Notes on duality theories of abelian groups

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Introduction

In December 2005, I had the good fortune of spending a week as the guest of Reinhard Winkler at Technische Universität Wien, and had the honour of being asked to deliver a mini-course on the topic of duality theory of abelian groups. During this visit, I came to realize that the only elementary text on this topic that I could refer a graduate student to, namely the one by Pontryagin [30], was published 20 years ago, and its well-organized content is half a century old. This is not to say that there are no excellent books that contain a lot of results on abelian topological groups (such as [12], [33], [26], or [2]), but my feeling is that they depart from the aim of presenting basic notions of duality theory in a self-contained and elementary manner. Therefore, encouraged by my host and his research group in Vienna, I decided to write up the notes that I prepared for the mini course (Chapters 1 and 2), and to keep developing it as a longer term project. In order to provide a smoother presentation, some elementary results were collected in the Appendix.

I use italics font for results that appear to be new and not part of the ”common knowledge,” while slanted font is used for all other statements. Let me know if I am wrong about any of them.

Past and future work

This notebook is under development, so all suggestions, comments, and questions are warmly welcome.

Chapter 1. This is definitely the only more-or-less ready piece, but I am not happy with my treatment of locally compact abelian groups. (Similarly to Roeder’s approach (cf. [32]), the only result borrowed from functional analysis is the Peter-Weyl theorem. I am still missing a nice proof of the structure theorem of compactly generated LCA groups, and a complete proof of the Pontryagin duality.)

Chapter 2. I have only a sketch. After a categorical introduction, which explains why cartesian closed categories are so interesting, I hope to cover k-groups (of Noble) and convergence groups in this chapter.

Appendix. It evolves as the chapters develop. It is a collection of results many readers might be familiar with, so I saw no point in including them into the chapters.

Future topics. Precompact (abelian) groups [the problem is that it requires the reader to be familiar with notions such as C -embedded subsets, etc]; examples of pathological or otherwise interesting groups [although [12] is probably the best source for this]; localic groups [it would be a nice example of group objects in a category].
Topics left out. Nuclear groups, topological vector spaces [these are both important, but I am not sure if I can present them in an elementary way].

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Chapter I

The Pontryagin dual

I.A. The evaluation homomorphism

I.1. Pontryagin dual. For Hausdorff topological groups \( G \) and \( H \), we denote by \( H(G;H) \) the space of continuous homomorphisms \( \gamma: G \rightarrow H \) equipped with the compact-open topology. (For a brief review of the compact-open topology, see Appendix B.) Since the property of being a homomorphism is equationally defined, \( H(G;H) \) is a closed subspace of \( C(G;H) \). Indeed,

\[
H(G;H) = \bigcap_{g_1, g_2 \in G} \{ f \in C(G;H) : f(g_1 g_2) = f(g_1) f(g_2) g \} \quad (I.1)
\]

and each of the sets in the intersection is closed. Topological groups have natural uniform structures, and the compact-open topology can also be realized as the topology of uniform convergence on compacta: if \( \gamma \) in \( H(G;H) \) if \( \frac{1}{n} \) converges uniformly to \( \frac{1}{n} \) for every compact subset \( K \subseteq G \).

For \( A \in \text{Ab}(\text{Top}) \), a character of \( A \) is a homomorphism \( \chi: A \rightarrow T \), where \( T = \mathbb{R}/\mathbb{Z} \). If \( \chi \) is continuous, then it must factor through the maximal Hausdorff quotient \( A = N_A \) of \( A \), because \( T \) is Hausdorff (cf. Proposition A.2). The Pontryagin dual \( \hat{A} \) of \( A \) is the group \( H(A;\mathbb{R}) \) of all continuous characters of \( A \) equipped with the compact-open topology. Since compact subsets of \( A = N_A \) are precisely the images of compact subsets in \( A \), one may view \( \hat{A} \) as the group \( H(A;T) \) both algebraically and topologically (cf. Corollary A.4).

We put \( A_d \) for the group \( A \) equipped with the discrete topology. The space \( C(\hat{A}_d;T) \) coincides with \( T^\hat{A}_d \), and thus compact. Therefore, its closed subspace \( \mathcal{K}_d \) is also compact; it carries the topology of pointwise convergence on \( A \), in other words, if \( \chi \) and only if \( (\alpha) \) for every \( \alpha \in A \). Since the compact-open topology is finer than the pointwise convergence one, \( \hat{A} \) ! \( \mathcal{K}_d \) is continuous.

I.2. Polar. The sets \( n = [\frac{1}{4n^2}; \frac{1}{4n}] \) form a base at \( 0 \) for \( T \), and have the property that if \( k \times 2 \) for \( k = 1; \ldots; n \), then \( x \in \mathbb{Z} \). (Here, \( [\frac{1}{4n}; \frac{1}{4n}] \) are identified with their images in \( T \).) For \( S \subseteq A \)}
and $\hat{A}$, their polar sets are defined as
\begin{align*}
S_B &= f \ 2 \hat{A} \ j \ (S) \ i_g; \\
C &= fa \ 2 \ A \ j8 \ 2; \ (a) \ 2 \ i_g = 1 \ (1); \quad (I.3)
\end{align*}

Since $S_B$ is closed in the topology of pointwise convergence, in particular, it is closed in the compact-open one. On the other hand, $C$ is closed in $A$ because each $2$ is continuous. It is immediately seen that $(C ; B)$ is a Galois-connection between subsets of $A$ and $\hat{A}$. Thus, $S_S S_B C$ and $C_B$.

I.3. Lemma. Let $\inj : A \to B$ be a morphism in $\text{Ab}(\text{Top})$.

(a) For $S \in A$, $\inj^1 (S_B) = \inj (S) Beth$.
(b) For $E$, $\inj^1 (C) = \inj (C) Beth$.
(c) For $S^0 \in B$, $\inj^1 (S_B^0) = \inj (S_B^0)$ Beth.
(d) For $S^0 \in \hat{A}$, $\inj (C) = \inj^1 (\inj^0 C)$ Beth.
(e) For $S \in A$, $\inj (S_B^C) = \inj (S_B^C)$ Beth.
(f) For $S^0 \in B$, $\inj^1 (S_B^0 C) = \inj^1 (S_B^0 C)$ Beth.

Proof. Since $\inj$ is continuous, it induces a continuous homomorphism $\inj : B \to \hat{A}$ defined by $\inj (C) = C$.

\begin{align*}
2 \inj (S_B) = (\inj (S)) 1 (1) + 2 S_B (1) 2 \inj^1 (S_B); \quad (I.4)
\end{align*}

and so (a) follows.

\begin{align*}
g 2 \inj (C) (1) + (g) 2 C (1) 8 2; \ (\inj (g)) 2 1 \\
(1) 8 2 \inj (1); \ (g) 2 1 (1) g 2 \inj (1) C; \quad (I.5)
\end{align*}

which shows (b).

(c) Since $\inj (\inj^1 (S^0))$ Beth, one has $S_B^0 \in (\inj^1 (S^0)_B)$. By (a) applied to $S = \inj^1 (S^0)_B$, $\inj (\inj^1 (S^0)_B) = \inj (\inj^1 (S^0)_B)$. Thus, $S_B^0 = \inj (\inj^1 (S^0)_B)$. (d) Since $\inj (\inj^1 (0))$ Beth, one has $\inj (\inj^1 (0))$. By (b) applied to $\inj (\inj^1 (0))$, $\inj (\inj^1 (0)) C = \inj (\inj^1 (0)) C$. Thus, $\inj (\inj^1 (0)) C = \inj (\inj^1 (0)) C$.

(e) follows from (a) and (d), and (f) follows from (b) and (c).

I.4. Lemma. Let $A$ be a subgroup, and let $H : A \to A$ be the canonical projection. Then:

(a) $H_B$ is a subgroup;
(b) $\hat{A} = H$ injective, and its image is $H_B$;
(c) $(6, 5)$ if $S^0 = A = H$ and $02 S^0$, then $H (S_B) = \hat{A} (S_B)$. 


Proof. (a) Since \( H^1 \) contains only the zero subgroup, \( (H) \) holds if and only if \( H \ ker T \). Thus, \( H^B \) is the annihilator subgroup of \( H \) in \( \hat{A} \).

(b) Since \( H \) is onto, \( \hat{H} \) is injective. Clearly, \( \text{Im} \hat{H} = H^B \), so in order to show the converse, let \( 2 H^B \). Then \( H \ ker T \), and thus induces a continuous character \( A = H \rightarrow T \) such that \( \hat{H} ( ) = H = \).

(c) One has \( H^{-1}(S^0) \) because \( 0 \leq S^0 \), and thus \( H^{-1}(S^0)^B \) \( H^B = \text{Im} \hat{H} \), by (b). Lemma I.3(c) applied to \( S = H^{-1}(S^0) \) yields \( H^{-1}(S^0)^B \) = \( H^{-1}(S^0) \) = \( S^B \). This completes the proof, because \( \hat{H} \) is injective.

For \( X \neq 2 \top \), we denote by \( K (X) \) the collection of compact subsets of \( X \). For \( G \neq 2 \top \), one puts \( N (G) \) for the collection of neighborhoods of \( e \) in \( G \).

I.5. Proposition. Let \( A \neq 2 \text{Ab}(\top) \).

(a) \( 2 \text{hom}(A; T) \) is continuous if and only if there is \( U \neq 2 \text{N} (A) \) such that \( (U) \).

(b) The collection \( \mathcal{K} \neq 2 \mathcal{K} (A) \) is a base at \( 0 \) to \( \hat{A} \).

(c) \( \hat{A} \) is equicontinuous if and only if there is \( U \neq 2 \text{N} (A) \) such that \( U^B \).

Proof. Observe that in order to establish continuity-like properties of homomorphisms of topological groups, it suffices to check them at a single point, and \( 0 \) is a convenient choice for that purpose.

(a) Necessity is obvious. So, in order to show sufficiency, let \( U \neq 2 \text{N} (A) \) be such that \( (U) \) \( 1 \). Because of the continuity of addition in \( A \), there is \( V \neq 2 \text{N} (A) \) such that \( V \vdash \{ z_{n \text{times}} \} + V \), and in particular, \( k (V) \) \( 1 \) for every \( k = 1; \ldots; n \). Therefore, \( (V) \) \( n \), as desired.

(b) Let \( C \neq 2 \mathcal{K} (A) \), and set \( K = C \neq 2 C \neq nC \). Then \( K \neq 2 \mathcal{K} (A) \), and \( (C) \) \( n \) for every \( C \mathcal{K} B \), because \( k (a) \) \( 1 \) for every \( a \in C \) and \( k = 1; \ldots; n \). Thus, we obtained \( K \mathcal{K} \neq 2 \mathcal{K} \hat{A} \mathcal{K} C \mathcal{K} \) \( n \), as desired.

(c) Because of the homogeneous structure of topological groups, equicontinuity of a family of characters can be checked at a single point. Equicontinuity at \( 0 \) means that for every neighborhood \( n \) of \( 0 \) there is \( V \neq 2 \text{N} (A) \) such that \( (V) \) \( n \) for every \( 2 \). By the argument presented in (a), this condition is satisfied for all \( n \) if and only if it is satisfied for \( n = 1 \). Since this condition for \( n = 1 \) is precisely \( U^B \), the proof is complete.

I.6. Evaluation homomorphism. Each \( a \neq 2 A \) gives rise to a continuous character \( \hat{a} \) of \( \text{hom}(A; T) \) given by evaluation: \( \hat{a} ( ) = (a) \). In particular, \( \hat{a} \) is a continuous character of \( \hat{A} \) (whose topology is finer than that of \( \text{hom}(A; T) \)). Thus, \( \hat{a} \neq 2 \hat{A} \), and the evaluation map

\[
\hat{a} : A \to \hat{A}
\]

(1.7)

\[
a \mapsto \hat{a}
\]

(1.8)

is a homomorphism of groups. In the sequel, we present necessary and sufficient conditions for \( \hat{A} \) to be continuous and an embedding (cf. Propositions I.7 and I.12), and it will become clear that in general, \( \hat{A} \) need not be continuous.
A collection $C$ of compact sets of a topological space $X$ is a **cobase** if for every compact subset $K \subseteq X$ there is $C \subseteq C$ such that $K \subseteq C$.

**I.7. Proposition.** ([28, 2.3]) Let $A \in \text{Ab}(\mathbb{T}_{\text{op}})$.  
(a) For every $U \in \mathcal{N}(\mathcal{A})$, $U^B$ is compact in $\mathcal{A}^\hat{}$.
(b) The following statements are equivalent:
   (i) $A$ is continuous;
   (ii) every compact subset of $\mathcal{A}^\hat{}$ is equicontinuous;
   (iii) $f^{U^B} j U \cap 2 N (\mathcal{A}) g$ is a cobase for $\mathcal{A}^\hat{}$;
   (iv) $f^{U^B \cap U \cap 2 N (\mathcal{A}) g}$ is a base for $\mathcal{A}^\hat{}$ at 0.
(c) $\ker A = \ker A^\hat{}$.

**Proof.** (a) As noted earlier, $U^B$ is closed in $\mathcal{A}^\hat{}$ (it is closed even in $\text{hom}(\mathcal{A};\mathbb{T})$). By Proposition I.5(c), it is also equicontinuous, and therefore it is compact by a standard Arzela-Ascoli type argument.

(b) The equivalence of (ii) and (iii) is an immediate consequence of Proposition I.5(c).

(i) Suppose that $A$ is continuous, and let $\mathcal{A}^\hat{}$ be compact. Then $B$ is a neighborhood of 0 in $\mathcal{A}^\hat{}$. Thus, by continuity of $A$, there is $U \in \mathcal{N}(\mathcal{A})$ such that $A \cap U \in B$, which is equivalent to $U \in B$. Therefore, by Proposition I.5(c), $A$ is equicontinuous.

(iii) If $V \in \mathcal{N}(\mathcal{A}^\hat{})$, then by Proposition I.5, there is a compact subset $\mathcal{A}^\hat{}$ such that $B \subseteq V$. Since $f^{U^B} j U \cap 2 N (\mathcal{A}) g$ is a cobase for $\mathcal{A}^\hat{}$, there is $U \in \mathcal{N}(\mathcal{A})$ such that $U \in B$. Therefore, $U \subseteq V$, as desired.

(iv) If $W \in \mathcal{N}(\mathcal{A}^\hat{})$, then there is $U \in \mathcal{N}(\mathcal{A})$ such that $U^B \subseteq W$, and therefore $U \subseteq W$. Hence, $A$ is continuous.

(c) For a $A$, one has

$$\ker A = 0$$

which completes the proof.

A combination of Propositions I.5(b) and I.7(b) yields:

**I.8. Corollary.** Let $A \in \text{Ab}(\mathbb{T}_{\text{op}})$. The following statements are equivalent:

(i) $A$ is continuous;
(ii) $f^{K} j K \cap 2 K \cap 2 (\mathcal{A}) g$ is a cobase for $\mathcal{A}^\hat{}$.

A map $f : X \rightarrow Y$ between Hausdorff spaces is said to be $k$-**continuous** if the restriction $f^{\hat{}}$ is continuous for every compact subset $K$ of $X$. Although $A$ need not be continuous (cf. Proposition I.7), its restriction to any compact subset of $\mathcal{A}$ is continuous.

**I.9. Theorem.** Let $A \in \text{Ab}(\mathbb{T}_{\text{haus}})$. The evaluation homomorphism $A$ is $k$-continuous.

**Proof.** By Corollary B.3, $e : A \rightarrow C (\mathcal{C}(\mathcal{A};\mathbb{T});\mathbb{T})$ is $k$-continuous. Since the image $e(\mathcal{A})$ is contained in $H (\mathcal{C}(\mathcal{A};\mathbb{T});\mathbb{T})$, and the map $H (\mathcal{C}(\mathcal{A};\mathbb{T});\mathbb{T}) \rightarrow H (\mathcal{C}(\mathcal{A};\mathbb{T});\mathbb{T}) = \mathcal{A}^\hat{}$ (given by restriction to $H (\mathcal{A};\mathbb{T})$) is continuous, $A$ is the composition of a $k$-continuous map with a continuous one.
I.10. Local quasi-convexity. Let $\mathcal{A} \in \text{Ab}(\mathcal{G})$. Following Banaszczyk, $S \in \mathcal{A}$ is quasi-convex if $S = S^{BC}$. The group $\mathcal{A}$ is locally quasi-convex (or briefly, LQC) if it admits a base of quasi-convex neighborhoods at 0, that is, if $\mathcal{F}U^{BC} \cup U \subseteq \mathcal{A}$ is a base at 0 (cf. [3]).

I.11. Proposition. ([6, 1]) The group $\hat{\mathcal{A}}$ is locally quasi-convex for every $\mathcal{A} \in \text{Ab}(\mathcal{G})$.

Proof. By Proposition I.5, the collection $f\mathcal{K} \cup \mathcal{K} \cap \mathcal{A}$ is a base at 0 at $\hat{\mathcal{A}}$. Each member of the collection is quasi-convex, because $K^{BC} = K^B$.

A variant of the next proposition appeared in [2, 6.10] and [9, 4.3], and seems to be a well-known result.

I.12. Proposition. Let $\mathcal{A} \in \text{Ab}(\mathcal{G})$.

(a) If $\mathcal{A}$ is an embedding, then $\mathcal{A}$ is locally quasi-convex.

(b) If $\mathcal{A}$ is locally quasi-convex and Hausdorff, then $\mathcal{A}$ is open onto its image.

(c) If $\mathcal{A}$ is locally quasi-convex and Hausdorff, then $\mathcal{A}$ is injective.

Proof. First, note that for $S \subseteq \mathcal{A}$, one has $S^{BC} = S^{BB} \setminus \mathcal{A}$. If $U \subseteq \mathcal{N} \subseteq \mathcal{A}$, then $U^B$ is compact in $\hat{\mathcal{A}}$ (cf. Proposition I.7(a)), and so $U^{BB} = \mathcal{U}$ is open in $\hat{\mathcal{A}}$ by Proposition I.5(b). Therefore, $\mathcal{A}(J^{BC})$ is open in $\mathcal{A}(\mathcal{A})$.

(a) Since $\mathcal{A}$ is an embedding, $\mathcal{A}(V) = \mathcal{A}(\mathcal{A})$ for every $V \subseteq \mathcal{N} \subseteq \mathcal{A}$. Thus, by Proposition I.7(b)(iv), there is $U \subseteq \mathcal{N} \subseteq \mathcal{A}$ such that $\mathcal{A}(J^{BC}) = U^{BB} \setminus \mathcal{A}(\mathcal{A}) \subseteq \mathcal{A}(V)$, which implies $U^{BC} = \mathcal{V}$, because $\mathcal{A}$ is injective.

(b) Let $V \subseteq \mathcal{N} \subseteq \mathcal{A}$, and using LQC pick $U \subseteq \mathcal{N} \subseteq \mathcal{A}$ such that $U^{BB} = \mathcal{V}$. Then one has $\mathcal{A}(J^{BC}) = \mathcal{A}(V)$, and $\mathcal{A}(J^{BC})$ is open in $\mathcal{A}(\mathcal{A})$.

(c) Let $a \neq 0$ be a non-zero element. Since $\mathcal{A}$ is Hausdorff and LQC, there exists $U \subseteq \mathcal{N} \subseteq \mathcal{A}$ such that $a \in U^{BC}$. Thus, there is $U \subseteq \mathcal{N}$ such that $\mathcal{A}(\mathcal{A}) = (a) \subseteq U$. Therefore, $\mathcal{A}(a) = 0$. Hence, $\mathcal{A}$ is injective.

I.13. Remarks. (1) In Proposition I.12(b), the map $\mathcal{A}$ need not be an embedding, because it is not necessarily continuous or injective.

(2) Following Neumann and Wigner, whenever $\mathcal{A}$ is injective, $\mathcal{A}$ is said to be maximally almost-periodic (or briefly, MAP; cf. [27]).

(3) For every $\mathcal{A} \in \text{Ab}(\mathcal{G})$, $\hat{\mathcal{A}}$ is injective. Indeed, if $\mathcal{A}(\mathcal{A}) = 0$, then for every $\hat{\mathcal{A}}$, $\mathcal{A}(\mathcal{A}) = 0$. Thus, for a $\mathcal{A}$ and $\mathcal{A}(\mathcal{A}) = 0$, one obtains $\mathcal{A}(a) = 0$. Therefore, $\mathcal{A}(\mathcal{A}) = 0$, as desired.

I.14. Corollary. Let $\mathcal{A} \in \text{Ab}(\text{Haus})$ be such that $\mathcal{A}$ is continuous. Then $\mathcal{A}$ is an embedding if and only if $\mathcal{A}$ is locally quasi-convex.

We denote by LQC the full subcategory of $\text{Ab}(\mathcal{G})$ formed by the locally quasi-convex groups, and present a functorial method of “turning” every group into an LQC group.

I.15. Theorem. LQC is an epireflective subcategory of $\text{Ab}(\mathcal{G})$, and the reflection $\mathcal{A}^H$ is given by equipping the group of $\mathcal{A}$ with a new group topology whose base at 0 is $fU^{BC} \cup U \subseteq \mathcal{A}$.
In less categorical language, Theorem I.15 states that for every \( A \in \text{Ab}(\mathcal{T}^\text{op}) \):

1. \( A^M \) is a topological group;
2. there is a continuous surjective homomorphism \( \sim_A : A \rightarrow A^M \) that is natural in \( A \);
3. \( \kappa^M : A^M \rightarrow B^M \) is continuous whenever \( \kappa : A \rightarrow B \) is a continuous homomorphism;
4. Every continuous homomorphism \( \kappa : A \rightarrow C \) into an LQC group \( C \) factors uniquely through \( A \), that is, there is a unique \( \kappa^M : A^M \rightarrow C \) such that \( \kappa = \kappa^M \circ \sim_A \).

**Proof.**

1. Each set of the form \( U^B = U \cap V \cap V \) is symmetric (i.e., \( U^B = (U \setminus V) \cap V \)), and it is easily seen that \( (U \setminus V)^B = U^B \setminus V^B \). Let \( U \) and \( V \) be such that \( U \cap V \neq \emptyset \). Then \( U^B \cap V^B \rightarrow U^B \cap V^B \). Therefore, the proposed candidate for \( \mathcal{A}^M \) defines a group topology on the underlying group of \( A \), as desired.
2. The topology of \( A^M \) is coarser than that of \( A \), so the identity homomorphism \( A \rightarrow A^M \) is continuous (and it is obviously natural in \( A \)).
3. Let \( U^2 = \mathcal{A}^M \). By Lemma I.3(f), \( \kappa^M : A^M \rightarrow A^M \) is continuous, and \( \kappa^M : A^M \rightarrow A^M \) is continuous, and therefore \( \kappa^M : A^M \rightarrow A^M \) is continuous.
4. Since \( A \) and \( A^M \) have the same underlying set, uniqueness of \( \kappa^M \) is clear. If \( C \in \text{LQC} \), then \( C = C^M \), and therefore \( \kappa = \kappa^M \), by (3).

**I.16. Corollary.** The limit of a family of locally quasi-convex groups formed in \( \text{Ab}(\mathcal{T}^\text{op}) \) is locally quasi-convex, and coincides with the limit formed in \( \text{LQC} \). In particular, \( \text{LQC} \) is closed under the formation of arbitrary products and subgroups.

**Proof.** The first statement is a well-known category theoretical property of reflective subcategories (cf. [24, IV.3, V.5]), and it implies the second one, as products are limits. For the third statement, let \( A \) be an LQC group and \( H \subset A \) be its subgroup. The reflection \( \mathcal{A}^H \) has the same underlying group as \( A^H \), and therefore

\[
\mathcal{A}^H = \text{Eq}(A \xrightarrow{H} \mathcal{A}^H); \tag{I.10}
\]

where \( \mathcal{A}^H \) is the canonical projection. Since equalizers are limits, this completes the proof.

**I.17. Corollary.** For every \( A \in \text{Ab}(\mathcal{T}^\text{op}) \), the underlying groups of \( \mathcal{A} \) and \( \mathcal{A}^M \) coincide.

**Proof.** By Proposition I.11 the group \( \hat{\mathcal{A}} = T \) is LQC. Thus, by Theorem I.15, every continuous character \( \kappa : A \rightarrow T \) gives rise to a continuous \( \kappa^M : A^M \rightarrow T \). Therefore, \( \mathcal{A} = \mathcal{A}^M \) as sets. The reverse inclusion is obvious, because \( A^M \) carries a coarser topology than \( A \).

**I.18. Proposition.** Let \( A \in \text{Ab}(\mathcal{T}^\text{op}) \) such that \( A \) is continuous. Then:

(a) \( \mathcal{A} \) and \( \mathcal{A}^M \) have the same compact subsets.
(b) \( A^M \) coincides with the group \( A \) equipped with the initial topology induced by \( A \);
(c) \( N_A^N = \ker A^N = \ker A \);
(d) \( A^N \) is continuous and open onto its image;
(e) \( \kappa_A : A \rightarrow A^M \) is an embedding;
Proof. Since \( \hat{\gamma}_A : \hat{K}^M \to \hat{A} \) is continuous, every compact subset of \( \hat{K}^M \) is also compact in \( \hat{A} \), and thus, by Proposition I.7(b), there exists \( U \subset N (A) \) such that \( U^B \).

(a) Suppose that \( \hat{\gamma}_A \) is compact. Then, by Proposition I.7(b), \( U^B \) for some \( U \subset N (A) \). By Proposition I.7(a), \( U^B = (U \times C)^B \) is compact in \( \hat{K}^M \). Since \( \hat{\gamma}_A \) is a closed subset of \( U^B \), is compact in \( \hat{K}^M \).

(b) By Proposition I.7(b), \( \bigcup U^B \bigcup U \subset N (A) \) is a base for \( \hat{A} \) at 0, because \( A \) is continuous. The statements follows from \( 1 (U^B) = U^B \).

(c) The first equality follows from (b) and Proposition A.2, because \( \hat{\gamma}_A \) is Hausdorff. For the second equality, observe that by Proposition I.7(c), \( \ker A = \bigcup \ker A \), and by (a), A and \( \hat{K}^M \) have the same continuous characters.

(d) By (a), polars of quasi-convex neighborhoods form a cobase to \( \hat{K}^M \), and thus \( \hat{\gamma}_A \) is continuous, by Proposition I.7(b). Therefore, the statement follows by Proposition I.12(b).

(e) By Proposition I.7(b), the collections \( \bigcup U^B \bigcup U \subset N (A) \) and \( \bigcup U^B \bigcup U \subset N (A) \) are bases at 0 to \( \hat{A} \) and \( \hat{K}^M \), respectively. Since every \( V \subset N (A) \) has the form of \( U^B \), this completes the proof.

I.19. Theorem. Let \( A \subset A \subset \text{Ab} (\text{Top}) \), and \( \hat{D} \) \( \hat{A} \) be a dense subgroup. Then:

(a) the underlying groups of \( \hat{D} \) and \( \hat{A} \) coincide;
(b) if \( \hat{D} \) is continuous, \( \hat{D} \) and \( \hat{A} \) have the same compact subsets, and \( \hat{A} \) is continuous;
(c) if \( \hat{D} \) is locally quasi-convex, then so is \( \hat{A} \).

Proof. (a) The inclusion \( \hat{\gamma}_D : \hat{D} \to \hat{A} \) induces a continuous homomorphism \( \hat{\gamma}_D : \hat{A} \to \hat{D} \), which is injective, because \( \hat{D} \) is dense in \( \hat{A} \). In order to show that \( \hat{\gamma}_D \) is surjective, let \( \hat{x} \in \hat{D} \). For each \( \hat{x} \in \hat{A} \), there is a net \( f(x) \in D \) such that \( x \to \hat{a} \). Thus, \( x \to \hat{a} \), and so \( x \to \hat{a} \). In other words, \( f(x) \in A \) is a Cauchy net in the complete group \( \hat{A} \), and therefore \( \lim_x \hat{a} \). Set \( (a) = \lim_x \hat{a} \); it is easily seen that \( a \) is well-defined and continuous. Hence, \( \hat{\gamma}_D (a) = a \), as desired.

(b) It suffices to show that \( \hat{\gamma}_D (a) = a \) is compact in \( \hat{A} \) for every compact subset \( \hat{D} \), because \( \hat{\gamma}_D \) is continuous. By Proposition I.7(b), since \( \hat{D} \) is continuous, \( \hat{D} \) is equicontinuous on the dense subgroup \( \hat{D} \), which means that there is \( U \subset N (A) \) such that \( (U \setminus \hat{D})^B \) (cf. Proposition I.5(b)). It follows from the density of \( \hat{D} \) that \( \text{Int} U \setminus \hat{D} \setminus U \), and therefore

\[
(U \setminus \hat{D})^B = (U \setminus \hat{D})^B \cap (U \setminus U)^B \quad (\text{I.}11)
\]

Hence, \( \hat{\gamma}_D (a) = a \) is a closed subset of the compact subset \( \text{Int} U^B \) of \( \hat{A} \) (cf. Proposition I.5(a)). Since \( \text{Int} U \subset N (A) \), this also shows that every compact subset of \( \hat{A} \) is contained in \( U^B \) for some \( U \subset N (A) \), which means, by Proposition I.7(b), that \( \hat{A} \) is continuous.

(c) Let \( V \subset N (A) \), and pick \( V \subset N (A) \) such that \( V_1 \supset V_1 \supset V \), so \( V_1 \cap \text{Int} V \) (cf. Proposition A.1(a)). There exists \( W_1 \subset N (D) \) such that \( W_1 \setminus D \supset V_1 \setminus D \), because \( V_1 \setminus D \subset 2 N (D) \) and \( D \) is LQC. (For \( \hat{D} \), \( \hat{D} \subset \hat{A} \) is \( \hat{\gamma}_D ( \bigcup \hat{D} \), because \( \hat{D} \subset \hat{A} \) as sets.) There is \( W \subset N (A) \) such that \( W_1 \setminus \hat{D} \), and since \( \hat{D} \) is dense, one has \( W_1 \setminus \hat{D} \), thus \( W_1 \setminus \hat{D} \). By (a), \( \hat{W}_1 \setminus \hat{D} \), and therefore

\[
\text{Int}(W_1^{\hat{B}}) \setminus \hat{D} = W_1^{\hat{B}} \setminus \hat{D} \quad V_1 = V \quad (\text{I.}12)
\]
For $U \in \mathcal{N}(\mathbb{A})$ such that $U + U \subseteq W$, one has $U^B + U^B \subseteq W^B$, and so $U^B = \text{Int}(W^B)$ (cf. Proposition A.1(a)). Hence, we found $U \in \mathcal{N}(\mathbb{A})$ such that $U^B \subseteq V$, as desired. \hfill \Box

1.20. Corollary. The completion of every locally quasi-convex group is locally quasi-convex. \hfill \Box

1.21. Precompactness. A subset $S \subseteq G$ of $G$ is said to be precompact if for every $U \in \mathcal{N}(\mathbb{A})$ there is a finite subset $F \subseteq G$ such that $S \subseteq FU$. Similarly to compactness, if $S \subseteq G$ is precompact and $S_1 \subseteq S$, then so are $S_1, S_1 + S_0$ and $\nu(S)$ for every continuous homomorphism $\nu : G \to H$. Our interest in this property arises from the following theorem on uniform spaces (cf. [17] and [13, 8.3.16]):

1.22. Theorem. A uniform space $(X;U)$ is compact if and only if it is complete and precompact.

1.23. Proposition. Let $A \in \text{Ab}(\text{Haus})$ be a locally quasi-convex group. Then:

(a) $\hat{A}(E)$ is precompact in $A$ for every equicontinuous subset $E \subseteq \hat{A}$;
(b) $K^B$ is precompact in $A$ for every compact subset $K \subseteq A$;
(c) if $K$ is continuous, then $\hat{A}(E)$ is precompact for every compact $F \subseteq \hat{A}$.

Proof. First, we note that by Proposition I.12, $\hat{A}(\mathbb{A})$ is continuous.

(a) By Proposition I.5(c), there is $V \in \mathcal{N}(\mathbb{A})$ such that $E \subseteq V^B$. By Proposition I.7(a), $V^B$ is compact in $\hat{A}$, and so $V^B \subseteq \hat{A}(\mathbb{A})$ is precompact. Thus, the continuous homomorphic image $\hat{A}(V^B \setminus \hat{A}(\mathbb{A}))$ is also precompact, and contains $\hat{A}(E)$. Therefore, $\hat{A}(E)$ is precompact.

(b) The set $K^B$ is open in $\hat{A}$, and thus $K^B$ is compact and equicontinuous in $\hat{A}$ (cf. Propositions I.7(a) and I.5(c)). Hence, by (a), $K^B = \hat{A}(K^B)$ is precompact in $\mathbb{A}$.

(c) By Propositions I.8, in this case, $FK^B \subseteq \hat{A}$ is a cobase for $\hat{A}$, and so the statement follows from (b). \hfill \Box

1.24. Quasi-convex compactness. It is a well-known that the closed convex hull of a weakly compact subset of a Banach space is weakly compact. In fact, the property of preservation of compactness under formation of closed convex hull characterizes completeness in the category of metrizable locally convex spaces (cf. [29, 2.4]). Motivated by this, one says that $A \in \text{Ab}(\text{Top})$ has the quasi-convex compactness property (or briefly, $A$ is QCP) if for every compact subset $K \subseteq A$, the quasi-convex hull $K^B$ (of $K$ in $A$) is compact.

1.25. Proposition. Let $A \in \text{Ab}(\text{Top})$ be such that $\mathbb{A}$ is continuous. Then $\hat{A}$ has the quasi-convex compactness property.

Proof. Let $\hat{A}$ be compact. By Proposition I.7(b), there is $V \in \mathcal{N}(\mathbb{A})$ such that $U^B$, and so $U^B = \text{Int}(W^B)$. Therefore, the closed set $U^B$ is a subset of the compact set $U^B$ (cf. Proposition I.7(a)). Hence, the result follows. \hfill \Box

1.26. Proposition. Let $A \in \text{Ab}(\text{Haus})$ be a locally quasi-convex group. If

(a) $A$ is complete, or
(b) $\mathbb{A}$ is surjective,
then $A$ has the quasi-convex compactness property.
Proof. By Proposition I.12, $\lambda^1: \hat{A} \to A$ is continuous.
(a) By Proposition I.23, $K^B C$ is precompact in $A$, and it, being a closed subspace, is complete. Therefore, by Theorem I.22, it is compact.
(b) Since $K^B$ is open in $\hat{A}$, $K^B C$ is compact in $\hat{A}$, and therefore $K^B C = \lambda^1 (K^B C)$ is compact, because $\lambda^1$ is continuous. □

I.27. Proposition. Let $A \in Ab(Haus)$ be such that $\hat{A}$ and $\hat{A}$ are continuous. The following statements are equivalent:
(i) $\lambda^1 (F)$ is compact for every compact subset $F \subset \hat{A}$;
(ii) $A$ has the quasi-convex compactness property.

Proof. (i) (ii) is immediate, because $K^B C$ is compact in $\hat{A}$ and $K^B C = \lambda^1 (K^B C)$.
(ii) (i): Let $F \subset \hat{A}$ be compact. Since $F$ is closed and $\lambda^1$ is continuous, $\lambda^1 (F)$ is also closed in $\hat{A}$. By Corollary I.8, there is a compact subset $K \subset A$ such that $F \subset K$ (because $\hat{A}$ is continuous). Thus, $\lambda^1 (F) = \lambda^1 (K^B C) = K^B C$, and $K^B C$ is compact because $A$ is QCP. Therefore, $\lambda^1 (F)$, being the closed subset of a compact set, is compact. □

I.B. Special groups and subgroups

I.28. Metrizable groups. Every Hausdorff topological group is completely regular (nice reference??), so by Tychonoff’s metrization theorem, if a Hausdorff group is second countable, then it is metrizable. But for topological groups, second countability is more than necessary in order to warrant metrizability. Theorem I.29 below (originally proved by Kakutani and Birkhoff in 1936) can also be obtained from well-known results on uniform spaces (cf. [13, 8.1.10, 8.1.21]).

I.29. Theorem. Every first countable Hausdorff topological group is metrizable. □

We turn to abelian metrizable groups. Recall that a Hausdorff space $X$ is a $k$-space if $F \subset X$ is closed in $X$ whenever $F \setminus K$ is closed for every $K \subset X$. (For details, see Appendix C.)

I.30. Theorem. ([10, Theorem 1]) Let $A \in Ab(Met)$. Then $\hat{A}$ is a $k$-space.

Although Theorem I.30 has a non-commutative generalization (cf. [23], [22, 3.4]), we provide here the original proof by Chasco, enriched with a few words of explanation.

Proof. In order to show that $\hat{A}$ is a $k$-space, let $\hat{A}$ be such that $\setminus$ is closed in $\hat{A}$ for every compact subset $\hat{A}$. We show that for every $\emptyset$, there exists a compact subset $K \subset A$ such that $\setminus = \emptyset$, and so $\setminus$ is closed in $\hat{A}$. Without loss of generality, we may assume that $\emptyset = 0$.

Since $A$ is metrizable, 0 has a decreasing countable base $\mathcal{F}_n$, and we set $U_0 = A$. Our aim is to construct inductively a family $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ of finite subsets of $A$ such that for every $n \in \mathbb{N}$,

$$F_n \setminus U_n,$$ (I.13)

$$\bigcap_{k=0}^{n} F_k \setminus U_{n+1} = \emptyset.$$ (I.14)
By Proposition I.7(a), $U^B_1$ is compact in $\hat{\mathcal{A}}$, and thus by our assumption, $U^B_1 \setminus x$ is closed in $U^B_1$. Thus, $U^B_1 \setminus x$ is compact in $\hat{\mathcal{A}}$, and so it is compact in the pointwise topology carried by $\hom(\mathcal{A}, T)$ (which is coarser than the compact-open one). In particular, $U^B_1 \setminus x$ is closed in the pointwise topology. A basic neighborhood of $0$ in the pointwise topology has the form $F^B$, where $F \in \mathcal{A}$ is finite, so $0 \notin U^B_1 \setminus F$ implies that there exists a finite subset $F_0$ of $\mathcal{A} = U_0$ such that $F_0 \setminus U^B_1 \setminus x = \emptyset$. This completes the proof for $n = 0$.

Suppose that we have already constructed $F_0; \ldots; F_{n-1}$ such that (I.13) and (I.14) hold. For each $x \in 2^U$, put

$$x = \bigcap_{k=0}^{n-1} F^B_k \setminus f(x)^B \setminus U^B_{n+1}.$$  \hfill (I.15)

Since $U_n \setminus U_{n+1}$, one has $U^B_n \setminus U^B_{n+1}$, and thus, by the inductive hypothesis,

$$\setminus x = \bigcap_{k=0}^{n-1} F^B_k \setminus U^B_n \setminus U^B_{n+1} = \bigcap_{k=0}^{n-1} F^B_k \setminus U^B_n = \emptyset.$$ \hfill (I.16)

The $x$ are closed subsets of the compact space $\hat{\mathcal{A}}$, and their intersection is empty. Therefore, there must be a finite subset $F_n \subseteq U_n$ such that $\setminus x = \emptyset$. Hence,

$$\setminus x = \bigcap_{k=0}^{n-1} F^B_k \setminus F^B_n \setminus U^B_{n+1} \emptyset.$$ \hfill (I.17)

as desired. Set $K = \bigcup_{n=0}^{\infty} F_n \setminus f(x)^B$. It follows from (I.13) that $K \cap U_n$ is finite for every $n \geq 2$, and thus $K$ is sequentially compact (because every sequence without a constant subsequence must converge to $0 \in K$). Therefore, $K$ is compact, because $\mathcal{A}$ is metrizable. By the construction of $K$, $K \setminus U^B_n \setminus x = \emptyset$ for every $n \geq 2$. By Proposition I.5(a), $\hat{\mathcal{A}} = \bigcup_{n=0}^{\infty} U^B_n$, and therefore $K \setminus x = \emptyset$, as desired.

I.31. Corollary. Let $\mathcal{A} \in 2^\mathcal{A} \setminus \mathcal{M} \setminus \mathcal{E}$. Then:

(a) $\mathcal{A}$ is continuous;
(b) $\kappa$ is continuous;
(c) ([10, Corollary 1]) $\hat{\mathcal{A}}$ is complete and metrizable.

Proof. (a) Since $\mathcal{A}$ is metrizable, its topology is determined by convergent sequences $x_n \to x_0$. For such a sequence, $f(x_n) \in 2^N \setminus \{f(x_0)\}$ is compact, and thus $\mathcal{A}$ is a $k$-space. Therefore, by Theorem I.9(b), $\mathcal{A}$ is continuous.

(b) By Theorem I.30, $\hat{\mathcal{A}}$ is a $k$-space, and so, by Theorem I.9(b), $\kappa$ is continuous.

(c) By Proposition I.7(b), $fU^B \cup 2^N \setminus (\mathcal{A})g$ is a base for $\hat{\mathcal{A}}$ at $0$, because $\mathcal{A}$ is continuous. Thus, $\hat{\mathcal{A}}$ is first-countable, because $\mathcal{A}$ is so. (Recall that for topological groups, being first-countable is equivalent to being metrizable.) By Theorem I.30, $\hat{\mathcal{A}}$ is a $k$-space, and thus $C(\hat{\mathcal{A}}, T)$ is
complete, because \( \mathbb{T} \) is so (cf. [20, 7.12]). Since \( \hat{A} \) is a closed subspace of \( C(\mathbb{R}; \mathbb{T}) \), this completes the proof.

A combination of Proposition I.27 and Corollary I.31 yields:

**I.32. Corollary.** Let \( A \) be an Abelian group. Then \( A \) has the quasi-convex compactness property if and only if \( A^1(\mathcal{F}) \) is compact in \( A \) for every compact \( \mathcal{F} \).

**Proof.** By Theorem I.19, \( \hat{A} \) and \( \hat{A} \) have the same underlying groups and compact subsets. On the other hand, by Theorem I.30, both \( \hat{A} \) and \( \hat{A} \) are k-spaces, because \( D \) and \( A \) are metrizable. Therefore, \( \hat{A} \) and \( \hat{A} \) have the same topology, because the topology of a k-space is determined by its compact subsets.

**I.33. Corollary.** ([10, Theorem 2], [22, 3.7]) Let \( A \) be an Abelian group, and let \( D \) be its dense subgroup. Then \( \hat{D} = \hat{A} \) as topological groups.

**Proof.** By Theorem I.19, \( \hat{D} \) and \( \hat{A} \) have the same underlying groups and compact subsets. On the other hand, by Theorem I.30, both \( \hat{D} \) and \( \hat{A} \) are k-spaces, because \( D \) and \( A \) are metrizable. Therefore, \( \hat{A} \) and \( \hat{A} \) have the same topology, because the topology of a k-space is determined by its compact subsets.

**I.34. Theorem.** Let \( A \) be a locally quasi-complete and complete.

(i) \( A \) is locally quasi-complete and complete;

(ii) \( A \) is a closed embedding;

(iii) \( A \) has the quasi-convex compactness property and \( \lambda \) is injective.

Ostling and Wilansky showed that a locally convex metrizable vector space is complete if and only if the absolutely convex closure of compact subset is compact (cf. [29, 2.4]). Hernández proved a far reaching generalization of this result for metrizable groups, namely, that if \( A \) is complete, let \( ^*A \) be its dense subgroup. Then \( \hat{D} = \hat{A} \) as topological groups.

**Proof.** (i) \( \Lambda \) is injective. Let \( K \) be compact. Then \( K^{BB} \) is compact in \( \hat{A} \), and since \( \Lambda^{-1}(\mathcal{F}) \) is closed, \( K^{BB} \setminus \Lambda^{-1}(\mathcal{F}) \) is also compact. Therefore, \( K^{BC} = A \setminus (K^{BB} \setminus \Lambda^{-1}(\mathcal{F})) \) is compact too.

(ii) \( \lambda \) is injective. Let \( K \) be compact. Then \( K^{BB} \) is compact in \( \hat{A} \), and since \( \Lambda^{-1}(\mathcal{F}) \) is closed, \( K^{BB} \setminus \Lambda^{-1}(\mathcal{F}) \) is also compact. Therefore, \( K^{BC} = \Lambda^{-1}(K^{BB} \setminus \Lambda^{-1}(\mathcal{F})) \) is compact too.

(iii) \( \lambda \) is injective. By Corollary I.32, QCP implies that \( A^1(\mathcal{F}) \) is compact for every compact subset \( \mathcal{F} \). So, if \( K \) is closed, then \( K \) is also compact, and thus \( \Lambda^{-1}(K) = \Lambda^{-1}(\mathcal{F}) \) is a compact subset of \( \mathcal{F} \), because \( \Lambda \) is continuous. Therefore, \( \Lambda^{-1}(\mathcal{F}) \) is closed, and hence, \( \Lambda^{-1}(\mathcal{F}) \) is k-continuous.

By Corollary I.31(c), \( \hat{A} \) is metrizable, and so is its subspace \( \hat{\Lambda}(\mathcal{F}) \). In particular, \( \hat{\Lambda}(\mathcal{F}) \) is a k-space, and therefore \( \Lambda^{-1}(\mathcal{F}) \) is continuous. This shows that \( \Lambda \) is an embedding, and hence, by Proposition I.12(a), \( A \) is LQC.

In order to show that \( \Lambda \) is complete, let \( B \) be the completion of \( A \). By Corollary I.33, \( \hat{A} = \hat{B} \) (as topological groups), because \( A \) is a dense subgroup of the metrizable group \( B \). Let \( f a_n g \) be a
Cauchy-sequence in $A$. Then $a_n \to b$ for some $b \in B$, and $K_1 = \{a_n \mid n \in \mathbb{N}\}$ is compact in $B$. So, $K_1 \cap 2 N = N (\hat{A})$, and by Proposition I.5(b), there is a compact subset $K \subseteq A$ such that $K \cap B = K_1$. Thus, $\{a_n \mid n \in \mathbb{N}\}$ is a compact Hausdorff, and so sequentially compact (because $A$ is metrizable). Therefore, $\{a_n \mid n \in \mathbb{N}\}$ has a convergent subsequence in $K$, which shows that $2 B \subseteq K$ is compact, and so sequentially compact (because $A$ is metrizable). Hence, $A$ is complete, as desired.

I.35. *Compact and discrete groups.* Let $A$ be a non-zero element, and let $x \in T$ be such that $\circ(a) = \circ(x)$. This defines a group homomorphism $a : \hat{A} \to T$ such that $\hat{A}(a) \notin 0$. Since $T$ is an injective abelian group, $a$ extends to $\hat{A} : A \to T$, and $A \to \hat{A}$, because $A$ is discrete. Thus, $\hat{A}(a) = \hat{A}(a) \notin 0$, which shows that $A$ is injective. By Lemma I.36(d), $\hat{A}$ is discrete, and thus $A$ is an isomorphism of topological groups.

**Proof.** (a) Let $a \in A$ be a non-zero element, and let $x \in T$ be such that $\circ(a) = \circ(x)$. This defines a group homomorphism $a : \hat{A} \to T$ such that $\hat{A}(a) \notin 0$. Since $T$ is an injective abelian group, $a$ extends to $\hat{A} : A \to T$, and $A \to \hat{A}$, because $A$ is discrete. Thus, $\hat{A}(a) = \hat{A}(a) \notin 0$, which shows that $A$ is injective. By Lemma I.36(d), $\hat{A}$ is discrete, and thus $A$ is an isomorphism of topological groups.

Suppose that $A$ is finite and generated. Then it decomposes into the direct sum of cyclic groups $A = \bigoplus_{n \in \mathbb{N}} a_n \mathbb{Z}$, where $a_n \mathbb{Z}$ is finite cyclic group are self-dual, and $\hat{A} = T$ and $\hat{A} = \mathbb{Z}$. Thus, surjectivity of $A$ follows, because the Pontryagin dual is an additive functor. (More details??)

In the general case, let $\hat{A}$ be a non-zero element, and let $x \in T$ be such that $\circ(a) = \circ(x)$. This defines a group homomorphism $a : \hat{A} \to T$ such that $\hat{A}(a) \notin 0$. Since $T$ is an injective abelian group, $a$ extends to $\hat{A} : A \to T$, and $A \to \hat{A}$, because $A$ is discrete. Thus, $\hat{A}(a) = \hat{A}(a) \notin 0$, which shows that $A$ is injective. By Lemma I.36(d), $\hat{A}$ is discrete, and thus $A$ is an isomorphism of topological groups.

(b) Since $A$ is compact Hausdorff, it is a $k$-space, and so $A$ is continuous (cf. Theorem I.9). By the Peter-Weyl theorem, for every non-zero $a \in A$, there is $A \to \hat{A}$ such that $\hat{A}(a) \notin 0$ (cf. [30, Thm. 32]). Thus, $A$ is injective, and therefore it is a closed embedding (because $A$ is compact). In order to show that $A$ is surjective, assume the contrary, that is, assume that $A$ is a proper closed subgroup of $A$. By Lemma I.36(c), $\hat{A}$ is compact, and thus so is the quotient
\( \widehat{\mathcal{A}} = \mathcal{A}(\mathcal{A}) \). By the Peter-Weyl theorem applied to \( \widehat{\mathcal{A}} = \mathcal{A}(\mathcal{A}) \), there exists a non-zero continuous character \( \chi : \widehat{\mathcal{A}} = \mathcal{A}(\mathcal{A}) \to \mathbb{T} \), which induces a non-zero \( \mathcal{A}\mathcal{H} \) such that \( \mathcal{A}(\mathcal{A}) = \mathcal{H}0g \) and \( 0 = 0 \). Since \( \mathcal{A}\mathcal{H} \) is discrete (cf. Lemma I.36(a)), by (a), \( \mathcal{A}\mathcal{H} \) is an isomorphism. In particular, there is a non-zero \( \mathcal{A}\mathcal{H} \) such that \( \mathcal{A}(\mathcal{A}) = \mathcal{H}0g \). Therefore, for a \( \mathcal{A}\mathcal{H} \),

\[
(a) = (\mathcal{A}(a))(\chi) = (\mathcal{A}(\chi))(\mathcal{A}(a)) = (\mathcal{A}(a))2 \quad (\mathcal{A}(\mathcal{A})) = \mathcal{H}0g; \tag{I.18}
\]

Hence, \( = 0 \) (and so \( = 0 \)) contrary to our assumption, which completes the proof. \( \square \)

I.38. Bohr-compactification. The Čech-compactification \( X \) of a Tychonoff space \( X \) has the property that every continuous function \( f : X \to K \) into a compact Hausdorff space \( X \) extends to \( X \), and thus factors uniquely through the dense embedding \( X \to X \). It appears to be less known, however, that \( X \) exists even when \( X \) is not Tychonoff, in which case the map \( X \to X \) is only continuous but need not be injective or open onto its image. (Every continuous map of \( X \) into a compact Hausdorff space still factors uniquely through \( X \).) In a categorical language, one says that the category \( \mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P} \) of compact Hausdorff spaces and their continuous maps is a reflective subcategory of \( \mathcal{T}\mathcal{O}\mathcal{P} \).

Similarly to the relationship between \( \mathcal{T}\mathcal{O}\mathcal{P} \) and \( \mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P} \), the full subcategory \( \mathcal{G}\mathcal{P}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) of compact Hausdorff groups is reflective in \( \mathcal{G}\mathcal{P}(\mathcal{T}\mathcal{O}\mathcal{P}) \) (topological groups and their continuous homomorphisms). The reflection is called the Bohr-compactification, and is denoted by \( \mathcal{G} : G \to \mathcal{B}G \). We show its existence for abelian groups, where the construction is rather simple

I.39. Theorem. For every \( 2 \mathcal{A} \) \( 2 \mathcal{A} \mathcal{H}(\mathcal{T}\mathcal{O}\mathcal{P}) \) there is \( \mathcal{B}A \), \( 2 \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) and a continuous homomorphism \( \mathcal{A} ! \mathcal{A} \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) such that every continuous homomorphism \( \mathcal{A} ! \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) factors uniquely through \( \mathcal{A} \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \).

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\pi} & \mathcal{K} \\
\mathcal{A} & \searrow & \\
& \mathcal{B} \mathcal{A}
\end{array}
\] (I.19)

Moreover, \( \mathcal{B} \mathcal{A} = (\mathcal{A})_d \), and \( \ker \mathcal{A} = \ker \mathcal{A} \).

Proof. Since \( \mathcal{B} \mathcal{A} \) is supposed to be compact, by Lemma I.36(a), \( \mathcal{B} \mathcal{A} \) must be discrete. Applying the universal property of \( \mathcal{B} \mathcal{A} \) to \( \mathcal{K} = \mathcal{T} \) and \( \mathcal{A} ! = \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) yields that \( \mathcal{A} \) and \( \mathcal{B} \mathcal{A} \) have the same continuous characters, that is, \( \mathcal{B} \mathcal{A} = (\mathcal{A})_d \) (the subscript \( d \) stands for the discrete topology). Using this necessary condition as a definition, one set \( \mathcal{B} \mathcal{A} = (\mathcal{A})_d \) and \( (\mathcal{A}(a))(\chi) = (\mathcal{A}(\chi))(\mathcal{A}(a)) = (\mathcal{A}(a))2 \) for every \( a \mathcal{A} \) and \( \mathcal{B} \mathcal{A} \) (I.18). By Proposition I.5(b), a basic open set in \( \mathcal{B} \mathcal{A} \) has the form \( ^b \mathcal{A} \), where \( \mathcal{A} \) is finite. Thus, \( \mathcal{A}^1(\mathcal{B}A) = \mathcal{A}^1(\mathcal{A}) \) is a neighborhood of \( 0 \) in \( \mathcal{A} \), and therefore \( \mathcal{A} \) is continuous.

If \( \mathcal{A} ! \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) is a continuous homomorphism, then it induces \( \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \). Since \( \mathcal{K} \) is discrete (cf. Lemma I.36(a)), \( (\mathcal{A})_d : \mathcal{K} \to \mathcal{A} \mathcal{H}(\mathcal{H}\mathcal{C}\mathcal{O}\mathcal{P}) \) is also continuous, and so \( (\mathcal{A})_d : (\mathcal{A})_d \to \mathcal{K} \) is continuous as well. By Theorem I.37, \( \mathcal{K} = \mathcal{K} \), and therefore \( \sim = (\mathcal{A})_d \). This consideration shows the uniqueness of \( \sim \) too, because \( \sim_k \) is uniquely determined by its dual. Since \( \mathcal{A}(a) \) and \( (\mathcal{A}(a)) \) are both evaluations at \( a \), the last statement follows. \( \square \)
A subgroup $H$ of $A$ such that $H^{BC} = H$ is said to be dually closed in $A$; if every continuous character of $H$ extends continuously to $A$, we say that $H$ is dually embedded in $A$. We saw in Theorem I.19(a) that every dense subgroup of a topological group is dually embedded.

I.40. Corollary. Let $A \in AB(Top)$ be such that $A$ is injective, and let $K$ be a compact subgroup of $A$. Then $K$ is dually closed and dually embedded in $A$.

Proof. Since $A$ is injective, $A$ must be Hausdorff. First, suppose that $A$ is compact. In order to show that $K$ is dually closed, let $x \neq A \cap K$. The quotient $A = K$ is also compact, and the image $x$ of $x$ in $A = K$ is non-zero. Thus, by Theorem I.37, there is $x \neq K = K$ such that $(x) \notin 0$. The character induces $2A$ by setting $a = (a)$ (where $a$ stands for the image of $a$ in $A = K$), and $K \ker a$, while $(a) \notin 0$. Therefore, $2K$, and hence $x \in K = K^{BC}$. (Note that by Lemma I.4, $K^{BC}$ is a subgroup of $A$, and so $K^{BC}$ coincides with its annihilator in $A$.) Therefore, $K = K^{BC}$. In order to show that $K$ is dually embedded, consider the discrete group $A$. By Lemma I.4(b), $A = K^{B}$! is injective and its image is $K^{BB}$. Since $A = A$ (cf. Theorem I.37), $K^{BB} = K^{BC} = K$. The quotient $A = K^{B}$ is discrete, and so $A = K^{B}$ is compact. Therefore, $A = K^{B}$! is an embedding onto $K$, that is, $K = A = K^{B}$. Hence, applying Theorem I.37 to the compact group $A = K^{B}$ yields $A = A = K = A^{B}$. In particular, $A = A^{B}$ given by restriction to $K$ is surjective, as desired.

In the general case, since $A$ continuous, $A$ is a compact subgroup of $A$. Therefore, by what we have shown so far, it is dually closed and dually embedded in $A$. Thus, for every $x \neq A \cap A(K)$, there is $2A = 2A$ such that $(A(K)) = 0$ and $(a) \notin 0$. In particular, this holds for every $x = A(a)$, where $a \neq A \cap K$. (Since $A$ is injective, $a \neq A \cap K$ implies $x \neq A \cap A(K)$.) Therefore, $2K^{B}$, and so $A = K^{BC}$. Hence, $K = K^{BC}$, as desired. For the second statement, observe that $K = A(K)$ (because $A$ is injective, $K$ is compact, and $A$ is Hausdorff). Thus, if $K^{B} = A(K)$, then it admits a continuous extension to $A$, because $A(K)$ is dually embedded in $A$. Therefore, $A$ is a continuous extension of $K$. Hence, $K$ is dually embedded in $A$, as desired. \hfill \□

I.41. Open and compact subgroups. Open subgroups of abelian groups are closely related to compact subgroups of the Pontryagin dual, and vice versa. We turn to establishing the properties of open and compact subgroup of abelian groups. Our presentation of these results was strongly influenced by the work of Banaszczyk, Chasco, Martín-Peinador, and Bruguera (cf. [4] and [7]).

I.42. Proposition. ([4, 2.2]) Let $A \in AB(Top)$, $U \subset A$ be an open subgroup, and consider the exact sequence

$$0 \rightarrow U \rightarrow A \rightarrow A = U \rightarrow 0$$

(I.20)

(a) The map $\hat{U} : \hat{A} \rightarrow U$! is an embedding and $\hat{A} = U^{B}$.

(b) The subgroup $U$ is dually embedded in $A$. In other words, $\hat{U} : \hat{A} \rightarrow U$ is surjective.

(c) $\hat{U}$ is open, and thus a quotient.
(d) The induced sequence

\[
\begin{array}{c}
0 \rightarrow \hat{A} = U \xrightarrow{\iota} \hat{A} \xrightarrow{\gamma} \hat{U} \rightarrow 0
\end{array}
\]  \hspace{1cm} (I.21)

is exact.

**Proof.** Note that \( A = U \) is discrete, because \( U \) is open, and so \( \hat{A} = U \) is compact by Lemma I.36(b).

(a) By Lemma I.4(b), \( \gamma_U \) is injective, and its image is \( U^B \). Thus, \( \gamma_U \) is an embedding, because it is continuous, its domain is compact and its codomain is Hausdorff.

(b) Let \( 2 \hat{U} \), and let \( V \supseteq 2 \mathbb{N} \cup \{1\} \) be such that \( (V) \neq 1 \) (cf. Proposition I.5(a)). Since \( T \) is an injective abelian group, admits an extension \( : A \rightarrow T \). Thus, \( (V) = (V) \neq 1 \), and \( V \supseteq A \), because \( U \) is open in \( A \). Therefore, by Proposition I.5(a), \( \gamma_U \) is continuous on \( A \).

(c) In order to make the proof more transparent, we decomposed it into three simple steps.

**Step 1:** \( A = U + b \hat{1} \) for some \( b \in 2 \hat{A} \). Let \( 1 \) be the order of \( b \) in \( A = U \) (possibly infinite). Given a compact subset \( K \subseteq A \), it can be covered by finitely many translates \( k_1 b + U; \ldots ; k_m b + U \) of \( U \), and without loss of generality we may assume that \( k_i < 1 \) since \( U \) is an open subgroup, it is also closed, and so \( \langle K \rangle = \{ k \in K \mid k \neq 1 \} \cup U \) is compact. Thus, \( C = \langle K \rangle \cdot b \) is compact subset of \( U \), and \( K \subseteq \langle k_1 b + C \rangle \subseteq U \cup b \in C \). Without loss of generality, we may assume that \( b \in 2 \mathbb{C} \) if \( b \notin 1 \). Set \( \tilde{W} = f 2 \hat{U} \cdot j \cdot C \) \( 2g_b \), and we show that \( \tilde{W} \subseteq U \subseteq U^B \).

To that end, let \( 2 \tilde{W} \). If \( 2 \hat{A} \) is an extension of \( f \), then

\[
\langle K \rangle \subseteq \langle k_1 b + C \rangle \subseteq C \subseteq \langle k_m b + C \rangle \subseteq K \subseteq U^B.
\]  \hspace{1cm} (I.22)

If \( 1 \) is finite, then \( \langle 2 \rangle \subseteq K \), and so \( \langle K \rangle \subseteq 2 \hat{A} \). Let \( 2 \mathbb{C} \). \( T \) be the closest point to 0 such that \( T \) is finite, then \( 2 \mathbb{C} \). \( T \) be the closest point to 0 such that \( T \neq 0 \). Set \( \mu(\mathbf{H} \cdot b) = \mu + n \mathbf{r} \). The character \( \mu \) is continuous on \( A \) because \( \mathbf{H} \cdot b \) is continuous, and \( U \) is an open subgroup. Furthermore, one has \( k_1 \cdot \langle b \rangle \subseteq 2 \hat{A} \) for each \( i \), because \( k_i < 1 \) and \( T \). \( L \). \( 1 \). Therefore, by (I.22), \( \langle K \rangle \subseteq 2 \hat{A} \), which means that \( 2 \hat{K} \subseteq U \). Hence, \( \tilde{W} \subseteq U \subseteq U^B \), as desired.

**Step 2:** \( A = U \) is finitely generated, that is, \( A = \mathbf{H} \cdot b_1 ; \ldots ; b_n \cdot \hat{1} \) for \( b_1 ; \ldots ; b_n \subseteq 2 \hat{A} \). Set \( U_0 = U \) and \( U_k = \mathbf{H} \cdot b_1 ; \ldots ; b_k \cdot \hat{1} \). Each \( U_k \) is an open subgroup of \( U_{k+1} \) (and of \( A \)), and thus by step 1, each \( \gamma_{U_k} : U_{k+1} \rightarrow U_k \) is open. Therefore, \( \gamma_{U} = \gamma_{U_{k+1}} \). \( U_k \) is open.

**Step 3:** In the general case, let \( K \subseteq A \) be compact. Then \( K \) can be covered by finitely many translates \( \mathbf{H} \cdot b \cup U \). \( \phi U \). \( U^0 = \mathbf{H} \cdot b_1 ; \ldots ; b_n \cdot \hat{1} \). Since \( K \subseteq U^0 \), the image \( \gamma_U : K \rightarrow U^0 \) coincides with the polar of \( K \) in \( \hat{U} \) with respect to \( U^0 \) and thus open. By step 2, \( \gamma_{U} : U^0 \rightarrow \hat{U} \) is open. Therefore, \( \gamma_{U} : K \rightarrow \hat{U} \) is open in \( \hat{U} \), as desired.

(d) Exactness at \( \hat{A} = U \) and \( \hat{U} \) were shown in (b) and (c), respectively. Exactness at \( \hat{A} \) also follows from (b), because \( \text{Im} \gamma_U = U^B = \ker \gamma_U \). \( \square \)

**I.43. Proposition.** Let \( A \in \text{Ab}(\mathbb{G}) \), \( K \subseteq A \) be a compact subgroup, and consider the exact sequence

\[
\begin{array}{c}
0 \rightarrow K \xrightarrow{k} A \xrightarrow{\gamma} A = K \rightarrow 0
\end{array}
\]  \hspace{1cm} (I.23)

(a) \( \text{The map } \gamma_K : A = K \rightarrow A \) is an embedding and \( \hat{K} = K \).
(b) The map $\gamma_k : \hat{A} \to \hat{K}$ is open onto its image.
(c) The induced sequence

$$0 \to \hat{A} = K \to \hat{A} \to \hat{K}$$

is exact.
(d) If $\gamma_k$ is injective, then $\gamma_k : \hat{A} \to \hat{K}$ is surjective, and so $\gamma_k$ is a quotient map.

**Proof.** Note that by Lemma I.36(a), $\hat{K}$ is compact.

(a) A basic neighborhood of $0$ in $\hat{A} = K$ is of the form $L^B$, where $L \subseteq A = K$ is compact (cf. Proposition I.5(b)). By Lemma A.3(b), $K(L)$ is a compact subset of $A$, and thus $K(L)^B$ is a basic neighborhood of $0$ in $\hat{A}$. Without loss of generality, we may assume that $0 \in L$, and then by Lemma I.4(c), $K(L)^B = K(L^B)$, as desired.

(b) Since $\ker \gamma_k = K^B$ is open in $\hat{A}$ and $\hat{K}$ is discrete, the statement is obvious.

(c) Exactness at $\hat{A} = K$ and at $\hat{K}$ follows from (b), because $\text{Im} \, \gamma_k = K^B = \ker \gamma_k$.

(d) By Corollary I.40, $K$ is dually embedded in $A$, which completes the proof. 

I.44. **Theorem.** ([4, 2.3]) Let $A \in Ab(Top)$, and let $U \subseteq A$ be an open subgroup. Then:

(a) $u$ is injective (resp., surjective) if and only if $\gamma_k$ is injective (resp., surjective);
(b) $u$ is an isomorphism of topological groups if and only if $\gamma_k$ is so.

**Proof.** By Proposition I.42, the exact sequence

$$0 \to U \to A \to A = U \to 0$$

(1.25)

gives rise to an exact sequence

$$0 \to \hat{A} = U \to \hat{A} \to \hat{U} \to 0$$

(1.26)

with $\hat{A} = U$ compact, $\gamma_u$ an embedding, and $\gamma_U$ a quotient map. Thus, the conditions of Proposition I.43 are fulfilled (by Remark I.13(3), $\hat{K}$ is injective), and so

$$0 \to \hat{U} \to \hat{A} \to \hat{U} \to 0$$

(1.27)

is also exact, $\hat{U}$ is an embedding, and $\hat{A}$ is a quotient map. Since the evaluation homomorphism is a natural transformation, we obtain a commutative diagram with exact rows:

$$0 \to U \to A \to A = U \to 0$$

(1.28)

$$\downarrow \gamma_k \quad \downarrow \gamma_k \quad \downarrow \gamma_k$$

The group $A = U$ is discrete, and thus $\gamma_k$ is a topological isomorphism by Theorem I.37. Therefore, (a) follows from the well-known Five Lemma for abelian groups. In order to show (b), observe that $u = a^{\frac{1}{2}}$ and $a^{-\frac{1}{2}} = a^{\frac{1}{2}}$, and that it suffices to check the continuity of a homomorphism on a neighborhood of $0$. Since $U$ is open, the statement follows. 

□
I.45. Theorem. ([4, 2.6]) Let \( A \in \text{Ab}(\text{Haus}) \) be such that \( A \) is injective, and let \( K \subset A \) be a compact subgroup. Then:

(a) \( A-K \) is injective;
(b) \( A-K \) is surjective if and only if \( A \) is so;
(c) \( A-K \) is an isomorphism of topological groups if and only if \( A \) is so.

I.46. Remark. Although in Proposition I.43 we assumed no separation axioms, here we do, because \( A \) being injective implies that \( A \) is Hausdorff.

Proof. Since \( K \) is injective, by Proposition I.43, the exact sequence in (I.23) gives rise to an exact sequence

\[
0 \rightarrow A-K \xrightarrow{\phi} \hat{A} \xrightarrow{\pi} A-K \rightarrow 0 \quad (I.29)
\]

where \( A-K \) is an open subgroup of \( \hat{A} \) and \( \phi \) is a quotient. Therefore, by Proposition I.42, the lower row of the commutative diagram below is exact:

\[
0 \rightarrow K \xrightarrow{\pi} A \xrightarrow{\pi} A-K \rightarrow 0 \quad (I.30)
\]

Since \( K \) is compact Hausdorff, \( A-K \) is an isomorphism of topological groups (cf. Proposition I.37), and thus (a) and (b) follow from the Five Lemma for abelian groups.

(c) If \( A \) is an isomorphism of topological groups, then \( A-K \) is a bijection by (a) and (b), and the topologies of \( A-K \) and \( \hat{A}-K \) coincide, because \( A-K = \hat{A}-K = A-K \).

Conversely, suppose that \( A-K \) is a topological isomorphism. Then \( A \) is bijective by (a) and (b). We show that \( A \) is continuous. To that end, let \( F \) be a filter converging to \( 0 \) in \( A \). Then \( K \cap F = K \), and so \( \hat{A}-K \) is continuous, because \( \hat{A}-K = A-K \). By Lemma A.3(a), the filter \( A \cap F \) has a cluster point \( x \in K \). Because \( \hat{A} \) is compact, \( A \) is continuous. Therefore, \( x \) is the unique cluster point of \( A \cap F \) for every filter \( F \). Hence, \( A \cap F \) is continuous, and \( A \) is continuous.

Since the situation is completely symmetric (one could invert \( K \), \( A \), and \( A-K \)), the proof of the continuity of \( A \) is similar.

I.47. Locally compact groups. We conclude this chapter with making a step toward the classical Pontryagin duality for locally compact Hausdorff abelian (LCA) groups. We denote by \( \text{LC} \) the category of locally compact Hausdorff spaces, and thus \( \text{Ab}(\text{LC}) \) is the category of LCA groups.

I.48. Proposition. Let \( A \in \text{Ab}(\text{LC}) \). Then:

(a) \( \hat{A} \in \text{Ab}(\text{LC}) \);
(b) \( A \) is continuous;
(c) \( A \) is injective.
Proof. (a) Clearly, $\hat{A}$ is Hausdorff. Let $U \subseteq N (A)$ such that $U$ is compact. By Proposition I.7(a), $U^B$ is compact in $\hat{A}$, and by Proposition I.5(a), $U^B$ is a basic neighborhood of $0$ in $\hat{A}$. Since $U^B = U^B$, this completes the proof.

(b) Let $V \subseteq N (A)$ be such that $V$ is compact. By Theorem I.9, $\hat{A}$ is k-continuous, and so $\hat{A}$ is continuous. In particular, $\hat{A}$ is continuous at $0$, and therefore it is continuous.

(c) Let $\hat{A}$ be a non-zero element.

First, suppose that $A$ is generated by $V \subseteq N (A)$ such that $V$ is compact. By replacing $V$ with \[ V \subseteq \cup_{n \in N} V \subseteq N (A) \] if necessary, we may assume that $A \subseteq V$ and $V = V$ from the outset. By [33, Lemma 2.42], there is a subgroup $H$ such that $H \subseteq V = f0g$, $H = 2^n$ for some $n \in N$, and $A = H$ is compact. The image $a + H$ of $a$ in $A = H$ is non-zero, because $A \cap V$ and $V \subseteq H = f0g$. Thus, by Theorem I.37(b), $A = H$ such that $(a + H) \subseteq 0$. Therefore, one has $A (a) = (a + H) \subseteq 0$, where $\hat{A} = H$.

Hence, $A (a) \subseteq 0$.

In the general case, pick $V \subseteq N (A)$ such that $V = V$, $V = V$, and $V$ is compact, and put $U = \hat{A}$, the subgroup generated by $V$. Then $U + V = U$, and thus $U = \hat{A}$ (by Proposition A.1(a)). Therefore, $U$ is an open subgroup of $A$. By what we proved so far, $U$ is injective, hence, by Theorem I.44(a), $A$ is injective. \[ \square \]

I.49. Remark. The proof of (c) falls short of being self-contained because it is based on a Lemma from [33]. I hope to find a simple proof of the relevant parts of the Lemma. Suggestions???

I.50. Lemma. Let $A \subseteq A (L C)$, and suppose that there is a countable family $\{ U_n \} \subseteq N (A)$ of neighborhoods of $0$ in $A$, such that $\bigcap U_n = f0g$. Then $A$ is metrizable.

Proof. By replacing each $U_n$ with $U_n^0 \subseteq N (A)$ such that $U_n^0 \subseteq U_n$ and $U_n^0$ is compact if necessary, we may assume that $\bigcap U_n = f0g$, and $U_1$ is compact (such $U_n^0$ exists, because $A$ is regular).

Furthermore, without loss of generality, we may assume that $U_{n+1} \subseteq U_n$. By Theorem I.29, it suffices to show that $\bigcap U_n$ is a base at $0$ for $A$. To that end, let $V \subseteq N (A)$. Then $\bigcap (U_n \cap V) = \{ \}$, and $\bigcap U_n \cap \text{Int} V$ is a decreasing family of compact closed subsets of the compact space $U_1$. Therefore, there is $n_0$ such that $U_{n_0} \cap \text{Int} V = \{ \}$. Hence, $U_{n_0} \cap \text{Int} V$, as desired. \[ \square \]

I.51. Lemma. Let $A \subseteq A (L C)$, and let $V \subseteq N (A)$ be such that $V$ is compact. Then $V$ contains a compact subgroup $K$ of $A$ such that $A = K$ is metrizable.

Proof. We construct a family $\{ V_n \}$ of neighborhoods of $0$ in $A$. Put $V_1 = (V \setminus V)$, and for $n \in N$, using continuity of addition in $A$, pick $V_{n+1} \subseteq N (A)$ such that $V_{n+1} \subseteq V_n$ and $V_{n+1} = V_n$. Put $K = \bigcap_{n=1}^{T} V_n$; $K$ is a subgroup of $A$, because $K \subseteq V_n \subseteq V_{n+1}$ and $V_{n+1} = V_n$.

One has $K + V_{n+1} \subseteq V_n$, and thus $K \subseteq V_n$ (cf. Proposition A.1(a)). Therefore, $K \subseteq V_n \subseteq V_n$, and $K$ is compact. The inclusion $K + V_{n+1} \subseteq V_n$ also yields $K = \bigcap_{n=1}^{T} (K + V_n)$, which means that $K (V_n) = K \subseteq K$. Hence, $f (V_n) = K \subseteq K$ is metrizable, as desired. \[ \square \]

I.52. Theorem. The following statements are equivalent:

1. $A$ is Hausdorff.
2. $A$ is k-continuous.
3. $A$ is continuous.
4. $A$ is continuous at $0$. 
5. $A$ is injective.
6. $A$ is metrizable.
(i) For every second countable \( A \in \text{Ab}(\mathcal{L}C) \), \( \lambda \) is an isomorphism of topological groups.

(ii) For every \( A \in \text{Ab}(\mathcal{L}C) \), \( \lambda \) is an isomorphism of topological groups.

**Proof.** (i) (ii): Let \( A \in \text{Ab}(\mathcal{L}C) \). First, suppose that \( A \) is generated by \( V \in \mathcal{N}(A) \) such that \( V \) is compact. By replacing \( V \) with \( V \setminus V \) if necessary, we may assume that \( V = V \) from outset. Then \( \lambda = \bigoplus_{n=1}^{\infty} \left( \bigoplus_{k=1}^{n} V_{k} \right) \) so \( \lambda \) is \(-\)compact, and in particular, \( \lambda \) is Lindelöf. Let \( K \subset V \) be a compact subgroup such that \( A = K \) is metrizable (cf. Lemma I.51). The quotient \( A = K \) is second countable, because it is a continuous image of the Lindelöf space \( A \). (The properties Lindelöf, separable, and second countable are equivalent for metrizable spaces.) Thus, \( \lambda = K \) is an isomorphism of topological groups by our assumption. By Proposition I.48(c), \( \lambda \) is injective, and therefore, by Theorem I.45(c), \( \lambda \) is an isomorphism of topological groups.

In the general case, pick \( V \in \mathcal{N}(A) \) such that \( V \) is compact, and put \( U = \langle V \rangle \), the subgroup generated by \( V \). Then \( U + V \subset U \), and thus \( U \subset \text{Int} U \) (by Proposition A.1(a)). Therefore, \( U \) is an open subgroup of \( A \). By what we proved so far, \( U \) is an isomorphism of topological groups, because \( U \) is generated by a compact neighborhood of \( 0 \). Hence, by Theorem I.44(b), \( \lambda \) is an isomorphism of topological groups. \( \square \)
Appendix

A. Separation properties of topological groups

For $G \in \mathfrak{Top}$, one puts $N(G)$ for the collection of neighborhoods of $e$ in $G$.

A.1. Proposition. Let $G \in \mathfrak{Top}$.

(a) If $V \subseteq U$ for $U, V, W \subseteq N(G)$, then $V \cap W \subseteq \text{Int} U$.

(b) $G$ is regular.

(c) The following statements are equivalent:
   (i) $G$ is $T_3$ (i.e., $T_1$ and regular);
   (ii) $G$ is Hausdorff;
   (iii) $G$ is $T_1$;
   (iv) $G$ is $T_0$.

Proof. (a) Set $W_1 = (\text{Int} W)^{-1}$, and let $x \in V$. Then $W_1 \subseteq N(G)$ (because the group inversion is continuous), and $V \setminus xW_1 \neq \emptyset$. Thus, there is $v \in W$ and $w_1 \in W_1$ such that $xw_1 = v$, or $x = vw_1^{-1}$. Therefore, $x \in V \setminus W_1 = V \setminus (\text{Int} W)$. Since $V \setminus (\text{Int} W)$ is open, it is contained in $\text{Int} (V \setminus W) \subseteq \text{Int} U$. Hence, $x \in \text{Int} U$, as desired.

(b) By continuity of the multiplication group $m : G \times G \to G$, for every $U \subseteq N(G)$ there is $V \subseteq N(G)$ such that $VV \subseteq U$, and thus $V \subseteq \text{Int} U$ by (a).

(c) Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. In order to complete the proof, observe that by (b), $G$ is regular, and every $T_0$ regular topological space $X$ is $T_1$. (Indeed, if $x, y \in X$ are distinct points, then one of them, say $x$, has a neighborhood $U_0$ such that $y \not\in U_0$. Then $F = X \cap \text{Int} U_0$ is a closed subset containing $y$ that does not contain $x$. Since $X$ is regular, there are disjoint open subsets $U$ and $V$ of $X$ such that $F \subseteq U$ and $x \in V$. This completes the proof.)

A.2. Proposition. Let $G \in \mathfrak{Top}$, and set $N_G = \cap N(G)$. Then:

(a) $N_G$ is a compact normal subgroup of $G$;

(b) $G = N_G$ is Hausdorff;

(c) for every continuous homomorphism $\phi : G \to H$ into a Hausdorff group $H$, $N_G$ is kernel. In particular, $\phi$ factors uniquely through $G$. $G = N_G$.

Proof. (a) Let $U \subseteq N(G)$ and $g \in G$. By continuity of the group operations, there is $V \subseteq N(G)$ such that $VV^{-1} \subseteq U$ and $g^{-1}Vg \subseteq U$. Thus, $N_GN_G^{-1} \subseteq VV^{-1} \subseteq U$ and $gN_Gg^{-1}N_Gg \subseteq g^{-1}Vg \subseteq U$. This is true for every $U \subseteq N(G)$, and therefore $N_GN_G^{-1} \subseteq N_G$ and $gN_Gg \subseteq N_G$. Hence, $N_G$ is a normal subgroup. Since $N_G$ is contained in every neighborhood of $e$, its compactness is clear.
(b) For every \( U \supseteq \mathcal{N}(G) \), there is \( V \supseteq \mathcal{N}(G) \) such that \( V \cap U \neq \emptyset \), and thus \( V \mathcal{N}(G) \). Thus, \( \mathcal{N}(G) \) is the reflection of \( G \rightarrow \mathcal{N}(G) \) in \( \mathcal{N}(G) \) is Hausdorff, by Proposition A.1.

(c) If \( H \) is Hausdorff, then \( f \circ g = \text{ker}^T(g) \cap \mathcal{N}(H) \), and so
\[
\ker^T(f) = f^{-1}(g) \cap \mathcal{N}(H) = \mathcal{N}(G) = \mathcal{N}(g); \tag{1}
\]
as desired.

The group \( G = \mathcal{N}(G) \) is the maximal Hausdorff quotient of \( G \). In categorical language, this means that \( G = \mathcal{N}(G) \) is the reflection of \( G \rightarrow \mathcal{N}(G) \) in \( \mathcal{N}(G) \) (Haus).

### A.3. Lemma

Let \( G \rightarrow \mathcal{N}(G) \), \( K \rightarrow G \) be a compact subgroup, and put \( K : G \rightarrow \mathcal{N}(G) \) for the canonical projection.

(a) If \( F \) is a filter in \( G \) such that \( K \) is a cluster point of \( \mathcal{T}(F) \) in \( G = \mathcal{N}(G) \), then \( F \) has a cluster point in \( K \).

(b) If \( L \rightarrow \mathcal{N}(G) \) is compact, then so is \( \mathcal{N}(F) \).

**Proof.** (a) Assume that \( F \) has no cluster point in \( K \). Then each point \( x \in K \) has a neighborhood \( U_x \) such that \( U_x \) does not mesh \( F \), that is, there is \( F \supseteq F \) such that \( U_x \setminus F = \emptyset \). The collection \( fU_x \mathcal{N}(K) \) is an open cover of \( K \), and thus it has a finite subcover \( U_{x_1}, \ldots, U_{x_n} \), because \( K \) is compact. Set \( U = U_{x_1} \cap \cdots \cap U_{x_n} \) and \( F = F_{x_1} \cap \cdots \cap F_{x_n} \). Since \( F \) is a filter, \( F \supseteq \mathcal{N}(G) \), and one has \( U \setminus F = \emptyset \) and \( K \subseteq U \). Using the group multiplication \( m : G \rightarrow \mathcal{N}(G) \), the last inclusion can be expressed as \( f \circ g \subseteq \mathcal{N}(G) \). Since \( m \) is continuous, \( \mathcal{N}(G) \) is open, and thus, by a tube-lemma type argument, there is \( V \supseteq \mathcal{N}(G) \) such that \( V \cap \mathcal{N}(G) \) is compact. Equivalently, \( K \) is a cluster point of \( \mathcal{N}(G) \), because \( L \) is compact. Equivalently, \( K \) is a cluster point of \( \mathcal{N}(G) \), and therefore \( x \mathcal{N}(G) \) has a cluster point in \( K \) (by (a)). Hence, \( F \) has a cluster point in \( x \mathcal{N}(G) \), which shows that \( \mathcal{N}(F) \) is compact.

(b) Let \( F \) be a filter meshing \( \mathcal{N}(G) \) (i.e., \( F \setminus \mathcal{N}(G) \) is for every \( F \)). Then \( \mathcal{N}(F) \) is a filter that meshes \( L \), and thus has a cluster point \( x \mathcal{N}(G) \), because \( L \) is compact. Equivalently, \( K \) is a cluster point of \( \mathcal{N}(G) \), and therefore \( x \mathcal{N}(G) \) has a cluster point in \( K \) (by (a)). Hence, \( F \) has a cluster point in \( x \mathcal{N}(G) \), which shows that \( \mathcal{N}(F) \) is compact.

A combination of Proposition A.2(a) and Lemma A.3 yields:

### A.4. Corollary

Compact subsets of \( G = \mathcal{N}(G) \) are precisely the images of compact subsets of \( G \).

---

## B. Exponentiability and the compact-open topology

Let \( T \) be a full subcategory of \( \mathcal{T}(G) \), the category of topological spaces and their continuous maps. Suppose that \( T \) has finite products; they may differ from the products in \( \mathcal{T}(G) \), but each product in \( T \) must have the same underlying set as in \( \mathcal{T}(G) \). Each function \( f : X \rightarrow Y \) gives rise to a map \( f : Y \rightarrow Z^X \) defined by \( f(y) = f_y \), where \( f_y Z \) is given by \( f_y(x) = f(x; y) \). One would like to equip the function set \( T = \mathcal{Z}(X; Z) \) with a suitable topology such that \( f \) is continuous whenever
is so. If we add the natural requirement of functoriality of the topology, we arrive at the concept of exponentiability. A space $X \in T$ is exponentiable (in $T$) if the functor $X : T ! T$ has a right adjoint. Since we assume $T$ to have finite products, it has a terminal object which must then be the one-point space $\mathfrak{g}$, because $T$ is a full subcategory. Thus, one expects $T (X ; \mathfrak{g})$ to be the underlying set of the hom-object. In other words, $X$ is exponentiable if there is a way of topologizing the set $T (X ; \mathfrak{g})$ for every $\mathfrak{g} \in T$ such that the resulting space is in $T$, and the map

$$T (X ; \mathfrak{g}) \mapsto T (\mathfrak{g} ; T (X ; \mathfrak{g}))$$

is a bijection that is natural in $\mathfrak{g}$ and $\mathfrak{g}$.

The question of describing exponentiable Hausdorff spaces was raised by Hurewicz in a personal communication with Fox, who was the first to give a partial answer to the question (cf. [14]). He proved that for every regular locally compact space $X$ and every $\mathfrak{g} \in \text{Haus}$ the set $\text{Haus}(X ; \mathfrak{g})$ can be topologized in a way that $f$ is continuous if and only if $f$ is so. Fox also showed that for every separable metric space $X$, it is possible to topologize $\text{Haus}(X ; \mathbb{R})$ in a way that (2) holds for every $\mathfrak{g}$ and only if $X$ is locally compact. Although Fox did not prove that the condition of local compactness is sufficient, and he showed necessity only for separable metric spaces $X$, he was actually very close to a complete solution.

For Hausdorff spaces $X$ and $\mathfrak{g}$, we denote by $C (X ; \mathfrak{g})$ the space $\text{Haus}(X ; \mathfrak{g})$ equipped with the compact-open topology: Its subbase is the family $f [K ; \mathfrak{g}] \upharpoonright K \times \text{compact}; \mathfrak{g} \times \text{open}$, where $[K ; \mathfrak{g}] = \{ f \in \text{Haus}(X ; \mathfrak{g}) | f(K) \subseteq \mathfrak{g} \}$. The compact-open topology was also invented by Fox, and following his work, Arens studied the separation properties of $C (X ; \mathfrak{g})$ (cf. [1]). Arens was not far from proving that local compactness is necessary for exponentiability in $\text{Haus}$ (cf. [1, Theorem 3]). Jackson proved that if $X$ is locally compact, then $C (X ; \mathfrak{g})$ is the right adjoint of $X$, moreover, the bijection

$$C (X ; \mathfrak{g}) \mapsto C (\mathfrak{g} ; C (X ; \mathfrak{g}))$$

is actually a homeomorphism for every Hausdorff space $\mathfrak{g}$ and $\mathfrak{g}$ (cf. [19]). It appears that Fox already recognized that (3) is a bijection, and Jackson’s main achievement is proving that it is a homeomorphism.

**B.1. Remark.** Some authors, including Isbell, erroneously credit Brown for proving (3) (cf. [18]). The fact is that Brown himself refers both to Fox and Jackson in his paper in question, and he lays no claim to the “classical” results on special cases of the exponential law (as a personal communication with him reveals). At the same time, Brown was the first to show that the category of Hausdorff $k$-spaces is cartesian closed, but unfortunately this seems to be somewhat forgotten (cf. [5, 3.3]).

Whitehead proved that if $X$ is a locally compact Hausdorff space, then $1_X \circ \mathfrak{g}$ is a quotient map in $\text{Haus}$ for every quotient map $\mathfrak{g}$ in $\text{Haus}$ (cf. [35]). Michael proved that the converse is also true: For a Hausdorff space $X$, the map $1_X \circ \mathfrak{g}$ is a quotient map in $\text{Haus}$ for every quotient map $\mathfrak{g}$ in $\text{Haus}$ if and only if $X$ is locally compact (cf. [25, 2.1]). In other words, the functor $X$ preserves quotients if and only if $X$ is locally compact. If $X$ is exponentiable, then $X$ must preserve
every colimit in $\text{Haus}([24, \text{V.5.1}])$, and in particular it has to preserve coequalizers in $\text{Haus}$ (which is equivalent to preservation of quotients; for details see [22, 1.1]). Thus, by Michael’s result the local compactness of $X$ follows. Therefore, exponentiable spaces in $\text{Haus}$ can be characterized as follows.

**B.2. Theorem.** A space $X \in \text{Haus}$ is exponentiable in $\text{Haus}$ if and only if $X$ is locally compact.

A proof of sufficiency of local compactness in Theorem B.2 is available in standard topology textbooks (cf. [13, 3.2]).

**B.3. Corollary.** Let $X, Y \in \text{Haus}$. The evaluation map

$$e : X ! C (C (X ; Y ), Y )$$

is $k$-continuous, that is, $e_{x}^{K}$ is continuous for every compact subset $K \subseteq X$.

**Proof.** Let $K \subseteq X$ be a compact subset. The map

$$E : K \ni K \subseteq Y \ni f$$

is continuous, because $E (K \ni K, \bar{W}) = \bar{W}$ for every open subset $\bar{W}$ of $Y$. Let $[\iota \cup]$, be a subbasic open subset of $C (C (X ; Y ), Y )$, where $C (X ; Y )$ is compact and $U \subseteq Y$ is open. By Theorem B.2, $e$ is exponentiable (because it is compact), and thus $E_{x}^{K} : K \ni Y$ gives rise to a continuous map $e_{x}^{K} : K \ni C (\iota ; Y )$ that is defined by $(e_{x}^{K}, \iota (K)) (f) = f (\iota)$. Therefore, $(e_{x}^{K})^{-1} ([\iota \cup]) = e_{x}^{K}^{-1} ([\iota \cup])$ is open in $K$, as desired.

Exponentiable spaces in $\text{Top}$ were characterized by Day and Kelly as so-called *core-compact* spaces (cf. [11]), but this result is beyond the scope of this summary. For a detailed review of exponentiability, we refer the reader to [22, Chapter 1].

**C. Hausdorff $k$-spaces**

A map $f : X \ni Y$ between Hausdorff spaces is said to be $k$-*continuous* if the restriction $f_{x}^{K}$ is continuous for every compact subset $K \subseteq X$. Theorem I.9 suggests that it might be helpful to consider spaces whose $k$-continuous maps are continuous.

**C.1. Proposition.** ([13, 3.3.18-21]) Let $X \in \text{Haus}$. The following statements are equivalent:

1. if $O \setminus K$ is open in $K$ for every $K \subseteq X$, then $O$ is open in $X$;
2. if $F \setminus K$ is closed for every $K \subseteq X$, then $F$ is closed;
3. every $k$-continuous map $f : X \ni Y$ into a Hausdorff space $Y$ is continuous.
Proof. The equivalence of (i) and (ii) is obvious, and clearly each of them implies (iii).

(iii) Let $kX$ be the underlying set of $X$ equipped with the topology

\[ f \circ X \cap O \setminus K \text{ is open in } K \text{ for every } K \supseteq K' (X) g: \]  

We call $kX$ the $k$-ification of $X$. Its topology is finer than the topology of $X$, and so $kX$ is Hausdorff. Furthermore, the topologies of $X$ and $kX$ coincide on compact subsets, and thus the identity map $X \to kX$ is $k$-continuous. Therefore, by (iii), it is continuous, and hence $X = kX$, as desired. 

A Hausdorff space $X$ is a $k$-space if it satisfies the equivalent conditions of Proposition C.1. The category of Hausdorff $k$-spaces and their continuous maps is denoted by $k\text{Haus}$. It is possible to define $k$-spaces without assuming any separation axioms (cf. [34] and [22, 1.1]), but it is unnecessary for our purposes.

C.2. Theorem.

(a) The category $k\text{Haus}$ is a coreflective subcategory of $\text{Haus}$ with coreflector $k$, the $k$-ification.

(b) The product of $X \times Y \in k\text{Haus}$ in $k\text{Haus}$ is the $k$-ification $k(X \times Y)$ of their product in $\text{Haus}$.

(c) ([5, 3.3]) The category $k\text{Haus}$ is cartesian closed, and the internal hom is given by the $k$-ification $kC(X \times Y)$ of the compact-open topology.

Proof. (a) First, note that by Proposition C.1, $kX$ is a Hausdorff $k$-space for every $X \in \text{Haus}$. Let $g: X \to Y$ be a continuous map between Hausdorff spaces $X$ and $Y$, and let $O \subseteq kY$ be open. Since $g$ is continuous, $g(K) \supseteq K' (Y)$ for every $K \supseteq K' (X)$. Thus, $g(K) \setminus O$ is open in $g(K)$, and hence $K \setminus g^{-1}(O) = g^{-1}(g(K) \setminus O)$ is open in $K$. Therefore, $kg: kX \to kY$ is continuous. This shows that $k: \text{Haus} \to k\text{Haus}$ is a functor (it is clear that it preserves $kX$ and composition). In order to show that $k$ is a coreflector, observe that if $X \in k\text{Haus}$ (i.e., $X = kX$), then $kf: X \to kY$ is the unique continuous map such that $f = x_Y \circ kf$, where $x_Y: kY \to Y$ is the identity.

(b) follows from (a) and [24, VI.3, V.5.1].

It is not hard to see that (c) follows from Corollary B.3, but for a detailed proof, we refer the reader to [24, VII.8.3].
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