SOLUTIONS OF FOUR-DIMENSIONAL FIELD THEORIES VIA $M$ THEORY

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$\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions are studied by formulating them as the quantum field theories derived from configurations of fourbranes, fivebranes, and sixbranes in Type IIA superstrings, and then reinterpreting those configurations in $M$ theory. This approach leads to explicit solutions for the Coulomb branch of a large family of four-dimensional $\mathcal{N} = 2$ field theories with zero or negative beta function.

March 1997

\textsuperscript{1} Research supported in part by NSF Grant PHY-9513835.
1. Introduction

Many interesting results about field theory and string theory have been obtained by studying the quantum field theories that appear on the world-volume of string theory and $M$ theory branes. One particular construction that was considered recently in $2+1$ dimensions \cite{1} and has been further explored in \cite{2} and applied to $\mathcal{N} = 1$ models in four dimensions in \cite{3} will be used in the present paper to understand the Coulomb branch of some $\mathcal{N} = 2$ models in four dimensions. The aim is to obtain for a wide class of four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry the sort of description obtained in \cite{4} for models with $SU(2)$ gauge group.

The construction in \cite{1} involved branes of Type IIB superstring theory – to be more precise the Dirichlet threebranes and the solitonic and Dirichlet fivebranes. One considers, for example, NS fivebranes with threebranes suspended between them (figure 1). The fivebranes, being infinite in all six of their world-volume directions, are considered to be very heavy and are treated classically. The interest focuses on the quantum field theory on the world-volume of the threebranes. Being finite in one of their four dimensions, the threebranes are macroscopically $2+1$ dimensional. The quantum field theory on this effective $2+1$ dimensional world has eight conserved supercharges, corresponding to $N = 4$ supersymmetry in three dimensions or $\mathcal{N} = 2$ in four dimensions. Many properties of such a model can be effectively determined using the description via branes.

To make a somewhat similar analysis of $3+1$ dimensional theories, one must replace the threebranes by fourbranes, suspended between fivebranes (and, as it turns out, also in
the presence of sixbranes). Since the fourbrane is infinite in four dimensions (and finite in the fifth), the field theory on such a fourbrane is $3 + 1$-dimensional macroscopically.

Type IIB superstring theory has no fourbranes, so we will consider Type IIA instead. Type IIA superstring theory has Dirichlet fourbranes, solitonic fivebranes, and Dirichlet sixbranes. Because there is only one brane of each dimension, it will hopefully cause no confusion if we frequently drop the adjectives “Dirichlet” and “solitonic” and refer to the branes merely as fourbranes, fivebranes, and sixbranes.

One of the main techniques in [1] was to use $SL(2, \mathbb{Z})$ duality of Type IIB superstrings to predict a mirror symmetry of the $2 + 1$ dimensional models. For Type IIA there is no $SL(2, \mathbb{Z})$ self-duality. The strong coupling limit of Type IIA superstrings in ten dimensions is instead determined by an equivalence to eleven-dimensional $M$ theory; this equivalence will be used in the present paper to obtain solutions of four-dimensional field theories. As we will see, a number of facts about $M$ theory fit together neatly to make this possible.

In section 2, we explain the basic techniques and solve models that are constructed from configurations of Type IIA fourbranes and fivebranes on $\mathbb{R}^{10}$. In section 3, we incorporate sixbranes. In section 4, we analyze models obtained by considering Type IIA fourbranes and fivebranes on $\mathbb{R}^{9} \times S^{1}$. Many novel features will arise, including a geometric interpretation of the gauge theory beta function in section 2 and a natural family of conformally invariant theories in section 4. As we will see, each new step involves some essential new subtleties, though formally the brane diagrams are analogous (and related by $T$-duality) to those in [1].

2. Models With Fourbranes And Fivebranes

In this section we consider fourbranes suspended between fivebranes in Type IIA superstring theory on $\mathbb{R}^{10}$. Our fivebranes will be located at $x^7 = x^8 = x^9 = 0$ and – in the classical approximation – at some fixed values of $x^6$. The worldvolume of the fivebrane is parametrized by the values of the remaining coordinates $x^0, x^1, \ldots, x^5$.

In addition, we introduce fourbranes whose world-volumes are parametrized by $x^0, x^1, x^2, x^3$, and $x^6$. However, our fourbranes will not be infinite in the $x^6$ direction. They will terminate on fivebranes. (Occasionally we will consider a semi-infinite fourbrane

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2 Compactification of such a brane system on a circle has been considered in [2] in the Type IIB context.
that terminates on a fivebrane at one end, and extends to $x^6 = \infty$ or $-\infty$ at the other end.) A typical picture is thus that of figure 2(a). As in [1], we will examine this picture first from the fivebrane point of view and then from the fourbrane point of view.

It will be convenient to introduce a complex variable $v = x^4 + ix^5$. Classically, every fourbrane is located at a definite value of $v$. The same is therefore also true for its possible endpoints on a fivebrane.

2.1. Theory On Fivebrane

A fact that was important in [1] is that on the worldvolume of a Type IIB fivebrane there propagates a $U(1)$ gauge field. A system of $k$ parallel but noncoincident fivebranes can be interpreted as a system with $U(k)$ gauge symmetry spontaneously broken to $U(1)^k$. Points at which Type IIB threebranes end on fivebranes carry magnetic charge in this spontaneously broken gauge theory.

Even though one draws similar brane pictures in the Type IIA case, the interpretation is rather different. Type IIA fivebranes do not carry gauge fields, but rather self-dual antisymmetric tensors. When parallel fivebranes become coincident, one gets not enhanced gauge symmetry but a strange critical point with tensionless strings [5], concerning which too little is known for it to be useful in the present paper.
However, the endpoints of a fourbrane on a fivebrane do behave as charges in an appropriate sense. A fivebrane on which fourbranes end does not really have a definite value of $x^6$ as the classical brane picture suggests. The fourbrane ending on a fivebrane creates a “dimple” in the fivebrane. What one would like to call the $x^6$ value of the fivebrane is really the $x^6$ value measured at $v = \infty$, far from the disturbances created by the fourbranes.

To see whether this makes sense, note that $x^6$ is determined as a function of $v$ by minimizing the total fivebrane worldvolume. For large $v$ the equation for $x^6$ reduces to a Laplace equation,

$$\nabla^2 x^6 = 0. \tag{2.1}$$

Here $\nabla^2$ is the Laplacian on the fivebrane worldvolume. $x^6$ is a function only of the directions normal to the fourbrane ends, that is only of $v$ and $\overline{v}$. Since the Green’s function of the Laplacian in two dimensions is a logarithm, the large $v$ behavior of $x^6$ is determined by (2.1) to be

$$x^6 = k \ln |v| + \text{constant} \tag{2.2}$$

for some $k$. Thus, in general, there is no well-defined large $v$ limit of $x^6$. This contrasts with the situation considered in [1] where (because of considering threebranes instead of fourbranes) $x^6$ obeys a three-dimensional Laplace equation, whose solution approaches a constant at infinity. The limiting value $x^6(\infty)$ is then the “$x^6$ value of the fivebrane” which appears in the classical brane diagram and was used in [1] to parametrize the configurations.

Going back to the Type IIA case, for a fivebrane with a single fourbrane ending on it from, say, the left, $k$ in (2.2) is an absolute constant that depends only on the fourbrane and fivebrane tensions (and hence the Type IIA string coupling constant). However, a fourbrane ending on a fivebrane on its right pulls in the opposite direction and contributes to $k$ with the opposite sign from a fivebrane ending on the left. If $a_i, i = 1, \ldots, q_L$ and $b_j, j = 1, \ldots, q_R$ are the $v$ values of fourbranes that end on a given fivebrane on its left and on its right, respectively, then the asymptotic form of $x^6$ is

$$x^6 = k \sum_{i=1}^{q_L} \ln |v - a_i| - k \sum_{j=1}^{q_R} \ln |v - b_j| + \text{constant}. \tag{2.3}$$

We see that $x^6$ has a well-defined limiting value for $v \to \infty$ if and only if $q_L = q_R$, that is if there are equal forces on the fivebrane from both left and right.
For any finite chain of fivebranes with fourbranes ending on them, as in figure 2(a), it is impossible to obey this condition, assuming that there are no semi-infinite fourbranes that go off to \( x^6 = \infty \) or \( x^6 = -\infty \). At least the fivebranes at the ends of the chain are subject to unbalanced forces. The “balanced” case, a chain of fivebranes each connected by the same number of fourbranes, as in figure 2(b), is most natural if one compactifies the \( x^6 \) direction to a circle, so that all fourbranes are finite in extent. It is very special and will be the subject of section 4.

Another important question is affected by a related infrared divergence. For this, we consider the motion of fourbranes. When the fourbranes move, the disturbances they produce on the fivebranes move also, producing a contribution to the fourbrane kinetic energy. We consider a situation in which the \( a_i \) and \( b_j \) vary as a function of the first four coordinates \( x^\mu, \mu = 0, \ldots, 3 \) (which are the “spacetime” coordinates of the effective four-dimensional field theories studied in this paper). The fivebrane kinetic energy has a term \( \int d^4 x d^2 v \sum_{\mu=0}^3 \partial_\mu x^6 \partial^\mu x^6 \). With \( x^6 \) as in (2.3), this becomes

\[
 k^2 \int d^4 x d^2 v \left[ \text{Re} \left( \sum_i \partial_\mu a_i \left( \frac{1}{v-a_i} \right) - \sum_j \partial_\mu b_j \left( \frac{1}{v-b_j} \right) \right) \right]^2.
\] (2.4)

The \( v \) integral converges if and only if

\[
 \partial_\mu \left( \sum_i a_i - \sum_j b_j \right) = 0,
\] (2.5)

so that

\[
 \sum_i a_i - \sum_j b_j = q_\alpha,
\] (2.6)

where \( q_\alpha \) is a constant characteristic of the \( \alpha \)th fivebrane. While the \( q_\alpha \) are constants that we will eventually interpret in terms of “bare masses,” the remaining \( a \)'s and \( b \)'s are free to vary; they are indeed “order parameters” which depend on the choice of quantum vacuum of the four-dimensional field theory.

The above discussion of the large \( v \) behavior of \( x^6 \) and its kinetic energy is actually only half of the story. From the point of view of the four-dimensional \( N = 2 \) supersymmetry of our brane configurations, \( x^6 \) is the real part of a complex field that is in a vector multiplet. The imaginary part of this superfield is a scalar field that propagates on the fivebrane. If Type IIA superstring theory on \( \mathbf{R}^{10} \) is reinterpreted as \( M \) theory on \( \mathbf{R}^{10} \times S^1 \), the scalar
in question is the position of the fivebrane in the eleventh dimension. We have labeled the ten dimensions of Type IIA as \(x^0, x^1, \ldots, x^9\), so we will call the eleventh dimension \(x^{10}\). In generalizing \((2.3)\) to include \(x^{10}\), we will use \(M\) theory units (which differ by a Weyl rescaling from Type IIA units used in \((2.3)\)). Also, we understand \(x^{10}\) to be a periodic variable with period \(2\pi R\).

With this understood, the generalization of \((2.3)\) to include \(x^{10}\) is

\[
x^6 + ix^{10} = R \sum_{i=1}^{q_L} \ln(v - a_i) - R \sum_{j=1}^{q_R} \ln(v - b_j) + \text{constant.} \tag{2.7}
\]

The fact that \(x^6 + ix^{10}\) varies holomorphically with \(v\) is required by supersymmetry. The imaginary part of this equation states that \(x^{10}\) jumps by \(\pm 2\pi R\) when one circles around one of the \(a_i\) or \(b_j\) in the complex \(v\) plane. In other words, the endpoints of fourbranes on a fivebrane behave as vortices in the fivebrane effective theory (an overall constant in \((2.7)\) was fixed by requiring that the vortex number is one). This is analogous, and related by \(T\)-duality, to the fact that the endpoint of a threebrane on a fivebrane looks like a magnetic monopole, with magnetic charge one, in the fivebrane theory; this fact was extensively used in \([1]\). The interpretation of brane boundaries as charges on other branes was originally described in \([5]\).

In terms of \(s = (x^6 + ix^{10})/R\), the last formula reads

\[
s = \sum_{i=1}^{q_L} \ln(v - a_i) - \sum_{j=1}^{q_R} \ln(v - b_j) + \text{constant.} \tag{2.8}
\]

### 2.2. Four-Dimensional Interpretation

Now we want to discuss what the physics on this configuration of branes looks like to a four-dimensional observer.

We consider a situation, shown in figure 2(a) in a special case, with \(n + 1\) fivebranes, labeled by \(\alpha = 0, \ldots, n\). Also, for \(\alpha = 1, \ldots, n\), we include \(k_\alpha\) fourbranes between the \((\alpha - 1)^{th}\) and \(\alpha^{th}\) fivebranes.

It might seem that the gauge group would be \(\prod_{\alpha=1}^{n} U(k_\alpha)\), with each \(U(k_\alpha)\) factor coming from the corresponding set of \(k_\alpha\) parallel fourbranes. However, \((2.6)\) means precisely that the \(U(1)\) factors are “frozen out.” To be more precise, in \((2.3)\), \(\sum_i a_i\) is the scalar part of the \(U(1)\) vector multiplet in one factor \(U(k_\alpha)\), and \(\sum_j b_j\) is the scalar part of the \(U(1)\) multiplet in the “next” gauge group factor \(U(k_{\alpha+1})\). \((2.6)\) means that the difference \(\sum_i a_i - \sum_j b_j\) is “frozen,” and therefore, by supersymmetry, an entire \(U(1)\) vector...
supermultiplet is actually missing from the spectrum. Since such freezing occurs at each point in the chain, including the endpoints (the fivebranes with fourbranes ending only on one side), the $U(1)$’s are all frozen out and the gauge group is actually $\prod_{\alpha=1}^{n} SU(k_\alpha)$.

What is the hypermultiplet spectrum in this theory? By reasoning exactly as in [1], massless hypermultiplets arise (in the classical approximation of the brane diagram) precisely when fourbranes end on a fivebrane from opposite sides at the same point in spacetime. Such a hypermultiplet is charged precisely under the gauge group factors coming from fourbranes that adjoin the given fivebrane. So the hypermultiplets transform, in an obvious notation, as $(k_1, k_2) \oplus (k_2, k_3) \oplus \ldots \oplus (k_{n-1}, k_n)$. The constants $q_\alpha$ in (2.6) determine the bare masses $m_\alpha$ of the $(k_\alpha, k_{\alpha+1})$ hypermultiplets, so in fact arbitrary bare masses are possible. The bare masses are actually

$$m_\alpha = \frac{1}{k_\alpha} \sum_i a_{i,\alpha} - \frac{1}{k_{\alpha+1}} \sum_j a_{j,\alpha+1},$$

(2.9)

where $a_{i,\alpha}$, $i = 1, \ldots, k_\alpha$ are the positions in the $v$ plane of the fourbranes between the $\alpha-1^{th}$ and $\alpha^{th}$ fivebrane. In other words, $m_\alpha$ is the difference between the average position in the $v$ plane of the fourbranes to the left and right of the $\alpha^{th}$ fivebrane. $m_\alpha$ is not simply a multiple of $q_\alpha$, but the $q_\alpha$ for $\alpha = 1, \ldots, n$ determine the $m_\alpha$.

Now, we come to the question of what is the coupling constant of the $SU(k_\alpha)$ gauge group. Naively, if $x_\alpha^6$ is the $x^6$ value of the $\alpha^{th}$ fivebrane, then the gauge coupling $g_\alpha$ of $SU(k_\alpha)$ should be given by

$$\frac{1}{g_\alpha^2} = \frac{x_\alpha^6 - x_{\alpha-1}^6}{\lambda},$$

(2.10)

where $\lambda$ is the string coupling constant.

We have here a problem, though. What precisely is meant by the objects $x_\alpha^6$? As we have seen above, these must be understood as functions of $v$ which in general diverge for $v \to \infty$. Therefore, we must interpret $g_\alpha$ as a function of $v$:

$$\frac{1}{g_\alpha^2(v)} = \frac{x_\alpha^6(v) - x_{\alpha-1}^6(v)}{\lambda}.$$  

(2.11)

We interpret $v$ as setting a mass scale, and $g_\alpha(v)$ as the effective coupling of the $SU(k_\alpha)$ theory at mass $|v|$. Then $1/g_\alpha^2(v)$ generally, according to (2.3), diverges logarithmically for $v \to \infty$. But that is familiar in four-dimensional gauge theories: the inverse gauge coupling of an asymptotically free theory diverges logarithmically at high energies. We thus interpret this divergence as reflecting the one loop beta function of the four-dimensional theory.
It is natural to include $x^{10}$ along with $x^6$, and thereby to get a formula for the effective theta angle $\theta_\alpha$ of the $SU(k_\alpha)$ gauge theory, which is determined by the separation in the $x^{10}$ direction between the $\alpha-1^{th}$ and $\alpha^{th}$ fivebranes. Set

$$\tau_\alpha = \frac{\theta_\alpha}{2\pi} + \frac{4\pi i}{g_\alpha^2}. \quad (2.12)$$

Then in terms of $s = (x^6 + ix^{10})/R$ (with distances now measured in $M$ theory units) we have

$$-i\tau_\alpha(v) = s_\alpha(v) - s_{\alpha-1}(v). \quad (2.13)$$

(A multiplicative constant on the right hand side has been set to one by requiring that under $x^{10}_\alpha \rightarrow x^{10}_\alpha + 2\pi R$, the theta angle changes by $\pm 2\pi$.) But according to (2.8), at large $v$ one has $s_\alpha(v) = (k_\alpha - k_{\alpha+1}) \ln v$, so

$$-i\tau_\alpha(v) \cong (2k_\alpha - k_{\alpha-1} - k_{\alpha+1}) \ln v. \quad (2.14)$$

The standard asymptotic freedom formula is $-i\tau = b_0 \ln v$, where $-b_0$ is the coefficient of the one-loop beta function. So (2.14) amounts to the statement that the one-loop beta function for the $SU(k_\alpha)$ factor of the gauge group is

$$b_{0,\alpha} = -2k_\alpha + k_{\alpha-1} + k_{\alpha+1}. \quad (2.15)$$

This is in agreement with a standard field theory computation for this model. In fact, for $N = 2$ supersymmetric QCD with gauge group $SU(N_c)$ and $N_f$ flavors, one usually has $b_0 = -(2N_c - N_f)$. In the case at hand, $N_c = k_\alpha$, and the $(k_{\alpha-1}, \overline{k}_\alpha) \oplus (k_\alpha, \overline{k}_{\alpha+1})$ hypermultiplets make the same contribution to the $SU(k_\alpha)$ beta function as $k_{\alpha-1} + k_{\alpha+1}$ flavors, so the effective value of $N_f$ is $k_{\alpha-1} + k_{\alpha+1}$.

2.3. Interpretation Via $M$ Theory

By now we have identified a certain class of models that can be constructed with fivebranes and fourbranes only. The remaining question is of course how to analyze these models. For this we will use $M$ theory.

First of all, the reason that one may effectively go to $M$ theory is that according to (2.10), a rescaling of the Type IIA string coupling constant, if accompanied by a rescaling of the separations of the fivebranes in the $x^6$ direction, does not affect the field theory coupling constant and so is irrelevant. One might be concerned that (2.10) is just a
classical formula. But in fact, we have identified in the brane diagram all marginal and relevant operators (the coupling constants and hypermultiplet bare masses) of the low energy $\mathcal{N} = 2$ field theory, so any additional parameters (such as the string coupling constant) really are irrelevant. Therefore we may go to the regime of large $\lambda$.

What will make this useful is really the following. A fourbrane ending on a fivebrane has no known explicit conformal field theory description. The end of the fourbrane is a kind of singularity that is hard to understand in detail. That is part of the limitation of describing this system via Type IIA superstrings. But in $M$ theory everything we need can be explicitly understood using only the low energy limit of the theory. The Type IIA fivebrane on $\mathbb{R}^{10}$ is simply an $M$ theory fivebrane on $\mathbb{R}^{10} \times S^1$, whose world-volume, roughly, is located at a point in $S^1$ and spans a six-manifold in $\mathbb{R}^{10}$. A Type IIA fourbrane is an $M$ theory fivebrane that is wrapped over the $S^1$ (so that, roughly, its world-volume projects to a five-manifold in $\mathbb{R}^{10}$). Thus, the four-brane and five-brane come from the same basic object in $M$ theory. The Type IIA singularity where the fourbrane appears to end on a fivebrane is, as we will see, completely eliminated by going to $M$ theory.

The Type IIA configuration of parallel fivebranes joined by fourbranes can actually be reinterpreted in $M$ theory as a configuration of a single fivebrane with a more complicated world history. The fivebrane world-volume be described as follows. (1) It sweeps out arbitrary values of the first four coordinates $x^0, x^1, \ldots, x^3$. It is located at $x^7 = x^8 = x^9 = 0$. (2) In the remaining four coordinates $x^4, x^5, x^6$, and $x^{10}$ – which parametrize a four-manifold $Q \cong \mathbb{R}^3 \times S^1$ – the fivebrane worldvolume spans a two-dimensional surface $\Sigma$. (3) If one forgets $x^{10}$ and projects to a Type IIA description in terms of branes on $\mathbb{R}^{10}$, then one gets back, in the limit of small $R$, the classical configuration of fourbranes and fivebranes that we started with. (4) Finally, $\mathcal{N} = 2$ supersymmetry means that if we give $Q$ the complex structure in which $v = x^4 + ix^5$ and $s = x^6 + ix^{10}$ are holomorphic, then $\Sigma$ is a complex Riemann surface in $Q$. This makes $\mathbb{R}^4 \times \Sigma$ a supersymmetric cycle in the sense of [6] and so ensures spacetime supersymmetry.

In the approximation of the Type IIA brane diagrams, $\Sigma$ has different components that are described locally by saying that $s$ is constant (the fivebranes) or that $v$ is constant (the fourbranes). But the singularity that appears in the Type IIA limit where the different components meet can perfectly well be absent upon going to $M$ theory; and that will be so generically, as we will see. Thus, for generic values of the parameters, $\Sigma$ will be a smooth complex Riemann surface in $Q$. 

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This smoothness is finally the reason that going to $M$ theory leads to a solution of the problem. For large $\lambda$, all distances characteristic of the Riemann surface $\Sigma$ are large and it will turn out that there are generically no singularities. So obtaining and analyzing the solution will require only a knowledge of the low energy long wavelength approximation to $M$ theory and its fivebranes.

**Low Energy Effective Action**

We will now work out the low energy four-dimensional physics that will result from such an $M$ theory configuration. The discussion is analogous to, but more elementary than, a situation considered in [7] where an $\mathcal{N} = 2$ theory in four dimensions was related to a brane of the general form $R^4 \times \Sigma$.

Vector multiplets will appear in four dimensions because on the worldvolume of an $M$ theory fivebrane there is a chiral antisymmetric tensor field $\beta$, that is, a two-form $\beta$ whose three-form field strength $T$ is self-dual. Consider in general a fivebrane whose worldvolume is $R^4 \times \Sigma$, where $\Sigma$ is a compact Riemann surface of genus $g$. According to [8], in the effective four-dimensional description, the zero modes of the antisymmetric tensor give $g$ abelian gauge fields on $R^4$. The coupling constants and theta parameters of the $g$ abelian gauge fields are described by a rank $g$ abelian variety which is simply the Jacobian $J(\Sigma)$.

These conclusions are reached as follows. Let

$$T = F \wedge \Lambda + *F \wedge *\Lambda,$$

where $F$ is a two-form on $R^4$, $\Lambda$ is a one-form on $\Sigma$, and $*$ is the Hodge star. This $T$ is self-dual, and the equation of motion $dT = 0$ gives Maxwell’s equations $dF = d*F = 0$ along with the equations $d\Lambda = d*\Lambda = 0$ for $\Lambda$. So $\Lambda$ is a harmonic one-form, and every choice of a harmonic one-form $\Lambda$ gives a way of embedding solutions of Maxwell’s equations on $R^4$ as solutions of the self-dual three-form $T$. If $\Sigma$ has genus $g$, then the space of self-dual (or anti-self-dual) $\Lambda$’s is $g$-dimensional, giving $g$ positive helicity photon states (and $g$ of negative helicity) on $R^4$. The low energy theory thus has gauge group $U(1)^g$. The terms quadratic in $F$ in the effective action for the gauge fields are obtained by inserting (2.16) in the fivebrane kinetic energy $\int_{R^4 \times \Sigma} |T|^2$; the Jacobian of $\Sigma$ enters by determining the integrals of wedge products of $\Lambda$’s and $*\Lambda$’s.

In our problem of $n + 1$ parallel Type IIA fivebranes joined by fourbranes, the $M$ theory fivebrane is $R^4 \times \Sigma$, where $\Sigma$ is not compact. So the above discussion does not immediately apply. However, $\Sigma$ could be compactified by adding $n + 1$ points. Indeed,
for a single fivebrane, Σ would be a copy of \( C \) (the \( v \) plane), which is \( \mathbb{C}P^1 \) with a point deleted. So if there are no fourbranes, we have just \( n+1 \) disjoint copies of \( \mathbb{C}P^1 \) with a point omitted from each. Including a fourbrane means cutting holes out of adjoining fivebranes and connecting them with a tube. This produces (if all \( k_\alpha \) are positive) a connected Riemann surface \( \Sigma \) which can be compactified by adding \( n+1 \) points. Note that the deleted points are “at infinity”; the metric on \( \Sigma \) that is obtained from its embedding in \( Q \) is complete and looks “near each puncture” like the flat complex plane with the puncture being the point “at infinity.”

The reason that noncompactness potentially modifies the discussion of the low energy effective action is that in (2.14), one must ask for \( \Lambda \) to be square-integrable, in the metric on \( \Sigma \) which comes from its embedding in \( Q \), as well as harmonic. Since the punctures are “at infinity,” square-integrability implies that \( \Lambda \) has vanishing periods on a contour that surrounds any puncture. (A harmonic one-form \( \Lambda' \) that has a non-vanishing period on such a contour would look near \( v = \infty \) like \( \Lambda' = dv/v \), leading to \( \int \Lambda' \wedge *\Lambda' = \int dv \wedge d\bar{v}/|v|^2 = \infty \).) Hence \( \Lambda \) extends over the compactification \( \overline{\Sigma} \) of \( \Sigma \). Since moreover the equation for a one-form on \( \overline{\Sigma} \) to be self-dual is conformally invariant and depends only on the complex structure of \( \overline{\Sigma} \), the square-integrable harmonic one-forms on \( \Sigma \) are the same as the harmonic one-forms on \( \overline{\Sigma} \). So finally, in our problem, the low energy effective action of the vector fields is determined by the Jacobian of \( \Sigma \).

It is thus of some interest to determine the genus of \( \overline{\Sigma} \). We construct \( \overline{\Sigma} \) beginning with \( n+1 \) disjoint copies of \( \mathbb{C}P^1 \), of total Euler characteristic \( 2(n+1) \). Then we glue in a total of \( \sum_{\alpha=1}^{n} k_\alpha \) tubes between adjacent \( \mathbb{C}P^1 \)'s. Each time such a tube is glued in, the Euler characteristic is reduced by two, so the final value is \( 2(n+1) - 2 \sum_{\alpha=1}^{n} k_\alpha \). This equals \( 2 - 2g \), where \( g \) is the genus of \( \overline{\Sigma} \). So we get \( g = \sum_{\alpha=1}^{n} (k_\alpha - 1) \). This is the expected dimension of the Coulomb branch for the gauge group \( \prod_{\alpha=1}^{n} SU(k_\alpha) \). In particular, this confirms that the \( U(1) \)'s are “missing”; for the gauge group to be \( \prod_{\alpha=1}^{n} U(k_\alpha) \), the genus would have to be \( \sum_{\alpha=1}^{n} k_\alpha \).

So far we have emphasized the effective action for the four-dimensional gauge fields. Of course, the rest of the effective action is determined from this via supersymmetry. For instance, the scalars in the low energy effective action simply determine the embedding of \( \mathbb{R}^4 \times \Sigma \) in spacetime, or more succinctly the embedding of \( \Sigma \) in \( Q \); and their kinetic energy is obtained by evaluating the kinetic energy for motion of the fivebrane in spacetime.

*The Integrable System*
In general, the low energy effective action for an $\mathcal{N} = 2$ system in four dimensions is determined by an integrable Hamiltonian system in the complex sense. The expectation values of the scalar fields in the vector multiplets are the commuting Hamiltonian flows; the orbits generated by the commuting Hamiltonian flows are the complex tori which determine the kinetic energy of the massless four-dimensional vectors. This structure was noticed in special cases [9-11] and deduced from the generalities of low energy supersymmetric effective field theory [12].

A construction of many complex integrable systems is as follows. Let $X$ be a two-dimensional complex symplectic manifold. Let $\Sigma$ be a complex curve in $X$. Let $W$ be the deformation space of pairs $(\Sigma', \mathcal{L}')$, where $\Sigma'$ is a curve in $X$ to which $\Sigma$ can be deformed and $\mathcal{L}'$ is a line bundle on $\Sigma'$ of specified degree. Then $W$ is an integrable system; it has a complex symplectic structure which is such that any functions that depend only on the choice of $\Sigma'$ (and not of $\mathcal{L}'$) are Poisson-commuting. The Hamiltonian flows generated by these Poisson-commuting functions are the linear motions on the space of $\mathcal{L}'$s, that is, on the Jacobian of $\Sigma'$.

This integrable system was described in [13], as a generalization of a gauge theory construction by Hitchin [14]; a prototype for the case of non-compact $\Sigma$ is the extension of Hitchin’s construction to Riemann surfaces with punctures in [15]. The same integrable system has appeared in the description of certain BPS states for Type IIA superstrings on K3 [16].

In general, fix a hyper-Kahler metric on the complex symplectic manifold $X$ (of complex dimension two) and consider $M$ theory on $\mathbb{R}^7 \times X$. Consider a fivebrane of the form $\mathbb{R}^4 \times \Sigma$, where $\mathbb{R}^4$ is a fixed linear subspace of $\mathbb{R}^7$ (obtained by setting three linear combinations of the seven coordinates to constants) and $\Sigma$ is a complex curve in $X$. Then the effective $\mathcal{N} = 2$ theory on $\mathbb{R}^4$ is controlled by the integrable system described in the last paragraph, with the given $X$ and $\Sigma$. This follows from the fact that the scalar fields in the four-dimensional theory parametrize the choice of a curve $\Sigma'$ to which $\Sigma$ can be deformed (preserving its behavior at infinity) while the Jacobian of $\Sigma'$ determines the couplings of the vector fields.

The case of immediate interest is the case that $X = Q$ and $\Sigma$ is related to the brane diagram with which we started the present section. The merit of this case (relative to an arbitrary pair $(X, \Sigma)$) is that because of the Type IIA interpretation, we know a gauge theory whose solution is given by this special case of the integrable model. Some generalizations that involve different choices of $X$ are in sections 3 and 4.
BPS States

The spectrum of massive BPS states in models constructed this way can be analyzed roughly as in [7], by using the fact that M theory twobranes can end on fivebranes [5,17]. BPS states can be obtained by considering suitable twobranes in $\mathbb{R}^7 \times X$. To ensure the BPS property, the twobrane world volume should be a product $\mathbb{R} \times D$, where $\mathbb{R}$ is a straight line in $\mathbb{R}^4 \subset \mathbb{R}^7$ (representing “the world line of the massive particle in spacetime”) and $D \subset X$ is a complex Riemann surface with a non-empty boundary $C$ that lies on $\Sigma$. By adjusting $D$ to minimize the area of $D$ (keeping fixed the holomogy class of $C \subset \Sigma$), one gets a twobrane worldvolume whose quantization gives a BPS state.

2.4. Solution Of The Models

We now come to the real payoff, which is the solution of the models.

The models are to be described in terms of an equation $F(s, v) = 0$, defining a complex curve in $Q$.

Since $s$ is not single-valued, we introduce

$$t = \exp(-s) = \exp(- x^6 + i x^{10})/R) \quad (2.17)$$

and look for an equation $F(t, v) = 0$.

Now if $F(t, v)$ is regarded as a function of $t$ for fixed $v$, then the roots of $F$ are the positions of the fivebranes (at the given value of $v$). The degree of $F$ as a polynomial in $t$ is therefore the number of fivebranes. To begin with, we will consider a model with only two fivebranes. $F$ will therefore be quadratic in $t$.

Classically, if one regards $F(t, v)$ as a function of $v$ for fixed $t$, with a value of $t$ that is “in between” the two fivebranes, then the roots of $F(t, v)$ are the values of $v$ at which there are fourbranes. We will set the number of fourbranes suspended between the two fivebranes equal to $k$, so $F(t, v)$ should be of degree $k$ in $v$. (If $t$ is “outside” the classical position of the fivebranes, the polynomial $F(t, v)$ still vanishes for $k$ values of $v$; these roots will occur at large $v$ and are related to the “bending” of the fivebranes for large $v$.)

So such a model will be governed by a curve of the form

$$A(v)t^2 + B(v)t + C(v) = 0, \quad (2.18)$$

with $A, B,$ and $C$ being polynomials in $v$ of degree $k$. We set $F = At^2 + Bt + C$. 13
At a zero of $C(v)$, one of the roots of (2.18) (regarded as an equation for $t$) goes to $t = 0$. According to (2.17), $t = 0$ is $x^6 = \infty$. Having a root of the equation which goes to $x^6 = \infty$ at a fixed limiting value of $v$ (where $C(v)$ vanishes) means that there is a semi-infinite fourbrane to the “right” of all of the fivebranes.

Likewise, at a zero of $A(v)$, a root of $F$ goes to $t = +\infty$, that is to say, to $x^6 = -\infty$. This corresponds to a semi-infinite fourbrane on the “left.”

Since there are $k$ fourbranes between the two fivebranes, these theories will be $SU(k)$ gauge theories. As in [1], a semi-infinite fourbrane, because of its infinite extent in $x^6$, has an infinite kinetic energy (relative to the fourbranes that extend a finite distance in $x^6$) and can be considered to be frozen in place at a definite value of $v$. The effect of a semi-infinite fourbrane is to add one hypermultiplet in the fundamental representation of $SU(k)$.

We first explore the “pure gauge theory” without hypermultiplets. For this we want no zeroes of $A$ or $C$, so $A$ and $C$ must be constants and the curve becomes after a rescaling of $t$

$$t^2 + B(v)t + 1 = 0.$$  \hspace{1cm} (2.19)

In terms of $\tilde{t} = t + B/2$, this reads

$$\tilde{t}^2 = \frac{B(v)^2}{4} - 1.$$ \hspace{1cm} (2.20)

By rescaling and shifting $v$, one can put $B$ in the form

$$B(v) = v^k + u_2 v^{k-2} + u_3 v^{k-3} + \ldots + u_k.$$ \hspace{1cm} (2.21)

(2.20) is our first success; it is a standard form of the curve that governs the $SU(k)$ theory without hypermultiplets [18,19].

We chose $F(t, v)$ to be of degree $k$ in $v$ so that, for a value of $t$ that corresponds to being “between” the fivebranes, there would be $k$ roots for $v$. Clearly, however, the equation $F(t, v) = 0$ has $k$ roots for $v$ for any non-zero $t$ (we recall that $t = 0$ is “at infinity” in the original variables). What is the interpretation of these roots for very large or very small $v$, to the left or right of the fivebranes? For $t$ very large, the roots for $v$ are approximately at

$$t \cong c \cdot v^k,$$ \hspace{1cm} (2.22)
and for \( t \) very small they are approximately at
\[
 t \cong c' \cdot v^{-k}; \tag{2.23}
\]
here \( c, c' \) are constants. The formulas \( t \cong v^{\pm k} \) are actually special cases of \((2.8)\); they represent the “bending” of the fivebranes as a result of being pulled on by fourbranes. The formulas \((2.22)\) and \((2.23)\) show that for \( x^6 \to \pm \infty \), the roots of \( F \), as a function of \( v \) for fixed \( t \), are at very large \( |v| \). These roots do not correspond, intuitively, to positions of fourbranes but are points “near infinity” on the bent fivebranes.

We can straightforwardly incorporate hypermultiplet flavors in this discussion. For this, we merely incorporate zeroes of \( A \) or \( C \). For example, to include \( N_f \) flavors we can take \( A = 1 \) and \( C(v) = f \prod_{j=1}^{N_f} (v - m_j) \) where the \( m_j \), being the zeroes of \( C \), are the positions of the semi-infinite fourbranes or in other words the hypermultiplet bare masses, and \( f \) is a complex constant. Equation \((2.20)\) becomes
\[
 \tilde{t}^2 = \frac{B(v)^2}{4} - f \prod_{j=1}^{N_f} (v - m_j). \tag{2.24}
\]
We set now
\[
 B(v) = e(v^n + u_2 v^{n-2} + u_3 v^{n-3} + \ldots + u_n) \tag{2.25}
\]
with \( e \) and the \( u_i \) being constants. We have shifted \( v \) by a constant to remove the \( v^{n-1} \) term. This is again equivalent to the standard solution \([20, 21]\) of the \( SU(k) \) theory with \( N_f \) flavors. As long as \( N_f \neq 2k \), one can rescale \( \tilde{t} \) and \( v \) to set \( e = f = 1 \). Of course, shifting \( v \) by a constant to eliminate the \( v^{k-1} \) term in \( B \) will shift the \( m_j \) by a constant. This is again a familiar part of the solution of the models.

Of special interest is the case \( N_f = 2k \) where the beta function vanishes. In this case, by rescaling \( \tilde{t} \) and \( v \), it is possible to remove only one combination of \( e \) and \( f \). The remaining combination is a modulus of the theory, a coupling constant. This is as expected: four-dimensional quantum Yang-Mills theory has a dimensionless coupling constant when and only when the beta function vanishes.

The coupling constant for \( N_f = 2k \) is coded into the behavior of the fivebrane for \( z, t \to \infty \). This behavior, indeed, is a “constant of the motion” for finite energy disturbances of the fivebrane configuration and hence can be interpreted as a coupling constant in the four-dimensional quantum field theory. The behavior at infinity for \( N_f = 2k \) is
\[
 t \cong \lambda_{\pm} v^k, \tag{2.26}
\]
where \( \lambda_\pm \) are the two roots of the quadratic equation

\[
y^2 + ey + f = 0. \tag{2.27}
\]

This follows from the fact that the asymptotic behavior of the equation is

\[
t^2 + e(v^k + \ldots)t + f(v^{2k} + \ldots) = 0. \tag{2.28}
\]

\( y \) can be identified as \( t/v^k \). The fact that the two fivebranes are parallel at infinity – on both branches \( t \cong v^k \) for \( v \to \infty \) – means that the distance between them has a limit at infinity, which determines the gauge coupling constant.

A rescaling of \( t \) or \( v \) rescales \( \lambda_\pm \) by a common factor, leaving fixed the function

\[ w = -4\lambda_+\lambda_-/(\lambda_+ - \lambda_-)^2 \]

which is also invariant under exchange of the \( \lambda \)'s. This function can be constructed as a product of cross ratios of the four distinguished points \( 0, \infty, \lambda_+ \) and \( \lambda_- \) on the \( y \) plane. Let \( \mathcal{M}_{0,4;2} \) be the moduli space of the following objects: a smooth Riemann surface of genus zero with four distinct marked points, two of which \( (0 \text{ and } \infty) \) are distinguished and ordered, while the others \( (\lambda_+ \text{ and } \lambda_-) \) are unordered. The choice of a value of \( w \) (not equal to zero or infinity) is the choice of a point in \( \mathcal{M}_{0,4;2} \). The point \( w = 1 \) is a \( \mathbb{Z}_2 \) orbifold point in \( \mathcal{M}_{0,4;2} \).

In the conventional description of this theory, one introduces a coupling parameter \( \tau \) appropriate near one component of “infinity” in \( \mathcal{M}_{0,4;2} \) – near \( w \to 0 \) (which corresponds for instance to \( \lambda_+ \to 0 \) at fixed \( \lambda_- \)), where the \( SU(k) \) gauge theory is weakly coupled.\footnote{The other possible degeneration is \( w \to \infty \) \( (\lambda_+ \to \lambda_-) \). This is in \( M \) theory the limit of coincident fivebranes, and a weakly coupled description in four dimensions is not obvious.}

In \cite{20}, the solution \((2.24)\) is expressed in terms of \( \tau \). Near \( w = 0 \) one has \( w = e^{2\pi i \tau} \); the inverse function \( \tau(w) \) is many-valued. The fact that the theory depends only on \( w \) and not on \( \tau \) is from the standpoint of weak coupling interpreted as the statement that the theory is invariant under a discrete group of duality transformations. This group is \( \Gamma = \pi_1(\mathcal{M}_{0,4;2}) \). It can be shown that \( \Gamma \) is isomorphic to the index three subgroup of \( SL(2,\mathbb{Z}) \) consisting of integral unimodular matrices

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

with \( b \) even; this group is usually called \( \Gamma_0(2) \).
The Case Of A Positive Beta Function

What happens when the $SU(k)$ gauge theory has positive beta function, that is for $N_f > 2k$? The fivebrane configuration (2.24) still describes something, but what? The first main point to note is that for $N_f > 2k$, the two fivebranes are parallel near infinity; both branches of (2.24) behave for large $v$ as $\tilde{t} \sim v^{N_f/2}$. I interpret this to mean that the four-dimensional theory induced from the branes is conformally invariant at short distances and flows in the infrared to the $SU(k)$ theory with $N_f$ flavors.

What conformally invariant theory is this? A key is that for $N_f \geq 2k + 2$, there are additional terms that can be added to (2.24) without changing the asymptotic behavior at infinity (and cannot be absorbed in redefining $\tilde{t}$ and $v$). Such terms really should be included because they represent different vacua of the same quantum system.

There are two rather different cases to consider. If $N_f = 2k'$ is even, the general curve with the given behavior at infinity is

$$\tilde{t}^2 = \frac{1}{4} e'(v^{k'} + \ldots)^2 - f \prod_{i=1}^{2k'} (v - m_i). \tag{2.30}$$

This describes the the $SU(k')$ theory with $2k'$ flavors, a theory that is conformally invariant in the ultraviolet and which by suitably adjusting the parameters can reduce in an appropriate limit (taking $e'$ to zero while rescaling $v$ and some of the other parameters) to the solution (2.24) for the $SU(k)$ theory with $N_f > 2k$ flavors. The $SU(k')$ theory with $2k'$ flavors has of course a conventional Lagrangian description, valid when the coupling is weak.

The other case is $N_f = 2k' + 1$, with $k' \geq k$. The most general curve with the same asymptotic behavior as (2.24) is then

$$\tilde{t}^2 = \frac{1}{4} e'(v^{k'} + \ldots)^2 - f \prod_{i=1}^{2k'+1} (v - m_i). \tag{2.31}$$

There is no notion of weak coupling here; the asymptotic behavior of the fivebranes is $\tilde{t} = \lambda_\pm v^{n'+1/2}$ with $\lambda_- = -\lambda_+$ so that $w$ has the fixed value 1. (We recall that this is the $\mathbb{Z}_2$ orbifold point on $M_{0,4;2}$.) (2.31) describes a strongly coupled fixed point with no obvious Lagrangian description and no dimensionless “coupling constant,” roughly along the lines of the fixed point analyzed in [22]. By specializing some parameters, it can flow in the infrared to the $SU(k)$ theory with $2k' + 1$ flavors for any $k < k'$. 

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Also, the $SU(k'+1)$ theory with $2k'+2$ flavors can flow in the infrared to the fixed point just described. This is done starting with (2.24) by taking one mass to infinity while shifting and adjusting the other variables in an appropriate fashion.

In the rest of this paper, we concentrate on models of zero or negative beta function. Along just the above lines, models of positive beta function can be derived from conventional fixed points like the one underlying (2.30) or unconventional ones like the one underlying (2.31); the conventional and unconventional fixed points are linked by renormalization group flows.

### 2.5. Generalization

Now we will consider a more general model, with a chain of $n+1$ fivebranes labeled from 0 to $n$, the $\alpha-1^{th}$ and $\alpha^{th}$ fivebranes, for $\alpha = 1, \ldots, n$, being connected by $k_\alpha$ fourbranes. We assume that there are no semi-infinite fourbranes at either end.

The gauge group is thus $\prod_{\alpha=1}^{n} SU(k_\alpha)$, and the coefficient in the one-loop beta function of $SU(k_\alpha)$ is

$$b_{0,\alpha} = -2k_\alpha + k_{\alpha+1} + k_{\alpha-1}. \quad (2.32)$$

(We understand here $k_0 = k_{n+1} = 0$.) We will assume $b_{0,\alpha} \leq 0$ for all $\alpha$. Otherwise, as in the example just treated, the model is not really a $\prod_\alpha SU(k_\alpha)$ gauge theory at short distances but should be interpreted in terms of a different ultraviolet fixed point. Note that $\sum_{\alpha=1}^{n} b_{0,\alpha} < 0$ (in fact $\sum_{\alpha=1}^{n} b_{0,\alpha} = -k_1 - k_n$), so it is impossible in a model of this type for all beta functions to vanish. (Models with vanishing beta function can be obtained by including semi-infinite fourbranes at the ends of the chain, as above, or by other generalizations made in sections three and four.)

If the position $t_\alpha(v)$ of the $\alpha^{th}$ fivebrane, for $\alpha = 0, \ldots, n$, behaves for large $v$ as

$$t_\alpha(v) \sim h_\alpha v^{a_\alpha} \quad (2.33)$$

with $a_0 \geq a_1 \geq a_2 \ldots \geq a_n$ and constants $h_\alpha$, then from our analysis of the relation of the beta function to “bending” of fivebranes, we have

$$a_\alpha - a_{\alpha-1} = -b_{0,\alpha}, \quad \text{for } \alpha = 1, \ldots, n. \quad (2.34)$$

---

4 By analogy with the $SU(k)$ theory with $N_f$ hypermultiplets treated in the last subsection, semi-infinite fourbranes would be incorporated by taking the coefficients of $t^{n+1}$ and $t^0$ in the polynomial $P(t, v)$ introduced below to be polynomials in $v$ of positive degree. This gives solutions of models that are actually special cases of models that will be treated in section 3.
The fivebrane worldvolume will be described by a curve \( F(v, t) = 0 \), for some polynomial \( F \). \( F \) will be of degree \( n + 1 \) in \( t \) so that for each \( v \) there are \( n + 1 \) roots \( t_\alpha(v) \) (already introduced in (2.33)), representing the \( v \)-dependent positions of the fivebranes. \( F \) thus has the general form

\[
F(t, v) = t^{n+1} + f_1(v)t^n + f_2(v)t^{n-1} + \ldots + f_n(v)t + 1. \tag{2.35}
\]

As in the special case considered in section 2.4, the coefficients of \( t^{n+1} \) and \( t^0 \) are non-zero constants to ensure the absence of semi-infinite fourbranes; those coefficients have been set to 1 by scaling \( F \) and \( t \). Alternatively, we can factor \( F \) in terms of its roots:

\[
F = \prod_{\alpha=0}^{n}(t - t_\alpha(v)). \tag{2.36}
\]

The fact that the \( t^0 \) term in (2.35) is independent of \( v \) implies that

\[
\sum_{\alpha=0}^{n} a_\alpha = 0, \tag{2.37}
\]

and this, together with the \( n \) equations (2.34), determines the \( a_\alpha \) for \( \alpha = 0, \ldots, n \). The solution is in fact

\[
a_\alpha = k_{\alpha+1} - k_\alpha \tag{2.38}
\]

with again \( k_0 = k_{n+1} = 0 \).

If the degree of a polynomial \( f(v) \) is denoted by \( [f] \), then the factorization (2.36) and asymptotic behavior (2.33) imply that

\[
[f_1] = a_0, \ [f_2] = a_0 + a_1, \ [f_3] = a_0 + a_1 + a_2, \ldots. \tag{2.39}
\]

Together with (2.38), this implies simply

\[
[f_\alpha] = k_\alpha, \text{ for } \alpha = 1, \ldots, n. \tag{2.40}
\]

If we rename \( f_\alpha \) as \( p_{k_\alpha}(v) \), where the subscript now equals the degree of a polynomial, then the polynomial \( F(t, v) \) takes the form

\[
F(t, v) = t^{n+1} + p_{k_1}(v)t^n + p_{k_2}(v)t^{n-1} + \ldots + p_{k_n}(v)t + 1. \tag{2.41}
\]

The curve \( F(t, v) = 0 \) thus describes the solution of the model with gauge group \( \prod_{\alpha=1}^{n} SU(k_\alpha) \) and hypermultiplets in the representation \( \sum_{\alpha=1}^{n-1}(k_\alpha, k_{\alpha+1}) \). The fact that
the coefficient of $t^i$, for $1 \leq i \leq n$, is a polynomial of degree $k_i$ in $v$ has a clear intuitive interpretation: the zeroes of $p_{k_{\alpha}}(v)$ are the positions of the $k_{\alpha}$ fourbranes that stretch between the $\alpha^{th}$ and $\alpha + 1^{th}$ fivebrane.

The polynomial $p_{k_{\alpha}}$ has the form

$$p_{k_{\alpha}}(v) = c_{\alpha,0}v^{k_{\alpha}} + c_{\alpha,1}v^{k_{\alpha}-1} + c_{\alpha,2}v^{k_{\alpha}-2} + \ldots + c_{\alpha,k_{\alpha}}.$$  \hspace{1cm} (2.42)

The leading coefficients $c_{\alpha,0}$ determine the asymptotic positions of the fivebranes for $v \to \infty$, or more precisely the constants $h_{\alpha}$ in (2.33). In fact by comparing the factorization $F(t, v) = \prod_{\alpha}(t - t_{\alpha}(v)) = \prod_{\alpha}(t - h_{\alpha}t^{v_{\alpha}} + O(t^{v_{\alpha}-1}))$ to the series (2.41) one can express the $h_{\alpha}$ in terms of the $c_{0,\alpha}$.

The $h_{\alpha}$ determine the constant terms in the asymptotic freedom formula

$$-i\tau_{\alpha} = -b_{0,\alpha}\ln v + \text{constant}$$ \hspace{1cm} (2.43)

for the large $v$ behavior of the inverses of the effective gauge couplings. Thus, the $c_{\alpha,0}$’s should be identified with the gauge coupling constants. Of course, one combination of the $c_{\alpha,1}$’s can be eliminated by rescaling the $v$’s; this can be interpreted as a renormalization group transformation via which (as the beta function coefficients $b_{0,\alpha}$ are not all zero) one coupling constant can be eliminated.

In particular, the $c_{\alpha,0}$ are constants that parametrize the choice of a quantum system, not order parameters that determine the choice of a vacuum in a fixed quantum system. The $c_{\alpha,1}$ are likewise constants, according to (2.3); they determine the hypermultiplet bare masses. (One of the $c_{\alpha,1}$ can be removed by adding a constant to $v$; in fact there are $n$ $c_{\alpha,1}$’s and only $n - 1$ hypermultiplet bare masses.) The $c_{\alpha,s}$ for $s = 2, \ldots, k_{\alpha}$ are the order parameters on the Coulomb branch of the $SU(k_{\alpha})$ factor of the gauge group.

3. Models With Sixbranes

3.1. Preliminaries

The goal in the present section is to incorporate sixbranes in the models of the previous section. The sixbranes will enter just like the D fivebranes in [1] and for some purposes can be analyzed quite similarly.

Thus we consider again the familiar chain of $n + 1$ fivebranes, labeled from 0 to $n$, with $k_{\alpha}$ fourbranes stretched between the $\alpha - 1^{th}$ and $\alpha^{th}$ fivebranes, for $\alpha = 1, \ldots, n$. But now
we place $d_\alpha$ sixbranes between the $\alpha - 1^{th}$ and $\alpha^{th}$ fivebranes, for $\alpha = 1, \ldots, n$. A special case is sketched in figure 3. In the coordinates introduced at the beginning of section two, each sixbrane is located at definite values of $x^4, x^5$, and $x^6$ and has a world-volume that is parametrized by arbitrary values of $x^0, x^1, \ldots, x^3$ and $x^7, x^8,$ and $x^9$.

Given what was said in section two and in [1], the interpretation of the resulting model as a four-dimensional gauge theory is clear. The gauge group is $\prod_{\alpha=1}^{n} SU(k_\alpha)$. The hypermultiplets consist of the $(k_\alpha, \overline{k}_\alpha + 1)$ hypermultiplets that were present without the sixbranes, plus additional hypermultiplets that become massless whenever a fourbrane meets a sixbrane. As in [1], these additional hypermultiplets transform in $d_\alpha$ copies of the fundamental representation of $SU(k_\alpha)$, for each $\alpha$. The bare masses of these hypermultiplets are determined by the positions of the sixbranes in $v = x^4 + ix^5$. As in [1], the positions of the sixbranes in $x^6$ decouple from many aspects of the low energy four-dimensional physics.

One difference from section two is that (even without semi-infinite fourbranes) there are many models with vanishing beta function. In fact, for each choice of $k_\alpha$ such that the models considered in section two had all beta functions zero or negative, there is upon inclusion of sixbranes a unique choice of the $d_\alpha$ for which the beta functions all vanish, namely

$$d_\alpha = 2k_\alpha - k_{\alpha+1} - k_{\alpha-1}$$  \hspace{1cm} (3.1)

(where we understand that $k_0 = k_{n+1} = 0$). By solving all these models, we will get a much larger class of solved $\mathcal{N} = 2$ models with zero beta function than has existed hitherto. For each such model, one expects to find a non-perturbative duality group generalizing the
duality group $SL(2,\mathbb{Z})$ of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. From the solutions we will get, the duality groups turn out to be as follows. Let $\mathcal{M}_{0,n+3;2}$ be the moduli space of objects of the following kind: a smooth Riemann surface of genus zero with $n+3$ marked points, two of which are distinguished and ordered while the other $n+1$ are unordered. Then the duality group of a model with $n+1$ fivebranes is the fundamental group $\pi_1(\mathcal{M}_{0,n+3;2})$. One can think roughly of the genus zero Riemann surface in the definition of $\mathcal{M}_{0,n+3;2}$ as being parametrized by the variable $t$ of section two, with the marked points being $0, \infty$, and the positions of the $n+1$ fivebranes.

In contrast to section two, we would gain nothing essentially new by incorporating semi-infinite fourbranes at the two ends of the chain. This gives hypermultiplets in the fundamental representation of the groups $SU(k_1)$ and $SU(k_n)$ that are supported at the ends of the chain; we will anyway generate an arbitrary number of such hypermultiplets via sixbranes. Another generalization that would give nothing essentially new would be to include fourbranes that connect fivebranes to sixbranes. Using a mechanism considered in [1], one can by moving the sixbranes in the $x^6$ direction reduce to the case that all fourbranes end on fivebranes. One could also add sixbranes to the left or to the right of all fivebranes. In fact, we will see how this generalization can be incorporated in the formulas. In the absence of fourbranes ending on them, sixbranes that are to the left or right of everything else simply decouple from the low energy four-dimensional physics.

Another generalization is to consider fourbranes that end on sixbranes at both ends. As in [1], such a fourbrane supports a four-dimensional hypermultiplet, not a vector multiplet, and configurations containing such fourbranes must be included to describe Higgs branches (and mixed Coulomb-Higgs branches) of these theories. We will briefly discuss the Higgs branches in section 3.5.

### 3.2. Interpretation In $M$ Theory

Since our basic technique is to interpret Type IIA brane configurations in $M$ theory, we need to know how to interpret the Type IIA sixbrane in $M$ theory. This was first done in [23].

Consider $M$ theory on $\mathbb{R}^{10} \times S^1$. This is equivalent to Type IIA on $\mathbb{R}^{10}$, with the $U(1)$ gauge symmetry of Type IIA being associated in $M$ theory with the rotations of the $S^1$. States that have momentum in the $S^1$ direction are electrically charged with respect to this $U(1)$ gauge field and are interpreted in Type IIA as Dirichlet zerobranes. The
sixbrane is the electric-magnetic dual of the zerobrane, so it is magnetically charged with respect to this same $U(1)$.

The basic object that is magnetically charged with respect to this $U(1)$ is the “Kaluza-Klein monopole” or Taub-NUT space. This is derived from a hyper-Kahler solution of the four-dimensional Einstein equations. The metric is asymptotically flat, and the space-time looks near infinity like a non-trivial $S^1$ bundle over $R^3$. The Kaluza-Klein magnetic charge is given by the twisting of the $S^1$ bundle, which is incorporated in the formula given below by the appearance of the Dirac monopole potential.

Using conventions of [23] adapted to the notation of the present paper, if we define a three-vector $\vec{r} = (x^4, x^5, x^6)R$, and set $r = |\vec{r}|$ and $\tau = x^{10}/R$, then the Taub-NUT metric is

$$ds^2 = \frac{1}{4} \left( \frac{1}{r} + \frac{1}{R^2} \right) d\vec{r}^2 + \frac{1}{4} \left( \frac{1}{r} + \frac{1}{R^2} \right)^{-1} (d\tau + \vec{\omega} \cdot d\vec{r})^2. \quad (3.2)$$

Here $\vec{\omega}$ is the Dirac monopole potential (which one can identify locally as a one-form obeying $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} (1/r)$).

To construct a sixbrane on $R^{10} \times S^1$, we simply take the product of the metric (3.2) with a flat metric on $R^7$ (the coordinates on $R^7$ being $x^0, \ldots, x^3$ and $x^7, \ldots, x^9$). We will be interested in the case of many parallel sixbranes, which is described by the multi-Taub-NUT metric [24]:

$$ds^2 = \frac{V}{4} d\vec{r}^2 + \frac{V^{-1}}{4} (d\tau + \vec{\omega} \cdot d\vec{r})^2, \quad (3.3)$$

where now

$$V = 1 + \sum_{a=1}^{d} \frac{1}{|\vec{r} - \vec{x}_a|} \quad (3.4)$$

and $\vec{\nabla} \times \omega = \vec{\nabla} V$. This describes a configuration of $d$ parallel sixbranes, whose positions are the $\vec{x}_a$.

The reason that by going to eleven dimensions we will get some simplification in the study of sixbranes is that, in contrast to the ten-dimensional low energy field theory in which the sixbrane core is singular, in $M$ theory the sixbrane configuration is described by the multi-Taub-NUT metric (3.4), which is complete and smooth (as long as the $\vec{x}_a$ are distinct). This elimination of the sixbrane singularity was in fact emphasized in [23]. In going from $M$ theory to Type IIA, one reduces from eleven to ten dimensions by dividing by the action of the vector field $\partial/\partial\tau$. This produces singularities at $d$ points at which $\partial/\partial\tau$ vanishes; those $d$ points are interpreted in Type IIA as positions of sixbranes. In
general in physics, appearance of singularities in a long wavelength description means that to understand the behavior of a system one needs more information. The fact that the sixbrane singularity is eliminated in going to \( M \) theory means that, if the radius \( R \) of the \( x^{10} \) circle is big\(^5\), the \( M \) theory can be treated via low energy supergravity. This is just analogous to what happened in section 2; the singularity of Type IIA fourbranes ending on fivebranes was eliminated upon going to \( M \) theory, as a result of which low energy supergravity was an adequate approximation. The net effect is that unlike either long wavelength ten-dimensional field theory or conformal field theory, the long wavelength eleven-dimensional field theory is an adequate approximation for the problem.

In this paper we will really not use the hyper-Kahler metric of the multi-Taub-NUT space, but only the structure (or more exactly one of the structures) as a complex manifold. If as before we set \( v = x^4 + ix^5 \), then in one of its complex structures the multi-Taub-NUT space can be described by the equation

\[
yz = \prod_{a=1}^{d} (v - e_a)
\]

(3.5)

in a space \( \mathbb{C}^3 \) with three complex coordinates \( y, z, \) and \( v \). Here \( e_a \) are the positions of the sixbranes projected to the complex \( v \) plane. Note that (3.5) admits the \( \mathbb{C}^* \) action

\[
y \rightarrow \lambda y, \quad z \rightarrow \lambda^{-1}z,
\]

(3.6)

which is the complexification of the \( U(1) \) symmetry of (3.3) that is generated by \( \partial/\partial \tau \). For the special case that there are no fivebranes, this \( \mathbb{C}^* \) corresponds to the transformation \( t \rightarrow \lambda t \) where \( t = \exp \left( -(x^6 + ix^{10})/R \right) \). Hence very roughly, for large \( y \) with fixed or small \( z \), \( y \) corresponds to \( t \) and for large \( z \) with fixed or small \( y \), \( z \) corresponds to \( t^{-1} \). (As there is a symmetry exchanging \( y \) and \( z \), their roles could be reversed in these assertions.)

In section 3.6, we will use the approach of [24] to show that the multi-Taub-NUT space is equivalent as a complex manifold to (3.5). The formulas in section 3.6 can also be used to make the asymptotic identification of \( y \) and \( z \) with \( t \) and \( t^{-1} \) more precise. For now, we note the following facts, which may orient the reader. When all \( e_a \) are coincident at, say, \( v = 0 \), (3.5) reduces to the \( A_{n-1} \) singularity \( yz = v^n \). A system of parallel and coincident sixbranes in Type IIA generates a \( U(n) \) gauge symmetry; the \( A_{n-1} \) singularity is the

\(^5\) We recall that we can assume this radius to be big since it corresponds to an “irrelevant” parameter in the field theory.
mechanism by which such enhanced gauge symmetry appears in the $M$ theory description. In general, (3.5) describes the unfolding of the $A_{n-1}$ singularity.

The complex structure (3.5) does not uniquely fix the hyper-Kahler metric, not even the behavior of the metric at infinity. The same complex manifold (3.5) admits a family of “asymptotically locally Euclidean” (ALE) metrics, which look at infinity like $\mathbb{C}^2/\mathbb{Z}_n$. (They are given by the same formula (3.3), but with a somewhat different choice of $V$.) The metrics (3.3) are not ALE but are “asymptotically locally flat” (ALF).

Even if one asks for ALF behavior at infinity, the hyper-Kahler metric involves parameters that do not appear in (3.5). The hyper-Kahler metric (3.3) depends on the positions $\mathbf{x}_a$ of the sixbranes, while in (3.5) one sees only the projections $e_a$ of those positions to the $v$ plane. From the point of view of the complex structure that is exhibited in (3.5), the $x^6$ component of the sixbrane positions is coded in the Kahler class of the metric (3.3).

In studying the Coulomb branch of $\mathcal{N} = 2$ models, we will really need only the complex structure (3.5); the $x^6$ positions of sixbranes will be irrelevant. This is analogous to the fact that in studying the Coulomb branch of $\mathcal{N} = 4$ models in three dimensions by methods of [1], the $x^6$ positions of Dirichlet fivebranes are irrelevant. As that example suggests, the $x^6$ positions are relevant for understanding the Higgs branches of these models.

In one respect, the description (3.5) of the complex structure is misleading. Whenever $e_a = e_b$ for some $a$ and $b$, the complex manifold (3.5) gets a singularity. The hyper-Kahler metric, however, becomes singular only if two sixbranes have equal positions in $x^6$ and not only in $v$. When two sixbranes have the same position in $v$ but not in $x^6$, the singular complex manifold (3.5) must be replaced by a smooth one that is obtained by blowing up the singularities, replacing each $A_k$ singularity by a configuration of $k$ curves of genus zero. This subtlety will be important when, and only when, we briefly examine the Higgs branches of these models.

3.3. $\mathcal{N} = 2$ Supersymmetric QCD Revisited

Now we want to solve for the Coulomb branch of a model that is constructed in terms of Type IIA via a configuration of fourbranes, fivebranes, and sixbranes. The only change from section 2 is that to incorporate sixbranes we must replace $Q = \mathbb{R}^3 \times S^1$, in which the $M$ theory fivebrane propagated in section 2, by the multi-Taub-NUT space $\tilde{Q}$ that was just introduced. We write the defining equation of $\tilde{Q}$ as

$$yz = P(v), \quad (3.7)$$

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Figure 4. A specific configuration of $k$ fourbranes (horizontal lines), two fivebranes (vertical lines) and $d$ sixbranes (depicted by the symbol $\otimes$) that gives a representation of $\mathcal{N} = 2$ supersymmetric QCD in four dimensions, with gauge group $SU(k)$ and $d$ hypermultiplet flavors in the fundamental representation.

with $P(v) = \prod_{a=1}^{d} (v - e_a)$. Type IIA fourbranes and fivebranes are described as before by a complex curve $\Sigma$ in $\tilde{Q}$. $\Sigma$ will be described by an equation $F(y, v) = 0$. Note that we can assume that $F$ is independent of $z$, because $z$ could be eliminated via $z = P(v)/y$.

For our first attempt to understand the combined system of fourbranes, fivebranes, and sixbranes, we consider the example in figure 4 of two parallel fivebranes connected by $k$ fourbranes, with $d$ sixbranes between them. We assume that there are no semi-infinite fourbranes extending to the left or right of the figure. This configuration should correspond to $\mathcal{N} = 2$ supersymmetric QCD, that is to an $SU(k)$ gauge theory with $d$ hypermultiplets in the fundamental representation of $SU(k)$.

As in section 2, the fact that there are two fivebranes means that the equation $F(y, v) = 0$, regarded as an equation in $y$ for fixed $v$, has generically two roots. Thus, $F$ is quadratic in $y$ and has the general form

$$A(v)y^2 + B(v)y + C(v) = 0. \quad (3.8)$$

By clearing denominators and dividing by common factors, we can assume that $A, B,$ and $C$ are relatively prime polynomials.

Now we must interpret the statement that there are no semi-infinite fourbranes. This means, as in section 2, that it is impossible for $y$ or $z$ (which correspond roughly to $t$ and $t^{-1}$ in the notation of section 2) to go to infinity at a finite value of $v$. The requirement that $y$ never diverges at finite $v$ means that – if $A, B,$ and $C$ are understood to have no common
factors – \( A(v) \) is a constant, which we can take to equal 1. So the defining equation of \( \Sigma \) reduces to
\[
y^2 + B(v)y + C(v) = 0. \tag{3.9}
\]
Now let us express this in terms of \( z = P(v)/y \). We get
\[
C(v)z^2 + B(v)P(v)z + P(v)^2 = 0. \tag{3.10}
\]
\( z \) will diverge at zeroes of \( C \) unless both \( BP \) and \( P^2 \) are divisible by \( C \). Such divergence would represent the existence of a semi-infinite fourbrane.

In particular, the absence of semi-infinite fourbranes implies that \( P^2 \) is divisible by \( C \). So any zero of \( C \) is a zero of \( P \), that is, it is one of the \( e_a \). Moreover, in the generic case that the \( e_a \) are distinct, each \( e_a \) can appear as a root of \( C \) with multiplicity at most two. Thus, we can label the \( e_a \) in such a way that \( e_a \) is a root of \( C \) with multiplicity 2 for \( a \leq i_0 \), of multiplicity 1 for \( i_0 < a \leq i_1 \), and of multiplicity 0 for \( a > i_1 \). We then have
\[
C = f \prod_{a=1}^{i_0} (v - e_a)^2 \prod_{b=i_0+1}^{i_1} (v - e_b) \tag{3.11}
\]
with some non-zero complex constant \( f \). The requirement that \( BP \) should be divisible by \( C \) now implies that the \( e_a \) of \( a \leq i_0 \) are roots of \( B \), so
\[
B(v) = \tilde{B}(v) \prod_{a \leq i_0} (v - e_a) \tag{3.12}
\]
for some polynomial \( \tilde{B} \).

The equation (3.8) now reduces to
\[
y^2 + \tilde{B}(v) \prod_{a \leq i_0} (v - e_a)y + f \prod_{a \leq i_0} (v - e_a)^2 \prod_{b=i_0+1}^{i_1} (v - e_b) = 0. \tag{3.13}
\]
In terms of \( \tilde{y} = y/ \prod_{a \leq i_0} (v - e_a) \), this is
\[
\tilde{y}^2 + \tilde{B}(v)\tilde{y} + f \prod_{a=i_0+1}^{i_1} (v - e_a) = 0. \tag{3.14}
\]
If \( \tilde{B}(v) \) is a polynomial of degree \( k \), this is (for \( i_1 - i_0 \leq 2k \); otherwise as at the end of section 2 one encounters a new ultraviolet fixed point) the familiar solution of the \( SU(k) \) gauge theory with \( i_1 - i_0 \) flavors in the fundamental representation, written in the same
form in which it appeared in section 2. The \( e_a \) with \( a \leq i_0 \) or \( a > i_1 \) have decoupled from the gauge theory.

This suggests the following interpretation: the sixbranes with \( a \leq i_0 \) are to the left of all fivebranes, the sixbranes with \( i_0 + 1 \leq a \leq i_1 \) are between the two fivebranes, and the sixbranes with \( a > i_1 \) are to the right of all fivebranes. If so then (in the absence of fourbranes ending on the sixbranes) the sixbranes with \( a \leq i_0 \) or \( a > i_1 \) would be decoupled from the four-dimensional gauge theory, and the number of hypermultiplet copies of the fundamental representation of \( SU(k) \) would be \( i_1 - i_0 \), as we have just seen. We will now justify that interpretation.

**Interpretation Of \( i_0 \) and \( i_1 \)**

The manifold \( \tilde{Q} \) defined by \( yz = P(v) \) maps to the complex \( v \) plane, by forgetting \( y \) and \( z \). Let \( Q_v \) be the fiber of this map for a given value of \( v \). For generic \( v \), the fiber is a copy of \( \mathbb{C}^* \). Indeed, whenever \( P(v) \neq 0 \), the fiber \( Q_v \), defined by

\[
yz = P(v),
\]

is a copy of \( \mathbb{C}^* \) (the complex \( y \) plane with \( y = 0 \) deleted). This copy of \( \mathbb{C}^* \) is actually an orbit of the \( \mathbb{C}^* \) action \((\ref{act})\) on \( \tilde{Q} \).

We recall from section 3.2 that if \( z \) or \( y \) is large with the other fixed, then the asymptotic relation between \( z, y \), and \( t = \exp\left(- (x^6 + ix^{10})/R\right) \) is \( y \approx t \) or \( z \approx t^{-1} \). \( t \to 0 \) means large \( x^6 \), which we call “being on the right”; \( t \to \infty \) means \( x^6 \to -\infty \), which we call “being on the left.” Thus \( z \) much larger than \( y \) or vice-versa corresponds to being on the right or on the left in \( x^6 \).

The surface \( \Sigma \) is defined by an equation \( F(y, v) = 0 \) where \( F \) is quadratic in \( y \); it intersects each \( Q_v \) in two points. (\( Q_v \) is not complete, but we have chosen \( F \) so that no root goes to \( y = \infty \) or \( z = \infty \) for \( v \) such that \( P(v) \neq 0 \).) These are the two points with five-branes, for the given value of \( v \).

Now consider the special fibers with \( F(v) = 0 \). This means that for some \( a, v \) is equal to \( e_a \), the position in the \( v \) plane of the \( a \)th sixbrane. The fiber \( F_v \) is for such \( v \) defined by

\[
yz = 0,
\]

and is a union of two components \( C_v \) and \( C'_v \) with, respectively, \( z = 0 \) and \( y = 0 \). The total number of intersection points of \( \Sigma \) with \( F_v \) is still 2, but some intersections lie on \( C_v \) and
some lie on $C'_v$. Without passing through any singularity, we can go to the case that the intersections on $C_v$ are at large $y$ and those on $C'_v$ are at large $z$. Hence, fivebranes that correspond to intersections with $C_v$ are to the left of the $a^{th}$ sixbrane ($y$ is much bigger than $z$ so they are at a smaller value of $x^6$) and fivebranes that correspond to intersections with $C'_v$ are to the right of the $a^{th}$ sixbrane (they are at a larger value of $x^6$).

The intersection points on $C_v$ are the zeroes of (3.13) which as $v \to e_a$ do not go to $y = 0$. The intersection points on $C'_v$ are likewise the zeroes of that polynomial that do vanish as $v \to e_a$. The number of such intersections with $C'_v$ is two if $a \leq i_0$, one if $i_0 + 1 \leq a \leq i_1$, and zero otherwise. This confirms that the number of sixbranes to the left of both fivebranes is $i_0$, the number which are to the left of one and to the right of the other is $i_1 - i_0$, and the number which are to the right of both is $i_1$.

3.4. Generalization

We will now use similar methods to solve for the Coulomb branch of a more general model with $n + 1$ fivebranes, joined in a similar way by fourbranes and with sixbranes between them.

The curve $\Sigma$ will now be defined by the vanishing of a polynomial $F(y, v)$ that is of degree $n + 1$ in $y$:

$$y^{n+1} + A_1(v)y^n + A_2(v)y^{n-1} + \ldots + A_{n+1}(v) = 0. \quad (3.17)$$

The $A_\alpha(v)$ are polynomials in $v$. We assume that there are no semi-infinite fourbranes and therefore have set the coefficient of $y^{n+1}$ to 1. Substituting $y = P(v)/z$, we get

$$A_{n+1}z^{n+1} + A_nPz^n + A_{n-1}P^2z^{n-1} + \ldots + P^{n+1} = 0. \quad (3.18)$$

Hence absence of semi-infinite fourbranes implies that $A_\alpha P^{n+1-\alpha}$ is divisible by $A_{n+1}$ for all $\alpha$ with $0 \leq \alpha \leq n$. (In this assertion we understand $A_0 = 1$.) In particular, $P^{n+1}$ is divisible by $A_{n+1}$.

It follows that all zeroes of $A_{n+1}$ are zeroes of $P$, and occur (if the $e_a$ are distinct) with multiplicity at most $n + 1$. As in the example considered before, zeroes of $P$ that occur as zeroes of $A_{n+1}$ with multiplicity 0 or $n + 1$ make no essential contribution (they correspond to sixbranes that are to the left or the right of everything else and can be omitted). So we will assume that all zeroes of $P$ occur as zeroes of $A_{n+1}$ with some multiplicity between
1 and \(n\). There are therefore integers \(i_0, i_1, \ldots, i_n\) with \(i_0 = 0 \leq i_1 \leq i_2 \leq \ldots \leq i_{n-1} \leq i_n = n\) such that if for \(1 \leq s \leq n\)

\[
J_s = \prod_{a=i_{s-1}+1}^{i_s} (v - e_a)
\]

(3.19) then

\[
A_{n+1} = f \prod_{s=1}^{n} J_s^{n+1-s}
\]

(3.20)

with \(f\) a constant. By an argument along the lines given at the end of section 3.3, we can interpret \(i_\alpha\) as the number of sixbranes to the left of the \(\alpha\)th fivebrane. So \(d_\alpha = i_\alpha - i_{\alpha-1}\) is the number of sixbranes between the \(\alpha - 1\)th and \(\alpha\)th fivebranes. The number of hypermultiplets in the fundamental representation of the \(\alpha\)th factor of the gauge group will hence be \(d_\alpha\).

The requirement that \(A_\alpha P^{n+1-\alpha}\) is divisible by \(A_{n+1}\) is then equivalent to the statement

\[
A_\alpha = g_\alpha(v) \prod_{s=1}^{\alpha-1} J_s^{\alpha-s}
\]

(3.21)

with some polynomial \(g_\alpha(v)\). We interpret \(g_\alpha(v)\) as containing the order parameters for the \(\alpha\)th factor of the gauge group. So if \(g_\alpha(v)\) is of degree \(k_\alpha\), then the gauge group is

\[
G = \prod_{\alpha=1}^{n} SU(k_\alpha).
\]

(3.22)

The hypermultiplet spectrum consists of the usual \((k_\alpha, \bar{k}_\alpha+1)\) representations plus \(d_\alpha\) copies of the fundamental representation of \(SU(k_\alpha)\).

The curve describing the solution of this theory should thus be

\[
y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} + \ldots + g_\alpha(v) \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} \cdot y^{n+1-\alpha} + \ldots + f \prod_{s=1}^{n} J_s^{n+1-s} = 0.
\]

(3.23)

This of course reduces in the absence of sixbranes to the solution found in (2.41); it likewise gives back the standard solution of \(\mathcal{N} = 2\) supersymmetric QCD when there are precisely two fivebranes. As a further check, let us examine the condition on the \(d_\alpha\) and the \(k_\alpha\) under which the beta function vanishes. Note that the coefficient of \(y^n\) is of degree \(v^{k_1}\). All fivebranes will be parallel at large \(v\), and the beta function will vanish, if

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the coefficient of $y^{n+1-m}$ is of order $v^{mk_1}$ for $m = 1, \ldots, n + 1$. Those conditions can be evaluated to give $k_2 + d_1 = 2k_1$, $k_3 + k_1 + d_2 = 2k_2$, and so on – the standard conditions for vanishing beta function of the gauge theory.

In this case of vanishing beta function, let the polynomials $g_\alpha(v)$ be of the form $g_\alpha(v) = h_\alpha v^{k_\alpha} + O(v^{k_\alpha-1})$. Then the asymptotic behavior of the roots of (3.23) (regarded as an equation for $y$) is $y \sim \lambda_i v^{k_1}$, where the $\lambda_i$ are the roots of the polynomial equation

$$x^{n+1} + h_1 x^n + h_2 x^{n-1} + \ldots + h_n x + f = 0. \quad (3.24)$$

On the $x$ plane, there are $n + 3$ distinguished points, namely 0, $\infty$, and the $\lambda_i$. The $\lambda_i$ are of course defined only up to permutation and (as one could rescale $y$ and $x$) up to multiplication by a common complex scalar. A choice of the $\lambda_i$, modulo those equivalences, determines the asymptotic distances between fivebranes and hence the bare gauge coupling constants. The same choice also determines a point in the moduli space $M_{0,n+3;2}$ that was introduced in section 3.1. In any description by a Lagrangian field theory with coupling parameters $\tau_i$, the fundamental group $\pi_1(M_{0,n+3;2})$ would be interpreted as the group of discrete duality symmetries.

### 3.5. Higgs Branches

In this subsection, we will sketch how the transition to a Higgs branch (or a mixed Higgs-Coulomb branch) can be described from the present point of view.

![Figure 5. A configuration representing a mixed Coulomb-Higgs branch. Here as before fivebranes are shown as vertical solid lines and fourbranes as horizontal solid lines. But in contrast to figures (3) and (4), sixbranes are depicted (as in [1]) as vertical dashed lines. This makes it easier to visualize the hypermultiplet moduli of fourbranes that end on parallel sixbranes. Such a modulus appears whenever there is a fourbrane suspended between two sixbranes as in this example.](attachment:image.png)
We recall that the transition to a Higgs branch is a process in which the genus of \( \Sigma \) drops by one (or more) and a transition is made to a new branch of vacua in which there are massless hypermultiplets. In terms of Type IIA brane diagrams, massless hypermultiplets result (as in [1]) from fourbranes suspended between fivebranes, a configuration shown in figure 5.

For a transition to a Higgs branch to occur, it is necessary for two hypermultiplet bare masses to become equal. From the present point of view, this means that that the positions of two sixbranes in \( v \) become equal. It is not necessary for the two sixbranes to have equal positions in \( x^6 \). In fact, the semiclassical brane diagram of figure 5 cannot be drawn if the \( x^6 \) values of the sixbranes are equal.

The hypermultiplet bare masses are the roots of \( P(v) = \prod_a (v - e_a) \). We therefore want to consider the case that two \( e_a \) are coincident at, say, the origin. The other \( e_a \) will play no material role, and we may as well take the case of only two sixbranes. So we take \( P(v) = v^2 \). The equation \( yz = P(v) \) is in this case

\[
x y = v^2; \tag{3.25}
\]

and describes a manifold \( Q_0 \) which has a singularity at the point \( P \) with coordinates \( x = y = z = 0 \).

We recall, however, from the discussion in section 3.2 that in case two sixbranes coincide in \( v \) but not in \( x^6 \), such a singularity should be blown up. Thus, the multi-Taub-NUT manifold \( \tilde{Q} \) does not coincide with \( Q_0 \), but is a smooth surface obtained by blowing up the singularity in \( Q_0 \). In the blow-up, \( P \) is replaced by a smooth curve \( C \) of genus zero.

Now we consider a curve \( \Sigma \) in \( \tilde{Q} \) (or \( Q_0 \)) representing a point on the Coulomb branch of one of the models considered in this section. Let \( g \) be the generic genus of \( \Sigma \). Nothing essential will be lost if we consider the case of supersymmetric QCD – two fivebranes; gauge group \( SU(n) \). So \( \Sigma \) is defined by a curve of the form

\[
y^2 + By + fv^2 = 0. \tag{3.26}
\]

Nothing of interest will happen unless \( \Sigma \) passes through the singular point \( y = z = v = 0 \). That is so if and only if \( B \) vanishes at \( v = 0 \) (if \( B \) is non-zero at \( v = 0 \) then either \( y \) is non-zero for \( v \to 0 \), or \( y \sim v^2 \) for \( v \to 0 \) and \( z \) is non-vanishing at \( v = 0 \)), so generically \( B = bv + O(v^2) \) with a non-zero constant \( b \).
So near $P$, $\Sigma$ looks like
\[ y^2 + bvy + fv^2 = 0. \]  
(3.27)

This curve has a singularity at $y = v = 0$. In fact, the quadratic polynomial $y^2 + bvy + fv^2$ has a factorization as $(y + \gamma v)(y + \gamma' v)$. Generically, the two factors correspond, near $P$, to two branches of $\Sigma$ that meet “transversely” at $P$, giving the singularity. The genus of $\Sigma$ drops by one when this singularity appears. So $\Sigma$ now has genus $g - 1$.

We actually want to consider the case in which the two sixbranes are not coincident in $x^6$, so we must consider the curve defined by (3.26) not in the singular manifold $Q_0$ but in its smooth resolution $\tilde{Q}$. This curve has two components. One is a smooth curve $\Sigma'$ of genus $g - 1$ and the other is a copy of the genus zero curve $C$ in $\tilde{Q}$ that is obtained by the blowup of $P$. $\Sigma'$ is smooth (generically) because after the blowup the two branches $y + \gamma v = 0$ and $y + \gamma' v = 0$ of $\Sigma$ no longer meet. A copy of $C$ is present because the polynomial $y^2 + By + v^2$ vanishes on $P$ and hence (when pulled back to $\tilde{Q}$) on $C$.

At this point, by adding a constant to $B$, we could deform the two-component curve $\Sigma' + C$ (which is singular where $\Sigma'$ and $C$ meet) back to a smooth irreducible curve of genus $g$ that does not pass through $P$ or $C$. Instead, we want to make the transition to the Higgs branch.

We recall that in the present paper, the curve $\Sigma$ is really an ingredient in the description of a fivebrane in eleven dimensions. The fivebrane propagates in $R^7 \times \tilde{Q}$. $R^7$ has coordinates $x^0, x^1, \ldots, x^7$ and $x^8, x^9$. The fivebrane world-volume is of the form $R^4 \times \Sigma$, where $\Sigma$ is a curve in $\tilde{Q}$ and $R^4$ is a subspace of $R^7$ defined by (for instance) $x^7 = x^8 = x^9 = 0$.

The transition to the Higgs branch can be described as follows. When $\Sigma$ degenerates to a curve that is a union of two branches $\Sigma'$ and $C$, the fivebrane degenerates to two branches $R^4 \times \Sigma'$ and $R^4 \times C$. At this point, it is possible for the two branches to move independently in $R^7$. $R^4 \times C$ can move to $\tilde{R}^4 \times C$, where $\tilde{R}^4$ is a different copy of $R^4$ embedded in $R^7$. For unbroken supersymmetry, $\tilde{R}^4$ should be parallel to $R^4$, so it is defined in $R^7$ by $(x^7, x^8, x^9) = \bar{w}$ for some constant $\bar{w}$.

The four-dimensional field theory derived from a fivebrane on $\tilde{R}^4 \times C$ has no massless vector multiplets, as $C$ has genus zero. It has one massless hypermultiplet, whose components are $\bar{w}$ and $\int_C \beta$, where $\beta$ is the chiral two-form on the fivebrane worldvolume.

A motion of $R^4 \times \Sigma'$ in the $x^7, x^8, x^9$ directions, analogous to the above, is not natural because $\Sigma'$ is non-compact and such a motion would entail infinite action per unit volume.
on $\mathbb{R}^4$. The allowed motions of $\mathbb{R}^4 \times \Sigma'$ are the motions of $\Sigma'$ in $\tilde{Q}$ that determine the order parameters on the Coulomb branch and that we have been studying throughout this paper. The four-dimensional field theory derived from a fivebrane on $\mathbb{R}^4 \times \Sigma'$ has $g - 1$ massless vector multiplets, because $\Sigma'$ is a curve of genus $g - 1$, and one hypermultiplet. The combined system of fivebranes on $\mathbb{R}^4 \times \Sigma'$ and on $\tilde{\mathbb{R}}^4 \times C$ has $g - 1$ massless vector multiplets and one hypermultiplet.

There is no way to deform $\Sigma'$ to a curve of genus $g$. It is only $\Sigma' + C$ that can be so deformed. So once $C$ has moved to $\vec{w} = 0$, there is no way to regain the $g^{th}$ massless vector multiplet except by first moving $C$ back to $\vec{w} = 0$. The transition to the Higgs branch has been made.

3.6. Metric And Complex Structure

Finally, using the techniques of [24], we will briefly describe how to exhibit the complex structure (3.5) of the ALF manifold (3.3). In that paper, the formula (3.3) for the ALF hyper-Kahler metric is obtained in the following way.

Let $H$ be a copy of $\mathbb{R}^4$ with the flat hyper-Kahler metric. Let $M = H^d \times H$, with coordinates $q_a$, $a = 1, \ldots, d$, and $w$. Consider the action on $M$ of an abelian group $G$, locally isomorphic to $\mathbb{R}^m$, for which the hyper-Kahler moment map is

$$\mu_a = \frac{1}{2} r_a + y,$$

(3.28)

where $r = q_a q_a^*$ and $y = (w - \overline{w})/2$. Notation is as explained in [24]. $G$ is a product of $d$ factors; the $a^{th}$ factor, for $a = 1, \ldots, d$, acts on $q_a$ by a one-parameter group of rotations that preserve the hyper-Kahler metric, on $w$ by translations, and trivially on the other variables. The manifold defined as $\mu^{-1}(e)/G$, with an arbitrary constant $e$, carries a natural hyper-Kahler metric, which is shown in [24] to coincide with (3.3). The choice of $e$ determines the positions $\vec{x}_a$ of the sixbranes in (3.3).

To exhibit the structure of this hyper-Kahler manifold as a complex manifold, one may proceed as follows. In any one of its complex structures, $H$ can be identified as $\mathbb{C}^2$. One can pick coordinates so that each $q_a$ consists of a pair of complex variables $y_a, z_a$, and $w$ consists of a pair $v, v'$, such that the action of $G$ is

$$y_a \rightarrow e^{i\theta_a} y_a$$
$$z_a \rightarrow e^{-i\theta_a} z_a$$
$$v \rightarrow v$$
$$v' \rightarrow v' - \sum_{a=1}^d \theta_a$$

(3.29)
where the $\theta_a$ are real parameters.

Once a complex structure is picked, the moment map $\mu$ breaks up as a complex moment map $\mu_C$ and a real moment map $\mu_R$. A convenient way to exhibit the complex structure of the ALF manifold is the following. Instead of setting $\mu$ to a constant value and dividing by $G$, one can set $\mu_C$ to a constant value and divide by $G_C$, the complexification of $G$ (whose action is given by the formulas (3.29) with the $\theta_a$ now complex-valued). The advantage of this procedure is that the complex structure is manifest.

The components of $\mu_C$ are

$$\mu_{C,a} = y_a z_a - v. \quad (3.30)$$

Setting the $\mu_{C,a}$ to constants, which we will call $-e_a$, means therefore taking

$$y_a z_a = v - e_a. \quad (3.31)$$

Dividing by $G_C$ is accomplished most simply by working with the $G_C$-invariant functions of $y_a, z_a, v$, and $v'$. In other words, the $G_C$-invariants can be regarded as functions on the quotient $\tilde{Q} = \mu_C^{-1}(-e_a)/G_C$.

The basic invariants are $y = e^{iv'} \prod_{a=1}^d y_a$, $z = e^{-iv'} \prod_{a=1}^d z_a$, and $v$. The relation that they obey is, in view of (3.31),

$$yz = \prod_{a=1}^d (v - e_a), \quad (3.32)$$

which is the formula by which we have defined the complex manifold $\tilde{Q}$. This exhibits the complex structure of the ALF manifold, for generic sixbrane positions $e_a$.

---

6 The quotient should be taken in the sense of geometric invariant theory. This leads to the fact, exploited in section 3.5, that when two sixbranes coincide in $v$ but not in $x^6$, the ALF manifold (3.3) is equivalent as a complex manifold not to $yz = \prod_a (v - e_a)$ but to the smooth resolution $\tilde{Q}$ of that singular surface. We will treat the invariant theory in a simplified way which misses the precise behavior for $e_a = e_b$. The calculation we do presently with invariants really proves not that the ALF manifold is isomorphic to $\tilde{Q}$, but only that it has a holomorphic and generically one-to-one map to $\tilde{Q}$. When $\tilde{Q}$ is smooth (as it is for generic $e_a$), the additional fact that the ALF manifold is hyper-Kahler implies that it must coincide with $\tilde{Q}$.
4. Elliptic Models

4.1. Description Of The Models

In this section we compactify the $x^6$ direction to a circle, of radius $L$, and consider a chain of $n$ fivebranes arranged around this circle, as in figure 6. Let $k_\alpha$ be the number of fourbranes stretching between the $\alpha - 1$th and $\alpha$th fivebrane, and let $d_\alpha$ be the number of sixbranes localized at points between the $\alpha - 1$th and $\alpha$th fivebrane. The beta function of the $SU(k_\alpha)$ factor in the gauge group is then

$$b_{0,\alpha} = -2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha.$$  \hspace{1cm} (4.1)

Since $\sum_\alpha b_{0,\alpha} = \sum_\alpha d_\alpha$, and the $d_\alpha$ are all non-negative, the only case in which all beta functions are zero or negative is that case that all $b_{0,\alpha} = d_\alpha = 0$. Then writing $0 = \sum_\alpha k_\alpha(-2k_\alpha + k_{\alpha-1} + k_{\alpha+1}) = -\sum_\alpha (k_\alpha - k_{\alpha-1})^2$, we see that this occurs if and only if all $k_\alpha$ are equal to a fixed integer $k$. The present section will be devoted to analyzing this case.

The gauge group is $G = U(1) \times SU(k)^n$. Only the occurrence of a $U(1)$ factor requires special comment. The condition \textsuperscript{7} “freezes out” the difference between the $U(1)$ factors.

\textsuperscript{7} In the context of three-dimensional models with $N = 4$ supersymmetry, configurations of fivebranes arranged around a circle were studied in $\textsuperscript{3}$.
in the gauge group supported on alternate sides of any given fivebrane. In sections 2 and 3, we considered a finite chain of fivebranes with $U(1)$’s potentially supported only in the “interior” of the chain, and this condition sufficed to eliminate all $U(1)$’s. In the present case of $n$ fivebranes arranged around a circle with fourbranes connecting each neighboring pair, (2.6) eliminates $n - 1$ of the $U(1)$’s, leaving a single (diagonal) $U(1)$ factor in the gauge group.

Hypermultiplets arise from fourbranes that meet a single fivebrane at the same point in space from opposite sides. If the symbol $k_\alpha$ represents the fundamental representation of the $\alpha^{th}$ $SU(k)$ factor in $G$, then the hypermultiplets transform as $\oplus_{\alpha=1}^{n} k_\alpha \otimes \overline{k}_{\alpha+1}$. Note that all of these hypermultiplets are neutral under the $U(1)$, so that all beta functions vanish including that of the $U(1)$. The $U(1)$, while present, is thus completely decoupled in the model. The curve $\Sigma$ that we will eventually construct will have the property that its Jacobian determines the coupling constant of the $U(1)$ factor as well as the structure of the $SU(k)^n$ Coulomb branch.

A special case that merits some special discussion is the case $n = 1$. In that case the gauge group consists just of a single $SU(k)$ (times the decoupled $U(1)$) and the $k \otimes \overline{k}$ hypermultiplet consists of a copy of the adjoint representation of $SU(k)$ plus a neutral singlet. This in fact corresponds to the $\mathcal{N} = 4$ theory with gauge group $U(k)$; however, we will study it eventually in the presence of a hypermultiplet bare mass that breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$. Precisely this model has been solved in [12], and we will recover the description in that paper.

**Hypermultiplet Bare Masses**

Before turning to $M$ theory, we will analyze, in terms of Type IIA, the hypermultiplet bare masses.

Let $a_{i,\alpha}$, $i = 1, \ldots, k$ be the $v$ values of the fourbranes between the $\alpha - 1^{th}$ and $\alpha^{th}$ fivebranes. According to (2.3), the bare mass $m_\alpha$ of the $k_\alpha \otimes \overline{k}_{\alpha+1}$ hypermultiplet is

$$m_\alpha = \frac{1}{k} \left( \sum_i a_{i,\alpha} - \sum_j a_{j,\alpha+1} \right). \quad (4.2)$$

This formula seems to imply that the $m_\alpha$ are not all independent, but are restricted by $\sum_\alpha m_\alpha = 0$. However, that restriction can be avoided if one choses correctly the spacetime in which the branes propagate.
So far, we have described the positions of the fourbranes and fivebranes in terms of \( x^6 \) and \( v = x^4 + ix^5 \). Since we are now compactifying the \( x^6 \) direction to a circle, this part of the spacetime is so far \( T = S^1 \times C \), where \( S^1 \) is the circle parametrized by \( x^6 \) and \( C \) is the \( v \) plane.

We can however replace \( S^1 \times C \) by a certain \( C \) bundle over \( S^1 \). In other words, we begin with \( x^6 \) and \( v \) regarded as coordinates on \( R^3 = R \times C \), and instead of dividing simply by \( x^6 \rightarrow x^6 + 2\pi L \) for some \( L \), we divide by the combined operation

\[
\begin{align*}
x^6 & \rightarrow x^6 + 2\pi L \\
v & \rightarrow v + m,
\end{align*}
\] (4.3)

for an arbitrary complex constant \( m \). Starting with the flat metric on \( R^3 \), this gives a \( C \) bundle over \( S^1 \) with a flat metric; we call this space \( T_m \). Now when one goes all the way around the \( x^6 \) circle, one comes back with a shifted value of \( v \), as suggested in figure 6(b). The result is that the formula \( \sum_{\alpha} m_{\alpha} = 0 \) which one would get on \( R \times C \) is replaced on \( T_m \) by

\[
\sum_{\alpha} m_{\alpha} = m. \tag{4.4}
\]

Thus arbitrary hypermultiplet bare masses are possible, with a judicious choice of the spacetime.

4.2. Interpretation In M Theory

Now we want to study these models via M theory.

Going to \( M \) theory means first of all including another circle, parametrized by a variable \( x^{10} \) with \( x^{10} \cong x^{10} + 2\pi R \). Now because in the present section we are compactifying also the \( x^6 \) direction to a circle, we have really two circles. The metric structure, however, need not be a simple product \( S^1 \times S^1 \). Dividing \( x^6 \rightarrow x^6 + 2\pi L \) can be accompanied by a shift of \( x^{10} \), the combined operation being

\[
\begin{align*}
x^6 & \rightarrow x^6 + 2\pi L \\
x^{10} & \rightarrow x^{10} + \theta R
\end{align*}
\] (4.5)

with some angle \( \theta \). We also still divide by \( x^{10} \rightarrow x^{10} + 2\pi R \), as in uncompactified Type IIA. In the familiar complex structure in which \( s = x^6 + ix^{10} \) is holomorphic, the quotient of the \( s \) plane by these equivalences is a complex Riemann surface \( E \) of genus one which – by varying \( L \) and \( \theta \) for fixed \( R \) (that is fixed ten-dimensional Type IIA string coupling
constant) – can have an arbitrary complex structure. \( E \) also has a flat metric with an
area that (if we let \( R \) vary) is arbitrary; this, however, will be less important, since we are
mainly studying properties that are controlled by the holomorphic data.

The interpretation of this generalization for our problem of gauge theory on branes is
as follows. The \( \alpha \)th fivebrane has, in the \( M \) theory description, a position \( x_{10}^{\alpha} \) in the \( x^{10} \)
direction, as well as a position \( x_{6}^{\alpha} \) in the \( x^{6} \) direction. The theta angle \( \theta_{\alpha} \) of the \( \alpha \)th \( SU(k) \)
factor in the gauge group is

\[
\theta_{\alpha} = \frac{x_{10}^{\alpha} - x_{10}^{\alpha - 1}}{R}.
\]  

(4.6)

If metrically \( x^{6} - x^{10} \) space were a product \( S^1 \times S^1 \) (or in other words if \( \theta = 0 \) in (4.5))
then (4.6) would imply that \( \sum_{\alpha} \theta_{\alpha} = 0 \). Instead, via (4.5), we arrange that when one goes
around a circle in the \( x^{6} \) direction, one comes back with a shifted valued of \( x^{10} \); as a result
one has

\[
\sum_{\alpha} \theta_{\alpha} = \theta. 
\]

(4.7)

In a Type IIA description, one would not see the \( x^{10} \) coordinate. The fact that \( x^{10} \)
shifts by \( \theta \) under \( x^{6} \rightarrow x^{6} + 2\pi L \) would be expressed by saying that the holonomy around
the \( x^{6} \) circle of the Ramond-Ramond \( U(1) \) gauge field of Type IIA is \( e^{i\theta} \). The \( x^{10} \) positions
of a fivebrane would be coded in the value of a certain scalar field that propagates on the
fivebrane.

\section*{Duality Group}

In general, \( E \) is a (smooth) genus one Riemann surface with an arbitrary complex
structure, and the fivebranes are at \( n \) arbitrary points \( p_1, \ldots, p_n \) on \( E \). By varying in an
arbitrary fashion the complex structure of \( E \) and the choice of the \( p_{\sigma} \), the bare couplings
and theta angles of \( G' = \prod_{\alpha=1}^{k} SU(k) \) can be varied in arbitrarily. (The coupling and
theta angle of the \( U(1) \) factor in the full gauge group \( G = U(1) \times G' \) is then determined
in terms of those.) The duality group of these models can thus be described as follows.
Let \( \mathcal{M}_{1,n} \) be the moduli space of smooth Riemann surfaces of genus one with \( n \) distinct,
unordered marked points. The duality group is then \( \pi_1(\mathcal{M}_{1,n}) \). For \( n = 1 \), \( \pi_1(\mathcal{M}_{1,1}) \) is the
same as \( SL(2, \mathbb{Z}) \), and this becomes the usual duality group of \( \mathcal{N} = 4 \) super Yang-Mills
theory. For \( n > 1 \), \( \pi_1(\mathcal{M}_{1,n}) \) is a sort of hybrid of \( SL(2, \mathbb{Z}) \) and the duality group found
in section 3.

\section*{Incorporation Of v}
We now want to consider also the position of the fivebranes in \( v = x^4 + ix^5 \). An important special case is that in which the fivebranes propagate in \( X = E \times \mathbb{C} \), where \( \mathbb{C} \) is the complex \( v \) plane. However, from the discussion of (4.3), it is clear that in general we should consider not a product \( E \times \mathbb{C} \) but a \( \mathbb{C} \) bundle over \( E \). In general, we start with \( \mathbb{R} \times S^1 \times \mathbb{C} \) (with respective coordinates \( x^6, x^{10}, \) and \( v \)) and divide by the combined symmetry
\[
\begin{align*}
x^6 &\to x^6 + 2\pi L \\
x^{10} &\to x^{10} + \theta \\
v &\to v + m.
\end{align*}
\] (4.8)

The quotient is a complex manifold that we will call \( X_m \); it can be regarded as a \( \mathbb{C} \) bundle over \( E \). From the discussion at the Type IIA level, it is clear that the parameter \( m \) must be identified with the sum of the hypermultiplet bare masses.

The complex manifold \( X_m \) will actually not enter as an abstract complex manifold; the map \( X_m \to E \) (by forgetting \( \mathbb{C} \)) will be an important part of the structure. As a \( \mathbb{C} \) bundle over \( E \), \( X_m \) is an “affine bundle”; this means that the ﬁbers are all copies of \( \mathbb{C} \) but there is no way to globally deﬁne an “origin” in \( \mathbb{C} \), in a fashion that varies holomorphically. Such affine bundles over \( E \), with the associated complex line bundle (in which one ignores shifts of the ﬁbers) being trivial, are classiﬁed by the sheaf cohomology group \( H^1(E, \mathcal{O}_E) \), which is one-dimensional; the one complex parameter that enters is what we have called \( m \). If \( X_m \) is viewed just as a complex manifold with map to \( E \), \( m \) could be set to 1 (given that it is non-zero) by rescaling \( v \), but we prefer not to do that since the ﬁvebrane effective action is not invariant under rescaling of \( v \).

The complex manifold \( X_m \) appeared in [12], where the \( SU(k) \) theory with massive adjoint hypermultiplet – in other words, the \( n = 1 \) case of the series of models considered here – was described in terms of an appropriate curve in \( X_m \), rather as we will do below. Actually, in what follows we will consider curves in \( X_m \) that “go to infinity” at certain points, corresponding to the positions of fivebranes. In [12], a “twist” of \( X_m \) was made to keep the curve from going to infinity.

\footnote{As such it is isomorphic to \( \mathbb{C}^* \times \mathbb{C}^* \).}
4.3. Solution Of The Models

What remains is to describe the solution of the models. First we consider the special case that the sum of the hypermultiplet bare masses is zero,

$$
\sum_{\alpha} m_{\alpha} = 0,
$$

(4.9)

so that the model will be described by a curve $$\Sigma$$ in $$X = E \times \mathbb{C}$$. There are $$n$$ fivebranes at points $$p_1, p_2, \ldots, p_n$$ in $$E$$; and to use a classical Type IIA language (which we will presently reformulate in a way more suitable in M theory) each pair of adjacent five-branes is connected by $$k$$ fourbranes.

First of all, the elliptic curve $$E$$ can be described by a Weierstrass equation,

$$
zy^2 = 4x^3 - g_2xz^2 - g_3z^3
$$

in homogeneous coordinates $$x, y, z$$; $$g_2$$ and $$g_3$$ are complex constants. Usually we work in coordinates with $$z = 1$$ and write simply

$$
y^2 = 4x^3 - g_2x - g_3.
$$

(4.10)

$$E$$ admits an everywhere non-zero holomorphic differential

$$\omega = \frac{dx}{y}.
$$

(4.11)

To incorporate the classical idea that there are $$k$$ fourbranes between each pair of fivebranes, we proceed as follows. $$X$$ maps to $$E$$ by forgetting $$\mathbb{C}$$; under this map, the curve $$\Sigma \subset X$$ maps to $$E$$. Via the map $$\Sigma \rightarrow E$$, $$\Sigma$$ can be interpreted as a $$k$$-fold cover of $$E$$, the $$k$$ branches being the positions of the fourbranes in $$\mathbb{C}$$. In other words, $$\Sigma$$ is defined by an equation $$F(x, y, v) = 0$$, where $$F$$ is of degree $$k$$ in $$v$$:

$$F(x, y, v) = v^k - f_1(x, y)v^{k-1} + f_2(x, y)v^{n-2} \mp \ldots + (-1)^k f_k(x, y).
$$

(4.12)

The functions $$f_i(x, y)$$ are meromorphic functions on $$E$$ (and hence are rational functions of $$x$$ and $$y$$) obeying certain additional conditions that will be described.

The idea here is that for generic $$x$$ and $$y$$, the equation $$F(x, y, v)$$ has $$k$$ roots for $$v$$, which are the positions of the fourbranes in the $$v$$ plane. Call those roots $$v_i(x, y)$$. Unless the $$f_i$$ are all constants, there will be points on $$E$$ at which some of the $$f_i$$ have poles. At such a point, at least one of the $$v_i(x, y)$$ diverges.

We would like to interpret the poles in terms of positions of fivebranes. Let us first explain why such an interpretation exists. An $$M$$ theory fivebrane located at $$v = v_0$$ would
be interpreted in Type IIA as a fourbrane at \( v = v_0 \). A Type IIA fivebrane located at some point \( p \in E \) also corresponds to a fivebrane in Type IIA. The equation for such a fivebrane is, say, \( s = s_0 \) where \( s \) is a local coordinate on \( E \) near \( p \) and \( s = s_0 \) at \( p \). The combined Type IIA fourbrane-fivebrane system can be described in \( M \) theory by a fivebrane with the world-volume

\[
(v - v_0)(s - s_0) = 0. \tag{4.13}
\]

The space of solutions of this equation has two branches, \( v = v_0 \) and \( s = s_0 \); these are interpreted in Type IIA as the fourbrane and fivebrane, respectively. There is a singularity where the two branches meet. Now without changing the asymptotic behavior of the curve described in (4.13) – in fact, while changing only the microscopic details – one could add a constant to the equation, getting

\[
(v - v_0)(s - s_0) = \epsilon. \tag{4.14}
\]

The singularity has disappeared; what in Type IIA is a fourbrane and a fivebrane appears in this description as a single, smooth, irreducible object. On the other hand, if we solve (4.14) for \( v \) we get

\[
v = v_0 + \frac{\epsilon}{s - s_0}. \tag{4.15}
\]

We see that a fivebrane corresponds to a simple (first order) pole in \( v \).

Poles of the \( f_i \) will lead to singularities of the \( v_i \). It is now possible to determine what kind of singularities we should allow in the \( f_i \). At a point \( p_\sigma \) at which a fivebrane is located, one of the \( v_i \) should have a simple pole, analogous to that in (4.15), and the others should be regular. The \( v_i \) will behave in this way if and only if the \( f_i \) have simple poles at \( p_\sigma \). So the functions \( f_1, \ldots, f_k \) have simple poles at the points \( p_1, \ldots, p_n \) and no other singularities.

This then almost completes the description of the solution of the models: they are described by curves \( F(x, y, v) = 0 \) in \( E \times \mathbb{C} \), where \( F \) is as in (4.12) and the allowed functions \( f_i \) are characterized by the property just stated. What remains is to determine which parameters in the \( f_i \) are hypermultiplet bare masses and which ones are order parameters describing the choice of a quantum vacuum.

First let us count all parameters. By the Riemann-Roch theorem, the space of meromorphic functions on \( E \) with simple poles allowed at \( p_1, \ldots, p_n \) is \( n \)-dimensional. As we have \( k \) such functions, there are \( kn \) parameters in all. Of these, \( n - 1 \) should be hypermultiplet bare masses (because of (4.3) there are only \( n - 1 \) hypermultiplet bare masses),
leaving \( n(k - 1) + 1 \) order parameters. The gauge group \( G = U(1) \times SU(k)^n \) has rank \( n(k - 1) + 1 \), so \( n(k - 1) + 1 \) is the dimension of the Coulomb branch, and hence is the correct number of order parameters. It remains then to determine which \( n - 1 \) parameters are the hypermultiplet bare masses.

Let us note the following interpretation of the function \( f_1 \): in view of the factorization

\[
F(x, y, v) = \prod_{i=1}^{k} (v - v_i(x, y)),
\]

one has

\[
f_1(x, y) = \sum_{i=1}^{k} v_i(x, y).
\]

The generic behavior is that near any one of the \( p_\sigma \), all of the \( v_i \) except one remain finite, and the remaining one, say \( v_1(x, y) \), has a simple pole. So according to (4.16) the singular behavior of \( v_1 \) is the same as the singular behavior of \( f_1 \). In other words, the singular part of \( f_1 \) determines the behavior of \( \Sigma \) near infinity. Since hypermultiplet bare masses are always coded in the behavior of the curve \( \Sigma \) at infinity – as we saw in (2.5), that is why the bare masses are constant – the hypermultiplet bare masses must be coded in the singular part of \( f_1 \).

The singular part of \( f_1 \) depends only on \( n - 1 \) complex parameters. In fact, \( f_1 \) itself depends on \( n \) complex parameters, but as one is free to add a constant to \( f_1 \) without affecting its singular behavior, the singular part of \( f_1 \) depends on \( n - 1 \) parameters. Thus, fixing the hypermultiplet bare masses completely fixes the singular part of \( f_1 \). The additive constant in \( f_1 \) and the parameters in \( f_j, j > 1 \) are the order parameters specifying a choice of quantum vacuum. Actually, the additive constant in \( f_1 \) is the order parameter on the Coulomb branch of the \( U(1) \) factor in the gauge group; this constant can be shifted by adding a constant to \( v \) and so does not affect the Jacobian of \( \Sigma \), in agreement with the fact that the \( U(1) \) is decoupled. The order parameters of the \( SU(k)^n \) theory are the \( n(k - 1) \) coefficients in \( f_2, f_3, \ldots, f_n \).

To be more complete, one would like to know which functions of the singular part of \( f_1 \) are the hypermultiplet bare masses \( m_\alpha \). One approach to this question is to think about the integrable system that controls the structure of the Coulomb branch. We recall from section 2.3 that a point in the phase space of this integrable system is given by the choice of a curve \( \Sigma \subset E \times \mathbb{C} \) with fixed behavior at infinity together with the choice of a line bundle on the compactification of \( \Sigma \). As in section 17 of the second paper in [1], the cohomology class of the complex symplectic form on the phase space should vary linearly with the masses. How to implement this condition for integrable systems of the kind considered
here is explained in section 2 of [12]. The result is as follows: the hypermultiplet bare masses are the residues of the differential form $\beta = f_1(x, y)\omega$. Since the sum of the residues of a meromorphic differential form vanishes, this claim is in accord with (4.9).

4.4. Extension To Arbitrary Masses

What remains is to eliminate the restriction (4.9) and solve the models with arbitrary hypermultiplet bare masses. For this, as we have discussed in section 4.2, it is necessary to consider curves $\Sigma$ not in $X = E \times \mathbb{C}$, but in an affine bundle over $E$ that we have called $X_m$.

$X_m$ differs from the trivial product bundle $X = E \times \mathbb{C} \to E$ by twisting by an element of $H^1(E, \mathcal{O}_E)$. That cohomology group vanishes if a point is deleted from $E$. We can pick that point to be the point $p_\infty$ with $x = y = \infty$ in the Weierstrass model (4.10). To preserve the symmetry among the points $p_\sigma$ at which there are fivebranes, we take $p_\infty$ to be distinct from all of the $p_\sigma$. Because $X_m$ coincides with $X$ away from the fiber over $p_\infty$, we can describe the curve $\Sigma$ away from $p_\infty$ by the same equation as before, $F(x, y, v) = 0$ with

$$F(x, y, v) = v^k - f_1(x, y)v^{k-1} + f_2(x, y)v^{k-2} \mp \ldots \mp (-1)^k f_k(x, y). \quad (4.17)$$

Away from $x = y = \infty$, the functions $f_i(x, y)$ are subject to the same conditions as before – no singularities except simple poles at the points $p_\sigma$.

Previously, we required that the roots $v_i(x, y)$ were finite at $x = y = \infty$ (since there are no fivebranes there) and hence that the $f_i$ were finite at $x = y = \infty$. For describing a curve on $X_m$, that is not the right condition. The trivialization of the affine bundle $X_m$ over $E$ minus the point at infinity breaks down at $x = y = \infty$. A good coordinate near infinity is not $v$ but

$$\tilde{v} = v + \left(\frac{m}{2k}\right) \frac{y}{x}. \quad (4.18)$$

(Instead of $y/x$ one could use any other function with a simple pole at $x = y = \infty$. For the moment one should think of the $m/2k$ on the right hand side (4.18) as an arbitrary constant.) It is not $v$ but $\tilde{v}$ that should be finite at $x = y = \infty$.

Thus the restrictions on the $f_i$ that are needed to solve the model with arbitrary hypermultiplet bare masses can be stated as follows:

(1) The functions $f_i(x, y)$ are meromorphic functions on $E$ with no singularities except simple poles at the $p_\sigma$, $\sigma = 1, \ldots, n$, and poles (of order $i$) at $x = y = \infty$. 

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The singular part of the function $F(v, x, y)$ near $x = y = \infty$ disappears if this function is expressed in terms of $\tilde{v}$ instead of $v$.

The hypermultiplet bare masses $m_\alpha$ are the residues of the differential form $\beta = f_1 \omega$ at the points $p_\sigma$. Since the sum of the residues of $\beta$ will vanish, $\beta$ has a pole at $x = y = \infty$ with residue $-\sum_\alpha m_\alpha$. We can now relate this expression to the parameter $m$ in (4.18). Since condition (2) above implies that the singular behavior of $f_1$ is $f_1 = -my/2x + \ldots$, and since the differential form $(dx/y)(y/2x)$ has a pole at infinity with residue 1, the residue of $\beta$ is in fact $-m$, so we get

$$m = \sum_\alpha m_\alpha. \tag{4.19}$$

This relation between the coefficient $m$ by which $X_m$ is twisted and the hypermultiplet bare masses $m_\alpha$ was anticipated in (4.4).

Just as in the case $m = 0$ that we considered first, the order parameters on the Coulomb branch are the parameters not fixed by specifying the singular part of $f_1$.

In [12], the solution of this model for the special case $n = 1$ was expressed in an equivalent but slightly different way. Since – to adapt the discussion to the present language – there was only one fivebrane, the fivebrane was placed at $p_\infty$ without any loss of symmetry.

In place of conditions (1) and (2), the requirements on the $f_i$ were the following:

(1’) The functions $f_i(x, y)$ are meromorphic functions on $E$ with no singularities except a pole of order at most $i$ at $x = y = \infty$.

(2’) After the change of variables (4.18), the singularity of the function $F(x, y, v)$ at $x = y = \infty$ is only a simple pole.

These conditions were used as the starting point for fairly detailed calculations of the properties of the model.

For the general case of $n$ fivebranes, if we choose one of the fivebrane locations, say $p_1$, to equal $p_\infty$, then (1) and (2) can be replaced by the following conditions:

(1’’) The functions $f_i(x, y)$ are meromorphic functions on $E$ whose possible singularities are simple poles at $p_2, \ldots, p_n$ and a pole of order $i$ at $x = y = \infty$.

(2’’) After the change of variables (4.18), the singularity of the function $F(x, y, t)$ at $x = y = \infty$ is only a simple pole.

These conditions are equivalent to (1) and (2), up to a translation on $E$ that moves $p_1$ to infinity and a change of variables $v \to v + a(x, y)$ for some function $a$. 45
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