Rational top and its classical $r$-matrix

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Abstract
We construct a rational integrable system (the rational top) on a co-adjoint orbit of $SL_N$ Lie group. It is described by the Lax operator with spectral parameter and classical non-dynamical skew-symmetric $r$-matrix. In the case of the orbit of minimal dimension the model is gauge equivalent to the rational Calogero–Moser (CM) system. To obtain the results we represent the Lax operator of the CM model in two different factorized forms—without spectral parameter (related to the spinless case) and another one with the spectral parameter. The latter gives rise to the rational top while the first one is related to generalized Cremmer–Gervais $r$-matrices. The gauge transformation relating the rational top and CM model provides the classical rational version of the IRF-Vertex correspondence. From the geometrical point of view it describes the modification of $SL(N, \mathbb{C})$-bundles over degenerated elliptic curve. In view of the Symplectic Hecke Correspondence the rational top is related to the rational spin CM model. Possible applications and generalizations of the suggested construction are discussed. In particular, the obtained $r$-matrix defines a class of KZB equations.

Keywords: integrable systems, rational $r$-matrix, IRF-Vertex correspondence, Calogero model
1. IRF-Vertex and/or symplectic hecke correspondences

We propose the classical rational version of the IRF-Vertex (or Face-Vertex) correspondence. In the elliptic and trigonometric cases it was found in [1] and [2] respectively. In those papers authors suggested explicit formulae for the twists relating non-dynamical [3–5] and dynamical [6, 7] quantum $R$-matrices (with spectral parameter) by representing $L$-operators in the factorized forms [8–10]. The phenomenon of the IRF-Vertex correspondence [3, 11] was observed many times at different levels [12–18]. We are motivated by geometric approach to integrable systems [19, 20], where the Lax operators with spectral parameter are considered as sections of some bundles over algebraic curves. In this approach the IRF-Vertex twists are treated as modifications of bundles (or Hecke transformations) [15]. The latter act on the Lax operator as special gauge transformations degenerated at some point and give rise to the Symplectic Hecke Correspondence [20]. In this way, one can describe sets of integrable systems with gauge equivalent Lax representations corresponding to different characteristic classes of bundles. In addition to both sides of the IRF-Vertex correspondence this geometric description provides intermediate integrable models and related $R$-matrices [21, 22]. In this paper we deal with the Hecke operator given by the following matrix:

$$
\Xi_i(z, \tilde{q}) = \left(z + \tilde{q}_i\right)^{N_i}, \quad i, j = 1 \ldots N, \quad \sum_{i=1}^{N} \tilde{q}_i = 0
$$

(1.1)

(see (A.9)–(A.10)). It is the rational analogue of the Hasegawa’s elliptic twist [1]. On the other hand it is the modification of bundles over degenerated elliptic curve. Some details of underlying geometric construction will be given in [23].

The paper is organized as follows. In the following section we give explicit formula for non-dynamical $r$-matrix with spectral parameter. Next, we describe the underlying integrable system $\tilde{N}$- the rational top (section 3). It is defined on a co-adjoint orbit of $SL_N$. Then we show that in the case of the orbit of minimal dimension the defined model is gauge equivalent to CM one (section 4). The result is achieved by presenting the Lax operator in the factorized form using (1.1). Finally we give precise relations between the non-dynamical $r$-matrix and that of the CM model (section 5), with possible applications and generalizations are discussed in section 6. In particular, the obtained $r$-matrix defines a class of KZB equations. Some useful formulae and details of proofs are given in the appendices.

2. Classical rational $r$-matrix

The purpose of the paper is to describe at classical level the rational version of the IRF-Vertex correspondence obtained in [1]. While the IRF side goes with the spin extension [24] of $sl_N$, Calogero–Moser model (CM) [25], the Vertex side corresponds to the rational top. We show that the top is described by the following non-dynamical $r$-matrix:

$$
r^{\text{top}}(z, w) = r^{\text{top}}(x) = \frac{1}{N_x} \sum_{j=1}^{N} E_{\tilde{q}_j} \otimes
$$

(2.1)

$^{6}$ The title is analogous to the elliptic top [20] which appears from the Belavin–Drinfeld elliptic $r$-matrix [26]. A naive rational limit from the latter gives $r(z, w) = P_{12}(z - w)$, where $P_{12}$ is the permutation operator.
\[
\left[ \sum_{j=0}^{N-1} (\varphi(i) \varphi(j)) E_{\varphi^{-1}(\varphi(i)),j} \right] = -N \sum_{j=0}^{N-1} \delta_{\varphi^{-1}(\varphi(i)),j} \left( \varphi(i) \varphi(j) \right) \left( s+j \right) - N \sum_{j=0}^{N-1} \delta_{\varphi^{-1}(\varphi(i)),j} \left( \varphi(i) \varphi(j) \right) \left( s+j \right) - N \sum_{j=0}^{N-1} \delta_{\varphi^{-1}(\varphi(i)),j} \left( \varphi(i) \varphi(j) \right) \left( s+j \right) - N \sum_{j=0}^{N-1} \delta_{\varphi^{-1}(\varphi(i)),j} \left( \varphi(i) \varphi(j) \right) \left( s+j \right)
\]

where \( x = z - w \). The following statement holds:

**Theorem 1.** 1. The r-matrix (2.1) is skew-symmetric and satisfies the classical Yang–Baxter equation

\[
\left[ r_{12}^{\text{top}}(z, w), r_{13}^{\text{top}}(z, u) \right] + \left[ r_{12}^{\text{top}}(z, w), r_{23}^{\text{top}}(w, u) \right] + \left[ r_{13}^{\text{top}}(z, u), r_{23}^{\text{top}}(w, u) \right] = 0.
\]

2. The r-matrix (2.1) is gauge equivalent to the r-matrix of rational \( sl_N \) Calogero–Moser model with the gauge transformation

\[
g = \Xi(z, \underline{q}) D^{-1}(\underline{q}),
\]
where $\Xi(z, q)$ is given by (1.1),

$$D_i(q) = D_i(q) = \delta_{ij} \prod_{k \neq i} (\bar{q}_i - \bar{q}_k)$$

and $\bar{q} = (\bar{q}_1, ..., \bar{q}_N)$ is the set of $\text{sl}_N$ CM particles coordinates, $\bar{q}_j = q_j - \frac{1}{N} \sum_{k=1}^{N} q_k$.

The proof is direct. In fact, the first part follows from the second one, and the second is precisely described in section 5. The skew-symmetry $r_{ij}(z, w) = -r_{ij}(w, z)$ (see (3.10)) can be verified separately.

Presumably, (2.1) is some non-trivial limit from the Belavin–Drinfeld elliptic $r$-matrix. The case $N = 2$ (2.2) was found in [27] by considering bundles over degenerated (cuspidal) elliptic curves. Independently, the same answer was also obtained at the level of Lax operators [28] (see also [29, 30]) using special limiting procedure starting from the elliptic case. Let us mention that in [31] the non-dynamical $\text{sl}_N$ $r$-matrix was obtained in a different form $r(z, w) = \frac{r_{12}}{z - w} + r^r + r^r$ using the approach of [32], where $P_{12}$ is the permutation operator while $r^r$ and $r^r$ are some non-trivial constants. From the results of [33] it is natural to expect that the answers (2.1) and the one from [31] are gauge equivalent. However, this question deserves further elucidation.

The non-dynamical form of CM $r$-matrix was studied in [17] (see also [18, 34, 35]) in the cases without spectral parameters (including the rational one). Authors of [17] used gauge transformation (2.5) with $\Xi(z, q)$ replaced by the Vandermonde matrix (it was originally found in [36])

$$V_{ij}(q) = q_j^{i-1}, i, j = 1 ... N.$$ (2.7)

It was shown that the spinless CM $r$-matrix without spectral parameters is related to the Cremmer–Gervais one [5]. We also discuss this case in section 4.

### 3. Rational top

In this paper we use $\text{gl}_N^*$-variables $S_{ij}$ dual to generators $E_{ij}$ of Lie algebra $\text{gl}_N$: $(E_{ij})_{ab} = \delta_{ia} \delta_{bj}$. Then the $\text{sl}_N$-variables are naturally defined as

$$S_j = S_j - \frac{1}{N} \delta_{ij} \text{ tr } S \in \text{sl}_N, \quad S = \sum_{i,j} E_{ij} S_{ij} \in \text{gl}_N,$$ (3.1)

i.e. we use ‘bar’ in $\text{sl}_N$ case. In the same manner this notation is used for canonical CM variables (4.9)

$$\bar{q}_j = q_j - \frac{1}{N} \sum_{k=1}^{N} q_k, \quad \bar{p}_j = p_j - \frac{1}{N} \sum_{k=1}^{N} p_k.$$ (3.2)

Let us start with the main statement.

**Theorem 2.** The $r$-matrix (2.1) defines the classical integrable system (the rational top) on a co-adjoint orbit of $\text{SL}(N, \mathbb{C})$ Lie group. The phase space is parameterized by $S \in \text{sl}_N^*$ with the Poisson-Lie structure

$$\{ S_{ij}, S_{kl} \} = \delta_{ij} S_{kl} - \delta_{kl} S_{ij}$$ (3.3)
and fixed eigenvalues of \( \tilde{S} \). The \( \text{sl}_N \)-valued Lax matrix

\[
\tilde{L}_1^{\text{top}}(z) = \text{tr}_2 \left( \overline{n}_2^{\text{top}}(z) \bar{S}_2 \right)
\]

obeys

\[
\begin{bmatrix}
\tilde{L}_1^{\text{top}}(z), & \tilde{L}_2^{\text{top}}(w) \\
\end{bmatrix} = \left[ \overline{n}_2^{\text{top}}(z - w), \tilde{L}_1^{\text{top}}(z) + \tilde{L}_2^{\text{top}}(w) \right]
\]

and provides the set of integrals of motion (with respect to the Poisson structure (3.3)) as coefficients of the spectral curve \( \det(\lambda - \tilde{L}^{\text{top}}(z)) = 0 \).

The proof of the theorem follows from the Yang–Baxter equation (2.4).

Explicit form of the Lax matrix (3.4) follows from (2.1) and (3.4):

\[
\tilde{L}^{\text{top}}_{ij}(z) = \frac{1}{Nz^2} \chi
\]

\[
\begin{align*}
\sum_{\gamma=0}^{q(i)} \left( q(i) \right)_\gamma & S_{\gamma^j} e^{q(i)(-\gamma)} z^{-j} \sum_{\rho=0}^{q(i) - N - j} \left( q(i) \right)_\rho N_{\rho} \sum_{\gamma=0}^{q(i) - N - j - 1} \left( q(i) \right)_\gamma \left( 1 \right)_{\gamma (j)} + j \gamma - 2 \left( j \gamma + 1 \right) \sum_{\rho=0}^{q(i) - N - j - 1} \gamma \rho z^{-j}.
\end{align*}
\]

It has the following structure:

\[
\tilde{L}^{\text{top}}(z) = \frac{1}{Nz^2} \bar{S} + \frac{1}{N} \sum_{k=0}^{2N-1} \bar{z}^k \mathcal{F}(S) = \frac{1}{N} \sum_{k=0}^{2N-1} \bar{z}^k \mathcal{F}(S), \quad \mathcal{F}(S) = \bar{S}.
\]

where the coefficients of the expansion are defined by linear constant operators on \( \text{gl}(N, \mathbb{C}) \):

\[
\mathcal{F}(S) = \sum_{i,j=1}^{N} \sum_{n=1}^{N} \mathcal{F}_{ij, mn} S_{mn} E_{ij}.
\]

Using these notations the \( r \)-matrix (2.1) takes the form:

\[
\tilde{r}^{\text{top}}(z, w) = \frac{1}{N} \sum_{i,j=1}^{N} \sum_{m=1}^{N} \sum_{k=1}^{2N-1} (z - w)^k \bar{z}^k \mathcal{F}_{ij, mn} E_{ij} \otimes E_{mn}.
\]

The property of skew-symmetry means that

\[
\mathcal{F}_{ij, mn} = (-1)^{i+j} \mathcal{F}_{mn, ij}.
\]
For $N = 2$ and $N = 3$ (3.6) takes the form:

$\mathfrak{sl}(2, \mathbb{C})$: $\hat{L}^{\text{op}}(z) = \frac{1}{2} \left( \begin{array}{c}
\frac{1}{2z} \left( S_{11} - S_{22} \right) - z S_{12} \\
\frac{1}{z} \left( S_{11} - S_{22} \right) - z^2 S_{12} \\
\frac{1}{2z} \left( S_{22} - S_{11} \right) + z S_{12}
\end{array} \right) \quad (3.11)$

$\mathfrak{sl}(3, \mathbb{C})$: $\hat{L}^{\text{op}}(z) = \frac{1}{3} \left( \begin{array}{c}
\frac{1}{z} \left( 2 S_{11} - S_{22} - S_{33} \right) + S_{12} - 3 z S_{23} + 2 z^2 S_{33} \\
\frac{1}{z} \left( S_{11} - S_{22} - S_{33} \right) + 3 z S_{12} - 3 z^2 S_{13} - 3 z^3 S_{33} \\
\frac{1}{z} \left( S_{22} - S_{11} - S_{33} \right) - 3 z S_{12} + 3 z^2 S_{13} - 3 z^3 S_{33} \\
\frac{1}{z} \left( S_{33} \right) + S_{12} - 3 z S_{23} + 2 z^2 S_{33} \\
+ 2 z^2 S_{12} + 3 z^3 S_{23} + 2 z^4 S_{33} \\
+ z^2 S_{12} - 6 z S_{23} - 3 z^4 S_{33} \\
- S_{12} + 3 z S_{23} + z^2 S_{33}
\end{array} \right) \quad (3.12)

It follows from theorem 2 that $\{ \text{tr} \left( \hat{L}^{\text{op}}(z)^{\mu} \right), \text{tr} \left( \hat{L}^{\text{op}}(w)^{\nu} \right) \} = 0$. Therefore, traces of powers of (3.6) are the generating functions of the Hamiltonians. The coefficients behind the highest order poles are the Casimir functions:

$$\text{tr} \left( \hat{L}^{\text{op}}(z)^{\mu} \right) = \frac{1}{(\text{par}Nz)^{\mu}} \text{tr} S^{\mu} + \ldots, \ m = 1 \ldots N \quad (3.13)$$

Let us recall standard arguments for the Liouville-Arnold integrability of the models with the Lax matrices of type (3.7). The phase space is a generic $SL(N, \mathbb{C})$-orbit. Its dimension equals $N^2 - N$ because only eigenvalues of $\hat{S}$ are fixed. It is easy to see that the number of coefficients behind the non-negative powers of $z$ in the set (3.13) equals $N(N + 1)/2$. Indeed, expansion of (3.13) gives

$$\frac{1}{m} \text{tr} \left( \hat{L}^{\text{op}}(z)^{\mu} \right)^{\nu} = \frac{1}{z^m} H_{m, m}^{\text{op}} + \frac{1}{z^{m-1}} H_{m, m-1}^{\text{op}} + \ldots + H_{m, 0}^{\text{op}} + \ldots, \ m = 1 \ldots N \quad (3.14)$$

The Hamiltonians $H_{m, 0}^{\text{op}} = 0$ are trivial due to (3.10). Then the total number of the non-trivial Hamiltonians can be computed as $\sum_{m=1}^{N} k = N(N + 1)/2$. Subtracting then the number of the Casimir functions $H_{m, m}^{\text{op}}$ (equal to $N(N + 1)/2$) we get exactly half the dimension of the phase space. Equation (3.5) guarantees that the coefficients are the Poisson commuting Hamiltonians, i.e. $\{ H_{1}^{\text{op}}, H_{m, n}^{\text{op}} \} = 0$. Verifying that these Hamiltonians are independent is a more complicated task. We hope to prove it in our future publications.

The coefficients $\mathcal{J}$ (3.8) can be found from explicit formula (3.6). This allows us to compute Hamiltonians. For example, the quadratic Hamiltonian is generated by $\text{tr} \left( \hat{L}^{\text{op}}(z)^2 \right)$. It can be written as

$$H_{2, 0}^{\text{op}} = \frac{1}{2} \text{tr} \left( \mathcal{J} (S) \mathcal{J} (S) \right) + \text{tr} \left( \hat{S} \mathcal{J} (S) \right) \quad (3.15)$$

7 In fact, all coefficients $H_{k, k, z}^{\text{op}}$ behind positive powers of $z$ vanish because $\hat{L}^{\text{op}}(z)$ is gauge equivalent to spin CM model which Lax matrix has only simple pole in $z$ and no positive powers. We prove this statement elsewhere. It follows from this argument that $\mathcal{J} (S)$ satisfy a set of relations like $\text{tr} \left( \mathcal{J} (S) \mathcal{J} (S) \right) = 0, \ldots.$
For $N = 2$

\[ \text{sl}(2, \mathbb{C}) : H_{2,0}^{\text{top}} = -S_{12} \].

(3.16)

For $N = 3$

\[ \text{sl}(3, \mathbb{C}) : H_{3,0}^{\text{top}} = \frac{1}{3}S_{13}^2 - S_{11}S_{12} + S_{23}(S_{31} - S_{11}) \].

(3.17)

The dimension of generic $\text{SL}_3$-orbit equals 6. Hence, we need two more Hamiltonians. They are

\[ H_{3,0}^{\text{top}} = \frac{1}{27} \left( S_{12}^2S_{21} + 3S_{21}S_{22}S_{23} - 3S_{11}S_{12}S_{13} - 5S_{12}S_{11}S_{31} + S_{21}S_{22}S_{31} - 2S_{12}S_{11}S_{23} + 6S_{11}S_{23}S_{21} - 2S_{12}S_{22}^2 - 3S_{11}S_{21}^2 + 3S_{11}S_{23}^2 - 2S_{11}S_{11}^2 \right), \]

(3.18)

We will describe some details of equations of motion and Lax pairs ($M$-operators) in [23].

4. Factorized L-operators and Calogero–Moser model

**Theorem 3.** In the case when the co-adjoint orbit is of minimal dimension

\[ \mathcal{O}^{\text{min}} : \text{Spec}(\tilde{\mathcal{S}}) = (\nu, ..., \nu, - (N - 1)\nu), \quad \dim \mathcal{O}^{\text{min}} = 2N - 2 \]

(4.1)

the Lax operator (3.4) is gauge equivalent to the Lax operator of the rational Calogero–Moser model with the coupling constant $\nu$. The gauge transformation is given by

\[ g = \Xi(z, \mathbf{q})D^{-1}(\mathbf{q}). \quad \Xi(z, \mathbf{q}) = \Xi(z, \mathbf{q}) \frac{1}{\left(\det \Xi(z, \mathbf{q})\right)^{1/N}}. \]

(4.2)

It provides the following change of variables for the case (4.1):

\[ \tilde{S}_i(\mathbf{p}, \mathbf{q}, \nu) = N\text{Res}_{z=0}L_{ij}^{\text{top}}(z) = (-1)^{p_i} \sum_{m=1}^{N} q_m^{\nu(i)} p_m - \nu \frac{\partial}{\partial q_m} q_m^{\nu(i)} \frac{1}{\prod_{l \neq m} \left( q_m - q_l \right)} \sigma_{\nu(i)}(\mathbf{q}) \]

(4.3)

where $\sigma_{\nu}(\mathbf{q})$ are the elementary symmetric functions (A.19)–(A.21).

**Minimal orbit.** Again, it is convenient to start with $\text{gl}_N$ case. Consider the co-adjoint orbit of $\text{GL}_N$ of minimal dimension, i.e. let

\[ \mathcal{O}^{\text{min}} : \text{Spec}(S) = (0, ..., 0, - N\nu), \quad \dim \mathcal{O}^{\text{min}} = 2N - 2 \]

(4.4)

It means that $S$ is represented as a product of vector by covector

\[ S = \alpha^i \times \beta^i \quad \text{or} \quad S_{ij} = \alpha_i \beta_j. \]

(4.5)

The Poisson brackets

\[ \{ S_i, S_{ij} \} = \delta_{ij}S_i - \delta_{ij}S_{ij}. \]

(4.6)
are realized via the Poisson brackets between components of $\alpha$ and $\beta$. They are given by bivector $\{\alpha_i, \beta_j\}$ (while $\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0$) which is $N \times N$ matrix

$$
\|\{\alpha_i, \beta_j\}\| = \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
\beta_1 & \beta_2 & \cdots & \beta_{N-2} & \beta_{N-1} & 0
\end{pmatrix}
$$

(4.7)

It differs from the (anti)canonical one $-\mathbb{1}_{N \times N}$ by the non-trivial row $\{\alpha_i, \beta_j\}_N$. The latter comes from the Poisson (Dirac) reduction generated by constrains $\sum_{k=1}^N \alpha_i \beta_k^* = -\nu N$ and $\beta_N = 1$.

The proof of theorem 3 is based on the factorized form of the Lax matrix of Calogero–Moser model (CM).

**Rational CM without spectral parameter.** Let $p_i$ and $q_i$, $i = 1 \ldots N$ are canonically conjugated momenta and coordinates of CM particles

$$
\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0
$$

and

$$
\bar{p}_i = q_i - \frac{1}{N} \sum_{k=1}^N q_k, \quad \bar{p}_i = p_i - \frac{1}{N} \sum_{k=1}^N p_k.
$$

(4.9)

are those in the center of mass frame. The Lax matrix in the $\mathfrak{s}\mathfrak{l}_N$ case with the coupling constant $\nu$ can be written as follows:

$$
L_{ij}^{CM} = \delta_{ij} \left( \bar{p}_i - \nu \sum_{k \neq i}^N \frac{1}{q_i - q_k} \right) + \left(1 - \delta_{ij}\right) \nu \frac{V_i}{v_i - q_j}
$$

(4.10)

It differs from the custom one by the canonical map

$$
p_i \rightarrow p_i + a \sum_{k \neq i}^N \frac{1}{q_i - q_k} = p_i + \partial_j \log \det V^a
$$

(4.11)

with the constant $a = -\nu$. It can be verified directly (see [23]) that the Lax matrix (4.10) is represented in the following form:

$$
L^{CM} = \bar{P} - \nu \partial_0 g_0^{-1} \partial_0 g_0, \quad g_0 = g_0(z, \bar{q}) = VD^{-1}
$$

(4.12)

where

$$
\bar{p}_i = \delta_{ij} \bar{p}_i, \quad D_{ij} = \delta_{ij} \prod_{k \neq i} \left( q_i - q_k \right),
$$

(4.13)

$$
V_i = V_i(z, \bar{q}) = \left( z + \bar{q} \right)^{-1}.
$$

(4.14)

Equivalently,

$$
L^{CM} = \bar{P} - \nu DV^{-1} C_0 VD^{-1},
$$

(4.15)
since \( \partial V = C_0 V \). In spite of the fact that matrix \( V (4.14) \) depends on \( z \) explicitly, the Lax matrix (4.10) is independent of \( z \). It happens because

\[
V(z, q) = B(z) V(0, q), \quad B_j(z) = \delta_{j\rho} \left( \frac{i - 1}{j - 1} \right) z^{i-j}, \quad B(z) = e^{c_0}. \quad (4.17)
\]

Therefore,

\[
B^{-1}(z) = B(-z), \quad B^{-1}(z) \partial B(z) = C_0. \quad (4.18)
\]

**Rational CM with spectral parameter.** In this case the Lax matrix is given by naive rational limit from the elliptic one suggested in [37]. In \( \mathfrak{gl}_N \) case it can be written as

\[
L_{ij}^{CM}(z) = \delta_{ij} \left( \rho_i - \nu \sum_{k=1}^{N} \frac{1}{q_i - q_k} \right) + \left( 1 - \delta_{ij} \right) \frac{\nu}{q_i - q_j} - \frac{1}{N z} = L_{ij}^{CM} - \nu \frac{1}{N z}. \quad (4.19)
\]

The diagonal part here also differs from the custom one by the canonical map (4.11). In the \( \mathfrak{sl}_N \) case we have

\[
L_{ij}^{CM}(z) = \delta_{ij} \left( \rho_i - \nu \sum_{k=1}^{N} \frac{1}{q_i - q_k} \right) + \left( 1 - \delta_{ij} \right) \frac{\nu}{q_i - q_j} - \frac{1}{N z}. \quad (4.20)
\]

Similarly to (4.12) the Lax matrices are presented in the forms:

\[
L^{CM}(z) = P - \nu g^{-1} \partial g, \quad g = Z(z, q) D^{-1}(q) \quad (4.21)
\]

where \( D \) is from (4.13) and \( Z \) is from (1.1) or (A.9) (c.f. (2.5)). And

\[
L^{CM}(z) = \bar{P} - \nu \bar{g}^{-1} \partial \bar{g}, \quad (4.22)
\]

where \( \bar{g} \) is from (4.2).

**Hecke transformation** acts on (4.22) in a very simple way—by gauge transformation (see [20]):

\[
\bar{L}^{CM}(z) \rightarrow \bar{g}(z) \bar{L}^{CM}(z) \bar{g}^{-1}(z) = \bar{g}(z) \bar{P} \bar{g}^{-1}(z) - \nu \partial \bar{g}(z) \bar{g}^{-1}(z) = \quad (4.23)
\]

\[
= \bar{Z}(z) \bar{P} \bar{Z}^{-1}(z) - \nu \partial \bar{Z}(z) \bar{Z}^{-1}(z)
\]

The statement of the theorem 3 is that this matrix coincides with (3.6) in the case (4.1) with the change of variables (4.3):

\[
\bar{L}^{top}(z) \big|_{\rho_{\text{min}}}^{(4.3)} = \bar{g}(z) \bar{L}^{CM}(z) \bar{g}^{-1}(z). \quad (4.24)
\]

The proof is similar to the one of theorem 4 which is given in Appendix B. Let us just mention that the residue of (4.24) is easily calculated via (A.23). For variables (4.5) it gives the following ÒbosonizationÓ formulae:

\[^8\text{In quantum case one can quantize the Poisson brackets (4.5)-(4.7) as } \alpha_j \rightarrow \hbar \partial \alpha_j,\]
One can verify that the Poisson bivector (4.7) is reproduced via the canonical Poisson brackets (4.8). Then (c.f.(4.3))

\[
S_i(\hat{p}, \hat{q}, \nu) = \alpha \beta_j = (-1)^{\nu} \sum_{m=1}^{N} \hat{q}_m^{\nu} \hat{p}_m - \nu \hat{p}_m \partial_{\nu m} \hat{q}_m^{\nu} - \sigma_{\nu m}(\hat{q}).
\]  

(4.26)

The obtained bosonization formulae (4.3) are naturally generalized to the case of differential operators. Replace the momenta \(p_m\) in (4.3) as \(p_m \rightarrow \hbar \partial_{m}\). Then the corresponding differential operators \(\hat{S}_i\) commute as generators of the Lie algebra \(\text{gl}_N\):

\[
\delta_{ij} \delta_{kl} = \delta_{ik} \delta_{jl}.
\]

The elliptic case when the matrix \(\Xi\) is given as in (6.4) this type of formulae was obtained in [1].

**Cremmer–Gervais top.** Let us perform a similar gauge transformation in the case of CM without spectral parameter (4.12)–(4.14):

\[
L^{\text{CM}} \rightarrow S_0(\nu) L^{\text{CM}} S_0^{-1}(\nu) = L^{\text{CG}}(\nu) = V(\nu, \hat{q}) \bar{P} V^{-1}(\nu, \hat{q}) - \nu \partial \nu V(\nu, \hat{q}) V^{-1}(\nu, \hat{q}).
\]

(4.27)

Using (4.17)–(4.18) we see that

\[
L^{\text{CG}}(\nu) = V(\nu, \hat{q}) \bar{P} V^{-1}(\nu, \hat{q}) - \nu C_0 = B(\nu)L^{\text{CG}}(0)B^{-1}(\nu),
\]

(4.28)

\[
L^{\text{CG}}(0) = V(0, \hat{q}) \bar{P} V^{-1}(0, \hat{q}) - \nu C_0.
\]

(4.29)

It means that the spectral parameter is fictive here, and can be gauged away by non-dynamical gauge transformation \(B(\nu)\). The latter reflects the fact that initial model (CM) describes only \(N - 1\) degrees of freedom. It appears that the Lax matrix (4.29) can be also written in terms of \(\text{gl}_N\)-variables \(S_{ij}\) (4.26).

**Theorem 4.** The Lax matrix (4.29) can be written in terms of \(\text{gl}_N\)-variables \(S_{\nu m}\) (4.26) as follows:

\[
L^{\text{CG}}(0) = \delta_{\nu \nu} \sum_{i=1}^{N} \left( S_{\nu i} q^{-i} + \sum_{j=1}^{N} S_{\nu j} q^{-j} \right)
\]

\[
- \delta_{\nu \nu} \sum_{i=1}^{N} \sum_{j=1}^{N} S_{\nu i} q^{-i} q^{-j} - \delta_{\nu \nu} \frac{1}{N} \sum_{k=1}^{N} \sum_{l=1}^{N} S_{\nu k} q^{-k} q^{-l} + 1,
\]

(4.30)

where \(q^{-1}\) is the inverse of \(q(A.10)\) and the sum is taken over all indices for which \(q^{-1}\) is defined (i.e. when its arguments are not equal to \(N - 1\)).

The proof is achieved by substitution of (4.26) into (4.29). It is given in appendix B.

This kind of result was obtained in [17] at the level of \(r\)-matrices. The obtained non-dynamical \(r\)-matrix was shown to be the Cremmer–Gervais one [5]. This is why we call the obtained model the Cremmer–Gervais top. Conjugating by \(B(\nu)\) (4.17) we get \(L^{\text{CG}}(\nu)\) which is related to Jordanian (generalized) \(R\)-matrices of Cremmer–Gervais type [18, 38] (see also [39]). Notice that its Hamiltonians coincide with subset of rational top Hamiltonians (3.14):
This is due to (4.19). However, this model has only $N - 1$ independent Hamiltonians. It corresponds to the minimal orbit case (4.1), while the rational top has more Hamiltonians and describes generic orbit.

Let us give (4.30) for $N = 2$ and $N = 3$:

$$H_{k}^{\text{CG}} = \frac{1}{k} \text{tr} \left( L^{\text{CG}} \right)^{k} = H_{k,0}^{\text{top}}, \quad k = 1 \ldots N. \quad (4.31)$$

$\text{sl}(2, \mathbb{C})$: $L^{\text{CM}}(0) = \begin{pmatrix} 0 & S_{32} \\ -S_{11} & 0 \end{pmatrix}$

$\text{sl}(3, \mathbb{C})$: $L^{\text{CM}}(0) = \begin{pmatrix} \frac{1}{3} S_{12} & S_{23} & S_{13} \\ -S_{11} & -\frac{1}{3} S_{12} & S_{23} \\ -S_{21} & S_{33} & \frac{1}{3} S_{12} \end{pmatrix}$

5. Rational classical IRF-Vertex correspondence

Let us make a precise assertion of the second part of theorem 1. Recall that the classical $r$-matrix structure for the rational Calogero–Moser model is given as follows [40, 41]:

$$\left\{ L_{1}^{\text{CM}}(z), L_{2}^{\text{CM}}(w) \right\} = \left[ r_{12}^{\text{CM}}(z, w), L_{1}^{\text{CM}}(z) \right] - \left[ r_{21}^{\text{CM}}(w, z), L_{2}^{\text{CM}}(w) \right]. \quad (5.1)$$

$$r_{12}^{\text{CM}}(z, w) = r_{12}^{0}(z, w) + \frac{1}{Nw} \sum_{i} E_{i} \otimes E_{i} + \sum_{i \neq j} \left( \frac{1}{q_{i} - q_{j}} + \frac{1}{Nw} \right) E_{i} \otimes E_{j}, \quad (5.2)$$

where $r_{12}^{0}$ is the $r$-matrix of the spin Calogero model:

$$r_{12}^{0}(z, w) = \frac{1}{Nz - w} \sum_{i \neq j} E_{i} \otimes E_{i} - \sum_{i \neq j} \frac{1}{q_{i} - q_{j}} E_{i} \otimes E_{j}. \quad (5.3)$$

Using notation (see (4.20))

$$l_{ij}^{\text{CM}}(z) \equiv \left( D^{\varepsilon^{-1}} \partial D^{-1} \right)_{ij} = \delta_{ij} \left( \frac{1}{Nz} + \sum_{k \neq i} \frac{1}{q_{k} - q_{i}} \right) - \left( 1 - \delta_{ij} \right) \left( \frac{1}{q_{i} - q_{j}} - \frac{1}{Nz} \right), \quad (5.4)$$

we have

$$r_{12}^{\text{CM}}(z, w) = \frac{1}{Nz - w} \sum_{i} E_{i} \otimes E_{i} + \sum_{i \neq j} l_{ij}^{\text{CM}}(z - w) E_{i} \otimes E_{j} + l_{ij}^{\text{CM}}(w) E_{j} \otimes E_{j} \quad (5.5)$$

For our purposes we need

$$\tilde{r}_{12}^{\text{CM}}(z, w) = r_{12}^{\text{CM}}(z, w) - \frac{1}{N} \otimes l^{\text{CM}}(w) - \frac{1}{Nz - w} \otimes 1 \quad (5.6)$$

which also satisfies (5.1). The latter redefinition can be obtained by going to $\mathfrak{sl}_{N}$-valued generators together with simple dynamical twist $r \rightarrow r + \delta r$, $\delta r = \sum_{i=1}^{N} \partial_{i} \log D \otimes E_{i}$ with $D$ from (4.13) (see, e.g. Lemma 1 in [42]). The non-dynamical $r$-matrix appears via the gauge
transformation (4.21). It gives
\[ r_{12}^{\text{top}}(z, w) = g(z)g(w) \left( r_{12}^{\text{CM}}(z, w) + g_1^{-1}(z) \right) g_1^{-1}(z)g_2^{-1}(w), \] (5.7)

where the second term is easily computed:
\[ g_1^{-1}(z) \left( g(z), L_2^{\text{CM}}(w) \right) = \frac{1}{N} e^{\text{CM}}(z) \otimes 1 - \frac{1}{N z} 1 \otimes 1 + \sum_{i,j} \left( E_{ai} - E_{aj} \right) \otimes E_{bj} I_i^{\text{CM}}(z). \] (5.8)

After cumbersome calculations one can obtain the resultant r-matrix (2.1). We will give the proof of the statement at quantum level in our next paper.

6. Applications and remarks

- Having non-dynamical skew-symmetric r-matrix such that \( r(z, w) = r(z - w) \) one can naturally define the Knizhnik–Zamolodchikov–Bernard (KZB) equations [43]. Consider tensor product \( V^{\otimes n} \) of \( n \) sl\(_2\)-modules. The \( r(z, w) \)-matrix acts on \( V^{\otimes 2} \). Let \( r(ab)(z_a, z_b) \) act on \( a \)-th and \( b \)-th components of \( V^{\otimes n} \). Then the KZB equations for conformal block \( \psi \) are defined as
\[ V_a \psi = 0 , \quad V_a = \partial_{z_a} + \sum_{\alpha \geq n} r^{\alpha}(z_a, z_c), \quad a = 1, ..., n . \] (6.1)

(6.1) are compatible \( \{ V_a, V_b \} = 0 \) due to the classical Yang–Baxter equation (2.4). The r-matrix (2.1) satisfies the abovementioned conditions (it is skew-symmetric and \( r(z, w) = r(z - w) \)). Therefore, the KZB equations with r-matrix (2.1) are well defined.

- The KZB equations (6.1) are known to describe quantization of the Schlesinger system [44]. At the level of classical mechanics it is easy to construct generalizations of the rational top of Gaudin–Schlesinger type. Let the phase space be a direct product of \( n \) co-adjoint orbits, i.e. we have the variables \( \tilde{S}^a, a = 1, ..., n \) with the Poisson structure be a direct sum of (4.6):
\[ \{ S^a_j, S^b_j \} = \delta^{ab} \left( \delta_{ij} S^a_i - \delta_{ij} S^a_i \right). \] (6.2)

The Lax operator of the Gaudin model is constructed via the one of the rational top (3.6):
\[ \mathcal{L}^{\text{Gaudin}}(z) = \sum_{a=1}^n L^{\alpha\alpha}(z - z_a, S^a). \] (6.3)

It also satisfies the classical exchange relations (3.5), and hence defines an integrable system. It differs from the standard rational Gaudin model and provides an alternative limit from the elliptic Gaudin model [45]. Similarly, the Schlesinger type model appears by replacing (6.3) with the connection along the curve \( \partial_z + \mathcal{L}^{\text{Gaudin}}(z) \). It leads to an alternative rational limit of the elliptic Schlesinger system [46]. We are going to describe these models in details in our future publications. It is interesting to compare the models with those considered in [47] for sl\(_1\) case using non-dynamical r-matrices [48].

- In the trigonometric case the R-matrix is known at quantum level from [2], where the trigonometric analogue of \( \Xi (1.1) \) was found. The analogue of (4.12) is known as well [5, 7, 49]:
\[ V_q(x) = e^{i(\xi)_{ij}}; \quad D_q = \delta_{ij} \prod_{\ell \neq i} (e^{\theta_{ij} - e^{\theta_{\ell j}}}); \quad \Xi_q(x) = \left\{ \begin{array}{ll} e^{i(\xi)_{ij}} & , \quad i = 1, \ldots, N - 1, \\ e^{i(\xi)_{ij}} + (-1)^{N-1}e^{-i(\xi)_{ij}} & , \quad i = N, \end{array} \right. \quad (6.4) \]

where \( x_j = \bar{q}_j + z \). In the elliptic case the spectral parameter is crucially important. The elliptic \( \Xi \) was found in [1] at quantum level:

\[ \Xi_q(z, q | r) = \theta \left[ \frac{\frac{1}{N} - \frac{1}{2}}{\frac{N}{2}} \right] \left( z - N\bar{q}_j, N\bar{r} \right), \quad D_q = \delta_{ij} \prod_{\ell \neq i} \theta (q_i - q_{\ell}). \quad (6.5) \]

- As we mentioned the rational top is actually not a rational model but rather a degenerated elliptic one [28]. It is defined on the bundle over the curve \( y^2 = z^4 \) [27]. We hope our results may shed light on possible elliptic generalizations of the recently investigated dualities in integrable systems [50, 51].
- While the CM models possess relativistic generalization after Ruijsenaars [52], the top-like models have also (group) extensions of this type [53]. These type of models can be described explicitly using the factorized form of \( L \)-operators. In the group case we have the following representations without and with spectral parameter for the Ruijsenaars-Schneider models:

\[ L^{RS} = g_0^{-1}(z)g_0(z - \nu \eta)e^{\nu \eta}, \quad L^{RS}(z) = g^{-1}(z)g(z - \nu \eta)e^{\nu \eta} \quad (6.6) \]

with \( g_0 \) or \( g \) given by (4.12), (4.21) and (6.4)–(6.5). Here \( \eta \) is the inverse light speed.
- It is an interesting question—for which \( g \) the expression (6.6) gives Lax matrix of an integrable model? A general answer is that the \( \Xi(z) \) matrix should be a modification of the underlying bundle. We hope to clarify this question in [23].
- The structure of \( \bar{q} \) argument of \( \Xi(z, \bar{q}) \) is naturally formulated in terms of \( A_{N-1} \) root system and corresponding fundamental weights. Therefore, one can await extensions to other roots systems of simple Lie algebras. Presumably, the \( r \)-matrix is the same (in accordance with [36]) while the \( r \)-matrix structure can be of reflection type.
- Consideration of concrete (2.2), (2.3) of (2.1) show some cancellations in the complicated expression (2.1). We hope that the obtained answer for the classical \( r \)-matrix can be written in a compact form.

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Appendix A. Rational modification

Here we collect some simple algebraic facts related to matrix (1.1). We start with Vandermonde matrix.
\[ V'_j(x) = x_j^{i-1}, \quad i, j = 1, \ldots, N. \quad (A.1) \]

It has the following determinant

\[ \det V = \prod_{1 \leq i < j \leq N} (x_i - x_j) \quad (A.2) \]

and inverse

\[ V_{kl}^{-1} = \frac{1}{(l - 1)!} \frac{1}{\partial^l_{\mu} \prod_{r \neq k}^{N} \frac{\mu - x_r}{x_k - x_r}} \left|_{\mu=0} \quad (A.3) \right. \]

The latter formula can be easily obtained by considering the set of polynomials of degree \( N - 1 \):

\[ f_k(\zeta) = f_k(\zeta, x_1, \ldots, x_N) = \prod_{r \neq k}^{N} \frac{\zeta - x_r}{x_k - x_r}, \quad k = 1, \ldots, N. \quad (A.4) \]

From obvious property

\[ f_k(x_j) = \delta_{kj} = \sum_{l=1}^{N} V_{kl}^{-1} x_j^{l-1} \quad (A.5) \]

and the Taylor expansion

\[ f_k(\zeta) = \sum_{l=1}^{N} \frac{1}{(l - 1)!} \frac{1}{\partial^l_{\mu} \prod_{r \neq k}^{N} \frac{\mu - x_r}{x_k - x_r}} \zeta^{l-1} \quad (A.6) \]

we get (A.3) as

\[ V_{kl}^{-1} = \frac{1}{(l - 1)!} \frac{1}{\partial^l_{\mu} f_k(\mu)} \left|_{\mu=0} \quad (A.7) \right. \]

Multiplying (A.5) by \( \zeta^{k-1} \) and summing up over \( k \) we come to identify:

\[ \sum_{k=1}^{N} x_k^{m-1} f_k(\zeta) = \zeta^{m-1}, \quad m = 1, \ldots, N. \quad (A.8) \]

uf matrix is the main object:

\[ \Xi(z, q) = \begin{pmatrix}
1 & z + \bar{q}_1 & z + \bar{q}_2 & \ldots & z + \bar{q}_N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(z + \bar{q}_1)^{N-2} & (z + \bar{q}_2)^{N-2} & \ldots & (z + \bar{q}_N)^{N-2} \\
(z + \bar{q}_1)^{N} & (z + \bar{q}_2)^{N} & \ldots & (z + \bar{q}_N)^{N}
\end{pmatrix} \quad (A.9) \]

Using function

\[ \varphi(i) = \begin{cases}
   i - 1 & \text{for } 1 \leq i \leq N - 1, \\
   i & \text{for } i = N.
\end{cases} \quad \varphi^{-1}(i) = \begin{cases}
   i + 1 & \text{for } 0 \leq i \leq N - 2, \\
   i & \text{for } i = N.
\end{cases} \quad (A.10) \]
it takes the form (1.1). Consider
\[ \Xi_j(x) = \chi_j^{(l)}. \]  
(A.11)

The determinant equals
\[ \det \Xi(x) = \det V(x) \sum_{j=1}^{N} x_j = \left( \sum_{j=1}^{N} x_j \right) \prod_{1 \leq i < j \leq N} (x_i - x_j). \]  
(A.12)

In order to get (A.12) consider \( N + 1 \times N + 1 \) Vandermonde matrix \( V^{\text{N+1}} \) (depending on \( N + 1 \) variables). The matrix \( \Xi \) is obtained from \( V \) by deleting the \( N \)th row and \( N + 1 \)-th column. Therefore,
\[ \det \Xi = (-1)^{N+1} \left[ \begin{array}{c} (N+1) \\ V^{(N+1)} \end{array} \right] \det V \]  
(A.13)

Substituting here (A.2) and (A.3) taken for \( N := N + 1 \) we get (A.12).

Consider the set of polynomials of degree \( N \):
\[ h_k(\zeta) = \left( 1 + \sum_{j=1}^{N} x_j \right) \prod_{\substack{x_k=x_j \text{ for } j \neq k}}^{N} (\zeta - x_j) = \prod_{\substack{x_k=x_j \text{ for } j \neq k}}^{N} (\zeta - x_k) = -\sum_{j=1}^{N} x_j. \]  
(A.14)

From (A.5) we have
\[ h_k(x_j) = \delta_{kj}. \]  
(A.15)

The analogues of (A.6)-(A.7) are easily obtained:
\[ \Xi^{-1}_j(x) = \frac{1}{\phi(j)!} \frac{\partial^{\phi(j)} h_j(\zeta, x)}{\partial \zeta^{\phi(j)}} \bigg|_{\zeta=0}. \]  
(A.16)

or
\[ \sum_{j=1}^{N} \Xi^{-1}_j(x) \zeta^{(j)} = h_j(\zeta, x). \]  
(A.17)

The analogue of (A.8) reads as follows\(^9\):
\[ \sum_{k=1}^{N} \frac{x_m^k}{x_k} h_k(\zeta) = \zeta^m + \prod_{i=1}^{N} (\zeta - x_i) \frac{\delta_{m,N-1}}{\sum_{i=1}^{N} x_i}, \quad m = 1, ..., N. \]  
(A.18)

It is also convenient to use the elementary symmetric functions. They appear from the expansion
\[ \mathcal{H}(\zeta, x) = \prod_{k=1}^{N} (\zeta - x_k) = \sum_{k=0}^{N} (-1)^k \zeta^k \sigma_k(x_1, ..., x_N) \]  
(A.19)

\(^9\) Notice that the second term in the r.h.s. vanishes for \( m = \phi(m') \).
\[ \sigma_{N-d}(x) = (-1)^N \sum_{1 \leq i < j \leq N} x_i x_\ldots x_{d} \ , \ d = 0, \ldots , N \]  
(A.20)

In the same way define the set \( k \sigma_i(x) \) by

\[ - \prod_{m \neq k} \left( \zeta - x_m \right) = \partial_i \mathcal{H}(\zeta, x) = \sum_{j=0}^{N-1} (-1)^j \zeta \sigma_j(x). \]  
(A.21)

In this notation

\[ V_{kj}^{-1}(x) = (-1)^j \frac{\partial_{j-1}(x)}{\prod_{m \neq k} (x_k - x_j)}. \]  
(A.22)

Set also \( k \sigma_i(x) = \partial_{i-1}(x) = 0 \). Since \( h_k(\zeta, x) = -\left( \frac{\zeta - x_k}{x_k - x_j} \prod_{i \neq k} \frac{1}{x_i - x_j} \right) \partial_i \mathcal{H}(\zeta, x) \), then

\[ \Xi_{kj}^{-1}(x) = (-1)^{\nu ij} \frac{\sum_{i=1}^N x_i}{\prod_{m \neq k} (x_k - x_i)} \left( \frac{k}{\sigma_{ij-1}(x)} + x_k \frac{k}{\sigma_{ij}(x)} \right). \]  
(A.23)

The following set of identities holds:

\[ x_k \sigma_0(x) = \sigma_k(x), \]
\[ \vdots \]
\[ \sigma_{j-1}(x) + x_k \sigma_j(x) = \sigma_j(x), \ \forall \ k \text{ and } j = 1 \ldots N - 1, \]

\[ \vdots \]
\[ \sigma_{N-1}(x) = \sigma_N(x). \]  
(A.24)

Hence,

\[ \Xi_{kj}^{-1}(x) = (-1)^{\nu ij} \frac{\sigma_{ij}(x)}{\sum_{i=1}^N x_i \prod_{m \neq k} (x_k - x_i)} = (-1)^{\nu ij} \frac{\sigma_{ij}(x)}{\prod_{m \neq k} (x_k - x_i)} , \ \forall \ k, j . \]  
(A.25)

Equations (A.24) can be considered as linear system for expressing \( k \sigma_i \) in terms of \( \sigma_j \). It can be done in two ways—using expansion in positive powers of \( x_k \)

\[ m \sigma_j(x) = \sum_{i=0}^{N-j-1} (-x_m \sigma_{j+i+1}(x)) \text{ or } m \sigma_j(x) = \sum_{i=j+1}^{N} (-x_m)^{j-i} \sigma_i(x) \]  
(A.26)

or in negative powers

\[ m \sigma_j(x) = -\sum_{i=0}^{j} (-x_m)^{-j-i} \sigma_{j-i}(x). \]  
(A.27)
In terms of the symmetric functions the inverse of $\Xi(z, \bar{q})$ (A.9) has the following form:

$$
\Xi_{mj}^{-1}(z, \bar{q}) = \frac{1}{Nz} \prod_{j \neq m} (q_m - q_j) \left( \sigma_{0j}(\bar{q}) + \sum_{s=1}^{N-j} \sum_{s=1}^{N-j} \sigma_{s,j-1}(\bar{q}) \left( s + j - 1 \right) \right) - N\sigma_{s,j-1}(\bar{q}) \left( s + j - 2 \right) \right) \right) \right) \left( N - j \right) \bar{\Theta}_{N-j-1}^{(m)}(\bar{q}) \left( j - 1 \right)) \right). \tag{A.28}
$$

**Appendix B. proof of Theorem 4**

Let us start with the case $\nu = 0$. Substitution of $V^{-1}$ (A.22) into (4.27) gives

$$
L_{ij}^{CG} (\nu = 0) = \sum_{m=1}^{N} (-1)^{i+1} q_{m}^{i-1} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i}(\bar{q}), \quad j \neq i - 1, \tag{B.1}
$$

In order to represent it as some linear combination of $S_{ab} (4.3)$ we use (A.26) or (A.27). The choice between these two possibilities comes from the requirement to have the power of $\bar{q}$ in the interval $0 ... N$:

$$
L_{ij}^{CG} (\nu = 0) = \begin{cases} 
\sum_{m=1}^{N} \sum_{n=1}^{N-j} (-1)^{i+c} q_{m}^{i+c-1} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i}(\bar{q}), \quad j \neq i - 1, \\
- \sum_{m=1}^{N} \sum_{n=1}^{j-1} (-1)^{j-c} q_{m}^{j-c-2} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i-1}(\bar{q}), \quad j < i - 1,
\end{cases} \tag{B.2}
$$

In formula (4.3) the index of $\sigma(\bar{q})$ as well as the power of $\bar{q}$ is an image of $p$-function. Therefore, we should exclude somehow the terms corresponding to the indices $N - 1$ (they have no preimages). Dropping of terms with $\sigma_{N-1}(\bar{q})$ does not change the sum since $\sigma_{N-1}(\bar{q}) = 0$. This gives $(1 - \delta_{c,N-j-1})$ in the upper line of (4.30) and $(1 - \delta_{j,N})$ in the lower one. The exclusion of terms with $\bar{q}^{N-1}$ is not so simple. These terms exist in the upper line of (B.2) for $j = i - 1$ and $j \neq i$. In the case $j = i - 1$, $\bar{q}^{N-1}$ is set to zero since it is multiplied by $\sigma_{N-1}(\bar{q}) = 0$. In the case $j \neq i$ all such terms are equal to each other for all $i$ (with $c = N - i$ in the sum), i.e. the terms with $\bar{q}^{N-1}$ form the scalar matrix with the same diagonal elements value $\sum_{m=1}^{N} (-1)^{i} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i}(\bar{q})$. Since $\text{tr} \bar{P} = 0$ then $\text{tr} L_{ij}^{CG} (\nu = 0) = 0$. Thus, the sum of the unwanted term with the rest of the sum should be traceless. This condition allows us to compute the unwanted term:

$$
\sum_{m=1}^{N} (-1)^{i} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i}(\bar{q}) = - \frac{1}{N} \sum_{n=1}^{N} \sum_{i=0}^{N-n-2} \sum_{c=0}^{N-n-2} \bar{q}_{m}^{i+c-1} \frac{\bar{P}_{m}}{\prod_{j \neq m} (q_m - q_j)} \sigma_{j,i}(\bar{q}). \tag{B.3}
$$

It provides the last one summand in (4.30):

$$
-\delta_{ij} \frac{1}{N} \sum_{k=1}^{N} \sum_{c=0}^{N-k-2} \sum_{c=0}^{N-k-2} S_{k+c, k+c+1} = - \delta_{ij} \frac{1}{N} \sum_{k=1}^{N} \sum_{c=0}^{N-k-2} \sum_{c=0}^{N-k-2} S_{k+c, k+c+1}
$$

In the case $\nu \neq 0$ the computation is made in a similar way. In fact, the answer is the same as in $\nu = 0$ case. The only remark—one should use $\text{sl}_N$ bosonization formulae instead of $\text{gl}_N (4.5)$.
since \( tr S = - N \nu \) (4.4). The latter means that in the obtained formulae for \( \nu = 0 \) we need to replace \( S_{ab} \rightarrow S_{ab} - \delta_a^b \sum_i S_i \). It is easy to see that the Cartan part is contained in the line \( j = i - 1 \). The corresponding correction is made in the upper line of (4.30). □

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