The Entropy Method in Large Deviation Theory

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November 4, 2022

Abstract

This paper illustrates the power of the entropy method in addressing problems from large deviation theory. We provide and review entropy proofs for most fundamental results in large deviation theory, including Cramer’s theorem, the Gärtner–Ellis theorem, and Sanov’s theorem. Moreover, by the entropy method, we also strengthen Sanov’s theorem to the strong version.

1 Introduction

Information theory was fundamentally established by the works of Harry Nyquist and Ralph Hartley, in the 1920s, and Claude Shannon in the 1940s, which successfully addressed the theoretic limit of information communication. Since then, information theory gradually influenced other branches of mathematics. For example, the notion of (information) entropy introduced by Shannon after he consulted von Neumann, the father of the computer, was widely employed in probability theory, functional analysis, ergodic theory, graph theory and so on.

Information theory concerns realizing reliable and efficient communication of a source over a noisy channel. Entropy and mutual information were introduced by Shannon to determine the theoretical limit of such kind of communication. Entropy and mutual information are special cases of a more general concept, known as the relative entropy. For a nonnegative $\sigma$-finite measure $\mu$ and a probability measure $Q$ defined on the same space such that

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\( Q \ll \mu, \) the relative entropy of \( Q \) with respect to (w.r.t.) \( \mu \) is
\[
D(Q\|\mu) = \int \log \left( \frac{dQ}{d\mu} \right) dQ
\]
which is the expectation of the information density \( \tau_{Q\|\mu} = \log \left( \frac{dQ}{d\mu} \right) \). When \( \mu \) is the counting measure on a countable set, \( H(X) = -D(Q\|\mu) \) is the Shannon entropy of \( X \sim Q \). When \( \mu \) is the Lebesgue measure on an Euclidean space, \( h(X) = -D(Q\|\mu) \) is the differential entropy of \( X \sim Q \). The relative entropy was also widely used in the case that \( \mu \) is a probability measure. In this case, for two probability measures \( Q \ll P \),
\[
\tag{1}
D(Q\|P) = \int \log \left( \frac{dQ}{dP} \right) dQ.
\]
If \( Q \) is not absolutely continuous w.r.t. \( P \), then \( D(Q\|P) := +\infty \). The mutual information between \( X \) and \( Y \) with \( (X,Y) \sim P_{XY} \) is \( I(X;Y) = D(P_{XY}\|P_X \otimes P_Y) \). So, the relative entropy is a general concept incorporating the entropy and mutual information. The relative variance entropy of \( Q \) with respect to (w.r.t.) \( \mu \) is
\[
V(Q\|\mu) = \text{Var}_Q \left[ \tau_{Q\|\mu}(X) \right] = \text{Var}_Q \left[ \log \frac{dQ}{d\mu}(X) \right].
\]

1.1 Organization
In Section 2, we introduce the concept of information projection (or shortly, I-projection), which will be then used to characterize the asymptotic exponent in the large deviation theory. In Section 3, we prove several versions of Cramer’s theorem by the entropy method. In particular, the entropy proof for the simple version of Cramer’s theorem in Section 3.1 is considered as a paradigm to illustrate the power of the entropy method. Still using the entropy method, we next generalize the Cramer’s theorem to the non-i.i.d. setting, and obtain the Gärtner–Ellis theorem in Section 4. We then focus on another fundamental theorem in large deviation theory—Sanov’s theorem in Section 5. The entropy proof for this theorem is given in Section 5.1, and the entropy proof for the strong version is given in Section 5.2. As an application, the entropy proof of Gibbs conditioning principle is introduced in Section 6.

1.2 Notations
Let \( \mathcal{X} \) be a Hausdorff topological space, and \( \mathcal{B}_\mathcal{X} \) is the Borel \( \sigma \)-algebra on \( \mathcal{X} \). Let \( P_\mathcal{X} \) (or shortly \( P \)) be a probability measure on \( \mathcal{X} \). We also use
The probability measures $P_X, Q_X, R_X$ can be thought as the push-forward measures (or the distributions) induced jointly by the same measurable function $X$ (random variable) from an underlying measurable space to $\mathcal{X}$ and by different probability measures $P, Q, R$ defined on the underlying measurable space. Without loss of generality, we assume that $X$ is the identity map, and $P, Q, R$ are the same as $P_X, Q_X, R_X$. So, $P_X, Q_X, R_X$ could be independently specified to arbitrary probability measures. We say that all probability measures induced by the underlying measure $P$, together with the corresponding measurable spaces, constitute the $P$-system. So, $P_X$ is in fact the distribution of the random variable $X$ in the $P$-system, where the letter "$P$" in the notation $P_X$ refers to the $P$-system and the subscript "$X$" refers to the random variable. When emphasizing the random variables, we write $X \sim P_X$ to indicate that $X$ follows the distribution $P_X$ in the $P$-system.

We use $P_X^n$ to denote the $n$-fold product of $P_X$. For a probability measure $P_X$ and a regular conditional distribution (transition probability or Markov kernel) $P_{Y|X}$ from $\mathcal{X}$ to $\mathcal{Y}$, we denote $P_X P_{Y|X}$ as the joint probability measure induced by $P_X$ and $P_{Y|X}$. For a distribution $P_X$ on $\mathcal{X}$ and a measurable subset $A \subseteq \mathcal{X}$, $P_X(\cdot|A)$ denotes the conditional probability measure given $A$. For brevity, we write $P_X(x) := P_X(\{x\}), x \in \mathcal{X}$. In particular, if $X \sim P_X$ is discrete, the restriction of $P_X$ to the set of singletons corresponds to the probability mass function of $X$ in the $P$-system. We use $Q_X \ll P_X$ to denote that the distribution $Q_X$ is absolutely continuous w.r.t. $P_X$. We use $X^n$ to denote a random vector $(X_1, X_2, \ldots, X_n)$ taking values on $(\mathcal{X}^n, \mathcal{B}_\mathcal{X}^n)$, and use $x^n := (x_1, x_2, \ldots, x_n)$ to denote its realization. For an $n$-length vector $x^n$, we use $x^i$ to denote the subvector consisting of the first $i$ components of $x^n$, and $x^n_{i+1}$ to denote the subvector consisting of the last $n - i$ components. For a probability measure $P_{X^n}$ on $\mathcal{X}^n$, we use $P_{X_k|X^{k-1}}$ to denote the regular conditional distribution of $X_k$ given $X^{k-1}$ induced by $P_{X^n}$. For a measurable function $f : \mathcal{X} \to \mathbb{R}$, sometimes we adopt the notation $P_X(f) := \mathbb{E}_{P_X}[X] := \int_{\mathcal{X}} f \, dP_X$. For a conditional probability measure $P_{X|Y}$, define the conditional expectation operator induced by $P_{X|Y}$ as $P_{X|Y}(f)(y) := \int f dP_{X|Y = y}$ for any measurable function $f : (\mathcal{X}, \mathcal{B}_\mathcal{X}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ if the integral is well-defined for every $y$. When the random variable $X$ is clear from the texture, we briefly denote $P_X$ as $P$. For example, we denote the expectation $\mathbb{E}_{P_X}[X]$ as $\mathbb{E}_{P}[X]$.

The relative entropy is defined in (1). The conditional relative entropy is defined as

$$D(Q_X|W \| P_X|W Q_W) = D(Q_X|W Q_W \| P_X|W Q_W).$$
Given \( n \geq 1 \), the empirical measure for a sequence \( x^n \in \mathcal{X}^n \) is

\[
\mathbb{L}_{x^n} := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}
\]

where \( \delta_x \) is Dirac mass at the point \( x \in \mathcal{X} \).

Denote \( d_{\mathcal{P}}(P, Q) = \inf \{ \delta > 0 : P(A) \leq Q(A_{\delta}) + \delta, \forall \text{ closed } A \subseteq \mathcal{Z} \} \) with \( A_{\delta} := \bigcup_{z \in A} \{ z' \in \mathcal{Z} : d(z, z') < \delta \} \), where \( \mathcal{Z} \) is \( \mathcal{X} \) or \( \mathcal{Y} \). Here, \( d_{\mathcal{P}} \) is known as the Lévy–Prokhorov metric on \( \mathcal{P}(\mathcal{Z}) \) which is compatible with the weak topology on \( \mathcal{P}(\mathcal{Z}) \). We use \( \overline{A}, A^o, \) and \( A^c := \mathcal{Z} \setminus A \) to respectively denote the closure, interior, and complement of the set \( A \subseteq \mathcal{Z} \). Denote the sublevel set of the relative entropy (or the divergence “ball”) as \( D_{\epsilon}(P_X) := \{ Q_X : D(Q_X \| P_X) \leq \epsilon \} \) for \( \epsilon \geq 0 \). The Lévy–Prokhorov metric, the TV distance, and the relative entropy admit the following relation: For any \( Q_X, P_X \),

\[
\sqrt{2D(Q_X \| P_X)} \geq \|Q_X - P_X\|_{TV} \geq d_{\mathcal{P}}(Q_X, P_X),
\]

which implies for \( \epsilon \geq 0 \),

\[
D_{\sqrt{2\epsilon}}(P_X) \subseteq B_{\epsilon}(P_X).
\]

The first inequality in (2) is known as Pinsker’s inequality, and the second inequality follows by definition [7].

We denote \( \inf \emptyset := +\infty, \sup \emptyset := -\infty \). Given two positive sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \), we say that \( a_n \preceq b_n \) if \( \lim_{n \to \infty} a_n/b_n \leq 1 \), and \( a_n \sim b_n \) if \( a_n \preceq b_n \) and \( b_n \preceq a_n \).

## 2 Preliminaries: Information Projection

For a set \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \), define

\[
D(\Gamma \| P) := \inf_{R \in \Gamma} D(R \| P).
\]

Any probability measure \( R^* \) attaining the infimum above is called the information projection or I-projection of \( P \) on \( \Gamma \).

**Lemma 1.** If \( \Gamma \) is closed in the weak topology, then the I-projection exists.
Proof. This lemma follows by the lower semicontinuity of \( R \mapsto D(R\|P) \) and the compactness of the sublevel sets of \( R \mapsto D(R\|P) \) under the weak topology.

If \( \Gamma \) is convex and such \( R^* \) exists, the convexity of \( \Gamma \) guarantees its uniqueness since \( D(R\|P) \) is strictly convex in \( R \). Another important property of the I-projection is the following equivalence.

**Theorem 1.** [3, Theorem 2.2] A probability measure \( Q \in \Gamma \cap D_\infty(P) \) is the I-projection of \( P \) on the convex set \( \Gamma \) of probability measures iff every \( R \in \Gamma \) satisfies

\[
D(R\|P) \geq D(R\|Q) + D(Q\|P). \tag{3}
\]

The proof of this theorem is based on differentiating \( D(P_\lambda\|P) \) w.r.t. \( \lambda \) at \( \lambda = 0 \), where \( P_\lambda = \lambda P + (1 - \lambda)Q \) with \( \lambda \in [0, 1] \); see details in [3, Theorem 2.2]. The inequality (3) can be written as

\[
\int \left( \log \frac{dQ}{dP} \right) dR \geq D(Q\|P),
\]

or

\[
\int \left( \log \frac{dQ}{dP} \right) d(R - Q) \geq 0.
\]

So, for the I-projection \( Q \), the set

\[
\mathcal{H} := \left\{ R : \int \left( \log \frac{dQ}{dP} \right) d(R - Q) = 0 \right\}
\]

constitutes a “supporting hyperplane” of \( \Gamma \). Moreover, the function \( 1 + \log \frac{dQ}{dP} \) can be seen as the “gradient” of \( Q \mapsto D(Q\|P) \), and \( \mathcal{H} \) is the “tangent hyperplane” of \( Q \mapsto D(Q\|P) \). Analogously to Euclidean spaces, considering \( D(Q\|P) \) as a “distance”, it holds that \( Q \) is the projection of \( P \) to \( \mathcal{H} \), and

\[
D(R\|P) = D(R\|Q) + D(Q\|P), \quad \forall R \in \mathcal{H}.
\]

As a consequence of Theorem 1, when \( \Gamma \) is specified by linear constraints, the I-projection can be written out explicitly. To illustrate this point, we let \( f : \mathcal{X} \to \mathbb{R} \) be a measurable function. Consider the following I-projection problem:

\[
\gamma(\alpha) := \inf_{Q:Q(f)=\alpha} D(Q\|P).
\]

To give the I-projection for \( \gamma(\alpha) \), we first introduce the following lemma, which can be easily verified by definition.
**Lemma 2.** For a measurable function \( f \) and \( \lambda \in \mathbb{R} \) such that \( P(e^{\lambda f}) < +\infty \), define a probability measure \( Q_\lambda \) with density 
\[
\frac{dQ_\lambda}{dP} = \frac{e^{\lambda f}}{P(e^{\lambda f})},
\]
then for any \( R \),
\[
D(R\|P) - D(Q_\lambda\|P) = D(R\|Q_\lambda) + \lambda (R(f) - Q_\lambda(f)) 
\geq \lambda (R(f) - Q_\lambda(f)).
\]

If there is \( \lambda^* \in \mathbb{R} \) such that \( P(e^{\lambda^* f}) < +\infty \) and \( Q_{\lambda^*}(f) = \alpha \), then by the lemma above, we have for any \( R \in \Gamma \),
\[
D(R\|P) = D(R\|Q_{\lambda^*}) + D(Q_{\lambda^*}\|P),
\]
which is an analogue of Pythagoras’ theorem for information distance. Further, by Theorem 1, \( Q_{\lambda^*} \) is the I-projection for \( \gamma(\alpha) \). For this case,
\[
\gamma(\alpha) = \lambda^* \alpha - \log P(e^{\lambda^* f}).
\]
In fact, the RHS above is equal to
\[
\gamma^*(\alpha) := \sup_{\lambda \in \mathbb{R}} \lambda \alpha - \log P(e^{\lambda f}). \tag{4}
\]
In other words, when \( \lambda^* \) exists, it attains the supremum in (4). We next show that \( \gamma(\alpha) = \gamma^*(\alpha) \) always holds, even when \( \lambda^* \) does not exist.

**Theorem 2** (Duality for the I-Projection (Equality Constraint)). *It holds that for any \( \alpha \in \mathbb{R} \),
\[
\gamma(\alpha) = \gamma^*(\alpha).
\]

The proof of this theorem is deferred to Section 7, since it is long and technical.

We now consider the inequality constraint. Define
\[
\gamma_+(\alpha) := \inf_{Q:Q(f) \geq \alpha} D(Q\|P),
\]
and
\[
\gamma^*_+(\alpha) := \sup_{\lambda \geq 0} \lambda \alpha - \log P(e^{\lambda f}).
\]
Following proof steps similar to those of Theorem 2, we obtain the following duality.

**Theorem 3** (Duality for the I-Projection (Inequality Constraint)). *Let \( f : \mathcal{X} \to \mathbb{R} \) be a measurable function. Then, it holds that for any \( \alpha \in \mathbb{R} \),
\[
\gamma_+(\alpha) = \gamma^*_+(\alpha).
\]
3 Cramer’s Theorems

3.1 Cramer’s Theorem (Simple Version)

The entropy method is a mathematical method involving relative entropies (or entropies). To illustrate the power of the entropy method, we now provide an entropy proof of Cramer’s theorem. Cramer’s theorem is the most fundamental result in the large deviation theory, which characterizes the decay exponent of the tail probability for normalized sum of i.i.d. real-valued random variables.

Theorem 4 (Cramer’s Theorem). Let $X_i \sim P, i = 1, 2, \ldots$ be i.i.d. real-valued random variables. For any $\alpha \in \mathbb{R}$,

$$\lim_{n \to \infty} -\frac{1}{n} \log P\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \geq \alpha \right\} = \gamma(\alpha),$$

where

$$\gamma_+(\alpha) := \inf_{Q: \mathbb{E}_Q[X] \geq \alpha} D(Q\| P).$$

Here $\gamma$ is known as the rate function, which admits the following dual formula: For any $\alpha$,

$$\gamma_+(\alpha) = \gamma_+^*(\alpha) := \sup_{\lambda \geq 0} \lambda \alpha - \log \mathbb{E}_P[e^{\lambda X}].$$

Denote $\alpha_{\text{max}} := \sup\{\alpha : \gamma_+(\alpha) < +\infty\}$. Since $\gamma_+$ is convex, nonnegative, and nondecreasing on $\mathbb{R}$, it holds that $\gamma_+(\alpha)$ is continuous in $\alpha < \alpha_{\text{max}}$. It is easy to see that $\alpha_{\text{max}} = \text{ess}\sup P X$ and $\gamma_+(\alpha_{\text{max}}) = -\log P\{\alpha_{\text{max}}\}$.

Proof. Proof of “$\geq$”: To this end, we can denote $A_n := \{ x^n : \frac{1}{n} \sum_{i=1}^{n} x_i \geq \alpha \}$. Then, the probability we are going to estimate is $P^{\otimes n}(A_n)$. Define an auxiliary probability measure $Q_{X^n} := P^{\otimes n}(\cdot|A_n)$. It is easily verified that

$$-\frac{1}{n} \log P^{\otimes n}(A_n) = \frac{1}{n} D(Q_{X^n}\| P^{\otimes n}) \geq \frac{1}{n} \sum_{i=1}^{n} D(Q_{X_i}\| P) = D(Q_{X^n\| K}\| P_{K}) \geq D(Q_X\| P),$$

(5)
where the first inequality follows by the superadditivity of relative entropy, the auxiliary random variable \( K \sim Q_K := \text{Unif}[n] \) denotes a random time index independent of \( X^n \) (under both \( P, Q \)), and in the last line, \( X := X_K \). On the other hand, since \( Q_X^n \) is concentrated on \( A_n \), we have \( \mathbb{E}_Q \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] \geq \alpha \), i.e., \( \mathbb{E}_Q [X] \geq \alpha \). We hence have

\[
-\frac{1}{n} \log P^{\otimes n}(A_n) \geq \gamma_+(\alpha).
\]

Proof of “\( \leq \)”: It is verified that

\[
-\frac{1}{n} \log P^{\otimes n}(A_n) = \inf_{Q_X^n : Q_X^n(A_n) = 1} \frac{1}{n} D(Q_X^n \| P^{\otimes n}),
\]

where the optimal \( Q_X^n \) attaining the infimum is \( P^{\otimes n}(\cdot | A_n) \) since for any \( Q_X^n \) concentrated on \( A_n \),

\[
D(Q_X^n \| P^{\otimes n}) = D(Q_X^n \| P^{\otimes n}(\cdot | A_n)) - \frac{1}{n} \log P^{\otimes n}(A_n).
\]

Given any \( Q_X \) such that \( \mathbb{E}_Q [X] > \alpha \), denote auxiliary probability measures \( R_X^n := Q_X^{\otimes n}(\cdot | A_n) \) and \( \bar{R}_X^n := Q_X^{\otimes n}(\cdot | A_n^c) \). Denote \( p_n := Q_X^{\otimes n}(A_n) \), which converges to 1 by the law of large numbers (LLN). Furthermore,

\[
nD(Q_X \| P) = p_n D(R_X^n \| P^{\otimes n}) + (1 - p_n) D(\bar{R}_X^n \| P^{\otimes n}) - H_2(p_n)
\geq p_n D(R_X^n \| P^{\otimes n}) - H_2(p_n),
\]

where \( H_2 : t \in [0,1] \mapsto -t \log_2 t - (1 - t) \log_2 (1 - t) \) is the binary entropy function and the equality above can be easily verified by definition. That is,

\[
D(R_X^n \| P^{\otimes n}) \leq \frac{nD(Q_X \| P) + H_2(p_n)}{p_n}
\]

Since \( R_X^n \) is a feasible solution to the infimization in (6),

\[
\limsup_{n \to \infty} \frac{-1}{n} \log P^{\otimes n}(A_n) \leq \limsup_{n \to \infty} \frac{nD(Q_X \| P) + H_2(p_n)}{np_n}
= D(Q_X \| P).
\]

Since \( Q_X \) satisfying \( \mathbb{E}_Q [X] > \alpha \) is arbitrary,

\[
\limsup_{n \to \infty} \frac{-1}{n} \log P^{\otimes n}(A_n) \leq \inf_{Q_X : \mathbb{E}_Q [X] > \alpha} D(Q_X \| P)
\leq \lim_{\delta \downarrow 0} \gamma_+(\alpha + \delta).
\]
By the continuity, for \( \alpha \neq \alpha_{\text{max}} \), \( \limsup_{n \to \infty} -\frac{1}{n} \log P^\otimes n(A_n) \leq \gamma_+ (\alpha) \), and hence \( \lim_{n \to \infty} -\frac{1}{n} \log P^\otimes n(A_n) = \gamma_+ (\alpha) \).

For \( \alpha = \alpha_{\text{max}} \), \( \gamma_+ (\alpha_{\text{max}}) = -\log P_X \{ \alpha_{\text{max}} \} \). For this case, \( P^\otimes n(A_n) \geq P \{ \alpha_{\text{max}} \}^n \), and hence \( \lim_{n \to \infty} -\frac{1}{n} \log P^\otimes n(A_n) = \gamma_+ (\alpha) \) still holds.

In fact, common proofs for Cramer’s theorem, e.g., the one given in [6], are from the dual perspective, for which the resultant rate function is expressed by the Fenchel–Legendre transform of the logarithmic moment generating function. However, the entropy proof here is from the primal perspective, for which the resultant rate function is characterized by an information projection problem (the minimization of the relative entropy over a convex set). Furthermore, the entropy proof here is short, since we do not need to divide the proof into many cases, e.g., whether the expectation of \( X_i \) exists. The entropy proof neither requires the technique of change of measure, since the auxiliary probability measure \( Q_X \) in the proof of “\( \leq \)” is chosen independently of the original probability measure \( P^\otimes n \), which plays the role of changing of measure.

Generally speaking, the entropy method typically consists of three steps:

1. First, introduce auxiliary probability measures (or auxiliary random variables),
2. Then, express the problem in terms of relative entropies of these auxiliary probability measures,
3. Lastly, derive bounds by using properties of relative entropies.

Moreover, for the first step, in a probability measure space, if the extreme problem that we consider is about sets, then the auxiliary measures are usually defined as conditional probability measures given these sets; if the extreme problem is about nonnegative integrable functions, then the auxiliary measures are usually defined as probability measures with densities proportional to these functions (or their variants). The first step is unnecessary if the probability measures that we want are already given in the problem.

The example above illustrates the power of the entropy method, and this example is not an exceptional case. In fact, this method is simple, general, and powerful in the sense that it does not only apply to many probabilistic, combinatorial, and functional-analytic problems, but also works for general probability measure spaces and usually yields exponentially tight bounds.
3.2 Cramer’s Theorem (General Version)

The simple version of Cramer’s theorem was proven in the previous section. We now prove the general version of this theorem.

Let $X = \mathbb{R}$. Consider a random vector $X^n$, consisting of i.i.d. real-valued random variables $X_i \sim P, i \in [n]$. Let $\mu_n$ denote the law of $S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

**Definition 1.** A sequence of probability measures $\{\nu_n\}$ on $(\mathcal{X}, \mathcal{B})$ is said to satisfy the large deviation principle with a rate function $I$ if, for all $\Gamma \in \mathcal{B}$,

$$\inf_{x \in \Gamma} I(x) \leq \liminf_{n \to \infty} -\frac{1}{n} \log \nu_n(\Gamma) \leq \limsup_{n \to \infty} -\frac{1}{n} \log \nu_n(\Gamma) \leq \inf_{x \in \Gamma^c} I(x).$$

Define

$$\gamma(\alpha) := \inf_{Q: \mathbb{E}_Q[X] = \alpha} D(Q\|P) = \sup_{\lambda \in \mathbb{R}} \lambda \alpha - \log \mathbb{E}_P[e^{\lambda X}],$$

where the duality follows by Theorem 2.

**Theorem 5** (Cramer’s Theorem). [6, Theorem 2.2.3] The sequence of measures $\{\mu_n\}$ satisfies the LDP with the convex rate function $\gamma$, namely:

(a) For any closed set $F \subseteq \mathbb{R}$,

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \inf_{x \in F} \gamma(x).$$

(b) For any open set $G \subseteq \mathbb{R}$,

$$\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \inf_{x \in G} \gamma(x).$$

**Proof.** Proof of (a): We extend the domain of $\gamma$ to $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$ by the continuous extension. Let $\alpha^* \in \mathbb{R}$ be the minimum point of $\gamma$, i.e., $\gamma(\alpha^*) \leq \gamma(\alpha), \forall \alpha \in \bar{\mathbb{R}}$. For a set $F \subseteq \mathbb{R}$, let $F_1 := F \cap (-\infty, \alpha^*)$ and $F_2 := F \cap (\alpha^*, +\infty)$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\alpha_1 = \sup F_1$ and $\alpha_2 = \inf F_2$. Define

$$\gamma_-(\alpha) := \inf_{Q: \mathbb{E}_Q[X] \leq \alpha} D(Q\|P)$$

$$\gamma_+(\alpha) := \inf_{Q: \mathbb{E}_Q[X] \geq \alpha} D(Q\|P).$$
Observe that $\gamma$ is convex, and $\gamma_-(\alpha) = \inf_{x \leq \alpha} \gamma(x)$ and $\gamma_+ (\alpha) = \inf_{x \geq \alpha} \gamma(x)$. So,

$$
\gamma_- (\alpha) = \begin{cases} 
\gamma(\alpha) & \alpha \leq \alpha^* \\
\gamma(\alpha^*) & \alpha > \alpha^*
\end{cases},
$$

$$
\gamma_+ (\alpha) = \begin{cases} 
\gamma(\alpha) & \alpha \geq \alpha^* \\
\gamma(\alpha^*) & \alpha < \alpha^*
\end{cases}.
$$

Then,

$$
-\frac{1}{n} \log \mu_n(F) \geq -\frac{1}{n} \log (\mu_n((\infty, \alpha_1]) + \mu_n([\alpha_2, \infty)))
$$

$$
\geq \min \{\gamma_-(\alpha_1), \gamma_+(\alpha_2)\} - \epsilon_n
$$

$$
= \min \{\gamma(\alpha_1), \gamma(\alpha_2)\} - \epsilon_n
$$

$$
\geq \inf_{x \in F} \gamma(x) - \epsilon_n,
$$

where $\epsilon_n = \frac{1}{n} \log 2$, and the second inequality follows by Theorem 4 (more specifically, the proof of Theorem 4).

Proof of (b): For any open set $G \subseteq \mathbb{R}$, let $A_n = \{x^n : \frac{1}{n} \sum_{i=1}^n x_i \in G\}$. Given $Q$ such that $\mathbb{E}_Q [X] \in G$, denote $R_{X^n} := Q^\otimes n (|A_n)$ and $R_{X^n} := Q^\otimes n (|A_n^c)$. Denote $p_n := Q^\otimes n (A_n)$, which converges to 1 by the law of large numbers (LLN). Furthermore, following steps similar to the proof of Theorem 4, it holds that

$$
-\frac{1}{n} \log P^\otimes n (A_n) \leq \inf_{Q: \mathbb{E}_Q [X] \in G} \frac{n D(Q\|P) + H_2(p_n)}{np_n}
$$

$$
\rightarrow \inf_{Q: \mathbb{E}_Q [X] \in G} D(Q\|P)
$$

$$
= \inf_{x \in G} \gamma(x).
$$

3.3 Cramer’s Theorem for $\mathbb{R}^d$

We now consider $X = \mathbb{R}^d$. Consider a random sequence $X^n$, consisting of i.i.d. random vectors $X_i \sim P, i \in [n]$. Let $\mu_n$ denote the law of $S_n = \frac{1}{n} \sum_{i=1}^n X_i$. We next extend the general version of Cramer’s theorem to this case.

Denote $\Lambda(\lambda) := \log \mathbb{E}_P [e^{\lambda \cdot X}]$. Denote its effective domain $\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty\}$. For simplicity, suppose $\mathcal{D}_\Lambda = \mathbb{R}^d$. So, $\Lambda(b e_i) \rightarrow +\infty$ as $b \rightarrow \infty$, where $e_i$ is a $d$-length vector with 1 as the $i$th coordinate and 0’s as other coordinates.
Define for $\alpha \in \mathbb{R}^d$,
\[
\gamma(\alpha) := \inf_{Q: E_Q[X] = \alpha} D(Q\|P)
= \sup_{\lambda \in \mathbb{R}^d} \langle \lambda, \alpha \rangle - \Lambda(\lambda).
\] (8)

**Theorem 6** (Cramer’s Theorem for $\mathbb{R}^d$). [6, Theorem 2.2.30] The sequence of measures $\{\mu_n\}$ satisfies the LDP with the convex rate function $\gamma$, namely:

(a) For any closed set $F \subseteq \mathbb{R}^d$,
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \inf_{x \in F} \gamma(x).
\]

(b) For any open set $G \subseteq \mathbb{R}^d$,
\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \inf_{x \in G} \gamma(x).
\]

**Definition 2.** A family of probability measures $\mu_n$ on $\mathcal{X}$ is exponentially tight if for every $b \in (0, \infty)$, there is a compact set $K_b \subseteq \mathcal{P}(\mathcal{X})$ such that
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(K_b) \geq b.
\]

**Proof of Theorem 6.** Proof of (a): Suppose $F$ is a closed ball which is obviously convex. Then, we denote $A_n := \{x^n: \frac{1}{n} \sum_{i=1}^n x_i \in F\}$. Define $Q_{X^n} := P^\otimes n(\cdot|A_n)$. Similarly to (5), it holds that
\[
-\frac{1}{n} \log P^\otimes n(A_n) \geq D(Q_X\|P),
\]
where $X := X_K$ and the auxiliary random variable $K \sim Q_K := \text{Unif}[n]$ denotes a random time index independent of $X^n$ (under both $P, Q$). On the other hand, since $Q_{X^n}$ is concentrated on $A_n$, by Jensen’s inequality, we have $E_Q \left[\frac{1}{n} \sum_{i=1}^n X_i\right] \in F$, i.e., $E_Q [X] \in F$. We hence have
\[
-\frac{1}{n} \log P^\otimes n(A_n) \geq \inf_{x \in F} \gamma(x).
\]

Since compact sets can be covered by an appropriate finite collection of small enough balls, Statement (a) for compact sets follows by the union of events bound and the lower semicontinuity of the rate function (seen from (8)). We can also extend Statement (a) to closed sets by the fact that $\mu_n$ is an exponentially tight family of probability measures.
Lemma 3 (Exponential Tightness). [6, pp. 38-39]

\[ \lim_{\rho \to \infty} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(\mathbb{R}^d \setminus [-\rho, \rho]^d) = +\infty. \]

In other words, \( \mu_n \) is exponentially tight.

Proof of (b): Before proving (b), we first prove the following lemma. Here without loss of generality, we equip \( \mathbb{R}^d \) with the metric \( \|x - y\|_\infty \).

Lemma 4 (Concentration on Balls). For any \( Q \),

\[ \liminf_{n \to \infty} -\frac{1}{n} \log Q^\otimes n \{ x^n : \frac{1}{n} \sum_{i=1}^n x_i \in B_\epsilon(\alpha)^c \} > 0, \]

where \( \alpha := \mathbb{E}_Q [X] \). In other words, \( \mathbb{P} \{ \frac{1}{n} \sum_{i=1}^n X_i \in B_\epsilon(\alpha)^c \} \to 0 \) exponentially fast as \( n \to \infty \).

This lemma is proven as follows. We have

\[ Q^\otimes n \{ x^n : \frac{1}{n} \sum_{i=1}^n x_i \in B_\epsilon(\alpha)^c \} \leq Q^\otimes n \{ x^n : \left\| \frac{1}{n} \sum_{i=1}^n x_i - \alpha \right\|_\infty \geq \epsilon \} \]

\[ \leq \sum_{j=1}^d Q_{X(j)}^\otimes n \{ x^n(j) : \left| \frac{1}{n} \sum_{i=1}^n x_i(j) - \alpha(j) \right| \geq \epsilon \} \]

\[ \to 0 \]

exponentially fast as \( n \to \infty \), where \( x(j) \) denotes the \( j \)-th coordinate of \( x \) and the convergence in the last line follows by the simple version of Cramer’s theorem in Theorem 4. So, Lemma 4 holds.

We turn back to proving (b). It suffices to assume that \( G \) is open. Denote \( A_n := \{ x^n : \frac{1}{n} \sum_{i=1}^n x_i \in G \} \) and choose \( Q \) such that \( \alpha := \mathbb{E}_Q [X] \in G \). Then, \( B_\epsilon(\alpha) \subseteq G \) for sufficiently small \( \epsilon > 0 \). Denote \( p_n := Q^\otimes n(A_n) \). Then, by Lemma 4,

\[ p_n \geq Q^\otimes n \{ x^n : \frac{1}{n} \sum_{i=1}^n x_i \in B_\epsilon(\alpha) \} \]

\[ = 1 - Q^\otimes n \{ x^n : \frac{1}{n} \sum_{i=1}^n x_i \in B_\epsilon(\alpha)^c \} \]

\[ \to 1 \] as \( n \to \infty \).
Following steps similar to those in the proof of Theorem 4, we have

\[
\limsup_{n \to \infty} -\frac{1}{n} \log P^\otimes n(A_n) \leq \inf_{Q_X : \mathbb{E}_Q[X] \in G} D(Q_X \| P) = \inf_{x \in G} \gamma(x).
\]

\[
\square
\]

### 3.4 Strong Cramer’s Theorem

The simple version of Cramer’s theorem given in Section 3.1 can be further strengthened. The resultant result is known as the **strong Cramer’s theorem** [1]. We next present the proof in [1] for the strong Cramer’s theorem. Although the proof is not new, the strong Cramer’s theorem is rewritten in terms of relative entropies and relative variances.

Let \( \mathcal{X} = \mathbb{R} \). Consider a random vector \( X^n \), consisting of i.i.d. real-valued random variables \( X_i \sim P, i \in [n] \). Suppose that \( Q \) attains \( \gamma(\alpha) \) defined in (7) which satisfies

\[
d_{Q} d_{P}(x) = e^{\lambda^* x} \mathbb{E}_{P}[e^{\lambda^* X}]
\]

for some \( \lambda^* \in \mathbb{R} \) such that \( \mathbb{E}_{Q}[X] = \alpha \). Then, we have

\[
\mathbb{P}\{\sum_{i=1}^{n} X_i \geq n\alpha\}
= \int 1_{\{\sum_{i=1}^{n} x_i \geq n\alpha\}} \frac{dP^\otimes n}{dQ^\otimes n}
= e^{-nD(Q \| P)} \int 1_{\{\sum_{i=1}^{n} x_i \geq n\alpha\}} e^{-\left(\log \frac{dQ^\otimes n}{dP^\otimes n} - nD(Q \| P)\right)} dQ^\otimes n
= e^{-nD(Q \| P)} E \left[ 1_{\{\sum_{i=1}^{n} W_i \geq 0\}} e^{-\sum_{i=1}^{n} W_i}\right],
\]

(9)

where \( W_i := \log \frac{dQ}{dP}(X_i) - D(Q \| P) \) with i.i.d. \( X_i \sim Q \). Note that \( \mathbb{E}[W_i] = 0 \) and \( \text{Var}[W_i] = V(Q \| P) = \lambda^{*2} \text{Var}_{Q}[X] \).

To estimate the expectation in (9), we need to use the central limit theorems. Intuitively, \( \frac{1}{\sqrt{nV(Q \| P)}} \sum_{i=1}^{n} W_i \) asymptotically follows the standard normal distribution. By simply replacing it with a standard normal random variable \( Z \), we obtain

\[
\mathbb{E}\left[ 1_{\{Z \geq 0\}} e^{-\sqrt{nV(Q \| P)} Z}\right] \sim \frac{1}{\sqrt{2\pi nV(Q \| P)}}.
\]

14
In fact, this intuition is true when \( X_i \sim P \) are non-lattice. When \( X_i \sim P \) are lattice, an additional factor will appear at the RHS. These claims, stated in the following lemma, can be proven by using Berry–Esséen expansion; see e.g., \([6]\).

**Lemma 5.** For i.i.d. \( W_i \) with mean zero and variance \( V \), it holds that

\[
E \left[ \mathbb{1}_{\sum_{i=1}^n W_i \geq 0} e^{-\sum_{i=1}^n W_i} \right] \sim \frac{c}{\sqrt{2\pi n V}},
\]

where \( c = 1 \) if \( W_i \) are non-lattice, and \( c = \frac{d}{1-e^{-d}} \) if \( W_i \) are lattice with maximal step\(^1\) \( d \) and\(^2\) \( 0 < P(W_1 = 0) < 1 \).

So, we obtain the following strong LD theorem. Note that if \( X_i \sim P \) are lattice with maximal step \( d \), then \( W_i \) are lattice with maximal step \( \lambda^*d \).

**Theorem 7** (Strong Cramer’s Theorem). \([1]\) Let \( \alpha > 0 \). Suppose that \( Q \) attain \( \gamma(\alpha) \). Then,

\[
\mathbb{P} \left\{ \sum_{i=1}^n X_i \geq n\alpha \right\} \sim \frac{c}{\sqrt{2\pi n V(Q\|P)}} e^{-nD(Q\|P)},
\]

where \( c = 1 \) if \( X_i \sim P \) are non-lattice, and \( c = \frac{\lambda^*d}{1-e^{-\lambda^*d}} \) if \( X_i \sim P \) are lattice with maximal step \( d \) and \( 0 < \mathbb{P}(X_1 = \alpha) < 1 \).

## 4 General Principle and Gärtner–Ellis Theorem

### 4.1 General Principle

We now extend Cramer’s theorem to the non-i.i.d. setting. Suppose that \( \mathcal{X} \) is a metric space, and \( \mathcal{B}_\mathcal{X} \) is the Borel \( \sigma \)-algebra on \( \mathcal{X} \). Consider a sequence of probability measures \( \{\mu_n\} \) on \( \mathcal{X} \). Define for \( x \in \mathcal{X} \),

\[
\gamma_-(x) := \lim_{\epsilon \downarrow 0} \liminf_{n \to \infty} E_{n,\epsilon}(x)
\]

\[
\gamma_+(x) := \lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} E_{n,\epsilon}(x),
\]

where

\[
E_{n,\epsilon}(x) := -\frac{1}{n} \log \mu_n(B_\epsilon(x)).
\]

\(^1\)That is, for some \( x_0, d, \) the random variable \( d^{-1}(W_1 - w_0) \) is (a.s.) an integer number, and \( d \) is the largest number with this property.

\(^2\)The condition \( 0 < \mathbb{P}(W_1 = 0) < 1 \) implies that \( w_0/d \) is an integer and that \( V > 0 \).
Note that $\gamma_-$ and $\gamma_+$ can be written in terms of relative entropies since
\[ E_{n,\epsilon}(x) = \inf_{\nu_n : \nu_n(B(x)) = 1} \frac{1}{n} D(\nu_n \| \mu_n). \tag{10} \]

**Theorem 8** (General Principle). Assume that $\{\mu_n\}$ is an exponentially tight sequence of probability measures. Then, it holds that:

(a) For any closed set $F \subseteq \mathcal{X}$,
\[ \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \inf_{x \in F} \gamma_-(x). \]

(b) For any open set $G \subseteq \mathcal{X}$,
\[ \limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \inf_{x \in G} \gamma_+(x). \]

(c) Suppose $\gamma_-(x) = \gamma_+(x) =: \gamma(x)$ for all $x \in \mathcal{X}$. Then, $\{\mu_n\}$ satisfies the LDP with the rate function $\gamma$.

In fact, the parameter $n$ can be replaced by any positive $a_n$ such that $a_n \to \infty$ as $n \to \infty$, or even replaced by $1/\epsilon$ in which case, all limits are taken as $\epsilon \downarrow 0$.

**Proof.** Proof of (a): Since any compact set $F$ can be covered by an appropriate finite collection of small enough balls $\{B_{\epsilon}(x_i)\}_{i=1}^k$ with $x_i \in F$, Statement (a) for compact sets follows by the union of events bound. Namely,
\[ \mu_n(F) \leq \sum_{i=1}^k \mu_n(B_{\epsilon}(x_i)) \leq k \max_{1 \leq i \leq k} \mu_n(B_{\epsilon}(x_i)), \]
and hence,
\[ \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \min_{1 \leq i \leq k} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{\epsilon}(x_i)) \]
\[ \geq \inf_{x \in F} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{\epsilon}(x)). \tag{11} \]
Let $y_i \in F$ attain the infimum in (11) within a gap $1/i$. Passing to a subsequence, we assume $y_i \to y^* \in F$ as $i \to \infty$. So, the bound in (11) is equal to
\[ \lim_{i \to \infty} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{\epsilon}(y_i)). \tag{12} \]
For sufficiently large $i$, $d(y^*, y_i) < \epsilon$. So, (12) is further lower bounded by

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{2\epsilon}(y^*)) \geq \gamma_-(y^*) \geq \inf_{x \in F} \gamma_-(x).$$

Substituting these into (11) yields

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{2\epsilon}(y^*)) \geq \gamma_-(y^*) \geq \inf_{x \in F} \gamma_-(x).$$

Finally, by the assumption that $\mu_n$ is an exponentially tight family of probability measures, we can extend Statement (a) from compact sets to all closed sets.

Proof of (b): For open set $G$, let $x \in G$. Then, $B_{\epsilon}(x) \subseteq G$ for sufficiently small $\epsilon > 0$. We have

$$\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \limsup_{\epsilon \downarrow 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(B_{\epsilon}(x)) = \gamma_+(x).$$

Since $x \in G$ is arbitrary,

$$\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \inf_{x \in G} \gamma_+(x).$$

\[ \square \]

### 4.2 Gärtner–Ellis Theorem

We now suppose that $X = \mathbb{R}^d$. Let $\mu_n$ denote the law of $Z_n \in \mathbb{R}^d$. Denote $\Lambda_n(\lambda) := \log \mathbb{E}[e^{\langle \lambda, Z_n \rangle}]$. Denote $\Lambda(\lambda) := \lim_{n \to \infty} \frac{1}{n} \Lambda_n(n\lambda)$ where the limit is supposed to exist. Denote its effective domain $D_\Lambda := \{ \lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty \}$. Let $\Lambda^*(\cdot)$ be the Fenchel–Legendre transform of $\Lambda(\cdot)$, with $D_{\Lambda^*} := \{ x \in \mathbb{R}^d : \Lambda^*(x) < +\infty \}$.

Define for $x \in \mathbb{R}^d$,

$$\gamma_n(x) := \inf_{\nu_n : \mathbb{E}_{\nu_n}[X] = x} \frac{1}{n} D(\nu_n \| \mu_n) = \sup_{\lambda \in \mathbb{R}^d} \langle \lambda, x \rangle - \frac{1}{n} \Lambda_n(n\lambda).$$

**Assumption:** For each $\lambda \in \mathbb{R}^d$, the logarithmic moment generating function $\Lambda(\lambda)$ exists as an extended real number. Further, the origin belongs to the interior of $D_{\Lambda^*}$. 

17
Definition 3. \( y \in \mathbb{R}^d \) is an exposed point of \( \Lambda^* \) if for some \( \lambda \in \mathbb{R}^d \) and all \( x \neq y \),
\[
\langle \lambda, y \rangle - \Lambda^*(y) > \langle \lambda, x \rangle - \Lambda^*(x).
\] (13)
\( \lambda \) in (13) is called an exposing hyperplane.

Theorem 9 (Gärtner–Ellis Theorem). [6, Theorem 2.3.6] Let the assumption above hold.
(a) For any closed set \( F \subseteq \mathbb{R}^d \),
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(F) \geq \inf_{x \in F} \Lambda^*(x).
\]
(b) For any open set \( G \subseteq \mathbb{R}^d \),
\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(G) \leq \inf_{x \in G \cap F} \Lambda^*(x),
\]
where \( F \) is the set of exposed points of \( \Lambda^* \) whose exposing hyperplane belongs to \( D^*_\Lambda \).

Definition 4. A convex function \( \Lambda : \mathbb{R}^d \to (-\infty, \infty] \) is essentially smooth if: (a) \( D^*_\Lambda \) is non-empty. (b) \( \Lambda(\cdot) \) is differentiable throughout \( D^*_\Lambda \). (c) \( \Lambda(\cdot) \) is steep, namely, \( \lim_{n \to \infty} |\nabla \Lambda(\lambda_n)| = \infty \) whenever \( (\lambda_n) \) is a sequence in \( D^*_\Lambda \) converging to a boundary point of \( D^*_\Lambda \). Theorem 9 implies the following statement [6, Theorem 2.3.6],
(c) If \( \Lambda \) is an essentially smooth, lower semicontinuous function, then the LDP holds with the good rate function \( \Lambda^*(\cdot) \).

Proof. Proof of (a): For any measure \( \nu_n, \nu_n(B_\epsilon(x)) = 1 \) implies \( \nu_n[X] \in B_\epsilon(x) \), since a closed ball is (completely) convex (due to the fact that the norm \( \| \cdot \|_p \) with \( p \geq 1 \) is convex). So, by (10),
\[
\gamma_- (x) \geq \liminf_{\epsilon \downarrow 0} \liminf_{n \to \infty} \inf_{y \in B_\epsilon(x)} \gamma_n(y) = \liminf_{\epsilon \downarrow 0} \liminf_{n \to \infty} \sup_{y \in B_\epsilon(x)} (\lambda, y) - \frac{1}{n} \Lambda_n(n\lambda) \geq \sup_{\lambda \in \mathbb{R}^d} \liminf_{\epsilon \downarrow 0} \inf_{y \in B_\epsilon(x)} (\lambda, y) - \limsup_{n \to \infty} \frac{1}{n} \Lambda_n(n\lambda) = \sup_{\lambda \in \mathbb{R}^d} (\lambda, x) - \Lambda(x) = \Lambda^*(x),
\] (14)
where (14) follows since the linear functional \( x \mapsto \langle \lambda, x \rangle \) is continuous.

The proof of (a) is complete by showing that \( \{\mu_n\} \) is an exponentially tight sequence of probability measures.

**Lemma 6** (Exponential Tightness). \([6, pp. 48-49]\) Under that assumption that the origin is in \( \mathcal{D}^\Lambda \), it holds that

\[
\lim_{\rho \to \infty} \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(\mathbb{R}^d \setminus [-\rho, \rho]^d) = +\infty.
\]

Proof of (b): Fix \( y \in \mathcal{F} \cap G \) and let \( \eta \in \mathcal{D}^\Lambda \) denote an exposing hyper-plane for \( y \). Then, for all \( n \) large enough, \( \Lambda_n(n\eta) < \infty \) and the associated probability measures \( \tilde{\mu}_n \) are well-defined via

\[
\frac{d\tilde{\mu}_n}{d\mu_n}(z) = e^{\langle n\eta, z \rangle - \Lambda_n(n\eta)}.
\]

**Lemma 7** (Concentration on Balls). \([6, pp. 49-50]\) For any \( \epsilon > 0 \),

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \tilde{\mu}_n(B_\epsilon(y)^c) > 0.
\]

In other words, \( \tilde{\mu}_n(B_\epsilon(y)^c) \to 0 \) exponentially fast as \( n \to \infty \).

Given any \( \tilde{\mu}_n \), denote an auxiliary probability measure \( \pi_n := \tilde{\mu}_n(\cdot|B_\epsilon(y)) \). Denote \( p_n := \tilde{\mu}_n(B_\epsilon(y)) \), which converges to 1 by Lemma 7. So,

\[
\frac{d\pi_n}{d\mu_n}(z) = \frac{e^{\langle n\eta, z \rangle - \Lambda_n(n\eta)} \mathbb{1}_{B_\epsilon(y)}(z)}{p_n}.
\]

Since \( \pi_n \) is a feasible solution to the infimization in (10),

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(B_\epsilon(y)) \leq \limsup_{n \to \infty} \frac{D(\pi_n||\mu_n)}{n} = \limsup_{n \to \infty} \frac{1}{n} \int \log \frac{e^{\langle n\eta, z \rangle - \Lambda_n(n\eta)}}{p_n} d\pi_n(z) = \limsup_{n \to \infty} \mathbb{E}_{\pi_n}[\langle \eta, Z \rangle] - \Lambda(\eta) \leq \sup_{z \in B_\epsilon(y)} \langle \eta, z \rangle - \Lambda(\eta).
\]

So,

\[
\lim \limsup_{\epsilon \downarrow 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_n(B_\epsilon(y)) \leq \limsup_{\epsilon \downarrow 0} \sup_{z \in B_\epsilon(y)} \langle \eta, z \rangle - \Lambda(\eta) = \langle \eta, y \rangle - \Lambda(\eta) \leq \Lambda^*(y),
\]

where (15) follows since the linear functional \( z \mapsto \langle \eta, z \rangle \) is continuous. □
It is also possible to generalize the Gärtner–Ellis theorem to the abstract version, i.e., the Baldi theorem in [6, Theorem 4.5.20], by using the entropy method.

5 Sanov’s Theorems

5.1 Sanov’s Theorem

The empirical sum is in fact determined by the empirical measure. We next consider the LD theory of empirical measures and prove Sanov’s theorem by the entropy method. For the finite alphabet case, Sanov’s theorem can be proven by another information-theoretic method, known as the method of types. In fact, Csiszár found that the method of types, combined with discretization techniques, can be also used to prove Sanov’s theorem for the general alphabet case; see [5]. Another proof based on discretization is given in [2]. However, our proof given below is based on Csiszár’s works on I-projections in [3, 4].

Recall that \( \mathcal{X} \) is a Hausdorff topological space (so that all singletons are closed and hence Borel), and \( \mathcal{B}_\mathcal{X} \) is the Borel \( \sigma \)-algebra on \( \mathcal{X} \). Recall that \( L_{\mathcal{X}^n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \) denotes the empirical measure of \( X^n \sim P^\otimes n \). Consider the weak topology on \( P(\mathcal{X}) \). Let \( \mathcal{B}^w \) denote the Borel \( \sigma \)-algebra generated by the weak topology. Sanov’s theorem says that under \( \mathcal{B}^w \), \( L_{\mathcal{X}^n} \) satisfies the large deviation principle.

Definition 5. For a set of probability measures \( \mathcal{A} \subseteq P(\mathcal{X}) \), the completely convex hull of \( \mathcal{A} \), denoted by \( c\text{conv} \mathcal{A} \), is the set of probability measures \( Q_X \) such that \( Q_X = Q_Z \circ Q_{\mathcal{X}|Z} \) for some probability space \( (Z, \Sigma_Z, Q_Z) \) and Markov kernel \( Q_{\mathcal{X}|Z} \) from \( (Z, \Sigma_Z) \) to \( (\mathcal{X}, \mathcal{B}_\mathcal{X}) \) satisfying \( Q_{\mathcal{X}|Z}=z \in \mathcal{A} \) for each \( z \in \mathcal{Z} \).

Definition 6. [4] A set of probability measures \( \mathcal{A} \subseteq P(\mathcal{X}) \) is completely convex if \( \mathcal{A} = c\text{conv} \mathcal{A} \).

Note that a completely convex set is obviously convex, but the converse is not true. In fact, the definition above reduces to the one of “convex” if we restrict \( \mathcal{Z} \) to be finite. An example of completely convex sets is sets specified by linear constraints, e.g., \{ \( Q : Q(f) = \alpha \) \} and \{ \( Q : Q(f) \geq \alpha \) \} for any measurable \( f \) and real number \( \alpha \). Another example is closed balls in \( P(\mathcal{X}) \) under the Lévy–Prokhorov metric.

Lemma 8. Any closed ball \( B_{\epsilon}(P_X) \) in \( P(\mathcal{X}) \) under the Lévy–Prokhorov metric is completely convex. Moreover, if \( \mathcal{X} \) is Polish, then the completely convex
hull of a convex set \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \) satisfies that \( \Gamma \subseteq \text{cconv}\Gamma \subseteq \Gamma \) where \( \Gamma \) is the closure of \( \Gamma \) under the Lévy–Prokhorov metric.

Lemma 8 implies that if \( \mathcal{X} \) is Polish, then for a (not necessarily convex) set \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \), \( \text{cconv}\Gamma = \text{conv}\Gamma \) where \( \text{conv}\Gamma \) is the convex hull of \( \Gamma \) under the Lévy–Prokhorov metric. In other words, under the weak topology, the closed completely convex hull of a set in \( \mathcal{P}(\mathcal{X}) \) is just its closed convex hull.

To prove Lemma 8, we first observe that the closure of a set can be written as the intersection of all its enlargements.

**Lemma 9.** For a set \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \), \( \Gamma = \inf_{\epsilon > 0} \Gamma_{\epsilon} \), where \( \Gamma_{\epsilon} := \bigcup_{P_X \in \Gamma} B_\epsilon(P_X) \) for \( \epsilon > 0 \) are closed enlargements of \( \Gamma \) under the Lévy–Prokhorov metric.

**Proof.** First, \( \Gamma \subseteq \Gamma_{\epsilon} \) for all \( \epsilon > 0 \). Then, \( \Gamma \subseteq \inf_{\epsilon > 0} \Gamma_{\epsilon} \). On the other hand, if a point \( R \in \Gamma_{\epsilon} \) for all \( \epsilon > 0 \), then there is a sequence \( \{R_i\} \subseteq \Gamma \) such that \( d_{P}(R, R_i) \to 0 \), i.e., \( R \in \Gamma \) or \( R \) is a limit point of \( \Gamma \). Hence, \( \Gamma \supseteq \inf_{\epsilon > 0} \Gamma_{\epsilon} \).

We now use Lemma 9 to prove Lemma 8.

**Proof of Lemma 8.** If \( Q_{X|Z=z} \in B_{\epsilon}(P_X), \forall z \), then for any \( \delta > \epsilon \),

\[
Q_{X|Z=z}(A) \leq P_X(A_\delta) + \delta, \forall \text{ closed } A \subseteq \mathcal{X}, \forall z,
\]

which, by taking expectations w.r.t. \( Q_Z \), further implies

\[
Q_X(A) \leq P_X(A_\delta) + \delta, \forall \text{ closed } A \subseteq \mathcal{X},
\]

i.e., \( d_{P}(Q_X, P_X) \leq \delta \). Since \( \delta > \epsilon \) is arbitrary, we have \( d_{P}(Q_X, P_X) \leq \epsilon \), i.e., \( Q_X \in B_{\epsilon}(P_X) \).

To prove the second statement, by Lemma 9, it suffices to prove \( \text{cconv}\Gamma \subseteq \Gamma_{\epsilon} \) for all \( \epsilon > 0 \). We first prove that \( \Gamma_{\epsilon} \) is convex. If \( Q_{X|Z=z} \in \Gamma_{\epsilon} \), then there is \( P_X^{(z)} \in \Gamma \) such that \( d_{P}(Q_{X|Z=z}, P_X^{(z)}) \leq \epsilon \), which implies for any \( \delta > \epsilon \),

\[
Q_{X|Z=z}(A) \leq P_X^{(z)}(A_\delta) + \delta, \forall \text{ closed } A \subseteq \mathcal{X}.
\] (16)

Taking expectations for (16) w.r.t. discrete distribution \( Q_Z \) with finite support, we have for any \( \delta > \epsilon \),

\[
Q_X(A) \leq P_X(A_\delta) + \delta, \forall \text{ closed } A \subseteq \mathcal{X},
\]

where \( P_X := \sum_z Q_Z(z)P_X^{(z)} \). Hence, \( d_{P}(Q_X, P_X) \leq \epsilon \). By convexity of \( \Gamma \), we have \( P_X \in \Gamma \). So, \( Q_X \in \Gamma_{\epsilon} \), i.e., \( \Gamma_{\epsilon} \) is convex.
If \( \mathcal{X} \) is Polish, then \( \mathcal{P}(\mathcal{X}) \) with the Lévy–Prokhorov metric forms a Polish metric space. Let \( \hat{\mathcal{P}} \) be a countable dense subset \( \mathcal{P}(\mathcal{X}) \). Denote \( \hat{\Gamma} := \Gamma_{\epsilon} \cap \hat{\mathcal{P}} \). Then, \( \hat{\Gamma} = \{ R_i \}_{i \geq 1} \) is a countable dense subset of \( \Gamma_{\epsilon} \). Denote \( \hat{\Gamma}_{\delta} \) as an closed enlargement of \( \hat{\Gamma} \). Then, \( \Gamma_{\epsilon} \subseteq \hat{\Gamma}_{\delta} \subseteq \Gamma_{\epsilon + \delta} \). Write \( \hat{\Gamma}_{\delta} \) as a partition \( \{ B_j \} \) of \( \hat{\Gamma}_{\delta} \) given by \( B_i := B_{\delta}(R_i) \setminus \bigcup_{j=1}^{i-1} B_j, i \geq 1 \). Denote \( Z_i := \{ z : Q_{X_{\lfloor Z = z}} \in B_i \} \) and \( q_i := Q_{Z}(Z_i) \). Since \( Z_i \) is not necessarily measurable in \( Z \), to make it measurable, we consider a finer \( \sigma \)-algebra which is generated by the sets \( Z_i \) and the sets in the \( \sigma \)-algebra of \( Z \). Then, under the new \( \sigma \)-algebra, denote \( Q_{Z} := Q_{X_{\lfloor Z_i}} \circ Q_{X_{\lfloor Z}} \) and \( Q_{X} := Q_{Z} \circ Q_{X_{\lfloor Z}} = \sum_{i \geq 1} q_i Q_{Z_i} \). Denote \( R_{X} := \sum_{i \geq 1} q_i R_i \) and \( R_{X}^{(n)} := \sum_{i=1}^{n} q_i R_i \).

Since \( B_i \) is a subset of \( B_{\delta}(R_i) \), by the first statement of this lemma, \( d_{P}(Q_{Z_i}, R_i) \leq \delta \), which implies that \( d_{P}(Q_{X_{\lfloor Z_i}}, R_i) \leq \delta \). Moreover, by the definition of the Lévy–Prokhorov metric, \( d_{P}(R_{X}, R_{X}^{(n)}) \leq \sum_{i \geq n+1} q_i \), which vanishes as \( n \to \infty \). So, \( d_{P}(Q_{X_{\lfloor Z_i}} \circ Q_{X_{\lfloor Z}}, R_i) \leq \delta + \sum_{i \geq n+1} q_i \). Since \( R_i \in \Gamma_{\epsilon} \) and \( R_{X}^{(n)} \) is a convex combination of \( R_i \)'s, by the convexity of \( \Gamma_{\epsilon} \), we have \( R_{X}^{(n)} \in \Gamma_{\epsilon} \). So, \( Q_{X} \in \Gamma_{\epsilon + 2\delta} \). Finally, we obtain \( \text{cconv}_{\epsilon} \Gamma \subseteq \text{cconv}_{\epsilon + 2\delta} \Gamma \subseteq \text{cconv}_{\epsilon} \Gamma \). Since \( \epsilon, \delta > 0 \) are arbitrary, \( \text{cconv}_{\epsilon} \Gamma \subseteq \text{cconv}_{\epsilon + 2\delta} \Gamma = \Gamma \). \( \square \)

For (completely) convex Borel sets, the following version of Sanov’s theorem holds.

**Theorem 10** (Sanov’s Theorem for (Completely) Convex Sets). The following hold.

1. \cite[Theorem 1]{4} Let \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \) be a completely convex Borel set. Then, it holds that

\[
\mathbb{P}\{ L_{X_{\Gamma}} \in \Gamma \} \leq e^{-n D(\Gamma||P)}.
\] (17)

2. Moreover, if \( \mathcal{X} \) is Polish, then (17) also holds for closed convex sets \( \Gamma \).

The first statement of the theorem above and the proof below are due to Csiszár \cite[Theorem 1]{4}. However, the essentially same technique was previously used by Massey \cite{8}, whose result in fact yields the finite alphabet version of the theorem above.

**Proof.** Proof of Statement 1: Denote \( A = L^{-1}(\Gamma) \), and \( Q_{X_{\Gamma}} = P^{\otimes n}(\cdot|A) \). Then, we have

\[
- \log P^{\otimes n}(A) = D(Q_{X_{\Gamma}}||P^{\otimes n}).
\]
By the chain rule and the fact that conditioning increases the relative entropy, the RHS is lower bounded by
\[ nD(Q_{X,J|X^{J-1},J} \| P|Q_{X^{J-1}}) \geq nD(Q_{X,J} \| P), \]
where \( J \sim \text{Unif}[n] \) is a random time-index independent of \( X^n \).

On the other hand, \( L_{X^n} \in \Gamma \) holds \( Q_{X^n} \)-a.s. We immediately obtain
\[ E_{Q_{X^n}}[L_{X^n}] \in \Gamma, \]
which can be rewritten as \( Q_{X,J} \in \Gamma \).

Relaxing \( Q_{X,J} \) to an arbitrary distribution in \( \Gamma \), we obtain that for any \( n \geq 1 \),
\[ -\frac{1}{n} \log P^{\otimes n} (A) \geq D(\Gamma \| P). \]

Proof of Statement 2: By Lemma 8, if \( \mathcal{X} \) is Polish, then under the weak topology, a closed convex set is also a closed completely convex set. Then, applying Statement 1, we obtain Statement 2. \( \square \)

In fact, Sanov’s theorem can be extended to any Borel sets.

**Theorem 11** (Sanov’s Theorem). [10, 6] Assume \( \mathcal{X} \) is Polish. For the \( \sigma \)-algebra \( \mathcal{B}^n \), the empirical measures \( L_{X^n} \) satisfy the LDP in \( \mathcal{P}(\mathcal{X}) \) equipped with the weak topology with the good, convex rate function \( D(\cdot \| P) \).

Before proving Theorem 11, we introduce the fact that the law of \( L_{X^n} \) is exponentially tight.

**Lemma 10** (Exponential Tightness). [6, Lemma 6.2.6] For each \( b \in (0, \infty) \), there is a compact set \( K_b \subseteq \mathcal{P}(\mathcal{X}) \) such that
\[ \liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{P}\{L_{X^n} \in K_b^c\} \geq b. \]
In other words, the law of \( L_{X^n} \) is exponentially tight.

For completeness, the proof of this lemma is given below.

**Proof of Lemma 10.** Choose a non-decreasing sequence \( \{K_j : j \geq 1\} \) of compact subsets of \( \mathcal{X} \) so that \( P(K_j) \leq (e - 1) \cdot e^{-2j} \), and set \( V = \sum_{j=0}^{\infty} \mathbb{1}_{\mathcal{X} \setminus K_j} \). Then \( V < j \) on \( K_j \), and so,
\[ P(e^V) = \int_{\bigcup_{j=1}^{\infty} K_j} e^V dP = \lim_{j \to \infty} \int_{K_j} e^V dP = \sum_{j=1}^{\infty} \int_{K_j \setminus K_{j-1}} e^V dP \leq (e - 1) \sum_{j=1}^{\infty} e^{-2j} e^j = 1. \]
At the same time, \( V \geq j \) off \( K_j \), and so for any probability measure \( Q \), \( Q(V) \leq b \implies Q(\mathcal{X} \setminus K_j) \leq \frac{b}{2} \). By [9, Theorem 1.12], \( \mathcal{K}_b = \{ Q : Q(V) \leq b \} \) is relatively compact. In addition, because \( V \) is lower semi-continuous, \( \mathcal{K}_b \) is closed. Hence, for each \( b > 0 \), \( \mathcal{K}_b \) is compact in \( \mathcal{P}(\mathcal{X}) \). Finally, \[
abla \{ L_{X^n} \in \mathcal{K}_b^c \} = \mathbb{P}\{ L_{X^n}(V) > b \} \leq e^{-n b} \mathbb{P}[e^{n L_{X^n}(V)}] = e^{-n b} \mathbb{E}_P[e^{V(X)}^n] = e^{-n b}.
\]

**Proof of Theorem 11.** By the general principle given in Theorem 8 and the exponential tightness given in Lemma 10, to prove Theorem 11, it suffices to show that \( \gamma_-(Q) = \gamma_+(Q) = D(Q\|P) \) for all \( Q \in \mathcal{P}(\mathcal{X}) \), where

\[
\gamma_-(Q) = \lim_{\epsilon \downarrow 0} \liminf_{n \to \infty} E_{n, \epsilon}(Q),
\]

\[
\gamma_+(Q) = \lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} E_{n, \epsilon}(Q),
\]

and

\[
E_{n, \epsilon}(Q) = -\frac{1}{n} \log \mathbb{P}\{L_{X^n} \in B_\epsilon(Q)\} = \inf_{R^{(n)} : R^{(n)}(L^{-1}(B_\epsilon(Q))) = 1} \frac{1}{n} D(R^{(n)}\|P^{\otimes n}).
\]

Proof of \( \gamma_- \geq D(\cdot\|P) \): Since the closed ball \( B_{\epsilon/2}(Q) \) is completely convex, by Theorem 10,

\[
-\frac{1}{n} \log \mathbb{P}\{L_{X^n} \in B_\epsilon(Q)\} \geq D(B_{\epsilon/2}(Q)\|P).
\]

Taking \( \lim_{\epsilon \downarrow 0} \liminf_{n \to \infty} \) and by the lower semicontinuity of \( D(\cdot\|P) \), we have \( \gamma_-(Q) \geq D(Q\|P) \).

Proof of \( \gamma_+ \leq D(\cdot\|P) \): Before proving the upper bound in Theorem 11, we first prove the following lemma.

**Lemma 11 (Concentration on Balls).** For any \( \epsilon > 0 \),

\[
\liminf_{n \to \infty} -\frac{1}{n} \log Q^{\otimes n}(L^{-1}(B_\epsilon(Q)^c)) \geq \epsilon^2 / 2.
\]
This lemma is proven as follows. Since $B_ε(Q)^c$ is closed, by Statement (a) of Theorem 8 and the lower bound proven above,
\[
\liminf_{n \to \infty} -\frac{1}{n} \log Q^\otimes n(\mathbf{L}_X)^{-1}(B_ε(Q)^c) \geq \inf_{R \in B_ε(Q)^c} \gamma_-(R) \geq D(B_ε(Q)^c \| Q) \geq \epsilon^2/2,
\]
where the last inequality follows by (2). So, Lemma 11 holds.

We turn back to proving $\gamma_+ \leq D(\| P)$. Denote $A_n = \mathbf{L}_X^{-1}(B_ε(Q))$. Denote $p_n := Q^\otimes n(A_n)$, which, by Lemma 11, converges to 1 as $n \to \infty$.

Following steps similar to those in the proof of Theorem 4,
\[
\limsup_{n \to \infty} -\frac{1}{n} \log P^\otimes n(A_n) \leq \limsup_{n \to \infty} \frac{nD(Q_X \| P) + H_2(p_n)}{np_n} = D(Q_X \| P).
\]
Therefore, $\gamma_+(Q) \leq D(Q \| P)$.

Combining the two inequalities proven above with the fact that $\gamma_- \leq \gamma_+$, we obtain $\gamma_-(Q) = \gamma_+(Q) = D(Q \| P)$ for all $Q \in \mathcal{P}(\mathcal{X})$.

5.2 **Strong Sanov’s Theorem**

We next strengthen Sanov’s theorem for convex sets. The resultant result is called the **strong Sanov’s theorem**. Such a result incorporates the strong Cramer’s theorem as a special case. Recall that $\mathcal{X}$ is a Hausdorff topological space.

**Theorem 12** (Strong Sanov’s Theorem for Convex Sets). *Let $\Gamma$ be a convex Borel set. Assume that $D(\| P)$ is attained by some $Q^*$ (i.e., the I-projection exists). Denote $P_\Gamma$ as the law of $t_{Q^* \| P}(X)$ with $X \sim Q^*$. Suppose that $P_\Gamma$ is non-lattice, or it is lattice with maximal step $d$ such that $0 < P_\Gamma(D(Q^* \| P)) < 1$. Denote $c = 1$ if $P_\Gamma$ is non-lattice, and $c = \frac{d}{1-e^{-d}}$ if $P_\Gamma$ is lattice.*

1. It holds that
\[
\mathbb{P}(\mathbf{L}_X \in \Gamma) \leq e^{-nD(Q^* \| P)},
\]
and
\[
\mathbb{P}(\mathbf{L}_X \in \Gamma) \leq \frac{c}{\sqrt{2\pi nV(Q^* \| P)} e^{-nD(Q^* \| P)}}.
\]
2. Moreover, if additionally, $B_{\epsilon}(Q^*) \cap \mathcal{H}_+ \subseteq \Gamma$, then

$$
P\{L_{X^n} \in \Gamma\} \sim \frac{e^{-nD(Q^* \| P)}}{\sqrt{2\pi n V(Q^* \| P)}} e^{-nD(Q^* \| P)}, \quad \text{(20)}$$

where

$$
\mathcal{H}_+ := \left\{ R : \int \left( \log \frac{dQ^*}{dP} \right) dR \geq D(Q^* \| P) \right\}.
$$

In fact, (18) is restatement of Statement 2 of Theorem 10.

**Proof.** We first prove (19). Then, similarly to (9), we have

$$
P\{L_{X^n} \in \Gamma\} = e^{-nD(Q^* \| P)} E \left[ 1_{\{L_{X^n} \in \Gamma\}} e^{-\sum_{i=1}^n W_i} \right], \quad \text{(21)}$$

where $X^n \sim Q^{*\otimes n}$ and $W_i = \log \frac{dQ^*}{dP}(X_i) - D(Q^* \| P)$. Note that $E[W_i] = 0$ and $\text{Var}[W_i] = V(Q^* \| P)$.

A completely convex set must be convex. For convex $\Gamma$ and any $R \in \Gamma$, it holds that

$$
D(R \| P) \geq D(R \| Q^*) + D(Q^* \| P).
$$

For $x^n$ such that $L_{x^n} \in \Gamma$, substituting $L_{x^n}$ into the inequality above, we have

$$
E_{L_{x^n}} \left[ \log \frac{dQ^*}{dP}(X) \right] \geq D(Q^* \| P). \quad \text{(22)}
$$

The LHS is $\frac{1}{n} \sum_{i=1}^n \log \frac{dQ^*}{dP}(x_i)$. Hence, under i.i.d. $X_i \sim Q^*$, (22) reduces to $\sum_{i=1}^n W_i \geq 0$. This implies that

$$
E \left[ 1_{\{L_{x^n} \in \Gamma\}} e^{-\sum_{i=1}^n W_i} \right] \leq E \left[ 1_{\{\sum_{i=1}^n W_i \geq 0\}} e^{-\sum_{i=1}^n W_i} \right] \leq 1.
$$

Moreover, by Lemma 5, we obtain (19).

We next prove (20). We now lower bound the expectation in (21):

$$
E \left[ 1_{\{L_{x^n} \in \Gamma\}} e^{-\sum_{i=1}^n W_i} \right] \\
\geq E \left[ 1_{\{L_{x^n} \in B_{\epsilon}(Q^*) \cap \mathcal{H}_+\}} e^{-\sum_{i=1}^n W_i} \right] \\
\geq E \left[ 1_{\{L_{x^n} \in \mathcal{H}_+\}} e^{-\sum_{i=1}^n W_i} \right] \\
- E \left[ 1_{\{L_{x^n} \notin \mathcal{H}_+ \cap B_{\epsilon}(Q^*)\}} e^{-\sum_{i=1}^n W_i} \right].
$$
Note that, as shown above, \( L_{X^n} \in \mathcal{H}_+ \) is equivalent to \( \sum_{i=1}^{n} W_i \geq 0 \). So, the first expectation in the last line above satisfies

\[
\mathbb{E} \left[ \mathbf{1}_{\{L_{X^n} \in \mathcal{H}_+\}} e^{-\sum_{i=1}^{n} W_i} \right] = \mathbb{E} \left[ \mathbf{1}_{\{\sum_{i=1}^{n} W_i \geq 0\}} e^{-\sum_{i=1}^{n} W_i} \right] \sim \frac{e}{\sqrt{2\pi n V(Q^\star \| P)}} \text{ as } n \to \infty.
\]

The second expectation in the last line above is upper bounded by

\[
\mathbb{P}\{L_{X^n} \in \mathcal{H}_+ \cap B_\varepsilon(Q^\star)^c\} \leq \mathbb{P}\{L_{X^n} \in B_\varepsilon(Q^\star)^c\}.
\]

By Lemma 11, the RHS decays exponentially fast as \( n \to \infty \).

So, we finally obtain

\[
\mathbb{P}\{L_{X^n} \in \Gamma\} \geq \frac{e}{\sqrt{2\pi n V(Q^\star \| P)}} e^{-nD(Q^\star \| P)}.
\]

Combining this with (19) yields (20).

\[\square\]

6 Gibbs Conditioning Principle

The large deviation result is closely related to the Gibbs conditioning principle. The latter concerns an important question on conditional distributions in statistical mechanics. Let \( X_1, X_2, ..., X_n \) be a sequence of i.i.d. random variables with each follows \( P \) on a Polish space \( \mathcal{X} \). Denote \( L_{X^n} \) as its empirical measure. Given a set \( \Gamma \subseteq \mathcal{P}(\mathcal{X}) \) and a constraint \( L_{X^n} \in \Gamma \), what is the conditional law of \( X_1 \) when \( n \) is large? In other words, what are the limit points, as \( n \to \infty \), of the conditional probability measures

\[
Q_n(B) = \mathbb{P}\{X_1 \in B|L_{X^n} \in \Gamma\}, B \in \mathcal{B}_\mathcal{X}.
\]

**Theorem 13** (Gibbs’s Principle). \cite[Theorem 1]{Gibbs} Suppose that \( \Gamma \) is a completely convex Borel set such that \( I_\Gamma := D(\Gamma^0 \| P) = D(\Gamma \| P) \). Then, \( \{Q_n\} \) converges to the unique \( Q^\star \) attaining \( D(\Gamma \| P) \) as \( n \to \infty \) under the relative entropy (and hence also in the weak topology), i.e., \( \lim_{n \to \infty} D(Q_n \| Q^\star) = 0 \).

This theorem is a consequence of Theorem 10. An example of \( \Gamma \) is \( \Gamma = \{Q : \mathbb{E}_Q[X] \geq \alpha\} \) with \( \alpha \geq \mathbb{E}_P[X] \). The \( Q^\star \) in this case is

\[
dQ^\star = e^{\lambda x} \left( \frac{dQ}{dP} \right)^{\alpha} = \frac{e^{\lambda x}}{\mathbb{E}_P[e^{\lambda X}]} \]

for some \( \lambda \geq 0 \) such that \( \mathbb{E}_Q^\star[X] = \alpha \).
Proof. Denote $A_n = \{ x^n : x^n \in \Gamma \}$, $Q_{X^n} = P^{\otimes n}(\cdot | A_n)$, and $J \sim \text{Unif} [n]$ is a random time-index independent of $X^n$. Then, it is easily verified that $Q_{X_j} = Q_{X_1} = Q_n$. By the proof of Theorem 10,
\[
D(\Gamma^0 \| P) \geq \limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{P}\{L_{X^n} \in \Gamma\} \\
\geq \limsup_{n \to \infty} D(Q_n \| P) \\
\geq \liminf_{n \to \infty} D(Q_n \| P) \geq D(\Gamma \| P) \geq D(\Gamma \| P).
\]
So, $\lim_{n \to \infty} D(Q_n \| P) = I_{\Gamma}$.

For a convex Borel set $\Gamma$ and any $R \in \Gamma$, it holds that
\[
D(R \| P) \geq D(R \| Q^*) + D(Q^* \| P).
\]
So,
\[
D(Q_n \| P) \geq D(Q_n \| Q^*) + D(Q^* \| P).
\]
Taking limit, we obtain
\[
I_{\Gamma} \geq \lim_{n \to \infty} D(Q_n \| Q^*) + I_{\Gamma}.
\]
Hence, we obtain $\lim_{n \to \infty} D(Q_n \| Q^*) = 0$. 

7 Appendix: Proof of Theorem 2

Denote $\Lambda (\lambda) := \log P(e^{\lambda f})$. Denote its effective domain $\mathcal{D}_\Lambda := \{ \lambda \in \mathbb{R} : \Lambda (\lambda) < +\infty \}$. Then,
\[
\gamma^* (\alpha) = \sup_{\lambda \in \mathcal{D}_\Lambda} \lambda \alpha - \log P(e^{\lambda f}).
\]
Note that $\Lambda (0) = 0$ and hence, $0 \in \mathcal{D}_\Lambda$. If $\mathcal{D}_\Lambda = \{0\}$, then $\gamma^* (\alpha) = 0$ for any $\alpha$. For this case, $P(f)$ does not exist, i.e., $P(f^+) = P(f^-) = +\infty$ where $f^+ = \max \{ f, 0 \}$ and $f^- = -\min \{ f, 0 \}$ [6, Lemma 2.2.5].

Denote $R_0 = P(\cdot | [a_0, b_0])$ and $R_1 = P(\cdot | [a_1, b_1])$, where we choose $-a_0, b_0, -a_1, b_1$ sufficiently large and satisfying $a_0 \leq a_1 \leq b_0 \leq b_1$, $R_0(f) \leq \alpha$ and $R_1(f) \geq \alpha$. Denote $R_\theta = \theta R_0 + (1-\theta)R_1$ with $\theta \in [0, 1]$ chosen such that $R_\theta(f) = \alpha$. Here $\bar{\theta} = 1 - \theta$. Since $R_\theta$ is feasible for the infimization in $\gamma^*(\alpha)$, we have
\[
\bar{\gamma}(\alpha) \geq D(R_\theta \| P) \\
= -P([a_1, b_0]) \log (P([a_1, b_0])) \\
- \bar{\theta} P([a_0, a_1]) \log (\bar{\theta} P([a_0, a_1])) \\
- \theta P([b_0, b_1]) \log (\theta P([b_0, b_1])).
\]
Note that as \(-a_0, b_0, -a_1, b_1 \to \infty\), it holds that \(P([a_1, b_0]) \to 1\) and \(P([a_0, a_1]), P([b_0, b_1]) \to 0\). So, \(D(R_\theta \| P) \to 0\), which implies \(\gamma(\alpha) = 0\). Hence, \(\gamma^*(\alpha) = \gamma(\alpha)\) for the case \(\mathcal{D}_\Lambda = \{0\}\).

We next consider the case \(\mathcal{D}_\Lambda \neq \{0\}\). For this case, \(P(f)\) exists (possibly it is equal to \(+\infty\) or \(-\infty\)).

Denote \(\lambda_{\text{max}} := \sup \mathcal{D}_\Lambda\) and \(\lambda_{\text{min}} := \inf \mathcal{D}_\Lambda\). Then, \(\mathcal{D}_\Lambda\) is an interval with endpoints \(\lambda_{\text{min}}, \lambda_{\text{max}}\). Note that \(\Lambda\) is convex on \(\mathbb{R}\), and continuously differentiable on \(\mathcal{D}_\Lambda^\prime\). Moreover, \(\Lambda(0) = 0\) and \(\Lambda'(\lambda) = Q_\lambda(f)\) where \(\frac{dQ_\lambda}{df} = \frac{e^{\lambda f}}{P(e^{\lambda f})}\).

Decompose the integral into two parts: \(P(e^{\lambda f}) = \int_{\{f > 0\}} e^{\lambda f} dP + \int_{\{f \leq 0\}} e^{\lambda f} dP\). By Lebesgue’s dominated convergence theorem, \(\lambda \mapsto \int_{\{f > 0\}} e^{\lambda f} dP\) is continuous for \(\lambda \geq 0\) since \(e^{\lambda f} \leq 1\) for any \(\lambda \geq 0\). By Lebesgue’s dominated convergence theorem, \(\lambda \mapsto \int_{\{f \leq 0\}} e^{\lambda f} dP\) is continuous for \(0 \leq \lambda < \lambda_{\text{max}}\), and by the monotone convergence theorem, it is left-continuous at \(\lambda_{\text{max}}\) if \(\lambda_{\text{max}} < +\infty\) (even if \(\Lambda'(\lambda_{\text{max}}) = +\infty\)). So, \(\Lambda\) is continuous on \([0, +\infty)\) if \(\lambda_{\text{max}} = +\infty\), and \([0, \lambda_{\text{max}}]\) if \(\lambda_{\text{max}} < +\infty\). Similarly, \(\Lambda\) is continuous on \((-\infty, 0]\) if \(\lambda_{\text{min}} = -\infty\), and \([\lambda_{\text{min}}, 0]\) if \(\lambda_{\text{min}} > -\infty\). Hence, \(\Lambda\) is continuous on \(\mathcal{D}_\Lambda\). Similarly, \(\lambda \mapsto Q_\lambda(f)\) is also continuous on \(\mathcal{D}_\Lambda\).

For any \(Q\) such that \(Q(f) = \alpha\) and \(\lambda \in \mathcal{D}_\Lambda\),

\[
D(Q\|P) \geq \lambda (Q(f) - Q_\lambda(f)) + D(Q_\lambda\|P) = \lambda (\alpha - Q_\lambda(f)) + D(Q_\lambda\|P) = \lambda \alpha - \log P(e^{\lambda f}),
\]

which implies \(\gamma(\alpha) \geq \gamma^*(\alpha)\).

We next prove \(\gamma(\alpha) \leq \gamma^*(\alpha)\). We now divide the proof into four cases.

Case I: We consider the case in which there exists \(\lambda^* \in \mathcal{D}_\Lambda\) such that \(Q_{\lambda^*}(f) = \alpha\), where \(Q_\lambda\) a probability measure with density

\[
\frac{dQ_{\lambda^*}}{dP} = \frac{e^{\lambda^* f}}{P(e^{\lambda^* f})}.
\]

For this case, \(Q_{\lambda^*}\) is a feasible solution to the infimization in the definition of \(\gamma(\alpha)\). So,

\[
\gamma(\alpha) \leq D(Q_{\lambda^*}\|P) = \lambda^* \alpha - \log P(e^{\lambda^* f}) \leq \gamma^*(\alpha).
\]

We next consider other cases, in which there is no such \(\lambda^*\) described above, i.e., \(Q_\lambda(f) > \alpha, \forall \lambda \in \mathcal{D}_\Lambda\) or \(Q_\lambda(f) < \alpha, \forall \lambda \in \mathcal{D}_\Lambda\).

Case II: We consider the case in which \(\lambda_{\text{max}} = +\infty\) and \(Q_\lambda(f) < \alpha, \forall \lambda \in \mathcal{D}_\Lambda\), which implies that \(\lim_{\lambda \to +\infty} Q_\lambda(f) \leq \alpha\), i.e., \(\text{esssup}_P f \leq \alpha\).
If \( \alpha > \text{esssup}_PF \), then both \( \gamma(\alpha) \) and \( \gamma^*(\alpha) \) are equal to \( +\infty \).

If \( \alpha = \text{esssup}_PF \) and \( P(f^{-1}(\alpha)) > 0 \), then define \( Q_\infty = P(\cdot | f^{-1}(\alpha)) \). For this case, \( \gamma(\alpha) = D(Q_\infty \| P) \) and \( \gamma^*(\alpha) = \lim_{\lambda \to \infty} -\log P(e^{\lambda(f-\alpha)}) \). Denote \( A_\delta := \{ f \geq \alpha - \delta \} \) for \( \delta > 0 \) and \( A = f^{-1}(\alpha) \). Then, \( P(A) \leq P(e^{\lambda(f-\alpha)}) \leq P(A_\delta) + e^{-\lambda\delta} \). Hence, \( P(A) \leq \lim_{\lambda \to \infty} P(e^{\lambda(f-\alpha)}) \leq P(A_\delta) \) for any \( \delta > 0 \), which implies \( \lim_{\lambda \to \infty} P(e^{\lambda(f-\alpha)}) \leq \lim_{\delta \to 0} P(A_\delta) = P(\lim_{\delta \to 0} A_\delta) = P(A_0) \).

Here, we use the obvious fact \( \lim_{\delta \to 0} A_\delta = A_0 \). Moreover, by the definition of essential supremum, \( P(A) = P(A_0) \). Therefore, \( \gamma^*(\alpha) = -\log P(A) = D(Q_\infty \| P) = \gamma(\alpha) \).

If \( \alpha = \text{esssup}_PF \) and \( P(f^{-1}(\alpha)) = 0 \), then \( P(f \geq \alpha) = 0 \) which implies that for any \( Q \ll P, Q(f \geq \alpha) = 0 \). Since the probability measure is continuous in events, we have \( \lim_{\alpha' \to \alpha} Q(f < \alpha') = Q(f < \alpha) = 1 \). So, for any \( \delta > 0 \), there is \( \alpha' < \alpha \) such that \( 1 - \delta < Q(f < \alpha') \leq 1 \), which implies \( Q(f) < \alpha \). Hence, \( \gamma(\alpha) = +\infty \). On the other hand, \( \gamma^*(\alpha) = \lim_{\lambda \to \infty} -\log P(e^{\lambda(f-\alpha)}) \).

Similarly, for any \( \delta > 0 \), there is \( \alpha' < \alpha \) such that \( 1 - \delta < P(f < \alpha') \leq 1 \), which implies for any real \( \lambda, P(e^{\lambda f}) < e^{\lambda\alpha'}(1-\delta) + e^{\lambda\alpha}\delta \). Hence, \( \lim_{\lambda \to \infty} P(e^{\lambda(f-\alpha)}) \leq \lim_{\lambda \to \infty} e^{\lambda(\alpha'-\alpha)(1-\delta)} + \delta = \delta \), which could be arbitrarily small since \( \delta > 0 \) is arbitrary. So, \( \gamma^*(\alpha) = +\infty \).

Case III: We now consider the case in which \( \lambda_{\text{max}} < +\infty \) and \( Q_\lambda(f) < \alpha, \forall \lambda \in D_A \), which implies that \( \lambda_{\text{max}} \in D_A \) since otherwise, \( \Lambda'(\lambda) = Q_\lambda(f) \to \infty \) as \( \lambda \uparrow \lambda_{\text{max}} \). For this case, obviously,

\[
\gamma^*(\alpha) = \lambda_{\text{max}} \alpha - \log P(e^{\lambda_{\text{max}} f}).
\]

We next prove that \( \gamma(\alpha) \) is also equal to this expression.

For \( b > 0 \), define \( f_b(x) = \min\{f(x), b\} \). Denote \( Q_{b,\lambda} \) as a probability measure with density

\[
\frac{dQ_{b,\lambda}}{dP} = \frac{e^{bf_b}}{P(e^{bf_b})}.
\]

**Fact 1.** For all sufficiently large \( b \), \( Q_{b,\lambda_{\text{max}}}(f) < \alpha \).

**Fact 2.** Given any \( \lambda > \lambda_{\text{max}} \), it holds that \( Q_{b,\lambda}(f) \to +\infty \) as \( b \to \infty \).

Fact 1 follows since by the monotone convergence theorem, \( \lim_{b \to \infty} Q_{b,\lambda_{\text{max}}}(f) = \lim_{b \to \infty} \frac{P(e^{\lambda_{\text{max}}(b-f)})}{P(e^{\lambda_{\text{max}}f})} = \frac{P(e^{\lambda_{\text{max}}f})}{P(e^{\lambda_{\text{max}}f})} = Q_{\lambda_{\text{max}}}(f) < \alpha \).

We now prove Fact 2. Note that by the monotone convergence theorem,
$P(e^{λf_b}) \to P(e^{λf}) = +\infty$ as $b \to \infty$. Furthermore, for any $b_0 < b$,

$$Q_{b,λ}(f) \geq Q_{b,λ}(f_b) = \frac{P(e^{λf_b})}{P(e^{λf_b})} = \frac{\int_{\{f \leq b_0\}} e^{λf} f dP + \int_{\{b_0 < f \leq b\}} e^{λf_b} f_b dP}{\int_{\{f \leq b_0\}} e^{λf} dP + \int_{\{b_0 < f \leq b\}} e^{λf_b} dP} \geq \frac{\int_{\{f \leq b_0\}} e^{λf} f dP + b_0 \int_{\{b_0 < f \leq b\}} e^{λf_b} f_b dP}{\int_{\{f \leq b_0\}} e^{λf} dP + \int_{\{b_0 < f \leq b\}} e^{λf_b} dP}.$$  

For fixed $b_0$, both the first integrals in numerator and denominator at the last line are finite, and the integral $\int_{\{b_0 < f \leq b\}} e^{λf_b} f_b dP$ tends to infinity as $b \to \infty$. So, $\liminf_{b \to \infty} Q_{b,λ}(f_b) \geq b_0$. Since $b_0$ is arbitrary, $\lim_{b \to \infty} Q_{b,λ}(f_b) = +\infty$, completing the proof of Fact 2.

By Facts 1 and 2 and the continuity of $λ \mapsto Q_{b,λ}(f)$, for sufficiently large $b$, there is $λ_b > λ_{max}$ such that $Q_{b,λ_b}(f) = α$. So,

$$γ(α) \leq λ_b α - \log P(e^{λ_b f_b}).$$

Letting $b \to \infty$, it holds that $λ_b \downarrow λ_{max}$ and $f_b \uparrow f$, which implies that the RHS converges to $λ_{max} α - \log P(e^{λ_{max} f})$. Hence, $γ(α) \leq λ_{max} α - \log P(e^{λ_{max} f})$.

Case IV: We consider the case $Q_λ(f) > α$, $∀λ \in D_Λ$. Following arguments similar to those in Cases II and III, $γ(α) = γ^*(α)$ still holds for this case.

**Acknowledgements**

The author would like to thank Prof. Vincent Y. F. Tan from the National University of Singapore for his suggestion to investigate the Gärtner–Ellis theorem by using the entropy method.

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