NATURAL DENSITY AND PROBABILITY, CONSTRUCTIVELY.

SAMUELE MASCHIO

Dipartimento di Matematica “T. Levi Civita”. Univ. di Padova. Via Trieste, 63. Padova. Italy
e-mail address: maschio@math.unipd.it

ABSTRACT. We give here a constructive account of the frequentist approach to probability, by means of natural density. Using this notion of natural density, we introduce some probabilistic versions of the Limited Principle of Omniscience. Finally we give an attempt general definition of probability structure which is pointfree and takes into account abstractely the process of probability assignment.

1. INTRODUCTION

Natural density \cite{9} provides a notion of size for subsets of natural numbers. Classically the density of $A \subseteq \mathbb{N}^+$ is defined as

$$\delta(A) = \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

provided this limit exists. Since classically subsets of $\mathbb{N}$ are in bijective correspondence with sequences of 0s and 1s, this notion of density still works for such sequences and can provide a constructive account of frequentist probability. Here we study such a notion of frequentist probability in a constructive framework and we introduce some weak probabilistic forms of the limited principles of omniscience. Our treatment is informal à la Bishop, but it can be formalized in the extensional level of the Minimalist Foundation \cite{8, 7} with the addition of the axiom of unique choice. Finally we will propose a more general pointfree notion of probabilistic structure inspired by the properties which hold in our constructive treatment of frequentist probability and in other classical situations related to Kolmogorov’s axiomatic probability and to fuzzy sets.

2. A CONSTRUCTIVE ACCOUNT OF NATURAL DENSITY

2.1. Potential events. A potential event $e$ is a sequence of 0s and 1s.

$$e(n) \in \{0, 1\} \ [n \in \mathbb{N}^+]$$

Potential events form a set with extensional equality, that is two potential events $e$ and $e'$ are equal if $e(n) = e'(n) \ [n \in \mathbb{N}^+]$. This set, which we denote with $\mathcal{P}$, can be endowed with a structure of boolean algebra as follows:

(1) the bottom $\bot$ is $\lambda n.0$;
(2) the top $\top$ is $\lambda n.1$;
(3) the conjunction $e \land e'$ of $e$ and $e'$ in $\mathcal{P}$ is $\lambda n.(e(n)e'(n))$;
(4) the disjunction of $e \lor e'$ of $e$ and $e'$ in $\mathcal{P}$ is $\lambda n.(e(n) + e'(n) - e(n)e'(n))$;
(5) the negation $\neg e$ of $e \in \mathcal{P}$ is $\lambda n. (1 - e(n))$;
(6) for every $e, e' \in \mathcal{P}$, $e \leq e'$ if and only if $e(n) \leq e'(n)$ for all $n \in \mathbb{N}^+$.

Potential events can be understood as sequences of outcomes ("success" or "failure") in a sequence of iterated trials.

For every potential event $e$, one can define a sequence of rational numbers
$$\Phi(e)(n) \in \mathbb{Q} \ [n \in \mathbb{N}^+]$$
taking $\Phi(e)(n)$ to be $\frac{\sum_{i=1}^{n} e(i)}{n}$. This sequence is called the sequence of rates of success (or frequencies) of the potential event $e$.

2.2. Actual events. Actual events are those potential events for which the sequence of rates of success can be shown constructively to be convergent.

**Definition 2.1.** An actual event is a pair $(e, \gamma)$ where $e$ is a potential event and $\gamma$ is a strictly increasing sequence of natural numbers such that
$$|\Phi(e)(\gamma(n) + i) - \Phi(e)(\gamma(n) + j)| \leq \frac{1}{n} \bigg[ n \in \mathbb{N}^+, \ i, j \in \mathbb{N} \bigg]$$

We denote with $\hat{\mathcal{A}}$ the set of actual events and two actual events $(e, \gamma)$ and $(e', \gamma')$ are equal, $(e, \gamma) = \hat{\mathcal{A}} (e', \gamma')$, if $e =_\mathcal{P} e'$.

Let us now recall the definition of Bishop real numbers from [2].

**Definition 2.2.** A Bishop real $x$ is a sequence $x(n) \in \mathbb{Q} \ [n \in \mathbb{N}^+]$ of rational numbers such that
$$|x(n) - x(m)| \leq \frac{1}{n} + \frac{1}{m} \bigg[ n \in \mathbb{N}^+, \ m \in \mathbb{N}^+ \bigg]$$

Two Bishop reals $x$ and $y$ are equal if $|x(n) - y(n)| \leq \frac{2}{n} \bigg[ n \in \mathbb{N}^+ \bigg]$. The set of Bishop reals is denoted with $\mathbb{R}$.

We now show that one can well define a notion of probability on actual events.

**Proposition 2.3.** If $(e, \gamma)$ is an actual event, then $\Phi(e) \circ \gamma$ is a Bishop real number.

*Proof.* Suppose $m \leq n \in \mathbb{N}^+$. Then $\gamma(m) \leq \gamma(n)$ and hence
$$|\Phi(e \circ \gamma)(n) - \Phi(e \circ \gamma)(m)| = |\Phi(e)(\gamma(n)) - \Phi(e)(\gamma(m))| \leq \frac{1}{m} < \frac{1}{n} + \frac{1}{m}.$$ 

This implies that $\Phi(e) \circ \gamma$ is a Bishop real. \hfill $\Box$

**Proposition 2.4.** If $(e, \gamma)$ and $(e', \gamma')$ are equal actual events, then $\Phi(e) \circ \gamma$ and $\Phi(e') \circ \gamma'$ are equal Bishop reals.

*Proof.* Suppose $(e, \gamma)$ and $(e', \gamma')$ are equal actual events. Since for every $n \in \mathbb{N}^+$, $\gamma(n) \leq \gamma'(n)$ or $\gamma'(n) \leq \gamma(n)$ and $e =_\mathcal{P} e'$, then
$$|\Phi(e \circ \gamma)(n) - (\Phi(e' \circ \gamma')(n)| = |\Phi(e)(\gamma(n)) - \Phi(e)(\gamma'(n))| \leq \frac{1}{n} < \frac{2}{n} \bigg[ n \in \mathbb{N}^+ \bigg].$$

From this it follows that $\Phi(e) \circ \gamma$ and $\Phi(e') \circ \gamma'$ are equal Bishop reals. \hfill $\Box$
As a consequence of the previous two propositions we can give the following

**Definition 2.5.** The function $P : \tilde{A} \to \mathbb{R}$ is defined as follows: if $(e, \gamma)$ is an actual event, then

$$P(e, \alpha) := \Phi(e) \circ \gamma.$$ 

2.3. **Properties of actual events.** We show here some basic properties of $P$. However, before proceeding, we need a trivial but useful result:

**Lemma 2.6.** If $(e, \gamma)$ is an actual event and $\gamma'$ is an increasing sequence of positive natural numbers such that $\gamma(n) \leq \gamma'(n)$ for every $n \in \mathbb{N}^+$, then $(e, \gamma')$ is an actual event.

**Proof.** This is an immediate consequence of the definition of actual event. □

We first show that impossible events are actual null events, i.e. the so called strictness.

**Proposition 2.7.** $(\bot, \lambda n. n)$ is an actual event and $P(\bot, \lambda n. n) =_R 0$.

**Proof.** $\Phi(\bot)(n) = 0$ for every $n \in \mathbb{N}^+$ and so in particular $|\Phi(\bot)(n) - \Phi(\bot)(m)| = 0$ for every $n, m \in \mathbb{N}^+$. This means that $(\bot, \lambda n. n)$ is an actual event and $P(\bot, \lambda n. n) := \Phi(\bot) \circ (\lambda n. n) =_R \lambda n. 0 =_R 0$. □

Then we show that actual events and $P$ satisfy involution.

**Proposition 2.8.** If $(e, \gamma)$ is an actual event, then $(\neg e, \gamma)$ is an actual event and

$$P(\neg e, \gamma) =_R 1 - P(e, \gamma).$$

**Proof.** Let $(e, \gamma)$ be an actual event. It is immediate to see that

$$\Phi(\neg e)(n) = 1 - \Phi(e)(n)$$

for every $n \in \mathbb{N}^+$. From this it follows that

$$|\Phi(\neg e)(\gamma(n) + m) - \Phi(\neg e)(\gamma(n) + m')| = |1 - \Phi(e)(\gamma(n) + m') - 1 + \Phi(e)(\gamma(n) + m')| =

= |\Phi(e)(\gamma(n) + m') - \Phi(e)(\gamma(n) + m)| \leq \frac{1}{n} \left[ n \in \mathbb{N}^+, \ m \in \mathbb{N}, \ m' \in \mathbb{N} \right].$$

Hence $(\neg e, \gamma)$ is an actual event. Moreover for every $n \in \mathbb{N}^+$

$$|P(\neg e, \gamma)(n) - (1 - P(e, \gamma))(n)| = |1 - P(e, \gamma)(n) - (1 - P(e, \gamma)(2n))| =

= |P(e, \gamma)(2n) - P(e, \gamma)(n)| \leq \frac{1}{n} + \frac{1}{2n} < \frac{2}{n}$$

so $P(\neg e, \gamma) =_R 1 - P(e, \gamma)$. □

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1We follow here the definitions of the operations and relations between reals given in Bishop’s book [2].
Moreover actual events are closed under disjunction of incompatible actual events:

**Proposition 2.9.** If \((e, \gamma)\) and \((e', \gamma')\) are actual events with \(e \land e' = \bot\), then if 
\[
\eta := \lambda n. (\gamma(2n) + \gamma'(2n))
\]
the pair \((e \lor e', \eta)\) is an actual event.

**Proof.** Since \(e \land e' = \bot\), we have that \(e \lor e' = e + e'\). Moreover \(\eta\) is clearly strictly increasing, as \(\gamma\) and \(\gamma'\) are so. Moreover for every \(n \in \mathbb{N}^+\), it is immediate to see that 
\[
\Phi(e + e', n) = \Phi(e, n) + \Phi(e', n).
\]

In particular for every \(n \in \mathbb{N}^+, i, j \in \mathbb{N}\)
\[
\left| \Phi(e + e')(\eta(n) + i) - \Phi(e + e')(\eta(n) + j) \right| 
\leq 
\left| \Phi(e)(\gamma(2n) + \gamma'(2n) + i) - \Phi(e)(\gamma(2n) + \gamma'(2n) + j) \right| 
+ 
\left| \Phi(e')(\gamma(2n) + \gamma'(2n) + i) - \Phi(e')(\gamma(2n) + \gamma'(2n) + j) \right| 
\leq 
\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.
\]
Hence \((e \lor e', \eta)\) is an actual event. \(\square\)

We now show that the set of actual events with null probability is downward closed:

**Proposition 2.10.** Let \((e, \gamma)\) be an actual event with \(\mathbb{P}(e, \gamma) = \mathbb{R} 0\) and let \(e'\) be a potential event such that \(e' \leq e\), then \((e', \lambda n. \gamma(6n))\) is an actual event.

**Proof.** For every \(n \in \mathbb{N}^+\) and \(i, j \in \mathbb{N}\)
\[
\left| \Phi(e')(\gamma(6n) + i) - \Phi(e')(\gamma(6n) + j) \right| 
\leq 
\left| \Phi(e')(\gamma(6n) + i) \right| + \left| \Phi(e')(\gamma(6n) + j) \right| 
\leq 
\left| \Phi(e)(\gamma(6n) + i) \right| + \left| \Phi(e)(\gamma(6n) + j) \right| 
\leq 
\left| \Phi(e)(\gamma(6n) + i) \right| - \left| \Phi(e)(\gamma(6n)) \right| + \left| \Phi(e)(\gamma(6n) + j) \right| - \left| \Phi(e)(\gamma(6n)) \right| \leq 
\frac{1}{6n} + \frac{1}{6n} + \frac{2}{6n} = \frac{1}{n}
\]
by using the fact that \((e, \gamma)\) is an actual event and \(\mathbb{P}((e, \gamma)) = \mathbb{R} 0\). Hence \((e', \lambda n. \gamma(6n))\) is an actual event. \(\square\)

Moreover \(\mathbb{P}\) satisfies monotonicity:

**Proposition 2.11.** If \((e, \gamma)\) and \((e', \gamma')\) are actual events and \(e \leq e'\), then 
\[
\mathbb{P}(e, \gamma) \leq \mathbb{P}(e', \gamma').
\]

**Proof.** First of all if we take \(\eta\) to be the sequence \(\lambda n. (\gamma(n) + \gamma'(n))\), then \((e, \eta)\) and \((e', \eta)\) are actual events equal to \((e, \gamma)\) and \((e', \gamma')\) respectively as a consequence of proposition 2.6.

In order to show that \(\mathbb{P}(e, \eta) \leq \mathbb{P}(e', \eta)\), we must prove that for every \(n \in \mathbb{N}^+\)
\[
(\mathbb{P}(e, \eta) - \mathbb{P}(e', \eta))(n) \leq \frac{1}{n}
\]
But 
\[
(\mathbb{P}(e, \eta) - \mathbb{P}(e', \eta))(n) = (\mathbb{P}(e, \eta))(2n) - (\mathbb{P}(e', \eta))(2n) = \Phi(e)(\eta(2n)) - \Phi(e')(\eta(2n)).
\]
Hence we must prove that 
\[
\Phi(e)(\eta(2n)) \leq \Phi(e')(\eta(2n)) + \frac{1}{n} \left[ n \in \mathbb{N}^+ \right]
\]
But this is trivially true, because from \(e \leq e'\) we can deduce that 
\[
\Phi(e)(n) \leq \Phi(e')(n) \left[ n \in \mathbb{N}^+ \right].
\]
\(\square\)
Finally we prove that $\mathbb{P}$ satisfies a form of modularity:

**Proposition 2.12.** If $(e, \alpha)$, $(e', \beta)$, $(e \land e', \gamma)$, $(e \lor e', \delta)$ are actual events, then

$$\mathbb{P}(e \lor e', \delta) + \mathbb{P}(e \land e', \gamma) = \mathbb{P}(e, \alpha) + \mathbb{P}(e', \beta).$$

**Proof.** Using proposition 2.6 if $\varepsilon := \lambda n.(\alpha(n) + \beta(n) + \gamma(n) + \delta(n))$, then

1. $\mathbb{P}(e \lor e', \delta) + \mathbb{P}(e \land e', \gamma) = \mathbb{P}(e \lor e', \varepsilon) + \mathbb{P}(e \land e', \varepsilon)$
2. $\mathbb{P}(e, \alpha) + \mathbb{P}(e', \beta) = \mathbb{P}(e, \varepsilon) + \mathbb{P}(e', \varepsilon)$

So we must prove that $\mathbb{P}(e, \varepsilon) + \mathbb{P}(e', \varepsilon) = \mathbb{P}(e \lor e', \varepsilon) + \mathbb{P}(e \land e', \varepsilon)$. However this is immediate as for every $n \in \mathbb{N}^+$

$$e(n) + e'(n) = (e \lor e')(n) + (e \land e')(n)$$

from which it immediately follows that for every $n \in \mathbb{N}^+$

$$\Phi(e, n) + \Phi(e', n) = \Phi(e \land e', n) + \Phi(e \lor e', n).$$

$\square$

### 2.4. Regular events.

Among potential events, there are some events which can be considered sort of “deterministic”: their sequence of outcomes of trials have a periodic behaviour, up to a possible finite number of accidental errors in the recording of the results: these events are here called *regular*:

**Definition 2.13.** Let $\alpha$ and $\pi$ be two finite lists of elements of $\{0, 1\}$ with length $\ell(\alpha)$ and $\ell(\pi) > 0$, respectively. We define the potential event $\|\alpha, \pi\|$ as follows

\[
\begin{cases}
\|\alpha, \pi\|(i) := \alpha_i & \text{if } i \in \mathbb{N}^+, i \leq \ell(\alpha) \\
\|\alpha, \pi\|(i) := \pi_{m(i-\ell(\alpha) - 1, \ell(\pi) + 1)} & \text{if } i > \ell(\alpha)
\end{cases}
\]

where $\alpha_i$ and $\pi_i$ denote the $i$th component of $\alpha$ and $\pi$, respectively and where for all $a \in \mathbb{N}$ and $b \in \mathbb{N}^+$, $rm(a, b)$ is the remainder of $a$ divided by $b$. The potential events of this form are called *regular*.

The goal of the following propositions is to show that regular events are actual. We first show that regular events without errors are actual events and their probability is equal to the frequency of their period.

**Proposition 2.14.** For every finite non-empty list $\pi = [\pi_1, ..., \pi_m]$, $(\|[]\|, |\pi|), \lambda n.4nm$ is an actual event and

$$\mathbb{P}([\|[]\|, \pi|, \lambda n.4nm) = \mathbb{P}([\|[]\|, \pi|, \lambda n.4nm).$$

**Proof.** For every $n \in \mathbb{N}^+$ and every $i, j, \in \mathbb{N}$

$$\left| \Phi([\|[]\|, \pi|, (4nm + i) \land \Phi([\|[]\|, \pi|, (4nm + j) \right| =$$

$$= \left| \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(i, m))m}{4nm + i} \right| + \left| \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(j, m))m}{4nm + j} \right| \leq \left| \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(i, m))m}{4nm + i} - \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(j, m))m}{4nm + j} \right| \leq \left| \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(i, m))m}{4nm + i} - \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(j, m))m}{4nm + j} \right| + \left| \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(j, m))m}{4nm + i} - \frac{\sum_{k=1}^{m} \pi_k}{m} \frac{(4n + q\ell(j, m))m}{4nm + j} \right| \leq$$
Suppose $\Phi(\mathbf{\alpha})$ is a potential event defined by $\mathbf{\alpha} = (\alpha_1, \ldots, \alpha_m)$, then $\mathbf{\alpha} \subseteq \mathbb{N}$. However this sum is equal to $\prod_{k=1}^{m} \pi_k$.

Hence $\prod_{k=1}^{m} \pi_k = \prod_{k=1}^{m} p_k \leq \prod_{k=1}^{m} \pi_k$.

Hence $\mathbb{P}(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n) = \prod_{k=1}^{m} p_k$.

The next step consists in proving that regular events definitely equal to 0 are actual events and their probability is 0.

**Proposition 2.15.** If $\mathbf{\alpha}$ is a finite list of 0s and 1s with length $m$, then $(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n)$ is an actual event and $\mathbb{P}(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n) = 0$.

**Proof.** Suppose $n \in \mathbb{N}^+$ and $i, j \in \mathbb{N}$ with $i \geq j$. Then

$\Phi(\prod_{k=1}^{m} \pi_k)(2n + i) - \Phi(\prod_{k=1}^{m} \pi_k)(2n + j) \leq \mathbb{P}(\prod_{k=1}^{m} \pi_k)(2n + i) + \mathbb{P}(\prod_{k=1}^{m} \pi_k)(2n + j) = \frac{\alpha_1 + \ldots + \alpha_m}{2n + i} + \frac{\alpha_1 + \ldots + \alpha_m}{2n + j} \leq \frac{2m}{2n} = \frac{1}{n}$

Hence $(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n)$ is an actual event. Moreover for every $n \in \mathbb{N}^+$

$\mathbb{P}(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n)(n) = \frac{\alpha_1 + \ldots + \alpha_m}{2n} \leq \frac{1}{2n} < \frac{2}{n}

Hence $\mathbb{P}(\prod_{k=1}^{m} \pi_k, \mathbb{N}.2n) = 0$.

Finally we prove that actual events are closed under shift to the right of terms and that probability is preserved by these shifts.

**Definition 2.16.** If $e$ is a potential event, then $e^+$ is the potential event defined by

$$
\begin{align*}
    e^+(1) &:= 0 \\
e^+(n+1) &:= e(n) \text{ for } n \in \mathbb{N}^+
\end{align*}
$$

**Proposition 2.17.** If $(e, \alpha)$ is an actual event, then $(e^+, \mathbb{N}(\gamma(3n)+1))$ is an actual event and $\mathbb{P}(e^+, \mathbb{N}(\gamma(3n)+1)) = \mathbb{P}(e, \gamma)$.

**Proof.** If $n \in \mathbb{N}^+$ and $i, j \in \mathbb{N}$, then $|\Phi(e^+)(\gamma(3n) + 1 + i) - \Phi(e^+)(\gamma(3n) + 1 + j)|$ is less or equal than the sum of $|\Phi(e^+)(\gamma(3n) + 1 + i) - \Phi(e)(\gamma(3n) + i)|, |\Phi(e)(\gamma(3n) + i) - \Phi(e)(\gamma(3n) + j)|$ and $|\Phi(e)(\gamma(3n) + j) - \Phi(e^+)(\gamma(3n) + 1 + j)|$. However this sum is equal to

$$
\frac{\Phi(e)(\gamma(3n) + i)}{\gamma(3n) + i + 1} + \frac{\Phi(e)(\gamma(3n) + i) - \Phi(e)(\gamma(3n) + j)}{\gamma(3n) + j + 1}
$$

$$
\leq \frac{1}{\gamma(3n) + j + 1} + \frac{1}{3n} + \frac{1}{\gamma(3n) + j + 1} \leq 3 \frac{1}{3n} = \frac{1}{n}
$$

Hence $(e, \alpha)$ is an actual event. Moreover

$$
|\Phi(e^+)(\gamma(3n) + 1) - \Phi(e)(\gamma(n))| = \left| \Phi(e)(\gamma(3n)) \frac{\gamma(3n)}{\gamma(3n) + 1} - \Phi(e)(\gamma(n)) \right| \leq
$$
\[
|\Phi(e)(\gamma(3n)) - \Phi(e)(\gamma(n))| + \frac{1}{\gamma(3n) + 1} \leq \frac{1}{n} + \frac{1}{3n + 1} < \frac{2}{n}
\]

Hence \( \mathbb{P}(e^+, \lambda n. (\gamma(3n) + 1)) = R \mathbb{P}(e, \gamma) \). \qedhere

Putting these results together we obtain the following:

**Theorem 2.18.** For every regular event \( \| \alpha, \pi \| \), there exists a strictly increasing sequence of natural numbers \( \gamma \) such that \((\| \alpha, \pi \|, \gamma)\) is an actual event and \( \mathbb{P}(\| \alpha, \pi \|, \gamma) = \frac{\sum \ell(\pi) \cdot \pi_k}{\ell(\pi)} \).

**Proof.** First of all, notice that \( \| \alpha, \pi \| = \perp \| \alpha, [0] \| \lor (\ldots \| [\pi] \| \ldots)^{\perp} \) where \( ^{\perp} \) is applied \( \ell(\alpha) \) times. By proposition 2.15, we know that there exists \( \gamma_1 \) such that \((\| \alpha, [0] \|, \gamma_1)\) is an actual event and, as a consequence of propositions 2.14 and 2.17, there exists \( \gamma_2 \) such that \( ((\ldots \| [\pi] \| \ldots)^{\perp}, \gamma_2) \) is an actual event.

Moreover it is clear that \( \| \alpha, [0] \| \lor (\ldots \| [\pi] \| \ldots)^{\perp} = \perp \), thus, using proposition 2.3, we obtain that there exists \( \gamma \) such that \((\| \alpha, \pi \|, \gamma)\) is an actual event.

Finally, \( \mathbb{P}(\| \alpha, \pi \|, \gamma) = \mathbb{P}(\| \alpha, [0] \|, \gamma_1) + \mathbb{P}((\ldots \| [\pi] \| \ldots)^{\perp}, \gamma_2) - \mathbb{P}(\perp, \lambda n. n) \) by propositions 2.12 and 2.17 and, by propositions 2.15, 2.17, 2.14 and 2.7, this is equal to \( \frac{\sum \ell(\pi) \cdot \pi_k}{\ell(\pi)} \). \( \square \)

We conclude this section with the following result:

**Theorem 2.19.** Regular events form a boolean algebra with the operations inherited by the algebra of potential events.

**Proof.** First of all \( \perp = \| [], [0] \| \) and \( \top = \| [], [1] \| \). Suppose now that \( \| \alpha, \pi \| \) and \( \| \beta, \psi \| \) are regular events. Then \( \neg \| \alpha, \pi \| := \| \neg \alpha, \neg \pi \| \) where \( \neg \alpha \) and \( \neg \pi \) are obtained by changing each term \( x \) of the finite lists in \( 1 - x \). Without loss of generality, we can suppose that \( \ell(\beta) \geq \ell(\alpha) \). It is immediate to check that

\[
\| \alpha, \pi \| \land \| \beta, \psi \| = \| \gamma, \rho \|
\]

\[
\| \alpha, \pi \| \lor \| \beta, \psi \| = \| \gamma', \rho' \|
\]

where \( \ell(\gamma) = \ell(\gamma') = \ell(\beta), \ell(\rho) = \ell(\rho') = \ell(\pi)\ell(\psi), \).

\[
\begin{cases}
\gamma_i := \alpha_i \land \beta_i & \text{if } i \leq \ell(\alpha) \\
\gamma_i := \pi_{rm(i - \ell(\alpha) - 1, \ell(\pi)) + 1} \land \beta_i & \text{if } \ell(\alpha) < i \leq \ell(\beta) \\
\gamma'_i := \alpha_i \lor \beta_i & \text{if } i \leq \ell(\alpha) \\
\gamma'_i := \pi_{rm(i - \ell(\alpha) - 1, \ell(\pi)) + 1} \lor \beta_i & \text{if } \ell(\alpha) < i \leq \ell(\beta) \\
\rho_i := \pi_{rm(\ell(\beta) - \ell(\alpha) + i - 1, \ell(\pi)) + 1} \lor \psi_{rm(i - 1, \ell(\psi)) + 1} \\
\rho'_i := \pi_{rm(\ell(\beta) - \ell(\alpha) + i - 1, \ell(\pi)) + 1} \land \psi_{rm(i - 1, \ell(\psi)) + 1} & \text{for every } 1 \leq i \leq \ell(\pi)\ell(\psi).
\end{cases}
\]

\( \square \)
3. Probabilistic limited principles of omniscience

The limited principle of omniscience, for short LPO, is a non-constructive principle which is weaker than the law of excluded middle (see [1], [2]) which is very important for constructive reverse mathematics (see [5]). It can be formulated in our framework as follows

\[ \text{LPO : } (\forall e \in P)((\forall n \in \mathbb{N}^+)(e(n) = 0) \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)) \]

Having introduced a notion of probability \( \mathbb{P} \) on binary sequences, we can weaken this axiom by restricting it to some subset of potential events or by weakening the disjunction.

Let us first introduce some abbreviations: we will write

1. \( \text{List}^+(A) \) for the set of non-empty lists of elements of a set \( A \);
2. \( \text{Incr}(\mathbb{N}^+, \mathbb{N}^+) \) for the set of increasing sequences of positive natural numbers.
3. \( e \in \mathbb{A} \) for (\( \exists \gamma \in \text{Incr}(\mathbb{N}^+, \mathbb{N}^+) \))(\( (e, \gamma) \in \mathbb{A} \));
4. \( \phi(\mathbb{P}[e]) \) for \( e \in \mathbb{A} \land (\forall \gamma \in \text{Incr}(\mathbb{N}^+, \mathbb{N}^+))(\{e, \gamma\} \in \mathbb{A} \rightarrow \phi(\mathbb{P}(e, \gamma))) \) if \( \phi(x) \) depends on a real number \( x \);
5. \( e \in \mathbb{R} \) for (\( \exists \alpha \in \text{List}((0, 1]))(\exists \pi \in \text{List}^+((0, 1]))(e =_\mathbb{P} ||\alpha, \pi||)\);
6. \( e \in \mathbb{N} \) for \( e \in \mathbb{A} \land \mathbb{P}[e] =_\mathbb{R} 0 \).

In such a way we can define subsets \( \mathbb{A}, \mathbb{R} \) and \( \mathbb{N} \) of \( P \) consisting of actual, regular and null events, respectively.

Before proceeding, let us prove a simple, but very important fact.

**Proposition 3.1.** Let \( e \in P; \)

1. if \( \mathbb{P}[e] > 0 \), then (\( \exists n \in \mathbb{N}^+ \))(\( e(n) = 1 \));
2. if (\( \forall n \in \mathbb{N}^+ \))(\( e(n) = 0 \)), then \( \mathbb{P}[e] =_\mathbb{R} 0 \).

**Proof.** The second point is immediate since (\( \forall n \in \mathbb{N}^+ \))(\( e(n) = 0 \)) means exactly \( e =_\mathbb{P} \perp \).

Suppose now that (\( e, \gamma \)) \( \in \mathbb{A} \) and (\( \exists n \in \mathbb{N}^+ \))(\( e(n) = 1 \)), by definition of \( > \) between real numbers, there exists \( m \in \mathbb{N} \) such that \( \mathbb{P}(e, \gamma)(m) > \frac{1}{m} \), i.e.,

\[ \sum_{i=1}^{\gamma(m)} me(i) > \gamma(m) > 0 \]

Hence there exists \( n \) such that \( 1 \leq n \leq \gamma(m) \) and \( e(n) = 1 \). Thus (\( \exists n \in \mathbb{N}^+ \))(\( e(n) = 1 \)). \( \square \)

We can now introduce a new family of “probabilistic” versions of the limited principle of omniscience.

**Definition 3.2.** For every subset \( \mathcal{E} \) of \( P \) we define the following principles:

1. \( \text{LPO}[\mathcal{E}] \) \( (\forall e \in P)[e \in \mathcal{E} \rightarrow (\forall n \in \mathbb{N}^+)(e(n) = 0) \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)] \)
2. \( \text{PP-LPO}[\mathcal{E}] \) \( (\forall e \in P)[e \in \mathcal{E} \rightarrow \mathbb{P}(e) = 0 \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)] \)
3. \( \text{PPP-LPO}[\mathcal{E}] \) \( (\forall e \in P)[e \in \mathcal{E} \rightarrow \mathbb{P}(e) = 0 \lor \mathbb{P}(e) > 0] \)

First of all, one can notice that \( \text{LPO}[P] \) is exactly \( \text{LPO} \). Let us now study the relation between these probabilistic versions of \( \text{LPO} \). As a direct consequence of proposition 3.1 we have the following

**Proposition 3.3.** If \( \mathcal{E} \) is a subset of \( P \), then

1. \( \text{PP-LPO}[\mathcal{E}] \rightarrow \text{PP-LPO}[\mathcal{E}] \)
2. \( \text{LPO}[\mathcal{E}] \rightarrow \text{PP-LPO}[\mathcal{E}] \)

**Proof.** Points (1) and (2) follow from points (1) and (2) in proposition 3.1 respectively. \( \square \)
Moreover we have

**Proposition 3.4.** If $\mathcal{E} \subseteq \mathcal{E}'$ are subsets of $\mathcal{P}$, then

1. $\text{LPO}[\mathcal{E}'] \rightarrow \text{LPO}[\mathcal{E}]$
2. $\text{P-LPO}[\mathcal{E}'] \rightarrow \text{P-LPO}[\mathcal{E}]$
3. $\text{PP-LPO}[\mathcal{E}'] \rightarrow \text{PP-LPO}[\mathcal{E}]$

**Proof.** This is an obvious consequence of the definitions.

Some of these principles can be proven constructively.

**Proposition 3.5.** The following hold:

1. $\text{LPO}[\mathcal{R}]$
2. $\text{P-LPO}[\mathcal{R}]$
3. $\text{PP-LPO}[\mathcal{R}]$
4. $\text{PP-LPO}[\mathcal{N}]$
5. $\text{P-LPO}[\mathcal{N}]$
6. $\neg \text{PP-LPO}[\mathcal{P}]$

**Proof.** (1) $\text{LPO}[\mathcal{R}]$ holds as it reduces to control, for each $e = \parallel \alpha, \pi \parallel$ in $\mathcal{R}$, a finite number of entries (those in $\alpha$ and $\pi$). (2) is a consequence of (1) and proposition 3.3. For point (3), if $e = \parallel \alpha, \pi \parallel$ in $\mathcal{R}$, then $\mathbb{P}(e) = \sum_{j=1}^{\ell(\pi)} \pi_j \in \mathbb{Q}$. Hence we can decide whether $\mathbb{P}(e) = 0$ or $\mathbb{P}(e) > 0$. Point (4) is obvious, since $e \in \mathcal{N}$ means, by definition, that $\mathbb{P}(e) = 0$. (5) is a consequence of (4) and proposition 3.3. For point (6), there exist potential events $e$ for which there is no $\gamma$ such that $(e, \gamma) \in \tilde{\mathcal{A}}$. Consider for example the sequence 10110011110000... which alternates a group of $2^n$ ones and a group of $2^n$ zeros increasing $n$ at each step.

Moreover we have the following result.

**Proposition 3.6.** For every subset $\mathcal{E}$ of $\mathcal{P}$

$$(\text{P-LPO}[\mathcal{E}] \land \text{LPO}[\mathcal{N}]) \rightarrow \text{LPO}[\mathcal{E}]$$

**Proof.** Suppose $\mathbb{P} - \text{LPO}[\mathcal{E}]$ holds. Then $(\forall e \in \mathcal{P}) [e \in \mathcal{E} \rightarrow \mathbb{P}(e) = 0 \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)]$ which is equivalent to $(\forall e \in \mathcal{P}) [e \in \mathcal{E} \rightarrow e \in \mathcal{N} \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)]$. If $\text{LPO}[\mathcal{N}]$ holds too, then $(\forall e \in \mathcal{P}) [e \in \mathcal{E} \rightarrow ((\forall n \in \mathbb{N}^+)(e(n) = 0) \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)) \lor (\exists n \in \mathbb{N}^+)(e(n) = 1)]$ which is equivalent to $(\forall e \in \mathcal{P}) [e \in \mathcal{E} \rightarrow ((\forall n \in \mathbb{N}^+)(e(n) = 0) \lor (\exists n \in \mathbb{N}^+)(e(n) = 1))]$ which is $\text{LPO}[\mathcal{E}]$. In particular, since $\mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{P}$ and propositions 3.3 and 3.4 hold, (1), (2) and (3) follow.
Hence we can summarize the situation through the following diagram.

\[
\begin{array}{cccc}
\text{LPO} & \equiv & \text{LPO}[\mathcal{P}] & \longrightarrow \quad \mathbb{P} - \text{LPO}[\mathcal{P}] \\
\downarrow & & \downarrow & \\
\text{LPO}[\mathcal{A}] & \longrightarrow & \mathbb{P} - \text{LPO}[\mathcal{A}] & \quad \mathbb{P} \mathbb{P} - \text{LPO}[\mathcal{A}] \\
\downarrow & & \downarrow & \\
\text{LPO}[\mathcal{N}] & \longrightarrow & \top & \\
\end{array}
\]

We conclude by recalling that both \( \text{LPO} \equiv \text{LPO}[\mathcal{P}] \) and \( \mathbb{P} \mathbb{P} - \text{LPO}[\mathcal{A}] \) can be proven classically.

4. Toward a stratified notion of probability?

As we saw in section 2, there is a 3-layers structure arising from the constructive frequentist approach to probability via natural density: an algebra of actual events is carved out of an algebra of potential events by extending the notion of frequency of finite lists of 0s and 1s (which are identified, up to a finite number of errors, with regular events).

We now recall other situations from classical mathematics, in which a notion of probability is “constructed” from a simpler notion and in which such a 3-layers structure emerges.

4.1. Definition of a probability using a cumulative distribution function. A cumulative distribution function is a function \( F : \mathbb{R} \rightarrow [0, 1] \) such that

1. \( \lim_{x \rightarrow +\infty} F(x) = 1 \);
2. \( \lim_{x \rightarrow -\infty} F(x) = 0 \);
3. \( F \) is non-decreasing;
4. \( F \) is right continuous.

Such a function determines a probability \( \mathbb{P} \) on the algebra \( \mathcal{R} \) of subsets of \( \mathbb{R} \) which are finite unions of left-open right-closed intervals (including unbounded intervals and the empty set). One just have to define \( \mathbb{P}((a, b]) := F(b) - F(a) \), \( \mathbb{P}((\infty, b]) := F(b) \), \( \mathbb{P}((a, +\infty)) := 1 - F(a) \), \( \mathbb{P}(\emptyset) := 0 \) and \( \mathbb{P}(\mathbb{R}) := 1 \) and, then, for finite disjoint unions of such intervals \( \mathbb{P}(\bigcup_{i=1}^{n} I_i) = \sum_{i=1}^{n} \mathbb{P}(I) \).

If \( \mathcal{B}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R}) \) is the \( \sigma \)-algebra of Borel subsets of \( \mathbb{R} \), i.e. the \( \sigma \)-algebra generated by open subsets of \( \mathbb{R} \) (which coincides with the \( \sigma \)-algebra generated by \( \mathcal{R} \)), then one can prove (see e.g. [3]) that there exists a unique probability measure \( \mathbb{P} \) on \( \mathcal{B}(\mathbb{R}) \) such that \( \mathbb{P}(E) = \mathbb{P}(E) \) for every \( E \in \mathcal{R} \).

4.2. Caratheodory construction. If \( \mathbb{P}^* : \mathcal{P}(\Omega) \rightarrow [0, 1] \) is an outer probability measure, i.e. if \( \mathbb{P}^* \) satisfies the following:

1. \( \mathbb{P}^*(\emptyset) = 0 \) and \( \mathbb{P}^*(\Omega) = 1 \)
2. if \( E \subseteq E' \), then \( \mathbb{P}^*(E) \leq \mathbb{P}^*(E') \)
3. \( \mathbb{P}^*(\bigcup_{n \in \mathbb{N}} E_n) \leq \sum_{n \in \mathbb{N}} \mathbb{P}^*(E_n) \)
then there is a standard way to produce the largest $\sigma$-algebra on which $P^*$ is a probability measure: this $\sigma$-algebra is the so-called Caratheodory algebra (see e.g. [3])

$$C := \{E \in P(\Omega)\mid \text{for every } X \in P(\Omega) \ (P^*(X) = P^*(X \setminus E) + P^*(X \cap E))\}$$

If $\Omega$ can be endowed with a metric $\delta$ such that for every $E, E' \subseteq \Omega$ with $\delta(E, E') > 0$

$$P^*(E \cup E') = P^*(E) + P^*(E') \ (\ast)$$

then the $\sigma$-algebra of Borel subsets $B(\Omega, \delta)$ is included in $A$. Concretely, $P^*$ is often defined via a valuation on open sets, e.g. in the case of Lebesgue measure in the interval $[0, 1]$.

### 4.3. Fuzzy subsets of a probability space.

Let $(\Omega, \mathcal{R}, \mathbb{P})$ be a Kolmogorov probability space, i.e. $\mathcal{R}$ is a $\sigma$-algebra of subsets of $\Omega$ (which is a Boolean algebra) and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{R})$. A fuzzy subset (see [10]) of $\Omega$ is a function $\varphi : \Omega \to [0, 1]$.

Fuzzy subsets of $\Omega$ form a Heyting algebra with

1. $\varphi \leq \psi$ if and only if $\varphi(\omega) \leq \psi(\omega)$ for every $\omega \in \Omega$;
2. $\bot(\omega) := 0$ as bottom;
3. $\top(\omega) := 1$ as top;
4. $\varphi \land \psi$ defined by $(\varphi \land \psi)(\omega) := \inf(\varphi(\omega), \psi(\omega))$ as infimum of $\varphi$ and $\psi$;
5. $\varphi \lor \psi$ defined by $(\varphi \lor \psi)(\omega) := \sup(\varphi(\omega), \psi(\omega))$ as supremum of $\varphi$ and $\psi$;
6. $\varphi \rightarrow \psi$ defined by

\[
\begin{cases}
(\varphi \rightarrow \psi)(\omega) := 1 & \text{if } \varphi(\omega) \leq \psi(\omega) \\
(\varphi \rightarrow \psi)(\omega) := \psi(\omega) & \text{otherwise}
\end{cases}
\]

as Heyting implication of $\varphi$ and $\psi$.

Moreover fuzzy subsets of $\Omega$ form a De Morgan algebra with involution $\neg$ defined by

$$\neg(\varphi)(\omega) := 1 - \varphi(\omega)$$

for every fuzzy subset $\varphi$ of $\Omega$ and every $\omega \in \Omega$.

Among fuzzy subsets of $\Omega$, there are some $\varphi$ which are integrable, i.e. for which the following integral is defined:

$$\int_{\Omega} \varphi \ d\mathbb{P}$$

Integrable fuzzy subsets are closed under $\land$, $\lor$, $\neg$ and include $\bot$ and $\top$, hence they form a de Morgan algebra. Moreover, if we define $\overline{\mathbb{P}}(\varphi) := \int_{\Omega} \varphi \ d\mathbb{P}$ for every integrable fuzzy subset $\varphi, \psi$ we have that

1. $\overline{\mathbb{P}}(\bot) = 0$.
2. $\overline{\mathbb{P}}(\neg\varphi) = 1 - \overline{\mathbb{P}}(\varphi)$;
3. $\overline{\mathbb{P}}(\varphi \land \psi) + \overline{\mathbb{P}}(\varphi \lor \psi) = \overline{\mathbb{P}}(\varphi) + \overline{\mathbb{P}}(\psi)$;
4. if $\varphi \leq \psi$, then $\overline{\mathbb{P}}(\varphi) \leq \overline{\mathbb{P}}(\psi)$.

If for every $E \in \mathcal{R}$ with $\mathbb{P}(E) = 0$, every subset $E'$ of $E$ is in $\mathcal{R}$, then the following property holds for fuzzy subsets of $\Omega$: if $\varphi$ is integrable, $\overline{\mathbb{P}}(\varphi) = 0$ and $\psi \leq \varphi$ is a fuzzy subset, then $\psi$ is integrable.

One can also notice that the set of those integrable fuzzy subsets $\varphi$ for which $\neg\varphi = \varphi \rightarrow \bot$ is exactly the set of characteristic functions of subsets in $\mathcal{R}$. Since subsets in $\mathcal{R}$ can be identified with their characteristic functions, $\overline{\mathbb{P}}$ can be considered as an extension of $\mathbb{P}$ as for every $E \in \mathcal{R}$, $\mathbb{P}(E) = \int_{\Omega} \chi_{\{E\}} \ d\mathbb{P} = \overline{\mathbb{P}}(\chi_{\{E\}})$. 

4.4. An attempt of stratified definition of probability structure. Inspired by the similarities in the situations above, we give here an attempt definition of probability structure.

Definition 4.1. If $\mathcal{P} = (\mathcal{P}, \leq, \land, \lor, \rightarrow, \bot, \top)$ is a Heyting algebra representing an algebra of potential events for which there exists a unary operator $\neg$ for which $(\mathcal{P}, \leq, \land, \lor, \neg, \bot, \top)$ is a de Morgan algebra, a probability structure on $\mathcal{P}$ can be defined as a triplet $(A, \mathcal{P}, R)$ such that

$$
\begin{align*}
A & \text{ set } & R & \subseteq A & A & \subseteq \mathcal{P} & \mathcal{P} & \in A \rightarrow \mathbb{R} \\
\bot & \in R & \top & \in R & \frac{e \in \mathcal{P}}{e \land e' \in R} & \frac{e \in \mathcal{P}}{e \lor e' \in R} & \frac{e \in \mathcal{P}}{-e \in R} \\
& e' \in A & \mathcal{P}(e') = 0 & e \leq e' & \frac{e \in A}{e \in A} & \frac{e \in A}{e \land e' \in A} & \frac{e \in A}{e \lor e' \in A} & \frac{e \in A}{-e \in A} \\
& \mathcal{P}(\bot) = 0 & \frac{e \in A}{e' \in A} & \frac{e \in \mathcal{P}}{e \leq e'} & \frac{e \in A}{e \lor e' \in A} & \frac{e \in A}{e \land e' \in A} & \frac{e \in A}{\mathcal{P}(e) \leq \mathcal{P}(e')} & \frac{\mathcal{P}(-e) = 1 - \mathcal{P}(e)}{\mathcal{P}(e \lor e') + \mathcal{P}(e \land e') = \mathcal{P}(e) + \mathcal{P}(e')}
\end{align*}
$$

$(R, \leq, \land, \lor, \neg, \bot, \top)$ is a boolean algebra.

Every Kolmogorov measure space $(\Omega, \mathcal{E}, \mathcal{P})$ is trivially an example of this structure (one just have to identify $\mathcal{P}$, $A$ and $R$ with the algebra of events $\mathcal{E}$). The definition of a probability (4.1) using a cumulative distribution function determines a probability structure by simply taking $\mathcal{P}$ and $A$ to be the algebra of Borel subsets of reals, $R$ to be the algebra $\mathcal{R}$, and $\mathcal{P}$ as probability function. Caratheodory construction (4.2) determines a probability structure by taking $\mathcal{P}$ to be the powerset algebra of a metric space $(\Omega, \delta)$, $A$ to be the Caratheodory algebra determined by an outer probability measure on $\mathcal{P}(\Omega)$ satisfying condition $(\ast)$, $R$ to be the algebra of Borel subsets of $(\Omega, \delta)$, and $\mathcal{P}^*$ as probability function. Also the construction in (4.3) determines a probability structure when for every $E \in R$ with $\mathcal{P}(E) = 0$, every subset $E'$ of $E$ is in $\mathcal{R}$: one can take $\mathcal{P}$ to be the set of fuzzy sets, $A$ to be the set of integrable fuzzy sets, $R$ to be the set of (characteristic functions of) subsets in $\mathcal{R}$, and $\mathcal{P}$ as probability function. Finally, our constructive natural density determines a probability structure: $\mathcal{P}$, $A$ and $R$ are the algebras of potential, actual and regular events, while $\mathcal{P}$ is the natural density defined in section 2.

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