**Einstein billiards and spatially homogeneous cosmological models**

Sophie de Buyl¹, Gaïa Pinardi and Christiane Schomblond

Physique Théorique et Mathématique,  
Université Libre de Bruxelles,  
C.P. 231, B-1050, Bruxelles, Belgium.  
sdebuyl@ulb.ac.be, gpinardi@ulb.ac.be, cschomb@ulb.ac.be

**Abstract**

In this paper, we analyse the Einstein and Einstein-Maxwell billiards for all spatially homogeneous cosmological models corresponding to 3 and 4 dimensional real unimodular Lie algebras and provide the list of those models which are chaotic in the Belinskii, Khalatnikov and Lifschitz (BKL) limit. Through the billiard picture, we confirm that, in $D = 5$ spacetime dimensions, chaos is present if off-diagonal metric elements are kept: the finite volume billiards can be identified with the fundamental Weyl chambers of hyperbolic Kac-Moody algebras. The most generic cases bring in the same algebras as in the inhomogeneous case, but other algebras appear through special initial conditions.

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1 Introduction

It has recently been shown [1, 2] that the classical dynamics of the spatial scale factors (and of the dilatons if any) of $D$-dimensional gravity coupled to $p$-forms can be described, in the vicinity of a spacelike singularity, as a billiard motion, i.e. as the relativistic motion of a ball inside a region of hyperbolic space bounded by hyperplanes on which it undergoes elastic bounces. These results generalize the work by Belinskii, Khalatnikov and Lifshitz (BKL) [3] that discovered the chaotic behaviour of the generic solution of the vacuum Einstein equation, in four dimensional spacetime, near such a singularity. These authors showed that, near a spacelike singularity, a decoupling of the spatial points occurs which hugely simplifies the dynamical equations of the spatial metric, in the sense that, asymptotically, they cease to be partial differential equations to simply become, at each spatial point, a set of second-order non linear ordinary differential equations with respect to time.

These equations coincide with the dynamical equations of some spatially homogeneous cosmological models which possess some of the qualitative properties of more generic solutions. For $D = 4$, the spatially homogeneous vacuum models that share the chaotic behaviour of the more general inhomogeneous solutions are labelled as Bianchi type IX and VIII; their homogeneity groups are respectively $SU(2)$ and $SL(2, \mathbb{R})$. In higher spacetime dimensions, i.e. for $5 \leq D \leq 10$, one also knows chaotic spatially homogeneous cosmological models but none of them is diagonal [4]. In fact, diagonal models are too restrictive to be able to reproduce the general oscillatory behaviour but, as shown e.g. in [5], chaos is restored when non-diagonal metric elements are taken into account.

A systematic way to study the asymptotic behaviour of solutions of Einstein’s equations in the neighborhood of a spacelike singularity and in any spacetime dimension is provided by the Hamiltonian approach developed in [6] in which gravity is coupled to a collection of $p$-forms. In this framework, a billiard description of the asymptotic evolution of the scale factors is naturally set up and chaos follows from the fact that the billiard’s volume is finite. In many interesting examples, some of them related to supergravity models [6, 8], the billiard can be identified with the fundamental Weyl chamber of a Kac-Moody algebra and the reflections against the billiard walls with the fundamental Weyl reflections which generate the Weyl group (or Coxeter group): accordingly, the finiteness of the volume, hence also chaos, rely upon the hyperbolic character of the underlying Kac-Moody algebra;
this last property is established through its Cartan matrix or equivalently in its Dynkin diagram.

The purpose of this paper is to analyse the billiard evolution of spatially homogeneous non-diagonal cosmological models in $D = 4$ or 5 spacetime dimensions, in the Hamiltonian formalism mentioned hereabove. Since one knows that the full field content of the theory is important in the characterization of the billiard, we compare the pure Einstein gravity construction to that of the coupled Einstein-Maxwell system. The gravitational models we are interested in are in a one-to-one correspondence with the real Lie algebras - a complete classification based on their structure constants exists for $d = 3$ and $d = 4$, $(D = d + 1)$\cite{9, 12} - and we restrict our analysis to the unimodular ones, because only for such models can the symmetries of the metric be prescribed at the level of the action. We proceed along the lines defined in \cite{6} but with a special concern about the rôle played by the constraints. Indeed, while in the general inhomogeneous case, the constraints essentially assign limitations on the spatial gradients of the fields without having an influence on the generic form of the BKL Hamiltonian, in the present situation, they precisely relate the coefficients that control the walls in the potential. Consequently, the question arises whether they can enforce the disappearing of some (symmetry or electric or magnetic) walls as already do the vanishing structure constants with gravitational walls. They could prevent the generic oscillatory behaviour of the scale factors. The answer evidently depends on the Lie algebra considered and on its dimension: for example, while going from the Bianchi IX model in $d = 3$ to the corresponding $SU(3, S)3$ model in $d = 4$, the structure constants remain the same but the momentum constraints get less restrictive. Hence generic behaviour is easier to reach when more variables enter the relations. We find that, except for the Bianchi IX and VIII cases in $D = 4$, symmetry walls (hence off-diagonal elements) are needed to close the billiard table: thereby confirming, in the billiard picture, previous results about chaos restoration. Moreover, we find that when the billiard has a finite volume in hyperbolic space, it can again be identified with the fundamental Weyl chamber of one of the hyperbolic Kac-Moody algebras. In the most generic situation, these algebras coincide with those already relevant in the general inhomogeneous case. However, in special cases, new rank 3 or 4 simply laced algebras are exhibited.

\footnote{We will use the notations of MacCallum, we refer to \cite{12} for translation to other notations}
The paper is organized as follows. We first adapt to the spatially homogeneous case that part of the general Hamiltonian formalism set up in [6] necessary to understand how, at the BKL limit, the billiard walls arise in the potential. We explicitly write down the form of the momentum constraints in the generalized Iwasawa variables and analyse their meaning for each of the 3 and 4 real unimodular Lie algebras as well as their impact on the billiard’s shape. For the finite volume billiards, we compute the scalar products of the gradients of the dominant walls using the metric defined by the kinetic energy and show that the matrix

$$A_{AB} = 2 \left( \frac{w_A | w_B}{w_A | w_A} \right)$$

where

$$\left( w_A | w_B \right) = G^{ab} w_{Aa} w_{Bb}$$

is the generalized Cartan matrix of an hyperbolic Kac-Moody algebra.

2 General setting

2.1 Spatially homogeneous models, Hamiltonian

In this paper, we are specially interested in $d = 3$ and $d = 4$ dimensional spatially homogeneous models equipped with a homogeneity group simply transitively acting; these models are known to be in a one-to-one correspondence with the 3 and 4 dimensional real Lie algebras and have been completely classified [12, 13]. We restrict our analysis to the unimodular algebras, i.e. those whose adjoint representation is traceless$^3$, that is $C_{i k} = 0$, since only for these homogeneous models do the equations of motion follow from a reduced Hamiltonian action in which the symmetry of the metric is enforced before taking variational derivatives.

We work in a pseudo-Gaussian gauge defined by vanishing shift $N^i = 0$ and assume the $D = d + 1$ dimensional spacetime metric of the form

$$\text{d}s^2 = -(N \text{d}x^0)^2 + g_{ij}(x^0) \text{d}x^i \text{d}x^j$$

where $x^0$ is the time coordinate, $t$ is the proper time, $\text{d}t = -N \text{d}x^0$, and $N$ is the lapse. For definiteness and in agreement with the choice made in [6], we will assume that the spatial singularity occurs in the past, for $t = 0$.  

$^3$The group Adjoint representation is unimodular.
The gravitational dynamical variables \( g_{ij} \) are the components of the \( d \) dimensional spatial metric in the time-independent co-frame \( \{ \omega^i = \omega^i_j dx^j \} \) invariant under the group transformations

\[
d\omega^i = -\frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k; \tag{2.2}
\]
the \( C^i_{jk} \)'s are the group structure constants. The metric \( g_{ij}(x^0) \) depends only on time and may contain off-diagonal elements. With use of \( g \equiv \det g_{ij} \), one defines the rescaled lapse as \( \tilde{N} = N/\sqrt{g} \).

In the spatially homogeneous Einstein-Maxwell system, there is besides the metric, an electromagnetic 1-form potential \( A \) and its 2-form field strength \( F = dA \). In the temporal gauge \( A_0 = 0 \), the potential reduces to

\[
A = A_j \omega^j. \tag{2.3}
\]

In the Hamiltonian framework, we assume the potential itself to be spatially homogeneous\(^4\) so that its space components in the \( \omega^j \) frame are functions of \( x^0 \) only: \( A_j = A_j(x^0) \). Accordingly, its field strength takes the special form

\[
F = dA = \partial_0 A_j dx^0 \wedge \omega^j - \frac{1}{2} A_i C^i_{jk} \omega^j \wedge \omega^k \tag{2.4}
\]
which shows the links between the components of the magnetic field and the structure constants. Hence, from the Jacobi identity, one infers that

\[
F_{i|j} C^i_{k\ell} = 0. \tag{2.5}
\]

The first order action for the homogeneous Einstein-Maxwell system can be obtained from the \( D \) dimensional Hilbert-Einstein action in ADM form after space integration has been carried out; this operation brings in a constant space volume factor that will be ignored hereafter. The action is given by

\[
S[g_{ij}, \pi^{ij}, A_j, \pi^j] = \int dx^0 (\pi^{ij} \dot{g}_{ij} + \pi^j \dot{A}_j - \tilde{N} H). \tag{2.6}
\]

\(^4\)This is more restrictive than requiring spatial homogeneity of the field strength; in the present analysis the difference only arises with regard to the magnetic walls which are always subdominant.
The Hamiltonian \( \tilde{N}H \) reads as

\[
H = K + M \quad (2.7)
\]

\[
K = \pi^{ij} \pi_{ij} - \frac{1}{d - 1} \pi^i_i \pi^j_j + \frac{1}{2} \pi^i_j \pi^j_i \quad (2.8)
\]

\[
M = -gR + \frac{1}{4} F_{ij} F^{ij} \quad (2.9)
\]

where \( R \) is the spatial curvature scalar defined, in the unimodular cases, by the following combination of structure constants and metric coefficients

\[
R = -\frac{1}{2} \left( C^{ijk} C_{jik} + \frac{1}{2} C^{ijk} C_{ijk} \right), \quad (2.10)
\]

where

\[
C_{ijk} = g_{\ell i} C^{\ell j} \quad \text{and} \quad C^{ijk} = g^{\ell i} g^{km} C_{i\ell m}. \quad (2.11)
\]

The equations of motion are obtained by varying the action (2.6) with respect to the spatial metric components \( g_{ij} \), the spatial 1-form components \( A_j \) and their respective conjugate momenta \( \pi^{ij} \) and \( \pi^j \). The dynamical variables still obey the following constraints:

\[
H \approx 0 \quad \text{(Hamiltonian constraint)} \quad (2.12)
\]

\[
H_i = -C^{j \ell} i k \pi^k_j + \pi^j F_{ij} \approx 0 \quad \text{(momentum constraints)}; \quad (2.13)
\]

notice that the Gauss law for the electric field is identically satisfied on account of the unimodularity condition.

### 2.2 Generalized Iwasawa variables

In order to develop the billiard analysis, it is necessary to change the variables, i.e. to replace the metric components \( g_{ij} \) by the new variables \( \beta^a \) and \( \mathcal{N}^a_i \), defined through the Iwasawa matrix decomposition [6]

\[
g = \mathcal{N}^T \mathcal{A}^2 \mathcal{N} \quad (2.14)
\]

where \( \mathcal{N} \) is an upper triangular matrix with 1’s on the diagonal and \( \mathcal{A} \) is a diagonal matrix with positive entries parametrized as

\[
\mathcal{A} = \exp(-\beta), \quad \beta = \text{diag}(\beta^1, \beta^2, ..., \beta^d). \quad (2.15)
\]
The explicit form of (2.14) reads

\[ g_{ij} = \sum_{a=1}^{d} e^{-2\beta^a} N^a_i N^a_j. \] (2.16)

The \( \beta \)'s are often referred to as the scale factors although they more precisely describe their logarithms. The \( N^a_i \)'s measure the strength of the off-diagonal metric components and define how to pass from the invariant \( \{\omega^i\} \) co-frame to the Iwasawa co-frame \( \{\theta^a\} \) in which the metric is purely diagonal

\[ \theta^a = N^a_j \omega^j. \] (2.17)

In this basis, one has for the components of the 1-form

\[ A_j = A_a N^a_j. \] (2.18)

The changes of variables (2.16) and (2.18) are continued to the momenta as canonical point transformations in the standard way via

\[ \pi^{ij} \dot{g}_{ij} + \pi^j \dot{A}_j = \pi_a \dot{\beta}^a + \sum_{a<j} P^j_a \dot{N}^a_j + \mathcal{E}^a \dot{A}_a. \] (2.19)

In this expression, \( P^j_a \) denotes the momentum conjugated to \( N^a_j \) and is defined for \( a < j \), \( \mathcal{E}^a \) denotes the momentum conjugated to \( A_a \). The Iwasawa components of the electric and magnetic fields are given by

\[ \mathcal{E}^a \equiv N^a_j \pi^j, \quad \mathcal{F}_{ab} \equiv F_{ij} N^i_a N^j_b \] (2.20)

where \( N^i_a \) denotes the element on line-\( j \), column-\( a \) of the inverse matrix \( N^{-1} \). This matrix enters the definition the vectorial frame \( \{ e_a \} \) dual to the co-frame \( \{ \theta^a \} \) by

\[ e_a = X_j N^j_a. \] (2.21)

While shifting to the Iwasawa basis and co-basis, the structure constants of the group, which also define the Lie brackets of the vectorial frame \( \{ X_i \} \) dual to the invariant co-frame \( \{ \omega^i \} \)

\[ [X_i, X_j] = -X_k C^k_{ij}, \] (2.22)

transform as the components of a \( (\frac{1}{2}) \)-tensor so that

\[ [e_b, e_c] = -e_a C^a_{bc}, \quad \text{with} \quad C^a_{bc} = N^a_i N^j_b N^k_c C^i_{jk}. \] (2.23)
2.3 Splitting of the Hamiltonian

Following [6], we next split the Hamiltonian $H$ into two parts: the first one, denoted by $H_0$, is the kinetic term for the local scale factors $\beta^a$; the second one, denoted by $V$, is the potential and contains all the other contributions. Thus

$$H = H_0 + V$$  (2.24)

and

$$H_0 = \frac{1}{4} C^{ab}_{\pi_a \pi_b}.$$  (2.25)

The total potential naturally splits into

$$V = V_S + V_G + V_{(p)}^{el} + V_{(p)}^{magn}$$  (2.26)

where

$$V_S = \frac{1}{2} \sum_{a<b} e^{-2(\beta^b - \beta^a)} (\mathcal{P}_a \mathcal{N}_b)^2.$$  (2.27)

is quadratic in the $\mathcal{P}'$s and as such related to the kinetic energy of the off-diagonal metric components; one refers to it as to the ”symmetry” potential. $V_S$ vanishes in the case of pure diagonal $g_{ij}$. Next comes the gravitational or curvature potential

$$V_G = -gR = \frac{1}{2} e^{-2\sum d \beta^d} \sum_{a,b,c} (e^{2\beta_c} C^{\alpha}_{bc} C^{\beta}_{ac} + \frac{1}{2} e^{-2\beta^a + 2\beta^b + 2\beta^c} (C^{\alpha}_{bc})^2)$$  (2.28)

involving the structure constants in the Iwasawa basis defined by (2.23). The last two terms in the potential correspond to the electric and magnetic energy:

$$V^{el} = \frac{1}{2} e^{2\sum e_a} (E^a)^2, \quad V^{magn} = \frac{1}{4} e^{-2\sum e_a} (F_{ab})^2.$$  (2.29)

where, with the notations of [6],

$$e_a = \beta^a, \quad m_{ab} = \sum_{c \not\in \{ab\}} \beta^c.$$  (2.30)
2.4 BKL Limit, billiard walls

As the above formulae explicitly show, i) the total potential exhibits the general form

\[ V = V(\beta, N, P, E, F) = \sum_A c_A(N, P, E, F) \exp(-2w_A(\beta)) \]  (2.31)

where \( w_A(\beta) = w_{Ab}\beta^b \) are linear forms in the scale factors and ii) apart from the first term in the right-hand side of the curvature potential (which will introduce subdominant walls), the prefactors \( c_A \) are all given by the square of a real polynomial, implying \( c_A \geq 0 \). As explained in [6], with the following gauge choice, \( \tilde{N} = \rho^2 = -\beta_a\beta^a = -\rho^2\gamma_a\gamma^a \), and in the BKL limit corresponding to \( \rho \to \infty \), the exponentials terms in the potential \( \tilde{N}V \) become sharp walls and may be replaced by

\[ \lim_{\rho \to \infty}[c_A \rho^2 e^{-2\rho w_A(\gamma)}] = \Theta_\infty(-2w_A(\gamma)) \]  (2.32)

where \( \Theta_\infty \) is defined through

\[ \Theta_\infty(x) = \begin{cases} 
0 & x < 0 \\
\infty & x > 0 
\end{cases} \]  (2.33)

and has the property of being invariant under multiplication by a positive factor. Accordingly, in this limit, the positive \( c_A \)'s can be absorbed by the \( \Theta_\infty \) and the null ones can simply be dropped out; once this has been done, the Hamiltonian no longer depends on the variables \( \chi \)

\[ \chi \in \{N, P, E, F\} \]  (2.34)

that enter these coefficients. This also means that the BKL Hamiltonian only contains the scale factors \( \beta^a \) and their conjugate momenta \( \pi_a \) hence, as can be immediately inferred from their equations of motion, all the \( \chi \)'s must asymptotically tend to constants. These constants are arbitrary until and unless the constraints enter the play.

Another way to see that, before the constraints are taken into account, the asymptotic values of the \( \chi \)'s are arbitrary constants is to consider the map

\[ \phi : \chi_0 \to \chi^\infty = \phi(\chi_0) \]  (2.35)
which sends the initial values $\chi_0$ on their asymptotic BKL values, for a given set of initial data $\{\beta_0, \pi_0\}$, and to use the fact that, at least locally, this map must be invertible. This statement is clearly true for an Hamiltonian which does not at all depend on the $\chi$’s. It is reasonable to assume that this property remains valid in the more difficult situation that we face here, where the $\chi$’s only asymptotically get out of the Hamiltonian.

In the general inhomogeneous case, the constraints are essentially conditions on the space gradients of the dynamical variables and, as such, they introduce no limitation on the generic BKL Hamiltonian: that’s why generically the Hamiltonian contains all the walls. The same happens of course when the constraints are absent. In the spatial homogeneity context, however, they have to be taken into account since they can influence the billiard’s shape. Of course, the prefactors controlling the presence of the curvature walls crucially depend on the homogeneity group under consideration: the more the group “looks” abelian, the less curvature walls are present. The momentum constraints further establish linear relations between the other wall coefficients $c_A$’s in which all together the structure constants, the $\chi$’s and even the variables $\{\beta, \pi\}$ are mixed up; so the question arises whether, asymptotically, they are equivalent to the condition that some of the $c_A$’s vanish forcing the corresponding walls to disappear. That is what we will systematically investigate in the following.

3 $d = 3$ homogeneous models

3.1 Iwasawa variables

In spatial dimension $d = 3$, using the simplified notations

$$\mathcal{N}_2^1 = n_1, \quad \mathcal{N}_3^1 = n_2, \quad \mathcal{N}_3^2 = n_3$$

$$\mathcal{P}_1^2 = p_1, \quad \mathcal{P}_1^3 = p_2, \quad \mathcal{P}_2^3 = p_3,$$

the prefactors $(\mathcal{P}_a^j \mathcal{N}_j^b)^2$ of the possible symmetry walls $e^{-2(\beta^b - \beta^a)}, b > a$, in (3.1), read

for $a = 1, b = 2$ : $c_{12} = (p_1 + n_3p_2)^2$ (3.3)

for $a = 1, b = 3$ : $c_{13} = (p_2)^2$ (3.4)

for $a = 2, b = 3$ : $c_{23} = (p_3)^2.$ (3.5)
The Iwasawa decomposition (2.14) provides explicitly
\[
g_{11} = e^{-2\beta_{1}}, \quad g_{12} = n_{1}e^{-2\beta_{1}}, \quad g_{13} = n_{2}e^{-2\beta_{1}} \tag{3.6}
\]
\[
g_{22} = n_{1}^{2}e^{-2\beta_{1}} + e^{-2\beta_{2}}, \quad g_{23} = n_{1}n_{2}e^{-2\beta_{1}} + n_{3}e^{-2\beta_{2}} \tag{3.7}
\]
\[
g_{33} = n_{2}^{2}e^{-2\beta_{1}} + n_{3}^{2}e^{-2\beta_{2}} + e^{-2\beta_{3}} \tag{3.8}
\]
and its canonical extension reads
\[
2\pi_{11} = -(\pi_{1} + 2n_{1}p_{1} + 2n_{2}p_{2})e^{2\beta_{1}} - (n_{1}^{2}\pi_{2} + 2n_{1}n_{3}p_{3} - 2n_{1}n_{2}p_{3})e^{2\beta_{3}}
- (n_{2}^{2} + n_{2}^{2}n_{3} - 2n_{1}n_{2}n_{3})\pi_{3}e^{2\beta_{3}} \tag{3.9}
\]
\[
2\pi_{12} = p_{1}e^{2\beta_{1}} + (n_{1}\pi_{2} + 2n_{1}n_{3}p_{3} - n_{2}p_{3})e^{2\beta_{2}}
+ (n_{1}n_{3}^{2} - n_{2}n_{3})\pi_{3}e^{2\beta_{3}} \tag{3.10}
\]
\[
2\pi_{13} = p_{2}e^{2\beta_{1}} - n_{1}p_{3}e^{2\beta_{2}} + (n_{2} - n_{1}n_{3})\pi_{3}e^{2\beta_{3}} \tag{3.11}
\]
\[
2\pi_{22} = -(\pi_{2} + 2n_{2}p_{3})e^{2\beta_{2}} - n_{2}^{2}\pi_{3}e^{2\beta_{3}} \tag{3.12}
\]
\[
2\pi_{23} = p_{3}e^{2\beta_{2}} + n_{3}\pi_{3}e^{2\beta_{3}} \tag{3.13}
\]
\[
2\pi_{33} = -\pi_{3}e^{2\beta_{3}}. \tag{3.14}
\]
In order to easily translate the constraints in terms of the Iwasawa variables,
we also mention the following usefull formulae
\[
2\pi_{1} = p_{1} \tag{3.15}
\]
\[
2\pi_{2}^{3} = p_{2} \tag{3.16}
\]
\[
2\pi_{2} = n_{1}p_{2} + p_{3} \tag{3.17}
\]
\[
2\pi_{2}^{1} = n_{1}(\pi_{2} - \pi_{1}) + (e^{-2(\beta_{2} - \beta_{1})} - n_{1}^{2})p_{1} + (n_{3}e^{-2(\beta_{2} - \beta_{1})} - n_{1}n_{2})p_{2}
+ (n_{1}n_{3} - n_{2})p_{3} \tag{3.18}
\]
\[
2\pi_{3}^{1} = n_{2}(\pi_{3} - \pi_{1}) + n_{1}n_{3}(\pi_{2} - \pi_{3}) + [n_{3}e^{-2(\beta_{2} - \beta_{1})} - n_{1}n_{2}]p_{1}
+ [e^{-2(\beta_{3} - \beta_{1})} + n_{2}e^{-2(\beta_{2} - \beta_{1})} - n_{2}^{2}]p_{2}
+ [n_{1}n_{3}^{2} - n_{2}n_{3} - n_{1}e^{-2(\beta_{3} - \beta_{2})}]p_{3} \tag{3.19}
\]
\[
2\pi_{3}^{2} = -n_{3}(\pi_{2} - \pi_{3}) + n_{2}p_{1} + (e^{-2(\beta_{3} - \beta_{2})} - n_{3}^{2})p_{3} \tag{3.20}
\]

### 3.2 \( d = 3 \) pure gravity billiards

These spatially homogeneous models are known in the literature as the class-
A Bianchi models; they are classified according to their real, unimodular,
isometry Lie algebra [9, 13].

1. **Bianchi-type I**: $C_{ij}^k = 0, \forall i, j, k.$
   This is the abelian algebra. There is no spatial curvature and the constraints are identically verified. Accordingly, the billiard walls are only made of symmetry walls $\beta^i - \beta^j, i > j,$ among which the two dominant ones are $w_{32} = \beta^3 - \beta^2$ and $w_{21} = \beta^2 - \beta^1.$ This is the infinite volume non-diagonal Kasner billiard. Its projection on the Poincaré disc is represented by the shaded area in figure 1.

2. **Bianchi-type II**: $C_{23}^1 = 1.$
   This case is particularly simple because the constraints are easy to analyse. Indeed, the momentum constraints read
   \[
   \pi_1^2 = 0 \quad \text{and} \quad \pi_3^2 = 0 \quad \iff \quad p_1 = 0 \quad \text{and} \quad p_2 = 0. \tag{3.21}
   \]
   That means, referring to (3.3), (3.4) and (3.5), that $c_{12} = 0$ and $c_{13} = 0$ and that they clearly eliminate the symmetry walls $w_{21} = \beta^2 - \beta^1$ and $w_{31} = \beta^3 - \beta^1;$ the last $p_3$ remains free so the symmetry wall $w_{32} = \beta^3 - \beta^2$ is present. Moreover, the only non zero structure constants being $C_{23}^1 = 1,$ one single curvature wall survives which is $2\beta^1.$ Since two walls can never close the billiard, its volume is also infinite. Its projection on the Poincaré disc is given by the shaded area in figure 1.

3. **Bianchi-type VIo**: $C_{23}^1 = 1 = C_{13}^2.$
   The momentum constraints read
   \[
   \pi_2^3 = 0, \quad \pi_1^3 = 0 \quad \text{and} \quad \pi_1^2 + \pi_2^1 = 0. \tag{3.22}
   \]
They are equivalent to

\[\begin{align*}
0 &= p_2 \quad (3.23) \\
0 &= p_3 \quad (3.24) \\
0 &= n_1(\pi_2 - \pi_1) + (e^{-2(\beta^2 - \beta^1)} - n_1^2 + 2)p_1. \quad (3.25)
\end{align*}\]

The first two, according to (3.4) and (3.5), clearly tell that the symmetry walls \(w_{32} = \beta^3 - \beta^2\) and \(w_{31} = \beta^3 - \beta^1\) are absent from the potential. With (3.24) put in (3.3), one sees that the coefficient of the third symmetry wall \(w_{21}\) becomes \(c_{12} = (p_1)^2\) and the question arises whether the constraint (3.25) implies \(p_1^\infty = 0?\) In order to answer this question, we will exhibit an asymptotic solution for which \(p_1^\infty\) is a constant \(\neq 0\). Hence, \(p_1^\infty\) can be different from zero and the symmetry wall \(w_{21} = \beta^2 - \beta^1\) that it multiplies is generically present.

Let us build such a solution. We know that after a finite number of collisions \(\Pi\) against the curvature walls, the ball will never meet them again. At such a time, one can find a solution of Kasner’s type with ordered exponents. Let the diagonal Kasner solution be\(^5\)

\[ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2q_i}(dx^i)^2 \quad (3.26)\]

this is an asymptotic solution of the Bianchi VI type if \(q_1 > 0\) and \(q_2 > 0\). On this diagonal asymptotic solution, we next perform a linear transformation which generates another asymptotic solution which is no longer diagonal: let

\[G = L^T G_{\text{Kasner}} L; \quad (3.27)\]

\(G\) is the new metric, \(G_{\text{Kasner}}\) is the diagonal metric given above. We select \(L\) in the form of a Lorentz boost\(^6\) in the plane \((1,2)\) such as to produce but a single off-diagonal element, namely \(g_{12}:\)

\(^5\)Here, the Kasner exponents will be denoted \(q_i\) in order to avoid confusion with \(p_i\), the momenta conjugate to \(n_i\).

\(^6\)L is not really restricted here to define a Lie algebra automorphism, that is, the structure constants need not be conserved because, at the limit considered, the curvature becomes negligible and remains negligible after the transformation; however the \(L\)-transformation here chosen conserves the Bianchi VI structure constants and is a gauge transformation \([10]\).
\[
L = \begin{pmatrix}
\cosh \alpha & \sinh \alpha & 0 \\
\sinh \alpha & \cosh \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (3.28)

Accordingly, the new metric elements are
\[
g_{11} = t^{2q_1} \cosh^2 \alpha + t^{2q_2} \sinh^2 \alpha \quad (3.29)
\]
\[
g_{12} = (t^{2q_1} + t^{2q_2}) \sinh \alpha \cosh \alpha \quad (3.30)
\]
the other stay unchanged. From these expressions and (3.6), (3.7), (3.8), one immediately extracts the \(t\)-dependence of \(n_1\) and of the \(\beta\)'s which explicitly gives
\[
n_1 = \frac{g_{12}}{g_{11}} = \frac{(t^{2q_1} + t^{2q_2}) \cosh \alpha \sinh \alpha}{t^{2q_1} \cosh^2 \alpha + t^{2q_2} \sinh^2 \alpha} \quad (3.31)
\]
and
\[
e^{2(\beta^2 - \beta^1)} = \frac{(t^{2q_1} \cosh^2 \alpha + t^{2q_2} \sinh^2 \alpha)^2}{t^{2q_1+2q_2}}. \quad (3.32)
\]

In order to discover the time behaviour of \(p_1\), we make use of the equation of motion for \(n_1\) which, since \(p_1\) just appears in the Hamiltonian through the coefficient \(c_{12}\), simply reads
\[
\dot{n}_1 = \frac{dn_1}{d\tau} = \frac{dn_1}{dt} \frac{dt}{d\tau} = p_1 e^{-2(\beta^2 - \beta^1)}, \quad (3.33)
\]
remembering that \(\sqrt{\mathcal{g}} \, dt = -\sqrt{\mathcal{g}} d\tau\) and that the sum of the Kasner exponents is 1 such that \(\sqrt{\mathcal{g}} = t\). Hence, the momentum \(p_1\) is given by
\[
p_1 = -\dot{n}_1 e^{2(\beta^2 - \beta^1)}; \quad \text{assuming } q_1 < q_2, \text{ one finds that, in the vicinity of } t = 0, \text{ it behaves as}
\]
\[
p_1 \simeq C + B t^{2(q_2 - q_1)} + \ldots \quad (3.34)
\]
where \(C = p_1^\infty\) is not constrained to be zero. This result allows us to state that, in general, the symmetry wall \(w_{21} = \beta^2 - \beta^1\) is present, beside the two curvature walls \(2\beta^1\) and \(2\beta^2\); the dominant walls being \(w_{21} = \beta^2 - \beta^1\) and \(2\beta^1\), they don’t close the billiard.

Notice that, for \(t \to 0\), the exponential in the constraint (3.25) goes to zero like
\[
e^{-2(\beta^2 - \beta^1)} \simeq \cosh^{-4} \alpha t^{2(q_2 - q_1)}, \quad (3.35)
\]
as announced by the BKL analyzis when the coefficient of the wall \(c_{12} \neq 0\) and also that the limit \(t \to 0\) of the product \(p_1 e^{2(\beta^2 - \beta^1)}\) exists and is zero. Consequently, (3.25) admits a clear limit when \(t \to 0\).
Had one taken advantage of the gauge freedom to assign the initial values $n_0^i$ and $p_0^i$ such as $n_1^\infty = 0$, then the constraint (3.25) would have imposed the last symmetry wall to be absent and the billiard, here limited by the two curvature walls, to be open.

These examples show that the presence/absence of symmetry walls may depend on initial conditions or on gauge conditions; nevertheless, in the cases mentioned hereabove, a billiard with an infinite volume remains a billiard with an infinite volume.

4. **Bianchi-type VII**: $C_{23}^1 = 1, \quad C_{13}^2 = -1$.

This case is very similar to the preceding one. The only changes are i) that the third constraint in (3.22) has to be replaced by $(\pi_2^2 - \pi_1^2 = 0)$ and ii) that its translation into the Iwasawa variables in (3.23) now reads

$$0 = n_1(\pi_2 - \pi_1) + (e^{-2(\beta^2 - \beta^1}) - n_1^2 - 2)p_1.$$ (3.36)

Accordingly, we can apply the same reasoning as before and conclude that generically, $p_1^\infty \neq 0$ and that the symmetry wall $w_{21} = \beta^2 - \beta^1$ appears in the potential.

5. **Bianchi-type IX**: $C_{23}^1 = 1, \quad C_{31}^2 = 1, \quad C_{12}^3 = 1$.

This case and the next one deserve a particular treatment because the structure constants are such that all curvature walls, namely $2\beta^1$, $2\beta^2$, $2\beta^3$, appear and these three gravitational walls already form a finite volume billiard.

---

$^7$This situation is very specific to the homogeneous models in $D = 4$; in higher space-time dimensions, the number of curvature walls allowed by the structure constants is not sufficient to produce a finite volume billiard.
Let us first discuss the generic case. The momentum constraints take the form
\[ \pi_2^3 - \pi_3^2 = 0, \quad \pi_1^3 - \pi_3^1 = 0, \quad \pi_2^1 - \pi_1^2 = 0; \quad (3.37) \]
and in terms of the Iwasawa variables, they become
\[
\begin{align*}
    n_3(\pi_2 - \pi_3) &- n_2p_1 + n_1p_2 \\
        &+ \left[-e^{-2(\beta^3 - \beta^2)} + n_3^2 + 1\right]p_3 = 0 \quad (3.38)
\end{align*}
\]
\[
\begin{align*}
    n_2(\pi_3 - \pi_1) &+ n_1n_3(\pi_2 - \pi_3) + [n_3e^{-2(\beta^3 - \beta^1)} - n_1n_2]p_1 \\
        &+ [e^{-2(\beta^3 - \beta^1)} + n_3^2e^{-2(\beta^2 - \beta^1)} - n_2^2 - 1]p_2 \\
        &+ [n_1n_3^2 - n_2n_3 - n_1e^{-2(\beta^3 - \beta^2)}]p_3 = 0 \quad (3.39)
\end{align*}
\]
\[
\begin{align*}
    n_1(\pi_2 - \pi_1) &+ [e^{-2(\beta^2 - \beta^1)} - n_1^2 - 2]p_1 \\
        &+ [n_3e^{-2(\beta^2 - \beta^1)} - n_1n_2]p_2 \\
        &+ [n_1n_3 - n_2]p_3 = 0. \quad (3.40)
\end{align*}
\]
Remember that we already know that the billiard has a finite volume, hence the question is no longer to state between chaos or non chaos but rather to define more precisely the shape of the billiard. We cannot copy the reasoning made for Bianchi VI since we have here to account for an infinite number of collisions.

In order to study the implications of the constraints (3.38) - (3.40), we shall rely on the heuristic estimates made in [6], where the asymptotic behaviour of the variables is analysed in the BKL limit. From that analysis, it follows that, when \( \rho \to \infty \): i) the \( \pi_a \)'s go to zero as powers of \( 1/\rho \), ii) the \( n_i \)'s and the \( p_i \)'s tend to constants \( n_i^\infty \) and \( p_i^\infty \) up to additive terms which also go to zero as powers of \( 1/\rho \) and iii) that the exponentials either vanish (if the walls are present) or oscillate between zero and \( \infty \) (if they are absent). What we infer from this information is that if the \( p_i^\infty \) were not strictly zero then the constraint system hereabove would have no BKL limit. This is obviously wrong because a constraint need be obeyed all the time; so the limit must exist. Now we know that the \( p_i \)'s go to zero, can we say something more about the walls that they multiply? Compared to the exponential growth of the wall, the power decrease of its prefactor is not fast enough to prevent the symmetry wall to appear in the Hamiltonian.
Accordingly, the billiard’s edge is formed by the leading symmetry walls $w_{32}$ and $w_{21}$ and by the curvature wall $2\beta^1$. On the Poincaré disc, it is one of the six small triangles included in the larger one bordered by the curvature walls. Its Cartan matrix is that of the hyperbolic Kac-Moody algebra $AE_3 = A_1^{\wedge\wedge}$

$$
\begin{pmatrix}
 2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -2 & 2 \\
\end{pmatrix}
$$

already relevant in the general inhomogeneous case. Its Dynkin diagram is given in figure 3.

![Dynkin diagram](image)

Figure 3: Dynkin diagrams of the $A_1^{\wedge\wedge}$ Kac-Moody algebra

Other interesting cases exist with less symmetry walls, which require specific initial conditions.

![Bianchi IX Billiards](image)

Figure 4: Bianchi IX Billiards

1. No symmetry wall at all. This situation is the one mentioned above; it happens when the metric is assumed diagonal always, hence for $n_1 = 0, n_2 = 0, n_3 = 0$; the solution of the momentum constraints is then $p_1 = 0, p_2 = 0, p_3 = 0$. This assumption is consistent with the equations of motion. The Cartan matrix of this billiard is given by

$$
\begin{pmatrix}
 2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2 \\
\end{pmatrix}
$$

and its corresponding Dynkin diagram is number 1 in figure 5; the associated Kac-Moody algebra is hyperbolic, it has number 7 in the enumeration provided in reference [14].
2. One symmetry wall. This happens when one chooses the initial data such that \( n_1 = 0, n_2 = 0 \) and \( p_1 = 0, p_2 = 0 \); then, according to the equations of motion, these variables remains zero all the time. The billiard is closed by two curvature walls \( 2\beta^1 \) and \( 2\beta^2 \) and the symmetry wall \( w_{32} \). On the Poincaré disc, its volume is half of that of the triangle made of curvature walls. The Cartan matrix is given by

\[
\begin{pmatrix}
2 & 0 & -2 \\
0 & 2 & -2 \\
-2 & -2 & 2 \\
\end{pmatrix}
\]  \hspace{1cm} (3.43)

and its Dynkin diagram is number 2 in figure 5. It characterizes the third rank 3 Lorentzian Kac-Moody algebra in the classification given in \([13]\).

![Figure 5: Dynkin diagrams of special algebras met in Bianchi IX models](image)

These last two Kac-Moody algebras are subalgebras of \( A_1^{\wedge\wedge} \). \([11]\). Remark: let us recall that the reflections on the faces of the billiard generate a Coxeter group which is identified with the Weyl group of the associated Kac-Moody algebra. The larger the set of walls, the larger the number of generators. Since in the non generic cases the set of walls is a subset of the one in the generic case, the associate Coxeter group is a subgroup of the generic one.

6. **Bianchi-type VIII**: \( C^1_{23} = 1, \quad C^2_{31} = 1, \quad C^3_{12} = -1. \)

The analysis of this case follows closely the previous one. The sign change in the structure constants modifies the constraints as follows

\[
\pi_2^3 + \pi_3^2 = 0, \quad \pi_1^3 + \pi_3^1 = 0, \quad \pi_2^2 - \pi_1^1 = 0 \hspace{1cm} (3.44)
\]

and induces some sign changes in their Iwasawa counterparts. The conclusions are those of the Bianchi IX model: the generic billiard is the one of the algebra \( A_1^{\wedge\wedge} \).
3.3 $d = 3$ Einstein-Maxwell billiards

The momentum constraints (2.13) of the Einstein-Maxwell homogeneous models generally (except for the abelian Bianchi I) mix gravitational and one-form variables; in comparison with the pure gravity case, i) no constraint remains which clearly forces the prefactor of a symmetry wall to be zero and ii) additional terms of the type $\pi(1)F(1)$ enter in the constraints system.

Accordingly, generically, besides the curvature walls allowed by the structure constants, one expects all symmetry, electric and magnetic walls to be present. The dominant ones are the symmetry walls $w_{21} = \beta^2 - \beta^1$, $w_{32} = \beta^3 - \beta^2$ and the electric wall $e_1 = \beta^1$ which replaces the curvature wall of the pure gravity case. They close the billiard whose Cartan matrix is

$$
\begin{pmatrix}
2 & -4 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}.
$$

The Dynkin diagram is displayed in figure 6; the corresponding Kac-Moody algebra is the hyperbolic $A_2^{(2)\wedge}$ algebra which is the Lorentzian extension of the twisted affine algebra $A_2^{(2)}$ also encountered in [7].

Figure 6: Dynkin diagram of the $A_2^{(2)\wedge}$ algebra

4 $d = 4$ homogeneous models

Our billiard analysis of the four dimensional spatially homogeneous cosmological models has been carried out along the same lines as for $d = 3$: results relative to pure gravity models and those relative to the Einstein-Maxwell homogeneous system will be given separately.
4.1 Extension of the notations and Iwasawa variables

In spatial dimension \( d = 4 \), we introduce the matrix variables

\[
\mathcal{N} = \begin{pmatrix}
1 & n_{12} & n_{13} & n_{14} \\
0 & 1 & n_{23} & n_{24} \\
0 & 0 & 1 & n_{34} \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

(4.1)

The Iwasawa decomposition of the metric extends beyond the 3-dimensional formulae listed in (3.6), (3.7), (3.8), through the additional components

\[
g_{14} = n_{14} e^{-2\beta^1} \\
g_{24} = n_{12} n_{14} e^{-2\beta^1} + n_{24} e^{-2\beta^2} \\
g_{34} = n_{13} n_{14} e^{-2\beta^1} + n_{23} n_{24} e^{-2\beta^2} + n_{34} e^{-2\beta^3} \\
g_{44} = n_{14}^2 e^{-2\beta^1} + n_{24}^2 e^{-2\beta^2} + n_{34}^2 e^{-2\beta^3} + e^{-2\beta^4}.
\]

(4.2) (4.3) (4.4) (4.5)

The momentum conjugate to \( \beta^a \) is written as \( \pi^a \), as before; the momentum conjugate to \( \mathcal{N}^a_j = n_{aj} \) is denoted \( \mathcal{P}^j_a = p^{ja} \) and is only defined for \( j > a \).

The prefactors \( c_{ab} = (\mathcal{P}^j_a \mathcal{N}^b_j)^2 \) of the symmetry walls \( e^{-2(\beta^b - \beta^a)} \), \( b > a \), in the Hamiltonian are explicitly given by

\[
c_{12} = (p^{31} n_{23} + p^{41} n_{24})^2 \\
c_{13} = (p^{31} + p^{41} n_{34})^2 \\
c_{14} = (p^{41})^2 \\
c_{23} = (p^{32} + p^{42} n_{34})^2 \\
c_{24} = (p^{42})^2 \\
c_{34} = (p^{43})^2.
\]

(4.6) (4.7) (4.8) (4.9) (4.10) (4.11)

Once the change of dynamical variables has been continued in a canonical point transformation, the momentum contraints still express linear relations among the \( \pi^i_j \)'s which translate into linear relations on the momenta \( \pi^a, p^{ja} \); their coefficients are polynomials in the \( n_{ai} \)'s times exponentials of the type \( e^{-2(\beta^b - \beta_a)} \), with \( b > a \). The explicit form of the constraints depends on the model considered.

4.2 \( d = 4 \) Pure gravity

We label the various 4 dimensional spatially homogeneous models according to the classification of the 4 dimensional real unimodular Lie algebras given
by M. MacCallum [12]; they are

1. **Class** $U1[1,1,1]$: $C_{14}^1 = \lambda, C_{24}^2 = \mu, C_{34}^3 = \nu$, with $\lambda + \mu + \nu = 0$.

   One can still set $\lambda = 1$ except in the abelian case where $\lambda = \mu = \nu = 0$.

2. **Class** $U1[Z,\bar{Z},1]$: $C_{14}^1 = -\frac{\mu}{2}, C_{24}^2 = 1, C_{24}^1 = -1, C_{24}^2 = -\frac{\mu}{2}$,

   $C_{34}^3 = \mu$. $\mu = 0$ is a special case.

3. **Class** $U1[2,1]$: $C_{14}^1 = -\frac{\mu}{2}, C_{14}^2 = 1, C_{24}^2 = -\frac{\mu}{2}, C_{34}^3 = \mu$ and $\mu$ is 0 or 1.

4. **Class** $U1[3]$: $C_{14}^2 = 1, C_{24}^3 = 1$.

5. **Class** $U3I0$: $C_{23}^4 = 1, C_{31}^2 = 1, C_{12}^3 = -1$.

6. **Class** $U3I2$: $C_{23}^4 = -1, C_{31}^2 = 1, C_{12}^3 = 1$.

7. **Class** $U3S1$ or $sl(2) \oplus u(1)$: $C_{23}^1 = 1, C_{31}^2 = 1, C_{12}^3 = -1$.

8. **Class** $U3S3$ or $su(2) \oplus u(1)$: $C_{23}^1 = 1, C_{31}^2 = 1, C_{12}^3 = 1$.

Let us mention that for all of the four dimensional algebras, except of course the abelian one, the structure constant $C_{134}^0 \neq 0$ and consequently that the curvature wall $2\beta_1 + \beta_2$ is always present.

Our analysis leads to the conclusion that, from the billiard point of view, the previous models can be collected into two sets: the first one has an open billiard, the second one has a finite volume billiard whose Cartan matrix is that of the hyperbolic Kac-Moody algebra $A_2^{\Lambda}$ exactly as in the general inhomogeneous situation.

The first set contains the abelian algebra and $U1[1,1,1]|_{\mu \neq 0,-1}$, $U1[2,1]$ and $U1[Z,\bar{Z},1]|_{\mu \neq 0}$; all the others belong to the second set. Because explicit developments soon become lengthy and since the reasonings always rest on similar arguments, we have chosen not to review systematically all cases as for $d=3$ but rather to illustrate the results on examples taken in both sets:

1. As a representative of the first set, we take $U1[Z,\bar{Z},1]|_{\mu \neq 0}$.

   The momentum constraints read

   $$\pi_1^4 = \pi_2^4 = \mu \pi_3^4 = -\frac{\mu}{2}(\pi_1^1 + \pi_2^2 - 2\pi_3^3) + \pi_2^1 - \pi_1^2 = 0; \quad (4.12)$$
in terms of the Iwasawa variables, they become

\[ p^{41} = p^{42} = \mu p^{43} = 0 \]  

(4.13)

and

\[
\begin{align*}
- \frac{\mu}{2}(2\pi_3 - \pi_1 - \pi_2) + n_{12}(\pi_2 - \pi_1) + \\
+ \left( e^{-2(\beta^2 - \beta^1)} - n_{12}^2 - 1 \right) p^{21} + (n_{23}e^{-2(\beta^2 - \beta^1)} - n_{12}n_{13} + \frac{3n_{13}\mu}{2})p^{31} \\
+ (n_{12}n_{23} - n_{13} + \frac{3n_{23}\mu}{2})p^{32} = 0.
\end{align*}
\]

(4.14)

The first three constraints (4.13) clearly indicate that the symmetry walls \( \beta^4 - \beta^3, \beta^4 - \beta^2, \beta^4 - \beta^1 \) are absent from the potential. What information can one extract from the fourth constraint (4.14)? In order to answer this question, we will proceed as for the Bianchi VI model: we will exhibit an asymptotic solution with prefactors \( c_{23} = (p^{32}_\infty)^2 \) and \( c_{12} = (p^{31}_\infty + n_{23}^\infty p^{31}_\infty)^2 \) different from zero. We here again face a situation in which the billiard motion is made of a finite number of collisions against the curvature walls. After the last bounce, the motion can be described by a Kasner type solution

\[ ds^2 = -dt^2 + \sum_{i=1}^{4} t^{2q_i} (dx^i)^2 \]

(4.15)

which is a suitable asymptotic solution at the conditions \( q_1 > 0, q_2 > 0 \) and \( q_3 > 0 \) (no condition on \( q_4 \) is required because there is no curvature wall involving \( \beta^4 \)). We next perform a linear transformation

\[ G = L^T G_{\text{Kasner}} L \]

(4.16)

in order to get another asymptotic solution which is non-diagonal. Here, for simplicity, \( L \) can be taken in the form that produces the off-diagonal elements we are interested in,

\[
L = \begin{pmatrix}
  l_1 & l_2 & l_3 & 0 \\
  m_1 & m_2 & m_3 & 0 \\
  r_1 & r_2 & r_3 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}.
\]

(4.17)

Then using the equations of motion for \( n_{12}, n_{13} \) and \( n_{23} \) we obtain

\[ p^{21} + n_{23}p^{31} = \dot{n}_{12} e^{2(\beta^2 - \beta^1)} \]

(4.18)
and
\[ p^{32} = n_{23} e^{2(\beta^3 - \beta^2)}. \] (4.19)

By direct application of the formulae given in reference [6], section (4.2), assuming the following order \( q_1 < q_2 < q_3 \) for the Kasner exponents, we find that for \( t \to 0 \), (4.18) and (4.19) tend to constants that depend on the parametrization of \( L \) but that will generically be different from zero.

Consequently, our conclusion is that, in the generic case, the dominant walls of the billiard are the symmetry walls \( w_{32} = \beta^3 - \beta^2 \), \( w_{21} = \beta^2 - \beta^1 \) and the curvature wall \( 2\beta^1 + \beta^2 \); it is indeed an open billiard.

Notice that in all cases of the first set, the non vanishing structure constants are all of the form \( C_a^b \) with \( a, b = 1, 2, 3 \). It is easy to check that this forbids all curvature wall containing \( \beta^4 \) and therefore that the remaining curvature walls cannot be expected to close the billiard.

2. As a first representative of the second set, we take \( U1[Z, Z, 1]_{\mu=0} \).

The constraints are given by (1.12), (1.13) and (1.14) for \( \mu = 0 \) and one immediately sees that, compared to the preceding case, one constraint drops out leaving the symmetry wall \( w_{43} = \beta^4 - \beta^3 \) in place. To conclude the analysis of the last constraint, we can also provide a solution with non vanishing prefactors for the symmetry walls. Since the structure constants do not play an important rôle in the explicit construction hereabove, the solution obtained with \( \mu \neq 0 \) can also be used for \( \mu = 0 \) if one assigns particular initial condition such that \( p^{43} = 0 \) and \( n_{34} = 0 \). Again, we end up here with the following list of dominant walls: \( w_{43} = \beta^4 - \beta^3 \), \( w_{32} = \beta^3 - \beta^2 \), \( w_{21} = \beta^2 - \beta^1 \) and \( 2\beta^1 + \beta^2 \). The Cartan matrix is that of the algebra \( A_2^{\wedge} \) as previously announced.

![Diagram](image)

Figure 7: Dynkin diagram of the \( A_2^{\wedge} \) algebra

3. Another interesting example of the second set is provided by \( U3S3 \).

Its homogeneity group is the direct product \( SU(2) \times U(1) \) so that this model appears as the four dimensional trivial extension of the Bianchi IX model: since the structure constants are the same as in the 3-dimensional
case, the momentum constraints do not change, when expressed in terms of the metric components and their momenta

\[ \pi_3^1 - \pi_1^3 = 0, \quad \pi_2^1 - \pi_1^2 = 0, \quad \pi_3^2 - \pi_1^3 = 0; \quad (4.20) \]

their number does not change either but, in terms of Iwasawa variables, they involve much more terms than their \( d = 3 \) counterpart given in (3.38), (3.39) and (3.40)

\[ \begin{align*}
- n_{13}(\pi_1 - \pi_3) + n_{12}n_{23}(\pi_2 - \pi_3) - (n_{12}n_{13} - n_{23}e^{-2(\beta^2 - \beta^1)})p^{21} \\
- (n_{13}^2 - n_{23}e^{-2(\beta^2 - \beta^1)} - e^{-2(\beta^1 - \beta^3)} - 1)p^{31} \\
+ (n_{12}n_{23} - n_{13}n_{23} - n_{12}e^{-2(\beta^1 - \beta^3)})p^{32} \\
- (n_{13}n_{14} - n_{23}e^{-2(\beta^2 - \beta^1)}n_{24} - n_{34}e^{-2(\beta^1 - \beta^3)})p^{41} \\
+ (n_{12}n_{23}n_{24} - n_{14}n_{23} - n_{12}n_{34}e^{-2(\beta^1 - \beta^2)})p^{42} \\
- (n_{12}n_{23}n_{34} - n_{12}n_{24} - n_{13}n_{34} + n_{14})p^{43} = 0 \\
\end{align*} \quad (4.21) \]

\[ \begin{align*}
- n_{12}(\pi_1 - \pi_2) - (n_{12}^2 - e^{-2(\beta^2 - \beta^1)} - 1)p^{21} \\
- (n_{12}n_{13} - n_{23}e^{-2(\beta^2 - \beta^1)})p^{31} + (n_{12}n_{23} - n_{13})p^{32} \\
- (n_{12}n_{14} - n_{24}e^{-2(\beta^2 - \beta^1)})p^{41} + (n_{12}n_{24} - n_{14})p^{42} = 0 \\
\end{align*} \quad (4.22) \]

\[ \begin{align*}
- n_{23}(\pi_2 - \pi_3) + n_{13}p^{21} - n_{12}p^{31} - (n_{23}^2 - e^{-2(\beta^3 - \beta^2)} + 1)p^{32} \\
- (n_{23}n_{24} - n_{34}e^{-2(\beta^3 - \beta^2)})p^{42} + (n_{23}n_{34} - n_{24})p^{43} = 0. \\
\end{align*} \quad (4.23) \]

These constraints are of course i) linear in the momenta, ii) the only exponentials which enter these expressions are build of \( w_{32} = \beta^3 - \beta^2, \ w_{31} = \beta^3 - \beta^1 \) and \( w_{21} = \beta^2 - \beta^1 \) and iii) as before, their coefficients are exactly given by the square root of the corresponding \( c_4 \)’s in the potential, namely \( \sqrt{c_{12}}, \sqrt{c_{13}} \) and \( \sqrt{c_{23}} \). Accordingly, the question arises whether these equations are equivalent to \( c_{12} = 0, c_{23} = 0 \) and \( c_{13} = 0 \). If, asymptotically, the exponentials go to zero and can be dropped out of the constraints, the remaining equations can be solved for \( p^{21}, p^{31}, p^{32} \) in terms of the other variables among which figure now, not only the \( \pi_i - \pi_j \) already present in the 3-dimensional case which asymptotically go to zero, but also the asymptotic values of the \( p^{4i} \)’s which remain unconstrained. It follows that none of the solutions of the above system is generically forced to vanish so that all the symmetry walls
are expected to be present. The absence of a wall can only happen in non
generic situations with very peculiar initial conditions.

We expect this result to become the rule in higher dimensions for trivial
extensions like \( SU(2) \times SU(2) \times \ldots \times U(1) \times \ldots \times U(1) \); the billiard will then
become that of the Kac-Moody algebra \( A_r^{\wedge, \wedge} \) relevant in the general Einstein
theory.

We can nevertheless provide a particular solution with initial data obeying
\( n_{12} = n_{13} = n_{14} = 0 \) and \( p^{21} = p^{31} = p^{41} = 0 \). These values are conserved
in the time evolution. In this case, the leading walls are the symmetry walls
\( w_{13}, w_{32} \) and the curvature ones \( 2 \beta^2 + \beta^3, 2 \beta^1 + \beta^2 \). The Cartan matrix is
\[
\begin{pmatrix}
2 & -1 & -1 & 0 \\
-1 & 2 & -1 & -1 \\
-1 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{pmatrix}.
\]
(4.24)

The billiard is characterized by the rank 4 Lorentzian Kac-Moody algebra
which bears number 2 in the classification given in [14]; its Dynkin diagram
is drawn in figure 8.

![Figure 8: Dynkin diagram of algebra number 2 in the subset of rank 4](image)

The \( D = 5 \) results are summarized in the following table:

| \( U1[1, 1, 1]_{\mu=\nu=\lambda=0} \) | non chaotic |
| \( U1[1, 1, 1]_{\mu\neq0,-1} \) | non chaotic |
| \( U1[Z, \bar{Z}, 1]_{\mu\neq0} \) | non chaotic |
| \( U1[1, 1, 1]_{\mu=-1} \) | chaotic |
| \( U1[Z, \bar{Z}, 1]_{\mu=0} \) | chaotic |
| \( U1[2, 1] \) | chaotic |
| \( U1[3] \) | chaotic |
| \( U3I0 \) | chaotic |
| \( U3I2 \) | chaotic |
| \( U3S3 \) | chaotic |
| \( U3S1 \) | chaotic |
Chaos or non chaos for $D = 5$ models

4.3 $d = 4$ Einstein-Maxwell models

As in the three dimensional case, because of the presence of the electromagnetic field in the expression of the momentum constraint (2.13), no symmetry wall can be eliminated. Moreover, the electric walls always prevail over the curvature ones if any. The billiard is in all cases characterized by the following set of dominant walls $w_{43} = \beta^4 - \beta^3$, $w_{32} = \beta^3 - \beta^2$, $w_{21} = \beta^2 - \beta^1$ and $e_1 = \beta^1$. Its Cartan matrix is that of the hyperbolic Kac-Moody algebra $G_2^{\wedge}$.

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ
\end{array}
\]

Figure 9: Dynkin diagram of the $G_2^{\wedge}$ algebra

In the general analysis of the billiards attached to coupled gravity + p-forms systems, one could assume $2p < d$ without loss of generality, because the complete set of walls is invariant under electric-magnetic duality. This invariance may however not remain in some spatially homogeneous cases due to the vanishing of some structure constants which lead to incomplete sets of walls so that other, a priori unexpected Kac-Moody algebras, might appear. An illustrative and interesting example is given in $D = 5$ by the Einstein + 2-form system governed by the abelian algebra. Here, the dominant walls are i) the symmetry walls $\beta^4 - \beta^3$, $\beta^3 - \beta^2$ and $\beta^2 - \beta^1$ and ii) the electric wall $\beta^1 + \beta^2$. This billiard brings in the new hyperbolic Kac-Moody algebra carrying number 20 in [14], whose Dynkin diagram is given in figure 10 hereafter.

\[
\begin{array}{c}
\circ \quad \circ \quad \circ
\end{array}
\]

Figure 10: Dynkin diagram of the algebra relevant for the Einstein + 2-form system
5 Conclusions

In this paper, we have analyzed the Einstein and Einstein-Maxwell billiards for all the spatially homogeneous cosmological models in 3 and 4 dimensions. In the billiard picture, we confirm that in spacetime dimensions $5 \leq D \leq 10$, diagonal models are not rich enough to produce the never ending oscillatory behaviour of the generic solution of Einstein’s equations and that chaos is restored when off-diagonal metric elements are kept. Chaotic models are characterized by a finite volume billiard which can be identified with the fundamental Weyl chamber of an hyperbolic Kac-Moody algebra: in the most generic chaotic situation, the algebra coincides with the one already relevant in the inhomogeneous case; this remains true after the addition of an generic homogeneous electromagnetic field. Other algebras can also appear for special initial data or gauge choices: in fact, these are all the simply-laced known hyperbolic Kac-Moody algebras of ranks 3 and 4, except the one which has number 3 among those of rank 4 listed in [14]. The billiard of a non chaotic model is even not a simplex.

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