ANALYSIS IN $J_2$

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Abstract. This is an expository paper in which I explain how core mathematics, particularly abstract analysis, can be developed within a concrete countable set $J_2$ (the second set in Jensen’s constructible hierarchy). The implication, well-known to proof theorists but probably not to most mainstream mathematicians, is that ordinary mathematical practice does not require an enigmatic metaphysical universe of sets. I go further and argue that $J_2$ is a superior setting for normal mathematics because it is free of irrelevant set-theoretic pathologies and permits stronger formulations of existence results.

Perhaps many mathematicians would admit to harboring some feelings of discomfort about the ethereal quality of Cantorian set theory. Yet draconian alternatives such as intuitionism, which holds that simple number-theoretic statements like the twin primes conjecture may have no definite truth value, probably violate the typical working mathematician’s intuition far more severely than any vague unease he may feel about remote cardinals such as, say, $\aleph_{\omega}$.

I believe that ordinary mathematical practice is actually most compatible with an intermediate foundational stance I call conceptualism. This is a modernized version of the predicativist philosophy originally formulated by Henri Poincaré and Bertrand Russell, which treats elementary number theory from an essentially platonistic point of view but in the realm of set theory admits only those sets that can be constructed in a very concrete way. One consequence of this restriction is that on the conceptualist account all sets are countable. While this would seem to render the bulk of ordinary mathematics illegitimate, work begun by Hermann Weyl and continued by many others has gradually shown that in fact most if not all core mathematics can be developed within surprisingly minimalist systems. This point has been emphasized in $\mathbb{R}$. However, early work of this type tended to be presented in a somewhat abstruse or idiosyncratic way, while more recent work is generally couched in axiomatic frameworks that mainstream mathematicians might find unintuitive. The goal of this paper is to show how to develop core mathematics in a concrete countable domain called $J_2$ which should be relatively easily appreciated by mainstream mathematicians with no special training in logic. (The first two sections of the paper are probably the main hurdle; however, they might not need to be read in detail before proceeding to later sections.) Our central novelty is not any major technical advance but rather the choice of an approach that is closer in spirit to the classical style, and hence, perhaps, a more congenial environment for doing normal mathematics than one finds in previous work of this type (e.g., [1, 3, 4, 7, 11, 14, 15]). Since a really thorough exposition could fill several volumes I have chosen to present the foundational material at a fairly high level of detail, followed by a moderately detailed treatment of basic real analysis and successively more synoptic treatments of more advanced topics. I focus on abstract
analysis because it tends to have a more set-theoretic flavor than other general areas of mainstream mathematics, and hence it presents a greater challenge to being formalized in $J_2$.

Since conceptualism admits very little, if any, of the classical set-theoretic universe beyond what is used in ordinary mathematics, a case can be made that not only is it more compatible with the working mathematician’s intuition, it also provides a better fit with actual mathematical practice. Furthermore, existence results actually become stronger: in our setting for example, they become proofs of existence in $J_2$. An analogy can be drawn with the transition from the use of naive infinitesimals to the epsilon-delta method, where the primary motivation for making the switch is to eliminate reliance on ill-defined metaphysical entities, but a side benefit, apparent only after one has adopted the new view, is a genuinely deeper and more precise understanding of the subject matter. For these reasons I believe that conceptualist systems may come to be seen as a superior setting for doing normal mathematics.

I will make some brief comments about the philosophical content of conceptualism in §1.4 but otherwise will let the mathematical development speak for itself. For a more thorough discussion of the ideas of conceptualism including its philosophical justification, see [12] (for general readers) or [13] (for readers with some background in proof theory).

1. The mini-universe $J_2$

The main goal of this section is to introduce a countable set $J_2$ which will play the role of a miniature universe in which core mathematics can be developed. We also formulate a “first definability principle” according to which $J_2$ is closed under all normal mathematical constructions. We briefly discuss some philosophical motivation in §1.4.

1.1. Rudimentary functions. In order to define $J_2$ we need a way to construct new sets from an existing repertoire. This is done by means of a class of “rudimentary” functions. For more on this material see Chapter VI of [2].

**Definition 1.1.** Rudimentary functions from (tuples of) sets to sets are constructed according to the following conditions:

1. The functions
   
   \[
   \begin{align*}
   F(x_1, \ldots, x_n) & = x_i \\
   F(x_1, \ldots, x_n) & = \{x_i, x_j\} \\
   F(x_1, \ldots, x_n) & = x_i - x_j \quad \text{(set-theoretic difference)},
   \end{align*}
   \]

   where $1 \leq i, j \leq n$, are rudimentary.

2. If $G$, $H$, and $H_1, \ldots, H_k$ are rudimentary then so are the functions
   
   \[
   \begin{align*}
   F(x_1, \ldots, x_n) & = G(H_1(x_1, \ldots, x_n), \ldots, H_k(x_1, \ldots, x_n)) \\
   F(x_1, \ldots, x_n) & = \bigcup_{y \in \{x_1, \ldots, x_n\}} H(y, x_2, \ldots, x_n).
   \end{align*}
   \]

3. All rudimentary functions are generated by (1) and (2).
Definitions such as the preceding probably have little meaning the first time one sees them. Therefore, the reader is invited to build up his intuition by proving a few cases of the following proposition.

**Proposition 1.2.** ([2], Lemma VI.1.1) The following functions are rudimentary:

- \( F(x) = x \)
- \( F(x) = \bigcup_{y \in x} y \)
- \( F(x, y) = x \cup y \)
- \( F(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\} \)
- \( F(x, y) = \langle x, y \rangle = \{\{x\}, \{x, y\}\} \)
- \( F(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n \rangle = \langle \langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle \)
- \( F(x_1, \ldots, x_n) = \{G(y, x_2, \ldots, x_n) : y \in x_1\} \) if \( G \) is rudimentary
- \( F(x) = \{z \in \bigcup_{y \in x} y : z \in y \text{ for all } y \in x\} \)
- \( F(x, y) = x \cap y \)
- \( F(x) = \text{1st}(x) = \begin{cases} y & \text{if } x = \langle y, z \rangle \text{ for some } z \\ \emptyset & \text{otherwise} \end{cases} \)
- \( F(x) = \text{2nd}(x) = \begin{cases} z & \text{if } x = \langle y, z \rangle \text{ for some } y \\ \emptyset & \text{otherwise} \end{cases} \)
- \( F(x) = \text{dom}(x) = \{y : \langle y, z \rangle \in x \text{ for some } z\} \)
- \( F(x) = \text{ran}(x) = \{z : \langle y, z \rangle \in x \text{ for some } y\} \)
- \( F(x, y) = x \times y \)
- \( F(x) = x|_y = x \cap (y \times \text{ran}(x)) \)
- \( F(x) = \{(z, y) : \langle y, z \rangle \in x\} \).

Rudimentary functions have an alternative characterization which is slightly more concrete, if less elegant ([2], Lemma VI.1.11): a function is rudimentary if and only if it can be built up from the nine functions

- \( F_0(x, y) = \{x, y\} \)
- \( F_1(x, y) = x - y \)
- \( F_2(x, y) = x \times y \)
- \( F_3(x, y) = \{(u, z, v) : z \in x \text{ and } \langle u, v \rangle \in y\} \)
- \( F_4(x, y) = \{(z, v, u) : z \in x \text{ and } \langle u, v \rangle \in y\} \)
- \( F_5(x, y) = \{\text{ran}(x|_z) : z \in y\} \)
- \( F_6(x) = \bigcup_{y \in x} y \)
- \( F_7(x) = \text{dom}(x) \)
- \( F_8(x) = \{\langle u, v \rangle : u, v \in x \text{ and } u \in v\} \)

by composition. We will not use this fact, however. In general, we can avoid dealing with the mechanics of rudimentary functions by invoking either Proposition 1.2 or Corollary 1.8 below.
1.2. The set $J_2$. We now define the central object of this paper.

**Definition 1.3.** (a) The *rudimentary closure* of a set $x$ is the smallest set $y$ such that $x \subseteq y$, $x \in y$, and $y$ is closed under application of all rudimentary functions (or equivalently, $y$ is closed under application of the functions $F_0, \ldots, F_8$ listed above).

(b) $J_0 = \emptyset$; $J_1$ is the rudimentary closure of $J_0$; $J_2$ is the rudimentary closure of $J_1$.

These are the first few terms of a transfinite sequence known as *Jensen’s hierarchy of constructible sets*. Again, for more on this material the reader is referred to [2].

$J_1$ has a straightforward description: it is just the set of all “hereditarily finite” sets, i.e.,

$$J_1 = \emptyset \cup P(\emptyset) \cup P(P(\emptyset)) \cup \cdots$$

where $P$ denotes the power set operation. Every finite tree gives rise to a hereditarily finite set by labelling terminal nodes with $\emptyset$ and labelling a general node with the set of labels of its immediate successors; then the label of the root node (indeed, every label) is hereditarily finite, and conversely every hereditarily finite set can be obtained in this way from a finite tree.

We cannot expect to have any simple description of $J_2$, but at least we have the following basic facts. A set $x$ is said to be *transitive* if $y \in x$ and $z \in y$ implies $z \in x$, i.e., every element of $x$ is a subset of $x$.

**Proposition 1.4.** $J_2$ is countable and transitive.

Countability follows from the fact that $J_1$ is countable since there are only countably many rudimentary functions. For transitivity, see Lemma VI.1.7 of [2]. The basic idea is to use induction on the minimum number of times clause (2) of Definition 1.1 is used in the definition of a rudimentary function. If we call this the “degree” of the function then we can prove that if a transitive set is closed under application of all rudimentary functions of degree at most $k$ and we add the value of a rudimentary function of degree $k + 1$ applied to some tuple of elements, we do not lose transitivity. From this it is possible to inductively infer that for any $k$ the closure of a transitive set under application of all rudimentary functions of degree at most $k$ is transitive, which we can then use to infer transitivity of $J_2$ from transitivity of $J_1$.

1.3. **Definable subsets.** A fundamental property of $J_2$ is the fact that it is closed under the formation of definable subsets of a set. In order to explain this concept precisely we need to specify a formal language for set theory. The symbols of the language we will use are the following:

Logical symbols: $\wedge$ (and); $\vee$ (or); $\neg$ (not); $\Rightarrow$ (implies); $\Leftrightarrow$ (if and only if); $\forall$ (for all); $\exists$ (there exists).

Set-theoretic symbols: $\in$; $\subseteq$; $=$.

Additional symbols: parentheses; a countable list of variables $v, w, \ldots$.

This list is redundant in that several symbols could be eliminated in favor of more complicated expressions involving the other symbols. But for our purposes there is no particular need to do this.

*Atomic formulas* are expressions of the form $v \in w$, $v \subseteq w$, or $v = w$ for any variables $v$ and $w$. All legal expressions (formulas of the language) can be built up from the atomic ones by using the logical symbols in a predictable way.
The interpretation of such an expression is straightforward except for the following point. In order to interpret a quantifier $\forall v$ or $\exists v$ we need to know the range of possible values of $v$. Every use of a formula $\phi$ therefore requires that we specify the intended universe of discourse $x$. We then say that $\phi$ is relativized to $x$.

**Definition 1.5.** Let $x$ be a set and let $\phi(v, v_1, \ldots, v_k)$ be a formula of the language described above with free (i.e., unquantified) variables $v, v_1, \ldots, v_k$. For each $1 \leq i \leq k$ fix a set $x_i \in x$. Then the set

$$\{ y \in x : \phi^x(y, x_1, \ldots, x_k) \}$$

of $y \in x$ which make the expression $\phi$ true when $v_i$ takes the value $x_i$ for $1 \leq i \leq k$ is a definable subset of $x$.

I will try to help the reader develop some intuition for the notion of definability in $J_2$.

We have the following result:

**Theorem 1.6.** If $x \in J_2$ and $y$ is a definable subset of $x$ then $y \in J_2$.

See Lemma VI.1.17 of [2]. This can be proven by induction on the complexity of the formula $\phi$ which is used to define $y$. Actually, the right way to prove the theorem is to prove the stronger statement that if $\phi$ has free variables $w_1, \ldots, w_j, v_1, \ldots, v_k$ and if $x_1, \ldots, x_k \in x$ then the set

$$\{ \langle y_1, \ldots, y_j \rangle \in x^j : \phi^x(y_1, \ldots, y_j, x_1, \ldots, x_k) \}$$

belongs to $J_2$. One can use specific rudimentary functions to infer this statement given that it holds for formulas of lower complexity (in particular, the formulas $\psi_1$ and $\psi_2$ if $\phi = \psi_1 \lor \psi_2$, the formula $\psi$ if $\phi = (\exists v)\psi$, etc.).

Theorem 1.6 is more powerful than it might appear. For example, suppose we want to define a subset of $x \in J_2$ using parameters $x_1, \ldots, x_k \in J_2$ some of which might not be elements of $x$. This can be accomplished using Theorem 1.6 by replacing $x$ with $x \cup \{ x_1, \ldots, x_k \}$, which still belongs to $J_2$ and makes the $x_i$ available for use as parameters. We may also need to modify $\phi$ so as to ensure that quantified variables continue to range only over $x$; this can be done using the bounded quantifiers $(\forall v \in x)$ and $(\exists v \in x)$. These are abbreviations, defined by

$$(\forall v \in x)A \equiv (\forall v)(v \in x \Rightarrow A)$$

and

$$(\exists v \in x)A \equiv (\exists v)(v \in x \land A).$$

Even more is true. For example, since $J_2$ is closed under cartesian products (Proposition 1.2) we can use $x \times y$ in place of $x$ to obtain definable sets of ordered pairs. Proposition 1.2 gives some indication of the variety of other constructions which are available in $J_2$. This leads to the following principle.

**Definition 1.7.** A $\Delta_0$ formula is a formula built up from atomic formulas using the logical connectives $\land, \lor, \neg, \Rightarrow, \leftrightarrow$ and the bounded quantifiers $(\forall v \in w)$ and $(\exists v \in w)$ defined above (for any variables $v$ and $w$).
Corollary 1.8. First Definability Principle (FDP): If $x_1, \ldots, x_k, y_1, \ldots, y_n \in J_2$, $F : J_2^n \to J_2$ is rudimentary, and $\phi$ is a $\Delta_0$ formula with free variables $v, v_1, \ldots, v_k$, then

$$\{ z \in F(y_1, \ldots, y_n) : \phi(z, x_1, \ldots, x_k) \}$$

belongs to $J_2$.

(This follows from Lemma VI.1.6 (v) of [2]. Note that we need not relativize $\phi$ because all of its quantifiers are bounded.)

Initially one can only use the FDP by laboriously writing out $\phi$ in order to ensure that the property one has in mind really can be expressed as a $\Delta_0$ formula, but it should not take long to develop a good intuition for which properties are of this type. Very roughly speaking, a property can be expressed by a $\Delta_0$ formula if one can determine its truth for given values of the free variables using only sets that appear as elements of those values. Working through the results of Section 2 should give the reader a basic sense of when this is possible.

1.4. Motivation. According to the conceptualist philosophy all sets have to be built up explicitly from other previously constructed sets. Just what should count as an “explicit” construction may be open to debate, but constructions using the functions $F_i$ of §1.1 are certainly acceptable. Consequently $J_2$ is available for conceptualist mathematics. In contrast, for example, passing from $\mathbb{N}$ to its power set would not be considered explicit because, although one can understand what subsets of $\mathbb{N}$ are, one has no clear idea of how to concretely generate all of them. For more on these issues see [12].

Although there is perhaps no way to establish this decisively, it seems reasonable to suppose that applying the functions $F_i$, or (what is equivalent in the long run) passing from a set to its definable subsets, exhausts the conceptually acceptable means of set construction provided one is allowed to do this transfinately many times. This claim is similar in spirit to the Church-Turing thesis which formalizes the informal concept of an algorithm. One argument for it is empirical since no one has yet found any other reasonably concrete constructions which cannot be reduced to these. Another argument can be made on the basis of the naturality of alternative characterizations of the definability construction in terms of infinitary recursion theory [8, 10].

One’s first reaction to the definability construction may be that it is artificial to exclude sets which cannot be explicitly defined in some special language. But we are not using an arbitrary language, we are using the standard language of set theory, and the reason it is the standard language of set theory is because broadly speaking it is able to express everything that we can imagine concretely doing with sets. Thus, the possibility that there might exist “impredicative” sets that one in principle cannot imagine actually constructing hinges on the philosophical question of whether there is some sense in which the universe of sets can be conceived of as a well-defined, independently existing, canonical entity to which we have, even in principle, only limited access.

In any case, a conceptualist should be able to work not only with $J_2$, but also with $J_\omega$, $J_{\omega^2}$, $J_{\omega^\omega}$, etc. Just how far into the transfinite he should be willing to go is a rather subtle question that I discuss in detail in [13]. However, from the point of view of ordinary mainstream mathematics the question is not pressing since for these purposes virtually everything one can do in any $J_\alpha$ can already be done in
Some readers may be more familiar with Gödel’s hierarchy $L_\alpha$ of constructible sets, so I should mention that we could have used these instead (with $\alpha$ a limit ordinal). The $J_\alpha$ are merely a little more convenient since they have better closure properties.

2. Set Theory

From here on we will be working within $J_2$. We begin in this section by defining notions of “sets” and “classes”, which we call “ι-sets” and “ι-classes”, that are appropriate to this context. We formulate a second definability principle relevant to ι-classes and define and establish basic facts about ι-functions and ι-relations. In §2.3 and §2.4 we observe that in the world of $J_2$ all sets are countable and there exists a universal well-ordering.

2.1. Sets and classes. According to §1.4, conceptually the set-theoretic universe may be conceived as being built up in stages $J_\alpha$ indexed by ordinals $\alpha$. Thus, if we stop the construction at any $\alpha$, our notion of “set” at that point will coincide with the elements of $J_\alpha$. Just as in classical set theory, it is also convenient to have a notion of “classes” which may or may not be sets. It might seem natural in this setting to take as classes all definable subsets of the current universe. However, this is not a good definition because, for example, $\mathbb{N}$ can be realized as a set in $J_2$ (see §2.3) and the fact that $J_2$ is countable permits a diagonalization argument by which we can construct a subset of $\mathbb{N}$ which is a definable subset of $J_2$ but not an element of $J_2$. If we do not want proper classes to possibly be contained in sets, the right definition seems to be the following.

**Definition 2.1.** An ι-set is an element of $J_2$. An ι-subset is a subset that is an ι-set. An ι-class is a definable subset of $J_2$ whose intersection with every ι-set is an ι-set, and a proper ι-class is an ι-class that is not an ι-set. An ι-subclass is a subset that is an ι-class.

The requirement that the intersection of an ι-class with any ι-set must be an ι-set imposes a kind of uniformity condition on ι-classes. The presence of extra uniformity conditions will be a recurring motif throughout this paper.

**Proposition 2.2.** The following hold.

(a) Every ι-set is an ι-class, every element of an ι-class is an ι-set, and every ι-class contained in an ι-set is an ι-set.

(b) Let $x$ and $y$ be ι-sets. Then $F(x,y)$ is an ι-set for any rudimentary function $F$; in particular, $x \cup y$, $x \cap y$, $x - y$, $x \times y$, $\bigcup_{z \in x} z$, and (provided $x \neq \emptyset$) $\bigcap_{z \in x} z$ are ι-sets.

(c) Let $X$ and $Y$ be ι-classes. Then $X \cup Y$, $X \cap Y$, $J_2 - X$, and $X \times Y$ are ι-classes.

(d) If $X$ is an ι-class then so is

$$P_1(X) = \{ x \in J_2 : x \subseteq X \}.$$

(e) If $x$ is an ι-set and $Y \subseteq x \times J_2$ is an ι-class then $\bigcup Y_\alpha$ and $\bigcap Y_\alpha$ are ι-classes, where $Y_\alpha = \{ y : \langle a, y \rangle \in Y \}$ for all $a \in x$. 

$J_2$. But it is good to keep in mind the possibility of going further should the need arise. For instance, this might be necessary in order to carry out some transfinite construction which is not covered by the FDP.
(f) If $x$ is an $\iota$-set and $Y \subseteq x \times J_2$ is an $\iota$-class then

$$\Pi Y_a = \{f \in J_2 : f \text{ is a function with domain } x \text{ such that } f(a) \in Y_a \text{ for all } a \in x\}$$

is an $\iota$-class.

Proof. (a) The first statement follows from transitivity of $J_2$ (Proposition 1.2) since for any $x \in J_2$ we have

$$x = \{y \in J_2 : y \in x\},$$

which shows that $x$ is a definable subset of $J_2$. Since the intersection of any two $\iota$-sets is an $\iota$-set (see part (b)), this implies that $x$ is an $\iota$-class. The second and third assertions are trivial.

(b) By definition, $J_2$ is closed under application of all rudimentary functions, so this follows immediately from the definition of $\iota$-sets.

(c) Let $X$ and $Y$ be $\iota$-classes. We will first show that $X \times Y \subseteq J_2$ is definable; the other cases are similar but easier.

Since $X$ and $Y$ are $\iota$-classes, we have

$$X = \{x \in J_2 : \phi^J_2(x, x_1, \ldots, x_k)\}$$

and

$$Y = \{y \in J_2 : \psi^J_2(y, y_1, \ldots, y_l)\}$$

for some $x_1, \ldots, x_k, y_1, \ldots, y_l \in J_2$ and some formulas $\phi$ and $\psi$. Then

$$X \times Y = \{z \in J_2 : (\exists x, y \in J_2)(z = \langle x, y \rangle \land \phi^J_2(x, x_1, \ldots, x_k) \land \psi^J_2(y, y_1, \ldots, y_l))\}.$$  

This works because every ordered pair in $X \times Y$ belongs to $J_2$ by the fact that the function $F(x, y) = \langle x, y \rangle$ is rudimentary (Proposition 1.2).

In the above formula we used the statement “$z = \langle x, y \rangle$”. We must verify that this could be expressed in the language specified in Proposition 1.4. It can be written out as follows:

$$(\exists v)(\exists w)(\forall u)(u \in v \iff u = x) \land (\forall u)(u \in w \iff (u = x \lor u = y)) \land (\forall u)(u \in z \iff (u = v \lor u = w)).$$

This asserts that $z = \{v, w\}$ where $v = \{x\}$ and $w = \{x, y\}$.

We omit the simple proofs that $X \cup Y$, $X \cap Y$, and $J_2 - X$ are definable. The fact that the intersection of each of these with any $\iota$-set is an $\iota$-set follows easily from part (b) together with the fact that this is true of $X$ and $Y$. So each is an $\iota$-class.

To see that $X \times Y$ is an $\iota$-class, consider its intersection with an $\iota$-set $x$. By Proposition 1.2, $\text{dom}(x)$ and $\text{ran}(x)$ are $\iota$-sets, and we have

$$(X \times Y) \cap x = ((X \cap \text{dom}(x)) \times (Y \cap \text{ran}(x))) \cap x.$$ 

Since $X$ and $Y$ are $\iota$-classes, part (b) now shows that the final expression is an $\iota$-set, as desired.

(d) To see that $\mathcal{P}(X) \subseteq J_2$ is definable, say $X = \{x \in J_2 : \phi^J_2(x, x_1, \ldots, x_k)\}$ for some $x_1, \ldots, x_k \in J_2$ and some formula $\phi$. Then

$$\mathcal{P}(X) = \{y \in J_2 : (\forall x \in J_2)(x \in y \Rightarrow \phi^J_2(x, x_1, \ldots, x_k))\}.$$
Now let \( x \) be an \( \iota \)-set. Then \( X \cap \bigcup_{y \in x} y \) is an \( \iota \)-set by part (b) and hence
\[
\mathcal{P}_\iota(X) \cap x = \{ z \in x : z \subseteq X \} = \{ z \in x : z \subseteq X \cap \bigcup_{y \in x} y \}
\]
is an \( \iota \)-set by the FDP (Corollary 1.3), so \( \mathcal{P}_\iota(X) \) is an \( \iota \)-class.

(e) The proof that \( \bigcup Y_a \) and \( \bigcap Y_a \) are definable is similar to the proof of the corresponding fact for \( X \times Y \) given in part (c). To show that \( \bigcup Y_a \) is an \( \iota \)-class, let \( z \) be an \( \iota \)-set; then
\[
z \cap \bigcup Y_a = \text{ran}(Y \cap (x \times z)),
\]
which is an \( \iota \)-set by part (b) and the fact that \( Y \) is an \( \iota \)-class. Applying this result to the \( \iota \)-class \( Z = (x \times J_\mathfrak{a}) - Y \) yields that \( \bigcap Y_a \) is an \( \iota \)-class since \( z \cap \bigcap Y_a = z - (z \cap \bigcup Z_a) \).

(f) The proof of definability of \( \prod Y_a \) is again similar to the corresponding proof in part (c). Now consider its intersection with an \( \iota \)-set \( z \). Let
\[
y = Y \cap \left( x \times \text{ran} \left( \bigcup_{a \in z} a \right) \right);
\]
this is an \( \iota \)-set by part (b) of this proposition and the fact that \( Y \) is an \( \iota \)-class. Then
\[
z \cap \prod Y_a = \{ f \in z : f \text{ is a function with domain } x \text{ such that } f \subseteq y \}.
\]
Thus, to show that \( z \cap \prod Y_a \) is an \( \iota \)-set it will suffice to show that \( \{ f \in z : f \text{ is a function with domain } x \} \) is an \( \iota \)-set, since its intersection with \( \mathcal{P}_\iota(y) \) will then also be an \( \iota \)-set by the fact that \( \mathcal{P}_\iota(y) \) is an \( \iota \)-class (part (d)). We can do this using the FDP because \( \{ f \in z : f \text{ is a function with domain } x \} \) equals
\[
\left\{ f \in z : f \subseteq x \times \text{ran} \left( \bigcup_{y \in z} y \right) \land (\forall v \in x) \left( \exists w \in \text{ran} \left( \bigcup_{y \in z} y \right) \right) (\langle v, w \rangle \in f) \right\}.
\]
Here \( (\exists w \in v)\mathcal{A}(w) \) abbreviates
\[
(\exists w \in v)(\mathcal{A}(w) \land (\forall a \in v)(\mathcal{A}(a) \Rightarrow a = w))
\]
and we can express \( \langle v, w \rangle \in f \) in \( \Delta_0 \) form as
\[
(\exists b \in f)(\exists c, d \in b)[v \in c \land v \in d \land w \in d \land (\forall e \in b)(e = c \lor e = d) \land (\forall e \in c)(e = v) \land (\forall e \in d)(e = v \lor e = w)].
\]
This completes the proof.

In general, verifying that a putative \( \iota \)-class is a definable subset of \( J_2 \) is usually simply a matter of writing out its description in the formal language specified in (1.3). The only common complication is the possible need to use ordered pairs, which can be handled as in the proof of Proposition 2.2 (c) above. So from here on I will generally simply say “verifying definability is straightforward” when this conclusion is needed.

Checking that the intersection of a putative \( \iota \)-class with any \( \iota \)-set is an \( \iota \)-set is not always quite so simple. Usually the quickest way to do this is to intersect with an arbitrary \( \iota \)-set and apply the FDP. We also have the following general tool which is sometimes useful.
Proposition 2.3. Second Definability Principle (SDP): Let \( X \) be an \( \iota \)-class and let \( Y \) be a subset of \( X \) which is definable by means of a \( \Delta_0 \) formula with parameters from \( J_2 \). Then \( Y \) is an \( \iota \)-class.

If \( X = J_2 \) then this is a straightforward consequence of the FDP which applies that principle to \( Y \cap x \) for arbitrary \( x \in J_2 \); see (2.1, Lemma VI.1.6 (v)). In the general case we have \( Y = X \cap Y' \) where \( Y' \) is the set which results from replacing \( X \) with \( J_2 \) in the definition of \( Y \). So we reduce to the previous case using Proposition 2.2 (c).

2.2. Functions and relations. Along with \( \iota \)-sets and \( \iota \)-classes we have analogous notions of \( \iota \)-relations and \( \iota \)-functions. I use the notations \( F[X] = \{F(a) : a \in X\} \) and \( F^{-1}[Y] = \{a \in X : F(a) \in Y\} \). The graph of a function \( F : X \to Y \) is the set

\[
\Gamma(F) = \{(a, b) \in X \times Y : b = F(a)\}
\]

(literally the same thing as \( F \), but I will follow ordinary mathematical usage which distinguishes them).

Definition 2.4. An \( \iota \)-relation on an \( \iota \)-class \( X \) is an \( \iota \)-class contained in \( X \times X \). An \( \iota \)-function from an \( \iota \)-class \( X \) to an \( \iota \)-class \( Y \) is a function \( F : X \to Y \) such that (1) \( \Gamma(F) \) is an \( \iota \)-class and (2) for each \( \iota \)-subset \( x \subseteq X \), \( F[x] \) is contained in an \( \iota \)-set. An \( \iota \)-injection (surjection) is an \( \iota \)-function which is injective (surjective). An \( \iota \)-bijection is a bijection which is an \( \iota \)-function in both directions.

A small \( \iota \)-relation is an \( \iota \)-relation which is an \( \iota \)-set and a small \( \iota \)-function is an \( \iota \)-function whose graph is an \( \iota \)-set.

We can usually verify condition (2) on \( \iota \)-functions using the FDP, since \( F[x] \) will in fact be an \( \iota \)-set (Proposition 2.2 (b)).

We collect basic properties of \( \iota \)-relations and \( \iota \)-functions in the following two propositions.

Proposition 2.5. Let \( X \) be an \( \iota \)-class and let \( R \) be an \( \iota \)-relation on \( X \).

(a) \( R \) is small if and only if \( \text{dom}(R) \) and \( \text{ran}(R) \) are \( \iota \)-sets.

(b) \( R^{-1} = \{(b, a) : (a, b) \in R\} \) is also an \( \iota \)-relation on \( X \).

(c) If \( Y \subseteq X \) is an \( \iota \)-class then \( R \cap (Y \times Y) \) is an \( \iota \)-relation on \( Y \).

Proof. (a) If \( R \) is small then its domain and range are \( \iota \)-sets by Proposition 2.2 (b) since the domain and range functions are rudimentary (Proposition 1.2). Conversely, if its domain and range are \( \iota \)-sets then their product is an \( \iota \)-set by Proposition 2.2 (b) and \( R \) is then an \( \iota \)-set by Proposition 2.2 (a).

(b) Definability of \( R^{-1} \) is straightforward. Now let \( F(x) = \{(b, a) : (a, b) \in x\} \); by Proposition 1.2 this is a rudimentary function. Fix an \( \iota \)-set \( x \). Then \( R^{-1} \cap x = F(R \cap F(x)) \). Now \( F \) takes \( \iota \)-sets to \( \iota \)-sets because it is rudimentary, and \( R \) is an \( \iota \)-class, so this shows that \( R^{-1} \cap x \) is an \( \iota \)-set. We conclude that \( R^{-1} \) is an \( \iota \)-class.

(c) This follows from the intersection and product clauses of Proposition 2.2 (c).

Proposition 2.6. Let \( X \) and \( Y \) be \( \iota \)-classes and let \( F : X \to Y \) be an \( \iota \)-function.

(a) \( F \) is a small \( \iota \)-function if and only if \( X \) is an \( \iota \)-set.

(b) If \( X_0 \subseteq X \) is an \( \iota \)-class then \( F|_{X_0} \) is an \( \iota \)-function. If \( x \subseteq X \) is an \( \iota \)-set then \( F[x] \) is an \( \iota \)-set.

(c) If \( Y_0 \subseteq Y \) is an \( \iota \)-class then \( F^{-1}[Y_0] \) is an \( \iota \)-class. If \( F \) is a small \( \iota \)-function and \( y \subseteq Y \) is an \( \iota \)-set then \( F^{-1}[y] \) is an \( \iota \)-set.
(d) If $Z$ is an $\iota$-class and $G : Y \to Z$ is an $\iota$-function then $G \circ F : X \to Z$ is an $\iota$-function.

(e) If $Z$ is an $\iota$-class then the identity map from $Z$ to itself is an $\iota$-function, as is any constant function on $Z$.

Proof. (a) If $F$ is a small $\iota$-function then $X = \text{dom}(\Gamma(F))$ is an $\iota$-set since the domain function is rudimentary (Proposition 2.2). Conversely, if $X$ is an $\iota$-set then we can find an $\iota$-set $y$ such that $F[X] \subseteq y$; then $\Gamma(F) \subseteq X \times y$ is an $\iota$-set by Proposition 2.2 (a) and (b).

(b) $F|_{X_0}$ trivially satisfies condition (2) for $\iota$-functions given that $F$ does, and its graph is an $\iota$-class because $\Gamma(F|_{X_0}) = \Gamma(F) \cap (X_0 \times J_2)$ is an $\iota$-class by Proposition 2.2 (c). It follows from this and part (a) that if $x \subseteq X$ is an $\iota$-set then $F|_x$ is a small $\iota$-function; then since $F[x] = \text{ran}(\Gamma(F|_x))$ and the range function is rudimentary (Proposition 1.2) we infer that $F[x]$ is an $\iota$-set.

(c) We prove the second statement first. If $F$ is a small $\iota$-function then $X$ is an $\iota$-set by part (a); if $y \subseteq Y$ is also an $\iota$-set then $\Gamma(F) \cap (X \times y)$ is an $\iota$-set with domain $F^{-1}[y]$. This is then an $\iota$-set since the domain function is rudimentary.

Definability of $F^{-1}[Y_0]$ is straightforward. Now consider its intersection with an $\iota$-set $x$. Without loss of generality assume $x \subseteq X$ and find an $\iota$-set $y$ such that $F[x] \subseteq y$. Then

$$F^{-1}[Y_0] \cap x = F|_x^{-1}[Y_0 \cap y];$$

since $F|_x$ is a small $\iota$-function (by parts (a) and (b)) and $Y_0 \cap y$ is an $\iota$-set, this is an $\iota$-set by what we just proved.

(d) First suppose $F$ is a small $\iota$-function. Then $X, y = F[X]$, and $z = G[y]$ are $\iota$-sets and $G|_y$ is a small $\iota$-function by parts (a) and (b), and this implies that $\Gamma(G \circ F)$ is an $\iota$-set by the FDP since $\Gamma(G \circ F) = \{a \in X \times z : (\exists u \in X)(\exists v \in y)(\exists w \in z)((u, v) \in \Gamma(F) \land (v, w) \in \Gamma(G|_y) \land a = (u, w))\}$.

Here ordered pairs are handled as in the proof of Proposition 2.2 (f).

Now suppose $F$ is an $\iota$-function. Definability of $\Gamma(G \circ F)$ is shown by a straightforward modification of the above expression and condition (2) on $\iota$-functions is trivial. To verify that $\Gamma(G \circ F)$ is an $\iota$-class, fix an $\iota$-set $x$; then

$$\Gamma(G \circ F) \cap x = \Gamma(G \circ F|_{\text{dom}(x)}) \cap x.$$

But $F|_{\text{dom}(x)}$ is a small $\iota$-function since the domain function is rudimentary (Proposition 1.2), so $\Gamma(G \circ F|_{\text{dom}(x)})$ is an $\iota$-set by what we proved first. Hence $\Gamma(G \circ F) \cap x$ is an $\iota$-set.

(e) Let $F$ be the identity map on $Z$. It is straightforward to check that $\Gamma(F) = \{(a, a) : a \in Z\}$ is definable, and $F$ trivially satisfies condition (2) on $\iota$-functions. To see that $\Gamma(F)$ is an $\iota$-class, let $x$ be an $\iota$-set. Then

$$\Gamma(F) \cap x = \{(a, a) : a \in Z \cap \text{dom}(x)\}$$

is an $\iota$-set, as desired. Any constant function $F$ whose image is an $\iota$-set has graph $Z \times \{a\}$, which is an $\iota$-class by Proposition 2.2 (c), and it easily follows that $F$ is an $\iota$-function.

Proposition 2.7. Every rudimentary function is an $\iota$-function.

This follows from ([2], Lemma VI.1.3).
2.3. Natural numbers and countability. We encode the natural numbers as sets in the standard way by inductively setting
\[ 0 = \emptyset \quad \text{and} \quad n + 1 = \{0, \ldots, n\}. \]

**Proposition 2.8.** \( \mathbb{N} = \{0, 1, 2, \ldots\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\} \) is an \( \iota \)-set. The standard operations \(+\) and \(\cdot\) on \(\mathbb{N}\) are small \(\iota\)-functions and the standard ordering \(\leq\) on \(\mathbb{N}\) is a small \(\iota\)-relation.

**Proof.** The first claim can be proven using Theorem 1.6 by showing that \(\mathbb{N}\) is a definable subset of \(J_1\). It is clear that \(\mathbb{N} \subseteq J_1\) from the characterization of \(J_1\) as the set of all hereditarily finite sets. Moreover, every \(n \in \mathbb{N}\) is transitive (\(x \in n\) and \(y \in x\) implies \(y \in n\)) and linearly ordered by \(\in\) (for any \(x, y \in n\), either \(x \in y\) or \(y \in x\) or \(x = y\)), and it is not hard to see that any hereditarily finite set with these properties belongs to \(\mathbb{N}\). (Start by observing that the least element of such a set must be \(\emptyset\).) Thus one can write a formula \(\phi\) that expresses transitivity and linearity and obtain \(\mathbb{N} = \{x \in J_1 : \phi^J(x)\}\), and hence \(\mathbb{N}\) is an \(\iota\)-set by Theorem 1.6.

The standard ordering \(\leq\) on \(\mathbb{N}\) actually coincides with \(\subseteq\), so one sees that \(\leq\) is an \(\iota\)-set by writing
\[ \leq = \{x \in \mathbb{N} \times \mathbb{N} : (\exists m, n \in \mathbb{N})(x = (m, n) \wedge m \subseteq n)\} \]
and using the FDP. (As usual, the ordered pair notation is an abbreviation; see the proof of Proposition 2.2 (f).) Realizing the graphs of \(+\) and \(\cdot\) as \(\iota\)-sets is a little more complicated (but this immediately implies that they are \(\iota\)-functions); I will describe a method for getting the graph of \(+\) using the FDP. First, the defining formula for the graph of \(+\) should assert that \(x = \langle a, b, c\rangle\) is an ordered triple of elements of \(\mathbb{N}\). Next, it should assert that if \(b = 0\) then \(a = c\). Finally, there should be an inductive clause that handles the case \(b > 0\) which should state “if \(b \neq 0\) then there exists a function \(f \in J_1\) with domain \(a + 1\) (\(= a \cup \{a\}\)) such that \(f(0) = b, f(a) = c\), and for any \(n < a\) we have \(f(n + 1) = f(n) \cup \{f(n)\}\).” The existence of such a function ensures that \(a + b = c\), and the point of taking \(f \in J_1\) is that \(J_1\) contains all finite partial functions from \(\mathbb{N}\) to \(\mathbb{N}\), so that if \(a + b = c\) then there is such a function \(f\) in \(J_1\). This shows that \(+\) is a small \(\iota\)-function, and the argument for \(\cdot\) is analogous; its defining formula can be built up using the graph of \(+\) as a parameter. \(\square\)

Now that the familiar operations of arithmetic are available in \(J_2\), it is easier to get some idea of the scope of the concept of definability. For example, the set of even numbers is defined by the expression
\[ \{v \in \mathbb{N} : (\exists w \in \mathbb{N})(2 \cdot w = v)\}, \]
the set of prime numbers is defined by the expression
\[ \{u \in \mathbb{N} : \neg(u = 1) \wedge (\forall v, w \in \mathbb{N})[v \cdot w = u \Rightarrow (v = 1 \lor w = 1)]\}, \]
and so on. Of course any finite set of natural numbers is definable by the formula
\[ \{v \in \mathbb{N} : (v = \_\_\_) \lor \cdots \lor (v = \_\_\_)\} \]
with arbitrary numbers inserted in the blank spots. In fact, practically all sets of natural numbers that we can in any sense explicitly describe are in \(J_2\). But not all: by enumerating all \(x \in J_2\) contained in \(\mathbb{N}\) (recall that \(J_2\) is countable) and diagonalizing we can get \(X \subseteq \mathbb{N}\) with \(X \in J_3\) but \(X \not\in J_2\), and we can go on to
get subsets of $\mathbb{N}$ in $J_4$ but not $J_3$, etc. Such sets can be specified precisely and in some sense explicitly but they are rather unusual and not likely to occur in ordinary mathematical practice.

One might be concerned that even if we rarely if ever need to explicitly specify such an apparently pathological object as a set of numbers that is not in $J_2$, these might still arise as solutions to problems of interest. We must ask whether any standard existence proofs fail in $J_2$ because the solutions which they nonconstructively identify might not belong to $J_2$. The answer to this question is that in general even nonconstructive arguments will not lead one out of $J_2$; this is not an absolute fact but rather an empirical observation about the kinds of arguments used in ordinary mathematics. In other words, even nonconstructive existence proofs generally do identify definable solutions to the problems they address. This is the meaning of the comment I made in the introduction about existence results being strengthened when one does mathematics in $J_2$.

We now want to observe that within $J_2$ every $\alpha$-set is countable, i.e., in bijection either with a natural number or with $\mathbb{N}$. We already know this is true “from the outside” as it were by Proposition 1.13 but the new claim is that for every $\alpha$-set in $J_2$ there is such a bijection within $J_2$, i.e., an $\alpha$-bijection. A key tool in the proof is the following result:

**Lemma 2.9.** Suppose $x, y \in J_2$, $x$ is an infinite subset of $y$, and there is a surjective $\alpha$-function $f$ from $\mathbb{N}$ onto $y$. Then there is an $\alpha$-bijection from $\mathbb{N}$ onto $x$.

The proof of this lemma uses the technique introduced in the construction of $+$ in the proof of Proposition 2.8. Namely, we define the graph of an $\alpha$-function $g$ from $\mathbb{N}$ onto $x$ using $f$, $x$, and $\mathbb{N}$ as parameters by writing a formula $\varphi(z, f, x, \mathbb{N})$ that asserts “$z \in \mathbb{N} \times x$, and if $z = \langle a, b \rangle$ then there exists a function $h : a + 1 \to \mathbb{N}$ in $J_1$ such that $f(h(a)) = b$; if $k \leq h(a)$ satisfies $f(k) \in x$ then there exists $n \leq a$ such that $f(h(n)) = f(k)$; and for each $n \leq a$ we have $f(h(n)) \in x$ and $f(h(n)) \neq f(k)$ for any $k < h(n)$.” The key point is that $J_1$ contains the graphs of all functions from $a + 1 = \{0, \ldots, a\}$ into $\mathbb{N}$, so that the required partial enumeration of $g$ is guaranteed to exist.

**Theorem 2.10.** Let $x$ be an infinite $\alpha$-set. Then there is an $\alpha$-bijection from $\mathbb{N}$ onto $x$.

I omit details of the proof. The result can be established using the sets $S_\alpha$ defined on p. 252 of [2] with $\alpha = \omega + n$; one can show inductively that (1) for every $n$ there is an $\alpha$-surjection from $\mathbb{N}$ onto $S_{\omega+n}$ and (2) for every $n$ there exists $m$ such that $S_{\omega+m}$ contains the transitive closure of $S_{\omega+n}$. Since $J_2 = \bigcup S_{\omega+n}$, this plus Lemma 2.9 implies the theorem.

**Corollary 2.11.** (a) If $X$ is an $\alpha$-class then so is

$$P_{\text{fin}}(X) = \{ y \in J_2 : y \subseteq X \text{ is finite} \}.$$ 

If $x$ is an $\alpha$-set then so is $P_{\text{fin}}(x)$.

(b) If $X$ is an $\alpha$-class then so is the set of all finite sequences in $X$ (i.e., the graphs of all functions from $\{0, \ldots, n\}$ into $X$ for arbitrary $n \in \mathbb{N}$). If $x$ is an $\alpha$-set then so is the set of all finite sequences in $x$.

**Proof.** (a) It is straightforward to verify that $P_{\text{fin}}(X)$ is an $\alpha$-class for any $\alpha$-class $X$. Now suppose $x$ is a set. The claim follows from Proposition 2.2 (d) if $x$
is finite, so suppose \( x \) is infinite and by Theorem 2.10 let \( f : N \to x \) be an \( \iota \)-bijection. Observe that \( \mathcal{P}_{\text{fin}}(N) = \{ x \in J_1 : x \subseteq N \} \) is an \( \iota \)-set by the FDP. Then \( \mathcal{P}_{\text{fin}}(x) = F_5(\Gamma(f), \mathcal{P}_{\text{fin}}(N)) \) where \( F_5 \) is the rudimentary function from 2.11 so \( \mathcal{P}_{\text{fin}}(x) \) is an \( \iota \)-set.

(b) Again, the statement about \( \iota \)-classes is straightforward. To show that the set of finite sequences in \( x \) is an \( \iota \)-set, observe that \( \mathcal{P}_{\text{fin}}(N \times x) \) is an \( \iota \)-set by part (a); as this contains the set of all finite sequences in \( x \), the desired result follows by the FDP.

2.4. Well-ordering and quotients. The following result is basic but its proof is a little involved.

**Theorem 2.12.** There is an \( \iota \)-relation \( \preceq_H \) which well-orders \( J_2 \) in such a way that each initial segment is an \( \iota \)-set and each \( \iota \)-set is contained in an initial segment.

This result can be extracted from Lemma VI.2.7 of \([2]\) using the fact mentioned above that the transitive closure of each \( S_{\omega+\iota} \) is contained in some \( S_{\omega+\iota+\iota} \). I will call \( \preceq_H \) the universal well-ordering on \( J_2 \).

Theorem 2.12 can be seen as a strong form of the axiom of choice which yields a well-ordering not only of every \( \iota \)-set but of the universal \( \iota \)-class. One basic way in which this is useful is in allowing us to form quotients by equivalence relations. If we are given an equivalence relation on an \( \iota \)-set which is an \( \iota \)-relation, we can form a quotient \( \iota \)-set in the traditional way as the set of blocks of the equivalence relation. But if we have an equivalence relation on a proper \( \iota \)-class some of whose blocks are proper \( \iota \)-classes, then we clearly cannot do this because proper \( \iota \)-classes cannot be elements of \( \iota \)-classes. Instead, we can use the universal well-ordering to define a version of the quotient by extracting a distinguished element from each block.

**Definition 2.13.** An equivalence \( \iota \)-relation is an \( \iota \)-relation which is an equivalence relation. Let \( X \) be an \( \iota \)-class and let \( \sim \) be an equivalence \( \iota \)-relation on \( X \). We define the quotient \( X/\sim \) to be

\[
X/\sim = \{ a \in X : (\forall b)((b \in X \land a \sim b) \Rightarrow a \preceq_H b) \}
\]

\[
= \{ a \in J_2 : \phi^{J_2}(a) \land (\forall b \in J_2)[(\psi^{J_2}(b) \land \psi^{J_2}([a,b])) \Rightarrow \sigma^{J_2}([a,b])] \}
\]

where \( \phi \) is a formula that defines \( X \), \( \psi \) is a formula that defines \( \sim \), and \( \sigma \) is a formula that defines \( \preceq_H \) (suppressing parameters).

**Proposition 2.14.** The quotient of an \( \iota \)-set by an equivalence \( \iota \)-relation is an \( \iota \)-set. The quotient of an \( \iota \)-class by an equivalence \( \iota \)-relation is an \( \iota \)-class.

**Proof.** Since \( \preceq_H \) is an \( \iota \)-relation its restriction to any \( \iota \)-set is a small \( \iota \)-relation (Proposition 2.2 (a), (c)). Thus, if \( X \) and \( \sim \) are \( \iota \)-sets then the definition of the quotient can be carried out within \( J_2 \) and hence it is an \( \iota \)-set by the FDP.

Now let \( X \) be an \( \iota \)-class and let \( \sim \) be an equivalence \( \iota \)-relation on \( X \). Definability of \( X/\sim \) was exhibited in Definition 2.13. For any \( \iota \)-set \( x \), by Theorem 2.12 we can find an \( \iota \)-set \( y \) which contains \( x \) and is an initial segment of the universal well-ordering. Then

\[
(X/\sim) \cap x = (X/\sim) \cap y \cap x = [(X \cap y)/\sim'] \cap x
\]

where \( \sim' \) is the restriction of \( \sim \) to \( X \cap y \). Now \( X \cap y \) is an \( \iota \)-set and \( \sim' \) is an equivalence \( \iota \)-relation, so \((X \cap y)/\sim'\) is an \( \iota \)-set by the first part of the proposition.
Its intersection with $x$ is then an $\iota$-set by Proposition 2.2 (b), and this shows that $X/\sim$ is an $\iota$-class.

3. The real line

We now define the $\iota$-real line $\mathbb{R}_\iota$ in $J_2$. It is a proper $\iota$-class, the standard ordering is an $\iota$-relation, and the standard operations are $\iota$-functions. Various definitions of $\mathbb{R}_\iota$ are equivalent. In fact all of the standard classical functions from $\mathbb{R}_\iota$ to $\mathbb{R}_\iota$ are $\iota$-functions.

We define $\iota$-open and $\iota$-closed $\iota$-classes in $\mathbb{R}_\iota$ and establish their basic properties. The important notion of an $\iota$-set being a “proxy” for an $\iota$-class is introduced here. We then define $\iota$-compact and $\iota$-connected $\iota$-classes and discuss $\iota$-continuous $\iota$-functions.

3.1. Definition of $\mathbb{R}_\iota$. From here on I will usually invoke the FDP and the SDP without giving any details; the reader should convince himself that these uses are legitimate, which he should be able to do by verifying legitimacy in detail in a few selected examples. In future sections I will use these principles without even mentioning them.

The usual construction of $\mathbb{Z}$ as the set of ordered pairs of natural numbers modulo equivalence $\langle m, n \rangle \sim \langle m', n' \rangle \Leftrightarrow m+n' = m'+n$ can be straightforwardly carried out in $J_2$, as can the usual construction of $\mathbb{Q}$ in terms of ordered pairs of integers modulo equivalence. There is likewise no obstacle to defining the usual algebraic operations and order relation on $\mathbb{Q}$.

Definition 3.1. A (lower) Dedekind $\iota$-cut is an $\iota$-subset of $\mathbb{Q}$ which is neither $\emptyset$ nor $\mathbb{Q}$, has no greatest element, and contains every element less than any element it contains. We also call a Dedekind $\iota$-cut an $\iota$-real. The $\iota$-real line $\mathbb{R}_\iota$ is the set of all Dedekind $\iota$-cuts. The functions $+, \cdot : \mathbb{R}_\iota^2 \to \mathbb{R}_\iota$ and the ordering $\leq \subseteq \mathbb{R}_\iota^2$ are defined in the standard way.

We have the following basic facts.

Theorem 3.2. $\mathbb{R}_\iota$ is an $\iota$-class, $+$ and $\cdot$ are $\iota$-functions from $\mathbb{R}_\iota^2$ to $\mathbb{R}_\iota$, and $\leq$ is an $\iota$-relation on $\mathbb{R}_\iota$. The subfield of $\mathbb{R}_\iota$ generated by any $\iota$-subset is an $\iota$-set. Every nonempty $\iota$-subset of $\mathbb{R}_\iota$ that is bounded above has a least upper bound and the map that takes each nonempty $\iota$-subset that is bounded above to its least upper bound is an $\iota$-function.

Proof. $\mathbb{R}_\iota$ is an $\iota$-class by the SDP. Likewise $\leq_{\mathbb{R}_\iota}$ is an $\iota$-relation by the SDP and $+$ and $\cdot$ are $\iota$-functions by the SDP (used to check that they are $\iota$-classes) and the FDP (used to check condition (2) on $\iota$-functions).

Given any nonempty $\iota$-subset $x \subseteq \mathbb{R}_\iota$, use Theorem 2.10 to find an $\iota$-surjection $f : \mathbb{N} \to x$ and let $a$ be the $\iota$-set of (some nice encoding of) all words in the variables $v_0, v_1, \ldots$ and the field operations. We can then use the FDP to show that the set $b$ of ordered pairs $\langle r, q \rangle \in a \times \mathbb{Q}$ such that $q < r(f(0), f(1), \ldots)$ (i.e., $r$ evaluated with $v_n = f(n)$) is an $\iota$-set. Finally, we can use the rudimentary function $F(a) = \{G(y) : y \in a\}$ from Proposition 2.2 with $G(y) = \{y\}$ to show that $\{\{r\} : r \in a\}$ is an $\iota$-set; applying the rudimentary function $F_3$ from 2.1 to these two $\iota$-sets yields that the subfield generated by $x$ is an $\iota$-set.

Next, $\bigcup_{y \in x} y$ is an $\iota$-set by Proposition 2.2 (b), and it belongs to $\mathbb{R}_\iota$ provided $x$ is bounded above. Since $\leq_{\mathbb{R}_\iota}$ is just the inclusion relation, it follows that $\bigcup_{y \in x} y$
is a least upper bound for $x$. Moreover, according to Proposition 2.7, the function $x \mapsto \bigcup_{y \in X} y$ is rudimentary and hence it is an $\iota$-function by Proposition 2.6 so its restriction (call this $F$) to $\mathcal{P}_i(R_i)$ is an $\iota$-function by Proposition 2.6 (d) and Proposition 2.6 (b); finally, the desired function $F|_{\mathcal{P}_i(R_i)}$ is an $\iota$-function by Proposition 2.6 (b) and (e).

The other standard constructions of $R_i$ are also available. For example, a construction via Cauchy $\iota$-sequences (see Definition 3.4) can be carried out using Definition 2.13. One can prove the analog of Theorem 3.2 here too. In fact, all standard constructions are equivalent by the following result. Let an $\iota$-field be an $\iota$-class equipped with $\iota$-functions which make it a field, such that the subfield generated by any $\iota$-subset is an $\iota$-set. It is $\iota$-ordered if it is an ordered field such that the partial order is an $\iota$-relation, and it is $\iota$-complete if every nonempty $\iota$-subset that is bounded above has a least upper bound and the map which takes each nonempty $\iota$-subset that is bounded above to its least upper bound is an $\iota$-function.

**Theorem 3.3.** Every $\iota$-complete $\iota$-ordered $\iota$-field is isomorphic to $R_i$ via an $\iota$-bijection.

The first part of the proof involves showing that every element of $\mathcal{F}$ corresponds to a Dedekind $\iota$-cut. To see this, let $x \in \mathcal{F}$. Then the fact that $\leq_{\mathcal{F}}$ is an $\iota$-relation implies that $\{ p \in Q_{\mathcal{F}} : p <_{\mathcal{F}} x \}$ is an $\iota$-set, where $Q_{\mathcal{F}}$ is the canonical copy of $Q$ in $\mathcal{F}$. Since $Q_{\mathcal{F}}$ is the subfield of $\mathcal{F}$ generated by $1_{\mathcal{F}}$, it is an $\iota$-set. Moreover, $Q_{\mathcal{F}}$ is isomorphic to $Q$ via an $\iota$-bijection, so we infer that every element of $\mathcal{F}$ determines an $\iota$-cut. Conversely, every $\iota$-cut determines an element of $\mathcal{F}$ since $\mathcal{F}$ is $\iota$-complete. Density of $Q_{\mathcal{F}}$ in $\mathcal{F}$ is proven just as in the classical case. This shows that there is an order-isomorphism between $\mathcal{F}$ and $R_i$. The fact that this map respects $+$ and $\cdot$ is proven just as in the classical case. Finally, the fact that the graph of the isomorphism is an $\iota$-class can be shown using the FDP and the fact that it is an $\iota$-function in both directions uses the hypothesis that the map which takes each nonempty upper bounded $\iota$-set to its least upper bound is an $\iota$-function.

The next result can be used to show that all of the standard functions from $R_i$ to $R_i$ are $\iota$-functions. We use the following terminology.

**Definition 3.4.** Let $X$ and $Y$ be $\iota$-classes. An $\iota$-sequence in $X$ is an $\iota$-function from $N$ to $X$. An $\iota$-sequence of $\iota$-functions from $X$ to $Y$ is an $\iota$-function from $N \times X$ to $Y$.

**Lemma 3.5.** If $(a_n)$ is a Cauchy $\iota$-sequence in $R_i$ then it converges to a limit in $R_i$. The map that takes (graphs of) Cauchy $\iota$-sequences in $R_i$ to their limits is an $\iota$-function.

**Proof.** Regarding elements of $R_i$ as Dedekind $\iota$-cuts, we have

$$\lim a_n = \bigcup_{n \in N} \bigcap_{k \geq n} (a_k - 1/n).$$

Using this expression one can write a formula for $\lim a_n$ (as a subset of $Q$) in terms of the graph of $(a_n)$, and it follows from the FDP that the limit is an $\iota$-real. It is then fairly routine to verify that the set of Cauchy $\iota$-sequences is an $\iota$-class and that the map taking such a sequence to its limit is an $\iota$-function.

**Proposition 3.6.** Let $X \subseteq R_i$ be an $\iota$-class. Then any pointwise limit of an $\iota$-sequence of $\iota$-functions from $X$ to $R_i$ is an $\iota$-function.
Proof. Let \( F : \mathbb{N} \times X \to \mathbb{R}_1 \) be an \( \iota \)-sequence of \( \iota \)-functions and suppose that for each \( a \in X \) the \( \iota \)-sequence \( (F(\cdot, a)) \) is Cauchy. We claim that the map \( G \) which takes \( a \in X \) to the graph of the \( \iota \)-sequence \( (F(\cdot, a)) \) is an \( \iota \)-function. Definability of its graph is straightforward. To show that \( \Gamma(G) \) is an \( \iota \)-class it is sufficient to show that its intersection with \( x \times J_2 \) is an \( \iota \)-set for every \( \iota \)-subset \( x \) of \( X \), since for any \( \iota \)-set \( x \) we have

\[
\Gamma(G) \cap x = (\Gamma(G) \cap (\text{dom}(x) \times J_2)) \cap x.
\]

So given an \( \iota \)-set \( x \subseteq X \), find an \( \iota \)-subset \( y \) of \( \mathbb{R}_1 \) such that \( F[\mathbb{N} \times x] \subseteq y \); then \( \Gamma(G) \cap (x \times J_2) \) is an \( \iota \)-subset of \( x \times J_2 \) by the FDP, using the \( \iota \)-set \( \Gamma(F) \cap (\mathbb{N} \times x \times y) \) as a parameter. The second condition on \( \iota \)-functions is proven similarly and we conclude that \( G \) is an \( \iota \)-function. The desired result then follows from the lemma since any composition of \( \iota \)-functions is an \( \iota \)-function (Proposition 2.6(d)).

\[\Box\]

Corollary 3.7. All of the standard functions on \( \mathbb{R}_1 \) are \( \iota \)-functions.

All polynomials on \( \mathbb{R}_1 \) are \( \iota \)-functions since \( + \) and \( \cdot \) are \( \iota \)-functions (Theorem 2.4), the constant functions and the identity map on \( \mathbb{R}_1 \) are \( \iota \)-functions (Proposition 2.6(e)), and compositions of \( \iota \)-functions are \( \iota \)-functions (Proposition 2.6(d)). All standard continuous functions \( \sin t, \cos t, \ln t, e^t, \) etc. are \( \iota \)-functions by Proposition 3.3 together with this observation. Standard discontinuous functions such as the jump function \( f(t) = 0 \) for \( t \leq 0 \) and \( 1 \) for \( t > 0 \) can also be seen to be \( \iota \)-functions via pointwise approximation by polynomials, or more simply by a direct proof.

3.2. Open \( \iota \)-classes. We now consider \( \iota \)-classes \( U \subseteq \mathbb{R}_1 \), which are “open”. No such \( \iota \)-class will be an \( \iota \)-set unless it is empty, because every nonempty open \( \iota \)-class will contain an interval of positive length, and no such interval can be contained in an \( \iota \)-set. The right definition seems to be the following. Let \( \mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty) \) and \( \mathbb{R}_1^+ = \mathbb{R}_1 \cap (0, \infty) \).

Definition 3.8. An \( \iota \)-class \( U \subseteq \mathbb{R}_1 \) is \( \iota \)-open if there is an \( \iota \)-set \( u \subseteq \mathbb{Q} \times \mathbb{Q}^+ \) such that for any \( \iota \)-real number \( r \in \mathbb{R}_1 \), we have \( r \in U \) if and only if there exists \( \langle p, q \rangle \in u \) with \( |r - p| < q \). We call \( u \) a proxy for \( U \).

That is, every \( \iota \)-open \( \iota \)-class is a union of an \( \iota \)-set of balls with rational centers and rational radii. Note that this is not the same as saying that for every \( r \in U \) there exists a ball containing \( r \) and contained in \( U \). The latter condition is weaker since we require that \( u \) be an \( \iota \)-set (but see 3.4).

The concept of a proxy is useful if we want to make a statement about an \( \iota \)-set of \( \iota \)-open \( \iota \)-classes. This does not make literal sense because a proper \( \iota \)-class can never be a member of another \( \iota \)-set or \( \iota \)-class, but we can often give such a statement a reasonable meaning using proxies. For example, in part (a) of the next result we assert that the union of any \( \iota \)-set of \( \iota \)-open \( \iota \)-classes is \( \iota \)-open; this should be understood as abbreviating the assertion “given an \( \iota \)-set of proxies of \( \iota \)-open \( \iota \)-classes, the union of the corresponding \( \iota \)-open \( \iota \)-classes is \( \iota \)-open”.

Proposition 3.9. (a) The union of any \( \iota \)-set of \( \iota \)-open \( \iota \)-classes is an \( \iota \)-open \( \iota \)-class.
(b) Any intersection of finitely many \( \iota \)-open \( \iota \)-classes is an \( \iota \)-open \( \iota \)-class.
(c) Let \( u \subseteq \mathbb{Q} \times \mathbb{Q}^+ \) be an \( \iota \)-set. Then

\[ U = \{ r \in \mathbb{R}_1 : |r - p| < q \text{ for some } \langle p, q \rangle \in u \} \]

is an \( \iota \)-open \( \iota \)-class.
Theorem 3.13. by the FDP. We conclude that \( \lim \) is covered by finitely many balls indexed by \( u \).

(b) The intersection of any two balls with rational radii and rational centers is again a ball with a rational radius and a rational center. Thus if \( U \) and \( V \) are two \( \iota \)-open \( \iota \)-classes, a proxy for their intersection is obtained by intersecting every ball in a proxy for \( U \) with every ball in a proxy for \( V \). This is enough.

(c) \( U \) is an \( \iota \)-class by the SDP, and it is then \( \iota \)-open by definition. \( \square \)

Corollary 3.10. Let \( u \subseteq \mathbb{R}_c \times \mathbb{R}_c^+ \) be an \( \iota \)-set. Then
\[
U = \{ r \in \mathbb{R}_c : |r - p| < q \text{ for some } (p, q) \in u \}
\]
is an \( \iota \)-open \( \iota \)-class.

Proof. Define
\[
v = \{(p, q) \in \mathbb{Q} \times \mathbb{Q}^+ : q + |p - r| \leq s \text{ for some } (r, s) \in u \}.
\]
This is an \( \iota \)-set by the FDP. By Proposition 3.9 (c), \( v \) is a proxy for an \( \iota \)-open \( \iota \)-class, which by inspection equals \( U \). \( \square \)

In particular, the \( \iota \)-open ball
\[
B_q(p) = \{ r \in \mathbb{R}_c : |r - p| < q \}
\]
is an \( \iota \)-open \( \iota \)-class for any \( p \in \mathbb{R}_c \) and \( q \in \mathbb{R}_c^+ \).

3.3. Closed \( \iota \)-classes.

Definition 3.11. Let \( c \) be an \( \iota \)-subset of \( \mathbb{R}_c \). The \( \iota \)-closure of \( c \) is the set of all \( \iota \)-real numbers that are limits of Cauchy \( \iota \)-sequences in \( c \). An \( \iota \)-class \( C \subseteq \mathbb{R}_c \) is \( \iota \)-closed if it is the \( \iota \)-closure of an \( \iota \)-set \( c \), in which case \( c \) is a proxy for \( C \).

Proposition 3.12. Any Cauchy \( \iota \)-sequence in an \( \iota \)-closed \( \iota \)-class converges to an element of that \( \iota \)-class.

Proof. Let \( C \) be an \( \iota \)-closed \( \iota \)-class with proxy \( c \) and let \((a_n)\) be a Cauchy \( \iota \)-sequence in \( C \). We can define a Cauchy sequence \((b_n)\) with values in \( c \) that has the same limit by letting \( b_n \) be the first element of \( c \), with respect to \( \preccurlyeq_U \), whose distance to \( a_n \) is less than \( 1/n \). This is an \( \iota \)-sequence because its graph is an \( \iota \)-subset of \( \mathbb{N} \times c \) by the FDP. We conclude that \( \lim a_n \) belongs to \( C \). \( \square \)

Theorem 3.13. An \( \iota \)-class \( C \subseteq \mathbb{R}_c \) is \( \iota \)-closed if and only if \( \mathbb{R}_c - C \) is \( \iota \)-open.

Proof. Suppose \( C \subseteq \mathbb{R}_c \) is \( \iota \)-closed and let \( c \) be a proxy for \( C \). Then
\[
u = \{(p, q) \in \mathbb{Q} \times \mathbb{Q}^+ : q \leq |p - r| \text{ for all } r \in c \}
\]
is an \( \iota \)-set by the FDP. Let \( U \) be the corresponding \( \iota \)-open \( \iota \)-class. It is clear that \( U \subseteq \mathbb{R}_c - C \). Conversely, for any \( r \in \mathbb{R}_c - C \) there must exist \( \epsilon > 0 \) such that \( B_\epsilon(r) \cap c = \emptyset \); otherwise we could find a Cauchy \( \iota \)-sequence in \( c \) converging to \( r \) by the method used to construct \((b_n)\) in Proposition 3.12. Density of \( \mathbb{Q} \) in \( \mathbb{R}_c \) now implies \( r \in U \). We conclude that \( \mathbb{R}_c - C = U \) is \( \iota \)-open.

Now let \( U \subseteq \mathbb{R}_c \) be \( \iota \)-open; we must show that its complement is \( \iota \)-closed. We may assume \( U \) is nonempty. Let \( u \) be a proxy for \( U \). Then let \( x \) be the set of ordered pairs of rationals \((p, q)\) with the property that \( p < q \) and the interval \([p, q]\) is covered by finitely many balls indexed by \( u \). That is, \((p, q) \in x \) if and only if \( p < q \) and there exists a finite sequence \((p_1, q_1), \ldots, (p_n, q_n)\) in \( u \) such that (1)
Some simple manipulations using rudimentary functions (in particular the function FDP, \( F \)) can be used to show that given an \( n \)-sequence (since every Cauchy \( n \)-sequence is eventually constant), this will not be an \( n \)-closed \( n \)-class. For example, it is possible to construct a proper \( n \)-class that contains exactly one \( n \)-real number in the interval \([n, n+1] \) for each \( n \in \mathbb{N} \). Although it is closed under limits of Cauchy \( n \)-sequences (since every Cauchy \( n \)-sequence is eventually constant), this will not be an \( n \)-closed \( n \)-class according to Definition 3.11 because it is not the \( n \)-closure of an \( n \)-set. Its complement will be an \( n \)-open \( n \)-class by Proposition 3.14 (c), so the \( n \)-closure of \( c \) is an \( n \)-closed \( n \)-class by Theorem 3.13.

### Corollary 3.14
(a) The intersection of any \( n \)-set of \( n \)-closed \( n \)-classes is an \( n \)-closed \( n \)-class.
(b) Any union of finitely many \( n \)-closed \( n \)-classes is an \( n \)-closed \( n \)-class.
(c) The \( n \)-closure of any \( n \)-subset of \( \mathbb{R}_n \) is an \( n \)-closed \( n \)-class.

**Proof.** Given an \( n \)-set of proxies for \( n \)-closed \( n \)-classes, the construction of \( U \) from \( X \) yields an \( n \)-set of proxies for the complementary \( n \)-open \( n \)-classes. (Assuming \( x \) is an infinite \( n \)-set of proxies for \( n \)-closed \( n \)-classes, use an \( n \)-bijection between \( \mathbb{N} \) and \( x \) to find an \( n \)-subset of \( \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \), whose restriction to each \( n \in \mathbb{N} \) is a proxy for the \( n \)-open \( n \)-class that is complementary to the corresponding \( n \)-closed \( n \)-class in \( x \). Then use the rudimentary function \( F_5 \) from (1.11) to get the desired \( n \)-set of proxies for the complementary \( n \)-open \( n \)-classes.) The result now follows from Proposition 3.15 (a) and Theorem 3.13.

(b) This follows directly from Proposition 3.15 (b) and Theorem 3.13.

(c) Let \( c \subseteq \mathbb{R}_n \) be an \( n \)-set. Then the proof of Theorem 3.13 shows that the complement of its \( n \)-closure is the union of an \( n \)-set of balls with rational centers and radii. The latter is an \( n \)-open \( n \)-class by Proposition 3.15 (c), so the \( n \)-closure of \( c \) is an \( n \)-closed \( n \)-class by Theorem 3.13.

### 3.4 Separable \( n \)-classes
As I mentioned at the beginning of 3.2, for an \( n \)-subclass of \( \mathbb{R}_n \), to be \( n \)-open it is not sufficient that every element of the \( n \)-class be contained in a ball contained in the \( n \)-class. For example, it is possible to construct a proper \( n \)-class that contains exactly one \( n \)-real number in the interval \([n, n+1] \) for each \( n \in \mathbb{N} \). Although it is closed under limits of Cauchy \( n \)-sequences (since every Cauchy \( n \)-sequence is eventually constant), this will not be an \( n \)-closed \( n \)-class according to Definition 3.11 because it is not the \( n \)-closure of an \( n \)-set. Its complement will be an \( n \)-class that is not \( n \)-open according to Definition 3.8 but which does have the ball property mentioned above.

However, the familiar equivalences do hold if we make additional separability assumptions.

**Proposition 3.15.** Let \( X \subseteq \mathbb{R}_n \) be an \( n \)-class and let \( x \subseteq X \) be an \( n \)-set. Then the following are equivalent:
(a) Every \( n \)-open ball about a point of \( X \) intersects \( x \).
(b) Every \( n \)-open ball with rational center and radius that intersects \( X \) also intersects \( x \).
(c) Every \( \iota \)-open \( \iota \)-class that intersects \( X \) also intersects \( x \).
(d) \( X \) is contained in the \( \iota \)-closure of \( x \).

Proof. The equivalence of (a), (b), and (c) is trivial, as is the fact that (d) implies them. The final implication is proven by constructing a Cauchy \( \iota \)-sequence in \( x \) that converges to a given point in \( X \) by the technique used in the proof of Proposition 3.12.

Definition 3.16. Let \( X \subseteq \mathbb{R}_\iota \) be an \( \iota \)-class and let \( x \subseteq X \) be an \( \iota \)-set. We say that \( x \) is \( \iota \)-dense in \( X \) if any of the equivalent conditions of Proposition 3.15 holds. We say that \( X \) is \( \iota \)-separable if it contains an \( \iota \)-dense \( \iota \)-subset.

Lemma 3.17. Every \( \iota \)-open \( \iota \)-class is \( \iota \)-separable, as is every \( \iota \)-closed \( \iota \)-class.

Proposition 3.18. Let \( X \subseteq \mathbb{R}_\iota \) be an \( \iota \)-separable \( \iota \)-class and let \( Y = \mathbb{R}_\iota - X \).
(a) \( X \) is \( \iota \)-closed if and only if it is closed under limits of all Cauchy \( \iota \)-sequences in \( X \).
(b) \( Y \) is \( \iota \)-open if and only if for every \( r \in Y \) there exists \( \epsilon > 0 \) such that \( B_\epsilon(r) \subseteq Y \).

Proof. (a) The forward direction was Proposition 3.12. For the reverse direction, let \( c \) be an \( \iota \)-dense \( \iota \)-subset of \( X \); then \( X \) is automatically contained in the \( \iota \)-closure of \( c \), and it contains the \( \iota \)-closure of \( c \) because it is closed under limits of Cauchy \( \iota \)-sequences. So \( X \) is \( \iota \)-closed.
(b) The forward direction is easy. For the reverse direction, let \( c \) be an \( \iota \)-dense \( \iota \)-subset of \( X \) and suppose every element of \( Y \) is contained in a ball contained in \( Y \). Let \( C \) be the \( \iota \)-closure of \( c \); it is then easy to verify that \( Y = \mathbb{R}_\iota - C \), which implies that \( Y \) is \( \iota \)-open by Theorem 3.13.

3.5. Compactness and connectedness.

Definition 3.19. Let \( K \subseteq \mathbb{R}_\iota \) be an \( \iota \)-class. We say that \( K \) is \( \iota \)-compact if any \( \iota \)-set of \( \iota \)-open \( \iota \)-classes which covers \( K \) has a finite subcover. It is \( \iota \)-connected if there do not exist \( \iota \)-open \( \iota \)-classes \( U \) and \( V \) such that \( K \subseteq U \cup V \), \( K \cap U \neq \emptyset \), \( K \cap V \neq \emptyset \), and \( U \cap V = \emptyset \).

Recall our convention about phrases like “\( \iota \)-set of \( \iota \)-open \( \iota \)-classes” from §3.3; the above really means that whenever \( x \) is an \( \iota \)-set of proxies of \( \iota \)-open \( \iota \)-classes and the union of the corresponding \( \iota \)-classes contains \( K \), there is a finite subset of \( x \) with the same property.

Theorem 3.20. Let \( K \) be an \( \iota \)-separable \( \iota \)-subclass of \( \mathbb{R}_\iota \). Then the following are equivalent:
(i) \( K \) is an \( \iota \)-closed and bounded \( \iota \)-class;
(ii) \( K \) is \( \iota \)-compact;
(iii) \( K \) is bounded and contains the limits of all of its Cauchy \( \iota \)-sequences;
(iv) every \( \iota \)-sequence in \( K \) has an \( \iota \)-subsequence which converges to a limit in \( K \).

Proof. (i) \( \Rightarrow \) (ii): This reduces to the assertion that \([0,1]\) is \( \iota \)-compact in the usual way. Since every \( \iota \)-open \( \iota \)-class is a union of an \( \iota \)-set of balls with rational centers and radii, we further reduce to covers of \([0,1]\) by such balls. The proof of \( \iota \)-compactness is now the standard one, already essentially given in the last part of the proof of Theorem 3.18.
Theorem 3.21. Suppose $K$ is $\iota$-compact. If $K$ were not bounded then $\{B_n(0) : n \in \mathbb{N}\}$ would be an $\iota$-set of $\iota$-open $\iota$-classes which covers $K$ but has no finite subcover. So $K$ is bounded. Similarly, if there were a Cauchy $\iota$-sequence $(r_n)$ in $K$ whose limit did not belong to $K$ then $\{B_q(p) : p, q \in \mathbb{Q} \text{ and } q < |p - r|/2\}$ would contradict $\iota$-compactness, where $r = \lim r_n$. So $K$ is closed under convergence of Cauchy $\iota$-sequences.

(iii) $\Rightarrow$ (iv): Suppose $K$ is bounded and contains the limits of all of its Cauchy $\iota$-sequences and let $(r_n)$ be an $\iota$-sequence in $K$. For each $k$ let $I_k$ be the leftmost interval of the form $[j/2^k, (j + 1)/2^k]$ which contains infinitely many terms of the $\iota$-sequence $(r_n)$ and let $n_k$ be the first index greater than $n_k - 1$ such that $r_{n_k} \in I_k$. Then $(r_{n_k})$ is a Cauchy $\iota$-subsequence of $(r_n)$ since the $I_k$ are nested, and hence it converges to a limit in $K$.

(iv) $\Rightarrow$ (i) Suppose every $\iota$-sequence in $K$ has an $\iota$-subsequence which converges to a limit in $K$ and let $c$ be an $\iota$-dense $\iota$-subset of $K$. If $c$ were unbounded we could find an $\iota$-sequence in $c$ which diverges to $\pm \infty$ and hence has no convergent $\iota$-subsequence, so $c$ must be bounded. This implies that $K$ is bounded. Now $K$ is automatically contained in the $\iota$-closure of $c$; conversely, any Cauchy $\iota$-sequence in $c$ must converge to a limit in $K$ because by assumption it has an $\iota$-subsequence which converges to a limit in $K$. Thus $K$ is the $\iota$-closure of $c$ and hence $K$ is $\iota$-closed.

(Note that only the proof of (iv) $\Rightarrow$ (i) used $\iota$-separability.)

The characterization of $\iota$-connected $\iota$-classes in $\mathbb{R}_\iota$ is proven in just the same way as in the classical case.

Theorem 3.21. An $\iota$-subclass of $\mathbb{R}_\iota$ is $\iota$-connected if and only if it is an interval.

3.6 Continuous $\iota$-functions.

Definition 3.22. Let $X$ be an $\iota$-subclass of $\mathbb{R}_\iota$. An $\iota$-function $F : X \to \mathbb{R}_\iota$ is $\iota$-continuous if the inverse image of every $\iota$-open $\iota$-class in $\mathbb{R}_\iota$ is the intersection of an $\iota$-open $\iota$-class with $X$.

Theorem 3.23. Let $X$ be an $\iota$-separable $\iota$-subclass of $\mathbb{R}_\iota$ and let $F : X \to \mathbb{R}_\iota$ be an $\iota$-function. Then the following are equivalent:

(a) $F$ is $\iota$-continuous.

(b) The inverse image of every $\iota$-closed $\iota$-class under $F$ is the intersection of an $\iota$-closed $\iota$-class with $X$.

(c) For any $\iota$-set $c \subseteq X$ with $\iota$-closure $C$, the $\iota$-closure of $F[c]$ contains $F[C \cap X]$.

(d) $F$ preserves convergence of $\iota$-sequences.

(e) For every $x \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|r - s| < \delta$ implies $|F(r) - F(s)| < \varepsilon$.

If any of these conditions holds then there is an $\iota$-function which takes any proxy $u$ of an $\iota$-open $\iota$-class $U \subseteq \mathbb{R}_\iota$ to a proxy $v$ of an $\iota$-open $\iota$-class $V \subseteq \mathbb{R}_\iota$ such that $F^{-1}[U] = X \cap V$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d): The classical proofs carry over without alteration.

(d) $\Rightarrow$ (e): Suppose (e) fails and choose $r, \varepsilon$ such that for every $\delta > 0$ there exists $s$ with $|r - s| < \delta$ and $|F(r) - F(s)| \geq \varepsilon$. This implies that there is a sequence in $X$ which converges to $r$ but whose image does not converge to $F(r)$. However, it is not obvious that there is an $\iota$-sequence with this property so the contradiction is not immediate.
Let $x$ be an $\iota$-dense $\iota$-subset of $X$. If for every $k \in \mathbb{N}$ there exists $s \in x$ such that $|r - s| < 1/k$ and $|F(r) - F(s)| \geq \epsilon/2$ then we can find an $\iota$-sequence which converges to $r$ whose image does not converge to $F(r)$. This would falsify (d) and we would be done. Therefore we now assume that there exists $k \in \mathbb{N}$ such that $|F(r) - F(s)| < \epsilon/2$ for every $s \in x$ such that $|r - s| < 1/k$.

By the choice of $r$ we know there exists $s \in X$ with $|r - s| < 1/k$ and $|F(r) - F(s)| \geq \epsilon$. But we also know that $x$ is $\iota$-dense in $X$ and hence there is an $\iota$-sequence $(s_n)$ in $x$ that converges to $s$, and $|r - s_n| < 1/k$ for sufficiently large $n$. But then the choice of $k$ yields $|F(r) - F(s_n)| < \epsilon/2$ for sufficiently large $n$, which implies that $F(s_n) \not\rightarrow F(s)$, and we have falsified (d).

$(e)$ \Rightarrow $(a)$: Suppose $(e)$ holds and let $U \subset \mathbb{R}_i$ be an $\iota$-open $\iota$-class with proxy $u$; we must show that $F^{-1}(U)$ is the intersection of an $\iota$-open $\iota$-class with $X$. Let $x$ be an $\iota$-dense $\iota$-subset of $X$ and let $v$ be the $\iota$-set of pairs $(p, q) \in Q \times Q^+$ such that $F(x \cap B_q(p)) \subseteq B_{q'-1/k}(p')$ for some $(p', q') \in v$ and some $k \in \mathbb{N}$. We claim that $v$ is a proxy for an $\iota$-open $\iota$-class $V$ whose intersection with $X$ equals $F^{-1}(U)$.

First we show that $V$ contains $F^{-1}(U)$. To see this let $r \in F^{-1}(U)$. Find $(p', q') \in v$ and $k \in \mathbb{N}$ such that $F(r) \in B_{q'-1/k}(p')$ and let $\epsilon = q'-1/k - |p' - F(r)|$. By (e) we can then find $\delta > 0$ such that $|r - s| < \delta$ implies $|F(r) - F(s)| < \epsilon$. Finally we can find $(p, q)$ such that $r \in B_q(p) \subseteq B_k(r) \subseteq B_{q'-1/k}(p')$, which implies that $(p, q) \in v$, and we conclude that $r \in V$. This shows one containment.

Now we must show that $F^{-1}(U)$ contains $V \cap X$. Suppose not and fix $r \in V \cap X$ such that $F(r) \notin U$. Fix $(p, q) \in v$ such that $r \in B_q(p)$; by (e) and density of $x$ in $X$ we can then find, for every $k \in \mathbb{N}$, an $s \in x \cap B_q(p)$ such that $|F(r) - F(s)| < 1/k$. But since $F(r) \notin U$ this contradicts the fact that $F(x \cap B_q(p))$ is contained in some $B_{q'-1/k}(p')$. We conclude that $V \cap X \subseteq F^{-1}(U)$. This completes the proof that $(e)$ implies $(a)$.

In the proof of $(e) \Rightarrow (a)$ the proxy $v$ is definable from the proxy $u$ and this shows that the map $u \mapsto v$ is an $\iota$-function by the FDP. \hfill \box

The proof of the next result is no different from the classical case.

**Theorem 3.24.** The composition, sum, and product of any two $\iota$-continuous $\iota$-functions is $\iota$-continuous.

**Theorem 3.25.** (a) The image of an $\iota$-separable $\iota$-compact $\iota$-class under an $\iota$-continuous $\iota$-function is an $\iota$-separable $\iota$-compact $\iota$-class.

(b) The image of an $\iota$-connected $\iota$-class under an $\iota$-continuous $\iota$-function is an $\iota$-connected $\iota$-class.

**Proof.** (a) Let $K$ be an $\iota$-separable $\iota$-compact $\iota$-class and let $F$ be an $\iota$-continuous $\iota$-function. One proves the $\iota$-compactness property of $F[K]$ just as in the classical case, using the last part of Theorem 3.23 to pull an $\iota$-open cover of $F[K]$ back to an $\iota$-open cover of $K$. However, we must also show that $F[K]$ is an $\iota$-separable $\iota$-class. Let $c$ be an $\iota$-dense $\iota$-subset of $K$; we claim that $F[K]$ equals the $\iota$-closure of $F[c]$. The forward containment follows from Theorem 3.23 (c). For the reverse containment, let $r$ belong to the $\iota$-closure of $F[c]$. If $r \notin F[K]$ then $\{B_q(p) : p, q \in Q$ and $q < |p - r|/2\}$ would be an $\iota$-open cover of $F[K]$ with no finite subcover, which would pull back to an $\iota$-open cover of $K$ with no finite subcover. This contradicts $\iota$-compactness of $K$ and we conclude that $F[K]$ is the closure of $F[c]$, so it is an $\iota$-class by Corollary 3.14 (c).
(b) Again, one proves $\iota$-connectedness of the image just as in the classical case. This does not use the fact that the image is an $\iota$-class, but it implies that the image is an interval and hence it is in fact an $\iota$-class.

Corollary 3.26. Any $\iota$-continuous $\iota$-function on an $\iota$-separable $\iota$-compact $\iota$-subclass of $\mathbb{R}$, is bounded and achieves its maximum and minimum. Any $\iota$-continuous $\iota$-function on an $\iota$-connected $\iota$-subclass of $\mathbb{R}$, achieves all intermediate values.

4. Topology

We define $\iota$-metric and $\iota$-topological spaces and discuss their basic properties.

4.1. Metric spaces. Most of the following material on metric spaces is a straightforward generalization of material in Section 3.

Definition 4.1. An $\iota$-metric space is an $\iota$-class $X$ together with an $\iota$-function $D : X \times X \to [0, \infty)$ which satisfies the usual metric axioms, such that the map that takes convergent $\iota$-sequences to their limits is an $\iota$-function. It is $\iota$-separable if it contains an $\iota$-subset $x$ which intersects every ball and it is $\iota$-complete if every Cauchy $\iota$-sequence converges.

Definition 4.2. Let $X$ be an $\iota$-metric space. We define the completion of $X$ to be the $\iota$-class of Cauchy $\iota$-sequences in $X$ modulo the standard equivalence, with distance $\iota$-function defined in the usual way.

Proposition 4.3. The completion of any $\iota$-metric space $X$ is an $\iota$-complete $\iota$-metric space. The canonical embedding of $X$ in its completion is an $\iota$-bijection between $X$ and its image.

In Definition 4.1 we included an extra assumption stating that the map that takes convergent $\iota$-sequences to their limits is an $\iota$-function. This assumption is harmless because the standard construction of the $\iota$-completion ensures this condition. That is, if we apply the $\iota$-completion construction to an $\iota$-class $X$ equipped with any $\iota$-function $D : X \times X \to [0, \infty)$ which satisfies the usual metric axioms, the result is an $\iota$-complete $\iota$-metric space. The image of $X$ in its completion will also be an $\iota$-metric space, though if $X$ is not an $\iota$-metric space then the canonical embedding will not be an $\iota$-bijection of $X$ with its image because the inverse map will not be an $\iota$-function.

Proposition 4.4. In any $\iota$-metric space, any pointwise limit of an $\iota$-sequence of $\mathbb{R}$-valued $\iota$-functions is an $\iota$-function.

Definition 4.5. Let $X$ be an $\iota$-metric space. An $\iota$-class $U \subseteq X$ is $\iota$-open if there is an $\iota$-set $u \subseteq X \times \mathbb{R}^+$ such that for any $r \in X$ we have $r \in U$ if and only if there exists $(p, q) \in u$ with $D(r, p) < q$. An $\iota$-class $C \subseteq X$ is $\iota$-closed if its complement is $\iota$-open. We call $u$ a proxy both for the $\iota$-open $\iota$-class $U$ and the $\iota$-closed $\iota$-class $X \setminus U$.

Theorem 4.6. Let $X$ be an $\iota$-separable $\iota$-metric space.

(a) The union of any $\iota$-set of $\iota$-open $\iota$-classes is $\iota$-open and the intersection of any $\iota$-set of $\iota$-closed $\iota$-classes is $\iota$-closed.
(b) Any intersection of finitely many \( \iota \)-open \( \iota \)-classes is \( \iota \)-open and any union of finitely many \( \iota \)-closed \( \iota \)-classes is \( \iota \)-closed.

(c) Every \( \iota \)-subset of \( X \times \mathbb{R}^+_1 \) is a proxy for an \( \iota \)-open class (and for its complementary \( \iota \)-closed class).

(We need \( \iota \)-separability of \( X \) in part (b), and also to ensure that \( X \) is \( \iota \)-open.)

**Definition 4.7.** The \( \iota \)-closure of an \( \iota \)-class \( Y \) in an \( \iota \)-metric space \( X \) is the set of all limits of convergent \( \iota \)-sequences in \( Y \). \( Y \) is \( \iota \)-dense in \( X \) if its \( \iota \)-closure equals \( X \).

**Definition 4.8.** (a) An \( \iota \)-class \( K \) contained in an \( \iota \)-metric space is \( \iota \)-totally bounded if there is an \( \iota \)-function \( f : \mathbb{N} \rightarrow \mathcal{P}_{\text{fin}}(K) \) such that for each \( k \in \mathbb{N} \) the \( \iota \)-class \( K \) is covered by the balls of radius \( 2^{-k} \) about the elements of \( f(k) \).

(b) An \( \iota \)-subclass of an \( \iota \)-metric space is \( \iota \)-compact if any \( \iota \)-set of \( \iota \)-open \( \iota \)-classes which covers it has a finite subcover.

**Theorem 4.9.** Let \( X \) be an \( \iota \)-separable \( \iota \)-metric space. The following are equivalent:

(i) \( X \) is \( \iota \)-compact;

(ii) \( X \) is \( \iota \)-complete and \( \iota \)-totally bounded;

(iii) Every \( \iota \)-sequence in \( X \) has a convergent \( \iota \)-subsequence.

**Proof.** (i) \( \Rightarrow \) (ii): Suppose \( X \) is not \( \iota \)-complete and let \((r_n)\) be a Cauchy \( \iota \)-sequence with no limit. Let \( x \) be an \( \iota \)-dense \( \iota \)-subset of \( X \) and let \( y \subseteq x \times \mathbb{Q}^+ \) be the \( \iota \)-set of pairs \((p,q)\) such that \( 2q < \lim D(p,r_n) \). Then it is easy to see that \( \{B_y(p) : (p,q) \in y\} \) is an \( \iota \)-set of \( \iota \)-open balls that covers \( X \) but has no finite subcover. We conclude that any \( \iota \)-compact space is \( \iota \)-complete.

Now suppose \( X \) is \( \iota \)-compact. By Theorem 2.10 we may write \( x = \{r_n : n \in \mathbb{N}\} \) for some \( \iota \)-sequence \((r_n)\). We claim that for each \( k \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that every \( r \in x \) satisfies \( D(r,r_n) < 2^{-k} \) for some \( n \leq m \). If such an \( m \) exists for each \( k \) then we can verify total boundedness by setting \( f(k) = \{r_1, \ldots, r_m\} \). But if there were no such \( m \) for some value of \( k \) then the balls \( B_{2^{-k-1}}(r_n) \) would cover \( X \) yet have no finite subcover, contradicting \( \iota \)-compactness. So \( X \) is \( \iota \)-totally bounded.

(ii) \( \Rightarrow \) (iii): Suppose \( X \) is \( \iota \)-complete and \( \iota \)-totally bounded and let \((r_n)\) be an \( \iota \)-sequence in \( X \). Let \( f \) be an \( \iota \)-function which verifies total boundedness. For each \( k \) let \( g(k) \) be the set of \( r \in f(k) \) such that \( B_{2^{-k}}(r) \) contains infinitely many terms of the \( \iota \)-sequence \((r_n)\). Observe that \( g(k) \) is nonempty for each \( k \) and that for all \( r \in g(k) \) and all \( l > k \) there exists \( s \in g(l) \) with \( D(r,s) < 2^{-k} + 2^{-l} \), since the balls of radius \( 2^{-l} \) about the elements of \( f(l) \) within this distance of \( r \) cover \( B_{2^{-k}}(r) \). We can now define a Cauchy \( \iota \)-sequence \((s_k)\) by letting \( s_{k+1} \) be the \( \preceq_{\iota \iota} \)-minimal element of \( g(k+1) \) such that \( D(s_k,s_{k+1}) < 2^{-k} + 2^{-k-1} \). Finally, define an \( \iota \)-subsequence of \((r_n)\) by letting \( n_{k+1} \) be the smallest index larger than \( n_k \) such that \( D(s_{k+1},r_{n_{k+1}}) < 2^{-k} \). This will be a Cauchy \( \iota \)-sequence and hence it will converge.

(iii) \( \Rightarrow \) (i): Let \( z \) be an \( \iota \)-set of proxies of \( \iota \)-open \( \iota \)-classes which cover \( X \). Let \( x \) be an \( \iota \)-dense \( \iota \)-subset of \( X \) and let \( y \subseteq x \times \mathbb{Q}^+ \) be the \( \iota \)-set of pairs \((p,q)\) which satisfy \( D(p,p') + q \leq q' \) for some \((p',q')\) in one of the proxies in \( z \). Then the balls \( B_y(p) \) with \((p,q) \in y \) cover \( X \) and it will suffice to find a finite subset of \( y \) with the same property. Assuming \( y \) is infinite, by Theorem 2.10 we may
write \(y = \{(p_n, q_n) : n \in \mathbb{N}\}\) for some \(\iota\)-sequences \((p_n)\) and \((q_n)\). If there exists \(n\) such that every \(r \in x\) satisfies \(D(r, p_j) < q_j - 1/n\) for some \(j \leq n\) then the balls \(\{B_{q_j}(p_j) : j \leq n\}\) cover \(X\) and we are done. Otherwise, for each \(n\) let \(s_n\) be the \(\geq_{\iota\iota}\)-least element of \(x\) such that \(D(s_n, p_j) \geq q_j - 1/n\) for all \(j \leq n\). If (iii) holds then we may extract a convergent \(\iota\)-subsequence \((s_n)\), and its limit \(s\) must be contained in some ball \(B_{q_j}(p_j)\), which yields a contradiction. This completes the proof. □

**Definition 4.10.** An \(\iota\)-separable \(\iota\)-metric space \(X\) is **boundedly \(\iota\)-compact** if every \(\iota\)-closed ball \(\overline{B}_a(r) = \{s \in X : D(r, s) \leq a\}\) \((r \in X, a \geq 0)\) is \(\iota\)-compact.

**Proposition 4.11.** In any \(\iota\)-separable \(\iota\)-metric space the \(\iota\)-closure of any \(\iota\)-set is an \(\iota\)-closed \(\iota\)-class. In any \(\iota\)-separable boundedly \(\iota\)-compact \(\iota\)-metric space every \(\iota\)-closed \(\iota\)-class is \(\iota\)-separable.

The proof of the second part of Proposition 4.11 is similar to the second part of the proof of Theorem 3.13 augmented by the König’s lemma technique used in the proof of (ii) ⇒ (iii) in Theorem 3.13. (The observation that König’s lemma plays an important role in arguments of this type must be credited to the reverse mathematics school.) The problem is to find an \(\iota\)-dense \(\iota\)-subset \(c\) of the complement of an \(\iota\)-open \(\iota\)-class \(U\), and this is done in terms of an \(\iota\)-dense subset \(x\) of the ambient \(\iota\)-metric space \(X\). We construct \(c\) so as to contain a point in \(\overline{B}_{x_0}(p) \cap (X - U)\) for each \(r \in x\) and \(p \in \mathbb{Q}^+\) such that \(\overline{B}_p(r) \not\subseteq U\). This is possible because bounded \(\iota\)-compactness allows us to diagnose whether \(\overline{B}_p(r)\) is contained in \(U\) by checking whether it is contained in finitely many balls \(B_q(p')\) with \(p', q'\) in a proxy for \(U\). Doing this for \(\overline{B}_{2^{-n}}(s)\) for all \(n \in \mathbb{N}\) and \(s \in x \cap B_{x_0}(r)\), we can then use a König’s lemma argument together with the universal well-ordering to simultaneously extract a Cauchy sequence of centers of such balls for each \(r\) and \(p\) such that \(\overline{B}_p(r) \not\subseteq U\). We finally pass to the limits of the Cauchy sequences using the fact that the map that takes convergent \(\iota\)-sequences to their limits is an \(\iota\)-function.

**Definition 4.12.** Let \(X\) and \(Y\) be \(\iota\)-metric spaces. An \(\iota\)-function \(F : X \to Y\) is **\(\iota\)-continuous** if the inverse image of every \(\iota\)-open \(\iota\)-class in \(Y\) is an \(\iota\)-open \(\iota\)-class in \(X\). \(F\) is an **\(\iota\)-homeomorphism** if it is an \(\iota\)-bijection and both \(F\) and \(F^{-1}\) are \(\iota\)-continuous.

**Theorem 4.13.** Let \(X\) be an \(\iota\)-separable \(\iota\)-metric space, let \(Y\) be an \(\iota\)-metric space, and let \(F : X \to Y\) be an \(\iota\)-function. Then the following are equivalent:

(a) \(F\) is \(\iota\)-continuous.
(b) The inverse image of every \(\iota\)-closed \(\iota\)-class in \(Y\) is an \(\iota\)-closed \(\iota\)-class in \(X\).
(c) For any \(\iota\)-set \(c \subseteq X\) with \(\iota\)-closure \(C\), the \(\iota\)-closure of \(F[c]\) contains \(F[C]\).
(d) \(F\) preserves convergence of \(\iota\)-sequences.
(e) For every \(r \in X\) and every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(|r - s| < \delta\) implies \(|F(r) - F(s)| < \epsilon\).

If any of these conditions holds then there is an \(\iota\)-function which takes any proxy \(u\) of an \(\iota\)-open \(\iota\)-class \(U \subseteq Y\) to a proxy \(v\) of an \(\iota\)-open \(\iota\)-class \(V \subseteq X\) such that \(F^{-1}[U] = V\).

**Proposition 4.14.** Let \(X\) be an \(\iota\)-compact \(\iota\)-metric space and let \(Y\) be an \(\iota\)-metric space.

(a) Every \(\iota\)-closed \(\iota\)-subclass of \(X\) is \(\iota\)-compact.
(b) If \( X \) is \( \iota \)-separable and \( F : X \to Y \) is an \( \iota \)-continuous \( \iota \)-function then \( F[X] \) is an \( \iota \)-separable \( \iota \)-compact \( \iota \)-subclass of \( Y \).

(c) If \( X \) is \( \iota \)-separable and \( F : X \to Y \) is an \( \iota \)-continuous \( \iota \)-bijection then it is an \( \iota \)-homeomorphism.

(In the proof of part (c) use parts (a) and (b) of this proposition together with the second part of Proposition 4.11 to show that the image of any \( \iota \)-closed \( \iota \)-subclass of \( X \) is an \( \iota \)-separable \( \iota \)-compact \( \iota \)-subclass of \( Y \), then show that \( Y \) must be \( \iota \)-separable using Theorem 4.13 (c), and then use the first part of Proposition 4.11 to verify that any \( \iota \)-separable \( \iota \)-compact \( \iota \)-subclass of \( Y \) must be \( \iota \)-closed.)

**Theorem 4.15.** (Baire category theorem) The intersection of any \( \iota \)-set of \( \iota \)-open \( \iota \)-dense \( \iota \)-classes in an \( \iota \)-separable \( \iota \)-complete \( \iota \)-metric space is \( \iota \)-dense.

**Proof.** Recall that any \( \iota \)-set of \( \iota \)-open \( \iota \)-dense \( \iota \)-classes is countable by Theorem 2.10. So let \( X \) be an \( \iota \)-separable \( \iota \)-complete \( \iota \)-metric space and let \( (u_n) \) be an \( \iota \)-sequence of proxies for \( \iota \)-open \( \iota \)-dense \( \iota \)-classes \( U_n \) in \( X \). Let \( x \) be an \( \iota \)-dense \( \iota \)-subset of \( X \). For each \( r \in x \) and each \( q \in \mathbb{Q}^+ \) we can find a point \( s_{r,q} \in B_q(r) \cap U_n \) as follows.

As in the classical proof, recursively define an \( \iota \)-sequence \( (r_n) \) in \( x \) together with an \( \iota \)-sequence of radii \( (q_n) \) in \( \mathbb{Q}^+ \) such that \( q_n \leq 2^{-n} \) and \( B_{2q_n+1}(r_{n+1}) \subseteq B_{q_n}(r_n) \cap U_n \). Then let \( s_{r,q} \) be the limit of this \( \iota \)-sequence. Using \( \preceq_{\mathfrak{U}} \) and the fact that the map taking convergent \( \iota \)-sequences to their limits is an \( \iota \)-function, this construction can be carried out simultaneously for all \( r \) and \( q \), so the \( s_{r,q} \) constitute an \( \iota \)-set which is contained in the intersection of the \( U_n \) and whose \( \iota \)-closure is an \( \iota \)-closed \( \iota \)-class (by Proposition 4.11) which contains \( x \). Since \( x \) is \( \iota \)-dense in \( X \), we are done. \( \square \)

### 4.2. Topological spaces

In order to define a topology on an \( \iota \)-class \( X \) we must specify which of its \( \iota \)-subclasses are open. This is naturally done by specifying a single \( \iota \)-subclass \( \mathcal{T} \) of \( T \times X \) where the elements of \( T \) serve as proxies for the open \( \iota \)-classes. Thus each \( a \in T \) is a proxy for the \( \iota \)-class \( U_a = \{ r \in X : \langle a, r \rangle \in \mathcal{T} \} \). It is convenient to simply take \( T = J_2 \); there is no loss in generality since we always have \( T \times X \subseteq J_2 \times X \).

Another feature of the following definition might require explanation. Classically the family of open sets is closed under arbitrary unions and finite intersections. However, verifying these closure properties in any given example would generally be done by in effect determining proxies for the union and intersection. Thus it is natural in our setting, and it bars no important examples, to require the existence of \( \iota \)-functions which evaluate unions and intersections of proxies.

**Definition 4.16.** An \( \iota \)-topological space (or just \( \iota \)-space) is an \( \iota \)-class \( X \) together with an \( \iota \)-subclass \( \mathcal{T} \) of \( J_2 \times X \) with the following properties:

(i) \( \emptyset = U_a \) and \( X = U_b \) for some \( a, b \in J_2 \);

(ii) there is an \( \iota \)-function \( \kappa : \mathcal{P}_s(J_2) \to J_2 \) such that for any \( \iota \)-set \( x \) we have

\[
\bigcup_{a \in x} U_a = U_{\kappa(x)};
\]

(iii) there is an \( \iota \)-function \( \lambda : J_2 \times J_2 \to J_2 \) such that for any \( \iota \)-sets \( a \) and \( b \) we have

\[
U_a \cap U_b = U_{\lambda(a,b)}
\]

where \( U_a = \{ r \in X : \langle a, r \rangle \in \mathcal{T} \} \) for any \( a \in J_2 \). The \( U_a \) are the \( \iota \)-open \( \iota \)-classes in \( X \) and \( \mathcal{T} \) is an \( \iota \)-topology on \( X \). The \( \iota \)-closed \( \iota \)-classes are the complements \( X - U_a \).
Example 4.17. (a) Let \( x \) be an \( \iota \)-set and let \( \mathcal{T} \) be the \( \iota \)-class of all ordered pairs \( \langle y, r \rangle \in J_2 \times x \) such that \( r \in x \cap y \). This is the discrete \( \iota \)-topology on \( x \).
(b) Let \( X \) be an \( \iota \)-class and let \( \mathcal{T} \) be the \( \iota \)-class of all ordered pairs \( \langle x, r \rangle \in J_2 \times X \) such that \( x = (0, y) \) for some \( y \in J_2 \) and \( r \in X - y \). This is the co-countable \( \iota \)-topology on \( X \).

It is straightforward to verify that the discrete and co-countable \( \iota \)-topologies satisfy Definition 4.16.

Definition 4.18. Let \( X \) be an \( \iota \)-space with \( \iota \)-topology \( \mathcal{T} \subseteq J_2 \times X \) and let \( Y \) be an \( \iota \)-subclass of \( X \). Then \( \mathcal{T}' = \mathcal{T} \cap (J_2 \times Y) \) is the relative \( \iota \)-topology on \( Y \) and \( X \) equipped with this \( \iota \)-topology is an \( \iota \)-subspace of \( X \).

Next we introduce a basic tool for constructing \( \iota \)-topologies. If \( \mathcal{B} \subseteq B \times X \) and \( a \in B \) then we write \( B_a = \{ r \in X : \langle a, r \rangle \in \mathcal{B} \} \).

Proposition 4.19. Let \( X \) and \( B \) be \( \iota \)-classes and let \( \mathcal{B} \) be an \( \iota \)-subclass of \( B \times X \) such that \( B_a = X \) for some \( a \in B \). Suppose also that there exists an \( \iota \)-function \( \lambda_0 : B \times B \to \mathcal{P}_i(B) \) such that for any \( a, b \in B \) we have \( B_a \cap B_b = \bigcup_{c \in \lambda_0(a,b)} B_c \). Then the \( \iota \)-class \( \mathcal{T} \subseteq J_2 \times X \) consisting of the ordered pairs \( \langle x, r \rangle \) such that \( x \in \mathcal{P}_i(B) \) and \( r \in \bigcup_{a \in x} B_a \) is an \( \iota \)-topology on \( X \).

Definition 4.20. The \( \iota \)-topology defined in Proposition 4.19 is the \( \iota \)-topology generated by \( \mathcal{B} \).

Example 4.21. Let \( X \) be an \( \iota \)-separable \( \iota \)-metric space and let \( x \subseteq X \) be an \( \iota \)-dense \( \iota \)-subset. Let \( B = x \times \mathbb{Q}^+ \) and define \( \mathcal{B} \subseteq B \times X \) to be \( \{ \langle r, q, s \rangle : D(r, s) < q \} \). This satisfies the conditions of Proposition 4.19 with

\[
\lambda_0(\{ \langle r_1, q_1 \rangle, \langle r_2, q_2 \rangle \}) = \{ \langle r, q \rangle \in x \times \mathbb{Q}^+ : D(r, r_i) + q \leq q_i \text{ for } i = 1, 2 \}.
\]

The \( \iota \)-open classes for the \( \iota \)-topology generated by \( \mathcal{B} \) are precisely the \( \iota \)-open classes identified in Definition 4.3. This is the metric \( \iota \)-topology on \( X \).

Definition 4.22. (a) An \( \iota \)-family of \( \iota \)-topological spaces consists of an \( \iota \)-set \( x \), an \( \iota \)-class \( \mathcal{X} \subseteq x \times J_2 \), and an \( \iota \)-class \( \mathcal{T} \subseteq x \times J_2 \times J_2 \) with the property that for each \( a \in x \) the \( \iota \)-class

\[
\mathcal{T}_a = \{ \langle b, r \rangle : \langle a, b, r \rangle \in \mathcal{T} \}
\]

is an \( \iota \)-topology on \( X_a = \{ r \in J_2 : \langle a, r \rangle \in \mathcal{X} \} \). We will write \( \mathcal{X} = \{ X_a : a \in x \} \).

(b) Given an \( \iota \)-family of \( \iota \)-topological spaces \( \{ X_a : a \in x \} \), let \( B \) be the \( \iota \)-class of functions from finite subsets of \( x \) into \( J_2 \) and let \( \mathcal{B} \subseteq B \times \prod X_a \) be the \( \iota \)-class of pairs \( \langle h, f \rangle \) such that for each \( a \in \text{dom}(h) \) we have \( \langle a, h(a), f(a) \rangle \in \mathcal{T} \). The product \( \iota \)-topology on \( \prod X_a \) is the \( \iota \)-topology generated by \( \mathcal{B} \).

4.3. Continuity.

Definition 4.23. Let \( X \) and \( Y \) be \( \iota \)-topological spaces with \( \iota \)-topologies \( \mathcal{T}_X \) and \( \mathcal{T}_Y \). An \( \iota \)-function \( F : X \to Y \) is \( \iota \)-continuous if there is an \( \iota \)-function \( \widetilde{F} : J_2 \to J_2 \) such that for each \( b \in J_2 \) we have \( F^{-1}[V_b] = U_{\widetilde{F}(b)} \), where \( U_a = \{ r \in X : \langle a, r \rangle \in \mathcal{T}_X \} \) and \( V_b = \{ s \in Y : \langle b, s \rangle \in \mathcal{T}_Y \} \). An \( \iota \)-homeomorphism is an \( \iota \)-continuous \( \iota \)-bijection whose inverse is also \( \iota \)-continuous.

Proposition 4.24. Compositions of \( \iota \)-continuous \( \iota \)-functions are \( \iota \)-continuous.
Proposition 4.27. Let $\kappa$ with the function $\iota$ is sufficient to consider this case by Proposition 4.24) then the function from $\mathcal{T}$ to $\mathcal{F}$ is $\iota$-second countable if it has an $\iota$-base for which $B$ is an $\iota$-set.

Proposition 4.26. Let $X$ and $Y$ be $\iota$-topological spaces and let $F : X \to Y$ be an $\iota$-function. Suppose $\mathcal{B} \subseteq B \times Y$ is an $\iota$-base for $Y$. Then $F$ is $\iota$-continuous if and only if there is an $\iota$-function $\bar{F}_0 : B \to J_2$ such that for each $b \in B$ we have $F^{-1}[V_b] = U_{\bar{F}_0(b)}$, where $U_a = \{r \in X : \langle a, r \rangle \in \mathcal{T}_X\}$ and $V_b = \{s \in Y : \langle b, s \rangle \in \mathcal{T}_Y\}$.

The key observation in the proof of Proposition 4.26 is that if $\mathcal{B}$ generates $\mathcal{T}_Y$ (it is sufficient to consider this case by Proposition 4.24) then the function from $\mathcal{P}_1(B)$ to $J_2$ which takes an $\iota$-subset $x \subseteq B$ to $\{F(b) : b \in x\}$ is an $\iota$-function. We can then define an $\iota$-function $\bar{F}$ that verifies $\iota$-continuity by composing this function with the function $\kappa$ of Definition 4.18 (relative to $\mathcal{T}_X$).

Proposition 4.27. Let $\{X_a : a \in J_2\}$ be an $\iota$-family of $\iota$-topological spaces and let $Y$ be an $\iota$-topological space. For each $b \in J_2$ let $\pi_b : \prod_a X_a \to X_b$ be the natural projection.

(a) Each $\pi_b$ is $\iota$-continuous.

(b) Let $F : Y \to \prod_a X_a$ be an $\iota$-function. Then $F$ is $\iota$-continuous if and only if the $\iota$-function $G : Y \times x \to \prod_a X_a (= \text{the disjoint union of the } X_a)$ defined by $G(a, r) = F(r)(a) \in X_a$ is $\iota$-continuous. Here we give $x$ the discrete $\iota$-topology, $Y \times x$ the product $\iota$-topology, and $\prod_a X_a$ the $\iota$-topology generated by the $\iota$-topologies on the individual $\iota$-spaces.

At this point we could go on to develop general topology in the $J_2$ setting. Most standard results go through, although usually additional hypotheses such as separability or second countability are required. However, this seems somewhat extraneous to the development of core mathematics since most topological spaces that appear in ordinary settings are separable and metrizable and for these spaces most of the basic results in general topology are easy. Those spaces not of this type which do appear in mainstream settings (e.g., the Zariski topology) do not seem to require deep results from general topology. The weak* topology on the dual of a separable Banach space is typically not metrizable but its restriction to the unit ball is, and this coupled with the Krein-Smullian theorem appears to render metric space theory sufficient for most applications.

5. Other topics

We sketch ways of developing various other topics from abstract analysis within $J_2$.

5.1. Measure and integration. We can define $\iota\sigma$-algebras in a manner analogous to the definition of $\iota$-topologies (Definition 4.18). Thus, an $\iota\sigma$-algebra on an $\iota$-class $X$ is an $\iota$-subspace $M$ of $J_2 \times X$ for which there exist $\iota$-functions $\kappa : J_2 \to J_2$ and $\lambda : \mathcal{P}_1(J_2) \to J_2$ such that for any $a$ and $x$ we have $X - M_a = M_{\kappa(a)}$ and $\bigcup_{a \in J_2} M_a = M_{\lambda(x)}$, where $M_a = \{r \in X : \langle a, r \rangle \in M\}$ for any $a \in J_2$. We call the sets $M_a$ the measurable $\iota$-subsets of $X$ and we call $a$ a proxy for $M_a$. 
Most $\sigma$-algebras of interest are defined in terms of a generating set, and it may not be obvious how to generate an $\nu$-$\sigma$-algebra from a given family of $\nu$-classes because classically this involves a recursive construction along the set of all countable ordinals, which are not available in $J_2$. However, it can be done in the following way.

Let $X$ and $B$ be $\nu$-classes and let $B \subseteq B \times X$ be an $\nu$-class. We describe a way to construct an $\nu$-$\sigma$-algebra $M$ generated by the sets $B_a = \{r \in X : \langle a, r \rangle \in B\}$.

We may assume that the set $\{B_a : a \in B\}$ is an algebra, i.e., it is closed under complements and finite unions. It also simplifies matters slightly to observe that we only need closure under the single operation which, for any $\nu$-set of $\nu$-measurable $\nu$-subclasses of $X$, forms the complement of their union. Applying this operation to an $\nu$-set containing only one $\nu$-class yields the complement of that $\nu$-class, and composing the operation with complementation then produces unions. Thus, proxies for $\nu$-measurable $\nu$-classes will be given by trees whose terminal nodes are elements of $B$ (i.e., proxies for generating $\nu$-classes) and each of whose non-terminal nodes will represent the complement of the union of the $\nu$-classes corresponding to its immediate successors.

Specifically, define a $B$-tree to be a small $\nu$-function $f$ whose range is contained in $\mathbb{N}$ and which satisfies the following conditions. For all $a \in \text{dom}(f)$, if $f(a) = 0$ then $a \in B$; if $f(a) = 1$ then $a$ is infinite; and if $f(a) \geq 1$ then $a \subseteq \Gamma(f)$ and we have $f(a) = 1 + \sup_{b \in \text{dom}(a)} f(b)$. We also require that there exist exactly one element on which $f$ attains a maximal value.

We now define $M \subseteq J_2 \times X$. If $a \in J_2$ is not a $B$-tree then let $M_a = \emptyset$. For any $B$-tree $f$, we define $M_f$ to be the set of $r \in X$ for which there exists an $\nu$-function $g : \text{dom}(f) \rightarrow \{0, 1\}$ such that (1) if $f(a) = 0$ then $g(a) = 1$ if and only if $r \in B_a$, (2) if $f(a) > 0$ then $g(a) = 1$ if and only if $g(b) = 0$ for all $b \in \text{dom}(a)$, and (3) $g(a) = 1$ for the element $a$ on which $f$ attains its maximum value.

What makes this construction work is the condition that $a$ be infinite if $f(a) = 1$. This means that either the domain of $f$ consists of a single element in $B$ or else, in set-theoretic terms, $f$ has infinite rank. An easy induction shows that every set in $J_2$ has rank less than $2\omega$, so it follows that any $\nu$-set of $B$-trees contains only $B$-trees of at most some maximal finite height. Thus, any $\nu$-set of $B$-trees can be amalgamated into a single $B$-tree and this can be used to show that $M$ is closed under complements of unions.

We define an $\nu$-measure on an $\nu$-class $X$ equipped with an $\nu$-$\sigma$-algebra $M$ to be an $\nu$-function $\mu$ from $J_2$ into $[0, \infty]$ which satisfies $\mu(a) = \mu(b)$ if $M_a = M_b$, $\mu(a) = 0$ if $M_a = \emptyset$, and $\mu(\kappa(x)) = \sum_{a \in x} \mu(a)$ if the $\nu$-classes $M_a (a \in x$) are disjoint. If $M$ is generated by an algebra $B$ and $\mu_0$ is a premeasure on $B$ then we can extend $\mu_0$ to an $\nu$-measure $\mu$ by a standard inner/outer measure construction. Specifically, if $\mu_0(X) < \infty$ then we can show by induction on $B$-trees that for every $M_a$ there exists a pair of $\nu$-sequences $(a_n)$ and $(b_n)$ such that each $a_n$ is an $\nu$-subset of $B$ with $M_a \subseteq \bigcup_{b \in a_n} B_b$; each $b_n$ is an $\nu$-subset of $B$ with $X - M_a \subseteq \bigcup_{b \in b_n} B_b$; and $\sum_{b \in a_n} \mu_0(b) + \sum_{b \in b_n} \mu_0(b) \leq \mu_0(X) + 2^{-n}$. This can then be used to define $\mu$. The case $\mu_0(X) = \infty$ introduces no fundamental difficulties.

The theory of integration seems to carry over with no major complications. We define an $\nu$-measurable $\nu$-function from $X$ to $Y$ to be an $\nu$-function $F : X \rightarrow Y$ such that there exists an $\nu$-function $\tilde{F}$ from proxies of $\nu$-measurable $\nu$-classes $N_b$ in $Y$ to proxies of $\nu$-measurable $\nu$-classes $M_a$ in $X$ which satisfies $F^{-1}[N_b] = M_{\tilde{F}(b)}$ for all $b \in J_2$. This makes approximation by simple functions possible since, e.g.,
\((-\infty,k/n] \cap \mathbb{R}_+ : k \in \mathbb{Z}\)} is an \(\iota\)-set of \(\iota\)-measurable \(\iota\)-classes in \(\mathbb{R}_+\) for each \(n\), and the family of all such \(\iota\)-sets is an \(\iota\)-set, so the fact that \(\tilde{F}\) is an \(\iota\)-function would permit the construction of a corresponding \(\iota\)-set of simple \(\iota\)-functions which approximate \(F\) and have range contained in \(\{k/n : k \in \mathbb{Z}\}\).

5.2. Banach spaces. We define an \(\iota\)-real \(\iota\)-vector space to be an \(\iota\)-class \(V\) equipped with \(\iota\)-functions \(+: V \times V \to V\) and \(\cdot: \mathbb{R}_+ \times V \to V\) which satisfy the usual vector space axioms, such that for any \(\iota\)-set \(x \subseteq V\) and any small \(\iota\)-subfield \(y \subseteq \mathbb{R}_+\), the \(y\)-linear span of \(x\) is an \(\iota\)-set. An \(\iota\)-real \(\iota\)-Banach space is an \(\iota\)-real \(\iota\)-vector space equipped with an \(\iota\)-function \(\| \cdot \|: V \to [0, \infty)\) which satisfies the norm axioms and such that \(D(r, s) = \|r - s\|\) is an \(\iota\)-complete \(\iota\)-metric on \(V\). We can define \(\iota\)-complex \(\iota\)-Banach spaces analogously.

Much of the general theory of Banach spaces seems to carry over fairly easily. There is no problem proving versions of the open mapping and closed graph theorems and the principle of uniform boundedness. The Hahn-Banach theorem presents a difficulty, however. Suppose we want to extend a bounded linear functional on an \(\iota\)-separable \(\iota\)-subspace of an \(\iota\)-separable \(\iota\)-real \(\iota\)-Banach space \(V\) to the entire space without increasing its norm. Extending by a single dimension can be done in the usual way, but it is not obvious that a sequence of one-dimensional extensions will give rise to a (small) \(\iota\)-function on an \(\iota\)-dense \(\iota\)-subset of \(V\). The problem is that at each step we define a new \(\iota\)-real number and it is not clear that the sequence of \(\iota\)-reals so obtained must be an \(\iota\)-set. One way to get around this problem is by using only rational numbers to define the extensions and allowing the norm to increase by a small amount. In fact, if we use rationals for the extensions we can simultaneously, for all \(n\), define extensions which increase the norm by at most \(2^{-n}\) and which converge pointwise to a single extension that does not increase the norm. In this way we can ensure that the extension to \(V\) is an \(\iota\)-function.

If \(V\) is an \(\iota\)-separable \(\iota\)-Banach space then we define its \(\iota\)-dual to be the \(\iota\)-class of bounded linear \(\iota\)-functions from an \(\iota\)-dense \(\iota\)-subset of \(V\) (which is without loss of generality a vector space over \(\mathbb{Q}\), say) into the scalar field. Each such \(\iota\)-function \(f\) can be regarded as a proxy for a bounded linear \(\iota\)-function \(F\) from \(V\) into the scalars.

5.3. Function spaces. Let \(X\) be an \(\iota\)-separable \(\iota\)-compact \(\iota\)-metric space. We can define \(C(X)\) to be the \(\iota\)-class of uniformly \(\iota\)-continuous \(\iota\)-functions from an \(\iota\)-dense \(\iota\)-subset of \(X\) into the scalars. We want to think of each such function as a proxy for a continuous function on \(X\) and it is clear how to do this. The following seems like a good general definition. An \(\iota\)-function space over \(X\) is an \(\iota\)-subclass \(F\) of \(V \times X \times \mathbb{F}_\iota\), where \(V\) is an \(\iota\)-Banach space, \(\mathbb{F}_\iota\) is the scalar field, for each \(v \in V\) the \(\iota\)-class \(\{\langle r, s \rangle \in X \times \mathbb{F}_\iota : \langle v, r, s \rangle \in F\}\) is the graph of an \(\iota\)-function \(F_v : X \to \mathbb{F}_\iota\), and we have \(F_{v+w} = F_v + F_w\) and \(F_{rv} = rF_v\) for all \(v, w \in V\) and \(r \in \mathbb{F}_\iota\). That is, the vector space operations in \(V\) correspond to pointwise operations on the \(\iota\)-functions \(F_v\).

This definition encompasses not only standard function spaces like \(C(X)\), but also the \(\iota\)-dual of an \(\iota\)-Banach space as defined in the preceding section. However, it does not work with “function” spaces like \(L^p(X)\) whose elements are actually equivalence classes of functions. One way to handle these would be to weaken the \(\iota\)-function space definition so that \(F_{v+w} = F_v + F_w\) and \(F_{rv} = rF_v\) hold off of an \(\iota\)-class with \(\iota\)-measure zero, for each \(v, w, r\).
Versions of the Stone-Weierstrass and Tietze extension theorems hold for $C(X)$ with $X$ an $\iota$-separable $\iota$-compact $\iota$-metric space. In the former case, the hypothesis must include that the $\iota$-dense subalgebra of $C(X)$ be $\iota$-separable. We can then work exclusively with an $\iota$-dense $\iota$-set of functions in the subalgebra, which permits execution of the compactness arguments needed in the proof.

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