RANDOM LIPSCHITZ-KILLING CURVATURES: REDUCTION PRINCIPLES, INTEGRATION BY PARTS AND WIENER CHAOS

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ABSTRACT. In this survey we collect some recent results regarding the Lipschitz-Killing curvatures (LKCs) of the excursion sets of random eigenfunctions on the two-dimensional standard flat torus (arithmetic random waves) and on the two-dimensional unit sphere (random spherical harmonics). In particular, the aim of the present survey is to highlight the key role of integration by parts formulae in order to have an extremely neat expression for the random LKCs. Indeed, the main tool to study local geometric functionals of random waves on manifold is to exploit their Wiener chaos decomposition and show that (often), in the so-called high-energy limit, a single chaotic component dominates their behavior. Moreover, reduction principles show that the dominant Wiener chaotic component of LKCs of random waves’ excursion sets at threshold level \( u \neq 0 \) is proportional to the integral of \( H_2(f) \), \( f \) being the random field of interest and \( H_2 \) the second Hermite polynomial. This will be shown via integration by parts formulae.

1. Introduction

The aim of the present survey is to sum up several results presented recently in a number of articles about the local geometry of so-called random waves, that are random eigenfunctions of the Laplacian on a compact smooth Riemannian manifold, and giving the reader some evidences for broadly understanding a specific part of this stream of literature. By this, we mean to give some insights of the methodologies and techniques behind most of the proofs, creating, as much as possible, a common thread that unifies the examined works. Lastly, in order to achieve the above-mentioned goal, we will present some alternative proofs of some known results.

1.1. Background and notation. In order to introduce our framework, let us fix some preliminary notation. Consider a smooth Riemannian manifold \((M, g)\) and a random eigenfunction \( f_n : M \to \mathbb{R} \) of the Laplacian defined with respect to the Riemannian metric \( g \), denoted \( \Delta_g \), that is \( f_n \) almost surely solves the Helmholtz equation \( \Delta_g f_n + \lambda_n f_n = 0 \), where \( -\lambda_n \leq 0 \) is its eigenvalue. Now fix a level \( u \in \mathbb{R} \), we are interested in the geometric properties of the excursion sets of \( f_n \), i.e.

\[ E_u(f_n, M) := \{ x \in M : f_n(x) \geq u \} , \]

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in the high-energy (or high-frequency) limit $\lambda_n \to \infty$. Indeed, starting from the seminal work by [6], the subject has recently attracted great interest, in particular as a consequence of the author conjecturing that, as $\lambda_n \to \infty$, local geometric functionals of a planar random eigenfunction $f_n$ reflect the behavior of a typical deterministic Laplace eigenfunction on any generic manifold.

Indeed, the geometry of excursion sets for deterministic, and hence more challenging, eigenfunctions of the Laplacian on a smooth, compact Riemannian manifold have been studied intensively for some time, see among others [17, 9, 61, 19] and the recent remarkable articles solving the Yau’s conjecture [27, 28]; as a consequence, the Berry random wave model attracted several researchers and many articles were written as a consequence of Berry’s conjecture, see e.g. [46, 55, 60].

Another interesting fact that arose from a work by Berry [7], is that the fluctuations of the boundary length of $E_u(f_n, D)$ at $u = 0$, $D$ being a smooth subset of $\mathbb{R}^2$, that is of the so-called nodal length, have an unexpected logarithmic order; this was due to a cancellation into the computations whose meaning at that time was considered obscure by the author. This curious phenomenon attracted many researchers to work on local geometric functionals associated to the Berry random wave model on compact manifolds, in particular on their nodal length, see e.g. [24, 60], and the cancellation phenomenon was connected to an exact simplification of several terms appearing in the Kac-Rice formula, which is used to compute the so-called two-point correlation function of the random geometric functional and hence its variance. Later on, [37, 38] related the cancellation phenomena also to the disappearance, for $u = 0$, of the quadratic term in the Hermite expansion of the area of $E_u(f_n, S^2)$, $S^2$ be the two-dimensional unit sphere, but only in [34] the authors elected the Wiener chaos decomposition as fundamental to study the second order behavior of nodal lines on the torus and in general of these local geometric functionals, see Section 3.

In this survey we want to focus on two cases, when $\mathcal{M}$ is either the two-dimensional standard flat torus or the two-dimensional unit sphere, denoted by $T^2$ and $S^2$ respectively. In two dimensions, the so-called Lipschitz-Killing curvatures (see [1]), in the sequel often abbreviated as LKCs, of the excursion sets of the random field $f_n$ characterize its local geometry, those are the Euler-Poincaré characteristic $L^{f_n}_0(E_u(f_n, \mathcal{M}))$, the boundary length $L^{f_n}_1(E_u(f_n, \mathcal{M}))$ and the area $L^{f_n}_2(E_u(f_n, \mathcal{M}))$.

The above mentioned cancellation phenomena depend on the threshold $u$ and happen at $u = 0$ for $L^{f_n}_0(E_u(f_n, \mathcal{M}))$ and $L^{f_n}_1(E_u(f_n, \mathcal{M}))$, at $u = -1, 0, 1$ for $L^{f_n}_0(E_u(f_n, \mathcal{M}))$. As a consequence, there is a fundamental difference between what happen at $u = 0$, that is for the geometry of the so-called nodal sets, and what happen at levels $u \neq 0$. In the sequel, we will show explicitly how these cancellations become evident if one consider the Wiener chaos decompositions of the Lipschitz-Killing curvatures, see Remark 3.5 as well as Section 4.

Let us briefly give the bibliographic references of the results obtained for LKCs in the case of both random spherical harmonics ($\mathcal{M} = S^2$) and arithmetic random waves ($\mathcal{M} = T^2$) in the past years.
1.2. Nodal case. Marinucci and Wigman [37] studied the variance of the so-called defect in the case of $\mathcal{M} = S^2$, which is closely related to the area of the excursion sets at level $u = 0$, $\mathcal{L}_2^f (E_0(f_n, S^2))$, while in [50, 51] Rossi obtained quantitative central limit theorems for the defect in the case $\mathcal{M} = S^d$, $d \geq 2$, $S^d$ being the $d$-dimensional unit-sphere, and showed that its high-energy limit behavior only depend on the odd Wiener chaoses, as the even chaoses vanish at $u = 0$. The defect on the two-dimensional standard flat torus was only very recently studied in [25], where the authors proved results on the variance and on its spatial distribution.

Now let us focus on past works involving $\mathcal{L}_1^f (E_0(f_n, \mathcal{M}))$ and related functionals. Number theorists, namely the authors of [46] and [55], were the first researchers writing on the zeroes of arithmetic random waves. This fact is a direct consequence of the structure of Laplace eigenspaces on the torus, which is inextricably linked to arithmetic considerations, like e.g. that of enumerating lattice points on circles, see Section 2.1. In particular, the authors of [46] and [55] studied the expectation and variance of the Leray measure of the nodal sets and of the nodal volume, respectively. While Rudnick and Wigman [55] only provided a bound for the variance of the volume, the celebrated article [24] by Krishnapur, Kurlberg and Wigman gives an exact asymptotic, showing the non-universality of the limit. Such non-universality was reconfirmed by Marinucci, Peccati, Rossi and Wigman [34], who provided a non-central limit theorem for the nodal length (see also [18], where the authors showed that the non-universality is preserved by the so-called phase-singularities). On the unit sphere, some preliminary results were obtained already in [5] and [62], where the expectations of the nodal lengths of the long and small energy window random functions were computed. However, the first results on local geometric functionals on $S^2$ that can be compared to the ones obtained in [24] on $T^2$ were presented by Wigman [60], who computed the expectation and variance of the nodal length of the spherical harmonics. Moreover, Marinucci, Rossi and Wigman [36] obtained a central limit theorem, in the high frequency limit (see also the survey [50] as well as the interesting monograph [33]).

The study of the Euler-Poincaré characteristic at level $u = 0$ is still open; what is known is that a cancellation is occurring at levels $u = -1, 0, 1$, see [11, 12] for the results on the sphere and on the torus respectively.

1.3. Non-zero levels. Let us now consider non-zero level Lipschitz-Killing curvatures. Regarding the excursion area, that is $\mathcal{L}_2^f (E_u(f_n, \mathcal{M}))$ when $u \neq 0$. In the case of the two dimensional sphere, Marinucci and Wigman [38] computed the variance and obtained a CLT, while Marinucci and Rossi [35] extended the results to $S^d$, $d \geq 2$. For arithmetic random waves, analogous results can be found in the recent article [12].

Let us now focus on the boundary length for $u \neq 0$: the computation of the variance and a CLT on $S^2$ can be found once again in Rossi PhD thesis [50], while analogous results on $T^2$ can be found in [Remark 2.4] [34], as well as [52] and [12], showing that the universality is preserved in the non-nodal case. Indeed, it is important to underline that, while for the spherical harmonics both the nodal length and the non-zero-level length converge in distribution to a normal random variable,

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1Roughly speaking, the defect of a random eigenfunction is the difference between its positive and negative area.
on the torus the nodal length converges in law to a linear combination of independent chi-squares (where the coefficients depend on the chosen subsequence of energy levels) and the non-zero-level length to a Gaussian random variable.

Regarding the EPC at \( u \neq 0 \), complete results, namely on the variance and on the high-energy limit distribution, with a quantitative CLT, for spherical harmonics can be found in [11], while for arithmetic random waves once again in [12].

Related results on the same type of functionals but on other manifolds can be found in [54], [10], [30], [29], [4], [18], [53], [31], [32], [39], [16], [35], [13], [14], [8], [15], [57], [58], [14], [35], [31], [32], [39], [16], [35], [13], [14], [8], [15], [57], [58], [44, 49, 59], [3], [41].

1.4. Motivations and aim of the survey. Even if the Wiener chaos expansion was already used to study random geometric functionals, for instance in [23], [2], [20], the work by Marinucci, Peccati, Rossi and Wigman [34] was fundamental in presenting the remarkable idea that has allowed to prove limit theorems for nodal (and then also non-nodal) statistics of random waves on compact manifolds: first one has to derive their chaos decompositions, then prove that, most of the time, a single chaotic projection is dominant. On the sphere and on the torus, the dominating chaotic projections can often be represented as explicit functionals of a finite collection of independent Gaussian coefficients and one can make use of the standard CLT or its possible generalizations

\[^2\] Moreover, first in [11] and then in [36, 12], the authors were able to prove some reduction theorems in which they show that the dominant chaotic component could even have a neater expression, given by a deterministic function of the threshold \( u \) times the integral of a single Hermite polynomial evaluated on the field \( f_n \). Some immediate reduction principles were already proved via integration by parts formulae, in particular the Green identity, for the boundary length at \( u \neq 0 \) in [50], [52], in order to prove CLTs via usual tools and show the exact cancellation of the second chaos at \( u = 0 \). However, the extension to the Euler-Poincaré characteristics in [11] and [12] rely on a different approach requiring more involved computations. Regarding the reduction principle for the nodal length on the sphere in [30] (see also [59] for analogous results on the plane), the situation is not directly comparable since it is an asymptotic full correlation result, i.e. the authors were able to prove that the behavior of the single dominant chaotic component is asymptotically the same of a simpler statistic, the so-called sample trispectrum.

The aim of the present survey is to present the fundamental steps that, after the seminal work [34], allowed to prove high-energy limit results in the above mentioned works, focussing on Wiener chaos expansions and reduction principles. In particular, we will not present the asymptotic results regarding the variance and the distributional limit of the three LKCs, but only show the crucial role of the Wiener chaos expansion of these local geometric functionals, together with their reduction theorems that can be beautifully obtained via integration by parts formulae for the non-zero level case, \( u \neq 0 \). Finally, we will show that the reduction principles obtained in [11] and [12], that is for the Euler-Poincaré characteristics on the sphere and on the torus respectively, can also be proved via integration by parts formulae.

\[^2\] Another well-known strategy could be the use of the so-called Fourth Moment Theorem by [45] and in its quantitative form by [42]. This can be crucial in situations in which the standard CLT cannot be applied.
Remark 1.1. In this survey, by reduction principles we mean not only the fact that a single chaotic component dominates the functional of interest, but also that this single chaotic component can be written as a function of the level \( u \neq 0 \) times the integral of \( H_2\left(f_n\right) \). Moreover, it is important to stress that the reduction principles for the boundary length on both manifolds and for the Euler-Poincaré characteristic on the torus hold for each \( n \), namely are non-asymptotic results. The high-energy asymptotic regime starts to play a fundamental role when one wants to prove that a single component in the Wiener chaos expansion dominates the Lipschitz-Killing curvature of interest. See also Remarks 2.1 and 4.1.

Remark 1.2. Note that in [41], an integration by parts formula for some general functionals of independent random field is presented. However, this formula does not cover the case of the Euler-Poincaré characteristic.

Remark 1.3. At the end of this introduction, it is worth pointing out in which sense the geometric functionals considered here have a local nature, as often mentioned before: the crucial point is that the three Lipschitz-Killing curvatures satisfy some additivity properties. Indeed, it is always possible to exploit the fact that, if \( A, B \) are two closed convex subsets of \( \mathcal{M} \) and \( A \cap B = \emptyset \), then \( L_j(A \cup B) = L_j(A) + L_j(B) \), \( j = 0, 1, 2 \). This property fails for so-called global geometric quantities associated with excursion sets, whose study becomes more challenging. To mention some results of this type, Nazarov and Sodin [40] studied the expectation of the number of connected components of nodal sets of generic Gaussian random functions of several real variables, while, for the so-called random band-limited functions, in [56] is showed that topologies and nestings of the zero and nodal sets have universal laws of distribution.

Plan of the survey. In Section 2 we briefly present the construction of arithmetic random waves and random spherical harmonics, and we set most of the notations. In Section 3 we give a brief compendium on Wiener chaos and we show the chaotic decompositions of the three LKCs. Finally, in Section 4 we present the reduction principles together with their proofs, in particular giving an alternative proof for the reduction formula of the Euler-Poincaré characteristic.

2. Random eigenfunctions

In Section 1.1 we gave a brief definition of random eigenfunctions of the Laplacian. Since we will focus on \( \mathcal{M} \) being either the two-dimensional standard flat torus or the two-dimensional unit sphere, here we want to introduce in more detail random eigenfunctions of the Laplacian on these two specific smooth compact Riemannian manifolds or, more precisely, we will present how these eigenfunctions are constructed in the Gaussian random framework. In particular, we will see that, in the high-energy limit, i.e. when the eigenvalues diverge, the covariance structure of the Gaussian random eigenfunction converges in some sense to the one of the Berry random wave model on the Euclidean plane, whose covariance kernel is, for \( x, y \in \mathbb{R}^2 \),

\[
\text{Cov} (f(x), f(y)) = J_0 (\|x - y\|),
\]

\( J_0 \) being the Bessel function of order 0 (see [22 Section 1.71]) and \( \|\cdot\| \) denoting the Euclidean norm. It is worth stressing that this model, according to Berry, predict the local behavior of deterministic eigenfunction on generic chaotic surfaces for large eigenvalues, see [6, 7].
2.1. **Arithmetic random waves.** It is a standard fact that the Laplacian eigenvalues for the two-dimensional standard flat torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ are of the form $-\lambda_n = -4\pi^2 n$, where $n$ is an integer that can be written as a sum of two squares, i.e.

$$n \in S := \{ n = a^2 + b^2 : a, b \in \mathbb{Z} \},$$

this is why the eigenspaces of the Laplacian on $\mathbb{T}^2$ are related to the theory of lattice points on circles. In order to introduce the eigenspace associated to each $\lambda_n$, we need to define the set of frequencies

$$(2.2) \quad \Lambda_n := \{ \xi \in \mathbb{Z}^2 : \|\xi\| = \sqrt{n} \}, \quad n \in S,$$

denoting $\Lambda_n$ its cardinality, so that $\mathcal{N}_n$ is the multiplicity of $\lambda_n$. In particular, via the set $\Lambda_n$ in (2.2) and denoting $\delta_z$ the Dirac mass at $z \in \mathbb{R}^2$, one can define a probability measure $\mu_n$ on the unit circle $S^1 \subset \mathbb{R}^2$

$$\mu_n := \frac{1}{N_n} \sum_{\xi \in \Lambda_n} \delta_{\xi/\sqrt{n}},$$

see [24] for a more detailed discussion.

The eigenspace $\mathcal{E}_n$ associated with $\lambda_n$ is spanned by the $L^2$-orthonormal set of functions $\{ e^{2\pi i \langle \xi, \cdot \rangle} \}_{\xi \in \Lambda_n}$, so that, for $n \in S$, the arithmetic random wave of order $n$ is a Gaussian randomization of functions living in the eigenspace $\mathcal{E}_n$, or, being more precise, a random linear combination of the following form, see [55]:

$$(2.3) \quad f_n(x) := \frac{1}{\sqrt{N_n}} \sum_{\xi \in \Lambda_n} a_\xi e^{i2\pi \langle \xi, x \rangle}, \quad x \in \mathbb{T}^2,$$

where $\{a_\xi\}_{\xi \in \Lambda_n}$ is a family of identically distributed standard complex Gaussian random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and independent except for the relation $\overline{a_\xi} = a_{-\xi}$ that ensures $f_n$ to be real. Equivalently, the random field $f_n$ can be defined as the centered Gaussian function on $\mathbb{T}^2$ whose covariance kernel is, for $x, y \in \mathbb{T}^2$,

$$(2.4) \quad \text{Cov}(T_n(x), T_n(y)) = \frac{1}{N_n} \sum_{\xi \in \Lambda_n} e^{i2\pi \langle \xi, x-y \rangle}$$

and by this alternative definition it is clear that the law of $f_n$ is invariant under translations, namely that $f_n$ is stationary. Note there exists a density one subsequence $\{n_j\}_j \subset S$ of energy levels such that, as $j \to +\infty$,

$$\mu_{n_j} \Rightarrow d\theta/2\pi,$$

$d\theta$ denoting the uniform measure on $S^1$, see [24]. From (2.4) we have, for $x, y \in \mathbb{T}^2$, as $j \to +\infty$,

$$\text{Cov}(T_{n_j}(x/2\pi \sqrt{n_j}), T_{n_j}(y/2\pi \sqrt{n_j})) = \int_{S^1} e^{i\theta \langle x-y \rangle} d\mu_{n_j}(\theta) \to J_0(\|x-y\|),$$

$J_0$ still denoting the Bessel function of order zero, showing the convergence to the Berry Random Wave Model [21]. A partial classification of the others possible weak-* limits of subsequences of $\{\mu_n\}_{n \in S}$ can be found in [17].

**Remark 2.1.** The result for $\mathcal{M} = \mathbb{T}^2$ presented in this survey are non-asymptotic, however, they are the preamble of most of the high-energy limit results obtained in the works mentioned in the introduction. On the torus, by high-energy limit one means the asymptotic behavior as $n \to \infty$ such that $\mathcal{N}_n \to \infty$, which is not granted
as for the spherical case (see the next section). Anyway we can say that \( \mathcal{N}_n \) grows on average as \( \sqrt{\log n} \), see [26]. As a consequence, talking about high-energy limit on the torus, one has to take the extra assumption that \( \mathcal{N}_n \) grows to infinity with \( n \). This assumption will be tacit in the sequel. See also Remark 4.1.

2.2. Random spherical harmonics. On the two-dimensional unit sphere \( S^2 \), the Laplacian eigenvalues are of the form \( -\lambda_n = -n(n+1) \), where \( n \in \mathbb{N} \), the multiplicity of the \( n \)-th eigenvalue is \( 2n+1 \) and the family \( \{Y_{n,m}\}_{m=-n}^{n} \) of deterministic functions, which are a base of the so-called spherical harmonics [33, Section 3.4], represents an orthonormal basis for the eigenspace \( E_n \) corresponding to \( \lambda_n \). As a consequence, one can choose an arbitrary \( L^2 \)-orthonormal basis for \( E_n \) and construct the \( n \)-th random spherical harmonic on \( S^2 \) in the following way, see [60]:

\[
 f_n(x) := \sqrt{\frac{4\pi}{2n+1}} \sum_{m=-n}^{n} a_{n,m} Y_{n,m}(x), \quad x \in S^2,
\]

where \( \{a_{n,m}\}_{m=-n}^{n} \) is a family of identically distributed standard complex Gaussian random variables, and independent except for the relation \( a_{n,m} = (-1)^n a_{n,-m} \) that ensures \( f_n \) to be real. Also on the sphere, the random field \( f_n \) can be equivalently defined as the centered Gaussian function on \( S^2 \) whose covariance kernel is, for \( x, y \in S^2 \),

\[
 \text{Cov}(f_n(x), f_n(y)) = P_n(\cos d(x, y)),
\]

where \( P_n \) denotes the \( n \)-th Legendre polynomial, see [33, Section 13.1.2], and \( d(x, y) \) the geodesic distance between the two points \( x, y \in S^2 \). Thanks to Hilb’s asymptotic formula [22, Theorem 8.21.12], which states that, uniformly for \( \theta \in [0, \pi - \varepsilon] \) (\( \varepsilon > 0 \)), as \( n \to +\infty \),

\[
 P_n(\cos \theta) \sim \sqrt{\frac{\theta}{\sin \theta}} J_0((n + 1/2)\theta),
\]

we have again some heuristics regarding the above-mentioned Berry’s conjecture [21].

**Remark 2.2.** The reader probably already noticed the unified notation \( f_n, -\lambda_n \) for the Gaussian random eigenfunction and its eigenvalue, respectively, that lives either on the two dimensional unit sphere or on the two dimensional standard flat torus. Since the results presented in this survey are independent of the manifold, \( f_n \) will denote, from now on, both the random fields. In particular, if \( M = \mathbb{T}^2 \) one has to think about arithmetic random waves and the fact that \( n \in \mathbb{N} \) will be implied, otherwise, if \( M = S^2 \) one has to think about random spherical harmonics.

3. Lipschitz-Killing curvatures and Wiener chaos

Let us recall once again that for a random field with a two-dimensional domain, Lipschitz-Killing curvatures are the three quantities that characterize any local geometric functional associated with its excursion sets, for a detailed discussion see [1, Ch.6, Section 6.3] and in particular Theorem 6.3.1. Moreover, several recent articles have shown that the Lipschitz-Killing curvatures for the excursion sets of both random spherical harmonics and arithmetic random waves are often dominated, in the high-energy limit \( n \to \infty \), by a single Wiener chaotic component. More precisely, for the Lipschitz-Killing curvatures of the excursion sets at level \( u \neq 0 \), this dominant chaotic component is the projection onto the second order Wiener chaos.
and it can be written, thanks to the reduction principles proved in \[50\] \[11\] \[12\], as a simple explicit function of the threshold parameter \(u\) times the centered norm of these random fields, see also Section \[1\] this is why its disappearance results in a smaller order variance and on a different limiting behavior. For this reason, in Section \[3.1\] we will present a short compendium on Wiener chaos and then in Section \[3.2\] we will show results regarding the Wiener chaotic decompositions for the three Lipschitz-Killing curvatures, namely the excursion area, the boundary length and the Euler Poincaré characteristic, that were obtained by the various authors that worked on this topic.

We stress that, regarding nodal Lipschitz-Killing curvatures, the dominant term is not always a single chaotic component and it can be of different order, for instance the fourth chaos for the nodal length and odd chaoses for the defect.

3.1. Wiener chaos. As previously remarked, since \[54\], the Wiener chaos decomposition of local geometric functionals such as LKCs plays a fundamental role in understanding their asymptotic behavior, so here we introduce the concept of Wiener chaos both on the torus and on the sphere, trying as much as possible to unify the framework. For a complete discussion on Wiener chaos see [Section 2.2] \[43\] and the references therein. From now on, \(\mathcal{M}\) will denote either the two-dimensional sphere \(S^2\) or the two-dimensional torus \(T^2\).

Denote by \(\{H_k\}_{k \geq 0}\) the sequence of Hermite polynomials on \(\mathbb{R}\); these polynomials are defined recursively as follows: \(H_0 \equiv 1\) and

\[
H_k(t) = tH_{k-1}(t) - H'_{k-1}(t), \quad k \geq 1.
\]

Recall that \(\mathbb{H} := \{(k!)^{-1/2}H_k, k \geq 0\}\) forms a complete orthonormal system in the space of square integrable real functions \(L^2(\gamma)\) with respect to the standard Gaussian density \(\gamma\) on the real line, see [Section 1.4] \[43\].

Arithmetic Random Waves (2.3) are generated from a family of complex-valued Gaussian random variables \(\{a_\xi\}_{\xi \in \mathbb{Z}^2}\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and verifying the following properties: (1) every \(a_\xi\) has the form \(a_\xi = \Re(a_\xi) + i\Im(a_\xi)\) where \(\Re(a_\xi)\) and \(\Im(a_\xi)\) are two independent real-valued, centred, Gaussian random variables with variance 1/2, (2) the \(a_\xi\)'s are stochastically independent, save for the relations \(a_{-\xi} = \overline{a_\xi}\) in particular making \(f_n\) real-valued. In the case \(\mathcal{M} = T^2\), let us define the space \(\mathcal{A}\) to be the closure in \(L^2(\mathbb{P})\) of all real finite linear combinations of random variables \(\zeta\) of the form

\[
\zeta = z a_\xi + \overline{\zeta} a_{-\xi},
\]

where \(\xi \in \mathbb{Z}^2\) and \(z \in \mathbb{C}\).

Random spherical harmonics (2.5) are generated from a family of complex-valued Gaussian random variables \(\{a_{\ell,m} : \ell = 0, 1, 2, \ldots, m = -\ell, \ldots, \ell\}\) such that (a) every \(a_{\ell,m}\) has the form \(x_{\ell,m} + iy_{\ell,m}\), where \(x_{\ell,m}\) and \(y_{\ell,m}\) are two independent real-valued Gaussian random variables with mean zero and variance 1/2; (b) \(a_{\ell,m}\) and \(a_{\ell',m'}\) are independent whenever \(\ell \neq \ell'\) or \(m' \notin \{m, -m\}\), and (c) \(a_{\ell,m} = (-1)^\ell a_{\ell,-m}\) In the case \(\mathcal{M} = S^2\), define the space \(\mathcal{A}\) to be the closure in \(L^2(\mathbb{P})\) of all real finite linear combinations of random variables \(\zeta\) of the form

\[
\zeta = z a_{\ell,m} + \overline{z} (-1)^\ell a_{\ell,-m}, \quad z \in \mathbb{C}.
\]

In both cases, \(\mathcal{A}\) is a real centered Gaussian Hilbert subspace of \(L^2(\mathbb{P})\).
Let us fix now an integer $q \geq 0$; the $q$-th Wiener chaos $C_q$ associated with $A$ is defined as the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables of the type

$$H_{p_1}(\xi_1) \cdot H_{p_2}(\xi_2) \cdots H_{p_k}(\xi_k)$$

for $k \geq 1$, where the integers $p_1, \ldots, p_k \geq 0$ satisfy $p_1 + \cdots + p_k = q$, and $(\xi_1, \ldots, \xi_k)$ is a standard real Gaussian vector extracted from $A$ (in particular, $C_0 = \mathbb{R}$).

Taking into account the orthonormality and completeness of $H$ in $L^2(\gamma)$ (see e.g. [43, Theorem 2.2.4]), it is possible to prove that $C_q \perp C_m$ in $L^2(\mathbb{P})$ for every $q \neq m$, and moreover

$$L^2(\Omega, \sigma(A), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q,$$

that is, every real-valued functional $F$ of $A$ can be (uniquely) represented as a series, converging in $L^2$, of the form

$$F = \sum_{q=0}^{\infty} F[q],$$

where $\text{Proj}[F|q]$ stands for the projection of $F$ onto $C_q$, and the series converges in $L^2(\mathbb{P})$. Plainly, $\text{Proj}[F|0] = \mathbb{E}[F]$.

In the sequel, for $i, j = 1, 2$, we will denote by $\partial_i f_n(x) = \partial_i f_n(x_1, x_2)$ the partial derivative with respect to $x_i$ and denote $\partial_{ij} f_n(x) = \partial_{ij} f_n(x_1, x_2)$ the second partial derivative with respect to $x_i$ and $x_j$. For $\mathcal{M} = \mathbb{S}^2$, note that $x = (x_1, x_2) = (\theta, \phi)$, $\theta \in (0, \pi)$, $\phi \in [0, 2\pi)$, and in this system of coordinates the gradient is given by $\nabla = \left(\frac{\partial}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}\right)$. The random fields $f_n, \partial_j f_n, \partial_{ij} f_n$ viewed as collections of Gaussian random variables indexed by $x \in \mathcal{M}$ are all lying in $A$, i.e. for every $x \in \mathcal{M}$ we have

$$f_n(x), \partial_j f_n(x), \partial_{ij} f_n(x) \in A.$$

3.2. Chaotic expansions of Lipschitz-Killing curvatures. We recall that, from now on, $\mathcal{M}$ denotes either $\mathbb{T}^2$ or $\mathbb{S}^2$, and it will not be specified when not needed. The Lipschitz-Killing curvatures are finite-variance functionals of $A$, hence applying (3.1) we get the series expansion

$$L^f_k(\mathcal{E}_u(f_n, \mathcal{M})) = \sum_{q=0}^{\infty} \text{Proj}[L^f_k(\mathcal{E}_u(f_n, \mathcal{M}))|q].$$

Let us be more precise.

The excursion area has the following integral representation, see also [Section 3][37],

$$L^f_2(\mathcal{E}_u(f_n, \mathcal{M})) = \int_{\mathcal{M}} 1_{\{f_n(x) \geq u\}} \, dx$$

which guarantees that $L^f_2(\mathcal{E}_u(f_n, \mathcal{M})) \in L^2_2(\mathbb{P})$. In [37] and [12] one can find the following result, which is proven in detail in [Section 3][37].
Proposition 3.1 ([37, 12]). For every \( n \) such that \( f_n \) is an eigenfunction of \( \Delta_g \) with eigenvalue \( \lambda_n \) and \( u \in \mathbb{R} \), the chaotic decomposition of \( \mathcal{L}_2^n(\mathcal{E}_u(f_n, \mathcal{M})) \) is given by

\[
\mathcal{L}_2^n(\mathcal{E}_u(f_n, \mathcal{M})) = \sum_{q=0}^{\infty} \frac{\gamma_q(u)}{q!} \int_{\mathcal{M}} H_q(f_n(x)) \, dx,
\]

where \( \gamma_q(u) := H_{q-1}(u)\phi(u) \), and the convergence of the series is in \( L^2(\mathbb{P}) \).

The boundary length has the following formal integral representation, see [Section 7.2.1] [50]

\[
\mathcal{L}_1^n(\mathcal{E}_u(f_n, \mathcal{M})) = \frac{1}{2} \int_{\mathcal{M}} \delta_u(f_n(x)) \| \nabla f_n(x) \| \, dx,
\]

where \( \delta_u \) is the Dirac mass in \( u \), and \( \nabla f_n \) is the gradient of \( f_n \). Let us introduce two collections of coefficients \( \{ \alpha_{2n,2m} : n, m \geq 1 \} \) and \( \{ \beta_l(u) : l \geq 0 \} \), that are needed in order to state the chaotic expansion of \( \mathcal{L}_1^n(\mathcal{E}_u(f_n, \mathcal{M})) \) and are related to the Hermite expansion of the norm \( \| \cdot \| \) in \( \mathbb{R}^2 \) and the (formal) Hermite expansion of the Dirac mass \( \delta_u(\cdot) \) respectively, see [50]. These are given by

\[
\beta_l(u) := H_l(u)\phi(u),
\]

where \( H_l \) still denotes the \( l \)-th Hermite polynomial and

\[
\alpha_{2n,2m} := \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m} p_{n+m}\left(\frac{1}{4}\right)},
\]

where for \( N \in \mathbb{N} \) and \( x \in \mathbb{R} \)

\[
p_N(x) := \sum_{j=0}^{N} (-1)^j \cdot (-1)^N \binom{N}{j} \frac{(2j+1)!}{(j!)^2} x^j,
\]

the ratio \( \frac{(2j+1)!}{(j!)^2} \) being the so-called swinging factorial restricted to odd indices.

Proposition 3.2 ([30, 34, 12]). For every \( n \) such that \( f_n \) is an eigenfunction of \( \Delta_g \) with eigenvalue \( \lambda_n \) and \( u \in \mathbb{R} \) the chaotic expansion of \( \mathcal{L}_1^n(\mathcal{E}_u(f_n, \mathcal{M})) \) is

\[
\mathcal{L}_1^n(\mathcal{E}_u(f_n, \mathcal{M})) = \frac{1}{2} \sqrt{\frac{\lambda_n}{2}} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \sum_{k=0}^{u} \frac{\alpha_{2k,2u-2k} \beta_{q-2u}(u)}{(2k)!(2u-2k)!(q-2u)!} \times \int_{\mathcal{T}} H_{q-2u}(f_n(x)) H_{2k}(\tilde{\partial}_1 f_n(x)) H_{2u-2k}(\tilde{\partial}_2 f_n(x)) \, dx,
\]

where the convergence of the series is in \( L^2(\mathbb{P}) \), and \( \tilde{\partial}_j f_n, j = 1, 2, x \) denotes normalized first derivatives.

The Euler-Poincaré characteristic has the following formal representation

\[
\mathcal{L}_0^n(\mathcal{E}_u(f_n, \mathcal{M})) = \int_{\mathcal{T}} \det(\nabla^2 f_n(x)) \mathbf{1}_{\{f_n(x) \geq u\}} \delta_0(\nabla f_n(x)) \, dx,
\]

where \( \nabla^2 f_n \) is the Hessian matrix of \( f_n \), and abusing notation \( \delta_0 \) denotes the Dirac mass in \( (0, 0) \).

The following result presents the chaotic expansion of \( \mathcal{L}_0^n(\mathcal{E}_u(f_n, \mathcal{M})) \), note that [11] and [12] do not give explicit expressions for chaotic coefficients but those corresponding to the zero-th and second Wiener chaoses.
Proposition 3.3 \((11)\) \((12)\). For \(n \in S\) and \(u \in \mathbb{R}\), the chaotic expansion of \(L_0^{f_n}(\mathcal{E}_u(f_n, \mathcal{M}))\) is
\[
L_0^{f_n}(\mathcal{E}_u(f_n, \mathcal{M})) = 2\lambda_n \sum_{q=0}^{+\infty} \sum_{a+b+c+2d+2e=q} \eta^{(n)}_{a,b,c}(u) \frac{\beta_{2d}\beta_{2e}}{a!b!c!(2d)!(2e)!} \int_{\mathbb{R}} H_a \left( \frac{\partial_1 f_n(x)}{\kappa_3} \right) \\
\times H_b \left( \frac{\partial_2 f_n(x)}{\kappa_4} \right) H_c \left( \frac{\partial_3 f_n(x)}{\kappa_5} \right) H_{2d} \left( \frac{\partial_1 f_n(x)}{\kappa_1} \right) H_{2e} \left( \frac{\partial_2 f_n(x)}{\kappa_1} \right) \ dx,
\]
for some coefficients \(\eta^{(n)}_{a,b,c}(u) \in \mathbb{R}, a, b, c \in \mathbb{N}\), where the series converges in \(L^2(\mathbb{P})\)
\[
(3.9) \quad \beta_q := \beta_q(0) = \phi(0) \mathcal{H}_q(0)
\]
as defined in \((3.5)\), and \(\kappa_1, \ldots, \kappa_5\) are for \(\mathcal{M} = S^2\),
\[
\kappa_1 = \frac{\sqrt{\lambda_n}}{\sqrt{2}} \quad \kappa_2 = \frac{\sqrt{\lambda_n(\lambda_n + 2)}}{2\sqrt{2}} \quad \kappa_3 = \frac{\sqrt{\lambda_n\sqrt{3}\lambda_n - 2}}{2\sqrt{2}} \\
\kappa_4 = \frac{\sqrt{\lambda_n\sqrt{3}\lambda_n - 2}}{2\sqrt{2}} \quad \kappa_5 = \frac{\lambda_n\sqrt{\lambda_n - 2}}{\sqrt{3\lambda_n - 2}}
\]
while for \(\mathcal{M} = T^2\),
\[
\kappa_1 = \frac{\lambda_n}{2} \quad \kappa_2 = \frac{\lambda_n}{2\sqrt{2}} \sqrt{\frac{1 - \mu_n(4)}{3 + \mu_n(4)}} \quad \kappa_3 = \frac{\lambda_n}{2\sqrt{2}} \sqrt{3 + \mu_n(4)} \\
\kappa_4 = \frac{\lambda_n}{2\sqrt{2}} \sqrt{1 - \mu_n(4)} \quad \kappa_5 = \lambda_n \sqrt{\frac{1 + \mu_n(4)}{3 + \mu_n(4)}}
\]
where \(\mu_n(4)\) is the fourth Fourier coefficient of \(\mu_n\).

Remark 3.4. The two formal representations \((3.4)\) and \((3.7)\) are justified by the use of the following \(\varepsilon\)-approximating random variables, \(\varepsilon > 0\),
\[
(3.10) \quad L_0^{f_n,\varepsilon}(\mathcal{E}_u(f_n, \mathcal{M})) := \frac{1}{2} \int_{\mathcal{M}} \frac{1}{2\varepsilon} 1_{[u-\varepsilon,u+\varepsilon]}(f_n(x)) \| \nabla f_n(x) \| \ dx \\
(3.11) \quad L_0^{f_n,\varepsilon}(\mathcal{E}_u(f_n, \mathcal{M})) := \int_{\mathbb{T}} \det(\nabla^2 f_n(x)) 1_{[f_n(x)\geq u]} \frac{1}{(2\varepsilon)^2} 1_{[-\varepsilon,\varepsilon]^2}(\nabla f_n(x)) \ dx
\]
that, uniformly in \(n\), converge to the first and the zero-th Lipschitz-Killing curvature respectively. The chaotic expansions are then computed for \((3.10)\) and \((3.11)\), and then extended to the original LKCs, see the detailed \((12)\) Section 4.2 and the references therein.

4. Reduction principles and integration by parts formulae

For the case \(u \neq 0\), the three Lipschitz-Killing Curvatures are dominated by their second-order chaotic components, that are given by
\[
\text{Proj}[L_0^{f_n}(\mathcal{E}_u(f_n, \mathcal{M}))][2] = 2\lambda_n \sum_{a+b+c+2d+2e=2} \eta^{(n)}_{a,b,c}(u) \frac{\beta_{2d}\beta_{2e}}{a!b!c!(2d)!(2e)!} \int_{\mathbb{R}} H_a \left( \frac{\partial_1 f_n(x)}{\kappa_3} \right) \\
\times H_b \left( \frac{\partial_2 f_n(x)}{\kappa_4} \right) H_c \left( \frac{\partial_3 f_n(x)}{\kappa_5} \right) H_{2d} \left( \frac{\partial_1 f_n(x)}{\kappa_1} \right) H_{2e} \left( \frac{\partial_2 f_n(x)}{\kappa_1} \right) \ dx.
\]
\[ \times H_b \left( \frac{\partial_2 f_n(x)}{\kappa_4} \right) H_c \left( \frac{\partial_{22} f_n(x)}{\kappa_5} - \frac{\kappa_2}{\kappa_5 \kappa_3} \partial_1 f_n(x) \right) \]
\[ \times H_{2d} \left( \frac{\partial_1 f_n(x)}{\kappa_1} \right) H_{2e} \left( \frac{\partial_2 f_n(x)}{\kappa_1} \right) \, dx , \]
\[ \text{Proj}[\mathcal{L}_1^f(\mathcal{E}_u(f_n, \mathcal{M})]|2] = \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_{0000}}{2!} \right) \int_{\mathcal{M}} H_2(f_n(x)) \, dx \]
\[ + \frac{\beta_0 a_{20}}{2!} \int_{\mathcal{M}} \left\{ H_2 \left( \partial_1 f_n(x) \right) + H_2 \left( \partial_2 f_n(x) \right) \right\} \, dx , \]
\[ \text{Proj}[\mathcal{L}_2^f(\mathcal{E}_u(f_n, \mathcal{M})]|2] = \frac{1}{2} u \phi(u) \int_{\mathcal{M}} (f_n(x)^2 - 1) \, dx ; \]

note that for the boundary length and the Euler-Poincaré characteristic, the second chaos seems to depend on the derivatives of the field \( f_n \). However, in \[50, 11\] on the sphere, in \[12\] on the torus, it was shown that all the expressions of the projections onto the second order Wiener chaos can be reduced to the following beautiful formula, which involves a deterministic function of the level \( u \) and the integral of the second Hermite polynomial \( H_2 \), evaluated on the field \( f_n \):
\[ \text{Proj}[\mathcal{L}_k^f(\mathcal{E}_u(f_n, \mathcal{M})]|2] = c_k(u) \left( \frac{\lambda_n}{2} \right)^{2-k} \int_{\mathcal{M}} H_2(f_n(x)) \, dx + O_{L^2(\mathcal{M})}(1) \cdot \delta_0^k \delta_M^2 , \]
\[ c_2(u) = \frac{1}{2} H_1(u) \phi(u), \quad c_1(u) = \frac{1}{2} \sqrt{\frac{\pi}{8}} H_1(u)^2 \phi(u), \quad c_0(u) = \frac{1}{2} H_1(u) H_2(u) \phi(u) \frac{1}{2 \pi} , \]

for every \( k = 0, 1, 2 \) and \( u \in \mathbb{R} \). While for the excursion area \( \mathcal{L}_2^f(\mathcal{E}_u(f_n, \mathcal{M}) \) the second chaos is immediately proportional to the integral of \( H_2(f_n(x)) \), on the contrary, more computations are needed in order to show that for the boundary lengths and the Euler-Poincaré characteristics it is also the case. Indeed, as one can see in \[4.1\] and \[4.2\], simply using chaotic decomposition, the boundary length depends on both the level and its gradient, while the Euler-Poincaré characteristic depends on both first and second derivatives of the field. For the boundary length, the Green’s formula (IBP) is used to prove that its second chaos is proportional to the integral of \( H_2 \) of the level, see \[50\] Section 7.3 for the computations on the sphere and both [remark 2.4][34] and [Proposition 3.2][12] for statements on the torus. A unified discussion can also be found in \[52\], in particular Section 4, and we show it again in Section \[4.1\] for sake of completeness. For the EPC, via some analytic computations, \[11\] and \[12\] show that formula \[4.3\] holds. Here we will show that this can also be done using only IBP as for the boundary length, see Section \[4.2\].

Remark 4.1. Note that in formula \[4.3\] the high-energy regime is present only for \( k = 0 \) and \( \mathcal{M} = S^2 \). This means that in all other cases the formula is exact, in the sense that is non-asymptotic.

Regarding the nodal case, that is for \( u = 0 \), only the boundary length is dominated by a single chaotic component, the fourth one; this is shown in \[36\] for the sphere and the reduction principle consists in having the dominant term asymptotically proportional to the sample trispectrum, which is the integral of \( H_4(f_n) \), \( H_4 \) being the fourth Hermite polynomial. Analogous results on the torus are shown in
where, however, the dominant fourth chaos is not proportional to the sample trispectrum.

4.1. The boundary length. In this section we show the crucial role of Green’s formula in order to see the cancellation of the second order Wiener chaos for the boundary length \( L^f_n(\mathcal{E}_n(f_n, \mathcal{M})) \) when \( u = 0 \). This was shown for the first time in [50] in the case of \( \mathcal{M} = S^2 \) and since the proof is completely independent of the manifold, as pointed out in [52, Section 4.1.2], as well as very short and interesting, we represent it here in a unified way, using our generic notation. In fact, this was the very first reduction principle for LKCs and it was proven via some integration by parts formula. Recalling the Green identity on manifolds

\[
\int_{\mathcal{M}} f_n(x) \Delta f_n(x) dx = - \int_{\mathcal{M}} \langle \nabla f_n(x), \nabla f_n(x) \rangle dx,
\]

one can apply it as follows,

\[
\text{Proj}[\mathcal{L}^f_n(\mathcal{E}_n(f_n, \mathcal{M}))][2] = \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_2\alpha_{00}}{2!} \int_{\mathcal{M}} H_2(f_n(x)) \, dx \right.
\]

\[
+ \frac{\beta_0(u)\alpha_{20}}{2!} \int_{\mathcal{M}} \left\{ H_2(\partial_1 f_n(x)) + H_2(\partial_2 f_n(x)) \right\} \, dx
\]

\[
= \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_2\alpha_{00}}{2!} \int_{S^2} (f_n(x)^2 - 1) \, dx \right.
\]

\[
+ \frac{\beta_0(u)\alpha_{20}}{2!} \int_{S^2} \left( \frac{2}{(\ell + 1)} (\nabla f_n(x), \nabla f_n(x)) - 2 \right) \, dx
\]

\[
= \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_2\alpha_{00}}{2!} \int_{S^2} (f_n(x)^2 - 1) \, dx \right.
\]

\[
+ \frac{\beta_0(u)\alpha_{20}}{2!} \int_{S^2} \left( - \frac{2}{(\ell + 1)} f_n(x) \Delta f_n(x) - 2 \right) \, dx
\]

\[
= \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_2\alpha_{00}}{2!} \int_{S^2} (f_n(x)^2 - 1) \, dx \right.
\]

\[
+ \frac{\beta_0(u)\alpha_{20}}{2!} \int_{S^2} \left( 2f_n(x)^2 - 2 \right) \, dx
\]

\[
= \sqrt{\frac{\lambda_n}{2}} \left( \frac{\beta_2\alpha_{00}}{2!} + \beta_0(u)\alpha_{20} \right) \int_{S^2} H_2(f_n(x)) \, dx,
\]

which is (4.3) in the case of \( k = 1 \).

4.2. The Euler-Poincaré characteristic. In this section we want to give an alternative proof of the reduction principles for Euler-Poincaré characteristics given in [Theorem 1.11] and [Theorem 2.4] [12]. This alternative proof is independent of the manifold \( \mathcal{M} \), except for the constants involved in the computations, that are different on the torus and on the sphere, see also Proposition 3.3 in Section 3. For this reason, we will prove (4.3) in the case \( k = 0 \) setting \( \mathcal{M} = T^2 \), avoiding to repeat analogous computations but with different constants for \( \mathcal{M} = S^2 \).

Remark 4.2. To be precise, the main difference regarding the computations to reach (4.3) in the case \( k = 0 \) on the two different manifolds, is the fact that on the sphere one has to consider covariant derivatives instead of flat derivatives. The flat geometry of the two-dimensional standard flat torus gives a neater expression for the
projection onto the second order Wiener chaos of the Euler-Poincaré characteristic, by neater we mean an expression without a reminder – see (4.3).

From [Section 6.1] [2], we know that the projection of $\mathcal{L}_0^f(E_n(f_n, T^2))$ onto the second order Wiener chaos can be compactly written as follows, see also [Section 3.2] [1] for the case of $S^2$,

$$\text{Proj}[\mathcal{L}_0(n; u)|2] = h_{35}(u; n) \int_{T^2} Y_3(x)Y_5(x)dx + \frac{1}{2} \sum_{i=1}^{5} h_i(u; n) \int_{T^2} H_i(Y_i(x))dx,$$

where

$$Y_1(x) = \frac{1}{\kappa_1} \partial_1 f_n(x) \quad Y_2(x) = \frac{1}{\kappa_1} \partial_2 f_n(x) \quad Y_3(x) = \frac{1}{\kappa_3} \partial_{11} f_n(x)$$

$$Y_4(x) = \frac{1}{\kappa_4} \partial_{12} f_n(x) \quad Y_5(x) = \frac{1}{\kappa_5} \partial_{22} f_n(x) - \frac{\kappa_2}{\kappa_3 \kappa_5} \partial_{11} f_n(x),$$

recalling that (see Proposition 3.3)

$$\kappa_1 = \sqrt{\frac{\lambda_n}{2}} \quad \kappa_2 = \frac{\lambda_n}{2 \sqrt{2}} \frac{1 - \hat{\mu}_n(4)}{\sqrt{3 + \hat{\mu}_n(4)}} \quad \kappa_3 = \frac{\lambda_n}{2 \sqrt{2}} \sqrt{3 + \hat{\mu}_n(4)}$$

$$\kappa_4 = \frac{\lambda_n}{2 \sqrt{2}} \sqrt{1 - \hat{\mu}_n(4)} \quad \kappa_5 = \frac{\lambda_n}{2 \sqrt{2}} \frac{\sqrt{1 + \hat{\mu}_n(4)}}{\sqrt{3 + \hat{\mu}_n(4)}},$$

while

$$h_{35}(u; n) = \frac{\lambda_n}{2 \sqrt{2} \pi} \sqrt{1 + \hat{\mu}_n(4)} u \phi(u)(1 + u^2) + (3 + \hat{\mu}_n(4)) \Phi(-u),$$

and

$$h_1(u; n) = h_2(u; n) = -\frac{\lambda_n}{4 \pi} u \phi(u),$$

$$h_3(u; n) = \frac{\lambda_n}{4 \pi} \left[ 2 u (1 + u^2) \phi(u) + \Phi(-u) (1 - \hat{\mu}_n(4)) \right],$$

$$h_4(u; n) = -\frac{\lambda_n}{4 \pi} (1 - \hat{\mu}_n(4)) \Phi(-u),$$

$$h_5(u; n) = \frac{\lambda_n}{4 \pi} \frac{u (1 + u^2) (1 + \hat{\mu}_n(4)) \phi(u)}{3 + \hat{\mu}_n(4)}.$$

Let us now show that, starting from (4.4), we can simply use integration by parts to arrive at (4.3), also for the case of $k = 0$, that is for the Euler-Poincaré characteristic:

$$\text{Proj}[\mathcal{L}_0(n; u)|2] = h_{35}(u; n) \int_{T^2} \frac{1}{\kappa_3} \partial_{11} f_n(x) \left( \frac{1}{\kappa_5} \partial_{22} f_n(x) - \frac{\kappa_2}{\kappa_3 \kappa_5} \partial_{11} f_n(x) \right) dx$$

$$+ \frac{h_1(u; n)}{2} \int_{T^2} H_2 \left( \frac{1}{\kappa_1} \partial_1 f_n(x) \right) dx + \frac{h_1(u; n)}{2} \int_{T^2} H_2 \left( \frac{1}{\kappa_1} \partial_2 f_n(x) \right) dx$$

$$+ \frac{h_3(u; n)}{2} \int_{T^2} H_2 \left( \frac{1}{\kappa_4} \partial_{11} f_n(x) \right) dx + \frac{h_4(u; n)}{2} \int_{T^2} H_2 \left( \frac{1}{\kappa_4} \partial_{12} f_n(x) \right) dx$$

$$+ \frac{\kappa_5(u; n)}{2} \int_{T^2} H_2 \left( \frac{1}{\kappa_5} \partial_{22} f_n(x) - \frac{\kappa_2}{\kappa_3 \kappa_5} \partial_{11} f_n(x) \right) dx$$
\[ = h_{35}(u; n) \int_{T^2} \left( \frac{1}{\kappa_3 k_5} \partial_{22} f_n(x) \partial_{11} f_n(x) - \frac{\kappa_2}{\kappa_3 k_5} (\partial_{11} f_n(x))^2 \right) dx \]

\[ + \frac{\kappa_1(u; n)}{2} \int_{T^2} \left( \frac{1}{\kappa_1^2} (\partial_{1} f_n(x))^2 - 1 \right) dx + \frac{\kappa_1(u; n)}{2} \int_{T^2} \left( \frac{1}{\kappa_1^2} (\partial_{2} f_n(x))^2 - 1 \right) dx \]

\[ + \frac{\kappa_3(u; n)}{2} \int_{T^2} \left( \frac{1}{\kappa_3^2} (\partial_{11} f_n(x))^2 - 1 \right) dx + \frac{\kappa_4(u; n)}{2} \int_{T^2} \left( \frac{1}{\kappa_4^2} (\partial_{12} f_n(x))^2 - 1 \right) dx \]

\[ + \frac{\kappa_5(u; n)}{2} \int_{T^2} \left( \frac{1}{\kappa_5^2} \left( \partial_{22} f_n(x) - \frac{\kappa_2}{\kappa_3} \partial_{11} f_n(x) \right)^2 - 1 \right) dx. \]

Now we use the fact that

\[ \int_{T^2} (\partial_{12} f_n(x))^2 \, dx = \int_{T^2} \partial_{11} f_n(x) \partial_{22} f_n(x) \, dx \]

to have

\[ \text{Proj} \{ \chi(A_n(f_n; T^2)) \} = \]

\[ = \frac{1}{\kappa_3 k_5} h_{35}(u; n) + \frac{1}{k_1^2} h_4(u; n) \int_{T^2} \partial_{1, x}^2 f_n(x) \partial_{11} f_n(x) \, dx \]

\[ + \left[ \frac{h_3(u; n)}{2k_3^2} - \frac{\kappa_2}{\kappa_3 k_5} \frac{h_{35}(u; n)}{k_3^2} \right] \int_{T^2} (\partial_{11} f_n(x))^2 \, dx + \frac{h_1(u; n)}{2\kappa_1^2} \int_{T^2} \| \nabla f_n(x) \|^2 \, dx \]

\[ + \frac{h_5(u; n)}{2} \int_{T^2} \left[ \frac{1}{\kappa_5^2} (\partial_{22} f_n(x))^2 + \frac{\kappa_2}{\kappa_3 k_5} (\partial_{11} f_n(x))^2 - 2 \frac{\kappa_2}{\kappa_3 k_5} \partial_{22} f_n(x) \partial_{11} f_n(x) \right] \, dx \]

\[ - \left[ h_1(u; n) + h_3(u; n) + h_4(u; n) + h_5(u; n) - \frac{\kappa_2}{\kappa_3 k_5} \right] \]

\[ = A(u; n) \int_{T^2} \partial_{2, x}^2 f_n(x) \partial_{11} f_n(x) \, dx + B(u; n) \int_{T^2} (\partial_{11} f_n(x))^2 \, dx \]

\[ + C(u; n) \int_{T^2} \| \nabla f_n(x) \|^2 \, dx + D(u; n) \int_{T^2} (\partial_{22} f_n(x))^2 \, dx - E(u; n). \]

After using integration by parts, now we just have to compute the constants in front of the integral terms:

\[ A(u; n) = \frac{1}{\kappa_3 k_5} h_{35}(u; n) + \frac{1}{k_1^2} h_4(u; n) - \frac{\kappa_2}{\kappa_3 k_5} h_5(u; n) = \]

\[ = \frac{2\sqrt{2}}{\lambda_n^2 \sqrt{1 + \mu_n(4)}} \frac{\lambda_n}{2\sqrt{2}\pi} \frac{\mu(1 + u^2)}{3 + \mu_n(4)} \Phi(-u) \]

\[ - \frac{8}{\lambda_n^2 (1 - \mu_n(4)) \Phi(-u)} \]
As a consequence, we easily have that

\[ B(u; n) = \frac{h_3(u; n)}{2\kappa_5^2} - \frac{\kappa_2 h_35(u; n)}{\kappa_5^2 \kappa_5} + \frac{\kappa_5^2 h_5(u; n)}{2} \]

\[ = \frac{1}{\kappa_5^2} \times \left( \frac{h_3(u; n)}{2} - \frac{\kappa_2 h_35(u; n)}{\kappa_5} + \frac{\kappa_5^2 h_5(u; n)}{2} \right) \]

\[ = \frac{8}{\lambda_n^2 (3 + \tilde{\mu}_n(4))^2} \times \left( \frac{\lambda_n}{8\pi} \left[ \frac{2u(1 + u^2)\phi(u)}{3 + \tilde{\mu}_n(4)} + \Phi(-u)(1 - \tilde{\mu}_n(4)) \right] \right. \]

\[ - (1 - \tilde{\mu}_n(4)) \left( \frac{\lambda_n}{8\pi} u(1 + u^2)\phi(u) \right) \]

\[ + \frac{(1 - \tilde{\mu}_n(4))^2 \lambda_n}{3 + \tilde{\mu}_n(4)} \frac{64\pi u(1 + u^2)\phi(u)}{8\lambda_n} \]

\[ = \frac{u(1 + u^2)\phi(u)}{8\lambda_n} \]

\[ C(u; n) = \frac{1}{\kappa_5^2} \frac{h_5(u; n)}{2} = \frac{u(1 + u^2)\phi(u)}{8\lambda_n} \]

\[ D(u; n) = \frac{h_1(u; n)}{2\kappa_5^2} = -\frac{u \phi(u)}{4\pi} \]

\[ E(u; n) = h_1(u; n) + h_3(u; n) + h_4(u; n) + h_5(u; n) \]

As a consequence, we easily have that

\[ \text{Proj}[\chi(A_u(f_n; T^2))] = \frac{\phi(u)(1 + u^2)}{4\lambda_n\pi} \int_{\mathbb{T}^2} \partial_{x,y}^2 f_n(x) \partial_{x,y} f_n(x) dx \]

\[ + \frac{u(1 + u^2)\phi(u)}{8\lambda_n\pi} \int_{\mathbb{T}^2} (\partial_{x,y} f_n(x))^2 dx \]

\[ - \frac{u \phi(u)}{4\pi} \int_{\mathbb{T}^2} \|\nabla f_n(x)\|^2 dx + \frac{u(1 + u^2)\phi(u)}{8\lambda_n\pi} \int_{\mathbb{T}^2} (\partial_{x,y} f_n(x))^2 dx \]

\[ - \frac{\lambda_n}{8\pi} H_1(u) H_2(u) \phi(u) \]

\[ = \frac{u(1 + u^2)\phi(u)}{8\lambda_n\pi} \int_{\mathbb{T}^2} (\Delta_{x,y} f_n(x))^2 dx - \frac{u \phi(u)}{4\pi} \int_{\mathbb{T}^2} \|\nabla f_n(x)\|^2 dx \]

\[ - \frac{\lambda_n}{8\pi} H_1(u) H_2(u) \phi(u) \]
and recalling the basic (Green-Stokes) identity
\[\int_{T^2} \|\nabla f_n\|^2 \, dx = -\int_{T^2} f_n \Delta_{T^2} f_n \, dx\]
we obtain
\[
\text{Proj}\left[\chi\left(A_u(f_n; T^2)\right)\right] = u(1 + u^2)\phi(u) \int_{T^2} (\lambda_n f_n(x))^2 \, dx
+ \frac{u \phi(u)}{4\pi} \int_{T^2} f_n(x) \Delta_{T^2} f_n(x) \, dx - \frac{\lambda_n}{8\pi} H_1(u) H_2(u) \phi(u)
\]
\[= u(1 + u^2)\phi(u) \int_{T^2} \lambda_n^2 f_n(x)^2 \, dx - \frac{2u \phi(u)}{8\pi} \int_{T^2} \lambda_n f_n(x)^2 \, dx
- \frac{\lambda_n}{8\pi} H_1(u) H_2(u) \phi(u)
\]
\[= \frac{H_1(u) H_2(u) \phi(u) \lambda_n}{8\pi} \int_{T^2} H_2(f_n(x)) \, dx,
\]
which is the desired formula.

References
1. R. J. Adler and J. E. Taylor, Random fields and geometry, Springer Monographs in Mathematics, Springer-Verlag New York, 2007.
2. J. M. Azaïs and J. R. León, CLT for crossings of random trigonometric polynomials, Electron. J. Probab. 18 (2013), no. none, 1–17.
3. D. Beliaev, V. Cammarota, and I. Wigman, Two point function for critical points of a random plane wave, Int. Math. Res. Notices (2017), rnx197.
4. J. Benatar and R. W. Maffucci, Random waves on \(\mathbb{R}^3\): Nodal area variance and lattice point correlations, Int. Math. Res. Notices (2017), rnx220.
5. P. Bérard, Volume des ensembles nodaux des fonctions propres du laplacien, Bony-Sjostrand-Meyer seminar, École Polytechnique, Palaiseau Exp. No. 14 (1984–1985), 10 pp.
6. M.V. Berry, Regular and irregular semiclassical wavefunctions, J. Phys. A 10 (1977), no. 12, 2083–2092.
7. , Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature, J. Phys. A 35 (2002), no. 13, 3025–3038.
8. S. Bourguin, C. Durastanti, D. Marinucci, and G. Peccati, Approximate normality of high-energy hyperspherical eigenfunctions, Journal of Mathematical Analysis and Applications 436 (2016), no. 2, 1121–1148.
9. J. Brüning, über knoten von eigenfunktionen des laplace-beltrami-operators, Mathematische Zeitschrift 158 (1978), 15–21.
10. V. Cammarota, Nodal area distribution for arithmetic random waves, Transactions of the A. M. S. 372 (2019), 3539–3564.
11. V. Cammarota and D. Marinucci, A quantitative central limit theorem for the Euler-Poincaré characteristic of random spherical eigenfunctions, Ann. Probab. 46 (2018), no. 6, 3188–3228.
12. V. Cammarota, D. Marinucci, and M. Rossi, Lipschitz-killing curvatures for arithmetic random waves, arXiv:2010.14165, October 2020.
13. V. Cammarota, D. Marinucci, and I. Wigman, Fluctuations of the euler-poincaré characteristic for random spherical harmonics, Proc. Amer. Math. Soc. 144 (2016), 4759–4775.
14. , On the distribution of the critical values of random spherical harmonics, The Journal of Geometric Analysis 26 (2016), 3252–3324.
15. V. Cammarota and I. Wigman, Fluctuations of the total number of critical points of random spherical harmonics, Stochastic Process. Appl. 127 (2017), no. 12, 3825–3869.
16. Simon Campese, Domenico Marinucci, and Maurizia Rossi, Approximate normality of high-energy hyperspherical eigenfunctions, Journal of Mathematical Analysis and Applications 461 (2018), no. 1, 500–522.
17. S.-Y. Cheng, Eigenfunctions and nodal sets, Commentarii mathematici Helvetici 51 (1976), 43–56.
18. F. Dalmao, I. Nourdin, G. Peccati, and M. Rossi, Phase singularities in complex arithmetic random waves, Electron. J. Probab. 24 (2019), 45 pp.
19. H. Donnell and C. Fefferman, Nodal sets of eigenfunctions on riemannian manifolds, Inventiones Mathematicae 63 (1988), 161–183.
20. A. Estrade and J. R. León, A central limit theorem for the Euler characteristic of a Gaussian excursion set, The Annals of Probability 44 (2016), no. 6, 3849 – 3878.
21. L. Fainsilber, P. Kurlberg, and B. Wennberg, Lattice points on circles and discrete velocity models for the boltzmann equation, SIAM J. Math. Anal. 37 (2006), no. 6, 1903–1922.
22. Szego G., Orthogonal polynomials, American Mathematical Society, 1975.
23. M. F. Kratz and J. R. León, Level curves crossings and applications for Gaussian models, Extremes 13 (2010), no. 3, 315–351.
24. M. Krishnapur, P. Kurlberg, and I. Wigman, Non-universality of nodal length distribution for arithmetic random waves, Ann. Math. 177 (2013), no. 2, 699–737.
25. P. Kurlberg, I. Wigman, and N. Yesha, The defect of toral laplace eigenfunctions and arithmetic random waves, arXiv:2006.11644, June 2020.
26. E. Landau, Über die eindeutung der positiven zahlen nach vier klassen nach der mindestzahl der zu ihrer addition zusammensetzung erforderlichen quadrate, Archiv der Math. und Physik 13 (1908), no. 3, 305–312.
27. A. Logunov, Nodal sets of laplace eigenfunctions: proof of nadirashvili’s conjecture and of the lower bound in yau’s conjecture, Ann. Math. 187 (2018), no. 1, 241–262.
28. A. Logunov, E. Malinnikova, N. Nadirashvili, and F. Nazarov, The sharp upper bound for the area of the nodal sets of dirichlet laplace eigenfunctions, arXiv:2104.09012, April 2021.
29. R. W. Maffucci, Nodal intersections for random waves against a segment on the 3-dimensional torus, Journal of Functional Analysis 272 (2017), no. 12, 5218 – 5254.
30. , Nodal intersections of random eigenfunctions against a segment on the 2-dimensional torus, Monatshefte für Mathematik 183 (2017), no. 2, 311–328.
31. , Nodal intersections for arithmetic random waves against a surface, preprint, 2018.
32. D. Marinucci and G. Peccati, Ergodicity and gaussianity for spherical random fields, Journal of Mathematical Physics 51 (2010), no. 4, 043301.
33. , Random fields on the sphere: Representation, limit theorems and cosmological applications, London Mathematical Society Lecture Note Series, Cambridge University Press, 2011.
34. D. Marinucci, G. Peccati, M. Rossi, and I. Wigman, Non-universality of nodal length distribution for arithmetic random waves, GAFA 3 (2016), 926–960.
35. D. Marinucci and M. Rossi, Stein-Malliavin Approximations for Nonlinear Functionals of Random Eigenfunctions on $\mathbb{S}^d$, J. Funct. Anal. 268 (2015), no. 8, 2379–2420.
36. D. Marinucci, M. Rossi, and I. Wigman, The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics, Ann. Inst. Henri Poincaré Probab. Stat. 56 (2020), no. 1, 374–390.
37. D. Marinucci and I. Wigman, The defect variance of random spherical harmonics, J. Phys. A 44 (2011), no. 35, 355206.
38. , On the area of excursion sets of spherical gaussian eigenfunctions, Journal of Mathematical Physics 52 (2011), no. 9, 093301.
39. , On nonlinear functionals of random spherical eigenfunctions, Comm. Math. Phys. 327 (2014), no. 3, 849–872.
40. F. Nazarov and M. Sodin, Asymptotic laws for the spatial distribution and the number of connected components of zero sets of gaussian random functions, J. Math. Phys. Anal. Geom. 12 (2016), no. 3, 205–278.
41. M. Notarnicola, Fluctuations of nodal sets on the 3-torus and general cancellation phenomena, ALEA Lat. Am. J. Probab. Math. Stat. 18 (2021), 1127–1194.
42. I. Nourdin and G. Peccati, Stein’s method on Wiener chaos, Probab. Theory Related Fields 1–2 (2009), no. 145, 75–118.
43. , Normal approximation with malliavin calculus: From stein’s method to universality, Cambridge University Press, 2012.
44. I. Nourdin, G. Peccati, and M. Rossi, Nodal statistics of planar random waves, Comm. Math. Phys. 369 (2019), no. 1, 99–151.
45. D. Nualart and G. Peccati, *Central limit theorems for sequences of multiple stochastic integrals*, Ann. Probab. **33** (2005), no. 1, 177–193.

46. F. Oravecz, D. Rudnick, and I. Wigman, *The Lévy measure of nodal sets for random eigenfunctions on the torus*, Annales de l’institut Fourier **58** (2008), no. 1, 299–335.

47. Kurlberg P. and Wigman I., *On probability measures arising from lattice points on circles*, Mathematische Annalen **367** (2017), no. 3-4, 1057–1098.

48. G. Peccati and M. Rossi, *Quantitative limit theorems for local functionals of arithmetic random waves*, Abel Symposium 2016 (Springer), 2018, pp. 659–689.

49. G. Peccati and A. Vidotto, *Gaussian random measures generated by Berry’s nodal sets*, J. Stat. Phys. **178** (2020), no. 4, 996–1027.

50. M. Rossi, *The geometry of spherical random fields*, Ph.D.-Thesis University of Rome Tor Vergata, 2015.

51. , *The defect of random hyperspherical harmonics*, J. Theor. Probab. **32** (2019), 2135–2165.

52. , *Random nodal lengths and wiener chaos*, Probabilistic methods in geometry, topology and spectral theory (Y. Canzani, L. Chen, and D. Jakobson, eds.), Contemp. Math., vol. 739, Centre Rech. Math. Proc., Amer. Math. Soc., Providence, RI, 2019, pp. 155–169.

53. M. Rossi and I. Wigman, *Asymptotic distribution of nodal intersections for arithmetic random waves*, Nonlinearity **31** (2018), no. 10, 4472.

54. D. Rudnick, I. Wigman, and N. Yesha, *Nodal intersections for random waves on the 3-dimensional torus*, Annales de l’institut Fourier **66** (2016), no. 6, 2455–2484.

55. Z. Rudnick and I. Wigman, *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*, Ann. Henri Poincaré **9** (2008), no. 1, 109–130.

56. , *Topologies of nodal sets of random band limited functions*, Advances in the Theory of Automorphic Forms and Their L-functions, Contemporary Mathematics, vol. 664, 2016.

57. A. P. Todino, *A quantitative central limit theorem for the excursion area of random spherical harmonics over subdomains of $S^2$*, J. Math. Phys. **60** (2019), no. 2.

58. , *Nodal lengths in shrinking domains for random eigenfunctions on $S^2$*, Bernoulli **26** (2020), no. 4, 3081–3110.

59. A. Vidotto, *A note on the reduction principle for the nodal length of planar random waves*, Statist. Probab. Lett. **174** (2021).

60. I. Wigman, *Fluctuations of the nodal length of random spherical harmonics*, Comm. Math. Phys. **298** (2010), no. 3, 787–813.

61. S.T. Yau, *Seminar on differential geometry*, Annals of mathematics studies, Princeton University Press, 1982.

62. Z. Zelditch, *Real and complex zeros of riemannian random waves*, Spectral Analysis in Geometry and Number Theory, Contemporary Mathematics, vol. 484, 2009.

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