One-parameter family of closed, radiation-filled Friedmann-Robertson-Walker

“quantum” universes

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Abstract

Using as an illustrative example the $p=1$ operator-ordered Wheeler-DeWitt equation for a closed, radiation-filled Friedmann-Robertson-Walker universe, we introduce and discuss the supersymmetric double Darboux method in quantum cosmology. A one-parameter family of “quantum” universes and the corresponding “wavefunctions of the universe” for this case are presented.

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Many interesting results have been obtained mostly in one-dimensional quantum mechanics by means of Darboux-Witten (DW) supersymmetric procedures; for a recent review see [1]. These are, essentially, factorizations of the one-dimensional Schrödinger differential operator, first performed in the supersymmetric context by Witten in 1981 [2], and in mathematics literature, in the broader sense of the Darboux transformation, by Darboux, as early as 1882 [3]. Constructing families of isospectral potentials is an important supersymmetric topic which may

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have many physical applications \[1\]. Since in quantum cosmology the Wheeler-DeWitt (WDW) equation is a Schrödinger-type equation one might think of introducing the isospectral issue in this field as well. This is our purpose in the present work. To accomplish it, we shall use the simple example of a radiation-filled, closed Friedmann-Robertson-Walker (FRW) “quantum” universe.

We briefly review the well-established supersymmetric double DW technique of deleting followed by “reinstating” a nodeless, radial (half-line) ground-state, say \(u_0(\tilde{a})\), of a potential \(V^- (\tilde{a})\) by means of which one can generate a one-parameter family of isospectral potentials \(V_{iso}(\tilde{a}; \lambda)\), where \(\lambda\) is a labeling, real parameter of each member potential in the set. The family reads explicitly

\[
V_{iso}(\tilde{a}; \lambda) = V^- (\tilde{a}) - 2[\ln(J_0 + \lambda)]'' = V^- (\tilde{a}) - \frac{4u_0u_0'}{J_0 + \lambda} + \frac{2u_0^4}{(J_0 + \lambda)^2},
\]

(1)

where the primes are derivatives with respect to the radial variable \(\tilde{a}\) and

\[
J_0(\tilde{a}) \equiv \int_0^{\tilde{a}} u_0^2(y)dy.
\]

The result of Eq. (1) is obtained if one factorizes the one-dimensional Schrödinger equation with the operators \(A = \frac{d}{d\tilde{a}} + W(\tilde{a})\) and \(A^\dagger = -\frac{d}{d\tilde{a}} + W(\tilde{a})\), where the superpotential function is given by \(W = -u_0'/u_0\). Then, in the DW scheme the potential and superpotential enter an initial ‘bosonic’ Riccati equation \(V^- = W^2 - W'\). At the same time, one can build a ‘fermionic’ Riccati equation \(V^+ = W^2 + W'\), corresponding to a ‘fermionic’ Schrödinger equation for which the operators \(A\) and \(A^\dagger\) are applied in reversed order. Thus, the ‘fermionic’ potential is found to be \(V^+ = V^- (\tilde{a}) - 2 \left( \frac{u_0'}{u_0} \right)'.\) This potential does not have \(u_0\) as the ground state eigenfunction. However, it is possible to “reintroduce” the \(u_0\) solution into the spectrum by means of the general superpotential solution of the fermionic Riccati equation. The general Riccati solution reads

\[
W_{gen} = W(\tilde{a}) + \frac{d}{d\tilde{a}} \ln[J_0(\tilde{a}) + \lambda].
\]

(3)

The way to obtain Eq. (3) is simple and well-known \[3\]. From Eq. (3) one can easily get Eq. (1) by just using the bosonic Riccati equation. But, is there any price for this procedure? Yes, there is. The new wavefunction differs from the initial one by a ‘normalization’, \(\lambda\)-dependent constant.
The important thing which was noticed in the literature was the damping nature of the integral $J_0$, both for the family of potentials as for the wavefunctions. The parameter $\lambda$ is acting as a “damping distance” which is signaling the importance of the integral term. Finally another useful formula gives the family of $\lambda$-dependent (non-normalizable) wavefunctions as follows

$$u_{\text{gen}}(\tilde{a}; \lambda) = \frac{u_0(\tilde{a})}{J_0 + \lambda},$$

(4)

We choose now one of the most simple cosmological metrics, a closed FRW model filled only with radiation, in order to clearly illustrate the supersymmetric isospectral issue in quantum cosmology. Kung [4] has recently discussed the WDW equation for the closed, $k > 0$ FRW metric with various combinations of cosmological constant and matter. The WDW equation used by Kung reads

$$\left[-\tilde{a}^{-p} \frac{\partial}{\partial \tilde{a}} \tilde{a}^{p} \frac{\partial}{\partial \tilde{a}} + \left(\frac{3\pi}{2G}\right)^2 \frac{1}{k^2} \left(k\tilde{a}^2 - \frac{8\pi G}{3} \rho a^4\right)\right] u(\tilde{a}) = 0,$$

(5)

where the variable $a$ is the cosmological scale factor. The parameter $p$ enters as a consequence of the ambiguity in the ordering of $a$ and $\partial/\partial a$; $p = 1$ is the so-called Laplacian ordering, whereas $p = 2$ is convenient for the WKB approximation.

In the matter sector various contributions may be present but we shall consider only the radiation case, i.e.,

$$\frac{8\pi G}{3} \rho \rightarrow \frac{8\pi G}{3} \left[\rho_r \left(\frac{a_0}{a}\right)^4\right],$$

(6)

where $\rho_r$ is the radiation energy density. The most compact form of the WDW equation in the FRW metric can be written down as follows,

$$\left[-\tilde{a}^{-p} \frac{\partial}{\partial \tilde{a}} \tilde{a}^{p} \frac{\partial}{\partial \tilde{a}} + V(\tilde{a})\right] u(\tilde{a}) = 0,$$

(7)

where the tilde variables are rescaled and dimensionless ones. Thus, only with the radiation matter sector the cosmological potential turns out to be an oscillator potential of the form

$$V(\tilde{a}) \equiv \tilde{a}^2 - \tilde{\beta}^2,$$

(8)
where \( \tilde{a}^2 = \frac{3\pi}{2Gk}a^2 \) is the tilde scale factor of the universe, and \( \tilde{\beta}^2 = \frac{4\pi^2}{k^2}\rho_0 a_0^4 \) expresses the radiation effect on the cosmological expansion.

With the ansatz \( u(\tilde{a}) \equiv g(\tilde{a})e^{-\tilde{a}^2/2} \), and with the further change of variable \( x = \tilde{a}^2 \), one gets the following confluent hypergeometric equation for \( g(x) \),

\[
x \frac{d^2 g}{dx^2} + \left( \frac{1 + p}{2} - x \right) \frac{dg}{dx} + \left( \frac{\tilde{\beta}^2 - 1 - p}{4} \right) g = 0. \tag{9}
\]

Using the \( p = 1 \) factor ordering and with the natural assumption that the physical \( u \) must vanish asymptotically for \( x \to \infty \), Eq. (9) turns into an eigenvalue problem for the Laguerre polynomials \( g(x) = L_0^{\tilde{\beta}^2-2}(x) \), where \( n \in N_+ \). Thus, in this case, the wavefunctions of the universe are of the type

\[
u_n(\tilde{a}) = e^{-\tilde{a}^2/2}L_0^{\tilde{\beta}^2-2}(\tilde{a}^2). \tag{10}\]

One can also see that the radiation energy density is quantized through the condition \( \tilde{\beta}^2 = 4n + 2 \). Thus, \( \tilde{\beta} = \sqrt{2} \) is the lowest value, corresponding to the Gaussian ground state wavefunction \( u_0(\tilde{a}) = e^{-\tilde{a}^2/2} \). Other factor orderings lead to various confluent hypergeometric functions \( g \) and to a change of the quantization condition [4].

In the following, we shall perform the double Darboux construction using the Gaussian nodeless wavefunction of the closed, radiation-filled FRW universe. One can easily obtain \( \mathcal{W} = \tilde{a} \), that is the superpotential is the dimensionless scale factor of the universe, and \( V^+ = V^- + 2 = \tilde{a}^2 - \tilde{\beta}^2 + 2 = \tilde{a}^2 \).

In this case, the one-parameter family of bosonic cosmological potentials can be written as follows,

\[
V_{iso} = \tilde{a}^2 - 2 + \frac{4\tilde{a}u_0^2}{\lambda_{s\text{erf}}(\tilde{a}) + \lambda} + \frac{2u_0^4}{(\lambda_{s\text{erf}}(\tilde{a}) + \lambda)^2}. \tag{11}
\]

In order to avoid singularities one should take \( |\lambda| > \lambda_s \equiv \sqrt{\pi}/2 \) [4]. We have plotted some members of the family equation (11) in Fig. 1. For \( \lambda \to \pm\infty \) the members are very close in shape to the original \( V^- \) potential.
The one-parameter family of wavefunctions of the universe are

\[ u_{\text{gen}}(\tilde{a}; \lambda) = \frac{u_0}{\lambda \text{erf}(\tilde{a}) + \lambda}. \]  

(12)

The members of the family of wavefunctions given by Eq. (12) are plotted in Fig. 2 for the same \( \lambda \) values as for the potentials. They can also be written as “Gaussians” \( u_{\text{gen}} = e^{-\tilde{a}^2/2\sigma^2(\tilde{a})} \) with a dispersion depending on the scale factor of the universe

\[ \sigma^2(\tilde{a}) = \frac{\tilde{a}^2}{\tilde{a}^2 + \ln(\lambda \text{erf}(\tilde{a}) + \lambda)^2}. \]  

(13)

We recall that Kiefer \[6\] has considered wavepackets with time-dependent dispersion within the WKB approximation to minisuperspace models.

In conclusion, we have introduced concepts like isospectral cosmological potentials and a corresponding one-parameter family of wavefunctions of the universe for the simple case of closed, radiation-filled FRW “quantum” universe.

The work must be considered as merely a brief introduction of the isospectral supersymmetric problem in quantum cosmology. Its further applications in this field will be fully explored in future investigations.

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References

[1] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. 251, 267 (1995)
[2] E. Witten, Nucl. Phys. B 188, 513 (1981)
[3] G. Darboux, C.R. Acad. Sci. 94, 1456 (1882)
[4] J.H. Kung, Gen. Rel. Grav. 27, 35 (1995)
[5] B. Mielnik, J. Math. Phys. 25, 3387 (1984); M.M. Nieto, Phys. Lett. B 145, 208 (1984); J. Pappademos, U. Sukhatme and A. Pagnamenta, Phys. Rev. A 48, 3525 (1993)
[6] C. Kiefer, Phys. Rev. D 38, 1761 (1988)
Members of the one-parameter isospectral family of closed, radiation-filled FRW potentials for the following values of the $\lambda$-parameter (from the right to the left): (a) -.8863; (b) -.8903; (c) -.9303 together with the original potential (the evolvent) $V^- = \dot{a}^2 - 2$. 

Fig. 1
Fig. 2
Squares of the “wavefunction of the universe” for the same values (right to left) of the $\lambda$ parameter as in Fig. 1. The largest peak is scaled down forty times.