FOURIER SERIES ON COMPACT SYMMETRIC SPACES:
K-FINITE FUNCTIONS OF SMALL SUPPORT

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Abstract. The Fourier coefficients of a function \( f \) on a compact symmetric space \( U/K \) are given by integration of \( f \) against matrix coefficients of irreducible representations of \( U \). The coefficients depend on a spectral parameter \( \mu \), which determines the representation, and they can be represented by elements \( \hat{f}(\mu) \) in a common Hilbert space \( \mathcal{H} \).

We obtain a theorem of Paley-Wiener type which describes the size of the support of \( f \) by means of the exponential type of a holomorphic \( \mathcal{H} \)-valued extension of \( \hat{f} \), provided \( f \) is \( K \)-finite and of sufficiently small support. The result was obtained previously for \( K \)-invariant functions, to which case we reduce.

1. Introduction.

The present paper is a continuation of our article [19]. We consider a Riemannian symmetric space \( X \) of compact type, realized as the homogeneous space \( U/K \) of a compact Lie group \( U \). Up to covering, \( U \) is the connected component of the group of isometries of \( X \). As an example, we mention the sphere \( S^n \), for which \( U = \text{SO}(n+1) \) and \( K = \text{SO}(n) \). In the cited paper, we considered \( K \)-invariant functions on \( U/K \). The Fourier series of a \( K \)-invariant function \( f \) is

\[
\sum_{\mu} a_{\mu} \psi_{\mu}(x),
\]

where \( \psi_{\mu} \) is the zonal spherical function associated with the representation of \( U \) with highest weight \( \mu \), and where the Fourier coefficients \( a_{\mu} \) are given by

\[
a_{\mu} = d(\mu) \hat{f}(\mu) = d(\mu) \int_{U/K} f(x) \overline{\psi_{\mu}(x)} \, dx,
\]

with \( d(\mu) \) being the representation dimension, and \( dx \) being the normalized invariant measure on \( U/K \). The main result of [19] is a local Paley-Wiener theorem, which gives a necessary and sufficient condition on the coefficients in the series (1.1) that it is the Fourier series of a smooth \( K \)-invariant function \( f \) supported in a geodesic ball of a given sufficiently small radius \( r \) around the origin in \( U/K \). The condition is, that \( \mu \mapsto a_{\mu} \) extends to a holomorphic function of exponential type \( r \) satisfying certain invariance under the action of the Weyl...
In the present paper we consider the general case where the $K$-invariance is replaced by $K$-finiteness. Instead of being scalars, the Fourier coefficients take values in the Hilbert space $\mathcal{H} = L^2(K/M)$, where $M$ is a certain subgroup of $K$. In case of $U/K = S^n$, we have $K/M = S^{n-1}$. Our main result is Theorem 7.2 below, which describes the set of Fourier coefficients of $K$-finite smooth functions on $U/K$, supported in a ball of a given sufficiently small radius. The corresponding result for Riemannian symmetric spaces of the non-compact type is due to Helgason, see [10].

Our method is by reduction to the $K$-invariant case. For the reduction we use Kostant’s description of the spherical principal series of a semisimple Lie group [15]. A similar reduction was found by Torasso [29] for Riemannian symmetric spaces of the non-compact type, thus providing an alternative proof of the mentioned theorem of Helgason.

2. Basic notation

We recall some basic notation from [19]. We are considering a Riemannian symmetric space $U/K$, where $U$ is a connected compact semisimple Lie group and $K$ a closed symmetric subgroup. By definition this means that there exists a nontrivial involution $\theta$ of $U$ such that $K_0 \subset K \subset U^\theta$. Here $U^\theta$ denotes the subgroup of $\theta$-fixed points, and $K_0 := U_0^\theta$ its identity component. The base point in $U/K$ is denoted by $x_0 = eK$.

The Lie algebra of $U$ is denoted $\mathfrak{u}$, and by $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ we denote the Cartan decomposition associated with the involution $\theta$. We endow $U/K$ with the Riemannian structure induced by the negative of the Killing form on $\mathfrak{q}$.

Let $a \subset \mathfrak{q}$ be a maximal abelian subspace, $a^*$ its dual space, and $a^*_C$ the complexified dual space. The set of non-zero weights for $a$ in $\mathfrak{u}_C$ is denoted by $\Sigma$. The roots $\alpha \in \Sigma \subset a^*_C$ are purely imaginary valued on $a$. The corresponding Weyl group, generated by the reflections in the roots, is denoted $W$. We make a fixed choice of a positive system $\Sigma^+$ for $\Sigma$, and define $\rho \in i\mathfrak{a}^*$ to be half the sum of the roots in $\Sigma^+$, counted with multiplicities. The centralizer of $a$ in $K$ is denoted $M = Z_K(a)$.

Some care has to be taken because we are not assuming $K$ is connected. We recall that if $U$ is simply connected, then $U^\theta$ is connected and $K = K_0$, see [13], p. 320. We recall also that in general $K = MK_0$, see [19], Lemma 5.2.

In the following we shall need to complexify $U$ and $U/K$. Since $U$ is compact there exists a unique (up to isomorphism) connected complex Lie group $U_C$ with Lie algebra $\mathfrak{u}_C$ which contains $U$ as a real Lie subgroup. Let $\mathfrak{g}$ denote the real form $\mathfrak{k} \oplus i\mathfrak{q}$ of $\mathfrak{u}_C$, and let $G$ denote the connected real Lie subgroup of $U_C$ with this Lie algebra. Then $\mathfrak{g}_C = \mathfrak{u}_C$ as complex vector spaces, and $U_C$ complexifies $G$ as well as $U$. In particular, the almost complex structures that $\mathfrak{u}$ and $\mathfrak{g}$ induce on $U_C$ are identical. For this reason we shall denote $U_C$ also by $G_C$. The Cartan involutions of $\mathfrak{u}$ and $U$ extend to involutions of $\mathfrak{g}_C$ of $G_C$, which we shall denote again by $\theta$, and which leave $\mathfrak{g}$ and $G$ invariant. The corresponding Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} + i\mathfrak{q}$. It follows that $K_0 = G^\theta$. 


is maximal compact in \(G\), and \(G/K_0\) is a Riemannian symmetric space of the non-compact type.

We denote by \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}\) and \(G = K_0 AN\) the Iwasawa decompositions of \(\mathfrak{g}\) and \(G\) associated with \(\Sigma^+\). Here \(A = \exp(\mathfrak{a})\) and \(N = \exp \mathfrak{n}\). Furthermore, we let \(H : G \rightarrow i\mathfrak{a}\) denote the Iwasawa projection

\[
K_0 AN \ni k \exp Y n = g \mapsto H(g) = Y.
\]

Let \(K_{0C}, A_C,\) and \(N_C\) denote the connected subgroups of \(G_C\) with Lie algebras \(\mathfrak{k}_C, \mathfrak{a}_C\) and \(\mathfrak{n}_C\), and put \(K_C = K_{0C} K\). Then \(G_C/K_C\) is a symmetric space, and it carries a natural complex structure with respect to which \(U/K\) and \(G/K_0\) are totally real submanifolds of maximal dimension.

**Lemma 2.1.** There exists an open \(K_C \times K\)-invariant neighborhood \(V^\alpha\) of the neutral element \(e\) in \(G_C\), and a holomorphic map

\[
H : V^\alpha \rightarrow \mathfrak{a}_C,
\]

which agrees with the Iwasawa projection on \(V^\alpha \cap G\), such that

\[
u \in K_C \exp(H(u)) N_C
\]

for all \(u \in V^\alpha\).

**Proof.** (See [23] or [24].) We first assume that \(K = U^\theta\). Then \(K_C = G_C^\theta\). Since \(\mathfrak{g}_C = \mathfrak{n}_C \oplus \mathfrak{a}_C \oplus \mathfrak{k}_C\), there exist an open neighborhood \(T_{n_C} \times T_{a_C}\) of \((0,0)\) in \(\mathfrak{n}_C \times \mathfrak{a}_C\) such that the map

\[
\mathfrak{n}_C \times \mathfrak{a}_C \ni (X,Y) \mapsto \exp X \exp Y \cdot x_0 \in G_C/K_C
\]

is a biholomorphic diffeomorphism of \(T_{n_C} \times T_{a_C}\) onto an open neighborhood \(\mathcal{V}\) of \(x_0 = eK_C\) in \(G_C/K_C\). We assume, as we may, that \(T_{n_C}\) and \(T_{a_C}\) are invariant under the complex conjugation with respect to the real form \(\mathfrak{g}\).

We denote by \(V^\alpha\) the open set \(\{ x \mid x^{-1} K_C \in \mathcal{V} \} \subset G_C\). The map

\[
K_C \times T_{a_C} \times T_{n_C} \ni (k,Y,X) \mapsto k \exp Y \exp X \in V^\alpha \subset G_C
\]

is then a biholomorphic diffeomorphism.

In particular, the map \(H : V^\alpha \rightarrow \mathfrak{a}_C\) defined by

\[
k \exp Y \exp X \mapsto Y
\]

for \(k \in K_C, Y \in T_{a_C}\) and \(X \in T_{n_C}\), is holomorphic and satisfies (2.2).

The conjugation with respect to \(\mathfrak{g}\) lifts to an involution of \(G_C\) that leaves \(G\) pointwise fixed. Moreover, since this conjugation commutes with \(\theta\), it stabilizes \(K_C\). Hence it stabilizes \(V^\alpha\). Let \(u \in V^\alpha \cap G\) and write \(u = k \exp Y \exp X\) with \(k \in K_C, Y \in T_{a_C}\) and \(X \in T_{n_C}\). It follows that \(k, Y\) and \(X\) are fixed by the conjugation. In particular, \(Y \in \mathfrak{a}_C\) and \(X \in \mathfrak{n}_C\), and hence \(k = u \exp(-X) \exp(-Y) \in G \cap K_C = K_0\). Therefore, \(u = k \exp Y \exp X\) is the Iwasawa decomposition, and \(H(u) = Y\) the Iwasawa projection, of \(u\).

We postpone the condition of right-\(K\)-invariance and consider the general case where \(K_0 \subset K \subset U^\theta\). We retain the sets \(T_{n_C}\) and \(T_{a_C}\) from above and recall that \(K_C = K_{0C} K\) is an open subgroup of the previous \(K_C\). Again we define \(V^\alpha = K_C \exp(T_{a_C}) \exp(T_{n_C})\). This is an open subset of the previous \(V^\alpha\).
The restriction of the previous $H$ to this set is obviously holomorphic, agrees with Iwasawa on $\mathcal{V}^a \cap G$, and it is easily seen to satisfy (2.2).

Finally, we note that $\mathcal{V}^a$ contains an $\text{Ad}K$ invariant open neighborhood $V$ of $e$ in $G_C$. Hence, for each $k \in K$, the set $\mathcal{V}^a k$ is left-$K_C$-invariant and contains $V$. The intersection $\cap_{k \in K} \mathcal{V}^a k$ is $K \times K$ invariant and contains $V$. The interior of this set has all the properties requested of $\mathcal{V}^a$.

We call the map in (2.1) the complexified Iwasawa projection. A particular set $\mathcal{V}^a$ as above can be constructed as follows. Let

$$\Omega = \{X \in a \mid (\forall \alpha \in \Sigma) |\alpha(X)| < \pi/2\}.$$ 

The set $\mathcal{V} = \text{Cr}(G/K_0) = G \exp \Omega K_C \subset G_C/K_C$, called the complex crown of $G/K_0$, was introduced in [1]. Its preimage in $G_C$ is open and contained in $N_C A_C K_C \subset G_C$. This is shown for all classical groups in [16], Theorem 1.8, and in general in [14], Theorem 3.21. See also [8], [18]. Let $\mathcal{V}^a = \{x^{-1} \mid x \in \mathcal{V}\} \subset G_C$. The existence of the holomorphic Iwasawa projection $\mathcal{V}^a \rightarrow a_C$ is established in [16], Theorem 1.8, with a proof that can be repeated in the general case. It follows that $\mathcal{V}^a$ has all the properties mentioned in Lemma 2.1.

One important property of the crown is that it is $G$-invariant and that all the spherical functions on $G/K$ extends to a holomorphic function on the crown (it is in fact maximal with this property, see [17], Theorem 5.1). However, this property plays no role in the present article, where we shall just assume that $\mathcal{V}^a$ has the properties in Lemma 2.1, and $\mathcal{V} = (\mathcal{V}^a)^{-1}$.

3. Fourier analysis

In this section we develop a local Fourier theory for $U/K$ based on elementary representation theory. The theory essentially originates from Sherman [23].

An irreducible unitary representation $\pi$ of $U$ is said to be spherical if there exists a non-zero $K$-fixed vector $e_\pi$ in the representation space $V_\pi$. The vector $e_\pi$ (if it exists) is unique up to multiplication by scalars. After normalization to unit length we obtain the matrix coefficient

$$\psi_\pi(u) = \langle \pi(u) e_\pi, e_\pi \rangle$$

which is the corresponding zonal spherical function.

From the point of view of representation theory it is natural to define the Fourier transform of an integrable function $f$ on $U/K$ to be the map that associates the vector

$$\pi(f) e_\pi = \int_U f(u \cdot x_0) \pi(u) e_\pi \, du = \int_{U/K} f(x) \pi(x) e_\pi \, dx \in V_\pi,$$

to each spherical representation, with a fixed choice of the unit vector $e_\pi$ for each $\pi$ (see [20] for discussion on the noncompact case). The corresponding Fourier series is

$$(3.1) \quad \sum_\pi d(\pi) \langle \pi(f) e_\pi, \pi(x) e_\pi \rangle$$
Lemma 3.1. Let \( \pi \) be a \( U \)-stable \( \pi \) in \( G \), proved in \cite{12}, p. 535, in the case that \( U \) is simply connected.

In the case of the sphere \( S^2 \), the expansion of \( f \) in spherical harmonics \( Y^m_\ell(x) \) (with integral labels \( |m| \leq l \)) is obtained from this expression when we express \( \pi(x)e_\pi \) by means of an orthonormal basis for the \((2l + 1)\)-dimensional representation space \( V_\pi = V_l \).

For the purpose of Fourier analysis it is convenient to embed all the representation spaces \( V_\pi \), where \( \pi \) is spherical, in a common Hilbert space \( \mathcal{H} \), independent of \( \pi \), such that \( \hat{f} \) can be viewed as an \( \mathcal{H} \)-valued function on the set of equivalence classes of irreducible spherical representations. This can be achieved as follows.

Recall that in the classification of Helgason, a spherical representation \( \pi = \pi_\mu \) is labeled by an element \( \mu \in a_C^* \), which is the restriction, from a compatible maximal torus, of the highest weight of \( \pi \) (see \cite{12}, p. 538). We denote by \( \Lambda^+(U/K) \subset a_C^* \) the set of these restricted highest weights, so that \( \mu \rightarrow \pi_\mu \) sets up a bijection from \( \Lambda^+(U/K) \) onto the set of equivalence classes of irreducible \( K \)-spherical representations. According to the theorem of Helgason, every \( \mu \in \Lambda^+(U/K) \) satisfies

\[
\langle \mu, \alpha \rangle = \langle \alpha, \alpha \rangle \in \mathbb{Z}^+,
\]

for all \( \alpha \in \Sigma^+ \), where the brackets denote the inner product induced by the Killing form. Furthermore, if \( U \) is simply connected, then an element \( \mu \in a_C^* \) belongs to \( \Lambda^+(U/K) \) if and only if it satisfies \( 3.2 \). For the description in the general case, one must supplement \( 3.2 \) by both the assumption that \( \pi_\mu \) descends to \( U \), and that the \( K_0 \)-fixed vector is also \( K \)-fixed.

For each \( \mu \in \Lambda^+(U/K) \) we fix an irreducible unitary spherical representation \( (\pi_\mu, V_\mu) \) of \( U \) and a unit \( K \)-fixed vector \( e_\mu \in V_\mu \). Furthermore, we fix a highest weight vector \( v_\mu \) of weight \( \mu \), such that \( \langle v_\mu, e_\mu \rangle = 1 \). The following lemma is proved in \cite{12}, p. 535, in the case that \( U \) is simply connected.

**Lemma 3.1.** Let \( \mu \in \Lambda^+(U/K) \). Then \( \pi_\mu(m)v_\mu = v_\mu \) for all \( m \in M \), and the vectors \( \pi_\mu(k)v_\mu \), where \( k \in K_0 \), span the space \( V_\mu \).

**Proof.** Let \( m \in M \) be given. Since \( m \) centralizes \( a \) and normalizes \( n \), it follows that \( \pi_\mu(m)v_\mu \) is again a highest weight vector of the same weight. Hence \( \pi_\mu(m)v_\mu = cv_\mu \). By taking inner products with \( e_\mu \), which is \( M \)-fixed, it follows that \( c = 1 \). The statement about the span follows directly from the Iwasawa decomposition \( G = K_0AN \).

It follows from Lemma 3.1 that the map \( V_\mu \rightarrow L^2(K/M) \), \( v \mapsto \langle v, \pi_\mu(\cdot)v_\mu \rangle \), is injective. We shall use the space \( \mathcal{H} = L^2(K/M) \) as our common model for the spherical representations. It will be convenient to use an anti-linear embedding of \( V_\mu \). Hence we define for \( \mu \in \Lambda^+(U/K) \)

\[
h_v(k) = \langle \pi_\mu(k)v_\mu, v \rangle, \quad (k \in K)
\]

and \( \mathcal{H}_\mu = \{ h_v \mid v \in V_\mu \} \). Then \( v \mapsto h_v \) is a \( K \)-equivariant anti-isomorphism \( V_\mu \rightarrow \mathcal{H}_\mu \subset \mathcal{H} \).
Notice that \( \mu = 1 \), the constant function on \( K/M \). Hence 1 belongs to \( H_\mu \) for all \( \mu \in \Lambda^+(U/K) \). Although we shall not use it in the sequel, we also note that every \( K \)-finite function in \( H = L^2(K/M) \) belongs to \( H_\mu \) for some \( \mu \) (this can be seen from results explained below, notably Lemma 4.1 and equation (7.4), where for a given \( K \)-type \( \delta \) one chooses \( \mu \) such that \( P(-\mu - \rho) \) is nonsingular).

According to the chosen embedding of \( V_\mu \) in \( H \), we define the Fourier transform of an integrable function \( f \) on \( U/K \) by

\[
\tilde{f}(\mu) = \int_{U/K} f(u) h_{\pi_\mu(u)e_\mu} \, du \in H
\]

for \( \mu \in \Lambda^+(U/K) \), that is

\[
(3.4) \quad \tilde{f}(\mu, b) = \int_{U/K} f(u) \langle \pi_\mu(k)v_\mu, \pi_\mu(u)e_\mu \rangle \, du,
\]

for \( b = kM \in K/M \). If \( f \) is \( K \)-invariant, then \( \tilde{f}(\mu) \) is independent of \( b \). Integration over \( K \) then shows that this definition agrees with the spherical Fourier transform in (1.2).

It is easily seen that the Fourier transform \( f \mapsto \tilde{f}(\mu) \) is intertwining for the left regular actions of \( K \) on \( U/K \) and \( K/M \), respectively. In particular, it maps \( K \)-finite functions on \( U/K \) to \( K \)-finite functions on \( K/M \).

We now invoke the complex group \( G_C \) and the complexified Iwasawa projection defined in the preceding section. Let \( V^a \subset G_C \) and \( H: V^a \to a_C \) be as in Lemma 2.1 and let \( \mu \in \Lambda^+(U/K) \). Since \( \pi_\mu \) extends to a holomorphic representation of \( G_C \), it follows from Lemma 2.1 that

\[
\langle \pi_\mu(k)v_\mu, e_\mu \rangle = e^{\mu(H(u))}
\]

for all \( u \in V^a \). Let \( V = \{ x^{-1} \mid x \in V^a \} \subset G_C \). Then

\[
(3.5) \quad \langle \pi_\mu(k)v_\mu, \pi_\mu(u)e_\mu \rangle = e^{\mu(H(u^{-1}k))}
\]

for \( k \in K, u \in U \cap V \) and \( \mu \in \Lambda^+(U/K) \).

**Lemma 3.2.** Let \( f \) be an integrable function on \( U/K \) with support in \( U \cap V \). Then

\[
(3.6) \quad \tilde{f}(\mu, k) = \int_{U/K} f(u) e^{\mu(H(u^{-1}k))} \, du,
\]

for all \( k \in K/M \), and the Fourier transform \( \mu \mapsto \tilde{f}(\mu) \) extends to a holomorphic \( H \)-valued function on \( a_C^* \), also denoted by \( \tilde{f} \), satisfying the same equation (3.6). Moreover,

\[
(3.7) \quad \pi_\mu(f)e_\mu = \int_{K/M} \tilde{f}(-\mu - 2\rho, k)\pi_\mu(k)v_\mu \, dk
\]

for all \( \mu \in \Lambda^+(U/K) \).

The measure on \( K/M \) used in (3.7) is the quotient of the normalized Haar measures on \( K \) and \( M \).

**Proof.** The expression (3.6) follows immediately from (3.4) and (3.5). The integrand in (3.6) depends holomorphically on \( \mu \), locally uniformly with respect to \( u \) and \( k \). Hence an analytic continuation is defined by this formula.
In order to establish the identity (3.7) it suffices to show that
\[ \pi_\mu(u)e_\mu = \int_{K/M} e^{-(\mu + 2\rho)H(u^{-1}k)}\pi_\mu(k)v_\mu\,dk \]
for \( u \in U \cap V \). The latter identity is easily shown to hold for \( u \in G \) (use [12], p. 197, Lemma 5.19, and the fact that \( K/M = K_0/(M \cap K_0) \)). By analytic continuation it then holds for \( u \in V_0 \), the identity component of \( V \). Since \( V = V_0K_\mathbb{C} \), it follows for all \( u \in V \).

**Corollary 3.3.** (Sherman) Assume \( f \in L^2(U/K) \) has support contained in \( U \cap V \). Then the sum
\[ \sum_{\mu \in \Lambda^+(U/K)} d(\mu) \int_{K/M} \tilde{f}(\mu - 2\rho, k) \langle \pi_\mu(k)v_\mu, \pi_\mu(x)e_\mu \rangle\,dk, \quad x \in U/K, \]
converges to \( f \) in \( L^2(U/K) \), and it converges uniformly if \( f \) has a sufficient number of continuous derivatives.

**Proof.** (See [23]). Follows immediately from (3.1) by insertion of (3.7). \( \square \)

In [24] the inversion formula of Corollary 3.3 is extended to a formula for functions on \( U/K \) without restriction on the support (for symmetric spaces of rank one). We shall not use this extension here. For the special case of the sphere \( U/K = S^n \), see also [22], [26] and [31].

### 4. The spherical principal series

The space \( \mathcal{H} = L^2(K/M) = L^2(K_0/(M \cap K_0)) \) is the representation space for the spherical principal series for \( G \). We denote by \( \sigma_\lambda \) this series of representations, given by
\[ \langle \sigma_\lambda(g)\psi(k) \rangle = e^{-(\lambda + \rho)H(g^{-1}k)}\psi(\kappa(g^{-1}k)) \]
for \( \lambda \in \mathfrak{a}_\mathbb{C}^*, \; g \in G, \; \psi \in \mathcal{H} \) and \( k \in K_0 \). Here \( \kappa: G \to K_0 \) is the Iwasawa projection \( k \mapsto g \).

Let \( \mu \in \Lambda^+(U/K) \). By extending \( \pi_\mu \) to a holomorphic representation of \( \check{G}_\mathbb{C} \) and then restricting to \( G \), we obtain a finite dimensional representation of \( G \), which we again denote by \( \pi_\mu \). We now have the following well-known result. It relates the embedding of \( V_\mu \) into \( \mathcal{H} \), which motivated (3.3), to the principal series representations.

**Lemma 4.1.** Let \( \mu \in \Lambda^+(U/K) \). The map \( v \mapsto h_v \) defined by (\ref{4.3}) provides a \( G \)-equivariant embedding of the contragredient of \( \pi_\mu \) into \( \sigma_{-\mu-\rho} \).

**Proof.** Recall that the contragredient representation can be realized on the conjugate Hilbert space \( \check{V}_\mu \) by the operators \( \pi_\mu(g^{-1})^* \), and notice that \( v \mapsto h_v \) is linear from \( \check{V}_\mu \) to \( \mathcal{H} \). Since \( v_\mu \) is a highest weight vector it follows easily from (\ref{4.1}) that
\[ \sigma_{-\mu-\rho}(g)h_v = h_{\pi_\mu(g^{-1})v} \]
for \( g \in G \). \( \square \)
The space $C^\infty(K/M) \subset \mathcal{H}$ carries the family of representations, also denoted by $\sigma_\lambda$, of $\mathfrak{g}_C$ obtained by differentiation and complexification. Thus, although the representations $\sigma_\lambda$ of $G$ in general do not complexify to global representations of $U$, the infinitesimal representations $\sigma_\lambda$ of $u_C$ are defined for all $\lambda \in \mathfrak{a}_C^*$. We denote by $\mathcal{H}_\lambda^\infty$ the space $C^\infty(K/M)$ equipped with the representation $\sigma_\lambda$ of $u_C = \mathfrak{g}_C$, and with the left regular representation of $K$.

**Lemma 4.2.** The Fourier transform $f \mapsto \tilde{f}(\mu)$ defines a $(u, K)$-homomorphism from $C^\infty(U/K)$ to $\mathcal{H}_{-\mu - \rho}^\infty$ for all $\mu \in \Lambda^+(U/K)$. Moreover, the holomorphic extension, defined in Lemma 3.2, restricts to a $(u, K)$-homomorphism from $\{f \in C^\infty(U/K) \mid \text{supp} f \subset U \cap V\}$ to $\mathcal{H}_{-\mu - \rho}^\infty$ for all $\mu \in \mathfrak{a}_C^*$.

**Proof.** Since $\pi_\mu$ is a unitary representation of $U$ it follows from Lemma 4.1 that $\sigma_{-\mu - \rho}(X) h_v = h_{\pi_\mu(X)} v$ for $X \in u, v \in V_\mu$. The first statement now follows, since

$$\tilde{f}(\mu) = \int_{U/K} f(u) h_{\pi_\mu(u)e}\, du.$$ 

It follows from Lemma 5.2 and Theorem 6.1 below, that the second statement can be derived from the first by analytic continuation with respect to $\mu$, provided the support of $f$ is sufficiently small. However, we prefer to give an independent proof, which only requires assumptions on the support of $f$ as stated in the lemma.

Since the Fourier transform in (3.6) is clearly $K$-equivariant, it suffices to prove the intertwining property

$$(4.2) \quad [L(X)f](-\mu - \rho)(\mu) = \tilde{f}(\mu)$$

for $X \in \mathfrak{q}$. By definition

$$[L(X)f](u) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX)u)$$

and hence by invariance of the measure

$$[L(X)f](-\mu, k) = \int_{U/K} f(u) \left. \frac{d}{dt} \right|_{t=0} e^{\mu(H(u^{-1}\exp(-tX)k))} \, du.$$ 

Let $\mathfrak{p} = i\mathfrak{q}$ so that $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, and write $X = iY$ for $Y \in \mathfrak{p}$. Since the complexified Iwasawa map $H$ is holomorphic, it follows that

$$\left. \frac{d}{dt} \right|_{t=0} e^{H(u^{-1}\exp(-tX)k)} = \left. \frac{d}{dt} \right|_{t=0} e^{H(u^{-1}\exp(-tY)k)}.$$ 

Furthermore

$$H(u^{-1}\exp(-tY)k) = H(u^{-1}\kappa(\exp(-tY)k)) + H(\exp(-tY)k)$$
and hence we derive
\[
[L(X)f](\mu, k) \\
= \left. i \frac{d}{dt} \right|_{t=0} \left[ e^{\mu(H(\exp(-tY)k))} \int_{U/K} f(u) e^{\mu(H(u^{-1}\kappa(\exp(-tY)k)))} du \right] \\
= \left. i \frac{d}{dt} \right|_{t=0} \left[ e^{\mu(H(\exp(-tY)k))} \tilde{f}(\mu, \kappa(\exp(-tY)k)) \right].
\]
Since by definition \(\sigma_{-\mu-\rho}(X) = i\sigma_{-\mu-\rho}(Y)\), the last expression is exactly \(\sigma_{-\mu-\rho}(X) \tilde{f}(\mu)\) evaluated at \(k\).

We recall that there exist normalized standard intertwining operators between the principal series:
\[
\mathcal{A}(w, \lambda): \mathcal{H} \to \mathcal{H}, \quad w \in W,
\]
such that
\[
(4.3) \quad \sigma_{w\lambda}(g) \circ \mathcal{A}(w, \lambda) = \mathcal{A}(w, \lambda) \circ \sigma_{\lambda}(g)
\]
for all \(g \in G\). The normalization is such that
\[
(4.4) \quad \mathcal{A}(w, \lambda)1 = 1
\]
for the constant function 1 on \(K/M\). The map \(\lambda \mapsto \mathcal{A}(w, \lambda)\) is meromorphic with values in the space of bounded linear operators on \(\mathcal{H}\).

We need the following property of the Poisson kernel, which is defined for \(x \in G\) and \(k \in K_0\) by \(e^{-(\lambda+\rho)H(x^{-1}k)}\). By Lemma 2.1 it is defined also for \(x \in V\) and \(k \in K\).

**Lemma 4.3.** The identity
\[
(4.5) \quad \mathcal{A}(w, \lambda)e^{-(\lambda+\rho)H(x^{-1})} = e^{-(w\lambda+\rho)H(x^{-1})},
\]
of functions in \(\mathcal{H}\), holds for all \(x \in V\).

**Proof.** The identity is well-known for \(x \in G\). In fact in this case it follows easily from (1.1), (1.3) and (1.4). The map \(x \mapsto e^{\rho(H(x^{-1}))}\) is holomorphic \(\mathcal{H}\)-valued on \(V\) for each \(\mu \in a^*_C\), because the complexified Iwasawa projection is holomorphic. Hence (1.2) holds for \(x \in V_0\) by analytic continuation, and then for \(x \in V\) by the obvious left-\(K_C\)-invariance of both sides with respect to \(x^{-1}\).

5. **The \(K\)-finite Paley-Wiener space**

For each irreducible representation \(\delta\) of \(K_0\) we denote by \(\mathcal{H}_\delta\) the finite dimensional subspace of \(\mathcal{H}\) consisting of the functions that generate an isotypical representation of type \(\delta\). Likewise, for each finite set \(F\) of \(K_0\)-types, we denote by \(\mathcal{H}_F\) the sum of the spaces \(\mathcal{H}_\delta\) for \(\delta \in F\). Obviously, the intertwining operators \(\mathcal{A}(w, \lambda)\) preserve each subspace \(\mathcal{H}_F\). Although we do not need it in the sequel, we remark that \(\lambda \mapsto \mathcal{A}(w, \lambda)|_{\mathcal{H}_F}\) is a rational map from \(a^*_C\) into the space of linear operators on the finite dimensional space \(\mathcal{H}_F\), for each \(F\), see [30].
Note that since $K/K_0$ is finite, a function on $K/M = K_0/(K_0 \cap M)$ is $K_0$-finite if and only if it is $K$-finite. We use the notations $\mathcal{H}_\delta$ and $\mathcal{H}_F$ also for an irreducible representation $\delta$ of $K$, and for a set $F$ of $K$-types.

**Definition 5.1.** For $r > 0$ the $K$-finite Paley-Wiener space $\text{PW}_{K,r}(a)$ is the space of holomorphic functions $\varphi$ on $\mathfrak{a}_C^*$ with values in $\mathcal{H} = L^2(K/M)$ satisfying the following.

(a) There exists a finite set $F$ of $K$-types such that $\varphi(\lambda) \in \mathcal{H}_F$ for all $\lambda \in \mathfrak{a}_C^*$.

(b) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that
\[
\|\varphi(\lambda)\| \leq C_k (1 + |\lambda|)^{-k} e^{r|\text{Re} \lambda|}
\]
for all $\lambda \in \mathfrak{a}_C^*$.

(c) The identity $\varphi(w(\mu + \rho) - \rho) = A(w, -\mu - \rho)\varphi(\mu)$ holds for all $w \in W$, and for generic $\mu \in \mathfrak{a}_C^*$.

We note that the norm on $\mathfrak{a}_C^*$ used in (b) is induced by the negative of the Killing form on $\mathfrak{a}$. In particular we see that $\text{PW}_{K,r}(a) = \text{PW}_{K_0,r}(a)$, that is, the $K$-finite Paley-Wiener space is the same for all the spaces $U/K$ where $K_0 \subset K \subset U^\theta$.

Notice that the Paley-Wiener space $\text{PW}_r(a)$ defined in [19] can be identified with the space of functions $\varphi$ in $\text{PW}_{K,r}(a)$, for which $\varphi(\lambda)$ is a constant function on $K/M$ for each $\lambda$. This follows from the normalization (4.4).

The functions in the Paley-Wiener space are uniquely determined by their restriction to $\Lambda^+(U/K)$, at least when $r$ is sufficiently small. This is seen in the following lemma.

**Lemma 5.2.** There exists $R > 0$ such that if $\varphi \in \text{PW}_{K,r}(a)$ for some $r < R$ and $\varphi(\lambda) = 0$ for all $\mu \in \Lambda^+(U/K)$, then $\varphi = 0$.

**Proof.** The relevant value of $R$ is the same as in [19] Thm. 4.2 (iii) and Remark 4.3. The lemma follows easily from application of [19], Section 7, to the function $\lambda \mapsto \langle \varphi(\lambda, \cdot), \psi \rangle$ for each $\psi \in \mathcal{H}$. \hfill $\square$

Obviously $\text{PW}_{K,r}(a)$ is $K$-invariant, where $K$ acts by the left regular representation on functions on $K/M$. The following lemma shows that it is also a $(\mathfrak{u}, K)$-module.

**Lemma 5.3.** Let $r > 0$, $\varphi \in \text{PW}_{K,r}(a)$ and $X \in \mathfrak{u}_C$. Then the function $\psi = \sigma(X)\varphi$ defined by
\[
\psi(\lambda) = \sigma_{-\lambda - \rho}(X)(\varphi(\lambda)) \in \mathcal{H}
\]
for each $\lambda \in \mathfrak{a}_C^*$, belongs to $\text{PW}_{K,r}(a)$.

**Proof.** Recall that $\sigma_{-\lambda - \rho}(X)$ is defined by complexification of the infinitesimal action of $\mathfrak{g}$ on the smooth functions in $\mathcal{H}$, and note that $\varphi(\lambda)$ is smooth on $K/M$, since it is $K$-finite. Hence we may assume $X \in \mathfrak{g}$. It is easily seen that $\psi(\lambda)$ is $K_0$-finite, of types which occur in the tensor product of the adjoint representation $\text{Ad}$ of $K_0$ on $\mathfrak{g}$ with types from $F$. Hence condition (a) is valid for the function $\psi$. Condition (c) follows immediately from the intertwining property of $A(w, \lambda)$. It remains to verify holomorphicity in $\lambda$, and the estimate in (b) for $\psi$. 

By definition both the holomorphicity and norm in the estimate (b) refer to the Hilbert space $\mathcal{H} = L^2(K/M)$. However, because of condition (a) and since $\mathcal{H}_F$ is finite dimensional, it is equivalent to require holomorphicity of $\psi(\lambda)(x)$ pointwise for each $x \in K/M$, and likewise to require the exponential estimate for $\psi(\lambda)(x)$ pointwise with respect to $x$. Thus let an element $x = kM \in K/M$ be fixed, where $k \in K_0$.

Note that by (4.1) 
\[
(\sigma(X)\varphi)(\lambda)(k) = \left. \frac{d}{dt} \right|_{t=0} e^{-\lambda + \rho(H(\exp(-tX)k))} \varphi(\lambda)(\kappa(\exp(-tX)k)) .
\]
Differentiating with the Leibniz rule, we obtain a sum of two terms. The first term is
\[
(5.1)\left. \frac{d}{dt} \right|_{t=0} (e^{-\lambda + \rho(H(\exp(-tX)k))}) \varphi(\lambda)(k).
\]
Let $\alpha(Z) = H(\exp(Z)k) \in i\mathfrak{a}$ for $Z \in \mathfrak{g}$, then $\alpha(0) = 0$ and it follows that (5.1) equals
\[
(\lambda + \rho)(d\alpha_0)(\varphi(\lambda))(k)
\]
where $d\alpha_0$ is the differential of $\alpha$ at 0. It is now obvious that (5.1) is holomorphic and satisfies the same the growth estimate as $\varphi(\lambda)(k)$. Hence (b) is valid for the first term.

The second term is
\[
(5.2)\left. \frac{d}{dt} \right|_{t=0} \varphi(\lambda)(\kappa(\exp(-tX)k)) ,
\]
which we rewrite as follows. Let
\[
\beta(Z) = \kappa(\exp(Z)k)k^{-1} \in K_0
\]
for $Z \in \mathfrak{g}$, then $\beta(0) = e$ and
\[
\varphi(\lambda)(\kappa(\exp(-tX)k)) = \varphi(\lambda)(\beta(-tX)k).
\]
It follows that (5.2) equals
\[
L(d\beta_0)(\varphi(\lambda))(k)
\]
where $d\beta_0(X) \in T_{e}K_0 = \mathfrak{t}$. The linear operator $L(d\beta_0(X))$ preserves the finite dimensional space $\mathcal{H}_F$ and hence restricts to a bounded linear operator on that space. It follows that (5.2) is holomorphic in $\lambda$ and satisfies (b). □

6. Fourier transform maps into Paley-Wiener space

In this section we prove the following result. Let $C^\infty_K(U/K)$ denote the space of $K$-finite smooth functions on $U/K$, and for each $r > 0$ let
\[
C^\infty_{K,r}(U/K) = \{ f \in C^\infty_K(U/K) \mid \text{supp} f \subset \text{Exp}(\overline{B_r(0)}) \}
\]
where $\overline{B_r(0)}$ denotes the closed ball in $\mathfrak{q}$ of radius $r$ and center 0, and $\text{Exp}$ denotes the exponential map of $U/K$.

**Theorem 6.1.** There exists a number $R > 0$ such that $\text{Exp}(\overline{B_R(0)}) \subset U \cap V$ and such that the following holds for every $r < R$:

*If $f \in C^\infty_{K,r}(U/K)$, then the holomorphic extension of $\tilde{f}$ from Lemma 5.2 belongs to $\text{PW}_{K,r}(\mathfrak{a})$.***
In the proof we shall reduce to the case where $K = K_0$. The following lemma prepares the way for this reduction.

The projection $p : U/K_0 \to U/K$ is a covering map. Hence we can choose $R > 0$ such that $p$ restricts to a diffeomorphism of the open ball $\text{Exp}(B_R(0))$ in $U/K_0$ onto the open ball $\text{Exp}(B_R(0))$ in $U/K$. It follows that for each $r < R$ a bijection $F \mapsto f$ of $C^\infty_{K_0,r}(U/K_0)$ onto $C^\infty_{K,r}(U/K)$ is defined by

$$f(u) = \sum_{v \in K/K_0} F(uv), \quad u \in U$$

for $F \in C^\infty_{K_0,r}(U/K_0)$, where for each $u$ at most one term is non-zero. The inverse map is given by

$$F(u) = \begin{cases} f(p(u)), & u \in \text{Exp}(B_R(0)), \\ 0, & \text{otherwise}, \end{cases}$$

for $f \in C^\infty_{K,r}(U/K)$. Let $\mathcal{V}_a \subset G_C$ be as in Lemma 2.1, and note that this set also satisfies the assumptions of that lemma for the symmetric space $U/K_0$. As before, let $\mathcal{V} = \{x^{-1} | x \in \mathcal{V}_a\}$.

**Lemma 6.2.** Let $f \in C^\infty_{K,r}(U/K)$ and $F \in C^\infty_{K_0,r}(U/K_0)$ be as above. Then $f$ is supported in $U \cap \mathcal{V}$ if and only if $F$ is supported in $U \cap \mathcal{V}$. In this case, the analytically continued Fourier transforms of these functions satisfy

$$\tilde{f}(\mu) = c\tilde{F}(\mu)$$

for all $\mu \in \mathfrak{a}_C^*$, where $c$ is the index of $K_0$ in $K$.

**Proof.** It follows from the definition of the map $F \mapsto f$ that

$$\tilde{f}(\mu, k) = \int_U \sum_{v \in K/K_0} F(uv)e^{\mu(H(u^{-1}k))} \, du = c\tilde{F}(\mu, k)$$

by right-$K$-invariance of the Haar measure and left-$K$-invariance of $H$. \qed

We can now give the proof of Theorem 6.1.

**Proof.** Property (a) in Definition 5.1 follows immediately from the fact that the Fourier transform is $K$-equivariant. Moreover, the transformation law for the Weyl group in Property (c) follows easily from Lemma 4.3 by integration over $U \cap \mathcal{V}$ against $f(u)$.

For the proof of Property (b), with $r$ bounded by a suitable value $R$, we reduce to the case that $K$ is connected. We assume that $R$ is sufficiently small as described above Lemma 6.2. Then according to the lemma, given a function $f \in C^\infty_{K,r}(U/K)$, the function $F \in C^\infty_{K_0,r}(U/K_0)$ has the same Fourier transform up to a constant. The reduction now follows since $PW_{K,r}(a) = PW_{K_0,r}(a)$, as mentioned below Definition 5.1. For the rest of this proof we assume $K = K_0$.

It is known from [19], Thm. 4.2(i), that the estimate in Property (b) holds for $K$-invariant functions on $U/K$. We prove the property in general by reduction to that case. In particular, we can use the same value of $R > 0$ (see [19], Remark 4.3).

Fix an irreducible $K$-representation $(\delta, V_\delta)$. It suffices to prove the result for functions $f$ that transform isotypically under $K$ according to this type.
We shall use Kostant’s description in [15] of the $K$-types in the spherical principal series. We draw the results we need directly from the exposition in [11], Chapter 3. In particular, we denote by $H_δ^*$ the finite dimensional subspace of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ which is the image under symmetrization of the space of harmonic polynomials on $\mathfrak{p}$ of type $δ$, and we denote by $E_δ$ the space

$$E_δ = \text{Hom}_K(V_δ, H_δ^*),$$

of linear $K$-intertwining maps $V_δ \to H_δ^*$. It is known that $E_δ$ has the same dimension as $V_δ^M$.

We denote by $\text{Hom}^*(V_δ^M, E_δ)$ the space of anti-linear maps $V_δ^M \to E_δ$. The principal result we need is Theorem 2.12 of [11], p. 250, according to which there exists a rational function $P = P_δ$ on $\mathfrak{a}^*_C$ with values in $\text{Hom}^*(V_δ^M, E_δ)$ such that

$$\int_{K/M} e^{-\langle \lambda + \rho, H(x^{-1}k) \rangle} \langle v, \delta(k)v' \rangle dk = [L(P(\lambda)(v'))\varphi_\lambda](x)$$

for all $v \in V_δ, v' \in V_δ^M$ and $x \in G/K$, and for $\lambda \in \mathfrak{a}^*_C$ away from the singularities of $P(\lambda)$. Here $L$ denotes the action of the enveloping algebra from the left on functions on $G/K$, and $\varphi_\lambda$ denotes the spherical function

$$\varphi_\lambda(x) = \int_{K/M} e^{-\langle \lambda + \rho, H(x^{-1}k) \rangle} dk$$
on $G/K$.

The equality (6.1) is valid for $x \in U \cap V$ by analytic continuation. Let $f \in C^\infty_K(U/K)_δ$, where $r < R$ and the subscript $δ$ indicates that $f$ is $K$-finite of this type. Then

$$f(x) = d(δ) \int_K \chi_δ(l)f(lx) dl$$

for all $x \in U$, where $\chi_δ$ is the character of $δ$. It follows that

$$\tilde{f}(μ, k) = d(δ) \int_{U/K} \int_K \chi_δ(l)f(lu) dl e^{\mu(H(u^{-1}k))} du$$

and hence by Fubini and invariance of measures

$$\tilde{f}(μ, k) = d(δ) \int_{U/K} \int_{K/M} \int_M \chi_δ(lmk^{-1}) dm e^{\mu(H(u^{-1}l))} dl f(u) du.$$The inner expression $\int_M \chi_δ(lmk^{-1}) dm$ is a finite sum of matrix coefficients of the form $\langle \delta(l)v, \delta(k)v' \rangle$ with $v \in V_δ$ and $v' \in V_δ^M$, and hence it follows from (6.1) that $\tilde{f}(μ, k)$ for generic $μ \in \mathfrak{a}^*_C$ is a finite sum of expressions of the form

$$\int_{U/K} [L(P(-μ - ρ)(\delta(k)v')(v))\varphi_{-μ - ρ}](u)f(u) du$$

with $v$ and $v'$ independent of $μ$ and $k$. In these expressions the right invariant differential operators $L(P(-μ - ρ)(\delta(k)v')(v)$ can be thrown over, by taking adjoints. Since the spherical function is $K$-invariant, we finally obtain

$$\int_{U/K} \varphi_{-μ - ρ}(u) \int_K [L(P(-μ - ρ)(\delta(k)v')(v))^* f](yu) dy du.$$
Notice that (6.2) is the spherical Fourier transform from [19], Section 6. It follows that \( \tilde{f}(\mu, k) \), for \( \mu \) generic and \( k \in K \), is a finite sum in which each term has the form of the spherical Fourier transform applied to the \( K \)-integral of a derivative of \( f \) by a differential operator with coefficients that depends rationally on \( \mu \) and continuously on \( k \). The application of a differential operator to \( f \) does not increase the support, hence it follows from the estimates in [19] that each term is a rational multiple of a function of \( \mu \) of exponential type, with estimates which are uniform with respect to \( k \). It then follows from [11] Lemma 5.13, p. 288, and its proof, that the Fourier transform \( \tilde{f}(\mu, k) \) itself is of the same exponential type. We have established Property (b) in Definition 5.1 for \( \tilde{f} \). \( \square \)

7. Fourier transform maps onto Paley-Wiener space

Let \( \varphi \in \text{PW}_{K,r}(a) \) for some \( r > 0 \) and consider the function \( f \) on \( U/K \) defined by the Fourier series

\[
(7.1) \quad f(x) = \sum_{\mu \in \Lambda^+/(U/K)} d(\mu) \int_{K/M} \varphi(-\mu - 2\rho, k) \langle \pi_{\mu}(k)v_{\mu}, \pi_{\mu}(x)e_{\mu} \rangle \, dk.
\]

It follows from the estimate in Property (b) of Definition 5.1 that the sum converges and defines a smooth function on \( U/K \) (see [27]).

**Theorem 7.1.** There exists a number \( R > 0 \) such that \( \text{Exp}(\bar{B}_R(0)) \subset U \cap V \) and such that the following holds for every \( r < R \). For each \( \varphi \in \text{PW}_{K,r}(a) \) the function \( f \) on \( U/K \) defined by (7.1) belongs to \( C_{\infty}^K(U/K) \) and has Fourier transform \( \tilde{f} = \varphi \).

**Proof.** Again we first reduce to the case that \( K \) is connected. Assuming that the theorem is valid in that case, we find a number \( R > 0 \) such that every function \( \varphi \in \text{PW}_{K_0,r}(a) \), where \( r < R \), is of the form \( \tilde{F} \) for some \( F \in C_{\infty}^{K_0}(U/K_0) \). We may assume that \( R \) is as small as explained above Lemma 6.2.

Let \( \varphi \in \text{PW}_{K,r}(a) \) be given and recall that \( \text{PW}_{K,r}(a) = \text{PW}_{K_0,r}(a) \). Let \( F \in C_{\infty}^{K_0}(U/K_0) \) with \( \tilde{F} = c^{-1}\varphi \), and construct \( f \in C_{\infty}^{K,r}(U/K) \) as in Lemma 6.2.

It follows from the lemma that \( \tilde{f} = c\tilde{F} = \varphi \), and then it follows from Corollary 3.3 that \( f \) is the function given by (7.1). This completes the reduction.

For the rest of this proof, we assume that \( K = K_0 \). The value of \( R \) that we shall use is the same as in [19], Thm. 4.2(ii) and Remark 4.3. We may assume that \( \varphi(\lambda, \cdot) \) is isotypical of a given \( K \)-type \( \delta \) for all \( \lambda \in a_C^\ast \).

For \( v \in V_\delta \) and \( v' \in V_\delta^M \) we denote by \( \psi_{v,v'} \) the matrix coefficient

\[
\psi_{v,v'}(k) = \langle v, \delta(k)v' \rangle
\]

on \( K/M \). By the Frobenius reciprocity theorem it follows that these functions \( \psi_{v,v'} \) span the space \( \mathcal{H}_\delta \). Moreover, it follows from the definition of the standard intertwining operators by means of integrals over quotients of \( \theta(N) \), that these operators act on each function \( \psi_{v,v'} \) only through the second variable. That is, there exists a linear map

\[
B(w, \lambda) : V_\delta^M \to V_\delta^M
\]
such that
\[
(A(w, \lambda)\psi_{v,v'} = \psi_{v,B(w,\lambda)v'}. \tag{7.2}
\]
for all \(v, v'.\) Notice that the dependence of \(B(w, \lambda)\) on \(\lambda\) is anti-meromorphic.

It follows (by using a basis for \(V_{\delta}\)) that we can write \(\varphi(\mu, k)\) as a finite sum of functions of the form
\[
\psi_{v,v'(\mu)}(k)
\]
where \(v \in V_{\delta}\) is fixed and where \(v': \mathfrak{a}_{\mathbb{C}}^* \to V_{\delta}^M\) is anti-holomorphic of exponential type \(r\) and satisfies the transformation relation in Definition \(5.1\) (c), that is,
\[
v'(w(\mu + \rho) - \rho) = B(w, -\mu - \rho)v'(\mu) \tag{7.3}
\]
for \(w \in W\).

Since the Poisson transformation for \(G/K\) is equivariant for the left action and injective for generic \(\lambda\), it follows from \(6.1\), by applying the inverse Poisson transform on both sides, that
\[
\psi_{v,v'}(\sigma_{\lambda}(P(\lambda)(v')(v)))1 \tag{7.4}
\]
for all \(v \in V_{\delta}, v' \in V_{\delta}^M\) (see also \[11\], Thm. 3.1, p. 251), and for all \(\lambda\) for which \(P(\lambda)\) is non-singular. Here 1 denotes the constant function with value 1 on \(K/M\). We apply \(7.2\) for \(\lambda = -\mu - \rho\) generic to the function \(\psi_{v,v'(\mu)}\) and thus obtain our Paley-Wiener function \(\varphi(\mu, \cdot)\) as a finite sum of elements of the form
\[
\sigma_{-\mu - \rho}(P(-\mu - \rho)(v'(\mu)))(v)))1.
\]

The functions \(P: \mathfrak{a}_{\mathbb{C}}^* \to \text{Hom}^*(V_{\lambda}^M, E_{\delta})\) satisfy the following transformation property
\[
P(w\lambda) \circ B(w, \lambda) = P(\lambda). \tag{7.5}
\]
Indeed, it follows from \(7.3\), \(7.2\) and \(4.3\) that
\[
\sigma_{w\lambda}(P(w\lambda)(B(w, \lambda)v')(v)))1 = \sigma_{w\lambda}(P(\lambda)(v')(v))1
\]
for all \(v\) and \(v'\), and generic \(\lambda\). The identity \(7.5\) follows, since the map \(u \mapsto \sigma_{\nu}(u)1\) is injective from \(H_{\delta}^*\) to \(H\) for generic \(\nu\) according to \[11\], Thm. 3.1, p. 251 (alternatively, \(7.5\) follows from \[11\], Thm. 3.5, p. 254).

It follows from \(7.5\) combined with \(6.3\) that the function
\[
\mu \mapsto u(\mu) := P(-\mu - \rho)(v'(\mu))(v) \in H_{\delta}^*
\]
satisfies \(u(w(\mu + \rho) - \rho) = u(\mu)\) for generic \(\mu\), that is, the shifted function \(\lambda \mapsto u(\lambda - \rho)\) is \(W\)-invariant. Notice that \(u\) is a rational multiple of a holomorphic function of \(\mu\), since \(P(-\mu - \rho)\) is antilinear in \(v'\), and \(v'\) is antiholomorphic in \(\mu\).

It follows from \[11\], Prop. 4.1, p. 264, that \(\lambda \mapsto P(-\lambda)\) is non-singular on an open neighborhood of the set where
\[
\text{Re} \langle \lambda, \alpha \rangle \geq 0
\]
for all roots \(\alpha \in \Sigma^+.\) Hence \(u(\lambda - \rho)\) is holomorphic on this set. By the above-mentioned \(W\)-invariance the function is then holomorphic everywhere. Since it is a rational multiple of a function of exponential type \(r\), we conclude from \[11\], Lemma 5.13, p. 288, that it has exponential type \(r\).
Since $H_\delta^*$ is finite dimensional we thus obtain an expression for $\varphi(\lambda, \cdot)$ as a finite sum of functions of the form

$$\varphi_i(\lambda)\sigma_\lambda(u_i)1,$$

with scalar valued functions $\varphi_i$ on $\mathfrak{a}_C^*$ which are $W$-invariant (for the action twisted by $\rho$) and of exponential type $r$, and with $u_i \in H_\delta^*$.

According to the theorem proved in [19], each function $\varphi_i$ is the spherical Fourier transform of a $K$-invariant smooth function $f_i \in C_\infty^{\infty}(U/K)$. The function $L(u_i)f_i$ also belongs to $C_\infty^{\infty}(U/K)$, and by Lemma 4.2 it has Fourier transform $\varphi_i(\lambda)\sigma_\lambda(u_i)1$. We conclude that if $f$ is the sum of the $L(u_i)f_i$, then $\hat{f} = \varphi$, as desired.

Finally, it follows from Corollary 3.3 that $f$ is identical to the function given by the Fourier series (7.1).

We combine Theorems 6.1 and 7.1 to obtain the following.

**Theorem 7.2.** There exists a number $R > 0$ such that the Fourier transform is a bijection of $C_\infty^{\infty}(U/K)$ onto $PW_{K,r}(a)$ for all $r < R$.

We note the following corollary, which is analogous to a result of Torasso in the non-compact case (see [11], Cor. 5.19, p. 291).

**Corollary 7.3.** There exists $r > 0$ such that each function in $C_\infty^{\infty}(U/K)$ is a finite linear combination of derivatives of $K$-invariant functions in $C_\infty^{\infty}(U/K)$ by members of $U(g)$, acting from the left.

**Proof.** More precisely, the proof above shows that if $f \in C_\infty^{\infty}(U/K)$ is $K$-finite of isotype $\delta$, then $f = \sum_i L(u_i)f_i$ with $u_i \in H_\delta^*$ and $f_i \in C_\infty^{\infty}(U/K)^K$.

8. Final remarks

Every function $f \in C_\infty^{\infty}(U/K)$ can be expanded in a sum of $K$-types,

$$f = \sum_{\delta \in \hat{K}} f_\delta$$

where $f_\delta \in C_\infty^{\infty}(U/K)$ is obtained from $f$ by left convolution with the character of $\delta$ (suitably normalized). It is easily seen that $f$ is supported in a given closed geodesic ball $B$ around $x_0$, if and only if each $f_\delta$ is supported in $B$. The following is then a consequence of Theorem 7.2.

**Corollary 8.1.** There exists $R > 0$ with the following property. Let $f \in C_\infty^{\infty}(U/K)$ and $r < R$. Then $f \in C_\infty^{\infty}(U/K)$ if and only if the Fourier transform $\hat{f}_\delta$ of each of the functions $f_\delta$ allows a holomorphic continuation satisfying the growth estimate (b) of Definition 5.1 (with constants depending on $r$).

For example, in the case of the sphere $S^2$, the expansion (8.1) of $f$ reads $f = \sum_{m \in \mathbb{Z}} f_m$, and the Fourier transform of $f_m$ is the map

$$l \mapsto \begin{cases} a_{l,m} & \text{for } l \geq |m| \\ 0 & \text{for } 0 \leq l < |m| \end{cases}$$
where $c_{m,l}$ are the coefficients of the spherical harmonics expansion

$$f = \sum_{l=0}^{\infty} (2l + 1) \sum_{|m| \leq l} c_{l,m} Y_{l}^{m}.$$  

The condition in Corollary 8.1 is thus that the map (8.2) has a holomorphic extension to $l \in \mathbb{C}$ of the proper exponential type, for each $m \in \mathbb{Z}$.

It is an obvious question, whether the assumption of $K$-finiteness can be removed in Theorem 7.2. It is not difficult to remove it from Theorem 7.1. Assume that $\varphi$ satisfies Properties (b) and (c) in Definition 5.1 for a suitably small value of $r$. Define a function $f : U/K \to \mathbb{C}$ by (7.1). Using the arguments from [27, 28] it follows that $f \in C^\infty(U/K)$. By expanding $f$ as in (8.1) it follows from Corollary 8.1 that $f$ has support inside the ball of radius $r$. It also follows that $\tilde{f} = \varphi$.

The nontrivial part would be to remove the assumption from Theorem 6.1. At this point we do not know if the Fourier transform actually maps all non-$K$-finite functions of small support into the space of functions satisfying the estimate in Property (b). The ingredients in our proof, in particular the matrices $P(\lambda)$, depend on the $K$-types. We would like to point out that for the noncompact dual $G/K$, this direction is proved in [11], p. 278, using the Radon transform. It has been suggested to us by Simon Gindikin that [6] might be used in such an argument for $U/K$.

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