Heat Kernels and Hardy Spaces on Non-Tangentially Accessible Domains with Applications to Global Regularity of Inhomogeneous Dirichlet Problems

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Abstract. Let $n \geq 2$ and $\Omega$ be a bounded non-tangentially accessible domain (for short, NTA domain) of $\mathbb{R}^n$. Assume that $L_D$ is a second-order divergence form elliptic operator having real-valued, bounded, measurable coefficients on $L^2(\Omega)$ with the Dirichlet boundary condition. The main aim of this article is threefold. First, the authors prove that the heat kernels $\{K_{t_D}^{L_D}\}_{t > 0}$ generated by $L_D$ are Hölder continuous. Second, for any $p \in (0, 1]$, the authors introduce the ‘geometrical’ Hardy space $H^p_{r}(\Omega)$ by restricting any element of the Hardy space $H^p(\mathbb{R}^n)$ to $\Omega$, and show that, when $p \in \left(\frac{n}{n+\delta_0}, 1\right]$, $H^p_{r}(\Omega) = H^p(\Omega) = H^p_{L_D}(\Omega)$ with equivalent quasi-norms, where $H^p(\Omega)$ and $H^p_{L_D}(\Omega)$ respectively denote the Hardy space on $\Omega$ and the Hardy space associated with $L_D$, and $\delta_0 \in (0, 1]$ is the critical index of the Hölder continuity for the kernels $\{K_{t_D}^{L_D}\}_{t > 0}$. Third, as applications, the authors obtain the global gradient estimates in both $L^p(\Omega)$, with $p \in (1, p_0)$, and $H^p_{z}(\Omega)$, with $p \in \left(\frac{n}{n+1}, 1\right]$, for the inhomogeneous Dirichlet problem of second-order divergence form elliptic equations on bounded NTA domains, where $p_0 \in (2, \infty)$ is a constant depending only on $n, \Omega$, and the coefficient matrix of $L_D$. Here, the ‘geometrical’ Hardy space $H^p_{r}(\Omega)$ is defined by restricting any element of the Hardy space $H^p(\mathbb{R}^n)$ supported in $\overline{\Omega}$ to $\Omega$, where $\overline{\Omega}$ denotes the closure of $\Omega$ in $\mathbb{R}^n$. It is worth pointing out that the range $p \in (1, p_0)$ for the global gradient estimate in the scale of Lebesgue spaces $L^p(\Omega)$ is sharp and the above results are established without any additional assumptions on both the coefficient matrix of $L_D$, and the domain $\Omega$.

1 Introduction

The study of elliptic value problems on non-smooth domains of $\mathbb{R}^n$ has a long history (see, for instance, [21, 39, 43] and the references therein). In recent years, the research of the global regularity for elliptic equations with rough coefficients on non-smooth domains of $\mathbb{R}^n$ has aroused great interest (see, for instance, [11, 21, 22, 23, 24, 25, 30, 57, 59]). The global regularity estimates of elliptic equations with rough coefficients on the non-smooth domain $\Omega$ of $\mathbb{R}^n$ in the scale of Lebesgue spaces $L^p(\Omega)$, with $p \in (1, \infty)$, have been extensively studied in the existing literatures (see, for instance, the recent survey article [21], the monograph [57], and the references therein).

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However, there exist very few literatures on global regularity estimates of elliptic equations with rough coefficients on the non-smooth domain $\Omega$ of $\mathbb{R}^n$ in the scale of Hardy spaces $H^p(\Omega)$ with $p \in (0, 1]$.

Let $n \geq 2$ and $\Omega$ be a bounded non-tangentially accessible domain (for short, NTA domain) of $\mathbb{R}^n$. Assume that $L_D$ is a second-order divergence form elliptic operator having real-valued, bounded, measurable coefficients on $L^2(\Omega)$ with the Dirichlet boundary condition. The main aim of this article is threefold. First, we prove that the heat kernels $\{K^{L_D}_{t}\}_{t>0}$ generated by $L_D$ are Hölder continuous. Second, for any $p \in (0, 1]$, we introduce the ‘geometrical’ Hardy space $H^p(\Omega)$ by restricting any element of the Hardy space $H^p(\mathbb{R}^n)$ to $\Omega$, and show that, when $p \in (\frac{n}{n+\delta_0}, 1]$, $H^p(\Omega) = H^p(\Omega) = H^p_{L_D}(\Omega)$ with equivalent quasi-norms, where $H^p(\Omega)$ and $H^p_{L_D}(\Omega)$ respectively denote the Hardy space on $\Omega$ and the Hardy space associated with $L_D$, and $\delta_0 \in (0, 1]$ is the critical index of the Hölder continuity for the kernels $\{K^{L_D}_{t}\}_{t>0}$. Third, as applications, for the inhomogeneous Dirichlet boundary value problem

\begin{equation}
\begin{cases}
-\text{div}(Au) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where the matrix $A$ is real-valued, bounded, and measurable, and satisfies the uniform ellipticity condition [see (1.3) below for the details], and $\partial \Omega$ denotes the boundary of $\Omega$, we obtain the global gradient estimates of the weak solution $u$ in both Lebesgue spaces $L^p(\Omega)$, with $p \in (1, p_0)$, and Hardy spaces $H^p(\Omega)$, with $p \in (\frac{n}{n+\delta_0}, 1]$, where $p_0 \in (2, \infty)$ is a constant depending only on $n$, $\Omega$, and the coefficient matrix $A$. Here, the ‘geometrical’ Hardy space $H^p(\Omega)$ is defined by restricting any element of the Hardy space $H^p(\mathbb{R}^n)$ supported in $\overline{\Omega}$ to $\Omega$, where $\overline{\Omega}$ denotes the closure of $\Omega$ in $\mathbb{R}^n$. Meanwhile, it is worth pointing out that the range $p \in (1, p_0)$ of $p$ for the global gradient estimate in the scale of the Lebesgue space $L^p(\Omega)$ is sharp [see Remark 1.10(i) below for the details].

Compared with the global regularity estimate of elliptic equations on the non-smooth domain $\Omega$ of $\mathbb{R}^n$ in Lebesgue spaces $L^p(\Omega)$ established in [1, 11, 20, 22, 59], we obtain the global regularity estimate for the Dirichlet problem (1.1) without any additional assumptions on both the coefficient matrix $A$ and the domain $\Omega$. Recall that the global gradient estimate in $L^p(\Omega)$ with any given $p \in (1, \infty)$ for the Dirichlet problem (1.1), with $f$ replaced by div($f$), was established by Di Fazio [20], under the assumptions that $A \in \text{VMO}(\mathbb{R}^n; \mathbb{R}^n)$ (see, for instance, [56]) and $\partial \Omega \in C^{1,1}$, which was weakened to $\partial \Omega \in C^1$ by Auscher and Qafsaoui [1]. Moreover, the global gradient estimate in $L^p(\Omega)$ with any given $p \in (1, \infty)$ for the problem (1.1), with $f$ replaced by div($f$), was obtained by Byun and Wang [11], under the assumptions that $A$ satisfies the $(\delta, R)$-BMO condition (see, for instance, [11] or Definition 2.6 below for its definition) for sufficiently small $\delta \in (0, \infty)$, and that $\Omega$ is a bounded Reifenberg flat domain of $\mathbb{R}^n$ (see, for instance, [53, 64] or Remark 2.5(i) below for its definition). Furthermore, for the Dirichlet problem (1.1) with $f$ replaced by div($f$), the global gradient estimate in $L^p(\Omega)$ with any given $p \in (1, \infty)$ was established by Dong and Kim [22, 23], under the assumptions that $A$ has partial sufficiently small BMO coefficients and that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with small Lipschitz constant, or a bounded Reifenberg flat domain. Meanwhile, for the problem (1.1) with $f$ replaced by div($f$), the global gradient estimate in $L^p(\Omega)$, with any given $p \in (\frac{4n}{n+3}, 3+\epsilon)$ when $n \geq 3$, or $p \in (\frac{4n}{n+3}, 4+\epsilon)$ when $n = 2$, was obtained by Shen [59], under the assumptions that $A \in \text{VMO}(\mathbb{R}^n; \mathbb{R}^n)$ and that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz...
domain, where \( \varepsilon \in (0, \infty) \) is a constant depending only on \( n \) and \( \Omega \).

Moreover, NTA domains considered in this article were originally introduced by Jerison and Kenig [38] when studying the boundary behavior of harmonic functions. We point out that NTA domains have a wide generality and contain Lipschitz domains, BMO\(_1\) domains, quasi-spheres, and some Reifenberg flat domains as special examples (see, for instance, \([38, 44, 64]\)). Furthermore, NTA domains are closely related to the theory of quasi-conformal mappings (see, for instance, \([38, 41]\) and the references therein).

To describe the main results of this article, we first recall some necessary notions. Let \( \Omega \) be a bounded NTA domain of \( \mathbb{R}^n \) as in Definition 2.1 below (see also \([38]\)) and \( p \in (0, \infty) \). Recall that the Lebesgue space \( L^p(\Omega) \) is defined by setting

\[
L^p(\Omega) := \left\{ f \text{ is measurable on } \Omega : \|f\|_{L^p(\Omega)} := \left[ \int_{\Omega} |f(x)|^p \, dx \right]^{1/p} < \infty \right\}.
\]

Moreover, for any given \( m \in \mathbb{N} \), let

\[
L^p(\Omega; \mathbb{R}^m) := \{ f := (f_1, \ldots, f_m) : \text{ for any } i \in \{1, \ldots, m\}, f_i \in L^p(\Omega) \}
\]

and

\[
\|f\|_{L^p(\Omega; \mathbb{R}^m)} := \sum_{i=1}^{m} \|f_i\|_{L^p(\Omega)}.
\]

Denote by \( W^{1,p}(\Omega) \) the Sobolev space on \( \Omega \) equipped with the norm

\[
\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega; \mathbb{R}^n)},
\]

where \( \nabla f \) is the distributional gradient of \( f \). Furthermore, \( W^{1,p}_0(\Omega) \) is defined to be the closure of \( C_c^\infty(\Omega) \) in \( W^{1,p}(\Omega) \), where \( C_c^\infty(\Omega) \) denotes the set of all infinitely differentiable functions on \( \Omega \) with compact support contained in \( \Omega \).

For any given \( x \in \Omega \), let \( A(x) := [a_{ij}(x)]_{i,j=1}^n \) denote an \( n \times n \) matrix with real-valued, bounded, and measurable entries. Then \( A \) is said to satisfy the uniform ellipticity condition if there exists a positive constant \( \mu_0 \in (0, 1) \) such that, for any \( x \in \Omega \) and \( \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \),

\[
\mu_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu_0^{-1}|\xi|^2.
\]

Denote by \( L_D \) the maximal-accretive operator (see, for instance, \([51, p. 23, \text{Definition 1.46}]\) for the definition) on \( L^2(\Omega) \) with the largest domain \( \mathcal{D}(L_D) \subset W^{1,2}_0(\Omega) \) such that, for any \( f \in \mathcal{D}(L_D) \) and \( g \in W^{1,2}_0(\Omega) \),

\[
(L_D f, g) = \int_{\Omega} A(x)\nabla f(x) \cdot \nabla g(x) \, dx,
\]

where \((\cdot, \cdot)\) denotes the interior product in \( L^2(\Omega) \). In this sense, for any \( f \in \mathcal{D}(L_D) \), we write

\[
L_D f := -\text{div}(A\nabla f).
\]
Let $\{K_t^{L_D}\}_{t>0}$ be the kernels of the semigroup $\{e^{-tL_D}\}_{t>0}$. By [19, Corollary 3.2.8] (see also [3]), we find that there exist positive constants $C$ and $c$ such that, for any $t \in (0, \infty)$ and $x, y \in \Omega$,

\[ |K_t^{L_D}(x, y)| \leq \frac{C}{t^{n/2}} \exp \left\{ -\frac{|x-y|^2}{ct} \right\}. \tag{1.5} \]

Furthermore, it is worth pointing out that the upper and the lower bound estimates, and the Hölder continuity of the heat kernels play a key roles in the study of the well-posedness of some parabolic partial differential equations, real-variable characterizations of some function spaces, and some Sobolev-type inequalities (see, for instance, [19, 33, 55]).

Now, we state the main results of this article as follows; see Definitions 2.6 and 2.4 below, respectively, for the definitions of both the $(\gamma, R)$-BMO condition and the $(\gamma, \sigma, R)$ quasi-convex domain.

**Theorem 1.1.** Let $n \geq 2$, $\Omega$ be a bounded NTA domain of $\mathbb{R}^n$, the real-valued, bounded, and measurable matrix $A$ satisfy (1.3), and $L_D$ be as in (1.4). Denote by $\{K_t^{L_D}\}_{t>0}$ the heat kernels generated by $L_D$.

(i) Then there exists a constant $\delta_0 \in (0, 1]$, depending only on $n$, $A$, and $\Omega$, such that, for any given $\delta \in (0, \delta_0)$, there exist constants $C$, $c \in (0, \infty)$ such that, for any $t \in (0, \infty)$ and $x, \gamma_1, \gamma_2 \in \Omega$ with $|\gamma_1 - \gamma_2| \leq \sqrt{t}/2$,

\[ |K_t^{L_D}(x, \gamma_1) - K_t^{L_D}(x, \gamma_2)| \leq \frac{C}{t^{n/2}} \left[ \frac{|\gamma_1 - \gamma_2|}{\sqrt{t}} \right]^\delta \exp \left\{ -\frac{|x-\gamma_1|^2}{ct} \right\}. \tag{1.6} \]

(ii) For any given $\delta_0 \in (0, 1]$, there exists a constant $\gamma_0 \in (0, \infty)$, depending only on $\delta_0$, $n$, and $\Omega$, such that, if $A$ satisfies the $(\gamma, R)$-BMO condition and $\Omega$ is a $(\gamma, \sigma, R)$ quasi-convex domain for some $\gamma \in (0, \gamma_0)$, $\sigma \in (0, 1)$, and $R \in (0, \infty)$, then, for any given $\delta \in (0, \delta_0)$, there exist constants $C$, $c \in (0, \infty)$ such that, for any $t \in (0, \infty)$ and $x, \gamma_1, \gamma_2 \in \Omega$ with $|\gamma_1 - \gamma_2| \leq \sqrt{t}/2$, (1.6) holds true.

**Remark 1.2.** We point out that, when $\Omega$ is a bounded Lipschitz domain of $\mathbb{R}^n$, the conclusion of Theorem 1.1(i) is well known (see, for instance, [3]). Moreover, when $\Omega$ is a bounded semi-convex domain of $\mathbb{R}^n$ (see, for instance, [48, 49] or Remark 2.5(ii) below for the details), and $L_D := -\Delta_D$ with $\Delta_D$ being the Laplace operator with the Dirichlet boundary condition on $\Omega$, Theorem 1.1(ii) was obtained in [26, Lemma 2.7]. Recall that the bounded semi-convex domain $\Omega$ is a $(\gamma, \sigma, R)$ quasi-convex domain for any $\gamma \in (0, 1)$, some $\sigma \in (0, 1)$, and some $R \in (0, \infty)$ (see, for instance, [69]). Thus, Theorem 1.1(ii) essentially improves [26, Lemma 2.7] by weakening the assumptions on both the matrix $A$ and the domain $\Omega$.

When $n \geq 3$, we prove Theorem 1.1(i) by using an upper estimate for the Green function associated with $L_D$ in terms of distance functions (see, for instance, [35, Remark 4.9]), the Harnack inequality (see, for instance, [31, Theorem 8.22]), and the functional calculus associated with $L_D$. Precisely, using the upper estimate for the Green function of $L_D$ in terms of distance functions, and the Harnack inequality, and borrowing some ideas from the proof of Grüter and Widman [32,
Theorem (1.9)), we prove the Hölder continuity of the Green function associated with \( L_D \). Moreover, applying the Hölder continuity of the Green function, and the functional calculus associated with \( L_D \), and borrowing some ideas from the proofs of Duong et al. [26, Lemmas 2.6 and 2.7], we prove Theorem 1.1(i) in the case of \( n \geq 3 \). When \( n = 2 \), we show Theorem 1.1(i) via establishing the global gradient estimate for the Dirichlet problem (1.1) in \( L^p(\Omega) \) with some \( p \in (2, \infty) \) (see Lemma 3.6 below), and using the Sobolev embedding theorem. Furthermore, we prove Theorem 1.1(ii) by establishing the global gradient estimate for the Dirichlet problem (1.1) in \( L^p(\Omega) \) for sufficiently large \( p \in (n, \infty) \) (see Lemma 3.9 below), and applying the Sobolev embedding theorem.

Next, we recall the definitions of the Hardy space \( \mathcal{H}^p(\mathbb{R}^n) \), the ‘geometrical’ Hardy spaces \( \mathcal{H}^p_0(\Omega) \) and \( \mathcal{H}^p(\Omega) \), the Hardy space \( \mathcal{H}^p(\Omega) \), and the Hardy space \( \mathcal{H}^p_{L_D}(\Omega) \) associated with \( L_D \).

Denote by \( \mathcal{S}(\mathbb{R}^n) \) the space of all Schwartz functions equipped with the well-known topology determined by a countable family of norms, and by \( \mathcal{S}'(\mathbb{R}^n) \) its dual space (namely, the space of all tempered distributions) equipped with the weak-* topology. Let \( \mathcal{D}(\Omega) \) denote the space of all infinitely differentiable functions with compact support in \( \Omega \) equipped with the inductive topology, and \( \mathcal{D}'(\Omega) \) its topological dual equipped with the weak-* topology, which is called the space of distributions on \( \Omega \).

In what follows, for any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), we always let \( B(x, r) := \{ y \in \mathbb{R}^n : |y - x| < r \} \).

**Definition 1.3.** Let \( p \in (0, 1] \) and \( \Omega \) be a domain of \( \mathbb{R}^n \).

(i) The Hardy space \( \mathcal{H}^p(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that \( f^+ \in L^p(\mathbb{R}^n) \) equipped with the quasi-norm \( ||f||_{\mathcal{H}^p(\mathbb{R}^n)} := ||f^+||_{L^p(\mathbb{R}^n)} \), where the radial maximal function \( f^\ast \) of \( f \) is defined by setting, for any \( x \in \mathbb{R}^n \),

\[
f^\ast(x) := \sup_{t \in (0, \infty)} |e^{-t\Delta}(f)(x)|.
\]

Here, \( \{e^{-t\Delta}\}_{t \geq 0} \) denotes the heat semigroup generated by the Laplace operator \( \Delta \) on \( \mathbb{R}^n \).

(ii) The Hardy space \( \mathcal{H}^p_0(\Omega) \) is defined by setting

\[
\mathcal{H}^p_0(\Omega) := \{ f \in \mathcal{H}^p(\mathbb{R}^n) : f = 0 \text{ on } \overline{\Omega}^C \} / \{ f \in \mathcal{H}^p(\mathbb{R}^n) : f = 0 \text{ on } \Omega \}.
\]

Here and thereafter, \( \overline{\Omega} \) and \( \overline{\Omega}^C \) denote, respectively, the closure of \( \Omega \) in \( \mathbb{R}^n \), and the complementary set of \( \Omega \) in \( \mathbb{R}^n \). Moreover, the quasi-norm of the element in \( \mathcal{H}^p_0(\Omega) \) is defined to be the quotient norm, namely, for any \( f \in \mathcal{H}^p_0(\Omega) \),

\[
||f||_{\mathcal{H}^p_0(\Omega)} := \inf \left\{ ||F||_{\mathcal{H}^p(\mathbb{R}^n)} : F \in \mathcal{H}^p(\mathbb{R}^n), F = 0 \text{ on } \overline{\Omega}^C, \text{ and } F|_{\Omega} = f \right\},
\]

where the infimum is taken over all \( F \in \mathcal{H}^p(\mathbb{R}^n) \) satisfying \( F = 0 \) on \( \overline{\Omega}^C \), and \( F = f \) on \( \Omega \).

(iii) A distribution \( f \in \mathcal{D}'(\Omega) \) is said to belong to the Hardy space \( \mathcal{H}^p(\Omega) \) if \( f \) is the restriction to \( \Omega \) of a distribution \( F \) in \( \mathcal{H}^p(\mathbb{R}^n) \), namely,

\[
\mathcal{H}^p(\Omega) := \{ f \in \mathcal{D}'(\Omega) : \text{ there exists an } F \in \mathcal{H}^p(\mathbb{R}^n) \text{ such that } F|_{\Omega} = f \}.
\]
where and \( B \) operators, has aroused great interests (see, for instance, [10, 28, 36, 37, 60] for Hardy spaces on \( \mathbb{H} \) in various fields of analysis and partial differential equations. In recent years, the study on the real-variable theory of Hardy spaces on domain \( \Omega \) was initiated by Stein and Weiss [62] and then systematically developed by Fefferman and Stein [29], plays important roles in various fields of analysis and partial differential equations. In recent years, the study on the real-variable theory of Hardy spaces on \( \mathbb{H} \) or its domains, associated with different differential operators, has aroused great interests (see, for instance, [10, 28, 36, 37, 60] for Hardy spaces on \( \mathbb{H} \), and [2, 7, 9, 12, 26, 46, 65, 66, 68] for Hardy spaces on domains). Moreover, the Hardy space \( H^p(\Omega) \) on the domain \( \Omega \) of \( \mathbb{H} \) was introduced and studied by Miyachi [50]. Furthermore, when \( \Omega \) is a Lipschitz domain of \( \mathbb{H} \), the ‘geometrical’ Hardy spaces \( H^p_\Sigma(\Omega) \) and \( H^p_\Gamma(\Omega) \) on domains were

\[
H^p(\mathbb{R}^n) = \{ F \in H^p(\mathbb{R}^n) : F = 0 \text{ on } \Omega \}.
\]

Moreover, for any \( f \in H^p(\Omega) \), the quasi-norm \( \|f\|_{H^p(\Omega)} \) of \( f \) in \( H^p(\Omega) \) is defined by setting

\[
\|f\|_{H^p(\Omega)} := \inf \{ \|F\|_{H^p(\mathbb{R}^n)} : F \in H^p(\mathbb{R}^n) \text{ and } F|_{\Omega} = f \},
\]

where the infimum is taken over all \( F \in H^p(\mathbb{R}^n) \) satisfying \( F = f \) on \( \Omega \).

(iv) Let \( \phi \in C_c^\infty(\mathbb{R}^n) \) be a nonnegative function such that

\[
\text{supp} (\phi) := \{ x \in \mathbb{R}^n : \phi(x) \neq 0 \} \subset B(0, 1)
\]

and \( \oint_{\mathbb{R}^n} \phi(x) \, dx = 1 \), here and thereafter, \( 0 \) denotes the origin of \( \mathbb{R}^n \). For any \( f \in \mathcal{D}'(\Omega) \), the radial maximal function \( f^+_\Omega(x) \) is defined by setting, for any \( x \in \Omega \),

\[
f^+_\Omega(x) := \sup_{t \in (0, \delta(x)/2)} |\phi_t * f(x)|,
\]

where, for any \( x \in \Omega \), \( \delta(x) := \text{dist} (x, \Omega^c) \) and, for any \( t \in (0, \infty) \), \( \phi_t(\cdot) := \frac{1}{t^n} \phi\left(\frac{\cdot}{t}\right) \). Then the Hardy space \( H^p(\Omega) \) is defined by setting

\[
H^p(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : \|f\|_{H^p(\Omega)} := \left\| f^+_\Omega \right\|_{L^p(\Omega)} < \infty \right\}.
\]

Let \( p \in (0, 1] \). From the definitions of both \( H^p_\Sigma(\Omega) \) and \( H^p_\Gamma(\Omega) \), it follows that \( H^p_\Sigma(\Omega) \subset H^p_\Gamma(\Omega) \). Moreover, by [15], we find that \( H^p_\Sigma(\Omega) \subset H^p(\Omega) \).

**Definition 1.4.** Let \( n \geq 2 \), \( \Omega \) be a bounded NTA domain of \( \mathbb{R}^n \), \( p \in (0, 1] \), and \( L_D \) be as in (1.4). For any \( f \in L^2(\Omega) \), the Lusin area function, \( S_{L_D}(f) \), associated with \( L_D \), is defined by setting, for any \( x \in \Omega \),

\[
S_{L_D}(f)(x) := \left[ \int_{\Gamma(x)} \left| r^2 L_D e^{-r^2 L_D(f)}(y) \right|^2 \frac{dy \, dt}{|B_\Omega(x, t)|t} \right]^{1/2},
\]

where \( \Gamma(x) := \{ (y, t) \in \Omega \times (0, \infty) : |x - y| < t \} \) and \( B_\Omega(x, t) := B(x, t) \cap \Omega \).

A function \( f \in L^2(\Omega) \) is said to be in the set \( \mathbb{H}^p_{L_D}(\Omega) \) if \( S_{L_D}(f) \in L^p(\Omega) \); moreover, for any \( f \in \mathbb{H}^p_{L_D}(\Omega) \), define \( \|f\|_{H^p_{L_D}(\Omega)} := \|S_{L_D}(f)\|_{L^p(\Omega)} \). Then the Hardy space \( H^p_{L_D}(\Omega) \) is defined to be the completion of \( \mathbb{H}^p_{L_D}(\Omega) \) with respect to the quasi-norm \( \| \cdot \|_{H^p_{L_D}(\Omega)} \).

It is well known that the real-variable theory of Hardy spaces on \( \mathbb{H} \), initiated by Stein and Weiss [62] and then systematically developed by Fefferman and Stein [29], plays important roles in various fields of analysis and partial differential equations. In recent years, the study on the real-variable theory of Hardy spaces on \( \mathbb{H} \) or its domains, associated with different differential operators, has aroused great interests (see, for instance, [10, 28, 36, 37, 60] for Hardy spaces on \( \mathbb{H} \), and [2, 7, 9, 12, 26, 46, 65, 66, 68] for Hardy spaces on domains). Moreover, the Hardy space \( H^p(\Omega) \) on the domain \( \Omega \) of \( \mathbb{H} \) was introduced and studied by Miyachi [50]. Furthermore, when \( \Omega \) is a Lipschitz domain of \( \mathbb{H} \), the ‘geometrical’ Hardy spaces \( H^p_\Sigma(\Omega) \) and \( H^p_\Gamma(\Omega) \) on domains were
introduced by Chang et al. [15, 16] which naturally appear in the study of the regularity of the Dirichlet and the Neumann boundary value problems of second-order elliptic equations (see, for instance, [2, 13, 15, 26]).

Then we have the following equivalence relation between $H^p_0(\Omega)$, $H^p(\Omega)$, and $H^p_{L_D}(\Omega)$.

**Theorem 1.5.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, and $\delta_0 \in (0, 1]$ as in Theorem 1.1. Then, for any given $p \in (\frac{n}{n+1}, 1]$, $H^p_0(\Omega) = H^p(\Omega)$ with equivalent quasi-norms. Moreover, for any given $p \in (\frac{n}{n+1}, 1]$ and $\Omega$ is a bounded Lipschitz domain, the equivalence of the spaces $H^p_{L_D}(\Omega)$, $H^p(\Omega)$, and $H^p(\Omega)$ in Theorem 1.5 was established by Bui et al. [9, Theorem 4.4 and Remark 4.5(c)] (see also [68, Corollary 4.5]). Thus, Theorem 1.5 improves the results of Auscher and Russ [2, Theorem 1 and Proposition 5] and Bui et al. [9, Theorem 4.4 and Remark 4.5(c)] by weakening the assumptions on both the domain $\Omega$ and the operator $L_D$.

Furthermore, when $\Omega$ is a bounded Lipschitz domain, the equivalence of $H^p_0(\Omega)$ and $H^p(\Omega)$ for any given $p \in (\frac{n}{n+1}, 1]$ was essentially proved by Chang et al. [15, Theorem 2.7]. Recall that NTA domains contain Lipschitz domains as special examples (see, for instance, [38, 64] or Remark 2.5(iii) below). Therefore, the equivalence of $H^p_0(\Omega)$ and $H^p(\Omega)$ for any given $p \in (\frac{n}{n+1}, 1]$ obtained in Theorem 1.5 improves [15, Theorem 2.7] via weakening the assumption on the domain $\Omega$ under consideration.

By subtly using some geometrical properties of NTA domains, obtained in Lemma 2.2 below, the reflection technology related to NTA domains, the atomic characterizations of both $H^p(\mathbb{R}^n)$ and $H^p(\Omega)$, and the Hölder continuity of the heat kernels $\{K^{L_D}_{r}\}_{r>0}$ given in Theorem 1.1, we show Theorem 1.5.

Let $n \geq 2$, $p \in (1, \infty)$, and $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain. Assume that

\begin{equation}
(1.7) \quad p_* := \begin{cases} 
\frac{n p}{n + p} & \text{when } p \in \left(\frac{n}{n - 1}, \infty\right), \\
1 + \epsilon & \text{when } p \in \left(1, \frac{n}{n - 1}\right],
\end{cases}
\end{equation}

where $\epsilon \in (0, \infty)$ is an arbitrary given constant. Let $p \in (1, \infty)$, $f \in L^p(\Omega)$, and the real-valued, bounded, and measurable matrix $A$ satisfy (1.3). Then a function $u$ is called a weak solution of the Dirichlet boundary value problem (1.1) if $u \in W^{1,p}_0(\Omega)$ and, for any $\varphi \in C^\infty_c(\Omega)$,

\begin{equation}
(1.8) \quad \int_\Omega A(x) \nabla u(x) \cdot \nabla \varphi(x) \, dx = \int_\Omega f(x) \varphi(x) \, dx.
\end{equation}

Moreover, the Dirichlet problem (1.1) is said to be uniquely solvable if, for any $f \in L^p(\Omega)$, there exists a unique $u \in W^{1,p}_0(\Omega)$ such that (1.8) holds true for any $\varphi \in C^\infty_c(\Omega)$.

**Remark 1.7.** Assume that $f \in L^2(\Omega)$, where 2, is as in (1.7) with $p$ replaced by 2. Then, by the Sobolev inequality (see, for instance, [5, Theorem 1.1]) and the Lax–Milgram theorem (see, for
instance, [31, Theorem 5.8]), we find that the Dirichlet problem (1.1) is uniquely solvable and the weak solution \( u \) satisfies

\[
\|\nabla u\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)},
\]

where \( C \) is a positive constant independent of \( u \) and \( f \).

For the Dirichlet problem (1.1) on bounded NTA domains, we have the following global regularity estimates in both Lebesgue spaces \( L^q(\Omega) \) with \( q \in (1, p_0) \), and Hardy spaces \( H^p_2(\Omega) \) with \( q \in \left( \frac{n}{n+1}, 1 \right) \), where \( p_0 \in (2, \infty) \) is a constant depending only on \( n \), the domain \( \Omega \), and the matrix \( A \).

**Theorem 1.8.** Let \( n \geq 2 \), \( \Omega \) be a bounded NTA domain of \( \mathbb{R}^n \), and the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3).

(i) Then there exists a \( p_0 \in (2, \infty) \), depending only on \( n \), \( A \), and \( \Omega \), such that, for any given \( q \in \left( \frac{n}{n+1}, p_0 \right) \) and \( p \in (1, n) \) satisfying \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), and any \( f \in L^p(\Omega) \cap L^2(\Omega) \), the weak solution \( u \) of the problem (1.1) uniquely exists and satisfies \( \nabla u \in L^q(\Omega; \mathbb{R}^n) \) and

\[
\|\nabla u\|_{L^q(\Omega; \mathbb{R}^n)} \leq C\|f\|_{L^p(\Omega)},
\]

where \( C \) is a positive constant independent of \( u \) and \( f \).

(ii) Let \( q \in (1, \frac{n}{n+1}] \) and \( p \in (\frac{n}{n+1}, 1] \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), and \( f \in H^p_{LD}(\Omega) \cap L^2(\Omega) \). Then the weak solution \( u \) of the problem (1.1) uniquely exists and satisfies \( \nabla u \in L^q(\Omega; \mathbb{R}^n) \) and

\[
\|\nabla u\|_{L^q(\Omega; \mathbb{R}^n)} \leq C\|f\|_{H^p_{LD}(\Omega)},
\]

where \( C \) is a positive constant independent of \( u \) and \( f \).

(iii) Let \( q \in (\frac{n}{n+1}, 1] \) and \( p \in (\frac{n}{n+2}, \frac{n}{n+1}] \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), and \( f \in H^p_{LD}(\Omega) \cap L^2(\Omega) \). Then the weak solution \( u \) of the problem (1.1) uniquely exists and satisfies \( \nabla u \in H^q_{2}(\Omega; \mathbb{R}^n) \) and

\[
\|\nabla u\|_{H^q_{2}(\Omega; \mathbb{R}^n)} \leq C\|f\|_{H^p_{LD}(\Omega)},
\]

where \( C \) is a positive constant independent of \( u \) and \( f \). Here and thereafter, the space \( H^p_{2}(\Omega; \mathbb{R}^n) \) is defined as \( L^p(\Omega; \mathbb{R}^n) \) via \( L^p(\Omega) \) replaced by \( H^p_{2}(\Omega) \) [see (1.2)].

We prove Theorem 1.8 by the following strategy. We first obtain the global gradient estimate for the problem (1.1) in \( L^q(\Omega) \) with \( q \in (2, p_0) \), by using (1.9), a reverse Hölder inequality for the gradient of the weak solution of some local Dirichlet boundary value problems (see Lemma 3.8 below for the details), and a real-variable argument for \( L^q(\Omega) \) estimates, essentially established by Shen [58, Theorem 3.4] (see also [57, Theorem 4.2.6] and [59, Theorem 3.3]). Then we show (i) by the global gradient estimate in \( L^q(\Omega) \) with \( q \in (2, p_0) \), the conclusion of (ii), and the complex interpolation theory of Hardy spaces (see, for instance, [42, Theorem 8.1 and (9.3)]). Moreover, we prove (ii) via establishing some fine estimate for the kernels \( \{K_t^{L,p}\}_{t>0} \) (see Lemma 5.1 below), and using the global gradient estimate in \( L^q(\Omega) \) with \( q \in (2, p_0) \), and the molecular characterization of \( H^p_{LD}(\Omega) \). Finally, we show (iii) via using the global gradient estimate in \( L^q(\Omega) \) with \( q \in (2, p_0) \), and the molecular characterization of both \( H^p_{LD}(\Omega) \) and \( H^p(\mathbb{R}^n) \).

As a corollary of Theorems 1.5 and 1.8, we have the following conclusion.
Corollary 1.9. Let \( n \geq 2 \), \( \Omega \) be a bounded NTA domain of \( \mathbb{R}^n \), and \( \delta_0 \in (0, 1] \) as in Theorem 1.1. Assume that \( q \in (1, \frac{n}{n-1}] \) and \( p \in (\frac{n}{n+\delta_0}, 1] \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \). Let \( f \in H^p_0(\Omega) \cap L^2(\Omega) \). Then the weak solution \( u \) of the problem (1.1) uniquely exists and satisfies \( \nabla u \in L^q(\Omega; \mathbb{R}^n) \) and
\[
\|\nabla u\|_{L^q(\Omega; \mathbb{R}^n)} \leq C\|f\|_{H^p_0(\Omega)},
\]
where \( C \) is a positive constant independent of \( u \) and \( f \).

Remark 1.10. (i) By an example given in [4, p. 120] (which is essentially due to C. E. Kenig, as was pointed out in [4, p. 119, Theorem 7]), we find, that even when \( \Omega \subset \mathbb{R}^2 \) is a bounded smooth domain, there exists a real-valued, bounded, and measurable matrix \( A \) satisfying (1.3) such that, for any given \( p \in (2, \infty) \), the weak solution \( u \) of the problem (1.1) with some \( f \in L^2(\Omega) \) satisfies \( |\nabla u| \notin L^p(\Omega) \). Therefore, the range \( q \in (\frac{n}{n-1}, p_0) \) of \( q \) obtained in Theorem 1.8(i) is sharp.

(ii) When \( A := I \) (the identity matrix) and \( \Omega \) is a bounded Lipschitz domain of \( \mathbb{R}^n \), Dahlberg [18, Theorem 1] obtained Theorem 1.8(i) with \( p_0 := 3 + \epsilon_0 \) when \( n \geq 3 \), and with \( p_0 := 4 + \epsilon_0 \) when \( n = 2 \), where \( \epsilon_0 \in (0, \infty) \) is a constant depending only on \( n \) and \( \Omega \). Meanwhile, Dahlberg [18] also showed that the range \( q \in (\frac{n}{n-1}, p_0) \) of \( q \) is sharp in this case. Thus, Theorem 1.8(i) extends the results of Dahlberg [18, Theorem 1] via weakening the assumptions on both the coefficient matrix \( A \) and the domain \( \Omega \).

Moreover, to the best of our knowledge, the global gradient estimates obtained in (ii) and (iii) of Theorem 1.8 and Corollary 1.9 are new even when \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain.

The organization of the remainder of this article is as follows.

In Section 2, we present the definitions of both NTA domains and quasi-convex domains, some geometrical properties of NTA domains, the definition of the \((\gamma, R)\)-BMO condition, and the atomic and the molecular characterizations of the Hardy spaces \( H^p(\mathbb{R}^n) \), \( H^p(\Omega) \), and \( H^p_{L^p}(\Omega) \) associated with \( L_p \).

Section 3 is devoted to the proof of Theorem 1.1. The proofs of Theorems 1.5 and 1.8 are presented, respectively, in Sections 4 and 5.

Finally, we make some conventions on notation. Throughout the whole article, we always denote by \( C \) or \( c \) a positive constant which is independent of the main parameters, but it may vary from line to line. We also use \( C(\gamma, \beta, \ldots) \) to denote a positive constant depending on the indicated parameters \( \gamma, \beta, \ldots \). The symbol \( f \lesssim g \) means that \( f \leq Cg \). If \( f \leq g \) and \( g \leq f \), then we write \( f \sim g \). If \( f \leq Cg \) and \( g = h \) or \( g \leq h \), we then write \( f \leq g \sim h \) or \( f \sim g \leq h \), rather than \( f \leq g \leq h \) or \( f \leq g \leq h \). For each ball \( B := B(x_B, r_B) \) in \( \mathbb{R}^n \), with \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), and \( \alpha \in (0, \infty) \), let \( \alpha B := B(x_B, \alpha r_B) \); furthermore, denote by \( B_\Omega \) the ball \( B \cap \Omega \) in \( \Omega \) with \( B \) being a ball of \( \mathbb{R}^n \). For any subset \( E \) of \( \mathbb{R}^n \), we denote by \( E^c \) the set \( \mathbb{R}^n \setminus E \), and by \( 1_E \) its characteristic function. We also let \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \). The symbol \( [s] \) for any \( s \in \mathbb{R} \) denotes the largest integer not greater than \( s \). For any multi-index \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+ := (\mathbb{Z}_+)^n \), let \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). For any ball \( B \) of \( \mathbb{R}^n \) or of \( \Omega \), let \( S_j(B) := (2^j+1)B \setminus (2^j)B \) for any \( j \in \mathbb{N} \), and \( S_0(B) := 2B \). Moreover, for any \( q \in [1, \infty] \), we denote by \( q' \) its conjugate exponent, namely, \( 1/q + 1/q' = 1 \). Finally, for any measurable set \( E \subset \mathbb{R}^n \) with \( |E| < \infty \), and any \( f \in L^1(E) \), we let
\[
\int_E f(x) \, dx := \frac{1}{|E|} \int_E f(x) \, dx.
\]
2 Preliminaries

In this section, we first recall the definitions of NTA domains, quasi-convex domains, and the \((\gamma, R)\)-BMO condition, and then give some geometrical properties of NTA domains. Moreover, we present the atomic and the molecular characterizations of the Hardy spaces \(H^p(\mathbb{R}^n)\), \(H^p(\Omega)\), and \(H^p_{LD}(\Omega)\) associated with \(L_D\).

2.1 NTA Domains

In this subsection, we first recall the definitions of NTA domains introduced by Jerison and Kenig [38] (see also [44, 64]) and quasi-convex domains introduced by Jia et al. [40], and then state some geometrical properties of NTA domains. We begin with recalling several notions. For any given \(x \in \mathbb{R}^n\) and measurable subset \(E \subset \mathbb{R}^n\), let \(d \operatorname{ist}(x, E) := \inf\{|x - y| : y \in E\}\). Meanwhile, for any measurable subsets \(E, F \subset \mathbb{R}^n\), let \(\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}\) and \(\operatorname{diam}(E) := \sup\{|x - y| : x, y \in E\}\).

**Definition 2.1.** Let \(n \geq 2\), \(\Omega \subset \mathbb{R}^n\) be a domain which means that \(\Omega\) is a connected open set, and \(\Omega^C := \mathbb{R}^n \setminus \Omega\).

(i) The domain \(\Omega\) is said to satisfy the interior [resp., exterior] corkscrew condition if there exist constants \(R \in (0, \infty)\) and \(\sigma \in (0, 1)\) such that, for any \(x \in \partial\Omega\) and \(r \in (0, R)\), there exists a point \(x_0 \in \Omega\) [resp., \(x_0 \in \Omega^C\)], depending on \(x\), such that \(B(x_0, \sigma r) \subset \Omega \cap B(x, r)\) [resp., \(B(x_0, \sigma r) \subset \Omega^C \cap B(x, r)\)].

(ii) The domain \(\Omega\) is said to satisfy the Harnack chain condition if there exist a constant \(m_1 \in (1, \infty)\) and a constant \(m_2 \in (0, \infty)\) such that, for any \(x_1, x_2 \in \Omega\) satisfying

\[
M := \frac{|x_1 - x_2|}{\min\{\operatorname{dist}(x_1, \partial\Omega), \operatorname{dist}(x_2, \partial\Omega)\}} > 1,
\]

there exists a chain \(\{B_i\}_{i=1}^N\) of open Harnack balls, with \(B_i \subset \Omega\) for any \(i \in \{1, \ldots, N\}\), that connects \(x_1\) to \(x_2\); namely, \(x_1 \in B_1, x_2 \in B_N, B_i \cap B_{i+1} \neq \emptyset\) for any \(i \in \{1, \ldots, N - 1\}\), and, for any \(i \in \{1, \ldots, N\}\),

\[
m_1^{-1}\operatorname{diam}(B_i) \leq \operatorname{dist}(B_i, \partial\Omega) \leq m_1\operatorname{diam}(B_i),
\]

where the integer \(N\) satisfies \(N \leq m_2 \log_2 M\).

(iii) The domain \(\Omega\) is called a non-tangentially accessible domain (for short, NTA domain) if \(\Omega\) satisfies the interior and the exterior corkscrew conditions, and the Harnack chain condition.

We point out that NTA domains include Lipschitz domains, BMO\(_1\) domains, Zygmund domains, quasi-spheres, and some Reifenberg flat domains as special examples (see, for instance, [38, 44, 64]). Moreover, it is well known that NTA domains are \(W^{1,p}\)-extension domains with \(p \in [1, \infty)\) (see, for instance, [34, 41]). Meanwhile, NTA domains have the following geometrical properties.

**Lemma 2.2.** Let \(n \geq 2\) and \(\Omega\) be a bounded NTA domain of \(\mathbb{R}^n\).
(i) Then there exists a constant \( C \in (0, 1) \), depending only on \( n \) and \( \Omega \), such that, for any ball \( B \subset \mathbb{R}^n \) with the center \( x_B \in \overline{\Omega} \) and the radius \( r_B \in (0, \text{diam}(\Omega)) \),

\[
|B \cap \Omega| \geq C|B|.
\]

(ii) Let \( B := B(x_B, r_B) \subset \mathbb{R}^n \) be a ball satisfying that \( 2B \subset \Omega \) and \( 4B \cap \Omega^C \neq \emptyset \). Then there exists a ball \( \bar{B} \subset \Omega^C \) such that \( r_{\bar{B}} \sim r_B \) and \( \text{dist}(B, \bar{B}) \sim r_B \), where the positive equivalence constants are independent of both \( B \) and \( \bar{B} \).

(iii) For any ball \( B \subset \mathbb{R}^n \) with the center \( x_B \in \partial \Omega \) and the radius \( r_B \in (0, \text{diam}(\Omega)) \), there exists a ball \( \bar{B} \subset \Omega^C \cap B \) such that \( r_{\bar{B}} \sim r_B \), where the positive equivalence constants are independent of both \( B \) and \( \bar{B} \). In particular, there exists a constant \( C \in (0, 1) \), depending only on \( n \) and \( \Omega \), such that, for any ball \( B \subset \mathbb{R}^n \) with the center \( x_B \in \partial \Omega \) and the radius \( r_B \in (0, \text{diam}(\Omega)) \),

\[
|B \cap \Omega^C| \geq C|B|.
\]

Proof. We first show (i). By the fact that \( \Omega \) is a \( W^{1,p} \)-extension domain with \( p \in [1, \infty) \), and [34, Theorem 2], we conclude that there exist constants \( R_0 \in (0, \infty) \) and \( C_1 \in (0, 1] \) such that, for any given ball \( B := B(x_B, r_B) \) of \( \mathbb{R}^n \) with \( x_B \in \overline{\Omega} \) and \( r_B \in (0, R_0) \),

\[
(2.1) \quad |B \cap \Omega| \geq C_1|B|.
\]

If \( \text{diam}(\Omega) \leq R_0 \), then, from (2.1), it follows that (i) holds true in this case. If \( \text{diam}(\Omega) > R_0 \), then, for any ball \( B := B(x_B, r_B) \) of \( \mathbb{R}^n \) with \( x_B \in \overline{\Omega} \) and \( r_B \in (R_0, \text{diam}(\Omega)) \),

\[
|B \cap \Omega| \geq |B(x_B, R_0) \cap \Omega| \geq C_1|B(x_B, R_0)|
\]

\[
= C_1 \left[ \frac{R_0}{\text{diam}(\Omega)} \right]^n |B(x_B, \text{diam}(\Omega))| \geq C_1 \left[ \frac{R_0}{\text{diam}(\Omega)} \right]^n |B|,
\]

which, together with (2.1), further implies that (i) holds true in this case. This finishes the proof of (i).

Now, we prove (ii). Let \( y_B \in \partial \Omega \) be such that \( |x_B - y_B| = \text{dist}(x_B, \partial \Omega) \). Define \( \ell := |x_B - y_B| \). Then \( \ell \in (2r_B, 4r_B) \). By the exterior corkscrew condition of \( \Omega \), we find that there exists a ball \( \bar{B} := B(x_0, \sigma r) \subset [\Omega^C \cap B(y_B, r)] \), where \( r \in \left( \frac{1}{4} \min\{R, \ell\}, \min\{R, \ell\} \right) \), and \( \sigma \) and \( R \) are as in Definition 2.1. From this, we deduce that

\[
\frac{r_B}{r_{\bar{B}}} \leq \max \left\{ \frac{\text{diam}(\Omega)}{2\sigma R}, \frac{1}{\sigma} \right\}
\]

and

\[
\frac{r_B}{r_{\bar{B}}} \geq \frac{1}{4\sigma},
\]

which implies that \( r_B \sim r_{\bar{B}} \). Moreover, it is easy to see that \( \text{dist}(B, \bar{B}) \geq r_B \) and

\[
\text{dist}(B, \bar{B}) \leq |x_B - y_B| + r = \ell + \frac{1}{\sigma} r_{\bar{B}} \leq r_B.
\]

Therefore, \( \text{dist}(B, \bar{B}) \sim r_B \). This finishes the proof of (ii).
Finally, we show (iii). Let $B := B(x_B, r_B)$ be a ball of $\mathbb{R}^n$ with the center $x_B \in \partial \Omega$ and the radius $r_B \in (0, \text{diam(}\Omega))$. Then, when $r_B \in (0, \min\{R, \text{diam(}\Omega)\})$, where $R \in (0, \infty)$ is as in Definition 2.1, from the exterior corkscrew condition of $\Omega$, we deduce that there exists a ball $\bar{B} := B(x_0, \sigma r_B) \subset [\Omega^C \cap B(x_B, r_B)]$, which further implies that $r_B \sim r_B$ and

$$\left| B(x_B, r_B) \cap \Omega^C \right| \geq |B(x_0, \sigma r_B)| \geq |B(x_B, r_B)|.$$

Thus, (iii) holds true in the case that $r_B \in (0, \min\{R, \text{diam(}\Omega)\})$.

Moreover, when $r_B \in (\min\{R, \text{diam(}\Omega)\}, \text{diam(}\Omega))$, let $\ell := \min\{R, \text{diam(}\Omega)\}/2$. By the exterior corkscrew condition of $\Omega$ again, we conclude that there exists a ball $\bar{B} := B(x_0, \sigma \ell) \subset [\Omega^C \cap B(x_B, \ell)]$, which implies that $r_B \sim r_B$ and

$$\left| B(x_B, r_B) \cap \Omega^C \right| \geq |B(x_B, \ell) \cap \Omega^C| \geq |B(x_B, \ell)| \sim |B(x_B, \text{diam(}\Omega))| \geq |B(x_B, r_B)|.$$  

From this, it follows that (iii) holds true in the case that $r_B \in (\min\{R, \text{diam(}\Omega)\}, \text{diam(}\Omega))$. This finishes the proof of (iii) and hence of Lemma 2.2.

\[ \square \]

**Remark 2.3.** Let $n \geq 2$ and $\Omega$ be a bounded NTA domain of $\mathbb{R}^n$. By Lemma 2.2(i), we find that $(\Omega, |\cdot|, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [17], where $|\cdot|$ denotes the usual norm on $\mathbb{R}^n$, and $dx$ the Lebesgue measure on $\mathbb{R}^n$. Moreover, as a space of homogeneous type, the collection of all balls of $\Omega$ is given by the set

$$\left\{ B_\Omega := B \cap \Omega : \text{any ball } B \subset \mathbb{R}^n \text{ satisfying } x_B \in \overline{\Omega} \text{ and } r_B \in (0, \text{diam(}\Omega)) \right\},$$

where $x_B$ denotes the center of $B$ and $r_B$ the radius of $B$.

Now, we recall the definition of the $(\gamma, \sigma, R)$ quasi-convex domain as follows.

**Definition 2.4.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a domain, $\gamma, \sigma \in (0, 1)$, and $R \in (0, \infty)$. Then $\Omega$ is called a $(\gamma, \sigma, R)$ quasi-convex domain if, for any $x \in \partial \Omega$ and $r \in (0, R]$,

(a) there exists an $x_0 \in \Omega$, depending on $x$, such that $B(x_0, \sigma r) \subset [\Omega \cap B(x, r)]$;
(b) there exists a convex domain $V := V(x, r)$, depending on $x$ and $r$, such that $[B(x, r) \cap \Omega] \subset V$ and

$$d_H(\partial(B(x, r) \cap \Omega), \partial V) \leq \gamma r,$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff distance which is defined by setting, for any non-empty measurable subsets $E_1$ and $E_2$ of $\mathbb{R}^n$,

$$d_H(E_1, E_2) := \max \left\{ \sup_{x \in E_1} \inf_{y \in E_2} |x - y|, \sup_{x \in E_2} \inf_{y \in E_1} |x - y| \right\}.$$

The concept of quasi-convex domains was introduced by Jia et al. [40] to study the global regularity of second-order elliptic equations. Roughly speaking, a quasi-convex domain is a domain satisfying that the local boundary is close to be convex at small scales. It is easy to find that, if $\Omega$ is a convex domain, then $\Omega$ is a $(\gamma, \sigma, R)$ quasi-convex domain for any $\gamma \in (0, 1)$, some $\sigma \in (0, 1)$, and some $R \in (0, \infty)$. 


Remark 2.5.  
(i) Let \( n \geq 2, \gamma \in (0,1), \) and \( R \in (0,\infty). \) A domain \( \Omega \subset \mathbb{R}^n \) is called a \((\gamma,R)\)-Reifenberg flat domain if, for any \( x_0 \in \partial \Omega \) and \( r \in (0,R_0], \) there exists a system \( \{y_1, \ldots, y_n\} \) of coordinates, which may depend on \( x_0 \) and \( r, \) such that, in this coordinate system, \( x_0 = 0 \) and

\[
[B(0,r) \cap \{y \in \mathbb{R}^n : y_n > \gamma r\}] \subset [B(0,r) \cap \Omega] \subset [B(0,r) \cap \{y \in \mathbb{R}^n : y_n > -\gamma r\}].
\]

The Reifenberg flat domain was introduced by Reifenberg [53], which naturally appears in the theory of both minimal surfaces and free boundary problems. A typical example of Reifenberg flat domains is the well-known Van Koch snowflake (see, for instance, [64]). Moreover, it was shown by Kenig and Toro [44, Theorem 3.1] that there exists a \( \gamma_0(n) \in (0,1), \) depending only on \( n, \) such that, if \( \Omega \) is a \((\gamma,R)\)-Reifenberg flat domain for some \( \gamma \in (0,\gamma_0(n)) \) and \( R \in (0,\infty), \) then \( \Omega \) is an NTA domain. Meanwhile, it is easy to see that a \((\gamma,R)\)-Reifenberg flat domain is also a \((\gamma,\sigma,R)\) quasi-convex domain for some \( \sigma \in (0,1). \) In recent years, boundary value problems of elliptic or parabolic equations on Reifenberg flat domains have been widely concerned and studied (see, for instance, [8, 11, 23, 24, 25]).

(ii) It is known that, for any open set \( \Omega \subset \mathbb{R}^n \) with compact boundary, \( \Omega \) is a semi-convex domain of \( \mathbb{R}^n \) if and only if \( \Omega \) is a Lipschitz domain satisfying a uniform exterior ball condition (see, for instance, [48, 49, 67] for the definitions of both the semi-convex domain and the uniform exterior ball condition). It is worth pointing out that convex domains of \( \mathbb{R}^n \) are semi-convex domains and (semi-)convex domains are special cases of Lipschitz domains (see, for instance, [48, 49, 67]). Furthermore, a (semi-)convex domain is also a \((\gamma,\sigma,R)\) quasi-convex domain for any \( \gamma \in (0,1), \) some \( \sigma \in (0,1), \) and some \( R \in (0,\infty) \) (see, for instance, [69]).

(iii) On NTA domains, Lipschitz domains, quasi-convex domains, Reifenberg flat domains, \( C^1 \) domains, and (semi-)convex domains, we have the following relations (see, for instance, [38, 40, 44, 64, 67, 69]).

(a) class of \( C^1 \) domains
   \( \subset \) class of Lipschitz domains with small Lipschitz constants
   \( \subset \) class of Lipschitz domains
   \( \subset \) class of NTA domains;
(b) class of \( C^1 \) domains
   \( \subset \) class of Lipschitz domains with small Lipschitz constants
   \( \subset \) class of Reifenberg flat domains
   \( \subset \) class of quasi-convex domains;
(c) class of (semi-)convex domains \( \subset \) class of quasi-convex domains.

Next, we recall the definition of the \((\gamma,R)\)-BMO condition (see, for instance, [11]). For any given domain \( \Omega \subset \mathbb{R}^n, \) denote by \( L^1_{\text{loc}}(\Omega) \) the set of all locally integrable functions on \( \Omega. \)

Definition 2.6. Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a domain, and \( \gamma, R \in (0,\infty). \) A function \( f \in L^1_{\text{loc}}(\Omega) \) is said to satisfy the \((\gamma,R)\)-BMO condition if

\[
\|f\|_{\gamma,R} := \sup_{B(x,r) \subset \Omega, r \in (0,R)} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \leq \gamma,
\]
where the supremum is taken over all balls \( B(x, r) \subset \Omega \) with \( r \in (0, R) \), and \( f_{B(x,r)} := \int_{B(x,r)} f(z) \, dz \).

Moreover, a matrix \( A := \{a_{ij}\}_{i,j=1}^n \) is said to satisfy the \((\gamma,R)\)-BMO condition if, for any \( i, j \in \{1, \ldots, n\}, a_{ij} \) satisfies the \((\gamma,R)\)-BMO condition.

## 2.2 Hardy Spaces

In this subsection, we recall the atomic and the molecular characterizations of the Hardy spaces \( H^p(\mathbb{R}^n), H^p(\Omega), \) and \( H^p_{L_0}(\Omega) \) associated with \( L_0 \).

**Definition 2.7.** Let \( p \in (0,1], q \in (1,\infty), \) and \( s \in \mathbb{Z}_+ \) with \( s \geq \lfloor n(\frac{1}{p} - 1) \rfloor \).

(i) A function \( a \in L^q(\mathbb{R}^n) \) is called a \((p, q, s)\)-atom if there exists a ball \( B \) of \( \mathbb{R}^n \) such that

\[ \text{supp}(a) \subset B; \]

\[ \|a\|_{L^q(\mathbb{R}^n)} \leq |B|^{1/q - 1/p}; \]

\[ \frac{1}{|B|} \int_{\mathbb{R}^n} a(x) x^\alpha \, dx = 0 \quad \text{for any} \ \alpha \in \mathbb{Z}_n^+ \quad \text{with} \ |\alpha| \leq s. \]

(ii) The atomic Hardy space \( H^p_{at}(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) satisfying that \( f = \sum_{j=1}^\infty \lambda_j a_j \in \mathcal{S}'(\mathbb{R}^n) \), where \( \{a_j\}_{j=1}^\infty \) is a sequence of \((p, q, s)\)-atoms, and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) satisfies \( \sum_{j=1}^\infty |\lambda_j|^p < \infty \). Moreover, for any \( f \in H^p_{at}(\mathbb{R}^n) \), the quasi-norm \( \|f\|_{H^p_{at}(\mathbb{R}^n)} \) of \( f \) is defined by setting

\[ \|f\|_{H^p_{at}(\mathbb{R}^n)} := \inf \left\{ \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} \right\}, \]

where the infimum is taken over all decompositions of \( f \) as above.

**Definition 2.8.** Let \( p \in (0,1], q \in (1,\infty), s \in \mathbb{Z}_+ \) with \( s \geq \lfloor n(\frac{1}{p} - 1) \rfloor \), and \( \varepsilon \in (0,\infty) \).

(i) A function \( \alpha \in L^q(\mathbb{R}^n) \) is called a \((p, q, s, \varepsilon)\)-molecule associated with the ball \( B \) of \( \mathbb{R}^n \) if

\[ \text{supp}(\alpha) \subset B; \]

\[ \|\alpha\|_{L^q(\mathcal{C}(B))} \leq 2^{-j\varepsilon} |B|^{1/q - 1/p}; \]

\[ \frac{1}{|B|} \int_{\mathbb{R}^n} \alpha(x) x^\beta \, dx = 0 \quad \text{for any} \ \beta \in \mathbb{Z}_n^+ \quad \text{with} \ |\beta| \leq s. \]

(ii) The molecular Hardy space \( H^p_{mol}(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) satisfying that \( f = \sum_{j=1}^\infty \lambda_j a_j \in \mathcal{S}'(\mathbb{R}^n) \), where \( \{a_j\}_{j=1}^\infty \) is a sequence of \((p, q, s, \varepsilon)\)-molecules, and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) satisfies \( \sum_{j=1}^\infty |\lambda_j|^p < \infty \). Moreover, define

\[ \|f\|_{H^p_{mol}(\mathbb{R}^n)} := \inf \left\{ \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} \right\}, \]

where the infimum is taken over all decompositions of \( f \) as above.

Then we have the following atomic and molecular characterizations of the Hardy space \( H^p(\mathbb{R}^n) \), respectively (see, for instance, [45, 61, 63]).

**Lemma 2.9.** Let \( p \in (0,1], q \in (1,\infty), s \in \mathbb{Z}_+ \) with \( s \geq \lfloor n(\frac{1}{p} - 1) \rfloor \), and \( \varepsilon \in (\max\{n + s, n/p\}, \infty) \). Then the spaces \( H^p(\mathbb{R}^n) = H^p_{at}(\mathbb{R}^n) = H^p_{mol}(\mathbb{R}^n) \) with equivalent quasi-norms.
Definition 2.10. Let $n \geq 2$, $\Omega$ be a domain of $\mathbb{R}^n$, $p \in (0, 1]$, $q \in (1, \infty]$, and $s \in \mathbb{Z}_+$ with $s \geq \lfloor n(\frac{1}{p} - 1) \rfloor$.

(i) A ball $B \subset \Omega$ is called a type (a) ball of $\Omega$ if $4B \subset \Omega$, and a ball $B \subset \Omega$ is called a type (b) ball of $\Omega$ if $2B \cap \Omega^C = \emptyset$ but $4B \cap \Omega^C \neq \emptyset$.

(ii) A function $\alpha \in L^q(\Omega)$ is called a type (a) $(p, q, s)_{\Omega}$-atom if there exists a type (a) ball $B \subset \Omega$ such that $\text{supp}(\alpha) \subset B$ and $\alpha$ is a $(p, q, s)$-atom.

Moreover, a function $b \in L^q(\Omega)$ is called a type (b) $(p, q)_{\Omega}$-atom if there exists a type (b) ball $B \subset \Omega$ such that $\text{supp}(b) \subset B$ and $\|b\|_{L^q(\Omega)} \leq |B|^{1/q-1/p}$.

(iii) The atomic Hardy space $H^p_{\text{at}}(\Omega)$ is defined to be the set of all distributions $f \in \mathcal{D}'(\Omega)$ satisfying that there exist two sequences $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ and $\{\kappa_j\}_{j=1}^{\infty} \subset \mathbb{C}$, a sequence $\{a_j\}_{j=1}^{\infty}$ of type (a) $(p, q, s)_{\Omega}$-atoms, and a sequence $\{b_j\}_{j=1}^{\infty}$ of type (b) $(p, q)_{\Omega}$-atoms such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{j=1}^{\infty} \kappa_j b_j$$

in $\mathcal{D}'(\Omega)$, and

$$\sum_{j=1}^{\infty} |\lambda_j|^p + \sum_{j=1}^{\infty} |\kappa_j|^p < \infty.$$ 

Furthermore, for any given $f \in H^p_{\text{at}}(\Omega)$, let

$$\|f\|_{H^p_{\text{at}}(\Omega)} := \inf \left\{ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} + \left( \sum_{j=1}^{\infty} |\kappa_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all decompositions of $f$ as in (2.2).

Then we have the following atomic characterization of the Hardy space $H^p(\Omega)$ on the domain $\Omega$ (see, for instance, [50, Theorem 1]).

Lemma 2.11. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a proper domain, $p \in (0, 1]$, $q \in (1, \infty]$, and $s \in \mathbb{Z}_+$ with $s \geq \lfloor n(\frac{1}{p} - 1) \rfloor$. Then $H^p(\Omega) = H^p_{\text{at}}(\Omega)$ with equivalent quasi-norms.

Definition 2.12. Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, $L_D$ as in (1.4), and $p \in (0, 1]$. Assume that $q \in (1, \infty]$, $M \in \mathbb{N}$, $\epsilon \in (0, \infty)$, $B := B(x_B, r_B)$ with $x_B \in \Omega$ and $r_B \in (0, \text{diam} (\Omega))$, and $B_{\Omega} := B \cap \Omega$.

(i) A function $\alpha \in L^q(\Omega)$ is called a $(p, q, M, \epsilon)_{L_D}$-molecule associated with the ball $B_{\Omega}$ if, for any $k \in \{0, \ldots, M\}$ and $j \in \mathbb{Z}_+$, it holds true that

$$\left\| (r_B^{-2}L_D) \alpha \right\|_{L^q(S_{j}(B_{\Omega})))} \leq 2^{-\epsilon j} |2^j B_{\Omega}|^{1/q} |B_{\Omega}|^{-1/p}.$$ 

(ii) For any $f \in L^2(\Omega)$,

$$f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$$

$$f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$$
is called a molecular \((p, q, M, \epsilon)_L\)-representation of the function \(f\) if, for any \(j \in \mathbb{N}\), \(\alpha_j\) is a \((p, q, M, \epsilon)_L\)-molecule associated with the ball \(B_{\Omega, j} \subset \Omega\), the summation (2.3) converges in \(L^2(\Omega)\), and \(\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}\) satisfies \(\sum_{j=1}^\infty |\lambda_j|^p < \infty\).

Let

\[
\mathbb{H}^{p,q,M,\epsilon}_{L,D,\text{mol}}(\Omega) := \left\{ f \in L^2(\Omega) : f \text{ has a molecular \((p, q, M, \epsilon)_L\)-representation} \right\}
\]

equipped with the quasi-norm

\[
\|f\|_{\mathbb{H}^{p,q,M,\epsilon}_{L,D,\text{mol}}(\Omega)} := \inf \left\{ \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} : \{\lambda_j\}_{j=1}^\infty \text{ is a \((p, q, M, \epsilon)_L\)-representation of } f \right\},
\]

where the infimum is taken over all molecular \((p, q, M, \epsilon)_L\)-representations of \(f\) as above.

Then the molecular Hardy space \(H_{L,D,\text{mol}}^{p,q,M,\epsilon}(\Omega)\) is then defined as the completion of the set \(\mathbb{H}^{p,q,M,\epsilon}_{L,D,\text{mol}}(\Omega)\) with respect to the quasi-norm \(\| \cdot \|_{\mathbb{H}^{p,q,M,\epsilon}_{L,D,\text{mol}}(\Omega)}\).

By Remark 2.3, we find that bounded NTA domains of \(\mathbb{R}^n\) are spaces of homogeneous type. Thus, for the Hardy space \(H_{L,D}^p(\Omega)\), we have the following molecular characterization (see, for instance, [6]).

**Lemma 2.13.** Let \(n \geq 2\), \(\Omega \subset \mathbb{R}^n\) be a bounded NTA domain, \(L_D\) as in (1.4), and \(p \in (0, 1]\). Then, for any \(q \in (1, \infty)\), \(M \in \mathbb{N} \cap \left( \frac{n}{2p}, \infty \right)\), and \(\epsilon \in (\frac{n}{p}, \infty)\), \(H_{L,D}^p(\Omega) = H_{L,D,\text{mol}}^{p,q,M,\epsilon}(\Omega)\) with equivalent quasi-norms.

### 3 Proof of Theorem 1.1

In this section, we show Theorem 1.1. Let \(L_D\) be as in (1.4). It was shown by Grütter and Widman [32] that, when \(n \geq 3\), the Green function associated with the operator \(L_D\) exists; in this case, denote by \(G(\cdot, \cdot)\) the Green function associated with \(L_D\). Then we have the following estimates for \(G(\cdot, \cdot)\) which are known (see, for instance, [35, Remark 4.9]).

**Lemma 3.1.** Let \(n \geq 3\) and \(\Omega\) be a bounded NTA domain of \(\mathbb{R}^n\). Then there exist constants \(C \in (0, \infty)\) and \(\beta \in (0, 1]\), depending only on \(n\), the matrix \(A\) appeared in the definition of \(L_D\), and \(\Omega\), such that, for any \(x, y \in \Omega\),

\[
G(x, y) \leq C |\delta(y)|^{\beta} |x - y|^{2-n-\beta},
\]

here and thereafter, \(\delta(y) := \text{dist}(y, \partial\Omega)\) with \(\partial\Omega\) being the boundary of \(\Omega\).

Now, we prove the following Hölder continuity of the Green function \(G\) via using Lemma 3.1 and the Harnack inequality (see, for instance, [31, Theorem 8.22]).

**Lemma 3.2.** Let \(n \geq 3\) and \(\Omega\) be a bounded NTA domain of \(\mathbb{R}^n\). Then there exist constants \(C \in (0, \infty)\) and \(\beta \in (0, 1]\), depending only on \(n\), the matrix \(A\) appeared in the definition of \(L_D\), and \(\Omega\), such that, for any \(x, y_1, y_2 \in \Omega\),

\[
|G(x, y_1) - G(x, y_2)| \leq C |y_1 - y_2|^{\beta} \left[ |x - y_1|^{2-n-\beta} + |x - y_2|^{2-n-\beta} \right].
\]
Proof. We show the present lemma via borrowing some ideas from the proof of [32, Theorem (1.9)]. Fix \( x, y_1, y_2 \in \Omega \). Without loss of generality, we may assume that \( G(x, y_1) \geq G(x, y_2) \). Now, we prove the present lemma by considering the following three cases on \( |y_1 - y_2| \).

Case 1) \( |y_1 - y_2| \geq |x - y_1|/2 \). In this case, by the upper estimate that \( G(x, z) \leq |x - z|^{2-n} \) for any \( x, z \in \Omega \) (see, for instance, [32, (1.8)]) and \( 1 \leq |y_1 - y_2||x - y_1|^{-1} \), we conclude that, for any given \( \beta \in (0, 1] \),

\[
|G(x, y_1) - G(x, y_2)| = G(x, y_1) - G(x, y_2) \leq G(x, y_1) \leq |x - y_1|^{2-n} \\
\leq |y_1 - y_2|^\beta |x - y_1|^{2-n-\beta} \\
\leq |y_1 - y_2|^\beta \left[ |x - y_1|^{2-n-\beta} + |x - y_2|^{2-n-\beta} \right].
\]

Case 2) \( |y_1 - y_2| < |x - y_1|/2 \) and \( \delta(y_1) \leq |y_1 - y_2| \). In this case, from Lemma 3.1 and \( \delta(y_1) \leq |y_1 - y_2| \), it follows that there exists a \( \beta \in (0, 1] \) such that

\[
|G(x, y_1) - G(x, y_2)| \leq G(x, y_1) \leq [\delta(y_1)]^\beta |x - y_1|^{2-n-\beta} \\
\leq |y_1 - y_2|^\beta \left[ |x - y_1|^{2-n-\beta} + |x - y_2|^{2-n-\beta} \right].
\]

Case 3) \( |y_1 - y_2| < |x - y_1|/2 \) and \( \delta(y_1) > |y_1 - y_2| \). In this case, let \( R := \min\{|\delta(y_1)|, |x - y_1|/2\} \). It is easy to see that \( L_D^2 G(x, \cdot) = 0 \) in \( B(y_1, R) \). Applying the Harnack inequality (see, for instance, [31, Theorem 8.22]), we find that there exists a \( \beta \in (0, 1] \) such that, for any 0 < \( r_1 \leq r_2 \leq R \),

\[
\max_{B(y_1, r_1)} G(x, \cdot) - \min_{B(y_1, r_2)} G(x, \cdot) \leq \left( \frac{r_1}{r_2} \right)^\beta \max_{B(y_1, R)} G(x, \cdot),
\]

which further implies that

\[
|G(x, y_1) - G(x, y_2)| \leq |y_1 - y_2|^\beta R^{-\beta} \max_{B(y_1, R)} G(x, \cdot).
\]

If \( R = |x_1 - y|/2 \), then, by (3.4) and the upper estimate \( G(x, y_1) \leq |x - y_1|^{2-n} \), we conclude that

\[
|G(x, y_1) - G(x, y_2)| \leq |y_1 - y_2|^\beta |x - y_1|^{2-n-\beta} \\
\leq |y_1 - y_2|^\beta \left[ |x - y_1|^{2-n-\beta} + |x - y_2|^{2-n-\beta} \right].
\]

If \( R = \delta(y_1) \), then \( \delta(y_1) \leq |x - y_1|/2 \). From this and Lemma 3.1, we deduce that, for any \( z \in B(y_1, R) \),

\[
G(x, z) \leq [\delta(z)]^\beta |x - z|^{2-n-\beta} \leq [|z - y_1| + \delta(y_1)]^\beta [|x - y_1| - |y_1 - z|]^{2-n-\beta} \\
\leq [\delta(y_1)]^\beta |x - y_1|^{2-n-\beta},
\]

which, combined with (3.4), further implies that

\[
|G(x, y_1) - G(x, y_2)| \leq |y_1 - y_2|^\beta \left[ |x - y_1|^{2-n-\beta} + |x - y_2|^{2-n-\beta} \right].
\]

This, together with (3.2), (3.3), and (3.5), then finishes the proof of Lemma 3.2. \( \square \)
Moreover, since \( L_D \) is a \( \omega_0 \)-accretive operator on \( L^2(\Omega) \) with some \( \omega_0 \in [0, \pi/2) \) (see, for instance, [4, 51] for the definition of the \( \omega_0 \)-accretive operator). For any given \( \theta \in [0, \pi) \), let \( \Gamma_\theta := \{ z \in \mathbb{C} | \arg z < \theta \} \). It is well known that \( -L_D \) generates a holomorphic semigroup \( \{ e^{-tL_D} \}_{t \in \mathbb{R}} \) in \( \Gamma_{\frac{\pi}{2}-\omega_0} \) (see, for instance, [4, 51]).

Moreover, it is known that, for any \( \lambda \in \Gamma_{\pi-\omega_0} \), the operator \( L_D + \lambda I \) is bounded on \( L^2(\Omega) \), and has the integral kernel \( G_\lambda \), where \( I \) denotes the identity operator. Namely, for any given \( f \in L^2(\Omega) \) and \( x \in \Omega \),
\[
(L_D + \lambda I)^{-1} f(x) = \int_{\Omega} G_\lambda(x,y)f(y) \, dy.
\]

**Lemma 3.3.** Assume that \( n \geq 3 \) and \( \Omega \) is a bounded NTA domain of \( \mathbb{R}^n \). Let \( \mu \in (\omega_0, \pi/2) \). Then there exists a \( \delta \in (0, 1] \) such that, for any \( \lambda \in \Gamma_{\pi-\mu} \) and \( x, y_1, y_2 \in \Omega \) with \( x \neq y_i \) for \( i \in \{1, 2\} \),
\[
|G_\lambda(x,y_1) - G_\lambda(x,y_2)| \leq C \max_{i \in \{1,2\}} e^{-\gamma \sqrt{\mu|x-y_i|}} |y_1 - y_2|^\beta \left| \frac{1}{|x-y_1|^{n-2+\delta}} + \frac{|\lambda|^{\beta/2}}{|x-y_2|^{n-\delta}} \right|,
\]
where \( C \) is a positive constant depending only on \( \mu, n, \) and \( \Omega \), and \( \gamma \) is a positive constant depending only on \( \mu \) and \( \delta \).

**Proof.** Let \( \lambda \in \Gamma_{\pi-\mu} \). By the resolvent identity (see, for instance, [52, p. 36, (9.1)]), we conclude that, for any \( y, z \in \Omega \) with \( y \neq z \),
\[
(3.6) \quad G_\lambda(y,z) = G(y,z) + \lambda \int_{\Omega} G(t,z)G_\lambda(y,t) \, dt.
\]

Moreover, since \( L_D \) generates a holomorphic semigroup \( \{ e^{-tL_D} \}_{t \in \mathbb{R}} \) (see, for instance, [51, Theorem 1.52]), and the kernels \( K_\lambda^L \) of the semigroup \( \{ e^{-tL_D} \}_{t \geq 0} \) satisfy the Gaussian upper bound estimate (1.5), similarly to the proof of [26, Lemma 2.3], it follows that there exists a positive constant \( c \) such that, for any \( y, z \in \Omega \) with \( y \neq z \),
\[
(3.7) \quad |G_\lambda(y,z)| \leq e^{-c \sqrt{\mu |y-z|}} \frac{1}{|y-z|^{n-2}}.
\]

Thus, by (3.6), (3.7), and Lemma 3.2, we find that there exists a \( \beta \in (0, 1] \) such that, for any given \( x, y_1, y_2 \in \Omega \) with \( x \neq y_i \) for any \( i \in \{1, 2\} \),
\[
(3.8) \quad |G_\lambda(x,y_1) - G_\lambda(x,y_2)|
\]
\[
\leq |G(x,y_1) - G(x,y_2)| + |\lambda| \int_{\Omega} |G(z,y_1) - G(z,y_2)|G_\lambda(x,z) \, dz
\]
\[
\leq |y_1 - y_2|^\beta \left[ |x-y_1|^{2-n-\beta} + |x-y_2|^{2-n-\beta} \right]
\]
\[
+ |\lambda||y_1 - y_2|^\beta \int_{\Omega} \frac{e^{-c \sqrt{\mu |x-z|}}}{|x-z|^{n-2}} \left| |z-y_1|^{2-n-\beta} + |z-y_2|^{2-n-\beta} \right| \, dz.
\]

For any \( i \in \{1, 2\} \), let \( \Omega_1 := \{ z \in \Omega : |x-z| \leq |z-y_i| \} \) and \( \Omega_2 := \{ z \in \Omega : |x-z| > |z-y_i| \} \). Then, for any \( i \in \{1, 2\} \), we have
\[
(3.9) \quad \int_{\Omega} \frac{e^{-c \sqrt{\mu |x-z|}}}{|x-z|^{n-2}} \frac{1}{|z-y_i|^{n-2+\beta}} \, dz
\]
\[ \int_{\Omega_1} e^{-c \sqrt{|z-x|}} \frac{1}{|z-y_1|^{n+2+\beta}} \, dz + \int_{\Omega_2} \cdots =: I_1 + I_2. \]

For $I_1$, from the fact that, for any $z \in \Omega_1$, $2|z-y_1| \geq |x-z| + |z-y_1| \geq |x-y_1|$, we deduce that

\begin{equation}
I_1 \leq \frac{2}{|x-y_1|^{n+2+\beta}} \int_{\Omega_1} e^{-c \sqrt{|z-x|}} \frac{1}{|z-y_1|^{n+2+\beta}} \, dz \leq \frac{1}{|x-y_1|^{n+2+\beta}} \frac{1}{|A|}.
\end{equation}

Moreover, it is easy to see that, for any $z \in \Omega_2$, $2|x-z| \geq |x-z| + |z-y_1| \geq |x-y_1|$. By this, we find that

\begin{equation}
I_2 \leq \frac{2}{|x-y_1|^{n+2}} \int_{\Omega_2} e^{-c \sqrt{|z-x|}} \frac{1}{|z-y_1|^{n+2+\beta}} \, dz \leq \frac{1}{|x-y_1|^{n+2}} \frac{1}{|A|^{1-\beta/2}}.
\end{equation}

To finish the proof of the present lemma, we fix $x, y_1, y_2 \in \Omega$ with $x \neq y_1$ for any $i \in \{1, 2\}$, and, without loss of generally, we may assume that $|x-y_1| \leq |x-y_2|$. Thus, from (3.8), (3.9), (3.10), (3.11), and $|x-y_1| \leq |x-y_2|$, it follows that

\begin{equation}
|G_d(x, y_1) - G_d(x, y_2)|
\leq |y_1 - y_2|^{\beta/2} \left[ \frac{1}{|x-y_1|^{n+2+\beta}} + \frac{1}{|x-y_2|^{n+2+\beta}} \right] + |y_1 - y_2|^{\beta/2} \frac{|A|^{\beta/2}}{|x-y_1|^{n+2}} + \frac{|A|^{\beta/2}}{|x-y_2|^{n+2}} \leq |y_1 - y_2|^{\beta/2} \left[ \frac{1}{|x-y_1|^{n+2+\beta}} + \frac{|A|^{\beta/2}}{|x-y_2|^{n+2}} \right].
\end{equation}

Furthermore, by (3.7) and $|x-y_1| \leq |x-y_2|$, we conclude that

\[ |G_d(x, y_1) - G_d(x, y_2)| \leq \sum_{i=1}^2 e^{-c \sqrt{|x-y_i|}} \frac{1}{|x-y_i|^{n+2}} \leq e^{-c \sqrt{|x-y_1|}} \frac{1}{|x-y_1|^{n+2}}, \]

which, together with (3.12), further implies that, for any $\delta \in (0, \beta)$,

\[ |G_d(x, y_1) - G_d(x, y_2)| \leq \left( |y_1 - y_2|^{\beta/2} \left[ \frac{1}{|x-y_1|^{n+2+\beta}} + \frac{|A|^{\beta/2}}{|x-y_1|^{n+2}} \right] \right)^{\delta/\beta} e^{-c \sqrt{|x-y_1|}} \frac{1}{|x-y_1|^{n+2}} \leq e^{-c \sqrt{|x-y_1|}} \frac{1}{|x-y_1|^{n+2}} \left[ 1 + |x-y_1|^{\delta/2} |A|^{\delta/2} \right]. \]
We prove the present lemma via borrowing some ideas from the proof of [26, Lemma 2.6].

Lemma 3.4. Let $n \geq 3$, $\Omega$ be a bounded NTA domain of $\mathbb{R}^n$, $\mu \in (\omega_0, \pi/2)$, and $\lambda \in \Gamma_{\mu}$. Then, for any $m \in \mathbb{N}$ with $m > (n + 2)/2$, the operator $(L_D + \lambda I)^{-m}$ has a kernel $R_{\lambda, m}$ and there exists a constant $\delta \in (0, 1)$ such that, for any $x, y_1, y_2 \in \Omega$ with $x \neq y_i$ for $i \in \{1, 2\},$

$$|R_{\lambda, m}(x, y_1) - R_{\lambda, m}(x, y_2)| \leq C|\lambda|^{-m+\frac{3}{2}}|y_1 - y_2|^{\delta} \left[ \max_{[0,1,2]} e^{-c \sqrt{|\lambda|} |x-y|} \right],$$

where $C$ is a positive constant depending only on $\mu$, $\delta$, $n$, and $\Omega$, and $c$ is a positive constant depending only on $\mu$ and $\delta$.

Proof. We prove the present lemma via borrowing some ideas from the proof of [26, Lemma 2.6]. Similarly to the proof of [27, Theorem 1], using (1.5), we find that, for any $m \in \mathbb{N}$ with $m > n/2$, $(L_D + \lambda I)^{-m}$ has a kernel $R_{\lambda, m}$ and there exists a positive constant $\kappa$ such that, for any $x, y \in \Omega$ with $x \neq y$,

$$|R_{\lambda, m}(x, y)| \leq |\lambda|^{-m+\frac{3}{2}} e^{-\kappa \sqrt{|\lambda|} |x-y|}.$$  

(3.13)

Let $m \in \mathbb{N}$ and $m > (n + 2)/2$. Then, from $(L_D + \lambda I)^{-(m+1)} = (L_D + \lambda I)^{-m}(L_D + \lambda I)^{-1}$, we deduce that, for any $x, y \in \Omega$ with $x \neq y$,

$$R_{\lambda, m+1}(x, y) = \int_{\Omega} R_{\lambda, m}(x, z) G_D(z, y) \, dz.$$  

(3.14)

Therefore, by (3.13), (3.14), and Lemma 3.3, we find that there exists a $\delta \in (0, 1)$ such that, for any $x, y_1, y_2 \in \Omega$ with $x \neq y_i$ for any $i \in \{1, 2\},$

$$|R_{\lambda, m+1}(x, y_1) - R_{\lambda, m+1}(x, y_2)|$$

$$\leq \int_{\Omega} |R_{\lambda, m}(x, z)| |G_D(z, y_1) - G_D(z, y_2)| \, dz$$

$$\leq |\lambda|^{-m+\frac{3}{2}} |y_1 - y_2|^\delta \int_{\Omega} e^{-\kappa \sqrt{|\lambda|} |x-z|} e^{-\gamma \sqrt{|\lambda|} |z-y_1|} \frac{1}{|z-y_1|^{n-2+\delta}} \, dz$$

$$+ |\lambda|^{-m+\frac{3}{2}} |y_1 - y_2|^\delta \int_{\Omega} e^{-\kappa \sqrt{|\lambda|} |x-z|} e^{-\gamma \sqrt{|\lambda|} |z-y_2|} \frac{|\lambda|^{\delta/2}}{|x-y_1|^{n-2+\delta}} \, dz$$

$$+ |\lambda|^{-m+\frac{3}{2}} |y_1 - y_2|^\delta \int_{\Omega} e^{-\kappa \sqrt{|\lambda|} |x-z|} e^{-\gamma \sqrt{|\lambda|} |z-y_2|} \frac{1}{|z-y_2|^{n-2+\delta}} \, dz$$

$$+ |\lambda|^{-m+\frac{3}{2}} |y_1 - y_2|^\delta \int_{\Omega} e^{-\kappa \sqrt{|\lambda|} |x-z|} e^{-\gamma \sqrt{|\lambda|} |z-y_2|} \frac{|\lambda|^{\delta/2}}{|x-y_2|^{n-2}} \, dz$$

$$=: II_1 + II_2 + II_3 + II_4.$$  

(3.15)

Let $c := \min(\kappa, \gamma/2)$, where $\kappa$ is as in (3.13). From the fact that, for any $z \in \Omega, |x-z| + |z-y_1| \geq |x-y_1|$, it follows that

$$\int_{\Omega} e^{-\kappa \sqrt{|\lambda|} |x-z|} e^{-\gamma \sqrt{|\lambda|} |z-y_1|} \frac{1}{|z-y_1|^{n-2+\delta}} \, dz$$
Theorem 7.7), we find that, for any given \( \lambda \), which, combined with (3.16) and (3.15), implies that
\[
|\lambda|^{-1+\frac{2}{m}} \int_{\Omega} e^{-\frac{2}{m} |z-y_1|^2} \frac{1}{|z-y_1|^\mu} \, dz 
\]
which further implies that
\[
(3.16) \quad II_1 \leq |\lambda|^{-(m+1)+\frac{2}{m}+\frac{2}{\mu}} |y_1 - y_2|^\delta e^{-c \sqrt{|\lambda|} |x-y_1|}.
\]

Similarly to the estimation of (3.16), for any \( j \in \{2, 3, 4\} \), we also have
\[
II_j \leq |\lambda|^{-(m+1)+\frac{2}{m}+\frac{2}{\mu}} |y_1 - y_2|^\delta e^{-c \sqrt{|\lambda|} |x-y_1|},
\]
which, combined with (3.16) and (3.15), implies that
\[
|R_{\lambda,m+1}(x,y_1) - R_{\lambda,m+1}(x,y_2)| \leq |\lambda|^{-(m+1)+\frac{2}{m}+\frac{2}{\mu}} |y_1 - y_2|^\delta e^{-c \sqrt{|\lambda|} |x-y_1|}.
\]

This finishes the proof of Lemma 3.4.

\( \square \)

Lemma 3.5. Let \( n \geq 3 \), \( \Omega \) be a bounded NTA domain of \( \mathbb{R}^n \), the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3), and \( L_D \) be as in (1.4). Then there exists a constant \( \delta_0 \in (0, 1] \), depending only on \( n \), \( A \), and \( \Omega \), such that, for any given \( \delta \in (0, \delta_0) \), there exist constants \( C, c \in (0, \infty) \) such that, for any \( t \in (0, \infty) \) and \( x, y_1, y_2 \in \Omega \) with \( |y_1 - y_2| \leq \sqrt{t}/2 \),
\[
|K_{L_D}(x,y_1) - K_{L_D}(x,y_2)| \leq C \frac{|y_1 - y_2|^\delta}{t^{\delta/2}} \exp \left\{ -\frac{|x-y_1|^2}{ct} \right\}.
\]

Proof. For any given \( \mu \in (\pi/2, \pi - \omega_0) \) and \( R \in (0, \infty) \), define
\[
\Gamma_1 := \{ re^{i\mu} : r \geq R \}, \quad \Gamma_2 := \{ Re^{i\phi} : |\phi| \leq \mu \},
\]
\[
\Gamma_3 := \{ re^{i\mu} : r \geq R \}, \quad \Gamma_R := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\]
Let \( m \in \mathbb{N} \) and \( m \geq \frac{n+3}{2} \). Using the inverse Laplace transform (see, for instance, [52, p. 30, Theorem 7.7]), we find that, for any given \( t \in (0, \infty) \) and any \( x, y \in \Omega \) with \( x \neq y \),
\[
(3.17) \quad K_{L_D}^{L_D}(x,y) = (-1)^m \frac{(m-1)!}{2\pi i t^{m-1}} \int_{\Gamma_R} e^{\mu} R_{\lambda,m}(x,y) \, d\lambda,
\]
where \( R \in [R(x,y,t), \infty) \) and
\[
(3.18) \quad R(x,y,t) := \max \left\{ \frac{1}{t}, \frac{|x-y|^2}{t^2} \right\}.
\]
Fix $x, y_1, y_2 \in \Omega$ with $x \neq y_i$ for any $i \in \{1, 2\}$, a $t \in (0, \infty)$, and an $R \in \{\max \{R(x, y_1, t), R(x, y_2, t)\}, \infty\}$, where $R(x, y_1, t)$ and $R(x, y_2, t)$ are as in (3.18). Then, by (3.17) and Lemma 3.4, we conclude that

$$(3.19) \quad [K^L_t(x, y_1) - K^L_t(x, y_2)]$$

$$\leq \sum_{i=1}^{2} \frac{1}{m-1} \int_{1}^{\infty} e^{R(1)} |\alpha|^{-m+\frac{\delta}{2}+\frac{c}{2}} e^{-c \sqrt{\alpha} |x-y|} |d| |\alpha|$$

where $\delta \in (0, 1)$ is as in Lemma 3.4. From the assumption that $R \geq \max\{\frac{1}{t}, \frac{|x-y|^2}{t^2}\}$, it follows that there exists a positive constant $\tilde{c}$ such that

$$\leq e^{-c \sqrt{\alpha} |x-y|} e^{-\frac{c}{2}Rt} \int_{1}^{\infty} e^{-\frac{c}{2}s} s^{-m+\frac{\delta}{2}+\frac{c}{2}} ds$$

which together with (3.20), further implies that

$$E \leq t^{-\frac{c}{2}} e^{-c_1 \frac{|y_1-y_2|^2}{t^{1/2}}} \left[\frac{|y_1-y_2|}{t^{1/2}}\right]^6.$$

Similarly to the estimation of (3.21), we also have

$$F \leq t^{-\frac{c}{2}} e^{-c_1 \frac{|y_1-y_2|^2}{t^{1/2}}} \left[\frac{|y_1-y_2|}{t^{1/2}}\right]^6.$$

From this, (3.21), and (3.19), we deduce that, for any $t \in (0, \infty)$ and $x, y_1, y_2 \in \Omega$ satisfying $|y_1 - y_2| \leq \sqrt{t}/2$,

$$[K^L_t(x, y_1) - K^L_t(x, y_2)] \leq t^{-\frac{c}{2}} \left(\max_{i \in [1, 2]} e^{-c_1 \frac{|y_i-y|^2}{t^{1/2}}} \right) \left[\frac{|y_1-y_2|}{t^{1/2}}\right]^6 \leq t^{-\frac{c}{2}} e^{-c_1 \frac{|y_1-y|^2}{t^{1/2}}} \left[\frac{|y_1-y_2|}{t^{1/2}}\right]^6.$$

This finishes the proof of Lemma 3.5.
To show Theorem 1.1, we also need the following global regularity estimate for the Dirichlet problem (1.1).

**Lemma 3.6.** Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded NTA domain, and the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3). Then there exists a positive constant \( p_0 \in (2, \infty) \), depending only on \( n, A, \) and \( \Omega \), such that, for any given \( p \in [2, p_0) \), the Dirichlet problem (1.1), with \( f \in L^{p_0}(\Omega) \), is uniquely solvable and, moreover, for any weak solution \( u \) of the problem (1.1), \( u \in W^{1,p}_0(\Omega) \) and there exists a positive constant \( C \), depending only on \( n, p, \) and \( \Omega \), such that

\[
\|\nabla u\|_{L^p(\Omega; \mathbb{R}^n)} \leq C\|f\|_{L^{p_0}(\Omega)}.
\]

To prove Lemma 3.6, we need the global regularity estimate for the following Dirichlet problem (3.23).

Let \( p \in (1, \infty) \) and \( f \in L^p(\Omega; \mathbb{R}^n) \). Then a function \( u \) is called a weak solution of the following Dirichlet boundary value problem

\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
-\text{div}(A\nabla u) = \text{div}(f) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{array}
\right.
\end{aligned}
\]

if \( u \in W^{1,p}_0(\Omega) \) and, for any \( \varphi \in C^\infty_c(\Omega) \),

\[
\int_\Omega A(x)\nabla u(x) \cdot \nabla \varphi(x) \, dx = -\int_\Omega f(x) \cdot \nabla \varphi(x) \, dx.
\]

Moreover, the Dirichlet problem (3.23) is said to be uniquely solvable if, for any \( f \in L^p(\Omega; \mathbb{R}^n) \), there exists a unique \( u \in W^{1,p}_0(\Omega) \) such that (3.24) holds true for any \( \varphi \in C^\infty_c(\Omega) \).

By the Lax–Milgram theorem, we conclude that, when \( p = 2 \), the Dirichlet problem (3.23), with \( f \in L^2(\Omega; \mathbb{R}^n) \), is uniquely solvable and the weak solution \( u \) satisfies

\[
\|\nabla u\|_{L^2(\Omega; \mathbb{R}^n)} \leq \mu_0^{-1}\|f\|_{L^2(\Omega; \mathbb{R}^n)},
\]

where \( \mu_0 \) is as in (1.3).

**Lemma 3.7.** Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded NTA domain, and the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3). Then there exists a positive constant \( p_0 \in (2, \infty) \), depending only on \( n, A, \) and \( \Omega \), such that, for any given \( p \in (p'_0, p_0) \) with \( 1/p'_0 + 1/p_0 = 1 \), the Dirichlet problem (3.23), with \( f \in L^{p_0}(\Omega; \mathbb{R}^n) \), is uniquely solvable and, moreover, for any weak solution \( u \) of the problem (3.23), \( u \in W^{1,p}_0(\Omega) \) and there exists a positive constant \( C \), depending only on \( n, p, \) and \( \Omega \), such that

\[
\|\nabla u\|_{L^p(\Omega; \mathbb{R}^n)} \leq C\|f\|_{L^{p_0}(\Omega; \mathbb{R}^n)}.
\]

Via replacing [69, Lemma 4.3] by the following Lemma 3.8, using a real-variable argument for \( L^p(\Omega) \) estimates, which was essentially established by Shen [58, Theorem 3.4] (see also [57, Theorem 4.2.6] and [59, Theorem 3.3]), and then repeating the proof of [69, Theorem 1.10], we can prove Lemma 3.7; we omit the details here.
Lemma 3.8. Let \( n \geq 2 \), \( \Omega \subset \mathbb{R}^n \) be a bounded NTA domain, and \( B(x_0, r) \) a ball such that \( r \in (0, r_0/4) \) and either \( x_0 \in \partial \Omega \) or \( B(x_0, 2r) \subset \Omega \), where \( r_0 \in (0, \text{diam} (\Omega)) \) is a constant. Assume that the real-valued, bounded, and measurable matrix \( A \) satisfies (1.3), and \( u \in W^{1,2}(B_\Omega(x_0, 2r)) \) is a weak solution of the following Dirichlet problem

\[
\begin{aligned}
\begin{cases}
\text{div} (A\nabla u) = 0 & \text{in } B_\Omega(x_0, 2r), \\
u = 0 & \text{on } B(x_0, 2r) \cap \partial \Omega.
\end{cases}
\end{aligned}
\]

Then there exists a constant \( p_0 \in (2, \infty) \), depending on \( \Omega \), \( n \), and \( \mu_0 \) in (1.3), such that

\[
\left[ \int_{B_\Omega(x_0, r)} |\nabla u(x)|^{p_0} \, dx \right]^{1/p_0} \leq C \left[ \int_{B_\Omega(x_0, 2r)} |\nabla u(x)|^2 \, dx \right]^{1/2},
\]

where \( C \) is a positive constant depending only on \( n \), \( \Omega \), and \( p_0 \).

Lemma 3.8 was established in [47, Lemma 3.2 and Corollary 4.1].

Now, we show Lemma 3.6 by using Lemma 3.7.

Proof of Lemma 3.6. Let \( p_0 \in (2, \infty) \) be as in Lemma 3.8, \( p \in [2, p_0) \), and \( f \in L^{p*}(\Omega) \). Consider the Dirichlet problem

\[
\begin{aligned}
\begin{cases}
-\text{div} (A^* \nabla v) = \text{div} (g) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

with \( g \in L^{p'}(\Omega; \mathbb{R}^n) \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then, by Lemma 3.7 and \( g \in L^{p'}(\Omega; \mathbb{R}^n) \) with \( p' \in (p_0', p_0) \), we conclude that the Dirichlet problem (3.25) is uniquely solvable and the weak solution \( v \) satisfies

\[
\| \nabla v \|_{L^{p'}(\Omega; \mathbb{R}^n)} \leq \| g \|_{L^{p'}(\Omega; \mathbb{R}^n)}.
\]

Moreover, from the assumptions that \( p \geq 2 \) and \( f \in L^{p}(\Omega) \), and Remark 1.7, we deduce that there exists a weak solution \( u \in W^{1,2}_0(\Omega) \) for the Dirichlet problem (1.1) with \( f \in L^{p}(\Omega) \). Moreover, we have

\[
\int_{\Omega} \nabla u(x) \cdot g(x) \, dx = - \int_{\Omega} A^*(x) \nabla v(x) \cdot \nabla u(x) \, dx = - \int_{\Omega} A(x) \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\Omega} f(x) v(x) \, dx,
\]

which, together with the Hölder inequality, the Sobolev embedding theorem on NTA domains (see, for instance, [5, Theorem 1.1]), and (3.26), further implies that

\[
\| \nabla u \|_{L^{p}(\Omega; \mathbb{R}^n)} = \sup_{\|g\|_{L^{p'}(\Omega; \mathbb{R}^n)} \leq 1} \left| \int_{\Omega} \nabla u(x) \cdot g(x) \, dx \right| = \sup_{\|g\|_{L^{p'}(\Omega; \mathbb{R}^n)} \leq 1} \left| \int_{\Omega} f(x) v(x) \, dx \right|
\]

\[
\leq \sup_{\|f\|_{L^{p}(\Omega)} \leq 1} \| f \|_{L^{p}(\Omega)} \| \nabla v \|_{L^{p'}(\Omega; \mathbb{R}^n)}
\]

\[
\leq \sup_{\|f\|_{L^{p}(\Omega)} \leq 1} \| f \|_{L^{p}(\Omega)} \| \nabla v \|_{L^{p'}(\Omega; \mathbb{R}^n)}
\]
Proof of Theorem 1.1. Then, for any $n \in \mathbb{N}$, the solution $u$ of the problem (1.1) on $(\gamma, \sigma, R)$ quasi-convex domains.

Lemma 3.9. Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, and the real-valued, bounded, and measurable matrix $A$ satisfy (1.3). Assume that $p \in (2, \infty)$. Then there exists a positive constant $\gamma_0 \in (0, 1)$, depending only on $n$, $p$, and $\Omega$, such that, if $A$ satisfies the $(\gamma, R)$-BMO condition and $\Omega$ is a $(\gamma, \sigma, R)$ quasi-convex domain for some $\gamma \in (0, \gamma_0)$, $\sigma \in (0, 1)$, and $R \in (0, \infty)$, then the Dirichlet problem (1.1), with $f \in L^p(\Omega)$, is uniquely solvable and, moreover, for any weak solution $u$ of the problem (1.1), $u \in W^{1, p}_0(\Omega)$ and there exists a positive constant $C$, depending only on $n$, $p$, and $\Omega$, such that

$$
\|\nabla u\|_{L^p(\Omega; \mathbb{R}^n)} \leq C\|f\|_{L^p(\Omega)}.
$$

Lemma 3.10. Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, and $r_0 \in (0, \text{diam}(\Omega))$. Assume that the matrix $A$ is as in Lemma 3.9. Let $v \in W^{1, 2}(B_\Omega(x_0, 4r))$ be a weak solution of the equation $\text{div}(A\nabla v) = 0$ in $B_\Omega(x_0, 4r)$ satisfying $v = 0$ on $B(x_0, 4r) \cap \partial \Omega$, where $r \in (0, r_0/4)$ and either $x_0 \in \partial \Omega$ or $B(x_0, 4r) \subset \Omega$. Then, for any given $p \in (2, \infty)$, there exists a constant $\gamma_0 \in (0, 1)$, depending only on $n$, $p$, and $\Omega$, such that, if $A$ satisfies the $(\gamma, R)$-BMO condition and $\Omega$ is a $(\gamma, \sigma, R)$ quasi-convex domain for some $\gamma \in (0, \gamma_0)$, $\sigma \in (0, 1)$, and $R \in (0, \infty)$, then the weak reverse Hölder inequality holds true, where $C$ is a positive constant independent of $v$, $x_0$, and $r$.

Lemma 3.10 was established in [69].

Proof of Lemma 3.9. Via replacing Lemma 3.8 by Lemma 3.10, and repeating the proof of Lemma 3.6, we prove the present lemma, which completes the proof of Lemma 3.9. 

Finally, we prove Theorem 1.1 by using Lemmas 3.5, 3.6, and 3.9.

Proof of Theorem 1.1. We first show (i). When $n \geq 3$, by Lemma 3.5, we find that the desired conclusion of (i) holds true in this case.

Now, we assume that $n = 2$. Denote by $L^*_D$ the adjoint operator of $L_D$. Then $L^*_D = -\text{div}(A^*\nabla \cdot)$, where $A^*$ is the transpose of the matrix $A$. Denote by $\{e^{-tL^*_D}\}_{t>0}$ the semigroup generated by $L^*_D$. Then, for any $t \in (0, \infty)$, $(e^{-tL^*_D})^* = e^{-tL_D}$ (see, for instance, [52, p. 41, Corollary 10.6]), where $(e^{-tL_D})^*$ denotes the adjoint operator $e^{-tL_D}$, which implies that, for any $t \in (0, \infty)$ and $x, y \in \Omega$, $K^L_D(t, x, y) = K^L_D(t, y, x)$. Here and thereafter, $\{K^L_D(t, x, y)\}_{t>0}$ denote the kernels of the semigroup $\{e^{-tL_D}\}_{t>0}$. Thus, to prove (i) in the case of $n = 2$, it suffices to show that there exists a $\delta_0 \in (0, 1]$ such that.
for any given $\delta \in (0, \delta_0)$, there exists a constant $c \in (0, \infty)$ such that, for any $t \in (0, \infty)$ and $x, y_1, y_2 \in \Omega$ with $|y_1 - y_2| \leq \sqrt{t}/2$,
\begin{equation}
(3.27) \quad \left| K_t^{L^p}(y_1, x) - K_t^{L^p}(y_2, x) \right| \leq \frac{1}{p t^{1/2}} \left[ |y_1 - y_2|^{\delta} \right] \exp \left\{ -\frac{|x - y_1|^2}{ct} \right\}.
\end{equation}

For any given $t \in (0, \infty)$ and $x \in \Omega$, let $u(\cdot) := K_t^{L^p}(\cdot, x)$ and $f(\cdot) := -\frac{d}{dt} K_t^{L^p}(\cdot, x)$. From the facts that, for any $t \in (0, \infty)$ and $x \in \Omega$, $K_t^{L^p}(\cdot, x), \frac{d}{dt} K_t^{L^p}(\cdot, x) \in \mathcal{D}(L^p) \subset W^{1,2}(\Omega)$, and $L_t^* e^{-tL_t^*} g = -\frac{d}{dt} e^{-tL_t^*} g$ for any $g \in L^2(\Omega)$, we deduce that
\begin{equation}
(3.28) \quad \begin{cases}
\text{div} (A^* \nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Let $p_0 \in (2, \infty)$ be as in Lemma 3.6, and $\delta_0 := 1 - 2/p_0$. For any given $\delta \in (0, \delta_0)$, take $\delta_1 \in (\delta, \delta_0)$ sufficiently large such that there exists a $p_1 \in (2, p_0)$ satisfying $\delta_1 := 1 - 2/p_1$. By (3.28) and Lemma 3.6, we conclude that
\begin{equation}
(3.29) \quad \left\| \nabla K_t^{L^{p_1}}(\cdot, x) \right\|_{L^{p_1}(\Omega; \mathbb{R}^n)} \leq \left| \frac{d}{dt} K_t^{L^{p_1}}(\cdot, x) \right|_{L^{p_1}(\Omega)}.
\end{equation}

Then, from $n = 2$, the Sobolev embedding theorem (see, for instance, [31, Theorem 7.26]), the Poincaré inequality (see, for instance, [5, Theorem 1.1]), and (3.29), it follows that $K_t^{L^{p_1}}(\cdot, x) \in C^{0, \delta_1}(\Omega)$ and, for any $t \in (0, \infty)$ and $x \in \Omega$,
\begin{equation}
(3.30) \quad \left\| K_t^{L^{p_1}}(\cdot, x) \right\|_{C^{0, \delta_1}(\Omega)} \leq \left\| K_t^{L^{p_1}}(\cdot, x) \right\|_{W^{1,p_1} (\Omega)} \leq \left\| \nabla K_t^{L^{p_1}}(\cdot, x) \right\|_{L^{p_1}(\Omega; \mathbb{R}^n)} \leq \left| \frac{d}{dt} K_t^{L^{p_1}}(\cdot, x) \right|_{L^{p_1}(\Omega)}.
\end{equation}

Here, $C^{0, \delta_1}(\Omega)$ denotes the Hölder space on $\Omega$, which is defined by setting
\[ C^{0, \delta_1}(\Omega) := \left\{ g \text{ is bounded and continuous on } \Omega : \|g\|_{C^{0, \delta_1}(\Omega)} < \infty \right\} \]

with \[ \|g\|_{C^{0, \delta_1}(\Omega)} := \sup_{x, y \in \Omega, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\delta_1}. \]

Furthermore, by [51, Theorem 6.17] and (1.5), we find that, for any $t \in (0, \infty)$ and $x, y \in \Omega$,
\[ \left| \frac{d}{dt} K_t^{L^{p_1}}(x, y) \right| \leq \frac{1}{p_1 t^{1+n/2}} \exp \left\{ -\frac{|x - y|^2}{ct} \right\}, \]
which further implies that
\begin{equation}
(3.31) \quad \left| \frac{d}{dt} K_t^{L^{p_1}}(\cdot, x) \right|_{L^{p_1}(\Omega)} \leq \left( \int_\Omega \frac{1}{t^{1+n/2}} e^{-\frac{|x - y|^2}{ct}} dy \right)^{1/(p_1)} \leq \frac{1}{t^{1+n/2}}.
\end{equation}
From (3.30), (3.31), and \( \frac{1}{(p_1)^1} - \frac{1}{p_1} = \frac{1}{n} \), we deduce that, for any \( t \in (0, \infty) \) and \( x \in \Omega \),
\[
(3.32) \quad \left\| K_t^{L_0} (\cdot, x) \right\|_{C^{0,1} (\Omega)} \leq \frac{1}{t^{n/2}} \frac{1}{t^{1+(\frac{n}{p_1})}} \sim \frac{1}{t^{n+\delta_1/2}}.
\]
Thus, by (3.32), we conclude that, for any \( t \in (0, \infty) \) and \( x, y_1, y_2 \in \Omega \),
\[
(3.33) \quad \left| K_t^{L_0} (y_1, x) - K_t^{L_0} (y_2, x) \right| \leq \frac{1}{t^{n/2}} \left\| y_1 - y_2 \right\|^{\delta_1}.
\]
On the other hand, from (1.5), it follows that, for any \( t \in (0, \infty) \) and \( x, y_1, y_2 \in \Omega \) with \( |y_1 - y_2| \leq \sqrt{t}/2 \),
\[
\left| K_t^{L_0} (y_1, x) - K_t^{L_0} (y_2, x) \right| \leq \left| K_t^{L_0} (y_1, x) \right| + \left| K_t^{L_0} (y_2, x) \right| \leq \frac{1}{t^{n/2}} \exp \left\{ -\frac{|x-y_1|^2}{ct} \right\},
\]
which, together with (3.33), further implies that
\[
\left| K_t^{L_0} (y_1, x) - K_t^{L_0} (y_2, x) \right| \leq \frac{1}{t^{n/2}} \left\| y_1 - y_2 \right\|^{\delta_1} \exp \left\{ -\frac{|x-y_1|^2}{ct} \right\}.
\]
This finishes the proof of (3.27), and hence of (i).

Next, we show (ii). To prove this, similarly to the proof of (i) in the case of \( n = 2 \), it suffices to show that, for any given \( \delta_0 \in (0, 1) \), there exists a constant \( \gamma_0 \in (0, \infty) \), depending on \( \delta_0, n, \) and \( \Omega \), such that, if \( A \) satisfies the \((\gamma, R)\)-BMO condition and \( \Omega \) is a \((\gamma, \sigma, R)\) quasi-convex domain for some \( \gamma \in (0, \gamma_0) \), \( \sigma \in (0, 1) \), and \( R \in (0, \infty) \), then, for any given \( \delta \in (0, \delta_0) \), there exists a positive constant \( c \in (0, \infty) \) such that, for any \( t \in (0, \infty) \) and \( x, y_1, y_2 \in \Omega \) with \( |y_1 - y_2| \leq \sqrt{t}/2 \),
\[
(3.34) \quad \left| K_t^{L_0} (y_1, x) - K_t^{L_0} (y_2, x) \right| \leq \frac{1}{t^{n/2}} \left\| y_1 - y_2 \right\|^{\delta} \exp \left\{ -\frac{|x-y_1|^2}{ct} \right\}.
\]
For any given \( \delta_0 \in (0, 1) \), let \( p \in (n, \infty) \) be such that \( \delta_0 = 1 - n/p \). By Lemma 3.9, we find that there exists a constant \( \gamma_0 \in (0, 1) \), depending on \( p, n, \) and \( \Omega \), such that, if \( A \) satisfies the \((\gamma, R)\)-BMO condition and \( \Omega \) is a \((\gamma, \sigma, R)\) quasi-convex domain for some \( \gamma \in (0, \gamma_0) \), \( \sigma \in (0, 1) \), and \( R \in (0, \infty) \), then
\[
(3.35) \quad \left\| \nabla K_t^{L_0} (\cdot, x) \right\|_{L^p (\Omega; \mathbb{R}^n)} \leq \left\| \frac{d}{dt} K_t^{L_0} (\cdot, x) \right\|_{L^p (\Omega)}.
\]
Replacing (3.29) by (3.35), and repeating the estimation of (3.27), we conclude that (3.34) holds true. This finishes the proof of (ii), and hence of Theorem 1.1. \( \square \)

4 Proof of Theorem 1.5

In this section, we show Theorem 1.5. We begin with recalling some concepts on the tent space \( T^p (\Omega \times (0, \infty)) \) (see, for instance, [54]).
Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain and $p \in (0, \infty)$. Then the tent space $T^p(\Omega \times (0, \infty))$ is defined to be the set of all measurable functions $f$ on $\Omega \times (0, \infty)$ such that $\|f\|_{T^p(\Omega \times (0, \infty))} := \|\mathcal{A}(f)\|_{L^p(\Omega)} < \infty$, where, for any $x \in \Omega$,

$$\mathcal{A}(f)(x) := \left[ \int_0^\infty \int_{\Gamma(x)} |f(y, t)|^2 \frac{dy dt}{|B_\Omega(x, t)|^p} \right]^{1/2},$$

where $\Gamma(x) := \{y \in \Omega : |x - y| < t\}$.

Moreover, assume that $O$ is an open subset of $\Omega$. Then the tent over $O$, denoted by $T(O)$, is defined by setting

$$T(O) := \{ (x, t) \in \Omega \times (0, \infty) : \text{dist} (x, O^c \cap \Omega) \geq t \}.$$

Let $p \in (0, 1]$ and $q \in (1, \infty)$. A measurable function $a$ on $\Omega \times (0, \infty)$ is called a $(p, q)$-atom if there exists a ball $B_\Omega$ of $\Omega$, which means $B_\Omega := B \cap \Omega$ and $B := B(x_B, r_B)$ is a ball of $\mathbb{R}^n$ with $x_B \in \Omega$ and $r_B \in (0, \text{diam} (\Omega))$, such that

(i) $\text{supp} (a) := \{(x, t) \in \Omega \times (0, \infty) : a(x, t) \neq 0\} \subset T(B_\Omega)$;

(ii) $\|a\|_{L^q(\Omega \times (0, \infty))} \leq |B_\Omega|^{1/q - 1/p}$.

Then we have the following atomic decomposition of $T^p(\Omega \times (0, \infty))$, which is a special case of [54, Theorem 1.1].

Lemma 4.2. Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain, $p \in (0, 1)$, and $q \in (1, \infty)$. Then, for any $f \in T^p(\Omega \times (0, \infty))$, there exist $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_j\}_{j \in \mathbb{N}}$ of $(p, q)$-atoms such that, for almost every $(x, t) \in \Omega \times (0, \infty)$, $f(x, t) = \sum_{j \in \mathbb{N}} \lambda_j a_j(x, t)$ and

$$C^{-1} \|f\|_{T^p(\Omega \times (0, \infty))} \leq \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{T^p(\Omega \times (0, \infty))},$$

where $C$ is a positive constant independent of $f$. Moreover, if $f \in T^p(\Omega \times (0, \infty)) \cap T^2(\Omega \times (0, \infty))$, then $f(x, t) = \sum_{j \in \mathbb{N}} \lambda_j a_j(x, t)$ holds true in both $T^p(\Omega \times (0, \infty))$ and $T^2(\Omega \times (0, \infty))$.

Next, we prove Theorem 1.5 by using Lemmas 2.2, 2.9, 2.11, 2.13, and 4.2.

Proof of Theorem 1.5. Let $p \in (\frac{n}{n+1}, 1]$. We first prove that

$$H^p_{\ell^p}(\Omega) = H^p(\Omega)$$

with equivalent quasi-norms. Let $f \in H^p_{\ell^p}(\Omega)$. Then there exists an $F \in H^p(\mathbb{R}^n)$ such that $F|_{\Omega} = f$ and

$$\|f\|_{H^p_{\ell^p}(\Omega)} \sim \|F\|_{H^p(\mathbb{R}^n)}.$$

Moreover, by Lemma 2.9, we find that there exist a sequence $\{a_j\}_{j \in \mathbb{N}}$ of $(p, \infty, 0)$-atoms, and $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ satisfying $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ such that $F = \sum_{j=1}^{\infty} \lambda_j a_j$ in $S'(\mathbb{R}^n)$, and

$$\|F\|_{H^p(\mathbb{R}^n)} \sim \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}.$$
To show $f \in H^p(\Omega)$ and $\|f\|_{H^p(\Omega)} \leq \|f\|_{H^p_0(\Omega)}$, it suffices to prove that, for any $(p, \infty, 0)$-atom $a$ supported in the ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, $a \in H^p(\Omega)$ and

\begin{equation}
\|a\|_{H^p(\Omega)} \leq 1.
\end{equation}

Indeed, if (4.4) holds true, from (4.2), (4.3), and (4.4), it follows that $f \in H^p(\Omega)$ and

$$
\|f\|_{H^p(\Omega)}^p = \|F\|_{H^p(\Omega)}^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p \|a_j\|_{H^p(\Omega)}^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p \sim \|f\|_{H^p_0(\Omega)}^p,
$$

which further implies that $\|f\|_{H^p(\Omega)} \leq \|f\|_{H^p_0(\Omega)}$.

Now, we prove (4.4) by considering the following two cases on the ball $B$.

**Case 1** $4B \subset \Omega$. In this case, by the definitions of $(p, \infty, 0)$-atoms, we conclude that $a$ is a type (a) $(p, \infty, 0)_{\Omega}$-atom, which, combined with Lemma 2.11, implies that $a \in H^p(\Omega)$ and (4.4) holds true.

**Case 2** $4B \cap \partial \Omega \neq \emptyset$. In this case, let $\phi$ be as in Definition 1.3(iv). We first claim that $\text{supp}(a_{\Omega}^\circ) \subset 8B_\Omega$. We prove this claim via borrowing some ideas from the proof of [7, Theorem 3.1]. Indeed, for any $x \in (8B_\Omega)^c$,

\begin{equation}
a_{\Omega}^\circ(x) = \sup_{r \in (0, \delta(x)/2)} \left| \int_{B_\Omega} \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) a(y) \, dy \right|
\leq \left[ \sup_{r \in (0, 7r_B)} + \sup_{r \in (7r_B, \delta(x)/2)} \right] \left| \int_{B_\Omega} \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) a(y) \, dy \right|
=: I_1 + I_2,
\end{equation}

where $\delta(x) := \text{dist}(x, \partial \Omega)$ with $\partial \Omega$ being the boundary of $\Omega$. From the fact that $|x-y| > 7r_B$ for any $y \in B_\Omega$ and $x \in (8B_\Omega)^c$, we deduce that, for any $t \in (0, 7r_B)$, $y \in B_\Omega$, and $x \in (8B_\Omega)^c$, $\phi\left(\frac{x-y}{t}\right) = 0$, which implies that $I_1 = 0$. Moreover, it is easy to see that, when $\delta(x) \leq 14r_B$, $I_2 = 0$. We assume that $\delta(x) > 14r_B$. In this case, we find that, for any $y \in B_\Omega$ and $x \in (8B_\Omega)^c$ with $\delta(x) > 14r_B$,

\begin{equation}
|x-y| \geq \delta(x) - \delta(y).
\end{equation}

Since $4B \cap \partial \Omega \neq \emptyset$, it follows that, for any $y \in B_\Omega$, $\delta(y) \leq 4r_B$, which, together with (4.6) implies that, for any $y \in B_\Omega$ and $x \in (8B_\Omega)^c$ with $\delta(x) > 14r_B$,

$$
|x-y| \geq \delta(x) - \delta(y) \geq \delta(x) - 4r_B > \delta(x) - \delta(x)/2 = \delta(x)/2 > t.
$$

By this, we conclude that, for any $t \in [7r_B, \delta(x)/2)$, $y \in B_\Omega$, and $x \in (8B_\Omega)^c$, $\phi\left(\frac{x-y}{t}\right) = 0$, which further implies that $I_2 = 0$. This, combined with (4.5) and $I_1 = 0$, concludes that, for any $x \in (8B_\Omega)^c$, $a_{\Omega}^\circ(x) = 0$. Thus, $\text{supp}(a_{\Omega}^\circ) \subset 8B_\Omega$.

Recall that the Hardy–Littlewood maximal operator $M$ on $\mathbb{R}^n$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$
M(f)(x) := \sup_{B \ni x} \int_B |f(y)| \, dy.
$$
where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing $x$. Then, from $\text{supp } (a^+_\Omega) \subset 8B$, the fact that $a^+_\Omega \leq M(a)$, the Hölder inequality, and the boundedness of $M$ on $L^q(\mathbb{R}^n)$ with $q \in (1, \infty)$, we deduce that

$$
\|a\|_{H^p(\Omega)} = \|a^+_\Omega\|_{L^p(\Omega)} = \|a^+_\Omega\|_{L^p(8B)} \leq \|M(a)\|_{L^q(8B)} |8B|^{1/p-1/q} \\
\leq \|a\|_{L^q(B)} |B|^{1/q-1/p} |B|^{1/p-1/q} \sim 1.
$$

Therefore, (4.4) holds true.

Next, we assume that $f \in H^p(\Omega)$. By Lemma 2.11, we find that there exist two sequences $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$, a sequence $\{a_j\}_{j=1}^\infty$ of type $(a)$ $(p, \infty, 0)_\Omega$-atoms, and a sequence $\{b_j\}_{j=1}^\infty$ of type $(b)$ $(p, \infty)_\Omega$-atoms such that

$$
f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j
$$

in $\mathcal{D}'(\Omega)$, and

$$
\|f\|_{H^p(\Omega)} \sim \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} + \left( \sum_{j=1}^\infty |\kappa_j|^p \right)^{1/p}.
$$

Let $b$ be a type $(b)$ $(p, \infty)_\Omega$-atom supported in the ball $B := B(x_B, r_B) \subset \Omega$ with $x_B \in \Omega$ and $r_B \in (0, \infty)$. From Lemma 2.2(ii), it follows that there exists a ball $\tilde{B}(x_{\tilde{B}}, r_{\tilde{B}}) \subset \Omega^C$ with $x_{\tilde{B}} \in \Omega^C$ and $r_{\tilde{B}} \in (0, \infty)$ such that $r_{\tilde{B}} \sim r_B$ and $\text{dist}(B, \tilde{B}) \sim r_B$. Assume that $B_0(x_{B_0}, r_{B_0}) \subset \mathbb{R}^n$ with $x_{B_0} \in \mathbb{R}^n$ and $r_{B_0} \in (0, \infty)$ is a ball such that $B \cup \tilde{B} \subset B_0$ and $r_{B_0} \sim r_B$. Let

$$
(4.8) \quad \overline{b} := b - \frac{1}{|B|} \int_B b(x) \, dx \mathbf{1}_{\overline{B}}.
$$

Then $|\overline{b}|_\Omega = b$, $\text{supp } (\overline{b}) \subset B_0$, and $\int_{\mathbb{R}^n} \overline{b}(x) \, dx = 0$. Moreover, it is easy to see that

$$
\left\|\overline{b}\right\|_{L^\infty(\mathbb{R}^n)} \leq |b|_{L^\infty(B)} \left[ 1 + \frac{|B|}{|B|} \right] \leq \|b\|_{L^\infty(B)} \leq |B|^{-1/p} \sim |B_0|^{-1/p}.
$$

Therefore, $\overline{b}$ is a harmless constant multiple of a $(p, \infty, 0)$-atom supported in the ball $B_0$.

For any $j \in \mathbb{N}$, let $\tilde{b}_j$ be as in (4.8) with $b$ replaced by $b_j$. Define

$$
\tilde{f} := \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j.
$$

Then $\tilde{f}|_{\Omega} = f$, $\tilde{f} \in H^p(\mathbb{R}^n)$, and

$$
\left\|\tilde{f}\right\|_{H^p(\mathbb{R}^n)} \leq \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} + \left( \sum_{j=1}^\infty |\kappa_j|^p \right)^{1/p}.
$$
which, together with (4.7), further implies that \( f \in H^p_0(\Omega) \) and \( \|f\|_{H^p_0(\Omega)} \leq \|f\|_{H^p(\Omega)} \). This finishes the proof of (4.1).

Let \( p \in \left( \frac{m+1}{m}, 1 \right] \), where \( \delta_0 \in (0, 1) \) is as in Theorem 1.1. Now, we prove that \( H^p_0(\Omega) = H^p_{\delta_0}(\Omega) \) with equivalent quasi-norms. To this end, we first show that

\[
(4.9) \quad \left[ H^p_0(\Omega) \cap L^2(\Omega) \right] \subset \left[ H^p_{\delta_0}(\Omega) \cap L^2(\Omega) \right].
\]

Let \( f \in [H^p_0(\Omega) \cap L^2(\Omega)] \). By the definition of \( H^p_0(\Omega) \), we conclude that there exists an \( \tilde{f} \in H^p(\mathbb{R}^n) \) such that \( \tilde{f} \mid_{\Omega} = f \) and

\[
(4.10) \quad \|\tilde{f}\|_{H^p(\mathbb{R}^n)} \leq \|f\|_{H^p_0(\Omega)}.
\]

Then there exist a sequence \( \{a_j\}_{j \in \mathbb{N}} \) of \( (p, \infty, 0) \)-atoms, and \( \{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) such that

\[
(4.11) \quad \tilde{f} = \sum_{j \in \mathbb{N}} \lambda_j a_j
\]

in both \( S'(\mathbb{R}^n) \) and \( (H^p(\mathbb{R}^n))^* \), and

\[
(4.12) \quad \|\tilde{f}\|_{H^p(\mathbb{R}^n)} \sim \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p},
\]

where \( (H^p(\mathbb{R}^n))^* \) denotes the dual space of \( H^p(\mathbb{R}^n) \). Denote by \( \{H^p_{L^0}\}_{l>0} \) the kernels of the family \( \{tL_D e^{-lD} \}_{l>0} \) of operators. From the fact that, for any \( t \in (0, \infty) \),

\[
tL_D e^{-lD} = 2 \left( t \frac{l}{2} L_D e^{-l/2L_D} \right) e^{-l/2L_D},
\]

we deduce that, for any \( t \in (0, \infty) \) and \( x, y \in \Omega \),

\[
(4.13) \quad H^p_{L^0}(x, y) = 2 \int_\Omega H^p_{L^0}(x, z) K^p_{L^0}(y, z) dz.
\]

Moreover, by [51, Theorem 6.17] and (1.5), we find that, for any \( t \in (0, \infty) \) and \( x, y \in \Omega \),

\[
(4.14) \quad |H^p_{L^0}(x, y)| \leq \frac{1}{p^{n/2}} \exp \left\{ -\frac{|x-y|^2}{ct} \right\},
\]

which, combined with (4.13) and Theorem 1.1, implies that, for any given \( \delta \in (0, \delta_0) \), and any \( t \in (0, \infty) \) and \( x, y, z \in \Omega \) with \( |y-z| \leq \sqrt{t}/2 \),

\[
(4.15) \quad |H^p_{L^0}(x, y) - H^p_{L^0}(x, z)| \leq \frac{1}{p^{n/2}} \left[ \frac{|y-z|}{\sqrt{t}} \right]^{\delta} \exp \left\{ -\frac{|x-y|^2}{ct} \right\}.
\]

For any \( t \in (0, \infty) \) and \( x \in \Omega \), denote by \( \tilde{H}^p_{L^0}(x, \cdot) \) the zero extension of \( H^p_{L^0}(x, \cdot) \) from \( \Omega \) to \( \mathbb{R}^n \). Let \( \gamma \in (0, \infty) \). Then the Campanato space \( \mathcal{L}^{\gamma, 1.0}(\mathbb{R}^n) \) is defined by setting

\[
\mathcal{L}^{\gamma, 1.0}(\mathbb{R}^n) := \left\{ g \in L^1_{\text{loc}}(\mathbb{R}^n) : \|g\|_{\mathcal{L}^{\gamma, 1.0}(\mathbb{R}^n)} < \infty \right\},
\]
where
\[
\|g\|_{L^{p}(\mathbb{R}^{n})} := \sup_{B \subset \mathbb{R}^{n}} \left| B \right|^{-\gamma} \int_{B} |g(x) - g_{B}| \, dx
\]
with the supremum taken over all balls \( B \subset \mathbb{R}^{n} \). Then, similarly to the proof of [14, Lemma 3.9], via applying Theorem 1.1 and Lemma 2.2, we find that, for any given \( x \in \Omega \), and \( \delta \in (0, \delta_{0}) \), \( H_{L}^{B}(x, \cdot) \in L^{2/n, 1}(\mathbb{R}^{n}) \). From this, the fact that \( L^{2/n, 1}(\mathbb{R}^{n}) \) is the dual space of \( H^{p}(\mathbb{R}^{n}) \) (see, for instance, [61]), \( p \in \left( \frac{n}{n+\delta_{0}}, 1 \right) \), and (4.11), it follows that, for any \( t \in (0, \infty) \) and \( x \in \Omega \),
\[
\int_{\Omega} H_{L}^{B}(x, y) f(y) \, dy = \int_{\mathbb{R}^{n}} \overline{H}_{L}^{B}(x, y) \overline{f}(y) \, dy = \sum_{j \in \mathbb{N}} \lambda_{j} \int_{\mathbb{R}^{n}} \overline{H}_{L}^{B}(x, y) a_{j}(y) \, dy
\]
\[
= \sum_{j \in \mathbb{N}} \lambda_{j} \int_{\Omega} H_{L}^{B}(x, y) a_{j}(y) \, dy,
\]
which, together with the boundedness of \( S_{L_{D}} \) on \( L^{2}(\Omega) \) (see, for instance, [6, Theorem 2.13]), further implies that, for almost every \( x \in \Omega \),
\[
(4.16) \quad S_{L_{D}}(f)(x) = \sum_{j \in \mathbb{N}} |\lambda_{j}| S_{L_{D}}(a_{j})(x).
\]
To prove (4.9), we only need to show that, for any \((p, \infty, 0)\)-atom \( a \) supported in the ball \( B := B(x_{B}, r_{B}) \) with \( x_{B} \in \mathbb{R}^{n} \) and \( r_{B} \in (0, \infty) \),
\[
(4.17) \quad \int_{\Omega} \left[ S_{L_{D}}(a)(x) \right]^{p} \, dx \leq 1.
\]
Indeed, if (4.17) holds true, then, by (4.16), (4.17), (4.10), (4.11), and (4.12), we conclude that \( f \in H_{L_{D}}^{p}(\Omega) \) and
\[
\|f\|_{H_{L_{D}}^{p}(\Omega)} \leq \left\{ \sum_{j \in \mathbb{N}} |\lambda_{j}|^{p} \int_{\Omega} \left[ S_{L_{D}}(a_{j})(x) \right]^{p} \, dx \right\}^{1/p} \leq \left( \sum_{j \in \mathbb{N}} |\lambda_{j}|^{p} \right)^{1/p}
\]
\[
\sim \|f\|_{H^{p}p}(\mathbb{R}^{n}) \leq \|f\|_{L^{p}(\mathbb{R}^{n})}.
\]
Thus, (4.9) holds true.

Next, we show (4.17) by considering the following three cases on the ball \( B \).

Case 1) \( B \cap \Omega = \emptyset \). In this case, we find that, for any \( x \in \Omega \),
\[
t^{2} L_{D} e^{-t^{2} L_{D}}(a)(x) = \int_{B \cap \Omega} H_{L}^{B}(x, y) a(y) \, dy = 0,
\]
which implies that \( S_{L_{D}}(a) = 0 \). From this, we deduce that (4.17) holds true in this case.

Case 2) \( B \subset \Omega \). In this case, by the fact that \( S_{L_{D}} \) is bounded on \( L^{q}(\Omega) \) with \( q \in (1, \infty) \) (see, for instance, [6, Theorem 2.13]), we conclude that
\[
(4.18) \quad \int_{4B_{\Omega}} \left[ S_{L_{D}}(a)(x) \right]^{p} \, dx \leq \|S_{L_{D}}(a)\|_{L^{q}(4B_{\Omega})}^{p} |4B_{\Omega}|^{1/\overline{q}} \leq \|a\|_{L^{p}(4B_{\Omega})}^{p} |4B_{\Omega}|^{1/\overline{q}} \leq 1.
\]
Furthermore, for any \( x \in (4B)^C \cap \Omega \), we have

\[
(4.19) \quad [S_{LD}(a)(x)]^2 = \int_0^r \int_{|y-x|<t} |t^2 L_De^{-t^2 L_D}(a)(y)|^2 \frac{dy \, dt}{|B(x,t)|} + \int_0^\infty \int_{|y-x|<t} \cdots = E + F.
\]

It is easy to see that, for any \( x \in (4B)^C \cap \Omega \), and any \( t \in (0, \infty) \) and \( y, z \in \Omega \) satisfying \( |x-y| < t \) and \( |z - x_B| < r_B \),

\[
(4.20) \quad t + |y - z| \geq t + |x - x_B| - |x - y| - |x_B - z| > |x - x_B| - r_B \geq \frac{3|x - x_B|}{4}.
\]

Take \( \delta_1 \in (0, \delta) \) such that \( p > \frac{n}{n + \delta_1} \). From (4.14), (4.20), and the Hölder inequality, it follows that

\[
(4.21) \quad E = \int_0^r \int_{|y-x|<t} \left\| \int_B H_{L_D}^D(y,z)a(z) \, dz \right\|^2 \frac{dy \, dt}{|B(x,t)|} \\
\leq \int_0^r \int_{|y-x|<t} \left\| \int_B \frac{1}{t^n} e^{-\frac{|y-z|^2}{ct}} a(z) \, dz \right\|^2 \frac{dy \, dt}{t^{n+1}} \\
\leq \int_0^r \int_{|y-x|<t} \left\| \int_B \frac{r_B^{\delta_1}}{|y-z|^{n+\delta_1}} |a(z)| \, dz \right\|^2 \frac{dy \, dt}{t^{n+1}} \\
\leq \frac{1}{|x - x_B|^{2(n+\delta_1)}} \left[ \int_B |a(z)| \, dz \right]^2 \int_0^r t^{2\delta_1-1} \, dt \\
\leq \left[ \frac{r_B}{|x - x_B|^{2(n+\delta_1)}} \right]^{2(n+\delta_1)} |B|^{-2/p}.
\]

Next, we deal with the term \( F \). By \( \int_B a(x) \, dx = 0 \), (4.15), (4.20), \( t > r_B \), and the Hölder inequality, we find that, for any \( y \in \Omega \),

\[
|t^2 L_De^{-t^2 L_D}(a)(y)| \\
\leq \int_B \left| H_{L_D}^D(y,z) - K_{L_D}^D(y,x_B) \right| |a(z)| \, dz \\
\leq \int_B \frac{1}{t^n} \left| \frac{|z-x_B|^\delta}{t} \right| e^{-\frac{|y-z|^2}{ct}} |a(z)| \, dz \\
\leq \int_B \left[ \frac{r_B^{\delta_1}}{|y-z|^{n+\delta_1}} |a(z)| \, dz \right] \\
\leq \int_B \left[ \frac{r_B^{\delta_1}}{t^n} \right] |a(z)| \, dz \leq \frac{r_B^{2n+\delta}}{|x-x_B|^{2(n+\delta_1)} |B|^{-1/p}}.
\]

From this, we deduce that

\[
F \leq \frac{r_B^{2n+\delta}}{|x-x_B|^{2(n+\delta_1)}} \int_0^r \int_{|y-x|<t} t^{-2(2\delta_1)} \frac{dy \, dt}{|B(x,t)|}.
\]
Let $f(x)$ and $t$ which implies that $\omega \in L^2(\Omega)$, $\mathcal{H}^{\ldots}(x,y) = 0$, which implies that, for any given $t \in (0,\infty)$ and any $x \in \Omega$, $\int_{\Omega(4B_2)} [S_L^a(a(x))]^p \, dx = \sum_{j=2}^{\infty} [S_L^a(a(x))]^p \, dx 
leq \sum_{j=2}^{\infty} 2^{-p(n+\delta_1)j}[B]^{-1} \, dx \nleq \sum_{j=2}^{\infty} 2^{-p(n+\delta_1)j}[B]^{-1} \, dx \approx 1,$

which, together with (4.18), further implies that (4.17) holds true in this case.

Case 3) $B \cap \Omega \neq \emptyset$. In this case, take $y_B \in B \cap \Omega$. From the fact that, for any given $t \in (0,\infty)$ and $x \in \Omega$, $\mathcal{H}^{\ldots}(x,y) = 0$, which implies that, for any given $t \in (0,\infty)$ and any $x \in \Omega$, $\int_{B_2} [H^{\ldots}(x,y)-H^{\ldots}(x,y)] \, a(y)dy.$

The remaining estimations are similar to those of Case 2, we omit the details here. This finishes the proof of (4.17), and hence of (4.9).

Now, we prove that

(4.22) \[ \mathcal{H}^{\ldots}(\Omega) \subset \mathcal{H}^{\ldots}(\Omega) \subset \mathcal{H}^{\ldots}(\Omega). \]

Let $f \in [\mathcal{H}^{\ldots}(\Omega) \cap L^2(\Omega)]$. Then, by the $H^{\ldots}$-functional calculus associated with $L_{\Omega}$ (see, for instance, [4, p. 8]), we find that

(4.23) \[ f = 8 \int_{0}^{\infty} \left( t^2 L_{\Omega} e^{-t^2 L_{\Omega}} \right) \left( t^2 L_{\Omega} e^{-t^2 L_{\Omega}} \right) (f) \, dt, \]

in $L^2(\Omega)$. From the assumption $f \in [\mathcal{H}^{\ldots}(\Omega) \cap L^2(\Omega)]$, we deduce that $S_L^a(f) \in L^p(\Omega) \cap L^2(\Omega)$, which implies that

(4.24) \[ t^2 L_{\Omega} e^{-t^2 L_{\Omega}} (f) \in T^p(\Omega) \times (0,\infty) \cap T^2(\Omega) \times (0,\infty) \]

and

(4.25) \[ \|f\|_{\mathcal{H}^{\ldots}(\Omega)} = \|t^2 L_{\Omega} e^{-t^2 L_{\Omega}} (f)\|_{T^p(\Omega) \times (0,\infty)} . \]

By (4.24) and Lemma 4.2, we conclude that there exist $\lambda_j \in \mathbb{C}$ and a sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ of $(p, 2)$-atoms associated, respectively, with the balls $\{B_j \cap \Omega\}_{j \in \mathbb{N}}$ of $\Omega$ such that, for almost every $(x,t) \in \Omega \times (0,\infty)$,

(4.26) \[ t^2 L_{\Omega} e^{-t^2 L_{\Omega}} (f) = \sum_{j \in \mathbb{N}} \lambda_j a_j \]
and

\[ \left\| t^2 L_D e^{-t^2 L_D} (f) \right\|_{L^p(\Omega \times (0, \infty))} \sim \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}. \]

For any \( j \in \mathbb{N} \), let \( \alpha_j := 8 \int_0^\infty t^2 L_D e^{-t^2 L_D} (a_j(\cdot, t)) \frac{dt}{t} \). Then, from (4.23) and (4.26), it follows that

\[ f = \sum_{j \in \mathbb{N}} \lambda_j \alpha_j \]
in \( L^2(\Omega) \).

For any \((p, 2)\)-atom \( a \) associated with the ball \( B_\Omega \), let

\[ \alpha := 8 \int_0^\infty t^2 L_D e^{-t^2 L_D} (a) \frac{dt}{t}. \]

To show (4.22), it suffices to prove that there exist a function \( \overline{\alpha} \) on \( \mathbb{R}^n \) such that

\[ \overline{\alpha}|_\Omega = \alpha, \]
and a sequence \( \{\kappa_i\} \subset \mathbb{C} \) and a sequence \( \{b_i\} \) of \((p, 2, 0)\)-atoms such that \( \overline{\alpha} = \sum_i \kappa_i b_i \) in \( L^2(\mathbb{R}^n) \), and

\[ \sum_i |\kappa_i|^p \lesssim 1. \]

Indeed, if (4.29) and (4.30) hold true, then, by (4.28), we find that, for any \( j \in \mathbb{N} \), there exists a function \( \overline{\alpha}_j \) on \( \mathbb{R}^n \) such that \( \overline{\alpha}_j|_\Omega = \alpha_j \). Let

\[ \overline{f} := \sum_{j \in \mathbb{N}} \lambda_j \overline{\alpha}_j. \]

Then \( \overline{f}|_\Omega = f \). Furthermore, from (4.30), we deduce that there exist a sequence \( \{\kappa_{i,j}\} \subset \mathbb{C} \) and a sequence \( \{b_{i,j}\} \) of \((p, 2, 0)\)-atoms such that

\[ \overline{f} = \sum_{j \in \mathbb{N}} \sum_i \lambda_j \kappa_{j,i} b_{j,i} \]
and

\[ \sum_{j \in \mathbb{N}} |\lambda_j \kappa_{j,i}|^p \lesssim \sum_{j \in \mathbb{N}} |\lambda_j|^p. \]

By this, Lemma 2.9, (4.25), and (4.27), we conclude that \( \overline{f} \in H^p(\mathbb{R}^n) \) and

\[ \left\| \overline{f} \right\|_{H^p(\mathbb{R}^n)} \sim \left\| \overline{f} \right\|_{H^{p,2,0}(\mathbb{R}^n)} \lesssim \|f\|_{H^p_{L_D}(\Omega)}. \]

Thus, \( f \in H^p_{L_D}(\Omega) \) and

\[ \|f\|_{H^p_{L_D}(\Omega)} \lesssim \|f\|_{H^p_{L_D}(\Omega)}. \]
which, combined with the arbitrariness of \( f \in H^p (\Omega) \cap L^2 (\Omega) \), implies that (4.22) holds true.

Next, we prove (4.29) and (4.30) by considering the following two cases on the ball \( B := B(x_B, r_B) \), with \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), which appears in the support of \( a \).

**Case 1)** \( 8B \cap \Omega^C \neq \emptyset \). In this case, let \( S_0(B_\Omega) := \bigcup_{j=0}^2 S_j(B_\Omega) \) and

\[
J_\Omega := \{ k \in \mathbb{N} : k \geq 3, |S_k(B_\Omega)| > 0 \}.
\]

Moreover, assume that \( 1_0 := 1_{S_0(B_\Omega)} \), \( m_0 := \int_{S_0(B_\Omega)} \alpha(x) \, dx \), and, for any \( k \in J_\Omega \), \( 1_k := 1_{S_k(B_\Omega)} \) and \( m_k := \int_{S_k(B_\Omega)} \alpha(x) \, dx \). Then

\[
\alpha = \alpha 1_0 + \sum_{k \in J_\Omega} \alpha 1_k
\]

holds true almost everywhere and also in \( L^2 (\Omega) \). From the assumption \( 8B \cap \Omega^C \neq \emptyset \), it follows that there exists a \( y_B \in \partial \Omega \) such that \( B(y_B, 16r_B) \supset 8B \). By this and Lemma 2.2(iii), we find that there exists a ball \( B \subset \Omega^C \) such that \( r_B \sim r_B \) and \( \text{dist} (B_\Omega, B) \sim r_B \). Therefore, there exists a ball \( B_0^* \subset \mathbb{R}^n \) such that \( (B \cup \bar{B}) \cap B_0^* \) and

\[
r_B^* \sim r_B.
\]

Let

\[
b_0 := \alpha 1_0 - \left[ \frac{1}{|B|} \int_{S_0(B_\Omega)} \alpha(x) \, dx \right] 1_{\bar{B}}.
\]

Then \( \int_{\mathbb{R}^n} b_0(x) \, dx = 0 \) and \( \text{supp} (b_0) \subset B_0^* \). Moreover, it is known that \( \| \alpha \|_{L^2 (\Omega)} \leq \| \alpha \|_{L^2 (\Omega; (0, \infty))} \) (see, for instance, [6, Proposition 4.5]), which further implies that

\[
\| \alpha \|_{L^2 (\Omega)} \leq \| \alpha \|_{L^2 (\Omega; (0, \infty))} \leq |B_\Omega|^{1/2-1/p}.
\]

From this, \( r_B^* \sim r_B \), Lemma 2.2(i), and (4.31), we deduce that

\[
\| b_0 \|_{L^2 (\mathbb{R}^n)} \leq \| \alpha \|_{L^2 (\Omega)} + \| \alpha \|_{L^2 (\Omega; (0, \infty))} |\bar{B}|^{-1/2} |B|^{1/2} \leq \| \alpha \|_{L^2 (\Omega)} \leq |B_\Omega|^{1/2-1/p} \sim |B_0^*|^{1/2-1/p}.
\]

Therefore, \( b_0 \) is a harmless constant multiple of a \((p, 2, 0)\)-atom.

For any \( k \in J_\Omega \), by the definition of \( J_\Omega \), we find that \( 2^k r_B \leq \text{diam} (\Omega) \) and \( 2^k B \cap \partial \Omega \neq \emptyset \). By this, we conclude that there exists a \( y_B \in \partial \Omega \) such that \( B(y_B, 2^{k+1} r_B) \supset 2^k B \), which, together with Lemma 2.2(iii), further implies that there exists a ball \( \bar{B}_k \subset \Omega^C \) such that \( r_{\bar{B}_k} \sim 2^k r_B \) and \( \text{dist} (S_k(B_\Omega), \bar{B}_k) \leq 2^k r_B \). Let the ball \( B_k^* \subset \mathbb{R}^n \) satisfy that \( \bar{B}_k \cup S_k(B_\Omega) \subset B_k^* \) and \( r_{B_k^*} \sim r_{2^k B} \). Assume that, for any \( k \in J_\Omega \),

\[
b_k := \alpha 1_k - \left[ \frac{1}{|\bar{B}_k|} \int_{S_k(B_\Omega)} \alpha(x) \, dx \right] 1_{\bar{B}_k}.
\]

Then, for any \( k \in J_\Omega \), \( \text{supp} (b_k) \subset B_k^* \) and \( \int_{\mathbb{R}^n} b_k(x) \, dx = 0 \). Furthermore, from the Hölder inequality and Lemma 2.2(i), it follows that

\[
\| b_k \|_{L^2 (\mathbb{R}^n)} \leq \| \alpha \|_{L^2 (\Omega; (0, \infty))} + \| \alpha \|_{L^2 (\Omega; (0, \infty))} |\bar{B}_k|^{-1/2} |S_k(B_\Omega)|^{1/2}
\]
Moreover, we have, for any $k \in J_\Omega$ with $k \geq 3$, and any $x \in S_k(B_\Omega)$,
\begin{equation}
|\alpha(x)| \leq \int_0^s \int_{B_\Omega} |H_{r_0^k}^k(x, y)| |a(y, t)| \frac{dy dt}{t} \\
\leq \int_0^s \int_{B_\Omega} \frac{1}{r_0^k} e^{-\frac{|x-y|^2}{r_0^k}} |a(y, t)| \frac{dy dt}{t} \\
\leq \|a\|_{T^2(\Omega \times (0, \infty))} \left\{ \int_0^s \int_{B_\Omega} \frac{r_0^k}{|x-y|^{2(n+1)}} \frac{dy dt}{t} \right\}^{1/2} \\
\leq |x-x_B|^{-(n+1)} r_B |B_\Omega|^{1/2} \|a\|_{T^2(\Omega \times (0, \infty))} \lesssim 2^{-(n+1)} |B_\Omega|^{1/p},
\end{equation}
which, together with (4.33), further implies that
\begin{equation}
\|b_k\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-2(n/2+1)} |B_\Omega|^{1/2-1/p} \lesssim 2^{-2s_0} |B_k|^{1/2-1/p},
\end{equation}
where $s_0 := n \left( 1 + \frac{1}{n} - \frac{1}{p} \right)$. By this, supp $(b_k) \subset B_k^r$, and $\int_{\mathbb{R}^n} b_k(x) \, dx = 0$, we conclude that $2^{s_0} b_k$ is a harmless constant multiple of a $(p, 2, 0)$-atom. Let
\[
\overline{\alpha} := b_0 + \sum_{k \in J_\Omega} 2^{-s_0} (2^{s_0} b_k).
\]
Then $\overline{\alpha}|_{\Omega} = \alpha, \overline{\alpha} \in H^p(\mathbb{R}^n)$, and $\|\overline{\alpha}\|_{H^p(\mathbb{R}^n)} \lesssim \frac{1}{p}$. 

Case 2) $8B \subset \Omega$. In this case, let $k_0 \in \mathbb{N}$ be such that $2^{k_0} B \subset \Omega$ but $(2^{k_0+1} B) \cap \partial \Omega \neq \emptyset$. Then $k_0 \geq 3$. Let 
\[
J_{\Omega, k_0} := \{ k \in \mathbb{N} : k \geq k_0 + 1, |S_k(B_\Omega)| > 0 \}.
\]
For any $k \in \mathbb{Z}_+$, let $1_k := 1_{S_k(B_\Omega)}$,
\[
m_k := \int_{S_k(B_\Omega)} \alpha(x) \, dx,
\]
\[
M_k := \alpha 1_k - m_k |S_k(B_\Omega)|^{-1} 1_k,
\]
and $\overline{M}_k := \alpha 1_k$. Then
\[
\alpha = \sum_{k=0}^{k_0} M_k + \sum_{k \in J_{\Omega, k_0}} M_k + \sum_{k=0}^{k_0} m_k |S_k(B_\Omega)|^{-1} 1_k.
\]
For any $k \in \{ 0, \ldots, k_0 \}$, from the definition of $M_k$, we deduce that 
\[
\int_{\mathbb{R}^n} M_k(x) \, dx = 0 \quad \text{and} \quad \text{supp} (M_k) \subset 2^{k+1} B.
\]
Moreover, if $k = 0$, by the Hölder inequality and (4.32), we find that
\begin{equation}
\|M_0\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(\Omega)} \lesssim \|\alpha\|_{L^2(\Omega)} \lesssim |B|^{1/2-1/p}
\end{equation}
and, if $k \in \{ 1, \ldots, k_0 \}$, from (4.34), it follows that
\begin{equation}
\|M_k\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(S_k(B_\Omega))} \lesssim 2^{-2s_0} |2^{k+1} B|^{1/2-1/p},
\end{equation}

\[
\lesssim \|\alpha\|_{L^2(S_k(B_\Omega))}.
\]
where $s_0$ is as in (4.35). Thus, for any $k \in \{0, \ldots, k_0\}$, $2^{s_0 k} M_k$ is a harmless constant multiple of a $(p, 2, 0)$-atom.

For any $k \in J_{\Omega, k_0}$, by the definitions of both $k_0$ and $J_{\Omega, k_0}$, we have $2^k r_B \leq \text{diam} (\Omega)$ and $2^k B \cap \partial \Omega \neq \emptyset$. By this, we conclude that there exists a $y_B \in \partial \Omega$ such that $B(y_B, 2^{k+1} r_B) \supset B^k$, which, combined with Lemma 2.2(iii), implies that there exists a ball $\overline{B_k} \subset \Omega^C$ such that $r_{\overline{B_k}} \sim 2^k r_B$ and $\text{dist} (S_k(B_\Omega), \overline{B_k}) \lesssim 2^k r_B$. Then there exists a ball $B_k^*$ such that $\overline{B_k} \cup S_k(B_\Omega) \subset B_k^*$ and $r_{B_k^*} \sim 2^k r_B$. Let

$$a_k := \alpha 1_{S_k(B_\Omega)} - \frac{1}{|B_k|} \int_{S_k(B_\Omega)} \alpha (x) \, dx \, 1_{\overline{B_k}}.$$ 

Then $\text{supp} (a_k) \subset B_k^*$ and $\int_{\mathbb{R}^n} a_k (x) \, dx = 0$. Moreover, from (4.34) and Lemma 2.2(i), we deduce that

$$\|a_k\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(S_k(B_\Omega))} \lesssim 2^{-s_0 k} |B_k^*|^{1/2 - 1/p}.$$

Thus, for any $k \in J_{\Omega, k_0}$, $2^{s_0 k} a_k$ is a harmless constant multiple of a $(p, 2, 0)$-atom. For any $j \in \{0, \ldots, k_0\}$, let $N_j := \sum_{k=j}^{k_0} m_k$. It is easy to see that

$$\sum_{k=0}^{k_0} m_k |S_k(B_\Omega)|^{-1} 1_k = \sum_{k=1}^{k_0} \left[ |S_k(B_\Omega)|^{-1} 1_k - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1} \right] N_k + N_0 |B|^{-1} 1_0.$$

For any $k \in \{1, \ldots, k_0\}$, by

$$\|S_k(B_\Omega)|^{-1} 1_k - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1} \| \leq |2^k B|^{-1},$$

the Hölder inequality, and (4.34), we find that

$$\left\| \left| S_k(B_\Omega) \right|^{-1} 1_k - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1} \right\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(S_k(B_\Omega))} |S_j(B_\Omega)|^{1/2} \sum_{j=k}^{k_0} \left( \sum_{l=0}^{k_0} m_l |S_l(B_\Omega)|^{-1} 1_l \right)^{1/2} \lesssim \|\alpha\|_{L^2(S_k(B_\Omega))} |S_j(B_\Omega)|^{1/2} \sum_{j=k}^{k_0} \left( \sum_{l=0}^{k_0} m_l |S_l(B_\Omega)|^{-1} 1_l \right)^{1/2} \lesssim 2^{-s_0 k} |2^k B|^{1/2 - 1/p} \ ,$$

where $s_0$ is as in (4.35), which, together with

$$\int_{\mathbb{R}^n} \left| \left| S_k(B_\Omega) \right|^{-1} 1_k (x) - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1} (x) \right| \, dx = 0$$

and $\text{supp} (|S_k(B_\Omega)|^{-1} 1_k - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1}) \subset 2^k B$, implies that, for any $k \in \{1, \ldots, k_0\}$, the function $2^{s_0 k} |S_k(B_\Omega)|^{-1} 1_k - |S_{k-1}(B_\Omega)|^{-1} 1_{k-1} |N_k$ is a harmless constant multiple of a $(p, 2, 0)$-atom.

Finally, we deal with $N_0 |B|^{-1} 1_0$. From $2^{s_0 - 1} r_0 < \text{dist} (x_0, \partial \Omega) \leq 2^{s_0} r_0$, it follows that there exist a positive integer $K$ and a sequence $\{B_{0,i}\}_{i=1}^K$ of balls such that

(i) $K \sim 2^{s_0}$.
(ii) for any $i \in \{1, \ldots, K\}$, $r_{B_{0,i}} = 2r_0$ and $B_{0,i} \subset \Omega$;

(iii) for any $i \in \{1, \ldots, K - 1\}$, $B_{0,i} \cap B_{0,i+1} \neq \emptyset$ and $\text{dist}(B_{0,i}, \partial \Omega) \geq \text{dist}(B_{0,i+1}, \partial \Omega)$;

(iv) $2B_{0,K} \cap \partial \Omega \neq \emptyset$.

By Lemma 2.2(ii), we conclude that there exists a ball $B_{0,K+1} \subset \Omega^C$ such that $r_{B_{0,K+1}} \sim r_0$ and $\text{dist}(B_{0,K}, B_{0,K+1}) \sim r_0$. Let

$$a_{0,1} := N_0|2B|^{-1}1_0 - N_0|B_{0,1}|^{-1}1_{B_{0,1}}$$

and

$$a_{0,i} := N_0|B_{0,i-1}|^{-1}1_{B_{0,i-1}} - N_0|B_{0,i}|^{-1}1_{B_{0,i}}$$

with $i \in \{2, \ldots, K + 1\}$. Obviously, for any $i \in \{1, \ldots, K + 1\}$, from the definition of $a_{0,i}$, we deduce that $\int_{\mathbb{R}^n} a_{0,i}(x) \, dx = 0$ and there exists a ball $B_{0,i}^* \subset \mathbb{R}^n$ such that $\text{supp}(a_{0,i}) \subset B_{0,i}^*$ and

$$r_{B_{0,i}^*} \sim r_B.$$

Moreover, similarly to the estimation of [65, (3.66)], we have

$$|N_0| \lesssim 2^{-\frac{n+1}{2}k_0}|B|^{1-1/p}.$$  

For any $i \in \{1, \ldots, K + 1\}$, by the definition of $a_{0,i}$, (4.40), and (4.41), we conclude that

$$\|a_{0,i}\|_{L^2(\mathbb{R}^n)} \lesssim |N_0||B|^{-1/2} \lesssim 2^{-\frac{n+1}{2}k_0}|B|^{-1/2} \sim 2^{-\frac{n+1}{2}k_0}|B_{0,i}^*|^{1/2-1/p},$$

which, combined with the facts that $\int_{\mathbb{R}^n} a_{0,i}(x) \, dx = 0$ and $\text{supp}(a_{0,i}) \subset B_{0,i}^*$, further implies that $2^{-\frac{n+1}{2}k_0}a_{0,i}$ is a harmless constant multiple of a $(p, 2, 0)$-atom. Let

$$\bar{\alpha} := \sum_{k=1}^{k_0} M_k + \sum_{k \in J_{\Omega}, k_0} a_k + \sum_{k=1}^{k_0} \left[|S_k(B_{0,1})|^{-1}1_k - |S_{k-1}(B_{0,1})|^{-1}1_{k-1}\right]N_k + \sum_{i=1}^{K+1} a_{0,i}.$$ 

It is easy to find that $\bar{\alpha}|_\Omega = \alpha$. Moreover, from (4.36), (4.37), (4.38), (4.39), and (4.42), it follows that $\bar{\alpha}$ has the following atomic decomposition

$$\bar{\alpha} = \sum_{k=1}^{k_0} 2^{-s_0k} \left(2^{s_0k} M_k + \sum_{k \in J_{\Omega}, k_0} 2^{-s_0k} a_k\right)$$

$$+ \sum_{k=1}^{k_0} 2^{-s_0k} \left[2^{s_0k} \left|S_k(B_{0,1})\right|^{-1}1_k - |S_{k-1}(B_{0,1})|^{-1}1_{k-1}\right]N_k$$

$$+ \sum_{i=1}^{K+1} 2^{-\frac{n+1}{2}k_0} \left[2^{\frac{n+1}{2}k_0}a_{0,i}\right]$$

and

$$\sum_{k=1}^{k_0} 2^{s_0k} p + \sum_{k \in J_{\Omega}, k_0} 2^{s_0k} p + \sum_{k=1}^{k_0} 2^{s_0k} p + \sum_{i=1}^{K+1} 2^{\frac{n+1}{2}k_0}p$$
Proof. Assume that \( \eta \) problem (1.1). Let (5.1)
\[ -H \]
which, together with the fact that (1.8) and (4.30), and hence of (4.!2).
By (4.9) and (4.22), we obtain
\[ \left[ H^p_{L^p_0}(\Omega) \cap L^2(\Omega) \right] = \left[ H^p_{L^p_0}(\Omega) \cap L^2(\Omega) \right], \]
which, together with the fact that \( H^p_{L^p_0}(\Omega) \cap L^2(\Omega) \) and \( H^p_{L^p_0}(\Omega) \cap L^2(\Omega) \) are dense, respectively, in the spaces \( H^p_{L^p_0}(\Omega) \) and \( H^p_{L^p_0}(\Omega) \), and a density argument, implies that \( H^p_{L^p_0}(\Omega) \) and \( H^p_{L^p_0}(\Omega) \) coincide with equivalent quasi-norms. This finishes the proof of Theorem 1.5. \( Q.E.D. \)

5 Proof of Theorem 1.8

In this section, we prove Theorem 1.8. We begin with establishing the following estimates for the kernels of the family \( \{(tL_D)^k e^{-tL_D}\}_{t>0} \) of operators.

**Lemma 5.1.** Let \( n \geq 2, \, \Omega \subset \mathbb{R}^n \) be a bounded NTA domain, the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3), and \( L_D \) be as in (1.4). Assume that \( p_0 \in (2, \infty) \) as in Lemma 3.6 and \( q \in (2, p_0) \). For any given \( k \in \mathbb{N} \), denote by \( \{K^L_{t,k}\}_{t>0} \) the kernels of the family \( \{(tL_D)^k e^{-tL_D}\}_{t>0} \) of operators. Then there exist positive constants \( C \) and \( c \), depending on \( n, \, p, \, k, \) and \( \Omega \), such that, for any \( r \in (0, \text{diam} (\Omega)), \, y \in \Omega, \) and \( t \in (0, \infty), \)
\[
\frac{\int_{|x-y| \leq 2r} |\nabla_x K^L_{t,k}(x, y)|^q dx}{r^{\frac{nq}{2}} e^{-\frac{Cq}{2}}} \leq Cr^{\frac{nq}{2}} e^{-\frac{Cq}{2}}.
\]

**Proof.** Assume that \( f \in L^2(\Omega) \) and \( u \in W^{1,2}_0(\Omega) \) is a weak solution of the Dirichlet boundary value problem (1.1). Let \( \eta \in C^\infty_c(\mathbb{R}^n) \). Then
\[
-(\nabla (A\nabla (u\eta))) = -\nabla (u A \nabla \eta) - A \nabla u \cdot \nabla \eta + f \eta
\]
in the sense of (1.8). Indeed, for any \( \varphi \in C^\infty_c(\Omega), \)
\[
\int_{\Omega} A(x) \nabla (u \eta)(x) \cdot \nabla \varphi(x) \, dx
\]
\[
= \int_{\Omega} A(x) \eta(x) \nabla u(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} A(x) u(x) \nabla \eta(x) \cdot \nabla \varphi(x) \, dx
\]
\[
= \int_{\Omega} A(x) \nabla u(x) \cdot \nabla (\eta \varphi)(x) \, dx - \int_{\Omega} A(x) \nabla u(x) \cdot \nabla \eta(x) \varphi(x) \, dx
\]
\[
+ \int_{\Omega} A(x) u(x) \nabla \eta(x) \cdot \nabla \varphi(x) \, dx
\]
\[
= \int_{\Omega} f(x) \eta(x) \varphi(x) \, dx - \int_{\Omega} A(x) \nabla u(x) \cdot \nabla \eta(x) \varphi(x) \, dx
\]
+ \int_\Omega A(x)u(x) \nabla \eta(x) \cdot \nabla \varphi(x) \, dx,

which implies that (5.1) holds true.

Let \( t \in (0, \infty), y \in \Omega, r \in (0, \text{diam}(\Omega)), u_t := K_{t,k}(\cdot, y), \) and \( f := -\frac{d}{dt}K_{t,k}(\cdot, y). \) Then \( L_D u_t = f \) in the sense of (1.8). Take \( \eta \in C_c^\infty(\mathbb{R}^n) \) satisfying \( \eta \equiv 1 \) on \( \{ x \in \mathbb{R}^n : r \leq |x - y| \leq 2r \}, \)

\[
\text{supp} \ (\eta) \subset \left\{ x \in \mathbb{R}^n : \frac{5r}{6} \leq |x - y| \leq \frac{13r}{6} \right\},
\]

and \( |\nabla \eta| \leq r^{-1}. \) Assume that \( q \in (2, p_0) \) and \( p \in (1, n) \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n}. \) Then, by Lemmas 3.6 and 3.7, and (5.1), we conclude that

\[
\|\nabla (u_t \eta)\|_{L^q(\Omega; \mathbb{R}^n)} \leq \|u_t A \nabla \eta\|_{L^q(\Omega; \mathbb{R}^n)} + \|A \nabla u_t \cdot \nabla \eta\|_{L^p(\Omega)} + \|f \eta\|_{L^p(\Omega)}.
\]

Let \( S_1 := \{ x \in \Omega : r \leq |x - y| \leq 2r \} \) and \( S_2 := \{ x \in \Omega : \frac{5r}{6} \leq |x - y| \leq \frac{13r}{6} \}. \) Then, from (5.2), we deduce that

\[
\|\nabla u_t\|_{L^p(S_1; \mathbb{R}^n)} \leq r^{-1} \|u_t\|_{L^p(S_2)} + r^{-1} \|\nabla u_t\|_{L^p(S_2; \mathbb{R}^n)} + \left\| \frac{dt}{dt} \right\|_{L^p(S_2)}.
\]

We first assume that \( p \leq 2. \) Similarly to the proof of [3, Proposition 16], we have

\[
\|\nabla u_t\|_{L^2(S_2; \mathbb{R}^n)} \leq t^{-\frac{\frac{1}{2} - \frac{1}{q}}{2}} \left( \frac{I}{t^{1/2}} \right) \leq e^{-\frac{r^2}{\alpha}},
\]

which, together with the Hölder inequality, implies that

\[
r^{-1} \|\nabla u_t\|_{L^p(S_2; \mathbb{R}^n)} \leq r^{-1} \|\nabla u_t\|_{L^2(S_2; \mathbb{R}^n)} |S_2|^{\frac{1}{2} - \frac{1}{q}} \leq r^{\frac{n}{q} - 1} t^{-\frac{n}{2q}} e^{-\frac{r^2}{\alpha}}.
\]

Furthermore, by [51, Theorem 6.17] and (1.5), we find that, for any \( x \in \Omega, \)

\[
|u_t(x)| + t \left| \frac{dt}{dt} u_t(x) \right| \leq t^{-\frac{n}{2q}} e^{-\frac{|\eta(x)|^2}{\alpha}},
\]

which further implies that

\[
r^{-1} \|u_t\|_{L^p(S_2)} + \left\| \frac{dt}{dt} \right\|_{L^p(S_2)} \leq r^{-1} t^{\frac{n}{2q}} t^{-\frac{n}{2q}} e^{-\frac{r^2}{\alpha}} + t^{-1} r^2 t^{-\frac{n}{2q}} e^{-\frac{r^2}{\alpha}} \leq r^{\frac{n}{q} - 1} t^{-\frac{n}{2q}} e^{-\frac{r^2}{\alpha}}.
\]

From this, (5.3), and (5.4), it follows that

\[
\|\nabla u_t\|_{L^p(S_1; \mathbb{R}^n)} \leq r^{\frac{n}{q} - 1} t^{-\frac{n}{2q}} e^{-\frac{r^2}{\alpha}}.
\]

This finishes the proof of the present lemma in the case that \( p \leq 2. \)

When \( p > 2, \) take \( i_0 \in \mathbb{N} \) and \( 1 < p_{i_0} \leq 2 < p_{i_0 - 1} < \cdots < p_1 := p < q \) such that \( \frac{1}{p_{i_0 + 1}} - \frac{1}{p_i} = \frac{1}{n} \) for any \( i \in \{1, \ldots, i_0 - 1 \}. \) Then, using a simple iteration argument, (5.3), (5.5), and (5.6), we conclude that (5.6) also holds true in this case that \( p > 2. \) This finishes the proof of Lemma 5.1. □
Moreover, to show Theorem 1.8, we need the following uniform boundedness of the family \( \{ \sqrt{\nabla e^{-|L_D|}} \}_{t>0} \) of operators on \( L^p(\Omega) \), whose proof is similar to that of [4, Proposition 22]; we omit the details here.

**Lemma 5.2.** Let \( n \geq 2, \Omega \subset \mathbb{R}^n \) be a bounded NTA domain, the real-valued, bounded, and measurable matrix \( A \) satisfy (1.3), and \( L_D \) be as in (1.4). Then there exists an \( \varepsilon_0 \in (0, \infty) \), depending only on \( n \) and \( \mu_0 \), such that, for any given \( p \in (2-\varepsilon_0, 2+\varepsilon_0) \), the family \( \{ \sqrt{\nabla e^{-|L_D|}} \}_{t>0} \) of operators is uniformly bounded on \( L^p(\Omega) \).

Now, we prove Theorem 1.8 by using Lemmas 3.6, 5.1, and 5.2.

**Proof of Theorem 1.8.** We first show (i). By Lemma 3.6, we find that the conclusion of (i) holds true when \( q \in (2, p_0) \). We assume that (ii) holds true, and use the conclusion of (ii) to finish the proof of (i). Define a linear operator \( \overline{\nabla L_D^{-1}} \) on \( \mathbb{R}^n \) as follows. Let \( p \in [1, n] \) and \( q \in [\frac{n}{n-p}, \infty) \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \). For any \( f \in L^p(\mathbb{R}^n) \) when \( p \in (1, n) \), or \( f \in H^1(\mathbb{R}^n) \), and any \( x \in \mathbb{R}^n \), let

\[
\overline{\nabla L_D^{-1}}(f)(x) := \begin{cases} \nabla L_D^{-1}(f|\Omega)(x) & \text{when } x \in \Omega, \\ 0 & \text{when } x \in \Omega^C. \end{cases}
\]

Then, from Lemma 3.6, it follows that, when \( q \in (2, p_0) \),

\[
(5.7) \quad \left\| \overline{\nabla L_D^{-1}}(f) \right\|_{L^q(\mathbb{R}^n)} = \left\| \nabla L_D^{-1}(f|\Omega) \right\|_{L^q(\Omega; \mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)}.
\]

Thus, the operator \( \overline{\nabla L_D^{-1}} \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n, \mathbb{R}^n) \). Moreover, by (ii), we conclude that

\[
(5.8) \quad \left\| \overline{\nabla L_D^{-1}}(f) \right\|_{L^q(\mathbb{R}^n; \mathbb{R}^n)} = \left\| \nabla L_D^{-1}(f|\Omega) \right\|_{L^q(\Omega; \mathbb{R}^n)} \leq \| f \|_{L^p(\Omega)} \leq \| f \|_{H^1(\mathbb{R}^n)},
\]

which further implies that \( \overline{\nabla L_D^{-1}} \) is bounded from \( H^1(\mathbb{R}^n) \) to \( L^\frac{n}{n-p}(\mathbb{R}^n; \mathbb{R}^n) \). Recall that, for any \( q \in (1, \infty) \), the Hardy space \( H^q(\mathbb{R}^n) \) is just the Lebesgue space \( L^q(\mathbb{R}^n) \). From this, (5.7), (5.8), and the complex interpolation theory of Hardy spaces on \( \mathbb{R}^n \) (see, for instance, [42, Theorem 8.1 and (9.3)]), we deduce that \( \overline{\nabla L_D^{-1}} \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n, \mathbb{R}^n) \) for any given \( p \in (1, \frac{nq}{n-p}) \) and \( q \in (\frac{n}{n-1}, p_0) \) satisfying \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), which, combined with the definition of \( \overline{\nabla L_D^{-1}} \), implies that the operator \( \nabla L_D^{-1} \) is bounded from \( L^p(\Omega) \) to \( L^q(\Omega; \mathbb{R}^n) \) for any given \( p \in (1, \frac{nq}{n-p}) \) and \( q \in (\frac{n}{n-1}, p_0) \) satisfying \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \). This finishes the proof of (i).

Now, we prove (ii). Let \( p \in (\frac{n}{n-1}, 1] \) and \( q \in (1, \frac{n}{n-1}] \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), and \( f \in H^0_{L_D}(\Omega) \cap L^2(\Omega) \). Take \( 1 < p_1 < 2 < q_1 < \min(p_0, 2+\varepsilon_0) \) such that \( \frac{1}{q_1} - \frac{1}{p_1} = \frac{1}{q} - \frac{1}{p} \), where \( p_0 \) and \( \varepsilon_0 \) are, respectively, as in Lemmas 3.6 and 5.2. Let \( \varepsilon \in (\frac{2}{p}, \infty) \) and \( M \in \mathbb{N} \) \((\max(\frac{2}{2p}, \frac{1+\varepsilon}{2}), \infty) \). Then there exist \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) and a sequence \( \{\alpha_j\}_{j=1}^{\infty} \) of \((p, q_1, M, \varepsilon)_{L_D}\)-molecules associated, respectively, with the balls \( \{B_{\Omega,j}\}_{j=1}^{\infty} \) such that

\[
(5.9) \quad f = \sum_{j=1}^{\infty} \lambda_j \alpha_j.
\]
in $L^2(\Omega)$, and

\begin{equation}
(5.10) \quad \|f\|_{H^p_{L_D}(\Omega)} \approx \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p},
\end{equation}

where, for any $j \in \mathbb{N}$, $B_{\Omega} := B_j \cap \Omega$, and $B_j := B(x_j, r_j)$ with $x_j \in \Omega$ and $r_j \in (0, \text{diam} (\Omega))$ is a ball of $\mathbb{R}^n$. To finish the proof of (ii), it suffices to prove that, for any $(p, q, M, \epsilon)_{L_D}$-molecule $\alpha$ associated with the ball $B_\Omega := B \cap \Omega$,

\begin{equation}
(5.11) \quad \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^q(\Omega; \mathbb{R}^n)} \leq 1.
\end{equation}

Indeed, if (5.11) holds true, then, by (5.9), (5.10), (5.11), and $q > 1 \geq p$, we find that

\begin{align*}
\left\| \nabla u \right\|_{L^q(\Omega; \mathbb{R}^n)} & = \left\| \nabla L_D^{-1}(f) \right\|_{L^q(\Omega; \mathbb{R}^n)} \leq \sum_{j=1}^{\infty} |\lambda_j| \left\| \nabla L_D^{-1}(\alpha_j) \right\|_{L^q(\Omega; \mathbb{R}^n)} \\
& \leq \sum_{j=0}^{\infty} |\lambda_j| \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} \sim \|f\|_{H^p_{L_D}(\Omega)},
\end{align*}

which implies that (ii) holds true.

Next, we prove (5.11). From Lemma 3.6, it follows that $\nabla L_D^{-1}$ is bounded from $L^{p_1}(\Omega)$ to $L^{q_1}(\Omega; \mathbb{R}^n)$, which, together with the H"{o}lder inequality and $\frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q}$, implies that

\begin{equation}
(5.12) \quad \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^{q_1}(\Omega; \mathbb{R}^n)} \leq \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^{q_1}(\Omega; \mathbb{R}^n)} \|B_\Omega\|_{\mathbb{R}^n}^{\frac{1}{q_1} - \frac{1}{q}} \leq \|\alpha\|_{L^{p_1}(\Omega; \mathbb{R}^n)} \|B_\Omega\|_{\mathbb{R}^n}^{\frac{1}{p} - \frac{1}{q}} \leq 1.
\end{equation}

For any $j \in \mathbb{N}$ with $j \geq 3$, let $S_j(B_\Omega) := (2^{j+3} B_\Omega) \setminus (2^{j-3} B_\Omega)$ and $E_j(B_\Omega) := \Omega \setminus S_j(B_\Omega)$. Then, by the equation

\begin{equation}
\nabla L_D^{-1}(\alpha) = \int_0^{\infty} \nabla e^{-t L_D}(\alpha) \, dt
\end{equation}

and the Minkowski inequality, we conclude that

\begin{equation}
(5.13) \quad \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^q(S_j(B_\Omega); \mathbb{R}^n)} \leq \int_0^{2^\frac{q}{q-1}} \left\| \nabla e^{-t L_D}(\alpha) \right\|_{L^q(S_j(B_\Omega); \mathbb{R}^n)} \, dt + \int_{2^\frac{q}{q-1}}^{\infty} \cdots \approx I + II.
\end{equation}

For the term I, we have

\begin{equation}
(5.14) \quad I = \int_0^{2^\frac{q}{q-1}} \left\| \nabla e^{-t L_D}(\alpha 1_{S_j(B_\Omega)}) \right\|_{L^q(S_j(B_\Omega); \mathbb{R}^n)} \, dt \\
+ \int_0^{2^\frac{q}{q-1}} \left\| \nabla e^{-t L_D}(\alpha 1_{E_j(B_\Omega)}) \right\|_{L^q(S_j(B_\Omega); \mathbb{R}^n)} \, dt =: I_1 + I_2.
\end{equation}
From Lemma 5.2, the Hölder inequality, and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, we deduce that

$$I_1 \leq \int_0^{r_B^2} \left\| \nabla e^{-tL_D} \left( \alpha 1_{S_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} dt \leq 2^{-j\epsilon} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} \int_0^{r_B^2} t^{-1/2} dt \leq 2^{-j(\epsilon - \frac{\alpha}{q})}.$$

Moreover, by Lemma 5.1, the Minkowski inequality, and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, we conclude that

$$I_2 \leq \int_0^{r_B^2} \left\| \nabla e^{-tL_D} \left( \alpha 1_{E_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} dt \leq 2^{-j\epsilon} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} \int_0^{r_B^2} \int_{E_j(B_0)} \left\| \nabla K_1^{L_D}(\cdot, y) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} |\alpha(y)| dy dt \leq 2^{-j(\epsilon - \frac{\alpha}{q})}.$$

For the term II, we have

$$II = \int_{r_B^2}^{\infty} \frac{1}{t^M} \left\| \nabla (tL_D)^M e^{-tL_D} \left( (L^{-M} \alpha) 1_{S_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} dt + \int_{r_B^2}^{\infty} \frac{1}{t^M} \left\| \nabla (tL_D)^M e^{-tL_D} \left( (L^{-M} \alpha) 1_{B_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} dt =: II_1 + II_2.$$

From the facts that, for any $t \in (0, \infty)$,

$$\sqrt{t} \nabla (tL_D)^M e^{-tL_D} = 2^{M+\frac{1}{2}} \left( \left( \frac{t}{2} \right)^{1/2} \nabla e^{-tL_D} \right) \left( \frac{t}{2} L_D \right)^M e^{-tL_D},$$

$\{(tL_D)^M e^{-tL_D} \}_{t>0}$ is uniformly bounded on $L^s(\Omega)$ for any given $s \in [1, \infty)$ (see, for instance, [51]), and Lemma 5.2, it follows that, for any given $s \in (2 - \epsilon_0, 2 + \epsilon_0)$, $\sqrt{t} \nabla (tL_D)^M e^{-tL_D} \}_{t>0}$ is uniformly bounded on $L^s(\Omega)$. By this and the Hölder inequality, we find that

$$II_1 \leq \int_{r_B^2}^{\infty} \frac{1}{t^M} \left\| \nabla (tL_D)^M e^{-tL_D} \left( (L^{-M} \alpha) 1_{S_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} dt \leq 2^{-j\epsilon} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} \int_{r_B^2}^{\infty} \frac{1}{t^{M+1/2}} dt \leq 2^{-j(\epsilon - \frac{\alpha}{q})}.$$

Moreover, from the Hölder inequality, the Minkowski inequality, and Lemma 5.1, we deduce that

$$II_2 \leq \int_{r_B^2}^{\infty} \frac{1}{t^M} \left\| \nabla (tL_D)^M e^{-tL_D} \left( (L^{-M} \alpha) 1_{E_j(B_0)} \right) \right\|_{L^q_1(S_j(B_0); \mathbb{R}^n)} \left| 2^j B_\Omega \right|^{\frac{1}{q} - \frac{1}{q_1}} dt \leq 2^{-j(\epsilon - \frac{\alpha}{q})}.$$
Thus, (5.11) holds true. This finishes the proof of (ii).

which, combined with (5.12) and

Thus, by (5.13) through (5.19), we conclude that, for any \( \epsilon > \frac{a}{p} > \frac{2}{q} \), further implies that

\[
\left\| \nabla L_D^{-1}(\alpha) \right\|_{L^p(\Omega; \mathbb{R}^n)} \leq \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^p(\partial B_0; \mathbb{R}^n)} + \sum_{j=3}^{\infty} \left\| \nabla L_D^{-1}(\alpha) \right\|_{L^p(\partial B_0; \mathbb{R}^n)} \leq 1.
\]

Thus, (5.11) holds true. This finishes the proof of (ii).

Finally, we show (iii). Let \( p \in \left( \frac{n}{n+2}, \frac{n}{n+1} \right] \) and \( q \in \left( \frac{n}{n+1}, 1 \right] \) satisfy \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \), and \( f \in H^{p_0}_{t_D}(\Omega) \cap L^2(\Omega) \). Take \( 1 < p_2 < 2 < q_2 < \min\{p_0, 2 + \epsilon_0\} \) such that \( \frac{1}{q_2} - \frac{1}{p_2} = \frac{1}{q} - \frac{1}{p} \) where \( p_0 \) and \( \epsilon_0 \) are, respectively, as in Lemmas 3.6 and 5.2. Let \( \epsilon > \left( \frac{a}{p}, \infty \right) \) and \( M \in \mathbb{N} \cap \max\{\frac{a}{q_2}, \frac{1+p}{2}, \infty\} \). Then there exist \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) and a sequence \( \{\alpha_j\}_{j=1}^{\infty} \) of \( (p, q_2, M, \epsilon)_{L_D}\)-molecules associated, respectively, with the balls \( \{B_{j, \Omega, 1}\}_{j=1}^{\infty} \) such that (5.9) and (5.10) hold true. To finish the proof of (iii), it suffices to show that, for any \( (p, q_2, M, \epsilon)_{L_D}\)-molecule \( \alpha \), the zero extension of \( \nabla L_D^{-1}(\alpha) \) from \( \Omega \) to \( \mathbb{R}^n \), denoted by \( \nabla L_D^{-1}(\alpha) \), is a harmless constant multiple of a \( (q_2, 2, 0, \epsilon)\)-molecule associated with the ball \( B \). Indeed, if this claim holds true, then, by this, \( p < q_2 \), (5.9), and (5.10), we find that the zero extension of \( \nabla L_D^{-1}(f) \) from \( \Omega \) to \( \mathbb{R}^n \), denoted by \( \nabla L_D^{-1}(f) \), belongs to \( H^p(\mathbb{R}^n; \mathbb{R}^n) \), and

\[
\int_{\mathbb{R}^n} \nabla L_D^{-1}(\alpha(x)) \, dx = \int_{\mathbb{R}^n} \nabla L_D^{-1}(\alpha(x)) \varphi(x) \, dx = - \int_{\mathbb{R}^n} \nabla L_D^{-1}(\alpha(x)) \nabla \varphi(x) \, dx = 0.
\]
Furthermore, by (i), we find that $\nabla L^{-1}$ is bounded from $L^{p_2}(\Omega)$ to $L^{q_2}(\Omega; \mathbb{R}^n)$, which, combined with $\frac{1}{q_2} - \frac{1}{p_2} = \frac{1}{q} - \frac{1}{p}$, implies that

$$\|\nabla L^{-1}(\alpha)\|_{L^{q_2}(4B; \mathbb{R}^n)} \lesssim \|\alpha\|_{L^{p_2}(\Omega)} \lesssim |B\Omega|^\frac{1}{q_2} - \frac{1}{p} \sim |B\Omega|^\frac{1}{q_2} - \frac{1}{p} \lesssim |4B|^{-\frac{1}{q_2} - \frac{1}{p}}. \quad (5.22)$$

Moreover, similarly to the proof of (5.20), for any $j \in \mathbb{N}$ with $j \geq 3$, we have

$$\|\nabla L^{-1}(\alpha)\|_{L^{q_2}(S_j(B); \mathbb{R}^n)} \lesssim 2^{-j\epsilon} |2jB|^{\frac{1}{q_2}} |B|^{-\frac{1}{q}}. \quad (5.23)$$

Thus, from (5.21), (5.22), and (5.23), it follows that $\nabla L^{-1}(\alpha)$ is a harmless constant multiple of a $(q, q_2, 0, \epsilon)$-molecule. This finishes the proof of (iii), and hence of Theorem 1.8. \hfill \Box

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