Propagators for spinless and spin-1/2 Aharonov-Bohm-Coulomb systems

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Abstract

The propagator of the spinless Aharonov-Bohm-Coulomb system is derived by following the Duru-Kleinert method. We use this propagator to explore the spin-1/2 Aharonov-Bohm-Coulomb system which contains a point interaction as a Zeeman term. Incorporation of the self-adjoint extension method into the Green’s function formalism properly allows us to derive the finite propagator of the spin-1/2 Aharonov-Bohm-Coulomb system. As a by-product, the relation between the self-adjoint extension parameter and the bare coupling constant is obtained. Bound-state energy spectra of both spinless and spin-1/2 Aharonov-Bohm-Coulomb systems are examined.
I. INTRODUCTION

A great deal of attention of the Aharonov-Bohm(AB) effect [1], which sheds light on the non-trivial physical significance of the scalar and vector potentials at quantum level, has been paid in recent years in the context of anyonic [2], cosmic string [3], and (2+1)D gravity theories [4]. Since anyon, a two-dimensional object, carries the magnetic flux [5], the dominant interaction between anyons is of AB type. Arovas et al have pointed out in their seminal paper [6] by calculating the second virial coefficient that the statistics of anyon system interpolates between bosons and fermions, which is the most important property of anyon system for attacking the high-$T_c$ phenomena in superconductivity.

Many authors investigated whether or not this property can be maintained when the spin degree of freedom is included. One remarkable difference of the spin-1/2 AB problem from the spinless case is a point interaction potential term which occurs as a mathematical description of the Zeeman interaction of spin with a magnetic flux tube. Gerbert dealt with this problem [7] by applying the self-adjoint extension method [8] to the partial wave. He claimed that when $|m + \alpha| < 1$, where $m$ and $\alpha$ are angular momentum quantum number and flux parameter respectively, both regular and irregular radial solutions at the origin are allowed. Being compatible with the boundary condition derived by self-adjoint extension technique, his solution naturally contains undetermined real parameter, say $\theta$, which is called as self-adjoint extension parameter.

Hagen also analyzed the same problem on the physical ground [9]. He chose the physically motivated expression of the flux tube

$$H \propto \lim_{R \to 0} \frac{1}{R} \delta(R - r)$$

where $H$ is a magnetic field, and solved the radial Schrödinger equation at the $r < R$ and $r > R$ regions separately. Upon applying the matching conditions at $r = R$ in the $R \to 0$ limit, he argued that when both $|m + \alpha| < 1$ and $|m| + |m + \alpha| = -\alpha s$ are satisfied simultaneously, where $s$ is twice the spin quantum number, the physically relevant wave is the irregular solution at the origin. Although the self-adjoint extension method does not yield the latter one, it is easily shown that Hagen’s result coincides with Gerbert’s when the self-adjoint extension parameter $\theta$ equals $\pi/4$. Hagen also calculated [10] the second virial
coefficient for the spin-1/2 AB system and showed that it is completely different from that for the spinless AB system.

A series of the above-mentioned papers on the spin-1/2 AB problem raises an issue on the physical meaning of the self-adjoint extension parameter. Some authors [11] tried to dig out its physical meaning in the internal structure of the magnetic flux tube. This kind of approach, however, encountered with the criticism [12] that their calculation within the framework of the Dirac equation tends not to be reliable because of the occurrence of Klein’s paradox. Jackiw [13] also approached the same issue from the different point of view. Analyzing the two- and three-dimensional delta-function potentials, he asserted that the self-adjoint extension parameter has a certain relation with a renormalized (or bare) coupling constant.

Recently his result was used for the derivation of propagator in the spin-1/2 AB system by one of us [14]. In Ref. [14] it is shown that the relation between the self-adjoint extension parameter and the renormalized coupling constant is more consistently derived by incorporating the self-adjoint extension method into the Green’s function formalism properly. The method suggested in Ref. [14] is also applied to the one-dimensional $\delta'$-potential case [15].

Since the AB interaction is accomplished between charge and magnetic flux, the interacting anyon system naturally involves the Coulomb modification. The effect of Coulomb potential in the spin-1/2 AB problem is discussed by Hagen in Ref. [12] by solving the Schrödinger equation. Since the Feynman propagator [16] is essential and useful for analyzing the time-dependent scattering and statistical property of anyon system, it is very important to derive the propagator of the Aharonov-Bohm-Coulomb (ABC) system. In the present paper we will derive the propagators of the spinless and spin-1/2 ABC systems by using the Duru-Kleinert pseudotime method [17] and the essential idea in Ref. [14].

The non-relativistic solution of hydrogen atom has been a long-standing problem in the path-integral framework. Duru and Kleinert solved the problem with a help of Kustaanheimo-Stiefel (KS) [18] transformation and through the introduction of the dimensional extension and pseudotime. Their essential idea is based on the following observations: if the system possesses a Feynman path-integral for the time evolution amplitude, it does so also for the fixed-energy amplitude $K[\vec{x}_b, \vec{x}_a; E]$. Their idea came up with the following
time-sliced formula

\[
K[\vec{x}_b, \vec{x}_a; E] = \lim_{N \to \infty} (N + 1) \int_0^\infty d\epsilon_s f_r(\vec{x}_b) f_l(\vec{x}_a) \int \left( \prod_{j=1}^N d\vec{x}_j \right) \int \left( \prod_{j=1}^{N+1} \frac{d\vec{p}_j}{(2\pi)^D} \right) e^{iA^N_E}.
\]

(1.1)

where

\[
A^N_E = \sum_{j=1}^{N+1} \left[ \vec{p}_j \cdot (\vec{x}_j - \vec{x}_{j-1}) - \epsilon_s f_l(\vec{x}_j) \{H(\vec{p}_j, \vec{x}_j) - E\} f_r(\vec{x}_{j-1}) \right].
\]

(1.2)

Here, \(H\) and \(E\) are the Hamiltonian of a given system and its eigenvalue respectively, and \(f_l(\vec{x})\) and \(f_r(\vec{x})\) regulating functions defined in chapter 12 of Ref.[19]. Pseudotime \(s\) is defined as \(ds/dt = f_l(\vec{x}) f_r(\vec{x})\) and \((N + 1)\epsilon_s = s_b - s_a \equiv s\).

In this paper the propagators of the spinless and spin-1/2 ABC systems are derived. Sec.II gives the derivation of the propagator of the spinless ABC system using Eq.(1.1) and Levi-Cività transformation that is a two-dimensional version of the KS transformation. In Sec.III we will review Ref.[14] briefly to prepare the calculation in spin-1/2 ABC system. In Sec.IV the derivation of the propagator of the spin-1/2 ABC system is presented by the appropriate incorporation of the self-adjoint extension method into the Green’s function formalism. A brief conclusion is given in Sec.V. Throughout this paper, we take \(\hbar = 1\) for simplicity.
II. PROPAGATOR FOR SPINLESS ABC SYSTEM

In this section we will derive the propagator for a spinless ABC system by following the method used by Kleinert[17, 19]. Let us start with the Hamiltonian

$$H = \frac{(\vec{p} - e\vec{A})^2}{2M} + \frac{\xi}{r}$$  \hspace{1cm} (2.1)

where the AB potential is

$$\vec{A} = \frac{\alpha}{e} \epsilon_{ij} \frac{r_j}{r^2}$$  \hspace{1cm} (2.2)

in Coulomb gauge and $\epsilon_{12} = 1$.

Then the momentum integration of Eq.(1.1) can be performed straightforwardly and the resultant fixed-energy amplitude is

$$K[\vec{x}_b, \vec{x}_a; E] = (N + 1) \int_0^\infty d\epsilon_s f_r(\vec{x}_b) f_l(\vec{x}_a) \left( \frac{M}{2\pi i \epsilon_s f_l(\vec{x}_a) f_l(\vec{x}_b)} \right) \frac{1}{\sqrt{\prod_{j=1}^N \frac{d\vec{x}_j}{(2\pi i \epsilon_s f_l(\vec{x}_j) f_r(\vec{x}_j)/M)^{1/2}}}}$$  \hspace{1cm} (2.3)

$$\times \exp \sum_{j=1}^{N+1} \left[ M \frac{\epsilon_s f_l(\vec{x}_j)}{2\epsilon_s f_l(\vec{x}_j) f_r(\vec{x}_j-1)} (\vec{x}_j - \vec{x}_{j-1})^2 + e\vec{A}_j \cdot (\vec{x}_j - \vec{x}_{j-1}) - \frac{\xi}{r_j} E f_r(\vec{x}_{j-1}) \right].$$

In deriving Eq.(2.3), we fixed $D = 2$.

With the following choice

$$f_l(\vec{x}) = r^{1-\lambda}$$  \hspace{1cm} (2.4)

$$f_r(\vec{x}) = r^\lambda$$

$K[\vec{x}_b, \vec{x}_a; E]$ can be simplified in a form

$$K[\vec{x}_b, \vec{x}_a; E] = (N + 1) \int_0^\infty d\epsilon_s \frac{M}{2\pi i \epsilon_s} \left( \frac{r_a}{r_b} \right)^{1-2\lambda} \int \prod_{j=2}^{N+1} \frac{M}{2\pi i \epsilon_s} d\Delta \vec{x}_j \right] e^{iS_{0,E}^N}$$  \hspace{1cm} (2.5)

where

$$S_{0,E}^N = \sum_{j=1}^{N+1} \left[ M \frac{\epsilon_s}{2\epsilon_s r_j^{1-\lambda} r_{j-1}^{\lambda}} (\vec{x}_j - \vec{x}_{j-1})^2 + e\vec{A}_j \cdot (\vec{x}_j - \vec{x}_{j-1}) - \frac{\xi}{r_j} E \frac{r_{j-1}}{r_j} \lambda \right]$$  \hspace{1cm} (2.6)
whose continuum limit is
\[ S_{0,E}[\vec{x}, \vec{x}'] = -\xi s + \int_0^s ds \left[ \frac{M}{2r} \vec{x}'^2 + eA \cdot \vec{x}' + Er \right] \] (2.7)
and \( \vec{x}' = d\vec{x}/ds \). Since the continuum limit of \( K[\vec{x}_b, \vec{x}_a; E] \) is independent of \( \lambda \) \([19]\), we set \( \lambda = 1/2 \), which yields
\[ K[\vec{x}_b, \vec{x}_a; E] = (N + 1) \int_0^\infty ds M \frac{M}{2\pi i\epsilon_s} \int \left[ \prod_{j=2}^{N+1} M \frac{M}{2\pi i\epsilon_s r_{j-1}} d\Delta \vec{x}_j \right] e^{iS_{0,E;\lambda=1/2}}. \] (2.8)

We now apply the Levi-Cività transformation
\[
\begin{pmatrix} x \\ y \end{pmatrix} = A(\vec{u}) \begin{pmatrix} u^1 \\ u^2 \end{pmatrix},
\] (2.9)
where
\[
A(\vec{u}) = \begin{pmatrix} u^1 - u^2 \\ u^2 \\ u^1 \end{pmatrix},
\] (2.10)
to the path-integral calculation of \( K[\vec{x}_b, \vec{x}_a; E] \). From the definition of Levi-Cività transformation \((2.9)\) and \((2.10)\) it is easily shown that
\[
x^2 + y^2 = \left[ (u^1)^2 + (u^2)^2 \right]^2 \] (2.11)
\[
\Delta x^2 + \Delta y^2 = 4 \left[ (u^1)^2 + (u^2)^2 \right] \left[ (\Delta u^1)^2 + (\Delta u^2)^2 \right]
\]
\[
\frac{\partial (x,y)}{\partial (u^1,u^2)} = 2^2 r.
\]
Furthermore, the Levi-Cività transformation of the AB potential term is
\[
\vec{A}_j \cdot d\vec{x}_j \equiv \alpha \frac{y_j \Delta x_j - x_j \Delta y_j}{r_j^2}
\] (2.12)
\[
= 2\alpha \frac{u_j^2 \Delta u_j^1 - u_j^1 \Delta u_j^2}{(u_j)^2}.
\]
Hence, the AB-potential term with flux \( \alpha \) in \((x,y)\)-space is changed into the same form with flux \( 2\alpha \) in \((u^1, u^2)\)-space. It is worthwhile to note that the Levi-Cività transformation is a mapping from flat \((x,y)\)-space to flat \((u^1, u^2)\)-space unlike KS transformation whose image space has a torsion and curvature \([19, 21]\). Therefore we can change the measure of Eq.(2.8) as \([19]\).
\[ \prod_{j=2}^{N+1} \left( \frac{M}{2\pi i \epsilon r_{j-1}} d\Delta \bar{x}_j \right) \Rightarrow \prod_{j=1}^{N} \left( \frac{M}{2\pi i \epsilon r_{j}} d\bar{x}_j \right). \] (2.13)

From Eqs. (2.11), (2.12), and (2.13) \( K[\bar{x}_b, \bar{x}_a; E] \) is changed into

\[ K[\bar{x}_b, \bar{x}_a; E] = (N + 1) \int_{0}^{\infty} d\epsilon_s \left( \frac{M}{2\pi i \epsilon_s} \right) \int \left[ \prod_{j=1}^{N} \frac{4M}{2\pi i \epsilon_s} d^2 \bar{u}_j \right] e^{iS_E^N[\bar{u}, \bar{u}']} \] (2.14)

where

\[ S_E^N[\bar{u}, \bar{u}'] = -\xi s + \sum_{j=1}^{N+1} \left[ 2M \left( \frac{(\bar{u}_j - \bar{u}_{j-1})^2}{\epsilon_s} + 2e\bar{A}_j \cdot \Delta \bar{u}_j + \epsilon_s E \bar{u}_j^2 \right) \right] \] (2.15)

and

\[ \bar{A}^u = \frac{\alpha}{e} \left( \frac{u^2}{(u^1)^2 + (u^2)^2} \right). \] (2.16)

The time-sliced action (2.13) represents the AB plus harmonic oscillator system whose mass and flux parameter are \( 4M \) and \( 2\alpha \) respectively, and angular frequency is

\[ \omega^2 = -\frac{E}{2M}. \] (2.17)

The well-known propagator [14,22] of the AB plus harmonic oscillator system can be used to calculate the fixed-energy amplitude directly. After treating the square root property of the Levi-Civit\`a transformation carefully that was explained nicely in chapter 13 of Ref.[19], \( K[\bar{x}_b, \bar{x}_a; E] \) becomes

\[ K[\bar{x}_b, \bar{x}_a; E] = \frac{1}{4} \int_{0}^{\infty} ds \left( K[\bar{u}_b, \bar{u}_a; s] + K[-\bar{u}_b, \bar{u}_a; s] \right) \] (2.18)

where

\[ K[\bar{u}_b, \bar{u}_a; s] = \sum_m e^{im(\phi_b - \phi_a)} K_m[\bar{u}_b, \bar{u}_a; s]. \] (2.19)

Here, \( \phi \) is defined as polar angle in \((u^1, u^2)\)-space

\[ u^1 = u \cos \phi \] (2.20)

\[ u^2 = u \sin \phi \]

and
\[ K_m[u_b, u_a; s] = \frac{(4M)\omega}{2\pi i \sin \omega s} e^{-i\xi s} \]
\[ \times \exp \left[ \frac{i(4M)\omega}{2}(u^2_a + u^2_b) \cot \omega s \right] I_{|m+2\alpha|} \left( \frac{-i(4M)\omega u_a u_b}{\sin \omega s} \right). \]

After inserting Eqs. (2.19) and (2.21) into Eq.(2.18) and representing the result in terms of the polar coordinate in \((x, y)\) space: \(\theta = 2\phi\) and \(r = u^2\), one arrives at the final form of fixed-energy amplitude

\[ K[\vec{x}_b, \vec{x}_a; E] = \sum_{-\infty}^{\infty} e^{i m(\theta_b - \theta_a)} K_m[x_b, x_a; E] \] (2.22)
where

\[ K_m[x_b, x_a; E] = \frac{M \omega}{\pi i} \int_0^\infty ds \frac{e^{-i\xi s}}{\sin \omega s} \exp \left[ 2iM \omega (x_a + x_b) \cot \omega s \right] I_{|m+2\alpha|} \left( \frac{-4iM \omega}{\sin \omega s} \sqrt{x_a x_b} \right). \] (2.23)

\(K_m[x_b, x_a; E]\) is a fixed-energy amplitude associated to sum over all possible paths within the \(m^{th}\) homotopy class.

In order to obtain the energy spectrum one has to check the poles of fixed-energy amplitude carefully. This is easily achieved by changing the variable \(v = -i/\sin \omega s\) and performing the \(v\)-integration explicitly using the integral formula [23]

\[ \int_0^\infty dx \frac{1}{\sqrt{x^2 + z^2}} e^{\mu \sqrt{x^2 + z^2}} \exp[-p\sqrt{x^2 + z^2}] I_\nu(cx) \] (2.24)
\[ = \frac{1}{cz} \Gamma \left( \frac{1+\nu+\mu}{2} \right) W_{\mu, \nu}(z_+) W_{\mu, \nu}(z_-) \]
where \(W_{a,b}\) and \(M_{a,b}\) are usual Whittaker functions and \(z_{\pm} = z(p \pm \sqrt{p^2 - c^2})\).

The \(v\)-integration makes \(K_m[x_b, x_a; E]\) to be

\[ K_m[x_b, x_a; E] = \frac{1}{4\pi i \omega \sqrt{x_a x_b}} \frac{\Gamma \left( \frac{1}{2} + \left| m + \alpha \right| + \frac{\xi}{2\omega} \right)}{\Gamma (1 + 2 \left| m + \alpha \right|)} \]
\[ \times W_{-\frac{\xi}{2\omega}, |m+\alpha|} (4M \omega \text{Max}(x_a, x_b)) W_{-\frac{\xi}{2\omega}, |m+\alpha|} (4M \omega \text{Min}(x_a, x_b)). \] (2.25)

The energy spectrum of the ABC system is deduced from the poles of the gamma function in the numerator:

\[ E_{n,m} = -\frac{1}{2} \frac{M \ell^2}{(n + \left| m + \alpha \right| - \frac{1}{2})^2}, \quad n = 1, 2, \ldots \] (2.26)
This spectrum is in agreement with the result corresponding to the regular solution obtained in Ref. [12].

In Sec. III The main idea of Ref. [14] for the evaluation of propagator in spin-1/2 ABC system will be briefly reviewed.
III. PROPER INCORPORATION OF THE SELF-ADJOINT EXTENSION METHOD INTO THE GREEN’S FUNCTION FORMALISM

In this section we will discuss how to incorporate the self-adjoint extension method into the Green’s function formalism briefly. A one-dimensional Hamiltonian below will be a good model for this purpose:

\[ H = H_V + v\delta(x) \]  

(3.1)

where \( H_V \) is

\[ H_V = \frac{p^2}{2} + V(x). \]  

(3.2)

Taking a Laplace transform to the well-known integral equation

\[ G[x_1, x_2; t] = G_V[x_1, x_2; t] - v \int_0^t ds \int dx G_V[x_1, x, t - s] \delta(x) G[x, x_2; s] \]  

(3.3)

where \( G[x_1, x_2; t] \) and \( G_V[x_1, x_2; t] \) are euclidean propagators of \( H \) and \( H_V \) respectively, it is straightforward to derive the relation

\[ \hat{G}[x_1, x_2; E] = \hat{G}_V[x_1, x_2; E] - v \hat{G}_V[x_1, 0; E] \hat{G}[0, x_2; E]. \]  

(3.4)

From Eq.(3.4) one can easily obtain

\[ \hat{G}[0, x_2; E] = \frac{\hat{G}_V[0, x_2; E]}{1 + v \hat{G}_V[0, 0; E]} \]  

(3.5)

by taking a \( x_1 \to 0 \) limit. Upon combining Eqs. (3.4) and (3.5) energy-dependent Green’s function \( \hat{G}[x_1, x_2; E] \) is easily calculated from \( \hat{G}_V[x_1, x_2; E] \):

\[ \hat{G}[x_1, x_2; E] = \hat{G}_V[x_1, x_2; E] - \frac{\hat{G}_V[x_1, 0, E] \hat{G}_V[0, x_2; E]}{1 + \hat{G}_V[0, 0; E]}. \]  

(3.6)

Applying the above method to the \( V(x) = 0 \) case which was done firstly in Ref.[24] produces the energy-dependent Green’s function for the \( \delta \)-function potential system

\[ \hat{G}[x, y, E] = \frac{e^{-\sqrt{2E}|x-y|}}{\sqrt{2E}} - v \frac{e^{-\sqrt{2E}(|x|+|y|)}}{\sqrt{2E}(\sqrt{2E}+v)} \]  

(3.7)

and euclidean time-dependent propagator by taking an inverse Laplace transform to Eq.(3.7)
\[ G(x, y; t) = G_0(x, y; t) - v \int_0^\infty dze^{-vz}G_0(x, |y| - |z|; t), \]

where \( G_0(x, y; t) \) is the time-dependent propagator for a one-dimensional free particle.

It is a simple matter to show that \( G(x, y; t) \) satisfies the well-known boundary condition in the Kronig-Penney model

\[ \frac{\partial G}{\partial x}(0^+, y; t) - \frac{\partial G}{\partial x}(0^-, y; t) = 2vG[0, y; t]. \]

An important observation that there are two different ways to calculate the bound-state spectrum from energy-dependent Green’s function will play a crucial role for the calculation of higher dimensional cases. One of the two ways to obtain the bound-state energy is checking the poles of energy-dependent Green’s function, which is an universal property of \( \hat{G}(x, y, E) \). The other is applying the boundary condition (3.9) to the energy-dependent Green’s function:

\[ \frac{\partial \hat{A}}{\partial x}[0^+, y; E] - \frac{\partial \hat{A}}{\partial x}[0^-, y; E] = 2v\hat{A}[0, y; E] \]

where \( \hat{A}(x, y, E) \equiv \hat{G}(x, y, E) - \hat{G}_0(x, y, E) \). This means that the boundary condition (3.9) plays an important role for the occurrence of a bound state. In Ref.[14] the use of these two different ways is examplified to derive the relation between the self-adjoint extension parameter and the bare coupling constant in two- and three-dimensional systems. Once this relation is obtained, the derivation of energy-dependent Green’s function is straightforward.

The above-mentioned method, however, cannot be applied directly to the spin-1/2 ABC system because the energy-dependent Green’s function of a spinless ABC system, which can be obtained from fixed-energy amplitude (2.22) and (2.25) through the relation

\[ \hat{G}(\vec{x}_b, \vec{x}_a; E) = iK[\vec{x}_b, \vec{x}_a; -E], \]

diverges when either \( \vec{x}_a \) or \( \vec{x}_b \) approaches zero. This is because of the magnetic flux tube located at the origin. In such a case as suggested in Ref.[14] Eq.(3.6) must be modified to be

\[ \hat{G}(\vec{x}_b, \vec{x}_a; E) = \hat{G}_V[\vec{x}_b, \vec{x}_a; E] - \frac{\hat{G}_V[\vec{x}_b, \vec{\epsilon}_1; E]\hat{G}_V[\vec{\epsilon}_2, \vec{x}_a; E]}{\frac{1}{v} + \lim_{\epsilon_2 \to \vec{\epsilon}_1} \hat{G}_V[\vec{\epsilon}_2, \vec{\epsilon}_1; E]} \]

In Eq.3.12 the limit \( \epsilon_1 \to 0 \) should be taken at the final stage of calculation.
It is shown in Ref.[14] through the explicit calculation that this method produces a physically relevant propagator consistent with the boundary condition deduced by self-adjoint extension method and derives the relationship between the self-adjoint extension parameter and the coupling constant consistently and convincingly.

In next section we will apply this method to the spin-1/2 ABC system and obtain the energy-dependent Green’s function explicitly.
IV. PROPAGATOR FOR SPIN-1/2 ABC SYSTEM

From Eqs. (2.22) and (2.25) the energy-dependent Green’s function \( \hat{G}_B[\vec{x}_b, \vec{x}_a; E] \) of the spinless ABC system is

\[
\hat{G}_B[\vec{x}_b, \vec{x}_a; E] = \sum_{m=-\infty}^{\infty} e^{i m (\theta_b - \theta_a)} \hat{G}_m^B[\vec{x}_b, \vec{x}_a; E] \tag{4.1}
\]

where

\[
\hat{G}_m^B[\vec{x}_b, \vec{x}_a; E] = \frac{1}{4\pi\Omega \sqrt{x_a x_b}} \frac{\Gamma\left(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega}\right)}{\Gamma(1 + 2 |m + \alpha|)} \times W_{\frac{\xi}{2\Omega} |m+\alpha|}(4M\Omega \text{Max}(x_a, x_b)) M_{\frac{\xi}{2\Omega} |m+\alpha|}(4M\Omega \text{Min}(x_a, x_b)) \tag{4.2}
\]

and

\[
\Omega^2 = \frac{E}{2M}. \tag{4.3}
\]

When spin degree of freedom is added to the ABC system, it is well known [7, 9, 13] that a delta-function potential appears because of the Zeeman interaction. The Hamiltonian, therefore, for a large component of Dirac spinor is

\[
H_F = H_B + v\delta(\vec{x}) \tag{4.4}
\]

where \( H_B \) is given in Eq. (2.1).

Now let us define the energy dependent Green’s function of spin-1/2 system in terms of each homotopy classes:

\[
\hat{G}^F[\vec{x}_b, \vec{x}_a; E] = \sum_{m=-\infty}^{\infty} e^{i m (\theta_b - \theta_a)} \hat{G}_m^F[\vec{x}_b, \vec{x}_a; E]. \tag{4.5}
\]

If one uses the asymptotic formula of the modified Bessel function \( I_\nu(z) \propto z^\nu \) in Eq. (2.23), one can realize easily that the application of the self-adjoint extension method must be restricted to the domain \( |m + \alpha| < 1/2 \) because of the normalizability condition. This is also conjectured in Ref. [25]. We, therefore, confine our attention to this domain for the time being.

Upon combining Eqs. (3.12), (4.1) and (4.5) one can get the following relation:

\[
\hat{A}_m[x_b, x_a; E] = f_m(\epsilon_1, \epsilon_2; E) \frac{W_{\frac{\xi}{2\Omega} |m+\alpha|}(4M\Omega x_a)}{\sqrt{x_a}} \frac{W_{\frac{\xi}{2\Omega} |m+\alpha|}(4M\Omega x_b)}{\sqrt{x_b}} \tag{4.6}
\]
where
\[ \hat{A}_m[x_b, x_a; E] \equiv \hat{G}_m^R[x_b, x_a; E] - \hat{G}_m^B[x_b, x_a; E] \] (4.7)
and
\[ f_m(\epsilon_1, \epsilon_2; E) = -\frac{1}{\epsilon_2 + \hat{G}_m^B[\epsilon_2, \epsilon_1; E]} \frac{1}{\sqrt{\epsilon_1 \epsilon_2 (4\pi \Omega)^2}} \left( \frac{\Gamma \left( \frac{1+2|m+\alpha|+\xi}{2} \right)}{\Gamma(1+2|m+\alpha|)} \right)^2 \] (4.8)
\[ \times M_{-\frac{\xi}{2\Omega}|m+\alpha|}(4M\Omega \epsilon_1)M_{-\frac{\xi}{2\Omega}|m+\alpha|}(4M\Omega \epsilon_2). \]

In order to apply the boundary condition at the origin deduced from the self-adjoint extension method, it is more convenient to express \( \hat{A}_m[x_b, x_a; E] \) in terms of hypergeometric function:
\[ \hat{A}_m[x_b, x_a; E] = f_m(\epsilon_1, \epsilon_2; E) g_m(x_a) g_m(x_b) \] (4.9)
where
\[ g_m(r) = (4M\Omega)^{\frac{1}{2}+|m+\alpha|} e^{-2M\Omega r|m+\alpha|} \] (4.10)
\[ \times U \left( \frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega}, 1 + 2|m + \alpha|; 4M\Omega r \right) \]
and
\[ U(a, b, z) = \frac{\pi}{\sin \pi b} \left[ \frac{F(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-a} F(1+a-b, 2-b, z) \Gamma(a)\Gamma(2-b) \right]. \] (4.11)

Now let us apply the boundary condition which is obtained from the self-adjoint extension method
\[ \lim_{x_a \to 0} x_a^{m+\alpha} \hat{A}_m[x_b, x_a; E] \] (4.12)
\[ = \lambda_m \lim_{x_a \to 0} \left( \frac{1}{x_a^{m+\alpha}} \left[ \hat{A}_m[x_b, x_a; E] - \hat{A}_m[x_b, x_a'; E] \right] \right) \]
where \( \lambda_m \) is a self-adjoint extension parameter. The boundary condition (4.12) is derived by one of us in Ref.[25].

After using the asymptotic formula of \( U(a, b, z) \) and performing a tedious calculation, one can show that the boundary condition (4.12) yields the following relation
\[
\frac{\Gamma\left(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega}\right)}{\Gamma\left(\frac{1}{2} - |m + \alpha| + \frac{\xi}{2\Omega}\right)} = \frac{1}{\lambda_m (4M\Omega)^{2|m+\alpha|}} \frac{\Gamma(2 |m + \alpha|)}{\Gamma(-2 |m + \alpha|)}. \tag{4.13}
\]

Hence, by solving Eq.(4.13) one can obtain the bound-state energy. Although Eq.(4.13) is too complicated to evaluate the bound-state energy explicitly, \(\lambda_m \to 0\) and \(\infty\) limiting features are interesting. In these cases the bound-state spectra are explicitly determined as poles of the gamma function, i.e.,

\[
E_{n,m} = \frac{1}{2} \frac{M \xi^2}{(n - \frac{1}{2} + |m + \alpha|)^2} \quad (\lambda_m = 0) \tag{4.14}
\]

\[
E_{n,m} = \frac{1}{2} \frac{M \xi^2}{(n - \frac{1}{2} - |m + \alpha|)^2} \quad (\lambda_m = \infty). \quad n = 1, 2, \ldots
\]

These bound-state energies coincide exactly with those of regular and singular solutions given in Ref.[12]. The absence of minus sign is due to the euclidean characteristic.

Now the relation between the self-adjoint extension parameter \(\lambda_m\) and the bare coupling constant \(v\) can be explored. This relation is easily obtained through the comparision of Eq.(4.13) with the poles of \(\hat{A}_m[x_b, x_a; E]\)

\[
\frac{1}{v} + \hat{G}_m^B[\epsilon_2, \epsilon_1; E] = 0. \tag{4.15}
\]

Counting on the asymptotic formula of \(U(a, b, z)\) one can show that the poles of \(\hat{A}_m[x_b, x_a; E]\) arise when the following relation is satisfied:

\[
\frac{\Gamma\left(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega}\right)}{\Gamma\left(\frac{1}{2} - |m + \alpha| + \frac{\xi}{2\Omega}\right)} = -\frac{1}{(4M\Omega)^{2|m+\alpha|}} \frac{\Gamma(2 |m + \alpha|)}{\Gamma(-2 |m + \alpha|)} \tag{4.16}
\]

\[
\times \frac{1}{(\epsilon_1 \epsilon_2)^{|m+\alpha|}} \left[ \frac{2\pi |m + \alpha|}{\lambda_m} + \left( \frac{\epsilon_1}{\epsilon_2} \right)^{|m+\alpha|} \right].
\]

Comparation of Eq.(4.16) with Eq.(4.13) enables one to get the relation between the self-adjoint extension parameter and the coupling constant

\[
\frac{1}{v} = -\frac{M}{2\pi |m + \alpha|} \left[ (\epsilon_1 \epsilon_2)^{|m+\alpha|} + \left( \frac{\epsilon_1}{\epsilon_2} \right)^{|m+\alpha|} \right]. \tag{4.17}
\]

This relation makes the denominator of Eq.(4.8) to be

\[
\frac{1}{v} + \hat{G}_m^B[\epsilon_2, \epsilon_1; E] = -\frac{M}{2\pi |m + \alpha|} (\epsilon_1 \epsilon_2)^{|m+\alpha|} \tag{4.18}
\]

\[
\times \left[ \frac{1}{\lambda_m} - (4M\Omega)^{2|m+\alpha|} \frac{\Gamma(-2 |m + \alpha|) \Gamma\left(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega}\right)}{\Gamma(2 |m + \alpha|) \Gamma\left(\frac{1}{2} - |m + \alpha| + \frac{\xi}{2\Omega}\right)} \right].
\]

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By inserting Eq.(4.18) into (4.8) and taking a limit $\epsilon_2 \to \epsilon_1^+$ and $\epsilon_1 \to 0$, $f_m(\epsilon_1, \epsilon_2; E)$ becomes the $\epsilon_1$- and $\epsilon_2$-independent finite quantity

$$
\lim_{\epsilon_1, \epsilon_2 \to 0} f_m(\epsilon_1, \epsilon_2; E) = \frac{2\pi |m + \alpha| (4M\Omega)^{1+2|m+\alpha|}}{M (4\pi\Omega)^2} \left( \frac{\Gamma(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega})}{\Gamma(1 + 2 |m + \alpha|)} \right)^2
\times \left[ \frac{1}{\lambda_m} - (4M\Omega)^{2|m+\alpha|} \frac{\Gamma(-2 |m + \alpha|)\Gamma(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega})}{\Gamma(2 |m + \alpha|)\Gamma(\frac{1}{2} - |m + \alpha| + \frac{\xi}{2\Omega})} \right]^{-1}.
$$

Substituting Eq.(4.19) into Eq.(4.6) directly $\hat{A}_m[x_b, x_a; E]$ is easily obtained. Hence, the energy-dependent Green’s function for the spin-1/2 system is

$$
G^F[\vec{x}_b, \vec{x}_a; E] = \sum_{|m+\alpha|>1/2} e^{im(\theta_b-\theta_a)} G_m^B[x_b, x_a; E] + \sum_{|m+\alpha|<1/2} e^{im(\theta_b-\theta_a)} \left[ G_m^B[x_b, x_a; E] + \frac{2\pi |m + \alpha| (4M\Omega)^{1+2|m+\alpha|}}{M (4\pi\Omega)^2} \left( \frac{\Gamma(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega})}{\Gamma(1 + 2 |m + \alpha|)} \right)^2
\times \left[ \frac{1}{\lambda_m} - (4M\Omega)^{2|m+\alpha|} \frac{\Gamma(-2 |m + \alpha|)\Gamma(\frac{1}{2} + |m + \alpha| + \frac{\xi}{2\Omega})}{\Gamma(2 |m + \alpha|)\Gamma(\frac{1}{2} - |m + \alpha| + \frac{\xi}{2\Omega})} \right]^{-1}
\times \frac{W - \frac{\xi}{2\pi|m+\alpha|} (4M\Omega x_a) W - \frac{\xi}{2\pi|m+\alpha|} (4M\Omega x_b)}{\sqrt{x_a} \sqrt{x_b}} \right].
$$

Of course we can obtain the fixed-energy amplitude from Eq.(4.20) by using the relation (3.11).
V. CONCLUSION

We derived the propagators of the spinless and spin-1/2 ABC systems explicitly. For the derivation of propagator in the spinless ABC system we applied the Duru-Kleinert method which was used firstly to evaluate the propagator for the case of hydrogen atom. It is found that the spinless ABC system with mass $M$ and flux parameter $\alpha$ is reduced to the AB plus harmonic oscillator system with mass $4M$, flux parameter $2\alpha$ and angular frequency $\sqrt{-E/2M}$ through the Levi-Cività transformation that is a two-dimensional counterpart of the KS transformation. The fact that the Levi-Cività transformation is mapping from flat space to flat space unlike the KS transformation makes the derivation of propagator in the spinless ABC system extremely simple. The final form of the propagator is expressed as an winding number representation and the bound-state spectrum deduced from it is in agreement with that corresponding to the regular solution at the origin suggested in the previous article.

With the propagator of the spinless ABC system we analyzed also the spin-1/2 ABC system. For the derivation of the propagator in the spin-1/2 ABC system the following two different approaches were used to get the bound-state spectrum:

1. by checking the poles of the energy-dependent Green’s function.
2. by requiring the boundary condition derived from the self-adjoint extension to the energy-dependent Green’s function.

Identification of these two different spectra naturally leads one to the relation between the self-adjoint extension parameter and the coupling constant. Substitution of this relation into the denominator of the energy-dependent Green’s function enabled us to obtain the finite propagator of the spin-1/2 ABC system.

Since AB interaction is acted on charge and magnetic flux tube, interaction between anyons requires a Coulomb modification. Although the expression of the two-dimensional Coulomb term may be disputable, reflecting from the fact that real world is three-dimensional one and two dimension is embedded in it, the propagator obtained in the present paper might be used for the study of the time-dependent scattering or statistical properties of anyon system.
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