Updating the Number of Crossings of Complete Geometric Graphs

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Abstract

Let \( S \) be a set of \( n \) points in general position in the plane. Let \( \text{cr}(S) \) be the number of pairs of edges that cross in a rectilinear drawing of the complete graph with \( S \) as its vertex set. Suppose that this number is known. In this paper we consider the problem computing \( \text{cr}(S') \), where \( S' \) comes from adding, removing or moving a point from \( S \).

1 Introduction

Let \( G \) be a graph on \( n \) vertices; let \( S \) be a set of \( n \) points in general position (no three of them collinear) in the plane. A rectilinear drawing of \( G \) is a drawing of \( G \) in the plane that satisfies the following. Its vertices are points in general position and its edges are drawn as straight line segments. The number of crossings of a rectilinear drawing is the number of pairs of its edges that cross. The rectilinear crossing number of \( G \) is the minimum number of crossings over all rectilinear drawings of \( G \); we denote it by \( \text{cr}(G) \). In the case that \( G \) is a complete graph, note that the number of crossings in a rectilinear drawing of \( G \), depends only on the position of its vertices. Let \( \text{cr}(S) \) be the number of crossings in a rectilinear drawing of the complete graph \( K_n \) with \( S \) as its vertex set. Therefore,

\[
\text{cr}(K_n) = \min\{\text{cr}(S) : S \text{ is a set of } n \text{ points in general position in the plane}\}.
\]

Since this value only depends on \( n \), for brevity we refer to \( \text{cr}(K_n) \) as \( \text{cr}(n) \). The current best lower and upper bounds on \( \text{cr}(n) \) are

\[
0.379972\binom{n}{4} < \text{cr}(n) < 0.380473\binom{n}{4} + \Theta(n^3).
\]
The lower bound was given by Ábrego Fernández-Merchant, Leaños and Salazar [AFMLS08]; the upper bound was given by Fabila-Monroy and López [FL14]. Historically, the upper bounds on $\text{cr}(n)$ have been given by finding arbitrarily large constructions with small rectilinear crossing number. These constructions start with a “small” set with few crossing and from this point set produce arbitrarily large point sets with few crossings. This approach has been refined over the years [Sin71, BDG03, AAK06, ´AFM07, ´ACFM +10]. With the current best such construction being that of Ábrego, Cetina, Fernández-Merchant, Leaños, and Salazar [ACFM+10].

In [FL14] a simple heuristic was used to improve many of the the best known sets of 27, . . . , 100 points with few crossings. In particular, they found a set of 75 points with 450492 crossings. This point set together with the construction of [ACFM+10] provide the current best upper bound on $\text{cr}(n)$.

The heuristic used in [FL14] is as follows. Choose a random point $p$ of $S$ and a random point $q$ near $p$. Afterwards, compute $\text{cr}(S \setminus \{p\} \cup \{q\})$. If this number is less or equal to $\text{cr}(S)$ then replace $p$ with $q$ in $S$. The improvements obtained in [FL14] were done by many iterations of this procedure. Experimentally, it seems that heuristics of these type work well in practice; recently in [BK15], Balko and Kyncl used simulated annealing to improve the best upper bound on $K_n$ of a parameter similar to the rectilinear crossing number called the pseudolinear crossing number.

The computation of $\text{cr}(S)$ can be done in $O(n^2)$ time. Since in the heuristic of [FL14] only one point of $S$ changes at each step, it is reasonable to consider the following question.

**Problem 1** Suppose that $S'$ comes from $S$ by moving a point. Can $\text{cr}(S')$ be computed in $o(n^2)$ time assuming that $\text{cr}(S)$ is already known?

We have also observed experimentally that removing or adding a point from a point set with few crossings tends to produce a point set with few crossings. Thus, we also consider the following two algorithmic questions.

**Problem 2** What is the time complexity of computing all the values of $\text{cr}(S \setminus \{p\})$ for every point $p \in S$?

**Problem 3** Let $C$ be a set of points disjoint from $S$, such that $S \cup C$ is in general position. What is the time complexity of computing the values of $\text{cr}(S \cup \{p\})$ for every point $p \in C$?

In this paper we prove the following theorems related to these problems, respectively.

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1These point sets were obtained from Oswin Aichholzer’s page [http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/)](http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/)
Theorem 1 Let $S$ be a set of $n$ points in the general position in the plane; let $C$ be a set of $O(n)$ points in the plane disjoint from $S$, such that $S \cup C$ is in general position. Let $p$ be a point in $S$. Then the set of values
\[ \{ \cr(S') : S' = S \setminus \{ p \} \cup \{ q \}, q \in C \} \]
can be computed in $O(n^2)$ time.

Theorem 2 Let $S$ be a set of $n$ points set in general position. Then the set of values
\[ \{ \cr(S \setminus \{ p \}) : p \in S \} \]
can be computed in $O(n^2)$ time.

Theorem 3 Let $S$ be a set of $n$ points in the general position in the plane; let $C$ be a set of $O(n)$ points in the plane disjoint from $S$, such that $S \cup C$ is in general position. Then the set of values
\[ \{ \cr(S \cup \{ q \}) : q \in C \} \]
can be computed in $O(n^2)$ time.

Note that in each of these theorems the amortized time per point is linear.

We implemented the algorithms implied by Theorems 1, 2 and 3. We used these implementations and the heuristic of [FL14] to improve many of the known sets for $n = 27, \ldots, 100$. Unfortunately, none of these sets is sufficient to improve the asymptotic value of $\cr(n)$. Our implementations are available at www.pydcg.org.

This paper organized as follows. In Section 2 we introduce some preliminaries. In Section 3 we prove Theorems 1, 2, and 3. In Section 4 we present a table with new upper bounds on $\cr(n)$ for some values of $n \leq 100$.

2 Preliminaries

In this section we prove a pair of lemmas that will be used to prove Theorems 1, 2, and 3. Afterwards, we recall the concept of the $\lambda$-matrix of a point set, and give a characterization the number of crossings of a point set in terms of its $\lambda$-matrix.

Lemma 4 Let $p$ be a point not in $S$. For every point $q$ in $S$, let $S(q)$ be the set of points in $S$ to the left of the directed line from $p$ to $q$. Suppose that each point $r$ in $S$ has a weight $w(r)$ assigned to it. If the counterclockwise order of the points in $S$ around $p$ is known. Then the following set of values can be computed in linear time.
\[ \left\{ \sum_{r \in S(q)} w(r) : q \in S \right\} \]
Proof. Let \( q_1 \) be a point of \( S \). Let \( \ell \) be a directed line from \( p \) to \( q_1 \). Rotate \( \ell \) counterclockwise around \( p \), and let \((q_1, \ldots, q_n)\) be the points of \( S \) in the order that \( \ell \) encounters them during this rotation. This order can be computed from the counterclockwise order of the points in \( S \) around \( p \) in \( O(n) \) time. Compute \( \sum_{r \in S} w(r) \) in \( O(n) \) time. Since \( \sum_{r \in S} w(r) \) can be computed from \( \sum_{r \in S(q_i)} w(r) \) in constant time, the result follows. \( \square \)

**Lemma 5** The set counterclockwise orders of \( S \setminus \{p\} \) around \( p \) for every \( p \) in \( S \) can be computed in \( O(n^2) \) time.

**Proof.** This is done by dualizing \( S \) to a set of \( n \) lines. The corresponding line arrangement can be constructed in time \( O(n^2) \) with standard algorithms. The clockwise orders of \( S \setminus \{p\} \) around each \( p \in S \) can then be extracted from the line arrangement in \( O(n^2) \) time. \( \square \)

**The \( \lambda \)-Matrix**

Let \( p \) and \( q \) be a pair of points not necessarily in \( S \). We denote by \( \lambda_S(p, q) \) the number of points of \( S \) that lie to the left of the directed line from \( p \) to \( q \); in the case that \( p = q \), we set \( \lambda_S(p, q) := 0 \). Let \( p_1, p_2, \ldots, p_n \) be the points in \( S \). The \( \lambda \)-matrix of \( S \) is the matrix whose \( (i, j) \)-entry is equal to \( \lambda_S(p_i, p_j) \). The following lemma is well known; it can be proven from Lemma 4 by assigning a weight equal to one to every point in \( S \).

**Lemma 6** The \( \lambda \)-matrix of \( S \) can be computed in \( O(n^2) \) time.

It is known that the \( \lambda \)-matrix of \( S \) determines \( \cr(S) \). This was shown independently by Lovász, Wagner, Welzl, and Wesztergombi [LL04], and by Ábrego and Fernández-Merchant [AFm03]. We now provide another characterization of \( \cr(S) \) in terms of the \( \lambda \)-matrix of \( S \). For two any finite sets of points \( P \) and \( Q \), define

\[
f_S(P, Q) := \sum_{p \in P} \sum_{q \in Q} \left( \frac{\lambda_S(p, q)}{2} \right).
\]

**Lemma 7**

\[
\cr(S) = f_S(S, S) - \frac{n(n-1)(n-2)(n-3)}{8}.
\]

**Proof.** Let \( p, q, r \) and \( s \) be four different points of \( S \). We call the tuple \((p, q, \{r, s\})\) a pattern. If the points \( r \) and \( s \) are both to the left of the directed line from \( p \) to \( q \), we say that \((p, q, \{r, s\})\) is a type A pattern, otherwise we call it a type B pattern. We denote by \( A(S) \) and \( B(S) \) the number of type A and type B patterns in \( S \), respectively.

Let \( P \) be a set of four points. If \( P \) is in convex position, then \( P \) determines 4 type A patterns and 8 type B patterns. If \( P \) is not in convex position, then \( P \) determines 3 type A patterns and 9 type B patterns. Let \( \square(S) \) denote the
number of subsets of $S$ of four points in convex position, and let $\triangle(S)$ denote the number of subsets of $S$ of four points not in convex position. Thus,

$$A(S) = 4\square(S) + 3\triangle(S) \text{ and } B(S) = 8\square(S) + 9\triangle(S).$$

Note that $A(S) + B(S) = n(n-1)(n-2)(n-3)/2$, $\mathfrak{cr}(S) = \square(S)$ and

$$A(S) = \sum_{p,q \in S} \binom{\lambda_S(p, q)}{2}.$$

Therefore,

$$\mathfrak{cr}(S) = A(S) - \frac{(A(S) + B(S))/4}{n(n-1)(n-2)(n-3)/8}.\tag{□}$$

### 3 Proof of Theorems 1, 2 and 3

#### Proof of Theorem 2

For every point $p \in S$, compute the clockwise order of $S \setminus \{p\}$ around $p$; afterwards, compute the $\lambda$-matrix of $S$. By Lemmas 5 and 6 this can be done in $O(n^2)$ time. Using the $\lambda$-matrix of $S$ compute $f_S(S, S)$, $\{f_S(\{p\}, S) \colon p \in S\}$ and $\{f_S(S, \{p\}) : p \in S\}$ in $O(n^2)$ time.

Note that by Lemma 7 for every $p \in S$ we have that

$$\mathfrak{cr}(S \setminus \{p\}) = f_{S \setminus \{p\}}(S \setminus \{p\}, S \setminus \{p\}) - \frac{(n-1)(n-2)(n-3)(n-4)}{8}.$$

Thus, it is enough to compute $\{f_{S \setminus \{p\}}(S \setminus \{p\}, S \setminus \{p\}) : p \in S\}$ in $O(n^2)$ time, For every $p \in S$, let

$$\nabla_p := f_{S \setminus \{p\}}(S \setminus \{p\}, S \setminus \{p\}) - f_S(S, S) + f_S(\{p\}, S) + f_S(S, \{p\}).$$

To compute $\{f_{S \setminus \{p\}}(S \setminus \{p\}, S \setminus \{p\}) : p \in S\}$ in $O(n^2)$ time, we compute $\{\nabla_p : p \in S\}$ in $O(n^2)$ time. Note that

$$\nabla_p = \sum_{q \in S \setminus \{p\}} \sum_{r \in S \setminus \{p\}} \left( \binom{\lambda_{S \setminus \{p\}}(q, r)}{2} - \binom{\lambda_S(q, r)}{2} \right).$$

Let $r$ be a point in $S \setminus \{p\}$. Note that

$$\binom{\lambda_{S \setminus \{p\}}(q, r)}{2} - \binom{\lambda_S(q, r)}{2} = 0.$$
if \( p \) is to the right of the directed line from \( q \) to \( r \), and
\[
\left( \frac{\lambda_{S\setminus\{p\}}(q,r)}{2} \right) - \left( \frac{\lambda_S(q,r)}{2} \right) = 1 - \lambda_S(q,r)
\]
if \( p \) is to the left of the directed line from \( q \) to \( r \). Moreover, \( p \) is to the left of the directed line from \( q \) to \( r \) if and only if \( r \) is to the left of the directed line from \( p \) to \( q \).

For every point \( q \in S \) do the following. To every point \( r \in S \setminus \{q\} \) assign the weight \( w_q(r) = 1 - \lambda_S(q,r) \). For every \( p \in S \setminus \{q\} \), let \( S_p(q) \) be the set of points of \( S \) to the left of the directed line from \( p \) to \( q \). Thus,
\[
\sum_{r \in S\setminus\{p\}} \left( \left( \frac{\lambda_{S\setminus\{p\}}(q,r)}{2} \right) - \left( \frac{\lambda_S(q,r)}{2} \right) \right) = \sum_{r \in S_p(q)} w_q(r).
\]
By Lemma \( 4 \) for a fixed \( q \in S \), the set of values
\[
\left\{ \sum_{r \in S_p(q)} w_q(r) : p \in S \setminus \{q\} \right\}
\]
can be computed in linear time. This implies that the set
\[
\left\{ \left\{ \sum_{r \in S_p(q)} w_q(r) : p \in S \setminus \{q\} \right\} : q \in S \right\}
\]
can be computed in \( O(n^2) \) time. Therefore,
\[
\{ \nabla_p : p \in C \}
\]
can be computed in \( O(n^2) \) time; the result follows.

**Proof of Theorem 3**

For every point \( p \in S \cup C \), compute the clockwise order of \((S \cup C) \setminus \{p\}\) around \( p \); by Lemma \( 5 \) this can be done in \( O(n^2) \) time. For every pair of points \( p, q \in S \cup C \), let \( H(p,q) \) be the subset of points of \( S \cup C \) to the left of the directed line from \( p \) to \( q \). To every point \( p \) in \( S \cup C \) assign a weight of \( w(p) = 1 \) if \( p \) is in \( S \), and a weight of \( w(p) = 0 \) if \( p \) is in \( C \). Use Lemma \( 4 \) to compute the set of values
\[
\left\{ \sum_{r \in H(p,q)} w(r) : p, q \in S \cup C \right\}
\]
in \( O(n^2) \) time. Note that for every pair of points \( p, q \in S \cup C \) we have that
\[
\lambda_S(p,q) = \sum_{r \in H(p,q)} w(r).
\]
Therefore, $f_S(S, S), \{f_S(p, S \cup \{p\}) : p \in S\}$ and $\{f_S(S \cup \{p\}, \{p\}) : p \in S\}$ can be computed in $O(n^2)$ time.

Note that by Lemma[7] for every $p \in C$ we have that

$$\mathcal{O}(S \cup \{p\}) = f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) - \frac{n(n + 1)(n - 1)(n - 2)}{8}.$$  

Thus, it is enough to compute $\{f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) : p \in C\}$ in $O(n^2)$ time. For every $p \in C$, let

$$\nabla_p := f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) - f_S(S, S) - f_S(\{p\}, S) - f_S(S, \{p\}).$$

To compute $\{f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) : p \in C\}$ in $O(n^2)$ time, we compute $\{\nabla_p : p \in S\}$ in $O(n^2)$ time. Note that

$$\nabla_p = f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) - f_S(S, S) - f_S(\{p\}, S) - f_S(S, \{p\})$$

$$= f_{S \cup \{p\}}(S \cup \{p\}, S \cup \{p\}) - f_S(S, S) - f_{S \cup \{p\}}(\{p\}, S) - f_{S \cup \{p\}}(S, \{p\})$$

$$= \sum_{q \in S} \sum_{r \in S} \left( \frac{\lambda_{S \cup \{p\}}(q, r)}{2} - \frac{\lambda_S(q, r)}{2} \right).$$

Let $r$ be a point in $S$. Note that

$$\left( \frac{\lambda_{S \cup \{p\}}(q, r)}{2} \right) - \left( \frac{\lambda_S(q, r)}{2} \right) = 0$$

if $p$ is to the right of the directed line from $q$ to $r$, and

$$\left( \frac{\lambda_{S \cup \{p\}}(q, r)}{2} \right) - \left( \frac{\lambda_S(q, r)}{2} \right) = \lambda_S(q, r)$$

if $p$ is to the left of the directed line from $q$ to $r$. Moreover, $p$ is to the left of the directed line from $q$ to $r$ if and only if $r$ is to the left of the directed line from $p$ to $q$.

For every point $q \in S$ do the following. To every point $r \in S$ assign the weight $w_q(r) = \lambda_S(q, r)$. For every point $p \in C$, let $S_p(q)$ be the set of points of $S$ to the left of the directed line from $p$ to $q$. Thus,

$$\sum_{r \in S_p(q)} \left( \left( \frac{\lambda_{S \cup \{p\}}(q, r)}{2} \right) - \left( \frac{\lambda_S(q, r)}{2} \right) \right) = \sum_{r \in S_p(q)} w_q(r).$$

By Lemma[3] for a fixed $q$, the set of values

$$\left\{ \sum_{r \in S_p(q)} w_q(r) : p \in C \right\}$$
can be computed in linear time. This implies that the set
\[
\left\{ \sum_{r \in S_p(q)} w_q(r) : p \in C \right\} : q \in S
\]
can be computed in \(O(n^2)\) time. Therefore,
\[
\{ \nabla_p : p \in C \}
\]
can be computed in \(O(n^2)\) time; the result follows.

**Proof of Theorem 1**

For this it is enough to apply Theorem 3 with \(S \setminus \{p\}\) as the starting set of points, and \(C\) as the set of possible new points.

### 4 New small sets with few crossings

| \(n\) | \(\overline{cr}(n) \leq\) | \(n\) | \(\overline{cr}(n) \leq\) | \(n\) | \(\overline{cr}(n) \leq\) | \(n\) | \(\overline{cr}(n) \leq\) |
|-----|----------------|-----|----------------|-----|----------------|-----|----------------|
| 70  | 339252         | 80  | 587284         | 88  | 867887         | 96  | 1238898        |
| 71  | 59645          | 81  | 617952         | 89  | 909846         | 97  | 1292664        |
| 72  | 380934         | 82  | 649861         | 90  | 951383         | 98  | 1348066        |
| 74  | 426411         | 83  | 682982         | 91  | 995484         | 99  | 1405050        |
| 76  | 477778         | 84  | 717278         | 92  | 1040952        | 100 | 1463967        |
| 77  | 502021         | 85  | 753011         | 93  | 1087919        |     |                |
| 78  | 529284         | 86  | 789919         | 94  | 1136592        |     |                |
| 79  | 557743         | 87  | 828129         | 95  | 1187161        |     |                |

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