Time-Fractional KdV Equation Describing the Propagation of Electron-Acoustic Waves in plasma

S. A. El-Wakil, E. M. Abulwafa, E. K. El-shewy and A. A. Mahmoud
Theoretical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract

The reductive perturbation method has been employed to derive the Korteweg-de Vries (KdV) equation for small but finite amplitude electron-acoustic waves. The Lagrangian of the time fractional KdV equation is used in similar form to the Lagrangian of the regular KdV equation. The variation of the functional of this Lagrangian leads to the Euler-Lagrange equation that leads to the time fractional KdV equation. The Riemann-Liouville definition of the fractional derivative is used to describe the time fractional operator in the fractional KdV equation. The variational-iteration method given by He is used to solve the derived time fractional KdV equation. The calculations of the solution with initial condition $A_0 \text{sech}(cx)^2$ are carried out. The result of the present investigation may be applicable to some plasma environments, such as the Earth’s magnetotail region.

Keywords: Electron-acoustic waves; Euler-Lagrange equation, Riemann-Liouville fractional derivative, fractional KdV equation, He's variational-iteration method.

PACS: 05.45.Df, 05.30.Pr.
1 Introduction

Because most classical processes observed in the physical world are non-conservative, it is important to be able to apply the power of variational methods to such cases. A method used a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton’s equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative. It was shown that such fractional derivatives in the Lagrangian describe nonconservative forces [1, 2]. Further study of the fractional Euler-Lagrange can be found in the work of Agrawal [3, 4], Baleanu and coworkers [5, 6] and Tarasov and Zaslavsky [7, 8]. During the last decades, Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Some of the areas where Fractional Calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory [9].

On the other hand, electron acoustic waves (EAWs) have been observed in the laboratory when the plasma consisted of two species of electrons with different temperatures, referred to as hot and cold electrons [10], or in an electron ion plasma with ions hotter than electrons [11]. Also its propagation plays an important role not only in laboratory but also in space plasma. For example, Bursts of broadband electrostatic noise (BEN) emissions have been observed in auroral and other regions of the magnetosphere, e.g. polar cusp, plasma sheet boundary layer (PSBL). see [12]. There are different methods to study nonlinear systems [13]. Washimi and Taniti [13] were the first to use reductive perturbation method to study the propagation of a slow modulation of a quasimonochromatic waves through plasma. And then the attention has been focused by many authors [14–15].

To the author’s knowledge, the problem of time fractional KdV equation in collisionless plasma has not been addressed in the literature before. So, our motive here is to study the effects of time fractional parameter on the electrostatic structures for a system of unmagnetized collisionless plasma consisting of a cold electron fluid and isothermal ions with two different temperatures obeying Boltzmann type distributions. We expect that the inclusion of time fractional parameter will change the properties as well as the regime of existence of solitons. Several methods have been used to solve fractional differen-
tial equations such as: the Laplace transform method, the Fourier transform method, the iteration method and the operational method [16]. Recently, there are some papers deal with the existence and multiplicity of solution of nonlinear fractional differential equation by the use of techniques of nonlinear analysis [17-18]. In this paper, the resultant fractional KdV equation will be solved using a variational-iteration method (VIM) firstly used by He [19].

This paper is organized as follows: Section 2 is devoted to describe the formulation of the time-fractional KdV (FKdV) equation using the variational Euler-Lagrange method. In section 3, variational-Iteration Method (VIM) is discussed. The resultant time-FKdV equation is solved approximately using VIM. Section 5 contains the results of calculations and discussion of these results.

2 Basic equations

We consider a homogeneous system of unmagnetized collisionless plasma consisted of a cold electron fluid and isothermal ions with two different temperatures obeying Boltzmann type distributions. Such system is governed by the following normalized equations in one dimension [15]:

\[
\frac{\partial}{\partial t} n_e(x, t) + \frac{\partial}{\partial x} [n_e(x, t) u_e(x, t)] = 0, \quad (1a) \\
\frac{\partial}{\partial t} u_e(x, t) + u_e(x, t) \frac{\partial}{\partial x} u_e(x, t) - \frac{\partial}{\partial x} \phi(x, t) = 0, \quad (1b) \\
\frac{\partial^2}{\partial x^2} \phi(x, t) - n_e(x, t) + n_{il}(x, t) + n_{ih}(x, t) = 0, \quad (1c)
\]

the two ions density \( n_{il}(x, t) \) and \( n_{ih}(x, t) \) are given by:

\[
n_{il}(x, t) = \mu \exp\left[\frac{-\phi(x, t)}{\mu + \nu \beta}\right], \quad (1d) \\
n_{ih}(x, t) = \nu \exp\left[\frac{-\beta \phi(x, t)}{\mu + \nu \beta}\right]. \quad (1e)
\]

In the earlier equations, \( n_e(x, t) \) is the cold electron density normalized by equilibrium value \( n_{e0} \), \( u_e(x, t) \) is the cold electron fluid velocity normalized by \( C_{eff} = (T_{eff}/m_e)^{1/2}, T_{eff} = \frac{2T}{\mu_{he} + \gamma l}, T_l \) is the temperature of low temperature
ion with initial normalized equilibrium density $\mu$, $T_h$ is the temperature of high temperature ion with initial normalized equilibrium density $\nu$, $\beta = \frac{T_h}{T_h}$ is the ions temperatures ratio, $\phi(x, t)$ is the electric potential normalized by $T_{eff}/e$, $m_e$ is the mass of electron, $e$ is the electron charge. $x$ is the space co-ordinate normalized to the effective Debye length $\lambda_{D_{eff}} = \left(\frac{T_{eff}}{4\pi e^2 n_c} \right)^{\frac{1}{2}}$ and $t$ is the time variable normalized by the inverse of the cold electron plasma frequency $\omega_{pe}^{-1}$, $[\omega_{pe} = \left(\frac{4\pi e^2 n_c}{m_e} \right)^{\frac{1}{2}}]$. The neutrality condition reads $\mu + \nu = 1$. Equations (1a) and (1b) represent the inertia of cold electron and equation (1c) is the Poisson’s equation needs to make the self consistent. The two ion-densities are described by Boltzmann type distributions given by equations (1d) and (1e).

3 Nonlinear small-amplitude

According to the general method of reductive perturbation theory [13], the slow stretched co-ordinates are introduced as:

\[ \tau = \epsilon^2 t, \quad \xi = \epsilon^2 (x - \lambda t), \]

where $\epsilon$ is a small dimensionless expansion parameter and $\lambda$ is the wave speed normalized by $C_{eff}$. All physical quantities appearing in (1) are expanded as power series in $\epsilon$ about their equilibrium values as:

\[ n_e(\xi, \tau) = 1 + \epsilon n_1(\xi, \tau) + \epsilon^2 n_2(\xi, \tau) + \epsilon^3 n_3(\xi, \tau) + ..., \quad (3a) \]
\[ u_e(\xi, \tau) = \epsilon u_1(\xi, \tau) + \epsilon^2 u_2(\xi, \tau) + \epsilon^3 u_3(\xi, \tau) + ..., \quad (3b) \]
\[ \phi(\xi, \tau) = \epsilon \phi_1(\xi, \tau) + \epsilon^2 \phi_2(\xi, \tau) + \epsilon^3 \phi_3(\xi, \tau) + ..., \quad (3c) \]

with the boundary conditions that as $|\xi| \to \infty$, $n_e = 1$, $u_e = 0$, $\phi = 0$.

Substituting (2) and (3) into (1) and equating the coefficients of like powers of $\epsilon$ lead, from the lowest and second-order equations in $\epsilon$, to the following KdV equation for the first-order perturbed potential:

\[ \frac{\partial \phi_1(\xi, \tau)}{\partial \tau} + A \phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^3 \phi_1(\xi, \tau)}{\partial \xi^3} = 0, \quad (4a) \]

where
\[ A = \frac{\lambda^3}{2} \left[ \frac{\mu + \nu\beta^2}{2} - \frac{3}{\lambda^4} \right], \quad B = \frac{\lambda^3}{2} \] and \( \lambda = \pm 1. \) (4b)

In equation (4a), \( \phi_1(\xi, \tau) \) is a field variable, \( \xi \) is a space coordinate in the propagation direction of the field and \( \tau \in T(= [0, T_0]) \) is the time coordinate. The resultant KdV equation (4a) can be converted into time-fractional KdV equation as follows:

Using a potential function \( V(\xi, \tau) \), where \( \phi_1(\xi, \tau) = V_\xi(\xi, \tau) = \Phi(\xi, \tau) \), gives the potential equation of the regular KdV equation (4a) in the form

\[ V_{\xi\tau}(\xi, \tau) + A V_\xi(\xi, \tau)v_{\xi\xi}(\xi, \tau) + B V_{\xi\xi\xi\xi}(\xi, \tau) = 0, \] (5)

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Lagrangian of this regular KdV equation (4a) can be defined using the semi-inverse method [20, 21] as follows:

The functional of the potential equation (5) can be represented by

\[ J(V) = \int_R d\xi \int_T d\tau \{ V(\xi, \tau)[c_1 V_{\xi\tau}(\xi, \tau) + c_2 AV_\xi(\xi, \tau)v_{\xi\xi}(\xi, \tau) + c_3 BV_{\xi\xi\xi\xi}(\xi, \tau)] \}, \] (6)

where \( c_1, c_2 \) and \( c_3 \) are constants to be determined. Integrating by parts and taking \( V_\tau|_R = V_\xi|_R = V_\xi|_T = 0 \) lead to

\[ J(V) = \int_R d\xi \int_T d\tau \{ V(\xi, \tau)[-c_1 V_\xi(\xi, \tau)V_\tau(\xi, \tau) - \frac{1}{2} c_2 A V_\xi^3(\xi, \tau) + c_3 B V_{\xi\xi}^2(\xi, \tau)] \}. \] (7)

The unknown constants \( c_i(i = 1, 2, 3) \) can be determined by taking the variation of the functional (7) to make it optimal. Taking the variation of this functional, integrating each term by parts and making the variation optimum give the following relation

\[ 2c_1 V_{\xi\tau}(\xi, \tau) + 3c_2 A V_\xi(\xi, \tau)v_{\xi\xi}(\xi, \tau) + 2c_3 BV_{\xi\xi\xi\xi}(\xi, \tau) = 0. \] (8)

As this equation must be equal to (5), the unknown constants are given as

\[ c_1 = 1/2, \quad c_2 = 1/3 \quad \text{and} \quad c_3 = 1/2. \] (9)
Therefore, the functional given by (7) gives the Lagrangian of the regular KdV equation as

\[ L(\tau, \xi, \xi) = -\frac{1}{2} V_\xi(\xi, \tau)V_\tau(\xi, \tau) - \frac{1}{6} AV_\xi^3(\xi, \tau) + \frac{1}{2} BV_\xi^2(\xi, \tau). \]  

(10)

Similar to this form, the Lagrangian of the time-fractional version of the KdV equation can be written in the form

\[ F(0D_\tau^\alpha V, \xi, \xi) = -\frac{1}{2} [0D_\tau^\alpha V(\xi, \tau)] V_\xi(\xi, \tau) - \frac{1}{6} AV_\xi^3(\xi, \tau) + \frac{1}{2} BV_\xi^2(\xi, \tau), \]

\[ 0 \leq \alpha < 1, \]  

(11)

where the fractional derivative is represented, using the left Riemann-Liouville fractional derivative definition as [16]

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(k - \alpha)} \int_a^t \frac{d}{dt}[ \int_a^t df(t - \tau)^{k-\alpha-1} f(\tau)] dt, \]

\[ k - 1 \leq \alpha \leq 1, t \in [a, b]. \]  

(12)

The functional of the time-FKdV equation can be represented in the form

\[ J(V) = \int_R d\xi \int_T d\tau F(0D_\tau^\alpha V, \xi, \xi), \]  

(13)

where the time-fractional Lagrangian \( F(0D_\tau^\alpha V, \xi, \xi) \) is defined by (11).

Following Agrawal’s method [3, 4], the variation of functional (13) with respect to \( V(\xi, \tau) \) leads to

\[ \delta J(V) = \int_R d\xi \int_T d\tau \{ \frac{\partial F}{\partial 0D_\tau^\alpha V} \delta_0D_\tau^\alpha V + \frac{\partial F}{\partial \xi} \delta \xi + \frac{\partial F}{\partial \xi} \delta \xi \}. \]  

(14)

The formula for fractional integration by parts reads [3, 16]

\[ \int_a^b dt f(t) D_t^\alpha g(t) = \int_a^t dt g(t) D_t^\alpha f(t), \quad f(t), g(t) \in [a, b]. \]  

(15)

where \( D_t^\alpha \), the right Riemann-Liouville fractional derivative, is defined by [16]
\begin{align}
\tau D^\alpha_0 f(t) &= \frac{(-1)^k}{\Gamma(k - \alpha)} \frac{d^k}{dt^k} [\int_t^b d\tau (\tau - t)^{k-\alpha-1} f(\tau)], \\
k - 1 &\leq \alpha \leq 1, t \in [a, b]. \tag{16}
\end{align}

Integrating the right-hand side of (14) by parts using formula (15) leads to

\[\delta J(V) = \int_R d\xi \int_T d\tau \left[ \tau D^\alpha_0 \left( \frac{\partial F}{\partial \partial_0 D^\alpha_0 V} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial V} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial V \xi} \right) \right] \delta V, \tag{17}\]

where it is assumed that \( \delta V|_T = \delta V|_R = \delta V|_R = 0 \).

Optimizing this variation of the functional \( J(V) \), i.e., \( \delta J(V) = 0 \), gives the Euler-Lagrange equation for the time-FKdV equation in the form

\[\tau D^\alpha_0 \left( \frac{\partial F}{\partial \partial_0 D^\alpha_0 V} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial V} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial V \xi} \right) = 0. \tag{18}\]

Substituting the Lagrangian of the time-FKdV equation (11) into this Euler-Lagrange formula (18) gives

\[-\frac{1}{2} \tau D^\alpha_0 V(\xi, \tau) + \frac{1}{2} \partial^\alpha_0 V(\xi, \tau) + AV(\xi, \tau)V(\xi, \tau) + BV(\xi, \tau)V(\xi, \tau) = 0. \tag{19}\]

Substituting for the potential function, \( V(\xi, \tau) = \phi_1(\xi, \tau) = \Phi(\xi, \tau) \), gives the time-FKdV equation for the state function \( \Phi(\xi, \tau) \) in the form

\[\frac{1}{2} \left[ \partial D^\alpha_0 \Phi(\xi, \tau) - \tau D^\alpha_0 \Phi(\xi, \tau) \right] + A \Phi(\xi, \tau) \Phi(\xi, \tau) + B \Phi(\xi, \tau) = 0, \tag{20}\]

where the fractional derivatives \( \partial D^\alpha_0 \) and \( \tau D^\alpha_0 \) are, respectively the left and right Riemann-Liouville fractional derivatives and are defined by (12) and (16).

The time-FKdV equation represented in (20) can be rewritten by the formula

\[\frac{1}{2} \partial D^\alpha_0 \Phi(\xi, \tau) + A \Phi(\xi, \tau) \frac{\partial}{\partial \xi} \Phi(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \Phi(\xi, \tau) = 0, \tag{21}\]
where the fractional operator $R_0^\alpha D_\tau^\alpha$ is called Riesz fractional derivative and can be represented by [4, 16]

$$R_0^\alpha D_\tau^\alpha f(t) = \frac{1}{2}[D_\tau^\alpha f(t) + (-1)^k D_\tau^\alpha f(t)]$$

$$= \frac{1}{2} \frac{1}{\Gamma(k - \alpha)} \frac{d^k}{dt^k} \int_a^t d\tau |\tau|^k - \alpha - 1 f(\tau),$$

$$k - 1 \leq \alpha \leq 1, \quad t \in [a, b]. \quad (22)$$

The nonlinear fractional differential equations have been solved using different techniques [16-20]. In this paper, a variational-iteration method (VIM) [21] has been used to solve the time-FKdV equation that is formulated using Euler-Lagrange variational technique.

### 4 Variational-iteration method

A general Lagrange multiplier method is constructed to solve non-linear problems, which was first proposed to solve problems in quantum mechanics [21]. The VIM is a modification of this Lagrange multiplier method [22]. The basic features of the VIM are as follows. The solution of the linear term of the problem or the initial (boundary) condition of the nonlinear problem is used as initial approximation or trail function. A more highly precise approximation can be obtained using iteration correction functional. Variational-iteration method (VIM) [21] has been used successfully to solve different types of integer nonlinear differential equations [22, 23]. Also, VIM is used to solve linear and nonlinear fractional differential equations [24, 25]. This VIM has been used in this paper to solve the formulated time-FKdV equation.

Considering a nonlinear partial differential equation consists of a linear part $\hat{LU}(x, t)$, nonlinear part $\hat{NU}(x, t)$ and a free term $f(x, t)$ represented as

$$\hat{LU}(x, t) + \hat{NU}(x, t) = f(x, t), \quad (23)$$

where $L$ is the linear operator and $N$ is the nonlinear operator. According to the VIM, the $(n + 1)^{th}$ approximation solution of (23) can be given by the iteration correction functional as [24, 25]
\[ U_{n+1}(x, t) = U_n(x, t) + \int_0^t d\tau \lambda(\tau)[\dot{L}U_n(x, \tau) + \dot{N}U_n(x, \tau) - f(x, \tau)], \quad n \geq 0, \]  

where \( \lambda(\tau) \) is a Lagrangian multiplier and \( \dot{U}_n(x, \tau) \) is considered as a restricted variation function, i.e.; \( \delta \dot{U}_n(x, \tau) = 0 \). Extreme the variation of the correction functional (24) leads to the Lagrangian multiplier \( \lambda(\tau) \). The initial iteration can be used as the solution of the linear part of (23) or the initial value \( U(x, 0) \). As \( n \) tends to infinity, the iteration leads to the exact solution of (23), i.e.;

\[ U(x, t) = \lim_{n \to \infty} U_n(x, t). \]  

For linear problems, the exact solution can be given using this method in only one step where its Lagrangian multiplier can be exactly identified.

5 Time-fractional KdV equation solution

The time-FKdV equation represented by (21) can be solved using the VIM by the iteration correction functional (24) as follows:

Affecting from left by the fractional operator \( \mathcal{R}_0^\alpha D_\tau^{\alpha-1} \) on (21) leads to

\[ \frac{\partial}{\partial \tau} \Phi(\xi, \tau) = \mathcal{R}_0^\alpha D_\tau^{\alpha-1} \Phi(\xi, \tau) |_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} - \mathcal{R}_0^\alpha D_\tau^{1-\alpha} [A \Phi(\xi, \tau) \frac{\partial}{\partial \xi} \Phi(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \Phi(\xi, \tau)], \]

\[ 0 \leq \alpha \leq 1, \quad \tau \in [0, T_0], \]  

where the following fractional derivative property is used [16]

\[ \mathcal{R}_a^\alpha D_b^\beta \left[ \mathcal{R}_a^\alpha D_b^\beta f(t) \right] = \mathcal{R}_a^\alpha D_b^{\alpha+\beta} f(t) - \sum_{j=1}^{k} \mathcal{R}_a^\alpha D_b^{\beta-j} f(t)|_{t=a} \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \quad k-1 \leq \beta < k. \]  

As \( \alpha < 1 \), the Riesz fractional derivative \( \mathcal{R}_0^\alpha D_\tau^{\alpha-1} \) is considered as Riesz fractional integral \( \mathcal{R}_0^{1-\alpha} \) that is defined by [16]
\[ R_0 I^\alpha f(t) = \frac{1}{2} [ \int_0^t I^\alpha f(t) + \tau I^\alpha f(t) ] = \frac{1}{2} \frac{1}{\Gamma(\alpha)} \int_a^b d\tau |t - \tau|^{\alpha-1} f(\tau), \quad \alpha > 0, \]  

where \( R_0 I^\alpha f(t) \) and \( \tau I^\alpha f(t) \) are the left and right Riemann-Liouville fractional integrals, respectively [16].

The iterative correction functional of equation (26) is given as

\[ \Phi_{n+1}(\xi, \tau) = \Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \left\{ \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') - R_0 I_{\tau'}^{1-\alpha} \Phi_n(\xi, \tau') \right\} |_{\tau' = 0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha - 1)} \]

\[ + R_0 D_{\tau'}^{1-\alpha} \left[ A \Phi_n(\xi, \tau') \frac{\partial}{\partial \xi} \Phi_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \Phi_n(\xi, \tau') \right] \]

where \( n \geq 0 \) and the function \( \Phi_n(\xi, \tau) \) is considered as a restricted variation function, i.e., \( \delta \Phi_n(\xi, \tau) = 0 \). The extreme of the variation of (29) using the restricted variation function leads to

\[ \delta \Phi_{n+1}(\xi, \tau) = \delta \Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') \]

\[ = \delta \Phi_n(\xi, \tau) + \lambda(\tau') \delta \Phi_n(\xi, \tau) - \int_0^\tau d\tau' \frac{\partial}{\partial \tau'} \lambda(\tau') \delta \Phi_n(\xi, \tau') = 0. \]

This relation leads to the stationary conditions \( 1 + \lambda(\tau) = 0 \) and \( \frac{\partial}{\partial \tau'} \lambda(\tau') = 0 \), which leads to the Lagrangian multiplier as \( \lambda(\tau') = -1 \). Therefore, the correction functional (29) is given by the form

\[ \Phi_{n+1}(\xi, \tau) = \Phi_n(\xi, \tau) - \int_0^\tau d\tau' \left\{ \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') - R_0 I_{\tau'}^{1-\alpha} \Phi_n(\xi, \tau') \right\} |_{\tau' = 0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha - 1)} \]

\[ + R_0 D_{\tau'}^{1-\alpha} \left[ A \Phi_n(\xi, \tau') \frac{\partial}{\partial \xi} \Phi_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \Phi_n(\xi, \tau') \right] \]
where \( n \geq 0 \).

In Physics, if \( \tau \) denotes the time-variable, the right Riemann-Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development [3]. Therefore, the right-derivative is used equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

\[
\Phi_0(\xi, \tau) = \Phi(\xi, 0) = A_0 \sec h^2(c\xi).
\]  

(31)

where \( A_0 = \frac{3v}{A} \) and \( c = \frac{1}{2} \sqrt{\frac{v}{B}} \) are constants.

Substituting this zero order approximation into (30) and using the definition of the fractional derivative (22) lead to the first order approximation as

\[
\Phi_1(\xi, \tau) = A_0 \sec h(c\xi)^2 + 2A_0c \sinh(c\xi) \sec h(c\xi)^3 \\
* \left[4c^2B + (A_0A - 12c^2B) \sec h(c\xi)^2 \right] \frac{\tau^\alpha}{\Gamma(\alpha + 1)}.
\]  

(32)

Substituting this equation into (30), using the definition (22) and the Maple package lead to the second order approximation in the form
\[ \Phi_2(\xi, \tau) = A_0 \sec h(c\xi)^2 + 2A_0c \sinh(c\xi) \sec h(c\xi)^3 \]
\[ \times \left[ 4c^2B + (A_0A - 12c^2B) \sec h(c\xi)^2 \right] \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \]
\[ + 2A_0c^2 \sec h(c\xi)^2 \]
\[ \times \left[ 32c^4B^2 + 16c^2B(5A_0A - 63c^2B) \sec h(c\xi)^2 \right] \frac{\tau^2\alpha}{\Gamma(2\alpha + 1)} \]
\[ + 2A_0c^2 \sec h(c\xi)^2 \]
\[ \times \left[ 4A_0c^3 \sinh(c\xi) \sec h(c\xi)^5 \right] \frac{\tau^3\alpha}{\Gamma(3\alpha + 1)} \]

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.

6 Results and discussion

Numerical studies have been made for a small amplitude electron-acoustic waves in an unmagnetized collisionless plasma consisted of a cold electron fluid and isothermal ions with two different temperatures obeying Boltzmann type distributions. We have derived the Korteweg-de Vries equation by using the reductive perturbation method [13]. The Riemann-Liouville fractional derivative [16] is used to describe the time fractional operator in the FKdV equation. He’s variational-iteration method [21] is used to solve the derived time-FKdV equation.

However, since one of our motivations was to study effects of initial equilibrium density \( \mu \) of low temperature ion and time fractional order \( \alpha \) on the existence of solitary waves. Our system can support two kinds of potential structure namely, compressive and rarefactive pulses. Depending on the sign
of the coefficient of the nonlinear term $A$, compressive soliton exists if $A > 0$ while rarefactive soliton exists if $A < 0$.

In Fig (1), profiles of the bell-shaped rarefactive and compressive solitary pulses are obtained due to the change of the range of $\alpha$. Figure (2) shows that both the amplitude and the width of the compressive solitons increase while both decrease for rarefactive solitons. Also, the time fractional order decreases the amplitude of the rarefactive and compressive solitons as shown in Fig (3).

In summary, it has been found that amplitude and width of the electron-acoustic waves as well as parametric regime where the solitons can exist is sensitive to the initial equilibrium density of low temperature ion. Moreover, the time fractional order plays the role of higher order perturbation theory in changing the soliton amplitude.

The application of our model might be particularly interesting in the new observations for the Earth’s plasma sheet boundary layer region. We have stressed out that it is necessary to study the critical case for $A = 0$, the amplitude of the solitary pulse tends to infinity and the time-FKdV equation is not appropriate for describing the system. This is beyond the scope of the further work.
References

[1] Riewe, F., Nonconservative Lagrangian and Hamiltonian mechanics, Physical Review E 53(2) (1996) 1890.

[2] Riewe, F., Mechanics with fractional derivatives, Physical Review E 55(3) (1997) 3581.

[3] Agrawal, O. P., Formulation of Euler-Lagrange equations for fractional variational problems, J. Mathematical Analysis and Applications 272 (2002) 368.

[4] Agrawal, O. P., Fractional variational calculus in terms of Riesz fractional derivatives J. Physics A: Mathematical and Theoretical 40 (2007) 6287.

[5] Baleanu, D. and Avkar, T., Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, Nuovo Cimento B 119 (2004) 73-79.

[6] Muslih, S. I., Baleanu, D. and Rabei, E., Hamiltonian formulation of classical fields within Riemann-Liouville fractional derivatives, Physica Scripta 73 (2006) 436-438.

[7] Tarasov, V. E. and Zaslavsky, G. M., Fractional Ginzburg-Landau equation for fractal media, Physica A: Statistical Mechanics and Its Applications 354 (2005) 249-261.

[8] Tarasov, V. E. and Zaslavsky, G. M., Nonholonomic constraints with fractional derivatives, J. Physics A: Mathematical and General 39 (2006) 9797-9815.

[9] Sabatier, J., Agrawal, O. P. and Tenreiro Machado, J. A. (editors), Advances in Fractional Calculus, (Springer, Dordrecht, The Netherlands, 2007).

[10] Ikezawa S, Nakamura Y. J. Phys. Soc. Jpn 1981; 50, 962.

[11] Fried, B. D. & Gould, R. W., Longitudinal Ion Oscillations in a Hot Plasma, 1961, Phys. Fluids, 4, 139-147.
[12] Pottelette R., Ergun R E., Treumann R A., Berthomier M., Carlson C. W., McFadden J. P., Roth I., Geophys. Res. Lett. 1999, 26, 2629.

[13] Washimi H. and Taniuti T., Phys. Rev. Lett. 1966; 17, 996.

[14] S. A. Elwakil, M. A. Zahran and E. K. El-Shewy, Phys. Scr 2007, 75: 803.

[15] Kakad, A. P., S. V. Singh, R. V. Reddy, G. S. Lakhina, G. S. Tagare, Advances in Space Research (2009), doi: 10.1016/j.asr.2009.03.005.

[16] Podlubny, I., Fractional Differential Equations, (Academic Press, San Diego, 1999).

[17] Babakhani, A. and Gejji, V. D., Existence of positive solutions of nonlinear fractional differential equations, J. Mathematical Analysis and Applications 278 (2003) 434–442.

[18] He, J-H., Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering 167 (1998) 57-68.

[19] He, J.-H., A new approach to nonlinear partial differential equations, Communication Nonlinear Science and Numerical Simulation 2(4) (1997) 230-235.

[20] He, J.-H., Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbo-machinery aerodynamics, Int. J. Turbo Jet-Engines 14(1) (1997) 23-28.

[21] He, J.-H., Variational principles foe some nonlinear partial differential equations with variable coefficients, Chaos, Solitons and Fractals 19 (2004) 847-851.

[22] He, J-H. and Wu, X-H. Construction of solitary solution and compacton-like solution by variational iteration method, Chaos, Solitons and Fractals 29 (2006) 108-113.

[23] Abulwafa, E. M., Abdou M. A., Mahmoud A. A., The Variational-Iteration Method to Solve the Nonlinear Boltzmann Equation, Zeitschrift für Naturforschung A 63a (2008) 131-139.
[24] Molliq, R. Y., Noorani, M. S. M. and Hashim, I., Variational iteration method for fractional heat- and wave-like equations, Nonlinear Analysis: Real World Applications 10 (2009) 1854-1869.

[25] Sweilam, N. H., Khader, M. M. and Al-Bar, R. F., Numerical studies for a multi-order fractional differential equation, Physics Letters A 371 (2007) 26–33.
Figure Captions

Fig (1): The electric potential \( \Phi(\xi, \tau) \) vs \( \xi \) and \( \tau \) for \( \lambda = 1, v = 0.04, \alpha = 0.5, \beta = 0.05 \), (a) \( \mu = 0.2 \) and (b) \( \mu = 0.3 \).

Fig (2): The electric potential \( \Phi(\xi, \tau) \) vs \( \xi \) and \( \mu \) for \( \lambda = 1, v = 0.04, \alpha = 0.5, \beta = 0.05 \) and \( \tau = 5 \): (a) 3 dimensions and (b) 2 dimensions.

Fig (3): The amplitude of the electric potential \( \Phi(0, \tau) \) vs \( \tau \) and \( \alpha \) for \( \lambda = 1, v = 0.04, \beta = 0.05 \) and \( \mu = 0.2 \): (a) 3 dimensions and (b) 2 dimensions.