DEFORMING A HYPERSURFACE BY A CLASS OF GENERALIZED FULLY
NONLINEAR CURVATURE FLOWS

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ABSTRACT. In this paper, we concern a generalized fully nonlinear curvature flow involving $k$-th elementary symmetric function for principal curvature radii in Euclidean space $\mathbb{R}^n$, $k$ is an integer and $1 \leq k \leq n - 1$. For $1 \leq k < n - 1$, based on some initial data and constrains on smooth positive function defined on the unit sphere $\mathbb{S}^{n-1}$, we obtain the long time existence and convergence of the flow. Especially, the same result shall be derived for $k = n - 1$ without any constraint on the smooth positive function.

1. Introduction

The geometric flows are deemed as an effective tool for promoting the research in some fields, such as geometric analysis, PDEs etc. Among a range of flows, one with the speed of functions involving principal curvatures have been widely studied. Firey [13] first introduced the Gauss curvature flow to describe shape of worn stones. Husiken [19] concerned with the mean curvature flow to characterise the asymptotic behaviour of the hypersurface. Chow [11], Andrews [1], Brendle-Choi-Dasklopoulos [6] considered the curvature flow at the speed of $\alpha$-power of Gauss curvature to analyse the deformation of hypersurface, for more reading, refer to [2, 7, 10, 23, 24]. Another main issue is the study of the problem on the existence of the prescribed polynomial of the principal curvature radii of the hypersurface. Urbas [29], Gerhart [15], Chow-Tsai [12], Bryan-Ivaki-Scheuer [4] shed light on the convergence for the flow with the speed of symmetric polynomial of the principal curvature radii of the hypersurface. In this paper, we consider a class of generalized fully nonlinear curvature flow of convex hypersurfaces $M_t$ parameterized by smooth map $X(\cdot, t) : \mathbb{S}^{n-1} \to \mathbb{R}^n$ satisfying

\[
\begin{aligned}
\frac{\partial X(x,t)}{\partial t} &= \left. \frac{1}{f(v)} \sigma_k(x, t) \varphi(X \cdot v) (X \cdot v) G(X) \right| v - X; \\
X(x, 0) &= X_0(x),
\end{aligned}
\]

where $\sigma_k$ is the $k$-th elementary symmetric function for principal curvature radii, $\varphi : (0, +\infty) \to (0, +\infty)$, $G : \mathbb{R}^n \to (0, +\infty)$ are two given continuous functions, and $v = x$ is the unit outer normal vector of $M_t$ at $X(\cdot, t)$. An important feature of our flow (1.1) is that the speed is a more general curvature function without homogeneity. If a positive self-similar solution of (1.1) exists,
it is a solution to the following fully nonlinear equation

$$\gamma \varphi(h)G(\nabla h)\sigma_k(x) = f(x) \quad \text{for } \gamma = 1.$$  \hspace{1cm} (1.2)

When $k = n - 1$, (1.2) reduces to the smooth case of dual Orlicz-Minkowski problem studied by [9, 25]. In the case $\varphi(s) = s^{1-p}$, $G(y) = |y|^{2-n}$, (1.2) is just the $L_p$ dual Minkowski problem and is the dual Minkowski problem for the index $p = 1$ characterized by the dual curvature measure, which was first proposed by [18]. When $1 \leq k < n - 1$, for $\varphi(s) = s^{1-p}$, $G(y) = 1$, (1.2) is called $L_p$-Christoffel-Minkowski problem. This problem is related to the problem of prescribing $k$-th $p$-area measures, Ivaki [20] and Sheng-Yi [27] gave the existence of smooth solutions in the case $p \geq k + 1$ from the perspective of geometric flows. For $G(y) = 1$, (1.2) becomes the Orlicz-Christoffel-Minkowski problem considered by [22].

To obtain the solvability of (1.2) via the flow (1.1), we need some constraints on $\varphi$ and $G$.

(A): Suppose $\varphi(s)\max_{|y| = s} G(y) < \beta_0 s^{-k-\epsilon}$ for some positive constants $\epsilon, \beta_0$ for $s$ near $+\infty$, and $\varphi(s)\min_{|y| = s} G(y) > \beta_1 s^{-k-\epsilon}$ for some positive constants $\epsilon, \beta_1$ for $s$ near 0. Here $k$ is the order of $\sigma_k$.

We are now in a position to state that the main aim of current work is to obtain the long time existence and convergence results of the flow (1.1). It is shown in the following theorem.

**Theorem 1.1.** Let $M_0$ be a smooth, and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin. Supposing $1 \leq k < n - 1$, $k$ is an integer. Let $\varphi, G$ be smooth functions satisfying A. Furthermore, for any $s > 0$, $y \in \mathbb{R}^n \setminus 0$,

$$\frac{\partial}{\partial s} \left( s \frac{\partial}{\partial s} (\log \varphi(s)) \right) \geq 0, \quad -\vartheta \leq s \frac{\partial}{\partial s} (\log \varphi(s)) \leq -1,$$

$$k + s \frac{\partial}{\partial s} (\log \varphi(s)) + \frac{\nabla G(y) \cdot y}{G} < 0, \quad \nabla G(y) \cdot y_T = 0,$$

where $\vartheta$ is a positive constant and $y_T$ is the tangential component of $y$. If $f$ is a smooth function on $\mathbb{S}^{n-1}$ such that

$$(k + 1)f^{-\frac{1}{2(n-k)}} \delta_{ij} + (k + \vartheta)\nabla_{ij}(f^{-\frac{1}{2(n-k)}})$$

is positive definite. Then there exists a smooth, strictly convex solution $M_t$ to flow equation (1.1) for all time $t > 0$, and it subconverges in $C^\infty$ to a smooth and strictly convex solution to equation (1.2) for $\gamma = 1$.

In view of Theorem 1.1, for $1 \leq k < n - 1$, if $\varphi(s) = s^{1-p}$, $G(y) = 1$ for $p \geq k + 1$, equipped with the above condition on $f$, Theorem 1.1 recovers the existence results to the $L_p$-Christoffel-Minkowski problem which have been derived by [17, 20, 27]. It should be remarked that, for $k = n - 1$, the same result shall be showed in Theorem 1.1 without constraint on $f$. 

Theorem 1.2. Let $M_0$ be a smooth, and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin, and $f$ be a smooth and positive function on $\mathbb{S}^{n-1}$. Supposing $k = n - 1$. Let $\varphi, G$ be smooth functions satisfying $A$. Then there exists a smooth, strictly convex solution $M_t$ to flow equation (1.1) for all time $t > 0$, and it subconverges in $C^\infty$ to a smooth and strictly convex solution to equation (1.2) for $\gamma = 1$.

Notice that Theorem 1.2 recovers the existence results to the dual Orlicz-Minkowski problem showed in [25] from the Gauss curvature flow point of view.

The organization of this paper goes as follows: In Section 2, we collect some basic knowledge about $k$-th Hessian operators and convex bodies. In Section 3, we introduce the geometric flow. In Section 4 and 5, we obtain the priori estimates of the solution to the relevant flow. In Section 6, we complete the proof of Theorem 1.1 and Theorem 1.2. At last, we shall provide a special uniqueness result to the solution of equation (1.2).

2. Preliminaries

We first give some basics on $k$-Hessian operators, recommended to see [16] for a good reference.

2.1. $K$-th Hessian Operators. The $k$-th elementary symmetric function for $\lambda = (\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ is defined as

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n-1.$$ 

Let $S_{n-1}$ be the set of all symmetric $(n - 1) \times (n - 1)$ metrics. The $k$-th elementary symmetric function for $A \in S_{n-1}$ is

$$\sigma_k(A) =: \sigma_k(\lambda(A)), \quad \lambda(A) \text{ is the eigenvalue of } A.$$ 

We say $A \in S_{n-1}$ belongs to $\Gamma_k$ if its eigenvalue $\lambda(A) \in \Gamma_k$. The Garding cone $\Gamma_k$ is defined as

$$\Gamma_k = \{ \lambda \in \mathbb{R}^{n-1} | \sigma_i(\lambda) > 0, \text{ for } 1 \leq i \leq k \}.$$ 

Here we denote by $\sigma_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$. Now, we list some standard formulas and properties of elementary symmetric functions that we shall use in what follows.

Proposition 2.1. Let $\lambda = (\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ and $k = 0, 1, \cdots, n - 1$. Then,

(i) $\sigma_{k+1}(\lambda) = \sigma_{k+1}(\lambda|i) + \lambda_i \sigma_k(\lambda|i), \quad \forall 1 \leq i \leq n - 1.$

(ii) $\sum_{i=1}^{n-1} \lambda_i \sigma_k(\lambda|i) = (k + 1) \sigma_{k+1}(\lambda).$

(iii) $\sum_{i=1}^{n-1} \sigma_k(\lambda|i) = (n - k - 1) \sigma_k(\lambda).$

(iv) $\frac{\partial \sigma_{k+1}(\lambda)}{\partial \lambda_i} = \sigma_k(\lambda|i).$
Proposition 2.2 (Newton-Maclaurin inequality). For $\lambda \in \Gamma_k$ and $1 \leq l \leq k \leq n - 1$, we have
\[
\left( \frac{\sigma_k(\lambda)}{C^k_{n-1}} \right)^{\frac{1}{k}} \leq \left( \frac{\sigma_l(\lambda)}{C^l_{n-1}} \right)^{\frac{1}{l}}.
\]

Proposition 2.3 (Concavity). For any $k > l \geq 0$, suppose $A = A_{ij} \in \Theta_{n-1}$ such that $\lambda(A) \in \Gamma_k$. Then, we have
\[
\left[ \frac{\sigma_k(A)}{\sigma_l(A)} \right]^{\frac{1}{k-l}}
\]
is a concave function in $\Gamma_k$.

Proposition 2.4 (Inverse Concavity). Suppose $F$ is a smooth symmetric function in $\Gamma_{n-1}$. $F$ is inverse concave, i.e.,
\[
F_*(\lambda_i) = \frac{1}{F(\lambda_i^{-1})}
\]
is concave.

2.2. Basics of convex bodies. For the good references on convex bodies, please refer to Gardner [14] and Schneider [28].

For $Y, Z \in \mathbb{R}^n$, $Y \cdot Z$ denotes the standard inner product. For $X \in \mathbb{R}^n$, we denote by $|X| = \sqrt{X \cdot X}$ the Euclidean norm. Let $\mathbb{S}^{n-1}$ be the unit sphere, and $C(\mathbb{S}^{n-1})$ be the set of continuous functions defined on the unit sphere $\mathbb{S}^{n-1}$. A compact convex set of $\mathbb{R}^n$ with non-empty interior is called a convex body.

If $\Omega$ is a convex body containing the origin in $\mathbb{R}^n$, for $x \in \mathbb{S}^{n-1}$, the support function of $\Omega$ (with respect to the origin) is defined by
\[
h(\Omega, x) = \max\{x \cdot Y : Y \in \Omega\}.
\]
Extending this definition to a homogeneous function of degree one in $\mathbb{R}^n \setminus \{0\}$ by the equation $h(\Omega, x) = |x|h(\Omega, \frac{x}{|x|})$. The radial function $\rho$ of $\Omega$ is given by
\[
\rho(\Omega, u) = \max\{|a > 0 : au \in \Omega\}, \quad u \in \mathbb{S}^{n-1}.
\]
Note that $\partial \Omega$ can be represented by its radial function,
\[
\partial \Omega = \{\rho(\Omega, u)u : u \in \mathbb{S}^{n-1}\}.
\]
The map $g : \partial \Omega \to \mathbb{S}^{n-1}$ denotes the Gauss map of $\partial \Omega$. Meanwhile, for $\omega \subset \mathbb{S}^{n-1}$, the inverse of Gauss map $g$ is expressed as
\[
g^{-1}(\omega) = \{X \in \partial \Omega : g(X) \text{ is defined and } g(X) \in \omega\}.
\]
For simplicity in the subsequence, we abbreviate $g^{-1}$ as $F$. In particular, for a convex body $\Omega$ being of class $C^2_+$, i.e., its boundary is of class $C^2$ and of positive Gauss curvature, the support function of $\Omega$ can be written as
\[
h(\Omega, x) = x \cdot F(x) = g(X) \cdot X, \quad \text{where } x \in \mathbb{S}^{n-1}, \ g(X) = x \text{ and } X \in \partial \Omega.
\] (2.1)
Let $e_1, e_2, \ldots, e_{n-1}$ be a local orthonormal frame on $\mathbb{S}^{n-1}$, $h_i$ be the first order covariant derivatives of $h(\Omega, \cdot)$ on $\mathbb{S}^{n-1}$ with respect to the frame. Differentiating (2.1) with respect to $e_i$, we get

$$h_i = e_i \cdot F(x) + x \cdot F_i(x).$$

Since $F_i$ is tangent to $\partial \Omega$ at $F(x)$, we obtain

$$h_i = e_i \cdot F(x).$$  \hspace{1cm} (2.2)

Combining (2.1) and (2.2), we have (see also [29, p. 97])

$$F(x) = h_i(\Omega, x)e_i + h(\Omega, x)x = \nabla_{\mathbb{S}^{n-1}} h(\Omega, x) + h(\Omega, x)x.$$  \hspace{1cm} (2.3)

Here $\nabla_{\mathbb{S}^{n-1}}$ is the spherical gradient. On the other hand, since we can extend $h(\Omega, x)$ to $\mathbb{R}^n$ as a $1$-homogeneous function $h(\Omega, \cdot)$, then restrict the gradient of $h(\Omega, \cdot)$ on $\mathbb{S}^{n-1}$, it yields that (see for example [16, p. 14-16])

$$\nabla h(\Omega, x) = F(x), \quad \forall x \in \mathbb{S}^{n-1},$$  \hspace{1cm} (2.4)

where $\nabla$ is the gradient operator in $\mathbb{R}^n$. Let $h_{ij}$ be the second order covariant derivatives of $h(\Omega, \cdot)$ on $\mathbb{S}^{n-1}$ with respect to the local frame. Then, applying (2.3) and (2.4), we have (see, e.g., [21, p. 382])

$$\nabla h(\Omega, x) = h_i e_i + hx, \quad F_i(x) = (h_{ij} + h\delta_{ij}) e_j.$$  \hspace{1cm} (2.5)

### 3. The geometric flow and relevant functional

In this section, we are in the place to introduce the geometric flow and the relevant functional.

Let $M_0$ be a smooth, origin symmetric and strictly convex body in $\mathbb{R}^n$, as presented above, we are concerned with a family of convex hypersurfaces $M_t$ parameterized by smooth map $X(\cdot, t): \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ satisfying the following flow equation,

$$\begin{cases}
\frac{\partial X(x, t)}{\partial t} = \frac{1}{f(x)} \sigma_k(x, t) \varphi(X \cdot v)(X \cdot v) G(X)v - X(x, t); \\
X(x, 0) = X_0(x),
\end{cases}$$  \hspace{1cm} (3.1)

Multiplying both sides of (3.1) by $v$, by means of the definition of support function, we describe the flow equation associated with the support function $h(x, t)$ of $\Omega_t$ as

$$\begin{cases}
\frac{\partial h(x, t)}{\partial t} = \frac{1}{f(x)} \sigma_k(x, t) \varphi(h(x, t)) h G(\nabla_{\mathbb{S}^{n-1}} h(x, t) + h(x, t)I) - h(x, t); \\
h(x, 0) = h_0(x),
\end{cases}$$  \hspace{1cm} (3.2)

Denote by $\rho(u, t)$ the radial function of $\Omega_t$ for any $u \in \mathbb{S}^{n-1}$. Let $u$ and $x$ be related by

$$\rho(u, t) u = \nabla_{\mathbb{S}^{n-1}} h(x, t) + h(x, t)x.$$  

So,

$$\rho(u, t)^2 = |\nabla_{\mathbb{S}^{n-1}} h(x, t)|^2 + h(x, t)^2.$$
and $\rho(u, t)$ satisfies (see [7])

$$\frac{1}{\rho(u, t)} \frac{\partial \rho(u, t)}{\partial t} = \frac{1}{h(x, t)} \frac{\partial h(x, t)}{\partial t}.$$ 

We are now in a position to give evolution equations of geometric quantities by using the flow equation (3.1). For simplicity, we write

$$\Theta = \frac{1}{f(x)} h(x, t) \varphi(h(x, t)) G(\nabla_{\nabla h} h(x, t) + h(x, t) I), \quad P = \Theta \sigma_k(x, t), \quad w_{ij} = h_{ij} + h\delta_{ij}.$$ 

Notice that the eigenvalue of $\{w_{ij}\}$ and $\{w^{ij}\}$ are respectively the principal radii and principal curvatures of $\partial \Omega$, (see [29]), where $\{w^{ij}\}$ is the inverse matrix of $\{w_{ij}\}$.

**Lemma 3.1.** The following evolution equations hold along the flow (3.1).

$$\partial_t w_{ij} - \Theta \sigma_k^{pq} \nabla_{pq} w_{ij}$$

$$= (k + 1) \Theta \sigma_k \delta_{ij} - \Theta \sigma_k^{pq} \delta_{pq} w_{ij} + \Theta(\sigma_k^{ip} w_{jp} - \sigma_k^{jp} w_{ip})$$

$$+ \Theta \sigma_k^{pq, mn} \nabla_j w_{pq} \nabla_i w_{mn} + \sigma_k \nabla_j \Theta + \nabla_j \nabla_k \Theta + \nabla_i \sigma_k \nabla_j \Theta - w_{ij}, \quad (3.3)$$

$$\partial_t w^{ij} - \Theta \sigma_k^{pq} \nabla_{pq} w^{ij}$$

$$= -(k + 1) \Theta \sigma_k w^{jp} w^{ip} + \Theta \sigma_k^{pq} \delta_{pq} w^{ij} - \Theta w^{ip} w^{jq}(\sigma_k^{rp} w_{rq} - \sigma_k^{rq} w_{rp})$$

$$- \Theta w^{il} w^{j} (\sigma_k^{pq, mn} + 2 \sigma_k^{pm} w^{pq}) \nabla_l w_{pq} \nabla_s w_{mn}$$

$$- w^{ip} w^{jq} (\sigma_k \nabla_p \Theta + \nabla_q \sigma_k \nabla_p \Theta + \nabla_p \sigma_k \nabla_q \Theta) + w^{ij}, \quad (3.4)$$

and

$$\partial_t \left( \frac{\rho(u, t)^2}{2} \right) = \Theta \sigma_k^{ij} \nabla_{ij} \left( \frac{\rho(u, t)^2}{2} \right)$$

$$= (k + 1) h \Theta \sigma_k - \rho^2 + \sigma_k \nabla_j h \nabla_j \Theta - \Theta \sigma_k^{ij} w_{mj} w_{mj}. \quad (3.5)$$

**Proof.** For the specific computations, one can refer to [20]. Here for reader’s convenience, we give the detail. For (3.3), since

$$\partial_t \nabla_{ij} h = \nabla_{ij} (\partial_t h) = \sigma_k \nabla_{ij} \Theta + \nabla_j \sigma_k \nabla_i \Theta + \nabla_i \sigma_k \nabla_j \Theta + \Theta \nabla_{ij} \sigma_k - h_{ij},$$

where

$$\nabla_{ij} \sigma_k = \sigma_k^{pq, mn} \nabla_j w_{pq} \nabla_i w_{mn} + \sigma_k^{pq} \nabla_{ij} w_{pq}.$$ 

By virtue of Ricci identity,

$$\nabla_{ij} w_{pq} = \nabla_{pq} w_{ij} + \delta_{ij} \nabla_{pq} w - \delta_{pq} \nabla_{ij} w + \delta_{pq} \nabla_{ij} w - \delta_{pq} \nabla_{ij} w.$$ 

So,

$$\partial_t h_{ij} = \Theta \sigma_k^{pq} \nabla_{pq} w_{ij} + k \Theta \sigma_k \delta_{ij} - \Theta \sigma_k^{pq} \delta_{pq} w_{ij} + \Theta(\sigma_k^{ip} w_{jp} - \sigma_k^{jp} w_{ip})$$

$$+ \Theta \sigma_k^{pq, mn} \nabla_j w_{pq} \nabla_i w_{mn} + \sigma_k \nabla_{ij} \Theta + \nabla_j \sigma_k \nabla_i \Theta + \nabla_i \sigma_k \nabla_j \Theta - h_{ij}.$$
This together with \( w_{ij} = h_{ij} + h\delta_{ij} \) gives (3.3). Then using (3.3), the evolution equation (3.4) of \( w^{ij} \) follows from
\[
\partial_t w^{ij} = -w^{im} w^{lj} \partial_t w_{ml}, \quad \nabla_{pq} w^{ij} = 2 w^{ik} w^{mn} w^{lj} w_{mp} - w^{im} w^{lj} \nabla_{pq} w_{ml}.
\]
For (3.5), due to \( \rho^2 = h^2 + |\nabla S_n - 1|^2 \), we directly compute
\[
\partial_t \left( \frac{\rho^2}{2} \right) - \Theta \sigma^i_j \nabla_i \left( \frac{\rho^2}{2} \right)
= h\partial_t h + \nabla_i h \nabla_i h - \Theta \sigma^i_j (\nabla^2 h_{ij} h + \nabla_i h \nabla_j h + \nabla_m h \nabla_j \nabla_m h + \nabla_m h \nabla_m h)
= h\partial_t h + \nabla_i h \nabla_j (\Theta \sigma^j_k - h) - \Theta \sigma^j_k (\nabla h \nabla_j h + \nabla_m h \nabla_j (w_{mi} - h\delta_{mi})]
- \Theta \sigma^j_k (w_{ij} - h\delta_{ij}) - \Theta \sigma^j_k (w_{mi} - h\delta_{mi})(w_{mj} - h\delta_{mj})
= (k + 1) h\Theta \sigma^j_k - \rho^2 + \sigma^j_k \nabla_i h \nabla_i \Theta - \Theta \sigma^j_k w_{mi} w_{mj}.
\]
Hence, the proof is completed. \( \square \)

4. \( C^0, C^1 \) estimates

In this section, we shall obtain the \( C^0, C^1 \) estimates of solutions to the flow (3.1). Let us begin with completing the \( C^0 \) estimate.

**Lemma 4.1.** Supposing \( 1 \leq k \leq n-1 \), \( k \) is an integer. Let \( f \) be a smooth and positive function on \( S^{n-1} \), \( M_t \) be a smooth and strictly convex solution satisfying the flow (3.1), and let \( \varphi, G \) be smooth functions satisfying A. Then
\[
\frac{1}{C} \leq h(x, t) \leq C, \quad \forall (x, t) \in S^{n-1} \times (0, \infty), \tag{4.1}
\]
and
\[
\frac{1}{C} \leq \rho(u, t) \leq C, \quad \forall (u, t) \in S^{n-1} \times (0, \infty). \tag{4.2}
\]
Here \( h(x, t) \) and \( \rho(u, t) \) are the support function and the radial function of \( M_t \).

**Proof.** Suppose the spatial maximum of \( h(x, t) \) is attained at \( x^0 \in S^{n-1} \). Then at \( x^0 \), we have
\[
\nabla_{S^{n-1}} h = 0, \quad \nabla_{S^{n-1}}^2 h \leq 0, \quad \rho = h,
\]
and
\[
\nabla_{S^{n-1}}^2 h + h I \leq h I.
\]
This illustrates that, at \( x^0 \),
\[
\frac{\partial h}{\partial t} = \frac{1}{f(x)} h\sigma^i_k G(\rho, u) \varphi(h) - h
\leq \frac{1}{f_{\min}} h[G(h, u) h^k \varphi(h) - f_{\min}].
\]
If $\varphi(s) \max_{y \in S} G(y) < \beta_0 s^{-k-\varepsilon}$ for some positive constant $\varepsilon$ and $\beta_0$ for $s$ tends to $+\infty$, then, at $x^0$, there is

\[
\frac{\partial h}{\partial t} \leq \frac{1}{f_{\min}} h[h^{-\varepsilon} \beta_0 - f_{\min}].
\] (4.3)

The right hand side of (4.3) is negative for $\max_{S^{n-1}} h(x, t)$ large enough, thus the upper bound of $\max_{S^{n-1}} h(x, t)$ follows.

Similarly, we can estimate the spatial minimum of $h(x, t)$. Suppose $\min_{S^{n-1}} h(x, t)$ is attained at $x^1$. Then at $x^1$, we have

\[
\nabla_{S^{n-1}} h = 0, \quad \nabla^2_{S^{n-1}} h \geq 0, \quad \rho = h,
\]

and

\[
\nabla^2_{S^{n-1}} h + h I \geq h I.
\]

Therefore, at $x^1$, we have

\[
\frac{\partial h}{\partial t} = \frac{1}{f(x)} \nabla_{S^{n-1}} h \sigma_k G(\rho, u) \varphi(h) - h \geq \frac{1}{f_{\max}} h[G(\rho, u) h^k \varphi(h) - f_{\max}].
\]

If $\varphi(s) \min_{y \in S} G(y) > \beta_1 s^{-k-\varepsilon}$ for some positive constants $\varepsilon$ and $\beta_1$ for $s$ near 0, then, at $x^1$, we have

\[
\frac{\partial h}{\partial t} \geq \frac{1}{f_{\max}} h[h^{-\varepsilon} \beta_1 - f_{\max}].
\] (4.4)

The right hand side of (4.4) is positive for $\min_{S^{n-1}} h(x, t)$ small enough and the lower bound of $\min_{S^{n-1}} h(x, t)$ follows. Due to $\rho(u, t) u = \nabla_{S^{n-1}} h(x, t) + h(x, t) x$, one has (see [8])

\[
\min_{S^{n-1}} h(x, t) \leq \rho(u, t) \leq \max_{S^{n-1}} h(x, t)\] (4.5)

Hence, the estimate (4.2) immediately holds via (4.5). The proof is completed.

The $C^1$ estimate is as follows.

**Lemma 4.2.** Supposing $1 \leq k \leq n-1$, $k$ is an integer. Let $f$ be a smooth and positive function on $S^{n-1}$, $M$ be a smooth and strictly convex solution satisfying the flow (3.1), and let $\varphi, G$ be smooth functions satisfying $A$. Then

\[
|\nabla_{S^{n-1}} h(x, t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times (0, \infty),
\]

and

\[
|\nabla_{S^{n-1}} \rho(u, t)| \leq C, \quad \forall (u, t) \in S^{n-1} \times (0, \infty)
\]

for some $C > 0$, independent of $t$. 
**Proof.** Lemma 4.1 and the following facts 

\[
\rho^2 = h^2 + |\nabla_{S^{n-1}} h|^2, \quad h = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla_{S^{n-1}} h|^2}}
\]

elaborate the results. \hfill \square

5. **$C^2$ Estimate**

**Lemma 5.1.** Supposing $1 \leq k \leq n - 1$, $k$ is an integer. Let $f$ be a smooth and positive function on $\mathbb{S}^{n-1}$, $M_t$ be a smooth and strictly convex solution satisfying the flow (3.1), and let $\varphi, G$ be smooth functions satisfying A. Then

\[
\sigma_k(x, t) \geq C, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times (0, \infty),
\]

for a positive constant $C$, independent of $t$.

**Proof.** Considering the auxiliary function

\[
Q = \log P - A \frac{\rho^2}{2},
\]

where $P = \Theta \sigma_k$, and $A$ is a positive constant to be determined later. Assume that the spatial minimum of $Q$ is achieved at $\tilde{x}_0$. Then at $\tilde{x}_0$, we have

\[
0 = \nabla_i Q = \frac{\nabla_i P}{P} - A \nabla_i \left( \frac{\rho^2}{2} \right),
\]

and

\[
0 \leq \nabla_{ij} Q = \frac{\nabla_{ij} P}{P} - \frac{\nabla_i P \nabla_j P}{P^2} - A \nabla_{ij} \left( \frac{\rho^2}{2} \right).
\]

Now, we compute

\[
\partial_t Q - \Theta \sigma_k \nabla_{ij} Q
= \frac{1}{P} (\partial_t P - \Theta \sigma_k \nabla_{ij} P) - A \left[ \partial_t \left( \frac{\rho^2}{2} \right) - \Theta \sigma_k \nabla_{ij} \left( \frac{\rho^2}{2} \right) \right] + \frac{\Theta}{P^2} \sigma_k \nabla_i P \nabla_j P. \tag{5.2}
\]

Clearly

\[
\partial_t P = \Theta \partial_t \sigma_k + \sigma_k \partial_t \Theta, \tag{5.3}
\]

in which

\[
\partial_t \sigma_k = \sigma_k \partial_i (\nabla_{ij} h + \delta_{ij} h) = \sigma_k \nabla_{ij} P - \sigma_k \nabla_{ij} h + \sigma_k \delta_{ij} P - \sigma_k \delta_{ij} h = \sigma_k \nabla_{ij} P - \sigma_k (w_{ij} - h \delta_{ij}) + \sigma_k \delta_{ij} P - \sigma_k \delta_{ij} h = \sigma_k \nabla_{ij} P + \sigma_k \delta_{ij} P - k \sigma_k,
\]
and
\[ \partial_t \Theta = \frac{1}{f} G \varphi \frac{\partial h}{\partial t} + \frac{1}{f} h \varphi \left( \nabla G \cdot \frac{\partial X}{\partial t} \right) + \frac{1}{f} h G \varphi \frac{\partial h}{\partial t} \]
\[ = \frac{1}{f} G \varphi (P - h) + \frac{1}{f} h \varphi (\nabla G \cdot (Pv - X)) + \frac{1}{f} h G \varphi' (P - h). \]

In turn, (5.3) becomes
\[ \partial_t P = \Theta (\sigma_{ij}^k \nabla_j P + \sigma_{ij}^k \delta_{ij} P - k \sigma_k) + \sigma_k \left[ \frac{1}{f} G \varphi (P - h) + \frac{1}{f} h \varphi (\nabla G \cdot (Pv - X)) + \frac{1}{f} h G \varphi' (P - h) \right] \]
\[ = \Theta \sigma_{ij}^k \nabla_j P + \Theta \sigma_{ij}^k \delta_{ij} P - kP + \frac{P}{h} (P - h) + \frac{P^2 \nabla G \cdot v}{G} - \frac{P \nabla G \cdot X}{G} + P (P - h) \varphi' \phi \]
\[ = \Theta \sigma_{ij}^k \nabla_j P + \Theta \sigma_{ij}^k \delta_{ij} P - P \left( k + 1 + \frac{\varphi' h}{\varphi} + \frac{\nabla G \cdot X}{G} \right) + \frac{P^2}{h} \left( 1 + \frac{\varphi' h}{\varphi} + \frac{(\nabla G \cdot hv)}{G} \right). \]

So, (5.2) turns into
\[ \partial_t Q - \Theta \sigma_{ij}^k \nabla_j Q \]
\[ = \frac{\Theta}{P^2} \sigma_{ij}^k \nabla_j P \nabla_j P + \Theta \sigma_{ij}^k \delta_{ij} - \left( k + 1 + \frac{\varphi' h}{\varphi} + \frac{\nabla G \cdot X}{G} \right) \]
\[ + \frac{P}{h} \left( 1 + \frac{\varphi' h}{\varphi} + \frac{h(\nabla G \cdot v)}{G} \right) - A(k + 1) h \Theta \sigma_k + A \rho - A \sigma_k \nabla_j \Theta + A \Theta \sigma_{ij}^k w_{mi} w_{mj} \]
\[ \geq A \rho^2 - \left( k + 1 + \frac{\varphi' h}{\varphi} + \frac{(\nabla G \cdot X)}{G} \right) + \frac{P}{h} \left( 1 + \frac{\varphi' h}{\varphi} + \frac{(\nabla G \cdot hv)}{G} \right) - A(k + 1) h \Theta \sigma_k - A \sigma_k \nabla_j \Theta. \]

(5.4)

By using the $C^0$, $C^1$ estimates, choosing $A$ large enough, the right hand side of (5.4) is strictly positive provided $\min_{S_n} \sigma_k \to 0$ ($Q$ is negatively large enough), thus the lower bound of $Q$ follows, hence $\sigma_k$ is uniformly bounded below away from zero.

**Lemma 5.2.** Supposing $1 \leq k \leq n - 1$, $k$ is an integer. Let $f$ be a smooth and positive function on $S^{n-1}$, $M_i$ be a smooth and strictly convex solution satisfying the flow (3.1), and let $\varphi, G$ be smooth functions satisfying A. Then
\[ \sigma_k (x, t) \leq \tilde{C}, \ \forall (x, t) \in S^{n-1} \times (0, \infty), \]

for a positive constant $\tilde{C}$, independent of $t$.

**Proof.** Using the $C^0$ estimate, there exists a positive constant $B$ such that
\[ B < \rho^2 < \frac{1}{B} \]
for all $t > 0$. Now, setting the auxiliary function as
\[ \chi(x, t) = \frac{1}{f} G \varphi \sigma_k \]
\[ = \frac{P}{h} \frac{1}{1 - \frac{\rho^2}{2}} \]
for all $t > 0$. Now, setting the auxiliary function as
Suppose the spatial maximum of \( \chi(x, t) \) at \( \tilde{x}_1 \). Then at \( \tilde{x}_1 \), we get

\[
0 = \nabla_i \chi = \nabla_i \left( \frac{P}{h} \right) \frac{1}{1 - \frac{Bp^2}{2}} + \frac{P}{h} \frac{B \nabla_i (\frac{\rho^2}{2})}{1 - \frac{Bp^2}{2}}.
\]

and

\[
0 \geq \nabla_i \chi = \nabla_i \left( \frac{P}{h} \right) \frac{1}{1 - \frac{Bp^2}{2}} + \frac{P}{h} \frac{B \nabla_i (\frac{\rho^2}{2})}{1 - \frac{Bp^2}{2}}.
\]

We are in a position to compute the following

\[
\partial_t \chi - \Theta \sigma_{ij}^k \nabla_i \chi
\]

\[
= \frac{1}{1 - \frac{Bp^2}{2}} \partial_t \left( \frac{P}{h} \right) + \frac{P}{h} \frac{B}{(1 - \frac{Bp^2}{2})^2} \nabla_i \left( \frac{\rho^2}{2} \right) - \Theta \sigma_{ij}^k \frac{1}{1 - \frac{Bp^2}{2}} \nabla_i \left( \frac{P}{h} \right) - \Theta \sigma_{ij}^k \frac{P}{h} \frac{B}{(1 - \frac{Bp^2}{2})^2} \nabla_i \left( \frac{\rho^2}{2} \right)
\]

\[
= \frac{1}{1 - \frac{Bp^2}{2}} \left[ \partial_t \left( \frac{P}{h} \right) - \Theta \sigma_{ij}^k \nabla_i \left( \frac{P}{h} \right) \right] + \frac{P}{h} \frac{B}{(1 - \frac{Bp^2}{2})^2} \left[ \partial_i \left( \frac{\rho^2}{2} \right) - \Theta \sigma_{ij}^k \nabla_i \left( \frac{\rho^2}{2} \right) \right] 
\]

in which

\[
\partial_t \left( \frac{P}{h} \right) = \frac{N \sigma_{ij}^k \nabla_i \left( \frac{P}{h} \right)}{h}
\]

\[
= \frac{1}{h} (\partial_x P - \Theta \sigma_{ij}^k \nabla_i P) - P \frac{\sigma_{ij}^k \nabla_i \nabla_j}{h} + \frac{2P^2}{h} \sigma_{ij}^k \tilde{\nabla} \tilde{\nabla} \frac{P}{h}
\]

\[
= \frac{1}{h} \left[ \Theta \sigma_{ij}^k \nabla_i \nabla_j \frac{P}{h} \right] + \frac{P^2}{h^2} \left( \frac{\nabla G \cdot X}{G} \right) \left[ 1 + \frac{\nabla G \cdot X}{G} \right]
\]

\[
= \frac{P}{h} \left( \frac{\nabla G \cdot X}{G} \right) \left[ 1 + \frac{\nabla G \cdot X}{G} \right] + \frac{P^2}{h^2} \left( \frac{\nabla G \cdot X}{G} \right) \left[ 1 + \frac{\nabla G \cdot X}{G} \right] + \frac{2P}{h} \sigma_{ij}^k \nabla_i \nabla_j \frac{P}{h}
\]

In turn, (5.5) becomes

\[
\partial_t \chi - \Theta \sigma_{ij}^k \nabla_i \chi
\]

\[
= \frac{1}{1 - \frac{Bp^2}{2}} \left[ \frac{P}{h} \left( k + \frac{\nabla G \cdot X}{G} \right) + \frac{P^2}{h^2} \left( k + \frac{\nabla G \cdot X}{G} \right) + \frac{2P}{h} \sigma_{ij}^k \nabla_i \nabla_j \frac{P}{h} \right]
\]

\[
+ \frac{P}{h} \frac{B}{(1 - \frac{Bp^2}{2})^2} \left[ (k + 1)h \Theta \sigma_{ij}^k \rho^2 + \sigma_{ij}^k \nabla_i \Theta - \Theta \sigma_{ij}^k w_m w_m \right].
\]
On one hand, at \( \tilde{x}_1 \), there satisfies
\[
\nabla_j \frac{P}{h} = -\frac{P}{h} \frac{B \nabla_j \rho^2}{1 - \frac{B \rho^2}{2}} = -\frac{P}{h} \frac{B w_{jm} h_m}{1 - \frac{B \rho^2}{2}}.
\]
(5.8)

On the other hand, by the inverse concavity of \( (\sigma_k)^{\frac{1}{k}} \) (see [3, 29]), we have
\[
(\sigma_k)^{\frac{1}{k}} w_{im} w_{jm} \geq (\sigma_k)^{\frac{1}{k+1}}
\]
which illustrates that
\[
\sigma_k^{ij} w_{im} w_{jm} \geq k(\sigma_k)^{1 + \frac{1}{k}}.
\]

So,
\[
\partial_t \chi - \Theta \sigma_k^{ij} \nabla^{ij} \chi 
\leq \frac{1}{1 - \frac{B \rho^2}{2}} \left[ -\frac{P}{h} \left( k + \frac{\varphi}{\varphi} h + \nabla G \cdot X \right) + \frac{P^2}{h^2} \left( k + \frac{\varphi}{\varphi} h + h(\nabla G \cdot v) \right) \right]
\]
\[
+ \frac{P}{h} \frac{B}{(1 - \frac{B \rho^2}{2})^2} \left[ (k + 1)h \Theta \sigma_k - \rho^2 + \sigma_k \nabla_i \Theta \nabla_i \Theta - \Theta (\sigma_k)^{1 + \frac{1}{k}} \right].
\]

This in conjunction with the \( C^0, C^1 \) estimates, we have
\[
\partial_t \chi \leq c_1 \chi + c_2 \chi^2 - c_3 \chi^{2 + \frac{1}{k}}
\]
for some positive constants \( c_1, c_2, c_3 \). Then, we see that \( \chi(x, t) \) is uniformly bounded from above. Hence, the upper bound of \( \sigma_k \) is obtained. \( \square \)

**Lemma 5.3.** Supposing \( 1 \leq k < n - 1 \), \( k \) is an integer. Let \( M_t \) be a smooth and strictly convex solution satisfying the flow (3.1). Suppose \( f, \varphi, G \) be smooth and positive functions satisfying the assumptions of Theorem 1.1 and A. Then
\[
\kappa_i(x, t) \leq \hat{C}, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times (0, \infty),
\]
(5.10)
where \( \hat{C} \) is a positive constant independent of \( t \).

**Proof.** Suppose the spatial maximum of the maximum eigenvalue of the matrix \( \left\{ \frac{w^i}{h} \right\} \) is attained at \( \tilde{x}_2 \in \mathbb{S}^{n-1} \). By a rotation, we may assume \( \{w_{ij}(\tilde{x}_2, t)\} \) is diagonal, and the maximum eigenvalue of the matrix \( \left\{ \frac{w^i}{h} \right\} \) at \( (\tilde{x}_2, t) \) is \( \frac{w^{11}}{h}(\tilde{x}_2, t) \). So, at \( \tilde{x}_2 \), we have
\[
\nabla_i \left( \frac{w^{11}}{h} \right) = 0,
\]
i.e.,
\[
\frac{w^{11}}{h}_{1i} = -\frac{h_i}{h},
\]
(5.11)
Moreover,
\[
\nabla_{ij} \left( \frac{w^{11}}{h} \right) \leq 0.
\]
Now, we compute the evolution equation of \( \sigma_k \) as

\[
\frac{\partial}{\partial t} \frac{w_{11}^k}{h} - \Theta \sigma_k \nabla \nabla \frac{w_{11}^k}{h} = \frac{2}{h} \Theta \sigma_k \nabla \nabla \frac{w_{11}^k}{h} - \Theta \sigma_k (w_{11}^k)^2 - \Theta \sigma_k f_{11} + \Theta \sigma_k f \frac{(5.12)}{h} \sigma_k (w_{11}^k)^2 + \frac{\Theta}{h} \sigma_k \delta_\nu \frac{w_{11}^k}{h} \]

- \frac{1}{h} (w_{11}^k)^2 (\sigma_k^{i j m n} + 2 \sigma_k^{i m} w_{n j}^k) \nabla \nabla \frac{w_{11}^k}{h} \nabla \nabla w_{m n}^k

By means of the inverse concavity of \((\sigma_k)^2\), there is (see [3, 29]),

\[
(\sigma_k^{i j m n} + 2 \sigma_k^{i m} w_{n j}^k) \nabla \nabla w_{m n}^k \geq \frac{k + 1}{k} \frac{(\nabla \nabla \sigma_k)^2}{\sigma_k}. \quad (5.12)
\]

Utilizing Schwartz inequality, we obtain

\[
2 |\nabla \nabla \sigma_k \nabla \nabla | \leq \frac{k + 1}{k} \frac{\Theta (\nabla \nabla \sigma_k)^2}{\sigma_k} + \frac{k}{k + 1} \frac{\sigma_k (\nabla \nabla \Theta)^2}{\sigma_k}. \quad (5.13)
\]

Using (5.12) and (5.13), at \( \bar{x}_2 \), there is

\[
\frac{\partial}{\partial t} \frac{w_{11}^k}{h} \leq - \frac{(w_{11}^k)^2}{h} \sigma_k \nabla \nabla - \frac{k + 1}{k + 1} \frac{(\nabla \nabla \Theta)^2}{\Theta} + (k + 1) \Theta (1 - k) \Theta \sigma_k \frac{w_{11}^k}{h} + \frac{2}{h} \sigma_k \frac{w_{11}^k}{h}. \quad (5.14)
\]

To estimate (5.14). Let \( t \) be the arc-length of the great circle passing through \( x_0 \) with the unit tangent vector \( e_1 \), then

\[
\nabla \nabla \Theta - \frac{k}{k + 1} \frac{(\nabla \nabla \Theta)^2}{\Theta} \Theta + (k + 1) \Theta (1 + k) \Theta \sigma_k \nabla \nabla \nabla \nabla \frac{w_{11}^k}{h} = (k + 1) \Theta (1 + k) \Theta \sigma_k \nabla \nabla \nabla \nabla \sigma_k \Theta \sigma_k \frac{w_{11}^k}{h}. \quad (5.15)
\]

On one hand,

\[
\Theta = (f^{-1}) \varphi h G + f^{-1} G \varphi h \left( 1 + \frac{\varphi}{\varphi} \right) + f^{-1} \varphi h G \frac{G}{\varphi} \varphi. \quad (5.16)
\]

On the other hand,

\[
\Theta = (f^{-1}) \varphi h G + 2 (f^{-1}) \varphi G h \left( 1 + \frac{\varphi}{\varphi} \right) + 2 (f^{-1}) \varphi h G \frac{G}{\varphi} \varphi + f^{-1} \varphi h G \left( 1 + \frac{\varphi}{\varphi} \right) + f^{-1} \varphi h G \frac{G}{\varphi} \varphi \frac{G}{\varphi} \varphi. \quad (5.17)
\]
In view of (5.15),

\[
1 + \Theta^{-\frac{1}{n}}(\Theta^{\frac{1}{n}})_{u} = 1 + \frac{1}{k+1} \Theta^{-1} u - \frac{k}{(k+1)^2} \Theta^{-2} u^2
\]

\[
= 1 + \frac{1}{k+1} f(f^{-1})_{u} + \frac{2 f}{(k+1)h} (f^{-1})_{h} h_{i} \left(1 + \frac{\varphi'}{\varphi} h\right) + \frac{\varphi'}{(k+1)\varphi} h_{i}^2 \left(1 + \frac{\varphi'}{\varphi}\right) + \frac{h_{u}}{(k+1)h} \left(1 + \frac{\varphi'}{\varphi}\right) + \frac{h_{i}^2}{(k+1)h} \left(1 + \frac{\varphi'}{\varphi}\right) + \frac{2}{k+1} (f^{-1})_{f} G_{i} G_{i} + \frac{G_{i} h_{i}}{(k+1)G h} \left(1 + \frac{\varphi'}{\varphi}\right)
\]

\[
+ \frac{\varphi'}{(k+1)\varphi G} h_{i} G_{i} + \frac{h_{i}}{(k+1)h G} G_{i} + \frac{G_{u}}{(k+1)G} - \frac{k}{(k+1)^2} f^2 (f^{-1})_{i}^2 - \frac{k}{(k+1)^2} h_{i}^2 \left(1 + \frac{\varphi'}{\varphi}\right)^2
\]

\[
= 1 + \frac{1}{k+1} f(f^{-1})_{u} + \frac{2 f}{(k+1)^2} (f^{-1})_{f} h_{i} \left(1 + \frac{\varphi'}{\varphi} h\right) + \frac{h_{u}}{(k+1)h} \left(1 + \frac{\varphi'}{\varphi}\right) + \frac{h_{i}^2}{(k+1)h} \left(1 + \frac{\varphi'}{\varphi}\right) + \frac{2}{k+1} (f^{-1})_{f} G_{i} G_{i} - \frac{2 k}{(k+1)^2} h_{i} G_{i} \left(1 + \frac{\varphi'}{\varphi}\right)
\]

\[
+ \frac{2 k (f^{-1})_{f} h_{i}}{h (k+1)^2} \left(1 + \frac{\varphi'}{\varphi} h\right) - \frac{k}{(k+1)^2} f^2 (f^{-1})_{i}^2 + \frac{h_{i}^2}{(k+1)^2 h^2} \left(1 + \frac{\varphi'}{\varphi}\right) (\frac{\varphi'}{\varphi} - k)
\]

\[
+ \frac{2}{k+1} (f^{-1})_{f} G_{i} G_{i} + \frac{G_{i} h_{i}}{(k+1)G h} \left(1 + \frac{\varphi'}{\varphi} h\right) + \frac{\varphi'}{(k+1)\varphi G} h_{i} G_{i} + \frac{h_{i}}{(k+1)h G} G_{i} + \frac{G_{u}}{(k+1)G}
\]

\[
- \frac{k}{(k+1)^2} G_{i}^2 G_{i}^2 - \frac{2 k f}{(k+1)^2} (f^{-1})_{f} G_{i} - \frac{2 k}{(k+1)^2} h_{i} G_{i} \left(1 + \frac{\varphi'}{\varphi} h\right)
\]

\[
= 1 + \frac{\varphi' h}{\varphi} h_{u} + \frac{h_{u}^2}{(k+1)h} \left(1 + \frac{\varphi'h}{\varphi}\right) - \frac{1}{h (k+1)^2 f} \left[ h_{i} \left(\frac{k - \varphi'h}{f h}\right)^{\frac{1}{2}} - (f^{-1})_{i} \left(\frac{h f}{k - \varphi'h}\right)^{\frac{1}{2}}\right]^2
\]

\[
+ \frac{1}{k+1} \left[ (k - \frac{\varphi'h}{\varphi}) - (f^{-1})_{i} f^2 \left(\frac{k}{k+1} + \frac{1}{k+1} + \frac{\varphi'h}{\varphi} - k\right) + (f^{-1})_{u} f \right]
\]

\[
+ \frac{2}{k+1} (f^{-1})_{f} G_{i} G_{i} + \frac{G_{i} h_{i}}{(k+1)G h} \left(1 + \frac{\varphi'}{\varphi} h\right) + \frac{\varphi'}{(k+1)\varphi G} h_{i} G_{i} + \frac{h_{i}}{(k+1)h G} G_{i} + \frac{G_{u}}{(k+1)G}
\]

\[
- \frac{k}{(k+1)^2} G_{i}^2 G_{i}^2 - \frac{2 k f}{(k+1)^2} (f^{-1})_{f} G_{i} - \frac{2 k}{(k+1)^2} h_{i} G_{i} \left(1 + \frac{\varphi'h}{\varphi}\right).
\]

(5.18)
Making use of the assumptions of Theorem 1.1, one see that \( (1 + \varphi' h)' \geq 0 \), and \( \frac{\varphi'}{\varphi} h \leq -1 \). (5.18) becomes

\[
1 + \Theta \frac{\partial}{\partial t} (\Theta^{\frac{-1}{k+1}})u
\geq \frac{1 + \varphi' h}{k + 1} h + \frac{1}{k + 1} \left[ k - \frac{\varphi'}{\varphi} \right] - (f^{-1})^2 \frac{k}{k + 1} + \frac{1 + \varphi' h}{k + 1} h + (f^{-1})u \frac{f}{\varphi}
\]

\[
+ \frac{2}{k + 1} (f^{-1})G_i G + \frac{G_i h_i}{G} + \frac{G_i h_i}{(k + 1) G} h + \frac{\varphi'}{G} h + h_i G_i + \frac{h_i}{G} h + G + G + G + G
\]

\[
- \frac{k}{(k + 1)^2 G} + 2 \frac{k}{k + 1} \frac{G_i G_i}{G} + \frac{2 k}{k + 1} \frac{h_i h_i G_i}{G} + \frac{h_i G_i}{(k + 1)^2} G_i + G + G
\]

Using again the assumptions of Theorem 1.1, explicitly,

\[
\left( k - \frac{\varphi'}{\varphi} \right) - (f^{-1})^2 \frac{k}{k + 1} h + (f^{-1})u h
\]

\[
\geq (k + 1) - (f^{-1})^2 \frac{k}{k + 1} h + (f^{-1})u h
\]

\[
= (k + 1) + (k + \vartheta) f^{-\frac{1}{k+\vartheta}} (f^{-\frac{1}{k+\vartheta}})_u
\]

\[
= f^{-\frac{1}{k+\vartheta}} \left[ (k + 1) + (k + \vartheta) f^{-\frac{1}{k+\vartheta}} + (k + \vartheta) f^{-\frac{1}{k+\vartheta}}_u \right]
\]

\[
\geq c_0,
\]

where \( c_0 \) is a positive constant depending on \( f \) and the minimum eigenvalue of \((k + 1) f^{-\frac{1}{k+\vartheta}} + (k + \vartheta)(f^{-\frac{1}{k+\vartheta}})_u\). On the other hand,

\[
G_i = (\nabla G \cdot e_i) w_{ii},
\]

and

\[
G_{ii} = ((\nabla (G \cdot e_i) \cdot e_i) w_{i}^2 - (\nabla G \cdot x) w_{ii} + (\nabla G \cdot e_i) w_{ii}.\]
Using $C^0, C^1$ estimates, then (5.21) and (5.22) imply that
\[
\begin{align*}
&\frac{2}{k+1} (f^{-1})_i G_i + \frac{G_i h_i}{(k+1)G} \left( 1 + \frac{\varphi'}{\varphi} h \right) + \frac{\varphi'}{(k+1)\varphi G} h_i G_i + \frac{h_i}{(k+1)G} G_i + \frac{G_i}{(k+1)G} \\
&- \frac{k}{(k+1)^2} \frac{G^2}{G^2} - \frac{2k}{k+1} \frac{f}{G} (f^{-1})_i G_i - \frac{2k}{(k+1)^2} \frac{h_i G_i}{h G} \left( 1 + \frac{\varphi'}{\varphi} h \right) \geq -c_1 w_{ii} - c_2 w_{ii}^2 + \frac{1}{k+1} \frac{\nabla G \cdot e_i}{G} w_{ii}.
\end{align*}
\]
(5.23)

Substituting (5.23) and (5.20) into (5.19), we get
\[
\partial_t \frac{w_{ii}}{h} \leq -\frac{(w_{ii})^2}{h} \sigma_s \Theta \left( c_0 - c_1 w_{ii} - c_2 w_{ii}^2 + \frac{2 - \theta - k}{h} w_{ii} + \frac{\nabla G \cdot e_i}{G} w_{1ii} \right) + \frac{2w_{ii}}{h}.
\]
(5.24)

Using (5.11) into (5.24), we have
\[
\partial_t \frac{w_{ii}}{h} \leq -C_1 \left( \frac{w_{ii}}{h} \right)^2 + C_2 \frac{w_{ii}}{h},
\]
(5.25)
where $C_1$ and $C_2$ are positive constants independent of $t$. This give the upper bound of $w_{ii}$. The proof is completed.

**Lemma 5.4.** Supposing $k = n - 1$. Let $f$ be a smooth and positive function on $S^{n-1}$, $M_t$ be a smooth and strictly convex solution satisfying the flow (3.1), and let $\varphi, G$ be smooth functions satisfying $A$. Then
\[
\kappa_i(x, t) \leq C, \quad \forall (x, t) \in S^{n-1} \times (0, \infty)
\]
where $C$ is a positive constant, independent of $t$.

**Proof.** We consider the auxiliary function,
\[
\bar{E}(x, t) = \log \lambda_{\max}(\{w^{ij}\}) - d \log h + \frac{l}{2} \rho^2,
\]
(5.26)
where $d$ and $l$ are positive constants to be specified later, and $\lambda_{\max}(\{w^{ij}\})$ is the maximal eigenvalue of $\{w^{ij}\}$.

For any fixed $t \in (0, \infty)$, we assume that the maximum of $\bar{E}(x, t)$ is attained at $x_0$ on $S^{n-1}$. By a rotation of coordinates, we may assume that $\{w^{ij}(x_0, t)\}$ is diagonal, and $\lambda_{\max}(\{w^{ij}(x_0, t)\}) = w_{ii}(x_0, t)$. Then, (5.26) turns into
\[
E(x, t) = \log w_{ii} - d \log h + \frac{l}{2} \rho^2.
\]

At $x_0$, we have
\[
0 = \nabla_i E = -w_{ii}^1 \nabla_i w_{ii} - \frac{h_i}{h} + l \rho_i
= -w_{ii}^1 (h_{ii} + h_1 \delta_{ii}) - \frac{h_i}{h} + l \rho_i,
\]
(5.27)
and

\[ 0 \geq \nabla_{ii} E = -w^{11} \nabla_{ii} w_{11} + 2w^{11} \sum \omega^{ik} (\nabla_{i} w_{ik})^2 - (w^{11})^2 (\nabla_{i} w_{11})^2 - d \left( \frac{h_{ii}}{h} - \frac{h_{ii}^2}{h^2} \right) + l \rho^2 + l \rho \rho_{ii}. \] (5.28)

In addition,

\[ \partial_{i} E = -w^{11} \partial_{i} w_{11} - d \frac{h_{t}}{h} + l \rho \partial_{i} t = -w^{11} (h_{11t} + h_{t}) - d \frac{h_{t}}{h} + l \rho \partial_{i} t. \] (5.29)

Now, recall that

\[ \log(h_{t} + h) = \log \sigma_{n-1} + \chi(x, t), \] (5.30)

where

\[ \chi(x, t) = \log \left( \frac{1}{f} hG(\nabla h) \varphi(h) \right). \]

Differentiating (5.30) gives

\[ \frac{h_{ji} + h_{ij}}{h_{t} + h} = w^{ik} \nabla_{j} w_{ik} + \nabla_{j} \chi \]
\[ = w^{ii} (h_{jii} + h_{i} \delta_{ij}) + \nabla_{j} \chi, \] (5.31)

and

\[ \frac{h_{11t} + h_{11}}{h_{t} + h} = \frac{(h_{11} + h_{11})^2}{(h_{t} + h)^2} \]
\[ \begin{align*}
&= \sum w^{ij} \nabla_{11} w_{ii} - \sum w^{ij} w^{ik} (\nabla_{11} w_{ik})^2 + \nabla_{11} \chi. \end{align*} \] (5.32)

The Ricci identity on sphere shows

\[ \nabla_{11} w_{ij} = \nabla_{ij} w_{11} - \delta_{ij} w_{11} + \delta_{1i} w_{1j} - \delta_{1j} w_{1i} + \delta_{1j} w_{1i}. \]
Hence, apply the Ricci identity, (5.28), (5.29), (5.31) and (5.32), at \( x_0 \), we obtain

\[
\frac{\partial_t E}{h_t + h} = -w_{11}^1(h_{11} + h_t) - d \frac{h_t}{h(h_t + h)} + l \frac{\rho \rho_t}{(h_t + h)} = -w_{11}^1 \left[ h_{11} + h + h_{11} - h_t - h + h + h_t \right] - d \frac{h_t}{h(h_t + h)} + l \frac{\rho \rho_t}{(h_t + h)}
\]

\[
= -w_{11}^1 \frac{h_t + h_{11}}{h_t + h} + \frac{1}{h + 1} + h_t - w_{11} - d \frac{h_t}{h + h} + l \frac{\rho \rho_t}{(h_t + h)}
\]

\[
= -w_{11}^1 \frac{h_t + h_{11}}{h_t + h} + (1 + d) - w_{11} - d \frac{h_t}{h} + l \frac{\rho \rho_t}{(h_t + h)}
\]

\[
\leq -w_{11}^1 \sum_{i} w^{\mu}_{1} \nabla_{1} w_{1 \mu} + w_{11} \sum_{i} w^{\mu}_{i} w^{\nu}_{i} \left( \nabla_{1} w_{1 \nu} \right)^2 - w_{11} \nabla_{1} \chi + \frac{(1 + d)}{h + h} + l \frac{\rho \rho_t}{(h_t + h)}
\]

\[
\leq \sum_{i} w^{\mu}_{i} \left( \nabla_{1} w_{1 \mu} \right)^2 - 2w_{11} \sum_{i} w^{\mu}_{i} w^{\nu}_{i} \left( \nabla_{1} w_{1 \nu} \right)^2 + \sum_{i} w^{\mu}_{i} \left( h_{i} - \frac{h_{i}^2}{h} \right) - \sum_{i} w^{\mu}_{i} l \rho_{i} - \sum_{i} w^{\mu}_{i} l \rho_{i i}
\]

\[
+ w_{11} \sum_{i} w^{\mu}_{i} w^{\nu}_{i} \left( \nabla_{1} w_{1 \nu} \right)^2 - w_{11} \nabla_{1} \chi + \frac{(1 + d)}{h + h} + l \frac{\rho \rho_t}{(h_t + h)} + \sum_{i} w^{\mu}_{i} - (n - 1)w_{11}
\]

\[
\leq -d \frac{w^{\mu}_{i}}{h} - w_{11} \nabla_{1} \chi + \frac{(1 + d)}{h + h} + l \left[ \frac{\rho \rho_t}{(h_t + h)} - \sum_{i} w^{\mu}_{i} (l \rho_{i} + \rho_{i i}) \right].
\]  

(5.33)

It is simple to calculate

\[
\rho_t = \frac{h h_t + \sum h_k h_{k i}}{\rho},
\]

\[
\rho_i = \frac{h h_t + \sum h_k h_{k i}}{\rho} = \frac{h w_{i i}}{\rho},
\]

\[
\rho_{i j} = \frac{h h_{i j} + h h + \sum h_k h_{k j} + \sum h_k h_{k j}}{\rho} = \frac{h h_{i j} w_{i i} w_{j j}}{\rho^3}.
\]  

(5.34)

Applying (5.34), we have

\[
\frac{\rho \rho_t}{h_t + h} - \sum w^{\mu}_{i} (l \rho_{i} + \rho_{i i}) = \frac{h h_t}{h_t + h} - h \sum w^{\mu}_{i} h_{i i} - w^{\mu}_{i} \sum h_{i i}^2
\]

\[
- \frac{\left| \nabla_{1} \chi \right|^2}{h_t + h} + \sum h_k \nabla_{k} \chi
\]

\[
= h - \frac{\rho^2}{h_t + h} - (n - 1) h - \sum w_{i i} + \sum h_k \nabla_{k} \chi
\]

\[
\leq C - \frac{\rho^2}{h_t + h} - \sum w_{i i} + \sum h_k \nabla_{k} \chi.
\]  

(5.35)
Substituting (5.35) into (5.33), we obtain
\[
\frac{\partial_i E}{h_t + h} \leq -d \sum w^{ii} + C(d + l) + \frac{(1 + d - l\rho^2)}{h_t + h} - l \sum w_{ii} - w^{11} \nabla_{11} \chi + l \sum h_k \nabla_k \chi. \tag{5.36}
\]
Since
\[
\nabla_k \chi = -\frac{f_k}{f} + \frac{h_k}{h} + \frac{\varphi'}{\varphi} h_k + \frac{(\nabla G \cdot e_i) w_{iij}}{G},
\]
and
\[
\nabla_{11} \chi = \frac{-ff_{11} + f^2_1}{f^2} + \frac{hh_{11} - h_1^2}{h^2} + \frac{\varphi'' h_1^2 + \varphi' h_{11}}{\varphi} - \frac{(\varphi')^2 h_1^2}{\varphi^2} = \frac{[\nabla G \cdot e_1 w_{111}]}{G^2} + \frac{(\nabla^2 G \cdot e_1) \cdot e_1 w_{111}}{G} - \frac{(\nabla G \cdot e_1) w_{111}}{G} - \frac{(\nabla G \cdot \chi) w_{111}}{G}.
\]
It is clear to see that
\[
l \sum h_k \nabla_k \chi = l \sum h_k \left[ -\frac{f_k}{f} + \frac{h_k}{h} + \frac{\varphi'}{\varphi} h_k + \frac{(\nabla G \cdot e_i) \cdot w_{iij}}{G} \right]
\leq C_1 l + l \frac{\nabla G \cdot e_k}{G} + \frac{\nabla G \cdot e_k}{G} - \rho \rho = C_2 l,
\tag{5.37}
\]
and
\[
-w^{11} \nabla_{11} \chi = -w^{11} \left( \frac{-ff_{11} + f^2_1}{f^2} + \frac{hh_{11} - h_1^2}{h^2} + \frac{\varphi'' h_1^2 + \varphi' h_{11}}{\varphi} - \frac{(\varphi')^2 h_1^2}{\varphi^2} \right) + \frac{[\nabla G \cdot e_1 w_{111}]}{G^2} - \frac{(\nabla^2 G \cdot e_1) \cdot e_1 w_{111}}{G} - \frac{(\nabla G \cdot e_1) w_{111}}{G} + \frac{\nabla G \cdot \chi}{G}.
\tag{5.38}
\]
From (5.27) and (5.34), we get
\[
w^{11} w_{111} = -d \frac{h_i}{h} + l \rho \rho_i.
\]
Hence, (5.38) becomes
\[
-w^{11} \nabla_{11} \chi \leq C_2 w^{11} + C_3 + C_4 w_{111} + C_5 l + C_6 d.
\tag{5.39}
\]
Combining (5.37) and (5.39), we obtain
\[
l \sum h_k \nabla_k \chi - w^{11} \nabla_{11} \chi \leq C_1 l + C_2 d + C_3 w^{11} + C_4 w_{111} + C_5 l + C_6.
\tag{5.40}
\]
Substituting (5.40) into (5.36), if we choose \( l \gg d \), then
\[
\frac{\partial_i E}{h_t + h} \leq -d \sum w^{ii} + C(d + l) + C_3 w^{11} + C_4 w_{111} + C_5 l + C_6.
\tag{5.41}
\]
In view of (5.41), if we take
\[
d > C_3,
\]
then for \( w^{11} \) large enough, there is
\[
\frac{\partial_i E}{h_t + h} < 0,
\]
which yields
\[ E(x_0, t) = \bar{E}(x_0, t) \leq C \]
for some \( C > 0 \), independent of \( t \). This implies that the principal curvature is bounded from above.

\[ \square \]

6. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Lemmas 4.1, 4.2, 5.1, 5.2 and 5.3 show that (3.1) is uniformly parabolic in \( C^2 \) norm space. Then, by means of the standard Krylov’s regularity theory [26] of uniform parabolic equation, the estimates of higher derivatives can be naturally obtained, it implies that the long-time existence and regularity of the solution of (3.1). Moreover, there exists a uniformly positive constant \( C \), independent of \( t \), such that
\[ ||h||_{C^i_j(S^{n-1} \times [0, \infty))} \leq C \]
for each pairs of nonnegative integers \( i \) and \( j \).

With the aid of Arzelà-Ascoli theorem and a diagonal argument, we can extract a subsequence \( \{t_k\} \subset (0, \infty) \) such that there exists a smooth function \( \tilde{h}(x) \) satisfying
\[ ||h(x, t_k) - \tilde{h}(x)||_{C^i_j(S^{n-1})} \to 0 \]
for each nonnegative integer \( i \) as \( k \to \infty \).

Observe that the problem \( \frac{1}{\varphi}G \varphi \sigma_k = 1 \) does not have any variational structure, so we may expect the convergence of solutions for all initial hypersurfaces. Following the similar lines in [5], utilizing the assumptions of Theorem 1.1, as computed in (5.6),
\[ \partial_t \left( \frac{P}{h} - 1 \right) - \Theta \sigma^{ij}_k \nabla_{ij} \left( \frac{P}{h} - 1 \right) \]
\[ = -\frac{P}{h} \left( k + \frac{\varphi' h}{\varphi} + \frac{\nabla G \cdot X}{G} \right) \frac{P^2}{h^2} \left( k + \frac{\varphi' h}{\varphi} + \frac{(\nabla G \cdot h v)}{G} \right) + 2 \frac{\Theta}{h} \sigma^{ij}_k \nabla_i h \nabla_j \left( \frac{P}{h} - 1 \right) \]
\[ = \left( k + \frac{\varphi' h}{\varphi} + \frac{\nabla G \cdot X}{G} \right) \left( \frac{P}{h} - 1 \right) \frac{P}{h} + 2 \frac{\Theta}{h} \sigma^{ij}_k \nabla_i h \nabla_j \left( \frac{P}{h} - 1 \right). \]

Since \( k + \frac{\varphi' h}{\varphi} + \frac{\nabla G \cdot X}{G} < 0 \), choosing the initial hypersurface \( M_0 \) satisfying \( (\frac{P}{h} - 1)_{M_0} > 0 \), using (6.3), one see that the positivity of \( \frac{P}{h} - 1 \) is preserved along the flow, so we obtain
\[ \partial_t h = P - h > 0 \quad \text{for all } t \geq 0. \]

By Lemma 4.1,
\[ C \geq h(x, t) - h(x, 0) = \int_0^t (P - h)(x, t) dt, \]
This implies that
\[ \int_0^\infty (P - h)(x, t) dt \leq C. \]
Then there exists a subsequence of \( t_k \to \infty \) such that
\[ (P - h)(x, t_k) \to 0 \text{ as } t_k \to \infty. \]
Namely,
\[ \left( \varphi(h(x, t_k))G(X_{h_k})\sigma_k(x, t_k)\frac{1}{f(x)} - h(x, t_k) \right) \to 0 \text{ as } t_k \to \infty. \]
Passing to the limit, we obtain
\[ \varphi(\tilde{h}(x))G(\nabla \tilde{h})\tilde{\sigma}(x)\frac{1}{f(x)} = 1. \]
Thus the hypersurface with support function \( \tilde{h}(x) \) is our desired solution to
\[ \varphi(h(x))G(\nabla h)\sigma(x)\frac{1}{f(x)} = 1. \]

**Proof of Theorem 1.2.** Similarly, Lemmas 4.1, 4.2, 5.1, 5.2, 5.4 reveal that (6.1) holds. We are in a position to establish the functional (see also [25]) corresponding to (3.1) with \( k = n - 1 \) as
\[ J(t) = \int_{\mathbb{R}^{n-1}} \left( \int_0^t \frac{1}{\varphi(s)} ds \right) f(x) dx - \int_{\mathbb{R}^{n-1}} \left( \int_0^t G(s, u)s^{n-1} ds \right) du. \] (6.6)
Under the flow (3.1), the monotonicity of \( J(t) \) is showed as in the following
\[ \frac{dJ(t)}{dt} = \int_{\mathbb{R}^{n-1}} \frac{f(x)}{\varphi(h)} \frac{\partial h}{\partial t} dx - \int_{\mathbb{R}^{n-1}} G(\rho, u)\rho^{n-1} \frac{\partial \rho}{\partial t} du \\
= \int_{\mathbb{R}^{n-1}} \frac{f(x)}{\varphi(h)} \frac{\partial h}{\partial t} dx - \int_{\mathbb{R}^{n-1}} G(\rho, u)\frac{\partial h}{\partial t} \rho \frac{\partial \rho}{\partial t} du \\
= \int_{\mathbb{R}^{n-1}} \frac{\frac{\partial h}{\partial t}}{\varphi(h)} \left( \frac{f(x)}{\varphi(h)} - G(\rho, u)\sigma_{n-1}(x, t) \right) dx \\
= - \int_{\mathbb{R}^{n-1}} \frac{G\sigma_{n-1}(x, t) h^{1/2} - h}{\frac{1}{f}\varphi h} dx \leq 0. \] (6.7)
Using (6.7) and Lemmas 4.1, 4.2, there is
\[ C \geq J(0) - J(t) = \int_0^t \int_{\mathbb{R}^{n-1}} \frac{\left( G\sigma_{n-1}(x, t) h^{1/2} - h \right)^2}{\frac{1}{f}\varphi h} dxdy, \] (6.8)
where \( C \) is a positive constant, independent of \( t \). This implies that
\[ \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \frac{\left( G\sigma_{n-1}(x, t) h^{1/2} - h \right)^2}{\frac{1}{f}\varphi h} dxdy \leq C, \] (6.9)
Then, there exists a subsequence of $t_k \to \infty$ such that
\[
\int_{S^{n-1}} \left( G\varphi \sigma_{n-1}(x, t_k) h^{\frac{1}{n-1}} - h \right)^2 \frac{1}{f(x)} \varphi(x, t_k) \, dx \to 0, \quad \text{as } t_k \to \infty.
\]
(6.10)
Taking a limit, we get
\[
\int_{S^{n-1}} \left( G(\nabla \tilde{h})\varphi(\tilde{h}) \tilde{\sigma}_{n-1}(x) \tilde{h}^{\frac{1}{n-1}} - \tilde{h} \right)^2 \frac{1}{f(\tilde{h})} \varphi(x) \, dx = 0.
\]
(6.11)
This illustrates that
\[
\frac{1}{f(x)} G(\nabla \tilde{h})\varphi(\tilde{h}) \tilde{\sigma}_{n-1}(x) = 1.
\]

7. The uniqueness of the solution

Observe that, for general $\varphi$ and $G$, there is no uniqueness result for solution to (1.2). In what follows, we shall give a special uniqueness result for (1.2).

**Theorem 7.1.** Suppose $G(y) = G(|y|)$. If
\[
\varphi(m_1)G(m_2) \leq \varphi(s_1)G(s_2)m^{-k}
\]
(7.1)
holds for some positive $s_1, s_2, m \geq 1$. Then the solution to the equation
\[
\frac{1}{f(x)} \sigma_k(x)\varphi(h)G(\nabla h) = 1
\]
(7.2)
is unique. Here $1 \leq k \leq n - 1$, $k$ is an integer.

**Proof.** Assume $h_1$ and $h_2$ be two solutions of equation (7.2). To prove $h_1 = h_2$, on the one hand, we take by contradiction, with satisfying $\max \frac{h_1}{h_2} > 1$. Suppose $\frac{h_1}{h_2}$ achieves its maximum at $x_0 \in S^{n-1}$. It follows $h_1(x_0) > h_2(x_0)$. Let $\Lambda = \log \frac{h_1}{h_2}$. So, at $x_0$, one has
\[
0 = \nabla_{S^{n-1}} \Lambda = \frac{\nabla_{S^{n-1}} h_1}{h_1} - \frac{\nabla_{S^{n-1}} h_2}{h_2},
\]
(7.3)
and applying (7.3), there is
\[
0 \geq \nabla_{S^{n-1}}^2 \Lambda
\]
\[
= \frac{\nabla_{S^{n-1}}^2 h_1}{h_1} - \frac{\nabla_{S^{n-1}} h_1 \otimes \nabla_{S^{n-1}} h_1}{h_1^2} - \frac{\nabla_{S^{n-1}}^2 h_2}{h_2} + \frac{\nabla_{S^{n-1}} h_2 \otimes \nabla_{S^{n-1}} h_2}{h_2^2}
\]
(7.4)
Since $h_1$ and $h_2$ are solutions of equation (7.2), using (7.2) and (7.4), at $x_0$, we obtain

$$1 = \frac{\sigma_k(\nabla_{g^n-1}^2 h_2 + h_2 I)G(\nabla_{g^n-1}^1 h_2 + h_2 x_1)\varphi(h_2)}{\sigma_k(\nabla_{g^n-1}^2 h_2 + h_1 I)G(\nabla_{g^n-1}^1 h_1 + h_1 x_1)\varphi(h_1)}$$

$$= \frac{h_2^k \sigma_k \left( \frac{\nabla_{g^n-1}^1 h_2}{h_2} + I \right) G \left( h_2 \sqrt{\frac{\nabla_{g^n-1}^1 h_2}{h_2}} + 1 \right) \varphi(h_2)}{h_1^k \sigma_k \left( \frac{\nabla_{g^n-1}^1 h_1}{h_1} + I \right) G \left( h_1 \sqrt{\frac{\nabla_{g^n-1}^1 h_1}{h_1}} + 1 \right) \varphi(h_1)}$$

$$\geq \frac{h_2^k G \left( h_2 \sqrt{\frac{\nabla_{g^n-1}^1 h_2}{h_2}} + 1 \right) \varphi(h_2)}{h_1^k G \left( h_1 \sqrt{\frac{\nabla_{g^n-1}^1 h_1}{h_1}} + 1 \right) \varphi(h_1)}$$

Set $h_2(x_0) = \delta h_1(x_0)$ and $s = h_1 \left( \sqrt{\frac{\nabla_{g^n-1}^1 h_1}{h_1}} + 1 \right) (x_0)$. Then

$$\frac{\delta^k G(\delta s)\varphi(h_2)}{G(s)\varphi(h_1)} \leq 1,$$

i.e.,

$$G(\delta s)\varphi(\delta h_1) \leq G(s)\varphi(h_1)\delta^{-k}.$$

In light of the assumption in Theorem 7.1, $\delta \geq 1$, it implies that $h_1(x_0) \leq h_2(x_0)$, which is a contradiction. This reveals

$$\max \frac{h_1}{h_2} \leq 1. \quad (7.6)$$

On the other hand, interchanging the role of $h_1$ and $h_2$, applying the same argument as above, we have

$$\max \frac{h_2}{h_1} \leq 1. \quad (7.7)$$

Combining (7.6) and (7.7), this illustrates that $h_1 = h_2$. So, we complete the proof.

\[\square\]

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DEFORMING A HYPERSURFACE BY A CLASS OF GENERALIZED FULLY NONLINEAR CURVATURE FLOWS

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