Wilson Loops from D-branes in $\text{AdS}_4 \times \text{CP}^3$ with $B_{\text{NS}}$ Holonomy

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Abstract: We study Wilson loops in $\text{N}=6$ superconformal Chern-Simons theory with gauge group $U(M) \times U(N)$ that is dual to $N$ M2-branes and $(M-N)$ fractional M2-branes, equivalently, discrete 3-form holonomy at $\text{C}4/\mathbb{Z}_k$ orbifold singularity. We give description of these Wilson loops in terms of macroscopic fundamental string and D6-branes in the dual $\text{AdS}_4 \times \text{CP}^3$ geometry with $B_{\text{NS}}$ holonomy turned on over $\text{CP}^1 \subset \text{CP}^3$.

Keywords: D-branes, AdS-CFT Correspondence
1. Introduction and Summary

The ABJM \cite{1} theory has been conjectured to be dual to M-theory on AdS$_4 \times S^7/Z_k$ with $N$ units of four-form flux which for $k << N << k^5$ can be compactified to type IIA theory on AdS$_4 \times CP^3$, where $k$ is the level of Chern-Simons theory with gauge group $U(N) \times U(N)$. The ABJM theory is weakly coupled for $\lambda << 1$, where $\lambda = N/k$ is the 't Hooft coupling. In continuation with this proposal, Aharony, Bergman, and Jafferis (ABJ) \cite{2} identified a further class of gauge-gravity duality with extended supersymmetry namely, a three dimensional $N = 6$ superconformal Chern-Simons theory with a gauge group $U(M)_k \times \overline{U(N)}_{-k}$, with $k$ being the level of the CS theory, is dual to type IIA string theory on AdS$_4 \times CP^3$ with $B_{NS}$ holonomy turned on over $CP^1 \subset CP^3$ \cite{13}. In gauge theory, Wilson loop operators are non-local gauge invariant operators in which the theory can be formulated. One defines a Wilson loop as the trace in an arbitrary representation $R$ of the gauge group $G$ of the holonomy matrix associated with parallel transport along a closed curve $C$ in spacetime. Since the beginning of the proposed AdS/CFT correspondence \cite{10}, it is known that Wilson loops in $N = 4$ SYM theory can be calculated in dual description using macroscopic strings \cite{11,12}. In a recent interesting paper \cite{13} a class of new open strings in AdS$_5$ were found arising out of the solutions to the equations of motion corresponding to fundamental strings and they describe Wilson loops in the fundamental representations. Some time back it was argued in very interesting paper \cite{14}, for type IIB theory, that Wilson loops have a gravitational dual description in terms of D5-branes or alternatively in terms of D3-branes in AdS$_5 \times S^5$ background \cite{15}.

Recently the Wilson loop operators have been investigated in the three-dimensional, N=6 superconformal Chern-Simons theory dual to IIA superstring theory on AdS$_4 \times CP^3$ in \cite{31,32,33,34}. For a given contour there are two linear combinations of Wilson loop operators transforming oppositely under time-reversal transformation. Like the AdS$_5 \times$
In the $S^5$ case the D-brane configurations wrapping various cycles in the $\mathbb{CP}^3$ has been shown to correspond to dual operators in the boundary theory. We generalize these to the case of ABJ theory, which has an additional $B_{NS}$ holonomy turned on over $\mathbb{CP}^1 \subset \mathbb{CP}^3$. We propose Wilson loops containing both gauge potential and a pair of bi-fundamental fields almost in the same way as in case of ABJM theory. However there is a subtle difference that now the gauge groups are $U(M)$ and $U(N)$ respectively. Further, the functional form of the Wilson loop operator is constrained by the requirement of affine symmetry along the contour $C$ and by superconformal symmetry $R^{1,2}$ and by gauge and $SU(4)$ symmetries. We generalize the analysis given in [33] and find out that there are two elementary Wilson loops operators. We further give Wilson loop operators in terms of objects in the dual supergravity on $\text{AdS}_4 \times \mathbb{CP}^3$ with $B_{NS}$ holonomy.

Firstly, the fundamental string description of Wilson loop is precisely the same as in case of ABJM theory and hence again these strings describe symmetric combinations of Wilson loops in fundamental representation. As it will be clear in what follows that the effect of the non-zero torsion is sub-leading in the supergravity description. Equivalently, the dynamics of the string with world-volume topology $\text{AdS}_2 \subset \text{AdS}_4$ is not affected by the NS-NS two form field. For the D2-brane description of Wilson loop that span the subspace $\text{AdS}_2 \times S^1 \subset \text{AdS}_4 \times \mathbb{CP}^3$, the presence of two form $B_{NS}$ does not affect this. This solution was studied in [34]. We study the Wilson loop solution corresponding to D6-brane with topology $\text{AdS}_2 \times \Sigma_4$ where $\Sigma_4 \subset \mathbb{CP}^3$. More precisely we consider D6-brane configuration that spans $\text{AdS}_2$ subspace of $\text{AdS}_4$ and couples to $C_5$. We find out that this D6-brane describes the Wilson loops in anti-symmetric representation.

The rest of the paper is organized as follows. In section-2 we review basic facts about the ABJ theory. Section-3 is devoted to review of basic properties of Wilson loops in the supersymmetric $U(M) \times \overline{U(N)}$ gauge theory. In section-4 we study these Wilson loops from the supergravity dual of $\text{AdS}_4 \times \mathbb{CP}^3$ with $B_{NS}$ holonomy and suggest their relations to the Wilson loops in ABJ theory.

2. Review of ABJ Theory

In this section we review basic facts about ABJ theory. This theory is a $N = 6$ supersymmetric Chern-Simons matter theory with two gauge groups of ranks $N, M$ and levels $k$ and $-k$ respectively. In addition to the gauge fields there are bosonic and fermionic field $Y^A$ and $\psi_A$ respectively. More precisely this theory has the following properties:

- Gauge and global symmetries:
  
  gauge symmetry: $U(M) \times \overline{U(N)}$

  global symmetry: $SU(4)$

- The field content of given theory is as follows:
  
  $A^{(L)}_\mu : \text{Adj}(U(M))$,  
  $A^{(R)}_\mu : \text{Adj}(\overline{U(N)})$ .

(2.1)
Further there is $M \times N$ matrix valued matter fields-4 complex scalar $Y^A (A = 1, 2, 3, 4)$ and their hermitian conjugates $Y_A^\dagger$. There are also $M \times N$ matrix valued fermions $\psi_A$ together with their hermitian conjugates $\psi^A$. Fields with superscript $A$ index transform in the 4 of $R$ symmetry $SU(4)$ group and those with subscript index transform in the $\overline{4}$ representations.

The corresponding Lagrangian has the following form $^3$

$$
\mathcal{L} = -\frac{k}{2\pi} \text{Tr}(D^\mu Y^\dagger A D_\mu Y^A) + \frac{i k}{2\pi} \text{Tr}(\psi^A \gamma^\mu D_\mu \psi_A) - V + \mathcal{L}_{CS} +
+ i \text{Tr}(\psi^A Y^\dagger_B A_B Y^B) - i \text{Tr}(\psi_A \psi^A Y^B Y^\dagger_B) +
+ 2i \text{Tr}(\psi_A \psi^B Y^A Y^\dagger_B) - 2i \text{Tr}(\psi^B \psi_A Y^\dagger_B Y^A) +
+ i \epsilon^{ABC} \text{Tr}(\psi_A Y^\dagger_B \psi_C Y^\dagger_D) - i \epsilon_{ABC} \text{Tr}(\psi^A Y^B \psi^C Y^D),
$$

(2.2)

where $\mathcal{L}_{CS}$ is a Chern-Simons term and $V(Y)$ is a sextic scalar potential

$$
\mathcal{L}_{CS} = \frac{k}{4\pi} \epsilon^{\mu \nu \lambda} \text{Tr}[A^{(L)}_\mu \partial_\nu A^{(L)}_\lambda + \frac{2i}{3} A^{(L)}_\mu A^{(L)}_\nu A^{(L)}_\lambda] -
- \frac{k}{4\pi} \epsilon^{\mu \nu \lambda} \text{Tr}[A^{(R)}_\mu \partial_\nu A^{(R)}_\lambda + \frac{2i}{3} A^{(R)}_\mu A^{(R)}_\nu A^{(R)}_\lambda],
$$

$$
V(Y) = -\frac{1}{3} \text{Tr}[Y^A Y^\dagger_A Y^B Y^\dagger_B Y^C Y^\dagger_C + Y^A Y^\dagger_A Y^B Y^\dagger_B Y^C Y^\dagger_C] +
+ 4 Y^A Y^\dagger_A Y^B Y^\dagger_B Y^C Y^\dagger_C - 6 Y^A Y^\dagger_A Y^B Y^\dagger_B Y^C Y^\dagger_C] .
$$

(2.3)

Further, the covariant derivatives for the scalars are defined as

$$
D_\mu Y^A = \partial_\mu Y^A + i A^{(L)}_\mu Y^A - i Y^A A^{(R)}_\mu, \quad D_\mu Y^\dagger_A = \partial_\mu Y^\dagger_A - i Y^\dagger_A A^{(L)}_\mu + i A^{(R)}_\mu Y^\dagger_A,
$$

(2.4)

and for fermions

$$
D_\mu \psi_A = \partial_\mu \psi_A + i A^{(L)}_\mu \psi_A - i \psi_A A^{(R)}_\mu, \quad D_\mu \psi^A = \partial_\mu \psi^A - i \psi^A A^{(L)}_\mu + i A^{(R)}_\mu \psi^A.
$$

(2.5)

In sharp contrast from $N = 4$ SYM theory, well known from the $AdS_5/CFT_4$ correspondence, the characteristic property of ABJ theory is that the coupling parameters are integer valued. This fact suggests that these coupling parameters cannot possibly run under renormalization group flow. Then it is convenient to introduce the parameters $\lambda, \overline{\lambda}$ and consider the generalized 't Hooft planar limit

$$
M, N, k \to \infty \quad \text{with} \quad \lambda \equiv \frac{N}{k}, \quad \overline{\lambda} \equiv \frac{M}{k}, \quad b = \frac{(M - N)}{k} \quad \text{fixed} .
$$

(2.6)

$^3$Note that we use the notation as in [36].
As was shown in [2] the number of fractional branes $M - N$ is limited to $0 \leq (M - N) \leq k$. This is an important fact since in the planar limit this implies that $b \in [0, 1]$. Let us discuss the parity transformation in dual theory. By definition the parity transformation maps the Chern-Simons parameters by $k$ to $-k$ while holding $M, N$ fixed. In case of ABJM theory the parity transformations are defined as

$$
P: \quad t \rightarrow t, \quad x^1 \rightarrow x^1, \quad x^2 \rightarrow -x^2, \quad (A^{(L)}_\mu, A^{(R)}_\mu, Y^A, Y^A_\dagger) \rightarrow (-A^{(R)}_\mu, -A^{(L)}_\mu, Y^A_\dagger, Y^A).$$

(2.7)

Since this parity exchanges $Y^A$ with $Y^A_\dagger$ it exchanges two isomorphic groups $U(N)$ and $\overline{U(N)}$. It is also clear that the ABJM theory is manifestly invariant under this parity transformation. On the other hand in the ABJ theory the parity transformation cannot be a symmetry of the theory since the two gauge groups are different and cannot be exchanged.

On the other hand as nicely argued in [2] and in [3] the parity transformation maps one ABJ theory with a given gauge group to another ABJ theory with a different gauge group. Explicitly, it was argued that there is an equivalence relations between these two ABJ theories:

$$U(M)_k \times \overline{U(N)}_{-k} \simeq U(N)_k \times \overline{U(M)}_{-k}.$$  (2.8)

3. Wilson loops in $N = 6$ Superconformal Chern-Simons Theory

Our goal is to find Wilson loop operators in ABJ theory. The proposed Wilson loops contain both gauge potential and a pair of bi-fundamental fields almost in the same way as in ABJM theory. However there is a subtle difference that now the gauge groups are $U(M)$ and $U(N)$ respectively. Further, it was carefully discussed in [33] that the functional form of the Wilson loop operator is constrained by the requirement of affine symmetry along the contour $C$, by superconformal symmetry along $R^{1,2}$, and by gauge and $SU(4)$ symmetries.

On the other hand since these Wilson loops are independent it is clear that the Wilson loops in ABJ theory have to obey the same conditions as that of ABJM theory [33].

Let us be more specific. We denote the coordinates of $R^{1,2}$ as $x^\mu$ and of $SU(4)$ internal space as $z^I, \overline{z}_I$. With two gauge fields $A^{(L)}_\mu$ of $U(M)$ and $A^{(R)}_\mu$ of $\overline{U(N)}$ we can construct two types of Wilson loops operators that are associated with each gauge groups. Let us consider $U(M)$ gauge group. Following [33] we propose the $U(M)$ Wilson loop operator as

$$W_M[C, P] = \frac{1}{M} \text{Tr} \mathcal{P} \exp i \int_C d\tau (\dot{x}^\mu A_\mu(x) + M^B_A Y^A(x) Y^A_\dagger(x)),$$  (3.1)

where the vector field $\dot{x}^\mu(\tau)$ specifies the path $C$ in $R^{1,2}$ and $M^B_A$ was defined in [33]. Finally, the Tr means the trace over $U(M)$ group.
In the same way we can propose the Wilson loop operator of $U(N)$ in the form
\[
\mathbb{W}_N[C, P] = \frac{1}{N} \text{Tr} P \exp i \int_C d\tau (\dot{x}^\mu A_\mu^R(\tau) + M_A^R(\tau) Y_B^\dagger(\tau) Y^A(\tau)) ,
\] (3.2)
where now Tr denotes the trace over $U(N)$ group. From the point of view of ABJ theory more Wilson loops operators are possible, for example we can consider linear combination
\[
W_M[C, P] + \mathbb{W}_N[C, P]
\] (3.3)
or the product of Wilson loop operators
\[
W_N[C, P] \cdot \mathbb{W}_N[C, P]
\] (3.4)
As was also nicely shown in [33] the form of Wilson loops is restricted by requirement that they preserve some fractions of $N = 6$ supersymmetry. However it turns out that this does not determine possible combinations of Wilson loops since supersymmetry acts on $W_M[C, P]$ and $\mathbb{W}_N[C, P]$ independently.

The fact that the ABJ theory possesses two sets of Wilson loops, one for $U(M)$ and the second one for $U(N)$ the dual holographic theory is puzzling. Explicitly, as in case of $AdS_5/CFT_4$ correspondence we expect that the Wilson loops in fundamental representation corresponds the fundamental string in the bulk [11, 12]. In case of ABJM theory this problem was resolved in [30, 31, 33] where it was argued that fundamental string corresponds to the symmetric combinations of Wilson loops in dual theory. The question is whether this picture changes in case of ABJ theory when the two gauge groups are different and when the dual background possesses discrete $B_{NS}$ holonomy. In particular, we would like to see whether dual description is sensitive to the parity breaking effects that are represented by the presence of non-trivial $B_{NS}$ holonomy.

4. Description of Wilson loops in Supergravity Dual $AdS_4 \times CP^3$ with $B_{NS}$ Holonomy

In this section we try to find description of the Wilson loop introduced in previous section in terms of macroscopic objects in dual $AdS_4 \times CP^3$ geometry.

We start by writing down the metric for $AdS_4 \times CP^3$, which in a particular parametrization reads
\[
ds^2 = \tilde{R}^2 (ds^2_{AdS_4} + 4 ds^2_{CP^3} , \\
ds^2_{AdS_4} = [\cosh^2 \rho \ dt^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) ] \\
ds^2_{CP^3} = \frac{1}{4} \left[ d\alpha^2 + \cos^2 \frac{\alpha}{2} (d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2) + \sin^2 \frac{\alpha}{2} (d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2) + \\
+ \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} (d\chi + \cos \theta_1 d\varphi_1 - \cos \theta_2 d\varphi_2)^2 \right] ,
\] (4.1)
where
\[
0 \leq \alpha, \theta_1, \theta_2 \leq \pi , \quad 0 \leq \varphi_1, \varphi_2 \leq 2\pi , \quad 0 \leq \chi \leq 4\pi ,
\] (4.2)
and where
\[ \tilde{R}^2 = \frac{R^3}{4k}, \quad e^{2\Phi_0} = \frac{R^3}{k^3}. \] (4.3)

While taking the limit \( \alpha' = 1 \), the curvature radius \( R \) is given by \( \tilde{R}^2 = \pi \sqrt{2\lambda} \). The 't Hooft coupling constant is \( \lambda \equiv \frac{N}{k} \) where \( k \) is the level of the 3-dimensional N=6 ABJM model.

Let us now consider Ramond-Ramond gauge fields. The first non-trivial one is the one-form potential \( C_1 \) that takes the form
\[ C_1 = \frac{k}{4} \left[ \cos \alpha d\chi + 2 \cos^2 \frac{\alpha}{2} \cos \theta_1 d\varphi_1 + 2 \sin^2 \frac{\alpha}{2} \cos \theta_2 d\varphi_2 \right]. \] (4.4)

Further there is a three-form potential \( C_3 \) in the form
\[ C_3 = \frac{1}{8} R^3 \cosh^3 u \cosh \rho dt \wedge d\rho \wedge d\phi, \] appropriate for a time-like curve,
\[ C_3 = \frac{1}{8} R^3 \cosh^3 u \sinh \rho d\psi \wedge d\rho \wedge d\phi, \] appropriate for a space-like curve. (4.5)

The Hodge dual of \( F_4 = dC_3 \) is a six form \( F_6 \) that is proportional to the volume form of \( CP^3 \)
\[ F_6 = \ast F_4 = \frac{3}{28} k \sin^3 \alpha \sin \theta_1 \sin \theta_2 d\alpha \wedge d\theta_1 \wedge d\theta_2 \wedge d\chi \wedge d\varphi_1 \wedge d\varphi_2. \] (4.6)

Then we can define the five form \( C_5 \) as
\[ C_5 = -\frac{k}{28} (\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2) \sin \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\chi \wedge d\varphi_1 \wedge d\varphi_2 \] (4.7)
that obeys the relation \( F_6 = dC_5 \). The characteristic property of the background dual to ABJ theory is an existence of an additional non-trivial NS-NS two form
\[ \mathcal{B}_{NS} = -\frac{b}{4} \sin \alpha d\alpha \wedge (d\chi + \cos \theta_1 d\varphi_1 - \cos \theta_2 d\varphi_2) - \] \[ \quad - \frac{b}{2} \left( \cos^2 \frac{\alpha}{2} \sin \theta_1 d\theta_1 \wedge d\varphi_1 + \sin^2 \frac{\alpha}{2} \sin \theta_2 d\theta_2 \wedge d\varphi_2 \right). \] (4.8)

Let us now consider Dp-brane in given background. Generally the low energy dynamics of Dp-brane is governed by following action
\[ S = S_{DBI} + S_{WZ}, \]
\[ S_{DBI} = -\tau_p \int d^{p+1}x \sqrt{-\det A}, \]
\[ A_{\alpha\beta} = \partial_\alpha x^M \partial_\beta x^N G_{MN} + 2\pi F_{\alpha\beta}, \]
\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - (2\pi)^{-1} B_{MN} \partial_\alpha x^M \partial_\beta x^N, \]
\[ S_{WZ} = \tau_p \int e^{2\pi \mathcal{F}} \wedge C, \] (4.9)
where \( \tau_p \) is D\( p \)-brane tension, \( \xi^\alpha, \alpha = 0, 1, \ldots, p \) are the \( (p+1) \) world-volume coordinates and where \( A_\alpha \) is gauge field living on the world-volume of D\( p \)-brane. Further \( \Phi, G_{MN} \) and \( B_{MN} \) are space-time dilaton, graviton and NS two form field respectively. Finally \( C \) in the last line in (4.9) means collection of Ramond-Ramond fields. In order to study Wilson lines we perform an analytic continuation when we introduce \( \tau \) as \( t = i\tau \) so that the Euclidean form of D\( p \)-brane action takes the form

\[
S = S_{DBI} + S_{WZ},
\]

\[
S_{DBI} = \tau_p \int d^p\xi d\tau e^{-\Phi_0} \sqrt{\det A},
\]

\[
A_{\alpha\beta} = \partial_{\alpha} x^M \partial_{\beta} x^N G_{MN} + 2\pi F_{\alpha\beta},
\]

\[
F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} - (2\pi)^{-1} B_{MN} \partial_{\alpha} x^M \partial_{\beta} x^N,
\]

\[
S_{WZ} = \tau_p \int e^{2\pi i F} \wedge C.
\]

Our goal is to analyze description of the Wilson lines in ABJ theory in terms of objects in supergravity dual of \( AdS_4 \times CP_3 \) with \( B_{NS} \) holonomy.

Let us start with the situation of Wilson loop description using fundamental string. In fact, it turns out that fundamental string description of Wilson loop is the same as in case of ABJM theory and hence these strings describe symmetric combinations of Wilson loops in fundamental representation. We will see below that the effect of the non-zero parameter \( b \) is sub-leading in supergravity description. Equivalently, since the string with world-volume topology \( AdS_2 \) is stretched in \( AdS_4 \) completely it is clear that its dynamics is not affected by NS-NS two form field.

Now we consider the Wilson loop in symmetric representation. In case of \( AdS_5/CFT_4 \) correspondence these objects are described by D3-branes that span \( AdS_2 \) subspace of \( AdS_5 \) together with some \( \Sigma_2 \) subspace of \( S^5 \) \cite{14, 24}. In case of ABJM theory the corresponding object is D2-brane that spans the subspace \( AdS_2 \times S^1 \) inside \( AdS_4 \times CP^3 \). However since in this case the D2-brane wraps \( S^1 \) in \( CP^3 \), it is clear that the presence of two form \( B_{NS} \) holonomy does not affect this configuration at all.

Let us now consider the final example of D6-brane with topology \( AdS_2 \times \Sigma_4 \) where \( \Sigma_4 \subset CP^3 \). According to standard arguments these D6-branes should correspond to Wilson loops in anti-symmetric representation. More precisely we consider D6-brane that spans \( AdS_2 \) subspace of \( AdS_4 \) and couples to \( C_5 \). The similar calculation with zero \( B_{NS} \) holonomy has been performed in \cite{31} and we closely follow this approach. Firstly, it is easy to see that the WZ term takes the form

\[
S_{WZ} = \tau_6 \int 2\pi F \wedge C^{(5)}
\]

since the contribution \( B \wedge C^{(5)} \) vanishes due to the fact that D6-brane is extended in \( AdS_4 \) as well. On the other hand the pullback of \( B \) field will appear in the DBI action. More precisely, let us consider following embedding ansatz

\[
\xi^1 = \rho, \quad \xi^2 = \psi, \quad \xi^3 = -\frac{1}{2} \chi, \quad \xi^4 = \varphi_1, \quad \xi^5 = \theta_1, \quad \xi^6 = \varphi_2, \quad \xi^7 = \theta_2,
\]
\[ F_{\rho \psi} = -F_{\psi \rho} = f(\rho) , \quad u = 0 , \quad \alpha = \text{const} . \]

(4.12)

For this parameterization we obtain following non-zero components of the matrix \( A \equiv A_0 + B \)

\[
\begin{align*}
A_{\rho\rho} &= \tilde{R}^2 , \quad A_{\rho\psi} = -A_{\psi\rho} = 2\pi f , \quad A_{\psi\psi} = \tilde{R}^2 \sinh^2 \rho , \\
A_{33} &= 4\tilde{R}^2 \sin^2 \frac{\alpha}{2} \cos \frac{\alpha}{2} , \\
A_{34} &= A_{43} = -2\tilde{R}^2 \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \cos \theta_1 , \\
A_{36} &= A_{63} = 2\tilde{R}^2 \sin^2 \frac{\alpha}{2} \cos \theta_2 , \\
A_{44} &= \tilde{R}^2 \left[ \cos \frac{\alpha}{2} \sin^2 \theta_1 + \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \cos \theta_1 \right] , \\
A_{45} &= -A_{54} = -\frac{b}{4} \cos \frac{\alpha}{2} \sin \theta_1 , \\
A_{46} &= -\tilde{R}^2 \sin^2 \frac{\alpha}{2} \cos \frac{\alpha}{2} \cos \theta_1 \cos \theta_2 , \\
A_{55} &= \tilde{R}^2 \cos^2 \frac{\alpha}{2} , \quad A_{77} = \tilde{R}^2 \sin^2 \frac{\alpha}{2} , \\
A_{66} &= \tilde{R}^2 \left[ \sin^2 \frac{\alpha}{2} \sin^2 \theta_2 + \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \cos^2 \theta_2 \right] , \\
A_{67} &= -\frac{b}{4} \sin^2 \frac{\alpha}{2} \sin \theta_2 = -A_{76} ,
\end{align*}
\]

(4.13)

where \( A_0 \) are components of the matrix \( A \) that do not depend on \( b \) and where \( B \) are the matrix components proportional to \( b \).

In principle we can then evaluate \( \det A \). However the resulting expression is very complicated and is not very interesting. For that reason we simplify the analysis using the following observation. We know that \( b \in [0, 1) \). Then since \( g \) is proportional to \( \tilde{R}^2 \) we can presume that \( (A_0)_{ij} B^{jk} \ll (A_0)_{ik} \) and hence we can write

\[
\sqrt{\det A} \approx \sqrt{\det A_0} \left( 1 - \frac{1}{4} A_0^{jk} B_{kl} A_0^{lm} B_{mj} \right) .
\]

(4.14)

Then the form of the ansatz (4.12) implies

\[
\det A_0 = 4\tilde{R}^{10} \sin^2 \theta_1 \sin^2 \theta_2 \sin^6 \frac{\alpha}{2} \cos^6 \frac{\alpha}{2} (\sinh^2 \rho + \beta^2 f^2)
\]

(4.15)

and

\[
B_{ij} A_0^{jk} B_{kl} A_0^{li} = 2B_{45} A_0^{55} B_{54} A_0^{44} + 2B_{67} A_0^{77} B_{76} A_0^{66} ,
\]

(4.16)
where
\[
A_{0}^{55} = \frac{1}{4\tilde{R}^2 \cos^2 \frac{\alpha}{2}}, \quad A_{0}^{44} = \frac{1}{\tilde{R}^2 \sin^2 \theta_1 \cos^2 \frac{\alpha}{2}},
\]
\[
A_{0}^{66} = \frac{1}{\tilde{R}^2 \sin^2 \frac{\alpha}{2} \sin^2 \theta_2}, \quad A_{0}^{77} = \frac{1}{4\tilde{R}^2 \sin^2 \frac{\alpha}{2}},
\]

(4.17)

and where the explicit form of the of $B_{ij}$ follow from (4.13). Using these results we finally find that (4.14) is equal to
\[
\sqrt{\det A} \approx \tilde{R}^5 \sin \theta_1 \sin \theta_2 \sin^3 \frac{\alpha}{2} \cos^3 \frac{\alpha}{2} \sqrt{\text{sinh}^2 \rho + \beta^2 f^2} \left(1 + \frac{b^2}{64\tilde{R}^4}\right).
\]

(4.18)

Consequently the DBI part of D6-brane action takes the form
\[
S_{DBI} = \tau_6 \tilde{R}^7 \frac{k}{2R} \int 2 \sin \theta_1 \sin \theta_2 \sin^3 \frac{\alpha}{2} \cos^3 \frac{\alpha}{2} \sqrt{\text{sinh}^2 \rho + \beta^2 f^2} \left(1 + \frac{b^2}{64\tilde{R}^4}\right) = \tau_6 \pi^3 \frac{R^9}{2^4 k^2} \int d\psi \sin^3 \frac{\alpha}{2} \cos^3 \frac{\alpha}{2} \sqrt{\text{sinh}^2 \rho + \beta^2 f^2} \left(1 + \frac{b^2}{64\tilde{R}^4}\right),
\]

(4.19)

where we have introduced the parameter $\beta$ defined as
\[
\beta = \frac{8\pi k}{\tilde{R}^3}
\]

(4.20)

and where we have performed the following integration
\[
\int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \int_0^{2\pi} d\chi \int_0^{\pi} d\theta_1 \sin \theta_1 \int_0^{\pi} d\theta_2 \sin \theta_2 = 2^5 \pi^3.
\]

(4.21)

Considering the WZ term we find
\[
S_{WZ} = i\tau_6 \int 2\pi F \wedge C^{(5)} = i\tau_6 \frac{R^9}{2^4 k^2} \beta \int 2f (\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2) \sin \theta_1 \sin \theta_2.
\]

(4.22)

Finally we perform the integration over five coordinates of CP$^3$ in (4.22) that again gives the factor $2^5 \pi^3$. Consequently the full D6-brane action takes the form
\[
S_{DBI} + S_{WZ} = \frac{R^9 \pi^3 \tau_6}{2^4 k^2} \int d\psi \sin^3 \alpha \sqrt{\text{sinh}^2 \rho + \beta^2 f^2} \left(1 + \frac{b^2 k^2}{4R^6}\right) + i\beta f (\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2).
\]

(4.23)
Using this form of the action we easily derive the equation of motion for $\alpha$

$$3\sin^2 \alpha \cos \alpha \sqrt{\sinh^2 \rho + \beta^2 f^2 (1 + \frac{b^2 k^2}{4R^6})} - 3i\beta f(\sin^3 \alpha) = 0.$$  \hfill (4.24)

Further the equations of motion for the components of the gauge fields $A_1, A_2$ imply an existence of conserved electric flux $\Pi$

$$\sin^3 \alpha \frac{\beta^2 f}{\sqrt{\sinh^2 \rho + \beta^2 f^2}} (1 + \frac{b^2 k^2}{4R^6}) + i\beta(\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2) = \Pi$$  \hfill (4.25)

that allows us to express $f$ as a function of $\Pi$ so that

$$f = \frac{(\Pi - i(\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2)) \sinh \rho}{\sqrt{\beta^4 \sin^6 \alpha (1 + \frac{b^2 k^2}{4R^6})^2 - \beta^2 (\Pi - i(\sin^2 \alpha \cos \alpha + 2 \cos \alpha - 2))^2}}.$$  \hfill (4.26)

Finally, using (4.24) we find the dependence of $\Pi$ on $\alpha$

$$\Pi = 2i\beta(\cos \alpha - 1) - i\beta \sin^2 \alpha \cos \alpha \frac{b^2 k^2}{2R^6},$$  \hfill (4.27)

where we neglected $\frac{b^4 k^4}{64 R^{12}}$. It is well known that the electric flux is proportional to the number of dissolved fundamental strings on the world-volume of D6-brane. Since the electric flux is aligned along $\rho, \psi$ directions we find that the number of strings that are stretched in these directions and that are localized at given $\alpha$ is equal to

$$Q = \frac{R^9 i\Pi}{8k^2} = \frac{N}{2} \left[ (1 - \cos \alpha) + \frac{1}{2} \sin^2 \alpha \cos \alpha \frac{b^2 k^2}{2R^6} \right],$$  \hfill (4.28)

where the factor $i$ was included as a consequence of the fact that we work in space with Euclidean signature and where the factor $2^5 \pi^3$ follows from the integration over $CP^3$. Observe that now the number of fundamental string is sensitive to the presence of $B_{NS}$ holonomy. Finally we determine the value of the action (4.23) for the ansatz (4.12) and we find

$$S = \frac{\pi^3 \tau_6 R^9}{8k^2} \int d\rho d\psi \sin \rho \left[ \frac{\sin^2 \alpha}{\sqrt{1 + \cos^2 \alpha \frac{b^2 k^2}{2R^6}}} + \frac{2 \cos \alpha (\cos \alpha - 1)}{\sqrt{1 + \cos^2 \alpha \frac{b^2 k^2}{2R^6}}} \right] \left[ 1 + \frac{b^2 k^2}{2R^6} \right]$$  \hfill (4.29)

However this is not full story since we should perform a Legendre transformation in this action in order to write it as a functional of $\Pi$ instead of $f$ [37]. To do this we add to it an expression

$$\delta S = -\frac{\pi^3 \tau_6 R^9}{2^4 k^2} \int d\rho d\psi (\Pi f) =$$

$$= -\frac{\pi^3 \tau_6 R^9}{2^4 k^2} \int d\rho d\psi \sinh \rho \left[ \frac{2 \cos \alpha (\cos \alpha - 1)}{\sqrt{1 + \cos^2 \alpha \frac{b^2 k^2}{2R^6}}} + \frac{\sin^2 \alpha \cos^2 \alpha \frac{b^2 k^2}{2R^6}}{\sqrt{1 + \cos^2 \alpha \frac{b^2 k^2}{2R^6}}} \right] \left[ 1 + \frac{b^2 k^2}{2R^6} \right].$$  \hfill (4.30)
Consequently the sum of (4.29) with (4.30) gives

\[ S_{L.T.} = S + \delta S = \frac{\pi^3 R^9 \tau_6}{24 k^2} \int d\rho d\psi \sinh \rho \sin^2 \alpha \left( 1 + \cos^2 \alpha \frac{b^2 k^2}{4 R^6} \right) \left( 1 + \frac{b^2 k^2}{2 R^6} \right). \]  

(4.31)

To proceed further we use (4.28) to express \( \cos \alpha \) as a function of \( Q, N \) and \( b \)

\[ \cos \alpha = \frac{N - 2Q}{N} \left( 1 + \frac{b^2 k^2}{4 R^6} \right) \]  

(4.32)

and following [31] we regularize the integration over AdS2 in (4.31) with the result \( \int d\rho d\psi \sinh \rho \text{reg} = -2\pi \). Then using the relations

\[ N = \lambda k, R^6 = 2^7 \pi^3 k^3 \sqrt{2} \lambda^{3/2} \]  

(4.33)

we find that (4.31) is equal to

\[ S_{L.T.} = -\pi \sqrt{2} \lambda \frac{Q(N - Q)}{N} \left( 1 + \frac{b^2 k^2}{4 R^6} \left( 1 - \frac{N}{2Q} \right) \right) \times \]

\[ \times \left( 1 + \frac{N - 2Q b^2 k^2}{N - Q} \frac{b^2 k^2}{8 R^6} \left( 1 + \frac{b^2 k^2}{4 R^6} \left( \frac{N - 2Q}{N} \right)^2 \right) \right) \left( 1 + \frac{b^2 k^2}{2 R^6} \right). \]  

(4.34)

It is easy to see that (4.34) is invariant under exchange \( Q \rightarrow (N - Q) \). The fact that the action is invariant under exchange \( Q \rightarrow N - Q \) implies that this D6-brane describes Wilson loop in anti-symmetric representation.

Now we would like to interpret these results from the point of view of Wilson loops in ABJ theory. Luckily we can use the results derived in case of ABJM theory [31, 33]. Let us again restrict to the case of circular Wilson line that should be non-zero. As was derived in [31, 33] the expectation values of these Wilson loops in weak coupling limit are equal to

\[ W_M = 1 + \frac{\pi^2 M^2}{k^2} - \frac{\pi^2 M^2}{6k^2}, \]

\[ W_N = 1 + \frac{\pi^2 N^2}{k^2} + \frac{\pi^2 N^2}{6k^2} \]  

(4.35)

so that their symmetric linear combination is equal to

\[ W^+ = \frac{1}{2} (W_M + W_N) = 1 + \pi^2 \lambda^2 (1 + \frac{5 b}{6 \lambda} - \frac{5 b^2}{12 \lambda^2}). \]  

(4.36)

If we extrapolate these results into the strong coupling limit when \( \lambda >> 1 \) and use the fact that \( b \in [0, 1) \) we find that all terms proportional to \( b \) are sub-leading. However this
result suggests that the dual description of the symmetric combinations of Wilson loops in fundamental representations is given by fundamental string since as we argued above it is not sensitive to the presence of $B_{NS}$ holonomy.

In the same way we can consider an anti-symmetric combinations of Wilson loops in fundamental representation and we find

$$W^{-} = \frac{1}{2}(W_M - W_N) = \pi^2 \lambda^2 \left[ -\frac{1}{6} + \frac{\pi^2}{2\lambda} + \frac{5\pi^2}{12} \frac{b^2}{\lambda^2} \right].$$

(4.37)

It is interesting that the anti-symmetric combination of the Wilson loops contain the even powers of $b$ only. Unfortunately it is not known the dual description of these Wilson loops.

Let us now consider Wilson loop in anti-symmetric representation. It was shown in [31] that in case of ABJM theory the D6-brane description of this object gives the result

$$W = -\frac{p(N-p)}{N} \pi \sqrt{2\lambda},$$

(4.38)

where the factor $\pi \sqrt{2\lambda}$ corresponds to the Wilson loop calculated in fundamental representation. Note that the value of Wilson line does not change when we replace $p$ with $N-p$ which is characteristic property of Wilson line in antisymmetric representation. Further, it was argued in [31] that the appropriate interpretation of (4.38) in the dual ABJM theory is given a linear combination of the Wilson loops in anti-symmetric representations where the first one corresponds the trace over $U(N)$ and the second one the trace over $U(N)$. Let us now return to the case of ABJ theory with the gauge group $U(N) \times U(M)$. We claim that the result derived in previous section correctly describes the Wilson loop in anti-symmetric representation of the rank $Q$ since as we have shown it is symmetric under exchange $Q \rightarrow N - Q$. We also mean that it is very interesting that this result is sensitive to the presence of non-trivial $B_{NS}$ holonomy even if the corrections are sub-leading in supergravity approximation. We can compare this fact with the calculations of the Wilson loops in symmetric representation when we argued that their supergravity description does not see the presence of $B_{NS}$ at all.

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