Regularity of symbolic and bracket powers of Borel type ideals

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Abstract

In this paper, we show that the regularity of the $q$-th symbolic power $I^{(q)}$ and the regularity of the $q$-th bracket power $I^{[q]}$ of a monomial ideal of Borel type $I$, satisfy the relations $\text{reg}(I^{(q)}) \leq q \text{ reg}(I)$, respectively $\text{reg}(I^{[q]}) \geq q \text{ reg}(I)$. Also, we give an upper bound for $\text{reg}(I^{[q]})$.

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Introduction

Let $K$ be an infinite field, and let $S = K[x_1, \ldots, x_n], n \geq 2$ the polynomial ring over $K$. Bayer and Stillman [1] note that Borel fixed ideals $I \subset S$ satisfy the following property:

\[(*) \quad (I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty) \text{ for all } j = 1, \ldots, n.\]

Herzog, Popescu and Vladoiu [7] define a monomial ideal $I$ to be of Borel type if it satisfies $(*)$. We mention that this concept appears in [3, Definition 1.3] as the so called weakly stable ideal. Also, this concept appears in [2, Definition 3.1], as the so called monomial ideal of nested type. We further studied this class of monomial ideals in [4] and [5].

In the first section, we recall some results regarding ideals of Borel type. Also, we discuss the relation between the sequential chain of an ideal of Borel type $I$, defined in [7], and the primary decomposition of $I$. In the second section, we prove that if $I$ is an ideal of Borel type, then $I^{(q)}$ and $I^{[q]}$ are also ideals of Borel type, where $I^{(q)}$ is the $q$-th symbolic power of $I$ and $I^{[q]}$ is the $q$-th bracket power of $I$. In [5], we proved that $\text{reg}(I^{(q)}) \leq q \text{ reg}(I)$. We give a similar result for the $q$-th symbolic power. More precisely, we prove that $\text{reg}(I^{(q)}) \leq q \text{ reg}(I)$, see Theorem 2.3. Also, we prove that $\text{reg}(I^{[q]}) \geq q \text{ reg}(I)$, see Theorem 2.5. In Proposition 2.10, we prove that $\text{reg}(I^{[q]}) \leq q \text{ reg}(I) + (q - 1)(n - 1)$.

1 Some basic facts on Borel type ideals.

Firstly, we recall the following equivalent characterizations of ideals of Borel type given in [7] and in [2].

Proposition 1.1. Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

(a) $I$ is an ideal of Borel type.
(b) For any $1 \leq j < i \leq n$, we have $(I : x_i^\infty) \subset (I : x_j^\infty)$.
(c) Each $P \in \text{Ass}(S/I)$ has the form $P = (x_1, \ldots, x_m)$ for some $1 \leq m \leq n$. 

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Let $I \subset S$ be a monomial ideal of Borel type. Since each prime ideal $P \in \text{Ass}(S/I)$ is of the form $P = (x_1, \ldots, x_m)$ for some $1 \leq m \leq n$, we can assume that $I$ has an irredundant primary decomposition:

$$I = \bigcap_{i=1}^{r} Q_i; \text{ such that } P_i := \sqrt{Q_i} = (x_1, \ldots, x_{n_i-1}), \ n \geq n_0 > n_1 > \cdots > n_{r-1} \geq 1. \ (1)$$

For each $0 \leq i \leq r - 1$, we define $I_i := \bigcap_{j=i+1}^{r} Q_j$. We claim that $I_{i+1} = (I_i : x_{n_i}^{\infty})$ for all $0 \leq i \leq r - 1$. Indeed, since $Q_{i+1}$ is $P_{i+1}$-primary, it follows that there exists a positive integer $k$ such that $x_{n_i}^k \in Q_{i+1}$. So $(I_i : x_{n_i}^{\infty}) \supseteq ((Q_{i+1} : I_{i+1}) : x_{n_i}^{\infty}) \supseteq (x_{n_i}^k : I_{i+1} : x_{n_i}^{\infty}) = I_{i+1}$.

For the converse inclusion, note that $(I_i : x_{n_i}^{\infty}) = (Q_{i+1} : I_{i+1} : x_{n_i}^{\infty})$.

Thus, the chain of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r := S$ is the sequential chain of $I$, as it was defined in [7]. Note that $n_i = \max\{j : x_j | u \text{ for some } u \in G(I_i)\}$, where we denoted by $G(I_i)$ the set of minimal monomial generators of $I_i$.

Let $J_i$ be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \ldots, x_n]$, $0 \leq i \leq r$. Then, the saturation $J_i^{sat} = (J_i : m_i^{\infty})$ is generated by the elements of $G(I_{i+1})$, where $m_i = (x_1, \ldots, x_n)S_i$. It follows that $I_{i+1}/I_i \cong (J_i^{sat}/J_i)[x_{n_i+1}, \ldots, x_n]$.

It would be appropriate to recall the definition of the Castelnuovo-Mumford regularity.

**Definition 1.2.** Let $K$ be an infinite field, and let $S = K[x_1, \ldots, x_n]$, $n \geq 2$ the polynomial ring over $K$. Let $M$ be a finitely generated graded $S$-module. The Castelnuovo-Mumford regularity $\text{reg}(M)$ of $M$ is

$$\max_{i,j}\{j - i : \beta_{ij}(M) \neq 0\},$$

where $\beta_{ij}(M) = \text{dim}_K(\text{Tor}_i(K, M))_j$ denotes the $ij$-th graded Betti number of $M$.

If $M$ is an artinian graded $S$-module, we denote $s(M) = \max\{t : M_t \neq 0\}$. Herzog, Popescu and Vlăduţ proved the following formula for the regularity of a monomial ideal of Borel type:

**Proposition 1.3.** [7, Corollary 2.7] If $I$ is a Borel type ideal, with the notations above, we have

$$\text{reg}(I) = \max\{s(J_0^{sat}/J_0), \ldots, s(J_{r-1}^{sat}/J_{r-1})\} + 1.$$

**Remark 1.4.** Let $I \subset S$ be a monomial ideal of Borel type with the irredundant primary decomposition (1), $I = \bigcap_{i=1}^{r} Q_i$. Let $q$ be a positive integer. We have

$$I^q = \left(\bigcap_{i=1}^{r} Q_i\right)^q = \bigcap_{a \in \mathbb{N}^r, |a| = q} \prod_{i=1}^{r} Q_i^{a(i)},$$

where $a = (a(1), \ldots, a(r))$ and $|a| := a(1) + \cdots + a(r)$. We fix a vector $a \in \mathbb{N}^r$ with $|a| = q$. We have

$$\sqrt[r]{\prod_{i=1}^{r} Q_i^{a(i)}} = \bigcap_{i=1}^{r} Q_i^{a(i)} = \bigcap_{i=1}^{r} \sqrt[r]{Q_i^{a(i)}} = P_{m(a)},$$

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where \( m(a) = \min\{i: \ a(i) > 0\} \). It follows that \( \text{Ass}(S/I^q) \subset \text{Ass}(S/I) \). In particular, we get another proof of the fact that \( I^q \) is an ideal of Borel type, than the one which we given in [3]. Moreover, \( I^q \) has a primary decomposition:

\[
I^q = Q_1^q \cap Q_2(Q_1 \cap Q_2)^{q-1} \cap \cdots \cap Q_r(Q_1 \cap \cdots \cap Q_r)^q.
\]

**Example 1.5.** We consider the ideal \( Q = (x_1^{a_1}, \ldots, x_m^{a_m}) \subset S \), where \( m \leq n \) is a positive integer and \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 1 \) are integers. According to Proposition 1.3, \( \text{reg}(Q) = s(S/Q) + 1 \), where \( S = \mathbb{K}[x_1, \ldots, x_m] \) and \( Q = \sqrt{Q} \cap S \). Since \( u = x_1^{a_1-1} \cdots x_m^{a_m-1} \in S \) is the monomial of the highest degree which is not contain in \( Q \), it follows that

\[
\text{reg}(Q) = \sum_{i=1}^{m} (a_i - 1) + 1 = a_1 + \cdots + a_m - m + 1.
\]

We consider the ideal \( Q^q = (x_1^{qa_1}, \ldots, x_m^{qa_m}, x_1^{(q-1)a_1}x_2^{a_2}, \ldots) \). Note that \( Q^q \cap \sqrt{Q} = \sqrt{Q^q} \) and therefore \( \text{reg}(Q^q) = s(S/Q^q) + 1 \). One can easily see that \( u = x_1^{qa_1-1}x_2^{a_2-1} \cdots x_m^{a_m-1} \) is the monomial of the highest degree which is not contain in \( Q^q \). Thus:

\[
\text{reg}(Q^q) = qa_1 - 1 + \sum_{i=2}^{m} (a_i - 1) + 1 = qa_1 + a_2 + \cdots + a_m - m + 1.
\]

Note that \( \text{reg}(Q^q) \leq q \text{reg}(Q) \), as we already know from [3, Corollary 1.8], and the equality holds if and only if \( a_2 = \cdots = a_m = 1 \).

## 2 Regularity of symbolic and bracket powers of Borel type ideals

Now, assume \( I \subset S \) is an arbitrary monomial ideal with the primary irredundant decomposition \( I = \bigcap_{i=1}^{r} Q_i \). Let \( q \) be a positive integer. The \( q \)-th symbolic power of \( I \) is, by definition, the ideal

\[
I^{(q)} := \bigcap_{i=1}^{r} Q_i^q.
\]

With this notation, we have the following lemma.

**Lemma 2.1.** If \( I \subset S \) is an ideal of Borel type and \( q \) is a positive integer, then \( \text{Ass}(S/I^{(q)}) \subset \text{Ass}(S/I) \). In particular, \( I^{(q)} \) is an ideal of Borel type.

**Proof.** Assume \( I = \bigcap_{i=1}^{r} Q_i \) is the irredundant primary decomposition of \( I \) given in (1). It follows that \( I^{(q)} := \bigcap_{i=1}^{r} Q_i^q \). This primary decomposition of \( I^{(q)} \) is not necessarily irredundant. However, since \( \sqrt{Q_i^q} = \sqrt{Q_i} \), it follows that \( \text{Ass}(S/I^{(q)}) \subset \text{Ass}(S/I) \). Therefore, by Proposition 1.1(c), \( I^{(q)} \) is an ideal of Borel type. \( \square \)
Example 2.2. We consider the following ideals, \( Q = (x^{10}, x^6y^3, x^2y^7, y^8) \subset S := K[x, y, z] \), \( Q' = Q + (x^4y^6) \subset S' \), and \( I := (Q, z^2) \cap Q' = (Q, x^4y^6z^2) \subset S \). Since \( Q \not\subseteq Q' \), it follows that \((Q, z^2) \cap Q'\) is the primary irredundant decomposition of \( I \). Therefore, \( \text{Ass}(S/I) = \{(x, y), (x, y, z)\} \). On the other hand, by straightforward computation, we get:

\[
Q^2 = Q'^2 = (x^{20}, x^{16}y^3, x^{12}y^6, x^{10}y^8, x^8y^{10}, x^4y^{14}, x^2y^{15}, y^{16}).
\]

It follows that \( I^{(2)} = Q'^2 \cap (Q, z^2)^2 = Q^2 \cap (Q^2 + z^2Q + (z^4)) = Q^2 \) and therefore \( \text{Ass}(S/I^{(2)}) = \{(x, y)\} \). One can easily check that \( s(K[x, y]/(Q \cap K[x, y])^2) = 21 \) and thus \( \text{reg}(I^{(2)}) = 22 \), according to Proposition 1.3.

Also, \( s(K[x, y]/(Q' \cap K[x, y]^2)) = 11 \) and \( s((Q, z^2x^4y^6)/Q') = 12 \). Therefore, by Proposition 1.3, we get \( \text{reg}(I) = 13 \).

Let \( I \subset S \) be a Borel type ideal with the primary decomposition \( I := \bigcap_{i=1}^r Q_i \) from (1). We consider the sequential chain \( I = I_0 \subset I_1 \subset \cdots \subset I_r = S/I \) of \( I \), where \( I_i := \bigcap_{j=i+1}^r Q_j \).

We have the following chain of ideals

\[
I^{(q)} = I_0^{(q)} \subset I_1^{(q)} \subset \cdots \subset I_r^{(q)} = S,
\]

where \( I_i^{(q)} = \bigcap_{j=i+1}^r Q_j^{(q)} \). In the chain above, we may have some equalities. Nevertheless, if we denote \( J_i \) be the monomial ideal generated by \( G(I_i) \) in \( S_i := K[x_1, \ldots, x_n] \), we have

\[
I_{i+1}^{(q)}/I_i^{(q)} \cong ((J_i^{(q)})^{sat}/J_i^{(q)})[x_{n+1}, \ldots, x_n].
\]

Also, the sequential chain of \( I_i^{(q)} \) is obtained from the previous chain of ideal, by removing those ideals \( I_i \) with \( I_i = I_{i-1} \). Thus, by Proposition 1.3,

\[
\text{reg}(I^{(q)}) = \max\{s((J_i^{(q)})^{sat}/J_i^{(q)}), 0 \leq i \leq r - 1\} + 1.
\]  

(2)

Now, we are able to prove the following Theorem.

Theorem 2.3. With the above notations, we have:

\[
\text{reg}(I^{(q)}) \leq q \cdot \text{reg}(I).
\]

Proof. We fix \( 0 \leq i \leq r - 1 \). Since \( I_i := \bigcap_{j=i+1}^r Q_j \), it follows that \( J_i = \bigcap_{j=i+1}^r \bar{Q}_j \), where \( \bar{Q}_j \) is the ideal generated by \( G(Q_j) \) in \( S_i \). On the other hand, since \( J_i^{sat} \) is generated by the elements of \( G(I_{i+1}) \), it follows that \( J_i^{sat} = \bigcap_{j=i+2}^r \bar{Q}_j \).

Note that

\[
s(J_i^{sat}/J_i) + 1 = \min\{j : m_i^j J_i^{sat} \subset J_i\}
\]

and therefore \( s(J_i^{sat}/J_i) + 1 = \min\{j : m_i^j \bar{Q}_k \subset \bar{Q}_{i+1} \text{ for all } k = i + 2, \ldots, r \} \).

Similarly, since \( I_i^{(q)} := \bigcap_{j=i+1}^r Q_j^{(q)} \), it follows that

\[
s((J_i^{(q)})^{sat}/J_i^{(q)}) + 1 = \min\{j : m_i^j \bar{Q}_k^{(q)} \subset \bar{Q}_{i+1}^{(q)} \text{ for all } k = i + 2, \ldots, r \}.
\]

Note that if \( m_i^j \bar{Q}_k \subset \bar{Q}_{i+1} \) then \( m_i^j \bar{Q}_k^{(q)} = (m_i^j \bar{Q}_k)^q \subset \bar{Q}_{i+1}^q \). Therefore, we get

\[
s((J_i^{(q)})^{sat}/J_i^{(q)}) + 1 \leq q \cdot (s(J_i^{sat}/J_i) + 1).
\]

By applying Proposition 1.3 to \( I \) and (2) we get the required conclusion. \( \square \)
Let \( I \subset S \) be a monomial ideal of Borel type. An interesting question is to find a relation between \( \text{reg}(I^q) \) and \( \text{reg}(I^{[q]}) \).

Let \( I \subset S \) be a monomial ideal and let \( q \) be a nonnegative integer. We define the \( q \)-th bracket power of \( I \), to be the ideal \( I^{[q]} \), generated by all monomials \( u^q \), where \( u \in I \) is a monomial. In particular, \( I^{[0]} = S \) and \( I^{[1]} = I \). Note that if \( G(I) = \{ u_1, \ldots, u_m \} \) is the set of minimal monomial generators of \( I \), then \( G(I^{[q]}) = \{ u_1^q, \ldots, u_m^q \} \). Note that \( I^{[q]} \subset I^q \) for all \( q \). In fact, when \( q \geq 2 \), the equality holds if and only if \( I \) is principal. Also, one can easily see that \( (I \cap J)^{[q]} = I^{[q]} \cap J^{[q]} \) for any monomial ideals \( I, J \subset S \).

Now, assume \( I = \bigcap_{i=1}^r Q_i \) is the irredundant primary decomposition of \( I \). We claim that \( I^{[q]} = \bigcap_{i=1}^r Q_i^{[q]} \) is the irredundant primary decomposition of \( I^{[q]} \), where \( q \) is a positive integer. In order to prove this, we fix an integer \( i \) with \( 1 \leq i \leq r \) and chose a monomial \( u \in Q_i \). Obviously, \( u^q \in Q_i^{[q]} \). We claim that \( u^q \not\in \bigcap_{j \neq i} Q_j \). Assume this is not the case. It follows that \( u^q = u_q^q w_j \) for some monomials \( u_j \in J_j \) and \( w_j \in S \), for all \( j \neq i \). Therefore, \( u_j \mid u \) for all \( j \neq i \). It follows that \( u \in \bigcap_{j \neq i} Q_j \), a contradiction.

As a consequence, we get the following Lemma.

**Lemma 2.4.** If \( I \subset S \) be a monomial ideal and \( q \) a positive integer, then \( \text{Ass}(S/I) = \text{Ass}(S/I^{[q]}) \). In particular, if \( I \) is of Borel type, then \( I^{[q]} \) is of Borel type.

Now, we are able to prove the following Theorem.

**Theorem 2.5.** Let \( I \subset S \) be a monomial ideal of Borel type. Then:

\[
\text{reg}(I^{[q]}) \geq q \cdot \text{reg}(I).
\]

**Proof.** We consider the primary irredundant decomposition \( \bigcap_{i=1}^r Q_i \) of \( I \) from (1) and the sequential chain \( I = I_0 \subset I_1 \subset \cdots \subset I_r := S \) of \( I \), where \( I_i = \bigcap_{j=i+1}^r Q_j \), for \( 0 \leq i \leq r-1 \). Note that the sequential chain of \( I^{[q]} \), is \( I^{[q]} = I_0^{[q]} \subset I_1^{[q]} \subset \cdots \subset I_r^{[q]} = S \). Indeed, all the inclusions are stricts.

We fix an integer \( 0 \leq i \leq r-1 \). Let \( J_i \) be the monomial ideal generated by \( G(I_i) \) in \( S_i := K[x_1, \ldots, x_n] \). We denote \( \bar{Q}_j \), the ideal generated by \( G(Q_j) \) in \( S_j \), for all \( 1 \leq j \leq r \). With these notations, we have \( J_i = \bigcap_{j=i+1}^r \bar{Q}_j \) and \( J_i^{[q]} = \bigcap_{j=i+1}^r \bar{Q}_j^{[q]} \). On the other hand, since \( J_i^{sat} \) is generated by the elements of \( G(I_{i+1}) \), it follows that \( J_i^{sat} = \bigcap_{j=i+1}^r \bar{Q}_j \).

Let \( u \in J_i^{sat} \setminus J_i \) be a nonzero monomial. We claim that \( x_i^{q-1} u^q \in (J_i^{[q]})^{sat} \setminus J_i^{[q]} \). It is clear that \( x_i^{q-1} u^q \in (J_i^{[q]})^{sat} \setminus J_i^{[q]} \). If we assume that \( x_i^{q-1} u^q \not\in J_i^{[q]} \), it follows that \( x_i^{q-1} u^q = v^q \cdot w \), where \( v \in J_i \) is a monomial and \( w \in S \) is a monomial. Since \( v^q \mid x_i^{q-1} u^q \), it follows that \( v^q \mid u \) and therefore \( u \in J_i \), a contradiction.

As a consequence, we get \( s((J_i^{[q]})^{sat}/J_i^{[q]}) \geq q \cdot s(J_i^{sat}/J_i) + q - 1 \). By applying Proposition 1.3, we get the required conclusion. \( \square \)

**Remark 2.6.** The conclusions of Theorem 2.3 and Theorem 2.5 hold for monomial ideals \( I \subset S \) with \( \text{Ass}(S/I) \) totally ordered by inclusion. Indeed, if \( I \) is such an ideal, we can define a ring isomorphism \( \varphi : S \to S \) given by a reordering of variables, such that \( \varphi(I) \) is an ideal of Borel type. Since the Castelnuovo-Mumford regularity is an invariant, it follows that \( \text{reg}(I) = \text{reg}(\varphi(I)) \).
Bermejo and Giemenez give in [2] a formula for the regularity of a Borel type ideal $I$, when the irredundant irreducible decomposition is known. More precisely, they proved the following Proposition.

**Proposition 2.7.** [2, Corollary 3.17] Let $I \subset S$ be a monomial ideal of Borel type. Assume $I = \bigcap_{i=1}^{m} C_i$ is the irredundant irreducible decomposition of $I$. Then:

$$\text{reg}(I) = \max\{\text{reg}(C_i) : i = 1, \ldots, m\}.$$  

Since $C_i$’s are irreducible monomial ideals, they are generated by powers of variables. Since $\sqrt{C_i} \in \text{Ass}(S/I)$ and $I$ is of Borel type, we may assume that $C_i = (x_1^{a_{i1}}, \ldots, x_r^{a_{ir_i}})$, where $r_i$ is an integer with $1 \leq r_i \leq n$ and $a_{ij}$ are some positive integers. Denote $S_i := K[x_1, \ldots, x_r]$. If we denote $\tilde{C}_i$ the ideal generated by $G(C_i)$ in $S_i$, then, by Proposition 1.3, as in Example 1.5, we have $\text{reg}(C_i) = s(S_i/\tilde{C}_i) + 1 = a_{i1} + \cdots + a_{ir_i} - r_i + 1$. Therefore, we get the following corollary.

**Corollary 2.8.** With the notations above,

$$\text{reg}(I) = \max\{a_{i1} + \cdots + a_{ir_i} - r_i + 1 : i = 1, \ldots, m\}.$$  

Let $q$ be a positive integer and consider the ideal $I^{[q]}$. Since $I = \bigcap_{i=1}^{m} C_i$, it follows that $I^{[q]} = \bigcap_{i=1}^{m} C_i^{[q]}$ and $C_i^{[q]} = (x_1^{qa_{i1}}, \ldots, x_r^{qa_{ir_i}})$. Note that $\bigcap_{i=1}^{m} C_i^{[q]}$ is the irredundant irreducible decomposition of $I^{[q]}$. Indeed, we can argue in the same way as we did for the irreducible primary decomposition of $I^{[q]}$. Therefore, by Corollary 2.8, we get the following.

**Corollary 2.9.** $\text{reg}(I^{[q]}) = \max\{qa_{i1} + \cdots + qa_{ir_i} - r_i + 1 : i = 1, \ldots, m\}.$

The above formula leads us to the following upper bound for $\text{reg}(I^{[q]})$.

**Proposition 2.10.** Let $I \subset S$ be an ideal of Borel type and let $q$ be a positive integer. Then:

$$\text{reg}(I^{[q]}) \leq q \text{reg}(I) + (q - 1)(n - 1) = \alpha q \text{reg}(I) - (n - 1),$$

where $\alpha = 1 + \frac{n - 1}{\text{reg}(I)}$.

**Proof.** With the above notations, we may assume $\text{reg}(I) = a_{i1} + \cdots + a_{ir_i} - r_i + 1$ for some $1 \leq i \leq m$. According to Corollary 2.8 and Corollary 2.9, $\text{reg}(I^{[q]}) = qa_{i1} + \cdots + qa_{ir_i} - r_i + 1$. Therefore, $\text{reg}(I^{[q]}) = q \text{reg}(I) + (q - 1)(r_i - 1)$. Since $r_i - 1 \leq n - 1$, we get the required inequality. The remaining equality is trivial. 

We conclude this paper, with the following example.

**Example 2.11.** Let $I = (x) \cap (x^2, y) = (x^2, xy) \subset S = K[x, y]$. Let $q$ be a positive integer. It follows that $I^q = (x^{2q}, x^{2q-1}y, \ldots, x^q y^q) = (x^q) \cap (x^{2q}, x^{2q-1}y, \ldots, x^q y^{q-1}, y^q)$.

Also, we obtain $I^{(q)} = (x^q) \cap (x^2, y^q) = (x^q) \cap (x^{2q}, x^{2q-2}y, \ldots, x^2 y^{q-1}, y^q) = (x^{2q}, x^{2q-2}y, \ldots, x^{q-2} y^q, x^q y^q, x^q y^{q+1})$.

On the other hand, $I^{[q]} = (x^q) \cap (x^{2q}, y^q) = (x^{2q}, x^q y^q)$. 

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We consider the sequential chain of $I$, $I := I_0 \subset I_1 \subset I_2 := S$, where $I_1 = (x)$. We have $J_0 = I \subset S$ and $J_1 = (x) \subset K[x]$. Therefore, $J_0^a = I^a$, $J_1^q = I^q$ and $J_0^{[q]} = I^{[q]}$. Also, $J_1^q = J_1^{[q]} = J_1^{[q]} = (x^q) \subset K[x]$. We get $J_0^{sat} = (x)S$, $(J_0^q)^{sat} = (J_0^{[q]})^{sat} = (x^q)S$ and $J_1^{sat} = (J_1^q)^{sat} = (J_1^{[q]})^{sat} = (J_1^{[q]})^{sat} = (x^q)S$

We have $s(J_1^{sat}/J_1) = 0$ and $s((J_1^q)^{sat}/J_1^q) = s((J_1^{[q]})^{sat}/J_1^{[q]}) = q - 1$.

Also, one can easily compute $s(J_0^{sat}/J_0) = 1$, $s((J_0^q)^{sat}/J_0^q) = 2q - 1$, $s((J_0^{[q]})^{sat}/J_0^{[q]}) = 2q - 1$ and $s((J_0^{[q]})^{sat}/J_0^{[q]}) = 3q - 2$. By Proposition 1.3, it follows that $\text{reg}(I) = 2$, $\text{reg}(I^q) = \text{reg}(I^{[q]}) = 2q$ and $\text{reg}(I^{[q]}) = 3q - 1$.

Since $I = (x) \cap (x^2, y)$ is also the irreducible irredundant decomposition of $I$, by Corollary 2.8 and Corollary 2.9, we can compute directly $\text{reg}(I) = \max\{1 - 1 + 1, 2 + 1 - 2 + 1\} = 2$
and, respectively, $\text{reg}(I^{[q]}) = \max\{q - 1 + 1, 2q + q - 2 + 1\} = 3q - 1$.

Note that $\text{reg}(I^{[q]}) = q \text{reg}(I) + (q - 1)(2 - 1)$ and therefore, the upper bound given in Proposition 2.10 is the best possible.

References

[1] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math 87 (1987), 1-11.

[2] I. Bermejo, P. Gimenez, *Saturation and Castelnuovo-Mumford regularity*, Journal of Algebra, 303 no. 2 (2006), 592-617.

[3] G. Caviglia, E. Sbarra, *Characteristic-free bounds for the Castelnuovo Mumford regularity*, Compos. Math. 141 no. 6 (2005), 1365-1373.

[4] M. Cimpoeas “A stable property of Borel type ideals”, Communications in Algebra, vol 36 no 2, 2008, p.674-677.

[5] M. Cimpoeas “Some remarks on Borel type ideals”, Communications in Algebra, vol 37, no 2, 2009, p.724-727.

[6] D. Eisenbud "Commutative algebra", Springer-Verlag, New York, 1995.

[7] J. Herzog, D. Popescu, M. Vladoiu, *On the Ext-Modules of ideals of Borel type*, Contemporary Math. 331 (2003), 171-186.

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