Couplings in Renormalizable Supersymmetric SO(10) Models

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Abstract

We study the most general renormalizable couplings containing Higgs $H(10)$, $D(120)$, $\overline{\Delta(126)} + \Delta(126)$, $A(45)$, $E(54)$ and $\Phi(210)$ in the supersymmetric SO(10) models. The Clebsch-Gordan Coefficients are calculated using the maximal subgroup $SU(5) \times U(1)_X$.

Keywords: GUT, symmetry breaking, Clebsch-Gordan Coefficients
1 Introduction

Supersymmetric (SUSY) Grand Unification Theories (GUTs) of SO(10)\textsuperscript{[1, 2]} are very important candidates for the new physics beyond the Standard Model (SM). Since the SM gauge group \(G_{321} = SU(3)_C \times SU(2)_L \times U(1)_Y\) is not a maximal subgroup of SO(10), there are many different routines to break SO(10) into \(G_{321}\) through approximately intermediate symmetries. Two maximal subgroups \(G_{422} = SU(4)_C \times SU(2)_L \times SU(2)_R\) and \(G_{51} = SU(5) \times U(1)_X\) are usually taken as the intermediate symmetries. In practice, even without any intermediate symmetry breaking scale, the maximal subgroups are used to distinguish different states which are in the same SO(10) representations but also have the same SM representations.

Among the many systematic studies on the SUSY SO(10) models\textsuperscript{[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]}, many of them focus on the so-called Minimal SUSY SO(10) (MSSO10)\textsuperscript{[9, 10, 11]} which contains the Higgs superfields \(H(10), \Delta(126) + \Delta(126)\) and \(\Phi(210)\). The simplest extension of MSSO10 which contains one more \(D(120)\) is studied in\textsuperscript{[19, 20, 21]}. There are also models which use \(A(45)\) and \(E(54)\) instead of \(\Phi(210)\) to break SO(10)\textsuperscript{[6, 23]}. Most of the previous studies use the \(G_{422}\) subgroup, except in\textsuperscript{[22]} where the MSSO10 is studied using the \(G_{51}\).

Using \(G_{422}\), the most general renormalizable couplings containing fields \(H(10), D(120), \Delta(126) + \Delta(126), A(45), E(54)\) and \(\Phi(210)\) have been studied in\textsuperscript{[12]}. This general model offers powerful tools for building realistic models which may need all these fields\textsuperscript{[24]}. However, the general model has not been checked entirely and an independent treatment using \(G_{51}\) subgroup is absent. The realization of gauge coupling unification\textsuperscript{[25, 26, 27, 28]} can be most easily discussed within the SU(5) subgroup of SO(10), even this SU(5) is only approximate. Also, in the missing partner model of SO(10)\textsuperscript{[29]}, the solution to the outstanding problem of doublet-triplet splitting depends on the group structure of SU(5). The recent study of the \(B - L = -2\) operators is also based on the SU(5) language\textsuperscript{[22]}. Thus a study of the most general renormalizable couplings containing fields \(H(10), D(120), \Delta(126) + \Delta(126), A(45), E(54)\) and \(\Phi(210)\) based on the \(G_{51}\) subgroup is highly desirable, both as an independent check on\textsuperscript{[12]} and as a tool for future model building, compliment to the \(G_{422}\) approach.

In the present work, we will study the SUSY SO(10) models using its maximal subgroup \(G_{51}\). Using the most general renormalizable superpotential as in\textsuperscript{[12]}, we will calculate the Clebsch-Gordan Coefficients (CGCs) using \(G_{51}\). After introducing notations and explaining the states in Section 2, we will give illustrative examples on the CGC calculations in Section 3, study the superpotential and the symmetry breaking conditions in Section 4, and present all the masses and mass matrices in Section 5. Then we will compare our results with those in the literature in Section 6 and summarize in Section 7.

2 Notations and states

A SO(2N) group can be studied using the SU(N) basis\textsuperscript{[30]}. Our SO(10) notations mainly follow\textsuperscript{[12]} in the SU(5) or the \(Y\) diagonal basis\textsuperscript{[5]} defined by \(1 \equiv 1' + 2'i, 2 \equiv 1' - 2'i, \ldots, 9 \equiv 9' + 0'i, 0 \equiv 9' - 0'i\), besides a normalization factor \(\frac{1}{\sqrt{2}}\). Here \(1', 2', \ldots, 9', 0'\) are the conventional SO(10) basis. Then \((1, 3, 5, 7, 9)\) and \((2, 4, 6, 8, 0)\) transform as \((5, 2)\) and \((\overline{5}, -2)\), respectively, under
SU(5) × U(1)\(_X\). Also, in larger SU(5) representations, they correspond to the superscripts and the subscripts, respectively, of tensors of higher ranks. We will take 1, 2, 3, 4 as the weak indices, and 6, 7, 8, 9, 0 as the color indices.

For \(H(10)\) of SO(10), it contains

\[
\hat{H}^{(1/2,0)}_{(5,2)} = [1, 3], \quad \hat{H}^{(3,1,1/2)}_{(5,2)} = [5, 7, 9], \\
\hat{H}^{(1,2,1/2)}_{(5,-2)} = [2, 4], \quad \hat{H}^{(3,1,1/2)}_{(5,-2)} = [6, 8, 0].
\]

(1)

Hereon we will denote a state as \(R_i^{(J, x)}\), where \(R\) and \(J\) are the SO(10) and SU(5) representations, respectively, \(x\) is the \(U(1)\)_\(X\) quantum number, and \(i\) is the SM representation.

The states in the anti-symmetric \(A(45)\) of SO(10) can be constructed from the product of two (different) \(H(10)\)s. \((5 \otimes \overline{5})_A = 1 \oplus 24\) gives

\[
\left(\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array}\right) \otimes \left(\begin{array}{ccccc} 2 & 4 & 6 & 8 & 0 \end{array}\right) - \left(\begin{array}{c} 2 \\ 4 \\ 6 \\ 8 \\ 0 \end{array}\right) \otimes \left(\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \end{array}\right) \right)^T = \left(\begin{array}{ccccc} 12 & 14 & 16 & 18 & 10 \\ 32 & 34 & 36 & 38 & 30 \end{array}\right),
\]

so that

\[
\hat{A}^{(1,1,0)}_{(1,0)} = \frac{i}{\sqrt{10}}(12 + 34 + 56 + 78 + 90),
\]

\[
\hat{A}^{(1,1,0)}_{(24,0)} = \frac{i}{\sqrt{60}}(3 \times [12 + 34] - 2 \times [56 + 78 + 90]),
\]

\[
\hat{A}^{(1,3,0)}_{(24,0)} = \frac{i}{\sqrt{2}}(14, 32, \frac{12 - 34}{\sqrt{2}}),
\]

\[
\hat{A}^{(8,1,0)}_{(24,0)} = \frac{i}{\sqrt{2}}(58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{2}}[56 - 78], \frac{1}{\sqrt{6}}[56 + 78 - 2[90]]),
\]

\[
\hat{A}^{(3,2,1/2)}_{(24,0)} = \frac{i}{\sqrt{2}}[25, 27, 29, 45, 47, 49] = \frac{i}{\sqrt{2}}(5, 7, 9),
\]

\[
\hat{A}^{(3,2,1/2)}_{(24,0)} = -\frac{i}{\sqrt{2}}[1, 3] [6, 8, 0],
\]

(2)

where the parenthesis means anti-symmetrization \((ab) = ab - ba\), and the square bracket are used for grouping of indices into complete SU(2)_L and SU(3)_C representations. In the brackets, the italicized integer numbers stand for numerical factors to be distinguished from the SO(10) indices. The anti-symmetric part of the SU(5) product \((5 \otimes \overline{5})_A = 10\) gives

\[
\left(\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array}\right) \otimes \left(\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \end{array}\right) - \left(\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array}\right) \otimes \left(\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \end{array}\right) \right)^T = \left(\begin{array}{ccccc} 0 & 13 & 15 & 17 & 19 \\ 31 & 0 & 35 & 37 & 39 \end{array}\right),
\]

(3)

\[
\left(\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array}\right) \otimes \left(\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \end{array}\right) - \left(\begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array}\right) \otimes \left(\begin{array}{ccccc} 1 & 3 & 5 & 7 & 9 \end{array}\right) \right)^T = \left(\begin{array}{ccccc} 0 & 13 & 15 & 17 & 19 \\ 31 & 0 & 35 & 37 & 39 \end{array}\right),
\]

(4)
Under the complex conjugation (c.c), we have
\[
\mathring{\Delta}_{(10,4)}^{(1,1,1)} = \frac{i}{\sqrt{2}} (13),
\]
\[
\mathring{\Delta}_{(10,4)}^{(3,2,\pm)} = \frac{i}{\sqrt{2}} (15, 17, 19, 35, 37, 39) = \frac{i}{\sqrt{2}} \left( [1, 3][5, 7, 9] \right),
\]
\[
\mathring{\Delta}_{(10,4)}^{(3,1,\mp)} = \frac{i}{\sqrt{2}} (79, 95, 57),
\]
and \((\bar{5} \otimes \bar{5})_A = 10\) gives
\[
\begin{pmatrix}
2 \\
4 \\
6 \\
8 \\
0
\end{pmatrix} \otimes \begin{pmatrix}
2 \\
4 \\
6 \\
8 \\
0
\end{pmatrix} - \begin{pmatrix}
2 \\
4 \\
6 \\
8 \\
0
\end{pmatrix} \otimes \begin{pmatrix}
2 \\
4 \\
6 \\
8 \\
0
\end{pmatrix} = \begin{pmatrix}
0 & (24) & (26) & (28) & (20) \\
(42) & 0 & (46) & (48) & (40)
\end{pmatrix},
\]
so that
\[
\mathring{\Delta}_{(10,4)}^{(1,1,1)} = -\frac{i}{\sqrt{2}} (24),
\]
\[
\mathring{\Delta}_{(10,4)}^{(3,2,\pm)} = -\frac{i}{\sqrt{2}} (26, 28, 20, 46, 48, 40) = -\frac{i}{\sqrt{2}} \left( [2, 4][6, 8, 0] \right),
\]
\[
\mathring{\Delta}_{(10,4)}^{(3,1,\mp)} = -\frac{i}{\sqrt{2}} (80, 06, 80).
\]
In \((3,5,7)\) the factor \(i\) or \(-i\) are included in accord with the notations in [12]. The states in \(E(54)\) of \(SO(10)\) can be constructed in a similar way, \(e.g.,\)
\[
\mathring{\hat{E}}_{(24,0)}^{(1,1,0)} = \frac{1}{\sqrt{60}} \{ 3 \times [12 + 34] - 2 \times [56 + 78 + 90] \}.
\]
Here the curly brackets stand for symmetrization \([ab] = ab + ba\).

Higher ranked representations of \(SO(10)\) can be constructed similarly. \(D(120)\) can be constructed from anti-symmetrizing \(10 \otimes 45\), and \(\Delta(126) - \mathring{\Delta}(126)\) from \(45 \otimes 120\). For example,
\[
\mathring{\hat{D}}_{(10,6)}^{(1,1,-1)} = \frac{1}{N} \left( \mathring{\hat{H}}_{(5,2)}^{(3,1,-\frac{1}{2})} \otimes \mathring{\hat{A}}_{(10,4)}^{(3,1,-\frac{1}{2})} \right)_A = \frac{1}{N} \left( [5, 7, 9] \otimes (79, 95, 57) \right)_A = \frac{1}{\sqrt{6}} (579),
\]
\[
\mathring{\hat{\Delta}}_{(1,10)}^{(1,1,0)} = \frac{1}{N'} \left( \mathring{\hat{D}}_{(10,6)}^{(1,1,-1)} \otimes \mathring{\hat{A}}_{(10,4)}^{(1,1,1)} \right)_A = \frac{1}{N'} \left( (579) \otimes (13) \right)_A = \frac{1}{\sqrt{120}} (13579),
\]
where \(N, N'\) are normalization factors which are only determined at the ends of the identities. Under the complex conjugation (c.c), we have \(\mathring{T} = 2, \mathring{3} = 4, \mathring{5} = 6, \mathring{7} = 8, \mathring{9} = 0,\) and vice versa.
So, we have
\[
\mathring{\Delta}_{(1,-10)}^{(1,1,0)} = \frac{1}{\sqrt{120}} (13579) = \frac{1}{\sqrt{120}} (24680).
\]
As can be easily verified, \( \hat{\Delta}^{(1,1,1)}_{(1,1,1,0)} \) and \( \hat{\Delta}^{(1,1,1)}_{(1,1,1,-10)} \) satisfy

\[
\begin{align*}
\hat{\Delta}^{(1,1,1)}_{(1,1,1,0)} & = \hat{\Delta}^{(1,1,1)}_{(1,1,1,-10)}, \\
\hat{\Delta}^{(1,1,1)}_{(1,1,1,0)} & = -\hat{\Delta}^{(1,1,1)}_{(1,1,1,-10)},
\end{align*}
\]

(11)

where \( i\varepsilon_{1234567890} = 1 \).

\( \Phi(210) \) can be constructed from \( 45 \otimes 45 \), e.g.,

\[
\begin{align*}
\hat{\phi}^{(1,1,1,0)}_{(1,0)} & = \frac{1}{N_1} \hat{A}^{(1,1,1,0)}_{(1,0)} \otimes \hat{\Delta}^{(1,1,1,0)}_{(1,0)} \\
& = -\frac{1}{\sqrt{240}} (1234 + 1256 + 1278 + 1296 + 3456 + 3478 + 3496 + 5678 + 5690 + 7890) \\
& = -\frac{1}{\sqrt{240}} (1234 + [5678 + 5690 + 7890] + [12 + 34][56 + 78 + 90]),
\end{align*}
\]

(12)

\[
\begin{align*}
\hat{\phi}^{(1,1,1,0)}_{(24,0)} & = \frac{1}{N_2} \hat{A}^{(1,1,1,0)}_{(24,0)} \otimes \hat{A}^{(1,1,1,0)}_{(24,0)} \\
& = -\frac{1}{N_2} ([12 + 34] + [56 + 78 + 90]) \otimes (6 \times [12 + 34] - 2 \times [56 + 78 + 90]) \\
& = -\frac{1}{\sqrt{90 \times 24}} (6 \times [1234] + [12 + 34][56 + 78 + 90] - 4 \times [5678 + 5690 + 7890]),
\end{align*}
\]

(13)

where \( N_{1,2} \) are normalization factors, and

\[
\hat{A}^{(1,1,1,0)}_{(24,0)} \otimes \hat{A}^{(1,1,1,0)}_{(24,0)} = (9 \times [1234] - 6[12 + 34][56 + 78 + 90] + 4 \times [5678 + 5690 + 7890])
\]

contains not only \( \hat{\phi}^{(1,1,1,0)}_{(1,0)} \) and \( \hat{\phi}^{(1,1,1,0)}_{(24,0)} \), but also a state orthogonal to them which is

\[
\hat{\phi}^{(1,1,1,0)}_{(75,0)} = -\frac{1}{\sqrt{18 \times 24}} (9 \times [1234] - [12 + 34][56 + 78 + 90] + [5678 + 5690 + 7890]).
\]

(14)
The SM singlets which break SO(10) into its SM subgroup when they get VEVs are:

\[ \hat{a}_1 \equiv \hat{A}^{(1,1,0)}_{(1,0)} = \frac{i}{\sqrt{10}}(12 + 34 + 56 + 78 + 90), \]

\[ \hat{a}_2 \equiv \hat{A}^{(1,1,0)}_{(24,0)} = \frac{i}{\sqrt{60}}(3 \times [12 + 34] - 2 \times [56 + 78 + 90]), \]

\[ \hat{E} \equiv \hat{E}^{(1,1,0)}_{(24,0)} = \frac{1}{\sqrt{60}}\{3 \times [12 + 34] - 2 \times [56 + 78 + 90]\}, \]

\[ \hat{V}_R \equiv \hat{\Delta}^{(1,1,0)}_{(1,-10)} = \frac{1}{\sqrt{120}}(24680), \]

\[ \hat{V}_R \equiv \hat{\Delta}^{(1,1,0)}_{(1,10)} = \frac{1}{\sqrt{120}}(13579), \]

\[ \hat{\phi}_1 \equiv \hat{\Phi}^{(1,1,0)}_{(1,0)} = -\frac{1}{\sqrt{240}}(1234 + [5678 + 5690 + 7890] + [12 + 34][56 + 78 + 90]), \]

\[ \hat{\phi}_2 \equiv \hat{\Phi}^{(1,1,0)}_{(24,0)} = -\frac{1}{\sqrt{90 \times 24}}(6 \times [1234] - 4 \times [5678 + 5690 + 7890] + [12 + 34][56 + 78 + 90]), \]

\[ \hat{\phi}_3 \equiv \hat{\Phi}^{(1,1,0)}_{(75,0)} = -\frac{1}{\sqrt{18 \times 24}}(3 \times [1234] + [5678 + 5690 + 7890] - [12 + 34][56 + 78 + 90]). \]

A minus sign is included in (12-14) following [12]. The normalization factors before are chosen according to the requirement of \( \hat{R}_{(I,x)}\hat{R}^*_{(I,-x)} = 1 \). Note that the extra factors ±i in A and −1 in \( \phi_{1,2,3} \) follow [12], so that there may exist extra minus sign in some CGCs involving them which actually do not violate the symmetries in the CGCs.

The other states are summarized in Table 1-5, where different states of the same SM representations in a same SO(10) field are distinguished by their different representations under \( G_{51} \). In Table 1 all would-be Goldstone states are given.
Table 1: States in the would-be Goldstone modes.

| State Parameters | States in the Goldstone modes |
|------------------|-------------------------------|
| $(1, 1, 1) + c.c.$ | $A_{(10,4)}^{(1,1,1)}$, $\hat{D}_{(10,6)}^{(1,1,1)}$, $\Delta_{(10,6)}^{(1,1,1)}$, $\Phi_{(10,4)}^{(1,1,1)}$ | $\frac{1}{\sqrt{2}} (13) + c.c.$, $\frac{1}{\sqrt{2}} (680) + c.c.$, $\frac{1}{\sqrt{240}} ([12 + 34] 680) + c.c.$, $\frac{1}{\sqrt{240}} (13 [56 + 78 + 90]) + c.c.$ |
| $(3, 1, \frac{2}{3}) + c.c.$ | $A_{(10,4)}^{(3,1,\frac{2}{3})}$, $\Delta_{(10,6)}^{(3,1,\frac{2}{3})}$, $\Phi_{(10,4)}^{(3,1,\frac{2}{3})}$ | $\frac{1}{\sqrt{2}} (80, 06, 68) + c.c.$, $\frac{1}{\sqrt{2}} (13 [5, 7, 9]) + c.c.$, $\frac{1}{\sqrt{240}} (13 [78 + 90], 7 [56 + 90], 9 [56 + 78]) + c.c.$, $\frac{1}{\sqrt{144}} (2 [5680, 7806, 9068] - [12 + 34] [80, 06, 68]) + c.c.$ |
| $(3, 2, -\frac{5}{6}) + c.c.$ | $A_{(24,0)}^{(3,2,-\frac{5}{6})}$, $\hat{D}_{(24,0)}^{(3,2,-\frac{5}{6})}$, $\hat{\Phi}_{(24,0)}^{(3,2,-\frac{5}{6})}$, $\Phi_{(75,0)}^{(3,2,-\frac{5}{6})}$ | $\frac{1}{\sqrt{2}} ([2, 4] [5, 7, 9]) + c.c.$, $\frac{1}{\sqrt{2}} ([2, 4] [5, 7, 9]) + c.c.$, $\frac{1}{\sqrt{2}} (234, 124 [5, 7, 9] + [2, 4] [578 + 90], 7 [56 + 90], 9 [56 + 78]) + c.c.$, $\frac{1}{\sqrt{240}} (2 \times [234, 124] [5, 7, 9] - [2, 4] [578 + 90], 7 [56 + 90], 9 [56 + 78]) + c.c.$ |
| $(3, 2, \frac{1}{6}) + c.c.$ | $A_{(10,4)}^{(3,2,\frac{1}{6})}$, $\Delta_{(10,6)}^{(3,2,\frac{1}{6})}$, $\Phi_{(10,4)}^{(3,2,\frac{1}{6})}$ | $\frac{1}{\sqrt{2}} ([1, 3] [5, 7, 9]) + c.c.$, $\frac{1}{\sqrt{2}} ([1, 3] [5, 7, 9]) + c.c.$, $\frac{1}{\sqrt{6}} ([-4, 2] [80, 06, 68]) + c.c.$, $\frac{1}{\sqrt{240}} \left( [124, 234] [80, 06, 68] + ([-4, 2] [5680, 7806, 9068]) \right) + c.c.$, $\frac{1}{\sqrt{144}} (2 \times [134, 123] [5, 7, 9] + [1, 3] [578 + 90], 7 [56 + 90], 9 [56 + 78]) + c.c.$, $\frac{1}{\sqrt{144}} (2 \times [134, 123] [5, 7, 9] - [1, 3] [578 + 90], 7 [56 + 90], 9 [56 + 78]) + c.c.$ |
Table 2: States in \([\{(1, 2, \frac{1}{2}) + \text{c.c.}\}].

| \((1, 2, \frac{1}{2}) + \text{c.c.}\) | \(\hat{H}_{(5, 2)}^{(1, 2, \frac{1}{2})}\) | \(\hat{D}_{(5, 2)}^{(1, 2, \frac{1}{2})}\) | \(\hat{D}_{(45, 2)}^{(1, 2, \frac{1}{2})}\) | \(\hat{\Delta}_{(45, 2)}\) | \(\hat{\Phi}_{(5, -8), (5, \frac{1}{2})}\) | \([1, 3] + \text{c.c.}\) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \(\frac{1}{\sqrt{24}} \left(\left[134, 123\right] + [1, 3] \left[56 + 78 + 90\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{24}} \left(3 \times \left[134, 123\right] - [1, 3] \left[56 + 78 + 90\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{720}} \left(\left[134, 123\right] \left[56 + 78 + 90\right] - \left[1, 3\right] \left[5678 + 5690 + 7890\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{720}} \left(\left[134, 123\right] \left[56 + 78 + 90\right] + \left[1, 3\right] \left[5678 + 5690 + 7890\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{720}} \left([-4, 2] \times 680\right) + \text{c.c.}\) |

Table 3: States in \([\{(3, 1, -\frac{1}{3}) + \text{c.c.}\}].

| \((3, 1, -\frac{1}{3}) + \text{c.c.}\) | \(\hat{H}_{(5, 2)}^{(3, 1, -\frac{1}{3})}\) | \(\hat{D}_{(5, 2)}^{(3, 1, -\frac{1}{3})}\) | \(\hat{D}_{(45, 2)}^{(3, 1, -\frac{1}{3})}\) | \(\hat{\Delta}_{(45, 2)}\) | \(\hat{\Phi}_{(5, -8), (5, -1)}\) | \([5, 7, 9] + \text{c.c.}\) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \(\frac{1}{\sqrt{24}} \left(\left[12 + 34\right] \left[5, 7, 9\right] + \left[5, 78 + 90\right], 7 \left[56 + 90\right], 9 \left[56 + 78\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{24}} \left(\left[12 + 34\right] \left[5, 7, 9\right] - \left[5, 78 + 90\right], 7 \left[56 + 90\right], 9 \left[56 + 78\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{240}} \left(\left[1234 \left[5, 7, 9\right]\right] - \left(57890, 56790, 56789\right)\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{240}} \left(\left[1234 \left[5, 7, 9\right]\right] + \left(57890, 56790, 56789\right)\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{1440}} \left(\left[12 + 34\right] \left[5, 78 + 90\right], 7 \left[56 + 90\right], 9 \left[56 + 78\right]\right) + \text{c.c.}\) | \(\frac{1}{\sqrt{24}} \left(24 \left[80, 06, 68\right]\right) + \text{c.c.}\) |
Table 4: States in the other $G_{321}$ multiplets including $\Delta$ or $\overline{\Delta}$.

| Multiplet | States |
|-----------|--------|
| $\left(\frac{1}{2}, 1, \frac{1}{2}\right) + c.c.$ | $\Delta^{(1,1,2)}_{(50,-2)} \frac{1}{\sqrt{120}} (13680) + c.c.$ |
| $\left(1, 3, 1\right) + c.c.$ | $\mathcal{E}^{(1,3,1)}_{(15,4)} \mathcal{\Delta}^{(1,3,1)}_{(15,-6)} \left[11, -33, -\frac{13}{\sqrt{2}}\right] + c.c.$ |
| $\left(3, 1, -\frac{4}{3}\right) + c.c.$ | $\mathcal{D}^{(3,1,-\frac{4}{3})}_{(45,-2)} \mathcal{\Delta}^{(3,1,-\frac{4}{3})}_{(45,-2)} \frac{1}{\sqrt{240}} (24 [5 \{78 + 90\}, 7 \{56 + 90\}, 9 \{56 + 78\}]) + c.c.$ |
| $\left(3, 2, \frac{7}{6}\right) + c.c.$ | $\mathcal{D}^{(3,2,\frac{7}{6})}_{(35,-2)} \mathcal{\Delta}^{(3,2,\frac{7}{6})}_{(35,-2)} \frac{1}{\sqrt{240}} ([134, 123] \{80, 06, 68\} - [1, 3] \{5680, 7806, 9068\}) + c.c.$ |
| $\left(6, 1, -\frac{2}{3}\right) + c.c.$ | $\mathcal{D}^{(6,1,-\frac{2}{3})}_{(15,4)} \mathcal{\Delta}^{(6,1,-\frac{2}{3})}_{(15,-6)} \frac{1}{\sqrt{120}} (55, 77, 99, \frac{179}{\sqrt{2}}, \frac{95}{\sqrt{2}}, \frac{57}{\sqrt{2}}) + c.c.$ |
| $\left(6, 1, \frac{1}{3}\right) + c.c.$ | $\mathcal{D}^{(6,1,\frac{1}{3})}_{(35,-2)} \mathcal{\Delta}^{(6,1,\frac{1}{3})}_{(35,-2)} \frac{1}{\sqrt{120}} (580, 670, 689, \frac{60 [90-78]}{\sqrt{2}}, \frac{8 [56-90]}{\sqrt{2}}, \frac{0 [78-56]}{\sqrt{2}}) + c.c.$ |
| $\left(6, 1, \frac{2}{3}\right) + c.c.$ | $\mathcal{D}^{(6,1,\frac{2}{3})}_{(50,2)} \mathcal{\Delta}^{(6,1,\frac{2}{3})}_{(50,2)} \frac{1}{\sqrt{120}} (13 \{580, 670, 689, 5 [90-78], 8 [56-90], 0 [78-56]\}) + c.c.$ |
| $\left(6, 3, \frac{1}{3}\right) + c.c.$ | $\mathcal{D}^{(6,3,\frac{1}{3})}_{(50,-2)} \mathcal{\Delta}^{(6,3,\frac{1}{3})}_{(50,-2)} \mathcal{\Delta}^{(1,3,1)}_{(15,6)} \frac{1}{\sqrt{120}} (14, 32, \frac{12-34}{\sqrt{2}}, \frac{6 [90-78], 8 [56-90], 0 [78-56]}{\sqrt{2}}) + c.c.$ |
| $\left(8, 2, \frac{1}{2}\right) + c.c.$ | $\mathcal{D}^{(8,2,\frac{1}{2})}_{(15,4)} \mathcal{\Delta}^{(8,2,\frac{1}{2})}_{(15,6)} \frac{1}{\sqrt{120}} (14, 32, \frac{12-34}{\sqrt{2}}, \frac{6 [90-78], 8 [56-90], 0 [78-56]}{\sqrt{2}}) + c.c.$ |

Note: The states are represented in a simplified form, and the full expressions for the states are given in the table.
Table 5: States in the other $G_{321}$ multiplets including $\Phi$.

| (1, 2, $\frac{3}{2}$) + c.c. | $\hat{\Phi}^{(1,2,\frac{3}{2})}_{(24,0)}$ | $\frac{1}{\sqrt{2}} ([1, 3] 680) + c.c.$ |
| (1, 3, 0) | $\hat{A}^{(1,3,0)}_{(24,0)}$ | $\frac{1}{\sqrt{2}} \left( 14, 32, \frac{12-34}{\sqrt{2}} \right)$ |
| | $\hat{E}^{(1,3,0)}_{(24,0)}$ | $\frac{1}{\sqrt{2}} \left( 14, 32, \frac{12-34}{\sqrt{2}} \right)$ |
| | $\hat{\Phi}^{(1,3,0)}_{(24,0)}$ | $\frac{1}{\sqrt{2}} \left( 14, 32, \frac{12-34}{\sqrt{2}} \right)$ |
| (3, 1, $\frac{5}{3}$) + c.c. | $\hat{\Phi}^{(3,1,\frac{5}{3})}_{(75,0)}$ | $\frac{1}{\sqrt{2}} (13 [80, 06, 68]) + c.c.$ |
| (3, 3, $\frac{2}{3}$) + c.c. | $\hat{\Phi}^{(3,3,\frac{2}{3})}_{(40,0)}$ | $\frac{1}{\sqrt{2}} \left( 14, 32, \frac{12-34}{\sqrt{2}} \right)$ |
| (6, 2, $-\frac{1}{6}$) + c.c. | $\hat{\Phi}^{(6,2,-\frac{1}{6})}_{(40,-4)}$ | $\frac{1}{\sqrt{2}} \left( [2, 4] 580, 670, 689, \frac{6(90-78)}{\sqrt{2}}, \frac{8(56-90)}{\sqrt{2}}, \frac{0(78-56)}{\sqrt{2}} \right) + c.c.$ |
| (6, 2, $\frac{5}{6}$) + c.c. | $\hat{\Phi}^{(6,2,\frac{5}{6})}_{(75,0)}$ | $\frac{1}{\sqrt{2}} \left( [1, 3] 580, 670, 689, \frac{6(90-78)}{\sqrt{2}}, \frac{8(56-90)}{\sqrt{2}}, \frac{0(78-56)}{\sqrt{2}} \right) + c.c.$ |
| (8, 1, 0) | $\hat{A}^{(8,1,0)}_{(24,0)}$ | $\frac{1}{\sqrt{2}} \left( 58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{2}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
| | $\hat{E}^{(8,1,0)}_{(24,0)}$ | $\frac{1}{\sqrt{2}} \left( 58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{2}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
| | $\hat{\Phi}^{(8,1,0)}_{(24,0)}$ | $\frac{1}{\sqrt{12}} \left( 5890, 5078, 7056, 7690, 9678, 9856, \frac{1}{\sqrt{2}} [5690 - 7890], \frac{1}{\sqrt{6}} [5678] - 5690 - 7890 \right)$ |
| | | + $[12 + 34] \left( 58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{6}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
| | | $\frac{1}{\sqrt{6}} [5678] - 5690 - 7890 \right)$ |
| | | $- [12 + 34] \left( 58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{6}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
| (8, 1, 1) + c.c. | $\hat{\Phi}^{(8,1,1)}_{(70,4)}$ | $\frac{1}{\sqrt{24}} \left( 13 [58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{2}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
| (8, 3, 0) | $\hat{\Phi}^{(8,3,0)}_{(75,0)}$ | $\frac{1}{\sqrt{24}} \left( 14, 32, \frac{12-34}{\sqrt{2}} \right) [58, 50, 70, 76, 96, 98, \frac{1}{\sqrt{2}} [56 - 78], \frac{1}{\sqrt{6}} [56 + 78 - 2[90]] \right)$ |
3 Examples of the CGC calculations

Using the states in (15) and in Table 1-5 all the CGCs can be calculated directly. For example, in the coupling \[5, 12\]

\[-iA\hat{\Delta} = -iA_{a'b'}\hat{\sum}_{a'd'e'f'}\Delta_{a'd'e'f'} = -iA_{abcdef}\Delta_{abcdef}, \tag{23}\]

we have

\((-i)(a_1\hat{a}_1) \cdot (\hat{V}_R \hat{V}_R) \cdot (V_R \hat{V}_R) \equiv a_1 V_R \hat{V}_R (-i)(\hat{a}_1 \cdot \hat{V}_R \cdot \hat{V}_R) = a_1 V_R \hat{V}_R \frac{1}{\sqrt{10} \sqrt{120} \sqrt{120}} (12 + 34 + 56 + 78 + 90) \cdot (13579) \cdot (24680) = a_1 V_R \hat{V}_R \frac{1}{120 \sqrt{10} \sqrt{10}} \times 5 \times (-21) \cdot (13579) \cdot (24680) = a_1 V_R \hat{V}_R \frac{1}{120 \sqrt{10} \sqrt{10}} \times 5 \times (-1) \times 4! = a_1 V_R \hat{V}_R (-\frac{1}{\sqrt{10}}). \tag{24}\]

For another example, in the coupling

\[\frac{1}{120}\varepsilon \Phi^2 = \frac{1}{120} (-i)\varepsilon_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} A_{a_1 a_2} \Phi_{a_3 a_4 a_5 a_6} \Phi_{a_7 a_8 a_9 a_0}, \tag{25}\]

we have

\[\frac{1}{120}\varepsilon (a_2\hat{a}_2) \cdot (\Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \Phi^{(1,2,-\frac{1}{2})}_{(5,8)}) \cdot (\Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \Phi^{(1,2,-\frac{1}{2})}_{(5,8)}) = a_2 \Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \begin{vmatrix} \frac{i}{\sqrt{60}} \times (\frac{1}{\sqrt{24}})^2 \times (-i)(1234567890) \\ (3 \times [12 + 34] - 2 \times [56 + 78 + 90]) \cdot (-3579, 1579) \cdot (-4680, 2680) \\ 24 \times 120 \sqrt{60} \times 3 \times (1234567890) \cdot (12) \cdot (-3579) \cdot (-4680) \\ 24 \times 120 \sqrt{60} \times 3 \times 2! \times 4! \times 4! \\ \frac{1}{5} \sqrt{\frac{3}{5}} \end{vmatrix} = a_2 \Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \Phi^{(1,2,-\frac{1}{2})}_{(5,8)} \frac{1}{5} \sqrt{\frac{3}{5}}. \tag{26}\]

There are two directions in \(\Phi^{(1,2,-\frac{1}{2})}_{(5,8)}\) but only one of which needs to be counted in the calculations.

All the CGCs can be calculated following the examples given above. In the following, we will focus on the couplings with at least one field with large VEV, since they are relevant to the symmetry breaking and the masses. Only the CGCs with at least one SM singlet will be given, although the other CGCs can be calculated without problems.
4 The superpotential and the symmetry breaking

The most general renormalizable Higgs superpotential is \[ W = \frac{1}{2} m_1 \Phi^2 + m_2 \Delta^2 + \frac{1}{2} m_3 H^2 + \frac{1}{2} m_4 A^2 + \frac{1}{2} m_5 E^2 + \frac{1}{2} m_6 D^2 \]

\[
+ \lambda_1 \Phi^3 + \lambda_2 \Phi \Delta^2 + (\lambda_3 \Delta + \lambda_4 \Delta) H \Phi + \lambda_5 A^2 \Phi - i \lambda_6 A \Delta \Phi + \lambda_7 \frac{120}{120} \epsilon A \Phi^2
\]

\[
+ E \left( \lambda_8 E^2 + \lambda_9 A^2 + \lambda_1 \Phi^2 + \lambda_1 \Delta^2 + \lambda_2 \Delta^2 + \lambda_3 H^2 \right) + D^2 \left( \lambda_4 E + \lambda_5 \Phi \right)
\]

\[
+ D \left\{ \lambda_{16} H A + \lambda_{17} H \Phi + (\lambda_{18} \Delta + \lambda_{19} \Delta) A + (\lambda_{20} \Delta + \lambda_{21} \Delta) \Phi \right\},
\]

(27)

only the SM singlets are relevant to the symmetry breaking into the SM gauge group. Inserting the VEVs into \[(27), \] one obtains

\[
\langle W \rangle = \frac{1}{2} m_1 \left[ \phi_1^2 + \phi_2^2 + \phi_3^2 \right] + m_2 V_R\overline{V_R} + \frac{1}{2} m_4 (a_1^2 + a_2^2) + \frac{1}{2} m_5 E^2
\]

\[
+ \lambda_1 \left[ \phi_1^2 - \frac{1}{2} \frac{7}{54} \frac{4}{27} \sqrt{3} + \phi_3^2 \right] + 3 \phi_1 (\phi_2 - 2 \phi_3) \frac{4}{12 \sqrt{15}} - 3 \phi_2 \phi_3 \frac{5}{27 \sqrt{3}} + 3 \phi_3 \phi_3 \frac{5}{108 \sqrt{3}}
\]

\[
+ \lambda_2 \phi_1 V_R \overline{V_R} \frac{1}{2 \sqrt{15}}
\]

\[
+ \lambda_5 \left[ a_1^2 \phi_1 + a_2^2 \phi_1 \left( - \frac{1}{2 \sqrt{15}} \right) - a_2^2 \phi_2 \frac{1}{3 \sqrt{3}} + a_2^2 \phi_3 \frac{5}{6 \sqrt{3}} + 2 a_1 a_2 \phi_2 \frac{5}{10 \sqrt{10}} \right]
\]

\[
+ \lambda_6 a_1 V_R \overline{V_R} \left( - \frac{1}{2 \sqrt{10}} \right)
\]

\[
+ \lambda_7 \left[ a_1^2 \phi_1 \frac{6}{5 \sqrt{10}} - a_1^2 \phi_2 \frac{4}{5 \sqrt{10}} + a_1^2 \phi_3 \sqrt{5 \sqrt{10}}
\]

\[
- 2 a_2 \phi_1 \phi_2 \frac{3}{5 \sqrt{15}} + 2 a_2 \phi_2 \phi_3 \frac{1}{3 \sqrt{3}} + a_2^2 \phi_2 \frac{4}{15 \sqrt{15}} - a_2^2 \phi_3 \frac{16}{15 \sqrt{15}} \]

\[
+ \lambda_8 E^3 \frac{1}{2 \sqrt{15}} + \lambda_9 E \left[ a_1 a_2 \frac{2}{\sqrt{10}} + a_2^2 \frac{1}{2 \sqrt{15}} \right]
\]

\[
+ \lambda_{10} E \left[ \phi_1^2 \frac{1}{12 \sqrt{15}} + \phi_2^2 \frac{2}{3 \sqrt{15}} + 2 \phi_1 \phi_2 \frac{3}{4 \sqrt{15}} + 2 \phi_2 \phi_3 \frac{5}{12 \sqrt{3}} \right].
\]

(28)

Keeping SUSY at the high energy scale requires the D-flatness condition which is

\[
|V_R| = |\overline{V_R}|.
\]

(29)

These VEVs are determined by the F-flatness conditions required by SUSY,

\[
\left\{ \frac{\partial}{\partial \phi_1}, \frac{\partial}{\partial \phi_2}, \frac{\partial}{\partial \phi_3}, \frac{\partial}{\partial V_R}, \frac{\partial}{\partial \overline{V_R}}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial E} \right\} \langle W \rangle = 0.
\]

(30)
from which we have

\[
0 = m_1 \phi_1 + \lambda_1 \phi_1^2 \frac{3}{2\sqrt{15}} + \lambda_1 (\phi_2^2 - 2\phi_2^3) \frac{1}{2\sqrt{15}} + \lambda_2 V_R V_R \frac{1}{2\sqrt{15}} + \lambda_5 a_1^2 \frac{2}{\sqrt{15}} - \lambda_5 a_2^2 \frac{1}{2\sqrt{15}} \\
+ \lambda_7 a_1 \phi_1 \frac{12}{5\sqrt{10}} - \lambda_7 a_2 \phi_2 \frac{6}{\sqrt{15}} + \lambda_10 E \phi_2 \frac{3}{2\sqrt{15}},
\]

\[
0 = m_1 \phi_2 - \lambda_1 \phi_2^2 \frac{7}{18\sqrt{15}} + \lambda_1 \phi_1 \phi_2 \frac{1}{2\sqrt{15}} - \lambda_1 \phi_3^2 \frac{1}{9\sqrt{15}} + \lambda_1 \phi_2 \phi_3 \frac{5}{18\sqrt{3}} + \lambda_5 a_2^2 \frac{1}{3\sqrt{15}} + \lambda_5 a_1 a_2 \frac{2}{\sqrt{10}} \\
- \lambda_7 a_1 \phi_2 \frac{8}{5\sqrt{10}} - \lambda_7 a_2 \phi_1 \frac{6}{5\sqrt{15}} + \lambda_7 a_2 \phi_3 \frac{2}{3\sqrt{3}} + \lambda_7 a_2 \phi_2 \frac{8}{15\sqrt{15}} \\
+ \lambda_10 E \phi_2 \frac{1}{6\sqrt{15}} + \lambda_10 E \phi_1 \frac{3}{2\sqrt{15}} + \lambda_10 E \phi_3 \frac{5}{6\sqrt{3}},
\]

\[
0 = m_1 \phi_3 + \lambda_1 \phi_3^2 \frac{4}{9\sqrt{3}} - \lambda_1 \phi_1 \phi_3 \frac{1}{\sqrt{15}} - \lambda_1 \phi_2 \phi_3 \frac{8}{9\sqrt{15}} + \lambda_1 \phi_2^2 \frac{5}{36\sqrt{3}} + \lambda_5 a_2^2 \frac{5}{6\sqrt{3}} \\
+ \lambda_7 a_1 \phi_3 \frac{4}{5\sqrt{10}} + \lambda_7 a_2 \phi_2 \frac{2}{3\sqrt{3}} - \lambda_7 a_2 \phi_3 \frac{32}{15\sqrt{15}} + \lambda_10 E \phi_3 \frac{4}{3\sqrt{15}} + \lambda_10 E \phi_2 \frac{5}{6\sqrt{3}},
\]

\[
0 = V_R \text{ or } V_R \left[ m_2 + \lambda_2 \phi_1 \frac{1}{2\sqrt{15}} - \lambda_6 a_1 \frac{1}{\sqrt{10}} \right].
\]

\[
0 = m_4 a_1 + \lambda_5 a_1 \phi_1 \frac{4}{\sqrt{15}} + \lambda_5 a_2 \phi_2 \frac{2}{\sqrt{10}} - \lambda_6 V_R V_R \frac{1}{\sqrt{10}} + \lambda_7 \phi_1^2 \frac{6}{5\sqrt{10}} \\
- \lambda_7 \phi_2^2 \frac{4}{5\sqrt{10}} + \lambda_7 \phi_3^2 \frac{2}{5\sqrt{10}} + \lambda_7 E a_2 \frac{2}{\sqrt{10}},
\]

\[
0 = m_4 a_2 + \lambda_5 a_1 \phi_2 \frac{2}{\sqrt{10}} - \lambda_5 a_2 \phi_1 \frac{1}{\sqrt{15}} - \lambda_5 a_2 \phi_3 \frac{2}{3\sqrt{15}} + \lambda_5 a_2 \phi_3 \frac{5}{3\sqrt{3}} - \lambda_7 \phi_1 \phi_2 \frac{6}{5\sqrt{15}} \\
+ \lambda_7 \phi_2 \phi_3 \frac{2}{3\sqrt{3}} + \lambda_7 \phi_2^2 \frac{4}{15\sqrt{15}} - \lambda_7 \phi_3^2 \frac{16}{15\sqrt{15}} + \lambda_7 E a_1 \frac{2}{\sqrt{10}} + \lambda_7 E a_2 \frac{1}{\sqrt{15}},
\]

\[
0 = m_5 E + \lambda_8 E^2 \frac{3}{2\sqrt{15}} + \lambda_9 a_1 a_2 \frac{2}{\sqrt{10}} + \lambda_9 a_2^2 \frac{1}{2\sqrt{15}} \\
+ \lambda_10 a_2^2 \frac{1}{2\sqrt{15}} + \lambda_10 \phi_3^2 \frac{2}{3\sqrt{15}} + \lambda_10 \phi_1 \phi_2 \frac{3}{2\sqrt{15}} + \lambda_10 \phi_2 \phi_3 \frac{5}{6\sqrt{3}}.
\]

(31)

5 Masses and mass matrices

The masses and mass matrices for the many states in the general model can be given once the VEVs are determined by the superpotential parameters of the model. They are presented in the form

\[
M_{R,(1,i)}^{(j,y)} = M_R \delta_{RS} \delta_{I,J} \delta_{x,-y} + \sum_{T_K} \lambda_{RST} C_{R,(1,i)}^{(j,y)} T_{(1,i)}^{(j,y)} T_{(1,i)}^{(j,y)} < T_K >
\]

, where \{\lambda_{RST}\} = \{6 \lambda_1, 6 \lambda_8, 2 \lambda_5, 2 \lambda_7, 2 \lambda_9, 2 \lambda_{10}, 2 \lambda_{11}, 2 \lambda_{12}, 2 \lambda_{13}, 2 \lambda_{14}, 2 \lambda_{15}, \lambda_2, \lambda_3, \lambda_4, \lambda_6, \lambda_{16}, \lambda_{17}, \lambda_{18}, \lambda_{19}, \lambda_{20}, \lambda_{21}\}, and \(C_{R,(1,i)}^{(j,y)} T_{(1,i)}^{(j,y)}\) is a CGC calculated in the \(G_{51}\) basis which is preformed in the present study. We give the masses and mass matrices in the following, from which the CGCs calculated in the present work in the \(G_{51}\) basis can be read off.
\[(1, 1, 0)\]

\[c: \hat{A}^{(1, 1, 0)}_{(1, 0)}, \hat{A}^{(1, 1, 0)}_{(24, 0)}, \hat{E}^{(1, 1, 0)}_{(24, 0)}, \hat{\Delta}^{(1, 1, 0)}_{(1, -10)}, \hat{\Delta}^{(1, 1, 0)}_{(1, 10)}, \hat{\Phi}^{(1, 1, 0)}_{(1, 0)}, \hat{\Phi}^{(1, 1, 0)}_{(24, 0)}, \hat{\Phi}^{(1, 1, 0)}_{(75, 0)}\]

\[r: \hat{A}^{(1, 1, 0)}_{(1, 0)}, \hat{A}^{(1, 1, 0)}_{(24, 0)}, \hat{E}^{(1, 1, 0)}_{(75, 0)}, \Delta^{(1, 1, 0)}_{(1, 10)}, \Delta^{(1, 1, 0)}_{(1, -10)}, \Phi^{(1, 1, 0)}_{(1, 0)}, \Phi^{(1, 1, 0)}_{(24, 0)}, \Phi^{(1, 1, 0)}_{(75, 0)}\]

\[
\begin{pmatrix}
    m_{11}^{(1,1,0)} & m_{12}^{(1,1,0)} & \sqrt{\frac{2}{5}} a_2 \lambda_9 & -\frac{V_R \lambda_6}{\sqrt{10}} & -\frac{V_R \lambda_6}{\sqrt{10}} & m_{16}^{(1,1,0)} & m_{17}^{(1,1,0)} & \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_7 \phi_3 \\
    m_{12}^{(1,1,0)} & m_{22}^{(1,1,0)} & m_{23}^{(1,1,0)} & 0 & 0 & m_{26}^{(1,1,0)} & m_{27}^{(1,1,0)} & m_{28}^{(1,1,0)} \\
    \sqrt{\frac{2}{5}} a_2 \lambda_9 & m_{23}^{(1,1,0)} & m_{33}^{(1,1,0)} & 0 & 0 & \frac{1}{2} \sqrt{\frac{2}{5}} \lambda_10 \phi_2 & m_{37}^{(1,1,0)} & m_{38}^{(1,1,0)} \\
    -\frac{V_R \lambda_6}{\sqrt{10}} & 0 & 0 & m_{44}^{(1,1,0)} & 0 & \frac{V_R \lambda_2}{2\sqrt{15}} & 0 & 0 \\
    -\frac{V_R \lambda_6}{\sqrt{10}} & 0 & 0 & 0 & m_{44}^{(1,1,0)} & \frac{V_R \lambda_2}{2\sqrt{15}} & 0 & 0 \\
    m_{16}^{(1,1,0)} & m_{26}^{(1,1,0)} & \frac{1}{2} \sqrt{\frac{2}{5}} \lambda_10 \phi_2 & \frac{V_R \lambda_2}{2\sqrt{15}} & \frac{V_R \lambda_2}{2\sqrt{15}} & m_{66}^{(1,1,0)} & m_{67}^{(1,1,0)} & -\frac{\lambda_1 \phi_3}{\sqrt{15}} \\
    m_{17}^{(1,1,0)} & m_{27}^{(1,1,0)} & m_{37}^{(1,1,0)} & 0 & 0 & m_{67}^{(1,1,0)} & m_{77}^{(1,1,0)} & m_{78}^{(1,1,0)} \\
    \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_7 \phi_3 & m_{28}^{(1,1,0)} & m_{38}^{(1,1,0)} & 0 & 0 & -\frac{\lambda_1 \phi_3}{\sqrt{15}} & m_{78}^{(1,1,0)} & m_{88}^{(1,1,0)}
\end{pmatrix}
\]

(32)
where

\[
\begin{align*}
m_{11}^{(1,1,0)} &\equiv m_4 + \frac{4\lambda_5 \phi_1}{\sqrt{15}}, \\
m_{12}^{(1,1,0)} &\equiv \sqrt{\frac{2}{5}} E\lambda_9 + \sqrt{\frac{2}{5}} \lambda_5 \phi_2, \\
m_{16}^{(1,1,0)} &\equiv \frac{4a_1 \lambda_5}{\sqrt{15}} + \frac{6}{5} \sqrt{\frac{2}{5}} \lambda_7 \phi_1, \\
m_{17}^{(1,1,0)} &\equiv \sqrt{\frac{2}{5}} a_2 \lambda_5 - \frac{4}{5} \sqrt{\frac{2}{5}} \lambda_7 \phi_2, \\
m_{22}^{(1,1,0)} &\equiv m_4 + \frac{E\lambda_9}{\sqrt{15}} - \frac{\lambda_5 \phi_1}{3\sqrt{15}} - \frac{2\lambda_5 \phi_2}{3\sqrt{3}} + \frac{5\lambda_5 \phi_3}{3\sqrt{3}}, \\
m_{23}^{(1,1,0)} &\equiv \sqrt{\frac{2}{5}} a_1 \lambda_9 + \frac{a_2 \lambda_9}{\sqrt{15}}, \\
m_{26}^{(1,1,0)} &\equiv -\frac{a_2 \lambda_5}{\sqrt{15}} - \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7 \phi_2, \\
m_{27}^{(1,1,0)} &\equiv \sqrt{\frac{2}{5}} a_1 \lambda_5 - \frac{2a_2 \lambda_5}{3\sqrt{15}} - \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7 \phi_1 + \frac{8\lambda_7 \phi_2}{15\sqrt{15}} + \frac{2\lambda_7 \phi_3}{3\sqrt{3}}, \\
m_{28}^{(1,1,0)} &\equiv \frac{5a_2 \lambda_5}{3\sqrt{3}} + \frac{2\lambda_7 \phi_2}{3\sqrt{3}} - \frac{32\lambda_7 \phi_3}{15\sqrt{15}}, \\
m_{33}^{(1,1,0)} &\equiv m_5 + \sqrt{\frac{3}{5}} E\lambda_8, \\
m_{37}^{(1,1,0)} &\equiv \frac{1}{2} \sqrt{\frac{3}{5}} \lambda_1 \phi_1 + \frac{\lambda_1 \phi_2}{6\sqrt{15}} + \frac{5\lambda_1 \phi_3}{6\sqrt{3}}, \\
m_{38}^{(1,1,0)} &\equiv \frac{5\lambda_1 \phi_2}{6\sqrt{3}} + \frac{4\lambda_1 \phi_3}{3\sqrt{15}}, \\
m_{44}^{(1,1,0)} &\equiv m_2 - \frac{a_1 \lambda_6}{\sqrt{10}} + \frac{\lambda_2 \phi_1}{2\sqrt{15}}, \\
m_{66}^{(1,1,0)} &\equiv m_1 + \frac{6}{5} \sqrt{\frac{2}{5}} a_1 \lambda_7 + \sqrt{\frac{3}{5}} \lambda_1 \phi_1, \\
m_{67}^{(1,1,0)} &\equiv -\frac{2}{5} \sqrt{\frac{3}{5}} a_2 \lambda_7 + \frac{1}{2} \sqrt{\frac{3}{5}} E\lambda_10 + \frac{\lambda_1 \phi_2}{2\sqrt{15}}, \\
m_{77}^{(1,1,0)} &\equiv m_1 - \frac{4}{5} \sqrt{\frac{2}{5}} a_1 \lambda_7 + \frac{8a_2 \lambda_7}{15\sqrt{15}} + \frac{E\lambda_10}{6\sqrt{15}} + \frac{\lambda_1 \phi_1}{2\sqrt{15}} + \frac{7\lambda_1 \phi_2}{9\sqrt{15}} + \frac{5\lambda_1 \phi_3}{18\sqrt{3}}, \\
m_{78}^{(1,1,0)} &\equiv \frac{2a_2 \lambda_7}{3\sqrt{3}} + \frac{5E\lambda_10}{6\sqrt{3}} + \frac{5\lambda_1 \phi_2}{18\sqrt{3}} - \frac{8\lambda_1 \phi_3}{9\sqrt{15}}, \\
m_{88}^{(1,1,0)} &\equiv m_1 + \frac{2}{5} \sqrt{\frac{2}{5}} a_1 \lambda_7 - \frac{32a_2 \lambda_7}{15\sqrt{15}} + \frac{4E\lambda_10}{3\sqrt{15}} - \frac{\lambda_1 \phi_1}{3\sqrt{15}} - \frac{8\lambda_1 \phi_2}{9\sqrt{15}} + \frac{8\lambda_1 \phi_3}{9\sqrt{3}}.
\end{align*}
\]
\[ [(1, 1, 1) + \text{c.c.}] \]

c: \( \hat{A}^{(1,1,1)}_{(10,4)} \), \( \hat{D}^{(1,1,1)}_{(10,6)} \), \( \hat{\Delta}^{(1,1,1)}_{(10,6)} \), \( \hat{\Phi}^{(1,1,1)}_{(10,4)} \)

r: \( \hat{A}^{(1,1,1)}_{(10,6)}, \hat{D}^{(1,1,1)}_{(10,6)}, \hat{\Delta}^{(1,1,1)}_{(10,6)}, \hat{\Phi}^{(1,1,1)}_{(10,6)} \]

\[
\begin{pmatrix}
    m^{(1,1,1)}_{11} & -\frac{V_R\lambda_{10}}{\sqrt{10}} & m^{(1,1,1)}_{14} \\
    \frac{V_R\lambda_{10}}{\sqrt{10}} & m^{(1,1,1)}_{22} & -\frac{V_R\lambda_{6}}{2\sqrt{10}} \\
    -\frac{V_R\lambda_{6}}{\sqrt{10}} & m^{(1,1,1)}_{32} & m^{(1,1,1)}_{33} \\
    m^{(1,1,1)}_{14} & -\frac{V_R\lambda_{6}}{2\sqrt{10}} & m^{(1,1,1)}_{14}
\end{pmatrix}
\]

where

\[
\begin{align*}
    m^{(1,1,1)}_{11} & \equiv m_4 + \sqrt{\frac{3}{5}} E\lambda_9 + \frac{\lambda_5\phi_1}{\sqrt{15}} + \frac{2\lambda_6\phi_2}{\sqrt{15}} + \frac{\lambda_5\phi_3}{\sqrt{3}}, \\
m^{(1,1,1)}_{14} & \equiv -\sqrt{\frac{2}{5}} a_1\lambda_5 + \frac{2a_2\lambda_5}{\sqrt{15}} - \frac{2\sqrt{\frac{3}{5}} a_1\lambda_7}{\sqrt{15}} + \frac{8\lambda_7\phi_2}{5\sqrt{15}} - \frac{2\lambda_7\phi_3}{5\sqrt{3}}, \\
m^{(1,1,1)}_{22} & \equiv m_6 - \frac{2E\lambda_4}{\sqrt{15}} + \frac{\lambda_5\phi_1}{\sqrt{15}} - \frac{4a_1\phi_2}{3\sqrt{15}} + \frac{\lambda_5\phi_3}{\sqrt{3}}, \\
m^{(1,1,1)}_{23} & \equiv -\frac{1}{5} ia_1\lambda_{18} - \frac{1}{5} i\sqrt{\frac{3}{2}} a_2\lambda_{18} - \frac{1}{10}\sqrt{\frac{3}{2}} \lambda_2\phi_1 - \frac{\lambda_2\phi_2}{10\sqrt{6}} + \frac{\lambda_2\phi_3}{2\sqrt{30}}, \\
m^{(1,1,1)}_{32} & \equiv \frac{1}{5} ia_1\lambda_{19} + \frac{1}{5} i\sqrt{\frac{3}{2}} a_2\lambda_{19} - \frac{1}{10}\sqrt{\frac{3}{2}} \lambda_2\phi_1 + \frac{\lambda_2\phi_2}{10\sqrt{6}} + \frac{\lambda_2\phi_3}{2\sqrt{30}}, \\
m^{(1,1,1)}_{33} & \equiv m_2 - \frac{3a_1\lambda_6}{\sqrt{5}} + \frac{1}{5}\sqrt{\frac{3}{5}} a_2\lambda_6 + \frac{\lambda_2\phi_1}{\sqrt{5}} - \frac{\lambda_2\phi_2}{\sqrt{10\sqrt{15}}} - \frac{\lambda_2\phi_3}{10\sqrt{3}}, \\
m^{(1,1,1)}_{44} & \equiv m_1 + \frac{4}{\sqrt{5}} a_1\lambda_7 - \frac{8a_2\lambda_7}{\sqrt{5}} + \frac{E\lambda_1}{2\sqrt{15}} + \frac{1}{2}\sqrt{\frac{3}{5}} \lambda_1\phi_1 - \frac{\lambda_1\phi_2}{3\sqrt{15}} + \frac{5\lambda_1\phi_3}{6\sqrt{3}}.
\end{align*}
\]

\[
[(3, 1, \frac{7}{4}) + \text{c.c.}] \]

c: \( \hat{A}^{(3,1,\frac{7}{4})}_{(10,6)}, \hat{D}^{(3,1,\frac{7}{4})}_{(10,6)}, \hat{\Delta}^{(3,1,\frac{7}{4})}_{(10,6)}, \hat{\Phi}^{(3,1,\frac{7}{4})}_{(10,4)}, \hat{\Phi}^{(3,1,\frac{7}{4})}_{(10,4)} \)

r: \( \hat{A}^{(3,1,\frac{7}{4})}_{(10,4)}, \hat{D}^{(3,1,\frac{7}{4})}_{(10,6)}, \hat{\Delta}^{(3,1,\frac{7}{4})}_{(10,6)}, \hat{\Phi}^{(3,1,\frac{7}{4})}_{(10,4)}, \hat{\Phi}^{(3,1,\frac{7}{4})}_{(10,4)} \)

\[
\begin{pmatrix}
    m^{(3,1,\frac{7}{4})}_{11} & -\frac{V_R\lambda_{10}}{\sqrt{10}} & m^{(3,1,\frac{7}{4})}_{14} \\
    \frac{V_R\lambda_{10}}{\sqrt{10}} & m^{(3,1,\frac{7}{4})}_{22} & -\frac{V_R\lambda_{6}}{2\sqrt{10}} \\
    -\frac{V_R\lambda_{6}}{\sqrt{10}} & m^{(3,1,\frac{7}{4})}_{32} & m^{(3,1,\frac{7}{4})}_{33} \\
    m^{(3,1,\frac{7}{4})}_{14} & -\frac{V_R\lambda_{6}}{2\sqrt{10}} & m^{(3,1,\frac{7}{4})}_{14}
\end{pmatrix}
\]

\[ [(1, 1, 1) + \text{c.c.}] \]

c: \( \hat{A}^{(1,1,1)}_{(10,4)} \), \( \hat{D}^{(1,1,1)}_{(10,6)} \), \( \hat{\Delta}^{(1,1,1)}_{(10,6)} \), \( \hat{\Phi}^{(1,1,1)}_{(10,4)} \)

r: \( \hat{A}^{(1,1,1)}_{(10,6)}, \hat{D}^{(1,1,1)}_{(10,6)}, \hat{\Delta}^{(1,1,1)}_{(10,6)}, \hat{\Phi}^{(1,1,1)}_{(10,6)} \)
where

\[
\begin{align*}
m_{11}^{(3,1,\frac{2}{3})} &\equiv m_4 - \frac{2E\lambda_9}{\sqrt{15}} + \frac{\lambda_5\phi_1}{3\sqrt{15}} - \frac{4\lambda_5\phi_2}{3\sqrt{3}} + \frac{\lambda_6\phi_3}{3\sqrt{3}}, \\
m_{14}^{(3,1,\frac{2}{3})} &\equiv -\sqrt{\frac{2}{3}}a_1\lambda_5 - \frac{4a_2\lambda_5}{3\sqrt{15}} - \frac{2}{3}\sqrt{\frac{3}{5}}\lambda_7\phi_1 - \frac{16\lambda_7\phi_2}{15\sqrt{15}} - 2\lambda_7\phi_3, \\
m_{15}^{(3,1,\frac{2}{3})} &\equiv \frac{1}{3}\sqrt{\frac{10}{3}}a_2\lambda_5 - \frac{2}{3}\sqrt{\frac{2}{15}}\lambda_7\phi_2 - \frac{8}{15}\frac{2}{3}\lambda_7\phi_3, \\
m_{22}^{(3,1,\frac{2}{3})} &\equiv m_6 + \frac{4E\lambda_{14}}{\sqrt{15}} + \frac{\lambda_{15}\phi_1}{\sqrt{15}} + \frac{8\lambda_{15}\phi_2}{9\sqrt{15}} + \frac{\lambda_{15}\phi_3}{9\sqrt{3}}, \\
m_{23}^{(3,1,\frac{2}{3})} &\equiv -\frac{1}{5}ia_1\lambda_{19} + \frac{1}{5}i\sqrt{\frac{2}{3}}a_2\lambda_{19} + \frac{1}{10}\sqrt{\frac{3}{2}}\lambda_{21}\phi_1 - \frac{\lambda_{21}\phi_2}{15\sqrt{6}} - \frac{\lambda_{21}\phi_3}{6\sqrt{30}}, \\
m_{32}^{(3,1,\frac{2}{3})} &\equiv \frac{1}{5}ia_1\lambda_{18} - \frac{1}{5}i\sqrt{\frac{2}{3}}a_2\lambda_{18} + \frac{1}{10}\sqrt{\frac{3}{2}}\lambda_{20}\phi_1 - \frac{\lambda_{20}\phi_2}{15\sqrt{6}} - \frac{\lambda_{20}\phi_3}{6\sqrt{30}}, \\
m_{33}^{(3,1,\frac{2}{3})} &\equiv m_2 - \frac{3a_1\lambda_6}{5\sqrt{10}} - \frac{2a_2\lambda_6}{5\sqrt{15}} + \frac{\lambda_2\phi_1}{\sqrt{15}} + \frac{\lambda_2\phi_2}{\sqrt{30}} + \frac{\lambda_2\phi_3}{\sqrt{3}}, \\
m_{44}^{(3,1,\frac{2}{3})} &\equiv m_1 + \frac{4}{5}\sqrt{\frac{2}{3}}a_1\lambda_7 + \frac{16a_2\lambda_7}{15\sqrt{15}} - \frac{E\lambda_{10}}{3\sqrt{15}} + \frac{1}{2}\sqrt{\frac{3}{5}}\lambda_1\phi_1 + \frac{2\lambda_1\phi_2}{9\sqrt{15}} + \frac{5\lambda_1\phi_3}{18\sqrt{3}}, \\
m_{45}^{(3,1,\frac{2}{3})} &\equiv \frac{2}{3}\sqrt{\frac{2}{15}}a_2\lambda_7 - \frac{1}{3}\sqrt{\frac{5}{6}}E\lambda_{10} - \frac{1}{9}\sqrt{\frac{5}{6}}\lambda_1\phi_2 - \frac{2}{9}\sqrt{\frac{2}{3}}\lambda_1\phi_3, \\
m_{55}^{(3,1,\frac{2}{3})} &\equiv m_1 - \frac{2}{5}\sqrt{\frac{2}{3}}a_1\lambda_7 - \frac{28a_2\lambda_7}{15\sqrt{15}} - \frac{7E\lambda_{10}}{6\sqrt{15}} - \frac{1}{9}\sqrt{\frac{5}{3}}\lambda_1\phi_2 + \frac{5\lambda_1\phi_3}{9\sqrt{3}}. 
\end{align*}
\]

\[
[(3, 2, -\frac{5}{6}) + c.c.]
\]

**c:** \(\tilde{A}_{(24,0)}^{(3,2,-\frac{5}{6})}, \tilde{E}_{(24,0)}^{(3,2,-\frac{5}{6})}, \tilde{\Phi}_{(24,0)}^{(3,2,-\frac{5}{6})}, \tilde{\Phi}_{(75,0)}^{(3,2,-\frac{5}{6})}\)

**r:** \(\tilde{A}_{(24,0)}^{(3,2,\frac{5}{6})}, \tilde{E}_{(24,0)}^{(3,2,\frac{5}{6})}, \tilde{\Phi}_{(24,0)}^{(3,2,\frac{5}{6})}, \tilde{\Phi}_{(75,0)}^{(3,2,\frac{5}{6})}\)

\[
\begin{pmatrix}
m_{11}^{(3,2,-\frac{5}{6})} - \sqrt{\frac{2}{5}}a_1\lambda_9 - \frac{a_2\lambda_9}{2\sqrt{15}} & m_{13}^{(3,2,-\frac{5}{6})} & m_{14}^{(3,2,-\frac{5}{6})} & \\
\sqrt{\frac{2}{5}}a_1\lambda_9 - \frac{a_2\lambda_9}{2\sqrt{15}} & m_{11}^{(3,2,-\frac{5}{6})} - \frac{2E\lambda_9}{\sqrt{15}} & m_{23}^{(3,2,-\frac{5}{6})} & m_{23}^{(3,2,-\frac{5}{6})}
\end{pmatrix}, 
\]

\[
\begin{pmatrix}
\frac{\lambda_{10}\phi_1}{3\sqrt{6}} & m_{11}^{(3,2,-\frac{5}{6})} - \sqrt{\frac{2}{5}}a_1\lambda_9 - \frac{a_2\lambda_9}{2\sqrt{15}} & m_{13}^{(3,2,-\frac{5}{6})} & m_{14}^{(3,2,-\frac{5}{6})} \\
\sqrt{\frac{2}{5}}a_1\lambda_9 - \frac{a_2\lambda_9}{2\sqrt{15}} & m_{11}^{(3,2,-\frac{5}{6})} - \frac{2E\lambda_9}{\sqrt{15}} & m_{23}^{(3,2,-\frac{5}{6})} & m_{23}^{(3,2,-\frac{5}{6})}
\end{pmatrix} \frac{1}{5}\sqrt{\frac{5}{6}}\lambda_{10}\phi_2 + \frac{\lambda_{10}\phi_3}{3\sqrt{6}}, 
\]

(38)
where

\[ m_{11}^{(3,2,-\frac{5}{6})} = m_4 + \frac{E\lambda_9}{2\sqrt{15}} - \frac{\lambda_5\phi_1}{\sqrt{15}} - \frac{\lambda_5\phi_2}{3\sqrt{15}} + \frac{\lambda_5\phi_3}{3\sqrt{3}}, \]

\[ m_{13}^{(3,2,-\frac{5}{6})} = -\sqrt{\frac{2}{5}}a_1\lambda_5 + \frac{a_2\lambda_5}{3\sqrt{15}} + \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7\phi_1 - \frac{4\lambda_7\phi_2}{15\sqrt{15}} - \frac{2\lambda_7\phi_3}{15\sqrt{3}}, \]

\[ m_{14}^{(3,2,-\frac{5}{6})} = -\frac{1}{3} \sqrt{\frac{10}{3}}a_2\lambda_5 - \frac{2}{3} \sqrt{\frac{2}{15}} \lambda_7\phi_2 + \frac{4}{15} \sqrt{\frac{2}{3}} \lambda_7\phi_3, \]

\[ m_{23}^{(3,2,-\frac{5}{6})} = \frac{1}{2} \sqrt{\frac{3}{5}} \lambda_{10}\phi_1 + \frac{\lambda_{10}\phi_2}{12\sqrt{15}} + \frac{\lambda_{10}\phi_3}{6\sqrt{3}}, \]

\[ m_{33}^{(3,2,-\frac{5}{6})} = m_1 - \frac{4}{5} \sqrt{\frac{2}{5}} a_1\lambda_7 + \frac{4a_2\lambda_7}{15\sqrt{15}} + \frac{E\lambda_{10}}{12\sqrt{15}} + \frac{\lambda_{11}\phi_1}{2\sqrt{15}} - \frac{7\lambda_{11}\phi_2}{18\sqrt{15}} + \frac{\lambda_{11}\phi_3}{18\sqrt{3}}. \]

\[ m_{34}^{(3,2,-\frac{5}{6})} = \frac{2}{3} \sqrt{\frac{2}{15}} a_2\lambda_7 + \frac{1}{9} \sqrt{\frac{5}{6}} E\lambda_{10} + \frac{1}{9} \sqrt{\frac{2}{3}} \lambda_1\phi_2 - \frac{1}{9} \sqrt{\frac{2}{3}} \lambda_1\phi_3, \]

\[ m_{44}^{(3,2,-\frac{5}{6})} = m_1 + \frac{2}{5} \sqrt{\frac{2}{5}} a_1\lambda_7 - \frac{22a_2\lambda_7}{15\sqrt{15}} + \frac{11E\lambda_{10}}{12\sqrt{15}} - \frac{\lambda_{11}\phi_1}{\sqrt{15}} - \frac{11\lambda_{11}\phi_2}{18\sqrt{15}} + \frac{4\lambda_{11}\phi_3}{9\sqrt{3}}. \] (39)

\[ ([3, 2, \frac{1}{6}]) + c.c. \]

\[ \begin{pmatrix} m_{11}^{(3,2,\frac{1}{6})} & \frac{1}{2} \sqrt{\frac{2}{5}} a_2\lambda_9 & -\frac{V_R\lambda_9}{\sqrt{10}} & \frac{V_R\phi_9}{\sqrt{10}} & 0 & m_{16}^{(3,2,\frac{1}{6})} & m_{17}^{(3,2,\frac{1}{6})} \\ \frac{1}{2} \sqrt{\frac{2}{5}} a_2\lambda_9 & m_5 + \frac{1}{2} \sqrt{\frac{2}{5}} E\lambda_8 & 0 & 0 & -\frac{2}{5} \sqrt{\frac{2}{5}} \lambda_{10}\phi_2 & -\frac{\lambda_{10}\phi_2}{\sqrt{6}} \\ \frac{V_R\phi_9}{\sqrt{10}} & 0 & m_{33}^{(3,2,\frac{1}{6})} & m_{34}^{(3,2,\frac{1}{6})} & -\frac{ia_2\lambda_1}{2\sqrt{6}} & -\frac{\lambda_1\phi_2}{\sqrt{6}} \frac{E\lambda_{10}}{\sqrt{15}} & -\frac{1}{10} V_R\lambda_2 & 0 \\ \frac{V_R\lambda_9}{\sqrt{10}} & 0 & m_{33}^{(3,2,\frac{1}{6})} & m_{44}^{(3,2,\frac{1}{6})} & -\frac{ia_2\lambda_1}{2\sqrt{6}} & -\frac{\lambda_1\phi_2}{\sqrt{6}} \frac{E\lambda_{10}}{\sqrt{15}} & -\frac{1}{10} V_R\lambda_2 & 0 \\ \lambda_{10}\phi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \] (40)
where

\[ m^{(3.2, \frac{1}{6})}_{11} = m_4 + \frac{E \lambda_9}{2\sqrt{15}} + \frac{\lambda_5 \phi_1}{\sqrt{15}} + \frac{\lambda_5 \phi_2}{3\sqrt{15}} - \frac{\lambda_5 \phi_3}{3\sqrt{3}}, \]

\[ m^{(3.2, \frac{1}{6})}_{16} = -\sqrt{\frac{2}{5}} a_1 \lambda_5 + \frac{a_2 \lambda_5}{3\sqrt{15}} - \frac{2}{3} \sqrt{\frac{3}{5}} \lambda_7 \phi_1 + \frac{4\lambda_7 \phi_2}{15\sqrt{15}} + \frac{2\lambda_7 \phi_3}{15\sqrt{3}}, \]

\[ m^{(3.2, \frac{1}{6})}_{17} = -\frac{1}{3} \sqrt{\frac{10}{3}} a_2 \lambda_5 + \frac{2}{3} \sqrt{\frac{10}{15}} \lambda_7 \phi_2 - \frac{4}{3} \sqrt{\frac{10}{3}} \lambda_7 \phi_3, \]

\[ m^{(3.2, \frac{1}{6})}_{33} = m_6 - \frac{E \lambda_{14}}{3\sqrt{15}} + \frac{\lambda_{15} \phi_1}{\sqrt{15}} - \frac{2\lambda_{15} \phi_2}{9\sqrt{15}} - \frac{\lambda_{15} \phi_3}{9\sqrt{3}}, \]

\[ m^{(3.2, \frac{1}{6})}_{34} = -\frac{1}{5} ia_1 \lambda_{18} - \frac{ia_2 \lambda_{18}}{10\sqrt{6}} - \frac{1}{10} \sqrt{\frac{3}{2}} \lambda_2 \phi_1 - \frac{\lambda_2 \phi_2}{60\sqrt{3}} - \frac{\lambda_2 \phi_3}{6\sqrt{30}}, \]

\[ m^{(3.2, \frac{1}{6})}_{43} = \frac{1}{5} ia_1 \lambda_{19} + \frac{ia_2 \lambda_{19}}{10\sqrt{6}} - \frac{1}{10} \sqrt{\frac{3}{2}} \lambda_{21} \phi_1 - \frac{\lambda_{21} \phi_2}{60\sqrt{3}} - \frac{\lambda_{21} \phi_3}{6\sqrt{30}}, \]

\[ m^{(3.2, \phi)}_{44} = m_2 - \frac{3a_1 \lambda_6}{5\sqrt{10}} + \frac{a_2 \lambda_6}{10\sqrt{15}} + \frac{\lambda_2 \phi_1}{5\sqrt{15}} - \frac{\lambda_2 \phi_2}{60\sqrt{3}} - \frac{\lambda_2 \phi_3}{30\sqrt{3}}, \]

\[ m^{(3.2, \phi)}_{55} = m_2 + \frac{3a_1 \lambda_6}{5\sqrt{10}} - \frac{a_2 \lambda_6}{10\sqrt{15}} + \frac{\lambda_2 \phi_1}{5\sqrt{15}} - \frac{\lambda_2 \phi_2}{20\sqrt{15}}, \]

\[ m^{(3.2, \phi)}_{66} = m_1 + \frac{4}{3} \sqrt{\frac{2}{5}} a_1 \lambda_7 - \frac{4a_2 \lambda_7}{15\sqrt{15}} + \frac{E \lambda_{10}}{12\sqrt{15}} + \frac{1}{2} \sqrt{\frac{3}{5}} \lambda_1 \phi_1 - \frac{\lambda_1 \phi_2}{18\sqrt{15}} - \frac{5\lambda_1 \phi_3}{18\sqrt{3}}, \]

\[ m^{(3.2, \phi)}_{67} = -\frac{2}{3} \sqrt{\frac{2}{5}} a_2 \lambda_7 + \frac{1}{3} \sqrt{\frac{5}{6}} E \lambda_{10} + \frac{1}{9} \sqrt{\frac{5}{6}} \lambda_1 \phi_2 - \frac{1}{3} \sqrt{\frac{5}{6}} \lambda_1 \phi_3, \]

\[ m^{(3.2, \phi)}_{77} = m_1 - \frac{2}{5} \sqrt{\frac{5}{6}} a_1 \lambda_7 + \frac{22a_2 \lambda_7}{15\sqrt{15}} + \frac{11E \lambda_{10}}{12\sqrt{15}} - \frac{1}{3} \sqrt{\frac{5}{6}} \lambda_1 \phi_2 + \frac{\lambda_1 \phi_3}{9\sqrt{3}}. \]  

\[ (1, 2, \frac{1}{2}) + c.c. \]

\[ c: \hat{H}^{(1.2, \frac{1}{2})}_{(5, 2)}, \hat{D}^{(1.2, \frac{1}{2})}_{(5, 2)}, \hat{D}^{(1.2, \frac{1}{2})}_{(45, 2)}, \hat{\Delta}^{(1.2, \frac{1}{2})}_{(45, 2)}, \hat{\Delta}^{(1.2, \frac{1}{2})}_{(5, 2)}, \hat{\Phi}^{(1.2, \frac{1}{2})}_{(5, -8)} \]

\[ r: \hat{H}^{(1.2, -\frac{1}{2})}_{(5, -2)}, \hat{D}^{(1.2, -\frac{1}{2})}_{(5, -2)}, \hat{D}^{(1.2, -\frac{1}{2})}_{(45, -2)}, \hat{\Delta}^{(1.2, -\frac{1}{2})}_{(45, -2)}, \hat{\Delta}^{(1.2, -\frac{1}{2})}_{(5, -2)}, \hat{\Phi}^{(1.2, -\frac{1}{2})}_{(5, -8)} \]

\[
\begin{pmatrix}
\begin{array}{cccccccc}
\frac{m_{3} + \sqrt{\frac{2}{5}} \lambda_{13} E}{1} & \frac{m_{21}}{1} & \frac{m_{22}}{1} & \frac{m_{23}}{1} & \frac{m_{24}}{1} & \frac{m_{25}}{1} & \frac{m_{26}}{1} & \frac{m_{27}}{1} \\
\frac{m_{21}}{1} & \frac{m_{22}}{1} & \frac{m_{23}}{1} & \frac{m_{24}}{1} & \frac{m_{25}}{1} & \frac{m_{26}}{1} & \frac{m_{27}}{1} & \frac{m_{28}}{1} \\
\frac{m_{24}}{1} & \frac{m_{25}}{1} & \frac{m_{26}}{1} & \frac{m_{27}}{1} & \frac{m_{28}}{1} & \frac{m_{29}}{1} & \frac{m_{30}}{1} & \frac{m_{31}}{1} \\
\frac{m_{28}}{1} & \frac{m_{29}}{1} & \frac{m_{30}}{1} & \frac{m_{31}}{1} & \frac{m_{32}}{1} & \frac{m_{33}}{1} & \frac{m_{34}}{1} & \frac{m_{35}}{1} \\
\frac{m_{31}}{1} & \frac{m_{32}}{1} & \frac{m_{33}}{1} & \frac{m_{34}}{1} & \frac{m_{35}}{1} & \frac{m_{36}}{1} & \frac{m_{37}}{1} & \frac{m_{38}}{1} \\
\frac{m_{35}}{1} & \frac{m_{36}}{1} & \frac{m_{37}}{1} & \frac{m_{38}}{1} & \frac{m_{39}}{1} & \frac{m_{40}}{1} & \frac{m_{41}}{1} & \frac{m_{42}}{1} \\
\frac{m_{38}}{1} & \frac{m_{39}}{1} & \frac{m_{40}}{1} & \frac{m_{41}}{1} & \frac{m_{42}}{1} & \frac{m_{43}}{1} & \frac{m_{44}}{1} & \frac{m_{45}}{1} \\
\frac{m_{42}}{1} & \frac{m_{43}}{1} & \frac{m_{44}}{1} & \frac{m_{45}}{1} & \frac{m_{46}}{1} & \frac{m_{47}}{1} & \frac{m_{48}}{1} & \frac{m_{49}}{1} \\
\end{array}
\end{pmatrix}
\]  

\[ (42) \]
where

\[
\begin{align*}
    m_{12}^{(1,2, \frac{1}{2})} &\equiv - \frac{2i \alpha_1 \lambda_6}{\sqrt{15}} + \frac{i \alpha_2 \lambda_6}{2 \sqrt{10}} - \frac{\lambda_1 \phi_1}{\sqrt{10}} - \frac{3 \lambda_1 \phi_2}{4 \sqrt{10}}, \\
    m_{13}^{(1,2, \frac{1}{2})} &\equiv - \frac{1}{2} i \sqrt{\frac{5}{6}} \alpha_2 \lambda_6 - \frac{1}{4} \sqrt{\frac{5}{6}} \lambda_1 \phi_2 - \frac{\lambda_1 \phi_3}{\sqrt{6}}, \\
    m_{15}^{(1,2, \frac{1}{2})} &\equiv - \frac{1}{5} \sqrt{3} \lambda_4 \phi_1 + \frac{1}{10} \sqrt{3} \lambda_4 \phi_2, \\
    m_{21}^{(1,2, \frac{1}{2})} &\equiv \frac{2i \alpha_1 \lambda_6}{\sqrt{15}} - \frac{i \alpha_2 \lambda_6}{2 \sqrt{10}} - \frac{\lambda_1 \phi_1}{\sqrt{10}} - \frac{3 \lambda_1 \phi_2}{4 \sqrt{10}}, \\
    m_{22}^{(1,2, \frac{1}{2})} &\equiv m_6 + \frac{\lambda_{14} E}{2 \sqrt{15}} + \frac{\lambda_{15} \phi_1}{\sqrt{15}} - \frac{\lambda_{15} \phi_2}{2 \sqrt{15}}, \\
    m_{23}^{(1,2, \frac{1}{2})} &\equiv \frac{1}{6} \sqrt{5} \lambda_{14} E + \frac{1}{18} \sqrt{5} \lambda_{15} \phi_2 - \frac{\lambda_{15} \phi_3}{9}, \\
    m_{24}^{(1,2, \frac{1}{2})} &\equiv \frac{i \alpha_2 \lambda_6}{4 \sqrt{2}} + \frac{\lambda_{20} \phi_2}{24 \sqrt{2}} + \frac{\lambda_{20} \phi_3}{6 \sqrt{10}}, \\
    m_{25}^{(1,2, \frac{1}{2})} &\equiv - \frac{1}{5} i \sqrt{3} a_1 \lambda_1 + \frac{3 i \alpha_2 \lambda_6}{20 \sqrt{2}} + \frac{\lambda_{21} \phi_1}{10 \sqrt{2}} - \frac{3 \lambda_{21} \phi_2}{40 \sqrt{2}}, \\
    m_{31}^{(1,2, \frac{1}{2})} &\equiv \frac{1}{2} i \sqrt{\frac{5}{6}} \alpha_2 \lambda_6 - \frac{1}{4} \sqrt{\frac{5}{6}} \lambda_1 \phi_2 - \frac{\lambda_1 \phi_3}{\sqrt{6}}, \\
    m_{33}^{(1,2, \frac{1}{2})} &\equiv m_6 + \frac{13 \lambda_{14} E}{6 \sqrt{15}} - \frac{\lambda_{15} \phi_1}{3 \sqrt{15}} - \frac{7 \lambda_{15} \phi_2}{18 \sqrt{15}} + \frac{2 \lambda_{15} \phi_3}{9 \sqrt{3}}, \\
    m_{34}^{(1,2, \frac{1}{2})} &\equiv - \frac{1}{5} i \alpha_1 \lambda_1 + \frac{13 i \alpha_2 \lambda_6}{20 \sqrt{6}} + \frac{\lambda_{20} \phi_1}{10 \sqrt{6}} - \frac{\lambda_{20} \phi_2}{120 \sqrt{6}} - \frac{\lambda_{20} \phi_3}{3 \sqrt{30}}, \\
    m_{35}^{(1,2, \frac{1}{2})} &\equiv \frac{i \alpha_2 \lambda_6}{4 \sqrt{6}} - \frac{\lambda_{21} \phi_2}{24 \sqrt{6}} - \frac{\lambda_{21} \phi_3}{6 \sqrt{30}}, \\
    m_{42}^{(1,2, \frac{1}{2})} &\equiv \frac{i \alpha_2 \lambda_6}{4 \sqrt{2}} + \frac{\lambda_{21} \phi_2}{24 \sqrt{2}} + \frac{\lambda_{21} \phi_3}{6 \sqrt{10}}, \\
    m_{43}^{(1,2, \frac{1}{2})} &\equiv \frac{1}{5} i \alpha_1 \lambda_1 - \frac{13 i \alpha_2 \lambda_6}{20 \sqrt{6}} + \frac{\lambda_{21} \phi_1}{10 \sqrt{6}} - \frac{\lambda_{21} \phi_2}{120 \sqrt{6}} - \frac{\lambda_{21} \phi_3}{3 \sqrt{30}}, \\
    m_{44}^{(1,2, \frac{1}{2})} &\equiv m_2 + \frac{a_1 \lambda_6}{5 \sqrt{10}} + \frac{1}{10} \sqrt{\frac{3}{5}} a_2 \lambda_6 - \frac{\lambda_2 \phi_2}{6 \sqrt{15}} + \frac{\lambda_2 \phi_3}{15 \sqrt{3}}, \\
    m_{52}^{(1,2, \frac{1}{2})} &\equiv \frac{1}{5} i \sqrt{3} a_1 \lambda_1 - \frac{3 i \alpha_2 \lambda_6}{20 \sqrt{2}} + \frac{\lambda_{20} \phi_1}{10 \sqrt{2}} + \frac{3 \lambda_{20} \phi_2}{40 \sqrt{2}}, \\
    m_{53}^{(1,2, \frac{1}{2})} &\equiv - \frac{i \alpha_2 \lambda_6}{4 \sqrt{6}} - \frac{\lambda_{20} \phi_2}{24 \sqrt{6}} - \frac{\lambda_{20} \phi_3}{6 \sqrt{30}}, \\
    m_{55}^{(1,2, \frac{1}{2})} &\equiv m_2 - \frac{a_1 \lambda_6}{5 \sqrt{10}} - \frac{1}{10} \sqrt{\frac{3}{5}} a_2 \lambda_6 + \frac{\lambda_2 \phi_1}{5 \sqrt{15}} - \frac{\lambda_2 \phi_2}{10 \sqrt{15}}, \\
    m_{66}^{(1,2, \frac{1}{2})} &\equiv m_1 + \frac{2}{5} \sqrt{\frac{2}{5}} a_1 \lambda_7 + \frac{2}{5} \sqrt{\frac{3}{5}} a_2 \lambda_7 - \frac{1}{4} \sqrt{\frac{3}{5}} \lambda_{10} E + \sqrt{\frac{3}{5}} \lambda_1 \phi_1 - \frac{1}{2} \sqrt{\frac{3}{5}} \lambda_1 \phi_2. \\
\end{align*}
\]
\[
\left[ (3, 1, -\frac{1}{3}) + c.c. \right]
\]

c: \( \hat{H}^{(3,1,-\frac{1}{3})}_{(5,2)} \), \( \hat{D}^{(3,1,-\frac{1}{3})}_{(5,2)} \), \( \hat{\Delta}^{(3,1,-\frac{1}{3})}_{(45,2)} \), \( \hat{\Delta}^{(3,1,-\frac{1}{3})}_{(5,2)} \), \( \hat{\Delta}^{(3,1,-\frac{1}{3})}_{(50,2)} \), \( \Phi^{(3,1,-\frac{1}{3})}_{(5,-8)} \)

\[
\begin{pmatrix}
m_3 - \frac{2\lambda_4E}{\sqrt{15}}_{(3,1,-\frac{1}{3})} & m_{12}^{(3,1,-\frac{1}{3})} & m_{13}^{(3,1,-\frac{1}{3})} & m_{14}^{(3,1,-\frac{1}{3})} & m_{15}^{(3,1,-\frac{1}{3})} & -\sqrt{\frac{2}{15}}\lambda_4\phi_3 & \frac{\lambda_4\sqrt{V_R}}{\sqrt{5}} \\m_{21}^{(3,1,-\frac{1}{3})} & m_{22}^{(3,1,-\frac{1}{3})} & m_{23}^{(3,1,-\frac{1}{3})} & m_{24}^{(3,1,-\frac{1}{3})} & m_{25}^{(3,1,-\frac{1}{3})} & 0 & \frac{\lambda_3\sqrt{V_R}}{\sqrt{5}} \\m_{31}^{(3,1,-\frac{1}{3})} & m_{32}^{(3,1,-\frac{1}{3})} & m_{33}^{(3,1,-\frac{1}{3})} & m_{34}^{(3,1,-\frac{1}{3})} & m_{35}^{(3,1,-\frac{1}{3})} & \frac{2\lambda_4E}{\sqrt{3}} & \frac{\lambda_3\sqrt{V_R}}{\sqrt{30}} \\m_{41}^{(3,1,-\frac{1}{3})} & m_{42}^{(3,1,-\frac{1}{3})} & m_{43}^{(3,1,-\frac{1}{3})} & m_{44}^{(3,1,-\frac{1}{3})} & m_{45}^{(3,1,-\frac{1}{3})} & \frac{2\lambda_4E}{\sqrt{3}} & \frac{2\lambda_4E}{\sqrt{3}} \\m_{51}^{(3,1,-\frac{1}{3})} & m_{52}^{(3,1,-\frac{1}{3})} & m_{53}^{(3,1,-\frac{1}{3})} & m_{54}^{(3,1,-\frac{1}{3})} & m_{55}^{(3,1,-\frac{1}{3})} & \frac{2\lambda_4E}{\sqrt{3}} & \frac{\lambda_3\sqrt{V_R}}{\sqrt{30}} \\m_{61}^{(3,1,-\frac{1}{3})} & m_{62}^{(3,1,-\frac{1}{3})} & m_{63}^{(3,1,-\frac{1}{3})} & m_{64}^{(3,1,-\frac{1}{3})} & m_{65}^{(3,1,-\frac{1}{3})} & \frac{2\lambda_4E}{\sqrt{3}} & \frac{\lambda_3\sqrt{V_R}}{\sqrt{30}} \\m_{71}^{(3,1,-\frac{1}{3})} & m_{72}^{(3,1,-\frac{1}{3})} & m_{73}^{(3,1,-\frac{1}{3})} & m_{74}^{(3,1,-\frac{1}{3})} & m_{75}^{(3,1,-\frac{1}{3})} & \frac{2\lambda_4E}{\sqrt{3}} & \frac{\lambda_3\sqrt{V_R}}{\sqrt{30}} \end{pmatrix}
\]

(44)
where

\begin{align*}
m_{12}^{(3,1,\frac{1}{2})} & \equiv \frac{-2ia_1\lambda_{16}}{\sqrt{15}} - \frac{ia_2\lambda_{16}}{3\sqrt{10}} - \frac{\lambda_{17}\phi_1}{\sqrt{10}} + \frac{\lambda_{17}\phi_2}{2\sqrt{10}}, \\
m_{13}^{(3,1,\frac{1}{2})} & \equiv -\frac{1}{3}i\sqrt{\frac{5}{2}}a_2\lambda_{16} - \frac{1}{6}\sqrt{\frac{5}{2}}\lambda_{17}\phi_2 + \frac{\lambda_{17}\phi_3}{3\sqrt{2}}, \\
m_{14}^{(3,1,\frac{1}{2})} & \equiv -\frac{1}{3}\lambda_3\phi_2 - \frac{\lambda_3\phi_3}{3\sqrt{5}}, \\
m_{15}^{(3,1,\frac{1}{2})} & \equiv -\frac{1}{5}\sqrt{3}\lambda_4\phi_1 - \frac{\lambda_4\phi_2}{5\sqrt{3}}, \\
m_{16}^{(3,1,\frac{1}{2})} & \equiv \frac{2ia_1\lambda_{16}}{\sqrt{15}} + \frac{ia_2\lambda_{16}}{3\sqrt{10}} - \frac{\lambda_{17}\phi_1}{\sqrt{10}} + \frac{\lambda_{17}\phi_2}{2\sqrt{10}}, \\
m_{21}^{(3,1,\frac{1}{2})} & \equiv m_6 - \frac{\lambda_{14}E}{3\sqrt{15}} + \frac{\lambda_{15}\phi_1}{\sqrt{15}} + \frac{\lambda_{15}\phi_2}{3\sqrt{15}}, \\
m_{22}^{(3,1,\frac{1}{2})} & \equiv \frac{1}{3}\sqrt{\frac{5}{3}}\lambda_{14}E + \frac{1}{9}\sqrt{\frac{5}{3}}\lambda_{15}\phi_2 + \frac{\lambda_{15}\phi_3}{9\sqrt{3}}, \\
m_{23}^{(3,1,\frac{1}{2})} & \equiv m_6 + \frac{\lambda_{20}\phi_1}{10\sqrt{2}} - \frac{\lambda_{20}\phi_3}{10\sqrt{2}} - \frac{\lambda_{21}\phi_2}{20\sqrt{2}}, \\
m_{24}^{(3,1,\frac{1}{2})} & \equiv \frac{-i\sqrt{3}a_1\lambda_{19} - ia_2\lambda_{19}}{10\sqrt{2}} - \frac{\lambda_{21}\phi_1}{10\sqrt{2}} - \frac{\lambda_{21}\phi_2}{10\sqrt{2}} - \frac{\lambda_{21}\phi_3}{20\sqrt{2}}, \\
m_{31}^{(3,1,\frac{1}{2})} & \equiv \frac{1}{3}\sqrt{\frac{5}{2}}a_2\lambda_{16} - \frac{1}{6}\sqrt{\frac{5}{2}}\lambda_{17}\phi_2 + \frac{\lambda_{17}\phi_3}{3\sqrt{2}}, \\
m_{32}^{(3,1,\frac{1}{2})} & \equiv m_6 - \frac{\lambda_{14}E}{3\sqrt{15}} - \frac{\lambda_{15}\phi_1}{3\sqrt{15}} - \frac{\lambda_{15}\phi_2}{3\sqrt{15}} + \frac{4\lambda_{15}\phi_3}{9\sqrt{3}}, \\
m_{33}^{(3,1,\frac{1}{2})} & \equiv -\frac{1}{5}ia_1\lambda_{18} - \frac{ia_2\lambda_{18}}{10\sqrt{2}} + \frac{\lambda_{20}\phi_1}{10\sqrt{2}} + \frac{\lambda_{20}\phi_2}{20\sqrt{2}}, \\
m_{34}^{(3,1,\frac{1}{2})} & \equiv \frac{i\sqrt{3}a_1\lambda_{19}}{6\sqrt{2}} - \frac{\lambda_{21}\phi_2}{36\sqrt{2}} + \frac{18\sqrt{10}}{18}, \\
m_{35}^{(3,1,\frac{1}{2})} & \equiv -\frac{1}{3}ia_2\lambda_{19} + \frac{\lambda_{21}\phi_2}{18} - \frac{\lambda_{21}\phi_3}{9\sqrt{5}}, \\
m_{36}^{(3,1,\frac{1}{2})} & \equiv \frac{i\sqrt{3}a_1\lambda_{19}}{6\sqrt{2}} + \frac{\lambda_{21}\phi_2}{18} - \frac{\lambda_{21}\phi_3}{9\sqrt{5}}, \\
m_{42}^{(3,1,\frac{1}{2})} & \equiv \frac{i\sqrt{3}a_1\lambda_{19}}{6\sqrt{2}} + \frac{\lambda_{21}\phi_2}{36\sqrt{2}} - \frac{\lambda_{21}\phi_3}{18}, \\
m_{43}^{(3,1,\frac{1}{2})} & \equiv \frac{1}{5}ia_1\lambda_{19} + \frac{ia_2\lambda_{19}}{10\sqrt{2}} + \frac{\lambda_{21}\phi_1}{10\sqrt{2}} - \frac{\lambda_{21}\phi_2}{20\sqrt{2}}, \\
m_{44}^{(3,1,\frac{1}{2})} & \equiv m_2 + \frac{a_1\lambda_6}{5\sqrt{10}} - \frac{a_2\lambda_6}{5\sqrt{15}}, \\
m_{52}^{(3,1,\frac{1}{2})} & \equiv \frac{1}{5}i\sqrt{3}a_1\lambda_{18} + \frac{ia_2\lambda_{18}}{10\sqrt{2}} + \frac{\lambda_{20}\phi_1}{10\sqrt{2}} - \frac{\lambda_{20}\phi_2}{20\sqrt{2}}, \\
m_{53}^{(3,1,\frac{1}{2})} & \equiv -\frac{ia_2\lambda_{18}}{6\sqrt{2}} - \frac{\lambda_{20}\phi_2}{36\sqrt{2}} + \frac{\lambda_{20}\phi_3}{18\sqrt{10}}, \\
m_{55}^{(3,1,\frac{1}{2})} & \equiv m_2 - \frac{a_1\lambda_6}{5\sqrt{10}} + \frac{a_2\lambda_6}{5\sqrt{15}} + \frac{\lambda_{20}\phi_1}{5\sqrt{15}} + \frac{\lambda_{20}\phi_2}{15\sqrt{15}}, \\
m_{63}^{(3,1,\frac{1}{2})} & \equiv \frac{1}{3}ia_2\lambda_{18} + \frac{\lambda_{20}\phi_2}{18} - \frac{\lambda_{20}\phi_3}{9\sqrt{5}}, \\
m_{66}^{(3,1,\frac{1}{2})} & \equiv m_2 - \frac{a_1\lambda_6}{5\sqrt{10}} + \frac{a_2\lambda_6}{5\sqrt{15}} - \frac{\lambda_{20}\phi_1}{10\sqrt{15}} - \frac{\lambda_{20}\phi_2}{30\sqrt{15}} + \frac{\lambda_{20}\phi_3}{15\sqrt{3}},
\end{align*}
\[(1, 1, 2) + \text{c.c.}\]

c: \(\hat{\Delta}^{(1,1,2)}_{(55, -2)}\)

r: \(\hat{\Delta}^{(1,1,-2)}_{(50,2)}\)

\[
m_2 - \frac{\lambda_6 a_1}{5\sqrt{10}} + \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_6 a_2 - \frac{\lambda_2 \phi_1}{10\sqrt{15}} - \frac{\lambda_2 \phi_2}{5\sqrt{15}} + \frac{\lambda_2 \phi_3}{5\sqrt{3}}. \tag{46}
\]

\[(1, 2, \frac{3}{2}) + \text{c.c.}\]

c: \(\hat{\Phi}^{(1,2,\frac{3}{2})}_{(40,-4)}\)

r: \(\hat{\Phi}^{(1,2,-\frac{3}{2})}_{(48,4)}\)

\[
m_1 - \frac{1}{4}\sqrt{\frac{3}{5}} \lambda_{10} E - \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_7 a_1 - \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7 a_2 - \frac{1}{2}\sqrt{\frac{5}{3}} \lambda_1 \phi_2 + \frac{\lambda_1 \phi_3}{\sqrt{3}}. \tag{47}
\]

\[(1, 3, 0)\]

c: \(\hat{A}^{(1,3,0)}_{(24,0)}, \hat{E}^{(1,3,0)}_{(24,0)}, \hat{\Phi}^{(1,3,0)}_{(24,0)}\)

r: \(\hat{A}^{(1,3,0)}_{(24,0)}, \hat{E}^{(1,3,0)}_{(24,0)}, \hat{\Phi}^{(1,3,0)}_{(24,0)}\)

\[
\begin{pmatrix}
m_{11}^{(1,3,0)} & \sqrt{\frac{2}{5}} \lambda_9 a_1 + \sqrt{\frac{3}{5}} \lambda_9 a_2 & m_5^{(1,3,0)} + 3 \sqrt{\frac{3}{5}} \lambda_8 E \\
m_{13}^{(1,3,0)} & m_{23}^{(1,3,0)} & m_{33}^{(1,3,0)}
\end{pmatrix}, \tag{48}
\]

where

\[
m_{11}^{(1,3,0)} \equiv m_4 + \sqrt{\frac{3}{5}} \lambda_9 E - \frac{\lambda_5 \phi_1}{\sqrt{15}} - \frac{2\lambda_5 \phi_2}{\sqrt{15}} - \frac{\lambda_5 \phi_3}{\sqrt{3}},
\]

\[
m_{13}^{(1,3,0)} \equiv -\sqrt{\frac{2}{5}} \lambda_5 a_1 + \frac{2\lambda_5 a_2}{\sqrt{15}} + \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7 \phi_1 + \frac{8\lambda_7 \phi_2}{5\sqrt{15}} + \frac{2\lambda_7 \phi_3}{5\sqrt{3}},
\]

\[
m_{23}^{(1,3,0)} \equiv -\frac{1}{2}\sqrt{\frac{3}{5}} \lambda_{10} \phi_1 - \frac{\lambda_{10} \phi_2}{2\sqrt{15}} + \frac{\lambda_{10} \phi_3}{2\sqrt{3}},
\]

\[
m_{33}^{(1,3,0)} \equiv m_1 + \frac{\lambda_{10} E}{2\sqrt{15}} - \frac{4}{5} \sqrt{\frac{2}{5}} \lambda_7 a_1 + \frac{8\lambda_7 a_2}{5\sqrt{15}} + \frac{\lambda_1 \phi_1}{2\sqrt{15}} - \frac{7\lambda_1 \phi_2}{3\sqrt{15}} - \frac{\lambda_1 \phi_3}{6\sqrt{3}}. \quad \tag{49}
\]
\[ (1, 3, 1) + \text{c.c.} \]

\[ \text{c: } \widehat{E}^{(1,3,1)}_{(15,4)} \triangleq \Delta^{(1,3,1)}_{(15,-6)} \]

\[ \text{r: } \widehat{E}^{(1,3,-1)}_{(15,-4)} \triangleq \Delta^{(1,3,-1)}_{(15,6)} \]

\[
\left( \begin{array}{c}
\frac{m_5 + 3\sqrt{\frac{2}{5}}\lambda_8 E}{2\lambda_1 V_R} \\
2m_2 + \frac{3\lambda_6 a_1}{5\sqrt{10}} - \frac{\lambda_2 \phi_1}{10\sqrt{15}} - \frac{1}{10\sqrt{5}} \left( \frac{2\lambda_2 \phi_2}{5} \right)
\end{array} \right).
\]

\[ [(3, 1, -\frac{4}{3}) + \text{c.c.}] \]

\[ \text{c: } \widehat{D}^{(3,1,-\frac{4}{3})}_{(45,-2)} \triangleq \Delta^{(3,1,-\frac{4}{3})}_{(45,-2)} \]

\[ \text{r: } \widehat{D}^{(3,1,\frac{4}{3})}_{(45,2)} \triangleq \Delta^{(3,1,\frac{4}{3})}_{(45,2)} \]

\[
\left( \begin{array}{cc}
m^{(3,1,-\frac{4}{3})}_{11} & m^{(3,1,-\frac{4}{3})}_{12} \\
m^{(3,1,-\frac{4}{3})}_{21} & m^{(3,1,-\frac{4}{3})}_{22}
\end{array} \right),
\]

where

\[
m^{(3,1,-\frac{4}{3})}_{11} \equiv m_6 + \frac{4\lambda_{14} E}{3\sqrt{15}} - \frac{\lambda_{15} \phi_1}{3\sqrt{15}} + \frac{4\lambda_{15} \phi_2}{9\sqrt{15}} + \frac{5\lambda_{15} \phi_3}{9\sqrt{3}},
\]

\[
m^{(3,1,-\frac{4}{3})}_{12} \equiv \frac{i\lambda_{18} a_1}{5} - \frac{i\lambda_{18} a_2}{5\sqrt{10}} - \frac{\lambda_{20} \phi_1}{10\sqrt{6}} + \frac{\lambda_{20} \phi_2}{5\sqrt{6}} + \frac{\lambda_{20} \phi_3}{2\sqrt{30}},
\]

\[
m^{(3,1,-\frac{4}{3})}_{21} \equiv \frac{i\lambda_{19} a_1}{5} + \frac{i\lambda_{19} a_2}{5\sqrt{10}} - \frac{\lambda_{21} \phi_1}{10\sqrt{6}} + \frac{\lambda_{21} \phi_2}{5\sqrt{6}} + \frac{\lambda_{21} \phi_3}{2\sqrt{30}},
\]

\[
m^{(3,1,-\frac{4}{3})}_{22} \equiv m_2 + \frac{\lambda_6 a_1}{5\sqrt{10}} + \frac{4\lambda_6 a_2}{5\sqrt{15}} + \frac{\lambda_2 \phi_3}{10\sqrt{3}}.
\]

\[ [(3, 1, \frac{2}{3})] \]

\[ \text{c: } \widehat{\Phi}^{(3,1,\frac{2}{3})}_{(75,0)} \]

\[ \text{r: } \widehat{\Phi}^{(3,1,\frac{2}{3})}_{(75,0)} \]

\[
m_1 + \frac{\lambda_{10} E}{2\sqrt{15}} + \frac{2\sqrt{2}}{5} \frac{\lambda_7 a_1}{5\sqrt{15}} - \frac{4\lambda_7 a_2}{5\sqrt{15}} - \frac{\lambda_1 \phi_1}{3\sqrt{15}} + \frac{\lambda_1 \phi_2}{3\sqrt{3}} + \frac{4\lambda_1 \phi_3}{3\sqrt{3}}.
\]

24
\[ [(3, 2, \frac{7}{6}) + c.c.] \]

c: \( \hat{D}_{(45,-2)}^{(3,2,\frac{7}{6})}, \Delta^{(3,2,\frac{7}{6})}_{(45,-2)} \)

r: \( \hat{D}_{(45,2)}^{(3,2,\frac{7}{6})}, \Delta^{(3,2,\frac{7}{6})}_{(50,2)} \)

\[
\begin{pmatrix}
\frac{m_{11}^{(3,2,\frac{7}{6})}}{\lambda_{14} E}, & \frac{m_{12}^{(3,2,\frac{7}{6})}}{3}, & \frac{m_{13}^{(3,2,\frac{7}{6})}}{3} \\
\frac{m_{21}^{(3,2,\frac{7}{6})}}{\lambda_{14} E}, & \frac{m_{22}^{(3,2,\frac{7}{6})}}{3}, & \frac{m_{23}^{(3,2,\frac{7}{6})}}{3} \\
\frac{m_{31}^{(3,2,\frac{7}{6})}}{\lambda_{14} E}, & \frac{m_{32}^{(3,2,\frac{7}{6})}}{3}, & \frac{m_{33}^{(3,2,\frac{7}{6})}}{3}
\end{pmatrix},
\]  

where

\[ m_{11}^{(3,2,\frac{7}{6})} \equiv m_6 - \frac{\lambda_{14} E}{3 \sqrt{15}} - \frac{\lambda_{15}}{3 \sqrt{15}} - \frac{2 \lambda_{15} \phi_2}{3 \sqrt{15}} + \frac{\lambda_{15} \phi_3}{3 \sqrt{15}}; \]

\[ m_{21}^{(3,2,\frac{7}{6})} \equiv \frac{i \lambda_{19} a_2}{2 \sqrt{6}} - \frac{\lambda_{21} \phi_2}{12 \sqrt{6}} + \frac{1}{3} \sqrt{\frac{2}{15}} \lambda_{21} \phi_3; \]

\[ m_{31}^{(3,2,\frac{7}{6})} \equiv \frac{i \lambda_{18} a_1}{5} + \frac{i \lambda_{18} a_2}{10 \sqrt{6}} - \frac{\lambda_{20} \phi_1}{10 \sqrt{6}} + \frac{13 \lambda_{20} \phi_2}{60 \sqrt{6}} + \frac{\lambda_{20} \phi_3}{6 \sqrt{30}}; \]

\[ m_{12}^{(3,2,\frac{7}{6})} \equiv \frac{i \lambda_{18} a_2}{2 \sqrt{6}} - \frac{\lambda_{20} \phi_2}{12 \sqrt{6}} + \frac{1}{3} \sqrt{\frac{2}{15}} \lambda_{20} \phi_3; \]

\[ m_{22}^{(3,2,\frac{7}{6})} \equiv m_2 - \frac{\lambda_{6} a_1}{5 \sqrt{10}} + \frac{7 \lambda_{6} a_2}{10 \sqrt{15}} - \frac{\lambda_{2} \phi_1}{10 \sqrt{15}} - \frac{7 \lambda_{2} \phi_2}{60 \sqrt{15}} + \frac{\lambda_{2} \phi_3}{15 \sqrt{3}}; \]

\[ m_{13}^{(3,2,\frac{7}{6})} \equiv \frac{i \lambda_{19} a_1}{5} - \frac{i \lambda_{19} a_2}{10 \sqrt{6}} - \frac{\lambda_{21} \phi_1}{10 \sqrt{6}} + \frac{13 \lambda_{21} \phi_2}{60 \sqrt{6}} + \frac{\lambda_{21} \phi_3}{6 \sqrt{30}}; \]

\[ m_{33}^{(3,2,\frac{7}{6})} \equiv m_2 + \frac{\lambda_{6} a_1}{5 \sqrt{10}} - \frac{7 \lambda_{6} a_2}{10 \sqrt{15}} - \frac{\lambda_{2} \phi_2}{12 \sqrt{15}} + \frac{\lambda_{2} \phi_3}{30 \sqrt{3}}. \]  

\[ [(3, 3, -\frac{1}{3}) + c.c.] \]

r: \( \hat{D}_{(45,2)}^{(3,3,-\frac{1}{3})}, \Delta^{(3,3,-\frac{1}{3})}_{(45,2)} \)

\[
\begin{pmatrix}
\frac{m_{11}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{12}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{13}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{21}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{22}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{23}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{31}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{32}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{33}^{(3,3,-\frac{1}{3})}}{3}
\end{pmatrix},
\]  

\[
\begin{pmatrix}
\frac{m_{11}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{12}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{13}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{21}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{22}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{23}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{31}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{32}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{33}^{(3,3,-\frac{1}{3})}}{3}
\end{pmatrix},
\]  

\[
\begin{pmatrix}
\frac{m_{11}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{12}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{13}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{21}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{22}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{23}^{(3,3,-\frac{1}{3})}}{3} \\
\frac{m_{31}^{(3,3,-\frac{1}{3})}}{\lambda_{14} E}, & \frac{m_{32}^{(3,3,-\frac{1}{3})}}{3}, & \frac{m_{33}^{(3,3,-\frac{1}{3})}}{3}
\end{pmatrix}.
\]  

25
where

\[
m^{(3,3,-\frac{1}{3})}_{11} \equiv m_6 + \frac{4\lambda_{14}E}{3\sqrt{15}} - \frac{\lambda_{15}\phi_1}{3\sqrt{15}} - \frac{2\lambda_{15}\phi_2}{3\sqrt{15}} - \frac{\lambda_{15}\phi_3}{3\sqrt{3}},
\]

\[
m^{(3,3,-\frac{1}{3})}_{21} \equiv \frac{i\lambda_{19}a_1}{5} - \frac{i}{5} \sqrt{2} \frac{2}{3} \lambda_{19}a_2 + \frac{\lambda_{21}\phi_1}{10\sqrt{6}} - \frac{1}{15} \sqrt{\frac{2}{3}} \lambda_{21}\phi_2 + \frac{\lambda_{21}\phi_3}{6\sqrt{30}},
\]

\[
m^{(3,3,-\frac{1}{3})}_{12} \equiv -\frac{i\lambda_{18}a_1}{5} + \frac{i}{5} \sqrt{2} \frac{2}{3} \lambda_{18}a_2 + \frac{\lambda_{20}\phi_1}{10\sqrt{6}} - \frac{1}{15} \sqrt{\frac{2}{3}} \lambda_{20}\phi_2 + \frac{\lambda_{20}\phi_3}{6\sqrt{30}},
\]

\[
m^{(3,3,-\frac{1}{3})}_{22} \equiv m_2 + \frac{\lambda_6 a_1}{5\sqrt{10}} - \frac{\lambda_6 a_2}{5\sqrt{15}} - \frac{\lambda_2\phi_2}{6\sqrt{15}} - \frac{\lambda_2\phi_3}{30\sqrt{3}}.
\]

\[
[(3, 3, \frac{2}{3}) + c.c.]
\]

\[
c: \hat{\Phi}^{(3,3,\frac{2}{3})}_{(40,-4)}
\]

\[
r: \hat{\Phi}^{(3,3,\frac{2}{3})}_{(50,4)}
\]

\[
m_1 + \frac{\lambda_{10}E}{2\sqrt{15}} - \frac{2}{5} \frac{2}{5} \lambda_7 a_1 + \frac{4\lambda_7 a_2}{5\sqrt{15}} - \frac{1}{5} \frac{5}{3} \lambda_1\phi_2 - \frac{\lambda_1\phi_3}{3\sqrt{3}}.
\]

\[
[(6, 1, -\frac{2}{3}) + c.c.]
\]

\[
c: \hat{E}^{(6,1,-\frac{2}{3})}_{(15,4)}, \hat{\Delta}^{(6,1,-\frac{2}{3})}_{(15,-6)}
\]

\[
r: \hat{E}^{(6,1,-\frac{2}{3})}_{(15,-4)}, \hat{\Delta}^{(6,1,-\frac{2}{3})}_{(15,6)}
\]

\[
\left( m_5 - \frac{2}{5} \frac{3}{5} \lambda_8 E \right) m_2 + \frac{3\lambda_8 a_1}{5\sqrt{10}} + \frac{2\lambda_8 a_2}{5\sqrt{15}} + \frac{\lambda_2\phi_1}{10\sqrt{15}} + \frac{\lambda_2\phi_2}{5\sqrt{15}}.
\]

\[
[(6, 1, \frac{4}{3}) + c.c.]
\]

\[
c: \hat{D}^{(6,1,\frac{4}{3})}_{(45,-2)}, \hat{\Delta}^{(6,1,\frac{4}{3})}_{(45,-2)}
\]

\[
r: \hat{D}^{(6,1,\frac{4}{3})}_{(45,2)}, \hat{\Delta}^{(6,1,\frac{4}{3})}_{(45,2)}
\]

\[
\left( \begin{array}{ccc}
m^{(6,1,\frac{4}{3})}_{11} & m^{(6,1,\frac{4}{3})}_{12} & m^{(6,1,\frac{4}{3})}_{21} \\
m^{(6,1,\frac{4}{3})}_{12} & m^{(6,1,\frac{4}{3})}_{22} & m^{(6,1,\frac{4}{3})}_{22} \\
m^{(6,1,\frac{4}{3})}_{21} & m^{(6,1,\frac{4}{3})}_{22} & m^{(6,1,\frac{4}{3})}_{22} \\
\end{array} \right),
\]

\[
26
\]
where

\[ m_{11}^{(6,1,1/3)} \equiv m_6 - \frac{2\lambda_{14}E}{\sqrt{15}} - \frac{\lambda_{15}\phi_1}{3\sqrt{15}} + \frac{4\lambda_{15}\phi_2}{9\sqrt{15}} - \frac{\lambda_{15}\phi_3}{9\sqrt{3}}, \]
\[ m_{21}^{(6,1,1/3)} \equiv \frac{i\lambda_{18} a_1}{5} + \frac{i}{5}\sqrt{\frac{3}{2}}\lambda_{18} a_2 - \frac{\lambda_{20}\phi_1}{10\sqrt{6}} + \frac{\lambda_{20}\phi_2}{30\sqrt{6}} + \frac{\lambda_{20}\phi_3}{6\sqrt{30}}, \]
\[ m_{12}^{(6,1,1/3)} \equiv -\frac{i\lambda_{19} a_1}{5} - \frac{i}{5}\sqrt{\frac{3}{2}}\lambda_{19} a_2 - \frac{\lambda_{21}\phi_1}{10\sqrt{6}} - \frac{\lambda_{21}\phi_2}{30\sqrt{6}} + \frac{\lambda_{21}\phi_3}{6\sqrt{30}}, \]
\[ m_{22}^{(6,1,1/3)} \equiv m_2 + \frac{\lambda_6 a_1}{5\sqrt{10}} - \frac{\lambda_6 a_2}{5\sqrt{15}} + \frac{\lambda_2\phi_1}{10\sqrt{15}} + \frac{\lambda_2\phi_2}{15\sqrt{15}} + \frac{\lambda_2\phi_3}{15\sqrt{3}}. \]  \hspace{1cm} (61)

\[(6, 1, 4/3) + c.c.\]

c: \( \hat{\Delta}^{(6,1,4/3)}_{(50,2)} \)

r: \( \hat{\Delta}^{(6,1,-4/3)}_{(50,-2)} \)

\[ m_2 = -\frac{\lambda_6 a_1}{5\sqrt{10}} - \frac{4\lambda_6 a_2}{5\sqrt{15}} - \frac{\lambda_2\phi_1}{10\sqrt{15}} + \frac{2\lambda_2\phi_2}{15\sqrt{15}} + \frac{\lambda_2\phi_3}{15\sqrt{3}}. \]  \hspace{1cm} (62)

\[(6, 2, -1/6) + c.c.\]

c: \( \hat{\Phi}^{(6,2,-1/6)}_{(40,-4)} \)

r: \( \hat{\Phi}^{(6,2,1/6)}_{(30,4)} \)

\[ m_1 - \frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E - \frac{2}{5}\sqrt{\frac{2}{5}}\lambda_7 a_1 - \frac{2}{5}\sqrt{\frac{3}{5}}\lambda_7 a_2 + \frac{1}{6}\sqrt{\frac{5}{3}}\lambda_1 \phi_2 - \frac{\lambda_1 \phi_3}{3\sqrt{3}}. \]  \hspace{1cm} (63)

\[(6, 2, 5/6) + c.c.\]

c: \( \hat{\Phi}^{(6,2,5/6)}_{(75,0)} \)

r: \( \hat{\Phi}^{(6,2,-5/6)}_{(75,0)} \)

\[ m_1 - \frac{1}{4}\sqrt{\frac{3}{5}}\lambda_{10}E + \frac{2}{5}\sqrt{\frac{2}{5}}\lambda_7 a_1 + \frac{2}{5}\sqrt{\frac{3}{5}}\lambda_7 a_2 - \frac{\lambda_1 \phi_1}{\sqrt{15}} + \frac{\lambda_1 \phi_2}{2\sqrt{15}}. \]  \hspace{1cm} (64)

\[(6, 3, 1/4) + c.c.\]

c: \( \hat{\Delta}^{(6,3,1/4)}_{(50,-2)} \)

r: \( \hat{\Delta}^{(6,3,-1/4)}_{(50,2)} \)

\[ m_2 = -\frac{\lambda_6 a_1}{5\sqrt{10}} + \frac{\lambda_6 a_2}{5\sqrt{15}} + \frac{\lambda_2\phi_1}{10\sqrt{15}} - \frac{\lambda_2\phi_2}{30\sqrt{15}} - \frac{\lambda_2\phi_3}{15\sqrt{3}}. \]  \hspace{1cm} (65)
\[(8, 1, 0)\]

\[
\text{c: } \hat{A}^{(8,1,0)}, \hat{E}^{(8,1,0)}, \phi^{(8,1,0)}, \Phi^{(8,1,0)}
\]

\[
\text{r: } \hat{A}^{(8,1,0)}, \hat{E}^{(8,1,0)}, \phi^{(8,1,0)}, \Phi^{(8,1,0)}
\]

\[
\begin{pmatrix}
    m_{11}^{(8,1,0)} & \sqrt{\frac{2}{5}} a_1 \lambda_9 - \frac{2a_2 \lambda_5}{\sqrt{15}} \\
    m_{13}^{(8,1,0)} & \sqrt{\frac{2}{5}} a_1 \lambda_5 - \frac{2a_2 \lambda_7}{3 \sqrt{15}} \\
    m_{14}^{(8,1,0)} & \frac{1}{3} \sqrt{\frac{2}{5}} \lambda_{10} \phi_2 - \frac{1}{3} \sqrt{\frac{2}{3}} \lambda_{10} \phi_3 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
    m_{5}^{(8,1,0)} & \sqrt{\frac{3}{5}} \lambda_8 E \\
    m_{23}^{(8,1,0)} & \frac{1}{3} \sqrt{\frac{2}{5}} \lambda_{10} \phi_2 - \frac{1}{3} \sqrt{\frac{2}{3}} \lambda_{10} \phi_3 \\
    m_{34}^{(8,1,0)} & \frac{1}{3} \sqrt{\frac{2}{5}} \lambda_{10} \phi_2 - \frac{1}{3} \sqrt{\frac{2}{3}} \lambda_{10} \phi_3 \\
\end{pmatrix}
\]

\[(66)\]

where

\[
m_{11}^{(8,1,0)} \equiv m_4 - \frac{2\lambda_9 E}{\sqrt{15}} - \frac{\lambda_5 \phi_1}{\sqrt{15}} + \frac{4\lambda_5 \phi_2}{3 \sqrt{15}} - \frac{\lambda_5 \phi_3}{3 \sqrt{3}},
\]

\[
m_{13}^{(8,1,0)} \equiv -\sqrt{\frac{2}{5}} a_1 \lambda_5 - \frac{4a_2 \lambda_5}{3 \sqrt{15}} + \frac{2}{5} \sqrt{\frac{3}{5}} \lambda_7 \phi_1 + \frac{16 \lambda_7 \phi_2}{15 \sqrt{15}} + \frac{2 \lambda_7 \phi_3}{15 \sqrt{3}},
\]

\[
m_{14}^{(8,1,0)} \equiv \frac{1}{3} \sqrt{\frac{8}{5}} \lambda_{10} \phi_2 - \frac{1}{3} \sqrt{\frac{2}{3}} \lambda_{10} \phi_3,
\]

\[
m_{23}^{(8,1,0)} \equiv \frac{1}{2} \sqrt{\frac{3}{5}} \lambda_{10} \phi_1 + \frac{\lambda_{10} \phi_2}{3 \sqrt{15}} + \frac{\lambda_{10} \phi_3}{6 \sqrt{3}},
\]

\[
m_{33}^{(8,1,0)} \equiv m_1 - \frac{4}{5} \sqrt{\frac{2}{5}} a_1 \lambda_7 - \frac{16a_2 \lambda_7}{15 \sqrt{15}} - \frac{\lambda_{10} E}{3 \sqrt{15}} + \frac{\lambda_1 \phi_1}{2 \sqrt{15}} + \frac{14 \lambda_1 \phi_2}{9 \sqrt{15}} - \frac{\lambda_1 \phi_3}{18 \sqrt{3}},
\]

\[
m_{34}^{(8,1,0)} \equiv \frac{2}{3} \sqrt{\frac{2}{5}} a_2 \lambda_7 - \frac{1}{3} \sqrt{\frac{5}{6}} \lambda_{10} E - \frac{1}{9} \sqrt{\frac{5}{6}} \lambda_1 \phi_2 - \frac{2}{9} \sqrt{\frac{2}{3}} \lambda_1 \phi_3,
\]

\[
m_{44}^{(8,1,0)} \equiv m_1 + \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_7 a_1 + \frac{28 \lambda_7 a_2}{15 \sqrt{15}} - \frac{7 \lambda_{10} E}{6 \sqrt{15}} - \frac{\lambda_1 \phi_1}{\sqrt{15}} + \frac{7 \lambda_1 \phi_2}{9 \sqrt{15}} + \frac{2 \lambda_1 \phi_3}{9 \sqrt{3}}.
\]

\[(67)\]

\[
[(8, 1, 1) + \text{c.c.}]
\]

\[
\text{c: } \hat{\phi}^{(8,1,1)}
\]

\[
\text{r: } \hat{\phi}^{(8,1,-1)}
\]

\[
m_1 + \frac{\lambda_{10} E}{2 \sqrt{15}} - \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_7 a_1 + \frac{4 \lambda_7 a_2}{5 \sqrt{15}} + \frac{1}{3} \sqrt{\frac{5}{3}} \lambda_1 \phi_2 + \frac{\lambda_1 \phi_3}{3 \sqrt{3}}.
\]

\[(68)\]
\[(8, 2, \frac{1}{2}) + c.c.\]

c: \(\hat{D}(8, 2, \frac{1}{2}) \equiv \Delta(8, 2, \frac{1}{2}) \equiv \Delta(8, 2, \frac{1}{2})\)

r: \(\hat{D}(8, 2, -\frac{1}{2}) \equiv \Delta(8, 2, -\frac{1}{2}) \equiv \Delta(8, 2, -\frac{1}{2})\)

\[
\begin{pmatrix}
  m_{11}^{(8,2,\frac{1}{2})} & m_{12}^{(8,2,\frac{1}{2})} & m_{13}^{(8,2,\frac{1}{2})} \\
  m_{21}^{(8,2,\frac{1}{2})} & m_{22}^{(8,2,\frac{1}{2})} & m_{23}^{(8,2,\frac{1}{2})} \\
  m_{31}^{(8,2,\frac{1}{2})} & m_{32}^{(8,2,\frac{1}{2})} & m_{33}^{(8,2,\frac{1}{2})}
\end{pmatrix}
\]

(69)

where

\[
\begin{align*}
  m_{11}^{(8,2,\frac{1}{2})} & \equiv m_6 - \frac{\lambda_{14} E}{3\sqrt{15}} - \frac{\lambda_{15} \phi_1}{3\sqrt{15}} + \frac{4\lambda_{15} \phi_2}{9\sqrt{15}} - \frac{\lambda_{15} \phi_3}{9\sqrt{3}}, \\
  m_{12}^{(8,2,\frac{1}{2})} & \equiv -\frac{1}{5} i a_1 \lambda_{18} - \frac{i a_2 \lambda_{18}}{10\sqrt{6}} + \frac{\lambda_{20} \phi_1}{10\sqrt{6}} + \frac{7\lambda_{20} \phi_2}{60\sqrt{6}} + \frac{\lambda_{20} \phi_3}{6\sqrt{30}}, \\
  m_{13}^{(8,2,\frac{1}{2})} & \equiv -\frac{ia_2 \lambda_{19}}{2\sqrt{6}} + \frac{\lambda_{21} \phi_2}{12\sqrt{6}} + \frac{\lambda_{21} \phi_3}{3\sqrt{30}}, \\
  m_{21}^{(8,2,\frac{1}{2})} & \equiv \frac{1}{5} i a_1 \lambda_{19} + \frac{i a_2 \lambda_{19}}{10\sqrt{6}} + \frac{\lambda_{21} \phi_1}{10\sqrt{6}} + \frac{7\lambda_{21} \phi_2}{60\sqrt{6}} + \frac{\lambda_{21} \phi_3}{6\sqrt{30}}, \\
  m_{22}^{(8,2,\frac{1}{2})} & \equiv m_2 + \frac{a_1 \lambda_6}{5\sqrt{10}} + \frac{1}{10} \sqrt{\frac{3}{5}} a_2 \lambda_6 + \frac{\lambda_{20} \phi_2}{12\sqrt{15}} - \frac{\lambda_{20} \phi_3}{30\sqrt{3}}, \\
  m_{23}^{(8,2,\frac{1}{2})} & \equiv \frac{ia_2 \lambda_{18}}{2\sqrt{6}} + \frac{\lambda_{20} \phi_2}{12\sqrt{6}} + \frac{\lambda_{20} \phi_3}{3\sqrt{30}}, \\
  m_{31}^{(8,2,\frac{1}{2})} & \equiv m_3 - \frac{a_1 \lambda_6}{5\sqrt{10}} - \frac{1}{10} \sqrt{\frac{3}{5}} a_2 \lambda_6 - \frac{\lambda_{20} \phi_1}{10\sqrt{15}} + \frac{\lambda_{20} \phi_3}{20\sqrt{15}}, \\
  m_{33}^{(8,2,\frac{1}{2})} & \equiv m_3 + \frac{\lambda_{10} E}{2\sqrt{15}} + \frac{2}{5} \sqrt{\frac{2}{5}} \lambda_{7} a_1 - \frac{4\lambda_{7} a_2}{5\sqrt{15}} - \frac{\lambda_{1} \phi_1}{\sqrt{15}} - \frac{\lambda_{1} \phi_2}{3\sqrt{15}} - \frac{2\lambda_{1} \phi_3}{3\sqrt{3}}.
\end{align*}
\]

(70)

\((8, 3, 0)\)

c: \(\hat{\Phi}^{(8,3,0)}\)

r: \(\hat{\Phi}^{(8,3,0)}\)

(71)

6 Discussions

The results presented in the above Sections can be compared with previous studies, provided suitable transformations in notations are made. Firstly, in a recent study[22], \(G_{51}\) is used in
the MSSO10 with $126 + \overline{126} + 210$ breaking the SO(10). Relations of the VEVs are

$$
\phi_1 = -\frac{1}{4} \sqrt{3} S_{1210}, \quad \phi_2 = \frac{1}{2} \sqrt{\frac{3}{5}} S_{24_{210}}, \quad \phi_3 = \frac{\sqrt{3}}{2} S_{75_{210}},
$$

$$
V_R = \sqrt{\frac{15}{2}} S_{126}, \quad \bar{V}_R = \sqrt{\frac{15}{2}} S_{\overline{126}}.
$$

However, we find that the coefficient before $S^2_{75_{210}} S_{1210}$ coupling should be $\frac{3}{100}$ instead of $\frac{1}{40}$ in their (3).

We can also compare our results with [12] which uses $G_{422}$ as the maximal subgroup. When the following transformations are made,

$$
\begin{pmatrix}
\hat{A}^{(1,1,0)}_{(1,0)} & \hat{A}^{(1,1,0)}_{(1,1,0)} \\
\hat{A}^{(24,0)}_{(1,0)} & \hat{A}^{(1,1,0)}_{(1,1,1)}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\
\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}}
\end{pmatrix}
\begin{pmatrix}
\hat{A}^{(1,1,0)}_{(1,1,0)} & \hat{A}^{(1,1,0)}_{(1,1,1)} \\
\hat{A}^{(1,1,0)}_{(24,0)} & \hat{A}^{(1,1,0)}_{(15,1,3)}
\end{pmatrix},
$$

$$
\begin{pmatrix}
\hat{\Phi}^{(3,1,\frac{-2}{3})}_{(24,0)} \\
\hat{\Phi}^{(3,1,\frac{-2}{3})}_{(10,2,2)}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\
\sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}}
\end{pmatrix}
\begin{pmatrix}
\hat{\Phi}^{(3,1,\frac{-2}{3})}_{(15,1,1)} & \hat{\Phi}^{(3,1,\frac{-2}{3})}_{(10,2,2)} \\
\hat{\Phi}^{(3,1,\frac{-2}{3})}_{(10,4)} & \hat{\Phi}^{(3,1,\frac{-2}{3})}_{(10,2,2)}
\end{pmatrix},
$$

$$
\begin{pmatrix}
\hat{D}^{(1,2,\frac{1}{2})}_{(5,2)} \\
\hat{D}^{(1,2,\frac{1}{2})}_{(45,2)}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{1}{4}} & \sqrt{\frac{3}{4}} \\
\sqrt{\frac{3}{4}} & -\sqrt{\frac{1}{4}}
\end{pmatrix}
\begin{pmatrix}
\hat{D}^{(1,2,\frac{1}{2})}_{(1,2,2)} & \hat{D}^{(1,2,\frac{1}{2})}_{(15,2,2)} \\
\hat{D}^{(1,2,\frac{1}{2})}_{(5,2)} & \hat{D}^{(1,2,\frac{1}{2})}_{(10,1,1)}
\end{pmatrix}.
$$

$$
\begin{pmatrix}
\hat{D}^{(3,1,\frac{-1}{2})}_{(5,2)} \\
\hat{D}^{(3,1,\frac{-1}{2})}_{(45,2)}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\
\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
\hat{D}^{(3,1,\frac{-1}{2})}_{(6,1,3)} & \hat{D}^{(3,1,\frac{-1}{2})}_{(3,1,\frac{-1}{2})} \\
\hat{D}^{(3,1,\frac{-1}{2})}_{(5,2)} & \hat{D}^{(3,1,\frac{-1}{2})}_{(10,1,1)}
\end{pmatrix},
$$
constrictions. So we need to redefine the couplings to absorb the extra factor \( 3! \) coming from the exchange symmetry of the three \( \Phi \)s, and \( 2! \) coming from the index.

Furthermore, as is pointed out in [31], not all of dimensionless couplings in the superpotential (27) can be taken as order 1 in numerical calculations. This can be seen in the the conventional SO(10) basis of \( 1', 2', \cdots, 9', 0' \) where the fields are defined as

\[
\hat{H} \equiv a' \quad \hat{A} \equiv \frac{1}{\sqrt{2}} (a'b') \quad \hat{E} \equiv \frac{1}{\sqrt{2}} \{a'b' \} \quad \hat{D} \equiv \frac{1}{\sqrt{3}} (a'b'c')
\]

and in [12] the SO(10) invariants are defined as, e.g.,

\[
\lambda_1 \Phi^3 \equiv \lambda_1 (\Phi_{a'bc'd'} \hat{\Phi}_{a'bc'd'} \cdot (\Phi_{a'b'c'} \hat{\Phi}_{a'b'c'}) \cdot (\hat{\Phi}_{a'b'c'} \cdot \hat{\Phi}_{a'b'c'})) \\
\equiv \lambda_1 (\Phi_{a'bc'd'} \Phi_{a'b'c'} \Phi_{a'b'c'} \cdot \hat{\Phi}_{a'bc'd'} \cdot (\hat{\Phi}_{a'bc'd'} \cdot \hat{\Phi}_{a'bc'd'})) \\
\equiv \lambda_1 (\Phi_{a'bc'd'} \Phi_{a'b'c'} \Phi_{a'b'c'} \Phi_{a'b'c'}) \times 3! \times (\frac{1}{\sqrt{24}})^3 \times \frac{1}{\sqrt{6}^3} \times (a'b'c') \cdot (a'b'c') \cdot (c'd'c') \\
\equiv \lambda_1 (\Phi_{a'bc'd'} \Phi_{a'b'c'} \Phi_{a'b'c'} \Phi_{a'b'c'}) \times 3! \times (\frac{1}{\sqrt{24}})^3 \times (2!)^3 \\
= \frac{1}{\sqrt{6}} \lambda_1 \Phi_{a'bc'd'} \Phi_{a'b'c'} \Phi_{a'b'c'} \Phi_{a'b'c'} \equiv \lambda_1 \Phi_{a'bc'd'} \Phi_{a'b'c'} \Phi_{a'b'c'} \Phi_{a'b'c'} \Phi_{a'b'c'}
\]

where the \( 3! \) comes from the exchange symmetry of the three \( \Phi \)s, and \( 2!^3 \) comes from index. So we need to redefine the couplings to absorb the extra factor \( \frac{1}{\sqrt{6}} \). We give the redefined couplings in Table 6. When these new couplings are taken as order 1, and all the masses and mass matrices agree with those given in [12], except in [12] the coefficients of \( 24 \) and \( 42 \) elements in the \([(1, 1, 1) + c.c] \) mass matrix should be \( -\frac{1}{2\sqrt{10}} \) instead of \( -\frac{1}{\sqrt{10}} \).

In summary, we have studied the renormalizable SUSY SO(10) model which contains many important Higgs fields used previously. We use \( G_{51} \) as the maximal subgroup and our results are highly consistent with those given in \( G_{422} \) basis. The results of this work can be used for reference in model building.

We thank Z.-X. Ren for many discussions. ZYC also thanks Feng-Shu Jin for many helps.

7 Summary

In summary, we have studied the renormalizable SUSY SO(10) model which contains many important Higgs fields used previously. We use \( G_{51} \) as the maximal subgroup and our results are highly consistent with those given in \( G_{422} \) basis. The results of this work can be used for reference in model building.

We thank Z.-X. Ren for many discussions. ZYC also thanks Feng-Shu Jin for many helps.
Table 6: Normalized couplings $\lambda$'s which are order one.

| Old       | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ | $\lambda_8$ | $\lambda_9$ | $\lambda_{10}$ | $\lambda_{11}$ |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|----------------|------------------|
| New       | $\sqrt{6} \lambda_1'$ | $10 \sqrt{6} \lambda_2'$ | $\sqrt{5} \lambda_3'$ | $\sqrt{5} \lambda_4'$ | $\frac{\sqrt{6}}{2} \lambda_5'$ | $5 \sqrt{2} \lambda_6'$ | $\frac{\sqrt{2}}{5} \lambda_7'$ | $\frac{\sqrt{3}}{7} \lambda_8'$ | $\sqrt{2} \lambda_9'$ | $2 \sqrt{2} \lambda_{10}'$ | $\frac{\sqrt{5}}{2} \lambda_{11}'$ |

| Old       | $\lambda_{12}$ | $\lambda_{13}$ | $\lambda_{14}$ | $\lambda_{15}$ | $\lambda_{16}$ | $\lambda_{17}$ | $\lambda_{18}$ | $\lambda_{19}$ | $\lambda_{20}$ | $\lambda_{21}$ |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| New       | $\frac{2 \sqrt{2}}{5} \lambda_{12}'$ | $\sqrt{2} \lambda_{13}'$ | $\frac{2 \sqrt{2}}{5} \lambda_{14}'$ | $\frac{3 \sqrt{2}}{5} \lambda_{15}'$ | $\sqrt{3} \lambda_{16}'$ | $2 \lambda_{17}'$ | $\sqrt{10} \lambda_{18}'$ | $\sqrt{10} \lambda_{19}'$ | $2 \sqrt{30} \lambda_{20}'$ | $2 \sqrt{30} \lambda_{21}'$ |

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