Correlation between conserved charges in PNJL Model with multi-quark interactions

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We present a study of correlations among conserved charges like baryon number, electric charge and strangeness in the framework of 2+1 flavor Polyakov loop extended Nambu-Jona-Lasinio model at vanishing chemical potentials, up to fourth order. Correlations up to second order have been measured in Lattice QCD which compares well with our estimates given the inherent difference in the pion masses in the two systems. Possible physical implications of these correlations and their importance in understanding the matter obtained in heavy-ion collisions are discussed. We also present comparison of the results with the commonly used unbound effective potential in the quark sector of this model.

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I. INTRODUCTION

It is now well established that strongly interacting matter may exhibit a variety of phases depending on the ambient thermodynamic conditions. We are on the way to draw the phase diagram of quantum chromodynamics (QCD) which is the theory of strong interactions. The difficulty that we still encounter is to work with relatively strong coupling strengths in the theory. The best way to go about is to perform numerical simulation on the discretized version of QCD - the so called Lattice QCD (LQCD). This formulation however has not got rid of all its inherent technical problems and it would take some time to find the final answers [1–14]. Meanwhile one can look into the properties of strongly interacting matter through effective models of QCD. Polyakov loop extended Nambu-Jona-Lasinio (PNJL) model is one such model that successfully captures various properties of strongly interacting matter [15–30]. The question as to exactly what extent can this model emulate QCD is still a matter under investigation. The best way to judge it is to measure several sensitive quantities in this model and contrast some of them to that available in the LQCD measurements. The correlations among conserved charges

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are some such quantities that we intend to investigate here. Essentially, the fluctuations and correlations of conserved charges and their higher order cumulants provide information about the degrees of freedom of strongly interacting matter. These can be extracted from the PNJL model and LQCD through the study of diagonal and off-diagonal susceptibilities respectively. They can also provide information about the existence of critical behavior, if any. This phenomenological study is important as there is a rapid progress in the experimental front at the facilities at CERN and Brookhaven heavy-ion colliders.

On the Lattice, some of these correlators have been measured at zero chemical potential. With 2 flavors it was shown \[31, 34\] that the fluctuations rise rapidly around the crossover region from hadronic matter to quasi-free quark matter. The higher order cumulants show non-monotonic behavior \[7, 35\]. Similar measurements have been made in 2 flavor PNJL model with three-momentum cutoff regularization \[21, 36, 38\]. Quark number susceptibility (QNS) at finite density has been estimated in some works within 2 flavor PNJL model \[39\]. Also, QNS has been studied in Hard Thermal Loop approximation \[40, 42\]. Recently the idea of the numerical Taylor expansion in terms of chemical potential for PNJL model has been used within the constraint that the net strange quark density is zero, which is the case in ultra relativistic heavy ion collision \[43\].

For 2+1 flavors, fluctuations have been recently measured in LQCD \[44-46\] as well as in PNJL model both with the usual unbound effective potential (UEP) \[18, 47, 48\] and with the bound effective potential (BEP) \[49\]. Similar calculations have been carried out in Polyakov loop coupled quark-meson (PQM) model \[50-53\] and its renormalization group improved version \[54\].

In this work we investigate the off-diagonal susceptibilities which give the correlations among different conserved charges. Our paper is organized as follows. In Sec. II, we discuss the basic formalism of the PNJL model as well as the method of extracting the Taylor expansion coefficients of pressure that gives the various susceptibilities. In section III we present and discuss our results together with a comparison with the data obtained in LQCD. The last section contains a summary and our conclusions.

II. FORMALISM

A. Thermodynamic Potential

2+1 flavor PNJL model with unbound effective potential has been studied elaborately in a number of recent works \[13, 16, 20, 22\]. To introduce a bound in the effective potential eight quark interaction terms have been introduced in 2+1 flavor NJL model \[24, 27\] and in the 2 flavor PNJL model \[28, 29\]. We developed the 2+1 flavor PNJL model with bound effective potential to study finite temperature and chemical potential properties \[30\] within three-momentum cutoff regularization scheme. Comparing with available LQCD data the model was shown to reproduce various aspects of QCD thermodynamics quite satisfactorily. We shall be using this model in the present work. The relevant thermodynamic potential in the mean
field approximation can be written as [30],

$$\Omega = U' [\Phi, \bar{\Phi}, T] + 2g_{\text{S}} \sum_{f=u,d,s} \sigma_f^2 - \frac{g_D}{2} \sigma_u \sigma_d \sigma_s + 3 \frac{g_1}{2} ( \sum_{f=u,d,s} \sigma_f^2 )^2$$

$$+ 3g_2 \sum_{f=u,d,s} \sigma_f^2 - 6 \sum_{f=u,d,s} \int_0^\Lambda \frac{d^3p}{(2\pi)^3} E_f \Theta ( \Lambda - |\vec{p}| )$$

$$- 2T \sum_{f=u,d,s} \int_0^\infty \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 3( \Phi + \bar{\Phi} e^{-\frac{E_f - \mu_f}{T}} + e^{-\frac{E_f + \mu_f}{T}} ) \right]$$

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(1)

where $g_{\text{S}}$ and $g_D$ are the four quark and six quark coupling constant and $g_1$ and $g_2$ are the eight quark coupling constant. Here $\sigma_f = \langle \bar{\psi}_f \psi_f \rangle$ denotes chiral condensate of the quark with flavor $f$ and $E_f = \sqrt{\vec{p}^2 + M_f^2}$ is the single quasi-particle energy. Here, constituent mass $M_f$ of flavor $f$ is given by the self-consistent gap equation;

$$M_f = m_f - 2g_{\text{S}} \sigma_f + \frac{g_D}{2} \sigma_{f+1} \sigma_{f+2} - 2g_1 \sigma_f ( \sigma_u^2 + \sigma_d^2 + \sigma_s^2 ) - 4g_2 \sigma_f^3$$

where $f$, $f+1$ and $f+2$ take the labels of flavor $u$, $d$ and $s$ in cyclic order. In the above expression, the vacuum part integral has an ultraviolet cutoff $\Lambda$. For fixing the parameters $m_{\text{S}}$, $\Lambda$, $g_{\text{S}}$, $g_D$, $g_1$, $g_2$ we have used the following physical conditions [30]:

$$m_\pi = 138 \text{ MeV} \quad m_K = 494 \text{ MeV} \quad m_0 = 480 \text{ MeV} \quad m_{\eta'} = 957 \text{ MeV}$$

$$f_\pi = 93 \text{ MeV} \quad f_K = 117 \text{ MeV}$$

and $m_u$ is kept fixed at 5.5 MeV. The parameters are given in table [II] for UEP and BEP.

The Polyakov loop $\Phi$ and its charge conjugate $\bar{\Phi}$ are defined as,

$$\Phi = (\text{Tr}_c L)/N_c, \quad \bar{\Phi} = (\text{Tr}_c L^\dagger)/N_c$$

where, $L$ is the Wilson line given by,

$$L = \left[ \mathcal{P} \exp \left( i \int_0^\beta A_4 d\tau \right) \right] = \exp \left[ i \frac{A_4}{T} \right]$$

The Polyakov loop potential $U'$ with the Vandermonde (VdM) term can be expressed as [22],

$$U' (\Phi, \bar{\Phi}, T) / T^4 = U (\Phi, \bar{\Phi}, T) / T^4 - \kappa \ln [J (\Phi, \bar{\Phi})]$$

(2)

where $U (\Phi, \bar{\Phi}, T)$ is the Landau-Ginsburg type potential given by [16],

$$\frac{U (\Phi, \bar{\Phi}, T)}{T^4} = - \frac{b_2 (T)}{2} \Phi \bar{\Phi} - \frac{b_1}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\Phi \bar{\Phi})^2$$

(3)

with,

$$b_2 (T) = a_0 + a_1 \frac{T_0}{T} + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3,$$

(4)
and \( b_3, b_4 \) are constants. \( T_0 \) is the deconfinement temperature in a pure gauge theory. The VdM determinant in eqn. 2 is given by

\[
J[\Phi, \bar{\Phi}] = (27/24\pi^2)(1 - 6\Phi \bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2)
\]

Here \( \kappa \) is a phenomenological constant which is determined by reproducing the pressure calculated on Lattice. For the Polyakov loop potential we choose the parameters which reproduce the Lattice data of pure gauge thermodynamics [1]. According to pure SU(3) lattice gauge theory value of \( T_0 \) is found to be 270 MeV. However we took \( T_0 \) as 190 MeV to get the crossover temperature (\( T_c \)) consistent with the LQCD data. Various thermodynamic quantities like scaled pressure, entropy and energy density are reproduced extremely well in Polyakov loop model using the ansatz (3) and (4) with parameters summarized below,

\[
a_0 = 6.75, a_1 = -1.95, a_2 = 2.625, a_3 = -7.44, b_3 = 0.75, b_4 = 7.5, T_0 = 190\text{MeV}
\]

There have been some work [55] where the Polyakov loop parameters have been obtained by fitting with the full LQCD results rather than with the pure gauge theory results as done here. However, since we have fitted the pressure obtained in our model with that obtained in the full LQCD to obtain the parameter \( \kappa \), the effect of full QCD is incorporated. In this work the Polyakov loop is a global object. One can improve on it by taking in to the consideration of quantum and local corrections as done in [56, 57]. However, in this work, we are mainly looking at the trends of different observables rather than exact matching with the LQCD results. As more and more refined LQCD results are coming up we understand that exact quantitative status is going to change. Hence we do not incorporate those involved calculations in this work.

### B. Taylor expansion of pressure

The pressure of the strongly interacting matter can be written as,

\[
P(T, \mu_B, \mu_Q, \mu_S) = -\Omega(T, \mu_B, \mu_Q, \mu_S), \tag{5}
\]

where \( T \) is the temperature, \( \mu_B \) is the baryon (B) chemical potential, \( \mu_Q \) is the charge (Q) chemical potential and \( \mu_S \) is the strangeness (S) chemical potential. From the usual thermodynamic relations the first derivative of pressure with respect to quark chemical potential \( \mu_q \) is the quark number density and the second derivative corresponds to the QNS.

Our first job is to minimize the thermodynamic potential numerically with respect to the fields \( \sigma_u, \sigma_d, \sigma_s, \Phi \) and \( \bar{\Phi} \). Using these values of the fields we get the mean field value for pressure using the equation (5). The scaled pressure obtained in a given range of chemical potential at a particular temperature can be expressed in a Taylor series as,

\[
\frac{p(T, \mu_B, \mu_Q, \mu_S)}{T^4} = \sum_{n=i+j+k} c_{i,j,k}^{B,Q,S}(T)\left(\frac{\mu_B}{T}\right)^i\left(\frac{\mu_Q}{T}\right)^j\left(\frac{\mu_S}{T}\right)^k \tag{6}
\]
where,

\[ c_{B,Q,S}^{i,j,k}(T) = \left. \frac{1}{i!j!k!} \frac{\partial^i}{\partial \mu_B^i} \frac{\partial^j}{\partial \mu_Q^j} \frac{\partial^k}{\partial \mu_S^k} \right|_{\mu_q,Q,S=0} \frac{\partial^i}{\partial \mu_B^i} \frac{\partial^j}{\partial \mu_Q^j} \frac{\partial^k}{\partial \mu_S^k} \]  

(7)

The flavor chemical potentials \( \mu_u, \mu_d, \mu_s \) are related to \( \mu_B, \mu_Q, \mu_S \) by,

\[
\begin{align*}
\mu_u &= \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q, \\
\mu_d &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q, \\
\mu_s &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_S
\end{align*}
\]  

(8)

Here the odd terms vanish due to CP symmetry at vanishing chemical potential and the correlation functions with \( i + j + k \) even are nonzero. In this work we evaluate the correlation coefficients up to fourth order which are generically given by:

\[ c_{X,Y}^{i,j} = \left. \frac{1}{i!j!} \frac{\partial^i}{\partial \mu_X^i} \frac{\partial^j}{\partial \mu_Y^j} P/T^4 \right|_{\mu_q,Q,S=0} \frac{\partial^i}{\partial \mu_X^i} \frac{\partial^j}{\partial \mu_Y^j} \]  

(9)

where, X and Y each stands for B, Q and S with \( X \neq Y \). To extract the Taylor coefficients, first the pressure is obtained as a function of different combinations of chemical potentials for each value of \( T \) and fitted to a polynomial about zero chemical potential using the gnuplot fit program [58]. Stability of the fit has been checked by varying the ranges of fit and simultaneously keeping the values of least squares to \( 10^{-10} \) or even less.

III. RESULTS AND DISCUSSION

We now set out to present the results obtained for correlation among different conserved charges. First the leading order correlations are shown and compared with those of LQCD. Later we discuss the behavior of some higher order correlations predicted from PNJL model.

Let us consider the baryon-strangeness (BS) correlation. In Fig.1(a) leading order BS correlation is shown and compared with LQCD data. Though there is good qualitative agreement the results have significant difference quantitatively. This is not very surprising as the inherent physical masses of the constituents in our model and those on the Lattice are substantially different - the ratio of pion to kaon mass is about a factor of 2 larger on the Lattice than the physical value. As we shall see this kind of departure remains for almost all the correlations measured on the Lattice and in PNJL model.

The BS correlation normalized to the strangeness and baryon number fluctuations respectively are given by

\[ C_{BS} = \frac{\chi_{BS}}{\chi_{SS}} = -\frac{1}{2} \frac{c_{11}^{BS}}{c_2^2} \]  

\[ C_{SB} = \frac{\chi_{BS}}{\chi_{BB}} = -\frac{1}{2} \frac{c_{11}^{BS}}{c_2^2} \]

where we have used the notation; \( \chi_{XY} = \frac{\partial^2 P}{\partial \mu_X \partial \mu_Y} \) and \( \chi_{XX} = \frac{\partial^2 P}{\partial \mu_X^2} \). It was argued in Ref. [59] that \( C_{BS} \) has entirely different behavior in hadron gas and in QGP, and therefore this can be a reasonable diagnostic tool for identifying the nature of the matter formed in heavy-ion collisions through event-by-event fluctuations. In quark phase, baryon number and strangeness
are strongly correlated through the strange quark indicating $C_{BS}$ should approach its Stefan-Boltzmann (SB) limit as soon as quark quasi-particles are dominant. The corresponding value for the ratio $\frac{c_{BS}^{11}}{c_{S}^{2}}$ is $-\frac{3}{2}$. From Fig.1(b) we see that closely above $T_C$ the ratio reaches its SB limit.

At low temperatures in the hadronic phase the situation is different. Numerator of $C_{BS}$ has contributions from strange baryons only, whereas the denominator has contributions from all strange hadrons. So the ratio approaches zero as the temperature is decreased.

A similar behavior is found in the ratio $C_{SB}$ which gives the BS correlation normalized to the fluctuation of baryon number. This is shown in Fig.1(c). Here again the high temperature behavior is consistent with a quark quasi-particle picture and at low temperatures the strange baryon correlation is much smaller than the baryon fluctuation due to the large mass of the strange baryons. For both these ratios we see a nice qualitative agreement with lattice data though quantitative disagreement persist.

Going a step further we show the behavior of some fourth order correlations - $c_{BS}^{22}$, $c_{31}^{BS}$ and $c_{13}^{BS}$ in Fig.2. At low $T$, in the hadronic phase, all three correlations go to zero. On the other hand the correlations approach their SB limit at temperature close to $2.5T_C$. For $c_{22}^{BS}$ and
There are two cusps, one corresponding to the chiral transition in the light quark sector and the other corresponding to the strange sector. Similar features for diagonal correlators were discussed by us in Ref. [49]. For $c_{BS}^{\mu}$ the strange sector completely overwhelms the light quark sector as expected, and so there is only one peak at $1.5T_c$. Therefore, if somehow these three correlations freeze out earlier than thermal and chemical freeze-out in heavy-ion collision experiments, they will not only be good indicators of the crossover but can also draw the explicit distinction between the chiral transitions in the light and strange quark sectors.

We now turn to baryon-charge (BQ) correlation. In Fig. 3(a) leading order correlation is shown. At both very low and at very high temperatures $c_{11}^{BQ}$ is zero. This is because at low temperatures the contributions from heavy baryons decrease. On the other hand in the high temperature weakly interacting phase the baryon and charge quantum numbers are completely independent of each other. The peak in $c_{11}^{BQ}$ occurs slightly above $T_c$ and is a clean indicator of the crossover. We show $c_{11}^{BQ}$ normalized with respect to $c_2^B$ and $c_2^Q$ respectively in other two
FIG. 3: Leading order baryon-charge (BQ) correlation as a function of $T/T_C$. Lattice data taken from Ref.[45].

These can be expressed in terms of the following:

\[ C_{BQ} = \frac{\chi_{BQ}}{\chi_{QQ}} = \frac{1}{2} \frac{c_{11}^{BQ}}{c_2^{QQ}} \]

\[ C_{QB} = \frac{\chi_{BQ}}{\chi_{BB}} = \frac{1}{2} \frac{c_{11}^{BQ}}{c_2^{BB}} \]

Both the ratios go to zero at high $T$. At low $T$ the ratios show different behavior. While $c_2^B$ becomes small at low temperatures due to heavy baryons, $c_2^Q$ is not so small due to the contributions from light mesons. Thus $C_{QB}$ remains non-zero whereas $C_{BQ}$ goes to zero. Interestingly, in case of a complete thermal equilibrium, in the fireball created in heavy-ion collisions, these two quantities, if measured, would give valuable insight into the quantitative aspects of the PNJL model at low temperatures in the hadronic phase. The difference in Lattice and PNJL model studies which is more pronounced below $T_c$, would then have to confront this measurement in the experiment.

In Fig.4 we have plotted the 3 fourth order correlation coefficients $c_{22}^{BQ}$, $c_{31}^{BQ}$ and $c_{13}^{BQ}$. All of them show a pronounced peak close to $T_c$. Among these $c_{31}^{BQ}$ is non-zero only at $T_c$ and so is
the best indicator of the crossover. These correlators do not show the double peak structure as contribution from strange sector is subdominant.

In Fig. 4 the leading order QS correlation is shown. As in the case of baryon number, charge is also strongly correlated to strangeness through strange quarks and therefore at high temperature $c_{11}^{QS}$ approaches its SB limit. At low temperatures, the strangeness carriers become heavier and hence the correlation decreases. Interesting features arise in the ratios,

$$C_{QS} = \frac{\chi_{QS}}{\chi_{SS}} = \frac{1}{2} \frac{c_{11}^{QS}}{c_2}$$

$$C_{SQ} = \frac{\chi_{QS}}{\chi_{QQ}} = \frac{1}{2} \frac{c_{11}^{QS}}{c_2}$$

The ratio $C_{SQ}$ decreases with decreasing temperature until it reaches $T_c$. There it shows a plateau both in PNJL and Lattice measurements as most likely this is where hadronic degrees of freedom start to become dominant. Thereafter the ratio again decreases. This seems to indicate that as we raise the temperature, the strangeness content and hence its correlation keeps on increasing reaching a saturation. This is suddenly broken by the liberation of new
degrees of freedom in terms of quark quasi-particles just above $T_c$. More detailed investigation is necessary to confirm this picture. In retrospect we do expect and see a similar behavior for $C_{SB}$ and $C_{QB}$ though not highly prominent.

At large temperatures $C_{QS}$ takes up non-zero value \[60\]. However it increases with decreasing temperature due to the same reason as $C_{QB}$, i.e., the strangeness fluctuations decrease at a faster rate at lower temperatures. The saturation effect near $T_c$ is normalized out.

In Fig.6 we have plotted the fourth order correlations. All three fourth order correlations $c_{22}^{QS}$, $c_{31}^{QS}$ and $c_{13}^{QS}$ show similar behavior as for the BS correlations. Therefore these two sets can be used complementarily to understand the state of affairs in heavy ion collisions.

**IV. CONCLUSION**

We have studied the correlations between different conserved charges in the PNJL model. The baryon-strange (BS), baryon-charge (BQ) and the charge-strange (QS) correlations were obtained by fitting the pressure in a Taylor series expansion around vanishing chemical potential. The ratio of these leading order correlators to the respective quadratic diagonal correlators were
obtained. The results were shown both for the physically bound effective PNJL model (BEP) as well as the conventional PNJL model without a physical bound (UEP). Lattice QCD data exists for the leading order correlators and were contrasted against our calculations.

In general the comparison with Lattice data gave excellent qualitative agreement reproducing all the physical features in all the coefficients that could be compared. However quantitative differences remain which we believe is mainly due to the difference in effective masses. Interestingly the UEP seems to be closer to the Lattice results. One can therefore say that the PNJL model is standing quite strong as an effective model of QCD.

As we discussed in the main text the combined studies of various correlators would give us a clearer picture of the matter created in heavy-ion collision experiments. The leading order coefficients can be most useful in identifying if the QGP is formed, while the higher order coefficients could identify the crossover region.

We have noted a slight saturation of charge and strangeness through the saturation of the ratios $C_{SQ}$, $C_{SB}$ and $C_{QB}$ in a small temperature region just below $T_c$. This is found both in the PNJL model and in LQCD. Perhaps this is when the degrees of freedom crossover from hadronic to the partonic ones.

The higher order correlators containing strangeness show two cusps at around $T_c$ and around

FIG. 6: Fourth order charge-strange correlation coefficients as a function of $T/T_c$. Arrows on the right indicate the corresponding SB limit.
$1.5T_c$, corresponding to the temperatures where the chiral crossover in the light and strange quark sectors occur. This widens the temperature range where the fluctuations remain much larger than the values at low temperature thus increasing the scope of identifying the possible existence of the high temperature phase in heavy-ion collisions. On the other hand the higher order BQ correlators have a sharp peak and is an accurate tool to decide the crossover temperature.

Even if we are not lucky enough to identify the high temperature phase from the correlations which need to get frozen much before thermal or chemical freeze-out, the various equilibrium thermodynamic measurements of the correlators would help us in determining the finite temperature behavior of the hadronic sector studied theoretically using PNJL model and Lattice QCD.

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Note Added: After finishing this work we learned of a recent work by W. Fu and Y. Wu (arxiv: 1010.0892 [hep-ph]). However they have calculated the off-diagonal susceptibilities only for the unbound effective potential of the PNJL model. Furthermore, they have not included the Vandermonde term in the Polyakov loop potential.
