We study the structure of superselection sectors of an arbitrary perturbation of a conformal field theory. We describe how a restriction of the q-deformed $\widehat{sl}(2)$ affine Lie algebra symmetry of the sine-Gordon theory can be used to derive the S-matrices of the $\Phi^{(1,3)}$ perturbations of the minimal unitary series. This analysis provides an identification of fields which create the massive kink spectrum. We investigate the ultraviolet limit of the restricted sine-Gordon model, and explain the relation between the restriction and the Fock space cohomology of minimal models. We also comment on the structure of degenerate vacuum states. Deformed Serre relations are proven for arbitrary affine Toda theories, and it is shown in certain cases how relations of the Serre type become fractional spin supersymmetry relations upon restriction.

*Keywords*: sine-Gordon model, perturbation of conformal field theory, affine quantum group.

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1. Introduction

The domain of integrable massive quantum field theory in 1+1 dimensions has broadened considerably over the past few years to include many new models. This is due primarily to the idea that massive theories can be formulated as integrable perturbations of conformal field theories (CFT), as originally put forward by A. Zamolodchikov[1]. Consequently, if one perturbs relatively novel conformal field theories such as the minimal conformal series [2][3], one is certain to obtain new and interesting massive theories. Exact S-matrices have now been proposed for certain perturbations of many infinite series of unitary and non-unitary conformal field theories. Some unitary examples include the \( \Phi^{(1,3)} \) perturbations [4][5][6][7], \( \Phi^{(1,2)} \) perturbations [9] of the \( c < 1 \) minimal unitary models, W-invariant theories [10][11][12][13], the cosets \( G_k \otimes G_l / G_k + l \) [10], \( Z_k \) parafermions [14], and \( Z_k \) parafermions coupled to a single boson (fractional supersymmetric sine-Gordon theories), which includes the special cases of \( N=2 \) superconformal series and level-\( k \) Wess-Zumino-Witten theories [15][16]. Perturbations of \( N=2 \) superconformal field theories have also been studied in a Landau-Ginzburg approach [17]. Many of these models are quantum field theory versions of solvable models of lattice statistical mechanics [18][19]. Indeed some of the above S-matrices were obtained in [20] from a lattice Bethe-ansatz approach, which is close in spirit to the statistical mechanics methodology.

In this paper we will be concerned with the \( \Phi^{(1,3)} \) perturbation of the minimal unitary series, with central charge

\[
c = 1 - \frac{6}{p(p+1)} \quad p = 3, 4, ...
\]  

This model is formally described by the Euclidean action

\[
S = S_{\text{CFT}} + \frac{\lambda}{2\pi} \int d^2 z \ \Phi^{(1,3)}(z, \bar{z}).
\]  

The conjectured properties of these models are as follows. The massive spectrum consists of kinks \( K_{jk}, j, k \in \{0, 1/2, 1, ..., p/2 - 1\}, j = k \pm 1/2 \). The scattering of these kinks is described by an exact S-matrix of the so-called RSOS form, which can be derived by a quantum group restriction of the conjectured S-matrix for sine-Gordon solitons. These kinks have the appealing physical interpretation as connecting \( p - 1 \) degenerate vacua in a Landau-Ginzburg description of the conformal model [21][8]. These models, which have come to be known as the restricted sine-Gordon theories (RSG), are further characterized...
by some fractional spin supersymmetries \[21\][8]. Recent support for these conjectures has been provided in studies of the thermodynamic Bethe ansatz \[22\][23].

Though the above properties of the RSG theories are generally accepted, a complete quantum field theoretic basis for some of the essentially on-shell arguments was missing in the original works. This was due in part to the absence of a satisfactory non-perturbative derivation of the usual (unrestricted) sine-Gordon soliton S-matrix \[24\]. More recently in \[15\][27] a derivation of this latter S-matrix was given based on the construction of some non-local conserved currents in the sine-Gordon (SG) model that generate the q-deformation of \(sl(2)\) affine Lie algebra \(U_q \left( \widehat{sl}(2) \right)\) \[28\][29]. Quantum affine algebras in connection with \(\Phi^{(1,3)}\) perturbed minimal CFT was also discussed in \[30\]\[3\].

The purpose of this paper is to provide a quantum field theoretic derivation of the RSG S-matrices based on an analysis of their non-local conserved currents. These quantum symmetries are inherited from the \(U_q \left( \widehat{sl}(2) \right)\) symmetry of the SG model, and are related to the fractional supersymmetries described in \[8\], in a way we will make precise. However the algebra satisfied by the conserved charges is no longer \(U_q \left( \widehat{sl}(2) \right)\), but rather a ‘restriction’ of it. Our treatment identifies the fields which create the quantum kinks as the intertwiners for the \textit{chiral} fields \(\phi^{(2,1)}\) or \(\phi^{(2,1)}\). Based on the study of the thermodynamic Bethe ansatz equations of Al. Zamolodchikov \[22\], Klassen and Melzer \[23\] conjectured that the degenerate vacua were associated with the local CFT states \(\Phi^{(n,n)}(0)|0\rangle\). We offer a justification of this identification within our framework.

The general theory of superselection sectors, as outlined in \[31\][32], provides the proper conceptual understanding of some aspects of the RSG quantum field theory. We provide a simple characterization of the superselection sectors of an arbitrary perturbed CFT in the sequel. To our knowledge the RSG theories represent the first example of a model of massive particles with non-abelian sectors.

---

1 The standard non-perturbative framework for quantizing the SG theory is the quantum inverse scattering method (QISM) \[25\], which is an algebraization of Bethe-ansatz methods. Unfortunately the soliton sectors are not easily dealt with in this framework. For a Bethe-ansatz derivation of the SG soliton S-matrix in the Thirring description, see \[26\].

2 However there are significant differences in the proposals of \[27\] and \[30\] that are not easily reconciled. The differences are apparently due to the fact that in \[27\] and the present work, one is only concerned with true symmetries of the theory that are generated by quantum conserved charges which commute with the Hamiltonian and S-matrix. On the other hand the quantum affine structure in \[30\] is more in the spirit of the quantum group symmetry of CFT, which encodes certain aspects of the fusion rules, and is not a symmetry in the above sense.
In the next section we will complete the identification of the $U_q\left(sl(2)\right)$ symmetry in the SG theory by proving that the conserved charges satisfy the appropriate deformed Serre relations. We also extend this analysis to the generalization of SG theory to an arbitrary affine Toda theory. This computation is similar but not identical to the computations in [33][30]; rather our computation is closer to the formulation of twisted homology [34]. Sections 3 and 4 contain the main results outlined above.

2. Quantum Affine Symmetry of the Sine-Gordon and Affine Toda Theories

2a. Quantum Affine Algebras

We review a few basis facts about the quantum affine algebras[28][29]. Let $\tilde{\alpha}_i, i = 1,..,\text{rank}(\hat{G})$ denote a basis of simple roots of an arbitrary, possibly twisted, affine Lie algebra $\hat{G}$, and $a_{ij} = 2\tilde{\alpha}_i \cdot \tilde{\alpha}_j / |\tilde{\alpha}_i|^2$ its generalized Cartan matrix. Untwisted $\hat{G}$ are obtained from a simple Lie algebra $G$ by appending the maximal root $\tilde{\alpha}_0$ of $G$ to the simple roots of $G$, such that $\text{rank}(\hat{G}) = \text{rank}(G) + 1$. The quantum affine algebra $U_q\left(\hat{G}\right)$ is a deformation of the universal enveloping algebra of $\hat{G}$ generated by $H_i, E_i^\pm, i = 1,..,\text{rank}(\hat{G})$ satisfying the relations

\[
\begin{align*}
[H_i, H_j] &= 0 \\
[H_i, E_j^\pm] &= \pm \tilde{\alpha}_i \cdot \tilde{\alpha}_j E_j^\pm \\
[E_i^+, E_j^-] &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}}
\end{align*}
\]

where $q_i \equiv q^{|\tilde{\alpha}_i|^2}/2$, and $q$ is a free parameter. The complete set of relations includes the additional deformed Serre relations

\[
\sum_{\nu=0}^{1-a_{ij}} (-)^\nu \left[ \frac{1-a_{ij}}{\nu} \right]_{q_i} (E_i^\pm)^{1-a_{ij}-\nu} E_j^\pm (E_i^\pm)^\nu = 0
\]

where

\[
\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [m]_q! = \prod_{1 \leq i \leq m} [i]_q, \quad [i]_q = \frac{q^i - q^{-i}}{q - q^{-1}}.
\]
The algebra \( \mathcal{U}_q(\hat{G}) \) is a Hopf algebra equipped with comultiplication \( \Delta : \mathcal{U}_q(\hat{G}) \to \mathcal{U}_q(\hat{G}) \otimes \mathcal{U}_q(\hat{G}) \), counit \( \varepsilon : \mathcal{U}_q(\hat{G}) \to \mathbb{C} \), and antipode \( s : \mathcal{U}_q(\hat{G}) \to \mathcal{U}_q(\hat{G}) \), with the following properties:

\[
\Delta(a)\Delta(b) = \Delta(ab) \quad (2.3a) \\
(\Delta \otimes \text{id}) \Delta(a) = (\text{id} \otimes \Delta) \Delta(a) \quad (2.3b) \\
(\varepsilon \otimes \text{id}) \Delta(a) = (\text{id} \otimes \varepsilon) \Delta(a) = a \quad (2.3c) \\
m(s \otimes \text{id}) \Delta(a) = m(\text{id} \otimes s) \Delta(a) = \varepsilon(a) \quad (2.3d)
\]

for \( a, b \in \mathcal{U}_q(\hat{G}) \) and \( m \) the multiplication map: \( m(ab) = ab \). Eq. (2.3a) implies \( \Delta \) is a homomorphism of \( \mathcal{U}_q(\hat{G}) \) to \( \mathcal{U}_q(\hat{G}) \otimes \mathcal{U}_q(\hat{G}) \), (2.3b) is the coassociativity, and (2.3c, d) are the defining properties of the counit and antipode. These operations have the following specific form

\[
\begin{align*}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i \\
\Delta(E_i^\pm) &= E_i^\pm \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^\pm \\
s(E_i^\pm) &= -q_i^\pm E_i^\pm, \quad s(H_i) = -H_i \\
\varepsilon(E_i^\pm) &= \varepsilon(H_i) = 0.
\end{align*}
\]

Let us define an adjoint action \( \text{ad}_a : \mathcal{U}_q(\hat{G}) \to \mathcal{U}_q(\hat{G}) \) for \( a \in \mathcal{U}_q(\hat{G}) \) as follows. If

\[
\Delta(a) = \sum_i a_i \otimes a^i 
\]

then

\[
\text{ad}_a(b) = \sum_i a_i b s(a^i). 
\]

(Above \( a_i, a^i, b \in \mathcal{U}_q(\hat{G}) \).) This adjoint action is an action of \( \mathcal{U}_q(\hat{G}) \) on itself, i.e. if \( m_{c}^{e} \) are the ‘structure constants’ of \( \mathcal{U}_q(\hat{G}) \), then

\[
\begin{aligned}
e_a e_b &= m_{c}^{e} e_c \quad \Rightarrow \quad \text{ad}_{e_a} \text{ad}_{e_b} = m_{c}^{e} \text{ad}_{e_c}.
\end{aligned}
\]

(See [35] [36].) For ordinary Lie algebras, \( \text{ad} \) is just the commutator.

For reasons that will become apparent, let us define a new basis in \( \mathcal{U}_q(\hat{G}) \) generated by \( H_i, Q_i, \overline{Q}_i \):

\[
Q_i \equiv E_i^+ q^{H_i/2}, \quad \overline{Q}_i \equiv E_i^- q^{-H_i/2}.
\]

(2.8)

4
They have the following properties as a consequence of (2.4):

\[
\begin{align*}
\Delta(Q_i) &= Q_i \otimes 1 + q^{H_i} \otimes Q_i \\
\Delta(Q_i^{-1}) &= Q_i^{-1} \otimes 1 + q^{H_i} \otimes Q_i^{-1} \\
s(Q_i) &= -q^{-H_i}Q_i, \quad s(Q_i^{-1}) = -q^{-H_i}Q_i^{-1} \\
\varepsilon(Q_i) &= \varepsilon(Q_i^{-1}) = 0.
\end{align*}
\] (2.9)

The relations (2.1) can now be written as

\[
[Q_i, Q_j] = \bar{\alpha}_i \cdot \bar{\alpha}_j Q_j \quad [Q_i, Q_j^{-1}] = -\bar{\alpha}_i \cdot \bar{\alpha}_j Q_j^{-1}
\]

\[
ad_{Q_i}(Q_j) = Q_i Q_j - q_i^{-2} Q_i Q_j = \frac{1 - q^{2H_i}}{q_i^{-2} - 1}.
\] (2.10)

It can be shown by explicit computation that the deformed Serre relations can be expressed in the remarkably simple form

\[
ad_{Q_i}^{1-a_{ij}}(Q_j) = ad_{Q_i^{-1}}^{1-a_{ij}}(Q_j) = 0.
\] (2.11)

The above result is a variation of the presentation defined in [37].

For \( U_q(\hat{sl}(2)) \) the simple roots may be chosen such that \( \bar{\alpha}_0 = -\bar{\alpha}_1, \ |\bar{\alpha}_1|^2 = 2 \). The central element \( k = H_0 + H_1 \) of the affine Lie algebra \( \hat{sl}(2) \) is called the level. The \( U_q(\hat{sl}(2)) \) loop algebra is obtained by setting \( k = 0 \).

2b. The \( U_q(\hat{sl}(2)) \) Conserved Currents of the Sine-Gordon Theory

We now review how the \( U_q(\hat{sl}(2)) \) loop algebra symmetry is realized in the sine-Gordon theory [27]. The SG theory may be treated as a massive perturbation of the \( c = 1 \) conformal field theory corresponding to a single real scalar field [3].

\[
S = \frac{1}{4\pi} \int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{\pi} \int d^2z : \cos(\tilde{\beta} \Phi) :.
\] (2.12)

With the above normalization of the kinetic term, the free fermion and Kosterlitz-Thouless points occur at \( \tilde{\beta} = 1, \sqrt{2} \) respectively [3]. The theory has a well-known conserved topological current

\[
\mathcal{J}_{\mu}^{\text{top}}(x) = \frac{\tilde{\beta}}{2\pi} \varepsilon_{\mu\nu} \partial_{\nu} \Phi(x)
\] (2.13)

3. \( z, \bar{z} \) are Euclidean light-cone coordinates: \( z = (t + i\bar{z})/2, \bar{z} = (t - i\bar{z})/2 \).

4. \( \tilde{\beta} \) is related to the conventional coupling \( \beta \) in [38] by \( \tilde{\beta} = \beta/\sqrt{4\pi} \).
which defines the topological charge

\[ T = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \Phi(x). \] (2.14)

With the above \( \beta \)-dependent normalization of the topological charge, the soliton states are known to have \( T = \pm 1 \) \[38\][39]. We will give a different derivation of this normalization in the sequel where we describe superselection sectors of perturbed CFT. For applications to perturbations of the minimal unitary CFT’s, we need only consider the theory in the range \( 1 \leq \beta \leq \sqrt{2} \), where the only particles in the spectrum are solitons.

Define the quasi-chiral components \( \varphi, \bar{\varphi} \) of \( \Phi \) as

\[
\varphi(x, t) = \frac{1}{2} \left( \Phi(x, t) + \int_{-\infty}^{x} dy \, \partial_t \Phi(y, t) \right)
\]

\[
\bar{\varphi}(x, t) = \frac{1}{2} \left( \Phi(x, t) - \int_{-\infty}^{x} dy \, \partial_t \Phi(y, t) \right),
\] (2.15)
such that \( \Phi = \varphi + \bar{\varphi} \). When \( \lambda = 0, \varphi = \varphi(z) \), and

\[
< \varphi(z) \varphi(w) > = -\log(z - w).
\]

Similarly for \( \bar{\varphi} \). We will make use of the following braiding relations:

\[
\exp(i a \varphi(x, t)) \exp(i b \varphi(y, t)) = e^{\pm i \pi a b} \exp(i b \varphi(y, t)) \exp(i a \varphi(x, t)) \; \text{ for } x < y \quad (2.16a)
\]

\[
\exp(i a \bar{\varphi}(x, t)) \exp(i b \bar{\varphi}(y, t)) = e^{\mp i \pi a b} \exp(i b \bar{\varphi}(y, t)) \exp(i a \bar{\varphi}(x, t)) \; \text{ for } x < y \quad (2.16b)
\]

\[
\exp(i a \varphi(x, t)) \exp(i b \bar{\varphi}(y, t)) = e^{i \pi a b} \exp(i b \bar{\varphi}(y, t)) \exp(i a \varphi(x, t)) \; \forall \; x, y,
\] (2.16c)

which are derived using

\[
[\Phi(x, t), \partial_t \Phi(y, t)] = 4\pi i \delta(x - y).
\]

The topological charge of these fields follows from the relation

\[
[T, \exp(i a \varphi + i b \bar{\varphi})] = \beta(a - b) \exp(i a \varphi + i b \bar{\varphi}). \quad (2.17)
\]

The conformal dimensions \((h, \bar{h})\) of the field \( \exp(i a \varphi + i b \bar{\varphi}) \) are \((a^2/2, b^2/2)\), and its Lorentz spin (eigenvalue under Lorentz boosts) is \( h - \bar{h} \).
It can be shown that the model (2.12) has the following non-local quantum conserved currents:

\[ \partial_z J_\pm = \partial_z H_\pm ; \quad \partial_z \overline{J}_\pm = \partial_z \overline{H}_\pm, \]  

(2.18)

where

\[ J_\pm(x,t) = \exp \left( \pm \frac{2i}{\beta} \varphi(x,t) \right), \quad \overline{J}_\pm(x,t) = \exp \left( \mp \frac{2i}{\beta} \overline{\varphi}(x,t) \right) \]  

(2.19a)

\[ H_\pm(x,t) = \lambda \frac{\beta^2}{\beta^2 - 2} \exp \left[ \pm i \left( \frac{2}{\beta} - \hat{\beta} \right) \varphi(x,t) \mp i\hat{\beta} \overline{\varphi}(x,t) \right] \]  

(2.19b)

\[ \overline{H}_\pm(x,t) = \lambda \frac{\beta^2}{\beta^2 - 2} \exp \left[ \mp i \left( \frac{2}{\beta} - \hat{\beta} \right) \overline{\varphi}(x,t) \pm i\hat{\beta} \varphi(x,t) \right]. \]  

(2.19c)

These currents define the following conserved charges

\[ Q_\pm = \frac{1}{2\pi i} \left( \int dz J_\pm + \int dz \overline{H}_\pm \right) \]  

(2.20)

\[ \overline{Q}_\pm = \frac{1}{2\pi i} \left( \int dz \overline{J}_\pm + \int dz \overline{H}_\pm \right). \]

The conserved charges have non-trivial Lorentz spin:

\[ \frac{1}{\gamma} \equiv \text{spin} (Q_\pm) = -\text{spin} (\overline{Q}_\pm) = \frac{2}{\beta^2} - 1. \]  

(2.21)

Consequently the non-local conserved currents have non-trivial braiding relations among themselves and with other fields. In applications to quantum field theory, many of the essential properties of quantum groups find their origin in these braiding relations [40, 27]. This fact was also recognized by Gomez and Sierra in their study of quantum group symmetry in the minimal conformal models [41]. Consider more generally some conserved currents \( \partial_\mu J^a_\mu = 0 \), defining some conserved charges \( Q^a = \frac{1}{2\pi i} \int dx J^a_0(x,t) \), and let us suppose the following braiding relations with a set of fields \( \Phi^i(x) \):

\[ J^a_\mu(x,t) \Phi^k(y,t) = \sum_{b,l} R^a_{bl} \Phi^l(y,t) J^b_\mu(x,t) ; \quad \text{for } x < y. \]  

(2.22)

(The above ansatz for the braiding relations is not completely general. For certain kinds of non-abelian braiding, the appropriate generalization will appear in the sequel.) The action of the charges \( Q^a \) on the fields, which we will denote as \( \text{ad}_{Q^a} \), can be defined as follows

\[ \text{ad}_{Q^a} (\Phi^k(y)) = \frac{1}{2\pi i} \int_{\gamma(y)} dz_\nu \varepsilon^{\nu\mu} J^a_\mu(z) \Phi^k(y), \]  

(2.23)
where the contour $\gamma(y)$ is drawn in figure 1.

Fig. 1. The contour of integration defining the adjoint action in (2.23).

Using the braiding relations (2.22) one derives

$$\text{ad}_{Q^a} \left( \Phi^k(y) \right) = Q^a \Phi^k(y) - R^{ak}_{bl} \Phi^l(y) Q^b. \quad (2.24)$$

If $R^{ab}_{dc}$ is the braiding matrix of the currents with themselves as in (2.22), then the integrated version of (2.24) is

$$\text{ad}_{Q^a} (Q^b) = Q^a Q^b - R^{ab}_{dc} Q^c Q^d. \quad (2.25)$$

The adjoint action of $Q^a$ on a product of two fields at different spacial locations is again defined as in (2.23), where now the contour surrounds the locations of both fields. Using the braiding relations to pass the current through the first field before acting on the second, one finds that this action has a non-trivial comultiplication

$$\Delta (Q^a) = Q^a \otimes 1 + \Theta^a_b \otimes Q^b, \quad (2.26)$$

where $\Theta^a_b$ is the braiding operator which acts on the vector space spanned by the fields $\Phi^i$, i.e. $\Theta^a_b$ has the matrix elements $\Theta^{ak}_{bl} = R^{ak}_{bl}$.

Returning now to the SG theory, and using the braiding relations

$$J_\pm(x, t) \mathcal{J}_\mp(y, t) = q^{-2} \mathcal{J}_\mp(y, t) J_\pm(x, t) \quad ; \quad \forall \; x, y \quad (2.27)$$

where

$$q = \exp(-2\pi i / \hat{\beta}^2) = -\exp(-i \pi / \gamma), \quad (2.28)$$
one can show that $Q_\pm, \overline{Q}_\pm$ and $T$ together generate the $U_q\left(\hat{sl}(2)\right)$ relations (2.10). The isomorphism is

$$
Q_+ = c Q_1 \quad ; \quad Q_- = c Q_0 \\
\overline{Q}_- = c \overline{Q}_1 \quad ; \quad \overline{Q}_+ = c \overline{Q}_0
$$

(2.29)

$$
T = H_1 = - H_0,
$$

where $c$ is a constant ($c = \frac{\lambda}{2\pi} \left( \frac{\hat{\delta}^2}{\beta^2 - 1} \right)^2 (q^{-2} - 1)$). The last relation implies the level is zero and we are actually dealing with the loop algebra. We emphasize that the algebraic relations (2.10) for the conserved charges were derived in [27] using the quantum field theory expressions (2.23)(2.25); the RHS of (2.10) was found in closing the contour in (2.23). Thus it is important to notice that for the particular conserved charges we are considering, the adjoint action defined in the quantum field theory (2.23) and the formal adjoint action (2.6) are equivalent. We will have more to say about this below.

The asymptotic soliton states of topological charge ±1 are denoted $| \pm 1/2, \theta \rangle$, where $\theta$ is the rapidity:

$$
p_0(\theta) = m \cosh(\theta) \quad p_1(\theta) = m \sinh(\theta),
$$

(2.30)

and $T | \pm 1/2, \theta \rangle = \pm | \pm 1/2, \theta \rangle$. A set of chiral fields of topological charge ±1 with non-vanishing matrix elements between the states and the vacuum can be taken to be

$$
\Psi_\pm(x,t) = \exp\left( \pm \frac{i}{\beta} \varphi(x,t) \right) \\
\overline{\Psi}_\pm(x,t) = \exp\left( \mp \frac{i}{\beta} \varphi(x,t) \right).
$$

(2.31)

The fields $\Psi_\pm$ and $\overline{\Psi}_\pm$ do not create independent particle states, for the usual reasons.

The representation of $U_q\left(\hat{sl}(2)\right)$ on the space of one-particle states can be shown to be

$$
Q_\pm = c e^{\theta/\gamma} \sigma_\pm q^{\pm \sigma_3/2} \\
\overline{Q}_\pm = c e^{-\theta/\gamma} \sigma_\pm \overline{q}^{-\sigma_3/2} \\
T = \sigma_3,
$$

(2.32)

5 e.g. a free Dirac fermion in 2 dimensions has 4 components, which create 2 independent particle states.
where $\sigma_\pm, \sigma_3$ are the Pauli spin matrices. The on-shell operators $\exp(\theta/\gamma)$ are a consequence of the Lorentz spin of the conserved charges, since the Lorentz boost generator is represented as $-\partial_\theta$ on-shell.

The above representation of the $\mathcal{U}_q(\widehat{sl}(2))$ loop algebra is in the so-called principal gradation. Gradations of loop algebras are only meaningful mathematically up to inner automorphisms. The standard principal gradation of the $\widehat{sl}(2)$ loop algebra is $E_1^+ = xE_+, E_1^- = x^{-1}E_-, E_0^+ = xE_-, E_0^- = x^{-1}E_+, H_1 = -H_0 = H$, where $E_\pm, H$ generate the finite dimensional $sl(2)$ Lie algebra, and $x$ is a ‘spectral’ parameter. Another gradation we will make use of is the homogeneous one, defined by the automorphism

$$\sigma^{-1} a_{\text{prin.}}(x) \sigma = a_{\text{homo.}}(x^2), \quad (2.33)$$

where $\sigma = x^{H/2}$, and $a(x) \in \widehat{sl}(2)$. In this homogeneous gradation one has $E_1^+ = E_+, E_1^- = E_-, E_0^+ = x^2E_-, E_0^- = x^{-2}E_+$. For our particular application, $x$ is identified with $\exp(\theta/\gamma)$ and is a consequence of the Lorentz spin of the conserved charges. Thus, though the choice of gradation is not intrinsically meaningful mathematically, different gradations are certainly to be distinguished physically, since they reflect scaling dimensions of the conserved currents involved.

The action of the conserved charges on the multiparticle states is provided by the comultiplication which follows from (2.20) and the braiding of the non-local currents with the soliton fields. This comultiplication derived in the quantum field theory is equivalent to (2.3). The two-particle to two-particle S-matrix $\hat{S}$ is an operator from $V_1 \otimes V_2$ to $V_2 \otimes V_1$, where $V_i$ are the vector spaces spanned by $| \pm 1/2, \theta_i \rangle$. The $\mathcal{U}_q(\widehat{sl}(2))$ symmetry of the S-matrix is the condition

$$\hat{S}_{12}(\theta_1 - \theta_2; q) \Delta_{12}(a) = \Delta_{21}(a) \hat{S}_{12}(\theta_1 - \theta_2; q) \quad a \in \mathcal{U}_q(\widehat{sl}(2)). \quad (2.34)$$

Explicitly:

$$\left[ \hat{S}(\theta; q), \sigma_3 \otimes 1 + 1 \otimes \sigma_3 \right] = 0$$

$$\hat{S}(\theta; q) \left( e^{\theta_1/\gamma} \sigma_+ q^{\pm\sigma_3/2} \otimes 1 + q^{\pm\sigma_3} \otimes e^{\theta_2/\gamma} \sigma_+ q^{\pm\sigma_3/2} \right) = \left( e^{\theta_2/\gamma} \sigma_+ q^{\pm\sigma_3/2} \otimes 1 + q^{\pm\sigma_3} \otimes e^{\theta_1/\gamma} \sigma_+ q^{\pm\sigma_3/2} \right) \hat{S}(\theta; q) \quad (2.35)$$

$$\hat{S}(\theta; q) \left( e^{-\theta_1/\gamma} \sigma_+ q^{\mp\sigma_3/2} \otimes 1 + q^{\mp\sigma_3} \otimes e^{-\theta_2/\gamma} \sigma_+ q^{\mp\sigma_3/2} \right) = \left( e^{-\theta_2/\gamma} \sigma_+ q^{\mp\sigma_3/2} \otimes 1 + q^{\mp\sigma_3} \otimes e^{-\theta_1/\gamma} \sigma_+ q^{\mp\sigma_3/2} \right) \hat{S}(\theta; q).$$

\[\begin{array}{c|c|c}
6 & \sigma_3 &=& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\sigma_+ &=& \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \\
\sigma_- &=& \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}
\end{array}\]
(θ = θ₁ − θ₂.) The minimal solution to these symmetry equations is the conjectured S-matrix of SG solitons[24].

Readers familiar with the QISM will recognize the above equations as those which characterize the $R(x, q)$ matrix for the SG model, except for the identification of the spectral parameter $x$ and the parameter $q$. Indeed, solutions to the above equations were first obtained in [25]. However in the QISM $R(x, q)$ describes the commutation relations of the monodromy matrix, thus $R(x, q)$ is not a priori related to the S-matrix; in fact in the QISM $x$ is a formal parameter, and $q$ is different from (2.28). The non-local conserved charges provide an explanation for why $\hat{S}$ is related to $R$, and fixes the non-perturbative dependence on the coupling $\hat{\beta}$.

2c. Deformed Serre Relations

In this subsection we show how the deformed Serre relations are derived from braiding relations and from the knowledge of the ultraviolet behavior of the model. Let us consider the general setting of the preceding subsection, but with ‘abelian’ braiding. That is, we suppose to have a conserved non-local current $J_\mu(x, t)$ with braiding relations

$$J_\mu(x, t) J_\nu(y, t) = e^{i\pi \theta} J_\nu(y, t) J_\mu(x, t), \quad x < y,$$

and a field $\Phi$ such that

$$J_\mu(x, t) \Phi(y, t) = e^{i\pi \eta} \Phi(y, t) J_\mu(x, t), \quad x < y.$$ 

Let $Q$ be the conserved charge corresponding to $J_\mu$. The expression

$$\text{ad}_Q^n (\Phi(y, t)) = \frac{1}{(2\pi i)^n} \int \prod_{j=1}^{n} dz_j J_\nu(z_j) \Phi(y, t)$$

is a multiple integral over a product of loops going from $-\infty$ to $-\infty$ around $y$. If this integration domain could be shrunk to a region close to $y$, as one can do in the case of local currents, then we would use our knowledge of the short distance behavior of the theory to do the computation. When is this possible? The answer can be formulated in terms of twisted homology [34], but here we choose a more direct approach. Let us replace the point $-\infty$ where the integration contours originate and end by a point $(x, t)$ with the same time coordinate as $(y, t)$ (figure 2), and ask when the resulting integral (2.38) is independent of $x$. 

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Let us compute the derivative with respect to $x$. Both upper and lower integration bounds of the integral over $z_j$ are equal to $x$. Thus the derivative with respect to $x$ is a sum of boundary terms, two for each integration variable, which are all equal up to a phase to

$$J_t(x, t) \int \prod_{j=1}^{n-1} (J^\mu(z_j) \varepsilon_{\mu\nu} dz_j^\nu) \Phi(y, t).$$  \hspace{1cm} (2.39)

The phase of the lower integration bound of the $j$-th variable is obtained by braiding $J^\mu(z_j)$ through $J(z_1) \cdots J(z_{j-1})$ and is thus $e^{i\pi(j-1)\vartheta}$. Similarly, we get the phase $e^{2\pi i(n-j)\vartheta+2\pi i\eta e^{i\pi(j-1)\vartheta}}$ from the upper integration bound. Summing up, the derivative with respect to $x$ is equal to (2.39) times

$$\left( e^{2\pi i\eta+i\pi \vartheta(n-1)} - 1 \right) \frac{e^{i\pi \eta \vartheta} - 1}{e^{i\pi \vartheta} - 1}.$$  \hspace{1cm} (2.40)

We conclude that if the braiding phases are related by the formula

$$\eta + \vartheta \frac{n - 1}{2} = 0 \mod 1,$$  \hspace{1cm} (2.41)

the integration domain can be shrunk to a product of loops as in figure 2 with $x$ arbitrarily close to $y$.

We can thus argue using the short distance behavior of the theory. Suppose that $J, \Phi$ have ultraviolet scaling dimension $h_J, h_\Phi$. The field $\text{ad}_Q^n(\Phi)$ has then scaling dimension

$$n(h_J - 1) + h_\Phi.$$  \hspace{1cm} (2.42)

This information, plus the knowledge of the quantum numbers of the operators, can be sufficient to identify $\text{ad}_Q^n(\Phi)$ with a field in the conformal field theory describing the ultraviolet behavior. In particular if no operator exists with the given quantum numbers and scaling dimension (2.42), we conclude that $\text{ad}_Q^n(\Phi) = 0$.  

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Let us apply these ideas to the Serre relations. They are in this case

\[ \text{ad}^3_{Q_{\pm}}(Q_{\pm}) = \text{ad}^3_{Q_{\pm}}(\overline{Q}_{\pm}) = 0. \quad (2.43) \]

Consider the expression \( \text{ad}^{n}_{Q_{+}}(J_{\mu}(y,t)) \). We see that we are in the situation described above with

\[ \vartheta = -\frac{4}{\beta^2}, \quad \eta = \frac{4}{\beta^2} \quad (2.44) \]

and condition (2.41) is precisely satisfied when \( n = 3 \). Shrinking the integration domain we can reduce the computation to free conformal field theory: the scaling dimension of the component of \( \text{ad}^3_{Q_{+}}(J_{\mu}(y,t)) \) of leading order in \( \lambda \) has ultraviolet scaling dimension \( 8/\beta^2 - 3 \). On the other hand the result is in the conformal family of the vertex operator \( \exp(i4\varphi/\beta) \). But the fields in this family have scaling dimension equal to \( 8/\beta^2 \) or higher.

We conclude that to leading order in \( \lambda \), \( \text{ad}^3_{Q_{+}}(J_{\mu}(y,t)) = 0 \). The terms of higher order in \( \lambda \) are treated in a similar way: the conformal weights are in all cases lower that the minimal values in the corresponding conformal families, and we have

\[ \text{ad}^3_{Q_{+}}(J_{\mu}(y,t)) = 0, \quad (2.45) \]

implying the first Serre relation (2.43). The other relations in (2.43) are proven in the same way.

The above proof of the Serre relations extends easily to an arbitrary quantum affine Lie algebra. Let \( \bar{\Phi} \) denote a vector of fields valued in the Cartan subalgebra of an arbitrary affine Lie algebra \( \hat{G} \), and consider the affine Toda theory:

\[ S = \frac{1}{4\pi} \int d^2z \, \partial_x \Phi \cdot \partial_x \Phi + \frac{\lambda}{2\pi} \int d^2z \sum_{\alpha_j \text{ simple}} \exp \left( -i\beta \frac{2}{|\alpha_j|^2} \overline{\alpha}_j \cdot \Phi \right). \quad (2.46) \]

This theory has non-local conserved charges \( Q_j \) generated by the purely chiral components

\[ J_{\overline{\alpha}_j}(x) = \exp \left( i \frac{\beta}{\beta} \overline{\alpha}_j \cdot \Phi \right), \quad (2.47) \]

and also charges \( \overline{Q}_j \) generated by the purely anti-chiral components

\[ \overline{J}_{\overline{\alpha}_j}(x) = \exp \left( i \frac{\beta}{\beta} \overline{\alpha}_j \cdot \overline{\varphi} \right). \quad (2.48) \]
(ϕ and ϕ̅ are quasi-chiral components of Φ as in (2.13). Define the topological charges

\[ H_i = \frac{\beta}{2\pi i} \int dx \bar{\alpha}_i \cdot \partial_x \Phi(x,t). \]  

(2.49)

Using the braiding relations

\[ J_{\bar{\alpha}_j}(x) J_{-\bar{\alpha}_j}(y) = q^{-|\bar{\alpha}_j|^2} J_{-\bar{\alpha}_j}(y) J_{\bar{\alpha}_j}(x) \quad \forall \ x, y, \]  

(2.50)

where \( q = \exp(-i\pi/\beta^2) \), and results from [27], one can show that these charges satisfy the \( \mathcal{U}_q(\hat{G}) \) algebra in the form (2.10).

Consider now the deformed Serre relations (2.11). From

\[ J_{\bar{\alpha}_i}(x) J_{\bar{\alpha}_j}(y) = \exp \left( -\frac{i\pi}{\beta^2} \bar{\alpha}_i \cdot \bar{\alpha}_j \right) J_{\bar{\alpha}_j}(y) J_{\bar{\alpha}_i}(x) \quad x < y, \]  

(2.51)

one sees that we can apply the general result described above with

\[ \vartheta = -\frac{|\bar{\alpha}_i|^2}{\beta^2}, \quad \eta = -\frac{\bar{\alpha}_i \cdot \bar{\alpha}_j}{\beta^2}. \]  

(2.52)

The solution to (2.41) is thus \( n = 1 - a_{ij} \), where \( a_{ij} \) is the generalized Cartan matrix of \( \hat{G} \). Using scaling arguments similar to the \( \mathcal{U}_q(s\bar{l}(2)) \) case, one establishes (2.11).

### 3. Superselection Sectors in Massive Perturbations of CFT

As we will see, the proper general framework for understanding certain features of the RSG quantum field theory is the general theory of superselection sectors, in particular the case of non-abelian sectors, as outlined in [31][32]. The basic ingredients of a theory with superselection sectors are as follows. The complete Hilbert space \( \mathcal{H} \) of the theory is decomposed into ‘charge’ sectors \( \mathcal{H}_i : \mathcal{H} = \sum_i \mathcal{H}_i \). Fields \( \Psi(x) \) are operators from \( \mathcal{H} \rightarrow \mathcal{H} \).

In particular, the charged fields \( \Psi_{ji}^{(k)}(x) \) intertwine the spaces \( \mathcal{H}_i \) and \( \mathcal{H}_j \):

\[ \Psi_{ji}^{(k)}(x) : \mathcal{H}_i \rightarrow \mathcal{H}_j. \]  

(3.1)

The superselection sectors of a CFT can be summarized as follows. One has chiral and anti-chiral primary fields \( \phi^{(i)}(z) \) and \( \bar{\phi}^{(i)}(\bar{z}) \), and their associated states |i⟩ = |

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7 We follow the convention that \( \phi^{(i)}, \bar{\phi}^{(i)} \) refer to chiral and anti-chiral fields respectively, whereas the local spinless fields are \( \Phi^{(i)} = \phi^{(i)} \bar{\phi}^{(i)} \).
\( \phi^{(i)}(0)|0\rangle, \mid \bar{i} \rangle = \overline{\phi^{(i)}}(0)|0\rangle \). The sectors \( \mathcal{H}_i \) (\( \overline{\mathcal{H}}_i \)) correspond to the states \(|i\rangle \) (\(|\bar{i}\rangle \)) and their descendents with respect to the chiral algebra. The complete Hilbert space is given by

\[
\mathcal{H}^{\text{CFT}} = \sum_{i,j} \mathcal{H}_i^{\text{CFT}} \otimes \overline{\mathcal{H}}_j^{\text{CFT}}.
\] (3.2)

Note that we do not assume that CFT consists only of local fields, which corresponds to \( \mathcal{H}^{\text{CFT}} = \sum_i \mathcal{H}_i^{\text{CFT}} \otimes \overline{\mathcal{H}}_i^{\text{CFT}} \). This is important for massive perturbations of CFT, since non-local fields are generally required for example to create particle states in the charge sectors. Let \( N^{ij}_k \) denote the fusion algebra of the CFT: \( i \times j = \sum_k N^{ij}_k \), and similarly for the anti-chiral fields. For every non-zero \( N^{ij}_k \) there exists an intertwiner (or chiral vertex operator in this context, see [12][13][14] for precise definitions):

\[
\phi^{(i)}_{kj} : \mathcal{H}^{\text{CFT}}_j \rightarrow \mathcal{H}^{\text{CFT}}_k \quad ; \quad \overline{\phi^{(i)}}_{kj} : \overline{\mathcal{H}}^{\text{CFT}}_j \rightarrow \overline{\mathcal{H}}^{\text{CFT}}_k.
\] (3.3)

Consider now a massive perturbation of the CFT, formally described by the action

\[
S = S^{\text{CFT}} + \frac{\lambda}{2\pi} \int d^2z \, \Phi_{\text{pert.}}(z, \bar{z}).
\] (3.4)

The perturbing field \( \Phi_{\text{pert.}}(z, \bar{z}) \) is a local spinless field, and can be generally expressed as \( \Phi_{\text{pert.}} = \phi_{\text{pert.}} \overline{\phi}_{\text{pert.}} \), where \( \phi_{\text{pert.}} \) (\( \overline{\phi}_{\text{pert.}} \)) is a (anti-)chiral field in the CFT, with scaling dimensions \( h(\phi_{\text{pert.}}) = \overline{h}(\overline{\phi}_{\text{pert.}}) \). The superselection sectors of the perturbed CFT are generally fewer than those of the CFT. We propose their following characterization:

The superselection sectors of the perturbed CFT are intertwined by the set of fields which are local with respect to the perturbing field \( \Phi_{\text{pert.}} \).

One is led to the above characterization by the following considerations. Let \( \mathcal{F} \) denote the space of fields of the CFT, which includes all possible, not necessarily spinless, products of chiral with anti-chiral fields, and \( \mathcal{F}_L \subset \mathcal{F} \) the space of fields which are local with respect to \( \Phi_{\text{pert.}} \). From general principles, one expects that the intertwining fields should be local with respect to observables. For the perturbed CFT (3.4), the trace of the energy-momentum tensor is equal to \( \beta(\lambda) \Phi_{\text{pert.}} \), where \( \beta(\lambda) \) is the beta-function [14]. Thus locality with respect to the energy-momentum tensor requires the intertwining fields to be in \( \mathcal{F}_L \).

Alternatively, one may argue as follows. Let \( \Psi(x) \in \mathcal{F} \) and consider a correlation function involving this field in perturbation theory:

\[
\langle \Psi(x) \cdots \rangle = \langle \Psi(x) \cdots \rangle^{\text{CFT}} + \lambda \int d^2z \langle \Phi_{\text{pert.}}(z, \bar{z}) \Psi(x) \cdots \rangle + O(\lambda^2) + \ldots
\] (3.5)
Unless \( \Psi(x) \in \mathcal{F}_L \), the integral over \( z, \overline{z} \) is not well-defined. Indeed, if \( \Psi(z) \) is a chiral field in \( \mathcal{F}_L \), then \( \partial_z \Psi \) is completely well-defined and computable in perturbation theory\(^[1]\).

In the language of algebraic quantum field theory, the concept of superselection sectors is based on the choice of an observable algebra of (bounded functions of) local fields, of which the superselection sectors are representation spaces. Intertwining fields are then local with respect to the observable algebra. In conformal field theory, it is natural to choose as observable algebra the algebra generated by the energy momentum tensor, or possibly some larger local chiral algebra. When one considers perturbations of conformal models by some local operator, parametrized by a coupling constant \( \lambda \), the above reasoning shows that one should adjoin the perturbing field to the observable algebra. One effect of this procedure is, as we have seen, that some sectors of the CFT disappear, because their intertwining fields are not local with respect to the perturbation. Another effect is that some CFT sectors are welded because the larger algebra does not leave them invariant.

In a theory in which we have a description in terms of microscopic degrees of freedom (e.g. the field \( \Phi \) in the SG model or the order parameter in a Landau-Ginzburg theory) one has an absolute notion of locality: local fields at a point \( x \) are functions of the microscopic degrees of freedom in an infinitesimal neighborhood of \( x \). In order to describe the topological behaviour of the model one should not distinguish between sectors which are intertwined by local fields. Let us call the larger sectors obtained this way topological sectors. A good example where the distinction can be made is the free massless bosonic field. Sectors are labeled by two numbers, the magnetic (topological) charge and the electric charge. In each sector we have a different representation of the current algebra generated by \( \epsilon_{\mu\nu} \partial^\nu \Phi \) and \( \partial_\mu \Phi \). The intertwining fields for the electric charge are exponentials of the local field \( \Phi \) whereas the intertwining fields carrying magnetic charge are non local. Thus we see that the topological sectors are labeled by the magnetic charge only, so that the sectors of different electric charge but with the same magnetic charge are welded into a single topological sector.

Let us specify the sectors of the perturbed theory more precisely. Define an index set \( L \), such that for \( i \in L \), \( \phi^{(i)}(z) \) is local with respect to \( \phi_{\text{pert.}}(z) \) in the CFT. Denote by \( \phi^{(i)}(\overline{z}) \) the field conjugate to \( \phi^{(i)} \). The field \( \phi^{(i)}(\overline{z}), i \in L \) is also local with respect to \( \phi_{\text{pert.}}(\overline{z}) \). Thus \( \mathcal{F}_L \) is spanned by \( \phi^{(i)}, \overline{\phi}^{(i)}, i \in L \), and any product of these fields with local fields. We consider the diagonal theory in which the local primary fields are \( \phi^{(i)} \overline{\phi}^{(i)} \). The sectors corresponding to fields \( \phi^{(i)} \) and \( \overline{\phi}^{(i)} \) are intertwined by local fields, thus they do
not intertwine distinct topological sectors; the chiral and anti-chiral CFT sectors are thus welded. The topological sectors of the perturbed theory are thus

$$\mathcal{H}_{pert} = \sum_{i \in L} \mathcal{H}_{i}^{pert} ,$$

and they are intertwined as follows

$$\phi_{kj}^{(i)}, \phi_{kj}^{(i)} : \mathcal{H}_{j}^{pert} \rightarrow \mathcal{H}_{k}^{pert}, \quad i, j, k \in L .$$

Since braiding is a property of the short distance behavior of a quantum field theory, the braiding of the intertwiners in the CFT and in the perturbed theory are the same.

Let us illustrate these ideas by using them to re-derive the superselection sectors of the SG theory. Fröhlich established the existence of sectors of integer topological charge (2.14) from the general principles of algebraic quantum field theory [31]; one important point of this work is that it did not rely on any semi-classical arguments, as opposed to [39]. Since our characterization of sectors is rather different, it is important to check it in this simple example. The chiral primary fields of the $c = 1$ free boson theory are $e^{i\alpha \phi(z)}$, which defines the sector $\mathcal{H}_{\alpha}^{CFT}$, where $\alpha$ is any real parameter. The perturbing field is $\cos(\hat{\beta} \Phi)$. From the operator product expansion

$$\exp \left( i \frac{n}{\beta} \phi(z) \right) \exp \left( -i \hat{\beta} \phi(w) \right) = \frac{1}{(z - w)^{n}} \exp \left( i \frac{n}{\beta} - \hat{\beta} \right) \phi(w) + \ldots$$

one deduces that $e^{in\phi/\hat{\beta}} \in \mathcal{F}_{L}$, for $n$ an integer. Thus the sectors of the SG theory are $\mathcal{H}_{SG} = \sum_{n \in \mathbb{Z}} \mathcal{H}_{n}^{SG}$. From (2.17) one sees that the fields $e^{in\phi/\hat{\beta}}$ and $e^{-in\phi/\hat{\beta}}$ have precisely topological charge $n$. They intertwine the sectors according to $\mathcal{H}_{m} \rightarrow \mathcal{H}_{m+n}$. Thus our conclusions are in complete agreement with the results in [31]. Indeed, the above analysis provides an alternative derivation of the normalization of the topological current (2.13).

We turn now to the sectors of the RSG theory. This theory is a perturbation of the minimal unitary series by the operator $\Phi^{(1,3)}$. The chiral primary fields of the minimal model are $\phi^{(n,m)}(z)$ with $1 \leq n \leq p - 1, 1 \leq m \leq p$, with conformal dimension

$$h_{nm} = \frac{[(p + 1)n - pm]^{2} - 1}{4p(p + 1)} .$$
From these dimensions one finds that $h_{n,1} + h_{1,3} - h_{n,3} = n - 1$, thus the fields $\phi^{(n,1)}$ are in $\mathcal{F}_L$. Relabel the integers $n$ of $\phi^{(n,1)}$ as $n = 2j + 1$ and let $\phi^{(n,1)} \equiv \phi^{(j)}$. The superselection sectors of the RSG theory are thus

$$H_{RSG} = \sum_{j \in \{0, 1/2, 1, \ldots, p/2 - 1\}} H^R_{j_{RSG}}.$$ (3.10)

The fusion rules are the standard ones

$$j_1 \times j_2 = \frac{\min(j_1 + j_2, p - j_1 - j_2)}{2} \sum_{j = |j_1 - j_2|} j.$$ (3.11)

Let us relabel the intertwiners $\phi^{(n,1)}_{(n_2,1)(n_1,1)}$ in a similar way as $\phi^{(j)}_{j_2j_1}$. Then the sectors are intertwined as follows

$$\phi^{(j)}_{j_2j_1}, \quad \phi^{(j)}_{j_2j_1} : \quad \mathcal{H}_{j_1} \to \mathcal{H}_{j_2}.$$ (3.12)

In the SG theory, the soliton states are created by the intertwining fields. Thus on general grounds, one expects qualitatively that the spectrum of massive kink states in the RSG theory should intertwine the sectors $\mathcal{H}_j$. This will be made explicit in the next section.

As a further example, consider the $su(2)$ Wess-Zumino-Witten model at level $k$ perturbed by the marginal operator $\Phi_{pert.} = \sum_a J^a \tilde{T}^a$, where $J^a, \tilde{T}^a$ are the chiral and anti-chiral $su(2)$ affine currents. This is a massive model due to its non-zero beta-function (for $k = 1$ this is the chiral Gross-Neveu model). Since all primary fields are local with respect to $J^a$, the sectors of the perturbed theory in this case are just a welding of chiral and anti-chiral CFT sectors: $\mathcal{H}^{pert.} = \sum_{j \leq k/2} \mathcal{H}^{pert.}_j$. The conjectured spectrum of these models\cite{15}\cite{10} indeed reflects the existence of these sectors.

4. The Restricted Quantum Affine Symmetry

In this section we will describe the restriction of the SG theory from the vantage of the quantum affine symmetry described above. We will show that the restriction of the SG theory can be described as the restriction to fields and asymptotic states that form a quotient of the space of singular vectors with respect to an $\mathcal{U}_q(sl(2))$ subalgebra of $\mathcal{U}_q\left(\widehat{sl(2)}\right)$. We will describe the restriction in two steps. First we construct fields and states which are covariant with respect to $\mathcal{U}_q(sl(2))$. Then we project onto the restricted
space of fields and states. We then describe how the residual \( \mathcal{U}_q \left( \widehat{\mathfrak{sl}(2)} \right) \) symmetry acts on the restricted model.

4a. Quantum Group Covariant Fields and States

The \( \mathcal{U}_q \left( \widehat{\mathfrak{sl}(2)} \right) \) algebra has two \( \mathcal{U}_q (\mathfrak{sl}(2)) \) subalgebras generated by \( \{Q_+, \overline{Q}_-, T\} \) and \( \{Q_-, \overline{Q}_+, T\} \). (These two subalgebras are not independent.) The energy-momentum tensor of the SG theory is not invariant under \( \mathcal{U}_q (\mathfrak{sl}(2)) \), as first noticed by Reshetikhin and Smirnov [7]. To see this explicitly, for simplicity consider the conformal limit, where \( T_{zz} = -\partial_z \varphi \partial_z \varphi/2 \) and similarly for \( T_{\overline{z}\overline{z}} \). Then from explicit computation one sees that \([Q_+, T_{zz}(z)] \neq 0\). However familiar results from the Feigin-Fuchs construction [46][47], indicate that \( T_{zz} \) can be made invariant with respect to \( Q_+ \) by modifying it to include a background charge term:

\[
T'_{zz} = -\frac{1}{2} \partial_z \varphi \partial_z \varphi + i\sqrt{2} \alpha_0 \partial^2_z \varphi. \tag{4.1}
\]

By choosing \( \alpha_0 \) appropriately such that \( Q_+ \) becomes identified with a screening operator, one can ensure that

\[
[Q_+, T'_{zz}] = 0. \tag{4.2}
\]

The condition that \( Q_+ \) is a screening operator is equivalent to the requirement that it have scaling dimension 0. The operators \( e^{i\alpha \varphi} \) now have dimension \( \alpha^2/2 - \sqrt{2} \alpha \alpha_0 \) with respect to \( T'_{zz} \). Thus \( \alpha_0 \) is fixed to be a solution of

\[
1 = \frac{2}{\beta^2} - \frac{2\sqrt{2}}{\beta} \alpha_0. \tag{4.3}
\]

As usual the background charge term in \( T'_{zz} \) contributes to the conformal anomaly \( c = 1 - 24\alpha_0^2 \). For

\[
\frac{\widehat{\beta}}{\sqrt{2}} = \sqrt{\frac{p}{p + 1} }, \tag{4.4}
\]

\( \alpha_0 \) is such that \( c \) is that of the minimal unitary series \([1.1]\). A similar analysis applies to \( \overline{Q}_- \) and \( T_{\overline{z}\overline{z}} \). The above inclusion of a background charge does not affect the fact that \( Q_+, \overline{Q}_-, T \) generate the algebra \( \mathcal{U}_q (\mathfrak{sl}(2)) \). Note that it is not possible to simultaneously demand \( T'_{zz} \) to be invariant under both \( \mathcal{U}_q (\mathfrak{sl}(2)) \) subalgebras, since \( Q_+ \) and \( Q_- \) cannot simultaneously be given dimension 0. For the remainder of this section, \( \mathcal{U}_q (\mathfrak{sl}(2)) \) will refer to the subalgebra generated by \( Q_+, \overline{Q}_-, \) and \( T \). We also rescale the charges \( Q_+, \overline{Q}_- \rightarrow \)}
$c^{-1}Q_{-}, c^{-1}\overline{Q}_{-}$, where the constant $c$ is defined in (2.29), so that the resulting charges satisfy the usual $\mathcal{U}_q(sl(2))$ relations.

In order to make certain arguments, let us follow the Feigin-Fuchs construction and define the vertex operator fields

$$V_{nm}(x) = e^{i\sqrt{2}\alpha_{nm}\varphi}, \quad \overline{V}_{nm}(x) = e^{i\sqrt{2}\alpha_{nm}\overline{\varphi}}, \quad \text{(4.5)}$$

where

$$\alpha_{nm} = \frac{1}{2}(1-n)\alpha_+ + \frac{1}{2}(1-m)\alpha_-$$

$$\alpha_+ = \frac{\sqrt{2}}{\beta}, \quad \alpha_- = -\frac{\bar{\beta}}{\sqrt{2}} \quad \text{(4.6)}$$

$1 \leq n \leq p-1, \quad 1 \leq m \leq p$.

These fields represent the minimal model primary fields $\phi^{(n,m)}, \overline{\phi}^{(n,m)}$. The currents $J_{-}$ and $\overline{J}_{+}$ in (2.19) are associated with the primary fields $\phi^{(3,1)}$ and $\overline{\phi}^{(3,1)}$. Note also that the soliton fields $\Psi_{-}$ and $\overline{\Psi}_{+}$ are associated with $\phi^{(2,1)}$ and $\overline{\phi}^{(2,1)}$, whereas $\Psi_{+}$ and $\overline{\Psi}_{-}$ are not in the usual spectrum of primary fields. Note that of the two operators $\exp(\pm i\bar{\beta}\Phi)$ that define $\cos(\bar{\beta}\Phi)$ in (2.12), one becomes a screening operator, the other becomes the field $\Phi^{(1,3)}$ of dimension $(p-1)/(p+1)$, so that we are actually describing the model (1.2).

Define the fields $\phi^{(j)}_{-j}(x)$ and $\overline{\phi}^{(j)}_{j}(x)$ as

$$\phi^{(j)}_{-j}(x) \equiv V_{2j+1,1}(x) = \exp\left(\frac{-2i}{\beta} j \varphi(x)\right)$$

$$\overline{\phi}^{(j)}_{j}(x) \equiv \overline{V}_{2j+1,1}(x) = \exp\left(\frac{-2i}{\bar{\beta}} j \overline{\varphi}(x)\right). \quad \text{(4.7)}$$

Now consider the fields obtained by adjoint action with $Q_{+}$ and $\overline{Q}_{-}$:

$$\phi^{(j)}_{-m}(x) = \text{ad}_{Q_{+}}^{j-m} \left(\phi^{(j)}_{-j}(x)\right)$$

$$\overline{\phi}^{(j)}_{m}(x) = \text{ad}_{\overline{Q}_{-}}^{j-m} \left(\overline{\phi}^{(j)}_{j}(x)\right). \quad \text{(4.8)}$$

The adjoint action in (4.8) is defined in the quantum field theory as an integral of the current $J_{+}$ or $\overline{J}_{-}$ along a contour surrounding $x$, as in (2.23). These fields were considered by Gomez and Sierra in their study of the quantum group symmetry of minimal conformal models and shown to comprise spin-$j$ $\mathcal{U}_q(sl(2))$ multiplets. Related results can be found in [34]. We will establish these properties using other arguments.
It is not difficult to see from the results of section 2c that through the adjoint action (4.8) one indeed obtains a finite number of fields. Namely, from the braiding relations

\[
J_+(x) J_+(y) = e^{-4\pi i/\beta^2} J_+(y) J_+(x) \quad x < y
\]

(4.9)

and the general condition (2.41), one finds that contours defining \( \phi^{(j)}_{j+1} \) can be closed. Then using arguments as for establishing the Serre relations one finds

\[
\phi^{(j)}_{j+1}(x) = \overline{\phi}^{(j)}_{-j}(x) = 0.
\]

(4.10)

Thus the fields \( \phi^{(j)}_m, \overline{\phi}^{(j)}_m \), \(-j \leq m \leq j \) define \( 2j + 1 \) dimensional multiplets.

We now consider the transformation properties of the fields \( \phi^{(j)}_m, \overline{\phi}^{(j)}_m \) with respect to \( U_q(sl(2)) \). From the braiding relations

\[
J_+(x) \phi^{(j)}_m(y) = q^{-2m} \phi^{(j)}_m(y) J_+(x) \quad x < y,
\]

(4.11)

and (2.24) one finds

\[
\text{ad}_{Q_+} \left( \phi^{(j)}_m(x) \right) = Q_+ \phi^{(j)}_m(x) - q^{-2m} \phi^{(j)}_m(x) Q_+.
\]

(4.12)

Using the fact that the field \( \phi^{(j)}_m \) has topological charge \( T = -2m \), (4.12) can be expressed as

\[
\text{ad}_{Q_+} \left( \phi^{(j)}_m(x) \right) = Q_+ \phi^{(j)}_m(x) + q^T \phi^{(j)}_m(x) s(Q_+),
\]

(4.13)

where the antipode of \( Q_+ \) is given in (2.9): \( s(Q_+) = -q^{-T} Q_+ \).

The adjoint action expressed in the form (4.13) has some important properties which we now explain. Consider an arbitrary Hopf algebra \( \mathcal{A} \) with the properties (2.3). Let \( a \in \mathcal{A} \) and express its comultiplication as in (2.5). Define the adjoint action on a field or product of fields as

\[
\text{ad}_a(\Phi(x_1) \cdots \Phi(x_n)) = \sum_i a_i \Phi(x_1) \cdots \Phi(x_n) s(a_i).
\]

(4.14)

Then

1. Fields related through adjoint action form a representation of \( \mathcal{A} \). More precisely let \( \Phi_v(x) \) denote the set of fields so obtained, where \( v \) spans a vector space \( V \), and let \( \rho_V(a) \) denote a representation of \( \mathcal{A} \) on \( V \). Then

\[
\text{ad}_a(\Phi_v(x)) = \Phi_{\rho_V(a)v}(x).
\]

(4.15)
2.

\( \text{ad}_a(\Phi_{v_1}(x_1) \Phi_{v_2}(x_2)) = \sum_i \text{ad}_{a_i}(\Phi_{v_1}(x_1)) \text{ad}_{a_i}(\Phi_{v_2}(x_2)). \)  

(4.16)

These properties are a consequence of the Hopf algebra properties (2.3). The relations (4.13) (4.16) apply as well to spacetime independent operators such as conserved charges.

Thus the fields \( \phi_{m}^{(j)} \) indeed form \( 2j + 1 \) dimensional representations of \( U_q(sl(2)) \).

Similar results apply to the fields \( \overline{\phi}_{m}^{(j)} \). We will need the braiding relations of these fields. It follows from general principles that

\[
\phi_{m_1}^{(j_1)}(x_1) \phi_{m_2}^{(j_2)}(x_2) = \sum_{m_1', m_2'} (R^{j_1 j_2})_{m_1 m_2}^{m_1' m_2'} \phi_{m_1'}^{(j_1)}(x_1) \phi_{m_2'}^{(j_2)}(x_2) \quad x_1 < x_2
\]

\[
\overline{\phi}_{m_1}^{(j_1)}(x_1) \overline{\phi}_{m_2}^{(j_2)}(x_2) = \sum_{m_1', m_2'} (R^{j_1 j_2})_{m_1 m_2}^{m_1' m_2'} \overline{\phi}_{m_1'}^{(j_2)}(x_2) \overline{\phi}_{m_2'}^{(j_1)}(x_1) \quad x_1 < x_2,
\]

(4.17)

where \( R^{j_1 j_2} \) is the universal \( R \)-matrix \[48] for \( U_q(sl(2)) \) evaluated in the representations indicated, and \( \overline{R} = R^{-1} \). The universal \( R \)-matrix has the defining relation

\[
R \Delta(a) = \Delta'(a) R
\]

(4.18)

where \( \Delta' = P \Delta \), and \( P \) is the permutation operator. The relations (4.17) are easily established by applying \( \text{ad}_a \) to both sides and using (4.16) to show that \( R^{j_1 j_2} \) must satisfy its defining relations. Alternatively these braiding relations can be verified by explicit computation as was done in [11]. From (2.16) one also has

\[
\overline{\phi}_{m_1}^{(j_1)}(x_1) \phi_{m_2}^{(j_2)}(x_2) = \sum_{m_1', m_2'} (R^{j_1 j_2})_{m_1 m_2}^{m_1' m_2'} \phi_{m_1'}^{(j_2)}(x_2) \overline{\phi}_{m_2'}^{(j_1)}(x_1) \quad \forall \ x_1, x_2.
\]

(4.19)

We now apply these results to the RSG theory by first identifying which of the above fields are especially meaningful in our context. Since the charges \( Q_{\pm} \) and \( \overline{Q}_{\pm} \) are conserved in the SG theory, and we have the identifications \( J_- = \phi^{(1)}_{-1}, J_+ = \overline{\phi}^{(1)}_{1} \), the fields

\[
J_m(x) \equiv \phi_{m}^{(1)}(x) \quad \overline{J}_m(x) = \overline{\phi}_{m}^{(1)}(x)
\]

(4.20)

generate some non-local conserved currents and charges associated with the spin-1 representation of \( U_q(sl(2)) \). Namely,

\[
\partial_x J_m = \partial_x H_m \quad \partial_x \overline{J}_m = \partial_x \overline{H}_m \quad m = \pm 1, 0
\]

(4.21)

\[8\] The \( R \)-matrices, and also the q-Clebsch-Gordan coefficients and q-6j symbols below, can be computed from the results in [48]. Most of the relevant ones were evaluated explicitly in [8].
for some $\mathcal{H}_m, \overline{\mathcal{H}}_m$, and

\[
Q^{(1)}_m = \frac{1}{2\pi i} \left( \int dz J_m + \int dz \mathcal{H}_m \right)
\]
\[
\overline{Q}^{(1)}_m = \frac{1}{2\pi i} \left( \int dz \overline{J}_m + \int dz \overline{\mathcal{H}}_m \right)
\]

(4.22)

are conserved. The Lorentz spin of these charges follows from the dimension of the currents $J_-, \overline{J}_+$ (with respect to $T'$) and the fact that $Q_+, \overline{Q}_-$ have spin 0:

\[
\text{spin} \left( Q^{(1)}_m \right) = -\text{spin} \left( \overline{Q}^{(1)}_m \right) = 2/\gamma = 2/p,
\]

(4.23)

where $\gamma$ is defined in (2.21). Note that the spin is precisely doubled due to the inclusion of the background charge.

The SG soliton fields $\Psi_{\pm}(x), \overline{\Psi}_{\pm}(x)$ do not form spin-1/2 $U_q(sl(2))$ multiplets. Therefore we define new soliton fields:

\[
K_m(x) \equiv \phi^{(1/2)}_m(x), \quad \overline{K}_m(x) \equiv \overline{\phi}^{(1/2)}_m(x).
\]

(4.24)

These fields have topological charge $\pm 1$, however whereas $K_{-1/2} = \Psi_-$ and $\overline{K}_{1/2} = \overline{\Psi}_+$, $K_{1/2} \neq \Psi_+, \overline{K}_{-1/2} \neq \overline{\Psi}_-$. Recall that the fields $\Psi_+, \overline{\Psi}_-$ were not in the spectrum of primary fields of the minimal models; we have thus replaced them by more appropriate ones.

We define asymptotic states $|K_m(\theta)\rangle$ to be created by the fields $K_m(x)$ or $\overline{K}_m(x)$. They can be normalized such that

\[
\langle 0|K_m(x)|K_{-m}(\theta)\rangle = \langle 0|\overline{K}_m(x)|K_{-m}(\theta)\rangle = e^{-ip(\theta) \cdot x}.
\]

(4.25)

The S-matrix for the scattering of the states $|K_m(\theta)\rangle$ can be determined exactly by using its various non-local symmetries. We first determine the action of the charges $Q^{(1)}_m, \overline{Q}^{(1)}_m$ on the states. The essential features of this action are a simple consequence of the $U_q(sl(2))$ symmetry. Consider more generally a Hopf algebra $\mathcal{A}$. Let $V'$ and $V''$ denote representations of $\mathcal{A}$, and $\mathcal{O}$ an operator from $V'$ to $V''$: $\mathcal{O} \in \text{Hom}(V', V'')$. As usual $\mathcal{A}$ acts on $\mathcal{O}$ as

\[
\text{ad}_a(\mathcal{O}) = \sum_i a_i \mathcal{O} s(a^i)
\]

(4.26)

for $a \in \mathcal{A}$ and $\Delta(a)$ of the form (2.5). For all $v' \in V'$, one has

\[
a \mathcal{O} v' = \sum_i \text{ad}_{a_i}(\mathcal{O}) a^i v'.
\]

(4.27)
The above relation is easily proven using the properties (2.3). Namely, let \( \Delta(a_i) = \sum_j a_{ij} \otimes a_{ij} \) and \( \Delta(a^i) = \sum_j a^i_j \otimes a^i_j \). The coassociativity implies \( \sum_{i,j} a_{ij} \otimes a_{ij} \otimes a^i = \sum_{i,j} a_i \otimes a^i_j \otimes a^j \). Thus the RHS of (4.27) is

\[
\sum_i \text{ad}_{a_i} (\mathcal{O}) a^i v' = \sum_{i,j} a_{ij} \mathcal{O}(a^i_j) a^i v' = \sum_{i,j} a_i \mathcal{O}(a^i_j) a^i v'.
\]

(4.28)

Now using the fact that \( \sum_j s(a^i_j) a^i_j = \varepsilon(a^i) \) and \( \sum_i a_i \varepsilon(a^i) = a \), one has established (4.27).

Suppose that \( \mathcal{O} \) transforms in some representation \( V \) of \( \mathcal{A} \). Let \( V \) be spanned by vectors \( v \), and denote the operators \( \{ \mathcal{O} \} \) as \( \mathcal{O}_v \). The operators \( \mathcal{O}_v \) are maps \( V \rightarrow \text{Hom}(V', V'') \), and from the property analogous to (4.15) one has

\[
\text{ad}_a (\mathcal{O}_v) = \mathcal{O}_{av}.
\]

(4.29)

The map \( C \):

\[
C : \quad V \otimes V' \rightarrow V'' : v \otimes v' \rightarrow \mathcal{O}_v v'
\]

(4.30)

by (4.27) is a homomorphism of \( \mathcal{A} \)-modules. More specifically

\[
C \Delta(a) v \otimes v' = C a_i v \otimes a^i v' = \mathcal{O}_{av} a^i v' = a C v \otimes v'.
\]

(4.31)

Thus the map \( C \) is, by definition, a Clebsch-Gordan projector. In the \( sl(2) \) case it is unique up to proportionality. The results just described may be thought of as a generalization of the Wigner-Eckart theorem to an arbitrary Hopf algebra.

The above reasoning implies the following structure for matrix elements:

\[
\langle K_{\beta}(\theta) | Q_m^{(1)} | K_{\alpha}(\theta) \rangle = \hat{c} e^{2\theta/\gamma} \left[ \begin{array}{cc} 1 & 1/2 \\ m & \beta \\ \alpha \end{array} \right]_q
\]

\[
\langle K_{\beta}(\theta) | \overline{Q}_m^{(1)} | K_{\alpha}(\theta) \rangle = \hat{c} e^{-2\theta/\gamma} \left[ \begin{array}{cc} 1 & 1/2 \\ m & \beta \\ \alpha \end{array} \right]_q
\]

(4.32)

where \( \hat{c} \) is some constant, and the brackets refer to the \( U_q(sl(2)) \) Clebsch-Gordan coefficients. The factors \( e^{\pm 2\theta/\gamma} \) are a consequence of (4.23).
Consider now the action of the charges $Q^{(1)}_m, \overline{Q}^{(1)}_m$ on multi-soliton states. This action is defined by a comultiplication $\Delta$, which is determined from the braiding relations of the currents $J_m, \overline{J}_m$ with the fields $K_m$ or $\overline{K}_m$, as in (2.23). As explained in [27], the contours defining the action of charges on particle states can be closed if one considers the action of chiral charges on anti-chiral soliton fields, and vice-versa. From the braiding relations (4.17) (4.19) one therefore deduces

\begin{equation}
\Delta \left( Q^{(1)}_m \right) = Q^{(1)}_m \otimes 1 + \left( R^{1,1/2} \right)_m^n \otimes Q^{(1)}_n
\end{equation}

(4.33)

Alternatively, these relations can be derived purely quantum group theoretically. Since $Q^{(1)}_- = \text{ad}_{Q^+} Q^-$ is given in terms of generators of $U_q \left( \hat{sl}(2) \right)$, we can in principle compute the coproduct explicitly. The computation is simplified by the following observation [30]. It follows from the general form of the Hopf algebra (2.9) that $\Delta(Q^{(1)}_m)$ has the form (2.26). The matrix $\Theta \in \text{End} (V_1 \otimes V_{1/2})$ satisfies the intertwiner relation $\Theta \Delta = \Delta' \Theta$. It must therefore be proportional to the $R$-matrix $R^{1,1/2}$. The proportionality constant is determined by looking at the $n = m = 1$ matrix element.

Finally, the charges $Q^+, \overline{Q}^-, T$ also have a well defined action on states. Since the Lorentz spin of these charges is zero, and they generate the $U_q \left( sl(2) \right)$ algebra, one deduces that they have the following representation on states

\begin{equation}
Q^+ = c \sigma^+ q^{\sigma_3/2}, \quad \overline{Q}^- = c \sigma^- q^{\sigma_3/2}, \quad T = \sigma_3,
\end{equation}

(4.34)

and $\Delta$ as in (2.9).

The 2-particle to 2-particle S-matrix as usual is an operator $\hat{S}_{12} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ where $V_{1,2}$ are the 2-dimensional vector spaces spanned by $|K_m(\theta)\rangle$. It is required to be a solution to the symmetry equations

\begin{equation}
\hat{S}_{12}(\theta) \Delta_{12}(a) = \Delta_{21}(a) \hat{S}_{12}, \quad \text{for} \quad a = T, Q^+, \overline{Q}^-, Q^{(1)}_m, \overline{Q}^{(1)}_m.
\end{equation}

(4.35)

It is a remarkable fact that a solution to these symmetry equations can be obtained from the usual S-matrix of SG solitons by a simple change of gradation. This is ultimately the explanation for why it is possible to obtain the S-matrix for perturbations of the minimal models from the S-matrix of the SG theory. Let $\hat{S}^{\text{prin.}}$ be the minimal solution to the
equations (2.35), which is known to be the usual SG S-matrix. Define a new S-matrix \( \hat{S}_{\text{homo}} \).

\[
\sigma_{12}^{-1} \hat{S}^{\text{prin}}. \sigma_{12} = \hat{S}_{\text{homo}},
\]

(4.36)

where \( \sigma_{12} = \sigma \otimes \sigma \), and \( \sigma = e^{i \sigma_3 / 2 \gamma} \). The conjugation by \( \sigma \) has the effect of changing the representation of \( Q_+, \overline{Q}_- \) from the principal gradation (2.32) to the homogeneous one. Namely, let us refer to the representation in (2.32) as \( Q_+^{\text{prin}}, \overline{Q}_-^{\text{prin}} \), and the representation (4.34) as \( Q_+^{\text{homo}}, \overline{Q}_-^{\text{homo}} \). Then

\[
\sigma^{-1} Q_+^{\text{prin}}. \sigma = Q_+^{\text{homo}}, \quad \sigma^{-1} \overline{Q}_-^{\text{prin}}. \sigma = \overline{Q}_-^{\text{homo}}.
\]

(4.37)

In the present context the need to change gradation is simply a consequence of the background charge in the energy-momentum tensor, which alters the Lorentz spin of conserved charges. That \( \hat{S}_{\text{homo}} \) is a solution to the symmetry equations (4.35) was shown explicitly in [8].

We emphasize that the solution \( \hat{S}_{\text{homo}} \) to the equations (4.35) does not represent a physically meaningful S-matrix as it stands, in contrast to \( \hat{S}^{\text{prin}} \). Indeed, only \( \hat{S}^{\text{prin}} \) can be made crossing symmetric. The S-matrix \( \hat{S}_{\text{homo}} \) is to be viewed as an intermediate step in the construction of the S-matrix of the RSG theory, which we describe in the next section. The restriction of the SG theory relies on the \( U_q(sl(2)) \) invariance; to describe it one needs an S-matrix for \( U_q(sl(2)) \) covariant states and with covariant symmetries, which is what \( \hat{S}_{\text{homo}} \) represents. Indeed the construction in this section is very much in the spirit of the work of Moore and Reshetikhin on the \( U_q(sl(2)) \) symmetry of CFT [49].

4b. The Restriction

We will now use the results of the last subsection to provide a derivation of the RSG S-matrix based on its residual symmetries. The primary mathematical technique from the theory of quantum groups we will employ is the so-called vertex-RSOS correspondence [50], which finds its origin in classical lattice statistical mechanics [18] [19].

The restricted space of states is a subquotient of the space \( \mathcal{H} \) of states of the sine-Gordon model. The restriction consists of two steps: the first restriction is to solid-on-solid (SOS) states and the second to restricted SOS (RSOS) states. The SOS space is by definition the direct sum of sectors \( \mathcal{H}_j \) of \( U_q(sl(2)) \) singular vectors of topological charge \(-2j, (j = 0, 1/2, 1, \ldots)\),

\[
\mathcal{H}_j = \{ \psi \in \mathcal{H} : \overline{Q}_- \psi = 0, \quad T \psi = -2j \psi \}
\]

(4.38)
such that the representation of $\mathcal{U}_q(sl(2))$ generated by these singular vectors is irreducible of non-vanishing $q$-dimension. For the specific value of the SG coupling $\hat{\beta}$ (4.4),

$$q = -e^{-i\pi/p},$$  

(4.39)

the latter condition puts a limitation on the allowed values of the quantum spin $j$:

$$j \leq \frac{p}{2} - 1.$$  

(4.40)

This restriction of the allowed values of $j$ is implicit throughout this section.

It is convenient to represent the sector $\mathcal{H}_j$ as

$$\mathcal{H}_j = \text{Hom}_{\mathcal{U}_q(sl(2))}(V_j, \mathcal{H})$$  

(4.41)

where $V_j$ is the irreducible spin $j$ representation of $\mathcal{U}_q(sl(2))$. The isomorphism of (4.41) with (4.38) is the identification of a singular vector with the subrepresentation it generates.

It is clear from the representation (4.41) that we can also characterize the space in terms of highest weight singular vectors (i.e. vectors in the kernel of $Q_+$) of positive topological charge, rather than lowest weight singular vectors (4.38).

Of course $\mathcal{H}$ is highly reducible as $\mathcal{U}_q(sl(2))$-module and we can restrict our attention to the invariant subspace spanned by products of fundamental fields at given positions $x_1, \ldots, x_N$ (or rather smeared with a given test function) applied to the vacuum:

$$K_{\alpha_1}(x_1) \cdots K_{\alpha_N}(x_N)|0\rangle, \quad \alpha_i = \pm 1/2.$$  

(4.42)

The SOS states in the space (4.42) are therefore given by

$$\bigoplus_{j=0}^{p/2-1} \text{Hom}(V_j, V_{1/2}^\otimes N),$$  

(4.43)

where Hom in this section denotes the space of $\mathcal{U}_q(sl(2))$ module homomorphisms. The asymptotic states corresponding to (4.42) are

$$|K_{\alpha_1}(\theta_1) \cdots K_{\alpha_N}(\theta_N)\rangle$$  

(4.44)

and the asymptotic $N$-particle SOS states with given rapidities are also described by the space (4.43). Both classes of states (4.42) and (4.44) are supposed to form a dense subspace

9 This limitation on the allowed spins is a more restricted definition of SOS states than the usual one in statistical mechanics
of the SG Hilbert space. The space spanned by the vectors (4.44) with fixed rapidities
is invariant under the full \( U_q(\widehat{sl}(2)) \). In the sequel we fix \( \theta_1, \ldots, \theta_N \) and consider the
vector space spanned by the vectors (4.44).

The Shapovalov bilinear form on \( V_j \) with \( (v, v) = 1 \) for the highest weight vector \( v \) and
such that \( (E_1^\pm u, w) = (u, E_1^\mp w) \), \( u, w \in V_j \), induces a bilinear form on the space (4.43). The
RSOS restriction of an \( U_q(sl(2)) \) module \( M \) is the direct sum over \( j \in \{0, 1/2, 1, \ldots, p/2-1\} \)
of the quotient \( \text{Hom}'(V_j, M) \) of \( \text{Hom}(V_j, M) \) by the kernel of \( (, ) \). If \( M \) is a tensor product
of irreducible modules, it can be described explicitly as restricted path space (see below).

In going to the restricted model we have lost the \( U_q(sl(2)) \) symmetry, but it is to
be expected that there is still a residual symmetry coming from the rest of \( U_q(\widehat{sl}(2)) \). In
particular, we expect that the spin 1 multiplet \( Q_m^{(1)} \), when suitably projected onto RSOS states, provides conserved charges for the restricted model.

Before describing how this works, we have to make the notion of conserved charge in
a theory with superselection sectors \( H_j \) more precise. It is natural to define a conserved
charge as an operator \( Q_{ij} \) from \( H_j \) to \( H_i \) such that

\[ S_i Q_{ij} = Q_{ij} S_j. \]  \( (4.45) \)

Here \( S_i \) is the time evolution operator the \( S \) matrix in the sector \( H_i \). Conserved charges
can be multiplied only if the sectors match, so they do not build an algebra but rather the
space of morphism of a category whose objects are the sectors of the model.

Such charges \( Q_{ij} \) can be easily constructed starting from the multiplet \( Q_m^{(1)} \). Fix for
each pair of spin values \( i, j \) compatible with the fusion rules a \( U_q(sl(2)) \) Clebsch-Gordan operator

\[ C_{ij} \in \text{Hom}(V_i, V_1 \otimes V_j). \]  \( (4.46) \)

In our \( sl(2) \) example, this operator is unique up to proportionality. Denote by \( U_q(\widehat{sl}(2))^{ad} \)
the space \( U_q(\widehat{sl}(2)) \) with adjoint action of \( U_q(sl(2)) \). View \( Q_m^{(1)} \) as

\[ Q^{(1)} \in \text{Hom}(V_1, U_q(\widehat{sl}(2))^{ad}), \]  \( (4.47) \)

so that \( (Q^{(1)} \otimes 1)C_{ij} \in \text{Hom}(V_i, U_q(\widehat{sl}(2))^{ad} \otimes V_j) \) acts naturally as a linear operator

\[ Q_{ij} : \text{Hom}'(V_j, M) \to \text{Hom}'(V_i, M), \]  \( (4.48) \)

for any \( U_q(\widehat{sl}(2)) \)-module \( M \). For \( \psi \in \text{Hom}(V_j, M) \), \( Q_{ij} \psi \) is defined as the composition

\[ V_i \to U_q(\widehat{sl}(2)) \otimes V_j^{1 \otimes \psi} U_q(\widehat{sl}(2)) \otimes M \to M. \]  \( (4.49) \)
As $M$ we take $V_{1/2}^{\otimes N}$ or more generally $V_{1/2}^{\otimes N} \otimes V_{k}$. A parallel construction can be done for the operators $\Theta_{m}^{n}$, viewed as an element of $\text{End}_{C}V \otimes \mathcal{U}_{q}(\hat{sl}(2))$ with the intertwining property

$$\Theta \Delta(X) = \Delta'(X) \Theta, \quad X \in \mathcal{U}_{q}(sl(2)).$$

(4.50)

We get linear operators

$$\Theta_{ij}^{kl} : \text{Hom}'(V_{i}, V_{j}^{\otimes N} \otimes V_{k}) \rightarrow \text{Hom}'(V_{i}, V_{j}^{\otimes N} \otimes V_{k}).$$

(4.51)

The comultiplication rule (4.33) translates into an RSOS relation. For this we need the isomorphism [52]

$$\text{Hom}'(V_{j}, V_{1/2}^{\otimes N} \otimes V_{l}) \simeq \oplus_{k} \text{Hom}'(V_{j}, V_{1/2}^{\otimes N_{1}} \otimes V_{k}) \otimes \text{Hom}'(V_{k}, V_{1/2}^{\otimes N_{2}} \otimes V_{l}),$$

valid if $N_{1} + N_{2} = N$. On the $k$th summand in this decomposition we have

$$NQ_{ij} = N_{1}Q_{ij} \otimes N_{2} 1 + \sum_{l} N_{1} \Theta_{ij}^{lk} \otimes N_{2} Q_{lk},$$

(4.52)

(4.53)

where the dependence on the tensor power is displayed as a left subscript. Iterating the decomposition (4.52) we obtain the restricted path space decomposition

$$\text{Hom}'(V_{j}, V_{1/2}^{\otimes N} \otimes V_{l}) = \oplus_{\text{paths}} \otimes_{s=0}^{N-1} \text{Hom}(V_{j_{s+1}}, V_{1/2} \otimes V_{j_{s}}).$$

(4.54)

The direct sum is over all paths $j_{0}, \ldots, j_{s}$ with $|j_{s+1} - j_{s}| = 1/2$, $j_{0} = j$, $j_{N} = l$, and $j_{s} \leq p/2 - 1$.

A completely parallel construction can be done for the multiplet $Q^{(1)}$. In particular we have conserved charges $\overline{Q}_{ij}$ with RSOS “comultiplication”

$$N\overline{Q}_{ij} = N_{1}\overline{Q}_{ij} \otimes N_{2} 1 + \sum_{l} N_{1} \overline{\Theta}_{ij}^{lk} \otimes N_{2} \overline{Q}_{lk},$$

(4.55)

In an abstract setting, what we have done is the following: we have a Hopf algebra $A$ with a Hopf subalgebra $B$ and an $A$-module $M$. The restricted model is given by sectors $\mathcal{H}_{j} = \text{Hom}_{B}(V_{j}, M)$, for some class of $B$-modules $\{V_{j}\}$. One then considers the category whose objects are the modules $V_{i}$ and whose space of morphisms between $V_{i}$ and $V_{j}$ is $\text{Hom}_{B}(V_{i}, A^{ad} \otimes V_{j})$. This category acts by symmetries on the sectors in the sense that there is a contravariant functor to the category whose objects are the sectors and whose morphisms are linear maps commuting with the time evolution. In the above construction
we have specialized this general setting to the elements of $\text{Hom}_B(V_i, A^{ad} \otimes V_j)$ that come from a homomorphism $Q^{(1)}$ from a particular $B$-module to $A^{ad}$.

Let now see how the construction works in explicit terms. From the fields $K_\alpha(x)$ and $\mathcal{J}_m(x)$ in (1.24) and (1.21) let us define new fields $K_{j_1 j_2 j_1}(x), j_1 = j_2 \pm 1/2$ and $J_{j_1 j_2 j_1}(x), j_2 = \{j_1, j_1 \pm 1\}$, for $j_2, j_1 \in \{0, 1/2, 1, \ldots\}$ in the following way:

\[
K_{j_1 j_2 j_1}(x) K_{j_1 j_2 j_1}(x) \cdots K_{j_1 j_2 j_1}(x)|0\rangle \\
= \sum_{m_i} K_{j_1 j_2 j_1}(x)^{m_1} K_{j_1 j_2 j_1}(x)^{m_2} \cdots K_{j_1 j_2 j_1}(x)^{m_i}|0\rangle
\]

\[
J_{j_1 j_2 j_1}(x) J_{j_1 j_2 j_1}(x) \cdots J_{j_1 j_2 j_1}(x)|0\rangle \\
= \sum_{m_i} J_{j_1 j_2 j_1}(x)^{m_1} J_{j_1 j_2 j_1}(x)^{m_2} \cdots J_{j_1 j_2 j_1}(x)^{m_i}|0\rangle
\]

where $j_0 = m_0 = 0$,

\[
K_{j_1 j_2 j_1}(x) = \sum_{\alpha} \left[ \begin{array}{ccc} j_2 & 1/2 & j_1 \\ m_2 & \alpha & m_1 \end{array} \right]_q K_\alpha(x)
\]

\[
J_{j_1 j_2 j_1}(x) = \sum_{m} \left[ \begin{array}{ccc} j_2 & 1 & j_1 \\ m_2 & m & m_1 \end{array} \right]_q \mathcal{J}_m(x).
\]

The fields $K_{j_1 j_2 j_1}(x)$ and $J_{j_1 j_2 j_1}(x)$ are defined similarly from $K_\alpha(x)$ and $\mathcal{J}_m(x)$. More generally, one may define fields $\phi^{(j)}_m(x)$ and $\overline{\phi}^{(j)}_m(x)$ from $\phi^{(j)}_m, \overline{\phi}^{(j)}_m$.

The product of fields in (4.56) is characterized as being a lowest weight vector in a $U_q(sl(2))$ representation and gives an explicit basis of the path space (4.54). Note that $K_{1/2}(0)|0\rangle = \phi^{(1/2)}(-1/2)(0)|0\rangle$ and

\[
K_{j_1 j_2 j_1}(x) K_{1/2}(0)|0\rangle = \sum_{m} K_{j_1 j_2 j_1}(x)^{1/2} K_{1/2}(0) \phi_m^{(1/2)}(0)|0\rangle.
\]

The state $\phi^{(1/2)}(0)|0\rangle \equiv |1/2\rangle_{CFT}$ is a state of the CFT, namely the one corresponding to the field $\phi^{(2,1)}$. More generally define

\[
\phi^{(j)}_m(0)|0\rangle \equiv |j\rangle_{m}^{CFT}.
\]

The states $|j\rangle_{-j}^{CFT}$ are the primary states $|j \pm 1, 1\rangle = \phi^{(2j+1,1)}(0)|0\rangle$ of the minimal CFT. (These states are not to be confused with the massive particle-like asymptotic states of the perturbed theory.) The generalization of (1.58) is

\[
K_{j_1 j_2 j_1}(x)|j_1\rangle = \sum_{m_1} K_{j_1 j_2 j_1}(x)^{j_2} j_1)^{CFT}.
\]
The fields $K_{j_2j_1}(x)$ expressed in the form (1.60) are nothing other than the intertwiners (chiral vertex operators) for the minimal model field $\phi^{(2,1)}(x)$ [11 34]. Similar arguments apply to the fields $J_{j_2j_1}$. More precisely we have the identifications

$$K_{j_2j_1}(x) \sim \phi^{(2,1)}_{(2j_2+1,1)(2j_1+1,1)}(x)$$

$$J_{j_2j_1}(x) \sim \phi^{(3,1)}_{(2j_2+1,1)(2j_1+1,1)}(x)$$

where the fields on the RHS are the chiral vertices for $\phi^{(2,1)}, \phi^{(3,1)}$. The restriction of allowed spins (1.33) is of course consistent with the limitation of the $(n, m)$ indices of the primary fields $\phi^{(n,m)}$.

We will need the braiding relations for the fields in (1.56). By using (1.56), the braiding relations (1.17)-(1.19), and the identity [18]:

$$\sum_{m_2, \alpha_1, \alpha_2} \left[ \begin{array}{ccc} j_2 & j & j' \\ m_2 & m_1 & m_3 \\ \alpha_2 & \alpha_1 & \alpha' \end{array} \right]_q \left( R^{j'j} \right)^{\alpha_1 \alpha_2}_{\alpha_1 \alpha_2} q^{j(j+1)} = \sum_{j_4, m_4} \left( \Theta^{j'j} \right)^{j_4j_1}_{j_3j_2} \left[ \begin{array}{ccc} j_3 & j & j' \\ m_3 & m_4 & m_2 \\ \alpha_1 & \alpha_2 & \alpha' \end{array} \right]_q \left\{ j_4 \right\}^q$$

where $C_j = j(j + 1)$, and $\{*\}_q$ are q-6j symbols, one can show that the fields satisfy the non-abelian RSOS braiding relations

$$\phi^{(j')}_{j_3j_2}(x) \phi^{(j)}_{j_2j_1}(y) = \sum_{j_4} \left( \Theta^{j'j} \right)^{j_4j_1}_{j_3j_2} \phi^{(j)}_{j_3j_4}(y) \phi^{(j')}_{j_4j_1}(x) \quad x < y$$

$$\bar{\phi}^{(j')}_{j_3j_2}(x) \phi^{(j)}_{j_2j_1}(y) = \sum_{j_4} \left( \bar{\Theta}^{j'j} \right)^{j_4j_1}_{j_3j_2} \bar{\phi}^{(j)}_{j_3j_4}(y) \bar{\phi}^{(j')}_{j_4j_1}(x) \quad x < y$$

$$\bar{\phi}^{(j')}_{j_3j_2}(x) \phi^{(j)}_{j_2j_1}(y) = \sum_{j_4} \left( \Theta^{j'j} \right)^{j_4j_1}_{j_3j_2} \phi^{(j)}_{j_3j_4}(y) \bar{\phi}^{(j')}_{j_4j_1}(x) \quad \forall \ x, y$$

where

$$\left( \Theta^{j'j} \right)^{j_4j_1}_{j_3j_2} = (-)^{j_1+j_3+j_2-j_4} q^{C_{j_2}+C_{j_4}-C_{j_3}} \left\{ j' \right\}^q$$

The 1-1 q-6j symbols are presented in the appendix. Smirnov has also considered fields with such braiding relations in the present context [53].
Just as the asymptotic states $|K_{\alpha}(\theta)\rangle$ are created by the fields $K_{\alpha}(x)$, $\overline{K}_{\alpha}(x)$, we define states created by the fields $K_{j_{2}j_{1}}(x)$, $\overline{K}_{j_{2}j_{1}}(x)$:

$$|K_{j_{n}j_{n-1}}(\theta_{n})K_{j_{n-1}j_{n-2}}(\theta_{n-1})\cdots K_{j_{1}j_{0}}(\theta_{1})\rangle = \sum_{m_{i}}|K_{j_{n}j_{n-1}}(\theta_{n})\rangle_{m_{n}}^{j_{n}} |K_{j_{n-1}j_{n-2}}(\theta_{n-1})\rangle_{m_{n-1}}^{j_{n-1}} \cdots |K_{j_{1}j_{0}}(\theta_{1})\rangle_{m_{0}}^{j_{1}}$$

(4.66)

$(j_{0} = m_{0} = 0)$ where

$$|K_{j_{2}j_{1}}(\theta)\rangle_{m_{1}}^{j_{2}} = \sum_{\alpha} \left[ \begin{array}{c c c} j_{2} & 1/2 & j_{1} \\ m_{2} & \alpha & m_{1} \end{array} \right]_{q} |K_{\alpha}(\theta)\rangle.$$  

(4.67)

The fields $K_{j_{2}j_{1}}$, $\overline{K}_{j_{2}j_{1}}$ are not the unique fields that create the above states; the product of these fields with any field in the vacuum sector can also create them.

Since the currents $J_{m}, \overline{J}_{m}$ are conserved in the perturbed theory, so are $J_{j_{2}j_{1}}, \overline{J}_{j_{2}j_{1}}$, which define conserved charges $Q_{j_{2}j_{1}}, \overline{Q}_{j_{2}j_{1}}$, with Lorentz spin given in (4.23). These charges have expressions in terms of $Q_{m}^{(1)}$, $\overline{Q}_{m}^{(1)}$ obtained by integrating (4.57). Indeed one can show in an intrinsic minimal model description [8] (i.e. without reference to the Feigin-Fuchs construction) that the fields $\phi^{(3,1)}, \overline{\phi}^{(3,1)}$ define conserved currents for the action (1.2), i.e.

$$\partial_{\overline{\tau}}\phi^{(3,1)} = \lambda C \partial_{\overline{z}}\left(\phi^{(3,1)}\overline{\phi}^{(1,3)}\right),$$

(4.68)

where $C$ is a structure constant, which gives the conserved charges

$$Q = \int \frac{dz}{2\pi i} \phi^{(3,1)} + \int \frac{d\overline{z}}{2\pi i} \left(\lambda C \phi^{(3,3)}\overline{\phi}^{(1,3)}\right)$$

$$\overline{Q} = \int \frac{d\overline{z}}{2\pi i} \overline{\phi}^{(3,1)} + \int \frac{dz}{2\pi i} \left(\lambda C \overline{\phi}^{(3,3)}\phi^{(1,3)}\right).$$

(4.69)

The charges $Q_{j_{2}j_{1}}, \overline{Q}_{j_{2}j_{1}}$ represent the above charges in specific superselection sectors.

In the CFT the current $\phi^{(3,1)}$ plays a special role: for each $p$ it generates a fractional spin chiral algebra with a series of minimal unitary representations, where the usual $p-th$ minimal model is the lowest member of this series [34]. For recent results regarding this chiral algebra see [33].

We now derive the implications of the conserved charges $Q_{j_{2}j_{1}}, \overline{Q}_{j_{2}j_{1}}$ for the S-matrix of the states (4.66). Consider the action of the charges on a 1-kink state. One has

$$(Q_{j_{3}j_{2}}|K_{j_{2}j_{1}}(\theta)\rangle)^{m_{3}}_{m_{1}} = \sum_{m_{2}, m, \alpha} \left[ \begin{array}{c c c} j_{3} & 1 & j_{2} \\ m_{3} & m & m_{2} \end{array} \right]_{q} \left[ \begin{array}{c c c} j_{2} & 1/2 & j_{1} \\ m_{2} & \alpha & m_{1} \end{array} \right]_{q} Q_{m}^{(1)}|K_{\alpha}(\theta)\rangle.$$  

(4.70)
Using the matrix elements (4.32), and the identity \[48\]
\[
\sum_{m,m_2,\alpha} \left[ \begin{array}{ccc} j_3 & 1 & j_2 \\ m_3 & 1 & m_2 \end{array} \right]_q \left[ \begin{array}{ccc} j_2 & 1/2 & j_1 \\ m_2 & \alpha & m_1 \end{array} \right]_q \left[ \begin{array}{ccc} 1 & 1/2 & 1/2 \\ 1 & \beta & \alpha \end{array} \right]_q = \left\{ \begin{array}{ccc} j_3 & 1 & j_2 \\ 1/2 & j_1 & 1/2 \end{array} \right\} \left[ \begin{array}{ccc} j_3 & 1/2 & j_1 \\ m_3 & \beta & m_1 \end{array} \right]_q,
\]
\] one finds
\[
Q_{j_3j_2} | K_{j_2j_1}(\theta) \rangle = e^{2\theta/p} \left\{ \begin{array}{ccc} j_3 & 1 & j_2 \\ 1/2 & j_1 & 1/2 \end{array} \right\}_q | K_{j_3j_1}(\theta) \rangle. \tag{4.72}
\]
Similarly
\[
\overline{Q}_{j_3j_2} | K_{j_2j_1}(\theta) \rangle = e^{-2\theta/p} \left\{ \begin{array}{ccc} j_3 & 1 & j_2 \\ 1/2 & j_1 & 1/2 \end{array} \right\}_q | K_{j_3j_1}(\theta) \rangle. \tag{4.73}
\]

This formula is the expression in the path space basis of the action of \(Q_{j_3j_2}\) on \(\text{Hom}_{U_q(sl(2))}(V_{j_2}, V_{j_1/2} \otimes V_{j_1})\).

Using the comultiplication formula (4.53), and the path space decomposition (4.54), one can compute the action of \(Q\) on arbitrary path spaces. One has for instance
\[
Q_{j_3j_2} | K_{j_2j_1}(\theta_2)K_{j_1j_0}(\theta_1) \rangle = (Q_{j_3j_2} | K_{j_2j_1}(\theta_2) \rangle) | K_{j_1j_0}(\theta_1) \rangle + \sum_{j_4} \left( \Theta^{j_{4j_1}}_{j_3j_2} | K_{j_2j_1}(\theta_2) \rangle \right) (Q_{j_4j_1} | K_{j_1j_0}(\theta_1) \rangle), \tag{4.74}
\]
where
\[
\hat{\Theta}^{j_{4j_1}}_{j_3j_2} | K_{j_2j_1}(\theta) \rangle = \left( \Theta_{j_3j_2}^{1,1/2} \right)^{j_{4j_1}}_{j_3j_2} | K_{j_3j_4}(\theta) \rangle. \tag{4.75}
\]

The RSG S-matrix \(S_{j_2j_3}^{j_1j_0}(\theta)\) was conjectured to describe the 2-kink to 2-kink scattering process
\[
| K_{j_2j_1}(\theta_2)K_{j_1j_0}(\theta_1) \rangle \rightarrow | K_{j_2j_3}(\theta_1)K_{j_3j_0}(\theta_2) \rangle \tag{4.76}
\]
\((j_0\) is no longer necessarily 0.) As usual this S-matrix is required to commute with the conserved charges \(Q_{j_2j_1}, \overline{Q}_{j_2j_1}\), whose action is given above. Starting from the conjectured RSG S-matrix, it was shown in \([8]\) that this S-matrix does indeed possess these on-shell symmetries. Our analysis thus provides a derivation of the RSG S-matrix if it happens to be the minimal \(^{10}\) solution to the symmetry equations. This is likely to be the case

\(^{10}\) The solutions to the symmetry equations can only be unique up to overall scalar functions of rapidity. For minimal solutions, these overall factors are the minimal ones that are required for crossing symmetry and unitarity.
since the symmetries we have constructed are inherited from the \( U_q \left( \widehat{\mathfrak{sl}(2)} \right) \) symmetry of the SG theory, and this latter symmetry does provide a unique minimal solution. For completeness we present the solution

\[
S_{j_1 j_2 j_3}^{j_1 j_0} (\theta) = \frac{u(\theta)}{2\pi i} \left( \frac{[2j_1 + 1]_q [2j_3 + 1]_q}{[2j_2 + 1]_q [2j_0 + 1]_q} \right)^{-\theta/2\pi i} \times \left\{ \sinh(\theta/p) \delta_{j_2 j_0} \left[ \frac{[2j_1 + 1]_q [2j_3 + 1]_q}{[2j_2 + 1]_q [2j_0 + 1]_q} \right]^{1/2} + \sinh \left( \frac{i\pi - \theta}{p} \right) \delta_{j_1 j_3} \right\},
\]

where \( u(\theta) \) is a scalar function defined in (8).

4c. The Restricted Quantum Affine Symmetry

As explained in the last sub-section, after restriction the remnant of the quantum affine symmetry is the conserved charges \( Q, \overline{Q} \) defined in (1.69). Evidently, these charges no longer satisfy the \( U_q \left( \widehat{\mathfrak{sl}(2)} \right) \) relations. The residual symmetries obey new relations which we now characterize.

The relations satisfied by the charges should be characterized by braided commutators, as explained for the \( U_q \left( \widehat{\mathfrak{sl}(2)} \right) \) case in section 2b. From the braiding relations (1.64), one finds the RSOS analog of (2.23), (2.24):

\[
\text{ad}_{Q_{j_3 j_2}} \left( \phi_{j_2 j_1}^{(j)} (y) \right) = Q_{j_3 j_2} \phi_{j_2 j_1}^{(j)} (y) - \sum_{j_4} \left( \Theta^{1j}_{j_3 j_2} \phi_{j_4 j_1}^{(j)} (y) \right) Q_{j_4 j_1},
\]

(4.78a)

\[
\text{ad}_{Q_{j_3 j_2}} \left( \overline{\phi}_{j_2 j_1}^{(j)} (y) \right) = Q_{j_3 j_2} \overline{\phi}_{j_2 j_1}^{(j)} (y) - \sum_{j_4} \left( \overline{\Theta}^{1j}_{j_3 j_2} \overline{\phi}_{j_4 j_1}^{(j)} (y) \right) Q_{j_4 j_1},
\]

(4.78b)

and similarly for \( \text{ad}_{Q} \left( \phi^{(j)} (y) \right) \) and \( \text{ad}_{Q} \left( \phi^{(j)} (y) \right) \). We will also need the adjoint action on a product of fields, which is given by the comultiplication in (1.53), (1.55):

\[
\text{ad}_{Q_{j_3 j_2}} \left( \phi_{j_2 j_1}^{(j)} (x) \phi_{j_1 j_0}^{(j')} (y) \right) = \text{ad}_{Q_{j_3 j_2}} \left( \phi_{j_2 j_1}^{(j)} (x) \right) \phi_{j_1 j_0}^{(j')} (y) + \sum_{j_4} \hat{\Theta}_{j_3 j_2}^{j_4 j_1} \left( \phi_{j_2 j_1}^{(j)} (x) \right) \text{ad}_{Q_{j_4 j_1}} \left( \phi_{j_1 j_0}^{(j')} (y) \right),
\]

(4.79)

where

\[
\hat{\Theta}_{j_3 j_2}^{j_4 j_1} \left( \phi_{j_2 j_1}^{(j)} (x) \right) = \left( \Theta^{1j}_{j_3 j_2} \phi_{j_3 j_4}^{(j)} (x) \right).
\]

(4.80)
We first consider the analog of the $U_q\left(\widehat{sl}(2)\right)$ relations (2.10). As shown in [27], one can always close the contour in the adjoint action of chiral on anti-chiral conserved charges; using this result one can show that

$$\text{ad}_{Q^j_{3j_2}}(Q^j_{3j_1}) = Q^j_{3j_2} Q^j_{3j_1} - \sum_{j_4} (\mathcal{H}^{11})^{j_4j_1}_{j_3j_2} Q^j_{3j_4} Q^j_{3j_1} = \tilde{T}^{j_3j_1}_{j_3j_1} \quad (4.81)$$

where $\tilde{T}^{j_3j_1}_{j_3j_1}$ is an intertwiner for the topological charge of the conserved current $\lambda C e^{\mu\nu} \partial \Phi^{(3,3)}(x)$:

$$\tilde{T} = \frac{\lambda C}{2\pi i} \int dx \partial_x \Phi^{(3,3)}(x). \quad (4.82)$$

A more interesting question is whether there are additional relations of the Serre type. The way to answer this question was outlined in section 2c. Namely consider $\text{ad}_{Q^j_{3j_2}}(J^\mu(y))$, where $J^\mu(y) = (\phi^{(3,1)}, \lambda C\phi^{(3,3)}\phi^{(1,3)})$ is the non-local current for the charge $Q$. Displaying sectors, one considers

$$\text{ad}_{Q^j_{n+1}j_n} \cdots \text{ad}_{Q^j_{3j_2}} \left(\text{ad}_{Q^j_{3j_1}}(J^1_{j_1j_0}(y))\right). \quad (4.83)$$

In the abelian case the condition for closure of the contours was (2.41). Using similar arguments, it is not difficult to see that the condition for closure can now be formulated as follows. For the remainder of this section let $\Theta^{j_4j_1}_{j_3j_2} \equiv (\mathcal{H}^{11})^{j_4j_1}_{j_3j_2}$. Let $v(j_n, \ldots, j_1)$, where $j_{n+1} = j_n, j_n \pm 1$, span a vector space. Define a matrix $M(j_{n+1}, j_0)$ which acts on this vector space, with the matrix elements:

$$\left(M(j_{n+1}, j_0)\right)^{j_n', \ldots, j_1'}_{j_n, \ldots, j_1} = \sum_k \Theta^{j_n'_{n+1}, j_n}_{j_n+1, j_n} \cdots \Theta^{j_4'j_3, j_4}_{j_3, j_2} \Theta^{j_2'j_1, j_2}_{j_2, j_1} \Theta^{k, j_0}_{j_2j_1}. \quad (4.84)$$

The spins $j_{n+1}, j_0$ correspond to the initial and final sectors. $M(j_{n+1}, j_0)$ represents the product of braiding factors which is the non-abelian generalization of the first summand of the term in braces in (2.40), and is represented graphically in figure 3.
Fig. 3. The product of braiding matrices representing the matrix $M_{(jn+1,j_0)}$.

The braiding matrices $\Theta^{j_3j_0}_{j_2j_1}$ in the definition of $M$ are of course subject to consistency with the fusion rules, i.e. $j_{i+1}$ must appear in the fusion $j_i \times 1$, for $i = 0, 1, 2, 3; j_4 \equiv j_0$.

Let $\sum_{j_1,\ldots,j_n} c^{j_n\cdots j_1} v(j_n, \ldots, j_1)$ be an eigenvector of $M$ with eigenvalue 1. Then the contours may be closed in the expression

$$\sum_{j_1,\ldots,j_n} c^{j_n\cdots j_1} \text{ad}_{Q_{j_{n+1}j_n}} \cdots \text{ad}_{Q_{j_3j_2}} (\text{ad}_{Q_{j_2j_1}} (J_{j_1j_0}(y)))$$

(4.85)

to give a well-defined operator.

The above closure condition is difficult to implement in practice, so we only present a few interesting examples. It is clear from the explicit values of the braiding matrices $\Theta$ that the closure condition is not fulfilled for a fixed $n$ for all $p$. The simplest example occurs at $p = 4$. The relevant fusion rules are $0 \times 1 = 1, 1 \times 1 = 0, 1/2 \times 1 = 1/2$. Using the q-6j symbols in the appendix one finds for $q = -\exp(-i\pi/4)$ that

$$\Theta_{01}^{10} = \Theta_{10}^{01} = \Theta_{1/2}^{1/2} = -1, \quad (p = 4),$$

(4.86)

thus the braiding is effectively abelian. Using the fusion rules one finds three eigenvectors of the matrix $M$ for $n = 1$: $v(0, 1, 0), v(1, 0, 1), v(1/2, 1/2, 1/2)$, and by (4.86) these have eigenvalue 1. Thus the contour in (4.85) can be closed for $n = 1$. Since $Q$ and $J_{j_1j_0}(y)$ have scaling dimension $1/2$ and $3/2$ respectively, and $1 \times 1 = 0$, $\text{ad}_Q (J)$ must be a dimension 2 operator in the identity sector. Thus we conclude that

$$\text{ad}_{Q_{j_2j_1}} (J_{j_1j_0}(y)) = T_{zz}(y), \quad \text{ad}_{Q_{j_2j_1}} (\overline{J}_{j_1j_0}(y)) = T_{zz} \overline{y}$$

(4.87)

$$\Rightarrow Q^2 = P, \quad \overline{Q}^2 = \overline{P},$$

(4.88)

where $P, \overline{P}$ are the light-cone momentum operators. This resulting supersymmetry is not unexpected since the $p = 4$ minimal CFT is known to be supersymmetric; the $\Phi^{(1, 3)}$ perturbation preserves the supersymmetry, as first noticed by A. Zamolodchikov [21]. One can check that the on-shell representation (4.72) indeed satisfies (4.88), since $P = e^\theta, \overline{P} = e^{-\theta}$.

A more non-trivial example occurs at $p = 6$. Now the fusion rules with spin-1 are $0 \times 1 = 1, 1 \times 1 = 0 + 1 + 2, 2 \times 1 = 1$. Using these we found three eigenvectors of
M for \( n = 2 \) that are spanned by single vectors: \( v(0,1,1,0), v(2,1,1,2), \) and \( v(2,1,1,0) \). Moreover, for \( q = -\exp(-i\pi/6) \) these eigenvectors have eigenvalue 1:

\[
\Theta_{21}^{11} \Theta_{11}^{10} \Theta_{11}^{10} = \Theta_{01}^{11} \Theta_{11}^{10} \Theta_{11}^{10} = \Theta_{11}^{21} \Theta_{11}^{12} \Theta_{11}^{21} = 1.
\] (4.89)

Scaling arguments indicate that the operator in (4.85) for \( n = 2, p = 6 \) is again of dimension 2. Comparing the initial and final sectors for the eigenvectors \( v(0,1,1,0) \) and \( v(2,1,1,2) \), one concludes from the fusion rules that the operator in (4.85) for these two cases is necessarily proportional to the energy-momentum tensor. Thus

\[
ad_{Q_{01}} (ad_{Q_{11}} (Q_{10})) = ad_{Q_{21}} (ad_{Q_{11}} (Q_{12})) = \tilde{c} P,
\]

(4.90)

for some constant \( \tilde{c} \), and similarly for \( ad_{Q_{21}}^2 (Q) \). The above relations may be written out explicitly using the integrated versions of (4.78) and (4.79). We illustrate the first relation in (4.90):

\[
\sum_{j,k,l} Q_{01} Q_{11} Q_{10} - \Theta_{11}^{j0} Q_{01} Q_{1j} Q_{j0} - \Theta_{01}^{k1} \Theta_{k1}^{l0} Q_{0k} Q_{kl} Q_{l0} + \Theta_{11}^{j0} \Theta_{01}^{k1} \Theta_{kj}^{l0} Q_{0k} Q_{kl} Q_{l0} = \tilde{c} P.
\]

(4.91)

4d. The Ultraviolet Limit of the Restricted Model

The ultraviolet limit of the sine Gordon model for \( \hat{\beta} < \sqrt{2} \) is given by a free boson field. Indeed the cosine perturbation is relevant in this range and is therefore invisible at short distances. The scaling dimensions of the exponentials are modified by the additional term in the energy-momentum tensor (4.1) of the restricted SG model, in such a way that one of the exponential becomes marginal. The ultraviolet limit is then the Liouville model with imaginary coupling

\[
S = \frac{1}{4\pi} \int d^2 z \partial \Phi \partial \Phi + \frac{\lambda}{\pi} \int d^2 z : \exp \left( i\hat{\beta} \Phi \right) :
\]

(4.92)

This theory is known to be conformally invariant. For the discrete series of values of the coupling (4.4) it is believed to be equivalent to the unitary minimal models when suitably restricted [56] [57] [58].

Let us compare the restriction defined in this paper with the restriction appearing in the free field representation of minimal models. We discuss this for holomorphic chiral fields. A similar discussion applies to antiholomorphic fields. The primary fields \( \phi^{(2j+1,1)} \) of the CFT are identified with the operators (4.8). Note that since \( V_{2j+1,1} \) is local with
respect to the Liouville perturbation and has regular operator product expansion with it, the corresponding primary fields is holomorphic. However, the screening operator, which is identified with the current \( \exp(-2i\phi/\hat{\beta}) \), has a double pole singularity in its operator product with the perturbation. Hence it is conserved but not holomorphic. Thus this construction gives a slightly different but equivalent description of minimal models.

Correlation functions of fields (4.8) in the conformal limit form a vector space isomorphic to a tensor product of \( U_q(sl(2)) \) representations. They do not in general obey the conformal Ward identities. The violation is due to boundary terms in the integration over the position of the screening operators. It was remarked in [59][41][34] that the linear combinations of such correlators that obey the conformal Ward identities are in fact identified with \( U_q(sl(2)) \) singular vectors in the tensor product. Indeed, the step operator is identified topologically with a boundary operator acting on relative homology groups [34], so that absolute cycles are in its kernel. This gives part of the SOS restriction, namely the restriction to singular vectors.

The rest of the RSOS restriction in the UV limit is understood in terms of the Fock space cohomology defined in [47]. The unitary truncation of the Fock space of (topological) charge \( 1 - m \) is the cohomology group \( \text{Ker} \ Q^m_+ / \text{Im} \ Q^p-m_+ \). It was noted by Pasquier and Saleur [51] that this cohomology on the space of singular vectors in the tensor product of spin 1/2 representations gives precisely the restricted path space described above. In other words, we see that the discrete series of restricted SG models are given in their UV limit by minimal models in their free field representation. The restriction in the UV limit consists of two parts: the topological part ensures the validity of conformal Ward identities. The RSOS restriction is equivalent to the Fock space cohomology, which selects irreducible unitary Virasoro representations in the Fock space.

4e. Remarks on the Degenerate Vacuum Structure

For \( p = 4, 6 \) A. Zamolodchikov has argued from a microscopic analysis that the \( \Phi^{(1,3)} \) perturbation of the minimal conformal models has 3 and 5 degenerate vacua respectively, and interpreted the kinks \( |K_{j_2 j_1}(\theta)\rangle \) as interpolating the two vacua labeled \( j_1 \) and \( j_2 \). The 3-fold vacuum degeneracy was checked numerically for \( p = 4 \) in [10], by using the truncated conformal space approach. The qualitative aspects of this picture was extended to all \( p \) in [8] by examining the topological charge of the vertex operators \( V_{nm} \). Namely, since the topological charge of the operators \( V_{n1} \) is \( 1 - n \), and \( 1 \leq n \leq p - 1 \), the minimum topological charge is \( 2 - p \). On the other hand classical field configurations of topological charge \( n \)
satisfy $\hat{\beta} (\Phi(\infty) - \Phi(-\infty)) = 2\pi n$, and thus connect the wells in the $\cos(\hat{\beta} \Phi)$ SG potential. Thus the wells are effectively $p - 1$ in number. The problem with the latter argument is that it does not allow an intrinsic description of these degenerate vacua in terms of the usual properties of the minimal conformal models. Recently, based on the thermodynamic Bethe-ansatz equations of Al. Zamolodchikov, Klassen and Melzer conjectured that the lowest excited states in finite volume are the local states associated with $\Phi^{(n,n)}(0)|0\rangle$. In infinite volume these states become degenerate with the vacuum, and there are precisely $p - 1$ in number. The 1-kink states we have described in the previous section are associated with the chiral CFT states $\phi^{(2,1)}(0)|0\rangle$, and these are to be interpreted as the first excited states above the vacua in infinite volume.

We now offer some justification for the above identification of the vacua in the approach we are developing. To do this one must identify a suitable order parameter. For the SG theory, one may take the SG field $\Phi$ itself. The infinitely many degenerate vacua of the $\cos(\hat{\beta} \Phi)$ potential occur at $\Phi = 2\pi n/\hat{\beta}, n \in \mathbb{Z}$. However a different choice of order parameter is more suitable for generalization. In [27] it was shown that the q-affine charges satisfy

$$Q_\pm Q_\mp = q^{-2} Q_\mp Q_\pm = \frac{\lambda}{2\pi i} \left( \frac{\hat{\beta}^2}{\hat{\beta}^2 - 2} \right)^2 \int dx \partial_x \exp \left( \pm i \left( \frac{2}{\hat{\beta}} - \hat{\beta} \right) \Phi(x,t) \right). \quad (4.93)$$

The RHS of the above equation may be considered as a generalized topological charge (which is actually a function of the usual topological charge). So let us define the order parameters of the SG theory as $O_{\pm}^{SG} = \exp \left( \pm i \left( \frac{2}{\hat{\beta}} - \hat{\beta} \right) \Phi(x) \right)$. The fields $O_{\pm}^{SG}$ are local scalar fields which define topological currents $\tilde{J}_\mu^\pm(x) = \varepsilon^{\mu\nu} \partial_\nu O_\pm^{SG}(x)$. This choice of order parameter is in a sense more fundamental than $\Phi$ since its topological charge appears in the symmetry algebra. Thus, if the symmetry algebra has a well-defined representation on the states, this implies that $O_{\pm}^{SG}(\infty) - O_{\pm}^{SG}(-\infty)$ is well-defined on these states. If we further suppose that the topological particle spectrum has the ‘kink’ property of connecting degenerate vacua, then $O_{\pm}^{SG}$ must probe these vacua. At the location of the degenerate vacua $\Phi = 2\pi n/\hat{\beta}$, the order parameters take the values $O_{\pm}^{SG} = \exp(\pm 4\pi in/\hat{\beta}^2)$. For irrational values of the coupling constant $\hat{\beta}$, all of the vacua are distinguished by this order parameter.

Let $Q, \overline{Q}$ be the residual quantum affine charges in (4.69). For the perturbed minimal CFT the relation (4.93) becomes (4.81). This suggests that an appropriate order parameter for the RSG theory is $\Phi^{(3,3)}(x)$. Recall that $\Phi^{(3,3)} = (\Phi^{(2,2)})^2$, thus $\Phi^{(2,2)}$ is superior
as an order parameter, since it is more refined. In the Landau-Ginzburg description of the p-th minimal model, the Landau-Ginzburg scalar field was identified precisely with $\Phi^{(2,2)}$, and the Landau-Ginzburg potential is of degree $2(p - 1)$ and can support $p - 1$ degenerate vacua. Thus our identification of the order parameter is entirely consistent with the Landau-Ginzburg picture. Considering the Feigin-Fuchs representation of the field $\Phi^{(2,2)}$, one sees that it takes the values $(1, -q, (-q)^2, \ldots)$ at the wells of the SG potential. Since $:(\Phi^{(2,2)})^k := \Phi^{(1+k,1+k)}$, one can thus associate the degenerate vacua with the states $\Phi^{(n,n)}(0)|0\rangle$. However the precise connection between the minima of the SG cosine potential and the minima of the Landau-Ginzburg potential has not been clarified and is an interesting problem.

Note that in finite volume (on the cylinder) one should not expect that the degeneracy of the classical vacua of the LG potential corresponds to degenerate ground states of the quantum theory. Indeed tunneling between the minima of the potentials induces a splitting of the energy levels, and this explains why the states $\Phi^{(n,n)}(0)|0\rangle$ have non-degenerate energies in the CFT. The true ground state is described by a wave function on constant field configurations without nodes. The first excited states are then described by wave functions with nodes, and are obtained from the ground state by applying suitable polynomials in the order parameter at imaginary time $-\infty$. These polynomials are the normal ordered powers of the order parameter $\Phi^{(1+k,1+k)}(0)$ in the conformal limit, and the coefficients of these polynomials are precisely chosen so that the corresponding states are eigenvectors of the hamiltonian. In the infinite volume limit tunneling is suppressed, the splitting disappears and in the low temperature phase degenerate vacua appear.

5. Conclusions

The techniques we have described in this work should extend to the other models listed in the introduction, though the details have not been worked out. In particular, the formulation of the fractional supersymmetric sine-Gordon models and their restrictions to the perturbed cosets $su(2)_k \otimes su(2)_l/su(2)_{k+l}$ lends itself straightforwardly to the above treatment. Again, our characterization of superselection sectors provides a qualitative understanding of the spectrum. For example, for the current-current perturbation of the Wess-Zumino-Witten theories at level-$k$, the massive kink spectrum $K^{\pm}_{j_2,j_1}$ is created by the chiral intertwiners $\phi^{(1/2)}_{j_2,j_1}$ for the spin-1/2 primary field of the CFT.
The new frontier of this subject is the determination of the off-shell properties of the models, namely the form factors and correlation functions, from symmetry principles. Since the quantum affine symmetries and their restrictions are large enough to characterize the S-matrices, they are likely to characterize off-shell information as well.

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Appendix A. Some q-6j symbols.

Here we present explicit expressions for spin 1-1 q-6j symbols. The other q-6j symbols that are needed above can be found in [8]. Below, $[n]$ denotes $[n]_q$, defined in (2.2).

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j \\
1 & j & j \\
\end{array}
\end{align*}_q = \frac{[2j]}{[2j+2]} ([2j+3][2j-1] - 1)
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j+1 & j+1 \\
1 & j & j+1 \\
\end{array}
\end{align*}_q = \frac{1}{[1]}
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j+1 & j+1 \\
\end{array}
\end{align*}_q = -\frac{[2]}{[2j+2]}
\]  

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j & j \\
\end{array}
\end{align*}_q = \frac{[2]}{[2j+2]} \sqrt{\frac{[2j+3]}{[2j+1]}}
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j-1 \\
1 & j & j \\
\end{array}
\end{align*}_q = -\frac{[2]}{[2j]} \sqrt{\frac{[2j-1]}{[2j+1]}}
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j & j+1 \\
\end{array}
\end{align*}_q = \frac{[2]}{[2j+2][2j+1]}
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j+2 & j+1 \\
\end{array}
\end{align*}_q = 1
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j+1 & j \\
\end{array}
\end{align*}_q = \frac{[1]}{[1]}
\begin{align*}
\begin{array}{ccc}
1 & j+1 & j \\
1 & j & j+1 \\
\end{array}
\end{align*}_q = \frac{[1]}{[1]}
\begin{align*}
\begin{array}{ccc}
1 & j+1 & j \\
1 & j & j+1 \\
\end{array}
\end{align*}_q
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j & j+1 \\
1 & j+1 & j \\
\end{array}
\end{align*}_q = \frac{[1]}{[1]}
\begin{align*}
\begin{array}{ccc}
1 & j+1 & j \\
1 & j+1 & j+1 \\
\end{array}
\end{align*}_q
\]

\[
\begin{align*}
\begin{array}{ccc}
1 & j+\frac{3}{2} & j+\frac{3}{2} \\
1 & j+\frac{1}{2} & j+\frac{3}{2} \\
\end{array}
\end{align*}_q = \frac{1}{[2j+2] \sqrt{[2j][2j+4]}}
\]

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