Abstract. We prove Kollár’s injectivity theorem for globally $F$-regular varieties.

1. Introduction

The following injectivity theorem for semi-ample line bundles was proved by Kollár [Kol, Theorem 2.2]:

Theorem 1.1 (Kollár’s injectivity theorem). Let $X$ be an $n$-dimensional smooth projective variety over an algebraically closed field of characteristic zero and $\mathcal{L}$ be a semi-ample line bundle on $X$, that is, $\mathcal{L}^{\otimes r}$ is base point free for some positive integer $r$. Let $s$ be a non-zero section in $H^0(X, \mathcal{L}^{\otimes \ell})$ for some positive integer $\ell$. Then the map

$$\times s : H^i(X, \omega_X \otimes \mathcal{L}^{\otimes m}) \to H^i(X, \omega_X \otimes \mathcal{L}^{\otimes (\ell+m)})$$

induced by the tensor product with $s$ is injective for all $m \geq 1$ and $i \geq 0$, where $\omega_X$ is the canonical line bundle on $X$.

Kollár’s injectivity theorem can be viewed as a generalization of the Kodaira vanishing theorem. Indeed, combining Theorem 1.1 with the Serre vanishing theorem, we can easily recover the Kodaira vanishing theorem. The reader is referred to [Fuj2] for a recent development of such an injectivity theorem in characteristic zero.

This paper discusses what happens if the variety is defined over a field of positive characteristic. Kollár’s proof of Theorem 1.1 depends on the Hodge decomposition and does not work in positive characteristic. Indeed, Raynaud [Ray] gave a counterexample to the Kodaira vanishing theorem in positive characteristic. On the other hand, Mehta-Ramanathan [MR] proved that the Kodaira vanishing theorem holds for (Cohen-Macaulay) globally $F$-split varieties, that is, for projective varieties in positive characteristic whose absolute

2010 Mathematics Subject Classification. 14F17, 13A35.

Keywords and phrases. injectivity theorem, vanishing theorem, globally $F$-regular varieties.
Frobenius morphisms split globally. Moreover, Fujino [Fuj1] proved that Kollár’s injectivity theorem holds for projective toric varieties in any characteristic. Since projective toric varieties are globally $F$-split in positive characteristic, it is natural to ask the following question:

**Question 1.2.** Does Kollár’s injectivity theorem hold for globally $F$-split varieties?

Inspired by the theory of $F$-singularities, Smith [Sm] introduced the notion of globally $F$-regular varieties (see Definition 2.1 for the definition), a special class of globally $F$-split varieties. Projective toric and Schubert varieties in positive characteristic are examples of globally $F$-regular varieties (see [Sm], [LRPT] and [Has]). A weak form of the Kawamata-Viehweg vanishing theorem [Sm, Corollary 4.4], which is stronger than the Kodaira vanishing theorem, holds for globally $F$-regular varieties. In this paper, as a partial positive answer to Question 1.2, we prove that Kollár’s injectivity theorem holds for globally $F$-regular varieties:

**Theorem 1.3** (cf. Theorem 3.1). Let $X$ be an $n$-dimensional globally $F$-regular projective variety over an $F$-finite field of characteristic $p > 0$ and $L$ be a semi-ample line bundle on $X$. Let $s$ be a non-zero section in $H^0(X, L^{\otimes \ell})$ for some positive integer $\ell$. Then the map

$$\times s : H^i(X, \omega_X \otimes L^{\otimes m}) \to H^i(X, \omega_X \otimes L^{\otimes (\ell+m)})$$

induced by the tensor product with $s$ is injective for all $m \geq 1$ and $i \geq 0$. Moreover, $H^i(X, \omega_X \otimes L^{\otimes m}) = 0$ for all $m \geq 1$ and $j \neq n - \kappa$, where $\kappa = \kappa(X, L)$ is the Iitaka dimension of $L$.

We can also prove a relative version of Theorem 1.3 (see Theorem 3.1). The proof is inspired by Fujino’s idea of using multiplication maps to prove various vanishing theorems on toric varieties (see [Fuj1]), but we focus more attention on the Cohen-Macaulayness of the higher direct images of the adjoint bundle $\omega_X \otimes L^{\otimes m}$ under the semi-ample fibration induced by $L$.

**Acknowledgments.** The authors were partially supported by JSPS KAKENHI #26707002, 15H03611, 16H02141 and 17H02831. They are grateful to Sho Ejiri, Nobuo Hara, Kenta Sato and Ken-ichi Yoshida for helpful comments. The authors are also indebted to the referee for thoughtful suggestions. The first author would like to thank the organizers of “Workshop in Algebraic Geometry” held in Hanga Roa, Chile on December 18–22, 2016.

**Notation.** Throughout this paper, all rings are commutative rings with unity. A variety over a field $k$ means an integral separated
scheme of finite type over $k$. We use without explanation standard notation and conventions of the book [KM].

2. Preliminaries and lemmas

In this section, we briefly review the definition and basic properties of globally $F$-regular varieties introduced by Smith [Smi].

Recall that a field $k$ of characteristic $p > 0$ is said to be $F$-finite if the extension degree $[k : k^p]$ is finite.

**Definition 2.1** ([MR], [Smi], [HX]). Let $X$ be a normal variety over an $F$-finite field $k$ of characteristic $p > 0$.

(i) $X$ is said to be **globally $F$-split** if the Frobenius map $O_X \to F^e_* O_X$ splits as an $O_X$-module homomorphism.

(ii) $X$ is said to be **globally $F$-regular** if for every effective Weil divisor $D$ on $X$, there exists an integer $e \geq 1$ such that the composite map

$$O_X \to F^e_* O_X \hookrightarrow F^e_* O_X(D)$$

of the $e$-times iterated Frobenius map $O_X \to F^e_* O_X$ and the natural inclusion $F^e_* O_X \hookrightarrow F^e_* O_X(D)$ splits as an $O_X$-module homomorphism.

(iii) Let $f : X \to T$ be a projective morphism to a variety $T$ over $k$. We say that $X$ is **globally $F$-regular over $T$** if there exists an affine covering $\{U_i\}_{i \in I}$ of $T$ such that $f^{-1}(U_i)$ is globally $F$-regular for all $i \in I$.

**Remark 2.2.** Let $X$, $T$ and $k$ be as in Definition 2.1.

(1) It follows from an argument similar to the proof of [HH, Theorem 3.1] that Definition 2.1 (iii) is independent of the choice of the affine covering $\{U_i\}_{i \in I}$. In particular, when $T$ is affine, $X$ is globally $F$-regular over $T$ if and only if $X$ is globally $F$-regular.

(2) Let $T \to S$ be a projective morphism of varieties over $k$. If $X$ is globally $F$-regular over $S$, then $X$ is globally $F$-regular over $T$.

(3) Globally $F$-regular varieties are Cohen-Macaulay (see for example [Smi, Proposition 4.1]).

The following lemma is proved by an argument similar to [Smi, Corollary 4.3], but we include the proof here for the sake of completeness.

**Lemma 2.3.** Let $f : X \to T$ be a projective morphism from a normal variety $X$ to a variety $T$ over an $F$-finite field $k$ of characteristic $p > 0$. Suppose that $\mathcal{L}$ is an $f$-nef line bundle on $X$. If $X$ is globally $F$-regular over $T$, then $R^i f_* \mathcal{L} = 0$ for all $i > 0$. 
Proof. We may assume that $T$ is affine, say $T = \text{Spec } R$, and $X$ is globally $F$-regular. Let $H$ be an $f$-ample effective Cartier divisor on $X$. First, we show that $H^i(X, \mathcal{L}^{\text{can}}(H)) = 0$ for all $m \geq 0$ and $i > 0$. Indeed, $H^i(X, (\mathcal{L}^{\text{can}}(H))^{\text{op}}) = 0$ for all $i > 0$ and all sufficiently large $e$ by the Serre vanishing theorem. On the other hand, since $X$ is globally $F$-regular, we have an injective $R$-homomorphism

$$H^i(X, \mathcal{L}^{\text{can}}(H)) \hookrightarrow H^i(X, F^i_! O_X \otimes (\mathcal{L}^{\text{can}}(H))) \cong H^i(X, (\mathcal{L}^{\text{can}}(H))^{\text{op}})$$

for all $i \geq 0$ and all $e \geq 1$. Thus, we obtain the desired vanishing.

Now we will show that $H^i(X, \mathcal{L}) = 0$ for all $i > 0$. There exists an integer $e \geq 1$ such that we have an injective $R$-homomorphism

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, F^i_! O_X(H) \otimes \mathcal{L}) \cong H^i(X, (\mathcal{L}^{\text{can}}(H))^{\text{op}}(H))$$

for all $i \geq 0$, because $X$ is globally $F$-regular. Therefore, it follows from the above vanishing that $H^i(X, \mathcal{L}) = 0$ for all $i > 0$. \hfill $\Box$

As a corollary of Lemma 2.3, we obtain a Grauert-Riemenschneider type vanishing theorem for globally $F$-regular varieties.

**Corollary 2.4.** Let $f : X \rightarrow Y$ be a surjective projective morphism of normal varieties over an $F$-finite field of characteristic $p > 0$ with $f_* \mathcal{O}_X = \mathcal{O}_Y$. Suppose that $X$ is globally $F$-regular over $Y$. Then $R^i f_* \mathcal{O}_X = 0$ for all $i \geq 1$. Moreover, it holds that $R^{n-m} f_* \omega_X = \omega_Y$ and $R^i f_* \omega_X = 0$ for all $i \neq n - m$, where $\dim X = n$ and $\dim Y = m$.

**Proof.** The former statement immediately follows from Lemma 2.3. We will show the latter statement. We may assume that $Y$ is affine and $X$ is globally $F$-regular. Note that by Remark 2.2 (3) and [HIWY, Proposition 1.2], $X$ and $Y$ are Cohen-Macaulay. Let $\omega_Y^\bullet$ and $\omega_X^\bullet := f^! \omega_Y^\bullet$ be normalized dualizing complexes on $Y$ and $X$, respectively. Since $X$ is Cohen-Macaulay,

$$R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_X^\bullet) \cong \omega_X^\bullet \cong \omega_X[n].$$

Similarly, since $Y$ is Cohen-Macaulay and $\mathcal{O}_Y \cong R f_* \mathcal{O}_X$,

$$R \mathcal{H}om_{\mathcal{O}_Y}(R f_* \mathcal{O}_X, \omega_Y^\bullet) \cong R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y[m]) \cong \omega_Y[m].$$

Thus, by the Grothendieck duality, we have natural isomorphisms

$$R f_* \omega_X[n] \cong R f_* R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\bullet) \cong R \mathcal{H}om_{\mathcal{O}_X}(R f_* \mathcal{O}_X, \omega_Y^\bullet) \cong \omega_Y[m],$$

which implies that $R^i f_* \omega_X = 0$ for all $i \neq n - m$ and $R^{n-m} f_* \omega_X = \omega_Y$. \hfill $\Box$
We also use the following elementary and purely algebraic lemma in the proof of the main theorem.

**Lemma 2.5.** Let \((R, m)\) be a Noetherian local domain with dualizing complex and \(M\) be a finitely generated \(R\)-module. Then there exists a nonzero element \(c \in R\) such that \(c \cdot H^i_m(M) = 0\) for all \(i < \dim R\).

**Proof.** If \(\dim M < \dim R\), then the annihilator \(\text{ann}_R(M)\) of \(M\) is a nonzero ideal and its nonzero element is a desired element. If \(\dim M = \dim R\), then the assertion follows from [Sch, p.46]. □

### 3. Main Theorem

In this section, we prove our main theorem, that is, Theorem 3.1. Theorem 1.3 is the special case of Theorem 3.1 where \(S = \text{Spec} \, k\).

**Theorem 3.1.** Let \(\pi : X \to S\) be a surjective projective morphism of varieties over an \(F\)-finite field \(k\) of characteristic \(p > 0\), and suppose that \(X\) is globally \(F\)-regular over \(S\). Let \(L\) be a \(\pi\)-semi-ample line bundle on \(X\) and \(\kappa\) denote the relative Iitaka dimension of \(L\), that is, the Iitaka dimension \(\kappa(X_\eta, L_\eta)\) of the generic fiber \(X_\eta\) of \(\pi\). Then the following holds:

(i) \(R^i \pi_*(\omega_X \otimes L) = 0\) for all \(i \neq n - \kappa\), where \(n = \dim X_\eta\).

(ii) \(R^{n-\kappa} \pi_*(\omega_X \otimes L)\) is a torsion-free \(O_S\)-module.

(iii) Let \(U\) be a non-empty Zariski open subset of \(S\) and \(s\) be a non-zero section in \(H^0(U, \pi_*(L^{\otimes \ell}))\) for some integer \(\ell \geq 1\). Then the map

\[
\times s : R^{n-\kappa} \pi_*(\omega_X \otimes L^{\otimes m})|_U \to R^{n-\kappa} \pi_*(\omega_X \otimes L^{\otimes (\ell+m)})|_U
\]

induced by the tensor product with \(s\) is injective for every \(m \geq 1\).

**Proof.** We may assume that \(S = \text{Spec} \, A\) is a \(d\)-dimensional affine variety and \(X\) is a \((d + n)\)-dimensional globally \(F\)-regular variety, because (i) and (ii) are local properties. Then it suffices to show that

(i') \(H^i(X, \omega_X \otimes L) = 0\) for all \(i \neq n - \kappa\),

(ii') \(H^{n-\kappa}(X, \omega_X \otimes L)\) is a torsion-free \(A\)-module, and

(iii') if \(s\) is a non-zero section in \(H^0(X, L^{\otimes \ell})\) for some integer \(\ell \geq 1\), then the map

\[
\times s : H^{n-\kappa}(X, \omega_X \otimes L^{\otimes m}) \to H^{n-\kappa}(X, \omega_X \otimes L^{\otimes (\ell+m)})
\]

induced by the tensor product with \(s\) is injective for every \(m \geq 1\) as an \(A\)-module homomorphism.

Since \(L\) is semi-ample over \(S\), there exist a surjective projective morphism \(f : X \to Y\) over \(S\) with \(f_* O_X = O_Y\) and a Cartier divisor \(H\) on \(Y\) such that \(H\) is ample over \(S\) and \(f^* O_Y(H) \cong L^{\otimes r}\) for some integer

\[r.\]
that there exists a nonzero element of the local cohomology \( H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-m)}) = 0 \) vanishes for all \( i < \dim Y = d + \kappa \). It follows from Lemma 2.5 that there exists a nonzero element \( c \in R \) such that \( c \cdot H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-b)}) = 0 \) for all \( i < d + \kappa \) and all \( 0 \leq b \leq r - 1 \). Put \( D = \text{div}_X(c) \). Since \( X \) is globally \( F \)-regular, there exists an integer \( e \geq 1 \) such that the composite map
\[
\mathcal{O}_X \to f_\ast \mathcal{O}_X \leftarrow F^e \mathcal{O}_X(D)
\]
splits as an \( \mathcal{O}_X \)-module homomorphism. Then tensoring with \( L^{\mathcal{O}(-m)} \) and taking the direct image by \( f \) yield the splitting of the following composite map
\[
f_\ast L^{\mathcal{O}(-m)} \to F^e f_\ast L^{\mathcal{O}(-mp')} \xrightarrow{\text{aff}} F^e f_\ast L^{\mathcal{O}(-mp')}
\]
as an \( R \)-module homomorphism. Furthermore, taking the \( i \)-th local cohomology, we find that the following composite map
\[
H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-m)}) \to H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-mp')}) \xrightarrow{\text{aff}} H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-mp')})
\]
is injective. We write \( mp' = ar + b \) for integers \( a \) and \( b \) with \( 0 \leq b \leq r - 1 \). Since \( f_\ast L^{\mathcal{O}(-mp')} \cong f_\ast L^{\mathcal{O}(-b)} \otimes \mathcal{O}_Y(-aH) \), we have an isomorphism \( H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-mp')}) \cong H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-b)}) \). Therefore, for each \( i < d + \kappa \),
\[
H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-mp')}) \xrightarrow{\text{aff}} H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-mp')})
\]
is the zero map by the choice of \( c \), which implies by the injectivity of the map \((\ast)\) that \( H^i_{\{y\}}(Y, f_\ast L^{\mathcal{O}(-m)}) = 0 \) for all \( i < d + \kappa \).

\text{Step 2.} For all integers \( m \geq 1 \), we show the following two properties:

(1) \( R^i f_\ast(\omega_X \otimes \mathcal{O}^{\mathcal{L}^m}) = 0 \) for all \( i \neq n - \kappa \).

(2) \( R^{r - \kappa} f_\ast(\omega_X \otimes \mathcal{L}^m) \) is a maximal Cohen-Macaulay \( \mathcal{O}_Y \)-module, so in particular a torsion-free \( \mathcal{O}_Y \)-module.

Let \( \omega_Y \) and \( \omega_X := f^\ast \omega_Y \) be normalized dualizing complexes on \( Y \) and \( X \), respectively. Since \( X \) and \( Y \) are Cohen-Macaulay by Remark 2.2 (3), we have isomorphisms \( \omega_X \cong \omega_X[d + n] \) and \( \omega_Y \cong \omega_Y[d + \kappa] \).
Note also that $R^nf_*L^\otimes(m) \cong f_*L^\otimes(m)$ by Lemma 2.3. Then, thanks to the Grothendieck duality,

\[
R^{n-k+j}f_*(\omega_X \otimes L^\otimes(m)) = H^{i-d-k}(R^nf_*(\omega_X^* \otimes L^\otimes(m))) \\
\cong H^{i-d-k}(Rf_*\mathcal{H}om_{O_Y}(L^\otimes(m), \omega_X^*)) \\
\cong H^{i-d-k}(\mathcal{H}om_{O_Y}(f_*L^\otimes(m), \omega_Y^*)) \\
\cong H^i(f_*\mathcal{H}om_{O_Y}(f_*L^\otimes(m), \omega_Y)) \\
\cong \mathcal{E}xt^i_{O_Y}(f_*L^\otimes(m), \omega_Y).
\]

It follows from Step 1 and [BH, Theorem 3.3.10] that the Hom sheaf $\mathcal{H}om_{O_Y}(f_*L^\otimes(m), \omega_Y)$ is a maximal Cohen-Macaulay $O_Y$-module and the Ext sheaf $\mathcal{E}xt^i_{O_Y}(f_*L^\otimes(m), \omega_Y)$ vanishes for all $j > 0$. Therefore, $R^{n-k}f_*(\omega_X \otimes L^\otimes(m))$ is a maximal Cohen-Macaulay module and $R^{n-k+j}f_*(\omega_X \otimes L^\otimes(m)) = 0$ for all $j \neq 0$.

**Step 3.** We now prove the assertions (ii') and (iii').

We consider the Leray spectral sequence:

\[
E^{2i}_2 = H^i(Y, R^jf_*(\omega_X \otimes L^\otimes(m))) \Rightarrow H^{i+j}(X, \omega_X \otimes L^\otimes(m)).
\]

By Step 2 (1), this spectral sequence induces an isomorphism

\[
H^i(Y, R^{n-k}f_*(\omega_X \otimes L^\otimes(m))) \cong H^{i+n-k}(X, \omega_X \otimes L^\otimes(m)) \tag{**}
\]

for all $i$. Then we obtain (ii') from Step 2 (2), because the direct image of a torsion-free sheaf by a surjective morphism is torsion-free.

Take a nonzero element $t \in H^0(Y, O_Y(\ell H))$ such that $f^*t = s^r$. The map induced by the tensor product with $t$

\[
\times t : H^0(Y, R^{n-k}f_*(\omega_X \otimes L^\otimes(m))) \to H^0(Y, R^{n-k}f_*(\omega_X \otimes L^\otimes(m)) \otimes O_Y(\ell H))
\]

is injective by Step 2 (2) again. Since

\[
R^{n-k}f_*(\omega_X \otimes L^\otimes(m)) \otimes O_Y(\ell H) \cong R^{n-k}f_*(\omega_X \otimes L^\otimes(r+\ell m))
\]

by the projection formula, the map $\times t$ can be identified via the isomorphism (**) with the map

\[
\times s^r : H^{n-k}(X, \omega_X \otimes L^\otimes(m)) \to H^{n-k}(X, \omega_X \otimes L^\otimes(r+\ell m)).
\]

Thus, (iii') follows from the injectivity of the map $\times t$.

**Step 4.** We finally show the assertion (i').

First, we verify that there exists an integer $e_0 \geq 1$ such that

\[
H^{i+n-k}(X, \omega_X \otimes L^\otimes(p^0)) \cong H^i(Y, R^{n-k}f_*(\omega_X \otimes L^\otimes(p^0))) = 0
\]
for all \( i \neq 0 \). The first isomorphism is nothing but \((\star \star)\). If \( e \geq 1 \) is an integer and we write \( p^e = ar + \beta \) for integers \( a \) and \( \beta \) with \( 0 \leq \beta \leq r - 1 \), then one has an isomorphism
\[
R^{n-\kappa} f_*(\omega_X \otimes L^{\otimes p^e}) \cong R^{n-\kappa} f_*(\omega_X \otimes \mathcal{O}_X(aH)).
\]
Thus, the existence of such an \( e_0 \) follows from the Serre vanishing theorem. On the other hand, by the global \( F \)-regularity of \( X \), the map \( \mathcal{O}_X \to F^e \mathcal{O}_X \) splits for every integer \( e > 0 \). Taking its \( \omega_X \)-dual, we see that the trace map \( Tr^e : F^e \omega_X \to \omega_X \) of Frobenius is surjective and splits. The splitting of the trace map \( Tr^e \) induces the splitting of
\[
R^i \pi_! O_X(\omega_X \otimes L^{\otimes p^e}) \cong R^i \pi_! \omega_X \otimes L \to \omega_X \otimes L,
\]
which yields an injective map
\[
H^{i+n-\kappa}(X, \omega_X \otimes L) \to H^{i+n-\kappa}(X, \omega_X \otimes L^{\otimes p^e}).
\]
for all \( e > 0 \) and all \( i \). Therefore, we obtain \((i')\) from the vanishing of \( H^{i+n-\kappa}(X, \omega_X \otimes L^{\otimes p^e}) \) for all \( i \neq 0 \).

We can reformulate Theorem 3.1 in the following way:

**Corollary 3.2** (cf. [EV], [Fuj2, Theorem 2.12]). Let \( \pi : X \to S \) be a projective morphism of varieties over an \( F \)-finite field of characteristic \( p > 0 \) and \( L = \mathcal{O}_X(H) \) be a \( \pi \)-semi-ample line bundle on \( X \). Suppose that \( X \) is globally \( F \)-regular over \( S \). Let \( D \) be an effective Weil divisor on \( X \) such that \( tH \sim_{\mathbb{Q}, \pi} D + D' \) for some rational number \( t > 0 \) and some effective Weil divisor \( D' \) on \( X \). Then the natural homomorphism
\[
R^i \pi_* O_X(\omega_X \otimes L) \to R^i \pi_* O_X(\omega_X \otimes L(D))
\]
determined by \( D \) is injective for all \( i \geq 0 \).

**Proof.** Replacing \( S \) by the image of \( \pi \), we may assume that \( \pi \) is surjective. Take positive integers \( a \) and \( b \) such that \( aH \sim_{\mathbb{Z}, \pi} bD + bD' \). In order to prove the assertion of the corollary, it is enough to show that the map
\[
R^i \pi_* O_X(\omega_X \otimes L) \to R^i \pi_* O_X(\omega_X \otimes L(aH))
\]
\[
\cong R^i \pi_* O_X(\omega_X \otimes L^{\otimes (a+1)})
\]
is injective. However, once we take a section \( s \in H^0(X, L^{\otimes a}) \) corresponding to \( aH \), it immediately follows from Theorem 3.1.

**Remark 3.3.** (1) Since projective toric varieties are globally \( F \)-regular in positive characteristic (see for example [Smi, Proposition 6.4]), Theorem 3.1 is a generalization of [Fuj1, Theorem 4.1] when the variety is projective over an \( F \)-finite field.
A projective normal variety $X$ is said to be of Fano type if there exists an effective $\mathbb{Q}$-Weil divisor $\Delta$ on $X$ such that $(X, \Delta)$ is klt with $K_X + \Delta$ anti-ample. Schwede-Smith proved in [SS, Theorem 5.1] that if $X$ is a projective variety of Fano type over a field of characteristic zero, then its modulo $p$ reduction is globally $F$-regular for almost all $p$. Thus, Theorem 3.1 gives an alternative proof of Kollár’s injectivity theorem [Kol, Theorem 2.2] for varieties of Fano type in characteristic zero.

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