(u_{k-1}, \theta)(u_{k-1}(\theta) + \lambda_k(\theta)d_k) - F(\theta)]\} = 0
\]

which corresponds to

\[
C_k d_k = Q_k
\]

where \(C_k = E\{\lambda_k^2(\theta)K(u_{k-1}, \theta)\}\), \(Q_k = E\{\lambda_k(\theta)[F(\theta) - K(u_{k-1}, \theta)u_{k-1}(\theta)]\}\). The size of \(C_k\) is same to that of the original \(K(\theta)\), and therefore can be solved by available methods of FEM [1]. Random variable \(\lambda_k(\theta)\) is then updated via the similar procedure as

\[
(d_k^T K(u_{k-1}, \theta) d_k) \lambda_k(\theta) = d_k^T [F(\theta) - K(u_{k-1}, \theta)u_{k-1}(\theta)]
\]

Here we show the application of polynomial chaos expansion (PCE) [14] for determining \(\lambda_k(\theta)\): random variable \(\lambda_k(\theta)\) can be represented in terms of a
set of stochastic basis functions $H_m(\theta)$ as

$$
\lambda_k(\theta) = \sum_{m=0}^{P-1} c_{km} H_m(\theta)
$$

(29)

where $c_{km}$ are coefficients of $H_m(\theta)$, and $P$ is the number of expansion terms which is related to approximation accuracy. Substituting Eq. (29) into Eq. (28) and utilizing stochastic Galerkin criterium yields

$$
R_k c_k = S_k
$$

(30)

where $S_{kn} = d_k^T E \{H_n(\theta) [F(\theta) - K(u_{k-1}, \theta) u_{k-1}(\theta)]\}$, $m, n = 0, \ldots, P - 1$ and $R_{kmn} = d_k^T E \{H_n(\theta) K(u_{k-1}, \theta) H_m(\theta)\} d_k$. The number of $H_m(\theta)$ is $P = (M + p)!/(M!p!)$ if the dimension of $\theta$ is $M$ and the order of PC expansion is $p$. In this case, the computational burden for determining $c_k$ will be significantly prohibitive for high stochastic dimensions of $\theta$ and large order of $p$.

4. Solving the couple $(\lambda_1(\theta), d_1)$

Similar as the determination of $d_k$, the following Eq. (26)-like equation can be formulated in the determination of $d_1$,

$$
E \{\lambda_1(\theta) [K(\lambda_1(\theta) d_1, \theta) \lambda_1(\theta) d_1 - F(\theta)]\} = 0
$$

(31)

which can be simplified as

$$
C_1 d_1 = Q_1
$$

(32)

where $C_1 = E \{\lambda_1(\theta) K(\lambda_1(\theta) d_1, \theta) \lambda_1(\theta)\}$, $Q_1 = E \{\lambda_1(\theta) F(\theta)\}$. Obviously, available methods for nonlinear FEM [2] can be readily used to solve
Eq. (32). Similarly, the following Eq. (28)-like equation can be obtained to determine $\lambda_1$,

$$d_1^T K (\lambda_1 (\theta) d_1, \theta) d_1 \lambda_1 (\theta) - d_1^T F (\theta) = 0 \quad (33)$$

which can be considered as a nonlinear ODE and be solved by numerous available methods [22].

5. SFE equation of the SPDE

The variational formulation of SPDE can be written as: find $u \in V (\theta)$ such that

$$A (u, v, \theta) = B (v, \theta), \forall v \in V (\theta) \quad (34)$$

holds, where $A (u, v, \theta) = \int_\Omega (a (u) \nabla u \nabla v + buv) \, dx$, $B (v, \theta) = \int_\Omega f v \, dx$. Based on Eq. (34), Eq. (1) can be obtained by virtue of Galerkin method

$$K_{mn} (u, \theta) = \int (a (u) \nabla \varphi_m \nabla \varphi_n + b \varphi_m \varphi_n) \, dx, F_n (\theta) = \int f \varphi_n \, dx, m, n = 1, \cdots, N \quad (35)$$

The random field $a (x, \theta)$ and $b (x, \theta)$ are represented as

$$a (x, \theta) = \sum_{i=0}^{M_1} \xi_i (\theta) a_i (x), b (x, \theta) = \sum_{l=0}^{M_2} \eta_l (\theta) b_l (x) \quad (36)$$

For the linear case, substituting $a (u, x, \theta) = a (x, \theta)$ and Eq. (36) into Eq. (35) yields

$$K (\theta) = \sum_{i=0}^{M} \xi_i (\theta) K_i \quad (37)$$

where we let $\{ \xi_i (\theta) \}_{i=M_1+1}^{M} = \{ \eta_i (\theta) \}_{i=1}^{M_2}$ and $K_{i,mn} = \int a_i (x) \nabla \varphi_m \nabla \varphi_n \, dx, i = 1, \cdots, M_1$, $K_{i,mn} = \int b_i (x) \nabla \varphi_m \nabla \varphi_n \, dx, i = M_1 + 1, \cdots, M$, $K_{0,mn} = \int (a_0 (x) \nabla \varphi_m \nabla \varphi_n + b_0 (x) \varphi_m \varphi_n) \, dx$. 28
For the nonlinear case, similar to Eq. (37), substituting \( a(u, x, \theta) = a(x, \theta) u(x, \theta) \) and Eq. (36) into Eq. (35) yields

\[
K(u_k, \theta) = \sum_{j=1}^{k} \lambda_j(\theta) \sum_{i=0}^{M_1} \xi_i(\theta) K_i(d_j) + \sum_{l=0}^{M_2} \eta_l(\theta) K_l
\]

where \( K_i(d_j) = \int a_i(x) d_j \nabla \varphi_m \nabla \varphi_n dx \), \( K_{t,mn} = \int b_l(x) \varphi_m \varphi_n dx \).

6. Stochastic Euler Bernoulli Beam

A five-term KL truncation is used to expand random bending rigidity \( \omega(x, \theta) \), as shown in Fig.7.

![Figure 7: Five-term eigenfunctions (Left) and eigenvalues (Right)](image)

Fig.8 (Left) shows a good convergence in step 4 of Algorithm 1, demonstrating the efficiency of our algorithm. From Fig.8 (Right), it is seen that accuracy of the resulting approximate PDF of point \( A \) increases with the number of components in \( u(\theta) \), and the result from four-term approximation is in a very good accordance with that of Monte Carlo simulations with \( 1 \times 10^5 \) realizations.
7. Stochastic Burgers Equation

A five-term KL truncation is used to expand the modeled randomness $\rho(x, \theta)$, as shown in Fig.9.

From Fig.10 (Left), it is clear that Algorithm 1 has good convergence for nonlinear problems. Fig.10 (Right) compares the resulting approximate PDF of point $x = 0.2$ and that from the Monte Carlo simulation with $1 \times 10^5$ s realizations, which once again verifies the accuracy of our algorithm.
8. Stochastic Kirchhoff-Love Plate

A ten-term KL truncation is used to expand the Young modulus $E(x, y, \theta)$, as shown in Fig.11.

Fig.12 and Fig.13 show the obtained $\{d_{\omega k}, d_{y k}, d_{x k}\}_{k=1}^{6}$ and the resulting PDFs of $\{\lambda_k(\theta)\}_{k=1}^{6}$. Given the initial random variable samples $\{\lambda_k^{(0)}(\theta^{(m)})\}_{m=1}^{1 \times 10^4}$, PDFs of the corresponding random variables $\{\lambda_k(\theta)\}$ can be determined via the linear case of Algorithm 1. The number of couples $\{\lambda_k(\theta), d_k\}$ that constitute the stochastic solution is adopted as $k = 6$. 

Figure 10: Iteration error (Left) and PDF of the point $x = 0.2$ (Right)

Figure 11: Ten-term eigenfunctions (Left) and eigenvalues (Right)
Fig. 12: Solution3: \( \{d_{wk}\}_{k=1}^{6} \) (1-2 Row), \( \{d_{yk}\}_{k=1}^{6} \) (3-4 Row) and \( \{d_{xk}\}_{k=1}^{6} \) (5-6 Row)

Fig. 13: Solution3: PDFs of \( \{\lambda_k(\theta)\}_{k=1}^{6} \)

Fig. 14 shows the iteration errors for each couple \((\lambda_k(\theta), d_k)\). The convergence criterias in Algorithm 1 are set as \( \varepsilon = 10^{-6} \) (Left) and \( \varepsilon = 10^{-10} \) (Right), respectively. Fig. 14 (Left) illustrates the good convergence of the
proposed algorithm, and Fig. 14 (Right) indicates that the proposed algorithm can achieve very high accuracy with a small number of retained terms.

![Figure 14: Iteration error: $\varepsilon = 10^{-6}$ (Left) and $\varepsilon = 10^{-10}$ (Right)](image)

Fig. 15 shows the resulting PDFs of $u_\omega$ (Left), $u_y$ (Mid) and $u_x$ (Right) at node 325 obtained from our algorithm and that from Monte Carlo simulation with $1 \times 10^5$ realizations, which verifies the good accuracy of our algorithm.

![Figure 15: PDFs of $u_\omega$ (Left), $u_y$ (Mid) and $u_x$ (Right) at the node 325, the red line $\Delta$ and the blue line $\oplus$ denote the computed PDFs obtained by the proposed algorithm and the reference PDFs obtained from $1 \times 10^5$ Monte Carlo simulations, respectively](image)

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