A Compendium of Hopf-Like Bifurcations in Piecewise-Smooth Dynamical Systems.

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Abstract

This Letter outlines 20 geometric mechanisms by which limit cycles are created locally in two-dimensional piecewise-smooth systems of ODEs. These include boundary equilibrium bifurcations of hybrid systems, Filippov systems, and continuous systems, and limit cycles created from folds and by the addition of hysteresis or time-delay. Scaling laws for the amplitude and period of the limit cycles are compared to (classical) Hopf bifurcations.

Hopf bifurcations form perhaps the simplest mechanism by which limit cycles (isolated periodic orbits) are created in systems of ODEs. They occur when the real part of a complex conjugate pair of eigenvalues associated with an equilibrium changes sign as parameters of the system are varied [1]. The limit cycle grows out of the equilibrium with an amplitude asymptotically proportional to the square root of the parameter change. The period of the limit cycle varies from $\frac{2\pi}{\omega}$, where $\pm i\omega$ are eigenvalues of the equilibrium at the bifurcation.

In order for Hopf bifurcations to occur in a generic fashion, the ODEs must be $C^3$ (have continuous third derivatives), at least locally. Piecewise-smooth ODEs are commonly used to model physical systems with impacts, switches, or other abrupt processes [2]. For such systems, the presence of switching manifolds, where the ODEs are not smooth, allows limit cycles to be created in a wide variety of Hopf-like bifurcations (HLBs).

This Letter briefly summarises and compares HLBs. Details will be provided in a subsequent publication [3]. For simplicity only two-dimensional systems are treated. In higher dimensions HLBs are expected to occur in essentially the same way (with the same scaling laws), but additional complexities are possible.

For each type of HLB, one limit cycle is created locally. Suppose a HLB occurs at $\mu = 0$, where $\mu \in \mathbb{R}$ is a parameter, and that the limit cycle exists for small $\mu > 0$. Then there exist
| # | Description                        | $a$  | $b$ |
|---|-----------------------------------|------|-----|
| 1 | focus/focus BEB                   | 1    | 0   |
| 2 | focus/node BEB                    | 1    | 0   |
| 3 | generic BEB                       | 1    | 0   |
| 4 | degenerate BEB                    | 1    | 0   |
| 5 | slipping foci                     | 1    | 0   |
| 6 | slipping focus/fold               | 1/2  | 1/2 |
| 7 | slipping folds                    | 1/2  | 1/2 |
| 8 | fixed foci                        | 1    | 0   |
| 9 | fixed focus/fold                  | 1    | 0   |
|10 | fixed folds                       | 1/2  | 1/2 |
|11 | impacting admissible focus        | 1    | 0   |
|12 | impacting virtual focus           | 1    | 0   |
|13 | impacting virtual node            | 1    | 0   |
|14 | impulsive                         | 1    | 0   |
|15 | hysteretic pseudo-equilibrium     | 1    | 1   |
|16 | time-delayed pseudo-equilibrium   | 1    | 1   |
|17 | hysteretic two-fold               | 1/3  | 1/3 |
|18 | time-delayed two-fold             | 1/2  | 1/2 |
|19 | intersecting discontinuity surfaces | 1   | 1   |
|20 | square-root singularity           | 1    | 0   |

Table 1: The exponents in the scaling laws (1) for Hopf bifurcations and 20 Hopf-like bifurcations.

The exponents $a$ and $b$ are determined by the type of HLB bifurcation; the coefficients $k_1$ and $k_2$ are system specific. For Hopf bifurcations, $a = 1/2$ and $b = 0$. For a physical system, values for $a$ and $b$ can often be estimated from experimental data. The results here could aid model selection in that models giving HLBs with incorrect scaling laws would be eliminated.

Table 1 lists the HLBs and their values of $a$ and $b$. Mostly $a = 1$ (linear growth) because many of the HLBs are governed by piecewise-linear ODEs. Indeed, $a \neq 1$ only for HLBs that involve two folds (a fold is a point on a switching manifold where one smooth component of the ODEs is tangent to the switching manifold). HLBs with the same values of $a$ and $b$ can be distinguished by qualitative features, such as the shape of the limit cycle in relation to the switching manifold. Below, for each type of HLB (numbered 1–20), we give a description and two typical phase portraits (one for each side of the bifurcation) in cases for which the limit cycle is stable. Fig. 1 shows such phase portraits for the Hopf bifurcation.
The first four HLBs in Table 1 are boundary equilibrium bifurcations (BEBs) where an equilibrium collides with a switching manifold. In each case, if the limit cycle is stable, it must encircle an unstable focus. First, consider a system that is continuous on a switching manifold, and, at least locally, can be put in the form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{cases}
F_L(x, y; \mu), & x \leq 0, \\
F_R(x, y; \mu), & x \geq 0,
\end{cases}
\]

where \(x\) and \(y\) are the state variables and \(x = 0\) is the switching manifold. By assumption, (2) is continuous but non-differentiable on \(x = 0\), so at the BEB the eigenvalues associated with the equilibrium typically change discontinuously. For HLB 1, see Fig. 2, the equilibrium changes from an unstable focus to a stable focus. In order for a stable limit cycle to be created, the attraction of the stable focus must dominate the repulsion of the unstable focus. Specifically we need \(\alpha < 0\), where

\[
\alpha = \frac{\lambda_L}{\omega_L} + \frac{\lambda_R}{\omega_R},
\]

and \(\lambda_L \pm i\omega_L\) and \(\lambda_R \pm i\omega_R\), with \(\lambda_L > 0, \lambda_R < 0, \omega_L > 0, \) and \(\omega_R > 0\), are the eigenvalues associated with the equilibria at the bifurcation [4, 5]. For HLB 2, the equilibrium changes to a stable node and there is no such criticality condition [6, 7]. Both bifurcations are governed by the linear terms in a piecewise expansion of (2), and so \(a = 1\). Also, \(b = 0\), but unlike Hopf bifurcations the limiting value of the period is not given by a simple expression. These HLBs have been identified in the McKean neuron model [8] and other piecewise-linear models of excitable systems [9, 10].

Next we consider Filippov systems of the form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{cases}
F_L(x, y; \mu), & x < 0, \\
F_R(x, y; \mu), & x > 0,
\end{cases}
\]

which are discontinuous on \(x = 0\). Subsets of \(x = 0\) at which \(F_L\) and \(F_R\) both point towards \(x = 0\) are attracting sliding regions. When an orbit reaches an attracting sliding region (as time increases), it subsequently evolves on \(x = 0\). For Filippov systems such sliding motion is governed by the convex combination of \(F_L\) and \(F_R\) tangent to \(x = 0\) [11].

Figure 1: Phase portraits of a smooth two-dimensional ODE system at parameter values either side of a supercritical Hopf bifurcation.
When an unstable focus of (4) collides with $x = 0$ in a BEB, it may turn into an attracting pseudo-equilibrium (an equilibrium of the sliding motion). In this case a limit cycle exists with the focus (HLB 3). The limit cycle involves only one side of the switching manifold and has a segment of sliding motion. This type of HLB occurs in, for instance, the Gause predator-prey model [12].

Generic codimension-one BEBs in Filippov systems involve one equilibrium and one

![Phase portraits for boundary equilibrium bifurcations (BEBs). The switching manifold is indicated by a green line. Equilibria are shown as circles; folds are shown as triangles. Stable equilibria, stable limit cycles, and attracting sliding regions are coloured blue. Unstable equilibria and repelling sliding regions are coloured red.](image)

Figure 2: Phase portraits for boundary equilibrium bifurcations (BEBs). The switching manifold is indicated by a green line. Equilibria are shown as circles; folds are shown as triangles. Stable equilibria, stable limit cycles, and attracting sliding regions are coloured blue. Unstable equilibria and repelling sliding regions are coloured red.
pseudo-equilibrium \[13\]. For symmetric Filippov systems of the form (1), the point on \(x = 0\) at which the bifurcation occurs can be an equilibrium of both \(F_L\) and \(F_R\), such as for a circuit system given in \[14\]. Such BEBs resemble that of continuous systems, except sliding motion is possible. If an unstable focus transitions to a stable focus with \(\alpha < 0\), both foci have the same direction of rotation, and, at least locally, the system has no attracting sliding regions when the unstable focus exists, then a unique stable limit cycle exists around the unstable focus (HLB 4). If attracting sliding regions are present, up to three nested limit cycles may be created at the BEB simultaneously \[15\] \[16\].

Again consider (4), but now suppose \(F_L\) and \(F_R\) each have either a focus or a fold on \(x = 0\) for all values of \(\mu\) in a neighbourhood of 0. Furthermore, suppose that the foci/folds ‘slip’ along \(x = 0\) as the value of \(\mu\) is varied, and collide at \(\mu = 0\), Fig. 3. A local limit cycle can be created at \(\mu = 0\), and there are three cases: (i) two foci (HLB 5), (ii) one focus and one fold (HLB 6), and (iii) two folds (HLB 7). In the last case the amplitude of the limit cycle is asymptotically proportional to \(\sqrt{\mu}\). This bifurcation is generic and occurs in a prototypical model of balancing via on-off control \[17\].

Now suppose \(F_L\) and \(F_R\) each have either a focus or a fold fixed at the same point on \(x = 0\), Fig. 4. This point is an equilibrium, or may be treated as one, and its stability may

Figure 3: Phase portraits for slipping folds and foci.
change, say at $\mu = 0$, as the value of $\mu$ is varied. Generically a limit cycle is created at $\mu = 0$. As with slipping foci and folds, there are three cases: (i) two foci (HLB 8), (ii) one focus and one fold (HLB 9), and (iii) two folds (HLB 10). Again only in the case of two folds does the amplitude of the limit cycle have nonlinear asymptotic growth. The case of two foci was identified in a model of a car braking system in [18]. In this model the amplitude of the limit cycle has square-root growth because the nonlinear terms are non-generic (cubic instead of quadratic).

Next we consider hybrid systems of the form
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = F(x, y; \mu), \text{ for } x < 0,
\]
\[
y \mapsto -\phi(y; \mu), \text{ when } x = 0.
\]

Let $F_1$ denote the first component of $F$. We assume that $F_1(0, y; \mu) > 0$ for all $y > 0$, and $F_1(0, y; \mu) < 0$ for all $y < 0$. We also assume $\phi(0; \mu) = 0$, and $\phi(y; \mu) > 0$ for all $y > 0$. These conditions ensure that applications of the map $\phi$ are always followed by motion in $x < 0$.

Systems of this form are commonly used to model mechanical systems with hard impacts.

Figure 4: Phase portraits for fixed folds and foci.
by assuming impacting components undergo instantaneous velocity reversals [19]. In (5), \( \phi \) represents the impact law and \( F \) describes motion between impacts.

Suppose an equilibrium of (5) collides with \( x = 0 \) (necessarily at \( x = y = 0 \)) when \( \mu = 0 \), Fig. 5. This is a BEB and we say that the equilibrium changes from admissible (when its \( x \)-value is negative) to virtual (when its \( x \)-value is positive). A limit cycle is created at \( \mu = 0 \) in three distinct scenarios. Specifically, a stable limit cycle can coexist with (i) an admissible unstable focus (HLB 11), (ii) a virtual stable focus (HLB 12), or (iii) a virtual stable node (HLB 13). In each case, \( a = 1 \) and \( b = 0 \), as with BEBs in continuous systems and Filippov

![Figure 5: Phase portraits for HLBs in impacting and impulsive systems. Dashed curves indicate the action of the map \( \phi \).](image-url)
systems. Indeed the bifurcations can be analysed by defining a vector field in \( x > 0 \) that mimics the action of \( \phi \) [20].

Now consider a hybrid system with a map \( \phi \) from one manifold, say \( x = 0 \), to a different manifold (see already HLB 14 in Fig. 5). Systems of this form are often used to model impulsive systems, where \( \phi \) describes the action of an impulse [21]. Here we describe a HLB in such a system following [22]. Suppose the system has a boundary equilibrium on \( x = 0 \), and that at this point the magnitude of the impulse is zero. Similar to the fixed foci and folds of Fig. 4 as parameters are varied the stability of the equilibrium can change and a limit cycle be created (HLB 14). A general algebraic condition determining the onset of the bifurcation is more complicated than for HLBs 11–13, because \( \phi \) provides a rotation by an arbitrary angle (not simply 180°).

Next we introduce perturbations. Filippov systems are useful mathematical models of switched systems, particularly control systems and electrical systems, when the time between switches is small relative to the overall time-scale of the dynamics. Such models may be made more realistic by incorporating hysteresis or time-delay to capture individual switching events. This regularises the switching manifold, replacing sliding motion with rapid switching.

If a Filippov system of the form (4) has a stable pseudo-equilibrium at \((x, y) = (0, 0)\), say, then upon replacing the switching condition at \( x = 0 \) with hysteretic conditions at \( x = \pm \mu \), we generate a limit cycle [23] (HLB 15). By instead supposing that orbits switch at a time \( \mu > 0 \) after crossing \( x = 0 \), we also generate a limit cycle (HLB 16). In both cases, the period of the limit cycle is asymptotically proportional to \( \mu \) (i.e. \( b = 1 \)).

Now suppose (4) has a stable invisible-invisible two-fold at \((x, y) = (0, 0)\), see Fig. 6. By adding hysteresis or time-delay as above we again generate a stable limit cycle. Interestingly, hysteresis (HLB 17) gives \( a = b = \frac{1}{3} \) [24]. This is because the Poincaré map on \( x = -\mu \) has the form

\[
P(y; \mu) = \sqrt{y^2 + c_1 \mu + c_2 y^3 + \cdots},
\]

and so the solution to the fixed point equation, \( y = P(y; \mu) \), involves a cube-root. Time-delay (HLB 18) instead gives \( a = b = \frac{1}{2} \), as with earlier HLBs involving two folds.

Now we consider a Filippov system with two switching manifolds (modelling, say, a switched system with two independent switches). Since switching manifolds are codimension-one surfaces, in two dimensions two switching manifolds generically intersect at a point. This point may behave like an equilibrium, and, as with HLBs 8–10, under parameter variation the intersection point may change stability and a limit cycle be created. This is HLB 19, see Fig. 7. This requires orbits to spiral around the intersection point and occurs in the neuron model of [25], where two different discontinuous functions are used to model the firing rates of excitatory and inhibitory neurons.

Finally, in [26] the authors study BEBs in a piecewise-smooth continuous neuron model that involves a square-root singularity. By this we mean that, in the form (2), one component, say \( F_R \), has a \( \sqrt{|x|} \)-term. As in the usual continuous scenario (HLBs 1–2), both \( F_L \) and \( F_R \) have an equilibrium, and these coincide at the BEB. Here, however, in order to generate a limit cycle locally, both equilibria must be foci (HLB 20). The bifurcation therefore closely resembles HLB 1, however the square-root term prevents the limit cycle from deeply penetrating the \( x > 0 \) half plane. While the amplitude of the limit cycle is asymptotically
proportional to $\mu$, its maximum $x$-value is asymptotically proportional to $\mu^2$. For this reason, HLB 20 is perhaps best viewed an intermediary of HLB 1 and HLB 3.

The HLBs presented here are not intended to form a complete list but hopefully cover the most fundamental scenarios and those reported in mathematical models. Other bifurcations of piecewise-smooth systems that involve limit cycles, but less closely resemble a Hopf bifurcation, include discontinuity induced bifurcations at which two limit cycles are created simultaneously \cite{27, 28}. BEBs can mimic saddle-node bifurcations in that two equilibria (one of which is a pseudo equilibrium in the case of Filippov systems) may collide and annihilate at the bifurcation. Here a local limit cycle is created at the same time if the limiting

\begin{figure}
\centering
\includegraphics[width=\textwidth]{phase_portraits.png}
\caption{Phase portraits for systems with hysteresis or time-delay.}
\end{figure}
Figure 7: Phase portraits for systems with intersecting switching manifolds and a square-root singularity.

A piecewise-linear system satisfies a certain global property \[7, 13\]. Limit cycles can also be created in global bifurcations such as ‘canard super-explosions’ \[9, 10\], and certain bifurcations of piecewise-linear systems with three or more components \[29, 30\].

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