Multilevel Monte-Carlo for computing the SCR with the standard formula and other stress tests

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Abstract

This paper studies the multilevel Monte-Carlo estimator for the expectation of a maximum of conditional expectations. This problem arises naturally when considering many stress tests and appears in the calculation of the interest rate module of the standard formula for the SCR. We obtain theoretical convergence results that complements the recent work of Giles and Goda [14] and gives some additional tractability through a parameter that somehow describes regularity properties around the maximum. We then apply the MLMC estimator to the calculation of the SCR at future dates with the standard formula for an ALM savings business on life insurance. We compare it with estimators obtained with Least Square Monte-Carlo or Neural Networks. We find that the MLMC estimator is computationally more efficient and has the main advantage to avoid regression issues, which is particularly significant in the context of projection of a balance sheet by an insurer due to the path dependency. Last, we discuss the potentiality of this numerical method and analyse in particular the effect of the portfolio allocation on the SCR at future dates.

1 Introduction

Solvency II is a regulatory framework introduced in Europe in the period post-financial crisis of 2008. Solvency II establishes the requirements to be met to exercise the insurance or reinsurance activity in Europe and aims to protect policyholders and to give stability in the financial sector of the European Union.

One of the advantages of the Solvency II directive is that the computation required to evaluate the Solvency Required Capital (SCR) considers the specific risks borne by the insurers in comparison to the previous rules where the need of own funds ignored, for example, part of risks embedded in the asset side of the balance sheet. Indeed, under Solvency II the evaluation of the SCR amounts either to use the standard formula by applying shocks to each asset class or to calculate a quantile of the conditional law of the profits and losses for variations of the initial state of the market, given a portfolio of contracts.

Today, the SCR indicator is one of the most important Key Performance Indicator used by companies to monitor the activity. In particular, the so-called Solvency II ratio computed as the ratio between the “Eligible Own Fund” (EOF) and the SCR measures the solvency capacity of the insurers and it is followed by analysts to evaluate them in financial markets. Nevertheless, it is important to remark that the SCR corresponds to the amount of required capital in a 1 year horizon. Then, at time \( t \) to have an idea of the total amount of required capital during the life of a product or over the duration of the business, it is not only necessary to compute the \( SCR_t \) but also to estimate the SCR in future dates (\( SCR_{t+1}, SCR_{t+2}, etc. \)).

The aim of this work is to deal with the problem of computing SCR at future dates which has several practical applications to real problems that arise in the insurance industry. One of the first applications that should be cited comes from the regulatory side and is called ORSA (Own Risk and Solvency Assessment) process which aims to evaluate from a prospective point of view the solvency needs related to the specific risk profile of the insurance

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companies. In order to do that, the computations of the SCR at future dates is necessary to ensure that the insurer is able to integrate the regulatory constraints in terms of solvency during the strategic plan horizon.

Other important applications appear when the notion of cost of capital is concerned. In general, the cost of capital refers to the desired return on the immobilized capital during the life of a product or over the duration of the business and can be written under the following form:

\[
\text{Targeted return} \times \text{Immobilized Capital}
\]

where Targeted return represents an expected return, for example, from a shareholder standpoint. It is possible to exhibit a relation between the Immobilized Capital and the SCR computed at futures dates by considering the simple following reasoning. At \( t = 0 \), the \( \text{SCR}_0 \) is the amount the shareholder needs to immobilize at inception in order to pay its liabilities in unfavorable cases. At \( t = 1 \), the insurance company pays the expected return to the shareholder on \( \text{SCR}_0 \) and lends from him the amount \( \text{SCR}_1 \) to continue to exercise the insurance activity. By repeating this mechanism at each time step until the end of the business or the product maturity (time \( T \)), the total immobilized capital at \( t = 0 \) corresponds to

\[
\text{Immobilized Capital}_0 = \text{SCR}_0 + PV\left( \sum_{t=1}^{T} \text{SCR}_t - \text{Targeted return} \times \text{SCR}_{t-1} \right)
\]

where \( PV(x) \) stands for the present value of \( x \) and given that \( \text{SCR}_T = 0 \) (it is assumed that the company closes its business at the end of the year \( T \) and then there is no need of capital between \( T \) and \( T + 1 \)). Thus, among the applications related to the cost of capital, one can mention:

(i) Applications for ALM (Asset Liability Management) when a Strategic Asset Allocation needs to be computed for a given portfolio: to evaluate the optimality of an asset allocation, a criterion based on the sum of the present values of shareholder margins minus the amount of cost of capital generated by the asset allocation is usually studied. The idea of this approach is to analyse if the future gains generated by the portfolio meet the shareholder’s expectations in terms of cost of capital.

(ii) Applications for pricing, when evaluating if future margins pay the return expected by the shareholders. Before launching a new product, the insurers evaluate the profitability of that product and then compare expected future shareholder margins with the need of capital generated by the new business.

Finally, the computation of future SCR can be used as a tool for studying the solvency of the company under different economic scenarios. For example, the current low interest rates environment yields in several questions on the solvency of the insurance companies and on the sustainability of the Savings business. In addition, the SCR computation at future dates allows to better understand the pattern of cash-flows generated by a product during the lifetime of the business. In particular, the approach based on shocks employed in this work as, for example, the shocks on the market conditions is useful to study the evolution of the balance sheet and the policyholder behavior under those shocked conditions.

In general, today, the computations where SCR at futures dates are needed are based on rough estimations from the initial SCR which can ignore the evolution of the risk profile of the insurer and may lead to bad decisions impacting the business. Thus, the goal of the present work is to develop numerical methods for the calculation of the SCR required in the future. We focus here on the calculation of SCR with the standard formula, which is fully described by the documents of the European Insurance and Occupational Pensions Authority (EIOPA) [21, 22]. Basically, this standard formula consists in applying different shocks on the different market sectors: the impact on the portfolio of each shock is evaluated in a risk-neutral world, and the SCR is then evaluated by using an aggregation formula from these impacts.

Many works in the literature deals with the numerical computation of the SCR so that we cannot be exhaustive. Devineau and Loisel [9], Bauer et al. [3] have investigated numerical methods based on nested simulations. Bauer et al. [2], Krah et al. [23] and Floryszczak et al. [11] have used Least Square Monte-Carlo (regress now) methods.
for the risk while Pelsser and Schweizer [25], Cambou and Filipović have developed the replicating portfolio (or regress later) approach. Recently, Cheredito et al. [7] and Fernandez-Arjona and Filipović [10] have proposed to use neural networks to approximate the conditional expectation. Up to our knowledge, there are however no dedicated study on the use of multilevel Monte-Carlo estimators for the calculation of the SCR with practical application in an insurance context. This paper fills this gap. Besides, most of the paper deals with the quantile formulation of the SCR and focus on the calculation of the current value of the SCR (there are few exceptions such as Vedani and Devineau [26]). Here, we consider instead the calculation of the SCR with the standard formula at future dates. Last, most of the literature either use simple Markovian underlying models or consider instead models from insurance companies that are black boxes, which makes difficult the reproducibility of the results. Here, we are in between and make our experiments on a synthetic ALM model that we recently developed and fully presented in [1] which takes into account many path-dependent features of the ALM for life insurance.

We now describe the formal mathematical framework and consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(X\) and \(Y\) be two random variables such that \(X\) takes values in a general measurable space \((G, \mathcal{G})\) and \(Y\) takes values in \(\mathbb{R}^P\), \(P \in \mathbb{N}^*\). We make the following assumptions:

(A.1) \(Y\) is square integrable \(\mathbb{R}^P\)-valued random variable,

(A.2) \(\phi : G \to \mathbb{R}\) is a measurable real-valued function \(\phi\) such that \(\phi(X)\) is square integrable.

For the financial application that we consider in this paper, \(X\) generally represents the market information up to some time, and therefore may be the realization of asset paths. For example, we may take \(G = \mathcal{C}([0, t], \mathbb{R}^d)\) if we consider a market with \(d \in \mathbb{N}^*\) continuous assets up to time \(t > 0\). However, under some Markovian assumption, the market information at time \(t\) may simply be sum up by the current value of the assets, in which case we may take \(G = \mathbb{R}^d\). We are interested in the problem of computing nested expectations of the form:

\[
I = \mathbb{E}\left[ h \left( \mathbb{E} \left[ Y^1 | X \right], \ldots, \mathbb{E} \left[ Y^P | X \right] \right) \phi(X) \right],
\]

where \(h : \mathbb{R}^P \to \mathbb{R}\) is a measurable function with sublinear growth (i.e. \(\exists C > 0, \forall x \in \mathbb{R}^d, |h(x)| \leq C(1 + |x|)\)), which ensures by Assumptions (A.1) and (A.2) that \(I\) is well defined. In Formula (1), \(\mathbb{E}[Y^1|X]\) typically represents the expected loss at time \(t\) if one implements the shock (or stress test) number \(i \in \{1, \ldots, P\}\) immediately after time \(t\). The function \(h\) describes the aggregation of the shocks in terms of own funds while the function \(\phi\) weights the different events up to \(t\).

The calculation of \(I\) is usually made by using a nested Monte-Carlo method: one simulates \(J\) independent samples of \(X\) called primary scenarios and then, for each primary scenarios, one simulates independently \(K\) independent samples of \(Y\) to approximate the conditional expectations involved in (1) by the corresponding empirical mean. This method has been investigated by Gordy and Juneja [18] and Broadie et. al [5] to calculate the probability of large losses and the Value-at-Risk. Under mild assumptions, the optimal tuning to approximate \(I\) with a precision of \(\varepsilon > 0\) is to take \(J = O(\varepsilon^{-2})\) primary scenarios and \(K = O(\varepsilon^{-1})\) secondary scenarios. Thus, the overall complexity is in \(O(\varepsilon^{-3})\). The multilevel Monte-Carlo method (MLMC) developed by Giles [12] has been applied to the calculation of nested expectations by Haji-Ali [20], Bujok et al. [6] and Giles [13]. Under some regularity assumptions on \(h\), they show that the antithetic MLMC estimator achieves a precision \(\varepsilon > 0\) with a computational cost of \(O(\varepsilon^{-2})\). Under additional regularity assumptions on \(h\) or on the probability density function of \((X, Y)\), Giorgi et al. [16] have applied the Richardson-Romberg Multilevel method developed by Lemaire and Pagès [24] that improves the convergence of the MLMC estimator.

In this work, we focus on the case where \(h\) is the maximum function that is sublinear, but does not satisfy the standard regularity assumptions. We are thus interested in the problem of computing nested expectations of the form

\[
I = \mathbb{E}\left[ \max \left\{ \mathbb{E} \left[ Y^1 | X \right], \ldots, \mathbb{E} \left[ Y^P | X \right] \right\} \phi(X) \right].
\]

Such kind of expectation arises in the standard formula of the interest rate module of the SCR. More generally, the problem of computing (2) occurs when one has to determine the worst of a set of \(P\) shocks (or stress tests) on a portfolio of securities at some future time \(t\) called risk horizon. For the interest rate module of the SCR, one rather has to compute \(\mathbb{E}\left[ \max \left\{ \mathbb{E} \left[ Y^1 | X \right], \ldots, \mathbb{E} \left[ Y^P | X \right], 0 \right\} \phi(X) \right]\), which amounts to add a zero coordinate to \(Y\). When the function \(\phi\) is nonnegative and such that \(\mathbb{E}[\phi(X)] = 1\), \(\phi(X)\) can be seen as a change
of probability on the different events up to time $t$. The function $\phi(X)$ does not add any technical difficulty in our study, but it enables us to perform the evolution up to time $t$ under the real probability and the evaluation of the losses under the risk-neutral probability, as it is recommended by Solvency II. Studies of MLMC estimators for nested expectations for irregular functions $h$ with applications to risk management have recently been made by Giles and Haji-Ali [15], Bourgey et al. [4] and Giorgi et al. [16]. In a very recent work, Giles and Goda [14] have studied precisely the problem of computing (2) with the MLMC method.

The contribution of this paper is twofold. First, we provide an original mathematical analysis of the MLMC estimator for the calculation of (2) that completes the result obtained by Giles and Goda [14]. Our analysis relies on different arguments and the required assumptions are therefore also different. In particular, Giles and Goda make some technical assumptions to control the probability of two elements being close to the maximum. These assumptions are replaced in our analysis by an integrability assumption involving a parameter $\eta \in (0, 1)$ that gives some additional flexibility in the application of the MLMC estimator. Our second contribution is to apply this method to an ALM model for life insurance that takes into account the main characteristic of the business: book values, profit-sharing mechanism, minimum guaranteed rate, etc. Thus, the model is truly path-dependent so that the conditional expectation at time $t$ really involves the past dynamics making the use of regression techniques more delicate. One of the main advantage of the MLMC estimator is to calculate directly $I$ and skip the question of regression. The second main advantage is that it provides an estimator with accuracy $\varepsilon$ and with a computational cost in $O(\varepsilon^{-2})$: it is thus asymptotically as efficient as a Monte-Carlo method for plain expectations. In our numerical study, we compare the estimation of $I$ with MLMC, Least Square Monte-Carlo (LSMC) estimator and the use of Neural Networks (NN), and demonstrate the main advantages of the MLMC estimator.

The paper is organized as follows. Section 2 presents the mathematical results on the estimation of $I$ with nested Monte-Carlo and MLMC. Technical proofs are postponed to Appendix A. Section 3 then deals with the application to ALM. Subsections 3.1 and 3.3 present the ALM model for life insurance business that we developed in [1] while Subsection 3.2 recalls the calculation of the SCR with the standard formula. Subsection 3.4 compares the numerical performance of the MLMC estimator with estimators obtained with LSMC or NN. Last, Subsection 3.5 shows the interest of analysing the SCR at future dates, exhibiting some interesting properties such as the dependence of the SCR on the portfolio allocation or on the market risk premia.

## 2 Mathematical analysis of Monte-Carlo estimators of $I$

### 2.1 Nested Monte-Carlo estimator

In order to compute $I$ defined by (2), the classical approach is to approximate the inner and outer expectation using Monte-Carlo estimators. The procedure consists in generating an i.i.d sample $(X_1, \ldots, X_J)$ of $X$ called outer (or primary) scenarios. Then, conditionally on $X_i$, we sample $(Y_{i,1}, \ldots, Y_{i,K})$ called inner (or secondary scenarios) following the conditional law of $Y$ given $X = X_i$ and approximate the conditional expectation $E[Y^p|X = X_i]$, for $p \in \{1, \ldots, P\}$, by

$$\hat{E}_{i,K}^p = \frac{1}{K} \sum_{k=1}^K Y_{i,k}^p$$  \hspace{1cm} (3)

The outer expectation is then approximated using the standard MC estimator :

$$\hat{I}_{J,K} = \frac{1}{J} \sum_{j=1}^J \max \left\{ \hat{E}_{j,K}^1, \ldots, \hat{E}_{j,K}^P \right\} \phi(X^j)$$  \hspace{1cm} (4)

This estimator has been studied for example by Gordy and Juneja [18] in the context of portfolio risk measurement. The nested simulation procedure introduces two level of error since we combine the estimates from the outer and inner levels of simulation to compute $I$. In a standard way, we analyse the Mean-Square Error (MSE) of the estimate $\text{MSE}(\hat{I}_{J,K}) = E[|\hat{I}_{J,K} - I|^2]$ and use the bias-variance decomposition:

$$\text{MSE}(\hat{I}_{J,K}) = \text{bias}^2(\hat{I}_{J,K}) + \text{Var}(\hat{I}_{J,K}),$$
where \( \text{bias}(\hat{I}_{J,K}) = E[\hat{I}_{J,K}] - I \).

**Notation.**
- We set \( E^p_X := E[Y^p|X] \) for \( p \in \{1, \ldots, P\} \) and \( M^p_X = \max\{E[Y^1|X], \ldots, E[Y^p|X]\} \).
- Let \( K \in \mathbb{N}^* \) and \( Y_1, \ldots, Y_K \) be an i.i.d. sample following the conditional law of \( Y \) given \( X \). Then, we set
  \[
  \forall p = 1, \ldots, P, \quad \hat{E}^p_K = \frac{1}{K} \sum_{k=1}^{K} Y^p_k \quad \text{and} \quad \hat{M}^p_K = \max\{\hat{E}^1_K, \ldots, \hat{E}^p_K\}. \tag{5}
  \]
- Besides, when \( K \) is even, we define
  \[
  \forall p = 1, \ldots, P, \quad \hat{E}^{p,j}_{K/2} = \frac{2}{K} \sum_{k=K/2+1}^{K} Y^p_k \quad \text{and} \quad \hat{M}^{p,j}_{K/2} = \max(\hat{E}^{1,j}_{K/2}, \ldots, \hat{E}^{p,j}_{K/2}). \tag{6}
  \]

From the LLN, we have \( \hat{E}^p_K \to E^p_X \) and \( \hat{M}^p_K \to M^p_X \) almost surely as \( K \to +\infty \). The next theorem analyses the MSE of the nested estimator and provides estimates that will be then useful for the analysis of the MLMC estimator.

**Theorem 1.** Let \( P \geq 2 \) and \( \eta \in (0,1) \). Let \( X, Y \) be random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We assume that (A.1) and (A.2) hold, and we define, for \( p \in \{1, \ldots, P\} \),

\[
\sigma_p(X) = \sqrt{\text{Var}(Y^p|X)}, \quad \Sigma^p(X) = \sum_{i=1}^{P} \sigma^1_{i+\eta}(X).
\]

Then, we have

\[
C_p(X) = 2^\eta \Sigma^p(X) \sum_{p'=2}^{P} \frac{1}{|E^{p'}_X - M^{p'-1}_X|^\eta}.
\]

Assume that the following condition holds:

(i) \( \forall p = 2, \ldots, P, \quad \mathbb{P}\left(M^{p-1}_X = E^p_X\right) = 0, \)

(ii) \( \Sigma^2 = E[\Sigma^2_p(X)\phi^2(X)] < \infty \) and \( C = E[C_p(X)||\phi(X)||] < \infty \).

Then, we have

\[
\left| E\left( (\hat{M}^p_K - M^p_X) \phi(X) \right) \right| \leq \frac{C}{K^{1+\eta}} \quad \text{and} \quad E\left( (\hat{M}^p_K - M^p_X)^2 \phi^2(X) \right) \leq \frac{\Sigma^2}{K}. \tag{7}
\]

Besides, if \( V = \text{Var}(M^p_X \phi(X)) < \infty \), we get

\[
\text{MSE}(\hat{I}_{J,K}) \leq \frac{C^2}{K^{1+\eta}} + \frac{2V}{J} + \frac{2\Sigma^2}{JK}. \tag{8}
\]

With this upper bound, taking \( K = O(e^{-\frac{2}{2\pi}}) \) and \( J = O(\varepsilon^{-2}) \) is an asymptotically optimal choice to get \( \text{MSE}(\hat{I}_{J,K}) = O(\varepsilon^2) \) while minimizing the computation cost \( JK \).

**Remark 1.** Let us note that the assumptions (i) and \( C < \infty \) of Theorem 1 are only needed to improve the upper bound on the bias. If it does not hold, we still have

\[
\left| E\left( (\hat{M}^p_K - M^p_X) \phi(X) \right) \right| \leq E\left( (\hat{M}^p_K - M^p_X) \phi(X) \right) \leq \frac{\Sigma}{\sqrt{K}}.
\]

Note that the speed in \( O(K^{-1/2}) \) is optimal. Consider the example where \( Y = (Y^1, Y^2) \) and, given \( X, Y^1 \) and \( Y^2 \) are independent normal distribution with unit variance and the same mean \( m(X) \). Then, \( M^2_X = m(X) \) and, given \( X, M^2_K - M^2_X \) has the same law as \( \frac{1}{\sqrt{K}} \max(G^1, G^2) \) where \( G^1 \) and \( G^2 \) are independent standard normal variables.
Remark 2. For practical applications such as the standard formula for the SCR interest rate module, one usually considers the positive part of the maximum. This amounts to add the coordinate $Y^p+ = \max \{Y^p, 0\}$ in our framework. Thus, if we assume in addition that $P(M^p_X = 0) = 0$ and $\tilde{C} = \mathbb{E} \left[ (C_p(X) + \frac{2^{p+\epsilon} \sigma^p \eta(X)}{\sigma^p}) | \phi(X) | \right] < \infty$, then

$$\left| \mathbb{E} \left( (\tilde{M}^p_K)^+ - (M^p_X)^+ \right) \phi(X) \right| \leq \frac{\tilde{C}}{K^{\frac{1+\eta}{\eta}}} \text{ and } \mathbb{E} \left( (\tilde{M}^p_K)^+ - (M^p_X)^+ \right)^2 \phi^2(X) \leq \frac{\Sigma^2}{K}. $$

Remark 3. Let us assume for simplicity that $\phi \equiv 1$ and there exists $\sigma, \tilde{\sigma} \in \mathbb{R}^*_+$ such that for all $p \in \{1, \ldots, P\}$,

$$\sigma \leq \sigma_p(X) \leq \tilde{\sigma}, \text{ a.s.}$$

Then, the integrability condition (ii) of Theorem 1 is equivalent to have $\mathbb{E}[|E^p_X - M^p_X|^{-\eta}] < \infty$ for all $p \in \{2, \ldots, P\}$. Suppose now that $E^p_X - M^p_X \rightarrow 0$. Then, the integrability condition near $0$ gives

$$\int_{-\epsilon}^{\epsilon} |x|^{-\eta} f_p(x) dx < \infty \iff \eta < 1.$$ 

This indicates that, in a quite general framework, condition (ii) of Theorem 1 is not satisfied for $\eta = 1$ but may be satisfied for any $0 < \eta < 1$.

The proof of Theorem 1 is a consequence of the next lemma, whose proof is postponed to Appendix A.2. The analysis is rather standard, but the difficulty is to handle in the bias analysis the singularity of the maximum when two (or more) arguments equal. This is why we need Assumption (i) and the finiteness of $C$ in Assumption (ii). These assumptions are different from Assumptions 2 and 3 that are used by Giles and Goda [14] in a similar context.

With their assumptions, they obtain a bias in $O(1/K^{1-\delta})$ for any arbitrary $0 < \delta < 1$. Here, we directly see the link between the integrability assumption and the bias in $O(1/K^{1+\frac{\rho}{2}})$. Besides, let us note that we do need to assume the boundedness of any moments of $Y^p \phi(X)$, $p \in \{1, \ldots, P\}$ (Assumption 1 of [14]) since we are using a different approach that does not make use of the Burkholder-Davis-Gundy inequality.

Lemma 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\eta \in (0, 1]$. Besides the random variable $X$, we consider real valued random variables $\tilde{\theta}_K^i$ and functions $\varphi_i(X)$, with $i \in \{1, 2\}$ that satisfy the following conditions:

(i) $\tilde{\theta}_K^i \xrightarrow[\infty]{} \varphi_i(X)$ a.s.

(ii) There are nonnegative measurable functions $C_i$ and $\sigma_i^2$ such that for all $K \in \mathbb{N}^*$:

$$\left| \mathbb{E} \left[ \tilde{\theta}_K^i - \varphi_i(X) | X \right] \right| \leq \frac{C_i(X)}{K^{\frac{1+\eta}{\eta}}}, \tag{9}$$

$$\mathbb{E} \left[ \left( \tilde{\theta}_K^i - \varphi_i(X) \right)^2 | X \right] \leq \frac{\sigma_i^2(X)}{K}. \tag{10}$$

(iii) Setting $\varphi_2(X) = \varphi_2(X) - \varphi_1(X)$, we have $\mathbb{P}(\{|\varphi_2(X)| = 0\}) = 0$.

Then, we have with

$$C(X) = \mathbb{1}_{\varphi_2(X) < 0} C_1(X) + \mathbb{1}_{\varphi_2(X) > 0} C_2(X) + 2^p \sigma_1^{1+\eta}(X) + \sigma_2^{1+\eta}(X), \tag{11}$$

$$\sigma^2(X) = \sigma_1^2(X) + \sigma_2^2(X), \tag{12}$$

the following estimates:

$$\left| \mathbb{E} \left[ \max \{\tilde{\theta}_K^1, \tilde{\theta}_K^2\} - \max \{\varphi_1(X), \varphi_2(X)\} | X \right] \right| \leq \frac{C(X)}{K^{\frac{1+\eta}{\eta}}}, \tag{13}$$

$$\mathbb{E} \left[ \left( \max \{\tilde{\theta}_K^1, \tilde{\theta}_K^2\} - \max \{\varphi_1(X), \varphi_2(X)\} \right)^2 | X \right] \leq \frac{\sigma^2(X)}{K}. \tag{14}$$
Proof of Theorem 1. We first prove by induction on $P \geq 2$ that

\[ \left| \mathbb{E} \left( \hat{M}_K^P - M_X^P \right| X \right| \leq \frac{C_P(X)}{K^{1+\eta}} \text{ and } \mathbb{E} \left( \left( \hat{M}_K^P - M_X^P \right)^2 | X \right) \leq \frac{\Sigma_P^2(X)}{K}. \]  

(15)

We apply Lemma 2 noticing that $\mathbb{E}[\hat{M}_K^p - E_X^p | X] = 0$ and $\mathbb{E}[(\hat{M}_K^p - E_X^p)^2 | X] = \frac{\sigma^2_p(X)}{K}$ for $p \in \{1, \ldots, P\}$. First, this gives the result for $P = 2$. Second, with the induction hypothesis for $P$, Lemma 2 gives that (15) is satisfied for $P + 1$ with

\[ C_{P+1}(X) = C_P(X) + 2^\eta \frac{\Sigma_{P+1}^1(X) + \sigma_{P+1}(X)}{|E_{K+1}^P - M_X|^{1+\eta}} \text{ and } \Sigma_{P+1}^2(X) = \Sigma_P^2(X) + \sigma_{P+1}(X), \]

which gives the claim.

Since $\text{bias}(\hat{I}_{J,K}) = \mathbb{E} \left[ (\hat{M}_K^P - M_X^P) \phi(X) \right]$, we get $|\text{bias}(\hat{I}_{J,K})| \leq \frac{\mathbb{E}[\phi(X)]}{C_P(X)} = \frac{C_P(X)}{K}$. Similarly, we have

\[ \text{Var}(\hat{I}_{J,K}) = \frac{1}{J} \sum_{l=0}^J \text{Var}(\hat{M}_l^P \phi(X)) \leq \frac{2}{J} \text{Var} \left[ \left( \hat{M}_K^P - M_X^P \right) \phi(X) \right] + \frac{2}{J} \text{Var}[\hat{M}_X^P \phi(X)] \]

\[ \leq \frac{2}{J} \mathbb{E} \left[ \left( \hat{M}_K^P - M_X^P \right)^2 \phi^2(X) \right] + \frac{2}{J} \text{Var}[\hat{M}_X^P \phi(X)], \]

which leads to (8).

Last, we notice that for $c_1, c_2 > 0$, the minimization of $J K \frac{c_1}{K^{1+\eta}} + \frac{c_2}{J} = \varepsilon^2$ leads to $J = \frac{c_1}{(1+\eta)c_2} K^{1+\eta}$ and thus $K = O(\varepsilon^{-\frac{1}{1+\eta}})$ and $J = O(\varepsilon^{-2})$. Since $\frac{1}{J} \leq \frac{1}{J} \leq \frac{1}{J} \leq \frac{1}{J}$, this choice is asymptotically optimal: it gives $MSE(\hat{I}_{J,K}) = O(\varepsilon^2)$ with a computational cost in $O(\varepsilon^{-3-\frac{1}{1+\eta}})$.

2.2 The Multilevel Monte-Carlo estimator

We now present the Multilevel Monte-Carlo (MLMC) estimator of $I$. We consider $L \in \mathbb{N}$ that represents the number of levels. Let $J_0, \ldots, J_L \in \mathbb{N}^*$ and $K_0, \ldots, K_L \in \mathbb{N}^*$ be such that

\[ \forall l \in \{1, \ldots, L\}, \ K_l = K_0 2^l. \]  

(16)

For each level $l \in \{0, \ldots, L\}$, we consider $(X_{l,j}, 1 \leq j \leq J_l)$ i.i.d. random variables having the same distribution as $X$, and random variables $(Y_{l,j,k}, 1 \leq j \leq J_l, 1 \leq k \leq K_l)$ that are independent given $(X_{l,j}, 1 \leq j \leq J_l)$ and such that $Y_{l,j,k}$ follows the distribution of $Y$ given $X = X_{l,j}$. These random variables are assumed to be independent between levels, i.e. $(X_{l,j}, Y_{l,j,k}, 1 \leq j \leq J_l, 1 \leq k \leq K_l)_{l=0, \ldots, L}$ are independent. Then, we define for $l \in \{0, \ldots, L\}$ and $p \in \{1, \ldots, P\}$:

\[ \hat{E}_{l,j,K}^p = \frac{1}{K} \sum_{k=1}^K Y_{l,j,k}^p, K \in \{1, \ldots, K_l\} \]  

(17)

\[ \hat{M}_{l,j,K}^p = \max(\hat{E}_{l,j,K}^p, \ldots, \hat{E}_{l,j,K}) \]  

(18)

Then, the MLMC estimator of $I$ is defined by

\[ \hat{I}_{MLMC} = \frac{1}{J_0} \sum_{l=1}^{J_0} \hat{M}_{l,0,K_0}^p \phi(X_{0,j}) + \frac{1}{J_1} \sum_{l=1}^{J_1} \sum_{l=1}^{J_1} (\hat{M}_{l,j,K_l}^p - \hat{M}_{l,j,K_l-1}^p) \phi(X_{l,j}). \]  

(19)

Let us assume that the assumptions of Theorem 1 hold. We have bias($\hat{I}_{MLMC}$) = $\mathbb{E} \left[ (\hat{M}_K^P - M_X^P) \phi(X) \right] = O(K_l^{-1+\eta}) = O(2^{-\frac{1+\eta}{2} L})$. Besides, we have

\[ \text{Var} \left( \left( \hat{M}_{K_l}^P - \hat{M}_{K_l-1}^P \right) \phi(X) \right) \leq 2 \text{Var} \left( \left( \hat{M}_{K_l}^P - M_X^P \right) \phi(X) \right) + 2 \text{Var} \left( \left( \hat{M}_{K_l-1}^P - M_X^P \right) \phi(X) \right) = O(K_l^{-1}) = O(2^{-1}) \]

and the computational cost of $(\hat{M}_{l,j,K_l}^p - \hat{M}_{l,j,K_l-1}^p) \phi(X_{l,j})$ is $O(K_l) = O(2^l)$. We can thus apply Theorem 1 [13], which leads to the following result.
Proposition 3. Let us assume that the assumptions of Theorem 1 hold for some \( \eta \in (0, 1) \). Then, by taking when \( \varepsilon \to 0 \)

\[
L = \left\lfloor \frac{2}{1 + \eta} \frac{\varepsilon}{\log(2)} \right\rfloor, \quad J_0 = 2 \left\lfloor \frac{2}{1 + \eta} \frac{|\log(\varepsilon)|}{\log(2)} \right\rfloor = O(\varepsilon^{-2} \log(\varepsilon)) \quad \text{and} \quad J_l = J_0 2^{-l}, l \in \{1, \ldots L\},
\]

we have \( \text{MSE}(\hat{P}_{\text{LMC}}) = \mathbb{E}[\hat{P}_{\text{LMC}} - P]^2 = O(\varepsilon^2) \) with a computational cost in \( O(\varepsilon^{-2} \log^2(\varepsilon)) \).

If only the assumption \( \Sigma^2 < \infty \) of Theorem 1 holds, the same conclusion holds by taking \( \eta = 0 \) in (20).

Proof. We just check that the parameters achieve the claim. From the bias-variance decomposition, we get by using Theorem 1, (16) and (20) that there is a positive constant \( C \) such that

\[
\text{MSE}(\hat{P}_{\text{LMC}}) \leq C \left( \frac{1}{K^1_{K_0} \cdot \varepsilon} + \frac{1}{J_0} + \sum_{l=1}^L \frac{1}{J_i K_i} \right) = C \left( \frac{2 - (1 + \eta) L}{K^1_{K_0} \cdot \varepsilon} + \frac{1}{J_0} + \frac{L}{J_0 K_0} \right).
\]

The choice of \( L \) gives \( 2^{-(1+\eta)L} \leq \varepsilon^2 \) and the choice of \( J_0 \) then gives \( \frac{L}{J_0} = O(\varepsilon^2) \). Last the computational cost is given by \( \sum_{l=0}^L J_i K_i = L J_0 K_0 = O(\varepsilon^{-2} \log^2(\varepsilon)) \). In the case where we only know \( \Sigma^2 < \infty \), only the second statement of Equation (7) holds, and we get

\[
\left| \mathbb{E} \left[ \left( \hat{M}_{K_0}^p - M_X^p \right) \phi(X) \right] \right| \leq \sum_{\sqrt{K_L}} \sum_{\sqrt{K_0}} 2^{-L/2},
\]

which gives the second claim with the same arguments.

Remark 4. Let us note that the analysis of the computational cost gives that it is asymptotically bounded by \( C \varepsilon^{-2} \log^2(\varepsilon) \) for some constant \( C > 0 \), but it does not analyse precisely this constant. Nonetheless, since this cost is \( L J_0 K_0 \), this constant can be chosen to be proportional to the number of levels.

Thus, the analysis of the bias given by Theorem 1 under the integrability assumption \( \mathbb{E}[C_P(X)|\phi(X)|] < \infty \) enables to reduce the number of levels and then to reduce this constant.

It is however possible to construct a better estimator using the MLMC antithetic estimator

\[
\hat{P}_{\text{LMC}} = \frac{1}{J_0} \sum_{j=1}^J \hat{M}^p_{0, j, K_0} \phi(X_{0,j}) + \sum_{l=1}^L \frac{1}{J_l} \sum_{j=1}^{J_l} \left( \hat{M}^p_{l, j, K_l} - \frac{\hat{M}^p_{l, j, K_l-1}}{2} \right) \phi(X_{l,j}), \quad (21)
\]

where we set for \( p \in \{1, \ldots, P\} \),

\[
\hat{P}^p_{l, j, K_{l-1}} = \frac{1}{K_{l-1}} \sum_{k=K_{l-1}+1}^{K_l} Y^p_{l, j, k} \text{ and } \hat{M}^p_{l, j, K_{l-1}} = \max(\hat{E}^p_{l, j, K_{l-1}}, \ldots, \hat{E}^p_{l, j, K_{l-1}}). \quad (22)
\]

This is a rather natural idea to reduce the variance contribution of each level, see Section 9.1 of [13]. However, the irregularity of the maximum function makes the analysis of the variance more delicate as if it were a smooth function. Giles and Goda [14] give an analysis of the variance that require again the boundedness of any moments of \( Y^p \phi(X) \), \( p \in \{1, \ldots, P\} \) (Assumption 1 of [14]) and assumptions to control the probability that another component is close to the maximum (Assumptions 2 and 3 of [14]). Here, our proof relies on a different argument and only requires a moment condition that quantifies in a different way the probability that two or more arguments in the maximum are close to the maximum. Details are in the Appendix (see Proposition 12).

Remark 5. For the calculation of (1) with a general function \( h \), the antithetic MLMC estimator is defined by

\[
\frac{1}{J_0} \sum_{j=1}^J h(\hat{E}^1_{0, j, K_0}, \ldots, \hat{E}^P_{0, j, K_0}) \phi(X_{0,j}) + \sum_{l=1}^L \frac{1}{J_l} \sum_{j=1}^{J_l} \left( h(\hat{E}^1_{l, j, K_l}, \ldots, \hat{E}^P_{l, j, K_l}) - \frac{h(\hat{E}^1_{l, j, K_{l-1}}, \ldots, \hat{E}^P_{l, j, K_{l-1}})}{2} \right) \phi(X_{l,j}).
\]

In particular, it is possible to estimate by MLMC the value of (1) for different functions \( f \) with the same simulations.
Theorem 4. Let $\eta \in (0, 1]$. We assume that the assumptions of Theorem 1 hold and besides that
\[ \forall p \in \{2, \ldots, P\}, \ E \left[ \frac{D_{2+\eta}(X)}{|E_X - M_X|^{1+\eta}} \phi^2(X) \right] < \infty, \]
where $D_{2+\eta}(X) = E[|Y^p - E[Y^p|X]|^{2+\eta}|X]$. Then, by taking when $\varepsilon \to 0$
\[ L = \left[ \frac{2}{1+\eta} \left| \log(\varepsilon) \right| \right], \ J_0 = \left[ \frac{2 \left| \log(\varepsilon) \right|}{\log(\varepsilon)} \right] = O(\varepsilon^{-2}) \text{ and } J_l = \left[ J_0 2^{-(1+\frac{2}{l})} \right], l \in \{1, \ldots, L\}, \]
we have $MSE(\hat{I}_A^{\text{MLMC}}) = E[\hat{I}_A^{\text{MLMC}} - I]^2] = O(\varepsilon^2)$ with a computational cost in $O(\varepsilon^{-2})$.

Proof. We have $\text{bias}(\hat{I}_A^{\text{MLMC}}) = \text{bias}(\hat{I}_A^{\text{MLMC}}) = O(2^{-\frac{1+4}{2}L})$. By Proposition 12, the variance of each level satisfies
\[ \text{Var} \left( \frac{\hat{M}_{K_l}^p - \hat{M}_{K_l-1}^p + \hat{M}_{K_l}^p}{2} \right) = O(K^{-1+\frac{2}{l}}) = O(2^{-l(1+\frac{2}{l})}), \]
and the computational cost of $\left( \frac{\hat{M}_{K_l}^p - \hat{M}_{K_l-1}^p + \hat{M}_{K_l}^p}{2} \right) \phi(X)$ is in $O(K_l) = O(2^l)$. We are thus in the framework of Theorem 1 of [13], and we just check that the choice of parameters (23) gives the claim. By using the bias variance decomposition, we have
\[ MSE(\hat{I}_A^{\text{MLMC}}) \leq C \left( 2^{-(1+\eta)L} + 1 + \sum_{l=1}^{L} \frac{1}{J_l K_l^{1+\frac{2}{l}}} \right) \leq C \left( \varepsilon^2 + \varepsilon^2 \sum_{l=0}^{L} 2^{-\frac{2}{l}} \right). \]
Since $\sum_{l=0}^{L} 2^{-\frac{2}{l}} \leq \sum_{l=0}^{\infty} 2^{-\frac{2}{l}} = \frac{1}{2-\frac{1}{2}}$, we indeed have $MSE(\hat{I}_A^{\text{MLMC}}) = O(\varepsilon^2)$. Observing that for $\varepsilon \in \mathbb{R}_+$ small enough, we have $J_0 2^{-(1+\frac{2}{l})} \geq 1$ and thus $J_l \leq 2J_0 \times 2^{-(1+\frac{2}{l})}$ for $l \in \{0, \ldots, L\}$, we can upper bound the computational cost as follows
\[ \sum_{l=0}^{L} J_l K_l \leq 2J_0 K_0 \sum_{l=0}^{L} 2^{-\frac{2}{l}} \leq \frac{4K_0 \varepsilon^{-2}}{1-2^{-\frac{1}{2}}}. \]

Remark 6. We can easily extend Theorem 4 if we assume that the assumption of Theorem 1 is true for some $\eta_1 \in (0, 1]$ and that
\[ \forall p \in \{2, \ldots, P\}, \ E \left[ \frac{D_{2+\eta_2}(X)}{|E_X - M_X|^{1+\eta_2}} \phi^2(X) \right] < \infty, \]
for some $\eta_2 > 0$. If we then take
\[ L = \left[ \frac{2}{1+\eta_1} \left| \log(\varepsilon) \right| \right], \ J_0 = \left[ \frac{2 \left| \log(\varepsilon) \right|}{\log(\varepsilon)} \right] = O(\varepsilon^{-2}) \text{ and } J_l = \left[ J_0 2^{-(1+\frac{\eta_2}{2})} \right], l \in \{1, \ldots, L\}, \]
we get in the same way that $MSE(\hat{I}_A^{\text{MLMC}}) = O(\varepsilon^2)$ with a computational cost in $O(\varepsilon^{-2})$. However, roughly speaking, the integrability assumption of Theorem 1 for the bias deals with the integrability of $\frac{1}{|E_X - M_X|^{1+\eta}}$ when $|E_X - M_X|$ is close to 0, similarly as the assumption for the variance estimate. Thus, it is rather natural to consider $\eta_1 = \eta_2$, and we state Theorem 4 in this case for sake of simplicity.

2.3 Least-Square Monte Carlo techniques for Nested Expectations

In this paragraph, we aim at presenting briefly the classical technique of regression in our context, i.e. for the calculation of $I$. For simplicity, we only consider here regressors that are indicator functions.
Let $N_r \in \mathbb{N}^*$ be the number of regressors. We consider $B_1, \ldots, B_{N_r} \in \mathcal{G}$ disjoint measurable sets of the space where $X$ takes values, and we define for $n \in \{1, \ldots, N_r\}$ and $p \in \{1, \ldots, P\}$,

$$
\alpha_n^p = \mathbb{E}[Y^p | X \in B_n] = \frac{\mathbb{E}[Y^p \mathbb{1}_{X \in B_n}]}{\mathbb{P}(X \in B_n)}
$$

(with the convention $0/0 = 0$).

Then we have

$$
\forall p \in \{1, \ldots, P\}, \mathbb{E} \left[ \left( Y^p - \sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n} \right)^2 \right] = \min_{\alpha_1, \ldots, \alpha_{N_r} \in \mathbb{R}} \mathbb{E} \left[ \left( Y^p - \sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X \in B_n} \right)^2 \right],
$$

i.e. $\sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n}$ is the $L^2$ projection of $Y^p$ on $\{\sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X \in B_n} : \alpha_1, \ldots, \alpha_{N_r} \in \mathbb{R}\}$. It is a natural proxy of $E_X^p$, which is the $L^2$ projection on the larger space of $\sigma(X)$-measurable random variables. We then define $\gamma_n^p = \max_{p=1, \ldots, P} \alpha_n^p$, so that $\sum_{n=1}^{N_r} \gamma_n^p \mathbb{1}_{X \in B_n}$ approximates $M^p_X$.

Let us consider $(X_j, Y_j)_{1 \leq j \leq J}$ an i.i.d. sample following the distribution of $(X, Y)$. We define

$$
\hat{\gamma}_{n,j}^p = \frac{\sum_{j=1}^{J} Y_j^p \mathbb{1}_{X_j \in B_n}}{\sum_{j=1}^{J} \mathbb{1}_{X_j \in B_n}}
$$

(with the same convention $0/0 = 0$)

and have similarly

$$
\forall p \in \{1, \ldots, P\}, \frac{1}{J} \sum_{j=1}^{J} \left( Y_j^p - \sum_{n=1}^{N_r} \hat{\gamma}_{n,j}^p \mathbb{1}_{X_j \in B_n} \right)^2 = \min_{\alpha_1, \ldots, \alpha_{N_r} \in \mathbb{R}} \frac{1}{J} \sum_{j=1}^{J} \left( Y_j^p - \sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X_j \in B_n} \right)^2.
$$

We define $\hat{\gamma}_{n,j}^p = \max_{p=1, \ldots, P} \hat{\gamma}_{n,j}^p$, so that $\sum_{n=1}^{N_r} \hat{\gamma}_{n,j}^p \mathbb{1}_{X_j \in B_n}$ approximates $M^p_{X,j}$. Thus, we define the Least Square Monte-Carlo estimator of $I$ by

$$
\hat{I}_{LSMC}^{p} = \frac{1}{J} \sum_{j=1}^{J} \phi(X_j) \sum_{n=1}^{N_r} \hat{\gamma}_{n,j}^p \mathbb{1}_{X_j \in B_n}.
$$

(24)

We are interested in estimating the MSE of this estimator. The next proposition gives a framework to analyse it, which is useful to determine asymptotically the number of regressors and the number of Monte-Carlo samples to reach a given precision $\varepsilon > 0$.

**Proposition 5.** For $p = 1, \ldots, P$, we set $\sigma_p(x) = \text{Var}(Y^p | X = x)$, and assume that there exists $\sigma, \overline{\sigma} \in \mathbb{R}^*_+$ such that for all $x \in G$, $\sigma_p^2(x) \leq \overline{\sigma}^2$ and $|\phi(x)| \leq \overline{\phi}$. Then, we have

$$
\mathbb{E}[|\hat{I}_{LSMC}^{p} - I|^2] \leq 2\overline{\phi}^2 \left( \frac{\overline{\sigma}^2 N_r P + \mathbb{E}[|M^p_X|^2]}{J} + \sum_{p=1}^{P} \mathbb{E} \left[ \left( E_X^p - N_r \alpha_n^p \mathbb{1}_{X \in B_n} \right)^2 \right] \right).
$$

We now suppose in addition that:

1. $G = [0, 1]^d$, $N_r = n_r^d$ and for any $n \in \{1, \ldots, N_r\}$,

$$
B_n = \left[ \frac{i_1}{n_r}, \frac{i_1 + 1}{n_r} \right) \times \cdots \times \left[ \frac{i_d}{n_r}, \frac{i_d + 1}{n_r} \right).
$$

where $i_1, \ldots, i_d \in \{0, \ldots, n_r - 1\}$ are the unique integers determined by the decomposition in base $n_r$ of $n - 1$, i.e. $n - 1 = \sum_{k=1}^{d} i_k n_r^{k-1}$,

2. for any $p \in \{1, \ldots, P\}$, the function $G \ni x \mapsto E_X^p = \mathbb{E}[Y^p | X = x]$ is Lipschitz continuous with constant $L$ for the $\| \cdot \|_\infty$ norm on $\mathbb{R}^d$. 


Then, we have
\[ E[(\hat{I}_{LSMC} - I)^2] \leq 2\varphi^2 \left( \frac{\sigma^2 N \rho}{J} + \frac{PL^2}{N_r^{2/d}} \right). \]

With this upper bound, taking \( J \sim c\varepsilon^{-d-2} \) and \( N_r \sim c'\varepsilon^{-d} \), for some constants \( c, c' > 0 \), is an asymptotic optimal choice to have \( E[(\hat{I}_{LSMC} - I)^2] = O(\varepsilon^2) \), with an overall computational cost in \( O(\varepsilon^{-d-2}) \).

In comparison with the MLMC estimator, it is worth to notice that \( \hat{I}_{LSMC} \) suffers from the curse of dimensionality. The larger is the dimension of \( G \) (the space where \( X \) takes values), the more it requires computational effort. As we will see, for the problem of the calculation of the SCR for ALM management, this is particularly detrimental.

**Proof.** From the definition of \( \hat{I}_{LSMC} \) (24), we get by using \((a + b)^2 \leq 2a^2 + 2b^2\) and Jensen’s inequality:
\[
E[(\hat{I}_{LSMC} - I)^2] = E \left[ \left( \hat{I}_{LSMC} - \frac{1}{J} \sum_{j=1}^{J} \phi(X_j) M_{X_j}^P + \frac{1}{J} \sum_{j=1}^{J} \phi(X_j) M_{X_j}^P - I \right)^2 \right]
\leq E \left[ \frac{2}{J} \sum_{j=1}^{J} \phi^2(X_j) \left( \sum_{n=1}^{N_r} \hat{\gamma}_{n,j} \mathbb{1}_{X_j \in B_n} - M_{X_j}^P \right)^2 \right] + 2 \text{Var}(\phi(X)M_{X_j}^P)\]
\leq 2\varphi^2 \left( \frac{1}{J} \sum_{j=1}^{J} \left( \sum_{n=1}^{N_r} \hat{\gamma}_{n,j} \mathbb{1}_{X_j \in B_n} - M_{X_j}^P \right)^2 + \frac{E[(M_{X_j}^P)^2]}{J} \right).

Now, Theorem 8.2.4 [17] gives
\[
E \left[ \frac{1}{J} \sum_{j=1}^{J} \left( E_{X_j}^p \mathbb{1}_{X_j \in B_n} - \sum_{n=1}^{N_r} \hat{\alpha}_{n,j} \mathbb{1}_{X_j \in B_n} \right)^2 \right] \leq \frac{\sigma^2 N_r}{J} + E \left[ \left( E_X^P - \sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{x \in B_n} \right)^2 \right].
\]

We recall that \( \hat{\gamma}_{n,j} = \max_{p=1, \ldots, P} \hat{\alpha}_{n,j} \) and observe that for \( X_j \in B_n \), we have
\[
(M_{X_j}^P - \max_{p=1, \ldots, P} \hat{\alpha}_{n,j})^2 \leq \max_{p=1, \ldots, P} (E_{X_j}^P - \hat{\alpha}_{n,j})^2 \leq \sum_{p=1}^{P} (E_{X_j}^P - \hat{\alpha}_{n,j})^2
\]
since \( \max_{p=1, \ldots, P} a_p - \max_{p=1, \ldots, P} b_p \leq \max_{p=1, \ldots, P} |a_p - b_p| \) for any \( a, b \in \mathbb{R}^P \). This gives the first upper bound.

We now consider the case \( G = [0, 1]^d \) with the related assumptions. Then, for \( X \in B_n \), we have for any \( p \)
\[
|E_{X}^P - \alpha_n^P| = \left| E_{X}^P - \int_{x \in B_n} E_{X}^P \mathbb{P}(X \in dx | X \in B_n) \right| \leq \int_{x \in B_n} |E_{X}^P - E_{x}^P| \mathbb{P}(X \in dx | X \in B_n) \leq \frac{L}{n_r} = \frac{L}{N_r^{1/d}},
\]
since \( \|X - x\|_\infty \leq \frac{1}{n_r} \) for \( X, x \in B_n \). This gives the second bound. To have this upper bound smaller than \( C\varepsilon^2 \) for some constant \( C > 0 \), one must at least have \( N_r \geq c_1 \varepsilon^{-d} \) and \( J \geq c_2 N_r \varepsilon^{-2} \) for some constants \( c_1, c_2 > 0 \), which leads to take \( N_r \sim c'\varepsilon^{-d} \) and \( J \sim \varepsilon^{-d-2} \).

Last, we observe that the computational cost to find \( n \) such that \( x \in B_n \) is constant since \( i_k = |n_r x_k| \) and \( n = 1 + \sum_{k=1}^{d} i_k n_r^{-k-1} \). Therefore, computing all the \( 2N_r \) sums \( \sum_{j=1}^{J} Y^P_j \mathbb{1}_{X_j \in B_n} \) and \( \sum_{j=1}^{J} \mathbb{1}_{X_j \in B_n} \) that define can be achieved with a computational cost of \( O(J) \), and the calculation of (24) costs similarly \( O(J) \). Since \( J \sim c_3 \varepsilon^{-d-2} \), we get the claim. \( \square \)
2.4 Numerical results on a toy example: the Butterfly Call Option with the Black-Scholes model

The goal of this section is to illustrate the theoretical results on a simple case where the conditional expectations are known explicitly. Thus, we consider an asset following the Black-Scholes model:

\[ S_t = S_0 \exp \left( \sigma W_t - \frac{\sigma^2 t}{2} \right), \quad t \geq 0, \]

where \( W \) is a standard Brownian motion and \( \sigma > 0 \) is the volatility. We consider a butterfly option with payoff at time \( T > 0 \):

\[ \psi(S_T) = (S_T - K_1)^+ + (S_T - K_2)^+ - 2 \left( S_T - \frac{K_1 + K_2}{2} \right)^+, \]

where \( 0 < K_1 < K_2 \). The price of this butterfly option at time \( t \in [0, T] \) is given by

\[ \mathbb{E}[\psi(S_T)|S_t] = \text{Call}^{BS}(T - t, S_t, K_1) + \text{Call}^{BS}(T - t, S_t, K_2) - 2 \text{Call}^{BS}(T - t, S_t, K_1 + K_2) =: \text{Butterfly}(T - t, S_t), \]

with \( \text{Call}^{BS}(t, s, K) = s \mathcal{N}(\frac{1}{\sqrt{t}} \ln(s/K) + \frac{\sigma}{2} \sqrt{t}) - K \mathcal{N}(\frac{1}{\sqrt{t}} \ln(s/K) - \frac{\sigma}{2} \sqrt{t}) \), where \( \mathcal{N} \) is the cumulative distribution function of the standard normal distribution.

Now, we consider multiplicative upward and downward shocks \( s^{up/down} \) that occur instantaneously at time \( t \), we want to compute the worst loss between these shocks when it is positive. Since the Black-Scholes model is multiplicative with respect to the spot value, this shocks amounts to multiply the asset by \((1 \pm s^{up/down})\). Hence, setting \( X = S_t \), \( Y^1 = (\psi(S_T) - \psi((1 + s^{up})S_T)) \) and \( Y^2 = (\psi(S_T) - \psi((1 + s^{down})S_T)) \) we want to compute the following quantity:

\[ I = \mathbb{E} \left[ \max \left\{ \mathbb{E}[Y^1|X], \mathbb{E}[Y^2|X], 0 \right\} \right]. \]

We are thus indeed in our general framework with \( P = 3 \) and \( Y^3 = 0 \) and \( \phi(x) = 1 \), and we have

\[ M^3_X = \max \{ \text{Butterfly}(T - t, (1 + s^{up})X), \text{Butterfly}(T - t, (1 + s^{down})X), 0 \}. \]

Since \( X \) follows a log-normal distribution, the exact value of \( I \) can be thus obtained by numerical integration.

**Numerical values.** In all our numerical experiments, we consider the initial price \( S_0 = 100 \), the volatility \( \sigma = 0.3 \), the strikes \( K_1 = S_0 + a \) and \( K_2 = S_0 - a \) with \( a = 50 \), the option maturity \( T = 2 \) years and perform the shocks at \( t = 1 \) year. In our tests, we take \( s^{up} = 0.2 \) and \( s^{down} = -0.2 \).

Figure 1 illustrates the bias \( \mathbb{E}[M^3_X - M^3_X] \) in function of \( K \) with a log-scale. The expectation is approximated by the Nested Monte-Carlo with \( J = 10^3 \) to get a negligible statistical error. As a comparison, the function \( K \mapsto 1/K \) is drawn, and we observe that two curves are quite parallel, which indicates that the bias behaves asymptotically like \( c/K \). Also, we have drawn in Figure 2 the variance of \( \hat{M}^3_{K_1} - \frac{\hat{M}^3_{K_1 - 1} + \hat{M}^3_{K_1 + 1}}{2} \) in function of \( K_1 \), and we observe a behaviour in \( K_1^{-3/2} \). Thus, it is reasonable to apply then the Multilevel method with \( \eta = 1 \) to determine the parameters in Equation (23). We have drawn in Figure 3 the RMSE in function of the computational cost (defined by \( \sum_{l=0}^L J_k K_l \)) for different values of \( \eta \leq 1 \). We observe a behaviour in \( \varepsilon^{-2} \), which is in line with Theorem 4. The RMSE is calculated empirically by running many times the Multilevel method.

We now present the implementation of LSMC estimator. We note that \( X' = \frac{1}{\sigma} \log(X) + \sigma/2 \) is a standard normal distribution and therefore takes with a probability greater than 99% its values in \([-3, 3]\). We notice that \( \mathbb{E}[Y^p|X] = \mathbb{E}[Y^p|X'] \) and take the following regressors

\[ 1_{X' \in [-3+6j/N_r, -3+6(j+1)/N_r]}, \quad j = 0, \ldots, N_r - 1. \]

Up to a translation, we are thus in the framework of Proposition 5. In Figure 4, we have plotted the RMSE as a function of the number of samples \( J \), which is also the computational cost of the method. The behaviour is in line with the theoretical result given by Proposition 5.

We already see on this one-dimensional example that the MLMC estimator has some benefit in terms of convergence with respect to the LSMC estimator. As we will see in the next section for the \( SCR \) estimation, this benefit is much more important when \( X \) takes values in a high-dimensional space.
3 Calculation of the SCR with the Standard Formula in an ALM model

In this section, we want to illustrate how to compare the MLMC and LSMC methods on a more realistic example for the application in insurance. Namely, we consider the case of Asset Liability Management (ALM) for life insurance contracts and are interested in the calculation of the SCR with the standard formula after $t$ years. This example is of practical interest and the conditional expectations that are at stake are typically high-dimensional. In fact, the process that determines the strategy is really path-dependent and involves book values, market values, crediting rates, etc. Here, we will use the recent ALM model that we have developed in [1], and we focus on the interest rate module of the SCR with the standard formula.
3.1 The ALM model in a nutshell

In this section, we briefly present the ALM model developed in [1] and refer to this paper for the full details. We consider an insurance company that handles a life insurance, namely a General Account guaranteed with participation contracts. We consider a runoff portfolio with an initial Mathematical Reserve $MR_0$ corresponding to policyholders’ deposit. The Capitalization Reserve (a buffer for capital gains on bonds imposed by the French legislation) and the Profit Sharing Reserve (a buffer for capital gains on stocks to smooth the crediting rate) are empty at time 0, i.e. $CR_0 = 0$ and $PSR_0 = 0$. At time 0, the insurance company invests $MR_0$ in two asset classes, stocks and riskless bonds, with respective weights $w^S_0 \in [0,1]$ and $w^b_0 = 1 - w^S_0$. Thus the initial Market Value and Book Value in stock (resp. in bonds) is given by

$$MV^S_0 = BV^S_0 = w^S_0 MR_0 \quad \text{(resp.} \quad MV^b_0 = BV^b_0 = w^b_0 MR_0).$$

During all the ALM strategy, the insurance company invests in an equity asset $(S_t)_{t \geq 0}$ that may be a stock index or more generally an average of stocks with weights corresponding to the investment of the insurance company. It therefore has $\phi^S_0 = MV^S_0 / w^S_0$ equity assets at time 0. The insurance company also invests in bonds, and we assume that this investment is made with an equally weighted portfolio of bonds with maturities $1, \ldots, n$. We introduce some notation to precise this: we denote $P(t,t+i)$ the price at time $t$ of a Zero-Coupon bond at time $t$ with maturity $t+i$, $B(t, n, c) = \sum_{i=1}^n c P(t, t+i) + P(t, t+n)$ the price at time $t$ of a bond with constant coupon $c \in \mathbb{R}$, unit nominal value and maturity $n$ and for $c = (c^i)_{i \in \{1, \ldots, n\}} \in \mathbb{R}^n$ we denote by

$$\bar{B}(t, n, c) = \frac{1}{n} \sum_{i=1}^n B(t, i, c^i)$$

the value of an equally weighted portfolio of bonds with coupons $c$. During all the ALM strategy, we assume that bonds are bought at par with the swap rate $c_{\text{swap}}(t, n) = \frac{1-P(t+1)}{\sum_{i=1}^n P(t+1)}$. We set $c_0 = (c_{\text{swap}}(0, i))_{i \in \{1, \ldots, n\}}$ and have $\bar{B}(0, n, c_0) = 1$. At time 0, the insurance company has then $\phi^b_0 = MV^b_0 = MV^b_0 / \bar{B}(0, n, c_0)$ assets $\bar{B}(0, n, c_0)$.

We assume that the portfolio is handled up to time $T \in \mathbb{N}^*$ and that it is static on each period $(t-1, t)$, $t \in \{1, \ldots, T\}$. At each time $t$, it is reallocated in such a way to have at the end of the reallocation

$$\phi^S_t = \frac{w^S_t MV_t}{S_t}, \quad \phi^b_t = \frac{w^b_t MV_t}{\bar{B}(t, n, c_t)}$$

quantities of equity assets and bonds, where $MV_t$ denotes the market value of the portfolio at time $t$ and $w^S_t = 1 - w^b_t \in [0,1]$ is the target weight decided for ALM strategy. The coupons $c_t \in \mathbb{R}^n$ are determined by the reallocation procedure that we describe now and takes into account the specificities of life insurance contracts. We decompose this reallocation in five steps:

1. Calculation of the cash inflows and book value movements related to the bonds. Since the portfolio composition is unchanged on $(t-1, t)$, the insurer receives $\frac{\phi^b_t}{n} (1 + \sum_{i=1}^n c^i_{t-1})$ corresponding to the nominal value of the expiring bonds and the coupons. The value of the matured bonds $\frac{\phi^b_t}{n}$ is removed from the book value of bonds $BV^b_t$.

2. Payment of the policyholders that exit their contract. The proportion of policyholders that exit on $(t-1, t]$ is denoted by $p^{t}_{t-1}$. It is modelled as the sum of a deterministic part related to the relevant life table and of a dynamic part modelling surrenders $DSR(\Delta_{t-1}) = DSR_{\text{max}} \mathbb{1}_{\Delta_{t} \leq \alpha} + DSR_{\text{max}} \frac{\beta - \Delta_{t}}{\beta - \alpha} \mathbb{1}_{\alpha < \Delta_{t} < \beta}$, where $\Delta_{t-1}$ is the difference between the crediting rate to policyholders $r_{ph}(t-1)$ and a competitor rate $r_{t-1}^{\text{comp}}$. We assume that policyholders exit uniformly on $(t-1, t]$ and the amount to pay is thus $p^{t}_{t-1} MR_{t-1}(1 + r_G/2)$, where $r_G$ is the minimum guaranteed rate. This means that they are remunerated with this rate on the last period.

3. Reallocation step. At this step, the market value of the portfolio is given by

$$MV_t = G_t + \phi^S_{t-1} S_t + \frac{\phi^b_{t-1}}{n} \sum_{i=1}^{n-1} B(t, i, c^i_{t-1}),$$

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where $G_t$ is the liquidity gap that corresponds to the difference between the cash inflows and outflows of the two first steps. The second term $\phi_{i-1}^S S_t$ represents the market value of equity assets, and the last term the market value of bond assets. Note that a bond at time $t-1$ with maturity $i+1$ and coupon $c^i_{t-1}$ becomes at time $t$ a bond with maturity $i$ with the same coupon.

The portfolio is reallocated with the prescribed weights $w^S_i \in [0,1]$ and $w^b = 1 - w^S_i$ given by the ALM strategy. The amount of equity assets to hold is thus given by $\phi^S_i = w^S_i M V_t / S_t$. If this quantity is greater than $\phi^S_{i-1}$, there is a purchase of $\phi^S_i - \phi^S_{i-1}$ equity assets which increases the book value $B V^S_t$ by $(\phi^S_i - \phi^S_{i-1}) S_t$. If this quantity is lower than $\phi^S_{i-1}$, there is a sell of $\phi^S_{i-1} - \phi^S_i$ equity assets which decreases the book value $B V^S_t$ by the factor $\phi^S_i / \phi^S_{i-1}$. This generates capital gain or loss on stocks that is registered, since capital gain has to be redistributed to policyholders with a participation rate $\pi$.

The reallocation in bonds follows the same principles but is more involved. At the end of this step, the portfolio in bonds is made with $\phi^b_i$ combinations of bonds $B(t,n,c_t)$. Since the bonds are bought at par, there is a precise relation between $c_t$, $c_{t-1}$ and the swap rates at time $t$. According to the French legislation rules, the capital gain or loss on bonds is stored in the Capitalization Reserve and is separated from the ALM portfolio. Details can be found in [1].

4. **Determination of the crediting rate.** This step determines the policyholders’ earning rate $r_{ph}(t)$ on the period $(t-1,t)$. Due to regulatory constraint, it has to be greater than the minimum guaranteed rate $r_G$ and the amount distributed to policyholders has to be greater than the proportion $\pi_{pr}$ of the gains (participation rate). Besides the insurance company compares $r_{ph}(t)$ with a competitor rate $r^{comp}_t$ (typically the market short rate) and tries at best to have $r_{ph}(t) \geq r^{comp}_t$ to avoid dynamic surrenders. We call “target rate”, the maximum rate given by these three constraints.

The amount to distribute is typically made with the coupons, the capital gain or loss on stocks and possibly dividends. To smooth these gains along the years, the insurance company uses a Profit Sharing Reserve. In addition, the insurance company may also want to realize a part of latent gain or loss on stocks. In the model developed in [1], the amount to distribute depends on all these quantities. We distinguish four cases (from the best to the worst).

(A) The target rate can be distributed without using latent gain or by realizing all the latent loss on stocks.

(B) The target rate can be distributed by using latent gain or without realizing all the latent loss on stocks. The proportion of gain or loss is determined accordingly.

(C) The target rate cannot be reached with the available amount, but the minimum guaranteed rate can be distributed. The insurance company then uses all the latent gain in order to serve the best possible rate.

(D) The minimum guaranteed rate cannot be reached with the available amount. Then, the insurance company clears out the Profit Sharing Reserve and credit the policyholders with the lowest rate above $r_G$ that also satisfies participation rate constraints.

Once $r_{ph}(t)$ is determined, the Mathematical Reserve of the remaining policyholders is updated accordingly: $MR_t = MR_{t-1}(1 - p^g_{t-1})(1 + r_{ph}(t))$. The Profit Sharing Reserve and the book value of stocks are also modified according to the case. The shareholder’s margin can be calculated as well as the profit and loss $P&L_t$ generated on the period $(t-1,t)$, which is defined as the sum of the shareholder’s margin and the interest generated by the Capitalization Reserve. Again, all the details can be found in [1].

5. **Externalization of the Capitalization Reserve and of Shareholders’ margin from the accounting.** This last step is a technical accounting operation that slightly change the quantities of assets and the book values, while keeping unchanged the target weights $w^S_i$ and $w^b_i$.

The last step at the final time $T$ follows the same lines: instead of being reallocated, the portfolio is cleared and policyholders get back the remaining Mathematical Reserve $MR_T$. 

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3.2 The Solvency Capital Requirement with the standard formula

We now present the main lines of the SCR calculation with the standard formula as indicated by the EIOPA [21, 22]. Let us denote by \((\mathcal{F}_t, t \geq 0)\) the filtration representing the market information at time \(t \geq 0\) and \(\mathbb{Q}\) the pricing measure. We consider a short-rate model \((r_t, t \geq 0)\) for interest rates and define at time \(t \in \{0, \ldots, T-1\}\) the Basic Own Funds by

\[
BOF_t = \mathbb{E}^\mathbb{Q}\left[ \sum_{u=t+1}^{T} e^{-\int_{u}^{t} r_s \, ds} P&L_u | \mathcal{F}_t \right],
\]

i.e. the expected value of the discounted future profits and losses. The principle of the standard formula is to apply shocks on each asset class (equity, interest rate, etc.) and evaluate the variation of Basic Own Funds. Then, the SCR on market risk is obtained by using a given formula that aggregates all risk modules. In this paper, we focus on the interest rate module, where upward and downward shocks are prescribed by the regulator. The methodology to apply these shocks is described in Section 2.5 of [1]. We have used in our simulations the shocks specified in [21]. At time \(t\), the SCR value of the interest module is then defined by

\[
SCR_{int}^t = \max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\},
\]

where shocks are applied at time \(t\) on the interest-rate curve. We also set

\[
SCR_{up}^t = \max\{BOF_t - BOF_t^{\text{upward shock}}, 0\} \quad \text{and} \quad SCR_{down}^t = \max\{BOF_t - BOF_t^{\text{downward shock}}, 0\},
\]

so that \(SCR_{int}^t = \max(SCR_{up}^t, SCR_{down}^t)\). At time \(t = 0\), \(SCR_{int}^0\) is a number that can be calculated by Monte-Carlo. This has been investigated in [1]. However, for the ALM strategy, it may be useful to have quantitative insights on the evolution of the SCR along the time to assess the cost of capital. Thus, in this paper we are interested with the valuation of

\[
I = \mathbb{E}^\mathbb{P}\left[ \max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\} \right],
\]

the average value under the historical (or real) probability of the SCR at time \(t\). If we denote by \(\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t}\) the change of probability, we have

\[
I = \mathbb{E}^\mathbb{Q}\left[ \frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t}\right. \max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\},
\]

and we are precisely in the framework of Section 2 if \(X\) denotes a random variable that represents all the market information up to time \(t\) (i.e. \(\sigma(X) = \mathcal{F}_t\)). The equity module of the SCR is similarly defined by

\[
SCR_{eq}^t = \max\{BOF_t - BOF_t^{\text{equity shock}}, 0\},
\]

where the equity shock amounts to a strong decrease of \(S\) immediately after \(t\). Usually, the maximum with zero is useless since the shock is always negative. Last the standard formula that defines the SCR on market risk as follows (see Articles 164 and 165 of [8]):

\[
SCR_{mkt}^{int} = \sqrt{\left(SCR_{eq}^t\right)^2 + SCR_{int}^t} + 2\varepsilon SCR_{eq}^t SCR_{int}^t
\]

where \(\varepsilon = 0\) if the interest-rate exposure is due to the upward-shock on interest rates and \(\varepsilon = \frac{1}{2}\) if it is due to the downward shock of the interest rate module. Thus, the expected value of the SCR is given by:

\[
\mathbb{E}^\mathbb{P}[SCR_{mkt}^{int}] = \mathbb{E}^\mathbb{Q}\left[ \frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t}\sqrt{\left(SCR_{eq}^t\right)^2 + \left(SCR_{int}^t\right)^2 + 2\varepsilon SCR_{eq}^t SCR_{int}^t}\right].
\]
3.3 The stock and short-rate models

We consider \((W_t, Z_t)_{t \geq 0}\) a standard two-dimensional Brownian motion under \(\mathbb{Q}\). Following [1], we assume that the equity assets follows a Black-Scholes model and that the short interest rate follows a Vasicek++ (or Hull and White) model:

\[
\frac{dS_t}{S_t} = r_t dt + \sigma_S dW_t^p
\]

\[
r_t = x_t + \varphi(t), \quad \text{with} \quad dx_t = k(\theta - x_t)dt + \sigma_r(\gamma dW_t^p + \sqrt{1 - \gamma^2}dZ_t^p),
\]

where \(\gamma \in [-1, 1]\) tunes the dependence between equity and interest rates. We assume \(k, \theta, \sigma_S, \sigma_r > 0\). As explained in [1], the shift function \(\varphi : \mathbb{R}_+ \to \mathbb{R}\) is particularly convenient to implement the shocks prescribed by the EIOPA. Mainly, shocks amounts to modify the shift, leaving the dynamics of \(x\) unchanged, which makes easy to calculate the ALM strategies in both normal and shocked cases on each sample.

A new feature with respect to [1] is that we now also consider the dynamics under the real probability \(\mathbb{P}\). We assume here for simplicity the following basic change of probability

\[
\left. \frac{d\mathbb{P}}{d\mathbb{Q}} \right|_{\mathbb{P}_t} = \exp \left( \lambda^W W_t + \lambda^Z Z_t - \frac{1}{2} \left( (\lambda^W)^2 + (\lambda^Z)^2 + 2\gamma \lambda^W \lambda^Z \right) t \right) =: L_t,
\]

with \(\lambda^W, \lambda^Z \in \mathbb{R}\). By the Cameron-Martin theorem, \(dW_t^p = W_t - \lambda^W dt\) and \(dZ_t^p = Z_t - \lambda^Z dt\) are independent Brownian motions under \(\mathbb{P}\). We then have the following dynamics under \(\mathbb{P}\):

\[
\frac{dS_t}{S_t} = (r_t + \lambda^W \sigma_S) dt + \sigma_S dW_t^p
\]

\[
r_t = x_t + \varphi(t), \quad \text{with} \quad dx_t = k \left( \theta + \sigma_r \frac{\gamma \lambda^W + \sqrt{1 - \gamma^2} \lambda^Z}{k} - x_t \right) dt + \sigma_r (\gamma dW_t^p + \sqrt{1 - \gamma^2} dZ_t^p),
\]

To run this asset model with the ALM model described in Subsection 3.1, we have to be able to sample \(S_t, r_t\) and the change of probability \(L_t\) at each time \(t \in \mathbb{N}\). It is possible to do it exactly by using the following recurrence formula

\[
S_t = S_{t-1} \exp \left( \int_{t-1}^t x_u du + \int_{t-1}^t \varphi(u) du + \sigma_S (W_t - W_{t-1}) - \frac{\sigma_S^2}{2} \right),
\]

\[
x_t = x_{t-1} e^{-k} + \theta(1 - e^{-k}) + \sigma_r \int_{t-1}^t e^{-k(t-u)} (\gamma dW_u + \sqrt{1 - \gamma^2} dZ_u),
\]

\[
L_t = L_{t-1} \exp \left( \frac{\lambda^W}{k} (W_t - W_{t-1}) + \lambda^Z (Z_t - Z_{t-1}) - \frac{1}{2} \left( (\lambda^W)^2 + (\lambda^Z)^2 + 2\gamma \lambda^W \lambda^Z \right) \right),
\]

and

\[
\int_{t-1}^t x_u du = \frac{1}{k} (x_{t-1} - x_t) + \theta + \frac{\sigma_r}{k} [\gamma (W_t - W_{t-1}) + \sqrt{1 - \gamma^2} (Z_t - Z_{t-1})].
\]

The law of \((W_t - W_{t-1}, Z_t - Z_{t-1}, \int_{t-1}^t e^{-k(t-u)} (\gamma dW_u + \sqrt{1 - \gamma^2} dZ_u))\) is a centered Normal distribution with covariance

\[
\begin{bmatrix}
1 & 0 & \gamma \frac{1 - e^{-k}}{k} \\
0 & 1 & \sqrt{1 - \gamma^2} \frac{1 - e^{-k}}{k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\gamma \frac{1 - e^{-k}}{k} & 0 & \sqrt{1 - \gamma^2} \frac{1 - e^{-k}}{k} \\
0 & 1 & \frac{1 - e^{-2k}}{2k}
\end{bmatrix}
\]

This is the same law as \((G_1, G_2, \gamma \frac{1 - e^{-k}}{k} G_1 + \sqrt{1 - \gamma^2} \frac{1 - e^{-k}}{k} G_2 + \sqrt{1 - e^{-2k}} - \left( \frac{1 - e^{-k}}{k} \right)^2 G_3)\), where \(G_1, G_2, G_3\) are independent standard Normal variables. Once this triplet is sampled exactly, we can calculate easily \((S_t, x_t, L_t)\) using the formulas above.
3.4 Numerical experiments I: comparison between methods to calculate $E[SCR^{int}_t]$ 

We now present numerical results on the calculation of $I$ defined by (25). We use the following parameters for the ALM model and for the asset model. They are summarized in Tables 1 and 2. Unless specified, we also consider $\mathbb{P} = \mathbb{Q}$, i.e. that the real and risk-neutral probability are the same. We will discuss however later on the impact of this change of probability for the SCR.

| Stock model | Short-rate model |
|-------------|-----------------|
| $S_0 = 1$ | $r_0 = \theta = 0.02$ |
| $\sigma_S = 0.1$ | $\sigma_r = 0.01$ |
| $\gamma = 0$ | $k = 0.2$ |

Table 1: Market-model parameters

| Management Parameters | Liability Parameters |
|-----------------------|----------------------|
| Target allocation in stock $w^t_S = 0.05$ | Dynamic surrenders triggering thresholds $\beta = -0.01$ and $\alpha = -0.05$ |
| Target allocation in bond $w^t_B = 0.95$ | Maximum lapse dynamic surrender rate $DSR_{max} = 0.3$ |
| Participation rate $\pi_{pr} = 0.9$ | Deterministic constant exit rate $p = 0.05$ |
| Minimum guaranteed rate $r^G = 0.015$ | Time horizon: $T = 30$ years |
| Competitor rate $r^{comp}_t = r_t$ | |
| Smoothing coefficient of the PSR: $\bar{\rho} = 0.5$ | |
| Bond portfolio maximal maturity $n = 20$ | |
| Projection Horizon $T = 30$ | |

Table 2: Liability and management parameters

The implementation of the MLMC antithetic estimator is easily made by using (23). Instead, the implementation of the LSMC raises some issues. The main one is to choose the regressors. In fact, the ALM model presented in Subsection is truly path-dependent, and one needs to know $(r^t, S^t)_{t' \in \{1, \ldots, t\}}$ to determine the book values, the different reserves and the Bond portfolio at time $t$. Thus, $SCR^{int}_t$ depends on all the past before $t$. For $t = 1$ the dimension of the regression space is equal to 2 and the choice of the regressors $r_1$ and $S_1$ is obvious. When $t$ gets larger, this is no longer the case and in view of the theoretical complexity result of Proposition 5 one cannot afford to use all the $2t$ regressors. It is then important to select few regressors. We explain now the procedure that we have used.

3.4.1 Selection of the regressors for the LSMC estimator

In Table 3 we have listed 12 relevant risk-factors for the insurance company. We will select the most relevant ones for the SCR interest rate module by using a forward selection procedure.

To do so, we sample $J_v$ scenarios up to time $t$ of the ALM model. This produces in particular $J_v$ samples of $(X^1_t, \ldots, X^{12}_t)_{j = 1, \ldots, J_v}$. Then, we approximate for each scenario the value of the interest rate module of the SCR, $SCR^{int}_t$, by using a Nested Monte-Carlo with $K$ of secondary scenarios. We note $\widehat{SCR}^{Nested,j}_t$ these approximations (we drop for readability the superscript “int” in Paragraph 3.4.1). In our numerical application, we have taken $J_v = 2000$ validation scenarios and $K = 10^4$ inner scenarios. Let $\widetilde{SCR} : \mathbb{R}^{12} \to \mathbb{R}$ be a function approximating the SCR from the values of $X$. We now consider the empirical RMSE, i.e.

$$\sqrt{\frac{1}{J_v} \sum_{j=1}^{J_v} (\widehat{SCR}^{Nested,j}_t - \widetilde{SCR}(X^j_t))^2}$$
as a criterion to assess the accuracy of the regression function \( \widehat{SCR} \).

We start by selecting the first variable. Up to a linear rescaling of the sample we may assume without loss of generality that all the variables are in \([0, 1]\). We consider the 12 possible regressor functions for \( l \in \{1, \ldots, 12\} \),

\[
\sum_{i=0}^{n_r-1} \hat{\alpha}^l_i X_j^i \in [i/n_r, (i+1)/n_r), \quad \text{with } \hat{\alpha}^l_i = \frac{\sum_{j=1}^{J_v} \widehat{SCR}^{Nested,j}_t X_j^i \in [i/n_r, (i+1)/n_r)}{\sum_{j=1}^{J_v} \mathbb{1}_{X_j^i \in [i/n_r, (i+1)/n_r)}},
\]

and select \( l^*_1 \in \{1, \ldots, 12\} \) that achieves the lowest RMSE. Once \( l_1 \) is selected, we consider the following 11 regressor functions for \( l \in \{1, \ldots, 12\} \setminus \{l_1\} \):

\[
\sum_{i_1, i_2=0}^{n_r-1} \hat{\alpha}^{l_1,l_2}_{i_1, i_2} \mathbb{1}_{X_j^{i_1} \in [i_1/n_r, (i_1+1)/n_r)} X_j^{i_2} \in [i_2/n_r, (i_2+1)/n_r)},
\]

with \( \hat{\alpha}^{l_1,l_2}_{i_1, i_2} = \frac{\sum_{j=1}^{J_v} \widehat{SCR}^{Nested,j}_t X_j^{i_1} \in [i_1/n_r, (i_1+1)/n_r)}{\mathbb{1}_{X_j^{i_2} \in [i_2/n_r, (i_2+1)/n_r)} \sum_{j=1}^{J_v} \mathbb{1}_{X_j^{i_1} \in [i_1/n_r, (i_1+1)/n_r)}}.\]

We then select the regressor \( l_2 \in \{1, \ldots, 12\} \setminus \{l_1\} \) that gives the smallest RMSE. We then proceed similarly to select the next variables. We have run this selection for \( t = 10 \) with \( n_r = 5 \). Table 4 shows the result of this algorithm and indicate the Book values of bonds as the more significant variable to approximate the SCR module on interest rates. We notice that the RMSE is significantly reduced by using the second variable. In contrast, the third variable moderately improves the criterion. Since the number of variables is also a limitation then for the use of the LSMC estimator, we do not go further in the selection procedure. Figure 5 illustrates the approximation of the values of \( \widehat{SCR}^{Nested,j}_t \) by the regression function with the two first regressors.

| Attribute                  | Risk-factor description                  |
|----------------------------|------------------------------------------|
| \( X_1^t = S_t \)         | Equity asset value                      |
| \( X_2^t = r_t \)         | Short rate                               |
| \( X_3^t = \phi^S_t \)    | Position in Stock                        |
| \( X_4^t = \phi^B_t \)    | Position in bonds                        |
| \( X_5^t = BV^S_t \)      | Book value of bonds                      |
| \( X_6^t = BV^B_t \)      | Book value of equity assets              |
| \( X_7^t = MR_t \)        | Mathematical Reserve                     |
| \( X_8^t = PSR_t \)       | Profit sharing reserve                   |
| \( X_9^t = CR_t \)        | Capitalization Reserve                   |
| \( X_{10}^t = MV_t \)     | Portfolio market value                   |
| \( X_{11}^t = \phi^B_t B(t, n, c_t) \) | Market value of bonds            |
| \( X_{12}^t = \phi^S_t S_t \) | Market value of equity assets         |

Table 3: Non-exhaustive list of risk factors

| Best attribute | RMSE  |
|----------------|-------|
| First variable | \( BV^S_t \) (\( l_1 = 5 \)) | 0.8739 |
| Second variable| \( r_t \) (\( l_2 = 2 \))    | 0.5292 |
| Third variable | \( S_t \) (\( l_3 = 1 \))    | 0.5084 |

Table 4: Result of the Forward selection procedure for \( SCR^{int}_t \) with \( t = 10 \).
3.4.2 Comparison between MLMC and LSMC estimators

We focus on the calculation of $\mathbb{E}[\widehat{SCR}_{10}^{int}]$ with $t = 10$ years. We now compare and test numerically the MLMC antithetic estimator $\widehat{I}_{A}^{\text{MLMC}}$ defined by (21) with the LSMC estimator $\widehat{I}_{A}^{\text{LSMC}}$ defined by (24) using the local cube basis and the regressors selected by the procedure described in Paragraph 3.4.1.

In Figure 6, we have drawn the Root Mean Square Error of the estimator $\widehat{I}_{A}^{\text{MLMC}}$ and of the estimators $\widehat{I}_{A}^{\text{LSMC}}$ using the first, the two first and the three first selected regressors. In order to derive the RMSE of the different estimators, as no closed formulas is available in this framework, we rely on a full nested Monte-Carlo procedure based on a fixed simulation budget of $\Gamma = 10^8$ sample paths to approximate the true value of $I$. The allocation between primary and secondary scenarios correspond to $M \approx \frac{\Gamma}{3}$ primary samples and $K \approx \frac{\Gamma}{3}$ inner scenarios, as prescribed by Theorem 1 for $\eta = 1$. To compute the RMSE of the different estimators, we produce $N_{\text{batch}} = 10$ independent simulations $(\tilde{I}_{j})_{j=1}^{N_{\text{batch}}}$ and indicate the empirical RMSE $\sqrt{\frac{1}{N_{\text{batch}}} \sum_{j=1}^{N_{\text{batch}}} |\tilde{I}_{j} - I|^2}$. We plot the empirical RMSE’s of the different estimators as a function $J$ (with $J := \sum_{l=0}^{L} J_{l} K_{l}$ for the MLMC estimator). This represents the number of samples, as well as the computational cost (in log-scale) that is in $O(J)$ for both estimators.

Concerning the LSMC estimators, we notice that the estimators with two regressors does much better with the estimator with one regressor. Instead, the interest of using a third regressor is tiny. We also observe that the RMSE does not really decrease after $10^4$ samples on our example. This is due to the regression error: since we approximate $SCR_{10}^{int}$ by a function of two or three variables, there is no way to go beyond a certain level of precision. This is particularly noticeable for $t \geq 10$ years: the projection of the balance sheet through the ALM model is truly non-Markovian and the history up to time $t$ cannot be summed up by two or three variables. In
comparison, the convergence of the MLMC antithetic estimator is in line with Theorem 4 and is asymptotically more accurate than the LSMC estimator. Besides, the MLMC estimator avoids the step of selecting regressors that requires computational time and may be determinant for the accuracy of the LSMC estimator. Last, we notice that for a same level of precision, the computational time required by the MLMC is slightly smaller than the one required by the LSMC estimator. More precisely, the computational time needed for $J = 2 \times 10^5$ (where the three estimators have quite the same accuracy) are 9950 seconds for $\hat{I}_{\text{MLMC}}$, 11230 seconds for $\hat{I}_{\text{LSMC}}$ with two regressors ($n_r = 23$) and 12650 seconds for $\hat{I}_{\text{LSMC}}$ with three regressors ($n_r = 13$).

We now make a comment on the choice of the parameter $\eta$ for the MLMC antithetic estimator. We recall that $\eta$ is, roughly speaking, related to the probability that two (or more) arguments of the maximum function are close to the maximum, see Theorems 1 and 4. Heuristically, the smaller is this probability, the larger can be $\eta$, which then reduces the number of levels and then the computational cost. In Figure 7, we have plotted the convergence of the MLMC antithetic estimator for $\eta \in \{1/2, 3/4, 1\}$ in function of the theoretical computational cost $\sum_{l=0}^L J_l K_l$. Basically, the three estimators converge, but the one obtained with $\eta = 1$ does not seem to be asymptotically in $O(\varepsilon^{-2})$ while the two others are in line with the theoretical convergence in $O(\varepsilon^{-2})$. This shows the interest of the parameter $\eta$ in a practical application and explains why we have chosen to take $\eta = 3/4$ is our experiments in Figure 6.

### 3.4.3 Comparison between LSMC and the use of Neural Network

In Paragraph 3.4.2, we have noticed that a significant drawback of the LSMC estimator with respect to the MLMC antithetic estimator is that it requires first to select regressors. Beyond the computational time needed by this selection, there is a significant regression error. A natural idea to skip the selection step in the regression is to use Neural Networks (NN). We have implemented a feedforward neural network with one hidden-layer. The hidden-layer is made with 10 or 50
Figure 7: Convergence of the MLMC estimator for different values of $\eta$ in function of the computational cost $\sum_{l=0}^{L} J_l K_l$.

Table 5: Input feature of the Neural Network Algorithm

| Input Feature                      |
|-----------------------------------|
| $X_1^t$ Bond Book-Value $BV^b_t$  |
| $X_2^t$ Stock Book-value $BV^s_t$ |
| $X_3^t$ Position in bond $\phi^b_t$ |
| $X_4^t$ Position in stock $\phi^s_t$ |
| $X_5^t$ Profit-Sharing Reserve level $PSR_t$ |
| $X_6^t$ Mathematical Reserve level $MR_t$ |
| $X_7^t$ Capitalization Reserve level $CR_t$ |
| $X_8^t$ Spread crediting rate/competing rate $\Delta_t$ |
| $X_9^t$ Stock price $S_t$         |
| $X_{10}^t$ Interest-rate $r_t$    |

Table 6 indicates the RMSE of the estimator with the different methods. First, we notice that there is no need on our example to consider many neurons: a simple layer with 10 neurons is enough and do as well as the NN with 50 neurons in terms of RMSE. We notice also that the estimator given by the NN is slightly better than the one

neurons. The activation function used is the sigmoid function. To train the network, we generate $J$ number of primary outer scenarios $(X_{t,j})_{1\leq j \leq J}$ for which only one inner simulation is performed to obtain $Z_j$ that represents the maximal variation of the discounted P&L due to the shocks. Then, one minimizes

$$\frac{1}{J} \sum_{j=1}^{J} (Z_j - \text{NN}(X_{t,j}))^2,$$

where NN is the function generated by the neural network, so that it approximates the desired conditional expectation. The input features of the network have not been pre-processed: the optimization above has to be enough for detecting the relevant variables for the approximation of the conditional expectation. Once the NN has been obtained, we then estimate $E[SCR_{int}^t]$ simply by the empirical mean $\hat{I}_{NN}^t := \frac{1}{J} \sum_{j=1}^{J} \text{NN}(X_{t,j})$. Since we only use the NN on the training sample, the standard problem of overfitting is not an issue for our application. We compare the RMSE of this estimator with the RMSE obtained with the LSMC method. The aim of our procedure is to assess if a Neural Network feeding with a whole range of input features is able to select relevant attributes and to compare with the LSMC method with well-chosen features.

Table 6 indicates the RMSE of the estimator with the different methods. First, we notice that there is no need on our example to consider many neurons: a simple layer with 10 neurons is enough and do as well as the NN with 50 neurons in terms of RMSE. We notice also that the estimator given by the NN is slightly better than the one...
obtained with the LSMC with two or three regressors when the training sample gets large. However, the use of neural networks present serious drawbacks. First, it requires to store all the samples to achieve the minimization of (31) while the LSMC (and also MLMC) estimator only uses once each sample. Second, the time needed by the minimization (indicated in Table 7) is important, making at the end this method less competitive than MLMC. Note that one could be then tempted to train the NN on a smaller size of samples and then use it for large : one would then face the problem of overfitting, which we want to avoid.

| $J$   | LSMC dim 1 | LSMC dim 2 | LSMC dim 3 | NN: 10 neurons | NN: 50 neurons |
|-------|------------|------------|------------|----------------|---------------|
| 500   | 1.0e-3     | 3.36e-4    | 3.50e-4    | 7.075e-4       | 7.56e-4       |
| $10^3$| 1.0e-3     | 3.75e-4    | 4.28e-4    | 6.46e-4        | 3.017e-4      |
| $5 \times 10^3$| 9.52e-4 | 1.23e-4    | 1.46e-4    | 1.63e-4        | 1.8153e-4     |
| $5 \times 10^4$| 1.0e-3     | 1.12e-4    | 1.24e-4    | 6.60e-5        | 7.29e-5       |
| $10^5$| 9.97e-4    | 1.068e-4   | 1.19e-4    | 6.32e-5        | 6.92e-5       |

Table 6: RMSE of $E[SCR^t_{\text{int}}]$ for $t = 10$ given by the Neural Network (one hidden layer with the indicated number of neurons) and the LSMC in function of $J$

To sum up, the use of NN can indeed be useful to reduce the approximation error observed by using LSMC estimators. However, it both demands memory and computational time, making the gain with respect to the LSMC not obvious. The MLMC estimator presents the clear advantage to avoid this issue of function approximation, and to avoid any storage of data.

3.5 Numerical experiments II: some insights on the ALM

We now present some applications of the MLMC antithetic estimator for the ALM. One of the major issue in ALM is to determine the optimal asset allocation between the different asset class backed to the insurance portfolio. For that reason, it is crucial to evaluate precisely the amount of SCR required by the strategy. We are interested in calculating $E[SCR^t_{\text{mkt}}]$, $E[SCR^t_{\text{eq}}]$ and $E[SCR^t_{\text{int}}]$, see Subsection 3.2 for the definition of these modules. Note that at time $t > 0$, the $SCR^q_t$ and $SCR^m_t$ are random variables so that we cannot use the aggregation formula to get directly $E[SCR^m_t]$ from $E[SCR^q_t]$ and $E[SCR^m_t]$. Note that by using the MLMC Antithetic estimator, it possible and easy to calculate at the same time these expectations: see Remark 5 for the general expression of these estimators that we use for different functions $h$. At each level $l$, one simulates $J_l$ primary scenarios up to time $t$. Then, one simulates for each primary scenario $K_l$ secondary scenarios, on which we perform four different evolutions: the first one without any shock, the second one with the equity shock, the third one with the upward shock on interest rates and the last one with the downward shock on interest rates. Then, one computes the corresponding empirical means related to the calculation of $SCR^q_t$ and $SCR^m_t$ and use the aggregation formula (27) for $SCR^{\text{mkt}}_t$.

Let us note that the discontinuity induced by the coefficient $\varepsilon$ may in principle deteriorate the MLMC estimation. However, we have thus run the MLMC estimator with a regularization of this coefficient and noticed a tiny impact of the regularization . This can be heuristically understood from Figure 11: the activation of $\varepsilon$ may occur on a wide range (perhaps the whole range) of values of $SCR^{\text{int}}$, which smooths the phenomenon.

Figures 8, 9 and 10 illustrate respectively the different values of the SCR modules $E[SCR^{\text{mod}}_t]$ for mod $\in \{\text{mkt, eq, int, up, down}\}$ in function of the constant allocation weight $w^S$ in equity for $t = 0$, $t = 10$ and $t = 20$ years (at $t = 0$, we can remove the expectation). As one may expect, these values globally decrease with respect to the time since we are considering a run-off portfolio with an exit rate greater than 5%. We notice several interesting points.
At time $t = 10$, $\mathbb{E}[\text{SCR}^{int}_t]$ is significantly larger than $\max(\mathbb{E}[\text{SCR}^{up}_t], \mathbb{E}[\text{SCR}^{down}_t])$, which shows that deterministic proxy values of SCR modules may induce errors. We no longer observe this phenomenon at time $t = 20$ because the greater shock is always given by the upward shock and we have then $\text{SCR}^{int}_{20} \approx \text{SCR}^{down}_{20}$ (the green and blue curves coincide). This is explained in the next point.

The main effects of the shocks on the interest rate are the following. The upward (resp. downward) shock leads to an immediate decrease (resp. increase) of the portfolio market value, but on the long run higher (resp lower) rates gives a better (resp. worse) profitability. Here, we are considering a run-off portfolio with final maturity $T = 30$. Thus, as $t$ increases, the effect on the long run of this shocks get less important making the immediate effect on market value dominant. Thus, at $t = 20$ the downward shock is harmless while the upward shock gets painful. This explains why we observe $\text{SCR}^{int}_{20} \approx \text{SCR}^{down}_{20}$.
Then, we can do this easily if one has computed the values of the sensitivities \( E \) the new value of \( E \) For example, suppose that we have computed the value of

These sensitivities are interest-

\[ \frac{\delta S}{\delta r_0} \] and \( \frac{\delta S}{\delta S_0} \), where implicitly all values are kept constant but respectively \( r_0 \) and \( S_0 \). Note that these sensitivities can be computed with the MLMC antithetic estimator with the same samples that are needed for the estimation of \( SCR^{mkt} \). Table (8) indicates the sensitivities obtained with our parameters with \( \delta r_0 = 0.001 \) and \( \delta S_0 = 0.01 \).

| \( \frac{\delta S}{\delta S_0} \) | \( \frac{\delta S}{\delta r_0} \) |
|-----------------|-----------------|
| 0.0228          | -0.0845         |

Table 8: Sensitivities of the \( SCR_{10}^{mkt} \) with \( w^S = 0.05 \).

Last, the MLMC estimation is a tool for example to analyse how the SCR depends on the risk premia of stocks and interest rates. If the evaluation of the SCR has to be performed for regulatory reasons under a risk neutral framework, it is more relevant for ALM to calculate \( E[SCR_{t}^{mod}] \) under the real probability, which corresponds to the average value of the own funds that will be necessary at time \( t \). From equations (26) (generalized to any other SCR module) and (30), it is possible to see the impact of the risk premia \( \lambda^W \) and \( \lambda^Z \) on each SCR module. In Figures 12 and 13, we have indicated the more remarkable ones: the dependence of \( E[SCR_{t}^{int}] \) on \( \lambda^Z \) and of \( E[SCR_{t}^{eq}] \) on \( \lambda^W \). We notice that the larger is \( \lambda^Z \) the larger is \( E[SCR_{t}^{int}] \). This can be understood as follows. A higher \( \lambda^Z \) leads to a higher mean reverting level for the short rate \( r \). Thus, under the real probability measure (on time \([0, t] \)), bonds are better remunerated and at time \( t \) the amount of savings (mathematical reserve) is higher. Since the evaluation of \( SCR_{t}^{int} \) is risk neutral, \( \lambda^Z \) has then no incidence on this evaluation. Thus, we observe a higher value of \( E[SCR_{t}^{int}] \) simply because the mathematical reserve at time \( t \) is higher because of better returns. The same interpretation holds for \( E[SCR_{t}^{eq}] \): the higher is \( \lambda^W \), the higher is the amount of savings at time \( t \) and therefore the higher is \( E[SCR_{t}^{eq}] \).
A Technical proofs for Theorems 1 and 4

A.1 Preliminary results

In this section, we gather elementary but useful results for the analysis of the nested and the multilevel Monte-Carlo estimators.

Proposition 6. Let $g_0(u) = \max\{u, 0\}$ and, for $\varepsilon > 0$, $g_\varepsilon(u) = \frac{u^2}{2\varepsilon} \mathbf{1}_{u \in [0, \varepsilon]} + (u - \varepsilon) \mathbf{1}_{u > \varepsilon}$. The function $g_\varepsilon$ is $C^1$ and piecewise $C^2$ with

$$g'_\varepsilon(u) = \frac{u}{\varepsilon} \mathbf{1}_{u \in [0, \varepsilon]} + \mathbf{1}_{u > \varepsilon}, \quad g''_\varepsilon(u) = \frac{1}{\varepsilon} \mathbf{1}_{u \in [0, \varepsilon]}.$$

Moreover, $g_\varepsilon$ is $1$-Lipschitz and we have

$$g_\varepsilon \leq g_0 \leq g_\varepsilon + \frac{\varepsilon}{2}.$$

In addition, for any $\theta, \hat{\theta} \in \mathbb{R}$ we have:

$$\forall a \in \mathbb{R}, 0 \leq \int_{\theta}^{\hat{\theta}} g''_\varepsilon(t + a) \, dt \leq \int_{\mathbb{R}} g''_\varepsilon(t) \, dt = 1. \quad (32)$$

Finally, the following asymptotic properties holds : $\forall u \in \mathbb{R} \ g_\varepsilon(u) \xrightarrow{\varepsilon \to 0} g_0(u)$, $\forall u \in \mathbb{R} \ g'_\varepsilon(u) \xrightarrow{\varepsilon \to 0} \mathbf{1}_{u > 0}$ and $\int_{\mathbb{R}} g''_\varepsilon(u) \varphi(u) \, du \xrightarrow{\varepsilon \to 0} \varphi(0)$ for any function $\varphi : \mathbb{R} \to \mathbb{R}$ that is right-continuous at 0.

Lemma 7. Let $\theta, \hat{\theta} \in \mathbb{R}$ and $\{a_\varepsilon\}$ an arbitrary function that converges to $a$ as $\varepsilon \to 0$, then:

$$\limsup_{\varepsilon \to 0} \left| \int_{\theta}^{\hat{\theta}} g''_\varepsilon(t + a_\varepsilon) \, dt \right| \leq \left| \mathbf{1}_{\theta \leq -a \leq \hat{\theta}} - \mathbf{1}_{\hat{\theta} \leq -a \leq \theta} \right| \quad (33)$$

Proof of Lemma 7. Without loss of generality, we assume that $\theta < \hat{\theta}$. First we have that :

$$\int_{\theta}^{\hat{\theta}} g''_\varepsilon(t + a_\varepsilon) \, dt = \frac{1}{\varepsilon} \int_{A_\varepsilon} 1 \, dt$$
where $A_{\varepsilon} = [\theta, \hat{\theta}] \cap [-a_{\varepsilon}, \varepsilon - a_{\varepsilon}]$. Hence, if $-a < \theta$ or $\hat{\theta} < -a$, it exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0], A_{\varepsilon} = \emptyset$. In this case:

$$\limsup_{\varepsilon \to 0} \left| \int_{\theta}^{\hat{\theta}} g''_{\varepsilon}(t + a_{\varepsilon}) dt \right| = 0$$

Otherwise (i.e., if $\theta \leq -a \leq \hat{\theta}$) we always have: $A_{\varepsilon} \subset [-a, \varepsilon - a_{\varepsilon}]$. Therefore, we have $\limsup_{\varepsilon \to 0} \left| \int_{\theta}^{\hat{\theta}} g''_{\varepsilon}(t + a_{\varepsilon}) dt \right| \leq 1$ by (32). Thus we obtain

$$\limsup_{\varepsilon \to 0} \left| \int_{\theta}^{\hat{\theta}} g''_{\varepsilon}(t + a_{\varepsilon}) dt \right| \leq 1_{\{\theta < -a \leq \hat{\theta}\}}. \tag{34}$$

**Lemma 8.** Let $\theta, \hat{\theta} \in \mathbb{R}$. Then, we have

$$1_{\theta < -a < \hat{\theta}} - 1_{\theta < 0, \hat{\theta} > 0} \leq 1_{\theta \hat{\theta} < 0} \tag{35}$$

**Proof.** Inequality (34) is an equality when $\theta \neq 0$ or $\theta = 0, \hat{\theta} \neq 0$ and an obvious inequality when $\theta = \hat{\theta} = 0$. The right hand side of (35) is nonnegative. When $\theta < 0$ and $\hat{\theta} \geq 0$ (resp. $\theta > 0$ and $\hat{\theta} \leq 0$), we have $|\hat{\theta} - \theta| = -\hat{\theta} - \theta \geq -\theta = |\theta|$ (resp. $|\hat{\theta} - \theta| = -\theta + \hat{\theta} \geq \theta = |\theta|$). \[
\]

**Lemma 9.** Let $(Z_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence of square integrable real valued random variables. Let $\mu = \mathbb{E}[Z_1]$ and $\sigma = \sqrt{\text{Var}[Z_1]}$. For $\gamma \in [1, 2]$, we define $D_{\gamma} = \sigma^{\gamma}$ and for $\gamma > 2$, $D_{\gamma} = \mathbb{E}[|Z_1 - \mu|^\gamma] \in [0, +\infty]$. Then, we have

$$\mathbb{E}\left[\frac{1}{K} \sum_{k=1}^{K} Z_k - \mu \right]^\gamma \leq C_{\gamma} \frac{D_{\gamma}}{K^{\gamma/2}}$$

with $C_{\gamma} = 1$ for $\gamma \in [1, 2]$ and $C_{\gamma} = \left(2^{\gamma-1}\right)^{\gamma}$ for $\gamma > 2$.

**Proof.** For $\gamma \in [1, 2]$, We have from Jensen inequality $\mathbb{E}\left[\left(\frac{1}{K} \sum_{k=1}^{K} Z_k - \mu \right)^\gamma\right] \leq \mathbb{E}\left[\left(\frac{1}{K} \sum_{k=1}^{K} Z_k - \mu \right)^2\right]^{\gamma/2} = \sigma^{\gamma}/K^{\gamma/2}$. For $\gamma > 2$, this result is stated in Corollary 2.5 [19]. \[
\]

**A.2 Nested Monte-Carlo estimator**

**Proof of Lemma 2.** Let $\tilde{b}(X) = \mathbb{E}\left[\max\{\hat{\theta}_K, \hat{\theta}_R\} - \max\{\varphi_1(X), \varphi_2(X)\}\right | X]$. Since $\max\{a, b\} = a + g_0(b - a)$ for $a, b \in \mathbb{R}$, we get

$$\tilde{b}(X) = \mathbb{E}\left[\hat{\theta}_K - \varphi_1(X) + g_0(\hat{\theta}_R) - g_0(\varphi_2(X)) \right | X] \tag{36}$$

Now, we observe that

$$\tilde{b}_e(X) = \lim_{\varepsilon \to 0} \tilde{b}_e(X) = \mathbb{E}\left[\hat{\theta}_K - \varphi_1(X) + g_e(\hat{\theta}_R) - g_e(\varphi_2(X)) \right | X] \tag{37}$$

Since $g_e$ is $1$-Lipschitz, we have

$$|g_e(\hat{\theta}_R) - g_e(\varphi_2(X))| \leq |\hat{\theta}_R - \varphi_2(X)| \leq |\hat{\theta}_K - \varphi_1(X)| + |\hat{\theta}_R - \varphi_2(X)|.$$

Then, we get (37) by using the integrability assumption (ii) and Lebesgue’s dominated convergence theorem. Since $g_e$ is $C^1$ and piecewise $C^2$, we can make a Taylor expansion to obtain:

$$\tilde{b}_e(X) = \mathbb{E}\left[\hat{\theta}_K - \varphi_1(X) + g_e'(\varphi_2(X))(\hat{\theta}_R - \varphi_2(X)) + \int_{\varphi_2(X)}^{\hat{\theta}_R} g_e''(t) dt \right | X]$$

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Then, since \( g'(\varphi_21(X)) \) is \( \sigma(X) \)-measurable, using Proposition 6 we get
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \hat{\theta}_K - \varphi_1(X) + g'(\varphi_21(X)) \left( \hat{\theta}_K^{21} - \varphi_21(X) \right) |X \right] = \mathbb{E} \left[ \hat{\theta}_K - \varphi_1(X) + 1_{\varphi_21(X)>0} \left( \hat{\theta}_K^{21} - \varphi_21(X) \right) |X \right]
\]
Using condition (ii) we get :
\[
\mathbb{E} \left[ 1_{\varphi_21(X)\leq 0} \left( \hat{\theta}_K - \varphi_1(X) \right) + 1_{\varphi_21(X)>0} \left( \hat{\theta}_K^{21} - \varphi_2(X) \right) |X \right] \leq \frac{1_{\varphi_21(X)\leq 0}C_1(X) + 1_{\varphi_21(X)>0}C_2(X)}{K^{1+\eta}} \tag{38}
\]
Now we focus on the remainder in the Taylor decomposition. Using Lemma 7 and the dominated convergence theorem and then Lemma 8 with \( \varphi_21(X) \neq 0 \text{ a.s.} \) (Assumption (iii)), we get
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \int_{\varphi_21(X)}^{\hat{\theta}_K^{21}} \left( \hat{\theta}_K^{21} - \varepsilon \right) g''(\varepsilon) \, d\varepsilon |X \right] \leq \mathbb{E} \left[ \hat{\theta}_K^{21} - \hat{\theta}_K \right] \limsup_{\varepsilon \to 0} \mathbb{E} \left[ \int_{\varphi_21(X)}^{\hat{\theta}_K^{21}} g''(\varepsilon) \, d\varepsilon |X \right]
\]
and the convexity of \( x \mapsto x^{1+\eta} \) to get
\[
\mathbb{E} \left[ \hat{\theta}_K^{21} - \varphi_21(X) \right]^{1+\eta} |X \right] \leq 2^{\eta} \left( \mathbb{E} \left[ \hat{\theta}_K - \varphi_1(X) \right]^{1+\eta} |X \right] + \mathbb{E} \left[ \hat{\theta}_K - \varphi_2(X) \right]^{1+\eta} |X \right]
\]
and (37) and (39) give the bias estimate (13).

We now focus on the variance. The proof is straightforward using condition (ii) and the inequality \( \max\{a_1, a_2\} - \max\{b_1, b_2\} \leq \max\{|a_1 - b_1|, |a_2 - b_2|\} \) :
\[
\mathbb{E} \left[ \max\{\hat{\theta}_K, \hat{\theta}_K^{21}\} - \max\{\varphi_1(X), \varphi_2(X)\} \right]^{2} |X | \leq \mathbb{E} \left[ \max\{\hat{\theta}_K - \varphi_1(X) \}, \hat{\theta}_K^{21} - \varphi_2(X) \right]^{2} |X | \leq \mathbb{E} \left[ \hat{\theta}_K - \varphi_1(X) \right]^{2} |X | + \mathbb{E} \left[ \hat{\theta}_K^{21} - \varphi_2(X) \right]^{2} |X | \leq \frac{\sigma^2(X)}{K \cdot 1+\eta}
\]
\[
\text{A.3 Antithetic MLMC estimator}
\]
In this section we prepare the proof of Theorem 4 and start with a useful preliminary lemma.

**Lemma 10.** Let \( p \geq 2 \) and \( K \in 2\mathbb{N}^* \). With the notation introduced in (5) and (6), the following property holds:
\[
\mathbb{E} \left[ \left( \hat{M}_K^{\frac{p-1}{2}} - \hat{M}_K^{\frac{p-1}{2}} \right)^2 |X \right] \leq 2\mathbb{E} \left[ \left( \hat{M}_K^{p-1} - \hat{M}_K^{p-1} \right)^2 |X \right] + 2\mathbb{E} \left[ h^2 \left( \hat{M}_K^{p-1} - \hat{M}_K^{p-1} \right) |X \right],
\]
where \( h(x, y) = \left( \frac{x+y}{2} \right) - \left( \frac{x+y}{2} \right) \).
Proof of Lemma 10. Observing that \( \forall a, b \in \mathbb{R}, \max\{a, b\} = a + (b - a)^+ \), we deduce that
\[
\hat{M}_K^p - \frac{\hat{M}_{K/2}^p + \hat{M}_{K/2}^{p,'}}{2} = \max\{\hat{E}_K^p, \hat{M}_{K-1}^p\} - \frac{\max\{\hat{E}_{K/2}^p, \hat{M}_{K-1/2}^p\} + \max\{\hat{E}_{K/2}^{p,'}, \hat{M}_{K-1/2}^{p,'}\}}{2}
\]
\[
= \hat{E}_K^p + (\hat{M}_{K-1}^p - \hat{E}_K^p) + \frac{\hat{E}_{K/2}^p + (\hat{M}_{K-1/2}^p - \hat{E}_{K/2}^p) + \hat{E}_{K/2}^{p,'} + (\hat{M}_{K-1/2}^{p,'} - \hat{E}_{K/2}^{p,'})}{2}
\]
\[
= (\hat{M}_{K-1}^p - \hat{E}_K^p) + \frac{(\hat{M}_{K-1/2}^p - \hat{E}_{K/2}^p) + (\hat{M}_{K-1/2}^{p,'} - \hat{E}_{K/2}^{p,'})}{2}
\]
\[
= (\hat{M}_{K-1}^p - \hat{E}_K^p) + \left(\frac{\hat{M}_{K/2}^{p-1} + \hat{M}_{K/2}^{p-1,'}}{2} - \hat{E}_K^p\right)^+ + h\left(\hat{M}_{K-1}^p - \hat{E}_{K/2}^p, \hat{M}_{K-1/2}^{p,'} - \hat{E}_{K/2}^{p,'}\right),
\]
using that \( \hat{E}_K^p = \frac{\hat{E}_{K/2}^p + \hat{E}_{K/2}^{p,'}}{2} \) for the third and fourth equality. We conclude using that \((a + b)^2 \leq 2(a^2 + b^2)\).

The following lemmas give a bound on \( h \)

Lemma 11. Let \( h(x, y) = \left(\frac{x+y}{2}\right)^+ - \frac{(x)^+ + (y)^+}{2} \) for \( x, y \in \mathbb{R} \). Then, we have \( h(x, y) = -\frac{|x|\wedge|y|}{2} 1_{xy \leq 0} \).

Proof. By distinguishing the cases as follows, we get the claim:
\[
h(x, y) = \begin{cases} y/2 & \text{if } x + y \geq 0, x > 0, y < 0, \\ x/2 & \text{if } x + y \geq 0, x < 0, y > 0, \\ -x/2 & \text{if } x + y < 0, x > 0, y < 0, \\ -y/2 & \text{if } x + y < 0, x < 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Proposition 12. We use the notation introduced in (5) and (6). Let \( \eta > 0 \) and \( D_{2+\eta}^p(X) = \mathbb{E}[|Y^p - \mathbb{E}[Y^p|X]|^{2+\eta}|X] \). We assume that \( \mathbb{P}(E_X^p = M_X^{-1}) = 0 \) for all \( p \in \{2, \ldots, P\} \) and
\[
\forall p \in \{2, \ldots, P\}, \quad \mathbb{E}\left[\frac{D_{2+\eta}^p(X)}{|E_X^p - M_X^{-1}|^{\eta}} \phi^2(X)\right] < \infty.
\]
Then, there exist a constant \( C \in \mathbb{R}_+^* \) such that
\[
\text{Var}\left(\hat{M}_K^p - \frac{\hat{M}_{K/2}^p + \hat{M}_{K/2}^{p,'}}{2}\right) \phi(X) \leq C K^{1+\eta/2}.
\]

Proof. Let us define for \( p = 1, \ldots, P \),
\[
U_K^p = \mathbb{E}\left[\hat{M}_K^p - \frac{\hat{M}_{K/2}^p + \hat{M}_{K/2}^{p,'}}{2} | X\right],
\]
\[
\varepsilon_K^p = \mathbb{E}\left[h^2 \left(\hat{M}_K^{p-1} - \hat{E}_K^{p} - \frac{\hat{M}_{K/2}^{p-1} - \hat{E}_{K/2}^{p}}{2}\right) | X\right].
\]
We notice that \( U_K^1 = 0 \), and Lemma 10 gives \( U_K^p \leq 2(U_K^{p-1} + \varepsilon_K^p) \) for \( p = 2, \ldots, P \). A straightforward induction leads to
\[
U_K^p \leq \sum_{p=2}^{P} 2^{P+1-p} \varepsilon_K^p.
\]
(40)
The variance being smaller than the expectation of the square, we get by using the tower property of the conditional expectation
\[
\text{Var}\left(\hat{M}_K^p - \frac{\hat{M}_{K/2}^p + \hat{M}_{K/2}^{p,'}}{2}\right) \phi(X) \leq \sum_{p=2}^{P} 2^{P+1-p} \mathbb{E}[\varepsilon_K^p \phi^2(X)].
\]
(41)
For $p = 2, \ldots, P$, we define the following random variables

$$H_p^p = M_{p-1}^p - E_{p-1}^p, \quad \hat{H}_{K/2}^p = \hat{M}_{p-1}^p - \hat{E}_{p-1}^p, \quad \hat{H}_{K/2}^{p'} = \hat{M}_{p-1}^{1'} - \hat{E}_{p}^{p'}.$$ 

We now use Lemma 9 and get

$$1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0 + 1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^{p'} \hat{H}_{K/2}^p < 0 \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0$$

that is true a.s. since $\mathbb{P}(H_p^p = 0) = 0$ to get

$$\mathbb{E}[\varepsilon_K^p \phi^2(X)] = \frac{1}{4} \left[ \left( \min \left( \hat{H}_{K/2}^p, \hat{H}_{K/2}^{p'} \right) \right)^2 \phi^2(X) 1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0 \right] + \frac{1}{4} \mathbb{E} \left[ \left( \min \left( \hat{H}_{K/2}^p, \hat{H}_{K/2}^{p'} \right) \right)^2 \phi^2(X) 1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0 \right] \leq \frac{1}{4} \left( \mathbb{E} \left[ \left( \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} \right)^2 \phi^2(X) 1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0 \right] \right) = \frac{1}{2} \mathbb{E} \left[ \left( \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} \right)^2 \phi^2(X) 1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0 \right],$$

since $\hat{H}_{K/2}^p$ and $\hat{H}_{K/2}^{p'}$ have the same law given $X$. Now, we use that $\hat{H}_{K/2}^p \leq \hat{H}_{K/2}^p - H_X^p$ on $\{\hat{H}_{K/2}^p H_X^p < 0\}$ and Lemma 8 gives $1_{\hat{H}_{K/2}^p \hat{H}_{K/2}^{p'} < 0} \leq \frac{|\hat{H}_{K/2}^p - H_X^p|^n}{|H_X^p|^n}$ for $\eta > 0$. This leads to

$$\mathbb{E}[\varepsilon_K^p \phi^2(X)] \leq \frac{1}{2} \mathbb{E} \left[ \frac{\hat{H}_{K/2}^p - H_X^p}{{|H_X^p|^n}} \phi^2(X) \right].$$

We now use Lemma 9 and get $\mathbb{E}[|\hat{H}_{K/2}^p - H_X^p|^{2+\eta}|X] \leq C_{2+\eta} |D_{2+\eta}(X)| \sum_{p=2}^{P} 2^{p+1-\eta} \mathbb{E} \left[ \frac{|D_{2+\eta}(X)|}{|H_X^p|^\eta} \phi^2(X) \right].$

Using this bound in (41), we get the claim with $C = 2^{p/2} C_{2+\eta} \sum_{p=2}^{P} 2^{p+1-\eta} \mathbb{E} \left[ \frac{|D_{2+\eta}(X)|}{|H_X^p|^\eta} \phi^2(X) \right].$

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