AN INTEGRAL FUNCTIONAL DRIVEN BY FRACTIONAL BROWNIAN MOTION

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ABSTRACT. Let $B^H$ be a fractional Brownian motion with Hurst index $0 < H < 1$ and the weighted local time $L^H(\cdot, t)$. In this paper, we consider the integral functional

$$C^H_t(a) := \lim_{\varepsilon \to 0} \int_0^t 1_{\{|B^H_s - a| > \varepsilon\}} \frac{1}{B^H_s - a} ds^{2H} \equiv \frac{1}{\pi} \mathcal{H} \mathcal{L}^H(\cdot, t)(a)$$

in $L^2(\Omega)$ with $a \in \mathbb{R}$, $t \geq 0$ and $\mathcal{H}$ denoting the Hilbert transform. We show that

$$C^H_t(a) = 2 \left( (B^H_t - a) \log |B^H_t - a| - B^H_t + a \log |a| - \int_0^t \log |B^H_s - a| \delta B^H_s \right)$$

for all $a \in \mathbb{R}$, $t \geq 0$ which is the fractional version of Yamada’s formula, where the integral is the Skorohod integral. Moreover, we introduce the following occupation type formula:

$$\int_{\mathbb{R}} C^H_t(a) g(a) da = 2H\pi \int_0^t (\mathcal{H} g)(B^H_s) s^{2H-1} ds$$

for all continuous functions $g$ with compact support.

1. Introduction

Given $H \in (0, 1)$, a fractional Brownian motion (fBm) $B^H = \{B^H_t, 0 \leq t \leq T\}$ with Hurst index $H$ is a mean zero Gaussian process such that

$$E \left[ B^H_t B^H_s \right] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right]$$

for all $t, s \geq 0$. For $H = 1/2$, $B^H$ coincides with the standard Brownian motion $B$. $B^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $B^H$. As a Gaussian process, one can construct the stochastic calculus of variations with respect to $B^H$. Some surveys and complete literatures for fBm could be found in Biagini et al [5], Decreusefond-Üstünel [12], Hu [19], Mishura [24], Nourdin [25], Nualart [26] and the references therein.

Let now $F$ be an absolutely continuous function such that the Skorohod integral

$$\int_0^t F'(B^H_s - a) \delta B^H_s$$

is well-defined and the second derivative $F'' = f$ exists in the sense of Schwartz’s distribution. Then the process

$$K^H_t(a) := 2 \left( F(B^H_t - a) - F(-a) - \int_0^t F'(B^H_s - a) \delta B^H_s \right),$$

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exists for all \(a \in \mathbb{R}\). Denote

\[
\chi_t^H(a) := \int_0^t f(B_s^H - a)ds^{2H}
\]

for all \(t \geq 0, a \in \mathbb{R}\). By Itô’s formula one can find the following questions:

- if the Lebesgue integral (1.2) converges,
  \[K_t^H(a) = \chi_t^H(a)\]?
- how to characterize the process \(K_t^H(a)\) if the Lebesgue integral (1.2) diverges?

Clearly, the first question is positive by approximating. However, the second question is not obvious even if \(H = \frac{1}{2}\) and \(f\) is a special function. Thus, the question arises again:

- for which functions does the Lebesgue integral (1.2) diverge?

When \(H = \frac{1}{2}\), \(B_t^H\) coincides with the standard Brownian motion \(B\) and by the Engelbert-Schmidt zero-one law, the Lebesgue integral (1.2) diverges if \(F'' = f\) is not locally integrable, i.e.

\[
\int_{-M}^{M} |f(x - a)|dx = \infty
\]

for some \(M > 0\). Thus, when

\[
|f(x)| \geq C|x|^{-\alpha}
\]

for \(x \in \mathbb{R}\), the Lebesgue integral (1.2) diverges, where \(C > 0, \alpha \geq 1\). For some special functions \(F\), for examples,

\[
F''(x) = |x|^{-\gamma}\text{sign}(x)
\]

with \(1 \leq \gamma < \frac{3}{2}\), one studied the characterization and properties of the process \(K_t^\frac{1}{2}(a)\). Itô–McKean \cite{21} first considered the process \(K_t^\frac{1}{2}(a)\) and the Lebesgue integral (1.2) for \(F\) satisfying (1.3) with \(\gamma = 1\). For the process \(K_t^\frac{1}{2}(a)\) and the Lebesgue integral (1.2) driven by the function \(F\) satisfying (1.3), some systematic studies are due to Biane-Yor \cite{3}, Yamada \cite{29, 30, 31} and Yor \cite{35}, and some extensions and limit theorems are established by Bertoin \cite{3, 4}, Cherny \cite{7}, Csaki et al. \cite{9, 10}, Csaki-Hu \cite{20}, Hu \cite{20}, Fitzsimmons-Getoor \cite{14, 15}, Mansuy-Yor \cite{23}, Yor \cite{37} and the references therein. However, those researches apply only to Markov process, and for non-Markov processes there has only been little investigation on the integral functional. See Eddahbi-Vives \cite{13}, Gradinaru et al. \cite{17}, Yan \cite{32} and Yan-Zhang \cite{35}.

When \(H \neq \frac{1}{2}\) the second and third questions above are not trivial. The main difficulty consists in the fact that the stochastic integral

\[
\int_0^t F'(B_s^H - a)\delta B_s^H
\]

is a Skorohod integral with respect to the fBm and the integrand is not smooth. Therefore, its control is not obvious and one needs sharp estimates. The \(L_2\) norm of this stochastic Skorohod integral involves the Malliavin derivatives of the integrand and tedious estimation on the joint density of the fBm. Moreover, for a nonsmooth function \(f\) it is not easy to give an exact calculus of the moment of order 2 for the Skorohod integral (1.4) even if the simple functions \(F'(x) = \log |x|\) and \(F''(x) = |x|^{-\alpha}\text{sign}(x)\) with \(\alpha > 0\). But, when \(H = \frac{1}{2}\), the integral (1.4) is Itô’s integral and its existence is obvious. On the other hand, it is unclear whether the Engelbert-Schmidt zero-one law actually holds for fBm \(B_t^H\). Thus, it seems interesting to study the process \(K_t^H(a)\) and the Skorohod integral (1.4) with the singular integrand for \(H \neq \frac{1}{2}\). In this paper, as a start reviewing the object and continued to Yan \cite{32}, we consider the integrals

\[
\int_0^t \frac{ds^{2H}}{B_s^H - a}, \quad a \in \mathbb{R}
\]
and the processes

\[ C_t^H(a) := 2 \left( F(B_t^H-a) - F(-a) - \int_0^t F'(B_s^H-a) \delta B_s^H \right), \quad a \in \mathbb{R} \]

with \( t \geq 0 \), where the integral in \( \text{1.6} \) is the Skorohod integral and \( F(x) = x \log|x| - x \). In the present paper we will consider the functional and discuss some related questions. We will divide the discussion as two parts since the research methods of the case \( \frac{1}{2} < H < 1 \) is essentially different with the case \( 0 < H < \frac{1}{2} \). In Section 6 we study the case \( 0 < H < \frac{1}{2} \) and the case \( \frac{1}{2} < H < 1 \) is considered in Section 3. Section 4 and Section 5.

This paper is organized as follows. In Section 2 we present some preliminaries for fBm. In Section 3 we consider the existence of \( C_t^H(a) \) for \( \frac{1}{2} < H < 1 \). In fact, by smoothness approximating one can prove the existence of the Skorohod integral

\[ \int_0^t \log |B_s^H-a| \delta B_s^H, \]

however, it is not easy to give the exact estimates of the moments. To give the existence and exact estimates of the moments, we define the function

\[ \Psi_{s,r,a,b}(x,y) := \varphi_{s,r}(x,y) - \varphi_{s,r}(x,b) \theta(1+b-y) \]

\[ - \varphi_{s,r}(y,a) \theta(1+a-x) + \varphi_{s,r}(a,b) \theta(1+a-x) \theta(1+b-y) \]

with \( x, y, a, b \in \mathbb{R}, s, r > 0 \), where \( \theta(x) = 1_{\{x>0\}} \) and \( \varphi_{s,r}(x,y) \) is the density function of \((B_s^H, B_t^H))\), and show that the identity

\[ \text{1.7} \]

\[ E \left[ G_1'(B_t^H-a) G_2'(B_t^H-b) \right] = \int_{\mathbb{R}^2} G_1'(x-a) G_2'(y-b) \Psi_{s,r,a,b}(x,y) dxdy \]

holds for all \( G_1, G_2 \in C^\infty(\mathbb{R}) \) with compact supports and \( G_1(1) = G_2(1) = 0 \). By using \( \text{1.7} \) we show that the integral \( \int_0^t F_t^+(B_s^H-a) \delta B_s^H \) exists and

\[ E \left[ \int_0^t F_t^+(B_s^H-a) \delta B_s^H \right]^2 = \int_0^t \int_0^t E \left[ F_t^+(B_s^H-a) F_t^+(B_r^H-a) \right] \phi(s,r) dsdr \]

\[ + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\xi) \phi(r,\eta) \int_0^\infty \Psi_{s,\xi,\eta}(x,y) \frac{dy}{(y-x)} \]

with \( \phi(s,r) = H(2H-1)|s-r|^{2H-2} \), where the integral \( \int_0^t F_t^+(B_s^H-a) \delta B_s^H \) is the Skorohod integral and

\[ F_t^+(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x \log x - x, & \text{if } x > 0. \end{cases} \]

In Section 4 for \( \frac{1}{2} < H < 1 \) we show that the representation

\[ \text{1.8} \]

\[ C_t^H(a) = \lim_{\epsilon \downarrow 0} \int_0^t 1_{\{|B_s^H-a| \geq \epsilon\}} \frac{2H s^{2H-1} B_s^H - a}{B_s^H - a} ds, \quad a \in \mathbb{R}, t \geq 0 \]

holds in \( L^2(\Omega) \), which points out that \( a \mapsto \frac{1}{\pi} C_t^H(a) \) coincides with the Hilbert transform of the weighted local time

\[ a \mapsto \mathcal{L}^H(a,t) = 2H \int_0^t \delta(B_s^H-a) s^{2H-1} ds \]

and the fractional version of Yamada’s formula

\[ (B_t^H-a) \log |B_t^H-a| - (B_t^H-a) = -a \log |a| + a + \int_0^t \log |B_s^H-a| \delta B_s^H + \frac{1}{2} C_t^H(a) \]

holds. In section 5 we introduce the so-called occupation type formula

\[ \text{1.9} \]

\[ \int_{\mathbb{R}} C_t^H(a) g(a) da = 2H \pi \int_0^t (\mathcal{H} g)(B_s^H) s^{2H-1} ds \]
for all continuous function \( g \) with compact support and \( \frac{1}{2} < H < 1 \), where \( \mathcal{H} \) denotes Hilbert transform. In Section 6 we study the case \( 0 < H < \frac{1}{2} \) by using the generalized quadratic covariation introduced in Yan et al [33].

2. Preliminaries

2.1. Cauchy principal value. It is known that the Cauchy principal value, named after Augustin Louis Cauchy, is a method for assigning values to certain improper integrals which would otherwise be undefined. Depending on the type of singularity in the integrand \( f \), the Cauchy principal value is defined as one of the following:

\[
\lim_{\epsilon \downarrow 0} \left( \int_{a}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{b} f(x)dx \right) = \lim_{\epsilon \downarrow 0} \int_{a}^{b} 1_{\{|c-x| \geq \epsilon\}} f(x),
\]

where \( c \in (a,b) \) is a unique point such that \( \int_{a}^{b} f(x)dx = \infty \).

The limiting operation given in (2.1) is called the (Cauchy) principal value of the divergent integral \( \int_{a}^{b} f(x)dx \) and the limiting process displayed in (2.1) is denoted as

\[\text{v.p.} \int f(x)dx.\]

The notation v.p. (valeur principale) is seen in European writings. We have, as an example,

\[\text{v.p.} \int_{a}^{b} \frac{dx}{c-x} = \log \frac{c-a}{b-c}\]

for all \( a < c < b \). Moreover, for a Borel function \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R} \) with

\[
\int_{a}^{a+1} \frac{\varphi(x) - \varphi(a)}{x-a} dx + \int_{a+1}^{\infty} \frac{\varphi(x)}{x-a} dx < \infty,
\]

we can define the Cauchy’s principal value

\[\text{v.p.} \int_{a}^{\infty} \frac{\varphi(x)}{x-a} dx : = \int_{a}^{a+1} \frac{\varphi(x) - \varphi(a)}{x-a} dx + \int_{a+1}^{\infty} \frac{\varphi(x)}{x-a} dx\]

\[= \lim_{\epsilon \downarrow 0} \left( \int_{a+\epsilon}^{\infty} \frac{\varphi(x)}{x-a} dx + \varphi(a) \log \epsilon \right).\]

Recall that the Hilbert transform \( \mathcal{H} f \) of \( f \in L^2(\mathbb{R}) \) is defined as follows

\[
\mathcal{H} f(a) := \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} 1_{\{|x-a| \geq \epsilon\}} \frac{f(x)dx}{x-a} = \frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{f(x)dx}{x-a} \equiv \frac{1}{\pi} \text{v.p.} \frac{1}{\pi} * f(x),
\]

where \( * \) denotes the convolution in the theory of distributions, which plays an important role in real and complex analysis. It is also important to note that \( \mathcal{H} f \) belongs to \( L^2 \) and

\[
\int_{\mathbb{R}} (\mathcal{H} f(x))^2 dx = \int_{\mathbb{R}} f^2(x)dx
\]

holds, and moreover, if \( f \) is a Hölder continuous function with compact support, then the limit in (2.2) exists for every \( x \in \mathbb{R} \). For more aspects on these material we refer to King [22].
2.2. Fractional Brownian motion. In this subsection, we briefly recall some basic definitions and results of fBm. For more aspects on these material we refer to Alós et al. [1], Biagini et al [3], Decreusefond-Üstünel [1], Hu [13], Mishura [21], Nourdin [23], Nualart [26] and the references therein. Throughout this paper we assume that $0 < H < 1$ is arbitrary but fixed and we let $B^H = \{B_t^H, 0 \leq t \leq T\}$ be a one-dimensional fBm with Hurst index $H$ defined on $(\Omega, \mathcal{F}^H, \mathbb{P})$.

Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}$, $t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right].$$

The application $\varphi \in \mathcal{E} \to B^H(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^H$ and it can be extended to $\mathcal{H}$. When $\frac{1}{2} < H < 1$ the Hilbert space $\mathcal{H}$ can be written as

$$\mathcal{H} = \{ \varphi : [0,T] \to \mathbb{R} \mid \|\varphi\|_{\mathcal{H}} < \infty \},$$

where

$$\|\varphi\|_{\mathcal{H}}^2 := \int_0^T \int_0^T \varphi(s)\varphi(r)\phi(s,r)dsdr$$

with $\phi(s,r) = H(2H - 1)|s - r|^{2H - 2}$. Notice that the elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order (see, for instance, Pipiras-Taqqu [27]). Denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)),$$

where $f \in C^\infty_b(\mathbb{R}^n)$ ($f$ and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The derivative operator $D^H$ (the Malliavin derivative) of a functional $F$ of the form (2.3) is defined as

$$D^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)) \varphi_j.$$

The derivative operator $D^H$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2}.$$}

The divergence integral $\delta^H$ is the adjoint of derivative operator $D^H$. That is, we say that a random variable $u$ in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $\delta^H$, denoted by Dom($\delta^H$), if

$$E \left| (D^H F, u)_{\mathcal{H}} \right| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta^H(u)$ is defined by the duality relationship

$$E \left[ F \delta^H(u) \right] = E\langle D^H F, u \rangle_{\mathcal{H}}$$

for any $u \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset$ Dom($\delta^H$), and when $\frac{1}{2} < H < 1$ we have

$$\left| \delta^H(u) \right|^2 = E\|u\|_{\mathcal{H}}^2 + E \int_{[0,T]^4} D^H_{\xi} u_r D^H_{\eta} u_s \phi(\eta, r, s)dsdrd\xi d\eta$$

for any $u \in \mathbb{D}^{1,2}$. By the duality between $D^H$ and $\delta^H$ one have that the following result for the convergence of divergence integrals which is given in Nualart [26].

**Proposition 2.1.** Let $\{u_n, n = 1, 2, \ldots\} \subset$ Dom($\delta^H$) such that $u_n \to u$ in $L^2(\Omega; \mathbb{H})$ for some $u \in L^2(\Omega; \mathbb{H})$. If that there exists $U \in L^2(\Omega)$ such that

$$\delta^H(u_n) \to U$$

in $L^2(\Omega; \mathbb{H})$, as $n \to \infty$. Then, $u$ belongs to Dom($\delta^H$) and $\delta^H(u) = U$. 

We will use the notation
\[ \delta^H(u) = \int_0^T u_s \delta B_s^H \]
to express the Skorohod integral of a process \( u \), and the indefinite Skorohod integral is defined as \( \int_0^t u_s \delta B_s^H = \delta^H(u_{1[0,t]}) \). Recall the Itô type formula for fBm \( B^H \),
\[ f(B_t^H) = f(0) + \int_0^t f'(B^H_s) \delta B^H_s + H \int_0^t f''(B^H_s) s^{2H-1} ds \]
for any \( f \in C^2(\mathbb{R}) \). Also recall that \( B^H \) has a local time \( \mathcal{L}^H(x,t) \) continuous in \( (x,t) \in \mathbb{R} \times [0,\infty) \) which satisfies the occupation formula (see Geman-Horowitz \( \cite{16} \))
\begin{equation}
\int_0^t \Phi(B^H_s) ds = \int_\mathbb{R} \Phi(x) \mathcal{L}^H(x,t) dx
\end{equation}
for every nonnegative bounded function \( \Phi \) on \( \mathbb{R} \), and such that
\[ \mathcal{L}^H(x,t) = \int_0^t \delta(B^H_s - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(s \in [0,t], |B^H_s - x| < \epsilon), \]
where \( \lambda \) denotes Lebesgue measure and \( \delta(x) \) is the Dirac delta function. It is well-known that the local time \( \mathcal{L}^H(x,t) \) has Hölder continuous paths of order \( \gamma \in (0,1-H) \) in time, and of order \( \kappa \in (0, \frac{1-H}{2}) \) in the space variable, provided \( H \geq \frac{1}{3} \). Define the so-called weighted local time \( \mathcal{L}_c^H(x,t) \) of \( B^H \) at \( x \) as follows
\[ \mathcal{L}_c^H(x,t) = 2H \int_0^t s^{2H-1} \mathcal{L}^H(x,s) ds \equiv 2H \int_0^t \delta(B^H_s - x)s^{2H-1} ds. \]
The Hölder continuity properties of \( \mathcal{L}^H(x,t) \) can be transferred to the weighted local time \( \mathcal{L}_c^H(x,t) \), and then \( \mathcal{L}_c^H \) has a compact support in \( x \), and the following Tanaka formula holds (see Coutin et al \( \cite{8} \) and Hu et al \( \cite{18} \)):
\begin{equation}
(B^H_t - x)^- = (-x)^- - \int_0^t 1_{\{B^H_s < x\}} \delta B^H_s + \frac{1}{2} \mathcal{L}^H(x,t).
\end{equation}
At the end of this section we will establish some technical estimates associated with fractional Brownian motion. For simplicity we let \( C \) stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to \( c \).

**Lemma 2.1.** For all \( r,s \in [0,T], \ s \geq r \) and \( 0 < H < 1 \) we have
\begin{equation}
\frac{1}{2}(2 - 2^H)r^{2H}(s-r)^{2H} \leq s^{2H}r^{2H} - \mu_{s,r}^2 \leq 2r^{2H}(s-r)^{2H},
\end{equation}
where \( \mu_{s,r} = E(B^H_s B^H_r) \).

By the local nondeterminacy of fBm we can prove the lemma (Yan et al \( \cite{34} \)), and Yan et al \( \cite{33} \) gave an elementary proof by using the inequality
\begin{equation}
(1 + x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha
\end{equation}
with \( 0 \leq x, \alpha \leq 1 \). It is important to note that inequality \( \eqref{2.9} \) is stronger than the well known (Bernoulli) inequality
\[ (1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha, \]
because \( 2^\alpha - 1 \leq \alpha \) for all \( 0 \leq \alpha \leq 1 \).

**Lemma 2.2.** For all \( s > r > 0 \) and \( \frac{1}{2} < H < 1 \) we have
\begin{equation}
c_H(s-r)^{rs^{2H-2}} \leq \mu - r^{2H} \leq C_H(s-r)^{rs^{2H-2}}
\end{equation}
and
\begin{equation}
c_H(s-r)^{rs^{2H-1}} \leq s^{2H} - \mu \leq C_H(s-r)^{rs^{2H-1}}
\end{equation}
where \( \mu_{s,r} = E(B^H_t B^H_r) \).

**Proof.** For the inequalities (2.11) we have

\[
\mu - r^{2H} = \frac{1}{2} (s^{2H} - r^{2H} - (s - r)^{2H}) = \frac{1}{2} s^{2H} (1 - x^{2H} - (1 - x)^{2H})
\]

with \( x = \frac{r}{s} \). By the continuity of the functions

\[
f_1(x) = \frac{1 - x^{2H} - (1 - x)^{2H}}{x(1 - x)}, \quad f_2(x) = \frac{x(1 - x)}{1 - x^{2H} - (1 - x)^{2H}}
\]

for \( x \in (0, 1) \) and \( \lim_{x \to 0} f_i(x) = \lim_{x \to 1} f_i(x) = 2H \) for \( i = 1, 2 \), we see that there exists a constant \( C > 0 \) such that

\[
\frac{1}{C} x(1 - x) \leq 1 - x^{2H} - (1 - x)^{2H} \leq C x(1 - x)
\]

for all \( x \in [0, 1] \), which gives the inequalities (2.11). The inequalities (2.11) is clear. \( \square \)

3. The existence of \( C^H(a) \)

Beside on the smooth approximation one can prove the existence of \( C^H \). In order to use the smooth approximation, we define the function \((x, y) \mapsto \Psi_{s,r,a,b}(x, y) \) on \( \mathbb{R}^2 \) by

\[
\Psi_{s,r,a,b}(x, y) := \varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) \theta(1 + b - y) - \varphi_{s,r}(a, y) \theta(1 + a - x) + \varphi_{s,r}(a, b) \theta(1 + a - x) \theta(1 + b - y)
\]

with \( s, r > 0 \) and \( a, b \in \mathbb{R} \), where \( \theta(x) = \mathbbm{1}_{\{x > 0\}} \) and \( \varphi_{s,r}(x, y) \) denotes the density function of \( (B^H_t, B^H_r) \). That is,

\[
\varphi_{s,r}(x, y) = \frac{1}{2 \pi \rho_{s,r}} \exp \left\{ -\frac{1}{2 \rho_{s,r}^2} \left( r^{2H} x^2 - 2 \mu_{s,r} x y + s^{2H} y^2 \right) \right\},
\]

where \( \mu_{s,r} = E(B^H_t B^H_r) \) and \( \rho_{s,r}^2 = (rs)^{2H} - \mu_{s,r}^2 \). Denote the density function of \( B^H_t \) by \( \varphi_s(x) \). The following Lemmas give some properties and estimates of \( \Psi_{s,r,a,b}(x, y) \). The first lemma is a simple calculus exercise.

**Lemma 3.1.** Let \( G_i \in C^\infty(\mathbb{R}) \) have compact supports for \( i = 1, 2 \). Then we have

\[
\int_{\mathbb{R}^2} G'_1(x - a)G_2(y - b)\varphi_{s,r}(x, y)dxdy
\]

\[
= \int_{\mathbb{R}^2} G'_1(x - a)G_2(y - b)\Psi_{s,r,a,b}(x, y)dxdy
\]

\[
- G_2(1) \int_{\mathbb{R}} G_1(x - a) \frac{\partial}{\partial x} \varphi_{s,r}(x, b)dx
\]

\[
- G_1(1) \int_{\mathbb{R}} G_2(y - b) \frac{\partial}{\partial y} \varphi_{s,r}(a, y)dy - \varphi_{s,r}(a, b)G_1(1)G_2(1)
\]

for all \( r, s > 0 \) and \( a, b \in \mathbb{R} \), and moreover, if \( G_i(1) = 0 \) for \( i = 1, 2 \), we then have

\[
\int_{\mathbb{R}^2} G'_1(x - a)G_2(y - b)\varphi_{s,r}(x, y)dxdy = \int_{\mathbb{R}^2} G'_1(x - a)G_2(y - b)\Psi_{s,r,a,b}(x, y)dxdy
\]

for all \( r, s > 0 \) and \( a, b \in \mathbb{R} \).

**Lemma 3.2.** For any \( x, y, z \in \mathbb{R} \) and \( \beta \in [0, 1] \) we have

\[
|\varphi_{s,r}(x, y) - \varphi_{s,r}(z, y)| \leq \frac{r^{\beta H}}{\rho_{s,r}^{1+\beta}} |x - z|^\beta e^{-\frac{y^2}{2 \pi \rho_{s,r}^2}}
\]
and
\[
\varphi_{s,r}(x,y) - \varphi_{s,r}(x,z) \leq \frac{s^2H}{\rho_{s,r}} |y-z|^\beta e^{-\frac{s}{2\pi r^2} y^2}.
\]

Proof. We have
\[
\|\varphi_{s,r}(x,y) - \varphi_{s,r}(x,z)\| \leq \frac{1}{2\pi \rho_{s,r}} \left| e^{-\frac{1}{2\rho_{s,r}} (r^2H x^2 - 2\mu_{s,r} xy + s^2H y^2)} - e^{-\frac{1}{2\rho_{s,r}} (r^2H z^2 - 2\mu_{s,r} zy + s^2H y^2)} \right| \beta
\]
for all \( \beta \in [0,1] \). It follows from Mean Value Theorem that
\[
|\varphi_{s,r}(x,y) - \varphi_{s,r}(z,y)|
\leq \frac{1}{2\pi \rho_{s,r}} \left| e^{-\frac{1}{2\rho_{s,r}} (r^2H x^2 - 2\mu_{s,r} xy + s^2H y^2)} - e^{-\frac{1}{2\rho_{s,r}} (r^2H z^2 - 2\mu_{s,r} zy + s^2H y^2)} \right| \beta
\]
for some \( \xi \) between \( z \) and \( x \). Combining this with the fact \(|x|e^{-x^2} \leq 1\), we get
\[
|\varphi_{s,r}(x,y) - \varphi_{s,r}(z,y)| \leq \frac{1}{2\pi \rho_{s,r}} |x-z|^\beta e^{-\frac{s}{2\pi r^2} y^2}
\]
for all \( \beta \in [0,1] \). Similarly, one can obtain the estimate (3.5). \( \square \)

Lemma 3.3. The estimate
\[
\Lambda_1(s,r,a,b) := \int_a^\infty \int_b^\infty |\Psi_{s,r,a,b}(x,y)| dx dy \leq \frac{C_{H,T,\beta}s^\beta H/2}{r(1+\beta)(s-r)(1+\beta)H}
\]
holds for all \( \beta \in (0,1), 0 < r < s \leq T \) and \( a, b \in \mathbb{R} \).

Proof. We have
\[
\Lambda_1(s,r,a,b) = \int_a^{\frac{a+1}{1+\beta}} \int_b^{\frac{b+1}{1+\beta}} \frac{1}{(x-a)(y-b)} |\Psi_{s,r,a,b}(x,y)| dx dy
\]
and
\[
\leq \int_a^{\frac{a+1}{1+\beta}} dx \int_b^{\frac{b+1}{1+\beta}} \frac{1}{(x-a)(y-b)} \Big| \varphi_{s,r}(x,y) - \varphi_{s,r}(x,b) - \varphi_{s,r}(x,a) + \varphi_{s,r}(a,b) \Big| dy
\]
\[
+ \int_a^{\frac{a+1}{1+\beta}} dx \int_b^\infty \frac{1}{(x-a)(y-b)} \Big| \varphi_{s,r}(x,y) - \varphi_{s,r}(x,b) \Big| dy
\]
\[
+ \int_a^\infty dx \int_b^{\frac{b+1}{1+\beta}} \frac{1}{(x-a)(y-b)} \Big| \varphi_{s,r}(x,y) - \varphi_{s,r}(a,y) \Big| dy
\]
\[
+ \int_a^\infty dx \int_b^\infty \frac{1}{(x-a)(y-b)} \varphi_{s,r}(x,y) dy = \Lambda_{11}(s,r,a,b) + \Lambda_{12}(s,r,a,b) + \Lambda_{13}(s,r,a,b) + \Lambda_{14}(s,r,a,b).
\]
Clearly, $\Lambda_{14}(s, r, a, b) \leq 1$ and we have

$$
\Lambda_{12}(s, r, a, b) + \Lambda_{13}(s, r, a, b) \leq \frac{2^{\beta H}}{\rho_{s, r}} \int_a^{a+1} dx \int_b^{b+1} dy \frac{1}{(x-a)(y-b)^{1-\beta}} e^{-\frac{\beta}{2\pi} y^2} dy \\
+ \frac{2^{\beta H}}{\rho_{s, r}} \int_a^{a+1} dx \int_b^{b+1} dy \frac{1}{(x-a)^{1-\beta}(y-b)} e^{-\frac{\beta}{2\pi} y^2} dy \\
\leq C_{H, \beta} \frac{8(1+\beta)H}{r(1+\beta)(s-r)(1+\beta)H}
$$

and $a, b \in \mathbb{R}$ by Lemma 3.2 with $\beta \in (0, 1)$ and Lemma 2.1. In order to estimate $\Lambda_{11}(s, r, a, b)$, by Lemma 3.2 we have

$$(3.7) \quad |\varphi_{s, r}(x, y) - \varphi_{s, r}(x, b) - \varphi_{s, r}(a, y) + \varphi_{s, r}(a, b)| \leq 2 \frac{s^H}{\rho_{s, r}} |y - b|^\beta$$

and

$$(3.8) \quad |\varphi_{s, r}(x, y) - \varphi_{s, r}(x, b) - \varphi_{s, r}(a, y) + \varphi_{s, r}(a, b)| \leq 2 \frac{r^H}{\rho_{s, r}} |x - a|^\beta$$

for all $a, b, x, y \in \mathbb{R}$, which give

$$(3.9) \quad |\varphi_{s, r}(x, y) - \varphi_{s, r}(x, b) - \varphi_{s, r}(a, y) + \varphi_{s, r}(a, b)| \leq 2 \frac{(s r)^{\beta H/2}}{\rho_{s, r}} |(x-a)(y-b)|^{\beta/2}.$$

It follows from Lemma 2.1 that

$$\Lambda_{11}(s, r, a, b) = \int_a^{a+1} dx \int_b^{b+1} dy |\varphi_{s, r}(x, y) - \varphi_{s, r}(x, b) - \varphi_{s, r}(a, y) + \varphi_{s, r}(a, b)|$$

$$\leq 2 \frac{sr^{\beta H/2}}{\rho_{s, r}} \int_a^{a+1} dx \int_b^{b+1} dy \frac{1}{(x-a)^{1-\beta}(y-b)^{1-\beta}} dy$$

$$\leq C_{H, \beta} \frac{8\beta H/2}{r(2+\beta)H/2(s-r)(1+\beta)H}$$

for all $\beta \in (0, 1)$ and $a, b \in \mathbb{R}$. This completes the proof.

The next proposition shows the process

$$(3.10) \quad C_{t+}^H(a) := 2 \left( F_+(B_t^H - a) - F_+(-a) - \int_0^t F_+^r(B_s^H - a) \delta B_s^H \right)$$

exists in $L^2(\Omega)$, where

$$(3.11) \quad F_+(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x \log x - x, & \text{if } x > 0. \end{cases}$$

**Proposition 3.1.** Let the function $F_+$ be given as above. Then the random variable

$$(3.12) \quad \int_0^t F_+^r(B_s^H - a) \delta B_s^H$$

exists and

$$E \left| \int_0^t F_+^r(B_s^H - a) \delta B_s^H \right|^2 = \int_0^t \int_0^t E[F_+^r(B_s^H - a)F_+^r(B_r^H - a)] \phi(s, r) ds dr$$

$$+ \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \int_a^{\infty} \int_a^{\infty} \Psi_{s, r, \alpha, \beta}(x, y) dx dy$$

for all $t \geq 0$ and $a \in \mathbb{R}$. 
By smooth approximation we can obtain the statement. Define the function \( \zeta \) on \( \mathbb{R} \) by

\[
\zeta(x) := \begin{cases} 
  ce^{-(x-1)^2/2}, & x \in (0, 2), \\
  0, & \text{otherwise},
\end{cases}
\]

where \( c \) is a normalizing constant such that \( \int_{\mathbb{R}} \zeta(x) \, dx = 1 \). Define the mollifiers

\[
\zeta_n(x) := n \zeta(nx), \quad n = 2, \ldots
\]

and the sequence of smooth functions

\[
G_n(x) := \int_{\mathbb{R}} F'_+(y) \zeta_n(x-y) \, dy = n \int_{x-\frac{2}{n}}^{x} F'_+(y) \zeta(n(x-y)) \, dy
\]

\[
= \int_{0}^{2} F'_+(x - \frac{y}{n}) \zeta(y) \, dy, \quad n = 2, \ldots.
\]

Then \( G_n \in C^\infty(\mathbb{R}) \) with compact support for all \( n \geq 2 \).

**Lemma 3.4.** Let the functions \( G_n, n \geq 2 \) be defined as above. Then we have

\[
|G_n(x)| \leq \psi_1(x) := \begin{cases} 
  0, & \text{if } x \leq 0, \\
  C(1 + |\log x|), & \text{if } x > 0
\end{cases}
\]

for all \( x \in \mathbb{R} \), and

\[
G_n(x) \to F'_+(x)
\]

for all \( x \neq 0 \), as \( n \) tends to infinity.

**Proof.** Clearly, \( G_n(x) = 0 \) for \( x \leq 0 \) and

\[
G_n(x) = n \int_{(x-\frac{2}{n}) \vee 0}^{x} \zeta(n(x-y)) \log y \, dy
\]

for \( x > 0 \).

When \( 0 < x \leq \frac{2}{n} \), we have

\[
|G_n(x)| \leq n \int_{0}^{x} \zeta(n(x-y)) |\log y| \, dy \leq -n \int_{0}^{x} \log y \, dy = nx(1 - \log x) \leq 2(1 - \log x).
\]

On the other hand, by (3.17) we get

\[
G_n(x) = n \int_{0}^{2} \zeta(z) \log(x - \frac{z}{n}) \, dz = \int_{0}^{2} \zeta(z) \log[1 - \frac{z}{nx}] \, dz
\]

\[
= \log x \int_{0}^{2} \zeta(z) \, dz + \int_{0}^{2} \zeta(z) \log(1 - \frac{z}{nx}) \, dz
\]

\[
= \log x + \int_{0}^{2} \zeta(z) \log(1 - \frac{z}{nx}) \, dz
\]

for \( x > \frac{2}{n} \), which gives

\[
|G_n(x)| \leq C \left( 1 + |\log x| \right)
\]

for \( x > \frac{2}{n} \) since

\[
\int_{0}^{2} |\log(1 - \frac{z}{u})|e^{-\frac{1}{1-(1-z)^2}} \, dz < \infty
\]

with \( u > 2 \).
Finally, the convergence (3.18) follows from \( G_n(x) = F'_n(x) = 0 \) for \( x < 0 \) and the next estimate:

\[
|G_n(x) - F'_n(x)| \leq \alpha^{-1} \int_0^2 \zeta(y) \left| F'_n(x - \frac{y}{n}) - F'_n(x) \right| dy
\]

\[
= \alpha^{-1} \int_0^2 \left| \log(x - \frac{y}{n}) - \log x \right| \zeta(y) dy
\]

\[
\leq \alpha^{-1} \int_0^2 \log \left( 1 + \frac{y}{nx - y} \right) \zeta(y) dy
\]

\[
\leq \alpha \log \left( 1 + \frac{2}{nx - 2} \right) \int_0^2 \zeta(y) dy = \frac{1}{\alpha} \log \left( 1 + \frac{2}{nx - 2} \right)
\]

for all \( x > \frac{2}{n} \). This completes the proof. \( \square \)

**Lemma 3.5.** Let the functions \( G_n, n \geq 2 \) be defined as above. Then we have

\[
|G'_n(x)| \leq \psi_2(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ C x^{-1} (1 + |\log x|), & \text{if } x > 0 \end{cases}
\]

for any \( x \in \mathbb{R} \), and

\[
(3.20) \quad G'_n(x) \to F''_n(x)
\]

for all \( x \neq 0 \), as \( n \) tends to infinity.

**Proof.** Clearly, \( G'_n(x) = 0 \) for \( x \leq 0 \), and we have for \( x > \frac{2}{n} \),

\[
G'_n(x) = \int_0^2 F''_n(x - \frac{y}{n}) \zeta(y) dy = \int_0^2 \zeta(y) \frac{1}{x - \frac{2}{n}} dy
\]

\[
= \int_0^2 \zeta(y) \frac{n}{nx - y} dy = \frac{1}{x} \int_0^2 \zeta(y) \left( 1 + \frac{y}{nx - y} \right) dy
\]

by (3.16). It follows that

\[
|G'_n(x)| \leq \frac{1}{x} \int_0^2 \zeta(y) \left( 1 + \frac{y}{2 - y} \right) dy \leq \frac{C}{x}
\]

for \( x > \frac{2}{n} \). On the other hand, for \( 0 < x \leq \frac{2}{n} \) we have

\[
G'_n(x) = n \int_{\mathbb{R}} F'_n(y) \frac{\partial}{\partial x} \zeta(n(x - y)) dy
\]

\[
= -2n^2 \int_{x - \frac{2}{n}}^{x} \frac{1 - n(x - y)}{(1 - (1 - n(x - y))^2) x^2} \zeta(n(x - y)) F'_n(y) dy.
\]

Combining this with the fact

\[
x^2 e^{-x} \leq 2 \quad (x \geq 0)
\]

lead to

\[
|G'_n(x)| \leq 4n^2 \int_0^x |F'_n(y)||1 - n(x - y)| dy \leq 8n^2 \int_0^x |F'_n(y)| dy
\]

\[
= -8n^2 \int_0^x \log y dy = 8n^2 x (1 - \log x) \leq \frac{32}{x} (1 - \log x)
\]

for \( 0 < x \leq \frac{2}{n} \), which gives the estimates of \( G'_n(x) \) with \( 0 < x \leq \frac{2}{n} \).

Finally, by the estimate

\[
\int_0^2 \zeta(y) \left( 1 + \frac{y}{nx - y} \right) dy \leq \int_0^2 \zeta(y) \left( 1 + \frac{y}{2 - y} \right) dy < \infty
\]
for all $x > \frac{2}{n}$ and Lebesgue’s dominated convergence theorem we have

$$\lim_{u \to \infty} \int_0^2 \zeta(y) \left(1 + \frac{y}{u - y}\right) dy = 1.$$  

Combining this with (3.21), we get the convergence (3.20) since $G'_n(x) = F'_r(x) = 0$ for all $x < 0$. This completes the proof. \qed

**Lemma 3.6.** Let $\psi_2$ be defined in Lemma 3.4. Then the estimate

$$\int_a^\infty \int_b^\infty \psi_2(x - a)\psi_2(y - b)|\Psi_{s,r,a,b}(x,y)|dx dy \leq \frac{C_{H, t, s} \gamma^{H/2}}{r(1+\gamma)(s-r)(1+\gamma)H}$$

holds for all $\gamma \in (0, 1), 0 < r < s \leq t$ and $a, b \in \mathbb{R}$.

**Proof.** Similar to Lemma 3.3 one can obtain the estimate since

$$|\log x| \leq C(x^{-\beta} + x^{\beta})$$

for all $x > 0$ and all $0 < \beta < 1$. \qed

**Lemma 3.7.** Let $\psi_1$ be defined in Lemma 3.4. Then we have

$$\int_{\mathbb{R}} \psi_1(x - a) \left|\frac{\partial}{\partial x} \varphi_{s,r}(x, a)\right| dx \leq C_{H, t, \alpha}(s-r)^{-(1+\alpha)H}r^{-(1+\alpha)H}$$

and

$$\int_{\mathbb{R}} \psi_1(y - a) \left|\frac{\partial}{\partial y} \varphi_{s,r}(a, y)\right| dy \leq C_{H, t, \alpha}(s-r)^{-(1+\alpha)H}r^{-(1+2\alpha)H}$$

for all $a \in \mathbb{R}, 0 < r < s \leq t$ and $1 - H < \alpha < 1$.

**Proof.** Given $a \in \mathbb{R}$ and $0 < r < s \leq t$. Make the substitution $\frac{rH}{\rho}(x - \frac{rH}{\rho}a) = y$. Then

$$\int_a^\infty \left|\log(x - a)\right| \left|\frac{\partial}{\partial x} \varphi_{s,r}(x, a)\right| dx$$

$$= \frac{rH}{\rho} \int_a^\infty \left|\log(x - a)\right| \left|\frac{rH}{\rho} \left(x - \frac{rH}{\rho}a\right)\right| \varphi_{s,r}(x, a) dx$$

$$\leq \frac{1}{2\pi \rho} \left(\int_{\frac{rH}{\rho}a}^{\infty} \left|\log \left(y + \frac{\mu - r^{2H}}{rH - a}\right)\right| |y| e^{-\frac{y^2}{2}} dy + \frac{\mu - r^{2H}}{rH - a} \int_{-\frac{rH}{\rho}a}^{\infty} \left|\log \left(y + \frac{\mu - r^{2H}}{rH - a}\right)\right| |y| e^{-\frac{y^2}{2}} dy\right)$$

$$\equiv \frac{1}{\rho} \left(\Delta_1(s, r, a) + \Delta_2(s, r, a)\right).$$

By Lemma 2.21 and the fact $|\log x| \leq x + x^{-\alpha}$ for all $x > 0$ and $\alpha \in (0, 1)$ we see that

$$\Delta_1(s, r, a) \leq \left(\frac{\rho}{rH}\right)^{-\alpha} + \frac{\rho}{rH} \int_{\frac{rH}{\rho}a}^{\infty} |y| e^{-\frac{y^2}{2}} dy \leq \frac{r^{H\alpha}}{\rho^\alpha} + \frac{\rho}{rH} \leq \frac{C_{H, t, \alpha}}{(s-r)^{H\alpha}}$$

and

$$\Delta_2(s, r, a) \leq \left(\frac{\rho}{rH}\right)^{-\alpha} + \frac{\rho}{rH} \int_{-\frac{rH}{\rho}a}^{\infty} |y| e^{-\frac{y^2}{2}} dy \leq \frac{r^{H\alpha}}{\rho^\alpha} + \frac{\rho}{rH} \leq \frac{C_{H, t, \alpha}}{(s-r)^{H\alpha}}.$$
\[ \Delta_2(s, r, a) \leq e^{-\frac{a^2}{2\pi}} \int_{-\infty}^{\infty} \left( y + \frac{\mu - r2H}{\rho r} a \right) |y| e^{-\frac{1}{2}y^2} dy + e^{-\frac{a^2}{2\pi}} \int_{-\infty}^{\infty} \left( y + \frac{\mu - r2H}{\rho r} a \right) \alpha^{-1} |y| e^{-\frac{1}{2}y^2} dy \\
\equiv \Delta_{21}(s, r, a) + \Delta_{22}(s, r, a). \]

Now, let us estimate \( \Delta_{21}(s, r, a) \) and \( \Delta_{22}(s, r, a) \). We have

\[ \Delta_{21}(s, r, a) \leq \int_{-\infty}^{\infty} |y| e^{-\frac{1}{2}y^2} dy + \left| \frac{a}{r} \right| e^{-\frac{1}{2}y^2} \int_{-\infty}^{\infty} \left( \frac{\mu - r2H}{\rho r} \right) \int_{-\infty}^{\infty} |y| e^{-\frac{1}{2}y^2} dy \]

by the fact \( |y| e^{-\frac{1}{2}y^2} \leq 1 \). On the other hand, we have also

\[ \Delta_{22}(s, r, a) 1_{\{a \geq 0\}} \leq e^{-\frac{a^2}{2\pi}} \left( \int_{-\infty}^{0} \left( y + \frac{\mu - r2H}{\rho r} a \right)^{\alpha} |y| e^{-\frac{1}{2}y^2} dy + \int_{0}^{\infty} y^{\alpha+1} |e^{-\frac{1}{2}y^2} dy \right) 1_{\{a \geq 0\}} \]

\[ \leq \frac{1}{\alpha} e^{-\frac{a^2}{2\pi}} \left( \frac{\mu - r2H}{\rho r} \alpha \right) 1_{\{a \geq 0\}} + C_\alpha 1_{\{a \geq 0\}} \]

and

\[ \Delta_{22}(s, r, a) 1_{\{a < 0\}} \leq 1_{\{a < 0\}} \int_{-\infty}^{\infty} \left( y + \frac{\mu - r2H}{\rho r} a \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy dy + 1_{\{a < 0\}} \int_{-\infty}^{\infty} \left( \frac{\mu - r2H}{\rho r} \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy dy \]

\[ \leq \alpha^{-1} \left( \frac{\mu - r2H}{\rho r} \right)^{\alpha} 1_{\{a < 0\}} + \frac{(\mu - r2H)^{\alpha-1}}{(\rho r)^{\alpha-1}} 1_{\{a < 0\}} \]

by the fact \( |y|^{\alpha} e^{-\frac{1}{2}y^2} \leq 1 \) with \( 0 \leq \alpha \leq 1 \). It follows from Lemma 2.1 and Lemma 2.2 that

\[ \Delta_2(s, r, a) = \Delta_{21}(s, r, a) + \Delta_{22}(s, r, a) \]

\[ \leq C_\alpha \left( 1 + \frac{\mu - r2H}{\rho} + \frac{(\mu - r2H)^{\alpha}}{\rho^\alpha} + \frac{(\mu - r2H)^{\alpha-1}}{(\rho r)^{\alpha-1}} \right) \]

\[ \leq C_{H, a, t}(s - r)^{-(1-H)(1-\alpha)} r^{-\alpha H} \]
for all $0 < r < s \leq t$. Combining this with Lemma 2.1 we have

\[
\int_{\mathbb{R}} \psi_1(x-a) \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx = \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx + \int_{a}^{\infty} \left| \log(x-a) \right| \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx \\
\leq \frac{CH_{a,t}}{\rho} (1 + \Delta_1(s,r,a) + \Delta_2(s,r,a)) \\
\leq \frac{CH_{a,t}}{\rho} \left( 1 + (s-r)^{-(1-H)(1-a)} \right) \\
\leq C_{H,a,t} (s-r)^{-(1+a)H}
\]

for all $0 < r < s \leq t$ and $1 - H < a \leq 1$. Similarly, we can obtain the estimate (3.23). \qed}

Now, we can prove Proposition 3.1.

\textbf{Proof of Proposition 3.1.} Let $G_n, n \geq 2$ be defined in (3.16). Then

\[ E \left| G_n(B^H_s - a) - F'_n(B^H_s - a) \right|^2 \longrightarrow 0 \quad (n \to \infty) \]

for all $s \geq 0$ and $a \in \mathbb{R}$ by Lemma 3.4, Lebesgue’s dominated convergence theorem and the next estimate:

(3.24) \[ E[\psi_1(B^H_s - a)^2] = \int_{a}^{\infty} (1 + |\log(x-a)|)^2 \varphi_s(x) \, dx < \infty \]

for all $s \geq 0$ and $a \in \mathbb{R}$. Thus, it is sufficient to show that the sequence

\[ Y^H_t(n) := \int_{0}^{t} G_n(B^H_s - a) \delta B^H_s, \quad n \geq 2 \]

is a Cauchy sequence in $L^2(\Omega)$. Denote $\tilde{G}_{n,m} = G_n - G_m$ for all $n, m \geq 2$. Then $Y^H_t(n)$ is a Cauchy sequence in $L^2(\Omega)$ if and only if

\[ E \left| Y^H_t(n) - Y^H_t(m) \right|^2 = E \left| \int_{0}^{t} \tilde{G}_{n,m}(B^H_s - a) \delta B^H_s \right|^2 \\
= \int_{0}^{t} \int_{0}^{t} E\tilde{G}_{n,m}(B^H_s - a)\tilde{G}_{n,m}(B^H_r - a)\phi(s,r)drds \\
+ \int_{0}^{t} ds \int_{0}^{s} d\xi \int_{0}^{t} dr \int_{0}^{r} d\eta \phi(s,\eta)\phi(r,\xi) \\
\cdot E \left[ \tilde{G}_{n,m}(B^H_s - a)\tilde{G}_{n,m}(B^H_r - a) \right] \\
\equiv \Lambda_{n,m}(1) + \Lambda_{n,m}(2) \longrightarrow 0 \]

as $n, m \to \infty$.

On the one hand, we have

\[ |\tilde{G}_{n,m}(x)| \leq C\psi_1(x) \quad (x \in \mathbb{R}) \]

and $\tilde{G}_{n,m}(x) \to 0$ for all $x \in \mathbb{R}$, as $n, m$ tends to infinity, by Lemma 3.4. Accrediting with the estimate

\[ \Lambda_2(s,r,a,a) := \int_{0}^{t} \int_{0}^{t} E[\psi_1(B^H_s - a)\psi_1(B^H_r - a)]\phi(s,r)drds \\
= \int_{0}^{t} \int_{0}^{t} \phi(s,r)drds \int_{a}^{\infty} \int_{a}^{\infty} (1 + |\log(x-a)|) \\
\cdot (1 + |\log(y-a)|)\varphi_{s,r}(x,y) \, dxdy < \infty \]
and Lebesgue’s dominated convergence theorem, we give the convergence

\[(3.26) \quad \Lambda_{n,m}(1) = \int_0^t \int_0^t \phi(s,r) dr ds \int_{\mathbb{R}^2} \tilde{G}_{n,m}(x-a) \tilde{G}_{n,m}(y-a) \varphi_s,r(x,y) dx dy \rightarrow 0\]

for \(a \in \mathbb{R}\), as \(n, m\) tend to infinity.

On the other hand, by Lemma 3.1 we have

\[(3.27) \quad \Lambda_{n,m}(2) = \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \]

\[\cdot \int_{\mathbb{R}^2} \tilde{G}_{n,m}'(x-a) \tilde{G}_{n,m}'(y-a) \Psi_{s,r,a,a}(x,y) dx dy + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \Theta_{n,m}(s,r,a,a)\]

for all \(n, m, t \geq 0 \) and \(a \in \mathbb{R}\), where

\[\Theta_{n,m}(s,r,a,b) = -\tilde{G}_{n,m}(1) \int_{\mathbb{R}} \tilde{G}_{n,m}(x-a) \frac{\partial}{\partial x} \varphi_{s,r}(x,b) dx \]

\[-\tilde{G}_{n,m}(1) \int_{\mathbb{R}} \tilde{G}_{n,m}(y-b) \frac{\partial}{\partial y} \varphi_{s,r}(a,y) dy - \varphi_{s,r}(a,b) \left(\tilde{G}_{n,m}(1)\right)^2.\]

Noting that

\[\int_0^r |s - \xi|^{2H - 2} d\xi = \frac{1}{2H - 1} \left(s^{2H - 1} + |s - r|^{2H - 1} \text{sign}(r - s)\right),\]

we get

\[(3.28) \quad \int_0^s d\xi \int_0^r |r - \xi|^{2H - 2} |s - \eta|^{2H - 2} d\eta \leq \frac{2}{(2H - 1)^2} r^{2H - 1} s^{2H - 1}.\]

It follows from Lemma 3.4 and Lemma 3.7 with \(1 - H < \alpha < \frac{1-H}{H} \) that

\[\int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) |\Theta_{n,m}(s,r,a,a)|\]

\[\leq C_{H,t} \tilde{G}_{n,m}(1) \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \frac{\phi(s,\eta) \phi(r,\xi)}{|s - r|^{H(1+\alpha)(1+\alpha)}|s \wedge r|^{(1+\alpha)H}}\]

\[+ C_{H,t} \tilde{G}_{n,m}(1) \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \frac{\phi(s,\eta) \phi(r,\xi)}{|s - r|^{H(1+\alpha)}|s \wedge r|^{(1+2\alpha)H}}\]

\[+ \left(\tilde{G}_{n,m}(1)\right)^2 \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \frac{1}{\rho} \rightarrow 0 \quad (n, m \rightarrow \infty)\]

for all \(t \geq 0\) and \(a \in \mathbb{R}\). Moreover, (3.28) and Lemma 3.6 imply that

\[(3.30) \quad \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \]

\[\cdot \int_{\mathbb{R}^2} \psi_2(x-a) \psi_2(y-a) \Psi_{s,r,a,a}(x,y) dx dy < \infty\]

for all \(t \geq 0\) and \(a \in \mathbb{R}\). Combining this with (3.27), (3.29), Lemma 3.5 and Lebesgue’s dominated convergence theorem that the convergence, we have

\[\Lambda_{n,m}(2) \rightarrow 0,\]
as \( n, m \) tend to infinity. Thus, we have showed that \( Y_t^H(n), n = 1, 2, \ldots \) is a Cauchy sequence in \( L^2(\Omega) \) and the process

\[
\lim_{n \to \infty} \int_0^t G_n(B_s^H - a) \delta B_s^H = \int_0^t F'_+(B_s^H - a) \delta B_s^H, \quad t \geq 0
\]

exists in \( L^2(\Omega) \).

Denote

\[
\tilde{\Theta}_n(s, r, a, b) := -G_n(1) \int_{\mathbb{R}} G_n(x - a) \frac{\partial}{\partial x} \varphi_{s,r}(x, b) \, dx
\]

\[
- G_n(1) \int_{\mathbb{R}} G_n(y - b) \frac{\partial}{\partial y} \varphi_{s,r}(a, y) \, dy - \varphi_{s,r}(a, b) G_n(1) G_n(1)
\]

for all \( a, b \in \mathbb{R} \) and \( 0 < r < s \). Then, for all \( t \geq 0 \) and \( a \in \mathbb{R} \) we have

\[
E \left| \int_0^t G_n(B_s^H - a) \delta B_s^H \right|^2 = \int_0^t \int_0^t E G_n(B_s^H - a) G_n(B_r^H - a) \phi(s, r) \, dr \, ds
\]

\[
+ \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta(\phi(s, \eta) \phi(r, \xi) - \varphi_{s,r}(a, b) G_n(1) G_n(1)) \Omega_n(s, r, a, a)
\]

\[
= \int_0^t \int_0^t E G_n(B_s^H - a) G_n(B_r^H - a) \phi(s, r) \, dr \, ds
\]

\[
+ \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \varphi_{s,r}(a, b) G_n(1) G_n(1) \Omega_n(s, r, a, a)
\]

by Lemma 3.2. Notice that

\[
\int_0^t ds \int_0^s d\xi \int_0^r d\eta \varphi_{s,r}(a, b) G_n(1) G_n(1) \Omega_n(s, r, a, a) \to 0,
\]

as \( n \) tends to infinity, by Lemma 3.4, Lemma 3.7 and 3.28. We introduce the identity (3.13) by taking the limit in \( L^2(\Omega) \) and the proposition follows.

Finally, by considering the function on \( \mathbb{R}^2 \)

\[
\tilde{\Psi}_{s,r,a,b}(x, y) := \varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) \theta(y - 1 - b)
\]

\[
- \varphi_{s,r}(a, y) \theta(x - 1 - a) + \varphi_{s,r}(a, b) \theta(x - 1 - a) \theta(y - 1 - b)
\]

with \( s, r > 0 \) and \( a, b \in \mathbb{R} \), and in a same way as proof of Proposition 3.1, we can show that the integral

\[
\int_0^t F'_-(B_s^H - a) \delta B_s^H, \quad t \geq 0
\]

and the process

\[
C_t^{-H}(a) := 2 \left( F_-(B^H_t - a) - F_-(a) - \int_0^t F'_-(B_s^H - a) \delta B_s^H \right), \quad t \geq 0
\]

exist in \( L^2(\Omega) \) for all \( a \in \mathbb{R} \), where

\[
F_-(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ x \log(-x) - x, & \text{if } x < 0. \end{cases}
\]
Proposition 3.2. For all $a \in \mathbb{R}$ the integral

\begin{equation}
X_t^H(a) := \int_0^t \log |B^H_s - a| \delta B^H_s,
\end{equation}

and

\begin{equation}
E \left| \int_0^t \log |B^H_s - a| \delta B^H_s \right|^2 = \int_0^t \int_0^t E[\log |B^H_s - a| \log |B^H_t - a|] \phi(s, r) ds dr
\end{equation}

\begin{equation}
+ \int_0^t ds \int_0^s d\xi \int_0^r dr \int_0^\infty d\eta \phi(s, \eta) \phi(r, \xi) \int_{\mathbb{R}^2} \frac{\Psi_{s, r, a, a}(x, y) dxdy}{(x - a)(y - a)}
\end{equation}

for all $t \geq 0$ and $a \in \mathbb{R}$ and the process

\begin{equation}
C_t^H(a) := 2 \left( F(B_t^H - a) - F(-a) - \int_0^t \log |B^H_s - a| \delta B^H_s \right), \ t \geq 0
\end{equation}

are well defined, where $F(x) = x \log |x| - x$

Proof. Clearly, $F'(x) = \log |x|$, and the proposition follows from $F' = F'_+ + F'_-$.

\[\Box\]

4. A REPRESENTATION OF THE FUNCTIONAL $C^H(a)$

In this section we will consider the representation of the functionals $C^{+, H}(a), C^{-, H}(a)$ and $C^H(a)$, which point out that $\frac{1}{\pi} C^H_t(\cdot)$ is the Hilbert transform of weighted local time $\mathcal{L}^H(\cdot, t)$. 

Lemma 4.1. For any $0 < \varepsilon < 1$, $0 < r < s \leq t$ and $\beta \in (0, 1)$ we have

\begin{equation}
A_3(s, r, a) := \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left( \log (x - a) - \left( \frac{1}{\varepsilon} \log \varepsilon \right)(x - a) \right)
\end{equation}

\begin{equation}
\cdot \left( \log (y - a) - \left( \frac{1}{\varepsilon} \log \varepsilon \right)(y - a) \right) \varphi_{s, r}(x, y) dxdy
\end{equation}

\begin{equation}
\leq C_H(sr)^{-H/2} \varepsilon^H
\end{equation}

and

\begin{equation}
A_4(s, r, a) := \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left( \frac{1}{x - a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left( \frac{1}{y - a} - \frac{1}{\varepsilon} \log \varepsilon \right) |\Psi_{s, r, a, a}(x, y)| dxdy
\end{equation}

\begin{equation}
\leq C_{H, t, \beta} \frac{s^{\beta H/2}}{r^{(1+\frac{\beta}{2})H}(s-r)^{(1+\beta)H}} \varepsilon^\beta (1 + \log^2 \varepsilon).
\end{equation}

Proof. The estimate 4.1 is clear. In order to prove 4.2, we have

\begin{equation}
\int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left( \frac{1}{x - a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left( \frac{1}{y - a} - \frac{1}{\varepsilon} \log \varepsilon \right)
\cdot [(x - a)(y - a)]^{\beta/2} dxdy \leq C_\beta \varepsilon^\beta (1 + \log^2 \varepsilon)
\end{equation}

for all $\beta \in (0, 1)$, which gives

\begin{equation}
\int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left( \frac{1}{x - a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left( \frac{1}{y - a} - \frac{1}{\varepsilon} \log \varepsilon \right) |\Psi_{s, r, a, a}(x, y)| dxdy
\end{equation}

\begin{equation}
= \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left( \frac{1}{x - a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left( \frac{1}{y - a} - \frac{1}{\varepsilon} \log \varepsilon \right)
\cdot |\varphi_{s, r}(x, y) - \varphi_{s, r}(x, b) - \varphi_{s, r}(a, y) + \varphi_{s, r}(a, b)| dxdy
\end{equation}

\begin{equation}
\leq C_{H, t, \beta} \frac{s^{\beta H/2}}{r^{(1+\frac{\beta}{2})H}(s-r)^{(1+\beta)H}} \varepsilon^\beta (1 + \log^2 \varepsilon)
\end{equation}

by (3.3) and Lemma 2.1 This completes the proof.

\[\Box\]
Lemma 4.2. Let $\frac{1}{2} < H < 1$ and $M > 0$. We then have

\begin{equation}
E \left| \mathcal{L}^H(b,t) - \mathcal{L}^H(a,t) \right|^2 \leq C_{H,a,t,M} |b - a|^\alpha
\end{equation}

for all $0 < \alpha < \frac{1}{2H}$, $t \geq 0$ and $a,b \in [-M,M]$.

The lemma is a direct consequence of Hölder continuity of $x \mapsto \mathcal{L}^H(x,t)$. Here, we shall use other method to prove it.

Proof of Lemma 4.2. Without loss of generality we may assume that $0 < a < b$. Define the function $f_{a,b}(x) = 1_{(a,b]}(x)$ and denote

\[ \tilde{B}_t^H(x) := \int_0^t \mathbf{1}_{\{B_s^H > x\}} \delta B_s^H \]

and

\[ \psi_t(x) := (B_t^H - x)^+ - (-x)^+ \]

for all $x \in \mathbb{R}$. Then the function $x \mapsto \psi_t(x)$ is Lipschitz continuous with Lipschitz constant 2, and we have

\[ |\psi_t(x) - \psi_t(y)| \leq 2|x - y| \]

for all $x,y \in \mathbb{R}$ and

\[ \mathcal{L}^H(x,t) = 2 \left( \psi_t(x) - \tilde{B}_t^H(x) \right) \]

by Tanaka formula, which deduces

\[ E \left| \mathcal{L}^H(b,t) - \mathcal{L}^H(a,t) \right|^2 \leq 8(b - a)^2 + 4E \left| \tilde{B}_t^H(b) - \tilde{B}_t^H(a) \right|^2 \]

for all $t \geq 0$.

On the other hand, similar to the proof of (3.13) by approximating the function $f(x) = 1_{(a,b]}(x)$ by smooth functions we can obtain

\begin{align}
E \left( \tilde{B}_t^H(b) - \tilde{B}_t^H(a) \right)^2 &= E \left( \int_0^t f_{a,b}(B_s^H) \delta B_s^H \right)^2 \\
&= \int_0^t \int_0^t E \left[ f_{a,b}(B_s^H) f_{a,b}(B_r^H) \right] \phi(s,r) ds dr \\
&\quad + \int_0^t ds \int_0^s dx \int_0^t dr \int_0^r d\eta \Lambda_5(s,r,a,b) \phi(s,\eta) \phi(r,\xi) \\
&= G_1(a,b) + G_2(a,b)
\end{align}

for all $\frac{1}{2} < H < 1$, where

\[ \Lambda_5(s,r,a,b) = \varphi_{s,r}(a,a) - \varphi_{s,r}(a,b) - \varphi_{s,r}(b,a) + \varphi_{s,r}(b,b). \]

For the first term, we have

\[ E \left[ f_{a,b}(B_s^H) f_{a,b}(B_t^H) \right] = \int_a^b \int_a^b \frac{1}{2\pi \rho_{s,r}} \exp \left( -\frac{1}{2\rho_{s,r}^2} (r^{2H} x^2 - 2\mu_{s,r} x y + s^{2H} y^2) \right) dx dy \]

\[ = \frac{1}{2\pi} \int_0^b dx \int_0^b \frac{e^{-\frac{1}{2} x^2}}{\rho_{s,r}} \int_0^{\frac{r^{2H} - \mu_{s,r} x}{\rho_{s,r}}} e^{-\frac{1}{2} y^2} dy \]

\[ \leq \frac{1}{\sqrt{2\pi}} \int_0^b dx \int_0^b \frac{e^{-\frac{1}{2} x^2}}{\rho_{s,r}} \left( \frac{1}{\sqrt{2\pi}} \int_0^{\frac{r^{2H} - \mu_{s,r} x}{\rho_{s,r}}} e^{-\frac{1}{2} y^2} dy \right)^{\beta} \]

\[ \leq \left( \frac{r^H (b - a)}{\rho_{s,r}} \right)^{\beta} \int_0^{\frac{1}{r^H}} dx \int_0^b e^{-\frac{1}{2} x^2} dx \leq \frac{r^{(\alpha-1)H}}{\rho_{s,r}} (b - a)^{1+\beta}, \]
for all $s,r > 0$ and $\beta \in (0,1)$. It follows from Lemma 2.1 that

\begin{equation}
G_1(a,b) = \int_0^t \int_0^t E \left[ f_{a,b}(B^H_s) f_{a,b}(B^H_r) \right] \phi(s,r) ds dr \\
\leq C_{H,\beta} t^{H(1-\beta)} (b-a)^{1+\beta}
\end{equation}

for all $\frac{1}{2} < H < 1$ and $0 \leq \beta < \frac{2H-1}{H}$.

For the second term, we have also by (3.8) and Lemma 2.1

\begin{equation}
G_2(a,b) = \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \Lambda_5(a,b,s,r) \phi(s,\eta) \phi(r,\xi) \\
\leq C_H (b-a)^{\alpha} \int_0^t \int_0^t \frac{(sr)^{2H-1-\alpha H}}{\rho_{s,r}} dr ds \leq C_{H,\alpha} (b-a)^{\alpha t^{H(2-\alpha)}}
\end{equation}

for all $0 < \alpha < \frac{1-H}{H}$, and the lemma follows.

\hfill \Box

The main object of this section is to prove the following theorem.

**Theorem 4.1.** The convergence

\begin{equation}
C_{t,H}^\varepsilon(a) = \lim_{\varepsilon \to 0} \left\{ \frac{1}{(\log \varepsilon)_2} \mathcal{L}^H(a,t) + \int_0^t 1_{\{B^H_s - a \geq \varepsilon \}} \frac{2H s^{2H-1}}{B^H_s - a} ds \right\}
\end{equation}

holds in $L^2(\Omega)$ for all $t \geq 0$.

**Proof.** Let $t \geq 0$ and $a \in \mathbb{R}$. We split the proof in three steps.

**Step I.** Define the function $F_{\varepsilon}$ as follows

\[ F_{\varepsilon}(x) = \begin{cases} 0, & x \leq 0, \\
\frac{1}{2\varepsilon} (x^2 \log \varepsilon), & 0 < x \leq \varepsilon, \\
\varepsilon - \frac{1}{2} (\varepsilon \log \varepsilon) + x \log x - x, & x > \varepsilon. \end{cases} \]

Then $F_{\varepsilon} \in C^1(\mathbb{R})$, and

\[ F_{\varepsilon}'(x) = \begin{cases} 0, & x \leq 0, \\
\frac{1}{\varepsilon} (x \log \varepsilon), & 0 < x \leq \varepsilon, \\
\log x, & x > \varepsilon, \end{cases} \quad F_{\varepsilon}''(x) = \begin{cases} 0, & x < 0, \\
\frac{1}{\varepsilon}, & 0 < x < \varepsilon, \\
\frac{1}{x}, & x > \varepsilon. \end{cases} \]

for all $\varepsilon \in (0,1)$. We shall show that the Itô formula

\begin{equation}
F_{\varepsilon}(B^H_t - a) - F_{\varepsilon}(-a) - \int_0^t F_{\varepsilon}'(B^H_s - a) \delta B^H_s \\
= \frac{\log \varepsilon}{\varepsilon} \int_0^t 1_{\{0 \leq B^H_s - a \leq \varepsilon \}} H s^{2H-1} ds + \int_0^t 1_{\{B^H_s - a \geq \varepsilon \}} \frac{H s^{2H-1}}{B^H_s - a} ds
\end{equation}

holds for $\varepsilon \in (0,1)$. Define the sequence of smooth functions

\begin{equation}
f_{n,\varepsilon}(x) := \int_{\mathbb{R}} F_{\varepsilon}(x-y) \zeta_n(y) dy = \int_0^2 F_{\varepsilon}(x-\frac{y}{n}) \zeta(y) dy, \quad n = 1,2,\ldots
\end{equation}

for all $\varepsilon \in (0,1)$, where $\zeta$ is defined by (3.14) and $\zeta_n(x) = n \zeta(nx)$. Then $f_{n,\varepsilon} \in C^\infty(\mathbb{R})$ and

\[ f_{n,\varepsilon}(B^H_t - a) = f_{n,\varepsilon}(-a) + \int_0^t f_{n,\varepsilon}'(B^H_s - a) \delta B^H_s + H \int_0^t f_{n,\varepsilon}''(B^H_s - a) s^{2H-1} ds \]

for all $n = 1,2,\ldots$. Notice that

\[ f_{n,\varepsilon}(x) \to F_{\varepsilon}(x), \quad f_{n,\varepsilon}'(x) \to F_{\varepsilon}'(x) \]

as $n \to \infty$, uniformly in $\mathbb{R}$, and

\[ |f_{n,\varepsilon}''(x)| \leq \frac{1}{\varepsilon} |\log \varepsilon|, \quad \forall x \in \mathbb{R}, \]
and \( f''_{n,\varepsilon}(x) \rightarrow F''_{\varepsilon}(x) \) pointwise (besides 0 and \( \varepsilon \)), as \( n \rightarrow \infty \) by Lebesgue’s dominated convergence theorem. We get

\[
\int_0^t f'_{n,\varepsilon}(B^H_s - a)\delta B^H_s = f_{n,\varepsilon}(B^H_t - a) - f_{n,\varepsilon}(-a) - H \int_0^t f''_{n,\varepsilon}(B^H_s - a)s^{2H-1}ds
\]

\[
\rightarrow F_{\varepsilon}(B^H_t - a) - F_{\varepsilon}(-a) - H \int_0^t F''_{\varepsilon}(B^H_s - a)s^{2H-1}ds \quad \text{in } L^2(\Omega)
\]

\[
= F_{\varepsilon}(B^H_t - a) - F_{\varepsilon}(-a) - H\varepsilon^{-1} \int_0^t 1_{\{0 < B^H_t - a < \varepsilon\}}s^{2H-1}ds \quad \text{a.s.},
\]

as \( n \rightarrow \infty \), which implies that Itô’s formula

\[
(4.9) \quad F_{\varepsilon}(B^H_t - a) = F_{\varepsilon}(-a) + \int_0^t F'_{\varepsilon}(B^H_s - a)\delta B^H_s + H \int_0^t F''_{\varepsilon}(B^H_s - a)s^{2H-1}ds
\]

holds for all \( \varepsilon \in (0, 1) \). This gives \( (1.7) \).

**Step II.** We show that the limit

\[
(4.10) \quad \lim_{\varepsilon \downarrow 0} \left\{ F_{\varepsilon}(B^H_t - a) - F_{\varepsilon}(-a) - \int_0^t F'_{\varepsilon}(B^H_s - a)\delta B^H_s \right\}
\]

exists in \( L^2(\Omega) \), and is equal to

\[
\frac{1}{2} C^{+,H}_t(a) = F_+(B^H_t - a) - F_+(a) - \int_0^t F'_{+}(B^H_s - a)\delta B^H_s,
\]

where \( F_+ \) is given by \( (3.11) \). We have

\[
E \left| \frac{1}{2} C^{+,H}_t(a) + F_{\varepsilon}(-a) - F_{\varepsilon}(B^H_t - a) + \int_0^t F'_{\varepsilon}(B^H_s - a)\delta B^H_s \right|^2 
\]

\[
\leq 3E |F(B^H_t - a) - F_{\varepsilon}(B^H_t - a)|^2 + 3|F_{\varepsilon}(-a) - F_+(a)|^2 
\]

\[
+ 3E \left| \int_0^t [F'(B^H_s - a) - F'_{\varepsilon}(B^H_s - a)]\delta B^H_s \right|^2.
\]

The first and second term of the right-hand side in \( (4.11) \) tends to 0 as \( \varepsilon \rightarrow 0 \) because

\[
|F_+(x) - F_{\varepsilon}(x)| \leq \varepsilon - \frac{1}{2} \varepsilon \log \varepsilon
\]

for all \( \varepsilon \in (0, 1) \). To estimate the third term, we consider the approximation of the function \( F_{\varepsilon}' \) as follows

\[
\widehat{G}_{n,\varepsilon}(x) = \int_\mathbb{R} F_{\varepsilon}'(y)\zeta_n(x - y)dy, \quad n \geq 2
\]

for all \( \varepsilon \in (0, 1) \), where \( \zeta_n, n \geq 2 \) is given by \( (5.15) \). Then \( G_{n,\varepsilon}, n \geq 2 \) are smooth functions with compact supports. Denote

\[
G_{n,\varepsilon}(x) := G_n(x) - \widehat{G}_{n,\varepsilon}(x)
\]

for \( x \in \mathbb{R} \), where \( G_n \) is defined by \( (3.16) \). Similar to proofs of Lemma \( (3.4) \) and Lemma \( (3.5) \), we can obtain the next statements:

\[
|G_{n,\varepsilon}(x)| \leq C\psi_1(x), \quad |G'_{n,\varepsilon}(x)| \leq C\psi_2(x)
\]

for all \( x \in \mathbb{R}, \varepsilon \in (0, 1) \) and

\[
G_{n,\varepsilon}(x) \rightarrow F'_{+}(x) - F'_{\varepsilon}(x), \quad G'_{n,\varepsilon}(x) \rightarrow F''_{+}(x) - F''_{\varepsilon}(x) \quad (n \rightarrow \infty)
\]
for all \( x \neq 0 \) and \( \varepsilon \in (0, 1) \). Thus, in a same way as the proof of Proposition 3.1, we can obtain

\[
E \left| \int_0^t \left[ F_{\varepsilon}'(B^H_s - a) - F_{\varepsilon}'(B^H_s - a) \right] \delta B^H_s \right|^2 \\
= \int_0^t \int_0^t \Lambda_3(s, r, a) \phi(s, r) ds dr + \int_0^t \int_0^t \frac{d \Lambda_4(s, r, a) \phi(s, \eta) \phi(r, \xi)}{dr} \int_0^r d\eta \Lambda_4(s, r, a) \phi(s, \eta) \phi(r, \xi) \\
= \int_0^t \int_0^t \Lambda_3(s, r, a) \phi(s, r) ds dr + \int_0^t \int_0^t (sr)^{2H-1} dr ds \Lambda_4(s, r, a) \phi(s, \eta) \phi(r, \xi) \\
\leq C_H(t^H + t^{2-(2+\beta)H}) \varepsilon^{3+H}(1 + \log^2 \varepsilon) \longrightarrow 0
\]

with \( 0 < \beta < \frac{1-H}{H} \) by Lemma 3.1. It follows from the Itô formula (4.7) that

\[
C_t^{+H}(a) = 2 \lim_{\varepsilon \downarrow 0} \left\{ F_{\varepsilon}'(B^H_t - a) - F_{\varepsilon}'(-a) - \int_0^t F_{\varepsilon}'(B^H_s - a) \delta B^H_s \right\} \\
= \lim_{\varepsilon \downarrow 0} J_t^H(\varepsilon, a)
\]

in \( L^2(\Omega) \), where

\[
J_t^H(\varepsilon, a) = \frac{\log \varepsilon}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq B^H_s - a \leq \varepsilon\}} 2H s^{2H-1} ds + \int_0^t \mathbb{1}_{\{B^H_s - a \geq \varepsilon\}} 2H s^{2H-1} ds.
\]

**Step III.** To end the proof, we decompose \( J_t^H(\varepsilon, a) \) as

\[
J_t^H(\varepsilon, a) = I_t(\varepsilon, a) + \left\{ \int_0^t \mathbb{1}_{\{B^H_s - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B^H_s - a} ds + (\log \varepsilon) \mathcal{L}^H(a, t) \right\},
\]

where

\[
I_t(\varepsilon, a) := \frac{\log \varepsilon}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq B^H_s - a \leq \varepsilon\}} 2H s^{2H-1} ds - (\log \varepsilon) \mathcal{L}^H(a, t).
\]

According to Lemma 4.2 we get

\[
E[I_t(\varepsilon, a)]^2 : = (\log \varepsilon)^2 E \left| \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq B^H_s - a \leq \varepsilon\}} 2H s^{2H-1} ds - \mathcal{L}^H(a, t) \right|^2 \\
= (\log \varepsilon)^2 E \left| \frac{1}{\varepsilon} \int_0^\varepsilon \mathcal{L}^H(x + a, t) dx - \mathcal{L}^H(a, t) \right|^2 \\
\leq (\log \varepsilon)^2 \frac{1}{\varepsilon} \int_0^\varepsilon E[\mathcal{L}^H(x + a, t) - \mathcal{L}^H(a, t)]^2 dx \\
\leq C_{H, t, a} \varepsilon^\alpha (\log \varepsilon)^2 \longrightarrow 0 \quad (\varepsilon \rightarrow 0)
\]

for all \( 0 < \alpha < \frac{1-H}{H} \) and \( t \geq 0 \), which shows that

\[
C_t^{+H}(a) = \lim_{\varepsilon \downarrow 0} \left\{ \int_0^t \mathbb{1}_{\{B^H_s - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B^H_s - a} ds + (\log \varepsilon) \mathcal{L}^H(a, t) \right\} \quad \text{in } L^2(\Omega)
\]

for all \( t \geq 0 \), and the theorem follows.

**Theorem 4.2.** The convergence

\[
C_t^H(a) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{\{B^H_s - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B^H_s - a} ds \equiv \text{v.p.} \int_0^t \frac{2H s^{2H-1}}{B^H_s - a} ds
\]

holds in \( L^2(\Omega) \).
Proof. In the same way as the proof of (4.6), we can show that the convergence

\[ C_{t}^{-H} (a) = \lim_{\varepsilon \downarrow 0} \left\{ -(\log \varepsilon) \mathcal{L}^{H} (a, t) + \int_{0}^{t} 1_{\{B_{s}^{H} - a \leq -\varepsilon\}} \frac{2H s^{2H-1}}{B_{s}^{H}} ds \right\} \]

holds in $L^{2}(\Omega)$. Thus, (4.13) follows from $F = F_{\pm} + F_{-}$, where $F(x) = x \log |x| - x$. □

According to the occupation formula we get

\[ C_{t}^{H} (a) = \lim_{\varepsilon \downarrow 0} \int_{0}^{t} 1_{\{|B_{s}^{H} - a| \geq \varepsilon\}} \frac{\mathcal{L}^{H} (x, t)}{x - a} dx \quad \text{in} \ L^{2}(\Omega) \]

(4.15)

\[ = v.p. \int_{\mathbb{R}} \frac{\mathcal{L}^{H} (x, t)}{x - a} dx = \pi \left( \mathcal{H} \mathcal{L}^{H} (\cdot, t) \right) (a) \]

for all $t \geq 0$ and $a \in \mathbb{R}$. As two natural results we get the fractional version of Yamada’s formula

\[(B_{t}^{H} - a) \log |B_{t}^{H} - a| - (B_{t}^{H} - a)\]

\[= -a \log |a| + a + \int_{0}^{t} \log |B_{s}^{H} - a| \delta B_{s}^{H} + \frac{1}{2} v.p. \int_{\mathbb{R}} \frac{\mathcal{L}^{H} (x, t)}{x - a} dx \]

for all $t \geq 0$ and $a \in \mathbb{R}$, and

\[ C_{t}^{H} (b) - C_{s}^{H} (a) \]

\[= \int_{0}^{\infty} \left[ \mathcal{L}^{H} (b + x, t) - \mathcal{L}^{H} (b - x, t) - \mathcal{L}^{H} (a + x, s) + \mathcal{L}^{H} (a - x, s) \right] \frac{dx}{x} \]

for all $a, b \in \mathbb{R}$ and $s, t \geq 0$. Recall that the local time $\mathcal{L}^{H} (x, t)$ admits a compact support and it is Hölder continuous of order $\gamma \in (0, 1 - H)$ in time, and of order $\kappa \in (0, \frac{1-H}{2H})$ in the space variable (see Geman-Horowitz [16]). We see that the process $(a, t) \mapsto C_{t}^{H} (a)$ admits Hölder continuous paths. In particular, we have

Proposition 4.1. Let $\frac{1}{2} < H < 1$. For all $t' \geq t \geq 0$, we have

\[ E \left[ \left| C_{t'}^{H} - C_{t}^{H} \right|^{2} \right] \leq C (t' - t)^{2H_{0}}, \]

where

\[ H_{0} = \begin{cases} H, & \text{if } \frac{1}{2} < H \leq \frac{2}{3}, \\ 1 - \frac{1}{2} H, & \text{if } \frac{2}{3} < H < 1. \end{cases} \]

Proof. Given $\varepsilon > 0$ and denote

\[ C_{t}^{H, \varepsilon} = \int_{0}^{t} 1_{\{|B_{s}^{H}| > \varepsilon\}} \frac{ds^{2H}}{B_{s}^{H}} \]
for $t \geq 0$. We have

$$E \left[ \mathbf{1}_{|B^H_t| > \varepsilon} \frac{1}{B^H_s B^H_r} \right] = \int_{\mathbb{R}^2} 1_{\{|x| > \varepsilon\}} 1_{\{|y| > \varepsilon\}} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy$$

$$= \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy + \int_{-\infty}^{\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy$$

$$+ \int_{\varepsilon}^{\infty} \int_{-\infty}^{-\varepsilon} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy + \int_{-\infty}^{\varepsilon} \int_{-\infty}^{-\varepsilon} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy$$

$$= \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} [\varphi_{s,r}(x,y) - \varphi_{s,r}(-x,y) - \varphi_{s,r}(x,-y) + \varphi_{s,r}(-x,-y)] dxdy$$

$$= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} [\varphi_{s,r}(x,y) - \varphi_{s,r}(-x,y)] dxdy$$

$$= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} \left(1 - e^{-\frac{2H}{\rho_{s,r}} \mu_{s,r} \xi y} \right) \varphi_{s,r}(x,y) dxdy$$

$$= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \left(\int_{0}^{\rho_{s,r}^2} e^{-xy \xi} d\xi \right) \varphi_{s,r}(x,y) dxdy$$

$$= 2 \int_{0}^{\rho_{s,r}^2} d\xi \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} e^{-xy \xi} \varphi_{s,r}(x,y) dxdy$$

for all $s, t \geq 0$. An elementary calculus can show that

$$\int_{0}^{\infty} \int_{0}^{\infty} e^{-xy \xi} \varphi_{s,r}(x,y) dxdy$$

$$= \frac{1}{2\pi \rho_{s,r}} \int_{0}^{\infty} e^{-\frac{1}{2\pi \rho_{s,r}} (1+2\mu_{s,r} \xi - \rho_{s,r}^2 \xi^2)} dy \int_{0}^{\infty} e^{-\frac{2H}{\rho_{s,r}^2} \xi} \left(1 - e^{-\frac{1}{2\pi \rho_{s,r}} \mu_{s,r} \xi y} \right) dy$$

$$= \frac{1}{4\sqrt{1 + 2\mu_{s,r} \xi - \rho_{s,r}^2 \xi^2}}$$

for all $\xi > 0$, which implies that

$$E \left[ \mathbf{1}_{|B^H_t| > \varepsilon} \frac{1}{B^H_s B^H_r} \right] = \int_{\mathbb{R}^2} 1_{\{|x| > \varepsilon\}} 1_{\{|y| > \varepsilon\}} \frac{1}{xy} \varphi_{s,r}(x,y) dxdy$$

$$\leq \int_{0}^{\rho_{s,r}^2} d\xi = \frac{1}{4\rho_{s,r}} \arcsin \frac{\mu_{s,r}}{\sqrt{\rho_{s,r}^2 + \mu_{s,r}^2}}$$

$$= \frac{1}{\rho_{s,r}} \arcsin \frac{\mu_{s,r}}{(2H)^2} \leq \frac{1}{\rho_{s,r}}$$

It follows that

$$E \left[ |C_{t'}^{H,\varepsilon} - C_{t}^{H,\varepsilon}|^2 \right] \leq \int_{t}^{t'} \int_{t}^{t'} \frac{1}{\rho_{s,r}} ds \frac{1}{\rho_{s,r}} dr \frac{1}{2H} \leq \frac{C(t' - t)^{2H}}{2H}$$

for all $0 < t < t' < T$ and $\varepsilon > 0$. This shows that

$$E \left[ |C_{t'}^{H} - C_{t}^{H}|^2 \right] \leq C(t' - t)^{2H}$$

and the proposition follows. \( \square \)

**Remark 4.1.** The above continuity results for the process $(x,t) \mapsto C^H(x,t) := C^H_t(x)$ are some reminders to us that we may consider the following integrals:

$$\int_{0}^{t} u_s dC^H_s, \quad \int_{\mathbb{R}} f(x) C^H(dx,t), \quad \int_{0}^{t} \int_{\mathbb{R}} f(x,s) C^H(dx,ds),$$
where \( u \) is an adapted process, and \((x,t) \mapsto f(x,t)\) and \(x \mapsto f(x)\) Borel functions on \(\mathbb{R} \times [0,T]\) and \(\mathbb{R}\), respectively. These will be considered in the other paper.

5. The occupation formula associated with \(\mathcal{C}^H(a)\)

From the previous sections we know that the process \((a,t) \mapsto \mathcal{C}^H_t(a)\) is Hölder continuous and in this section our main object is to expound and prove the next theorem which is an analogue of the occupation formula.

**Theorem 5.1.** Let \(\frac{1}{2} < H < 1\) and let \(g\) be a continuous function with compact support. We then have, almost surely,

\[
\int_{\mathbb{R}} \mathcal{C}^H_t(x)g(x)dx = 2H\pi \int_{0}^{t} (\mathcal{H}g)(B_s^H) s^{2H-1}ds
\]

and

\[
2H\pi \int_{0}^{t} f(B_s^H) s^{2H-1}ds = \int_{\mathbb{R}} \mathcal{C}^H_t(x)(\mathcal{H}^{-1} f)(x)dx
\]

for all \(t \geq 0\), where the operator \(\mathcal{H}^{-1}\) means the inverse transform of Hilbert transform \(\mathcal{H}\).

In order to prove the theorem we need some preliminaries.

**Lemma 5.1.** Let \(F(x) = x \log |x| - x\) and let \(g\) be a continuous function with compact support. Then the integral

\[
\int_{0}^{t} (F' \ast g)(B_s^H) \delta B_s^H
\]

evaluates in \(L^2(\Omega)\) for all \(t \geq 0\) and the process

\[
\mathcal{X}^g_t := (F \ast g)(B_t^H) - (F \ast g)(0) - \int_{0}^{t} (F' \ast g)(B_s^H) \delta B_s^H, \quad t \geq 0
\]

is well-defined.

**Proof.** From Lemma 3.1 it follows that

\[
E \left| \int_{0}^{t} (G \ast g)(B_s^H) \delta B_s^H \right|^2 = \int_{0}^{t} \int_{0}^{t} E \left[ (G \ast g)(B_s^H)(G \ast g)(B_r^H) \right] \phi(s,r)dsdr
\]

\[
+ \int_{0}^{t} ds \int_{0}^{t} dr \int_{0}^{s} d\xi \int_{r}^{\xi} d\eta \phi(s,\eta)\phi(r,\xi) E \left[ (G' \ast g)(B_s^H)(G' \ast g)(B_r^H) \right]
\]

\[
= \int_{0}^{t} \int_{0}^{t} \phi(s,r)dsdr \int_{\mathbb{R}^2} g(u)g(v)dudv \int_{\mathbb{R}^2} G(x-u)G(y-v)\varphi_{s,r}(x,y)dxdy
\]

\[
+ \int_{0}^{t} ds \int_{0}^{t} dr \int_{0}^{s} d\xi \int_{r}^{\xi} d\eta \phi(s,\eta)\phi(r,\xi)
\]

\[
\cdot \int_{\mathbb{R}^2} g(u)g(v)dudv \int_{\mathbb{R}^2} G'(x-u)G'(y-v)\Psi_{s,r,u,v}(x,y)dxdy
\]

\[
+ \int_{0}^{t} ds \int_{0}^{t} dr \int_{0}^{s} d\xi \int_{r}^{\xi} d\eta \phi(s,\eta)\phi(r,\xi) \int_{\mathbb{R}^2} g(u)g(v)\Lambda_7(s,r,u,v)dudv
\]

for all \(t > 0, u, v \in \mathbb{R}\), and all \(G \in C^\infty(\mathbb{R})\) with compact support, where

\[
\Lambda_7(s,r,u,v) = -G(1) \int_{\mathbb{R}} G(x-u) \frac{\partial}{\partial x} \varphi_{s,r}(x,v)dx
\]

\[
- G(1) \int_{\mathbb{R}} G(y-v) \frac{\partial}{\partial y} \varphi_{s,r}(u,y)dy - \varphi_{s,r}(u,v) G(1) G(1).
\]

Decompose \(F\) as

\[
F(x) = F_+(x) + F_-(x),
\]
where $F_+$ and $F_-$ are given in Section 3. Clearly, we have
\[
\int_{\mathbb{R}^2} |g(u)g(v)|dudv \int_{\mathbb{R}^2} |F'_+(x-u)F'_+(y-v)|\varphi_{s,r}(x,y)dx dy
\]
\[
\leq \int_{\mathbb{R}^2} |g(u)g(v)|dudv \left( \int_{u}^{\infty} \log^2(x-u)\varphi_{s}(x)dx + \int_{v}^{\infty} \log^2(y-v)\varphi_{r}(y)dy \right)^{1/2}
\]
\[
\leq \int_{\mathbb{R}^2} |g(u)g(v)|dudv \left( s^{-H} + s^H + |u| \right)^{1/2} \left( r^{-H} + r^H + |v| \right)^{1/2}
\]
\[
\leq C_H \left( r^{-H} + s^H + 1 \right) \left( \int_{\mathbb{R}} |g(u)| \left( \sqrt{|u|} + 1 \right)du \right)^2,
\]
and
\[
\int_{\mathbb{R}^2} |g(u)g(v)|dudv \int_{u}^{\infty} dx \int_{v}^{\infty} \frac{|\Psi_{s,r,u,v}(x,y)|dy}{(x-u)(y-v)}
\]
\[
\leq C_{H,T,\beta} \frac{t}{r^{(1+\beta)H}(s-r)^{(1+\beta)H}} \int_{\mathbb{R}^2} |g(u)g(v)|dudv
\]
by Lemma 3.3. Thus, similar to the proof of Proposition 3.1, by approximating the function $F'_+(x)$ by smooth functions with compact support, we can show that the integral $\int_0^t (F'_+ * g)(B^H_s)\delta B^H_s$ exists in $L^2(\Omega)$ for all $t \geq 0$ and
\[
E \left| \int_0^t (F'_+ * g)(B^H_s)\delta B^H_s \right|^2 = \int_0^t \int_0^t dsdr \phi(s,r) \int_{\mathbb{R}^2} dudv g(u)g(v)
\]
\[
\cdot \int_{\mathbb{R}^2} F'_+(x-u)F'_+(y-v)\varphi_{s,r}(x,y)dx dy
\]
\[
+ \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta)\phi(r,\xi)
\]
\[
\cdot \int_{\mathbb{R}^2} g(u)g(v)dudv \int_{u}^{\infty} dx \int_{v}^{\infty} \frac{\Psi_{s,r,u,v}(x,y)dy}{(x-u)(y-v)}.
\]
Similarly, we can also show that the integral $\int_0^t (F'_- * g)(B^H_s)\delta B^H_s$ exists in $L^2(\Omega)$ for all $t \geq 0$, and the lemma follows since $F = F_+ + F_-$. \hfill \square

**Lemma 5.2.** Let $F(x) = x \log |x| - x$ and let $g$ be a continuous function with compact support. Then
\[
(5.2) \quad \int_0^t \left( \int_{\mathbb{R}} F'(B^H_s - x)g(x)dx \right) \delta B^H_s = \int_{\mathbb{R}} \left( \int_0^t F'(B^H_s - x)\delta B^H_s \right) g(x)dx
\]
for all $0 \leq t \leq T$.

By using the divergence operator $\delta^H$ we can rewrite (5.2) as
\[
\delta^H (F' * g)(B^H) = \int_{\mathbb{R}} g(a)da \int_0^t F'(B^H - a)\delta B^H_s.
\]

**Proof of Lemma 5.2.** Clearly, we have
\[
F' * g = (F * g)', \quad (F * g)' = v.p. \frac{1}{x} * g.
\]
Moreover, the functional
\[
x \mapsto \int_0^t F'(B^H_s - x)\delta B^H_s
\]
is Borel measurable for every $t \geq 0$ and the right-hand side in (5.2) exists also in $L^2(\Omega)$ by Proposition 3.1.
Denote by \( X \) the process concerning the right hand in (5.2) and let
\[
    u_t = \int_{\mathbb{R}} F'(B_t^H - x)g(x)dx
\]
for \( t \geq 0 \). Then, the process \( X \) and \( u \) are measurable. Thus, it is enough to show that the following duality relationship holds:
\[
    E[U X_T] = E[\langle D^H U, u \rangle_{\mathcal{F}}]
\]
for all \( U \in \mathbb{D}^{1,2} \) by Lemma 5.1. This is clear. In fact, noting that
\[
    \int_{\mathbb{R}} \left( \int_0^T (D_s^H U) F'(B_s^H - x) \phi(s, r) d^s r \right) g(x) dx
    = \int_0^T \int_0^T (D_s^H U) \left( \int_{\mathbb{R}} F'(B_s^H - x) g(x) dx \right) \phi(s, r) d^s r, \quad \text{a.s.}
\]
for all \( U \in \mathbb{D}^{1,2} \), we have
\[
    E[U X_T] = E \left[ U \int_{\mathbb{R}} \left( \int_0^T F'(B_s^H - x) \delta B_s^H \right) g(x) dx \right]
    = \int_{\mathbb{R}} E \left[ U \int_0^T F'(B_s^H - x) \delta B_s^H \right] g(x) dx
    = \int_{\mathbb{R}} \left( E \langle D^H U, F'(B_s^H - x) \rangle_{\mathcal{F}} \right) g(x) dx
    = E \int_{\mathbb{R}} \left( \int_0^T (D_s^H U) F'(B_s^H - x) \phi(s, r) d^s r \right) g(x) dx
    = E \int_0^T \int_0^T (D_s^H U) \left( \int_{\mathbb{R}} F'(B_s^H - x) g(x) dx \right) \phi(s, r) d^s r = E[\langle D^H U, u \rangle_{\mathcal{F}}]
\]
for all \( U \in \mathbb{D}^{1,2} \), and the lemma follows. \( \square \)

**Proof of Theorem 5.1.** Let \( F(x) = x \log |x| - x \). Then second derivative \( (F * g)'' = F'' * g \) exists in the sense of Schwartz’s distribution, and similar to Theorem 4.1 we have
\[
    \mathcal{X}_t^g = H \int_0^t \text{v.p.} \frac{1}{x} * g(B_s^H) s^{2H-1} ds
\]
for all \( t \geq 0 \), where \( \mathcal{C}^g \) is defined in Lemma 5.1. It follows from Lemma 5.2 that
\[
    \frac{1}{2} \int_{\mathbb{R}} C_t^H(x) g(x) dx = \int_{\mathbb{R}} \left( F(B_t^H - x) - F(-x) - \int_0^t F'(B_s^H - x) \delta B_s^{H} \right) g(x) dx
    = F * g(B_t^H) - F * g(0) - \int_0^t F' * g(B_s^H) \delta B_s^H
    = F * g(B_t^H) - F * g(0) - \int_0^t (F * g)'(B_s^H) \delta B_s^H
    = H \int_0^t \text{v.p.} \frac{1}{x} * g(B_s^H) s^{2H-1} ds
    = H \pi \int_0^t \mathcal{H}^g(B_s^H) s^{2H-1} ds
\]
for all \( t \geq 0 \). This completes the proof. \( \square \)

**Corollary 5.1.** Let \( \frac{1}{4} < H < 1 \) and let \( g, g_n \in L^2(\mathbb{R}) \) be continuous with compact supports. If \( g_n \to g \) in \( L^2(\mathbb{R}) \), as \( n \) tends to infinity, we then have
\[
    \lim_{n \to \infty} \int_{\mathbb{R}} C_t^H(x) g_n(x) dx = \int_{\mathbb{R}} C_t^H(x) g(x) dx
\]
for all $t \geq 0$, in the $L^2(\Omega)$.

Proof. The convergence follows from the identity
\[
\int_{\mathbb{R}} (g_n(x) - g(x))^2 \, dx = \int_{\mathbb{R}} (\mathcal{H} g_n(x) - \mathcal{H} g(x))^2 \, dx.
\]
and Theorem 5.1. \hfill \square

6. The case $0 < H < \frac{1}{2}$

In the final section we consider the process $C^H$ with $0 < H < \frac{1}{2}$. Recall that for $0 < H < \frac{1}{2}$, Yan et al \cite[p. 33]{} obtained the generalized quadratic covariation of $f(B^H)$ and $B^H$ defined by
\[
[f(B^H), B^H]_t^{(H)} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \{ f(B_{s+\varepsilon}^H) - f(B_s^H) \} (B_{s+\varepsilon}^H - B_s^H) \, ds^2 H
\]
in probability, where $f$ is a Borel function. In Yan et al \cite[p. 33]{} one constructed the Banach space $\mathcal{H} = L^2(\mathbb{R}, \mu(dx))$ with
\[
\mu(dx) = \left( \int_0^T e^{-\frac{2}{2\varepsilon^2} \frac{2H ds}{\sqrt{2\pi s^{1-H}}} } \right) \, dx
\]
and
\[
|f|^2_{\mathcal{H}} = \int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{2}{2\varepsilon^2} \frac{2H dx ds}{\sqrt{2\pi s^{1-H}}} } = E \left( \int_0^T |f(B_s^H)|^2 \, ds^2 H \right),
\]
such that the generalized quadratic covariation $[f(B^H), B^H]^{(H)}$ exists in $L^2(\Omega)$ and
\[
E \left| [f(B^H), B^H]_t^{(H)} \right|^2 \leq C\|f\|^2_{\mathcal{H}},
\]
provided $f \in \mathcal{H}$. Moreover, the Bouleau-Yor identity takes the form
\[
[f(B^H), B^H]_t^{(H)} = - \int_0^t f(x) \mathcal{L}^H(dx,t)
\]
for all $f \in \mathcal{H}$. By using the generalized quadratic covariation Yan et al \cite[p. 33]{} obtained the next Itô formula:
\[
F(B^H_t) = F(0) + \int_0^t f(B^H_s) \delta B^H_s + \frac{1}{2} [f(B^H), B^H]_t^{(H)}
\]
for all $0 < H < \frac{1}{2}$, where $F$ is an absolutely continuous function such that $F' = f \in \mathcal{H}$ is left (right) continuous. It is important to note that the method used in Yan et al \cite[p. 33]{} is inefficacy for $\frac{1}{2} < H < 1$ in general and the similar results for $\frac{1}{2} < H < 1$ is unknown so far.

**Corollary 6.1.** Let $0 < H < \frac{1}{2}$ and let $F(x) = x \log |x| - x$. Then $F' \in \mathcal{H}$ and the Itô type formula
\[
F(B^H_t - a) = F(-a) + \int_0^t \log |B^H_s - a| \delta B^H_s + \frac{1}{2} [\log(B^H - a), B^H]_t^{(H)}
\]
holds and
\[
C^H_t(a) = \left[ \log |B^H - a|, B^H \right]_t^{(H)}
\]
for all $t \geq 0$ and $a \in \mathbb{R}$.

**Proof.** Let $F_+$ and $F_-$ be defined in Section 3. Then $F' \in \mathcal{H}$ is left continuous, and
\[
F_+(B^H_t - a) = F_+(-a) + \int_0^t F'_+(B^H_s - a) \delta B^H_s + \frac{1}{2} [F'_+(B^H - a), B^H]_t^{(H)}
\]
by Itô’s formula \cite[p. 33]{}. Similarly, we have
\[
F_-(B^H_t - a) = F_-(a) + \int_0^t F'_-(B^H_s - a) \delta B^H_s + \frac{1}{2} [F'_-(B^H - a), B^H]_t^{(H)}
\]
By integration by parts we have
\[ \Theta_{\varepsilon}(t, a) := \mathcal{L}^{H}(a - \varepsilon, t)F'(\varepsilon) - \mathcal{L}^{H}(a + \varepsilon, t)F'(-\varepsilon) = \left[ \mathcal{L}^{H}(a - \varepsilon, t) - \mathcal{L}^{H}(a + \varepsilon, t) \right] \log \varepsilon. \]

Denote
\[ \Theta_{\varepsilon}(t, a) := \mathcal{L}^{H}(a - \varepsilon, t)F'(\varepsilon) - \mathcal{L}^{H}(a + \varepsilon, t)F'(-\varepsilon) = \left[ \mathcal{L}^{H}(a - \varepsilon, t) - \mathcal{L}^{H}(a + \varepsilon, t) \right] \log \varepsilon. \]

By integration by parts we have
\[
C_{t}^{H}(a) = \left[ \log |B^{H} - a|, B^{H} \right]_{t}^{(H)} = - \int_{\mathbb{R}} \log |x - a| \mathcal{L}^{H}(dx, t)
= - \lim_{\varepsilon \downarrow 0} \left( \int_{a + \varepsilon}^{\infty} \log |x - a| \mathcal{L}^{H}(dx, t) + \int_{-\infty}^{a - \varepsilon} \log |x - a| \mathcal{L}^{H}(dx, t) \right)
= \lim_{\varepsilon \downarrow 0} \left( \int_{a + \varepsilon}^{\infty} \mathcal{L}^{H}(x, t) dx + \int_{-\infty}^{a - \varepsilon} \mathcal{L}^{H}(x, t) dx \right) + \lim_{\varepsilon \downarrow 0} \Theta_{\varepsilon}(t, a)
= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\mathcal{L}^{H}(x, t)}{x - a} dx + \lim_{\varepsilon \downarrow 0} \Theta_{\varepsilon}(t, a)
= \text{v.p.} \int_{\mathbb{R}} \frac{\mathcal{L}^{H}(x, t)}{x - a} dx = \pi \mathcal{H} \mathcal{L}^{H}(\cdot, t)(x)
\]
almost surely and in \(L^{2}(\Omega)\), for all \(t \geq 0\) and \(a \in \mathbb{R}\) since \(x \mapsto \mathcal{L}^{H}(x, \cdot)\) is Hölder continuous and has the compact support.

**Corollary 6.2.** Let \(F\) be given by Corollary 6.1 and let \(g\) be a continuous function with compact support. Then the integral \(F' \ast g \in \mathcal{H}\), and for all \(0 < H < \frac{1}{2}\), the process
\[
2 \left( (F \ast g)(B_{t}^{H}) - (F \ast g)(0) - \int_{0}^{t} (F' \ast g)(B_{s}^{H}) \delta B_{s}^{H} \right)
\]
is well-defined in \(L^{2}(\Omega)\) and is equal to
\[
[(F' \ast g)(B^{H}), B^{H}]_{t}^{(H)}
\]
for all \(t \in [0, T]\).

Thus, similar to proof of Theorem 5.1 we can obtain the following occupation formula.

**Theorem 6.1.** Let \(0 < H < \frac{1}{2}\) and let \(g\) be a continuous function with compact support. Then we have, almost surely,
\[
\int_{\mathbb{R}} C_{t}^{H}(x) g(x) dx = 2H \pi \int_{0}^{t} \left( \mathcal{H} g \right)(B_{s}^{H}) s^{2H - 1} ds
\]
and
\[
2H \pi \int_{0}^{t} g(B_{s}^{H}) s^{2H - 1} ds = \int_{\mathbb{R}} C_{t}^{H}(x)(\mathcal{H}^{-1} g)(x) dx
\]
for all \(t \in [0, T]\).

**Proof.** Let \(F(x) = x \log |x| - x\). By Corollary 6.1 and 6.1 we have
\[
E \left\| \int_{\mathbb{R}} \left( \int_{0}^{t} F'(B_{s}^{H} - x) \delta B_{s}^{H} \right) g(x) dx \right\|^{2} < \infty,
\]
since \(g\) admits a compact support. Thus, similar to Lemma 5.2 we can show that the Fubini theorem
\[
\int_{0}^{t} \left( \int_{\mathbb{R}} F'(B_{s}^{H} - x) g(x) dx \right) \delta B_{s}^{H} = \int_{\mathbb{R}} \left( \int_{0}^{t} F'(B_{s}^{H} - x) \delta B_{s}^{H} \right) g(x) dx
\]
holds for all $0 \leq t \leq T$. It follows from Corollary 6.1 and Corollary 6.2 that
\[
\frac{1}{2} \int_{\mathbb{R}} C_t^H(x) g(x) \, dx = \int_{\mathbb{R}} \left( F(B_s^H - x) - F(-x) - \int_0^t F'(B_s^H - x) \, dB_s^H \right) g(x) \, dx \\
= F \ast g(B_t^H) - F \ast g(0) - \int_0^t F' \ast g(B_s^H) \, dB_s^H \\
= -\frac{1}{2} \int_{\mathbb{R}} (F' \ast g)(x) \mathcal{L}^H(dx, t)
\]
On the other hand, by the Hölder continuity of $(x,t) \mapsto \mathcal{L}^H(x,t)$ and Lebesgue’s dominated convergence theorem we have
\[
\int_{\mathbb{R}} g(a) \, da \int_{\mathbb{R}} F'(x-a) \mathcal{L}^H(dx, t) \\
= \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left( \int_{a+\varepsilon}^{\infty} F'(x-a) \mathcal{L}^H(dx, t) + \int_{-\infty}^{a-\varepsilon} F'(x-a) \mathcal{L}^H(dx, t) \right) \, da \\
= \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left( \Theta_\varepsilon(t,a) - \int_{-\infty}^{a-\varepsilon} 1_{\{|x-a|>\varepsilon\}} F''(x-a) \mathcal{L}^H(x,t) \, dx \right) \, da \\
= -2H \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left( \int_0^t \int_{|B_s^H-a|>\varepsilon} g(a) \mathcal{L}^H(B_s^H-a) \, da \right) \, ds \\
= -2H \pi \int_0^t s^{2H-1} ds \int_{\mathbb{R}} 1_{\{|B_s^H-a|>\varepsilon\}} \frac{g(a)}{B_s^H-a} \, da \\
= -2H \pi \int_0^t H \mathcal{H} g(B_s^H) s^{2H-1} ds
\]
almost surely and in $L^2(\Omega)$, for all $t \in [0,T]$. This shows that
\[
\frac{1}{2} \int_{\mathbb{R}} C_t^H(x) g(x) \, dx = H \pi \int_0^t \mathcal{H} g(B_s^H) s^{2H-1} ds
\]
and the theorem follows. \hfill \Box

**Remark 6.1.** When $0 < H < \frac{1}{2}$, from the discussion in this section, we have fund that for all non-locally integrable Borel functions $f \in \mathcal{H}$, the identities
\[
K_t^H(f,a) := \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H-a|>\varepsilon\}} f(B_s^H) \, ds^{2H} = [f(B^H),B^H]^H_t
\]
in $L^2(\Omega)$ (almost surely) and
\[
\int_{\mathbb{R}} K_t^H(f,x) g(x) \, dx = 2H \int_0^t v.p.(f' \ast g)(B_s^H) s^{2H-1} ds
\]
hold for all continuous functions $g$ with compact supports, provided
\[
\mathcal{L}^H(a-\varepsilon,t)f(-\varepsilon) - \mathcal{L}^H(a+\varepsilon,t)f(\varepsilon) \rightarrow 0,
\]
in $L^2(\Omega)$ (almost surely), as $\varepsilon$ tends to zero.

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