CROSSING MATRICES OF POSITIVE BRAIDS

MAURICIO GUTIERREZ AND ZBIGNIEW NITECKI

Abstract. The crossing matrix of a braid on \( N \) strands is the \( N \times N \) integer matrix with zero diagonal whose \( i, j \) entry is the algebraic number (positive minus negative) of crossings by strand \( i \) over strand \( j \). When restricted to the subgroup of pure braids, this defines a homomorphism onto the additive subgroup of \( N \times N \) symmetric integer matrices with zero diagonal—i.e., it represents the abelianization of this subgroup. As a function on the whole \( N \)-braid group, it is a derivation defined by the action of the symmetric group on square matrices. The set of all crossing matrices can be described using the natural decomposition of any braid as the product of a pure braid with a “permutation braid” in the sense of Thurston, but the subset of crossing matrices for positive braids is harder to describe. We formulate a finite algorithm which exhibits all positive braids with a given crossing matrix, if any exist, or declares that there are none.

1. Introduction

The notion of a crossing matrix was formulated in [3]: for a geometric braid \( b \) with \( N \) strands, the crossing matrix \( C(b) \) is an \( N \times N \) matrix whose \( i, j \) entry is the (algebraic) number of crossings of strand \( i \) over strand \( j \). This matrix is invariant under the braid relations on geometric braids and hence is well-defined as a function on the braid group \( \mathcal{B}_N \). In section 2 we review the basic properties of this function and the relatively easy characterization of its range. The current paper focuses on the deeper problem of characterizing those matrices that arise as crossing matrices of positive braids—i.e., geometric braids whose crossings all go in the same direction (left over right). The reason for asking for such a characterization is that in this case there is no cancellation of a left-over-right crossing with a right-over-left crossing by the same strands—all crossings “show up” in the matrix. Examples show that a crossing matrix with all entries non-negative may nonetheless not represent any positive braid. In this paper, we explain several obstructions to being the crossing matrix of a positive braid, and give an algorithm which displays all the positive braids with a given matrix as their crossing matrix (including deciding when there are none).\(^1\) We have not been able to come up with a conceptual characterization of which

\(^1\)This algorithm has been successfully implemented in a Mathematica program, which can handle examples up to about size \( 7 \times 7 \) on a MacBook Pro with 8GB of memory.
matrices arise as crossing matrices of positive braids in general, although we conjecture a characterization for pure positive braids, resting on a very specific (conjectured) lemma.

2. Crossing Matrices of Braids

2.1. Definition of crossing matrices. We think of a braid with $N$ strands as the homotopy class, modulo endpoints, of a geometric braid: an ensemble of $N$ differentiable paths $p_i(t)$, $i = 1, \ldots, N$ in the plane (the strands of the braid) whose tangent is never horizontal; the set of (distinct) initial positions $\{p_i(0) : i = 1, \ldots, N\}$ and the set of final positions $\{p_i(1) : i = 1, \ldots, N\}$ differ only in their common vertical coordinate, which we take to be 1 for the initial and 0 for the final points; it is not required that the initial and final positions of any particular strand be horizontally aligned. We assume for convenience that these paths are pairwise transverse, so that there are only finitely many crossings between strands, and each such crossing is assigned crossing data: it is regarded as positive (resp. negative) depending on whether it is left-over-right or right-over-left as we move down the path.

Given a geometric braid $b$, we define its crossing matrix as the $N \times N$ matrix $C(b)$ whose $i, j$ entry is the algebraic crossing number (number of positive crossings minus number of negative crossings) for strand $i$ crossing over strand $j$. By definition, the diagonal entries of $C(b)$ are zero.

It is easy to check that the braid relations (which do modify some crossings) don’t affect the crossing matrix. So the crossing matrix $C(b)$ can be thought of as defined on the braid represented by $b$. The description of a geometric braid in terms of crossings of strands is a point of view used by Thurston [4], and contrasts with the point of view of Artin (in terms of generators and relations) in his original expositions of the braid group $\mathfrak{B}_N$ [1, 2].

2.2. The permutation associated to a braid. Attached to each geometric braid is the permutation on the set of horizontal coordinates of the starting points of strands which takes (the horizontal coordinate of) the starting point of each strand to its ending point.

We use Greek letters to denote permutations, regarded as rearrangements—that is, permutations act on positions rather than elements: a permutation $\pi \in \Sigma_N$ will be specified by the word\(^{2}\) $(12\cdots N)^{\pi} = \pi_1\pi_2\cdots\pi_N$ in the numbers $1, 2, \ldots, N$ resulting from the action of $\pi$ on the word $12\cdots N$. This contrasts with regarding a permutation as a bijective mapping $\pi: \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ on a set of $N$ elements and denoting it by the $N$-tuple $(\pi(1), \pi(2), \ldots, \pi(N))$ of images of the individual elements $1, 2, \ldots, N$ under this mapping. In fact the word $\pi(1)\pi(2)\cdots\pi(N)$ in our “rearrangement” notation denotes the inverse permutation\(^{3}\). For example, the rearrangement $\pi \in \Sigma_4$ which acts on the positions $1, 2, 3, 4$ via

$$\pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 4, \quad \pi(4) = 2$$

takes the word $abcd$ to

$$(abcd)^{\pi} = bdac \quad (e.g., (1234)^{\pi} = \pi_1\pi_2\pi_3\pi_4 = 2413)$$

\(^2\)Permutations will act on words on the right, denoted by superscript.

\(^3\)The inverse of a permutation or braid will be denoted by an overbar: $\bar{\pi}$. 

(π(1), π(2), π(3), π(4)) = (3, 1, 4, 2).

We extend the “rearrangement” action of permutations to matrices: If $A = ((a_{ij}))$ is an $N \times N$ matrix and $\pi \in \Sigma_N$, then $A^\pi$ is the matrix obtained by rearranging the rows as well as the columns of $A$ according to $\pi$.

When a matrix is the crossing matrix of a braid, say $A = C(a)$, then the permutation $\pi_a$ associated to $a$ can be read off of $A$: every (positive) crossing of strand $i$ over another strand moves its position one place to the right, while every (positive) crossing of another strand over the $i^{th}$ moves it to the left one space. From this it follows that $\pi_a$ is defined (as a mapping $\pi_a: \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$) by

$$
\pi_a(i) = i + \sum_{j=1}^N A_{ij} - \sum_{k=1}^N A_{ki}.
$$

2.3. **Crossing matrix of a product of braids.** With this notation we can explain the relation between the crossing matrices $C(a), C(b)$ of two braids and the crossing matrix $C(ab)$ of their product in the braid group, which we think of as represented by the geometric braid $a$ followed by the geometric braid $b$. The main observation is that the numbering of the strands of $b$ is changed when we premultiply by $a$: the $i^{th}$ strand of $b$ becomes an extension of the strand of $a$ which landed at the $i^{th}$ position—that is, in $ab$ it continues the $\pi(i)^{th}$ strand. From this it follows that a crossing of the $i^{th}$ strand of $b$ over its $j^{th}$ strand appears in $ab$ as a crossing of the $\pi(i)^{th}$ strand over the $\pi(j)^{th}$ strand. With this renumbering of strands in $b$, the crossings add, so we have

**Proposition 1.** For any two braids $a, b \in \mathcal{B}_N$,

$$
C(ab) = C(a) + C(b)^{\pi_a}.
$$

We will denote this *crossing product* operation on (crossing) matrices by a circled asterisk:

$$
A \odot B := A + B^{\pi_A}.
$$

2.4. **Order reversal sets.** In [4, §9.1], Thurston defined the *order reversal set* of a permutation $\pi \in \Sigma_N$:

$$
OR(\pi) := \{(i, j) \in \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\} \mid i < j \text{ but } \pi(i) > \pi(j)\}.
$$

He characterized the sets $S \subset \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\}$ which are order reversal sets for some permutation $\pi \in \Sigma_N$ via two properties which are easily seen to be necessary, and with a little more work are sufficient:

**Proposition 2** (Thurston). A subset $S \subset \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\}$ equals $OR(\pi)$ for some $\pi \in \Sigma_N$ if and only if the following properties both hold:

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4For an arbitrary matrix, this formula does not necessarily define a permutation, only a mapping—which might even map $\{1, 2, \ldots, N\}$ to a different set of integers. However, we shall show that in the context we consider it is always a permutation (Remark 7).
When these properties hold, the permutation $\pi$ is uniquely determined by $S$.

Proof. The necessity of these two conditions for an order reversal set is more easily seen when they are replaced with their contrapositives:

contra(1): If, for some $k$ between $i$ and $j$, neither $(i,k)$ nor $(j,k)$ belongs to $S$, then neither does $(i,j)$,

contra(2): If $(i,j) \notin S$ given $i < k < j$, then at most one of $(i,k)$ and $(k,j)$ belongs to $S$.

For the first, given $i < k < j$, if $\pi(i) < \pi(k)$ and $\pi(k) < \pi(j)$, then of course $\pi(i) < \pi(j)$; for the second, $\pi(i) < \pi(j)$ means we can’t have both $\pi(k) < \pi(i)$ and $\pi(k) > \pi(j)$.

To establish sufficiency, we consider the mapping $\sigma$: $\{1,2,\ldots,N\} \rightarrow \{1,2,\ldots,N\}$ defined by the analogue of Equation 1,

\begin{equation}
\sigma(i) = i + \#\{k \mid (i,k) \in S\} - \#\{k \mid (k,i) \in S\}.
\end{equation}

1. $(i,j) \in S \Rightarrow \sigma(i) > \sigma(j)$: Given $(i,j) \in S$, the first hypothesis insures that

$$\#\{k < j \mid (i,k) \in S\} + \#\{k > i \mid (k,j) \in S\} > j - i$$

since every $k$ between $i$ and $j$ appears in at least one of these two sets, and $j$ (resp. $i$) appears in the first (resp. second) set. This inequality can be rewritten as

$$i + \#\{k < j \mid (i,k) \in S\} > j - \#\{k > i \mid (k,j) \in S\}. $$

Now, the second hypothesis tells us that for $k > j$, $(j,k) \in S$ implies (since $(i,j) \in S$) that also $(i,k) \in S$, hence

$$\#\{k > j \mid (i,k) \in S\} \geq \#\{k > j \mid (j,k) \in S\}$$

and similarly, for $k < i$ if $(k,i) \in S$ then also $(k,j) \in S$, hence

$$-\#\{k < i \mid (k,i) \in S\} \geq -\#\{k < i \mid (k,j) \in S\}. $$

Adding these inequalities, we obtain

$$i + \#\{k < j \mid (i,k) \in S\} + \#\{k > j \mid (i,k) \in S\} - \#\{k < i \mid (k,i) \in S\} > j - \#\{k > i \mid (k,j) \in S\} + \#\{k > j \mid (j,k) \in S\} - \#\{k < i \mid (k,j) \in S\}$$

or

$$\sigma(i) := i + \#\{k \mid (i,k) \in S\} - \#\{k \mid (k,i) \in S\} > j - \#\{k \mid (k,j) \in S\} - \#\{k \mid (k,j) \in S\} := \sigma(j).$$

2. $i < j \& (i,j) \notin S \Rightarrow \sigma(i) < \sigma(j)$: Replacing the first (resp. second) hypothesis with its contrapositive, which is to say arguments parallel to the above yield the opposite inequalities and hence the desired conclusion, that for $i < j$, if $(i,j) \notin S$ then $\sigma(i) < \sigma(j)$.
The definition of \(\sigma(i)\) immediately guarantees that \(1 \leq \sigma(i) \leq N\), and the two inequalities above show that \(\sigma\) is injective, hence a permutation. Finally, it is easy to see that different permutations have different order reversal sets, giving the uniqueness statement in Proposition 2.

Thurston then formulated the notion of a permutation braid:

**Definition 3.** A permutation braid is a positive braid in which no pair of strands crosses more than once.

Given a permutation \(\pi \in \Sigma_N\)—which is to say, given the ending position of each strand, we can construct a geometric braid \(\pi^+\) by joining \((i, 1) \in \mathbb{R}^2\) to \((\pi(i), 0)\) by a straight line segment and making all crossings positive (in case this yields more than two such line segments crossing at the same point, we can perturb a little to get only pairwise crossings). It is clear that no pair of strands crosses more than once. This shows that every \(\pi \in \Sigma_N\) is the permutation associated to some permutation braid; moreover any permutation braid can be “straightened out” so as to be a \(\pi^+\). Thus

**Remark 4.** The mapping \(p : \Sigma_N \to \mathcal{B}_N\) taking a permutation \(\pi \in \Sigma_N\) to the braid \(\pi^+\) defined above is a bijection onto the set of permutation braids.

We caution the reader that this map is not a homomorphism: for example a braid with a single crossing is a permutation braid, but its square is not.

### 2.5. Characterization of crossing matrices.

The crossing matrix \(R_\pi\) of the permutation braid \(\pi^+\) is clearly the same as the matrix \(R\) with \(i, j\) entry 1 if \((i, j) \in OR(\pi)\) and 0 otherwise. Since all pairs \((i, j) \in OR(\pi)\) have \(i < j\), \(R\) is strictly upper triangular (\(R_{ij} = 0\) for \(i \geq j\)). The conditions in Proposition 2 characterizing the orientation-reversing set of a permutation can be reinterpreted as conditions on the matrix \(R\), which we formulate in

**Definition 5.** We say a square matrix \(A\) is \(T_0\) (resp. \(T_1\)) if

- **T0:** For any triple of indices \(i < k < j\), if \(A_{ik} = 0\) and \(A_{kj} = 0\), then \(A_{ij} = 0\).
- **T1:** For any triple of indices \(i < k < j\), if \(A_{ik} \neq 0\) and \(A_{kj} \neq 0\), then \(A_{ij} \neq 0\).

Note that both conditions refer only to entries above the diagonal of \(A\), and can be viewed as limitations on the distribution of zero entries (above the diagonal) in \(A\).

**Definition 6.** An \(R\)-matrix is a strictly upper triangular matrix, all of whose nonzero entries are 1, and which is both \(T_0\) and \(T_1\).

For any strictly upper triangular \(N \times N\) matrix \(R\), the positions of its nonzero entries form a set \(S\) of pairs \((i, j)\) with \(1 \leq i < j \leq N\), and Proposition 2 tells us when \(S\) is an OR set. The sufficiency argument in the proof of Proposition 2 amounts to saying that, for an \(R\)-matrix, Equation 2 defines a permutation \(\sigma \in \Sigma_N\). It is easy to see that, for an \(R\)-matrix, the term in Equation 2 which is added to \(i\) equals the \(i^{th}\) row sum

\[
\# \{k \mid (i, k) \in S\} = \# \{k > i \mid (i, k) \in S\} = \sum_{k=1}^{N} R_{ik}
\]
and the subtracted term equals the $i^{th}$ column sum

$$\# \{ k \mid (k.i) \in S \} = \# \{ k < i \mid (k.i) \in S \} = \sum_{k=1}^{N} R_{ki}$$

so Equation 2 really is the same as Equation 1 applied to the $R$-matrix $R$, therefore does not change the permutation defined by Equation 1. Thus, a corollary of Proposition 2 is

**Remark 7.** For any $N \times N$ matrix $A = S + R$, where $S$ is symmetric and $R$ is an $R$-matrix, the formula

$$\pi_A(i) = i + \sum_{j=1}^{N} A_{ij} - \sum_{k=1}^{N} A_{ki}$$

defines a permutation $\pi_A \in \Sigma_N$.

To characterize the matrices which arise as crossing matrices (of some, not necessarily positive, braid) we note two necessary conditions:

**Proposition 8.** For any crossing matrix $A = ((a_{ij})) = \mathcal{C}(b)$, $b \in \mathcal{B}_N$,

1. the diagonal entries are all zero:

$$a_{ii} = 0 \text{ for } i = 1, \ldots, N;$$

2. each entry above the diagonal is either equal to its symmetric twin, or exceeds it by one:

$$a_{ij} - a_{ji} \in \{0, 1\} \text{ for } 1 \leq i < j \leq N.$$

**Proof.**

(1) The first equation is the observation that no strand crosses itself.

(2) Suppose $1 \leq i < j \leq N$. Since strand $i$ starts to the left of strand $j$, if any crossings of these two strands occur, the first one must move $i$ to the right of $j$ either via a positive crossing of $i$ over $j$, or via a negative crossing of $j$ over $i$; the effect of this is to contribute an increase by one to the difference $a_{ij} - a_{ji}$. A second crossing must move $i$ back to the left of $j$, either via a negative crossing of $i$ over $j$ or via a positive crossing of $j$ over $i$; this decreases the difference $a_{ij} - a_{ji}$ by one.

As long as crossings of $i$ with $j$ continue, these two situations will alternate strictly, so the difference $a_{ij} - a_{ji}$ will oscillate between 0 and 1.

A braid is $b$ called **pure** if its permutation $\pi_b$ (as defined in §2.3) is the identity permutation. It is easy to see that this condition can be expressed by saying that the $i^{th}$ row sum equals the $i^{th}$ column sum for $i = 1, \ldots, N$. If for each pair of indices $i, j$ with $1 \leq i < j \leq N$ we set $x_{ij} = a_{ij} - a_{ji}$, this condition becomes the system of $N - 1$ equations in $\frac{(N-1)(N-2)}{2}$ unknowns of the form $\sum_{j=i+1}^{N} x_{ij} = 0$, $i = 1, \ldots, N - 1$. This system has many integer solutions, for example the matrix

$$\begin{bmatrix}
0 & 2 & 0 \\
1 & 0 & 2 \\
1 & 1 & 0
\end{bmatrix}$$
has each row sum equal to the corresponding column sum, but if we also throw in the earlier requirement that \( a_{ij} - a_{ji} := x_{ij} \in \{0, 1\} \) we see that the only solution is \( x_{ij} = 0 \) for all \( i < j \). Thus

**Lemma 9.** The crossing matrix of every pure braid is symmetric.

We note that the pure braids form a subgroup \( \mathfrak{P}_N \) of the braid group \( \mathfrak{B}_N \), and that the restriction of the crossing matrix mapping to pure braids is a homomorphism to the additive group \( \mathfrak{S}_N^0[\mathbb{Z}] \) of symmetric \( N \times N \) integer matrices with zero diagonal. This map is also surjective; to see this, note that each of the symmetric matrices \( S_{ij} \) whose only nonzero entries are a “1” in the \( i, j \) and \( j, i \) positions is the crossing matrix of the braid \( s_{ij} \) in which strand \( i \) crosses over all intermediate strands, “hooks” strand \( j \), and then crosses back over all the intermediate strands\(^5\) to return to its initial position.

Since the \( N \times N \) matrices \( C(s_{ij}) = S_{ij} \) for \( 1 \leq i < j \leq N \) generate the additive group \( \mathfrak{S}_N^0[\mathbb{Z}] \), this shows

**Proposition 10.** The crossing matrix map takes the subgroup \( \mathfrak{P}_N \) of pure \( N \)-strand braids onto the additive group \( \mathfrak{S}_N^0[\mathbb{Z}] \) of \( N \times N \) symmetric integer matrices with zero diagonal:

\[
C(\mathfrak{P}_N) = \mathfrak{S}_N^0[\mathbb{Z}].
\]

To extend our characterization of crossing matrices to all braids, we note that if \( b \in \mathfrak{B}_N \) is a braid with permutation \( \pi_b \), then the braid \( s(b) := b(\pi_b^+)^{-1} \) is a pure braid, and so we have a unique factoring \( b = s(b)\pi_b^+ \) as a pure braid followed by a permutation braid. It follows that we can write the crossing matrix of \( b \) as

\[
C(b) = C(s(b)\pi_b^+) = S \oplus R = S + R,
\]

where \( S = C(s(b)) \in \mathfrak{S}_N^0[\mathbb{Z}] \) (so \( \pi_S = id \)) and \( R = R_{\pi_b} \) is an \( R \)-matrix. Since \( R \)-matrices are upper triangular, the expression \( A = S + R \) is uniquely determined (if it exists) for any matrix; we will refer to it as the **SR decomposition** of \( A \). We can therefore complete our characterization of crossing matrices for (general) braids:

**Theorem 11.** An \( N \times N \) matrix \( A \) is the crossing matrix of some braid if and only if it has an **SR decomposition**.

We denote the set of all \( N \times N \) integer matrices with zero diagonal which have an **SR decomposition** by \( \mathcal{SR}_N \).

It can sometimes be difficult to immediately visualize the **SR decomposition** of an integer matrix, so we have adopted a tableau notation to make this clearer: given an \( N \times N \) matrix with an **SR decomposition** \( A = S + R \), we exhibit each entry of \( A \) strictly above the diagonal (\( 1 \leq i < j \leq N \)) as the sum of the corresponding entries \( s_{ij} \) of \( S \) and \( r_{ij} \) of \( R \), in the form “\( s_{ij}S + r_{ij}R \)”; note that

\[
s_{ij} = a_{ji}, \quad r_{ij} = a_{ij} - a_{ji}.
\]

\(^5\)(Note that the latter set of crossings is negative.)
For example, if

$$A = \begin{bmatrix}
0 & 1 & 3 & 1 \\
0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
2 & 0 & 0 & -1 \\
0 & 1 & -1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

then the *SR tableau* encoding this data is

$$SRT(A) = \begin{array}{cccc}
1 & R & 2S + R & R \\
2 & 0 & S & \ \\
3 & -S + R & \ \\
4 & & & \\
\end{array}$$

To complete this picture, we note that the crossing matrix map is not injective. First, we observe that the restriction of the crossing matrix map to the subgroup $\mathfrak{P}_N$ is the abelianization of that group; it follows that its kernel is the commutator subgroup of $\mathfrak{P}_N$. In general, two braids $a$ and $b$ with the same crossing matrix will have the same permutation (according to the formula in Equation 1) so $c = ab$ is a pure braid with zero crossing matrix; it follows that $b = ac$, where $c$ belongs to the commutator subgroup of the pure braid group—that is, $c$ can be written as a product of finitely many braids of the form $[p, q] := p_iq_i\bar{p}_i\bar{q}_i$. We formalize this observation:

**Proposition 12.** Two braids $a, b \in \mathfrak{B}_N$ satisfy $C(a) = C(b)$ if and only if $a = bc$, where $c$ belongs to the commutator subgroup of $\mathfrak{P}_N$; that is, $c = [p_1, q_1] \cdots [p_k, q_k]$ where $p_i, q_i \in \mathfrak{P}_N$ for $i = 1, \ldots, k$.

### 3. Crossing Matrices of Positive Braids

We turn now to the focus of this paper, the crossing matrices of positive braids; we will adapt all of our notation to this case by using a superscript “plus” to denote positivity: $\mathfrak{B}_N^+$ (resp. $\mathfrak{P}_N^+$) will denote the positive (resp. pure positive) braids. Again, our interest in this special case is prompted by the fact that there is no cancellation of crossings in the crossing matrix: for $b \in \mathfrak{B}_N^+$, every crossing is accounted for in $C(b)$.

#### 3.1. Two examples.

It would be natural to expect, in view of Theorem 11, that $C(\mathfrak{B}_N^+)$ consists of all matrices in $S\mathcal{R}_N$ with all entries non-negative. However, this is false; the following are examples of elements of $S\mathcal{R}_N$ which, while they have non-negative entries and are crossing matrices of some braids, are not crossing matrices of any positive braids.

It will prove easier to understand these and other examples using their $SR$ tableaux:

$$SRT(G) = \begin{array}{cccc}
1 & R & S & 0 \\
2 & 0 & S & \ \\
3 & R & \ \\
4 & & & \\
\end{array}$$
and

\[
SRT(K) = \begin{pmatrix}
1 & 0 & S & 0 & 0 \\
2 & R & R & 0 \\
3 & R & S \\
4 & 0 \\
5
\end{pmatrix}
\]

We will explain why these two matrices are not crossing matrices of any positive braids after developing some properties of such crossing matrices.

3.2. The positive realization problem. The positive braids form a semigroup which includes all permutation braids, so any product of permutation braids, while it may no longer be a permutation braid, will be a positive braid. Conversely, since the standard generators of the braid group are the permutation braids for transpositions of adjacent strands and a positive braid is given by a positive word in these generators, any realization of a matrix \( A \) as the crossing matrix of some positive braid can always be expressed as a product of permutation braids, and this corresponds to a factoring of \( A \) into a product (with respect to the crossing product operation \( \otimes \)) of \( R \)-matrices.

Remark 13. An \( N \times N \) matrix \( A \) is the crossing matrix of some positive braid if and only if it can be factored as a product

\[
A = R_{\pi_1} \otimes R_{\pi_2} \otimes \cdots \otimes R_{\pi_k}
\]

of \( R \)-matrices.

Our problem, then, is how to tell, given a matrix \( A \), whether such a factoring is possible. Since any permutation braid is a product of single-crossing braids (i.e., any permutation is a product of transpositions) we can focus on factorization into crossing matrices of transpositions.\(^6\)

We note that in addition to the fact that all entries in the crossing matrix of a positive braid are non-negative integers, there is an immediate further restriction on such crossing matrices:

Remark 14. If a braid is positive, \( b \in \mathcal{B}_N^+ \), then its crossing matrix \( C(b) \) must

1. have non-negative integer entries and
2. satisfy the \( T_0 \) property (Definition 5)

To see the second requirement, note that the argument for \( T_0 \) in the context of \( R \)-matrices (the necessity of the first condition in Proposition 2) is based only on the assumption that permutation braids are positive. By contrast, the argument for \( T_1 \) in Proposition 2 is based on the assumption that in a permutation braid no pair of strands crosses more than once; this is not necessarily the case for general positive braids.

\(^6\)By abuse of notation, we will refer to “factoring into transpositions”.
In view of Remark 14, we define $\mathcal{SR}_N^+$ to be the set of (non-negative integer, zero diagonal) T0 matrices which can be written as the sum of a (non-negative integer) symmetric matrix and an $R$-matrix:

$$\mathcal{SR}_N^+ := \{ A \in \mathbb{Z}^+ \mid A \text{ is T0 and } A = S + R, \text{ where } S \in \mathcal{S}_N^0[\mathbb{Z}], \text{ and } R \text{ is an } R - \text{ matrix} \}$$

We caution that the T0 property for $A$ does not necessarily require $S$, the symmetric part of the decomposition, to be T0 on its own (unless, of course, $R = 0$, so the braid is pure).

Note that the two examples in Section 3.1 fit this description: $G \in \mathcal{SR}_4^+$ and $K \in \mathcal{SR}_5^+$.

3.3. **Left division by permutations.** To study the factorization problem posed by Remark 14, we formulate a partial inverse to the crossing product operation $\otimes$. If we solve the equation

$$A = R_\pi \otimes B := R_\pi + B^\pi$$

for $B$ in terms of $A$, we obtain a “division on the left”, defined by

$$B = R_\pi \setminus A := (A - R_\pi)^\pi$$

which we will usually refer to as **left division by $\pi$**:

$$B = \pi \setminus A.$$  

If $A$ and $B$ are to be crossing matrices of positive braids, we must check conditions which insure, for $A \in \mathcal{SR}_N^+$, that $B \in \mathcal{SR}_N^+$.

The first requirement is the obvious one, that all entries of the matrix $A - R_\pi$ are non-negative; this is the same as requiring that $R_\pi \leq A$ entrywise. When this is the case, we will say that the permutation $\pi$ is **subordinate** to $A$.

The second requirement is that $B$ be T0. To understand how this can fail, we take a detour and develop a way to “localize” some of the effects of multiplication ($R_\pi \otimes B$) and division ($\pi \setminus A$) on a matrix $B \in \mathcal{SR}_N^+$ (resp. $A \in \mathcal{SR}_N^+$).

3.3.1. **Configurations.** Given a braid $b$ on $N$ strands, we can pick out a proper subset $\mathcal{I}$ of $m < n$ strands and study the relations just between these selected strands by considering the “subbraid” $b_\mathcal{I} \in \mathcal{B}_m$ which results from erasing all the other strands of $b$. The (crossing) matrix analogue of this is, given $A \in \mathcal{SR}_N^+$, to look at the $m \times m$ submatrix $A_\mathcal{I}$ consisting of elements whose indices are both in $\mathcal{I}$. Note that if $A \in \mathcal{SR}_N^+$ then $A_\mathcal{I} \in \mathcal{SR}_m^+$.

We can abstract this:

**Definition 15** (Configuration). *Given an $N \times N$ matrix $A$ and an index set $\mathcal{I} = \{k_1, k_2, \ldots, k_m\} \subset \{1, 2, \ldots, N\} \ (m \leq N)$, the configuration of $A$ on $\mathcal{I}$ is the $m \times m$ matrix $A_\mathcal{I}$ consisting of the rows and columns of $A$ whose indices belong to $\mathcal{I}$.*

When $A$ is acted on by a permutation $\pi$ (for example as part of a product or left division), the entries of $A_\mathcal{I}$ change position, in two ways: the set of new positions is the image $\pi(\mathcal{I})$ of $\mathcal{I}$ (as a mapping of $\{1, 2, \ldots, N\}$ to itself), and their relative order may be
scrambled; the scrambling action is given by the permutation \( \pi_I \) whose order-reversal set is the intersection of \( OR(\pi) \) with \( I \times I \):

\[
OR(\pi_I) = OR(\pi) \cap (I \times I).
\]

We refer to the permutation \( \pi_I \) (by abuse of terminology) as the \textbf{restriction of} \( \pi \) to \( I \).

It is easy to confirm that the total effect of \( \pi \) on configurations is given by

\[
A \rightarrow A^\pi = (A^\pi)_I \pi_I.
\]

Remark 16. \textit{For any} \( N \times N \) \textit{matrix} \( A \), \textit{any subset} \( I \subset \{1, \ldots, N\} \) \textit{and any permutation} \( \pi \) \textit{of} \( \{1, \ldots, N\} \),

\[
(A^\pi)_I = (A^\pi)_{I \pi_I}.
\]

The point here is that, although the set of indices \( I \) \textit{is} not in general invariant under \( \pi \), it is still true that if \( k_i < k_j \) are two elements of \( I \), then \( \pi(k_i) > \pi(k_j) \) only if \( \pi_I(i) > \pi_I(j) \).

Remark 16 lets us trace a configuration of crossings corresponding to a sub-braid through operations like “division by a permutation” without regard to how crossings outside that configuration are affected.

Remark 17. \textit{Given a permutation} \( \pi \) \textit{and an index set} \( I = \{k_1, \ldots, k_m\} \),

\[
R_{\pi_I} = (R_\pi)_I.
\]

In particular, \textit{given a matrix} \( A \), \textit{a permutation} \( \pi \) \textit{and the index set} \( I \) \textit{of} \( \pi \), \textit{the effect on} \( \text{the configuration} \ A_I \) \textit{corresponding to} \( I \) \textit{of multiplying} \( A \) \textit{by} \( R = R_\pi \) \textit{is given by}

\[
[(R \odot A)_I]_{\pi_i, \pi_j} = A_{i,j} + R_{i,j} \quad \text{for} \quad i,j \in I
\]

while the effect of left-dividing \( A \) \textit{by} \( \pi \) (\textit{provided} \( \pi \leq A \)) \textit{is expressed by}

\[
[(\pi \odot A)_I]_{\pi_i, \pi_j} = A_{i,j} - R_{i,j} \quad \text{for} \quad i,j \in I.
\]

A consequence of Remark 17 is that for any configuration \( A_I \) of \( A \), if the restriction \( \pi_I \) to \( I \) of a permutation \( \pi \) is the identity (that is, its order-reversal set is disjoint from \( I \)), then the configurations \( (R_\pi \odot A)_I \) and \( (\pi \odot A)_I \) both equal the configuration \( A_I \); in other words, the relative positions of these entries do not change, even though this matrix may be embedded in the corresponding larger matrix \( R_\pi \odot A \) (resp. \( \pi \odot A \)) in different ways.

3.3.2. \textit{Multiplication and division by transpositions.} Any permutation can be expressed as a product of transpositions of adjacent positions. We denote by \( \tau_i \in \Sigma_N \) (\( 1 < i < n \)) the permutation which interchanges positions \( i \) and \( i + 1 \) and leaves every other position alone. Its crossing matrix \( t_i = R_{\tau_i} \) is the matrix with 1 in position \( i \), \( i + 1 \) and zero everywhere else. Note that \( \tau_i \) is its own inverse.\(^7\)

\textit{Multiplication by} \( \tau_i \): \textit{Given} \( A \in \mathcal{SR}_N^+ \), \( t_i \odot A \) has rows (resp. columns) \( i \) and \( i + 1 \) interchanged, in particular the \textbf{subdiagonal} entry of \( t_i \odot A \) in position \( i + 1, i \) is the same as the \textbf{superdiagonal} entry at \( i, i + 1 \) in \( A \), while the \( i, i + 1 \) entry of \( t_i \odot A \) is one more than the \( i + 1, i \) entry of \( A \). In terms of the \( SR \) tableau, again aside from the \( i, i + 1 \) entry, multiplication by \( t_i \) interchanges rows \( i \) and \( i + 1 \) with each other, and columns \( i \) and \( i + 1 \) with each other. In position \( i, i + 1 \), a zero (resp. \( R \)) in \( A \) becomes an \( R \) (resp. \( S \)) in \( \tau_i \odot A \).

\(^7\)\text{Caution: this is not true of the corresponding permutation braid.}
Division by $\tau_i$: The transposition $\tau_i$ is subordinate to the matrix $A \in S\mathcal{R}_N^+$ if and only if the entry $a_{i,i+1}$ on the first superdiagonal of $A$ is nonzero, and in this case left division by $\tau_i$ also interchanges rows (resp. columns) $i$ and $i + 1$ except that the (subdiagonal) $i + 1, i$ entry of $\tau_i \setminus A$ is one less than the (superdiagonal) $i, i + 1$ entry of $A$. In terms of the $SR$ tableau, division by $\tau_i$ also interchanges the $i^\text{th}$ and $(i + 1)^\text{st}$ rows (resp. columns) of $A$ and, in the $i, i + 1$ position of the tableau, changes an $S$ to an $R$ or an $R$ to a zero.

3.3.3. Mirror symmetry. It will simplify some arguments to note that the symmetry on $m \times m$ matrices defined by
\[(A^*)_{i,j} := A_{(m+1-i),(m+1-j)}\]
preserves realizability. This consists of reversing the order of the rows (and columns): it is the analogue of transpose, but “flips” the matrix about its antidiagonal instead of its diagonal. We will refer to $A^*$ as the “mirror” of $A$. It is useful to have this operation not just for the full $N \times N$ matrices, but for their sub matrices as well (which is why it is defined above for $m \times m$ instead of just $N \times N$ matrices).

If $M = C(b)$, then $M^* = C(b^*)$, where $b^*$ is the geometric braid obtained by looking at $b$ from “behind”; equivalently, $b^*$ is obtained from $b$ by numbering the strands right-to-left instead of left-to-right. (Note that a positive crossing remains positive if viewed from “behind”.) Thus, a matrix $A \in S\mathcal{R}_N^+$ is realizable as the crossing matrix of a (positive) braid if and only if $A^*$ is.

3.3.4. Creating $T0$ violations. We want to understand how left division of $A \in S\mathcal{R}_N^+$ by a transposition $\pi = \tau_i$ subordinate to $A$ results in $\pi \setminus A \not\in S\mathcal{R}_N^+$, a situation which we have seen can only happen if $B = \pi \setminus A$ fails to be $T0$. A violation of $T0$ means a $3 \times 3$ configuration in $B$ of the form
\[
B_{(I)^e;\tau} = \begin{bmatrix}
0 & 0 & b \\
ap & 0 & 0 \\
b' & a' & 0
\end{bmatrix}
\]
with $b \neq 0$. If $\pi(I) = \{p_1, p_2, p_3\}$ with $p_1 < p_2 < p_3$ and $A \in S\mathcal{R}_N^+$ (so that the corresponding configuration $A_I$ is different from $B_{\pi(I)}$), then $\pi$, if it is a transposition, must interchange either $p_1$ and $p_2$, or $p_2$ and $p_3$. This requires that the interchanged pair of indices be adjacent. In short, there are only two possible ways that left dividing $A$ by a subordinate transposition can change $A_I$: either $\pi_I = \tau_1$ (and $p_2 = p_1 + 1$) or $\pi_I = \tau_2$ (and $p_3 = p_2 + 1$). Notice that these two situations are mirrors of each other, so we can concentrate on the case that $\pi_I = \tau_1$ (and $p_2 = p_1 + 1$). Then it follows from Remark 17 that
\[
A_I = \tau_1 \otimes B_{\pi(I)} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \otimes \begin{bmatrix}
0 & 0 & b \\
ap & 0 & 0 \\
b' & a' & 0
\end{bmatrix} = \begin{bmatrix}
0 & a + 1 & 0 \\
0 & 0 & b \\
a' & b' & 0
\end{bmatrix}.
\]
Since \( A \) (and hence \( A_T \)) has an \( SR \) decomposition, we must have \( a = a' = 0 \) and \( b = b' \) or \( b' + 1 \); thus
\[
A_T = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & b \\
0 & b' & 0
\end{bmatrix}
\]
with \( b = b' \) or \( b = b' + 1 \). This means the \( SR \) tableau of \( A_T \) is one of two possibilities:
\[
SRT(A_T) = \begin{bmatrix}
p_1 & R & 0 \\
p_2 & bS & 0 \\
p_3 & 0 & 0
\end{bmatrix}
\text{ or }
\begin{bmatrix}
p_1 & R & 0 \\
p_2 & 0 & bS + R \\
p_3 & 0 & 0
\end{bmatrix}
\]
But the second tableau violates the requirement that the \( R \)-matrix in the \( SR \) decomposition must be \( T_1 \), so only the first can belong to \( \mathcal{SR}_N^+ \).

We therefore adopt the following

**Definition 18.** A **blockage** in the matrix \( A \in \mathcal{SR}_N^+ \) is a \( 3 \times 3 \) configuration of the form
\[
SRT(A_T) = \begin{bmatrix}
\tau & R & 0 \\
\tau & bS & 0 \\
\tau & 0 & 0
\end{bmatrix}
\text{ or its mirror image }
\]
\[
SRT(A_T) = \begin{bmatrix}
\tau & bS & 0 \\
\tau & R & 0 \\
\tau & 0 & 0
\end{bmatrix}
\]
with \( b \neq 0 \). In either of these situations, we say that the “\( R \)” entry is **blocked** (by the “\( S \)” entry).

The preceding discussion shows that division of a matrix in \( \mathcal{SR}_N^+ \) by a subordinate transposition results in a \( T0 \) violation precisely if that transposition corresponds to a “blocked \( R \)” in the matrix:

**Proposition 19** (Division by a Blocked “\( R \)”). If \( B = \tau \setminus A \notin \mathcal{SR}_N^+ \) where \( A \in \mathcal{SR}_N^+ \) and \( \tau \in \Sigma_N \) is a transposition subordinate to \( A \), then \( A \) has a blockage \( A_T \) whose “\( R \)” entry lies on the first superdiagonal of \( A \), at the position corresponding to \( \tau \): either

1. \( \tau = \tau_i, \; j = i + 1 \) and
\[
SRT(A_T) = \begin{bmatrix}
i & R & 0 \\
i + 1 & bS & 0 \\
j & 0 & 0
\end{bmatrix}
\]
or
\[(2) \ \tau = \tau_j, \ k = j + 1 \text{ and} \]
\[
\begin{array}{ccc}
\tau & = & \tau_j, \ k = j + 1 \\
SRT(A) & = & \begin{bmatrix}
i & bS & 0 \\
j & R & j+1 \\
k & & \\
\end{bmatrix}
\end{array}
\]

In either case, the resulting $T_0$ violation is
\[
SRT(B) = \begin{bmatrix}
i & 0 & bS \\
j & 0 & \\
k & & \\
\end{bmatrix}
\]

Proposition 19 can also be understood geometrically, in terms of possible realizations of the matrix. Suppose $A \in \mathcal{S} \mathcal{R}_N^+$ has a blockage, not necessarily embedded with the “$R$” on the first superdiagonal of $A$. Then, in any possible realization of $A$ as the crossing matrix of a positive braid, the subbraid corresponding to this configuration consists of three strands, with the middle strand “hooking” one of the outer strands but crossing the other outer strand only once. In such a (sub)braid, all the “hooks” must precede the crossing, because once the crossing has occurred, the formerly outside strand separates the formerly middle strand from the other outer strand (see Figure 1).

In the situation where the “$R$” does lie on the first superdiagonal, this reasoning shows that attempting at that stage to introduce the transposition corresponding to that “$R$” cannot lead to a realization of the matrix. However, if via other divisions we can move the blockage so that the “$S$” entry is on the first superdiagonal, then we can divide by that transposition, thereby destroying the blockage, and continue.

\[\text{Figure 1. Geometric Interpretation of a Blockage}\]

The two examples in Subsection 3.1 both have the property that every nonzero entry in the first superdiagonal is a blocked “$R$”: the configurations $G_{124}, G_{134}, K_{235}$ and $K_{134}$ are all blockages whose “$R$” is embedded in the first superdiagonal of the respective matrix. Therefore, it is impossible to factor either of these matrices into permutation matrices—
equivalently, neither matrix is the crossing matrix of any positive braid. We call a matrix $A \in SR_N^+$ with this property a **totally blocked** matrix: it is impossible to left divide it by a transposition subordinate to the matrix (see below).

**Remark 20.** If $A \in SR_N^+$ is totally blocked—that is, every nonzero entry in the first superdiagonal of $A$ is a blocked “R”—then $A$ is not the crossing matrix of any positive braid.

As we pointed out above, the presence of a blockage somewhere in $A \in SR_N^+$ does not *a priori* mean that $A$ is not the crossing matrix of some positive braid—in fact, there are factorizations (equivalently, sequences of left divisions by transpositions) which create blockages, but then these blockages move around the matrix until they land with the “S” term on the first superdiagonal.

When the “S” entry lies on the first superdiagonal, division by the transposition subordinate to it yields a configuration of the form

$$
\begin{array}{c|c|c|c|c}
\hline
i & 0 & R & j & i + 1 \\
\hline
j & R + (b - 1)S & j + 1 & j + 1 & j + 1 \\
\hline
\end{array}
$$

(or its mirror image) and a further division by the same transposition yields

$$
\begin{array}{c|c|c|c|c}
\hline
i & R & 0 & j & i + 1 \\
\hline
j & (b - 1)S & j + 1 & j + 1 & j + 1 \\
\hline
\end{array}
$$

(*resp.* its mirror image); repeating this process $2(b-1)$ more times ends in the configuration

$$
\begin{array}{c|c|c|c|c}
\hline
i & 0 & R & j & i + 1 \\
\hline
j & R & j + 1 & j + 1 & j + 1 \\
\hline
\end{array}
$$

thus eliminating the blockage.

However, a matrix $A \in SR_N^+$ need not be totally blocked to fail to be a crossing matrix of some positive braid. Consider for example the $6 \times 6$ matrix

$$
V = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
$$
The two “R”’s on the first superdiagonal are blocked \((V_{236}, V_{145})\). Thus the only allowed left division is by \(\tau_1\), and this results in

\[
\begin{array}{cccccc}
1 & R & R & 0 & 0 \\
2 & R & 0 & R & 0 \\
3 & 0 & S & S \\
4 & R & S \\
5 & 0 & S \\
6 & 0 & S
\end{array}
\]

which is totally blocked (the “S” in position 2, 4 blocks the “R”’s in positions 1, 2 and 4, 5, while the “S” in position 3, 5 blocks the “R” in position 2, 3). One might say that the original \(V\) (as well as \(V_1\)) is \textit{virtually totally blocked}—every sequence of allowed left divisions by transpositions eventually leads to a totally blocked matrix. Such a matrix clearly cannot represent a positive braid.

This leads to a brute-force scheme for determining whether a given matrix \(A \in SR_N^+\) is the crossing matrix of some positive braid: quite simply, one tries every allowable sequence of left divisions by transpositions until one reaches either the zero matrix (in which case the original matrix has been factored into \(R\)-matrices, which exhibits a realization of \(A\)) or one reaches a totally blocked matrix. Since left division by a transposition subordinate to \(A \in SR_N^+\) reduces \(N(A)\), the sum of the entries of \(A\), by 1, any string of allowable left divisions will terminate in one of these two possibilities after at most \(N(A)\) steps. Of course the number of allowable sequences of left divisions is \textit{a priori} on the order of \(N(A)!\), but we have implemented this scheme in Mathematica code which can handle matrices of size up to about \(7 \times 7\) on a MacBook Pro with 8GB of memory.\(^8\)

Of course, this scheme provides an algorithmic characterization of crossing matrices for positive braids:

**Theorem 21.** The crossing matrix of any positive braid on \(N\) strands belongs to \(SR_N^+\): it is a non-negative integer matrix with an \(SR\) decomposition.

Conversely, every such matrix \(A \in SR_N\) is either the crossing matrix of some (perhaps many) positive braid on \(N\) strands, or it is virtually totally blocked (i.e., every sequence of left divisions by transpositions corresponding to positions in the first superdiagonals of successive left quotients terminates in a totally blocked matrix).

\(^8\)We record our deep thanks to our colleague Bruce Boghosian, who spent many hours helping us develop this code, as well as our former student Dan Fortunato, who helped us at the beginning of this process.
However, this fails to be the kind of conceptual characterization of such crossing matrices which we would like to see: a criterion which can be applied directly to a matrix without an exhaustive search through allowable sequences of left divisions. We have not succeeded in formulating even a conjectural version of such a criterion.

4. THE REALIZABILITY PROBLEM FOR SYMMETRIC MATRICES

The fact that the crossing product operation $\odot$ is simply matrix addition when restricted to $\mathcal{S}_{N}[\mathbb{Z}]$ makes it easier to think about realizability in this case—for example,

Remark 22. A sum of realizable symmetric matrices is automatically realizable as a composition of the individual realizations.

Also, we note that for any $R$-matrix $R$ (which by the discussion in Subsection 2.4 is the crossing matrix $R_{\pi}$ of some permutation $\pi \in \Sigma_{N}$) the symmetrization $S_{\pi}$ of $R$ is realized by $\pi^{\dagger}(\pi)^{\dagger}$. Combined with Remark 22 this shows

Remark 23. If $S \in \mathcal{S}_{N}[\mathbb{Z}^{+}]$ is $T_{0}$ and $T_{1}$, it is (positively) realizable.

Using these observations together with a case-by-case argument, we can show that for $N = 4$, any $T_{0}$ matrix in $\mathcal{S}_{N}[\mathbb{Z}^{+}]$ is (positively) realizable;

Theorem 24. A symmetric $4 \times 4$ non-negative integer matrix with zero diagonal is positively realizable if and only if it is $T_{0}$.

It is natural, based on this and our experimental evidence using the algorithm resulting from Theorem 21, that Theorem 24 extends to all $N$.

An approach to trying to prove the analogue of Theorem 24 for all $N$ might be via an induction on the number of nonzero entries in the matrix, using Theorem 24 to establish an initial case, and then the following idea, which we find plausible but have not succeeded in proving or disproving. We call an entry $a_{ij}$ of the $T_{0}$ matrix $A$ fully supported (even if that entry is zero) if, for every $k$ between $i$ and $j$, at least one of the entries $a_{ik}$ and $a_{kj}$ is nonzero.

Conjecture 25. Suppose $A$ is a positively realizable symmetric matrix, and $a_{ij} = 0$ but the position is fully supported in $A$. Then there is some realization of $A$ in which strands $i$ and $j$ become adjacent somewhere, so that changing $a_{ij}$ from zero to one results in a positively realizable matrix.

We caution that the word “some” is necessary here: the tableau

\[
\begin{bmatrix}
1 & S & 0 & 0 \\
2 & S & 0 \\
3 & S \\
4 &
\end{bmatrix}
\quad \oplus
\begin{bmatrix}
1 & S & 0 & 0 \\
2 & 0 & 0 \\
3 & 0 \\
4 &
\end{bmatrix}
\quad \oplus
\begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 0 & 0 \\
3 & S \\
4 &
\end{bmatrix}
\]

can be realized as a composition of three “hooks” in any order. If the middle one in the sum above occurs before or after both of the others (Figure 2), strands 1 and 4 come into adjacent positions, but if it occurs between them (Figure 3), then at any stage either strand
Figure 2. Middle hook after the others: strands 1 and 4 adjacent

Figure 3. Middle between the others: strands 1 and 4 separated by strands 2 and 3
2 or strand 3 separates these two.

Given this conjecture as a lemma, an inductive argument goes as follows: in any $T_0$ matrix, there are nonzero entries which do not support any other element (since all supports of an entry live in lower-numbered superdiagonals). “Erasing” one such entry yields a $T_0$ matrix with a lower entry sum; by induction on this sum, the latter is realizable, and hence by the conjectured lemma, so is our given matrix.

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E-mail address: magutierrez002@gmail.com

Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155

E-mail address: znitecki@tufts.edu