A family of étale coverings of the affine line

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1 Introduction

This note was inspired by a colloquium talk given by S. S. Abhyankar at the Tata Institute on the work of Abhyankar, Popp and Seiler (see [2]). It was pointed out in this talk that classical modular curves can be used to construct (by specialization) coverings of the affine line in positive characteristic. In this “modular” optic it seemed natural to consider Drinfel’d modular curves for constructing coverings of the affine line. This note is a direct outgrowth of this idea.

While it is trivial to see that affine line in characteristic zero has no non-trivial étale coverings, in [1] it was shown that the situation in positive characteristic is radically different and far more interesting. Let us, for the sake of definiteness, work over a field $k$ of characteristic $p > 0$. In [1], Abhyankar conjectured that any finite group whose order is divisible by $p$ and which is generated by its $p$-Sylow subgroups (such a finite group is sometimes called a “quasi-$p$-group”), occurs as quotient of the algebraic fundamental group of the affine line. It is customary to write $\pi_{1,\text{alg}}(\mathbb{A}^1_k)$ to denote the algebraic fundamental group of the affine line. While Abhyankar’s conjecture indicates that the algebraic fundamental group of the affine line is quite complicated, our result perhaps illustrates its cyclopean proportions. The result we prove (see Theorem 3 below) is the analogue of the following well-known classical result, which falls out of the theory of elliptic modular curves over the field of complex numbers: there is a continuous quotient $\pi_{1,\text{alg}}(\mathbb{P}^1_{\mathbb{C}} - \{0, 1728, \infty\}) \to \prod_p SL_2(\mathbb{Z}_p)/\{\pm 1\}$, where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers.

1 During his visit in the month of December, 1992
We would like to thank N. Mohan Kumar for numerous suggestions and conversations; he also explained to us a variant of Abhyankar’s Lemma, which is crucial to our argument. We would also like to thank E.-U. Gekeler, Dipendra Prasad for electronic correspondence while this note was being written— their remarks and suggestions have been extremely useful; in particular we would like to point out that the conjecture stated at the end of this note was formulated with the help of Dipendra Prasad. We would also like to thank M. V. Nori for useful comments and Dinesh Thakur for encouragement.

2 Resumé of Drinfel’d modules and their moduli

In this section we recall a few of the standard facts about Drinfel’d modules. Since the basic theory of Drinfel’d modules and their moduli is well documented we will be brief; all the facts which we will need can be found in the following standard references: [3], [8], [5]. Since we do not need the full strength of Drinfel’d’s work, we will work with a very special situation which is required for our purpose. In this section, we will outline this special situation.

Let us fix some notations. Let $F_q$ denote a finite field with $q = p^m$ elements and of characteristic $p$. We will write $A = F_q[T]$, $K = F_q(T)$. Further denote by $K_\infty$, the completion of $K$ along the valuation corresponding to $1/T$. Denote by $C$ the completion of the algebraic closure of $K_\infty$. The field $C$ is thus a “universal domain” of characteristic $p$. There is a natural inclusion $K \hookrightarrow C$.

Let $G_a$ denote the additive group scheme. Then it is easy to see that the ring of endomorphisms of $G_a$ defined over $C$, denoted by $\text{End}_C(G_a)$, is a noncommutative ring generated by the Frobenius endomorphism. More precisely, for an indeterminate $\tau$, consider the noncommutative ring $C\{\tau_p\}$ of all polynomials in $\tau$ with coefficients in $C$, and where the multiplication rule is given by $\tau_pa = a^p\tau_p$ for all $a \in C$. Then one checks that $\text{End}_C(G_a) \simeq C\{\tau_p\}$. Observe that this isomorphism gives rise to a ring homomorphism $\partial : \text{End}_C(G_a) \rightarrow C$ which sends an endomorphism of $G_a$ to the constant term of the polynomial in $\tau$ associated to it.

A Drinfel’d module $\phi$ over $C$ is a ring homomorphism $\phi : A \rightarrow \text{End}_C(G_a)$
such that composite map $\partial \phi$ is the natural inclusion of $A \to C$. It is easy to check that any such map factors through the subring $C\{\tau_p^m\} \subset C\{\tau_p\}$. For simplicity of notation, we will write $\tau = \tau_p^m$. Thus for any $a \in A$ we have an endomorphism $\phi_a$ of $G_a$ defined over $C$. Moreover, note that, any such $\phi$ is $F_q$ linear. So as $A$ is generated as an $F_q$-algebra by $T$, giving a $\phi$ thus amounts to specifying a single endomorphism of $G_a$ corresponding to $T \in A$, $\phi_T = \sum_{i=0}^r a_i \tau^i$, with $a_0 = T$. Then this can be extended to all of $A$. The $\tau$-degree of $\phi_T$ is called the rank of the Drinfel’d module $\phi$.

If $\phi, \phi'$ are two Drinfel’d modules then a morphism of Drinfel’d modules is an endomorphism $u \in \text{End}_C(G_a)$ such that for all $a \in A$ we have $\phi_a u = u \phi'_a$. It is easy to see that any such $u$ is in fact contained in $C\{\tau\}$.

For any $a \in A$ a Drinfel’d module $\phi$ specifies a closed subgroup-scheme of $G_a$: the kernel of the endomorphism $\phi_a$, $\ker(\phi_a)$. For example, if $a = T$, then the kernel of $\phi_T$ is simply the roots of the polynomial $T + \sum_{i=1}^r a_i X^{q^i-1} = 0$ together with 0. Moreover, for any ideal $I \subset A$, we can define a subgroup scheme of $G_a$ using the ring structure. One checks that $\ker(\phi_I)(C)$ is a free $(A/I)$-module of rank $r$, where $r$ is the rank of $\phi$. This lets us define a notion of an $I$-level structure on $\phi$. Drinfel’d has shown (see [3]) that there is a moduli of Drinfel’d modules with level structure (in general we have only a “coarse moduli scheme”). Our main interest is the case of rank two Drinfel’d modules. And henceforth, we shall work with Drinfel’d modules of rank two.

The theory of Drinfel’d modules of rank two behaves like the theory of elliptic curves. The fact that the moduli of Drinfel’d modules of rank two over $C$, is a smooth affine curve over $C$ is a very special case of a fundamental result of Drinfel’d (see [3]). We need several facts about these Drinfel’d modular curves. The first fact we need is the following:

**Theorem 1** The “coarse moduli” of rank two Drinfel’d modules over $C$ is the affine line over $C$.

**Proof:** For a proof see [4] paragraph 1.32, or [3]. \hfill \Box

Thus the above situation is analogous to the classical situation for elliptic curves. In fact the identification with the line is given by a “j-invariant”. If $\phi_T = T + aT + bT^2$, is a rank 2 module then $b \neq 0$ and then its j-invariant is $a^{q+1}/b$. 3
Before we need some notations. For any $F_q$ algebra $R$, let

$$G_1(R) = \left\{ g \in GL_2(R) \mid \det(g) \in F_q^\times \right\},$$

$$G(R) = G_1(R)/Z(F_q),$$

where $Z(F_q)$ is the group of $F_q$-valued scalar matrices with nonzero determinant.

The following result is really the heart of our construction.

**Theorem 2** For every non-zero ideal $I \subset A$, there is a “coarse moduli” of Drinfel’d modules of rank two with full $I$-level structure exists and is an affine curve over $C$. There is a “forget the level structure” morphism to $A_1^1 C$. This map is branched over $0 \in A_1^1 C$. The covering is Galois and the Galois group is $G(A/I)$. The ramification index of any point lying over 0 is $q + 1$ and is independent of $I$. In particular, the ramification is tame.

**Proof:** As mentioned earlier, the existence of the moduli is due to Drinfel’d (see [3]). These curves have been studied in great detail by Goss and Gekeler. In the our case the Galois group can easily be calculated, a convenient reference for it is [4], for instance see Lemma 1.4 on page 79, also see the first section of [4]. The ramification information is computed in [6], Lemma 4.2. One also finds it computed in [5], on page 87.

As in the classical situation, the ramification takes place over the Drinfel’d module with extra automorphisms. From the definitions, it is clear that an automorphism of a Drinfel’d module of rank two over $C$ is firstly an automorphism $u$ of $G_a$, defined over $C$. Clearly any such automorphism must be an invertible element of $C$. Then a simple calculation shows that if a nonzero element of $C$ is an automorphism of a Drinfel’d module then it must be a root of unity. One checks that with the exception of the module with $j$-invariant equal to zero, the automorphism group of the Drinfel’d module is $F_q^\times$. The module with $j$-invariant equal to zero has automorphism group $F_{q^2}^\times$. Thus in particular, the ramification is tame. These facts are easily proved by explicit calculations. □
3 The main theorem

We are now ready to state and prove our main theorem. One should note that most of the work has already gone in the construction and analysis the of the moduli of Drinfel’d modules.

**Theorem 3** If $p = 2, 3$ then assume that $q = p^m, m \geq 2$. There is a family of étale coverings, $Y_I$ of $\mathbb{A}_C^1$, indexed by the nonzero proper ideals $I \subset A$. The curves $Y_I$ are affine curves over $C$ and the covering $Y_I \rightarrow \mathbb{A}_C^1$ is Galois with Galois group $SL_2(A/I)/\{\pm 1\}$. Moreover, these coverings form an inverse system indexed by $I$. Thus in particular we have a continous quotient

$$\pi_1^\text{alg}(\mathbb{A}_C^1) \rightarrow \lim_{\longleftarrow I}(SL_2(A/I)/\{\pm 1\}).$$

**Proof**: By Theorem 2, we have a tamely ramified covering of the affine line which is branched over one point. Also note that the ramification index of any point over $0 \in \mathbb{A}_C^1$ is $q + 1$, independent of the ideal $I \subset A$.

Now we apply a suitable variant of Abhyankar’s Lemma to remove the tame ramification. The crucial thing is to ensure, if possible, that the “pull back” coverings remain irreducible. The following variant of Abhyankar’s Lemma (see Lemma 1) which was pointed out to me by N. Mohan Kumar, gives an explicit criterion to check irreducibility, then we apply this criterion to the case at hand. This is an easy exercise in elementary group theory. We have stated all the necessary results as a sequence lemmas, and since the proofs of all the individual statements are easy, we leave the details to the reader. \(\square\)

**Lemma 1** Let $k$ be an algebraically closed field of characteristic $p$. Let $X \rightarrow \mathbb{A}_k^1$ be a finite Galois cover defined over $k$, with Galois group $G$. Further assume that the cover is branched over $0 \in \mathbb{A}_k^1$, and any point lying over it is tamely ramified with ramification index $n$. Let $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ be a $\mathbb{Z}/n$ covering ramified completely at $0$, and unramified elsewhere. Let $X'$ be the normalization of the fibre product $X \times_{\mathbb{A}_k^1} \mathbb{A}_k^1$. Suppose that there are no nontrivial homomorphisms $G \rightarrow \mathbb{Z}/n$. Then $X'$ is irreducible, and the Galois group of the covering $X' \rightarrow \mathbb{A}_k^1$ is $G$. 

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Proof: Clearly one is reduced to proving the following field theory statement:

Let \( L/k(t) \) be a finite Galois extension with Galois group \( G \). Let \( E = k(t^{1/n}) \). Then \( L/k(t) \) and \( E/k(t) \) are linearly disjoint over \( k(t) \) if and only if there are no nontrivial homomorphisms \( G \to \mathbb{Z}/n \).

And the above statement is immediate from the fact that \( E/k(t) \) is a Kummer extension. This finishes the proof. \( \square \)

We now need some elementary group theoretic lemmas to apply the above criterion. Since the proofs are easy we will state the lemmas without proofs.

**Lemma 2** Let \( \wp \neq 0 \) is a prime ideal of \( A \). Then for all \( k \geq 1 \) and for all \( n \geq 2 \), the natural morphism

\[
GL_k(A/\wp^n) \to GL_k(A/\wp^{n-1})
\]

is surjective.

Recall that for any \( \mathbb{F}_q \) algebra \( R \), we had defined the groups

\[
G_1(R) = \left\{ g \in GL_2(R) \mid \det(g) \in \mathbb{F}_q^* \right\},
\]

\[
G(R) = G_1(R)/Z(\mathbb{F}_q),
\]

where \( Z(\mathbb{F}_q) \) is the group of \( \mathbb{F}_q \)-valued scalar matrices with nonzero determinant. Further, for any prime ideal \( \wp \neq 0 \) of \( A \), write \( G_1^n = G_1(A/\wp^n) \), and \( G^n = G(A/\wp^n) \), for all \( n \geq 1 \). The results which follow are valid for any non-zero prime ideal \( \wp \), so we have suppressed \( \wp \) in our notations.

**Lemma 3** The natural map \( G_1^n \to G_1^{n-1} \) is surjective for all \( n \geq 2 \).

**Lemma 4** The natural map \( G^n \to G^{n-1} \) is surjective for all \( n \geq 2 \).

Denote by \( G^{(n,n-1)} \) the kernel of the map \( G^n \to G^{n-1} \), similarly write \( G_1^{(n,n-1)} \), for the kernel of the corresponding map for \( G_1 \).

**Lemma 5** Let \( q = 2^m \), \( m \geq 2 \). Let \( F/\mathbb{F}_q \) be any finite extension. Then there are no nontrivial morphisms \( G(F) \to \mathbb{Z}/(q + 1) \).
Lemma 6 Let \( q = 2^m, m \geq 2 \). Then for any \( n \geq 1 \) there are no nontrivial maps \( G^n \to \mathbb{Z}/(q + 1) \).

Proof: The proof is by induction on \( n \). For \( n = 1 \), we are done by the previous lemma. Now show that the kernel of any map \( G^n \to \mathbb{Z}/(q + 1) \) contains \( G^{(n,n-1)} \). Then we are done by induction. \( \square \)

Lemma 7 Let \( q = p^m, p \neq 2 \). If \( p = 3 \) then \( m \geq 2 \). Then there is a canonical morphism \( G^n \to \mathbb{F}_q^*/\mathbb{F}_q^{*2} \). In particular we have a natural map \( G^n \to \mathbb{Z}/2 \), obtained by identifying \( \mathbb{F}_q^*/\mathbb{F}_q^{*2} \) with \( \mathbb{Z}/2 \).

Lemma 8 Let \( q = p^m, p \neq 2 \). If \( p = 3 \) then \( m \geq 2 \). Let \( F/\mathbb{F}_q \) be any finite extension. Then any morphism \( G(F) \to \mathbb{Z}/(q + 1) \) factors through the canonical morphism given by the above lemma, followed by the inclusion of \( \mathbb{Z}/2 \to \mathbb{Z}/(q + 1) \).

Lemma 9 Let \( q = p^m, p \neq 2 \). If \( p = 3 \), \( m \geq 2 \). Then any nontrivial morphism \( G^n \to \mathbb{Z}/(q + 1) \) factors through the canonical morphism.

Proof: This is again proved by induction on \( n \). As before, one checks that any such map is trivial on \( G^{(n,n-1)} \). \( \square \)

Now we can identify our Galois groups.

Lemma 10 Let \( q = 2^m, m \geq 2 \). For any non zero prime ideal \( \wp \subset A \), we have:
\[
G^n = SL_2(A/\wp^n)
\]

Lemma 11 Let \( q = p^m, p \neq 2 \), if \( p = 3 \) then \( m \geq 2 \). Let \( \wp \) be any non zero prime ideal in \( A \). For any \( n \geq 1 \), let
\[
\tilde{G}^n = ker(G^n \to \mathbb{Z}/2).
\]
Then \( \tilde{G}^n = SL_2(A/\wp^n)/\{\pm 1\} \).
Thus we can now prove our main theorem. If $p = 2$, then the pull back coverings remain irreducible and the Galois group is $G^n$. If $p$ is odd, then there is a quadratic subfield in common. And the Galois group is $\tilde{G}^n$. Then we are done by the above lemmas.

After the result of Madhav Nori (see [7]) and the one proved above, we would like to advance the following conjecture:

**Conjecture 1** Let $G/K$ be any isotropic, semisimple, simply-connected algebraic group over $K = \mathbb{F}_q(T)$, with center $Z$. Let $\mathbb{A}_K^{\text{fin}}$ denote the finite adeles of $K$. Then any maximal compact subgroup of $G(\mathbb{A}_K^{\text{fin}})/Z(\mathbb{F}_q)$ occurs as a continuous quotient of the fundamental group of the affine line over $\mathbb{C}$.

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