CATEGORIFICATION OF A FRIEZE PATTERN DETERMINANT

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Abstract. Brion, Crowe and Isaacs have computed the determinant of a matrix associated to a Conway-Coxeter frieze pattern. We generalise their result to the corresponding frieze pattern of cluster variables arising from the Fomin-Zelevinsky cluster algebra of type $A$. We give a representation-theoretic interpretation of this result in terms of certain configurations of indecomposable objects in the root category of type $A$.

1. Introduction

In [Cox, CoCo1, CoCo2] Conway and Coxeter studied frieze patterns of integers in the plane. Such a frieze pattern consists of a finite number of infinite rows of integers, with each row interlacing its neighbouring rows, and satisfies the unimodular rule, which states that for every four adjacent numbers forming a square:

\[
\begin{array}{ccc}
  b & & \\
  a & d & \\
  c & & \\
\end{array}
\]

the relation $ad - bc = 1$ is satisfied. The entries in the first and last row are zero; the entries in the second and penultimate rows are 1, and all other entries should be positive. It is said to be of order $n$ if there are $n + 1$ rows (we take $n \geq 3$). Under these assumptions it is shown in [Cox] that the pattern is necessarily periodic. More precisely, it is invariant under a glide, the product of a translation of $n/2$ steps horizontally (where a single step takes a number to its horizontal neighbour) and a reflection in the horizontal. Observe that a fundamental domain for this transformation can be obtained by taking an entry in the second row together with all the entries in the triangle bounded by the two diagonals leading downwards from the entry and the bottom row.

An example of a frieze pattern of order 7 is given in Figure 2. A fundamental domain as above is shown, together with its images under the glide and its inverse.

We note that frieze patterns of integers of various kinds have been studied recently; see, for example [ARS, BM2, BR, KS, MOT, Pro].

In [BCI] the following result is shown:

Theorem 1.1. [BCI] Let $M$ be the symmetric matrix whose upper triangular part is obtained from a frieze pattern as above by choosing a fundamental domain for the glide reflection as described above and rotating it 45 degrees clockwise. Then

$$\det(M) = -(2)^{n-2}.$$
We describe this more precisely in the next section. A connection between cluster algebras and frieze patterns was established in [CaCh], who showed that each frieze pattern can be obtained by choosing an appropriate cluster in the cluster algebra of type $A_{n-3}$, writing each cluster variable as a function of the elements of the cluster (necessarily a Laurent polynomial by the Laurent Phenomenon [FZ1, 3.1]). The cluster variables are parametrized by the diagonals in a regular $n$-sided polygon $P_n$ and, when the variables in the cluster are specialised to 1, the resulting integers, when arranged correctly, produce the corresponding frieze pattern. We choose the cluster algebra of type $A_{n-3}$ which is given by the homogeneous coordinate ring of the Grassmannian of 2-planes in an $n$-dimensional vector space [FZ2, 12.6]. It has cluster coefficients $u_{ij}$ where $i, j$ are the end-points of a boundary edge.

Our main result is to show that a result corresponding to Theorem 1.1 holds in the cluster algebra case:

**Theorem 1.2.** Let $\pi$ be a triangulation of $P_n$. Let $U(\pi)$ be the matrix of cluster variables (with $i, j$ entry containing the cluster variable corresponding to the diagonal with end-points $i$ and $j$) written in terms of the cluster corresponding to $\pi$. Then

$$\det(U(\pi)) = -(-2)^{n-2}u_{12}u_{23} \cdots u_{n-1,n}u_{11}.$$

We then show that this result can be given a categorical interpretation in terms of the root category of type $A_{n-1}$. By interpreting this category as a category of oriented edges between vertices of $P_n$ (using methods similar to those in [CCS]) we show that the above determinant can be reinterpreted as a sum over configurations of indecomposable objects in the root category. Each configuration is a maximal collection of indecomposable objects such that no object lies in the frame of the other (see Section 7 for the definition of frame) which is also of maximal cardinality.

2. Frieze patterns and cluster algebras

Let $n \geq 3$ be an integer and let $P_n$ be a regular $n$-sided polygon, with vertices $1, 2, \ldots, n$ numbered in cyclic order, clockwise around the boundary (thus we work with the vertices modulo $n$, with representatives in $\{1, 2, \ldots, n\}$). In [CoCo1, CoCo2] it is shown that every frieze pattern can be obtained from a triangulation $\pi$ of $P_n$ in the following way. For each pair of integers $i, j \in \{1, 2, \ldots, n\}$, define an integer $m_{ij}$ in the following way. Set $m_{ii} = 0$ and $m_{i,i+1} = 1$ for all $i$. Then let $m_{i-1,i+1}$ be the number of triangles in $\pi$ incident with vertex $i$. Define $m_{ij}$ for all $i \leq j$ inductively using the formula

$$m_{i-1,j+1} = m_{i-1,j}m_{i+1,j} - 1.$$

Set $m_{jj} = m_{jj}$ for all $i \leq j$. Then, for $0 \leq r \leq n - 1$, row $r$ of the frieze from the bottom contains the sequence of integers obtained by repeatedly cycling $m_{1,i+r}, m_{2,i+r}, \ldots, m_{n,n+r}$, arranged so that $m_{i,i+r}$ is between the entries $m_{i,i+r-1}$ and $m_{i+1,i+r}$ on the row below it. Note that the entries $m_{ij}$ lying in a triangle below $m_{1,n}$ form a fundamental domain for a glide reflection preserving the frieze (this is the glide reflection mentioned above).

For example, the frieze pattern corresponding to the triangulation in Figure 1 is shown in Figure 2. The middle triangle indicates the fundamental domain mentioned above, so $m_{17}$ is the top entry in the middle triangle; the bottom row is $m_{11}, m_{22}, \ldots, m_{77}$.

Theorem 1.1 can then be stated as follows:

**Theorem 2.1.** [HC] Let $n \geq 3$, let $\pi$ be a triangulation of $P_n$ and let $M(\pi) = (m_{ij})$ be the symmetric matrix defined above. Then $\det(M) = -(-2)^{n-2}$. 

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Figure 1. A triangulation of $P_7$

Figure 2. The frieze pattern corresponding to the triangulation in Figure 1 showing a fundamental domain for the glide together with its images under the glide and its inverse.

For the example in Figure 1, we have

$$M(\pi) = \begin{pmatrix}
0 & 1 & 2 & 1 & 2 & 1 \\
1 & 0 & 1 & 1 & 2 & 5 & 3 \\
2 & 1 & 0 & 1 & 3 & 8 & 5 \\
1 & 1 & 1 & 0 & 1 & 3 & 2 \\
1 & 2 & 3 & 1 & 0 & 1 \\
2 & 5 & 8 & 3 & 1 & 0 & 1 \\
1 & 3 & 5 & 2 & 1 & 1 & 0
\end{pmatrix}.$$ 

which has determinant $-(-2)^5 = 32$. In addition, a geometric interpretation of all of the entries in the frieze in terms of $\pi$ is given in [BCI].

We consider the cluster algebra $A$ of type $A_{n-3}$ associated to $P_n$. This cluster algebra appears in [FZ1] and is described in detail in [FZ3, 12.2]; see alternatively [Pro, 3.2] (using a perfect matching model for frieze patterns due to Gabriel Carroll and Gregory Price; see [loc. cit.] for details). We consider the version over the complex numbers.

Let $\pi$ be a triangulation of $P_n$ and let $F$ be the field of rational functions in the variables $u_{ij}$ where $i, j$ are the end-points of a diagonal in $\pi$ or a boundary edge of $P_n$ (regarding $u_{ij}$ and $u_{ji}$ as equal). Define elements $u_{ij} \in F$, for $i, j$ the end-points of an arbitrary diagonal of $P_n$, inductively as follows (again with $u_{ij} = u_{ji}$). If $i, j, k, l$ are vertices of $P_n$, clockwise around the boundary, and $u_{ij}, u_{jk}, u_{kl}, u_{li}$ and $u_{ik}$ are defined but $u_{jl}$ is not, define $u_{jl}$ by the following exchange relation:

(1) $u_{ij}u_{kl} + u_{jk}u_{li} = u_{ik}u_{jl}$.

It turns out that the elements $u_{ij}$ are well-defined. The cluster algebra $A$ is the $\mathbb{C}$-subalgebra of $F$ generated by the $u_{ij}$ for $i, j$ the end-points of any diagonal or boundary edge of $P_n$. The first kind of generator is known as a *cluster variable* and
the second as a cluster coefficient. For $n = 3$ there are no cluster variables, only coefficients. It is known that the cluster variables are Laurent polynomials in the $u_{ij}$ for $i, j$ the end-points of a diagonal in $\pi$ with coefficients given by polynomials in the cluster coefficients [FZ1, 3.1].

The cluster algebra $\mathcal{A}$ is independent (up to isomorphism) of the choice of $\pi$. In fact, it is isomorphic to the homogeneous coordinate ring of the Grassmannian of 2-planes in an $n$-dimensional vector space; see [FZ2, 12.6].

The exchange relation (1) holds for any choice of $a, b, c, d$ clockwise around the boundary of $P_n$. These relations are sometimes referred to as Ptolemy relations. In the above realisation, they are the Plücker relations.

Fix a triangulation $\pi$ of $P_n$. Setting $u_{ii} = 0$ for all $i$, we consider the symmetric matrix $U(\pi) = (u_{ij})$, where the $u_{ij}$ are regarded as Laurent polynomials in the $u_{ij}$ for $i, j$ the end-points of an edge in $\pi$ with coefficients given by polynomials in the cluster coefficients. By [CaCh, 5.2] (see also [Pro, §2, §3]), if the $u_{ij}$ for $i, j$ end-points of an edge in $\pi$ and the cluster coefficients are all specialised to 1, $U(\pi)$ becomes the matrix $M(\pi)$ defined above.

### 3. A determinantal result

In this section, we prove our main result:

**Theorem 3.1.** Let $n \geq 3$ and let $\pi$ be a triangulation of $P_n$. Let $U(\pi) = (u_{ij})$ be the matrix of cluster variables of $\mathcal{A}$ regarded as Laurent polynomials in the $u_{ij}$ for $i, j$ end-points of a diagonal in $\pi$ with coefficients in the polynomial ring of cluster coefficients. Then

$$\det(U(\pi)) = -(-2)^{n-2}u_{12}u_{23} \cdots u_{n-1,n}u_{n1}.$$  

**Proof:** We use induction on $n$, showing that the exchange relations are sufficient to imply the result. For $n = 3$ there is only one triangulation and the matrix $U$ is of the form:

$$U(\pi) = \begin{pmatrix} 0 & u_{12} & u_{13} \\ u_{12} & 0 & u_{23} \\ u_{13} & u_{23} & 0 \end{pmatrix},$$

which has determinant $2u_{12}u_{23}u_{13}$ as required. Now suppose that $n \geq 4$ and that the result is true for $n - 1$. A special vertex of $\pi$ is one which is incident with only one triangle of the triangulation. By [BCI, Lemma 1], $\pi$ must have at least two special vertices; choose one, say $a$. Let $b$ be a vertex of $\pi$ distinct from $a - 1, a, a + 1$. From the quadrilateral in Figure 3 we have the exchange relation:

$$u_{a-1,a}u_{a+1,b} + u_{a,a+1}u_{a-1,b} = u_{a-1,a+1}u_{ab}.$$
It follows that:

\[
\frac{u_{a-1,a}}{u_{a-1,a+1}}u_{a+1,b} + \frac{u_{a,a+1}}{u_{a-1,a+1}}u_{a-1,b} = u_{ab}.
\]

Let \( R_a \) denote the \( a \)th row of \( U(\pi) \). From the above, it follows that if we replace \( R_a \) with the row:

\[
R_a - \frac{u_{a-1,a}}{u_{a-1,a+1}}R_{a+1} - \frac{u_{a,a+1}}{u_{a-1,a+1}}R_{a-1},
\]

then the \( a,b \) entry \( u'_{a,b} \) in the new matrix will be 0 for any \( b \neq a-1, a \) or \( a+1 \).

Noting that \( u_{a-1,a-1} = 0 \), we also have:

\[
u'_{a,a-1} = \frac{u_{a-1,a}}{u_{a-1,a+1}}u_{a+1,a-1} - \frac{u_{a,a+1}}{u_{a-1,a+1}}u_{a-1,a-1} = 0,
\]

and, similarly, \( u'_{a,a+1} = 0 \), while

\[
u'_{aa} = \frac{u_{a-1,a}}{u_{a-1,a+1}}u_{a+1,a} - \frac{u_{a,a+1}}{u_{a-1,a+1}}u_{a-1,a} = -2\frac{u_{a-1,a}u_{a,a+1}}{u_{a-1,a+1}}.
\]

Expanding along the \( a \)th row, we obtain that the determinant of \( U(\pi) \) is

\[
\det(U(\pi)) = (-1)^a \frac{(-2)u_{a-1,a}u_{a,a+1}}{u_{a-1,a+1}} \det(U_a),
\]

where \( U_a \) denotes the matrix obtained by removing the \( a \)th row and the \( a \)th column from \( U \). It is clear that \( U_a = U(\pi_a) \) where \( \pi_a \) is the triangulation obtained from \( \pi \) by removing the vertex \( a \) and the edges incident with it — this has the effect of removing the triangle with vertices \( a-1, a \) and \( a+1 \). By the induction hypothesis,

\[
\det(U_a) = (-2)^{a-3}u_{a-1,a+1}u_{12}u_{23} \cdots u_{a-1,a}u_{a,a+1} \cdots u_{n-1,n}u_{n1},
\]

where the hats indicate omission. Combining this with equation \( \text{(2)} \) we obtain that

\[
\det(U(\pi)) = (-2)^{n-2}u_{12}u_{23} \cdots u_{n-1,n}u_{n1},
\]

as required. \( \square \)

4. An Example

Here we give an example of the result in the previous section. Let \( \pi \) be the triangulation of a pentagon shown in Figure 4. Then the corresponding matrix is given by:

\[
U(\pi) = \begin{pmatrix}
0 & u_{12} & 0 & 0 & 0 \\
u_{12} & 0 & u_{13} & 0 & 0 \\
u_{13} & u_{23} & 0 & u_{14} & 0 \\
u_{14} & u_{24} & u_{34} & 0 & u_{15} \\
u_{15} & u_{25} & u_{35} & u_{45} & 0
\end{pmatrix},
\]

where

\[
u = \frac{u_{12}u_{13}u_{14} + u_{12}u_{15}u_{34} + u_{14}u_{15}u_{23}}{u_{13}u_{14}}.
\]
Example 5.2. The translation quiver $\Gamma(5)$.

By Theorem 5.1, we have that
\[
\det(U(\pi)) = 8u_{12}u_{23}u_{34}u_{45}u_{51}.
\]

5. A geometric model of the root category

Let $n \geq 3$ be an integer and let $Q$ be a quiver of type $A_{n-1}$. Let $k$ be an algebraically closed field. Let $D^b(kQ)$ denote the bounded derived category of modules over $kQ$, with shift functor $[1]$. Let $R_n = D^b(kQ)/[2]$ denote the quotient of $D^b(kQ)$ by the square $[2]$ of the shift. In this section we shall exhibit a geometric construction of this category (along the lines of [CCS]). We remark that this category is sometimes referred to as the root category (of type $A$) since its objects can be put into one-to-one correspondence with the roots in the corresponding root system (by Gabriel’s Theorem). It was considered in [HI].

We now consider oriented edges between vertices of $P_n$, denoting the edge oriented from $i$ to $j$ by $[i, j]$, for any $1 \leq i, j \leq n$ with $i \neq j$ (thus boundary edges are included).

Recall that a translation quiver is a pair $(\Gamma, \tau)$ where $\Gamma$ is a locally finite quiver and $\tau : \Gamma_0' \to \Gamma_0$ is an injective map defined on a subset $\Gamma_0'$ of the vertices of $\Gamma$ such that for any $X \in \Gamma_0, Y \in \Gamma_0'$, the number of arrows from $X$ to $Y$ is the same as the number of arrows from $\tau(Y)$ to $X$. If $\Gamma_0' = \Gamma_0$ and $\tau$ is bijective, $(\Gamma, \tau)$ is called a stable translation quiver.

Let $\Gamma = \Gamma(n)$ be the quiver defined as follows. The vertices, $\Gamma_0$ are the set of all possible oriented edges between vertices of $P_n$ as above. The arrows, $\Gamma_1$, are of the form $[i, j] \to [i, j + 1]$ and $[i, j] \to [i + 1, j]$ (where $j + 1$ is interpreted as $1$ if $j = n + 1$ and similarly for $i + 1$), whenever $[i, j + 1]$ (respectively, $[i + 1, j]$) is a vertex of $\Gamma$. Thus an arrow comes from rotating an oriented edge clockwise about one of its end-points so that the other end-point moves to an adjacent vertex on the boundary of $P_n$.

Let $\tau$ be the automorphism of $\Gamma$ obtained by rotating $P_n$ through $2\pi/n$ anti-clockwise; thus $\tau([i, j]) = [i - 1, j - 1]$.

**Lemma 5.1.** The pair $(\Gamma, \tau)$ is a stable translation quiver.

The proof is as in [BNI] 2.2: note that this proof also works in the oriented case we have here.

**Example 5.2.** We consider the case when $n = 5$, so $P_n$ is a pentagon. The translation quiver $(\Gamma(5), \tau)$ is given in Figure 3.

By [K] 9.9, $R_n$ is a triangulated category, and, by [BMRRT] 1.3, it has Auslander-Reiten triangles and its Auslander-Reiten quiver, $\Gamma(R_n)$ is the quotient of the
Auslander-Reiten quiver of $D^b(kQ)$ by the automorphism induced by $[2]$. We have the following:

**Proposition 5.3.** The translation quiver $\Gamma(n)$ is isomorphic to $\Gamma(R_n)$.

**Proof.** Suppose that $Q$ is a linearly oriented quiver of type $A_{n-1}$, with arrows $i \leftarrow i + 1, 1 \leq i \leq n - 2$. Then, up to isomorphism, the indecomposable modules for $kQ$ are of the form $M_{ij}, 1 \leq i < j \leq n$, where $M_{ij}$ has socle $S_i$ (the simple module corresponding to vertex $i$) and length $j - i$. So $M_{i,i+1} = S_i$. For $i < j$, we also set $M_{ji} = M_{ij}[1]$. Then the map $[i,j] \mapsto M_{ij}$, for $1 \leq i, j \leq n$, $i \neq j$, gives a bijection between oriented edges between vertices of $P_n$ and isomorphism classes of indecomposable objects of $R_n$. The fact that this is an isomorphism of translation quivers follows from the description of the Auslander-Reiten quiver of $D^b(kQ)$ in [H2]. We just need to check that the mesh beginning at corresponding vertices is the same in each quiver. The only non-trivial cases are the meshes beginning with $M_{ij}$ where $j = n$ or $i = n$. In the first case, the mesh in $\Gamma(R_n)$ is:

![Diagram of mesh with $M_{i,1} = P_{i-1}[1]$](image)

and in the second case, the mesh in $\Gamma(R_n)$ is:

![Diagram of mesh with $M_{n,i+1} = I_{i+1}[1]$](image)

noting that in $R_n$, $X \cong X[2]$ for any object $X$. These meshes are the images of the corresponding meshes in $\Gamma(n)$, so we are done. □

**Remark 5.4.** We note that the induced subquiver of $\Gamma(n)$ on vertices of form $[i,j]$ with $i < j$ (with $\tau([i,j])$ undefined if $i = 1$) is isomorphic to the Auslander-Reiten quiver of $kQ \mod$.

We note that, as for the cluster category (see [BMRRT, §1]), the category $R_n$ is standard. We thus have the following corollary of Proposition 5.3 giving a geometric realisation of $R_n$.

**Corollary 5.5.** The root category $R_n$ is equivalent to the additive hull of the mesh category of $\Gamma(n)$.

We shall identify indecomposable objects in $R_n$, up to isomorphism, with the corresponding oriented edges between vertices of $P_n$, in the sequel, and we shall freely switch between objects and oriented edges between vertices in $P_n$. 
Figure 6. The two types of crossing between $X$ and $Y$.

### 6. Dimensions of Extension Groups

In this section we indicate how the dimensions of $\text{Ext}^1$-groups between indecomposable objects of $\mathcal{R}_n$ can be read off from the geometric model. We fix $i, j$ with $1 \leq i, j \leq n$, $i \neq j$, and consider the corresponding indecomposable object $[i, j]$.

Consider the two rectangles $R_B = R_B(i, j)$, with corners $[j, i], [j, j-1], [i-1, j-1]$ and $[i-1, i]$, and $R_F = R_F(i, j)$, with corners $[i+1, j+1], [i+1, i], [j, i]$ and $[j, j+1]$, in $\mathcal{R}_n$. Using [Bo] or the mesh relations and the Auslander-Reiten formula in $\mathcal{R}_n$ directly, we see that:

**Lemma 6.1.** Let $X$ and $Y$ be indecomposable objects in $\mathcal{R}_n$. Then:

(a) The space $\text{Ext}^1_{\mathcal{R}_n}([i, j], Y)$ is non-zero if and only if $Y$ lies in $R_B$. If it is non-zero then it is one-dimensional.

(b) The space $\text{Ext}^1_{\mathcal{R}_n}(X, [i, j])$ is non-zero if and only if $X$ lies in $R_F$. If it is non-zero then it is one-dimensional.

As an example, we show in Figure 7 (by underlining) those indecomposable objects $Y$ such that $\text{Ext}^1_{\mathcal{R}_5}([3, 1], Y) \neq 0$ for the case $n = 5$, and (by overlining) those indecomposable objects $X$ such that $\text{Ext}^1_{\mathcal{R}_5}(X, [3, 1]) \neq 0$. In each case the objects in question are written in bold font. Note that $[1, 3]$ is the only object satisfying both conditions.

Let $X, Y$ be two indecomposable objects of $\mathcal{R}_n$, regarded as oriented edges between vertices of $\mathcal{P}_n$. Then $X$ and $Y$ may cross each other in two different ways. If the tangents to $X, Y$ (in that order) form a pair of axes corresponding to the usual orientation on $\mathbb{R}^2$ we say that the crossing of $X$ and $Y$ is positive, otherwise negative. See Figure 6

Lemma 6.1 can be reinterpreted geometrically as follows.

**Proposition 6.2.** Let $X, X'$ be indecomposable objects in $\mathcal{R}_n$. Then $\dim \text{Ext}^1_{\mathcal{R}_n}(X, X')$ is equal to 1 if and only if one of the following conditions holds, and is zero otherwise:

(a) The crossing of $X, X'$ is positive;

(b) The terminal vertex of $X$ coincides with the initial vertex of $X'$ and $X'$ lies to the left of $X$ in $\mathcal{P}_n$;

(c) The initial vertex of $X$ coincides with the terminal vertex of $X'$ and $X'$ lies to the right of $X$ in $\mathcal{P}_n$;

(d) $X'$ is the reverse of $X$.

As an example, consider the objects $Y$ such that $\text{Ext}^1_{\mathcal{R}_5}([3, 1], Y) \neq 0$, displayed in Figure 7 (by underlining). Note that the crossing of $[3, 1]$ with $[2, 5]$ or with $[2, 4]$ is positive; both $[1, 4]$ and $[1, 5]$ start at the terminal vertex of $[3, 1]$ and lie to its left; $[2, 3]$ has terminal vertex coinciding with the initial vertex of $[3, 1]$ and lies to its right, and $[1, 3]$ is the reverse of $[3, 1]$.
7. Starting and ending frames

In this section we consider starting and ending frames of indecomposable objects in $\mathcal{R}_n$ (following [BMRRT, 8.4]). Let $\text{ind}(\mathcal{R}_n)$ denote the set of (isomorphism classes of) indecomposable objects of $\mathcal{R}_n$. Let $X$ be an indecomposable object in $\mathcal{R}_n$. Then the **starting frame** $S(X)$ of $X$ is the set

$$S(X) = \{Y \in \text{ind}(\mathcal{R}_n) : \text{Hom}_{\mathcal{R}_n}(X, Y) \neq 0, \text{Ext}^1_{\mathcal{R}_n}(Y, X) = 0\}.$$  

The **ending frame** $E(X)$ of $X$ is the set

$$E(X) = \{Y \in \text{ind}(\mathcal{R}_n) : \text{Hom}_{\mathcal{R}_n}(Y, X) \neq 0, \text{Ext}^1_{\mathcal{R}_n}(X, Y) = 0\}.$$  

We define the **frame** $F(X)$ of $X$ to be the union:

$$F(X) = S(X) \cup E(X).$$

It is easy to see (by direct calculation, or using Proposition 6.2 and the Auslander-Reiten formula) that the following hold:

**Proposition 7.1.**  
(a) $Y \in S(X)$ if and only if $Y$ and $X$ share a common terminal vertex and $Y$ lies to the left of $X$, or $Y$ and $X$ share a common initial vertex and $Y$ lies to the right of $X$, or $Y = X$. 
(b) $Y \in E(X)$ if and only if $Y$ and $X$ share a common initial vertex and $Y$ lies to the left of $X$, or $Y$ and $X$ share a common terminal vertex and $Y$ lies to the right of $X$, or $Y = X$. 
(c) $Y \in F(X)$ if and only if $X$ and $Y$ share a common initial vertex or share a common terminal vertex (or both).

8. Categorification of the determinantal result

Fix again a triangulation $\pi$ of $\mathcal{P}_n$. For $1 \leq i, j \leq n$ with $i \neq j$, we associate the indecomposable object $[i, j]$ (or oriented edge) with the $i, j$ position in the $n \times n$ matrix $U(\pi)$ considered in Section 1. The indecomposable $kQ$-modules (with $i < j$) correspond to the part of $U(\pi)$ above the leading diagonal and their shifts (with $i > j$) correspond to the part of $U(\pi)$ below the leading diagonal.

Reinterpreting Proposition 7.1 in these terms, we obtain:

**Lemma 8.1.** The frame of an indecomposable object $X$ corresponds to the union of the row and column of $U(\pi)$ containing $X$ (apart from the diagonal entries).

Define a **frame-free** configuration of $\mathcal{R}_n$ to be a maximal collection $C$ of (isomorphism classes of) indecomposable objects of $\text{ind} \mathcal{R}_n$ such that $Y \notin F(X)$ for all
Thus frame-free configurations of $\mathcal{R}_n$ correspond to maximal collections of positions in $U(\pi)$ which do not lie in the same row or column as each other and contain no diagonal entries.

**Lemma 8.2.** Let $C$ be a frame-free configuration of $\mathcal{R}_n$. Then the cardinality of $C$ is either $n - 1$ or $n$.

**Proof.** Since frame-free configurations cannot have objects in the same row or column, the maximum cardinality is $n$. If the cardinality of a configuration is $n - k$ where $k \geq 2$, there are at least two rows and two columns of $U(\pi)$ containing no elements of the configuration, a contradiction to its maximality (as at least two elements could be added to the configuration, at the non-diagonal intersections of the empty rows and columns). The result follows. □

Given a fixed-point free permutation, $\sigma$ (sometimes known as a derangement), let $C(\sigma)$ be the set of objects $[i, \sigma(i)]$ for $1 \leq i \leq n$. Since $\sigma$ is fixed-point free, it follows from Proposition 7.1(c) that $C(\sigma)$ is a frame-free configuration; it has cardinality $n$. It is clear that this gives a bijection between fixed-point free permutations and frame-free configurations of cardinality $n$.

Thus, as a collection of oriented edges between vertices of $\mathcal{P}_n$, a frame-free configuration in $\mathcal{R}_n$ of cardinality $n$ is a union of oriented cycles (with no cycles of cardinality 1).

Similarly, a permutation $\sigma$ with one fixed point, $p$, say, corresponds to a configuration of cardinality $n - 1$ consisting of the objects $[i, \sigma(i)]$ for $i \neq p$. This gives a bijection between the permutations with a single fixed point and the frame-free configurations of cardinality $n - 1$.

The number of permutations with a given number of fixed points is well-known (see e.g. [S, A008290]). We thus have the following:

**Lemma 8.3.** The number of frame-free configurations of $\mathcal{R}_n$ of cardinality $n$ is

$$!n := n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

(known as the subfactorial of $n$) while the number of frame-free configurations of $\mathcal{R}_n$ of cardinality $n - 1$ is $!n + (-1)^{n-1}$.

Thus the number of frame-free configurations in $\mathcal{R}_n$ of cardinality $n$ for $n = 2, 3, \ldots$, is $1, 2, 9, 44, 265, 1854$ (see [S A000166]) and the number of frame-free configurations in $\mathcal{R}_n$ of cardinality $n - 1$ is $0, 3, 8, 45, 264, 1855$ (see [S A000240]). The 9 frame-free configurations of $\mathcal{R}_4$ of cardinality 4 are shown in Figure 8 as collections of vertices in the AR-quiver (filled in vertices indicate those indecomposable objects in the collection) and as collections of oriented arcs between vertices of a square.

We see that the frame-free configurations of $\mathcal{R}_n$ of cardinality $n$ correspond bijectively to the non-zero terms in the expansion

$$\det(U(\pi)) = \sum_{\sigma \in \Sigma_n} (-1)^{\ell(\sigma)} u_{1,\sigma(1)} \cdots u_{n,\sigma(n)}$$

of the determinant of $U(\pi)$ (since $U(\pi)$ has zeros along its leading diagonal).

Given a frame-free configuration $C$ of $\mathcal{R}_n$ of cardinality $n$, define its sign $\varepsilon(C)$ to be

$$\varepsilon(C) = \prod_\gamma (-1)^{\ell(\gamma)-1}$$

where the product is over the oriented cycles in the representation of $C$ as a collection of oriented edges between vertices of $\mathcal{P}_n$, and $\ell(\gamma)$ is equal to the number
Figure 8. The 9 frame-free configurations of cardinality 4 in the root category of type $A_3$ and the corresponding terms in the expansion of $\text{det}(U(\pi))$. 

of vertices in $\gamma$ for a cycle $\gamma$. It is easy to see that this is equal to the sign of the corresponding permutation. Set $\alpha(C)$ to be the product of the entries of $U(\pi)$ corresponding to the elements of $C$.

We therefore have:

$$\text{det}(U(\pi)) = \sum_C \varepsilon(C) \alpha(C),$$

where the sum is over all frame-free configurations of $\mathcal{R}_n$ of cardinality $n$. The monomials in this expansion are shown for each frame-free configuration in the example in Figure 8.

We can reinterpret our main result, Theorem 3.1, representation-theoretically as follows:

**Theorem 8.4.** Let $n \geq 3$, let $\pi$ be a triangulation of $P_n$, and let $U(\pi) = (u_{ij})$ be the matrix of cluster variables in $\mathcal{A}$ regarded as Laurent polynomials in the $u_{ij}$ for $i, j$ end-points of diagonals in $\pi$ with coefficients given by polynomials in the cluster coefficients. Then we have:

$$\sum_C \varepsilon(C) \alpha(C) = -(-2)^{n-2} u_{12} u_{23} \cdots u_{n-1,n} u_{n,1},$$

where the sum is over all frame-free configurations of $\mathcal{R}_n = D^b(kQ)/[2]$ of maximum cardinality.

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