Propagation dynamics of Fisher–KPP equation with time delay and free boundaries

Ningkui Sun¹ · Jian Fang²

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Abstract
Incorporating free boundary into time-delayed reaction–diffusion equations yields a compatible condition that guarantees the well-posedness of the initial value problem. With the Fisher–KPP type nonlinearity we then establish a vanishing–spreading dichotomy result. Further, when the spreading happens, we show that the spreading speed and spreading profile are nonlinearly determined by a delay-induced nonlocal semi-wave problem. It turns out that time delay slows down the spreading speed.

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1 Introduction

In the pioneer work of Fisher [17], and Kolmogorov, Petrovski and Piskunov [22], it was shown that

\[ u_t = u_{xx} + f(u), \quad x \in \mathbb{R} \]  

(1.1)

with

\[ f \in C^1(\mathbb{R}, \mathbb{R}), \quad f(0) = 0 = f(1), \quad f(s) \leq f'(0)s, \quad s \geq 0, \]  

(1.2)

admits traveling waves solutions of the form \( u(t, x) = \phi(x - ct) \) satisfying \( \phi(-\infty) = 1 \) and \( \phi(+\infty) = 0 \) if and only if \( c \geq c_0 := 2\sqrt{f'(0)} \). In 1970s’, Aronson and Weinberger [2]
proved that the minimal wave speed \( c_0 \) is also the asymptotic speed of spread (spreading speed for short) in the sense that
\[
\lim_{t \to \infty} \sup_{|x| \leq (c_0 + \epsilon) t} u(t, x) = 0, \quad \lim_{t \to \infty} \inf_{|x| \geq (c_0 - \epsilon) t} u(t, x) = 1
\]  
(1.3)
for any small \( \epsilon > 0 \) provided that the initial function \( u(0, x) \) is compactly supported. These works have stimulated volumes of studies for the propagation dynamics of various types of evolution systems. Among others, of particular interest to the Fisher–KPP equation (1.1)–(1.2) with time delay or free boundary are two typical ones.

Schaaf [32] studied the following time-delayed reaction–diffusion equation
\[
\frac{du}{dt}(t, x) = \frac{d^2u}{dx^2}(t, x) + f(u(t, x), u(t - \tau, x)), \quad x \in \mathbb{R}, \ t > 0,
\]  
(1.4)
where \( \tau > 0 \) is the time delay. With the Fisher–KPP condition on \( \tilde{f}(s) := f(s, s) \) and the quasi-monotone condition \( \partial_2 f \geq 0 \), it was shown that the minimal wave speed \( c_0 = c_0(\tau) \) exists and it is determined by the system of two transcendental equations
\[
F(c, \lambda) = 0, \quad \frac{\partial F}{\partial \lambda}(c, \lambda) = 0,
\]  
(1.5)
where
\[
F(c, \lambda) = \lambda^2 + c\lambda + \partial_1 f(0, 0) + \partial_2 f(0, 0)e^{-\lambda \tau}.
\]  
(1.6)
The delay-induced spatial non-locality was brought to attention by So et al. [28], where they derived the following time-delayed reaction–diffusion model equation with nonlocal response for the study of age-structured population
\[
\frac{du}{dt} = u_{xx} - du + \gamma \int_{\mathbb{R}} b(u(t - \tau, x - y))k(y)dy, \quad x \in \mathbb{R}, \ t > 0,
\]  
(1.7)
where \( u \) represents the density of mature population, \( \tau > 0 \) is the maturation age, \( d \) is the death rate, \( b \) is the birth rate function, \( \gamma \) is the survival rate from newborn to being mature, and \( k \) is the redistribution kernel during the maturation period. As such, introducing time delay into diffusive equation usually gives rises to spatial non-locality due to the interaction of time lag (for maturation) and diffusion of immature population. In the extreme case where the immature population does not diffuse, the kernel \( k \) becomes the Dirac measure, and hence (1.7) reduces to (1.4). We refer to the survey article [21] for the delay-induced nonlocal reaction–diffusion problems. In [28], the authors obtained the minimal wave speed \( c_0(\tau) \) that is determined by a similar system to (1.5) provided that \( b \) is nondecreasing and \( b(s) - ds \) is of Fisher–KPP type. Wang et al. [37] proved that \( c_0(\tau) \) is decreasing in \( \tau \). Liang and Zhao [23] showed that \( c_0(\tau) \) is also the spreading speed for the solutions satisfying the following initial condition
\[
u(\theta, x) \text{ is continuous and compactly supported in } \theta \in [-\tau, 0) \text{ and } x \in \mathbb{R}.
\]  
(1.8)
Similar to the classical Fisher–KPP equation, the spreading speed \( c_0(\tau) \) for time-delayed reaction–diffusion equation is still linearly determined for both local and nonlocal problems thanks to the Fisher–KPP type condition.

We refer to [26] for more properties that are induced by time delay in reaction–diffusion equations, including the well-posedness of initial value problems as well as the role of the quasi-monotone condition on the comparison principle, and [14,15] for the delay-induced weak compactness of time-\( t \) solution maps when \( t \in (0, \tau] \) as well as its role in the study of wave propagation.
Recently, Du and Lin [10] proposed a Stefan type free boundary to the Fisher–KPP equation
\[
\begin{align*}
  u_t &= u_{xx} + u(1-u), & g(t) < x < h(t), & t > 0, \\
  u(t, g(t)) &= 0, & g'(t) &= -\mu u_x(t, g(t)), & t > 0, \\
  u(t, h(t)) &= 0, & h'(t) &= -\mu u_x(t, h(t)), & t > 0,
\end{align*}
\tag{1.9}
\]
where the free boundaries \( x = g(t) \) and \( x = h(t) \) represent the spreading fronts, which are determined jointly by the gradient at the fronts and the coefficient \( \mu \) in the Stefan condition. For more background of proposing such free boundary conditions, we refer to [4,10]. It was proved in [10] that the unique global solution \((u, g, h)\) has a spreading–vanishing dichotomy property as \( t \to \infty \): either \((g(t), h(t)) \to \mathbb{R} \) and \( u \to 1 \) (spreading case), or \( g(t) \to g_{\infty}, h(t) \to h_{\infty} \) with \( h_{\infty} - g_{\infty} \leq \pi \), and \( u \to 0 \) (vanishing case). Moreover, it was also proved that when spreading happens, there is a constant \( k_0 > 0 \) such that \(-g(t) \) and \( h(t) \) behave like a straight line \( k_0 t \) for large time, where \( k_0 \) is called the asymptotic speed of spread (spreading speed for short). Different from the classical Fisher–KPP speed, \( k_0 \) is the unique value of \( c \) such that the following nonlinear semi-line problem is solvable:
\[
\begin{align*}
  q'' - c q' + q(1-q) &= 0, & z > 0, \\
  q(\infty) &= 1, & \mu q_+'(0) &= c, & q(z) > 0, & z \leq 0, \\
  q(z) &= 0, & z \leq 0,
\end{align*}
\tag{1.10}
\]
where \( q_+'(0) \) is the right derivative of \( q(z) \) at 0. In particular, as \( \mu \) increases to infinity, \( k_0 \) increases to the classical Fisher–KPP speed 2. Later on, Du and Lou [11] obtained a rather complete characterization on the asymptotic behavior of solutions for (1.9) with some general nonlinear terms. For further related work on free boundary problems, we refer to [8,9,12] and the references therein.

In this paper, we aim to explore how to incorporate time delay and free boundary into the Fisher–KPP equation (1.1)–(1.2) so that the problem is well-posed, and then study their joint influence on the propagation dynamics.

Keeping a smooth flow for the organizations of the paper, we write down here the problem of interest while leaving in the next section the derivation details, including the emergence of the compatible condition (1.12) for the well-posedness of the initial value problem.
\[
\begin{align*}
  u_t(t, x) &= u_{xx}(t, x) - du(t, x) + f(u(t - \tau, x)), & x \in (g(t), h(t)), & t > 0, \\
  u(t, g(t)) &= 0, & g'(t) &= -\mu u_x(t, g(t)), & t > 0, \\
  u(t, h(t)) &= 0, & h'(t) &= -\mu u_x(t, h(t)), & t > 0, \\
  u(\theta, x) &= \phi(\theta, x), & g(\theta) \leq x \leq h(\theta), & \theta \in [-\tau, 0],
\end{align*}
\tag{P}
\]
where \( d \) and \( \tau \) are two positive constants, the nonlinear function \( f \) satisfies
\[
\begin{align*}
  f(\tilde{v}) \in C^{1+\tilde{v}}([0, \infty)) & \text{ for some } \tilde{v} \in (0, 1), \quad f(0) = 0, \quad f'(0) - d > 0; \\
  f(s) - ds &= 0 \text{ has a unique positive constant root } u^*; \\
  \frac{f(s)}{s} \text{ is monotonically increasing in } s \in [0, u^*],
\end{align*}
\tag{H}
\]
and the initial data \((\phi(\theta, x), g(\theta), h(\theta))\) satisfies
\[
\begin{align*}
  \phi(\theta, x) &\in C^{1,2}([-\tau, 0] \times [g(\theta), h(\theta)]), \\
  0 < \phi(\theta, x) &\leq u^* \text{ for } (\theta, x) \in [-\tau, 0] \times (g(\theta), h(\theta)), \\
  \phi(\theta, x) &\equiv 0 \text{ for } \theta \in [-\tau, 0], \quad x \notin (g(\theta), h(\theta)).
\end{align*}
\tag{1.11}
\]
as well as the compatible condition
\[ [g(\theta), h(\theta)] \subset [g(0), h(0)] \quad \text{for} \quad \theta \in [-\tau, 0]. \tag{1.12} \]

Assumption (H) ensures the Fisher–KPP structure as well as the comparison principle. Due to the nature of delay differential equations, the initial value, including the initial domain, has to be imposed over the history period \([-\tau, 0]\), as in (1.11). The interaction of time delay and free boundary gives rise to the compatible condition (1.12) that is essential for the well-posedness of the problem. If \(\tau = 0\), then the compatible condition (1.12) becomes trivial and problem \((P)\) reduces to (1.9). First, we obtain the following existence and uniqueness theorem for \((P)\).

**Theorem 1.1** (Well-posedness) For any given initial data \((\phi(\theta, x), g(\theta), h(\theta))\) satisfying (1.11) and (1.12), there exists a unique triple \((u, g, h)\) solving \((P)\) with \(u \in C^{1,2}((0, \infty) \times [g(t), h(t)])\) and \(g, h \in C^1([0, \infty))\).

With the compatible condition (1.12) we can cast problem \((P)\) into a fixed boundary problem and then apply the Schauder fixed point theorem to establish the local existence of solutions. The extension to all positive time is based on some a priori estimates.\(^1\)

From the maximum principle and (H), it follows that when \(t > 0\) the solution \(u > 0\) for \(x \in (g(t), h(t))\), and \(u_x(t, g(t)) > 0 > u_x(t, h(t))\), and hence, \(g'(t) < 0 < h'(t)\) for all \(t > 0\). Therefore, we can denote
\[ g_\infty := \lim_{t \to \infty} g(t) \quad \text{and} \quad h_\infty := \lim_{t \to \infty} h(t). \]

The next theorem gives the long-time behavior of the solution, which is stated as follows.

**Theorem 1.2** (Spreading-vanishing dichotomy) Let \((u, g, h)\) be the solution of \((P)\) with the initial data \((\phi(\theta, x), g(\theta), h(\theta))\) satisfying (1.11) and (1.12). Then the following alternative holds:

*Either*

(i) **Spreading:** \((g_\infty, h_\infty) = \mathbb{R}\) and
\[ \lim_{t \to \infty} u(t, x) = u^* \text{ locally uniformly in } \mathbb{R}, \]

*or*

(ii) **Vanishing:** \((g_\infty, h_\infty)\) is a finite interval with length no bigger than \(\frac{\pi}{\sqrt{f'(0)-d}}\) and
\[ \lim_{t \to \infty} \max_{t \leq x \leq h(t)} u(t, x) = 0. \]

In Sect. 4, we give some sufficient conditions for spreading or vanishing, and see that the number \(\frac{\pi}{\sqrt{f'(0)-d}}\) plays a key role for the spreading process: If \(h(0) - g(0) \geq \frac{\pi}{\sqrt{f'(0)-d}}\), then spreading will happen regardless of the choice of \(u(\theta, x)\); while if \(h(0) - g(0) < \frac{\pi}{\sqrt{f'(0)-d}}\) and \(u(\theta, x)\) is small, then vanishing will happen. We refer to Lemmas 4.2 and 4.4 for the details.

When spreading happens, we characterize the spreading speed and profile of the solutions. The nonlinear and nonlocal semi-wave problem
\[
\begin{align*}
q'' - cq' - dq + f(q(z - c\tau)) &= 0, \quad z > 0, \\
q(\infty) &= u^*, \quad \mu q'(0) = c, \quad q(z) > 0, \quad z > 0, \\
q(z) &= 0, \quad z \leq 0,
\end{align*}
\tag{1.13}
\]

\(^1\) We sincerely thank Professor Avner Friedman for his valuable comments and suggestions on the proof of the well-posedness.
will play an important role. If $\tau = 0$ then (1.13) reduces to the local form (1.10). We obtain the following result.

**Theorem 1.3** Problem (1.13) admits a unique solution $(c^*, q_{c^*})$ and $c^* = c^*(\tau)$ is decreasing in delay $\tau \geq 0$.

Due to the presence of time delay, the proof of Theorem 1.3 highly relies on the distribution of complex solutions of the following transcendental equation

$$\lambda^2 - c\lambda - d + f'(0)e^{-\lambda c\tau} = 0.$$ (1.14)

We refer to Lemma 3.2 and Proposition 3.3, which are independently of interest.

With the semi-wave established above, we can construct various super- and subsolutions to estimate the spreading fronts $h(t), g(t)$ and the spreading profile as $t \to \infty$, and obtain the following theorem.

**Theorem 1.4** (Spreading profile) Let $u$ be a solution satisfying Theorem 1.2(i). Then there exist two constants $H_1$ and $G_1$ such that

$$\lim_{t \to \infty} [h(t) - c^*t] = H_1, \quad \lim_{t \to \infty} h'(t) = c^*,$$

$$\lim_{t \to \infty} [g(t) + c^*t] = G_1, \quad \lim_{t \to \infty} g'(t) = -c^*,$$

$$\lim_{t \to \infty} \|u(t, \cdot) - q_{c^*}(c^*t + H_1 - \cdot)\|_{L^\infty([0,h(t)])} = 0,$$

$$\lim_{t \to \infty} \|u(t, \cdot) - q_{c^*}(c^*t - G_1 + \cdot)\|_{L^\infty([g(t),0])} = 0,$$ (1.15)-(1.16)

where $(c^*, q_{c^*})$ is the unique solution of (1.13).

The rest of this paper is organized as follows. In Sect. 2 we derive the compatible condition (1.12), with which we formulate problem (P) and then establish the well-posedness as well as the comparison principle. Section 3 is devoted to the study of the semi-wave problem (1.13). In Sect. 4, we establish the spreading-vanishing dichotomy result. Finally in Sect. 5, we characterize the spreading speed and profile of spreading solutions of (P).

## 2 The compatible condition, well-posedness and comparison principle

### 2.1 The compatible condition

To formulate problem (P), we start from the age-structured population growth law

$$p_t + p_a = D(a)p_{xx} - d(a)p,$$ (2.1)

where $p = p(t, x; a)$ denotes the density of species of age $a$ at time $t$ and location $x$, $D(a)$ and $d(a)$ denote the diffusion rate and death rate of species of age $a$, respectively.

Next we consider the scenario that the species has the following biological characteristics.

(A1) The species can be classified into two stages by age: mature and immature. An individual at time $t$ belongs to the mature class if and only if its age exceeds the maturation time $\tau > 0$. Within each stage, all individuals share the same behavior.

(A2) Immature population does not move in space.
The total mature population $u$ at time $t$ and location $x$ can be represented by the integral

$$u(t, x) = \int_{t}^{\infty} p(t, x; a) da. \quad (2.2)$$

We assume that the mature population $u$ lives in the habitat $[g(t), h(t)]$, vanishes in the boundary

$$u(t, g(t)) = 0 = u(t, h(t)), \quad t > 0, \quad (2.3)$$

and extends the habitat by obeying the Stefan type moving boundary conditions:

$$h'(t) = -\mu u_x(t, h(t)), \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \quad (2.4)$$

where $\mu$ is a given positive constant. Note that the immature population does not contribute to the extension of habitat due to their immobility, as assumed in (A2).

According to (A1) we may assume that

$$D(a) = \begin{cases} 1, & a \geq \tau, \\ 0, & 0 \leq a < \tau, \end{cases} \quad d(a) = \begin{cases} d, & a \geq \tau, \\ d_1, & 0 \leq a < \tau, \end{cases}$$

where $d$ and $d_1$ are two positive constants. Differentiating the both sides of the Equation (2.2) in time yields

$$u_t = \int_{t}^{\infty} p_t da = \int_{t}^{\infty} [-p_a + p_{xx} - dp] da = u_{xx} - du + p(t, x; \tau) - p(t, x; \infty). \quad (2.5)$$

Since no individual lives forever, it is natural to assume that

$$p(t, x; \infty) = 0. \quad (2.6)$$

To obtain a closed form of the model, one then needs to express $p(t, x; \tau)$ by $u$ in a certain way. Indeed, $p(t, x; \tau)$ denotes the newly matured population at time $t$, and it is the evolution result of newborns at $t-\tau$. In other words, there is an evolution relation between the quantities $p(t, x; \tau)$ and $p(t-\tau, x; 0)$. Such a relation is obeyed by the growth law (2.1) for $0 < a < \tau$, and hence it is the time-$\tau$ solution map of the following equation

$$\begin{cases} q_s = -d_1 q, & x \in \mathbb{R}, \quad 0 \leq s \leq \tau, \\ q(0, x) = p(t - \tau, x; 0), & x \in \mathbb{R}. \end{cases} \quad (2.7)$$

Thus, $p(t, x; \tau) = q(t, x) = e^{-d_1 \tau} p(t - \tau, x; 0)$. Further, the newborns $p(t - \tau, x; 0)$ is given by the birth $b(u(t - \tau, x))$, where $b$ is the birth rate function with $b(0) = 0$. Consequently,

$$p(t, x; \tau) = e^{-d_1 \tau} b(u(t - \tau, x)). \quad (2.8)$$

Combining (2.3)–(2.6) and (2.8), we are led to the following system:

$$\begin{align*}
 u_t(t, x) &= u_{xx}(t, x) - du(t, x) + e^{-d_1 \tau} b(u(t - \tau, x)), \quad t > 0, x \in [g(t - \tau), h(t - \tau)], \\
 u_t(t, x) &= u_{xx}(t, x) - du(t, x), \quad t > 0, x \in [g(t), h(t)][g(t - \tau), h(t - \tau)], \\
 u(t, g(t)) &= 0 = u(t, h(t)), \quad t > 0, \\
 h'(t) &= -\mu u_x(t, h(t)), \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0. \end{align*} \quad (2.9)$$

For $t > 0$, outside the habitat $(g(t), h(t))$ the mature population does not exist, that is,

$$u(t, x) \equiv 0 \quad \text{for} \quad t > 0, \ x \notin (g(t), h(t)). \quad (2.10)$$
Clearly, since the habitat is expanding for \( t > 0 \), we have
\[
[g(t - \tau), h(t - \tau)] \subset [g(t), h(t)], \quad t \geq \tau.
\] (2.11)

Hence, the first two equations in (2.9) can be written as the following single one
\[
 u_t(t, x) = u_{xx}(t, x) - du(t, x) + e^{-d_I t} b(u(t - \tau, x)), \quad t > 0, x \in [g(t), h(t)],
\] (2.12)
provided that (2.11) holds for \( t \geq 0 \). As such, in view of (2.11) we need an additional condition
\[
[g(t - \tau), h(t - \tau)] \subset [g(t), h(t)], \quad t \in [0, \tau).
\] (2.13)

Note that \([g(0), h(0)] \subset [g(t), h(t)]\) for \( t > 0 \). And as the coefficient \( \mu \to 0 \) we have \([g(t), h(t)] \to [g(0), h(0)]\) uniformly for \( t \in [0, \tau] \). Then, regardless of the influence of \( \mu \), (2.13) is strengthened to be
\[
[g(\theta), h(\theta)] \subset [g(0), h(0)] \quad \text{for} \quad \theta \in [-\tau, 0],
\]
which is the aforementioned compatible condition (1.12).

Setting \( f(s) := e^{-d_I s} b(s) \) in (2.9), we obtain problem (P).

## 2.2 Well-posedness

We employ the Schauder fixed point theorem to establish the local existence of solutions to (P), and prove the uniqueness, then extend the solutions to all time by an estimate on the free boundary.

**Theorem 2.1** Suppose (H) holds. For any \( \alpha \in (0, 1) \), there is a \( T > 0 \) such that problem (P) with the initial data \((\phi(\theta, x), g(\theta), h(\theta))\) satisfying (1.11) and (1.12), admits a solution
\[
 (u, g, h) \in C^{(1+\alpha)/2, 1+\alpha}([0, T] \times [g(t), h(t)]) \times C^{1+\alpha/2}([0, T]) \times C^{1+\alpha/2}([0, T]).
\]

**Proof** We divide the proof into three steps.

**Step 1.** We use a change of variable argument to transform problem (P) into a fixed boundary problem with a more complicated equation which is used in [5, 10]. Denote \( l_1 = g(0) \) and \( l_2 = h(0) \) for convenience, and set \( h_0 = \frac{1}{2} (l_2 - l_1) \). Let \( \xi_1(y) \) and \( \xi_2(y) \) be two nonnegative functions in \( C^3(\mathbb{R}) \) such that
\[
 \xi_1(y) = 1 \quad \text{if} \quad |y - l_2| < \frac{h_0}{4}, \quad \xi_1(y) = 0 \quad \text{if} \quad |y - l_2| > \frac{h_0}{2}, \quad |\xi_1'(y)| < \frac{6}{h_0} \quad \text{for} \quad y \in \mathbb{R};
\]
\[
 \xi_2(y) = 1 \quad \text{if} \quad |y - l_1| < \frac{h_0}{4}, \quad \xi_2(y) = 0 \quad \text{if} \quad |y - l_1| > \frac{h_0}{2}, \quad |\xi_2'(y)| < \frac{6}{h_0} \quad \text{for} \quad y \in \mathbb{R}.
\]

Define \( y = y(t, x) \) through the identity
\[
 x = x + \xi_1(y)(h(t) - l_2) + \xi_2(y)(g(t) - l_1) \quad \text{for} \quad t > 0,
\]
\[
 x = x \quad \text{for} \quad -\tau \leq t \leq 0,
\]
and set
\[
 w(t, y) := u(t, y + \xi_1(y)(h(t) - l_2) + \xi_2(y)(g(t) - l_1)) = u(t, x) \quad \text{for} \quad t > 0,
\]
\[
 w(\theta, y) := \phi(\theta, y) \quad \text{for} \quad -\tau \leq \theta \leq 0.
\]
Then the free boundary problem \((P)\) becomes
\[
\begin{aligned}
&w_t - A(g, h, y)w_{yy} + B(g, h, y)w_y = f(w(t - \tau, y)) - dw, \quad y \in (l_1, l_2), \quad t > 0, \\
&w(t, l_i) = 0, \quad t > 0, \quad i = 1, 2, \\
&w(\theta, y) = \phi(\theta, y), \quad y \in [l_1, l_2], \quad \theta \in [-\tau, 0],
\end{aligned}
\]
(2.14)
and
\[
g'(t) = -\mu w_y(t, l_1), \quad h'(t) = -\mu w_y(t, l_2), \quad t > 0,
\]
(2.15)
with \(f(w(t - \tau, y)) = f(u(t - \tau, y))\) and \([A(g, h, y) = [1 + \xi_1'(y)(h(t) - l_2) + \xi_2'(y)(g(t) - l_1)]^{-2}]\).

\[
B(g, h, y) = [\xi_1''(y)(h(t) - l_2) + \xi_2''(y)(g(t) - l_1)]A(g, h, y)^3
\]
\[
- [\xi_1(y)h'(t) + \xi_2(y)g'(t)]A(g, h, y)^{\frac{1}{2}}.
\]

Denote \(h_1 = -\mu (u_0)_y(0, l_2)\), and \(h_2 = \mu (u_0)_y(0, l_1)\). For \(0 < T \leq \min \left\{ \frac{h_0}{T_0(1 + \bar{h}_1 + h_2)}, \tau \right\}\), we define \(\Omega_T := [0, T] \times [l_1, l_2]\),

\[
\mathcal{D}_T^h = \{ h \in C^1([0, T]) : h(0) = l_2, \quad h'(0) = h_1, \quad \| h' - h_1 \|_{C([0, T])} \leq 1 \},
\]
\[
\mathcal{D}_T^g = \{ g \in C^1([0, T]) : g(0) = l_1, \quad g'(0) = -h_2, \quad \| g' + h_2 \|_{C([0, T])} \leq 1 \}.
\]

Clearly, \(\mathcal{D} := \mathcal{D}_T^h \times \mathcal{D}_T^g\) is a bounded and closed convex set of \(C^1([0, T]) \times C^1([0, T])\).

Noting that the restriction on \(T\), it is easy to see that the transformation \((t, y) \rightarrow (t, x)\) is well defined. By a similar argument as in [36], applying standard \(L^p\) theory and the Sobolev embedding theorem, we can deduce that for any given \((g, h) \in \mathcal{D}\), problem (2.14) admits a unique \(w(t, y; g, h) \in W^{1, 2}_p(\Omega_T) \hookrightarrow C^{1+\sigma, 1+\sigma}([0, T])\), which satisfies
\[
\| w \|_{W^{1, 2}_p(\Omega_T)} + \| w \|_{C^{1+\sigma, 1+\sigma}([0, T])} \leq C_1,
\]
(2.16)
where \(p > 1\) and \(C_1\) is a constant dependent on \(g(\theta), h(\theta), \alpha, p\) and \(\| \phi \|_{C^{1, 2}([0, T] \times [g(\theta), h(\theta)])}\).

Defining \(\hat{h}\) and \(\hat{g}\) by \(\hat{h}(t) = l_2 - \int_0^t \mu w_y(s, l_2)ds\) and \(\hat{g}(t) = l_1 - \int_0^t \mu w_y(s, l_1)ds\), respectively, then we have
\[
\hat{h}'(t) = -\mu w_y(t, l_2), \quad \hat{h}(0) = l_2, \quad \hat{h}'(0) = -\mu w_y(0, l_2) = h_1,
\]
and thus \(\hat{h} \in C^{\frac{\alpha}{2}}([0, T])\), which satisfies
\[
\| \hat{h}' \|_{C^{\frac{\alpha}{2}}([0, T])} \leq \mu C_1 =: C_2.
\]
(2.17)

Similarly \(\hat{g}' \in C^{\frac{\alpha}{2}}([0, T])\), which satisfies
\[
\| \hat{g}' \|_{C^{\frac{\alpha}{2}}([0, T])} \leq \mu C_1 =: C_2.
\]
(2.18)

Step 2. For any given triple \((g, h) \in \mathcal{D}\), we define an operator \(\mathcal{F}\) by
\[
\mathcal{F}(g, h) = (\hat{g}, \hat{h}).
\]
Clearly, \(\mathcal{F}\) is continuous in \(\mathcal{D}\), and \((g, h) \in \mathcal{D}\) is a fixed point of \(\mathcal{F}\) if and only if \((w, g, h)\) solves (2.14) and (2.15). We will show that if \(T > 0\) is small enough, then \(\mathcal{F}\) has a fixed point by using the Schauder fixed point theorem.

Firstly, it follows from (2.17) and (2.18) that
\[
\| \hat{h}' - h_1 \|_{C([0, T])} \leq C_2 T^{\frac{\alpha}{2}}, \quad \| \hat{g}' + h_2 \|_{C([0, T])} \leq C_2 T^{\frac{\alpha}{2}}.
\]
Thus if we choose $T \leq \min \left\{ \frac{h_0}{10(1+h_1+h_2)}, \, \tau, \, C_2^{-\frac{2}{\alpha}} \right\}$, then $\mathcal{F}$ maps $\mathcal{D}$ into itself. Consequently, $\mathcal{F}$ has at least one fixed point by using the Schauder fixed point theorem, which implies that (2.14) and (2.15) have at least one solution $(w, g, h)$ defined in $[0, T]$. Moreover, by the Schauder estimates, we have additional regularity for $(w, g, h)$ as a solution of (2.14) and (2.15), namely,

$$(w, g, h) \in C^{1+\alpha/2,2+\alpha}((0, T) \times [l_1, l_2]) \times C^{1+\alpha/2}((0, T)) \times C^{1+\alpha/2}(0, T))$$

and for any given $0 < \varepsilon < T$, there holds

$$\|w\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T] \times [l_1,l_2])} \leq C_3,$$

where $C_3$ is a constant dependent on $\varepsilon, g(\theta), h(\theta), \alpha$ and $\|\phi\|_{C^{1,2}}$. Thus we deduce a local classical solution $(u, g, h)$ of (P) by $(w, g, h)$, and $u \in C^{1+\alpha/2,2+\alpha}((0, T) \times [g(t), h(t)])$ satisfies

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([g(t),h(t)])} \leq C_3.$$

**Step 3.** We will prove the uniqueness of solutions of (P). Let $(u_i, g_i, h_i), i = 1, 2$, be two solutions of (P) and set

$$w_i(t, y) := u_i(t, y + \xi_1(y)(h_i(t) - l_2) + \xi_2(y)(g_i(t) - l_1)).$$

Then it follows from (2.16), (2.17) and (2.18) that

$$\|w_i\|_{W^{1,2}_p(\Omega_T)} + \|w_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T)} \leq C_1, \quad \|h_i\|_{C^{\frac{\alpha}{2}}(0,T)} \leq C_2, \quad \|g_i\|_{C^{\frac{\alpha}{2}}(0,T)} \leq C_2.$$

Set

$$\tilde{w}(t, y) := w_1(t, y) - w_2(t, y), \quad \tilde{g}(t) := g_1(t) - g_2(t), \quad \tilde{h}(t) := h_1(t) - h_2(t),$$

then we find that $\tilde{w}(t, y)$ satisfies

$$\begin{cases}
\tilde{w}_t - A_2(t, y)\tilde{w}_{yy} + B_2(t, y)\tilde{w}_y = \tilde{f}(t, y), \quad y \in (l_1, l_2), \quad t \in (0, T), \\
\tilde{w}(t, l_1) = \tilde{w}(t, l_2) = 0, \quad \tilde{w}(\theta, y) = 0, \\
y \in [l_1, l_2], \quad \theta \in [-\tau, 0],
\end{cases}$$

(2.19)

where

$$\tilde{f}(t, y) = (A_1 - A_2)(w_1)_{yy} - (B_1 - B_2)(w_1)_y + f(w_1(t - \tau, y)) - f(w_2(t - \tau, y)) - d\tilde{w},$$

and $A_i$ and $B_i$ are the coefficients of problem (2.14) with $(w_i, g_i, h_i)$ instead of $(w, g, h)$.

Recalling that $T \leq \tau$, then $f(w_1(t - \tau, y)) - f(w_2(t - \tau, y)) = 0$ for all $(t, y) \in \Omega_T$, thus

$$\tilde{f}(t, y) = (A_1 - A_2)(w_1)_{yy} - (B_1 - B_2)(w_1)_y - d\tilde{w}.$$

Thanks to this, we can apply the $L^p$ estimates for parabolic equations to deduce that

$$\|\tilde{w}\|_{W^{1,2}_p(\Omega_T)} \leq C_4(\|\tilde{g}\|_{C^1(0,T)} + \|\tilde{h}\|_{C^1(0,T)}),$$

(2.20)

with $C_4$ depending on $C_1$ and $C_2$. By a similar argument as in [36], we obtain that

$$\|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T)} \leq C\|\tilde{w}\|_{W^{1,2}_p(\Omega_T)}$$

for some positive constant $C$ independent of $T^{-1}$. Thus

$$\|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T)} \leq C C_4(\|\tilde{g}\|_{C^1(0,T)} + \|\tilde{h}\|_{C^1(0,T)}).$$

(2.21)
Since \( \tilde{h}'(0) = h'_1(0) - h'_2(0) = 0 \), then
\[
\|\tilde{h}'\|_{C^1([0, T])} = \mu \|\tilde{\omega}\|_{C^{1,0}(\Omega_T)} \leq \mu \|\tilde{\omega}\|_{C^{1,1+\alpha}(\Omega_T)}.
\]
This, together with (2.21), implies that
\[
\|\tilde{h}\|_{C^1([0, T])} \leq 2T^{\frac{\alpha}{2}} \|\tilde{h}'\|_{C^1([0, T])} \leq C_5 T^{\frac{\alpha}{2}} (\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])}),
\]
where \( C_5 = 2\mu CC_4 \). Similarly, we have
\[
\|\tilde{g}\|_{C^1([0, T])} \leq C_5 T^{\frac{\alpha}{2}} (\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])}).
\]
As a consequence, we deduce that
\[
\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])} \leq 2C_5 T^{\frac{\alpha}{2}} (\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])}).
\]
Hence for
\[
T := \min \left\{ \frac{h_0}{16(1 + h_1 + h_2)}, \tau, \frac{C_2^{2-\alpha}}{\alpha}, (4C_5)^{-\frac{2}{\alpha}} \right\},
\]
we have
\[
\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])} \leq \frac{1}{2} (\|\tilde{g}\|_{C^1([0, T])} + \|\tilde{h}\|_{C^1([0, T])}).
\]
This shows that \( \tilde{g} \equiv 0 = \tilde{h} \) for \( 0 \leq t \leq T \), thus \( \tilde{w} \equiv 0 \) in \( [0, T] \times [l_1, l_2] \). Consequently, the uniqueness of solution of (\( P \)) is established, which ends the proof of this theorem. \( \square \)

Lemma 2.2 Assume that (H) holds. Then every positive solution \( (u, g, h) \) of problem (\( P \)) exists and is unique for all \( t \in (0, \infty) \).

**Proof** Let \( [0, T_{\max}] \) be the maximal time interval in which the solution exists. In view of Theorem 2.1, it remains to show that \( T_{\max} = \infty \). We proceed by a contradiction argument and assume that \( T_{\max} < \infty \). Thanks to the choice of the initial data, the comparison principle implies that \( u(t, x) \leq u^* \) for \( (t, x) \in (0, T_{\max}) \times [g(t), h(t)] \). Construct the auxiliary function
\[
\tilde{u}(t, x) = u^* \left[ 2M(h(t) - x) - M^2(h(t) - x)^2 \right], \quad t \in [-\tau, T_{\max}], \ x \in [h(t) - M^{-1}, h(t)],
\]
where
\[
M := \max \left\{ \sqrt{d}, \frac{2}{h(-\tau) - g(-\tau)}, \frac{4}{3u^*} \max_{-\tau \leq \theta \leq 0} \|\phi(\theta, \cdot)\|_{C^1([g(\theta), h(\theta)])} \right\}.
\]
It follows the proof of [10, Lemma 2.2] to prove that there is a constant \( C_0 \) independent on \( T_{\max} \) such that \( h'(t) \leq C_0 \) for \( t \in (0, T_{\max}) \). The proof for \( -g'(t) \leq C_0 \) for \( t \in (0, T_{\max}) \) is parallel.

Let us now fix \( \epsilon \in (0, T_{\max}) \). Similar to the proof of Theorem 2.1, by standard \( L^p \) estimate, the Sobolev embedding theorem and the Hölder estimates for parabolic equation, we can find \( C_1 > 0 \) depending only on \( \epsilon, T_{\max}, u^*, h_0, \|\phi\|_{C^{1,2}([-\tau, 0] \times [g(\theta), h(\theta)])} \) and \( C_0 \) such that
\[
\|u\|_{C^{1,\alpha/2+\alpha}([\epsilon, T_{\max}] \times [g(t), h(t)])} \leq C_1.
\]
This implies that \( (u, g, h) \) exists on \( [0, T_{\max}] \). Choosing \( t_n \in (0, T_{\max}) \) with \( t_n \uparrow T_{\max} \), and regarding \( (u(t_n - \theta, x), g(t_n - \theta), h(t_n - \theta)) \) for \( \theta \in [0, \tau] \) as the initial function, it then follows from the proof of Theorem 2.1 that there exists \( s_0 > 0 \) depending on \( C_0, C_1 \) and \( u^* \) independent of \( n \) such that problem (\( P \)) has a unique solution \( (u, g, h) \) in \( [t_n, t_n + s_0] \).
This yields that the solution \((u, g, h)\) of (P) can be extended uniquely to \([0, t_n + s_0]\). Hence 
\(t_n + s_0 > T_{\text{max}}\) when \(n\) is large. But this contradicts the assumption, which ends the proof of this lemma.

\[\square\]

**Proof of Theorem 1.1:** Combining Theorem 2.1 and Lemma 2.2, we complete the proof. \[\square\]

### 2.3 Comparison principle

In this subsection, we establish the comparison principle, which will be used in the rest of this paper. Let us start with the following result.

**Lemma 2.3** Suppose that (H) holds, \(T \in (0, \infty), \overline{g}, \overline{h} \in C^1([-\tau, T]), \overline{u} \in C(\overline{D_T}) \cap C^{1,2}(D_T)\) satisfies \(\overline{u} \leq u^*\) in \(\overline{D_T}\) with \(D_T = \{(t, x) \in \mathbb{R}^2 : -\tau < t \leq T, \overline{g}(t) < x < \overline{h}(t)\}\), and

\[
\begin{align*}
\overline{u}_t &\geq \overline{u}_{xx} - d\overline{u} + f(\overline{u}(t - \tau, x)), 
0 < t \leq T, \overline{g}(t) < x < \overline{h}(t), \\
\overline{u} &= 0, \quad \overline{g}(t) \leq \mu \overline{u}_x, 
0 < t \leq T, \quad x = \overline{g}(t), \\
\overline{u} &= 0, \quad \overline{h}'(t) \geq \mu \overline{u}_x, 
0 < t \leq T, \quad x = \overline{h}(t).
\end{align*}
\]

If \([g(\theta), h(\theta)] \subseteq [\overline{g}(\theta), \overline{h}(\theta)]\) for \(\theta \in [-\tau, 0]\) and \(\overline{u}(\theta, x) \in C^{1,2}([-\tau, 0] \times [g(\theta), h(\theta)])\) satisfies

\[
\phi(\theta, x) \leq \overline{u}(\theta, x) \leq u^* \text{ in } [-\tau, 0] \times [g(\theta), h(\theta)],
\]

then the solution \((u, g, h)\) of problem (P) satisfies \(g(t) \geq \overline{g}(t), h(t) \leq \overline{h}(t)\) in \((0, T)\), and

\[
\begin{align*}
u(t, x) &\leq \overline{u}(t, x) \text{ for } (t, x) \in (0, T) \times (g(t), h(t)).
\end{align*}
\]

**Proof** We integrate the ideas of [10, Lemma 5.7] and [26, Corollary 5] to deal with free boundary and time delay.

Firstly, for small \(\epsilon > 0\), let \((u_\epsilon, g_\epsilon, h_\epsilon)\) denote the unique solution of (P) with \(g(\theta)\) and \(h(\theta)\) replaced by \(g_\epsilon(\theta) := g(\theta)(1 - \epsilon)\) and \(h_\epsilon(\theta) := h(\theta)(1 - \epsilon)\) for \(\theta \in [-\tau, 0]\), respectively, with \(\mu\) replaced by \(\mu_\epsilon := \mu(1 - \epsilon)\), and with \(\phi(\theta, x)\) replaced by some \(\phi_\epsilon(\theta, x) \in C^{1,2}([-\tau, 0] \times [g_\epsilon(\theta), h_\epsilon(\theta)])\), satisfying

\[
0 < \phi_\epsilon(\theta, x) \leq \phi(\theta, x), \quad \phi_\epsilon(\theta, g_\epsilon(\theta)) = \phi(\theta, h_\epsilon(\theta)) = 0
\]

for \(\theta \in [-\tau, 0], x \in [g_\epsilon(\theta), h_\epsilon(\theta)]\), and for any fixed \(\theta \in [-\tau, 0]\) as \(\epsilon \to 0, \phi_\epsilon(\theta, x) \to \phi(\theta, x)\) in the \(C^2([g(\theta), h(\theta)])\) norm.

We claim that \(h_\epsilon(t) \leq \overline{h}(t)\), \(g_\epsilon(t) > \overline{g}(t)\) and \(u_\epsilon(t, x) \leq \overline{u}(t, x)\) for all \(t \in [0, T]\) and \(x \in [g_\epsilon(t), h_\epsilon(t)]\). Obviously, this is true for all small \(t > 0\). Now, let us use an indirect argument and suppose that the claim does not hold, then there exists a first \(t^* \in (0, T)\) such that

\[
u_\epsilon(t, x) < \overline{u}(t, x) \text{ for } t \in [0, t^*), x \in [g_\epsilon(t), h_\epsilon(t)] \subset (\overline{g}(t), \overline{h}(t)),
\]

and there is some \(x^* \in [g_\epsilon(t^*), h_\epsilon(t^*)]\) such that \(u_\epsilon(t^*, x^*) = \overline{u}(t^*, x^*)\).

Later, let us compare \(u_\epsilon\) and \(\overline{u}\) over the region

\[
\Omega_{t^*} := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq t^*, g_\epsilon(t) < x < h_\epsilon(t)\}.
\]

An direct computation shows that for \((t, x) \in \Omega_{t^*},
\[
(\overline{u} - u_\epsilon)_t - (\overline{u} - u_\epsilon)_{xx} + d(\overline{u} - u_\epsilon) \geq f(\overline{u}(t - \tau, x)) - f(u_\epsilon(t - \tau, x)) \geq 0.
\]
it then follows from the strong maximum principle that
\[ u_\epsilon(t, x) < \overline{u}(t, x) \quad \text{in } \Omega_{t^*}. \tag{2.22} \]

Thus either \( x^* = h_\epsilon(t^*) \) or \( x^* = g_\epsilon(t^*) \). Without loss of generality we may assume that \( x^* = h_\epsilon(t^*) \), then \( \overline{u}(t^*, h_\epsilon(t^*)) = u_\epsilon(t^*, h_\epsilon(t^*)) = 0 \). This, together with (2.22), implies that \( \overline{u}_h(t^*, h_\epsilon(t^*)) \leq (u_\epsilon)_h(t^*, h_\epsilon(t^*)) \), from which we obtain that
\[ h'_\epsilon(t^*) = -\mu \epsilon (u_\epsilon)_h(t^*, h_\epsilon(t^*)) < -\mu \overline{u}_h(t^*, h_\epsilon(t^*)) = \overline{h}'(t^*). \tag{2.23} \]

As \( h_\epsilon(t) < \overline{h}(t) \) for \( t \in [0, t^*) \) and \( h_\epsilon(t^*) = \overline{h}(t^*) \), then \( h'_\epsilon(t^*) \geq \overline{h}'(t^*) \), which contradicts (2.23). This proves our claim.

Finally, thanks to the unique solution of \((P)\) depending continuously on the parameters in \((P)\), as \( \epsilon \to 0 \), \( (u_\epsilon, g_\epsilon, h_\epsilon) \) converges to \((u, g, h)\), the unique solution of \((P)\). The desired result then follows by letting \( \epsilon \to 0 \) in the inequalities \( u_\epsilon < \overline{u} \), \( g_\epsilon \geq \overline{g} \) and \( h_\epsilon < \overline{h} \). \(\square\)

By slightly modifying the proof of Lemma 2.3, we obtain a variant of Lemma 2.3.

**Lemma 2.4** Suppose that \((H)\) holds, \( T \in (0, \infty) \), \( \overline{g}, \overline{h} \in C^1([-\tau, T]) \), \( \overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T) \) satisfies \( \overline{u} \leq u^* \) in \( \overline{D}_T \) with \( D_T = \{(t, x) \in \mathbb{R}^2 : -\tau < t \leq T, \ \overline{g}(t) < x < \overline{h}(t)\} \), and
\[
\begin{align*}
\overline{u}_t &\geq \overline{u}_{xx} - d\overline{u} + f(\overline{u}(t - \tau, x)), \quad 0 < t \leq T, \ \overline{g}(t) < x < \overline{h}(t), \\
\overline{u} &\geq u, \quad 0 < t \leq T, \ \overline{g}(t) < x < \overline{h}(t), \\
\overline{u} = 0, \quad \overline{h}'(t) &\geq -\mu \overline{u}_x, \quad 0 < t \leq T, \ \overline{g}(t) < x < \overline{h}(t),
\end{align*}
\]
with \( \overline{g}(t) \geq g(t) \) in \([0, T]\), \( h(\theta) \leq \overline{h}(\theta) \), \( \phi(\theta, x) \leq \overline{u}(\theta, x) \) for \( \theta \in [-\tau, 0] \) and \( x \in [\overline{g}(\theta), h(\theta)] \), where \((u, g, h)\) is a solution to \((P)\). Then
\[ h(t) \leq \overline{h}(t) \text{ in } (0, T], \quad u(x, t) \leq \overline{u}(x, t) \text{ for } (x, t) \in (0, T] \times (g(t), h(t)). \]

**Remark 2.5** The function \( \overline{u} \), or the triple \((\overline{u}, \overline{g}, \overline{h})\), in Lemmas 2.3 and 2.4 is often called a supersolution to \((P)\). A subsolution can be defined analogously by reversing all the inequalities. There is a symmetric version of Lemma 2.4, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.

### 3 Semi-waves

This Sect. is devoted to proving the existence and uniqueness of a semi-wave \( q(z) \) of (1.13), which will be used to construct some suitable sub- and supersolutions to study the asymptotic profiles of spreading solutions of \((P)\). Let us consider the following nonlocal elliptic problem
\[
\begin{align*}
q'' - cq' - dq + f(q(z - c\tau)) &= 0, \quad z > 0, \\
q(z) &= 0, \quad z \leq 0,
\end{align*}
\tag{3.1}
\]
where \( c \geq 0 \) is a constant.

If \( z \) is understood as the time variable, then we may regard problem (3.1) as a time-delayed dynamical system in the phase space \( C([-c\tau, 0], \mathbb{R}^2) \). When \( c\tau = 0 \), the phase space reduces to \( \mathbb{R}^2 \) and it follows from the phase plane analysis that (3.1) admits a unique positive solution \( q_0(z) \), which is increasing in \( z \) and \( q_0(z) \to u^* \) as \( z \to \infty \). When \( c\tau > 0 \), the phase space is of infinite dimension and the positivity and boundedness of the unique solution are not clear.
Proposition 3.1 Suppose (H) holds. For any given constant $c > 0$, problem (3.1) has a maximal nonnegative solution $q_c$. Moreover, either $q_c(z) \equiv 0$ or $q_c(z) > 0$ in $(0, \infty)$. Furthermore, if $q_c > 0$, then it is the unique positive solution of (3.1), $q'_c(z) > 0$ in $(0, \infty)$ and $q_c(z) \to u^*$ as $z \to \infty$, in addition, for any given constant $c_1 < c$, one has $q_c(z) < q_{c_1}(z)$ for $z \in (0, \infty)$, and $(q_c)'_+(0) < (q_{c_1})'_+(0)$.

Proof We divide the proof into four steps.

Step 1. Problem (3.1) always has a maximal nonnegative solution $\overline{q}$ and it satisfies

$$\overline{q} \leq u^* \quad \text{for } z \in [0, \infty).$$

Clearly, $0$ is a nonnegative solution of (3.1). For any $l > 0$, consider the following problem:

$$\begin{cases}
    w'' - cw' - dw + f(w(z - ct)) = 0, & 0 < z < l, \\
    w(l) = u^*, \quad w(z) = 0, & z \leq 0.
\end{cases}$$

(3.2)

It is well known problem (3.2) admits a unique solution $w^l(z) > 0$ for $z \in (0, l]$. Applying the maximal principle, we can deduce that $w^l(z) \leq u^*$ for $z \in [0, l]$. Moreover, it is easy to check that $w^l(z)$ is decreasing in $l > 0$ and increasing in $z \in [0, l]$ and

$$w^l(z) \to W(z) \quad \text{as } l \to \infty,$$

where $W(z)$ is a nonnegative solution of problem (3.1) and it satisfies $W(z) \leq u^*$ for $z \in [0, \infty)$.

In what follows, we want to prove that $W$ is the maximal nonnegative solution of (3.1). Let $q$ be an arbitrary nonnegative solution of (3.1), then $q(z) \leq u^*$ for $z \in [0, \infty)$. If $q \equiv 0$, then $q \leq W$. Suppose now $q \geq, \neq 0$, then $q > 0$ in $(0, \infty)$. Let us show $q(z) \leq W(z)$ for $z \in [0, \infty)$.

Firstly, for any fixed $l > 0$ we can find $M > 0$ large such that $Mw^l(z) \geq q(z)$ for $z \in [0, l]$. We claim that the above inequality holds for $M = 1$. On the contrary, define

$$M_0 := \inf\{M > 0 : \ Mw^l(z) \geq q(z) \quad \text{for } z \in [0, l]\},$$

then $M_0 > 1$ and $M_0w^l(z) \geq, \neq q(z)$ for $z \in [0, l]$. Thanks to the monotonicity of $w^l(z)$ in $z \in [0, l]$, then there is $z_0 \in (0, l)$ such that $M_0w^l(z_0) = u^*$ and $M_0w^l(z) < u^*$ for $z \in [0, z_0)$. It is easy to check that $q(z_0) < u^*$. Then the strong maximal principle yields that $M_0q^l(0) > q^l(0)$ and $M_0w^l(z) > q(z)$ for $z \in (0, z_0)$. Thus we can find a constant $0 < \epsilon \ll 1$ such that

$$M_1 := M_0(1 + \epsilon)^{-1} > 1, \quad M_1w^l(z) > q(z) \quad \text{for } z \in (0, z_0].$$

(3.3)

and $M_1w^l(z_0 + \tilde{z}) > u^*$ for $\tilde{z} = \min\{ct, \ l - z_0\}$. So there is $z_1 \in (0, \tilde{z}]$ such that $M_1w^l(z_0 + z_1) = u^*$ and $M_1w^l(z_0 + z) > u^*$ for $z \in (z_1, l - z_0]$.

Later, we want to prove that $M_1w^l(z) > q(z)$ for all $z \in (z_0, l]$. Combining the definition of $z_1$, we only need to prove $M_1w^l(z) \geq q(z)$ for all $z \in (z_0, z_0 + z_1]$. Since $M_1w^l(z) \geq q(z)$ for $z = z_0 + z_1$ and $z = z_0$, and for $z \in (z_0, z_0 + z_1]$,

$$(M_1w^l - q)'' - c(M_1w^l - q)' - d(M_1w^l - q)$$

$$= f(q(z - ct)) - M_1f(w^l(z - ct)) \leq f(q(z - ct)) - f(M_1w^l(z - ct)) \leq 0,$$

where the monotonicity of $f(v)/v$ and $f(v)$ in $v \in [0, u^*]$ and the fact where $M_1w^l(z - ct) \geq q(z - ct)$ for $z \leq z_0 + z_1$ are used. The comparison principle yields that $M_1w^l(z) \geq q(z)$.
for all \( z \in [z_0, z_0 + z_1] \). This, together with the definition of \( z_1 \) and \( (3.3) \), yields that \( M_1 u'(z) \geq q(z) \) for all \( z \in (0, l] \), which contradicts the definition of \( M_0 \). Then \( u'(z) \geq q(z) \) for \( z \in [0, l] \) is proved.

Finally, letting \( l \to \infty \), we deduce that

\[
W(z) \geq q(z) \quad \text{for } z \in [0, \infty),
\]
as we wanted. Thus Step 1 is proved.

Step 2. For any \( c \geq 0 \), if \( q \) is a positive solution of \( (3.1) \), then \( q'_+(0) > 0, q'(z) > 0 \) for \( z \in (0, \infty) \), and \( q(z) \to u^* \) as \( z \to \infty \).

Since \( q > 0 \) for \( z > 0 \), then the Hopf lemma can be used to deduce \( q'_+(0) > 0 \), it follows that \( q'(z) > 0 \) for all small \( z > 0 \). Setting

\[
y^* := \sup \{ \gamma > 0 : q(2\gamma - z) > q(z) \text{ for } z \in [0, \gamma), \quad q'(z) > 0 \text{ for } z \in (0, \gamma) \}.
\]

In the following, we shall show \( y^* = \infty \). Suppose by way of contradiction that \( y^* \in (0, \infty) \), then

\[
q(2y^* - z) \geq q(z) \quad \text{and} \quad q'(z) \geq 0 \quad \text{for } z \in [0, y^*].
\]

Define \( \tilde{q}(z) = q(2y^* - z) \) for \( z \in [y^*, 2y^*] \), then

\[
\tilde{q}'' - c\tilde{q}' - d\tilde{q} + f(\tilde{q}(z - c\tau)) = -2cq\xi, \quad \xi = 2y^* - z \in [0, y^*].
\]

Let us set

\[
Q(z; y^*) = Q(z) = \tilde{q}(z) - q(z) = q(\xi) - q(2y^* - \xi).
\]

Then \( Q \leq 0 \) for \( z \in [y^*, 2y^*] \) and it satisfies

\[
\begin{align*}
Q'' - cQ' - dQ &= f(q(z - c\tau)) - f(\tilde{q}(z - c\tau)) - 2cq\xi \leq 0, \\
Q(y^*) &= 0, \\
Q(2y^*) &= -q(2y^*) < 0.
\end{align*}
\]

(3.4)

The strong maximal principle and the Hopf lemma imply that

\[
Q(z) < 0, \quad z \in (y^*, 2y^*], \\
Q'(y^*) < 0.
\]

It follows the continuity that for all small \( \varepsilon \geq 0, \)

\[
Q'(y^* + \varepsilon; y^* + \varepsilon) < 0, \quad Q(z; y^* + \varepsilon) < 0 \quad \text{for } z \in (y^* + \varepsilon, 2y^* + 2\varepsilon),
\]

which implies that \( q(2y^* + 2\varepsilon - \xi) > q(\xi) \) for \( \xi \in [0, y^* + \varepsilon) \). Moreover, since \( Q'(y^* + \varepsilon; y^* + \varepsilon) = -2q'(y^* + \varepsilon) \), it then follows that \( q'(y^* + \varepsilon) > 0 \). But these facts contradict the definition of \( y^* \). Thus the monotonicity of positive solutions of \( (3.1) \) is established.

Next, we consider the asymptotic behavior of positive solution \( q \) of \( (3.1) \). The monotonicity of \( q \) in \( z > 0 \) implies that there is a constant \( a > 0 \) such that \( \lim_{z \to \infty} q(z) = a \). We claim that \( a = u^* \). For any sequence \( \{z_n\} \) with \( z_n \to \infty \) as \( n \to \infty \), define \( q_n(z) = q(z + z_n) \). Then \( q_n \) solves the same equation as \( q \) but over \(( -z_n, \infty) \). Since \( q_n \leq u^* \), it follows that there is a subsequence of \( \{q_n\} \) (still denoted by \( \{q_n\} \)) such that \( q_n \to \hat{q} \) locally in \( C^2(\mathbb{R}) \) as \( n \to \infty \), and \( \hat{q} \) solves

\[
v'' - cv' - dv + f(v(z - c\tau)) = 0, \quad z \in \mathbb{R}.
\]

On the other hand, it follows from \( \lim_{z \to \infty} q(z) = a \) that \( \hat{q} \equiv a \), which implies that \( a = u^* \), as we wanted. Thus this completes the proof of Step 2.

Step 3. We show that problem \( (3.1) \) has at most one positive solution.
Suppose problem (3.1) has two positive solutions \( q_1 \) and \( q_2 \), then \( 0 < q_i < u^* \) in \((0, \infty)\), and \( q_i(z) \to u^* \) as \( z \to \infty \) for \( i = 1, 2 \). Define
\[
\rho^* := \inf \left\{ \frac{q_1(z)}{q_2(z)} : z > 0 \right\}.
\]
From Step 2 we have \((q_i)^{\prime}_+(0) > 0, i = 1, 2 \). Then by L’Hôpital’s rule we obtain
\[
\lim_{z \to 0} \frac{q_1(z)}{q_2(z)} > 0,
\]
which together with \( \lim_{z \to +\infty} \frac{q_1(z)}{q_2(z)} = 1 \) implies that \( \rho^* \in (0, 1] \). Next we show \( \rho^* = 1 \). Indeed, assume for the sake of contraction that \( \rho^* \in (0, 1) \). Define
\[
w(z) := q_1(z) - \rho^* q_2(z).
\]
Then \( w(z) \geq 0 \) for \( z \geq 0 \), \( w(0) = 0 \), \( w(+\infty) = (1 - \rho^*)u^* > 0 \) and
\[
w^{\prime\prime} - cw^{\prime} - dw = -f(q_1(z - c\tau)) + \rho^* f(q_2(z - c\tau)) \leq 0,
\]
where the sub-linearity and monotonicity of \( f(z) \) for \( z \in (0, u^*) \) are used. By Hopf’s lemma, we see that \( 0 < w^\prime(0) = (q_1)^\prime_+(0) - \rho^* (q_2)^\prime_+(0) \), which implies that \( \lim_{z \to 0} \frac{q_1(z)}{q_2(z)} > \rho^* \).

Thus, in view of the definition of \( \rho^* \), we have an \( z_0 \in (0, +\infty) \) such that \( w(z_0) = 0 \). By the elliptic strong maximum principle, we infer that \( w(z) \equiv 0 \) for \( z > 0 \), a contradiction to \( w(+\infty) > 0 \). Therefore, \( \rho^* = 1 \), and hence, \( q_1(z) \geq q_2(z) \). Changing the role of \( q_1 \) and \( q_2 \) and repeating the above arguments, we obtain \( q_2(z) \geq q_1(z) \). The uniqueness is proved.

**Step 4.** Let us consider the monotonicity of positive solutions in \( c \).

Assume that \( q_c \) is a positive solution of (3.1). Choose \( c_1 < c \) and let \( q_{c_1} \) be the maximal nonnegative solution of (3.1) with \( c = c_1 \). Since \( u^* \) is a supersolution of (3.1), and by Step 2 we know that \( q_c \) is a subsolution of (3.1) with \( c = c_1 \), in view of the uniqueness of positive solution of this problem, then we see that \( q_{c_1}(z) \geq q_c(z) \) for \( z \in [0, \infty) \). It thus follows from the maximum principle and the Hopf lemma that

\[
q_{c_1}(z) > q_c(z) \quad \text{for} \quad z \in (0, \infty), \quad \text{and} \quad (q_{c_1})^{\prime}_+(0) > (q_c)^{\prime}_+(0).
\]

The proof of this proposition is complete now. \( \square \)

Next we give a necessary and sufficient condition for the existence of a positive solution of (3.1). For this purpose, we need the following property on the distribution of complex solutions to a transcendental equation.

**Lemma 3.2** Let \( c \geq 0 \) and \( \tau \geq 0 \). Define
\[
\Delta_c(\lambda, \tau) = \lambda^2 - c\lambda - d + f^{\prime}(0)e^{-\lambda c\tau}.
\]
Then there exists \( c_0(\tau) \in (0, 2\sqrt{f^{\prime}(0) - d}) \) such that the following statements hold:

(i) \( \Delta_c(\lambda, \tau) = 0 \) has a positive solution if and only if \( c \geq c_0(\tau) \);

(ii) \( \Delta_c(\lambda, \tau) = 0 \) has a complex solution in the domain
\[
\Omega := \left\{ \lambda \in \mathbb{C} : \text{Re}\lambda > 0, \text{Im}\lambda \in \left(0, \frac{\pi}{c\tau}\right) \right\},
\]
provided that \( c \in (0, c_0(\tau)) \).

Before the proof, we note that if \( \tau = 0 \) then \( \Delta_c(\lambda, \tau) = 0 \) reduces to a polynomial equation of order 2. It admits at least one positive solution if and only if \( c \geq 2\sqrt{f^{\prime}(0) - d} \) and exactly a pair of complex eigenvalues in \( \Omega \) when \( c \in (0, 2\sqrt{f^{\prime}(0) - d}) \).
We employ a continuation method with \( \tau \). It then follows from the implicit function theorem that for small \( \alpha, \beta \), \( \alpha^2 - \beta^2 - c \alpha - d + f'(0)e^{-c\tau \alpha} \cos c \tau \beta = 0 \), \( 2 \alpha \beta - c \beta - f'(0)e^{-c\tau \alpha} \sin c \tau \beta = 0 \), when \( |c| > 0 \), which is continuous in \( 0 < \tau < \tau^* \) and has no solution. Indeed, At \( \tau = 0 \), (3.7) admits \( \alpha(\tau) \) such that \( \alpha(\tau) = \alpha(0) \) where the equality holds if and only if

\[
\alpha' = \frac{c}{2}, \quad \beta = \frac{c}{2}. 
\]

We proceed with four steps.

**Step 1.** If \( \tau \) is small enough, then there is a solution in \( \Omega \). Indeed, At \( \tau = 0 \), (3.7) admits a solution \( (\alpha, \beta) = \left( \frac{c}{2}, \frac{\sqrt{|c|^2 - (f'(0))}^2}{2} \right) \). Note that

\[
\det \left( \begin{array}{cc} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{array} \right) \bigg|_{\tau=0} = \det \left( \begin{array}{cc} 2\alpha - c & -2\beta \\ 2\beta & 2\alpha + c \end{array} \right) > 0. 
\]

It then follows from the implicit function theorem that for small \( \tau \), \( \Delta_c(\lambda, \tau) \) admits a complex solution near \( \frac{c}{2} + i \frac{\sqrt{|c|^2 - (f'(0))}^2}{2} \), and hence, in the open domain \( \Omega \).

**Step 2.** For any \( \tau > 0 \), \( \Delta_c(\lambda, \tau) \) admits no solution with \( \beta = 0 \) or \( \beta = \frac{\pi}{c \tau} \) when \( c \tau > 0 \). It follows from statement (i) that there is no solution with \( \beta = 0 \) when \( c < c_0(\tau) \). If \( \beta \) equals \( \frac{\pi}{c \tau} \), then from the second equation of (3.7) we can infer that \( \alpha = \frac{c}{2} \). Substituting \( \alpha = \frac{c}{2} \) and \( \beta = \frac{\pi}{c \tau} \) into the first equation of (3.7), we obtain

\[
0 = -\frac{1}{4} c^2 - \left( \frac{\pi}{c \tau} \right)^2 - d - f'(0)e^{-c^2 \tau/2},
\]

a contradiction.

**Step 3.** If a solution \( \alpha(\tau) + i \beta(\tau) \) touches pure imaginary axis at some \( \tau = \tau^* > 0 \), then \( \alpha'(\tau^*) > 0 \). We use the implicit function theorem. By direct computations, we have

\[
\det \left( \begin{array}{cc} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{array} \right) \bigg|_{\tau=\tau^*} = \det \left( \begin{array}{cc} -c - c \tau f'(0) \cos c \tau \beta & -2\beta - c \tau f'(0) \sin c \tau \beta \\ 2\beta + c \tau f'(0) \sin c \tau \beta & -c - c \tau f'(0) \cos c \tau \beta \end{array} \right) = [ -c - c \tau f'(0) \cos c \tau \beta ]^2 + [2\beta + c \tau f'(0) \sin c \tau \beta]^2 \geq 0,
\]

where the equality holds if and only if \( -c - c \tau f'(0) \cos c \tau \beta = 0 \) and \( 2\beta + c \tau f'(0) \sin c \tau \beta = 0 \). Taking these two relations into (3.7) with \( \alpha = 0 \), we obtain

\[
\begin{cases} 
-\beta^2 - d - \frac{1}{\tau} = 0, \\
-c \beta + \frac{2\beta}{c \tau} = 0,
\end{cases}
\]

which is not solvable for \( \beta \). Therefore,

\[
\det \left( \begin{array}{cc} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{array} \right) \bigg|_{\tau=\tau^*} > 0.
\]

On the other hand,

\[
\left( \begin{array}{c} \partial_\tau F_1 \\ \partial_\tau F_2 \end{array} \right) \bigg|_{\tau=\tau^*} = -c \beta f'(0) \left( \begin{array}{c} \sin c \tau \beta \\ \cos c \tau \beta \end{array} \right).
\]
Consequently, by the implicit function theorem we have
\[
\left( \frac{\alpha'(\tau^*)}{\beta'(\tau^*)} \right)_{|\tau=\tau^*} = - \left( \frac{\partial_x F_1}{\partial_x F_2} \right) \left( \frac{\partial_{\beta} F_1}{\partial_{\beta} F_2} \right)^{-1} \left|_{\tau=\tau^*} \right),
\]
from which we compute to have
\[
\alpha'(\tau^*) = \frac{(2\beta^4 + 2d\beta^2 + c^2)c}{\det \left( \frac{\partial_x F_1}{\partial_x F_2} \frac{\partial_{\beta} F_1}{\partial_{\beta} F_2} \right)_{|_{\tau=\tau^*}}} > 0.
\]

\textbf{Step 4.} Completion of the proof. In Steps 2 and 3, we have verified that the perturbed solution at Step 1 cannot escape \(\Omega\) continuously as \(\tau\) increases from 0 to \(\infty\). Therefore, it always stays in \(\Omega\). \(\Box\)

Based on the above results, we are ready to give the following necessary and sufficient condition for (3.1) to have a unique positive solution.

\textbf{Proposition 3.3} Suppose \((H)\) holds. Problem (3.1) has a unique positive solution \(q \in C^2([0, \infty))\) if and only if \(c \in [0, c_0(\tau)]\), where \(c_0(\tau)\) is given in Lemma 3.2.

\textbf{Proof} Firstly, let us employ the super- and subsolution method to show that problem (3.1) admits a unique positive solution when \(c \in [0, c_0(\tau)]\). The case where \(c \tau = 0\) is trivial and the proof is omitted. Fix \(c \in (0, c_0(\tau))\). It follows from Lemma 3.2 that there exists \(\gamma > 0\) such that
\[
\tilde{L}(\lambda) = \lambda^2 - c\lambda - d + (1 - \gamma)f'(0)e^{-\lambda\tau} = 0
\]
has a solution \(\lambda = \alpha + i\beta\) in \(\Omega\).

\textbf{Claim.} The function
\[
\psi(x) := \begin{cases} 
\delta e^{\alpha x} \cos \beta x, & \beta x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \\
0, & \text{elsewhere},
\end{cases}
\]
is a subsolution provided that \(\delta\) is small enough.

Indeed, for \(\beta x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)\), we have
\[
L[\psi](x) := \psi''(x) - c\psi'(x) - d\psi(x) + f(\psi(x - c\tau)) = \psi(x) \left[ \alpha^2 - \beta^2 - ca - d - [2\alpha\beta - c\beta] \tan \beta x \right] + f(\psi(x - c\tau))
\]
\[
= -\psi(x) \frac{1}{\cos \beta x} (1 - \gamma)f'(0)e^{-\lambda\tau} \cos(b(x - c\tau)) + f(\psi(x - c\tau))
\]
\[
= -(1 - \gamma)f'(0)\delta e^{\alpha(x - c\tau)} \cos(b(x - c\tau)) + f(\psi(x - c\tau)).
\]
Choose \(\delta > 0\) sufficiently small such that
\[
f(\psi(x - c\tau)) \geq (1 - \gamma)f'(0)\psi(x - c\tau),
\]
with which we obtain
\[
L[\psi](x) \geq (1 - \gamma)f'(0)[\psi(x - c\tau) - \delta e^{\alpha(x - c\tau)} \cos(b(x - c\tau))].
\]
Clearly, if \(\beta(x - c\tau) \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)\), then \(\psi(x - c\tau) = \delta e^{\alpha(x - c\tau)} \cos(b(x - c\tau))\), and hence,
\[
L[\psi](x) \geq 0. \text{ If } \beta(x - c\tau) \notin \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \text{ then } \psi(x - c\tau) = 0, \text{ and hence,}
\]
\[
L[\psi](x) \geq -(1 - \gamma)f'(0)\delta e^{\alpha(x - c\tau)} \cos(b(x - c\tau)).
\]

with $\beta(x - c\tau) \in \left(\frac{3\pi}{2} - \beta c\tau, \frac{5\pi}{2} - \beta c\tau\right) \setminus \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$. Since $\beta c\tau \leq \pi$ (as proved in Lemma 3.2), we obtain $\cos \beta(x - c\tau) \leq 0$ when $\beta(x - c\tau) \in \left(\frac{3\pi}{2} - \beta c\tau, \frac{5\pi}{2} - \beta c\tau\right) \setminus \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$. To summarize, $L[y](x) \geq 0$ for $\beta x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ and $L[y](x) = 0$ for $\beta x \not\in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$. The claim is proved.

Having such a subsolution, we can infer that (3.1) admits a positive solution when $c \in [0, c_0(\tau))$. The proof of uniqueness of the solution of (3.1) follows from Proposition 3.1.

Next we show that (3.1) does not admit a positive solution when $c \geq c_0(\tau)$. We employ a sliding argument. Assume for the sake of contradiction that there is a solution $q(z)$. Since $c \geq c_0(\tau)$, $\Delta c(\lambda, \tau) = 0$ admits a positive solution $\lambda_1$. Define $w(z) = \sigma e^{\lambda_1 z} - q(z)$, $\sigma > 0$. Since $q(0) = 0$ and $q(+\infty) = u^*$, we may choose $\sigma$ such that $w(z) \geq 0$ for $z \geq 0$ and $w(z)$ vanishes at some $z \in (0, +\infty)$. Note that $f'(u) \leq f'(0)u$. It then follows that

$$w''(z) - cw'(z) - dw(z) = -f'(0)w(z-c\tau) + [f(q(z-c\tau)) - f'(0)q(z-c\tau)] \leq 0, \quad z \geq 0.$$  

(3.13)

By the elliptic strong maximum principle, we obtain $w(z) = 0$ for $z \geq 0$, a contradiction. The nonexistence is proved.

Based on the above results, we obtain the solvability of (1.13).

**Theorem 3.4** For any given $\tau > 0$, let $c_0(\tau)$ be given in Lemma 3.2. For each $\mu > 0$, there exists a unique $c^* = c^*_\mu(\tau) \in (0, c_0(\tau))$ such that $(q_{c^*})'_+(0) = \frac{c^*}{\mu}$, where $q_{c^*}(z)$ is the unique positive solution of (3.1) with $c$ replaced by $c^*$. Moreover, $c^*_\mu(\tau)$ is increasing in $\mu$ with

$$\lim_{\mu \to \infty} c^*_\mu(\tau) = c_0(\tau).$$

**Proof** From Propositions 3.1 and 3.3, it is known that for each $c \in [0, c_0(\tau))$, problem (3.1) admits a unique solution $q_c(z) > 0$ for $z > 0$, and for any $0 \leq c_1 < c_2 \leq c_0(\tau)$, $q_{c_1}(z) > q_{c_2}(z)$ in $(0, \infty)$. Define

$$P(0; c, \tau) := (q_c)'_+(0).$$  

(3.14)

Then $P(0; c, \tau) > 0$ for all $c \in [0, c_0(\tau))$ and it decreases continuously in $c \in [0, c_0(\tau))$. Let $c_n \uparrow c_0(\tau)$. For each $c_n$ problem (3.1) admits a unique solution $q_{c_n}(z)$. Clearly, $q_{c_n}$ converges to some $q^*$ and $(q_{c_n})'$ converges to $(q^*)'$ locally uniformly in $z \in [0, +\infty)$, and $q^*$ solves (3.1) with $c = c_0(\tau)$. By the nonexistence established in Proposition 3.3 we obtain $q^* \equiv 0$. In particular,

$$\lim_{c \uparrow c_0(\tau)} (q_c)'_+(0) = (q^*)'_+(0) = 0.$$  

(3.15)

We now consider the continuous function

$$\eta(c; \tau) := \eta_\mu(c; \tau) := P(0; c, \tau) - \frac{c}{\mu}, \quad c \in [0, c_0(\tau)).$$

By the above discussion we know that $\eta(c; \tau)$ is strictly decreasing in $c \in [0, c_0(\tau))$. Moreover, $\eta(0; \tau) = P(0; 0, \tau) > 0$ and $\lim_{c \uparrow c_0(\tau)} \eta(c; \tau) = -c_0(\tau)/\mu < 0$. Thus there exists a unique $c^* = c^*_\mu(\tau) \in (0, c_0(\tau))$ such that $\eta(c^*; \tau) = 0$, which means that

$$(q_{c^*})'_+(0) = \frac{c^*}{\mu}.$$  

Next, let us view $(c^*_\mu, c^*_\mu/\mu)$ as the unique intersection point of the decreasing curve $y = P(0; c, \tau)$ with the increasing line $y = c/\mu$ in the $c\mu$-plane, then it is clear that $c^*_\mu(\tau)$ increases to $c_0(\tau)$ as $\mu$ increases to $\infty$. The proof is complete. □
Remark 3.5 In [11], the authors considered the case $\tau = 0$. They obtained that for each $\mu > 0$, there is a unique $c^* = c^*_\mu(0) \in (0, c_0(0))$ such that $(q_c)_\mu(0) = \frac{c^*}{\mu}$, where $q_c(z)$ is the unique solution of (3.1) with $\tau = 0$ and $c = c^*$, and $c_0(0) = 2\sqrt{f'(0) - d}$. Moreover, $c^*_\mu(0)$ is increasing in $\mu$ with $$\lim_{\mu \to \infty} c^*_\mu(0) = c_0(0).$$

In the rest of this part, we study the monotonicity of $c^*_\mu(\tau)$ in $\tau$. For any given $\tau \geq 0$, the unique positive solution of (3.1) with $c \in (0, c_0(\tau))$ may be denoted by $q_c(z; \tau)$. Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3: For $\tau \geq 0$ and $\mu > 0$, let $c^*_\mu(\tau)$ be given in Theorem 3.4 and Remark 3.5 for $\tau > 0$ and $\tau = 0$, respectively. By Propositions 3.1 and 3.3, we see that for $\tau \geq 0$ and $c \in (0, c_0(\tau))$, problem (3.1) admits a unique positive solution $q_c(z; \tau)$. Moreover, $q_c(z; \tau)$ is increasing in $z > 0$ and decreasing in $c \in (0, c_0(\tau))$. Let $P(0; c, \tau)$ be defined as in (3.14).

Claim. For $0 \leq \tau_1 < \tau_2$, $P(0; c, \tau_1) > P(0; c, \tau_2)$ when $c \in (0, c_0(\tau_2))$.

We postpone the proof of the claim and reach the conclusion in a few lines. Note that $c^*_\mu(\tau)$ is the unique positive solution of $P(0; c, \tau) - \frac{c}{\mu} = 0$. In view of $\lim_{c \uparrow c_0(\tau_2)} P(0; c, \tau_2) = 0$, we have $c^*_\mu(\tau_2) \in (0, c_0(\tau_2))$. If $c^*_\mu(\tau_1) \geq c_0(\tau_2)$, then we are done. Otherwise, $c^*_\mu(\tau_1) \in (0, c_0(\tau_2))$, which, together with the claim, implies that $$\frac{c^*_\mu(\tau_1)}{\mu} = P(0; c^*_\mu(\tau_1), \tau_1) > P(0; c^*_\mu(\tau_1), \tau_2).$$

This further implies that $c^*_\mu(\tau_1) > c^*_\mu(\tau_2)$, due to the monotonicity of $P(0; c, \tau_2) - \frac{c}{\mu}$ in $c \in (0, c_0(\tau_2))$. Thus, $c^*_\mu(\tau)$ is decreasing in $\tau \geq 0$.

Proof of the claim. Since $c_0(\tau)$ is decreasing in $\tau \geq 0$, we see that $P(0; c, \tau_1)$ is well-defined when $c \in (0, c_0(\tau_2))$. By the monotonicity of $q_c(z; \tau_2)$ in $z > 0$, we have $q_c(z - c\tau_2; \tau_2) < q_c(z - c\tau_1; \tau_2)$, this, together with the monotonicity of $f(v)$ in $v$, implies that $f(q_c(z - c\tau_2; \tau_2)) < f(q_c(z - c\tau_1; \tau_2))$. Consequently,

$$q_c''(z; \tau_2) - cq_c'(z; \tau_2) - dq_c(z; \tau_2) + f(q_c(z - c\tau_1; \tau_2)) > 0, \quad z > 0.$$ 

Consider the initial value problem

$$\begin{align*}
v_t &= v_{zz} - cv_z - dv + f(v(t, z - c\tau_1)), \quad t > 0, \quad z > 0, \\
v(t, z) &= 0, \quad t > 0, \quad z \leq 0, \\
v(0, z) &= q_c(z; \tau_2).
\end{align*}$$

By the maximum principle we know that $v(t, z)$ is nondecreasing in $t \geq 0$ and its limit $v^*(z)$ as $t \to \infty$ satisfies (3.1) with $\tau = \tau_1$. By the uniqueness established in Proposition 3.1, we obtain $v^*(z) = q_c(z; \tau_1)$. Therefore,

$$q_c(z; \tau_2) = v(0, z) \leq v(t, z) \leq v(+\infty, z) = v^*(z) = q_c(z; \tau_1).$$

The claim is proved. $\square$

4 Long time behavior of the solutions

In this section we assume that the initial date $(\phi(\theta, x), g(\theta), h(\theta))$ satisfies (1.11)–(1.12) and study the asymptotic behavior of solutions of (P). Firstly, we give some sufficient conditions
for vanishing and spreading. Next, we prove the spreading-vanishing dichotomy result of (P). Let us start this section with the following equivalent conditions for vanishing.

**Lemma 4.1** Assume that (H) holds. Let \((u, g, h)\) be a solution of (P). Then the following three assertions are equivalent:

\begin{align*}
(i) & \ h_\infty \text{ or } g_\infty \text{ is finite}; \\
(ii) & \ h_\infty - g_\infty \leq \frac{\pi}{\sqrt{f'(0) - d}}; \\
(iii) & \ \lim_{t \to \infty} \|u(t, \cdot)\|_{L^\infty([g(t), h(t)])} = 0.
\end{align*}

**Proof** “(i)⇒(ii)”. Without loss of generality we assume \(h_\infty < \infty\) and prove (ii) by contradiction. Assume that \(h_\infty - g_\infty > \frac{\pi}{\sqrt{f'(0) - d}}\), then there exists \(t_1 > 1\) such that

\[h(t_1) - g(t_1) > \frac{\pi}{\sqrt{f'(0) - d}}.\]

Let us consider the following auxiliary problem:

\[
\begin{cases}
  v_t = v_{xx} - dv + f(v(t - \tau, x)), & t > t_1, \ x \in (g(t_1), \xi(t)), \\
  v(t, \xi(t)) = 0, \ \xi'(t) = -\mu v_x(t, \xi(t)), & t > t_1, \\
  v(t, g(t_1)) = 0, & t > t_1, \\
  \xi(t_1) = h(t_1), \ v(s, x) = u(s, x), & s \in [t_1 - \tau, t_1], \ x \in [g(s), h(s)].
\end{cases}
\]

(4.1)

It is easy to check that \(v\) is a subsolution of (P), then \(\xi(t) \leq h(t)\) and \(\xi(\infty) < \infty\) by our assumption. Using a similar argument as in [9, Lemma 3.3] one can show that

\[\|v(t, \cdot) - V(\cdot)\|_{C^0([g(t_1), \xi(t_1)])} \to 0, \quad \text{as } t \to \infty,\]

where \(V(x)\) is the unique positive solution of the problem

\[V'' - dV + f(V) = 0 \quad \text{for } x \in (g(t_1), \xi(\infty)), \quad V(g(t_1)) = V(\xi(\infty)) = 0.\]

Thus,

\[\lim_{t \to \infty} \xi'(t) = -\mu \lim_{t \to \infty} v_x(t, \xi(t)) = -\mu V'(\xi(\infty)) = \delta,\]

for some \(\delta > 0\), which contradicts the fact that \(\xi(\infty) < \infty\).

“(ii)⇒(iii)”. It follows from the assumption and [39, Proposition 2.9] that the unique positive solution of the following problem

\[
\begin{cases}
  v_t = v_{xx} - dv + f(v(t - \tau, x)), & t > 0, \ x \in [g_\infty, h_\infty], \\
  v(t, g_\infty) = v(t, h_\infty) = 0, & t > 0, \\
  v(\theta, x) \geq 0, & \theta \in [-\tau, 0], \ x \in [g_\infty, h_\infty],
\end{cases}
\]

(4.2)

with \(v(\theta, x) \geq \phi(\theta, x)\) in \([-\tau, 0] \times [g(\theta), h(\theta)]\), satisfies \(v \to 0\) uniformly for \(x \in [g_\infty, h_\infty]\) as \(t \to \infty\). Then the conclusion (iii) follows easily from the comparison principle.

“(iii)⇒(ii)”: Suppose by way of contraction argument that for some small \(\varepsilon > 0\) there exists \(t_2 > 1\) such that \(h(t) - g(t) > \frac{\pi}{\sqrt{f'(0) - d}} + 3\varepsilon\) for all \(t > t_2 - \tau\). Let \(l_\varepsilon := \frac{\pi}{\sqrt{f'(0) - d}} + \varepsilon\), it is well known that the following eigenvalue problem

\[
\begin{cases}
  -\varphi_{xx} + d\varphi - f'(0)\varphi = \lambda_1 \varphi, & 0 < x < l_\varepsilon, \\
  \varphi(0) = \varphi(l_\varepsilon) = 0,
\end{cases}
\]

has a negative principal eigenvalue, denoted by \(\lambda_1\), whose corresponding positive eigenfunction, denoted by \(\varphi\), can be chosen positive and normalized by \(\|\varphi\|_{L^\infty} = 1\). Set

\[w(t, x) := \varepsilon \varphi(x) \quad \text{for } x \in [0, l_\varepsilon],\]
with \( \epsilon > 0 \) small such that
\[
 f(\epsilon \varphi) \geq f'(0)\epsilon \varphi + \frac{1}{2} \lambda_1 \epsilon \varphi \quad \text{in } [0, l_\epsilon].
\]
It is easy to compute that for \( x \in [0, l_\epsilon] \),
\[
w_t - w_{xx} + dw - f(w(t - \tau, x)) = \epsilon \varphi [f'(0) + \lambda_1] - f(\epsilon \varphi) \leq 0.
\]
Moreover one can see that
\[
0 \leq w(x) = \epsilon \varphi(x) < u(t_2 + s, x + g(t_2 + s) + \epsilon), \quad x \in [0, l_\epsilon], \quad s \in [-\tau, 0]
\]
provided that \( \epsilon \) is sufficiently small. Then we can apply the comparison principle to deduce
\[
u(t + t_2, x + g(t_2 + \epsilon)) \geq w(x) > 0, \quad (t, x) \in [0, \infty) \times (0, l_\epsilon),
\]
contradicting (iii).

“\((ii) \Rightarrow (i)\)”. When (ii) holds, (i) is obvious. This proves the lemma. \(\Box\)

Next, we give a sufficient condition for vanishing, which indicates that if the initial domain and initial function are both small, then the species dies out eventually in the environment.

**Lemma 4.2** Assume that (H) holds. Let \((u, g, h)\) be a solution of (P). Then vanishing happens provided that \(h(0) - g(0) < \frac{\pi}{\sqrt{f'(0) - d}}\) and \(\|\phi\|_{L^\infty([-\tau, 0] \times [g(\theta), h(\theta)])}\) is sufficient small.

**Proof** Set
\[
h_0 = \frac{h(0) - g(0)}{2},
\]
then \(h_0 < \pi/(2\sqrt{f'(0) - d})\), so there exists a small \(\epsilon > 0\) such that
\[
\frac{\pi^2}{4(1 + \epsilon)^2 h_0^2} - (f'(0) + \epsilon)e^{\epsilon \tau} + d \geq \epsilon. \tag{4.3}
\]
For such \(\epsilon\), we can find a small positive constant \(\delta\) such that
\[
\pi \mu \delta \leq \epsilon^2 h_0^2, \quad f(v) \leq (f'(0) + \epsilon)v \quad \text{for } v \in [0, \delta].
\]

Define
\[
k(t) := h_0(1 + \epsilon - \frac{\epsilon}{2}e^{-\epsilon t}), \quad w(t, x) := \delta e^{-\epsilon t} \cos \left(\frac{\pi x}{2k(t)}\right), \quad t > 0, \quad x \in [-k(t), k(t)],
\]
\[
k(\theta) \equiv k_0 := h_0(1 + \frac{\epsilon}{2}), \quad w(\theta, x) \equiv w_0(x) := \delta \cos \left(\frac{\pi x}{h_0(2 + \epsilon)}\right),
\]
\[
\theta \in [-\tau, 0], \quad x \in [-k_0, k_0],
\]
and extend \(w(t, x)\) by 0 for \(t \in [-\tau, \infty), x \in (-\infty, -k(t)] \cup [k(t), \infty)\).

A direct calculation shows that for \(t > 0, x \in (-k(t), k(t))\),
\[
w_t - w_{xx} + dw - f(w(t - \tau, x))
\]
\[
= \left[\frac{\pi^2}{4k^2(t)} - \epsilon + d - (f'(0) + \epsilon) \frac{w(t - \tau, x)}{w(t, x)} \right] + \frac{\pi x k'(t)}{2k^2(t)} \tan \left(\frac{\pi x}{2k(t)}\right)
\]
\[
\geq \left[\frac{\pi^2}{4k^2(t)} + d - (f'(0) + \epsilon) \frac{w(t - \tau, x)}{w(t, x)} \right] w,
\]
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where we have used $k'(t) > 0, k(t) > 0$ for $t > 0$ and $y \tan y \geq 0$ for $y \in (−\pi/2, \pi/2)$.

When $t \geq \tau$ and $x \in (−k(t), k(t))$, it is easy to check that

$$
A := -\varepsilon + \frac{\pi^2}{4k^2(t)} + d - (f'(0) + \varepsilon) \frac{w(t-\tau, x)}{w(t, x)}
$$

$$
\geq -\varepsilon + \frac{\pi^2}{4h_0^2(1+\varepsilon)^2} + d - (f'(0) + \varepsilon)e^{\varepsilon \tau} \geq 0,
$$

where the fact that $\cos\left(\frac{\pi x}{2k(t-\tau)}\right) \leq \cos\left(\frac{\pi x}{2k(t)}\right)$ for $(t, x) \in [\tau, \infty) \times [−k(t), k(t)]$ and the monotonicity of $k(t)$ in $t \in [0, \infty)$ are used. If $t \in [0, \tau)$ and $x \in (−k(t), k(t))$, we have that

$$
A \geq -\varepsilon + \frac{\pi^2}{4h_0^2(1+\varepsilon)^2} + d - (f'(0) + \varepsilon)e^{\varepsilon \tau} \geq 0.
$$

Thus we have

$$
w_t - w_{xx} + dw - f(w(t-\tau, x)) \geq 0 \text{ in } (0, \infty) \times (-k(t), k(t)).
$$

On the other hand,

$$
k'(t) = \frac{\varepsilon^2 h_0}{2} e^{-\varepsilon t} \geq \frac{\pi \mu \delta}{2h_0} e^{-\varepsilon t} \geq -\mu w_x(t, k(t)) = \mu w_x(t, -k(t)).
$$

As a consequence, $(w(t, x), -k(t), k(t))$ will be a supersolution of $(P)$ if $w(\theta, x) \geq \phi(\theta, x)$ in $[-\tau, 0] \times [g(\theta), h(\theta)]$. Indeed, choose $\sigma_1 := \delta \cos\frac{\pi}{2\pi+\varepsilon}$, which depends only on $\mu, h_0, d$ and $f$. Then when $\|\phi\|_{L^\infty([-\tau, 0] \times [g(\theta), h(\theta)])} \leq \sigma_1$ we have $\phi(\theta, x) \leq \sigma_1 \leq w(\theta, x)$ in $[-\tau, 0] \times [g(\theta), h(\theta)]$, since $h_0 < k(0) = h_0(1 + \frac{\varepsilon}{2})$. It follows from the comparison principle that $h(t) \leq k(t)$, so

$$
h_\infty \leq h_0(1 + \varepsilon).
$$

This, together with the previous lemma, implies that vanishing happens.

\[ \square \]

**Remark 4.3** When $\tau = 0$, the proof of Lemma 4.2 reduces to that of [11, Theorem 3.2(i)].

We now present a sufficient condition for spreading, which reads as follows.

**Lemma 4.4** Assume that (H) holds. If $h(0) - g(0) \geq \pi / \sqrt{f'(0)} - d$, then spreading happens for every positive solution $(u, g, h)$ of $(P)$.

**Proof** Since $g'(t) < 0 < h'(t)$ for $t > 0$, we have $h(t) - g(t) \geq \pi / \sqrt{f'(0)} - d$ for any $t > 0$. So the conclusion $-g_\infty = h_\infty = \infty$ follows from Lemma 4.1. In what follows we prove

$$
\lim_{t \to \infty} u(t, x) = u^* \text{ locally uniformly in } \mathbb{R}.
$$

(4.4)

First, it is well known that for any $L > \pi/(2\sqrt{f'(0)} - d)$, the following problem

$$
W_{xx} - dW + f(W) = 0, \quad x \in (-L, L), \quad W(\pm L) = 0,
$$

holds. If $h(0) - g(0) \geq \pi / \sqrt{f'(0)} - d$, then spreading happens for every positive solution $(u, g, h)$ of $(P)$. 

\[ \square \]
admits a unique positive solution $W_L$, which is increasing in $L$ and satisfies
\[
\lim_{L \to \infty} W_L(x) = u^* \quad \text{locally uniformly in } \mathbb{R}.
\] (4.5)

Moreover we can find an increasing sequence of positive numbers $L_n$ with $L_n \to \infty$ as $n \to \infty$ such that $L_n > \pi/\sqrt{f'(0) - d}$ for all $n \geq 1$. Since $W_{L_n}$ converges to $u^*$ locally uniformly in $\mathbb{R}$, we can choose $t_n$ such that $h(t) \geq L_n$ and $g(t) \leq -L_n$ for $t \geq t_n$. It then follows from [39] the following problem
\[
\begin{cases}
w_t = w_{xx} - dw + f(w(t - \tau, x)), & t \geq t_n + \tau, \quad x \in [-L_n, L_n], \\
w(t, \pm L_n) = 0, & t \geq t_n + \tau, \\
w(s, x) = u(s, x), & s \in [t_n, t_n + \tau], \quad x \in [-L_n, L_n],
\end{cases}
\]
has a unique positive solution $w_n(t, x)$, which satisfies that
\[
w_n(t, x) \to W_{L_n}(x) \quad \text{uniformly for } x \in [-L_n, L_n] \text{ as } t \to \infty.
\]

Applying the comparison principle we have $w_n(t, x) \leq u(t, x)$ for all $t \geq t_n + \tau$, $x \in [-L_n, L_n]$. This, together with (4.5), yields that
\[
\liminf_{t \to \infty} u(t, x) \geq u^* \quad \text{locally uniformly for } x \in \mathbb{R}.
\] (6.6)

Later, since the initial data $u_0(s, x)$ satisfies $0 \leq u_0(s, x) \leq u^*$ for $(s, x) \in [-\tau, 0] \times [g(s), h(s)]$, it thus follows from the comparison principle that
\[
\limsup_{t \to \infty} u(t, x) \leq u^* \quad \text{locally uniformly for } x \in \mathbb{R}.
\]

Combining with (4.6), one can easily obtain (4.4), which ends the proof of this lemma. \(\square\)

Now we are ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2** It is easy to see that there are two possibilities: (i) $h_\infty - g_\infty \leq \pi/\sqrt{f'(0) - d}$; (ii) $h_\infty - g_\infty > \pi/\sqrt{f'(0) - d}$. In case (i), it follows from Lemma 4.1 that $\lim_{t \to \infty} ||u(t, \cdot)||_{L^\infty(g(t), h(t))} = 0$. For case (ii), it follows from Lemma 4.4 and its proof that $(g_\infty, h_\infty) = \mathbb{R}$ and $u(t, x) \to u^*$ as $t \to \infty$ locally uniformly in $\mathbb{R}$, which ends the proof. \(\square\)

### 5 Asymptotic profiles of spreading solutions

Throughout this section we assume that (H) holds and $(u, g, h)$ is a solution of (P) for which spreading happens. In order to determine the spreading speed, we will construct some suitable sub- and supersolutions based on semi-waves. Let $c^*$ and $q_{c^*}(z)$ be given in Theorem 3.4. The first subsection covers the proof of the boundedness for $|h(t) - c^*t|$ and $|g(t) + c^*t|$. Based on these results, we prove Theorem 1.4 in the second subsection.

#### 5.1 Boundedness for $|h(t) - c^*t|$ and $|g(t) + c^*t|$.

Let us begin this subsection with the following estimate.

**Lemma 5.1** Let $(u, g, h)$ be a solution of (P) for which spreading happens. Then for any $c \in (0, c^*)$, there exist small $\beta^* \in (0, d - f'(u^*))$, $T > 0$ and $M > 0$ such that for $t \geq T$, ...
(i) $[g(t), h(t)] \supseteq [-ct, ct]$;
(ii) $u(t, x) \leq u^*(1 - Me^{-\beta t})$ for $x \in [-ct, ct]$;
(iii) $u(t, x) \leq u^*(1 + Me^{-\beta t})$ for $x \in [g(t), h(t)]$.

**Proof** In order to prove conclusions (i) and (ii), inspired by [16], we will use the wave $q_c^*$ to construct the suitable subsolution. Here we mainly use the the monotonicity and exponential convergence of $q_c^*$.

(i) Since $q_c^*(z)$ is the unique positive solution of

$$
\begin{aligned}
q_c''(z) - c^*q_c'(z) - dq_c^* + f(q_c^*(z - c^*\tau)) &= 0, \\
q_c^*(z) &= 0, \\
\mu(q_c')_{+} &= c^*, \\
q_c^*(\infty) &= u^*,
\end{aligned}
$$

then it is easy to check that $(q_c')_{+}(0) > 0$. Since $q_c''(z) > 0$ for $z \geq 0$ and $q_c^*(z) \to u^*$ as $z \to \infty$, thus there is $z_0 > 1$ such that $q_c''(z) < 0$ for $z > z_0$. Thus there exists $\hat{z} \in (0, \infty)$ such that $q_c''(\hat{z}) = 0$ and $q_c''(z) > 0$ for $z \in [0, \hat{z}]$. This means that $q_c'(z)$ is increasing in $z \in [0, \hat{z}]$. Let $\hat{p}_0 \in (0, q_c^*(\hat{z}))$ be small. Define

$$
G(u, p) = \begin{cases} 
  d + [f(u - p) - f(u)]/p, & p > 0, \\
  d - f'(u), & p = 0,
\end{cases}
$$

for $p > 0$ and $u > p$. Then $G(u, p)$ is a continuous function for $0 \leq p \leq \hat{p}_0$ and $G(u^*, p) > 0, G(u^*, 0) = d - f'(u^*) > 0$, thus there exists $0 < \gamma \ll d$ such that $G(u^*, p) \geq 2\gamma$ for $0 \leq p \leq \hat{p}_0$. By continuity, there exists $\rho > 0$ small such that $G(u, p) \geq \gamma$ for $u^* - \rho \leq u \leq u^*, 0 \leq p \leq \hat{p}_0$. Furthermore, as $f(u^*) = du^*$, then there is a constant $b > 0$ such that

$$
f(v) - dv \leq b(u^* - v) \quad \text{for} \quad v \in [u^* - \rho, u^*].
$$

Inspired by [16], let us construct the following function:

$$
\underline{u}(t, x) := \max\{0, q_c^*(x + c^*t + \xi(t)) + q_c^*(c^*t - x + \xi(t)) - u^* - p(t)\}, \quad t > 0,
$$

and denote $g(t)$ and $h(t)$ be the zero points of $\underline{u}(t, x)$ with $t > 0$, that is

$$
\underline{u}(t, g(t)) = \underline{u}(t, h(t)) = 0.
$$

In the following, we will show that $(\underline{u}, g, h)$ is a subsolution of problem $(P)$. We only prove the case where $x \geq 0$, since the other is analogous. For any function $J$ depended on $t$, we write $J(t) := J(t - \tau)$ if no confusion arises. For simplicity of notations, we will write

$$
\xi^-(t) := -x + c^*t + \xi(t), \quad \xi^+(t) := x + c^*t + \xi(t), \quad \xi_{\tau}^- := \xi^-(t - \tau), \quad \xi_{\tau}^+ := \xi^+(t - \tau).
$$

Firstly, a direct calculation shows that for $(t, x) \in (\tau, \infty) \times [0, h(t)]$,

$$
\mathcal{N}[u] := \nabla u - u_{xx} + du - f(u(t - \tau, x)) = \xi[q_c^*(\xi^-) + q_c^*(\xi^+)] + f(q_c^*(\xi^-)) + f(q_c^*(\xi^+)) - f(q_c^*(\xi^+)) - q_c^*(\xi^+ - u^* - p(t)) - d(u^* + p) - p'.
$$

Assume that $\xi'(t) \leq 0$, and choose $\xi$ large such that $u^* - \frac{\xi}{2} \leq q_c^*(\xi^+) \leq u^*$ in $(\tau, \infty) \times [0, h(t)]$. The monotonicity of $q_c^*$ and its exponential rate of convergence to $u^*$ at $\infty$ imply that if we choose $\xi$ sufficiently large, then there exist positive constants $\nu, K_0$ and $K$ such that

$$
\frac{\nu}{2}u^* - q_c^*(\xi^+) \leq K_0e^{-\nu \xi^+} \leq Ke^{-\nu(\xi(t) + c^*)t}.
$$
Set \( p(t) = p_0 e^{-\beta t} \) with \( p_0 := \frac{1}{2} \min\{ \tilde{p}_0, \frac{\nu}{2} \} \) and \( \beta := \frac{1}{2} \min\{ \nu c^*, \alpha_0 \} \), where \( \alpha_0 \) is the unique zero point of
\[
d(e^{Ty} - 1) - y e^{Ty} + y = 0.
\]
Thus, when \( q_c^*(\zeta^-) \in [u^* - \rho, u^*] \) and \((t, x) \in (\tau, \infty) \times [0, h(t)]\), since \( q_c^*(z) \geq 0 \), then
\[
N[u] = \tilde{\xi}'[q_c^*(\zeta^-) + q_c^*(\zeta^+)] + f(q_c^*(\zeta^-)) + f(q_c^*(\zeta^+))
- f(q_c^*(\zeta^-) + q_c^*(\zeta^+) - u^* - p) - d(u^* + p - p')
\leq \gamma[q_c^*(\zeta^-) - u^* - p] + b[u^* - q_c^*(\zeta^+)] + d(p - p') - p'
\leq b[u^* - q_c^*(\zeta^+)] + d(p - p') - p' - \gamma p,
\]
provided that \( \tilde{\xi} \) is sufficiently large.

For the part \( q_c^*(\zeta^-) \in [0, u^* - \rho] \), then for \((t, x) \in (\tau, \infty) \times [0, h(t)]\) and sufficiently large \( \tilde{\xi} \), there are two positive constants \( d_1 \) and \( d_2 \) where \( d_1 < 1 \) such that \( q_c^*(\zeta^-) + q_c^*(\zeta^+) \geq d_1 \), and
\[
f(q_c^*(\zeta^-)) - f(q_c^*(\zeta^-) + q_c^*(\zeta^+) - u^* - p)
+ d[q_c^*(\zeta^+) - u^* - p] \leq d_2[u^* + p - q_c^*(\zeta^+)],
\]
thus we have
\[
N[u] = \tilde{\xi}'[q_c^*(\zeta^-) + q_c^*(\zeta^+)] + f(q_c^*(\zeta^-)) + f(q_c^*(\zeta^+))
- f(q_c^*(\zeta^-) + q_c^*(\zeta^+) - u^* - p) - d(u^* + p - p')
\leq d_1 \tilde{\xi}' + d_2[u^* + p - q_c^*(\zeta^+) + b[u^* - q_c^*(\zeta^+)] + d(p - p') - p'
\leq d_1 \tilde{\xi}' + d_2 + bKe^{-v(\tilde{\xi}+\epsilon)} + p_0 e^{-\beta t}[d_2 e^{\beta t} + d(e^{\beta t} - 1) + \beta]
\leq d_1 \tilde{\xi}' + p_0 e^{-\beta t}[d_2 e^{\beta t} + d(e^{\beta t} - 1) + 2\beta].
\]

Now let us choose \( \tilde{\xi} \) satisfies
\[
d_1 \tilde{\xi}' + \kappa p_0 e^{-\beta t} = 0
\]
with \( \tilde{\xi}(0) = \tilde{\xi}_0 \) sufficiently large, and \( \kappa := d_2 e^{\beta t} + d(e^{\beta t} - 1) + 2\beta \), then \( \tilde{\xi}'(t) \leq 0 \). Hence from the above we obtain that \( N[u] \leq 0 \) in this part.

Next, let us check the free boundary condition. When \( x = h(t) \), we set \( \zeta_1(t) = -h(t) + c^* t + \tilde{\xi}(t) \) and \( \zeta_2(t) = h(t) + c^* t + \tilde{\xi}(t) \), then
\[
q_c^*(\zeta_1(t)) + q_c^*(\zeta_2(t)) = u^* + p(t).
\]
We differentiate \((5.3)\) with respect to \( t \) to obtain
\[
[q_c^*(\zeta_2) - q_c^*(\zeta_1)](h'(t) - c^*) = p' - 2c^* q_c^*(\zeta_2) - [q_c^*(\zeta_2) + q_c^*(\zeta_1)] \tilde{\xi}'.
\]
By shrinking \( p_0 \) and enlarge \( \tilde{\xi}_0 \) if necessary, then we can see that \( \zeta_2(2) \gg 1 \), and \( q_c^*(\zeta_2(t)) \approx u^* \). This, together with \((5.3)\), yields that \( q_c^*(\zeta_1(t)) \approx p(t) \). Since \( q_c''(z) > 0 \) for \( 0 \ll z \ll 1 \) and \( q_c''(y) \) for \( y \gg 1 \) and \( q_c''(z) \ll 0 \) as \( z \to \infty \), thus we have
\[
0 < q_c''(\zeta_2) < (q_c')_+(0) < q_c''(\zeta_1).
\]

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Thanks to the choice of $\xi(t)$, we can compute that
\[ p' - 2c* q_+ (\xi_2) - [q_+ (\xi_2) + q_+ (\xi_1)] \xi' \geq \left( \frac{\kappa(q_+)'(0)}{d_1} - \beta \right) p_0 e^{-\beta t} - 2c* K_1 e^{-\nu(\xi(t)) + c* t} \geq 0, \]
where $K_1$ is a positive constant, $\kappa := d_2 e^{\beta t} + d(e^{\beta t} - 1) + 2\beta > 2\beta$ and we have used that by shrinking $d_1$ if necessary, then $\kappa(q_+)'(0) > \beta d_1$.

It follows from (5.4)–(5.6) and the monotonicity of $q_+ (z)$ in $z$ that
\[ h(t) \leq e^\kappa = \mu(q_+)'(0) \leq \mu[q_+ (\xi_1) - q_+ (\xi_2)] = -\mu u_c (t, h(t)). \]

Using (5.3) again, it is easy to see that $\xi_1(t)$ is non-increasing in $t \geq T_1$, thus for all $t \geq T_1$,
\[ h(t) - c* t \geq C_0 := h(T_1) - c* T_1 + \xi (\infty) - \xi (0). \]  (5.7)

Since $(u, g, h)$ is a spreading solution of $(P)$, then there exists $T_2 > 0$ such that
\[ u(T_1 + T_2 + s, x) \geq g(T_1 + \tau, x) \text{ for } \bar{s} \in [0, \tau], \ x \in [g(\tau), h(\tau)], \]
\[ g(T_1 + T_2) \leq g(T_1 + \tau) \text{ and } h(T_1 + T_2) \geq h(T_1 + \tau). \]

Consequently, $(u, g, h)$ is a subsolution of problem $(P)$, then we can apply the comparison principle to conclude that $u(t + T_1 + T_2, x) \geq u(t + T_1, x), h(t + T_1 + T_2) \geq h(t + T_1)$ for $t > 0$, $x \in [0, h(t)]$. This, together with (5.7), implies that
\[ h(t) - c* t \geq -C_1 \text{ for } t > 0, \]
with $C_1 := |C_0| + h(T_1 + T_2 + \tau) + c* (T_1 + T_2 + \tau)$. Similarly, by enlarging $C_1$ if necessary, we can have $h(t) + c* t \leq C_1$ for $t > 0$. Thus result (i) holds for large $T_1$.

(ii) From the proof of (i), it is easy to see that $u(t + T_2) \geq u(t, x)$ for $t > T_1$. The monotonicity of $q_+$ and its exponential rate of convergence to $u^*$ at $\infty$ can be used again to conclude that for any $c \in (0, c*)$ there exist constants $\nu, K > 0$ such that for any $x \in [0, ct]$ and $t > 0$,
\[ u^* - q_+(x + c* t + \xi(t)) \leq u^* - q_+(c* t + \xi(t)) \leq K e^{-\nu(c* t + \xi(t))}, \]
\[ q_+(x + c* t + \xi(t)) > q_+((c* - c)t + \xi(t)) \geq u^* - K e^{-\nu((c* - c)t + \xi(t))}. \]

Based on above results, we can find $T_3 > T_1 + T_2$ large such that for $t > T_3$ and $x \in [0, ct]$,
\[ u(t, x) \geq q_+(x + c* (t - T_3) + \xi(t - T_3)) + q_+(-x + c* (t - T_2) + \xi(t - T_2)) = u^* - p_0 e^{\beta t - (T_2)} \]
\[ \geq u^* - 2K e^{-\nu((c* - c)(t - T_2) + \xi(t - T_2))} - p_0 e^{\beta t - (T_2)} \geq u^* - Mu^* e^{-\beta* t}, \]
where $M > 0$ is sufficiently large and $\beta^* := \frac{1}{2} \min \{ \nu(c* - c), \beta, d - f'(u^*) \}$. The case where $x \in [-ct, 0]$ can be proved by a similar argument as above. The proof of (ii) is now complete.

(iii) Thanks to the choice of the initial data, we know that for any given $\beta^* > 0$ and $M > 0$,
\[ u(t, x) \leq u^* + Mu^* e^{-\beta^* t} \text{ for } (t, x) \in [0, \infty) \times [g(t), h(t)]. \]
This completes the proof. \qed

Next we prove the boundedness of $h(t) - c* t$ and show that $u(t, \cdot) \approx u^*$ in the domain $[0, h(t) - Z]$, where $Z > 0$ is a large number.
**Proposition 5.2** Assume that spreading happens for the solution \((u, g, h)\). Then

(i) there exists \(C > 0\) such that

\[
|h(t) - c^* t| \leq C \quad \text{for all} \quad t \geq 0; \tag{5.8}
\]

(ii) for any small \(\varepsilon > 0\), there exist \(Z_\varepsilon > 0\) and \(T_\varepsilon > 0\) such that

\[
\|u(t, \cdot) - u^*\|_{L^\infty([0, h(t) - Z_\varepsilon])} \leq u^* \varepsilon \quad \text{for} \quad t > T_\varepsilon. \tag{5.9}
\]

**Proof** In order to prove conclusions in this proposition, inspired by [12], we will use the semi-wave \(q_{c^*}\) to construct the suitable sub- and supersolution. Compared with [12], our problem considers the case where \(\tau > 0\). Due to \(\tau > 0\), there will be some space-translation of the semi-wave \(q_{c^*}\), which make our problem difficult to deal with. To overcome this difficulty, we mainly use the the monotonicity and exponential convergence of \(q_{c^*}\). Moreover, this idea also be used in the proof of Lemma 5.6. For clarity we divide the proof into several steps.

**Step 1.** To give some upper bounds for \(h(t)\) and \(u(t, x)\).

Fix \(c \in (0, c^*)\). It follows from Lemma 5.1 that there exist \(\beta^* \in (0, d - f'(u^*))\), \(M > 0\), and \(T > 0\) such that for \(t \geq T\), Lemma 5.1 (i), (ii) and (iii) hold. Thanks to (H), by shrinking \(\beta^*\) if necessary, we can find \(\rho > 0\) small such that

\[
d - f'(v)e^{\beta^* T} \geq \beta^* \quad \text{for} \quad v \in [u^* - \rho, u^* + \rho]. \tag{5.10}
\]

For any \(T_s > T + \tau\) large satisfying \(M u^* e^{-\beta^*(T_s - \tau)} < \frac{\rho}{2}\), there is \(M' > M\) such that \(M' u^* e^{-\beta^*(T_s - \tau)} < \rho\). Since \(q_{c^*}(z) \to u^*\) as \(z \to \infty\), we can find \(Z_0 > 0\) such that

\[
(1 + M' e^{-\beta^*(T_s + \tau)}) q_{c^*}(Z_0) \geq u^*. \tag{5.11}
\]

Now we construct a supersolution \((\bar{u}, g, \bar{h})\) to \((P)\) as follows:

\[
\bar{h}(t) := c^*(t - T_s) + h(T_s + \tau) + K M' (e^{-\beta^* T_s} - e^{-\beta^* T}) + Z_0 \quad \text{for} \quad t \geq T_s,
\]

\[
\bar{u}(t, x) := \min \left\{(1 + M' e^{-\beta^* T}) q_{c^*}(\bar{h}(t) - x), u^*\right\} \quad \text{for} \quad t \geq T_s, \quad x \leq \bar{h}(t),
\]

where \(K\) is a positive constant to be determined below.

Clearly, for all \(t \geq T_s\), \(\bar{u}(t, g(t)) > 0 = u(t, g(t)), \bar{u}(t, \bar{h}(t)) = 0\), and

\[
-\mu \bar{u}_x(t, \bar{h}(t)) = \mu (1 + M' e^{-\beta^* T}) q_{c^*}(0) = (1 + M' e^{-\beta^* T}) c^*
\]

\[
< c^* + M' K \beta^* e^{-\beta^* T} = \tilde{h}'(t),
\]

if we choose \(K\) with \(K \beta^* > c^*\). By the definition of \(\tilde{h}\) we have \(h(T_s + s) < \tilde{h}(T_s + s)\) for \(s \in [0, \tau]\). It then follows from (5.11) that for \((s, x) \in [0, \tau] \times [g(T_s + s), h(T_s + s)]\),

\[
(1 + M' e^{-\beta^*(T_s + s)}) q_{c^*}(\tilde{h}(T_s + s) - x) \geq (1 + M' e^{-\beta^*(T_s + \tau)}) q_{c^*}(Z_0) \geq u^*,
\]

which yields that \(\bar{u}(T_s + s, x) = u^* \geq u(T_s + s, x)\) for \((s, x) \in [0, \tau] \times [g(T_s + s), h(T_s + s)]\).

We now show that

\[
N[\bar{u}] := \bar{u}_t - \bar{u}_{xx} + d \bar{u} - f(\bar{u}(t - \tau, x)) \geq 0, \quad x \in [g(t), \bar{h}(t)], \quad t > T_s + \tau. \tag{5.12}
\]

Thanks to the definition of \(\bar{u}(t, x)\) and the monotonicity of \(q_{c^*}(z)\) in \(z\), we can find a decreasing function \(\eta(t) < \bar{h}(t)\) for \(t > T_s\), such that

\[
(1 + M' e^{-\beta^* T}) q_{c^*}(\bar{h}(t) - x) \begin{cases} > u^*, & x < \eta(t), \\ = u^*, & x = \eta(t), \\ < u^*, & x \in (\eta(t), \bar{h}(t)]. \end{cases}
\]
which implies that
\[ \dddot{u}(t, x) = u^* \] for \( x \leq \eta(t) \), and \( \dddot{u}(t, x) = (1 + M'e^{-\beta t})q_{c*}(\dddot{h}(t) - x) \) for \( x \in [\eta(t), \dddot{h}(t)] \).

As \( \mathcal{N}u^* = 0 \), thus in what follows, we only consider the case \( x \in [\eta(t), \dddot{h}(t)] \). Set \( q_{\tau} := q_{c*}(\dddot{h}_\tau - x) \) for convenience. A direct calculation shows that, for \( t > T_* + \tau \),

\[
\mathcal{N}[\dddot{u}] := \dddot{u}_t - \dddot{u}_{xx} + d \dddot{u} - f(\dddot{u}(t - \tau, x)) = -\beta^* M'e^{-\beta^* t} q_{c*} + (1 + M'e^{-\beta^* t}) \{ K \beta^* M'e^{-\beta^* t} q_{c*} + f(q_{\tau}) \}
- f((1 + M'e^{-\beta^* t(\tau-\tau)})q_{\tau})
= M'e^{-\beta^* t} \left\{ f(q_{\tau}) + K \beta^*(1 + M'e^{-\beta^* t}) q_{c*} - \beta^* q_{c*} \right\} + f(q_{\tau})
- f((1 + M'e^{-\beta^* t(\tau-\tau)})q_{\tau})
\geq M'e^{-\beta^* t} \left\{ K \beta^*(1 + M'e^{-\beta^* t}) q_{c*}'
- \left[ \left( f'((1 + \theta M'e^{-\beta^* t(\tau-\tau)})q_{\tau}) e^{\beta^* t} - d \right) q_{\tau} - \beta^* q_{c*} \right] \right\},
\]
for some \( \theta \in (0, 1) \). Since

\[
q_{c*}(z) \to u^* \quad \text{and} \quad \frac{(q_{c*}(z) - u^*)'}{q_{c*}(z) - u^*} \to k^* \quad \text{as} \quad z \to \infty,
\]
(5.13)

where \( k^* := c^* - \sqrt{(c^*)^2 + 4(d - f'(u^*))} < 0 \), there are \( z_0 > 0 \) and \( k_1 > 0 \) such that

\[
q_{c*}'(z) < 0, \quad q_{c*}(z) \geq u^* - \rho \quad \text{and} \quad q_{c*}'(z - 2c^* \tau) \leq k_1 q_{c*}'(z) \quad \text{for} \quad z > z_0.
\]
(5.14)

Moreover, we can compute that

\[
\Delta \dddot{h}(t) := \dddot{h}(t) - \dddot{h}_\tau(t) = c^* \tau + K M'e^{\beta^* t}(e^{\beta^* t} - 1).
\]

For any given \( K > 0 \), by enlarging \( T_* \) if necessary, we have that

\[
\Delta \dddot{h}(t) \in [e^{c^* \tau}, 2e^{c^* \tau}] \quad \text{for} \quad t \geq T_*.
\]
(5.15)

When \( \dddot{h}_\tau - x \leq z_0 \) and \( t > T_* + \tau \), it then follows that

\[
B := K \beta^*(1 + M'e^{-\beta^* t}) q_{c*}' - \left[ \left( f'((1 + \theta M'e^{-\beta^* t(\tau-\tau)})q_{\tau}) e^{\beta^* t} - d \right) q_{\tau} - \beta^* q_{c*} \right]
\geq \left[ d - f'((1 + \theta M'e^{-\beta^* t(\tau-\tau)})q_{\tau}) e^{\beta^* t} - \beta^* \right] q_{\tau} + K \beta^* q_{c*}' + \beta^* (q_{\tau} - q_{c*})
\geq K \beta^* q_{c*}'(\dddot{h}(t) - x) - \beta^* q_{c*}'(\dddot{h}(t) - x - \delta \Delta \dddot{h}(t)) \Delta \dddot{h}(t) \quad \text{(with} \quad \delta \in (0, 1))
\geq (K - 2k_1 c^* \tau) \beta^* q_{c*}'(\dddot{h}(t) - x) \geq 0
\]

provided that \( K \) is sufficiently large, and we have used \( M'e^{-\beta^* t(\tau-\tau)} u^* \leq \rho \) for \( t > T_* \), \( q_{c*}'(z) > 0 \) for \( z > 0 \), (5.10), (5.14) and (5.15). Thus \( \mathcal{N}[\dddot{u}] \geq 0 \) in this case.

When \( 0 \leq \dddot{h}_\tau - x \leq z_0 \) and \( t > T_* + \tau \), for sufficiently large \( K \), we have

\[
\mathcal{N}[\dddot{u}] \geq M'e^{-\beta^* t} \left[ K \beta^* D_1 - D_2 u^* e^{\beta^* t} - \beta^* u^* \right] \geq 0,
\]

where \( D_1 := \min_{z \in [0, z_0 + 2c^* \tau]} q_{c*}'(z) > 0 \), \( D_2 := \max_{z \in [0, 2c^* \tau]} f'(v) \), and (5.15) are used.

Summarizing the above results we see that \((\dddot{u}, g, \dddot{h})\) is a supersolution of \((P)\). Thus we can apply the comparison principle to deduce

\[
h(t) \leq \dddot{h}(t) \quad \text{and} \quad u(t, x) \leq \dddot{u}(t, x) \leq u^* + M'u^* e^{-\beta t} \quad \text{for} \quad x \in [g(t), h(t)], \quad t > T_*.\]
By the definition of $\tilde{h}$ we see that, for $C_\tau := h(T_\tau + \tau) + Z_0 + KM'$, we have
\[
h(t) < c^* t + C_\tau \quad \text{for all } t \geq 0.
\] (5.16) 
For any $\varepsilon > 0$, if we choose $T_1(\varepsilon) > T_\ast$ large such that $M' e^{-\beta^* T_1(\varepsilon)} < \varepsilon$, then we have
\[
u(t, x) \leq \tilde{u}(t, x) \leq u^*(1 + \varepsilon), \quad x \in [g(t), h(t)], \quad t > T_1(\varepsilon),
\] (5.17)
which ends the proof of Step 1.

**Step 2.** To give some lower bounds for $h(t)$ and $u(t, x)$.

Let $c$, $M$, $T$ and $\beta^*$ be as before. By shrinking $c$ if necessary, we can find $T^* > T + \tau$ large such that
\[
Mu^* e^{-\beta^* (t-\tau)} \leq \frac{\rho}{2} \quad \text{for } t \geq T^* \quad \text{and} \quad h(T^*) - cT^* \geq c^* \tau.
\] (5.18)
We will define the following functions
\[
\nu(t) = ct, \quad \bar{h}(t) = c^*(t - T^*) + cT^* - \sigma M (e^{-\beta^* T^*} - e^{-\beta^* t}), \quad t \geq T^*,
\]
\[
u(t, x) = (1 - Me^{-\beta^* t})q_{c^*}(\bar{h}(t) - x), \quad t \geq T^*, \quad x \in [g(t), \bar{h}(t)],
\]
where $\sigma$ is a positive constant to be determined later.

We will prove that $(\nu, \bar{h}, \bar{g})$ is a subsolution to $(P)$ for $t > T^*$. Firstly, for $t \geq T^*$,
\[
u(t, g(t)) = \nu(t, -ct) \leq u^* - Mu^* e^{-\beta^* t} \leq \nu(t, -ct) = \nu(t, \bar{g}(t)).
\]
Next, we check that $\bar{h}$ and $\nu$ satisfy the required conditions at $x = \bar{h}(t)$. It is obvious that $\nu(t, \bar{h}(t)) = 0$. If we choose $\sigma$ satisfying $\sigma \beta^* \geq c^*$, then
\[
-\mu \nu_x(t, \bar{h}(t)) = \mu(1 - Me^{-\beta^* t})(q_{c^*}')(0) = c^*(1 - Me^{-\beta^* t})
\]
\[
> c^* - \sigma M \beta^* e^{-\beta^* t} = \nu'(t).
\]

Later, let us check the initial conditions. From Lemma 5.1, it is easy to see that
\[
\bar{h}(T^* + s) \leq cT^* + c^* \tau \leq h(T^* + s),
\]
\[
u(T^* + s, x) \leq u^*(1 - Me^{-\beta^* (T^* + s)}) \leq \nu(T^* + s, x),
\]
for $s \in [0, \tau]$ and $x \in [g(T^* + s), h(T^* + s)]$.

Finally we will prove that $u_x - u_{xx} + du - f(\nu(t, \tau), x) \leq 0$ for $t \geq T^* + \tau$. Put $z = \bar{h}(t) - x$ and $\tau = q_{c^*}(\bar{h}(t - \tau) - x)$. It is easy to check that
\[
N[u] := u_x - u_{xx} + du - f(\nu(t, \tau), x)
\]
\[
\leq Me^{-\beta^* t} \left\{ \beta^* q_{c^*} - \sigma \beta^* (1 - Me^{-\beta^* t})q_{c^*}' + \left[ f'((1 - \theta_1 Me^{-\beta^*(t-\tau)})q_{c^*})e^{\beta^* t} - d \right]q_{c^*} \right\},
\]
for some $\theta_1 \in (0, 1)$. It follows from (5.13) that there are two constants $z_1 > 0, k_2 > 0$ such that
\[
q''_{c^*}(z) < 0, \quad q_{c^*}(z) \geq u^* - \frac{\rho}{2} \quad \text{and} \quad q_{c^*}(z - c^* \tau) \leq k_2 q_{c^*}'(z) \quad \text{for } z > z_1.
\] (5.19)
Moreover, we can compute that
\[
\Delta \bar{h}(t) := \bar{h}(t) - \bar{h}_\tau(t) = c^* \tau - \sigma Me^{-\beta^* t}(e^{\beta^* \tau} - 1).
\]
For any given $\sigma > 0$, by enlarging $T^*$ if necessary, we have that
\[
\Delta \bar{h}(t) \in [0, c^* \tau] \quad \text{for } t \geq T^*.
\] (5.20)
When $h_{\tau} - x > z_1$ and $t \geq T^* + \tau$, it then follows that
\[
C := \beta^* q_{c^*} - \sigma \beta^*(1 - Me^{-\beta^*t})q_{c^*}e^{\beta^*t} - d]
\leq \left[ \frac{f'(1 - \theta_1 Me^{-\beta^*(t+\tau)})}{q_{c^*}e^{\beta^*t} - d + \beta^*} \right] q_{c^*} \geq -\sigma^* \beta^* q_{c^*}(h(t) - x) + \beta^* q_{c^*}(h(t) - x - \theta_1 \Delta h(t)) \Delta h(t) \quad \text{(with } \theta_1 \in (0, 1))
\leq (k C^* - \sigma^*) \beta^* q_{c^*}(h(t) - x) \leq 0
\]
provided that $C$ is sufficiently large, and we have used $(1 - \theta_1 Me^{-\beta^*(t+\tau)})q_{c^*} \in [u^* - \rho, u^*]$ and (5.18) for $t \geq T^*$, and (5.10), (5.19), (5.20). Thus $N[u] \leq 0$ in this case.

When $0 \leq h_{\tau} - x \leq z_1$ and $t \geq T^* + \tau$, for sufficiently large $\sigma$, we have
\[
N[u] \leq Me^{-\beta^*t} \left[ \beta^* u^* - \sigma \beta^* \left(1 - \frac{\rho}{2u^*e^{-\beta^*t}}\right) D_1 + D_2 u^* e^{\beta^*t} \right] \leq 0,
\]
where $D_1 := \min_{z \in [0, z_1 + c^*rt]} q_{c^*}^*(z) > 0$, $D_2 := \max_{v \in [0, 2u^*]} f'(v)$ and (5.20) are used.

Consequently, $(u, g, h)$ is a subsolution to $P$, then the comparison principle implies that
\[
h(t) \leq h(t), \quad u(t, x) \leq u(t, x) \quad \text{for } t \geq T^*, \quad x \in [g(t), h(t)],
\]
which yields that
\[
h(t) \geq h(t) - \max_{t \in [0, T^*]} |h(t) - h(t)| \geq c^*t - C_l \quad \text{for all } t \geq 0,
\] (5.21)
where $C_l = \max_{t \in [0, T^*]} |h(t) - h(t)| + c^*T^* + \sigma M$. Combining with (5.16) we obtain (5.8).

On the other hand, for any $\varepsilon > 0$, since $q_{c^*}(\infty) = u^*$, there exists $Z_1(\varepsilon) > 0$ such that
\[
q_{c^*}(z) \geq u^*(1 - \frac{\varepsilon}{2}) \quad \text{for } z \geq Z_1(\varepsilon).
\]

It follows from (5.21) and (5.16) that
\[
h(t) - x \geq c^*t - C_l - x \geq h(t) - C_r - C_l - x \geq Z_1(\varepsilon) \quad \text{for } t > T^*,
\]
which yields that for $(t, x) \in \Phi_1 := \{(t, x) : ct \leq x \leq h(t) - C_r - C_l - Z_1(\varepsilon), \quad t > T^*\}$,
\[
u(t, x) \geq u(t, x) \geq (1 - Me^{-\beta^*t})q_{c^*}(Z_1(\varepsilon)) \geq u^*(1 - Me^{-\beta^*t}) \left(1 - \frac{\varepsilon}{2}\right).
\]

Moreover, if we choose $T_2(\varepsilon) > T^*$ such that $2Me^{-\beta^*T_2(\varepsilon)} < \varepsilon$, then
\[
u(t, x) \geq u^* \left(1 - \frac{\varepsilon}{2}\right)^2 > u^*(1 - \varepsilon) \quad \text{for } (t, x) \in \Phi_1 \quad \text{and } t > T_2(\varepsilon),
\] (5.22)
which completes the proof of Step 2.

Step 3. Completion of the proof of (5.9). Denote $T_\varepsilon := T_1(\varepsilon) + T_2(\varepsilon)$ and $Z_\varepsilon := C_r + C_l + Z_1(\varepsilon)$, then by (5.17) and (5.22) we have
\[
u(t, x) - u^* \leq u^* \varepsilon \quad \text{for } 0 \leq x \leq h(t) - Z_\varepsilon, \quad t > T_\varepsilon.
\]
This yields the estimate in (5.9), which completes the proof of this proposition.

Using a similar argument as above we can obtain the following result.

**Proposition 5.3** Assume that spreading happens for the solution $(u, g, h)$. Then

(i) there exists $C' > 0$ such that
\[
u(t, x) \leq C' \quad \text{for all } t \geq 0;
\] (5.23)
(ii) for any small $\varepsilon > 0$, there exists $Z'_\varepsilon > 0$ and $T'_\varepsilon > 0$ such that

$$\|u(t, \cdot) - u^*\|_{L^\infty([g(t) + Z'_\varepsilon, 0])} \leq u^* \varepsilon \quad \text{for } t > T'_\varepsilon.$$  

(5.24)

5.2 Asymptotic profiles of the spreading solutions

This subsection is devoted to the proof of Theorem 1.4. We will prove this theorem by a series of results. Firstly, it follows from Proposition 5.2 that there exist positive constant $C$ such that

$$-C \leq h(t) - c^* t \leq C \quad \text{for } t \geq 0.$$  

Let us use the moving coordinate $y := x - c^* t + 2C$ and set

$$h_1(t) := h(t) - c^* t + 2C, \quad g_1(t) := g(t) - c^* t + 2C \quad \text{for } t \geq 0,$$

and $u_1(t, y) := u(t, y + c^* t - 2C)$ for $y \in [g_1(t), h_1(t)]$, $t \geq 0$.

Then $(u_1, g_1, h_1)$ solves

$$\begin{cases}
(u_1)_t = (u_1)_{yy} + c^*(u_1)_y - du_1 + f(u_1(t - \tau, y + c^* \tau)), &g_1(t) < y < h_1(t), \quad t > 0, \\
u_1(t, y) = 0, &y = g_1(t), \quad t > 0, \\
u_1(t, y) = 0, &y = h_1(t), \quad t > 0.
\end{cases}$$  

(5.25)

Let $t_n \to \infty$ be an arbitrary sequence satisfying $t_n > \tau$ for $n \geq 1$. Define

$$v_n(t, y) = u_1(t + t_n, y), \quad H_n(t) = h_1(t + t_n), \quad k_n(t) = g_1(t + t_n).$$

Lemma 5.4 Subject to a subsequence,

$$H_n(t) \to H \text{ in } C^{1+\varepsilon}_{\text{loc}}(\mathbb{R}) \text{ and } \|v_n - V\|_{C^{1+\varepsilon}_{\text{loc}}(\Omega_n)} \to 0,$$  

(5.26)

where $v \in (0, 1)$, $\Omega_n = \{(t, y) \in \Omega : y \leq H_n(t)\}$, $\Omega = \{(t, y) : -\infty < y \leq H(t), t \in \mathbb{R}\}$, and $(V(t, y), H(t))$ satisfies

$$\begin{cases}
V_t = V_{yy} + c^* V_y - dV + f(V(t - \tau, y + c^* \tau)), & (t, y) \in \Omega, \\
V(t, H(t)) = 0, \quad H'(t) = -\mu V_y(t, H(t)) - c^*, & t \in \mathbb{R}.
\end{cases}$$  

(5.27)

Proof It follows from the proof of Lemma 2.2 that there is $C_0 > 0$ such that $0 < H'(t) \leq C_0$ for all $t > 0$. One can deduce that

$$-c^* < H'(t) \leq C_0 \quad \text{for } t + t_n \text{ large and every } n \geq 1.$$  

Define

$$z = \frac{y}{H_n(t)}, \quad w_n(t, z) = v_n(t, y),$$

and direct computations yield that

$$\begin{aligned}
(w_n)_t &= \frac{1}{H_n^2(t)}(w_n)_{zz} + \frac{c^* + z H_n'(t)}{H_n(t)}(w_n)_z - d w_n + f\left(w_n(t - \tau, \frac{H_n(t)z + c^* \tau}{H_n(t - \tau)}\right)
\end{aligned}$$  

for $k_n(t) / H_n(t) < z < 1$, $t > \tau - t_n$, and

$$w_n(t, 1) = 0, \quad H_n'(t) = -\mu \frac{(w_n)_z(t, 1)}{H_n(t)} - c^*, \quad t > \tau - t_n.$$
Since $w_n \leq u^*$, then $f\left(w_n\left(t - \tau, \frac{H_n(t)z + c^*\tau}{H_n(t-\tau)}\right)\right)$ is bounded. For any given $Z > 0$ and $T_0 \in \mathbb{R}$, using the partial interior-boundary $L^p$ estimates and the Sobolev embedding theorem (see [13,18]), for any $\nu' \in (0, 1)$, we obtain
\[
\|w_n\|_{C^{1+\nu',1+\nu'}([T_0, \infty) \times [-Z, 1])} \leq C_Z \quad \text{for all large } n,
\]
where $C_Z$ is a positive constant depending on $Z$ and $\nu'$ but independent of $n$ and $T_0$. Thanks to this, we can find a positive constant $C_1$ independent of $n$ and $T_0$ such that
\[
\|H_n\|_{C^{1+\nu'}([T_0, \infty))} \leq C_1 \quad \text{for all large } n.
\]
Hence by passing to a subsequence we may assume that as $n \to \infty$,
\[
w_n \to W \quad \text{in } C^{1+\nu}_{loc}(\mathbb{R} \times (-\infty, 1]), \quad H_n \to H \quad \text{in } C^{1+\nu}_{loc}(\mathbb{R}),
\]
with $\nu \in (0, \nu')$. Based on above results, we can see that $(W, H)$ satisfies that
\[
\begin{cases}
W_t = \frac{W_z}{H^2(t)} + \frac{c^* + c'H(t)}{H(t)} W_z - dW + f(W(t - \tau, H(t)z + c^*\tau)), \quad (t, z) \in (-\infty, 1] \times \mathbb{R}, \\
W(t, 1) = 0, \quad H'(t) = -\mu \frac{W_z(t,1)}{H(t)} - c^*, \quad t \in \mathbb{R}.
\end{cases}
\]
Define $V(t, y) = W(t, \frac{y}{H(t)})$. It is easy to check that $(V, H)$ satisfies (5.27), and (5.26) holds. \hfill \square

Later, we show by a sequence of lemmas that $H(t) \equiv H_0$ is a constant and hence
\[
V(t, y) = q_{c^*}(H_0 - y).
\]
Since $C \leq h(t) - c^*t + 2C \leq 3C$ for all $t \geq 0$, then $C \leq H(t) \leq 3C$ for $t \in \mathbb{R}$. Denote $\phi(z) := q_{c^*}(-z)$ for $z \in \mathbb{R}$, it follows from the proof of Proposition 5.2 that for $x \in [(c - c^*)(t + t_n), H_n(t)]$ and $t + t_n$ large,
\[
(1 - Me^{-\beta^*(t+t_n)})\phi(y - C) \leq v_n(t, y) \leq \min\left\{(1 + M'e^{-\beta^*(t+t_n)})\phi(y - 3C), \quad u^*\right\}.
\]
Letting $n \to \infty$ we have
\[
\phi(y - C) \leq V(t, y) \leq \phi(y - 3C) \quad \text{for all } t \in \mathbb{R}, \quad y < H(t).
\]
Define
\[
X^* := \inf\{X : V(t, y) \leq \phi(y - X) \quad \text{for all } (t, y) \in D\}
\]
and
\[
X_* := \sup\{X : V(t, y) \geq \phi(y - X) \quad \text{for all } (t, y) \in D\},
\]
Then
\[
\phi(y - X_*) \leq V(t, y) \leq \phi(y - X^*) \quad \text{for all } (t, y) \in D,
\]
and
\[
C \leq X_* \leq \inf_{t \in \mathbb{R}} H(t) \leq \sup_{t \in \mathbb{R}} H(t) \leq X^* \leq 3C.
\]
By a similar argument as in [13], we have the following result.
Lemma 5.5  \( X^* = \sup_{t \in \mathbb{R}} H(t), \)  \( X_* = \inf_{t \in \mathbb{R}} H(t), \)  and there exist two sequences \( \{s_n\}, \{\bar{s}_n\} \subset \mathbb{R} \) such that 

\[
H(t + s_n) \to X^*, \quad V(t + s_n, y) \to \phi(y - X^*) \quad \text{as} \quad n \to \infty
\]

uniformly for \( (t, y) \) in compact subsets of \( \mathbb{R} \times (-\infty, X^*], \) and 

\[
H(t + \bar{s}_n) \to X_*, \quad V(t + \bar{s}_n, y) \to \phi(y - X_*) \quad \text{as} \quad n \to \infty
\]

uniformly for \( (t, y) \) in compact subsets of \( \mathbb{R} \times (-\infty, X_*] \).

Based on Lemma 5.5, we have the following lemma.

Lemma 5.6  \( X^* = X_*, \)  and hence \( H(t) \equiv H_0 \) is a constant, which yields \( V(t, y) = \phi(y - H_0) \).

Proof  Argue indirectly we may assume that \( X_* < X^* \). Choose \( \epsilon = (X^* - X_*)/4 \). We will show next that there is \( T_\epsilon > 0 \) such that 

\[
H(t) - X^* \geq -\epsilon \quad \text{and} \quad H(t) - X_* \leq \epsilon \quad \text{for} \quad t \geq T_\epsilon.
\]  

(5.28)

which implies that \( X^* - X_* \leq 2\epsilon \). This contraction would complete the proof.

To complete the proof, we need to prove that for given \( \epsilon = (X^* - X_*)/4 \), there exist \( n_1(\epsilon) \) and \( n_2(\epsilon) \) such that 

\[
H(t) - X^* \geq -\epsilon \quad (\forall t \geq s_{n_1}), \quad H(t) - X_* \leq \epsilon \quad (\forall t \geq \bar{s}_{n_2}).
\]

It follows from \( \phi(y - X_*) \leq V(t, y) \leq \phi(y - X^*) \) that there exist \( C_1 > 0 \) and \( \beta_1 > 0 \) such that 

\[
|u^* - V(t, y)| \leq C_1 e^{\beta_1 y}.
\]

By Lemma 5.5, for any \( \epsilon > 0 \), there exist \( K > 0, T > 0 \) such that for \( \bar{s}_n > T + \tau \) and \( s \in [0, \tau] \), 

\[
\sup_{y \in (-\infty, K)} |V(\bar{s}_n + s, y) - \phi(y - X^*)| < \epsilon.
\]  

(5.29)

Set \( G(t) = H(t) + c^* t \) and \( U(t, y) = V(t, y - c^* t) \), then \((W, G)\) satisfies 

\[
\begin{cases}
U_t = U_{yy} - dU + f(U(t - \tau, y)), & t \in \mathbb{R}, \quad y \leq G(t), \\
U(t, G(t)) = 0, & G'(t) = -\mu U_y(t, G(t)), \quad t \in \mathbb{R}.
\end{cases}
\]  

(5.30)

It follows from Lemma 5.5 and (5.29) that there is \( n_1 = n_1(\epsilon) \) such that for \( n \geq n_1 \), 

\[
H(\bar{s}_n + s) \leq X_* + \epsilon \quad \text{for} \quad s \in [0, \tau].
\]  

(5.31)

\[
V(\bar{s}_n + s, y) \leq \phi(y - X_* - \epsilon) + \epsilon \quad \text{for} \quad s \in [0, \tau], \quad y \leq X_*.
\]  

(5.32)

Thanks to (H), for \( \beta_0 \in (0, \beta^*) \) small with \( \beta^* \) is given in the proof of Proposition 5.2, there is \( \eta > 0 \) small such that 

\[
d - f'(v)e^{\beta_0 \tau} \geq \beta_0 \quad \text{for} \quad v \in [u^* - \eta, u^* + \eta],
\]  

(5.33)

and we can find \( N > 1 \) independent of \( \epsilon \) satisfies 

\[
\phi(y - X_* - \epsilon) + \epsilon \leq (1 + N\epsilon e^{-\beta_0 \tau})\phi(y - X_* - N\epsilon) \quad \text{for} \quad y \leq X_* + \epsilon.
\]

Let us construct the following supersolution of problem (5.30):

\[
\begin{align*}
\bar{G}(t) &:= X_* + N\epsilon + c^* t + N\epsilon \phi(1 - e^{-\beta_0 (t - \bar{s}_n)}), \\
\bar{U}(t, y) &:= \min \left\{ (1 + N\epsilon e^{-\beta_0 (t - \bar{s}_n)})\phi(y - \bar{G}(t), u^*) \right\}.
\end{align*}
\]
Since \( \lim_{y \to -\infty} \left( 1 + N\varepsilon e^{-\beta_0(t - \tilde{s}_n)} \right) \phi(y - \tilde{G}(t)) > u^* \), then there is a smooth function \( \tilde{K}(t) \) of \( t \geq \tilde{s}_n \) such that \( \tilde{K}(t) \to -\infty \) as \( t \to \infty \) and \( (1 + N\varepsilon e^{-\beta_0(t - \tilde{s}_n)}) \phi(\tilde{K}(t) - \tilde{G}(t)) = u^* \).

We will check that \( (\tilde{U}, \tilde{K}, \tilde{G}) \) is a supersolution for \( t \geq \tilde{s}_n + \tau \) and \( y \in [\tilde{K}(t), \tilde{G}(t)] \). We note that

\[
\tilde{U}(t, y) = (1 + N\varepsilon e^{-\beta_0(t - \tilde{s}_n)}) \phi(y - \tilde{G}(t)) \quad \text{when} \quad y \in [\tilde{K}(t), \tilde{G}(t)].
\]

Firstly, it follows from (5.31) that for \( s \in [0, \tau] \),

\[
G(\tilde{s}_n + s) \leq X_s + \varepsilon + c^*(\tilde{s}_n + s) \leq X_s + N\varepsilon + c^*(\tilde{s}_n + s) \leq \tilde{G}(\tilde{s}_n + s).
\]

In view of (5.32), we have

\[
\tilde{U}(\tilde{s}_n + s, y) = (1 + N\varepsilon e^{-\beta_0s}) \phi(y - \tilde{G}(\tilde{s}_n + s)) \geq (1 + N\varepsilon e^{-\beta_0\tau}) \phi(y - X_s - \varepsilon - c^*(\tilde{s}_n + s)) + \varepsilon \geq V(\tilde{s}_n + s, y - c^*(\tilde{s}_n + s)) = U(\tilde{s}_n + s, y),
\]

for \( s \in [0, \tau] \) and \( y \leq G(\tilde{s}_n + s) \). By definition \( \tilde{U}(t, \tilde{G}(t)) = 0 \) and direct computation yields

\[
-\mu \tilde{U}_y(t, \tilde{G}(t)) = c^*(1 + N\varepsilon e^{-\beta_0(t - \tilde{s}_n)}) < c^* + N\varepsilon \beta_0 e^{-\beta_0(t - \tilde{s}_n)} = \tilde{G}'(t),
\]

if we choose \( \sigma \) with \( \sigma \beta_0 > c^* \). Since \( U \leq u^* \), it then follows from the definition of \( \tilde{K}(t) \) that \( \tilde{U}(t, \tilde{K}(t)) = u^* \geq U(t, \tilde{K}(t)) \).

Finally, let us show

\[
\mathcal{N}[\tilde{U}] := \tilde{U}_t - \tilde{U}_{yy} + d\tilde{U} - f(\tilde{U}(t - \tau, y)) \geq 0, \quad y \in [\tilde{K}(t), \tilde{G}(t)], \quad t > \tilde{s}_n + \tau. \tag{5.34}
\]

Put \( z := y - \tilde{G}(t), \zeta(t) := N\varepsilon e^{-\beta_0(t - \tilde{s}_n)} \) and \( \phi_\tau := \phi(y - \tilde{G}(t - \tau)) \). It is easy to compute that

\[
\mathcal{N}[\tilde{U}] = \zeta \left\{ f(\phi_\tau) - \beta_0 \phi - \sigma \beta_0 (1 + \zeta) \phi' - f''(1 + \theta_2 \xi e^{\beta_0 t}) \phi_\tau e^{\beta_0 t} \phi_\tau \right\} \quad \text{(with} \ \theta_2 \in (0, 1))
\]

\[
\geq \zeta \left\{ -\sigma \beta_0 (1 + \zeta) \phi' - \left[ f''(1 + \theta_2 \xi e^{\beta_0 t}) \phi_\tau e^{\beta_0 t} - d \right] \phi_\tau - \beta_0 \phi \right\}.
\]

Since

\[
\phi(z) \to u^* \quad \text{and} \quad \frac{(\phi(z) - u^*)'}{\phi(z) - u^*} \to k^* \quad \text{as} \quad z \to -\infty,
\]

where \( k^* := c^* - \sqrt{(c^*)^2 + 4(d - f''(u^*))} < 0 \), there are two constants \( \zeta_\eta < 0 \) and \( k_0 \) such that

\[
\phi''(z) > 0, \quad \phi(z) \geq u^* - \eta \quad \text{and} \quad \phi(z - 2c^* \tau) \geq k_0 \phi'(z) \quad \text{for} \quad z < \zeta_\eta. \tag{5.35}
\]

Moreover, we can compute that

\[
\Delta \tilde{G}(t) := \tilde{G}(t) - \tilde{G}(t - \tau) = c^* \tau + N \sigma \varepsilon e^{-\beta_0(t - \tilde{s}_n)}(e^{\beta_0 \tau} - 1).
\]

For any given \( \sigma > 0 \), by shrinking \( \varepsilon \) if necessary, we have that

\[
\Delta \tilde{G}(t) \in [c^* \tau, 2c^* \tau] \quad \text{for} \quad t > \tilde{s}_n + \tau. \tag{5.36}
\]
For \( y - \tilde{G}(t - \tau) \leq z_\eta \) and \( t > \tilde{s}_n + \tau \), direct calculation implies

\[
\mathcal{N}[\tilde{U}] \geq \zeta \left[ -\sigma \beta_0 (1 + \zeta) \phi' - \left[ f'((1 + \theta_2 \zeta e^{\beta_0 \tau}) \phi_\tau) e^{\beta_0 \tau} - d \right] \phi_\tau - \beta_0 \phi \right] \\
\geq \zeta \left[ \left[ d - f'((1 + \theta_2 \zeta e^{\beta_0 \tau}) \phi_\tau) e^{\beta_0 \tau} - \beta_0 \right] \phi_\tau - \sigma \beta_0 \phi' + \beta_0 (\phi_\tau - \phi) \right] \\
\geq \zeta \left[ \beta_0 \phi' (y - \tilde{G}(t) + \hat{\theta}_2 \Delta \tilde{G}(t)) \Delta \tilde{G}(t) - \sigma \beta_0 \phi' (y - \tilde{G}(t)) \right] \quad \text{(with } \hat{\theta}_2 \in (0, 1)) \\
\geq (2k_0 c^\alpha \tau - \sigma) \beta_0 \phi' (y - \tilde{G}(t)) \geq 0
\]

provided that \( \sigma \) is sufficiently large, and we have used \((1 + \theta_2 \zeta e^{\beta_0 \tau}) \phi_\tau \in [u^* - \eta, u^* + \eta] \) for \( t > \tilde{s}_n + \tau \), \((5.33)\), \( \phi'(z) \leq 0 \) for \( z \leq z_\eta \), \((5.35)\) and \((5.36)\).

When \( z_\eta \leq y - \tilde{G}(t - \tau) \leq 0 \) and \( t > \tilde{s}_n + \tau \), for sufficiently large \( \sigma \), we have

\[
\mathcal{N}[\tilde{U}] \geq \zeta \left[ -\sigma \beta_0 C_z - u^* e^{\beta_0 \tau} C_f - \beta_0 u^* \right] \geq 0,
\]

where \( C_z := \max_{z \in [0, z_\eta + 2c^* \tau]} \phi'(z) < 0 \), \( C_f := \max_{v \in [0, u^* \tau]} f'(v) \), and \((5.36)\) are used.

Thus \((5.34)\) holds, then we can apply the comparison principle to conclude that

\[
U(t, y) \leq \tilde{U}(t, y), \quad G(t) \leq \tilde{G}(t) \quad \text{for } y \in [\bar{K}(t), \tilde{G}(t)] \text{ and } t > \tilde{s}_n + \tau.
\]

This, together with the definition of \( H(t) \), yields that \( H(t) \leq X_\alpha + N \epsilon (1 + \sigma) \) for \( t > \tilde{s}_n + \tau \). By shrinking \( \epsilon \) if necessary, we obtain

\[
H(t) \leq X_\alpha + \epsilon \quad \text{for } t > \tilde{s}_n + \tau \text{ and } n > n_1.
\]

In the following, we show \( H(t) \geq X^* - \epsilon \) for all large \( t \). As in the construction of supersolution, for any \( \epsilon > 0 \), there exists \( n_2 = n_2(\epsilon) \) such that, for \( n \geq n_2 \),

\[
H(s_n + s) \geq X^* - \epsilon \quad \text{for } s \in [0, \tau],
\]

\[
V(s_n + s, y) \geq \phi(y - X^* + \epsilon) - \epsilon \quad \text{for } s \in [0, \tau], \quad y \leq X^* - \epsilon.
\]

We also can find \( N_0 > 1 \) independent of \( \epsilon \) such that

\[
\phi(y - X^* + \epsilon) - \epsilon \geq (1 - N_0 \epsilon e^{-\beta_0 \tau}) \phi(y - X^* + N_0 \epsilon) \quad \text{for } y \leq X^* - \epsilon.
\]

We can define a subsolution as follows:

\[
\begin{align*}
G(t) &:= X^* - N_0 \epsilon c^* t - N_0 \sigma \epsilon (1 - e^{-\beta_0 (t - s_n)}), \\
\tilde{U}(t, y) &:= (1 - N_0 \epsilon e^{-\beta_0 (t - s_n)}) \phi(y - \tilde{G}(t)).
\end{align*}
\]

Since \( U(t, y) \geq \phi(y - X_\alpha) \), there are \( C_0 \) and \( \alpha > 0 \) such that \( V(t, y) \geq u^* - C_0 \epsilon \alpha \) for all \( y \leq 0 \), which implies that \( U(t, y) \geq u^* - C_0 \epsilon \alpha (y - c^* t) \). Let us fix \( c \in (0, c^*) \) such that \( \beta_0 \leq \alpha (c + c^*) \). By enlarging \( n \) if necessary we may assume that \( C_0 \leq u^* N_0 \epsilon e^{\beta_0 s_n} \). Denote \( K(t) := -ct \).

By a similar argument as above and in Step 2 of Proposition 5.2, we can show that \((\tilde{U}, \tilde{G}, K)\) is a subsolution of problem \((5.30)\) by taking \( \sigma > 0 \) sufficiently large. The comparison principle can be used to conclude that

\[
\tilde{U}(t, y) \leq U(t, y), \quad \tilde{G} \leq G(t) \quad \text{for } t \geq s_n + \tau, \quad y \in [-ct, \tilde{G}(t)],
\]

which implies that \( G(t) \geq X^* - N_0 \epsilon (1 + \sigma) \) for \( t \geq s_n + \tau \). By shrinking \( \epsilon \) if necessary, we have

\[
X^* - \epsilon \leq G(t) \quad \text{for } t \geq s_n + \tau \text{ and } n \geq n_2.
\]

This completes the proof of this lemma. \( \Box \)
Theorem 5.7 Assume that (H) and spreading happens. Then there exists $H_1 \in \mathbb{R}$ such that

$$
\lim_{t \to \infty} [h(t) - c^* t] = H_1, \quad \lim_{t \to \infty} h'(t) = c^*,
$$

$$
\lim_{t \to \infty} \|u(t, \cdot) - q_{c^*} (c^* t + H_1 - \cdot)\|_{L^\infty([0, h(t)])} = 0,
$$

(5.40)  

(5.41)

where $(c^*, q_{c^*})$ is the unique solution of (1.13).

**Proof** It follows from Lemmas 5.4 and 5.6 that for any $t_n \to \infty$, by passing to a subsequence, 

$$
h(t + t_n) - c^* (t + t_n) \to H_1 := H_0 - 2C \quad \text{in } C^{1+\frac{1}{2}}_{\text{loc}}(\mathbb{R}).
$$

The arbitrariness of $(t_n)$ implies that 

$$
h(t) - c^* t \to H_1 \quad \text{and } h'(t) \to c^* \quad \text{as } t \to \infty,
$$

which proves (5.40).

In what follows, we use the moving coordinate $z := x - h(t)$ to prove (5.41). Set

$$
g_2(t) := g(t) - h(t), \quad u_2(t, z) := u(t, z + h(t)) \quad \text{for } z \in [g_2(t), 0], \ t \geq \tau,
$$

$$
g_n(t) := g(t + t_n) - h(t + t_n), \quad \tilde{h}_n(t) = h(t + t_n), \quad \tilde{u}_n(t, z) = u_2(t + t_n, z),
$$

then the pair $(\tilde{u}_n, \tilde{g}_n, \tilde{h}_n)$ solves

$$
\begin{cases}
(\tilde{u}_n)_t = (\tilde{u}_n)_z + \tilde{h}_n (\tilde{u}_n)_z + f(\tilde{u}_n(t - \tau, z + \tilde{h}_n(t) - \tilde{h}_n(t - \tau)) - d\tilde{u}_n, \ z \in (\tilde{g}_n(t), 0), \ t > \tau, \\
\tilde{u}_n(t, 0) = 0, \quad \tilde{g}_n(t) = -\mu(\tilde{u}_n)(t, z) - \tilde{h}_n'(t), \quad z = \tilde{g}_n(t), \ t > \tau, \\
\tilde{u}_n(t, 0) = 0, \quad \tilde{h}_n'(t) = -\mu(\tilde{u}_n)(t, 0), \quad t > \tau.
\end{cases}
$$

(5.42)

By the same reasoning as in the proof of Lemma 5.4, the parabolic regularity to (5.42) plus the Sobolev embedding theorem can be used to conclude that, by passing to a further subsequence if necessary, as $n \to \infty$, $\tilde{u}_n \to W$ in $C^{1+\frac{1}{2}}_{\text{loc}}(\mathbb{R} \times (-\infty, 0])$, and $W$ satisfies, in view of $\tilde{h}_n'(t) \to c^*$,

$$
\begin{cases}
W_t = W_{zz} + c^* W_z - dW + f(W(t - \tau, z + c^* \tau)), \ -\infty < z < 0, \ t \in \mathbb{R}, \\
W(t, 0) = 0, \quad c^* = -\mu W_z(t, 0), \quad t \in \mathbb{R}.
\end{cases}
$$

This is equivalent to (5.27) with $V = W$ and $H = 0$. Hence we can conclude

$$
W(t, z) = \phi(z) \quad \text{for } (t, z) \in \mathbb{R} \times (-\infty, 0].
$$

Thus we have proved that, as $n \to \infty$,

$$
\lim_{t \to \infty} [u(t + t_n, z + h(t + t_n)) - q_{c^*}(-z)] = 0 \quad \text{in } C^{1+\frac{1}{2}}_{\text{loc}}(R \times (-\infty, 0]).
$$

This, together with the arbitrariness of $(t_n)$, yields that

$$
\lim_{t \to \infty} [u(t, z + h(t)) - q_{c^*}(-z)] = 0 \quad \text{uniformly for } z \text{ in compact subsets of } (-\infty, 0].
$$

Then, for any $L > 0$,

$$
\|u(t, \cdot) - q_{c^*} (h(t) - \cdot)\|_{L^\infty([h(t) - L, h(t)])} \to 0 \quad \text{as } t \to \infty.
$$

Using the limit $h(t) - c^* t \to H_1$ as $t \to \infty$ we obtain

$$
\|u(t, \cdot) - q_{c^*} (c^* t + H_1 - \cdot)\|_{L^\infty([h(t) - L, h(t)])} \to 0 \quad \text{as } t \to \infty.
$$

(5.43)

Finally we prove (5.41). For any given small $\varepsilon > 0$, it follows from (5.9) in Proposition 5.2 that there exist two positive constants $Z_{\varepsilon}$ and $T_{\varepsilon}$ such that

$$
|u(t, x) - u^*| \leq u^* \varepsilon \quad \text{for } 0 \leq x \leq h(t) - Z_{\varepsilon}, \ t > T_{\varepsilon}.
$$
Since \( q_{c^*}(z) \rightarrow u^* \) as \( z \rightarrow \infty \), there exists \( Z^*_e > Z_e \) such that
\[
|q_{c^*}(c^*t + H_1 - x) - u^*| \leq u^*\varepsilon \quad \text{for} \quad x \leq c^*t + 2H_1 - Z^*_e.
\]
Taking \( T^*_e > T_e \) large such that \( h(t) < c^*t + 2H_1 \) for \( t > T^*_e \), then we obtain
\[
|u(t, x) - q_{c^*}(c^*t + H_1 - x)| \leq 2u^*\varepsilon \quad \text{for} \quad 0 \leq x \leq h(t) - Z^*_e, \quad t > T^*_e.
\]
Taking \( L = Z^*_e \) in (5.43) we see that for some \( T^{**}_e > T^*_e \),
\[
|u(t, x) - q_{c^*}(c^*t + H_1 - x)| \leq u^*\varepsilon \quad \text{for} \quad h(t) - Z^*_e \leq x \leq h(t), \quad t > T^{**}_e.
\]
This completes the proof of (5.41).

Taking use of a similar argument as above one can obtain the following result.

**Theorem 5.8** Assume that (H) and spreading happens. Then there exists \( G_1 \in \mathbb{R} \) such that
\[
\lim_{t \to \infty} [g(t) + c^*t] = G_1, \quad \lim_{t \to \infty} g'(t) = -c^*, \tag{5.44}
\]
\[
\lim_{t \to \infty} \|u(t, \cdot) - q_{c^*}(c^*t - G_1 + \cdot)\|_{L^\infty([g(t), 0])} = 0, \tag{5.45}
\]
where \((c^*, q_{c^*})\) is the unique solution of (1.13).

**Proof of Theorem 1.4.** The results in Theorem 1.4 follow from Theorems 5.7 and 5.8.

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