On a question of Goss

Sangtae Jeong

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Abstract

In this note we answer the question raised by D. Goss in [Applications of non-Archimedean integration to the L-series of τ-sheaves, J. Number Theory, 110 (2005), no. 1, 83–113] by proving that the group of locally analytic endomorphisms on the 1-units of a locally compact field of characteristic $p > 0$ is isomorphic to the $p$-adic integers.

After his proof of the analytic continuation of characteristic $p$ valued $L$-function, D. Goss remarked that the proof depends crucially on locally analyticity of exponential functions $u^y$ with $p$-adic integers $y$ on the 1-units of a locally compact field of characteristic $p > 0$. He also pointed out that his proof would automatically work for any other type of locally analytic endomorphism on the group of 1-units, if such an endomorphism exists. Then the following question was raised by Goss, who thought it to reasonable to expect a negative answer:

Question: Let $\mathbb{F}_r$ be a finite field of $r$ elements and $r$ be a power of a prime integer $p$. Let $\mathbb{F}_r((1/T))$ be the completion of a rational function field $\mathbb{F}_r(T)$ at the infinite prime and $U_1$ be the group of 1-units of $\mathbb{F}_r((1/T))$. Does there exist a locally analytic endomorphism $f : U_1 \to U_1$ which is not of the form $u \mapsto u^y$ for some $p$-adic integer $y$?

The purpose of this note is to give a negative answer to the Question above in very great generality. Indeed, we use the Hasse derivatives to give a proof that the group of locally analytic endomorphisms on the group of 1-units of a locally compact field of characteristic $p > 0$ is isomorphic to the $p$-adic integers. As a consequence, we show that the group of locally analytic automorphisms on the group of 1-units of a locally compact field of characteristic $p > 0$ is isomorphic to the $p$-adic units.

1 Notations and statements of results

Let $K$ be a non-Archimedean, locally compact local field of characteristic $p > 0$, with maximal compact subring of $R_K$ and associated maximal ideal $M_K$. Let $\mathbb{F}$ be the field of constants of $K$ and $\pi$ be a prime element of $K$. Then $K$ can be identified with the field of the Laurent series in $\pi$ over the finite field $\mathbb{F}$:

$$K = \mathbb{F}((\pi)).$$

We fix an absolute value $|\cdot|$ associated to the additive valuation $v$ on $K$ so that $|x| = q^{-v(x)}$ where $q$ denotes the order of $\mathbb{F}$. Let $\rho \in R_K$ with $0 < t = |\rho|$ and $\alpha$ be another element of $R_K$. The closed ball $B_{\alpha,t}$ around $\alpha$ of radius $t$ is defined by

$$B_{\alpha,t} = \{ u \in R_K | \ |u - \alpha| \leq t \}.$$
Let $U := U_1$ denote the group of 1-units of $K$:

$$U = 1 + M_K = 1 + \pi F[[\pi]].$$

Then $U \subset R_K$ is the closed ball around 1 of radius $1/q$, i.e., $U = B_{1,1/q}$.

We say that a continuous function $f : U \to K$ is analytic on $U$ if and only if $f$ may be expressed as the Taylor series of the form:

$$f(u) = \sum_{n=0}^{\infty} b_n \left(\frac{u-1}{\pi}\right)^n,$$

where $\{b_n\}_{n \geq 0} \subset K$ and $b_n \to 0$ as $n \to \infty$.

A continuous function $f : U \to K$ is said to be locally analytic on $U$ if for each $\alpha \in U$ there exists $t_{\alpha} > 0$ such that $f$ is analytic on $B_{\alpha,t_{\alpha}}$ in $U$.

Hence, an analytic function $f$ on $B_{\alpha,t_{\alpha}}$ can be expressed as the Taylor series of the form:

$$f(u) = \sum_{n=0}^{\infty} c_n \left(\frac{u-\alpha}{\rho}\right)^n,$$

where $\{c_n\}_{n \geq 0}$ is a null sequence in $K$ and $t_{\alpha} = |\rho|$.

Throughout, for $u \in U$ we set $u = 1 + x$ with $x \in M_K = \pi F[[\pi]]$ and view a locally analytic function $f$ on $U$ as a function of $x$ but not $u$, unless otherwise specified. By a change of variable we see from Equation (2) that a locally analytic function $f$ on $U$ may be rewritten as

$$f(u) = f(1 + x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{for some } \{a_n\}_{n \geq 0} \text{ in } K. \quad (3)$$

We remark that if the sequence of coefficients $\{a_n\}_{n \geq 0}$ in $K$ is bounded, the series in Equation (3) converges on $M_K$ since $a_n x^n \to 0$ as $n \to \infty$. In what follows we denote by $\Lambda$ the set of locally analytic endomorphisms $f$ from $U$ to itself:

$$\Lambda = \{f : U \to U| f \text{ is locally analytic on } U \text{ and } f(uv) = f(u)f(v) \text{ for all } u, v \in U\}.$$

Then it is obvious that $\Lambda$ has two binary operations, usual multiplication and composition. We also see easily that $\Lambda$ is a group under multiplication but not under composition. On the other hand, the $p$-adic integers $\mathbb{Z}_p$ act naturally on $U$ (see [2, 5]), so that the set $\{u^y : y \in \mathbb{Z}_p\}$ is a subgroup of $(\Lambda, \cdot)$ since $u^y$ is analytic on $U$, hence locally analytic on $U$. We here refer the reader to [1, 3] for details on locally analytic functions.

The following theorem answers the Goss’s question:

**Theorem 1.1.** Let $\Lambda$ be the group of locally analytic endomorphisms on $U$. Then $(\Lambda, \cdot)$ is isomorphic to $(\mathbb{Z}_p, +)$.

We might be wondering what happens if endomorphisms in $\Lambda$ are replaced with automorphisms.

The next result answers this question:

**Corollary 1.2.** Let $\Lambda_0$ be the group of locally analytic automorphisms on $U$. Then $(\Lambda_0, \circ)$ is isomorphic to $(\mathbb{Z}_p^*, \cdot)$.

We have an interesting result on ultimately periodic sequences which arise from $p$-adic binomial coefficients. By an ultimately periodic sequence $\{s_n\}_{n \geq 0}$ we mean here that it is periodic from some index $n$ on i.e., there exist integers $r > 0$ and $\omega \geq 0$ such that $s_{n+r} = s_n$ for all $n \geq \omega$.

**Corollary 1.3.** The following are equivalent: For $y \in \mathbb{Z}_p$,

1. $(1 + x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n \in F_p[[x]]$ is rational in $x \in M_K$.
2. $\{\binom{y}{n}\}_{n \geq 0}$ is an ultimately periodic sequence in $F_p$.
3. $y$ is in $\mathbb{Z}$. 

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2 Proofs

We begin by recalling that \((\Lambda, \cdot)\) is the group of locally analytic endomorphisms on \(U\). We first see that any locally analytic function \(f \in \Lambda\) has expansion coefficients which is bounded. Indeed, the coefficients belong to the prime subfield \(\mathbb{F}_p\) of \(K\) in the following lemma.

**Lemma 2.1.** If \(f(1 + x) = \sum_{n=0}^{\infty} a_n x^n \in 1 + xK[[x]]\) is a locally analytic endomorphism on \(U\) in \(\Lambda\) then \(a_n \in \mathbb{F}_p\) for all \(n = 0, 1, \cdots\).

**Proof.** Since a \(p\)th power mapping on \(U\) is linear, and a locally analytic function \(f \in \Lambda\) is an endomorphism on \(U\) we obtain \(f(1 + x^p) = f((1 + x)^p) = f(x + 1)^p\). By equating expansion coefficients of two end series we have \(a_n^p = a_n\) for all \(n = 0, 1, \cdots\). Hence the result follows.

**Lemma 2.2.** Let \(f(1 + x) = \sum_{n=0}^{\infty} a_n x^n \in 1 + x\mathbb{F}_p[[x]]\) be a locally analytic endomorphism in \(\Lambda\). Then \(a_n\) is zero if and only if \(D^{(n)} f(1 + x)\) is identically zero, where \(D^{(n)}\) denotes the Hasse derivative of order \(n\) of \(f\). In particular, \(a_1\) is zero if and only if \(f(1 + x) = g(1 + x)^p\) for some \(g(1 + x)\) in \(\Lambda\).

**Proof.** Since a locally analytic function \(f \in \Lambda\) is an endomorphism on \(U\) we can write, for two variables \(x\) and \(y\) in \(M_K\),

\[
 f(1 + x) f(1 + y) = f(1 + xy + x + y).
\]

By Equation (3) this can be rewritten as

\[
 \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} a_n y^n = \sum_{n=0}^{\infty} a_n (xy + x + y)^n. \tag{4}
\]

Some computations with Equation (4) give an crucial identity for \(x\):

For each \(m \geq 0\),

\[
 a_m \sum_{n=0}^{\infty} a_n x^n = \left( \sum_{n=m}^{\infty} a_n \binom{n}{m} x^{n-m} \right) (x + 1)^m. \tag{5}
\]

A moment’s thought gives that the series in the parenthesis of Equation (5) is nothing but the Hasse derivative of order \(m\) of \(f(1 + x)\), for which we denote \(D^{(m)} f(1 + x)\) (see [4, 6]). Moreover, the function \(f(1 + x)\) also appears on the same equation, so that Equation (5) can be rewritten as

\[
 a_m f(1 + x) = D^{(m)} f(1 + x)(1 + x)^m, \quad (m \geq 0). \tag{6}
\]

Since both \(f\) and \((1 + x)^m\) take values in \(U\) the preceding identity implies the first part of the result. The second follows from the first part, as \(a_n \in \mathbb{F}_p\) for all \(n\).

Let \(f(1 + x) = \sum_{n=0}^{N} a_n x^n \in 1 + x\mathbb{F}_p[[x]]\) be a polynomial of degree \(N\) in \(\Lambda\). Since \(D^{(N)} f(1 + x) = a_N\), Equation (6) with \(m = N\) gives the following result.

**Lemma 2.3.** If a locally analytic endomorphism \(f(1 + x) = \sum_{n=0}^{\infty} a_n x^n \in 1 + x\mathbb{F}_p[[x]]\) in \(\Lambda\) has coefficients \(a_n = 0\) for all but finitely many \(n\), then \(f(1 + x) = (1 + x)^N\) for some integer \(N \geq 0\).

We next have the same result as Lemma 2.3 for locally analytic functions \(f \in \Lambda\) whose expansion coefficients are ultimately periodic.

**Lemma 2.4.** If a locally analytic endomorphism \(f(1 + x) = \sum_{n=0}^{\infty} a_n x^n \in 1 + x\mathbb{F}_p[[x]]\) in \(\Lambda\) has coefficients \(\{a_n\}_{n \geq 0}\), which are ultimately periodic, then \(f(1 + x) = (1 + x)^N\) for some integer \(N\).
Let \( f(1+x) = \sum_{n=0}^{\infty} a_n x^n \) has ultimately periodic coefficients \( \{a_n\}_{n\geq 0} \) then \( f(1+x) \) is a rational function in \( x \). Now write \( f(1+x) = \frac{P(x)}{Q(x)} \) for two polynomials \( P(x) \) and \( Q(x) \), which are relatively prime in \( \mathbb{F}_p[x] \). Since \( f(1) = \frac{P(0)}{Q(0)} = 1 \) we may take \( P(0) = Q(0) = 1 \), if necessary, by cancelling out the same constant. Also we may here assume that \( Q(x) \) is not identically 1 otherwise we reduce to Lemma 2.3. Since \( f \) is an endomorphism on \( U \) then the following equation has to be satisfied:

\[
\frac{P(x) P(y)}{Q(x) Q(y)} = \frac{P(xy + x + y)}{Q(xy + x + y)}.
\]

From this equation we have

\[
P(x)P(y)Q(xy + x + y) = Q(x)Q(y)P(xy + x + y)
\]

as polynomials in the polynomial ring \( \mathbb{F}_p[x, y] \) in two variables over \( \mathbb{F}_p \). Since \( \mathbb{F}_p[x, y] \) is a unique factorization domain, we see \( P(x)P(y) \) divides \( P(xy + x + y) \) and \( Q(x)Q(y) \) divides \( Q(xy + x + y) \).

The result now follows from Lemma 2.3 along with both \( P(x)P(y) = P(xy + x + y) \) and \( Q(x)Q(y) = Q(xy + x + y) \), which come by comparing the highest degrees of polynomials on both sides of the two preceding equations.

Before proceeding to prove Theorem 1.1 we mention that it follows from the general theory of formal groups (see [7, Cor. 20.2.14]) as the formal multiplicative group \( G_m \) has height 1, along with Lemma 2.3 below. But our goal in this paper is to give another proof of this result in a neighborhood of 1 in \( U \), which is very well suited for \( G_m \) and which uses calculus in a profound way.

**Proof of Theorem 1.1** Since \( U \) is a module over \( \mathbb{Z}_p \) there is a well defined homomorphism from \( (\mathbb{Z}_p, +) \rightarrow (\Lambda, \cdot) \) given by \( u \mapsto u^p \). It is relatively straightforward to check that the map is injective. Suppose that \( u^p = 1 \) for some nonzero \( y \in \mathbb{Z}_p \). We may here assume that \( p \) does not divide \( y \) or \( y \) is a \( p \)-adic unit otherwise a \( p \)th power mapping reduces to the present case. Since \( u \) is arbitrarily taken in \( U \), consider the expansion of \( (1 + \pi)^p = 1 \), which is impossible to happen since \( y \) is a \( p \)-adic unit. It now remains to show the map is surjective, which is the main point of this note. Take any locally analytic function \( f \in \Lambda \) and we may assume by Lemmas 2.3 and 2.4 that \( f(1+x) \in 1 + x\mathbb{F}_p[x] \) is not rational in \( x \). Let \( \nu_0 \) be the maximal nonnegative integer such that \( f \) is a \( p^{\nu_0} \)th power in \( \mathbb{F}_p[x] \). Write

\[
f(1+x) = f_0(1+x)^{\nu_0} \quad \text{with} \quad f_0 = \sum_{n=0}^{\infty} a_n^{(0)} x^n \in 1 + x\mathbb{F}_p[x] \quad \text{in} \quad \Lambda.
\]

We here note that \( a_1^{(0)} \neq 0 \) in \( \mathbb{F}_p \) otherwise Lemma 2.2 forces us to contradict the maximality of \( \nu_0 \). Let \( g_0(1+x) \) be a locally analytic endomorphism on \( U \) so that

\[
f_0(1+x) = (1+x)^{a_1^{(0)}} g_0(1+x).
\]

Then we easily check that \( g_0(1+x) \) is a locally analytic endomorphism whose linear coefficient is 0. Hence by Lemma 2.2 there is a maximal positive integer \( \nu_1 \) such that \( g_0(1+x) = f_1(1+x)^{p^{\nu_1}} \) with \( f_1 = \sum_{n=0}^{\infty} a_1^{(1)} x^n, a_1^{(1)} \neq 0 \). Take \( g_1 \in \Lambda \) so that \( f_1(1+x) = (1+x)^{a_1^{(1)}} g_1(1+x) \). Then again by Lemma 2.2 \( g_1(1+x) \) is a \( p \)th power in \( \Lambda \). Now by applying the same argument above to \( g_1 \) and by iterating this process repeatedly we have two sequences of locally analytic functions in \( \Lambda \), \( \{f_n = \sum_{i=0}^{\infty} a_i^{(n)} x^i, a_i^{(n)} \neq 0 \}_{n \geq 0}, \{g_n \}_{n \geq 0} \) and a sequence of positive integers \( \{\nu_n\}_{n \geq 1} \) such that

\[
f_n(1+x) = (1+x)^{a_i^{(n)}} g_n(1+x) \quad \text{and} \quad g_{n-1}(1+x) = f_n(1+x)^{p^{\nu_n}}.
\]

If we plug in \( \{f_i\}_{0 \leq i \leq n} \) and \( \{g_i\}_{0 \leq i \leq n} \) into the function \( f \) we have

\[
f(1+x) = (1+x)^{\nu_n} g_n^{\lambda_n}
\]
with \( \lambda_n = \nu_0 + \cdots + \nu_n \) and \( y_n = a^{(0)}_1 p^{\lambda_0} + a^{(1)}_1 p^{\lambda_1} + \cdots + a^{(n)}_1 p^{\lambda_n} \). Since \( \{\nu_n\}_{n \geq 1} \) is a sequence of positive integers, as \( n \to \infty \), \( \lambda_n \to \infty \) so \( g_{n}^{\lambda_n} \) converges to 1. At the same time, \( y_n \) converges to some \( p \)-adic integer \( y \). Therefore \( f(u) = u^y \) for some \( y \in \mathbb{Z}_p \). The discussion above concludes that if \( f \) is a locally analytic endomorphism on \( U \), then in some open neighborhood of the identity \( f \) is of the form \( f(1 + x) = (1 + x)^y \) for some \( p \)-adic integer. By Lemma 2.5 below it follows that the map is surjective. \( \square \)

**Lemma 2.5.** If a locally analytic endomorphism \( f \) on \( U \) is of the form \( f(1 + x) = (1 + x)^y \) for some \( y \in \mathbb{Z}_p \), in some open neighborhood of \( 1 \), then it is identically \( (1 + x)^y \).

**Proof.** Let \( u = 1 + x \) be an arbitrary element of \( U \) and let \( V \) be some open neighborhood of \( 1 \). Then we see for some positive integer \( n \) that \( (1 + x)^p^n = 1 + x^{p^n} \) is in \( V \). Thus we must have

\[
(1 + x)^{p^n} = f((1 + x)^{p^n}) = f(1 + x^{p^n}) = (1 + x^{p^n})^y = ((1 + x)^y)^{p^n},
\]

for some \( p \)-adic integer \( y \). Therefore the injection of \( p \)th power mappings gives \( f(1 + x) = (1 + x)^y \) on \( U \).

\( \square \)

**Proof of Corollary 1.2** We note that \( f(1 + x) = \sum_{n=0}^{\infty} a_n x^n \in 1 + x \mathbb{F}_p[[x]] \) in \( \Lambda \) is a locally analytic automorphism on \( U \) if and only if \( a_1 \neq 0 \). We here leave a justification of this assertion to the reader. From the proof of Theorem 1.1 we have \( f(u) = u^y \) for some \( p \)-adic unit \( y \). So we are done. \( \square \)

Now the question arises of whether or not exponential functions \( u^y \) \( (y \in \mathbb{Z}_p \setminus \mathbb{Q}) \) have ultimately periodic coefficients. To this end, we first state the well known result for the \( p \)-adic numbers.

**Proposition 2.6.** A \( p \)-adic number \( a = \sum_{n=\omega \in \mathbb{Z}} a_n p^n \) \( (0 \leq a_i < p) \) is rational in \( \mathbb{Q} \) if and only if the sequence of digits \( \{a_n\} \) is ultimately periodic.

**Proof.** See [5, p. 147]. \( \square \)

We here have a function field analogue of Proposition 2.6.

**Proposition 2.7.** Let \( \mathbb{F}_q((x)) \) be the field of formal Laurent series in one variable \( x \) over a finite field \( \mathbb{F}_q \). Then \( f(x) = \sum_{n=\omega \in \mathbb{Z}} a_n x^n \) is a rational function in \( x \) if and only if the sequence of elements \( \{a_n\} \subset \mathbb{F}_q \) is ultimately periodic.

**Proof.** The translation to \( \mathbb{F}_q((x)) \) of the arguments [5, p. 147] in the proof of Proposition 2.6 goes in a parallel way as in the classical case. For it must rely on the fact that \( \mathbb{F}_q \) is finite, hence we have the well known analogue of Euler's theorem (see [9, Prop.1.8 p. 5]). \( \square \)

**Proof of Corollary 1.3** The equivalence of (1) and (2) follows from Proposition 2.7. And we see that the equivalence of (1) and (3) follows from the proof of Theorem 1.1. \( \square \)

We close the paper with several remarks.

1. The methodology of Theorem 1.1 reminds us of the well known result in calculus that every differentiable function \( y \) with \( y' = y \) must be a multiple of \( e^x \). We take one such \( h \) and divide by \( e^x \) and then show the derivative of the resulting function is identically 0. We can also use the Hasse derivatives or divided derivatives to calculate something similar in characteristic \( p \).

2. It is straightforward to see that the result for \( \mathbb{Z}_p \) analogous to Theorem 1.1 is trivial. But it might be nontrivial to have some parallel results on the ring of integers of finite extensions of \( \mathbb{Q}_p \).

3. (1) \( \iff \) (3) in Corollary 1.3 also follows from the known result by Mendes France and van der Poorten [8] using the finite automata tool: \((1 + x)^y\) is algebraic over \( \mathbb{F}_q(x) \) if and only if \( y \) is in
\[ Z_p \cap \mathbb{Q} \]. There their result also holds if \( 1 + x \) is replaced with an algebraic formal power series with constant term equal to 1.

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Department of Mathematics, Inha University, Incheon, Korea 402-751

E-mail address: stj@inha.ac.kr