Rotating Boson Stars with Large Self-interaction in (2+1) dimensions

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Abstract

Solutions for rotating boson stars in (2+1) dimensional gravity with a negative cosmological constant are obtained numerically. The mass, particle number, and radius of the (2 + 1) dimensional rotating boson star are shown. Consequently we find the region where the stable boson star can exist.

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Self-gravitating systems have been investigated in various situations. Boson stars (for reviews) have a very simple constituent, a complex scalar field which is bound by gravitational attraction. Thus the boson star provides us with the simplest model of relativistic stars.

The solutions for relativistic boson stars are only numerically obtained in four dimensions. In \((2 + 1)\) dimensions, static equilibrium configurations have been argued in Einstein gravity with a negative cosmological constant. In the previous paper, we obtained an exact solution for nonrotating boson star in \((2 + 1)\) dimensional gravity with a negative cosmological constant. We consider that the scalar field has a strong self-interaction. An infinitely large self-interaction term in the model leads to much simplifications as in the \((3+1)\) dimensional case.

In the present paper, we obtain numerical solutions for a rotating boson star in \((2 + 1)\) dimensional gravity with a negative cosmological constant. We assume that the scalar field has a strong self-interaction as in the previous paper. The rotating boson star in the \((3+1)\) dimensional case has been studied numerically by F. D. Ryan. We wish to study the similarity and the difference, between the \((2 + 1)\) dimensional model and the model in other dimensions. The study of the rotating boson star will lead to a new aspect of gravitating systems and clarify the similarity and/or the difference among the other dimensional cases.

We consider a complex scalar field with mass \(m\) and a quartic self-coupling constant \(\lambda\). The action for the scalar field coupled to gravity can be written down as

\[
S = \int d^3x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( R + 2C \right) - \left| \nabla_\mu \varphi \right|^2 - m^2 \left| \varphi \right|^2 - \frac{\lambda}{2} \left| \varphi \right|^4 \right],
\]

where \(R\) is the scalar curvature and the positive constant \(C\) stands for the (negative) cosmological constant. \(G\) is the Newton constant.

Varying the action with respect to the scalar field and the metric yields equations of motion. The equation of motion for the scalar field is

\[
\nabla^2 \varphi - m^2 \varphi - \lambda \left| \varphi \right|^2 \varphi = 0,
\]

while the Einstein equation is
\begin{equation}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \left[ 2\text{Re} \left( \nabla_\mu \varphi^* \nabla_\nu \varphi - \frac{1}{2} |\nabla \varphi|^2 g_{\mu\nu} \right) - m^2 |\varphi|^2 g_{\mu\nu} - \frac{\lambda}{2} |\varphi|^4 g_{\mu\nu} \right] + C g_{\mu\nu}. \tag{3} \end{equation}

We assume the three-dimensional metric for a circularly symmetric spacetime as
\begin{equation}
ds^2 = -e^{-2\delta(r)} \Delta(r) dt^2 + \frac{1}{\Delta(r)} dr^2 + r^2 (d\theta - \Omega(r) dt)^2, \tag{4} \end{equation}
where \(\delta\), \(\Delta\), and \(\Omega\) are functions of the radial coordinate \(r\) only. We will later use \(\Delta = Cr^2 - 8GM(r)\), where \(M\) is a function of \(r\).

We also assume that the complex scalar has the following dependence on the coordinates,
\begin{equation}
\varphi = e^{-i\omega t + i\ell \theta} \varphi(r), \tag{5} \end{equation}
where \(\varphi(r)\) is a function of the radial coordinate \(r\) only and \(\omega\) and \(\ell\) are constants.

Then the equations (2) and (3) are written as
\begin{equation}
\frac{1}{2r} \frac{d\Delta}{dr} + \frac{1}{4} r^2 e^{2\delta} \left( \frac{d\Omega}{dr} \right)^2 = -8\pi G \left[ \Delta \left( \frac{d\varphi}{dr} \right)^2 + \frac{\ell^2}{r^2} \varphi^2 + \frac{e^{2\delta}}{\Delta} (\omega - \ell \Omega)^2 \varphi^2 
+ m^2 \varphi^2 + \frac{\lambda}{2} \varphi^4 \right] + C, \tag{6} \end{equation}

\begin{equation}
\frac{\Delta}{r} \frac{d\delta}{dr} = -16\pi G \left[ \Delta \left( \frac{d\varphi}{dr} \right)^2 + \frac{e^{2\delta}}{\Delta} (\omega - \ell \Omega)^2 \varphi^2 \right], \tag{7} \end{equation}

\begin{equation}
-\frac{1}{2r^2 \sqrt{\Delta}} \frac{d}{dr} \left( r^3 e^\delta \frac{d\Omega}{dr} \right) = 16\pi G \frac{e^\delta}{\sqrt{\Delta}} (\omega - \ell \Omega)^2 \varphi^2, \tag{8} \end{equation}

\begin{equation}
\frac{1}{r e^{-\delta}} \frac{d}{dr} \left( r e^{-\delta} \Delta \frac{d\varphi}{dr} \right) = \frac{\ell^2}{r^2} \varphi - \frac{e^{2\delta}}{\Delta} (\omega - \ell \Omega)^2 \varphi + m^2 \varphi + \lambda \varphi^3. \tag{9} \end{equation}

For convenience, we rescale the variables as
\begin{equation}
\tilde{r} = mr, \quad \tilde{\varphi} = \sqrt{8\pi G} \varphi, \quad \Lambda = \frac{\lambda}{8\pi G m^2}, \quad \tilde{\Omega} = \frac{1}{m} (\Omega - \frac{\omega}{\ell}). \tag{10} \end{equation}

We also denote \(\Delta = Cr^2 - 8GM = (C/m^2)\tilde{r}^2 - 8GM\).

Using these variables, we can rewrite the equations as
\begin{equation}
-\frac{1}{2\tilde{r}} \frac{d(8GM)}{d\tilde{r}} + \frac{1}{4} \tilde{r}^2 e^{2\delta} \left( \frac{d\tilde{\Omega}}{d\tilde{r}} \right)^2 = \left[ \Delta \left( \frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 + \frac{\ell^2}{\tilde{r}^2} \tilde{\varphi}^2 + \frac{e^{2\delta}}{\Delta} \ell^2 \tilde{\Omega}^2 \tilde{\varphi}^2 + \tilde{\varphi}^2 + \frac{\Lambda}{2} \tilde{\varphi}^4 \right], \tag{11} \end{equation}

3
\[ \frac{\Delta d\delta}{\bar{r} dr} = -2 \left[ \Delta \left( \frac{d\bar{\phi}}{dr} \right)^2 + \frac{e^{2\delta}}{\Delta} \ell^2 \bar{\Omega}^2 \bar{\phi}^2 \right], \]  
\(12\)

\[-\frac{1}{2r^2} \frac{d}{d\bar{r}} \left( r^3 e^{\delta} \frac{d\bar{\Omega}}{d\bar{r}} \right) = -2 \frac{e^{\delta}}{\Delta r} \ell^2 \bar{\Omega} \bar{\phi}^2, \]
\(13\)

\[\frac{1}{\bar{r} e^{-\delta}} \frac{d}{d\bar{r}} \left( \bar{r} e^{-\delta} \frac{d\bar{\phi}}{d\bar{r}} \right) = \frac{\ell^2}{\bar{r}^2} \bar{\phi} - \frac{e^{2\delta}}{\Delta} \ell^2 \bar{\Omega}^2 \bar{\phi} + \bar{\phi} + \Lambda \bar{\phi}^3. \]
\(14\)

Next we rescale the variables again, to study the limit of large self-interaction. New variables are:

\[ r_* = \bar{r} / \sqrt{\Lambda}, \quad \varphi_* = \sqrt{\Lambda} \bar{\phi}, \quad L^2 = \frac{\ell^2}{\Lambda}, \quad \Omega_* = \sqrt{\Lambda} \bar{\Omega}, \]
\(15\)

and \( \Delta = Cr^2 - 8GM = C_* r_*^2 - 8GM \), where \( C_* = CA/m^2 \).

Then the equations can be written as

\[-\frac{1}{2r_*} (8GM') + \frac{1}{4} r_*^2 \varphi_*^2 (\Omega_*)^2 = \left[ \frac{1}{\Lambda} \Delta (\varphi_*')^2 + \frac{L^2}{r_*^2} \varphi_*^2 + \frac{e^{2\delta}}{\Delta} L^2 \Omega_*^2 \varphi_*^2 + \varphi_*^2 + \frac{1}{2} \varphi_*^4 \right], \]
\(16\)

\[\frac{\Delta}{r_*} \delta' = -2 \left[ \frac{1}{\Lambda} \Delta (\varphi_*')^2 + \frac{e^{2\delta}}{\Delta} L^2 \Omega_*^2 \varphi_*^2 \right], \]
\(17\)

\[-\frac{1}{2r_*^2} \left( r_*^3 e^{\delta} \Omega_* \right)' = 2 \frac{e^{\delta}}{\Delta r_*} L^2 (-\Omega_*) \varphi_*^2, \]
\(18\)

\[\frac{1}{\Lambda r_* e^{-\delta}} \left( r_* e^{-\delta} \Delta \varphi_* \right)' = \frac{L^2}{r_*^2} \varphi_* - \frac{e^{2\delta}}{\Delta} L^2 \Omega_*^2 \varphi_* + \varphi_* + \varphi_*^3, \]
\(19\)

where \( ' \) denotes the derivative with respect to \( r_* \).

Now we consider the limit of large self-interaction. The study of boson stars with the finite value of self-coupling is very difficult. Thus we simplify the field equations by the limit of large self-interaction. For the limit of large self-coupling \( \Lambda \rightarrow \infty \), these equations will be reduced to\(^1\)

\[ \frac{1}{2r_*} \bar{M}' = \frac{1}{4L^2} \frac{1}{r_*^2} e^{2\delta} (\Omega')^2 + \frac{L^2}{r_*^2} \varphi_*^2 + \frac{e^{2\delta}}{\Omega} \varphi_*^2 + \varphi_*^2 + \frac{1}{2} \varphi_*^4, \]
\(20\)

\(^1\) For a finite \( C_* \), the actual value of \( C = m^2 C_* / \Lambda \) becomes infinitely small if the limit \( \Lambda = \infty \) is taken literally. We can however interpret the limit as an approximation of a large self-coupling, and \( \Lambda \) is not simply taken as a mathematical infinity.
\[
\frac{1}{r_*} \delta' = -2 e^{2\delta} \Omega^2 \varphi_*^2,
\]
\[
\frac{1}{r_*} \left( r_*^3 e^{2\delta} \Omega \right)' = 4 \frac{e^{3\delta}}{\Delta} L^2 \Omega \varphi_*^2,
\]
\[
\varphi_*^3 = \varphi_* \left( \frac{e^{2\delta}}{\Delta} \Omega^2 - \frac{L^2}{r_*^2} - 1 \right),
\]

where \( \bar{M} = 8GM \) and \( \bar{\Omega} = L\Omega_* \).

Now, let us solve the set of an algebraic equation and differential equations (20-23).

We can solve Eq.(23) for \( \varphi_* \) easily. A trivial solution is

\[
\varphi_*^2 = 0.
\]

This corresponds to the region of no bosonic matter field, i.e., the vacuum outside a star. The interior solution for \( \varphi_* \) is

\[
\varphi_*^2 = \frac{e^{2\delta}}{\Delta} \bar{\Omega}^2 - \frac{L^2}{r_*^2} - 1.
\]

This describes the configuration of the boson field inside a star.

The term \( L^2 / r_*^2 \) yields a remarkable consequence. For nonzero \( L^2 \), any positive value for the solution become impossible in the vicinity of the origin. Thus, for a sufficiently small value for \( r_* \), the solution for \( \varphi_*^2 \) must be \( \varphi_*^2 = 0 \). In other words, a rotating boson star has a vacuum “hole” in its central region! We note that the rotating boson star in (3+1) dimensions also has a similar structure in the central region [6].

If we require that there is no singularity in the central vacuum region, the spacetime must be a usual anti-de Sitter spacetime. Therefore in the central region

\[
\Delta = 1 + Cr^2 = 1 + C_* r_*^2.
\]

This determines the value of \( \bar{M} \) in the central region as \( \bar{M} = -1 \).

Unfortunately, a set of equations (20-23) cannot be solved analytically. Thus, we solve these equations numerically here.

To solve the full solution numerically, we must specify the boundary conditions. We consider that the central vacuum region is the range \( 0 < r_* < r_{si} \). The value for \( r_{si} \) gives the location of the inside boundary of a rotating boson star. Here
\[ M(r_{si}) = -1, \]  

(27)

from the previous consideration. Obviously \( \delta \) has a constant value in the vacuum regions. Thus the value for \( \delta \) in the vacuum regions can be additively modified by redefinition of the time coordinate, etc.. Therefore we take

\[ \delta(r_{si}) = 0, \]  

(28)

without any loss of generality. Because of the continuity of the solution for \( \varphi_s \), a condition is required:

\[ \frac{\bar{\Omega}_i^2}{C_s r_{si}^2 + 1} - \frac{L^2}{r_{si}^2} - 1 = 0, \]  

(29)

where \( \bar{\Omega}_i = \bar{\Omega}(r_{si}) \).

Now we will solve the field equations numerically from \( r_* = r_{si} \). In the numerical solutions, \( \varphi_s^2 \) becomes zero again at \( r_* = r_{so} > r_{si} \). We consider that the region \( r_* > r_{so} \) is outside of a rotating boson star, that is, the vacuum region where \( \varphi_s^2 = 0 \). In this region, we can analytically solve the equations by using the boundary value for variables at \( r_* = r_{so} \).

In the outside vacuum region, the metric is described by BTZ solution \[ [7]. \]

By these conditions, we can solve the field equations numerically. Now the arbitrary parameter set is three conditions, \( \bar{\Omega}_i, C_s \) and \( L \). Numerical solutions for some parameter sets are shown Fig. 1.

In Fig. 1 \( \varphi_s^2 \) as a function of the radial coordinate \( r_* \) is shown. As seen before, we see that the boson star has a vacuum hole in the center of the star. \( L \) is the parameter that indicates the magnitude of the rotation. For a small \( L \), the configuration of the boson star is similar to the nonrotating boson star. This nonrotating boson star has already been studied \[ [4]. \]

For a large \( L \), the boson star has a large vacuum hole and a large radius. The region where the matter exists becomes narrow.

We cannot, however, calculate numerical values of \( \varphi_s^2, \Delta, \delta, \) and \( \bar{\Omega} \) for all set of the conditions \( \bar{\Omega}_i, L \) and \( C_s \). For some parameter sets, \( \varphi_s^2 \) diverges before connecting the outside vacuum region. Scalar curvature of the rotating boson star is the following:
\[ R = -16\pi G \left[ 2m^2 \varphi^2 + \frac{\lambda}{2} \varphi^4 \right]. \]  

(30)

If \( \varphi^2 \) diverge, scalar curvature also diverge. Then there is a singularity. Therefore it is worth noting that the parameter region for the possible solutions is restricted.

For the possible solutions, we can calculate the physical quantities. Now, we consider the mass of the boson star. Since the external solutions of boson stars correspond to BTZ solutions, the mass of the boson star is identified with the BTZ mass. We define the total mass of the boson star with the mass of the boson star observed at infinity. For the outside of the boson star, the equation (20) and (22) are shown

\[ \frac{1}{2r_*} \tilde{M}' = \frac{1}{4L^2} r_*^2 e^{2\delta} \left( \bar{\Omega}' \right)^2, \]  

(31)

\[ \frac{1}{r_*} \left( r_*^3 e^{\delta} \bar{\Omega}' \right)' = 0. \]  

(32)

From the equation (32),

\[ r_*^3 e^{\delta} \bar{\Omega}' = r_*^3 e^\delta \bar{\Omega}'_o = \text{Const.}, \]  

(33)

where \( r_\ast = r_{*o} \) is the external boundary of the boson star, and \( \delta_o = \delta(r_{*o}), \bar{\Omega}'_o = \bar{\Omega}'(r_{*o}). \)

Then we find from the equation (31),

\[ \frac{1}{2r_*} \tilde{M}' = \frac{1}{4L^2} r_*^2 e^{2\delta} \left( \bar{\Omega}' \right)^2 = \frac{1}{4L^2} r_*^2 e^{2\delta} \left( \frac{r_*^3 e^{2\delta_o}}{r_*^3 e^\delta} \bar{\Omega}'_o \right)^2, \]  

(34)

\[ \tilde{M}' = \frac{1}{2L^2 r_*^3} r_*^6 e^{2\delta_o} (\bar{\Omega}'_o)^2. \]  

(35)

We integrate this in the external region of the boson star:

\[ \int_{r_{*o}}^\infty \tilde{M}' dr_* = \frac{1}{2L^2} r_*^6 e^{2\delta_o} (\bar{\Omega}'_o)^2 \int_{r_{*o}}^\infty \frac{1}{r_*^3} dr_* = \frac{r_*^4}{4L^2} e^{2\delta_o} (\bar{\Omega}'_o)^2. \]  

(36)

Here, we note that

\[ \int_{r_{*o}}^\infty \tilde{M}' dr_* = \tilde{M}(\infty) - \tilde{M}(r_{*o}). \]  

(37)
Thus the total mass of the boson star is given,

$$8G M_{BTZ} = \bar{M}(\infty) = \bar{M}_o + \frac{r_{s_o}^4}{4L^2} e^{2\delta_o} (\Omega_o')^2,$$

where $\bar{M}_o = \bar{M}(r_{s_o})$.

The particle number of the boson star is given,

$$N = - \int d^2 x \sqrt{-g} \ i \ (\varphi \nabla^i \varphi^* - \varphi^* \nabla^i \varphi)$$

$$= \int d^2 x e^{-\delta} \ i \left[ \frac{e^{2\delta}}{\Delta} (2 i \omega \varphi^2) + \frac{e^{2\delta}}{\Delta} (-2 i \ell \varphi^2) \right]$$

$$= 2\pi \int dr \frac{r e^\delta}{\Delta} 2\varphi^2 (\Omega \ell - \omega)$$

$$= \frac{1}{2Gm} \int dr \frac{r_s \varphi^2_e e^\delta \Omega}{C_s r_s^2 - \bar{M}}$$

$$= \frac{1}{2Gm} \int dr \frac{r_s e^\delta}{\Delta} \Omega \varphi^2_e.$$  (39)

By using the obtained mass and particle number, the binding energy can be defined. The value of BTZ mass $M_{BTZ}$ is negative when the matter of boson star is a little. In particular, in the limit of “no matter”, $M_{BTZ}$ approaches $-1/(8G)$. Thus we will take the binding energy as

$$E = M_{BTZ} + \frac{1}{8G} - mN.$$  (40)

The binding energy must be negative for the stable boson star. Therefore the region of stable solution is restricted. The region of the stable and possible solution is shown in Fig. 2. In this parameter region, the stable boson star can exist. In Fig. 2 the maximum value of the cosmological constant is $C_s = C_{crit} \approx 2.5268$ for $L \to 0$. For a nonrotating boson star, the maximum value of the cosmological constant is also $C_s = C_{crit} \approx 2.5268$ [4]. This is trivial, because a rotating solution approaches a nonrotating solution when $L \to 0$. From Fig. 2 one can find the region where the stable boson star can exist is restricted.

These parameters, particularly $L$ and $\Omega_i$, is not the meaningful physical parameters. Thus we rewrite these parameters in the meaningful physical parameters. Instead of these,
we will use the angular momentum of the boson star \( J_* \), the mass of the boson star \( M_{\text{BTZ}} \) and the cosmological constant \( C_* \) as the meaningful physical parameters.

We will consider the angular momentum of the boson star. Since the external solution of the boson star corresponds to BTZ solution, \( \Delta \) is given as

\[
\Delta = Cr^2 - 8GM_{\text{BTZ}} + \frac{(8GJ)^2}{4r^2} = C_\ast r_\ast^2 - 8GM_{\text{BTZ}} + \frac{(8GJ_\ast)^2}{4r_\ast^2},
\]

where \( J_\ast^2 = (m^2/\Lambda)J^2 \) is the angular momentum of the boson star. Thus the angular momentum of the boson star \( J_* \) is given by

\[
8GJ_\ast = \frac{r_\ast^3}{\mathcal{L}} e^{\delta_o} (\dot{\Omega}_o). \tag{42}
\]

Using these parameter, \( J_\ast, M_{\text{BTZ}} \) and \( C_* \), the region of the stable boson star is shown in Fig. 3. Simplifying Fig. 3, the region of the stable boson star is shown with \( \sqrt{C_*J_*} \) and \( M_{\text{BTZ}} \) in Fig. 4.

In the region of Fig. 4, the stable boson star can exist. To understand Fig. 4, we consider the case of the black hole physics. In the case of the black hole, it is known that the black hole with the larger angular momentum than its mass has the naked singularity and cannot be a usual black hole. In an opposite case, as the larger mass than the angular momentum, the black hole can be a usual black hole. Thus, if the boson star become the black hole adiabatically, the boson star with \( \sqrt{C_*J_*} < M_{\text{BTZ}} \) can become the usual black hole.

In Fig. 4, there is the relation \( \sqrt{C_*J_*} = M_{\text{BTZ}} \) when the mass is large. This fact is interesting. Because the black hole, which has such the same value of the angular momentum and the mass, is the extreme black hole. Thus, the boson star with the large mass can certainly become the extreme black hole. In the case of boson stars, how the behavior of the extreme black hole is appeared is very interesting. But, if the mass of boson star is larger, the singularity is more often caused and we cannot calculate the field equations numerically. Therefore the numerical analysis of boson stars with the sufficient large mass is very difficult.
To summarize, we have obtained the numerical solutions describing the rotating boson stars with a very large self-coupling constant in (2+1) dimensions. We found that the rotating boson star has a vacuum hole in the center of the star. And the region of the solution for the stable boson star is shown. There is the maximum value of the cosmological constant $C^* = C^*_{\text{crit}} \approx 2.5268$, where $C^* = (\lambda/(8\pi G m^4))C$. This corresponds to the nonrotating case. When the mass of boson star is large, there is the relation between the angular momentum and the mass of boson star, $\sqrt{C^*} J^* = M_{\text{BTZ}}$.

The future plan for study of boson stars is the following. The analysis of boson star with the actually finite value of self-coupling is of much interest and will be necessary. More general cases including such as a $|\varphi|^6$ coupling may exhibit more complicated results, but the analysis of them can be carried out in the same manner as in the present study. On the other hand, the model of the spinning boson star in (3+1) dimensions has been studied [3]. The similarity and the difference, between our model and the model in the other dimensions, must be further studied. Particularly, it is interesting to study the shape of the boson star with a large angular momentum in various dimensions.

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FIG. 1. $\varphi_*^2$ as a function of $r_*$, where (A) $C_* = 0.1$, $\bar{\Omega}_i = 1.1$, $L = 0.1$, (B) $C_* = 2.526$, $\bar{\Omega}_i = 2.54$, $L = 0.001$ and (C) $C_* = 0.1$, $\bar{\Omega}_i = 1.45$, $L = 1.4$. 
FIG. 2. The region of the stable solution. These points are shown in the region of stable and possible solutions.

FIG. 3. The region of the stable boson star showing with the angular momentum of boson star $J_*$, the mass of boson star $M_{BTZ}$ and the cosmological constant $C_*$
FIG. 4. The region of the stable boson star showing with $\sqrt{C_*} J_*$ and $M_{BTZ}$