Exact Computation of the Hypergraph Turán Function for Expanded Complete 2-Graphs

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Abstract

Let \( l > k \geq 3 \). Let the \( k \)-graph \( H_l^{(k)} \) be obtained from the complete 2-graph \( K_l^{(2)} \) by enlarging each edge with a new set of \( k - 2 \) vertices. Mubayi [“A hypergraph extension of Turán’s theorem”, to appear in J. Combin. Th. (B)] computed asymptotically the Turán function \( \text{ex}(n, H_l^{(k)}) \). Here we determine the exact value of \( \text{ex}(n, H_l^{(k)}) \) for all sufficiently large \( n \), settling a conjecture of Mubayi.

1 Introduction

For \( k, l \geq 2 \) let \( \mathcal{K}_l^{(k)} \) be the family of all \( k \)-graphs \( F \) with at most \( \binom{l}{2} \) edges such that for some \( l \)-set \( L \) (called the core) every pair \( x, y \in L \) is covered by an edge of \( F \). Let the \( k \)-graph \( H_l^{(k)} \in \mathcal{K}_l^{(k)} \) be obtained from the complete 2-graph \( K_l^{(2)} \) by enlarging each edge with a new set of \( k - 2 \) vertices.

These \( k \)-graphs were recently studied by Mubayi [13] in the context of the Turán ex-function which is defined as follows. Let \( \mathcal{F} \) be a family of \( k \)-graphs. We say that a
A k-graph $G$ is $\mathcal{F}$-free if no $F \in \mathcal{F}$ is a subgraph of $G$. (When we talk about subgraphs, we do not require them to be induced.) Now, the Turán function $\text{ex}(n, \mathcal{F})$ is the maximum size of an $\mathcal{F}$-free $k$-graph $G$ on $n$ vertices. Also, let

$$
\pi(\mathcal{F}) = \lim_{n \to \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{k}}.
$$

(The limit is known to exist, see Katona, Nemetz, and Simonovits [9].)

To obtain the $k$-graph $T^{(k)}(n, l)$, $l \geq k$, partition $[n] = \{1, \ldots, n\}$ into $l$ almost equal parts (that is, of sizes $\lceil \frac{n}{l} \rceil$ and $\lfloor \frac{n}{l} \rfloor$) and take those edges which intersect every part in at most one vertex. Let us, for notational convenience, identify $k$-graphs with their edge sets and, for a $k$-graph $F$, write $\text{ex}(n, F)$ for $\text{ex}(n, \{F\})$, etc.

Mubayi [13, Theorem 1] proved the following result.

**Theorem 1 (Mubayi)** Let $n \geq l \geq k \geq 3$. Then $\text{ex}(n, \mathcal{K}_{l+1}^{(k)}) = |T^{(k)}(n, l)|$, and $T^{(k)}(n, l)$ is the unique maximum $\mathcal{K}_{l+1}^{(k)}$-free $k$-graph of order $n$.

It follows from Theorem 1 and the super-saturation technique of Erdős and Simonovits [3] that $\pi(H_{l+1}^{(k)}) = \pi(\mathcal{K}_{l+1}^{(k)})$, see [13, Theorem 2]. This gave us the first example of a non-degenerate $k$-graph with known Turán’s density for every $k$. (Previously, Frankl [5] did this for all even $k$.) Settling a conjecture posed in [13], we prove that the Turán functions of $H_{l+1}^{(k)}$ and $\mathcal{K}_{l+1}^{(k)}$ coincide for all large $n$.

**Theorem 2** For any $l \geq k \geq 3$ there is $n_0(l, k)$ such that for any $n \geq n_0(l, k)$ we have $\text{ex}(n, H_{l+1}^{(k)}) = |T^{(k)}(n, l)|$, and $T^{(k)}(n, l)$ is the unique maximum $H_{l+1}^{(k)}$-free $k$-graph of order $n$.

**Remark.** Theorem 2 is true for $k = 2$ by the Turán theorem [21]. If $k \geq 3$ and $2 \leq l < k$, then Theorem 2 is false: $\text{ex}(n, \mathcal{K}_{l+1}^{(k)}) = 0$ while $\text{ex}(n, H_{l+1}^{(k)}) > 0$.

**Remark.** We do not compute an explicit upper bound on $n_0(l, k)$ as this would considerably lengthen the paper. (For one thing, we would have to reproduce some proofs from [13] in order to calculate an explicit dependence between the constants there.)
2 Stability of $H_l^{(k)}$

Two $k$-graphs $F$ and $G$ of the same order are $m$-close if we can add or remove at most $m$ edges from the first graph and make it isomorphic to the second; in other words, for some bijection $\sigma : V(F) \to V(G)$ the symmetric difference between $\sigma(F) = \{\sigma(D) : D \in F\}$ and $G$ has at most $m$ edges.

Mubayi [13, Theorem 5] proved that $K_l^{(k)}$ is stable, meaning for the purpose of this article that for any $\varepsilon > 0$ there are $\delta > 0$ and $n_0$ such that any $K_l^{(k)}$-free $k$-graph $G$ of order $n \geq n_0$ and size at least $(\pi(K_l^{(k)}) - \delta) {n \choose k}$ is $\varepsilon {n \choose k}$-close to $T_l^{(k)}(n, l - 1)$. Here we prove the same statement for the single forbidden graph $H_l^{(k)}$, which we will need in the proof of Theorem 2.

Lemma 3 For any $l > k \geq 3$ the $k$-graph $H_l^{(k)}$ is stable, that is, for any $\varepsilon > 0$ there are $\delta = \delta(k, l, \varepsilon) > 0$ and $n_0 = n_0(k, l, \varepsilon)$ such that any $H_l^{(k)}$-free $k$-graph $G$ of order $n \geq n_0$ and size at least $(\pi(H_l^{(k)}) - \delta) {n \choose k}$ is $\varepsilon {n \choose k}$-close to $T_l^{(k)}(n, l - 1)$.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ which establishes the stability of $K_l^{(k)}$ with respect to $\frac{\varepsilon}{2}$. Assume that $\delta \leq \varepsilon$. Let $n$ be large and $G$ be an $H_l^{(k)}$-free $k$-graph on $[n]$ of size at least $(\pi(H_l^{(k)}) - \frac{\delta}{2}) {n \choose k}$.

Let us call a pair $\{x, y\}$ of vertices sparse if it is covered by at most

$$m = \left(l + (k - 2) \binom{l}{2} \right) \binom{n}{k - 3}$$

edges of $G$. Let $G'$ be obtained from $G$ by removing all edges containing sparse pairs, at most $\binom{n}{2} \times m < \frac{\delta}{2} {n \choose k}$ edges.

Let us show that the $k$-graph $G'$ is $K_l^{(k)}$-free. Suppose on the contrary that every pair from some $l$-set $L$ is covered by an edge of $G'$. It follows that every pair $\{x, y\} \subseteq L$ is not sparse with respect to $G$, that is, $G$ has more than $m$ edges containing $\{x, y\}$. This means that if we have a partial embedding of $H_l^{(k)}$ into $G$ with the core $L$, then we can always find a $G$-edge $D \ni x, y$ such that $D \setminus \{x, y\}$ is disjoint from the rest of the embedding. Thus $G$ has an $H_l^{(k)}$-subgraph with the core $L$, a contradiction.
We have $|G'| \geq (\pi(H^{(k)}_l) - \delta)\binom{n}{k}$. By the stability of $K^{(k)}_l$, $G'$ is $\frac{\delta}{2}\binom{n}{k}$-close to $T^{(k)}(n,l-1)$. The triangle inequality implies that $G$ is $(\frac{\delta}{2} + \frac{\varepsilon}{2})\binom{n}{k}$-close to $T^{(k)}(n,l-1)$. As $\delta \leq \varepsilon$, this finishes the proof of the lemma. 

3 Exactness

Proof of Theorem 2. Let us choose, in this order, positive constants $c_1, \ldots, c_5$, each being sufficiently small depending on the previous constants. Then, let $n_0$ be sufficiently large. In fact, we can take some simple explicit functions of $k, l$ for $c_1, \ldots, c_5$. However, $n_0$ should also be at least as large as the function $n_0(k, l + 1, c_5)$ given by Lemma 3.

Let $G$ be a maximum $H^{(k)}(n,l+1)$-free graph on $[n]$ with $n \geq n_0$. We have

$$|G| \geq |T^{(k)}(n,l)| \geq \frac{l(l-1)\ldots(l-k+1)}{lk} \binom{n}{k} = \pi(H^{(k)}_{l+1})\binom{n}{k},$$

where the first inequality follows from the fact that $T^{(k)}(n,l)$ is $H^{(k)}_{l+1}$-free while the second inequality can be shown directly. (For example, a simple averaging shows that the function $|T^{(k)}(n,l)|/\binom{n}{k}$ is decreasing in $n$.)

Let $V_1 \cup \cdots \cup V_l$ be a partition of $[n]$ such that

$$f = \sum_{D \in G} \left| \{i \in [l] : D \cap V_i \neq \emptyset \} \right|$$

is maximum possible. Let $T$ be the complete $l$-partite $k$-graph on $V_1 \cup \cdots \cup V_l$. Clearly, $f \geq k|T \cap G|$. As $n$ is sufficiently large, Lemma 3 implies that $G$ is $c_5\binom{n}{k}$-close to $T^{(k)}(n,l)$. (The value of $\delta > 0$ returned by Lemma 3 is not significant here because of the lower bound on the size of $G$.) The choice of $T$ implies that $f \geq k(|G| - c_5\binom{n}{k})$.

On the other hand, $f \leq k|G| - |G \setminus T|$. It follows that

$$|G \setminus T| \leq c_5k\binom{n}{k}.$$

Thus we have $|T| \geq |T^{(k)}(n,l)| - c_5k\binom{n}{k}$. This bound on $|T|$ can be easily shown to imply (or, alternatively, see Claim 1 in Proof of Theorem 5) that for each $i \in [l]$ we have, for example,

$$|V_i| \geq \frac{n}{2l}.$$
Let us call the edges in \( T \setminus G \) missing and the edges in \( G \setminus T \) bad. As \( |T| \leq |T^{(k)}(n, l)| \) with equality if and only if \( T \) is isomorphic to \( T^{(k)}(n, l) \), see [13, Equation (1)], the number of bad edges is at least the number of missing edges. It also follows that if \( G \subset T \), then we are done. Thus, let us assume that \( B \) is non-empty, where the 2-graph \( B \) consists of all bad pairs, that is, pairs of vertices which come from the same part \( V_i \) and are covered by an edge of \( G \).

For vertices \( x, y \) coming from two different parts \( V_i \), call the pair \( \{x, y\} \) sparse if \( G \) has at most
\[
m = \left( \binom{l+1}{2} (k-2) + l + 1 \right) \binom{n}{k-3}
\]
edges containing both \( x \) and \( y \); otherwise \( \{x, y\} \) is called dense.

Note that there are less than \( c_4 n^2 \) sparse pairs for otherwise we get a contradiction to (2): each sparse pair generates at least
\[
\left( \frac{n}{2l} \right)^{k-2} - m \geq \frac{1}{2} \left( \frac{n}{2l} \right)^{k-2}
\]
missing edges by [3] while each missing edge contains at most \( \binom{k}{2} \) sparse pairs.

Take any bad pair \( \{x_0, x_1\} \), where, for example, \( x_0, x_1 \in V_1 \) are covered by \( D \in G \). The number of vertices in \( H^{(k)}_{l+1} \) is \( \binom{l+1}{2} (k-2) + l + 1 \). Therefore, if we have a partial embedding of \( H^{(k)}_{l+1} \) into \( G \) such that a pair of vertices \( x, y \) from the core is dense, then we can find a \( G \)-edge containing both \( x, y \) and disjoint from the rest of the embedding.

It follows that for any choice of \((x_2, \ldots, x_l)\), where \( x_i \in V_i \setminus D \) for \( 2 \leq i \leq l \), at least one pair \( \{x_i, x_j\} \) with \( \{i, j\} \neq \{0, 1\} \) is sparse. Since \( x_0 \) and \( x_1 \) are fixed, each such sparse pair \( \{x_i, x_j\} \) is counted, very roughly, at most \( n^{l-3} \) times if \( \{x_i, x_j\} \cap \{x_0, x_1\} = \emptyset \), and at most \( n^{l-2} \) times if \( \{x_i, x_j\} \cap \{x_0, x_1\} \neq \emptyset \).

Since we have at most \( c_4 n^2 \) sparse pairs, the number of times the former alternative occurs is at most
\[
c_4 n^2 \times n^{l-3} \leq \frac{1}{2} \left( \frac{n}{2l} \right)^{l-1}.
\]
That is, by [3], for at least half of the choices of \((x_2, \ldots, x_l)\), the obtained sparse pair intersects \( \{x_0, x_1\} \). Let \( A \) consist of those \( z \in V(G) \) which are incident to at least \( c_1 n \) sparse pairs. Since \( \frac{1}{4} \left( \frac{n}{2l} - k \right)^{l-1}/n^{l-2} \geq c_1 n \), at least one of \( x_0 \) and \( x_1 \) belongs to \( A \). Thus, in summary, we have proved that every bad pair intersects \( A \).
Considering the sparse pairs, we obtain by (4) at least
\[
\left|A\right| \times c_1 n \times \frac{1}{2} \left(\frac{n}{2!}\right)^{k-2} \times \left(\frac{k}{2}\right)^{-1} \geq \left|A\right| \times c_2 n^{k-1}.
\]
missing edges and, consequently, at least \(\left|A\right| \times c_2 n^{k-1}\) bad edges. Let \(\mathcal{B}\) consist of the pairs \((D, \{x, y\})\), where \(\{x, y\} \in B, D \in G\) and \(x, y \in D\). (Thus \(D\) is a bad edge.) As each bad edge contains at least one bad pair, we conclude that \(|\mathcal{B}| \geq |A| \times c_2 n^{k-1}\). For any \((D, \{x, y\}) \in \mathcal{B}\), we have \(\{x, y\} \cap A \neq \emptyset\). If we fix \(x\) and \(D\), then, obviously, there are at most \(k - 1\) ways to choose a bad pair \(\{x, y\} \subset D\). Hence, some vertex \(x \in A\), say \(x \in V_1\), belongs to at least
\[
\frac{|\mathcal{B}| \times (k - 1)}{c_2 n^{k-1}} \geq \frac{c_2}{k-1} n^{k-1},
\]
bad edges, each intersecting \(V_1\) in another vertex \(y\).

Let \(Y \subset V_1\) be the neighborhood of \(x\) in the 2-graph \(B\). We have
\[
|Y| \geq \frac{c_2}{k-1} n^{k-1} \times \left(\frac{n}{k-2}\right)^{-1} \geq c_3 n.
\]
For \(j \in [2, l]\) let \(Z_j\) consist of those \(z \in V_j\) for which \(\{x, z\}\) is dense.

Suppose first that \(|Z_j| \geq c_3 n\) for each \(j \in [2, l]\). In this case we do the following. For every \(y \in Y\), fix some \(D_y \in G\) containing both \(x\) and \(y\). Consider an \((l + 1)\)-tuple \(L = (x, y, z_2, z_3, \ldots, z_l)\), where \(y \in Y\) and \(z_j \in Z_j \setminus D_y\) are arbitrary. We can find a partial embedding of \(H_{l+1}^{(k)}\) with core \(L\) such that every pair containing \(x\) is covered: the pair \(\{x, y\}\) is covered by \(D_y\) while each pair \(\{x, z_i\}\) is dense. Since \(G\) is \(H_{l+1}^{(k)}\)-free, at least one pair from the set \(\{y, z_2, \ldots, z_l\}\) is sparse. Since there are at least \((c_3 n - k)^l\) choices of \(L\) (note that \(x\) is fixed), this gives us at least \((c_3 n - k)^l/n^{l-2} > c_4 n^2\) sparse pairs, which is a contradiction as we already know.

Hence, assume that, for example, \(|Z_2| < c_3 n\). This means that all but at most \(c_3 n\) pairs \(\{x, z\}\) with \(z \in V_2\) are sparse, that is, there are at most
\[
c_3 n \times \left(\frac{n}{k-2}\right) + n \times m \leq c_3 n^{k-1}
\]
\(G\)-edges containing \(x\) and intersecting \(V_2\). Let us contemplate moving \(x\) from \(V_1\) to \(V_2\). Some edges of \(G\) may decrease their contribution to \(f\) by 1. But each such edge must contain \(x\) and intersect \(V_2\) so the corresponding total decrease is at most \(c_3 n^{k-1}\) by (4).
On the other hand, the number of edges of $G$ containing $x$, intersecting $V_1 \setminus \{x\}$, and disjoint from $V_2$ is at least $\frac{c_2}{k-1} n^{k-1} - c_3 n^{k-1}$ by (5) and (6). As $c_3$ is much smaller than $c_2$, we strictly increase $f$ by moving $x$ from $V_1$ to $V_2$, a contradiction to the choice of the parts $V_i$. The theorem is proved. 

4 Concluding Remarks

Lemma 3 also follows from the following more general Lemma 4. In order to state the latter result, we need some further definitions.

Let us call a family $F$ of $k$-graphs $s$-stable if for any $\varepsilon > 0$ there are $\delta > 0$ and $n_0$ such that for arbitrary $F$-free $k$-graphs $G_1, \ldots, G_{s+1}$ of the same order $n \geq n_0$, each of size at least $(\pi(F) - \delta)\binom{n}{k}$, some two are $\varepsilon\binom{n}{k}$-close. Please note that if $F$ is $s$-stable for some $s$ then it is also $t$-stable for any $t > s$. Lemma 3 implies that $H_i^{(k)}$ is 1-stable. Let $F[t]$ denote the $t$-blowup of a $k$-graph $F$, where each vertex $x$ is replaced by $t$ new vertices and each edge is replaced by the corresponding complete $k$-partite $k$-graph. Clearly, $|F[t]| = t^k |F|$.

Lemma 4 Let $t \in \mathbb{N}$. Let $F$ be a finite family of $k$-graphs which is $s$-stable. Let $\mathcal{H}$ be another (possibly infinite) $k$-graph family such that for each $F \in \mathcal{F}$ there is $H \in \mathcal{H}$ such that $H \subset F[t]$. If $\pi(\mathcal{H}) \geq \pi(F)$, then $\pi(\mathcal{H}) = \pi(F)$ and $\mathcal{H}$ is $s$-stable.

Proof. Our proof uses the following theorem of Rödl and Skokan [18, Theorem 7.1] which in turn relies on the Hypergraph Regularity Lemma of Rödl and Skokan [19] and the Counting Lemma of Nagle, Rödl, and Schacht [16] (see also Gowers [8]).

Theorem 5 (Rödl and Skokan) For all integers $l > k \geq 2$ and a real $\varepsilon > 0$ there exist $\mu = \mu(k, l, \varepsilon) > 0$ and $n_1 = n_1(k, l, \varepsilon) \in \mathbb{N}$ such that the following statement holds.

Given a $k$-graph $F$ with $v \leq l$ vertices, suppose that a $k$-graph $G$ with $n > n_1$ vertices contains at most $\mu n^v$ copies of $F$ as a subgraph. Then one can delete at most $\varepsilon \binom{n}{k}$ edges of $G$ to make it $F$-free. 

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Let \( \varepsilon > 0 \) be arbitrary. Let \( \delta > 0 \) and \( n_0 \) be constants satisfying the \( s \)-stability assumptions for \( F \) and \( \delta \). Assume that \( \delta \leq \varepsilon \). Let \( l \) be the maximum order of a \( k \)-graph in \( F \) and \( m = |F| \). Let \( \mu = \mu(k,l,\frac{\delta}{3m}) \) and \( n_1 = n_1(k,l,\frac{\delta}{3m}) \) be given by Theorem 5.

Also, assume that \( n_2 \) is so large that for every \( F \in F \) any \( F[t] \)-free \( k \)-graph of order \( n \geq n_2 \) contains at most \( \mu n^{v(F)} \) copies of \( F \), where \( v(F) \) denotes the number of vertices in \( F \). Such \( n_2 \) exists because any \( F[t] \)-free \( k \)-graph \( G \) of order \( n \) has at most \( o(n^{v(F)}) \) copies of \( F \), which follows from a theorem of Erdős [4]. Let \( n_3 = \max(n_0, n_1, n_2) \).

Let \( n \geq n_3 \) and let \( G_1, \ldots, G_{s+1} \) be arbitrary \( \mathcal{H} \)-free \( k \)-graphs each having \( n \) vertices and at least \( (\pi(F) - \frac{\delta}{3}) \binom{n}{k} \) edges. By Theorem 5 (and the choice of \( n_1 \) and \( n_2 \), for each \( F \in F \) each \( G_i \) can be made \( F \)-free by removing at most \( \frac{\delta}{3m} \binom{n}{k} \) edges. Hence, we can transform \( G_i \) into an \( \mathcal{F} \)-free \( k \)-graph \( G'_i \subset G_i \) by removing at most \( |F| \frac{\delta}{3m} \binom{n}{k} \leq \frac{\delta}{3} \binom{n}{k} \) edges.

We conclude that \( \pi(F) \geq \pi(\mathcal{H}) - \frac{\varepsilon}{3} \). As \( \varepsilon > 0 \) was arbitrary, we have \( \pi(F) = \pi(\mathcal{H}) \). Thus the density of each \( G'_i \) is at least \( \pi(\mathcal{H}) - \frac{\delta}{2} - \frac{\delta}{3} > \pi(F) - \delta \). By the \( s \)-stability of \( F \), some two of these graphs, for example, \( G'_i \) and \( G'_j \), are \( \varepsilon \binom{n}{k} \)-close. It follows that \( G_i \) and \( G_j \) are \( \varepsilon \binom{n}{k} \)-close. Thus the constants \( \frac{\delta}{2} \) and \( n_3 \) demonstrate the \( s \)-stability of \( \mathcal{H} \), proving Lemma 4.1.

The line of argument we used in this article might be useful for computing the exact value of \( \text{ex}(n, F) \) for other forbidden \( k \)-graphs \( F \). The approach in general could be the following.

1. Find a suitable \( k \)-graph family \( \mathcal{F} \ni F \) for which we can compute \( \pi(F) \) and prove the stability of \( \mathcal{F} \).

2. Deduce from Lemma 4 that \( \pi(F) = \pi(\mathcal{F}) \) and \( F \) is stable too.

3. Using the stability, obtain the exact value of \( \text{ex}(n, F) \). (The fact that stability often helps in proving exact results for the hypergraph Turán problem was observed and used by Füredi and Simonovits [7], Keevash and Sudakov [12, 11], and others.)

Extending the results by Sidorenko [20], the author [17] has successfully applied the above approach to computing the exact value of \( \text{ex}(n, T^{(4)}) \) for \( n \geq n_0 \), where the \( k \)-graph
$T^{(k)}$ consists of the following three edges: $[k]$, $[2, k+1]$, and $\{1\} \cup [k+1, 2k-1]$. The exact value of $\text{ex}(n, T^{(3)})$ was previously computed by Frankl and Füredi [6] (see also Bollobás [2], Keevash and Mubayi [10]).

Lemma 3 has an interesting application. Namely, the method of Mubayi and the author [14] (combined with Lemma 3) shows that the pair $(H_{k+2}^{(k)}, K_{k+1}^{(k)})$ is non-principal for any $k \geq 3$, that is,

$$\pi \left( \{H_{k+2}^{(k)}, K_{k+1}^{(k)}\} \right) < \min \left\{ \pi(H_{k+2}^{(k)}), \pi(K_{k+1}^{(k)}) \right\},$$

where $K_m^{(k)}$ denotes the complete $k$-graph of order $m$. This completely answers a question of Mubayi and Rödl [15] (cf. also Balogh [1]). We refer the Reader to [14] for further details.

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