Exact Matching and the Top-k Perfect Matching Problem

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Abstract

The aim of this note is to provide a reduction of the Exact Matching problem to the Top-k Perfect Matching Problem. Together with earlier work by El Maalouly, this shows that the two problems are polynomial-time equivalent.

The Exact Matching Problem is a well-known 40 years old problem for which a randomized, but no deterministic poly-time algorithm has been discovered. The Top-k Perfect Matching Problem is the problem of finding a perfect matching which maximizes the total weight of the $k$ heaviest edges contained in it.

2012 ACM Subject Classification Theory of computation → Design and analysis of algorithms; Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases Perfect Matching, Exact Matching, Independence Number, Parameterized Complexity.

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding Lasse Wulf: Supported by the Austrian Science Fund (FWF): W1230.

1 Reduction

Exact Matching (EM), defined in 1982 by Papadimitriou and Yannakakis [2], is one of only few natural problems which is known to be solvable in randomized polynomial time, but for which no deterministic poly-time algorithm is known so far.

\begin{tabular}{|l|}
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  **Exact Matching (EM)** \\
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  \textbf{Input:} A graph $G$, where every edge is colored blue or red, and an integer $k$. \\
  \textbf{Task:} Decide whether there exists a perfect matching $M$ in $G$ with exactly $k$ red edges. \\
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\end{tabular}

Given a weight function $w : E \rightarrow \mathbb{R}$ which assigns a weight $w(e)$ to every edge of a graph, and given a subset $F \subseteq E$ of the edges, we order the elements of $F$ by their weight. For a given integer $k$ we let $w^k(F)$ denote the sum of the weight of the $k$ heaviest elements in $F$. The function $w^k(\cdot)$ is called the top-$k$ weight function.

We show that EM can be reduced (in deterministic polynomial time) to the following optimization problem defined and studied in [1].

\begin{tabular}{|l|}
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  **Top-$k$ Perfect Matching (TkPM)** \\
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  \textbf{Input:} A weighted graph $G$ and integer $k$. \\
  \textbf{Task:} Find a perfect matching $M$ in $G$ maximizing the top-$k$ weight function $w^k(M)$. \\
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\end{tabular}

\textbf{Lemma 1.} $EM \leq_p TkPM$, even if the edge weights in the TkPM instance are bounded by a constant.

\textbf{Proof.} Consider an instance of EM, which is given by a graph $G := (V, E)$ and a red-blue coloring of its edges, and an integer $k$. We describe how to obtain in polynomial time an instance of TkPM given by a graph $G' := (V', E')$ and a weight function $w : E' \rightarrow \mathbb{R}$ and an
integer \( k' \). We start with \( G \) and subdivide all edges four times, i.e. every edge is replaced by a path of length 5. For every edge \( e \in E \), let \( P_e \subseteq G' \) be the path replacing it. In addition to this, we add \( 2k \) new vertices and \( k \) independent edges forming a perfect matching of these \( 2k \) vertices to the graph \( G' \). Let \( E_k \) be the set of these \( k \) edges. This completes the description of \( G' \). Observe that any perfect matching in \( G' \) must contain the set \( E_k \).

For the weight function \( w : E' \to \mathbb{R} \), we let \( w(e) = 2 \) for all \( e \in E_k \). For all the edges on some path \( P_e \), we distinguish the case whether \( e \) is colored red or blue in \( G \). If \( e \) is blue, all edges of \( P_e \) get weight 0. If \( e \) is red, the middle edge of \( P_e \) gets weight 3, the two edges adjacent to it get weight 2 and the two remaining outer edges get weight 0.

Finally, let \( k' := 2|R(G)| \) where \( |R(G)| \) is the number of red edges in \( G \). This completes our description of the TkPM instance. Let \( \alpha := 4|R(G)| + k \). We now claim that there is a PM in \( G \) with exactly \( k \) red edges if and only if there is a PM \( M' \) in \( G' \) with \( w^{k'}(M') \geq \alpha \).

To show this, observe that there is a one-to-one correspondence between perfect matchings in \( G \) and perfect matchings in \( G' \) where an edge is in the perfect matching of \( G \) if and only if the middle edge of \( P_e \) is in the perfect matching of \( G' \). Let \( M \subseteq G \) and \( M' \subseteq G' \) be two such perfect matchings which correspond to each other. Let \( r = |R(M)| \) be the number of red edges in the matching \( M \). Note that

\[
w^{k'}(M') \leq 3r + 2(k' - r) = 4|R(G)| + r.
\]

This inequality is due to the fact that the maximum weight of an edge is 3, but every edge of weight 3 is a middle edge of some path \( P_e \) where \( e \in E(G) \) is colored red. Therefore the \( k' \) heaviest edges in \( M' \) can contain at most \( r \) edges of weight 3 and at most \( k' - r \) edges of weight 2.

This shows that for the matching \( M' \) to achieve \( w^{k'}(M') \geq \alpha \), we need at least \( k \) red edges in \( M \). Finally suppose that \( M \) has at least \( k \) red edges, that is \( r \geq k \). Consider all the edges of non-zero weight in \( M' \). These are exactly the edges of weight 3 corresponding to the middle of a path \( P_e \) where \( e \) is red and \( e \in M \), and all pairs of edges of weight 2 corresponding to a path \( P_e \) where \( e \) is red and \( e \notin M \), and all the edges in \( E_k \). (Observe that paths \( P_e \) where \( e \) is blue have weight 0). We count the number of non-zero weight edges and observe that this number is \( r + 2(|R(G)| - r) + k = 2|R(G)| + k - r \). Using the assumption \( r \geq k \) we have that the number of non-zero edges is smaller or equal to \( k' \), so every non-zero edge is included in \( w^{k'}(M') \). In total, we have \( w^{k'}(M') = 3 \cdot r + 2k + 4 \cdot (|R(G)| - r) = 4|R(G)| - r + 2k \). This number is equal to \( \alpha \) in the case \( r = k \) and smaller than \( \alpha \) in the case \( r > k \).

We conclude that for all pairs \((M, M')\) of corresponding perfect matchings we have \( w^{k'}(M') \geq \alpha \) if and only if \( M \) has exactly \( k \) red edges.

In combination with the results of [1], we get that EM and TkPM are polynomially equivalent. Note that the reduction described there is not a Karp-reduction.

References

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