PERIODIC INTEGRAL TRANSFORMS
AND C*-ALGEBRAS

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Abstract. We construct canonical integral transforms, analogous to the Fourier transform, that have periods six and three. The existence of this transform is shown to arise naturally from the expectation that the Schwartz space on the real line, viewed as the Heisenberg module of Rieffel and Connes over the rotation C*-algebra, should extend to a module action over the crossed product of the latter by the canonical automorphisms of orders three and six (which does in fact happen and is shown here).

§1. INTRODUCTION.
It is a well known classical fact that the Fourier transform of a Schwartz function
\[
\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e(-tx) dx,
\]
has period four and extends to a unitary operator on $L^2(\mathbb{R})$. (Throughout the paper we write $e(t) := e^{2\pi i t}$. This stems from the fact that $\hat{f}(t) = f(-t)$. In this paper we show that if the product $tx$ in (1) is replaced by a suitable quadratic, then one obtains transforms of period three and six. More specifically, one has a one-parameter family of hexic transforms
\[
(Hf)(t) = \frac{i^{1/6}}{\sqrt{2\mu}} \int_{-\infty}^{\infty} f(x) e(2\mu tx - \mu x^2) dx,
\]
for $\mu > 0$ and $f$ in the Schwartz space $S(\mathbb{R})$. Thus, $H$ extends to a unitary operator on $L^2(\mathbb{R})$ of period six (i.e., $H^6 = I$). (The “ideal” transform is when $\mu = \frac{1}{2}$ as is explained in Remark 2 below.) Note that $H$ is a composition of the multiplication operator by the complex Gaussian $e(-\mu x^2)$ and an inverse Fourier transform (up to scaling), and hence is itself a unitary operator on $L^2(\mathbb{R})$ that leaves invariant $S(\mathbb{R})$.

Theorem 1. One has $(H^3f)(t) = f(-t)$ for all $f \in S(\mathbb{R})$, so that the transform $H$ has period six and extends to a unitary operator on $L^2(\mathbb{R})$. Further, its square $H^2$ (the cubic transform) is given by
\[
(H^2f)(t) = \frac{\sqrt{2\mu}}{i^{1/6}} e(\mu t^2) \int_{-\infty}^{\infty} f(x) e(2\mu tx) dx.
\]
Remark 1. It is interesting to note that although the Fourier transform has period four, if it is composed with multiplication by a complex Gaussian, as in (3), it can be made to have period three. Though this may seem a little surprising, it can be shown that from the C*-algebra point of view it is not (see Section 3).

Remark 2. By analogy with the fact that $e^{-\pi x^2}$ is invariant under the Fourier transform, one can easily check that $e^{-\pi(\sqrt{3} - i)\mu x^2}$ is invariant under $H$ (and hence also under the cubic transform). (It can be checked that this is the only function among the Gaussian exponentials, up to scalars, that is invariant under the cubic or hexic transform.) The reason we referred to $\mu = 1/2$ as the “ideal” case is that in this case one has $\frac{1}{2}(\sqrt{3} - i) = i^{-1/3}$ is of modulus 1 so that one has the invariant Gaussian $e^{-\pi i^{-1/3}x^2}$.

Remark 3. The Fourier transform is a “canonical” transform in the sense that it intertwines the translation and phase multiplication operators in the well known way. Similarly, the hexic transform, with $\mu = \frac{1}{6}$, is also canonical. In fact, letting $(T_x f)(t) = f(t - x)$ and $(E_x f)(t) = e(-xt)f(t)$, one checks the following relations (see §3 below):

$$T_x H = H E_x, \quad E_x H T_x = e(-\frac{1}{2}x^2)T_x H.$$  

Since the group $SL(2, \mathbb{Z})$ is known to contain finite order elements only of orders 2, 3, 4, and 6, it follows that a periodic canonical transform can only have these orders. This therefore gives us canonical transforms for each allowed order.

Remark 4. It may be worthwhile investigating properties that the above transforms have that are analogous to those that are well known to hold of the Fourier transform (as for example in Rudin [8]). For example, is there a multiplication $#$ on the space of Schwartz functions (or $L^1$ functions) such that $H(f \# g) = H(f)H(g)$? (For the Fourier transform this multiplication is convolution.) It may also be of interest to explore the extension of the transforms $H$ and $H^2$ to $\mathbb{R}^n$, or even to locally compact Abelian groups.

An application of Theorem 1 is the existence of finitely generated projective modules over crossed products $6_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$ and $3_\theta := A_\theta \rtimes_\rho \mathbb{Z}_3$, where $A_\theta$ is the rotation C*-algebra and $\rho$ is the canonical order six automorphism on $A_\theta$ (see §3). These modules will give rise to primary classes in the corresponding $K_0$-groups $K_0(6_\theta)$ and $K_0(3_\theta)$. We write $6_\theta^\infty$ and $3_\theta^\infty$ for the respective canonical smooth dense *-subalgebras. It is well known [5] that there are natural isomorphisms $K_+(6_\theta) = K_+(6_\theta^\infty)$ and $K_+(3_\theta) = K_+(3_\theta^\infty)$. (See Section 3.)

**Theorem 2.** Under the action (2), the Schwartz space $S(\mathbb{R})$ is a finitely generated projective right module $\mathcal{M}_6$ over $6_\theta^\infty$ (thus giving rise to a class in $K_0(6_\theta^\infty)$). Similarly, under the action of the order three unitary given by (3), the Schwartz space $S(\mathbb{R})$ is a finitely generated projective right module $\mathcal{M}_3$ over $3_\theta^\infty$. Therefore, $H^{-1} = H^5 = H^3H^2$ and by (3) one gets the formula for the inverse hexic transform

$$(H^{-1} f)(t) = \frac{\sqrt{2\mu}}{i^{1/6}} e(\mu t^2) \int_{-\infty}^{\infty} f(x)e(-2\mu tx)dx.$$  

One similarly gets a formula for the inverse cubic transform $(H^{-2} f) = (H^3 H f)(t) = (H f)(-t)$.
projective right module $\mathcal{M}_j$ over $\mathbb{Z}_0^\infty$ (thus giving rise to a class in $K_0(\mathbb{Z}_0^\infty)$). Further, one has

$$\tau_*[\mathcal{M}_6] = \frac{\theta}{6}, \quad \tau_*[\mathcal{M}_3] = \frac{\theta}{3},$$

for $j = 0, 1, \ldots, 5$, where $\tau_*$ is the induced map by the canonical trace $\tau$ on $K_0$.

In [1], Buck and the author compute the Connes-Chern characters of the hexic and cubic modules $\mathcal{M}_6, \mathcal{M}_3$ and show that there are explicit injections $\mathbb{Z}^{10} \to K_0(6\theta)$ and $\mathbb{Z}^8 \to K_0(3\theta)$ for each $\theta > 0$. The author believes that, just as in the Fourier case [10], these injections will turn out to be isomorphisms (at least for a dense $G_\delta$ set of $\theta$).

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§2. PROOF OF THEOREM 1.

We will make free use of the following identity

$$\int_{-\infty}^{\infty} e(Ax)e^{-\pi bx^2}dx = \frac{1}{\sqrt{b}}e^{-\pi A^2/b}$$

which holds for $b, A \in \mathbb{C}$, $\text{Re}(b) > 0$, and $\sqrt{b}$ is the principal square root.

The theorem follows once we show that:

(A) the set of Gaussians $f_\alpha(x) := e(-\alpha x)e^{-2\pi \mu x^2}$, where $\alpha \in \mathbb{R}$, is a total set in $L^2(\mathbb{R})$,
(B) $(H f_\alpha)(t) = f_\alpha(-t)$ for all $t, \alpha$,
(C) equality (3) holds for each $f_\alpha$.

Proof of (A). It is enough to show that if $g \in L^2(\mathbb{R})$ is such that

$$\int_{-\infty}^{\infty} g(x)e(-\alpha x)e^{-2\pi \mu x^2}dx = 0$$

for each $\alpha$, then $g = 0$. Setting $g(x)e^{-2\pi \mu x^2} = h(x)$ we note that $h$ is in $L^1(\mathbb{R})$ since it is a product of two $L^2$ functions. Hence one has

$$0 = \int_{-\infty}^{\infty} h(x)e(-\alpha x)dx = \hat{h}(\alpha)$$

for each $\alpha$. Therefore, $\hat{h} = 0$ and hence $h = 0$, i.e., $g = 0$. This proves (A) and shows that the set of linear combinations of functions of the form $f_\alpha$ is a dense subspace of $L^2(\mathbb{R})$.

Proof of (B). One has

$$(H f_\alpha)(t) = i^{1/6} \sqrt{2\mu} \int_{-\infty}^{\infty} e(-\alpha x)e^{-2\pi \mu x^2}e(2\mu tx - \mu x^2)dx$$

$$= i^{1/6} \sqrt{2\mu} \int_{-\infty}^{\infty} e((2\mu t - \alpha)x) e^{-2\pi \mu(1+i)x^2}dx$$

$$= \frac{i^{1/6}}{\sqrt{1+i}} e^{-\pi(2\mu t - \alpha)^2/(2\mu(1+i))}.$$
Applying $H$ again gives

$$(H^2 f_\alpha)(t) = \frac{i^{1/3} \sqrt{2\mu}}{\sqrt{1+i}} \int_{-\infty}^{\infty} e^{-\pi (2\mu x - \alpha)^2/(2\mu(1+i))} e(2\mu tx - \mu x^2) \, dx$$

$$= \frac{i^{1/3} \sqrt{2\mu}}{\sqrt{1+i}} e(C) \int_{-\infty}^{\infty} e((2\mu t + D)x) e^{-\pi \beta x^2} \, dx$$

$$= \frac{i^{1/3} \sqrt{2\mu}}{\sqrt{(1+i)\beta}} e(C) e^{-\pi (2\mu t + D)^2/\beta}$$

$$= \frac{i^{1/3}}{\sqrt{i}} e(C) e^{-\pi (2\mu t + D)^2/\beta}$$

where

$$\beta = \mu(1+i), \quad C = \frac{1}{8\mu} (i+1)\alpha^2, \quad D = -\frac{(i+1)}{2} \alpha.$$

A third iteration gives

$$(H^3 f_\alpha)(t) = \sqrt{2\mu} \ e(C) \int_{-\infty}^{\infty} e^{-\pi (2\mu x + D)^2/\beta} e(2\mu tx - \mu x^2) \, dx$$

$$= \sqrt{2\mu} \ e^{-\pi \alpha^2/2\mu} \int_{-\infty}^{\infty} e((2\mu t - i\alpha)x) e^{-2\pi \mu x^2} \, dx$$

$$= e^{-\pi \alpha^2/2\mu} \ e^{-\pi (2\mu t - i\alpha)^2/(2\mu)}$$

$$= e(\alpha t) e^{-2\pi \mu t^2}.$$
The corresponding crossed product $6_\theta := A_\theta \rtimes \rho Z_6$ is the universal C*-algebra generated by unitaries $U, V, W$ enjoying the commutation relations
\[ VU = \lambda UV, \quad WUW^{-1} = V, \quad WVW^{-1} = \lambda^{-1/2}U^{-1}V, \quad W^6 = I. \tag{5} \]

One may view the crossed product $3_\theta = A_\theta \rtimes \kappa Z_3$ as the C*-subalgebra of $6_\theta$ generated by $U, V, W$. We write $6_\theta^\infty$ and $3_\theta^\infty$ for their respective canonical smooth dense *-subalgebras. (For example, the elements of $6_\theta^\infty$ consist of sums of terms of the form $aW^j$ where $a \in A_\theta^\infty$.) Using Rieffel’s Theorem 2.15 [7] (with an appropriate lattice group in $\mathbb{R} \times \hat{\mathbb{R}}$) one obtains a smooth Heisenberg module structure on the Schwartz space $S(\mathbb{R})$, with $A_\theta^\infty$ acting on the right, given by
\[ (fU)(t) = f(t - \alpha), \quad (fV)(t) = e(-\alpha t)f(t), \]
where $\alpha = \sqrt{\theta}$. To extend this action so as to obtain a right $6_\theta^\infty$-module action on $S(\mathbb{R})$, we need $W$ to act as an integral transform
\[ (fW)(t) = \int_{-\infty}^{\infty} f(x)K(x, t)dx, \]
for suitable kernel function $K$, so that the relations (5) are satisfied. Rewrite the commutation relations in (5) involving $W$ in the form
\[ WU = VW, \quad VW = \lambda^{1/2}UWV, \quad W^6 = I. \tag{6} \]

For the second of these relations one has
\[ (fUWV)(t) = e(-\alpha t)(fUW)(t) = e(-\alpha t)\int_{-\infty}^{\infty} f(x)K(x + \alpha, t)dx \]
and
\[ (fVW)(t) = \int_{-\infty}^{\infty} e(-\alpha x)f(x)K(x, t)dx. \]

Hence, doing the same thing for the first relation in (6), one gets the relations
\[ K(x, t - \alpha) = e(-\alpha x)K(x, t), \quad \lambda^{1/2}K(x + \alpha, t) = e(\alpha t - \alpha x)K(x, t). \]

Now it is easy to check that the kernel function $K(x, t) = i^{1/6}e(tx - \frac{1}{2}x^2)$ (of the transform $H$ above with $\mu = \frac{1}{2}$) satisfies these relations. Therefore, by Theorem 1 we can define the right action of $W$ on $S(\mathbb{R})$ by:
\[ (fW)(t) = i^{1/6}\int_{-\infty}^{\infty} f(x)e(tx - \frac{1}{2}x^2)dx, \]
so that the three relations in (6) hold. This, together with the above actions of $U, V$ gives rise to a right $6_\theta^\infty$ module structure on $S(\mathbb{R})$. We shall denote this module by $\mathcal{M}_6$ and call it the hexic module. Also by Theorem 1, one has the order three action
\[ (fW^2)(t) = i^{-1/6}e(\frac{1}{4}t^2)\int_{-\infty}^{\infty} f(x)e(-tx)dx = i^{-1/6}e(\frac{1}{4}t^2)\widehat{f}(t), \]
which makes $S(\mathbb{R})$ into a right $3_R^\infty$-module—we call it the cubic module and denote by $\mathcal{M}_3$. In view of Rieffel’s inner product formulas [7], the $A_\theta^\infty$-valued inner on the Heisenberg module $S(\mathbb{R})$ can be defined by

$$\langle f, g \rangle_{A_\theta^\infty} = \sum_{m,n} \langle f, g \rangle_{A_\theta^\infty}(m,n) \cdot V^n U^m,$$

where $f, g \in S(\mathbb{R})$ and

$$\langle f, g \rangle_{A_\theta^\infty}(m,n) = \int_{-\infty}^{\infty} f(t + \alpha m)g(t)e(-\alpha nt) \, dt.$$

The associated $6_R^\infty$ and $3_R^\infty$-valued inner product $\langle \quad, \quad \rangle_{6_R^\infty}$, $\langle \quad, \quad \rangle_{3_R^\infty}$ are defined by symmetrization

$$\langle f, g \rangle_{6_R^\infty} = \sum_{j=0}^{5} \langle f, gW^{-j} \rangle_{A_\theta^\infty} W^j, \quad \langle f, g \rangle_{3_R^\infty} = \sum_{j=0}^{2} \langle f, gW^{-2j} \rangle_{A_\theta^\infty} W^{2j}.$$

With these inner products, and exactly as was done in the proof for the Fourier module in [9], one sees that the hexic and cubic modules are finitely generated projective, giving classes in the corresponding $K_0$-group, and therefore one obtains Theorem 2.

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