COMPACT SPACES WITH A $\mathbb{P}$-DIAGONAL

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Abstract. We prove that compact Hausdorff spaces with a $\mathbb{P}$-diagonal are metrizable. This answers problem 4.1 (and the equivalent problem 4.12) from [2].

Introduction

The purpose of this note is to show that a compact space with a $\mathbb{P}$-diagonal is metrizable.

To explain the meaning of this statement we need to introduce a bit of notation and define a few notions. For a space $M$ (always assumed to be at least completely regular) we let $K(M)$ denote the family of compact subsets of $M$. Following [5] we say that a space $X$ is $M$-dominated if there is a cover $\{C_K : K \in K(M)\}$ of $X$ by compact subsets with the property that $K \subseteq L$ implies $C_K \subseteq C_L$.

In the case that we deal with, namely where $M$ is the space of irrational numbers, we can simplify the cover a bit and make it more amenable to combinatorial treatment. The space of irrationals is homeomorphic to the product space $\omega^\omega$, where $\omega$ carries the discrete topology. We shall reserve the letter $\mathbb{P}$ for this space.

The set $\mathbb{P}$ is ordered coordinatewise: $f \leq g$ means $(\forall n)(f(n) \leq g(n))$. Using this order we simplify the formulation of $\mathbb{P}$-dominated as follows. If $K$ is a compact subset of $\mathbb{P}$ then the function $f_K$, given by $f_K(n) = \max\{g(n) : g \in K\}$, is well-defined. Using this one can easily verify that a space $X$ is $\mathbb{P}$-dominated iff there is a cover $\{K_f : f \in \mathbb{P}\}$ of $X$ by compact sets such that $f \leq g$ implies $K_f \subseteq K_g$. We shall call such a cover an order-preserving cover by compact sets.

Finally then we say that a space $X$ has a $\mathbb{P}$-diagonal if the complement of the diagonal, $\Delta$, in $X^2$ is $\mathbb{P}$-dominated. Problem 4.1 from [2] asks whether a compact space with a $\mathbb{P}$-diagonal is metrizable. The authors of that paper proved that the answer is positive if $X$ is assumed to have countable tightness, or in general if $\text{MA}(\aleph_1)$ is assumed. The latter proof used that assumption to show that $X$ has a small diagonal, which in turn implies that $X$ has countable tightness so that the first result applies. Thus, Problem 4.12 from [2], which asks if a compact space with a $\mathbb{P}$-diagonal has a small diagonal, is a natural reformulation of Problem 4.1.

The property of $\mathbb{P}$-domination arose in the study of the geometry of topological vector space; in [1] it was shown that if a locally convex space has a form of $\mathbb{P}$-domination then its compact sets are metrizable. The paper [2] contains more information and results leading up to its Problem 4.1.
The main result of [4] states that compact spaces with a $\mathcal{P}$-diagonal are metrizable under the assumption of the Continuum Hypothesis. The proof establishes that a compact space with a $\mathcal{P}$-diagonal that has uncountable tightness maps onto the Tychonoff cube $[0,1]^{\omega_1}$ and no compact space with a $\mathcal{P}$-diagonal maps onto the cube $[0,1]^\kappa$.

The principal result of this paper closes the gap between $\aleph_1$ and $\omega_1$ by establishing that no compact space with a $\mathcal{P}$-diagonal maps onto $[0,1]^{\omega_1}$.

Furthermore we would like to point out that Lemma 3 establishes a Baire category type property of $2^{\omega_1}$: in an order-preserving cover by compact sets there are many members with non-empty interior in the $G_\delta$-topology.

Some preliminaries

In the proof of the main lemma, Lemma 3, we need to consider three cases, depending on the values of the familiar cardinals $b$ and $\delta$. These are defined in terms of the mod finite order on $\mathcal{P}$: we say $f \leq^* g$ if $\{n : g(n) < f(n)\}$ is finite.

Then $b$ is the minimum size of a subset of $\mathcal{P}$ that is uncounctable with respect to $\leq^*$, and $\delta$ is the minimum size of a dominating (i.e., cofinal) set with respect to $\leq^*$.

Interestingly, $\delta$ is also the minimum size of a dominating set with respect to the coordinatewise order $\leq_1$; we shall use this in the proof of the main lemma. We refer to Van Douwen’s [5] for more information.

Since we shall be working with the Cantor cube $2^{\omega_1}$ we fix a bit of notation. If $I$ is some subset of $\omega_1$ then $\text{Fn}(I,2)$ denotes the set of finite partial functions from $I$ to 2. We let $2^{<\omega_1}$ denote the binary tree of countable sequences of zeros and ones. If $s \in \text{Fn}(\omega_1,2)$ then $[s]$ denotes $\{x \in 2^{\omega_1} : s \subseteq x\}$; the family $\{[s] : s \in \text{Fn}(\omega_1,2)\}$ is the standard base for the product topology of $2^{\omega_1}$. Similarly, if $\rho \in 2^{<\omega_1}$ then $[\rho] = \{x \in 2^{\omega_1} : \rho \subseteq x\}$, and the family $\{[\rho] : \rho \in 2^{<\omega_1}\}$ is the standard base for what is called the $G_\delta$-topology on $2^{\omega_1}$; a set dense with respect to this topology will be called $G_\delta$-dense.

When working with powers of the form $I^{\omega_1}$, where $I = \omega$ or $I = 2$, we use $\pi_\delta$ to denote the projection of $I^{\omega_1}$ onto $I^{\omega_1 \setminus \delta}$.

In the proof of Lemma 3 we shall need the following result, due to Todorcević.

**Lemma 1 ([6] Theorem 1.3).** If $b = \aleph_1$ then $\omega^{\omega_1}$ has a subset, $X$, of cardinality $\aleph_1$ such that for every $A \subseteq X^{\aleph_1}$ there are $D \subseteq [A]^{\aleph_0}$ and $\delta \in \omega_1$ such that $\pi_\delta[D] = \{d \in [\omega_1 \setminus \delta] : d \in D\}$ is dense in $\omega^{\omega_1 \setminus \delta}$. □

Theorem 1.3 of [6] is actually formulated as a theorem about $b$: drop the assumption $b = \aleph_1$ and replace every $\omega_1$ and $\aleph_1$ by $b$. As explained in [6] this shows that there are natural versions of the S-space problem that do have ZFC solutions.

The lemma also holds with $\omega$ replaced by 2, simply map $\omega^{\omega_1}$ onto $2^{\omega_1}$ by taking all coordinates modulo 2. In that case the density of $\pi_\delta[D]$ can be expressed by saying that for every $s \in \text{Fn}(\omega_1 \setminus \delta,2)$ the intersection $D \cap [s]$ is nonempty.

**BIG sets in $2^{\omega_1}$**

Let us call a subset, $Y$, of $2^{\omega_1}$ BIG if it is compact and projects onto some final product, that is, there is a $\delta \in \omega_1$ such that $\pi_\delta[Y] = 2^{\omega_1 \setminus \delta}$. The latter condition can be expressed without mentioning projections as follows: there is a $\delta \in \omega_1$ such that for every $s \in \text{Fn}(\omega_1 \setminus \delta,2)$ the intersection $Y \cap [s]$ is nonempty (and a dense set that is closed is equal to the whole space).
BIG sets are also big combinatorially, in the following sense.

**Lemma 2.** If $Y$ is a BIG subset of $2^{<\omega_1}$ then there is a node $\rho$ in the tree $2^{<\omega_1}$ such that $[\rho] \subseteq Y$.

*Proof.* Let $Y$ be BIG and fix a $\delta$ witnessing this. After reindexing we can assume $\delta = \omega$ and we let $B_t = \{x \in 2^{<\omega_1}: t \subseteq x\}$ and $Y_t = Y \cap B_t$ for $t \in 2^{<\omega}$.

Starting from $t_0 = \langle \rangle$ and $s_0 = \emptyset$ we build a sequence $\langle t_n : n \in \omega \rangle$ in $2^{<\omega}$ and a sequence $\langle s_n : n \in \omega \rangle$ in $\text{Fn}(\omega_1 \setminus \omega, \omega)$ such that $[s_n] \subseteq \pi_\delta[Y_{t_n}]$ for all $n$.

Given $t_n$ we can choose $i_n < 2$, and set $t_{n+1} = t_n \ast i_n$, such that $[s_n] \cap \pi_\delta[Y_{t_{n+1}}]$ has nonempty interior. Then choose an extension $s_{n+1}$ of $s_n$ such that $[s_{n+1}] \subseteq \pi_\delta[Y_{t_{n+1}}]$. With a bit of bookkeeping one can ensure that $\bigcup_n \text{dom} s_n$ is an initial segment of $\omega_1 \setminus \omega$. Let $\rho$ be the concatenation of $\bigcup_n t_n$ and $\bigcup_n s_n$.

To see that $\rho$ is as required let $x \in [\rho]$. By construction we have $x \in [s_n]$ for all $n$, so that, again for all $n$, there is $y_n \in Y_{t_n}$ such that $y_n$ and $x$ agree above $\text{dom} \rho$. If $s \in \text{Fn}(\omega_1, 2)$ determines a basic neighbourhood of $x$ then there is an $m$ such that $\text{dom} s \cap \text{dom} \rho$ is a subset of $\text{dom} t_m \cup \text{dom} s_m$. Then $y_n \in [s]$ for all $n \geq m$, so that the sequence $\langle y_n : n \in \omega \rangle$ converges to $x$, which shows that $x \in Y$. \qed

**Existence of BIG sets**

It is clear that a compact space is $\mathbb{P}$-dominated: simply let $K_f$ be the whole space for all $f$. However, in our proof we shall encounter $\mathbb{P}$-dominating covers that may consist of proper subsets. Our next result shows that such a cover of $2^{<\omega_1}$ by compact sets must contain a BIG subset.

**Lemma 3.** If $\langle K_f : f \in \mathbb{P} \rangle$ is an order-preserving cover of $2^{<\omega_1}$ by compact sets then there is an $f$ such that $K_f$ is BIG.

*Proof.* We consider three cases.

First we assume $\mathfrak{d} = \aleph_1$. In this case we show outright that there are $\rho \in 2^{<\omega_1}$ and $f \in \mathbb{P}$ such that $[\rho] \subseteq K_f$. Let $\langle f_\alpha : \alpha \in \omega_1 \rangle$ be a sequence that is $\leq^*\text{-dominating}$. Working toward a contradiction we assume no $\rho$ and $f$, as desired, can be found. This implies that for every $\rho$ and every $f$ the intersection $K_f \cap [\rho]$ is nowhere dense in $[\rho]$. Indeed, if such an intersection has interior then there is $s \in \text{Fn}(\omega_1, 2)$ such that $[s] \cap [\rho]$ is nonempty and contained in $K_f$. It would then be an easy matter to find $\sigma \in 2^{<\omega_1}$ that extends both $\rho$ and $s$, and then $[\sigma] \subseteq K_f$.

This allows us to choose an increasing sequence $\langle \rho_\alpha : \alpha \in \omega_1 \rangle$ in $2^{<\omega_1}$ such that $[\rho_\alpha] \cap K_{f_\alpha} = \emptyset$ for all $\alpha$. Then the point $x = \bigcup_\alpha \rho_\alpha$ does not belong to any $K_f$ because the $K_{f_\alpha}$ are cofinal in the whole family.

Next we assume $\mathfrak{d} > \aleph_1$. We apply $\mathfrak{b} = \aleph_1$ to find a special subset $X$ of $2^{<\omega_1}$ as in the comment after Lemma 1. In what follows, when $t \in \omega^{<\omega}$ we let $K(t)$ denote the union $\bigcup \{K_f : t \subseteq f\}$.

We choose an increasing sequence $\langle t_n : n \in \omega \rangle$ in $\omega^{<\omega}$, together with, for each $n$, an uncountable subset $A_n$ of $X$, a countable subset $D_n$ of $A_n$, and $\delta_n \in \omega_1$ such that $A_n \subseteq K(t_n)$ and for all $s \in \text{Fn}(\omega_1 \setminus \delta_n, 2)$ the intersection $D_n \cap [s]$ is nonempty. Simply use that $K(t) = \bigcup_k K(t \ast k)$ for all $t$.

Let $\delta = \sup \delta_n$ and enumerate each $D_n$ as $\langle d(n, m) : m \in \omega \rangle$.

For each $s \in \text{Fn}(\omega_1 \setminus \delta, 2)$ each $D_n$ intersects $[s]$ so that we can define $h_s \in \omega^\omega$ by $h_s(n) = \min \{d(n, m) : m \in [s]\}$.

By $\mathfrak{d} > \aleph_1$ there is $g \in \omega^\omega$ such that $\{n : h_s(n) < g(n)\}$ is infinite for all $s$. 
Now let \( E = \{ (d(n, m) : m < g(n), n \in \omega) \} \) and observe that \( E \) meets \([s]\) for every \( s \in \text{Fu}(\omega_1 \setminus \delta, 2) \), so that \( \pi_{\delta}[E] \) is dense in \( 2^{\omega_1 \setminus \delta} \).

For each \( n \) there is \( f_n \in \mathbb{P} \) that extends \( t_n \) and is such that \( \{d(n, m) : m < g(n)\} \) is a subset of \( K_{f_n} \). As \( f_m(n) = t_{n+1}(n) \) if \( m > n \) we may define \( f \in \mathbb{P} \) by \( f(n) = \max\{ f_m(n) : m \in \omega \} \) for all \( n \). Thus we find a single \( f \) such that \( E \subseteq K_f \), which immediately implies that \( K_f \) is BIG.

Our last case is when \( b > \aleph_1 \). We let \( A \) be the set of members, \( t \), of \( \omega^{<\omega} \) for which there is a \( \rho \in 2^{\omega_1} \) such that \( K(t) \cap [\rho] \) is \( G_\delta \)-dense in \([\rho]\).

As \( K(\emptyset) = 2^{\omega_1} \) we have \( \emptyset \in A \).

We show that if \( t \in A \), as witnessed by \( \rho \), then there is an \( m_t \) such that \( t * n \in A \) whenever \( n \geq m_t \); as \( K(t * m) \subseteq K(t * n) \) whenever \( m \leq n \) it follows that we need to find just one \( n \) such that \( t * n \in A \). Build, recursively, an increasing sequence \( \rho = \rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots \) in \( 2^{\omega_1} \) such that \( \rho_0 = \rho \) and, if possible, \([\rho_{n+1}] \cap K(t * n) = \emptyset \); if such an \( \rho_{n+1} \) cannot be found then \( K(t * n) \cap [\rho_n] \) is \( G_\delta \)-dense in \([\rho_n] \) and we are done. So assume that the recursion does not stop and set \( \bar{\rho} = \bigcup_n \rho_n \); then \([\bar{\rho}] \) is disjoint from \( \bigcup_n K(t * n) \), which is equal to \( K(t) \). This would contradict \( G_\delta \)-density of \( K(t) \) in \([\rho]\).

We can define \( h \in \mathbb{P} \) recursively by \( h(n) = m_{h|n} \), together with an increasing sequence \( \langle n_h : n \in \omega \rangle \) in \( 2^{\omega_1} \) such that \( K(h \upharpoonright n) \cap [\rho_n] \) is \( G_\delta \)-dense in \([\rho_n] \). Let \( \rho = \bigcup_n \rho_n \), then \( K(h \upharpoonright n) \cap [\rho] \) is \( G_\delta \)-dense in \([\rho] \) for all \( n \).

Let \( \delta = \text{dom} \rho \) and let \( s \in \text{Fu}(\omega_1 \setminus \delta, 2) \). We know that \( K(h \upharpoonright n) \cap [\rho] \cap [s] \neq \emptyset \) for all \( n \). So for every \( n \) we can take \( h_{s,n} \in \mathbb{P} \) that extends \( h \upharpoonright n \) and is such that \( K_{h_{s,n}} \cap [\rho] \cap [s] \neq \emptyset \). Because \( h_{s,n}(m) = h(m) \) if \( n > m \) we can define \( h_s \in \mathbb{P} \) by \( h_s(m) = \max h_{s,n}(m) \).

As \( b > \aleph_1 \) we can find \( f \geq h \) such that \( h_s \leq^* f \) for all \( s \). We claim that \( K_f \cap [\rho] \cap [s] \neq \emptyset \) for all \( s \), so that \([\rho] \subseteq K_f \) (the closed set \( K_f \cap [\rho] \) is dense in \([\rho]\)).

To see this take an \( s \) and let \( n \) be such that \( f(m) \geq h_{s,n}(m) \) for \( m \geq n \). It follows that \( f(m) \geq h_s(m) = h_{s,n}(m) \) for \( m \leq n \) and \( f(m) \geq h(m) \geq h_{s,n}(m) \) for \( m \geq n \). This implies that \( K_f \) meets \([\rho] \cap [s]\).

\(\square\)

**Remark 4.** The previous result is valid for all BIG sets: simply work inside \([\rho]\), where \( \rho \) is as in the conclusion of Lemma 2.

**Remark 5.** Lemma 3 generalises itself to the following situation: let \( X \) be compact, let \( \varphi : X \rightarrow 2^{\omega_1} \) be continuous and onto, and let \( \langle K_f : f \in \mathbb{P} \rangle \) be an order-preserving cover of \( X \) by compact sets. Then there is an \( f \) such that \( \varphi[K_f] \) is BIG.

One can go one step further: take a closed subset \( Y \) of \( X \) such that \( \varphi[Y] \) is BIG and conclude that for some \( f \in \mathbb{P} \) the image \( \varphi[Y \cap K_f] \) is BIG. Simply take \( \rho \) such that \( [\rho] \subseteq \varphi[Y] \) and work in the compact space \( Y \cap \varphi^{-*}[\langle [\rho] \rangle] \).

**Remark 6.** The reader may have pondered the need to consider three cases in the proof of Lemma 3. The cases \( \delta = \aleph_1 \) and \( b > \aleph_1 \) lead to fairly straightforward arguments because each give one a definite handle on things, be it a cofinal set of size \( \aleph_1 \) or the knowledge that all \( \aleph_1 \)-sized sets are bounded. The intermediate case, with just one unbounded set of size \( \aleph_1 \), is saved by Todorčević’s non-trivial translation of such a set into a subset of \( 2^{\omega_1} \) that is already quite big.

It would be interesting to see if Lemma 3 can be proved using just one argument.
THE MAIN RESULT

Now we show that that a compact space with a \( \mathbb{P} \)-diagonal does not admit a continuous map onto \([0, 1]^{\omega_1}\) and deduce our main result.

**Theorem 7.** Assume \( X \) is a compact space that maps onto \( 2^{\omega_1} \). Then \( X \) does not have a \( \mathbb{P} \)-diagonal.

**Proof.** Let \( \varphi : X \to 2^{\omega_1} \) be continuous and onto. We use Remark 5 and say that a closed subset, \( Y \), of \( X \) is BIG if its image \( \varphi[Y] \) is. That is, \( Y \) is BIG if there is a \( \delta \in \omega_1 \) such that \( Y \cap \varphi^{-1}[[s]] \neq \emptyset \) for all \( s \in \text{Fn}(\omega_1 \setminus \delta, 2) \).

We observe the following: if \( Y \) is BIG, as witnessed by \( \delta \), then for every \( s \in \text{Fn}(\omega_1 \setminus \delta, 2) \) the intersection \( Y \cap \varphi^{-1}[[s]] \) is BIG as well; this will be witnessed by any \( \gamma \) that contains the domain of \( s \).

In order to prove our theorem we assume that \( X \) does have a \( \mathbb{P} \)-diagonal, witnessed by \( \langle K_f : F \in \mathbb{P} \rangle \), and reach a contradiction.

In order for the final recursion in the proof to succeed we need some preparation. Enumerate \( \omega^{<\omega} \) in a one-to-one fashion as \( \langle t_n : n \in \omega \rangle \), say in such a way that \( t_m \subseteq t_n \) implies \( m \leq n \) (so that \( t_0 = 1 \)). We set \( Z_0 = X \) and given a BIG set \( Z_n \) we determine a BIG set \( Z_{n+1} \) as follows. We check if there is a BIG subset \( Z \) of \( Z_n \) with the property that for no point \( z \) in \( Z \) there is a BIG subset \( Y \) of \( Z \) and an \( f \in \mathbb{P} \) with \( t_n \subset f \) such that \( \{z\} \times Y \subseteq K_f \). If there is such a \( Z \) then every BIG subset of \( Z_{n+1} \) also has this property so we can pick one that is a proper subset of \( Z_{n+1} \) and let it be \( Z_{n+1} = Z_n \). In the end we set \( Y = \bigcap_n Z_n \). The set \( Y \) is BIG: for each \( n \) we have \( \gamma_n \in \omega_1 \) witnessing BIGness of \( Z_n \), then \( \bar{\delta}_n = \sup \gamma_n \) will witness BIGness of \( Y \).

Pick \( y_0 \in Y \), take \( i_0 \in 2 \) distinct from \( \varphi(y_0)(\bar{\delta}_0) \), let \( s_0 = \{\bar{\delta}_0, i_0\} \), and set \( Y_0 = Y \cap \varphi^{-1}[[s_0]] \). By the observation above, \( Y_0 \) is BIG. Also: \( \varphi(y_0) \notin \varphi[Y_0] \), so that \( \{y_0\} \times Y_0 \) is disjoint from the diagonal, \( \Delta \), of \( X \). By Remark 5 we can find a BIG subset \( Y_1 \) of \( Y_0 \) and \( f_0 \in \mathbb{P} \) such that \( \{y_0\} \times Y_1 \subseteq K_{f_0} \).

The point \( y_0 \) belongs to all \( Z_n \) and for any \( n \) such that \( t_n \supseteq f_0 \) (meaning that \( t_n(i) \supseteq f_0(i) \) for \( i \in \text{dom}(t_n) \)) it, the point \( y_0 \), witnesses that \( Z_{n+1} = Z_n \) in the following sense. The reason for having \( Z_{n+1} \) be a proper subset of \( Z_n \) would be that for all \( z \in Z \) and all BIG \( Z' \subseteq Z \) and all \( f \in \mathbb{P} \) with \( t_n \subset f \) we would have \( \{z\} \times Z' \subseteq K_f \). However, \( y_0 \) and \( Y_1 \) and \( f_0 \) show that this did not happen.

The conclusion therefore is that for every such \( t_n \) we know that every BIG \( Z \subseteq Y \) does have an element \( z \) and a BIG subset \( Z' \) such that \( \{z\} \times Z' \subseteq K_f \) for some \( f \in \mathbb{P} \) that extends \( t_n \).

This allows us to construct sequences \( \langle y_n : n \in \omega \rangle \) (points in \( Y \)), \( \langle Y_n : n \in \omega \rangle \) (BIG subsets of \( Y \)), and \( \langle f_n : n \in \omega \rangle \) (in \( \mathbb{P} \)) such that

1. \( y_n \in Y_n \), except for \( n = 0 \),
2. \( Y_{n+1} \subseteq Y_n \),
3. \( \{y_n\} \times Y_{n+1} \subseteq K_{f_n} \),
4. \( f_{n+1} \supseteq f_n \) and \( f_{n+1} \supseteq f_n | (n+1) \)

As before we note that \( f_m(n) = f_n(n) \) whenever \( m \geq n \), so we can define a function \( f \in \mathbb{P} \) by \( f(n) = \max\{f_m(n) : m \in \omega\} \). Note that \( f \supseteq f_n \) for all \( n \) so that

\[ \{y_n\} \times Y_{n+1} \subseteq K_{f_n} \subseteq K_f \]

for all \( n \).
It follows that $\langle y_m, y_n \rangle \in K_f$ whenever $m < n$. This shows that $\langle y_m, y \rangle \in K_f$ whenever $m \in \omega$ and $y$ is a cluster point of $\langle y_n : n \in \omega \rangle$. But then $\langle y, y \rangle \in K_f$ for every cluster point $y$ of $\langle y_n : n \in \omega \rangle$. However, $K_f$ was assumed to be disjoint from the diagonal of $X$. □

We collect all previous results in the proof of our main theorem.

**Theorem 8.** Every compact space with a $\mathbb{P}$-diagonal is metrizable.

**Proof.** As noted in the introduction the authors of [4] proved that a non-metrizable compact space with a $\mathbb{P}$-diagonal will map onto the Tychonoff cube $[0, 1]^{\omega_1}$ or, equivalently, that it has a closed subset that maps onto $2^{\omega_1}$.

However that closed subset would be a compact space with a $\mathbb{P}$-diagonal that does map onto $2^{\omega_1}$. Theorem 7 says that this is impossible. □

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