INTRINSIC SCALING PROPERTIES FOR NONLOCAL OPERATORS

MORITZ KASSMANN AND ANTE MIMICA

Abstract. We study growth lemmas and questions of regularity for generators of Markov processes. The generators are allowed to have an arbitrary order of differentiability less than 2. In general, this order is represented by a function and not by a number. The approach enables a careful study of regularity issues up to the phase boundary between integro-differential (positive order of differentiability) and integral operators (nonnegative order of differentiability). The proof is based on intrinsic scaling properties of the underlying operators and stochastic processes.

1. Introduction

One key argument in the regularity theory of differential equations of second order is the so-called growth lemma. Here is an example which is by now classical. Let $A$ be an elliptic operator of second order, e.g. $Au = \sum_{i,j} a_{ij}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} u$ for $u : \mathbb{R}^d \to \mathbb{R}$ where $(a_{ij}(\cdot))_{i,j}$ is uniformly positive definite and bounded. One could also consider nonlinear examples. The following growth lemma holds true in many cases:

**Lemma 1.1.** There is a constant $\theta \in (0,1)$ such that, if $R > 0$ and $u : \mathbb{R}^d \to \mathbb{R}$ with 

$$- Au \leq 0 \text{ in } B_{2R}, \quad u \leq 1 \text{ in } B_{2R}, \quad |(B_{2R} \setminus B_R) \cap \{u \leq 0\}| \geq \frac{1}{2} |B_{2R} \setminus B_R|,$$

then $u \leq 1 - \theta$ in $B_R$.

Such lemmas are systematically studied and applied in [Lan71]. Their importance is underlined in the article [KS79], in which the authors establish a priori bounds for elliptic equations of second order with bounded measurable coefficients. Nowadays they form a standard tool for the study of various questions of nonlinear partial differential equations of second order, cf. [CC95] and [DGV12]. Note that the property formulated in Lemma 1.1 is also referred to as expansion of positivity which describes the corresponding property for $1 - u$.

In the case of a linear differential operator $A$ the above lemma can be established with the help of the Markov process it generates. Let $X$ be the strong Markov process associated with the operator $A$, i.e. we assume that the martingale problem has a unique solution. Denote by $T_A, \tau_A$ the hitting resp. exit time for a measurable set $A \subset \mathbb{R}^d$ and by $\mathbb{P}_x$ the measure on the path space with $\mathbb{P}_x(X_0 = x) = 1$. The following property is a key to the above growth lemma.

**Proposition 1.2.** There is a constant $c \in (0,1)$ such that for every $R > 0$ and every measurable set $A \subset B_{2R} \setminus B_R$ with $|(B_{2R} \setminus B_R) \cap A| \geq \frac{1}{2} |B_{2R} \setminus B_R|$ and $x \in B_R$

$$\mathbb{P}_x(T_A < \tau_{B_{2R}}) \geq c. \quad (1.1)$$
The aim of this work is to establish a result like Proposition 1.2 and regularity estimates for a general class of operators and stochastic processes. The article [KS79] deals with a very specific case: operators of second order. Another very specific case, operators of fractional order \( \alpha \in (0, 2) \), is treated in [BL02]. Therein it is shown that Proposition 1.2 holds true for jump processes \( X \) generated by integral operators \( L: C^2_b(\mathbb{R}^d) \to C(\mathbb{R}^d) \) of the form

\[
Lu(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x + h) - u(x) - \langle \nabla u(x), h \rangle \mathbf{1}_{B_1}(h) \right) K(x, h) \, dh
\]

under the assumption \( K(x, h) = K(x, -h) \) and \( K(x, h) \asymp |h|^{-d-\alpha} \) for all \( x \) and \( h \) where \( \alpha \in (0, 2) \) is fixed. Note that this class includes the case \( Lu = -\Delta^{\alpha/2} u \) and versions with bounded measurable coefficients. As [KS79] does, the article [BL02] establishes a priori estimates in Hölder spaces. Results like Lemma 1.1 have been obtained for operators in the case \( K(x, h) \asymp |h|^{-d-\alpha} \) also for nonlinear problems, cf. [Sil06], [CS09] and [GS12].

The starting point of our research is the observation that Proposition 1.2 fails to hold for several interesting cases. One example is given by \( L \) as in (1.2) with \( K(x, h) = k(h) \asymp |h|^{-d} \) for \( |h| \leq 1 \) and some appropriate condition for \( |h| > 1 \). For example, the geometric stable process with its generator \(-\ln(1 + (-\Delta)^{\alpha/2})\), \( 0 < \alpha \leq 2 \), can be represented by (1.2) with a kernel \( K(x, h) = k(h) \) with such a behaviour for \( |h| \) close to zero. The operator resp. the corresponding stochastic process can be shown not to satisfy a uniformly hitting estimate like (1.1). This leads to the question whether a priori estimates can be obtained by this approach at all.

Given a linear operator with bounded measurable coefficients of the form (1.2), the main idea of this article is to determine an intrinsic scale which allows to establish a modification of (1.1).

We choose a measure different from the Lebesgue measure for the assumption \(|(B_{2R}\setminus B_{R}) \cap A| \geq \frac{1}{2}|B_{2R}\setminus B_{R}|\).

Let us formulate our assumptions and results. Assume \( 0 \leq \alpha < 2 \) and let \( K: \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty) \) be a measurable function satisfying the following conditions:

\[
\begin{align*}
(K_1) & \quad \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^2) K(x, h) \, dh \leq K_0, \\
(K_2) & \quad K(x, h) = K(x, -h) \quad (x \in \mathbb{R}^d, h \in \mathbb{R}^d), \\
(K_3) & \quad \kappa^{-1} \frac{\ell(|h|)}{|h|^2} \leq K(x, h) \leq \kappa \frac{\ell(|h|)}{|h|^2} \quad (0 < |h| \leq 1)
\end{align*}
\]

for some numbers \( K_0 > 0, \kappa > 1 \) and some function \( \ell: (0, 1) \to (0, \infty) \) which is locally bounded and varies regularly at zero with index \(-\alpha \in (-2, 0)\). Possible examples could be \( \ell(s) = 1 \), \( \ell(s) = s^{-3/2} \) and \( \ell(s) = s^{-\beta} \ln(\frac{1}{s})^2 \) for some \( \beta \in (0, 2) \), see Appendix A for a more detailed discussion.
Suppose that there exists a strong Markov process \( X = (X_t, \mathbb{P}_x) \) with trajectories that are right continuous with left limits associated with \( \mathcal{L} \) in the sense that for every \( x \in \mathbb{R}^d \)

(i) \( \mathbb{P}_x(X_0 = x) = 1 \);
(ii) for any \( f \in C_b^2(\mathbb{R}^d) \) the process \( \{ f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s) \, ds \mid t \geq 0 \} \) is a martingale under \( \mathbb{P}_x \).

Note that the existence of such a Markov process comes for free in the case when \( K(x,h) \) is independent of \( x \), see Section 2. In the general case it has been established by many authors in different general contexts, see the discussion in [AK09]. Denote by \( \tau_A = \inf \{ t > 0 \mid X_t \notin A \} \), \( T_A = \inf \{ t > 0 \mid X_t \in A \} \) the first exit time resp. hitting time of the process \( X \) for a measurable set \( A \subset \mathbb{R}^d \).

**Definition 1.3.** A bounded function \( u: \mathbb{R}^d \to \mathbb{R} \) is said to be harmonic in an open subset \( D \subset \mathbb{R}^d \) with respect to \( X \) (and \( \mathcal{L} \)) if for any bounded open set \( B \subset \overline{B} \subset D \) the stochastic process \( \{ u(X_{\tau_{B \setminus A}}) \mid t \geq 0 \} \) is a \( \mathbb{P}_x \)-martingale for every \( x \in \mathbb{R}^d \).

Before we can formulate our results we need to introduce an additional quantity. Note that \((K_1)\) and \((K_3)\) imply that \( \int_0^1 s \ell(s) \, ds \leq c \) holds for some constant \( c > 0 \). Let \( L: (0,1) \to (0,\infty) \) be defined by \( L(r) = \int_0^1 \ell(s) \, ds \). The function \( L \) is well defined because \( L(r) \leq r^{-2} \int_r^1 s^2 \ell(s) \, ds \leq cr^{-2} \). See Appendix A for several examples. We note that the function \( L \) is always decreasing. Our main result concerning regularity is the following result:

**Theorem 1.4.** There exist constants \( c > 0 \) and \( \gamma \in (0,1) \) so that for all \( r \in (0,\frac{1}{2}) \) and \( x_0 \in \mathbb{R}^d \)

\[
|u(x) - u(y)| \leq c \| u \|_{\infty} \frac{L(|x-y|)^{-\gamma}}{L(r)^{-\gamma}}, \quad x, y \in B_{r/4}(x_0) \tag{1.4}
\]

for all bounded functions \( u: \mathbb{R}^d \to \mathbb{R} \) that are harmonic in \( B_r(x_0) \) with respect to \( \mathcal{L} \).

Let us comment on this result. It is important to note that the result trivially holds if the function \( L: (0,1) \to (0,\infty) \) satisfies \( \lim_{r \to 0^+} L(r) < +\infty \). This is equivalent to the condition

\[
\int_{B_1} \frac{\ell(\|h\|)}{\|h\|^d} \, dh < +\infty, \tag{1.5}
\]

which, in the case \( K(x,h) = k(h) \), means that the Lévy measure is finite. Thus, for the proof, we can concentrate on cases where (1.5) does not hold. One feature of this article is that our result holds true up to and across the phase boundary determined by whether the kernel \( K(x,\cdot) \) is integrable (finite Lévy measure) or not.

Furthermore, note that the main result of [BL02] is implied by Theorem 1.4 since the choice \( \ell(s) = s^{-\alpha}, \alpha \in (0,2) \), leads to \( L(r) \propto r^{-\alpha} \). Given the whole spectrum of possible operators covered by our approach, this choice is a very specific one. It allows to use scaling methods in the usual way which are not at our disposal here. Table 1 in Appendix A contains several admissible examples one of which leads to \( L(0) < +\infty \) which means, as explained above, that (1.4) becomes pointless.

The main ingredient in the proof of Theorem 1.4 is a new version of Proposition 1.2 which we provide now. For \( r \in (0,1) \) we define a measure \( \mu_r \) by

\[
\mu_r(dx) = \frac{\ell(|x|)}{L(|x|)} \frac{1}{|x|^d} \mathbb{1}_{B_r \setminus B_r}(x) \, dx. \tag{1.6}
\]
Moreover, for $a > 1$, we define a function $\varphi_a : (0, 1) \to (0, 1)$ by $\varphi_a(r) = L^{-1}(\frac{1}{a}L(r))$. The following result is our modification of Proposition 1.2.

**Proposition 1.5.** There exists a constant $c > 0$ such that for all $a > 1$, $r \in (0, \frac{1}{2})$ and measurable sets $A \subset B_{\varphi_a(r)} \setminus B_r$ with $\mu_r(A) \geq \frac{1}{2} \mu_r(B_{\varphi_a(r)} \setminus B_r)$

$$\mathbb{P}_x(T_A < \tau_{B_{\varphi_a(r)}}) \geq \mathbb{P}_x(X_{\tau_{B_r}} \in A) \geq c \frac{\ln a}{a}$$

holds true for all $x \in B_{r/2}$.

The main novelties of Proposition 1.5 are that the measure $\mu_r$ depends on $r$ and that its density carries the factor $|x|^{-d}$. These two changes allow us to deal with the classical cases as well as with critical cases, e.g. given by $K(x, h) \approx |h|^{-d}1_{B_1}(h)$.

The article is organised as follows: In Section 2 we review the relation between translation invariant nonlocal operators and semigroups/Lévy processes. Presumably, Proposition 2.1 is interesting to many readers since it establishes a one-to-one relation between the behavior of a Lévy measure at zero and the multiplier of the corresponding generator for large values of $|\xi|$. In Section 3 we establish all tools needed to prove Proposition 1.5 which is a special case of Proposition 3.4. Section 4 contains the proof of Theorem 1.4. The last section is Appendix A in which we collect important properties of regularly resp. slowly varying functions. Moreover, the appendix contains a table with six examples which illustrate the range of applicability of our approach.

Throughout the paper we use the notation $f(r) \asymp g(r)$ to denote that the ration $f(r)/g(r)$ stays between two positive constants as $r$ converges to some value of interest.

2. Translation invariant operators

The aim of this section is to discuss properties of the operator $\mathcal{L}$ from (1.2) in the translation invariant case, i.e. when $K(x, h)$ does not depend on $x \in \mathbb{R}^d$. In this case there is a one-to-one correspondence between $\mathcal{L}$ and multipliers, semigroups and stochastic processes. One aim is to prove how the behavior of $\ell(|h|)$ for small values of $|h|$ translates into properties of the multiplier or characteristic exponent $\psi(|\xi|)$ for large values of $|\xi|$. This is achieved in Proposition 2.1. We add a subsection where we discuss which regularity results are known in critical cases of the (much simpler) translation invariant case. Note that our set-up, although allowing for a irregular dependence of $K(x, h)$ on $x \in \mathbb{R}^d$, leads to new results in these critical cases.

2.1. Generators of convolution semigroups and Lévy processes. In this section we consider space homogeneous kernels of the form $K(x, h) = k(h)$ satisfying $(K_1)$–$(K_3)$. As we will see, the underlying stochastic process belongs to the class of Lévy processes.

A stochastic process $X = (X_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it has stationary and independent increments, $\mathbb{P}(X_0 = 0) = 1$ and its paths are $\mathbb{P}$-a.s. right continuous with left limits. For $x \in \mathbb{R}^d$ we define a $\mathbb{P}_x$ to be the law of the process $X + x$. In particular, $\mathbb{P}_x(X_t \in B) = \mathbb{P}(X_t \in B - x)$ for $t \geq 0$ and measurable sets $B \subset \mathbb{R}^d$.

Due to stationarity and independence of increments, the characteristic function of $X_t$ is given by

$$\mathbb{E}[e^{i\langle \xi, X_t \rangle}] = e^{-t\psi(|\xi|)},$$
where $\psi$ is called characteristic exponent of $X$. It has the following Lévy-Khintchine representation

$$\psi(\xi) = \frac{1}{2} \langle A\xi, \xi \rangle + \langle b, \xi \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\langle \xi, h \rangle} + i\langle \xi, h \rangle\mathbb{1}_{B_1}(h)\right)\nu(dh),$$

where $A$ is a symmetric non-negative definite matrix, $b \in \mathbb{R}^d$ and $\nu$ is a measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |y|^2)\nu(dy) < \infty$ called the Lévy measure of $X$.

The converse also holds; that is, given $\psi$ as in the Lévy-Khintchine representation (2.1), there exists a Lévy process $X = \{X_t\}_{t \geq 0}$ with the characteristic exponent $\psi$. Details about Lévy processes can be found in [Ber96, Sat99].

To make a connection with our set-up, let $\nu$ be a measure defined by $\nu(dh) = k(h)\, dh$. It follows from $(K_1)$–$(K_3)$ that $\nu$ is a symmetric Lévy measure. Let $X = \{X_t\}_{t \geq 0}$ be a Lévy process corresponding to the characteristic exponent $\psi$ as in (2.1) with $A = 0$, $b = 0$ and the Lévy measure $\nu(dh) = k(h)\, dh$.

Now, $P_t f(x) := \mathbb{E}_x[f(X_t)]$ defines a strongly continuous contraction semigroup of operators $(P_t)_{t \geq 0}$ on the space $L^\infty(\mathbb{R}^d)$ equipped with the essential-supremum norm. Moreover, it is a convolution semigroup, since

$$P_t f(x) = \mathbb{E}_0[f(x + X_t)] = \int_{\mathbb{R}^d} f(x + y) \mu_t(dy),$$

where $(\mu_t)_{t \geq 0}$ is a convolution semigroup of (probability) measures defined by $\mu_t(B) := \mathbb{P}(X_t \in B)$.

The infinitesimal generator $\mathcal{L}$ of the semigroup $(P_t)_{t \geq 0}$ is given by

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d \setminus \{0\}} (u(x + h) - u(x) - \langle \nabla u(x), h \rangle\mathbb{1}_{B_1}(h))k(h)\, dh$$

(cf. proof of [Sat99, Theorem 31.5]).

Since $\left\{ u(X_t) - u(X_0) - \int_0^t \mathcal{L}u(X_s)\, ds : t \geq 0 \right\}$ is a martingale (with respect to the natural filtration) for every $u \in C_0^\infty(\mathbb{R}^d)$ (cf. proof of [RY05, Proposition VII.1.6]), it follows that $X$ is the process which corresponds to the kernel $K(x, h) = k(h)$ in our set-up.

It is worth of mentioning that there is a connection between the characteristic exponent and the symbol of the operator $\mathcal{L}$. To be more precise, if $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x)\, dx$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, then

$$\hat{\mathcal{L}}f(\xi) = -\psi(-\xi)\hat{f}(\xi)$$

for any $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space (cf. [Ber96, Proposition I.2.9]). Hence $-\psi(-\xi)$ is the symbol (multiplier) of the operator $\mathcal{L}$.

We finish this section with the result that reveals connection between the characteristic exponent $\psi$ and the function $L$.

**Proposition 2.1.** Let $\mathcal{L} : \mathcal{S} \to \mathcal{S}$ be given by (2.2). Assume $K(x, h) := k(h)$ satisfies $(K_1)$–$(K_3)$. There is a constant $c > 0$ such that

$$c^{-1}L(|\xi|^{-1}) \leq \psi(\xi) \leq cL(|\xi|^{-1}) \quad \text{for } \xi \in \mathbb{R}^d, \ |\xi| \geq 5.$$

**Proof.** Note first that, by $(K_3)$,

$$\kappa^{-1}j(|h|) \leq k(h) \leq \kappa j(|h|), \quad |h| \leq 1,$$
where \( j(s) := s^{-d} \ell(s), \ s \in (0, 1). \)

Since \( 1 - \cos x \leq \frac{1}{4} x^2 \), it follows from \((K_1)\) and \((K_3)\) that
\[
\psi(\xi) \leq \frac{1}{2} |\xi|^2 \int_{|h| \leq |\xi|^{-1}} |h|^2 j(|h|) \, dh + 2 \int_{|\xi|^{-1} < |h| \leq 1} j(|h|) \, dh + 2 \int_{|h| > 1} j(|h|) \, dh
\]
\[
\leq c_1 \left[ |\xi|^2 \int_0^{[\xi]^{-1}} s \ell(s) \, ds + L(|\xi|^{-1}) + 1 \right]
\]
\[
\leq c_2 (\ell(|\xi|^{-1}) + L(|\xi|^{-1})) \leq c_3 L(|\xi|^{-1}),
\]
where in the first integral of the penultimate inequality Karamata’s theorem has been used, while in the last inequality we have used that \( \ell(s) \leq c_3 L(s) \) for \( s \in (0, 1), \) cf. property \((1)\) in Appendix A.

To prove the lower bound first we choose an orthogonal transformation of the form \( Oe_1 = |\xi|^{-1} \xi, \) where \( e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^d. \) Then a change of variable yields
\[
\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot h)) j(|h|) \, dh = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos (|\xi| h_1)) j(|h|) \, dh
\]
\[
\geq \int_{[-1,1]^d} (1 - \cos (|\xi| h_1)) j(|h|) \, dh
\]

By the Fubini theorem,
\[
\psi(\xi) \geq 2 \int_0^1 (1 - \cos (|\xi| r)) F(r) \, dr,
\]
where \( F(r) := \int_{[-1,1]^{d-1}} j(\sqrt{|z|^2 + r^2}) \, dz, \ r \in (0, \frac{1}{2}). \) It follows from Potter’s theorem (cf. property \((4)\) in Appendix A) that there is a constant \( c_4 > 0 \) so that \( j(r) \geq c_4 j(s) \) for all \( 0 < r \leq s < 1. \) This implies
\[
F(r) \geq c_4 F(s), \quad 0 < r \leq s < 1.
\]

Hence,
\[
\psi(\xi) \geq 2 \sum_{k=0}^{[\pi^{-1}(\frac{2}{4} - \frac{1}{2})]} \int_{|\xi|^{-1}(\frac{2\pi}{4} + 2k\pi)} |\xi|^{-1}(\frac{3\pi}{4} + 2k\pi)) (1 - \cos (|\xi| r)) F(r) \, dr \geq c_4 \pi \sum_{k=0}^{[\pi^{-1}(\frac{2}{4} - \frac{1}{2})]} F(|\xi|^{-1}(\frac{3\pi}{4} + 2k\pi))
\]
\[
\geq c_4 \sum_{k=0}^{[\pi^{-1}(\frac{2}{4} - \frac{1}{2})]} \int_{|\xi|^{-1}(\frac{3\pi}{4} + 2k\pi) + (2k+1)\pi} |\xi|^{-1}(\frac{5\pi}{4} + 2k\pi)) F(r) \, dr \geq c_4 \int_{\frac{3\pi}{4} |\xi|^{-1}}^{1} F(r) \, dr
\]
\[
\geq c_5 \int_{\frac{3\pi}{4} |\xi|^{-1} \leq |h| \leq 1} j(|h|) \, dh = c_6 L(\frac{3\pi}{4} |\xi|^{-1}) \geq c_7 L(|\xi|^{-1}),
\]
where, in the last inequality, we have used property \((4)\) from Appendix A. Note that [Grz13] uses a similar trick to bound \( \psi \) from below.

### 2.2. Known results in the translation invariant case.

Let us explain which results, related to Theorem 1.4, have been obtained in the case where \( K(x, h) \) is independent of \( x \in \mathbb{R}^d. \)

Hölder estimates of harmonic functions are obtained for the Lévy process with the characteristic exponent \( \psi(\xi) = \frac{|\xi|^2}{m_1(1 + |\xi|^2)} - 1 \) in [Mim13a] by establishing a Krylov-Safonov type estimate replacing the Lebesgue measure with the capacity of the sets involved. Recently, regularity
3. Probabilistic estimates

Proposition 3.1. There exists a constant $C_1 > 0$ such that for $x_0 \in \mathbb{R}^d$, $r \in (0, 1)$ and $t > 0$
\[ P_{x_0}(\tau_{B_r(x_0)} \leq t) \leq C_1 t L(r). \]

Proof. Let $x_0 \in \mathbb{R}^d$, $0 < r < 1$ and let $f \in C^2(\mathbb{R}^d)$ be a positive function such that
\[ f(x) = \begin{cases} \frac{|x - x_0|^2}{r^2}, & |x - x_0| \leq \frac{r}{2} \\ \frac{|x - x_0|}{r}, & |x - x_0| \geq r \end{cases} \]
and for some $c_1 > 0$
\[ |f(x)| \leq c_1 r^2, \quad \left| \frac{\partial f}{\partial x_i}(x) \right| \leq c_1 r \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| \leq c_1. \]
By the optional stopping theorem we get
\[ \mathbb{E}_x f(X_{t \wedge \tau_{B_r(x_0)}}) - f(x_0) = \mathbb{E}_x \int_0^{t \wedge \tau_{B_r(x_0)}} \mathcal{L} f(X_s) \, ds, \quad t > 0. \tag{3.1} \]
Let $x \in B_r(x_0)$. We estimate $\mathcal{L} f(x)$ by splitting the integral in (1.2) into three parts.
\[ \int_{B_r} (f(x + h) - f(x) - \nabla f(x) \cdot h 1_{|h| \leq 1}) K(x, h) \, dh \leq \frac{c_2}{r^2} \int_{B_r} |h|^2 K(x, h) \, dh \leq c_2 \kappa \int_{B_r} |h|^{2-d} \ell(|h|) \, dh \leq c_3 r^2 \ell(r), \]
where in the last line we have used Karamata’s theorem, cf. property (2) in Appendix A. On the other hand, on $B_r^c$ we have
\[ \int_{B_r^c} (f(x + h) - f(x)) K(x, h) \, dh \leq 2\|f\|_{\infty} \int_{B_r^c} K(x, h) \, dh \leq 2\|f\|_{\infty} \left( \kappa \int_{B_1 \setminus B_r} |h|^{-d} \ell(|h|) \, dh + \int_{B_r^c} K(x, h) \, dh \right) \leq c_4 r^2 L(r) \, dr, \]
where we applied property (5) from Appendix A. Last, we estimate
\[ \int_{B_1 \setminus B_r} h \cdot \nabla f(x) K(x, h) \, dh \leq c_1 r \int_{B_1 \setminus B_r} |h| K(x, h) \, dh \leq c_1 \kappa r \int_{B_1 \setminus B_r} |h|^{-d+1} \ell(|h|) \, dh \leq c_5 r^2 \ell(r), \]
by Karamata’s theorem again. Therefore, by property (1) from Appendix A we conclude that there is a constant $c_6 > 0$ such that for all $x \in B_r(x_0)$ and $r \in (0, 1)$ we have
\[ \mathcal{L} f(x) \leq c_6 r^2 L(r). \tag{3.2} \]
Let us look again at (3.1). On \( \{ \tau_{B_r(x_0)} \leq t \} \) we have \( X_{t \wedge \tau_{B_r(x_0)}} \in B_r(x_0)^c \) and so \( f(X_{t \wedge \tau_{B_r(x_0)}}) \geq r^2 \). Thus, by (3.2) and (3.1) we get
\[
\mathbb{P}_{x_0}(\tau_{B_r(x_0)} \leq t) \leq c_6 L(r)t.
\]

\[\square\]

**Proposition 3.2.** There are constants \( C_2 > 0 \) and \( C_3 > 0 \) such that for \( x_0 \in \mathbb{R}^d \)
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \tau_{B_r(x_0)} \leq \frac{C_2}{L(r)}, \quad r \in (0, 1/2)
\]
and
\[
\inf_{x \in B_{r/2}(x_0)} \mathbb{E}_x \tau_{B_r(x_0)} \geq \frac{C_3}{L(r)}, \quad r \in (0, 1)
\]

**Proof.** The proof is similar to the proof of the exit time estimates in [BL02].

(a) First we prove the upper estimate for the exit time. Let \( x \in \mathbb{R}^d, r \in (0, 1/2) \) and let
\[
S = \inf\{ t > 0 | |X_t - X_{t-}| > 2r \}
\]
be the first time of a jump larger than \( 2r \). With the help of the Lévy system formula (cf. [BL02, Proposition 2.3]) and \((K_3)\) we can deduce
\[
\mathbb{P}_x(S \leq L(r)^{-1}) = \mathbb{E}_x \sum_{t \leq L(r)^{-1} \wedge S} 1_{\{|X_t - X_{t-}| > 2r\}} = \mathbb{E}_x \int_0^{L(r)^{-1} \wedge S} \int_{B_{2r}} K(X_s, h) dh ds
\]
\[
\geq c_1 \mathbb{E}_x[L(r)^{-1} \wedge S] \int_{2r}^1 \frac{\ell(s)}{s} ds.
\]

Since \( L \) is regularly varying at zero,
\[
\mathbb{E}_x[L(r)^{-1} \wedge S] \geq L(r)^{-1} \mathbb{P}_x(S > L(r)^{-1}) \geq c_2 L(2r)^{-1}(1 - \mathbb{P}_x(S \leq L(r)^{-1}))
\]
and so it follows from (3.3) that
\[
\mathbb{P}_x(S \leq L(r)^{-1}) \geq c_3
\]
with \( c_3 = \frac{c_1 c_2}{c_1 c_2 + 1} \in (0, 1) \). The strong Markov property and (3.3) lead to
\[
\mathbb{P}_x(S > m L(r)^{-1}) \leq (1 - c_3)^m, \quad m \in \mathbb{N}.
\]

Since \( \tau_{B_r(x_0)} \leq S \),
\[
\mathbb{E}_x \tau_{B_r(x_0)} \leq \mathbb{E}_x S \leq L(r)^{-1} \sum_{m=0}^{\infty} (m + 1) \mathbb{P}_x(S > L(r)^{-1} m)
\]
\[
\leq L(r)^{-1} \sum_{m=0}^{\infty} (m + 1)(1 - c_3)^m.
\]

(b) Now we prove the lower estimate of the exit time. Let \( r \in (0, 1) \) and \( y \in B_{r/2}(x_0) \). By Proposition 3.1,
\[
\mathbb{P}_y(\tau_{B_r(x_0)} \leq t) \leq \mathbb{P}_y(\tau_{B_{r/2}(y)} \leq t) \leq C_1 t L(r/2), \quad t > 0,
\]
since \( B_{r/2}(y) \subset B_r(x_0) \). Choose \( t = \frac{1}{2c_1L(r/2)} \). Then
\[
\mathbb{E}_y \tau_{B_r(x_0)} \geq \mathbb{E}_y[\tau_{B_r(x_0)}; \tau_{B_r(x_0)} > t] \geq t \mathbb{P}_y(\tau_{B_r(x_0)} > t) \geq t(1 - C_1L(r/2)t) = \frac{1}{4C_1L(r/2)}.
\]
By (3) from Appendix A we know that \( L \) is regularly varying at zero. Hence there is a constant \( c_1 > 0 \) such that \( L(r/2) \leq c_1L(r) \) for all \( r \in (0, 1/2) \). Therefore \( \mathbb{E}_y \tau_{B_r(x_0)} \geq \frac{1}{4C_1c_1L(r)} \).

Proposition 3.3. There is a constant \( C_4 > 0 \) such that for all \( x_0 \in \mathbb{R}^d \) and \( r, s \in (0, 1) \) satisfying \( 2r < s \)
\[
\sup_{x \in B_r(x_0)} \mathbb{P}_x(X_{\tau_{B_r(x_0)} \wedge t} \not\in B_s(x_0)) \leq C_4 \frac{L(s)}{L(r)}.
\]

Proof. Let \( x_0 \in \mathbb{R}^d \), \( r, s \in (0, 1) \) and \( x \in B_r(x_0) \). Set \( B_r := B_r(x_0) \). By the Lévy system formula, for \( t > 0 \)
\[
\mathbb{P}_x(X_{\tau_{B_r} \wedge t} \not\in B_s) = \mathbb{E}_x \sum_{s \leq \tau_{B_r} \wedge t} \mathbb{I}_{(X_s \in B_r, X_t \in B_s)} = \mathbb{E}_x \int_0^{\tau_{B_r} \wedge t} K(X_s, z - X_s) dz ds.
\]
Let \( y \in B_r \). Since \( s \geq 2r \), it follows that \( B_{s/2}(y) \subset B_s \) and hence
\[
\int_{B_s} K(y, z - y) dz \leq \int_{B_{s/2}(y)} K(y, z - y) dz \leq c_1 \int_{s/2} |u| du + c_2 \leq c_3L(s).
\]
where in the last inequality we have used that \( L \) varies regularly at zero and that \( \lim_{r \to 0^+} L(r) > 0 \), cf. (5) in Appendix A.

The above considerations together with Proposition 3.2 imply
\[
\mathbb{P}_x(X_{\tau_{B_r} \wedge t} \not\in B_s) \leq c_3L(s)\mathbb{E}_x \tau_{B_r} \leq c_4 \frac{L(s)}{L(r)}.
\]
Letting \( t \to \infty \) we obtain the desired estimate.

For \( x_0 \in \mathbb{R}^d \) and \( r \in (0, 1) \) we define the following measure
\[
\mu_{x_0,r}(dx) = \frac{\ell(|x - x_0|)}{L(|x - x_0|)} |x - x_0|^{-d} \mathbb{I}_{\{r \leq |x - x_0| < 1\}} dx.
\] (3.5)
Define \( \varphi_a(r) = L^{-1}(\frac{1}{a}L(r)) \) for \( r \in (0, 1) \) and \( a > 1 \). The following property is important for the construction below:
\[
r = L^{-1}(L(r)) \leq L^{-1}(\frac{1}{a}L(r)) = \varphi_a(r).
\] (3.6)
Now we can prove a Krylov-Safonov type hitting estimate which includes Proposition 1.5 as a special case.

Proposition 3.4. There exists a constant \( C_5 > 0 \) such that for all \( x_0 \in \mathbb{R}^d \), \( a > 1 \), \( r \in (0, \frac{1}{2}) \) and \( A \subset B_{\varphi_a(r)}(x_0) \setminus B_r(x_0) \) satisfying \( \mu_{x_0,r}(A) \geq \frac{1}{2} \mu_{x_0,r}(B_{\varphi_a(r)}(x_0) \setminus B_r(x_0)) \)
\[
\mathbb{P}_y(T_A < \tau_{B_{\varphi_a(r)}(x_0)}) \geq \mathbb{P}_y(X_{\tau_{B_r(x_0)}} \in A) \geq C_5 \frac{\ln a}{a}, \quad y \in B_{r/2}(x_0).
\]
Proof. Consider $x_0 \in \mathbb{R}^d$, $a > 1$, $r \in (0, \frac{1}{2})$ and a set $A \subset B_{\varphi_0}(x_0) \setminus B_r(x_0)$ satisfying $\mu_{x_0,r}(A) \geq \ell \mu_{x_0,r}(B_{\varphi_0}(x_0))$. Set $\mu := \mu_{x_0,r}$, $\varphi := \varphi_0$, $B_0 := B_{x_0}(x_0)$ and let $y \in B_{r/2}$. The first inequality follows from $\{X_{\tau_B} \in A\} \subset \{T_A < \tau_{B_0}\} \subset A \subset \mathbb{R}^d \setminus B_r$.

By the Lévy system formula, for $t > 0$,

$$
\mathbb{P}_y(X_{\tau_{B_r} \wedge t} \in A) = \mathbb{E}_y \sum_{s \leq \tau_{B_r} \wedge t} \mathbbm{1}_{\{X_s \in B_r, X_s \in A\}} = \mathbb{E}_y \int_0^{\tau_{B_r} \wedge t} K(X_s, z - X_s) \, dz \, ds.
$$

(3.7)

Since $|z - x| \leq |x_0 - x| + |x_0 - x| \leq |z - x_0| + r \leq 2|z - x_0|$ for $x \in B_r$ and $z \in B_r^c$,

$$
\mathbb{E}_y \int_0^{\tau_{B_r} \wedge t} K(X_s, z - X_s) \, dz \, ds \geq c_1 \mathbb{E}_y[\tau_{B_r} \wedge t] \int_0^{\tau_{B_r} \wedge t} \frac{\ell(|z - x_0|)}{|z - x|^2} \, dz,
$$

(3.8)

where we have used property (4) given in Appendix A.

Since $L$ is decreasing,

$$
\int_0^{\tau_{B_r} \wedge t} \frac{\ell(|z - x_0|)}{|z - x_0|^2} \, dz \geq \int_0^{\tau_{B_r} \wedge t} L(|z - x_0|) \mu(dz) \geq L(\varphi(r)) \mu(A) \geq \frac{L(r)}{2a} \mu(B_\varphi(r) \setminus B_r).
$$

(3.9)

Noting that

$$
\mu(B_\varphi(r) \setminus B_r) = c_2 \int_0^{\varphi(r)} \frac{1}{L(s)} \ell(s) \, ds = -c_2 \ln L(s)|_{\varphi(r)} = c_2 \ln a,
$$

we conclude from (3.7)–(3.9) that

$$
\mathbb{P}_y(T_A < \tau_{B_\varphi(r)}(x_0)) \geq c_3 L(r) \frac{\ln a}{a} \mathbb{E}_y[\tau_{B_r} \wedge t].
$$

Letting $t \to \infty$ and using the lower bound in Proposition 3.2 we get

$$
\mathbb{P}_y(T_A < \tau_{B_\varphi(r)}(x_0)) \geq c_3 L(r) \frac{\ln a}{a} \mathbb{E}_y[\tau_{B_r}] \geq c_3 L(r) \frac{\ln a}{a} C_3 \frac{L(r)}{r} \geq c_3 C_3 \frac{\ln a}{a}.
$$

4. Regularity of harmonic functions

Proof of Theorem 1.4. Let $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$, $x \in B_{r/2}(x_0)$. Using (4) from Appendix A with $\delta = 1$, we see that there is a constant $c_0 \geq 1$ so that

$$
\frac{L(s)}{L(s')} \leq c_0 \left( \frac{s}{s'} \right)^{-\alpha - 1}, \quad 0 < s < s' < 1.
$$

(4.1)

Define for $n \in \mathbb{N}$

$$
r_n := L^{-1}(L(\frac{n}{2})\alpha^{n-1}) \quad \text{and} \quad s_n := 3||u||_\infty b^{-(n-1)}
$$

for some constants $b \in (1, \frac{3}{2})$ and $a > c_0 2^{a+1}$ that will be chosen in the proof independently of $n$, $r$ and $u$. As we explained in the introduction, Theorem 1.4 trivially holds true of $\lim_{r \to 0^+} L(r)$ is finite. Thus, we can assume $\lim_{r \to 0^+} L(r)$ to be infinite. This implies that $r_n \to 0$ for $n \to \infty$ as it should be.
We will use the following abbreviations:

\[ B_n := B_{r_n}(x), \quad \tau_n := \tau_{B_n}, \quad m_n := \inf_{B_n} u, \quad M_n := \sup_{B_n} u. \]

We are going to prove

\[ M_k - m_k \leq s_k \quad (4.2) \]

for all \( k \geq 1 \).

Assume for a moment that \( (4.2) \) is proved. Then, for any \( r \in (0, \frac{1}{2}) \) and \( y \in B_{r/4}(x_0) \subset B_{r/2}(x) \) we can find \( n \in \mathbb{N} \) so that

\[ r_{n+1} \leq |y - x| < r_n. \]

Furthermore, since \( L \) is decreasing, we obtain with \( \gamma = \frac{\ln b}{\ln a} \in (0, 1) \)

\[ |u(y) - u(x)| \leq s_n = 3b\|u\|_\infty a^{-n/\ln a} = 3b\|u\|_\infty \left[ \frac{L(r_{n+1})}{L(\frac{1}{2})} \right]^{\frac{\ln b}{\ln a}} \leq 3b\|u\|_\infty \left[ \frac{L(|x - y|)}{L(\frac{1}{2})} \right]^{-\gamma}, \]

which proves our assertion. Thus it remains to prove \( (4.2) \).

We are going to prove \( (4.2) \) by an inductive argument. Obviously, \( M_1 - m_1 \leq 2\|u\|_\infty \leq s_1 \).

Since \( 1 < b < \frac{3}{2} \), it follows that

\[ M_2 - m_2 \leq 2\|u\|_\infty \leq 3\|u\|_\infty b^{-1} = s_2. \]

Assume now that \( (4.2) \) is true for all \( k \in \{1, 2, \ldots, n\} \) for some \( n \geq 2 \).

Let \( \varepsilon > 0 \) and take \( z_1, z_2 \in B_{n+1} \) so that

\[ u(z_1) \leq m_{n+1} + \frac{\varepsilon}{2}, \quad u(z_2) \geq M_{n+1} - \frac{\varepsilon}{2}. \]

It is enough to show that

\[ u(z_2) - u(z_1) \leq s_{n+1}, \quad (4.3) \]

since then

\[ M_{n+1} - m_{n+1} - \varepsilon \leq s_{n+1}, \]

which implies \( (4.2) \) for \( k = n + 1 \), since \( \varepsilon > 0 \) was arbitrary.

By the optional stopping theorem,

\[ u(z_2) - u(z_1) = \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1)] \\
= \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1) ; X_{\tau_n} \in B_{n-1}] \\
+ \sum_{i=1}^{n-2} \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1) ; X_{\tau_n} \in B_{n-i-1} \setminus B_{n-i}] \\
+ \mathbb{E}_{z_2}[u(X_{\tau_n}) - u(z_1) ; X_{\tau_n} \in B_n] = I_1 + I_2 + I_3. \]

Let \( A = \{ z \in B_{n-1} \setminus B_n | u(z) \leq \frac{m_n + M_n}{2} \} \). It is sufficient to consider the case \( \mu_{x, r_n}(A) \geq 1 \frac{\mu_{x, r_n}(B_{n-1} \setminus B_n)}{2} \), where \( \mu_{x, r} \) is the measure defined by \( (3.5) \). In the remaining case we would use \( \mu_{x, r_n}((B_{n-1} \setminus B_n) \setminus A) \geq \frac{1}{2} \mu_{x, r_n}(B_{n-1} \setminus B_n) \) and could continue the proof with \( \|u\|_\infty - u \) and

\[ (B_{n-1} \setminus B_n) \setminus A = \left\{ z \in B_{n-1} \setminus B_n | \|u\|_\infty - u(z) \leq \frac{\|u\|_\infty - m_n + \|u\|_\infty - M_n}{2} \right\} \]

instead of \( u \) and \( A \).
The estimate (4.1) implies $a = \frac{L(r_{n+1})}{L(r_n)} \leq c_0(r_{n+1})^{-1} - r_n^{-1}$, from where we deduce $r_{n+1} \leq r_n(c_0a^{-1})^{-1}$ because of $a > c_02^{a+1}$. Next, we make use of the following property:

$$r_{n-1} = L^{-1}(L(\frac{r}{a})a^{n-2}) = L^{-1}(\frac{1}{a}L(r_n)) = L^{-1}(\frac{1}{a}L(r_n)) = \varphi_a(r_n).$$  \hspace{1cm} (4.4)

Then by Proposition 3.4 (with $r = r_n$ and $x_0 = x$) we get

$$p_n := \mathbb{P}_{z_2}(X_{r_n} \in A) \geq C_5 \frac{\ln a}{a}.$$

Hence,

$$I_1 = \mathbb{E}_{z_2}[u(X_{r_n}) - u(z_1); X_{r_n} \in B_{n-1}] = \mathbb{E}_{z_2}[u(X_{r_n}) - u(z_1); X_{r_n} \in A] + \mathbb{E}_{z_2}[u(X_{r_n}) - u(z_1); X_{r_n} \in B_{n-1} \setminus A]$$

$$\leq \left(\frac{m_n + M_n}{2} - m_n\right) p_n + s_{n-1}(1 - p_n) \leq \frac{1}{2}s_np_n + s_{n-1}(1 - p_n) \leq s_{n-1}(1 - \frac{C_5}{2a} \ln a).$$

By Proposition 3.3,

$$I_2 \leq \sum_{i=1}^{n-2} s_{n-i-1}\mathbb{P}_{z_2}(X_{r_n} \notin B_{n-i}) \leq C_4 \sum_{i=1}^{n-2} s_{n-i-1} \frac{L(r_{n-i})}{L(r_n)}$$

$$\leq 3C_4\|u\|_\infty \sum_{i=1}^{n-2} b^{-(n-i-2)}a^n a^{-i} \leq 3C_4\|u\|_\infty s_{n+3} / a-a-b$$

$$\leq C_4 \frac{b^3}{n^4} s_{n+1}.$$

Similarly, by Proposition 3.3,

$$I_3 \leq 2\|u\|_\infty \mathbb{P}_{z_2}(X_{r_n} \notin B_1) \leq 2C_4\|u\|_\infty \frac{L(r_1)}{L(r_n)} = \frac{2C_4}{3}b \left(\frac{1}{a}a^{n-1} s_{n+1} \leq C_4 \frac{b^2}{a} s_{n+1}.$$

Hence,

$$u(z_2) - u(z_1) \leq s_{n+1}b^2 \left[1 - \frac{C_5}{2a} \ln a + \frac{C_4}{a-b} + \frac{C_4}{a}\right].$$

Since $a-b \geq a$ for $b \in (1, \frac{3}{2})$ and $a > c_02^{a+1} \geq 2$, it follows that

$$q := 1 - \frac{C_5}{2a} \ln a + \frac{C_4}{a-b} + \frac{C_4}{a} \leq 1 - \frac{C_5}{2a} \ln a - 14C_4 = 1 - \frac{C_5}{2a} \ln a - 14C_4.$$

Next, we choose $a > c_02^{a+1}$ so large that $C_5 \ln a - 14C_4 > 0$. Thus $q < 1$. Finally, we choose $b \in (1, \frac{3}{2})$ sufficiently small so that $b^2q < 1$.

Hence, (4.3) holds, which finishes the proof of the inductive step and the theorem. \hspace{1cm} \Box

**APPENDIX A. SLOW AND REGULAR VARIATION**

In this section we collect some properties of slowly resp. regularly varying functions that are used in our main arguments. Moreover we list several examples which illustrate the range of application of our approach.

**Definition A.1.** A measurable and positive function $\ell: (0, 1) \to (0, \infty)$ is said to vary regularly at zero with index $\rho \in \mathbb{R}$ if for every $\lambda > 0$

$$\lim_{\lambda \to 0^+} \frac{\ell(\lambda r)}{\ell(r)} = \lambda^\rho.$$
If a function varies regularly at zero with index 0 it is said to vary slowly at zero. For simplicity, we call such functions regularly varying resp. slowly varying functions.

Note that slowly resp. regularly varying functions include functions which are neither increasing nor decreasing. By [BGT87, Theorem 1.4.1 (iii)] it follows that any function $\ell$ that varies regularly with index $\rho \in \mathbb{R}$ is of the form $\ell(r) = r^\rho \ell_0(r)$ for some function $\ell_0$ that varies slowly. Assume $\int_0^1 s \ell(s) \, ds \leq c$ for some $c > 0$. Let $L: (0,1) \to (0,\infty)$ be defined by

$$L(r) = \int_0^r \frac{\ell(s)}{s} \, ds.$$  

The function $L$ is well defined because $L(r) = r^{-2} \int_0^1 r^2 \frac{\ell(s)}{s} \, ds \leq r^{-2} \int_0^1 s \ell(s) \, ds \leq cr^{-2}$. Note that ($K_2$) and ($K_3$) imply that $\int_0^1 s \ell(s) \, ds \leq c$ does hold in our setting. We note that the function $L$ is always decreasing.

Let us list further properties which are making use of in our proofs. Note that they are established [BGT87] for functions which are slowly resp. regularly varying at the point $+\infty$. By a simple inversion we adopt the results to functions which are slowly resp. regularly varying at the point 0.

1. If $\ell$ is slowly varying, then [BGT87, Proposition 1.5.9a] $L$ is slowly varying with

$$\lim_{r \to 0^+} L(r) = +\infty \quad \text{and} \quad \lim_{r \to 0^+} \frac{\ell(r)}{L(r)} = 0.$$  

2. If $\ell$ is slowly varying and $\rho > -1$, then Karamata’s theorem [BGT87, Proposition 1.5.8] ensures

$$\lim_{r \to 0^+} \frac{\int_0^r s^\rho \ell(s) \, ds}{r^{\rho+1} \ell(r)} = (\rho + 1)^{-1}.$$  

3. If $\ell$ is regularly varying of order $-\alpha < 0$ (in our case $0 < \alpha < 2$), then [BGT87, Theorem 1.5.11]

$$\lim_{r \to 0^+} \frac{L(r)}{\ell(r)} = \alpha^{-1}.$$  

In particular, if $\ell$ is regularly varying of order $-\alpha < 0$, then so is $L$.

4. Assume $\ell$ is regularly varying of order $-\alpha \leq 0$ and stays bounded away from 0 and $+\infty$ on every compact subset of $(0,1)$. Then Potter’s theorem [BGT87, Theorem 1.5.6 (ii)] implies that for every $\delta > 0$ there is a constant $C = C(\delta) \geq 1$ such that for $r, s \in (0,1)$

$$\frac{\ell(r)}{\ell(s)} \leq C \max \left\{ \left( \frac{r}{s} \right)^{-\alpha-\delta}, \left( \frac{s}{r} \right)^{-\alpha+\delta} \right\}.$$  

5. Since $L$ is nonincreasing, we observe $\lim_{r \to 0^+} L(r) \in (0, +\infty]$.

Let us look at different choices for the function $\ell$, given in Table 1. Here $\beta \in (0,2)$, $a > 1$ are fixed. We list six examples of a function $s \mapsto \ell_i(s)$ together with $s \mapsto L_i(s)$ and $s \mapsto \varphi_a(s) = L_i^{-1}\left(\frac{1}{a} L_i(s)\right)$. Recall that the function $\varphi_a$ appears in Proposition 1.5 and determines the scaling that we are using, see also property (4.4) and the definition of $r_n$ in the proof of Theorem 1.4. Note that case No. 6 is significantly different from the other cases. Both, the integral $\int_{B_1} |h|^{-d} \ell_6(|h|) \, dh$ and the expression $\lim_{s \to 0^+} L_6(s)$ are finite. Moreover, the limit $\lim_{s \to 0^+} L_6^{-1}\left(\frac{1}{a} L_6(s)\right)$ is not equal to zero. These differences reflect the fact that the corresponding
Table 1. Different choices for the function $\ell$ when $\beta \in (0, 2)$, $a > 1$.

| No. (i) | $\ell_i(s)$ | $L_i(s)$ | $\varphi_a(s) = L_i^{-1}(\frac{1}{a} L_i(s))$ |
|---------|-------------|----------|-----------------------------------------------|
| 1       | $s^{-\beta} \ln(\frac{2}{s})^2$ | $\asymp s^{-\beta} \ln(\frac{2}{s})^2$ | $\asymp s$ |
| 2       | $s^{-\beta}$ | $\frac{1}{\beta}(s^{-\beta} - 1)$ | $\asymp s$ |
| 3       | $\ln(\frac{2}{s})$ | $\asymp \ln^2(\frac{2}{s})$ | $\asymp s^{1/\sqrt{2}}$ |
| 4       | 1 | $\ln(\frac{1}{a})$ | $s^{1/a}$ |
| 5       | $\ln(\frac{2}{s})^{-1}$ | $\asymp \ln(\ln(\frac{2}{s}))$ | $\asymp \exp(-\left(\frac{\ln(\frac{2}{s})}{\alpha}\right)^{1/\alpha})$ |
| 6       | $\ln(\frac{2}{s})^{-2}$ | $\ln(2)^{-1} - \ln(\frac{2}{s})^{-1}$ | $\asymp \exp\left(-\frac{\ln(\frac{2}{s})}{\alpha} + \frac{1}{\alpha \ln(2/s)}\right)^{-1}$ |

operator in (1.2) has an integrable kernel. Recall that Proposition 2.1 relates the behavior of the function $L$ close to the origin to the behaviour of the multiplier of the operator (in the case of constant coefficients) for large values of $|\xi|$. In the case No. 6 the multiplier stays bounded.

Acknowledgements: We thank T. Grzywny for a helpful comment on the limit case $\alpha = 2$.

References

[AK09] H. Abels and M. Kassmann, The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels, Osaka J. Math. 46 (2009), no. 3, 661–683. MR 2583323 (2011d:35505)

[Ber96] J. Bertoin, Lévy processes, Cambridge University Press, Cambridge, 1996.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation, Cambridge University Press, Cambridge, 1987.

[BL02] R. F. Bass and D. Levin, Harnack inequalities for jump processes, Potential Anal. 17 (2002), 375–388.

[CC95] Luis A. Caffarelli and Xavier Cabrè, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR 1351007 (96h:35046)

[CS09] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), no. 5, 597–638. MR 2494809 (2010d:35376)

[DGV12] E. DiBenedetto, U. Gianazza, and V. Vespri, Harnack’s inequality for degenerate and singular parabolic equations, Springer Monographs in Mathematics, Springer, New York, 2012. MR 2865434

[Grz13] T. Grzywny, On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes, Potential Anal. (2013), to appear.

[GS12] N. Guillen and R. W. Schwab, Aleksandrov-Bakelman-Pucci type estimates for integro-differential equations, Arch. Ration. Mech. Anal. 206 (2012), no. 1, 111–157. MR 2968592

[KST97] N. V. Krylov and M. V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure, Dokl. Akad. Nauk SSSR 245 (1979), no. 1, 18–20.

[Lan71] E. M. Landis, Uravneniya eotorogo poryadka ellipticheskogo i parabolicheskogo tipov, Izdat. “Nauka”, Moscow, 1971. MR 0320507 (47 #9044)

[Mim13a] A. Mimica, Harnack inequality and Hölder regularity estimates for a Lévy process with small jumps of high intensity, J. Theor. Probab. 26 (2013), 329–348.

[Mim13b] A. Mimica, On harmonic functions of symmetric Lévy processes, Ann. Inst. H. Poincaré Probab. Statist. (2013), to appear.

[RY05] D. Revuz and M. Yor, Continuous martingales and Brownian motion, Springer, Berlin, 2005.
[Sat99] K.-I. Sato, Lévy processes and infinitely divisible distributions, Cambridge University Press, Cambridge, 1999.

[Sil06] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace, Indiana Univ. Math. J. 55 (2006), no. 3, 1155–1174. MR 2244602 (2007b:45022)

[ŠSV06] H. Šikić, R. Song, and Z. Vondraček, Potential theory of geometric stable processes, Probab. Theory Related Fields 135 (2006), no. 4, 547–575. MR 2240700 (2008h:60319)

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: moritz.kassmann@uni-bielefeld.de

Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia

E-mail address: amimica@math.hr