Functional Renormalization Description of the Roughening Transition

Anusha Hazareesing\textsuperscript{1,2} and Jean-Philippe Bouchaud\textsuperscript{1}

\textsuperscript{1} Service de Physique de l’État Condensé, Centre d’Études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France
\textsuperscript{2} Laboratoire de Physique Théorique de l’Ecole Normale Supérieure *, 24 rue Lhomond, 75231 Paris Cedex 05, France

Abstract

We reconsider the problem of the static thermal roughening of an elastic manifold at the critical dimension $d = 2$ in a periodic potential, using a perturbative Functional Renormalization Group approach. Our aim is to describe the effective potential seen by the manifold below the roughening temperature on large length scales. We obtain analytically a flow equation for the potential and surface tension of the manifold, valid at all temperatures. On a length scale $L$, the renormalized potential is made up of a succession of quasi parabolic wells, matching onto one another in a singular region of width $\sim L^{-6/5}$ for large $L$. We also obtain numerically the step energy as a function of temperature, and relate our results to the existing experimental data on $^4$He. Finally, we sketch the scenario expected for an arbitrary dimension $d < 2$ and examine the case of a non local elasticity which is realized physically for the contact line.

LPTENS preprint 99/XX

Electronic addresses: anusha@spec.saclay.cea.fr; bouchaud@spec.saclay.cea.fr

1 Introduction

The roughening transition has been studied in great detail, both theoretically and experimentally \cite{1,2}. Direct analogies with the (two dimensional) $XY$-model or the Coulomb gas furthermore make this problem particularly enticing \cite{3}. More recently, the role of disorder on the roughening transition or on the properties of the $XY$ model, has attracted considerable interest \cite{4,5,6,7,8}. In particular, replica calculations and Functional Renormalization
Group (FRG) methods have been applied to this problem, with sometimes conflicting results \[9\]. In this paper, we wish to reconsider the problem of the roughening transition in the absence of disorder, from a FRG point of view, where the flow is not \textit{a priori} projected onto the first harmonic of the periodic potential. Within a local renormalization scheme, we establish exact equations for the evolution of the full periodic potential \( V(\phi) \), and the surface tension \( \gamma \) with the length scale \( L = e^\ell \), which we analyze both numerically and analytically, in the low temperature phase. If we start with a sinusoidal periodic potential, the shape of the fixed point potential \( V^*(\phi) \) evolves to a nearly parabolic shape with matching points becoming more and more singular as the length scale increases. The nature of the singularity is investigated in detail close to the fixed point, that is for small values of the rescaled temperature \( \overline{T} = \frac{T}{2\pi \gamma \lambda^2} \) where \( \lambda \) is the periodicity of the potential and \( \gamma \) the elastic stiffness. We find that the width \( \Delta \phi \) of the singular region scales as \( L^{-3g(T)/5} \), where \( g(T) \) governs the scaling of the surface stiffness with the length scale according to \( \gamma(L) \sim L^{g(T)} \). The exponent \( g(T) \) tends towards 2 with negative corrections which we calculate, when \( \overline{T} \) goes to 0 (i.e. for \( L \to \infty \)).

The paper is organized as follows. In section 2, we introduce the model: we outline the calculations involved and discuss the differences with the approach of Nozières and Gallet (NG), and briefly examine the problem for \( d < 2 \). We then explain in section 3, by a mean field argument the origin of the singularity that develops during the renormalization flow. In section 4, we present a scaling form for the renormalized potential, around its maxima and close to the fixed point, which accounts for the nature of the singularity. Using our renormalization group flow, we compute in section 5 the step energy as a function of temperature. Finally, in section 6, we look at the case of a contact line in a periodic potential, as this is a physical realization of a non local elastic stiffness.

## 2 Model and functional renormalization group

We consider an elastic interface whose height fluctuations are described by a profile \( \Phi(x) \), where \( x \) is a \( d \)-dimensional vector, in the presence of a deterministic periodic potential \( V \). Supposing that the slope of the interface is everywhere small, the energy of the system is:

\[
H[\Phi] = \frac{\gamma}{2} \int d^d x \ (\nabla \Phi(x))^2 + \int d^d x \ V\left(\frac{\Phi(x)}{\lambda}\right)
\]  

where \( \gamma \) is the elastic stiffness and \( \lambda \) the periodicity of the potential. In the absence of periodic potential, the height fluctuations of the surface on a length scale \( L \) scale as \( L^{2-d} \). For \( d > 2 \), the interface is therefore always flat. For the critical dimension \( d = 2 \), the interface is rough only if the temperature exceeds a certain critical temperature \( T_R \). When the potential \( V \) is harmonic, this model is the continuous version of the Sine-Gordon model. NG have studied the statics of this problem using a two-parameter renormalization group scheme, and have written flow equations for \( \gamma(L) \) and the amplitude \( v_o(L) \) of the periodic potential. They suppose that during the flow, \( v_o \) remains small compared with the temperature and
neglect all higher harmonics of the potential. Correspondingly, within this procedure, the renormalization scheme ceases to be valid when \( v_o \) becomes of the order of the temperature. In the low temperature ‘flat’ phase, this occurs after a finite renormalization since \( v_o \) grows with distance.

In our calculation, we consider a general periodic function with the only constraint that it should be sufficiently smooth (we shall explain this more quantitatively in the following). Since we re-sum the whole perturbation expansion in \( v_o/T \), there is however no constraint on the amplitude of the potential, and the renormalization procedure can be carried on any length scale without interruption. The relevant coupling constant appears to be \( v_o/\gamma \) rather than \( v_o/T \). During the renormalization flow, we keep track of the whole function \( V(\varphi) \) instead of projecting onto the first harmonic, so that we have a more quantitative knowledge of the behaviour of the potential for low temperatures.

Technically, we proceed by considering the partition function:

\[
Z = \int d\Phi(x) \ e^{-\beta H[\Phi(x)]}
\]  

(2)

We perform the renormalization procedure by splitting the field \( \Phi \) into a slowly-varying and a rapidly-varying part as:

\[
\Phi(x) = \Phi^<(x) + \Phi^>(x)
\]  

(3)

The Fourier modes \( k \) of \( \Phi^< \) are such that \( 0 \leq |k| \leq |\Lambda|/s \), and those of \( \Phi^> \), such that \( |\Lambda|/s \leq |k| \leq |\Lambda| \), where \( s = e^{d\ell} \), \( |\Lambda| \) being a high momentum cut-off, of the order of \( 1/a \), where \( a \) is the lattice spacing. We integrate over the fast modes in the partition function and retain only the terms that renormalize the gradient term and the potential term. The other terms that are generated are discarded as irrelevant. Within this renormalization scheme, our calculation is exact. After some algebra detailed in appendix A, we obtain a set of flow equations for \( d = 2 \), for \( \overline{V} = \frac{V}{\gamma \lambda^2|\Lambda|^2} \) and \( \overline{T} = \frac{T}{2\pi \gamma \lambda^2} \), the rescaled potential and temperature (note that \( \overline{V} \) and \( \overline{T} \) are dimensionless):

\[
\frac{d\overline{V}}{d\ell} = (2 - g)\overline{V} - \pi \frac{\overline{V}^2}{(1 + \overline{V'})} + \frac{T}{2} \ln(1 + \overline{V''})
\]

\[
\frac{d\gamma}{d\ell} = g\gamma
\]

\[
\frac{dT}{d\ell} = -g\overline{T}
\]

(4)

where \( g \) is given by:

\[
g = 4\pi \int_0^1 d\varphi \frac{\overline{V}^2(\varphi)\overline{V''}(\varphi)}{(1 + \overline{V'}(\varphi))^5} + \frac{T}{4} \int_0^1 d\varphi \frac{\overline{V''}(\varphi)}{(1 + \overline{V'}(\varphi))^4}
\]

(5)
These equations call for some comments.

- The relevant perturbative parameter appears to be \( V \), rather than \( V/T \). In the limit \( V \ll 1 \), and in the case where the potential is purely harmonic (i.e. \( V(\varphi) = v_o \cos(2\pi \varphi) \)), the RG equations read:

\[
\frac{d u_o}{d\ell} = \left(2 - \frac{\pi T}{\gamma \lambda^2}\right) u_o
\]

\[
\frac{d\gamma}{d\ell} = 2\pi^4 \left(\frac{2\pi T}{\gamma \lambda^2}\right) \frac{v_o^2}{\gamma^2 \lambda^4}
\]

where \( u_o = v_o/|\Lambda|^2 \). The first equation is trivial and identical to the one in NG, and immediately leads to the value of the roughening transition temperature: \( T_R = 2\gamma_{\infty} \lambda^2 / \pi \), where \( \gamma_{\infty} \) is the renormalized value of \( \gamma \). The second is close to, but different from the one obtained in the particular renormalization scheme used by NG: near the critical temperature \( T_R \), the coefficient between parenthesis is equal to 4 in our case and to 0.4 according to NG.

- The renormalization of the surface tension, as measured by \( g \), is always positive. One can check that, as has been pointed out by NG, if the initial potential is parabolic (i.e. \( V(\phi) = v_0 \phi^2 \)), then the coefficient \( g \) vanishes identically, and there is no renormalization of the surface tension. This is indeed expected since in this (quadratic) case, all modes are decoupled.

- The flow equations only make sense if \( V'' > -1 \). We have checked numerically that if this condition is satisfied at the beginning, it prevails throughout the flow. On the other hand, if the initial potential is so steep that this condition is violated, the perturbative calculation is meaningless. This comes from the fact that metastable states, where the surface zig-zags between nearby minima of the potential, appear at the smallest length scales. In this respect, it is useful to note that the last term of the flow equation on \( V \) comes from the integration of the Gaussian fluctuations of the fast field around the slow field. The condition \( V'' > -1 \) is a stability condition for these fast modes. If the unrenormalized potential is harmonic (i.e. \( V(\varphi) = v_o \cos(2\pi \varphi) \)), and the unrenormalized surface tension given by \( \gamma_o \), then this condition reads

\[
\frac{v_o}{\gamma} \left(\frac{2\pi}{\lambda|\Lambda|}\right)^2 < 1,
\]

which simplifies to \( \frac{v_o}{\gamma} < 1 \) in the case where \( \lambda = a \). If the initial value of the potential is too large, one actually expects the transition to become first order (but see [1]). Actually, a variational calculation indeed predicts the transition to become first order when \( \frac{v_o}{\gamma} \geq 1 \) [11].

- The fundamentally new term in the above equation is the second term, proportional to \( V''^2 \), and independent of temperature. This term leads to the appearance of singularities
in the flow equation: up to second order in $V$, this equation is close to the Burgers’ equation (see below) for which it is well known that shocks develop in time. The fact that this term survives even in the zero temperature limit is at first sight strange, since one could argue that for $T = 0^+$, there are no longer any thermal fluctuations, and thus no renormalization. This argument is not correct because we are computing a partition function, thereby implicitly assuming that the infinite time limit is taken before the zero temperature limit. Such a non trivial renormalization has also been found in the context of pinned manifolds [11, 12], and can be understood very simply using a mean-field approximation, which we detail in the next section.

For completeness let us consider the case $1 < d < 2$. By simple scaling arguments, we can see that for $d < 1$ the interface is always rough. For $1 < d < 2$, we have a roughening transition between a flat phase and a rough phase. In this case, we allow $\Phi$ to renormalize and suppose that $\lambda$ renormalizes in the same way according to:

$$\frac{d\lambda}{d\ell} = \zeta \lambda$$

(7)

The other flow equations in terms of the rescaled parameters $\overline{V} = \frac{V}{\gamma \lambda^2 |\Lambda|^2}$ and $\overline{T} = \frac{K_d |\Lambda|^d T}{\gamma \lambda^2}$ (with $K_d = S_d/(2\pi)^d$ where $S_d$ is the $d$-dimensional sphere) now read:

$$\frac{d\overline{V}}{d\ell} = (d - g_d - 2\zeta)\overline{V} - \pi \frac{\overline{V}^2}{(1 + \overline{V}^2)} + \frac{\overline{T}}{2} \ln(1 + \overline{V}^2)$$

$$\frac{d\gamma}{d\ell} = (g_d + d - 2)\gamma$$

(8)

$$\frac{d\overline{T}}{d\ell} = -(g_d + d - 2 + 2\zeta)\overline{T}$$

where $g_d$ is given by:

$$g_d = \frac{2}{d} g.$$  

(9)

with $g$ given by equation (3) above. These equations have a non trivial fixed point for $g_d = 2 - d$ and $\zeta = 0$. This corresponds to a rescaled temperature $\overline{T}_R$ and a renormalized rescaled potential $\overline{V}$ such that equations (8) and (9) are satisfied. For $1 < d < 2$, we obtain $\overline{T}_R$ numerically by proceeding as follows: we self-consistently solve the differential equation on $\overline{V}$ obtained by putting $\frac{d\overline{V}}{d\ell} = 0$ for different fixed rescaled temperatures $\overline{T}$, imposing that $g$ is given by equation (3). This enables us to plot $g$ as a function of $\overline{T}$. The rescaled temperature $\overline{T}_R$ corresponding to the transition
Figure 1: $g$ as a function of the rescaled temperature $\overline{T}$ for $d=3/2$. The point where $g = 2 - d = 1/2$ determines the transition temperature.

The temperature is such that $g_d = 2 - d$. In the figure (1), we have plotted the result for $d = 3/2$, with $\gamma_o = 1$. In that case, $\overline{T} \simeq 1.2$.

### 3 Mean field analysis and effective potential

In this section, we show on a simplified mean field version of the model how the non linear term $V''$ arises in the flow equation of the potential. Using a discrete formulation of the problem and replacing the local elasticity modeled by the surface tension term by a coupling to all neighbours, we can rewrite the energy as:

$$H_{mf}(\{\Phi\}_i, \overline{\Phi}) = \frac{\gamma}{2a} \sum_i (\Phi_i - \overline{\Phi})^2 + a \sum_i V(\Phi_i)$$  \hspace{1cm} (10)

where $\overline{\Phi}$ is the center of mass of the system, $a$ the lattice spacing and $L = Na$. Implementing the constraint $\overline{\Phi} = 1/N \sum \Phi_i$ by means of a Lagrange multiplier in the partition function, we have:
\[ Z[\Phi] = \int d\eta \int \prod_i d\Phi_i \ e^{-\beta H_m^i(\{\Phi\}_i; \Phi) - \eta(N\Phi - \sum_i \Phi_i)} \tag{11} \]

which can also be expressed as:

\[ Z[\Phi] = \int d\eta \ e^{N \log z(\Phi, \eta) + N \frac{\eta^2 a}{2\beta\gamma}} \tag{12} \]

where

\[ z(\Phi, \eta) = \int d\Phi \ e^{-\frac{\beta\gamma}{2a} \left( \Phi - \Phi + \frac{\eta a}{\beta\gamma} \right)^2 - \beta a V(\Phi)} = z\left( \Phi + \frac{\eta a}{\beta\gamma} \right) \tag{13} \]

We are left with a simpler problem since we now have a one-body problem. We introduce the auxiliary partition function \( z_R(\Psi, \tau) \) defined as:

\[ z_R(\Psi, \tau) = \sqrt{\frac{\beta\gamma}{2\pi a\tau}} \int d\Phi \ e^{-\frac{\beta\gamma}{2a} (\Psi - \Phi)^2 - \tau - \beta a V(\Phi)} \tag{14} \]

Up to a multiplicative constant, one has \( z\left( \Phi + \frac{\eta a}{\beta\gamma} \right) = z_R\left( \Phi + \frac{\eta a}{\beta\gamma}, \tau = 1 \right) \), where \( z_R(\Psi, \tau) \) verifies the diffusion equation:

\[ \frac{\partial z_R}{\partial \tau} = \frac{a}{\beta\gamma} \frac{\partial^2 z_R}{\partial \Psi^2} \tag{15} \]

with an initial condition given by:

\[ z_R(\Psi, \tau = 0) = e^{-\beta a V(\Psi)} \tag{16} \]

Defining now the effective pinning potential \( V_R \) as:

\[ aV_R\left( \Phi + \frac{\eta a}{\beta\gamma}, \tau \right) = -T \log z_R\left( \Phi + \frac{\eta a}{\beta\gamma}, \tau \right) \tag{17} \]

we can then easily show that \( V_R \) is the Hopf-Cole solution of the non linear Burgers’ equation \[12\]:

\[ \frac{\partial V_R}{\partial (\tau a)} = \frac{T}{\gamma} \frac{\partial^2 V_R}{\partial \Psi^2} - \frac{a}{\gamma} \left( \frac{\partial V_R}{\partial \Psi} \right)^2 \tag{18} \]

where the temperature independent non linear term \( V_R^0 \) indeed appears.

It is easy to show that when \( N \to \infty \) or \( T \to 0 \), the original partition function can be solved by a saddle point method, leading after a change of variables to an effective potential per unit length:

\[ \frac{\partial^2 V_R}{\partial \Psi^2} = \frac{T}{\gamma} \left( \frac{\partial V_R}{\partial \Psi} \right)^2 \]
\[ V_{\text{eff}}(\Phi) = V_R(u, \tau = 1) - \frac{\gamma}{2a^2}(\Phi - u)^2 \]  

(19)

where \( u \) is given by:

\[ \frac{\gamma}{2a}(u - \Phi) = \frac{\partial V_R}{\partial \phi}(\phi, \tau = 1) \bigg|_{\phi = u} \]  

(20)

Now, by changing \( V_R \) to \(-V_R\), one can see that \(-V_{\text{eff}}\) can also be written as the solution of a Burgers’ equation. It is known from results on the Burgers’ equation, that, with ‘time’ \( \tau \), the effective potential \( V_R \) develops shocks, smoothed out at finite temperature, between which it has a parabolic shape. The appearance of singularities is due to the non linear term in the partial differential equation, which indeed survives in the limit \( T = 0 \). It is interesting to see how this ‘toy’ renormalization group captures some important features of the full scheme, such as the one shown above for a non disordered potential.

### 4 Analysis for small \( \overline{T} \)

In this section, we go back to the model introduced in section 2 and analyze the nature of \( \overline{V} \) close to the low temperature fixed point, that is for small values of the rescaled temperature \( \overline{T} \). Since \( g > 0 \) in the low temperature phase, this corresponds to the large scale structure of the renormalized potential for all temperatures \( T < T_R \).

Expanding \( \overline{V} \) around one of its minima as \( \overline{V}(\phi) = \overline{V}_m + \frac{1}{2}\kappa(\phi - \phi^*)^2 \), and replacing \( \overline{V} \) in the flow equation(4), we have

\[ \frac{d\overline{V}_m}{d\ell} = (2 - g)\overline{V}_m + \frac{T}{2}\ln(1 + \kappa) \]

\[ \frac{d\kappa}{d\ell} = (2 - g)\kappa - 2\pi \frac{\kappa^2}{1 + \kappa} \]  

(21)

One can actually check that a parabolic shape for \( \overline{V} \) is exactly preserved by the renormalization flow. However, since the potential has to be periodic, these parabolas should match periodically around each maximum that is for \( \phi - \phi^* = 0, 1, 2, ... \). The region of the maximum is therefore expected to be singular. To investigate the nature of the renormalized periodic potential around its maximum value, we will thus make a scaling ansatz on \( \overline{V}'' \) for small \( \overline{T} \).

For our perturbative calculation to be valid, we expect \( \overline{V}''(0) \) to be \( > -1 \). Now since we expect a singularity to develop as \( \overline{T} \) goes to 0, it is probable (and actually self-consistently checked) that \( \overline{V}''(0) \) should tend towards \(-1\). As \( \overline{T} \) goes to 0, we thus make the scaling ansatz:

\[ 1 + \overline{V}''(\phi) = \overline{T}^\delta \mathcal{F}' \left( \frac{\phi}{T^\gamma} \right) \]  

(22)

where \( \mathcal{F}'(0) > 0 \). This means that the width of the singular region behaves as \( \Delta \phi \sim \overline{T}^\gamma \). Hence, in the scaling region:
\[ V(\varphi) = -\varphi + T^{3+\alpha} F\left(\frac{\varphi}{T}\right) \]  

(23)

with \( F(0) = 0 \) to ensure that \( \varphi = 0 \) is a maximum of \( V \). Integrating once more the above equation, one finds:

\[ V(\varphi) = V_M - \frac{\varphi^2}{2} + T^{\delta+2\alpha} G\left(\frac{\varphi}{T}\right) \]  

(24)

with \( G' = F \). Replacing this last equation in the flow equation for \( V \), we obtain:

\[ \frac{dV_M}{d\ell} = (2 - g)V_M + \frac{T}{2} \log T \]  

(25)

Suppose that equations (21) have a fixed point as \( T \) goes to zero, and that close to the fixed point one can neglect the left hand side of these equations. This leads to the relation \( \kappa = (2 - g)/(2\pi - 2 + g) \). Supposing moreover that the parabolic solution extends almost over a whole period and that the correction brought about by the rounding off of the singularity around the maxima of the potential is negligible, we also have

\[ (V_M - V_m) \simeq \frac{\kappa}{2} = \frac{2 - g}{4\pi - 4 + 2g} \]  

(26)

Now, subtracting equation (25) from equation (21), we find in the limit \( T \to 0 \):

\[ \frac{d}{d\ell}(V_M - V_m) = (2 - g)(V_M - V_m) + \frac{\delta}{2} T \log T \]  

(27)

Combining equations (26) and (27), we find that for the previous equation to have a fixed point as \( T \to 0 \), \( g \to 2 \) with negative corrections as:

\[ (2 - g) \simeq \sqrt{2\pi \delta T \log \frac{1}{T}} \]  

(28)

This result is independent of the way we calculate \( g \), the correction to the surface tension. In particular, it shows that at zero temperature, the surface tension diverges as \( (L/a)^2 \), where \( L \) is the size of the system.

We can deduce an equation satisfied by \( F' \), by plugging the ansatz for the derivatives of \( V \) in the flow equation for \( V' \):

\[ \frac{dV'}{d\ell} = (2 - g - 2\pi)V' + 2\pi \frac{V'}{1 + V'} + \pi V'^2 \frac{V''}{(1 + V')^2} + \frac{T}{2} \frac{V''}{1 + V'} \]  

(29)

Close to the fixed point, we again suppose that to leading order in \( T \), \( \frac{dV}{d\ell} = 0 \) in the above equation. Plugging in the ansatz for \( V' \) and \( V'' \), and considering the leading term in \( T \), we get to lowest order in \( T \):
\[-\frac{2\pi u}{\mathcal{F}'(u)} T^{\alpha-\delta} + \frac{\pi u^2 \mathcal{F}''(u)}{\mathcal{F}'(u)} T^{\alpha-\delta} + \frac{\mathcal{F}''(u)}{2\mathcal{F}'(u)} T^{1-\alpha} = 0 \quad (30)\]

We can show that necessarily \( \alpha - \delta = 1 - \alpha \). Indeed, if \( 1 - \alpha < \alpha - \delta \), \( \mathcal{F}'' \) would be equal to zero while if \( 1 - \alpha > \alpha - \delta \), \( \mathcal{F}'(0) \) would be equal to zero, both alternatives being thus impossible. Hence, equation (30) can be rewritten as

\[
\frac{1}{2} \frac{d}{du} \log \mathcal{F}' - \frac{\pi}{2} \frac{du^2}{\mathcal{F}'} = 0 \quad (31)
\]

which yields after integration:

\[
\mathcal{F}'(u) \log \left( \frac{\mathcal{F}'(u)}{\mathcal{F}'(0)} \right) = 2\pi u^2 \quad (32)
\]

At this stage, we can note that the exponent relation

\[
2\alpha - \delta = 1 \quad (33)
\]

is independent of the scheme used to calculate of \( g \) (see Appendix).

In the rest of this section, we calculate the exponents \( \alpha \) and \( \gamma \), and using \( g(\bar{T} \to 0) = 2 \), we also obtain \( \mathcal{F}'(0) \). These results now somewhat depend on the precise renormalization scheme we use to calculate the correction \( g \) to the surface tension. Replacing the derivatives of \( \mathcal{V} \) by their expressions in terms of \( \mathcal{F}' \) and \( \mathcal{F}'' \), in equation (30), and changing variables from \( \varphi \) to \( u = \frac{\varphi}{\bar{T}^\alpha} \), we get to lowest order in \( \bar{T} \):

\[
g(\bar{T}) \simeq \bar{T}^{\alpha-3\delta} 8\pi \int_0^\infty du \frac{u^2 \mathcal{F}''^2(u)}{\mathcal{F}'^4(u)} + \bar{T}^{1-\alpha-2\delta} \frac{1}{2} \int_0^\infty du \frac{\mathcal{F}''^2(u)}{\mathcal{F}'^4(u)} \quad (34)
\]

From the exponent relation \( 2\alpha - \delta = 1 \) derived previously, and the fact that \( g(\bar{T} \to 0) \) is finite, we have another exponent relation \( \alpha = 3\delta \), so that \( \alpha = 3/5 \) and \( \delta = 1/5 \). We can also show that \( g(\bar{T} \to 0) \) can be expressed in terms of \( \mathcal{F}'(0) \). From expression (32), one can see that \( \mathcal{F}' \) is a strictly increasing function on \([0, \infty]\) taking its values in \([e, \infty]\), and so we can change variables from \( \mathcal{F}' \) to its inverse function. Defining a new variable \( x \) as

\[
x = \frac{e\mathcal{F}'(u)}{\mathcal{F}'(0)} \quad (35)
\]

we have:

\[
u = \left( \frac{\mathcal{F}'(0)}{2\pi e} \right)^{1/2} (x \log(x))^{1/2} \quad (36)
\]

and

\[
\frac{du}{dx} = \frac{1}{2} (x \log(x))^{1/2} \left( \frac{2\pi e}{\mathcal{F}'(0)} \right)^{1/2} \quad (37)
\]

We can now express \( g(\bar{T} \to 0) \) in terms of an integral over \( x \) from \( e \) to \( \infty \) as
\begin{align*}
g(0) &= \frac{2}{\pi^2} \left( \frac{2\pi e}{F'(0)} \right)^{5/2} \left\{ \int_e^\infty dx \frac{(x \log(x/e))^{3/2}}{x^5 \log(x)} + \int_e^\infty dx \frac{(x \log(x/e))^{1/2}}{x^4 \log(x)} \right\} \tag{38}
\end{align*}

Hence, using the fact that \( g(T \to 0) = 2 \), we finally find the constant \( F'(0) \approx 1.38 \).

## 5 Step energy as a function of temperature

From physical considerations we know that below the roughening temperature, the interface grows by forming terraces. An important quantity governing the kinetics of growth is therefore the step energy. The width \( \xi \) of a step and its energy per unit length \( \beta_S \) can be obtained by comparing the elastic energy and the potential energy of a profile \( \Phi(x) \) which changes by one period over the length \( \xi \). Requiring that these two energies are of the same order of magnitude leads to: \( \gamma/\xi^2 \sim v_o \) where \( v_o \) is the amplitude of the periodic potential, or \( \xi \sim \sqrt{\gamma/v_o} \), and a step energy which scales as \( \beta_S \propto \sqrt{v_o \gamma} \). Since a step profile include Fourier modes such that \( \xi^{-1} < k < |\Lambda| \), it is natural to use in the above equations the values of \( \gamma \) and \( v_o \) calculated for the length \( L = ae^k = \xi \). Since \( \xi(L) \sim L/\sqrt{\gamma_o(L)} \), one sees that this corresponds to stopping the renormalization procedure when \( \gamma_o(L) \sim 1 \). We have integrated numerically the RG flow, starting from \( \gamma = 1 \) and from harmonic potentials of various amplitudes \( v_o \ll 1 \), and stopping for an arbitrary value \( \gamma_o \), chosen here to be \( \gamma_o = 0.4 \). The resulting step energy as a function of temperature is plotted in Figure (2). For \( T \) close to \( T_R \), one finds that \( \xi \) diverges as \( e^{1/\sqrt{T_R - T}} \), as it should since our RG flow essentially boils down to the standard one [1]. For small temperatures, however, we find that \( \beta_S \) tends to a finite value with a linear slope in temperature. This slope is seen to decrease as the initial amplitude of the potential \( \gamma_o \) increases. For \( \gamma_o = 0.01 \), \( \beta_S \) decreases by \( \sim 30\% \) when \( T \) increases from 0 to 0.25 \( T_R \). This decrease falls to \( \sim 10\% \) for \( \gamma_o = 0.1 \).

Experiments on Helium 4, on the other hand, have established that the step energy does only depend very weakly on temperature at small temperature, by not more than 5\% when the temperatures varies from 0.05 \( T_R \) to 0.25 \( T_R \). This suggests that the initial amplitude of the potential is of the same order as \( \gamma_o \): in this case, the width of the step is of order \( a \), and the bare parameters are not renormalized except possibly very close to \( T_R \). The conclusion that experiments must be in the regime \( \gamma_o \sim 1 \) is in agreement with [2], where \( \gamma_o \) is called \( t_c \) (up to a numerical prefactor); \( v_o/\gamma_o \) was estimated to be \( \sim 0.05 \). Since our RG flow is different from the one obtained by Nozières and Gallet, the values of the physical parameters obtained by a fit of our theory to the experiments will actually differ.

## 6 Case of the contact line

In this section, we repeat the previous analysis for the case of a contact line on a periodic substrate [13, 14]. The roughness of a contact line on a disordered substrate, at zero

\footnote{Other values of \( \gamma_o \) would not change the qualitative features reported below, provided \( \gamma_o \) is not too large.}
Figure 2: $\beta_S / \sqrt{v_o}$ as a function of the rescaled temperature for three different values of the bare periodic potential $\overline{v}_o$, with $\gamma_o = 1$. 
temperature, has been studied analytically and compared with the experimental predictions for the case of superfluid helium on a disordered cesium substrate, where the disorder arises from randomly distributed wettable heterogeneities which are oxydized areas of the substrate [15, 16]. A physical realization of the theoretical situation we consider here could be achieved by preparing a substrate with equally spaced oxydized lines which would act as periodic pinning grooves. In this case the critical dimension is $d = 1$. We denote by $\Phi$ the position of the line with respect to a mean position. The energy of the system is the sum of an elastic term and a potential term given by:

$$H[\Phi] = \frac{\gamma}{2} \int \frac{dk}{2\pi} |k||\Phi(k)|^2 + \int_0^L dx \ V(\Phi(x))$$  

(39)

where $L$ is the length of the substrate and $\gamma$ the stiffness.

The renormalization procedure is carried as before except that the propagator is now given by $G(k) = \frac{1}{\beta \gamma |k|}$. Moreover the renormalization of the stiffness now only comes from the scale change leading to the much simpler flow equation for $\gamma$:

$$\frac{d\gamma}{d\ell} = \gamma$$  

(40)

Defining as before the rescaled parameters $\overline{V}$ and $\overline{T}$ with $\overline{V} = \frac{V}{\gamma \lambda^2 |\Lambda|}$ and $\overline{T} = \frac{2T}{\gamma \lambda^2}$, the flow equations for $\overline{V}$ and $\overline{T}$ read:

$$\frac{d\overline{V}}{d\ell} = \overline{V} - \frac{\overline{V}^2}{1 + \overline{V}'} + \frac{T}{2} \log(1 + \overline{V}')$$  

(41)

and

$$\frac{d\overline{T}}{d\ell} = -\overline{T}$$  

(42)

During the flow, $\overline{T}$ flows to zero and the renormalized rescaled potential $\overline{V}$ develops shocks between which it has a parabolic shape. We characterize the singularities that develop around the maxima of $\overline{V}$ by the following scaling ansatz:

$$1 + \overline{V}'(\varphi) = T^\delta e^{-\frac{\varphi}{T^\alpha e^{-\frac{\varphi}{T}}}} F'\left(\frac{\varphi}{T^{\alpha} e^{-\frac{\varphi}{T}}}ight)$$  

(43)

where $F'(0) > 0$ and $F(0) = 0$. Putting $u = \frac{\varphi}{T^{\alpha} e^{-\frac{\varphi}{T}}}$, this implies that for $u \sim 1$,

$$\overline{V}(u) = -T^\alpha e^{-\frac{B}{T}} u + T^{\delta + \alpha} e^{-\frac{A + B}{T}} F(u)$$  

(44)

and

$$\overline{V}(u) = \overline{V}_M - T^{2\alpha} e^{-\frac{2B}{T}} \frac{u^2}{2} + T^{\delta + 2\alpha} e^{-\frac{A + 2B}{T}} G(u)$$  

(45)
with $G' = F$. Plugging the previous expressions into the flow equation $V'$:

$$
\frac{dV'}{d\ell} = -V' + 2V' \frac{V''}{1 + V'} + V'^2 \frac{V'''}{(1 + V')^2} + \frac{T}{2} \frac{V'''}{1 + V'}
$$

(46)

and supposing that $\frac{dV}{d\ell} = 0$, to leading order as $T$ goes to zero, we have to leading order in $\overline{T}$:

$$
-\frac{2u}{F'(u)} T^{3-\delta} e^{\frac{A-B}{T}} + \frac{u^2 F''(u)}{F'^2(u)} T^{3-\delta} e^{\frac{A-B}{T}} + \frac{F''(u)}{2F(u)} T^{1-\alpha} e^{\frac{B}{T}} = 0
$$

(47)

Since $F'(0) > 0$, we obtain a non trivial solution only if

$$
2\alpha - \delta = 1 \quad \text{and} \quad A - B = B
$$

(48)

and $F'$ is again solution of equation (32). We can obtain the values of the parameters $A$ and $B$ by considering separately the singular part and the regular part of the renormalized rescaled potential $\overline{V}$ close to the fixed point. We expand $\overline{V}$ around one of its minima $\varphi^*$ as:

$$
\overline{V}(\varphi) = \overline{V}_m + \frac{\kappa}{2} (\varphi - \varphi^*)^2
$$

(49)

and plug the resulting expression in the flow equation for $\overline{V}$. This yields:

$$
\frac{dV_m}{d\ell} = V_m + \frac{T}{2} \log(1 + \kappa)
$$

(50)

$$
\frac{d\kappa}{d\ell} = \kappa - \frac{2\kappa^2}{1 + \kappa}
$$

We note that the fixed point value for $\kappa$ is now finite as $T$ goes to zero and is given by $\kappa^* = 1$. Similarly the flow equation for $\overline{V}(0) = \overline{V}_M$ is:

$$
\frac{d\overline{V}_M}{d\ell} = \overline{V}_M + \frac{T}{2} \log\left(T^\delta e^{-\frac{A}{T} F'(0)}\right)
$$

(51)

Combining equations (50) and (51), the flow equation for the amplitude of $\overline{V}$ is given to leading order as $\overline{T} \to 0$ by:

$$
\frac{d}{d\ell} (\overline{V}_M - \overline{V}_m) = (\overline{V}_M - \overline{V}_m) - \frac{A}{2}
$$

(52)

Now, we expect that the singularity brings but a small correction to the parabolic part of the rescaled potential $\overline{V}$, so that at the fixed point the amplitude of $\overline{V}$ is given by $\kappa^*$, leading to $A = 1$. The value of the exponents $\alpha$ and $\delta$ would require the analysis of subdominant terms. The conclusion of this section is that in the case of the contact line, the width of the singular region of the renormalized potential decreases exponentially with length scale: the potential quickly becomes a succession of matched parabolas.
7 Conclusion

In this paper, we studied the problem of the thermal roughening transition using a FRG formalism. We have shown that below the roughening temperature, the periodic potential on large length scales cannot be described by its lowest harmonic and that during the flow shocks are generated in the effective pinning potential. We expect that this result is more generally valid, and also holds in the case of a disordered pinning potential [11, 12]. By performing a resummation of our perturbation expansion, our results are in principle valid in the strong coupling regime, where the coupling constant is proportional to $V/\gamma$ (rather than $V/T$). Correspondingly, we stop the renormalisation procedure not when $V(L) \sim T$ (as in NG), but rather when $L$ reaches the size of the objects under investigation (for example the width of the steps). By comparing our numerical result with the experimental determination of the step energy of liquid Helium 4, we have concluded that the surface of Helium 4 crystals are such that the coupling to the lattice is of the same order of magnitude as the surface tension. This is in qualitative agreement with Balibar et al. [2], who estimate $v_o/\gamma_o \sim 0.05$.

Acknowledgements

We wish to thank Sebastien Balibar, T. Emig and M. Mézard for very interesting discussions.
A Derivation of the flow equations

In this appendix, we sketch the procedure to obtain the flow equations \((4)\). We consider the partition function

\[
Z = \int d[\Phi] \ e^{-\beta H(\Phi(x))}
\]  \hspace{1cm} (53)

where \(H\) is the hamiltonian given by \((1)\). We split the field \(\Phi\) into a fast moving and a slow moving component and average over the fast moving part. We can rewrite the partition function, up to a multiplicative constant, as:

\[
Z = \int d[\Phi^<] e^{-\beta \int <(2\pi)^d|k|^2|\Phi^<(k)|^2} \left< e^{-\beta \int d^dx \ V \left( \frac{\Phi(x)}{\lambda} \right)} \right>_o
\]  \hspace{1cm} (54)

where \(< ... >_o\) represents the thermal average with respect to the gaussian weight:

\[
e^{-\beta \int > (2\pi)^d|k|^2|\Phi^>(k)|^2}
\]  \hspace{1cm} (55)

In the rest of this section we denote \(\int d^d k \) by \(\tilde{d}k\).

A.1 Renormalization of the periodic potential \(V\)

We look for contributions to the potential \(V\) resulting from the above averaging, which are of the same form as the terms present in the hamiltonian before starting the renormalization procedure and which are of order \(d\ell\). These terms are represented by connected graphs and are all obtained by expanding the potential term with respect to \(\Phi^>\) up to second order. There are only two ways of obtaining such graphs of order \(d\ell\):

- By contracting \(p\) two-legged terms \(-\frac{\beta}{2} \int d^d x \left( \frac{\Phi^>(x)}{\lambda} \right)^2 V'' \left( \frac{\Phi^<(x)}{\lambda} \right)\) with \(1 \leq p \leq \infty\).

We must calculate:

\[
\frac{1}{p!} \left( -\frac{\beta}{2\lambda^2} \right)^p \int \prod_{j=1}^p d^d x_j \ V'' \left( \frac{\Phi^<(x_j)}{\lambda} \right) \int \prod_{j=1}^p \tilde{d}k_j \tilde{d}k'_j \ e^{i(k_1+k'_1)x_1+...+i(k_p+k'_p)x_p} \left< \Phi^>(k_1)...\Phi^>(k'_p) \right>_o
\]  \hspace{1cm} (56)

which gives after averaging over the fast modes:

\[
\frac{(-1)^p}{2^p} \left( \frac{1}{\gamma \lambda^2} \right)^p \int \prod_{j=1}^p d^d x_j V'' \left( \frac{\Phi^<(x_j)}{\lambda} \right) \int \prod_{j=1}^p \tilde{d}k_j \ e^{i(k_1(x_1-x_2)+...+k_p(x_p-x_1))} \frac{1}{|k_1|^2...|k_p|^2}
\]  \hspace{1cm} (57)
In the above expression, the space dependence of $V''$ is slowly varying, and since the integral is dominated by the region where the $x_j$'s are close to one another, we can with little error, treat these terms as approximately equal to $V''(\Phi^< (x_1) / \lambda)$. After integrating over the rest of the $x_j$'s and summing up over $p$, we are left with:

$$- \beta d\ell \frac{K_d |\Lambda| dT}{2} \int d^d x \log \left( 1 + \frac{V''}{\gamma \lambda^2 |\Lambda|^2} \right)$$

(58)

- By contacting 2 one-legged terms $-\beta \int d^d x \left( \frac{\Phi^> (x)}{\lambda} \right) \Phi^< (\phi^< (x) / \lambda)$ with $p$ two-legged terms $-\frac{\beta}{2} \int d^d x \left( \frac{\Phi^> (x)}{\lambda} \right)^2 \Phi^< (\phi^< (x) / \lambda)$ with $0 \leq p \leq \infty$. To illustrate our method, we begin with $p = 0$. Expressing the fast modes in Fourier space, we have in discrete space:

$$\frac{1}{2!} \beta^2 \left( \frac{1}{\lambda^2} \right) \left( \frac{a}{L} \right)^{2d} \sum_{x} \sum_{y} V' \left( \frac{\Phi^< (x)}{\lambda} \right) \Phi^< \left( \frac{\Phi^< (y)}{\lambda} \right) \sum_{k} \sum_{k'} e^{i k x + i k' y} \langle \Phi^> (k) \Phi^> (k') \rangle_o (59)$$

which gives after averaging over the fast modes:

$$\frac{\beta}{2} \left( \frac{1}{\gamma \lambda^2} \right) a^{2d} \sum_{x} \sum_{y} V' \left( \frac{\Phi^< (x)}{\lambda} \right) \Phi^< \left( \frac{\Phi^< (y)}{\lambda} \right) \frac{1}{L} \sum_{k} \frac{e^{i k (x-y)}}{|k|^2}$$

(60)

In the above expression, the main contribution comes from the part $x = y$, while the part with $x \neq y$, has a phase which averages to almost zero. Using the fact that $|\Lambda| = \frac{2\pi}{a}$, the result is:

$$\beta d\ell \frac{K_d (2\pi)^d}{2} \int d^d x \frac{\nabla^2 (\Phi^< (x) / \lambda)}{\gamma \lambda^2 |\Lambda|^2}$$

(61)

Proceeding in a similar way for $1 \leq p \leq \infty$, we have

$$\beta^2 \left( \frac{-\beta}{2} \right)^p \frac{1}{(p+2)!} \frac{1}{2} \left( \frac{1}{\lambda^2} \right)^{p+1} \left( \frac{a}{L} \right)^{(p+2)d} \sum_{x,y,x_1 \ldots x_p} V' \left( \frac{\Phi^< (x)}{\lambda} \right) V' \left( \frac{\Phi^< (y)}{\lambda} \right) \Phi^< \left( \frac{\Phi^< (x_1)}{\lambda} \right) \ldots \Phi^< \left( \frac{\Phi^< (x_p)}{\lambda} \right)$$

$$\sum_{k,k',k_1 \ldots k_p} e^{i k x + i k' y + i (k_1 + k'_1) x_1 + \ldots + i (k_p + k'_p) x_p} \langle \Phi^> (k) \Phi^> (k') \Phi^> (k_1) \ldots \Phi^> (k_p) \rangle_o (62)$$

which yields after retaining the $x = y$ part and averaging over the fast modes:
\[
\frac{\beta}{2} (-1)^p a^d \left( \frac{1}{\lambda^2} \right)^{p+1} \int \prod_{j=1}^p d^d x_j d^d x_j V'^2(\Phi^<(x)) \prod_{j=1}^p V''(\Phi^<(x_j)) \]
\[
\left( \frac{\lambda_2^2}{\Lambda} \right)^{p+2} \int \prod_{j=1}^p \hat{d}k_j \hat{d}k \frac{e^{i k(x-x_1)+i k_1(x_1-x_2)+...+i k_p(x_p-x)}}{|k|^2 |k_1|^2 ... |k_p|^2} \]

Treating the above expression as equation (63) we are left with:
\[
\beta d\ell (-1)^p K_d (2\pi)^d \frac{1}{2} \int d^d x \left( \frac{V''(\Phi^<(x)/\lambda)}{\gamma \lambda^2 |\Lambda|^2} \right)^p \]

Summing up over \( p \), we finally obtain:
\[
\beta d\ell K_d (2\pi)^d \frac{1}{2} \int d^d x \frac{V'^2(\Phi^<(x)/\lambda)}{\gamma \lambda^2 |\Lambda|^2} \left( 1 + \frac{V''(\Phi^<(x)/\lambda)}{\gamma \lambda^2 |\Lambda|^2} \right) \]

Taking into account the rescaling of the potential term, and supposing we are in the flat phase so that \( \Phi \) and \( \lambda \) are not rescaled, we obtain for \( d = 2 \):
\[
\frac{dV}{d\ell} = 2V - \pi \frac{\frac{V'^2}{\gamma \lambda^2 |\Lambda|^2}}{1 + \frac{V''}{\gamma \lambda^2 |\Lambda|^2}} + \frac{T}{4\pi \gamma} \log \left( 1 + \frac{V''}{\gamma \lambda^2 |\Lambda|^2} \right) \]

This flow equation can be rewritten in terms of the rescaled parameters \( \nabla = \frac{V}{\gamma \lambda^2 |\Lambda|^2} \) and \( \mathcal{T} = \frac{T}{2\pi \gamma \lambda^2} \). Putting \( g = \frac{1}{\gamma} \frac{d\gamma}{d\ell} \), we have:
\[
\frac{d\nabla}{d\ell} = (2 - g)\nabla - \pi \frac{\nabla'^2}{(1 + \nabla'^2)} + \frac{\mathcal{T}}{2} \log (1 + \nabla'') \]

A.2 Renormalization of the surface tension \( \gamma \)

The contributions to the gradient term are obtained from equations (57) and (63).

- Consider first the contribution due to equation (63). Here, \( x \) is fixed since we have imposed that it should be equal to \( y \). We can thus expand the \( V'' \) terms with respect to this variable as:
\[
V'' \left( \frac{\Phi^>(x_j)}{\lambda} \right) = V'' \left( \frac{\Phi^>(x)}{\lambda} \right) + \frac{1}{\lambda} (x_j - x) \cdot \nabla \Phi^>(x) V'' \left( \frac{\Phi^>(x)}{\lambda} \right) \]

The contribution of equation (63) to the gradient term is given by:
\[ -\frac{\beta\gamma}{2} a^d (-1)^{p+1} \left( \frac{1}{\gamma\lambda^2} \right)^{p+2} \sum_{n<m} \int \prod_{j=1}^{p} d^d \tilde{x}_j d^d x \]

\[ (\tilde{x}_n \cdot \nabla \Phi(x))(\tilde{x}_m \cdot \nabla \Phi(x))V' \left( \frac{\Phi^>(x)}{\lambda} \right) V'' \left( \frac{\Phi^>(x)}{\lambda} \right) \]

\[ \sum_{n<m} \int \prod_{j=1}^{p} \bar{d}k_j \bar{d}k \frac{e^{i(k+\tilde{k}) + ik_1(\tilde{x}_1 - \tilde{x}_2) + \ldots + ik_p \tilde{x}_p}}{|k|^2 |k_1|^2 \ldots |k_p|^2} \]

After integrating over \( \tilde{x}_i \) for \( i \neq m, n \), we are left with:

\[ -\frac{\beta\gamma}{2} a^d (-1)^{p+1} \frac{1}{\gamma} \int \prod_{n<m} d^d x \]

\[ N \sum_{\nu=1}^{N} \tilde{x}_m \tilde{x}_n (\nabla \Phi(x))_{\nu}^2 \]

\[ V'^2 \left( \frac{\Phi^>(x)}{\lambda} \right) V'' \left( \frac{\Phi^>(x)}{\lambda} \right) \]

\[ \int \prod_{j=1}^{p} \bar{d}k_j \bar{d}k'' \frac{e^{i(k+\tilde{k}) + ik_1(\tilde{x}_1 - \tilde{x}_2) + \ldots + ik_p \tilde{x}_p}}{|(k')^n |(k'')^m - n |(k'')^2| p+1 - m} \]

Using the fact that \( ix^\nu e^{ikx} = \frac{\partial}{\partial k^\nu} e^{ikx} \), and an integration by parts, we finally get after summing over \( n < m \) and over \( p \):

\[ \int d^d x \int \frac{\nabla'^2 \left( \frac{\Phi^<(x)}{\lambda} \right) \nabla'' \left( \frac{\Phi^<(x)}{\lambda} \right)}{\left( 1 + \nabla'' \left( \frac{\Phi^<(x)}{\lambda} \right) \right)^5} \]

Since we are looking for the contribution to the gradient term, only the projection of the periodic function \( \frac{\nabla'^2 \nabla''}{(1 + \nabla'')^5} \) on the zeroth harmonic counts. The contribution of equation (63) to the elastic constant \( \gamma \) is thus:

\[ \gamma d\ell \frac{4}{d} K_d(2\pi)^d \int d^d x \frac{\nabla'^2 \left( \frac{\Phi^<(x)}{\lambda} \right) \nabla'' \left( \frac{\Phi^<(x)}{\lambda} \right)}{\left( 1 + \nabla'' \left( \frac{\Phi^<(x)}{\lambda} \right) \right)^5} \]

Consider now equation (57). In order to obtain a term of the form \( (\nabla \Phi^>(x))^2 \), we have to expand two \( V'' \) terms. We proceed as follows: if we choose to expand \( V''(\Phi^<(x_n)/\lambda) \) and \( V''(\Phi^<(x_m)/\lambda) \) with \( m < n \), we perform the expansion with respect to \( (x_m + x_n)/2 \). The \( V'' \) terms thus give:

\[ \gamma d\ell \frac{4}{d} K_d(2\pi)^d \int d\varphi \frac{\nabla'^2 (\varphi) \nabla'' (\varphi)}{\left( 1 + \nabla'' (\varphi) \right)^5} \]
\[- \frac{1}{4} (x_n - x_m) \cdot \nabla \Phi < \left( \frac{x_n + x_m}{2} \right)^2 V''(\frac{x_n + x_m}{2}) V''(\frac{x_n + x_m}{2}) \]  

Integrating over $x_j$ for $j \neq m, n$, we get:

\[- \frac{\beta \gamma}{2} \frac{(-1)^p T}{p} \frac{1}{\gamma^{\lambda^2}} \sum_{m<n} \int d^d x_m d^d x_n \]  

\[\left( \nabla \Phi < \left( \frac{x_n + x_m}{2} \right) \right)^2 V''(\frac{x_n + x_m}{2}) V''(\frac{x_n + x_m}{2}) \]  

Writing $(x_m - x_n)^2 e^{i(k-k')(x_m-x_n)} = \frac{\partial}{\partial k_{\nu}} \frac{\partial}{\partial k'_{\nu}} e^{i(k-k')(x_m-x_n)}$, and performing the rest of the calculation as described above, we find that the contribution to the elastic term in terms of the rescaled parameters $\bar{V}$ and $\bar{T}$ is given by

\[\gamma d \ell \frac{\bar{T}}{2d} \int_0^1 d\varphi \frac{\bar{V}''(\varphi)}{\left(1 + \bar{V}''(\varphi)\right)^4} \]  

finally leading, for $d = 2$, to the renormalization of $\gamma$ given in the main text. If we had chosen another expansion scheme to obtain the contribution to the gradient term, for instance if we had expanded the terms with respect to the centre of mass, we would have obtained a somewhat different value for $g$. This would only affect the precise value of the exponents $\alpha$ and $\delta$ obtained in the text, but not the qualitative features of the solution.
References

[1] Ph. Nozières, F. Gallet, J. Physique (France) 48 (1987) 353. Ph. Nozières, in Solids Far From Equilibrium, C. Godrèche Edt., Cambridge University Press (1991).

[2] S. Balibar, C. Guthmann, E. Rolley, J. Phys I (France), 3 1475 (1993), E. Rolley, C. Guthmann, E. Chevalier, S. Balibar, J. Low. Temp. Physics, 99, 851 (1995).

[3] J.V. José, L.P. Kadanoff, S. Kirkpatrick, D.R. Nelson, Phys. Rev. B 16(3) 1217 (1977)

[4] J. Cardy, S. Ostlund, Phys. Rev. B 25 6899 (1982)

[5] J.-P. Bouchaud, A. Georges, Phys. Rev. Lett. 68, 3908 (1992).

[6] H. Orland, Y. Shapir, Europhys. Lett. 30, 203-208 (1995)

[7] D. Carpentier, P. Le Doussal, cond-mat/9802083.

[8] T. Emig, T. Nattermann, Phys. Rev. Lett. 81, 1469 (1998), see also cond-mat/9808318.

[9] A. Hazareesing, J.-P. Bouchaud, Phys. Rev. Lett. 81(26), 5953 (1998)

[10] Y. Saito, Z. Phys. B 32, 75 (1978)

[11] L. Balents, D.S. Fisher, Phys. Rev. B 48 5949 (1993)

[12] L. Balents, J.-P. Bouchaud, M. Mézard, J. Physique I 6 1007 (1996).

[13] P.G. de Gennes, Reviews of Modern Physics, Vol. 57, No 3, Part I , (1985)

[14] J.F. Joanny and P.G. de Gennes, J.Chem. Phys. 81, pp 552, 1984

[15] E. Rolley, C. Guthmann, R. Gombrowicz, V. Repain, Phys. Rev. Lett. 80, 2865-2868 (1998)

[16] A. Hazareesing, M. Mézard, To appear in Phys. Rev. E