Darboux integrable discrete equations possessing an autonomous first-order integral

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Abstract
Darboux integrable difference equations on the quad-graph are completely described in the case of the equations that possess autonomous first-order integrals in one of the characteristics. A generalization of the discrete Liouville equation is obtained from a subclass of this equation via a non-point transformation. The detailed proof of the general proposition on the symmetry structure for the quad-graph equations is given as an auxiliary result.

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1. Introduction and basic definitions

Let us consider difference equations of the form

$$u_{(i+1,j+1)} = F(u_{(i,j)}, u_{(i+1,j)}, u_{(i,j+1)}),$$

where $u$ is a function of two integers, the lower multi-index denotes values of the arguments for this function, and the equation holds true for $(i, j) \in \mathbb{Z}^2$. The function $F$ is assumed here to be single-valued. Integrable (in various senses) equations of such form are actively studied in recent years (e.g. see [1–5] and references within). The present work is devoted to the equations for which there exist functions $I$ and $J$ remaining unchanged after the shifts in $j$ and $i$, respectively, i.e. such that the relations

$$I(u_{(i,j)}, u_{(i+1,j)}, \ldots, u_{(i+k,j)}) = I(u_{(i,j+1)}, u_{(i+1,j+1)}, \ldots, u_{(i+k,j+1)}),$$

$$J(u_{(i,j)}, u_{(i,j+1)}, \ldots, u_{(i,j+l)}) = J(u_{(i+1,j)}, u_{(i+1,j+1)}, \ldots, u_{(i+1,j+l)})$$

1 www.researcherid.com/rid/D-1158-2009
hold true for any solution \( u_{(i,j)} \) of the equation. Such equations are called Darboux integrable and can be considered as a special case of C-integrable ones (in accordance with the term of [6]). The simplest example of a Darboux integrable equation is

\[
u_{(i+1,j+1)} = u_{(i+1,j)} + u_{(i,j+1)} - u_{(i,j)}. \tag{4}
\]

Here \( I = u_{(i+1,j)} - u_{(i,j)} \), \( J = u_{(i,j+1)} - u_{(i,j)} \). This equation is a difference analogue of the partial differential wave equation \( u_{xy} = 0 \). Another example is the equation

\[
u_{(i+1,j+1)} = \frac{(u_{(i+1,j)} - 1)(u_{(i,j+1)} - 1)}{u_{(i,j)}} \tag{5}
\]

from [7]. It is similar to the well-known Liouville equation \( u_{xy} = e^u \) in properties and, according to [8], the corresponding functions \( I \) and \( J \) are defined by the formulas

\[
I = \left( \frac{u_{(i+2,j)}}{u_{(i+1,j)}} - 1 \right) + \left( \frac{u_{(i,j)} - 1}{u_{(i+1,j)}} + 1 \right),
\]

\[
J = \left( \frac{u_{(i,j+2)}}{u_{(i,j+1)}} - 1 \right) + \left( \frac{u_{(i,j)} - 1}{u_{(i,j+1)}} + 1 \right).
\]

In general, the equations (1) can be regarded as difference analogues of the partial differential equations

\[
u_{xy} = F(u, u_x, u_y). \tag{6}
\]

The concept of the Darboux integrability was initially introduced for partial differential equations back in the 19th century. Searching for Darboux integrable equations of the form (6) was started in classical works such as [9], and the most recent and complete classification result was obtained in [10] more than a century later. At present, a classification is absent for Darboux integrable equations of the form (6) was started in classical works such as [9], and the most recent and complete classification result was obtained in [10] more than a century later. At present, a classification is absent for Darboux integrable equations of the form (6) was started in classical works such as [9], and the most recent and complete classification result was obtained in [10] more than a century later. At present, a classification is absent for Darboux integrable equations (1) and only separate examples of such equations are known (see, for instance, [11–13]). This is why a classification problem for a subclass of the equations (1) looks reasonable and may be a natural part of the future complete classification.

In the present paper we consider the case when \( l = 1 \) in (3) (and there are no restrictions on \( k \) in (2)). In this case, the defining relation for \( J \) takes the form

\[
J(u_{(i,j)}, u_{(i,j+1)}) = J(u_{(i+1,j)}, F(u_{(i,j)}, u_{(i+1,j)}, u_{(i,j+1)})) \tag{7}
\]

and implicitly (maybe not uniquely) defines the right-hand side \( F \) of (1) if \( J \) is known. The main result of the paper can be formulated in the following way. If equation (1) is Darboux integrable and \( F \) satisfies (7) so that (1) is Darboux integrable at least in a local sense (i.e. (2) holds true for some function \( I \) and any solution with the property \( (u_{(i+s,j)}, u_{(i+s,j+1)})) \in U \), \( s = 0, k \), and the set of such solutions is not empty). Although it is often not obvious whether a given \( J \) satisfies (8), this result allows to obtain Darboux integrable equations by choosing the functions \( \xi, \alpha, \beta \) and \( \gamma \). For example, the choice \( \xi(z) = e^z, \alpha(J) = J^{-1}, \beta(J) = J^{-2}, \gamma(J) = -J^{-1} \) generates equation (4). Other examples of such kind and the main result are contained in section 3.

To prove this main result, we need some auxiliary statements. One of them, the proposition on splitting symmetries of equation (1) into summands involving either shifts only in \( i \) or shifts only in \( j \), is useful not only in the context of the present paper and known to specialists but,
to the best author’s knowledge, has no published detailed proof for the general form of this statement. This proof is given in section 2.1.

Let us introduce designations, formulate assumptions and reformulate the Darboux integrability definition in accordance with these designations.

From now on, we will use the notation \( u_{m,n} := u_{(i+m,j+n)} \), \( u := u_{(i,j)} \) to omit \( i \) and \( j \) for brevity. According this notation, equation (1) reads

\[
u_{1,1} = F(u, u_{1,0}, u_{0,1}). \tag{9}\]

We assume that

\[
\frac{\partial F}{\partial u} \neq 0, \quad \frac{\partial F}{\partial u_{1,0}} \neq 0, \quad \frac{\partial F}{\partial u_{1,0}} \neq 0. \tag{10}\]

These conditions allow us to express any argument of the function \( F \) in terms of the others for rewriting (9), after appropriate shifts in \( i \) and \( j \), in any of the following forms

\[
u_{-1,-1} = F(u, u_{-1,0}, u_{0,-1}), \tag{11}\]
\[
u_{1,-1} = \tilde{F}(u, u_{1,0}, u_{0,-1}), \tag{12}\]
\[
u_{-1,1} = \tilde{F}(u, u_{-1,0}, u_{0,1}). \tag{13}\]

Because we use the local implicit function theorem to obtain (11)–(13), our considerations are mostly local (i.e. valid in an enough small neighborhood of any arbitrary selected solution of (9), which is considered in a finite number of the points \((i, j)\)). In general, the right-hand sides of (11)–(13) are not uniquely defined (may vary with \( i, j \) and with a solution in a neighborhood of which we consider these functions). We assume only that \( \tilde{F} \) is uniquely defined (i.e. has only one branch with values in the image of \( F \)). This assumption is used in a small part of the reasonings and we indicate the places of the paper where we apply the single-valuedness of \( \tilde{F} \) (thus, all propositions without such indication remain valid when \( \tilde{F} \) is multi-valued). In fact, we do not need equations (11) and (13) to obtain the above mentioned main result. But we use them to prove some auxiliary statements in the form that is more general than really needed for the purpose of the present paper. In these proofs we need only to know what variables do expressions with \( F \) and \( \tilde{F} \) depend on. And this does not change when \( F \) and \( \tilde{F} \) are varied. Therefore, it is no matter that \( F \) and \( \tilde{F} \) may depend on \( i \) and \( j \).

Using (9)–(13) and their consequences derived by shifts in \( i \) and \( j \), we can express any ‘mixed shift’ \( u_{m,n}, mn \neq 0 \), in terms of \( u_{(i,j)}, u_{(i+1,j+2)}, k, l \in \mathbb{Z} \), which are called dynamical variables. (A more detailed explanation of the dynamical variables and the recursive procedure of the mixed shift elimination can be found, for example, in [4].)

Values of the dynamical variables for fixed \( i \) and \( j \) serve as boundary conditions of the Goursat problem for the equation (9) and can be selected in an arbitrary way. Therefore, we treat the dynamical variables as functionally independent in the relationships that are valid for any solution of the equation (9), i.e. in all relationships stated below. The notation \( g[u] \) means that the function \( g \) depends on a finite number of the dynamical variables. In addition, \( f[i, j, u] \) designates that the function \( f \) may explicitly depend on \( i, j \) and a finite set of the dynamical variables which is the same for all \( i \) and \( j \). All functions are assumed to be analytical.

Now let \( T_i \) and \( T_j \) denote the operators of the forward shifts in \( i \) and \( j \) by virtue of the equation (9). The inverse (backward) shift operators are denoted by \( T_i^{-1} \) and \( T_j^{-1} \). We use a shift operator with a superscript \( k \) to designate the \( k \)-fold application of this operator (e.g. \( T_j^3 := T_j \circ T_j \circ T_j \), \( T_j^{-2} := T_j^{-1} \circ T_j^{-1} \) and \( T_j^1 := T_j \)). For a more compact notation we also set
any operator with the zero superscript equal to the operator of multiplication by unit (i.e. the identity mapping). In these notations the shift operators are defined by the following rules:

\[ T_i^0(f(i, j, a, b, \ldots)) = f(i + k, j, T_i(a), T_i(b), \ldots) \quad \Rightarrow \quad T_i^0(u_{m,0}) = u_{m+k,0}, \]

\[ T_j^0(f(i, j, a, b, \ldots)) = f(i, j + k, T_j(a), T_j(b), \ldots) \quad \Rightarrow \quad T_j^0(u_{0,m}) = u_{0,m+k}, \]

for any function \( f \) and any integers \( k \) and \( m \); for any \( n \in \mathbb{N} \) the relationships

\[ T_i(u_{0,n}) = T_i^{-1}(F), \quad \quad T_i(u_{0,-n}) = T_i^{1-n}(	ilde{F}), \]

\[ T_j(u_{n,0}) = T_j^{-1}(F), \quad \quad T_j(u_{-n,0}) = T_j^{1-n}(	ilde{F}), \]

\[ T_i^{-1}(u_{0,n}) = T_i^{n-1}(F), \quad \quad T_i^{-1}(u_{0,-n}) = T_i^{1-n}(\tilde{F}), \]

\[ T_j^{-1}(u_{n,0}) = T_j^{n-1}(F), \quad \quad T_j^{-1}(u_{-n,0}) = T_j^{1-n}(\tilde{F}) \]

hold true (i.e. mixed variables \( u_{\pm n, \pm 1}, u_{\pm n, -1} \) and \( u_{\pm n, -1} \) are expressed in terms of the dynamical variables by virtue of (9)–(13)). As mentioned above, \( F, \tilde{F} \) and the expressions for \( u_{-n, \pm 1}, u_{-1, \pm n} \) may explicitly depend on \( i \) and \( j \) but we omit these dependences for brevity. Generally speaking, the shift operators are well-defined only locally because both \( F \) and \( \tilde{F} \) may be different for different solutions of (9). This is why we add the phrase ‘for any solution’ in definitions 1 and 2 to emphasize that defining relations must be valid for any values of the dynamical variables and any choices of \( F \).

**Definition 1.** An equation of the form (9) is called Darboux integrable if there exist functions \( I[u] \) and \( J[u] \) such that the relations \( T_i(I) = I \) and \( T_j(J) = J \) hold true for any solution of (9) and each of the functions essentially depends on at least one of the dynamical variables. In this case, the functions \( I[u] \) and \( J[u] \) are respectively called an i-integral and a j-integral of the equation (9).

It is easy to check (see lemma 1 below) that i-integrals cannot depend on the dynamical variables of the form \( u_{0,p} \) and j-integrals—on the dynamical variables of the form \( u_{q,0} \). Thus, i- and j-integrals have the form \( I(u_{m,0}, u_{m+1,0}, \ldots, u_{k,0}) \) and \( J(u_{0,n}, u_{0,n+1}, \ldots, u_{0,1}) \), respectively. The numbers \( k - m \) and \( l - n \) are called order of the corresponding integral. We can set \( n = m = 0 \) without loss of generality because \( T_i^{-n} \) and \( T_j^{-m} \) respectively map any i- and j-integrals into i- and j-integrals again. Thus, equations (11)–(13) are in fact not needed for the above definition. In addition, this also means that an equation of the form (9) possesses a first-order j-integral and satisfies conditions (10) if and only if it can be defined by a relationship of the form

\[ T_j(\phi(u, u_{0,1})) = \phi(u, u_{0,1}), \quad \frac{\partial \phi(u, u_{0,1})}{\partial u} \frac{\partial \phi(u, u_{0,1})}{\partial u_{0,1}} \neq 0. \quad (14) \]

This relationship is the main subject of the present paper. The first equation of (14) coincides, up to notation, with (7) and serves as a defining relation for the single-valued right-hand side of (9). Therefore, it differs from the equation \( \phi(u_{1,0}, u_{1,1}) = \phi(u, u_{0,1}) \) that may generate multi-valued forward evolution (difficulties of equations with such evolution are discussed, for example, in [14]).

It should be noted that the present paper (in contrast to, for instance, [13, 15]) deals only with the autonomous integrals, i.e. with the integrals that do not depend explicitly on the discrete variables \( i \) and \( j \). As demonstrated in [13], equations (9) may have non-autonomous integrals despite the lack of an explicit dependence on \( i \) and \( j \) in \( F \). For example, if \( J = \phi(u, u_{0,1}) - iC \), then the defining relation \( T_i(J) = J \) remains autonomous and takes the
form. The last equation implies (14) with $\phi = \sin(2\pi \psi/C)$ and other examples of non-autonomous first-order integrals in [13] can also be reduced to (14). But it is not clear whether this is true for any equation (9) admitting a first-order non-autonomous $j$-integral. Therefore, the main result of the present paper does not pretend to give an exhaustive description of all Darboux integrable equations (9) possessing first-order $j$-integrals.

A part of equations (14) can be rewritten in the form $T_j(a(u, u_{1,0})) = b(u, u_{1,0})$, where the functions $a$ and $b$ are functionally independent, and hence admits the non-point invertible transformation $v = a(u, u_{1,0})$ (see [11] for more details). In section 3 we show that such equations exist among Darboux integrable equations (14) too and the transformation $v = a(u, u_{1,0})$ maps them into a family of Darboux integrable equations, which can be considered as a generalization of the equation (5). It can be proved that this family contains all Darboux integrable equations of the form $T_j(\Omega(u, u_{0,1})) = \Psi(u, u_{0,1})$ possessing second-order autonomous $j$-integrals but this is beyond the scope of the present paper.

2. Auxiliary statements

Before going on, we need to introduce the operator

$$L = T_i T_j - \frac{\partial F}{\partial u_{1,0}} - \frac{\partial F}{\partial u_{0,1}} = \frac{\partial F}{\partial u}$$

and to define a new term.

**Definition 2.** An equation $u_t = f[i, j, u]$ is called a symmetry of equation (9) if the relation $L(f) = 0$ holds true for any solution of (9) and any $i, j$.

It should be noted that the general definition of symmetries is given to formulate theorem 1 below in the most general form. Outside of section 2.1 we do not need non-autonomous symmetries and consider symmetries of the form $u_t = f[u]$ only.

The sketch of further reasonings can be summarized as follows. According to [8], if an equation of the form (9) is Darboux integrable and possesses a $j$-integral $\phi$, then there exists an operator $R = \sum_{q=0}^{r} \lambda_q[u]T^q_j, \lambda_r \neq 0$, such that

$$u_t = R(\xi(T^p_j(\phi), T^p_j(\phi), T^q_j(\phi), \ldots))$$

is a symmetry of this equation for any integer $p$ and any function $\xi$ depending on a finite number of the arguments. The single-valuedness of $F$ was, in fact, assumed in [8] to construct the operator $R$ (the case of multi-valued $F$ needs more carefully checking or using an alternative approach like that was applied in [16]). In section 2.2 we show that any symmetry of the equation (9) is mapped into an equation of the form $v_t = g[v]$ by the substitution $v = \phi[u]$ if $\phi[u]$ is the integral of smallest order. On the other hand, the work [17] (which is partially reproduced in section 2.3 for the reader convenience) completely describes the substitutions of the form $v = \phi(u, u_{0,1})$ for the equations (16) with the right-hand side independent of variables $u_{l,0}$, $l \in \mathbb{Z} \setminus \{0\}$. This gives us a necessary condition of Darboux integrability for the equations (14) if we show that their symmetries (16) do not depend on variables of the form $u_{l,0}$. The latter is done in section 2.1 by proving a general proposition on symmetry structure.

2.1. Symmetry structure

**Theorem 1.** Any symmetry $u_t = f[i, j, u]$ of equation (9) has the form

$$u_t = \hat{f}(i, j, u_{m,0}, u_{m+1,0}, u_{m+2,0}, \ldots) + \hat{f}(i, j, u_{0,n}, u_{n+1,0}, u_{n+2,0}, \ldots)$$

This theorem is due to Y. A. Startsev and is reproduced in section 2.3 for the reader convenience.
i.e. \( f[i, j, u] \) is the sum of two terms such that the first one does not depend on \( u_{0,k} \) and the second one does not depend on \( u_{k,0} \) for any non-zero \( k \in \mathbb{Z} \).

Let us recall that, according to the previously introduced notation, the set of possible arguments of \( f[i, j, u] \) is finite and the same for all \( i, j \). The above theorem is not true without this limitation (we can use the relation \( L(f) = 0 \) as a recurrent formula with selecting ‘initial values’ of \( f \) on the lines \( i = 0 \) and \( j = 0 \) in an arbitrary way if this limitation is excluded).

The statement of the theorem is not new. Specialists in symmetries of discrete equations believe it is true because the analogous proposition for the equations (6) is well-known and proved in [18] (note parenthetically that the same was done for semi-discrete equations in [19]). But the proof of theorem 1 is available, for example, in [20] for only a simple special case of symmetries depending on five dynamical variables and, to the author’s best knowledge, has been absent for symmetries of the general form and an arbitrary high order. The general form of the above theorem is also contained in [21] but the proof is, in fact, omitted in this work. It is convenient to give the proof by using the following simple proposition.

**Lemma 1.** Let \( A[i, j, u], B[i, j, u] \) and \( C[i, j, u] \) do not depend on \( u_{0,k} \) for any non-zero integer \( k \neq 1 \) and \( AB \neq 0 \) for any \( i \) and \( j \). Then a function \( g[i, j, u] \) does not depend on \( u_{0,k} \) for any non-zero \( k \in \mathbb{Z} \) if \( g \) satisfies the relationship

\[
AT_0(g) + Bg = C
\]

for all \( j \) and \( j \).

From now on, we will use the notation

\[
f'_p := \frac{\partial f}{\partial u_{p,0}}, \quad f'_q := \frac{\partial f}{\partial u_{0,q}}, \quad f''_{p,q} := \frac{\partial^2 f}{\partial u_{p,q} \partial u_{0,0}}
\]

to denote the partial derivatives of a function \( f[i, j, u] \) in in-line formulas.

**Proof.** Assume the contrary. Let \( l \) and \( s \) be respectively the largest positive and the smallest negative integers for which \( g[i, j, u] \) depends on \( u_{0,l} \) and \( u_{0,s} \). (Naturally, a function \( h[i, j, u] \) is considered as dependent on a variable \( z \) if \( h_z \neq 0 \) in at least one point \((i, j, u)\).) Differentiation of (18) with respect to \( u_{0,l+1} \) and \( u_{0,s} \) gives rise to \( T_j(g'_j) = 0 \) and \( g''_s = 0 \), respectively. Thus, we arrive to a contradiction that proves the lemma. \( \square \)

**Proof of theorem 1.** It is obvious that the symmetry \( u_t = f[i, j, u] \) has the form (17) if and only if \( f''_{p,q} = 0 \) for all non-zero integers \( p \) and \( q \). Assume the contrary. Let \( l \) and \( s \) be respectively the largest and the smallest non-zero integers for which there exist non-zero integers \( \delta \) and \( \sigma \) such that \( f''_{\delta \sigma} \neq 0 \) and \( f''_{\sigma 0} \neq 0 \). Then \( f = f[i, j, u] + \ddot{f}[i, j, u] \), where \( f''_s = 0 \) for all non-zero integers \( r \) and \( \ddot{f}_k = 0 \) for all non-zero integers \( k \neq [s, l] \).

If \( l > 0 \), then the differentiation of the relationship \( L(\ddot{f} + \ddot{f}) = 0 \) with respect to \( u_{l+1,0} \) gives us

\[
\frac{\partial T^l_{\delta} (\ddot{F})}{\partial u_{l+1,0}} T_j(g) - \frac{\partial F}{\partial u_{l,0}} g + \frac{\partial L(\ddot{f})}{\partial u_{l+1,0}} = 0,
\]

where \( g = T_j(\ddot{f}_j) \). According to lemma 1, the functions \( g \) and \( \ddot{f}_k = T^{-1}_k(g) \) do not depend on \( u_{0,q} \) for any non-zero integer \( q \). And this contradicts the assumption \( \ddot{f}'_{\delta \sigma} \neq 0 \).

If \( l < 0 \), then \( s < 0 \) too. Differentiating \( L(\ddot{f} + \ddot{f}) = 0 \) with respect to \( u_{s,0} \), we obtain

\[
\frac{\partial F}{\partial u_{s,0}} \frac{\partial T^l_{\sigma} (\ddot{F})}{\partial u_{s,0}} T_j(g) + \frac{\partial F}{\partial u_{s,0}} g - \frac{\partial L(\ddot{f})}{\partial u_{s,0}} = 0,
\]

where \( g = \ddot{f}_k \) and \( \ddot{F} \) is the right-hand side of (13). Hence, the assumption \( \ddot{f}'_{\sigma 0} \neq 0 \) contradicts lemma 1. \( \square \)
Corollary 1. Let an equation of the form (9) possess a j-integral $\phi$ and there exists an operator $R = \sum_{k=0}^{r} \lambda_k [u] T_j^k$ such that (16) is a symmetry of this equation for any integer $p$ and any function $\xi$ depending on a finite number of the arguments. Then the coefficients $\lambda_k$ of the operator $R$ do not depend on $u_{q,0}$ for any non-zero integer $q$.

Proof. Let $n$ be the largest integer for which $\phi_{n} \neq 0$. Then we set $\xi = T_j^p(\phi)$, where $p$ is selected so that all $\lambda_k$ do not depend on $u_{q,0}$ for any $m \geq n + p$.

Now assume the contrary again. Let $l$ be the largest number for which there exists a non-zero integer $s$ such that $\lambda_s$ depend on $u_{s,0}$. Then

$$\frac{\partial R(T_j^p(\phi))}{\partial u_{l,0}} = \sum_{k=0}^{l} \frac{\partial \lambda_k}{\partial u_{k,0}} T_j^{p+k}(\phi), \quad \frac{\partial^2 R(T_j^p(\phi))}{\partial u_{l,0} \partial u_{n,p+l}} = \frac{\partial \lambda_k}{\partial u_{k,0}} \frac{\partial T_j^{p+l}(\phi)}{\partial u_{n,p+l}} \neq 0.$$ 

The last inequality contradicts theorem 1.

2.2. Integrals as substitutions

Let us consider the chain of differential–difference equations

$$u_t = g(u_{0,k}, u_{0,k+1}, \ldots, u_{0,n}). \quad (19)$$

It should be noted that any equation of the form $u_t = f[u]$ generates the differentiation $\partial_t$ with respect to $t$ by virtue of this equation. On the functions of the dynamical variables, the differentiation $\partial_j$ is defined by the formula $\partial_j(h[u]) = h_s$, where

$$h_s = \sum_{q=-\infty}^{\infty} \frac{\partial h}{\partial u_{q,0}} T_j^q + \sum_{q=0}^{\infty} \frac{\partial h}{\partial u_{0,q}} T_j^q,$$

i.e. $h_s$ is the linearization operator (Frechét derivative) of $h$.

Definition 3. We say that equation (19) admits a difference substitution

$$v = \phi(u_{0,1}, u_{0,2}, \ldots, u_{0,n}) \quad (20)$$

into an equation of the form $v_t = 2g(u_{0,k}, v_{0,k+1}, \ldots, v_{0,n})$ if the function $\phi$ depends on at least two dynamical variables and the relation

$$\partial_s(\phi) = g(T_j^k(\phi), T_j^{k+1}(\phi), \ldots, T_j^n(\phi)) \quad (21)$$

holds true (i.e. $v$ is a solution of the equation $v_t = 2g$ for any solution of (19)).

We call (20) a Miura-type substitution if there exist operators

$$R = \sum_{q=0}^{r} \lambda_q (u_{0,q}, u_{0,q+1}, \ldots, u_{0,t}) T_j^q, \quad \lambda_r \neq 0, \quad (22)$$

$$\hat{R} = \sum_{q=0}^{r+m} \lambda_q (v_{0,q}, v_{0,q+1}, \ldots, v_{0,t}) T_j^q$$

such that the equation $u_t = R(\xi(T_j^p(\phi), T_j^{p+1}(\phi), \ldots))$ admits the substitution (20) into the equation $v_t = \hat{R}(\xi(v_{0,p}, v_{0,p+1}, \ldots))$ for any integer $p$ and any function $\xi$ depending on a finite number of the arguments.

It is easy to see that the above definition in no way uses equation (9) because the shift operator $T_j$ is applied here to the functions depending only on the variables of the form $u_{0,p}$. However, the integrals of equations (9) can be interpreted as substitutions for equations (19).

To show this, we use the two lemmas below, which allow us to transfer appropriate reasoning for equations (6) from the work [10] to the case of difference equations with almost no changes.
Lemma 2. Let \( \phi[u] \) be a \( j \)-integral of the smallest order for an equation of the form (9). Then for any \( j \)-integral \( J[u] \) of this equation there exists a function \( \xi \) such that
\[
J[u] = \xi(T^p_j(\phi), T_j^{p+1}(\phi), T_j^{p+2}(\phi), \ldots, T^q_j(\phi))
\]
for some integers \( p \) and \( q \).

The proof of lemma 2 is omitted because it coincides with the proof of the analogous proposition for semi-discrete equations in [15] (see theorem 3.2 within). It is obvious that the converse is also true: the right-hand side of (23) is a \( j \)-integral for any \( \xi, p \) and \( q \) because \( T_j^s \) commutes with \( T_i \) for any \( s \in \mathbb{Z} \) and therefore maps \( j \)-integrals into \( j \)-integrals again.

Lemma 3. If the right-hand side \( \hat{F} \) of (12) is single-valued, then for any function \( h[u] \) there exists an operator \( H = \sum_{r \in \mathbb{Z}} h_r(u)T_j^r \) such that
\[
(T_i(h))_s = T_i \circ h_s = H \circ L,
\]
where \( \circ \) denotes the composition of operators and \( L \) is defined by (15).

It is easy to check in the proof below that (24) holds true for multi-valued \( \hat{F} \) too, but the coefficients \( h_r \) for negative \( s \) may explicitly depend on \( i, j \) and may be different for different solutions of (9) in this case. The single-valuedness \( \hat{F} \) can also be excluded from the lemma if \( h_s \neq 0 \) for all \( s < 0 \).

In particular, lemma 3 implies that the differentiation \( \partial_i \) commutes with \( T_i \) if \( u_i = f[u] \) is a symmetry of equation (9). This corollary remains valid in the case of multi-valued \( \hat{F} \) too.

Proof. It is easy to check that \( T_j \circ \eta[u]_s = T_j(\eta[u])_s = T_j(\eta'_l)_L \) if \( \eta'_l \neq 0 \) for all non-zero integers \( k \neq 1 \). This implies
\[
T^n_j(\hat{F})_s = T^n_j \circ \hat{F}_s = \sum_{r=1}^{n} T_j^r \left( \frac{\partial T^{n-r}_j(\hat{F})}{\partial u_{1,0}} \right) T^{r-1}_j \circ L = \forall n > 0.
\]

Differentiating the consequence \( T^{-1}_j(F) = F(u_{0,-1}, \hat{F}, u) = u_{1,0} \) of (9) with respect to \( u, u_{0,-1} \) and \( u_{1,0} \) and denoting \( \theta := -1/F'_1 \), we obtain
\[
\frac{\partial \hat{F}}{\partial u} = T_j^{-1} \left( \theta \frac{\partial F}{\partial u_{0,1}} \right), \quad \frac{\partial \hat{F}}{\partial u_{0,-1}} = T_j^{-1} \left( \theta \frac{\partial F}{\partial u} \right), \quad \frac{\partial \hat{F}}{\partial u_{1,0}} = -T_j^{-1}(\theta),
\]
and hence \( T_j^{-1}T_j - \hat{F}_s = T_j^{-1} \circ \theta L \). This gives us \( T_j^{-1} \circ \eta[u]_s = T_j^{-1}(\eta[u])_s = T_j^{-1} \circ \theta \eta'_l_1L \) if \( \eta'_l \neq 0 \) for all non-zero integers \( k \neq 1 \), and
\[
T^n_j(\hat{F})_s = T^n_j \circ \hat{F}_s = \sum_{r=1}^{n} \left( \theta \frac{\partial T^{n-r}_j(\hat{F})}{\partial u_{1,0}} \right) T^{r-1}_j \circ L = \forall n > 0.
\]

At the same time, the left-hand side of (24) is equal to
\[
\sum_{r=-\infty}^{\infty} \left( T_j \left( \frac{\partial h}{\partial u_{0,1}} \right) (T_j^{r-1}(F)_s - T_jT_j^r) + T_j \left( \frac{\partial h}{\partial u_{0,-1}} \right) (T_j^{r-1}(\hat{F})_s - T_jT_j^r) \right)
\]
\[
= \sum_{r=-\infty}^{\infty} \left( T_j \left( \frac{\partial h}{\partial u_{0,1}} \right) (T_j^{r-1}(F)_s - T_j^{r-1} \circ F_s - T_j^{r-1} \circ L) + T_j \left( \frac{\partial h}{\partial u_{0,-1}} \right) (T_j^{r-1}(\hat{F})_s - T_j^{r-1} \circ \hat{F}_s - T_j^{r-1} \circ \theta L) \right).
\]
Taking (25) and (26) into account, we therefore obtain (24).
Theorem 2. Assume that equation (9) possesses \( j \)-integrals and a symmetry of the form (19). Let \( \phi[u] \) be a \( j \)-integral of the smallest order for this equation. Then the equation (19) admits the substitution \( v = \phi[u] \) into an equation of the form \( v_t = \hat{g}(v_{0,k}, v_{0,k+1}, \ldots, v_{0,n}) \). If, in addition, the equation (9) is Darboux integrable and the right-hand side \( F \) of (12) is single-valued, then \( v = \phi[u] \) is a Miura-type substitution.

Proof. According to lemma 3, if \( T(\phi) = \phi \) and \( L(g) = 0 \), then \( T(\partial_x(\phi)) = \partial_x(\phi) \). Thus, \( \partial_x(\phi) \) is a \( j \)-integral, and lemma 2 implies that (21) holds true for some function \( \hat{g} \) (it is easy to check that \( p \) and \( q \) in (23) must coincide with \( k \) and \( n \) if \( J = \partial_x(\phi) \)).

If the equation (9) is Darboux integrable, then it possesses symmetries (16). The operator \( R \) in (16) has the form (22) by corollary 1. Thus, \( v = \phi[u] \) is a Miura-type substitution.

2.3. Miura-type difference substitutions

For the reader convenience, we reproduce a part of the work [17] in this section. This part is based on an association of the first-order difference substitutions with differential–difference equations of the form

\[
(u_t)_x = a(u, u_1)u_x, \quad a \neq 0, \tag{27}
\]

where \( a \) depends on an integer \( j \) and a real variable \( x \), and \( u_t \) designates \( u(x, j+n) \). The \( n \)th derivative of \( u \) with respect to \( x \) is denoted by \( u^{(n)} \).

Let \( T \) designate the operator of the forward shift in \( j \) by virtue of (27). This operator is defined by the following rules: \( T(f(x, b, c, \ldots)) = f(x, T(b), T(c), \ldots) \) for any function \( f \); \( T(u_t) = u_{t+1}; T(u^{(n)}) = D_x^{n-1}(au_t) \), where

\[
D_x = \frac{\partial}{\partial x} + \sum_{k=0}^\infty u^{(k+1)} + \sum_{k=1}^\infty (T^{k-1}(au_t) - \frac{\partial}{\partial u_k} + T^{1-k} \left( \frac{u_x}{a(u_{x-1}, u)} \right) \frac{\partial}{\partial u_{x-k}}),
\]

i.e. \( D_x \) is the operator of the total derivative with respect to \( x \) by virtue of equation (27). The inverse shift operator \( T^{-1} \) is defined in a similar way.

Definition 4. An equation of the form (27) is called Darboux integrable if there exist functions \( J(x, u, u_1, \ldots, u_m) \) and \( X(x, u, u_1, \ldots, u^{(n)}) \) such that

\[
\frac{\partial J}{\partial u_k} \neq 0, \quad \frac{\partial J}{\partial u_m} \neq 0, \quad \frac{\partial X}{\partial u^{(n)}} \neq 0
\]

and the relationships \( D_x(J) = 0, T(X) = X \) hold true. In this case, the functions \( J \) and \( X \) are respectively called a \( j \)-integral and an \( x \)-integral of the equation (27). The numbers \( m - k \) and \( n \) are called order of the corresponding integral.

Definition 5. An equation of the form

\[
\omega = f(u_h, u_{h+1}, \ldots, u_m, u, u^{(1)}, \ldots, u^{(n)})
\]

is called a symmetry of (27) if the relation \( \mathcal{L}(f) = 0 \) holds true, where

\[
\mathcal{L} = TD_x - aD_x - \frac{\partial a}{\partial u_1} T - u_k \frac{\partial a}{\partial u}. \tag{28}
\]
The analogue of lemma 3 can be proved for differential–difference equations too, and the differentiation \( \partial_f \), which is defined by the formula
\[
\partial_f(h) = h_x(f) = \sum_{q=1}^{\infty} \frac{\partial h}{\partial u^q} D^q_s(f) + \sum_{q=-\infty}^{\infty} \frac{\partial h}{\partial u^q} T^q(f),
\]
commutes with \( T \) if \( u_t = f \) is a symmetry of (27).

It is easy to see that \( T_j \) coincides, up to the change of notation \( u_{0,j} \rightarrow u_j \), with \( T \) on the functions depending only on shifts of \( u \) in \( j \). Therefore, we can replace \( u_{0,j} \) and \( T_j \) with \( u_j \) and \( T \), respectively, in definition 3. Taking this into account, we can formulate the following proposition.

**Lemma 4.** If an equation of the form \( u_t = g(u_k, u_{k+1}, \ldots, u_n) \) admits the substitution \( v = \phi(u, u_1) \) into an equation \( v_t = \hat{g}(v_k, v_{k+1}, \ldots, v_n) \), then \( u_t = g \) is a symmetry of the equation
\[
(u_1)_x = -\frac{\phi_v}{\phi_u} u_x.
\]

**Proof.** It is not difficult to see that \( \phi \) is a \( j \)-integral of the equation (29). In addition, the relationship
\[
\phi_u T(g) + \phi_g = \hat{g}(T^k(\phi), T^{k+1}(\phi), \ldots, T^n(\phi))
\]
holds true by the definition of a substitution. The straightforward calculations gives us that the operator \( L \) for (29) satisfies the formula \( L = \phi_u^{-1} D_\phi (\phi_u T + \phi_w) \), and hence \( L(g) = \phi_u^{-1} D_\phi (\hat{g}) = 0 \).

According to [17] (see corollary 3 within), an equation of the form (27) has an autonomous \( x \)-integral (and therefore is Darboux integrable) if it possesses symmetries (16) with \( R \) of the form (22) (naturally, \( u_{0,j} \) and \( T_j \) must be respectively replaced with \( u_j \) and \( T \) in (16), (22)). Thus, the classification problem for the Miura-type substitutions \( v = \phi(u, u_1) \) is reduced to searching the equations (27) that possess \( x \)-integrals.

**Lemma 5.** Equation (27) is Darboux integrable if and only if there exists a function \( G(u) \) such that
\[
X = \frac{u^{(3)}}{u^{(1)}} - \frac{3}{2} \left( \frac{u^{(2)}}{u^{(1)}} \right)^2 + G(u)(u^{(1)})^2
\]
is an \( x \)-integral of this equation.

It should be noted that the proof of the lemma coincides almost word-by-word with the analogical proof for the equation \( u_x = a(x, u, u_t) u_t \) from the work [22].

**Proof.** Let \( Q \) be the \( x \)-integral of the smallest order for (27), and let \( n \) denote the order of \( Q \). It is easy to check by straightforward calculation that \( u_t = \Omega u_t \) is a symmetry of (27) for any \( x \)-integral \( \Omega \). This is why the differentiation \( \partial_f \), where \( f = \Omega u_t \), commutes with the operator \( T \), and \( \partial_f(Q) = Q_s(f) \) is also an \( x \)-integral. Thus, we obtain that the operator \( Q_s \circ u_t = \sum_{s=0}^{n} z_s D^s_x \), where
\[
z_s = \sum_{k=n}^{n} C_k^s Q_{u^k} u^{(k+1-s)},
\]
maps \( x \)-integrals into \( x \)-integrals again. And this is possible only if all the coefficients \( z_s \) of this operator lie in the kernel of \( T - 1 \).
\( \text{Let } n > 1 \). Then, according to [15], we can choose the integral \( Q \) so that it depends linearly on \( u^{(n)} \). In view of (31), this means that the order of \( z_n \) is less than \( n \). Therefore, \( z_n \) cannot be an integral and is a function depending just on \( x \). Since we can multiply \( Q \) by arbitrary non-zero functions of \( x \), we can assume \( z_n = u_n Q_{\alpha^n} = 1 \) without loss of generality. It yields that \( Q_{\alpha^n} \) does not depend on \( u^{(n)} \) for \( k = 2, n \), and \( z_n \) is also independent of \( u^{(n)} \) for \( s = \frac{n}{2}, n \) by formula (31). The latter means that \( \xi_n = \xi_n(x) \) if \( s > 1 \). Finding now \( Q_{\alpha^n} \) from (31), we obtain

\[
Q_{\alpha^n} = \left( \xi_n(x) - \sum_{k=n+1}^{n} C_k^{(n)} Q_{\alpha^n} u^{(k+1-s)} \right) u^{-1}, \quad 1 < s < n. \tag{32}
\]

Taking \( Q_{\alpha^n} = u^{-1}_n \) into account and sequentially finding \( Q_{\alpha^{n-1}}, Q_{\alpha^{n-2}}, \ldots \) by formula (32), we obtain that \( Q_{\alpha^n} \) is independent of \( u^{(l)} \) for all \( l > n - s + 1 \) and \( s > 1 \).

Let us prove now that \( n \leq 3 \). In order to do this, we assume the contrary and consider separately the cases of even and odd \( n > 3 \).

Let \( n = 2m \) and \( m > 1 \). In this case formula (32) gives

\[
Q_{\alpha^{2m+1}} = -C_m^{m+1} u^{(m)} u^{-2} + g(x, u, u, \ldots, u^{(m-1)}),
\]

\[
Q_{\alpha^{2m}} = -C_m^{m+1} u^{(m+1)} u^{-2} + h(x, u, u, \ldots, u^{(m)}),
\]

\[
Q_{\alpha^{2m-1}} = -C_m^{m+1} u^{-2} = -C_m^{m+1} u^{-2},
\]

But it is easy to check that the identity \( C_n^{m+1} = C_n^{m-1} \) can hold true only for odd \( n \).

In the same way, for \( n = 2m - 1 \) and \( m > 2 \) we have

\[
Q_{\alpha^{2m+1}} = -C_m^{m+1} u^{(m-1)} u^{-2} + g(x, u, u, \ldots, u^{(m-2)}),
\]

\[
Q_{\alpha^{2m}} = -C_m^{m+1} u^{(m+1)} u^{-2} + h(x, u, u, \ldots, u^{(m)}),
\]

\[
Q_{\alpha^{2m-1}} = -C_m^{m+1} u^{-2} = -C_m^{m+1} u^{-2},
\]

while \( C_n^{m+1} = C_n^{m-1} \) can be valid only for even \( n \).

Thus, \( n \leq 3 \), and it remains to treat independently the cases \( n = 1, 2, 3 \) for completing the proof. For \( n = 3 \), as it is shown above,

\[
Q = \frac{u^{(3)}}{u^{(1)}} + g(x, u, u^{(1)}, u^{(2)}).
\]

In view of this, the formulas for the coefficients \( z_j \) of the operator \( Q \circ u^{(1)} \) become

\[
z_0 = D_v(Q) - g_x,
\]

\[
z_1 = 2 \frac{u^{(3)}}{u^{(1)}} + 2 u^{(2)} g_w + g^{(1)} g_w,
\]

\[
z_2 = 3 \frac{u^{(2)}}{u^{(1)}} + u^{(1)} g_w.
\]

The relation \( z_2 = \xi(x) \) gives us

\[
g = -\frac{3}{2} \left( \frac{u^{(2)}}{u^{(1)}} \right)^2 + \xi \left( \frac{u^{(2)}}{u^{(1)}} \right) + h(x, u, u^{(1)}).
\]

Substituting \( g \) into the expressions for \( z_1 \) and \( z_0 \), we obtain

\[
z_1 = 2Q - 2h + u^{(1)} h_w - \xi \left( \frac{u^{(2)}}{u^{(1)}} \right)
\]

\[
\Rightarrow \quad (T - 1) \left( 2h - u^{(1)} h_w + \xi \left( \frac{u^{(2)}}{u^{(1)}} \right) \right) = 0
\]

\[
\Rightarrow \quad \xi = 0,
\]

\[
2h - u^{(1)} h_w = \eta(x);
\]

\[
z_0 = D_v(Q) - h_x \Rightarrow \quad (T - 1) (h_x) = 0
\]

\[
\Rightarrow \quad h_x = \xi'(x) \Rightarrow \quad h = \xi(x) + \hat{h}(u, u^{(1)}).
\]
Finally, we arrive at the equation $u^{(1)} \dot{h}_{u^{(1)}} = 2\dot{h} + 2\xi(x) + \eta(x)$, which implies $2\xi(x) + \eta(x) = c$, $\dot{h} = G(u)(a^{(1)})^2 - c/2$. Thus, equation (27) should possess an integral (30) if $n = 3$.

For $n = 2$ similar reasonings give us that $Q = u^{(2)}/u^{(1)} + C(u)u^{(1)}$ can be chosen as an $x$-integral of the smallest order. But then $D_x(Q) - Q^2/2$ reads as (30).

According to [8], the Laplace invariant $H_0 = a_n u_x + a T^{-1}(a_n u_x)$ of equation (27) must vanish if (27) possesses a first-order $x$-integral. Applying $T^{-1}$ to both the sides of (27), we obtain $T^{-1}(u_x) = u_x/T^{-1}(a)$. In view of this,

$$H_0 = u_x \left( a_n + a T^{-1} \left( \frac{d_n}{a} \right) \right) = 0.$$ 

Differentiating the last relation with respect to $u_{-1}$, we obtain $(\ln(a))_{u_{-1}} = 0$ and, therefore, $a = \xi(u_1) \eta(u)$. Substituting it into the expression for $H_0$, we get

$$\eta(u) \xi(u) + \eta(u) \xi'(u) = 0 \quad \Rightarrow \quad \xi(u) = \frac{c}{\eta(u)}, \quad a = c \frac{\eta(u)}{\eta(u_1)}.$$ 

Thus, the relation $D_x(\xi(u)-c \zeta(u)) = 0$, where $\zeta'(u) = \eta(u)$, holds true for any equation (27) possessing a first-order $x$-integral. This implies that (27) possesses the integral

$$X = \frac{D_x^2(\xi(u))}{D_x(\xi(u))} - \frac{3}{2} \left( \frac{D_x^2(\xi(u))}{D_x(\xi(u))} \right)^2,$$

which, as one can check easily, has the form (30). \quad \Box

**Theorem 3.** If $v = \phi(u, u_{1})$ is a Miura-type substitution, then there exist functions $\alpha$, $\beta$, $\gamma$ and $\xi$ such that

$$\zeta(u_{1}) = \alpha(\phi(u, u_{1})) + \frac{\beta(\phi(u, u_{1}))}{\gamma(\phi(u, u_{1}))} - \zeta(u), \quad \beta \zeta' \neq 0.$$ \quad (33)

It is obvious that (33) can hold true only if $|\alpha'| + |\beta'| + |\gamma'| \neq 0$.

**Proof.** According to lemma 4 above and corollary 3 from [17], the equation (29) is Darboux integrable if $v = \phi(u, u_{1})$ is a Miura-type substitution. Therefore, this equation has an $x$-integral of the form (30) by lemma 5.

The change of variables $u = \xi(\bar{u})$ maps (29) into the equation

$$(\bar{u}_{1})_{x} = \frac{\varphi}{\varphi_{\bar{u}_{1}}} \bar{u}_{x},$$ \quad (34)

where $\varphi = \phi(\xi(\bar{u}), \xi(\bar{u}_{1}))$, and (30) – into the integral

$$\bar{X} = \frac{\bar{u}_{(3)}}{\bar{u}_{(1)}} \left( \frac{\bar{u}_{(2)}}{\bar{u}_{(1)}} \right)^2 + \frac{\xi''(\bar{u})}{\xi'(\bar{u})} \left( \frac{\xi'(\bar{u})}{\xi''(\bar{u})} \right)^2 G(\xi(\bar{u})) (\xi'(\bar{u}))^2 \left( \bar{u}_{(1)} \right)^2$$

of (34). We can choose $\xi$ so that the coefficient at $(\bar{u}_{(1)})^2$ vanishes and (34) possesses the $x$-integral

$$\bar{X} = \frac{\bar{u}_{(3)}}{\bar{u}_{(1)}} - \frac{3}{2} \left( \frac{\bar{u}_{(2)}}{\bar{u}_{(1)}} \right)^2.$$ \quad (35)

Since $\varphi, \bar{u}$ and $\bar{u}_{1}$ are functionally dependent, we can express $\bar{u}_{1}$ as $\bar{u}_{1} = \theta(\bar{u}, \varphi)$. Successively differentiating this expression, we obtain

$$D_x(\bar{u}_{1}) = \theta_{\bar{u}} \bar{u}_{(1)}^{(1)}, \quad D_x^2(\bar{u}_{1}) = \theta_{\bar{u} \bar{u}} (\bar{u}_{(1)})^{2} + \theta_{\bar{u} \bar{u}_{(1)}} \bar{u}_{(2)}^{(1)},$$

$$D_x^3(\bar{u}_{1}) = \theta_{\bar{u} \bar{u} \bar{u}} (\bar{u}_{(1)})^{3} + 3 \theta_{\bar{u} \bar{u} \bar{u}_{(1)}} \bar{u}_{(2)}^{(1)} + \theta_{\bar{u} \bar{u} \bar{u}_{(1)}} \bar{u}_{(3)}^{(1)}.$$
These formulas and (35) imply that
\[ T(\tilde{X}) = \tilde{X} + \left( \frac{\theta_{u_{\tilde{a}}}}{\theta_u} - \frac{3}{2} \left( \frac{\theta_{u_{\tilde{a}}}}{\theta_u} \right)^2 \right) (\tilde{u}^{(1)})^2 \Rightarrow \left( \frac{\theta_{u_{\tilde{a}}}}{\theta_u} \right) = \frac{1}{2} \left( \frac{\theta_{u_{\tilde{a}}}}{\theta_u} \right)^2. \]
Solving the latter equation, we obtain \( \tilde{u}_1 = \alpha(\phi) + \beta(\phi)/(\gamma(\phi) - \tilde{u}) \). The inverse change of variables \( \tilde{u} = \zeta(u) \) in the last relationship gives us (33).

\[ \square \]

3. The classification result and examples

Theorems 2 and 3 imply the following necessary condition of Darboux integrability.

**Theorem 4.** Let equation (9) be Darboux integrable, possess a \( j \)-integral \( \phi(u, u_{0,1}) \) and the right-hand side \( \hat{F} \) of (12) be single-valued. Then there exist functions \( \alpha, \beta, \gamma \) and \( \zeta \) such that the relationship (33) holds true.

It should be noted that we need single-valuedness of \( \hat{F} \) to only guarantee existence of symmetries (16). Thus, the above theorem remains valid without assumption about single-valuedness of \( \hat{F} \) in any situation when (9) admits symmetries of the form (16).

A converse to theorem 4 is also true. Because our considerations are local, this converse needs an accurate formulation.

**Theorem 5.** Let \( \phi \) satisfy a relationship of the form (33) for any \( (u, u_1) \) that lies in a domain \( U \) and the function \( F(u, u_{1,0}, u_{0,1}) \) be implicitly defined by the relationships

\[ \phi(u_{1,0}, F) = \phi(u, u_{0,1}), \quad F(z, z, w) = w. \] (36)

Then \( (u_{1,0}, F) \in U \) for any \( (u, u_{0,1}) \in U \) and enough small \( |u_{1,0} - u| \);

\[ \zeta(F) = \alpha(\phi(u, u_{0,1})) + \frac{\beta(\phi(u, u_{0,1}))}{\gamma(\phi(u, u_{0,1})) - \zeta(u_{1,0})} \] (37)

for any \( (u, u_{0,1}), (u_{1,0}, F) \in U \); and the relation \( T_j(I) = I \) with

\[ I = \frac{\zeta(u_{1,0}) - \zeta(u_{1,0})}{\zeta(u_{1,0})} \frac{\zeta(u_{2,0}) - \zeta(u)}{\zeta(u_{1,0}) - \zeta(u)} \] (38)

holds true for the equation (9) wherever \( F \) is defined by (33), (36) (i.e. for any \( (u_{0,0}, T_j(u_{k,0})) \in U, k = 0, 1, 3 \)).

**Proof.** By construction, the transformation \( (u, u_{0,1}) \rightarrow (u_{1,0}, F(u, u_{1,0}, u_{0,1})) \) becomes the identity mapping if \( u_{1,0} = u \). Therefore, \( (u_{1,0}, F) \in U \) if \( (u, u_{0,1}) \in U \) and \( u_{1,0} \) lies in a neighborhood of \( u \).

Replacing \( u \) and \( u_1 \) with \( u_{1,0} \) and \( F(u, u_{1,0}, u_{0,1}) \), respectively, in (33) and taking the first equation of (36) into account, we obtain (37). Equation (37) and \( T_j(\phi) = \phi \) imply that

\[ T_j(\zeta(u_{k,0})) = T_j^k(\zeta(u_{0,1})) = \alpha(\phi) + \frac{\beta(\phi)}{\gamma(\phi) - \zeta(u_{k,0})} \]

for any \( k \) and \( u_{k,0} \) such that \( (u_{k,0}, T_j(u_{k,0})) \in U \). Therefore, we have

\[ T_j(\zeta(u_{k,0}) - \zeta(u_{k,0})) = \beta(\phi) \frac{\zeta(u_{k,0}) - \zeta(u_{k,0})}{(\gamma(\phi) - \zeta(u_{k,0}))(\gamma(\phi) - \zeta(u_{k,0}))} \]

and \( T_j(I) = I. \)

\[ \square \]
The above theorems mean that any Darboux integrable equation (9) satisfying (36) is related via a point transformation \( \tilde{u} = \xi(u) \) to an equation of the form

\[
\tilde{u}_{1,1} = \alpha(\varphi) + \frac{\beta(\varphi)}{\gamma(\varphi) - \tilde{u}_{1,0}}, \quad \beta \neq 0,
\]

where \( \varphi(\tilde{u}, \tilde{u}_{0,1}) \) is defined by the relationship

\[
\tilde{u}_{0,1} = \alpha(\varphi) + \frac{\beta(\varphi)}{\gamma(\varphi) - \tilde{u}},
\]

and \( \phi = \varphi(\xi(u), \xi(u_{0,1})) \) is the \( j \)-integral of the equation (9).

Although we can assume without loss of generality that \( \zeta(u) = u \), another choice of \( \zeta \) is sometimes convenient to obtain a more simple form of the equation. For example, the equation

\[
\phi = \frac{u + 1}{u_{0,1}}, \quad \frac{u_{1,1}}{u_{0,1}} = \frac{u_{1,0} + 1}{u + 1}.
\]  

(39)

This equation was obtained in [13] and has the \( i \)-integral \((u_{2,0} - u_{1,0})/(u_{1,0} - u)\). The latter illustrates that some Darboux integrable equations (14) can possess \( i \)-integrals of the order less than 3, while (38) guarantees the existence of the third-order \( i \)-integral only.

Up to the point transformation, the above example is a particular case of the equation that is generated by choice \( \zeta(u) = u \), \( \alpha(\phi) = \delta \phi \), \( \beta(\phi) = D - C \phi - \delta(D \phi - B) \) and \( \gamma \neq A \phi - B \), where \( A, B, C, D \) and \( \delta \) are arbitrary constants such that \( |\delta| + |A| + |C| \neq 0 \), \( |\delta A| + |C - \delta B| + |D| \neq 0 \). Without loss of generality, we can assume that \( \delta \) equals 1 or 0.

Substituting this choice into (33) and solving it for \( \phi \), we obtain

\[
\phi = \frac{u_{0,1}(u + B) + D}{u_{0,1} + \delta u + C}.
\]

The corresponding equation (14) after solving for \( u_{1,1} \) takes the form

\[
u_{1,1} = \frac{(u + B)(\delta u_{1,0} + C) - AD)u_{0,1} + \delta D(u_{1,0} - u)}{A(u_{1,0} - u)u_{0,1} + (u_{1,0} + B)(\delta u + C) - AD}
\]

and can be rewritten as

\[
T_j \left( \frac{\delta D - u_{1,0}(A u + C - \delta B)}{u_{1,0} - u} \right) = \frac{(u + B)(\delta u_{1,0} + C) - AD}{u_{1,0} - u}.
\]  

(40)

The latter formula means that this equation admits the non-point invertible transformation (see [11] for more details). This allows us to derive another Darboux integrable equation from (40) (and this is why such \( \alpha, \beta \) and \( \gamma \) are chosen). Let

\[
v = \frac{\delta D - u_{1,0}(Au + C - \delta B)}{u_{1,0} - u}.
\]  

(41)

Then (40) implies that

\[
v_{0,1} := T_j(v) = \frac{(u + B)(\delta u_{1,0} + C) - AD}{u_{1,0} - u}.
\]  

(42)

Expressing \( u_{1,0} \) in terms of \( u \) and \( v \) from (41), we obtain

\[
u_{1,0} = \frac{uv + \delta D}{v + Au + C - \delta B}.
\]  

(43)
The substitution of (43) into (42) gives rise to the quadratic equation $P_2u^2 + P_1u + P_0 = 0$ with the coefficients

$$P_2 = \delta v + Av_{0,1} + AC,$$
$$P_1 = (C + \delta B)v + (C - \delta B)v_{0,1} + (BC - AD - \delta D)(A - \delta) + C^2,$$
$$P_0 = (BC - AD)v - \delta Dv_{0,1} + BC(C - \delta B) + D(\delta AB - AC + \delta^2 B).$$

Thus, the transformation (41) maps solutions of (40) into solutions of the equation

$$T_j(\theta(v, v_{0,1})) = \frac{v\theta(v, v_{0,1}) + \delta D}{A\theta(v, v_{0,1}) + v + C - \delta B},$$

(44)

where $\theta$ is a root of the polynomial $P_2\theta^2 + P_1\theta + P_0$. Repeating the above reasonings in the inverse order, it is easy to see that the transformation $u = \theta(v, v_{0,1})$ maps solutions of (44) back into solutions of (40). This implies that rewriting the integrals of (40) in terms of $v$ and its shifts gives us integrals of (44). Therefore,

$$J[v] = \phi(\theta, T_j(\theta)) = \frac{T_j(\theta)(\theta + B) + D}{AT_j(\theta) + \delta\theta + C}$$

is a $j$-integral of (44). Using (43) and its consequences, we can express $u_{k,0}, k = 1, 2, 3$, in terms of $u, v, v_{1,0}, v_{2,0}$ and substitute these expressions into (38). This gives rise to the formula

$$I[v] = \frac{(v_{2,0} + v_{1,0} + C - \delta B)(v_{1,0} + v + C - \delta B)}{v_{1,0}^2 + v_{1,0}(C - \delta B) - \delta AD}$$

for an $i$-integral of the equation (44).

It should be noted that (44) coincides with the discrete Liouville equation (5) if $C = -1$, $B \neq 0, A = \delta = 0$. The equation (44) also takes the form $v_{1,1}(v + 1) = v_{1,0}(v_{0,1} + 1)$ (compare (39)) if $C = D = 0, B = -1, A \neq -1, \delta = 1$. In general, any Darboux integrable equation of the form

$$T_j(\Omega(\tilde{v}, \tilde{v}_{0,1})) = \Psi(\tilde{v}, \tilde{v}_{0,1}), \quad \frac{\partial\Omega}{\partial\tilde{v}} \frac{\partial\Psi}{\partial\tilde{v}_{0,1}} - \frac{\partial\Omega}{\partial\tilde{v}_{0,1}} \frac{\partial\Psi}{\partial\tilde{v}} \neq 0,$$

possessing a second-order autonomous $j$-integral is related to an equation of the form (44) via a point transformation $v = \mu(\tilde{v})$. This can be proved by reasonings that is similar to those used in [17], but the rigorous proof of this proposition is beyond the scope of the present paper.

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