Two Partial Orders for Littlewood-Richardson Tableaux

Justyna Kosakowska, Markus Schmidmeier and Hugh Thomas

Abstract: In this manuscript we show that two partial orders defined on the set of Littlewood-Richardson fillings of a given shape \((\alpha, \beta, \gamma)\) are equivalent if \(\beta \setminus \gamma\) is a horizontal and vertical strip. In fact, we give two proofs for the equivalence of the box order and the dominance order for fillings, both are algorithmic. The first of these proofs emphasizes links to the Bruhat order for the symmetric group and the second gives more straightforward construction of box-moves. This work is motivated by the known result that the equivalence of the two combinatorial orders leads to a description of the geometry of the representation space of invariant subspaces of nilpotent linear operators.

Key words: partial order, Littlewood-Richardson filling, Bruhat order

1. Introduction

Let \(\alpha, \beta, \gamma\) be partitions. By \(\mathcal{T}_\beta^{\alpha, \gamma}\) we denote the set of all Littlewood-Richardson fillings (LR-fillings) of the shape \((\alpha, \beta, \gamma)\). On the set \(\mathcal{T}_\beta^{\alpha, \gamma}\) there are defined two partial orders \(\leq_{\text{box}}\) and \(\leq_{\text{dom}}\) (see Section 2 for definitions). These orders have combinatorial nature and are of importance in the theory of invariant subspaces of nilpotent linear operators. They control the geometry (in particular: degenerations and boundary of irreducible components) of varieties of invariant subspaces of nilpotent linear operators, see [3, 4, 5]. Therefore, it is important to investigate properties of these orders. One of the main results of the paper is the following theorem.

**Theorem 1.1.** Let \(\Gamma, \tilde{\Gamma}\) be LR-fillings of the shape \((\alpha, \beta, \gamma)\). If \(\beta \setminus \gamma\) is a horizontal and vertical strip then the following conditions are equivalent.

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It is easy to see that the box relation implies the dominance relation (see Lemma 2.1). We show in [5] that several other relations of geometric or of algebraic nature lie between the box and the dominance relations. If those two are equal, then all the relations coincide.

We present two different proofs of Theorem 1.1. Both proofs are constructive. The first one, presented in Section 3, shows connections of our problem with the Bruhat order in the symmetric group $S_n$. The second proof, given in Section 4, gives more straightforward algorithm. Given LR-fillings such that $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$, this algorithm computes a sequence of box-moves that convert $\Gamma$ to $\tilde{\Gamma}$. This proves that $\Gamma \leq_{\text{box}} \tilde{\Gamma}$.

Moreover, in this case in Section 5 we describe some properties of the poset $\mathcal{T}_{\alpha,\gamma}^\beta \leq_{\text{box}} = (\mathcal{T}_{\tilde{\alpha},\tilde{\gamma}}^\beta \leq_{\text{dom}})$. We prove that in the poset $(\mathcal{T}_{\alpha,\gamma}^\beta \leq_{\text{box}})$ there exists exactly one minimal and exactly one maximal element, and that all saturated chains have the same length.

2. Definitions and notation

Following [2, 6] we recall definitions and notation connected with LR-fillings.

**Notation:** Recall that a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a finite non-increasing sequence of natural numbers; we picture $\alpha$ by its Young diagram which consists of $k$ rows of length given by the parts of $\alpha$. The transpose $\alpha'$ of $\alpha$ is given by the formula

$$\alpha'_j = \# \{i : \alpha_i \geq j\},$$

it is pictured by the transpose of the Young diagram for $\alpha$. Two partitions $\alpha$, $\tilde{\alpha}$ are in the dominance partial order, in symbols $\alpha \leq_{\text{dom}} \tilde{\alpha}$, if the inequality

$$\alpha_1 + \cdots + \alpha_j \leq \tilde{\alpha}_1 + \cdots + \tilde{\alpha}_j$$

holds for each $j$.

Fix two partitions $\gamma \subseteq \beta$ such that the Young diagram for $\gamma$ is contained in the Young diagram for $\beta$. The skew diagram $\beta \setminus \gamma$ is said to be a vertical strip if $\beta_i \leq \gamma_i + 1$ holds for all $i$, and a horizontal strip if $\beta' \setminus \gamma'$ is a vertical strip. A rook strip is a horizontal and vertical strip.
Given three partitions $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta$, $\gamma$, we will consider fillings of $\beta \setminus \gamma$ which have $\alpha_1$ entries 1’s, $\alpha_2$ entries 2’s, etc. We describe such a filling as having the content $\alpha$ and the shape $(\alpha, \beta, \gamma)$. A filling is said to be an LR-filling if the following three conditions are satisfied

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- (lattice permutation property) for each $u > 1$ and for each column $c$: on the right hand side of $c$, the number of entries $u - 1$ is at least the number of entries $u$.

Example: Let $\alpha = (2, 2, 1)$, $\beta = (4, 3, 3, 2, 1)$, $\gamma = (3, 2, 2, 1)$. We have to fill the skew diagram $\beta \setminus \gamma$ with two 1’s, two 2’s, and one 3. Due to the conditions on an LR-filling, this can be done in exactly three ways.

In this example, $\beta \setminus \gamma$ is a vertical but not a horizontal strip.

Notation: One can represent an LR-filling $\Gamma$ by a sequence of partitions

$$\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$$

where $\gamma^{(i)}$ denotes the region in the Young diagram $\beta$ which contains the entries $\Box$, $\blacksquare$, $\blacklozenge$, $\blacklozenge$. If $\Gamma$ has shape $(\alpha, \beta, \gamma)$, then $\gamma = \gamma^{(0)}$, $\beta = \gamma^{(s)}$, and $\alpha_i = |\gamma^{(i)} \setminus \gamma^{(i-1)}|$ for $i = 1, \ldots, s$.

In the example above, the first filling is given by the sequence of partitions $\Gamma = [(3, 2, 2, 1), (4, 2, 2, 2), (4, 3, 2, 2, 1), (4, 3, 3, 2, 1)]$.

We introduce two partial orders on the set $T_{\alpha, \gamma}^\beta$ of all LR-fillings of the same shape.

Definition: Two LR-fillings $\Gamma = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$, $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \ldots, \tilde{\gamma}^{(s)}]$ of the same shape are in dominance order, in symbols $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$, if for each $i$, the corresponding partitions $\gamma^{(i)}$, $\tilde{\gamma}^{(i)}$ are in the dominance partial order, i.e. $\gamma^{(i)} \leq_{\text{dom}} \tilde{\gamma}^{(i)}$. 
Definition: Suppose Γ, ˜Γ are LR-fillings of the same shape which we assume to be a horizontal strip or a vertical strip. We say ˜Γ is obtained from Γ by a box move if after two entries in Γ have been exchanged in such a way that the smaller entry is in the higher position in ˜Γ, we obtain ˜Γ by re-sorting the entries in each row and in each column if necessary. We denote by ≤_box the partial order generated by box moves.

Here is an example:

Γ :

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 1 & 2 \\
\end{array}
\]

< _box

Γ :

\[
\begin{array}{ccc}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 1 & 2 \\
\end{array}
\]

To get ˜Γ we apply to Γ the following sequence of moves. First we exchange 3 and 1 in rows 3 and 4, then 2 and 1 in rows 2 and 3 and finally 3 and 2 in rows 4 and 5.

Note that here the short sequence of box moves (given by exchanging 3 and 1 in rows 4 and 2, and 2 and 3 in rows 5 and 3) is not admissible due to the lattice permutation property.

In the next example in Γ we exchange 3 and 1 in rows 3 and 4 and sort entries in the third column.

Γ :

\[
\begin{array}{ccc}
2 & 1 & 3 \\
4 & 1 & 2 \\
3 & 1 & 2 \\
\end{array}
\]

< _box

Γ :

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & 1 & 2 \\
3 & 1 & 2 \\
\end{array}
\]

We finish this section by establishing the following fact.

Lemma 2.1. For LR-fillings of the same shape, the ≤_box-order implies the ≤_dom-order.

Proof. Suppose that the LR-filling ˜Γ = [ ˜γ(0), . . . , ˜γ(s) ] is obtained from Γ = [ γ(0), . . . , γ(s) ] by a box move based on entries i and j with, say, i < j. The process of reordering the entries in each row or column will not affect entries less than i or larger than j, so the partitions γ(0), . . . , γ(j−1), and γ(j), . . . , γ(s) remain unchanged. The partitions γ(ℓ), ˜γ(ℓ) for i ≤ ℓ < j are different and satisfy γ(ℓ) < _dom ˜γ(ℓ) (since the defining partial sums can only increase). This shows that Γ < _dom ˜Γ.

□
3. The Bruhat order and the first proof of the main result

Notation: Let \( \alpha = (\alpha_1 \geq \cdots \geq \alpha_k) \) be a partition of \( n \). Let \( \beta \setminus \gamma \) be a skew diagram with \( n \) boxes, that is a rook strip. Given a filling \( X \) of the content \( \alpha \), we write \( \pi(X) = (x_1, \ldots, x_n) \) for a permutation in \( S_n \) as follows. We read the filling from top right to bottom left, and we read the 1’s as (in order) 1 up to \( \alpha_1 \), we read the 2’s as \( \alpha_1 + 1 \) to \( \alpha_1 + \alpha_2 \), and so forth.

Example: Let \( \beta = (5, 4, 3, 2, 1) \), \( \gamma = (4, 3, 2, 1) \) and \( \alpha = (2, 2, 1) \). Consider the following fillings of \( \beta \setminus \gamma \).

\[
X: \begin{array}{cccc}
1 & 2 & 3 & 1 \\
2 & & & \\
& 3 & & \\
& & 2 & \\
& & & 1
\end{array}, \quad \tilde{X}: \begin{array}{cccc}
1 & 2 & 3 & 1 \\
& & & \\
& 1 & & \\
& & 2 & \\
& & & 3
\end{array}
\]

Note that \( \pi(X) = (1, 3, 5, 2, 4) \) and \( \pi(\tilde{X}) = (1, 2, 3, 4, 5) \).

For any filling \( X \) of the content \( \alpha \), the permutation \( \pi(X) \) will satisfy that the numbers 1, 2, \ldots, \( \alpha_1 \) are in increasing order, the numbers \( \alpha_1 + 1 \), \ldots, \( \alpha_1 + \alpha_2 \) are in increasing order, and so forth. We denote such permutations by \( S^\alpha \). The map \( \pi \) is a bijection between fillings of given \( \beta \setminus \gamma \) of the content \( \alpha \) and \( S^\alpha \).

**Proposition 3.1.** Let \( X, Y \) be fillings of \( \beta \setminus \gamma \) of the content \( \alpha \). The relation \( Y <_{\text{dom}} X \) holds if and only if \( \pi(X) < \pi(Y) \) in the Bruhat order.

**Proof.** Note that \( x \preceq y \) in the Bruhat order in \( S_n \) if and only if for all \( i \), the increasing rearrangement of the first \( i \) letters of \( x \) has each letter less than or equal the corresponding rearrangement of the first \( i \) letters of \( y \). (This is \([11 \text{ Theorem 2.6.3}]\).) This condition is equivalent to the following. For each \( i \) and each \( j \), the number of elements less than or equal to \( j \) in the first \( i \) letters of \( x \) is greater than or equal to the corresponding number for \( y \).

Let \( x = \pi(X) \), \( y = \pi(Y) \) and suppose \( x \preceq y \) in the Bruhat order. Let \( i, r \in \{1, \ldots, k\} \). If we consider \( j = \alpha_1 + \cdots + \alpha_r \), we see that \( x \preceq y \) implies that the number of boxes labeled \( 1, \ldots, r \) in the first \( i \) boxes of \( X \) is greater than or equal to the corresponding number in \( Y \). This implies that \( Y \preceq_{\text{dom}} X \).

On the other hand, if \( Y \preceq_{\text{dom}} X \), then by the \( i \)-th box of \( X \), there have been at least as many boxes with entry 1 than in \( Y \), at least as many boxes with entry 1 or 2 in \( X \) than in \( Y \), and so on. This implies that the increasing
rearrangement of the first $i$ entries of $x$ has each letter less than or equal the corresponding entry of the increasing rearrangement of $y$. Finally $x \leq y$ in the Bruhat order. □

Let $L \subseteq S_n$ be the set of permutations which correspond to LR-fillings of the content $\alpha$.

Now we consider the “box move”. We recall that a decreasing box move on $X$ swaps two entries of $X$, so that the larger entry moves earlier, and the smaller entry moves later. We have already established, in Lemma 2.1, that applying a decreasing box move, moves us down in dominance order. We wish to prove the converse result, that if $X > Y$ in dominance order, then there is a sequence of decreasing box moves which take us from $X$ to $Y$, and such that at each intermediate step, the filling stays lattice.

_Notation:_ For any $i = 1, \ldots, n-1$, denote by $s_i = (i, i+1) \in S_n$ the adjacent transposition. Let $x \in S_n$ and let $x = s_{i_1} s_{i_2} \ldots s_{i_k}$ with $k$ minimal. The number $k$ is called the _length_ of $x$ and we denote it by $\ell(x) = k$.

We now state several elementary lemmas.

**Lemma 3.2.** $x \in S^\alpha$ if and only if $\ell(s_i x) > \ell(x)$ for all $i \not\in \{\alpha_1, \alpha_2 + \alpha_1, \ldots\}$.

_Proof._ Note that, $x = (x_1, \ldots, x_n) \in S^\alpha$ if and only if for all $i \not\in \{\alpha_1, \alpha_1 + \alpha_2, \ldots\}$, we have that $i$ appears to the left of $i+1$. The effect of left-multiplying $x$ by $s_i$ is to swap the positions of $i$ and $i+1$. This increases the length if and only if $i$ started to the left of $i + 1$. □

Thanks to the previous lemma, we recognize $S^\alpha$ as a parabolic quotient. It consists of the minimal-length coset representatives for $S_\alpha \backslash W$, where $S_\alpha$ permutes separately the entries $1, \ldots, \alpha_1$, the entries $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$, and so on.

For $\pi \in S_n$, define the inversions of $\pi$ to be pairs $a < b$ such that $\pi^{-1}(a) > \pi^{-1}(b)$.

**Lemma 3.3.** The _inversions_ of $\pi$ are the transpositions $t$ such that $\ell(t \pi) < \ell(\pi)$.

_Proof._ Left-multiplying by $t = (a \ b)$ has the effect of swapping $a$ and $b$ in the one-line notation for $\pi$. This decreases the length if and only if the larger number comes before the smaller, i.e., $\pi^{-1}(a) > \pi^{-1}(b)$. □
Notation: For \( x \in S_n \), define the \( \alpha \)-recording tableau of \( x \) to be the filling with entries \( x^{-1}(1), \ldots, x^{-1}(\alpha_1) \) in the first row, \( x^{-1}(\alpha_1 + 1), \ldots, x^{-1}(\alpha_1 + \alpha_2) \) in the second row, etc. A tableau of size \( n \) is said to be standard if it contains each of the numbers from 1 to \( n \) exactly once, and its entries increase along rows and down columns.

Example: Let \( x = (1, 3, 5, 2, 4) \) and \( \alpha = (2, 2, 1) \). The \( \alpha \)-recording tableau of \( x \) is the following.

\[
\begin{array}{ccc}
1 & 4 & \textbf{2} \\
2 & 5 & 3
\end{array}
\]

Lemma 3.4. For \( x \in S_n \), we have that \( x \in L \) if and only if the \( \alpha \)-recording tableau for \( x \) is standard.

Proof. The row-increasing condition for standardness is equivalent to \( x \in S^\alpha \). Suppose this is satisfied, and let \( X \) be the corresponding tableau. The number of entries \( \leq t \) in row \( i \) of the recording tableau is equal to the number of boxes in \( X \) labeled \( i \) we have seen in reading the first \( t \) entries of \( X \). Denote this number by \( \lambda_i^{(t)} \). The lattice condition is then equivalent to the condition that for each \( t \), the resulting shape \( (\lambda_1^{(t)}, \lambda_2^{(t)}, \ldots) \) is a partition, and this condition is easily seen to be equivalent to column-increasingness. \( \square \)

We can also think of a standard \( \alpha \)-recording tableau as the record of a process of adding a sequence of single boxes, resulting in the final shape \( \alpha \), such that at each step, the shape is a partition.

Lemma 3.5. If \( x \lessdot y \) is a cover in Bruhat order, such that \( x \) and \( y \) correspond to lattice fillings \( X \) and \( Y \) respectively, then there is a decreasing box move from \( X \) to \( Y \).

Proof. A cover in Bruhat order swaps two entries, and moves the larger number up. \( \square \)

By Lemma 3.5, our desired result follows if we show that, if \( x < y \) in Bruhat order, then there is a sequence of Bruhat covers from \( x \) to \( y \) which remains in the set \( L \). It is therefore enough to establish the following proposition.

Proposition 3.6. If \( x < y \) in Bruhat order, with \( x, y \in L \), then there is a cover \( x \lessdot z \) with \( z \in L \) such that \( z \leq y \).
Proof. Write an expression for \( y \) as a product of adjacent transpositions \( s_{i_1} \ldots s_{i_p} \). Since \( x < y \), there is a subword of this word which equals \( x \). As in the proof of \([1, \text{Lemma 2.2.1}]\), choose one such that the leftmost omitted reflection is as far to the right as possible (i.e. \( x = s_{i_1} \ldots \hat{s}_{i_{j_1}} \ldots s_{i_{j_k}} \ldots s_{i_p} \) with \( j_1 < \ldots < j_k \) such that \( j_1 \) maximal, and \( \hat{s} \) means that \( s \) is omitted). Define \( z \) to obtained from this subword by adding back in the leftmost simple reflection in the word for \( y \) not in the chosen subword for \( x \) (i.e. \( z = s_{i_1} \ldots \hat{s}_{j_2} \ldots \hat{s}_{j_k} \ldots s_{i_p} \)).

By the proof of \([1, \text{Lemma 2.2.1}]\), this is a reduced expression for a permutation which covers \( x \) in Bruhat order and lies below \( y \). Note that our conventions differ from those of \([1, \text{Lemma 2.2.1}]\) by reversing left and right.

The fact that \( z \in S_\alpha \) follows from the proof of \([1, \text{Theorem 2.5.5}]\), because we have identified \( S_\alpha \) as a certain parabolic quotient. (Note that in \([1]\), they generally take parabolic quotients with the subgroup acting on the right. Our reversal mentioned above is consistent with the fact that we take parabolic quotients with the subgroup acting on the left.) The proof of \([1, \text{Theorem 2.5.5}]\) uses the same construction as the previously-cited \([1, \text{Lemma 2.2.1}]\), it just proves an additional property under a stronger hypothesis.

We now verify that \( z \in L \). We can write \( z = xt \) for \( t \) a transposition. Suppose \( t = (a \ b) \) with \( a < b \). This has the effect that \( x \) and \( z \) take different values restricted to \( \{a, b\} \). Thus, \( x^{-1} \) and \( z^{-1} \) differ when restricted to \( \{x(a), x(b)\} \), where the values of \( x^{-1} \) and \( z^{-1} \) are \( a, b \). This means that the tableaux for \( x \) and \( z \) differ in that \( a \) and \( b \) swap places. The fact that \( \ell(x) < \ell(z) \) means that, as we read the tableau in usual order (left to right and top to bottom), we will encounter first \( a \), then \( b \) in \( x \), and first \( b \) then \( a \) in \( z \).

We refer to the box of the \( \alpha \)-recording tableau for \( x \) which contains the entry \( a \) as \( A \), and the box which contains the entry \( b \) as \( B \). We fix this nomenclature, so that even if we refer to a different tableau of shape \( \alpha \), the boxes \( A \) and \( B \) are the same (while now perhaps containing different values). We can thus say that in the tableau for \( z \), \( A \) contains the value \( b \), and \( B \) contains the value \( a \). By what we have already shown, \( B \) is in a row weakly below the row of \( A \).

We think of the \( \alpha \)-recording tableau of \( x \) as determining a sequence of boxes which we add, starting at the empty shape, and eventually reaching the shape \( \alpha \). We add the boxes in numerical order (according to the entry they are filled with). As we have pointed out already, the fact that the boxes can be added in this order such that at each step, the shape is a partition,
captures the condition of corresponding to a standard tableau, and thus of coming from an element of $L$, see proof of Lemma 3.4.

So, to establish the lattice condition for $z$, we must establish that if we modify the order of adding boxes determined by $x$, so that we add $B$ when we would have added $A$, and vice versa, each of the intermediate shapes will still be a partition.

Suppose the tableau for $x$ looks as below, where we have specified only the entries equal to $a$ and $b$. As we have already established, the tableau for $z$ is the same except that $a$ and $b$ change places.

```
+   
+   
+   
+   
   a
   b
```

Since the tableau corresponding to $x$ is standard, the entries in boxes marked with a $+$ must be greater than $a$, and the entries in boxes marked with a $-$ must be less than $b$. There are two ways that the tableau corresponding to $z$ could fail to be standard: if one of the entries in a $+$ box were less than or equal to $b$, or if one of the entries in a $-$ box were greater than or equal to $a$.

The fact that $\ell(z) = \ell(x) + 1$ implies that the entries between $a$ and $b$ in one-line notation for $x$ are all either greater than $b$ or less than $a$ [Lemma 2.1.4]. This means that between adding $A$ and $B$ in the sequence of partitions for $x$, we will only add boxes which are either (i) in a higher row than $A$, (ii) in the same row as, but to the left of $A$, (iii) in a row below $B$, or (iv) to the right of $B$ in the same row. Of these, (ii) is impossible because we are at the point where we can add $A$, so all the boxes to the left of it in the same row must already have been added. Also, (iv) is impossible because, in adding the boxes determined by $x$, we cannot add any boxes to the right of $B$ until we have added $B$. Thus, the boxes which will be added between adding $A$ and adding $B$ are all in rows of the $\alpha$-recording tableau either above the row of $A$ or below the row of $B$. This means that, as we construct the $x$ tableau, the boxes marked $+$ and $-$ will not be touched between when we add $A$ and when we add $B$. Since the boxes marked $+$ were empty when we added $A$, they are still empty when we added $B$, and thus they contain entries at least equal to $b$. Since the boxes marked $-$ are filled when we add $B$, they must have been filled when we added $A$, so they are filled with entries less than or equal to $a$.

Two bad possibilities remain: that the boxes $A$ and $B$ are actually ad-
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The fact that $z$ is in $S^n$, which we already checked, is equivalent to the fact that each row of the tableau corresponding to $z$ is in increasing order. Thus, $A$ and $B$ cannot be in the same row, since $a$ and $b$ would occur in the correct order in the tableau for $x$, and therefore out of order in the tableau for $z$.

We now rule out the possibility that $A$ and $B$ are adjacent in the same column. We can write $x = (x(a) \, x(b))z$. The effect of left-multiplying $z$ by $(x(a) \, x(b))$ can be described as removing the simple reflection which we added to obtain $z$ from $x$. Since, by the construction of $z$, we know that the words for $y$ and $z$ agree up to that simple reflection, $(x(a) \, x(b))y$ can also be obtained from $y$ by removing the same simple reflection. The tableau corresponding to $(x(a) \, x(b))y$ can be obtained from that of $y$ by swapping the entries in $A$ and $B$. The fact that this decreases the length of $y$ means that $A$ contains the larger value and $B$ contains the lower value in $y$. Since, in the case that we are considering, $A$ and $B$ are vertically adjacent, this contradicts our assumption that $y \in L$, which would imply that the tableau for $y$ is standard.

This shows that $z$ is lattice, as desired, which completes the proof. □

Remark: This proof for the implication $X <_{\text{dom}} \tilde{X} \implies X <_{\text{box}} \tilde{X}$ is constructive as it exhibits the first box move: Let $s_1 \cdots s_k$ be a reduced expression for $\pi(X)$. Write $\pi(\tilde{X})$ as a subword $s_{i_1} \cdots s_{i_{j_1}} \cdots s_{i_j} \cdots s_{i_k}$ such that $j_1$ is maximal. Then $s_{i_1} \cdots s_{i_{j_1-1}}s_{i_{j_1}}s_{i_{j_1-1}} \cdots s_{i_k} = (a, b)$ is a transposition and $(a, b) \pi(\tilde{X}) = \pi(\hat{X})$ defines an LR-filling $\hat{X}$ which satisfies $X \leq_{\text{dom}} \hat{X}$ and $\hat{X} <_{\text{box}} \tilde{X}$.

Example: Consider

\[ X : \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 1 & \end{array}, \quad \tilde{X} : \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 1 & \end{array} \]

We have $\pi(X) = 13524$, and $\pi(\tilde{X}) = 12345$. Let’s say we want to move down from $\tilde{X}$. We write $\pi(X) = (45)(23)(34) = s_4s_2s_3$. $\pi(\tilde{X}) = e$, so the subword of $s_4s_2s_3$ corresponding to $e$ is the empty subword. We add the leftmost reflection back in, so that is $(a, b) = s_4$. This gives us the permutation 12354, which is obtained from the filling $\hat{X}$ (reading down from the top.
right) 11232, which is indeed lattice. To get the next step up the chain, we go to $z' = s_4 s_2 = 13254$, which comes from the filling $\hat{X}$ with $\pi(\hat{X}) = 12132$, which is again lattice. Here we apply the transposition $(a, b) = s_4 s_2 s_4 = s_2$ to $\pi(\hat{X})$. In the final step we obtain $X$ by applying the transposition $(a, b) = s_4 s_2 s_3 s_2 s_4 = (2, 5)$ to $\pi(\hat{X})$.

4. The second proof of the main result and an algorithmic approach

Let $\Gamma, \tilde{\Gamma}$ be LR-fillings of shape $(\alpha, \beta, \gamma)$, where $\beta \setminus \gamma$ is a rook strip. By Theorem 1.1, $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$ implies $\Gamma \leq_{\text{box}} \tilde{\Gamma}$. However, the proof presented in Section 3 does not give an explicit procedure to find box-moves that transform $\Gamma$ into $\tilde{\Gamma}$. In this section we provide an algorithm that gives box-moves. In this section we assume that $\beta \setminus \gamma$ is a rook strip.

**Notation:** For an LR-filling $\Gamma$ we denote by $\omega(\Gamma)$ the list of entries when read from left to right, and each column from the bottom up. Clearly, $\Gamma$ is determined uniquely by its shape and by the list of entries. By $\omega(\Gamma)_{\leq c}$ we denote the initial segment of $\omega(\Gamma)$ consisting of the entries in the first $c$ columns.

For a filling $\Gamma$ of shape $(\alpha, \beta, \gamma)$, we denote by $e_c = \gamma_1 + \cdots + \gamma_c$ the number of empty boxes in the first $c$ columns, and by $\# \omega(\Gamma)_{\leq u} \leq c$ the number of entries at most $u$ in $\omega(\Gamma)_{\leq c}$.

**Lemma 4.1.** Suppose $\Gamma, \tilde{\Gamma}$ are LR-fillings of the same shape such that $\Gamma <_{\text{dom}} \tilde{\Gamma}$ holds. Then $\omega(\Gamma) < \omega(\tilde{\Gamma})$ in the lexicographical order.

**Proof.** Let $k \in \mathbb{N}$ be such that for $1 \leq i < k$ the entries $\omega(\Gamma)_i = \omega(\tilde{\Gamma})_i$ are equal, and $x = \omega(\Gamma)_k \neq \omega(\tilde{\Gamma})_k = z$. We need to show that $x < z$. Since $\omega(\tilde{\Gamma})_k = z$, we have $\tilde{\gamma}^{(z)}_k = \beta_k$ and $\tilde{\gamma}^{(z-1)}_k = \beta_k - 1 = \gamma_k$. Similarly, $\gamma^{(x)}_k = \beta_k$ and $\gamma^{(x-1)}_k = \beta_k - 1 = \gamma_k$. Moreover since $x \neq z$, we have $\gamma^{(z-1)}_k = \gamma^{(z)}_k$.

Suppose the LR-fillings are given by partition sequences $\Gamma = (\gamma^{(i)})_i$, $\tilde{\Gamma} = (\tilde{\gamma}^{(i)})$. Since $\Gamma <_{\text{dom}} \tilde{\Gamma}$, it follows from the definition that $\gamma^{(z)}_{\leq \text{dom}} \leq_{\text{dom}} \tilde{\gamma}^{(z)}_{\leq \text{dom}}$ holds, hence $\sum_{i \leq k} \gamma^{(z)}_i \geq \sum_{i \leq k} \tilde{\gamma}^{(z)}_i$. As $\sum_{i < k} \gamma^{(z)}_i = \sum_{i < k} \tilde{\gamma}^{(z)}_i$, we obtain $\gamma^{(z)}_k \geq \tilde{\gamma}^{(z)}_k = \beta_k$, hence $\gamma^{(z)}_k = \tilde{\gamma}^{(z)}_k$. Moreover $\gamma^{(z-1)}_k = \gamma^{(z)}_k = \gamma^{(x)}_k = \gamma^{(x-1)}_k + 1$. Thus $x < z$. 

□
Lemma 4.2. \( \Gamma \leq_{\text{dom}} \tilde{\Gamma} \) if and only if \( \#(\omega(\Gamma))_{\leq c}^{\leq u} \geq \#(\omega(\tilde{\Gamma}))_{\leq c}^{\leq u} \) holds for all \( u, c \).

Proof. First note that for an LR-filling \( \Gamma = [\gamma(i)] \), and for \( u, c \in \mathbb{N} \), the number of boxes at most \( u \) in the first \( c \) columns is \( \sum_{i \leq c} \gamma_i^{(u)} = ec + \#(\omega(\Gamma))_{\leq c}^{\leq u} \).

The following statements are equivalent.

- \( \Gamma \leq_{\text{dom}} \tilde{\Gamma} \)
- \( \gamma^{(u)} \leq_{\text{dom}} \tilde{\gamma}^{(u)} \), for all \( u \)
- \( \sum_{i \leq c} \gamma_i^{(u)} \geq \sum_{i \leq c} \tilde{\gamma}_i^{(u)} \), for all \( u, c \)
- \( \#(\omega(\Gamma))_{\leq c}^{\leq u} \geq \#(\omega(\tilde{\Gamma}))_{\leq c}^{\leq u} \), for all \( u, c \).

\( \square \)

Proof of the Theorem 1.1. We have already seen that the box relation implies the dominance relation for LR-fillings.

For the converse, assume that \( \Gamma, \tilde{\Gamma} \) are LR-fillings of the same shape \((\alpha, \beta, \gamma)\) such that \( \Gamma <_{\text{dom}} \tilde{\Gamma} \). We construct an LR-filling \( \hat{\Gamma} \) of shape \((\alpha, \beta, \gamma)\) such that

\[ \Gamma \leq_{\text{dom}} \hat{\Gamma} \quad \text{and} \quad \hat{\Gamma} <_{\text{box}} \tilde{\Gamma}. \]

Denote the word of \( \tilde{\Gamma} \) as \( \omega(\tilde{\Gamma}) = (a_1, \ldots, a_n) \). Let \( k \) be such that \( \omega(\Gamma) \) and \( \omega(\tilde{\Gamma}) \) agree for the first \( k - 1 \) letters and that \( x = \omega(\Gamma)_k \) differs from \( z = \omega(\tilde{\Gamma})_k \). By Lemma 4.1, \( x < z \).

Suppose the first occurrence of \( x \) in \((a_{k+1}, \ldots, a_n)\) is at position \( m \), so \( x = a_m \) with \( m \in \{k + 1, \ldots, n\} \) as small as possible. Let \( y = \min\{a_i > x : k \leq i < m\} \), here \( y = z \) is allowed. Choose an arbitrary \( \ell \) such that \( y = a_\ell \) and \( k \leq \ell < m \).

We can now define \( \hat{\Gamma} \) as the filling obtained from \( \tilde{\Gamma} \) by exchanging \( y \) in column \( \ell \) with \( x \) in column \( m \).

To verify that \( \hat{\Gamma} \) is an LR-filling, we need to check the lattice permutation property. Let \( c \) be a column; we may assume that \( \ell < c \leq m \). The interesting entries are \( u = y \) and \( u = x + 1 \). If \( u = y \) and \( y > x + 1 \), then by choice of \( y \), all entries \( y - 1 \) on the right hand side of column \( c \) actually are on the right hand side of column \( m \). If \( y = x + 1 \) the result follows by the choice of \( m \), since \( \Gamma \) is an LR-filling which agrees with \( \hat{\Gamma} \) in \((a_1, \ldots, a_{k-1})\). If \( u = x + 1 \)
note that since \( \ell < c \leq m \), this case is only possible if \( y = x + 1 \) and has been considered above.

Next we show that \( \Gamma \leq \text{dom} \hat{\Gamma} \). Using Lemma 4.2 we show that the matrix

\[
(\#\omega(\Gamma)^{\leq u}_{\leq c} - \#\omega(\hat{\Gamma})^{\leq u}_{\leq c})_{c,u}
\]

has only nonnegative entries. For this, we compare the following two matrices ("\( > \)" stands for a positive, "\( \geq \)" for a nonnegative entry).

\[
\begin{array}{cccccccc}
1 & k-1 & k & \ell-1 & \ell & m-1 & n & c \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
\geq \geq \geq \geq \geq \geq \geq \geq \\
\end{array}
\]

The entries 1 arise since the boxes \( x \) and \( y \) in columns \( m \) and \( \ell \) have been exchanged.

According to Lemma 4.2, since \( \Gamma < \text{dom} \hat{\Gamma} \), all entries in the matrix are nonnegative. Since \( \hat{\Gamma} \) and \( \Gamma \) have the same first \( k-1 \) columns, all corresponding entries are 0. The different entries in the \( k \)-th columns of \( \Gamma \) and \( \hat{\Gamma} \), respectively, give rise to the 1’s. Since there are no entries \( x, \ldots, y-1 \) in columns \( k, \ldots, m-1 \) in \( \hat{\Gamma} \), the matrix entries there are strictly positive.
We see that the difference of the two matrices pictured is nonnegative in every component, hence the matrix \((\# \omega(\Gamma)_{\leq c}^u - \# \omega(\tilde{\Gamma})_{\leq c}^u)_{c,u}\) has only nonnegative entries. This shows \(\Gamma \leq_{\text{dom}} \tilde{\Gamma}\).

Finally, we repeat the process of splitting off box moves until \(\tilde{\Gamma} = \Gamma\). This process terminates after finitely many steps since the sum of the entries in the matrix \((\# \omega(\Gamma)_{\leq c}^u - \# \omega(\tilde{\Gamma})_{\leq c}^u)_{c,u}\) is nonnegative and decreases with each box move. □

Algorithm: The proof presented above gives the following algorithm.

Input: LR-fillings \(\Gamma, \tilde{\Gamma}\) of shape \((\alpha, \beta, \gamma)\) such that \(\beta \setminus \gamma\) is a rook strip and \(\Gamma <_{\text{dom}} \tilde{\Gamma}\).

Output: LR-filling \(\hat{\Gamma}\) of shape \((\alpha, \beta, \gamma)\) such that \(\Gamma \leq_{\text{dom}} \hat{\Gamma}\) and \(\hat{\Gamma} <_{\text{box}} \tilde{\Gamma}\).

Step 1. Find the smallest \(k\) such that \(\omega(\Gamma)_k \neq \omega(\tilde{\Gamma})_k\) and put \(x = \omega(\Gamma)_k\).

Step 2. Choose minimal \(m \geq k + 1\) such that \(x = \omega(\tilde{\Gamma})_m\).

Step 3. Let \(y = \min\{\omega(\tilde{\Gamma})_i > x : k \leq i < m\}\).

Step 4. Choose an arbitrary \(k \leq l < m\) such that \(y = \omega(\tilde{\Gamma})_l\).

Step 5. Define \(\hat{\Gamma}\) such that \(\omega(\hat{\Gamma})_i = \omega(\tilde{\Gamma})_i\), for \(i \neq l, m\), and \(\omega(\hat{\Gamma})_l = x\), \(\omega(\hat{\Gamma})_m = y\).

Example: Let \(\beta = (6, 5, 4, 3, 2, 1)\), \(\gamma = (5, 4, 3, 2, 1)\) and \(\alpha = (3, 2, 1)\). Consider two LR-fillings \(\Gamma\) and \(\tilde{\Gamma}\) of shape \((\alpha, \beta, \gamma)\) such that \(\omega(\Gamma) = (1, 3, 2, 2, 1, 1)\) and \(\omega(\tilde{\Gamma}) = (2, 3, 2, 1, 1, 1)\). It is straightforward to check that \(\beta \setminus \gamma\) is a rook strip and \(\Gamma <_{\text{dom}} \tilde{\Gamma}\).

\[
\Gamma :  \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 2 \\
2 \quad 1 \\
3 \\
1 \\
<_{\text{dom}} \tilde{\Gamma} : \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \\
1 \\
3 \\
2 \\
1
\]

We apply the algorithm. Note that \(k = 1, x = 1\) and \(m = 4\). Now we can choose \(y = \omega(\Gamma)_1 = 2\) or \(y = \omega(\Gamma)_3 = 2\). If we choose \(y = \omega(\Gamma)_1\), then \(\hat{\Gamma} = \Gamma\).

In the second case, i.e. if \(y = \omega(\Gamma)_3\), we get \(\omega(\hat{\Gamma}) = (2, 3, 1, 2, 1, 1)\). It is easy to see that \(\Gamma <_{\text{dom}} \tilde{\Gamma}\) and we can continue.

5. Combinatorial properties of the order \(\leq_{\text{box}}\)

In this section we study combinatorial properties of the posets \((\mathcal{T}^\beta_{\alpha, \gamma}, \leq_{\text{box}})\) and \((\mathcal{T}^\beta_{\alpha, \gamma}, \leq_{\text{dom}})\), mainly in the case where \(\beta \setminus \gamma\) is a rook strip.
5.1. An example

Consider the poset \((T_{\alpha,\gamma}^\beta, \leq_{\text{box}})\), where \(\beta = (6, 5, 4, 3, 2, 1)\), \(\gamma = (5, 4, 3, 2, 1)\) and \(\alpha = (3, 2, 1)\). All LR-fillings of this shape have entries: 1, 1, 1, 2, 2, 3. The Hasse diagram of \((T_{\alpha,\gamma}^\beta, \leq_{\text{box}})\) is the following (instead of \(\Gamma\) we write \(\omega(\Gamma)\)):

![Hasse diagram]

Consider the LR-fillings in frames. One is obtained from the other by a single box-move, but there is no chain of neighboring moves, i.e. moves that exchange neighbors. Moreover note that:

- \(\beta \setminus \gamma\) is a rook strip;
- in this poset there exists exactly one maximal and exactly one minimal element;
- all saturated chains have the same length;
- this poset is not a lattice;

5.2. Maximal and minimal elements

We have seen in [4, Proposition 5.5] that the poset \((T_{\alpha,\gamma}^\beta, \leq_{\text{dom}})\) has a unique maximal and a unique minimal element in the case where all parts of \(\alpha\) are at most 2. We show that this statement also holds true if \(\beta \setminus \gamma\) is a rook strip.

**Lemma 5.1.** Assume that \(\beta \setminus \gamma\) is a rook strip. In the poset \((T_{\alpha,\gamma}^\beta, \leq_{\text{box}})\) there exists:
1. exactly one maximal element: the LR-filling $\Gamma$ such that the coefficients in $\omega(\Gamma)$ are in non-increasing order,

2. exactly one minimal element: the LR-filling $\Gamma$ such that $\omega(\Gamma)$ has the form $(p_{k_1}^{k_1}, p_{k_2}^{k_2}, \ldots, p_{k_n}^{k_n})$, for some $k_1, \ldots, k_n$, where $p_i = (i, i-1, \ldots, 1)$ and $p_i^k = (p, p, \ldots, p)$ ($k$ times).

Proof. 1. Let $\Gamma$ be such that the coefficients in $\omega(\Gamma)$ are not in non-increasing order. Then there exist $j$ such that $\omega_j - 1 < \omega_j$, where $\omega(\Gamma) = (\omega_1, \ldots, \omega_n)$.

Then $\Gamma <_{\text{box}} \tilde{\Gamma}$, where

$$\omega(\tilde{\Gamma}) = (\omega_1, \ldots, \omega_{j-2}, \omega_j, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_n)$$

and $\Gamma$ is not maximal.

2. Note that the element of the required form $(p_{k_1}^{k_1}, p_{k_2}^{k_2}, \ldots, p_{k_n}^{k_n})$ is created as follows. Starting from the right hand side we always take the largest possible entry. Let $\Gamma$ be such that $\omega(\Gamma)$ has not the required form. Choose $i$ maximal with the property that $\omega_i$ is not the largest possible entry. Choose $j < i$ maximal with the property that $\omega_j$ is the largest possible entry that can be on the place $i$. It follows that $j < i$ and $\omega_j > \omega_k$ for all $k = j-1, \ldots, i$.

We can replace $\omega_j$ and $\omega_{j-1}$ and get an LR-filling $\tilde{\Gamma}$ such that $\tilde{\Gamma} <_{\text{box}} \Gamma$. This finishes the proof.

Example: 1. The first example shows that the condition that $\beta \setminus \gamma$ be a horizontal strip is needed for the uniqueness of the maximal element in $T_{\alpha, \gamma}^\beta$. Consider the partition triple $\beta = (4, 3, 2, 2, 1)$, $\gamma = (3, 2, 1, 1)$ and $\alpha = (2, 2, 1)$. The Hasse diagram of the poset $(T_{\alpha, \gamma}^\beta, \leq_{\text{dom}})$ has the following shape:
2. The second example shows that in Theorem 1.1, the condition that $\beta \setminus \gamma$ be a vertical strip is necessary. We also see that for horizontal strips, the poset $(T_{\alpha, \gamma}^\beta, \leq_{\text{box}})$ may have several minimal and several maximal elements.

Let $\beta = (5, 4, 3, 1)$, $\gamma = (4, 3, 2, 1)$ and $\alpha = (2, 2, 1)$. There are two LR-fillings of type $(\alpha, \beta, \gamma)$:

They are incomparable in $\leq_{\text{box}}$ relation, but

5.3. Saturated chains

Theorem 5.2. If $\beta \setminus \gamma$ is a rook strip, then all saturated chains in the poset $(T_{\alpha, \gamma}^\beta, \leq_{\text{box}})$ have the same length.

Proof. Since covers in the Bruhat order increase the number of inversions by 1, the theorem follows from Theorem 1.1, Proposition 3.1, Lemma 3.5 and Proposition 3.6.

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Address of the authors:

Faculty of Mathematics and Computer Science  
Nicolaus Copernicus University  
ul. Chopina 12/18  
87-100 Toruń, Poland  
justus@mat.umk.pl

Department of Mathematical Sciences  
Florida Atlantic University  
777 Glades Road  
Boca Raton, Florida 33431  
markus@math.fau.edu

Department of Mathematics and Statistics  
University of New Brunswick  
9 MacAuley Lane  
Fredericton, NB E3B 5A3  
hugh.ross.thomas@gmail.com