Convergence Analysis for Regular Wireless Consensus Networks

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Abstract—Average consensus algorithms can be implemented over wireless sensor networks (WSN), where global statistics can be computed using communications among sensor nodes locally. Simple execution, robustness to global topology changes due to frequent node failures and underlying distributed philosophy has made consensus algorithms more suitable to WSNs. Since these algorithms are iterative in nature, their performance is characterized by convergence speed. We study the convergence of the average consensus algorithms for WSNs using regular graphs. We obtained the analytical expressions for optimal consensus and convergence parameters which decides the convergence time for $r$-nearest neighbor cycle and torus networks. We have also derived the generalized expression for optimal consensus and convergence parameters for $m$-dimensional $r$-nearest neighbor torus networks. The obtained analytical results agree with the simulation results and shown the effect of network dimension, number of nodes and transmission radius on convergence time. This work provides the basic analytical tools for managing and controlling the performance of average consensus algorithm in the finite sized practical networks.

Index Terms—Consensus networks, Sensor networks, Average consensus algorithms, Regular graphs, Convergence time

I. INTRODUCTION

Consensus algorithms have received a lot of attention due to their ability to compute the desired global statistics by exchanging information only with direct neighbors. Average consensus algorithms have been extensively studied in distributed agreement and synchronization problems in the multi agent systems and load balancing in parallel computers ([1], [2], [3]). In contrast to the centralized algorithms, the underlying distributed and decentralized philosophy avoids the need of any central fusion for collecting the information. This approach is particularly suitable in the following situations: 1) global network topology information is not known; 2) dynamic topology changes because of frequent node failures; 3) the nodes are computationally constrained and incapable to support the sophisticated routing techniques.

Distributed average consensus can be applied to wireless sensor networks for data fusion [4], [5], [7]. As the consensus algorithms are iterative in nature, convergence rate of the algorithms which greatly influence the performance of the WSNs and it is lower bounded by the second smallest eigen value of the graph Laplacian [16]. To make this algorithm useful in a sensor network context, it is necessary to maximize the convergence rate to reach the consensus as soon as possible. Thus, extensive study to increase the convergence rate has been done in the literature. In [9], authors considered the distributed average consensus when the topology is random and the communication in the channels is corrupted by additive noise. They proved that running the consensus for long time reduces the bias of the final average estimate but increases its variance. A closed form expression for the mean square error of the state and the optimum choice of parameters has been derived in [6] to guarantee fastest convergence. Consensus on small world and ramanujan networks has been studied in the [14], [15] and it has been proved that convergence rate is maximized in these topologies. Optimal topologies from the all possible topologies which increase the convergence rate has been studied in [18]. In our work, we study the convergence of the consensus algorithm for distance-regular networks with varying number of nearest neighbors. These networks represents the notion of geographical proximity in the wireless sensor networks. The main motivation for using the regular graph model is most of the practical networks are finite sized which cannot be studied by asymptotic results existing in the literature. Random geometric graphs which exactly depict the behavior of WSNs behave asymptotically as a regular graph [13], which are also convenient to analyze the wireless networks [11], [12], [17]. Our results are more precise and can be applied to most of the practical wireless networks, which are of finite sized, whereas asymptotic analyses are only applicable to very large sized networks. In $r$-nearest neighbor cycle and torus, an edge will be existed between every pair of neighbors that are $r$ hops away. If a node’s transmission radius is increased, it will able to communicate with more number of nodes. Under this assumption, we have considered a variable $r$ as the transmission radius. Node’s transmission radius or nearest neighbors will greatly effect the convergence rate of consensus algorithms. Effect of communication parameters on consensus algorithm’s convergence rate has been studied analytically in [10]. But this work does not provide the exact formulation of the convergence time of the consensus algorithms.

Despite the fact that distributed average consensus is simple to implement, it is generally difficult to predict its convergence time. Although there have been several studies of wireless consensus networks, analytical tools to control the network performance are still inadequate. In this paper, we derive a general formula to efficiently and exactly compute the optimal convergence, consensus parameter and convergence time of the average consensus algorithm. The advantage of this kind of analysis is the reduction in the dependence on thousands of simulation trails to get the exact performance and to produce the precise results. We model the WSN as distance regular graphs and derived the analytical formulas for optimal consensus and convergence parameters for $r$-nearest neighbor cycle, $r$-nearest neighbor torus, $m$-dimensional $r$-nearest neighbor torus networks.
In summary, this paper is organized as follows.  
1) In Section II, we have given brief review about the consensus algorithms.  
2) In Section III, we have discussed the distance regular graphs and derived the weight matrix and eigen value expressions for the r-nearest neighbor networks.  
3) In Section IV, exact formulas for optimal consensus and optimal convergence parameters have been derived for the convergence analysis of distance regular graphs.  
4) In Section V, simulation results have been presented and compared with the obtained analytical results.

II. CONSENSUS ALGORITHMS

Let \( G = (V,E) \), be an undirected graph with node set \( V = \{1, 2, \ldots, n\} \) and an edge set \( E \subseteq V \times V \). Further, let \( A \) be the \( n \times n \) symmetric adjacency matrix of the graph \( G \), each entry of adjacency matrix is represented by \( a_{ij} \), which is 1 if node \( i \) is connected to node \( j \), else it is 0. The degree matrix \( D \) is defined as the diagonal matrix whose entry is \( d_{ii} = \text{deg}(v_i) \). The laplacian matrix of the graph \( G \) is the \( n \times n \) symmetric matrix \( L = D - A \), whose entries are

\[
l_{ij} = \begin{cases} 
\text{deg}(v_i) & \text{if } j = i \\
-a_{ij} & \text{if } j \neq i 
\end{cases}
\]

Let \( x_i(0) \) be the real scalar assigned to node \( i \) at \( t = 0 \). The average consensus algorithm will compute the average \( \frac{\sum_{i=1}^{n} x_i(t)}{n} \) at every node. At each step, node \( i \) carries out its update based on its local state and communication with its direct neighbors.

\[
x_i(t+1) = x_i(t) + h \sum_{j \in N_i} (x_j(t) - x_i(t)), \quad i = 1, \ldots, n 
\]

This iterative method can be expressed as the simple linear iteration

\[
x(t+1) = W x(t), \quad t = 0, 1, 2, \ldots
\]

where

\[
W = I - hL,
\]

is the weight matrix.

Proposition 1: Consensus parameter \((h)\) [20] of the average consensus algorithm can be expressed as

\[
h = \frac{2}{\lambda_1(L) + \lambda_{n-1}(L)}.
\]

Let \( \lambda_1(L) \) and \( \lambda_{n-1}(L) \) are the second smallest, second largest eigen values of the laplacian matrix.

Proposition 2: Convergence parameter \((\gamma)\) [20] of the average consensus algorithm can be expressed as

\[
\gamma = \frac{\lambda_{n-1}(L) - \lambda_1(L)}{\lambda_{n-1}(L) + \lambda_1(L)}.
\]

Proposition 3: Convergence time \((T)\) [21] of the average consensus algorithm can be expressed as

\[
T = (1/\log(1/\gamma)).
\]

III. REGULAR GRAPHS

A. r-nearest neighbor cycle

The r-nearest cycle can be represented by a circulant matrix [8]. A circulant matrix is defined as

\[
\begin{bmatrix}
a_1 & a_2 & \ldots & a_{n-1} & a_n \\
a_n & a_1 & \ldots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_3 & a_4 & \ldots & a_1 & a_2 \\
a_2 & a_3 & \ldots & a_{n-1} & a_1
\end{bmatrix}
\]

and \(k\)-th eigen value of a circulant matrix can be expressed as

\[
\lambda_k = a_1 + a_2 \omega^k + \ldots + a_n \omega^{(n-1)k}
\]

where \(\omega\) be the \(n\)-th root of 1. Then \(\omega\) is the complex number:

\[
\omega = \cos \left( \frac{2\pi}{n} \right) + i \sin \left( \frac{2\pi}{n} \right) = e^{\frac{2\pi i}{n}}
\]

The 1-nearest cycle and 2-nearest cycle are shown in Fig.1 and Fig.2 respectively. Let the adjacency matrix \(A\) and the degree matrix \(D\) of 1-nearest cycle, then they can be written as

\[
A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 1 \\
1 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 1 & 0 \\
2 & 0 & 0 & \ldots & 0 & 0 \\
0 & 2 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 1 & \ldots & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 2 \\
0 & 0 & \ldots & \ldots & \ldots & 0
\end{bmatrix}
\]

Theorem 1: The eigenvalues \(\lambda_k\) of weight matrix \(W\) for 1-nearest neighbor cycle can be expressed as

\[
\lambda_k(W) = (1 - 2h) + 2h \cos \left( \frac{2\pi k}{n} \right)
\]

where \(k = 0, 1, \ldots, (n-1)\).

Proof: From (11) and (12), the laplacian matrix \(L\) for 1-nearest neighbor cycle can be written as

\[
L = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & -1 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -1 \\
-1 & 0 & 0 & \ldots & -1 & 2
\end{bmatrix}
\]

Similarly weight matrix \(W\) for 1-nearest neighbor cycle can be written as

\[
W = (I - Lh) = \begin{bmatrix}
(1 - 2h) & h & 0 & \ldots & 0 & h \\
h & (1 - 2h) & h & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (1 - 2h) & h \\
h & 0 & 0 & \ldots & h & (1 - 2h)
\end{bmatrix}
\]
Hence, by using (9) and (15), $k^{th}$ eigen value of weight matrix $W$, for 1-nearest neighbor cycle can be written as,

$$\lambda_k(W) = (1 - 2h) + 2h\cos\left(\frac{2\pi k}{n}\right)$$  (16)

**Theorem 2**: The eigenvalues $\lambda_i$ of weight matrix $W$ for $r$-nearest neighbor cycle can be expressed as,

$$\lambda_i(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi ij}{n}\right)$$  (17)

where $j = 0, 1, ... (n - 1)$.

**Proof**: From (9), we can observe that, the first row is enough to get the eigen values of any circulant matrix.

The first row of adjacency matrix (A), degree matrix (D) and laplacian matrix (L) can be written as,

$$A_{1n} = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & 1 \end{bmatrix}$$  (18)

$$D_{1n} = [2r \ 0 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 0]$$  (19)

$$L_{1n} = \begin{bmatrix} 2r & -1 & -1 & -1 & \cdots & 0 & \cdots & -1 & -1 & -1 \end{bmatrix}$$  (20)

From the (22), first row of a circulant matrix for $r$-nearest neighbor cycle can be written as

$$W_{1n} = \begin{bmatrix} (1 - 2rh) & h & h & h & \cdots & 0 & \cdots & h & h & h \end{bmatrix}$$  (21)

Hence, by using (9) and (21), $i^{th}$ eigen value of $W$ for $r$-nearest neighbor cycle can be written as

$$\lambda_i(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi ij}{n}\right)$$  (22)

### B. $r$-nearest neighbor torus

A torus can be represented by the $n \times n$ block circulant matrix $A$ as

$$A = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & \cdots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix}$$  (23)

Let the number of nodes $n = n_1^2$, then each block $A_i$, for $i = 0, 1, \ldots (n_1 - 1)$ represents $n_1 \times n_1$ circulant matrices.

Let $W_{k_1,k_2}$ is the weight matrix for $k_1 \times k_2$ torus (2), then it can be written as,

$$W_{k_1,k_2} = \begin{bmatrix} (W_{k_1} - 2rhI_{k_1}) & hI_{k_1} & \cdots & \cdots & hI_{k_1} \\ hI_{k_1} & (W_{k_1} - 2rhI_{k_1}) & hI_{k_1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (W_{k_1} - 2rhI_{k_1}) & \cdots & \cdots & \cdots & hI_{k_1} \end{bmatrix}_{k_1 \times k_1}$$  (24)

Where $W_{k_1}$ is the weight matrix of $r$-nearest neighbor cycle consists of $k_1$ number of nodes and $I_{k_1}$ is the identity matrix of order $k_1 \times k_1$.

**Theorem 3**: The eigen values $\lambda_{j_1,j_2}$ of weight matrix $W_{k_1,k_2}$ for 1-nearest neighbor torus can be expressed as

$$1 - 4h + 2h \cos\left(\frac{2\pi j_1}{k_1}\right) + 2h \cos\left(\frac{2\pi j_2}{k_2}\right)$$  (25)

where $j_1 = 0, 1, 2, \ldots (k_1 - 1)$, $j_2 = 0, 1, 2, \ldots (k_2 - 1)$.

**Proof**: From (9) and (24), we can write the eigen value expression for $W_{k_1,k_2}$ as,

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = \lambda(W_{k_1}) - 2hI_{k_1} + hI_{k_1} e^{i\frac{2\pi j_2}{k_2}} + hI_{k_1} e^{i\frac{2\pi j_1(k_1-1)}{k_2}}$$  (26)

Hence, from (16) we can write the

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = 1 - 4h + 2h \cos\left(\frac{2\pi j_1}{k_1}\right) + 2h \cos\left(\frac{2\pi j_2}{k_2}\right)$$  (27)
Theorem 4: The eigen values $\lambda_{j_1,j_2}$ of weight matrix $W_{k_1, k_2}$ for $r$-nearest neighbor torus can be expressed as

$$\lambda_{j_1,j_2}(W_{k_1, k_2}) = (1-4rh) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_2}{k_2} \right) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_1}{k_1} \right)$$

where $j_1 = 0, 1, 2, ...(k_1-1)$, $j_2 = 0, 1, 2, ...(k_2-1)$.

Proof: From (24), the first row of weight matrix for $r$-nearest neighbor torus can be written as,

$$W_{1n} = \left[ W_{k_1} - 2rhI_{k_1} hI_{k_1}, ......... hI_{k_1} hI_{k_1} \right]$$

(30)

From (9) and (30), we can write

$$\lambda(W_{k_1}) - 2rh + 2h \sum_{i=1}^{m} \cos \left( \frac{2\pi ij_2}{k_2} \right)$$

Therefore, by substituting (22) in (31), we get

$$\lambda_{j_1,j_2}(W_{k_1, k_2}) = (1-4rh) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_2}{k_2} \right) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_1}{k_1} \right)$$

(32)

Theorem 5: The eigen values $\lambda_{j_1,j_2,...,j_m}$ of weight matrix $W$ for $m$-dimensional $r$-nearest neighbor torus can be expressed as

$$\lambda_{j_1,j_2,...,j_m}(W) = (1-2mrh) + 2h \sum_{j=1}^{r} \sum_{i=1}^{m} \cos \left( \frac{2\pi ij_i}{k_i} \right)$$

(33)

where $j_i = 0, 1, 2, ...(k_i-1)$.

Proof 5: From theorem 4, for two dimensional $r$- nearest neighbor torus, the eigen value expression will be

$$\lambda_{j_1,j_2}(W_{k_1, k_2}) = (1-4rh) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_2}{k_2} \right) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_1}{k_1} \right)$$

(34)

By using the above results, we can write the eigen value expression for three dimensional $r$- nearest neighbor torus as,

$$\lambda_{j_1,j_2,j_3}(W_{k_1, k_2, k_3}) = (1-6rh) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_3}{k_3} \right) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_2}{k_2} \right) + 2h \sum_{i=1}^{r} \cos \left( \frac{2\pi ij_1}{k_1} \right)$$

(35)

By observing (34) and (35), the eigen value expression for $m$-dimensional $r$-nearest neighbor torus can be written as,

$$\lambda_{j_1,j_2,...,j_m}(W) = (1-2mrh) + 2h \sum_{j=1}^{r} \sum_{i=1}^{m} \cos \left( \frac{2\pi ij_i}{k_i} \right)$$

(36)

where $j_i = 0, 1, 2, ...(k_i-1)$.

A two dimensional nearest neighbor torus can be defined in various ways as shown in the Fig. 3 and Fig. 4. To know the nearest neighbors in two dimensions, $L^1$ norm, $L^2$ norm, $L^\infty$ norm can be used. For simple and convenient analysis [11], $L^1$ norm and $L^\infty$ norm are preferable, where $L^1$ norm can be used when there is an connection between nodes whose shortest path is within $r$ hops on the torus and $L^\infty$ norm can be used when vertical and horizontal distance are both within $r$ hops on the torus.

IV. CONVERGENCE ANALYSIS FOR REGULAR GRAPHS

After deployment of sensor nodes for sensing, nodes start exchanging the information with their direct neighbors. This process will continue until every node gets the average of all measurements at every time instant. This measurement value which they exchange is called consensus parameter and we are denoting as $h$, optimal value of $h$ decides the optimal convergence parameter $\gamma$, which influences the convergence time $T$ of the average consensus algorithm. The below procedure has been followed to calculate the convergence time $T$.

1: Determine $\gamma(W) = \max \{|1-h\lambda_2(L)|, |1-h\lambda_n(L)|\}$.
2: Calculate $h$ from $1-h\lambda_2(L) = -1+ h\lambda_n(L)$.
3: Substitute the obtained $h$ value in modulus of second largest eigen value $1-h\lambda_2(L)$ to determine the optimal $\gamma$.
4: Calculate the convergence time $T = 1/(\log(1/\gamma))$.

A. $r$-nearest neighbor cycle

Theorem 6: Given $r$-nearest neighbor cycle $C^r_n$ and $n$ is even integer, optimal consensus parameter is

$$h = \frac{1}{2r + 1 - \frac{1}{2} \left( \frac{\sin \left( \frac{2\pi(r+0.5)}{n} \right)}{\sin \left( \frac{\pi}{n} \right)} + \cos \left( \pi r \right) \right)}$$

(37)

Proof: The proof is technical and deferred to the Appendix A.
Theorem 7: Given $r$-nearest neighbor cycle $C_n^r$ and $n$ is odd integer, optimal consensus parameter is

$$ h = \frac{1}{2r+1 - \frac{1}{2} \left( \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{n}} - \cos \left( \frac{\pi (2r+1)}{2n} \right) \right)} \quad (38) $$

Proof: The proof is technical and deferred to the Appendix A.

Theorem 8: Given $r$-nearest neighbor cycle $C_n^r$ and $n$ is odd integer, optimal convergence parameter is

$$ \gamma = \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{4r+2 - \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{n}} + \cos \left( \frac{\pi (2r+1)}{2n} \right)} \quad (39) $$

Proof: The proof is technical and deferred to the Appendix A.

Theorem 9: Given $r$-nearest neighbor cycle $C_n^r$ and $n$ is odd integer, optimal convergence parameter is

$$ \gamma = \frac{\sin \left( \frac{(2r+1)\pi}{n} \right) + \cos \left( \frac{\pi (2r+1)}{2n} \right)}{4r+2 - \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{n}} + \cos \left( \frac{\pi (2r+1)}{2n} \right)} \quad (40) $$

Proof: The proof is technical and deferred to the Appendix A.

B. $r$-nearest neighbor torus

Theorem 10: Given a $r$-nearest neighbor torus for $k_1, k_2$ are even integers, optimal consensus parameter is

$$ h = \frac{1}{1.5 + 3r - \frac{1}{2} \left( \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} + 2 \cos (\pi r) \right)} \quad (41) $$

Proof: The proof is technical and deferred to the Appendix B.

Theorem 11: Given a $r$-nearest neighbor torus for $k_1, k_2$ are odd integers, optimal consensus parameter is

$$ h = \frac{1}{1.5 + 3r - 0.5 \left( \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} + \sum_{i=1}^{m} \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} \right)} \quad (42) $$

Proof: The proof is technical and deferred to the Appendix B.

Theorem 12: Given a $r$-nearest neighbor torus for $k_1, k_2$ are even integers, optimal convergence parameter is

$$ \gamma = \frac{1}{\sin \left( \frac{(2r+1)\pi}{n} \right) - \sin \left( \frac{(2r+1)\pi}{n} \right) + \sin \left( \frac{(2r+1)\pi}{n} \right) + \sin \left( \frac{(2r+1)\pi}{n} \right)} \quad (43) $$

Proof: The proof is technical and deferred to the Appendix B.

Theorem 13: Given a $r$-nearest neighbor torus for $k_1, k_2$ are odd integers, optimal convergence parameter is

$$ \gamma = \frac{r+0.5+0.5 \left( \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} + \sum_{i=1}^{m} \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} \right)}{1.5 \left( \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} + \sum_{i=1}^{m} \frac{\sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} \right)} - \frac{r+0.5}{2} \cos (\pi r) \quad (44) $$

Proof: The proof is technical and deferred to the Appendix B.

C. $m$-dimensional torus

Theorem 14: Given a $m$-dimensional $r$-nearest neighbor torus for $k_1, k_2, k_3...k_m$ are even integers, optimal consensus parameter is

$$ h = \frac{1}{(m+1)(r+0.5) - \frac{0.5 \sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} - \frac{m \cos \pi r}{2}} \quad (45) $$

Proof: The proof is technical and deferred to the Appendix C.

Theorem 15: Given a $m$-dimensional $r$-nearest neighbor torus and for $k_1, k_2, k_3...k_m$ are odd integers, optimal consensus parameter is

$$ h = \frac{1}{(m+1)(r+0.5) - \frac{0.5 \sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} - \frac{m \cos \pi r}{2}} \quad (46) $$

Proof: The proof is technical and deferred to the Appendix C.

Theorem 16: Given a $m$-dimensional $r$-nearest neighbor torus and for $k_1, k_2, k_3...k_m$ are even integers, optimal convergence parameter is

$$ \gamma = \frac{1}{(m+1)(r+0.5) - \frac{0.5 \sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} - \frac{m \cos \pi r}{2}} \quad (47) $$

Proof: The proof is technical and deferred to the Appendix C.

Theorem 17: Given a $m$-dimensional $r$-nearest neighbor torus for $k_1, k_2, k_3...k_m$ are odd integers, optimal convergence parameter is

$$ \gamma = \frac{1}{(m+1)(r+0.5) - \frac{0.5 \sin \left( \frac{(2r+1)\pi}{n} \right)}{\sin \frac{\pi}{2}} - \frac{m \cos \pi r}{2}} \quad (48) $$

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T values are increasing with the number of nodes expressions with simulation results. The values of $h$ becomes almost constant after $T$ parameter pact of transmission radius, number of nodes on convergence been agreed with simulation results. The results shown the im-
convergence time $T$ to $0$ approaching to the constant values
ingly, As number of nodes increasing, the values of $h$ is
h is
mentioned in [2], for large sized networks consensus parameter $h$ is $0.5$.

Fig. 7, Fig. 8 and Fig. 9 show the $h$, $\gamma$ and $T$ versus transmission radius ($r$) for $r$- nearest neighbor cycle. We can see that, $h$, $\gamma$ and $T$ values has been decreased with $r$. It means, as each nodes’ transmission radius or nearest neighbors increases, the nodes can easily exchange with more number of nodes, resulting in reduction of convergence time.

We can see the impact of dimensions $k_1$ and $k_2$ on $h$, $\gamma$ and $T$ in Fig. 10, Fig. 11 and Fig. 12. We have also compared the obtained theoretical values of $h$, $\gamma$ and $T$ with simulation results for two dimensional $r$- nearest neighbor torus network as shown in Fig. 13, Fig. 14, Fig. 15. Finally, Fig.16, Fig. 17 and Fig. 18 show the $h$, $\gamma$ and $T$ versus network dimension $m$, varying $r$ values from 1 to 6. From these figures, we can say, for higher dimensional networks with more transmission radius, convergence time will be decreased. When $r$ is changing from 1 to 2, convergence time is decreasing drastically. Similarly, when dimension $m$ is changing from 1 to 2 and 2 to 3 the convergence time $T$ is clearly decreasing, whereas $T$ becomes almost constant after $m=4$. Interestingly, at $r=6$, convergence time $T$ becomes almost constant, irrespective of increase in dimension values.

VI. CONCLUSIONS

Analytical formulas for consensus parameter, convergence parameter and convergence time are derived for $r$-nearest neighbor cycle, $r$-nearest neighbor torus and $m$-dimensional $r$-nearest neighbor torus. The obtained analytical results have been agreed with simulation results. The results shown the impact of transmission radius, number of nodes on convergence time of the average consensus algorithm. Convergence time is decreasing with the transmission radius and increasing with the number of nodes for all the networks discussed. We also shown that convergence time is increasing with the increase in network dimension. With the increase in nearest neighbors or node’s transmission radius for higher dimension networks, convergence time is not changing significantly.

REFERENCES

[1] N.A. Lynch, Distributed Algorithms, Morgan Kaufmann Publishers, San Francisco, CA, 1996.
[2] C.-Z. Xu, F. Lau, Load Balancing in Parallel Computers: Theory and Practice, Kluwer Academic Publishers, Dordrecht, MA, 1997.
[3] R. Olfati-Saber, J. A. Fax and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proc. IEEE, vol. 95, no. 1, pp. 215-233, Jan. 2007.
[4] L. Xiao, S. Boyd, S. Lall, A scheme for robust distributed sensor fusion based on average consensus, in: Proceedings of the Fourth International Conference on Information Processing in Sensor Networks, Los Angeles, California, USA, 2005, pp. 63-70.
[5] D.P. Spanos, R. Olfati-Saber, R.M. Murray, “Distributed sensor fusion using dynamic consensus,” in: Proceedings of the 16th IFAC World Congress, Prague, Czech, 2005.
[6] S.S. Pereira and A. Pages-Zamora, “Fast mean-square convergence of consensus algorithms in WSNs with random topologies”, Proc. IEEE ICASSP, 2213-2216, April 2009.
[7] S. Kar and J. Moura, “Sensor networks with random links: topology design for distributed consensus”, IEEE Trans. Sig. Proc., 56(7), 3315-3327, July 2008.
[8] D. Geller, A. Kra, A. Popescu, and S. Simanca, “On circulant matrices,” [http://www.math.sunysb.edu/sorin] lecture notes.
[9] S. Kar and J. Moura, “Distributed consensus algorithms in sensor networks with imperfect communication: link failures and channel noise,” IEEE Trans. Sig. Proc., 57(1), 355-369, Jan 2009.
[10] S. Vanka, V. Gupta, and M. Haenggi, “Power-delay analysis of consensus
\[ \lambda_1(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{n} \right) \] (50)

\[ \lambda_2(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos (\pi j) \] (51)

\[ \lambda_{(n-1)}(W) = (1 - 2hr) + 2h \sum_{i=1}^{r} \cos \left( \frac{\pi i(n-1)}{n} \right) \] (52)

\[ |\lambda_1(W)| = |\lambda_2(W)| \] (53)

After substituting the (50) and (51) in (53), we get the optimal \( h \) as,

\[ h = \frac{1}{2r + 0.5 - \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{n} \right) - \sum_{j=1}^{r} \cos (\pi j)} \] (54)

Note 1: We used the below trigonometric identity [19] to simplify the \( h \) values.

\[ 1 + 2 \sum_{j=1}^{r} \cos (jx) = \frac{\sin \left( r + \frac{1}{2} \right) x}{\sin \frac{x}{2}} \] (55)

By using (55), \( h \) can be rewritten as,

\[ h = \frac{1}{2r + 1 - \frac{1}{2} \left( \frac{\sin \left( \frac{2\pi (r+0.5)}{n} \right)}{\sin \frac{\pi}{n}} + \cos(\pi r) \right)} \] (56)

By substituting the above \( h \) value in (50), we get \( \gamma \) for \( n \) is even integer

\[ \gamma = \frac{\left( \frac{\sin \left( \frac{2\pi (r+1)}{n} \right)}{\sin \frac{\pi}{n}} - \cos(\pi r) \right)}{4r + 2 - \frac{\sin \left( \frac{2\pi (r+2)}{n} \right)}{\sin \frac{\pi}{n}} + \cos(\pi r)} \] (57)

Since \( \lambda_{n-1}(W) \) is the smallest eigen value, for \( n \) is odd integer, \( \gamma \) will be minimized when

\[ |\lambda_1(W)| = \left| \frac{\lambda_{(n-1)}(W)}{2} \right| \] (58)

After substituting the (50) and (52) in (58), we get the optimal \( h \) as,

\[ h = \frac{1}{2r - \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{n} \right) - \sum_{j=1}^{r} \cos \left( \frac{\pi j(n-2)}{n} \right)} \] (59)

By using (55), \( h \) can be rewritten as,

\[ h = \frac{1}{2r + 1 - \frac{1}{2} \left( \frac{\sin \left( \frac{2\pi (r+1)}{n} \right)}{\sin \frac{\pi}{n}} - \cos \left( \frac{\pi (2r+1)}{2n} \right) \right)} \] (60)

**APPENDIX A**

**\( r \)-NEAREST NEIGHBOR CYCLE**

Proofs of Theorem 6, Theorem 7, Theorem 8 and Theorem 9 are given below.

From Theorem 2, we can write the following eigen value expressions for weight matrix \( W \) as,

\[ \lambda_0(W) = (1 - 2rh) + 2h \] (49)
By substituting the above optimal $h$ value in (59), we get $\gamma$ for $n$ is odd integer

$$\gamma = \frac{\sum_{j=1}^{r} \cos \left( \frac{2\pi j}{n} \right) - \sum_{j=1}^{r} \cos \left( \frac{\pi j(n-2)}{n} \right)}{2r - \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{n} \right) - \sum_{j=1}^{r} \cos \left( \frac{\pi j(n-2)}{n} \right)}$$  \hspace{1cm} (61)

**APPENDIX B**

**r-NEAREST NEIGHBOR TWO DIMENSIONAL TORUS**

Proofs of Theorem 10, Theorem 11, Theorem 12 and Theorem 13 are given below.

From Theorem 4, we can write the following eigen value expressions of $W$ as,

$$\lambda_{0,1}(W) = (1 - 2hr) + 2h \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{k_2} \right)$$  \hspace{1cm} (62)

$$\lambda_{k_1,k_2}(W) = (1 - 4hr) + 4h \sum_{j=1}^{r} \cos \left( \pi j \right)$$  \hspace{1cm} (63)

$$\lambda_{(k_1-1), (k_2-1)}(W) = (1 - 4hr) + 2h \sum_{j=1}^{r} \cos \left( \frac{2\pi (k_1-1)}{k_2} \right) + 2h \sum_{j=1}^{r} \cos \left( \frac{2\pi (k_2-1)}{k_2} \right)$$  \hspace{1cm} (64)

$\lambda_{0,1}(W)$ and $\lambda_{k_1,k_2}(W)$ are the second largest and the smallest eigen values of weight matrix $W$ for $k_1, k_2$ are even integers. $\gamma$ is minimum when,

$$|\lambda_{0,1}(W)| = \left| \lambda_{k_1,k_2}(W) \right|$$  \hspace{1cm} (65)

After substituting the (62) and (63) in (65), we get the optimal $h$ as,

$$h = \frac{1}{3r - \sum_{i=1}^{r} \cos \left( \frac{2\pi i}{k_2} \right) - 2 \sum_{i=1}^{r} \cos \left( \pi i \right)}$$  \hspace{1cm} (66)

By using (55), $h$ can be rewritten as,

$$h = \frac{1}{1.5 + 3r - \frac{1}{2} \left( \frac{\sin \left( \frac{2\pi (r+1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} + 2 \cos (\pi r) \right)}$$  \hspace{1cm} (67)

By substituting the above optimal $h$ value in (62), we get $\gamma$ for $k_1, k_2$ are even integers,

$$\gamma = \frac{r + 0.5 + 0.5 \left( \frac{\sin \left( \frac{2\pi (r+1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} - 2 \cos (\pi r) \right)}{1.5 + 3r - 0.5 \left( \frac{\sin \left( \frac{2\pi (r+1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} + 2 \cos (\pi r) \right)}$$  \hspace{1cm} (68)

$\lambda_{(k_1-1), (k_2-1)}(W)$ is the smallest eigen value for $k_1, k_2$ are odd integers. So $\gamma$ is minimized when,

$$|\lambda_{0,1}(W)| = \left| \lambda_{(k_1-1), (k_2-1)}(W) \right|$$  \hspace{1cm} (69)

After substituting the (62) and (64) in (69), we get the optimal $h$ as,

$$h = \frac{1}{3r - \sum_{i=1}^{r} \cos \left( \frac{2\pi i}{k_2} \right) - 2 \sum_{i=1}^{r} \cos \left( \frac{\pi (k_1-1)}{k_2} \right) + \sum_{i=1}^{r} \cos \left( \frac{\pi (k_2-1)}{k_2} \right)}$$  \hspace{1cm} (70)

By using (55), $h$ can be rewritten as,

$$h = \frac{1}{1.5 + 3r - 0.5 \left( \frac{\sin \left( \frac{2\pi (r+1)(k_1-1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} + \frac{1}{\sin \left( \frac{\pi}{k_2} \right)} + \frac{\sin \left( \frac{2\pi (r+1)(k_2-1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} \right)}$$  \hspace{1cm} (71)

By substituting the above optimal $h$ value in (62), we get $\gamma$ for $n = odd$ integer.

$$\gamma = \frac{r + 0.5 + 0.5 \left( \frac{\sin \left( \frac{2\pi (r+1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} - 2 \cos (\pi r) \right)}{1.5 + 3r - 0.5 \left( \frac{\sin \left( \frac{2\pi (r+1)(k_1-1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} + \frac{1}{\sin \left( \frac{\pi}{k_2} \right)} + \frac{\sin \left( \frac{2\pi (r+1)(k_2-1)}{k_2} \right)}{\sin \left( \frac{\pi}{k_2} \right)} \right)}$$  \hspace{1cm} (72)

**APPENDIX C**

**m-DIMENSIONAL r-NEAREST NEIGHBOR TORUS**

Proofs of Theorem 14, Theorem 15, Theorem 16 and Theorem 17 are given below.

From Theorem 5, we can write the following eigen value expressions for weight matrix $W$ as,

$$\lambda_{1,0,0, \ldots, 0}(W) = (1 - 2hr) + 2h \sum_{j=1}^{r} \cos \left( \frac{2\pi j}{k_1} \right)$$  \hspace{1cm} (73)

$$\lambda_{k_1,k_2, \ldots, k_m}(W) = (1 - 2mhr) + 2mh \sum_{j=1}^{r} \cos \left( \pi j \right)$$  \hspace{1cm} (74)

$$\lambda_{(k_1-1), (k_2-1), \ldots, (k_m-1)}(W) = (1 - 2mhr) + 2h \sum_{j=1}^{r} \sum_{i=1}^{m} \cos \left( \frac{\pi (k_i-1)}{k_i} \right)$$  \hspace{1cm} (75)

$\lambda_{1,0,0, \ldots, 0}(W)$ and $\lambda_{k_1,k_2, \ldots, k_m}(W)$ are the second largest and smallest eigen values of weight matrix $W$, for $k_1, k_2, k_3, \ldots, k_m$ are even integers. $\gamma$ is minimum when,

$$|\lambda_{1,0,0, \ldots, 0}(W)| = \left| \lambda_{k_1,k_2, \ldots, k_m}(W) \right|$$  \hspace{1cm} (76)

After Substituting the (73) and (74) in (76), we get the optimal $h$ as,

$$h = \frac{1}{r(m+1) - \sum_{i=1}^{m} \cos \left( \frac{2\pi i}{k_1} \right) + m \sum_{i=1}^{m} \cos \left( \pi i \right)}$$  \hspace{1cm} (77)

By using (55), $h$ can be rewritten as,

$$h = \frac{1}{(m+1)(r+0.5) - \frac{1}{2} \left( \frac{\sin \left( \frac{2\pi (r+0.5)k_1}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} - \frac{m}{2} \cos (\pi r) \right)}$$  \hspace{1cm} (78)

By substituting the above optimal $h$ value in (73), we get the $\gamma$ for $k_1, k_2, k_3, \ldots, k_m$ are even integers.
\[ \gamma = \frac{(m-1)(r+0.5) + \frac{0.5 \sin \left( \frac{(2r+1)\pi}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} - \frac{m}{2} \cos(\pi r)}{(m+1)(r+0.5) - \frac{0.5 \sin \left( \frac{(2r+1)\pi}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} - \frac{m}{2} \cos(\pi r)} \] (79)

\[ \lambda_{1,0,0}, \ldots, \lambda_{k_1-1, k_2-1}, \ldots, \lambda_{k_m-1} \text{ are the second largest and smallest eigen values of weight matrix } W, \text{ for } k_1, k_2, k_3 \ldots k_m \text{ are odd integers. } \gamma \text{ is minimum when,} \]

\[ |\lambda_{1,0,0}, \ldots, 0(W)| = \left| \lambda_{(k_1-1)/2, (k_2-1)/2, \ldots, (k_m-1)/2} \right| \] (80)

After substituting the (73) and (75) in (80), we get the optimal \( h \) as,

\[ h = \frac{1}{r(m+1) - \sum_{i=1}^{r} \cos \left( \frac{2\pi i}{k_1} \right) - \sum_{j=1}^{m} \sum_{i=1}^{r} \cos \left( \frac{\pi i(k_j-1)}{k_j} \right)} \] (81)

By using (55), \( h \) can be rewritten as,

\[ h = \frac{1}{(m+1)(r+0.5) - \frac{\sin \left( \frac{(r+0.5)\pi}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} + \sum_{l=1}^{m} \sin \left( \frac{(r+0.5)\pi(k_l-1)}{k_l} \right) - \frac{m}{2} \cos(\pi r)} \] (82)

By substituting the above optimal \( h \) value in (73), we get \( \gamma \) for \( k_1, k_2, k_3 \ldots k_m \) are odd integers.

\[ \gamma = \frac{(m-1)(r+0.5) + \frac{\sin \left( \frac{(r+0.5)\pi}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} + \sum_{l=1}^{m} \sin \left( \frac{(r+0.5)\pi(k_l-1)}{k_l} \right)}{(m+1)(r+0.5) - \frac{\sin \left( \frac{(r+0.5)\pi}{k_1} \right)}{\sin \left( \frac{\pi}{k_1} \right)} + \sum_{l=1}^{m} \sin \left( \frac{(r+0.5)\pi(k_l-1)}{k_l} \right)} \] (83)