The noncommutative harmonic oscillator in more than one dimensions

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Abstract

The noncommutative harmonic oscillator in arbitrary dimension is examined. It is shown that the ⋆-genvalue problem can be decomposed into separate harmonic oscillator equations for each dimension. The noncommutative plane is investigated in greater detail. The constraints for rotationally symmetric solutions and the corresponding two-dimensional harmonic oscillator are solved. The angular momentum operator is derived and its ⋆-genvalue problem is shown to be equivalent to the usual eigenvalue problem. The ⋆-genvalues for the angular momentum are found to depend on the energy difference of the oscillations in each dimension. Furthermore two examples of assymetric noncommutative harmonic oscillator are analysed. The first is the noncommutative two-dimensional Landau problem and the second is the three-dimensional harmonic oscillator with symmetrically noncommuting coordinates and momenta.

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1 Introduction

Noncommutative spaces, within the formulation of the deformation quantization through Moyal product, have been receiving increasing attention in field theories on $R^d$, string theories, in which they arise naturally as D-brane configuration spaces, and M-theory.

There has been considerable progress towards the direction of understanding the interplay between the UV and IR divergencies stemming from the underlying noncommutativity of scalar field theories on flat space-time. Some authors partially motivated by the suggestion of [1] on the possible stringy origin of the intriguing UV/IR mixing, have explored the relationship between noncommutative field theory and string theory [2].

In string theory it was realized recently that the introduction of a constant $B$-field gives rise to noncommutative string position operators. The multiloop amplitudes were computed for this noncommutative bosonic string in [3].

The appearance of noncommutative spaces triggered the reformulation of Quantum Mechanics [4]. In this set up the one dimensional Schrödinger equation was replaced by the $\star$-genvalue equation, while the wave functions became Wigner functions. The ordinary product was replaced by the pivotal associative noncommutative $\star$-product.

In the present work we consider the $\star$-genvalue problem for the n-dimensional noncommutative harmonic oscillator. It is shown that again the $\star$-genvalue problem is equivalent to the Schrödinger problem in an appropriate representation. The energy eigenvalues and eigenfunctions are determined as functions of the noncommutativity parameters.

The case of the two dimensional harmonic oscillator is examined thoroughly. The angular momentum operator is derived in the rotationally symmetric case. It is shown that the $\star$-genvalues for this operator contain, apart from the usual angular momentum, a term that depends on the energy difference of the oscillations in the two dimensions. This is to be interpreted as the angular momentum of the system.

In the asymmetric case critical values of the parameters arise. For these values the energy spectrum becomes infinitely degenerate at every level. This problem can be identified with the noncommutative Landau problem.

The paper is organized as follows:

In section 2 we summarize known results namely the definition of the Moyal product of functions in phase-space through the Weyl ordering prescription and the equivalence of the stationary $\star$-genvalue problem to the corresponding Schrödinger problem.

Section 3 is dedicated to the study of the two-dimensional harmonic oscillator for nontrivial commutation relations. The $\star$-genvalues and functions are determined for the corresponding Hamiltonian by solving the imaginary and real part equations.

In section 4 the $\star$-genvalue problem for the Hamiltonian and the angular momentum in two-dimensions is investigated. It is shown, that the Schrödinger problem in suitably transformed phase-space variables is equivalent to the $\star$-genvalue problem. Furthermore the $\star$-genvalue problem for the angular momentum enables one to calculate the eigenvalues of the associated operator.

In section 5 the existence of rotationally symmetric $\star$-genfunctions impose constraints on the possible commutation relations for the n-dimensional harmonic oscillator. In two-dimensions these constraints are solved explicitly and the $\star$-genvalue problem for the Hamiltonian is solved. The angular momentum operator, which is the generator of rotations, is constructed and its $\star$-genvalues are computed.

Finally section 6 copes with the most general commutation relations one can possibly en-
counter. We show that it is possible, through orthogonal transformations, to bring the matrix realizing the commutation relations into a symplectic form. In this way we can give explicit results for the energy levels, eigenfunctions and the $\star$-commutation relation of creation and annihilation operators for the $n$-dimensional harmonic oscillator. As a non-rotationally symmetric application we consider the noncommutative Landau problem and find a critical point for the magnetic field at which the energy levels are infinitely degenerate \[5\]. As a further application the three-dimensional noncommutative harmonic oscillator is considered.

2 Overview of the Wigner functions and the $\star$-genvalue problem

Let us start with a classical Hamiltonian $H(q,p) = p^2/2m + V(q)$ in one dimension. Upon quantization the canonical variables $q,p$ become operators $\hat{q}, \hat{p}$ satisfying the canonical commutation relations $[\hat{q}, \hat{p}] = i\hbar$. Consider now monomials of the form $q^m p^n$ with $m, n$ positive integers. To define the corresponding operator product it is possible to use the Weyl ordering prescription \[6\]:

$$\hat{W}(q^m p^n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \hat{p}^{n-k} q^m \hat{p}^k\tag{1}$$

according to which the $\hat{W}(q^m p^n)$ operator is symmetrized in $\hat{q}$ and $\hat{p}$ by use of Heisenberg's commutation relation. This regularization scheme can be extended to act on arbitrary power series functions $f(q,p)$ through linearity. Thus, Weyl ordering is an invertible map from the space of functions on the phase-space to the space of quantum operators. We can use now Weyl ordering to define a new product between functions on the phase-space \[4\], \[6\]:

$$f(q,p) \star g(q,p) = W^{-1}(W(f)W(g))\tag{2}$$

This is the celebrated Moyal product which enjoys the properties of noncommutativity, associativity and uniqueness.

In phase-space one can define Wigner quasi-distribution functions to calculate matrix elements of observables. The time-independent Wigner function corresponding to a pair of eigenstates $|\psi_n\rangle, |\psi_m\rangle$ of the Schrödinger problem, $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, is represented in two phase-space dimensions by:

$$f_{mn}(q,p) = \frac{1}{2\pi} \int dy e^{-ipy} <q - \frac{\hbar}{2} y|\psi_n\rangle <\psi_m|q + \frac{\hbar}{2} y > f^*_m(q,p)\tag{3}$$

where $\star$ stands for complex conjugation. In this case, one can show by employing the classical Hamiltonian and the definition of the star product \[2\] that Wigner functions obey the $\star$-genvalue equations:

$$H(q,p) \star f_{mn}(q,p) = E_n f_{mn}(q,p) ; \quad f_{mn}(q,p) \star H(q,p) = E_m f_{mn}(q,p).\tag{4}$$
In (3) for complete sets $|\psi_n> \,$ one can also derive that:

$$\sum_{m,n} f_{mn}(q,p)f^*_{mn}(q',p') = \frac{1}{2\pi \hbar} \delta(q - q') \delta(p - p')$$  \hspace{1cm} (5)$$

which enables the construction of arbitrary phase-space functions in terms of $f_{mn}(q,p)$.

For the diagonal case ($m=n$ in (3)) one can prove that the Wigner functions satisfy the diagonal $*$-genvalue equation:

$$H(q,p) \ast f_n(q,p) = f_n(q,p) \ast H(q,p) = E_n f_n(q,p)$$  \hspace{1cm} (6)$$

Furthermore, the real solutions of (3) are required to be of the Wigner type for the wavefunctions of $\hat{H}|\psi_n> = E_n|\psi_n>$. So, instead of solving the Schrödinger equation one can try to solve the $*$-genvalue equation to determine directly the diagonal Wigner functions and the corresponding energy spectrum. The orthogonality relation satisfied by those functions takes the form:

$$f_m \ast f_n = \frac{1}{2\pi \hbar} \delta_{m,n} f_m.$$  \hspace{1cm} (7)$$

3 The two-dimensional harmonic oscillator

In this case we have four coordinates $(q_i, p_j)$, with $i, j = 1, 2$, in phase space. The free Hamiltonian for this model is:

$$H(q_i, p_j) = \frac{1}{2} \sum_{i=1}^{2} (q^2_i + p^2_i)$$  \hspace{1cm} (8)$$

where, without loss of generality, parameters have been absorbed in the phase-space variables. Quantum Mechanically, the position and momentum operators satisfy the Heisenberg commutation relations. Here we will extend this realization to include nontrivial commutation relations for $q_1, q_2$ and $p_1, p_2$:

$$[q_i, p_j] = i\hbar \delta_{ij} \hspace{0.5cm} [q_1, q_2] = i\theta \hspace{0.5cm} [p_1, p_2] = -i\theta$$  \hspace{1cm} (9)$$

where $\theta$ is a real constant. The related star product takes the form:

$$\ast = \exp \left[ \frac{i}{2} \left( \partial_{q_1}, \partial_{p_1}, \partial_{q_2}, \partial_{p_2} \right) \right] \hspace{0.5cm} \left( \begin{array}{cccc} 0 & \hbar & \theta & 0 \\ -\hbar & 0 & 0 & -\theta \\ -\theta & 0 & 0 & \hbar \\ 0 & \theta & -\hbar & 0 \end{array} \right) \left( \begin{array}{c} \partial_{q_1} \\ \partial_{p_1} \\ \partial_{q_2} \\ \partial_{p_2} \end{array} \right) \right].$$  \hspace{1cm} (10)$$

The resulting $*$-genvalue equation is:

$$\left[ \left( q_1 + \frac{i\hbar}{2} \partial_{p_1} + \frac{i\theta}{2} \partial_{q_1} \right)^2 + \left( p_1 - \frac{i\hbar}{2} \partial_{q_1} - \frac{i\theta}{2} \partial_{p_1} \right)^2 \right] + \left[ \left( q_2 + \frac{i\hbar}{2} \partial_{p_2} - \frac{i\theta}{2} \partial_{q_1} \right)^2 + \left( p_2 - \frac{i\hbar}{2} \partial_{q_2} + \frac{i\theta}{2} \partial_{p_1} \right)^2 \right] = 2E f(q_1, p_1, q_2, p_2).$$  \hspace{1cm} (11)$$

Equation (11) splits into an equation for the imaginary part,

$$\hbar (p_1 \partial_{q_1} - q_1 \partial_{p_1}) + \hbar (p_2 \partial_{q_2} - q_2 \partial_{p_2}) + \theta (q_2 \partial_{q_1} - q_1 \partial_{q_2}) + \theta (p_1 \partial_{p_2} - p_2 \partial_{p_1}) \hspace{0.5cm} f = 0$$  \hspace{1cm} (12)$$
and an equation for the real part:

\[
\left[ \left( q_1^2 + p_1^2 + q_2^2 + p_2^2 \right) - \frac{(h^2 + \theta^2)}{4} \left( \partial_{q_1}^2 + \partial_{p_1}^2 + \partial_{q_2}^2 + \partial_{p_2}^2 \right) \right] f = 2Ef. \tag{13}
\]

Equation (12) admits a solution of the form \( f(z) \) with \( z = 2(q_1^2 + p_1^2 + q_2^2 + p_2^2) = 4H \). The real part equation (13) transforms to the ordinary differential equation:

\[
\left[ z\partial_z^2 + 2\partial_z + \frac{1}{(h^2 + \theta^2)} \left( E - \frac{z}{4} \right) \right] f(z) = 0. \tag{14}
\]

The problem with this reduction is the inconsistency underlying the number of degrees of freedom. We started with the two dimensional harmonic oscillator and we ended up with one oscillation equation. So we need to search for a set of transformations that will preserve the number of degrees of freedom. The appropriate transformations are:

\[
\begin{align*}
\hat{q}_1 &= q_1 \\
\hat{p}_1 &= \frac{1}{\sqrt{h^2 + \theta^2}} (hp_1 + \theta q_2) \\
\hat{q}_2 &= \frac{1}{\sqrt{h^2 + \theta^2}} (hq_2 - \theta p_1) \\
\hat{p}_2 &= p_2.
\end{align*} \tag{15}
\]

The imaginary part equation is then written as:

\[
\left[ (\hat{p}_1 \partial_{q_1} - \hat{q}_1 \partial_{p_1}) + (\hat{p}_2 \partial_{q_2} - \hat{q}_2 \partial_{p_2}) \right] f = 0. \tag{16}
\]

This implies that \( f \) is a function of \( z_1 = 2(\hat{q}_1^2 + \hat{p}_1^2) \) and \( z_2 = 2(\hat{q}_2^2 + \hat{p}_2^2) \). The real part equation transforms under the new variables into:

\[
\left[ z_1\partial_{z_1}^2 + \partial_{z_1} + z_2\partial_{z_2}^2 + \partial_{z_2} + \frac{1}{(h^2 + \theta^2)} \left( E - \frac{(z_1 + z_2)}{4} \right) \right] f(z_1, z_2) = 0. \tag{17}
\]

The diagonal \( \star \)-genfunctions \( f_{nm}(z_1, z_2) \) associated with this equation are determined through products of Laguerre polynomials:

\[
f_{nm}(\tilde{z}_1, \tilde{z}_2) = e^{-\frac{1}{2}(\tilde{z}_1 + \tilde{z}_2)} L_n(\tilde{z}_1) L_m(\tilde{z}_2) \tag{18}
\]

where:

\[
L_m(\tilde{z}_i) = \frac{1}{m!} e^{\tilde{z}_i} \frac{d^m}{d^{m}\tilde{z}_i} \left( e^{-\tilde{z}_i} \tilde{z}_i^m \right) \tag{19}
\]

and \( L_0(\tilde{z}_i) = 1, \ L_1(\tilde{z}_i) = 1 - \tilde{z}_i, \ L_2(\tilde{z}_i) = 1 - 2\tilde{z}_i + \frac{\tilde{z}_i^2}{2}, \ldots \) with \( \tilde{z}_i = z_i/\sqrt{h^2 + \theta^2} \). The energies corresponding to these \( \star \)-genfunctions are:

\[
E_{nm} = \sqrt{h^2 + \theta^2} (n + m + 1). \tag{20}
\]

The annihilation and creation operators are given in terms of the transformed variables as:

\[
\begin{align*}
a_i &= \frac{1}{\sqrt{2}} (\hat{q}_i + i\hat{p}_i) \\
a_i^\dagger &= \frac{1}{\sqrt{2}} (\hat{q}_i - i\hat{p}_i).
\end{align*} \tag{21}
\]
It is possible to express these operators in terms of the original variables by reversing our transformations. They satisfy the following modified commutation relations

\[ a_i \star a_j^\dagger - a_j^\dagger \star a_i = \delta_{ij} \sqrt{\hbar^2 + \theta^2} \]  

and \( a_i \star f_{00}(\bar{q}_i, \bar{p}_i) = 0 \). So they generate the \( \star \)-Fock space of states as follows:

\[ f_{nm} \propto a_1^{\dagger n} a_2^{\dagger m} \star f_{00} \star a_2^m a_1^n. \]  

The Hamiltonian takes the following form in terms of annihilation-creation operators:

\[ H = \sum_{i=1}^{2} \left( a_i \star a_i + \frac{\sqrt{\hbar^2 + \theta^2}}{2} \right). \]

### 4 Quantum Mechanics and the \( \star \)-genvalue problem in 2D

In the one dimensional case the \( \star \)-genvalue equation is equivalent to the Schrödinger equation, as was shown in [4]. However, in the two dimensional case one has to search for a suitable representation of the commutation relations before writing down an equation. This problem can be overcome for the harmonic oscillator of section 3 by using the transformations (15), to transform the commutation relations into two sets of Heisenberg commutation relations with \( \hbar \) replaced by \( \sqrt{\hbar^2 + \theta^2} \). The Schrödinger equation with respect to the new variables becomes:

\[ \frac{1}{2} \sum_{i=1}^{2} \left( \hat{q}_i^2 + \hat{p}_i^2 \right) \psi(\bar{q}_i, \bar{p}_i) = E \psi(\bar{q}_i, \bar{p}_i) \]  

and its eigenfunctions are given by:

\[ \psi_{nm}(\bar{q}_i, \bar{p}_i) = \psi_n(\bar{q}_1, \bar{p}_1) \psi_m(\bar{q}_2, \bar{p}_2) \]  

where \( \psi_n \) are the usual eigenfunctions of the one dimensional harmonic oscillator. This equation splits into two one-dimensional harmonic oscillator equations. Making use of the one-dimensional equivalence of the Schrödinger problem to the \( \star \)-genvalue problem we obtain that equation (25) (with \( \hbar \) replaced by \( \sqrt{\hbar^2 + \theta^2} \)) is equivalent to:

\[ H(\bar{q}_i, \bar{p}_i) \star f(\bar{q}_i, \bar{p}_i) = f(\bar{q}_i, \bar{p}_i) \star H(\bar{q}_i, \bar{p}_i) = E f(\bar{q}_i, \bar{p}_i). \]

The \( \star \)-product, which is the transformed version of the \( \star \)-product defined in (13), is:

\[ \star = \exp \left[ \frac{i}{2} \sqrt{\hbar^2 + \theta^2} \sum_{i=1}^{2} \left( \overleftarrow{\partial}_{\bar{q}_i} \overrightarrow{\partial}_{\bar{p}_i} - \overleftarrow{\partial}_{\bar{p}_i} \overrightarrow{\partial}_{\bar{q}_i} \right) \right]. \]

Going back to the original variables the Schrödinger problem is translated into the \( \star \)-genvalue problem of section 3. The diagonal Wigner functions are given in terms of the wave functions by:

\[ f(\bar{q}_1, \bar{p}_2) = f(\bar{q}_1, \bar{p}_1) f(\bar{q}_2, \bar{p}_2) \]

\[ = \frac{1}{(2\pi)^2} \int d\bar{y}_1 \ d\bar{y}_2 \ e^{-i\bar{y}_1 \bar{p}_1} e^{-i\bar{y}_2 \bar{p}_2} \prod_{i=1}^{2} \Psi_i^*(\bar{q}_i - \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_i}) \Psi_i(\bar{q}_i + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_i}) \]

\[ = \frac{1}{(2\pi)^2} \int d\bar{y}_1 \ d\bar{y}_2 \ e^{-i\bar{y}_1 \bar{p}_1} e^{-i\bar{y}_2 \bar{p}_2} \prod_{i=1}^{2} \Psi_i^* \Psi_i. \]  

(29)
Expressing the Wigner functions in the original variables one has: \( f_0(q_i, p_i) \equiv f(\tilde{q}_i, \tilde{p}_i) \).

Next let us investigate what happens with the angular momentum. The operator that commutes with the Hamiltonian and generates rotations in the original variables is:

\[
L = \frac{\hbar}{\hbar^2 + \theta^2} \left[ \hbar (q_1 p_2 - q_2 p_1) + \frac{\theta}{2} (p_1^2 - q_1^2 + p_2^2 - q_2^2) \right].
\]

Transforming this to the new variables gives:

\[
\tilde{L}(\tilde{q}_i, \tilde{p}_j) = \frac{\hbar}{\hbar^2 + \theta^2} \left[ \hbar (\tilde{q}_1 \tilde{p}_2 - \tilde{q}_2 \tilde{p}_1) + \frac{\theta}{2} (\tilde{p}_2^2 - \tilde{p}_1^2 - \tilde{q}_2^2 + \tilde{q}_1^2) \right].
\]

The angular momentum \( \star \)-genvalue equation becomes:

\[
L (q_i, p_i) \star f_0(q_i, p_i) = \tilde{L}(\tilde{q}_i, \tilde{p}_j) \star f(\tilde{q}_i, \tilde{p}_j)
\]

\[
= \frac{1}{(2\pi)^2} \frac{\hbar}{(\hbar^2 + \theta^2)} \hbar \left[ \tilde{q}_1 \left( \tilde{p}_2 - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_2}} \right) - \tilde{q}_2 \left( \tilde{p}_1 - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right) \right]
\]

\[
+ \frac{\theta}{2} \left[ \tilde{q}_2^2 + \left( \tilde{p}_2 - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_2}} \right)^2 - \tilde{q}_1^2 - \left( \tilde{p}_1 - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right)^2 \right]
\]

\[
\times \int dy_1 \int dy_2 \ e^{-iy_1 \left( \tilde{p}_1 + \frac{\theta}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right)} e^{-iy_2 \left( \tilde{p}_2 + \frac{\theta}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_2}} \right)} \prod_{i=1}^{2} \Psi_i^* \Psi_i
\]

\[
= \frac{1}{(2\pi)^2} \frac{\hbar}{(\hbar^2 + \theta^2)} \int \int dy_1 \int dy_2 \ e^{-iy_1 \tilde{p}_1} e^{-iy_2 \tilde{p}_2}
\]

\[
\times \left[ \left( \tilde{q}_1 + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_1} \right) \left( -i \partial_{y_2} - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_2}} \right) \right]
\]

\[
- \left( \tilde{q}_2 + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_2} \right) \left( -i \partial_{y_1} - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right)
\]

\[
+ \frac{\theta}{2} \left[ \left( \tilde{q}_2 + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_2} \right)^2 \left( -i \partial_{y_1} - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right)^2 \right]
\]

\[
- \left( \tilde{q}_1 + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 y_1} \right)^2 \left( -i \partial_{y_1} - \frac{i}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_1}} \right)^2 \right] \prod_{i=1}^{2} \Psi_i^* \Psi_i
\]

\[
= \frac{\hbar}{(\hbar^2 + \theta^2)} \left[ \hbar \sqrt{\hbar^2 + \theta^2 m_z + \theta (E_2 - E_1)} \right] f(\tilde{q}_i, \tilde{p}_j).
\]

Here we have used \( p_i e^{-iy_i p_i} = i \partial_{y_i} e^{-iy_i p_i} \) and partial integration has been performed. Note that:

\[
\left( \partial_{y_i} + \frac{1}{2} \sqrt{\hbar^2 + \theta^2 \partial_{\tilde{q}_i}} \right) \Psi^* \left( \tilde{q}_i - \frac{\sqrt{\hbar^2 + \theta^2}}{2} y_i \right) = 0.
\]

The discrete values \( m_z \) are the eigenvalues of the angular momentum for the commutative two dimensional harmonic oscillator:

\[
\tilde{L} \Psi(\tilde{q}_i, \tilde{p}_j) = (\tilde{q}_1 \hat{p}_2 - \tilde{q}_2 \hat{p}_1) \Psi(\tilde{q}_i, \tilde{p}_j) = \sqrt{\hbar^2 + \theta^2 m_z} \Psi(\tilde{q}_i, \tilde{p}_j).
\]
5 Rotationally symmetric case

Up to this point we have considered only the case where $q_i$ commute with $p_j$, for $i \neq j$ and the $q_i$, $p_j$ behave symmetrically, that is $[q_1, q_2] = -[p_1, p_2]$. Lets consider the more general situation where the commutation relations are governed by a general antisymmetric matrix $M$. In this case the $\star$-product reads:

$$\star = \exp \left( \frac{i}{2} \left[ \frac{\partial^T}{\partial I} M_{IJ} \frac{\partial}{\partial J} \right] \right)$$

(34)

where $\partial^T_I = (\partial_{q_i}, \partial_{p_j})$, and $M_{IJ} = -M_{JI}^T$ are $2 \times 2$ matrices. Here the summation convention is assumed. The imaginary part of the $\star$-genvalue equation now becomes:

$$X^T_I M_{IJ} \partial_J f = 0$$

(35)

where $X_I^T = (q_I, p_I)$. If $f \equiv f(X^T_I X_I)$ then this equation is satisfied provided that $M$ is antisymmetric. Interestingly enough, if one starts with the $\star$-genvalue problem for this $f$, ignoring the commutation relations, equation (35) would require that the matrix $M$ be antisymmetric.

The real part equation turns into the form:

$$\left[ X^T_I X_I - \frac{1}{4} (M_{IK_1} \partial_{K_1})^T (M_{IK_2} \partial_{K_2}) \right] f = 2E f.$$  

(36)

Again demanding $f = f(X^T_I X_I)$ we are led to:

$$X^T_I X_I f(X^T_I X_I) - \frac{1}{2} tr(M_{IK_1} M_{IK_2}) f'(X^T_I X_I) - X^T_I K_1 M_{IK_1} M_{IK_2} X_{K_2} f''(X^T_I X_I) = 2E f(X^T_I X_I).$$  

(37)

If this equation is to have rotationally symmetric solutions we need:

$$M^T_{IK_1} M_{IK_2} = \alpha \delta_{K_1 K_2} I$$

(38)

with $I$ the identity matrix.

Condition (38) can be solved explicitly in the case of four phase-space dimensions. The most general matrix $M$ that admits rotationally symmetric $\star$-genfunctions is given by:

$$M_{11} = \pm M_{22} = \begin{pmatrix} 0 & \hbar \\ -\hbar & 0 \end{pmatrix}; \quad M_{12} = -M_{21} = \begin{pmatrix} \theta & \phi \\ \mp \phi & \mp \theta \end{pmatrix}$$

(39)

where the two signs correspond to the two possible solutions that exist. We are going to examine more closely the first case, since the second can be treated on equal footing. In the present case the transformations that lead to the Heisenberg commutation relations are: $\tilde{X} = R^T X$, where $R$ is given by:

$$R = \begin{pmatrix}
\sqrt{\hbar^2 + \phi^2} & 0 & 0 & -\frac{\theta}{\sqrt{\hbar^2 + \phi^2}} \\
\sqrt{\hbar^2 + \phi^2} & 0 & 0 & -\frac{\phi}{\sqrt{\hbar^2 + \phi^2}} \\
\frac{\hbar}{\sqrt{\hbar^2 + \phi^2}} & \sqrt{\hbar^2 + \phi^2} & 0 & 0 \\
\frac{\phi}{\sqrt{\hbar^2 + \phi^2}} & \sqrt{\hbar^2 + \phi^2} & 0 & \hbar \\
\end{pmatrix}$$

(40)

Certainly these transformations are not unique. They are defined up to symplectic rotations which preserve both the Hamiltonian and the commutation relations in the transformed variables.
The angular momentum is found to be:

\[ L = \frac{\hbar}{\hbar^2 + \theta^2 + \phi^2} \left[ \hbar(q_1p_2 - q_2p_1) + \frac{\theta}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2) - \phi(p_1q_1 + p_2q_2) \right] \tag{41} \]

and in the transformed variables is reexpressed as:

\[ \bar{L} = \frac{\hbar}{\hbar^2 + \theta^2 + \phi^2} \left[ \sqrt{\hbar^2 + \phi^2}(q_1\bar{p}_2 - q_2\bar{p}_1) + \frac{\theta}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2) - \phi(p_1q_1 + p_2q_2) \right]. \tag{42} \]

Again the \( \star \)-genvalue problem for the Hamiltonian is the same as the Schrödinger problem with \( \star \)-genvalues \( E_{n_1,n_2} = \sqrt{\hbar^2 + \theta^2 + \phi^2(n_1 + n_2 + 1)} \). The same equivalence is valid for the angular momentum \( \star \)-genvalue problem producing the \( \star \)-genvalues

\[ \frac{\hbar}{\hbar^2 + \theta^2 + \phi^2} \left[ \sqrt{\hbar^2 + \phi^2}\sqrt{\hbar^2 + \theta^2 + \phi^2}m_z + \theta(E_1 - E_2) \right]. \]

In the limit \( \theta, \phi \to 0 \) we can recover from these expressions the usual ones.

### 6 General Case

Let us assume now that the matrix \( M \) used to define the commutation relations is a general antisymmetric matrix. The following lemma holds [7]:

**Lemma 1** Let \((V, \omega)\) be a symplectic vector space and \( g : V \times V \to \mathbb{R} \) be an inner product. Then there exists a basis \( u_1, \ldots, u_n, v_1, \ldots, v_n \) of \( V \) which is both \( g \)-orthogonal and \( \omega \)-standard. Moreover, this basis can be chosen such that

\[ g(u_j, u_j) = g(v_j, v_j) \]

for all \( j \). This means that it is possible, by rescaling, to find an orthogonal transformation \( R \) so that:

\[ R^TMR = J(M) \tag{43} \]

where

\[ J(M)_{IJ} = \alpha_I \delta_{IJ} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{44} \]

From (34) we see that if we make the transformation \( \bar{X} = R^TX \), then the matrix \( M \) in the \( \star \)-product is replaced by \( J(M) \) and the Hamiltonian remains invariant because the transformation is orthogonal. So the \( \star \)-genvalue problem now becomes:

\[ \bar{H}\bar{\star}f = Ef \tag{45} \]

where \( \bar{\star} \) is

\[ \bar{\star} = \exp \left[ i \frac{\bar{T}}{2} \partial_I J(M)_{IK} \bar{\partial}_K \right]. \tag{46} \]

The imaginary part equation (35) becomes:

\[ \sum_i \alpha_i(q_i\partial_{\bar{p}_i} - \bar{p}_i\partial_{q_i})f(q_i, \bar{p}_i) = 0 \tag{47} \]
where \( \alpha_i = \alpha_I \). This equation is satisfied by \( f \equiv f(z_i) \) where \( z_i = 2(q_i^2 + p_i^2) \). For this \( f \), the real part equation takes the form:

\[
\sum_i \left[ z_i \partial^2_{z_i} + \partial_{z_i} - \frac{1}{\alpha_i^2} \left( \frac{z_i}{4} - E_i \right) \right] f(z_i) = 0
\]

(48)

where \( E = \sum_i E_i \). This equation can be separated into a set of equations for each \( z_i \). Solving these equations we get the eigenvalues \( E_i = \alpha_i(n_i + 1/2) \) and the eigenfunctions:

\[
f_{n_i}(z_i) = e^{-\frac{1}{\alpha_i^2} z_i L_{n_i}(z_i/\alpha_i)}
\]

(49)

The overall eigenfunctions are products of the \( f_{n_i}(z_i) \). The annihilation and creation operators again take the form (51). They satisfy the following commutation relations:

\[
a_i \star a_j^\dagger - a_j^\dagger \star a_i = \alpha_i \delta_{ij}
\]

(50)

and the Hamiltonian becomes:

\[
H = \sum_i \left( a_i^\dagger \star a_i + \frac{\alpha_i}{2} \right).
\]

(51)

Again, the \( \star \)-genvalue problem is equivalent to the Schrödinger problem in the transformed variables. This means that it is also equivalent in the original variables, if one uses the representation for the original variables that result from the usual representation of the transformed ones.

As an example of a non-rotationally symmetric case let us consider the noncommutative Landau problem. Assume for convenience that \( \hbar = 1 \). Here the commutation relations are:

\[
[q_i, p_j] = i \delta_{ij} ; \quad [q_i, q_j] = i \theta ; \quad [p_1, p_2] = i B.
\]

(52)

The star operator is:

\[
\star = \exp \left[ \frac{i}{2} \left( \overleftrightarrow{\partial}_{q_1}, \overleftrightarrow{\partial}_{p_1}, \overleftrightarrow{\partial}_{q_2}, \overleftrightarrow{\partial}_{p_2} \right) \right] \left( \begin{array}{cccc}
0 & 1 & \theta & 0 \\
-1 & 0 & 0 & B \\
-\theta & 0 & 0 & 1 \\
0 & -B & -1 & 0
\end{array} \right) \left( \begin{array}{c}
\overleftrightarrow{\partial}_{q_1} \\
\overleftrightarrow{\partial}_{p_1} \\
\overleftrightarrow{\partial}_{q_2} \\
\overleftrightarrow{\partial}_{p_2}
\end{array} \right).
\]

(53)

The \( \alpha_I \) take the form:

\[
\alpha_{\pm} = \frac{1}{2} \left( \sqrt{(\theta - B)^2 + 4 \pm (\theta + B)} \right).
\]

(54)

These \( \alpha_{\pm} \) correspond to the frequencies of the Landau harmonic oscillator. In this case there is no degeneracy in the \( \alpha_{\pm} \) as opposed to the rotationally symmetric case.

The transformation matrix \( R \) that corresponds to this case is

\[
R = \begin{pmatrix}
\frac{\alpha_-(1 + B\delta_-)}{\sqrt{(1 - B\theta)^2 + \alpha_-^2(1 + B\delta_-)^2}} & 0 & -\frac{\alpha_+(1 - B\delta_+)}{\sqrt{(1 - B\theta)^2 + \alpha_+^2(1 - B\delta_+)^2}} & 0 \\
0 & \frac{1}{\sqrt{1 + \delta_-^2}} & 0 & 0 \\
0 & 0 & \frac{\alpha_-(1 - B\delta_-)}{\sqrt{(1 - B\theta)^2 + \alpha_-^2(1 + B\delta_-)^2}} & 0 \\
-\frac{B\theta}{\sqrt{(1 - B\theta)^2 + \alpha_-^2(1 + B\delta_-)^2}} & 0 & \frac{1 - B\theta}{\sqrt{(1 - B\theta)^2 + \alpha_+^2(1 - B\delta_+)^2}} & 0
\end{pmatrix}
\]

(55)
where we have assumed that \( B\theta < 1 \) and
\[
\tilde{\alpha}_\pm = \frac{1}{2} \left( \sqrt{(\theta - B)^2 + 4 (\theta - B)} \right),
\]
(56)
Note that if \( B\theta = 1 \) then the fourth row in the transformation matrix becomes zero, so the transformation becomes degenerate, which is not permitted. So there is a critical value for the magnetic field \( B_0 = 1/\theta \). At \( B_0 \) the frequency \( \alpha_- = 0 \), so there is an infinite degeneracy at the energy levels, corresponding to the excitations of the \( \alpha_- \) oscillator. This problem was solved through a series of transformations in [5], without resorting to the \( \star \)-genvalue formalism.

As a further example consider the case of the noncommutative Landau problem with the additional commutation relations \([q_1, p_2] = i\phi_1 \) and \([p_1, q_2] = i\phi_2 \). In this case the frequencies become:
\[
\alpha_\pm = \frac{1}{2} \left( \sqrt{(B - \theta)^2 + 4(\phi_1 + \phi_2)^2} \pm \sqrt{(B + \theta)^2 + (\phi_1 - \phi_2)^2} \right)
\]
(57)
and a degeneracy is produced provided that:
\[
B\theta - \phi_1\phi_2 = 1.
\]
(58)
Note that in this case the matrix \( M \) in the \( \star \)-product becomes degenerate again.

As a final example consider the six phase space dimensional case. The commutation relations we consider are found to be:
\[
[q_i, p_j] = i\delta_{ij} \quad [q_1, q_2] = i\theta_3 \quad [q_1, q_3] = -i\theta_2 \quad [q_2, q_3] = i\theta_1.
\]
(59)
All other commutation relations are trivial. Now the frequencies \( \alpha_I \) are
\[
\alpha_{1,2} = \sqrt{1 + \theta_1^2 + \theta_2^2 + \theta_3^2} \quad \alpha_3 = 1.
\]
(60)
There is twofold degeneracy for the first frequency. The transformation matrix in this case is:
\[
R = \begin{pmatrix}
-\frac{\theta_1\theta_3}{\alpha_\beta} & \frac{\theta_2}{\beta} & 0 & \frac{\theta_1\theta_3}{\alpha_\gamma} & 0 & -\frac{\theta_1}{\gamma} \\
-\frac{\theta_2\theta_3}{\alpha_\beta} & 0 & -\frac{\theta_1\theta_2}{\beta\gamma} & \frac{\theta_2}{\beta} & \theta_1 & 0 \\
-\frac{\theta_2\theta_3}{\alpha_\beta} & -\frac{\theta_1}{\beta} & 0 & \frac{\theta_2\theta_3}{\alpha_\gamma} & 0 & -\frac{\theta_2}{\gamma} \\
\frac{\theta_1}{\alpha_\beta} & 0 & -\frac{\theta_2\theta_1}{\beta\gamma} & \frac{\theta_1}{\beta} & \theta_2 & 0 \\
\frac{\theta_1}{\alpha_\beta} & 0 & 0 & -\frac{\beta}{\alpha\gamma} & 0 & -\frac{\theta_1}{\gamma} \\
0 & 0 & \frac{\beta}{\gamma} & 0 & \frac{\theta_1}{\gamma} & 0
\end{pmatrix}
\]
(61)
where
\[
\alpha = \sqrt{1 + \theta_1^2 + \theta_2^2 + \theta_3^2}
\]
\[
\beta = \sqrt{\theta_1^2 + \theta_2^2}
\]
\[
\gamma = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}.
\]
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