Finite-element modeling of critical loads of buckling of structures considering of nonlinearity of pre-buckling state

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Abstract. An approach is proposed to determine the critical loads of buckling, taking into account the geometric nonlinearity of the subcritical stress-strain state of structures. The first stage involves the structure strain-stress state in the context of geometrical non-linearity at some preset loading level. Afterwards the found solution neighborhood is supplemented with linearized non-linear equation system and the solution for the obtained linearized system is found. Following is the second linearization of the initial equation system in the linearized system solution neighborhood, obtained at the previous stage. The condition of equation systems degeneracy obtained after the second linearization causes a generalized eigenvalue problem. The approach is implemented using finite element models for the calculation of shell structural elements.

1. Introduction
The procedure of finite-element-based finding buckling load involves two methods of increasing accuracy of a final finite-element solution with necessary computer resources being reduced at the same time. The first method is a development of refined approaches and algorithms as to finding buckling loads. The second one is a creation of new, more efficient finite-element models.

2. Formulation of problem
This work offers an algorithm of defining buckling loads, which enables to consider a geometrical non-linearity pre-buckling state of structure elements at issue. The first stage involves the structure strain-stress state in the context of geometrical non-linearity at some preset loading level. Afterwards the found solution neighborhood is supplemented with linearized non-linear equation system and the solution for the obtained linearized system is found. Following is the second linearization of the initial equation system in the linearized system solution neighborhood, obtained at the previous stage. The condition of equation systems degeneracy obtained after the second linearization causes a generalized eigenvalue problem. The algorithm is implemented on the basis of finite element models for the calculation of shells.
3. Approach to finding buckling loads in the context of stress-strain state geometrical non-linearity

Let us consider a shell exposed to statistical loads and temperature with set pre-stress \( \sigma_0 \) and initial strain \( \varepsilon_0 \). The physical law is assumed as linear [1]:

\[
\sigma = \sigma_0 + D (\varepsilon + \varepsilon_0 - \alpha T),
\]

where \( \sigma \) is a stress vector, \( \varepsilon \) is a strain vector, \( \sigma_0 \) and \( \varepsilon_0 \) are pre-stress and initial strain vectors corresponding to the initial state, \( \alpha \) is a linear thermal expansion coefficients vector, \( T \) is a temperature counted from the initial state, \( D \) is a symmetric positive definite matrix of material elastic constants.

A relation between nodal displacements \( q \) and strains \( \varepsilon \) is assumed as quadratic:

\[
\varepsilon_i = \varepsilon_i^0 + \varepsilon_i^f = B_{ij} q_j + C_{ijk} q_j q_k. \tag{2}
\]

Here and further, the indices for the vectors \( \sigma, \varepsilon, \alpha \) and for the matrix \( D \) correspond to the components of the stress and strain vectors, the indices for the vectors \( q \) correspond to the components of the vector of nodal displacements.

The type of \( C \) matrix is defined by a certain definite element type, and matrix \( B \) is a linear strain matrix. Its type is specified by reviewing certain finite elements. The formula (2) and further assume a summation over repeated dummy index. Strain energy density \( w \) is found through (1) as stresses \( \sigma \) acting on strains \( \varepsilon \):

\[
w = \int_0^\varepsilon \sigma^T d\varepsilon = \frac{1}{2} \varepsilon^T D \varepsilon + \sigma_0^T \varepsilon + \varepsilon_0^T D \varepsilon - T \alpha^T D \varepsilon. \]

By integrating over the volume using (2) and subtracting external forces energy \( R \) at displacements \( q \), we shall find a potential energy. Using Lagrange variation principle, we shall obtain equilibrium equations:

\[
\frac{\partial \Pi}{\partial q_i} \equiv 0 = -R_i + \int_V \sigma_{0k} B_{ki} dV + \left( \int_V \varepsilon_{0k} D_{kl} B_{li} dV - \int_V T \alpha_k D_{kl} B_{li} dV \right) + \int_V B_{ki} D_{kl} B_{lj} dV q_j + \int_V \sigma_{0k} (C_{kij} + C_{kji}) dV q_j + \frac{1}{2} \int_V C_{kij} D_{lm} C_{mnk} dV q_j q_k q_n + \frac{1}{2} \int_V C_{kij} D_{lm} C_{mnk} dV q_j q_k q_n
\]

Let us introduce the following notations:

\[
\begin{align*}
\int_V \sigma_{0k} B_{kl} dV &= R_{ki}, \\
\int_V \varepsilon_{0k} D_{kl} B_{li} dV &= R_{ki}^f, \\
\int_V T \alpha_k D_{kl} B_{li} dV &= R_{ki}^a, \\
\int_V B_{ki} D_{kl} B_{lj} dV &= C_{ij}, \\
\int_V \sigma_{0k} (C_{kij} + C_{kji}) dV &= C_{ij}^a, \\
\int_V \varepsilon_{0k} D_{kl} (C_{lij} + C_{lij}) dV &= C_{ij}^f, \\
\int_V T \alpha_k D_{kl} (C_{lij} + C_{lij}) dV &= C_{ij}^a, \\
\int_V (C_{rij} D_{mn} C_{mlk} + C_{nkj} D_{mn} C_{mil}) dV &= X_{ijkl}.
\end{align*}
\]

As matrix \( C_{ijk} \) is symmetric with respect to indices 2 and 3 \( (C_{ijk} = C_{ikj}) \), matrix \( Q_{ijk} \) is symmetric with respect to any pair of indices:

\[
Q_{ik} = Q_{iki} = Q_{ik} = Q_{ki} = Q_{kii};
\]
matrix $X_{ijkl}$ is symmetric with respect to a pair of indices $(i,k)$ and $(l,j)$; the equilibrium equations shall take the following form:

$$R_i + R_i^\alpha - R_i^\epsilon - R_i^\sigma = C_{ij} q_j + C_{ij}^\alpha q_j + C_{ij}^\epsilon q_j - C_{ij}^\sigma q_j + Q_{ijk} q_j q_k + X_{ijkl} q_j q_k q_l.$$ 

Let us assume the shell stressing as one-parameter and linearly dependent on parameter $\lambda$:

$$R_i = \overline{R}_i + \lambda \hat{R}_i, \quad \sigma_i = \overline{\sigma}_i + \lambda \hat{\sigma}_i, \quad \varepsilon_i = \overline{\varepsilon}_i + \lambda \hat{\varepsilon}_i, \quad T_i = \overline{T}_i + \lambda \hat{T}_i. \quad (4)$$

In this regard the equilibrium equations shall take the following form:

$$\overline{R}_i + \overline{R}_i^\alpha - \overline{R}_i^\epsilon - \overline{R}_i^\sigma = (C_{ij} + C_{ij}^\sigma + C_{ij}^\epsilon - C_{ij}^\alpha) q_j + \lambda \left( C_{ij}^\sigma + \hat{C}_{ij}^\sigma - \hat{C}_{ij}^\alpha \right) + Q_{ijk} q_j q_k + X_{ijkl} q_j q_k q_l. \quad (5)$$

The solution of non-linear equations system (5) defines the displacements of nodes of the shell finite-element model, which correspond to its equilibrium state. At some $\lambda$ the system (5) degenerates, thus being equivalent to the shell critical equilibrium.

Let us assume the critical values $\lambda$ are sufficiently small, and linearize the system (5) in its solution neighborhood corresponding to $\lambda = 0$. Suppose

$$\overline{R}_i + \overline{R}_i^\alpha - \overline{R}_i^\epsilon - \overline{R}_i^\sigma = (C_{ij} + C_{ij}^\sigma + C_{ij}^\epsilon - C_{ij}^\alpha) q_j^* + Q_{ijk} q_j^* q_k^* + X_{ijkl} q_j^* q_k^* q_l^*,$$

where $q_j^*$ is a solution of non-linear equation system (5) at $\lambda = 0$.

By linearizing the system (5) in neighborhood $q_j^*$ and neglecting the last summand we shall obtain the following:

$$\hat{R}_i = \mathcal{C}_{ij} \hat{q}_j + \hat{C}_{ij} q_i^* + \lambda \hat{C}_{ij} \hat{q}_j + (Q_{ijk} q_k^* + Q_{ij} q_k^* \hat{q}_{ij}) \hat{q}_j + \lambda Q_{ijk} \hat{q}_{ij} \hat{q}_k,$$

where

$$\mathcal{C}_{ij} = C_{ij} + C_{ij}^\sigma + C_{ij}^\epsilon - C_{ij}^\alpha, \hat{C}_{ij} = \hat{C}_{ij}^\sigma - \hat{C}_{ij}^\alpha, \hat{R}_i = \hat{R}_i + \hat{R}_i^\alpha - \hat{R}_i^\epsilon - \hat{R}_i^\sigma,$$

$q_j$ is an unknown increment of nodal displacement vector.

With $\lambda = 0$ let us find the following:

$$-\hat{C}_{ij} q_j^* + \hat{R}_i = (\mathcal{C}_{ij} + Q_{ijk} q_k^* + Q_{ik} q_k^* \hat{q}_{ij}) \hat{q}_j,$$

Now suppose $\hat{q}_j^* = q_j^* + \lambda \hat{q}_j$, where $q_j^*$ is a solution of the system (5) at $\lambda = 0$. Let us linearize the system (5) in neighborhood of vector $q_j^*$ by introducing the following notations:

$$R_i = \overline{R}_i + (\lambda + \mu) \hat{R}_i; \quad q_j = \hat{q}_j^* + \mu \hat{q}_j. \quad (6)$$

From (5) and (6) taken into account we shall obtain the following:

$$\mathcal{R}_i + (\lambda + \mu) \hat{R}_i = \mathcal{C}_{ij} \hat{q}_j + \hat{C}_{ij} q_i^* + (\lambda + \mu) \hat{C}_{ij} \hat{q}_j + \lambda \left( \hat{C}_{ij} \hat{q}_j + (Q_{ijk} q_k^* + Q_{ik} q_k^* \hat{q}_{ij}) \hat{q}_j + \lambda Q_{ijk} \hat{q}_{ij} \hat{q}_k \right). \quad (7)$$

Let us differentiate the relation (7) with respect to $\mu$:

$$\hat{R}_i = \mathcal{C}_{ij} \hat{q}_j + \hat{C}_{ij} \hat{q}_j + \lambda \left( \hat{C}_{ij} \hat{q}_j + (Q_{ijk} q_k^* + Q_{ik} q_k^* \hat{q}_{ij}) \hat{q}_j + \lambda Q_{ijk} \hat{q}_{ij} \hat{q}_k \right) + \ldots$$
At $\mu = 0$ we shall obtain the following:

$$\{ [C_{ij} + (Q_{ijk}q^*_k + Q_{ikj}q^*_k)] + \lambda \left[ \dot{C}_{ij} + (Q_{ijk}\dot{q}_k + Q_{ikj}\dot{q}_k) \right] \} \ddot{q}_j = \dot{R}_i - \dot{C}_{ij}q^*_j - \lambda \dot{C}_{ij}\dot{q}_j. \quad (8)$$

The degeneracy condition of the system (8) has the following form:

$$\det \left\{ [C_{ij} + (Q_{ijk}q^*_k + Q_{ikj}q^*_k)] + \lambda \left[ \dot{C}_{ij} + (Q_{ijk}\dot{q}_k + Q_{ikj}\dot{q}_k) \right] \right\} = 0 \quad (9)$$

Thus, the problem reduces to defining eigenvalues of a pair of symmetric matrixes.

Now we shall note some aspects of selecting the load constant component in equations (8). As is clear from the method of reducing the non-linear system (5) to problem (9), the results being obtained are the more trustworthy the less the absolute value $\lambda$ is, while the value sign is not important. Thus, it is useful to choose constant components close to expected critical values. If the solution of non-linear system (5) is not extremely expensive in terms of a computer-based system resources cost, the iterative improvement of buckling loads is essentially possible.

On the other hand, there is a number of cases of practical importance, when a shell deforms in a linear manner to the extent of reaching critical equilibrium. Such cases include, among others, the axial compression of a cylindrical shell. In this situation the result does not depend on the selection of load intercept.

4. Finite-element implementation of defining buckling loads as illustrated by rotational shell of Gaussian zero curvature

Let us assume that structure deformation with a reasonable degree of accuracy is described by models based on the Kirchhoff-Love hypotheses. As an example of defining the type of matrixes $C$, $B$ and $Q$ (2,3) we shall consider a Kirchhoff-Love hypotheses-based two-dimensional finite element designed to simulate thin conical and cylindrical shells. The element is a rectangular-plan quadrinodal finite element of a thin anisotropic rotational shell of Gaussian zero curvature [1,2]. Radial stress is approximated with incomplete bicubic polynomial, and tangential displacement is approximated with bilinear polynomial. Besides, the finite element functions explicitly include statements, which describe it as a rigid-body displacement, that significantly improves the solution convergence [3-5].

Let us write a strain vector for a conical shell as follows: $\varepsilon = \{ \varepsilon_1, \varepsilon_2, \varepsilon_{12}, \kappa_1, \kappa_2, \kappa_{12} \}^T$, where $\varepsilon, \kappa$ are tangential strains and curvatures. Taking into account non-linear components the strain vector components are linked to displacement through the following equations [6]:

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13} \right),$$

$$\varepsilon_2 = \frac{\partial v}{\partial x} + \frac{u}{r} \sin \gamma + \frac{w}{r} \cos \gamma + \frac{1}{2} \left( \varepsilon_{21} + \varepsilon_{22} + \varepsilon_{23} \right),$$

$$\varepsilon_{12} = \frac{\partial u}{\partial \beta} - \frac{v \sin \gamma}{r} + \frac{\partial w}{\partial x} + e_{11} e_{21} + e_{12} e_{12} + e_{13} e_{13},$$

$$\kappa_1 = \frac{-\partial^2 u}{\partial \beta^2},$$

$$\kappa_2 = \frac{-\partial^2 v}{\partial \beta^2} + \frac{u}{r^2} \cos \gamma - \frac{w}{r^2} \sin \gamma,$$

$$\kappa_{12} = \frac{-\partial^2 w}{\partial \beta^2} \cos \gamma + \frac{u}{r^2} \sin \gamma \cos \gamma,$$

where $u, v, w$ are displacements of the shell reference surface, $\gamma$ is a cone angle, $r = R_2 \cos \gamma$ [1],

$e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}$ for conical shell are defined as follows:

$$e_{11} = \frac{\partial u}{\partial x}, \quad e_{12} = \frac{\partial v}{\partial x}, \quad e_{13} = \frac{\partial w}{\partial x}, \quad e_{21} = \frac{\partial u}{r \partial \beta} - \frac{v \sin \gamma}{r},$$

$$e_{22} = \frac{\partial v}{r \partial \beta}, \quad e_{23} = \frac{\partial w}{r \partial \beta}.$$
\[ e_{22} = \frac{\partial v}{r \partial \beta} + \frac{u}{r} \sin \gamma + \frac{w}{r} \cos \gamma, \quad e_{23} = \frac{\partial w}{r \partial \beta} - \frac{v \cos \gamma}{r}. \]

As is seen from equations (10), non-linear components are taken into account only in case of tangential strains.

Let us take a quadrangular finite element of a thin anisotropic rotational shell of Gaussian zero curvature with nodes in vertexes, which is generated through the intersection of two planes perpendicular to the center line and two planes passing through the center line. Thus, the element has 4 nodes and 20 degrees of freedom - 5 in each node: \( u_i, v_i, \vartheta_i, \vartheta_{\beta i}, \) where \( \vartheta x = \frac{\partial w}{\partial \beta} - \frac{v \cos \gamma}{r}, \) is an angle of normal rotation around axis \( x, \vartheta_{\beta} = -\frac{\partial w}{\partial \beta} \) is an angle of normal rotation around axis \( \beta. \)

Let us define the element displacement vector:

\[ u = u^0 + u^d \]

where

\[ u^0 = \{u_0, v_0, w_0\}^T \]
\[ u^d = \{u_d, v_d, w_d\}^T \]

is a rigid element displacement vector, obtained in the similar manner [4, 5, 7 - 12],

\[ u_d = a_1 x + a_8 \beta + a_9 x \beta, \]
\[ v_d = a_{10} \beta + a_{11} x \beta, \]
\[ w_d = a_{12} x \beta + a_{13} x^2 + a_{14} \beta^2 + a_{15} x^2 \beta + a_{16} x \beta^2 + a_{17} x^3 + a_{18} \beta^3 + a_{19} x^3 \beta + a_{20} x \beta^3, \]

where \( a_i, i = 1, 2, \ldots, 20 \) are indefinite coefficients linked to nodal displacement through the following equation: \( q = Aq, \) where \( q = \{a_1, \ldots, a_{20}\}^T \) is an indefinite coefficient vector,

\[ q = (q_1, \ldots, q_{20})^T \]

is a nodal displacement vector, \( A \) is a 20 - by - 20 matrix.

As matrix the displacement vector shall be written as follows:

\[ u = N(x, \beta)a, \]

where \( u = (u, v, w)^T, N(x, \beta) \) is a basis function matrix.

By rearranging the latter equation in terms of nodal displacements we shall obtain the following:

\[ u = N(x, \beta)Aq. \] (11)

By substituting (11) in Cauchy equations (10), we shall obtain a strain vector non-linear component expression arranged using nodal displacement vector as follows:

\[ \varepsilon^l = \tilde{N}Aq, \]

where \( \tilde{N} = \Omega N(x, \beta), \) \( \Omega \) is an operators matrix [1], and for strain vector non-linear component:

\[ \varepsilon_n^l = \frac{1}{2} e_x^T e_x = \frac{1}{2} q^T A^T N_x^T N_x Aq, \]
\[ \varepsilon_n^{12} = \frac{1}{2} e_{\beta}^T e_{\beta} = \frac{1}{2} q^T A^T N_{\beta}^T N_{\beta} Aq, \]

where \( N_x, N_{\beta} \) are matrices obtained from the following equations:

\[ e_x = N_x(x, \beta)a, e_{\beta} = N_{\beta}(x, \beta)a, \]

where \( e_x = \{e_{11}, e_{12}, e_{13}\}^T, \)
\[ e_{\beta} = \{e_{21}, e_{22}, e_{23}\}^T. \] Hence we can obtain matrix \( C_{ijk} \) (2), linking non-linear displacements:

\[ C_{1ij} = 1/2 A^T N_x^T N_x A, C_{2ij} = 1/2 A^T N_{\beta}^T N_{\beta} A, C_{3ij} = A^T N_{x}^T N_{\beta} A, \]

\[ C_{ijk} = A^T C_{kij} A. \] Thus, matrix \( Q_{ijk} \) (7) shall have the following form:

\[ Q_{ijk} = \iiint_V \tilde{N}_p D_{lm} \tilde{C}_{mrg} dV (A_{pj} A_{rj} A_{qk} + A_{pk} A_{ri} A_{qj} + A_{pj} A_{rk} A_{qi}). \]
5. Conclusion
An approach, methodology and algorithms for determining critical loads of buckling taking into account the geometric nonlinearity of the subcritical stress-strain state of structures, have been developed. The proposed approach is implemented in the form of finite-element models for determining the critical loads of buckling of shells of revolution. The developed approach, methodology and algorithms make it possible to obtain refined values of critical loads of buckling in those cases in which the geometric nonlinearity of the subcritical stress-strain state is realized (the presence of cutouts, areas of significant weakening of the rigidity of the structure, etc.).

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References
[1] Bakulin V N, Repinskii V V 2001 Finite-Element Strain Models of One-Layer and Three-Layer Conical Shells Mat. Model. 13 6 39
[2] Repinskii V V 1997 Effective Finite Elements for Calculation of stability of Thin Anisotropic Rotational Shells Vopr. Oboron. Tekhniki Ser. 15 117 3
[3] Cantin G., Glagh R W 1968 A curved, cylindrical shell finite element. AIAA J. 6 6 1057-1062
[4] Bakulin V N, Demidov V I 1978 Three-Layered Natural Curved Finite Element. Proc. of High. Ed. Inst. ch. Build. 5 5-10
[5] Bakulin V N 1985 The Finite-Element Method for Studying the Stress-Strain State of Three-Layer Cylindrical Shells Proc. of High. Ed. Inst. ch. Build. 5 5-10
[6] Novozhilov V V 1948 The Fundamentals of Non-Linear Elasticity Theory (Leningrad: Gostekhizdat) 211
[7] Bakulin V N 2006 Construction of Approximations for Simulation of Stress-Strain State of Base Layers and Core Layers of Three-Layered Non-Axisymmetric Cylindrical Shells Mat. Model. 18 8 101-110
[8] Zhelezov L P, Kabanov V V 1990 Displacement Functions of Rigid Finite Elements of Rotational Shell Izv. RAN. MT 1 131
[9] Bakulin V N 2007 Effective Models in the refined Analysis of the Strain State for Three-Layer Non-Axisymmetric Cylindrical Shells Doklady Physics 52 6 330-334
[10] Bakulin V N 2007 Construction of Approximations and Models for Studying Strain-Stress State of Three-Layered Non-Axisymmetric Cylindrical Shells Mat. Model. 19 12 118
[11] Bakulin V N 2009 Testing of Finite-Element Model Designed for Studying Stress-Strain State of Irregular Sandwich Shells Mat. Model. 21 8 121
[12] Bakulin V N 2018 An Efficient Model for Layer-by-Layer Analysis of Sandwich Irregular Cylindrical Shells of Revolution Dokl. Ph. 63 1 23-27