MAXIMAL REPRESENTATIONS OF COMPLEX HYPERBOLIC LATTICES INTO $SU(m, n)$

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Abstract. Let $\Gamma$ denote a lattice in $SU(1, p)$, with $p$ greater than 1. We show that there exists no Zariski dense maximal representation with target $SU(m, n)$ if $n > m > 1$. The proof is geometric and is based on the study of the rigidity properties of the geometry whose points are isotropic $m$-subspaces of a complex vector space $V$ endowed with a Hermitian metric $h$ of signature $(m, n)$ and whose lines correspond to the $2m$ dimensional subspaces of $V$ on which the restriction of $h$ has signature $(m, m)$.

1. Introduction

Let $\Gamma$ be a finitely generated group and $G$ be a connected semisimple Lie group. It is an interesting problem to select and study some connected components of the representation variety $\text{Hom}(\Gamma, G)$ that consist of homomorphisms $\rho : \Gamma \to G$ that are well behaved and, ideally, reflect some interesting geometric properties of the group $\Gamma$. The best example of this framework is the case in which $\Gamma$ is the fundamental group $\Gamma_g$ of a closed surface of genus $g \geq 2$ and $G$ is $\text{PSL}_2(\mathbb{R})$. In this case the Teichmüller space arises as a component of $\text{Hom}(\Gamma_g, G)/G$ that can be selected by means of a cohomological invariant [Gol80].

In the more general setting in which $G$ is any Hermitian Lie group, the so-called maximal representations form a well studied union of connected components of the character variety $\text{Hom}(\Gamma_g, G)/G$ which generalize the Teichmüller component [BIW10, BGPG06]. In analogy with holonomy representations of hyperbolizations, maximal representations can be characterized as those representations that maximize an invariant, the Toledo invariant, that can be defined in terms of bounded cohomology. Such representations are discrete and faithful, and give rise to interesting geometric structures [BIW10, GW12]. Maximal representations in Hermitian Lie groups were...
first studied by Toledo in [Tol89] where he proves that a maximal representation \( \rho : \Gamma \to SU(1, q) \) fixes a complex geodesic, and by Hernandez [Her91] who studied maximal representations \( \rho : \Gamma \to SU(2, q) \) and showed that the image must stabilize a symmetric space associated to the group SU(2, 2). In general any maximal representation stabilizes a tube-type subdomain [BIW10]. Despite this, a remarkable flexibility result holds for maximal representations \( \rho \) of fundamental groups of surfaces: if the image of \( \rho \) is a Hermitian Lie group of tube type, then \( \rho \) admits a one parameter family of deformations consisting of Zariski dense representations [BIW10, KP09].

An analogue of the Toledo invariant was defined by Burger and Iozzi in [BI00] for representations of a lattice \( \Gamma \) in SU(1, p) with values in a Hermitian Lie group \( G \). This allows to select a union of connected components of Hom(\( \Gamma, G \)) consisting of maximal representations. These generalize maximal representations of fundamental groups of surfaces: the fundamental group of a surface is naturally a lattice in SU(1, 1) = \( PSL_2(\mathbb{R}) \). However, if \( p \) is greater than one, a different behavior is expected: Goldman and Milson proved local rigidity for the standard embedding of \( \Gamma \) in SU(1, q) [GM87], and Corlette proved that maximal representations of uniform complex hyperbolic lattices with values in SU(1, q) all come from the standard construction [Cor88]. The picture for rank one targets was completed independently by Burger and Iozzi [BI08] and Koziarz and Maubon [KM08a]: any maximal representation of a lattice in SU(1, p) with values in SU(1, q) admits an equivariant totally geodesic holomorphic embedding \( \mathbb{H}^p_\mathbb{C} \to \mathbb{H}^q_\mathbb{C} \). Koziarz and Maubon generalized this result to the situation in which the target group is classical of rank 2 and the lattice is cocompact [KM08b]. It is conjectured that every maximal representation of a complex hyperbolic lattice with target a Hermitian Lie group is superrigid, namely it extends, up to a representation of \( \Gamma \) in the compact centralizer of the image, to a representation of the ambient group SU(1, p).

In this article we show that the conjecture holds for Zariski dense representations in SU(m, n), with \( m \) different from \( n \):

**Theorem 1.1.** Let \( \Gamma \) be a lattice in SU(1, p) with \( p > 1 \). If \( m \) is different from \( n \), then every Zariski dense maximal representation of \( \Gamma \) into PU(m, n) is the restriction of a representation of SU(1, p).

Combined with the classification of maximal representations of SU(1, p) [Ham11, Ham12], this implies the following:

**Corollary 1.2.** Let \( \Gamma \) be a lattice in SU(1, p) with \( p > 1 \). There are no Zariski dense maximal representations of \( \Gamma \) into SU(m, n), if \( 1 < m < n \).

Exploiting results of [BIW09], we are able to use our main theorem to give a structure theorem for all the maximal representations \( \rho : \Gamma \to SU(m, n) \).

**Theorem 1.3.** Let \( \rho : \Gamma \to SU(m, n) \) be a maximal representation. Then the Zariski closure \( L = \overline{\rho(\Gamma)}^Z \) splits as the product \( SU(1, p) \times L_t \times K \) where
\( L_t \) is a Hermitian Lie group of tube type without irreducible factors that are virtually isomorphic to \( SU(1, 1) \), and \( K \) is a compact subgroup of \( SU(m, n) \).

Moreover there exists an integer \( k \) such that the inclusion of \( L \) in \( SU(m, n) \) can be realized as
\[
\Delta \times i \times \text{Id} : L \to SU(1, p)^{m-k} \times SU(k, k) \times K < SU(m, n)
\]
where \( \Delta : SU(1, p) \to SU(1, p)^{m-k} \) is the diagonal embedding, \( i : L_t \to SU(k, k) \) is a tight holomorphic embedding and \( K \) is contained in the compact centralizer of \( \Delta \times i(L) \).

It is possible to show that there are no tube-type factors in the Zariski closure of the image of \( \rho \) by imposing some non-degeneracy hypothesis on the associated linear representation of \( \Gamma \) into \( GL(C^{m+n}) \):

**Corollary 1.4.** Let \( \Gamma \) be a lattice in \( SU(1, p) \), with \( p > 1 \) and let \( \rho \) be a maximal representation of \( \Gamma \) into \( SU(m, n) \). Assume that the associated linear representation of \( \Gamma \) on \( C^{n+m} \) has no invariant subspace on which the restriction of the Hermitian form has signature \((k, k)\) for some \( k \). Then

1. \( n \geq pm \),
2. \( \rho \) is conjugate to \( \overline{\rho} \times \chi_\rho \) where \( \overline{\rho} \) is the restriction to \( \Gamma \) of the diagonal embedding of \( m \) copies of \( SU(1, p) \) in \( SU(m, n) \) and \( \chi_\rho \) is a representation \( \chi_\rho : \Gamma \to K \), where \( K \) is a compact group.

Recently Klingler proved that all the representations of uniform complex hyperbolic lattices that satisfy a technical algebraic condition are locally rigid [Kli11]. As a particular case his main theorem implies that if \( \Gamma \) is a cocompact lattice in \( SU(1, p) \) and \( \rho : \Gamma \to SU(m, n) \) is obtained by restricting to \( \Gamma \) the diagonal inclusion of \( SU(1, p) \) in \( SU(m, n) \), then \( \rho \) is locally rigid. Since the invariant defining the maximality of a representation is constant on connected components of the representation variety, we get a new proof of Klingler's result in our specific case, and the generalization of this latter result to non-uniform lattices:

**Corollary 1.5.** Let \( \Gamma \) be a lattice in \( SU(1, p) \), with \( p > 1 \), and let \( \rho \) be the restriction to \( \Gamma \) of the diagonal embedding of \( m \) copies of \( SU(1, p) \) in \( SU(m, n) \). Then \( \rho \) is locally rigid.

Our proof of Theorem 1.1 is inspired by Margulis' beautiful proof of superrigidity for higher rank lattices: in order to show that a representation \( \rho : \Gamma \to G \) extends to the group \( H \) in which \( \Gamma \) sits as a lattice, it is enough to exhibit a \( \rho \)-equivariant algebraic map \( \phi : H/P \to G/L \) for some parabolic subgroups \( P \) of \( H \) and \( L \) of \( G \). The existence of measurable \( \rho \)-equivariant boundary maps \( \phi : H/P \to G/L \) where \( P < H \) is a minimal parabolic subgroup and \( G \) is a linear algebraic group is by now well understood [B104, Fur81, BF14], and the crucial part in the proof of superrigidity for our representations is to show that such a measurable equivariant boundary map must indeed be algebraic. In general not every representation of a complex hyperbolic lattice is superrigid: for example Livne constructed in
his PhD dissertation a non-arithmetic lattice in $SU(1, 2)$ that surjects onto a free group (cfr. [DM93, Chapter 16]), hence in particular many representations of that complex hyperbolic lattice do not extend to $SU(1, 2)$. This implies that some additional information on the boundary map $\phi$ is needed in order to deduce its algebraicity.

We restrict our interest to maximal representations precisely to be able to gather some information on a measurable boundary map $\phi$. The maximality of a representation $\rho$ can be rephrased as a property of the induced pullback map $\rho^* : H^2_{cb}(G, \mathbb{R}) \to H^2_b(\Gamma, \mathbb{R})$ in bounded cohomology. One of the advantages of bounded cohomology with respect to ordinary cohomology is that it can be isometrically computed from the complex of $L^\infty$ functions on some suitable boundary of the group [BM02] and, in all geometric cases known so far [BI02], the pullback map in bounded cohomology can be implemented using boundary maps. In particular we exploit results of [BI09] and we show that the fact that the representation $\rho$ is maximal implies that a $\rho$-equivariant measurable boundary map must preserve some incidence structure on the boundary (this was proven in [BI08] in the case in which the image is of rank one).

To describe more precisely this incidence structure, recall that one of the key features of the complex hyperbolic space is the existence of complex geodesics tangent to any vector in $T^1 \mathbb{H}^p_C$: these are precisely the totally geodesic holomorphic embeddings of the Poincaré disc in $\mathbb{H}^p_C$. The boundaries of these subspaces produce a family of circles in $\partial \mathbb{H}^p_C$, the so-called chains, that form an incidence structure that was first studied by Cartan in [Car32]. Under many respects, the natural generalization to higher rank of the visual boundary of the complex hyperbolic space is the Shilov boundary of a Hermitian symmetric space and the generalization of a complex geodesic, when maximal representations are involved, is a maximal tube-type subdomain. All these objects have an explicit linear description: it is well known that the boundary of the complex hyperbolic space can be identified with the set of isotropic lines in $\mathbb{C}^{p+1}$, and it is easy to check that a triple of lines $x, y, z$ is contained in a chain if and only if $\dim \langle x, y, z \rangle = 2$. Similarly the Shilov boundary $S_{m,n}$ of $SU(m, n)$ can be described as the set of maximal isotropic subspaces of $\mathbb{C}^{m+n}$ and, again, a triple of transversal isotropic subspaces $x, y, z$ in $S_{m,n}$ is contained in the boundary of a tube-type subdomain precisely when $\dim \langle x, y, z \rangle = 2m$. In such case we will say that $x, y, z$ are contained in an $m$-chain.

As it turns out, if $\rho : \Gamma \to SU(m, n)$ is a maximal representation and $\phi : \partial \mathbb{H}^p_C \to S_{m,n}$ is a measurable $\rho$-equivariant boundary map, then $\phi$ induces a map from the chain geometry of $\partial \mathbb{H}^p_C$ to the geometry whose space is $S_{m,n}$ and whose lines are the $m$-chains. Therefore most of this paper is devoted to the study of these geometries. We generalize some results of Cartan [Car32] and Goldman [Gol99] and this allows us to prove a strong rigidity result for measurable maps that preserve this geometry, that is a higher rank analogue of the main theorem of [Car32]:

\begin{itemize}
\item \textbf{Theorem:} If $\rho : \Gamma \to SU(m, n)$ is a maximal representation and $\phi : \partial \mathbb{H}^p_C \to S_{m,n}$ is a measurable $\rho$-equivariant boundary map, then $\phi$ induces a map from the chain geometry of $\partial \mathbb{H}^p_C$ to the geometry whose space is $S_{m,n}$ and whose lines are the $m$-chains. Therefore most of this paper is devoted to the study of these geometries. We generalize some results of Cartan [Car32] and Goldman [Gol99] and this allows us to prove a strong rigidity result for measurable maps that preserve this geometry, that is a higher rank analogue of the main theorem of [Car32]:
\end{itemize}
Theorem 1.6. Let $p > 1$, $1 < m < n$ and let $\phi : \partial \mathbb{H}_C^p \to S_{m,n}$ be a measurable map whose essential image is Zariski dense. Assume that almost every pair $x, y$ in $\partial \mathbb{H}_C^p$ satisfies $\dim_C \langle x, y \rangle = 2m$ and that, for almost every triple with $\dim \langle x, y, z \rangle = 2$, it holds $\dim(\phi(x), \phi(y), \phi(z)) = 2m$. Then $\phi$ coincides almost everywhere with a rational map.

Outline of the paper. In Section 2, after recalling the relevant concepts about Hermitian symmetric spaces and continuous bounded cohomology, we prove that a measurable boundary map associated with a maximal representation induces a map between chain geometries. Sections 3 to 5 are devoted to prove Theorem 1.6: in Section 3 we study the chain geometry of $S_{m,n}$ and prove some properties of the incidence structure of chains; in Section 4 we show that the restriction to almost every chain of a measurable boundary map associated with a maximal representation is rational; in Section 5 we show that this information is already enough to conclude. We finish the article with Section 6, where we prove all the remaining theorems announced in this introduction.

2. Preliminaries

We recall in this section the background material that we will need in the paper. The first subsection is about generalities on Hermitian symmetric spaces, more details can be found in the survey article [Kor00], or in the books [Sat80, PS69]. All the notions introduced in this subsection will be described explicitly in the case of the symmetric space associated to SU$(m, n)$ in Section 3.

2.1. Hermitian Symmetric spaces. Let $G$ be a connected semisimple Lie group of noncompact type with finite center and let $K$ be a maximal compact subgroup. We will denote by $X = G/K$ the associated symmetric space. Throughout this article we will be only interested in Hermitian symmetric spaces, that is in those symmetric spaces that admit a $G$-invariant complex structure $J$. It is a classical fact [Kor00, Theorem III.2.6] that these symmetric spaces admit a bounded domain realization, that means that they are biholomorphic to a bounded convex subspace of $\mathbb{C}^n$ on which $G$ acts via biholomorphisms. An Hermitian symmetric space is said to be of tube-type if it is also biholomorphic to a domain of the form $V + i\Omega$ where $V$ is a real vector space and $\Omega \subset V$ is a proper convex open cone. Hermitian symmetric spaces were classified by Cartan [Car35], and are the symmetric spaces associated to the exceptional Lie groups $E_7(-25)$ and $E_6(-14)$ together with 4 families of classical domains: the ones associated to SU$(p, q)$, of type $I_{p,q}$ in Cartan’s terminology, the ones associated to SO$^*(2p)$, of type $II_p$, the symmetric spaces, $III_p$, of the groups Sp$(2p, \mathbb{R})$, and the symmetric spaces of the group $IV_p$ associated to SO$(2, p)$. It is well known that the only spaces that are not of tube type are the symmetric space of $E_6(-14)$ and the families $I_{p,q}$ with $q \neq p$ and $II_p$ with $p$ odd. It follows from the classification that any Hermitian symmetric space contains maximal tube-type subdomains,
and those are all conjugate under the $G$-action, are isometrically and holomorphically embedded and have the same rank as the ambient symmetric space.

The $G$-action via biholomorphism on the bounded domain realization of $\mathcal{X}$ extends continuously on the topological boundary $\partial \mathcal{X}$. If the real rank of $G$ is greater than or equal to two, $\partial \mathcal{X}$ is not an homogeneous $G$-space, but contains a unique closed $G$-orbit, the Shilov boundary $S_G$ of $\mathcal{X}$. If $\mathcal{X}$ is irreducible, the stabilizer of any point $s$ of $S_G$ is a maximal parabolic subgroup of $G$. Under this respect $S_G$ is the algebraic variety of maximal parabolic subgroups of $G$, and the action of $G$ on $S_G$ is algebraic. In the reducible case, if $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$ is the de Rham decomposition in irreducible factors whose isometry group is $G_i$, then $S_G$ splits as the product $S_{G_1} \times \ldots \times S_{G_n}$ as well. Moreover when $\mathcal{Y}$ is a maximal tube-type subdomain of $\mathcal{X}$, the Shilov boundary of $\mathcal{Y}$ embeds in the Shilov boundary of $\mathcal{X}$.

The diagonal action of $G$ on the pairs of points $(s_1, s_2) \in S_G^2$ has a unique open orbit corresponding to pairs of opposite parabolic subgroups. Two points in $S_G$ are transversal if they belong to this open orbit. Whenever a pair $(s_1, s_2)$ of transversal points of $S_G$ is fixed, there exists a unique maximal tube-type subdomain $\mathcal{Y} = G_T/K_T$ of $\mathcal{X}$ such that $s_i$ belongs to $S_{G_T}$. In particular this implies that the Shilov boundaries of maximal tube-type subdomains define a rich incidence structure in $S_G$.

Given three points in $S_G$ there won’t, in general, exist a maximal tube-type subdomain $\mathcal{Y}$ of $\mathcal{X}$ whose Shilov boundary contains all the three points. However it is possible to determine when this happens with the aid of the Kähler form. Recall that, since $\mathcal{X}$ is a Hermitian symmetric space, it is possible to define a differential two form via the formula

$$\omega(X, Y) = g(X, JY)$$

where $g$ denotes the $G$-invariant Riemannian metric normalized so that its minimal holomorphic sectional curvature is $-1$, and $J$ is the complex structure of $\mathcal{X}$. Since $\omega$ is $G$-invariant, it is closed: every $G$-invariant differential form on a symmetric space is closed. This implies that $\mathcal{X}$ is a Kähler manifold and $\omega$ is its Kähler form. Let $\mathcal{X}^{(3)}$ denote the triples of pairwise distinct points in $\mathcal{X}$ and let us consider the function

$$\beta_{\mathcal{X}} : \mathcal{X}^{(3)} \to \mathbb{R}$$

$$\quad (x, y, z) \to \frac{1}{\pi} \int_{\Delta(x,y,z)} \omega$$

where we denote by $\Delta(x, y, z)$ any smooth geodesic triangle having $(x, y, z)$ as vertices. Since $\omega$ is closed, Stokes theorem implies that $\beta_{\mathcal{X}}$ is a well defined continuous $G$-invariant cocycle and it is proven in [CØ03] that it extends continuously to the triples of pairwise transversal points in the Shilov boundary. If a triple $(s_1, s_2, s_3) \in S^3$ doesn’t consist of pairwise transversal points, the limit of $\beta_{\mathcal{X}}(x_1^i, x_2^i, x_3^i)$ as $x_j^i$ approaches $s_j$ is not well defined, but Clerc proved that, restricting only to some preferred sequences (the one that converge *radially* to $s_j$), it is possible to get a measurable extension of $\beta_{\mathcal{X}}$ to
the whole Shilov boundary. The obtained extension $\beta_S : S^3_G \to \mathbb{R}$ is called the Bergmann cocycle and it is a measurable strict cocycle. The maximality of the Bergmann cocycle detects when a triple of points is contained in the Shilov boundary of a tube-type subdomain:

**Proposition 2.1.**

1. $\beta_S$ is a strict alternating $G$-invariant cocycle with values in $[-\mathrm{rk} \mathcal{X}, \mathrm{rk} \mathcal{X}]$,
2. if $\beta_S(s_1, s_2, s_3) = \mathrm{rk} \mathcal{X}$ then the triple $(s_1, s_2, s_3)$ is contained in the Shilov boundary of a tube-type subdomain.

**Proof.** The first fact was proven in [Cle07], the second can be found in [BIW09, Proposition 5.6].

We will call a triple $(s_1, s_2, s_3)$ in $S^3_G$ satisfying $\beta_S(s_1, s_2, s_3) = m$ a maximal triple. In the case where $G$ is $\text{SU}(1, p)$, that is a finite cover of the connected component of the identity in $\text{Isom}(\mathbb{H}^p_\mathbb{C})$, the maximal tube-type subdomains are complex geodesics of $\mathbb{H}^p_\mathbb{C}$ and the Bergmann cocycle coincides with Cartan’s angular invariant $c_p$ [Gol99, Section 7.1.4]. Following Cartan’s notation we will denote by chains the boundaries of the complex geodesics.

2.2. **Measurable boundary maps.** Let us now focus on the only family of Hermitian symmetric spaces of rank one, namely the complex hyperbolic spaces $\mathbb{H}^p_\mathbb{C}$ that are the symmetric spaces associated to the groups $\text{SU}(1, p)$.

The bounded domain realization of $\mathbb{H}^p_\mathbb{C}$ can be chosen to be the unit ball in $\mathbb{C}^p$. Moreover, since these symmetric spaces are of rank one, the topological boundary of the bounded domain realization is an homogeneous $\text{SU}(1, p)$-space and can be naturally identified both with the visual boundary of $\mathbb{H}^p_\mathbb{C}$ considered as a $\text{CAT}(0)$ space, and with the Shilov boundary of $\mathbb{H}^p_\mathbb{C}$. We will fix on $\partial \mathbb{H}^p_\mathbb{C}$ the Lebesgue measure class $[\mu]$ coming from the explicit realization as a $2p - 1$ dimensional sphere.

If $\Gamma$ is a lattice in $\text{SU}(1, p)$, then the action of $\Gamma$ on $\partial \mathbb{H}^p_\mathbb{C}$ preserves the Lebesgue measure class $[\mu]$. Moreover the space $\partial \mathbb{H}^p_\mathbb{C}$ is an amenable and doubly ergodic $\Gamma$-space. The first assertion follows from the fact that $\Gamma$ is a lattice in $\text{SU}(1, p)$, and $\partial \mathbb{H}^p_\mathbb{C}$ is an homogeneous space for $\text{SU}(1, p)$ with amenable stabilizers [Zim84, Corollary 4.3.7], and the second fact is a consequence of Howe-Moore’s Theorem [Zim84, 2.2.20]. These two properties of the space $\partial \mathbb{H}^p_\mathbb{C}$ allow to construct equivariant boundary maps with respect to a Zariski dense representation of the group $\Gamma$ in an algebraic group:

**Proposition 2.2.** [BI04, Proposition 7.2] Let $\Gamma$ be a lattice in $\text{SU}(1, p)$, $G$ a Lie group of Hermitian type and let $\rho : \Gamma \to G$ be a Zariski dense representation. Then there exists a $\rho$-equivariant measurable map $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S^3_G$ such that, for almost every pair of points $x, y$ in $\partial \mathbb{H}^p_\mathbb{C}$, $\phi(x)$ and $\phi(y)$ are transversal.

Let us now fix a measurable map $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S^3_G$, and define the essential Zariski closure of $\phi$ to be the minimal Zariski closed subset $V$ of $S^3_G$ such
that $\mu(\phi^{-1}(V)) = 1$. Such a set exists since the intersection of finitely many closed subset of full measure has full measure and $\mathcal{S}_G$ is an algebraic variety, in particular it is Noetherian. We will say that a measurable boundary map $\phi$ is Zariski dense if its essential Zariski closure is the whole $\mathcal{S}_G$.

**Proposition 2.3.** Let $\rho$ be a Zariski dense representation, then $\phi$ is Zariski dense.

*Proof.* Indeed let us assume by contradiction that the essential Zariski closure of $\phi((\partial \mathcal{H}_c)_{\mathbb{P}})$ is a proper Zariski closed subset $V$ of $\mathcal{S}_G$. The set $V$ is $\rho(\Gamma)$-invariant: indeed for every element $\gamma$ in $\Gamma$, we get $\mu(\phi^{-1}(\rho(\gamma)V)) = \mu(\gamma \phi^{-1}(V)) = 1$, hence, in particular, $\rho(\gamma)V = V$ by minimality of $V$.

Let us now recall that the Shilov boundary $\mathcal{S}_G$ is an homogeneous space for $G$, and let us fix the preimage $W$ of $V$ under the projection map $G \rightarrow G/Q = \mathcal{S}_G$. $W$ is a proper Zariski closed subset of $G$, moreover if $g$ is any element in $W$, the Zariski dense subgroup $\rho(\Gamma)$ of $G$ is contained in $Wg^{-1}$ and this gives a contradiction. $\square$

### 2.3. Continuous (bounded) cohomology.

We introduce now the concepts we will need about continuous and continuous bounded cohomology, standard references are respectively [BW00] and [Mon01]. A quick introduction to the relevant aspects of continuous bounded cohomology can also be found in [BI09].

Throughout the section $G$ will be a locally compact second countable group, every finitely generated group fits in this class when endowed with the discrete topology. The *continuous cohomology* of $G$ with real coefficients, $H^*_c(G, \mathbb{R})$ is the cohomology of the complex $(C^*_c(G, \mathbb{R})^G, d)$ where

$$C^n_c(G, \mathbb{R}) = \{ f : G^{n+1} \rightarrow \mathbb{R} | f \text{ is a continuous function} \},$$

the invariants are taken with respect to the diagonal action, and the differential $d^n : C^n_c(G, \mathbb{R}) \rightarrow C^{n+1}_c(G, \mathbb{R})$ is defined by the expression

$$d^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f((g_0, \ldots, \hat{g}_i, \ldots, g_{n+1})).$$

Similarly the *continuous bounded cohomology* $H^n_{cb}(G, \mathbb{R})$ of $G$ is the cohomology of the subcomplex $(C^n_{cb}(G, \mathbb{R})^G, d)$ of $(C^n_c(G, \mathbb{R})^G, d)$ consisting of bounded functions. The inclusion $i : C^n_{cb}(G, \mathbb{R})^G \rightarrow C^n_c(G, \mathbb{R})^G$ induces, in cohomology, the so-called *comparison map* $c : H^n_{cb}(G, \mathbb{R}) \rightarrow H^n_c(G, \mathbb{R})$. The Banach norm on the cochain modules $C^n_{cb}(G, \mathbb{R})$ defined by

$$\|f\|_\infty = \sup_{(g_0, \ldots, g_n) \in G^{n+1}} |f(g_0, \ldots, g_n)|$$

induces a seminorm on $H^n_{cb}(G, \mathbb{R})$ that is usually referred to as the *canonical seminorm* or *Gromov’s norm*.

Most of the results about continuous and continuous bounded cohomology are based on the functorial approach to the study of these cohomological theories that is classical in the case of continuous cohomology and was developed
by Burger and Monod [BM99] in the setting of continuous bounded cohomology. This allows to show that the cohomology of many different complexes realize canonically the given cohomological theory. Since we will only need applications of this machinery that are already present in the literature we will not describe its any further here and we refer instead to [BW00, Mon01] for details on this nice subject.

A first notable application of this approach to continuous cohomology is van Est Theorem [vE53, Dup76] that realizes the continuous cohomology of a semisimple Lie group in terms of $G$-invariant differential forms on the associated symmetric space:

**Theorem 2.4** (van Est). Let $G$ be a semisimple Lie group, then

$$\Omega^n(\mathcal{X}, \mathbb{R})^G \cong H^n_c(G, \mathbb{R}).$$

Under this isomorphism the differential form $\omega$ corresponds to the class of the cocycle $c_\omega$ defined by the formula

$$c_\omega(g_0, \ldots, g_n) = \frac{1}{\pi} \int_{\Delta(g_0 x, \ldots, g_n x)} \omega$$

for any fixed basepoint $x$ in $\mathcal{X}$.

Let us now focus more specifically on the second bounded cohomology of a Hermitian Lie group $G$. By van Est isomorphism the module $H^2_{cb}(G, \mathbb{R})$ is isomorphic to the vector space of the $G$-invariant differential 2-forms on $\mathcal{X}$ which are generated, as a real vector space, by the Kähler classes of the irreducible factors of the symmetric space $\mathcal{X}$. The class corresponding via van Est isomorphism to the Kähler class $\omega$ of $\mathcal{X}$ is represented by the cocycle $c_\omega(g_0, g_1, g_2) = \beta_\mathcal{X}(g_0 x, g_1 x, g_2 x)$ where $x \in \mathcal{X}$ is any fixed point.

It was proven in [DT87] for the irreducible classical domains and in [CO03] in the general case that the absolute value of the cocycle $c_\omega$ is bounded by $\text{rk}(\mathcal{X})$, in particular the class $[c_\omega]$ is in the image of the comparison map $c : H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$. It was proven in [BM99] that, if $G$ is a connected semisimple Lie group with finite center and without compact factors, the comparison map $c$ is injective (hence an isomorphism) in degree 2. We will denote by $\kappa^b_G$, the bounded Kähler class, that is the class in $H^2_{cb}(G, \mathbb{R})$ satisfying $c(\kappa^b_G) = [c_\omega]$. The Gromov norm of $\kappa^b_G$ can be computed explicitly:

**Theorem 2.5** ([DT87, CO03, BIW09]). Let $G$ be a Hermitian Lie group with associated symmetric space $\mathcal{X}$ and let $\kappa^b_G$ be its bounded Kähler class. If $\| \cdot \|$ denotes the Gromov norm, then

$$\|\kappa^b_G\| = \text{rk}(\mathcal{X}).$$

Let now $M$ be a locally compact second countable topological group, $G$ a Lie group of Hermitian type, $\rho : M \to G$ a continuous homomorphism. Precomposing with $\rho$ at the cochain level induces a pullback map in bounded cohomology $\rho^*_b : H^2_{cb}(G, \mathbb{R}) \to H^2_{cb}(M, \mathbb{R})$ that is norm non-increasing. Tight representations were first defined in [BIW09], these are representations $\rho$ for
which the pullback map is norm preserving, namely \( \|\rho^*(\kappa^b_M)\| = \|\kappa^b_M\| \). In the same paper the following structure theorem is proven:

**Theorem 2.6** ([BIW09, Theorem 7.1]). Let \( L \) be a locally compact second countable group, \( G \) a connected algebraic group defined over \( \mathbb{R} \) such that \( G = G(\mathbb{R})^0 \) is of Hermitian type. Suppose that \( \rho : L \to G \) is a continuous tight homomorphism. Then

1. The Zariski closure \( H = \overline{\rho(L)}^Z \) is reductive.
2. The group \( H = H(\mathbb{R})^0 \) almost splits as a product \( H_{nc}H_c \) where \( H_c \) is compact and \( H_{nc} \) is of Hermitian type.
3. If \( \mathcal{Y} \) is the symmetric space associated to \( H_{nc} \), then \( \mathcal{Y} \) is holomorphically and isometrically embedded in \( \mathcal{X} \) and the Shilov boundary \( S_{H_{nc}} \) sits as a subspace of \( S_G \).

2.4. Tight homomorphisms between Hermitian Lie groups. We now restrict to tight homomorphisms \( \rho : M \to G \) between Hermitian Lie groups, and indeed throughout the subsection every group will be a Hermitian Lie group, unless otherwise specified. In his thesis Hamlet classified all such homomorphisms with the additional assumption that both \( M \) and \( G \) are simple and \( G \) is classical. We will now describe the part of his work that we will need in the sequel, and generalize his results to the case of homomorphisms \( \rho : M \to SU(m,n) \) without irreducibility assumptions for \( M \). This results will be needed at the very end of the paper, in the proof of Theorem 1.3.

There is a correspondence between Lie group homomorphisms \( \rho : M \to G \), Lie algebra homomorphisms \( d\rho : m \to g \), and totally geodesic maps \( f_\rho : \mathcal{X}_M \to \mathcal{X}_G \). We will say that an homomorphism \( \rho \) is *holomorphic* if the associated totally geodesic map is holomorphic, and that an homomorphism \( \rho : M \to G \) is *positive* if \( \rho^*\kappa^b_G \) is a linear combination with positive coefficients of the Kähler classes of the irreducible factors of \( M \). Similarly we will say that a Lie algebra homomorphism \( d\rho : m \to g \) is *tight* if its associated Lie group homomorphism is. When classifying all tight homomorphisms it useful to talk about the associated Lie algebra homomorphism, in order not to have to deal with the finite centers of the Hermitian Lie groups, this way the statements are neater. With a slight abuse of notation we will say that a group, or a Lie algebra, is of tube type if the associated symmetric space is.

The following proposition summarizes easy properties of tight maps in this context:

**Proposition 2.7.**

1. If \( \rho : M \to G \) is holomorphic and isometric then \( \rho \) is tight if and only if \( M \) and \( G \) have the same rank.
2. The diagonal embedding \( M \to M^n \) is tight.
3. A map \( \rho : M \to G_1 \times \ldots \times G_n \) is tight if and only if all the induced maps \( \rho_i : M \to G_i \) are tight and positive or tight and negative.
(4) Let $\rho : M \to G$ and $\tau : G \to L$ be homomorphisms. If the composition $\tau \circ \rho$ is tight then both $\rho$ and $\tau$ are tight, if $\rho$ is tight and $\tau$ is tight and positive, then $\tau \circ \rho$ is tight.

Proof. (1) If $\rho$ is holomorphic and isometric, and we denote by $\omega_G$ the normalized Kähler form of the symmetric space associated to $G$, then $f_\rho^*(\omega_G) = \omega_M$. By the naturality of van Est isomorphism this implies that $\rho^*(\kappa_G^b) = \kappa_M^b$, hence $\rho$ is tight if and only if $\text{rk}(\mathcal{X}_G) = \|\kappa_G^b\| = \|\kappa_M^b\| = \text{rk}(\mathcal{X}_M)$.

(2) It is easy to verify that in this case $f_\rho^*\omega_M^n = n\omega_M$.

(3) This follows from the additivity of the rank with respect to products: if we denote by $\kappa_i^b$ the bounded Kähler class of the group $G_i$ we get that $\kappa_G^b = \sum \kappa_i^b$. In particular $\|\kappa_M^b\|\rho^*\kappa_G^b = \|\kappa_M^b\|\sum \rho_I^*\kappa_i^b = \sum (-1)^{\epsilon_i}\|\rho_I^*\kappa_i^b\|\kappa_M^b$ where $\epsilon_i$ is 0 if $\rho_i$ is positive and 1 if it is negative. This implies that

$$\|\rho^*\kappa_G^b\| = \|\sum \rho_I^*\kappa_i^b\| \leq \sum \|\kappa_i^b\| = \sum \text{rk}(\mathcal{X}_i).$$

Since now $\|\kappa_G^b\| = \text{rk}(\mathcal{X}) = \sum \text{rk}(\mathcal{X}_i)$ there can only be equality if all representations $\rho_i$ are tight and all the signs $\epsilon_i$ are equal.

(4) Is analogue to (3). $\square$

We now turn to the classification of all tight representations of $\text{SU}(1,p)$ into $\text{SU}(m,n)$. We begin this analysis by providing some examples of tight representations: let us consider the vector space $\mathbb{C}^{n+m}$ endowed with a Hermitian form $h$ of signature $(m,n)$, and let us fix an orthogonal direct sum decomposition

$$\mathbb{C}^{n+m} = V_1 \oplus \ldots \oplus V_m \oplus W$$

where $V_i$ are subspaces of $V$ such that $h|_{V_i}$ has signature $(1,p)$. We will denote by $j$ the associated embedding

$$j : \text{SU}(1,p)^m \oplus \text{SU}(n-pm) \hookrightarrow \text{SU}(m,n).$$

If $\Delta$ denotes the diagonal embedding

$$\Delta : \text{SU}(1,p) \to \text{SU}(1,p)^m,$$

the composition $j \circ \Delta : \text{SU}(1,p) \to \text{SU}(m,n)$ is tight: indeed the diagonal embedding is tight and the inclusion $j$ is holomorphic and isometric, this implies that $j$ is tight and positive since $\text{SU}(1,p)^m \oplus \text{SU}(n-pm)$ and $\text{SU}(m,n)$ have the same rank. We will call the composition $\rho = j \circ \Delta$ and the restriction $\overline{\rho}$ of $\rho$ to a lattice $\Gamma$ in $\text{SU}(1,p)$ a standard representation. It is worth remarking that, at the level of Lie algebras, the representation $dj$ corresponds to the inclusion of a regular subalgebra $l = \mathfrak{su}(1,p)^m$ of $\mathfrak{g} = \mathfrak{su}(m,n)$. This means that the subalgebra $l$ is the subalgebra of $\mathfrak{g}$ associated to a subsystem of a restricted Dynkin system of $\mathfrak{g}$, in this case the one in which all arrows are forgotten (see [Iha67, Section 2.3] for more details). Regular subalgebras where introduced and used by Ihara [Iha67] in his classification of holomorphic totally geodesic embeddings of Hermitian Lie groups, generalizing the partial results in this direction by Satake [Sat65], and play a big role in Hamlet's classification of tight homomorphisms as well.
The standard representations are the only tight homomorphisms of $\text{SU}(1,p)$:

**Proposition 2.8.** Let $G$ be an irreducible classical Lie group of Hermitian type. Assume that $\rho : \text{SU}(1,p) \to G$ is tight, then $g = \text{su}(m,n)$ for some $n \geq pm$, and $\rho$ is virtually a standard representation.

**Proof.** It follows from [Ham12, Theorem 1.1] that since $g$ and $\text{su}(1,p)$ are simple Hermitian Lie algebras, then $\rho$ must be holomorphic. In particular, by a theorem of Ihara [Iha67, Theorem 2], the associated Lie algebra representation $d\rho$ can be written as the composition $d\rho_1 \circ d\rho_2$ where $d\rho_2$ corresponds to the inclusion of a regular subalgebra, and $d\rho_1$ is an $(\text{H}2)$-representation. Since $\rho$ is tight both $\rho_1$ and $\rho_2$ must be tight. It follows from [Ham11, Proposition 6.2] that the only tight irreducible $(\text{H}2)$-representation of $\text{su}(1,p)$ is the identity representation (up to isomorphism), the classification of tight regular subalgebras carried out in detail in [Ham11, Section 5] implies that the only Hermitian Lie algebra admitting a tight regular subalgebra of the form $\text{su}(1,p)^*$ is the algebra $\text{su}(s,n)$ with $n \geq sp$. This implies the desired statement. \hfill $\square$

We will also need to understand what are the possible tightly embedded subgroups $L \hookrightarrow \text{SU}(m,n)$. If the inclusion is holomorphic, the classification of [Ham12] applies:

**Proposition 2.9.** Let $i : l \to \text{su}(m,n)$ be a tight holomorphic homomorphism. Then each factor of $l$ is either isomorphic to $\text{su}(s,t)$ or is of tube type. Moreover if $l = l_t \times l_{nt}$ where $l_t$ is the product of all the irreducible factors of tube type, then there exists a regular subalgebra $\text{su}(k,k) \oplus \text{su}(m-k,n-k)$ of $\text{su}(m,n)$ such that $l_t$ is included in $\text{su}(k,k)$ and $l_{nt}$ is included in $\text{su}(m-k,n-k)$.

**Proof.** Once more the inclusion $i : l \to \text{su}(m,n)$ splits as the composition of an $(\text{H}2)$-representation and the inclusion of a regular subalgebra. The analysis of [Ham11, Section 5] gives that the only regular tight subalgebras of $\text{su}(m,n)$ are direct copies of $\text{su}(k_1,k_2)$, that satisfy that the sum of the ranks equals $m$. Since in Section 6 of the same paper it is proved that there are tight $(\text{H}2)$-representations of a Lie algebra $g$ in $\text{su}(k_1,k_2)$ different from the inclusion only if $g$ is of tube type, and in that case the image is contained in a copy of $\text{su}(k_1,k_1)$, we get the desired result. \hfill $\square$

In order to conclude our analysis we need to show that every tight inclusion $L \hookrightarrow \text{SU}(m,n)$ is holomorphic. This was proven by Hamlet in [Ham12] with the additional hypothesis that $L$ is irreducible. We will now generalize his techniques and remove this latter hypothesis.

We need a preliminary lemma that is somewhat analogue to [Ham12, Theorem 6.4]. In the study of homomorphisms of real Lie algebras $\rho : l \to \text{su}(m,n)$, it is useful to consider the associated linear representation $\overline{\rho}$ of $l$ on $\mathbb{C}^{m+n}$ that is composition of $\rho$ with the standard representation $\text{su}(m,n) \to$
$\mathfrak{gl}(\mathbb{C}^{m+n})$. This latter representation can be decomposed into irreducible representations, and it can be studied in terms of its weights.

If $\rho_i : l_i \to \mathfrak{gl}(V_i)$ are representations, the tensor product representation $\rho^1 \boxtimes \rho^2 : l_1 \oplus l_2 \to \mathfrak{gl}(V_1 \otimes V_2)$ is defined by the expression

$$\rho^1 \boxtimes \rho^2(l_1, l_2)(v_1 \otimes v_2) = \rho^1(l_1)v_1 \otimes v_2 + v_2 \otimes \rho^2(l_2)v_2.$$ 

It is well known and easy to check that every irreducible linear representation $\rho$ of $\mathfrak{sl}_2$ is of the form $\rho^1 \boxtimes \rho^2$ for some irreducible representations $\rho^1 : l_1 \to \mathfrak{gl}(V_1)$ and $\rho^2 : l_2 \to \mathfrak{gl}(V_2)$. It is possible to chose the decomposition of $\rho$ in irreducible factors so that each factor correspond to a representation $\rho^i : l_i \to \mathfrak{sl}_2$ and each representation $\rho^i$ has values in $\mathfrak{sl}(m_i, n_i)$ for a suitable Hermitian form on $V_i$ (cfr. [Ham12, Theorem 4.3]).

**Lemma 2.10.** Let $\rho = \rho^1 \boxtimes \rho^2 : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ be an homomorphism whose associated linear representation is irreducible. If $\rho$ is tight, then either $\rho^1$ is trivial or $\rho^2$ is the standard representation and $\rho^2$ is trivial. This implies that every tight homomorphism $\rho : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ is either holomorphic or antiholomorphic in the first factor.

**Proof.** In the proof of [Ham12, Theorem 6.5] it is shown that, for every irreducible representation $\rho^1 : \mathfrak{su}(1,2) \to \mathfrak{su}(p,q)$, if $\rho^1$ is not the standard representation nor the trivial one, then the composition of $\rho^1$ with a disc $d : \mathfrak{su}(1,1) \to \mathfrak{su}(1,2)$ is a representation of $\mathfrak{su}(1,1)$ that has even nonzero weights. Let us assume by contradiction that the representation $\rho^1$ is not the standard representation nor trivial and consider the restriction $\rho|$ of $\rho$ to the subalgebra $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ associated to $d$. We get that $\rho|$ decomposes as

$$\sum \rho_l \boxtimes \rho_k$$

where we denote by $\rho_j$ the irreducible representation of $\mathfrak{su}(1,1)$ of highest weight $j$. Since some $l$ is even and nonzero, some of the irreducible factors of $\rho|$ is non-tight: it is proven in [Ham12, Theorem 6.2] that the only tight representations of $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ are of the form $\rho_0 \boxtimes \rho_k$ or $\rho_k \boxtimes \rho_0$ for some $k$ odd. It is proven in [Ham12, Theorem 5.3] that if there exists a non-tight irreducible factor of the restriction of $\rho$ to $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$, then $\rho$ is itself not tight. This gives the desired contradiction and hence implies that $\rho^1$ is either trivial or the standard representation.

In case $\rho^1$ is the standard representation, we get, again restricting to the same subalgebra $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ and decomposing $\rho|$ into $\sum \rho_l \boxtimes \rho_k$, that in at least one factor $\rho_l$ is not trivial, in particular $k$ must be zero again by [Ham12, Theorem 6.2].

The last statement of the lemma (that every tight homomorphism $\rho : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ is either holomorphic or antiholomorphic in the first factor) follows from [Ham12, Theorem 4.3]: since the algebra $\mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1)$ has only simple factors of type $\mathfrak{su}(p,q)$, then the decomposition of $\mathbb{C}^{m+n} = U_1 \oplus \ldots \oplus U_k$ into irreducible linear representations of $\mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1)$ can be chosen in such a way that the restriction of the metric of $\mathbb{C}^{m+n}$ to the spaces $U_i$ is non-degenerate and the decomposition is orthogonal. In particular the decomposition of $\rho$ into irreducible factors corresponds to the
fact that the image of $\rho$ is contained in $\mathfrak{su}(m_1, n_1) \oplus \ldots \oplus \mathfrak{su}(m_k, n_k)$. From this fact one easily sees that if a representation $\rho$ is tight, all the irreducible factors must be tight, and hence either holomorphic or antiholomorphic in $\mathfrak{su}(1, 2)$. We conclude, using Proposition 2.7 (3), that they are all positive (hence all holomorphic) or all negative (hence all antiholomorphic).

Notice that if an homomorphism $\rho : L \to M$ is holomorphic then it is positive. In particular, in the next proposition we will need the hypothesis that $\rho$ is positive in order to deduce that $\rho$ is holomorphic. This is not a big restriction: whenever an homomorphism $\rho$ is given, it is always possible to get a positive homomorphism up to changing the complex structures on the irreducible factors of $L$.

**Proposition 2.11.** Let $L$ be a Hermitian Lie algebra without factors isomorphic to $\mathfrak{su}(1, 1)$. Then every tight and positive homomorphism $i : L \to \mathfrak{su}(m, n)$ is holomorphic.

**Proof.** Let $L = L_1 \times \ldots \times L_k$ be the decomposition of $L$ into irreducible factors. In order to show that $i$ is holomorphic, it is enough to find, for every irreducible factor $L_s$, an holomorphically embedded subalgebra $j_s : g_s \to L_s$ such that the composition $i \circ j_s : g_s \to \mathfrak{su}(m, n)$ is holomorphic. Once that is done it is easy to conclude using the irreducibility of $L_s$ under the adjoint action of $L_s$ on itself that the restriction of $i$ to $L_s$ is holomorphic (see [Ham12, Lemma 5.4]). Clearly if the restriction of $i$ to each irreducible factor is holomorphic, the same is true for the representation $i$.

Let us now fix a factor $L_s$. It is easy to check with a case by case argument that there exists a tight and holomorphic embedding $j_s : g_s \to L_s$ where $g_s$ is either $\mathfrak{su}(1, 2)$ in case $L_s = \mathfrak{su}(1, p)$, $\mathfrak{sp}(4, \mathbb{R})$ in case $\text{rk}(L_s)$ is even, $\mathfrak{sp}(4, \mathbb{R}) \oplus \mathfrak{su}(1, 1)$ in case $\text{rk}(L_s)$ is odd and greater than one (see [Ham12, Theorem 7.1]).

Wedenote by $\kappa^b_s \in H^2_{\text{cb}}(L, \mathbb{R})$ the Kähler class of the factor $L_s$, and let $\alpha_t$ be such that $i^*(\kappa^b_{\text{SU}(m, n)}) = \sum \alpha_t \kappa^b_t$. For every $t$ different from $s$, we consider the diagonal disc $d_t : \mathfrak{su}(1, 1) \to L_t$. Clearly $d_t^* \kappa^b_t = ||\kappa^b_t|| \kappa^b_{\text{SU}(1, 1)}$. Let us now assume first that $L_s$ has higher rank. Then we consider the homomorphism $\phi_s : \mathfrak{sp}(4) \oplus \mathfrak{su}(1, 1) \to L$ that is given by $(d_1, \ldots, j_s, \ldots, d_k)$. The composition $i \circ \phi_s : \mathfrak{sp}(4) \oplus \mathfrak{su}(1, 1) \to \mathfrak{su}(m, n)$ is tight:

$$(i \circ \phi_s)^* (\kappa^b_{\text{SU}(m, n)}) = \phi^*_s \left( \sum_{t=1}^{k} \alpha_t \kappa^b_t \right) = \left( \sum_{t \neq s} \alpha_t ||\kappa^b_t|| \kappa^b_{\text{SU}(1, 1)} \right) + \alpha_s j^*_s \kappa^b_s.$$

Since $j_s$ is tight we get $||j^*_s \kappa^b_s|| = ||\kappa^b_s||$ and hence $||(i \circ \phi_s)^* (\kappa^b_{\text{SU}(m, n)})|| = \sum \alpha_t ||\kappa^b_t|| = ||\kappa^b_{\text{SU}(m, n)}||$.

Every tight and positive homomorphism $\phi : \mathfrak{sp}(4) \oplus \mathfrak{su}(1, 1)$ is holomorphic in the first factor (see [Ham12, Theorem 6.4]). By the discussion in the first paragraph, this implies that the restriction of $i$ to $L_s$ is holomorphic.
The case in which $I_s$ is isomorphic to $\mathfrak{su}(1, p)$ follows analogously applying Lemma 2.10 instead of [Ham12, Theorem 6.2].

\[ \square \]

**Remark 2.12.** It is easy to show that this result implies that every tight positive representation $\rho : L \to G$ is holomorphic provided that none of the irreducible factors of $L$ is virtually isomorphic to $\text{SU}(1, 1)$ and $G$ is classical of tube type. Indeed it is enough to remember that every classical domain of tube type admits a tight and holomorphic embedding in $\text{SU}(m, m)$ for some $m$. Additional arguments are required to deal with the exceptional domains and for the domains associated with $\text{SO}^*(2p)$ with $p$ odd.

### 2.5. Maximal representations.

A key feature of bounded cohomology is that, whenever $\Gamma$ is a lattice in $G$, it is possible to construct a left inverse $T^*_b : H^\bullet_b(\Gamma) \to \Pi^\bullet_{cb}(G)$ of the restriction map. Indeed the bounded cohomology of $\Gamma$ can be computed from the complex $(C^\bullet_{cb}(G, \mathbb{R})^\Gamma, d)$ and the transfer map $T^*_b$ can be defined on the cochain level by the formula

\[
T^*_b(c(g_0, \ldots, g_k)) = \int_{\Gamma \setminus G} c(gg_0, \ldots, gg_k) d\mu(g)
\]

where $\mu$ is the measure on $\Gamma \setminus G$ induced by the Haar measure of $G$ provided it is normalized to have total mass one. It is worth remarking that when we consider instead the continuous cohomology (without boundedness assumptions), a transfer map can be defined with the very same formula only for cocompact lattices, but the restriction map is in general not injective if the lattice is not cocompact.

Let us fix a representation $\rho : \Gamma \to G$. Since $H^2_{cb}(\text{SU}(1, p), \mathbb{R}) = \mathbb{R}\kappa^b_{\text{SU}(1, p)}$, the class $T^*_b \rho^*(\kappa^b_{\Gamma})$ is a scalar multiple of the Kähler class $\kappa^b_{\text{SU}(1, p)}$. The **generalized Toledo invariant** of the representation $\rho$ is the number $i_\rho$ such that $T^*_b \rho^*(\kappa^b_{\Gamma}) = i_\rho \kappa^b_{\text{SU}(1, p)}$. A consequence of Theorem 2.5, and the fact that the transfer map is norm non-increasing, is that $|i_\rho| \leq \text{rk}(\mathcal{X})$. A representation $\rho$ is called **maximal** if $|i_\rho| = \text{rk}(\mathcal{X})$. Clearly maximal representation are in particular tight representations.

It is worth remarking that the definition of generalized Toledo invariant, and hence of maximal representation, we are giving here is different from the one that was first introduced by Burger and Iozzi in [BI00]. Indeed the original definition was based on continuous cohomology only and on the fact that the pullback in continuous cohomology factors via the $L^2$ cohomology of the associated locally symmetric space. However it is proven in [BI07, Lemma 5.3] that the invariant that was originally defined in [BI00], $i_\rho$ in the notation of that article, and the invariant we defined here, that there was denoted by $t_b(\rho)$, coincide. Since we will not need $L^2$-cohomology in the sequel we will stick to this equivalent definition.

The following lemma will be useful at the very end of the article, in the proof of Corollary 1.5:
Lemma 2.13. The generalized Toledo invariant is constant on connected components of the representation variety.

Proof. This is proven in [BI08, Page 4]. \hfill \Box

2.6. A formula for the Toledo invariant. One of the main differences of bounded cohomology with respect to ordinary cohomology is that the bounded cohomology of a group can be computed from a suitable boundary of the group itself. Indeed if \( Z \) is an amenable \( H \) space then the bounded cohomology of \( H \) can be computed isometrically from the complex \( (L^\infty_b(Z^*,\mathbb{R})^H, d) \) [BM02]. For example, if \( H \) is either SU(1,p) or one of its lattices, the boundary \( \partial \mathbb{H}^p_\mathbb{C} \) is an amenable \( H \)-space and hence the complex \( (L^\infty_b((\partial \mathbb{H}^p_\mathbb{C})^*,\mathbb{R})^H, d) \) realizes isometrically the bounded cohomology of \( H \). Moreover, since the action of \( H \) on \( \partial \mathbb{H}^p_\mathbb{C} \) is doubly ergodic by Howe-Moore Theorem, we get that \( L^\infty_b((\partial \mathbb{H}^p_\mathbb{C})^2,\mathbb{R})^H = 0 \). This implies that \( H^2_b(H,\mathbb{R}) \) is isometrically isomorphic to \( L^\infty_b((\partial \mathbb{H}^p_\mathbb{C})^3,\mathbb{R})^H \). It is not hard to verify that the image of \( k^b_{SU(1,p)} \) under the isomorphism \( H^2_b(SU(1,p),\mathbb{R}) \cong L^\infty_b((\partial \mathbb{H}^p_\mathbb{C})^3,\mathbb{R})^{SU(1,p)} \) is Cartan’s angular invariant \( c_p \).

In higher rank it is not anymore true that the Shilov boundary of \( G \) is an amenable \( G \) space, anyway \( \beta_S \) is a cocycle in the complex \( (B_{alt}(S_G^*,\mathbb{R})^G, d) \) where \( B_{alt}(S_G^*,\mathbb{R}) \) denotes the bounded alternating Borel functions, and there is a natural map \( m^*: H^b_\ast(B_{alt}(S_G^*,\mathbb{R})^G, d) \to H^b_\ast(G,\mathbb{R}) \) that has the property that \( m^*[\beta_S] = \kappa_{G} \).

It is possible to prove, exploiting once again functoriality properties of bounded cohomology, the following result:

Proposition 2.14 ([BI09, Theorem 2.41]). Let \( H \) be either SU(1,p) or one of its lattices and let \( G \) be a Hermitian Lie group. Let \( \rho: H \to G \) be a representation, \( \beta_S : (S_G^*)^3 \to \mathbb{R} \) the Bergmann cocycle and let \( \phi: \partial \mathbb{H}^p_\mathbb{C} \to G/Q \) be a measurable \( \rho \)-equivariant boundary map. Then \( \phi^*\beta_S \in L^\infty_b((\partial \mathbb{H}^p_\mathbb{C})^3,\mathbb{R})^H \) corresponds to the class \( \rho^*\kappa_{G}^b \) in \( H^2_b(G,\mathbb{R}) \). for almost every triple \((x,y,z)\) in \( \partial \mathbb{H}^p_\mathbb{C} \), the formula

\[
i_\rho c_p(x,y,z) = \int_{H\backslash SU(1,p)} \beta_S(\phi(gx),\phi(gy),\phi(gz))d\mu(g)\]

holds.

Notice that the measurable function \( \phi^*\beta_S \) is well defined because \( \beta_S \) is defined everywhere and is a strict cocycle: in general if \( \lambda_1 \) denotes the Lebesgue measure on \( \partial \mathbb{H}^p_\mathbb{C} \) and \( \lambda_2 \) is the Lebesgue measure on \( S_G \), then \( \phi_*\lambda_1 \) is not absolutely continuous with respect of \( \lambda_2 \), and hence considering the pullback under \( \phi \) of an element in \( L^\infty(S_G^3,\mathbb{R}) \) makes no sense. We will now show that, since \( \beta_S \) is a strict \( G \)-invariant cocycle and \( SU(1,p) \) acts transitively on pairs of distinct points of \( \partial \mathbb{H}^p_\mathbb{C} \), the equality holds for every triple of pairwise distinct points (this is an adaptation in our context of an argument due to Bucher: cfr. the proof of [BBI13, Proposition 3] in case \( n = 3 \)).
Lemma 2.15. The equality in Proposition 2.14 holds for every triple $(x, y, z)$ of pairwise distinct points.

Proof. The formula of Proposition 2.14 is an equality between $\text{SU}(1,p)$-invariant strict cocycles: clearly this is true for the left-hand side, moreover the expression on the right-hand side is a strict cocycle since $\beta_S$ is, and is $\text{SU}(1,p)$ invariant since $\phi$ is $\rho$-equivariant and $\beta_S$ is $G$-invariant.

Let us now fix a $\text{SU}(1,p)$-invariant full measure set $O \subseteq (\partial \mathbb{H}_C^p)^3$ on which the equality holds. Since $O$ is of full measure, an application of Fubini’s Theorem is that for almost every pair $(y_1, y_2) \in (\partial \mathbb{H}_C^p)^2$ the set of points $z \in \partial \mathbb{H}_C^p$ such that $(y_1, y_2, z) \in O$ is of full measure. Let us fix a pair $(y_1, y_2)$ for which this holds and denote by $W$ the set of points $z$ such that $(y_1, y_2, z) \in O$. Since the $\text{SU}(1,p)$ action on $\partial \mathbb{H}_C^p$ is transitive on pairs of distinct points, for every $i$ there exist an element $g_i$ such that $(x_i, x_{i+1}) = (g_iy_1, g_2y_2)$. Let us now fix a point $x_3$ in the full measure set $g_1W \cap g_2W \cap g_3W$. Since $x_3$ is in $g_1W$, we get that $g_i^{-1}x_3 \in W$, and hence $(x_i, x_{i+1}, x_3) = g_i(y_1, y_2, g_i^{-1}x_3) \in O$.

In particular, computing the cocycle identity on the 4tuple $(x_0, x_1, x_2, x_3)$ we get that the identity of Proposition 2.14 holds for the triple $(x_0, x_1, x_2)$:

$$
\begin{align*}
\int_{\text{SU}(1,p)/\Gamma} & \beta_S(\phi(gx_0), \phi(gx_1), \phi(gx_2)) - \beta_S(\phi(gx_0), \phi(gx_2), \phi(gx_3)) + \\
& + \beta_S(\phi(gx_1), \phi(gx_2), \phi(gx_3)) \, dg = \\
& = \int_{\text{SU}(1,p)/\Gamma} \beta_S(\phi(gx_0), \phi(gx_1), \phi(gx_2)) \, dg.
\end{align*}
$$

□

Corollary 2.16. Let $\rho : \Gamma \to G$ be a maximal representation and let $\phi : \partial \mathbb{H}_C^p \to S_G$ be a $\rho$-equivariant measurable boundary map. Then for almost every maximal triple $(x, y, z) \subseteq (\partial \mathbb{H}_C^p)^3$, the triple $(\phi(x), \phi(y), \phi(z))$ is contained in the Shilov boundary of a tube-type subdomain and is a maximal triple.

Proof. Let us fix a positively oriented triple $(x, y, z)$ of points on a chain. We know from Lemma 2.15 that the equality

$$
\int_{\text{SU}(1,p)/\Gamma} \beta_S(\phi(gx), \phi(gy), \phi(gz)) \, dg = \text{rk}(\mathcal{X})
$$

holds: since $\rho$ is maximal, then $i_\rho = \text{rk}(\mathcal{X})$, and since $(x, y, z)$ are on a chain then $c_\rho(x, y, z) = 1$.

Since $\|\beta_S\|_\infty = \text{rk}(\mathcal{X})$, it follows that $\beta_S(\phi(gx), \phi(gy), \phi(gz)) = \text{rk}(\mathcal{X})$ for almost every $g$ in $\text{SU}(1,p)$. By Proposition 2.1, this implies that for almost every $g \in \text{SU}(1,p)$, the triple $(\phi(gx), \phi(gy), \phi(gz))$ is contained in the boundary of a tube-type subdomain. Since maximal triples in $\partial \mathbb{H}_C^p$ form an $\text{SU}(1,p)$-orbit, the fact that the result holds for almost every element $g$ implies that the result holds for almost every triple of positively oriented points in a chain. The same argument applies for negatively oriented triples. □
We will say that a measurable map \( \phi \) with this property preserves the chain geometry: indeed it induces an almost everywhere defined morphism \((\phi, \hat{\phi})\) from the geometry \( \partial \mathbb{H}^p_C \times C \) whose points are points in \( \partial \mathbb{H}^p_C \) and whose lines are the chains, to the geometry \( S_G \times T \) whose points are points in \( S_G \) and whose lines are the Shilov boundaries of maximal tube-type subdomains of \( S_G \). The morphism \((\phi, \hat{\phi})\) has the property that it preserves the incidence structure almost everywhere.

Purpose of the next sections is to show that a measurable Zariski dense map \( \phi : \partial \mathbb{H}^p_C \to S_{SU(m,n)} \) that preserves the chain geometry coincides almost everywhere with an algebraic map.

3. The Chain geometry of \( S_{m,n} \)

For the rest of the paper we will restrict our attention to the Hermitian Lie group \( SU(m,n) \) consisting of complex matrices that preserve a non-degenerate Hermitian form of signature \((m,n)\). It is a classical result (see for example [PS69, Section 2.10]) that, if \( m \) is different from \( n \), the group \( SU(m,n) \) is a Hermitian Lie group which is not of tube type. The groups \( SU(1,p) \), that are finite index covers of the groups \( PU(1,p) = \text{Isom}^0(\mathbb{H}^p_C) \), form a subfamily of those groups. We will denote, for the sake of brevity, by \( S_{m,n} \) the Shilov boundary associated to the group \( SU(m,n) \). The same space was denoted in the previous subsection by \( S_{SU(m,n)} \). The purpose of this section is to understand some features of the incidence structure of the subsets of \( S_{m,n} \) that arise as Shilov boundaries of the maximal tube-type subdomains of the symmetric space associated to \( SU(m,n) \). For reasons that will be explained later we will call \( m \)-chains such subsets.

The main tool that we will introduce in our investigations is a projection map \( \pi_x \), depending on the choice of a point \( x \in S_{m,n} \). The map \( \pi_x \) associates to a point \( y \) that is transversal to \( x \) the uniquely determined \( m \)-chain that contains both \( x \) and \( y \). The projection \( \pi_x \) encodes a part of the incidence structure that is easier to study but yet contains enough information to draw conclusions on the rigidity of the geometry. We will call a \((m,k)\)-circle a geometric object that arises as projection of an \( m \)-chain that doesn’t contain the point \( x \). We will be mostly interested in understanding what is a good description of a \((m,k)\)-circle, what is the intersection of a fiber of \( \pi_x \) with an \( m \)-chain, and what is a parametrization of the different lifts of a fixed \((m,k)\)-circle. The central results of the section are Proposition 3.16 and Proposition 3.18.

3.1. The Grassmannian manifold. Both the symmetric space and the Shilov boundary of \( SU(m,n) \) have an explicit description as closed subsets of the Grassmanian of the \( m \)-planes in \( \mathbb{C}^{m+n} \). This will allow us to give an explicit formula for the map \( \pi_x \) mentioned above. Before going into the details, we recall here a few facts about the rational structure of the Grassmanian manifold and its affine charts that will be needed in the sequel.
In general, if $M$ is a complex algebraic variety, $M$ can be endowed with the structure of a real algebraic variety by choosing a real structure $x \mapsto \overline{x}$ on $\mathbb{C}$, and we will be generally interested in the real Zariski topology whose closed subsets are locally defined by polynomials in the variables $z_i$ and $\overline{z}_i$. A map $f : V \to W$ between real affine varieties is called regular if it is defined by polynomial equations, and rational if it is defined on a Zariski open subset of $V$ by quotients of polynomials.

If $V$ is a complex vector space of dimension $n + m$, the Grassmannian of $m$-dimensional complex subspaces of $V$ is the quotient of the frame variety, whose points correspond to ordered bases of $m$-dimensional subspaces of $V$, under the natural action of the group $\text{GL}_m(\mathbb{C})$. In particular, once a basis of $V$ is fixed, the frame variety identifies with $M^+ = M^+((m + n) \times m, \mathbb{C})$, the set of matrices with $(m + n)$ rows and $m$ columns that have maximal rank: in fact it is enough to interpret a matrix $X \in M^+$ as the ordered basis of the subspace spanned by its columns. Under this identification the action of the group $\text{GL}_m(\mathbb{C})$ on the frame variety is given by the action of $\text{GL}_m(\mathbb{C})$ on $M^+$ via right multiplication.

We will denote by

$$j : M^+/\text{GL}_m(\mathbb{C}) \to \text{Gr}_m(\mathbb{C}^{n+m})$$

the quotient map, understood with respect to the standard basis: the image under $j$ of a matrix $M \in M^+$ the subspace of $\mathbb{C}^{m+n}$ spanned by the columns of $M$. The map $j$ is algebraic, and to any $\text{GL}_m(\mathbb{C})$-invariant Zariski closed subset of $M^+$ corresponds a Zariski closed subset of $\text{Gr}_m(V)$. For example the set of points in $\text{Gr}_m(V)$ that are non-transversal to a given $k$-dimensional subspace $W$ of $V$ is Zariski closed since it corresponds to the Zariski closed subset of $M^+$ defined by the vanishing of all the $(m+k)$-minors of the matrix whose columns are given by a basis of $W$ and the columns of $M$.

Let us now fix an $n$-dimensional subspace $W$ of $V$ and consider the Zariski open set $\text{Gr}_m(V)^W$ consisting of points $x \in \text{Gr}_m(V)$ that are transversal to $W$. For any fixed point $y$ in $\text{Gr}_m(V)^W$ there is a natural identification

$$\psi_W : \text{Gr}_m(V)^W \to \text{Lin}(y, W)$$

defined by requiring, for any $t \in y$, that $\psi_W(x)t$ is the unique vector in $W$ such that $t + \psi_W(x)t$ belongs to $x$. The map $\psi_W$ gives an algebraic affine chart of the Grassmannian. The Zariski topology of $\text{Gr}_m(V)$ is induced by the Zariski topology on the charts that can in turn be identified with $M(n \times m)$ by choosing bases of $y$ and $W$.

If we fix a real structure $v \to \overline{v}$ on $V$, we can induce a real structure on the Grassmannian $\text{Gr}_m(V)$ and we will be interested in the structure of $\text{Gr}_m(V)$ as a real algebraic variety. We also choose real structure on $M^+$ and on $M(n \times m)$ in a compatible way, so that the Zariski closed subsets of $\text{Gr}_m(V)$ correspond to the subsets that, in the affine charts, are defined by polynomial equations involving only the coefficients of a matrix and its conjugates.
3.2. A model for \( S_{m,n} \). Let us consider the space \( \mathbb{C}^{n+m} \) endowed with the standard Hermitian form \( h \) of signature \((m, n)\), where we will always assume, without loss of generality, that \( m \leq n \). This means that \( h \) is represented, with respect to the standard basis, by the matrix \[
abla_{m} - \lambda_{n} \]. Moreover we consider on \( \mathbb{C}^{n+m} \) the real structure given by \((v_{1}, \ldots, v_{n+m}) \mapsto (\overline{v}_{1}, \ldots, \overline{v}_{n+m})\).

We realize \( SU(m, n) \) as the real algebraic subgroup of \( SL_{n+m}(\mathbb{C}) \) consisting of the elements of \( SL_{m+n}(\mathbb{C}) \) preserving \( h \). Of course \( SU(m, n) \) acts on \( Gr_{m}(\mathbb{C}^{m+n}) \) preserving the subset \( \mathcal{X}_{m,n} \) consisting of subspaces on which the restriction of \( h \) is positive definite:

\[
\mathcal{X}_{m,n} = \{ x \in Gr_{m}(\mathbb{C}^{n+m}) \mid h|_{x} > 0 \}.
\]

The manifold \( \mathcal{X}_{m,n} \) is a model of the symmetric space of \( SU(m, n) \): indeed it is an homogeneous \( SU(m, n) \)-space, and the stabilizer of the point \( o = (e_{1}, \ldots, e_{m}) \) is the group \( S(U(m) \times U(n)) \) that is a maximal compact subgroup of \( SU(m, n) \). The set \( \mathcal{X}_{m,n} \) is contained in the domain of the affine chart \( \psi_{W} \) associated with the subspace \( W = (e_{m+1}, \ldots, e_{m+n}) \). In particular the map \( \psi_{W} \) allows to realize \( \mathcal{X}_{m,n} \) as a bounded domain \( X_{m,n} \) in \( \mathbb{C}^{m \times (m+n)} \):

\[
X_{m,n} = \{ X \in M(n \times m, \mathbb{C}) \mid X^*X < \text{Id} \}.
\]

Here and in the sequel, by the expression \( X^*X < \text{Id} \), we mean that the Hermitian matrix \( \text{Id} - X^*X \) is positive definite. The domain \( X_{m,n} \) corresponds under \( \psi \) to \( \mathcal{X}_{m,n} \) since an explicit formula for the composition \( \psi_{W} \circ j \) is given by

\[
\begin{bmatrix}
X_{1} \\
X_{2}
\end{bmatrix} \mapsto X_{2}X_{1}^{-1}.
\]

It is possible to verify that the bounded domain realization \( X_{m,n} \) of the symmetric space associated to \( SU(m, n) \) we are giving here is what is classically known as the Harish-Chandra realization (cfr. [Kor00, Theorem III.2.6]).

It follows from the explicit formula for the chart \( \psi_{W} \) that the action of \( SU(m, n) \) on the bounded model \( X_{m,n} \) is by fractional linear transformations: \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot X = (AX + B)(CX + D)^{-1} \), in particular it is an action via biholomorphisms of the domain and it extends continuously to the boundary. The unique closed \( SU(m, n) \)-orbit in the topological boundary of \( X_{m,n} \) is the set

\[
S_{m,n} = \{ X \in M(n \times m, \mathbb{C}) \mid X^*X = \text{Id} \},
\]

that corresponds, via the affine chart \( \psi_{W} \), to the set of isotropic subspaces of \( (\mathbb{C}^{m+n}, h) \):

\[
S_{m,n} = \{ x \in Gr_{m}(\mathbb{C}^{n+m}) \mid h|_{x} \equiv 0 \}.
\]

In order to verify this last assertion, recall that an inverse of the composition \( \psi_{W} \circ j : M^{+} \rightarrow M(n \times m, \mathbb{C}) \) is given by the map

\[
\lambda : X \mapsto \begin{bmatrix} \text{Id}_{m} \\ X \end{bmatrix}.
\]
It is now clear that the matrix $X$ satisfies $X^*X = \text{Id}$ if and only if the subspace of $\mathbb{C}^{n+m}$ spanned by the columns of $\lambda(X)$ is isotropic for $h$.

**Proposition 3.1.** The Shilov boundary $S_{m,n}$ is a real algebraic subvariety of the Grassmannian variety. The action of $\text{SU}(m,n)$ on $S_{m,n}$ is algebraic and transitive.

**Proof.** The fact that $S_{m,n}$ is a real algebraic subvariety of the Grassmannian follows from the fact that $S_{m,n}$ is contained in the domain of the affine chart $\text{Lin}(\langle e_1, \ldots, e_m \rangle, \langle e_{m+1}, \ldots, e_{m+n} \rangle)$ and the image $S_{m,n}$ of $S_{m,n}$ in that chart is defined by polynomial equations involving only the coefficients of a matrix and their conjugates. Moreover the real algebraic group $\text{SU}(m,n)$ acts algebraically on the real algebraic variety $\text{Gr}_m(\mathbb{C}^{m+n})$, and the restriction of the action to the Zariski closed subset $S_{m,n}$ of $\text{Gr}_m(\mathbb{C}^{m+n})$ is algebraic. The transitivity of $\text{SU}(m,n)$ on the set of isotropic $m$-subspaces follows from Witt’s theorem. □

It is worth remarking that in [BI04] it is explicitly constructed a complex variety $S_{m,n} = S_{m,n}(\mathbb{R})$.

**Pairs of transversal points.** We now turn to the study of the action of $G = \text{SU}(m,n)$ on pairs of transversal points in $S_{m,n}$. The general theory of Hermitian symmetric spaces recalled in Section 1 tells us that there exists a unique open $G$-orbit for the diagonal $G$-action on $S_{m,n}$ and any pair of points in this orbit is called transversal. Indeed in this specific realization points are transversal if and only if the underlying vector spaces are transversal, as proven in the following lemma:

**Lemma 3.2.** The action of $\text{SU}(m,n)$ is transitive on pairs of transversal isotropic subspaces. In particular a pair $(x, y) \in S_{m,n}^2$ is transversal if and only if the underlying vector subspaces are transversal.

**Proof.** Let $x_\infty = \langle e_i - e_{i+m} \mid 1 \leq i \leq m \rangle$ and $x_0 = \langle e_i + e_{i+m} \mid 1 \leq i \leq m \rangle$. It is easy to compute that both $x_\infty$ and $x_0$ are isotropic subspaces, moreover the pair $(x_\infty, x_0)$ forms a pair of transversal subspaces. Let us now fix another pair $(x, y)$ of isotropic subspaces that are transversal. Then the linear span $\langle x, y \rangle$ is a $2m$ dimensional subspace of $\mathbb{C}^{m+n}$ on which the restriction of $h$ is non-degenerate. In particular we can find a basis $(x_1, \ldots, x_m)$ of $x$ (resp. $(y_1, \ldots, y_m)$ of $y$) such that $h(x_i, y_j) = 2\delta_{i,j}$. If $(z_{2m+1}, \ldots, z_{m+n})$ is an orthonormal basis of $\langle x, y \rangle^\perp$, then the linear operator $L$ sending $e_i - e_{i+m}$ to $x_i$, $e_i + e_{i+m}$ to $y_i$ and $e_j$ to $z_j$ induces an element in $U(m,n)$ sending $(x_\infty, x_0)$ to $(x, y)$. In order to get an element in $\text{SU}(m,n)$, it is enough to rescale the matrix representing $L$ so that its determinant is 1. □

From now on we will often identify a point $x$ in the Shilov boundary with its underlying vector subspace, and we will use the notation $x \pitchfork y$, that would be more suited for the linear setting, with the meaning that the pair $(x, y)$ is a pair of transversal points. Moreover we will use the notation $S_{m,n}^{(2)}$
for the set of pairs of transversal points, and we will denote the set of points
in $S_{m,n}$ that are transversal to a given point $x$ by

$$S_{m,n}^x := \{ y \in S_{m,n} \mid y \pitchfork x \}.$$ 

We now want to describe the maximal tube-type subdomains of $X_{m,n}$ it is
well known that any such subspace is associated with the group $SU(m,m)$. Indeed for every linear $2m$-dimensional subspace $V$ of $\mathbb{C}^{m,n}$ on which the
restriction of $h$ has signature $(m,m)$, the closed subset

$$X_V = \{ z \in Gr_m(V) \mid h|_z > 0 \} = X_{m,n} \cap Gr_m(V)$$

is a subspace of $X_{m,n}$ that is holomorphically and isometrically embedded in
$X_{m,n}$ and is a symmetric space whose isometry group, $SU(V)$, is isomorphic
to $SU(m,m)$. In particular $X_V$ is a maximal tube-type subdomain of $X_{m,n}$
whose Shilov boundary

$$S_V = \{ z \in Gr_m(V) \mid h|_z = 0 \} = S_{m,n} \cap Gr_m(V)$$

is naturally a subspace of $S_{m,n}$. As already anticipated in the introduction
of this section, we will use the term $m$-chain to denote the subspaces of the
form $S_V$ for some linear subspace $V$ of $\mathbb{C}^{m,n}$

Let us now fix a pair of transversal points $(x,y) \in S_{m,n}^{(2)}$. We have seen
in the proof of Lemma 3.2 that the linear span $\langle x,y \rangle \subset \mathbb{C}^{m,n}$ is a $2m$-dimensional subspace $V_{xy}$ of $\mathbb{C}^{m,n}$, on which the restriction of $h$ has signature $(m,m)$. Let $T_{xy} := S_{V_{xy}}$ be the subset of $S_{m,n}$ consisting of maximal isotropic subspaces of $V_{xy}$, then $T_{xy}$ is the unique Shilov boundary of a max-
imal tube-type subdomain that contains the points $x$ and $y$. We will call $T_{xy}$
the $m$-chain through $x$ and $y$.

In the case $m = 1$, that is $X_{m,n} = \mathbb{H}^n_C$, the 1-chains are boundaries of
complex geodesics or chains in Cartan’s terminology. This is the reason why
we chose to call the Shilov boundaries of maximal tube-type subdomains
$m$-chains. To be more consistent with Cartan’s notation, will omit the 1,
and simply call chains the 1-chains.

3.3. The Heisenberg model $H_{m,n}(x)$. We now want to give another model
for an open subset of $S_{m,n}$ that will be extremely useful to study the chain
geometry in $S_{m,n}$. The new model we are introducing is sometimes referred
to as a Siegel domain of genus two and was studied, for example, by Koranyi
and Wolf in [KW65]. In the case $m = 1$ this model is described in [Gol99,
Chapter 4] but our conventions here will be slightly different.

We will need, once again, a more concrete description in terms of the
Grassmannian manifold in order to be able to compute what is needed.
The Siegel model can be described with the aid of a conjugate form, in
$SL(m + n, \mathbb{C})$, of $SU(m,n)$. In particular let $F$ be the linear endomorphism
of $\mathbb{C}^{m+n}$ that is represented with respect to the standard basis by the matrix
MAXIMAL REPRESENTATIONS INTO \( \text{SU}(m,n) \)

\[ F = \begin{bmatrix}
\frac{1}{\sqrt{2}} \text{Id} & 0 & \frac{1}{\sqrt{2}} \text{Id} \\
-\frac{1}{\sqrt{2}} \text{Id} & 0 & -\frac{1}{\sqrt{2}} \text{Id} \\
0 & \text{Id} & 0 \\
\end{bmatrix}^m_{-m}. \]

It is easy to verify that \( F^* = F^{-1} \). We consider the Hermitian form \( \bar{h} = F^* h F \). With respect to the standard basis \( \bar{h} \) is represented by the matrix

\[ \bar{h} = \begin{bmatrix}
0 & 0 & \text{Id} \\
0 & -\text{Id} & 0 \\
\text{Id} & 0 & 0 \\
\end{bmatrix}^m_{n-m}. \]

By definition of \( \bar{h} \), the linear map \( F \) conjugates the group \( \text{SU}(\mathbb{C}^{m+n}, h) \) into \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \): namely, for every \( g \in \text{SU}(\mathbb{C}^{n+m}, h) \), the element \( F^{-1} g F \) belongs to \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \). Moreover the preimage \( \mathcal{S}_{m,n} \) of the set \( \mathcal{S}_{m,n} \) is the Shilov boundary for the group \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \), and consists of isotropic subspaces for the Hermitian form \( \bar{h} \).

Let us now focus on the maximal isotropic subspace

\[ v_\infty = \langle e_i | 1 \leq i \leq m \rangle \in \mathcal{S}_{m,n}, \]

and let us denote by \( Q \) the stabilizer in \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \) of \( v_\infty \). Since \( Q \) is the stabilizer of a point in the Shilov boundary, it is a maximal parabolic subgroup of \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \). We now need an explicit expression for the group \( Q \) and some of its subgroups. It is easy to verify with a direct computation that

\[ Q = \left\{ \begin{bmatrix} A & B & E \\ 0 & C & F \\ 0 & 0 & A^{-*} \end{bmatrix}^m_{n-m} \mid A \in \text{GL}_m(\mathbb{C}), C \in \text{U}(n-m), A^{-1}B - F^*C = 0, E^*A^{-*} + A^{-1}E - F^*F = 0, \det C \det A \det A^{-*} = 1 \right\}. \]

We now want to compute the Langland decomposition of \( Q \). This means that we want to write \( Q = L \ltimes N \) where \( L \) is reductive and \( N \) is nilpotent. We already proved that the group \( \text{SU}(\mathbb{C}^{m+n}, h) \) acts transitively on pairs of transversal subspaces in \( \mathcal{S}_{m,n} \). This implies that the conjugate group \( \text{SU}(\mathbb{C}^{m+n}, \bar{h}) \) acts transitively on pairs of transversal points of \( \mathcal{S}_{m,n} \). In particular this implies that the group \( Q \) acts transitively on the set of maximal isotropic subspaces of \( (\mathbb{C}^{n+m}, \bar{h}) \) that are transversal to \( v_\infty \).

The Levi factor \( L \) in the Langland decomposition for \( Q \) is not unique, but it depends on the choice of a point \( v_0 \) that is transversal to \( v_\infty \). We chose, for simplicity, \( v_0 \) to be the maximal isotropic subspace \( v_0 = \langle e_{n+i} | 1 \leq i \leq m \rangle \), which is clearly transversal to \( v_\infty \), and we denote by \( L \) the stabilizer in \( Q \) of \( v_0 \). It follows from the explicit expression for elements of \( Q \) that

\[ L = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & A^{-*} \end{bmatrix} \mid A \in \text{GL}_m(\mathbb{C}), C \in \text{U}(n-m), \det C \det A \det A^{-*} = 1 \right\} = \text{SU}(n-m) \times \text{GL}_m(\mathbb{C}). \]
L is a reductive subgroup of Q and it is a Levi factor for Q being the intersection \( Q \cap Q_0 \) of two opposite parabolic subgroups, the stabilizers of two transversal points in \( \mathcal{S}_{m,n} \). Here \( Q_0 \) is the stabilizer of \( v_0 \).

Let us now consider the group

\[
N = \left\{ \begin{bmatrix} \text{Id} & F^* & E \\ 0 & \text{Id} & F \\ 0 & 0 & \text{Id} \end{bmatrix} \mid E^* + E - F^*F = 0 \right\}
\]

\( N \) is a two step nilpotent subgroup that can be identified with the generalized Heisenberg group \( H_{m,n} = M((n-m) \times m, \mathbb{C}) \ltimes u(m) \) where \( u(m) \) denotes the group of antiHermitian matrices and the semidirect product structure is given by

\[
(F_1, E_1) \cdot (F_2, E_2) = (F_1 + F_2, E_1 + E_2 + \frac{F_1^* F_2 - F_2^* F_1}{2})
\]

The subgroup \( u(m) \) is the center of \( H_{m,n} \), and the identification of \( N \) with \( H_{m,n} \) is defined by the formula

\[
\begin{bmatrix} \text{Id} & F^* & E \\ 0 & \text{Id} & F \\ 0 & 0 & \text{Id} \end{bmatrix} \mapsto \left( F, E - \frac{E^*}{2} \right)
\]

Since any element of \( Q \) can be written as a product \( n l \) where \( n \in N \) and \( l \in L \), the decomposition \( Q = NL \) is a Langland decomposition for the maximal parabolic subgroup \( Q \).

**Lemma 3.3.** The group \( N \) acts simply transitively on the set of points in \( \mathcal{S}^{v_\infty}_{m,n} \) that are transversal to \( v_\infty \).

**Proof.** Let \( x \in \mathcal{S}^{v_\infty}_{m,n} \) represent any isotropic \( m \)-dimensional subspace of \( (\mathbb{C}^{m+n}, \gamma) \) which is transversal to \( v_\infty \). Since \( x \) is isotropic and transverse to \( v_\infty \), it is also transverse to \( v_\infty^* = \langle e_1, \ldots, e_n \rangle \): indeed \( v_\infty \) is the radical of \( v_\infty^* \). This implies that \( x \) admits a basis of the form \( \begin{bmatrix} A \\ 1_{1d} \end{bmatrix} \), where the requirement that \( x \) is isotropic implies that \( A^* + A - B^*B = 0 \). Let us now consider the element \( n \in N \) corresponding to the pair \( (B, \frac{A - A^*}{2}) \in H_{m,n} \). It follows from the very construction that \( n \cdot v_0 = x \) and in particular \( N \) acts transitively on \( \mathcal{S}^{v_\infty}_{m,n} \). Since \( N \cap Q_0 = \text{Id} \) we get that the action is simple. The thesis follows from the observation that \( N \) preserves the set \( \mathcal{S}^{v_\infty}_{m,n} \) since it is contained in the stabilizer of \( v_\infty \). \( \square \)

We just gave a way of identifying the abstract Heisenberg group \( H_{m,n} \), the nilpotent radical \( N \) of the stabilizer of the point \( v_\infty \) and the open \( N \)-orbit of the point \( v_0 \). From now on we will denote the open \( N \)-orbit of the point \( v_0 \) with the symbol \( \mathcal{H}_{m,n}(v_\infty) \). It is worth remarking that \( \mathcal{H}_{m,n}(v_\infty) \) consists precisely of the intersection of \( \mathcal{S}_{m,n} \) with the affine chart of the Grassmanian associated with the subspace \( v_\infty^i \) and the latter chart realizes both \( \mathcal{S}^{v_\infty}_{m,n} \) and \( \mathcal{A}_{m,n} \) as unbounded subsets of \( M(n \times m, \mathbb{C}) \). We will not need this
realization, that is a realization of $\mathcal{X}_{m,n}$ as a Siegel domain of second kind, but only the identification of $\mathcal{H}_{m,n}(v_\infty)$ with the Heisenberg group $H_{m,n}$.

Notice that in the proof of Lemma 3.3 we gave a particularly nice section of the projection from an algebraic subset of the frame variety to $\mathcal{H}_{m,n}(v_\infty)$: each point of $\mathcal{H}_{m,n}(v_\infty)$ admits a basis of the form $\begin{bmatrix} A \\ B \\ \text{Id} \end{bmatrix}$, and with a slight abuse of notation we will often identify a point $x$ in $\mathcal{H}_{m,n}(v_\infty)$ with the matrix representing its basis.

Our next goal is to understand the action of $Q$ on $\mathcal{H}_{m,n}(v_\infty)$. We know that $Q = LN$, moreover an explicit formula for the action of $N$ on $\mathcal{H}_{m,n}(v_\infty)$ is given by
\[
\begin{bmatrix}
1 & F^* & E \\
0 & 1 & F \\
0 & 0 & \text{Id}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
\text{Id}
\end{bmatrix}
= 
\begin{bmatrix}
X + F^*Y + E \\
y + F \\
\text{Id}
\end{bmatrix}.
\]

Similarly an element $q \in Q$ corresponds to a pair $(C, A) \in U(n - m) \times \text{GL}_m(\mathbb{C})$ and the action is given by
\[
\begin{bmatrix}
A & 0 & 0 \\
0 & C & 0 \\
0 & 0 & A^{-*}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
\text{Id}
\end{bmatrix}
= 
\begin{bmatrix}
AX \\
CY \\
A^{-*}
\end{bmatrix} \cong 
\begin{bmatrix}
AXA^* \\
CYA^* \\
\text{Id}
\end{bmatrix}.
\]

Notice that under the identification of $\mathcal{H}_{m,n}(v_\infty)$ with the $N$ orbit of $v_0$, the action of $L$ on $\mathcal{H}_{m,n}(v_\infty)$ corresponds to the action of $L$ on $N$ by conjugation: if $l$ is an element of $L$, for any $n$ in $N$, the element $lnl^{-1}$ is the unique element in $N$ such that $lnl^{-1}v_0 = lnv_0$ and indeed we have
\[
\begin{bmatrix}
A & 0 & 0 \\
0 & C & 0 \\
0 & 0 & A^{-*}
\end{bmatrix}
\begin{bmatrix}
\text{Id} & -X^* & Y \\
0 & 1 & X \\
0 & 0 & \text{Id}
\end{bmatrix}
\begin{bmatrix}
A^{-1} & 0 & 0 \\
0 & C^{-1} & 0 \\
0 & 0 & A^*
\end{bmatrix}
= 
\begin{bmatrix}
\text{Id} & -AX^*C^* & \text{AYA}^* \\
0 & 1 & CXA^* \\
0 & 0 & \text{Id}
\end{bmatrix}.
\]

**Maximal triples.** We now briefly turn to the study of maximal triples in $S_{m,n}$. In this case the Bergmann cocycle can be explicitly computed, and it is shown in [DT87] that, given three points $Z_1, Z_2, Z_3$ in the bounded domain realization $X_{m,n}$, and denoting by $Y_i$ the matrix $Y_i = \text{Id} - Z_i^*Z_{i+1}$, and by $(y^k_i)_{k=1}^n$ the eigenvalues of $Y_i$ counted with multiplicity, one gets
\[
\int_{\Delta(Z_1, Z_2, Z_3)} \omega = \pi \beta_{X}(Z_1, Z_2, Z_3) = -2 \sum_{i,k} \arg(y^k_i).
\]

Notice that for all $i, k$, $\arg(1 - y^k_i) \in [-\pi/2, \pi/2]$.

Since the Bergmann cocycle extends continuously to triples of pairwise transversal points in the Shilov boundary, we get that the same formula holds for transversal triples in the boundary. Let us now consider the isotropic
Lemma 3.4. The triple \((x_\infty, x_0, x_1)\) is maximal.

Proof. This is a direct computation. The matrix \(Z_1\) representing \(x_\infty\) in the bounded domain realization is \(Z_1 = \begin{bmatrix} -\text{Id} & 0 \\ 0 & 0 \end{bmatrix}\), similarly \(x_0 = Z_2 = \begin{bmatrix} \text{Id} & 0 \\ 0 & 0 \end{bmatrix}\), and \(x_1 = Z_3 = \begin{bmatrix} \text{Id} & 0 \\ 0 & 0 \end{bmatrix}\). This implies that \(Y_1 = 2\text{Id}, Y_2 = \text{Id} - i\text{Id}, Y_3 = \text{Id} - i\text{Id}\). In particular \(\arg(y_k^\alpha) = -\pi/4\) if \(k = 2, 3\) and 0 if \(k = 1\). One can now easily compute that \(\beta_{\mathcal{S}_{m,n}}(x_\infty, x_0, x_1) = m\) and hence the triple is maximal. \(\square\)

It is well known that the group \(\text{SU}(m, n)\) acts transitively on maximal triples. We check this and the fact that if the Bergmann cocycle is maximal then a triple is contained in an \(m\)-chain for the sake of completeness.

Lemma 3.5. Let \((Z_1, Z_2, Z_3)\) be a triple such that \(\beta_{\mathcal{S}_{m,n}}(Z_1, Z_2, Z_3) = m\). Then there exists an \(m\)-chain containing \(Z_i\) for all \(i\).

Proof. Since we know that \(\text{SU}(m, n)\) is transitive on pairs of transversal point it is enough to justify the statement for maximal triples \((Z_1, Z_2, Z_3)\) such that \(Z_1 = x_\infty\) and \(Z_2 = x_0\). The third element \(Z_3\) can be written as \(Z_3 = \begin{bmatrix} W \\ Z \end{bmatrix}\) where \(Z\) has \(n - m\) rows and \(m\) columns and \(W\) is a square \(m\)-dimensional matrix; since \(Z_3\) belongs to \(\mathcal{S}_{m,n}\), we get \(W^*W + Z^*Z = \text{Id}\), hence the eigenvalues of \(W\) have absolute value smaller or equal to one and have all absolute value equal to one if and only if \(Z = 0\) that is equivalent to the fact that \(x\) belongs to the \(m\)-chain containing \(x_0\) and \(x_\infty\). It is easy to compute that \(Y_1 = 2\text{Id}, Y_2 = \text{Id} - W\) and \(Y_3 = \text{Id} + W^*\).

If \(w_j = \lambda_j e^{t_j}\) are the eigenvalues of \(W\), we get that
\[
\beta(x_\infty, x_0, x) = -2 \sum_{j=1}^{m} \left( \arctg \left( \frac{-\lambda_j \sin t_j}{1 - \lambda_j \cos t_j} \right) + \arctg \left( \frac{-\lambda_j \sin t_j}{1 + \lambda_j \cos t_j} \right) \right).
\]

Computing the derivative in \(t_j\) of the argument of the sum, one checks that the function attains its minimum in \(t_j = \pi/2\) where it has value \(2 \arctg(-\lambda_j)\). Clearly this is always bigger that \(-\pi/2\) and is equal to \(-\pi/2\) precisely when \(\lambda_j\) equals one. This implies that, if the triple is maximal, the matrix \(Z\) must be zero, and hence \(x\) must belong to the \(m\)-chain through \(x_\infty\) and \(x_0\). In the case in which \(\lambda_j = 1\) we get that \(\beta(x_\infty, x_0, x)\) is maximal if and only if \(t_j \in [0, \pi]\) for all \(j\). \(\square\)

In order to show that \(\text{SU}(m, n)\) is transitive on maximal triples, it is easier to work in the Heisenberg model where we have easy formulae for the stabilizer of the pair \((v_\infty, v_0)\). Let us hence consider the images under \(F^*\) of the subspaces we are considering. It is easy to check that
\[
\begin{align*}
F^*x_\infty &= v_\infty = \langle e_1, \ldots, e_m \rangle \\
F^*x_0 &= v_0 = \langle e_{n+1}, \ldots, e_{n+m} \rangle \\
F^*x_1 &= v_1 = \langle e_1 + ie_{n+1}, \ldots, e_m + ie_{n+m} \rangle
\end{align*}
\]
moreover the image under $F^*$ of the subspace corresponding to the point $\begin{bmatrix} W \\ \Id \end{bmatrix}$ is the isotropic subspace that is spanned by the columns of the matrix $\begin{bmatrix} (\Id - W)(\Id + W)^{-1} & 0 & \Id \end{bmatrix}^T$.

**Lemma 3.6.** $SU(m, n)$ is transitive on maximal triple.

**Proof.** Indeed if $e^{it_j}$ are the eigenvalues of the matrix $W$, then the eigenvalues of the matrix $(\Id - W)(\Id + W)^{-1}$ are $\frac{1-e^{it_j}}{1+e^{it_j}} = \frac{-i\sin t_j}{1+\cos t_j}$. In particular the eigenvalues of $W$ are imaginary numbers and are negative precisely when the arguments of the eigenvalues of $W$ are in $[0, \pi]$. We know that the stabilizer $L$ in $SU(\mathbb{C}^{n+m}, \overline{h})$ of the pair $(v_0, v_{\infty})$ is isomorphic to $U(n-m) \times GL_m(\mathbb{C})$ and the image of the space $v_1$ under the element $(C, A)$ is the subspace spanned by the columns of $[-iA^*A \ 0 \ \Id]^T$. In particular every element $x$ such that $\beta(x_{\infty}, x_0, x)$ is maximal is in the orbit $L \cdot v_1$, and this concludes the proof. \(\square\)

3.4. The projection $\pi_x$. Let us now consider the projection on the first factor $\pi : H_{m,n} \rightarrow M((n-m) \times m, \mathbb{C})$ that is obtained by quotienting the center $u(m)$ of $H_{m,n}$. Purpose of this subsection is to give a geometric interpretation of the quotient space $M((n-m) \times m, \mathbb{C})$: it corresponds to a parametrization of the space of chains through the point $v_{\infty}$. In order to make this more precise let us consider the set

$$W_{v_{\infty}} = \{ V \in Gr_{2m}(\mathbb{C}^{n+m}) \mid v_{\infty} < V \text{ and } \overline{h} \mid V \text{ has signature } (m, m) \}.$$ 

The following lemma gives an explicit identification of $W_{v_{\infty}}$ with the quotient space $M((n-m) \times m, \mathbb{C}) = H_{m,n}/u(m)$:

**Lemma 3.7.** There exists a bijection between $M((n-m) \times m, \mathbb{C})$ and $W_{v_{\infty}}$ defined by the formula

$$i : M((n-m) \times m, \mathbb{C}) \rightarrow W_{v_{\infty}}$$

$$A \mapsto \begin{bmatrix} A^* \\ \Id \\ 0 \end{bmatrix}.$$ 

**Proof.** Let $V$ be a point in $W_{v_{\infty}}$. Then $V^\perp$ is a $(n-m)$ dimensional subspace of $\mathbb{C}^{m+n}$ that is contained in $v_{\infty}^\perp$. This implies that $V^\perp$ admits a basis of the form $\begin{bmatrix} A & B & 0 \end{bmatrix}^T$ where $A$ has $m$ rows and $n - m$ columns and $B$ is a square $n - m$ dimensional matrix. Since the restriction of $\overline{h}$ on $V$ has signature $(m, m)$, the restriction of $\overline{h}$ to $V^\perp$ is negative definite, in particular the matrix $B$ must be invertible. This implies that, up to changing the basis of $V^\perp$, we can assume that $B = \Id_{n-m}$. This gives the desired bijection. \(\square\)

$W_{v_{\infty}}$ parametrizes the $m$-chains containing the point $v_{\infty}$. We will call them vertical chains: the intersection $T^{v_{\infty}}$ of a vertical chain $T$ with the Heisenberg model $H_{m,n}(v_{\infty})$ consists precisely of the fiber of the point associated to $T$ in Lemma 3.7 under the projection on the first factor in the Heisenberg model.
Lemma 3.8. Let $T \subset \mathcal{S}_{m,n}$ be a vertical chain and $V$ be the associated linear subspace, then

(1) for every $x$ in $T^{v \infty}$, then $\pi_1(x) = i^{-1}(V) = p_T$
(2) $T^{v \infty} = \pi_1^{-1}(p_T)$,
(3) the center $M$ of $N$ acts simply transitively on $T^{v \infty}$.

Proof. (1) Let us denote by $p_T$ the matrix in $M((n-m)\times m, \mathbb{C})$ corresponding to $V$ under the isomorphism of Lemma 3.7. An element $w$ of $\mathcal{H}_{m,n}(x^{\infty})$ with basis $[X \ Y \ \text{Id}_m]^T$ belongs to the chain $T$ if and only if $Y = p_T$: indeed the requirement that $V^\perp$ is contained in $w^\perp$ restates as

$$0 = [X^* \ Y^* \ \text{Id}_m] \begin{bmatrix} 0 & 0 & \text{Id} \\ 0 & -\text{Id} & 0 \\ \text{Id} & 0 & 0 \end{bmatrix} \begin{bmatrix} p_T^* \\ \text{Id} \\ 0 \end{bmatrix} = p_T^* - Y^*.$$ 

This implies that for every $w \in T$, we have $\pi_1(w) = p_T$.

Vice versa if $\pi_1(w) = p_T$, then $w$ is contained in $V$ and this proves (2).

(3) The fact that $M$ acts simply transitively on $T^{v \infty}$ is now obvious: indeed $M$ acts on the Heisenberg model by vertical translation stabilizing every vertical chain. $\square$

The stabilizer $Q$ of $v_\infty$ naturally acts on the space $W_{v_\infty}$, and it is easy to deduce explicit formulae for the induced action on $M((n-m)\times m, \mathbb{C}) \cong W_{v_\infty}$:

Lemma 3.9. The action of $Q$ on $M((n-m)\times m, \mathbb{C})$ is described by

(1) The pair $(C, A) \in \text{SU}(n-m) \times \text{GL}_m(\mathbb{C}) \cong L$ acts via

$$(C, A) \cdot X = CXA^*.$$ 

(2) The pair $(Z, W) \in M((n-m)\times m, \mathbb{C}) \times \text{u}(m) \cong N$ acts via

$$(Z, W) \cdot X = X + Z.$$ 

In the sequel, where this will not cause confusion, we will identify $W_{v_\infty}$ with $M((n-m)\times m, \mathbb{C})$ considering implicit the map $i^{-1}$ and we will denote by $\pi_{v_\infty}$ the projection $\pi_1 : \mathcal{H}_{m,n}(v_\infty) \to M((n-m)\times m, \mathbb{C})$. This emphasizes the role of the point at infinity $v_\infty$ and will be useful to avoid confusions in Section 4 where we will need to consider at the same time Heisenberg models for two different Shilov boundaries, the image and the target of a measurable boundary map $\phi$.

It is worth remarking that everything we did so far doesn’t really depend on the choice of the point $v_\infty$, and a map $\pi_x : \mathcal{S}_{m,n} \to W_x$ can be defined for every point $x \in \mathcal{S}_{m,n}$. We decided to stick to the point $v_\infty$, since the formulae in the explicit expressions are easier.

3.5. Projections of chains. We now want to understand what are the possible images under the map $\pi_{v_\infty}$ of other chains. We define the intersection index of an $m$-chain $T$ with a point $x \in \mathcal{S}_{m,n}$ by

$$i_x(T) = \dim(x \cap V_T)$$
where $V_T$ is the $2m$ dimensional linear subspace of $\mathbb{C}^{m+n}$ associated to $T$. Clearly $0 \leq i_{v_{x_\infty}}(T) \leq m$, and $i_{v_{x_\infty}}(T) = m$ if and only if the chain $T$ is vertical. In general we will call $k$-vertical a chain whose intersection index is $k$: with this notation vertical chains are $m$-vertical. Sometimes we will call horizontal the chains that are 0-vertical (in particular they intersect $v_{\infty}$ just in the zero vector).

In our investigations it will be precious to be able to relate different situations via the action of the group $\mathcal{G} = \text{SU}(\mathbb{C}^{m+n}, \overline{h})$, under this respect the following lemma will be fundamental:

**Lemma 3.10.** For every $k \in \{0, \ldots, m\}$ the group $\mathcal{G}$ acts transitively on

1. the pairs $(x, T)$ where $x \in \mathfrak{S}_{m,n}$ is a point and $T$ is an $m$-chain with $i_x(T) = k$,
2. the triples $(x, y, T)$ where $x \cap y$, $y \in T$ and $i_x(T) = k$.

In particular the intersection index is a complete invariant of $m$-chains up to the $Q$-action.

**Proof.** We will prove directly the second statement, that clearly implies also the first one. Since $\mathcal{G}$ acts transitively on the Shilov boundary, we can assume that $x$ is $v_{\infty}$, hence it is enough to show that $Q$ is transitive on $k$-vertical chains.

Since the intersection index $i_{v_{x_\infty}}(T)$ equals $k$, the span $W = \langle v_{\infty}, T \rangle$ has dimension $3m - k$, moreover since $W$ contains $V_T$ that is a subspace on which $h$ has signature $(m, m)$, the restriction of $\overline{h}$ to $W$ has signature $(m, 2m - k)$.

Since $Q$ is transitive on $3m - k$ subspaces of $\mathbb{C}^{m+n}$ containing $v_{\infty}$ and on which the restriction of $h$ has signature $(m, 2m - k)$, we can reduce to the case $W = \langle e_1, \ldots, e_{2m-k}, e_{n+1}, \ldots, e_{n+m} \rangle$ and, even more, we can assume that $n = 2m - k$. Since $Q$ is transitive on points transversal to $v_{\infty}$ (Lemma 3.2), we can assume that the $m$-chain $T$ contains the point $v_0$.

In this case the orthogonal $V_T^\perp$ to $V_T$ is a $m - k$ dimensional space on which $\overline{h}$ is definite negative. A point $Z$ in the frame manifold which has the property that $jZ = V_T^\perp$ is represented by a matrix with $3m - k$ rows and $m - k$ columns that has a block structure $[Z_1 \ Z_2 \ Z_3]^T$ where $Z_1$ and $Z_3$ are matrices with $m$ rows and $Z_2$ is a square $(m - k)$-dimensional matrix.

Since, by our assumption, the point $v_0$ belongs to $T$, the vector space $V_T^\perp$ is contained in $v_0^\perp$, hence $Z_1 = 0$. Together with the fact that the restriction of $\overline{h}$ to $z$ must be negative definite, this implies that we can assume (up to changing the representative for $V_T^\perp$) that $Z$ has the form $\begin{bmatrix} 0 & 0 \\ \text{Id}_{m-k} & Z_3 \end{bmatrix}$ for some matrix $Z_3$.

We claim that the hypothesis that the intersection index of $T$ and $v_{\infty}$ is $k$ is equivalent to the fact that the rank of $Z_3$ is $m - k$. Indeed the intersection $v_{\infty} \cap V$ is $v_{\infty} \cap z^\perp$ hence corresponds, in the basis $\langle e_1, \ldots, e_m \rangle$, to the kernel
of

\[
\begin{bmatrix}
\text{Id} & 0 & 0 \\
0 & -\text{Id} & \text{Id} \\
\text{Id} & \text{Id} & \text{Id}
\end{bmatrix}
\begin{bmatrix}
0 \\
\text{Id} \\
Z_3
\end{bmatrix} = Z_3.
\]

This implies that the group \( L = \text{stab}(v_\infty, v_0) \) acts transitively on the set of \( m \)-chains through \( v_0 \) with intersection index \( k \), since \( \text{GL}_m(\mathbb{C}) \subset L \) is transitive on matrices with \( m \) rows, \( m - k \) columns and rank \( m - k \).

As explained at the beginning of the section we want to give a parametrization of a generic chain \( T \) and study the restriction of \( \pi_{v_\infty} \) to \( T \). In view of Lemma 3.10, it is enough to understand, for every \( k \), the parametrization and the projection of a single \( k \)-vertical chain. The \( k \)-vertical chain we will deal with is the chain with associated linear subspace

\[
T_k = \langle e_i, e_j + e_{m+j} + e_{n+j}, v_0 \mid 1 \leq i < k < j \leq m \rangle.
\]

**Lemma 3.11.** \( T_k \) is the linear subspace associated to a \( k \)-vertical chain.

**Proof.** \( T_k \) is a \( 2m \)-dimensional subspace containing \( v_0 \). Moreover \( T_k \) splits as the orthogonal direct sum

\[
T_k = T_k^0 \oplus T_k^1 = \langle e_i, e_{n+i} \mid 1 \leq i \leq k \rangle \oplus \langle e_j + e_{m+j} + e_{n+j} \mid k + 1 \leq j \leq m \rangle = \langle v_\infty \cap T_k, e_{n+i} \mid 1 \leq i \leq k \rangle \oplus \langle e_j + e_{m+j} + e_{n+j} \mid k + 1 \leq j \leq m \rangle.
\]

Since \( v_\infty \cap T_k = \langle e_1, \ldots, e_k \rangle \), we get that \( i_{v_\infty}(T_k) \) is \( k \). Since \( \overline{h}|_{T_k^0} \) has signature \( (k, k) \) and \( \overline{h}|_{T_k^1} \) has signature \( (m - k, m - k) \), we get that the restriction of \( \overline{h} \) on \( T_k \) has signature \( (m, m) \) and this concludes the proof. \( \square \)

**Lemma 3.12.** \( \mathcal{H}_{m,n}(v_\infty) \cap T_k \) consists precisely of those subspaces of \( \mathbb{C}^{m+n} \) that admit a basis of the form

\[
\begin{bmatrix}
E^*E + C & E^*X \\
E & \text{Id} + X
\end{bmatrix}_k
\]

\[
\begin{bmatrix}
m-k \\
k \\
m-k \\
m-2m \\
k \\
m-k
\end{bmatrix}
\]

\[E \in M((m - k) \times k, \mathbb{C}) \text{ with } X \in U(m - k), C \in u(k).\]

The projection of \( T_k \) is contained in a subspace of \( M((n-m) \times m, \mathbb{C}) \) of dimension \( m^2 - km \), and consists of the points of \( M((n-m) \times m, \mathbb{C}) \) that have expression

\[
\begin{bmatrix}
0 & 0 & 0 \\
E \text{Id} + X & 0 \\
0 & 0
\end{bmatrix}
\]

with \( E \in M((m - k) \times k, \mathbb{C}) \) and \( X \in U(m - k) \).

**Proof.** It is enough to check that the orthogonal to \( T_k \) is

\[
T_k^\perp = \langle e_{m+i}, e_{m+j} + e_{n+j}, e_{2m+l} \mid 1 \leq i < k \leq m, 1 \leq l \leq n - 2m \rangle.
\]
This implies that any $m$-dimensional subspace $z$ of $T_k$, that is transversal to $v_\infty$, has a basis of the form

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{array}{c}
k \\
m-k
\end{array}
\]

and the restriction of $\overline{h}$ to $z$ is zero if and only if

\[
\begin{bmatrix}
0 & 0 \\
0 & k
\end{bmatrix}
\begin{array}{c}
m-k \\
n-2m
\end{array}
\]

is zero.

Equation (22) restates as $Z_{22} = \text{Id} + X$ for some $X \in U(m-k)$: indeed a square matrix $Z$ satisfies the equation $Z^* + Z = Z^*Z$, if and only if the equation $(Z - \text{Id})^*(Z - \text{Id}) = Z^*Z - Z - Z^* + \text{Id} = \text{Id}$ holds, which means that $Z - \text{Id}$ belongs to $U(m-k)$.

This concludes the proof of the first part of the lemma: the $(m-k) \times k$ matrix $Z_{21}$ can be chosen arbitrarily, Equation (12) uniquely determines $Z_{12}$ in function of $Z_{21}$ and $Z_{22}$, and Equation (11) determines the Hermitian part of $Z_{11}$ in function of $Z_{21}$, but is satisfied independently on the antiHermitian part of $Z_{11}$. This proves the first part of the lemma.

The second part is a direct consequence of the parametrization of $T_k^{v_\infty}$ we just gave, together with the identification of $W_{v_\infty}$ and $M((n-m) \times m, \mathbb{C})$ given in Lemma 3.7.

We will call a subset of $W_x$ that is projection of a $k$-vertical chain a $(m,k)$-circle. The reason for the name circle is due to the fact that, in the case $(m,n) = (1,2)$ the projections of horizontal chains are Euclidean circles in $\mathbb{C}$. This fact was first observed and used by Cartan in [Car32]. In the general case it is important to record both the dimension of the $m$-chain that is projected and the dimension of the $U(m-k)$ factor in the projection. This explains the notation. We will call generalized circle any subset of $M((n-m) \times m, \mathbb{C})$ arising as a projection of an $m$-chain. In particular a generalized circle is an $(m,k)$-circle for some $k$.

The ultimate goal of this section is to understand the possible lifts of a given $(m,k)$-circle. We begin by analyzing the stabilizers in $\text{SU}(\mathbb{C}^{m+n}, \overline{h})$ of some configurations:

**Lemma 3.13.** The stabilizer in $\text{SU}(\mathbb{C}^{m+n}, \overline{h})$ of the triple $(v_\infty, v_0, T_k)$ is the subgroup of $S_0$ of $L \cong \text{SU}(n-m) \times \text{GL}_m(\mathbb{C})$ consisting of pairs of the form

\[
\begin{bmatrix}
C_{11} & 0 & C_{13} \\
0 & C_{22} & 0 \\
C_{31} & 0 & C_{33}
\end{bmatrix}
\begin{array}{c}
k \\
m-k \\
n-2m
\end{array}
\begin{bmatrix}
Y & X \\
0 & C_{22}
\end{bmatrix}
\begin{array}{c}
k \\
m-k
\end{array}
\]

\[
\begin{bmatrix}
C_{11} & C_{13} \\
C_{31} & C_{33}
\end{bmatrix}
\in U(n-2m+k)
\]

\[
C_{22} \in U(m-k)
\]

\[
X \in M(k \times (m-k), \mathbb{C})
\]

\[
Y \in \text{GL}_k(\mathbb{C})
\]

**Proof.** We determined in Section 3.3 that the stabilizer $L$ in $\text{SU}(\mathbb{C}^{m+n}, \overline{h})$ of the pair $(v_\infty, v_0)$ is isomorphic to $\text{SU}(n-m) \times \text{GL}_m(\mathbb{C})$. The stabilizer of
the triple \((v_\infty, v_0, T_k)\) is clearly contained in \(L\) and consists precisely of the elements of \(L\) stabilizing \(T_k^\perp\).

In the proof of Lemma 3.12 we saw that the subspace \(T_k^\perp\) has a basis of the form \(\begin{bmatrix} \text{Id}_{n-m} & 0 \\ X \end{bmatrix}\) where \(X\) denotes the \(m \times (n-m)\) matrix \(\begin{bmatrix} 0_k & 0 & 0 \\ 0 & \text{Id}_{m-k} & 0 \end{bmatrix}\).

From the explicit expression of elements in \(L\) we get
\[
\begin{bmatrix} A & C \\ A^{-*} & \text{Id} \end{bmatrix} \begin{bmatrix} 0 \\ \text{Id} \end{bmatrix} = \begin{bmatrix} 0 & C \\ A^{-*}X \end{bmatrix} \cong \begin{bmatrix} 0 \\ \text{Id} \end{bmatrix} \begin{bmatrix} A^{-*}X^{-1} \end{bmatrix}.
\]

In turn the requirement that \(A^{-*}X^{-1} = X\) that is \(X = A^*XC\), implies, in the suitable block decomposition for the matrices, that
\[
\begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} 0_k & 0 & 0 \\ 0 & \text{Id}_{m-k} & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}_k = m-k
\begin{bmatrix} A_{21}C_{21} & A_{21}C_{22} & A_{21}C_{23} \\ A_{22}C_{21} & A_{22}C_{22} & A_{22}C_{23} \end{bmatrix}.
\]

This implies that \(A_{22}^* = C_{22}^{-1}\) and \(C_{21} = C_{23} = A_{12} = 0\). Moreover since \(C\) is unitary, also \(C_{12}\) and \(C_{13}\) must be 0, and both \(C_{22}\) and the matrix \(\begin{bmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{bmatrix}\) must be unitary. This concludes the proof.

Let us now denote by \(o\) the point \(o = \pi_{v_\infty}(v_0) = 0\) in \(W_{v_\infty}\) and by \(C_k\) the \((m,k)\)-circle that is the projection of \(T_k\). We will denote by \(S_1\) the stabilizer in \(Q\) of the pair \((o, C_k)\).

**Lemma 3.14.** The stabilizer of the pair \((o, C_k)\) is the group
\[S_1 = \text{Stab}_Q(o, C_k) = M \rtimes S_0\]
where, as above, we denote by \(M\) the center of the nilpotent radical \(N\) of \(Q\) and by \(S_0\) the stabilizer in \(Q\) of the pair \((v_0, T_k)\).

**Proof.** Recall that any element in \(Q\) can be uniquely written as a product \(nl\) where \(n\) is in \(N\), and \(l\) belongs to \(L\), the Levi component of \(Q\). Since any element in \(S_1\) fixes, by assumption, the point \(o = \pi_{v_\infty}(v_0)\) and since any element in \(L\) fixes \(o\), if \(nl\) is in \(S_1\) then \(n(o) = o\) that, in turn, implies that \(n\) belongs to \(M\). Hence \(S_1\) is of the form \(M \rtimes S\) for some subgroup \(S\) of \(L\).

Let now \(X\) be a point in \(W_{v_\infty} = M((n-m) \times m, \mathbb{C})\). The action of \((C, A) \in L\) on \(W_x\) is \(X \mapsto CXA^*\) (cfr. Lemma 3.9). We want to show that if \(C_k\) is preserved then \((C, A)\) must belong to \(S_0\). We have proven in Lemma 3.12 that any point \(z \in \pi_{v_\infty}(T_k)\) can be written as \(\begin{bmatrix} 0 & E & Z \\ 0 & 0 & 0 \end{bmatrix}\) for some matrices \(E \in M((m-k) \times k, \mathbb{C})\) and \(Z \in U(m-k)\). Explicit computations give that
Moreover this can be easily seen, for example by choosing $ZM$ group preserving $U$ zero everywhere apart from the check that the subgroup of the matrices $S$ form of a genuine element of $S_0$. In particular $C_{22}$ is invertible. Since $C_{22}(EA_{21}^* + (Id + Z)A_{22}^*)$ must be an element of $Id + U(m - k)$ for every $E$, we get that $A_{21}^*$ must be zero.

The result now follows from Claim 3.15 below. □

**Claim 3.15.** Let $C \in U(l)$ and $A \in GL_l(\mathbb{C})$ be matrices and let $U$ denote the set $$U = \{Id + X \mid X \in U(l)\} \subset M(l \times l, \mathbb{C}).$$ If $CUA^* = U$ then $A = C$.

**Proof.** Let us consider the birational map 

$$i : \ M(l \times l, \mathbb{C}) \rightarrow M(l \times l, \mathbb{C})$$

$$X \mapsto X^{-1},$$

that is defined on a Zariski open subset $O$ of $M(l \times l, \mathbb{C})$.

The image, under the involution $i$, of $U$ is the set

$$\mathcal{L} = \{W \mid Id - W^*W = 0\} = \frac{1}{2}Id + u(l).$$

Moreover $i(CXA^*) = A^{-1}i(X)C^{-1}$, hence in order to show that the subgroup preserving $U$ consists precisely of the pairs $(A, A)$, it is enough to check that the subgroup of $U(l) \times GL_l(\mathbb{C})$ preserving $\mathcal{L}$ consists precisely of the pairs $(A, A)$ with $A \in U(l)$.

This last statement amounts to show that the only matrix $B \in GL_m(\mathbb{C})$ such that $Id - W^*B - BW = 0$ for all $W \in \mathcal{L}$ is the identity itself. Choosing $W$ to be $\frac{1}{2}Id$ we get that $B^* + B = 2Id$ hence in particular $B = Id + Z$ with $Z \in u(l)$. Since moreover $\mathcal{L} = \{\frac{1}{2}Id + M \mid M \in u(l)\}$ we have to show that if $ZM + M^*Z^* = ZM +MZ = 0$ for all $M$ in $u(l)$ then $Z$ must be zero, and this can be easily seen, for example by choosing $M$ to be the matrix that is zero everywhere apart from the $l$-th diagonal entry where it is equal to $i$. □

We now have all the ingredients we need to show the following proposition:

**Proposition 3.16.** Let $x \in S_{m,n}$ be a point, $T$ be a chain with $i_x(T) = k$, $t \in T$ be a point, $y = \pi_x(t) \in W_x$. Then

1. $T$ is the unique lift of the $(m, k)$-circle $\pi_x(T)$ through the point $t$,
2. for any point $t_1$ in $T_{x,t} = \pi_x^{-1}(y)$ there exists a unique $m$ chain through $t_1$ which lifts of $\pi_x(T)$. 

\[
\begin{bmatrix}
0 & 0 \\
E & Id + Z \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
E & Id + Z \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{11}^* & A_{21}^* \\
A_{12}^* & A_{22}^*
\end{bmatrix} = 
\begin{bmatrix}
C_{12}E & C_{12}(Id + \dot{Z}) \\
C_{22}E & C_{22}(Id + Z) \\
C_{32}E & C_{32}(Id + Z)
\end{bmatrix}
\begin{bmatrix}
A_{11}^* & A_{21}^* \\
A_{12}^* & A_{22}^*
\end{bmatrix}.
\]
Proof. (1) As a consequence of Lemma 3.10, in order to prove the statements, we can assume that the triple \((x,t,T)\) is the triple \((v_\infty,v_0,T_k)\). Let \(T'\) be another \(m\)-chain containing the point \(t\) that lifts the \((m,k)\)-circle \(C_k\), a consequence of Lemma 3.10 is that there exists an element \(g \in L\) such that 
\[(v_\infty,v_0,T_k) = g(v_\infty,v_0,T').\] Moreover, since \(\pi_{v_\infty}(T') = C_k\), we get that \(g \in S_1\). But we know that \(S_1 \cap L = S_0\) and this proves that \(T' = T_k\).

(2) This is a consequence of the first part, together with the transitivity of \(M\) on the vertical chain \(T_{v_0v_\infty}\).

We now want to determine what are the lifts of a point \(y\) that are contained in an \(m\)-chain \(T\). We will assume that the point \(x\) is \(v_\infty\): since the group action is transitive on \(\mathfrak{S}_{m,n}\), this is not a real restriction, but is convenient since we gave explicit formulae only for the stabilizer of \(v_\infty\). Let us fix an \(m\)-chain \(T\) and consider the subgroup 
\[M_T = \text{Stab}_M(T).\]

Here, as usual, \(M\) denotes the center of \(N\), the nilpotent radical of the stabilizer \(Q\) of \(v_\infty\). Clearly if \(t\) is a lift of a point \(y \in \mathcal{W}_{v_\infty}\) that is contained in \(T\), then all the orbit \(M_T \cdot t\) consists of lifts of \(y\) that are contained in \(T\). We want to show that also the other containment holds, namely that the lifts are precisely the \(M_T\) orbit of any point.

We first describe the group \(M_T\) in the case in which \(T = T_k\), and in this case we denote, for the sake of brevity, by \(M_k\) the group \(M_{T_k}\). We also define the subgroup of \(\mathfrak{u}(m)\)
\[E_k = \{X \in \mathfrak{u}(m)| X_{ij} = 0 \text{ if } i > k \text{ or } j > k\} = \left\{ \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \middle| X_1 \in \mathfrak{u}(k) \right\},
\] and let us consider the identification \(\alpha : M \to \mathfrak{u}(m)\) described in Section 3.3.

**Lemma 3.17.** In the notations above, we have 
\[\alpha(M_k) = E_k.\]

Proof. We already observed that the orthogonal to \(T_k\) is 
\[T_k^\perp = \langle e_{m+i}, e_{m+j} + e_{n+j}, e_i | 1 \leq i \leq k \leq j \leq m, 2m < l \leq n \rangle.\]

Moreover an element of \(M\) stabilizes \(T_k\) if and only if it stabilizes \(T_k^\perp\). If now \(m = \begin{bmatrix} \mathbf{1} & 0 & E \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}\) is an element of \(M\), then the image \(m \cdot (e_{m+j} + e_{n+j}) = \sum E_{ij} e_j + e_{m+j} + e_{n+j}\) that belongs to \(T_k^\perp\) if and only if the \(j\)-th column of the matrix \(E\) is zero. This implies that the image under \(\alpha\) of the subgroup of \(M\) that fixes \(T_k^\perp\) is contained in \(E_k\). Viceversa if \(V_k\) is the linear subspace corresponding to \(T_k\) it is easy to check that \(\alpha^{-1}(E_k)\) belongs to \(\text{SU}(V_k)\), in particular it preserves \(T_k\). \(\square\)

Another invariant of a \(k\)-vertical chain that will be relevant in the next section is the intersection of the linear subspace \(V_T\) underlying \(T\) with \(v_\infty\). We will denote it by \(Z_T\):
\[Z_T = v_\infty \cap V_T.\]
In the standard case in which $T = T_k$ we will denote by $Z^k$ the subspace $Z_{T_k}$ which equals to the span of the first $k$ vectors of the standard basis of $\mathbb{C}^m$.

**Proposition 3.18.** Let $T$ be a $k$-vertical chain, then

1. If $g \in Q$ is such that $gT = T_k$, then $M_T = g^{-1}M_kg$.
2. For any point $x \in T$, we have $\pi_{v_\infty}^{-1}(\pi_{v_\infty}(x)) \cap T = M_T x$.
3. If $n \in N$, then $M_{nT} = M_T$.
4. If $a \in GL(m)$ is such that $a(Z_T) = Z^k$, then $\alpha(M_T) = a^{-1}E_k a^{-*}$.

**Proof.** (1) This follows from the definition of $M_k$ and $M_T$ and the fact that $M$ is normal in $Q$.

(2) Let us first consider the case $T = T_k$. In this case the statement is an easy consequence of the explicit parametrization of the chain $T_k$ we gave in Lemma 3.12: any two points in $T_k$ that have the same projection are in the same $M_k$ orbit. The general case is a consequence of the transitivity of $Q$ on $k$-vertical chains: let $g \in Q$ be such that $gT = T_k$ and let us denote by $y$ the point $gx$. Then we know that $M_ky = \pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap T_k$. This implies that

$$M_T x = g^{-1}M_k gx = g^{-1}(M_k y) = g^{-1} \pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap T_k =$$

$$= g^{-1} \pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap g^{-1} T_k =$$

$$= \pi_{v_\infty}(\pi_{v_\infty}(x)) \cap T.$$

Where in the last equality we used that the $Q$ action on $\mathcal{H}_{m,n}(v_\infty)$ induces an action of $Q$ on $W_{v_\infty}$ so that the projection $\pi_{v_\infty}$ is $Q$ equivariant.

(3) This is a consequence of the fact that $M$ is in the center of $N$: $M_{nT} = nM_T n^{-1} = M_T$.

(4) By (3) we can assume that $T$ is a chain through the point $v_0$: indeed there exists always an element $n \in N$ such that $nT$ contains $v_0$, moreover both $M_{nT} = M_T$ and $Z_{nT} = Z_T$ (the second assertion follows from the fact that any element in $N$ acts trivially on $v_\infty$).

Since $v_0 \in T$ and we proved in Lemma 3.10 that $L$ is transitive on $k$-vertical chains through $v_0$, we get that there exists a pair $(C, A) \in U(n - m) \times GL_m(\mathbb{C})$ such that, denoting by $g$ the corresponding element in $L$, we have $gT = T_k$. It follows from (1) that $M_T = g^{-1}M_kg$, in particular we have

$$\begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & A^* \end{bmatrix} \begin{bmatrix} \text{Id} & 0 & E \\ 0 & \text{Id} & 0 \\ 0 & 0 & \text{Id} \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & A^{-*} \end{bmatrix} = \begin{bmatrix} \text{Id} & 0 & A^{-1}EA^{-*} \\ 0 & \text{Id} & 0 \\ 0 & 0 & \text{Id} \end{bmatrix}$$

and hence the subgroup $\alpha(M_T)$ is the group $A^{-1}E_kA^{-*}$. Moreover, since $gT = T_k$ we have in particular that $gZ_T = Z^k$ and hence $A(Z_T) = Z^k$ if we consider $Z_T$ as a subspace of $v_\infty$.

In order to conclude the proof it is enough to check that for every $a \in GL_m(\mathbb{C})$ with $a(Z_T) = Z^k$ the subgroups $a^{-1}E_k a^{-*}$ coincide. Indeed it is enough to check that for every element $a \in GL_m(\mathbb{C})$ such that $a(Z^k) = Z^k$ then $a^{-1}E_k a^{-*} = E_k$. But if $a$ satisfies this hypothesis, the matrix $a^{-*}$ has
the form $\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$. In particular we can compute:

$$a^{-1} X a^{-*} = \begin{bmatrix} A_1^* & A_2^* \\ 0 & A_3^* \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} = \begin{bmatrix} A_1^* X_1 A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the latter matrix still belongs to $E_k$. \hfill \Box

4. The restriction to a chain is rational

In this section we prove that the chain geometry defined in Section 3 is rigid in the following sense:

**Theorem 4.1.** Let $\phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,n}$ be a measurable, chain geometry preserving, Zariski dense map. Then for almost every chain $C$ in $\partial \mathbb{H}^p_C$ the restriction $\phi|_C$ coincides almost everywhere with a rational map.

This is the key step of the proof of Theorem 1.6.

Let us recall that, whenever a point $x \in \mathcal{S}_{m,n}$ is fixed, the center $M_x$ of the nilpotent radical $N_x$ of the stabilizer $Q_x$ of $x$ in $\text{SU}(\mathbb{C}^{m+n}, \mathfrak{t})$ acts on the Heisenberg model $\mathcal{H}_{m,n}(x)$. Moreover, for every $m$-chain $T$ containing the point $x$, the $M_x$ action is simply transitive on the Zariski open subset $T^x$ of $T$. The picture above is true for both $\partial \mathbb{H}^p_C \cong S_{1,p}$ and $\mathcal{S}_{m,n}$ where, if $x \in \partial \mathbb{H}^p_C$, the group $M_x$ can be identified with $u(1)$, and, if $\phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,n}$ is the boundary map, $M_{\phi(x)} \cong u(m)$.

The idea of the proof is to show that, for almost every point $x \in \partial \mathbb{H}^p_C$, the boundary map is equivariant with respect to a measurable homomorphism $h : M_x \to M_{\phi(x)}$. Since such homomorphism must be algebraic, we get that the restriction of $\phi$ to almost every chain through $x$ must be algebraic.

In order to define the homomorphism $h$ we will prove first that a map $\phi$ satisfying our assumptions induces a measurable map $\phi_x : W_x \to W_{\phi(x)}$. Here $W_x$ can be identified with $\mathbb{C}^{p-1}$ and $W_{\phi(x)}$ can be identified with $\mathcal{M}((n-m) \times m, \mathbb{C})$, both these identifications are non canonical but we fix them once and for all. We will then use the map $\phi_x$ to define a cocycle $\alpha : M_x \times (\partial \mathbb{H}^p_C)^x \to M_{\phi(x)}$ with respect to which $\phi$ is equivariant. We will then show that $\alpha$ is independent on the point $x$ and hence coincides almost everywhere with the desired homomorphism.

4.1. First properties of chain preserving maps. Recall from Section 2 that a map $\phi$ is Zariski dense if the essential Zariski closure of $\phi(\partial \mathbb{H}^p_C)$ is the whole $\mathcal{S}_{m,n}$, or, equivalently if the preimage under $\phi$ of any proper Zariski closed subset of $\mathcal{S}_{m,n}$ is not of full measure. Moreover, by definition, a measurable boundary map preserves the chain geometry if the image under $\phi$ of almost every pair of distinct points is a pair of transversal points, and the image of almost every maximal triple $(x_0, x_1, x_2)$ in $(\partial \mathbb{H}^p_C)^3$, is a maximal triple $(\phi(x_0), \phi(x_1), \phi(x_2))$, hence in particular it is contained in an $m$-chain.

We will denote by $\mathcal{T}_1$ the set of chains in $\partial \mathbb{H}^p_C$, and by $\mathcal{T}_m$ the set of $m$-chains of $\mathcal{S}_{m,n}$. The set $\mathcal{T}_1$ is a smooth manifold, indeed an open subset
of the Grassmannian $\text{Gr}_2(\mathbb{C}^{p+1})$, and we will endow $\mathcal{T}_1$ with its Lebesgue measure class.

The following lemma, an application in this context of Fubini’s theorem, gives the first property of a chain geometry preserving map:

**Lemma 4.2.** Let $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$ be a chain geometry preserving map. For almost every chain $C \in \mathcal{T}_1$ there exists an $m$-chain $\hat{\phi}(C) \in \mathcal{T}_m$ such that, for almost every point $x$ in $C$, $\phi(x) \in \hat{\phi}(C)$.

**Proof.** There is a bijection between the set $(\partial \mathbb{H}^p_\mathbb{C})^{(3)}$ consisting of triples of distinct points on a chain and the set $T_1^{(3)} = \{(C, x, y, z) | C \in \mathcal{T}_1, (x, y, z) \in C^{(3)} \}$. In turn the projection onto the first factor endows the manifold $T_1^{(3)}$ with the structure of a smooth bundle over $\mathcal{T}_1$. In particular Fubini’s theorem implies that, for almost every chain $C \in \mathcal{T}_1$ and for almost every triple $(x, y, z) \in C^{(3)}$, the triple $(\phi(x), \phi(y), \phi(z))$ belongs to the same $m$-chain $\hat{\phi}(C)$. Moreover $\hat{\phi}(C)$ has the desired properties again as a consequence of Fubini theorem. □

**Corollary 4.3.** Let $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$ be a measurable chain preserving map. Then there exists a measurable map $\hat{\phi} : \mathcal{T}_1 \to \mathcal{T}_m$ such that, for almost every pair $(x, T) \in \partial \mathbb{H}^p_\mathbb{C} \times \mathcal{T}_1$ with $x \in T$, then $\phi(x) \in \hat{\phi}(T)$.

**Proof.** The only thing that we have to check is that the map $\hat{\phi}$ is measurable, but this follows from the fact that the map associating to a pair $(x, y) \in S_{m,n}^{(2)}$ the $m$-chain $T_{xy}$ is algebraic. □

**Corollary 4.4.** For almost every $x \in \partial \mathbb{H}^p_\mathbb{C}$, almost every chain in $W_x$ satisfies Lemma 4.2. We will call a point $x$ that satisfies the hypotheses of Corollary 4.4 generic for the map $\phi$. Let us now fix a point $x$ that is generic for the map $\phi$ and...
consider the diagram

\[
\begin{array}{ccc}
\mathcal{H}_{1,p}(x) & \xrightarrow{\phi} & \mathcal{H}_{m,n}(\phi(x)) \\
\downarrow{\pi_x} & & \downarrow{\pi_{\phi(x)}} \\
W_x & \xrightarrow{\phi_x} & W_{\phi(x)}.
\end{array}
\]

**Lemma 4.5.** If \(x\) is generic for \(\phi\), there exists a measurable map \(\phi_{x}\) such that the diagram commutes almost everywhere. Moreover \(\phi_{x}\) induces a measurable map \(\hat{\phi}_{x}\) from the set of circles of \(W_{x}\) to the set of generalized circles of \(W_{\phi(x)}\) such that, for almost every chain \(T\), we have that \(\hat{\phi}(T)\) is a lift of \(\hat{\phi}_{x}(\pi_{x}(T))\).

**Proof.** The fact that a map \(\phi_{x}\) exists making the diagram commutative on a full measure set is a direct application of Corollary 4.4.

Since the set of horizontal chains in \(\partial \mathbb{H}^{p}_{C}\) is a smooth bundle over the set of Euclidean circles in \(W_{x} \cong \mathbb{C}^{p-1}\), we have that, for almost every Euclidean circle \(C\), the map \(\hat{\phi}\) is defined on almost every chain \(T\) with \(\pi_{x}(T) = C\). Moreover a Fubini-type argument implies that, for almost every circle \(C\), the diagram commutes when restricted to the preimage of \(C\).

This implies that the projections \(\hat{\phi}_{x}(C) := \pi_{\phi(x)}(\hat{\phi}(T))\) coincide for almost every lift \(T_{i}\) of \(C\) if \(C\) satisfies the hypotheses of the previous paragraph, and this concludes the proof.

\(\square\)

### 4.2. A measurable cocycle

Recall that if \(H,K\) are topological groups and \(Y\) is a Borel \(H\)-space, then a map \(\alpha : H \times Y \to K\) is a Borel cocycle if it is a measurable map such that, for every \(h_{1}, h_{2} \in H\) and for almost every \(y \in Y\), it holds \(\alpha(h_{1}h_{2}, y) = \alpha(h_{1}, h_{2} \cdot y)\alpha(h_{2}, y)\).

**Proposition 4.6.** Let \(\phi\) be a measurable, chain preserving map \(\phi : \partial \mathbb{H}^{p}_{C} \to S_{m,n}\). For almost every point \(x\) in \(\partial \mathbb{H}^{p}_{C}\) there exists a measurable cocycle \(\alpha : M_{x} \times (\partial \mathbb{H}^{p}_{C})^{x} \to M_{\phi(x)}\) such that \(\phi\) is \(\alpha\)-equivariant.

**Proof.** Let us fix a point \(x\) generic for the map \(\phi\). For almost every pair \((m, y)\) where \(m \in M_{x}\) and \(y \in \mathcal{H}_{1,p}(x)\), we have that the points \(\phi(y)\) and \(\phi(my)\) are on the same vertical chain in \(\mathcal{H}_{m,n}(\phi(x))\). In particular there exists an element \(\alpha(m, y) \in M_{\phi(x)}\) such that \(\alpha(m, y)\phi(y) = \phi(my)\). We extend \(\alpha\) by defining it to be 0 on pairs that do not satisfy this assumption. The function \(\alpha\) is measurable since \(\phi\) is measurable.

We now have to show that the map \(\alpha\) we just defined is actually a cocycle. In order to do this let us fix the set \(\mathcal{O}\) of points \(z\) for which \(\hat{\phi}(T_{xz})\) is \(m\)-vertical and \(\phi(z) \in \hat{\phi}(T_{xz})\). \(\mathcal{O}\) has full measure as a consequence of Lemma 4.2. Let us now fix two elements \(m_{1}, m_{2} \in M_{x}\). For every element \(z\) in the full measure set \(\mathcal{O} \cap m_{2}^{-1}\mathcal{O} \cap m_{1}m_{2}^{-1}\mathcal{O}\), the three points \(\phi(z), \phi(m_{2}z), \phi(m_{1}m_{2}z)\) belong to the same vertical \(m\)-chain, moreover, by definition of \(\alpha\), we have

\[
\begin{align*}
\alpha(m_{1}m_{2}, z)\phi(z) &= \phi(m_{1}(m_{2}z)) \\
&= \alpha(m_{1}, m_{2})\phi(m_{2}z) \\
&= \alpha(m_{1}, m_{2})\alpha(m_{2}, z)\phi(z).
\end{align*}
\]
The conclusion follows from the fact that the action of $M_{\phi(z)}$ on $H_{m,n}(\phi(x))$ is simply transitive.

**Proposition 4.7.** Let us fix a point $x$. Assume that there exists a measurable function $\beta : M_x \times W_x \to M_{\phi(x)}$ such that for every $m \in M_x$, for almost every $T$ in $W_x$ and for almost every $z$ in $T$, the equality $\alpha(m, z) = \beta(m, T)$ holds. Then the restriction of the boundary map $\phi$ to almost every chain through the point $x$ is rational.

**Proof.** We are assuming that for every $m < n$ and for almost every $z$ in $T$, the equality $\alpha(m, z) = \beta(m, T)$ holds. Fubini’s Theorem then implies that for every $T$ in a full measure subset $F$ of $W_x$, for almost every $m$ in $M_x$ and almost every $z$ in $T$ the equality $\alpha(m, z) = \beta(m, T)$ holds. In particular for every vertical chain $T$ in $F$ and almost every pair $(m_1, m_2)$ in $M_2^T$ we have $\beta(m_1, T)\beta(m_2, T) = \beta(m_1 m_2, T)$: it is in fact enough to choose $m_1$ and $m_2$ so that the equality of $\alpha(m_1, z)$ and $\beta(m_1, T)$ holds for almost every $z$ and compute the cocycle identity for $\alpha$ in a point $z$ that works both for $m_1$ and $m_2$.

It is classical that if $\pi : G \to J$ satisfies $\pi(xy) = \pi(x)\pi(y)$ for almost every pair $(x, y)$ in $G^2$, then $\pi$ coincides almost everywhere with an actual Borel homomorphism (cfr. [Zim84, Theorem B.2]). In particular for every $T$ in $F$, we can assume (up to modifying $\beta|_T$ on a zero measure subset) that the restriction of $\beta$ to $T$ is a measurable homomorphism $\beta_T : M_x \to M_{\phi(x)}$ and hence coincides almost everywhere with an algebraic map. Since the action of $M_x$ and $M_{\phi(x)}$ on each vertical chain is algebraic and simply transitive, we get that for almost every vertical chain $T$ the restriction of $\phi$ to $T$ is algebraic.

The fact that we let $\beta$ depend on the vertical chain $T$ might be surprising, and it is probably possible to prove that the cocycle $\alpha$ coincides almost everywhere with an homomorphism that doesn’t depend on the vertical chain $T$. However since it suffices to prove that the restriction of $\alpha$ to almost every vertical chain coincides almost everywhere with an homomorphism, and since this reduces the technicalities involved, we will restrict to this version.

The remainder of the section is devoted to prove, using the chain geometry of $\mathcal{S}_{m,n}$, that the hypothesis of Proposition 4.7 is satisfied whenever $\phi$ is a Zariski dense map and $m < n$. In the following proposition we deal with a preliminary easy case, in which the geometric picture behind the general proof should be clear.

**Proposition 4.8.** Let $\phi : \partial \mathbb{H}^p \to \mathcal{S}_{m,n}$ be a measurable, Zariski dense, chain geometry preserving map, and let $n \geq 2m$. Then the restriction of $\phi$ to almost every chain coincides almost everywhere with a rational map.

**Proof.** We want to apply Proposition 4.7 and show that the cocycle $\alpha : M_x \times \partial \mathbb{H}^p \to M_{\phi(x)}$ only depends on the vertical chain a point belongs to. We consider the set $F \subseteq W_x$ of chains $F$ such that
(1) \( \hat{\phi}(F) \) is an \( m \)-vertical chain, hence in particular \( \phi_x(F) \) is defined,

(2) for almost every circle \( C \) containing the point \( F \in W_x \), for almost every chain \( T \) lifting \( C \) the diagram of Lemma 4.5 commutes almost everywhere.

It follows from the proof of Lemma 4.5 that the set \( \mathcal{F} \) is of full measure, moreover, we get, applying Fubini, that if \( F \) is an element in \( \mathcal{F} \), for almost every point \( z \) in \( F \) and almost every chain \( T \) through \( z \) the diagram of Lemma 4.5 commutes almost everywhere when restricted to \( T \). In particular, using Fubini again, this implies that for almost every point \( w \) in \( \partial_{\text{top}} \mathcal{C} \), the diagram of Lemma 4.5 commutes almost everywhere when restricted to the chain \( T_{zw} \).

Let us now fix a chain \( F \in \mathcal{F} \) and denote by \( \mathcal{O} \) the full measure set of points in \( F \) for which that holds. For every element \( m \in M_s \) we also consider the full measure set \( \mathcal{O}_m = \mathcal{O} \cap m^{-1} \mathcal{O} \). We claim that given two elements \( z_1, z_2 \in \mathcal{O}_m \) the cocycle \( \alpha(m, z) \) has the same value \( \beta(m, F) \). In fact let us fix two points \( z_1, z_2 \) in \( \mathcal{O}_m \) and let us consider the set \( \mathcal{A}_{z_1, z_2, m} \subseteq \partial_{\text{top}} \mathcal{C} \) consisting of points \( w \) such that

1. \( \phi(w) \in \hat{\phi}(T_{zw_1}) \cap \hat{\phi}(T_{zw_2}) \)
2. \( \phi(mw) \in \hat{\phi}(T_{mwz_1}) \cap \hat{\phi}(T_{mwz_2}) \)
3. \( \dim(\phi(z_1), \phi(z_2), \phi(w)) = 3m. \)

We claim that the set \( \mathcal{A}_{z_1, z_2, m} \) is not empty. Indeed, by definition of \( \mathcal{O}_m \), the set of points \( w \) satisfying the first two assumption is of full measure. Moreover, since \( n \geq 2m \), the set \( C \) of points in \( \mathcal{S}_{m,n} \) such that \( \dim(\phi(z_1), \phi(z_2), \phi(w)) < 3m \) is a proper Zariski closed subset of \( \mathcal{S}_{m,n} \). Since the map \( \phi \) is Zariski dense, the preimage of \( C \) cannot have full measure, and this implies that \( \mathcal{A}_{z_1, z_2, m} \) has positive measure, in particular it contains at least one point. The third assumption on the point \( w \) implies that the \( m \)-chain containing \( \phi(w) \) and \( \phi(z_i) \) is horizontal for \( i = 1, 2 \).

Let us fix a point \( w \in \mathcal{A}_{z_1, z_2, m} \) and consider the \( m \)-chain \( \hat{\phi}(mT_{wz_1}) \) for \( i = 1, 2 \). The \( m \)-chain \( \hat{\phi}(mT_{wz_i}) \) is a lift of the \((m, 0)\)-circle \( C_i = \pi_{\phi(x)}(\hat{\phi}(T_{wz_i})) \) that contains both the points \( \phi(mz_i) \) and \( \phi(mw) \). In particular, since \( \alpha(m, z_i)\hat{\phi}(T_{wz_i}) \) is a lift of \( C_i \) containing \( z_i \) we get that \( \alpha(m, z_i)\hat{\phi}(T_{wz_i}) = \hat{\phi}(mT_{wz_i}) \). Similarly we get that \( \alpha(m, w)\hat{\phi}(T_{wz_i}) = \hat{\phi}(mT_{wz_i}) \). This gives that \( \alpha(m, z_i)^{-1}\alpha(m, w) \in E_{T_{wz_i}} \), but the latter group is the trivial group.
since we know that the chain $T_{wz_1}$ is 0-vertical. This implies that $\alpha(m, z_1) = \alpha(m, z_2)$. 

\section{Possible errors.} We know want to understand what information on the difference $\alpha(m, z_1) - \alpha(m, z_2)$ one can get by applying the same argument as in Proposition 4.8 with respect to a generic point $w$ in $\partial \mathbb{H}^p_\mathbb{C}$. The main idea is that even if the argument of Proposition 4.7 gives only partial information on the difference $\alpha(m, z_1) - \alpha(m, z_2)$, by running the same argument with many generic points it is possible to reconstruct the full information. As a consequence of Proposition 4.7 it is enough to be able to deduce that $\alpha(z_1, m) = \alpha(z_2, m)$ for pairs $(z_1, z_2)$ such that $(x, z_1, z_2)$ is maximal, and we will stick to this setting.

Let us now consider the maximal triple $(v_\infty, v_0, v_1) \in \mathcal{S}_{m,n}^3$ where, as before, $v_\infty = (e_1, \dots, e_m)$, $v_0 = (e_{n+1}, \dots, e_{n+m})$ and $v_1 = (e_j + ie_{n+j} | 1 \leq j \leq m)$. For every point $z$ in $\mathcal{S}_{m,n}$, we consider the integer $k(z)$ such that $\langle v_0, v_1, z \rangle = 3m - k(z)$. Both $m$-chains $T_{v_0, z}$ and $T_{v_1, z}$ are $k(z)$-vertical: in fact $\langle v_0, v_\infty, v_1, z \rangle = \langle v_0, v_1, z \rangle = \langle v_\infty, v_1, z \rangle = \langle v_\infty, v_0, z \rangle$.

In particular we can define a map

$$\beta : \mathcal{S}_{m,n} \to \text{Gr}(v_\infty)^2$$

$$z \mapsto (\langle v_0, z \rangle \cap v_\infty, \langle v_1, z \rangle \cap v_\infty).$$

Recall from Subsection 3.5 that we denote by $Z^k$ the subspace of $\mathbb{C}^m = v_\infty$ spanned by the first $k$ elements of the standard basis, and that $E_k$ denotes the subgroup of $u(m)$ consisting of matrices that can have nonzero values only in the first $k \times k$ block.

Let us now fix a point $z$ in $\mathcal{S}_{m,n}(v_\infty)$ and let $k$ be the integer $k(z)$. We pick elements $g_1, g_2$ in $\text{GL}(m)$ such that $g_i\beta_i(z) = (e_1, \dots, e_k)$ and we define a subgroup $E(z)$ of $u(m)$ by setting

$$E(z) = g_1^{-1}E_kg_1^{-*} + g_2^{-1}E_kg_2^{-*}.$$ 

\begin{lemma}
Let $x \in \partial \mathbb{H}_\mathbb{C}$ be a generic point and $m \in M_x$ be an element.
\begin{itemize}
\item For almost every chain $T \in W_x$ and almost every pair of points $z_1, z_2 \in T$, $z_i \in O_m$ and $(\phi(x), \phi(z_1), \phi(z_2))$ is maximal.
\item If we conjugate $\phi$ so that $\phi(x) = v_\infty$, $\phi(z_1) = v_0$ and $\phi(z_2) = v_1$ and we fix a point $w \in A_{z_1, z_2, m}$, we get that
\begin{equation}
\alpha(m, z_1) - \alpha(m, z_2) \in E(\phi(w)).
\end{equation}
\end{itemize}
\end{lemma}

\begin{proof}
Since $\phi$ sends almost every maximal triple to a maximal triple, it follows that, for almost every pair $(z_1, z_2)$ on a vertical chain, it satisfies $(\phi(x), \phi(z_1), \phi(z_2))$ is maximal. Since $\text{SU}(\mathbb{C}^{m+n}, T)$ is transitive on maximal triple, we can assume that $\phi(z_1) = v_0$ and $\phi(z_2) = v_1$.

Now, by definition of $w$, $\phi(w) \in \hat{\phi}(T_{w, z_1}) \cap \hat{\phi}(T_{w, z_2})$ and similarly $\phi(mw) \in \hat{\phi}(T_{mw, z_1}) \cap \hat{\phi}(T_{mw, z_2})$. Moreover $\hat{\phi}(T_{mw, z_1})$ and $\hat{\phi}(T_{w, z_1})$ have the same projection, hence by Proposition 3.18 we get that

$$\alpha(m, w) - \alpha(m, z_1) \in g_1^{-1}E_kg_1^{-*}.$$
In the same way one gets that
\[ \alpha(m, w) - \alpha(m, z_2) \in g_2^{-1}E_kg_2^{-*}. \]
And this concludes the proof. \(\square\)

Throughout the subsection we will fix the setting provided by Lemma 4.9: we fix \(m \in \mathbb{R}\) and we assume (up to conjugating the map \(\phi\)) that \(v_0, v_1\) belong to \(O_m\). The purpose of the rest of the section is to show that the intersection
\[ \bigcap_{w \in A} E(\phi(w)) \]
the trivial group. We will informally call errors the subgroups \(E(\phi(w))\), since they correspond to the possible defects of the cocycle \(\alpha\) to be an actual homomorphism. Our first goal is to prove a criterion to show that the intersection \(\bigcap_{i \in I} E(t_i) = \{0\}\) when \(t_i\) are points in \(S_{m,n}\) and \(I\) is a big enough finite set of indices.

In order to understand errors better, let us fix a pair of subspaces \((Z_1, Z_2)\) of \(\mathbb{C}^m\) and consider the subgroup of \(u(m)\) generated as a vector subspace of \(u(m)\) by the elementary matrices of the form \(z_1z_2^* - z_2z_1^*\) where \(z_i\) is a vector of \(Z_i\). We will denote by \(S(Z_1, Z_2)\) this subgroup:
\[ S(Z_1, Z_2) = \langle z_1z_2^* - z_2z_1^* | z_i \in Z_i \rangle < u(m). \]

**Lemma 4.10.** Let \(z\) be a point in \(S_{m,n}\). Then
\[ E(z) = S(\beta_1(z), \beta_1(z)) + S(\beta_2(z), \beta_2(z)) \]
where \(\beta_i\) denotes the \(i\)-th component of the map \(\beta\).

**Proof.** Clearly \(E_k\) coincides with \(S(Z^k, Z^k)\). Moreover, if \(g_i\) is an element such that \(g_i\beta_i(z) = Z^k\) then
\[ g_i^{-1}S(Z^k, Z^k)g_i^{-*} = S(g_i^{-1}Z^k, g_i^{-1}Z^k) = S(\beta_i(z), \beta_i(z)). \]
The assertion now follows from the definition of the subgroup \(E(z)\). \(\square\)

We will also need the following map

**Lemma 4.11.** The map
\[ \delta : \text{Gr}_k(v_\infty)^2 \rightarrow \text{Gr}(u(m)) \]
\[ (Z_1, Z_2) \mapsto S(Z_1^\perp, Z_2^\perp) \]
is algebraic.

**Proof.** This follows from the fact that both the map that associates to a vector space its orthogonal and the one that associates to a pair of vectors \((z_1, z_2)\) the antiHermitian matrix \(z_1z_2^* - z_2z_1^*\) are regular. \(\square\)

The standard Hermitian form on \(\mathbb{C}^m\) induces a positive definite Hermitian form \(H\) on \(u(m)\) that is defined by the formula
\[ H(A, B) = \text{tr}(AB^*). \]
In the next lemma we will introduce a subspace \( I(z) \) of \( \mathfrak{u}(m) \) that is easy to work with and has the property that is contained in the orthogonal, with respect to the Hermitian form \( H \), to the subgroup \( E(z) \) of \( \mathfrak{u}(m) \). The letter \( I \) stands for information, namely if two elements \( \alpha_1, \alpha_2 \) of \( \mathfrak{u}(m) \) differ by an element of \( E(z) \), then the orthogonal projections of \( \alpha_1 \) and \( \alpha_2 \) on \( I(z) \) are equal. In particular if \( w \) belongs to \( \mathcal{A}_{m,v_0,v_1} \), the group \( I(\phi(w)) \) measures some of the information on \( \alpha(m,z_1) \) we can get applying the argument of Proposition 4.8 to the point \( w \).

**Lemma 4.12.** The subgroup

\[
I(z) = \delta \circ \beta(z) = S(Z_1^\perp, Z_2^\perp)
\]

is contained in the orthogonal to \( E(z) \) in \( \mathfrak{u}(m) \) with respect to \( H \).

**Proof.** Recall that \( S(Z_1^\perp, Z_2^\perp) \) is generated by matrices of the form \( r = r_1r_2^* - r_2r_1^* \) where \( r_i \) belongs to the orthogonal to \( Z_i \). Similarly a generator of \( S(Z_1, Z_1) \) is of the form \( a = a_1a_2^* - a_2a_1^* \) for some \( a_i \) in \( Z_1 \). Explicit computations of the trace of the matrix \( r^*a \) give

\[
H(r,a) = \operatorname{tr}((r_1r_2^* - r_2r_1^*)(a_2a_1^* - a_1a_2^*)) =
-(r_2^*a_1)(a_2r_1) + (r_1^*a_1)(a_2r_2) + (r_2^*a_2)(a_1r_1) - (r_1^*a_2)(a_1r_2) = 0
\]

since \( r_1^*a_i = 0 \) for our choice of \( r_1 \). This proves that \( I(z) \) is orthogonal to \( S(Z_1, Z_1) \), the proof of the orthogonality to \( S(Z_2, Z_2) \) is analogue.

□

This allows us to prove the criterium we were looking for:

**Corollary 4.13.** Let \( t_i, i \in I \), be points in \( \mathcal{S}_{m,n} \). If \( \langle I(t_i) | i \in I \rangle = \mathfrak{u}(m) \), then \( \bigcap_{i \in I} E(t_i) = \{0\} \).

**Proof.** This is a consequence of Lemma 4.12: since \( I(t_i) \subseteq E(t_i)^\perp \), we have that \( \langle I(t_i) \rangle \subseteq (\bigcap E(t_i))^\perp \) and hence

\[
\bigcap_{i \in I} E(t_i) \subseteq \langle I(t_i) \rangle^\perp = \{0\}.
\]

□

The above lemma will give us an useful criterium to show that \( \bigcap_{i \in I} E(z_i) \) consists generically only the zero matrix if \( |I| > m^2 \). Here by generically we mean in an open set in the Zariski topology, namely that the set of tuples \( (z_1 \ldots z_s) \) such that \( \cap E(z_i) \) is not reduced to the zero matrix is a proper Zariski closed subset of \( (\mathcal{S}_{m,n})^s \) if \( s > m^2 \). The first step of the proof of this last result is the surjectivity of the map \( \beta \).
Surjectivity of $\beta$. Let us now consider the Zariski open subset $\mathcal{D}$ of $\mathcal{S}_{m,n}$ consisting of points that are transversal to $v_\infty$ and $v_0$ and $v_1$ and whose intersection with the subspace $\langle v_\infty, v_0 \rangle$ has minimal dimension. This means that the dimension of the intersection is 0 if $n > 2m$, and $k = 2m - n$ otherwise. In particular, to avoid trivialities, we will always assume $0 < k < m$, and hence $m < n < 2m$. Otherwise we are back in the case we already treated in Proposition 4.8. We also set $l = m - k = n - m$.

We will study the restriction to $\mathcal{D}$ of the rational map $\beta$

$$\beta: \mathcal{D} \subseteq \mathcal{S}_{m,n} \rightarrow \Gr_k(v_\infty)^2,$$

$$z \rightarrow (\langle v_0, z \rangle \cap v_\infty, \langle v_1, z \rangle \cap v_\infty).$$

The map $\beta$ factors via the map

$$\gamma: \mathcal{D} \rightarrow W_{v_0} \times W_{v_1},$$

$$z \rightarrow (\langle v_0, z \rangle, \langle v_1, z \rangle).$$

Lemma 4.14. Parametrizations of $W_{v_0}$ and $W_{v_1}$ are given by the maps

$$M(l \times m, \mathbb{C}) \rightarrow W_{v_0} \quad M(l \times m, \mathbb{C}) \rightarrow W_{v_1},$$

$$A_0 \rightarrow \begin{bmatrix} 0 \\ \Id_l \\ A_0^* \end{bmatrix}^\perp \quad A_1 \rightarrow \begin{bmatrix} A_1^* \\ \Id_l \\ iA_1^* \end{bmatrix}^\perp.$$

A point $A_i$ in $W_i$ correspond to a $k$-vertical chain if $\rk(A_i) = l$.

Proof. We will only carry out the details for $v_1$, the statement for $v_0$ is analogous and we already justified it in the proof of Lemma 3.10. Since $v_1 = \langle e_j + i e_{n+j} \rangle$, we get that $v_1^\perp$ consists of vectors $x \in \mathbb{C}^{n+m}$ such that $x_j = -ix_{n+j}$ for all $j = 1, \ldots, m$.

Let us fix a $2m$-dimensional subspace $V$ associated with an $m$-chain $T$ in $W_{v_1}$. Since $V^\perp$ is an $l$-dimensional subspace of $v_1^\perp$, it has a basis of the form $\begin{bmatrix} X_2 \\ 0 \\ iX_1 \end{bmatrix}$ where $X_1$ has dimension $l \times l$ and $X_2$ has dimension $m \times l$. The requirement that $V^\perp$ is transversal to $v_0$ implies that $X_1$ is invertible, hence we can assume that it equals the identity, up to changing basis. And this justifies the first assertion, since for every choice of a matrix $X_2$ we get an $m$-chain through $v_1$.

Since the intersection of $V$ with $v_\infty$ is the subspace of $v_\infty$ that is contained in $V$, it corresponds to the kernel of $X_2$ (with respect to the standard basis of $v_\infty$). In particular the requirement that $T$ is $k$-vertical restates as the requirement that $X_2$ has rank $l = m - k$. $\square$

Lemma 4.15. Under the parametrizations of Lemma 4.14, the image of $\gamma$ is the closed subset $\mathcal{C}$ of $M(l \times m, \mathbb{C}) \times M(l \times m, \mathbb{C})$ defined by the equations

$$\mathcal{C} = \{(A_0, A_1) | A_0A_1^*A_1A_0^* - A_1A_0^* - A_0A_1^* = 0\}.$$

Proof. Two $m$-chains $T_0, T_1$ intersect in $\mathcal{S}_{m,n}$ if and only if the intersection $V_0 \cap V_1$ of their underlying vector spaces $V_0, V_1$ contains a maximal isotropic
Lemma 4.14: has the following expression, with respect to the coordinates described in Proof.

Proposition 4.16. The map \( \pi \) has signature \((0, l)\). Indeed, since \( V_0^\perp \) has signature \((0, l)\) and is contained in \((V_0 \cap V_1)^\perp\), we get that the signature of any subspace of \((V_0 \cap V_1)^\perp\) is \((k_1, l + k_2)\) for some \(k_1, k_2\). On the other hand if \( V_0 \cap V_1 \) contains a maximal isotropic subspace \( z \), then \((V_0 \cap V_1)^\perp \subseteq z^\perp\) and the latter space has signature \((0, l)\). In particular the signature of \((V_0 \cap V_1)^\perp\) would be \((0, l)\), and clearly the orthogonal of a subspace of signature \((0, l)\) contains a maximal isotropic subspace.

Since \((V_0 \cap V_1)^\perp = (V_0^\perp, V_1^\perp)\), we are left to check that the requirement that signature of this latter subspace is \((0, l)\) is equivalent to the requirement that \( A_1A_0^* \) belongs to \( \text{Id} + U(l) \). If now we pick a pair \((A_0, A_1) \in M(l \times m, \mathbb{C}) \times M(l \times m, \mathbb{C})\) representing a pair of subspaces \((V_0, V_1) \in W_{v_0} \times W_{v_0}\) we have that the subspace \((V_0 \cap V_1)^\perp\) is spanned by the columns of the matrix:

\[
\begin{bmatrix}
0 & A_1^* \\
\text{Id} & \text{Id} \\
A_0 & i A_1^*
\end{bmatrix}
\]

It is easy to compute the restriction of \( h \) to the given generating system of \((V_0 \cap V_1)^\perp\):

\[
\begin{bmatrix}
0 & \text{Id} & A_0 \\
A_1 & \text{Id} & -i A_1
\end{bmatrix}
\begin{bmatrix}
A_0 & i A_1^* \\
\text{Id} & -\text{Id}
\end{bmatrix}
\begin{bmatrix}
0 & A_1^* \\
\text{Id} & \text{Id} \\
A_0 & i A_1^*
\end{bmatrix}
= \begin{bmatrix}
\text{Id} & \text{Id} \\
\text{Id} & -\text{Id}
\end{bmatrix}
\begin{bmatrix}
A_0 & \text{Id} \\
A_1 & \text{Id}
\end{bmatrix}
\]

The latter matrix is negative semidefinite and has rank \( l \) if and only if

\[(4.1)\quad A_0 A_1^* A_1 A_0^* - A_1 A_0^* A_0 A_1^* = (A_0 A_1^* - \text{Id})(A_1 A_0^* - \text{Id}) - \text{Id} = 0.
\]

In this case the restriction of \( h \) to \((V_0 \cap V_1)^\perp\) has signature \((0, l)\).

We now have all the ingredients we need to show that \( \beta \) is surjective.

Proposition 4.16. The map \( \beta \) is surjective.

Proof. If \( \zeta \) is a map such that \( \beta = \zeta \gamma \), then it is easy to check that the map \( \zeta \) has the following expression, with respect to the coordinates described in Lemma 4.14:

\[
\zeta : W_{v_0} \times W_{v_1} \to \text{Gr}_k(v_\infty)^2, \quad (A_0, A_1) \mapsto (\text{ker}(A_0), \text{ker}(A_1)).
\]

In order to conclude the proof it is enough to show that any pair of \( k \)-dimensional subspaces of \( v_\infty \) can be realized as the kernels of a pair of matrices satisfying Equation 4.1.

We first consider the case in which the subspaces \( V_0, V_1 \) intersect trivially, of course this can only happen if \( k \leq l \). In this case there exists an element \( g \in U(m) \) such that \( g V_0 = \bar{V}_0 = \langle e_1, \ldots, e_k \rangle \) and that \( \bar{V}_1 = g V_1 \) is spanned by
the columns of the matrix \( \begin{bmatrix} B & 0 \\ Id_k & 0 \end{bmatrix} \) where \( B \) is a matrix in \( M(k \times k, \mathbb{C}) \). Clearly \( V_i \) is the kernel of \( A_i \) if and only if \( \overline{V}_i \) is the kernel of \( \overline{A}_i = A_i g^{-1} \) and \( A_i g^{-1} \) satisfies the equation 4.1 if and only if \( A_i \) does. In particular it is enough to exhibit matrices \( \overline{A}_i \) whose kernel is \( \overline{V}_i \).

Let us first notice that, for any matrix \( B \in M(k \times k, \mathbb{R}) \) there exists a matrix \( X \in \text{GL}_k(\mathbb{C}) \) such that \( X B \) is a diagonal matrix \( D \) whose elements are only 0 or 1. Let us now consider the matrices

\[
\overline{A}_1 = \begin{bmatrix} 0 & 2\text{Id} & 0 \\ 0 & 0 & 2\text{Id} \end{bmatrix}_{k \times (k-l)}, \quad \overline{A}_2 = \begin{bmatrix} X & D & 0 \\ 0 & 0 & \text{Id} \end{bmatrix}_{k \times (k-l)}
\]

By construction \( \overline{V}_1 \) is the kernel of \( \overline{A}_1 \) and \( \overline{V}_2 \) is the kernel of \( \overline{A}_2 \), moreover we have that \( \overline{A}_1 \overline{A}_2 \) satisfies Equation 4.1:

\[
\overline{A}_1 \overline{A}_2 = \begin{bmatrix} 0 & 2\text{Id} & 0 \\ 0 & 0 & 2\text{Id} \end{bmatrix} \begin{bmatrix} X^* & 0 \\ D^* & 0 \\ 0 & \text{Id} \end{bmatrix} = \begin{bmatrix} 2D^* & 0 \\ 0 & 2\text{Id} \end{bmatrix} \in \text{Id} + U(m).
\]

This implies that there is a point \( z \in \mathcal{S}_{m,n} \) with \( \beta(z) = (\overline{V}_0, \overline{V}_1) \).

The general case, in which the intersection of \( V_i \) is not trivial, is analogous: we can assume, up to the \( U(m) \) action that \( V_0 \cap V_1 = \langle e_1, \ldots, e_s \rangle \) and we can restrict to the orthogonal to \( V_0 \cap V_1 \) with respect to the standard Hermitian form \( \lambda \).

An important corollary of the last proposition is the following

**Corollary 4.17.** For every proper subspace \( L \) of \( u(m) \) the set \( C(L) = \{ z \in \mathcal{S}_{m,n} \mid I(z) \subseteq L \} \) is a proper Zariski closed subset of \( \mathcal{D} \subseteq \mathcal{S}_{m,n} \).

**Proof.** The subspace \( C(L) \) is Zariski closed since the subspaces of \( u(m) \) that are contained in \( L \) form a Zariski closed subset of the Grassmanian \( \text{Gr}(u(m)) \) of the vector subspaces of \( u(m) \), moreover the subspace \( I(z) \) is obtained as the composition \( I(z) = \delta \circ \beta \) of two regular maps (cfr. Lemma 4.11).

In order to verify that \( C(L) \) is a proper subset, unless \( L = u(m) \), it is enough to verify that the subspaces of the form \( S(Z_1^+, Z_2^+) \) with \( Z_i \) transversal subspaces span the whole \( u(m) \). Once this is proven, the result follows from the the surjectivity of \( \beta \): since \( \beta \) is surjective, the preimage of a proper Zariski closed subset is a proper Zariski closed subset. In turn the fact that the span

\[
( z_1 z_2^* - z_2 z_1^* \mid z_1, z_2 \in \mathbb{C}^m \text{ linearly independent} )
\]

is the whole \( u(m) \) follows from the fact that every matrix of the form \( i z_1 z_1^* \) is in the span, since such a matrix can be obtained as the difference \( z_1 (z_2 - iz_1)^* - (z_2 - iz_1) z_1^* - (z_1 z_2^* - z_2 z_1^*) \). \( \square \)

4.4. **Proof of Theorem 4.1.** Let us now go back to the setting of Theorem 4.1: we fix a measurable, chain geometry preserving, Zariski dense map \( \phi : \partial \mathbb{H}^p_{+} \to \mathcal{S}_{m,n} \), a generic point \( x \in \partial \mathbb{H}^p_{+} \) such that for almost every chain \( t \in W_x \), for almost every point \( y \in t \), \( \phi(y) \in \phi(t) \). We want to show that the
measurable cocycle \( \alpha : \partial \mathbb{H}^p_C \setminus \{x\} \times \mathfrak{u}(1) \to \mathfrak{u}(m) \) coincides on almost every vertical chain with a measurable homomorphism.

As a consequence of Lemma 4.9, it is enough to show that, for almost every pair \( z_1, z_2 \) on a vertical chain, the intersection \( \bigcap_{w \in \mathcal{O}_{z_1,z_2,m}} E(\phi(w)) = \{0\} \). In fact this would imply that the restriction of \( \alpha \) to almost every chain essentially doesn’t depend on the choice of the point, hence coincides with a measurable homomorphism.

Let us now show that we can find \( m^2 \) points \( w_1, \ldots, w_{m^2} \) in \( \partial \mathbb{H}^p_C \) such that \( \bigcap E(\phi(w_i)) = \{0\} \). We work by induction: let us fix \( j \) points \( w_0, \ldots, w_j \). If the set

\[
L_j = \{ \langle I(\phi(w_i)) \rangle \mid i < j \}
\]

is equal to the whole \( \mathfrak{u}(m) \) we are done. Otherwise it follows from Corollary 4.17 that the subset \( C(L_j) \) of \( \mathcal{S}_{m,n} \) is a proper Zariski closed subset of \( \mathcal{D} \). Here, as above we denote by \( \mathcal{D} \) the Zariski open subset of \( \mathcal{S}_{m,n} \) consisting of points \( z \) that are transversal both to \( \phi(z_1) \) and to the linear subspace \( \langle \phi(z_1), \phi(z_2) \rangle \) and by \( C(L_j) \) the subset consisting of those points such that \( I(z) \subseteq L_j \).

In particular, since \( \phi \) is Zariski dense, its essential image cannot be contained in \( C(L_j) \cup (\mathcal{S}_{m,n} \setminus \mathcal{D}) \) that is a Zariski closed subset of \( \mathcal{S}_{m,n} \). This implies that we can find a point \( w_j \) in the full measure set \( A_{z_1,z_2,m} \) such that \( \delta \circ \beta(\phi(w)) \) is not contained in \( L_j \). In turn this implies that \( L_{j+1} = \{ \langle I(\phi(w_i)) \rangle \mid i \leq j \} \) strictly contains \( L_j \), hence has dimension strictly bigger. This shows the claim and completes the proof of Theorem 4.1.

5. The boundary map is rational

In this section we will show that a chain geometry preserving map \( \phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,n} \) whose restriction to almost every chain is rational, coincides almost everywhere with a rational map.

Assume that the chain geometry preserving map \( \phi \) is rational and let us fix a point \( x \). Since the projection \( \pi_{\phi(x)} : \mathcal{S}_{m,n}^{\phi(x)} \to W_{\phi(x)} \) is regular we get that the map \( \phi_x : W_x \to W_{\phi(x)} \) induced by \( \phi \) is rational as well. The first result of the section is that the converse holds, namely that if there exist two points \( x, y \) in \( \partial \mathbb{H}^p_C \) such that \( \phi_x \) and \( \phi_y \) are rational, then the original map \( \phi \) had to be rational as well.

Let us first consider the points \( v_0, v_\infty \in \mathcal{S}_{m,n} \) and define the Zariski open subset \( \mathcal{A}_{0,\infty} \) of \( \mathcal{S}_{m,n} \)

\[
\mathcal{A}_{0,\infty} = \{ v \in \mathcal{S}_{m,n} \mid v \pitchfork T_{v_0v_\infty}, v \pitchfork v_0, v \pitchfork v_\infty \}.
\]

More generally, if \( x \) and \( y \) in \( \mathcal{S}_{m,n} \) are transversal points, we will denote by \( \mathcal{A}_{x,y} \) the set of isotropic subspaces transversal to \( T_{xy} \), \( x \) and \( y \). Notice that \( \mathcal{A}_{0,\infty} \) corresponds to the set of isotropic subspaces \( v \) with the property that, if \( \begin{bmatrix} V_0 & V_1 \\ V_1 & V_2 \end{bmatrix} \) is a frame of \( v \), then the matrix \( V_1 \) has maximal rank, and both \( V_0 \) and \( V_2 \) are invertible: indeed \( v \) is transversal to \( T_{v_0v_\infty} = \)
\[ \langle e_1, \ldots, e_m, e_{n+1}, \ldots, e_{n+m} \rangle \text{ if and only if } V_1 \text{ has maximal rank, moreover, since } V \text{ is isotropic, if } a \in \ker V_0 \text{, then } a \in \ker V_1 \text{ as well and then } a \text{ is a vector in the intersection of } V \text{ with } v_0. \]

We now consider the parametrizations of \( W_{\pi_\infty} \) and \( W_{\pi_0} \) described in Lemma 3.7 and Lemma 4.14 respectively and consider the projection

\[
\pi_{\infty} \times \pi_0 : \mathcal{A}_{0,\infty} \rightarrow W_{\pi_\infty} \times W_{\pi_0}
\]

\[
\begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} \rightarrow (V_1 V_0^{-1}, V_1 V_2^{-1}).
\]

By definition of the set \( \mathcal{A}_{0,\infty} \) the projections have image the set \( M(m \times (n - m), \mathbb{C})^+ \) consisting of matrices that have maximal rank: indeed this is implied by the requirement that both matrices \( V_0 \) and \( V_2 \) are invertible and the matrix \( V_1 \) has maximal rank. Since \( \pi_{\infty}(v_\infty) \) is the zero matrix we will also denote the set \( M(m \times (n - m), \mathbb{C})^+ \) as \( W_{\pi_0}(v_\infty) \) namely it is the subset of points in \( W_{\pi_0} \) that are maximally transversal to \( \pi_{\infty}(v_0) \). With the same notation \( W_{\pi_\infty}(v_0) \) denotes the set of points in \( W_{\pi_\infty} \) corresponding to matrices of maximal rank in the chart we gave, in which \( \pi_{\infty}(v_0) = 0 \).

**Lemma 5.1.** The map \( \pi_{\infty} \times \pi_0 : \mathcal{A}_{0,\infty} \rightarrow W_{\pi_\infty}(v_0) \times W_{\pi_0}(v_\infty) \) gives a birational isomorphism of \( \mathcal{A}_{0,\infty} \) with the closed subset \( \mathcal{C}_{v_0,v_\infty} \) of \( W_{\pi_\infty}(v_0) \times W_{\pi_0}(v_\infty) \) consisting of pairs \((A,B)\) such that \( A \) has maximal rank, \( A = XB \) for some invertible matrix \( X \), and \( AB^*BA^* - AB^* - BA^* = 0 \).

**Proof.** A frame \([V_0 \quad V_1 \quad V_2]^T\) corresponds to an isotropic subspace of \((\mathbb{C}^{m+n}, \mathcal{N})\) if and only if \( V_0^*V_2 - V_1^*V_1 + V_2^*V_0 = 0 \). This implies that the pair \((V_1 V_0^{-1}, V_1 V_2^{-1})\) satisfies the equation \( AB^*BA^* - AB^* - BA^* = 0 \), indeed:

\[
\begin{align*}
V_1 V_2^{-1} V_0^{-1} V_1^* V_1 V_0^{-1} V_2^{-1} V_1^* - V_1 V_2^{-1} V_0^{-1} V_2^* V_1^* - V_1 V_0^{-1} V_2^{-1} V_2^* V_1^* &= \\
V_1 V_2^{-1} V_0^{-1} (V_1^* V_1 - V_2^* V_2) V_0^{-1} V_2^{-1} V_2^* V_1^* &= 0
\end{align*}
\]

Moreover it is clear that the the matrix \( A = V_1 V_0^{-1} \) satisfies \( A = BX \) for the invertible matrix \( X = V_0 V_2^{-1} \).

Vice versa given a pair \((A,B)\) satisfying these properties let us consider the subspace \( V \) spanned by the columns of \( \zeta(A,B) = \begin{bmatrix} X \\ A \\ 0 \end{bmatrix} \) where \( X \) is the matrix such that \( A = BX \). By construction \((\pi_{\infty} V, \pi_0 V) = (A,B)\). Moreover it is easy to verify that the subspace is isotropic.

The subset \( \mathcal{C}_{v_0,v_\infty} \) is a Zariski closed subset of \( W_{\pi_\infty}(v_0) \times W_{\pi_0}(v_\infty) \) since the requirement that \( A = BX \) for some invertible matrix \( X \) can be rephrased as the requirement that the rank of the matrix \([A \; B]\) is \( m \) if \( l \leq m \), and as the requirement that the rank of the matrix \([A \; B]\) is \( l \) if \( m \leq l \). Both these requirements determine a Zariski closed subset of \( W_{\pi_\infty}(v_0) \times W_{\pi_0}(v_\infty) \). Moreover the association \((A,B) \mapsto X\) is rational since it corresponds to the solution of a linear system that is a rational function in the data of the system. This implies that the map \( j \circ \zeta : \mathcal{C}_{v_\infty,v_0} \rightarrow \mathcal{A}_{0,\infty} \) is a regular map and is the inverse
of the projection. In particular the projection gives a birational isomorphism with its image.

An immediate consequence of this last lemma is the following:

**Lemma 5.2.** Let \( x, y \) be a pair of transversal points in \( S_{m,n} \). Then there is a closed subset \( C_{x,y} \) of the product \( W_x \times W_y \) and a birational isomorphism \( \beta_{x,y} : A_{x,y} \rightarrow C_{x,y} \subseteq W_x \times W_y \).

**Proof.** This follows from the fact that the action of \( SU(m,n) \) is transitive on pairs of transversal points and it is algebraic. \( \square \)

We now have all the ingredients we need to prove the following

**Proposition 5.3.** Let \( x, y \in \partial \mathbb{H}_C^p \) be generic in the sense of Lemma 4.2 and assume that \( \phi(x) \) and \( \phi(y) \) are transversal points of \( S_{m,n} \). If the maps \( \phi_x \) and \( \phi_y \) coincide almost everywhere with a rational map, the same is true for \( \phi \).

**Proof.** Let us consider the diagram

\[
\begin{array}{ccc}
A_{x,y} \subseteq \partial \mathbb{H}_C^p & \xrightarrow{\phi} & A_{\phi(x),\phi(y)} \subseteq S_{m,n} \\
\downarrow \beta_{x,y} & & \downarrow \beta_{\phi(x),\phi(y)} \\
C_{x,y} \subseteq W_x \times W_y & \xrightarrow{\phi_x \times \phi_y} & C_{\phi(x),\phi(y)} \subseteq W_{\phi(x)} \times W_{\phi(y)}
\end{array}
\]

A consequence of Lemma 4.5 and of the definition of the isomorphisms \( \beta_{x,y} \) defined in Lemma 5.2 is that the diagram commutes almost everywhere. In particular, since the isomorphisms \( \beta_{x,y} \) and \( \beta_{\phi(x),\phi(y)} \) are birational, we get that \( \phi \) coincides almost everywhere with a rational map. \( \square \)

Let us now fix a point \( x \) in \( \partial \mathbb{H}_C^p \), and identify the space \( W_x \) with \( \mathbb{C}^{p-1} \). We want to study the map \( \phi_x : \mathbb{C}^{p-1} \rightarrow W_{\phi(x)} \). It follows from Lemma 3.12 restricted to the case \( m = 1 \) that the projections of chains in \( \partial \mathbb{H}_C^p \) to \( \mathbb{C}^{p-1} \) are Euclidean circles \( C \subset \mathbb{C}^{p-1} \) (eventually collapsed to points).

**Lemma 5.4.** If \( x \) is generic in the sense of Lemma 4.2, the restriction of \( \phi_x \) to almost every Euclidean circle \( C \) of \( \mathbb{C}^{p-1} \) is rational.

**Proof.** Every Euclidean circle \( C \subset \mathbb{C}^{p-1} \) is a circle in our generalized definition, namely is the projection of some 1-chain of \( \partial \mathbb{H}_C^p \). Indeed we know from Lemma 3.12 that the Euclidean circle \( (1 + e^{it}, 0, \ldots, 0) \subset \mathbb{C}^{p-1} \) is the projection of the chain associated to the linear subspace \( \langle e_1 + e_2, e_{p+1} \rangle \) of \( \mathbb{C}^{p+1} \). Moreover the set of Euclidean circles is a homogeneous space under the group of Euclidean similarities of \( \mathbb{C}^{p-1} \) and the group \( Q = \text{stab}(v_\infty) \) acts on \( \mathbb{C}^{p-1} \) as the full group of Euclidean similarities.

It follows from the explicit parametrization of a chain given in Lemma 3.12 that, whenever a point \( t \) in \( \pi_{x}^{-1}(C) \) is fixed, the lift map \( l : C \rightarrow T \) is algebraic, where \( T \) is the unique lift of \( C \) containing \( t \).
In particular, if $T$ is a chain such that the restriction of $\phi$ to $T$ coincides almost everywhere with a rational map, the restriction of $\phi_x$ to $C = \pi_x(T)$ coincides almost everywhere with a rational map. We can now use a Fubini based argument to get that, for almost every circle $C$, the restriction of $\phi_x$ to $C$ is rational: for almost every chain $T$, the restriction to $T$ coincides almost everywhere with a rational map, and since the space of chains that do not contain $x$ is a full measure subset of the space of chains in $\partial \mathbb{H}_n^\mathbb{C}$ that forms a smooth bundle over the space of Euclidean circles in $\mathbb{C}^{p-1}$. □

An usual Fubini type argument implies now the following

**Corollary 5.5.** For almost every complex affine line $L \subseteq \mathbb{C}^m$, for almost every Euclidean circle $C$ contained in $L$, the restriction of $\phi_x$ to $C$ is algebraic. Moreover the same is true for almost every point $p$ in $L$ and almost every circle $C$ containing $p$.

In order to conclude the proof we will apply many times the following well known lemma that allows to deduce that a map is rational provided that the restriction to enough many subvarieties is rational. Given a map $\phi : A \times B \to C$ and given a point $a \in A$ we denote by $a\phi : B \to C$ the map $a\phi(b) = \phi(a, b)$ in the same way, if $b$ is a point in $B$, $b\phi$ will the note the map $b\phi(a) = \phi(a, b)$

**Lemma 5.6 ([Zim84, Theorem 3.4.4]).** Let $\phi : \mathbb{R}^{n+m} \to \mathbb{R}$ be a measurable function. Let us consider the splitting $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. Assume that for almost every $a \in \mathbb{R}^n$ the function $a\phi : \mathbb{R}^m \to \mathbb{R}$ coincides almost everywhere with a rational function and for almost every $b \in \mathbb{R}^m$ the function $b\phi : \mathbb{R}^n \to \mathbb{R}$ coincides almost everywhere with a rational function, then $\phi$ coincides almost everywhere with a rational function.

This easily gives that the restriction of $\phi_x$ to any complex affine line $L$ in $\mathbb{C}^{p-1}$ coincides almost everywhere with a rational map:

**Lemma 5.7.** For almost every affine complex line $L \subseteq \mathbb{C}^{p-1}$, the restriction $\phi_x|_L$ coincides almost everywhere with a rational map.

**Proof.** Let us fix a line $L$ satisfying the hypothesis of Corollary 5.5 and denote by $\phi_L : \mathbb{C} \to W_{\phi(x)}$ the restriction of $\phi_x$ to $L$, composed with a linear identification of $L$ with the complex plane $\mathbb{C}$. By the second assertion of Corollary 5.5, we can find a point $p \in \mathbb{C}$ such that for almost every Euclidean circle $C$ through $p$ the restriction of $\phi_L$ to $C$ coincides almost everywhere with a rational map. Let us now consider the birational map $i_p : \mathbb{C} \to \mathbb{C}$ defined by $i_p(z) = (z - p)^{-1}$, and let us denote by $\psi_L$ the composition $\psi_L = \phi_L \circ i_p^{-1}$. Since the image under $i_p$ of Euclidean circles through the point $p$ are precisely the affine real lines of $\mathbb{C}$ that do not contain $0$, we get that the restriction of $\psi_L$ to almost every affine line coincides almost everywhere with a rational map. In particular a consequence of Lemma 5.6 is that the map $\psi_L$ itself coincides almost everywhere with a rational map.
Since $\phi_C$ coincides almost everywhere with $\psi_C \circ i_p$ we get that the same is true for the map $\phi_C$ and this concludes the proof.

Applying Lemma 5.6 again we deduce the following proposition:

**Proposition 5.8.** Let $\phi : \partial \mathbb{H}^P_C \to S_{m,n}$ be a map with the property that for almost every chain $C$ the restriction of $\phi$ to $C$ coincides almost everywhere with a rational map. Then for almost every point $x \in \partial \mathbb{H}^P_C$ the map $\phi_x$ coincides almost everywhere with a rational map.

In turn this was the last missing ingredient to prove Theorem 1.6

6. Conclusion

The last step of Margulis’ original proof of superrigidity involves showing that if a Zariski dense representation $\rho : \Gamma \to H$ of a lattice $\Gamma$ in the algebraic group $G$ admits an algebraic boundary map, then it extends to a representation of $G$ (cfr. [Mar91] and [Zim84, Lemma 5.1.3]). The same argument applies here to deduce our main theorem:

**Proof of Theorem 1.1.** Let $\rho : \Gamma \to \text{PU}(m,n)$ be a Zariski dense maximal representation and let $\psi : \partial \mathbb{H}^P_C \to S_{m,n}$ be a measurable $\rho$ equivariant boundary map, that exists as a consequence of Proposition 2.2 (the difference between $\text{SU}(m,n)$ and $\text{PU}(m,n)$ plays no role here, since the action of $\text{SU}(m,n)$ on $S_{m,n}$ factors through the projection to the adjoint form of the latter group). The essential image of $\psi$ is a Zariski dense subset of $S_{m,n}$ as a consequence of Proposition 2.3, moreover Proposition 2.14 implies that $\psi$ preserves the chain geometry.

Since we proved that any measurable, Zariski dense, chain preserving boundary map $\psi$ coincides almost everywhere with a rational map (cfr. Theorem 1.6), we get that there exists a $\rho$-equivariant rational map $\phi : \partial \mathbb{H}^P_C \to S_{m,n}$. The $\rho$-equivariance follows from the fact that $\phi$ coincides almost everywhere with $\psi$ that is $\rho$ equivariant. In particular, for every $\gamma$ in $\Gamma$ the set on which $\phi(\gamma x) = \rho(\gamma) \phi(x)$ is a Zariski closed, full measure set, and hence is the whole $\partial \mathbb{H}^P_C$.

Since $\phi$ is $\rho$-equivariant and rational it is actually regular: indeed the set of regular points for $\phi$ is a non-empty, Zariski open, $\Gamma$-equivariant subset of $\partial \mathbb{H}^P_C$. Since, by Borel density [Zim84, Theorem 3.2.5], the lattice $\Gamma$ is Zariski dense in $\text{SU}(1,p)$ and $\partial \mathbb{H}^P_C$ is an homogeneous algebraic $\text{SU}(1,p)$ space, the only $\Gamma$-invariant proper Zariski closed subset of $\partial \mathbb{H}^P_C$ is the empty set, and this implies that the set of regular points of $\phi$ is the whole $\partial \mathbb{H}^P_C$.

In the sequel it will be useful to deal with complex algebraic groups and complex varieties in order to exploit algebraic results based on Nullstellsatz. This is easily achieved by considering the complexification. We will denote by $G$ the algebraic group $\text{SL}(p+1,\mathbb{C})$ and by $H$ the group $\text{PSL}(m+n,\mathbb{C})$ endowed with the appropriate real structures so that $\text{SU}(1,p) = G(\mathbb{R})$ and $\text{PU}(m,n) = H(\mathbb{R})$. Since $\partial \mathbb{H}^P_C$ and $S_{m,n}$ are homogeneous spaces that are projective varieties, there exist parabolic subgroups $P < \text{SL}(p+1,\mathbb{C})$ and
\[ Q < \text{PSL}(m+n, \mathbb{C}) \text{ such that } \partial \mathbb{H}^p_C = (G/P)(\mathbb{R}) = G(\mathbb{R})/(P \cap G(\mathbb{R})) \text{ and } S_{m,n} = (H/Q)(\mathbb{R}). \]

The algebraic \( \rho \)-equivariant map \( \phi : \partial \mathbb{H}^p_C \to S_{m,n} \) lifts to a map \( \bar{\phi} : G(\mathbb{R}) \to S_{m,n} \) and we can extend the latter map uniquely to an algebraic map \( T : G \to H/Q \) using the fact that \( G(\mathbb{R}) \) is Zariski dense in \( G \). The extended map \( T \) is \( \rho \)-equivariant since \( G(\mathbb{R}) \) is Zariski dense in \( G \); whenever an element \( \gamma \in \Gamma \) is fixed, the set \( \{ g \in G \mid T(\gamma g) = \rho(\gamma)T(g) \} \) is Zariski closed and contains \( G(\mathbb{R}) \).

Let us now focus on the graph of the representation \( \rho : \Gamma \to H \) as a subset \( \text{Gr}(\rho) \) of the group \( G \times H \). Since \( \rho \) is an homomorphism, \( \text{Gr}(\rho) \) is a subgroup of \( G \times H \), hence its Zariski closure \( \overline{\text{Gr}(\rho)}^Z \) is an algebraic subgroup.

The image under the first projection \( \pi_1 \) of \( \overline{\text{Gr}(\rho)}^Z \) is a closed subgroup of \( G \): indeed the image of a rational morphism (over an algebraically closed field) contains an open subset of its closure, since in our case \( \pi_1 \) is a group homomorphism, its image is an open subgroup that is hence also closed.

Moreover \( \pi_1(\overline{\text{Gr}(\rho)}^Z) \) contains \( \Gamma \) that is Zariski dense in \( G \) by Borel density, hence equals \( G \).

We now want to use the existence of the algebraic map \( T \) and the fact that \( \rho(\Gamma) \) is Zariski dense in \( H \) to show that \( \overline{\text{Gr}(\rho)}^Z \) is the graph of an homomorphism. In fact it is enough to show that \( \overline{\text{Gr}(\rho)}^Z \cap (\{e\} \times H) = (e, e) \).

Let \( (e, f) \) be an element in \( \overline{\text{Gr}(\rho)}^Z \cap (\{e\} \times H) \). Since \( H \) is absolutely simple being an adjoint form of a simple Lie group, and \( N = \bigcap_{h \in H} hQh^{-1} \) is a normal subgroup of \( H \), it is enough to show that \( f \in N \) or, equivalently, that \( f \) fixes pointwise \( H/Q \).

But, since \( T \) is a regular map, and the actions of \( G \) on itself and of \( H \) on \( H/Q \) are algebraic, we get that the stabilizer of the map \( T \) under the \( G \times H \)-action,

\[
\text{Stab}_{G \times H}(T) = \{(g, h) \mid ((g, h) \cdot T)(x) = h^{-1}T(gx) = T(x), \forall x \in G\},
\]

is a Zariski closed subgroup of \( G \times H \). Moreover \( \text{Stab}_{G \times H}(T) \) contains \( \text{Gr}(\rho) \) and hence also \( \overline{\text{Gr}(\rho)}^Z \). In particular \( (e, f) \) belongs to the stabilizer of \( T \), hence the element \( f \) of \( H \) fixes the image of \( T \) pointwise. Since the image of \( T \) is \( \rho(\Gamma) \)-invariant, \( \rho(\Gamma) \) is Zariski dense and the set of points in \( H/Q \) that are fixed by \( f \) is a closed subset, \( f \) acts trivially on \( H/Q \). \( \square \)

We can now prove Corollary 1.2 and Theorem 1.3:

\textbf{Proof of Corollary 1.2.} Let \( \Gamma \) be a lattice in \( \text{SU}(1,p) \). Let us assume, by contradiction, that there exists a Zariski dense maximal representation \( \rho : \Gamma \to \text{SU}(m,n) \). In particular, quotiating by the center of \( \text{SU}(m,n) \), we get a Zariski dense maximal representation \( \rho \) in \( \text{PU}(m,n) \). Let \( \hat{\rho} : \text{SU}(1,p) \to \text{PU}(m,n) \) be the extension of \( \rho \) to \( \text{SU}(1,p) \) and \( \hat{\phi} : \partial \mathbb{H}^p_C \to S_{m,n} \) be the \( \rho \)-equivariant regular boundary map. The map \( \hat{\phi} \) is indeed \( \hat{\rho} \)-equivariant since
the set of elements \( g \in \text{SU}(1,p) \) such that \( \hat{\phi}(gx) = \hat{\rho}(g)\hat{\phi}(x) \) is Zariski closed and contains the Zariski dense subgroup \( \Gamma \) of \( \text{SU}(1,p) \).

Proposition 2.14 implies that the class \( \hat{\rho}^\ast(\kappa^b_{\text{SU}(m,n)}) \) can be represented by the pullback \( \hat{\phi}^\ast \beta_S \); this implies that \( \hat{\rho} \) is tight since
\[
\|\kappa^b_{\text{SU}(m,n)}\| = \|T_b\hat{\rho}^\ast \kappa^b_{\text{SU}(m,n)}\| = \|T_b[\hat{\phi}^\ast \beta_S]\| \leq \|\hat{\phi}^\ast \beta_S\| \leq \|\hat{\rho}^\ast(\kappa^b_{\text{SU}(m,n)})\|.
\]

Here in the first equality we used the maximality of \( \rho \), and in the second we applied the results of Proposition 2.14 to the group \( \Gamma \). It follows from Hamlet’s work (cfr. Proposition 2.8) that there exists no Zariski dense tight representation of \( \text{SU}(1,p) \) in \( \text{PU}(m,n) \), since the standard representation is clearly not Zariski dense, and this gives the desired contradiction. \( \square \)

**Proof of Theorem 1.3.** Let now \( \rho : \Gamma \to \text{SU}(m,n) \) be a maximal representation and let \( L \) be the Zariski closure of \( \rho(\Gamma) \) in \( \text{SL}(m+n,\mathbb{C}) \). Here, as above, \( \text{SU}(m,n) = H(\mathbb{R}) \) with respect to a suitable real structure on \( H = \text{SL}(m+n,\mathbb{C}) \). Since the representation \( \rho \) is tight, we get, as a consequence of Theorem 2.6, that \( L(\mathbb{R}) \) almost splits a product \( L_{nc} \times L_c \) where \( L_{nc} \) is a semisimple Hermitian Lie group tightly embedded in \( \text{SU}(m,n) \) and \( K = L_c \) is a compact subgroup of \( \text{SU}(m,n) \).

Let us now consider \( L_1, \ldots, L_k \) the simple factors of \( L_{nc} \), namely \( L_{nc} \), being semisimple, almost splits as the product \( L_{nc} = L_1 \times \cdots \times L_k \) where \( L_k \) is simple Hermitian Lie groups. The first observation is that none of the groups \( L_i \) can be virtually isomorphic to \( \text{SU}(1,1) \). In that case the composition of the representation \( \rho \) with the projection \( L_{nc} \to L_i \) would be a maximal representation of a complex hyperbolic lattice with values in a group that is virtually isomorphic to \( \text{SU}(1,1) \) and this is ruled out by [BI08]: indeed Burger and Iozzi prove, as the last step in their proof of [BI08, Theorem 2], that there are no maximal representations of complex hyperbolic lattices in \( \text{PU}(1,1) \).

This implies that the inclusion \( i : L_{nc} \to \text{SU}(m,n) \) fulfills the hypotheses of Proposition 2.11 and in particular it is holomorphic. As a consequence of Hamlet’s classification of tight holomorphic homomorphisms between Hermitian Lie groups, it now follows that each factor of the product decomposition \( L_1 \times \cdots \times L_k \) is either of tube-type or is isomorphic to \( \text{SU}(m_i,n_i) \), with \( m_i < n_i \), and \( L_{nc} \) is contained in a subgroup of \( \text{SU}(m,n) \) of the form \( \text{SU}(k,k) \times \text{SU}(m-k,n-k) \).

Let us now fix a factor \( L_i \) which is not of tube-type. Since, by Corollary 1.2, there is no Zariski dense representation of \( \Gamma \) in \( \text{SU}(m_i,n_i) \) if \( 1 < m_i < n_i \), we get that \( m_i = 1 \). Moreover, since the only Zariski dense tight representation of \( \text{SU}(1,p) \) in \( \text{SU}(1,q) \) is the identity map, we get that \( n_i = p \) and the composition of \( \rho \) with the projection to \( L_i \) is conjugate to the inclusion. Assume now that more then one factor of \( L_{nc} \) is not of tube-type, then we would have that the diagonal embedding \( i : \text{SU}(1,p) \to \text{SU}(1,p)^r \) is Zariski dense and this is clearly a contradiction. \( \square \)
Proof of Corollary 1.4. We know that the Zariski closure of the representation $\rho$ is contained in a subgroup of $\text{SU}(m,n)$ isomorphic to $\text{SU}(1,p)^t \times \text{SU}(m-t,m-t) \times K$. The product $M = \text{SU}(1,p)^t \times \text{SU}(m-t,m-t)$ corresponds to a splitting $\mathbb{C}^{m,n} = V_1 \oplus \ldots \oplus V_t \oplus W \oplus Z$ where the restriction of $h$ to $V_i$ is non-degenerate and has signature $(1,p)$ and the restriction of $h$ to $W$ is non-degenerate and has signature $(m-t,m-t)$. The subspace $W$ is left invariant by $M$ hence also by $K$ (since $K$ commutes with $M$ and all the invariant subspaces for $M$ have different signature). In particular the linear representation of $\Gamma$ associated with $\rho$ leaves invariant a subspace on which $h$ has signature $(k,k)$ for some $k$ greater than 1 unless there are no factors of tube-type in the decomposition of $L$. This latter case corresponds to standard embeddings. 

Proof of Corollary 1.5. Let us denote by $\rho_0 : \Gamma \to \text{SU}(m,n)$ the standard representation. Since by Lemma 2.13 the generalized Toledo invariant is constant on components of the representation variety, we get that any other representation $\rho$ in the component of $\rho_0$ is maximal. By Theorem 1.3 this implies that $\overline{\rho(\Gamma)}^Z$ almost splits as a product $K \times L_t \times \text{SU}(1,p)$, and is contained in a subgroup of $\text{SU}(m,n)$ of the form $\text{SU}(1,p)^t \times \text{SU}(m-t,m-t) \times K$. If the group $L_t$ is trivial then $\rho$ is a standard embedding, and is hence conjugate to $\rho_0$ up to a character in the compact centralizer of the image of $\rho_0$. In particular this would imply that $\rho_0$ is locally rigid.

Let us then assume by contradiction that there are representations $\rho_i$ arbitrarily close to $\rho_0$ and with the property that the tube-type factor of the Zariski closure of $\rho_i(\Gamma)$ is non trivial. Up to modifying the representations $\rho_i$ we can assume that the compact factor $K$ in the Zariski closure of $\rho_i$ is trivial.

By Theorem 1.3 this implies that $\rho_i(\Gamma)$ is contained in a subgroup of $\text{SU}(m,n)$ isomorphic to $\text{SU}(m,pm-1)$, moreover we can assume, up to conjugate the representations $\rho_i$ in $\text{SU}(m,n)$, that the Zariski closure of $\rho_i$ is contained in the same subgroup $\text{SU}(m,pm-1)$ for every $i$. Since the representations whose image is contained in the subgroup $\text{SU}(m,pm-1)$ is a closed subspace of $\text{Hom}(\Gamma,G)/G$, we get that the image of $\rho_0$ is contained in $\text{SU}(m,pm-1)$ and this is a contradiction, since the image of the diagonal embedding doesn’t leave invariant any subspace on which the restriction of $h$ has signature $(m,pm-1)$.

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