Classical Trace Anomaly

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PACS number: 04.20-s
Keywords: Higher order gravity; Lovelock & Einstein tensors; Weyl anomaly.

Abstract

We seek an analogy of the mathematical form of the alternative form of Einstein’s field equations for Lovelock’s field equations. We find that the price for this analogy is to accept the existence of the trace anomaly of the energy-momentum tensor even in classical treatments. As an example, we take this analogy to any generic second order Lagrangian and exactly derive the trace anomaly relation suggested by Duff. This indicates that an intrinsic reason for the existence of such a relation should perhaps be, classically, somehow related to the covariance of the form of Einstein’s equations.

1 Introduction

Recently there has been interest in considering gravity in higher dimensional space-time. In this context, one may use a consistent theory of gravity with a more general action, e.g. the Einstein–Hilbert action plus higher order terms.† Especially, much interest has been in the Lovelock gravity, as the most general second order Lagrangian which still yields the field equations as second order equations, and its applications in cosmology, see e.g. Refs. [3] and references therein. This particular combination of higher order terms are ghosts–free. Nevertheless, in order to obtain the observed accelerated cosmic expansion at the present epoch, other higher order modifications of gravity have also recently been attracted, among which a particular case of $\frac{1}{R}$ term modification has been shown to lead to instabilities. Though, Ref. [8] claims that further modification of this modified gravity by $R^2$ or other higher terms may resolve the instabilities, or perhaps make them avoidable.

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* This work was partially supported by a grant from the MSRT/Iran.
† See e.g. Ref. [1] for a brief review of the history of inclusion of scalar Lagrangians quadratic in the curvature tensor.
In our previous work\textsuperscript{[1]}, we showed that the analogy of the Einstein tensor (i.e.,
the splitting feature of it into two parts, the Ricci tensor and the term proportional
to the curvature scalar, with the trace relation between them, which is also
a common feature of each homogeneous term in the Lovelock tensor\textsuperscript{‡}) can be gen-
eralized, via a generalized trace operator (which we defined and denoted by \textit{Trace}
for homogeneous tensors\textsuperscript{§}), to any inhomogeneous Euler–Lagrange expression if it
can be spanned linearly in terms of homogeneous tensors.

As an example, we demonstrated this analogy of the mathematical form of the
Einstein tensor for the Lovelock tensor, and showed that it can be written as

\[ G_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R}, \quad (1.5) \]

\[ G^{(n)}_{\alpha\beta} = - \sum_{0 < n < \frac{D}{2}} \frac{1}{2n + 1} c_n g_{\alpha\mu} \delta_{\beta_1\ldots\beta_{2n}} R_{\alpha_1\alpha_2 \ldots \alpha_{2n-1} \alpha_{2n}} \delta_{\beta_1\ldots\beta_{2n}} \equiv \sum_{0 < n < \frac{D}{2}} c_n G^{(n)}_{\alpha\beta}, \quad (1.1) \]

and where the cosmological term has been neglected, \( G^{(1)}_{\alpha\beta} = G_{\alpha\beta} \) the Einstein tensor,
\( \delta_{\beta_1\ldots\beta_p} \) is the generalized Kronecker delta symbol, which is identically zero if \( p > D \), and
the maximum value of \( n \) is related to the dimension of space-time by

\[ n_{\text{max}} = \begin{cases} \frac{D}{2} - 1 & \text{even } D \\ \frac{D-1}{2} & \text{odd } D. \end{cases} \quad (1.2) \]

Also, our conventions are a metric of signature +2, \( R^\mu_{\nu\alpha\beta} = -\Gamma^\mu_{\nu\alpha, \beta} + \cdots \), \( R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu} \),
and the homogeneity is taken with respect to the metric and its derivatives with the homo-
genicity degree number (HDN) conventions of \([+1, g^{\mu\nu}] \) and \([+1, g^{\mu\nu, \alpha}] \) (see Ref. \[1\] for details).
Hence, one can relate the orders \( n \) in any Lagrangian, as in \( L^{(n)} \), that represents its HDN.

\[ \text{Trace}^{[h]} A^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_M} := \begin{cases} \frac{1}{h - \frac{N}{2} + \frac{M}{2}} \text{trace}^{[h]} A^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_M} & \text{when } h - \frac{N}{2} + \frac{M}{2} \neq 0 \\ \text{trace}^{[h]} A^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_M} & \text{when } h - \frac{N}{2} + \frac{M}{2} = 0. \end{cases} \quad (1.3) \]

Hence, for example, when \( h \neq 0 \) and \( h' \neq 0 \), one gets

\[ \text{Trace}^{[h']} C^{[h]} A_{\mu\nu} = \begin{cases} \frac{h+1}{h' + h + 1} [h] C \text{Trace}^{[h]} A_{\mu\nu} & \text{for } h \neq -1 \\ \frac{1}{h'} [h'] C \text{Trace}^{[h]} A_{\mu\nu} & \text{for } h = -1. \end{cases} \quad (1.4) \]
with the relation
\[
\text{Trace } \mathcal{R}_{\alpha\beta} = \mathcal{R} ,
\] (1.6)
where
\[
\mathcal{R}_{\alpha\beta} \equiv \sum_{0<n<D} c_n R^{(n)}_{\alpha\beta} , \quad \mathcal{R} = \kappa^2 \mathcal{L} \equiv \sum_{0<n<D} c_n R^{(n)} = \kappa^2 \sum_{0<n<D} c_n L^{(n)} ,
\] (1.7)
where \( \mathcal{L} \) is the Lovelock Lagrangian, and \( R^{(n)}_{\alpha\beta} \) and \( R^{(n)} \) are defined as
\[
R^{(n)}_{\alpha\beta} \equiv \frac{n}{2n} \delta^{\alpha_1\alpha_2...\alpha_{2n}}_{\beta_1\beta_2...\beta_{2n}} R_{\alpha_1\alpha_2\beta_1\beta_2} R_{\alpha_3\alpha_4\beta_3\beta_4} ... R_{\alpha_{2n-1}\alpha_{2n}\beta_{2n-1}\beta_{2n}}
\] (1.8)
and
\[
R^{(n)} \equiv \frac{1}{2n} \delta^{\alpha_1...\alpha_{2n}}_{\beta_1...\beta_{2n}} R_{\alpha_1\alpha_2\beta_1\beta_2} ... R_{\alpha_{2n-1}\alpha_{2n}\beta_{2n-1}\beta_{2n}} ,
\] (1.9)
where also \( R^{(1)}_{\alpha\beta} = R_{\alpha\beta} \), \( R^{(1)} = R \), \( \kappa^2 = \frac{16\pi G}{c^4} \), \( c_1 \equiv 1 \) and the other \( c_n \) constants are of the order of Planck’s length to the power \( 2(n-1) \).

Hence, we stated the Lovelock gravitational field equations in the form of
\[
\mathcal{G}_{\alpha\beta} = \frac{1}{2} \kappa^2 T_{\alpha\beta} ,
\] (1.10)
and classified the Lovelock tensor/Lagrangian as a generalized Einstein tensor/Lagrangian, and called \( \mathcal{R}_{\alpha\beta} \) and \( \mathcal{R} \) generalized Ricci tensor and generalized curvature scalar, respectively.

Now in this work, still motivated by the principle of general covariance, we proceed further the analogy and also enforce the mathematical form of the alternative form of Einstein’s field equations (as a covariant form for gravitational field equations) for the relevant alternative form of Lovelock's field equations. From this, we find that the price for this analogy is to accept the existence of the trace anomaly of the energy-momentum tensor even in classical treatments. Then, as an example, we take this analogy to any generic second order Lagrangian and exactly derive the trace anomaly relation suggested by Duff.[10]

Besides the very well known classical successes of Einstein’s theory, the above analogy is a part of a programme to impose an analogy of Einstein’s gravity on Lovelock’s gravity,[11] wherein the latter is then considered as a generalized Einstein’s gravity. Actually, because of the non-linearity of the field equations, it is very difficult to find out non-trivial exact analytical solutions of Einstein’s equation with higher curvature terms. So, our ultimate proposal has been to construct,
and hence to achieve, a generalized counterpart for each essential term used in Einstein’s gravity, especially, the metric. This, we believe, would give a better view on higher order gravities, and would also let straightway to apply the results of Einstein’s theory to Lovelock’s theory.

In Section 2, we will review the alternative form of the Einstein field equations; and in Section 3, we will perform the same analogy in order to get the alternative form of the Lovelock field equations. In Section 4, we will perform some discussions mainly on an idea that the geometrical curvature inducing matter. Finally, in Section 5, as an example, we will take this analogy to any generic second order Lagrangian and derive Duff’s suggested\(^{[10]}\) relation for the trace anomaly.

## 2 Alternative Form of Einstein’s Field Equations

With an amount of manipulation one can put Einstein’s field equations, \(G_{\alpha\beta} = \frac{1}{2}\kappa^2 T_{\alpha\beta}\), into an alternative form. Contracting the indices (i.e. calculating the standard trace or the Trace, because the HDN of the Einstein tensor is zero),\(^{[1]}\) one gets

\[
\text{Trace} G_{\alpha\beta} = \left(1 - \frac{D}{2}\right)R = \frac{1}{2}\kappa^2 T .
\]  \hspace{1cm} (2.1)

Obtaining \(R\) from this equation and substituting it back into Einstein’s equations, one gets the equivalent forms of Einstein’s field equations,

\[
R_{\alpha\beta} = \frac{1}{2}\kappa^2 \left(T_{\alpha\beta} - \frac{1}{D-2}g_{\alpha\beta} T\right) \equiv \frac{1}{2}\kappa^2 S_{\alpha\beta} \quad \text{when } D \neq 2 ,
\]  \hspace{1cm} (2.2)

where, for convenience, we refer to \(S_{\alpha\beta}\) as Source tensor. The word form has been applied for the alternative gravitational field equations because of the pedagogical reasons, mainly, in order to emphasize that \(R_{\mu\nu}\) is not the gravitation tensor and \(S_{\mu\nu}\) is not the source term since otherwise they do not satisfy the requirement of vanishing covariant derivatives.

In a vacuum (where there is no matter, i.e. when \(T_{\mu\nu} = 0\)), one gets \(R_{\mu\nu} = 0\). But, in four or more dimensions, this does not mean flat space-time, and gravitational fields exist in empty space. That is, in the general relativity, the space-time has itself some essence independent of matter, contrary to the strong version of the Mach idea.\(^{[12]}\) The link to this issue, in our opinion, could be the relation between \(T\) and \(R\) as in equation (2.1), and the basic concept of matter. This is somehow a procedure that it may indicate a more compatibility between the general relativity and the strong version of the Mach idea.\(^{[13]}\)

From the definition of the Source tensor, it is evident that vacuum is also equivalent to the case when \(S_{\mu\nu}\) vanishes, as equation (2.2) then yields \(R_{\mu\nu} = 0\).
3 Alternative Form of Lovelock’s Field Equations

We will follow the same route of the previous section to get an alternative form of Lovelock’s field equations. However, for this case with either the trace or the Trace, one cannot easily substitute for $\Re$ in terms of the trace or the Trace of $T_{\alpha\beta}$, as to be shown in the following. Therefore, its alternative form will differ (slightly) from Einstein’s one.

Taking the Trace of (1.10), and using (1.6) and (1.3), one gets

$$\text{Trace} G_{\alpha\beta} = \sum_{0<n<D/2} \left(1 - \frac{D}{2n} \right) c_n R^{(n)} = \frac{1}{2} \kappa^2 \text{Trace} T_{\alpha\beta} .$$

(3.1)

Obviously, one cannot extract $\sum c_n R^{(n)}$, as $\Re$, out of the summation in the above equation.

By setting $\text{Trace} T_{\alpha\beta} \equiv T$, from equation (3.1) we have

$$T = -\kappa^{-2} \sum_{0<n<D/2} \left( \frac{D}{n} - 2 \right) c_n R^{(n)} .$$

(3.2)

A similar form of equation (2.2) for Lovelock’s field equations can now be found by the following procedure.

By adding and subtracting a term in equation (3.1) we obtain

$$\sum_{0<n<D/2} \left(1 - \frac{D}{2n} \right) c_n R^{(n)} + \sum_{0<n<D/2} \left( \frac{D}{2n} - \frac{D}{2n} \right) c_n R^{(n)} = \frac{1}{2} \kappa^2 T .$$

Substituting for the first term from equation (1.7) and solving for $\Re$ when $D \neq 2$, it yields

$$\Re = -\frac{\kappa^2}{D-2} T + \frac{D}{D-2} \sum_{0<n<D/2} \frac{n-1}{n} c_n R^{(n)} .$$

(3.3)

By putting this into the field equations (1.10), we finally get

$$\Re_{\alpha\beta} - \frac{D}{2(D-2)} g_{\alpha\beta} \sum_{0<n<D/2} \frac{n-1}{n} c_n R^{(n)} = \frac{1}{2} \kappa^2 S_{\alpha\beta} \quad \text{when } D \neq 2 .$$

(3.4)

This is an analogous form of equation (2.2) that can be achieved in this stage. However, by continuing this procedure below, one can write equation (3.4) in a
manifestly more analogous form with equation (2.2). At present, one should notice that in the second term of the left hand side of the above equation, there is no contribution for the case $n = 1$, i.e. the associated Einstein’s part. Comparing the appearance of equation (3.4) (or even equation (1.10)) with Einstein’s equation, it is likely (as an analogous demand) that its second term on the left hand side is responsible for changes in the trace of energy-momentum tensor.

Our purpose is to eliminate the case of $n = 2$ from the second term in the left hand side of equation (3.4) by the same procedure that we performed for equation (1.10) which then led to equation (3.4). We do this in the following manipulation.

Using the following relation
\[
\frac{n - 1}{2n} = \frac{(n - 1)^2}{n^2} - \frac{(n - 1)(n - 2)}{2n^2}
\]
in equation (3.4), we have
\[
\Re_{\alpha\beta} - \frac{D}{D - 2} g_{\alpha\beta} \sum_{0 < n < \frac{D}{2}} \left[ \frac{(n - 1)^2}{n^2} - \frac{(n - 1)(n - 2)}{2n^2} \right] c_n R^{(n)} = \frac{1}{2} \kappa^2 S_{\alpha\beta} .
\]

Taking the Trace of (3.4), and using equations (1.6) and (1.4), we get\^[1]\]
\[
\sum_{0 < n < \frac{D}{2}} \left( 1 - \frac{D^2}{D - 2} \times \frac{n - 1}{2n^2} \right) c_n R^{(n)} = \frac{1}{2} \kappa^2 S ,
\]

\* Almost the same thing is obtained if one considers $\text{trace} \mathcal{G}_{\alpha\beta}$ instead. Taking the trace of equation (1.10), and using $\frac{1}{D} \text{trace} R_{\alpha\beta}^{(n)} = R^{(n)}$, one has
\[
\text{trace} \mathcal{G}_{\alpha\beta} = \sum_{0 < n < \frac{D}{2}} (n - \frac{D}{2}) c_n R^{(n)} = \frac{1}{2} \kappa^2 \text{trace} T_{\alpha\beta} ,
\]
and then, one gets
\[
\Re_{\alpha\beta} - \frac{1}{D - 2} g_{\alpha\beta} \sum_{0 < n < \frac{D}{2}} (n - 1) c_n R^{(n)} = \frac{1}{2} \kappa^2 (T_{\alpha\beta} - \frac{1}{D - 2} g_{\alpha\beta} \text{trace} T_{\alpha\beta}) ,
\]
where its difference with equation (3.4) is due to the difference between $\text{trace} T_{\alpha\beta} \equiv T_{\mu}^{\mu}$ and $\text{Trace} T_{\alpha\beta} \equiv T$, when used for dealing with Lovelock’s field equations (i.e., $\mathcal{G}_{\alpha\beta}$ is an inhomogeneous tensor).
where $S \equiv \text{Trace } S_{\alpha\beta}$. \(^\dagger\)

For the first term of equation (3.7), we substitute from equation (3.3), while using relation (3.5), and for its second term we employ

\[
\frac{n - 1}{n^2} = \frac{(n - 1)^2}{n^2} - \frac{(n - 1)(n - 2)}{n^2}.
\] (3.9)

Then, after rearrangements of equation (3.7), we get

\[
-\frac{D}{D - 2} \sum_{0 < n < \frac{D}{2}} \frac{(n - 1)^2}{n^2} c_n R^{(n)} = \frac{\kappa^2}{D - 4} S + \frac{2\kappa^2}{(D - 2)(D - 4)} T - \frac{D}{D - 4} \sum_{0 < n < \frac{D}{2}} \frac{(n - 1)(n - 2)}{n^2} c_n R^{(n)} ,
\] (3.10)

when $D \neq 4$.

By substituting equation (3.10) for the second term of equation (3.6), we finally find that

\[
\Re_{\alpha\beta} - \frac{D^2}{2(D - 2)(D - 4)} g_{\alpha\beta} \sum_{0 < n < \frac{D}{2}} \frac{(n - 1)(n - 2)}{n^2} c_n R^{(n)}
\]

\[
= \frac{1}{2} \kappa^2 S_{\alpha\beta} - \frac{\kappa^2}{D - 4} g_{\alpha\beta} S - \frac{2\kappa^2}{(D - 2)(D - 4)} g_{\alpha\beta} T \equiv \frac{1}{2} \kappa^2 S_{\alpha\beta}^{(2)} ,
\] (3.11)

where, by using the definition of $S_{\alpha\beta}$, equation (2.2), we have

\[
S_{\alpha\beta}^{(2)} = T_{\alpha\beta} - \frac{1}{D - 2} g_{\alpha\beta} \left( T + T^{(2)} \right) ,
\] (3.12)

where

\[
T^{(2)} \equiv \frac{1}{D - 4} \left[ 2(D - 2) S + 4 T \right] .
\] (3.13)

Or equivalently, by substituting for $S$ and $T$ in the above equation from equations

\(^\dagger\) Note that, if one uses this $S$ and the definition of $S_{\alpha\beta}$, equation (2.2), then by the aid of equation (3.2), one gets \[^1\]

\[
S = T - \frac{-\kappa^{-2}}{D - 2} \sum_{0 < n < \frac{D}{2}} \left( \frac{D}{n} - 2 \right) c_n \text{Trace } \left( g_{\alpha\beta} R^{(n)} \right)
\]

\[
= -\kappa^{-2} \sum_{0 < n < \frac{D}{2}} \left( \frac{D}{n} - 2 \right) \left[ 1 - \frac{D}{n(D - 2)} \right] c_n R^{(n)} ,
\] (3.8)

which is obviously identical to equation (3.7).
(3.8) and (3.2) respectively, we obtain

\[ T^{(2)} = -\frac{2\kappa^{-2}D}{D-4} \sum_{0<n<D/2} \frac{n-1}{n} \left( \frac{D}{n} - 2 \right) c_n R^{(n)}. \]  

(3.14)

In this notation, \( S^{(2)}_{\alpha\beta} \) and \( T^{(2)} \) give our desired result up to the \( n_{\text{max}} = 2 \). Hence, we have also chosen \( S^{(1)}_{\alpha\beta} = S_{\alpha\beta} \), but \( T^{(1)} \equiv 0 \).

Equation (3.11) is similar to equation (3.4), however, in the second term of the left hand side there is no contribution for the cases of \( n = 1 \) and \( n = 2 \). If we continue this iteration to all orders of \( n \) we eventually get

\[ \mathcal{R}_{\alpha\beta} = \frac{1}{2} \kappa^2 S^{(n_{\text{max}})}_{\alpha\beta}, \]  

(3.15)

where

\[ S^{(n_{\text{max}})}_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} \mathcal{T} \]  

(3.16)

with

\[ \mathcal{T} \equiv T + T^{(n_{\text{max}})} = -\kappa^{-2} \left( D - 2 \right) \mathcal{R}, \]  

(3.17)

and

\[ T^{(n_{\text{max}})} = -\kappa^{-2} D \sum_{0<n<D/2} \frac{n-1}{n} c_n R^{(n)}. \]  

(3.18)

Of course, one can also go from equation (3.15) back to equation (1.10), so (1.10) and (3.15) should be regarded as entirely equivalent forms of Lovelock’s field equations. Equation (3.15) is the closest similar form of equation (2.2) that we have reached by iteration of the above procedure. But after all, it is clear that we have considered the second term in equation (3.4) as equivalent to the Trace of an energy-momentum tensor which we have tried to justify by this procedure. Indeed, this is (almost) the same as the case of the alternative form of Einstein’s gravitational theory, and the same argument as was used there to clarify the word form is also applied here. Although here, the definition of each of \( S^{(n)}_{\alpha\beta} \)'s, except \( S^{(n_{\text{max}})}_{\alpha\beta} \), is not unique.

We know that \( T \) is the Trace of the energy-momentum tensor, an entity independent of the gravitational characters, whose relation to the geometry is through the gravitational field equations as given by field equation (3.2). But, what is \( T^{(n_{\text{max}})} \)? Does it also have an entity independent of the gravitational characters? Or is it a symbol defined by equation (3.18)?
We obtained $T^{(n_{\text{max}})}$ in the same way that we got $T^{(2)}$, where the latter is defined by equation (3.13) with respect to $T$ and $S$, i.e. things that depend on $T_{\alpha\beta}$. So $T^{(n_{\text{max}})}$ should also have the same kind of relation to $T_{\alpha\beta}$, independently of the gravitational characters. But, we still cannot know its exact relation. For example, for $T^{(2)}$, one gets into difficulties once one wants to find out its exact relation with $T_{\alpha\beta}$. This is because as,

\[ S = \text{Trace } S_{\alpha\beta} = T - \frac{1}{D-2} \text{Trace}(g_{\alpha\beta} T) , \]

in order to proceed we need to substitute for $T$ from equation (3.2) (and then use the distributivity of the Trace). Otherwise, if one just considers the fact that $T_{\alpha\beta}$, and so $T$, does not depend on the metric (and hence its HDN is zero), one gets

\[ S = \frac{-2T}{D-2} = \frac{2\kappa^{-2}}{D-2} \sum_{0<n<\frac{D}{2}} \left( \frac{D}{n} - 2 \right) c_n R^{(n)} , \tag{3.19} \]

where we used equation (3.2) in the last step. Whereas, if we use equation (3.2) before evaluating the Trace, we have (from equation (3.8)):

\[ S = \frac{2\kappa^{-2}}{D-2} \sum_{0<n<\frac{D}{2}} \left( \frac{D}{n} - 2 \right) c_n R^{(n)} - \frac{D\kappa^{-2}}{D-2} \sum_{0<n<\frac{D}{2}} \left( \frac{D}{n} - 2 \right) \left( 1 - \frac{1}{n} \right) c_n R^{(n)} . \]

Obviously, only in the case of $n = 1$, i.e. Einstein’s gravity, is the above equation equal to equation (3.19).

Besides, we must warn that in the above procedure, wherever we have used equation (3.2) (or similar ones) for $T$ in order to evaluate a relevant Trace (e.g. Trace$(g_{\alpha\beta} T)$), this is not obviously valid when $T$ vanishes. These equations are (3.7), (3.8), (3.10), (3.11), (3.14), (3.15), (3.18) and the second part of (3.17). Therefore, the definitions of (3.12), (3.13), (3.16) and the first part of (3.17) are also undefined when $T_{\alpha\beta}$ (hence $T$) vanishes. That is, using the above procedure, we do not know the real values of $S_{\alpha\beta}^{(2)}$, $T^{(2)}$, $S_{\alpha\beta}^{(n_{\text{max}})}$ and $T^{(n_{\text{max}})}$ in a vacuum.

On the other hand, in Einstein’s gravitational theory, by equation (2.1) we have that $R \propto T$. Hence, as an analogous demand, in Lovelock’s gravitational theory we should get $\Re \propto T$. Indeed, we have given this proportionality above, equation (3.17), where we also showed that $\tilde{T}$ is not equal to Trace$T_{\alpha\beta}$, but can be equal to $T + T^{(n_{\text{max}})}$. But, from now on, we waive the details of the foregoing procedure, and base our reasoning on the latter analogous demand. Hence, we
take $\mathcal{T}$ as equivalent to $\mathcal{R}$ with the same ratio of equation (3.17). We also use $T'$ instead of $T^{(n_{\text{max}})}$ (i.e., $T' \equiv T^{(n_{\text{max}})}$), and alternative field equations (3.15) as

$$\mathcal{R}_{\alpha \beta} = \frac{1}{2} \kappa^2 S_{\alpha \beta}, \quad (3.20)$$

where $S_{\alpha \beta} \equiv S^{(n_{\text{max}})}_{\alpha \beta}$. We will refer to $S_{\alpha \beta}$ and $\mathcal{T}$ as generalized Source tensor and trace generalization of the energy-momentum tensor respectively.

Finally, for a better view into the separation of $\mathcal{R}$ into $T$ and $T'$, one can expand the summations in their relevant equations as follows:

$$\mathcal{R} \equiv \sum_{0 < n < \frac{D}{2}} c_n R^{(n)} = c_1 R^{(1)} + c_2 R^{(2)} + c_3 R^{(3)} + \cdots$$

$$\equiv -\frac{\kappa^2 T}{D - 2} + \frac{-\kappa^2 T'}{D - 2} \quad \text{by Eq. (3.17)}$$

$$= \sum_{0 < n < \frac{D}{2}} \frac{D - 2n}{n(D - 2)} c_n R^{(n)} + \sum_{0 < n < \frac{D}{2}} \frac{D(n - 1)}{n(D - 2)} c_n R^{(n)} \quad \text{by Eqs. (3.2) & (3.18)}$$

$$= \left[ c_1 R^{(1)} + \frac{D - 4}{2(D - 2)} c_2 R^{(2)} + \frac{D - 6}{3(D - 2)} c_3 R^{(3)} + \cdots \right]$$

$$+ \left[ \frac{D}{2(D - 2)} c_2 R^{(2)} + \frac{2D}{3(D - 2)} c_3 R^{(3)} + \cdots \right], \quad (3.21)$$

where $n_{\text{max}}$ is related to $D$ by equation (1.2). As can be seen, $R^{(1)}$ does not contribute to $T'$, and higher order gravities contribute more to $T'$ the higher their orders, i.e. $(0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n - 1}{n}) D$, see equation (3.18).

For example, if $D = 8$, then $n_{\text{max}} = 3$ and we have

$$\mathcal{R} = c_1 R^{(1)} + c_2 R^{(2)} + c_3 R^{(3)}$$

$$\equiv -\frac{1}{6} \kappa^2 T - \frac{1}{6} \kappa^2 T'$$

$$\equiv \left( c_1 R^{(1)} + \frac{1}{3} c_2 R^{(2)} + \frac{1}{9} c_3 R^{(3)} \right)$$

$$+ \left( \frac{2}{3} c_2 R^{(2)} + \frac{8}{9} c_3 R^{(3)} \right).$$
4 Discussions

The following table summarizes our results of the analogous demands between Einstein’s and Lovelock’s gravitational theory.

**Table 1**: Analogy between Einstein’s and Lovelock’s gravitational theory.

| Field equations: | Einstein’s Theory | Lovelock’s Theory |
|------------------|-------------------|------------------|
| $G_{\alpha\beta} = \frac{1}{2}\kappa^2 T_{\alpha\beta}$ | $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ | $\mathcal{G}_{\alpha\beta} = \frac{1}{2}\kappa^2 T_{\alpha\beta}$ | $\mathcal{G}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R}$ |
| $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ | $\mathcal{G}_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R}$ |
| Alternative forms: | $R_{\alpha\beta} = \frac{1}{2}\kappa^2 S_{\alpha\beta}$ | $\mathcal{R}_{\alpha\beta} = \frac{1}{2}\kappa^2 \mathcal{S}_{\alpha\beta}$ |
| where: | $S_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} T$ | $\mathcal{S}_{\alpha\beta} \equiv \mathcal{T}_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} \mathcal{T}$ |
| and: | $T = -\kappa^{-2} (D - 2) R$ | $\mathcal{T} \equiv -\kappa^{-2} (D - 2) \mathcal{R}$ |
| but: | . . . | $\mathcal{T} = T + \mathcal{T}'$ |
| where: | . . . | $T' = -\kappa^{-2} \sum \left( \frac{D}{n} - 2 \right) c_n R^{(n)}$ |
| | | $T' \equiv -\kappa^{-2} D \sum \frac{n-1}{n} c_n R^{(n)}$

As it is obvious from Table 1, there still remains some (slight) differences in our developing structure (which is based on analogy).

The effect of $T'$ is what that is zero in Einstein’s gravitational theory and is included only in higher order gravitational theories. This has also been verified in the quantum theory, as trace anomalies. In the quantum theory, one relates and justifies the presence of the trace anomaly, classically, to higher order gravities. Here, we have actually stated a classical view of gravitation which explicitly shows the presence of an extra (anomalous) trace for the energy-momentum tensor with an indication of the constitution of higher order gravities towards this trace anomaly (see equation (3.21)). In the next section, as an example, we will use this procedure for any generic second order Lagrangian to derive trace anomaly relation suggested by Duff.

In addition, one may speculate, using the analogous demand, that in those cases where higher order gravities dominate, space-time “behaves” as if its energy-momentum has been “transferred” into matter’s energy-momentum in the sense that in a universe devoid of “matter” there should be also no meaning for the existence of space-time, i.e. a strong version of Mach’s principle. Hence, if one adopts the view that the geometrical curvature induces matter, then, when $T_{\alpha\beta}$

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* Also note that, in Einstein’s gravity, true gravitational fields exist even in empty space, i.e. their source should be the space-time itself.
vanishes, by equation (3.20), the gravitational field equations are

\[ \mathcal{R}_{\alpha\beta} = \frac{1}{2} \kappa^2 \left( -\frac{1}{D-2} \right) g_{\alpha\beta} T', \quad (4.1) \]

and thereby seen to be more consistent with Machian ideas.† Usual matter just modifies an already existing space-time but the “matter”, which we described above, would be the only source for the structure of space-time.

It seems that the above view violates the fundamentals of establishing a gravitational theory in the sense that in its field equation,

\[(\text{gravitation tensor}) \propto (\text{source term}),\]

the left hand side should represent the functions of geometry and the right hand side should denote source terms independent of the gravitational characters. However, the “real” or “usual” physical phenomena, to which one refers, corresponds to an energy scale of less than the Planck energy, i.e. the low energy approximation scale in the superstring theory. Also, the applicability of higher order gravitational theories are restricted only by the energy scale. Therefore, generally speaking, these theories do not have to satisfy all the requirements imposed on the fundamental theory. In other words, as the coefficients of higher order gravities are very small, one cannot detect such their implications in “real” world. However, these effects are important in highly curved areas, such as the very early universe, or in quantum physics.

On the other hand, in the literature, there are also the following views:

(a) The field equations for (apparently) empty higher dimensions are, in fact, equations for gravity with matter in lower dimensions.\(^{[16]}\)

(b) A large class of higher order theories of gravitation are equivalent to general relativity plus additional matter fields with a new metric (often referred to as “dynamical universality” of Einstein’s gravity, see, for example, Ref. [17] and references therein).

(c) There are conjectures about the origin of inertia in which inertia, as in the Machian idea, is not an intrinsic property of matter (see, for example, Ref. [18]).

† Note that, we are considering the complete Lovelock’s gravity (as a generalized Einstein’s gravity) that is here compared with Einstein’s gravity. Whereas, for example, even \( \mathcal{R}_{\alpha\beta} = 0 \) is obviously different from \( R_{\alpha\beta} = 0 \), as the former is \( R_{\alpha\beta} + R^{(2)}_{\alpha\beta} + \cdots = 0 \).
5 Derivation of Duff’s Suggested Relation

The covariant property of equations is our main theme of search. That is, by analogy, we demand that the form of the linear Lagrangian theory of Einstein is left unchanged when one works with the non-linear Lagrangian theory of gravitation, i.e. the form of equations (1.5) and (1.6) must holds for any such a Lagrangian term. In this section, we take this analogy to any generic second order Lagrangian, i.e. any combination of geometrical scalar Lagrangian terms with the HDN of plus two.

The only possible such terms are

\[ R^2, \ R_{\mu\nu}R^\mu_\nu, \ R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}, \ R_{;\mu}^{\mu}, \ R^{\mu\nu;\mu\nu}, \ R_{\mu\nu\rho\tau}^*R^{\mu\nu\rho\tau}, \ R_{\mu\nu\rho\tau}^{**}R^{\mu\nu\rho\tau}, \]

where \( *R^{\mu\nu\rho\tau} \) is the dual of \( R^{\mu\nu\rho\tau} \) which in four dimensions is defined as

\[ *R^{\mu\nu\rho\tau} \equiv \frac{1}{2} R^{\mu\nu}_{\alpha\beta} \varepsilon^{\alpha\beta\rho\tau}, \]

and where \( \varepsilon^{\alpha\beta\rho\tau} \) is the Levi–Civita tensor density.

But, the fourth and fifth terms give no contribution to the variation of the action, as they are complete divergences. The Lagrangian \( R_{\mu\nu\rho\tau}^*R^{\mu\nu\rho\tau} \), due to the Bianchi identity, reduces to a boundary term and vanishes identically. Also, the last term is actually equal to

\[ R_{\mu\nu\gamma\delta}^{**}R^{\mu\nu\gamma\delta} = - \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} \right), \]

where this is the second term of the Lovelock Lagrangian, the well known Gauss–Bonnet combination or the so-called Lanczos Lagrangian, which in four dimensional space-time is a topological invariant and its variation with respect to the metric leads only to a complete divergence, and hence, vanishes identically (the so-called Gauss–Bonnet theorem).

Hence, the corresponding generic Lagrangian, in \( D \geq 3 \) dimension, \( ^* \) is

\[ L_{\text{generic}}^{(2)} = \frac{1}{\kappa^2} \left( a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right), \quad (5.1) \]

\( ^* \) Obviously, in three and four dimensions, only two of these three terms are effective.
and its corresponding Euler–Lagrange expression is

\[
G^{(2)}_{\text{(generic)}\alpha\beta} \equiv \kappa^2 \frac{\delta(L^{(2)}_{\text{generic}}\sqrt{-g})}{\delta g^{\alpha\beta}}
\]

\[
= 2 \left[ a_1 R_{\alpha\beta} - 2a_3 R_{\alpha\mu} R^\mu_\beta + a_3 R_{\alpha\rho\mu\nu} R^\rho_\beta R^{\mu\nu} + (a_2 + 2a_3) R_{\alpha\beta;\mu} R^\mu_\nu \\
- (a_1 + \frac{1}{2} a_2 + a_3) R;_{\alpha\beta} + \left( \frac{1}{2} a_2 + 2a_3 \right) R_{\alpha\beta;\mu} R^\mu_\nu \right] \\
- \frac{1}{2} g_{\alpha\beta} \left[ \left( a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R^{\lambda\mu\nu\gamma} R_{\lambda}^{\rho\mu\nu} \right) - (4a_1 + a_2) R;_{\mu} R^\mu_\nu \right],
\]

where it is obviously of the form of equation (1.5), i.e.

\[
G^{(2)}_{\text{(generic)}\alpha\beta} \equiv R^{(2)}_{\text{(generic)}\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(2)}_{\text{generic}}.
\]  

Now, the form of relation (1.6), i.e. the “trace” relation

\[
\text{Trace} R^{(2)}_{\text{(generic)}\alpha\beta} = R^{(2)}_{\text{generic}},
\]

holds trivially, for non-zero coefficients, if and only if

\[
3a_1 + a_2 + a_3 = 0.
\]

Therefore, the analogy of Einstein’s gravitational theory, including the “trace” relation, cannot be achieved for any combination in the generic Lagrangian case of \(L^{(2)}_{\text{generic}}\), unless the above relation between its constituent coefficients are valid. That is, when its constants are not all independent, but obey the above constraint, which gives two degrees of freedom, in \(D > 4\), to choose constituent coefficients.

The Lanczos Lagrangian does indeed satisfy the above condition, as expected. So, in this sense, one may say that the coefficients used in the Lanczos Lagrangian are specific to a equivalence class of any generic second order Lagrangians having the same analogy including the “trace” relation. However, the field equations of all these Lagrangians are fourth order with respect to the metric, except the Lanczos Lagrangian which is of second order.

Note that, when one considers a homogeneous Lagrangian, e.g. \(L^{(2)}_{\text{generic}}\) alone, then obviously, one gets its corresponding homogeneous Euler–Lagrange terms
with a uniform HDN. And therefore, one can work with the usual trace instead of the Trace operator, and obtains the same results if one demands the appropriate “trace” relation, i.e. \( \frac{1}{n} \text{trace} R_{(\text{generic})}^{(n)} = R_{\text{generic}}^{(n)} \). Though, the notion of Trace operator was introduced as to be able to deal with when one considers the Einstein–Hilbert Lagrangian plus higher order terms, i.e. when one works with an inhomogeneous Lagrangian (see Ref. [1] for details).

As mentioned before, the constraint (5.5) is related to the relevant constraint in the trace anomaly of the energy-momentum tensor. We first give a brief review of this subject and then explain the above relation.

In the absence of a viable theory of quantum gravity, quantum field theory in a curved background space-time has been used as the starting point for an approach to a complete quantization of gravity,\(^{[21]}\) though one may still argue about an internal inconsistency.\(^{[22]}\) In this procedural attempt for quantum aspects of gravity, while the Einstein gravitational field is retained as an unquantized, classical background curved metric, the matter fields are treated quantum mechanically and quantized in the usual way.

In this approach, one sets \( g_{\mu\nu} = g^c_{\mu\nu} + g_{\mu\nu} \), where \( g^c_{\mu\nu} \) is the classical metric of Einstein’s gravitational background space-time, and \( g_{\mu\nu} \) is taken to be a quantum field propagating in this background. Then, in this semi-classical theory, regularization techniques are used to compute a finite, renormalized quantum vacuum expectation value of the energy-momentum tensor, \( \langle T_{\mu\nu} \rangle = \frac{\langle \text{out}\,0 \mid T_{\mu\nu} \mid 0,\text{in} \rangle}{\langle \text{out}\,0 \mid 0,\text{in} \rangle} \). Where the computation of \( \langle T_{\mu\nu} \rangle \) can also be treated via its definition by an effective Lagrangian density\(^{[23]}\) for the quantum matter fields, i.e.

\[
\delta \hat{L}_{\text{eff}} \equiv -\frac{1}{2} \sqrt{-g} \langle T_{\mu\nu} \rangle \delta g^{\mu\nu} .
\] (5.6)

Using the path-integral quantization procedure, one finds\(^{[21]}\) that there are divergent terms at the one-loop level in the effective action that are local and independent of the state. Hence, the divergences in the effective Lagrangian, \( L_{\text{div}} \), are entirely geometrical, and are built out of local tensors, i.e., actually, the higher order Lagrangians. These are interpreted as a contribution to the gravitational rather than the quantum matter Lagrangian. Although, as the coefficients of these terms diverge as \( 1/(D - 2n) \) in the dimensional continuation method of regularization, in odd dimensions \( L_{\text{eff}} \) is finite, and hence there is no anomaly in odd dimensional space-times. So, one must therefore introduce, a priori, the relevant counterterms into the original Einstein’s Lagrangian with bare coefficients into which the divergent terms can be absorbed to yield renormalized coefficients.\(^{[24]}\) Hence, the renormalized effective Lagrangian, \( L_{\text{ren}} \equiv L_{\text{eff}} - L_{\text{div}} \), are then finite.
The inclusion of higher order Lagrangian terms have also appeared in the effect of string theory on classical gravitational physics by means of a low energy effective action which expresses gravity at the classical level.\[^{[25]}\] This effective action in general gives rise to fourth order field equations (and brings in ghosts), and in particular cases, i.e. in the form of dimensionally continued Gauss-Bonnet densities, it is exactly, as mentioned at the beginning, the Lovelock terms (and consequently no ghosts arise).\[^{[4]}\] The Gauss–Bonnet term also appears naturally in the next-to-leading order term of the heterotic string effective action and plays an essential role in the Chern–Simons gravitational theories.\[^{[26]}\]

Now, in the special cases when the classical action of the matter field is invariant under conformal transformations, e.g. in the conformally coupled massless scalar field, the effective action is\[^{[21]}\] also conformally invariant. But, it has been shown\[^{[21]}\] that before relaxing the regularization, the divergent term in the effective action, away from $D = 2n$ dimension, is not conformally invariant, although it is in the physical limit $D = 2n$. Apparently, an indication of this conformal breakdown survives in the physical quantities even when one relaxes the regularization at the end of the calculation.

In any case, the anomalies generally occur in any regularization method as a consequence of introducing a scale into the theory in order to regularize it.\[^{[21]}\] The contribution of $L_{\text{div}}$ to the trace of the energy-momentum tensor, $\langle T^\rho_\rho\rangle_{\text{div}}$, is one of the above consequences. Hence, the finite, renormalized $\langle T^\rho_\rho\rangle_{\text{ren}}$ must also have a nonvanishing trace, i.e. $\langle T^\rho_\rho\rangle_{\text{ren}} = -\langle T^\rho_\rho\rangle_{\text{div}}$, despite the fact that the classical energy-momentum tensors, for the conformally invariant classical actions, must be traceless. This is known as a conformal, or trace, or Weyl, anomaly. Note that, as $I_{\text{eff}}$ itself is a conformally invariant action, the expectation value of the trace of the total energy-momentum tensor is zero.

It is now known\[^{[21]}\] that all the regularization techniques give conformal anomalies for fields of arbitrary spin, but with the same value for the scalar field, and even in non-conformal theories the anomalies still survive, and in addition are accompanied by contributions through the lack of conformal invariance. It has also been clarified\[^{[14]}\] that the anomalous terms cannot be removed and hence the conformal anomalies are inevitable. Besides, the anomalies which cannot be absorbed by a finite local renormalization have also been referred to as non-local conformal anomalies.\[^{[27]}\]

Therefore, the Weyl conformal invariance displayed by classical (massless) fields in interaction with Einstein’s gravity no longer survives in the quantum theory, and is generally broken as a consequence of the appearance of the Weyl anomalies at the one-loop level. So, the Weyl conformal invariance, perhaps, is not a good symmetry.
beyond the classical level.

It has been shown \cite{27,28} that, the most general form of the anomalous trace of the energy-momentum tensor for classically conformally invariant fields of arbitrary spin and dimension, when terms only quadratic in the Riemann-Christoffel tensor and its contractions are considered, is

$$\langle T^\rho_\rho \rangle_{\text{ren}} = -\frac{hc}{180(4\pi)^2} \left( a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \delta R_{\rho\rho} \right),$$

(5.7)

where numerical coefficients $a_1, a_2, a_3$ and $\delta$ are dimensionless constants which also determine the counterterms and are expected \cite{29} to be unique.

In the process of re-examining the Weyl anomaly’s applications, Duff noticed \cite{10} that, there is a consistency condition on these numerical coefficients when the dimensional regularization is applied to a classically conformally invariant theory in arbitrary dimension. That is

$$4 a_1 + a_2 = a_1 - a_3 = -\delta,$$

(5.8)

which holds for several explicit calculations in the case of conformal scalars, massless spin-$\frac{1}{2}$ fermions and spin-one gauge fields in an external gravitational field. The calculations by most authors confirmed \cite{10} the above constraint for the values of $a_1, a_2$ and $a_3$ in all cases, but not always that of $\delta$. Apparently, this is due to the fact that the term $R_{\rho\rho}$ is a local anomaly.\cite{10} However, the absorption of this term by any of the above mentioned four dimensional actions breaks the conformal invariance of the action, and hence, the above constraint cannot be applied.

The first part of the above consistency condition (5.8) was later derived \cite{30,31} based on a cohomological point of view, using the Wess–Zumino consistency conditions, which was claimed to be the true reason for the existence of such a relation. And more recently, there are some works in the literature which claim that the AdS/CFT correspondence, namely the holographic conformal anomaly, maybe responsible for it, see e.g. Ref. \cite{32} and references therein.

In the previous sections, we argued that, based on the analogous demand, there is a trace generalization of the energy-momentum tensor i.e., $\mathcal{T} = T + T'$, and actually, we stated a classical view of gravitation which explicitly shows the presence of an extra trace for the energy-momentum tensor, namely $T'$ equation (3.18). The same argument for generic cases yields

$$T' = -\kappa^{-2} D \sum_{n \geq 1} \frac{n - 1}{n} c_n R_{\text{generic}}^{(n)} \equiv \sum_{n \geq 1} T'_n,$$

(5.9)

where $R_{\text{generic}}^{(1)} \equiv R$ (however, $T'_1 = 0$), and there is no upper limit for $n$. 
In this section, we considered the case of $n = 2$, and in this case one only has

$$T'_2 = -\frac{\kappa^{-2}}{2} D c_2 R^{(2)}_{\text{generic}}, \quad (5.10)$$

where $R^{(2)}_{\text{generic}}$, according to equations (5.2) and (5.3), is

$$R^{(2)}_{\text{generic}} \equiv \kappa^2 L^{(2)}_{\text{generic}} - (4 a_1 + a_2) R; \mu^\mu, \quad (5.11)$$

with constraint (5.5). Comparing it with equation (5.7), it obviously shows that

$$\delta = -(4 a_1 + a_2) \quad (5.12)$$

Hence, it completely gives the trace anomaly relations suggested by Duff. Also, by matching equation (5.7) with equation (5.10) in $D = 4$ dimensions, one gets $c_2 = \frac{1}{90(4\pi)} L_p^2$, as expected.

In the above mentioned semi-classical theory, the effective action is a covariant functional i.e., invariant under diffeomorphisms and local gauge transformations. Therefore, the approximation procedures for calculating the effective action have to preserve the general covariance at each order. Hence, conformal invariance is also sacrificed to the needs of general covariance. This is what we have actually performed in the classical theory of gravitation through preserving the covariant property of the linear Lagrangian theory of Einstein’s gravity for each order of non-linear Lagrangian theories of gravitation, by an analogous demand.

Hence, the origin of Duff’s suggested relation, equation (5.8), between the coefficients of the conformal anomalies may classically be interpreted due to the general covariance of Einstein’s theory. Though, it is somehow a naive conjecture, nevertheless, it gives almost an easy classical manner to grasp its result.

Taking the analogy, that is the existence of the trace anomaly of the energy-momentum tensor even in classical treatments, further to any type of the third order Lagrangian gives a more deep sight of this procedure. Actually, investigation shows that there is an interesting similarity between Weyl invariant combinations in six dimensions and the constraint relation that coefficients of any generic third order Lagrangian must satisfy in order to hold the desired analogy, which then, can be used as a criterion. This may also lead one to speculate that there should be an intrinsic property between the appropriate heat kernel coefficients and the covariance of the form of the Einstein equations, and or between the matter and the geometry. Probing these thoughts, by further extending the analogous procedure, to see whether there is a geometrical meaning behind this generalization, perhaps besides what have already been given in the literature, has been our most important task on which our search still continues.
Acknowledgment: The author is grateful to Prof. John M Charap and the Physics Department of Queen Mary & Westfield College University of London where some part of this work has been carried out.

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