On Backward Doubly Stochastic Differential Evolutionary System

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Abstract

In this paper, we are concerned with backward doubly stochastic differential evolutionary systems (BDSDESs for short). By using a variational approach based on the monotone operator theory, we prove the existence and uniqueness of the solutions for BDSDESs. We also establish an Itô formula for the Banach space-valued BDSDESs.

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1 Introduction

Starting from Bismut’s pioneering work [2, 3] and Pardoux and Peng’s seminal work [27], the theory of backward stochastic differential equations (BSDEs) is rather complete (for instance, see [6, 10, 19]). As a natural generalization of BSDEs, backward stochastic partial differential equations (BSPDEs) arise in many applications of probability theory and stochastic processes, for instance in the optimal control of processes with incomplete information, as an adjoint equation of the Duncan-Mortensen-Zakai filtration equation (for instance, see [1, 13, 17, 34, 39]), and naturally in the dynamic programming theory fully nonlinear BSPDEs as the so-called backward stochastic Hamilton-Jacobi-Bellman equations, are also introduced in the study of non-Markovian control problems (see Peng [29] and Englezos and Karatzas [14]).

In this work, we consider the following backward doubly stochastic differential evolutionary system:

\[
\begin{align*}
- du(t) &= F(t, u(t), v(t)) \, dt + J(t, u(t), v(t)) \, dB_t - v(t) \, dW_t, \quad t \in [0, T], \\
u(T) &= G,
\end{align*}
\]

which are first introduced by Pardoux and Peng [28] as backward doubly SDEs (BDSDESs, for short) to give a probabilistic representation for certain systems of quasilinear stochastic partial differential...
differential equations (SPDEs, for short). Similar arguments to Tang [35] yield that this class of BDSDES includes SDEs, BSDEs, BDSDEs, SPDEs, backward stochastic partial differential equations (BSPDEs, for short) and backward doubly SPDEs (BDSPDEs, for short) as particular cases.

On account of the connections between BDSDEs and SPDEs, many important results for SPDEs have been obtained: Buckdahn and Ma [8, 9] established a stochastic viscosity solution theory for SPDEs; through investigations into a class of generalized BDSDEs, Boufoussi, Castéren and Mrhardy [4] gave a probability representation for the stochastic viscosity solution of SPDEs with nonlinear Neumann boundary conditions; Zhang and Zhao [37] used the extended Feymann-Kac formula to study the stationary solutions of SPDEs; Ichihara [18] discussed the homogenization problem for SPDEs of Zakai type through the BDSDE theory; Matoussi and Stoica proved the existence and uniqueness result for the obstacle problem of quasilinear parabolic stochastic PDEs. Recently, Han, Peng and Wu [16] established a Pontryagin type maximum principle for the optimal control problems with the state process driven by BDSDEs. It is worth noting that all the BDSDESs involved in the above results are finite dimensional. For the infinite dimensional case, BDSPDEs are first introduced and studied in Tang [35] by using the method of stochastic flows, while the generalized solution theory for BDSPDEs is blank.

By using a variational approach based on the monotone operator theory, we investigate the BDSDESs and prove the existence and uniqueness of the solutions for BDSDESs. The results seem to be new both for the finite dimensional case (BDSDEs) and the infinite dimensional case. Moreover, our results are also expected to extend the results of the previous paragraph to the infinite dimensional cases, i.e., we may extend Feymann-Kac formula and establish the stochastic viscosity theory for SPDEs on Hilbert spaces, construct stationary solutions for SPDEs on infinite dimensional spaces and investigate the optimal control problems with state processes driven by infinite dimensional BDSDESs. As an application to quasi-linear BDSPDEs, we get a more general existence and uniqueness result both for BDSPDEs and BSPDEs, which fills up the gap of the generalized solution theory for BDSPDEs. For the Banach space-valued BDSDESs, we also prove an Itô formula, which plays an equally important role as that for SPDEs (for instance, see [20, 30, 32]).

Our paper is organized as follows. In the next section, we set notations, hypotheses, and the notion of the solution to BDSDES (1.1) and list the main theorem. In Section 3, we prepare several auxiliary results, including a generalized Itô formula for the Banach space-valued BDSDESs, and a useful lemma on the weak convergence which is proved through a variational approach on basis of the monotone operator theory and will be used frequently in the following context. In section 4, by using the Galerkin approximation, we prove our main theorem first for the finite dimensional case and then the infinite dimensional case. In section 5, we apply our results to several examples. Section 6 is the appendix in which we prove our Itô formula for the Banach space-valued BDSDESs.

2 Preliminaries

Let V be a real reflexive and separable Banach space, and H a real separable Hilbert space. The norm in V is denoted by \( \| \cdot \|_V \), and the inner product and norm in H is denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) respectively. In this work we always assume that V is dense, and continuously imbedded in
For $r$ is such that $\phi$ is $F$-measurable, for any $(t, \phi, \phi')$ we shall still denote the dual product between $\phi$ and $\phi'$ by $\langle \phi, \phi' \rangle_{\mathcal{F}}$. Since it follows that

$$\langle \phi, \phi' \rangle_{\mathcal{F}} = \langle \phi, \phi' \rangle_{\mathcal{F}}$$

we shall still denote the dual product between $\mathcal{F}$ and $\mathcal{F}'$ by $\langle \cdot, \cdot \rangle_{\mathcal{F}}$. Simply, we denote the above framework by $(\mathcal{F}, \mathcal{F}', \mathcal{F}^\prime; \mathcal{F})$.

Fix a finite time $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space on which are defined two mutually independent cylindrical Wiener processes $W = \{W_t : t \in [0, T]\}$ and $B = \{B_t : t \in [0, T]\}$ taking values on separable Hilbert spaces $(U_1, \langle \cdot, \cdot \rangle_{U_1}, \|\cdot\|_{U_1})$ and $(U_2, \langle \cdot, \cdot \rangle_{U_2}, \|\cdot\|_{U_2})$ respectively. Denote by $(L(U_1, H), \langle \cdot, \cdot \rangle_2, \|\cdot\|_2)$ the separable Hilbert space of all Hilbert-Schmidt operators from $U_i$ to $H$, $i = 1, 2$. Denote by $\mathcal{N}$ the set of all the $\mathbb{P}$-null sets in $\mathcal{F}$. For each $t \in [0, T]$, define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B$$

where for any process $\eta$, $\mathcal{F}_{s,t}^\eta := \sigma\{\eta_r - \eta_s : s \leq r \leq t\} \vee \mathcal{N}$ and $\mathcal{F}_{s,t}^\eta := \mathcal{F}_{0,t}^\eta$. Note that as a collection of $\sigma$-algebras, $\{\mathcal{F}_t, t \in [0, T]\}$ is not a filtration, since it is neither increasing nor decreasing. Consider BDSDES (1.1) and write it into the following integral form:

$$u(t) = G + \int_t^T F(s, u(s), v(s)) \, ds + \int_t^T J(s, u(s), v(s)) \, d\overline{B}_s$$

$$- \int_t^T v(s) \, dW_s, \quad t \in [0, T]$$

(2.1)

where for any $(t, \varphi, \phi) \in [0, T] \times V \times L(U_1, H)$,

$$F(\cdot, t, \varphi, \phi) : \Omega \rightarrow V$$

and $J(\cdot, t, \varphi, \phi) : \Omega \rightarrow L(U_2, H)$

are $\mathcal{F}_t$-measurable. Moreover, in (1.1) and (2.1) the integral with respect to $\{B_t\}$ is a backward Itô integral and the integral with respect to $\{W_t\}$ is a standard Itô integral (c.f. [25]).

For any $p, q \in [1, \infty]$ and any real separable Banach space $(U, \|\cdot\|_U)$, denote by $M^{p,q}(0, T; U)$ the totality of $\varphi \in L^p(\Omega, \mathcal{F}, L^q([0, T], B([0, T]), U))$ with

$$\|\varphi\|_{M^{p,q}(0, T; U)} := \|\varphi\|_{L^p(\Omega, \mathcal{F}, L^q([0, T], B([0, T]), U))}$$

such that $\varphi_t$ is $\mathcal{F}_t$-measurable, for a.e. $t \in [0, T]$. For simplicity, set

$$M^p(0, T; U) := M^{p,p}(0, T; U).$$

For $r \in [1, \infty)$ we denote by $S^r(0, T; U)$ the totality of $\phi \in L^r(\Omega, \mathcal{F}, C([0, T], U))$ such that $\phi_t$ is $\mathcal{F}_t$-measurable, for any $t \in [0, T]$. Define

$$\|\phi\|_{S^r(0, T; U)} := \left\{ E\left[ \sup_{t \in [0, T]} \|\phi(t)\|_U^r \right]\right\}^{1/r}, \quad \phi \in S^r(0, T; U).$$
All the spaces defined above are complete.

Moreover, Letting \( 0 \leq \tau \leq T \) be a stopping time with respect to the backward filtration \( \mathcal{F}^B_{t,T}, t \in [0,T] \), define

\[
M^p(\tau, T; U) := \{ 1_{[\tau,T]}u : u \in M^p(0, T; U) \}
\]
equipped with the norm \( \|u\|_{M^p(\tau, T; U)} = \|1_{[\tau,T]}u\|_{M^p(0, T; U)} \) and in a similar way, we define \( S^q(\tau, T; U) \) and \( M^{p,q}(\tau, T; U) \).

For simplicity, we always denote \( M^p(\tau, T; R) \) (\( S^p(\tau, T; R) \) and \( M^{p,q}(\tau, T; R) \), respectively) by \( M^p(\tau, T) \) (\( S^p(\tau, T) \) and \( M^{p,q}(\tau, T) \), respectively).

By convention, we always treat elements of spaces like \( M^p(0, T; U) \) defined above as functions rather than distributions or classes of equivalent functions, and if we know that a function of this class has a modification with better properties, then we always consider this modification. For example, if \( u \in M^p(0, T; V) \) and \( u \) has a modification lying in \( S^q(0, T; H) \), we always adopt the treatment \( u \in M^p(0, T; V) \cap S^q(0, T; H) \).

Consider our BDSDES (1.1). We define the following assumptions.

There exist constants \( 1 > \delta > 0, \alpha > 0, q > 1, \alpha_1, K, K_1, \beta \geq 0 \) and a nonnegative real-valued process \( \varsigma \in M^1(0, T) \) such that the following conditions hold for all \( v, v_1, v_2 \in V, \phi, \phi_1, \phi_2 \in L(U_1, H) \) and \( (\omega, t) \in \Omega \times [0, T] \).

(A1) (Hemicontinuity) The map \( s \mapsto \langle F(t, v_1 + sv_2, \phi), \ v \rangle \) is continuous on \( R \).

(A2) (Monotonicity)
\[
2\langle F(t, v_1, \phi_1) - F(t, v_2, \phi_2), \ v_1 - v_2 \rangle + \|J(t, v_1, \phi_1) - J(t, v_2, \phi_2)\|^2 \\
\leq K_1\|v_1 - v_2\|^2 + \delta\|\phi_1 - \phi_2\|^2;
\]

(A3) (Coercivity)
\[
2\langle F(t, v, \phi), \ v \rangle + \|J(t, v, \phi)\|^2 + \alpha\|v\|^q_v \leq \delta\|\phi\|^2_1 + K\|v\|^2 + \varsigma(t);
\]

(A4) (Growth)
\[
\|F(t, v, \phi)\|^2_v \leq \left[ \varsigma(t) + K \left( \|v\|^q_v + \|v\|^2 + \|\phi\|^2_1 \right) \right] \left( 1 + \|v\|^2 \right) ,
\]
\[
\|J(t, v, \phi)\|^2_2 \leq K \left( \varsigma(t) + \|v\|^q_v + \|v\|^2 + \|\phi\|^2_1 \right) , \quad \frac{1}{q} + \frac{1}{q} = 1;
\]

(A5) (Lipchitz Continuity)
\[
\|F(t, v, \phi_1) - F(t, v, \phi_2)\|_s \leq K\|\phi_1 - \phi_2\|_1,
\]
\[
\|J(t, v_1, \phi) - J(t, v_2, \phi)\|_2 \leq K\|v_1 - v_2\|_V;
\]

(A6)
\[
J(t, v, \phi)J^*(t, v, \phi) \leq \phi^* + K(\|J(t, 0, 0)\|^2_2 + \|v\|^2)I + \alpha_1\|v\|^q_v \wedge \|v\|^2_v J
\]
where \( J^* \) (\( \phi^* \), respectively) denotes the adjoint transformation of \( J \) (\( \phi \), respectively) and \( I \) is the identity operator on \( H \).
Remark 2.1. Actually, we can deduce from (A2) and (A3) that for all \( \phi_1, \phi_2 \in L(U_1, H) \), \( v \in V \) and \((\omega, t) \in \Omega \times [0, T] \), there hold
\[
\max\{\|J(t, 0, 0)\|_2^2, \|F(t, 0, 0)\|_2^q\} \leq \varsigma(t) \text{ and } \|J(t, v, \phi_1) - J(t, v, \phi_2)\|_2^2 \leq \delta \|\phi_1 - \phi_2\|_1^2.
\]

Remark 2.2. In view of (A4) and (A6), we see that the function \( \|J(t, \cdot, \phi)\|_V^q \) is defined on \( V \) and dominated by the norm \( \|\cdot\|_V^q \) in some sense. This property goes beyond the calculations of \([20, 21, 26, 30, 32, 38]\). Moreover, if \( J(t, v, \phi)J^*(t, v, \phi) \) does not depend on \( \|\phi\|_1 \) or \( \|v\|_V \), the assumption (A6) is not necessary in our work. In addition, as \( J(t, v, \phi) \) is Lipchitz continuous with respect to \( v \) on \( V \), it seems not so strange that \( J(t, v, \phi)J^*(t, v, \phi) \) is dominated by \( \|\|v\|_V^2 \).

Definition 2.1. We say a pair of \( V \times L(U_1, H) \)-valued processes \((u, v)\) is a solution of the backward doubly stochastic differential evolutionary system (2.1) if
\[
(u, v) \in \left( M^{pq/2,q}(0, T; V) \cap S^p(0, T; H) \right) \times M^{p,2}(0, T; L(U_1, H)), \quad \text{for some } p \geq 2, q > 1
\]
and (2.1) holds in the weak sense (called in the distributional sense as well), i.e. for any \( \varphi \in V \) there holds almost surely
\[
\begin{align*}
\langle \varphi, u(t) \rangle &= \langle \varphi, G \rangle + \int_t^T \langle F(s, u(s), v(s), \varphi) \rangle ds - \int_t^T \langle \varphi, v(s) \rangle dW_s \\quad && (2.2) \\
&+ \int_t^T \langle \varphi, J(s, u(s), v(s)) \rangle d\tilde{B}_s, \quad \forall t \in [0, T].
\end{align*}
\]

Now we show our main result as the following theorem:

Theorem 2.1. Suppose assumptions (A1)-(A5) hold. Let \( 0 \leq \alpha_1(p-2) < \alpha \), and \( \|F(\cdot, 0, 0)\|_V^q, \varsigma \in M^{p/2,1}(0, T) \) for some \( p \geq \beta + 2 \). Moreover, if \( p > 2 \), we assume (A6) holds. Then for any \( G \in L^p(\Omega, \mathcal{F}_T, H) \), BDSDES (2.1) admits a unique solution
\[
(u, v) \in \left( M^{pq/2,q}(0, T; V) \cap S^p(0, T; H) \right) \times M^{p,2}(0, T; L(U_1, H))
\]
such that
\[
\begin{align*}
\|u\|_{S^p(0,T;H)} + &\|u\|_{M^{pq/2,q}(0,T;V)}^{q/2} + \|v\|_{M^{p,2}(0,T;L(U_1,H))} \\
\leq &\ C \left\{ \|G\|_{L^p(\Omega, \mathcal{F}_T, H)} + \|\varsigma\|_{M^{p/2,1}(0,T)}^{1/2} \right\}
\end{align*}
\]
where \( C \) is a constant depending on \( T, K, q, p, \delta, \beta, \alpha_1 \) and \( \alpha \).

Here, we point out that we always denote by \( C > 0 \) a constant which may vary from line to line and moreover, we denote by \( C(a_1, a_2, \cdots) \) a constant which depends on the variables \( a_1, a_2, \cdots \) just like the one appearing in the following typical inequality
\[
ab \leq \varepsilon a^2 + C(\varepsilon) b^2, \quad \varepsilon, a, b > 0.
\]

3 Auxiliary results

First, we give a useful lemma with the sketch of its proof.
Lemma 3.1. For any given \( p \geq 1, q, d > 1, r \geq 2 \) and separable reflexive Banach spaces \( U \) and \( \bar{U} \), with \( U \) continuously and densely embedded into \( \bar{U} \), we assert that

(i) \( M^p(0,T;U), S^p(0,T;U) \) and \( M^{q,d}(0,T;L(U_i,H)), i = 1,2 \) are all separable Banach spaces, and moreover, \( M^q(0,T;U) \) and \( M^{q,d}(0,T;L(U_i,H)), i = 1,2 \) are reflexive;

(ii) let \( \{u_n, n \in \mathbb{N}\} \) converge weakly to \( u \) in \( M^p(0,T;U) \) and to \( \bar{u} \) in \( M^{q,d}(0,T;\bar{U}) \), then \( \bar{u}(\omega,t) = u(\omega,t) \) for \( \mathbb{P} \otimes dt \)-almost all \( (\omega,t) \in \Omega \times [0,T] \);

(iii) define two linear operators

\[
I(f) := \int_{t}^{T} f(s) \, ds, \quad f \in M^q(0,T;U);
\]

\[
J(h) := \int_{t}^{T} h(s) \, dW_s, \quad h \in M^{q,2}(0,T;L(U_1,H));
\]

then the linear operators \( I \) and \( J \) are continuous from \( M^q(0,T;U) \) to itself and from \( M^{q,2}(0,T;L(U_1,H)) \) to \( M^{q,2}(0,T;H) \) respectively, and moreover, they both are continuous with respect to the corresponding weak topologies;

(iv) letting \( u_n, f_n, h_n \) and \( z_n \) converge weakly to \( u, f, h, \) and \( z \) in spaces \( M^p(0,T;H), M^q(0,T;V'), M^{q,2}(0,T;L(U_2,H)) \) and \( M^{r,2}(0,T;L(U_1,H)) \) respectively, then we conclude from

\[
\lim_{n \to \infty} \|G^n - G\|_{L^q(\Omega,\mathcal{F},H)} = 0,
\]

\[
u_n(t) = G^n + \int_{t}^{T} f_n(s) \, ds + \int_{t}^{T} h_n(s) \, d\bar{B}_s - \int_{t}^{T} z_n(s) \, dW_s
\]

and \( \bar{u}(t) := G + \int_{0}^{T} f(s) \, ds + \int_{0}^{T} h(s) \, d\bar{B}_s - \int_{0}^{T} z(s) \, dW_s \)

that \( u(\omega,t) = \bar{u}(\omega,t) \) for \( \mathbb{P} \otimes dt \)-almost all \( (\omega,t) \in \Omega \times [0,T] \).

Proof. (i) is obvious. From the definition of weak convergence, it follows that

\[
\int_{\Omega \times [0,T]} 1_{\{A\}}(\omega,s) f(u(\omega,s) - \bar{u}(\omega,s)) \, \mathbb{P}(d\omega)ds = 0, \quad \forall A \in \mathcal{F} \otimes \mathcal{B}([0,T]), \forall f \in \bar{U}',
\]

which implies (ii). As for (iii), it follows from

\[
\|I(f)\|_{M^q(0,T;U)}^q = E \left[ \int_{0}^{T} \int_{t}^{T} f(s) \, ds \right] \leq T^q E \left[ \int_{0}^{T} \|f(s)\|_{U}^q \, ds \right]
\]

and

\[
\|J(h)\|_{M^{q,2}(0,T;H)}^q = E \left[ \left( \int_{0}^{T} \int_{t}^{T} h(s) \, dW_s \right)^2 \right]^{q/2}\]

\[
\leq T^{q/2} E \left[ \sup_{t \in [0,T]} \|\int_{t}^{T} h(s) \, dW_s\|^q \right]
\]

\[
\leq CE \left[ \left( \int_{0}^{T} \|h(s)\|_1^2 \, ds \right)^{q/2} \right].
\]

Finally, (iv) can be deduced from the above assertions (i), (ii) and (iii). \( \square \)
As in the theory on the forward stochastic evolutionary systems (c.f. [20, 32]), the following Itô formula plays a crucial role in the proof of our main result Theorem 2.1.

**Theorem 3.2.** Let \( \xi \in L^2(\Omega, \mathcal{F}_T, H), q > 1, q' = \frac{q}{q-1}, f \in M^q(0, T; V') \) and \( h \in M^2(0, T; L(U_2, H)) \). Assume \((u, v) \in M^q(0, T; V) \times M^2(0, T; L(U_1, H))\) and that the following BDSDES:

\[
\begin{align*}
    u(t) &= \xi + \int_t^T f(s) \, ds + \int_t^T h(s) \, dB(s) - \int_t^T v(s) \, dW_s, \quad t \in [0, T] \\
    u(t) &= \xi + \int_t^T (2f(s), u(s)) + \|h(s)\|_2^2 - \|v(s)\|_2^2 \, ds \\
    + \int_t^T 2\langle u(s), h(s) \, dB_s \rangle - \int_t^T 2\langle u(s), v(s) \, dW_s \rangle
\end{align*}
\]

holds in the weak sense of Definition 2.1. Then we assert that \( u \in S^2(0, T; H) \) and the following Itô formula holds almost surely

\[
    \|u(t)\|^2 = \|\xi\|^2 + \int_t^T (2\langle f(s), u(s) \rangle + \|h(s)\|_2^2 - \|v(s)\|_2^2) \, ds \\
    + \int_t^T 2\langle u(s), h(s) \, dB_s \rangle - \int_t^T 2\langle u(s), v(s) \, dW_s \rangle
\]

for all \( t \in [0, T] \).

Here, we note that some techniques to prove Theorem 3.2 are borrowed from [30, 32] and for the reader’s convenience, we give the proof in the appendix.

**Lemma 3.3.** The solution in Theorem 2.1 is unique.

**Proof.** Suppose \((u^1, v^1)\) and \((u^2, v^2)\) are two solutions of (1.1) in

\[
\left( M^{pq/2,q}(0, T; V) \cap S^p(0, T; H) \right) \times M^{p, 2}(0, T; L(U_1, H)).
\]

Letting \((\bar{u}, \bar{v}) = (u^1 - u^2, v^1 - v^2)\), then by the product rule, Itô formula and assumption (A2), we obtain

\[
\begin{align*}
    &E\left[ e^{K_1t} \|\bar{u}(t)\|^2 \right] \\
    &= E\left[ \int_t^T e^{K_1s} \left( 2\langle F(s, u^1(s), v^1(s)) - F(s, u^2(s), v^2(s)), \bar{u}(s) \rangle \\
    + \|J(s, u^1(s), v^1(s)) - J(s, u^2(s), v^2(s))\|_2^2 - K_1 \|\bar{u}(s)\|^2 - \|\bar{v}(s)\|_2^2 \right) \, ds \right] \\
    &\leq (\delta - 1) E\left[ \int_t^T e^{K_1s} \|\bar{v}(s)\|_1^2 \, ds \right], \quad t \in [0, T]
\end{align*}
\]

which implies

\[
E\left[ e^{K_1t} \|\bar{u}(t)\|^2 + \int_t^T e^{K_1s} \|\bar{v}(s)\|_1^2 \, ds \right] \leq 0.
\]

Thus, \((\bar{u}, \bar{v}) = 0 \, \mathbb{P} \otimes dt\text{-a.e.} \). The path-wise uniqueness follows from the path continuity of \( u^1, u^2 \) in \( H \). We complete the proof.

**Remark 3.1.** From the proof of Lemma 3.3 it follows that the uniqueness is only implied by assumptions (A2) and (A4).
In recent years, the monotonicity method (for instance, see [7, 24, 33, 36]) is generalized and intensively used to analyze SPDEs (for example, see [20, 21, 26, 30, 32, 38]) and BSPDEs (see [23, 31, 38]). In the present paper, we shall generalize it to investigate the BDSDEs. Now, we show a useful lemma which plays an important role in the variational approach and will be used frequently below.

**Lemma 3.4.** Let \( p \geq 2, q > 1 \) and \( \zeta \in M^{p/2,1}(\tau, T) \) with \( \tau (0 \leq \tau < T) \) being one stopping time with respect to the backward filtration \( \{\mathcal{F}^B_{t,T}, t \in [0, T]\} \). The pair \((F, J)\) satisfy assumptions (A1), (A2) and (A4) with \( 0 \leq \beta \leq p - 2 \) on \([\tau, T] := \{\omega, t \in [\tau(\omega), T]\}\). Moreover, we assume that there hold the following

(a) \( u^n \rightarrow u \) weakly in \( M^{pq/2,2}(\tau; T; V) \), as \( n \rightarrow \infty \);
(b) \( u^n \rightarrow u \) weakly star in \( L^p(\Omega, L^{\infty}([\tau, T], H)) \) as \( n \rightarrow \infty \);
(c) \( v^n \rightarrow v \) weakly in \( M^{p,2}(\tau; T; L(U_1, H)) \) as \( n \rightarrow \infty \);
(d) \( G^n \rightarrow G \) strongly in \( L^p(\Omega, \mathcal{F}_T, H) \) as \( n \rightarrow \infty \);
(e) \( F^n(\cdot, u^n(\cdot), v^n(\cdot)) \rightarrow F \) weakly in \( M^{\prime q}([\tau, T]; V^*) \) as \( n \rightarrow \infty \);
(f) for \( P \otimes dt\)-almost \((\omega, t) \in \Omega \times [0, T]\), \( \lim_{n \rightarrow \infty} \|F^n(\omega, t, \varphi, \xi) - F(\omega, t, \varphi, \xi)\|_{s, \omega} = 0 \)
and \( \lim_{n \rightarrow \infty} \|J^n(\omega, t, \varphi, \xi) - J(\omega, t, \varphi, \xi)\|_2 = 0 \) hold for all \( \varphi \in V \) and all \( \xi \in L(U_1, H) \);
(g) \( J^n(\cdot, u^n(\cdot), v^n(\cdot)) \rightarrow J \) weakly in \( M^2(\tau, T; L(U_2, H)) \) as \( n \rightarrow \infty \);
(h) for each \( n \in \mathbb{N} \),

\[
\begin{align*}
  u^n(t) &= G^n + \int_t^T F^n(s, u^n(s), v^n(s)) \, ds + \int_t^T J^n(s, u^n(s), v^n(s)) \, dB_s \\
  &\quad - \int_t^T v^n(s) \, dW_s, \text{ holds in the weak sense of Definition 2.1,}
\end{align*}
\]

where for each \( n \in \mathbb{N} \), the pair \((F^n, J^n)\) satisfies assumptions (A2) and (A4) on \([\tau, T]\).

Then \((u, v) \in (M^{pq/2,2}(\tau; T; V) \cap S^p(\tau; T; H)) \times M^{p,2}(0, T; L(U_1, H))\) is the unique solution to (1.1).

**Proof.** Without any loss of generality, we take \( \tau \equiv 0 \). Define

\[
\bar{u}(t) = G + \int_t^T F(s) \, ds + \int_t^T J(s) \, dB_s - \int_t^T v(s) \, dW_s, \quad t \in [0, T].
\]

From assertion (iv) of Lemma 3.1, it follows that \( u(\omega, t) = \bar{u}(\omega, t) \) for almost \( \mathbb{P} \otimes dt\)-\((\omega, t) \in [0, T]\). Identify \( u \) with its modification \( \bar{u} \). Then by Theorem 3.2, we conclude that \( u \) is an \( H \)-valued continuous process and thus \( u \in S^p(0, T; H) \). It remains for us to prove \((F(\cdot, u(\cdot), v(\cdot)), J(\cdot, u(\cdot), v(\cdot))) = (\bar{F}(\cdot), \bar{J}(\cdot)) \) \( \mathbb{P} \otimes dt\)-a.e.\text{.}

For every \((\varphi, \xi) \in \left(L^{pq/2}(\Omega, \mathcal{F}; L^q(0, T; V)) \cap L^p(\Omega, \mathcal{F}; L^{\infty}(0, T; H))\right) \times L^p(\Omega, \mathcal{F}; L^2(0, T; L(U_1, H)))\) it follows from (f) and the domination convergence theorem that

\[
\lim_{n \rightarrow \infty} E \left[ \int_0^T \left( \|J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s))\|_2^2 \\
+ 2\langle J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s)), J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ 2\langle F^n(s, \varphi(s), \xi(s)) - F(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \right) \, ds \right] = 0.
\]
On the other hand, we have

\[
\langle F^n(s, u^n(s), v^n(s)), u^n(s) \rangle \\
= \langle F^n(s, u^n(s), v^n(s)) - F^n(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ \langle F^n(s, \varphi(s), \xi(s)) - F(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ \langle F^n(s, u^n(s), v^n(s)) - F(s, \varphi(s), \xi(s)), \varphi(s) \rangle + \langle F(s, \varphi(s), \xi(s)), u^n(s) \rangle,
\]

\[
\|J^n(s, u^n(s), v^n(s))\|_2^2 \\
= \|J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s))\|_2^2 + \|J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s))\|_2^2 \\
+ 2\langle J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s)), J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ \langle J(s, \varphi(s), \xi(s)), J^n(s, u^n(s), v^n(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ \langle J^n(s, u^n(s), v^n(s)), J(s, \varphi(s), \xi(s)) \rangle_2,
\]

\[
\|v^n(s)\|_1^2 = \|v^n(s) - \xi(s)\|_1^2 + \langle u^n(s) - \xi(s), \xi(s) \rangle_1 + \langle \xi(s), v^n(s) \rangle_1. \\
\|u^n(s)\|_1^2 = \|u^n(s) - \varphi(s)\|_1^2 + \langle u^n(s) - \varphi(s), \varphi(s) \rangle + \langle \varphi(s), u^n(s) \rangle
\]

and in view of (A2),

\[
2\langle F^n(s, u^n(s), v^n(s)) - F^n(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ \|J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s))\|_2^2 - \|v^n(s) - \xi(s)\|_1^2 - K_1\|u^n(s) - \varphi(s)\|_2^2 \leq 0.
\]
Therefore, by Theorem 3.2 and the product rule, we have almost surely

\[
e^{KiT} \| u^n(t) \|^2 \\
= e^{KiT} \| G^n \|^2 + \int_t^T 2e^{Kis} \langle u^n(s), J^n(s, u^n(s), v^n(s)) \rangle \, d\tilde{B}_s \\
+ \int_t^T e^{Kis} \left[ 2\langle F^n(s, u^n(s), v^n(s)), u^n(s) \rangle - \| v^n(s) \|^2 + \| J^n(s, u^n(s), v^n(s)) \|^2 \right. \\
- K_1 \| u^n(s) \|^2 \] \, ds - \int_t^T 2e^{Kis} \langle u^n(s), v^n(s) \rangle \, dW_s \\
\leq e^{KiT} \| G^n \|^2 + \int_t^T 2e^{Kis} \langle u^n(s), J^n(s, u^n(s), v^n(s)) \rangle \, d\tilde{B}_s \\
+ \int_t^T e^{Kis} \left[ 2\langle F^n(s, u^n(s), v^n(s)) - F^n(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ \| J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s)) \|^2_2 - \| v^n(s) - \xi(s) \|^2_2 - K_1 \| u^n(s) - \varphi(s) \|^2 \right] \, ds \\
+ \int_t^T e^{Kis} \left[ [J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s))]_2 \\
+ 2\langle J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s)), J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ 2\langle F^n(s, \varphi(s), \xi(s)) - F(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ 2\langle F^n(s, u^n(s), v^n(s)) - F(s, \varphi(s), \xi(s)), \varphi(s) \rangle + 2\langle F(s, \varphi(s), \xi(s)), u^n(s) \rangle \\
+ \langle J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ \langle J^n(s, u^n(s), v^n(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
- \langle \xi(s), v^n(s) - \xi(s) \rangle_1 - \langle v^n(s), \xi(s) \rangle_1 - K_1(\varphi(s), u^n(s) - \varphi(s)) \\
- K_1(\varphi(s), v^n(s)) \] \, ds - \int_t^T 2e^{Kis} \langle u^n(s), v^n(s) \rangle \, dW_s \\
\leq e^{KiT} \| G^n \|^2 + \int_t^T 2e^{Kis} \langle u^n(s), J^n(s, u^n(s), v^n(s)) \rangle \, d\tilde{B}_s \\
+ \int_t^T e^{Kis} \left[ [J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s))]_2 \\
+ 2\langle J^n(s, u^n(s), v^n(s)) - J^n(s, \varphi(s), \xi(s)), J^n(s, \varphi(s), \xi(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ 2\langle F^n(s, \varphi(s), \xi(s)) - F(s, \varphi(s), \xi(s)), u^n(s) - \varphi(s) \rangle \\
+ 2\langle F^n(s, u^n(s), v^n(s)) - F(s, \varphi(s), \xi(s)), \varphi(s) \rangle + 2\langle F(s, \varphi(s), \xi(s)), u^n(s) \rangle \\
+ \langle J(s, \varphi(s), \xi(s)) \rangle_2 \\
+ \langle J^n(s, u^n(s), v^n(s)) - J(s, \varphi(s), \xi(s)) \rangle_2 \\
- \langle \xi(s), v^n(s) - \xi(s) \rangle_1 - \langle v^n(s), \xi(s) \rangle_1 - K_1(\varphi(s), u^n(s) - \varphi(s)) \\
- K_1(\varphi(s), v^n(s)) \] \, ds - \int_t^T 2e^{Kis} \langle u^n(s), v^n(s) \rangle \, dW_s, \ \forall t \in [0, T].
\]

Letting \( n \to \infty \), by [3.7] and the lower continuity of weak convergence, we obtain for every
nonnegative $\psi \in L^\infty([0,T],\mathbb{R}^+)$,
\[
E \left[ \int_0^T \psi(t) \left( e^{K_1 t} \|u(t)\|^2 - e^{K_1 T} \|G\|^2 \right) \, dt \right]
\leq \liminf_{n \to \infty} E \left[ \int_0^T \psi(t) \left( e^{K_1 t} \|u^n(t)\|^2 - e^{K_1 T} \|G^n\|^2 \right) \, dt \right]
\leq E \left[ \int_0^T \psi(t) \left( \int_t^T e^{K_1 s} \left( 2 \langle F(s) - F(s, \varphi(s), \xi(s)) , \varphi(s) \rangle 
+ 2 \langle F(s, \varphi(s), \xi(s)) , u(s) \rangle + \langle J(s, \varphi(s), \xi(s)) , J(s) - J(s, \varphi(s), \xi(s)) \rangle 
+ \langle J(s) , J(s, \varphi(s), \xi(s)) \rangle - \langle \xi(s) , v(s) - \xi(s) \rangle - \langle v(s) , \xi(s) \rangle 
- K_1 \langle \varphi(s) , u(s) - \varphi(s) \rangle - K_1 \langle u(s) , \varphi(s) \rangle \right) \, ds \right) \, dt \right].
\] 
As
\[
E \left[ e^{K_1 t} \|u(t)\|^2 - e^{K_1 T} \|G\|^2 \right] = E \left[ \int_0^T e^{K_1 s} \left( 2 \langle \bar{F}(s) , u(s) \rangle + \|\bar{J}(s)\|_2^2 - \|v\|_1^2 - K_1 \|u(s)\|^2 \right) \, ds \right],
\] 
by inserting (3.9) into (3.8) we obtain
\[
E \left[ \int_0^T \psi(t) \left( \int_t^T e^{K_1 s} \left[ 2 \langle \bar{F}(s) - F(s, \varphi(s), \xi(s)) , u(s) - \varphi(s) \rangle - \|v(s) - \xi(s)\|_1^2 
+ \|\bar{J}(s) - J(s, \varphi(s), \xi(s))\|_2^2 - K_1 \|u(s) - \varphi(s)\|^2 \right) \, ds \right) \, dt \right] \leq 0.
\] 
Taking $(\varphi, \xi) = (u, v)$ we obtain $J(\cdot, u(\cdot), v(\cdot)) = \bar{J}$. Finally, first applying (3.10) to $(\varphi, \xi) = (u - \varepsilon \phi h, v)$ for $\varepsilon > 0, \phi \in L^\infty(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}(\{0,T\}))$ and $h \in V$, then dividing both sides by $\varepsilon$ and letting $\varepsilon \downarrow 0$, by (A1), (A4) and the dominated convergence theorem, we obtain
\[
E \left[ \int_0^T \psi(t) \left( \int_t^T e^{K_1 s} \langle \bar{F}(s) - F(s, u(s), v(s)) , h \rangle \, ds \right) \, dt \right] \leq 0
\] 
which, together with the arbitrariness of $\psi, h$ and $\phi$, implies $\bar{F} = F(\cdot, u, v)$.

Hence $(u, v)$ is a solution of (3.1) and the uniqueness follows from Lemma 3.3 and Remark 3.1.

\[\square\]

**Remark 3.2.** In view of the proof of Lemma 3.1, we can replace the assumption (f) by the following one:

for $dt$-almost $t \in [0,T]$,
\[
\lim_{n \to \infty} E[\|F^n(\omega, t, \varphi, \xi) - F(\omega, t, \varphi, \xi)\|_q^2 + \|J^n(\omega, t, \varphi, \xi) - J(\omega, t, \varphi, \xi)\|_2^2] = 0,
\] 
holds for all $(\varphi, \xi) \in V \times L(U_1, H)$.

**Remark 3.3.** Indeed, instead of $(F^n, J^n)$ satisfying (A2), (A4) on $[\tau, T]$ and (f) (or Remark 3.2), in order to obtain the assertion of Lemma 3.1 we need only to find $(\bar{F}^n, \bar{F}^n)$ satisfying (A2), (A4), (e), (g) and (f) (or Remark 3.2) such that
\[
2 \langle F^n(s, u^n(s), v^n(s)) , u^n(s) \rangle + \|J^n(s, u^n(s), v^n(s))\|_2^2 
\leq 2 \langle \bar{F}^n(s, u^n(s), v^n(s)) , u^n(s) \rangle + \|\bar{J}^n(s, u^n(s), v^n(s))\|_2^2, \text{ a.e. } (\omega, s) \in \Omega \times [0,T].
\] 
We can verify this claim in a similar way to the proof of Lemma 3.4.
4 Proof of Theorem 2.1

4.1 The finite dimensional case

**Theorem 4.1.** let \(l,m,n \in \mathbb{N}\) and \(V = H = V' = \mathbb{R}^n, U_1 = \mathbb{R}^m, U_2 = \mathbb{R}^l\). Then under the assumptions in Theorem 2.1 there exists a unique solution pair \((u, v) \in (S^p(0, T; H) \cap M^{p/2,q}(0, T; V)) \times M^{p,2}(0, T; L(U_1, H))\) to BDSDES (1.1).

Before the proof of Theorem 4.1 we show the following lemma which gives the estimates to the solution pair \((u, v)\) of BDSDES (1.1) in Theorem 4.1.

**Lemma 4.2.** Under the assumptions in Theorem 4.1 if
\[(u, v) \in (S^p(0, T; H) \cap M^{p/2,q}(0, T; V)) \times M^{p,2}(0, T; L(U_1, H))\]
is a solution to the equation (2.1), there holds the following estimate
\[
\|u\|_{S^p(0, T; H)} + \|u\|_{M^q(0, T; V)} + \|u\|_{M^{p/2,q}(0, T; V)}^{q/2} + \|v\|_{M^{p,2}(0, T; L(U_1, H))} 
\leq C \{ \|G\|_{L^p(0, \bar{T}, H)} + \|\varsigma\|_{M^{p/2,1}(0, \bar{T})}^{1/2}, \}
\]
where \(C\) is a nonnegative constant depending on \(T, K, q, p, \alpha, \alpha_1\) and \(\alpha_1\).

**Proof.** By Itô formula, we have
\[
\|u(t)\|^p + \frac{p}{2} \int_t^T \|u(s)\|^{p-2} \|v(s)\|^2 ds + \frac{p(p-2)}{2} \int_t^T \|u(s)\|^{p-4} \langle u(s), v(s)v^*(s)u(s) \rangle ds \\
= \|G\|^p + p \int_t^T \|u(s)\|^{p-2} \langle F(s, u(s), v(s)), u(s) \rangle ds \\
+ p \int_t^T \|u(s)\|^{p-2} \langle J(s, u(s), v(s)) d\mathcal{B}_s \rangle - p \int_t^T \|u(s)\|^{p-2} \langle u(s), v(s) dW_s \rangle \\
+ \frac{p(p-2)}{2} \int_t^T \|u(s)\|^{p-4} \langle u(s), J(s, u(s), v(s))J(s, u(s), v(s))^* u(s) \rangle ds \\
+ \frac{p}{2} \int_t^T \|u(s)\|^{p-2} \langle J(s, u(s), v(s)), v(s) \rangle ds \\
\leq \|G\|^p + p \int_t^T \|u(s)\|^{p-2} \langle J(s, u(s), v(s)) d\mathcal{B}_s \rangle - p \int_t^T \|u(s)\|^{p-2} \langle u(s), v(s) dW_s \rangle \\
+ \frac{p(p-2)}{2} \int_t^T \|u(s)\|^{p-4} \left( K \|u(s)\|^2 + K \|J(s, 0, 0)\|^2 + \alpha_1 \|u(s)\|_V \right) + \|u\|^{p-4} \langle u(s), v(s)v^*(s)u(s) \rangle ds \\
+ \frac{p}{2} \int_t^T \|u(s)\|^{p-2} \langle K \|u(s)\|^2 + \delta \|v(s)\|^2 + \varsigma(s) - \alpha \|u(s)\|_V \rangle ds \\
\leq \|G\|^p + p \int_t^T \|u(s)\|^{p-2} \langle J(s, u(s), v(s)) d\mathcal{B}_s \rangle - p \int_t^T \|u(s)\|^{p-2} \langle u(s), v(s) dW_s \rangle \\
+ \frac{p(p-2)}{2} \int_t^T \|u(s)\|^{p-2} \|v(s)\|^2 ds + \frac{p(p-2)\alpha_1}{2} \int_t^T \|u(s)\|^{p-2} \|u(s)\|_V^q ds \\
+ C \left[ \left( \int_0^T \varsigma(s) ds \right)^{p/2} + \int_t^T \|u(s)\|^{p} ds \right] + \frac{1}{4} \sup_{s \in [t, T]} \|u(s)\|^p \\
+ \frac{p(p-2)}{2} \int_t^T \|u(s)\|^{p-4} \langle u(s), v(s)v^*(s)u(s) \rangle ds, t \in [0, T]\]
which together with the following

\[
E\left[ \sup_{t \in [\tau, T]} \left( \left| \int_{t}^{T} \|u(s)\|^{p-2}u(s), J(s, u(s), v(s))d\overline{B}_s \right| \right. \right. \\
\left. \left. + \left| \int_{t}^{T} \|u(s)\|^{p-2}v(s), dW_{s} \right| \right) \right] \\
\leq CE \left[ \int_{\tau}^{T} \|u(s)\|^{2p-2} \varsigma(s) + \|u(s)\|^{q} + \|v(s)\|^{2} \right] ds^{1/2} \\
+ \left| \int_{\tau}^{T} \|u(s)\|^{2p-2} \|v(s)\|^{2} ds \right|^{1/2} \\
\leq \varepsilon_{1}E \left[ \sup_{t \in [\tau, T]} \|u(t)\|^{p} \right] + C(\varepsilon_{1}, K, T)E \left[ \int_{\tau}^{T} \|u(s)\|^{p} \right] ds + \left( \int_{\tau}^{T} \varsigma(s) ds \right)^{p/2} \\
+ \int_{\tau}^{T} \|u(s)\|^{p-2} \|v(s)\|^{2} ds + \int_{\tau}^{T} \|u(s)\|^{p-2} \|u(s)\|^{q} ds, \tau \in [0, T]
\]

implies by Gronwall inequality and Young inequality that

\[
E\left[ \sup_{t \in [0, T]} \|u(t)\|^{p} \right] + E \left[ \int_{0}^{T} \|u(s)\|^{p-2} \left( \|v(s)\|^{2} + \|u(s)\|^{q} \right) ds \right] \\
\leq C \left\{ E[\|G\|^{p}] + \left( \int_{0}^{T} \varsigma(s) ds \right)^{p/2} \right\}.
\] (4.2)

By Itô formula, we have

\[
\|u(t)\|^{2} + \int_{t}^{T} \|v(s)\|^{2} ds \\
= \|G\|^{2} + 2 \int_{t}^{T} \langle F(s, u(s), v(s)), u(s) \rangle ds + 2 \int_{t}^{T} \langle u(s), J(s, u(s), v(s))d\overline{B}_s \rangle \\
- 2 \int_{t}^{T} \langle u(s), v(s)dW_{s} \rangle + \int_{t}^{T} \|J(s, u(s), v(s))\|^{2} ds \\
\leq \|G\|^{2} + 2 \int_{t}^{T} \langle u(s), J(s, u(s), v(s))d\overline{B}_s \rangle - 2 \int_{t}^{T} \langle u(s), v(s)dW_{s} \rangle \\
+ K \int_{t}^{T} \|u(s)\|^{2} ds + \delta \int_{t}^{T} \|v(s)\|^{2} ds - \alpha \int_{t}^{T} \|u(s)\|^{q} ds + \int_{t}^{T} \varsigma(s) ds, \tau \in [0, T].
\]

Taking $L^{p/2}(\Omega, \mathcal{F})$-norm on both sides and noticing that

\[
E\left[ \left( \int_{t}^{T} \langle u(s), v(s)dW_{s} \rangle \right)^{p/2} \right] + \left( \int_{t}^{T} \langle u(s), J(s, u(s), v(s))d\overline{B}_s \rangle \right)^{p/2} \right] \\
\leq CE \left[ \left( \int_{t}^{T} \|u(s)\|^{2} \|v(s)\|^{2} ds \right)^{p/4} \right] + \left( \int_{t}^{T} \|u(s)\|^{2} \|J(s, u(s), v(s))\|^{2} ds \right)^{p/4} \right] \\
\leq \varepsilon_{2} \left\{ \|v\|_{\mathcal{L}^{p/2}(\Omega; L^{2}(U, H))} + \|u\|_{\mathcal{L}^{p/2}(\Omega; L^{2}(T; V))}^{p/2} \right\} \\
+ C(\varepsilon_{2}, p) \left\{ \|u\|_{\mathcal{S}^{p}(0, T; H)} + \|v\|_{\mathcal{S}^{p/2}(0, T; V)}^{p/2} \right\}.
\]
by the Young inequality and letting $\varepsilon_2$ be small enough, we obtain
\[
\|v\|_{M^{p,2}(0,T;L(U_2,H))} + \|u\|_{M^{p/2,0}(0,T;V)}^{q/2} \\
\leq C \left\{ \|u\|_{S^p(0,T,H)} + \|\xi\|_{M^{p/2,1}(0,T)}^{1/2} + \|G\|_{L^p(\Omega,\mathcal{F}_T,H)} \right\},
\] (4.3)
which together with (4.2) implies our estimate (4.1). We complete the proof. \hfill \Box

To prove Theorem 1.4, we need the following lemma which can be viewed as a corollary of [28, Theorem 1.4]. It is very likely that this result has already appeared somewhere, but we have not seen it, so we provide a proof here for the reader’s convenience.

**Lemma 4.3.** Let $p \geq 2$, $H = V = V' = \mathbb{R}^n$, $U_1 = \mathbb{R}^m$ and $U_2 = \mathbb{R}^l$, $n, m, l \in \mathbb{N}$. Assume that $(f, h)$ satisfies (A2) and (A5), and that for all $(\omega, t) \in \Omega \times [0,T]$, $z \in \mathbb{R}^{n \times m}$, $y_1, y_2 \in \mathbb{R}^n$,
\[
\|f(\omega, t, y_1, z) - f(\omega, t, y_2, z)\| \leq K |y_1 - y_2|,
\] (4.4)
where the constant $K$ comes from assumption (A5). Moreover, if $p > 2$, we suppose (A6) holds for the pair $(f, h)$. Let $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ and
\[
E \left[ \left( \int_0^T \|f(t, 0, 0)\| \, dt \right)^p + \left( \int_0^T \|h(t, 0, 0)\|_2^2 \, dt \right)^{p/2} \right] < \infty.
\]
Then the backward doubly stochastic differential equation (BDSDE, for short)
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T h(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \quad t \in [0,T] \tag{4.5}
\]
admits a unique solution $(Y, Z) \in S^p(0,T;\mathbb{R}^n) \times M^{p,2}(0,T;\mathbb{R}^{n \times m})$ such that
\[
E \left[ \sup_{t \in [0,T]} \|Y(t)\|^p + \left( \int_0^T \|Z_s\|_2^2 \, ds \right)^{p/2} \right] \leq C \left\{ E[\|\xi\|^p] + \left( \int_0^T \|f(s, 0, 0)\| \, ds \right)^p + \left( \int_0^T \|h(s, 0, 0)\|_2^2 \, ds \right)^{p/2} \right\} \tag{4.6}
\]
where $C$ is a constant depending on $T, p, K, \delta, \alpha$ and $\alpha_1$.

**Proof.**
**Step 1.** In a similar way to the proof of Lemma 4.2, we prove our estimate (4.6). Indeed, the only difference lies in the fact that, by (A2) we have
\[
\langle Y_s, f(s, Y_s, Z_s) \rangle + \|h(s, Y_s, Z_s)\|_2^2 \leq K_1 \|Y_s\|^2 + \delta \|Z_s\|_2^2 + \|Y_s\| \|f(s, 0, 0)\| + \|h(s, 0, 0)\|_2^2
\]
instead of the assumption (A3) on the pair $(f, h)$.
**Step 2.** In a similar way to Lemma 4.3, we prove the uniqueness.
**Step 3.** We prove the existence of the solution. Let
\[
f^N(t, y, z) = f(t, y, z) - f(t, 0, 0) + f(t, 0, 0)1_{\{\|f(t, 0, 0)\| \leq N\}}, \quad (t, y, z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m}.
\]
Then by [28, Theorem 1.4], there exists a unique solution $(Y^N, Z^N) \in S^p(0, T; \mathbb{R}^n) \times M^{p,2}(0, T; \mathbb{R}^{n \times m})$ to BDSDE (4.5) with $f$ replaced by $f^N$. Let $N, N' \in \mathbb{N}$ and $N' > N$. Then through a similar procedure to Step 1., we obtain

\[
\sup_{N \in \mathbb{N}} \left\{ \|Y^N\|_{S^p(0,T;\mathbb{R}^n)} + \|Y^N\|_{M^2(0,T;\mathbb{R}^n)} + \|Z^N\|_{M^{p,2}(0,T;\mathbb{R}^{n \times m})} \right\} \leq C, \\
\|Y^N - Y^{N'}\|_{S^2(0,T;\mathbb{R}^n)} + \|Z^N - Z^{N'}\|_{M^2(0,T;\mathbb{R}^{n \times m})} \\
\leq C\|f(\cdot, 0, 0)1_{\{|f(\cdot, 0, 0)| \leq |N,N'|\|\|}_{M^{2,1}(0,T;\mathbb{R}^n)} \rightarrow 0 \text{ as } N, N' \to \infty
\]

with the constant $C$ independent of $N$.

Thus, $(Y^N, Z^N)_{N \in \mathbb{N}}$ is a Cauchy sequence in $S^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times m})$. Denote the limit by $(Y, Z)$. From the Lipchitz continuity of $(f(\cdot, y, z), g(\cdot, y, z))$ with respect to $(y, z)$, it follows that

\[
\|f^N(\cdot, Y^N, Z^N) - f(\cdot, Y, Z)\|_{M^{2,1}(0,T;\mathbb{R}^n)} + \|h(\cdot, Y^N, Z^N) - h(\cdot, Y, Z)\|_{M^{p,2}(0,T;\mathbb{R}^{n \times 2})} \to 0.
\]

Hence, $(Y, Z)$ is a solution of BDSDE (4.5) in $S^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times m})$.

On the other hand, in view of the first equation in (4.7), we have

\[
(Y, z) \in (L^p(\Omega, L^\infty([0, T], \mathbb{R}^n)) \cap S^2(0, T; \mathbb{R}^n)) \times M^{p,2}(0, T; \mathbb{R}^{n \times m}),
\]

which implies that $(Y, Z)$ lies in $S^p(0, T; \mathbb{R}^n) \times M^{p,2}(0, T; \mathbb{R}^{n \times m})$. We complete the proof. $\square$

**Proof of Theorem 4.7.** **Step 1.** First of all, let us reduce assumption (A2) to the case of $K_1 = 0$. Assume $(u, v)$ is a solution to Equation (2.1) and set

\[
\theta(t) := e^{\frac{tK_1}{2}}, \\
\bar{u}(\omega, t) := \theta(t)u(\omega, t), \\
\bar{v}(\omega, t) := \theta(t)v(\omega, t), \\
\bar{J}(\omega, t, \bar{u}(t), \bar{v}(t)) := \theta(t)J(\omega, t, \theta(t)^{-1}\bar{u}(t), \theta(t)^{-1}\bar{v}(t)), \\
\bar{F}(\omega, t, \bar{u}(t), \bar{v}(t)) := \theta(t)F(\omega, t, \theta(t)^{-1}\bar{u}(t), \theta(t)^{-1}\bar{v}(t)) - \frac{1}{2}K_1\bar{u}(t).
\]

Then through careful computations, we check that the pair $(\bar{F}, \bar{J})$ also satisfies the same assumptions given to the pair $(F, J)$ only with the constant $K$ ($K_1$, respectively) replaced by another nonnegative constant $\bar{K}$ (0, respectively). Hence, we may assume $K_1 = 0$ in the following proof.

**Step 2.** Take $r \geq p \vee (q + \beta) \vee \frac{\rho}{\lambda}$. Assume further that $\|F(\cdot, 0, 0)\|_{L^r(\Omega, L^1([0, T], \mathbb{R}))}$, $\|J(\cdot, 0, 0)\|_{2} \in M^{r,2}(0, T; \mathbb{R})$ and $G \in L^r(\Omega, \mathcal{F}_T, \mathbb{R}^n)$. Fix $\bar{v} \in M^{r,2}(0, T; \mathbb{R}^n \times \mathbb{R}^m)$. Consider the following backward doubly stochastic differential equation:

\[
u(t) = G + \int_t^T F(s, u(s), \bar{v}(s)) \, ds + \int_t^T J(s, u(s), v(s)) \, d\bar{B}_s - \int_t^T v(s) \, dW_s. \tag{4.8}
\]

In this case, for every $(\omega, t) \in \Omega \times [0, T], x \mapsto F(\omega, t, x, \bar{v}(\omega, t))$ is a continuous monotone function on $\mathbb{R}^n$. Let $F_\varepsilon(\omega, t, \cdot) (\varepsilon > 0)$ be the Yosida approximation of $F(\omega, t, \cdot, \bar{v}(\omega, t))$, i.e.

\[
F_\varepsilon(\omega, t, x) := e^{-\varepsilon}(H_\varepsilon(\omega, t, x) - x) = F(\omega, t, H_\varepsilon(\omega, t, x)) \\
H_\varepsilon(\omega, t, x) := (I - \varepsilon F(\omega, t, \cdot, \bar{v}(\omega, t)))^{-1}(x).
\tag{4.9}
\]
Then we conclude (c.f. [36]) that $x \mapsto H_\varepsilon(\omega, t, x)$ is a homeomorphism on $\mathbb{R}^n$ for each $(\omega, t)$ and that for any $x, y \in \mathbb{R}^n$

(a) $\|F_\varepsilon(\omega, t, x) - F_\varepsilon(\omega, t, y)\| \leq 2\varepsilon^{-1}\|x - y\|$;
(b) $\|F_\varepsilon(\omega, t, x)\| \leq \|F(\omega, t, x, \tilde{\nu}(\omega, t))\|$;
(c) $\langle F_\varepsilon(\omega, t, x) - F_\varepsilon(\omega, t, y), x - y \rangle \leq 0$;
(d) $\lim_{\epsilon \to 0} \|F_\varepsilon(\omega, t, x) - F(\omega, t, x, \tilde{\nu}(\omega, t))\| = 0$.

It follows from (b), (c) and (A5) that for any $x \in \mathbb{R}^n$

$$\langle F_\varepsilon(\omega, t, x), x \rangle \leq \|x\| \cdot \|F_\varepsilon(\omega, t, 0)\| \leq (\|F(\omega, t, 0, 0)\| + K\|\tilde{\nu}(\omega, t)\|) \cdot \|x\|. \tag{4.10}$$

By Lemma 4.3 there exists a unique solution $(u^\varepsilon, v^\varepsilon) \in S^r(0, T; \mathbb{R}^n) \times M^{r, 2}(0, T; \mathbb{R}^{n \times m})$ to the following BDSDE:

$$u^\varepsilon(t) = G + \int_t^T F_\varepsilon(s, u^\varepsilon(s)) \, ds + \int_t^T J(s, u^\varepsilon(s), v^\varepsilon(s)) \, d\tilde{B}_s - \int_t^T v^\varepsilon(s) \, dW_s. \tag{4.11}$$

Since $V = H = V' = \mathbb{R}^n$, in view of (A6) we have

$$J(t, \varphi, \phi)J^*(t, \varphi, \phi) \leq \phi \phi^* + C(K, \alpha_1) \left(\|J(t, 0, 0)\|_2^2 + \|\varphi\|^2\right) I, \quad \forall (\varphi, \phi) \in \mathbb{R}^n \times \mathbb{R}^{n \times m}.$$

Thus, by using (4.10) and in a similar way to the proof of Lemma 4.2, we obtain

$$\begin{align*}
\|u^\varepsilon\|_{S^r(0, T; \mathbb{R}^n)} + & \|v^\varepsilon\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times m})} \\
\leq & C \left\{ \|\zeta\|_{L^r(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R}^n)} + \|F(\cdot, 0, 0)\|_{M^{r, 1}(0, T; \mathbb{R}^n)} + \|J(\cdot, 0, 0)\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times l})} \\
& + \|\tilde{\nu}\|_{M^{r, 1}(0, T; \mathbb{R}^{n \times m})} \right\} \\
\leq & C \left\{ \|\zeta\|_{L^r(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R}^n)} + \|F(\cdot, 0, 0)\|_{M^{r, 1}(0, T; \mathbb{R}^n)} + \|J(\cdot, 0, 0)\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times l})} \\
& + T^{1/2}\|\tilde{\nu}\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times m})} \right\} \tag{4.12}
\end{align*}$$

with the constant $C$ independent of $\varepsilon$.

On the other hand, we have

$$\begin{align*}
E \left[ \int_0^T \|F_\varepsilon(t, u^\varepsilon(t))\| q' \, dt \right] \\
\leq E \left[ \int_0^T \|F(t, u^\varepsilon(t), \tilde{\nu}(t))\| q' \, dt \right] \\
\leq E \left[ \int_0^T (s(t) + K(\|u^\varepsilon(t)\|_V^q + \|u^\varepsilon(t)\|_V^2 + \|\tilde{\nu}(t)\|_V^2)) \left(1 + \|u^\varepsilon(t)\|^{2q} \right) \, dt \right] \\
\leq C(K, T, \|s\|_{M^{r/2, 1}(0, T)}, \|\tilde{\nu}\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times m})}, \|u^\varepsilon\|_{S^r(0, T; \mathbb{R}^n)}) \tag{4.13}
\end{align*}$$

and

$$\begin{align*}
E \left[ \int_0^T \|J(t, u^\varepsilon(t), v^\varepsilon(t))\|_2^2 \, dt \right] \\
\leq CE \left[ \int_0^T (\|J(t, 0, 0)\|_2^2 + \|u^\varepsilon(t)\|_V^2 + \|v^\varepsilon(t)\|_V^2) \, dt \right] \\
\leq C \left\{ \|J(\cdot, 0, 0)\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times m})} + \|u^\varepsilon\|_{M^{r, 2}(0, T; \mathbb{R}^{n \times m})} + \|u^\varepsilon\|_{S^r(0, T; \mathbb{R}^n)} \right\} \tag{4.14}
\end{align*}$$
Combining (4.12), (4.13) and (4.14), we conclude that there exists a sequence \( \varepsilon_k \downarrow 0 \) and \((\bar{u}, \bar{v}, \bar{F}, \bar{J})\) such that

\[
\begin{align*}
\text{Integrating both sides from } 0 \text{ to } T, \quad &

\text{we have}

\quad u^{\varepsilon_k} \to \bar{u} \text{ weakly in } L^r(\Omega, L^\infty(0, T; \mathbb{R}^n)); \\
\quad u^{\varepsilon_k} \to \bar{u} \text{ weakly in } M^r(0, T; \mathbb{R}^n); \\
\quad v^{\varepsilon_k} \to \bar{v} \text{ weakly in } M^{r,2}(0, T; \mathbb{R}^{n \times m}); \\
\quad F_{\varepsilon_k}(\cdot, u^{\varepsilon_k}(\cdot)) \to \bar{F} \text{ weakly in } M^{r'}(0, T; \mathbb{R}^n); \\
\quad J(\cdot, u^{\varepsilon_k}(\cdot), v^{\varepsilon_k}(\cdot)) \to \bar{J} \text{ weakly in } M^2(0, T; \mathbb{R}^{n \times I})
\end{align*}
\]

as \( k \to \infty \).

By Lemma 3.4 \((\bar{u}, \bar{v}) \in S^r(0, T; \mathbb{R}^n) \times M^{r,2}(0, T; \mathbb{R}^{n \times m})\) is the unique solution to (4.8).

Take \( t^0 = 0 \). We consider the following Picard iteration: for \( k \in \mathbb{N} \), let \((u^k, v^k) \in M^r(0, T; \mathbb{R}^n) \times M^{r,2}(0, T; \mathbb{R}^{n \times m})\) be the unique solution of (4.8) with \( \bar{v} \) replaced by \( v^{k-1} \) there. Set \((X^k, Z^k) := (u^{k-1} - u^k, v^{k-1} - v^k)\). In view of the assumption (A2) (with \( K_1 = 0 \)), we have

\[
E \left[ \|X^k(t)\|^2 + \int_t^T \|Z^k(s)\|_1^2 \, ds \right]
= E \left[ 2 \int_t^T \left\{ (X^k(s), F(s, u^{k+1}(s), v^{k+1}(s)) - F(s, u^k(s), v^{k-1}(s)) \right) \right.
+ \int_t^T \|J(s, u^{k+1}(s), v^{k+1}(s)) - J(s, u^k(s), v^k(s))\|_2^2 \, ds \n\right]
= E \left[ \int_t^T \left\{ 2(X^k(s), F(s, u^{k+1}(s), v^{k+1}(s)) - F(s, u^k(s), v^k(s)) \right) \right.
+ \int_t^T \|J(s, u^{k+1}(s), v^{k+1}(s)) - J(s, u^k(s), v^k(s))\|_2^2 \, ds \n\right]
\leq E \left[ \int_t^T \|Z^k(s)\|_1^2 \, ds + C_0 \int_t^T \|X^k(s)\|^2 \, ds + \frac{1 - \delta}{2} \int_t^T \|Z^k(s)\|_1^2 \, ds \n\right]
+ \frac{1 - \delta}{4} \int_t^T \|Z^{k-1}(s)\|_1^2 \, ds
\]

(4.16)

Thus, for \( \mu := 2C_0 / (1 - \delta) \)

\[
\begin{align*}
- \frac{d}{dt} \left( e^{\mu t} \int_t^T E[\|X^k(s)\|^2] \, ds \right) + e^{\mu t} \int_t^T E[\|Z^k(s)\|_1^2] \, ds 
\leq \frac{e^{\mu t}}{2} \int_t^T E[\|Z^{k-1}(s)\|_1^2] \, ds =: \frac{a_k(t)}{2}.
\end{align*}
\]

Integrating both sides from 0 to \( T \), we get

\[
\int_0^T E[\|X^k(s)\|^2] \, ds + \int_0^T a_{k+1}(t) \, dt \leq \frac{1}{2} \int_0^T a_k(t) \, dt \leq \frac{1}{2k} \int_0^T a_1(t) \, dt =: \frac{c_1}{2k}.
\]

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Then it follows from (4.16) that
\[
\int_0^T E[\|Z^k(s)\|^2_1] \, ds \leq \frac{c_1 \mu}{2k} + \frac{1}{2} \int_0^T E[\|Z^{k-1}(s)\|^2_1] \, ds
\]
which yields
\[
\int_0^T E[\|Z^k(s)\|^2_1] \, ds \leq \frac{(k\mu + 1)c_1}{2^k}.
\]
Therefore, there exists a pair \((u, v) \in M^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^{n \times m})\) such that
\[
\lim_{k \to \infty} \|u - u^k\|_{M^2(0, T; \mathbb{R}^n)} + \|v^k - v\|_{M^2(0, T; \mathbb{R}^{n \times m})} = 0. \tag{4.17}
\]
From (4.16) and the above estimates we also have
\[
\sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} E[\|u^k(t)\|^2_1] < +\infty. \tag{4.18}
\]
On the other hand, similar to (4.12) we have
\[
b_k(t) := \|u^k\|_{S^r(t, T; \mathbb{R}^n)} + \|v^k\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})} \\
\leq C \left\{ \|G\|_{L^r(\Omega, \mathcal{F}_T, \mathbb{R}^n)} + \|F(\cdot, 0, v^{k-1})\|_{M^{r,1}(t, T; \mathbb{R}^n)} + \|J(\cdot, 0, 0)\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})} \right\} \\
\leq C_0 \left\{ \|G\|_{L^r(\Omega, \mathcal{F}_T, \mathbb{R}^n)} + \|F(\cdot, 0, 0)\|_{M^{r,1}(t, T; \mathbb{R}^n)} + \|J(\cdot, 0, 0)\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})} \right\} \\
+ (T-t)^{1/2}\|v^{k-1}\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})} \\
\leq C_1 + C_1(T-t)^{1/2}\|v^{k-1}\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})}, \quad t \in [0, T)
\]
where the constants \(C, C_0, C_1\) are all independent of \(k\) and \(T-t\). Choose \(\tau = T - \frac{1}{4C_1T} \wedge T\). Then for any \(t \in [\tau, T]\), we have
\[
b_k(t) \leq C_1 + \frac{1}{2}b_{k-1}(t), \quad b_1(t) < \infty,
\]
which implies \(\sup_{k \in \mathbb{N}} \{\|u^k\|_{S^r(t, T; \mathbb{R}^n)} + \|v^k\|_{M^{r,2}(t, T; \mathbb{R}^{n \times m})}\} \leq 2C_1 + b_1(t) < \infty\).

Hence we have \((u, v) \in L^r(\Omega, L^\infty([\tau, T], \mathbb{R}^n)) \times M^{r,2}(\tau, T; \mathbb{R}^{n \times m})\). By induction, we conclude that
\[
(u, v) \in L^r(\Omega, L^\infty([0, T], \mathbb{R}^n)) \times M^{r,2}(0, T; \mathbb{R}^{n \times m}). \tag{4.20}
\]

We now show that \((\bar{u}, \bar{v})\) admits a version which is a solution to Eq. (2.1). In fact, let \((\bar{u}, \bar{v})\) solve the following equation:
\[
\bar{u}(t) = G + \int_t^T F(s, \bar{u}(s), v(s)) \, ds + \int_t^T J(s, \bar{u}(s), \bar{v}(s)) \, d\bar{B}_s - \int_t^T \bar{v}(s) \, dW_s.
\]

Similar to (4.16), it follows that
\[
E \left[ \|\bar{u}(t) - u^k(t)\|^2_1 \right] + (1 - \delta) \int_t^T E \left[ \|\bar{v}(s) - v^k(s)\|^2_1 \right] \, ds \\
\leq c_0 \int_t^T E \left[ \|\bar{u}(s) - u^k(s)\|^2_1 \right] \, ds + \frac{1 - \delta}{2} \int_t^T E \left[ \|\bar{v}(s) - v(s)\|^2_1 \right] \, ds \\
+ \frac{1 - \delta}{4} \int_t^T E \left[ \|v^k(s) - v^{k-1}(s)\|^2_1 \right] \, ds.
\]
In view of (4.17), we have
\[
\lim_{k \to \infty} \int_t^T E \left[ \|v^k(s) - v^{k-1}(s)\|^2 \right] ds = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_t^T E \left[ \|\tilde{v}(s) - v^k(s)\|^2 \right] ds = \int_t^T E \left[ \|\tilde{v}(s) - v(s)\|^2 \right] ds.
\]

Taking \( \eta(t) := \limsup_{k \to \infty} E[\|u^k(t) - \bar{u}(t)\|^2] \), by (4.18) and Fatou’s lemma, we obtain
\[
\eta(t) \leq c_0 \int_t^T \eta(s) ds - \lim_{k \to \infty} \int_t^T E \left[ (1 - \delta)\|\tilde{v}(s) - v^k(s)\|^2 - \frac{1 - \delta}{2}\|\tilde{v}(s) - v(s)\|^2 \right] ds
\]
\[
+ \frac{1 - \delta}{4} \lim_{k \to \infty} \int_t^T E \left[ \|v^k(s) - v^{k-1}(s)\|^2 \right] ds \leq c_0 \int_t^T \eta(s) ds
\]
which implies that \( \eta \equiv 0 \) by Gronwall’s inequality. Furthermore, we conclude
\[
\lim_{k \to \infty} \int_t^T E[\|\tilde{v}(s) - v^k(s)\|^2] ds = \int_t^T E[\|\tilde{v}(s) - v(s)\|^2] ds = 0.
\]

It follows that \((\bar{u}, \bar{v})\) is a modification of \((u, v)\). By Theorem 3.2, Lemma 3.3 and estimate (4.20), we conclude that \((\bar{u}, \bar{v})\) is in \(S^p(0, T; \mathbb{R}^n) \times M^{p,2}(0, T; \mathbb{R}^{n \times m})\) is the unique solution to BDSDES (27).

**Step 3.** For any \(N > 0\), denote
\[
F^N := F - F(\cdot, 0, 0)1_{\{\|F(\cdot, 0, 0)\|_{L^\infty} \geq N\}}, J^N := J_{1\{\|J(\cdot, 0, 0)\|_{L^\infty} \leq N\}} G^N := G_{1\{\|G\|_{L^\infty} \leq N\}}.
\]

Then in view of **Step 2**, there exists a unique solution \((u^N, v^N) \in (S^p(0, T; H) \cap M^{p,2}(0, T; V)) \times M^{p,2}(0, T; L(U_1, H))\) to the following BDSDES:
\[
u^N(t) = G^N + \int_t^T F^N(s, u^N(s), v^N(s)) ds + \int_t^T J^N(s, u^N(s), v^N(s)) dB_s
\]
\[- \int_t^T v^N(s) dW_s. \quad (4.21)
\]

Since
\[
\langle \varphi, F(t, 0, 0)1_{\{\|F(0, 0, 0)\|_{L^\infty} \geq N\} \rangle \leq \epsilon \|\varphi\|^q_{V} + C(\epsilon)\|F(t, 0, 0)\|^q_{V'}, \forall \varphi \in V,
\]
for any \((\varphi, \phi) \in V \times L(U_1, H)\), we have
\[
\langle \varphi, F^N(t, \varphi, \phi) \rangle + \|J^N(t, \varphi, \phi)\|^2 \leq - (\alpha - \epsilon)\|\varphi\|^q_{V} + \delta\|\phi\|^2 + C(\epsilon)\|F(t, 0, 0)\|^q_{V'} + \zeta(t)
\]
with the positive constant \(\epsilon\) waiting to be determined later. Then choosing \(\epsilon\) to be so small that \(\alpha - (p - 2)\alpha_1 - \epsilon > 0\), we check that the pair \((F^N, J^N)\) satisfies all the assumptions given to the pair \((F, J)\) with \(\zeta\) replaced by \(C(\epsilon)\|F(0, 0, 0)\|^q_{V'} + \zeta \leq (C(\epsilon) + 1)\zeta\). By Lemma 4.2, we have
\[
\|u^N\|_{S^p(0, T; H)} + \|u^N\|_{M^{p,2}(0, T; V)} + \|v^N\|_{M^{p,2}(0, T; L(U_1, H))}
\]
\[
\leq C \left\{ \|\varphi\|_{L^{p/2,1}(0, T; \Omega)}^{1/2} + \|G^N\|_{L^p(\Omega, \mathcal{F}_T, H)} \right\}
\]
\[
\leq C \left\{ \|\varphi\|_{M^{p,2,1}(0, T; \Omega)}^{1/2} + \|G\|_{L^p(\Omega, \mathcal{F}_T, H)} \right\}. \quad (4.22)
\]
On the other hand, in view of (A4), we have

\[
E \left[ \int_0^T \| F^N(t, u^N(t), v^N(t)) \|_{q^*}^q \, dt \right] \\
\leq C E \left[ \int_0^T \left( \| F(t, u^N(t), v^N(t)) \|_{q^*}^q + \| F(t, 0, 0) \|_{q^*}^q \right) \, dt \right] \\
\leq C \left\{ E \left[ \int_0^T (\varsigma(t) + K(\| u^N(t) \|_V^q + \| u^N(t) \|^2 + \| v^N(t) \|^2_1)) \left( 1 + \| u^N(t) \|^2 \right) \, dt \right] \right\} \\
\leq C \left\{ \| \varsigma \|_{L^{p/2,1}(0,T)}^{p/2} + \| v^N \|_{L^{p,q/2}(0,T;L(U_1,H))}^p + \| u^N \|_{S^p(0,T;H)}^{p/2} + \| u^N \|_{M^{pq/2,q}(0,T;V)}^q \right\}.
\]

(4.23)

and

\[
E \left[ \int_0^T \| J^N(t, u^N(t), v^N(t)) \|_2^2 \, dt \right] \\
\leq C E \left[ \int_0^T (\varsigma(t) + \| u^N(t) \|^2 + \| u^N(t) \|^2 + \| v^N(t) \|^2_1) \, dt \right] \\
\leq C \left\{ \| \varsigma \|_{L^{p/2,1}(0,T)} + \| v^N \|_{L^{p,q/2}(0,T;L(U_1,H))} + \| u^N \|_{S^p(0,T;H)}^2 + \| u^N \|_{M^{pq/2,q}(0,T;V)}^q \right\}.
\]

(4.24)

Thus, there exists a subsequence \( N_k \) and \( u, v, \bar{F}, \bar{J} \) such that

\[
\begin{align*}
&u^{N_k} \rightharpoonup u \text{ weakly star in } L^p(\Omega, L^\infty(0,T;H)); \\
u^{N_k} \rightharpoonup u \text{ weakly in } M^{pq/2,q}(0,T;V); \\
v^{N_k} \rightharpoonup v \text{ weakly in } M^{p,2}(0,T;L(U_1,H)); \\
F^{N_k}(\cdot, u^{N_k}(\cdot)) \rightharpoonup \bar{F} \text{ weakly in } M^{q'}(0,T;V'); \\
J(\cdot, u^{N_k}(\cdot), v^{N_k}(\cdot)) \rightharpoonup \bar{J} \text{ weakly in } M^2(0,T;L(U_2,H))
\end{align*}
\]

(4.25)
as \( k \to \infty \). It is clear that, for any \((\varphi, \phi) \in V \times L(U_1, H)\)

\[
\lim_{k \to \infty} \| F^{N_k}(\cdot, 0, 0) 1_{\{\| F(\cdot, 0, 0) \|_{L^2} \geq N_k \}} \|_{*} + \| J^{N_k}(\cdot, \varphi, \phi) 1_{\{\| J(\cdot, 0, 0) \|_{L^2} \geq N_k \}} \|_2 = 0, \text{ } \mathbb{P} \otimes dt \text{-a.e.}
\]

By Lemma 3.4

\[(u, v) \in \left( S^p(0,T;H) \cap M^{pq/2,q}(0,T;V) \right) \times M^{p,2}(0,T;L(U_1,H))\]

is the unique solution to (1.1) and we complete our proof. \( \square \)

4.2 Proof of Theorem 2.1

Let \( \{e_i\}_{i \in \mathbb{N}} \subset V \) be an orthonormal basis of \( H \) and let \( H_n := \text{span}\{e_1, \ldots, e_n\} \) such that \( \text{span}\{e_i\}_{i \in \mathbb{N}} \) is dense in \( V \). Let \( P_n : V' \to H_n \) be defined by

\[
P_n \phi := \sum_{i=1}^n \langle \phi, e_i \rangle e_i, \quad \phi \in V'.
\]

Obviously, \( P_n|H \) is just the orthogonal projection onto \( H_n \). Let \( \{g_1^1, g_2^1, \ldots\} \) be an orthogonal basis of \( U_1, i = 1, 2 \) and

\[
W^n(t) := P_n^1 W_t := \sum_{i=1}^n \langle W_t, g_i^1 \rangle U_1 g_i^1, \quad B^n(t) := P_n^2 B_t := \sum_{i=1}^n \langle B_t, g_i^2 \rangle U_2 g_i^2.
\]

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Consider the collections of \( \sigma \)-algebras on \( (\Omega, \mathcal{F}, P) \) given by
\[
\mathcal{F}^n_t = \sigma(W^n_s, 0 \leq s \leq t) \vee \mathcal{F}^B_{t,T}.
\]
We put, by definition, for any \( t \in [0, T] \)
\[
\tilde{F}^n(t, \phi, \varphi) = E \left[ F(t, \phi, \varphi) | \mathcal{F}^n_t \right], \quad F^n(t, \phi, \varphi) = P_n \tilde{F}^n(t, \phi, \varphi) \quad \text{and} \quad \tilde{J}^n(t, \phi, \varphi) = E \left[ J(t, \phi, \varphi) | \mathcal{F}^n_t \right], \quad J^n(t, \phi, \varphi) = P_n \tilde{J}^n(t, \phi, \varphi), \quad \phi \in V, \varphi \in L(U_1, H).
\]
\begin{equation}
\tag{4.26}
\end{equation}
For each \( n \in \mathbb{N} \) we consider the following backward doubly stochastic differential equation on \( H_n \):
\[
u^n(t) = G^n + \int_t^T F^n(s, u^n(s), v^n(s)) \, ds + \int_t^T J^n(s, u^n(s), v^n(s)) \, dB^n_s - \int_t^T v^n(s) \, dW^n_s.
\]
\begin{equation}
\tag{4.27}
\end{equation}
with \( G^n := E \left[ P_n G | \mathcal{F}^n_T \right] \).

**Proof of Theorem 2.1.** The uniqueness follows from Lemma 3.3 and it remains to prove the existence and estimate \( 4.28 \). First, for every \( n \in \mathbb{N} \) it can be checked that \( 4.27 \) satisfies all the conditions of Theorem 4.1 only with \( \zeta(t) \) replaced by \( \zeta^n(t) := E[\zeta(t) | \mathcal{F}^n_t] \). In view of Theorem 4.1 and Lemma 4.2 there exists a unique solution \( (u^n, v^n) \in (S^p(0, T; H) \cap M^{p,2}(0, T; L(U_1, H)) \times M^{p,2}(0, T; L(U_1, H))) \) to \( 4.27 \) such that
\[
\|u^n\|_{S^p(0,T;H)} + \|v^n\|_{M^{p,2}(0,T;L(U_1,H))} + \|u^n\|_{\mathcal{M}^{p,2/\eta}(0,T;V)} \leq C \left\{ \|\zeta\|^{1/2}_{M^{p,2}(0,T)} + \|G^n\|_{L^p(\Omega,\mathcal{F},H)} \right\}
\]
\begin{equation}
\tag{4.28}
\end{equation}
On the other hand,
\[
E \left[ \int_0^T \left( \|F^n(t, u^n(t), v^n(t))\|_{Y'}^2 + \|\tilde{F}^n(t, u^n(t), v^n(t))\|_{Y'}^2 \right) \, dt \right]
\leq CE \left[ \int_0^T \|F(t, u^n(t), v^n(t))\|_{Y'}^2 \, dt \right]
\leq CE \left[ \int_0^T (\zeta(t) + K(\|u^n(t)\|_{Y'}^2 + \|v^n(t)\|_{Y'}) \left( 1 + \|u^n(t)\|_{Y}^2 \right) \right) \, dt \right]
\leq C \left\{ \|\zeta\|_{M^{p/2,1}(0,T)} + \|u^n\|_{M^{p,2}(0,T;L(U_1,H))} + \|u^n\|_{S^p(0,T;H)} + \|u^n\|_{\mathcal{M}^{p,2/\eta}(0,T;V)} \right\}
\]
and
\[
E \left[ \int_0^T \left( \|J^n(t, u^n(t), v^n(t))\|_{L^2}^2 + \|\tilde{J}^n(t, u^n(t), v^n(t))\|_{L^2}^2 \right) \, dt \right]
\leq CE \left[ \int_0^T (\zeta(t) + \|v^n(t)\|_{L^2}^2 + \|u^n(t)\|_{L^2}^2 + \|v^n(t)\|_{L^2}^2) \, dt \right]
\leq C \left\{ \|\zeta\|_{M^{p/2,1}(0,T)} + \|u^n\|_{M^{p,2}(0,T;L(U_1,H))}^2 + \|u^n\|_{S^p(0,T;H)}^2 + \|u^n\|_{\mathcal{M}^{p,2/\eta}(0,T;V)}^q \right\},
\]
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where the constants $C$s are all independent of $n$. Thus, there exists a positive constant $C$ independent of $n$ such that

$$
C \geq \|u^n\|_{L^p(0,T;H)}^p + \|v^n\|_{M^p/2,q(0,T;V)}^p + \|v^n\|_{M^{p,2}(0,T;L(U_1,H))}
+ \|J^n(t,u^n(t),v^n(t))\|_{M^p(0,T;L(U_2,H))}^p + \|F^n(t,u^n(t),v^n(t))\|_{M^q(0,T;V)}^q
+ \|\tilde{J}^n(t,u^n(t),v^n(t))\|_{M^q(0,T;V)}^q
$$

(4.20)

from which it follows that there exists a subsequence $n_k \to \infty$ and $(u,v,\tilde{F},\tilde{J},\tilde{F},\tilde{J})$ such that

$$
u^{n_k} \rightharpoonup u \text{ weakly star in } L^p(\Omega, L^\infty(0,T;H));
u^{n_k} \rightharpoonup u \text{ weakly in } M^{p,q/2}(0,T;V);
v^{n_k} \rightharpoonup v \text{ weakly in } M^{p,2}(0,T;L(U_1,H));$$

$$
F^{n_k}(\cdot, u^{n_k}(\cdot), v^{n_k}(\cdot)) \rightharpoonup \tilde{F} \text{ weakly in } M'(0,T;V);
J(\cdot, u^{n_k}(\cdot), v^{n_k}(\cdot)) \rightharpoonup \tilde{J} \text{ weakly in } M^2(0,T;L(U_2,H));
\tilde{F}^{n_k}(\cdot, u^{n_k}(\cdot), v^{n_k}(\cdot)) \rightharpoonup \tilde{F} \text{ weakly in } M'(0,T;V);
\tilde{J}(\cdot, u^{n_k}(\cdot), v^{n_k}(\cdot)) \rightharpoonup \tilde{J} \text{ weakly in } M^2(0,T;L(U_2,H)).$$

Through a density argument, we check that $(\tilde{F},\tilde{J}) \equiv (\tilde{F},\tilde{J})$.

Through such a calculation as

$$
\lim_{n \to \infty} E\left[\|G - E[P_n G]_{\mathcal{F}_T^n}\|^p\right]
\leq 2^{p-1} \lim_{n \to \infty} E\left[\|G - E[\mathcal{F}_T^n]_n\|^p + \|E[G - P_n G]_{\mathcal{F}_T^n}\|^p\right]
\leq 2^{p-1} \lim_{n \to \infty} E\left[\|G - E[\mathcal{F}_T^n]_n\|^p + \|G - P_n G\|^p\right] = 0,
$$

we obtain

$$
G^{n_k} \rightharpoonup G \text{ strongly in } L^p(\Omega, \mathcal{F}_T,H)
$$

and for $dt$-almost all $t \in [0,T]$, $\forall (\varphi, \xi) \in V \times L(U_1,H),

$$
\lim_{n \to \infty} E[\|\tilde{F}^n(t,\varphi,\xi) - F(t,\varphi,\xi)\|_{\mathcal{F}_t}^q + \|\tilde{J}^n(t,\varphi,\xi) - J(t,\varphi,\xi)\|_{\mathcal{F}_t}^2] = 0.
$$

Then by Lemma 3.4, Remark 3.2 and 3.3 $(u,v)$ is the unique solution of BDSDES (1.1). Moreover, from (4.28) we deduce that estimate (2.23) holds. We complete the proof.

\[\square\]

5 Examples

First, let us consider the following quasi-linear BDSPDE:

$$
\begin{align*}
-du(t,x) &= \left[\partial_{x_j}\left(a^{ij}(t,x)\partial_{x_i}u(t,x) + \sigma^{ji}(t,x)v(t,x)\right)\right] + b^i(t,x)\partial_{x_j}u(t,x)
+ c(t,x)u(t,x) + \zeta^i(t,x)v(t,x) + g(t,x,u(t,x),\nabla u(t,x),v(t,x))
+ \partial_{x_j}f^i(t,x,u(t,x),\nabla u(t,x),v(t,x))
- v(t,x)dW^i_t
+ h'(t,x,u(t,x),\nabla u(t,x),v(t,x))\overline{B}^i_t, \; (t,x) \in Q := [0,T] \times \mathcal{O};
\end{align*}
$$

(5.1)

$$
u(T,x) = G(x), \; x \in \mathcal{O}.
$$

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Here and in the following we use Einstein’s summation convention, $T \in (0, \infty)$ is a fixed deterministic terminal time, $\mathcal{O} \subset \mathbb{R}^n$ is a domain with boundary $\partial \mathcal{O} \in C^1$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator in $\mathbb{R}^n$, and $\{W_t := (W_t^1, \ldots, W_t^n), t \in [0, T]\}$ and $\{B_t := (B_t^1, \ldots, B_t^m), t \in [0, T]\}$ are two mutually independent $m$-dimensional standard Brownian motions. Note that domain $\mathcal{O}$ can be chosen to be the whole space $\mathbb{R}^n$.

To BDSPDE [5.1], we give the following assumptions.

(B1) The triple

$$(f, g, h)(\cdot, t, \cdot, \varphi, y, z) : \Omega \times [0, T] \times \mathcal{O} \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l$$

are $\mathcal{F}_t \otimes \mathcal{B}(\mathcal{O})$-measurable for any $(t, \varphi, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. There exist nonnegative constants $\delta \in (0, 1), \kappa, \alpha, \beta$ and $L$ such that for all $(\vartheta_1, y_1, z_1), (\vartheta_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$,

$$|f(\omega, t, x, \vartheta_1, y_1, z_1) - f(\omega, t, x, \vartheta_2, y_2, z_2)| \leq L|\vartheta_1 - \vartheta_2| + \frac{\kappa}{2}|y_1 - y_2| + \beta^{1/2}|z_1 - z_2|,$$

$$|g(\omega, t, x, \vartheta_1, y_1, z_1) - g(\omega, t, x, \vartheta_2, y_2, z_2)| \leq L|\vartheta_1 - \vartheta_2| + |y_1 - y_2| + |z_1 - z_2|,$$

$$|h(\omega, t, x, \vartheta_1, y_1, z_1) - h(\omega, t, x, \vartheta_2, y_2, z_2)| \leq L|\vartheta_1 - \vartheta_2| + (\alpha)^{1/2}|y_1 - y_2| + (\delta)^{1/2}|z_1 - z_2|.$$  

(B2) For each $t \in [0, T]$, the functions $a(t), \sigma(t), b(t), c(t), \varsigma(t)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathcal{O})$-measurable. There exist constants $\varrho, \varrho' > 1$, and $\lambda, \Lambda > 0$ such that the following hold for all $\xi \in \mathbb{R}^n$ and $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$,

$$\lambda|\xi|^2 \leq (2a^{ij}(\omega, t, x) - \varrho^2 \sigma^{ij}(\omega, t, x))\xi^i \xi^j \leq \Lambda|\xi|^2;$$

$$|a(\omega, t, x)| + |\sigma(\omega, t, x)| + |b(\omega, t, x)| + |c(\omega, t, x)| + |\varsigma(\omega, t, x)| \leq \Lambda;$$

and $\lambda - \kappa - \varrho' \beta - \alpha > 0$ with $\frac{1}{\varrho} + \frac{1}{\varrho'} + \delta = 1$.

(B3) $G \in L^2(\Omega, \mathcal{F}_T, L^2(\mathcal{O}))$, and $h_0 := h(\cdot, \cdot, \cdot, 0, 0, 0) \in M^2(0, T; L^2(\mathcal{O} \cap \mathbb{R}^m))$,

$$f_0 := f(\cdot, \cdot, 0, 0, 0) \in M^2(0, T; L^2(\mathcal{O} \cap \mathbb{R}^m)), ~ g_0 := g(\cdot, \cdot, 0, 0, 0) \in M^2(0, T; L^2(\mathcal{O} \cap \mathbb{R}^m)).$$

Let $C_c^\infty(\mathcal{O})$ denote the set of all infinitely differentiable real-valued functions on $\mathcal{O}$ with compact support. For $p \in [1, \infty]$ and $\phi \in C_c^\infty(\mathcal{O})$, define

$$\|u\|_{H_b^{1,p}(\mathcal{O})} := \left( \int_\mathcal{O} (|\phi(x)|^p + |\nabla \phi(x)|^p) \, dx \right)^{1/p}. \quad (5.2)$$

Then the Sobolev space $H_b^{1,p}(\mathcal{O})$ is defined as the completion of $C_c^\infty(\mathcal{O})$ with respect to the norm $\|\cdot\|_{H_b^{1,p}(\mathcal{O})}$. As usual, we denote $H_b^{1,2}(\mathcal{O})$ and its dual space $H^{-1,2}(\mathcal{O})$ by $H_b^0(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ respectively. Here, Gelfand triple $(V, H, V')$ is realized as the triple $(H_b^0(\mathcal{O}), L^2(\mathcal{O}), H^{-1}(\mathcal{O}))$. Hence, by Theorem 2.1 we have

**Proposition 5.1.** Let the assumptions (B1)-(B3) hold. Then BDSPDE [5.1] admits a unique solution $(u, v) \in (M^2(0, T; L^2(\mathcal{O} \cap \mathbb{R}^m))) \times M^2(0, T; L^2(\mathcal{O} \cap \mathbb{R}^m))$ which satisfies

$$\|u\|_{S^2(0,T;L^2(\mathcal{O}))} + \|u\|_{M^2(0,T;H_b^0(\mathcal{O}))} + \|v\|_{M^2(0,T;L^2(\mathcal{O} \cap \mathbb{R}^m))} \leq C \left\{ \|\mathcal{G}\|_{L^2(\Omega, \mathcal{F}_T, L^2(\mathcal{O} \cap \mathbb{R}^m))} + \|f_0\|_{M^2(0,T;L^2(\mathcal{O} \cap \mathbb{R}^m))} + \|g_0\|_{M^2(0,T;L^2(\mathcal{O}))} + \|h_0\|_{M^2(0,T;L^2(\mathcal{O} \cap \mathbb{R}^m))} \right\}$$

with the constant $C$ depending on $\lambda, \alpha, \beta, \delta, L, \kappa, \varrho, \Lambda, \varrho'$ and $T$. 

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Remark 5.1. In Proposition[5.1] if we assume further that $h \equiv 0$, $G$ is $\mathcal{F}_t^W$-measurable, and for any $(\vartheta, y, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ $f(\cdot, \vartheta, y, z)$, $g(\cdot, \vartheta, y, z)$ and $h(\cdot, \vartheta, y, z)$ are all $\mathcal{F}_t^W$-adapted processes. Then our BDSDES (5.1) degenerates into a BSPDE on which some behavior properties of the solutions, on basis of Proposition 5.1, have been obtained by Qiu and Tang [31] under a more general framework.

Remark 5.2. In view of the whole proof of our main theorem 2.1, we deal in fact with a much more general class of BDSPDEs. Precisely, we solve the following BDSPDE:

$$u(t, x) = G(x) + \int_t^T \mathcal{L}u(s, x) + \langle \delta(s, x), Dv(s, x) \rangle + f(s, x, u(s, x), Du(s, x), v(s, x)) \, ds$$

$$+ \int_t^T h^r(s, x, u(s, x), Du(s, x), v(s, x)) \, d\mathcal{B}^r_s - \int_t^T v^r(s, x) \, dW^r_s$$

(5.3)

where $\text{ess sup}_{s \in [0, T]} \| \delta(x) \| \leq c_0 < 1$ and $\mathcal{L}$ is a non-positive self-adjoint sub-Markovian operator associated with a symmetric Dirichlet form defined on some space $L^2(Q, m(dx))$ and which admits a gradient $D$. One particular case of the previous example lies in the case where $Q = O$, $m(dx) = dx$, $Du(x) = \nabla u(x)\sigma(x)$ and

$$\mathcal{L}u(t, x) = \partial_{x_j} (a_{ij}(x) \partial_{x_i} u(t, x))$$

with $a = \sigma \sigma^* \geq 0$ which is not necessary to be uniformly positive definite as (B2) and is allowed to be degenerate. We refer to [5, 15, 22] for a detailed exposition and references to the theory of Dirichlet forms. We also refer to [12, 11] for a counterpart on the SPDE theory.

It is worthy noting that our BDSDESs like (1.1) include as particular cases the forward stochastic differential evolutionary systems listed in [30, Chapter 4, Page 55–91]. Consider the following BDSDES:

$$u(t) = G + \int_t^T A(s, u(s)) + \delta_1 v(s) + f(s) \, ds$$

$$+ \int_t^T \theta_2 v(s) + A_1(s, u(s)) \, d\mathcal{B}_s - \int_t^T v(s) \, dW_s$$

(5.4)

with $W$ and $B$ being one-dimensional Wiener processes and $(\delta_1, \delta_2) \in \mathbb{R} \times (0, 1)$. Let $A_1(t, u)$ be Lipschitzian continuous with respect to $u$ on $H$. Then $A(t, u)$ can be chosen to be any one listed in [30, Chapter 4, Page 59–73] with corresponding Gelfand triple. For example,

(a). $A(u) := -u|u|^{r-2} \text{ with } (V, H, V') := (L^r(O), L^2(O), L^r/(r-1)(O))$ and $r \in [2, \infty)$;

(b). $A(u) := \text{div} (|\nabla u|^{r-2} \nabla u) \text{ with } (V, H, V') := (H^1_0(O), L^2(O), (H^1_0(O))')$ and $r \in [2, \infty)$. Then the corresponding existence and uniqueness propositions are implied by Theorem 2.7.

6 Appendix

As in [20] and [30, 32], to prove Theorem 3.2 we need the following lemma. For abbreviation below we set

$$\mathcal{X} := L^q(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes dt; V).$$
Lemma 6.1. Let \( u \in X \). Then there exists a sequence of partitions \( I_l := \{0 = t^l_0 < t^l_1 < \cdots < t^l_{k_l} = T\} \) such that \( I_l \subset I_{l+1} \) and \( \pi(I_l) := \max_i (t^l_i - t^l_{i-1}) \to 0 \), \( u(t^l_i) \in V \) \( P \)-a.e. for all \( l \in \mathbb{N}, 1 \leq i \leq k_l - 1 \), and for

\[
\bar{u}^l := \sum_{i=2}^{k_l} 1_{[t^l_{i-1}, t^l_i]} u(t^l_i), \quad \bar{u}^i := \sum_{i=1}^{k_l-1} 1_{[t^l_{i-1}, t^l_i]} u(t^l_i), \quad l \in \mathbb{N},
\]

we have \( \bar{u}^l \) and \( \bar{u}^i \) belong to \( X \) such that

\[
\lim_{l \to \infty} \{ \| u - \bar{u}^l \|_X + \| u - \bar{u}^i \|_X \} = 0.
\]

Since the proof of Lemma 6.1 is standard (c.f. [30, Lemma 4.2.6] or [32, Lemma 4.1]), we omit it here.

Proof of Theorem 3.2. Step 1. Obviously, Equation (3.4) holds on \( V' \), i.e. (3.4) holds with both sides being \( V' \)-valued processes. Denote

\[
\bar{W}_t := \int_0^t v(r) \, dW_r \quad \text{and} \quad \bar{B}_t := \int_0^t h(r) \, dB_r, \quad t \in [0, T].
\]

Then \( \bar{W} \) and \( \bar{B} \) are continuous \( H \)-valued processes. Since \( f \in X' := M'((0, T; V')) \), both \( \int_0^T f(r) \, dr \) and \( u \) are continuous \( V' \)-valued processes. Through careful computations, we have

\[
\begin{align*}
\| u(s) \|^2 &= \| u(t) \|^2 - \| \bar{W}_t - \bar{B}_s \|^2 + \| \bar{B}_t - \bar{B}_s \|^2 + 2 \int_t^s \langle u(s), f(r) \rangle \, dr \\
&\quad - 2 \int_t^s \langle u(r), v(r) \, dW_r \rangle + 2 \int_t^s \langle u(s), h(r) \, dB_r \rangle \\
&\quad - \| u(t) - u(s) - \bar{W}_t + \bar{B}_s \|^2 \\
&\quad - 2 \langle u(t) - u(s) - \bar{W}_t + \bar{B}_s, \bar{W}_t - \bar{B}_s \rangle \\
&= (6.1)
\end{align*}
\]

holds for all \( t, s \in [0, T] \) such that \( t > s \) and \( u(t), u(s) \in V \). For any \( t = t^l_i \in I_l \backslash \{0, T\} \) given in Lemma 6.1 we have

\[
\begin{align*}
\| u(t) \|^2 - \| \xi \|^2 &= \sum_{j=i+1}^{k_l-1} (\| u(t^l_{j-1}) \|^2 - \| u(t^l_j) \|^2) \\
&= -2 \int_t^T \langle f(r), \bar{u}^l(r) \rangle \, dr - 2 \int_t^T \langle \bar{u}^l(r), v(r) \, dW_r \rangle + 2 \int_t^T \langle \bar{u}^l(r), h(r) \, dB_r \rangle \\
&\quad + 2 \langle \xi, \int_t^T h(r) \, dB_r \rangle + \sum_{j=i+1}^{k_l} \left( \| \bar{B}^l_j - \bar{B}^l_{j-1} \|^2 - \| \bar{W}^l_j - \bar{B}^l_{j-1} \|^2 \right) \\
&\quad + \sum_{j=i+1}^{k_l} \left( -2 \langle u(t^l_j) - u(t^l_{j-1}) - \bar{W}^l_j + \bar{B}^l_{j-1}, \bar{W}^l_j - \bar{B}^l_{j-1} \rangle - \| u(t^l_j) - u(t^l_{j-1}) - \bar{W}^l_j + \bar{B}^l_{j-1} \|^2 \right). \\
&\quad = (6.2)
\end{align*}
\]

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It can be checked that all the integral above are well defined. By Lemma 6.1 there holds

\[
E \left[ \int_0^T \| \hat{u}'(s), \ f(s) \| \ ds \right] \leq \| f \|_{X} \| \hat{u}' \|_{X} < c_1, \tag{6.3}
\]

where the constant \( c_1 > 0 \) is independent of \( l \). By BDG inequality, we have

\[
E \left[ \sup_{t \in [0, T]} \left| \int_t^T \langle \hat{u}'(s), \ v(s) \rangle \ dW_s \right| - \int_t^T \langle \hat{u}'(s), \ h(s) \rangle \ dB_s \right] \\
\leq 2E \left[ \sup_{t \in [0, T]} \left| \int_t^T \langle \hat{u}'(s), \ v(s) \rangle \ dW_s \right| \right] + E \left[ \sup_{t \in [0, T]} \left| \int_t^T \langle \hat{u}'(s), \ h(s) \rangle \ dB_s \right| \right] \\
\leq CE \left[ \int_0^T \left( \| v(s) \|_2^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 + \| h(s) \|_2^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 \right) \ ds \right]^{1/2} \tag{6.4}
\]

\[
\leq CE \left[ \int_0^T \left( \| v(s) \|_2^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 + \| h(s) \|_2^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 \right) \ ds \right]^{1/2}
\]

\[
\leq \frac{1}{4} E \left[ \sup_{1 \leq j \leq k_l} \| u(t_j^l) \|^2 \right] + CE \left[ \int_0^T \left( \| h(s) \|_2^2 + \| v(s) \|_1^2 \right) \ ds \right],
\]

with \( C \) being a generic constant independent of \( l \). On the other hand, we have

\[
\sum_{j=i+1}^{k_l} \left( -2 \langle u(t_j^l) - u(t_{j-1}^l), -\tilde{W}_{t_j^l} + \tilde{W}_{t_{j-1}^l}, \tilde{B}_{t_j^l} - \tilde{B}_{t_{j-1}^l} \rangle \right. \\
- \| u(t_j^l) - u(t_{j-1}^l) \|_{\mathbb{L}_2}^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 + \| \tilde{B}_{t_j^l} - \tilde{B}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 \right) \\
\leq \sum_{j=i+1}^{k_l} \left[ \| u(t_j^l) - u(t_{j-1}^l) \|_{\mathbb{L}_2}^2 \| \hat{u}'(s) \|_{\mathbb{L}_2}^2 + \| \tilde{B}_{t_j^l} - \tilde{B}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 \right. \\
\left. \| \tilde{W}_{t_j^l} - \tilde{W}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 \right] \\
= \sum_{j=i+1}^{k_l} \left( \| \tilde{W}_{t_j^l} - \tilde{W}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 \right) = \int_{t_i^l}^T \| v(s) \|_{\mathbb{L}_2}^2 \ ds, \tag{6.5}
\]

\[
E \left[ \sum_{j=i+1}^{k_l} \left( -\| \tilde{W}_{t_j^l} - \tilde{W}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 + \| \tilde{B}_{t_j^l} - \tilde{B}_{t_{j-1}^l} \|_{\mathbb{L}_2}^2 \right) \right] \\
= \sum_{j=i+1}^{k_l} E \left[ \int_{t_{j-1}^l}^{t_j^l} \left( -\| v(s) \|_{\mathbb{L}_2}^2 + \| h(s) \|_{\mathbb{L}_2}^2 \right) \ ds \right] \\
= E \left[ \int_{t_i^l}^T \left( -\| v(s) \|_{\mathbb{L}_2}^2 + \| h(s) \|_{\mathbb{L}_2}^2 \right) \ ds \right] \tag{6.6}
\]

and

\[
E \left[ \langle \xi, \int_{t_{i+1}^l}^T h(s) \ dW_s \rangle \right] \leq \left( E \left[ \| \xi \|_{\mathbb{L}_2}^2 \right] \right)^{1/2} \left( E \left[ \int_{t_{i+1}^l}^T \| h(s) \|_{\mathbb{L}_2}^2 \ ds \right] \right)^{1/2}. \tag{6.7}
\]
Hence, in view of (6.3)-(6.7), we obtain
\[
E \left[ \sup_{t \in I \setminus \{0\}} \|u(t)\|^2 \right] \leq c_2 < \infty
\]
for some constant \(c_2 > 0\) independent of \(l\). Therefore, setting \(I := \cup_{l \geq 1} I_l \setminus \{0\}\), with \(I_l\) as in Lemma 6.1 we have
\[
E \left[ \sup_{t \in I} \|u(t)\|^2 \right] \leq c_2,
\]
since \(I_l \subset I_{l+1}\) for all \(l \in \mathbb{N}\). Almost surely,
\[
\sum_{j=1}^{N} |\langle u(\omega, t), e_j \rangle|^2 \uparrow \|u(\omega, t)\|^2 \text{ as } N \uparrow \infty, \forall t \in [0, T]
\]
with \(\{e_j \mid j \in \mathbb{N}\} \subset V\) being an orthonormal basis of \(H\). For any \(x \in V' \setminus H\), set \(\|x\| = \infty\) as usual. Then, we conclude that \(t \to \|u(t)\|\) is lower semicontinuous almost surely. Since \(I\) is dense in \(0, T\], we arrive at \(\sup_{t \in [0, T]} \|u(t)\|^2 = \sup_{t \in I} \|u(t)\|^2\). Hence, we have
\[
E \left[ \sup_{t \in [0, T]} \|u(t)\|^2 \right] < \infty. \quad (6.8)
\]

**Step 2.** We prove the following approximating result:
\[
\lim_{l \to \infty} \sup_{t \in [0, T]} \left| \int_{t}^{T} \langle u(s) - \bar{u}^l(s), v(s) \, dW_s \rangle \right| = 0 \text{ in probability,}
\]
\[
\lim_{l \to \infty} \sup_{t \in [0, T]} \left| \int_{t}^{T} \langle u(s) - \bar{u}^l(s), h(s) \, d\tilde{B}_s \rangle \right| = 0 \text{ in probability.} \quad (6.9)
\]

As to (6.9), it is sufficient to prove the first equality, since the second follows similarly. As \(u\) is a continuous \(V'\)-valued process, we conclude from (6.8) that \(u\) is weakly continuous in \(H\). It follows that \(P_n u\) is continuous in \(H\) and thus
\[
\lim_{l \to \infty} \int_{0}^{T} \|P_n(u(s) - \bar{u}^l(s))\|^2 \|v(s)\|^2 \, ds = 0, \text{ a.s.,}
\]
where \(P_n\) is the orthogonal projection onto \(\text{span}\{e_1, \ldots, e_n\}\) in \(H\). It remains to prove that for each \(\varepsilon > 0\),
\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in [0, T]} \left| \int_{t}^{T} \langle (1 - P_n)u(s), v(s) \, dW_s \rangle \right| > \varepsilon \right) = 0,
\]
\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in [0, T]} \left| \int_{t}^{T} \langle (1 - P_n)u(s), v(s) \, dW_s \rangle \right| > \varepsilon \right) = 0. \quad (6.10)
\]
For each $n \in \mathbb{N}$, $\gamma \in (0,1)$ and $N > 1$, we have

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \left| \int_{0}^{T} \langle (1 - P_n) \bar{u}^l(s), v(s) dW_s \rangle \right| > \varepsilon \right) \\
\leq \mathbb{P} \left( \sup_{t \in [0,T]} \left| \int_{0}^{t} \langle (1 - P_n) \bar{u}^l(s), v(s) dW_s \rangle \right| > \frac{\varepsilon}{2} \right) \\
+ \mathbb{P} \left( \sup_{t \in [0,T]} \left| \int_{0}^{T} \langle (1 - P_n) \bar{u}^l(s), v(s) dW_s \rangle \right| > \frac{\varepsilon}{2} \right) \\
\leq \mathbb{P} \left( \sup_{t \in [0,T]} \left| \int_{0}^{t} \langle (1 - P_n) \bar{u}^l(s), v(s) dW_s \rangle \right| > \frac{\varepsilon}{2} \right) \\
+ \mathbb{P} \left( \left| \int_{0}^{T} \| \bar{u}^l(s) \|^2 d\langle (1 - P_n) \bar{W} \rangle_s \right| > \gamma^2 \right) \\
\leq \frac{2C}{\varepsilon} E \left[ \left| \int_{0}^{T} \| \bar{u}^l(s) \|^2 d\langle (1 - P_n) \bar{W} \rangle_s \right|^{1/2} \wedge \gamma \right] + \mathbb{P} \left( \left| \int_{0}^{T} \| \bar{u}^l(s) \|^2 d\langle (1 - P_n) \bar{W} \rangle_s \right| > \gamma^2 \right) \\
\leq \frac{2C\gamma}{\varepsilon} + \mathbb{P} \left( \sup_{t \in [0,T]} \| u(t) \| > N \right) + \frac{N^2}{\gamma^2} E \left[ \langle (1 - P_n) \bar{W} \rangle_T \right],
\]

where $C$ is a constant from BDG inequality and $\langle (1 - P_n) \bar{W} \rangle_t := \int_{0}^{t} \| (1 - P_n) v(s) \|^2 ds$. By letting $n \to \infty$, then $N \to \infty$ and finally $\gamma \to 0$, we complete the proof of the first equality of (6.10). The second equality of (6.10) follows similarly.

**Step 3.** We prove (3.5) holds for $t \in I$.

For this $t \in I$ fixed, we may assume that $t \neq T$. In this case, there exists a $N \in \mathbb{N}$ such that for any $l \geq N$, there exists a unique $0 < i < k_{l}$ satisfying $t = t_{i}^{l}$. In view of (6.6), (6.7), (6.9) and Lemma 6.11 taking limits in probability, we have

\[
\| u(t) \|^2 - \| \xi \|^2 \\
= 2 \int_{t}^{T} \langle f(s), u(s) \rangle ds + \int_{t}^{T} \langle u(s), h(s) dB_{s} \rangle - \int_{t}^{T} \langle u(s), v(s) dW_s \rangle \\
+ \int_{t}^{T} \| h(s) \|_{2} ds - \int_{t}^{T} \| v(s) \|_{1} ds + \gamma_{0} - \gamma_{1},
\]

where

\[
\gamma_{0} := \mathbb{P} \lim_{l \to \infty} \sum_{j=1}^{k_{l}} - 2 \langle u(t_{j}^{l}) - u(t_{j-1}^{l}), \bar{W}_{t_{j}^{l}} + \bar{W}_{t_{j-1}^{l}} + \bar{B}_{t_{j}^{l}} - \bar{B}_{t_{j-1}^{l}}, \bar{W}_{t_{j}^{l}} - \bar{W}_{t_{j-1}^{l}} \rangle, \\
\gamma_{1} := \mathbb{P} \lim_{l \to \infty} \sum_{j=1}^{k_{l}} \| u(t_{j}^{l}) - u(t_{j-1}^{l}) \|^2 - \bar{W}_{t_{j}^{l}} + \bar{W}_{t_{j-1}^{l}} + \bar{B}_{t_{j}^{l}} - \bar{B}_{t_{j-1}^{l}} \|
\]

exist and $\mathbb{P}$-lim denotes the limit in probability. Therefore, it remains to show that $\gamma_{0} = \gamma_{1} = 0$. In a similar way to the definition of $\bar{u}^l$ and $\bar{u}^l$, we define $\bar{W}^l, \bar{W}^l, \bar{B}^l$ and $\bar{B}^l$. For each $n \in \mathbb{N}$, we
have

\[ \gamma_1 = \mathbb{P} \lim_{t \to \infty} \left( \sum_{j=1+1}^{k_i} \| u(t_j') - u(t_{j-1}') - \tilde{W}_{t_j'} + \tilde{W}_{t_{j-1}'} + \tilde{B}_{t_j'} - \tilde{B}_{t_{j-1}'} \|^2 \right) \]

\[ = \mathbb{P} \lim_{t \to \infty} \left( \int_{t}^{T} \left\langle f(s), \hat{u}^l(s) - \tilde{u}^l(s) + P_n(\tilde{W}^l_s - \tilde{W}^l_s + \tilde{B}^l_s - \tilde{B}^l_s) \right\rangle ds \right) \]

\[ + \left\langle \xi - u(t_{k_i-1}') - \tilde{W}_{t_{k_i-1}'} + \tilde{W}_{t_{k_i-1}'} + \tilde{B}_{t_{k_i-1}'} - \tilde{B}_{t_{k_i-1}'} , \xi + P_n(\tilde{B}_T - \tilde{W}_T) \right\rangle \]

\[ + \sum_{j=1+1}^{k_i} \left\langle u(t_j') - u(t_{j-1}') - \tilde{W}_{t_j'} + \tilde{W}_{t_{j-1}'} + \tilde{B}_{t_j'} - \tilde{B}_{t_{j-1}'} , \right\rangle \]

\[ (1 - P_n) \left\langle -\tilde{W}_{t_j'} + \tilde{W}_{t_{j-1}'} + \tilde{B}_{t_j'} - \tilde{B}_{t_{j-1}'} \right\rangle \]

\[ = \mathbb{P} \lim_{t \to \infty} \left( A_1 + A_2 + A_3 \right) . \]

From Lemma 6.1, it follows that \( \mathbb{P} \lim_{t \to \infty} \left( \int_{t}^{T} (f(s), \hat{u}^l(s) - \tilde{u}^l(s)) ds \right) = 0. \) Since \( u \) is weakly continuous in \( H \), we have \( \mathbb{P} - \lim_{t \to \infty} A_2 = 0. \) Moreover, as \( P_n \tilde{W} \) and \( P_n \tilde{B} \) are continuous processes in \( V \),

\[ \mathbb{P} \lim_{t \to \infty} \left( \int_{t}^{T} \left\langle f(s), P_n(\tilde{W}^l_s - \tilde{W}^l_s + \tilde{B}^l_s - \tilde{B}^l_s) \right\rangle ds \right) = 0. \]

Thus, we have

\[ \gamma_1 \leq \mathbb{P} \lim_{t \to \infty} \left( \sum_{j=1+1}^{k_i} \| u(t_j') - u(t_{j-1}') - \tilde{W}_{t_j'} + \tilde{W}_{t_{j-1}'} + \tilde{B}_{t_j'} - \tilde{B}_{t_{j-1}'} \|^2 \right)^{1/2} \]

\[ \cdot \left( \sum_{j=1+1}^{k_i} \| (1 - P_n) \left( \tilde{W}_{t_{j-1}'} - \tilde{W}_{t_j'} + \tilde{B}_{t_{j-1}'} - \tilde{B}_{t_j'} \right) \|^2 \right)^{1/2} \]

\[ (6.12) \]

\[ = \gamma_1^{1/2} \left\langle (1 - P_n) \left( -\tilde{W} + \tilde{B} \right) \right\rangle_T^{1/2} . \]

By Lebesgue’s dominated convergence theorem, we have

\[ \lim_{n \to \infty} E \left[ \left\langle (1 - P_n)(\tilde{W} + \tilde{B}) \right\rangle_T \right] \]

\[ = \lim_{n \to \infty} E \left[ \int_{0}^{T} \left( \| (1 - P_n)h(s) \|^2 + \| (1 - P_n)v(s) \|^2 \right) ds \right] = 0. \]

Hence, \( \gamma_1 = 0. \)
Similarly,

\[
\gamma_0 = \mathbb{P} \lim_{l \to \infty} \sum_{j=i+1}^{k_l} -2 \left\langle u(t_j^l) - u(t_j^l-1), \bar{W}_t - \tilde{W}_t^l + \bar{B}_t - \tilde{B}_t^l \right\rangle
= 2\mathbb{P} \lim_{l \to \infty} \left( \int_0^T \left\langle f(s), + P_n \left( \tilde{W}_s - \bar{W}_s^l \right) \right\rangle ds + \left\langle \xi - u(t_{k_l-1}^l) - \bar{W}_t, + \tilde{W}_t^l + \bar{B}_t - \tilde{B}_t^l \right\rangle P_n \left( \bar{W}_T \right) + \sum_{j=i+1}^{k_l} \left\langle u(t_j^l) - u(t_j^l-1), \bar{W}_t - \tilde{W}_t^l + \bar{B}_t - \tilde{B}_t^l \right\rangle (1 - P_n) \left( \tilde{W}_t^l - \bar{W}_t^l \right) \right) \right)
\]

\[
\leq 2\mathbb{P} \lim_{l \to \infty} \left( \sum_{j=i+1}^{k_l} \left\| u(t_j^l) - u(t_j^l-1), \bar{W}_t - \tilde{W}_t^l + \bar{B}_t - \tilde{B}_t^l \right\|^2 \right)^{1/2} \cdot \left( \sum_{j=i+1}^{k_l} \left\| (1 - P_n) \left( \tilde{W}_t^l - \bar{W}_t^l \right) \right\|^2 \right)^{1/2}
\]

\[
\leq 2\gamma_0^{1/2} \left\langle \left( 1 - P_n \right) \left( \bar{W}_T \right) \right\rangle_T^{1/2},
\]

from which we deduce that \( \gamma_0 = \gamma_1 = 0 \).

**Step 4.** we prove (3.5) holds for all \( t \in [0, T] \). In view of **Step 2**, there exists \( \Omega' \subset \bar{\mathcal{F}} \) with probability 1 such that both the limits in (6.9) are point-wise ones in \( \Omega' \) for some subsequence (denoted again by \( l \to \infty \)) and (6.5) holds for all \( t \in I \) on \( \Omega' \). Fix \( t \in [0, T] \). In this case, for any \( l \in \mathbb{N} \) there exists a unique \( j(l) > 0 \) such that \( t \in [t_{j(l)-1}^l, t_{j(l)}^l] \). Letting \( t(l) := t_{j(l)}^l \), we have \( t(l) \downarrow t \) as \( l \to \infty \). By **Step 3**, for any \( l > m \) we have

\[
\left\| u(t(l)) - u(t(m)) \right\|^2
= 2 \int_{t(l)}^{t(m)} \left\langle f(s), u(s) - u(t(m)) \right\rangle ds + 2 \int_{t(l)}^{t(m)} \left\langle u(s) - u(t(m)), h(s) d\tilde{B}_s \right\rangle
- 2 \int_{t(l)}^{t(m)} \left\langle u(s) - u(t(l)), v(s) dW_s \right\rangle + \left\langle \tilde{W}_t^l, \bar{W}_t^l - \tilde{W}_t^l \right\rangle
- 2\langle u(t(l)) - u(t(m)) \rangle, \int_{t(l)}^{t(m)} v(s) dW_s - \langle \tilde{B}_t^l, \bar{B}_t^l \rangle
\]

(6.13)

By Lemma 6.1, selecting another subsequence if necessary, we conclude for some \( \Omega'' \subset \Omega' \) with probability 1 such that

\[
\lim_{m \to \infty} \int_0^T \left| \langle f(s), u(s) - \tilde{u}_m(s) \rangle \right| ds = 0.
\]
Since
\[
\sup_{l > m} \int_{t(l)}^{t(l) + t(m)} |\langle f(s), u(s) - \tilde{u}^m(s) \rangle| \, ds \leq \int_0^T |\langle f(s), u(s) - \tilde{u}^m(s) \rangle| \, ds,
\]
there holds that
\[
\lim_{m \to \infty} \sup_{l > m} \int_{t(l)}^{t(l) + t(m)} |\langle f(s), u(s) - \tilde{u}^m(s) \rangle| \, ds = 0
\]
on \Omega''$. Moreover, as
\[
2 \sup_{l > m} \left| \int_{t(l)}^{t(l) + t(m)} \langle u(s) - u(t(l)), h(s) \, dB_s \rangle \right| \leq 4 \sup_{t \in [0, T]} \left| \int_t^T \langle u(s) - \tilde{u}^m(s), h(s) \, dB_s \rangle \right|,
\]
in view of (6.8) and (6.9) (holding pointwise on $\Omega'$) and by the continuity of $\langle \tilde{W} \rangle_s$, $\langle \tilde{B} \rangle_s$ and $\tilde{W}_s$, we conclude that
\[
\lim_{m \to \infty} \sup_{l \geq m} \| u(t(l)) - u(t(m)) \|^2 = 0
\]
holds on $\Omega''$. Therefore, $(u(t(l)))_{t \in \mathbb{N}}$ is a Cauchy sequence in $H$ on $\Omega''$. As $u$ is a continuous $V'$-valued process, $\lim_{l \to \infty} \| u(t(l)) - u(t) \| = 0$ on $\Omega''$. Since (3.5) holds for $t(l)$ on $\Omega''$, letting $l \to \infty$, we get (3.5) for all $t / \in I$ on $\Omega''$.

**Step 5.** We complete our proof by proving that $u \in S^2(0, T; H)$.

From the continuity of the right-hand side of (3.5) on $\Omega''$, it follows that the map $t \mapsto \| u(t) \|$ is continuous on $[0, T]$. This together with (6.8) and the weak continuity of $u(t)$ in $H$ implies $u \in S^2(0, T; H)$.

**References**

[1] A. Bensoussan, Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions, *Stochastics*, 9 (1983), pp. 169–222.

[2] J. Bismut, contrôle des systèmes linéaires quadratiques, in *Applications de l'intégrale Stochastique*, Séminaire de Probabilité XII, vol. 649 of Lecture notes in Mathematics, Berlin, Heidelberg, New York, Springer, 1978, pp. 180–264.

[3] ______, An introductory approach to duality in optimal stochastic control, *SIAM Rev.*, 20 (1978), pp. 62–78.

[4] B. Boufoussi, J. V. Casteren, and N. Mrhardy, Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions, *Bernoulli*, 13 (2007), pp. 423–446.

[5] N. Bouleau and F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, *de Gruyter Stud. in Math.*, 14, Berlin: de Gruyter, 1991.

[6] P. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica, $L^p$ solutions of backward stochastic differential equations, *Stochastic Process. Appl.*, 108 (2003), pp. 604–618.

[7] F. E. Browder, Nonlinear elliptic boundary value problems, *Bull. Amer. Math. Soc.*, 69 (1963), pp. 862–874.
[8] R. Buchdahn and J. Ma, Stochastic viscosity solutions for nonlinear stochastic. Part I, Stochastic Processes and their Applications, 93 (2001), pp. 181–204.

[9] R. Buckdahn and J. Ma, Pathwise stochastic talor expansions and stochastic viscosity solutions for fully nonlinear stochastic pdes, The Annals of Probability, 30 (2002), pp. 1131–1171.

[10] F. Delbaen and S. Tang, Harmonic analysis of stochastic equations and backward stochastic differential equations, Probab. Theory Relat. Fields, 146 (2010), pp. 291–336.

[11] L. Denis, A general analytical result for non-linear SPDE’s and applications, Electronic Journal of Probability, 9 (2004), pp. 674–709.

[12] ———, Solutions of stochastic partial differential equations considered as Dirichlet processes, Bernoulli, 10 (2004), pp. 783–827.

[13] K. Du, J. Qiu, and S. Tang, $L^p$ theory for super-parabolic backward stochastic partial differential equations in the whole space, (2012). DOI: 10.1007/s00900-011-1554-9.

[14] N. Englezos and I. Karatzas, Utility Maximization with Habit Formation: Dynamic Programming and Stochastic PDEs, SIAM J. Control Optim., 48 (2009), pp. 481–520.

[15] M. Fukushima, Y. Oshima, and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, de Gruyter Stud. in Math. 19, Berlin: de Gruyter, 1994.

[16] Y. Han, S. Peng, and Z. Wu, Maximum principle for backward doubly stochastic control systems with applications, SIAM J. Control. Optim., 48 (2010), pp. 4224–4241.

[17] Y. Hu, J. Ma, and J. Yong, On semi-linear degenerate backward stochastic partial differential equations, Probab. Theory Relat. Fields, 123 (2002), pp. 381–411.

[18] N. Ichihara, Homogenization problem for stochastic partial differential equations of Zakai type, Stochastics and Stochastics Reports, 76 (2004), pp. 243–266.

[19] N. E. Karoui, S. Peng, and M. C. Quenez, Backward stochastic differential equations in finance, Math. Finance, 7 (1997), pp. 1–71.

[20] N. V. Krylov and B. L. Rozovskii, Stochastic evolution equations, J. Sov. Math., 16 (1981), pp. 1233–1277.

[21] W. Liu and M. Röckner, Spde in hilbert space with locally monotone coefficients, Journal of Functional Analysis, 259 (2010), pp. 2902–2922.

[22] Z. M. Ma and M. Röckner, Introduction to the Theory of (Non-symmetric) Dirichlet Forms, Springer, Berlin/ New York, 1992.

[23] A. M. Márquez-Durán and J. Real, Some results on nonlinear backward stochastic evolution equations, Stochastic Analysis and Applications, 22 (2004), pp. 1273–1293.

[24] G. J. Minty, Monotone (nonlinear) operators in hilbert space, Duke. Math. J., 29 (1962), pp. 341–346.
[25] D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands, *Probab. Theory Relat. Fields*, 78 (1988), pp. 535–581.

[26] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, *Stochastic.,* (1979), pp. 127–167.

[27] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, *Systems Control Lett.*, 14 (1990), pp. 55–61.

[28] E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, *Probab. Theory Relat. Fields*, 98 (1994), pp. 209–227.

[29] S. Peng, Stochastic Hamilton-Jacobi-Bellman equations, *SIAM J. Control Optim.*, 30 (1992), pp. 284–304.

[30] C. Prévôt and M. Röckner, A Concise Course on Stochastic Partial Differential Equations, *vol. 1905 of Lecture Notes in Mathematics*, Springer, 2007.

[31] J. Qiu and S. Tang, Maximum principles for backward stochastic partial differential equations, *Journal of Functional Analysis*, 262 (2012), pp. 2436–2480.

[32] J. Ren, M. Röckner, and F. Wang, Stochastic generalized porous media and fast diffusion equations, *Journal of Differential Equations*, 238 (2007), pp. 118–152.

[33] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, *vol. 49 of Mathematical Surveys and Monographs*, American Mathematical Society, 1996.

[34] S. Tang, The maximum principle for partially observed optimal control of stochastic differential equations, *SIAM J. Control Optim.*, 36 (1998), pp. 1596–1617.

[35] S. Tang, On backward stochastic partial differential equations, *tech. report, 34th SPA Conference, Osaka, September 2010.*

[36] E. Zeidler, Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators, *Springer*, 1990.

[37] Q. Zhang and H. Zhao, Stationary solutions of SPDEs and infinite horizon bdsdes, *J. Funct. Anal.*, 252 (2007), pp. 171–219.

[38] X. Zhang, On stochastic evolution equations with non-lipschitz coefficients, *Stochastics and Dynamics*, 9 (2009), pp. 549–595.

[39] X. Zhou, A duality analysis on stochastic partial differential equations, *Journal of Functional Analysis*, 103 (1992), pp. 275–293.