Quasiboson representations of $sl(n+1)$ and generalized quantum statistics

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Abstract

Generalized quantum statistics will be presented in the context of representation theory of Lie (super)algebras. This approach provides a natural mathematical framework, as is illustrated by the relation between para-Bose and para-Fermi operators and Lie (super)algebras of type $B$. Inspired by this relation, $A$-statistics is introduced, arising from representation theory of the Lie algebra $A_n$. The Fock representations for $A_n = sl(n+1)$ provide microscopic descriptions of particular kinds of exclusion statistics, which may be called quasi-Bose statistics. It is indicated that $A$-statistics appears to be the natural statistics for certain lattice models in condensed matter physics.

1 Introduction

During the last two decades quantum statistics became a field of increasing interest among field theorists and condensed matter theorists. Various new statistics were suggested, leading to generalizations or deviations from some of the first principles in quantum physics, such as the Heisenberg commutation relations, the Pauli exclusion principle and the commutativity of space-time.

Some of the generalizations of quantum statistics arose because of new developments in mathematics. For example, the theory of quantum groups led to the introduction of deformed Bose creation and annihilation operators (CAOs) [1]. In a similar spirit, multi-mode $q$-deformed CAOs were introduced, leading to the so-called quon algebra [2] and the related quon statistics. In the context of non-commutative geometry [3], deformations of the Heisenberg commutation relations have been studied too.

In condensed matter physics the discovery of the fractional quantum Hall effect in two-dimensional electron gasses led to the theoretical study of anyons (“particles” with fractional statistics) in two-dimensional systems [4].

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A further breakthrough in the area of quantum statistics was marked with the paper of Haldane [5], who proposed a generalized version of the Pauli exclusion principle. This new statistics is now referred to as (fractional) exclusion statistics. The validity of exclusion statistics has been tested on several examples, e.g. spinon excitations in a spin-$\frac{1}{2}$ quantum antiferromagnetic chain, anyon gas and anyons in a strong magnetic field, particles in a one-dimensional Luttinger liquid, ... Also in conformal field theories exclusion statistics is known to play a role [6].

Our own approach to generalized quantum statistics is inspired by Wigner’s ideas for noncanonical quantum systems [7] and the generalizations of Green in quantum field theory, which led to the introduction of para-Bose and para-Fermi statistics [8]. These statistics are now known to belong to the class of Lie (super)algebras of type $B$. Motivated by this relation, we introduce the more general concept of statistics belonging to a Lie (super)algebra. In particular, quantum statistics belonging to the Lie algebra $A_n$ is discussed here, and referred to as $A$-statistics. Some of the basic properties of $A$-statistics are summarized. It is shown that $A$-statistics has an interpretation as exclusion statistics. Furthermore, we show that $A$-statistics can be considered as a finite-dimensional approximation to Bose statistics. For this reason, it can also be referred to as quasi-Bose statistics, and the corresponding CAOs are called quasibosons or quasi-Bose operators. The quasi-Bose operators can be expressed as functions of the ordinary Bose operators, a fact which is related to the Holstein-Primakoff realization of $sl(n + 1)$. Finally, we present an example from condensed matter physics where the quasibosons play a role in the solution of the problem.

2 Para-Bose and para-Fermi statistics

In 1950, Wigner [7] showed that canonical quantum statistics can be generalized in a logically consistent way. In particular, abandoning the canonical commutation relation $[p, q] = -i$, one can search for all operators $q$ and $p$ such that the “classical” equations of motion $\dot{p} = -q$ and $\dot{q} = p$ are compatible (equivalent) with the Heisenberg equations $\dot{p} = -i[p, H]$, $\dot{q} = -i[q, H]$, where $H$ is the Hamiltonian of the system. Wigner solved this problem for the case of a one-dimensional oscillator ($H = \frac{1}{2}(p^2 + q^2)$), and showed that apart from the canonical solution there are infinitely many other solutions. For a $n$-dimensional oscillator, the compatibility conditions read:

$$\sum_{i=1}^{n} [[a^+_i, a^-_i], a^+_k] = \pm 2a^+_k.$$  \hspace{1cm} (1)

Herein, $a^+_k = \frac{1}{\sqrt{2}}(q_k + ip_k)$. The problem is thus reduced to finding all operators satisfying (1). Or, reformulated in a different way: find all (Hermitian) representations of the associative algebra generated by the elements $a^+_k$ subject to the relations (1).

One familiar solution is of course the canonical solution, since the ordinary Bose CAOs do satisfy (1). The corresponding representations space is the familiar boson
Another set of solutions are provided by the para-Bose operators. These operators \( b_i^\pm \) satisfy the triple relations

\[
[[b_i^\xi, b_j^\eta], b_k^\epsilon] = 2\epsilon \delta_{jk} \delta_{\epsilon, -\eta} b_i^\xi + 2\epsilon \delta_{ik} \delta_{\epsilon, -\xi} b_j^\eta,
\]

where \( i, j, k = 1, 2, \ldots, n \) and \( \xi, \eta, \epsilon = \pm, \pm 1 \). In quantum field theory, these operators were introduced by Green [8] as a possible generalization of statistics of integer-spin fields.

Green also generalized Fermi statistics to para-Fermi statistics by introducing the para-Fermi operators \( f_i^\pm \), satisfying

\[
[[f_i^\xi, f_j^\eta], f_k^\epsilon] = 2\delta_{jk} \delta_{\epsilon, -\eta} f_i^\xi - 2\delta_{ik} \delta_{\epsilon, -\xi} f_j^\eta.
\]

The para-Bose (resp. para-Fermi) algebra can be defined as the associative algebra generated by the elements \( b_i^\pm \) (resp. \( f_i^\pm \)) subject to the relations (2) (resp. (3)). It is then of importance to classify all irreducible (Hermitian Fock) representations of the para-Bose and para-Fermi algebra. For the para-Fermi algebra, it was realized by Kamefuchi and Takahashi [9], and independently by Ryan and Sudarshan [10], that the linear envelope of the elements

\[
f_i^\xi, \quad f_j^\eta, \quad (i, j, k = 1, \ldots, n, \xi, \eta, \epsilon = \pm)
\]

is the orthogonal Lie algebra \( B_n = so(2n+1) \). Thus the representation theory of the para-Fermi algebra reduces to a problem of representation theory of the Lie algebra \( B_n \).

In a similar way, Ganchev and Palev [11] showed that the linear envelope of the elements

\[
b_i^\xi, \quad \{b_j^\eta, b_k^\xi\}, \quad (i, j, k = 1, \ldots, n, \xi, \eta, \epsilon = \pm)
\]

is the orthosymplectic Lie superalgebra \( B(0,n) = osp(1/2n) \), thereby reducing the representation theory of the para-Bose algebra to a problem in representation theory of Lie superalgebras.

### 3 Statistics related to a Lie (super)algebra

As para-Bose statistics is belonging to the class of Lie superalgebras \( B(0,n) \) and para-Fermi statistics to the class of Lie algebras \( B_n \), one can wonder what type of statistics is belonging to the other classes of Lie (super)algebras.

Let \( \mathcal{A} \) be a Lie (super)algebra, with bracket \([ , ]\) (which could stand for a commutator \([ , ]\) or an anti-commutator \{ , \}).

**Definition 1** A set of root vectors \( a_1^\xi, \ldots, a_n^\xi \) (\( \xi = \pm \)) are creation and annihilation operators for \( \mathcal{A} \) if
• $\mathcal{A}$ is equal to the linear envelope of
  \[ a_\xi^\xi, \ [a_\eta^\eta, a_\epsilon^\epsilon], \quad (i, j, k = 1, \ldots, n, \ \xi, \eta, \epsilon = \pm); \]

• $a_i^+$ (resp. $a_i^-$) are negative (resp. positive) root vectors.

At the same time, one should also select the representations of the algebra that could play a role in physics by making the following requirements.

**Definition 2** The $\mathcal{A}$-module $W$ is a Fock space if it is a Hilbert space such that

• $(a_i^+)^\dagger = a_i^-$ (Hermiticity);

• there exists a vector $|0\rangle \in W$ such that $a_i^-|0\rangle = 0$ for all $i$ (existence of vacuum);

• $W$ is spanned on vectors of the type $a_1^+a_2^+\cdots a_m^+|0\rangle$ (irreducibility).

These definitions are a straightforward generalization of the properties that hold for para-Bose and para-Fermi operators and their representations.

One can now turn its attention to other classes of Lie (super)algebras. Of particular interest are the statistics related to Lie (super)algebras of type $A$. For $\mathcal{A} = A_n = sl(n+1)$, the statistics will be called $A$-statistics [12, 13]; its related CAOs will be called $A$-CAOs. For $\mathcal{A} = A(0, n) = sl(1/n)$, the statistics will be called $A$-superstatistics [14]; its related CAOs will be called $A$-super CAOs. The rest of this paper is devoted to $A$-statistics and its properties; for a more elaborate treatment of $A$-statistics, including proofs, see Ref. [13].

# 4 Properties of $A$-statistics

We begin this section by recalling some well-know facts about the Lie algebras $gl(n+1)$ and $sl(n+1)$. A basis for $gl(n+1)$ is given by the elements $e_{ij}$, $i, j = 0, 1, \ldots, n$, satisfying the relations

\[ [e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \]

The Cartan subalgebra $H$ of $gl(n+1)$ is spanned by the elements $h_i = e_{ii}$ ($i = 0, 1, \ldots, n$), and let $\epsilon_i$ be the dual basis of the dual space $H^*$. Then every element $e_{ij}$ ($i \neq j$) is a root vector with corresponding root $\epsilon_i - \epsilon_j$. The positive roots consist of $\epsilon_i - \epsilon_j$ with $i < j$, and the negative roots consist of $\epsilon_i - \epsilon_j$ with $i > j$.

The CAOs for the Lie algebra $sl(n+1) \subset gl(n+1)$ are chosen as follows:

\[ a_i^+ = e_{i,0}, \quad a_i^- = e_{0,i}, \quad (i = 1, 2, \ldots, n). \]

From the defining relations, it is easy to verify that the linear envelope of

\[ a_\xi^\xi, \ [a_\eta^\eta, a_\epsilon^\epsilon], \quad (i, j, k = 1, \ldots, n, \ \xi, \eta, \epsilon = \pm), \]
is indeed \( sl(n+1) = A_n \). Furthermore, it can be shown that the commutation relations of \( A_n \) are completely equivalent with the following set of relations:

\[
\begin{align*}
[a_i^+, a_j^+] &= [a_i^-, a_j^-] = 0, \\
[[a_i^+, a_j^-], a_k^+] &= \delta_{jk} a_i^+ + \delta_{ij} a_k^+, \\
[[a_i^+, a_j^-], a_k^-] &= -\delta_{jk} a_i^- - \delta_{ij} a_k^-.
\end{align*}
\]

These relations are similar to the para-Bose or para-Fermi relations in the sense that they also include triple relations. On the other hand, the actual structure of the relations is quite different and they will also lead to a different type of statistics. We shall refer to (4) as the relations of \( A \)-statistics, and call the corresponding CAOs \( A \)-operators. The description of \( A_n \) by means of the generators \( a_i^\pm \) and the relations (4) was already given in a paper by Jacobson [15]; therefore, the elements \( a_i^\pm \) could also be referred to as Jacobson generators of \( A_n \).

The Fock spaces of \( A \)-statistics can be classified using representation theory of \( A_n \) and the actual commutation relations between the generators. The following characterization holds:

**Theorem 3** \( W \) is a Fock space of \( A_n \) if and only if there exists a positive integer \( p \) such that

\[
a_i^- |0\rangle = 0, \quad \text{and} \quad a_i^- a_j^+ |0\rangle = p \delta_{ij} |0\rangle,
\]

for \( i, j = 1, 2, \ldots, n \). The Fock space is uniquely characterized by the number \( p \).

Because of the resemblance with Green’s definition of “parastatistics of order \( p \)”, we shall also refer here to \( p \) as the order of the \( A \)-statistics.

Let \( W_p \) be the Fock space characterized by \( p \). The following summarizes a number of properties of \( W_p \):

- \( W_p \) is spanned on vectors

\[
(a_1^+)^{l_1} (a_2^+)^{l_2} \cdots (a_n^+)^{l_n} |0\rangle
\]

with \( l_i = 0, 1, \ldots \). Any such vector is nonzero if and only if

\[
l_1 + l_2 + \cdots + l_n \leq p.
\]

- \( W_p \) is a finite dimensional highest weight module of \( A_n \) with highest weight \((p, 0, \ldots, 0)\) in the \( \epsilon_i \)-basis.

- The scalar product on \( W_p \) is completely characterized by

\[
\langle 0 | 0 \rangle = 1, \\
\langle v | a_i^+ w \rangle = \langle a_i^- v | w \rangle, \quad \forall v, w \in W_p.
\]

Thus an orthonormal basis for \( W_p \) is given by the vectors

\[
|p; l_1, \ldots, l_n\rangle = \sqrt{(p - \sum_{j=1}^n l_j)! (a_1^+)^{l_1} \cdots (a_n^+)^{l_n}} |0\rangle, \quad l_1 + l_2 + \cdots + l_n \leq p. \tag{5}
\]
• The transformation of the basis (5) under the action of the CAOs reads:

\[ a_i^+ |p; l_1, \ldots, l_i, \ldots, l_n \rangle = \sqrt{(l_i + 1)(p - \sum_{j=1}^{n} l_j)} \left| p; l_1, \ldots, l_i + 1, \ldots, l_n \right\rangle, \quad (6) \]

\[ a_i^- |p; l_1, \ldots, l_i, \ldots, l_n \rangle = \sqrt{l_i(p - \sum_{j=1}^{n} l_j + 1)} \left| p; l_1, \ldots, l_i - 1, \ldots, l_n \right\rangle. \quad (7) \]

The last equations indicate that the operators \( a_i^+ \) (resp. \( a_i^- \)) can be interpreted as creating (resp. annihilating) a “particle on the \( i^{th} \) orbital”. This can be further supported by introducing a Hamiltonian

\[ H = \sum_{i=1}^{n} \varepsilon_i h_i = \sum_{i=1}^{n} \varepsilon_i (a_i^+ a_i^- + h_0). \]

This Hamiltonian satisfies

\[ [H, a_i^\pm] = \pm \varepsilon_i a_i^\pm \]

and therefore the operators \( a_i^+ \) (resp. \( a_i^- \)) can be interpreted as creating (resp. annihilating) a “particle with energy \( \varepsilon_i \)”.

Clearly, \( H \) belongs to the Cartan subalgebra of \( gl(n+1) \), and not to \( sl(n+1) \). Therefore, it cannot be written as a function of the CAOs only. However, within a fixed irreducible representation \( W_p \), \( H \) can be represented as follows:

\[ H = \frac{1}{n+1} \sum_{i=1}^{n} \varepsilon_i \left( p + n[a_i^+, a_i^-] - \sum_{k \neq i=1}^{n} [a_k^+, a_k^-] \right). \]

5 The Pauli principle for \( A \)-statistics

Let us consider, as an example, \( A \)-statistics of order \( p = 4 \) with \( n = 6 \) orbitals, corresponding to 6 different energy levels. From (5), it follows that there is no restriction on the number of particles to be accommodated on a certain orbital or, which is the same, on a certain energy level, as long as the total number of particles in any configuration does not exceed \( p \). Hence, the following three states or configurations are allowed (the orbitals, i.e., the energy levels, are represented by lines, and the particles by dots):

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Note that the last two configurations are already “saturated” in the sense that no more particles can be added, since the total number of particles is already equal to \( p = 4 \). The following two configurations correspond to forbidden states:

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\]

Both of these states are not allowed since the total number of particles in the configuration exceeds \( p = 4 \).

This example clearly illustrates the accommodation properties of \( A \)-statistics of order \( p \). Because of this “exclusion principle”, \( A \)-statistics can be shown to be a special case of exclusion statistics. Haldane introduced (fractional) exclusion statistics by means of the relation \( \Delta d = -g \Delta N \) (for one kind of particles), where \( \Delta d \) is the change in dimension of a single-particle Hilbert space, \( \Delta N \) is the allowed increase of the number of particles, and \( g \) is the constant characterizing the exclusion statistics. One possible “integral” solution of this relation can be written as

\[
d(N) = n - g \cdot (N - 1). \tag{8}
\]

This should be interpreted as follows: let \( n \) be the total number of orbitals that are available for the first particle, and suppose \( N - 1 \) particles are already accommodated in the configuration, then \( d(N) \) expresses the dimension of the single-particle space for the \( N \)th particle (or the number of orbitals where the \( N \)th particle can be “loaded”). Bose statistics has \( g = 0 \), and Fermi statistics has \( g = 1 \).

If one accepts the natural assumption that (8) should hold for all admissible values of \( N \), i.e. one does not require (8) to be applicable for values of \( N \) which the system cannot accommodate, then \( A \)-statistics is a particular case of exclusion statistics, also with \( g = 0 \). \( A \)-statistics is similar to Bose statistics in the sense that there is no restriction on the number of particles on an orbital. The main difference comes from the fact that the total configuration should contain no more than \( p \) particles. For this reason, we shall refer to \( A \)-statistics also as quasi-Bose statistics.

### 6 Quasi-Bose creation and annihilation operators

Introducing new (representation dependent) CAOs by

\[
B(p)_i^\pm = a_i^\pm / \sqrt{p}, \quad i = 1, \ldots, n, \quad p \in \mathbb{N}, \tag{9}
\]

the transformation formulas (6) and (7) become

\[
B(p)_i^+ |p; l_1, \ldots, l_n\rangle = \sqrt{(l_i + 1)(1 - \frac{\sum_{k=1}^{n} l_k}{p})} |p; l_1, \ldots, l_i + 1, \ldots, l_n\rangle. \tag{10}
\]
rect way, by introducing the Hilbert space $\ell^2$ quasibosons. Holstein-Primakoff realization [17] of these operators is shown to be continuous operators over an appropriate dense subspace $\Omega$ of the Hilbert space. Then a certain strong topology can be introduced on the set of such operators so that $\lim_{p \to \infty} \mathbf{B}(p) \approx \mathbf{B}$. This is the main reason to refer to the operators $\mathbf{B}(p)$ as quasi-Bose CAOs, or quasi-bosons.

The above described approximation can be formulated in a mathematically correct way, by introducing the Hilbert space $\ell^2(\mathbb{Z}^n_+)$ with orthonormal basis $|\ell_1, \ldots, \ell_n\rangle$ ($\ell_i \in \mathbb{Z}^n_+$). On this Hilbert space, quasi-Bose operators $B_i$ and Bose operators $B$ are linear operators defined by means of the action $(p \in \mathbb{N})$

$$B_i(p)|\ell_1, \ldots, \ell_n\rangle = \sqrt{\ell_i(1 + 1 - \sum_{k=1}^{\ell_i} l_k \over p)} |\ell_1, \ldots, \ell_i + 1, \ldots, \ell_n\rangle, \quad \text{for} \quad \sum_k l_k \leq p,$$

$$B_i(p)|\ell_1, \ldots, \ell_n\rangle = \sqrt{l_i(1 + 1 - \sum_{k=1}^{l_i} l_k \over p)} |\ell_1, \ldots, \ell_i - 1, \ldots, \ell_n\rangle, \quad \text{for} \quad \sum_k l_k \leq p,$$

$$B_i(p)|\ell_1, \ldots, \ell_n\rangle = 0, \quad \text{for} \quad \sum_k l_k > p,$$

$$B_i^+(p)|\ell_1, \ldots, \ell_n\rangle = \sqrt{l_i + 1} |\ell_1, \ldots, \ell_i + 1, \ldots, \ell_n\rangle,$$

$$B_i^-(p)|\ell_1, \ldots, \ell_n\rangle = \sqrt{l_i} |\ell_1, \ldots, \ell_i - 1, \ldots, \ell_n\rangle.$$

These operators are shown to be continuous operators over an appropriate dense subspace $\Omega$ of the Hilbert space. Then a certain strong topology can be introduced on the set of such operators so that $\lim_{p \to \infty} B(p) \approx B$. This “bosonisation” of the quasi-Bose operators can be shown to be related to the Holstein-Primakoff realization [17] of $gl(n + 1)$ [18].
7 Quasi-Bose operators in physical model

The quasi-Bose operators introduced here should prove to be useful in physical boson models where finite-dimensional state spaces are required. As an example, consider a two-leg spin-1/2 quantum Heisenberg ladder \[18\] with Hamiltonian

\[
\hat{H} = \sum_i (J\hat{S}_i^+\hat{S}_{i+1}^- + J\hat{S}_i^-\hat{S}_{i+1}^+ + J_\perp\hat{S}_i^+\hat{S}_i^-).
\]

Herein, \(\hat{S}_i^\pm \equiv (\hat{S}_i^{\pm 1}, \hat{S}_i^{\pm 2}, \hat{S}_i^{\pm 3})\) are two commuting spin-1/2 vector operators “sitting” on site \(i\) of the chain \(\pm\) and the Hamiltonian is a scalar with respect to the total spin operator \(\hat{S} = \sum_i (\hat{S}_i^+ + \hat{S}_i^-)\):

\[
[\hat{S}_\alpha^\pm, \hat{S}_\beta^\pm] = i\sum_{\gamma} \epsilon_{\alpha\beta\gamma} \hat{S}_{\gamma i}^\pm, \quad [\hat{S}_\alpha^+, \hat{S}_\beta^-] = 0, \quad [\hat{H}, \hat{S}] = 0.
\]

If \(J_\perp \gg J\) (disordered phase) the state of the system is well described with the bond operator representation of spin operators \[19, 20\]. Following \[20\], the spin operators in \[18\] were expressed in terms of four pairs of bosons per site subject to an additional constraint. In \[21\] \(\hat{S}_i^\pm\) were realized via three pairs of Bose operators \(B_\alpha^\pm\) per site:

\[
\hat{S}_\alpha^\pm = \frac{1}{2} (\pm B_\alpha^- \pm B_\alpha^+ - i\epsilon_{\alpha\beta\gamma} B_{\beta i}^+ B_{\gamma i}^-), \quad \alpha, \beta, \gamma = 1, 2, 3. \tag{15}
\]

The introduction of such Bose operators is convenient for the solution of the model, but also creates certain problems. For instance, the local state space at site \(i\) becomes infinite dimensional (whereas it should be only 4-dimensional with the given spin-1/2 states). One way to overcome this problem is to introduce “by hands” an infinite on-site repulsion \[21\]. Another approach is to use the bond operator representation of Chubukov \[19\]. In this case the Bose operators \(B_\alpha^\pm\) in \[13\] are replaced by new operators \(b_\alpha^\pm\) as follows :

\[
B_\alpha^+ \to b_\alpha^+ = B_\alpha^+ \sqrt{1 - \sum_{\beta=1}^3 B_{\beta i}^+ B_{\beta i}^-}, \quad B_\alpha^- \to b_\alpha^- = B_\alpha^- \sqrt{1 - \sum_{\beta=1}^3 B_{\beta i}^+ B_{\beta i}^- B_{\alpha i}^-}.
\]

A close look at these expressions reveals that one is actually using the quasi-Bose operators of order \(p = 1\) in their Bose realization \[14\]. Replacing the Bose operators with \(p = 1\) quasi-Bose operators throughout the model, one obtains directly the physical state space and the correct expansions for the spin operators and the Hamiltonian. This shows the practical use of the quasi-Bose operators in this model. From the intrinsic properties of quasibosons, it is clear that the quasi-Bose operators should be useful in other physical models as well.
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