Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime

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Abstract

We establish the existence of local, covariant time ordered products of local Wick polynomials for a free scalar field in curved spacetime. Our time ordered products satisfy all of the hypotheses of our previous uniqueness theorem, so our construction essentially completes the analysis of the existence, uniqueness, and renormalizability of the perturbative expansion for nonlinear quantum field theories in curved spacetime. As a byproduct of our analysis, we derive a scaling expansion of the time ordered products about the total diagonal that expresses them as a sum of products of polynomials in the curvature times Lorentz invariant distributions, plus a remainder term of arbitrarily low scaling degree.

1 Introduction

In order to give a perturbative definition of a nonlinear quantum field theory in a globally hyperbolic, curved spacetime, it is necessary to define Wick polynomials and their time ordered products for the corresponding linear (i.e., non-self-interacting) field. In the case of a scalar field, a construction of these quantities was given recently by Brunetti, Fredenhagen and Köhler [2] and by Brunetti and Fredenhagen [3]. However, these authors did not impose a locality or covariance condition on the Wick polynomials or their time ordered products. In fact, the Wick polynomials were constructed in [4] by means of a normal ordering prescription with respect to an arbitrarily chosen Hadamard vacuum state. The Wick polynomials defined in this manner thereby possess an undesirable non-local dependence upon the choice of this vacuum state. Since no locality or covariance condition was imposed on the construction of time ordered-products of Wick polynomials in [3]—and, indeed, such conditions could not have been imposed since the Wick polynomials used in [3] were not local, covariant fields—the renormalization ambiguities were found to involve coupling functions rather than coupling constants.
In a recent paper \cite{12}, we introduced the notion of a *local, covariant quantum field* and we then imposed the requirement that the Wick polynomials and their time ordered products be local, covariant quantum fields. We also required that these quantities have a suitable continuous and analytic dependence upon the spacetime metric and have a suitable scaling behavior under scalings of the metric. (These latter notions are well defined only for local, covariant fields.) In addition, we required the Wick polynomials and their time ordered products to satisfy various additional properties, namely suitable commutation relations with the free field, a microlocal spectral condition, and (for the time ordered products) causal factorization and unitarity conditions. We refer the reader to \cite{12} for the precise statements of all of our conditions as well as a complete explanation of the algebraic framework within which our conditions were formulated.

In \cite{12}, uniqueness theorems were proven for both the Wick polynomials and their time ordered products. For the Wick polynomials, we showed that any two constructions that satisfy all of the above conditions can differ at most by a suitable sum of products of curvature terms of a definite scaling dimension multiplied by lower order Wick polynomials of the appropriate dimension. In particular, this implies that the ambiguity in defining Wick polynomials up to a given finite order is uniquely characterized by only a finite number of parameters. A similar uniqueness result was obtained for the time ordered products, thereby establishing that the ambiguity in defining these quantities up to any given finite order also is characterized by only a finite number of parameters. We then showed that $\lambda \varphi^4$-theory in curved spacetime is renormalizable in the sense that the ambiguities arising in the perturbative definition of this theory correspond precisely to the (finite number of) parameters appearing in the classical Lagrangian (provided that the possible curvature couplings of the appropriate dimension are included in this Lagrangian). Again, we refer the reader to \cite{12} for the precise statements and proofs of these results.

The above uniqueness theorems, of course, do not address the issue of whether there actually exists a construction of Wick polynomials and their time ordered products that satisfies all of our requirements. As already noted above, in \cite{4} Wick polynomials were constructed via a normal ordering prescription, but they fail to satisfy our requirement of being local and covariant. However, this deficiency can be repaired in a relatively straightforward manner by replacing the normal ordering prescription with respect to a (nonlocally defined) Hadamard vacuum state by a point-splitting prescription based upon a locally and covariantly defined Hadamard parametrix. It was proven in \cite{12} that such a construction does indeed satisfy all of our requirements, thus establishing the existence of local, covariant Wick polynomials.

One might hope that the construction of time ordered products given in \cite{3} could be similarly modified to yield local, covariant fields that satisfy all of our requirements. However, it is not at all obvious how to do this. As in \cite{3} (and as will be explicitly seen in subsection 3.1 below), the essential difficulty in defining time ordered products arises

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1 A general notion of local, covariant quantum fields has been given by \cite{4}.\[2\]
from the extension of certain multivariable distributions to the total diagonal; it is here that regularization/renormalization is needed. The usual momentum space methods of regularization are inapplicable in a curved, Lorentzian spacetime, but the Epstein-Glaser prescription [10] is well defined [3]. However, this prescription involves the modification of test functions by the subtraction of their truncated Taylor series multiplied by a “cutoff function”. The introduction of such a cutoff function makes the prescription inherently nonlocal. Consequently, the time ordered products defined by this prescription will fail to be local, covariant fields.

A similar difficulty with the Epstein-Glaser prescription occurs in Minkowski spacetime, where the introduction of the cutoff function makes the prescription fail to be Lorentz invariant. However, in Minkowski spacetime, a cohomology argument can then be used to establish existence of a satisfactory Lorentz invariant prescription [17]. We have not been able to generalize this argument to curved spacetime. For this reason, the issue of existence of local covariant time ordered products was left open in [12].

The main purpose of this paper is to prove the existence of local covariant time ordered products, thereby essentially completing\(^2\) the perturbative construction of nonlinear quantum field theory in curved spacetime. The basic idea of our construction is as follows. As already indicated above, our task is to extend certain distributions on \(M^{n+1} \setminus \Delta_{n+1}\) to all of \(M^{n+1}\) in a local, covariant manner, where \(M\) denotes the spacetime manifold, \(M^{n+1} = \times^{n+1} M\), and \(\Delta_{n+1}\) denotes the total diagonal of \(M^{n+1}\),

\[
\Delta_{n+1} = \{(x, x, \ldots, x) \mid x \in M\}.
\]  

The key idea which enables us to accomplish this is to analyze the scaling behavior of the unextended distributions near the total diagonal. To do so, we first introduce \(n\) “relative coordinates” \(y\) and show that we can view each unextended distribution as a distribution in \(y\) for each fixed \(x \in M\) (i.e., our distribution in \((n + 1)\) variables can be viewed as a distribution in the \(n\) relative coordinates that is parametrized by the point \(x\) on the total diagonal). We then show that near the total diagonal, each unextended distribution in question can be written as a finite sum of terms together with a “remainder term” with the following properties: (i) The terms in the finite sum are products of curvature terms in \(x\) times distributions, \(u\), in \(y\) that correspond to Lorentz invariant distributions in Minkowski spacetime\(^3\). (ii) The remainder term has a sufficiently low scaling degree under scaling of \(y\). The distributions, \(u\), may then be extended to the total diagonal by Minkowski spacetime methods, whereas the remainder term can be extended to the total

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\(^2\) As pointed out in [12], when defining Wick polynomials involving derivatives of the field, it is natural to require the vanishing of any Wick product that contains a factor of the wave operator applied to the field. This requirement was not imposed in [12], so the issue of existence of Wick polynomials that satisfy this additional condition remains open. We are currently investigating this issue; see also [16].

\(^3\) For the Feynman propagator and its powers, these terms would correspond to the momentum space expressions given in [3], since each \(u\) can be given a momentum space representation.
diagonal by continuity. The resulting extended distributions can then be shown to provide
a definition of local, covariant time ordered products that satisfy all of our requirements.

The paper is organized as follows. In section 2, we review our requirements on the
definition of time ordered products. These requirements are the ones previously given
in [12] except that we have replaced the continuity requirement of [12] under smooth
variations of the metric with a smoothness requirement. Further discussion of our new
smoothness requirement is given in Appendix A.

In section 3, we reduce the problem of constructing time ordered products to that of
extending certain scalar distributions to the total diagonal. In subsection 3.1, we proceed
inductively in the number, \( n \), of variables, and reduce the problem to the extension of
the time ordered products in \( n + 1 \) variables to the total diagonal. In section 3.2, we use
a local, covariant version of the Wick expansion to express these time ordered products
in \( n + 1 \) variables as sums of local Wick products times “c-number” distributions, \( t^0 \).
In subsection 3.3, we then translate our requirements on the definition of time ordered
products into requirements on the extensions of the distributions, \( t^0 \), to the total diagonal.

Section 4 is devoted to obtaining the desired extension of \( t^0 \). In subsection 4.1, we
introduce “relative coordinates”, \( y \), and then derive our scaling expansion of \( t^0 \) with the
properties indicated above. (Some properties of the distributions occurring in the scaling
expansion are obtained in Appendix B.) The scaling expansion is then used to extend \( t^0 \)
in subsection 4.2. Finally, in subsection 4.3, we show that the extended distributions, \( t \),
satisfy the properties listed in subsection 3.3, so that they define a notion of time ordered
products satisfying all of the requirements of section 2. Some concluding remarks are
given in section 5.

We will restrict consideration here to the theory of a scalar field \( \varphi \), but our basic meth-
ods and results should be applicable to other fields. As in [12], for notational simplicity
we restrict attention to time ordered products of Wick powers that do not contain deriva-
tives \( \varphi \). However, our results should extend straightforwardly to time ordered products
involving derivatives of the field, subject to the caveat mentioned in footnote 2 above. In
addition, for notational simplicity in treating the scaling behaviour, we restrict consider-
ation to the massless case, so that the free theory contains no dimensional parameters.
Again, our results can be straightforwardly generalized to the case where dimensional
parameters are present.

Notation and Conventions. Our notation and conventions are the same as in [12].
In particular, we define the Fourier transform on \( \mathbb{R}^m \) by \( \hat{u}(k) = (2\pi)^{-m/2} \int u(x)e^{ixk}d^m x \).
Multi-indices are denoted by \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \). If \( \alpha \) is an \( m \)-dimensional multi-
index, then we also use standard notations such as \( |\alpha| = \sum \alpha_i \), \( x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \) and \( \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \). We also use the “constant convention”, meaning that we use the same
symbol \( C \) for possibly different numerical constants in a chain of inequalities. The space of
compactly supported smooth functions on a space \( X \) with values in the complex numbers
is denoted by $\mathcal{D}(X)$ and the space of smooth functions on $X$ (not necessarily of compact support) by $\mathcal{E}(X)$. (For the definition of the topology on these spaces, see e.g. [19, Chap. V].) The corresponding topological dual spaces of distributions are denoted by $\mathcal{D}'(X)$ respectively $\mathcal{E}'(X)$. The elements in $\mathcal{E}'(X)$ are the distributions of compact support. The wave front set [14] of a distribution $u$ is denoted by $WF(u)$ and its analytic wave front set [14] (see also Appendix A) is denoted by $WF_A(u)$.

2 Required Properties of the Time Ordered Products

For the theory of a free scalar field, $\varphi(x)$, on an arbitrary globally hyperbolic spacetime, $(M, g)$, we previously defined [12] an “extended Wick-polynomial algebra”, $\mathcal{W}(M, g)$, which generalizes the construction of D"utsch and Fredenhagen [8] to curved spacetimes. This algebra is sufficiently large to contain elements corresponding to all Wick powers, $\varphi^k(x)$, (as distributions on compactly supported test functions on $M$) and their time ordered products

$$T = T(\varphi^{k_1}(x_1) \ldots \varphi^{k_n}(x_n)),$$

(as distributions on compactly supported smooth test functions on $M^n$). In [12] we imposed a set of requirements on both $\varphi^k$ and $T$ that uniquely determined these quantities up to certain well specified renormalization ambiguities. In [12], we also constructed Wick products satisfying all of our conditions, so in this paper we will view these quantities as known. Our task here is to construct time ordered products of Wick powers that satisfy the following list of requirements, which—apart from the smoothness condition T4 (see remark (1) at the end of this section)—correspond to the requirements previously given in [12]:

**T1 Locality/Covariance.** The time ordered products are local, covariant fields, as defined in [12].

**T2 Scaling** The time ordered products scale “almost homogeneously” under rescalings $g \to \lambda^{-2} g$ of the spacetime metric in the following sense. Let $\Phi$ be a local, covariant field in $n$ variables, and let $S_{\lambda} \Phi$ be the rescaled local, covariant field given by $S_{\lambda} \Phi[g] \equiv \lambda^{-4n} \sigma_{\lambda} \Phi[\lambda^{-2} g]$, where $\sigma_{\lambda} : \mathcal{W}(M, \lambda^{-2} g) \to \mathcal{W}(M, g)$ is the canonical isomorphism defined in [12]. The scaling dimension, $d_{\Phi}$, of a local covariant field is defined as

$$d_{\Phi} = \sup \{ \delta \in \mathbb{R} \mid \lim_{\lambda \to 0^+} \lambda^{-\delta} S_{\lambda} \Phi = 0 \}.$$

The scaling requirement on the time ordered product is then that

$$\lambda^{-d_T} S_{\lambda} T = T + \sum_{h=1}^{N} \ln h \lambda^{h} \Psi_h,$$

where $\Psi_h$ are the renormalization ambiguities.

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where \( d_T = \sum k_i \), \( N \) is some natural number and where \( \Psi_h \) are local, covariant fields with scaling dimension \( d_T \) which have fewer powers in the free field than \( T \).

**T3 Microlocal Spectrum condition.** Let \( \omega \) be any continuous state on \( \mathcal{W}(M, g) \), so that, as shown in [3], \( \omega \) has smooth truncated \( n \)-point functions for \( n \neq 2 \) and a two-point function \( \omega_2(x,y) = \omega(\varphi(x)\varphi(y)) \) of Hadamard from, i.e., \( \text{WF}(\omega_2) \subset C_+(M, g) \), where

\[
C_+(M, g) = \{ (x_1, k_1; x_2, -k_2) \in T^*M^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, k_2); k_1 \in (V^+)_{x_1} \}. \tag{5}
\]

Here the notation \( (x_1, k_1) \sim (x_2, k_2) \) means that \( x_1 \) and \( x_2 \) can be joined by a null-geodesic and that \( k_1 \) and \( k_2 \) are cotangent and coparallel to that null-geodesic. \( (V^+)_{x} \) is the future lightcone at \( x \). Furthermore, let

\[
\omega_T(x_1, \ldots, x_n) = \omega(T(\prod_{i=1}^n \varphi^{k_i}(x_i))). \tag{6}
\]

Then we require that

\[
\text{WF}(\omega_T) \subset C_T(M, g), \tag{7}
\]

where the set \( C_T(M, g) \subset T^* M^n \setminus \{0\} \) is described as follows (we use the graphological notation introduced in [2, 3]): Let \( G(p) \) be a “decorated embedded graph” in \( (M, g) \). By this we mean an embedded graph \( \subset M \) whose vertices are points \( x_1, \ldots, x_n \in M \) and whose edges, \( e \), are oriented null-geodesic curves. Each such null geodesic is equipped with a coparallel, cotangent covectorfield \( p_e \). If \( e \) is an edge in \( G(p) \) connecting the points \( x_i \) and \( x_j \) with \( i < j \), then \( s(e) = i \) is its source and \( t(e) = j \) its target. It is required that \( p_e \) is future/past directed if \( x_{s(e)} \notin J^\pm(x_{t(e)}) \). With this notation, we define

\[
C_T(M, g) = \left\{ (x_1, k_1; \ldots; x_n, k_n) \in T^*M^n \setminus \{0\} \mid \exists \text{ decorated graph } G(p) \text{ with vertices } \right. \left. x_1, \ldots, x_n \text{ such that } k_i = \sum_{e: s(e) = i} p_e - \sum_{e: t(e) = i} p_e \quad \forall i \right\}. \tag{8}
\]

**T4 Smoothness.** The functional dependence of the time ordered products on the spacetime metric, \( g \), is such that if the metric is varied smoothly, then the time ordered products vary smoothly, in the following sense. Consider a smooth one parameter family of metrics \( g^{(s)} \), let \( T^{(s)} \) be a corresponding family of time ordered products, and let \( C_T^{(s)} \) be given by eq. (8) for this family of metrics. Furthermore, let \( \omega^{(s)} \) be a family of Hadamard states with smooth truncated \( n \)-point functions \( (n \neq 2) \) depending smoothly on \( s \) and with two-point functions \( \omega_2^{(s)} \) depending smoothly on \( s \) in the sense that (see Appendix A)

\[
\text{WF}(\omega_2^{(s)}) \subset \left\{ (s, \rho, x_1, k_1; x_2, k_2) \in T^*(\mathbb{R} \times M^2) \setminus \{0\} \mid (x_1, k_1; x_2, k_2) \in C_+^{(s)} \right\}, \tag{9}
\]
where the family of cones \( C^{(s)}_+ \) is defined by eq. (5) in terms of the family \( g^{(s)} \). Then we require that the family of distributions given by

\[
\omega_T(s, x_1, \ldots x_n) = \omega^{(s)}(T^{(s)}(\prod_{i=1}^{n} \varphi^{k_i}(x_i)))
\]  

depends smoothly on \( s \) with respect to \( C^{(s)}_T \) in the sense that

\[
\text{WF}(\omega_T) \subset \left\{ (s, \rho; x_1, k_1; \ldots; x_n, k_n) \in T^*(\mathbb{R} \times M^n) \setminus \{0\} \mid (x_1, k_1; \ldots; x_n, k_n) \in C^{(s)}_T \right\}.
\]

**T5 Analyticity.** Similarly, we require that, for an analytic one-parameter family of analytic metrics, the expectation value of the time ordered products in an analytic family of states varies analytically in the same sense as in T4, but with the smooth wave front set replaced by the analytic wave front set.

**T6 Symmetry.** The time ordered products are symmetric under a permutation of the factors.

**T7 Unitarity.** We have \( T^* = \bar{T} \), where \( \bar{T} \) is the “anti-time-ordered” product, defined as

\[
\bar{T}(\varphi^{k_1}(x_1) \ldots \varphi^{k_n}(x_n)) = \sum_{I_1 \cup \cdots \cup I_j = \{1, \ldots, n\}} (-1)^{n+j}T(\prod_{i \in I_1} \varphi^{k_i}(x_i)) \ldots T(\prod_{i \in I_j} \varphi^{k_i}(x_i)),
\]  

where the sum runs over all partitions of the set \( \{1, \ldots, n\} \) into disjoint subsets \( I_1, \ldots, I_j \).

**T8 Causal Factorization.** In the case of a single factor, we require that \( T(\varphi^k(x)) = \varphi^k(x) \). For more than one factor, we require the time ordered product to satisfy the following causal factorization rule, which reflects the time-ordering of the factors. Consider a set of points \( (x_1, \ldots, x_n) \in M^n \) and a partition of \( \{1, \ldots, n\} \) into two non-empty disjoint subsets \( I \) and \( I^c \), with the property that no point \( x_i \) with \( i \in I \) is in the past of any of the points \( x_j \) with \( j \in I^c \), that is, \( x_i \notin \mathcal{J}^-(x_j) \) for all \( i \in I \) and \( j \in I^c \). Then the time ordered products factorize in the following way:

\[
T = T(\prod_{i \in I} \varphi^{k_i}(x_i)) T(\prod_{j \in I^c} \varphi^{k_j}(x_j)).
\]
T9 Commutator. The commutator of a time ordered product with a free field is given by lower order time ordered products times suitable commutator functions, namely

\[
T(\varphi^k_1(x_1) \ldots \varphi^k_n(x_n)), \varphi(y) = i \sum_{i=1}^n k_i \Delta(x_i, y) T(\varphi^k_1(x_1) \ldots \varphi^{k_i-1}(x_i) \ldots \varphi^k_n(x_n)), \quad (14)
\]

where \(\Delta\) is the causal propagator (commutator function), defined as the difference between the advanced and retarded fundamental solutions of the Klein-Gordon equation.

Remarks. (1) In our paper [12], we defined a notion of the continuous variation of a local covariant field under smooth variations of the metric, and we imposed this as a requirement on Wick powers and their time ordered products. We have replaced this requirement here with condition T4, which requires smooth (rather than continuous) dependence of the fields. It is easy to verify the the uniqueness results of [12] as well as the existence result of [12] for Wick powers go through without any essential change if the continuity requirement imposed there is replaced by condition T4. We prefer to work with condition T4 here because it is a much simpler condition to state, it is more general, and it closely parallels the analyticity requirement T5 that was previously imposed in [12]. Further discussion and explanation of conditions T4 and T5 is given in Appendix A.

(2) The microlocal spectrum condition is the same condition as formulated in [3]. It may be motivated by the fact that for noncoinciding points, \(\omega(T(\prod \varphi^k_i(x_i)))\) can be expressed in terms of Feynman graphs. A line in such a graph represents a Feynman propagator, \(\omega_F(x, y) \overset{\text{def}}{=} \omega(T(\varphi(x)\varphi(y))) = \omega_2(x, y) - i\Delta_{\text{adv}}(x, y)\), whose wave front set off the diagonal is given by [18]

\[
\text{WF}(\omega_F) = \{(x_1, k_1; x_2, -k_2) \mid (x_1, k_1) \sim (x_2, k_2); k_1 \in (V^{\pm})_{x_1} \Leftrightarrow x_2 \in J^{\pm}(x_1)\}. \quad (15)
\]

For non-coinciding points, the form of \(\text{WF}(\omega_T)\) follows from (13) and the rules for calculating the wave front set of a product of several distributions, see e.g. [4, Thm. 8.2.10]. For coinciding points, the form of \(\text{WF}(\omega_T)\) reflects the usual energy momentum conservation rules. On the total diagonal, \(\Delta_n\), the microlocal spectral condition reduces to

\[
\text{WF}(\omega_T) \upharpoonright_{\Delta_n} \perp T(\Delta_n). \quad (16)
\]

where the notation “\(\perp\)” means the following. If \(F \subset T^*X\) with \(X\) a manifold and \(Y \subset X\) a smooth submanifold, then \(F \upharpoonright Y \perp TY\) means that for any \((y, k) \in F \upharpoonright Y\) and any \((y, v) \in TY\) we have that \(k_\alpha v^\alpha = 0\).

(3) The “connected time ordered product”, \(T^c\), of \(n\) Wick-monomials is defined in terms
of the time ordered product by

$$T^c(\varphi^{k_1}(x_1) \ldots \varphi^{k_n}(x_n)) = \frac{\delta^n}{i^n \delta f_1(x_1) \ldots \delta f_n(x_n)} \ln S(f)\bigg|_{f_1=\ldots=f_n=0} = \sum_{I_1 \sqcup \ldots \sqcup I_j = \{1, \ldots, n\}} \frac{(-1)^{j+1}}{j} T(\prod_{i \in I_1} \varphi^{k_i}(x_i)) \ldots T(\prod_{i \in I_j} \varphi^{k_i}(x_i)),$$

where the $f_i$ are test functions of compact support and $S(f)$ is the formal $S$-matrix for the Lagrangian $L(x) = \sum_i f_i(x) \varphi^{k_i}(x)$,

$$S(f) = \sum_{n \geq 0} \frac{i^n}{n!} \int_{M^n} T(L(x_1) \ldots L(x_n)) \mu_g(x_1) \ldots \mu_g(x_n). \tag{17}$$

Our unitarity condition, T9, is equivalent to the condition $T^{c*} = (-1)^{n+1} T^c$ on the connected time ordered product.

(4) For Minkowski spacetime, condition T9 was given in [9, 1], where it was shown to be equivalent to the familiar Wick-expansion of the time ordered products (see subsection 3.2 below).

Our task is to construct time ordered products of Wick powers that satisfy conditions T1–T9. We shall proceed inductively in the number of factors, $n$, appearing in the time ordered product (2). By condition T8, for $n = 1$ the time ordered products are just the Wick powers, which were already constructed in [12]. Therefore, we may inductively assume that time ordered products with properties T1–T9 have been defined for any number of factors $\leq n$. The goal is to construct from these the time ordered products with $n + 1$ factors. In the next section, we reduce the problem (in close parallel with the analysis of [3]) to that of extending certain multivariable scalar distributions $t^0$ to the total diagonal.

3 Reduction to the problem of extending certain scalar distributions to the total diagonal

3.1 Construction of time ordered products up to the total diagonal

The key idea of causal perturbation theory is that the time ordered products with $n + 1$ factors are already uniquely determined as algebra-valued distributions on the manifold $M^{n+1}$ minus its total diagonal $\Delta_{n+1}$ by the causal factorization requirement T8 (see eq. (13)), once the time ordered products with less than or equal to $n$ factors are given. Following [3], this can be seen as follows:
Let $I$ be a proper subset of $\{1, 2, \ldots, n + 1\}$, and let $C_I$ be the subset of $M^{n+1}$ defined by

$$C_I = \{(x_1, x_2, \ldots, x_{n+1}) \mid x_i \notin J^+(x_j) \text{ for all } i \in I, j \in I^c\},$$

(18)

where $I^c$ is the complement of $I$. It can be seen that the sets $C_I$ are open and that the collection $\{C_I\}$ of these sets covers the manifold $M^{n+1} \setminus \Delta_{n+1}$. Let $\{f_I\}$ be a partition of unity subordinate to this covering. On the manifold $M^{n+1} \setminus \Delta_{n+1}$, we define the algebra-valued distributions $T_0$ by

$$T_0 = \sum_{I \subseteq \{1, \ldots, n+1\}, I \neq \emptyset} f_I T_I,$$

(19)

where

$$T_I = T(\prod_{i \in I} \varphi^k(x_i))T(\prod_{j \in I^c} \varphi^k(x_j)).$$

(20)

Using the causal factorization property T8 of the time ordered products with less or equal than $n$ factors, it can be seen that the definition of $T_0$ does not depend on the choice of the partition $\{f_I\}$, so $T_0$ is well defined. Property T8 applied to the time ordered products with $n + 1$ factors then requires that the restriction of $T$ to $M^{n+1} \setminus \Delta_{n+1}$ must agree with $T_0$. Thus, property T8 alone determines $T$ up to the total diagonal, as we desired to show.

We now claim that—assuming that time ordered products with less or equal than $n$ factors have been defined so as to satisfy properties T1–T9 on $M^n$—the fields $T_0$ with $n + 1$ factors automatically satisfy the restrictions of properties T1–T9 to $M^{n+1} \setminus \Delta_{n+1}$. Condition T8 can be immediately seen to hold by virtue of the definition of $T_0$. The proof that properties T1, T2, T6, T7 and T9 hold is relatively straightforward. A proof of the microlocal spectral condition, T3, can be given in exact parallel with reference [3]. A generalization of this argument can be used to prove that the smoothness and analyticity conditions, T4 and T5, also hold.

Our remaining task is to find an extension of each of the algebra-valued distributions $T_0$ in $n + 1$ factors from $M^{n+1} \setminus \Delta_{n+1}$ to all of $M^{n+1}$ in such a way that properties T1–T9 continue to hold for the extension. This step, of course, corresponds to renormalization. Condition T8 does not impose any additional conditions on the extension, so we need only satisfy T1–T7 and T9. However, it is not difficult to see that if an extension $T$ is defined that satisfies T1–T5 and T9, then that extension can be modified, if necessary, so as to also satisfy the symmetry and unitarity conditions, T6 and T7. Namely, if the

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4 Of course, if any $T_0$ failed to satisfy any of these properties on $M^{n+1} \setminus \Delta_{n+1}$, we would have a proof that no definition of time ordered products could exist that satisfies T1–T9.
extension, \( T \), of \( T^0 \) satisfied \( T1–T5 \) and \( T9 \) but failed to satisfy the symmetry condition, \( T6 \), we could define a new extension \( T' \) by symmetrization,

\[
T' = \frac{1}{(n+1)!} \sum_{\text{Perm } \pi} T(\phi^{k_{\pi(1)}}(x_{\pi(1)}) \ldots \phi^{k_{\pi(n+1)}}(x_{\pi(n+1)})).
\]  

(21)

The so obtained extension of \( T^0 \) then satisfies \( T1–T6 \) and \( T9 \). Similarly, suppose the extension, satisfied \( T1–T6 \) and \( T9 \) but failed to satisfy the unitarity condition, \( T7 \), so that the corresponding connected time ordered product, \( T^c \), fails to satisfy \( T^c = (-1)^n T^c \) (see remark (3) of section 2). Then we define \( T'^c = \frac{1}{2} (T^c + (-1)^n T^c) \) and redefine our extension by

\[
T' = T'^c(\prod_{i=1}^{n+1} \phi^{k_i}(x_i)) - \sum_{I_1 \sqcup \cdots \sqcup I_j = \{1, \ldots, n+1\}} \frac{(-1)^{j+1}}{j} T(\prod_{i \in I_1} \phi^{k_i}(x_i) \ldots T(\prod_{i \in I_j} \phi^{k_i}(x_i))).
\]  

(22)

This provides us with an extension of \( T^0 \) that satisfies \( T1–T7 \) and \( T9 \).

Thus, we have reduced the problem of defining time ordered products to the problem of extending the distributions \( T^0 \) defined by (19) from \( M^{n+1} \setminus \Delta_{n+1} \) to all of \( M^{n+1} \) so that properties \( T1–T5 \) and \( T9 \) continue to hold for the extension. In the next subsection, we will see that property \( T9 \) can be replaced by the requirement of a local Wick expansion for time ordered products.

### 3.2 Reduction to a c-number problem via a local Wick expansion

The next key simplification is to reduce the problem of defining the algebra valued distributions \( T \) to the problem of defining certain “c-number” distributions \( t \). As in [3], this is accomplished by means of a “Wick expansion”. The usual Wick expansion in Minkowski spacetime expresses time ordered products as a sum of normal ordered products with distributional coefficients. In the generalization to curved spacetime given in [3], the time ordered products are Wick expanded in terms of normal ordered products defined relative to an arbitrarily chosen quasi-free Hadamard state. However, such an expansion would not be useful here because the quasi-free Hadamard state—however it is chosen—has a nonlocal character. Consequently, the distributional coefficients occurring in the Wick expansion with respect to normal ordered products will fail to inherit the locality and covariance properties of the time ordered products themselves.

For this reason, we will employ here a Wick expansion of the time ordered products with respect to the local, covariant Wick products :\( \phi^{k_1} \ldots \phi^{k_n} :_H \) that were previously defined in [12] as follows: Let \( H(x, y) \) denote the local Hadamard parametrix

\[
H(x, y) = U(x, y)\sigma^{-1} + V(x, y)\ln \sigma,
\]  

(23)
where \( U, V \) are certain smooth functions defined in terms of the metric and the coupling parameters, \( \sigma \) is the signed squared geodesic distance and where the “i0” prescription for the singular terms is as for the two-point function in Minkowski space. (The formal power series defining \( V \) need not converge in smooth, non-analytic spacetimes, but a suitably modified convergent \( V \) can be used, as explained in [12, Sec. 5.2].) Following [12, Sec. 5.2], we then define in some neighborhood, \( U_n \), of the total diagonal, \( \Delta_n \), the algebra valued distributions

\[
\phi_{k_1}(x_1) \ldots \phi_{k_n}(x_n) : H = \left. \frac{\delta^{|k|}}{i^{k_1} \delta f(x_1)^{k_1} \ldots \delta f(x_n)^{k_n}} \exp \left( \varphi(f) + \frac{1}{2} H(f, f) \cdot \mathbb{1} \right) \right|_{f=0} \quad (24)
\]

with \(|k| = \sum k_i\). Our Wick-expansion is

\[
T(\prod_{i=1}^{n} \phi_{k_i}(x_i)) = \sum_{j_1 \ldots j_n} \binom{k}{j} t_{j_1 \ldots j_n}(x_1, \ldots, x_n) : \phi_{k_1-j_1}(x_1) \ldots \phi_{k_n-j_n}(x_n) : H, \quad (25)
\]

where \( t_{j_1 \ldots j_n} \) are c-number distributions on \( U_n \) and where \( \binom{k}{j} = \prod \binom{k_i}{j_i} \). Note that the Wick-expansion formula (25) is a different identity for different sets of exponents \( (k_1, \ldots, k_n) \), but that the same coefficients \( t_{j_1 \ldots j_n} \) appear in each identity.

We emphasize that since our local Wick-products (24) are defined only on a sufficiently small neighborhood, \( U_n \), of the total diagonal, our Wick-expansion will only make sense in this neighborhood. (This is in contrast with the Wick-expansion used in [3] which is based on a normal ordering prescription for Wick-products and therefore makes sense everywhere on \( M^n \).) This fact, however, will not cause any complications for our constructions, since we will need the Wick-expansion only for the purpose of extending \( T_0 \) to the total diagonal.

We claim now that any definition of time ordered products that satisfies requirements T3 and T9 must admit a Wick expansion of the form (25), with distributions coefficients satisfying

\[
\text{WF}(t_{j_1 \ldots j_n}) \subset C_T(M, g), \quad (26)
\]

where \( C_T(M, g) \) is the set specified in [3]. (Note that (24) implies in particular that the products of distributions implicit in our Wick-expansion formula actually exist and that the operator given by this formula defines—after smearing with a smooth test function—an element of our algebra \( \mathcal{W}(M, g) \).) To prove this claim, we note that eqs. (25) and (26) hold trivially for the time ordered product \( T(\varphi) \) of a single free field. Let us now inductively assume that eqs. (25) and (26) have been demonstrated for all time ordered products of the form \( T(\varphi_{k_1} \ldots \varphi_{k_n}) \), whenever \(|k| = \sum k_i < d\) for some \( d \geq 1\). We claim that they also hold for all multi-orders \( (k_1, \ldots, k_n) \) with \( \sum k_i = d\). To see this, we consider
the difference,

\[ D(x_1, \ldots, x_n) = T(\prod_{i=1}^{n} \varphi^{k_i}(x_i)) - \sum_{j_1 \ldots j_n, |j| < |k|} \binom{k}{j} t_{j_1 \ldots j_n}(x_1, \ldots, x_n) : \varphi^{k_1-j_1}(x_1) \ldots \varphi^{k_n-j_n}(x_n) : H, \]  

(27)

between the left side of eq. (25) and the expression on the right side of that equation, but with the term \( t_{k_1 \ldots k_n} \) omitted in the sum. (Note that this is precisely the term in eq. (25) which is not already known by the induction hypothesis.) We now commute \( D \) with a free field \( \varphi \). We use \( T^9 \) to evaluate the commutator with the time ordered product and we use the similar commutation relation that holds for the local Wick products occurring in the sum. If this is carried out, then one finds that \( [D(x_1, \ldots, x_n), \varphi(y)] = 0 \). Since the only elements of our algebra \( \mathcal{W}(M, g) \) that commute with all smeared field operators \( \varphi(f) \) are multiples of the identity \([12, Prop. 2.1]\), we thus find that \( D \) must in fact be given by a c-number distribution times the identity. We define \( t_{k_1 \ldots k_n} \) to be this c-number distribution.

Now \( t = t_{k_1 \ldots k_n} \) can trivially be written as \( t = \omega(D) \), for any Hadamard state, and each operator in the expression (27) for \( D \) satisfies \( \Omega T3 \). Hence, condition T3 holds also for \( D \), thus showing that \( \text{WF}(t_{k_1 \ldots k_n}) \subset C_T(M, g) \). We have therefore completed the induction step, thereby establishing that the Wick-expansion holds for all multi-orders \( (k_1, \ldots, k_n) \), provided only that T3 and T9 hold for the time ordered products.

Conversely, if a definition of time ordered products has been given that admits a Wick expansion of the form (25) with coefficients satisfying (26), then properties T3 and T9 will hold as well in the neighborhood of the total diagonal on which the Wick expansion is defined.

Since the distribution \( T^0 \) defined on \( M^{n+1} \setminus \Delta_{n+1} \) by (19) above satisfies properties T3 and T9 on \( M^{n+1} \setminus \Delta_{n+1} \), it also admits a local Wick expansion of the form (25), i.e., on \( U_{n+1} \setminus \Delta_{n+1} \) we have

\[ T^0(\prod_{i=1}^{n+1} \varphi^{k_i}(x_i)) = \sum_{j_1 \ldots j_{n+1}} \binom{k}{j} t^0_{j_1 \ldots j_{n+1}}(x_1, \ldots, x_{n+1}) : \varphi^{k_1-j_1}(x_1) \ldots \varphi^{k_{n+1}-j_{n+1}}(x_{n+1}) : H. \]  

(28)

In the next subsection, we will reformulate the problem of extending the algebra-valued distributions \( T^0 \) to the algebra-valued distributions \( T \) in terms of the extension of the c-number distributions \( t^0 \) appearing in (28) to the c-number distributions \( t \) appearing in (25).

5For the terms in the sum in eq. (27), this follows from inductive hypothesis eq. (26) on the \( t_{j_1 \ldots j_n} \) with \(|j| < |k|\), together with the fact that \( \omega(\varphi^{k_1}(x_1) \ldots \varphi^{k_n}(x_n):H) \) is smooth for all \( k_1, \ldots, k_n \).
3.3 Reformulation in terms of the extension of $t^0$

We return now to our inductive construction of time ordered products. We assume that all time ordered products involving $\leq n$ factors have been constructed so as to satisfy our assumptions T1–T9 and we consider an arbitrary time ordered product, $T$, in $(n + 1)$ factors. As noted in subsection 3.1, property T8 will hold if and only if $T$ is an extension to all of $M^{n+1}$ of the distribution $T^0$ on $M^{n+1} \setminus \Delta_{n+1}$ defined by (17). Since $T^0$ satisfies T1–T9 on $M^{n+1} \setminus \Delta_{n+1}$, we need only check that our extension preserves these properties. As noted at the end of subsection 3.1, we actually need only check that $T$ preserves properties T1–T5 and T9, since T8 does not provide any additional conditions on the extension and, by a suitable redefinition, it is straightforward to ensure that T6 and T7 are satisfied. Furthermore, as shown in the previous subsection, we may replace property T9 by the local Wick expansion (25). Thus, time ordered products satisfying all of our conditions will exist if and only if the c-number distributions $t^0$ on $U_{n+1} \setminus \Delta_{n+1}$ appearing in (28) can be extended to distributions $t$ on $U_{n+1}$ in such a way that the distribution $T$ defined by (25) continues to satisfy properties T1–T5. It is straightforward to check that this will be the case if and only if the extensions $t$ satisfy the following 5 corresponding conditions:

1. **Locality/Covariance.** The distributions $t$ are locally constructed from the metric in a covariant manner in the following sense. Let $\psi : N \to M$ be a causality-preserving isometric embedding, and let $f$ be a test function supported in a sufficiently small neighborhood in the total diagonal of $N^{n+1}$. Then we require that
   \[
   \psi^* t^[g_M](f) = t^[g_N](f),
   \]
   where $g_M$ and $g_N$ are the metrics on $M$ and $N$ respectively, so that $\psi^* g_M = g_N$.

2. **Scaling.** The distributions $t = t_{j_1\ldots j_{n+1}}$ scale homogeneously up to logarithmic terms, in the sense that there is an $N \in \mathbb{N}$ such that
   \[
   \lambda^{-d} t[\lambda^{-2} g] = t[g] + \sum_{h=1}^{N-1} \frac{\ln h}{h!} v_h[g],
   \]
   where the $v_h$ are certain local and covariant distributions, and where $d = \sum j_i$.

3. **Microlocal Spectral Condition.** $WF(t) \perp T(\Delta_{n+1})$.

4. **Smoothness.** Let $g^{(s)}$ be a smooth family of metrics on $M$, depending smoothly on a parameter, and view $t(s,x_1,\ldots,x_{n+1}) = t[g^{(s)}](x_1,\ldots,x_{n+1})$ as a distribution on $\mathbb{R} \times U_{n+1}$. Then we require that
   \[
   WF(t) \big|_{\mathbb{R} \times \Delta_{n+1}} \subset \{(s,\rho;x,k_1;\ldots;x,k_{n+1}) | \sum k_i = 0, \text{ not all } k_i = 0\}.
   \]
Analyticity. If $g^{(s)}$ is an analytic family of real analytic metric on $U_{n+1}$, then item t4 holds with the smooth wave front set replaced by the analytic wave front set.

Remarks. (1) If we apply the differential operator $\lambda \partial_\lambda = \frac{\partial}{\partial \ln \lambda}$ a total of $N$ times to both sides of eq. (30), then we obtain

$$ (\lambda \partial_\lambda - d)^N t[\lambda^{-2} g] = 0. \tag{32} $$

Moreover, if we apply $\lambda \partial_\lambda = \frac{\partial}{\partial \ln \lambda}$ only $h < N$ times to both sides of (30) and set $\lambda$ equal to one afterwards, the we find that the local covariant distributions $v_h$ are given by

$$ v_h[g] = (\lambda \partial_\lambda - d)^h t[\lambda^{-2} g]|_{\lambda=1}. \tag{33} $$

In fact, equation (32) is actually equivalent to (30), as one can see by rewriting the differential operator $\lambda \partial_\lambda - d$ as $\lambda^d \frac{\partial}{\partial \ln \lambda} \lambda^{-d}$ and then integrating (32) $N$ times.

(2) In formulating conditions t3–t5, we have taken advantage of the fact that on $U_{n+1} \setminus \Delta_{n+1}$, we have $t = t^0$, so $t$ is already known to satisfy the wave front set conditions corresponding to T3–T5 on $U_{n+1} \setminus \Delta_{n+1}$. Consequently, we need only require $t$ to satisfy the desired wave front set conditions on $\Delta_{n+1}$. Similarly, conditions t1 and t2 also are already known to hold on $U_{n+1} \setminus \Delta_{n+1}$, so we need only check that $t$ satisfies these conditions in an arbitrarily small neighborhood of $\Delta_{n+1}$.

In summary, in this section we have reduced the problem of defining time ordered products to the following question: Assume that time ordered products involving $\leq n$ factors have been constructed so as to satisfy our requirements T1–T9. Define $T^0$ by (19) and define the distributions $t^0$ on $U_{n+1} \setminus \Delta_{n+1}$ by (28). Can each $t^0$ be extended to a distribution $t$ on $U_{n+1}$ so as to satisfy requirements t1–t5?

4 Extension to the total diagonal

Thus far, our analysis of time ordered products corresponds closely to that given in [3]. The primary difference in our assumptions is that we have imposed the requirement that time ordered products be local, covariant fields (see T1) and that they satisfy certain additional requirements concerning scaling behavior (see T2), and smooth and analytic dependence on the metric (see T4 and T5). This has resulted in some important differences in our analysis as compared with [3]. In particular, as a consequence of the locality/covariance requirement, the Wick expansion of [3] in terms of normal ordered products with respect to a quasifree Hadamard state is not useful, so instead we introduced a local, covariant Wick expansion in subsection 3.2. Nevertheless, all of the steps in the analysis given in section 3 above are in close parallel with the analysis of [3].
As described at the end of section 3, our analysis will be completed if we can extend the distributions $t^0$ to the total diagonal so that they satisfy properties t1–t5. As is well known from quantum field theory in Minkowski spacetime, straightforward attempts to extend $t^0$ to the total diagonal give rise to formal expressions that do not make sense as distributions. Therefore, one normally proceeds by introducing some means of “regularizing” these formal expressions and then extracting a well defined “finite part” (up to renormalization ambiguities). In Minkowski spacetime, most approaches to regularization/renormalization involve the use of Euclideanization and/or momentum space methods, neither of which have a natural generalization to (non-static) curved Lorentzian spacetimes. For this reason, the authors of [3] employed the regularization procedure of Epstein and Glaser, which is “local” in the sense that it uses coordinate space methods that can be defined in a local region.

Nevertheless, the Epstein-Glaser method is not local in a strong enough sense for our purposes, since we need to ensure that the renormalized time ordered products will be local, covariant fields. A key step in the Epstein-Glaser regularization procedure is the introduction of certain “cutoff functions” of compact support in the “relative coordinates” that equal 1 in a neighborhood of the total diagonal. Since the prescription for the extension of $t^0$ depends upon the spacetime geometry throughout the region where the cutoff functions are non-zero, the extension, $t$, at a point $p \in \Delta_{n+1}$ will not depend only on the metric in an arbitrary small neighborhood of $p$ and, thus, will not depend locally and covariantly on the metric in the sense required by condition t1. There does not appear to be any straightforward way of modifying the Epstein-Glaser regularization procedure so that the resulting extension, $t$, will satisfy property t1. In particular, serious convergence difficulties arise if one attempts to shrink the support of the cutoff functions to the total diagonal. In addition, the cohomological argument of [17] also does not appear to admit a straightforward generalization to curved spacetime.

Consequently, we shall proceed by a different route here. Our approach to extend $t^0$ to the total diagonal is motivated by the idea (essentially the “equivalence principle”) that on sufficiently small scales a curved space “looks flat”, and that the divergences of $t^0$ in curved spacetime should be of the same nature as the corresponding $t^0$ in flat spacetime. However, this idea is not correct as just stated because a curved space is not actually flat (no matter on how small a scale one looks). Although it is true that the leading order divergences of $t^0$ will be essentially the same as in flat spacetime, in general there will be sub-leading-order divergences that are sensitive to the presence of curvature and are different from the divergences occurring for the corresponding $t^0$ in flat spacetime. Nevertheless, we will see in subsection 4.1 below that any local, covariant distributions that satisfies our scaling, smoothness, and analyticity conditions admits a “scaling expansion” about the total diagonal. This expansion expresses $t^0$ as a finite sum of terms plus a remainder term with the properties that (i) each term in the finite sum is a product of a curvature term times a distribution in the relative coordinates that corresponds to a Lorentz invariant distribution in Minkowski spacetime (which can be
extended to the total diagonal by Minkowski spacetime methods) and (ii) the remainder term admits a unique, natural extension to the diagonal by continuity. We shall thereby obtain an extension of $t^0$ in subsection 4.2. In subsection 4.3 we will then show that the resulting extension satisfies all of the required properties t1–t5.

4.1 The scaling expansion

As indicated above, the key step that will enable us to extend $t^0$ is to perform a scaling expansion of it about the total diagonal. However, \textit{a priori} it is not even clear what this means, since $t^0$ is a distribution in $n + 1$ variables, and it is not clear what it would mean to perform any kind of “expansion” of a distribution about the 4-dimensional submanifold $\Delta_{n+1}$.

The first step in obtaining the scaling expansion for $t^0$ is to show that it is possible to fix one of its $n + 1$ variables at a value $x$, and view it as a distribution in the remaining $n$ variables, $y$, which play the same role as “relative coordinates” in Minkowski spacetime. In other words, writing

$$x = x_1, \quad y = (x_2, \ldots, x_{n+1}).$$

we show that the (unextended) distribution $t^0$ possesses a well-defined restriction to the submanifold

$$C_x = \{ x \} \times (U^n \setminus (x, \ldots, x)), \quad (35)$$

where $U$ is a convex normal neighborhood of the point $x \in M$. In Minkowski spacetime, this result would follow as an immediate consequence of translation invariance. In our context, this result follows from the microlocal spectral condition: Since property T3 is known to hold for $T^0$, it follows that the wave front set of $t^0$ is contained in the set $C_T$. As can be seen from “energy-momentum conservation constraint” in eq. (8), $C_T$ does not contain any elements of the form $(x, k; y, 0)$. Since the conormal bundle, $N^*C_x$ of the submanifold $C_x$ is spanned precisely by such covectors, $WF(t^0)$ does not have any elements in common with $N^*C_x$. That the restriction exists is thus ensured by [14, Thm. 8.2.4]. This restriction may be identified with a distribution on $U^n \setminus (x, \ldots, x)$.

We now shall obtain our scaling expansion of $t^0$. The basic idea is to expand the $t^0[g](x, \cdot)$ at a fixed point $x$ in terms of the metric and its derivatives at $x$. The individual terms in the so-obtained series will be seen to be given by sums of products of local curvature terms at $x$ times Lorentz invariant distributions of the relative coordinates. The remainder for the suitably truncated series will not have this simple form, but will turn out to be regular enough to allow a unique extension.

To begin, we choose a convex normal neighborhood $U \subset M$ of $x$ and introduce Riemannian normal coordinates with respect to the metric $g$ around $x$. These coordinates are constructed by using the exponential map to identify $U$ with a subset of the tangent space
Thus, the Riemannian normal coordinates of a point $\xi \in U$ are given by

$$\alpha_x(\xi) = e \circ (\exp_x)^{-1}(\xi) \in \mathbb{R}^4,$$  \hfill (36)

However, when it is not likely to cause confusion, we will slightly abuse the notation by denoting all quantities—i.e., the point, its Riemannian normal coordinates, and the corresponding vector in Minkowski spacetime—simply by $\xi$. Similarly, the Riemannian normal coordinates of $y = (x_2, \ldots, x_{n+1})$ (see eq. (34) above) will be denoted $\alpha_x(y)$, but when it is not likely to cause confusion, we also will use $y$ to denote the Riemannian normal coordinates of these points or the corresponding vector in $\mathbb{R}^{4n}$.

The choice of isomorphism $e : T_xM \rightarrow \mathbb{R}^4$ is equivalent to a choice of an orthonormal tetrad, $e_\mu^a$, at the point $x$. Since any other orthonormal tetrad is of the form $\Lambda^\nu_\mu e^a_\nu$ for some Lorentz transformation $\Lambda$, the Riemannian normal coordinates, $\xi$, of a given point corresponding to the transformed tetrad at $x$ are then given in terms of the original normal coordinates by $\Lambda \xi$. Similarly, the Riemannian normal coordinates, $y$, of a point in $U^n$ are obtained by Lorentz transforming the coordinates of each point individually by $\Lambda$, the result of which we shall denote as $\Lambda y$.

Now let $g^{(s)}$ be the smooth 1-parameter family of metrics on $U$ whose coordinate components in Riemannian normal coordinates around $x$ are given by

$$g^{(s)}_{\mu\nu}(\xi) = g_{\mu\nu}(s\xi).$$  \hfill (37)

Note that if $\chi_s$ is the map from $U$ into itself given by $\xi \rightarrow s\xi$ in Riemannian normal coordinates about $x$, then this family of metrics can be alternatively written as

$$g^{(s)} = s^{-2}\chi_s^*g.$$  \hfill (38)

Note also that the definition of the above family of metrics does not depend on any additional data besides the specification of the point $x$ and the metric itself. In particular, it does not depend on our choice of tetrad at $x$.

By a slight generalization of the microlocal argument given at the beginning of this subsection, it follows from the fact that $T^0$ satisfies properties T3 and T4 on $U^{n+1} \setminus \Delta_{n+1}$ that $t^0[g^{(s)}](x, \cdot)$ makes sense as a family of distributions on $U^n \setminus (x, \ldots, x)$ that is parametrized by $(s, x)$. Furthermore, when smeared with a test function, $f$, in $y$, it follows that $t^0[g^{(s)}](x, f)$ is smooth in $(s, x)$. In addition, since differentiation does not increase the size of the wave front set, derivatives of $t^0$ with respect to $s$ also make sense as distributions on $U^n \setminus (x, \ldots, x)$ that are parametrized by $(s, x)$. Hence, for any $k$ and any arbitrary, but fixed $x \in M$, we can define a distribution on $U^n \setminus (x, \ldots, x)$ by

$$\tau_k[g](x, \cdot) = \frac{d^k}{ds^k}t^0[g^{(s)}](x, \cdot)\bigg|_{s=0}.$$  \hfill (39)
It follows that for any given natural number \( m \geq 0 \), we have the following Taylor expansion with remainder:

\[
t^0 = \sum_{k=0}^{m} \frac{1}{k!} \tau^0_k + r^0_m, \tag{40}
\]

where

\[
r^0_m[g](x, \cdot) = \frac{1}{m!} \int_0^1 (1 - s)^m \frac{d^{m+1} t^0}{ds^{m+1}} \left[ g^{(s)} \right] (x, \cdot) ds. \tag{41}
\]

Formula (40) is actually our desired scaling expansion of \( t^0 \). However, as it stands, (40) is merely an identity that would hold for any distribution in the variables \((s, x, y)\) that satisfies suitable wave front set conditions. The important properties of this formula for our distributions \( t^0 \) are stated in the following theorem.

**Theorem 4.1.**  
(i) \( \tau^0_k(x, \cdot) \) and \( r^0_m(x, \cdot) \) are distributions on \( U^n \setminus (x, \ldots, x) \) which are locally constructed from the metric in a covariant way in the sense that eq. (29) holds for all diffeomorphisms that leave the point \( x \) invariant.

(ii) We have the decomposition

\[
\tau^0_k(x, y) = \sum C(x) \cdot \alpha^*_x u^0(y), \tag{42}
\]

where the sum is finite. Here, \( C \equiv C^{\mu_1 \ldots \mu_l} \) denote the coordinate components of certain curvature tensors in Riemannian normal coordinates about \( x \) and the \( u^0 \equiv u^0_{\mu_1 \ldots \mu_l} \) are Lorentz-invariant tensor valued distributions defined on \( R^{4n} \) with the origin removed, that is,

\[
u^0_{\mu_1 \ldots \mu_l}(\Lambda \cdot) = \Lambda^{\nu_1}_{\mu_1} \ldots \Lambda^{\nu_l}_{\mu_l} u^0_{\nu_1 \ldots \nu_l}(\cdot)
\]

for any Lorentz-transformation \( \Lambda \). The local curvature tensors \( C \) arise as a sum of monomials in \( g_{ab}, R_{abcd}, \ldots, \nabla_{(e_1} \ldots \nabla_{e_{k-2})} R_{abcd} \). In the case considered here with no dimensional parameters, each monomial contains precisely \( k \) coordinate derivatives of the metric.

(iii) \( \tau^0_k \) and \( r^0_m \) scale almost homogeneously under rescalings of the metric with degree \( d \).

(iv) The distributions \( u^0 \) scale almost homogeneously with degree \( d - k \) under coordinate rescalings in the sense that there exists an \( N \in \mathbb{N} \) such that

\[
S^N_{d-k} u^0 = 0, \tag{44}
\]

where \( S^N = (\sum \xi_i \partial / \partial \xi_i + \rho)^N \).

(v) The scaling degree of \( r^0_m(x, \cdot) \) is less or equal than \( d - m - 1 \), i.e., the distributions \( \lambda^{d-m-1+\delta} r^0_m(x, \lambda \cdot) \), viewed as distributions on \( R^{4n} \setminus 0 \) via the pull-back by \( \alpha_x \), tend to the zero distribution as \( \lambda \searrow 0 \) for all \( x \) and all \( \delta > 0 \).
Remark. We note that (iv) means that $u^0$ scales homogeneously with degree $d - k$ under a rescaling of the coordinates, up to logarithmic terms. Namely, simple integration of (44) gives that
\[ \lambda^{d-k} u^0(\cdot) = u^0(\cdot) + \sum_{h=1}^{N-1} \frac{\ln^h \lambda}{h!} S^h_{d-k} u^0(\cdot). \] (45)

This implies in particular that the scaling degree of $u^0$ at the origin is $d - k$.

Proof. Item (i) follows directly from the fact that $t^0$ is local and covariant on its domain of definition.

To prove (ii), we consider, first, the case where all the components of $g$ in our Riemannian normal coordinates are polynomials in the Riemannian normal coordinates $\xi$ in a neighborhood of $x$. Then we may characterize $g$ by its components $g_{\mu\nu}$ at $x$ together with the components of the coordinate derivatives, $g_{\mu\nu,\sigma_1\sigma_2...}$ at $x$, only finitely many of which are nonzero. We may thus view $t^0$ as being a function of these quantities, and we express this by writing
\[ t^0(g)(x, \cdot) = t^0(g_{\mu\nu}, \ldots, g_{\mu_1\nu_1,\sigma_1\sigma_2...\sigma_l}, \ldots)(x, \cdot), \] (46)

Now smear with a test function, $f$, on $U^n \setminus (x, \ldots, x)$. Since $t^0(g)(x, f)$ depends smoothly on the metric—and hence is a smooth function of the finite number of variables
\[ g_{\mu\nu}(x), \ldots, g_{\mu_1\nu_1,\sigma_1\sigma_2...\sigma_l}(x), \ldots \]
on which it depends—we obtain
\[ \tau_k^0[g](x, f) = \partial^k t^0[g^{(s)}](x, f) \bigg|_{s=0} = \partial^k t^0[g_{\mu\nu}, \ldots, s^l g_{\mu_1\nu_1,\sigma_1\sigma_2...\sigma_l}, \ldots](x, f) \bigg|_{s=0} = k! \sum_{l_1+2l_2+...+ml_m=k} \frac{\partial^{l_1+...+l_m} t^0[\ldots]}{\partial^{l_1} g_{\mu_1\nu_1} \ldots \partial^{l_m} g_{\mu_m\nu_m,\sigma_1\sigma_2...\sigma_l}} \prod_j [g_{\nu_1,\sigma_1...\sigma_j}(x)]^{l_j}, \] (47)

where $[\ldots]$ stands for $[g_{\mu\nu}, 0, 0, \ldots]$.

We may rewrite this equation as
\[ \tau_k^0(x, y) = \sum C(x) \cdot \alpha_x^* u^0(y), \] (48)

where the $C \equiv C^{\mu_1...\mu_l}$ are monomials in $g_{\mu_1\nu_1,\sigma_1\sigma_2...\sigma_m}$, which have the property that the total number of derivatives of $g_{\mu\nu}$ appearing in each $C$ is equal to $k$, and where $u^0 \equiv u^0_{\mu_1...\mu_l}$.
are tensor-valued distributions on $\mathbb{R}^4n$ minus the origin, which are independent of $g$. Since we are working in Riemannian normal coordinates, the $m$-th coordinate derivates $g_{\mu\nu,\sigma_1\sigma_2...\sigma_m}$ of the metric tensor at $x$ can be rewritten as the coordinate components at $x$ of a local curvature term that is polynomially constructed from the metric, the curvature tensor and its derivatives at $x$. Moreover, such a curvature term must involve precisely $m$ derivatives of the metric. Hence, by our formula (47), we conclude that $C^{\mu_1...\mu_l}$ corresponds to a curvature term $C^{a_1...a_k}$ which arises as a sum of monomials in $R_{abcd}, \ldots, \nabla_{(e_1}\ldots \nabla_{e_k)}R_{abcd}$, each of which contains precisely $k$ derivatives of the metric.

We would next like to show that the distributions $u^0$ are Lorentz invariant. For this, we consider the diffeomorphism $\psi_\Lambda$ on $U \subset M$ given by $\xi \mapsto \Lambda \xi$ where $\Lambda$ is a Lorentz transformation, and where the point $\xi \in U$ has been identified with its Riemannian normal coordinates about $x$. By definition, this diffeomorphism will leave the point $x$ invariant, so we may apply item (i) to this diffeomorphism. From this, we get that

$$\sum C^{\mu_1...\mu_l}(x)u^0_{\mu_1...\mu_l}(\Lambda \cdot) = \sum C^{\mu_1...\mu_l}(x)\Lambda^{\nu_1}_{\mu_1} \ldots \Lambda^{\nu_l}_{\mu_l} u^0_{\nu_1...\nu_l}(\cdot).$$

(49)

Since this holds for all metrics, this means that $u^0$ must be Lorentz invariant. This proves (ii) for all metrics whose components in Riemannian normal coordinates are polynomials in the Riemannian normal coordinates $\xi$ in a neighborhood of $x$.

Now consider an arbitrary smooth metric $g$. In a compact neighborhood, $K$, of $x$, we can, for each $n$, find a metric $q^{(n)}$ that is polynomial in $\xi$ and is such that everywhere within $K$ we have $|g_{\mu\nu,\sigma_1...\sigma_m} - q^{(n)}_{\mu\nu,\sigma_1...\sigma_m}| < 2^{-n}$ for all $m \leq n$. Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth function with support in $[-1, 1]$ satisfying $\psi(-v) = \psi(v)$ and also satisfying $1 - \psi(v) = \psi(1 - v)$ for all $v \in [0, 1]$. Set $h^{(0)} = g$ and for $s \neq 0$ but in a sufficiently small neighborhood of 0 define

$$h^{(s)} = \sum_n \psi(|1/s| - n)q^{(n)}$$

(50)

(Note that at each $s$, there can be at most two terms in this sum which are nonvanishing.) Then it is straightforward to show that $h^{(s)}$ is a one parameter family of smooth metrics that depends smoothly on $s$. Consequently, each $\tau_k^0(h^{(s)})(x, \cdot)$ varies smoothly with $s$. However, we have already proven that eq. (42) holds for all $s \neq 0$. By the smoothness property t3 applied to $t^0$, it follows that eq. (42) continues to hold at $s = 0$, thus proving property (ii) for an arbitrary smooth metric $g$.

Property (iii) is a direct consequence of the fact that $t^0$ satisfies the scaling property t2. To see this, we note that

$$(\lambda \partial_\lambda - d)^N T_k^0[\lambda^{-2}g] = \partial_\lambda^N (\lambda \partial_\lambda - d)^N t^0[\lambda^{-2}g^{(s)}] \bigg|_{s=0} = 0,$$

(51)
since \( t^0 \) satisfies t2. This establishes (iii) for \( \tau_k^0 \). That \( r_m^0 \) satisfies (iii) then follows immediately from eq. (II).

To prove (iv), we note that (i) implies that

\[
\tau_k^0 \lambda^{-2} g = \partial_s^k t^0 \left[ \lambda^{-2} g(s) \right] \bigg|_{s=0} = \partial_s^k t^0 \left[ (\lambda s)^{-2} \chi_s g \right] \bigg|_{s=0} = \chi_{k-1} \partial_s^k t^0 \left[ g(\lambda s) \right] \bigg|_{s=0} = \lambda^k \chi_{k-1} \tau_k^0 \left[ g \right]
\]

(52)

By eq. (51), the differential operator \((\lambda \partial_{\lambda} - d)^N\) annihilates the left side of eq. (52). This implies that

\[
0 = \left( \lambda \partial_{\lambda} - d \right)^N \lambda^k \chi_{k-1} \tau_k^0 \left[ g \right] = \lambda^k \left( \lambda \partial_{\lambda} - d + k \right)^N \chi_{k-1} \tau_k^0 \left[ g \right].
\]

(53)

Substituting the decomposition of \( \tau_k^0 \) into the expression on the right side, we obtain

\[
0 = \sum C(x) \cdot \left( \lambda \partial_{\lambda} - d + k \right)^N u^0(\lambda \cdot) = \sum C(x) \cdot S_{d-k}^N u^0(\lambda \cdot).
\]

(54)

Since this holds for arbitrary metrics \( g \), it follows that \( S_{d-k}^N u^0 = 0 \), as we desired to show.

In order to establish the estimate on the scaling degree for \( r_m^0 \), item (v), we first use eq. (II) along with the same arguments as in (iv) to write

\[
r_m^0(x, \lambda \cdot) = \lambda^{m+1} \frac{1}{m!} \int_0^1 (1 - \mu)^m \partial_s^{m+1} t^0 \left[ \lambda^2 g(s) \right] (x, \cdot) \bigg|_{s=\lambda \mu} d\mu.
\]

(55)

Using the fact that \( t^0 \) satisfies property t2, we have (see eqs. (III) and (III))

\[
r_m^0(x, \lambda \cdot) = \lambda^{m+1-d} \sum_{l=0}^{N-1} \ln^l \lambda \psi_l^0(\lambda, x, \cdot),
\]

(56)

where

\[
\psi_l^0(\lambda, x, \cdot) \overset{\text{def}}{=} \frac{1}{l!m!} \int_0^1 (1 - \mu)^m \partial_s^{m+1} (v(\partial_v - d)^l t^0 \left[ \nu^2 g(s) \right] (x, \cdot) \bigg|_{s=\lambda \mu, \nu=1} d\mu.
\]

(57)

If \( f \) is a smooth test function on \( U^n \) whose support does not contain the point \((x, \ldots, x)\), then by wave front set arguments similar to those given above, it follows from the fact that \( T^0 \) satisfies conditions T3 and T4 that the quantities \( \psi_l^0(\lambda, x, f) \) are smooth in \( \lambda \) in a neighborhood of zero. This immediately implies (v). \( \square \)
Remarks. (1) As stated in property (v) of the above theorem, if we carry the scaling expansion, eq. (40), to higher order (i.e., larger \( m \)), the remainder term \( r_m^0 \) will have a lower scaling degree. However, it should be noted that the wave front set of \( t_0 \) is determined by the null geodesics of the curved spacetime metric \( g \) whereas the wave front set of each \( \tau_k \) is similarly determined by the null geodesics of the flat spacetime metric associated with the exponential map at \( x \). Since the null geodesics of these two metrics do not, in general, coincide (with the exception of the null geodesics passing through \( x \) itself), it is clear that \( r_m^0 \) remains fundamentally distributional in nature no matter how large \( m \) is chosen. It also should be noted that it is not claimed in Thm. 4.1 that \( r_m^0 \) converges to zero in any sense (even for an analytic spacetime) as \( m \to \infty \). Thus, eq. (40) should be viewed only as a “scaling expansion” with the properties specified in Thm. 4.1, not as a convergent power series.

(2) If we combine eqs. (40) and (42), we obtain an expansion of \( t_0 \) of the general form

\[
t_0(x, y) = \sum C(x) \cdot \alpha^*_x u^0(y) + r_m^0(x, y).
\]

If the terms in the sum in (58) are ordered by the engineering dimension of the curvature terms, \( C \), then the first term in the expansion has \( C = 1 \) and the corresponding distribution \( u^0 \) is the “scaling limit” at \( x \) of the distributions \( t^0 \) in the sense of Fredenhagen and Haag [11]. The higher order terms in the expansion then give corrections to the scaling limit, organized in powers of the curvature tensor and its derivatives. If dimensionful parameters are present in the theory, then the scaling expansion will be organized in terms of products of powers of the curvature and the dimensionful parameters. Our scaling expansion is also closely related to the “momentum space representation” of the Feynman propagator and its powers (see remark (3) below) given in [4], since the Lorentz invariant distributions, \( u^0 \), on Minkowski spacetime occurring in our expansion can be given a momentum space representation.

(3) For the Feynman propagator and its powers, the scaling expansion can be explicitly calculated from known properties of the Hadamard expansion. We will illustrate this with two examples. The first example is the simplest nontrivial time ordered product, \( T^0(\varphi(x)\varphi(y)) \). Its Wick-expansion is given by

\[
T^0(\varphi(x)\varphi(y)) = :\varphi(x)\varphi(y):_H + H_F(x, y) \mathbb{1},
\]

where \( H_F = H - i\Delta_{\text{adv}} \) is the “local Feynman parametrix”, where \( H \) is the Hadamard parametrix, eq. (23), and \( \Delta_{\text{adv}} \) is the advanced Green’s function. Thus, the only nontrivial distribution \( t^0 \) occurring in this expansion is

\[
t^0(x, y) = H_F(x, y) = U(x, y)(\sigma + i0)^{-1} + V(x, y) \ln(\sigma + i0),
\]

where \( U \) and \( V \) are as in the Hadamard parametrix (see eq. (23)). The first few terms, \( \tau_0^k \), in the scaling expansion for \( t^0 = H_F \) are easily found from the expansions for \( U \) and
given in \[7\] and many other references. Modulo an overall constant, one finds
\[
\tau_0(x, y) = (\eta_{\mu\nu} \xi^\mu \xi^\nu + i0)^{-1},
\]
\[
\tau_1(x, y) = 0,
\]
\[
\tau_2(x, y) = \frac{1}{12} R^\sigma{}_{\rho}(x) \xi_\sigma \xi_\rho (\eta_{\mu\nu} \xi^\mu \xi^\nu + i0)^{-1} - \frac{1}{24} \frac{1}{x^2} R(x) \ln(\eta_{\mu\nu} \xi^\mu \xi^\nu + i0),
\]
where, as above, \(\xi^\mu\) denotes the Riemannian normal coordinates of \(y\) relative to \(x\). Thus, in this example, our scaling expansion corresponds to the usual short distance approximation to the singular part of the Feynman propagator (see, e.g., \[6\]).

Our second example is the time ordered product \(T^0(\varphi^2(x)\varphi^2(y))\). Its Wick-expansion is given by
\[
T^0(\varphi^2(x)\varphi^2(y)) = \colon \varphi^2(x)\varphi^2(y) :_H + 2H_F(x, y) : \varphi(x)\varphi(y) :_H + H_F(x, y)^2 \mathbb{1}. \quad (61)
\]
The only new \(t^0\) arising in this expansion is the “fish graph”, \(t^0 = H_F^2\), a solution to the renormalization of which was found by B. S. Kay \[15\] prior to the commencement of the present work and played a role in the development of the present work. As can be seen from the above expansion for \(H_F\), the first few coefficients, \(\tau_k^0\), for the fish graph are, modulo an overall constant,
\[
\tau_0^0(x, y) = (\eta_{\mu\nu} \xi^\mu \xi^\nu + i0)^{-2},
\]
\[
\tau_1^0(x, y) = 0,
\]
\[
\tau_2^0(x, y) = \frac{1}{6} R^\sigma{}_{\rho}(x) \xi_\sigma \xi_\rho (\eta_{\mu\nu} \xi^\mu \xi^\nu + i0)^{-2} - \frac{1}{12} \frac{1}{x^2} R(x) (\eta_{\mu\nu} \xi^\mu \xi^\nu + i0)^{-1} \ln(\eta_{\sigma\rho} \xi^\sigma \xi^\rho + i0).
\]
It is easily seen that in both examples, the distributions \(\tau_k^0\) are local, covariant distributions of the form claimed in (ii)—i.e., they are sums of terms of the form \(C(x) \cdot \alpha^a u^0(y)\) with \(u^0\) a Lorentz-invariant Minkowski space distribution—and satisfy the scaling properties specified in Thm. \[4.1\].

(4) The above scaling expansion was carried out for the scalar distributions \(t^0\). It is straightforward to check that it also holds for the extended distributions \(t\) that will be defined in the next subsection. Much more generally, it should be possible to perform a similar scaling expansion for arbitrary local covariant fields that satisfy appropriate wave front set properties. This should yield a generalized operator product expansion in curved spacetime. We are currently investigating the properties of such an expansion.

### 4.2 Extension of \(t^0[g]\)

Theorem 4.1 of the previous subsection provides the necessary machinery to achieve our goal of extending \(t^0\) in such a way that properties t1–t5 are satisfied. The basic idea is simply to suitably extend each term in the scaling expansion, eq. (60). Each \(\tau_k^0\) in that equation is of the form \([12]\) and hence can be extended to the total diagonal by extending
the Minkowski spacetime distributions $u^0$ to the origin. This can be achieved by standard methods used in Minkowski spacetime. On the other hand, if $m$ is chosen sufficiently large, the remainder term $r^0_m$ will have sufficiently low scaling degree that it can be extended to the total diagonal by continuity. The proof that the so-obtained extension $t$ satisfies properties t1–t5 will be given in the next subsection.

The key result needed to extend each $\tau^0_k$ is the following:

**Lemma 4.1.** Let $u^0 \equiv u^0_{\mu_1...\mu_l}(y)$ with $y = (\xi_1, \ldots, \xi_n)$ be a Lorentz invariant tensor-valued distribution on $\mathbb{R}^{4n} \setminus 0$ which scales almost homogeneously with degree $\rho$ under coordinate rescalings, i.e.,

$$S^N_\rho u^0 = 0$$

where $S^N_\rho = (\sum \xi^\mu \partial / \partial \xi^\mu + \rho)^N$. Then $u^0$ has a Lorentz invariant extension, $u$, to a distribution on $\mathbb{R}^{4n}$ which also scales almost homogeneously with degree $\rho$ under rescalings of the coordinates.

**Proof.** We will first extend $u^0$ using the Epstein-Glaser prescription. This extension need not satisfy either the scaling or Lorentz invariance properties. However, we will show that the extension can be modified, if necessary, so as to scale almost homogeneously with degree $\rho$. We will then show that the resulting extension can be further modified, if necessary, so as to be Lorentz invariant while retaining the almost homogeneous scaling with degree $\rho$.

Choose an arbitrary smooth function $w$ of compact support on $\mathbb{R}^{4n}$ which is equal to one in a neighborhood of the origin. For any test function $f \in \mathcal{D}(\mathbb{R}^{4n})$ we set

$$(Wf)(y) = f(y) - w(y) \sum_{|\alpha| \leq \rho - 4n} y^\alpha \partial_\alpha f(0)/\alpha!,$$

where we use the usual multi-index notation. It follows from $S^N_\rho u^0 = 0$ that $u^0$ has scaling degree $\rho$, so by [3, Thm. 5.3], we can define an extension, $u$, of $u^0$ to $\mathbb{R}^{4n}$ by setting

$$u(f) = u^0(Wf).$$

It follows that the scaling degree of $u$ is $\rho$ [3, Thm. 5.3], but it need not hold that $u$ scales almost homogeneously with degree $\rho$, i.e., there is no guarantee that $S^M_\rho u = 0$ for some natural number $M$. However, one can calculate that

$$WS^N_\rho f(y) - S^N_\rho Wf(y) = \sum_{|\alpha| \leq \rho - 4n} \psi^\alpha(y) \partial_\alpha f(0)$$

For distributions with an exactly homogeneous scaling, this result has previously been obtained in [3, Thms. 3.2.3 and 3.2.4]. Thus, our theorem generalizes this result to the case of almost homogeneous scaling.
for some smooth functions \( \psi^\alpha \) whose support does not contain the origin. From this it follows immediately that

\[
S^N_\rho u = \sum_{|\alpha| \leq \rho - 4n} c^\alpha \partial_\alpha \delta, \tag{66}
\]

where \( c^\alpha = (-1)^{|\alpha|} u^0(\psi^\alpha) \).

We now define a modified distribution \( u' \) by

\[
u' = u - \sum_{|\alpha| \leq \rho - 4n} \frac{c^\alpha}{(\rho - 4n - |\alpha|)^N} \partial_\alpha \delta. \tag{67}\]

Using the fact that \( S_\rho \partial_\alpha \delta = (\rho - 4n - |\alpha|) \partial_\alpha \delta \), we find

\[
S^N_\rho u' = \sum_{|\alpha| = \rho - 4n} c^\alpha \partial_\alpha \delta. \tag{68}\]

If we apply the operator \( S_\rho \) to both sides of the above equation, then we get that

\[
S^N_\rho u' = 0, \quad \text{because} \quad S_\rho \partial_\alpha \delta = 0 \quad \text{for} \quad |\alpha| = \rho - 4n. \tag{69}\]

This means that \( u' \) is an extension of \( u^0 \) with the desired almost homogeneous scaling. For notational simplicity, we will drop the “prime” in the following and denote this modified extension as \( u \).

We now investigate the Lorentz transformation properties of \( u \). Restoring the tensor indices on \( u \), we find by a calculation similar to eq. (66) above that for any test function \( f \in D(\mathbb{R}^{4n}) \) and any Lorentz transformation, \( \Lambda \), we have

\[
u_{\mu_1 \ldots \mu_l}(f) - \Lambda^\nu_{\mu_1} \ldots \Lambda^\nu_{\mu_l} u_{\nu_1 \ldots \nu_l}(R(\Lambda)f) = \sum_{|\alpha| \leq \rho - 4n} b^\alpha_{\mu_1 \ldots \mu_l}(\Lambda) \partial_\alpha \delta(f), \tag{70}\]

where \( (R(\Lambda)f)(y) = f(\Lambda y) \) and the \( b^\alpha_{\mu_1 \ldots \mu_l}(\Lambda) \) are complex constants, which would vanish if and only if the distribution \( u \) were Lorentz invariant. We now apply the differential operator \( S^{N+1}_\rho \) to both sides of the above equation. Since \( S^N_\rho \) is itself a Lorentz invariant operator, we have \( R(\Lambda)S^N_\rho = S^N_\rho R(\Lambda) \). Therefore, since \( S^{N+1}_\rho u = 0 \), the operator \( S^{N+1}_\rho \) annihilates the left side of eq. (70), so we obtain

\[
0 = S^{N+1}_\rho \sum_{|\alpha| \leq \rho - 4n} b^\alpha_{\mu_1 \ldots \mu_l}(\Lambda) \partial_\alpha \delta = \sum_{|\alpha| \leq \rho - 4n} (\rho - 4n - |\alpha|)^{N+1} b^\alpha_{\mu_1 \ldots \mu_l}(\Lambda) \partial_\alpha \delta. \tag{71}\]

It follows immediately that \( b^\alpha_{\mu_1 \ldots \mu_l}(\Lambda) = 0 \), except possibly when \( |\alpha| = \rho - 4n \). Thus, we have

\[
u_{\mu_1 \ldots \mu_l}(f) - \Lambda^\nu_{\mu_1} \ldots \Lambda^\nu_{\mu_l} u_{\nu_1 \ldots \nu_l}(R(\Lambda)f) = b^\nu_{\mu_1 \ldots \mu_l}(\Lambda \mu_{\nu_1} \ldots \partial_{\nu_{\rho - 4n}} \delta(f) \tag{72}\]
for all \( f \) and all Lorentz-transformations \( \Lambda \). Using this equation, one finds the following transformation property for \( b(\Lambda) \),

\[
b(\Lambda_1 \Lambda_2) = b(\Lambda_1) + D(\Lambda_1)b(\Lambda_2),
\]

(73)

where we have now dropped the tensor-indices and where \( D \) denotes the tensor representation of the Lorentz-group on \((\otimes R^4)^* \otimes (\otimes^{\rho-4n} R^4)\). It then follows by the cohomological argument given in [17] that this relation implies that \( b \) can be written in the form

\[
b(\Lambda) = a - D(\Lambda)a \quad \forall \Lambda,
\]

(74)

where \( a \) is an element in \((\otimes l R^4)^* \otimes (\otimes \rho-4n R^4)\), not depending on \( \Lambda \). This enables us to define the modified extension

\[
u'_{\mu_1...\mu_l} = u_{\mu_1...\mu_l} - a_{\nu_1...\nu_{\rho-4n}} \partial_{\nu_1} \ldots \partial_{\nu_{\rho-4n}} \delta,
\]

(75)

where we have now restored the tensor indices. It is easily checked that \( u' \) is Lorentz invariant and satisfies \( S^{N+1}_{\rho} u' = 0 \). We have therefore accomplished the goal of constructing the desired extension of \( u^0 \).

\[\Box\]

Some analyticity properties of \( u \) and its Fourier transform that follow from its scaling behaviour are established in Appendix B. These results, however, will not be needed in our present analysis.

We now can give our prescription for extending \( t^0 \). Let \( d \) denote the scaling degree of \( t^0 \), let \( m = d - 4n \), and consider the expansion eq. (40). By theorem 4.1, each \( \tau_k^0 \) appearing in this expansion takes the form

\[
\tau_k^0(x, y) = \sum C(x) \cdot \alpha_x^* u^0(y)
\]

(76)

where the sum is finite. We extend \( \tau_k^0 \) to a distribution \( \tau_k \) on \( U^n \) by choosing an extension, \( u \), of each \( u^0 \) that satisfies the properties of Lemma [4.1] and defining

\[
\tau_k(x, y) = \sum C(x) \cdot \alpha_x^* u(y).
\]

(77)

Although \( \tau_k \) has been constructed as a distribution in \( y \) that is parametrized by \( x \), it is straightforward to check that \( \tau_k \) may also be viewed as a distribution jointly in \( x \) and \( y \).

On the other hand, we know by property (v) of Theorem 4.1 that the scaling degree of \( r_m^0 \) is less or equal to \( 4n - 1 \). Therefore we can apply [3, Thm. 5.2] to conclude that \( r_m^0(x, \cdot) \) has a unique extension, \( r_m(x, \cdot) \) to all of \( U^n \) with the same scaling degree for any given point \( x \). This extension is given by

\[
r_m(f) = \lim_{j \to \infty} r_m^0(\vartheta^{(j)} f),
\]

(78)
where $\vartheta^{(j)}$ is a sequence of smooth functions with support in $U^{n+1} \setminus \Delta_{n+1}$, which are identically one outside neighborhoods $U^{(j)}_{n+1}$ of $\Delta_{n+1}$, with $U^{(j)}_{n+1}$ shrinking to $\Delta_{n+1}$ as $j$ goes to infinity. By the scaling properties of $r^0_m$, this limit exists in the weak sense, and is independent of the particular choice of cutoff functions $\vartheta^{(j)}$ (see [3, Thm. 5.2]). Again, it can be shown that this extension defines a distribution jointly in $x$ and $y$.

Our extension, $t$, is then defined by

$$t = \sum_{k=0}^{m} \frac{1}{k!} \tau_k + r_m. \tag{79}$$

Our remaining task is to show that $t$ satisfies properties $t1$–$t5$.

### 4.3 Proof that $t$ satisfies properties $t1$–$t5$

As we now shall show, it is relatively straightforward to prove that the extension, $t$, of $t^0$ defined by eq. (79) above satisfies properties $t1$ and $t2$.

To show that $t1$ holds, we note that the prescription for extending $\tau^0_k$ clearly is local in the appropriate sense. However, it is not immediately obvious that the prescription yields a covariant extension $\tau_k$ in the sense required by $t1$ since the prescription involves $\alpha_x$, whose definition requires, in addition to the metric, a choice of a tetrad $e^\mu_\alpha$ at $x$. However, since any other tetrad at $x$ is related by a Lorentz-transformation, it follows immediately from the Lorentz invariance of the extensions $u$ in (77) that different choices of tetrad lead to the same distribution $\tau_k$. It follows that each $\tau_k$ is locally constructed from the metric in a covariant way in the sense required by $t1$.

In order to see that $r_m$ is local and covariant in the sense of $t1$, it is sufficient to show that $r_m[\psi^*g]$ is equal to $\psi^*r_m[g]$ for any diffeomorphism $\psi$ on $U$. We already know that this is true off the total diagonal $\Delta_{n+1}$, as the unextended distribution $r^0$ has this property. Thus, the difference between the two expressions must be a distribution supported on the total diagonal. Moreover, the scaling degree of this distribution must be less than $4n - 1$, by our choice $m = d - 4n$. It is well known that there are no such distributions apart from the zero distribution (essentially because the delta function and its derivatives have scaling degree $\geq 4n$). Therefore the difference must in fact be zero, showing that $r_m$ satisfies $t1$. Since all terms on the right side of eq. (79) satisfy $t1$, it follows that $t$ satisfies this property.

To establish $t2$, we first show that the extensions $\tau_k[g]$ have an almost homogeneous scaling under rescalings of the metric in the sense of $t2$. To see this, we consider a term $C \cdot \alpha^*_x u$ in the expansion (77). By Theorem 4.1, the curvature term $C$ will scale as $\lambda^{-k}$ under a rescaling of the metric by $\lambda^2$. On the other hand, for the term $\alpha^*_x u$, since $\alpha_x$ is just the inverse of the exponential map at $x$, a rescaling of the metric will correspond precisely to a coordinate rescaling by a factor of $\lambda$ in the distributions $u$. By Lemma 4.1, these distributions scale like $\lambda^{k-d}$ up to logarithmic corrections under such a
coordinate rescaling. Therefore, each individual term in formula (79) for $t_k$ has an almost homogeneous scaling with degree $d$ under rescalings of the metric. On the other hand, the almost homogeneous scaling of $r_m$ under a rescaling of the metric can be proven by an argument similar to the proof that $r_m$ is local and covariant. Consequently, we see that $t$ satisfies property t2.

It also is relatively straightforward to prove that each $\tau_k$ occurring in eq. (79) satisfies properties t3–t5. We know that $\tau_k$ is a finite sum of terms of the form $C(x) \cdot \alpha_x u(y)$, with $C(x)$ a polynomial in the curvature and its derivatives. Since $C(x)$ is smooth in $x$, we have

$$\text{WF}(C \cdot \alpha^* u) \subset \left\{ (x, \sum [\partial \alpha_x / \partial \xi_i]^t k_i; \xi_1, \left[ \partial \alpha_x / \partial \xi_i \right]^t k_1; \ldots; \xi_n, \left[ \partial \alpha_x / \partial \xi_i \right]^t k_n) \mid \right. \left( \alpha_x(\xi_1), k_1; \ldots; \alpha_x(\xi_n), k_n \right) \in \text{WF}(u) \right\}. \quad (80)$$

Here, we have written $y = (\xi_1, \ldots, \xi_n)$ and each $\xi_i$ denotes a point in a convex normal neighborhood of $x$, and not the Riemannian normal coordinates of that point. (This makes a difference here, since we are considering variations in $x$.) In eq. (80), $\partial \alpha_x / \partial \xi_i$ denotes the matrix of partial derivatives of $\alpha_x$ with respect to $\xi_i$ at fixed $x$, and $\partial \alpha_x / \partial x$ denotes the matrix of partial derivatives of $\alpha_x(\xi_i)$ with respect to $x$ at fixed $\xi_i$. However, at $\xi_i = x$ we have $\partial \alpha_x(\xi_i) / \partial \xi_i = -\partial \alpha_x(\xi_i) / \partial x$, since moving $\xi_i$ infinitesimally away from $\xi_i = x$ has the same effect on $\alpha_x(\xi_i)$ as moving $x$ infinitesimally by the same amount in the opposite direction. It follows that if $(x, k_1; \ldots; x, k_{n+1}) \in \text{WF}(C \cdot \alpha^* u)$, then $\sum k_i = 0$. This means precisely that $\text{WF}(C \cdot \alpha^* u) \mid_{\Delta_{n+1}} \perp T(\Delta_{n+1})$, i.e., the microlocal spectral condition, t3, is satisfied. Similarly, by using the fact that $C(x)$ is a polynomial in the curvature and $\alpha_x$ is the inverse of the exponential map—so that both $C(x)$ and $\alpha_x$ have appropriate smooth and analytic dependence on the metric—together with the fact that $u$ is independent of the metric, we find that the smoothness (t4) and analyticity (t5) conditions are satisfied by $\tau_k$.

Thus, we would be done if our expression (79) for $t$ corresponded to a suitably convergent power series. However, as already noted in remark (1) at the end of subsection 4.1, this is not the case, i.e., the remainder term, $r_m$, in eq. (79) is not expected to converge to zero in any sense useful for our purposes as $m \to \infty$. Therefore, in order to prove that $t$ satisfies properties t3–t5, it is necessary to explicitly analyze the remainder term $r_m$. This is technically quite cumbersome, since essentially the only thing useful that is known about $r_m$ is that it is the extension to $\Delta_{n+1}$ defined by eq. (78) of the expression $r_m^0$ given by eqs. (56) and (57). Equation (78) expresses $r_m$ as a weak limit of distributions whose wave front set properties are known, but wave front set properties are not preserved under weak convergence, so we must show that the sequence (78) converges in a suitably strong sense to enable us to prove that $r_m$ satisfies properties t3–t5. This will be accomplished in the proof of the following proposition, which—as will be explained in the remark following the statement of the proposition—will complete the proof that the
extensions \( t \) satisfy the properties t3–t5.

**Proposition 4.1.** Let \( g^{(s)} \) be a smooth one-parameter family of smooth metrics, and let \( r_m(s, x_1, \ldots, x_n) \) denote the remainder term in eq. (79), viewed as a distribution on \( \mathbb{R} \times U^n \). (Here \( m = d - 4n \), where \( d \) is the scaling dimension of \( t \).) Then the wave front set of \( r_m \) satisfies

\[
\text{WF}(r_m) \mid_{\mathbb{R} \times \Delta_n \perp T(\mathbb{R} \times \Delta_n)}, \tag{81}
\]

where the notation “\( \perp \)” was introduced below eq. (16). Similarly if \( g^{(s)} \) is an analytic one-parameter family of analytic metrics, then (81) holds for the analytic wave front set.

**Remark.** If we choose \( g^{(s)} = g \) for all \( s \), the above proposition implies that \( r_m \) satisfies the microlocal spectral condition t3. The proposition also implies that \( r_m \) satisfies the smoothness and analyticity conditions, t4 and t5. In fact, the proposition asserts a somewhat stronger version of these conditions, as it shows that the wave front set of \( r_m(s, x_1, \ldots, x_n) \) not only cannot contain any points of the form \( (s, \rho; x, k_1; \ldots; x, k_n) \) with \( \sum k_i \neq 0 \) but it also cannot contain any such points with \( \rho \neq 0 \). Since each \( \tau_k \) has already been shown above to satisfy t3–t5, it follows that \( t \) satisfies t3–t5 if \( r_m \) does. Thus, our construction of time ordered products satisfying properties T1–T9 of section 2 will be completed once we have completed the proof of this proposition.

**Proof.** We will give the proof only for the analytic case; the proof for the smooth case is similar, though somewhat simpler because the estimates needed to establish the wave front set properties are simpler in nature in the smooth case. As before, we proceed by induction in the number, \( n \), of variables \( (x_1, \ldots, x_n) \) on which \( r_m \) depends. We inductively assume that the analytic wave front set version of eq. (81) holds for all \( r_m \) that depend on \( n \) or fewer variables. By a slight generalization of the proof given above that \( \tau_k \) satisfies t3–t5, it can be shown that if \( g^{(s)} \) is an analytic one-parameter family of analytic metrics then each \( \tau_k \) also satisfies eq. (81). Consequently our inductive hypothesis implies that \( t(s, x_1, \ldots, x_n) \equiv t[g^{(s)}](x_1, \ldots, x_n) \) satisfies

\[
\text{WF}_A(t) \mid_{\mathbb{R} \times \Delta_n \perp T(\mathbb{R} \times \Delta_n)}, \tag{82}
\]

From the distributional coefficients \( t \) depending on \( n \) or fewer spacetime arguments and the real parameter \( s \) we obtain, by our inductive constructions, the distributional coefficients \( t^0(s, x_1, \ldots, x_{n+1}) \equiv t^0[g^{(s)}](x_1, \ldots, x_{n+1}) \) depending on \( n+1 \) spacetime arguments and the real parameter \( s \). These distributions are defined everywhere in \( \mathbb{R} \times U^{n+1} \), except for (\( \mathbb{R} \) times) the total diagonal, \( \Delta_{n+1} \). Their analytic wave front set \( \text{WF}_A(t^0) \) is therefore a subset of \( T^+(\mathbb{R} \times (U^{n+1} \setminus \Delta_{n+1})) \). By essentially the same arguments as given in [3, Sec. 7] (modulo a straightforward modification of those arguments with regard to
the additional parameter \( s \) on which the \( t^0 \) depend), the distribution \( t^0(s, x_1, \ldots, x_{n+1}) \) can be expressed as a finite sum of terms of the form

\[
t(s, \{x_i\}_{i \in I}) t(s, \{x_j\}_{j \in I^c}) \prod_{i \in I, j \in I^c} H_F(s, x_i, x_j)^{a_{ij}}.
\]  

(83)

on each of the open sets \( C_I \) introduced in eq. (18). Here, the \( a_{ij} \) are certain natural numbers, \( I \) is a nonempty proper subset of the set \( \{1, \ldots, n+1\} \) and \( I^c \) is its complement. (Note that since \( I \) is a nonempty proper subset, the expression (83) only involves the distributional coefficients \( t \) depending on \( n \) or fewer spacetime arguments.) Finally, \( H_F(s, x_1, x_2) \equiv H_F(g(s))(x_1, x_2) \) is the local Feynman parametrix introduced below eq. (59), for our analytic 1-parameter family of metrics. It can be seen by an explicit calculation that

\[
WF_A(H_F) |_{\mathbb{R} \times \Delta_2} \perp T(\mathbb{R} \times \Delta_2).
\]  

(84)

For each of the sets \( I \) described above, let us define a projection map \( \pi_I \) from \( \mathbb{R} \times U_{n+1} \) to \( \mathbb{R} \times U_{|I|} \) (with \( |I| \) the number of elements in \( I \)) by

\[
\pi_I : (s, x_1, \ldots, x_{n+1}) \rightarrow (s, \{x_i\}_{i \in I}).
\]  

(85)

Using the rules for calculating the analytic wave front set of products of distributions \[41\], we find from eq. (83) that the analytic wave front set of \( t^0 \) restricted to the open sets \( \mathbb{R} \times C_I \) is estimated by

\[
WF_A(t^0) |_{\mathbb{R} \times C_I} \subset (\pi_I^* WF_A(t) \cup \{0\}) + (\pi_{I^c}^* WF_A(t) \cup \{0\})
\]

\[
+ \sum_{i \in I, j \in I^c} a_{ij} \sum_{\{i,j\}} \pi_{\{i,j\}}^* WF_A(H_F) \cup \{0\}) \subset T^*(\mathbb{R} \times (U_{n+1} \setminus \Delta_{n+1})).
\]  

(86)

If we now take the closure in \( T^*(\mathbb{R} \times U_{n+1}) \) of the sets on both sides of the above relation (we denote this closure by an overbar), take the union over all \( I \), and use eqs. (82) and (84), then we obtain

\[
\overline{WF_A(t^0)} |_{\mathbb{R} \times \Delta_{n+1}} \perp T(\mathbb{R} \times \Delta_{n+1}).
\]  

(87)

From the properties of \( \tau_k \), it then follows that \( r_m^0(s, x_1, \ldots, x_{n+1}) \) also satisfies the same condition, i.e.,

\[
\overline{WF_A(r_m^0)} |_{\mathbb{R} \times \Delta_{n+1}} \perp T(\mathbb{R} \times \Delta_{n+1}),
\]  

(88)

Note that eq. (88) imposes a nontrivial restriction (beyond what we already know) on the wave front set of \( r_m^0 \). Our aim is to show that (88) continues to hold for the extension,
In order to simplify the discussion, we will show here only the weaker result that, for a fixed metric $g$, $r_m(x_1, \ldots, x_{n+1})$ satisfies

$$\text{WF}_A(r_m) \cap T(\Delta_{n+1}).$$

(89)

However, the arguments can be generalized straightforwardly to prove (81) in $n+1$ variables for an analytic one-parameter family of analytic metrics $g^{(s)}$.

As above, we choose relative coordinates $(x, y)$ around the total diagonal. We will identify the point $x \in M$ with its coordinates in some chart, and we will identify $y = (\xi_1, \ldots, \xi_n)$ with its Riemannian normal coordinates relative to $x$, so that the diagonal corresponds to $y = 0$. With this choice of coordinates, we identify $t^0$ (and likewise, $r^0_m$) with a distribution defined on $X \times (Y \setminus 0)$, where $X$ is an open set in $\mathbb{R}^4$ and $Y$ is an open neighborhood of the origin $\mathbb{R}^m$. Let $x_0$ be some fixed point in $X$. It is possible to construct a sequence of smooth functions of the form $\phi_N(x, y) = \phi'_N(x)\phi''_N(y)$, where $\phi'_N \in C^\infty_0(K')$ is 1 in a neighborhood of $x_0$, such that $\phi''_N$ vanishes in a neighborhood of 0 and is 1 outside some larger neighborhood $K''$, and where $\phi_N$ satisfies the estimate

$$|\partial^{\alpha+\beta} \phi_N| \leq C^{[\alpha]+1}_\alpha (N+1)^{[\beta]} \quad \forall \beta | N = 1, 2, \ldots .$$

(90)

If $f$ is a test function with support sufficiently close to $(x_0, 0)$, then the extension $r_m$ is defined by eq. (78). For our purposes, it is convenient to make the choice $\varphi^{(j)} = (\phi_N)^{2j}$, where the subscript $2^j$ means the pull-back by the function $(x, y) \to (x, 2^jy)$.

In order to show $\text{WF}_A(r_m) \cap T(\Delta_{n+1})$, we must demonstrate that $(x_0, k_0, 0, p_0)$ is in the complement of $\text{WF}_A(r_m)$ whenever $k_0 \neq 0$. It is not difficult to see that this will follow if we can show that it is possible to choose $K = K' \times K''$ so small that

$$|\left(\hat{\theta_N} \right)_{2^j} r^0_m(k, p)| \leq 2^{-j/2} C^{N+1}((N+1)/(|k| + |p|))^N$$

(91)

for all $(k, p)$ in some conic neighborhood $F$ of of $(k_0, p_0)$ and for all natural numbers $N$ and $j$. Here, $\theta_N \in C^\infty_0(K)$ is the cutoff function defined by $\hat{\phi}(x, y) - \phi_N(x, 2y)$. Note that the support of $\theta_N$ does not intersect the submanifold $X \times \{0\}$, and that the sequence of cutoff functions $\theta_N$ is again bounded in $\mathcal{E}'(K)$ and satisfies the estimate (78).

In order to analyze the Fourier transform on the left side of (91), we observe that

$$\left(\hat{\theta_N} \right)_{2^j} r^0_m(k, p) = 2^{-4nj} \theta_N(r^0)_{2^{-j}}(k, 2^{-j}p),$$

(92)

where $(r^0_m)_{2^{-j}}$ denotes the pull-back of the distribution $r^0_m$ by the map $(x, y) \to (x, 2^{-j}y)$, and where the factor $2^{-4nj}$ is due to the fact that $r^0_m$ transforms as a density. Recalling our choice $m = d - 4n$, we can write the quantity on the right side of this equation as

$$2^{-j} \sum_l (j \ln 2)^l \theta_N(2^{-l}k, 2^{-l}p),$$

(93)
where the \( \psi_0^l \in \mathcal{D}'(\mathbb{R} \times X \times (Y \setminus 0)) \) were defined in eq. (56), and where the Fourier transform is with respect to the variables \( x \) and \( y \).

We now claim that for any closed conic set \( \Gamma \) in \( \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^{4n} \) not containing elements of the form \((0,0,p)\) there is a neighborhood \( K_0 \subset \mathbb{R} \times X \times Y \) of \((0,x_0,0)\) such that for all \( l \)

\[
\text{WF}_A(\psi_0^l) \cap (K_0 \times \Gamma) = \emptyset. \tag{94}
\]

To prove this, we decompose \( \psi_0^l \) into simpler pieces, whose analytic wave front set is either known by the induction process or can be determined by elementary means. For this, we shall define a family of analytic metrics depending analytically on parameters \( s \equiv (v,\mu,x) \in P_1 \times P_2 \times P_3 \equiv P \), where \( P_1 \) is a small neighborhood of 1 in \( \mathbb{R} \), \( P_2 \) is a small neighborhood of 0 in \( \mathbb{R} \) and where \( P_3 \) is a convex normal neighborhood in \( M \) with respect to \( g \). In order to define this family, let \( \chi_{x,\mu} \) be the diffeomorphism which shrinks the Riemannian normal coordinates \( \xi \) of a spacetime point about the point \( x \in P_3 \) by a factor of \( \mu \). In terms of this family of diffeomorphism, our family of metrics is given by

\[
g^{(s)} = (v\mu)^{-2} \chi_{x,\mu}^* g. \tag{95}
\]

This is a real analytic family of analytic metrics (but with \( s \) now ranging over the 6-dimensional parameter space, \( P \), rather than over \( \mathbb{R} \)). We have already established that the analytic wave front set of the distribution \( t^0 \) on \( P \times (U^{n+1} \setminus \Delta_{n+1}) \) satisfies (87) (with \( \mathbb{R} \) replaced by \( P \)). In order to relate \( \psi_0^l \) to \( t^0 \), we let \( R^{(m)} \) be the map from test functions on \( \mathbb{R} \) to smooth functions on \( \mathbb{R} \) given by

\[
(R^{(m)} f)(\lambda) = \frac{1}{m!} \int_0^1 (1 - \mu)^m f^{(m+1)}(\lambda \mu) \, d\mu. \tag{96}
\]

Furthermore, we set

\[
D^{(l)} = \frac{1}{l!} (v \partial / \partial v + d)^l. \tag{97}
\]

Note that if \( f \) is a smooth function on \( \mathbb{R} \) with compact support, then we have \( \text{supp}(R^{(m)} f) \subset \text{supp}(f) \), where \( tR^{(m)} \) denotes the transpose of \( R^{(m)} \). Thus \( tR^{(m)} \) has proper support. It now follows straightforwardly from the definition of \( \psi_0^l \) that we can rewrite the action of the distributions \( \psi_0^l \) on test functions \( f \in C_0^\infty (\mathbb{R} \times (U^{n+1} \setminus \Delta_{n+1})) \) as

\[
\psi_0^l(f) = (j^*t^0) \left[ ((D_v^{(l)} \delta(\cdot - 1)) \otimes ((tR^{(m)}_{\mu} \otimes 1_{x_1,\ldots,x_{n+1}}) f) \right], \tag{98}
\]

where the subscripts on the operators indicate on which of the variables \((v,\mu,x_1,\ldots,x_{n+1})\) they act, and where \( j^*t^0 \) denotes the pull back of \( t^0 \) by the analytic map

\[
j : (v,\mu,x_1,\ldots,x_{n+1}) \rightarrow (v,\mu,x = x_1,\ldots,x_{n+1}) \in P \times U^{n+1}. \tag{99}
\]
The analytic wave front set of $\psi_0$ can now be estimated from eq. (98) using our knowledge of the analytic wave front set of $t^0$, eq. (87), together with the rules for calculating the wave front set of a distribution under composition with distribution kernels [14, Thm. 8.5.5], and under pull-back by analytic maps [14, Thm. 8.5.1]. For this, we only need to know the following additional facts: (i) The action of an analytic partial differential operator, such as $D(l)$, does not enlarge the analytic wave front set and (ii) the analytic wave front set of the distribution kernel of $R^{(m)}$ (viewed as a bidistribution on $\mathbb{R} \times \mathbb{R}$) does not contain any elements of the form $(\lambda, 0; \mu, \rho)$. The first statement is proven in [14, Thm. 8.4.7], and the second statement can be checked directly.

This information suffices to conclude from eq. (98) that
\[
\overline{\text{WF}_A(\psi_0^l)|_{\mathbb{R} \times \Delta_{n+1}}} \perp T(\mathbb{R} \times \Delta_{n+1}).
\]
If $\mathbb{R} \times (U^{n+1} \setminus \Delta_{n+1})$ is identified with subset of $\mathbb{R} \times X \times (Y \setminus 0)$ via the above choice of coordinates, then this means that
\[
\overline{\text{WF}_A(\psi_0^l)|_{\mathbb{R} \times X \times \{0\}}} \perp T(\mathbb{R} \times X \times \{0\}).
\]
As a consequence, the open set $T^*(\mathbb{R} \times X \times Y) \setminus \overline{\text{WF}_A(\psi_0^l)}$ contains a set of the form $K_0 \times \Gamma$ as claimed in eq. (94), provided that $K_0$ is chosen to be sufficiently sharply concentrated about the point $(0, x_0, 0)$.

We now blow up the sequence of cutoff functions $\theta_N \in C^\infty_0(K)$ to a bounded sequence cutoff functions in $C^\infty_0(K_0)$ which still satisfy the inequality (90), and which we shall denote by the same symbol. It then follows that with these cutoff functions,
\[
|\hat{\theta_N \psi_0^l}(\rho, k, p)| \leq C_N |(N + 1)/(|\rho| + |k| + |p|)|^N
\]
for all $(\rho, k, p) \in \Gamma$, provided that the support $K_0$ of $\theta_N$ is sufficiently sharply concentrated near $(0, x_0, 0)$, where the tilde denotes the Fourier transform in $\mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$. With our new choice for $\theta_N$, we can write (93) as
\[
(\hat{\theta_N}) \hat{\psi_0^l}(k, p) = (2\pi)^{-1/2}2^{-j} \sum_l (j \ln 2)^l \int_\mathbb{R} \hat{\theta_N \psi_0^l}(\rho, k, 2^{-j}p)e^{-i2^{-j}\rho} d\rho,
\]
where the sum is finite. Now the cone $\Gamma$ can be chosen such that $(\rho, k, 2^{-j}p) \in \Gamma$ for all points $(k, p)$ in the cone $F$, all $\rho$ and all $j$. Thus we can use (102) to estimate
\[
|((\hat{\theta_N}) \hat{r_0^m}(k, p)| \leq 2^{-j/2}C_N (N/|k|)^{N-1}.
\]
For $(k, p)$ in the cone $F$ it holds that $|k| > \epsilon|p|$ for some $\epsilon > 0$. This enables us to estimate the above expression further by
\[
\leq 2^{-j/2}C_N (N/(|k| + |p|))^{N-1}
\]
for all $(k, p)$ in $F$ and all natural numbers $N$ and $j$. This is what we wanted to show. □
Remark. The distributions $t$ have now been shown to have an analytic dependence on the metric in the sense of condition 5. This makes it possible to establish certain analyticity properties of the distributions $u$ in the scaling expansion for the $t$, as we will now show. We know that if $(s, \rho; x_1, k_1; \ldots; x_{n+1}, k_{n+1})$ is an element in $WF_A(t)$, then the element $(x_1, k_1; \ldots; x_{n+1}, k_{n+1})$ must necessarily be in the set $C_T^{(s)}$, given by eq. (8). By our scaling expansion, we know that the distributions $u$ are given in terms of $s$ derivatives of $t$ (evaluated at $s = 0$), so we can calculate $WF_A(u)$ from $WF_A(t)$ by the rules for the analytic wave front set under restriction and differentiation. It is straightforward to see that this gives

$$WF_A(u) \subset \left\{ (\xi_1, k_1; \ldots; \xi_n, k_n) \in T^*\mathbb{R}^{4n} \setminus \{0\} \mid \exists \text{ decorated graph } G(p) \text{ in } (\mathbb{R}^4, \eta) \text{ with vertices } 0, \xi_1, \ldots, \xi_n \text{ such that } k_i = \sum_{e : s(e) = i} p_e - \sum_{e : t(e) = i} p_e \text{ } \forall i \right\},$$

where we use the graph-theoretical notation introduced in T3, and where $\eta$ is the Minkowski metric.

5 Conclusions and Outlook

In this paper, we have given a construction of local, covariant time ordered products of an arbitrary number of local Wick powers. These local time ordered products were shown to satisfy properties T1–T9 of section 2. They therefore fulfill the assumptions of the uniqueness theorem of our previous paper \cite[Thm. 5.2]{12}. Consequently, for any given polynomial order in the free field, any other prescription for defining local time ordered products with the same properties will differ from the prescription given in the present paper by products of local curvature terms and lower order time ordered products, as specified precisely in our uniqueness theorem. Although in this paper we considered only a massless Klein-Gordon scalar field, our results can be generalized straightforwardly to allow mass, and we do not anticipate any difficulties in generalizing our results to fields with higher spin. Largely for notational simplicity, we also restricted consideration to time ordered products of Wick powers that do not contain derivatives of the field, but it should be straightforward to generalize our construction to allow Wick powers of differentiated fields (subject only to the caveat of footnote 3).

An important tool in our analysis was the scaling expansion introduced in subsection 4.1 for the distributions $t$ appearing in the local Wick expansion of time ordered products. In essence, this scaling expansion gives corrections to the “scaling limit” of \cite[14]{11}, organized in powers of the curvature (and the dimensionful parameters, if any are present). The scaling expansion generalizes to arbitrary $t$ in the local Wick expansion the usual “short distance expansion” for the Feynman propagator (see remark (3) at the end of subsection 4.1). Although we restricted consideration here to the distributions $t$, a similar scaling
expansion will exist for any local, covariant field which satisfies appropriate wave front set and scaling properties. The properties of the general scaling expansion for local covariant fields is currently under investigation.

The results of this paper essentially complete the analysis of the existence, uniqueness, and renormalizability of the perturbative expansion of nonlinear quantum fields (with polynomial self-interaction) in curved spacetime. It is natural to ask whether an “exact” formulation of nonlinear quantum field theory in curved spacetime can be given. The Wightman axioms and other similar systems cannot be straightforwardly generalized to curved spacetime on account of their essential usage of Poincare invariance and the existence of a preferred vacuum state. We are currently investigating the possibility that the notion of a local, covariant quantum field (together with suitable microlocal spectral conditions, etc.) may enable one to give a useful formulation of axiomatic quantum field theory in curved spacetime.

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A Smooth and analytic variation of distributions

A key requirement that we impose on our definition of Wick polynomials and their time ordered products is that they have appropriate smooth/analytic dependence on the spacetime metric. The purpose of this appendix is to elucidate the notion of smooth and analytic variation of distributions.

To begin, let $X$ be a manifold and for each $s \in \mathbb{R}$ let $u^{(s)} : X \to \mathbb{C}$ be a smooth (i.e., $C^\infty$) function. It is useful to view $u^{(s)}$ as a map $u : \mathbb{R} \times X \to \mathbb{C}$. We say that $u^{(s)}$ varies smoothly with $s$ if the map $u$ is smooth. Note that this requirement of (joint) smoothness of the map $u$ is stronger than the possible alternative requirement that $u^{(s)}(x)$ be a smooth function of $s$ for each fixed $x \in X$. This latter notion of (separate) smoothness in $s$ would not be a natural one in the context of this paper for the following reason: We consider one-parameter families of spacetimes $(M, g^{(s)})$ and there is no natural way of identifying spacetimes with different values of $s$. However, the notion of separate smoothness is not invariant under diffeomorphisms $\psi^{(s)} : X \to X$ that are (jointly) smooth in $(s, x)$.

Now, for each $s \in \mathbb{R}$ let $u^{(s)} \in \mathcal{D}'(X)$, i.e., $u^{(s)}$ is a distribution on $X$. We wish to define a notion of smooth variation of $u^{(s)}$ with $s$ that corresponds to the notion of (joint) smoothness of functions as defined in the previous paragraph. To do so, it is useful to
view \(u^{(s)}\) as a distribution, \(u\), on \(\mathbb{R} \times X\). The basic idea of our definition is to require \(u\) to be “not any more singular than each \(u^{(s)}\)”. One possible way of implementing this notion would be to demand that the wave front set of \(u\) be contained in the wave front set of \(u^{(s)}\) in the sense that \(WF(u) \subset \{(s, \rho; x, k) \mid \rho = 0, (x, k) \in WF(u^{(s)})\}\). However, this definition is unsatisfactory for the following two independent reasons. First, the requirement that \(\rho = 0\) is too strong in that it would, in particular, require the singularities of \(u^{(s)}\) to “remain in a fixed location in \(X\)” as \(s\) is varied. This would not be invariant under a one parameter family of diffeomorphisms \(\psi^{(s)} : X \to X\) that are (jointly) smooth in \((s, x)\). It should be noted that the distributions, \(u^{(s)}\), of interest in this paper have singularities on the light cones of \(g^{(s)}\) and, hence, their singularities cannot “remain in a fixed location” for non-conformal variations of \(g\). Consequently, we shall not require \(\rho = 0\) in our definition. Second, if \(u^{(s)}\) happens to be “less singular than normal” for some value of \(s\), then under the above proposed definition, \(u\) would fail to vary smoothly with \(s\) even if, in a naive sense, its variation with \(s\) was perfectly smooth. For example, \(u^{(s)}(x) = s\phi(x)\) would fail to be smooth at \(s = 0\) because \(WF(u^{(0)}) = \emptyset\) but \(WF(u)\) includes points with \(s = 0\). For this reason, we will define a more general notion of smoothness with respect to an arbitrary specified family of cones \(C^{(s)}\). (Here, a cone \(C\) is a subset of \(T^*X\setminus\{0\}\) having the property that if \((x, k) \in C\), then \((x, \lambda k) \in C\) for all \(\lambda > 0\).) For the definition to be nontrivial, we must choose \(C^{(s)}\) so that \(WF(u^{(s)}) \subset C^{(s)}\), but we need not choose \(C^{(s)} = WF(u^{(s)})\).

**Definition A.1.** Let \(u^{(s)}\) be a one-parameter family of distributions on a manifold \(X\) and let \(C^{(s)}\) be a family of cones. We say that \(u^{(s)}\) varies smoothly with \(s\) with respect to \(C^{(s)}\) if the wave front set of the corresponding distribution \(u\) on \(\mathbb{R} \times X\) satisfies

\[
WF(u) \subset \{(s, \rho; x, k) \in T^* (\mathbb{R} \times X) \setminus \{0\} \mid (x, k) \in C^{(s)}\}
\]

(107)

**Remarks.** (1) To illustrate the meaning of the above definition, let us consider the two extreme case, namely (a) when the cones are trivial, \(C^{(s)} = \emptyset\), and (b) when the cones are maximal, \(C^{(s)} = T^* X \setminus \{0\}\). In the first case (a) we immediately get that \(WF(u) = \emptyset\), so \(u\) is smooth jointly in \((s, x)\). In the second case (b), it might appear that our smoothness condition is in fact empty. However, this is not the case, since eq. (107) implies that no element of the form \((s, \rho; x, 0)\) can be in \(WF(u)\). Thus, for example, if \(u = v \otimes \phi\) with \(v\) a distribution on \(\mathbb{R}\) and \(\phi\) a distribution on \(X\), not depending on \(s\), eq. (107) requires that \(v\) is smooth.

(2) Let \(u_1^{(s)}\) and \(u_2^{(s)}\) be two families of distributions which are smooth with respect to cones \(C_1^{(s)}\) respectively \(C_2^{(s)}\). Then the rules for calculating the wave front set of a sum of two distributions gives that the family \(u_1^{(s)} + u_2^{(s)}\) is smooth with respect to the cones \(C_1^{(s)} \cup C_2^{(s)}\). Likewise, if \(\{0\} \notin C_1^{(s)} + C_2^{(s)}\), for each \(s\), then the product \(u_1^{(s)}u_2^{(s)}\) can be defined for each \(s\) and defines a distribution jointly in \((s, x)\). Moreover, the rules for calculating the wave front set of the product of two distributions gives that product family is smooth with respect to the cones \(C_1^{(s)} + C_2^{(s)}\).
The above definition allows us to define the notion of the smooth variation of a one parameter family, \( \omega^{(s)} \), of continuous states on the algebras \( \mathcal{W}(M, g^{(s)}) \) of the spacetimes \((M, g^{(s)})\): We say that \( \omega^{(s)} \) varies smoothly with \( s \) if each of the \( n \)-point functions \( \omega^{(s)}_{n} \) of \( \omega^{(s)} \)—viewed as a distribution on \( M^{n} \)—varies smoothly with \( s \) in the sense of Definition A.1 with \( C_{n} \) in \( \omega^{(s)} \) distributions, \( u \). This motivates the following definition. Let \( \mathcal{T} \) be a function on \( X \subset \mathbb{R}^{m} \), which is a real analytic in a neighborhood of a point \( x_{0} \) in \( \mathbb{R}^{m} \). Then it follows from Cauchy's integral formula, or rather its generalization to \( \mathbb{C}^{m} \), that

\[
|\partial^{\alpha} u| \leq C^{(\alpha) + 1} |\alpha| + 1 \quad \forall \alpha
\]

in a neighborhood of \( x_{0} \), where \( C \) is some constant and multi-index notation has been used. Conversely, if the above estimate holds for a function \( u \) in a neighborhood of \( x_{0} \), then \( u \) is real analytic in that neighborhood.

Condition (108) can be formulated equivalently in terms of Fourier transforms. Namely, one can show that an estimate of the form (108) holds if and only if there is a sequence \( u_{N} \) of compactly supported distributions equal to \( u \) in some open ball around \( x_{0} \), which is bounded in the space \( \mathcal{E}'(\mathbb{R}^{m}) \) of distributions of compact support, and which satisfies

\[
|\tilde{u}_{N}(k)| \leq C^{N+1}((N + 1)/|k|)^{N} \quad \forall N \in \mathbb{N}.
\]

This motivates the following definition. Let \( u \) be a distribution on \( X \subset \mathbb{R}^{m} \). The analytic wave front set \( \text{WF}_{A}(u) \) is defined to be the complement of the set of all points \((x_{0}, k_{0})\) in
\( X \times (\mathbb{R}^m \setminus 0) \) such that there is an open neighborhood \( U \) of \( x_0 \), a conic neighborhood \( \Gamma \) of \( k_0 \) and a bounded sequence \( u_N \in \mathcal{E}'(X) \) which is equal to \( u \) on \( U \) and satisfies \( (109) \) whenever \( k \in \Gamma \).

It is clear from the definition that \( u \) is given by a real analytic function in the neighborhood of points \( x_0 \) such that \( \text{WF}_A(u) \) contains no element of the form \((x_0, k_0)\). If \( f : X \to Y \) is an analytic one-to-one map, then the analytic wave front set of the pullback \( f^*u \) is given by \( \{(x, df^t(x)k) \mid (f(x), k) \in \text{WF}_A(u)\} \). This makes it possible, via localization in analytic charts, to define in an invariant way the analytic wave front set of a distribution on a real analytic manifold \( X \).

In practice it is useful that \( u_N \) can always be obtained as a product of \( u \) and suitable cutoff functions, see [14, Lem. 8.4.4]: Let \( \text{WF}_A(u) \cap (K \times F) = \emptyset \) for some compact subset \( K \) of \( X \) and some closed cone \( F \), and let \( \chi_N \) be a sequence of cutoff functions in \( C^\infty_0(K) \) such that for all \( \alpha \)

\[ |\partial^{\alpha+\beta} \chi_N| \leq C_\alpha^{[\beta]+1}(N + 1)^{|\beta|} \quad \forall |\beta| \leq N = 1, 2, \ldots \quad (110) \]

Then \( u_N = \chi_N u \) is bounded in \( \mathcal{E}'(X) \) and satisfies \( (109) \) for all \( k \in F \).

A one-parameter family of distributions, \( u^{(s)} \), on a real analytic manifold, \( X \), will be said to vary analytically with \( s \) with respect to the cones \( C^{(s)} \) if eq. \( (107) \) holds with \( \text{WF} \) replaced by \( \text{WF}_A \) everywhere in that equation. The notions of analytic variations of states and fields can then be defined in complete parallel with the definition of smooth variation given above. This agrees with the notions previously introduced in [12].

### B Properties of the distributions \( u \) in the scaling expansion

In the following proposition, we list some general properties which hold for any almost homogeneous distribution on \( \mathbb{R}^m \). The distributions \( u \) in our scaling expansion are particular examples of such distributions, and therefore the proposition applies to them. In particular, combining the upper bound \( (109) \) on the analytic wave front set with item (iii) in the proposition below, one can obtain detailed information about the analytic wave front set of the Fourier transforms \( \hat{u} \) of the distributions \( u \) in the scaling expansion. This information suffices to establish that \( \hat{u} \) is in fact an analytic function in a large portion of momentum-space, and that it is given by the boundary value of an analytic function for almost all momentum configurations (see [14, Thm. 8.4.15] for the appropriate criteria when a distribution can be written as the boundary value of an analytic function).

**Proposition B.1.** Let \( u \in \mathcal{D}'(\mathbb{R}^m) \) be an almost homogeneous distribution of degree \( \rho \), i.e., \( S^N_\rho u = 0 \) for some \( N \in \mathbb{N} \), where \( S^N_\rho = (\sum y^i \partial/\partial y^i + \rho)^N \). Then

\[ \text{WF}_A(u) \subset \{(y, k) \in T^*\mathbb{R}^m \setminus \{0\} \mid \sum y^ik_i = 0\}. \]
(ii) $u$ can be extended to test functions in Schwartz space and thereby defines a tempered distribution.

(iii) $\hat{u}$ is again an almost homogeneous distribution, with degree $m - \rho$. Furthermore, we have

\begin{align}
(x, k) \in \text{WF}_A(u) & \iff (k, -x) \in \text{WF}_A(\hat{u}) \quad \text{if } x \neq 0, k \neq 0,

x \in \text{supp}(u) & \iff (0, -x) \in \text{WF}_A(\hat{u}) \quad \text{if } x \neq 0,

k \in \text{supp}(\hat{u}) & \iff (0, k) \in \text{WF}_A(u) \quad \text{if } k \neq 0.
\end{align}

\[ (111) \]

**Proof.** Since $S^N u = 0$, and since $S^N$ has analytic coefficients, we have by [14, Thm. 8.6.1] that

\[ \text{WF}_A(u) \subset \text{Char}(S^N) = \{(y, k) \in T^* \mathbb{R}^m \setminus \{0\} \mid \sum y^i k_i = 0\}, \quad (112) \]

where $\text{Char}(P)$ is the characteristic set of a differential operator $P$, defined as the set of all $(y, k) \in T^* \mathbb{R}^m \setminus \{0\}$ such that $p(y, k) = 0$, where $p$ is the principal symbol of $P$. This proves (i).

Let $\chi_+$ and $\chi_-$ be smooth functions on $\mathbb{R}$ with the property that $\chi_+ + \chi_- = 1$, and such that $\chi_-(r) = 0$ for $r \leq r_0$ and $\chi_-(r) = 1$ for $r \geq 2r_0$ for some $r_0 > 0$. We can therefore write

\[ u(y) = \chi_+(|y|)u(y) + \chi_-(|y|)u(y). \quad (113) \]

The first distribution on the right side of this equation is by definition of compact support, and therefore trivially a tempered distribution. Thus (ii) will follow if we can show that also the second distribution on the right side is tempered. In order to prove this, we first show that it is possible to write $\chi_- u$ in “polar coordinates”. For this, we note that $\text{WF}(u) \cap N^*(\mathbb{S}^{m-1}) = \emptyset$ by (i), which implies by [14, Thm. 8.2.4] that $u$ has a well defined pull-back, $v$, to the unit sphere, $\mathbb{S}^{m-1}$, in $\mathbb{R}^m$. It follows from this, and the almost homogeneous scaling of $u$ that it is possible to write

\[ u(\chi_- f) = \sum_{j=0}^{N-1} \int_0^\infty \int_{\mathbb{S}^{m-1}} c_j \chi_-(r) r^{-\rho + m - 1} \ln^j r \cdot v(\hat{y}) f(r\hat{y}) dr d\mu(\hat{y}), \quad (114) \]

where $(r, \hat{y})$ denote polar coordinates in $\mathbb{R}^m$, $d\mu$ is the standard measure on $\mathbb{S}^{m-1}$ and the $c_j$ are unspecified complex constants. Since $v$ is a distribution on $\mathbb{S}^{m-1}$, there must exist differential operators $P_1, \ldots, P_k$ on $\mathbb{S}^{m-1}$ such that

\[ |v(h)| \leq C \sum_{l \leq k} \sup_{\hat{y} \in \mathbb{S}^{m-1}} |P_l h(\hat{y})| \quad (115) \]

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for all test functions $h \in \mathcal{D}(\mathbb{S}^{m-1})$. Moreover $\chi_-(r)r^{-\rho+m-1}\ln^j r$ is a smooth function on $\mathbb{R}$ which grows polynomially together with all its derivatives at infinity, and therefore is a tempered distribution. Combining these facts with eq. (114), we easily get the estimate

$$|u(\chi_\cdot f)| \leq C \sum_{|\alpha| \leq a,|\beta| \leq b} \sup_{y \in \mathbb{R}^m} |y^\alpha \partial^\beta f(y)|$$

(116)

for some $a, b \in \mathbb{N}$ and all $f$ in Schwartz space, thus showing that $\chi_- u$ is tempered.

We come to the proof of (iii). That the Fourier transform of $u$ scales almost homogeneously with degree $m - \rho$ follows directly from our definition. For the case of a distribution, $u$ that scales exactly homogeneously with degree $\rho$, the remaining three relations in eq. (111) correspond precisely to [14, Thm. 8.4.18]. The proof given in [14, Thm. 8.4.18] of the second and third relations in (111) can be applied without modification to distributions with almost homogeneous scaling. We therefore only have to prove the first relation in (111).

Since the Fourier transform $\hat{u}$ is again a tempered distribution which scales homogeneously up to logarithmic terms, the problem is symmetric and it therefore suffices to show that

$$(y_0, k_0) \notin \text{WF}_A(u) \Rightarrow (k_0, -y_0) \notin \text{WF}_A(\hat{u})$$

(117)

if $y_0 \neq 0, k_0 \neq 0$. Choose compact neighborhoods $K$ and $\hat{K}$ in $\mathbb{R}^m \setminus 0$ of $y_0$ and $k_0$ such that

$$\text{WF}_A(u) \cap (K \times \hat{K}) = \emptyset,$$

(118)

and a sequence of cutoff functions $\chi_N \in C_0^\infty(\hat{K})$ such that (110) is valid for every $\alpha$. We now estimate the Fourier transform of $v_N = \chi_N \hat{u}$ in a conic neighborhood of $y_0$. By Fourier’s inversion formula and the convolution theorem, we have

$$\hat{v}_N(-\lambda y) = \int_{\mathbb{R}^m} u(x)\hat{\chi}_N(x - \lambda y) d^m x.$$  

(119)

We now estimate expression (119) for $|y - y_0| < r$ and arbitrary $\lambda$. For this, we consider two cases, first $0 < \lambda \leq 1$, and second $\lambda > 1$. We begin with the first case. Since $u$ is a tempered distribution, we can estimate

$$|\hat{v}_N(-\lambda y)| \leq C \sum_{|\alpha| \leq a,|\beta| \leq b} \sup_{x \in \mathbb{R}^m} |x^\alpha \partial^\beta \hat{\chi}_N(x - \lambda y)|.$$  

(120)

Using (110), it is not difficult to estimate

$$|y^\alpha \partial^\beta \hat{\chi}_N(y)| \leq C(N + 1), \quad \forall N, |\alpha| \leq a, |\beta| \leq b$$

(121)

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From this one obtains the estimate

$$|\hat{v}_N(-\lambda y)| \leq C(N + 1) \leq C_M^{N-M+1}((N - M + 1)/\lambda)^{N-M}$$

(122)

for all $M < N$ and all $0 < \lambda \leq 1$.

In order to estimate (119) also for $\lambda > 1$, we use that $u$ scales almost homogeneous up to logarithms. This enables us to write

$$\hat{v}_N(-\lambda y) = \lambda^{p+m} \sum_{j \geq 0} \frac{\ln j \lambda}{j!} \int_{\mathbb{R}^m} u_j(x) \hat{\chi}_N(\lambda(x - y)) \, d^m x,$$

(123)

where the sum is finite and where $u_j = S^j_\rho u$, with $S_\rho = \sum y^i \partial \partial y^i + \rho$. Since $S_\rho$ is a partial differential operator with analytic coefficients, we conclude that $WF_A(u_j)$ is not bigger than $WF_A(u)$ and hence also has no intersection with $K \times \hat{K}$. We are now in a position to use exactly the same arguments as in the proof of [14, Thm. 8.4.18] (modulo a trivial additional estimate due to the logarithms), to show that there holds the estimate

$$|\hat{v}_N(-\lambda y)| \leq C^{N-M+1}((N - M + 1)/\lambda)^{N-M}, \quad \forall N, \lambda > 1, |y - y_0| < r,$$

(124)

for some natural number $M$. Together with (123) this shows that we have $\hat{v}_N(y) \leq C^{N-M+1}((N - M + 1)/|y|)^{N-M}$ for all $y$ in the conic neighborhood

$$\{-\lambda y \in \mathbb{R}^m \mid |y - y_0| \leq r, \lambda > 0\}$$

(125)

of $-y_0$ and for some fixed $M$. This proves the proposition.

References

[1] F. M. Boas: “Gauge theories in local causal perturbation theory,” [arXiv:hep-th/0001014]

[2] R. Brunetti, K. Fredenhagen and M. Köhler: “The microlocal spectrum condition and Wick polynomials on curved spacetimes,” Commun. Math. Phys. 180, 633-652 (1996)

[3] R. Brunetti, K. Fredenhagen: “Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on physical backgrounds,” Commun. Math. Phys. 208, 623-661 (2000)

[4] R. Brunetti, K. Fredenhagen and R. Verch: “The generally covariant locality principle: A new paradigm for local quantum physics,” [arXiv:math-ph/0112041].

[5] T. S. Bunch and L. Parker: “Feynman Propagator In Curved Space-Time: A Momentum Space Representation,” Phys. Rev. D20, 2499 (1979)
[6] S. M. Christensen: “Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method,” Phys. Rev. D14, 2490-2501 (1976)

[7] B. S. DeWitt and R. W. Brehme: “Radiation damping in a gravitational field,” Ann. Phys. 9, 220-259 (1960)

[8] M. Dütsch and K. Fredenhagen: “Algebraic quantum field theory, perturbation theory, and the loop expansion,” Commun. Math. Phys. 219 (2001) 5, [arXiv:hep-th/0001129]; “Perturbative algebraic field theory, and deformation quantization,” to appear in: Fields Inst. Commun., [arXiv:hep-th/0101079]

[9] M. Dütsch and K. Fredenhagen: “A local (perturbative) construction of observables in gauge theories: The example of QED,” Commun. Math. Phys. 203 (1999) 71, [arXiv:hep-th/9807078]

[10] H. Epstein and V. Glaser: “The rôle of locality in perturbation theory,” Ann. Inst. H. Poincaré Sec. A XIX, 211–295 (1973)

[11] K. Fredenhagen and R. Haag: “Generally covariant quantum field theory and scaling limits,” Commun. Math. Phys. 108, 91 (1987)

[12] S. Hollands and R. M. Wald: “Local Wick polynomials and time ordered products of quantum fields in curved spacetime,” Commun. Math. Phys. 223, 289 (2001) [arXiv:gr-qc/0103074].

[13] S. Hollands, W. Ruan: “The state space of perturbative interacting quantum field theories in curved spacetimes,” [arXiv:gr-qc/0108032]

[14] L. Hörmander: “The Analysis of Linear Partial Differential Operators I,” 2nd Edition, Springer-Verlag (1990)

[15] B.S. Kay: “Reducing the Renormalization Ambiguity in the Brunetti-Fredenhagen Approach to Interacting Quantum Field Theories on Curved Spacetime: The Example of $\lambda\varphi^4$ Theory,” in preparation.

[16] V. Moretti: “Comments on the stress-energy tensor operator in curved spacetime,” [arXiv:gr-qc/0109048]

[17] D. Prange: “Lorentz covariance in Epstein-Glaser renormalization,” [arXiv:hep-th/9904136]

[18] M. Radzikowski: “Microlocal approach to the Hadamard condition in quantum field theory on curved spacetime,” Commun. Math. Phys. 179, 529 (1996)
[19] M. Reed and B. Simon: “Methods of modern mathematical physics I,” Academic Press, New York (1973)