THE NUMBER OF INTEGER POINTS IN A FAMILY OF
ANISOTROPICALLY EXPANDING DOMAINS

YURI A. KORDYUKOV AND ANDREY A. YAKOVLEV

Abstract. We investigate the remainder in the asymptotic formula
for the number of integer points in a family of bounded domains in the
Euclidean space, which remain unchanged along some linear subspace
and expand in the directions, orthogonal to this subspace. We prove
some estimates for the remainder, imposing additional assumptions on
the boundary of the domain. We study the average remainder estimates,
where the averages are taken over rotated images of the domain by
a subgroup of the group SO(n) of orthogonal transformations of the
Euclidean space $\mathbb{R}^n$.

Using these results, we improve the remainder estimate in the adia-
batic limit formula for the eigenvalue distribution function of the Laplace
operator associated with a bundle-like metric on a compact manifold
equipped with a Riemannian foliation in the particular case when the
foliation is a linear foliation on the torus and the metric is the standard
Euclidean metric on the torus.

1. Preliminaries and main results

1.1. The setting of the problem. A classical problem on integer points
distribution consists in the study of the asymptotic behavior of the number
of points of the integer lattice $\mathbb{Z}^n$ in a family of homothetic domains in $\mathbb{R}^n$.
This problem is originated in the Gauss problem on the number of integer
points in the disk, where it is directly related with the arithmetic problem
on the number of representations of an integer as a sum of two squares,
and sufficiently well studied (see, for instance, books [2, 3, 4, 9] and the
references therein).

In this paper we investigate much less studied problem of counting integer
points in a family of anisotropically expanding domains. More precisely, let
$F$ be a $p$-dimensional linear subspace of $\mathbb{R}^n$ and $H = F^\perp$ the $q$-dimensional
orthogonal complement of $F$ with respect to the standard inner product
$(\cdot, \cdot)$ in $\mathbb{R}^n$, $p + q = n$. For any $\varepsilon > 0$, consider the linear transformation
$T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T_\varepsilon(x) = \begin{cases} x, & \text{if } x \in F, \\ \varepsilon^{-1}x, & \text{if } x \in H. \end{cases}$$

For any bounded set $S$ in $\mathbb{R}^n$, we put

$$(1.1) \quad n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$
The study of the asymptotic behavior of \( n_\varepsilon(S) \) as \( \varepsilon \to 0 \) for general domains in \( \mathbb{R}^n \) was started in [8]. In particular, the following asymptotic formula has been proved.

Let \( \Gamma = \mathbb{Z}^n \cap F \). \( \Gamma \) is a free abelian group. Denote by \( r = \text{rank} \Gamma \leq p \) the rank of \( \Gamma \). Let \( V \) be the \( r \)-dimensional subspace of \( \mathbb{R}^n \) spanned by the elements of \( \Gamma \). Observe that \( \Gamma \) is a lattice in \( V \), dual to the lattice \( \Gamma^* \): \( \Gamma^* = \{ \gamma^* \in V : (\gamma^*, \Gamma) \subset \mathbb{Z} \} \).

For any \( x \in V \), we denote by \( P_x \) the \((n-r)\)-dimensional affine subspace of \( \mathbb{R}^n \), passing through \( x \) orthogonal to \( V \).

**Theorem 1.1** ([8], Theorem 1.1). For any bounded open set \( S \) in \( \mathbb{R}^n \) with smooth boundary, we have

\[
n_\varepsilon(S) = \frac{\varepsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{\frac{1}{p-r+1-q}}), \quad \varepsilon \to 0.
\]

Here, if \( \Gamma \) is trivial, by definition we put \( \text{vol}(V/\Gamma) = 1 \).

Actually, as it can be easily seen from the proof of this theorem, it holds for any set \( S \subset \mathbb{R}^n \) such that for any \( \gamma^* \in \Gamma^* \) the intersection \( P_{\gamma^*} \cap S \) is a bounded open set in \( P_{\gamma^*} \) with Lipschitz boundary. Using this fact, one can easily prove a more general form of the formula (1.2).

**Theorem 1.2.** For any set \( S \) in \( \mathbb{R}^n \) such that for any \( \gamma^* \in \Gamma^* \) the intersection \( P_{\gamma^*} \cap S \) is a bounded open set in \( P_{\gamma^*} \) which is Jordan measurable, we have

\[
n_\varepsilon(S) \sim \frac{\varepsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S), \quad \varepsilon \to 0.
\]

In this paper, we continue the study of the remainder in the formula (1.3) given by

\[
R_\varepsilon(S) = n_\varepsilon(S) - \frac{\varepsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S).
\]

Remark that, in a slightly different context, the problem of counting integer points in a family of anisotropically expanding domains was also studied in considerable detail in [14, 15, 10, 11] (see also the introduction of [8]).

### 1.2. Main results.

The first goal of the paper is to obtain more precise estimates for \( R_\varepsilon(S) \), imposing some additional assumptions on \( S \).

**Theorem 1.3.** Let \( S \) be any subset of \( \mathbb{R}^n \) such that for any \( \gamma^* \in \Gamma^* \), the intersection \( P_{\gamma^*} \cap S \) is a bounded open set in \( P_{\gamma^*} \) with smooth boundary.

1. If, for any \( \gamma^* \in \Gamma^* \) and \( x \in F \cap V^\perp \), the intersection \( S \cap \{ \gamma^* + x + H \} \) is strictly convex, then we have

\[
R_\varepsilon(S) = O(\varepsilon^{\frac{2p}{p+r+1-q}}), \quad \varepsilon \to 0.
\]
(2) If, for any \( \gamma^* \in \Gamma^* \), the intersection \( P_{\gamma^*} \cap S \) is strictly convex, then:

\[
R_\varepsilon(S) = O(\varepsilon^{\frac{2q}{r+1-r}-q}), \quad \varepsilon \to 0.
\]

Remark that the first statement of Theorem 1.3 is a slight improvement of Theorem 1.2 in [8].

In [12, 13], Randol suggested in the case of a family of homothetic domains in \( \mathbb{R}^n \) to consider instead of the remainder for a domain \( S \) its averages over rotated or over rotated and translated images of \( S \). He observed that estimates for averages can be substantially smaller than individual estimates. In [12, 13], such results were proved for bounded convex open sets with analytic boundary. The results of Randol were extended by Varchenko [16] to arbitrary bounded open sets with smooth boundary, proving a conjecture by Arnold. Finally, in [1] the average remainder estimates were proved for any bounded convex open set \( S \) or in the case when the boundary of \( S \) is \( C^{3/2} \). We refer to [1, 3] for more information and references on this problem.

The second goal of the paper is to obtain similar results in the case under consideration. So we study averages of the remainder \( R_\varepsilon(S) \) taken over rotated images of \( S \) by a group of orthogonal transformations of the Euclidean space \( \mathbb{R}^n \). We will consider several subgroups of \( \text{SO}(n) \).

First, we consider the group \( \text{SO}(H) \) of orthogonal transformations of \( \mathbb{R}^n \), which fix any vector of \( F \) (and, as a consequence, take \( H \) to itself):

\[
\text{SO}(H) = \{ A \in \text{SO}(n) : A|_F = \text{Id} \}.
\]

We will denote by \( dh \) the Haar measure on \( \text{SO}(n) \) (and on any subgroup of \( \text{SO}(n) \) as well).

**Theorem 1.4.** For any subset \( S \) of \( \mathbb{R}^n \) such that for any \( \gamma^* \in \Gamma^* \) and \( x \in F \cap V^\perp \), the intersection \( S \cap \{ \gamma^* + x + H \} \) is a bounded open set in \( P_{\gamma^*} \) such that either \( S \cap \{ \gamma^* + x + H \} \) is convex or the boundary of \( S \cap \{ \gamma^* + x + H \} \) is \( C^{3/2} \), we have

\[
\int_{\text{SO}(H)} |R_\varepsilon(hS)| \, dh = O(\varepsilon^{\frac{2q}{r+1-r}-q}), \quad \varepsilon \to 0.
\]

Next, we consider the group \( \text{SO}(V^\perp) \) of orthogonal transformations of \( \mathbb{R}^n \), which fix any vector of \( V \):

\[
\text{SO}(V^\perp) = \{ A \in \text{SO}(n) : A|_V = \text{Id} \}.
\]

**Theorem 1.5.** For any subset \( S \) of \( \mathbb{R}^n \) such that for any \( \gamma^* \in \Gamma^* \), the intersection \( P_{\gamma^*} \cap S \) is a bounded open set in \( P_{\gamma^*} \) with smooth boundary, we have

\[
\int_{\text{SO}(V^\perp)} |R_\varepsilon(hS)| \, dh = O(\varepsilon^{\frac{2q}{r+1-r}-q}), \quad \varepsilon \to 0.
\]

Finally, Theorem 1.5 easily implies the result on the average remainder estimates for the full group \( \text{SO}(n) \).
Theorem 1.6. For any bounded open set $S$ in $\mathbb{R}^n$ with smooth boundary, we have

$$\int_{SO(n)} |R_\varepsilon(hS)| \, dh = O(\varepsilon^{-q_{n-r+1}}), \quad \varepsilon \to 0.$$ 

Remark 1.7. It appears that, using the results of [1], one can relax the assumptions on smoothness of the boundary of $S$ in Theorems 1.5 and 1.6.

1.3. Applications to adiabatic limits. It is well known that the Gauss problem on counting integer points in the disk is equivalent to the problem on the asymptotic behavior of the eigenvalue distribution function of some elliptic differential operator on a compact manifold, namely, of the Laplace operator on a torus. In the case under consideration, there is also an equivalent asymptotic spectral problem, namely, the problem on the asymptotic behavior of the eigenvalue distribution function of the Laplace operator on a torus in the adiabatic limit associated with a linear foliation.

As above, let $F$ be a $p$-dimensional linear subspace of $\mathbb{R}^n$ and $H = F^\perp$. Consider the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $F$ be the associated linear foliation on $\mathbb{T}^n$: the leaf $L_x$ of $F$ through $x \in \mathbb{T}^n$ has the form:

$$L_x = x + F \mod \mathbb{Z}^n.$$ 

The decomposition of $\mathbb{R}^n$ into the direct sum of subspaces $\mathbb{R}^n = F \oplus H$ induces the decomposition $g = g_F + g_H$ of the standard Euclidean metric $g$ on $\mathbb{R}^n$ into the sum of the tangential and transversal components. Define a one-parameter family $g_\varepsilon$ of Euclidean metrics on $\mathbb{R}^n$ by

$$g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.$$ 

We will also consider the metrics $g_\varepsilon$ as Riemannian metrics on $\mathbb{T}^n$.

Let $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$. For any $\varepsilon > 0$, consider the operator $H_\varepsilon$ in $C^\infty(\mathbb{T}^n)$ defined in the standard linear coordinates $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ by

$$H_\varepsilon = \sum_{j,\ell=1}^{n} g_\varepsilon^{j\ell} \left( \frac{\partial}{\partial x_j} - 2\pi i a_j \right) \left( \frac{\partial}{\partial x_\ell} - 2\pi i a_\ell \right),$$

where $g_\varepsilon^{j\ell}$ are the elements of the inverse matrix of $g_\varepsilon$. The operator $H_\varepsilon$ can be considered as the magnetic Schrödinger operator on the torus $\mathbb{T}^n$, associated with the metric $g_\varepsilon$ and the constant magnetic potential $A = \sum_{j=1}^{n} a_j dx_j$. It has a complete orthogonal systems of eigenfunctions

$$U_k(x) = e^{2\pi i (k, x)}, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{Z}^n,$$

with the corresponding eigenvalues

$$\lambda_k = (2\pi)^2 \|k - A\|_{g_\varepsilon}^{-2} = (2\pi)^2 \sum_{j,\ell=1}^{n} g_\varepsilon^{j\ell} (k_j - a_j) (k_\ell - a_\ell).$$

Denote by $N_\varepsilon(\lambda)$ the eigenvalue distribution function of $H_\varepsilon$:

$$N_\varepsilon(\lambda) = \{ k \in \mathbb{Z}^n : \lambda_k < \lambda \}, \quad \lambda \in \mathbb{R}.$$
It is easy to see that
\[ n_\varepsilon(B_{\sqrt{\lambda}}(A)) = N_\varepsilon(4\pi^2\lambda), \quad \lambda \in \mathbb{R}. \]
Thus, the problem on the asymptotic behavior of the number \( n_\varepsilon(B_{\sqrt{\lambda}}(A)) \) of integer points in the ellipsoid \( T_\varepsilon(B_{\sqrt{\lambda}}(A)) \) as \( \varepsilon \to 0 \) is equivalent to the problem on the asymptotic behavior of the eigenvalue distribution function \( N_\varepsilon(\lambda) \) as \( \varepsilon \to 0 \). The limiting procedure \( \varepsilon \to 0 \) is often called passing to adiabatic limit. This notion was introduced by Witten in 1985 in the study of global anomalies in string theory. We refer the reader to a survey paper [7] for some historic remarks and references.

In [5] (see also [6]), the first author computed the leading term of the asymptotics of the eigenvalue distribution function of the Laplace operator associated with a bundle-like metric on a compact manifold equipped with a Riemannian foliation, in adiabatic limit. The linear foliation on the torus is a Riemannian foliation, and a Euclidean metric on the torus is bundle-like.

A more precise estimate of the remainder in the asymptotic formula of [5] in this particular case was obtained in [8]. As a straightforward consequence of Theorem 1.3, we improve the remainder estimate of [8].

**Theorem 1.8.** For \( \lambda > 0 \), the following asymptotic formula holds as \( \varepsilon \to 0 \):
\[
N_\varepsilon(\lambda) = \varepsilon^{-q} \frac{\omega_{n-r}}{\text{vol}(V/T)} \sum_{\gamma^* \in \Gamma^*} \left( \frac{\lambda}{4\pi^2} - |\gamma^* - A|^2 \right)^{(n-r)/2} + O(\varepsilon^{2q \frac{n-r+1}{r+1}}),
\]
where \( \omega_{n-r} \) is the volume of the unit ball in \( \mathbb{R}^{n-r} \).

2. **Proof of the main results**

2.1. **Proof of Theorem 1.3.** We will follow the proof of Theorem 1.2 in [8]. Therefore, we will skip some details, referring the interested reader to [8]. First of all, we observe that we have the inclusion
\[
\mathbb{Z}^n \subset \bigcup_{\gamma^* \in \Gamma^*} P_{\gamma^*}.
\]
For any \( \gamma^* \in \Gamma^* \), denote
\[
\mathbb{Z}^n_{\gamma^*} = \mathbb{Z}^n \cap P_{\gamma^*} = \{ k \in \mathbb{Z}^n : \pi_V(k) = \gamma^* \}.
\]
We identify the affine subspace \( P_{\gamma^*} \) with the linear space \( V^\perp \), fixing an arbitrary point \( k_{\gamma^*} \in \mathbb{Z}^n_{\gamma^*} \):
\[
P_{\gamma^*} = k_{\gamma^*} + V^\perp.
\]
It is easy to see that
\[
\mathbb{Z}^n_{\gamma^*} = k_{\gamma^*} + \Gamma^\perp,
\]
where
\[
\Gamma^\perp = \mathbb{Z}^n \cap V^\perp.
\]
is a lattice in $V^\perp$. Observe that
\begin{equation}
\text{vol}(V^\perp/\Gamma^\perp) = \text{vol}(V/\Gamma).
\end{equation}

Thus, we can write
\begin{equation}
n_\varepsilon(S) = \sum_{\gamma^* \in \Gamma^\perp} n_\varepsilon(S, \gamma^*),
\end{equation}
where
\[ n_\varepsilon(S, \gamma^*) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n_{\gamma^*}). \]

Note that, since $S$ is bounded, the sum in the right hand side of (2.2) has finitely many non-vanishing terms.

Fix $\gamma^* \in \Gamma^\ast$. Let $\chi_{S, \gamma^*}$ be the indicator of the set $S_{\gamma^*} = S \cap P_{\gamma^*}$. It is easy to see that
\[
n_\varepsilon(S, \gamma^*) = \sum_{\gamma \in \Gamma^\perp} \chi_{S_{\gamma^*}}(k_{\gamma^*} + (T_{\varepsilon-1}(k_{\gamma^*}) - k_{\gamma^*}) + T_{\varepsilon-1}(\gamma)).
\]

The space $V^\perp$ decomposes into the direct sum
\begin{equation}
V^\perp = F_V \bigoplus H,
\end{equation}
where $F_V = F \cap V^\perp$. We will write the decomposition of $x \in V^\perp$, corresponding to (2.3), as follows:
\[
x = x_F + x_H, \quad x_F \in F_V, x_H \in H.
\]

Note that
\[
T_\varepsilon(x) = x_F + \varepsilon^{-1}x_H.
\]

Let $\rho \in C^\infty_c(\mathbb{R})$ be an even function such that $0 \leq \rho(x) \leq 1$ for any $x \in \mathbb{R}$ and $\text{supp} \rho \subset (-1, 1)$. For any $t_F > 0$ and $t_H > 0$, define a function $\rho_{t_F, t_H} \in C^\infty_0(V^\perp)$ by
\begin{equation}
\rho_{t_F, t_H}(x) = \frac{c}{t_F^{d_F} t_H^{d_H}} \rho \left( (t_F^{-2} x_F^2 + t_H^{-2} x_H^2)^{1/2} \right), \quad x \in V^\perp,
\end{equation}
where the constant $c > 0$ is chosen so that $\int_{V^\perp} \rho_{1,1}(x) \, dx = 1$. The function $\rho_{t_F, t_H}$ is supported in the ellipsoid
\[ B(0, t_F, t_H) = \left\{ x \in V^\perp : \frac{x_F^2}{t_F^2} + \frac{x_H^2}{t_H^2} < 1 \right\}. \]

Define the function $n_{\varepsilon, t_F, t_H}(S, \gamma^*)$ by
\[
n_{\varepsilon, t_F, t_H}(S, \gamma^*) = \sum_{k \in \mathbb{Z}^n_{\gamma^*}} (\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H})(k),
\]
where the function $\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H} \in C^\infty_0(P_{\gamma^*})$ is defined by
\[
(\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H})(y) = \int_{V^\perp} \chi_{T_\varepsilon(S_{\gamma^*})}(y - x) \rho_{t_F, t_H}(x) \, dx, \quad y \in P_{\gamma^*}.
\]
For any domain $D \subset P_{\gamma^*}$ and for any $t_F > 0$ and $t_H > 0$, denote
\[ D_{t_F, t_H} = \bigcup_{x \in D} (x + B(0, t_F, t_H)), \]
and
\[ D_{-t_F, -t_H} = P_{\gamma^*} \setminus (P_{\gamma^*} \setminus D)_{t_F, t_H}. \]
It is easy to see that, for any $\varepsilon > 0$, $t_F > 0$ and $t_H > 0$, one has
\[ T_{\varepsilon}(D_{t_F, t_H}) = (T_{\varepsilon}(D))_{t_F, t_H}. \]

**Lemma 2.1.** For any $\varepsilon > 0$, $t_F > 0$ and $t_H > 0$, we hav \[ n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) \leq n_{\varepsilon}(S, \gamma^*) \leq n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*). \]
For any $f \in S(V^\perp)$, define its Fourier transform $\hat{f} \in S(V^\perp)$ by\[ \hat{f}(\xi) = \int_{V^\perp} e^{-2\pi i (\xi, x)} f(x) \, dx. \]
Recall the Poisson summation formula
\[ \sum_{k \in \Gamma^\perp} f(k) = \frac{1}{\text{vol}(V/\Gamma)} \sum_{k' \in \Gamma'^{\perp}} \hat{f}(k'), \quad f \in S(V^\perp), \]
where $\Gamma'^{\perp} \subset V^\perp$ is the dual lattice of $\Gamma^\perp$, and we used the relation (2.1).
For any $N > 0$, we have the estimate
\[ |\hat{\rho}_{t_F, t_H}(\xi)| \leq C_N \frac{1}{1 + t_F|\xi|^N + t_H|\xi|^N}, \quad \xi \in V^\perp. \]
Therefore, we can apply (2.3) to the function
\[ f(x) = (\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, t_H})} \ast \rho_{t_F, t_H})(k_{\gamma^*} + x), \quad x \in V^\perp. \]
Using the relations
\[ \hat{T}_{\varepsilon}((S_{\gamma^*})_{t_F, t_H})(\xi) = e^{-q} e^{2\pi i \varepsilon((1 - T_{\varepsilon})(k_{\gamma^*}))} \hat{\chi}_{(S_{\gamma^*})_{t_F, t_H}}(T_{\varepsilon}(\xi)), \] and
\[ \hat{\rho}_{t_F, t_H}(\xi) = \hat{\rho}_{1,1}(t_F \xi_F + t_H \xi_H), \quad \xi \in V^\perp, \]
we obtain
\[ n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = \sum_{k \in \mathbb{Z}_{\gamma^*}} \chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, t_H})} \ast \rho_{t_F, t_H}(k) \]
\[ = \frac{e^{-q}}{\text{vol}(V/\Gamma)} \sum_{k \in \Gamma'^{\perp}} e^{2\pi i (k, (1 - T_{\varepsilon})(k_{\gamma^*}))} \hat{\chi}_{(S_{\gamma^*})_{t_F, t_H}}(T_{\varepsilon}(k)) \hat{\rho}_{1,1}(t_F k_F + t_H k_H). \]
We can write
\[ n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) + n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*), \]
where

\[ n_{\epsilon, t, \gamma}^f \left( (S_{\gamma^*})_{t_F, eH}, \gamma^* \right) = \frac{\epsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{k \in \Gamma^{i_*}, k_H = 0} \hat{\chi}(S_{\gamma^*})_{t_F, eH} (T_{\epsilon}(k)) \hat{\rho}_{1,1} (t_F k_F + t_H k_H), \]

and

\[ n_{\epsilon, t, \gamma}^a \left( (S_{\gamma^*})_{t_F, eH}, \gamma^* \right) = \frac{\epsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{k \in \Gamma^{i_*}, k_H \neq 0} \epsilon^{2\pi i (1-\epsilon^{-1})(k_H, k_{\gamma^*})} \hat{\chi}(S_{\gamma^*})_{t_F, eH} (T_{\epsilon}(k)) \hat{\rho}_{1,1} (t_F k_F + t_H k_H). \]

Let \( k \in \Gamma^{i_*} \) be such that \( k_H = 0 \). Then \( k \in F_V \). Since \( \Gamma^{i_*} \subset \mathbb{Q}^n \) and \( F_V \cap \mathbb{Q}^n = \{0\} \), we get \( k = 0 \). Thus, we have

\[ n_{\epsilon, t, \gamma}^f \left( (S_{\gamma^*})_{t_F, eH}, \gamma^* \right) = \frac{\epsilon^{-q}}{\text{vol}(V/\Gamma)} \text{vol}_{n-r}(P_{\gamma^*} \cap S) \]

\[ + \frac{\epsilon^{-q}}{\text{vol}(V/\Gamma)} \text{vol}_{n-r}((S_{\gamma^*})_{t_F, eH} \setminus S_{\gamma^*}). \]

Since

\[ \text{vol}_{n-r}((S_{\gamma^*})_{t_F, eH} \setminus S_{\gamma^*}) \leq C(t_F + t_H \epsilon), \]

we obtain that

\[ (2.9) \quad n_{\epsilon, t, \gamma}^f \left( (S_{\gamma^*})_{t_F, eH}, \gamma^* \right) = \frac{\epsilon^{-q}}{\text{vol}(V/\Gamma)} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(t_F \epsilon^{-q} + t_H \epsilon^{1-q}). \]

Consider the case when \( k \in \Gamma^{i_*} \) and \( k_H \neq 0 \). Here the arguments depend on the conditions on the domain \( S \) we have.

(1) Assume that for any \( x \in F \) the domain \( S \cap \{x + H \} \) is strictly convex.

For any \( t \in F_V \) and for any domain \( D \subset P_{\gamma^*} \), we denote

\[ D(t) = \{x_H \in H : k_{\gamma^*} + t + x_H \in D \} \subset H. \]

For any function \( \phi \in S(H) \), denote by \( F_H(\phi) \in S(H) \) its Fourier transform:

\[ [F_H(\phi)](\xi_H) = \int_H \phi(x_H) e^{-2\pi i (\xi_H, x_H)} \, dx_H, \quad \xi_H \in H. \]

It is easy to see that

\[ \hat{\chi}(S_{\gamma^*})_{t_F, eH} (T_{\epsilon}(k)) = \int_{F_V} e^{-2\pi i (k_F, x_F)} F_H[\chi(S_{\gamma^*})_{t_F, eH}(x_F)](\epsilon^{-1} k_H) \, dx_F, \]

and, therefore.

\[ (2.10) \quad |\hat{\chi}(S_{\gamma^*})_{t_F, eH} (T_{\epsilon}(k))| \leq \int_{F_V} |F_H[\chi(S_{\gamma^*})_{t_F, eH}(x_F)](\epsilon^{-1} k_H)| \, dx_F. \]
Thus, using the estimate (2.10), (2.11) and (2.6), we obtain that
\begin{equation}
|F_H[\chi(S_{\gamma^*})_{t_F,\varepsilon t_H}(x_F)](\xi)| = O(|\xi|^{-(q+1)/2}), \quad |\xi| \to \infty.
\end{equation}

Thus, using the estimate (2.10), (2.11) and (2.6), we obtain that
\begin{equation}
|n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*)| 
\leq C\varepsilon^{-q} \sum_{k \in \Gamma^{-1}, k_H \neq 0} \varepsilon^{(q+1)/2}|k_H|^{-(q+1)/2} \frac{1}{1 + t_F^N|k_F|^N + t_H^N|k_H|^N}
\leq C\varepsilon^{-q}\varepsilon^{(q+1)/2} \int_{V_{\perp}} |x_H|^{-(q+1)/2} \frac{dx_F dx_H}{1 + t_F^N|x_F|^N + t_H^N|x_H|^N}
\leq C\varepsilon^{-(q-1)/2} t_F^{-(p-r)} t_H^{-(q-1)/2}.
\end{equation}

Put $t_F = \varepsilon^{\alpha_F}$, $t_H = \varepsilon^{\alpha_H}$, where
\begin{align*}
\alpha_F &= \frac{2q}{q + 1 + 2(p - r)}, \quad \alpha_H = \frac{q - 1 - 2(p - r)}{q + 1 + 2(p - r)}.
\end{align*}

Using the estimates (2.9) and (2.12), we immediately conclude the proof of the statement (1).

(2) Assume that $P_{\gamma^*} \cap S$ is strictly convex. Then we have
\begin{equation}
|\hat{\chi}(S_{\gamma^*})_{t_F, \varepsilon t_H}(T_{\varepsilon}(k))| \leq C(|k_F| + |\varepsilon^{-1}k_H|)^{-\frac{n-r+1}{2}}.
\end{equation}

Using this fact, as in (2.12), we obtain
\begin{equation}
|n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*)| \leq C\varepsilon^{-q}\varepsilon^{(n-r+1)/2} t_F^{-(p-r)} t_H^{-(n-r+1)/2}.
\end{equation}

To complete the proof of the statement (2), we put $t_F = \varepsilon^{\alpha_F}$, $t_H = \varepsilon^{\alpha_H}$, where $\alpha_F \geq 0$ and $\alpha_H \geq -1$ are given by
\begin{align*}
\alpha_F &= \frac{2q}{n - r + 1}, \quad \alpha_H = \frac{q - p + r - 1}{n - r + 1}.
\end{align*}

2.2. Proof of Theorem 1.4. Let $h \in \text{SO}(H)$. We apply the formula (2.8) to the set $h(S)$. Since $T_{\varepsilon}h = hT_{\varepsilon}$, we have
\begin{equation}
(h(S)_{\gamma^*})_{t_F, \varepsilon t_H} = (h(S_{\gamma^*}))_{t_F, \varepsilon t_H} = h((S_{\gamma^*})_{t_F, \varepsilon t_H}).
\end{equation}

Therefore, the formula (2.8) reads as
\begin{equation}
n_{\varepsilon, t_F, t_H}((h(S)_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*)
= n_{\varepsilon, t_F, t_H}((h(S)_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) + n_{\varepsilon, t_F, t_H}((h(S)_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*),
\end{equation}
where, since $h$ preserves the volume in $\mathbb{R}^n$, the first term is independent of $h$ and satisfies the estimate
\begin{align*}
n_{\varepsilon, t_F, t_H}((h(S)_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) &= \frac{\varepsilon^{-q}}{\text{vol}(V/\Gamma)} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(t_F \varepsilon^{-q} + t_H \varepsilon^{-q}),
\end{align*}
and

\[
\eta_{\varepsilon, t_F, t_H}^\nu((h(S)_{\gamma^*})_{t_F, t_H}; \gamma^*) = \frac{\varepsilon^{-q}}{\text{vol}(V/\Gamma)} \sum_{k \in \Gamma^+, k_H \neq 0} e^{2\pi i (k_H(1-\varepsilon^{-1}h)k_{\gamma^*})} \times \nonumber
\]

\[
\times \hat{\chi}(S_{\gamma^*})_{t_F, t_H} (h^T_{\varepsilon}(k)) \hat{\rho}_{1,1} (t_F k_F + t_H k_H). \nonumber
\]

Here \(h^t = h^{-1}\) denotes the transpose of \(h\).

Consider the case when \(k \in \Gamma^+\) and \(k_H \neq 0\). We will keep notation used in the previous subsection. As in (2.10), we have

\[
|\hat{\chi}(S_{\gamma^*})_{t_F, t_H} (h^T_{\varepsilon}(k))| \leq \int_{F_V} |F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\varepsilon^{-1}h^t k_H)| dx_F. \nonumber
\]

Using the results of [1], we get

\[
\int_{\text{SO}(H)} |F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\varepsilon^{-1}h^t k_H)| dh \nonumber
\]

\[
\leq \left( \int_{\text{SO}(H)} |F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\varepsilon^{-1}h^t k_H)|^2 dh \right)^{1/2} = O(\varepsilon^{(q+1)/2}). \nonumber
\]

Proceeding as above (cf. (2.12)), we immediately complete the proof.

2.3. Proof of Theorem 1.5. Now we assume that \(h \in \text{SO}(V^\perp)\). Then we still have

\[
h(S)_{\gamma^*} = h(S_{\gamma^*}), \nonumber
\]

but the equality

\[
(h(S_{\gamma^*}))_{t_F, t_H} = h((S_{\gamma^*})_{t_F, t_H}) \nonumber
\]

only if

\[
t_F = \varepsilon t_H = t. \nonumber
\]

Therefore, the formula (2.14) holds for such \(t_F\) and \(t_H\).

Let \(k \in \Gamma^+\) with \(k_H \neq 0\). Then we have

\[
\hat{\chi}(S_{\gamma^*})_{t_F, t_H} (h^T_{\varepsilon}(k)) = \int_{V_\perp} \chi(S_{\gamma^*})_{t_F, t_H} (k_{\gamma^*} + x) e^{-2\pi i (h^T_{\varepsilon}(k), x)} dx \nonumber
\]

\[
= \int_{D_\varepsilon} e^{-2\pi i (T_{\varepsilon}(k), h(x-k_{\gamma^*}))} dx. \nonumber
\]

where, for simplicity of notation, we put \(D_\varepsilon = (S_{\gamma^*})_{t_F, t_H}\). By Stokes’ formula, we obtain

\[
\hat{\chi}(S_{\gamma^*})_{t_F, t_H} (h^T_{\varepsilon}(k)) = -\frac{1}{2\pi i |T_{\varepsilon}(k)|} \int_{\partial D_\varepsilon} e^{-2\pi i (T_{\varepsilon}(k), h(x-k_{\gamma^*}))} h^t n_x dx, \nonumber
\]
where \( n_\varepsilon = \frac{T_\varepsilon(k)}{|T_\varepsilon(k)|} \). We can write
\[
\int_{SO(V^\perp)} |\tilde{\chi}(S_{r^+})_{t_F,ct_H}(h^T_\varepsilon(k))|^2 \, dh = \frac{1}{4\pi^2|T_\varepsilon(k)|^2} \int_{\partial D_\varepsilon} \int_{\partial D_\varepsilon} \int_{SO(V^\perp)} e^{-2\pi i(k_F+\varepsilon-1h_F,h(x-y))} h^t_\varepsilon n_\varepsilon \, dx \wedge \eta h^t_\varepsilon n_\varepsilon \, dy \wedge dh.
\]

So this is an oscillating integral with the phase \( \Phi(x,y,h) = (k_H,h(x-y)) \), \( x,y \in \partial D_\varepsilon \subset V^\perp \), \( h \in SO(V^\perp) \).

If \((x_0,y_0,h_0) \in \partial D_\varepsilon \times \partial D_\varepsilon \times SO(V^\perp)\) is a critical point of \( \Phi \), then we have:

- for any \( v \in T_{x_0}\partial D_\varepsilon \) and \( w \in T_{y_0}\partial D_\varepsilon \)
  \[
  (k_H,h_0v) = (h_0^t k_H,v) = 0, \quad (k_H,h_0w) = (h_0^t k_H,w) = 0;
  \]
- for any \( X \in so(V^\perp) \)
  \[
  (k_H,h_0 X(x_0 - y_0)) = (h_0^t k_H, X(x_0 - y_0)) = 0.
  \]

By \( (2.15) \), it follows that \( x_0 - y_0 = \alpha h_0^t k_H \) with some \( \alpha \in \mathbb{R} \).

As in \[16\] Lemma 1, we have that at any critical point of \( \Phi \) on \( \partial D_\varepsilon \times \partial D_\varepsilon \times SO(V^\perp) \) the rank of its second differential is at least \( 2(n-r) - 2 \), that implies by a slight modification of \[16\] Lemma 2 that
\[
\int_{SO(V^\perp)} |\tilde{\chi}(S_{r^+})_{t_F,ct_H}(h^T_\varepsilon(k))|^2 \, dh \leq C|\varepsilon-1k_H|^{-(n-r+1)},
\]
and therefore
\[
\int_{SO(V^\perp)} |\tilde{\chi}(S_{r^+})_{t_F,ct_H}(h^T_\varepsilon(k))| \, dh \leq C|\varepsilon-1k_H|^{-(n-r+1)/2},
\]
As in \( (2.13) \), we obtain
\[
\int_{SO(V^\perp)} |n''_{t_F,ct_H}(h(S)_{\gamma^+})_{t_F,ct_H} \, \gamma^+| \, dh \\
\quad \leq C\varepsilon^{-q(n-r+1)/2} t_F^{-(p-r)/2} t_H^{(n-r+1)/2} = Ct^{-(n-r-1)/2}.
\]

To complete the proof, we put \( t = \varepsilon^\alpha \), where
\[
\alpha = \frac{2q}{n-r+1}.
\]

2.4. **Proof of Theorem 1.6.** First, we observe that there exists an invariant measure \( dh \) on the homogeneous space \( SO(V^\perp) \setminus SO(n) \) such that
\[
\int_{SO(n)} |R_\varepsilon(hS)| \, dh = \int_{SO(V^\perp) \setminus SO(n)} \left( \int_{SO(V^\perp)} |R_\varepsilon(h_1 h_2 S)| \, dh_1 \right) dh_2.
\]

For any \( h_2 \in SO(n) \), by Theorem [15] applied to the domain \( h_2 S \), we have
\[
\int_{SO(V^\perp)} |R_\varepsilon(h_1 h_2 S)| \, dh_1 \leq C\varepsilon^{-\frac{2q}{n-r+1}q}, \quad \varepsilon > 0.
\]
Moreover, it is easy to see from [16, Lemma 2] that the constant $C > 0$ can be chosen independent of $h_2 \in SO(n)$, that immediately completes the proof of Theorem 1.6.

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Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky str., 450008 Ufa, Russia

*E-mail address:* yurikor@matem.anrb.ru

Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky str., 450008 Ufa, Russia

*E-mail address:* yakovlevandrey@yandex.ru