THE PARTIAL TEMPERLEY–LIEB ALGEBRA
AND ITS REPRESENTATIONS

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Dedicated to the memory of Georgia Benkart

Abstract. We give a combinatorial description of a new diagram algebra, the partial Temperley–Lieb algebra, arising as the generic centralizer algebra $\text{End}_{U_q(\mathfrak{sl}_2)}(V^\otimes k)$, where $V = V(0) \oplus V(1)$ is the direct sum of the trivial and natural module for the quantized enveloping algebra $U_q(\mathfrak{gl}_2)$. It is a proper subalgebra of the Motzkin algebra (the $U_q(\mathfrak{sl}_2)$-centralizer) of Benkart and Halverson. We prove a version of Schur–Weyl duality for the new algebras, and describe their generic representation theory.

Introduction

The Temperley–Lieb algebra $\mathcal{TL}_k(\delta)$ arose in [TL71] in connection with the Potts model in mathematical physics. It was rediscovered by Vaughan Jones in his seminal work [Jon83, Jon85, Jon86, Jon87, Jon87b] on subfactors, in the guise of a von Neumann algebra, enabling spectacular applications to knot theory and many subsequent developments (see e.g. [Kau13, Mar91, GdlHJ89, Wes95, RSA14]). An important feature of these algebras is that when the ground ring is a field and $\delta = \pm (q + q^{-1})$,

\begin{equation}
\mathcal{TL}_k(\delta) \cong \text{End}_{U_q(\mathfrak{sl}_2)}(V(1)^\otimes k)
\end{equation}

for almost all values of the parameter $q$, where $V(1)$ is the 2-dimensional type-1 simple $U_q(\mathfrak{sl}_2)$-module (the "natural" module). Kauffman [Kau87] (see also [BW89]) found the now standard realization of $\mathcal{TL}_k(\delta)$ in terms of planar Brauer diagrams in the Brauer algebra.

The partial Brauer algebra $\mathcal{PB}_k(\delta, \delta')$, the span of all partition $k$-diagrams with blocksize at most two, was studied in [Hd14, MM14]. It comes naturally equipped with two independent parameters $\delta, \delta'$ of disparate topological significance. In [BH14] (see also [DEG17]) Benkart and Halverson introduced the Motzkin algebra $M_k(\delta, \delta')$, the subalgebra of $\mathcal{PB}_k(\delta, \delta')$ spanned by planar partial Brauer diagrams. It is known [DG22] that $\mathcal{PB}_k(\delta, \delta') \cong \mathcal{PB}_k(\delta, 1)$ for any $\delta' \neq 0$, so it is natural to restrict one’s attention to $\mathcal{PB}_k(\delta, 1)$ and $M_k(\delta, 1)$. For simplicity, set $M_k(\delta) := M_k(\delta, 1)$; this is the context of [BH14].

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When the ground ring is a field and $\delta = 1 \pm (q + q^{-1})$, they obtain an isomorphism
\[ M_k(\delta) \cong \text{End}_{U_q(sl_2)}(V^\otimes k) \]
for almost all values of $q$, where $V = V(1) \oplus V(0)$ is the direct sum of the natural module $V(1)$ (as above) and the trivial module $V(0)$. Actually, (2) is proved in [BH14] only for the case $\delta = 1 - (q + q^{-1})$, but it is easily extended to the above statement. Notice that the right hand side of (2) is independent of the choice of sign, so $M_k(1 + (q + q^{-1})) \cong M_k(1 - (q + q^{-1}))$.

The isomorphism (2) was unfortunately misstated in [BH14], where $U_q(gl_2)$ incorrectly appeared in place of $U_q(sl_2)$ on the right hand side. The two centralizers differ; indeed, their dimensions don’t agree (see §9).

The purpose of this paper is to identify a subalgebra $PTL_k(\delta)$ of $M_k(\delta)$, the partial Temperley–Lieb algebra of the title, such that when $\delta = 1 \pm (q + q^{-1})$,
\[ PTL_k(\delta) \cong \text{End}_{U_q(gl_2)}(V^\otimes k) \]
for almost all values of $q$. The algebra $PTL_k(\delta)$ has a basis indexed by the set of balanced Motzkin $k$-diagrams (diagrams with the same number of cups as caps). It is notable that the basis elements are not diagrams; instead, each basis element is an alternating sum of diagrams with a unique maximal balanced diagram as leading term. As the right hand side of (3) is again independent of the choice of sign, we have $PTL_k(1 + (q + q^{-1})) \cong PTL_k(1 - (q + q^{-1}))$.

The use of the adjective “partial” in describing the algebras $PTL_k(\delta)$ fits into a more general scheme of “partialization” that goes back at least to [Maz02].

When the ground ring is a field, all irreducible representations of a semisimple cellular algebra are absolutely irreducible. In other words, cellular algebras over a field are semisimple if and only if they are split semisimple, so the adjective “split” is often omitted in describing these algebras in the semisimple case. This applies to the algebras $TL_k(\delta)$, $M_k(\delta)$, and $PTL_k(\delta)$ appearing in this paper.

Our main results are as follows:

(i) In Theorem 3.1, we find two natural bases $\{\bar{d}\}$, $\{\tilde{d}\}$ of $PTL_k(\delta)$, each indexed by the set of balanced Motzkin $k$-diagrams. Theorem 2.5 works out a multiplication rule for each basis.

(ii) We show that $PTL_k(\delta)$ is an iterated inflation of Temperley–Lieb algebras, in the sense of [KX99, KX01, GP18], hence is cellular in the sense of [GL96]; see Remark 4.3. More precisely, in Theorem 4.2 we prove the stronger result that $PTL_k(\delta)$ is Morita equivalent to a direct sum of Temperley–Lieb algebras with parameter $\delta - 1$.

(iii) We construct the cell modules for $PTL_k(\delta)$, see Theorem 7.2, and prove that when the ground ring is a field and $TL_n(\delta - 1)$ is semisimple for all $n \leq k$ then the same is true of $PTL_k(\delta)$, see Theorem 4.4.
(iv) In Theorem 10.2, we slightly extend the aforementioned Schur–Weyl duality result of [BH14], showing that there are many choices in how to make $M_k(\delta)$ act on $V^{\otimes k}$, all of which imply that $M_k(\delta)$ is isomorphic to the $U_q(sl_2)$-centralizer of $V^{\otimes k}$ under suitable hypotheses.

(v) For $\delta = 1 \pm (q + q^{-1})$, under suitable hypotheses, we define a faithful action of $\text{PTL}_k(\delta)$ on $V^{\otimes k}$ commuting with the action of $U_q(gl_2)$, and prove the isomorphism (3) in Theorem 11.4. Thus we obtain a version of Schur–Weyl duality for $V^{\otimes k}$ regarded as a bimodule for $\text{PTL}_k(\delta)$, $U_q(gl_2)$.

Finally, in Appendix A, we reinterpret results of [GdlHJ89] to obtain a precise semisimplicity criterion for $\text{TL}_k(\delta)$, when $\delta = \pm (q + q^{-1})$ and the ground ring is a field.

1. SOME DIAGRAM ALGEBRAS

In this section, we define the standard diagram algebras needed for this paper. Unless stated otherwise, we work over an arbitrary unital commutative ring $k$.

1.1. Terminology. Let $[k] := \{1, \ldots, k\}$ and $[k]' := \{1', \ldots, k'\}$. The set $P_k$ is the collection of all set partitions (equivalence relations) on $[k] \cup [k]'$. If $d = \{B_1, \ldots, B_l\}$ belongs to $P_k$ where $B_1, \ldots, B_l$ are pairwise disjoint, we call the $B_i$ the blocks of $d$. Typically, $d$ is depicted by a graph on $2k$ vertices arranged in two parallel rows in a rectangle, with vertices in the top (resp., bottom) row indexed by $[k]$ (resp., $[k]'$) in order from left to right. Edges are drawn in the interior of the rectangle in any way such that the resulting connected components coincide with the blocks. Although this graphical depiction of elements of $P_k$ is not in general unique, the lack of uniqueness causes no difficulty. To be precise, we define a $k$-diagram $d$ to be the equivalence class of all graphs depicting its underlying set partition $d$, where two such depictions are equivalent if and only if they have the same blocks. Henceforth, we identify elements of $P_k$ with their corresponding $k$-diagrams.

If $d_1$, $d_2$ are $k$-diagrams, their composite configuration $\Gamma(d_1, d_2)$ is the graph obtained by placing $d_1$ above $d_2$ and identifying the corresponding vertices in the middle row. Let $d_1 \circ d_2$ be the diagram obtained by retaining the edges with endpoints in the union of the top and bottom rows of vertices in the composite configuration, along with those vertices, and discarding the rest of the configuration. The multiplication $\circ$ is associative, so $(P_k, \circ)$ is a monoid, the partition monoid.

One often identifies diagrams with morphisms in a suitable category. In this paper, the reader should think of a diagram as depicting a morphism going from its bottom to top row, so that products depict compositions in which morphisms act on the left of arguments.
1.2. The partition algebra. We refer the reader to [HR05] for basic properties of partition algebras. For any \( \delta \in k \), the partition algebra \( P_k(\delta) \) is a twisted semigroup algebra on \( P_k \). As a \( k \)-module, \( P_k(\delta) = k[P_k] \), the collection of \( k \)-linear combinations of elements of \( P_k \). Given \( k \)-diagrams \( d_1, d_2 \),

\[
d_1d_2 = \delta^N(d_1,d_2) d_3 = \delta^N(d_1,d_2) (d_1 \circ d_2)
\]

where \( N(d_1,d_2) \) is the number of interior blocks (connecting vertices in the middle row) in \( \Gamma(d_1,d_2) \) that get discarded in forming \( d_3 = d_1 \circ d_2 \). Extending this multiplication rule to linear combinations bilinearly as usual, the set \( P_k(\delta) \) becomes an associative algebra with unit.

Given diagrams \( d_1 \in P_k, d_2 \in P_l \), let \( d_1 \otimes d_2 \in P_{k+l} \) be the diagram obtained by placing \( d_1 \) to the left of \( d_2 \). (The notation \( \otimes \) in this context is not a tensor product.) The following basic diagrams

\[
1 := \vcenter{\hbox{\includegraphics{1_d1}}}, \quad p := \vcenter{\hbox{\includegraphics{p_d1}}}, \quad s := \vcenter{\hbox{\includegraphics{s_d1}}}, \quad b := \vcenter{\hbox{\includegraphics{b_d1}}}
\]

are fundamental building blocks for all partition diagrams. Notice that \( 1_k = 1 \otimes \cdots \otimes 1 \) (with \( k \) factors) is the identity element of \( P_k(\delta) \); in the sequel we will often abuse notation by writing 1 in place of \( 1_k \). In the set \( P_k \), define

\[
p_j := 1_{j-1} \otimes p \otimes 1_{k-j} = \vcenter{\hbox{\includegraphics{p_j_d1}}}
\]

with the isolated vertices in the \( j \)th column, for \( j = 1, \ldots, k \), and define

\[
s_i := 1_{i-1} \otimes s \otimes 1_{k-1-i} = \vcenter{\hbox{\includegraphics{s_i_d1}}}
\]

\[
b_i := 1_{i-1} \otimes b \otimes 1_{k-1-i} = \vcenter{\hbox{\includegraphics{b_i_d1}}}
\]

for \( i = 1, \ldots, k-1 \). The elements \( p_j, s_i, b_i \) form a set of generators of \( P_k(\delta) \); its defining relations are given in [HR05, Thm. 1.11]. (Note that \( b_i \) is denoted by \( p_{i+\frac{1}{2}} \) in that reference.) We also need the diagrams

\[
e := \vcenter{\hbox{\includegraphics{e_d1}}} \quad \text{and} \quad e_i := 1_{i-1} \otimes e \otimes 1_{k-1-i} = \vcenter{\hbox{\includegraphics{e_i_d1}}}
\]

for any \( i = 1, \ldots, k-1 \). The reader may easily check that the elements \( e_i \) satisfy the identity

\[
e_i = b_ip_ip_{i+1}b_i
\]

in \( P_k(\delta) \); this identity provides an alternative definition of \( e_i \).

1.3. The partial Brauer algebra. The subalgebra of \( P_k(\delta) \) spanned by the \( k \)-diagrams in which each block has cardinality at most 2 is the (one-parameter) partial Brauer algebra \( PB_k(\delta,\delta) \) of [MM13, H14], who studied
its more general two-parameter variant $\mathcal{PB}_k(\delta, \delta')$, with multiplication defined by

\begin{equation}
\begin{aligned}
d_1d_2 &= \delta^{N_1(d_1,d_2)}\delta'^{N_2(d_1,d_2)}d_3 = \delta^{N_1(d_1,d_2)}\delta'^{N_2(d_1,d_2)}(d_1 \circ d_2)
\end{aligned}
\end{equation}

where $N_1(d_1,d_2)$ (resp., $N_2(d_1,d_2)$) is the number of interior loops (resp., interior paths, including paths consisting of a single vertex) in the middle row of $\Gamma(d_1,d_2)$, and $d_3 = d_1 \circ d_2$ is the product in the partition monoid $P_k$.

Note that $e_i$, $s_i$ ($i \in [k-1]$) and $p_j$ ($j \in [k]$) belong to $\mathcal{PB}_k(\delta, \delta')$; in fact, this is a set of generators of that algebra. Both $e_i$ and $p_i$ are pseudoidempotents, satisfying

\begin{equation}
\begin{aligned}
e_i^2 &= \delta e_i, \\
p_i^2 &= \delta' p_i.
\end{aligned}
\end{equation}

For any $\delta' \neq 0$, we have $\mathcal{PB}_k(\delta, \delta') \cong \mathcal{PB}_k(\delta, 1)$; see (16) in §2. Thus it makes sense to focus on $\mathcal{PB}_k(\delta, 1)$.

1.4. The Motzkin algebra. A $k$-diagram that can be drawn without any intersections is said to planar. The *Motzkin algebra* $M_k(\delta, \delta')$ studied in [BH14] is the subalgebra of $\mathcal{PB}_k(\delta, \delta')$ spanned by the planar partial Brauer $k$-diagrams; that is, planar diagrams with blocks of cardinality at most 2.

The algebra $\mathcal{PB}_k(\delta, \delta')$ contains elements $r_i := p_is_i$ and $l_i := s_ip_i$, depicted by

\begin{equation}
\begin{aligned}
r_i &= 1_{i-1} \otimes r \otimes 1_{k-1-i} = \\
l_i &= 1_{i-1} \otimes l \otimes 1_{k-1-i} = 
\end{aligned}
\end{equation}

for $i \in [k-1]$, where $r, l$ in $P_2$ are given by

\begin{equation}
\begin{aligned}
r &= \begin{array}{c}
\cdot \\
\cdot
\end{array}, \\
l &= \begin{array}{c}
\cdot \\
\cdot
\end{array}.
\end{aligned}
\end{equation}

As the $r_i$, $l_i$ are planar, they belong to the Motzkin algebra $M_k(\delta, \delta')$. The diagrams $p_i$ are also planar, hence belong to $M_k(\delta, \delta')$. They satisfy

\begin{equation}
p_i = r_il_i = l_{i-1}r_{i-1}
\end{equation}

for all values of the indices for which the equalities are sensible. It is shown in [BH14] that

\begin{equation}
M_k(\delta, \delta') \text{ is generated by the } e_i, r_i, l_i \quad (i \in [k-1]).
\end{equation}

(The element $e_i$ is denoted by $t_i$ in [BH14].) A set of defining relations for these generators can be found in [HLP13].
1.5. The Temperley–Lieb algebra. The Brauer algebra $B_k(\delta)$ is the subalgebra of $P_k(\delta)$ spanned by the $k$-diagrams in which all blocks have exactly two elements. The Temperley–Lieb algebra $\mathcal{TL}_k(\delta)$ is the subalgebra of $B_k(\delta)$ spanned by the planar $k$-diagrams which are also in the Brauer algebra. It is the subalgebra generated by $e_1, \ldots, e_{k-1}$. As such, it is isomorphic to the algebra defined by those generators and satisfying the relations
\begin{equation}
e_i^2 = \delta e_i, \quad e_ie_{i+1}e_i = e_i, \quad e_ie_je_i = e_je_i \quad \text{if } |i-j| > 1.
\end{equation}
The rank of $\mathcal{TL}_k(\delta)$ over $k$ is equal to the $k$th Catalan number $C_k = \frac{1}{k+1} \binom{2k}{k}$.

It is noteworthy that the algebra morphism defined on generators by $e_i \mapsto -e_i$ for all $i$ defines an isomorphism of algebras
\begin{equation}
\mathcal{TL}_k(\delta) \cong \mathcal{TL}_k(-\delta).
\end{equation}
So the choice of sign of the parameter is purely a matter of convenience.

If $k$ is a field and $0 \neq q$ is an element of $k$ such that $q^2 \neq 1$, it is well known that
\begin{equation}
\mathcal{TL}_k(\pm(q + q^{-1})) \cong \text{End}_U(V(1)^\otimes k)
\end{equation}
where $V(1)$ is the 2-dimensional “natural” representation of the quantized enveloping algebra $U = U_q(\mathfrak{sl}_2)$. There is a (unique) copy of the trivial $U$-module in $V(1) \otimes V(1)$. In the isomorphism [L3], we identify
\[e_i = \pm(q + q^{-1}) 1^\otimes (i-1) \otimes \pi \otimes 1^\otimes (k-i-1)\]
where $\pi$ is an orthogonal projection onto that trivial module and 1 denotes the identity map. See Section [3] for details.

Finally, if $k$ is a field and $q$ a nonzero element of $k$ satisfying the condition $[1]_q [2]_q \cdots [k]_q \neq 0$, where $[k]_q$ is the balanced form of a quantum integer, then $\mathcal{TL}_k(\pm(q + q^{-1}))$ is semisimple over $k$; see Appendix [A]. In particular, this semisimplicity statement holds whenever $q$ is not a root of unity.

2. Alternating bases

Vaughan Jones [Jon94] (see also [HR05,BH19,BH19b]) introduced the orbit basis $\{o_d \mid d \in P_k\}$ of $P_k(\delta)$, defined as follows. Given $k$-diagrams $d_1, d_2$ write
\begin{equation}
d_1 \leq d_2 \iff \text{each block of } d_1 \text{ is contained in some block of } d_2.
\end{equation}
The relation $\leq$ is a partial order on the set $P_k$ of $k$-diagrams. The orbit basis $\{o_d \mid d \in P_k\}$ is defined by demanding that the unitriangular relation $d = \sum_{d \leq d'} o_{d'}$ hold for every $k$-diagram $d$.

A different basis $\{d \mid d \in P_k\}$, also in a unitriangular relation with the diagram basis, is defined by setting
\begin{equation}
d = \sum_{d' \leq d} (-1)^{\beta(d)-\beta(d')} d'
\end{equation}
for any $k$-diagram $d$, where $\beta(d)$ is the number of blocks of $d$. This basis is the alternating basis; it plays a crucial role in this paper.
The blocks of a partial Brauer diagram all have cardinality at most 2, so blocks are either singletons (isolated vertices) or edges (having two vertices as endpoints). For partial Brauer diagrams \( d, d' \) the relation \( d \leq d' \) defined in (14) holds if and only if all the edges of \( d \) are also edges of \( d' \). Equivalently, \( d \leq d' \) if and only if \( d \) is obtainable from \( d' \) by excising zero or more of its edges.

If \( d \) is a partial Brauer diagram then any term in the right hand side of the expansion (15) is (up to sign) also a partial Brauer diagram. The same holds for planar partial Brauer diagrams (that is, Motzkin diagrams). Hence, \( \bar{d} \in \text{PB}_k(\delta, \delta') \) for any partial Brauer diagram \( d \), and similarly \( \bar{d} \in \text{M}_k(\delta, \delta') \) for any planar partial Brauer diagram \( d \).

**Lemma 2.1.** For any unital commutative ring \( k \) and any \( \delta, \delta' \in k \), the sets

\[
\{ \bar{d} \mid d \text{ a partial Brauer diagram} \}, \quad \{ \bar{d} \mid d \text{ a Motzkin diagram} \}
\]

are bases of \( \text{PB}_k(\delta, \delta') \), \( \text{M}_k(\delta, \delta') \) respectively.

**Proof.** By the remarks preceding the lemma, the transition matrix expressing the \( \bar{d} \) in terms of the diagram basis in each algebra is unitriangular with respect to any linear order extending \( \leq \). \( \square \)

For any invertible \( \delta' \in k \), by [MM14, DG22] there is an algebra isomorphism

\[
\text{PB}_k(\delta, \delta') \cong \text{PB}_k(\delta, 1)
\]

defined by replacing the generator \( p_i \) by \( p_i/\delta' \). Thus, there is no loss of generality in setting \( \delta' = 1 \), so from now on we work in \( \text{PB}_k(\delta, 1) \) and in its subalgebra \( \text{M}_k(\delta, 1) \). This is convenient because \( p_i, 1 - p_i \) become a pair of commuting orthogonal idempotents.

For a given partial Brauer diagram \( d \), consider the subsets \([d], [d]'\) of \([k]\) (the bottom, top frame, respectively, of \( d \)) defined by

\[
[d] = \{ i \in [k] \mid \text{vertex } i' \text{ is non-isolated in } d \}, \quad [d]' = \{ i \in [k] \mid \text{vertex } i \text{ is non-isolated in } d \}.
\]

Elements of the set \([d]' \cup [d]\) form the frame of \( d \) and label the endpoints of the edges in \( d \); its complement in \([k]' \cup [k]\) labels the isolated vertices in \( d \).

**Proposition 2.2.** Let \( d \) be a partial Brauer diagram. The identities:

\[
(a) \quad \bar{d} = \prod_{i \in [d]} (1 - p_i) \prod_{i' \in [d]'} (1 - p_i)
\]

\[
(b) \quad \bar{d} = \rho_0(d) - \rho_1(d) + \rho_2(d) - \rho_3(d) + \cdots
\]

hold in \( \text{PB}_k(\delta, 1) \), where \( \rho_i(d) \) is the sum of all diagrams obtained from \( d \) by removing exactly \( i \) of its edges. If \( d \) is planar (i.e., a Motzkin diagram) then all terms on the right hand side of the identities are also planar, hence belong to \( \text{M}_k(\delta, 1) \).
Proof. (a) It follows from the definition of diagram multiplication that left (resp., right) multiplication by \( p_i \) removes the edge with endpoint \( i \) (resp., \( i' \)), for any \( i \in [d], i' \in [d] \). Expanding the products on the left and right of identity (a) thus gives

\[
\bar{d} = \sum_{d' \leq d} (-1)^{\text{edges}(d) - \text{edges}(d')} d'
\]

where \( \text{edges}(d) \) is the number of edges in \( d \). Horizontal edges are seemingly removed twice, once for each endpoint, but in fact the second multiplication by the appropriate \( p_i \) acts as the identity, thus doesn’t matter. For partial Brauer diagrams, the above expansion coincides with the expression in (15).

(b) This follows from the displayed equation in the proof of (a). \( \square \)

Remark 2.3. The formula in part (b) of the proposition says that, for partial Brauer diagrams \( d \), the element \( \bar{d} \) is obtained from \( d \) by inclusion-exclusion edge removal. As \( \rho_0(d) = d \), the leading term in the expansion is \( d \) itself.

An edge of a partial Brauer diagram \( d \) is a cup (resp., cap) if both of its endpoints are in \([k]\) (resp., in \([k']\) ). Such edges are also called horizontal. For a partial Brauer diagram \( d \), we define

\[
\tilde{d} := \prod_{i \in [d]} (1 - p_i) d \prod_{i' \in [d]} (1 - p_i)
\]

(17)

\[
\hat{d} := \prod_{i \in [d]} (1 - p_i) d \prod_{i' \in [d]} (1 - p_i)
\]

where \([d]_H, [d]_H \) respectively index the endpoints of horizontal edges in the top, bottom rows of \( d \), and similarly \([d]_V, [d]_V \) respectively index the endpoints of vertical edges in the top, bottom rows of \( d \). Notice that either product in the definition of \( \hat{d} \) may be omitted without changing the result. The element \( \tilde{d} \) (resp., \( \hat{d} \)) is obtained from \( d \) by inclusion-exclusion horizontal (resp., vertical) edge removal. We linearly extend the notations \( \bar{d}, \tilde{d}, \hat{d} \) to linear combinations of diagrams.

Proposition 2.4. Let \( d \) be a partial Brauer diagram. Then

(a) \( \tilde{d} = \hat{d} = \bar{d} \)

(b) \( \tilde{d} = \rho_0^H(d) - \rho_1^H(d) + \rho_2^H(d) - \cdots \)

(c) \( \hat{d} = \rho_0^V(d) - \rho_1^V(d) + \rho_2^V(d) - \cdots \)

where \( \rho_i^H(d) \) (resp., \( \rho_i^V(d) \)) is the sum of all diagrams obtained from \( d \) by removing \( i \) of its horizontal (resp., vertical) edges. Hence the sets

\[
\{ \tilde{d} \mid d \in \text{PB}_k \}, \quad \{ \hat{d} \mid d \in \text{M}_k \}
\]

are \( k \)-bases of \( \text{PB}_k(\delta, 1), \text{M}_k(\delta, 1) \), respectively.

Proof. (a) follows from the product formula in Proposition 2.2(a) and the fact that the \( p_i \) pairwise commute.
(b), (c) follow by expanding the right hand side in (17).

Applying the operator \(d \mapsto \tilde{d}\) to both sides of the identity in part (c) shows that if \(d\) is a partial Brauer diagram then \(\tilde{d}\) is expressible as an alternating sum of the form
\[
\tilde{d} = \tilde{d} - \tilde{d}_1 + \tilde{d}_2 - \cdots
\]
where each \(d_i < d\). So the transition between the sets \(\{\tilde{d} \mid d \in \mathcal{PB}_k\}\) and \(\{\tilde{d} \mid d \in \mathcal{PB}_k\}\) is unitriangular. The same holds if we restrict to Motzkin diagrams. The final claim now follows from Lemma 2.1. \(\square\)

We will need to consider various subalgebras of the Motzkin algebra \(\mathcal{M}_k(\delta)\). Let \(\mathcal{R}_k\) be the subalgebra generated by \(r_1, \ldots, r_{k-1}\) and \(\mathcal{L}_k\) the subalgebra generated by \(l_1, \ldots, l_{k-1}\). Write \(\mathcal{RL}_k\) for the subalgebra generated by \(\mathcal{R}_k\) and \(\mathcal{L}_k\). Notice that \(\tilde{d} = d\) for any \(d\) in \(\mathcal{RL}_k\), since such \(d\) have no horizontal edges.

**Theorem 2.5.** Suppose that \(d_1, d_2\) are partial Brauer \(k\)-diagrams. Let \(N_1(d_1, d_2)\) be the number of closed loops in the middle of \(\Gamma(d_1, d_2)\), as in equation (7), and set \(d_3 = d_1 \circ d_2\), the product in the partition monoid \(\mathcal{P}_k\). Let
\[
\Omega(d_1, d_2) = \left(\left([k] \setminus [d_1]\right) \cap [d_2]_H\right) \cup \left(\left([k] \setminus [d_2]\right) \cap [d_1]_H\right),
\]
the set indexing the vertices in the middle row of \(\Gamma(d_1, d_2)\) for which an isolated vertex in one diagram is identified with a horizontal edge endpoint in the other. Then:

\[
\begin{align*}
(a) \quad & \tilde{d}_1 \tilde{d}_2 = \left\{ \begin{array}{ll}
(\delta - 1)^{N_1(d_1, d_2)} d_3 & \text{if } [d_1] = [d_2] \\
0 & \text{otherwise.}
\end{array} \right. \\
(b) \quad & \tilde{d}_1 \tilde{d}_2 = \left\{ \begin{array}{ll}
(\delta - 1)^{N_1(d_1, d_2)} \prod_{i \in S}(1 - p_i) d_3 & \text{if } \Omega(d_1, d_2) \text{ is empty} \\
0 & \text{otherwise.}
\end{array} \right.
\end{align*}
\]

The set \(S\) in formula (b) is the set of indices on the top vertex of any through edge that snakes through some cups and caps in the middle row of \(\Gamma(d_1, d_2)\) before emerging to connect to a vertex in the bottom row.

**Proof.** (a) The proof is based on the formula in Proposition 2.2(a). First suppose that \([d_1]\) does not match \([d_2]\). Then there must be at least one isolated vertex that matches up with the endpoint of some edge. We can always insert a copy of \(p_i\) corresponding to that vertex, as multiplication by \(p_i\) is identity on an isolated vertex in the \(i\)th position. This shows that the product \(d_1 \circ d_2 = 0\).

From now on, suppose that \([d_1]\) matches \([d_2]\). There are four cases to consider. First, suppose that two propagating edges meet in the middle row of \(\Gamma(d_1, d_2)\) at the \(i\)th identified vertex. Then the idempotent \(1 - p_i\) can be commuted to both sides of \(d_1d_2\), as \(1 - p_j\) on the left and \(1 - p_m\) on the right, where the corresponding edge in \(d_3 = d_1d_2\) connects the \(j\)th vertex on the top row to the \(m\)th on the bottom.
Now suppose that there is an added cup in \( d_3 \) that is not present in \( d_1 \). This means two propagating edges in \( d_1 \) join up with a connected path in the middle of \( \Gamma(d_1, d_2) \) to form that additional cup. In this situation, we can commute all the interior idempotents on the path up to the top (to the left of \( d_3 \)).

The case of an added cap in \( d_3 \) that is not present in \( d_2 \) is analogous to the previous case.

It remains only to consider loops in the middle row of \( \Gamma(d_1, d_2) \). For each such loop, the product of the middle idempotents is equivalent to multiplication by the scalar \( \delta - 1 \). This completes the proof of (a).

(b) The proof of part (b) is similar to that for part (a), based on the formula in Proposition 2.4(b). \( \square \)

For example, if \( k = 3 \) one may verify that \( \tilde{e}_1 \tilde{e}_2 \) is equal to the 8-term linear combination

\[
\tilde{e}_1 \tilde{e}_2 = \begin{array}{cccc}
\cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

illustrating the equality \( \tilde{e}_1 \tilde{e}_2 = (1 - p_3) \tilde{e}_1 \tilde{e}_2 \) in Theorem 2.5(b).

**Corollary 2.6.** If \( x \) is a diagram in \( RL_k \) and \( d \) is a partial Brauer diagram then

\[
(a) \quad x \tilde{d} = \begin{cases}
\tilde{x}d & \text{if } [d]_H \subset [x] \\
0 & \text{otherwise.}
\end{cases}
\]

\[
(b) \quad \tilde{d}x = \begin{cases}
\tilde{d}x & \text{if } [d]_H \subset [x] \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** These formulas follow easily from part (b) of Theorem 2.5. We prove formula (a). Since \( x \) belongs to \( RL_k \), it has no horizontal edges, so \( [x]_H \) is empty. This means that \( \Omega(x, d) \) is empty if and only if \( [d]_H \subset [x] \), and furthermore, there are no interior closed loops in \( \Gamma(x, d) \). Formula (a) thus follows. The proof of formula (b) is symmetric. \( \square \)

### 3. The Partial Temperley–Lieb Algebra

We are now ready to define the main object of study for this paper. Henceforth we set \( M_k(\delta) := M_k(\delta, 1) \).

We say that a Motzkin diagram is *balanced* if it has the same number of cups as caps. Equivalently, a Motzkin diagram is balanced if the number of isolated vertices in each of its rows is the same. Let

\[
\mathcal{D}(k) := \{ d \mid d \text{ is a balanced Motzkin } k\text{-diagram} \}.
\]
We define the partial Temperley–Lieb algebra $PTL_k(\delta)$ to be the linear span of $\{ \bar{d} \mid d \in D(k) \}$. That $PTL_k(\delta)$ is a subalgebra of $M_k(\delta)$ follows from Theorem 2.5.

**Theorem 3.1.** Let $k$ be a commutative ring and fix $\delta \in k$. Then either of the sets $\{ \bar{d} \mid d \in D(k) \}$, $\{ \tilde{d} \mid d \in D(k) \}$ is a $k$-basis of $PTL_k(\delta)$.

**Proof.** The set $\{ \tilde{d} \mid d \in D(k) \}$ is linearly independent by Lemma 2.1. Since it spans the algebra, it is a basis. Furthermore, as in the final paragraph of the proof of Proposition 2.4, the transition matrices between the sets $\{ \tilde{d} \mid d \in D(k) \}$, $\{ \bar{d} \mid d \in D(k) \}$ are unitriangular, so $\{ \tilde{d} \mid d \in D(k) \}$ is also a basis. $\square$

Let $D_n(k)$ be the subset of $D(k)$ consisting of those balanced Motzkin $k$-diagrams having exactly $n$ edges. Notice that $D_k(k)$ is the set of Temperley–Lieb diagrams on $2k$ vertices. To any $d \in D_n(k)$, we associate a triple $(A,t,B)$, where $t \in D_n(n)$ is the unique Temperley–Lieb diagram on $2n$ vertices obtained by deleting the isolated vertices in $d$, and $A, B$ are the subsets of $[k]$ respectively indexing the non-isolated vertices in the top, bottom row of $d$. For example,

$$d = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}$$

corresponds to the triple $(A,t,B)$ where $A = \{1, 4, 5\}$, $B = \{2, 4, 6\}$, and

$$t = e_1e_2 = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}$$

in $D_3(3)$. Since any diagram $d$ is reconstructible from its triple, the following result is clear.

**Lemma 3.2.** The map $d \mapsto (A,t,B)$ defines a bijection between $D_n(k)$ and the set of triples of the above form.

From now on, write $d(A,t,B)$ for the $k$-diagram in $D_n(k)$ corresponding in the above lemma to a given triple $(A,t,B)$ such that $A, B$ are subsets of $[k]$ of cardinality $n$ and $t \in D_n(n)$. In other words, the map $(A,t,B) \mapsto d(A,t,B)$ is the inverse of the bijection in Lemma 3.2.

The cardinality of the set $TL_n$ of Temperley–Lieb $n$-diagrams is equal to the $n$th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. As $D(k) = \bigcup_{n=0}^k D_n(k)$ (disjoint union), by combining Lemma 3.2 with Theorem 3.1, we have

$$\text{rank}_k PTL_k(\delta) = |D(k)| = \sum_{n=0}^k \binom{k}{n}^2 C_n. \quad (18)$$

If $t \in D_n(n)$ is a Temperley–Lieb $n$-diagram, there are two equally natural ways to extend $t$ to a diagram in $D_n(k)$, for any $k \geq n$. Either of the maps

$$t \mapsto t \otimes \omega_{k-n}, \quad t \mapsto t \otimes 1_{k-n} \quad (19)$$
will do the job, where $\omega_j$ is the $j$-diagram in which all $2j$ vertices are isolated. The following observation is an immediate consequence of Theorem 2.5(a).

**Lemma 3.3.** Write $t_0 = t \otimes \omega_{k-n}$, $t_1 = t \otimes 1_{k-n}$ for the image of $t$ in $\mathcal{D}_n(n)$ under the map $\text{(19)}$. For any $n \leq k$, either of the linear maps such that $t \mapsto t_0$, $t \mapsto t_1$ defines an algebra isomorphism of $\mathcal{TL}_n(\delta-1)$ with a subalgebra of $\mathcal{M}_k(\delta)$.

Let $\mathcal{RP}_k$ (resp., $\mathcal{LP}_k$) be the subalgebra of $\mathcal{M}_k(\delta)$ generated by all $r_i$, $p_i$ (resp., all $l_i$, $p_i$). The following result is a variant of [BH14, (2.12)].

**Lemma 3.4.** Let $d = d(A, t, B)$ be a diagram in $\mathcal{D}_n(k)$, where $t$ is in $\mathcal{D}_n(n)$. In the Motzkin algebra $\mathcal{M}_k(\delta)$, we have the factorizations

$$d = r_A(t \otimes \omega_{k-n})l_B, \quad d = r_A(t \otimes 1_{k-n})l_B$$

where $r_A \in \mathcal{RP}_k$ (resp., $l_B \in \mathcal{LP}_k$) is the unique $k$-diagram with edges from the first $n$ bottom-row (resp., top-row) vertices connecting to the top-row (resp., top-row) vertices indexed by $A$ (resp., $B$), in order.

**Proof.** The second factorization is the RTL factorization in [BH14 §2]. It doesn’t matter in that argument if we replace the identity edges in $1_{k-n}$ by the isolated vertices in $\omega_{k-n}$, which yields the first factorization. \(\square\)

We illustrate the proof of Lemma 3.4. As an example of the RTL factorization in [BH14], we have

\[
\begin{aligned}
d &= \text{Diagram 1} \quad , \quad \text{Diagram 2} \\
&= \text{Diagram 3} \\
&= \text{Diagram 4}
\end{aligned}
\]

in which the middle diagram is the diagram $t \otimes 1_{1}$, where $t \in \mathcal{D}_7(7)$ is the diagram

\[
t = \text{Diagram 5}.
\]

It is clear that replacing $t \otimes 1_{1}$ in the middle of the above stacked product by $t \otimes \omega_{1}$ makes no difference in the product.

The next task is to identify a set of generators for $\mathcal{PTL}_k(\delta)$. To that end, we set

$$\varepsilon_i := \tilde{e}_i = (1 - p_i)e_i(1 - p_i)$$

We note that the equality $\varepsilon_i = (1 - p_i)e_i(1 - p_i)$ remains valid if we replace either or both factors of $(1 - p_i)$ by $(1 - p_{i+1})$. Since the diagrams $l_i$, $r_i$, and $p_i$ have no horizontal edges, it is clear that $\tilde{l}_i = l_i$, $\tilde{r}_i = r_i$, and $\tilde{p}_i = p_i$.

**Theorem 3.5.** $\mathcal{PTL}_k(\delta)$ is generated by the set $\{l_i, r_i, \varepsilon_i \mid i \in [k-1]\}$. 

Proof. Let $P$ be the subalgebra generated by the given set. By Theorem 3.1, it suffices to show that $\tilde{d}$ belongs to $P$, for any $d$ in $D(k)$. Let $d = d(A, t, B)$ as discussed in the paragraph after Lemma 3.2. By Lemma 3.4, we have $d = r_A(t \otimes \omega_{k-n}) \otimes l_B$. 

We will need to distinguish generators with differing number of vertices, so we temporarily (in this proof only) write $e_i(n)$ for the diagram $e_i$ in $D(n)$, for each $n \leq k$. For any $1 \leq n \leq k - 1$, let $e_{i,n}(k) = e_i(n) \otimes \omega_{k-n} = e_i(k)p_{n+1} \cdots p_k$. Then by Corollary 2.6, $\tilde{e}_{i,n}(k) = \tilde{e}_i(k)p_{n+1} \cdots p_k = e_i p_{n+1} \cdots p_k$, hence belongs to $P$. Now let $t = e_{i_1}(n_1) e_{i_2}(n_2) \cdots e_{i_m}(n_m)$ be any word that expresses $t$ in terms of the standard Temperley–Lieb generators of $TL_n(\delta)$. Then the multiplication rule in Theorem 2.5(b) along with Corollary 2.6 implies that the equation

$r_A \tilde{e}_{i_1,n_1}(k) \tilde{e}_{i_2,n_2}(k) \cdots \tilde{e}_{i_m,n_m}(k) \otimes l_B = \tilde{d} + \text{lower order terms}$

holds in $PTL_k(\delta)$. The left hand side of the above equation belongs to $P$. The lower order terms are a linear combination of $\tilde{u}$ such that $u$ belongs to $D_j(k)$ for some $j < n$. By induction, we may assume that each such $\tilde{u}$ belongs to $P$. Hence, the same conclusion holds for $\tilde{d}$, completing the proof. □

4. Semisimplicity of $PTL_k(\delta)$

In this section, we fix the ground ring $k$, $\delta \in k$, and $k$.

**Theorem 4.1.** Let $k$ be an arbitrary unital commutative ring. Let $X(n)$ be the $k$-linear span of $\{ \bar{d} \mid d \in D_n(k) \}$. Then the algebra $PTL_k(\delta)$ has the direct sum decomposition

$$PTL_k(\delta) = \bigoplus_{n=0}^{k} X(n)$$

into pairwise orthogonal two-sided ideals, in the sense that $xy = 0$ if $x \in X(n)$ and $y \in X(n')$, where $n \neq n'$.

**Proof.** This is an immediate consequence of Theorem 2.5 along with the fact that we are dealing only with balanced diagrams. □

Let $Q_n$ be the free $k$-module on the set of all cardinality $n$ subsets of $[k]$. Identify $\text{End}_k(Q_n)$ with the matrix algebra $\text{Mat}_{(n)}(k)$ by means of its basis. All tensor products in this paper are taken over the ground ring $k$, so we write $\otimes$ instead of $\otimes_k$. 

Theorem 4.2. For any $\delta \in k$, where $k$ is a commutative unital ring, the ideal $X(n)$ is isomorphic to the matrix algebra

$$\text{Mat}_{(n)}(\text{TL}_n(\delta - 1)) \cong \text{Mat}_{(n)}(k) \otimes \text{TL}_n(\delta - 1).$$

Furthermore, every $X(n)$-module is isomorphic to one of the form $Q_n \otimes N$, where $N$ is a $\text{TL}_n(\delta - 1)$-module.

Proof. Let $(A_i, d'_i, B_i)$ be the triples in Lemma 3.2 corresponding to diagrams $d_i \in D_n(k)$, for $i = 1, 2$. Theorem 4.2 implies that

$$\bar{d}_1 \bar{d}_2 = \begin{cases} (\delta - 1)^N \bar{d}_3 = 0 & \text{if } B_1 = A_2 \\ 0 & \text{otherwise} \end{cases}$$

where, in the nonzero case, $d'_2 = d'_1 \circ d'_2$ in the partition monoid $P_n$ and the triple corresponding to $d_3 \in D_n(k)$ is $(A_1, d'_3, B_2)$. Notice that $d'_2 d'_2 = (\delta - 1)^N d'_2$ in the Temperley–Lieb algebra $\text{TL}_n(\delta - 1)$. This proves the first claim. The second claim follows from the well known isomorphism

$$\text{Mat}_m(A) \cong \text{Mat}_m(k) \otimes A$$

as $k$-algebras, for any $k$-algebra $A$ and any $m$. The isomorphism is given for any $a_{ij} \in A$ by $\sum a_{ij} e_{ij} \mapsto \sum e_{ij} \otimes a_{ij}$, where $e_{ij}$ is the $(i, j)$th matrix unit. The final claim is a standard feature of Morita theory. The action of

$$\text{Mat}_{(n)}(k) \otimes \text{TL}_n(\delta - 1) \cong \text{End}_k(Q_n) \otimes \text{TL}_n(\delta - 1)$$

on $Q_n \otimes N$ is the obvious one, in which $(f \otimes t)(A \otimes v) = f(A) \otimes tv$, for any $f \in \text{End}_k(Q_n)$, $t \in \text{TL}_n(\delta - 1)$, $A \in Q_n$, $v \in N$. \qed

Remark 4.3. (i) Theorem 4.2 implies that $X(n)$ is Morita equivalent to $\text{TL}_n(\delta)$; that is, there is a category equivalence between their (left, or, equivalently, right) modules. In particular, this implies (rather trivially) that $\text{PTL}_k(\delta)$ is an iterated inflation of Temperley–Lieb algebras, in the sense of [KX99],[KX01],[GPT11].

(ii) The bijection in Lemma 3.2 between $D_n(k)$ and triples induces an isomorphism $X(n) \cong Q_n \otimes \text{TL}_n(\delta - 1) \otimes Q^*_n$, where $Q^*_n := \text{Hom}_k(Q_n, k)$ is the linear dual of $Q_n$. This is an isomorphism of $k$-modules, and also an isomorphism of $k$-algebras, where $Q^*_n \otimes Q_n$ is identified with $\text{End}_k(Q_n)$ by the usual isomorphism.

Theorem 4.4. Suppose that $k$ is a field. Then:

(a) $X(n)$ is semisimple if and only if the same is true of $\text{TL}_n(\delta - 1)$.
(b) If $X(n)$ is semisimple, its simple modules are of the form $Q_n \otimes \text{TL}^\lambda$, where $\text{TL}^\lambda$ is a simple $\text{TL}_n(\delta - 1)$-module.
(c) $\text{PTL}_k(\delta)$ is semisimple for any $\delta \in k$ such that $\text{TL}_n(\delta - 1)$ is semisimple for all $n = 0, 1, \ldots, k$. 

Proof. (a) follows from Theorem 4.2; see e.g. [Mun55, Lemma 4.5]. Part (b) is clear from the isomorphism in Theorem 4.2 and part (c) follows from part (a). □

Remark 4.5. By Theorem A.1, if \( q \) is a nonzero element of the field \( k \) such that \( \left\lceil \frac{n}{q} \right\rceil \neq 0 \), then \( \mathsf{PTL}_n(1 \pm (q + q^{-1})) \) is semisimple over \( k \). In particular, this holds whenever \( q \) is not a root of unity.

5. Representations of \( \mathsf{TL}_k(\delta) \)

The purpose of this section is to recall [GL96, Example 1.4] the standard combinatorial construction of the cell modules for a Temperley–Lieb algebra \( \mathsf{TL}_k(\delta) \). A planar involution with \( \lambda \) fixed points is a sequence of \( k \) points arranged in a line with \( k - \lambda \) of them joined in pairs and \( \lambda \) of them with an “end” attached. The pairs correspond to interchanges and ends to fixed points. The planar condition requires that the involution can be drawn in a half-plane if the ends are extended to infinity, without intersections. For example, the two diagrams

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\quad \quad
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\]



depict planar involutions on \( k = 7 \) points with 3 fixed points. One can think of planar involutions as products of disjoint cycles of length \( \leq 2 \), or as “half” Temperley–Lieb diagrams. Given two planar involutions with the same number of fixed points, there is a unique way to join the ends to create a Temperley–Lieb diagram. For instance, the above involutions join to make the Temperley–Lieb diagram

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\]



with the leftmost involution at the top and rightmost involution (inverted) at the bottom.

By labeling the fixed points and left endpoint of interchanges by 1 and labeling right endpoints of interchanges by \(-1\), a planar involution of length \( k \) produces a sequence \( a = (a_1, \ldots, a_k) \) such that

(i) \( a_i = \pm 1 \) for all \( 1 \leq i \leq k \).

(ii) Each partial sum \( a_1 + \cdots + a_n \geq 0 \) for all \( 1 \leq n \leq k \).

We call any sequence satisfying these conditions a Temperley–Lieb path. Given any Temperley–Lieb path, we can construct a unique planar involution that produces the sequence. To see this, pair each \( i \) such that some \( j > i \) exists with \( a_i + a_{i+1} + \cdots + a_j = 0 \) with the unique minimal such \( j \). Such pairings define the interchanges in the involution, and all unpaired vertices are fixed points. Thus, we have a bijection between the sets of planar involutions and Temperley–Lieb paths of the same length, so we may as well identify these sets.
A Temperley–Lieb diagram $t$ acts on a planar involution $a$ by the usual diagram multiplication, producing a multiple of another planar involution $b$ having at most as many fixed points as $a$. So the free $k$-module $W_0$ on the set of planar involutions of length $k$ is a $\mathrm{TL}_k(\delta)$-module. Let $W_0^{<\lambda}$ (resp., $W_0^{\leq\lambda}$) denote the span of the planar involutions with at most (resp., fewer than) $\lambda$ fixed points. Then

$$\mathrm{TL}^{\lambda} := W_0^{\leq\lambda}/W_0^{<\lambda}$$

is a $\mathrm{TL}_k(\delta)$-module, for any $\lambda$ such that $k - \lambda$ is an even number.

**Theorem 5.1** ([GL96]). Let $k$ be a commutative unital ring. The collection

$$\{\mathrm{TL}^{\lambda} \mid \lambda \equiv k \pmod{2}\}$$

is a complete set of cell modules for $\mathrm{TL}_k(\delta)$. If $k$ is a field and $\delta \in k$ is chosen such that $\mathrm{TL}_k(\delta)$ is semisimple, then the same set is a complete set of isomorphism classes of simple $\mathrm{TL}_k(\delta)$-modules.

**Remark 5.2.** There is another way to index the cell modules for $\mathrm{TL}_k(\delta)$, by the set $\Lambda_k$ of partitions of $k$ with at most two parts. The map $(\lambda_1, \lambda_2) \mapsto \lambda_1 - \lambda_2$ defines a bijection of $\Lambda_k$ with the set of integers in $\{0, 1, \ldots, k\}$ which are congruent to $k$ mod 2, where the value $\lambda_2 = 0$ is allowed. In this indexing scheme, for $\lambda = (\lambda_1, \lambda_2)$ in $\Lambda_k$, the notation $\mathrm{TL}^{\lambda} = W_0^{\leq m}/W_0^{<m}$ replaces the previous notation $\mathrm{TL}^{m} = W_0^{\leq m}/W_0^{<m}$, where $m = \lambda_1 - \lambda_2$. Here $\triangleright$ is the usual dominance order on partitions. We will switch to the partition notation starting in §7.

### 6. Motzkin paths and representations

We now recall the combinatorial construction of the cell modules for the Motzkin algebra, which generalizes the previous section. This will be applied in the next section to construct the cell modules for $\mathrm{PTL}_k(\delta)$. The remainder of this section closely follows [BH14, §4], to which the reader should refer for additional details.

A **Motzkin path** of length $k$ is a sequence $a = (a_1, \ldots, a_k)$ such that $a_i \in \{-1, 0, 1\}$ and each partial sum $a_1 + \cdots + a_n \geq 0$ for all $1 \leq n \leq k$. The **rank** of a Motzkin path $a = (a_i)$ is defined to be

$$\text{rank}(a) := a_1 + \cdots + a_k.$$  

For each index $i$ with $a_i = 1$, let $j$ be the smallest index (if any) such that $i < j \leq k$ and $a_i + a_{i+1} + \cdots + a_j = 0$. Whenever this happens, $(a_i, a_j) = (1, -1)$ are said to be **paired**; otherwise, $a_i = 1$ is **unpaired**. By omitting the zeros in a Motzkin path we obtain a Temperley–Lieb path in the sense of the previous section. Connecting paired indices by an edge and extending unpaired indices by a line to infinity, we recover the planar involution of that path. By including the discarded zeros as isolated vertices, we obtain a graph on $[k]$ called a **1-factor**. Since paired indices cancel one another in the sum,

$$\text{rank}(a) = \text{the number of fixed points (lines to infinity)}$$
in the corresponding 1-factor. For example, the graph

\[
\alpha = \begin{array}{c}
\bullet \\
\circ \circ \circ \\
\bullet \bullet \\
\end{array}
\]

is the 1-factor produced by the Motzkin path \((1, 1, 1, -1, 0, -1, 1, 1, 0, -1)\).

In general, a 1-factor on \(k\) vertices is a graph that gives a planar involution once its isolated points are removed. This means that fixed points of the involution may not appear between paired vertices. Labeling paired vertices in a 1-factor \(\alpha\) by 1, \(-1\) respectively, labeling fixed points by 1, and labeling all other vertices by 0 produces a Motzkin path \(a\) given by the sequence of labels whose corresponding 1-factor is \(\alpha\). So there is a bijection between Motzkin paths and 1-factors of the same length. From now on, we identify Motzkin paths with 1-factors by means of this bijection.

**Remark 6.1.** In [BH14], the fixed points of a 1-factor are depicted by white-colored vertices instead of lines to infinity.

For any given pair \((\alpha, \beta)\), where \(\alpha, \beta\) are 1-factors on \(k\) vertices of the same rank \(\lambda\), there is a unique Motzkin \(k\)-diagram \(C^\lambda_{\alpha, \beta}\) such that the fixed points in \(\alpha\) are connected to those in \(\beta\). For example, if \(\alpha\) is the 1-factor displayed above and \(\beta\) the 1-factor (of rank 2) below

\[
\beta = \begin{array}{c}
\bullet \\
\circ \circ \circ \\
\bullet \bullet \\
\end{array}
\]

then \(C^2_{\alpha, \beta}\) is obtained by reflecting \(\beta\) across its horizontal axis and then drawing edges connecting the fixed points in \(\alpha, \beta\) in order, which gives

\[
C^2_{\alpha, \beta} = \begin{array}{c}
\bullet \\
\circ \circ \circ \\
\bullet \bullet \bullet \\
\end{array}
\]

Notice that the zeros in the Motzkin paths corresponding to \(\alpha, \beta\) label the isolated vertices in the diagram \(C^\lambda_{C_{\alpha, \beta}}\). For any \(k\), the disjoint union

\[
\bigsqcup_{\lambda=0}^k \{C^\lambda_{\alpha, \beta} \mid \text{rank}(\alpha) = \text{rank}(\beta) = \lambda\}
\]

is a cellular basis of \(M_k(\delta)\). This is the basis of Motzkin \(k\)-diagrams.

Now we describe the cell modules for the Motzkin algebra. For a Motzkin diagram \(d\) and a Motzkin path \(a\) (viewed as a 1-factor) of the same length, the Motzkin path \(da\) is given by

\[
da = \delta^N(d, a)b
\]

where the multiplication is carried out by the usual graphical stacking procedure. The integer \(N(d, a)\) is the number of loops in the bottom row of \(\Gamma(d, a)\), and \(b\) is the resulting 1-factor in the top row of \(\Gamma(d, a)\), after erasing the second row of vertices.

For example, if \(d\) is the Motzkin diagram and \(a\) the 1-factor given by the pictures

\[
d = \begin{array}{c}
\end{array}
\]
then the resulting $\Gamma(d, a)$ is the configuration

\[
\Gamma(d, a) = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

and thus $da = \delta b$, where $b$ is the 1-factor

\[
b = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\]

There is a factor of $\delta$ in this example because there is a single loop in $\Gamma(d, a)$.

Let $W$ be the free $k$-module on the Motzkin paths of length $k$. The above action extends linearly to make $W$ into an $M_k(\delta)$-module. Since

\[
\text{rank}(da) \leq \min(\text{rank}(d), \text{rank}(a)),
\]

the $k$-submodule $W^{\leq \lambda}$ of $W$ spanned by the Motzkin paths of rank at most $\lambda$ is an $M_k(\delta)$-submodule, for any $\lambda = 0, 1, \ldots, k$. Thus we have a filtration

\[
(0) \subseteq W^{\leq 0} \subseteq W^{\leq 1} \subseteq \cdots \subseteq W^{\leq k} = W
\]

of $M_k(\delta)$-submodules. For each $\lambda$, the quotient module

\[
M^\lambda := W^{\leq \lambda} / W^{< \lambda}
\]

is an $M_k(\delta)$-module, where we set $W^{< \lambda} := W^{\leq \lambda-1}$ (and $W^{< 0} = (0)$).

**Theorem 6.2** ([BH14, Thms. 4.7, 4.16]). Let $k$ be a commutative unital ring. The cell modules for $M_k(\delta)$ are given by

\[
\{M^\lambda \mid 0 \leq \lambda \leq k\}.
\]

If $k$ is a field then $M^\lambda$ is indecomposable, and if $\delta \in k$ is chosen so that $M_k(\delta)$ is semisimple then the above set is a complete set of pairwise nonisomorphic simple $M_k(\delta)$-modules.

By restricting our attention to the Motzkin paths with no zeros, we obtain copies of the cell modules for $TL_k(\delta)$. The following was observed in [BH14, Rmk. 4.9].

**Theorem 6.3.** Let $T^\lambda$ be the $k$-span of the Motzkin paths $a$ in $M^\lambda$ with no zeros. The span of the Motzkin diagrams in $M_k(\delta)$ having no isolated vertices is a subalgebra isomorphic to $TL_k(\delta)$. When restricted to that subalgebra, the action of $M_k(\delta)$ on $M^\lambda$ makes $T^\lambda$ into a $TL_k(\delta)$-module which is isomorphic to the cell module $TL^\lambda$, for any $\lambda$ such that $k - \lambda$ is even.

7. **Representations of $PTL_k(\delta)$**

We continue to fix $k$ and $\delta \in k$, where $k$ is a given unital commutative ring. Recall from Theorem 4.1 that $PTL_k(\delta) = \bigoplus_{n=0}^k X(n)$. From now on, we will index representations by partitions of not more than two parts, instead of by integers as in the previous two sections; see Remark 5.2.
The type of a Motzkin path \( a = (a_1, \ldots, a_k) \) is the pair \( \lambda = (\lambda_1, \lambda_2) \) such that \( \lambda_1 \) (resp., \( \lambda_2 \)) be the number of \( i \) such that \( a_i = 1 \) (resp., \( a_i = -1 \)). Then \( \lambda_1 \geq \lambda_2 \geq 0 \) and \( \text{rank}(a) = \lambda_1 - \lambda_2 \). We set
\[
|\lambda| := \lambda_1 + \lambda_2.
\]
The 1-factor corresponding to \( a \) has \( \lambda_2 \) cups and \( a \) has \( k - |\lambda| \) zeros. Write
\[\Lambda := \{ (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq k \}\]
for the set of types that occur as the type of some Motzkin path. We identify elements of \( \Lambda \) with partitions of at most two parts. Thus \( \Lambda \) is the disjoint union of the \( \Lambda_n \) (see Remark 5.2) as \( n \) runs from 0 to \( k \).

For Motzkin paths \( a = (a_1, \ldots, a_k) \), \( a' = (a'_1, \ldots, a'_k) \) we write \( a' \preceq a \) if the following two conditions are satisfied:

(i) \( a_i = 0 \) implies that \( a'_i = 0 \).
(ii) \( a'_i = 0 \) if and only if \( a'_j = 0 \) whenever \( a_i, a_j \) are paired in \( a \).

In other words, \( a' \preceq a \) if and only if the 1-factor corresponding to \( a' \) is obtainable from the 1-factor corresponding to \( a \) by erasing zero or more edges or lines to infinity. Let \( \chi(a, a') \) be the number of such changes. Define \( \bar{a} \) to be the linear combination
\[
(24) \quad \bar{a} := \sum_{a' \preceq a} (-1)^{\chi(a, a')} a'.
\]
For example, if \( a = (1, -1, 1) \) then
\[
\bar{a} = (1, -1, 1) - (0, 0, 1) - (1, -1, 0) + (0, 0, 0).
\]
If \( i \) is any index such that \( a_i = \pm 1 \) then \( p_i a \) is the unique Motzkin path obtained from \( a \) by changing \( a_i \) to 0 (and also changing \( a_j \) to 0 if \( a_i, a_j \) are paired).

We define \( [a] \) to be the set of all \( i \) such that \( a_i = \pm 1 \). In the corresponding 1-factor, this set indexes the vertices that are endpoints of edges or lines to infinity. Notice that we have
\[
(25) \quad \bar{a} = \prod_{i \in [a]} (1 - p_i) a
\]
in the Motzkin algebra. Notice that \( \{ \bar{a} \mid a \text{ is a Motzkin path of length } k \} \) is a \( \mathbb{k} \)-basis for the free \( \mathbb{k} \)-module \( W \) on the set of Motzkin paths of length \( k \).

Theorem 7.1. Let \( \mathbb{k} \) be a commutative unital ring. Suppose that \( a \) is a Motzkin path in \( W \) of type \( \lambda \) in \( \Lambda \). Let \( b \) be the unique Motzkin path such that \( d a = \delta^N b \), for some \( N \), where \( d \in D(k) \) is balanced.
\[
\bar{d} \bar{a} = \begin{cases} 
(\delta - 1)^{N(d,a)} \bar{b} & \text{if } |d| = |a| \\
0 & \text{otherwise.}
\end{cases}
\]

If \( |d| = |a| \) then the type of \( b \) is some \( \mu \) in \( \Lambda \) such that \( |\mu| = |\lambda| \), and \( \text{rank}(a) - \text{rank}(b) \) is an even number.
Proof. Thanks to the identity \(\ref{identity}\), the action of \(\bar{d}\) by \(\bar{a}\) is given by precisely the same rule as the multiplication rule in Theorem \(\ref{thm1}\)(a), giving the first claim. Suppose that \(|d| = |a|\). Then the zeros in \(a\) appear at the same places as the isolated vertices in the bottom row of \(d\). Since \(d\) is balanced, it has the same number of isolated vertices in its top row, so \(|\mu| = |\lambda|\). The rank of \(b\) cannot be larger but may be strictly less than that of \(a\). (For instance, if \(k = 3\) and \(a = (1, 1, 1)\) then \(e_1a = b = (1, -1, 1)\), so \(e_1\bar{a} = \bar{b}\) where \(\text{rank}(a) = 3\), \(\text{rank}(b) = 1\).) The rank decreases only if one or more pairs of unpaired indices of 1 in \(a\) are replaced by pairings in \(b\), so the rank can decrease only in steps of 2. \(\square\)

Recall that the usual dominance order on partitions is defined by declaring that \(\lambda \triangleright \mu\) if and only if \(\lambda - \mu\) can be written as a sum of positive roots (in the root system of \(\mathfrak{gl}_n\)). In our situation, if \(|\lambda| = |\mu|\) and \(\lambda, \mu \in \Lambda\), this is equivalent to \(\lambda \triangleright \mu \iff \lambda - \mu = m(1, -1)\) for some integer \(m \geq 0\).

This is equivalent to \((\lambda_1 - \lambda_2) - (\mu_1 - \mu_2)\) being a nonnegative even integer. Write \(\lambda \triangleright \mu\) whenever \(\lambda \triangleright \mu\) but \(\lambda \neq \mu\).

For any \(\lambda\) in \(\Lambda\), let \(\bar{W}^{\ominus \lambda}\) and \(\bar{W}^{\ominus \lambda}\) be the \(k\)-span of the sets

\[\{\bar{a} \mid a \text{ has type } \mu \text{ and } \lambda \triangleright \mu\}\text{ and }\{\bar{a} \mid a \text{ has type } \mu \text{ and } \lambda \triangleright \mu\}\]

respectively. By Theorem \(\ref{thm2}\) both \(\bar{W}^{\ominus \lambda}\), \(\bar{W}^{\ominus \lambda}\) are PTL\((\delta)\)-submodules of \(W\). We define

\[\text{PTL}^\lambda := \bar{W}^{\ominus \lambda} / \bar{W}^{\ominus \lambda}\]

to be the corresponding quotient module. So the collection of \(\bar{a} + \bar{W}^{\ominus \lambda}\) such that \(a\) has type \(\lambda\) is a basis of PTL\(^\lambda\).

Theorem \(\ref{thm2}\) above and Theorems \(\ref{thm4} \ref{thm5}\) imply the following result.

**Theorem 7.2.** Let \(k\) be a commutative unital ring. The PTL\(^\lambda\) for \(\lambda\) in \(\Lambda\) are the cell modules for PTL\(_k(\delta)\). For any \(\lambda \in \Lambda\), with \(n = |\lambda|\) we have an isomorphism

\[\text{PTL}^\lambda \cong Q_n \otimes \text{TL}^\lambda\]

as \(X(n)\)-modules, where \(\text{TL}^\lambda\) is the cell module for \(\text{TL}_n(\delta - 1)\) indexed by \(\lambda\) in accordance with Remark \(\ref{rem1}\). Hence, if \(k\) is a field and \(\delta \in k\) is chosen so that PTL\(_k(\delta)\) is semisimple, then

\[\{\text{PTL}^\lambda \mid \lambda \in \Lambda\}\]

is a complete set of pairwise nonisomorphic simple PTL\(_k(\delta)\)-modules.

**Proof.** Suppose that \(a\) is a Motzkin path of type \(\lambda\), for \(\lambda\) in \(\Lambda\). Let \(A = \lfloor a \rfloor\). If \(n = |A|\) then \(A\) is an element of \(Q_n\). Let \(b\) be the Motzkin path obtained from \(a\) by removing all its zero entries; that is, \(b\) is the part of \(a\) supported by \(A\). Then \(b\) may be regarded as a Temperley–Lieb half diagram (a planar
involvement in the terminology of Graham and Lehrer; see [5]. Suppose that $|A| = n$. The linear map sending

$$a \mapsto A \otimes b$$

defines an isomorphism $W^{\otimes \lambda} \to Q_n \otimes W_0^{\otimes \lambda}$ that restricts to an isomorphism $W^{\otimes \lambda} \to Q_n \otimes W_0^{\otimes \lambda}$. Passing to quotients induces the desired isomorphism $PTL^\lambda \cong Q_n \otimes TL^\lambda$. This shows that the $PTL^\lambda$ are inflations of the various cell modules for $TL_n(\delta - 1)$, which proves the first claim. The last claim follows from the first two.

As a consequence of the last result, if $\lambda$ is in $\Lambda$ and satisfies $\lambda_1 - \lambda_2 = n$, we have

$$\text{rank}_k PTL^\lambda = \binom{k}{n} \text{rank}_k TL^\lambda.$$  

Remark 7.3. The subalgebra of $PTL_k(\delta)$ spanned by $\{\bar{d} \mid d \in D_k(k)\}$ is isomorphic to $TL_k(\delta - 1)$. For $\lambda \in \Lambda$ with $|\lambda| = k$, the action of $PTL_k(\delta)$ on $PTL^\lambda$, when restricted to that subalgebra, is isomorphic to $TL^\lambda$ as a $TL_k(\delta - 1)$-module.

8. $U_q(gl_2)$ AND $U_q(sl_2)$

We assume henceforth that $k$ is a field and that $0 \neq q \in k$ is not a root of unity. Let $U = U_q(gl_2)$ be the quantized enveloping algebra of the general linear Lie algebra $gl_2$. By definition, $U$ is the associative algebra with 1 generated by $E, F, K^\pm_1$ (for $i = 1, 2$) subject to the defining relations

\begin{align*}
K_1K_2 &= K_2K_1, \\
K_1K_i^{-1} &= K_i^{-1}K_i = 1 \quad (i = 1, 2) \\
K_1EK_i^{-1} &= qE, \\
K_2EK_i^{-1} &= q^{-1}E \\
K_1FK_i^{-1} &= q^{-1}F, \\
K_2FK_i^{-1} &= qF \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}, \quad \text{where } K := K_1K_2^{-1}.
\end{align*}

Furthermore, $U$ is a Hopf algebra with coproduct $\Delta : U \to U \otimes U$ and counit $\epsilon : U \to k$ given on generators by

\begin{align*}
\Delta(E) &= E \otimes K + 1 \otimes E \\
\Delta(F) &= F \otimes 1 + K^{-1} \otimes F \\
\Delta(K_i) &= K_i \otimes K_i \quad (i = 1, 2) \\
\epsilon(E) &= \epsilon(F) = 0, \quad \epsilon(K_i) = 1 \quad (i = 1, 2).
\end{align*}

The subalgebra of $U = U_q(gl_2)$ generated by $E, F, K^\pm_1$ is the quantized enveloping algebra $U_q(sl_2)$.

Remark 8.1. The coproduct $\Delta$ defined in (28) is the one used in [BH14]. It differs from the usual one in [Lus93,Jan96]. One could use either convention
in this paper, but for the sake of consistency, we stick with the choice made in [BH14].

We refer to [Jan96] Chap. 2 for basic facts about the representation theory of $U_q(\mathfrak{sl}_2)$. For each integer $n \geq 0$, by [Jan96] Thm. 2.6] there exist simple $U_q(\mathfrak{sl}_2)$-modules $L(n,+)$, $L(n,-)$ of dimension $n+1$. (In characteristic 2, $L(n,+)$ $\cong L(n,-)$.) Any simple $U_q(\mathfrak{sl}_2)$-module of dimension $n+1$ is isomorphic to either $L(n,+)$ or $L(n,-)$, and $L(n,+)$ is of type 1.

Lemma 8.2. Suppose that $k$ is a field and that $0 \neq q \in k$ is not a root of unity. For any $\lambda = (\lambda_1,\lambda_2) \in \mathbb{Z} \times \mathbb{Z}$ with $\lambda_1 - \lambda_2 \geq 0$, there is a unique $U_q(\mathfrak{gl}_2)$-module $V(\lambda)$ such that $V(\lambda) \cong L(\lambda_1 - \lambda_2, +)$ as $U_q(\mathfrak{sl}_2)$-modules, and

$$K_i v_+ = q^{\lambda_i} v_+ \text{ for } i = 1, 2$$

where $v_+$ is a highest weight vector of weight $\lambda_1 - \lambda_2$ for the $U_q(\mathfrak{sl}_2)$-module structure.

Proof. This follows from the results in [Jan96] Chap. 2, using the fact that any $U_q(\mathfrak{gl}_2)$-module is also (by restriction) a $U_q(\mathfrak{sl}_2)$-module. (See also [KS97 §7.3].) \hfill \Box

Under the same hypotheses, every finite dimensional $U_q(\mathfrak{gl}_2)$-module $M$ is semisimple. Furthermore, if $M$ is of type 1, it is isomorphic to a direct sum of modules of the form $V(\lambda)$ as in the above lemma.

The “polynomial” $U_q(\mathfrak{gl}_2)$-modules are direct sums of $V(\lambda)$ such that $\lambda = (\lambda_1,\lambda_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and $\lambda_1 \geq \lambda_2$. If $M$ is a simple polynomial $U_q(\mathfrak{gl}_2)$-module, we may (and do) identify its highest weight $\lambda$ with a partition of at most two parts. In particular, if $\lambda = (n)$ is a partition of one part, then we write $V(n)$ for $V(\lambda)$. Thus $V(n) \cong L(n, +)$ as $U_q(\mathfrak{sl}_2)$-modules.

Consider the simple $U_q(\mathfrak{gl}_2)$-modules $V(0)$, $V(1)$. Then $V(0) \cong k$ is the trivial module, with $U_q(\mathfrak{gl}_2)$ acting via the counit $\epsilon$. This means that if $v_0$ is a chosen basis of $V(0)$; then on $v_0$ the operators $E$, $F$ act as zero and each $K_i$ acts as 1. Fix a choice of $v_0$. Fix also a basis $\{v_1, v_{-1}\}$ of weight vectors of $V(1)$, where $v_1$ has weight $(1,0)$ and $v_{-1}$ has weight $(0,1)$ as $U_q(\mathfrak{gl}_2)$-modules, such that the action of $E$, $F$ is given by

$$Ev_1 = 0, \quad Ev_{-1} = v_1, \quad Fv_1 = v_{-1}, \quad Fv_{-1} = 0.$$

In other words, with respect to the basis $\{v_1, v_{-1}\}$, the matrices representing the action of $E$, $F$, $K_i$ are

$$E \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_1 \rightarrow \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}, \quad K_2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}.$$

Following [BH14], we set $V := V(0) \oplus V(1)$, with basis $\{v_1, v_0, v_{-1}\}$.

We will need the following general fact about tensor powers of bialgebra representations, for which we were unable to find a suitable reference. If $U$
is a bialgebra and \( V \) a \( U \)-module, then \( V^{\otimes k} \) is a \( U \)-module for any \( k > 1 \), with \( u \in U \) acting on \( V^{\otimes k} \) by

\[
\Delta^{(k)} : U \to U^{\otimes k},
\]

where \( \Delta \) is the coproduct on \( U \), lifted to \( \Delta^{(k)} \) inductively by (for example) defining \( \Delta^{(2)} = \Delta \) and \( \Delta^{(k+1)} = (\Delta \otimes 1^{\otimes (k-1)}) \Delta^{(k)} \) for \( k \geq 2 \).

**Lemma 8.3.** Let \( V \) be a \( U \)-module, where \( U \) is a bialgebra with coproduct \( \Delta \). Suppose that \( \psi \) is in \( \text{End}_U(V \otimes V) \); that is, \( \psi \Delta(u) = \Delta(u)\psi \), for any \( u \in U \). Then \( 1^{\otimes (i-1)} \otimes \psi \otimes 1^{k-1-i} \) commutes with the action of \( \Delta^{(k)}(u) \), for any \( u \in U \).

**Proof.** This is of course well known; we sketch a proof for completeness. It follows by induction from the coassociativity axiom

\[
(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta
\]

that

\[
\Delta^{(k)} = (1^{\otimes a} \otimes \Delta \otimes 1^{\otimes b})\Delta^{(k-1)}
\]

for any \( a, b \) such that \( a + b = k - 2 \). Taking \( a = i - 1 \), \( b = k - 1 - i \) proves the result. \( \square \)

9. **Structure of \( V \otimes V \) for \( V = V(0) \oplus V(1) \)**

We continue to assume in this section that \( k \) is a field and \( 0 \neq q \in k \) is not a root of unity. We wish to analyze the structure of \( V \otimes V \), as module for both \( U_q(\mathfrak{gl}_2) \) and \( U_q(\mathfrak{sl}_2) \). We have

\[
V \otimes V = (V(0) \otimes V(0)) \oplus (V(0) \otimes V(1)) \oplus (V(1) \otimes V(0)) \oplus (V(1) \otimes V(1))
\]

and since the first three direct summands on the right hand side are respectively isomorphic to \( V(0) \), \( V(1) \), and \( V(1) \), understanding the structure of \( V \otimes V \) reduces to understanding the structure of \( V(1) \otimes V(1) \).

To simplify notation, set \( v_{i,j} := v_i \otimes v_j \). Then \( \{ v_{i,j} \mid i,j \in \{1,0,-1\} \} \) is a basis of \( V \otimes V \), and \( \{ v_{i,j} \mid i,j \in \{1,-1\} \} \) is a basis of \( V(1) \otimes V(1) \). A simple direct computation shows that \( \{ v_{1,1}, q^{-1}v_{1,-1} + v_{-1,1}, v_{-1,-1} \} \) is a basis of weight vectors for a submodule of \( V(1) \otimes V(1) \) isomorphic to \( V(2) \). In order to pick a complement to this submodule, we observe that the vector

\[
Z_0 := -qv_{1,-1} + v_{-1,1}
\]

is orthogonal to the submodule, with respect to the standard bilinear form on \( V(1) \) extended to \( V(1) \otimes V(1) \), and this property uniquely determines \( Z_0 \) up to a scalar multiple. The line \( kZ_0 \) in \( V(1) \otimes V(1) \) is isomorphic to the trivial module \( V(0) \) as \( U_q(\mathfrak{sl}_2) \)-modules. Since \( K_i \) for \( i = 1, 2 \) both act as \( q \) on \( Z_0 \), it follows that \( kZ_0 \cong V(1,1) \) as \( U_q(\mathfrak{gl}_2) \)-modules. Hence

\[
V(1) \otimes V(1) \cong \begin{cases} V(2) \oplus V(1,1) & \text{as } U_q(\mathfrak{gl}_2) \text{-modules} \\ V(2) \oplus V(0) & \text{as } U_q(\mathfrak{sl}_2) \text{-modules.} \end{cases}
\]
It follows from (31) that

\[(32) \quad V \otimes V \cong \begin{cases} V(0) \oplus 2V(1) \oplus V(2) \oplus V(1,1) & \text{as } U_q(\mathfrak{gl}_2) \text{-modules} \\ 2V(0) \oplus 2V(1) \oplus V(2) & \text{as } U_q(\mathfrak{sl}_2) \text{-modules.} \end{cases}\]

From this it is immediate that

\[(33) \quad \dim \text{End}_{U_q(\mathfrak{gl}_2)}(V \otimes V) = 1^2 + 2^2 + 1^2 + 1^2 = 7 \]
\[(33) \quad \dim \text{End}_{U_q(\mathfrak{sl}_2)}(V \otimes V) = 2^2 + 2^2 + 1^2 = 9.\]

This dichotomy illustrates the error in [BH14], where it is implicitly assumed that the centralizers are the same in both cases.

The first decomposition given in equation (31) is a special case of the decomposition:

\[(34) \quad V(1) \otimes V(\lambda) \cong \bigoplus_{\mu} V(\mu)\]

where \(\lambda, \mu\) are partitions of at most two parts and \(\mu\) varies over the set of such partitions obtainable from \(\lambda\) by adding one box. The rule (34) is itself a special case of the standard Pieri rule. By repeated application of (34), we now construct the Bratteli diagram (see Figure 1) for the centralizer algebra \(Z_k(q) := \text{End}_{U_q(\mathfrak{gl}_2)}(V \otimes^k)\).

![Bratteli diagram](image)

**Figure 1.** Bratteli diagram for \(\text{End}_{U_q(\mathfrak{gl}_2)}(V \otimes^k)\), \(k \leq 4\)

Since \(q\) is not a root of unity, the dimension of the simple \(Z_k(q)\)-module indexed by a partition \(\lambda\) of at most two parts is the number of paths from
the top vertex to its label in the Bratteli diagram. The sum of the squares of those dimensions is the dimension of $Z_k(q)$, as tabulated below:

| $k$ | $\emptyset$ | (1) | (2) | (1$^2$) | (3) | (2, 1) | (4) | (3, 1) | (2, 2) | $\dim Z_k(q)$ |
|-----|-------------|-----|-----|---------|-----|--------|-----|--------|--------|-------------|
| 0   | 1           |     |     |         |     |        |     |        |        | 1           |
| 1   | 1           | 1   |     |         |     |        |     |        |        | 2           |
| 2   | 1           | 2   | 1   | 1       |     |        |     |        |        | 7           |
| 3   | 1           | 3   | 3   | 3       | 1   | 2      |     |        |        | 33          |
| 4   | 1           | 4   | 6   | 6       | 4   | 8      | 1   | 3      | 2      | 183         |

and they differ from the dimensions of $\text{End}_{U_q(\mathfrak{sl}_2)}(V^\otimes k)$ given in [BH14, Fig. 1]. Notice that the dimension of $Z_k(q)$ agrees with the dimension in (18) of the algebra $\text{PTL}_k(\delta)$, at least up to degree 4. We will prove in Theorem 11.4 that they agree in general, when $\delta = 1 \pm (q + q^{-1})$.

10. Schur–Weyl duality for the Motzkin algebras

We continue to assume that $k$ is a field, and $0 \neq q \in k$ is not a root of unity.

We endow $V = V(0) \oplus V(1)$ with the standard nondegenerate bilinear form such that $\langle v_i, v_j \rangle = \delta_{i,j}$ for all $i, j = 1, 0, -1$ and extend the form to $V \otimes V$ in the natural way. Let $\pi$ be the orthogonal projection of $V(1) \otimes V(1)$ onto the line $kZ \cong V(0)$, where $Z = -qv_{-1,1} + v_{-1,1}$ is the invariant in equation (39). With respect to the ordered basis $v_{1,1}, v_{1,1}, v_{-1,1}, v_{-1,1}$ the matrix of $\pi$ is

$$A(\pi) = \frac{1}{q + q^{-1}} \begin{bmatrix} 0 & -1 \\ -1 & q^{-1} \\ q^{-1} & 0 \end{bmatrix}$$

in which omitted entries should be interpreted as zero, as usual. Recall from §1.5 that the action of the generator $e_i$ in $TL_k(\pm(q + q^{-1}))$ on $V(1)^{\otimes k}$ is given by the operator $\pm(q + q^{-1}) 1^{\otimes (i-1)} \otimes \pi \pi 1^{\otimes (k-i-1)}$. This defines a faithful action of $TL_k(\pm(q + q^{-1}))$, and it is well known that

$$TL_k(\pm(q + q^{-1})) \cong \text{End}_{U_q(\mathfrak{sl}_2)}(V(1)^{\otimes k}) \cong \text{End}_{U_q(\mathfrak{sl}_2)}(V(1)^{\otimes k}).$$

In order to extend the above action to a faithful action of $M_k(1 \pm (q + q^{-1}))$ on $V^{\otimes k}$, we first consider how to do this for the case $k = 2$.

The algebra $M_2(1 \pm (q + q^{-1}))$ is generated by the elements $r, l$, and $e$ defined in §II. It has basis consisting of the nine Motzkin diagrams:

In order from left to right, the above elements are expressible in terms of the generators $r, l, e$ as follows:

$$1, r, l, lr, rl, e, re = le, er = el, r^2 = l^2 = rer = ler.$$

Since (in the partition algebra) the elements $r, l$ satisfy the identities

$$r = p_1 s = sp_2, \quad l = sp_1 = p_2 s$$
where \( s \) is the swap operator defined in \([31]\) and \( p_1, p_2 \) are projections onto \( V(0) \otimes V, V \otimes V(0) \) respectively, it is natural to define the action of \( r, l \) on basis elements by

\[
rv_{i,j} = \delta_{j,0}v_{0,i}, \quad lv_{i,j} = \delta_{i,0}v_{j,0}
\]

for all \( i, j \in \{1, 0, -1\} \), as in \([31]\) §3.4.

The lines \( \mathbb{k}v_{0,0}, \mathbb{k}Z_0 \) are isomorphic copies of the trivial \( \mathbb{U}_q(\mathfrak{sl}_2) \)-module in \( V \otimes V \), so it is natural to let \( e \) act as a projection onto some linear combination of the form

\[
Y_0 := v_{0,0} + \alpha Z_0 = v_{0,0} + \alpha(-qv_{1,-1} + v_{-1,1}) \quad (\alpha \neq 0).
\]

We demand that on restriction to \( V(1) \otimes V(1) \) (resp., \( V(0) \otimes V(0) \)) the action restricts to the Temperley–Lieb action defined above (resp., the trivial action). This forces \( \alpha \neq 0 \) and implies that each \( v_{i,j} \) is sent to a multiple \( \beta_{i,j}Y_0 \) of \( Y_0 \). The multipliers \( \beta_{i,j} \) are forced by our demands to be

\[
\beta_{1,-1} = \mp \alpha^{-1}, \quad \beta_{0,0} = 1, \quad \beta_{-1,1} = \pm \alpha^{-1}q^{-1}
\]

with all other \( \beta_{i,j} = 0 \). In other words, the matrix of the action of \( e \) on the 0-weight space \( (V \otimes V)_0 \) with respect to the ordered basis \( v_{1,-1}, v_{0,0}, v_{-1,1} \) is given by

\[
B(\alpha, \pm) := \begin{bmatrix}
\pm q & -\alpha q & \mp 1 \\
\mp \alpha^{-1} & 1 & \pm \alpha^{-1}q^{-1} \\
\mp 1 & \alpha & \pm q^{-1}
\end{bmatrix}
\]

The matrix \( B(\alpha, \pm) \) satisfies the relation

\[
B(\alpha, \pm)^2 = (1 \pm (q + q^{-1}))B(\alpha, \pm)
\]

so it is a scalar multiple of a projection in the above sense.

We now define two nondegenerate bilinear forms \( \langle -,- \rangle_t, \langle -,- \rangle_b \) on \( V \) by the rules:

\[
\langle v_1, v_{-1} \rangle_t = -\alpha q, \quad \langle v_0, v_{-1} \rangle_t = 1, \quad \langle v_{-1}, v_1 \rangle_t = \alpha
\]

\[
\langle v_1, v_{-1} \rangle_b = \mp \alpha^{-1}, \quad \langle v_0, v_{-1} \rangle_b = 1, \quad \langle v_{-1}, v_1 \rangle_b = \pm \alpha^{-1}q^{-1}
\]

with all other \( \langle v_i, v_j \rangle_t \), \( \langle v_i, v_j \rangle_b \) = 0. It is worth noticing that the nonzero values of \( \langle v_i, v_j \rangle_t \) (resp., \( \langle v_i, v_j \rangle_b \)) are encoded in the middle column (resp., middle row) of the matrix \( B(\alpha, \pm) \).

Just as in \([31]\) §3.4, the forms may be applied to give explicit formulas for the action of \( M_k(1 \pm (q + q^{-1})) \) on \( V^\otimes k \), as follows. Given a Motzkin \( k \)-diagram \( d \) and \( i_\alpha, j_\alpha \) in \( \{-1, 0, 1\} \) for \( \alpha = 1, \ldots, k \), label the top row vertices of \( d \) from left to right with basis elements \( v_{i_1}, \ldots, v_{i_k} \) and similarly label the bottom row vertices with \( v_{i_1}, \ldots, v_{i_k} \). The blocks of \( d \) are either isolated vertices or edges. Then:

\[
d(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (d)_{i_1, \ldots, i_k}^{j_1, \ldots, j_k} v_{j_1} \otimes \cdots \otimes v_{j_k}
\]
defines the action of \(d\), where the scalar \((d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}\) is the product of the weights taken over the various blocks of \(d\). The weight \((\beta)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}\) of a labeled block \(\beta\) of \(d\) is

\[
\begin{array}{c|c}
\delta_{i,0} & \text{if } \beta \text{ is an isolated vertex labeled by } v_i. \\
\delta_{i,j} & \text{if } \beta \text{ is a vertical edge with endpoints labeled by } v_i, v_j. \\
\langle v_i, v_j \rangle_t & \text{if } \beta \text{ is a horizontal top edge with left and right endpoints labeled by } v_i \text{ and } v_j, \text{ resp.} \\
\langle v_i, v_j \rangle_b & \text{if } \beta \text{ is a horizontal bottom edge with left and right endpoints labeled by } v_i \text{ and } v_j, \text{ resp.}
\end{array}
\]

Here \(\delta_{i,j}\) is the usual Kronecker delta function.

**Proposition 10.1.** For any \(\alpha \neq 0\), the above rules define a left action of \(M_k(\delta)\) on \(V^\otimes k\), where \(\delta = 1 \pm (q + q^{-1})\).

**Proof.** This is proved by the same argument as the proof of [BH14, Prop. 3.29]. In that argument, the notations \(v_i = v_{-1}, v_0 = v_0, v_i = v_i\) were defined. The argument depends on the following properties of the forms:

\[
\langle v_a^*, v_a \rangle_b \langle v_a, v_a^* \rangle_t = 1, \quad \langle v_a, v_a^* \rangle_t \langle v_a^*, v_a \rangle_b = \begin{cases} 
-q & \text{if } a = 1 \\
1 & \text{if } a = 0 \\
-q^{-1} & \text{if } a = -1
\end{cases}
\]

and only on those properties. One checks that for any \(\alpha \neq 0\), all of these properties still hold when we use equation (11) to define the forms. \(\square\)

The following is a slight generalization of results proved in [BH14, §3.4].

**Theorem 10.2.** Suppose that \(k\) is a field, \(0 \neq q \in k\) is not a root of unity, and \(0 \neq \alpha \in k\). For any \(i = 1,\ldots,k-1\), let \(e_i, r_i, l_i\) act on \(V^\otimes k\) in tensor positions \(i, i+1\) as the operator \(1^\otimes (i-1) \otimes g \otimes 1^\otimes (k-i-1)\), where \(g = e, r, l\) (as operators) respectively. This extends to an action of \(M_k(1 \pm (q + q^{-1}))\) that commutes with the action of \(U_q(sl_2)\). The corresponding representation \(\rho : M_k(1 \pm (q + q^{-1})) \to \text{End}_k(V^\otimes k)\) is faithful, thus induces an algebra isomorphism \(M_k(1 \pm (q + q^{-1})) \cong \text{End}_{U_q(sl_2)}(V^\otimes k)\).

**Proof.** The defining relations for \(M_k(\delta)\) are given in [HLP13]. It is a tedious yet elementary calculation to verify that our operators \(e_i, r_i, l_i\) satisfy precisely the same relations (for any \(\alpha \neq 0\)). It suffices to do the calculation in \(M_3(\delta)\). We used a computer algebra system to create explicit matrices for the generating operators and verified the defining relations accordingly. It follows that when \(\delta = 1 \pm (q + q^{-1})\), the action determines a representation

\[
\rho : M_k(1 \pm (q + q^{-1})) \to \text{End}_k(V^\otimes k).
\]
Another way to see this is to repeat the proof of [BH14, Prop. 3.29] with the appropriate substitutions. One can check by an elementary direct computation that $e$, $r$, and $l$ (as operators on $V \otimes V$) commute with the action of $U_q(sl_2)$; here again a computer algebra system is useful. It then follows from Lemma 8.3 that the action of any of the generators $e_i$, $r_i$, $l_i$ commutes with the action of $U_q(sl_2)$. Finally, the proof of [BH14, Thm. 3.31] also applies to our situation to show that $\rho$ is faithful, and the result follows.

**Remark 10.3.** (i) Working with $M_k(1 - q - q^{-1})$, Benkart and Halverson [BH14] choose to define the action of $e$ (which they denote by $t$) on $(V \otimes V)_0$ in terms of the matrix

$$B(q^{-1/2}, -) = \begin{pmatrix} -q & -q^{1/2} & 1 \\ q^{1/2} & 1 & -q^{-1/2} \\ 1 & q^{-1/2} & -q^{-1} \end{pmatrix}$$

and define their bilinear forms $\langle -,- \rangle_t$, $\langle -,- \rangle_b$ accordingly. In other words, they are setting $\alpha = q^{-1/2}$ and making a particular choice of sign. Thus they implicitly assume that $q^{1/2}$ exists in $k$. Our analysis shows that this assumption is avoidable by simply taking $\alpha = 1$ (or any other convenient nonzero value).

(ii) If we work instead in $M_k(1 + q + q^{-1})$, where $e^2 = (1 + q + q^{-1})e$, it is also possible to define the action of $e$ as a multiple of the orthogonal projection onto $Y_0$ (with respect to the standard bilinear form). The matrix giving the action of $e$ on $(V \otimes V)_0$ with respect to the same ordered basis as above is

$$B(\pm q^{-1/2}, +) = \begin{pmatrix} q & \mp q^{1/2} & -1 \\ \mp q^{1/2} & 1 & \mp q^{-1/2} \\ -1 & \pm q^{-1/2} & q^{-1} \end{pmatrix}$$

With this choice, the corresponding bilinear forms $\langle -,- \rangle_t$, $\langle -,- \rangle_b$ become identical.

### 11. Schur–Weyl duality for $\text{PTL}_k(\delta)$

We now turn to $\text{PTL}_k(\delta)$, with $\delta = 1 \pm (q + q^{-1})$, continuing to assume that $0 \neq q \in k$ is not a root of unity and $k$ is a field. This algebra acts faithfully on $V^\otimes k$ by restriction of the action of $M_k(\delta)$. Recall from (17) that

$$\epsilon_i = \tilde{\epsilon}_i = (1 - p_i)e_i(1 - p_i).$$

We have shown in Theorem 3.5 that $\text{PTL}_k(\delta)$ is generated by the $\epsilon_i$, $r_i$, $l_i$ ($i \in [k - 1]$). By Lemma 8.3 it suffices to understand the action in the case $k = 2$. Let $\epsilon$ be the operator on $V \otimes V$ given by the action of $\epsilon_1$ in $\text{PTL}_2(\delta)$. In terms of Motzkin diagrams we have the identity

$$\epsilon = \text{ } \text{ } \text{ } - \text{ } \text{ } - \text{ } + \text{ } .$$
An explicit calculation with the formulas in the preceding section reveals that $\varepsilon$ acts as
\[ v_{1,-1} \mapsto \pm (qv_{1,-1} - v_{-1,1}), \quad v_{-1,1} \mapsto \pm (-v_{1,-1} + q^{-1}v_{-1,1}) \]
with all other $v_{i,j} \mapsto 0$. This is independent of the choice of $\alpha \neq 0$. We note that $\varepsilon^2 = \pm (q + q^{-1})\varepsilon$. In other words, the restriction of $\varepsilon$ to $V(1) \otimes V(1)$ coincides with the standard action of the generator $e$ in $\mathbb{T}L_2(\pm (q + q^{-1}))$, as described at the beginning of [310].

Thus $\varepsilon_i$ acts on $V^\otimes k$ as the operator $1^\otimes(i-1) \otimes \varepsilon \otimes 1^{(k-i)-1}$, for any $i$ in $[k-1]$.

**Proposition 11.1.** The action of $\mathbb{P}TL_k(1 \pm (q + q^{-1}))$ on $V^\otimes k$ commutes with the action of $U_q(\mathfrak{gl}_2)$.

**Proof.** By Lemma [33] it suffices to check that the action of $\varepsilon$ on $V \otimes V$ commutes with that of $U_q(\mathfrak{gl}_2)$. We already know that it commutes with the action of $E$, $F$, so we only need to check commutation with the action of $K_1$, $K_2$. When restricted to $kV_{1,-1} \oplus kV_{-1,1}$, each $K_i$ acts as $q$, so $K_i$ acts as $q$ times the identity operator on that subspace. The result follows. \qed

With $\delta = 1 \pm (q + q^{-1})$, let $\varphi$ be the restriction of the representation $\rho : M_k(\delta) \to \text{End}_k(V^\otimes k)$ to $\mathbb{P}TL_k(\delta)$. Thus
\[ \varphi : \mathbb{P}TL_k(\delta) \to \text{End}_k(V^\otimes k) \]
is an injective algebra morphism.

**Proposition 11.2.** Suppose that $d = d(A,t,B)$ belongs to $\mathcal{D}(k)$, so that $d = r_A t_0 l_B$ as in Lemma [34] where $t_0 = t \otimes \omega_{k-n}$. Then $\bar{d} = r_A t_0 l_B$.

**Proof.** Observe that by the definition of the bar elements,
\[ \bar{t}_0 = \prod_{i=1}^n (1 - p_i) t_0 \prod_{i=1}^n (1 - p_i). \]
Then considerations similar to those in the proof of Theorem [35] show that
\[ r_A \prod_{i=1}^n (1 - p_i) t_0 \prod_{i=1}^n (1 - p_i) l_B = \prod_{i \in A} (1 - p_i) r_A t_0 l_B \prod_{i \in B} (1 - p_i) \]
and the result follows. \qed

To proceed, write $1 = \hat{p} + (1 - \hat{p})$, where $1 = 1_V$ is the identity map on $V$ and $\hat{p}$ is the projection map $V \to V(0)$. This decomposition (a sum of orthogonal idempotents) induces the defining decomposition
\[ V = \hat{p}V \oplus (1 - \hat{p})V = V(0) \oplus V(1). \]
As $V(0), V(1)$ are $U_q(\mathfrak{gl}_2)$-submodules of $V$, the operators $\hat{p}, 1 - \hat{p}$ commute with the action of $U_q(\mathfrak{gl}_2)$. Since $p_i$ acts as $\hat{p}$ on the $i$th tensor factor and as identity in the remaining factors, we can apply the same reasoning to the $i$th tensor position in $V^\otimes k$ to obtain the decomposition
\[ V^\otimes k = p_i V^\otimes k \oplus (1 - p_i) V^\otimes k = V^\otimes i-1 \oplus (V(0) \oplus V(1)) \otimes V^\otimes k-i. \]
Expanding the operator $1^\otimes k = (\hat{p} + (1-\hat{p}))^\otimes k$ binomially produces an identity
\[
1^\otimes k = \sum_{A \subseteq [k]} \prod_{i \in A} (1 - p_i) \prod_{j \in [k] \setminus A} p_j.
\]

Applying the above expansion to the space $V^\otimes k$ produces the decomposition
\[
V^\otimes k = \bigoplus_{A \subseteq [k]} V[A]
\]
where, for a given subset $A$ of $[k]$, $V[A] := V_1 \otimes V_2 \otimes \cdots \otimes V_k$ where $V_i = V(1)$ if $i \in A$ and $V_i = V(0)$ otherwise.

**Proposition 11.3.** Let $d \in \mathcal{D}(k)$ and let $(A, d', B)$ be the corresponding triple under the bijection in Lemma 3.2. If $n = |A| = |B|$ then the representation $\bar{d} \mapsto \varphi(\bar{d})$ maps $V[B]$ into $V[A]$ and maps all other $V[B']$ with $B' \neq B$ to zero.

**Proof.** Let $d = rtl$ be the factorization in Proposition 11.2 so that $\bar{d} = r\tilde{t}l$. Then $\varphi(\bar{d}) = \varphi(r)\varphi(\tilde{t})\varphi(l)$. Furthermore, $\varphi(r)$ induces an isomorphism $V([1, \ldots, n]) \to V[A]$ and $\varphi(r) = 0$ on all $V[Y]$ such that $Y \neq \{1, \ldots, n\}$. (The inverse map is obtained by flipping the diagram $r$ upside down.) Similarly, $\varphi(\tilde{t})$ induces an isomorphism $V'[B] \to V'[\{1, \ldots, n\}]$ and $\varphi(l) = 0$ on all $V[Y]$ such that $Y \neq B$. As $\varphi(\tilde{t})$ induces a map from $V'[\{1, \ldots, n\}] \cong V(1)^{\otimes n}$ into itself, and is zero on all other components, the result follows.

We are now ready to prove the following.

**Theorem 11.4.** Suppose that $k$ is a field and $0 \neq q \in k$ is not a root of unity. Set $\delta = 1 \pm (q + q^{-1})$. Then
\[
\text{End}_{U_q(\mathfrak{gl}_2)}(V^\otimes k) \cong \text{PTL}_k(\delta).
\]

Hence, $V^\otimes k$ satisfies Schur–Weyl duality with respect to the commuting actions of $U_q(\mathfrak{gl}_2)$, PTL$_k(\delta)$.

**Proof.** Since the actions commute, $\varphi(\text{PTL}_k(\delta))$ is contained in the commuting algebra $\text{End}_{U_q(\mathfrak{gl}_2)}(V^\otimes k)$. The action of PTL$_k(\delta)$ is faithful, so the desired isomorphism will follow once we show the inclusion is an equality. We do this by comparing dimensions. By the $U_q(\mathfrak{gl}_2)$-module decomposition in equation (43), we have
\[
\dim_k \text{End}_{U_q(\mathfrak{gl}_2)}(V^\otimes k) = \sum_{A, B} \dim_k \text{Hom}_{U_q(\mathfrak{gl}_2)}(V[A], V[B])
\]
where the sum is over all pairs $(A, B)$ of subsets of $[k]$. By classical Schur–Weyl duality, the simple $U_q(\mathfrak{gl}_2)$-modules appearing as constituents of $V[A] \cong V(1)^{\otimes n}$, where $|A| = n$, are all indexed by partitions of $n$ with not more than two parts. Thus $\text{Hom}_{U_q(\mathfrak{gl}_2)}(V[B], V[A]) = (0)$ unless $|A| = |B|$. Furthermore, if $|A| = |B| = n$ for subsets $A, B$ of $[k]$, we have
\[
\text{Hom}_{U_q(\mathfrak{gl}_2)}(V[B], V[A]) \cong \text{End}_{U_q(\mathfrak{gl}_2)}(V(1)^{\otimes n}) \cong \text{TL}_n(\delta - 1)
\]
by Schur–Weyl duality for Temperley–Lieb algebras \([35]\), so

\[
\dim_k \text{Hom}_{U_q(gl_2)}(V[A], V[B]) = \dim_k \text{TL}_n(\delta - 1) = C_n
\]

where \(C_n\) is the \(n\)th Catalan number. Putting these facts together yields the equality

\[
\dim_k \text{End}_{U_q(gl_2)}(V^{\otimes k}) = \sum_{n=0}^k \binom{k}{n}^2 C_n
\]

which by equation \([18]\) agrees with the dimension of \(\text{PTL}_k(\delta)\). This proves the first statement in the theorem. The remaining claims then follow by standard facts in the theory of semisimple algebras. \(\square\)

**Corollary 11.5.** Under the same hypotheses, we have the decomposition

\[
V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda} V(\lambda) \otimes \text{PTL}^\lambda
\]

as \((U_q(gl_2), \text{PTL}_k(\delta))\)-bimodules, where the indexing set \(\Lambda\) is the set of partitions of \(n\) of not more than two parts, for \(0 \leq n \leq k\), as in Section \([7]\).

**Proof.** This is a standard fact in semisimple representation theory. \(\square\)

**Remark 11.6.** (i) It makes sense to set \(q = 1\) in \(\text{PTL}_k(1 \pm (q + q^{-1}))\), thus obtaining \(\text{PTL}_k(1 \pm 2)\). If the field \(k\) has characteristic zero, the analogue of Theorem \([11.4]\) holds. In particular,

\[
\text{End}_{U(gl_2)}(V^{\otimes k}) \cong \text{PTL}_k(1 \pm 2)
\]

where \(U(gl_2)\) is the ordinary universal enveloping algebra of \(gl_2\). There is of course also a version of Corollary \([11.5]\) for this situation.

(ii) If \(q\) is a root of unity then Theorem \([11.4]\) (but not Corollary \([11.5]\)) still holds, provided that \(U_q(gl_2)\) is replaced by an appropriate \(k\)-form, but the proof is very different. One needs to work with the Lusztig “integral” form of the quantized enveloping algebra and to appeal to the paper \([DPS98]\), which established a version of Jimbo’s Schur–Weyl duality at roots of unity.

(iii) The image of \(U_q(gl_2)\) in \(\text{End}_k(V^{\otimes k})\) is isomorphic to a generalized \(q\)-Schur algebra in type \(A\), in the sense of \([Dot03]\), defined by the set \(\Lambda\).

We now derive explicit formulas for the action of \(\bar{d}, \tilde{d}\), where \(d \in \mathcal{D}(k)\). Recall from Section \([2]\) that

\[
\bar{d} = \prod_{i \in [d]} (1 - p_i) \quad \tilde{d} = \prod_{i' \in [d]} (1 - p_i)
\]

In the representation on tensor space, the element \(1 - p_i\) is the operator \(1^{\otimes (i-1)} \otimes (1 - \hat{p}) \otimes 1^{\otimes (k-i)}\), where \(\hat{p}\) is projection onto \(V(0)\) and hence \(1 - \hat{p}\) projects onto \(V(1)\). This observation gives the following result.

**Proposition 11.7.** Suppose that \(d\) is in \(\mathcal{D}(k)\). Given \(i_\alpha, j_\alpha\) in the set \([-1, 0, 1]\) for \(\alpha = 1, \ldots, k\), label the top row vertices of \(d\) from left to right
with \( v_{j_1}, \ldots, v_{j_k} \) and similarly label the bottom row vertices with \( v_{i_1}, \ldots, v_{i_k} \). Then the action \( \tilde{d} \) on \( V^\otimes k \) is given by

\[
\tilde{d}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} v_{j_1} \otimes \cdots \otimes v_{j_k}
\]

and similarly the action \( \tilde{d} \) on \( V^\otimes k \) is given by

\[
\tilde{d}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} v_{j_1} \otimes \cdots \otimes v_{j_k}
\]

where the scalars \((\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k}\) and \((\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k}\) are the product over the modified weights of the labeled blocks of \( \tilde{d} \). The modified weight of a block \( \beta \) in \( \tilde{d} \) is the same is its weight minus a correction term of \( \delta_{i,0} \delta_{j,0} \) applied to all (resp., all horizontal) edges of \( \tilde{d} \) in computing the action of \( \tilde{d} \) (resp., \( \tilde{d} \)).

**Proof.** This follows from the observation preceding the proposition and the table of weights preceding Proposition 10.11 by considering the cases separately. \( \square \)

### Appendix A. Semisimplicity Criterion for \( TL_k(\pm(q + q^{-1})) \)

The purpose of this Appendix is to highlight a precise semisimplicity criterion for Temperley–Lieb algebras that arose in the work of Vaughan Jones, following the exposition of [GdlHJ89]. Assume that \( k \) is a field. For \( q \in k \), let

\[
[n]_q := 1 + q + q^2 + \cdots + q^{n-1}
\]

in \( Z[q] \) be the usual quantum integer (regarded as an element of \( k \)) and define \([k]_q^1 = [1]_q [2]_q \cdots [k]_q\). If \( q \neq 0 \) then the balanced quantum integer \([n]_q \) in \( Z[q, q^{-1}] \) may be defined as

\[
[n]_q := q^{-(n-1)}[n]_{q^2}.
\]

We also define \([k]_q^1 = [1]_q [2]_q \cdots [k]_q\). Here is the criterion.

**Theorem A.1** ([GdlHJ89]). If \( k \) is a field and \( 0 \neq q \in k \) satisfies \([k]_q^1 \neq 0 \) then \( TL_k(\pm(q + q^{-1})) \) is semisimple over \( k \).

**Proof.** For \( 0 \neq \beta \in k \), the Jones algebra \( A_k(\beta) \) is the algebra with 1 on generators \( u_1, \ldots, u_{k-1} \) subject to the defining relations

\[
u_i^2 = u_i, \quad \beta u_i u_{i \pm 1} = u_i, \quad u_i u_j = u_j u_i \text{ if } |i - j| > 1.
\]

By Prop. 2.8.5(a) in [GdlHJ89], \( A_k(\beta) \) is (split) semisimple over \( k \) if

\[
P_1(\beta^{-1}) P_2(\beta^{-1}) \cdots P_{k-1}(\beta^{-1}) \neq 0,
\]

where the \( P_n(x) \) are polynomials in \( Z[x] \) satisfying the recursion \( P_0(x) = 1, \ P_1(x) = 1, \) and \( P_{n+1}(x) = P_n(x) - x P_{n-1}(x) \) for all \( n \geq 1 \). Choose \( q \) in \( k \).
such that $q \neq 0$, $q \neq -1$, and $\beta = q + q^{-1} + 2$. (Replace $k$ by a suitable quadratic extension if necessary.) By Prop. 2.8.3(iv) in [GdlHJ89],

$$P_n(\beta^{-1}) = \frac{1 + q + q^2 + \cdots + q^n}{(1 + q)^n} = \frac{[n+1]_q}{(1 + q)^n}.$$ 

Hence, $A_k(\beta) = A_k(q + q^{-1} + 2)$ is semisimple over $k$ if $[k]^1_q \neq 0$. Now, by setting $e_i = \delta u_i$ for all $i$ we recover the defining relations (11) if and only if $\beta = \delta^2$, so $A_k(\delta^2) \cong \text{TL}_k(\delta)$. We conclude that if $[k]^1_q \neq 0$ then $\text{TL}_k(\pm(q^{1/2} + q^{-1/2}))$ is semisimple over $k$. To obtain the final conclusion, we replace $q^{1/2}$ by $q$. This has the effect of replacing $[k]^1_q$ by $[k]^{1/q}_q$, up to a power of $q$.

\[\Box\]

**Remark A.2.** It makes sense to specialize $q$ to 1 in Theorem A.1. Then $[k]^{1}_1 = k!$ is the ordinary factorial of $k$, and the semisimplicity criterion coincides with the one appearing in Maschke’s theorem for finite symmetric groups. This is no accident, as $\text{TL}_k(\pm(q + q^{-1}))$ is a quotient of an appropriate Iwahori–Hecke algebra of type A.

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