$T\bar{T}$ deformation and the light-cone gauge

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Abstract

The homogeneous inviscid Burgers equation which determines the spectrum of a $T\bar{T}$ deformed model has a natural interpretation as the condition of the gauge invariance of the target space-time energy and momentum of a (non-critical) string theory quantised in a generalised uniform light-cone gauge which depends on the deformation parameter. As a simple application of the light-cone gauge interpretation we derive the $T\bar{T}$ deformed Lagrangian for a system of any number of scalars, fermions and chiral bosons with an arbitrary potential. We find that the $T\bar{T}$ deformation is driven by the canonical Noether stress-energy tensor but not the covariant one.

*Invited contribution to the special issue of the “Proceedings of the Steklov Institute of Mathematics” dedicated to the 80th anniversary of Andrei Slavnov.

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1 Introduction and summary

This paper is dedicated to Andrei Alekseevich Slavnov, the best supervisor one can hope for, on the occasion of his 80th birthday. He taught me the methods used in this paper.

The irrelevant deformation of a 2d field theory by the $T \bar{T}$ operator introduced in [1] has attracted a lot of attention after the $T \bar{T}$ deformation of integrable field theories was analysed in [2, 3], for recent lecture notes see [4]. The most interesting feature of a $T \bar{T}$ deformed model is that its spectrum is completely fixed by the spectrum of the undeformed model [1]. In the case where the momentum of a state is equal to 0, the energy of the state, as a function of the deformation parameter $\alpha$ and the circumference $R$ of the cylinder the theory is defined on, satisfies the homogeneous inviscid Burgers equation

$$\partial_\alpha \mathcal{E}_\alpha(R) + \mathcal{E}_\alpha(R) \partial_R \mathcal{E}_\alpha(R) = 0,$$

whose integrated form is

$$\mathcal{E}_\alpha(R) = \mathcal{E}_0(R - \alpha \mathcal{E}_\alpha(R)). \quad (1.1)$$

Introducing the circumference $R_0$ of the undeformed theory

$$R_0 = R - \alpha \mathcal{E}_\alpha(R) \quad \iff \quad R = R_0 + \alpha \mathcal{E}_0(R_0),$$

the eq. (1.1) takes the form

$$\mathcal{E}_\alpha(R_0 + \alpha \mathcal{E}_0(R_0)) = \mathcal{E}_0(R_0). \quad (1.2)$$

Thus, the homogeneous inviscid Burgers equation is just a statement that the energy of a $T \bar{T}$ deformed theory on a circle of circumference $R_0 + \alpha \mathcal{E}_0(R_0)$ is independent of the deformation parameter. If we now denote

$$\mathcal{E}_\alpha(R_0 + \alpha \mathcal{E}_0(R_0)) = H = \int_{P_0}^{P_-} d\sigma \mathcal{H}_{\text{ws}}, \quad R_0 + \alpha \mathcal{E}_0(R_0) = P_- = J + aH,$$
then we see that eq. (1.2) is the same as eq. (2.17) in the review [5], where the propagation of strings on a background with time and space isometries was analysed in a so-called uniform light-cone gauge introduced in [6]. This is a one-parameter generalisation of the standard light-cone gauge which corresponds to $a = 1/2$. Due to the two isometry directions, the target space-time energy $E$ and the total momentum $J$ are conserved. They are independent of a gauge choice if the total world-sheet momentum vanishes, and since $H = E - J$ is the gauge-fixed light-cone world-sheet Hamiltonian one immediately gets (1.2).

Thus, we conclude that the homogeneous inviscid Burgers equation which determines the spectrum of a $T\bar{T}$ deformed model at vanishing world-sheet momentum can be interpreted as the condition of gauge invariance of the target space-time energy and momentum of a (non-critical) string theory quantised in a uniform light-cone gauge. It is also clear that the deformation parameter $a$ should be related to the gauge parameter $\alpha$ as $a = 1/2 + \alpha$ because for $a = 1/2$ the light-cone strings in flat space are described by a free theory which is naturally taken as the undeformed model. The light-cone strings in $AdS_5 \times S^5$ space are not described by a free theory for any choice of the gauge parameter, and the most natural choice is $a = 0$ which explains the parametrisation chosen in [6].

Let us stress that the world-sheet Hamiltonian density $H_{ws}$ does depend on a gauge parameter in a very nontrivial way, and if one fixes the world-sheet size $R$ then the spectrum of the world-sheet Hamiltonian $H$ will depend on the gauge parameter too. That is why a gauge parameter can be treated as a deformation one.

If the world-sheet momentum does not vanish the target space-time energy $E$ and momentum $J$ are not gauge-invariant anymore. Still, in the case of Lorentz invariant models the relation to a light-cone gauge-fixed string sigma model can be used to derive the inhomogeneous inviscid Burgers equation governing the spectrum for any world-sheet momentum as we demonstrate in section 2.2.

The light-cone gauge interpretation gets more support if we note that the CDD factor $e^{-i\alpha m^2 \sinh \theta}$ which relates the deformed and undeformed models exactly coincides with the $\alpha$-dependent factor in eq. (8.9) of [7], see also eq. (3.94) of [5]. Indeed, introducing the rapidities $\theta_1$ and $\theta_2$ of the colliding particles, the $T\bar{T}$ CDD factor can be rewritten in the form

$$e^{-i\alpha m^2 \sinh(\theta_1 - \theta_2)} = e^{-i\alpha (p_1 \omega_2 - p_2 \omega_1)}, \quad p_k = m \sinh \theta_k, \quad \omega_k = m \cosh \theta_k,$$

used in [7]. The $T\bar{T}$ CDD factor also appeared in the study of effective bosonic string theory in flat space in [8], and its relation to the $T\bar{T}$ deformation was noticed in [9]. Recently, it was pointed out that it also describes the world-sheet scattering of light-cone strings on $AdS_5$ backgrounds without RR fields [10, 11, 12]. Note also that in the form (1.3) the $T\bar{T}$ CDD factor is also valid for massless particles and only affects the left-right

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1The derivation of the factor was not given in [7] due to its simplicity but it will be reviewed in subsection 2.2 for reader’s convenience.
scattering [10, 11, 12], and for a nonrelativistic (integrable) model with any dispersion relation.

It is often said that the $T\bar{T}$ deformation of free bosons is flat space string theory in static gauge with a deformation parameter dependent $B$ field. This is technically correct but it hides the actual origin of the $B$ field, and it provides no insight on how to find a $T\bar{T}$ deformed action. It is known [13, 14] that a uniform light-cone gauge-fixed action can be obtained by first T-dualising in the $x^-$ direction, then integrating out the world-sheet metric, and finally fixing the static gauge, $x^+ = \tau, \bar{x}^- = \sigma$ in the resulting Nambu-Goto action. The $B$ field appears as a result of the $T$-duality but there is no $B$ field in the string sigma model we start with. We do not see much (if any) technical advantages in performing the T-duality, and in this paper we only use the standard phase space approach to a light-cone gauge fixing which provides a crystal clear relation to the Burgers equation.

The relation of string theory in the uniform light-cone gauge to the $T\bar{T}$ deformation of conformal field theories is not new. It was mentioned in [10], and was successfully used to analyse $T\bar{T}$ deformation of free supersymmetric models in [16, 17]. Apparently, the full power of the light-cone gauge approach has not been fully appreciated by the $T\bar{T}$ deformed community [18].

In this paper we generalise this approach to any model and, as a simple application of the light-cone gauge interpretation, we derive the $T\bar{T}$ deformed Lagrangian for a (not necessarily Lorentz invariant) system of any number of scalars, fermions and chiral bosons with an arbitrary potential, see eq.(3.45). We find that the $T\bar{T}$ deformation is always driven by the canonical Noether stress-energy tensor but not the covariant tensor (obtained by the metric variation), see (3.52). A similar observation was made in [16, 17]. It is natural because for a non-Lorentz invariant model the covariant stress-energy tensor may not be defined.

The $T\bar{T}$ deformed Lagrangian (3.45) of a model with bosons is always of the square root Nambu-Goto form. However, it simplifies drastically for a purely fermionic model of $n^-_f$ right-moving real fermions $\theta^\prime_-$, and $n^+_f$ left-moving real fermions $\theta^\prime_+$, and is given by

$$\mathcal{L}_{AAF} = \frac{iK^+ + iK^- + \alpha(K^+K^- - K^+K^-) - V}{1 + \alpha V}. \quad (1.4)$$

Here

$$K^\pm_\gamma \equiv \theta^\prime_\gamma K^-_\gamma \partial_\gamma \theta^\prime_-, \quad K^-_\gamma \equiv \theta^\prime_\gamma K_\gamma \partial_\gamma \theta^\prime_+ + \partial_\pm = \partial_\tau \pm \partial_\sigma, \quad (1.5)$$

$K^\pm_\alpha$ are fermion kinetic matrices whose $\theta$-independent pieces are symmetric and can be diagonalised, and $V$ is an arbitrary potential. If we want a Lorentz invariant model, then $K^\pm_\alpha$ and $V$ may depend on the products $\theta^\prime_- \theta^\prime_+$ only.

The Lagrangian (1.4) is in fact a generalisation of the Alday-Arutyunov-Frolov (AAF) model [19] to any number of fermions and any potential. The AAF model is an integrable

\footnote{See eq.(2.5) for the definition of $x^\pm$.}

\footnote{The interpretation was also known to the author [19].}
model of a massive Dirac fermion with $\alpha = -1/2$. It describes the $\mathfrak{su}(1|1)$ sector of the $\text{AdS}_5 \times S^5$ superstring in the $a = 0$ light-cone gauge. In the modern terminology the AAF model is the $T\bar{T}$ deformation of a free massive Dirac fermion. In fact it was obtained in [19] in exactly the same way as the one described in section 3 of this paper. Some properties of the model were investigated in [20]-[25], and we will discuss the implications of these studies in Conclusions. It was later realised in [26] that quantising the $\mathfrak{su}(1|\!|1)$ sector in the standard $a = 1/2$ light-cone gauge leads to a free massive Dirac fermion, and the independence of the target space-time energy $E$ and momentum $J$ of a gauge choice was used to find the spectrum of $E$ in the semi-classical approximation which was the quantity of interest for the AdS/CFT correspondence [27].

Let us also mention that if we are not interested in Lorentz invariance then $K_{ab}^\pm$ and $V$ can have any dependence on the fermions. In particular, if we only consider the right-moving fermions and choose for simplicity $K_{ij}^+ = \delta_{ij}$ then (1.4) takes the following form

$$L_{\text{SYK}} = \frac{i\theta^k \partial_+ \theta_k^+ - V}{1 + \alpha V}.$$  

Forgetting about $x^-$ coordinate, one can interpret this Lagrangian as a $T\bar{T}$ deformation of the SYK model [28, 29]. It would be interesting to see how the properties of the SYK model are modified by the $T\bar{T}$ deformation.

The plan of the paper is as follows. In section 2.1 we review the construction of the uniform light-cone gauge-fixed action for bosonic strings propagating in a target manifold possessing time and space abelian isometries. In section 2.2 we explain how the inviscid Burgers equation and the $T\bar{T}$ CDD factor appear in the light-cone gauge approach. In section 2.3 the $T\bar{T}$ deformation of a Lorentz invariant sigma-model of bosonic fields with an arbitrary potential is considered. In section 3.1 we generalise the consideration in section 2.1 to a Green-Schwarz type sigma model with any number of scalars, fermions and chiral bosons. In section 3.2 the $T\bar{T}$ deformed Lagrangian for a system of any number of scalars, fermions and chiral bosons with an arbitrary potential is derived. Finally, in Conclusions we discuss open questions and generalisations of the light-cone gauge approach.

2 Light-cone gauge and inviscid Burgers equation

2.1 Uniform light-cone gauge

In this subsection we follow closely the review [3]. To explain the ideas in this subsection we only consider bosonic strings propagating in a $n + 2$-dimensional target Minkowski manifold $\mathcal{M}$ possessing (at least) two abelian isometries, one of which is in the time direction. We denote coordinates of $\mathcal{M}$ by $X^M$, $M = 0, 1, \ldots, n + 1$, the time and space isometry coordinates by $t \equiv X^0$ and $\phi \equiv X^{n+1}$, respectively, and the “transversal” coordinates by $x^\mu$, $\mu = 1, \ldots, n$. The two abelian isometries are realised by shifts of $t$ and $\phi$. If the variable $\phi$ is an angle then the range of $\phi$ is from 0 to $2\pi R_\phi$. Obviously,
the metric $G_{MN}$ of $\mathcal{M}$ does not depend on $t$ and $\phi$. We assume for simplicity that the components $G_{t\mu}$ and $G_{\phi\mu}$ vanish. Thus, the metric of $\mathcal{M}$ is of the form

$$ds^2 = G_{MN} dX^M dX^N = G_{tt} dt^2 + 2G_{t\phi} dtd\phi + G_{\phi\phi} d\phi^2 + G_{\mu\nu} dx^\mu dx^\nu,$$

(2.1)

where $G_{MN}$ is the target-space metric independent of $t$ and $\phi$.

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where $G_{MN}$ is the target-space metric independent of $t$ and $\phi$.

Then, the string action is given by

$$S = -\frac{1}{2} \int_{-r}^{r} d\sigma d\tau \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN},$$

(2.2)

where $\gamma^{\alpha\beta} = h^{\alpha\beta} \sqrt{-h}$ is the Weyl-invariant combination of the world-sheet metric $h_{\alpha\beta}$ with $\det \gamma = -1$.[4] The range of the world-sheet space coordinate $\sigma$ is $-r \leq \sigma \leq r$ where $r$ will be fixed by a generalised uniform light-cone gauge.

To impose a uniform light-cone gauge we transform the string action (2.2) to the first-order form

$$S = \int_{-r}^{r} d\sigma d\tau \left( p_M \dot{X}^M + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2} \frac{\gamma^{00}}{\gamma^{00}} C_2 \right).$$

(2.3)

Here $p_M$ are momenta canonically-conjugate to the coordinates $X^M$

$$p_M = \frac{\delta S}{\delta \dot{X}^M} = -\gamma^{0\beta} \partial_\beta X^N G_{MN}, \quad \dot{X}^M \equiv \partial_0 X^M ,$$

and $C_1$ and $C_2$ are the two Virasoro constraints

$$C_1 = p_M X'^M, \quad C_2 = G^{MN} p_M p_N + X'^M X'^N G_{MN}, \quad X'^M \equiv \partial_1 X^M ,$$

which are solved after imposing a light-cone gauge.

The string action invariance under the shifts of $t$ and $\phi$ implies the conservation of the target space-time energy $E$, and of the total (angular) momentum $J$ of the string in the $\phi$-direction

$$E = -\int_{-r}^{r} d\sigma p_t , \quad J = \int_{-r}^{r} d\sigma p_\phi.$$

(2.4)

It is clear that the charges $E$ and $J$ are gauge-independent.

To impose a uniform gauge we introduce the “light-cone” coordinates and momenta:

$$x^- = \phi - t , \quad x^+ = \frac{1}{2} (\phi + t) + \alpha x^- , \quad p_\phi = p_\phi + p_t , \quad p_- = \frac{1}{2} (p_\phi - p_t) - \alpha p_+ ,$$

(2.5)

4If the $B$-field has nonvanishing components only in the transverse directions $x^\mu$ then the analysis below is not really modified.

5In the conformal gauge $\gamma^{\alpha\beta} = \text{diag}(-1,1)$.
Here $\alpha$ is an arbitrary parameter of the most general light-cone coordinates such that the light-cone momentum $p_+$ is equal to $p_+ = p_{\phi} + p_t$. As a result, in the corresponding uniform light-cone gauge the world-sheet Hamiltonian is equal to $H_{ws} = E - J = -P_+$. As we will show, $\alpha$ is a $T \bar{T}$ deformation parameter used in [2].

The total light-cone momenta are found by using (2.4)

$$P_+ = \int_r^x \sigma p_+ = J - E, \quad P_- = \int_{-r}^x \sigma p_- = \frac{1}{2}(J + E) - \alpha P_+.$$  

Assuming that the target space-time metric is of the form (2.1), and by using the light-cone coordinates we write the action (2.3) in the form

$$S = \int_{-r}^x d\sigma d\tau \left( p_+ \dot{x}^+ + p_- \dot{x}^- + p_{\mu} \dot{x}^\mu + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2} \gamma^{00} C_2 \right),$$  

where

$$C_1 = p_+ x^+ + p_- x^- + p_{\mu} x^\mu,$$

and

$$C_2 = G_{++} p_+^2 + 2 G_{-+} p_+ p_- + G_{--} p_-^2 + G_{++}(x^+)^2 + 2 G_{+-} x^- x^+ + G_{--}(x^-)^2 + 2 \mathcal{H}_x.$$  

Here $\mathcal{H}_x$ is the part of the constraint which depends only on the transversal fields $x^\mu$ and $p_{\mu}$

$$\mathcal{H}_x = \frac{1}{2} \left( G_{\mu\nu} p_\mu p_\nu + x^\mu x^\nu G_{\mu\nu} \right),$$

and the light-cone components of the target space metric are given by

$$G_{++} = G_{\phi\phi} + 2 G_{t\phi}, \quad G_{--} = \frac{G_{++}}{\det G_{lc}}, \quad \det G_{lc} \equiv G_{tt} G_{t\phi} - G_{\phi\phi}^2,$$

$$G_{-+} = \left( \frac{1}{2} - \alpha \right) G_{\phi\phi} + \left( 2 \alpha^2 - \frac{1}{2} \right) G_{t\phi} + \left( \frac{1}{2} + \alpha \right)^2 G_{tt}, \quad G_{--} = \frac{G_{-+}}{\det G_{lc}};$$

$$G_{-+} = \left( \frac{1}{2} - \alpha \right) G_{\phi\phi} - 2 \alpha G_{t\phi} - \left( \alpha + \frac{1}{2} \right) G_{tt}, \quad G_{-+} = - \frac{G_{--}}{\det G_{lc}}.$$

A uniform light-cone gauge is defined by the two conditions

$$x^+ = \tau + a \frac{\pi}{r} m R \rho \sigma, \quad p_- = 1, \quad a = \frac{1}{2} + \alpha,$$  

As was mentioned in the Introduction, it is related to the parameter $a$ used in [6, 5] as $a = \frac{1}{2} + \alpha$. We have also slightly changed the notations in comparison to [3]: $x^\pm = x^\pm_{\text{there}}, p^\pm = p^\pm_{\text{there}}$, to make them closer to the commonly used.
where $m$ is an integer winding number which represents the number of times the string winds around the circle parametrised by $\phi$. It appears if $\phi$ is an angle variable with the range $0 \leq \phi \leq 2\pi R_\phi$ and, therefore, it obeys the constraint
\begin{equation}
\phi(r) - \phi(-r) = 2\pi m R_\phi, \quad m \in \mathbb{Z}.
\end{equation}

Integrating the gauge condition $p_- = 1$ over $\sigma$, we relate the constant $r$ to the total light-cone momentum
\begin{equation}
r = \frac{1}{2} P_-, \quad P_-.
\end{equation}

Thus, the world-sheet of the light-cone string model is a cylinder of circumference $P_-$. The gauge-fixed action is found by solving the Virasoro constraints. First, we use $C_1$ to find $x'^-$
\begin{equation}
C_1 = x'^- + a \frac{2\pi}{P_-} m R_\phi p_+ + p_\mu x'^\mu = 0 \quad \Rightarrow \quad x'^- = -a \frac{2\pi}{P_-} m R_\phi p_+ - p_\mu x'^\mu.
\end{equation}

Then, the solution is substituted into $C_2$, and the resulting equation $C_2 = 0$ is solved for $p_+$. Finally, having found these solutions we bring the string action (2.6) to the gauge-fixed form
\begin{equation}
S_\alpha = \int_{-r}^{r} d\sigma d\tau \left( p_\mu \dot{x}'^\mu - H_{ws} \right),
\end{equation}
where
\begin{equation}
H_{ws} = -p_+(p_\mu, x'^\mu, x'^\mu)
\end{equation}
is the density of the world-sheet Hamiltonian which depends only on the transversal fields $p_\mu, x'^\mu$ which are periodic, $x'^\mu(r) = x'^\mu(-r)$, because we assumed that the strings are closed. Thus, the gauge-fixed string action describes a two-dimensional model on a cylinder of circumference $2r = P_-$. Clearly, for generic values of $\alpha$ the gauge-fixed world-sheet Hamiltonian is of a square root Nambu-Goto type. The two-dimensional model is not in general Lorentz invariant on the world-sheet. However, it is invariant under the shifts of the world-sheet coordinate $\sigma$, and therefore, the total world-sheet momentum of the string is conserved
\begin{equation}
P_{ws} = -\int_{-r}^{r} d\sigma p_\mu x'^\mu.
\end{equation}

States of the resulting two-dimensional model may have any world-sheet momentum. However, only the physical states which satisfy the level-matching condition
\begin{equation}
\Delta x'^- = \int_{-r}^{r} d\sigma x'^- = a \frac{2\pi}{P_-} m R_\phi H_{ws} - \int_{-r}^{r} d\sigma p_\mu x'^\mu = a \frac{2\pi}{P_-} m R_\phi H_{ws} + P_{ws} = 2\pi m R_\phi,
\end{equation}
\end{equation}
have the target space-time energy $E$ and momentum $J$, and therefore the world-sheet energy $E_{ws} = E - J$, independent of the gauge parameter $\alpha$. Solving eq. (2.19) for $P_{ws}$, one finds

$$P_{ws} = \frac{2\pi m R_0 (P_- - aH_{ws})}{P_-} = \frac{2\pi m R_0 J}{P_-} = \frac{2\pi m k}{P_-},$$

(2.20)

where we have taken into account that in quantum theory the charge $J$ should be quantised: $J = k/R_0$, $k \in \mathbb{Z}$.

### 2.2 Inviscid Burgers equation

Now we are ready to derive the inviscid Burgers equation. We consider a physical state with momentum $P_{ws}$ given by (2.20), and energy $E_{ws}(R, \alpha)$ of a light-cone gauge-fixed model on a cylinder of circumference $P_-$. To simplify the notations we denote $R \equiv P_-$, and $E_\alpha(R) \equiv E_{ws}(R, \alpha)$. Then, we have

$$E_\alpha(R) = E_0(R_0) = E_0(R - \alpha E_\alpha(R)).$$

(2.21)

Thus,

$$E_\alpha(R) = E_0(R_0) = E_0 \left( R - \alpha E_\alpha(R) \right),$$

(2.22)

which is the integrated form of the homogeneous inviscid Burgers equation

$$\partial_t E_\alpha(R) + \frac{1}{2} \partial_R E_\alpha^2(R) = 0,$$

(2.23)

and $\alpha$ is indeed equal to the $T\bar{T}$ deformation parameter used in [2].

It is worth noting that if $m = 0$ then $H_{ws}$ has no dependence on $P_-$, and the dependence of the gauge-fixed world-sheet Hamiltonian $H_{ws} = \int_{-r}^r d\sigma H_{ws}$ on $P_-$ comes only through the integration bounds $\pm r$. In this situation we can consider the decompactification limit where $P_- = R \to \infty$, and get a two-dimensional model defined on a plane. Let us assume that the asymptotic states and S-matrix are well-defined. To find a relation between the deformed and undeformed S-matrices let us also assume that the models are integrable. Then, at large $R$ the spectrum of these models is determined by Bethe equations

$$\exp^{ip_i R} \prod_{k \neq i}^N S_{ik}(p_i, p_k) = 1, \quad \sum_{k=1}^N p_k = 0,$$

(2.24)

$$E_\alpha(R) = \sum_{i=1}^N \omega_i, \quad R \gg 1,$$

(2.25)

We assume for simplicity that the scattering matrices are diagonal. In general, the spectrum would be described by a nested Bethe ansatz. Since Bethe equations for auxiliary roots do not depend on $R$, they do not change the conclusion.
where $\omega_i$ is the dispersion relation of the $i$-th particle. Since $R = R_0 + \alpha \mathcal{E}_\alpha(R)$ we can rewrite (2.24) in the form

$$e^{i\mu R_0} \prod_{k \neq i} e^{i\alpha(p_i\omega_k - p_k\omega_i)} S_{ik}(p_i, p_k) = 1, \quad \sum_{k=1}^N p_k = 0. \quad (2.26)$$

These are Bethe equations of the undeformed model with the S-matrices related as

$$S_{ik}^{(0)}(p_i, p_k) = e^{i\alpha(p_i\omega_k - p_k\omega_i)} S_{ik}(p_i, p_k), \quad S_{ik}(p_i, p_k) = e^{-i\alpha(p_i\omega_k^{(0)} - p_k\omega_i^{(0)})} S_{ik}^{(0)}(p_i, p_k). \quad (2.27)$$

The CDD factor of the form $e^{-i\alpha(p_i\omega_k - p_k\omega_i)}$ appeared explicitly in eq.(8.9) of [7] in the study of the AdS$_5$ × S$^5$ world-sheet S-matrix. For a relativistic theory one has

$$\omega_i = m_i \cosh \theta_i, \quad p_i = m_i \sinh \theta_i, \quad p_i\omega_k - p_k\omega_i = m_i m_k \sinh(\theta_i - \theta_k), \quad (2.28)$$

and one reproduces the phase found in [2]

$$S_{ik}(\theta_{ik}) = e^{-i\alpha m_i m_k \sinh \theta_{ik}} S_{ik}^{(0)}(\theta_{ik}), \quad \theta_{ik} = \theta_i - \theta_k. \quad (2.29)$$

The relations (2.27) between the deformed and undeformed S-matrices are valid for non-integrable models if the energies of scattering particles are low enough. It has been argued in [34] that these relations are exact and valid for any model and arbitrary energies. It would be interesting to see if one could come to the same conclusion in the current approach.

Eq.(2.22) is valid only for physical states with the world-sheet momentum (2.20), for example in the zero-winding number case $P_{ws} = 0$. To have an arbitrary world-sheet momentum one has to consider string configurations whose target space-time image is an open string with end points moving in unison so that $\Delta x^-$ remains constant. This however leads to the dependence of the target space-time energy $E$ and momentum $J$ on $\alpha$. Indeed, taking into account that $\Delta x^+ = 0$, one finds (assuming $P_{ws} = 0$)

$$\Delta t = -\left(\frac{1}{2} + \alpha\right) \Delta x^-, \quad \Delta \phi = \left(\frac{1}{2} - \alpha\right) \Delta x^-, \quad (2.30)$$

and, therefore, different values of $\alpha$ correspond to different open string configurations. If the gauge-fixed model does not possess any symmetry relating the world-sheet energy and momentum then there seems to be no relation between the spectra of the undeformed and deformed models. However, if the models are Lorentz invariant on the world-sheet then the relation is relatively easy to find.

We assume again that a model is integrable but the total world-sheet momentum of a state does not vanish. We denote $P_\alpha \equiv P_{ws}$. The spectrum of energies $\mathcal{E}_\alpha(R, P_\alpha)$ is

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9For a nonrelativistic model the dispersion relations of deformed and undeformed models do not have to coincide.

9Its existence was also mentioned in [31, 32], and it was computed perturbatively to leading order in small momentum expansion in [33].

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determined by Bethe equations (2.24) where we have \( \sum_{k=1}^{N} p_k = P_{\alpha} \). We rewrite (2.24) in the form which generalises (2.26)

\[
e^{i p \cdot R_0 + i \alpha P_{\alpha} \omega} \prod_{k \neq i}^{N} e^{i \alpha (p_i \omega_k - p_k \omega_i)} S_{ik}(\theta_{ik}) = e^{im_i(R_0 \sinh \theta_i + \alpha P_{\alpha} \cosh \theta_i)} \prod_{k \neq i}^{N} S_{ik}^{(0)}(\theta_{ik}) = 1,
\]

(2.31)

\[\mathcal{E}_\alpha(R, P_\alpha) = \sum_{k=1}^{N} m_k \cosh \theta_k, \quad P_\alpha = \sum_{k=1}^{N} m_k \sinh \theta_k.\]

where we used Lorentz invariance, and that \( R = R_0 + \alpha \mathcal{E}_\alpha(R, P_\alpha) \). Let us now introduce the shifted rapidity \( \vartheta = \theta - \Delta \theta \) where \( \Delta \theta \) is found from the equation

\[
R_0 \sinh(\theta - \Delta \theta) = R_0 \sinh \theta + \alpha P_{\alpha} \cosh \theta \Rightarrow \quad \Delta \theta = \frac{R_0}{\alpha} \sinh \theta - R_0 \cosh \Delta \theta,
\]

(2.32)

\[
R_0^2 - \alpha^2 P_{\alpha}^2 = \mathcal{R}_0^2.
\]

(2.33)

Eq. (2.33) shows how the target space-time charge \( (J+E)/2 = \mathcal{R}_0 \) depends on the worldsheet momentum and the gauge parameter \( \alpha \). In terms of \( \vartheta \), (2.31) takes the standard form of the Bethe equations of the undeformed theory on a circle of circumference \( \mathcal{R}_0 \)

\[
e^{i \mathcal{R}_0 m_i \sinh \vartheta_i} \prod_{k \neq i}^{N} S_{ik}^{(0)}(\vartheta_{ik}) = 1,
\]

(2.34)

\[\mathcal{E}_0(\mathcal{R}_0, P_0) = \sum_{k=1}^{N} m_k \cosh \vartheta_k, \quad P_0 = \sum_{k=1}^{N} m_k \sinh \vartheta_k.\]

We also find the following relations

\[
\mathcal{E}_\alpha(R, P_\alpha) = \mathcal{E}_0(\mathcal{R}_0, P_0) \cosh \Delta \theta + P_0 \sinh \Delta \theta, \quad P_\alpha = P_0 \cosh \Delta \theta + \mathcal{E}_0(\mathcal{R}_0, P_0) \sinh \Delta \theta,
\]

(2.35)

(2.36)

and therefore

\[
\mathcal{E}_\alpha^2(R, P_\alpha) - P_\alpha^2 = \mathcal{E}_0^2(\mathcal{R}_0, P_0) - P_0^2.
\]

(2.37)

Eqs. (2.32–2.37) represent the integrated form of the inhomogeneous inviscid Burgers equation, see e.g. [3, 35].

\[
\partial_\alpha \mathcal{E}_\alpha(R, P_\alpha) + \frac{1}{2} \partial_R [\mathcal{E}_\alpha^2(R, P_\alpha) - P_\alpha^2] = 0.
\]

(2.38)

Note that due to the momentum quantisation \( P_{\alpha} = 2\pi k/R \), \( P_0 = 2\pi k/\mathcal{R}_0 \) where \( k \) is an \( \alpha \)-independent integer. The discussion above strictly speaking applies only for large \( R \). For finite \( R \) one can use the TBA equations which however have the same dependence on \( R \) and \( P_{\alpha} \), and therefore lead to the same Burgers equations, cf. [3, 35].

Another way (which we cannot fully justify) to derive the inhomogeneous inviscid Burgers equation is as follows. We consider a state with momentum \( P_{\alpha} \) and energy
\( \mathcal{E}_\alpha(R, P_\alpha) \) of a gauge-fixed model on a cylinder of circumference \( R \). Then, we go to the reference frame where the world-sheet momentum is zero. Due to the Lorentz invariance we get

\[
P_\alpha = \mathcal{E}_\alpha(R', 0) \sinh \psi, \quad \mathcal{E}_\alpha(R, P_\alpha) = \mathcal{E}_\alpha(R', 0) \cosh \psi, \tag{2.39}
\]

\[
\mathcal{E}^2_\alpha(R, P_\alpha) - P_\alpha^2 = \mathcal{E}^2_\alpha(R', 0), \tag{2.40}
\]

where \( R' \) is the circumference of the world-sheet cylinder in the new reference frame which due to the Lorentz contraction satisfies the equation

\[
\frac{dR'}{dR} = \frac{1}{\cosh \psi}.
\]

Now, taking into account that \( \mathcal{E}_\alpha(R', 0) \) satisfies the homogeneous inviscid Burgers equation (2.23), one derives (2.38).

### 2.3 Scalar fields with arbitrary potential

As the first example of usefulness of the light-cone gauge approach to the \( T\bar{T} \) deformation let us consider the deformation of a sigma-model of \( n \) scalar fields with the action

\[
S_0 = \int_{r}^{r} d\sigma d\tau \left( -\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} - V(x) \right), \tag{2.41}
\]

where \( \eta^{\alpha\beta} = \text{diag}(-1, 1) \), and \( V \) is an arbitrary potential. In the Hamiltonian formalism this action takes the form

\[
S_0 = \int_{-r}^{r} d\sigma d\tau \left( p_\mu \dot{x}^\mu - \mathcal{H}_0 \right), \quad \mathcal{H}_0 \equiv \mathcal{H}_x + V(x) \tag{2.42}
\]

where \( \mathcal{H}_x \) is given by (2.9). We want \( S_0 \) to be the light-cone gauge-fixed action for \( \alpha = 0 \) of a string sigma model on \( \mathcal{M} \). To have the two-dimensional Lorentz invariance we need to set the winding number \( m \) to 0, and therefore \( x^+ = \tau \). Then, the only way not to get a square-root type gauge-fixed Hamiltonian is to require that the constraint \( C_2 \) (2.8) is linear in \( p_+ \) at \( \alpha = 0 \), and therefore

\[
G^{++}|_{\alpha=0} = 0 = G_{--}|_{\alpha=0} \implies G_{t\phi} = \frac{G_{\phi\phi} + G_{tt}}{2}, \tag{2.43}
\]

where we have used (2.11). Solving the constraint \( C_2 \) (2.8) for \( p_+ \) with \( G_{t\phi} \) given by (2.43), we find the gauge-fixed Hamiltonian at \( \alpha = 0 \)

\[
\mathcal{H}_{ws}|_{\alpha=0} = \frac{G_{\phi\phi} - G_{tt}}{2} \mathcal{H}_x - 2 \frac{G_{\phi\phi} + G_{tt}}{G_{\phi\phi} - G_{tt}}. \tag{2.44}
\]

We see that to reproduce \( \mathcal{H}_0 \) we have to require

\[
G_{\phi\phi} - G_{tt} = 2, \quad G_{\phi\phi} + G_{tt} = -V(x). \tag{2.45}
\]
Thus, to obtain the $T\bar{T}$ deformation of the model (2.41), we need to use the following target space-time metric
\[
G_{tt} = -(1 + \frac{V}{2}), \quad G_{\phi\phi} = 1 - \frac{V}{2}, \quad G_{t\phi} = -\frac{V}{2}.
\] (2.46)

It is easy to check that under these conditions
\[
\det G_{lc} = G_{tt}G_{\phi\phi} - G_{t\phi}^2 = -1,
\] (2.47)
and
\[
G^{−−} = 2V = −G_{++}, \quad G^{++} = 2α(1 + αV) = −G_{−−}, \quad G^{−+} = 1 + 2αV = G_{−+}.
\] (2.48)

Then, for an arbitrary $α$ the constraint $C^2$ takes the form
\[
C^2 = α(1 + αV)p^2_+ + (1 + 2αV)p_+ + H_x + V − α(1 + αV)(x'^2),
\] (2.49)
where $x'^2 = −p_μx^μ$. Solving the constraint for $p_+$, we find the gauge-fixed Hamiltonian
\[
H_{ws} = \frac{1}{α} - \frac{1}{2\tilde{α}} - \frac{1}{2\tilde{α}}\sqrt{1 - 4\tilde{α}H_x + 4\tilde{α}^2(x'^2)^2},
\] (2.50)
where
\[
\tilde{α} = α(1 + αV).
\] (2.51)

Expanding the Hamiltonian, one gets the expected result for a $T\bar{T}$ deformed model
\[
H_{ws} = H_x + V + (H_x^2 - V^2 - (x'^2)^2)α + O(α^2).
\] (2.52)

To see that the model is Lorentz invariant we find the $T\bar{T}$ deformed Lagrangian
\[
L_{ws} = \frac{1}{2}(\dot{x}^2 - x'^2) - V + det(T^β_γ)α + O(α^2),
\] (2.53)
where
\[
\dot{x}^2 \equiv G_{μν}\dot{x}^μ\dot{x}^ν, \quad x'^2 \equiv G_{μν}x^μx^ν, \quad \ddot{x}x' \equiv G_{μν}\ddot{x}^μ\ddot{x}^ν.
\] (2.54)

Obviously it is Lorentz invariant. Expanding the Lagrangian, one gets
\[
L_{ws} = \frac{1}{2}(\dot{x}^2 - x'^2) - V + det(T^β_γ)α + O(α^2),
\] (2.55)
where
\[
T^β_γ = \frac{∂L_{ws}}{∂\dot{x}^γ}\partial_β x^μ - δ^β_γ L_{ws}, \quad det(T^β_γ) = T^0_0T^1_1 - T^0_1T^1_0,
\] (2.56)
\[
T_{αβ} = η_αρT^ρ_β = G_{μν}\partial_α x^μ\partial_β x^ν - \frac{1}{2}η_αβ(η^δ_γ\partial_γ x^μ\partial_δ x^ν G_{μν} + 2V),
\] (2.57)
is the canonical stress-energy tensor of the undeformed model (2.41) which for the scalar model coincides with the covariant one. This is the $T\bar{T}$ deformation with $T\bar{T} = det T^β_γ = - det T_β^γ$. The $T\bar{T}$ deformed Lagrangian (2.55) seems to agree with the one guessed in [35] (for Euclidean world-sheet theory).
3 Models with fermions and chiral bosons

3.1 Light-cone gauge for Green-Schwarz type models

To consider \( T \bar{T} \) deformed models with fermions and chiral bosons we can use string sigma models of the Green-Schwarz type. Let us assume that in addition to bosonic fields \( X^M \) we have \( n_f \) real fermionic variables \( \theta^a \) and \( n_c \) real chiral bosons \( \varphi^c \) which are world-sheet scalars. We combine the fermions and chiral bosons into

\[
\Psi^a = (\theta^a, \varphi^c), \quad \Psi^a = \theta^a, \quad a = 1, \ldots, n_f, \quad \Psi^a = \varphi^c, \quad a = n_f + 1, \ldots, n_f + n_c, \quad (3.1)
\]

and we call the index \( a \) fermionic if \( \Psi^a \) is a fermion, and bosonic if it is a boson.

We consider a reasonably general sigma model with the following action

\[
S = -\int_{-\tau}^{\tau} d\sigma d\tau \left( \frac{1}{2} \gamma^{\alpha\beta} \Pi^M_{\alpha} \Pi^N_{\beta} G_{MN} + i \epsilon^{\alpha\beta} \partial_{\alpha} X^M \Psi^a A_{Mab} \partial_{\beta} \Psi^b + i \frac{1}{2} \epsilon^{\alpha\beta} \partial_{\alpha} \Psi^a F_{ab} \partial_{\beta} \Psi^b \right),
\]

where the skew-symmetric Levi-Civita symbol is defined by \( \epsilon^{\tau\sigma} = 1 \), the world-sheet currents \( \Pi^M_{\alpha} \) are given by

\[
\Pi^M_{\alpha} = \partial_{\alpha} X^M + i \Psi^a \Gamma^M_{ab} \partial_{\alpha} \Psi^b, \quad (3.3)
\]

and \( G_{MN}, \Gamma^M_{ab}, A_{Mab} \) and \( F_{ab} \) are arbitrary functions of \( x^\mu \) and \( \Psi^a \) whose parity depends on whether \( a, b \) are fermionic or bosonic, e.g.

\[
G_{MN} = G_{MN}(x, \Psi) = G_{MN}(x, \varphi) + G_{MN,ab}(x, \varphi) \theta^a \theta^b + G_{MN,abcd}(x, \varphi) \theta^a \theta^b \theta^c \theta^d + \cdots, \quad (3.4)
\]

\[
\Gamma^M_{ab} = \Gamma^M_{ab}(x, \Psi) = \Gamma^M_{ab}(x, \varphi) + \Gamma^M_{ab,cd}(x, \varphi) \theta^c \theta^d + \Gamma^M_{ab,cdde}(x, \varphi) \theta^c \theta^d \theta^e \theta^f + \cdots, \quad (3.5)
\]

if \( a, b \) are both either fermionic or bosonic, and

\[
\Gamma^M_{ab} = \Gamma^M_{ab}(x, \Psi) = \Gamma^M_{abc}(x, \varphi) \theta^c + \Gamma^M_{abcd}(x, \varphi) \theta^c \theta^d \theta^e + \cdots, \quad (3.6)
\]

if the indices are of opposite parity, and similar expansions for \( A_{Mab} \) and \( F_{ab} \). These functions are real \( (G_{MN})^* = G_{MN}, \ (\Gamma^M_{ab})^* = \Gamma^M_{ab} \) if at least one of the indices \( a, b \) is fermionic, and imaginary if both indices are bosonic. This follows from the conjugation rule for fermions: \( (\theta^a \theta^b)^* = (\theta^b)^* (\theta^a)^* = -\theta^a \theta^b \). Note that \( F_{ab} \) is symmetric under the exchange of \( a, b \) if \( a, b \) are fermionic, and it is skew-symmetric if at least one of the indices \( a, b \) is bosonic.

Obviously, as in the purely bosonic case, the action is invariant under the shifts of \( t \) and \( \phi \). We do not assume any target space symmetry, and therefore \( \Gamma^M_{ab} \) are not in general related to gamma matrices. Then, we assume that the reparametrisation invariance is the only gauge symmetry of the model, and therefore it is not \( \kappa \)-invariant.\(^{11}\)

\(^{10}\)We call these bosons chiral because, as we will see, if in an undeformed theory they are free then they satisfy the equations of motion \( \partial_{x^\pm} \varphi^c = 0 \).

\(^{11}\)If one starts with a \( \kappa \)-invariant model, then one first imposes a kappa-symmetry gauge condition and reduces its action to the form (3.2).
Introducing auxiliary momentum-like variables $\pi_M$, we rewrite the action (3.2) in the first-order form
\[
S = \int_{-r}^{r} d\sigma d\tau \left( \pi_M \Pi_0^M + \frac{\gamma_{01}^{01}}{\gamma_{00}} C_1 + \frac{1}{2\gamma_{00}} C_2 
- i\epsilon^{\alpha\beta} \partial_\alpha X^M \Psi^a A_{Mab} \partial_\beta \Psi^b - \frac{i}{2} \epsilon^{\alpha\beta} \partial_\alpha \Psi^a F_{ab} \partial_\beta \Psi^b \right),
\]
(3.7)

Here $\pi_M$ satisfy the equations
\[
\pi_M = \frac{\delta S}{\delta \Pi_0^M} = -\gamma^{03} \Pi_0^N G_{MN},
\]
and $C_1$ and $C_2$ are the two Virasoro constraints
\[
C_1 = \pi_M \Pi_1^M, \quad C_2 = G^{MN} \pi_M \pi_N + \Pi_1^M \Pi_1^N G_{MN}.
\]
Collecting the terms with $\dot{X}^M$ and $\dot{\Psi}^a$, we bring the action (3.7) to the form
\[
S = \int_{-r}^{r} d\sigma d\tau \left( p_M \dot{X}^M + i p_\Psi \dot{\Psi}^b + \frac{\gamma_{01}^{01}}{\gamma_{00}} C_1 + \frac{1}{2\gamma_{00}} C_2 \right),
\]
(3.8)

where $p_M$ is the momentum conjugated to $X^M$ and related to $\pi_M$ as
\[
p_M = \pi_M - i \Psi^a A_{Mab} \Psi^b,
\]
(3.9)
and $p_\Psi^b$ is given by
\[
p_\Psi^b = \Psi^a \pi_M \Gamma_{ab}^M + X^M \Psi^a A_{Mab} + \Psi^a F_{ab}.
\]
(3.10)

Now, we can proceed in the same way as in section 2. We introduce the light-cone coordinates and momenta (2.5), and rewrite the action (3.8) in the form (2.6) with
\[
C_1 = \pi_+ \Pi_1^+ + \pi_- \Pi_1^- + \pi_\mu \Pi_1^\mu,
\]
(3.11)
\[
C_2 = G^{--} \pi_- \pi_- + 2G^{-+} \pi_- \pi_+ + G^{++} \pi_+ \pi_+ 
+ G_{-\mu} \Pi_1^\mu \Pi_1^- + 2G_{++} \Pi_1^+ \Pi_1^+ \Pi_1^+ + 2\mathcal{H}_x,
\]
(3.12)

and
\[
\mathcal{H}_x = \frac{1}{2} \left( G^{\mu\nu} \pi_\mu \pi_\nu + \Pi_1^\mu \Pi_1^\nu G_{\mu\nu} \right).
\]
(3.13)

Here $\pi_M$ are related to $p_M$ by (3.9), and the light-cone components of the target space metric are given by (2.10-2.12) with $G \to G$.

Next, we impose the uniform light-cone gauge (2.13) with the winding number $m = 0$ for simplicity. Then, we solve the Virasoro constraint $C_1$ for $x^-$
\[
C_1 = 0 \quad \Rightarrow \quad \Pi_1^- = - \frac{\pi_+ \Pi_1^+ + \pi_\mu \Pi_1^\mu}{\pi_-} \Rightarrow 
\]
\[
x^- = -i \Psi^a \Gamma_{ab}^- \Psi^b - \frac{i \pi_+ \Psi^a \Gamma_{ab}^+ \Psi^b + \pi_\mu \Pi_1^\mu}{\pi_-},
\]
(3.14)
where
\[
\pi_- = 1 + i \Psi^a A_{ab} \Psi^b, \quad \pi_+ = p_+ + i \Psi^a A_{+ab} \Psi^b, \quad \pi_\mu = p_\mu + i \Psi^a A_{\mu ab} \Psi^b, \\
\Pi_1^- = x^r - i \Psi^a \Gamma_{ab} \Psi^b, \quad \Pi_1^+ = i \Psi^a \Gamma_{ab} \Psi^b, 
\]
(3.15)
Note that \(x^r\) is a polynomial in \(\theta^s\) and \(\theta'^s\). Finally, we substitute the solution into \(C_2\), and solve the resulting equation for \(\pi_+\). The string action (3.8) then takes the gauge-fixed form
\[
S_\alpha = \int_{-r}^r ds d\tau \left( p_\mu \dot{x}^\mu + i p_b \dot{\Psi}^b - \mathcal{H}_\text{ws} \right), \quad \mathcal{H}_\text{ws} = -p_+ (p_\mu, x^\mu, x^\mu, \Psi^a, \Psi^a), 
\]
(3.16)
where
\[
-p_+ = -\pi_+ + i \Psi^a A_{+ab} \Psi^b, \\
p_b^\Psi = \pi_+ \Psi^a \Gamma_{ab} + \pi_\mu \Psi^a \Gamma_{\mu ab} + x^r \Psi^a A_{-ab} + \pi_\mu \Psi^a \Gamma_{\mu ab} + x^\mu \Psi^a A_{\mu ab} + \Psi^a F_{ab}. 
\]
(3.17)
(3.18)
This is the \(TT\) deformed action of the model with the action \(S_0\). It has a nontrivial Poisson structure. To see how one can handle such a structure efficiently see [19, 31, 5].

### 3.2 Scalars, fermions and chiral bosons with arbitrary potential

The formalism developed in the previous subsection can be used to derive a \(TT\) deformed action of a sigma model of \(n_b\) scalar fields \(x^\mu\), and \(n_f^r\) right-moving real fermions \(\theta^r\), and \(n_f^l\) left-moving real fermions \(\theta^l\), and \(n_c^r\) right-moving real chiral bosons \(\varphi^r\), and \(n_c^l\) left-moving real chiral bosons \(\varphi^l\). For Lorentz invariant models the fermions are world-sheet spinors, while the numbers of chiral bosons should be even and half of them should be world-sheet scalars and the other half world-sheet chiral vectors. We combine fermions and chiral bosons again into two fields
\[
\Psi^a_\pm = (\theta^l, \varphi^r), \quad \Psi^a_\pm = \theta^a_\pm, \quad a = 1, \ldots, n_f^\pm, \quad \Psi^a_\pm = \varphi^a_\pm, \quad a = n_f^\pm + 1, \ldots, n_f^\pm + n_c^\pm, 
\]
(3.19)
and consider the following undeformed action
\[
S_0 = \int_{-r}^r ds d\tau \left( -\frac{1}{2} \eta^{ab} \partial_a x^\mu \partial_b x^\nu G_{\mu \nu} + i \Psi^a K_{ij}^a \partial_+ \Psi^j_+ + i \Psi^a K_{rs}^a \partial_- \Psi^s_- - V \right). 
\]
(3.20)
Here
\[
\partial_\pm = \partial_\tau \pm \partial_x, \quad i, j = 1, \ldots, n_f^\pm + n_c^\pm, \quad r, s = n_f^\pm + n_c^\pm + 1, \ldots, n_f^\pm + n_c^\pm + n_f^\pm + n_c^\pm, 
\]
(3.21)
\(K^a_{ij}\) are real if at least one of the indices \(a, b\) is fermionic, and imaginary if both indices are bosonic. Then, \(V\) is an arbitrary potential. They all depend on \(x\) and \(\Psi\) (in a Lorentz-invariant way if necessary). In the first-order formalism this action takes the form
\[
S_0 = \int_{-r}^r ds d\tau \left( p_\mu \dot{x}^\mu + i \Psi^a K_{ij}^a \dot{\Psi}^j_+ + i \Psi^a K_{rs}^a \dot{\Psi}^s_- - \mathcal{H}_0 \right), \\
\mathcal{H}_0 = \mathcal{H}_x + V - i \Psi^a K_{ij}^a \dot{\Psi}^j_+ + i \Psi^a K_{rs}^a \dot{\Psi}^s_+, 
\]
(3.22)
where $H_x$ is given by (2.9). We want $S_0$ to be the light-cone gauge-fixed action for $\alpha = 0$ of a string sigma model on $\mathcal{M}$ with $n_f = n_f^+ + n_f^-$ fermions $\theta^a = (\theta^a_+, \theta^a_-)$, and $n_c = n_c^+ + n_c^-$ chiral bosons $\varphi^a = (\varphi^a_+, \varphi^a_-)$, so that $\Psi^a = (\Psi^a_+, \Psi^a_-)$. It is easy to see that just as for the bosonic case discussed in section 2.3 to reproduce $H_x + V$ we have to use the same target space-time metric

$$G_{tt} = -(1 + \frac{V}{2}), \quad G_{\phi\phi} = 1 - \frac{V}{2}, \quad G_{t\phi} = -\frac{V}{2}. \quad (3.23)$$

Then, the gauge-fixed Hamiltonian becomes

$$H_{ws}\big|_{\alpha = 0} = \pi_- V + \frac{\pi_- H_x}{\pi_-^2 - (\Pi_+^1)^2} - \frac{\pi_+ \Pi_+^0 \Pi_+^1}{\pi_-^2 - (\Pi_+^1)^2} + i \Psi^a A_{ab} \Psi^b \quad (3.25)$$

Comparing (3.25) with $H_0$, we see that we have to impose the conditions

$$\pi_-\big|_{\alpha = 0} = 1 \quad \Rightarrow \quad A_{ab}\big|_{\alpha = 0} = \frac{1}{2}(A_{\phi ab} - A_{t ab}) = 0 \quad \Rightarrow \quad (3.26)$$

$$A_{+ab} = 2A_{\phi ab}, \quad A_{ab} = -\alpha A_{+ab} = -2\alpha A_{\phi ab},$$

$$\Pi_+^1\big|_{\alpha = 0} = 0 \quad \Rightarrow \quad \Gamma_{+ab}\big|_{\alpha = 0} = \frac{1}{2}(\Gamma_{\phi ab} + \Gamma_{t ab}) = 0 \quad \Rightarrow \quad (3.27)$$

$$\Gamma_{-ab} = 2\Gamma_{\phi ab}^t, \quad \Gamma_{ab} = \alpha \Gamma_{-ab} = 2\alpha \Gamma_{\phi ab}^t, \quad A_{+ij} = 2A_{\phi ij} = -K_{ij}^+ \quad A_{+rs} = 2A_{\phi rs} = K_{rs}^+, \quad A_{\phi ir} = A_{\phi ri} = 0. \quad (3.28)$$

Next we need to equate $p_{\Psi}^b \dot{\Psi}^b\big|_{\alpha = 0}$ with the kinetic term in (3.22). We get

$$p_{\Psi}^b \dot{\Psi}^b\big|_{\alpha = 0} = \left(\Psi^0 \Gamma_0^- + \pi_+ \Psi^a \Gamma_{-}^a + x^{[a} \Psi^a A_{[ab]} + \Psi^{ab} F_{ab}\right) \dot{\Psi}^b = \Psi^i K_{ij}^+ \dot{\Psi}_j^+ + \Psi^- K_{rs}^- \dot{\Psi}_r^-,$$

which leads to the solution

$$\Gamma_{ij}^- = 2\Gamma_{ij}^+ = K_{ij}^+, \quad \Gamma_{rs}^- = 2\Gamma_{rs}^+ = K_{rs}^+, \quad \Gamma_{ir}^\phi = \Gamma_{ri}^\phi = 0, \quad (3.29)$$

$$\Gamma_{ab}^\mu = A_{\mu ab} = F_{ab} = 0. \quad (3.30)$$

Calculating the constraint $C_2$ for an arbitrary $\alpha$ with $\Pi_1$ given by (3.14), one gets

$$\frac{C_2}{2} = \alpha(1 + \alpha V)(1 - \frac{\Pi_+^1}{\pi_-^2})^2$$

$$+ \pi_- \left(1 + 2\alpha V - 2\alpha(1 + \alpha V) \frac{\pi_+ \Pi_+^0 \Pi_+^1}{\pi_-^2} - (1 + 2\alpha V) \frac{(\Pi_+^1)^2}{\pi_-^2}\right) \pi_+ \quad (3.31)$$

$$+ H_x + \pi_-^2 V - \alpha(1 + \alpha V) \frac{(\pi_+ \Pi_+^0)^2}{\pi_-^2} - (1 + 2\alpha V) \frac{\pi_+ \Pi_+^0 \Pi_+^1}{\pi_-^2} - V(\Pi_+^1)^2. \quad 17$$
Solving the constraint for $\pi_+$, one gets
\begin{equation}
-\pi_+ = \frac{\pi_-}{\alpha} - \frac{\pi_-}{2\alpha} - \frac{p_\mu x^\mu \Pi_+^\dagger}{\pi_- - (\Pi_+^\dagger)^2} - \frac{\pi_-}{2\alpha} \sqrt{1 - 4\tilde{\beta} \mathcal{H}_x + 4\tilde{\beta}^2 (p_\mu x^\mu)^2},
\end{equation}
where
\begin{equation}
\tilde{\alpha} = \alpha (1 + \alpha \mathcal{V}), \quad \tilde{\beta} = \frac{\tilde{\alpha}}{\pi_- - (\Pi_+^\dagger)^2},
\end{equation}
and due to (3.26-3.30)
\begin{equation}
\pi_- = 1 - i \alpha \Psi^a A_{+ab} \Psi^b, \quad \Pi_+^\dagger = i \alpha \Psi^a \Gamma_{ab} \Psi^b, \quad \pi_\mu = p_\mu, \quad \Pi_\mu^\dagger = x^\mu.
\end{equation}
The gauge-fixed Hamiltonian is then given by
\begin{equation}
\mathcal{H}_{ws} = -\pi_+ + i \Psi^a A_{+ab} \Psi^b,
\end{equation}
while the gauge-fixed Lagrangian in the first-order form is
\begin{equation}
\mathcal{L}_{ws} = p_\mu \dot{x}^\mu + i p_b \dot{\Psi}^b - \mathcal{H}_{ws},
\end{equation}
where
\begin{equation}
p_b = \pi_- \Psi^a \Gamma_{ab}^- + \alpha \Psi^a \Gamma_{ab}^- \pi_+ - \alpha x' \Psi^a A_{+ab},
\end{equation}
\begin{equation}
x' = - \frac{p_\mu x'^\mu}{\pi_-} - i \Psi^a \Gamma_{ab}^- \Psi^b - \frac{i \alpha \Psi^a \Gamma_{ab}^- \Psi^b}{\pi_-} \pi_+.
\end{equation}
The equations of motion for $p_\mu$ take the form
\begin{equation}
\dot{x}^\mu + Q x'^\mu = R \frac{p^\mu - 2\tilde{\beta} p_\mu x'^\mu x^\mu}{\sqrt{1 - 4\tilde{\beta} \mathcal{H}_x + 4\tilde{\beta}^2 (p_\mu x^\mu)^2}},
\end{equation}
where
\begin{equation}
R = \frac{\tilde{\beta}}{\alpha} \pi_- (1 + i \alpha \Psi^a \Gamma_{ab}^- \dot{\Psi}^b + i \alpha \frac{\Pi_+^\dagger}{\pi_-} \Psi^a A_{+ab} \dot{\Psi}^b),
\end{equation}
\begin{equation}
Q = \frac{i \alpha}{\pi_-} \Psi^a A_{+ab} \dot{\Psi}^b + \frac{\Pi_+^\dagger}{\pi_-} R,
\end{equation}
which can be used to eliminate $p_\mu$ from the Lagrangian (3.36).

To write the resulting Lagrangian in a compact form let us introduce the following notations
\begin{equation}
K_\gamma^+ \equiv \Psi^\gamma_+ \Gamma_{ij}^+ \partial_\gamma \Psi^j_-, \quad K^-_\gamma \equiv \Psi^\gamma_- \Gamma_{rs}^- \partial_\gamma \Psi^r_+,
\end{equation}
in terms of which we have
\begin{equation}
\Psi^a \Gamma^-_{ab} \partial_\gamma \Psi^b = K^+ - K^-_\gamma, \quad \Psi^a A_{+ab} \partial_\gamma \Psi^b = -K^+ + K^-_\gamma.
\end{equation}
We also use light-cone coordinates on the world-sheet

\[ \partial_{\pm} x^\mu = \partial_{\tau} x^\mu \pm \partial_{\sigma} x^\mu. \]  

(3.44)

Then, the gauge-fixed Lagrangian takes the form

\[ L_{ws} = iK_+^+ + iK_-^- + \alpha(K_+^- K_+^- - K_-^- K_+^-) \]

\[ \frac{1}{2(1 + \alpha V)} + \frac{1}{2\alpha} + \frac{1}{2\alpha} \sqrt{\Lambda}, \]

(3.45)

where

\[ \Lambda = (1 + i\alpha(K_+^+ + K_-^-)) + \alpha^2(K_+^- K_+^- - K_-^- K_+^-) \]

\[ + 2\bar{\alpha}\partial_{-}x\partial_{+}x(1 + i\alpha(K_+^+ + K_-^-) - \alpha^2(K_+^- K_+^- + K_-^- K_+^-)) \]

\[ + \bar{\alpha}^2 (\partial_{-}x\partial_{+}x)^2 - (\partial_{-}x)^2(\partial_{+}x)^2 \]

\[ - 2i\alpha\bar{\alpha}(\partial_{-}x)^2K_+^- + K_+^+(\partial_{+}x)^2 \]

\[ + 2\alpha^2\bar{\alpha}(\partial_{-}x)^2K_-^+ K_+^- + K_-^- (\partial_{+}x)^2, \]

(3.46)

and

\[ \partial_{-}x\partial_{+}x \equiv G_{\mu\nu}\partial_{-}\partial_{\mu}\partial_{+}\partial_{\nu}, \quad (\partial_{\pm}x)^2 \equiv G_{\mu\nu}\partial_{\mu}\partial_{\pm}\partial_{\nu}. \]

(3.47)

In the derivation of (3.45) we have never needed Lorentz invariance of the undeformed model. Let us assume that \( V, K_+^+ \) and \( K_-^- \) are Lorentz scalars while \( K_+^- \) transforms as \( (\partial_{-}x)^2 \) and \( K_-^+ \) as \( (\partial_{+}x)^2 \). Then, one sees from (3.45) that this Lagrangian is Lorentz invariant. It is also clear that if \( K_{+ab} \) are Lorentz scalars, then \( \theta^+_{-} \) and \( \theta^-_{+} \) are the world-sheet right and left Majorana-Weyl spinors, respectively. It is certainly well-known that in the Green-Schwarz sigma models, as a result of the light-cone gauge fixing, the world-sheet fermion scalars become the world-sheet spinors. To understand what happens with the chiral bosons let us consider the field-independent part of \( K_{+ab} \) which is necessarily skew-symmetric for bosonic indices. Then, by an orthogonal transformation it can be brought to the standard symplectic form

\[ iK_{+ab}^\pm = \begin{pmatrix} +1 & \text{if} & a = b + n_+^+/2 \\ -1 & \text{if} & b = a + n_-^+/2 \\ 0 & \text{otherwise} \end{pmatrix}. \]

(3.48)

Thus, up to quadratic order in the chiral bosons the bosonic parts of \( K_+^+ \) and \( K_-^- \) take the form

\[ iK_+^+ = \sum_{i=1}^{n_+^+/2} (A_i^+ \partial_+ \varphi_i^+ - \varphi_i^+ \partial_+ A_i^+) , \quad A_i^+ = \varphi_i^+ + n_+^+/2, \]

\[ iK_-^- = \sum_{i=1}^{n_-^+/2} (A_i^- \partial_- \varphi_i^- - \varphi_i^- \partial_- A_i^-) , \quad A_i^- = \varphi_i^- + n_-^+/2, \]

(3.49)

\[ ^{12}\text{The symmetric part of } K_{+ab} \text{ always describes an interaction of the chiral bosons between themselves and other fields.} \]
which shows that we can take $\varphi^i_\pm$ and $A^i_\pm$, $i = 1, \ldots, n^\pm_c/2$ to be world-sheet scalars and vectors, respectively.

Expanding the Lagrangian (3.45), one gets

$$L_{ws} = \frac{1}{2} \partial_- x \partial_+ x + i K^+_+ + i K^-_- - V + \text{det}(T^\beta_\gamma) \alpha + O(\alpha^2) , \quad (3.50)$$

where

$$T^\beta_\gamma = \frac{\partial L_{ws}}{\partial \partial_\beta_\gamma x^\mu} \partial_\gamma \Psi_a + \frac{\partial L_{ws}}{\partial \partial_\beta_\gamma x^\mu} \partial_\gamma \Psi_a - \delta^\beta_\gamma L_{ws} , \quad (3.51)$$

is the canonical stress-energy tensor of the undeformed model (2.41) which for the fermion model does not coincide with the covariant one. This agrees with the consideration of $TT$ deformed supersymmetric systems of free massless bosons and fermions in [16, 17]. Explicitly, one gets

$$T^-_- = V - i K^+_+ , \quad T^+_+ = V - i K^-_- ,$$

$$T^+_+ = \frac{1}{2} (\partial_- x)^2 + i K^+_+ , \quad T^-_- = \frac{1}{2} (\partial_+ x)^2 + i K^-_- , \quad (3.52)$$

which is not symmetric off-shell (we lower indices with $\eta_+ = \eta_- = 1/2$). For Lorentz invariant models the canonical and covariant tensors differ by terms vanishing on-shell, therefore there is an $\alpha$-dependent nonlinear field redefinition which transforms $T_{can}$ to $T_{cov}^{13}$. This redefinition can be found as a series in $\alpha$ but it would ruin the nice structure of the Lagrangian (3.45). It is also unclear if this redefinition can be implemented in quantum theory.

If there are no bosonic $x^\mu$ fields the Lagrangian (3.45) simplifies drastically, and one gets

$$L_{AAF} = \frac{i K^+_+ + i K^-_- + \alpha (K^+_+ K^-_- - K^+_+ K^-_-) - V}{1 + \alpha V} . \quad (3.53)$$

If in addition there are no chiral bosons then the expansion in powers of $V$ terminates, and one gets a generalisation of the AAF model [19] to any number of fermions and any potential. If the undeformed model is integrable then the AAF model is integrable too. A $T\bar{T}$ deformation of the massive Thirring model was discussed in [36]. However, the covariant stress-energy tensor was used there, and by this reason the Lagrangian obtained in [36] differs from (3.53), and has a much more complicated structure.

4 Conclusions

In this paper we have developed the light-cone gauge approach to the $T\bar{T}$ deformed two-dimensional models which may or may not be Lorentz invariant. To demonstrate the power of this approach we have found the $T\bar{T}$ deformed Lagrangian (3.45) of a very

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\[13\] I thank Alessandro Sfondrini for a discussion of this point.
general system of scalars, fermions and chiral bosons with an arbitrary potential. Let us mention some open questions and generalisations of the approach.

It is obvious that any two gauges of a string sigma model are related to each other by a field-dependent transformation of coordinates which is just a consequence of the reparametrisation invariance of the model. To our knowledge a change of coordinates relating uniform light-cone gauges with different gauge parameters is not known explicitly. It would be interesting to find it, and compare with the transformation in [37]. We suspect that the canonical Noether stress-energy tensor should be used for models with fermions in their formula.

It was argued in [34, 38] that a $\bar{T}T$ deformed model can be obtained if one couples an undeformed Lorentz invariant theory to the flat space Jackiw-Teitelboim (JT) gravity. It would be of interest to understand the relation of our approach to the JT one. In particular, the light-cone gauge approach is democratic – it is a matter of convenience which model is called undeformed and which one its deformation. Moreover, Lorentz invariance is not important at all. On the other hand, the JT approach distinguishes an undeformed model and, as far as we can see, requires Lorentz invariance.

Obviously, if an undeformed model is integrable the $\bar{T}T$ deformed model is integrable too. The simplest way to find a $\bar{T}T$ deformed Lax pair is to start with a reparametrisation invariant Lax pair for the underlying string sigma model, impose the light-cone gauge, express unphysical fields and two-dimensional metric in terms of physical fields, and substitute the expressions into the Lax pair. That was how Lax pairs for the bosonic part of the light-cone AdS$_5 \times$ S$^5$ superstring and the AAF model were found [39, 19]. If a Lax pair of the undeformed model is known then there should exist a way to get from it a reparametrisation invariant Lax pair.

As immediately follows from the light-cone gauge approach if an undeformed model is supersymmetric the $\bar{T}T$ deformed model is supersymmetric too but the supersymmetry may be realised nonlinearly, see [16, 17]. However, in the case of supersymmetric models it might be more natural to use the NSR strings coupled to two-dimensional supergravity. The super-Virasoro constraints would require to impose additional fermionic gauge conditions which would lead to extra gauge parameters. This may allow one to consider more general multi-parameter deformations which involve supercharges.

In the case of integrable models the $\bar{T}T$ deformation belongs to a class of more general higher-spin deformations introduced in [2]. It is natural to expect that they can be described in the light-cone gauge approach if one considers strings coupled to W gravity, see [40] for a review.

The light-cone gauge approach seems to be the most natural way to study $\bar{T}T$ type deformations which break Lorentz invariance of an undeformed model, the simplest example being the $\bar{J}T$ deformation [11], and its higher-spin generalisation recently discussed in [22]. In this case the undeformed model possesses an additional conserved $U(1)$ current $J$ which from the string sigma model point of view would be a consequence of an additional space isometry in one of the transverse directions. It seems one way
to get the $J\bar{T}$ deformation is to include the third isometry direction and its conjugate momentum in the light-cone gauge definition. This would produce a multi-parameter gauge condition, and the corresponding gauge-fixed model would be a multi-parameter deformation by $TT^\dagger$, $J\bar{T}$ and $\bar{J}T$ operators. Another way would be to couple the $U(1)$ current to a non-dynamical gauge field. 

In this paper we assumed that $G_{t\mu} = G_{\phi\mu} = 0$ which seems to be a necessary condition for Lorentz invariance. If $G_{t\mu}$ and $G_{\phi\mu}$ do not vanish then Lorentz invariance in general is broken. It might be interesting to analyse what kind of non-Lorentz invariant models one can get in this case, in particular if one can reproduce the results of [33].

The consideration in this paper was purely classical. Obviously, a $TT^\dagger$ deformed model is not renormalisable, and quantisation of such a model leads to infinitely many quantum versions corresponding to one and the same classical model. The quantum $TT^\dagger$ deformed model by definition is the one whose spectrum satisfies the inhomogeneous inviscid Burgers equation. It is unclear whether a $TT^\dagger$ quantisation scheme always exists. Even if the model is integrable this may not be the most natural quantisation. For example, the two-particle S-matrix of the AAF model [19] was calculated in [20] in a particular quantisation scheme, and it was shown to be factorisable in [21]. The S-matrix, however, is not equal to the $TT^\dagger$ CDD factor, and leads to a nontrivial spectrum. It would be interesting to find the $TT^\dagger$ quantisation scheme for the AAF model, and to understand in general for which models such a scheme exists.

Assuming the quantum $TT^\dagger$ deformed model exists, the next question is how to compute its form factors. It is a very hard problem, and it seems the only way to solve it is to understand in full detail the relation between two light-cone gauges with different gauge parameters. In particular it might be necessary to understand how the change of coordinates discussed above is implemented on the quantum level.

Acknowledgements

I would like to thank Tristan McLoughlin and Alessandro Sfondrini for fruitful discussions, and Alessandro Sfondrini for useful comments on the manuscript.

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