On arithmetical nature of Tichy-Uitz’s function

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In [1] R. F. Tichy and J. Uitz introduced a one parameter family $g_\lambda$, $\lambda \in (0,1)$, of singular functions. When $\lambda = 1/2$ the function $g_\lambda$ coincides with the famous Minkowski question mark function. In this paper we describe the arithmetical nature of the function $g_\lambda$ when $\lambda = \frac{3 - \sqrt{5}}{2}$.

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1 Stern-Brocot sequences

We remind the definition of Stern-Brocot sequences $F_n, n = 0,1,2,\ldots$.

Consider the two-point set $F_0 = \{\frac{0}{1},\frac{1}{1}\}$. Let $n \geq 0$ and

$$F_n = \left\{0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1\right\},$$

where $x_{j,n} = p_{j,n}/q_{j,n}$, $(p_{j,n}, q_{j,n}) = 1$, $j = 0, \ldots, N(n)$ and $N(n) = 2^n$. Then

$$F_{n+1} = F_n \cup Q_{n+1}$$

with

$$Q_{n+1} = \{x_{j-1,n} \oplus x_{j,n}, \quad j = 1, \ldots, N(n)\}.$$

Here

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + b}{c + d}$$

is the mediant of the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

The elements of $Q_n$ can be characterized in the following way. A rational number $\xi \in [0,1]$ belongs to $Q_n$ if and only if in the continued fraction expansion

$$\xi = [0; a_1, a_2, \ldots, a_m] = 0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_m}}}, \quad a_j \in \mathbb{N}, \ a_m \geq 2. \quad (1)$$

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the sum of partial quotients is exactly $n+1$:

$$S(\xi) := a_1 + \ldots + a_m = n + 1.$$ 

So $\mathcal{F}_n$ consists of all rational $\xi \in [0,1]$ such that $S(\xi) \leq n+1$.

## 2 Tichy-Uitz’s singular functions

In [1] R. F. Tichy and J. Uitz considered a one parameter family $g_\lambda$, $\lambda \in (0,1)$, of singular functions. In this section we describe the construction of $g_\lambda$ from [1]. This construction is an inductive one.

Given $\lambda \in (0,1)$ put

$$g_\lambda(0) = g_\lambda(0/1) = 0, \quad g_\lambda(1) = g_\lambda(1/1) = 1.$$ 

Suppose that $g_\lambda(x)$ is defined for all elements $x \in \mathcal{F}_n$. Then we define $g_\lambda(x)$ for $x \in \mathbb{Q}_{n+1}$. Each $x \in \mathbb{Q}_{n+1}$ is of the form $x = x_{j-1,n} \oplus x_{j,n}$ where $x_{j-1,n}$ and $x_{j,n}$ are consecutive elements from $\mathcal{F}_n$. Then

$$g_\lambda(x_{j-1,n} \oplus x_{j,n}) = g_\lambda(x_{j-1,n}) + (g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n})) \lambda.$$ 

So we have defined $g_\lambda$ for all rational numbers from $[0,1]$. One can see that the function $g_\lambda(x)$ is a continuous function from $\mathbb{Q} \cap [0,1]$ to $[0,1]$. So it can be extended to a continuous function from the whole segment $[0,1]$ to $[0,1]$.

For every $\lambda$ the function $g_\lambda(x)$ increases in $x \in [0,1]$. By the Lebesgue theorem $g_\lambda(x)$ is a differentiable function almost everywhere. Moreover, it is easy to see that $g_\lambda'(x) = 0$ almost everywhere (is the sense of Lebesgue measure). Certain properties of functions $g_\lambda(x)$ were investigated in [1]. Some related topics one can find in [6] and [7]. Here we should note that in the case $\lambda = 1/2$ the function $g_{1/2}(x)$ coincides with the famous Minkowski question mark function $?(x)$. This function may be considered as the limit distribution function for Stern-Brocot sequences $\mathcal{F}_n$. The aim of the present paper is to explain the arithmetical nature of the function $g_\lambda(x)$ when $\lambda = \frac{3-\sqrt{5}}{2}$.

## 3 Minkowski’s function $?(x)$

Let us consider the function $g_{1/2}(x) = ?(x)$. This function was introduced by Minkowski. As it follows from the definition of $g_\lambda$ for $\lambda = 1/2$:

$$?(0) = ?(0/1) = 0, \quad ?(1) = ?(1/1) = 1.$$
and for \( x_{j-1,n}, x_{j,n} \in \mathcal{F}_n \)

\[ \mathcal{M}(x_{j-1,n} \oplus x_{j,n}) = \frac{\mathcal{M}(x_{j-1,n}) + \mathcal{M}(x_{j,n})}{2}. \]

The definition of \( \mathcal{M}(x) \) for irrational \( x \) follows by continuity.

R. Salem in [2] found a new presentation for \( \mathcal{M}(x) \). If \( x \in (0, 1) \) is represented in the form of regular continued fraction

\[ x = [0; a_1, a_2, \ldots, a_m, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_m + \frac{1}{\ddots}}}}}, \quad (2) \]

then

\[ \mathcal{M}(x) = \frac{1}{2a_1 - 1} - \frac{1}{2a_1 + a_2 - 1} + \frac{1}{2a_1 + a_2 + a_3 - 1} - \ldots \quad (3) \]

For rational \( x \) the representation (2) and consequently (3) is finite.

Minkowski’s question mark function may be treated as the limit distribution function for Stern-Brocot sequences in the following sense:

\[ \mathcal{M}(x) = \lim_{n \to \infty} \frac{\# \{ \xi \in \mathcal{F}_n : \xi \leq x \}}{\# \mathcal{F}_n} = \lim_{n \to \infty} \frac{\# \{ \xi \in \mathcal{F}_n : \xi \leq x \}}{2^n + 1}. \quad (4) \]

A finite formula for the right hand side of (4) was given by T. Rivoal in the preprint [3]. Various properties of Minkowski question mark function were investigated in papers [8] by A. Denjoy, [3] by P. Viader, J. Paradis, L. Bibiloni and in [4] by A. A. Dushistova, I. D. Kan and N. G. Moshchevitin.

### 4 General form of formula (3)

The formula (3) can be generalized on the whole family of functions \( g_{\lambda} \) in the following way.

**Proposition** Let \( x, \lambda \in (0, 1) \) and \( x = [0; a_1, \ldots, a_m, \ldots] \) is the regular continued fraction expansion of \( x \), then

\[ g_{\lambda}(x) = \lambda^{a_1 - 1} - \lambda^{a_1 - 1}(1 - \lambda)^{a_2} + \lambda^{a_1 - 1}(1 - \lambda)^{a_2 - 1} \lambda^{a_3} - \ldots + \]

\[ + ( -1 )^{m+1} \lambda^{(1 \leq i \leq m, i \equiv 1 \mod 2)} \sum_{i=1}^{a_i - 1} (1 - \lambda)^{(1 \leq i \leq m, i \equiv 0 \mod 2)} a_i + \ldots. \quad (5) \]
Proof: By definition of \( g_\lambda \)

\[
g_\lambda(0) = 0, \quad g_\lambda(1) = 1
\]

and

\[
g_\lambda(x_{j-1,n} \oplus x_{j,n}) = g_\lambda(x_{j-1,n}) + (g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n})) \lambda,
\]

where \( x_{j-1,n} \) and \( x_{j,n} \) are consecutive elements from \( \mathcal{F}_n \). We can also rewrite the formula (6) in the form

\[
g_\lambda(x_{j-1,n} \oplus x_{j,n}) = g_\lambda(x_{j,n}) - (g_\lambda(x_{j,n}) - g_\lambda(x_{j-1,n})) (1 - \lambda).
\]

The equality

\[
g_\lambda(1/a_1) = \lambda^{a_1-1}
\]
follows from the formula (6) immediately since \( 1/a_1 = 0 \oplus \ldots \oplus 0 \oplus 1 \) \( \text{times} (a_1-1) \) times.

Suppose that the formula (5) is proved for \( x = [0; a_1, \ldots, a_m] \), then it is enough to prove it for \( y = [0; a_1, \ldots, a_m + 1] \) and for \( z = [0; a_1, \ldots, a_m, 2] \).

Let \( m \) is odd, then by applying formula (6) we get

\[
g_\lambda(y) = g_\lambda([0; a_1, \ldots, a_{m-1}] \oplus [0; a_1, \ldots, a_m]) =
\]

\[
= g_\lambda([0; a_1, \ldots, a_{m-1}]) + \lambda(\sum_{a_i \equiv 1 \pmod{2}} a_i - 1) (1 - \lambda) (1 - \lambda) \]

and by applying formula (7) we get

\[
g_\lambda(z) = g_\lambda([0; a_1, \ldots, a_{m+1}] \oplus [0; a_1, \ldots, a_m]) =
\]

\[
= g_\lambda([0; a_1, \ldots, a_m]) - (1 - \lambda)(g_\lambda([0; a_1, \ldots, a_m]) - g_\lambda([0; a_1, \ldots, a_m + 1])) =
\]

\[
= g_\lambda([0; a_1, \ldots, a_m]) - \lambda \sum_{a_i \equiv 1 \pmod{2} \pmod{2}} a_i (1 - \lambda)^2.
\]

For even \( m \) the proof is analogously.
5 Regular reduced continued fractions and
the main result

Any real number $x$ can be expressed uniquely in the form

$$x = [b_0; b_1, b_2, \ldots, b_l, \ldots] = b_0 - \frac{1}{b_1 - \frac{1}{b_2 - \ldots - \frac{1}{b_l - \ldots}}}, \quad b_i \geq 2, \quad (10)$$

which is known as regular reduced continued fraction (eine reduziert-regelmassige Kettenbrouch [10], [9]).

For a rational number $x \in (0, 1)$ the representation (10) takes the form:

$$x = [1; b_1, \ldots, b_l]. \quad (11)$$

For such $x$ we denote $L(x) = b_1 + \ldots + b_l$.

Analogously to the sequence $F_n$ we define the sequence $\Xi_n$:

$$\Xi_n := \{0, 1\} \cup \left( \bigcup_{1 \leq k \leq n} \Theta_k \right),$$

where $\Theta_k = \{x \in \mathbb{Q} : L(x) = k + 1\}, \ k \geq 1$.

We arrange the elements of $\Xi_n$ in the increasing order:

$$\Xi_k = \{0 = \xi_{1,n} < \xi_{2,n} < \ldots < \xi_{\sharp \Xi_n,n} = 1\}.$$

We would like to note that in the special case $\lambda = \tau^2$ formula (5) gives:

$$g_{\tau^2}(x) = \tau^{2a_1-2} - \tau^{2a_1+a_2-2} + \tau^{2a_1+a_2+a_3-2} - \ldots +$$

$$+ (-1)^{m+1} \tau^{\sum_{i=1}^{m} \alpha_i a_i - 2} + \ldots, \quad (12)$$

where

$$\alpha_m = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 2, & \text{if } m \text{ is odd}. \end{cases}$$

For rational $x$ the representation (12) is finite.

The Theorem 1 below is the main results of the present paper. It generalizes the formula [4] on the regular reduced continued fractions.
Theorem 1 Function \( g_\lambda \), where \( \lambda = \tau^2 = \frac{3-\sqrt{5}}{2} \), \( \tau = \frac{\sqrt{5}-1}{2} \) coincides with the distributional function of the sequence \( \Xi_n \), that is

\[
g_{\tau^2}(x) = \lim_{n \to \infty} \frac{\# \{ \xi \in \Xi_n : \xi \leq x \}}{\# \Xi_n}, \quad x \in (0, 1).
\]

Now we consider the function

\[
\mathcal{M}(x) := \lim_{n \to \infty} \frac{\# \{ \xi \in \Xi_n : \xi \leq x \}}{\# \Xi_n}, \quad x \in (0, 1).
\]

Our purpose is to prove that \( \mathcal{M}(x) = g_\lambda \). Function \( \mathcal{M}(x) \) is increasing as a distribution function, so it is enough to prove that \( \mathcal{M}(x) \) coincides with \( g_{\tau^2}(x) \) for rational \( x \), that is

\[
\mathcal{M}(x \oplus y) = \mathcal{M}(x) + (\mathcal{M}(y) - \mathcal{M}(x)) \tau^2.
\]

for any two consecutive elements of \( \Xi_n \) for any \( n \).

6 Auxiliary results

The following result is well known. We present it without a proof.

Lemma 1 Let \( x \) is represented in the form \([I]\) and in the form \([II]\). To get the set \((b_1, \ldots, b_l)\) from \((a_1, \ldots, a_m)\) we should replace \( a_i \) by

1. \(2\ldots2\) if \( i \) is odd (empty string if \( a_i = 1 \)).
2. \(a_i + 2\) if \( i \) is even and \( i \neq m \).
3. \(a_i + 1\) if \( i \) is even and \( i = m \).

Lemma 2 For the number of elements in \( \Theta_n \) one has

\[
\# \Theta_1 = 1, \ \# \Theta_2 = 1, \ \# \Theta_{n+1} = \# \Theta_n + \# \Theta_{n-1},
\]

so \( \# \Theta_n = F_n \), where

\[
F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}}
\]

is the \( n \)th Fibonacci number.
Proof. We prove the lemma by induction. Since \( \Theta_1 = \{1/2\}, \Theta_2 = \{2/3\} \), then the base of induction is true. Let us suppose that the lemma is true for \( k \leq n \) and \( x = [[1; b_1, \ldots, b_l]] \in \Theta_{n+1} \), then \( b_1 + \ldots + b_l = n + 2 \). There are two cases: either \( b_l = 2 \) or \( b_l \geq 2 \). In the first case \( b_1 + \ldots + b_{l-1} = n \), so \( [[1; b_1, \ldots, b_{l-1}]] \in \Theta_{n-1} \), in the second case \( b_1 + \ldots + b_l - 1 = n + 1 \), so \( [[1; b_1, \ldots, b_l - 1]] \in \Theta_n \). Thus we have one-one correspondence between \( \Theta_{n-1} \cup \Theta_n \) and \( \Theta_{n+1} \), and so \( \# \Theta_{n+1} = \# \Theta_n + \# \Theta_{n-1} \).

Definition 1 Let \( x, y, z \) be consecutive elements of \( \Xi_n \), \( y \in \Theta_n \). We denote the mediant \( x \oplus y \) by \( y^l \), the mediant \( y \oplus z \) we denote by \( y^r \).

Lemma 3 Let \( x, y, z \) be consecutive elements of \( \Xi_n \), \( y \in \Theta_n \), then \( y^l \in \Theta_{n+2} \), \( y^r \in \Theta_{n+1} \).

Proof. Let \( y = [[1; b_1, \ldots, b_s]] \). Then \( y^l = [[1; b_1, \ldots, b_s, 2]] \), \( y^r = [[1; b_1, \ldots, b_s + 1]] \).

Now let us construct an infinite tree \( D \) whose nodes are labeled by rationals in \((0, 1)\). We identify the nodes with the rationals they labeled by. The root is labeled by \( 1/2 \). From node \( x \) come two arrows: the left arrow goes to \( x^l \) and the right arrow goes to \( x^r \). The nodes of the tree \( D \) are partitioned into levels. \( 1/2 \) belongs to the level 1. If \( x \) belongs to the level \( n \), then \( x^r \) belongs to the level \( n + 1 \), and \( x^l \) belongs to the level \( n + 2 \) (figure 1).

![Figure 1: Infinite tree construction](image-url)
It follows from the construction of the tree that nodes from level \( n \) of \( D \) are marked by numbers from \( \Theta_n \). So \( x \) belongs to the level \( n \) if and only if \( x \in \Theta_n \).

The subtree of \( D \) with root in the node \( x \) we denote by \( D^{(x)} \). The set of nodes of \( D \) from level 1 to level \( n \) we denote by \( D_n \). The set of nodes of \( D^{(x)} \cap D_n \) we denote by \( D^{(x)}_n \). Note that there exist a levels preserving isomorphism between \( D \) and \( D^{(x)} \). If \( x \) belongs to the level \( n \), then
\[
\#D^{(x)}_n = \#D_{m-n+1}.
\]

Besides
\[
\#D_n = \#\Theta_1 + \#\Theta_2 + \ldots + \#\Theta_n = F_1 + F_2 + \ldots + F_n = F_{n+2} - 1.
\]

### 7 Proof of Theorem 1

We remind that it is enough to prove (13) for any consecutive elements of \( \Xi_n \) \( x \) and \( y \).

To prove the equality (13) we consider the subtree \( D^{(x \oplus y)} \) of \( D \). Note that
\[
\{ \xi \in D^{(x \oplus y)} \} \cup \{ y \} = \{ \xi \in Q : x < \xi \leq y \}.
\]

Consequently
\[
M(y) - M(x) = \lim_{m \to \infty} \frac{\#\{ \xi \in \Xi_m : x < \xi \leq y \}}{\#\Xi_m} = \lim_{m \to \infty} \frac{\#D^{(x \oplus y)}_m}{\#D_m}.
\]

On the other hand
\[
M(x \oplus y) - M(x) = \lim_{m \to \infty} \frac{\#\{ \xi \in \Xi_m : x < \xi \leq x \oplus y \}}{\#\Xi_m} = \lim_{m \to \infty} \frac{\#D^{(x \oplus y)}_m}{\#D_m}.
\]

Let \( x \oplus y \in \Theta_k \), then \( (x \oplus y)^l \in \Theta_{k+2} \). Therefore
\[
\frac{M(x \oplus y) - M(x)}{M(y) - M(x)} = \lim_{m \to \infty} \frac{\#D^{(x \oplus y)}_m}{\#D^{(x \oplus y)}_m} = \lim_{m \to \infty} \frac{\#D^{(x \oplus y)}_m}{\#D^{(x \oplus y)}_m} = \lim_{m \to \infty} \frac{\#D_{m-k+1}}{\#D_{m-k+1}} = \lim_{m \to \infty} \frac{F_{m-k+1}}{F_{m-k+3}} = \tau^2.
\]
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