The reducibility of optimal 1-planar graphs with respect to the lexicographic product

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Abstract

A graph is called 1-planar if it can be drawn on the plane (or on the sphere) such that each edge is crossed at most once. A 1-planar graph $G$ is called optimal if it satisfies $|E(G)| = 4|V(G)| − 8$. If $G$ and $H$ are graphs, then the lexicographic product $G \circ H$ has vertex set the Cartesian product $V(G) \times V(H)$ and edge set $\{(g_1,h_1)(g_2,h_2) : g_1g_2 \in E(G), \text{ or } g_1 = g_2 \text{ and } h_1h_2 \in E(H)\}$. A graph is called reducible if it can be expressed as the lexicographic product of two smaller non-trivial graphs. In this paper, we prove that an optimal 1-planar graph $G$ is reducible if and only if $G$ is isomorphic to the complete multipartite graph $K_{2,2,2,2}$. As a corollary, we prove that every reducible 1-planar graph with $n$ vertices has at most $4n − 9$ edges for $n = 6$ or $n \geq 9$. We also prove that this bound is tight for infinitely many values of $n$. Additionally, we give two necessary conditions for a graph $G \circ 2K_1$ to be 1-planar.

Keywords: drawing, 1-planar graph, optimal 1-planar graph, lexicographic product, reducibility

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1 Introduction

A drawing of a graph $G = (V, E)$ is a mapping $D$ that assigns to each vertex in $V$ a distinct point in the plane (or on the sphere) and to each edge $uv$ in $E$ a continuous arc connecting $D(u)$ and $D(v)$. A drawing of a graph is a 1-planar drawing if each edge is crossed at most once. A graph is called 1-planar if it has a 1-planar drawing. A graph, together with a 1-planar drawing is called 1-plane. 1-planar graphs were first

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studied by Ringel (1965) [30] in the connection with the problem of the simultaneous coloring of the vertices and faces of plane graphs. Since then, many properties of 1-planar graphs have been widely investigated; see [1, 6, 9, 14, 23, 24, 37] for examples, see also [26] for a survey, and see [34] for a more recent book.

A graph is maximal \(1\)-planar if we cannot add any edge from the complement so that the resulting graph is still 1-planar and simple. It is well-known every 1-planar graph with \(n \geq 3\) vertices has at most \(4n - 8\) edges. A 1-planar graph with \(n\) vertices is called optimal if it has exactly \(4n - 8\) edges. Clearly, any optimal 1-planar graph is a maximal 1-planar graph, but the inverse may not always be true [3, 11]. Optimal 1-planar graphs have also been studied extensively due to their interesting properties. Suzuki [32] proved that every optimal 1-planar graph is obtained by adding a pair of crossing edges to each face of a 3-connected quadrangulation on the sphere. Didimo [16] proved that no optimal 1-planar graph has a 1-planar straight-line drawing. Hudák et al. [25] proved that every optimal 1-planar graph with even order has a perfect matching. Moreover, the matching extendability of optimal 1-planar graphs was characterized [18, 36]. Suzuki [33] proved that every optimal 1-planar graph has a \(K_6\)-minor and characterized optimal 1-planar graphs having no \(K_7\)-minor. Lenhart et al. [28] proved that every optimal 1-plane graph has a red–blue edge coloring such that the blue subgraph is maximal plane while the red subgraph has vertex degree at most four.

Among studies of 1-planar graphs, determining whether a given graph is 1-planar is a fundamental problem. Unfortunately, it is NP-complete to test whether a given graph is 1-planar [20, 27], while testing planarity of a given graph is solvable in linear time. It is easy to see that every graph is a minor of a 1-planar graph. Suzuki [33] proved that any graph is a topological minor of an optimal 1-planar graph. These imply that 1-planarity cannot be characterized in terms of forbidden minors or forbidden topological minors. So it seems difficult to test whether a given graph is 1-planar. However, there are results for some special graph classes. Czap and Hudák [13] characterized 1-planarity of complete multipartite graphs. Czap et al. [15] studied 1-planarity of join products of graphs. Binucci et al. [7] investigated the feasibility of a 1-planarity testing and embedding algorithm based on a backtracking strategy, and their approach can be successfully applied to some non-planar graphs with up to 30 vertices. Optimal 1-planar graphs can be recognized in linear time [10], while general 1-planar graphs are NP-hard to recognize.

A lexicographic product \(G \circ H\) of two graphs \(G\) and \(H\) is a graph such that the vertex set of \(G \circ H\) is the Cartesian product \(V(G) \times V(H)\) and two vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent in \(G \circ H\) if and only if either \(g_1\) is adjacent to \(g_2\) in \(G\) or \(g_1 = g_2\) and \(h_1\) is adjacent to \(h_2\) in \(H\). We call \(G\) and \(H\) the left and right factors of \(G \circ H\), respectively. The lexicographic product is also known as graph substitution, since \(G \circ H\) can be obtained from \(G\) by substituting a copy \(H_u\) of \(H\) for every vertex \(u\) of \(G\) and then joining
all vertices of \( H_u \) with all vertices of \( H_v \) if \( uv \in E(G) \). Note that the lexicographic product is not commutative, that is \( G \circ H \neq H \circ G \) in general. A graph is trivial if it has just one vertex or no vertex. A graph is called reducible if it is the lexicographic product of two non-trivial graphs, otherwise is called irreducible. In general, it seems not easy to determine whether a graph is reducible. As Feigenbaum and Schäffer [17] showed, the problem of recognizing whether a graph is a lexicographic product is equivalent in complexity to the graph isomorphism problem (the graph isomorphism problem is the computational problem of determining whether two finite graphs are isomorphic). The graph isomorphism problem (and so is the problem of testing reducibility) is not known to be solvable in polynomial time or to be NP-complete [2, 31].

Let \( C_n \) and \( K_n \) denote the cycle and the complete graph with \( n \) vertices, respectively. The disjoint union of \( k \) copies of a graph \( G \) is denoted by \( kG \). A cactus is a connected graph in which every edge belongs to at most one cycle. In 2015, Bucko and Czap [12] discussed the 1-planarity of lexicographic products of graphs. The authors arose a conjecture and an open problem as follows.

**Conjecture 1** ([12]). The graph \( G \circ K_2 \) is 1-planar if and only if \( G \) is a cactus.

The “if”-part of the above conjecture was proved in [12]. Recently, Matsumoto and Suzuki [29] solved the “only if”-part of the conjecture, and thus Conjecture 1 has been settled. So the study of 1-planarity of the lexicographic products of graphs only leaves the following open problem.

**Problem 1** ([12]). Let \( G \) be a connected graph with maximum degree at least 3. Characterize the 1-planarity of \( G \circ 2K_1 \).

So far, Problem 1 remains unsolved. Our main motivations come from the open problem. In this paper, we characterize the reducibility of optimal 1-planar graphs. As a corollary, we prove that every reducible 1-planar graph with \( n \) vertices has at most \( 4n - 9 \) edges for \( n = 6 \) or \( n \geq 9 \). We also prove that this bound is tight for infinitely many values of \( n \). Also as corollaries, we shall give two necessary conditions for a graph \( G \circ 2K_1 \) to be 1-planar. We have some other potential motivations, such as finding more properties of optimal 1-planar graphs because there are many open problems for optimal 1-planar graphs. For example, Bekos et al. [5] studied experimentally the book embedding problem from a SAT-solving perspective, and they conjectured that optimal 1-planar graphs have book thickness four. For general 1-planar graphs, Gollin et al. [19] determined precisely the maximum total number of cliques as well as the maximum number of cliques of any fixed size. However, for optimal 1-planar graphs, the sharp upper bounds on the maximum number of cliques of size 3, size 4 or size 5 are not known [19]. More open problems for optimal 1-planar graphs can be seen in [26]. Clearly, the reducibility (i.e. decomposition) of graphs is an inverse process of graph products. However, in contrast to graph products, not much research has been done on the reducibility of (beyond planar) graphs to our best knowledge.
Our main results are as follows.

**Theorem 1.** An optimal 1-planar graph $G$ is reducible if and only if $G$ is isomorphic to $K_{2,2,2,2}$.

**Corollary 1.** Let $G$ be a reducible 1-planar graph with $n$ vertices and $m$ edges. Then following statements hold.

(i) $m \leq 24$ for $n = 8$, and “24” is tight;

(ii) $m \leq 4n - 9$ for $n = 6$ or $n \geq 9$, and this bound is tight.

**Corollary 2.** Let $G$ be a graph with $n \geq 2$ vertices and $m$ edges. If $G \circ 2K_1$ is 1-planar, then following statements hold.

(i) $m \leq 2n - 3$ for $n \neq 4$, and $m \leq 6$ for $n = 4$;

(ii) for any subgraph of $G$ with $s \geq 2$ vertices has at most $2s - 3$ edges for $s \neq 4$ and has at most 6 edges for $s = 4$.

**Corollary 3.** Let $G$ be a graph with $n$ vertices. If $G \circ 2K_1$ is 1-planar, then $G$ is 1-planar for any $n$ and is not maximal 1-planar for $n \geq 17$.

The remaining sections of this paper are organized as follows. In Section 2, we give some terminology and notations. In Section 3, we give some lemmas. In Section 4, we give proofs of Theorem 1 and Corollaries 1, 2 and 3. In Section 5, we raise some open problems.

## 2 Terminology and notation

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph $G$, respectively. The notation $|V(G)|$ is also known as the order and $|E(G)|$ as the size of $G$. All graphs considered here are simple and nontrivial, unless otherwise stated. A drawing is good if no edge crosses itself, no two edges cross more than once, and no two incident edges cross. All 1-planar drawings are good in this paper.

Let $G$ be 1-planar graph with a 1-planar drawing $D$. An edge $e$ of $G$ is called a crossing edge under $D$ if it crosses with any other edge under the drawing $D$, and is called a non-crossing edge under $D$ otherwise. A cycle $C$ of $G$ is called a non-crossing cycle under $D$ if each edge on $C$ is non-crossing under $D$, and is called a crossing cycle under $D$. The planar skeleton $S(D)$ of $D$ is the subgraph of $G$ by removing all crossing edges of $D$. Let $H$ be a subgraph of $G$. The subdrawing $D|H$ of $H$ induced by $D$ is called a restricted drawing of $D$. Let $\mathcal{P}$ be a plane. Similar to planar drawings, the 1-planar drawing $D$ also defines the faces, which are the connected parts of $\mathcal{P} \setminus D$. Each face $f$ contains on its boundary a number of vertices and crossings of $D$; these
are called the corners of $f$. A face is uncrossed if all incident corners are vertices, and is crossed otherwise. A $k$-face is a face whose boundary walk has a length of exactly $k$.

Let $L$ be a closed curve in $\mathcal{P}$ that does not cross itself. Thus, $L$ separates $\mathcal{P}$ into two open regions, the bounded one (i.e., the interior of $L$) and the unbounded one (i.e., the exterior of $L$). We denote by $L_{\text{int}}$ and $L_{\text{out}}$ the interior and exterior of $L$, respectively.

Let $G$ be a graph. We call a cycle $C$ of $G$ an even cycle if $C$ contains an even number of edges. If $S$ is a set of vertices of $G$, the vertex-induced subgraph $G[S]$ is the subgraph of $G$ that has $S$ as its set of vertices and contains all the edges of $G$ that have both end-vertices in $S$. We denote by $G \cong H$ that graphs $G$ and $H$ are isomorphic. We denoted by $K_{n_1,n_2,\ldots,n_k}$ a complete multipartite graph with $k$ partition classes of sizes $n_1,n_2,\ldots,n_k$, respectively.

The terms not defined here can be found in [35].

## 3 Preliminary results

In this section we give some lemmas. For making the structure clear, we divide these lemmas into three groups.

### 3.1 Properties for lexicographic products of graphs

We begin with two simple lemmas.

**Lemma 1 ([21]).** The number of edges of a graph $G \circ H$ is $|V(G)| \times |E(H)| + |V(H)|^2 \times |E(G)|$.

**Lemma 2 ([21]).** A graph $G \circ H$ is connected if and only if $G$ is connected.

The following Lemma 3 is obtained based on the definition of lexicographic products, which will be a useful tool in Section 4.

**Lemma 3.** Let $G$ and $H$ be graphs with vertex set $\{g_1,g_2,\ldots,g_s\}$ and vertex set $\{h_1,h_2,\ldots,h_l\}$, respectively, where $s = |V(G)|$ and $l = |V(H)|$. Let $G' = G \circ H$. Then the following statements are hold.

(i) For $1 \leq i \neq j \leq s$, both $G'[(g_i,h_1),(g_i,h_2),\ldots,(g_i,h_l)]$ and $G'[(g_j,h_1),(g_j,h_2),\ldots,(g_j,h_l)]$ are isomorphic to $H$.

(ii) If a vertex $u$ in $G'$ is adjacent to $(g_k,h_i)$ where $1 \leq k \leq s$ and $1 \leq i \leq l$, then $u$ is adjacent to every vertex in $\{(g_k,h_1),(g_k,h_2),\ldots,(g_k,h_l)\}$.

**Proof.** For (i), by the definition of lexicographic products, $G'[(g_i,h_1),(g_i,h_2),\ldots,(g_i,h_l)]$ and $G'[(g_j,h_1),(g_j,h_2),\ldots,(g_j,h_l)]$ are copies of $H$ corresponding to $g_i$ and $g_j$, respectively. Clearly, they are both isomorphic to $H$. 

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For (ii), we first note that the vertex set \( \{(g_k, h_1), (g_k, h_2), \ldots, (g_k, h_i), \ldots, (g_k, h_l)\} \) in \( V(G') \) corresponds to the vertex \( g_k \) of \( G \) (\( g_k \) is substituted by a copy of \( H \) in \( V(G') \)). Clearly, by the definition of lexicographic products, \( u \) is adjacent to all vertices in \( \{(g_k, h_1), (g_k, h_2), \ldots, (g_k, h_i), \ldots, (g_k, h_l)\} \), since \( u \) is adjacent to \( (g_k, h_i) \) in \( G' \).

### 3.2 Results for the 1-planarity of lexicographical products of graphs

We collect some results related to the 1-planarity of lexicographical products of graphs, which will be used in the upcoming proof.

**Lemma 4 ([12]).** Let \( G \) be a connected graph with at least 3 vertices and let \( H \) be a graph with at least 4 vertices. Then \( G \circ H \) is not 1-planar.

**Lemma 5 ([12]).** Let \( G = K_2 \) and let \( H \) be a graph with at least 5 vertices. Then \( G \circ H \) is not 1-planar.

**Lemma 6 ([12]).** Let \( G = K_2 \) and let \( H \) be a graph with at most 4 vertices. Then \( G \circ H \) is 1-planar if and only if \( H \) is a subgraph of \( C_4 \) or \( H \) is a subgraph of \( C_3 \).

**Lemma 7 ([12, 29]).** Let \( G \) be a graph. Then \( G \circ K_2 \) is 1-planar if and only if \( G \) is a cactus.

### 3.3 Properties for 1-planar graphs

We now present some properties of 1-planar graphs (including optimal 1-planar graphs).

**Lemma 8 ([8, 32]).** Let \( G \) be a 1-planar graph with \( n \) vertices. Then the maximum number of edges of \( G \) is given by

\[
\begin{align*}
\binom{n}{2} & \quad \text{if } n \leq 6, \\
4n - 9 & \quad \text{if } n \in \{7, 9\}, \\
4n - 8 & \quad \text{otherwise}.
\end{align*}
\]

Barát and Tóth [3] gave a lower bound on the size of maximal 1-planar graphs.

**Lemma 9 ([3]).** Let \( G \) be a maximal 1-planar graph with \( n \geq 4 \) vertices and \( m \) edges. Then \( m \geq \frac{2n}{3} - \frac{10}{3} \) edges.

The following lemma is well known and is given as an exercise in the textbook [35] (see page 160). We ignore its proof here.

**Lemma 10 ([35]).** Let \( G \) be a cactus. Then \( |E(G)| \leq \left\lfloor \frac{3|V(G)| - 1}{2} \right\rfloor \).

We say that two 1-planar drawings of a graph are isomorphic if there is a homeomorphism of the sphere that maps one drawing to the other.
Lemma 11. There are exactly two non-isomorphic 1-planar drawings $D'$ and $D''$ of $K_{3,3}$, as shown in Fig. 1.

![Fig. 1. Two non-isomorphic 1-planar drawings of $K_{3,3}$.](image)

**Proof.** As seen in Fig. 1, $D'$ and $D''$ are two non-isomorphic 1-planar drawings of $K_{3,3}$. In what follows, we prove that $K_{3,3}$ has exactly two 1-planar drawings $D'$ and $D''$ up to isomorphic. Let \{x, y, z\} and \{u, v, w\} be the different partial sets of $K_{3,3}$, respectively. Clearly, $K_{3,3}$ contains a subgraph, namely $H$, isomorphic to $K_{2,3}$. Without loss of generality, we denote the vertex set of $H$ by \{x, y, z, u, v\}. From [22], we know that $H$ (i.e. $K_{2,3}$) has three non-isomorphic 1-planar drawings, namely, $D_1$, $D_2$ and $D_3$, as shown in Fig. 2.

![Fig. 2. Three non-isomorphic 1-planar drawings of $H$.](image)

We note here that $w$ in $K_{3,3}$ is adjacent to all the three vertices $x, y$ and $z$. First, we claim that any 1-planar drawing of $K_{3,3}$, namely $D$, does not contain $D_3$ as a subdrawing. Otherwise no matter which region $w$ is placed in, one of $wx$, $wy$ and $wz$ will cause some edge in $D_3$ to be crossed at least twice, which contradicts that $D$ is 1-planar.

If a 1-planar drawing of $K_{3,3}$ contains $D_1$ as a subdrawing, then $w$ can be placed in $(uxvyu)_{int}$, $(uyvzu)_{int}$, or $(uxvzu)_{out}$. If $w$ is placed in $(uxvyu)_{int}$, then $wx$ and $wy$ must lie in $(uxvyu)_{int}$, and they do not cross each other (since 1-planar drawings are good). The edge $wz$ can be chosen to cross one of $ux$, $vx$, $vy$ and $uy$, but no matter
what we choose, the resulting 1-planar drawing of $K_{3,3}$ is isomorphic to $D'$. Similarly, when $w$ is placed in $(uyzu)_{int}$ or $(uxzu)_{out}$, we can see that any resulting 1-planar drawing of $K_{3,3}$ obtained from $D_1$ by adding $wx, wy,$ and $wz$ is also isomorphic to $D'$.

If a 1-planar drawing, namely $D$, of $K_{3,3}$ contains the subdrawing $D_2$, then clearly $w$ can only be placed in $(ucvxu)_{int}$ or $(uzcyvxu)_{out}$ where $c$ is the crossing point caused by $uy$ and $vz$. If $w$ is placed in $(ucvxu)_{int}$, then $wx$ will not cross any other edges. Furthermore, $wy$ must cross $vx$. If not, then $wy$ will cross $ux$, but now $wz$ can not be added such that $D$ is 1-planar, a contradiction. We then see that $wz$ will cross $ux$. By check, the final 1-planar drawing of $K_{3,3}$ is isomorphic to $D''$. For if $w$ is placed in $(uzcyvxu)_{out}$, then $wx$, $wy$ and $wz$ will be non-crossing under $D$, and thus we get a unique 1-planar drawing (up to non-isomorphic) of $K_{3,3}$ that is isomorphic to $D'$.

Therefore, $K_{3,3}$ has exactly two non-isomorphic 1-planar drawings $D'$ and $D''$, as desired.

**Lemma 12.** Let $G$ be an optimal 1-planar graph. If $G = G_1 \circ G_2$, then $G_1$ is connected.

**Proof.** As any optimal 1-planar graph is clearly connected, $G_1$ is a connected graph by Lemma 2.

**Lemma 13 ([32]).** Let $G$ be an optimal 1-planar graph. Then $G$ has a unique 1-planar drawing into the sphere, up to isomorphism.

A plane graph is known as a quadrangulation if every face is a 4-face. In [8] or [32], the authors gave a relationship between optimal 1-planar graphs and 3-connected quadrangulations.

**Lemma 14 ([8],[32]).** Let $D$ be a 1-planar drawing of an optimal 1-planar graph. Then $D$ is obtained by adding a pair of crossing edges to each face of a 3-connected quadrangulation.

By Lemma 13 and Lemma 14, we can see that optimal 1-planar graphs and 3-connected quadrangulations have the one-to-one correspondence.

**Lemma 15.** Let $G$ be an optimal 1-planar graph with a 1-planar drawing $D$. For any vertex $v$ of $G$, the adjacent edges of $v$ under $D$ alternate between crossing and non-crossing edges.

**Proof.** Without loss of generality, we assume that the neighbors of $v$ in clockwise order under $D$ are $v_0, v_1, \cdots, v_{k-1}$. By Lemma 14, only one of $vv_i$ and $v_{i+1}v$ belongs to $E(S(D))$ where indices are taken modulo $k$, as desired.

**Lemma 16.** Let $G$ be an optimal 1-planar graph and $D$ be a 1-planar drawing of $G$. If $v_0v_2$ crosses $v_1v_3$ at a crossing point $c$ under $D$, then the following two statements hold.
(i) $G[v_0, v_1, v_2, v_3] \cong K_4$ and $v_0v_1, v_1v_2, v_2v_3$ and $v_3v_1$ are non-crossing under $D$;

(ii) $cv_i v_{i+1}$ bounds a crossed face of $D$ with indices taken modulo 4.

Proof. For (i), it is clear to see that $v_0v_1v_2v_3v_0$ is a 4 cycle in the planar skeleton $S(D)$ by Lemma 14. So (i) holds.

For (ii), by Lemma 14, $v_0v_1v_2v_3v_0$ bounds a 4-face in $S(D)$ that only contains the pair of crossing edges $v_0v_2$ and $v_1v_3$. Hence, the region bounded by $v_i v_{i+1}, cv_i$ and $cv_{i+1}$ where $1 \leq i \leq 4$ cannot contain any vertex. So (ii) holds.

Lemma 17. Let $G$ be an optimal 1-planar graph with a 1-planar drawing $D$. Let $C$ be a cycle of $G$. If $C$ is non-crossing under $D$, then $C$ is an even cycle.

Proof. By Lemma 14, the planar skeleton $S(D)$ is a quadrangulation. Clearly, any cycle of $S(D)$ is even. Since $C$ is non-crossing under $D$, $C$ is a cycle in $S(D)$, and thus $C$ is an even cycle of $G$, as desired.

4 Proof of main results

In this section, we prove Theorem 1 and three corollaries.

Proof of Theorem 1. First, the “if” part is easily proved. We can check easily that $K_{2,2,2,2}$ is optimal 1-planar (see Fig. 3) and $K_{2,2,2,2}$ is reducible since clearly $K_{2,2,2,2} = K_4 \circ 2K_1$ or $K_{2,2,2,2} = K_2 \circ C_4$.

We now prove the “only if” part. We assume that $G = G_1 \circ G_2$ where $G_1$ and $G_2$ are two graphs such that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$. Furthermore, we discuss the reducibility of $G$ in terms of the number of vertices of (the possible) left factor $G_1$ and right factor $G_2$, which corresponds to the following Case 1, Case 2 and Case 3, respectively.

Fig. 3. A 1-planar drawing of $K_{2,2,2,2}$.

Case 1. $|V(G_1)| = 2$ and $|V(G_2)| \geq 2$.

In this case, we claim that $G \cong K_{2,2,2,2}$. As $G_1$ is connected by Lemma 12 and $G_1$ has exactly 2 vertices, $G_1$ is isomorphic to $K_2$. By Lemma 5, $G_2$ has at most 4 vertices.
As $G$ has at least 8 vertices by Lemma 8 and $G_1 \cong K_2, G_2$ has at least 4 vertices. So $G_2$ has exactly 4 vertices, and hence $G$ has exactly 8 vertices. By “optimality” of $G$, $G$ has 24 ($= 4 \times 8 - 8$) edges. Hence, $G_2$ has 4 edges by Lemma 1. So $G_2$ is isomorphic to $C_4$ or a $C_3$ with a pendant edge. Furthermore, $G_2$ must be isomorphic to $C_4$ by Lemma 6, and thus $G \cong K_{2,2,2,2}$. Note that $K_{2,2,2,2}$ is optimal 1-planar and $K_{2,2,2,2} = K_2 \circ C_4$.

Thus, the claim holds.

**Case 2.** $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$.

In this case, we will prove that $G$ is not the lexicographic product of $G_1$ and $G_2$. Suppose for a contradiction that $G = G_1 \circ G_2$. If $|V(G_2)| \geq 4$, then $G_1 \circ G_2$ is not 1-planar by Lemma 4, a contradiction. Hence, we shall consider $|V(G_2)| = 3$.

Let $\{u_1, u_2, \cdots, u_k\}$ where $k \geq 3$ and $\{v_1, v_2, v_3\}$ be vertex sets of $G_1$ and $G_2$, respectively. By Lemma 12, $G_1$ is connected. Hence, there is an edge $u_iu_j$ for $1 \leq i \neq j \leq k$ in $G_1$. Since $|V(G_2)| \geq 3$, then $G$ contains a subgraph $T$ isomorphic to $K_{3,3}$ where two parts of $T$ are $X := \{(u_i, v_1), (u_i, v_2), (u_i, v_3)\}$ and $Y := \{(u_j, v_1), (u_j, v_2), (u_j, v_3)\}$.

For brevity, in the following we relabel the vertices of $X$ as $u, v, w$, and relabel the vertices of $Y$ as $x, y, z$.

The following two claims are obtained directly from Lemma 3.

**Claim 1.** If a vertex $p$ of $G$ is adjacent to one vertex of $X$ (resp. $Y$), then $p$ is adjacent to all vertices of $X$ (resp. $Y$).

**Claim 2.** The induced graph $G[\{x, y, z\}] \cong G[\{u, v, w\}]$.

By Lemma 11, there are two possible restricted drawings (A1) and (A2) of $D|T$ as shown in Fig. 4.

![Fig. 4. Two possible restricted 1-planar drawings of $D|T$.](image)

First, we claim that the drawing $D$ does not contain (A2) as a subdrawing. Otherwise by Lemma 16 (i), $xyzx$ will be a non-crossing 3-cycle in $D$, which contradicts to Lemma 17. Hence, $D$ contains (A1) as a subdrawing. Furthermore, since $xy$ crosses $uy$ in $D|H$, $uw$ and $xy$ are edges of $G$, and they are non-crossing under $D$ by Lemma 16 (i). Similarly, $ux$ and $vy$ are also non-crossing under $D$. 


Let $H$ be the subgraph of $G$ obtained from $T$ by adding two edges $xy$ and $uv$. Based on the analysis above, $D$ contains the restricted drawing $D|H$ as shown in Fig 5.

Combing Lemma 16 (i) with the fact that $uv$ is non-crossing under $D$, at least one of $uz$ and $vz$ is crossing under $D$. Since $uz$ and $vz$ are symmetrical in $D|H$, we assume without loss of generality that $uz$ is crossed by some edge, denoted by $st$, under $D$.

By symmetry of the vertex-ends $s$ and $t$, we discuss 5 cases according to whether $s$ and $t$ are vertices of $H$. Based on the assumption that $D$ is a 1-planar drawing, we observe that if $s$ or $t$ is in $V(H)$, then they may only be $v$, $w$ or $x$.

**Case 2.1.** $s \notin V(H)$ and $t \notin V(H)$.

We may assume that $s$ lies in $(uvzu)_{int}$ and $t$ lies in $(uvzu)_{out}$, see Fig. 6 (1). As $st$ crosses $uz$, $su$ will be a non-crossing edge under $D$ by Lemma 16 (i). By Claim 1, $sw$ and $sv$ are edges of $G$. Since $uv$ is non-crossing under $D$ and $uz$ has been crossed by $st$, $sw$ must cross $vz$, and hence $sv$ is non-crossing under $D$ by Lemma 16. Thus, edges $uv$, $su$ and $sv$ form a non-crossing 3-cycle under $D$; this contradicts to Lemma 17.

**Case 2.2.** $s \notin V(H)$ and $t = w$.

In this case, $s$ must lie in $(uvzu)_{int}$, see Fig. 6 (2). The vertex $s$ is adjacent to $z$ in $G$ by Lemma 16 (i), since $sw$ is crossed by $uz$. By Claim 1, $sx$ is an edge of $G$. But clearly $sx$ will be crossed twice or will cause at least one edge on $D|H$ to be crossed at least twice; this contradicts that $D$ is 1-planar.

**Case 2.3.** $s \notin V(H)$ and $t = x$.

In this case, $s$ lies in $(uvzu)_{int}$, see Fig. 6 (3). Since $sx$ crosses $uz$, $sz$ and $su$ is an edge of $G$ by Lemma 16 (i). By Claim 1, $sy$ and $sw$ are edges of $G$. As $uz$ has been crossed and $uv$ is non-crossing under $D$, one of $sy$ and $sw$, say $sy$, must cross $vz$, and hence the other edge $sw$ will be crossed twice or will cause at least one edge on $D|H$ to be crossed at least twice; this contradicts that $D$ is 1-planar.

**Case 2.4.** $s = v$ and $t \notin V(H)$.

In this case, $uz$ is crossed by $tv$, and $t$ must lie in $(uxwzu)_{out}$. By Lemma 16 (i),
tz and tu are edges of G, and both vz and tu are non-crossing under D. Furthermore, by Claim 1, ty is an edge of G. We then see that sy crosses wz or wx. If ty crosses wz, then yz is a non-crossing edge under D by Lemma 16 (i), and hence vy, vz and yz form a non-crossing 3-cycle under D, see Fig. 6 (4); this contradicts to Lemma 17. If ty crosses wz, then tx are a non-crossing edge under D by Lemma 16 (i), see Fig. 6 (5). But now tu, ux and tx form a non-crossing 3-cycle under D; this contradicts to Lemma 17.

![Diagram](image1)

**Fig. 6.** The possible six subdrawings involving the crossing edge uz.

**Case 2.5.** s = v and t = w.

In this case, since vw crosses uz, uw is an edge of G and vz is non-crossing under D by Lemma 16 (i), see Fig. 6 (6). Now we observe that G[{u, v, w}] \cong C_3, and thus G[{x, y, z}] is also isomorphic to C_3 by Claim 2. Hence, yz and xy are edges of G. We see that xz must cross wy, and thus yz is non-crossing under D by Lemma 16. Recall that vy is also non-crossing under D. Thus, yvzy is a non-crossing 3-cycle under D; this contradicts to Lemma 17.

So far, we have discussed all possible cases, and they all derive contradictions. Therefore, G is not the lexicographic product of G_1 and G_2, as desired.

**Case 3.** |V(G_1)| ≥ 3 and |V(G_2)| = 2.
In this case, we shall prove that $G \cong K_{2,2,2,2}$ if $G = G_1 \circ G_2$. Since $|V(G)| \geq 8$ by Lemma 8 and $|V(G_2)| = 2$, we have $|V(G_1)| \geq 4$. First we claim that $G_2$ is not isomorphic to $K_2$. If $G_2 \cong K_2$, then combing Lemma 1 with the optimality of $G$, we have $|V(G_1)| + 4 \times |E(G_1)| = |E(G)| = 4 \times (2 \times |V(G_1)|) - 8$, which implies that

$$|E(G_1)| = \frac{7}{4}|V(G_1)| - 2. \quad (4.1)$$

By Lemma 7, $G_1$ must be a cactus. Then, by Lemma 10, we have

$$|E(G_1)| \leq \left\lfloor \frac{3(|V(G_1)| - 1)}{2} \right\rfloor. \quad (4.2)$$

Combining (4.1) with (4.2), we have $|V(G_1)| \leq 2$, a contradiction to that $|V(G_1)| \geq 4$. Hence, $G_2 \cong 2K_1$.

Let $V(G_1) := \{u_1, u_2, \ldots, u_n\}$ where $n \geq 4$ and $V(G_2) := \{v_1, v_2\}$. By Lemma 2, $G_1$ is connected. Hence, $G_1$ has an edge $u_iu_j$ for $1 \leq i \neq j \leq n$. Then vertices $u_i$ (resp. $u_j$) is substituted with two vertices $(u_i, v_1)$ and $(u_i, v_2)$ in $G$ (resp. $(u_j, v_1)$ and $(u_j, v_2)$ in $G$). For simplicity, we set $u := (u_i, v_1)$, $v := (u_i, v_2)$, $x := (u_j, v_1)$ and $y := (u_j, v_2)$. Denote the vertices of $V(G) \setminus \{u, v, x, y\}$ by $\{w_1, w_2, \ldots, w_{2n-4}\}$.

Below, we give three simple claims.

**Claim 3.** If a vertex $p$ of $G$ is adjacent to one vertex of $\{u, v\}$ (resp. of $\{x, y\}$), then $p$ is also adjacent to the other vertex of $\{u, v\}$ (resp. of $\{x, y\}$).

*Proof.* This claim follows directly from Claim 3 (ii). \qed

**Claim 4.** The edge $uv \notin E(G)$ and $xy \notin E(G)$.

*Proof.* By Lemma 3 (i), both $G[u, v]$ and $G[x, y]$ are isomorphic to $2K_1$, as desired. \qed

**Claim 5.** The graph $G[u, v, x, y] \cong C_4$.

*Proof.* By the definition of lexicographic products, $ux, xv, vy$ and $yu$ are edges of $G$. So these edges form a 4-cycle of $G$. By Claim 4, both $uv$ and $xy$ are not edges of $G$. Hence $G[u, v, x, y] \cong C_4$, as desired. \qed

Hence, we denote the 4-cycle $uxvyu$ of $G$ by $C$. Below our attention is directed toward the cycle $C$.

First we show that $C$ will not cross itself under $D$. Since $D$ is good, any two successive edges of $C$ will not cross each other. For if two non-successive edges of $C$ cross each other, then $G[u, v, x, y] \cong K_4$ by Lemma 16 (i), and thus $uv$ is an edge of $G$; this contradicts to Claim 4.

We now show that $C$ is crossing under $D$. Suppose that $C$ is non-crossing under $D$. We choose an adjacent edge of $u$, namely $uw_1$, such that $uw_1$ is closest to $ux$ in cyclic order (clockwise). By Lemma 15, $uw_1$ is crossing under $D$, and thus $w_1 \neq y$ (since $C$
is non-crossing under $D$). Furthermore, Claim 4 implies that $w_1$ is not $v$. So $w_1$ lies in $C_{\text{int}}$, see Fig 7 (a). Without loss of generality, we assume that $uw_1$ is crossed by $w_2w_3$.

We notice that any one of vertex-ends of $w_2w_3$ is not $v$; otherwise, $uv$ will be an edge by Lemma 16 (i), which contradicts to Claim 4. Furthermore, we discuss the following three possible cases, all of which will yield contradictions.

**Case 3.1.** $w_i \neq x$ and $w_i \neq y$ for $i = 2, 3$.

We denote by $c$ the crossing point created by $uw_1$ and $w_2w_3$. By Lemma 16 (i), $uw_2$ and $uw_3$ are non-crossing edges under $D$. If one of $uw_2$ and $uw_3$, say $w_2$, is drawn between $uw_1$ and $ux$ in clockwise, then $uw_2$ is closer to $ux$ in clockwise order, which contradicts the choice of $uw_1$, see Fig. 7 (b). If both $uw_2$ and $uw_3$ are not drawn between $uw_1$ and $ux$ in clockwise, then $(uw_2cu)_{\text{out}}$ (or $(uw_3cu)_{\text{out}}$) contains the vertex $x$; see Fig 7 (b), and thus $(uw_2cu)_{\text{out}}$ (or $(uw_3cu)_{\text{out}}$) will not be a crossed 3-face of $D$, and this contradicts to Lemma 16 (ii).

![Fig. 7. The non-crossing 4-cycle $C$ involving the crossing edge $uw_1$ under $D$.](image)

**Case 3.2.** $w_i \neq x$ and $w_j = y$ for $2 \leq i \neq j \leq 3$.

Similar to Case 3.1, it will also lead a contradiction. We leave the readers to prove it.

**Case 3.3.** $w_2 = x$ or $w_3 = x$.

Without loss of generality, we assume that $w_2 = x$. By Claim 4, $w_3 \neq y$, and thus $w_3$ must lie in $C_{\text{int}}$, see Fig. 7 (d). By Lemma 16 (i), $uw_3$ and $xw_1$ are non-crossing edges under $D$. By Claim 3, $w_1y$, $w_3v$ and $w_3y$ are edges of $G$. As $C$ is non-crossing under $D$, $w_1y$ must cross $w_3v$, and thus $w_3y$ is non-crossing under $D$ by Lemma 16 (i).
Observe that $uyw_3u$ is a non-crossing 3-cycle under $D$, which contradicts Lemma 17.

From the analysis above, we have seen that all possible cases derive contradictions. Therefore, $C$ is crossing under $D$. That is to say, there exists some edge of $C$ that is crossing under $D$. Without loss of generality, we assume that $ux$ is crossed by some edge, namely $w_1w_2$. Furthermore, we claim that $w_i \notin V(C)$ for $i = 1, 2$. As $D$ is good, $w_i \neq u$ and $w_i \neq x$ for $i = 1, 2$. If some end-vertex of $w_1w_2$, say $w_1$, is $v$, then $w_2 \neq y$, otherwise $C$ is self-crossing under $D$, a contradiction (earlier, we have proved that $C$ is not self-crossing under $D$). Thus, $w_2 \notin V(C)$, but now $uv$ must be an edge of $G$ by Lemma 16 (i), since $v(=w_1)w_2$ crosses $ux$; this contradicts to Claim 3. By symmetry of $w_1$ and $w_2$, we also have $w_2 \neq v$. Similarly, $w_i \neq y$ for $i = 1, 2$. So $w_i \notin V(C)$ for $i = 1, 2$. Thus, we may assume that $w_1$ and $w_2$ lie in the exterior and in the interior of $C$, respectively. We immediately get the following claim.

**Claim 6.** The 1-planar drawing $D$ contains the subdrawing $F$, as shown in Fig. 8.

**Proof.** By Lemma 16 (i), $w_1u$, $w_1x$, $w_2u$ and $w_2x$ are non-crossing edges under $D$. By Claim 3, $w_1y$, $w_1v$, $w_2v$ and $w_2y$ are edges of $G$. The edge $w_2v$ cannot cross $vx$ or $vy$ since $D$ is good, and $w_2v$ also cannot cross $uy$ otherwise $uv$ will be an edge of $G$ by Lemma 16 (i), which contradicts Claim 4. So $w_2v$ must be placed in $C_{int}$. Similarly, $w_2y$ is also placed in $C_{int}$. Now we observe that both $w_1y$ and $w_1v$ cannot cross by any one edge of $uy$, $vx$ and $vy$, and thus they lie in $(w_1uyw_1)_{out}$. Thus, $D$ contains the subdrawing $F$ as shown in Fig. 8. $\square$

![Fig. 8. The subdrawing $F$.](image)

**Remark 1.** Notice that the 1-planar drawing $F$ in Claim 6 may also be drawn as $F^1$ or $F^2$ as shown in Fig 9, but they are isomorphic to $F$. If we consider $F^1$ or $F^2$, then some insignificant details in the following proof need to be adjusted. For example, if we consider $F^1$, then $v$ is in the “interior” of $w_1uyw_1$ whereas $v$ is in the “exterior” of $w_1uyw_1$ if we consider $F$. But it does not affect the correctness of our proof. So in the following we only consider $F$. 

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We now show that the cycle $C$ has at least two crossing edges under $D$. Suppose that $C$ has the unique crossing edge $ux$ under $D$. That is to say, $uy$, $vx$ and $vy$ are non-crossing under $D$. First, by Claim 6, $D$ contains $F$ as a subdrawing. We claim that $w_2y$ is crossed by some edge, say $w_3w_4$; otherwise, $uw_2yu$ is a non-crossing 3-cycle under $D$ (note that $uy$ and $uw_2$ are non-crossing), a contradiction to Lemma 17. Furthermore, we claim that one of $w_3$ and $w_4$ is $u$. If not, then one of $w_3$ and $w_4$, say $w_3$, lies in $(uw_2yu)_{int}$. By Lemma 16 (i), $w_3y$ is an edge, and thus $w_3x$ will be an edge of $G$. But $w_3x$ will cross at least two times; this contradicts that $D$ is 1-planar. Thus, $w_3 = u$ or $w_4 = u$. Without loss generality, we assume that $w_3 = u$. By Claim 3, $w_4 \neq v$, and thus $w_4$ must lie in $(uw_2yu)_{int}$, see Fig. 10. Furthermore, $w_4y$ is a non-crossing edge under $D$ by Lemma 16 (i), and thus $w_4x$ is also an edge by Claim 3. Now we see that $w_4x$ must cross $w_2v$, and hence $w_4v$ is a non-crossing edge under $D$ by Lemma 16 (i). Note that $vy$ is also non-crossing under $D$. So $yvw_4y$ is a non-crossing 3-cycle under $D$, which contradicts Lemma 17. Therefore, $C$ has at least two crossing edges under $D$.

We now show that the cycle $C$ has at least two crossing edges under $D$. Suppose that $C$ has the unique crossing edge $ux$ under $D$. That is to say, $uy$, $vx$ and $vy$ are non-crossing under $D$. First, by Claim 6, $D$ contains $F$ as a subdrawing. We claim that $w_2y$ is crossed by some edge, say $w_3w_4$; otherwise, $uw_2yu$ is a non-crossing 3-cycle under $D$ (note that $uy$ and $uw_2$ are non-crossing), a contradiction to Lemma 17. Furthermore, we claim that one of $w_3$ and $w_4$ is $u$. If not, then one of $w_3$ and $w_4$, say $w_3$, lies in $(uw_2yu)_{int}$. By Lemma 16 (i), $w_3y$ is an edge, and thus $w_3x$ will be an edge of $G$. But $w_3x$ will cross at least two times; this contradicts that $D$ is 1-planar. Thus, $w_3 = u$ or $w_4 = u$. Without loss generality, we assume that $w_3 = u$. By Claim 3, $w_4 \neq v$, and thus $w_4$ must lie in $(uw_2yu)_{int}$, see Fig. 10. Furthermore, $w_4y$ is a non-crossing edge under $D$ by Lemma 16 (i), and thus $w_4x$ is also an edge by Claim 3. Now we see that $w_4x$ must cross $w_2v$, and hence $w_4v$ is a non-crossing edge under $D$ by Lemma 16 (i). Note that $vy$ is also non-crossing under $D$. So $yvw_4y$ is a non-crossing 3-cycle under $D$, which contradicts Lemma 17. Therefore, $C$ has at least two crossing edges under $D$.

Now, we will prove that both the two edges $vx$ and $uy$ (on $C$) are non-crossing under $D$. Note that $D$ contains the subdrawing $F$. As $vx$ and $uy$ are symmetric in $F$,
we need only consider \( vx \). We assume that \( vx \) is crossed by some edge, namely \( w_3w_4 \), see Fig. 11. We can observe that if \( w_i \in V(F) \) for \( i = 3 \) or \( i = 4 \), then they can only be \( w_1 \) or \( w_2 \). In the following, it is sufficient to consider three cases.

**Case 3.4.** \( w_3 \notin V(F) \) and \( w_4 \notin V(F) \).

Without loss of generality, we may assume that \( w_3 \) and \( w_4 \) lie in \( (xvw_1)_{\text{int}} \) and \( (xvw_2)_{\text{out}} \), respectively; see Fig 11 (a). Since \( w_3w_4 \) crosses \( vx \), \( w_3v \) is an edge of \( G \) by Lemma 16 (i), and thus \( w_3u \) is an edge of \( G \) by Claim 3. But now \( u_3u \) will be crossed at least two times, a contradiction.

**Case 3.5.** \( w_3 \notin V(F) \) and \( w_4 = w_1 \).

Similar to Case 3.3, \( u_3u \) will also be crossed at least two times, a contradiction; see Fig 11 (b).

**Case 3.6.** \( w_3 = w_2 \).

In this case, \( w_4 \notin V(F) \), otherwise \( G \) will have multiedges. Since \( D \) is 1-planar, \( w_4 \) must lie in \( (xvw_1)_{\text{int}} \), see Fig 11 (c). Since \( w_2 = w_3 \) crosses \( vx \), \( w_4v \) is an edge of \( G \) by Lemma 16 (i), and thus \( w_4u \) is an edge of \( G \) by Claim 3. But \( u_4u \) will be crossed at least two times, a contradiction.

Thus, \( xv \) is non-crossing under \( D \). Similarly, \( uy \) is also non-crossing under \( D \).

Note that earlier we proved that at least two edges of \( C \) are crossing under \( D \). Hence, the edge \( vy \) on \( C \) must be crossed by some edge, namely \( w_3w_4 \). We now prove that \( w_3 \notin V(F) \) and \( w_4 \notin V(F) \). Since \( w_3w_4 \) has been crossed by \( vy \), we see that if \( w_3 \) or \( w_4 \) belongs \( V(F) \), then they can only be \( w_1 \) or \( w_2 \). Without loss of generality, we may assume that \( w_3 = w_1 \), and thus \( w_4 \neq w_2 \) by simplify of \( G \). Thus, \( w_4 \) must lie in \( (w_2vyw_2)_{\text{int}} \), see Fig. 12 (a). By Lemma 16 (i), \( w_1v \) will be a non-crossing edge under \( D \). Note that we have proved that \( w_1x \) and \( uy \) are non-crossing under \( D \). So the cycle \( vw_1yv \) is a non-crossing 3-cycle under \( D \), which contradicts Lemma 17. As for if \( w_3 = w_2 \), then similarly, we will similarly find a non-crossing 3-cycle \( w_2uyw_2 \) under \( D \), a contradiction; see Fig. 12 (b). By symmetry, \( w_4 \neq w_i \) for \( i = 1, 2 \). Therefore,
$w_3 \notin V(F)$ and $w_4 \notin V(F)$.

Fig. 12. (a) The vertex $w_3 = w_1$. (b) The vertex $w_3 = w_2$.

So we may assume that $w_3$ and $w_4$ lie in $(w_2vyw_2)_{\text{out}}$ and $(w_2yvw_2)_{\text{int}}$, respectively. Furthermore, $w_3$ must lie in $(w_1yvw_1)_{\text{out}}$, since $D$ is 1-planar and $w_3w_4$ has been crossed by $vy$.

Finally, we prove that $G \cong K_{2,2,2,2}$. Similar to Claim 6, all vertices of $C$, together with $w_3$ and $w_4$ and their edges will also form a subdrawing of $D$ isomorphic to $F$. So $D$ contains a 1-planar drawing $F'$ as shown in Fig. 13.

Fig. 13. The 1-planar drawing $F'$.

In $F'$, since $w_1v$ crosses $w_3x$, $w_1w_3$ will be a non-crossing edge under $D$ by Lemma 16 (i). Similarly, $w_2w_4$ is also a non-crossing edge under $D$. We called final 1-planar drawing $F''$ obtained from $F'$ by adding two non-crossing edges $w_1w_3$ and $w_2w_4$. We notice that every face of $F'$ is a crossed 3-face. Hence, $G$ does not have any other new vertex by Lemma 16 (ii), and obviously we also cannot add any new edge in $F''$. So $D$ is isomorphic to $F''$, and thus $G$ is isomorphic to $K_{2,2,2,2}$. Finally, it is easy to see that $K_{2,2,2,2} = K_4 \circ 2K_1$, $|V(K_4)| = 4 \geq 3$ and $|V(2K_1)| = 2$. Therefore, we complete the proof of Case 3.

The “only if” part of Theorem 1 follows directly from Case 1, Case 2 and Case 3.
Therefore, we have proved Theorem 1. □

Here, we prove Corollaries 1, 2 and 3.

**Proof of Corollary 1.** For (i), if \( n = 8 \), then \( m \leq 4 \times 8 - 8 = 24 \) by Lemma 8. By Theorem 1, \( K_{2,2,2,2} \) is a reducible 1-planar graph with 8 vertices and 24 edges, and thus “24” is tight.

For (ii), if \( n \geq 10 \), then \( m \leq 4 \times n - 8 \) by Lemma 8. Theorem 1 implies that any optimal 1-planar graph is irreducible if it has at least 10 vertices, and thus \( m \leq 4n - 9 \) for \( n \geq 10 \) (since \( G \) is reducible). For \( n = 9 \), then any 1-planar graph with 9 vertices has at most \( 4 \times 9 - 9 \) edges, and thus \( m \leq 4 \times 9 - 9 \) by Lemma 8. As for \( n = 6 \), we have \( m \leq \binom{6}{2} = 15 = 4 \times 6 - 9 \) by Lemma 8. Below we show that the bound \( 4n - 9 \) is tight. Let \( P_{\frac{n}{3}} \) be a path with \( \frac{n}{3} \) vertices where \( n \geq 3 \) and \( n \equiv 0 \pmod{3} \). We denote the vertex set of \( P_{\frac{n}{3}} \) by \( \{u_1, u_2, \cdots , u_{\frac{n}{3}}\} \). Let \( H = P_{\frac{n}{3}} \circ C_3 \). Clearly, \( H \) is reducible and has \( 4n - 9 \) edges. We demonstrate a construction of 1-planar drawing of \( H \) in Fig. 14. In Fig. 14, \( C_{u_i} \) is a 3-cycle in \( H \) that corresponds to the vertex \( u_i \) in \( P_{\frac{n}{3}} \) for \( 1 \leq i \leq \frac{n}{3} \). Thus, the bound \( 4n - 9 \) is tight for every \( k \geq 2 \) and \( n = 3k \). □

![Fig. 14. A planar drawing of \( \frac{n}{3} \) disjoint copies of \( C_3 \) on the left and a 1-planar drawing of \( H \) on the right.](image)

**Proof of Corollary 2.** For (i), from the proof in Theorem 1, \( K_4 \circ 2K_1 \cong K_{2,2,2,2} \) and \( K_{2,2,2,2} \) is 1-planar, and thus \( K_4 \circ 2K_1 \) is 1-planar. So for any subgraph \( H \) of \( K_4 \), \( H \circ 2K_1 \) is 1-planar. Hence, for any graph \( G \) with exactly 2 vertices or 3 vertices, we have \( m \leq 2n - 3 \), and for any graph \( G \) with exactly 4 vertices, we have \( m \leq 6 \). For \( n \geq 5 \), by Theorem 1, \( G \circ 2K_1 \) is not optimal a 1-planar graph. Thus, \(|E(G \circ 2K_1)| \leq 4 \times (2n) - 9\). Note that \(|E(G \circ 2K_1)| = 4m \) by Lemma 1. Therefore, we have \( 4m \leq 8n - 9 \), and thus \( m \leq \lfloor \frac{8n-9}{4} \rfloor = 2n - 3 \), as desired.

For (ii), note that any subgraph of a 1-planar graph is 1-planar, and thus (ii) follows directly from (i). □

**Proof of Corollary 3.** Clearly, \( G \circ 2K_1 \) contains \( G \) as a subgraph by the definition of the lexicographic products. So if \( G \circ 2K_1 \) is 1-planar, then \( G \) is 1-planar (any subgraph...
of a 1-planar graph is also 1-planar). Suppose $G$ is maximal 1-planar with $n \geq 17$ vertices. Then $|E(G)| \geq \frac{20}{9}n - \frac{10}{3}$ by Lemma 9. But $\frac{20}{9}n - \frac{10}{3} > 2n - 3$ for any integer $n \geq 17$, a contradiction to Corollary 2. So $G$ cannot be a maximal 1-planar graph for $n \geq 17$. □

5 Further problems

In this article, we characterize the reducibility of optimal 1-planar graphs. Let $G$ be a graph with $n$ vertices and $m$ edges. As a result, we prove that if $G \circ 2K_1$ is 1-planar, then $m \leq 2n - 3$ for $n \geq 2$ and $n \neq 4$ and $m \leq 6$ (= $2n - 2$) for $n = 4$. Clearly, when $n = 2$ or $n = 3$, the upper bound $2n - 3$ can be achieved. However, for $n \geq 5$, the following problem remains:

**Problem 2.** In Corollary 2, is the upper bound “$2n - 3$” for $n \geq 5$ tight?

From the viewpoint of the reverse process of lexicographic products, the above problem is equivalent to asking whether there exist some 1-planar graphs with even $n$ vertices and $4n - 12$ edges that can be written as the lexicographic product of a graph and $2K_1$.

Corollary 3 shows us that for $n \geq 17$, $G$ is not maximal 1-planar if $G \circ 2K_1$ is 1-planar. As for $n \leq 4$, $G$ can be a complete graph by Corollary 2 if $G \circ 2K_1$ is 1-planar, and thus $G$ can be maximal 1-planar if $G \circ 2K_1$ is 1-planar. Then the following problem is natural.

**Problem 3.** For an integer $n$ where $5 \leq n \leq 16$, is there a maximal 1-planar graph $G$ with $n$ vertices such that $G \circ 2K_1$ is 1-planar?

In another direction, it is natural to consider extending our results (the reducibility of optimal 1-planar graphs) to optimal $k$-planar graphs for $k \geq 2$. In particular, the structure of optimal 2-planar graphs and optimal 3-planar graphs are somewhat clear, see [4]. Their reducibility might be characterized in the future. It is also interesting to characterize the reducibility of optimal $k$-planar graphs with respect to other three standard graph products namely, the Cartesian product, the direct product and the strong product.

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References

[1] E. Ackerman, A note on 1-planar graphs, Discrete Appl. Math. 175 (2014), 104–108.

[2] L. Babai, Graph isomorphism in quasipolynomial time, in: Proc. 48th Annu. ACM Symp. Theory Comput., pp. 684–697.

[3] J. Barát, G. Tóth, Improvements on the density of maximal 1-planar graphs, J. Graph Theory 88 (2018), 101–109.

[4] M. A. Bekos, M. Kaufmann, C. N. Raftopoulou, On optimal 2- and 3-planar graphs, in: SoCG 2017, in: LIPIcs, vol. 77, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017, pp. 16:1–16:16.

[5] M. A. Bekos, M. Kaufmann, C. Zielke, The book embedding problem from a SAT-solving perspective, in: Proc. 23rd Intl. Symp. on Graph Drawing and Network Visualization, in: LNCS, vol. 9411, Springer Verlag, 2015, pp. 125–138.

[6] T. Biedl, J. Wittnebel, Matchings in 1-planar graphs with large minimum degree, J. Graph Theory 99 (2022), 217–230.

[7] C. Binucci, W. Didimo, F. Montecchiani, 1-planarity testing and embedding: An experimental study, Comput. Geom. 108 (2023).

[8] R. Bodendiek, H. Schumacher, K. Wagner, Bemerkungen zu einem Sechsfarbenproblem von G. Ringel, Abh. Math. Semin. Univ. Hamb. 53 (1983), 41–52.

[9] O. V. Borodin, A new proof of the 6 color theorem, J. Graph Theory 19 (1995), 507–521.

[10] F. J. Brandenburg, Recognizing optimal 1-planar graphs in linear time, Algorithmica 80 (2018), 1–28.

[11] F. J. Brandenburg, D. Eppstein, A. Gleißner, M. T. Goodrich, K. Hanauer, J. Reislhuber, On the density of maximal 1-planar graphs, in: International Symposium on Graph Drawing, Springer, 2012, pp. 327–338.

[12] J. Bucko, J. Czap, 1-planar lexicographic products of graphs, Appl. Math. Sci. 9 (2015), 5441–5449.

[13] J. Czap, D. Hudák, 1-planarity of complete multipartite graphs, Discrete Appl. Math. 160 (2012), 505–512.

[14] J. Czap, D. Hudák, On drawings and decompositions of 1-planar graphs, Electron. J. Comb. 20 (2013), P54.
[15] J. Czap, D. Hudák, T. Madaras, Joins of 1-planar graphs, Acta Math. Sin., Engl. Ser. 30 (2014), 1867–1876.

[16] W. Didimo, Density of straight-line 1-planar graph drawings, Inf. Process. Lett. 113 (2013), 236–240.

[17] J. Feigenbaum, A. A. Schäffer, Recognizing composite graphs is equivalent to testing graph isomorphism, SIAM J. Comput. 15 (1986), 619–627.

[18] J. Fujisawa, K. Segawa, Y. Suzuki, The matching extendability of optimal 1-planar graphs, Graphs Combin. 34 (2018), 1089–1099.

[19] J. P. Gollin, K. Hendrey, A. Methuku, C. Tompkins, X. Zhang, Counting cliques in 1-planar graphs, arXiv preprint, arXiv:2109.02906.

[20] A. Grigoriev, H. L. Bodlaender, Algorithms for graphs embeddable with few crossings per edge, Algorithmica 49 (2007), 1–11.

[21] R. H. Hammack, W. Imrich, S. Klavžar, W. Imrich, S. Klavžar, Handbook of Product Graphs, Vol. 2, CRC press Boca Raton, 2011.

[22] H. Harborth, Parity of numbers of crossings for complete n-partite graphs, Math. Slovaca 26 (1976), 77–95.

[23] Y. Huang, Z. Ouyang, F. Dong, On the size of matchings in 1-planar graph with high minimum degree, SIAM J. Discrete Math. 36 (2022), 2570–2584.

[24] Y. Huang, L. Zhang, Y. Wang, The matching extendability of 7-connected maximal 1-plane graphs, Discuss. Math. Graph Theory (2022).

[25] D. Hudák, T. Madaras, Y. Suzuki, On properties of maximal 1-planar graphs, Discuss. Math. Graph Theory 32 (2012), 737–747.

[26] S. G. Kobourov, G. Liotta, F. Montecchiani, An annotated bibliography on 1-planarity, Comput. Sci. Rev. 25 (2017), 49–67.

[27] V. P. Korzhik, B. Mohar, Minimal obstructions for 1-immersions and hardness of 1-planarity testing, J. Graph Theory 72 (2013), 30–71.

[28] W. J. Lenhart, G. Liotta, F. Montecchiani, On partitioning the edges of 1-plane graphs, Theoret. Comput. Sci. 662 (2017), 59–65.

[29] N. Matsumoto, Y. Suzuki, Non-1-planarity of lexicographic products of graphs, Discuss. Math. Graph Theory 41 (4), 1103–1114.

[30] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, Abh. Math. Semin. Univ. Hambg. 29 (1965), 107–117.
[31] U. Schöning, Graph isomorphism is in the low hierarchy, J. Comput. System Sci. 37 (1988), 312–323.

[32] Y. Suzuki, Re-embeddings of maximum 1-planar graphs, SIAM J. Discrete Math. 24 (2010), 1527–1540.

[33] Y. Suzuki, $K_7$-minors in optimal 1-planar graphs, Discrete Math. 340 (2017), 1227–1234.

[34] Y. Suzuki, 1-Planar Graphs, in: Beyond Planar Graphs, S.-H. Hong and T. Tokuyama (Ed(s)), Springer, (2020), 47–68.

[35] D. B. West, Introduction to Graph Theory (second edition), Prentice Hall, 2001.

[36] J. Zhang, Y. Wu, H. Zhang, The maximum matching extendability and factor-criticality of 1-planar graphs, arXiv preprint, arXiv:2205.12122.

[37] X. Zhang, The edge chromatic number of outer-1-planar graphs, Discrete Math. 339 (2016), 1393–1399.