SIMPLE APPROACH TO THERMAL BETHE ANSATZ

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Lecture delivered at the XXI International Conference on Differential Geometry Methods in Theoretical Physics, Tianjin, P. R. of CHINA, June 5-9, 1992

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ABSTRACT

We report on a new approach to the calculation of thermodynamic functions for crossing-invariant models solvable by Bethe Ansatz. In the case of the XXZ Heisenberg chain we derive, for arbitrary values of the anisotropy, a single non-linear integral equation from which the free energy can be exactly calculated. These equations are shown to be equivalent to an infinite set of algebraic equations of Bethe type which provide alternatively the thermodynamics. The high-temperature expansion follows in a systematic and relatively simple way from our non-linear integral equations. For low temperatures we obtain the correct central charge and predict the analytic structure of the full expansion around \( T = 0 \). Furthermore, we derive a single non-linear integral equation describing the finite-size ground-state energy of the Sine–Gordon quantum field theory.

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The computation of thermodynamical functions for integrable models started with the seminal works of C.N. Yang and C.P. Yang [1], of M. Takahashi [2] and M. Gaudin [3]. In Refs. [2,3] the free energy of the Heisenberg spin chain is written in terms of the solution of an infinite set of coupled nonlinear integral equations, derived on the basis of the so-called “string hypothesis”. In ref.[4], we proposed a simpler way to solve the thermodynamics by means of a single, rigorously derived, nonlinear integral equation. This method can be applied to a wide class of models solvable by Bethe Ansatz (see ref.[5]).

The XXZ hamiltonian for a periodic chain with $2L$ sites (we assume periodic boundary conditions but generalization to other b.c. is possible) takes the form

$$H_{XXZ} = -J \sum_{n=1}^{2L} \left[ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y - \cos \gamma (\sigma_n^z \sigma_{n+1}^z + 1) \right]$$

The anisotropy parameter $\gamma$ is assumed here to lay in the real interval $(0, \pi)$ characteristic of the gapless regime. $H_{XXZ}$ is related to the diagonal–to–diagonal transfer matrix $T_L(\theta)$ of the six–vertex model by

$$T_L(\theta) \xrightarrow{\theta \to 0} 1 - \frac{\theta}{2J \sin \gamma} H_{XXZ} + O(\theta^2)$$

where

$$T_L(\theta) = R_{12} R_{34} \ldots R_{2L-12L} R_{23} R_{45} \ldots R_{2L1}$$

and

$$R_{nm} = \frac{a+c}{2} + \frac{a-c}{2} \sigma_n^z \sigma_m^z + \frac{b}{2} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y)$$

$$a = \frac{\sin(\gamma - \theta)}{\sin \gamma}, \quad b = \frac{\sin \theta}{\sin \gamma}, \quad c = 1$$

$\theta$ is known as spectral parameter while $a$, $b$, and $c$ are the conventional six–vertex Boltzmann weights.
The free energy per site is defined as usual

\[ f(\beta) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \log \left[ \text{Tr} \left( e^{-\beta H_{XXZ}} \right) \right] \] (5)

and from eq. (2) we read

\[ e^{-\beta H_{XXZ}} = \lim_{N \to \infty} \left[ T_L(2\tilde{\beta}/N) \right]^N \]

where \( \tilde{\beta} = \beta J \sin \gamma \). Hence we can write

\[ f(\beta) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \lim_{N \to \infty} \log Z_{LN}(2\tilde{\beta}/N) \] (6)

where \( Z_{LN}(\theta) \equiv \text{Tr} \left[ T_L(\theta) \right]^N \) is the six–vertex partition function on a periodic diagonal lattice with \( L \times N \) sites. The two limits in eq. (6) cannot be interchanged since the degeneracy of \( T_L(0) = 1 \), that is \( 2^{2L} \), is strongly \( L \)–dependent. However, the crossing invariance of the six–vertex \( R \)–matrix implies that under a rotation by \( \pi/2 \) of the entire lattice plus the substitution \( \theta \to \gamma - \theta \), the numerical value of the partition function does not change (see e.g. ref.6). Therefore \( Z_{LN}(\theta) = Z_{NL}(\gamma - \theta) \) and

\[ f(\beta) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \lim_{N \to \infty} \log Z_{NL}(\gamma - 2\tilde{\beta}/N) \] (7)

where \( Z_{NL}(\theta) \equiv \text{Tr} \left[ T_N(\theta) \right]^L \). Now, as the well–known Bethe Ansatz solution tells us, \( T_N(\gamma) \) has a non–degenerate largest eigenvalue for any finite \( N \). Then the two limits in eq. (7) commute and one finds [7]

\[-2\beta f(\beta) = \lim_{N \to \infty} \log \Lambda^\text{max}_N(\gamma - 2\tilde{\beta}/N) \] (8)

where \( \Lambda^\text{max}_N(\theta) \) denotes the largest eigenvalue of \( T_N(\theta) \). Using the results of ref.[8] for the eigenvalues of the diagonal–to–diagonal transfer matrix, the free energy...
becomes

\[ f(\beta) = \frac{1}{2\beta} \lim_{N \to \infty} \left[ E_N(\beta) - 2N \log \frac{\sin(2\tilde{\beta}/N)}{\sin \gamma} \right] \]  

(9)

where

\[ E_N(\beta) = i \sum_{j=1}^{N} \epsilon(\lambda_j, \beta) \]
\[ \epsilon(\lambda, \beta) = \phi(\lambda + i\gamma/2, \gamma/2 - \tilde{\beta}/N) - \phi(\lambda - i\gamma/2, \gamma/2 - \tilde{\beta}/N) \]  

(10)

\[ \phi(\lambda, x) \equiv i \log \frac{\sinh(ix + \lambda)}{\sinh(ix - \lambda)} \quad (\phi(0, x) = 0) \]

The real numbers \( \lambda_1, \ldots, \lambda_N \) are the roots of the Bethe Ansatz equations

\[ N \left[ \phi(\lambda_j, \tilde{\beta}/N) + \phi(\lambda_j, \gamma - \tilde{\beta}/N) \right] = \sum_{k=1}^{N} \phi(\lambda_j - \lambda_k, \gamma) + \pi(2j - N - 1) \]  

(11)

associated to the largest eigenvalue \( \Lambda_{N}^{\text{max}}(\gamma - 2\tilde{\beta}/N) \) (we assumed \( N \) even). We define as usual the counting function [9]

\[ z_N(\lambda) = \frac{1}{2\pi} \left[ \phi(\lambda, \tilde{\beta}/N) + \phi(\lambda, \gamma - \tilde{\beta}/N) - \frac{1}{N} \sum_{k=1}^{N} \phi(\lambda - \lambda_k, \gamma) \right] \]  

(12)

which by construction enjoys the properties \( z_N(\lambda) = -z_N(-\lambda) = \overline{z_N(\lambda)} \) and

\[ z_N(\lambda_j) = j - (N + 1)/2 \]  

(13)

Our aim is to derive an integral equation for \( z_N(\lambda) \). Let us start by assuming that the root distribution becomes the continuous density \( \sigma_c(\lambda) \) (normalized to 1) in the limit \( N \to \infty \). Then it could be identified with the derivative of \( z_N(\lambda) \) and at the same time used to replace the discrete sum in eq. (12) with an integration.
For large but finite $N$ we would therefore obtain the following approximate linear integral equation for $\sigma_c(\lambda)$

$$2\pi\sigma_c(\lambda) = \phi'(\lambda, \beta/N) + \phi'(\lambda, \gamma - \beta/N) - \int_{-\infty}^{+\infty} d\mu \phi'(\lambda - \mu, \gamma)\sigma_c(\mu)$$  \hspace{1cm} (14)$$

Fourier transforming leads to the explicit solution

$$\sigma_c(\lambda) = z'_c(\lambda), \quad z_c(\lambda) = \frac{1}{\pi} \arctan \left[ \frac{\sinh(\pi\lambda/\gamma)}{\sin(\pi\beta/\gamma N)} \right]$$  \hspace{1cm} (15)$$

We see that $\sigma_c(\lambda)$ is strongly peaked at $\lambda = 0$ for large $N$, reflecting the singularity which develops at the origin in the $N \to \infty$ limit of the source term in eqs. (11), (14). The density picture correctly describes the BA roots wherever $\sigma_c(\lambda)$ is of order 1. From eq. (15) we then find as validity interval

$$|\lambda| \leq O(\sqrt{\beta/N})$$  \hspace{1cm} (16)$$

which shrinks to zero when $N \to \infty$. Roots $|\lambda_j| > O(\sqrt{\beta/N})$ have a spacing of order larger than $O(1/N)$ and cannot be described by densities. In particular, the roots with largest magnitudes have finite $N \to \infty$ limits spread by $O(1)$ intervals (we checked this fact numerically too). Therefore, contrary to the usual situation [9], we must go beyond the density description to obtain a bulk quantity like the free energy per site. The sum in the last term of eq.(12) can be written as a contour integral

$$\left\{ \frac{1}{N} \right\} \sum_{j=1}^{N} \phi(\lambda - \lambda_j, \gamma) = \int \frac{dz}{2\pi i N} \phi(\lambda - z, \gamma) \frac{d\log[1 + a_N(z)]}{dz}$$  \hspace{1cm} (17)$$

where $a_N = \exp(2\pi i N z_N)$ and the contour in eq.(17) encircles counterclockwise all the real roots $\lambda_k, k = 1, ..., N$. Notice that $a_N(\lambda_k) + 1 = 0, k = 1, ..., N$. Inserting
eq.(17) in eq.(12) and integrating by parts yields

\[
\log a_N(\lambda) = iN \left[ \phi(\lambda, \tilde{\beta}/N) + \phi(\lambda, \gamma - \tilde{\beta}/N) \right] + i \int_{-\infty}^{+\infty} \frac{dx}{\pi} \phi'(\lambda-x, \gamma) \text{Im} \log[1+a_N(x+i0)]
\]

(18)

Convoluting eq.(18) with the kernel

\[
K^{-1} = \delta(\lambda) - p(\lambda)
\]

we find

\[
\log \frac{a_N(\lambda)}{a_c(\lambda)} = \int_{-\infty}^{+\infty} d\mu \log \frac{1+a_N(\mu+i0)}{1+a_N(\mu+i0)}
\]

(20)

which is the sought nonlinear integral equation. (Here \(a_c = \exp(2\pi i N z_c)\)). Notice that \(a_N(\bar{\lambda}) = 1/a_N(\lambda)\). We evaluate the sum in eq. (10) by a similar procedure, with the result

\[
L_N(\beta) \equiv E_N(\beta) - E_c(\beta) = \int_{-\infty}^{+\infty} d\lambda \frac{\sinh \pi \lambda/\gamma \cos \pi \tilde{\beta}/\gamma N}{\cosh 2\pi \lambda/\gamma - \cos 2\pi \tilde{\beta}/\gamma N} \log \frac{1+a_N(\lambda+i0)}{1+a_N(\lambda+i0)}
\]

(21)

where \(E_c(\beta)\) is that part of the energy that follows from the root density (15), that is

\[
E_c(\beta) = \int_{-\infty}^{+\infty} d\lambda \epsilon(\lambda, \gamma) \sigma_c(\lambda) = -2N \int_{-\infty}^{+\infty} \frac{dk}{k} \frac{\sinh(\pi - \gamma)k \sinh(\gamma - 2\tilde{\beta}/N)k}{\sinh \pi k \cosh \gamma k}
\]

(22)

If eqs. (20) and (21) are analytically continued to the axis \(\text{Im} \lambda = \pm \gamma/2\) they assume the same structure of those derived in ref. [12] by different methods.
Inserting eqs. (21) and (22) in eq. (9) yields

\[ f(\beta) = E_{XXZ}(\gamma) + \beta^{-1}L(\beta) \]  

(23)

where \( E_{XXZ}(\gamma) \) is the ground–state energy of (1), namely

\[ E_{XXZ}(\gamma) = 2J \left[ \cos \gamma - \sin \gamma \int_{-\infty}^{+\infty} dk \frac{\sinh(\pi - \gamma)k}{\sinh \pi k \cosh \gamma k} \right] \]  

(24)

while \( L(\beta) \) is the \( N \to \infty \) limit of (21), that is

\[ L(\beta) = \text{Im} \int_{-\infty}^{+\infty} d\lambda \frac{\log[1 + a(\lambda + i0)]}{\gamma \sinh \pi(\lambda + i0)/\gamma} \]  

(25)

Similarly, the function \( a(\lambda) \) in this expression is the \( N \to \infty \) limit of \( a_N(\lambda) \). It is therefore the solution of the \( N \to \infty \) limit of eq. (20), that is

\[ -i \log a(\lambda) = -\frac{2\pi \tilde{\beta}}{\gamma \sinh \pi \lambda/\gamma} + 2 \int_{-\infty}^{+\infty} d\mu p(\lambda - \mu) \text{Im} \log[1 + a(\mu + i0)] \]  

(26)

Thus the calculation of the free energy (5) has been reduced to the problem of solving the single nonlinear integral equation (26) and then evaluating the integral (25).

We find from the \( N = \infty \) limit of eq. (18) an alternative form for our eq. (26)

\[ z(\lambda) = \tilde{\beta}q(\lambda) + \frac{1}{2\pi^2} \text{Im} \int_{-\infty}^{+\infty} d\mu \phi'(\lambda - \mu, \gamma) \log \cos \pi z(\mu + i0) \]  

(27)

where

\[ q(\lambda) = \frac{1}{\pi} \left( \frac{\sinh 2\lambda}{\cosh 2\lambda - \cos 2\gamma} - \coth \lambda \right) \]  

(28)
and \( z(\lambda) \equiv \frac{1}{2\pi} \log a(\lambda) \) is related to the original counting function

\[
z(\lambda) = \lim_{N \to \infty} N \left[ z_N(\lambda) - \frac{1}{2} \text{sign} \lambda \right] \quad (\lambda \text{ real}) \quad (29)
\]

By the residue theorem we then obtain

\[
z(\lambda) = \tilde{\beta} q(\lambda) - \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \left[ \phi(\lambda - \xi_j, \gamma) - \phi(\lambda, \gamma) \right] \quad (30)
\]

where the real numbers \( \xi_j \), defined by \( z(\xi_j) = j - 1/2, j \in \mathbb{Z} \), can be identified with the \( N \to \infty \) limit of the original BA roots \( \lambda_1, \lambda_2, \ldots, \lambda_N \). The new algebraic BA equations \( z(\xi_j) = j - 1/2 \) embody all the information about the XXZ thermodynamics. Eq. (30) shows that \( z(\lambda) \) has periodicity \( i\pi \) and has, as unique singularity on the real axis, a simple pole at the origin with residue \( -\tilde{\beta}/\pi \). The new algebraic BA equations \( z(\xi_j) = j - 1/2, j \in \mathbb{Z} \) embody all the information about the XXZ thermodynamics. They form an infinite set of Bethe Ansatz type algebraic equations equivalent to our non-linear integral equations (26) and (27). We want to stress that these roots are \textbf{discrete} although we have already set \( N = \infty \). The root spacing is actually of order \( T \). In the \( T \to 0 \) limit we recover the continuous distribution of roots associated with the antiferromagnetic ground-state[9]. In terms of the roots \( \xi_k \), the free energy takes the form

\[
f(\beta) = J \cos \gamma - \beta^{-1} \ln 2 + \frac{1}{2\beta} \sum_{k \in \mathbb{Z}} \log \left( \frac{1 - \cos 2\gamma}{\cosh 2\xi_j - \cos 2\gamma} \right) \quad (31)
\]

The roots \( \xi_j \) have an accumulation point at \( \xi = 0 \). That is, \( \xi_j = -\tilde{\beta}/[\pi(j - 1/2)] \) for \( j \to \infty \). For low temperatures, the largest root, \( \xi_1 \), is of order \( (\gamma/\pi) \ln \tilde{\beta} \).

Let us now study \( f(\beta) \) for high temperatures. When \( \beta \) is small it is convenient to use the alternative form (27) plus the uniform expansion

\[
z(\lambda) = \sum_{k=1}^{\infty} \tilde{\beta}^k b_k(\lambda) \quad (32)
\]
Since the residue of \( z(\lambda) \) is linear in \( \tilde{\beta} \) and \( z(\lambda) \) is an odd function, we have

\[
b_1(\lambda) \xrightarrow{\lambda \to 0} -\frac{1}{\pi \lambda} + O(\lambda), \quad b_k(0) = 0 \quad (k \geq 2) \tag{33}
\]

Inserting eq. (32) in eq. (27) we find, up to fourth order in \( \tilde{\beta} \)

\[
b_1(\lambda) = q(\lambda) \quad b_2(\lambda) = \frac{1}{4\pi} \phi''(\lambda, \gamma)
\]
\[
b_3(\lambda) = 0 \quad b_4(\lambda) = \frac{1}{6\pi} \left( \frac{1}{3} - \frac{1}{\sin^2 \gamma} \right) \phi''(\lambda, \gamma) + \frac{1}{144\pi} \phi'''(\lambda, \gamma) \tag{34}
\]

Then, from eqs. (23) and (25),

\[
f(\beta) = -\beta^{-1} \ln 2 + J \cos \gamma - \beta J^2 \left( 1 + \frac{1}{2} \cos^2 \gamma \right) + \beta^2 J^3 \cos \gamma + O(\beta^3) \tag{35}
\]

which indeed agrees with the high \( T \) expansion [2]. We want to remark that eq. (27) generates the functions \( b_k(\lambda) \) recursively, with easy integrations which involve only delta functions and derivatives thereof. It is indeed a very efficient way to recover the high–temperature expansion from the Bethe Ansatz solution [5].

We shall consider now the low temperature regime. When \( \beta \gg 1 \) eq. (26) indicates that \( z(\lambda) \sim \beta \), so that \( \log[1 + a(\lambda)] = 2\pi i z(\lambda) \), at least as long as \( |\lambda| \leq (\gamma/\pi) \log \beta \). At dominant \( \beta \gg 1 \) order eq. (26) linearizes in the same way as it does for small \( \beta \). Inserting then \( z(\lambda) \simeq \tilde{\beta} q(\lambda) \) in eq. (25), yields \( \lim_{\beta \to \infty} L(\beta) = 0 \). Therefore eq. (23) tells us that

\[
f(\beta) \xrightarrow[\beta \to \infty]{\sim} E_{XXZ}(\gamma) + O(\beta^{-2}) \tag{36}
\]

The contributions of order \( \beta^{-2} \) and smaller come from values of \( \lambda \) larger than \( (\gamma/\pi) \log \beta \), where the previous assumption \( \log a \sim O(\beta) \) does not hold anymore.
It is then convenient to introduce the new function

\[ A(x) = a(\lambda), \quad x = \lambda - \frac{\gamma}{\pi} \log \frac{4\pi\bar{\beta}}{\gamma} \]  

(37)

Then eqs. (25) and (26) reduce, in the \( \beta \to \infty \) limit, to

\[ L(\beta) = \frac{2}{\pi\beta} \int_{-\infty}^{+\infty} dx e^{-\pi x/\gamma} \text{Im} \log[1 + A(x + i0)] \]  

(38)

and

\[ -i \log A(x) = -e^{-\pi x/\gamma} + 2 \int_{-\infty}^{+\infty} dy p(x - y) \text{Im} \log[1 + A(y + i0)] \]  

(39)

Fortunately, it is not necessary to solve this integral equation to calculate the integral (38). In fact, we can use the following lemma.

**Lemma.** Assume that \( F(x) \) satisfies the nonlinear integral equation

\[ -i \log F(x) = \varphi(x) + \int_{x_1}^{x_2} dx \varphi'(x) \log[1 + F(x + i0)] \]  

(40)

where \( \varphi(x) \) is real for real \( x \) and \( x_1, x_2 \) are real numbers. Then the following relation holds

\[ \text{Im} \int_{x_1}^{x_2} dx \varphi'(x) \log[1 + F(x + i0)] = \frac{1}{2} \text{Im} [\varphi(x_2) \log(1 + F_2) - \varphi(x_1) \log(1 + F_1)] \]  

\[ + \frac{1}{2} \text{Re} [\ell(F_1) - \ell(F_2)] \]  

(41)

where \( F_{1,2} = F(x_{1,2}) \) and \( \ell \) is a dilogarithm function

\[ \ell(t) \equiv \int_0^t du \left[ \frac{\log(1 + u)}{u} - \frac{\log u}{1 + u} \right] \]  

(42)
To prove the lemma we consider the relation

\[ \ell(F_2) - \ell(F_1) = \int_{x_1}^{x_2} dx \left\{ \log[1 + F(x)] \frac{d}{dx} \log F(x) - \log F(x) \frac{d}{dx} \log[1 + F(x)] \right\} \]

and then use eq. (40) and its derivative to substitute \( \log F(x) \) and \( \frac{d}{dx} \log F(x) \). After taking real part and a little algebra this yields eq. (41) (related identities were used in ref. [12]).

The integral in eq. (38) may now be exactly calculated by invoking the lemma with \( F(x) = A(x), \varphi = -\exp(-\pi x/\gamma) \) and \( x_1 = -\infty, x_2 = +\infty \). We have \( A(x_1) = 0, A(x_2) = 1 \) and

\[ 2\text{Im} \int_{-\infty}^{+\infty} dx e^{-\pi x/\gamma} \log[1 + A(x + i0)] = -\frac{\gamma}{\pi} \ell(1) = -\frac{\pi \gamma}{6} \]

Then the free energy for low temperature reads

\[ f(\beta) = E_{XXZ}(\gamma) - \frac{\gamma}{6J \sin \gamma} \beta^{-2} + o(\beta^{-2}) \quad (43) \]

in perfect agreement with refs. [2,10,11]. As we have just shown, both the high and the low temperature leading behaviors of the free energy can be derived without much effort from our non–linear integral equation (26). The higher order corrections for high \( T \) can be obtained in a very systematic way [5]. The situation for the \( o(\beta^{-2}) \) terms in the low \( T \) expansion (43) is more involved. Qualitatively, however, it is rather easy to establish, from the asymptotic behaviour of the kernel \( p(\lambda - \mu) \) in eq. (26) and from the \( i\pi \) periodicity implied by eq. (30), that these higher order terms must be integer powers of \( T^{\pi \gamma} \) and \( T^{\gamma/(\pi - \gamma)} \).

Let us now consider the problem of calculating the (properly subtracted) ground state energy \( E(L) \) of 2D integrable massive models of Quantum Field Theory in a finite 1–volume \( L \). This subject recently received much attention
in connection with Perturbed Conformal Field Theory. In this context \( E(L) \) is known as ground–state scaling function. Our starting point is the lattice regularization of such models provided by the light–cone approach to integrable vertex models [8]. Specializing to the six–vertex model, the ground state energy, on a ring of length \( L \) formed by \( N \) sites, takes a form similar to eq. (21):

\[
E_N(L) - E_c(L) = \frac{N}{\gamma L} \text{Im} \int_{-\infty}^{+\infty} d\lambda \left[ \text{sech} \frac{\pi}{\gamma} (\lambda - \Theta) - \text{sech} \frac{\pi}{\gamma} (\lambda + \Theta) \right] \log[1 + a_N(\lambda + i0)]
\]

where

\[
E_c(L) = \frac{N^2}{L} \left[ -2\pi + \int_{-\infty}^{+\infty} d\lambda \frac{\phi(\lambda + 2\Theta, \gamma/2)}{\gamma \cosh \pi\lambda/\gamma} \right]
\]

and \( a_N(\lambda + i0) \) obeys an equation like (20) with the new source term

\[
\log a_c(\lambda) = 2 \arctan \frac{\sinh \pi\lambda/\gamma}{\cosh \pi\Theta/\gamma}
\]

The \( L \)–dependence in these expressions is hidden in the light–cone parameter \( \Theta \), which tends to infinity, in the continuum limit \( N \to \infty \), as [8]

\[
\Theta = \frac{\gamma}{\pi} \log \frac{4N}{mL}
\]

with \( m \), the physical mass scale, held fixed. Define now

\[
\epsilon(\lambda) = -\lim_{N \to \infty} \log a_N(\gamma\lambda/\pi)
\]

then from eqs. (20), (42–45) we find

\[
E(L) \equiv \lim_{N \to \infty} [E_N(L) - E_c(L)] = -m \int_{-\infty}^{+\infty} \frac{d\lambda}{\pi} \sinh \lambda \text{Im} \log \left[ 1 + e^{-\epsilon(\lambda + i0)} \right]
\]
and

\[ \epsilon(\lambda) = m L \sinh \lambda + 2 \int_{-\infty}^{+\infty} d\mu G(\lambda - \mu) \text{Im} \log \left[ 1 + e^{-\epsilon(\mu+i0)} \right] \]  

(50)

where \( G(\lambda) = (\gamma/\pi)p(\gamma\lambda/\pi) \). Thanks to the correspondence between the light–cone six–vertex model and the sine–Gordon (or Massive Thirring) model [8], eqs. (49) and (50) give the ground state scaling function \( E(L) \) of the sine–Gordon field theory on a ring of length \( L \). (Here \( m \) stands for the fermion mass). This is a rigorous and simpler alternative to the standard Thermodynamic Bethe Ansatz [14]. Notice that all UV divergent terms are contained in \( E_c(L) \) (cf. eq. (45)). \( E_c(L) \) also contains the finite, scaling part \( m^2 L \cot^2 \pi^2/2\gamma \) (also found in ref. [13] by completely different means), which gives the ground–state energy density in the \( L \rightarrow \infty \) limit.

Let us now study eqs. (49) and (50) for \( mL \gg 1 \) and \( mL \ll 1 \). For small values of \( mL \) the calculation closely parallels that for low \( T \) in the thermodynamics of the XXZ chain. Shifting \( \lambda \rightarrow \lambda - \log mL \) and applying the lemma leads to

\[ E(L) \xrightarrow{mL \rightarrow 0} -\frac{\pi}{6L} + O(1) \]  

(51)

showing the expected value \( c = 1 \) for the UV central charge. The large \( mL \) regime easily follows from eq. (50) by iteration. One finds the behavior typical of a massive quantum field theory

\[ E(L) \xrightarrow{L \rightarrow \infty} -\frac{2m}{\pi} K_1(mL) + O(e^{-2mL}) \]  

(52)

with \( K_1(z) \) the modified Bessel function of order 1.

We would like to remark that, contrary to the traditional thermal Bethe Ansatz [2,3], our approach does not rely on the string hypothesis on the structure of the solutions of the BA equations characteristic of the Heisenberg chain. This makes our approach definitely simpler. Most notably, the whole construction of
the thermodynamics no longer depends on whether $\gamma/\pi$ is a rational or not, unlike Takahashi approach [2]. This applies equally well to the problem of the ground state scaling function of the sine–Gordon field theory, since the standard TBA approach requires the string hypothesis.

Generalizations of the present TBA approach to higher spin chains as well as to magnetic chains and quantum field theories associated to Lie algebras other than $A_1$ (nested BA solutions) are relatively straightforward provided crossing symmetry holds [5].

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