NECESSARY CONDITIONS FOR FRACTIONAL HARDY-SOBOLEV’S INEQUALITIES

E. Ostrovsky and L. Sirota

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan.

Emails: galo@list.ru; sirota@zahav.net.il

Abstract.

In this short article we obtain some necessary conditions for a so-called fractional Hardy-Sobolev’s inequalities in multidimensional case.

We also give some examples to show the sharpness of these inequalities.

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary 34A34, 65M20, 42B25.

Key words and phrases: Lebesgue-Riesz’s norm and spaces, rearrangement invariant (r.i.) Grand and ordinary Lebesgue Spaces, exact estimations, operators, dilation method, domain, weight.

1. Introduction. Statement of problem. Notations.

A. Ordinary Hardy-Sobolev’s fractional inequalities.

The following assertion is called Hardy-Sobolev’s (ordinary) difference inequality:

\[
\left[ \int_D |u(x)|^q |x|^{-\mu} \, dx \right]^{1/q} \leq K_{HS}(p, q) \times \left[ \int \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^\beta} \, dx \, dy \right]^{1/p}.
\]

We will write further \( \mu = \lambda dq \).

Here \( D \) is open domain with positive Lebesgue measure \( \mu(D) = \int_D dx \) in the whole Euclidean space \( \mathbb{R}^d; d = 1, 2, \ldots \) equipped with ordinary Euclidean norm \( |x|; \, x \in \mathbb{R}^d \), for instance, whole space \( \mathbb{R}^d \) or its half space or unit ball, \( u = u(\cdot) \) is an arbitrary function from the class \( C_0^\infty(D) \), \( \alpha(1), \alpha(2), \beta, \lambda = \text{const.} \), \( p, q = \text{const} \in (1, \infty) \), the finite positive (if there exists) ”constant” \( K_{HS}(p, q) = K_{HS}(p, q; \alpha(1), \alpha(2); d) \) dependent of the \( p, q; \alpha(1), \alpha(2), \beta, \mu; d \) but not of the function \( u(\cdot) \).

By means of approximation we can assume \( u \in W^{\mu/q, q} \), especially when we investigate the lower bounds for the constants.

The finiteness of integrals in the left-hand and right-hand sizes in (1.1a) for arbitrary function \( u(\cdot) \in C_0^\infty(D) \) entrusts the following conditions on the constants \( \alpha(1), \alpha(2), \beta, \mu : \)

\[
\mu < d, \ \alpha(1) > -d, \ \alpha(2) > -d, \ \beta < 1, \ \alpha(1) + \alpha(2) - \beta > -d.
\]

(Ca)

We assume also that

\[
\lambda \in (0, 1/(2d - 1)).
\]

(Cb)

We will suppose hereafter the conditions (Ca) and (Cb) are satisfied.
The following generalization of inequality (1.1a) is called Hardy-Sobolev’s weight difference inequality:

\[
\left[ \int_D |u(x)|^q W_{\mu}(x) \, dx \right]^{1/q} \leq K_{HS}(p, q; W) \times \left[ \int_D \left( \int_D \frac{|u(x) - u(y)|^p W_\alpha(x, y)}{W_\beta(|x - y|)} \, dx \, dy \right)^{1/p} \right].
\] (1.1b)

More general case appears if we write instead \( u(x) \) the function \( u(x) - u(0) \), for instance

\[
\left[ \int_D |u(x) - u(0)|^q |x|^{-\mu} \, dx \right]^{1/q} \leq K^{(0)}_{HS}(p, q) \times \left[ \int_D \left( \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^{\beta}} \, dx \, dy \right)^{1/p} \right].
\] (1.1c)

B. Mixed Hardy-Sobolev’s fractional inequalities.

The inequality of a view

\[
\left[ \int_D |u(x)|^r |x|^{-\mu} \, dx \right]^{1/r} \leq K_{M, HS}(p, q, r) \times \left\{ \int_D |y|^{\alpha(2)} \, dy \left( \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)}}{|x - y|^{\beta}} \, dx \right)^{q/p} \right\}^{1/q}
\] (1.2a)

is said to be (ordinary) Mixed Hardy-Sobolev’s fractional inequality.

The weight version of Mixed Hardy-Sobolev’s fractional inequality has a view

\[
\left[ \int_D |u(x)|^r W_{-\mu}(x) \, dx \right]^{1/r} \leq K_{MHS; W}(p, q, r) \times \left\{ \int_D dy \left( \int_D \frac{|u(x) - u(y)|^p W_\alpha(x, y)}{W_\beta(|x - y|)} \, dx \right)^{q/p} \right\}^{1/q}.
\] (1.2b)

The functions \( W_\alpha(\cdot), W_\beta(\cdot), W_{-\mu}(\cdot) \) are weight function, i.e. are measurable positive almost everywhere functions.

C. Hardy-Sobolev’s fractional derivative-difference inequalities.

By definition, the inequality of a view

\[
\left[ \int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^{\beta}} \, dx \, dy \right]^{1/p} \leq K_{DD, HS}(p, s) \times \int_D ||\nabla u(x)||^s |x|^{-\mu} \, dx \right]^{1/s}
\] (1.3c)

is called fractional derivative-difference inequality.

Here for the vector \( x = \{x_1, x_2, \ldots, x_d\} \)

\[\nabla u(x) = \text{grad } u \overset{\text{def}}{=} (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_d).\]
We omit here and in the next pilcrow the obvious weight generalization of these inequalities.

D. Surface Hardy-Sobolev’s fractional inequalities.

The following assertion is named as surface Hardy-Sobolev’s difference inequality:

\[
\left[ \int_S |u(x)|^q |x|^{-\mu} \sigma(dx) \right]^{1/q} \leq K_{S;HS}(p,q) \times \left[ \int_D \int_D |u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)} |x - y|^{-\beta} \right]^{1/p}.
\]

Here \( S \) is smooth (sub)surface of a boundary \( \partial D \) of a dimensional \( m; 1 \leq m \leq d - 1 \) with correspondent surface measure \( d\sigma(x) = \sigma(dx) \).

Our aim is finding of some necessary conditions for the fractional Hardy-Sobolev’s inequalities.

We obtain also the lower bounds for the constants \( K_{HS}(p,q), K_{M;HS}(p,q,r), K_{S;HS}(p,q), K_{DD;HS}(p,s) \) and consider some generalizations on the so-called Grand Lebesgue spaces instead classical Lebesgue-Riesz’s \( L_p \) spaces.

Some upper estimations for Hardy-Sobolev’s fractional inequalities see, e.g. in the works \[1, 5, 6, 7, 2, 3, 4, 30, 11, 12, 13, 14, 15, 16\] etc.

The one-dimensional case \( d = 1 \) for the ordinary Hardy-Sobolev’s inequality was investigated before by Jakovlev \[9\] and Grisvard \[8\].

About applications of these inequalities see, e.g. \[3, 4, 7, 12, 16\].

We use the symbols \( C(X,Y), C(p,q;\psi) \), etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like \( C_1(X,Y) \) and \( C_2(X,Y) \). The relation \( g(\cdot) \asymp h(\cdot), p \in (A,B), \) where \( g = g(p), h = h(p), g, h : (A,B) \to \mathbb{R}_+ \), denotes as usually

\[ 0 < \inf_{p \in (A,B)} h(p)/g(p) \leq \sup_{p \in (A,B)} h(p)/g(p) < \infty. \]

The symbol \( \sim \) will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

\[ I(x \in A) = 1, x \in A, I(x \notin A) = 0, \]

here \( A \) is a measurable set.

2. Main result: necessary conditions for the Hardy-Sobolev’s fractional inequalities

A. Ordinary Hardy-Sobolev’s fractional inequalities.

**Theorem 1A.** Let in the inequality (1.1a) \( D = \mathbb{R}^d \) and suppose (1.1a) be satisfied for some non-constant function \( u(\cdot) \in C^\infty \). Then
\[
\frac{d - \mu}{q} = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \tag{2.1a}
\]

**Proof** used the so-called dilation method belonging to G.Talenty \[35\]. Namely, let \( \theta = \text{const} \in (0, \infty) \). The dilation operator \( T_\theta \) may be defined as follows:

\[
u_\theta(x) = T_\theta u(x) \overset{\text{def}}{=} u(x/\theta).
\]

Note that if \( u(\cdot) \in C^\infty_0 \), then \( u_\theta(\cdot) \in C^\infty_0 \).

We obtain substituting the function \( u_\theta(\cdot) \) into inequality (1.1a) instead the function \( u(\cdot) \) after changing variables \( x = \theta z, y = \theta v \):

\[
\theta^{(d-\mu)/q} \left[ \int_D |u(x) - u(0)|^q |x|^{-\mu} \, dx \right]^{1/q} \leq K_{HS}(p, q) \times \theta^{(2d+\alpha(1)+\alpha(2)-\beta)/p} \left[ \int_D \int_D |u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)} \left| x - y \right|^{-\beta} \, dx \, dy \right]^{1/p}.
\]

Since the value \( \theta \) is arbitrary in the set \((0, \infty)\), we conclude

\[
(d - \mu)/q = (2d + \alpha(1) + \alpha(2) - \beta)/p,
\]

Q.E.D.

Analogously may be proved the following results of this section.

**B. Mixed Hardy-Sobolev’s fractional inequalities.**

**Theorem 1B.** Let in the inequality (1.2a) \( D = R^d \) and suppose (1.2a) be satisfied for some non-constant function \( u(\cdot) \in C^\infty_0 \). Then

\[
\frac{d - \mu}{r} = \frac{d + \alpha(2) - \beta}{p} + \frac{d + \alpha(1)}{q}. \tag{2.1b}
\]

**C. Hardy-Sobolev’s fractional difference-derivative inequalities**

**Theorem 1C.** Let in the inequality (1.3a) \( D = R^d \) and suppose (1.3a) be satisfied for some non-constant function \( u(\cdot) \in C^\infty_0 \). Then

\[
\frac{d - \mu}{s} - 1 = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \tag{2.1c}
\]

**D. Hardy-Sobolev’s surface fractional difference-derivative inequalities**

We consider here the case when

\[
x \in S \iff x_{m+1} = x_{m+2} = \ldots = x_n = 0.
\]

We obtain using as before at the same dilation method as when in the proof of inequality (1.4a) holds then

\[
\frac{m - \mu}{q} = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \tag{2.1d}
\]
3. Weight generalizations of Hardy-Sobolev’s fractional inequalities.

We consider in this section the Hardy-Sobolev’s weight difference inequality in the whole space $D = \mathbb{R}^d$:

$$\left[ \int_{\mathbb{R}^d} |u(x)|^q W_{-\mu}(x) \, dx \right]^{1/q} \leq K_{HS}(p, q; W) \times \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p W_\alpha(x, y)}{W_\beta(|x - y|)} \, dx \, dy \right]^{1/p}.$$  \hspace{1cm} (3.1)

A new notations. For some finite constant $\alpha_0, \beta_0, \mu_0; \alpha_\infty, \beta_\infty, \mu_\infty$ we introduce a functions, to be presumed non-zero and integrable:

$$W_{-\mu,0}(z) = \inf_{\theta \in (0,1)} \frac{W_\mu(\theta z)}{\theta^{-\mu_0}},$$

$$W_{\beta,0}(z) = \inf_{\theta \in (0,1)} \frac{W_\beta(\theta z)}{\theta^{\beta_0}},$$

$$W_{\alpha,0}(z, v) = \sup_{\theta \in (0,1)} \frac{W_\alpha(\theta z, \theta v)}{\theta^{\alpha_0}};$$

$$W_{-\mu,\infty}(z) = \inf_{\theta \in (1,\infty)} \frac{W_\mu(\theta z)}{\theta^{-\mu_\infty}},$$

$$W_{\beta,\infty}(z) = \inf_{\theta \in (1,\infty)} \frac{W_\beta(\theta z)}{\theta^{\beta_\infty}},$$

$$W_{\alpha,\infty}(z, v) = \sup_{\theta \in (1,\infty)} \frac{W_\alpha(\theta z, \theta v)}{\theta^{\alpha_\infty}}.$$

**Theorem 3.1.** If the inequality (3.1) is satisfied for any non-zero function $u \in C_0^\infty(\mathbb{R}^d)$, then

$$\frac{d - \mu_0}{q} \geq \frac{d + \alpha_0 - \beta_0}{p},$$ \hspace{1cm} (3.2a)

$$\frac{d - \mu_\infty}{q} \leq \frac{d + \alpha_\infty - \beta_\infty}{p}.$$ \hspace{1cm} (3.2b)

As a corollary: when $\mu_0 = \mu_\infty = \mu, \alpha_0 = \alpha_\infty = \alpha, \beta_0 = \beta_\infty = \beta$, then both the inequalities (3.2a) and (3.2b) reduced to the known relation

$$\frac{d - \mu}{q} = \frac{d + \alpha - \beta}{p}.$$ \hspace{1cm} (3.2c)

**Proof** is alike to the proof of theorem 2.1; it used the Talenty dilation method and splitting into two cases: $\theta \in (0, 1)$ and $\theta \in (1, \infty)$. 
4. Lower bounds for constants in Hardy-Sobolev’s fractional inequalities. Examples.

We denote by $K_{HS}, K_{M;HS}, K_{S;HS}, K_{DD;HS}$ the minimal values of the constants $K_{HS} = K_{HS}(p,q), K_{M;HS} = K_{M;HS}(p,q,r), K_{S;HS} = K_{S;HS}(p,q), K_{DD;HS} = K_{DD;HS}(p,s)$ in the (correspondingly) Hardy-Sobolev’s, Mixed Hardy-Sobolev’s, Surface Hardy-Sobolev’s, and Differential - Difference Hardy-Sobolev’s inequalities for the whole space $D = \mathbb{R}^d$.

Note that for some particular cases of domains $D$ (half-spaces etc.) the exact values of these constants was calculated by K.Bogdan and B.Dyda [1], R.Frank, R.Seiringer [2].

For instance, $K_{HS} = K_{HS}(p,q) = K_{HS}(p,q; \alpha(1), \alpha(2), \lambda) = \sup_{u \in C_0^\infty(\mathbb{R}^d)} \left\{ \left[ \int \int_D \left( |u(x)|^q |x|^{-\mu} \right)^{1/q} dx \right]^{1/p} : \left[ \int \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)} |x-y|^\beta}{|x-y|^\beta} dx dy \right]^{1/p} \right\}$. (4.0)

Recall that $\mu = \lambda dp, \lambda \in (0,1)$.

**Theorem 4.a.** Suppose the domain $D$ contains some ball

$$B(t) = \{ x : |x| \leq t \}, \quad B = B(1)$$

with the center in origin of the whole space $\mathbb{R}^d$.

Let also in the ordinary Hardy-Sobolev’s inequality $1 \leq p < d/\lambda$ and $\alpha(1) = \alpha(2) = 0, \beta = d(1 + \lambda p)$. Assume that the condition (2.1a) is satisfied. Then

$$C_1(\lambda; d) \left[ \frac{p}{|p - d/\lambda|} \right]^{\lambda} \leq K_{HS}(p,p,0,0,\lambda) \leq C_2(\lambda; d) \cdot \left[ \frac{p}{|p - d/\lambda|} \right] . \quad (4.1a)$$

**Proof.** The upper estimation contains in fact in [1]; see also [2]; see also [8], [9], [11].

Without loss of generality we can assume $d = 1$; the multidimensional case $d \geq 2$ is investigated analogously, with at the same (counter)example.

It is sufficient to consider the one-dimensional case $d = 1$ and to obtain the lower estimate to consider the following example in the case when the domain $D$ contains the unit ball of the set $\mathbb{R}^d$, with the center in origin:

$$u_0(x) = |\log |x|| \cdot I(|x| \leq 1).$$

We have consequently as $p \in [1, 1/\lambda), \quad p \to 1/\lambda - 0 :$

$$L^p \overset{\text{def}}{=} \int_0^1 x^{-\lambda p} |\log x| dx = \frac{\Gamma(p+1)}{|1-\lambda p|^{p+1}};$$

$$L \sim |p-1/\lambda|^{1+1/p} \sim |p-1/\lambda|^{1+\lambda}.$$

Further,

$$R^p := \left[ \int_D \int_D \frac{|\log |x| - \log |y||^p}{|x-y|^\beta} dx dy \right] \sim \left[ \int_B \frac{|\log |x| - \log |y||^p}{|x-y|^\beta} dx dy \right] \sim$$
\[
\int_0^1 \rho^{d-1-\beta} |\log \rho| \, d\rho \times \int_0^{2\pi} \frac{|\log |\tan \phi| |^p}{|\cos \phi - \sin \phi|^{\beta}} \, d\phi =: I_1 \cdot I_2.
\]

Note that the second integral \(I_2\) is bounded when \(1 \leq p < 1/\lambda\) and

\[I_1 \sim \frac{C}{|1 - \lambda p|}.
\]

This completes the proof of theorem 4.a.

**Remark 4.1a.** We obtain in more general case when \(\alpha(1) \geq 0, \alpha(2) \geq 0, \alpha := \alpha(1) + \alpha(2) > 0, \mu = \alpha - \lambda dp\) denoting

\[p_0 = \frac{1}{\lambda} + \frac{\alpha}{\lambda d}:
\]

\[C_1(\alpha(1), \alpha(2), \lambda; d) \left[ \frac{p}{|p - p_0|} \right]^{1/p_0} \leq K_{HS}(p, p; \alpha(1), \alpha(2), \lambda) \leq C_2(\alpha(1), \alpha(2), \lambda; d) \cdot \left[ \frac{p}{|p - p_0|} \right].
\]

Analogously may be obtained the following results.

**Theorem 4.b.** Let in the mixed ordinary Hardy-Sobolev’s inequality \(1 \leq r < d/\lambda\) and \(\beta = d(1 + \lambda p)\). Assume the condition (2.1b) is satisfied. Then

\[C_3(\alpha(1), \alpha(2), \lambda; d) \left[ \frac{1}{|r - d/\lambda|} \right]^{\lambda} \leq K_{M;HS}(p, q) \leq C_4(\alpha(1), \alpha(2), \lambda; d) \cdot \left[ \frac{1}{|r - d/\lambda|} \right]. \quad (4.1b)
\]

**Theorem 4.c.** Let in the differential-difference Hardy-Sobolev’s inequality \(\beta = d(1 + \lambda p)\) and let the condition (2.1c) be satisfied. Then

\[C_5(\alpha(1), \alpha(2), \lambda; d) \leq K_{DD;HS}(p, s) \leq C_6(\alpha(1), \alpha(2), \lambda; d). \quad (4.1c)
\]

**Theorem 4.d.** Let in the surface Hardy-Sobolev’s inequality \(1 \leq q < m/\lambda\) and \(\beta = d(1 + \lambda p)\). Assume also the condition (2.1d) is satisfied. Then

\[C_7(\alpha(1), \alpha(2), \lambda; d, m) \left[ \frac{q}{|q - m/\lambda|} \right]^{\lambda} \leq K_{S;HS}(p, q) \leq C_8(\alpha(1), \alpha(2), \lambda; d, m) \cdot \left[ \frac{q}{|q - m/\lambda|} \right]. \quad (4.1d)\]
5. Generalization on a Bilateral Grand Lebesgue Spaces

We recall briefly the definition and needed properties of these spaces. More details see in the works [22], [23], [25], [26], [33], [34], [29], [27], [28] etc.

For a and b constants, \(1 \leq a < b \leq \infty\), let \(\psi = \psi(p)\), \(p \in (a, b)\), be a continuous positive function such that there exists a limits (finite or not) \(\psi(a + 0)\) and \(\psi(b - 0)\), with conditions \(\inf_{p \in (a, b)} > 0\) and \(\min\{\psi(a + 0), \psi(b - 0)\} > 0\). We will denote the set of all these functions as \(\Psi(a, b)\).

The Bilateral Grand Lebesgue Space (in notation BGLS) \(G(\psi; a, b) = G(\psi)\) is the space of all measurable functions \(f : D \rightarrow R\) or \(f : R^d \rightarrow R\) endowed with the norm

\[ ||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (a,b)} \left[ \frac{|f|_p}{\psi(p)} \right], \]

if it is finite.

The \(G(\psi)\) spaces over some measurable space \((X, F, \mu)\) with condition \(\mu(X) = 1\) (probabilistic case) appeared in an article [29].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces \(L_1(R^d)\) and \(L_\infty(R^d)\) under real interpolation method [27].

It was proved also that in this case each \(G(\psi)\) space coincides with the so-called exponential Orlicz space, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions \(\phi(G(\psi; a, b); \delta)\), Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

Let \(g : X \rightarrow R\) be some measurable function such that \(\exists(a, b), 1 \leq a < b \leq \infty\), such that \(\forall p \in (a, b) \Rightarrow |g|_p < \infty\).

We can then introduce the non-trivial function \(\psi_g(p)\) as follows:

\[ \psi_g(p) \overset{def}{=} |g|_p, \quad p \in (a, b). \]  

This choosing of the function \(\psi_g(\cdot)\) will be called natural choosing.

Remark 1. If we introduce the discontinuous function

\[ \psi_r(p) = 1, \quad p = r; \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b) \]

and define formally \(C/\infty = 0\), \(C = \text{const} \in R^1\), then the norm in the space \(G(\psi_r)\) coincides with the \(L_r\) norm:

\[ ||f||_{G(\psi_r)} = |f|_r. \]

Thus, the Bilateral Grand Lebesgue spaces are the direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces \(L_r\).

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [22], [25], theory of probability in Banach spaces [31], [29], [33], in the modern non-parametrical statistics, for example, in the so-called regression problem [33].

Let us introduce the following linear operators acting on the function \(u(\cdot) \in C_0^\infty(R^d)\):
\[ \delta_\lambda[u](x, y) = \frac{u(x) - u(y)}{|x - y|^{\lambda d}}, \quad (5.4) \]

\[ S_\lambda[u](x) = \frac{u(x)}{|x|^{\lambda d}}. \quad (5.5) \]

Let also \( \psi_2 = \psi_2(p) \) be some function from the class \( \Psi(a, b; R^d \times R^d) \) relative a new potential measure

\[ \nu(dx, dy) = \frac{dxdy}{|x - y|^{\lambda d}} \quad (5.6) \]

and such that \( b \leq 1/\lambda \) or conversely \( a \geq 1/\lambda \).

Put

\[ \psi_1(p) = K_{HS}(p) \cdot \psi_2(p). \]

Theorem 5.1a.

\[ ||S_\lambda[u]||_{G\psi_2} \leq 1 \cdot ||\delta_\lambda[u]||_{x - y}^{-\lambda d}||G\psi_1, \quad (5.7) \]

where the constant "1" is the best possible.

**Proof** is very simple. Let

\[ \delta_\lambda[u] \cdot |x - y|^{-\lambda d} \in G\psi_2; \]

without loss of generality we can suppose \( ||\delta_\lambda[u]||_{x - y}^{-\lambda d}||G\psi_2 = 1. \)

From the direct definition of the norm in Grand Lebesgue spaces it follows

\[ \left[ \int_{R^d} \int_{R^d} |\delta_\lambda[u]|^p(x, y) \frac{dxdy}{|x - y|^{\lambda d}} \right]^{1/p} \leq \psi_2(p), \quad p \in (a, b). \]

We obtain using the Hardy-Sobolev’s inequality with \( q = p : \)

\[ \left[ \int_{R^d} |S_\lambda[u]|^p(x) \right]^{1/p} \leq K_{HS}(p) \cdot \psi_2(p) = \psi_1(p), \]

which is equivalent to the assertion of our theorem.

The precision of the constant "1" follows immediately from the main result of paper [17].

Note that it follows from upper estimation for the constant \( K_{HS}(p) \) that if we define a new function \( \psi_3(p) \) as follows:

\[ \psi_3(p) = \frac{p \psi_2(p)}{|1/\lambda - p|}, \]

then

\[ ||S_\lambda[u]||_{G\psi_3} \leq C \cdot ||\delta_\lambda[u]||_{G\psi_1}. \]

This result is weakly exact in the following sense. Let \( \psi_4(p) \) be each function from the class \( G\Psi(a, b) \), where either \( a = 1/\lambda \) or \( b = 1/\lambda \) for which

\[ \lim_{p \to 1/\lambda} \frac{p \psi_4(p)}{|p - 1/\lambda|^{\lambda}} = 0. \]
Then for the function \( u_0(x) = |\log |x|| \cdot I(|x| \leq 1) \)
\[
\lim_{p \to 1} \frac{||S_\lambda[u_0]||_{G^p\psi}}{||\delta_\lambda[u_0]||_{G^p\psi}} = \infty.
\]

The mixed, or anisotropic Grand Lebesgue Spaces was introduced in [18]. Indeed, let \( u = u(x), \ x \in \mathbb{R}^n \) be measurable function: \( u : \mathbb{R}^n \to \mathbb{R} \). Recall that the anisotropic Lebesgue space \( L_{\vec{p}} \) consists on all the functions \( f \) with finite norm
\[
||f||_{L_{\vec{p}}} := \left( \int_{\mathbb{R}^{m_1}} |\mu_1(dx_1)| \left( \int_{\mathbb{R}^{m_2}} |\mu_2(dx_2)| \left( \int_{\mathbb{R}^{m_3}} |f(x)|^{p_3} \mu_3(dx_3) \right)^{p_2/p_3} \right)^{p_1/p_2} \right)^{1/p_1}.
\]
Here \( m_j = \text{dim } x_j, \sum_j m_j = n \).

Note that in general case
\[
||f||_{L_{p_1,p_2}} \neq ||f||_{p_2,p_1},
\]
but
\[
||f||_{p,p} = ||f||_p.
\]
Observe also that if \( f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2) \) (condition of factorization), then
\[
||f||_{p_1,p_2} = ||g_1||_{p_1} \cdot ||g_2||_{p_2},
\]
(formula of factorization).

Let \( \nu = \nu(\vec{p}) \) be some continuous positive on the set \( Q \); \( \vec{p} \in Q \) function such that
\[
\inf_{\vec{p} \in Q} \nu(p) > 0, \ \nu(p) = \infty, \ p \notin Q.
\]
We denote the set all of such a functions as \( \Psi_Q \).

The (multidimensional, anisotropic) Grand Lebesgue Spaces \( \text{GLS} = G_Q(\nu) = G_Q\nu \) space consists on all the measurable functions \( f : \mathbb{R}^n \to \mathbb{R} \) with finite norms
\[
||f||_{G_Q(\nu)} \overset{\text{def}}{=} \sup_{\vec{p} \in Q} [||f||_{\vec{p}/\nu\vec{p}}].
\]
In the further considered case \( n = 2d \), \( \mu_1(dx) = \mu_2(dx) = dx \).

**Theorem 5.2.** Let for some \( \nu \in G\nu \)
\[
\frac{\delta_\lambda(x, y)}{|x - y|^d} \in G\nu(R^d \times R^d).
\]
Define the domain \( R_r, \ r \geq 1 \) (sub-domain in the plane \( R^2 \)) by the following way:
\[
R_r = R_r(\alpha_1, \alpha_2, \beta, \mu; d) = \{(p, q) : p, q \geq 1, \frac{d - \mu}{r} = \frac{d + \alpha_2 - \beta}{p} + \frac{d + \alpha_1}{q} \}
\]
and the function
\[
\psi_5(r) = \inf_{(p,q) \in R_r} [\nu(p,q)K_{M,H}(p,q)].
\]
Assertion:

\[ ||S_\lambda u||_{G\psi_5} \leq ||\delta_\lambda u \cdot |x - y|^{-\lambda d}||_{G\nu}. \quad (5.13) \]

**Proof.** Without loss of generality we can and do suppose \( ||f||_{G_Q(\nu)} = 1 \); then

\[ ||\delta_\lambda u||_{p,q} \leq \nu(p,q). \]

It follows from the definition of the norm in BGLS spaces and the mixed norm inequality (2.1b) that

\[ |S_\lambda u|_r \leq \nu(p,q)K_{M,HS}(p,q), \quad (p,q) \in R_r, \quad (5.14) \]

therefore

\[ |S_\lambda u|_r \leq \inf_{(p,q) \in R_r} [\nu(p,q)K_{M,HS}(p,q)] = \psi_5(r). \quad (5.15) \]

**References**

[1] K.Bogdan, B.Dyda. *The best constant in a fractional Hardy inequality.* Matematische Nachrichten, V. 284, Iss. 5-6, (2011), 629-638.

[2] R.Frank, R.Seiringer. *Sharp Fractional Hardy Inequalities in Half-Space.* In: Around Sobolev Spaces, Springer, New York, (2010), 161-167, MR 2723817.

[3] R.Frank, R.Seiringer. *Non-linear Ground State Representation and Sharp Hardy Inequalities.* J.Funk. Anal., 255, (2008), 3407-3430. MR 2469027.

[4] R.Frank, E.H.Lieb, R.Seiringer. *Hardy-Lieb-Thirring inequalities for fractional Schr"odinger operator.* Proc. of AMS, 288, (2010), 1757-1771.

[5] B.Dyda. *On fractional Hardy inequalities.* Nonlocal Operators and Partial Differential Equations, Bedlewo, June 27th - July 3th, (2010).

[6] B.Dyda. *A fractional order Hardy inequalities.* Ill. J. Math., 48(2), (2004), 575-588.

[7] B.Dyda. *Embedding Theorems for Lipshitz and Lorentz Spaces on lower Ahlfors regular sets.* Studia Nath., 197, (2010), 247-256.

[8] P.Grisvard. *Espaces intermediaries entre espaces de Sobolev avec pads.* Ann. Scuola Norm. Sup. Pisa, 23, (1969), 373-386.

[9] G.N.Jakovlev. *Boundary properties of functions from the space \( W_p^{(l)} \) on domain with angular points.* Dokl. Akad. Nauk SSSR (Russian), 40, (1961), 73-76.

[10] M.P.Heinig, A.Kufner, L.-E. Persson. *On some fractional Hardy inequalities.* Proc. AMS, 128, N°1 (1997), 25-46.

[11] L.-E. Persson and A.Kufner. *Same Difference Inequalities with Weight and Interpolation.* Matematical Inequalities Applications, V.1 N°1 (1998), 437-444.

[12] B.Opic,A.Kufner. *Hardy-type inequalities.* Longman, Harlow, (1990).

[13] M.Loss, C.Sloane. *Hardy Inequalities for fractional Integrals on general Domains.* J.Funk. Anal., 259, N°6, (2010), 1369-1379, MR 2659764.

[14] J.Maligranda. *Generalized Hardy inequalities in rearrangement invariant spaces.* J. Math. pures et appl., 59, (1980), 405-415.

[15] N.Krugljak,L.Maligranda, L.E.Persson. *On an elementary approach to the fractional Hardy inequalities.* Proc. AMS, 128, N°3, (2000), 727-734.

[16] V.Maz’ya, T.Shaposhnikova. *On the Bourgain, Brezis and Mirocucu theorem concerning limiting embeddings of fractional Sobolev Spaces.* J. Funk. Anal., 195, (2008), N°2, 230-238.

[17] E.Ostrovsky, L.Sirota. *Boundedness of Operators in Bilateral Grand Lebesgue Spaces.* arXiv:1104.2963v1 [math.FA] 15 Apr 2011.
[18] E. Ostrovsky, L. Sirota, E. Rogover. Integral Operators in Bilateral Grand Lebesgue Spaces. arXiv:0912.2538v1 [math.FA] 13 Dez 2009.
[19] C. Bennet and R. Sharpley, Interpolation of operators. Orlando, Academic Press Inc., 1988.
[20] S.-K. Chua. Weighted Sobolev inequalities on domains satisfying the chain condition. Proc. Amer. Math. Soc., 122(4), (2003), 1181-1190.
[21] D.E. Edmunds and W.D. Evans. Sobolev Embeddings and Hardy Operators. In: Vladimir Maz'ya (Editor), "Sobolev Spaces in Mathematics", Part 1, International Mathematical Series, Volume 8, Springer Verlag, Tamara Rozhkovskaya Publisher; (2009), New York, London, Berlin; p.153-184.
[22] A. Fiorenza. Duality and reflexivity in grand Lebesgue spaces. Collect. Math. 51(2000), 131–148.
[23] A. Fiorenza and G.E. Karadzhov, Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine”, Sezione di Napoli, Rapporto tecnico 272/03(2005).
[24] G.H. Hardy, J.E. Littlewood and G. Pólya. Inequalities. Cambridge, (1952).
[25] T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. Arch. Rat.Mech. Anal., 119(1992), 129–143.
[26] T. Iwaniec, P. Koskela and J. Onninen, Mapping of Finite Distortion: Monotonicity and Continuity. Invent. Math. 144(2001), 507–531.
[27] B. Jawerth and M. Milman, Extrapolation theory with applications. Mem. Amer. Math. Soc. 440(1991).
[28] G.E. Karadzhov and M. Milman, Extrapolation theory: new results and applications. J. Approx. Theory, 113(2005), 38-99.
[29] Yu.V. Kozatchenko and E.I. Ostrovsky, Banach spaces of random variables of subgaussian type. Theory Probab. Math. Stat., Kiev, 1985, 42-56 (in Russian).
[30] A. Kufner. Weighted Sobolev Spaces. John Wiley Dons, 1985.
[31] M. Ledoux and M. Talagrand. Probability in Banach Spaces. Springer, Berlin, 1991.
[32] V. Maz'ja. Sobolev Spaces. Kluvner Academic Verlag, (2002), Berlin-Heidelberg-New York.
[33] E.I. Ostrovsky, Exponential Estimations for Random Fields. Moscow - Obninsk, OINPE, 1999 (Russian).
[34] E. Ostrovsky and L. Sirota, Moment Banach spaces: theory and applications. HAIT Journal of Science and Engineering, C, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).
[35] G. Talenty. Best constant in Sobolev inequality. Instit. Mat. Univ. Firenze, 22, (1974-1975), p. 1 - 32.
[36] A. Wannebo. Hardy inequalities and imbeddings in domains generalizing $C^{0,\lambda}$ domains. Proc. Amer. Math. Soc., 117, (1993), 449 - 457; MR 93d:46050.