THE WELL-POSEDNESS OF THE COMPRESSIBLE NON-ISENTROPIC EULER-MAXWELL SYSTEM IN $\mathbb{R}^3$

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Abstract. We first construct the global unique solution by assuming that the initial data is small in the $H^3$ norm but the higher order derivatives could be large. If further the initial data belongs to $\dot{H}^{-s}$ $(0 \leq s < 3/2)$ or $\dot{B}^{-s}_{2,\infty}$ $(0 < s \leq 3/2)$, we obtain the various decay rates of the solution and its higher order derivatives. In particular, the decay rates of the density and temperature of electron could reach to $(1 + t)^{-13/4}$ in $L^2$ norm.

1. Introduction

In the present paper, we consider the compressible non-isenropic Euler-Maxwell system (nonconservative form) \[ \begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u + \nabla \Theta + \Theta \nabla \ln \rho &= -(E + u \times \tilde{B}), \\
\partial_t \Theta + u \cdot \nabla \Theta + \frac{2}{3} \Theta \text{div} u &= \frac{1}{3} |u|^2 - (\Theta - 1), \\
\partial_t E - \nabla \times \tilde{B} &= \rho u, \\
\partial_t \tilde{B} + \nabla \times E &= 0, \\
div E &= 1 - \rho, \\
\text{div} \tilde{B} &= 0,
\end{align*} \]

The unknown functions $\rho, u, \Theta, E, \tilde{B}$ represent the electron density, electron velocity, absolute temperature, electric field and magnetic field, respectively. In the motion of the fluid, due to the greater inertia the ions merely provide a constant charged background.

Although the compressible Euler-Maxwell system is more and more important in the researches of plasma physics and semiconductor physics, a small amount of results are obtained since its mathematical complexity. In a unipolar form: Chen, Jerome and Wang [2] showed the one-dimensional global existence of entropy weak solutions to the initial-boundary value problem for arbitrarily large initial data in $L^\infty(\mathbb{R})$; Guo and Tahvildar-Zadeh [10] showed a blow-up criterion for spherically symmetric Euler-Maxwell system; Recently, there are some results on the global existence and the large time behavior of smooth solutions with small perturbations, see Tan et al. [23], Duan [3], Ueda and Kawashima [26], Ueda et al. [27]; For the asymptotic limits that derive simplified models starting from the Euler-Maxwell system, we refer to [12] [19] [31] for the relaxation limit, [31] for the non-relativistic limit, [17] [18] for the quasi-neutral limit, [21] [25] for WKB asymptotics and the references therein. In a bipolar form: Duan et al. [4] showed the global existence and time-decay rates of solutions near constant steady states with the vanishing electromagnetic field; Xu et al. [32] studied the well-posedness in critical Besov spaces. Since the unipolar or bipolar Euler-Maxwell system is a symmetrizable hyperbolic system, the Cauchy problem in $\mathbb{R}^3$ has a local unique smooth solution when the initial data is smooth, see Kato [14] and Jerome [13] for instance. Besides, we can refer to [5] [28] for the non-isenropic case.

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In this paper, we will refine a global existence of smooth solutions near the constant equilibrium \((1, 0, 1, 0, B_\infty)\) to the compressible non-isentropic Euler-Maxwell system and show some various time decay rates of the solution as well as its spatial derivatives of any order. We should highlight that our results highly depend on the relaxation terms of velocity and temperature. The non-relaxation case is much more difficult, we refer to \([4, 8]\) for such a case. Compared with the compressible isentropic Euler-Maxwell system \([23]\), there are two main difficulties except the computational complexity. The first difficulty is that we must obtain the symmetric hyperbolic non-isentropic Euler-Maxwell system to do the effective energy estimates. We solve this problem by making good use of the positive upper and lower bounds of density and temperature \((2.1)\). The other difficulty is how to deduce the higher decay rates of density and temperature, and we highlight that our results highly depend on the relaxation terms of velocity and temperature.

In the proof of Theorem 1.1, the major difficulties are the influence of the temperature and the regularity-loss of the electromagnetic field. We will do the refined energy estimates stated in Lemma 2.8–2.9, which allow us to deduce

\[
\frac{d}{dt} \mathcal{E}_3 + \mathcal{D}_3 \lesssim \sqrt{\mathcal{E}_3 \mathcal{D}_3}
\]

Then the Euler-Maxwell system \((1.1)\) is reformulated equivalently as

\[
\begin{cases}
\partial_t n = -u \cdot \nabla n - (1 + n) \text{div} u, \\
\partial_t u + u \cdot \nabla u + \nabla \theta + u \times B_\infty = -u \cdot \nabla u - \frac{1 + \theta}{1 + n} \nabla n - u \times B, \\
\partial_t \theta + \theta = -u \cdot \nabla \theta - \frac{2}{3}(1 + \theta) \text{div} u + \frac{1}{3}|u|^2, \\
\partial_t E - \nabla \times B - u = nu, \\
\partial_t B + \nabla \times E = 0, \\
\text{div} E = -n, \quad \text{div} B = 0,
\end{cases}
\tag{1.2}
\]

For \(N \geq 3\), we define the energy functional by

\[
\mathcal{E}_N(t) := \sum_{l=0}^{N} \left\| \nabla^l (n, u, \theta, E, B) \right\|_{L^2}^2
\]

and the corresponding dissipation rate by

\[
\mathcal{D}_N(t) := \sum_{l=0}^{N} \left\| \nabla^l (n, u, \theta) \right\|_{L^2}^2 + \sum_{l=0}^{N-1} \left\| \nabla^l E \right\|_{L^2}^2 + \sum_{l=0}^{N-1} \left\| \nabla^l B \right\|_{L^2}^2.
\]

Our first main result about the global unique solution to the system \((1.2)\) is stated as follows.

**Theorem 1.1.** Assume the initial data satisfy the compatible conditions

\[
\text{div} E_0 = -n_0, \quad \text{div} B_0 = 0.
\]

There exists a sufficiently small \(\delta_0 > 0\) such that if \(\mathcal{E}_3(0) \leq \delta_0\), then there exists a unique global solution \((n, u, \theta, E, B)(t)\) to the Euler-Maxwell system \((1.2)\) satisfying

\[
\sup_{0 \leq t \leq \infty} \mathcal{E}_3(t) + \int_0^\infty \mathcal{D}_3(\tau) \, d\tau \leq C \mathcal{E}_3(0).
\tag{1.3}
\]

Furthermore, if \(\mathcal{E}_N(0) < +\infty\) for any \(N \geq 3\), there exists an increasing continuous function \(P_N(\cdot)\) with \(P_N(0) = 0\) such that the unique solution satisfies

\[
\sup_{0 \leq t \leq \infty} \mathcal{E}_N(t) + \int_0^\infty \mathcal{D}_N(\tau) \, d\tau \leq P_N(\mathcal{E}_N(0)).
\tag{1.4}
\]

In the proof of Theorem 1.1, the major difficulties are the influence of the temperature and the regularity-loss of the electromagnetic field. We will do the refined energy estimates stated in Lemma 2.8–2.9 which allow us to deduce

\[
\frac{d}{dt} \mathcal{E}_3 + \mathcal{D}_3 \lesssim \sqrt{\mathcal{E}_3 \mathcal{D}_3}
\]
and for $N \geq 4$,
\[
\frac{d}{dt} \mathcal{E}_N + \mathcal{D}_N \leq C_N \mathcal{D}_{N-1} \mathcal{E}_N.
\]

Then Theorem 1.1 follows in the fashion of [9, 29, 23].

Our second main result is on some various decay rates of the solution to the system (1.2) by making the much stronger assumption on the initial data.

**Theorem 1.2.** Assume that $(n, u, \theta, E, B)(t)$ is the solution to the Euler-Maxwell system (1.2) constructed in Theorem 1.1 with $N \geq 5$. There exists a sufficiently small $\delta_0 = \delta_0(N)$ such that if $E_N(0) \leq \delta_0$, and assuming that $(u_0, \theta_0, E_0, B_0) \in H^{-s}$ for some $s \in [0, 3/2]$ or $(u_0, \theta_0, E_0, B_0) \in \dot{B}^{-s}_{2, \infty}$ for some $s \in (0, 3/2]$, then we have
\[
\|(u, \theta, E, B)(t)\|_{\dot{H}^{-s}} \leq C_0
\]
or
\[
\|(u, \theta, E, B)(t)\|_{\dot{B}^{-s}_{2, \infty}} \leq C_0.
\]

Moreover, for any fixed integer $k \geq 0$, if $N \geq 2k + 2 + s$, then
\[
\left\|\nabla^k (n, u, \theta, E, B)(t)\right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k + s}{2}}.
\]

Furthermore, for any fixed integer $k \geq 0$, if $N \geq 2k + 4 + s$, then
\[
\left\|\nabla^k (n, u, \theta, E)(t)\right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k + 2 + s}{2}}.
\]

if $N \geq 2k + 6 + s$, then
\[
\left\|\nabla^k (n(t))\right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k + 2 + s}{2}}.
\]

if $N \geq 2k + 12 + s$ and $B_\infty = 0$,
\[
\left\|\nabla^k (n, \theta, \text{div} u)(t)\right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k + 7 + s}{2}}.
\]

In the proof of Theorem 1.2, we mainly use the regularity interpolation method developed in Strain and Guo [21], Guo and Wang [11] and Sohinger and Strain [20]. To prove the optimal decay rate of the dissipative equations in the whole space, Guo and Wang [11] developed a general energy method of using a family of scaled energy estimates with minimum derivative counts and interpolations among them. However, this method can not be applied directly to the compressible non-isentropic Euler-Maxwell system which is of regularity-loss. To overcome this obstacle caused by the regularity-loss of the electromagnetic field, we deduce from Lemma 2.8 [20] that
\[
\frac{d}{dt} \mathcal{E}^{k+2}_k + \mathcal{D}^{k+2}_k \leq C_k \|(n, u)\|_{L^\infty} \left\|\nabla^{k+2} (n, u)\right\|_{L^2} \left\|\nabla^{k+2} (E, B)\right\|_{L^2},
\]
where $\mathcal{E}^{k+2}_k$ and $\mathcal{D}^{k+2}_k$ with minimum derivative counts are defined by (3.9) and (3.10) respectively. Then combining the methods of [11, 20] and a trick of Strain and Guo [21] to treat the electromagnetic field, we are able to conclude the decay rate (1.7). If in view of the whole solution, the decay rate (1.7) can be regarded as be optimal. The higher decay rates (1.8)–(1.10) follow by revisiting the equations carefully. In particular, we will use a bootstrap argument to derive (1.10).

By Theorem 1.2 and Lemma 2.4–2.6, we have the following corollary of the usual $L^p$–$L^2$ type of the decay results:

**Corollary 1.3.** Under the assumptions of Theorem 1.2 except that we replace the $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2, \infty}$ assumption by that $(u_0, \theta_0, E_0, B_0) \in L^p$ for some $p \in [1, 2]$, then for any fixed integer $k \geq 0$, if $N \geq 2k + 2 + s_p$, then
\[
\left\|\nabla^k (n, u, \theta, E, B)(t)\right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k + s_p}{2}}.
\]

Here the number $s_p := 3 \left(\frac{1}{p} - \frac{1}{2}\right)$. 
Furthermore, for any fixed integer \( k \geq 0 \), if \( N \geq 2k + 4 + s_p \), then
\[
\left\| \nabla^k (n, u, \theta, E)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k+1+s_p}{2}},
\]
if \( N \geq 2k + 6 + s_p \), then
\[
\left\| \nabla^k n(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k+2+s_p}{2}};
\]
if \( N \geq 2k + 12 + s_p \) and \( B_\infty = 0 \), then
\[
\left\| \nabla^k (n, \theta, \text{divu})(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\left(\frac{k+2}{2}+s_p\right)}.
\]

The followings are several remarks for Theorem \( \text{I.1}\text{I} \text{I}\text{I} \) and Corollary \( \text{I.3} \).

**Remark 1.4.** In Theorem \( \text{I.1}\text{I} \text{I} \text{I} \), we only assume that the initial data is small in the \( H^3 \) norm but the higher order derivatives could be large. Notice that in Theorem \( \text{I.1}\text{I} \text{I} \text{I} \) the \( H^{-s} \) and \( B_{2,\infty}^{-s} \) norms of the solution are preserved along the time evolution, however, in Corollary \( \text{I.3} \) it is difficult to show that the \( L^p \) norm of the solution can be preserved. Note that the \( L^2 \) decay rate of the higher order spatial derivatives of the solution is obtained. Then the general optimal \( L^q \) (\( 2 \leq q \leq \infty \)) decay rates of the solution follow by the Sobolev interpolation.

**Remark 1.5.** In Theorem \( \text{I.1}\text{I} \text{I} \text{I} \) the space \( \hat{H}^{-s} \) or \( B_{2,\infty}^{-s} \) was introduced there to enhance the decay rates. By the usual embedding theorem, we know that for \( p \in (1,2) \), \( L^p \subset \hat{H}^{-s} \) with \( s = 3(\frac{1}{p} - \frac{1}{2}) \in [0,3/2) \). Meanwhile, we note that the endpoint embedding \( L^1 \subset B_{2,\infty}^{-\frac{3}{2}} \) holds. Hence the \( L^p-L^2 \) (\( 1 \leq p \leq 2 \)) type of the optimal decay results follows as a corollary.

**Remark 1.6.** We remark that Corollary \( \text{I.3} \) not only provides an alternative approach to derive the \( L^p-L^2 \) type of the optimal decay results but also improves the previous results of the \( L^p-L^2 \) approach in Feng et al. [3]. In Feng et al. [3], assuming that \( B_\infty = 0 \) and \( \| (u_0, E_0, B_0) \|_{L^1} \) is sufficiently small, by combining the energy method and the linear decay analysis, Feng showed that
\[
\left\| (n, \theta)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{3}{4}}, \quad \left\| (u, B)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{1}{2}} \quad \text{and} \quad \left\| E(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{1}{4}}.
\]
Notice that for \( p = 1 \), our decay rate of \( (n, \theta)(t) \) is \((1 + t)^{-13/4}\) in \( \text{I.I.I} \), and \( u(t) \) is \((1 + t)^{-5/4}\) in \( \text{I.I.I} \).

**Notations:** In this paper, we use \( H^s(\mathbb{R}^3), \ s \in \mathbb{R} \) to denote the usual Sobolev spaces with norm \( \| \cdot \|_{H^s} \) and \( L^p(\mathbb{R}^3), \ 1 \leq p \leq \infty \) to denote the usual \( L^p \) spaces with norm \( \| \cdot \|_{L^p} \). \( \nabla^\ell \) with an integer \( \ell \geq 0 \) stands for the usual any spatial derivatives of order \( \ell \). When \( \ell < 0 \) or \( \ell \) is not a positive integer, \( \nabla^\ell \) stands for the \( \Lambda^\ell \) defined by \( \Lambda^\ell f := \mathcal{F}^{-1}(|\xi|^\ell \mathcal{F} f) \), where \( \mathcal{F} \) is the usual Fourier transform operator and \( \mathcal{F}^{-1} \) is its inverse. We use \( \hat{H}^s(\mathbb{R}^3), \ s \in \mathbb{R} \) to denote the homogeneous Sobolev spaces on \( \mathbb{R}^3 \) with norm \( \| \cdot \|_{\hat{H}^s} \) defined by \( \| f \|_{\hat{H}^s} := \| \Lambda^s f \|_{L^2} \). We then recall the homogeneous Besov spaces. Let \( \phi \in C_c^\infty(\mathbb{R}_x^3) \) be such that \( \phi(\xi) = 1 \) when \( |\xi| \leq 1 \) and \( \phi(\xi) = 0 \) when \( |\xi| \geq 2 \). Let \( \varphi(\xi) = \phi(\xi) - \phi(2\xi) \) and \( \varphi_j(\xi) = \varphi(2^{-j}\xi) \) for \( j \in \mathbb{Z} \). Then by the construction, \( \sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1 \) if \( \xi \neq 0 \). We define \( \bar{\Delta} j f := \mathcal{F}^{-1}(\varphi_j) * f \), then for \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), we define the homogeneous Besov spaces \( \hat{B}^s_{p,r}(\mathbb{R}^3) \) with norm \( \| \cdot \|_{\hat{B}^s_{p,r}} \) defined by
\[
\| f \|_{\hat{B}^s_{p,r}} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \bar{\Delta} j f \|_{L^p} \right)^{\frac{1}{q}}.
\]
Particularly, if \( r = \infty \), then
\[
\| f \|_{\hat{B}^s_{p,\infty}} := \sup_{j \in \mathbb{Z}} 2^{jsq} \| \bar{\Delta} j f \|_{L^p}.
\]
Throughout this paper, we let \( C \) denote some positive (generally large) universal constants and \( \lambda \) denote some positive (generally small) universal constants. They do not depend on either \( k \) or \( N \); otherwise, we will denote them by \( C_k, C_N, \) etc. We will use \( a \lesssim b \) if \( a \leq C b \), and \( a \sim b \) means that \( a \lesssim b \) and \( b \lesssim a \). We use \( C_0 \) to denote the constants depending on the initial data and \( k, N, s \). For simplicity, we write \( \| (A, B) \|_X := \| A \|_X + \| B \|_X \) and \( \int f := \int_{\mathbb{R}^3} f \, dx \).
Lemma 2.5. Let $\alpha, m, \ell \geq 0$. Then we have
\[
\|\nabla^\alpha f\|_{L^p} \leq C_p \|\nabla^m f\|_{L^2}^{\frac{1}{p}} \|\nabla^\ell f\|_{L^2}^{\theta},
\]
Here $0 \leq \theta \leq 1$ (if $p = +\infty$, then we require that $0 < \theta < 1$) and $\alpha$ satisfies
\[
\alpha + 3 \left(\frac{1}{2} - \frac{1}{p}\right) = m(1 - \theta) + \ell \theta.
\]
Proof. For the case $2 \leq p < +\infty$, we refer to Lemma A.1 in [11]; for the case $p = +\infty$, we refer to Exercise 6.1.2 in [17] (pp. 421).

2. NONLINEAR ENERGY ESTIMATES

In this section, we will do the a priori estimate by assuming that $\|f(t)\|_{H^3} \leq \delta \ll 1$. Then by Sobolev’s inequality, we have
\[
\frac{1}{2} \leq 1 + n, \quad 1 + \theta \leq \frac{3}{2},
\]
(2.1)

2.1. Preliminary. In this subsection, we collect the analytic tools used later in the paper.

Lemma 2.1. Let $2 \leq p \leq +\infty$ and $\alpha, m, \ell \geq 0$. Then we have
\[
\|\nabla^\alpha f\|_{L^p} \leq C_p \|\nabla^m f\|_{L^2}^{\frac{1}{p}} \|\nabla^\ell f\|_{L^2}^{\theta}.
\]

Proof. The case $2 \leq p < +\infty$, we refer to Lemma A.1 in [11]; for the case $p = +\infty$, we refer to Exercise 6.1.2 in [17] (pp. 421).

We recall the following commutator estimate:

Lemma 2.2. Let $k \geq 1$ be an integer and define the commutator
\[
\left[\nabla^k, g\right] h = \nabla^k(gh) - g\nabla^k h.
\]
Then we have
\[
\left\|\left[\nabla^k, g\right] h\right\|_{L^2} \leq C_k \left(\|g\|_{L^\infty} \left\|\nabla^{k-1} h\right\|_{L^2} + \left\|\nabla^k g\right\|_{L^2} \|h\|_{L^\infty}\right),
\]
and
\[
\left\|\nabla^k(gh)\right\|_{L^2} \leq C_k \left(\|g\|_{L^\infty} \left\|\nabla^k h\right\|_{L^2} + \left\|\nabla^k g\right\|_{L^2} \|h\|_{L^\infty}\right).
\]

Proof. It can be proved by using Lemma 2.1. See Lemma 3.4 in [15] (pp. 98) for instance.

Notice that when using the commutator estimate in this paper, we usually will not consider the case that $k = 0$ since it is trivial.

Lemma 2.3. If the function $f(n)$ satisfies
\[
f(n) \sim n \quad \text{and} \quad \left|f^{(k)}(n)\right| \leq C_k \text{ for any } k \geq 1,
\]
then for any integer $k \geq 0$, we have
\[
\left\|\nabla^k f(n)\right\|_{L^2} \leq C_k \left\|\nabla^k n\right\|_{L^2}.
\]

Proof. See Lemma 2.2 in [23].

We have the $L^p$ embeddings:

Lemma 2.4. Let $0 \leq s < 3/2$, $1 < p \leq 2$ with $1/2 + s/3 = 1/p$, then
\[
\|f\|_{H^{-s}} \lesssim \|f\|_{L^p}.
\]

Proof. It follows from the Hardy-Littlewood-Sobolev theorem, see [7].

Lemma 2.5. Let $0 < s \leq 3/2$, $1 \leq p < 2$ with $1/2 + s/3 = 1/p$, then
\[
\|f\|_{B_{p}^{-s}} \lesssim \|f\|_{L^p}.
\]

Proof. It follows from the Hardy-Littlewood-Sobolev theorem, see [7].
Proof. See Lemma 4.6 in [20]. □

It is important to use the following special interpolation estimates:

**Lemma 2.6.** Let $s \geq 0$ and $\ell \geq 0$, then we have

$$
\|\nabla^\ell f\|_{L^2} \leq \left\|\nabla^{\ell+1} f\right\|_{1-\theta}^{1-\theta} \|f\|_{H^{-s}}, \quad \text{where} \quad \theta = \frac{1}{\ell+1+s}.
$$

Proof. It follows directly by the Parseval theorem and Hölder’s inequality. □

**Lemma 2.7.** Let $s > 0$ and $\ell \geq 0$, then we have

$$
\|\nabla^\ell f\|_{L^2} \leq \left\|\nabla^{\ell+1} f\right\|_{1-\theta}^{1-\theta} \|f\|_{B_{2,\infty}^{-s}}, \quad \text{where} \quad \theta = \frac{1}{\ell+1+s}.
$$

Proof. See Lemma 4.5 in [20]. □

### 2.2. Energy estimates

In this subsection, we will derive the basic energy estimates for the solution to the Euler-Maxwell system (1.2). We begin with the standard energy estimates.

**Lemma 2.8.** For any integer $k \geq 0$, we have

$$
\begin{align*}
\frac{d}{dt} \sum_{l=k}^{k+2} \left\|\nabla^l (n, u, \theta, E, B)\right\|_{L^2}^2 + \lambda \sum_{l=k}^{k+1} \left\|\nabla^l (u, \theta)\right\|_{L^2}^2 \\
\leq C_k F(n, u, \theta, B) \left( \sum_{l=k}^{k+2} \left\|\nabla^l (n, u, \theta)\right\|_{L^2}^2 + \sum_{l=k}^{k+1} \left\|\nabla^l E\right\|_{L^2}^2 + \left\|\nabla^{k+1} B\right\|_{L^2}^2 \right) \\
+ \left\|\nabla^{k+2} B\right\|_{L^2}
\end{align*}
$$

where $F(n, u, \theta, B) := \|\nabla(n, u, \theta)\|_{H^{2+k\infty}}^2 + \|\nabla n, u, \theta\|_{H^3}^2 + \|\nabla B\|_{L^2}^2$.

Proof. Applying $\nabla^l \ (l = k, k + 1, k + 2)$ to the first five equations in (1.2) and then multiplying the resulting identities by $\frac{1+\theta}{1+n} \nabla^l n, \ 1+n \nabla^l u, \ 3\frac{1+n}{2} \nabla^l \theta, \ \nabla^l E, \ \nabla^l B$ respectively, summing up and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{align*}
\frac{d}{dt} \int \frac{1+\theta}{1+n} |\nabla^l n|^2 + (1+n)|\nabla^l u|^2 + \frac{3+1+n}{2} |\nabla^l \theta|^2 + |\nabla^l (E, B)|^2 + \left\|\nabla^l (n, u, \theta)\right\|_{L^2}^2 \\
\leq \int \left( \frac{\partial n}{1+n} - \frac{1+\theta}{(1+n)^2} \partial_n \right) |\nabla^l n|^2 + \partial_n |\nabla^l u|^2 + \frac{3}{2} \left( \frac{\partial n}{1+\theta} - \frac{1+n}{(1+\theta)^2} \partial^2 \theta \right) |\nabla^l \theta|^2 \\
-2 \int \frac{1+\theta}{1+n} \nabla^l (u \cdot \nabla n) \nabla^l n + (1+n) \nabla^l (u \cdot \nabla u) \cdot \nabla^l u + \frac{3+1+n}{2} |\nabla^l (u \cdot \nabla \theta)\nabla^l \theta| \\
-2 \int \frac{1+\theta}{1+n} \nabla^l ((1+n) \delta \nabla^l n) \nabla^l u + (1+n) \nabla^l \left( \frac{1+\theta}{1+n} \nabla^l \nabla^l u \right) \\
-2 \int \frac{1+\theta}{1+n} \nabla^l ((1+n) \delta \nabla^l \theta) + (1+n) \nabla^l \nabla^l \theta \cdot \nabla^l u + \int \frac{1+n}{1+\theta} |\nabla^l \nabla^l (|u|^2) \\
-2 \int (1+n) \nabla^l (u \times B) \cdot \nabla^l u - 2 \int n \nabla^l u \cdot \nabla^l E - \nabla^l (nu) \cdot \nabla^l E \\
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{align*}
$$

First, by (2.10), (2.21), (2.23) and Sobolev’s embedding inequality, we easily obtain

$$
I_1 \lesssim \|\nabla (n, u, \theta)\|_{H^3} \|\nabla (n, u, \theta)\|_{L^2}^2.
$$

Next, we estimate the term $I_2$. We rewrite $I_2$ as

$$
I_2 = -2 \int \frac{1+\theta}{1+n} \nabla^l (u \cdot \nabla n) \nabla^l n - 2 \int (1+n) \nabla^l (u \cdot \nabla u) \cdot \nabla^l u - 3 \int \frac{1+n}{1+\theta} |\nabla^l (u \cdot \nabla \theta)\nabla^l \theta| \\
:= I_{21} + I_{22} + I_{23}.
$$
First, we estimate $I_{21}$. By the commutator notation (2.2) and (2.1), we have
\[
I_{21} = -2 \int \frac{1 + \theta}{1 + n} \nabla^l (u \cdot \nabla n) \nabla n = -2 \int \frac{1 + \theta}{1 + n} (u \cdot \nabla \nabla^l n + \left[ \nabla^l, u \right] \cdot \nabla n) \nabla n
\lesssim \left| \int u \cdot \nabla (\nabla^l n \nabla n) + \left[ \nabla^l, u \right] \cdot \nabla n \nabla^l n \right|.
\]
By integrating by parts, we have
\[
\int u \cdot \nabla (\nabla^l n \nabla n) = - \int \text{div}u \left| \nabla^l n \right|^2 \leq \|\text{div}u\|_{L^\infty} \|\nabla^l n\|_{L^2}^2.
\]
We employ the commutator estimate of Lemma 2.2 to bound
\[
\int \left[ \nabla^l, u \right] \cdot \nabla n \nabla n \leq C_l \left( \|\nabla u\|_{L^\infty} \|\nabla^{l-1} \nabla n\|_{L^2} + \|\nabla^l u\|_{L^2} \|\nabla n\|_{L^\infty} \right) \|\nabla^l n\|_{L^2}
\leq C_l \|\nabla(n, u)\|_{L^\infty} \|\nabla^l (n, u)\|_{L^2} \|\nabla^l n\|_{L^2}.
\]
Then applying the same arguments to $I_{22}$ and $I_{23}$, by Sobolev’s and Cauchy’s inequalities, we obtain
\[
I_2 \leq C_l \|(n, u, \theta)\|_{H^3} \left| \nabla^l (n, u, \theta) \right|_{L^2}^2.
\]
We now estimate $I_3$. By the commutator notation (2.2), we rewrite $I_3$ as
\[
I_3 = -2 \int (1 + \theta) \left( \nabla^l \text{div}u \nabla^l n + \nabla^l \nabla n \cdot \nabla^l u \right)
-2 \int \frac{1 + \theta}{1 + n} \left[ \nabla^l, 1 + n \right] \text{div}u \nabla n + (1 + n) \left[ \nabla^l, \frac{1 + \theta}{1 + n} \right] \nabla n \cdot \nabla u.
\]
By integrating by parts, we obtain
\[
-2 \int (1 + \theta) \left( \nabla^l \text{div}u \nabla^l n + \nabla^l \nabla n \cdot \nabla^l u \right) = -2 \int (1 + \theta) \text{div} \left( \nabla^l u \nabla^l n \right)
= 2 \int \nabla \theta \cdot \nabla^l u \nabla n \lesssim \|\nabla \theta\|_{L^\infty} \|\nabla^l u\|_{L^2} \|\nabla^l n\|_{L^2}.
\]
By Lemma 2.2 and Sobolev’s and Cauchy’s inequalities, we obtain
\[
-2 \int \frac{1 + \theta}{1 + n} \left[ \nabla^l, 1 + n \right] \text{div}u \nabla n + (1 + n) \left[ \nabla^l, \frac{1 + \theta}{1 + n} \right] \nabla n \cdot \nabla^l u
\lesssim C_l \|\nabla(n, u, \theta)\|_{L^\infty} \left| \nabla^l (n, u, \theta) \right|_{L^2}^2.
\]
In fact, there is a key estimate in the following.
\[
\left\| \left[ \nabla^l, \frac{1 + \theta}{1 + n} \right] \nabla n \right\|_{L^2} \leq C_l \left( \left\| \nabla \left( \frac{1 + \theta}{1 + n} \right) \right\|_{L^\infty} \|\nabla^l n\|_{L^2} + C_l \left\| \nabla^l \left( \frac{1 + \theta}{1 + n} \right) \right\|_{L^2} \|\nabla n\|_{L^\infty} \right.
\leq C_l \|\nabla(n, \theta)\|_{L^\infty} \|\nabla^l n\|_{L^2} + C_l \left( \frac{1}{1 + n} \right) \left\| \nabla^l \left( \frac{1}{1 + n} \right) \right\|_{L^2} \|\nabla n\|_{L^\infty}
\leq C_l \|\nabla(n, \theta)\|_{L^\infty} \|\nabla^l (n, \theta)\|_{L^2} + C_l \|\nabla n\|_{L^\infty} \|\nabla^l \left( 1 - \frac{1}{1 + n} \right) \|_{L^2}
\leq C_l \|\nabla(n, \theta)\|_{L^\infty} \left| \nabla^l (n, \theta) \right|_{L^2}.
\]
where we have used that $1 - \frac{1}{1 + n} \sim n$. Then applying the same arguments to $I_4$ and $I_5$, by Sobolev’s and Cauchy’s inequalities, we obtain
\[
I_3 + I_4 + I_5 \leq C_l \|(n, u, \theta)\|_{H^3} \left| \nabla^l (n, u, \theta) \right|_{L^2}^2.
\]
For the term $I_6$, as in (2.3), we have that for $l = k$
\[
I_6 \leq C_k \left( \|u\|_{H^{k+1}}^2 + \|\nabla B\|_{L^2} \right) \left( \|\nabla^k u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right). \tag{2.6}
\]
for $l = k + 1$
\[
I_6 \leq C_k \left( \|u\|_{H^{k+2}}^2 + \|\nabla B\|_{L^2} \right) \left( \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right);
\]
for $l = k + 2$
\[
I_6 \leq C_k \left( \|u\|_{H^{k+3}}^2 + \|\nabla B\|_{L^2} \right) \left( \|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+2} B\|_{L^2}^2 \right) + C \|u\|_{L^\infty} \left( \|\nabla^{k+2} B\|_{L^2} \right) \left( \|\nabla^{k+2} u\|_{L^2} \right).
\]
Next, we estimate the last term $I_7$. By Lemma 2.2, we easily obtain for $l = k$ or $k + 1$,
\[
I_7 \leq C_k \|(n, u)\|_{H^2} \left( \|\nabla^l (n, u)\|_{L^2}^2 + \|\nabla^l E\|_{L^2}^2 \right);
\]
for $l = k + 2$,
\[
I_7 \leq C_k \|(n, u)\|_{L^\infty} \left( \|\nabla^{k+2} (n, u)\|_{L^2} \right) \left( \|\nabla^{k+2} E\|_{L^2} \right).
\]
Consequently, plugging these estimates for $I_1 \sim I_7$ into (2.4) with $l = k, k + 1, k + 2$, and then summing up, we deduce (2.3) from (2.1). \qed

Note that in Lemma 2.8, we only derive the dissipative estimate of $u, \theta$. We now recover the dissipative estimates of $n, E$ and $B$ by constructing some interactive energy functionals in the following lemma.

**Lemma 2.9.** For any integer $k \geq 0$, we have that for any small fixed $\eta > 0$,
\[
\frac{d}{dt} \left( \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l n + \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B \right) + \lambda \left( \sum_{l=k}^{k+2} \|\nabla^l n\|_{L^2}^2 + \sum_{l=k}^{k+2} \|\nabla^l E\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right) \leq C \sum_{l=k}^{k+2} \|\nabla^l (n, u, \theta)\|_{L^2}^2 + C_k G(n, u, \theta, B) \left( \sum_{l=k}^{k+2} \|\nabla^l (n, u, \theta)\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2 \right), \tag{2.7}
\]
where $G(n, u, \theta, B)$ is defined by
\[
G(n, u, \theta, B) := \|(n, u, \theta)\|_{H^{k+1}}^2 + \|\nabla B\|_{L^2}^2.
\]

**Proof.** We divide the proof into four steps.

**Step 1:** Dissipative estimate of $n$.

Applying $\nabla^l$ ($l = k, k + 1$) to (1.2) and then taking the $L^2$ inner product with $\nabla \nabla^l n$, we obtain
\[
\int \partial_t \nabla^l u \cdot \nabla \nabla^l n + \int \frac{1 + \theta}{1 + n} |\nabla \nabla^l n|^2 \leq -\int \nabla^l E \cdot \nabla \nabla^l n + C \|\nabla^l (u, \theta)\|_{L^2} \|\nabla^{l+1} n\|_{L^2} + \|\nabla^l (u \cdot \nabla u + B)\|_{L^2} \|\nabla^{l+1} n\|_{L^2} - \int \left[ \nabla^l \left( \frac{1 + \theta}{1 + n} \nabla n \right) - \frac{1 + \theta}{1 + n} \nabla \nabla^l n \right] \nabla \nabla^l n. \tag{2.8}
\]
Similarly, we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla \nabla^n = \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^n - \int \nabla^l u \cdot \nabla \nabla^n + \int \nabla^l \text{div} u \nabla^l \partial_t n = \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^n - \int \nabla^l \text{div} u \nabla^l \partial_t n
\]
\[
\geq \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^n - C \left( \left\| \nabla^{l+1} u \right\|_{L^2}^2 - C \left\| \nabla^l (u \cdot \nabla n) \right\|_{L^2}^2 - C \left\| \nabla^l (\text{div} u) \right\|_{L^2}^2 \right).
\]
Using the commutator estimate of Lemma 2.2, we have
\[
\left\| \nabla^l (u \cdot \nabla n) \right\|_{L^2} \leq \left\| u \right\| \left\| \nabla^l \nabla^n \right\|_{L^2} + \left\| \nabla^l, u \right\| \cdot \left\| \nabla^n \right\|_{L^2}
\]
\[
\leq \left\| u \right\|_{L^\infty} \left\| \nabla^{l+1} n \right\|_{L^2} + C_l \left\| \nabla u \right\|_{L^\infty} \left\| \nabla^l n \right\|_{L^2} + C_l \left\| \nabla^l u \right\|_{L^2} \left\| \nabla n \right\|_{L^\infty}
\]
\[
\leq C_l \left\| (n, u) \right\|_{H^3} \left( \left\| \nabla^l (n, u) \right\|_{L^2} + \left\| \nabla^{l+1} u \right\|_{L^2} \right).
\] (2.9)
Similarly,
\[
\left\| \nabla^l (\text{div} u) \right\|_{L^2} \leq C_l \left\| (n, u) \right\|_{H^3} \left( \left\| \nabla^l (n, u) \right\|_{L^2} + \left\| \nabla^{l+1} u \right\|_{L^2} \right).
\] (2.10)
Hence, we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla \nabla^n \geq \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^n - C \left( \left\| \nabla^{l+1} u \right\|_{L^2}^2
\]
\[
- C_l \left\| (n, u) \right\|_{H^3}^2 \left( \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{l+1} (n, u) \right\|_{L^2}^2 \right).
\] (2.11)
Next, integrating by parts and using the equation (1.2), we have
\[
- \int \nabla^l E \cdot \nabla \nabla^n = \int \nabla^l \text{div} E \nabla^l n = - \left\| \nabla^n \right\|_{L^2}^2.
\] (2.12)
And as in (2.9)–(2.10), we have
\[
\left\| \nabla^l (u \cdot \nabla u) \right\|_{L^2} \leq C_l \left\| u \right\|_{H^3} \left( \left\| \nabla^l u \right\|_{L^2} + \left\| \nabla^{l+1} u \right\|_{L^2} \right).
\] (2.13)
From the estimate of $I_6$ in Lemma 2.8, we have that for $l = k$ or $k + 1$,
\[
\left\| \nabla^l (x \times B) \right\|_{L^2} \leq C_k \left( \left\| u \right\|_{H^{k+1} \cap H^2} + \left\| B \right\|_{L^2} \right) \left( \left\| \nabla^l u \right\|_{L^2} + \left\| \nabla^{k+1} B \right\|_{L^2} \right).
\] (2.14)
Lastly, by Lemma 2.8 and (2.6), we obtain
\[
- \int \left[ \nabla^l \left( \frac{1 + \theta}{1 + n} \nabla n \right) - \frac{1 + \theta}{1 + n} \nabla \nabla^n \right] \cdot \nabla^n \nabla^n
\]
\[
= - \int \left[ \nabla^l, \frac{1 + \theta}{1 + n} \right] \nabla n \cdot \nabla \nabla^n \leq C_l \left\| (n, \theta) \right\|_{L^\infty} \left( \left\| \nabla (n, \theta) \right\|_{L^2}^2 + \left\| \nabla^{l+1} n \right\|_{L^2}^2 \right). \] (2.15)
Plugging these estimates (2.11)–(2.15) into (2.8), by Cauchy’s inequality and (2.1), we obtain
\[
\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla \nabla^n + \lambda \sum_{l=k}^{k+2} \left\| \nabla^l n \right\|_{L^2}^2 \leq C \sum_{l=k}^{k+2} \left\| \nabla^l (u, \theta) \right\|_{L^2}^2
\]
\[
+ C_k \left( \left\| (n, u, \theta) \right\|_{H^{k+1} \cap H^3}^2 + \left\| B \right\|_{L^2} \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, \theta) \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right).
\] (2.16)
This completes the dissipative estimate for $n$.

**Step 2: Dissipative estimate of $E$.**
Applying $\nabla^l$ ($l = k, k + 1$) to (1.24) and then taking the $L^2$ inner product with $\nabla^l E$, we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla^l E + \|\nabla^l E\|^2_{L^2} \leq -\int \nabla^l \left( \frac{1 + \theta}{1 + n} \nabla n \right) \cdot \nabla^l E + C \left\| \nabla^l (u, \nabla \theta) \right\|_{L^2} \left\| \nabla^l E \right\|_{L^2} + \left\| \nabla^l (u \cdot \nabla u + u \times B) \right\|_{L^2} \left\| \nabla^l E \right\|_{L^2}.
\]

Again, the delicate first term on the left-hand side of (2.17) involves $\partial_t \nabla^l u$, and the key idea is to integrate by parts in the $t$-variable and use the equation (1.24) in the Maxwell system. Thus we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla^l E = \frac{d}{dt} \int \nabla^l u \cdot \nabla^l E - \int \nabla^l u \cdot \nabla^l \partial_t E
\]
\[
= \frac{d}{dt} \int \nabla^l u \cdot \nabla^l E - \int \nabla^l u \cdot \nabla^l E + \left\| \nabla^l u \right\|_{L^2}^2 - \int \nabla^l u \cdot \nabla^l (nu + \nabla \times B).
\]

By Lemma 2.2 we have
\[
\left\| \nabla^l (nu) \right\|_{L^2} \leq C_l \|(n, u)\|_{H^2} \left\| \nabla^l (n, u) \right\|_{L^2}.
\]

We must be much more careful about the remaining term in (2.18) since there is no small factor in front of it. The key is to use Cauchy’s inequality and distinct the cases of $l = k$ and $l = k + 1$ due to the weakest dissipative estimate of $B$. For $l = k$, we have
\[
- \int \nabla^{k+1} u \cdot \nabla \times \nabla^{k+1} B \leq \varepsilon \left\| \nabla^{k+1} B \right\|^2_{L^2} + C_{\varepsilon} \left\| \nabla^k u \right\|^2_{L^2};
\]
for $l = k + 1$, integrating by parts, we obtain
\[
- \int \nabla^{k+1} u \cdot \nabla \times \nabla^{k+1} B = - \int \nabla \times \nabla^{k+1} u \cdot \nabla^{k+1} B
\]
\[
\leq \varepsilon \left\| \nabla^{k+1} B \right\|^2_{L^2} + C_{\varepsilon} \left\| \nabla^{k+2} u \right\|^2_{L^2}.
\]

By Lemma 2.2 (1.26), (2.1) and (2.5), we have
\[
- \int \nabla^l \left( \frac{1 + \theta}{1 + n} \nabla n \right) \cdot \nabla^l E = - \int \left( \nabla^l \left( \frac{1 + \theta}{1 + n} \right) \nabla n + \frac{1 + \theta}{1 + n} \nabla \nabla^l n \right) \cdot \nabla^l E
\]
\[
= - \int \left[ \nabla^l \left( \frac{1 + \theta}{1 + n} \right) \nabla n \cdot \nabla^l E
\]
\[
+ \int \frac{1 + \theta}{1 + n} \nabla^l n \cdot \nabla^l \text{div} E + \int \left( \frac{1 + \theta}{1 + n} \right) \nabla n \cdot \nabla^l E \right] \leq \varepsilon \left\| \nabla^l E \right\|^2_{L^2} + C_{l, \varepsilon} \|(n, \theta)\|_{H^3} \left\| \nabla^l (n, \theta) \right\|^2_{L^2}.
\]

Plugging the estimates (2.18), (2.22) and (2.13)–(2.14) from Step 1 into (2.17), by Cauchy’s inequality, we then obtain
\[
\frac{d}{dt} \sum_{l=k}^{k+1} \left\| \nabla^l u \cdot \nabla^l E + \lambda \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|^2_{L^2} \leq \varepsilon \left\| \nabla^{k+1} B \right\|^2_{L^2} + C_{\varepsilon} \sum_{l=k}^{k+2} \left\| \nabla^l (u, \theta) \right\|^2_{L^2}
\]
\[
+ C_k \left\| (n, u, \theta) \right\|^2_{H^2+1 \cap H^3} + \| \nabla B \|^2_{L^2} \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, \theta) \right\|^2_{L^2} + \left\| \nabla^{k+1} B \right\|^2_{L^2} \right).
\]
This completes the dissipative estimate for $E$.

**Step 3: Dissipative estimate of $B$.**
Applying $\nabla^k$ to (1.24) and then taking the $L^2$ inner product with $-\nabla \times \nabla^k B$, we obtain
\[
- \int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B + \left\| \nabla \times \nabla^k B \right\|^2_{L^2}
\]
\[ \leq \left\| \nabla^k u \right\|_{L^2} \left\| \nabla \times \nabla^k B \right\|_{L^2} + \left\| \nabla^k (nu) \right\|_{L^2} \left\| \nabla \times \nabla^k B \right\|_{L^2}. \]  

(2.24)

Integrating by parts for both the \( t \)- and \( x \)-variables and using the equation \( \nabla \cdot \mathbf{F} = 0 \), we have

\[
- \int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B = -\frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B + \int \nabla \times \nabla^k E \cdot \nabla^k \partial_t B
\]

\[
= -\frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B - \left\| \nabla \times \nabla^k E \right\|_{L^2}^2. 
\]

(2.25)

Plugging the estimates \( (2.19) \) with \( l = k \) and \( (2.25) \) into \( (2.24) \) and by Cauchy’s inequality, since \( \text{div} B = 0 \), we then obtain

\[
-\frac{d}{dt} \int \nabla^k E \cdot \nabla^k \nabla \times B + \lambda \left\| \nabla^{k+1} B \right\|_{L^2}^2 
\]

\[
\leq C \left\| \nabla^k u \right\|_{L^2}^2 + C \left\| \nabla^{k+1} E \right\|_{L^2}^2 + C_k \left\| (n, u) \right\|_{H^2} \left\| \nabla^k (n, u) \right\|_{L^2}. 
\]

(2.26)

This completes the dissipative estimate for \( B \).

**Step 4: Conclusion.**

Multiplying \( (2.26) \) by a small enough but fixed constant \( \eta \) and then adding it with \( (2.23) \) so that the second term on the right-hand side of \( (2.26) \) can be absorbed, then choosing \( \varepsilon \) small enough so that the first term on the right-hand side of \( (2.23) \) can be absorbed; we obtain

\[
\frac{d}{dt} \left( \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B \right) + \lambda \left( \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2} + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right)
\]

\[
\leq C \sum_{l=k}^{k+2} \left\| \nabla^l (u, \theta) \right\|_{L^2}^2 + C_k G(n, u, \theta, B) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, \theta) \right\|_{L^2} + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right). 
\]

(2.27)

Here \( G(n, u, \theta, B) \) is well-defined. Adding the inequality above with \( (2.16) \), we get \( (2.27) \). \( \Box \)

2.3. **Negative Sobolev estimates.** In this subsection, we will derive the evolution of the negative Sobolev norms of \( (u, \theta, E, B) \). In order to estimate the nonlinear terms, we need to restrict ourselves to that \( s \in (0, 3/2) \). We will establish the following lemma.

**Lemma 2.10.** For \( s \in (0, 1/2] \), we have

\[
\frac{d}{dt} \left\| (u, \theta, E, B) \right\|_{H^{-s}}^2 + \lambda \left\| (u, \theta) \right\|_{H^{-s}}^2 
\]

\[
\lesssim \left( \left\| (n, u, \theta) \right\|_{H^2} + \left\| \nabla B \right\|_{H^1} \right) \left\| (u, \theta, E, B) \right\|_{H^{-s}} + \left\| E \right\|_{H^2}, 
\]

(2.27)

and for \( s \in (1/2, 3/2) \), we have

\[
\frac{d}{dt} \left\| (u, \theta, E, B) \right\|_{H^{-s}}^2 + \lambda \left\| (u, \theta) \right\|_{H^{-s}}^2 
\]

\[
\lesssim \left( \left\| (n, u, \theta) \right\|_{H^2} + \left\| B \right\|_{L^2}^{-1/2} \left\| \nabla B \right\|_{L^2}^{3/2-s} \left\| u \right\|_{L^2} \right) \left\| (u, \theta, E, B) \right\|_{H^{-s}} + \left\| E \right\|_{H^2}. 
\]

(2.28)

**Proof.** The \( \Lambda^{-s} \) \((s > 0)\) energy estimate of \( (1.2) \) yields

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| (u, E, B) \right\|_{H^{-s}}^2 + \frac{3}{2} \left\| \theta \right\|_{H^{-s}}^2 \right) + \left\| u \right\|_{H^{-s}}^2 + \frac{3}{2} \left\| \theta \right\|_{H^{-s}}^2
\]

\[
= -\int \Lambda^{-s} \left( u \cdot \nabla u + \frac{\theta \nabla n}{1+n} + u \times B \right) \cdot \Lambda^{-s} u - \int \Lambda^{-s} \left( \nabla n \right) \cdot \Lambda^{-s} u
\]

\[
+ \int \Lambda^{-s} (nu) \cdot \nabla E - \frac{3}{2} \int \Lambda^{-s} \left( u \cdot \nabla \theta + \frac{2}{3} \theta \text{div} u + \frac{1}{3} |u|^2 \right) \Lambda^{-s} \theta
\]

\[
\lesssim \left\| u \cdot \nabla u + \theta \nabla n + u \times B \right\|_{H^{-s}} + \left\| u \right\|_{H^{-s}} + \left\| nu \right\|_{H^{-s}} \left\| E \right\|_{H^{-s}} + \left\| \nabla n \right\|_{H^{-s}} + \left\| u \right\|_{H^{-s}}
\]

\[
+ \left\| u \cdot \nabla \theta + \theta \text{div} u + |u|^2 \right\|_{H^{-s}} \left\| \theta \right\|_{H^{-s}}. 
\]

(2.29)
We now restrict the value of \( s \) in order to estimate the other terms on the right-hand side of (2.29). If \( s \in (0, 1/2) \), then \( 1/2 + s/3 < 1 \) and \( 3/s \geq 6 \). Then applying Lemma 2.4 together with Hölder’s, Sobolev’s and Young’s inequalities, we obtain
\[
\|u \cdot \nabla u\|_{L^{2s/3}} \lesssim \|u \cdot \nabla u\|_{L^{2s/3}} \lesssim \|u\|_{L^{2s/3}} \|\nabla u\|_{L^2}
\]
\[
\lesssim \|\nabla u\|_{L^2}^{1/2 + s} \|\nabla^2 u\|_{L^2}^{1/2 - s} \|\nabla u\|_{L^2} \lesssim \|\nabla u\|_{H^1}^2 + \|\nabla u\|_{L^2}^2.
\]
Similarly, we obtain
\[
\|\theta \nabla n + n \times B + nu + u \cdot \nabla \theta + \theta \text{div} u + |u|^2\|_{L^{2s/3}} \lesssim \|(n, u, \theta)\|_{H^1}^2 + \|\nabla B\|_{H^1}^2.
\]
Now if \( s \in (1/2, 3/2) \), we have that \( 1/2 + s/3 < 1 \) and \( 2 < 3/s < 6 \). Then applying Lemma 2.4 and using (different) Sobolev’s inequality, we have
\[
\|u \cdot \nabla u\|_{L^{2s/3}} \lesssim \|u\|_{L^{2s/3}} \|\nabla u\|_{L^2} \lesssim \|u\|_{L^{3/2}} \|\nabla u\|_{L^2}
\]
\[
\lesssim \|u\|_{H^1}^2 + \|\nabla u\|_{L^2}^2;
\]
\[
\|u \times B\|_{L^{2s/3}} \lesssim \|B\|_{L^2}^{3/2 - s} \|\nabla B\|_{L^2} \|u\|_{L^2}.
\]
Similarly, we obtain
\[
\|\theta \nabla n + nu + u \cdot \nabla \theta + \theta \text{div} u + |u|^2\|_{L^{2s/3}} \lesssim \|(n, u, \theta)\|_{H^1}^2.
\]
Note that we fail to estimate the remaining last term on the right-hand side of (2.29) as above. To overcome this obstacle, the key point is to make full use of the Poisson equation (1.2). Indeed, using (1.2), we have
\[
\|\nabla n\|_{H^{-s}} \lesssim \|\Lambda^{-s} \text{div} E\|_{L^2} \lesssim \|E\|_{H^2}.
\]
Now collecting all the estimates we have derived, by Cauchy’s inequality, we deduce (2.24) for \( s \in (0, 1/2) \) and (2.28) for \( s \in (1/2, 3/2) \). \( \square \)

2.4. **Negative Besov estimates.** In this subsection, we will derive the evolution of the negative Besov norms of \((u, \theta, E, B)\). The argument is similar to the previous subsection.

**Lemma 2.11.** For \( s \in (0, 1/2) \), we have
\[
\frac{d}{dt} \left\|(u, \theta, E, B)\right\|_{B_{2s}^{-s}} \lesssim \left(\left\|\theta\right\|_{B_{2s}^{-s}}^2 + \left\|\theta\right\|_{B_{2s}^{-s}}^2 + \left\|(n, u, \theta)\right\|_{H^1}^2 + \left\|\nabla B\right\|_{H^1}^2\right) \left\|(u, \theta, E, B)\right\|_{B_{s}^{s}} + \left\|E\right\|_{H^2}^2;
\]
and for \( s \in (1/2, 3/2) \), we have
\[
\frac{d}{dt} \left\|(u, \theta, E, B)\right\|_{B_{2s}^{-s}} \lesssim \left(\left\|\theta\right\|_{B_{2s}^{-s}}^2 + \left\|\theta\right\|_{B_{2s}^{-s}}^2 + \left\|\nabla n\right\|_{H^{-s}}^2 + \left\|\nabla B\right\|_{H^1}^2 \left\|\theta\right\|_{B_{2s}^{-s}} + \left\|E\right\|_{H^2}^2\right) \left\|(u, \theta, E, B)\right\|_{B_{s}^{s}} + \left\|E\right\|_{H^2}^2.
\]

**Proof.** The \( \Delta_j \) energy estimates of (1.2) and (1.3) yield, with multiplication of \( 2^{-2js} \) and then taking the supremum over \( j \in \mathbb{Z} \),
\[
\frac{1}{2} \frac{d}{dt} \left(\left\|(u, E, B)\right\|_{B_{2s}^{-s}}^2 + \left\|\theta\right\|_{B_{2s}^{-s}}^2 \right) \lesssim \sup_{j \in \mathbb{Z}} 2^{-2js} \left( - \int \Delta_j \left( u \cdot \nabla u + \frac{\theta}{1 + n} \nabla n + u \times B \right) \cdot \Delta_j u \right)
\]
\[
+ \sup_{j \in \mathbb{Z}} 2^{-2js} \left( \int \Delta_j(nu) \cdot \Delta_j E - \int \Delta_j \left( \frac{\nabla n}{1 + n} \right) \cdot \Delta_j u \right)
\]
\[
+ \sup_{j \in \mathbb{Z}} 2^{-2js} \left( - \frac{3}{2} \int \Delta_j \left( u \cdot \nabla \theta + \frac{2}{3} \theta \text{div} u + \frac{1}{3} |u|^2 \right) \cdot \Delta_j \theta \right)
\]
\[
\lesssim \left\|u \cdot \nabla u + \theta \nabla n + nu \times B\right\|_{B_{2s}^{-s}} \left\|u\right\|_{B_{2s}^{-s}} + \left\|nu\right\|_{B_{2s}^{-s}} \left\|E\right\|_{B_{2s}^{-s}} + \left\|\nabla n\right\|_{B_{2s}^{-s}} \left\|u\right\|_{B_{2s}^{-s}} + \left\|u \cdot \nabla \theta + \theta \text{div} u + |u|^2\left\|\theta\right\|_{B_{2s}^{-s}}.
\]
Then the proof is exactly the same as the proof of Lemma 2.11 except that we should apply Lemma 2.5 instead to estimate the $\dot{B}_{2,\infty}^s$ norm. Note that we allow $s = 3/2$.

3. Proof of Theorems

3.1. Proof of Theorem 1.1 In this subsection, we will prove the unique global solution to the system (1.2), and the key point is that we only assume the $H^3$ norm of initial data is small.

Step 1. Global small $\mathcal{E}_3$ solution.

We first close the energy estimates at the $H^3$ level by assuming a priori that $\sqrt{\mathcal{E}_3(t)} \leq \delta$ is sufficiently small. Taking $k = 0, 1$ in (2.3) of Lemma 2.8 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{3} \left( \left\| \nabla^l(n, u, \theta, E, B) \right\|^2_{L^2} + \lambda \sum_{l=0}^{3} \left\| \nabla^l(u, \theta) \right\|^2_{L^2} \right) \leq \sqrt{\mathcal{E}_3 D_3} + \sqrt{D_3 \mathcal{E}_3} \lesssim \delta D_3. \tag{3.1}
$$

Taking $k = 0, 1$ in (2.7) of Lemma 2.9 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{3} \left( \left\| \nabla^l n \right\|^2_{L^2} + \left\| \nabla^l E \right\|^2_{L^2} + \left\| \nabla^l B \right\|^2_{L^2} \right) \lesssim \sum_{l=0}^{3} \left\| \nabla^l (u, \theta) \right\|^2_{L^2} + \delta^2 D_3. \tag{3.2}
$$

Since $\delta$ is small, we deduce from $3.2 \times \varepsilon + 3.1$ that there exists an instant energy functional $\tilde{\mathcal{E}}_3$ equivalent to $\mathcal{E}_3$ such that

$$
\frac{d}{dt} \tilde{\mathcal{E}}_3 + D_3 \leq 0.
$$

Integrating the inequality above directly in time, we obtain (1.3). By a standard continuity argument, we then close the a priori estimates if we assume at initial time that $\mathcal{E}_3(0) \leq \delta_0$ is sufficiently small. This concludes the unique global small $\mathcal{E}_3$ solution.

Step 2. Global $\mathcal{E}_N$ solution.

We shall prove this by an induction on $N \geq 3$. By (1.3), then (1.4) is valid for $N = 3$. Assume (1.4) holds for $N - 1$ (then now $N \geq 4$). Taking $k = 0, \ldots, N - 2$ in (2.3) of Lemma 2.8 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{N} \left\| \nabla^l (n, u, \theta, E, B) \right\|^2_{L^2} + \lambda \sum_{l=0}^{N} \left\| \nabla^l (u, \theta) \right\|^2_{L^2} \leq C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N} + \sqrt{D_{N-1} \mathcal{E}_N} \leq C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N} \tag{3.3}
$$

Here we have used the fact that $3 \leq \frac{N-2}{2} + 2 \leq N - 2 + 1$ since $N \geq 4$. Note that it is important that we have put the two first factors in (2.3) into the dissipation.

Taking $k = 0, \ldots, N - 2$ in (2.7) of Lemma 2.9 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{N-1} \left( \sum_{l=0}^{N-1} \left\| \nabla^l u \right\|^2_{L^2} + \sum_{l=0}^{N-1} \left\| \nabla^l E \right\|^2_{L^2} + \sum_{l=1}^{N-1} \left\| \nabla^l B \right\|^2_{L^2} \right) + \lambda \sum_{l=0}^{N-1} \left\| \nabla^l u \right\|^2_{L^2} + \lambda \sum_{l=0}^{N-1} \left\| \nabla^l E \right\|^2_{L^2} + \lambda \sum_{l=1}^{N-1} \left\| \nabla^l B \right\|^2_{L^2} \leq C \sum_{l=0}^{N} \left\| \nabla^l (u, \theta) \right\|^2_{L^2} + C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N}. \tag{3.4}
$$

We deduce from (3.3) $\times \varepsilon + 3.3$ that there exists an instant energy functional $\tilde{\mathcal{E}}_N$ equivalent to $\mathcal{E}_N$ such that, by Cauchy’s inequality,

$$
\frac{d}{dt} \tilde{\mathcal{E}}_N + D_N \leq C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N} \leq \varepsilon D_N + C_N \varepsilon D_{N-1} \mathcal{E}_N.
$$
This implies
\[
\frac{d}{dt} \mathcal{E}_N + \frac{1}{2} \mathcal{D}_N \leq C_N \mathcal{D}_{N-1} \mathcal{E}_N.
\]
We then use the standard Gronwall lemma and the induction hypothesis to deduce that
\[
\mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(\tau) \, d\tau \leq C \mathcal{E}_N(0) e^{C \int_0^t \mathcal{D}_{N-1}(\tau) \, d\tau} \leq C \mathcal{E}_N(0) e^{C N P_{N-1}(\mathcal{E}_{N-1}(0))}
\]
\[
\leq C \mathcal{E}_N(0) e^{C N P_{N-1}(\mathcal{E}_{N}(0))} \equiv P_N(\mathcal{E}_N(0)).
\]
This concludes the global \(\mathcal{E}_N\) estimate. The proof of Theorem 1.1 is completed. \(\square\)

3.2. Proof of Theorem 1.2. In this subsection, we will prove the various time decay rates of the unique global solution to the system (1.2) obtained in Theorem 1.1. Fix \(N \geq 5\). We need to assume that \(\mathcal{E}_N(0) \leq \delta_0 = \delta_0(N)\) is small. Then Theorem 1.1 implies that there exists a unique global \(\mathcal{E}_N\) solution, and \(\mathcal{E}_N(t) \leq P_N(\mathcal{E}_N(0)) \leq \delta_0\) is small for all time \(t\). Since now our \(\delta_0\) is relative small with respect to \(N\), we just ignore the \(N\) dependence of the constants in the energy estimates in the previous section.

**Step 1.** The \(H^{-s}\) or \(B^{-s}_{2,\infty}\) norm is preserved along time evolution. (1.5) and (1.6) indicate that the \(\dot{H}^{-s}\) and \(B^s_{2,\infty}\) norm of \((u, \theta, E, B)(t)\) is preserved along time evolution. First, we prove (1.5) by Lemma 2.10. However, we are not able to prove them for all \(s \in [0, 3/2]\) at this moment. We must distinguish the arguments by the value of \(s\). First, for \(s \in (0, 1/2]\), we choose \((3.5)\) in time, by (1.2) we obtain that for \(s \in (0, 1/2]\),
\[
\|u(t, \theta, E, B)(0)\|_{H^{-s}}^2 \lesssim \|(u(0, \theta_0, 0, B_0))\|_{H^{-s}}^2 + \int_0^t D_3(\tau) \left(1 + \|u(\theta, E, B)(\tau)\|_{H^{-s}}\right) \, d\tau
\]
\[
\leq C_0 \left[1 + \sup_{0 \leq \tau \leq t} \|u(\theta, E, B)(\tau)\|_{H^{-s}}\right].
\]
By Cauchy’s inequality, this together with (1.5) gives (1.5) for \(s \in [0, 1/2]\) and thus verifies (1.7) for \(s \in [0, 1/2]\). Next, we let \(s \in (1/2, 1)\). Observing that we have \((u_0, \theta_0, E_0, B_0) \in H^{-1/2}\) since \(H^{-s} \cap L^2 \subset H^{-s'}\) for any \(s' \in [0, s]\), we then deduce from what we have proved for (1.7) with \(s = 1/2\) that the following decay result holds:
\[
\left\|\nabla^k (u, \theta, E, B)(t)\right\|_{H^2} \leq C_0 (1 + t)^{-\frac{k+1}{2}} \quad \text{for} \ k = 0, 1.
\]
(3.5)

Here, since we have required \(N \geq 5\) and now \(s = 1/2\), so we have taken \(k = 1\) in (1.7). Thus by (3.5), (1.3) and Hölder’s inequality, we deduce from (2.27) that for \(s \in (1/2, 1)\),
\[
\|u(t, \theta, E, B)(t)\|_{H^{-s}}^2 \lesssim \|u(0, \theta_0, E_0, B_0)\|_{H^{-s}}^2 + \int_0^t D_3(\tau) \left(1 + \|u(\theta, E, B)(\tau)\|_{H^{-s}}\right) \, d\tau
\]
\[
+ \int_0^t \|B(\tau)\|_{L^2}^{s-1/2} \|\nabla B(\tau)\|_{L^2}^{3/2-s} \sqrt{D_3(\tau)} \|u(\theta, E, B)(\tau)\|_{H^{-s}} \, d\tau
\]
\[
\leq C_0 \left[1 + \left(1 + \int_0^t (1 + \tau)^{-2(1-s)/2} \, d\tau\right) \sup_{0 \leq \tau \leq t} \|u(\theta, E, B)(\tau)\|_{H^{-s}}\right]
\]
\[
\leq C_0 \left[1 + \sup_{0 \leq \tau \leq t} \|u(\theta, E, B)(\tau)\|_{H^{-s}}\right].
\]
(3.6)

Here we have used the fact \(s \in (1/2, 1)\) so that the time integral in (3.6) is finite. This gives (1.5) for \(s \in (1/2, 1)\) and thus verifies (1.7) for \(s \in (1/2, 1)\). Now let \(s \in [1, 3/2]\). We choose \(s_0\) such that \(s - 1/2 < s_0 < 1\). Hence, \((u_0, \theta_0, E_0, B_0) \in H^{-s_0}\). We then deduce from what we have proved for (1.7) with \(s = s_0\) that the following decay result holds:
\[
\left\|\nabla^k (u, \theta, E, B)(t)\right\|_{H^2} \leq C_0 (1 + t)^{-\frac{k+s_0}{2}} \quad \text{for} \ k = 0, 1.
\]
(3.7)

Here, since we have required \(N \geq 5\) and now \(s = s_0 < 1\), so we can have taken \(k = 1\) in (1.7). Thus by (3.7) and Hölder’s inequality, we deduce from (2.28) that for \(s \in [1, 3/2]\), similarly as
minimum derivative counts

similarly except that we use instead Lemma 2.11. Since

\[
\frac{d}{dt} \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, \theta, E, B) \right\|_{L^2}^2 + \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2.
\]

By Lemma 2.8, we have that for \( k = 0, \ldots, N - 2 \),

\[
\frac{d}{dt} \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, \theta, E, B) \right\|_{L^2}^2 + \lambda \sum_{l=k}^{k+2} \left\| \nabla^l (u, \theta) \right\|_{L^2}^2 \lesssim \sqrt{\delta_0 D_k^{k+2}} + \left\| (n, u) \right\|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2}.
\]

By Lemma 2.9, we have that for \( k = 0, \ldots, N - 2 \),

\[
\frac{d}{dt} \sum_{l=k}^{k+2} \int \nabla^l u \cdot \nabla^l n + \sum_{l=k}^{k+2} \int \nabla^l u \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla^k \nabla \times B 
\]

\[
+ \lambda \sum_{l=k}^{k+2} \left\| \nabla^l n \right\|_{L^2}^2 + \sum_{l=k}^{k+2} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \lesssim \sum_{l=k}^{k+2} \left\| \nabla^l (u, \theta) \right\|_{L^2}^2 + \delta_0 \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2.
\]

Since \( \delta_0 \) is small, we deduce from (3.12) \( \times \varepsilon + (3.11) \) that there exists an instant energy functional \( \tilde{\mathcal{E}}_{k+2} \) equivalent to \( \mathcal{E}_{k+2} \) such that

\[
\frac{d}{dt} \tilde{\mathcal{E}}_{k+2} + D_{k+2} \lesssim \left\| (n, u) \right\|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2}.
\]

Note that we can not absorb the right-hand side of (3.13) by the dissipation \( D_k^{k+2} \) since it does not contain \( \left\| \nabla^{k+2} (E, B) \right\|_{L^2}^2 \). We will distinct the arguments by the value of \( k \). If \( k = 0 \) or \( k = 1 \), we bound \( \left\| \nabla^{k+2} (E, B) \right\|_{L^2} \) by the energy. Then we have that for \( k = 0, 1 \),

\[
\frac{d}{dt} \tilde{\mathcal{E}}_{k+2} + D_{k+2} \lesssim \sqrt{D_{k+2}^{k+2}} \sqrt{\tilde{\mathcal{E}}_{k+2}} \lesssim \sqrt{\delta_0 D_{k+2}^{k+2}},
\]

which implies

\[
\frac{d}{dt} \tilde{\mathcal{E}}_{k+2} + D_{k+2} \lesssim 0.
\]

If \( k \geq 2 \), we have to bound \( \left\| \nabla^{k+2} (E, B) \right\|_{L^2} \) in term of \( \left\| \nabla^{k+1} (E, B) \right\|_{L^2} \) since \( \sqrt{D_{k+2}^{k+2}} \) can not control \( \left\| (n, u) \right\|_{L^\infty} \). The key point is to use the regularity interpolation method developed in
By Lemma 2.1, we have
\[
\|(n, u)\|_{L^\infty} \left\| \nabla^{k+2}(n, u) \right\|_{L^2} \leq \left\| (n, u) \right\|_{L^2} \left\| \nabla^{k+2}(E, B) \right\|_{L^2} \leq (k + 2)_{k} \left( \frac{1}{2} - \frac{3}{2k} \right) + \alpha \times \frac{3}{2k} \implies \alpha = \frac{5}{3}k + 1.
\]
Hence, for \( k \geq 2 \), if \( N \geq \frac{5}{3}k + 1 \leftrightarrow 2 \leq k \leq \frac{5}{3}(N - 1) \), then by (3.14), we deduce from (3.13) that
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \lesssim \sqrt{\tilde{E}_N^N D_k^{k+2}} \lesssim \sqrt{\delta_0 D_k^{k+2}},
\]
which allow us to arrive at that for any integer \( k \) with \( 0 \leq k \leq \frac{5}{3}(N - 1) \) (note that \( N - 2 \geq \frac{5}{3}(N - 1) \geq 2 \) since \( N \geq 5 \)), we have
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \leq 0. \tag{3.15}
\]
We now begin to derive the decay rate from (3.15). Using Lemma 2.6 and (1.5), we have that for \( s \geq 0 \) and \( k + s > 0 \),
\[
\left\| \nabla^k B \right\|_{L^2} \leq \|B\|_{H^{-s,2}} \left\| \nabla^{k+1} B \right\|_{L^2}^{\frac{1}{k+1}} \leq C_0 \left\| \nabla^{k+1} B \right\|_{L^2}^{\frac{1}{k+1}}.
\]
Similarly, using Lemma 2.7 and (1.0), we have that for \( s > 0 \) and \( k + s > 0 \),
\[
\left\| \nabla^k B \right\|_{L^2} \leq \|B\|_{H^{-s,\infty}} \left\| \nabla^{k+1} B \right\|_{L^2}^{\frac{1}{k+1}} \leq C_0 \left\| \nabla^{k+1} B \right\|_{L^2}^{\frac{1}{k+1}}.
\]
On the other hand, for \( k + 2 < N \), we have
\[
\left\| \nabla^{k+2}(E, B) \right\|_{L^2} \leq \left\| \nabla^{k+1}(E, B) \right\|_{L^2}^{\frac{N-k-2}{N-k-1}} \left\| \nabla^{N}(E, B) \right\|_{L^2}^{\frac{1}{N-k-1}} \leq C_0 \left\| \nabla^{k+1}(E, B) \right\|_{L^2}^{\frac{N-k-2}{N-k-1}}.
\]
Then we deduce from (3.16) that
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + \{E_k^{k+2}\}^{1+\vartheta} \leq 0,
\]
where \( \vartheta = \max \left\{ \frac{1}{k+s}, \frac{1}{N-k-2} \right\} \). Solving this inequality directly, we obtain in particular that
\[
E_k^{k+2}(t) \leq \left[ \left( E_k^{k+2}(0) \right)^{1+\vartheta} + \vartheta t \right]^{-\frac{1}{1+\vartheta}} \leq C_0(1 + t)^{-\min\{k+s, N-k-2\}}. \tag{3.16}
\]
Notice that (3.16) holds also for \( k + s = 0 \) or \( k + 2 = N \). So, if we want to obtain the optimal decay rate of the whole solution for the spatial derivatives of order \( k \), we only need to assume \( N \) large enough (for fixed \( k \) and \( s \)) so that \( k + s \leq N - k - 2 \). Thus we should require that
\[
N \geq \max \left\{ k + 2, \frac{5}{3}k + 1, 2k + 2 + s \right\} = 2k + 2 + s.
\]
This proves the optimal decay (1.7).

**Step 3. Further decay.**

We first prove (1.8) and (1.9). First, noticing that \(-n = \text{div} E\), by (1.7), if \( N \geq 2k + 4 + s \), then
\[
\left\| \nabla^{k} n(t) \right\|_{L^2} \leq \left\| \nabla^{k+1} E(t) \right\|_{L^2} \leq C_0(1 + t)^{-\frac{k+1}{k+2}}. \tag{3.17}
\]
Next, applying \( \nabla^{k} \) to (1.2), (1.23), (1.24), and then multiplying the resulting identities by \( \nabla^{k} u, \frac{3}{2} \nabla^{k} \theta, \nabla^{k} E \) respectively, summing up and integrating over \( \mathbb{R}^3 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \left\| \nabla^{k}(u, E) \right\|_{L^2}^2 + \frac{3}{2} \left\| \nabla^{k} \theta \right\|_{L^2}^2 + \left\| \nabla^{k} u \right\|_{L^2}^2 + \frac{3}{2} \left\| \nabla^{k} \theta \right\|_{L^2}^2.
\]
We thus complete the proof of (1.8). Notice that (1.9) now follows by (3.17) with the improved 

Substituting (2.18) with

where we required

Now we prove (1.10). Assuming

into (3.19), we may then have

Substituting (2.18) with \( l = k \) into (3.19), we may then have

Since \( \varepsilon \) is small, we deduce from (3.20) \( \times \varepsilon + 3.18 \) that there exists \( F_k(t) \) equivalent to

\[ \| \nabla^k (u, \theta, E)(t) \|_{L^2}^2 \] such that, by Cauchy’s inequality, Lemma (2.22), (2.30), (2.6), (1.7) and (3.17),

\[ \frac{d}{dt} F_k(t) + F_k(t) \lesssim \| \nabla^{k+1} \theta \|_{L^2}^2 + \| \nabla^k B \|_{L^2}^2 + \| \nabla^k \left( u \cdot \nabla u + \frac{1 + \theta}{1 + n} \nabla n + u \times B \right) \|_{L^2}^2 \\
+ \| \nabla^k \left( u \cdot \nabla \theta + \theta \text{div} u + |u|^2 \right) \|_{L^2}^2 + \| \nabla^k (nu) \|_{L^2}^2 \\
\lesssim \| \nabla^{k+1} (n, \theta, B) \|_{L^2}^2 + \| \nabla^k n \|_{L^2}^2 + \| \nabla\theta \|_{L^2}^2 + \| \theta \|_{L^\infty}^2 \| \nabla^k \left( n, u, \theta \right) \|_{L^\infty}^2 \| \nabla^{k+1} (n, u, \theta) \|_{L^2}^2 \\
\lesssim C_0(1 + t)^{- (k + 1 + s)} , \]

where we required \( N \geq 2k + 4 + s \). Applying the standard Gronwall lemma to (3.21), we obtain

\[ F_k(t) \leq F_k(0) e^{-t} + C_0 \int_0^t e^{-(t - \tau)} (1 + \tau)^{- (k + 1 + s)} d\tau \lesssim C_0(1 + t)^{- (k + 1 + s)} . \]

This implies

\[ \| \nabla^k (u, \theta, E)(t) \|_{L^2} \lesssim \sqrt{F_k(t)} \lesssim C_0(1 + t)^{- \frac{k + 1 + s}{2}} . \]

We thus complete the proof of (1.8). Notice that (1.9) now follows by (3.17) with the improved decay rate of \( E \) in (1.8), just requiring \( N \geq 2k + 6 + s \).

Now we prove (1.10). Assuming \( B_\infty = 0 \), then we can extract the following system from (1.2) \( 1 \sim 1.2 \), denoting \( \psi = \text{div} u, \)

\[ \begin{aligned}
\partial_t n + \psi &= -u \cdot \nabla n - n \text{div} u, \\
\partial_t \psi + \psi - n &= -\Delta \theta - \text{div} \left( u \cdot \nabla u + \frac{1 + \theta}{1 + n} \nabla n + u \times B \right).
\end{aligned} \]

(3.22)

Applying \( \nabla^k \) to (3.22) and then multiplying the resulting identities by \( \nabla^k n, \nabla^k \psi \), respectively, summing up and integrating over \( \mathbb{R}^3 \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int \| \nabla^k n \|_{L^2}^2 + \| \nabla^k \psi \|_{L^2}^2 + \| \nabla^k \psi \|_{L^2}^2 = -\int \nabla^k (u \cdot \nabla n + n \text{div} u) \nabla^k n - \int \nabla^k \Delta \theta \nabla^k \psi \]
Applying $\nabla^k$ to (3.22)_3 and then multiplying by $-\nabla^k n$, as before integrating by parts over $t$ and $x$ variables and using the equation (3.22)_1, we may obtain

$$- \frac{d}{dt} \int \nabla^k \psi \nabla^k n + \|\nabla^k n\|_{L^2}^2 = \|\nabla^k \psi\|_{L^2}^2 + \int \nabla^k n \nabla^k \psi + \int \nabla^k (u \cdot \nabla n + n \text{div} u) \nabla^k \psi$$

$$= \int \nabla^k \left[ \Delta \theta + \text{div} \left( u \cdot \nabla u + \frac{1 + \theta}{1 + n} \nabla n + u \times B \right) \right] \nabla^k n.$$  

(3.24)

Since $\varepsilon$ is small, we deduce from (3.24) $\times \varepsilon + (3.23)$ that there exists $G_k(t)$ equivalent to $\|\nabla^k(n, \psi)\|_{L^2}^2$ such that, by Cauchy’s inequality,

$$\frac{d}{dt} G_k(t) + G_k(t) \lesssim \|\nabla^{k+2} \theta\|_{L^2}^2 + \|\nabla^{k+1} (u \cdot \nabla u)\|_{L^2}^2 + \|\nabla^k \left( \frac{1 + \theta}{1 + n} \nabla n \right)\|_{L^2}^2$$

$$+ \|\nabla^{k+1} (u \times B)\|_{L^2}^2 + \|\nabla^k (u \cdot \nabla n)\|_{L^2}^2 + \|\nabla^k (\text{div} u)\|_{L^2}^2.$$  

(3.25)

By Lemma 2.2, (2.3) and Cauchy’s inequality, we obtain

$$\|\nabla^{k+1} \left( \frac{1 + \theta}{1 + n} \nabla n \right)\|_{L^2}^2 \lesssim \left[ \|\nabla^{k+1} \frac{1 + \theta}{1 + n} \nabla n\|_{L^2}^2 + \left\| \frac{1 + \theta}{1 + n} \nabla^{k+2} n \right\|_{L^2}^2 \right.$$  

$$\lesssim \|\nabla^{k+2} n\|_{L^2}^2 + \|\nabla (n, \theta)\|_{L^\infty} \|\nabla^{k+1} (n, \theta)\|_{L^2}^2,$$

and

$$\|\nabla^{k+1} (u \times B)\|_{L^2}^2 = \|u \times \nabla^{k+1} B\|_{L^2}^2 + \left\| \nabla^{k+1} u \times B \right\|_{L^2}^2 + \left\| \nabla^k (u \times B) \right\|_{L^2}^2$$

$$\lesssim \|u\|_{L^\infty}^2 \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 \|B\|_{L^\infty}^2.$$  

The other nonlinear terms on the right-hand side of (3.25) can be estimated similarly. Hence, we deduce from (3.25) that, by (1.4)–(1.9),

$$\frac{d}{dt} G_k(t) + G_k(t) \lesssim \|\nabla^{k+2} (n, \theta)\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|\nabla^{k+1} B\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2$$

$$+ \|\nabla (n, \theta)\|_{L^\infty}^2 \|\nabla^{k+1} (n, \theta)\|_{L^2}^2 \lesssim C_0 \left( (1 + t)^{-k+3+s} + (1 + t)^{-k+7/2+2s} + (1 + t)^{-k+11/2+2s} \right)$$

$$\leq C_0 \left( (1 + t)^{-k+3+s} \right),$$  

(3.26)

where we required $N \geq 2k + 8 + s$. Applying the Gronwall lemma to (3.26) again, we obtain

$$G_k(t) \leq G_k(0) e^{-t} + C_0 \int_0^t e^{-t - \tau} (1 + \tau)^{-k+3+s} d\tau \leq C_0 (1 + t)^{-k+3+s}.$$  

This implies

$$\|\nabla^k (n, \psi)(t)\|_{L^2} \lesssim \sqrt{G_k(t)} \leq C_0 (1 + t)^{-\frac{k+3+s}{2}}.$$  

(3.27)

Now we consider the following system which consists of (3.22) and (1.2)_3:

$$\begin{cases} 
\partial_t n + \psi = -u \cdot \nabla n - n \text{div} u, \\
\partial_t \psi + \psi - n = -\Delta \theta - \text{div} \left( u \cdot \nabla u + \frac{1 + \theta}{1 + n} \nabla n + u \times B \right), \\
\partial_t \theta + \theta = -u \cdot \nabla \theta - \frac{2}{3} (1 + \theta) \text{div} u + \frac{1}{3} |u|^2.
\end{cases}$$  

(3.28)
First, we have the standard energy identity for the system (3.28)

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^k n|^2 + |\nabla^k \psi|^2 + |\nabla^k \theta|^2 + \left\| \nabla^k (\psi, \theta) \right\|^2_{L^2} = - \int \nabla^k (u \cdot \nabla n + n \text{div} u) \nabla^k n - \int \nabla^k \Delta \theta \nabla^k \psi - \frac{2}{3} \int \nabla^k \psi \nabla^k \theta
\]

\[- \int \nabla^k \text{div} \left( u \cdot \nabla u + \frac{1 + \theta}{1 + n} \nabla n + u \times B \right) \nabla^k \psi \]

\[- \int \nabla^k \left( u \cdot \nabla \theta + \frac{2}{3} \theta \text{div} u - \frac{1}{3} |u|^2 \right) \nabla^k \theta. \tag{3.29}
\]

Since \( \varepsilon \) is small, we deduce from (3.21) \( \times \varepsilon \) + (3.29) that there exists \( \mathcal{H}_k(t) \) equivalent to \( \| \nabla^k (n, \psi, \theta) \|^2_{L^2} \) such that, by Cauchy’s inequality and (3.27),

\[
\frac{d}{dt} \mathcal{H}_k(t) + \mathcal{H}_k(t) \lesssim \left\| \nabla^{k+2} \theta \right\|^2_{L^2} + \left\| \nabla^{k+1} (u \cdot \nabla u) \right\|^2_{L^2} + \left\| \nabla^{k+1} \left( \frac{1 + \theta}{1 + n} \nabla n \right) \right\|^2_{L^2} + \left\| \nabla^{k+1} (u \cdot \nabla n) \right\|^2_{L^2} + \left\| \nabla^k (\text{div} u) \right\|^2_{L^2} + \left\| \nabla^k (u \cdot \nabla \theta) \right\|^2_{L^2} + \left\| \nabla^k (\text{div} u) \right\|^2_{L^2} + \left\| \nabla^k (\theta \text{div} u) \right\|^2_{L^2}
\]

\[
\leq C_0 \left( (1 + t)^{-(k+3+s)} + (1 + t)^{-(k+7/2+2s)} + (1 + t)^{-(k+11/2+2s)} \right)
\]

\[
\leq C_0(1 + t)^{-(k+3+s)}, \tag{3.30}
\]

where we required \( N \geq 2k + 8 + s \). Applying the Gronwall lemma to (3.30) again, we obtain

\[
\mathcal{H}_k(t) \leq \mathcal{H}_k(0) e^{-t} + C_0 \int_0^t e^{-(t-\tau)(1 + \tau)^{-(k+3+s)}} d\tau \leq C_0(1 + t)^{-(k+3+s)}.
\]

This implies

\[
\left\| \nabla^k (n, \psi, \theta)(t) \right\|^2_{L^2} \lesssim \sqrt{\mathcal{H}_k(t)} \leq C_0(1 + t)^{-\frac{k+3+s}{2}}. \tag{3.31}
\]

If required \( N \geq 2k + 12 + s \), then by (3.31), we have

\[
\left\| \nabla^{k+2} (n, \theta)(t) \right\|^2_{L^2} \lesssim C_0(1 + t)^{-\frac{k+5+s}{2}}.
\]

Having obtained such faster decay, we can then improve (3.26) to be

\[
\frac{d}{dt} \mathcal{G}_k(t) + \mathcal{G}_k(t) \leq C_0 \left( (1 + t)^{-(k+5+s)} + (1 + t)^{-(k+7/2+2s)} \right) \leq C_0(1 + t)^{-(k+7/2+2s)}.
\]

Applying the Gronwall lemma again, we obtain

\[
\left\| \nabla^k (n, \psi)(t) \right\|^2_{L^2} \lesssim \sqrt{\mathcal{G}_k(t)} \leq C_0(1 + t)^{-(k/2+7/4+s)}.
\]

In light of the faster decay for \( \nabla^k \psi \), we can then improve (3.30) to be

\[
\frac{d}{dt} \mathcal{H}_k(t) + \mathcal{H}_k(t) \leq C_0 \left( (1 + t)^{-(k+5+s)} + (1 + t)^{-(k+7/2+2s)} \right) \leq C_0(1 + t)^{-(k+7/2+2s)}.
\]

Applying the Gronwall lemma again, we obtain

\[
\left\| \nabla^k (n, \theta, \psi)(t) \right\|^2_{L^2} \lesssim \sqrt{\mathcal{H}_k(t)} \leq C_0(1 + t)^{-(k/2+7/4+s)}.
\]

We thus complete the proof of (1.10). The proof of Theorem 1.2 is completed.
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