SCOBO: Sparsity-Aware Comparison Oracle Based Optimization

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Abstract

We study derivative-free optimization for convex functions where we further assume that function evaluations are unavailable. Instead, one only has access to a \textit{comparison oracle}, which, given two points $x$ and $y$, and returns a single bit of information indicating which point has larger function value, $f(x)$ or $f(y)$, with some probability of being incorrect. This probability may be constant or it may depend on $|f(x) - f(y)|$. Previous algorithms for this problem have been hampered by a query complexity which is polynomially dependent on the problem dimension, $d$. We propose a novel algorithm that breaks this dependence: it has query complexity only logarithmically dependent on $d$ if the function in addition has low dimensional structure that can be exploited. Numerical experiments on synthetic data and the MuJoCo dataset show that our algorithm outperforms state-of-the-art methods for comparison based optimization, and is even competitive with other derivative-free algorithms that require explicit function evaluations.

1 Introduction

We consider the well-studied optimization problem:

$$\minimize_{x \in \mathbb{R}^d} f(x) \quad (1)$$

where $f(x)$ is a convex function under the extremely restrictive assumption that one only has access to $f(x)$ through a \textit{comparison oracle}:
Definition 1.1. We say \( C_f(\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \{-1, +1\} \) is a comparison oracle for \( f \) if:
\[
P[ C_f(x, y) = \text{sign} (f(y) - f(x)) ] = \frac{1}{2} + \min \{ \delta_0, \mu |f(y) - f(x)|^\kappa - 1 \}
\]
where \( 0 < \delta_0 \leq 1/2, \mu > 0 \) and \( \kappa \geq 1 \) are the oracle noise parameters.

This definition of comparison oracle is as in [1]. It is frequently used as a model for comparisons made by humans [2]. Informally, the oracle \( C_f(x, y) \) tells you which point has larger function value, \( f(x) \) or \( f(y) \), with some probability of being incorrect. Here, the value of \( \kappa \) is critical to the behavior of the oracle. If \( \kappa > 1 \) then this probability is less than 1/2 and raises to 1/2 as \( f(x) \rightarrow f(y) \). On the other hand, \( \kappa = 1 \) implies that the comparison oracle is incorrect with some constant probability, independent of \( |f(y) - f(x)| \). Because comparison oracle queries are typically costly, we shall evaluate the performance of our method in terms of the number of required comparison oracle queries.

The need for robust, fast and reliable comparison-based optimization algorithms has recently exploded. For example, in reinforcement learning, to find a good policy for a given task one typically searches for a policy that maximizes a hand-engineered reward function. However for complex, real-world tasks (such as “scramble an egg” as suggested in [3]) it seems futile to attempt to define a precise, real-valued reward. On the other hand, comparison oracle feedback for such tasks can easily be obtained from humans (“This attempt to scramble an egg was better than that attempt”). Thus, multiple groups of researchers have sought to incorporate comparison-based algorithms into reinforcement learning, in order to utilize this qualitative, comparison-based feedback [3, 4, 5, 6].

In an entirely different direction, it has recently been observed that the problem of generating adversarial attacks on image classifiers from hard-label feedback can be recast as a comparison-based optimization problem [7]. By hard-label feedback, we mean that the attacker only has access to the final output of the model (“this image is a cat”), but not the probability output. Given that this is the most applicable form of attack, understanding the feasibility and limitations of this approach is of pressing concern.

Finally, we mention a parallel line of work in the multi-armed bandit community on what is known as the duelling bandit problem [8]. Like our work, this problem also only considers comparison oracle feedback. Unlike us, work in this area typically assumes a finite search space and focuses on minimizing the regret: \( \sum_{k=1}^{K} f(x_k) - f^* \) instead of the final optimization error: \( f(x_K) - f^* \). This problem has interesting applications to tailoring the search output to user preferences [9], news recommendation [10] and optimizing wearable exoskeletons for user comfort [11].

1.1 Prior work

The first work to consider (1) with comparison oracle feedback was [1]. There, a coordinate descent style algorithm that satisfies
\[
\mathbb{E}[ f(x_K) ] - f^* \leq \varepsilon
\]
in \( \tilde{O}(d^{2\kappa - 1} \varepsilon^{2 - 2\kappa}) \) queries, when \( f(x) \) is smooth and strongly convex, is provided (Throughout this paper, \( f^* := \min_{x \in \mathbb{R}^d} f(x) \) and \( \tilde{O}(\cdot) \) is used to suppress logarithmic factors). This approach was later extended by [12] who provided an empirically faster algorithm albeit with the same order of convergence. [7] provides an algorithm, SignOPT, which uses comparison oracle feedback under an easier noise model, essentially equivalent to \( \kappa = 1 \) in Definition 1.1, to form a proxy for the gradient. They prove that SignOPT finds \( x_K \) satisfying (2) in \( O(d^3 \varepsilon^{-2}) \) queries. We highlight the following drawbacks of existing algorithms:
1. The polynomial dependence of the number of queries on \( d \) is prohibitive.

2. Existing algorithms are not monotone. In fact, the sequence of functions values \( f(x_1), f(x_2), \ldots \) can increase substantially before decreasing again (see Section 7). This makes it impossible to determine, using only comparison oracle feedback, whether one should terminate the algorithm after \( k \) iterations or keep going in the hope that the sequence starts descending again.

3. Existing results hold only in expectation.

1.2 Our contributions

In this paper we provide an algorithm, which we dub SCOBO\(^1\) for comparison oracle optimization which overcomes the three shortcomings mentioned above.

**Theorem** (Main results, informally stated). Suppose that \( f(x) \) satisfies Assumptions 1–4. Then SCOBO (Algorithm 3) finds \( x_K \) satisfying:

\[
P[|f(x_K) - f^*| \leq \varepsilon] = 1 - o(1)
\]

in \( \tilde{O}(s^{2\kappa - 1} \varepsilon^{1/2 - 2\kappa}) \) queries where \( s \) is the compressibility of \( \nabla f \) (see Assumption 1). Moreover, for all \( k \leq K \):

\[
P[f(x_k) - f(x_{k-1}) \leq 0] = 1 - o(1).
\]

By assuming \( f(x) \) has some low dimensional structure, we are able to reduce the query complexity to only logarithmic dependence on \( d \). Our key theoretical innovations are:

1. A novel gradient estimator which uses tools from 1-bit compressed sensing.
2. A novel analysis of normalized gradient descent using inexact gradients.

In practice, this query complexity can be dramatically reduced by using an appropriate line search heuristic, which we introduce in Section 6. We also discuss how to set the intrinsic dimension, \( s \), in practice. In Section 7.1 we benchmark SCOBO against the state-of-the-art, and find that it offers a substantial speed-up. Finally, we end with some promising results of SCOBO applied to real-world problems from the MuJoCo suite [13].

1.3 Assumptions and notation

We make the following assumptions on \( f(x) \):

**Assumption 1** (Compressible Gradients). For all \( x \in \mathbb{R}^d \), and for some fixed \( s < d \), we have that:

\[
\|\nabla f(x)\|_1 \leq \sqrt{s}\|\nabla f(x)\|_2
\]

This generalizes the “sparse gradients” assumption: \( \|\nabla f(x)\|_0 := |\{i : \nabla_i f(x) \neq 0\}| \leq s \) studied in [14, 15]. In fact, the compressible gradients commonly exist in a wide range of applications, e.g. asset management and ImageNet adversarial attack [16, Figure 1].

\(^1\)SCOBO stands for Sparsity-aware Comparison-Based Optimization. Also this algorithm name is inspired by the Latin vocabulary *scobo*: to seek, search or probe.
Assumption 2 (Lipschitz Differentiability). There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$:

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|$$

Note that a simple consequence of Assumption 2 is that $\|\nabla^2 f(x)\|_2 \leq L$ for all $x$. If $f(x)$ has sparse gradients then it cannot be strongly convex. Instead, the more appropriate condition in this context is:

Assumption 3 (Restricted Strong Convexity). We assume that $f$ is convex and restricted $\nu$-strongly convex. That is, for all $x \in \mathbb{R}^d$, the following inequality holds:

$$f(x) - \min f \geq \nu \|x - P_\nu(x)\|^2,$$

where $P_\nu(\cdot)$ is the projection operator onto the solution set.

Assumption 4 (Low Rank Hessian). We assume that for all $x \in \mathbb{R}^d$, $\text{rank}(\nabla^2 f(x)) \leq s$.

1.4 Why assume low intrinsic dimension?

Derivative-free optimization in general is an unfortunate victim of the curse of dimensionality, but for the comparison-oracle paradigm the effect is particularly bad: in [1] it is shown that the worst-case complexity of any comparison-based optimization algorithm is $\Omega(d^{2s-2}\kappa^{-2})$ for strongly convex $f(x)$. In order to make progress, one needs to make additional assumptions on $f(x)$ and exploit them. Low intrinsic dimension has successfully been incorporated into other derivative-free contexts [17, 14, 15, 16]. Moreover, it is often observed in applications such as hyperparameter tuning for neural networks [18] and combinatorial optimization [19], as well as complex, simulation based optimization [20, 21]. Combining low intrinsic dimension with comparison-based optimization thus seems like a natural step forward.

2 One-bit compressed sensing

One-bit compressed sensing, first introduced in [22], is a framework for recovering an unknown signal from highly quantized linear measurements. Specifically, we assume that $x \in \mathbb{R}^d$ is unknown and that we only have access to measurements $y_1, \ldots, y_m \in \{-1, +1\}$ which are correlated with $\text{sign}(z_i^T x)$. In the noise-free setting we assume that $y_i = \text{sign}(z_i^T x)$. More generally, we assume that $y_i = \xi_i \text{sign}(z_i^T x)$ where $\xi_i \in \{-1, 1\}$ and $\mathbb{P}[\xi_i = 1] = p > 1/2$ allows for a random bit flip. Remarkably, even in the presence of corruptions, one can still recover $x$ from the measurement vector $y = [y_1, \ldots, y_m] \in \{0,1\}^m$, as the following theorem quantifies. For notational convenience, we set $\tilde{y}_i := \text{sign}(z_i^T x)$.

Theorem 2.1 ([23]). Let $z_1, \ldots, z_m$ be sampled uniformly and i.i.d from the unit sphere $S^{d-1}$. Suppose that $\|x\|_1 \leq \sqrt{s}$ and $\|x\|_2 = 1$. If $y_i = \xi_i \tilde{y}_i$ with $\xi_i \in \{-1, 1\}$ i.i.d. and $\mathbb{P}[\xi_i = 1] = p$, then

$$\hat{x} := \arg \max_{\|x'\|_1 \leq \sqrt{s} \text{ and } \|x'\|_2 \leq 1} \sum_{i=1}^m y_i z_i^T x'$$

satisfies $\|\hat{x} - x\| \leq \sqrt{s}$ with probability at least $1 - 8 \exp\left(-c\delta^2 m\right)$ as long as:

$$m \geq C\delta^{-2}(p - 1/2)^{-2}s \log(2d/s).$$

Remark 2.2. The theorem is presented in [23] for $z_i$ Gaussian random vectors. However, one can check that the result holds for any rotationally invariant distribution.
3 A one-bit gradient estimator

Recall that one of our main contributions is the construction of a gradient estimator, \( \hat{g} \approx g \), using only the output of the comparison oracle \( C_f(\cdot, \cdot) \). This construction was inspired by the observation that:

\[
C_f(x, x + rz_i) \approx \text{sign}(f(x + rz_i) - f(x)) \approx \text{sign}(z_i^\top g) \tag{5}
\]

where \( r > 0 \) and \( z_i \in \mathbb{R}^d \) is a random perturbation. Thus, one may think of the \( y_i = C_f(x, x + rz_i) \) as approximate one-bit measurements of \( g \). Hence, one may use one-bit compressed sensing, as outlined in Section 2, to recover \( g \) from \( y = [y_1, \ldots, y_m] \top \in \mathbb{R}^m \). We present the resulting gradient estimation algorithm as Algorithm 1.

**Algorithm 1** 1BitGradEst

1: Inputs: \( x \): Current point, \( s \): target sparsity, \( m \): number of queries, \( r \): sampling radius
2: Generate \( z_1, \ldots, z_m \sim U(S^{d-1}) \).
3: \( y_i \leftarrow C_f(x, x + rz_i) \) for \( i = 1, \ldots, m \)
4: Solve the quadratic program:
   \[
   \hat{g} \leftarrow \arg\max_{\|\hat{g}'\|_1 \leq \sqrt{s}} \sum_{i=1}^{m} y_i z_i^\top \hat{g}' \tag{6}
   \]
5: Output: \( \hat{g} \)

Analysing the accuracy of Algorithm 1 requires quantifying the approximations (a) and (b) in (5). Quantifying (a) is tricky, and requires one to estimate the magnitude of \( |f(x + rz_i) - f(x)| \) for \( z_i \sim U(S^{d-1}) \) (Recall that \( P[C_f(x, y) = \text{sign}(f(y) - f(x))] \propto |f(y) - f(x)| \)). Addressing (b) is also subtle. From Taylor’s theorem:

\[
\text{sign} (f(x + rz_i) - f(x)) = \text{sign} \left( rz_i^\top g + \frac{1}{2} r^2 z_i^\top \nabla^2 f(x + t_0 z_i) z_i \right) \tag{7}
\]

for some \( t_0 \in (0, 1) \). Thus, one needs to choose \( r \) such that the quadratic term in (7) does not affect the sign. This requires a delicate application of ideas from high-dimensional probability. We present the result of this analysis as Lemma 3.1. Proofs, as well as precise expressions for various parameters (\( \varepsilon_p, m, \ldots \)) are deferred to Appendix B. Note that \( \bar{y}_i := \text{sign}(z_i^\top g) \), and we think of the \( y_i \) as potentially corrupted versions of the \( \bar{y}_i \).

**Lemma 3.1.** 1. Suppose that \( f(x) \) satisfies Assumptions 1,2 and 4, and that \( \kappa > 1 \) and \( \delta_0 = 0.5 \).
   
   For any \( p \in (1/2, 1] \) define:
   \[
   \varepsilon_p = O\left( \sqrt{s} \left( \frac{p}{2} \right)^{1/(2\kappa-2)} \right). \tag{8}
   \]
   
   Then, if \( \|g\| \geq \varepsilon_p \nu \) and \( r = O(\sqrt{d}/s) \) we have that:
   \[
   P[\bar{y}_i = y_i] \geq p
   \]
2. Suppose that \( f(x) \) satisfies Assumption 1 and \( \kappa = 1 \) and \( \delta_0 < 0.5 \). Then, for any \( \varepsilon > 0 \) we have that:

\[
\mathbb{P}[\hat{y}_i = y_i] \geq \frac{1}{2} + \delta_0
\]

as long as \( \|\mathbf{g}\| \geq \varepsilon \nu \), and \( r = \varepsilon \nu / (L \sqrt{d}) \) for any \( \varepsilon > 0 \).

Lemma 3.1 bounds the probability that the measurement \( y_i \) has been flipped. Recall from Theorem 2.1 that the fidelity of the solution to (6) depends on this probability. With Lemma 3.1 in hand we may now quantify how close \( \hat{\mathbf{g}} \) is to the normalized true gradient:

**Theorem 3.2.** Fix any \( \eta \in (0, 1) \) and \( \varepsilon > 0 \). Then, under either of the following sets of conditions:

1. \( f(x) \) satisfies Assumptions 1, 2 and 4; \( \kappa > 1 \) and \( \delta_0 = 0.5 \); \( \|\mathbf{g}\| \geq \varepsilon \nu \) and

\[
m = O \left( \eta^{-2} \varepsilon^4 - 4 \kappa^2 s^4 - 3 \log(d) \right)
\]

\[
r = O(\sqrt{d}/s)
\]

2. \( f(x) \) satisfies Assumptions 1 and 2; \( \kappa = 1 \) and \( \delta_0 < 0.5 \) and

\[
m = O \left( \eta^{-2} \delta_0^{-2} s \log(d) \right)
\]

\[
r = O \left( 1/\sqrt{d} \right)
\]

then \( \hat{\mathbf{g}} \) satisfies \( \|\mathbf{g} - \hat{\mathbf{g}}\| \leq \eta \) with overwhelming probability.

### 4 Inexact normalized gradient descent

**Algorithm 2** INGD

1: **Inputs:** \( x_0 \) : Initial point, \( \alpha \) : step size

2: **for** \( k = 1, \ldots, K \) **do**

3: **Obtain** \( \hat{\mathbf{g}}_k \) with \( \|\mathbf{g}_k - \hat{\mathbf{g}}_k\|_2 \leq \eta \)

4: \( x_{k+1} = x_k - \alpha \hat{\mathbf{g}}_k \)

5: **end for**

6: **Output:** \( x_K \)

Normalized gradient descent (NGD), defined by the iteration \( x_{k+1} = x_k - \mathbf{g}_k / \|\mathbf{g}_k\| \), was first analyzed in [24], where it was suggested as an algorithm for quasi-convex minimization. Recently, there has been renewed interest in NGD from the machine learning community, as it has been shown that NGD can efficiently avoid saddle points [25] as well as deal with issues of exploding gradients [26]. However most work in this area assumes one has noise-free access to \( \mathbf{g}_k \), although see [27] for an interesting stochastic extension of NGD to the empirical risk minimization problem. To the best of our knowledge, there is no prior work on inexact NGD (INGD), where one only has access to a biased estimator of \( \mathbf{g}_k / \|\mathbf{g}_k\| \), call it \( \hat{\mathbf{g}}_k \), satisfying \( \|\mathbf{g}_k - \hat{\mathbf{g}}_k\| \leq \eta \). Specifically we prove the following theorem:
Theorem 4.1. Suppose that \( f(x) \) satisfies Assumptions 2 and 3 and suppose that \( \eta < \nu/L \). Then INGD with \( \alpha = O(\sqrt{\varepsilon}) \) finds \( x_K \) satisfying \( f(x_K) - f(x^*) \leq O(\varepsilon) \) in \( O(\varepsilon^{-3/2}) \) iterations.

This theorem extends earlier work of [25] in two ways:

1. Theorem 4.1 allows for errors in the estimates of the normalized gradients.

2. Theorem 4.1 relaxes the strong convexity requirement to only requiring that \( f(x) \) be restricted strongly convex.

The details of the proof are contained in Appendix C. We highlight a curious feature of NGD: in order to achieve an accurate solution one needs to choose a small step-size. In general this cannot be avoided, although we refer to [26] for some ideas on adaptively choosing \( \alpha \) if one has access to \( \|g_k\| \). In Section 6 we discuss how to incorporate a line search that allows one to use larger step sizes.

5 The proposed algorithm

By combining INGD with 1BitGradEst, we arrive at our proposed algorithm, presented as Algorithm 3. Our main result is the following theorem:

Algorithm 3 SCBO

1: Inputs: \( x_0, s, m, r \) and \( K \)
2: for \( k = 0, \ldots, K - 1 \) do
3: \( \hat{g}_k \leftarrow 1\text{BitGradEst}(x, s, m, r) \)
4: Obtain \( \alpha_k \).
5: \( x_{k+1} = x_k - \alpha_k \hat{g}_k \)
6: Check Stopping Criterion \{(Optional)\}
7: end for
8: Output: \( x_K \)

Theorem 5.1. Suppose that \( f(x) \) satisfies Assumptions 1–3, but that \( f(x) \) is only accessible through a comparison oracle (see Definition 1.1) with parameters \( \delta_0, \mu \) and \( \kappa \).

1. If \( \kappa > 1 \) and \( \delta_0 = 0.5 \), and \( f(x) \) in addition satisfies Assumption 4 then SCBO with constant step size returns \( x_K \) satisfying:

\[
f(x_K) - f^* \leq \varepsilon
\]

using only \( O\left(\frac{s^{2\kappa-1} \log d}{\varepsilon^{2\kappa-1/2}}\right) \) queries.

2. If \( \kappa = 1 \) and \( \delta_0 < 0.5 \) then SCBO with constant step size returns \( x_K \) satisfying

\[
f(x_K) - f^* \leq \varepsilon
\]

using only \( O\left(\frac{s \log d}{\varepsilon^{3/2} \delta_0^2}\right) \) queries.
Both results hold with probability \( 1 - O(\varepsilon^{-3/2}d^{-s}) \).

Deducing Theorem 5.1 from Theorem 4.1 is non-trivial. This is because, according to Theorem 3.2, the quality of the gradient estimate \( \hat{g}_k \) depends on the magnitude of \( g_k \). Thus, care must be taken in choosing the parameters of the 1BitGradEst subroutine. The details can be found in Appendix D together with precise instructions for choosing \( m, r, \alpha \) and \( K \). Theorem 5.1 assumes that \( s \) is known, but we discuss how to choose \( s \) adaptively below. We also highlight the following consequence of Theorem 5.1:

**Corollary 5.2.** With assumptions as in Theorem 1.2, let \( x_1, x_2, \ldots \) be the sequence of iterates produced by SCOBO. Then with probability \( 1 - O(\varepsilon^{-3/2}d^{-s}) \) either:

1. \( f(x_{k+1}) \leq f(x_k) \), or:
2. \( f(x_k) \leq \varepsilon \)

holds for all \( 0 \leq k \leq K - 1 \).

In other words, with overwhelming probability, SCOBO is a descent algorithm until it hits the target accuracy. We verify this experimentally in Section 7. Finally, we emphasize that, even for moderate values of \( d \) and \( s \) such as \( d = 10^4 \) and \( s = 10^3 \), the term \( d^{-s} \) is microscopically small. Hence, the probability of failure in Theorem 5.1 and Corollary 5.2 can safely be regarded as 0 in practice.

### 5.1 Extension to compositions

As observed elsewhere [28, 12], one can easily extend Theorem 5.1 to compositions of functions:

**Corollary 5.3.** Theorem 5.1 part 2. holds as stated if we instead assume that \( f(x) = g(h(x)) \) with \( h(x) \) satisfying Assumptions 1–3 and \( g(x) : \mathbb{R} \to \mathbb{R} \) any monotonically increasing function.

### 5.2 A stopping criterion

An obvious consequence of Corollary 5.2 is that if \( f(x_{k+1}) > f(x_k) \) then we have reached our target accuracy and so should stop. To check whether \( f(x_{k+1}) > f(x_k) \) one can query the oracle. This leads to a simple stopping criterion for SCOBO. We present this condition for the case \( \kappa = 1 \). First, define the \( M \)-trial comparison oracle:

\[
C_f^M(x, y) = \left( \sum_{i=1}^{M} C_f(x, y)_{\text{i-th query}} \right) / M, \tag{9}
\]

which repeatedly queries the same pair of points.

**Algorithm 4** Early Stopping

1. **Inputs:** \( x_k, x_{k+1} \) : consecutive iterates of SCOBO, \( \delta_0 \).
2. \( M \leftarrow (5 + 10\delta_0) / \delta_0^2 \)
3. **if** \( C_f^M(x_{k+1}, x_k) < 0 \) **then**
4. **Terminate SCOBO**
5. **end if**

**Theorem 5.4.** Under the assumptions of Theorem 1.2 with \( \kappa = 1 \) and \( \delta_0 < 0.5 \), if Algorithm 4 terminates SCOBO at step \( k + 1 \), then \( f(x_{k+1}) \leq \varepsilon \) with probability greater than 0.99.
5.3 Choosing the sparsity parameter

One potential drawback of SCOBO as an algorithm is that it requires a priori knowledge of $s$. Fortunately, there is an easy heuristic for selecting $s$. Define $\hat{y}_i = \text{sign}(z_i^\top \hat{g})$ and $\hat{y} = [\hat{y}_1, \ldots, \hat{y}_m]^\top$, and recall that $y$ is the binary vector of received measurements: $y_i = C_f(x, x + rz_i)$. Recall the Hamming distance:

$$d_H(\hat{y}, y) = |\{i : u_i \neq v_i\}|$$

If $\hat{g}$ is a good estimate of $g/\|g\|_2$ we expect $d_H(\hat{y}, y) \leq cm$ for small $c$. If this is not the case, we conclude that no good estimator satisfying $\|\hat{g}\| \leq \sqrt{s}$ exists, and so we should increase $s$. Thus, we recommend the following heuristic:

1. $\hat{g}_k \leftarrow 1\text{BitGrad}(x, s, m, r)$
2. While $d_H(\hat{y}, y) \leq 0.3m$ and $s \leq 0.1d$
   a. $m \leftarrow 1.1m$ and $s \leftarrow 1.1s$
   b. $\hat{g}_k \leftarrow 1\text{BitGrad}(x, s, m, r)$

One can use this rule to initialize, $s$, to update $s$ periodically, or both. We emphasize that one can recompute $g_k$ without wasting prior queries, so each pass through step 2 only requires $0.1m$ additional queries. In practice we observe that SCOBO is robust to the choice of $s$, and works well even in situations where there is no obvious low dimensional structure (see Section 7.2).

6 Line search

Algorithm 5 Inexact line search for SCOBO

1: Input: $x$: current point; $\hat{g}_k$: estimated gradient; $\alpha_{\text{def}}$: default step size; $M$: number of trials for comparison; $\omega \geq 0$: confidence parameter; $\psi > 1$: searching parameter.
2: $\alpha = \alpha_{\text{def}}$
3: while $C_f^M(x + \alpha \hat{g}_k, x + \psi \alpha \hat{g}_k) \leq -\omega$ do
4: $\alpha = \psi \alpha$
5: end while
6: Output: $\alpha$

While Algorithm 1 gives a good estimate of the direction of the true gradient, the length of the true gradient is not recovered in SCOBO. In fact, by the nature of the comparison oracle, the gradient length can never be recovered. As shown in Theorem 5.1, we can guarantee the convergence of SCOBO with a fixed small step size; however, it appears that longer step sizes may be able to significantly accelerate the convergence, particularly in the earlier stages of SCOBO since the length of the true gradient is larger. Hence, we propose an inexact step size line search method, Algorithm 5.

The main challenge for our line search is the noisy comparison oracle. To overcome this, we consider the $M$-trial comparison oracle as defined in (9). When $M$ is large enough, we will have $\text{sign}(C_f^M(x, y)) = \text{sign}(f(y) - f(x))$ almost surely. In particular, when $\kappa = 1$ and $f(y) < f(x)$, take $M = \beta \delta_0^{-2}$, then $C_f^M(x, y) < -\delta_0$ with probability at least $1 - \exp(-\beta/2)$. If $\kappa > 1$ the
probability that \( C_f(x, y) = \text{sign}(f(y) - f(x)) \) depends on \(|f(y) - f(x)|\), which means the theoretical \( M \) required cannot be computed \textit{a priori} unless we in addition assume strong convexity as in [1]. In practice, we pick a fixed \( M \) and assign a confidence parameter \( \omega \geq 0 \) so that \( f(y) < f(x) \) with high probability when \( C_f^M(x, y) \leq -\omega \). Starting with an initial step size \( \alpha_{\text{def}} \), the line search algorithm will repeatedly increase the step size by some factor \( \psi > 1 \) until \( f(x + \alpha \hat{g}_k) \geq f(x + \psi \alpha \hat{g}_k) \). When Algorithm 5 stops, the output \( \alpha \) is unlikely optimal; however, with high probability, it satisfies

\[
\psi^{-1} \alpha_* < \alpha \leq \alpha_*
\]

where \( \alpha_* = \arg\min_\alpha f(x + \alpha \hat{g}_k) \). Since the estimated gradient is close to the normalized true gradient, we conclude \( \alpha_* \geq c_* \|g_k\|/L \) where \( c_* \) is a constant depending on \( \|\hat{g}_k - \frac{x_k}{\|g_k\|}\| \). Therefore, \( \alpha \in (\psi^{-1} c_* \|g_k\|/L, \alpha_*] \) is a reasonably good step size and ensures that \( \|x_k - \alpha \hat{g}_k\| \approx \|g_k\| \) If one wishes to estimate \( \alpha_* \) more accurately, one can further apply Fibonacci search on the interval \([\alpha, \psi \alpha]\). Either way, the query complexity of the inexact line search is \( O(M \log_\psi (\alpha_*/\alpha_{\text{def}})) \).

6.1 Warm started line search

According to Theorem 5.1, we can use a smaller default step size in line search to obtain a lower final error. However, Algorithm 5 may repeatedly waste a lot of queries in reaching the larger optimal step sizes in the earlier stage of SCOBO if the default step size is too tiny. In this case, we can gain some efficiency by using the estimated step size from the last iteration as the initialization for the current iteration. This saves a large number of queries if the optimal step sizes do not change rapidly between iterations. With the warm start, we must also include a mechanism to reduce step size from the initial \( \alpha \) since it can be longer than the optimal step size in the new iteration. Nevertheless, Theorem 5.1 still stands, so we have the ability to set a smaller default step size for improving the accuracy of SCOBO. The warm started inexact line search is summarized as Algorithm 6 in Appendix E.

6.2 Choosing the step size

We have experimented with constant step size, line search and decaying step sizes. SCOBO works well in all three cases. Line search provides faster convergence for convex functions (Section 7.1) while using decaying step sizes provides more stable performance for highly non-convex functions such as in the MuJoCo control problems (Section 7.2). If ensuring descent (i.e. \( f(x_{k+1}) \leq f(x_k) \)) at every step is crucial, we recommend using a small, fixed, step size.

7 Numerical experiments

In this section, we demonstrate the empirical performance of SCOBO on both synthetic examples and the MuJoCo dataset [13].

7.1 Synthetic examples

We benchmark SCOBO on four synthetic test cases:

(a) We consider the skewed-quartic function used in [29]. We embed the 20-dimensional skewed-quartic function into 500-dimensional space. The comparison oracle parameters are set to be \( \kappa = 1.5, \mu = 1 \) and \( \delta_0 = 0.5 \).
(b) We consider the squared sum of the 20 largest-in-magnitude elements in a 500-dimensional vector, i.e., $f(x) = \sum_{i=1}^{20} x^2_{m_i}$ where $x_{m_i}$ is the $i$-th largest-in-magnitude entry at the current point $x$. The comparison oracle parameters are set to be $\kappa = 1.5$, $\mu = 4$ and $\delta_0 = 0.5$.

(c) We use the same objective function as in case (a), but the comparison oracle parameters are set to be $\kappa = 1$, $\mu = 1$ and $\delta_0 = 0.3$.

(d) We use the same objective function as in case (b), but the comparison oracle parameters are set to be $\kappa = 1$, $\mu = 1$ and $\delta_0 = 0.3$.

All four test cases have $s = 20$ and $d = 500$. By Theorem 5.1, we sample $m = s^2 \log(2d/s)$ queries per iteration for the gradient estimation in this experiment. In cases (a) and (b), the flipping probability of $C_f(x,y)$ will rise when $|f(x) - f(y)|$ is small, so we set a fixed sampling radius $r = 1/2\sqrt{s}$ in these 2 cases. In contrast, the comparison oracle parameters in cases (c) and (d) imply $P[C_f(x,y) = \text{sign}(f(y) - f(x))] = 0.8$, so the flipping probability of $C_f(x,y)$ is independent from $|f(y) - f(x)|$. Thus, we may use a smaller sampling radius of $r = 10^{-4}$, which offsets the perturbation due to $\nabla^2 f$ (see (7)).

We first numerically verify our convergence theorem. In Figure 1, we plot the convergence trajectory of SCOBO with fixed step size (SCOBO-FS) in blue where $\alpha = 2$ was used in all four
Figure 2: Convergence comparison among fixed step size version, vanilla line search version and warm started line search version of SCOBO. Top-left: case (a). Top-right: case (b). Bottom-left: case (c). Bottom-right: case (d).

cases. For comparison, we also plot the fraction of flipped oracle queries in red. We further plot the theoretical error bound as a horizontal yellow dash line for reference.

In cases (a) and (b), SCOBO converges slowly yet smoothly; meanwhile, the fraction of flipped comparison oracle queries keeps relatively low in the early stage. The number of flipped comparison increase rapidly when the optimality gap getting smaller. While the expectation of flipping probability later rises to over 40% and 30% respectively, SCOBO stays stably under the theoretical bound.

In cases (c) and (d), the fraction of flipped oracle queries is constantly 20% in expectation. This may seem to create a harder 1-bit compressed sensing problem for Algorithm 1, but this difficulty is offset by the smaller sampling radius. Hence, the trajectory of SCOBO shows smooth monotonic descent to the theoretical bound in both cases.

Overall, we observe that SCOBO converges successfully to the theoretical error bound in all three cases, and once reaching this bound remains underneath it. This verifies our convergence theorem (Theorem 5.1).

We investigate the empirical performance of different versions of SCOBO: fixed step size (SCOBO-FS), vanilla line search (SCOBO-LS) and warm started line search (SCOBO-WSLS). For all three cases, we use the default step size $\alpha_{\text{def}} = 2$ for SCOBO-LS and $\alpha_{\text{def}} = 10^{-4}$ for SCOBO-WSLS. The line search parameters are set to be $M = 40$, $\omega = 0.05$ and $\psi = 2$. The results are shown in
Figure 2. We find both versions of inexact line search methods accelerate the convergence dramatically in all test cases. Furthermore, we see SCOBO-WSLS is able to convergent to higher accuracy since it can use a tiny default step size without wasting unnecessary queries on distant cold start line search. In summary, SCOBO can be stably accelerated with the proposed line search methods.

Finally, we compare SCOBO against two state-of-the-art comparison oracle based optimization methods: Pairwise comparison coordinate descent (PCCD) [1], and SignOPT [7]. We implemented PCCD by ourselves and hand turned its parameters for the best performance. The code of SignOPT is obtained from the authors’ website, we use the parameters suggested in the paper; in particular, we sample 200 random directions for their gradient estimator, which is recommended by the authors. We also emphasize that we use the same key parameters (e.g. sampling radius) for all three tested algorithms, so we do not gain advantage from the parameter setting. The empirical results are summarized in Figure 3.

SignOPT has some slight advantage in the early stage of test cases (a) and (c), but SCOBO stably converges to more accurate solutions. For the harder cases (b) and (d), SignOPT fails. Note that the support of the gradient is fixed in cases (a) and (c) while the gradient support varies in cases (b) and (d). The varying gradient support doesn’t affect SCOBO, but it is problematic for SignOPT.

PCCD has reasonable performance in test cases (a) and (b), but fails cases (c) and (d) where $\kappa = 1$. This is caused by the fact that PCCD uses a 1-trial comparison oracle for coordinate line
Table 1: Parameters for SCOBO applied to MuJoCo. Note that for each model the dimension is the dimension of the action space times the dimension of the observation space

| Model           | Dimension | m  | s  |
|-----------------|-----------|----|----|
| Swimmer-v2      | 16        | 10 | 5  |
| Reacher-v2      | 22        | 26 | 16 |
| HalfCheetah-v2  | 102       | 100| 50 |

search. When the fraction of flipped queries is constantly high this line search is unreliable. Using the $M$-trial comparison oracle (9) could improve the quality of line search, but is unlikely to improve the overall efficiency. This is because PCCD uses an unreliable search direction and spends all its queries on line search. Thus, using (9) will immediately increase the total queries $M$-fold without necessarily yielding more descent per iteration. In contrast, SCOBO starts with a very good search direction ($\approx g_k$), and thus it makes sense to invest more queries in a more thorough line search.

In conclusion, we find SCOBO has the best performance among the three tested algorithms, in terms of both query complexity and convergence stability.

7.2 MuJoCo policy optimization

In this section, we use SCOBO to learn a policy for simulated robot control, using only comparison oracle feedback, for several problems from the MuJoCo suite of benchmarks [13]. Inspired by [30], we use a simple class of policies (linear policies) and minimal computational resources. We note that the objective functions for these problems (i.e. the reward obtained given an input policy) are highly non-convex and possess no obvious low dimensional structure. Nevertheless, SCOBO performs well. Our experimental set-up is as follow:

- We use a horizon of 1000 iterations for each rollout.
- The only access to the reward function was through a comparison oracle with $\kappa = 2, \mu = 0.5$ and $\delta_0 = 0.3$.
- The values of $m$ and $s$ for each experiment are displayed in Table 1. Note that these values are somewhat arbitrary; empirically we observed good performance for a broad range of $m$ and $s$ values.
- We do not use line search. Instead, we implement an exponentially decaying learning rate schedule.
- We use the state normalization/whitening trick introduced in [30] to encourage more equal exploration across dimensions.

On all our tests, the mean rewards eventually exceed the reward threshold specified in the OpenAI Gym environment. Compared with reinforcement learning and gradient estimation approaches in the literature, SCOBO on Swimmer-v2 and Reacher-v2 yields surprisingly competitive convergence in terms of number of queries required. Note that typical approaches to reinforcement learning receive the reward function value, encoded as a 32-bit float, upon each query. In contrast, SCOBO only receives 1 bit per query. When performance is measured as the number of bits required...
Figure 4: Rewards v.s. number of comparison oracles. Blue solid lines and shaded regions represent mean and +/- sigma of rewards. **Left:** Swimmer-v2. **Middle:** HalfCheetah-v2. **Right:** Reacher-v2.

*to exceed the reward threshold*, the performance of SCOBO exceeds that of the state of the art. For example, TD3 and CEM-TD3 [31] require roughly 3.2 million bits for HalfCheetah-v2, whereas SCOBO requires only around 400 thousand bits.

**Broader Impact**

The results in this paper lay solid theoretical foundation for optimization based on a comparison oracle. Algorithmic results will advance the techniques to solve problems based on human inputs of the comparison type, as well as the duelling bandit problems. Insights gained from this project will also permeate benefits to a gamut of human-in-the-loop-learning problems. We do not believe that our research will cause any ethical issue, or put anyone at a disadvantage.

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A High-dimensional probability

Analysis of a random variable, \( z \), sampled uniformly from the unit sphere \( S^{d-1} \subset \mathbb{R}^d \) for large \( d \) is a key ingredient to our theoretical guarantees. Hence, for the convenience of the reader we recall a few well-known results in high-dimensional probability in this section. For completeness, we include proofs.

**Theorem A.1.** Let \( z \sim U(S^{d-1}) \), and let \( z_i \) be the \( i \)-th component of \( z \). Then:

1. \( \mathbb{E}[z_i] = 0 \)
2. \( \mathbb{E}[z_i^2] = 1/d \)
3. \( \mathbb{P}[|z_i| \geq 1/\sqrt{d}] \geq 1/2 \)

**Proof. Part 1.** Without loss of generality, we may assume that \( i = 1 \). Since the distribution of \( z_1 \) is symmetric about the origin, it follows that \( \mathbb{E}[z_1] = 0 \).

**Part 2.** The probability of \( z_1 > h \) is proportional to the area of the hyperspherical cap of height \( h \). That is, the area of the portion of \( S^{d-1} \) above the hyperplane with equation \( x_1 = h \). From [32] we get that:

\[
\mathbb{P}[z_1 \geq h] = \frac{\text{Area hyperspherical cap of height } h}{\text{Area of } S^{d-1}} = \frac{1}{2} I_{1-h^2} \left( \frac{d-1}{2}, \frac{1}{2} \right)
\]

Where \( I \) represents the regularized, incomplete Beta function. Equivalently, \( X = 1 - z_1^2 \) is a Beta \( \left( \frac{d-1}{2}, \frac{1}{2} \right) \) random variable, hence:

\[
\mathbb{E}[z_1^2] = 1 - \mathbb{E}[X] = 1 - \left( \frac{(d-1)/2}{(d-1)/2 + 1/2} \right) = 1 - \frac{d-1}{d} = \frac{1}{d}
\]

**Part 3.** From the above:

\[
\mathbb{P}[z_1 \geq \frac{1}{\sqrt{d}}] = \frac{1}{2} I_{1-1/d} \left( \frac{d-1}{2}, \frac{1}{2} \right)
\]

We note, as in [28], that the function \( d \rightarrow I_{1-1/d} \left( \frac{d-1}{2}, \frac{1}{2} \right) \) is increasing. Because:

\[
I_{1/2} \left( \frac{2-1}{2}, \frac{1}{2} \right) = I_{1/2} \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2}
\]
where for the second inequality we have used the fact that the distribution Beta \((\frac{1}{2}, \frac{1}{2})\) is equal to the arcsine distribution. The claim then follows by symmetry, as:

\[
\mathbb{P}\left[|z_1| \geq \frac{1}{\sqrt{d}}\right] = 2\mathbb{P}\left[z_1 \geq \frac{1}{\sqrt{d}}\right] = I_{1-1/d}\left(\frac{d-1}{2}, \frac{1}{2}\right) \geq \frac{1}{2}
\]

We also need the following result on sums:

**Lemma A.2.** Let \(z \sim U(S^{d-1})\) and let \(a_1, \ldots, a_s > 0\) be arbitrary non-negative real numbers for \(s < d\). Then:

\[
\mathbb{P}\left[\sum_{i=1}^{s} a_i z_i^2 \geq \frac{5s \max_i a_i}{d}\right] \leq e^{-s}
\]

**Proof.** Observe that:

\[
\mathbb{P}\left[\sum_{i=1}^{s} a_i z_i^2 \geq \frac{5s \max_i a_i}{d}\right] \leq \mathbb{P}\left[\sum_{i=1}^{s} z_i^2 \geq \frac{5s}{d}\right].
\]

The result then follows from Lemma 2.2 of [33].

---

**B Proofs for Section 3**

We shall repeatedly use the following Taylor expansion:

\[
\text{sign} (f(x + rz_i) - f(x)) = \text{sign}\left(r z_i^T g + \frac{1}{2} r^2 z_i^T \nabla^2 f(x + t_0 z_i) z_i\right)
\]

for some \(t_0 \in (0, 1)\).

**Lemma B.1.** Define \(e_i := z_i^T \nabla^2 f(x + t_0 z_i) z_i\). If \(f(x)\) satisfies Assumptions 2 and 4 then \(|e_i| \leq 5Ls/d\) with probability greater than \(e^{-s}\).

**Proof.** Suppose that \(\nabla^2 f(x + t_0 z_i) = U \Lambda_s U^T\) where \(\Lambda_s = \text{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0)\). Clearly, if \(z_i \sim U(S^{d-1})\) then \(w_i := Uz_i \sim U(S^{d-1})\) too. Hence:

\[
|e_i| = w_i^T \Lambda w_i = \sum_{j=1}^{s} \lambda_j w_{i,j}^2.
\]

The result then follows from Lemma A.2 and the fact that \(\max_i \lambda_i = L\).

**Lemma B.2.** Suppose that \(z_i \sim U(S^{d-1})\), then

\[
\mathbb{P}\left[|z_i^T g| \geq \|g\|/\sqrt{d}\right] \geq \frac{1}{2}
\]

**Proof.** Without loss of generality we may assume that \(g = \|g\| e_1\), where \(e_1\) denotes the first canonical basis vector. Then:

\[
\mathbb{P}\left[|z_i^T g| \geq \|g\|/\sqrt{d}\right] = \mathbb{P}\left[|z_i^T e_1| \geq 1/\sqrt{d}\right] = \mathbb{P}[|z_{i,1}| \geq 1/\sqrt{d}] \geq \frac{1}{2}
\]

where the final inequality is from Theorem A.1 Part 3.
Lemma B.3. Suppose that \( f(x) \) satisfies Assumptions 2 and 4. If \( z_i \sim \mathcal{U}(S^{d-1}) \), \( \|g\| \geq \varepsilon \nu \) and \( r = \varepsilon \nu \sqrt{d}/(Ls) \) then:

\[
P \left[ |f(x + rz_i) - f(x)| \geq \frac{\varepsilon^2 \nu^2}{10Ls} \right] \geq 0.49
\]

Proof. Using (10) we get:

\[
f(x + rz_i) - f(x) = rz_i^\top g + r^2 e_i
\]

\[
\Rightarrow |f(x + rz_i) - f(x)| \geq r |z_i^\top g| - r^2 |e_i|
\]

for all values of \( r > 0 \). From Lemma B.1 we get:

\[
P \left[ \frac{1}{2} r^2 |e_i| \geq \frac{5Lr^2s}{2d} \right] \leq e^{-s}
\]

while from Lemma B.2 we obtain:

\[
P \left[ r |z_i^\top g| \geq \frac{r \|g\|}{\sqrt{d}} \right] \geq \frac{1}{2}.
\]

Combining these estimates with (11) and appealing to the union bound:

\[
P \left[ |f(x + rz_i) - f(x)| \geq \frac{r \|g\|}{\sqrt{d}} - \frac{5Ls \nu^2}{2d} \right] \geq \frac{1}{2} - e^{-s} \geq 0.49
\]

as long as \( s \geq 5 \) which in practice is always the case. Because \( \|g\| \geq \varepsilon \nu \):

\[
P \left[ |f(x + rz_i) - f(x)| \geq \frac{r \nu}{\sqrt{d}} - \frac{5Ls \nu^2}{2d} \right] \geq 0.49
\]

Now maximizing the lower bound with respect to \( r \) yields the desired result.

\[
\square
\]

Remark B.4. Suppose that \( r |z_i^\top g| - r^2 |e_i| > 0 \). Then, one can easily verify that:

\[
\text{sign}(f(x + rz_i) - f(x)) = \text{sign}(rz_i^\top g + r^2 e_i) = \text{sign}(rz_i^\top g)
\]

Hence if the event described in Lemma B.3 occurs we automatically have \( \text{sign}(f(x + rz_i) - f(x)) = \text{sign}(z_i^\top g) \).

Clearly, as \( \varepsilon \to 0 \) the lower bound on \( |f(x + rz_i) - f(x)| \) given by Lemma B.3 gets worse. Accordingly, the reliability of the comparison oracle (see Definition 1.1) gets worse. Lemma 3.1, which we state formally below, quantifies this.

Lemma B.5. (Lemma 3.1, precisely stated)

1. Suppose that \( f(x) \) satisfies Assumptions 1, 2 and 4, and that \( \kappa > 1 \) and \( \delta_0 = 0.5 \). For any \( p \in (1/2, 1] \) define:

\[
\varepsilon_p = \sqrt{\frac{10Ls}{\nu}} (0.49\mu)^{\frac{1}{p-1/2}} \left( p - \frac{1}{2} \right)^{\frac{1}{p-1/2}}.
\]

Then, if \( \|g\| \geq \varepsilon_p \nu \) and \( r = \varepsilon_p \nu \sqrt{d}/(Ls) \) we have that:

\[
P [\hat{y}_i = y_i] \geq p
\]

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2. Suppose that \( f(x) \) satisfies Assumptions 1 and 2, and that \( \kappa = 1 \) and \( \delta_0 < 0.5 \). Then, for any \( \varepsilon > 0 \) if \( \|g\| \geq \varepsilon \nu \) and \( r = \varepsilon \nu / (L \sqrt{d}) \) we have that:

\[
P[y_i = \tilde{y}_i] \geq \frac{1}{2} + \frac{\delta_0}{2}\]

Proof of Lemma 3.1. As a notational convenience, we introduce \( \hat{y}_i = \text{sign}(f(x + rz_i) - f(x)) \). As outlined in (5) we can show that \( y_i = \tilde{y}_i \) by showing that \( y_i = \hat{y}_i \) and \( \hat{y}_i = \tilde{y}_i \). Note that we can also obtain \( y_i = \tilde{y}_i \) when \( y_i = -\hat{y}_i \) and \( \hat{y}_i = -\tilde{y}_i \). To start, observe that for any event \( A \):

\[
P[y_i = \tilde{y}_i] = P[y_i = \tilde{y}_i | A] P[A] + P[y_i = \tilde{y}_i | A^c] (1 - P[A]) \quad (13)
\]

where \( A^c \) denotes the complementary event.

**Part 1.** Let:

\[
A = \left\{ |f(x + rz_i) - f(x)| > \frac{\varepsilon^2 \nu^2}{10 L s} \right\}
\]

From Lemma B.3 we get that \( P[A] \geq 0.49 \). From Remark B.4 and the definition of the comparison oracle we obtain:

\[
P\left[y_i = \tilde{y}_i \bigg| f(x + rz_i) - f(x) > \frac{\varepsilon^2 \nu^2}{10 L s} \right] \geq P\left[\tilde{y}_i = \hat{y}_i \right. \left. \text{ and } y_i = \tilde{y}_i \bigg| f(x + rz_i) - f(x) > \frac{\varepsilon^2 \nu^2}{10 L s} \right] \\
\geq \frac{1}{2} + \mu \left( \frac{\varepsilon^2 \nu^2}{10 L s} \right)^{\kappa-1} \quad (14)
\]

On the other hand:

\[
P\left[y_i = \tilde{y}_i \bigg| f(x + rz_i) - f(x) \leq \frac{\varepsilon^2 \nu^2}{10 L s} \right] \\
= P\left[\tilde{y}_i = \hat{y}_i \text{ and } y_i = \tilde{y}_i \bigg| f(x + rz_i) - f(x) \leq \frac{\varepsilon^2 \nu^2}{10 L s} \right] \\
+ P\left[\tilde{y}_i = -\hat{y}_i \text{ and } y_i = \tilde{y}_i \bigg| f(x + rz_i) - f(x) \leq \frac{\varepsilon^2 \nu^2}{10 L s} \right] \\
\geq \frac{1}{2}
\]

Combining these estimates with (13) and Lemma B.3:

\[
P[y_i = \tilde{y}_i] \geq \left( \frac{1}{2} + \mu \left( \frac{\varepsilon^2 \nu^2}{10 L s} \right)^{\kappa-1} \right) (0.49) + \left( \frac{1}{2} \right) (1 - 0.49) \\
= \frac{1}{2} + 0.49\mu \left( \frac{\varepsilon^2 \nu^2}{10 L s} \right)^{\kappa-1}
\]
Hence if:

$$\varepsilon = \varepsilon_p = \frac{\sqrt{10Ls}}{\nu} (0.49\mu)^{\frac{1}{2\kappa-2}} \left( p - \frac{1}{2} \right)^{\frac{1}{2\kappa-2}}$$

we indeed obtain that $$\mathbb{P}[y_i = \tilde{y}_i] = p$$.

**Part 2.** By Remark B.4 we have that $$\tilde{y}_i = \hat{y}_i$$ if:

$$r|z_i^T g| - r^2|e_i| > 0$$

Using the bound $$||e_i|| \leq ||\nabla^2 f(x)||_2 \leq L$$ and the choice of $$r$$ given, this reduces to:

$$|z_i^T g| > \frac{\varepsilon \nu}{\sqrt{d}}$$

(15)

So, choose $$\mathcal{A} = \{ |z_i^T g| > \frac{\varepsilon \nu}{\sqrt{d}} \}$$. By Lemma B.2 $$\mathbb{P}[\mathcal{A}] \geq 1/2$$ as long as $$||g|| \geq \varepsilon \nu$$. By the argument just given, $$\mathbb{P}[	ilde{y}_i = \hat{y}_i | \mathcal{A}] = 1$$. From the definition of the comparison oracle:

$$\mathbb{P} [\hat{y}_i = y_i | \mathcal{A}] = \mathbb{P} [\hat{y}_i = y_i | \mathcal{A}^c] = \frac{1}{2} + \delta_0$$

Observe that:

$$\begin{align*}
\mathbb{P} [\tilde{y}_i = y_i | \mathcal{A}] \\
&\geq \mathbb{P} [\tilde{y}_i = \hat{y}_i \text{ and } \hat{y}_i = y_i | \mathcal{A}] \\
&= \mathbb{P} [\tilde{y}_i = \hat{y}_i | \mathcal{A}] \mathbb{P} [\hat{y}_i = y_i | \mathcal{A}] \\
&= (1)(\frac{1}{2} + \delta_0) = \frac{1}{2} + \delta_0
\end{align*}$$

Proceeding from here is more subtle. One can check that $$\mathbb{P} [\tilde{y}_i = \hat{y}_i | \mathcal{A}^c] > 1/2$$. Because $$\mathbb{P} [\tilde{y}_i = y_i | \mathcal{A}^c] = \frac{1}{2} + \delta_0 > 1/2$$, we deduce that:

$$\begin{align*}
\mathbb{P} [\tilde{y}_i = y_i | \mathcal{A}^c] = \mathbb{P} [\tilde{y}_i = \hat{y}_i | \mathcal{A}^c] \mathbb{P} [\hat{y}_i = y_i | \mathcal{A}^c] + \mathbb{P} [\tilde{y}_i = -\hat{y}_i | \mathcal{A}^c] \mathbb{P} [\hat{y}_i = -y_i | \mathcal{A}^c] \\
&\geq \frac{1}{2}
\end{align*}$$

From (13) we obtain:

$$\mathbb{P}[y_i = \tilde{y}_i] = \left( \frac{1}{2} + \delta_0 \right) \mathbb{P}[\mathcal{A}] + \frac{1}{2} \left( 1 - \mathbb{P}[\mathcal{A}] \right) \geq \frac{1}{2} + \frac{\delta_0}{2}$$

$$\square$$

**Theorem B.6** (Theorem 3.2, precisely stated). Fix any $$\eta \in (0, 1)$$ and $$\varepsilon > 0$$. Then, under either of the following sets of conditions:

1. $$f(x)$$ satisfies Assumptions 1, 2 and 4; $$\kappa > 1$$ and $$\delta_0 = 0.5$$; $$||g|| \geq \varepsilon \nu$$ and

$$m \geq C \eta^{-3} (0.49\mu)^2 \left( \frac{\sqrt{10Ls}}{\varepsilon \nu} \right)^{4\kappa-4} \frac{s \log(2d/s)}{Ls}$$

$$r = \frac{\varepsilon \nu \sqrt{d}}{Ls}$$

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2. \( f(x) \) satisfies Assumptions 1 and 2; \( \kappa = 1 \) and \( \delta_0 < 0.5; \|g\| \geq \varepsilon \nu \) and
\[
m \geq 4C\eta^{-2}\delta_0^{-2}s \log(2d/s)
\]
\[
r = \frac{\varepsilon \nu}{L \sqrt{d}}
\]
Then:
\[
\left\| \hat{g} - \frac{g}{\|g\|} \right\| \leq \eta
\]
with probability at least \( 1 - 8 \exp \left(-c\eta^4 s \log(2d/s)\right) \).  

Proof. Set \( \varepsilon_p = \varepsilon \) in Lemma B.5 and solve for \((p - 1/2)\):
\[
\left(p - \frac{1}{2}\right)^{-1} = (0.49\mu)^2 \left(\frac{10Ls}{\varepsilon \nu}\right)^{4\kappa - 4}
\]
By Lemma 3.1 we may write \( y_i = \xi_i \tilde{y}_i \) where the \( \xi_i \) are random variables with \( \mathbb{P}[\xi_i = 1] \geq p \) and \( \mathbb{P}[\xi = -1] \leq 1 - p \), for the value of \( p \) implicit in (16), as long as \( \|g\| \geq \varepsilon \nu \). Now substitute (16) into the requirement for \( m \) presented in Theorem 2.1:
\[
m \geq C\eta^{-2}(0.49\mu)^2 \left(\frac{10Ls}{\varepsilon \nu}\right)^{4\kappa - 4} s \log(2d/s)
\]
Conditioned on \( x \), the \( \xi_i \) are independent. So, take \( \delta = \sqrt{\eta} \) and conclude from Theorem 2.1 that indeed:
\[
\left\| \hat{g} - \frac{g}{\|g\|} \right\| \leq \eta
\]
with the stated probability. \( \square \)

C Proofs for Section 4

Throughout this section, we assume that \( x_{k+1} = x_k - \alpha \hat{g}_k \) where \( \hat{g}_k \approx \frac{g_k}{\|g_k\|} \). Before proceeding, it is convenient to introduce the following notation:
\[
e_k := \frac{g_k}{\|g_k\|} - \hat{g}_k
\]
\[
X = \left\{ x : x = \arg\min_{x \in B} f(x) \right\}
\]
\[
\mathbb{P}_*(x) = \arg\min_{z \in X} \|x - z\|_2
\]
\[
\Delta_k = \|x_k - \mathbb{P}_*(x_k)\|_2
\]
We shall use the following inequality repeatedly, so we isolate it as a lemma:
Lemma C.1. Suppose that $f(x)$ satisfies Assumptions 2 and 3. Suppose further that $\|e_k\| \leq \eta < \nu/L$. Then:

$$(\Delta_{k+1} - \alpha \eta)^2 \leq \Delta_k^2 - \frac{2\alpha \nu}{L} \Delta_k + \alpha^2$$

Proof. Observe that:

$$\Delta_{k+1} = \|x_{k+1} - P_*(x_{k+1})\| \leq \|x_{k+1} - P_*(x_k)\|$$
$$= \|x_k - \alpha g_k - P_*(x_k)\|$$
$$= \|x_k - \alpha \left( \frac{g_k}{\|g_k\|} + e_k \right) - P_*(x_k)\|$$
$$\leq \|x_k - \alpha \frac{g_k}{\|g_k\|} - P_*(x_k)\| + \alpha \|e_k\|$$

Hence:

$$(\Delta_{k+1} - \alpha \|e_k\|)^2 \leq \|x_k - \alpha \frac{g_k}{\|g_k\|} - P_*(x_k)\|^2$$ \hspace{1cm} (17)

We now handle the term on the right-hand side:

$$\|x_k - \alpha \frac{g_k}{\|g_k\|} - P_*(x_k)\|^2$$
$$= \|x_k - P_*(x_k)\|^2 - 2 \gamma \left( \frac{g_k}{\|g_k\|} \right) \cdot x_k - P_*(x_k) + \alpha^2 \|\frac{g_k}{\|g_k\|}\|^2$$
$$\leq \|x_k - P_*(x_k)\|^2 - \frac{2\alpha \nu}{L} \|x_k - P_*(x_k)\|^2 + \alpha^2$$
$$\leq \|x_k - P_*(x_k)\|^2 - \frac{2\alpha \nu}{L} \|x_k - P_*(x_k)\|^2 + \alpha^2$$
$$= \Delta_k^2 - \frac{2\alpha \nu}{L} \Delta_k + \alpha^2$$ \hspace{1cm} (18)

Where in (a) we have used restricted strong convexity (Assumption 3) while in (b) we have used smoothness (Assumption 2): $\|g_k\| \leq L \|x_k - P_*(x_k)\| \Rightarrow \|x_k - P_*(x_k)\| \geq \frac{1}{L}$. We combine equations (17), (18) and use the assumption that $\|e_k\| \leq \eta$ to complete the proof.

It is interesting to determine when Lemma C.1 guarantees descent i.e. $\Delta_{k+1} \leq \Delta_k$.

Lemma C.2. Assume that $f(x)$ satisfies Assumptions 2 and 3. Suppose further that $\|e_k\| \leq \eta < \nu/L$ and that $\Delta_k \geq \frac{\alpha (1 - \eta^2)}{2(1 - \eta)}$. Then $\Delta_{k+1} \leq \Delta_k$

Proof. Suppose that

$$- \frac{2\alpha \nu}{L} \Delta_k + \alpha^2 \leq -2\alpha \eta \Delta_k + \alpha^2 \eta^2$$ \hspace{1cm} (19)

Then from Lemma C.1 one obtains:

$$(\Delta_{k+1} - \alpha \eta)^2 \leq \Delta_k^2 - \frac{2\alpha \nu}{L} \Delta_k + \alpha^2$$
$$\leq \Delta_k^2 - 2\alpha \eta \Delta_k + \alpha^2 \eta^2 = (\Delta_k - \alpha \eta)^2$$
where we are using the fact that $\eta < \nu/L \leq 1$. Hence $\Delta_{k+1} \leq \Delta_k$. Solving (19) for $\Delta_k$, and assuming $\eta < \nu/L$, we get the condition:

$$\Delta_k \geq \frac{\alpha(1-\eta^2)}{2(\nu L - \eta)}$$  \hspace{1cm} (20)

**Lemma C.3** (No escape, after [25]). Assume that $f(x)$ satisfies Assumptions 2 and 3. Fix $K > 0$ and assume that $\|\hat{g}_k - g_k\|_2 \leq \eta < \nu/L$ for all $0 \leq k \leq K - 1$. If, for any $k < K$, we have that $\Delta_k \leq \alpha(1 + \rho^*)$ then:

$$\Delta_{k+t} \leq \alpha(1 + \rho^*) \quad \text{for all } 0 \leq t \leq K - k$$

where:

$$\rho^* = \frac{\eta^2 - \frac{2\nu}{L} + 1}{\frac{2\nu}{L} - \eta}$$

**Proof.** First, observe that:

$$\alpha \rho^* = \alpha \frac{\eta^2 - \frac{2\nu}{L} + 1}{\frac{2\nu}{L} - \eta} \geq \frac{\eta^2 - 2\eta + 1}{\frac{2\nu}{L} - \eta} \geq \frac{\alpha(1-\eta)^2}{2(\frac{2\nu}{L} - \eta)}$$

so, by Lemma C.2 we have that if $\Delta_k \geq \alpha \rho^*$ then $\Delta_{k+1} \leq \Delta_k$. On the other hand, if $\Delta_k < \alpha \rho^*$ then:

$$\Delta_{k+1} = \|x_{k+1} - P_*(x_{k+1})\| \leq \|x_{k+1} - P_*(x_k)\|$$

$$= \|x_k - \alpha \hat{g}_k - P_*(x_k)\|$$

$$\leq \|x_k - P_*(x_k)\| + \alpha \|\hat{g}_k\| \overset{(a)}{=} \Delta_k + \alpha$$

$$\leq \alpha(1 + \rho^*)$$

Where in (a) we are of course using the fact that $\|\hat{g}_k\|_2 = 1$. Thus, we obtain:

$$\Delta_{k+1} \leq \max\{\Delta_k, \alpha(1 + \rho^*)\} \text{ if } \left\|\hat{g}_k - \frac{g_k}{\|g_k\|}\right\|_2 \leq \eta$$

From this, it is easy to deduce that if $\Delta_k \leq \alpha(1 + \rho^*)$ then $\Delta_{k+1} \leq \alpha(1 + \rho^*)$, and the Lemma follows by induction. \qed

Observe that if $\eta = 0$ then $\rho^* = L/\nu$ and we recover, albeit with different notation, Lemma 6 of [25]. We now prove an elementary lemma that quantifies the *rate of descent* of sequences satisfying the type of recurrence as in Lemma C.1.

**Lemma C.4** (Sequence analysis). Consider a sequence $e_k \geq 0$ obeying $e_{k+1}^2 \leq e_k^2 - ae_k + b$ for $k = 0, 1, \ldots$ where $a, b > 0$. We have

$$e_k \leq \frac{\sqrt{2e_0^{3/2}}}{\sqrt{2e_0 + ak}}, \quad k \in \{t : e_0, \ldots, e_{t+1} \geq 2b/a\}$$.  \hspace{1cm} 25
Proof. Suppose that \( e_k \geq 2b/a \), then \( e_{k+1}^2 \leq e_k^2 - b \leq e_k^2 \). Dividing both sides of \( e_{k+1}^2 \leq e_k^2 - ae_k + b \) by \( e_{k+1}^2 \) we obtain:

\[
\frac{1}{e_k^2} \leq \frac{1}{e_{k+1}^2} - \frac{a}{e_{k+1}^2 e_k} + \frac{b}{e_{k+1}^2 e_k^2} \\
\leq \frac{1}{e_{k+1}^2} - \frac{a}{e_{k+1}^2 e_k} + \frac{a}{2e_{k+1}^2 e_k} \\
\Rightarrow \frac{1}{e_k^2} \leq \frac{1}{e_{k+1}^2} - \frac{a}{2e_k^2} \leq \frac{1}{e_{k+1}^2} - \frac{a}{2e_0^2}
\]

\[
\Rightarrow 0 \leq \frac{1}{e_0^2} \leq \frac{1}{e_K^2} - \frac{aK}{2e_0^2} \quad \text{(by summing)}
\]

\[
\Rightarrow e_K^2 \leq \frac{2e_0^3}{2e_0 + aK} \Rightarrow e_K \leq \frac{\sqrt{2e_0^3/2}}{\sqrt{2e_0 + aK}}
\]

Where (a) follows from the fact that the sequence is decreasing.

We now apply these results to deduce Theorem 4.1:

Theorem C.5 (Theorem 4.1, formally stated). Assume that \( f(x) \) satisfies Assumptions 2 and 3. Let:

\[
K = \frac{2(\Delta_0 - \alpha \eta)^3}{(\alpha \rho^*)^2 \left( \frac{2\alpha \nu}{L} - 2\alpha \eta \right)}
\]

Assume that:

\[
\frac{\| \tilde{g}_k - g_k \|_2}{\| g_k \|_2} \leq \frac{\eta}{L} \quad \text{for } 0 \leq k \leq K - 1
\]

Then:

\[
f(x_K) - f_* \leq \frac{L^2}{2} \Delta^2 (1 + \rho^*)
\]

Proof. Observe that it will suffice to show that \( \Delta_K \leq \alpha (1 + \rho^*) \) as then by Assumption 2:

\[
f(x_K) - f_* \leq \frac{L}{2} \Delta_K \leq \frac{La^2 (1 + \rho^*)^2}{2}
\]

From Lemma C.1 we have:

\[
(\Delta_{k+1} - \alpha \eta)^2 \leq \Delta_k^2 - \frac{2\alpha \nu}{L} \Delta_k + \alpha^2
\]  
(21)

Let \( e_k = \Delta_k - \alpha \eta \), then one may rewrite (21) as:

\[
e_{k+1}^2 \leq e_k^2 - \left( \frac{2\alpha \nu}{L} - 2\alpha \eta \right) e_k + \alpha^2 \left( \eta^2 - \frac{2\eta \nu}{L} + 1 \right)
\]

\[= a \]
\[= b \]

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We shall now use Lemma C.4. First, observe that if \( e_k < 2b/a \) for any \( 0 \leq k \leq K - 1 \) then by substituting in the definitions of \( e_k, a \) and \( b \):

\[
\Delta_k - \alpha \eta < \frac{2\alpha^2 (\eta^2 - \frac{2\nu}{L} + 1)}{2\alpha \nu \frac{\nu}{L} - 2\alpha \eta} = \alpha \frac{\nu}{L} - \eta = \alpha \rho^*
\]

and hence \( \Delta_k \leq \alpha (\eta + \rho^*) \leq \alpha (1 + \rho^*) \). By Lemma C.3 we then have \( \Delta_K \leq \alpha (1 + \rho^*) \) and the theorem is proved. So, assume that \( e_k \geq 2b/a \) for \( 0 \leq k \leq K - 1 \). From Lemma C.4 we then obtain:

\[
\Delta_K - \alpha \eta \leq \frac{\sqrt{2} (\Delta_0 - \alpha \eta)^{3/2}}{\sqrt{2} (\Delta_0 - \alpha \eta) + (2\alpha \nu \frac{\nu}{L} - 2\alpha \eta) K} \leq \frac{\sqrt{2} (\Delta_0 - \alpha \eta)^{3/2}}{\sqrt{(2\alpha \nu \frac{\nu}{L} - 2\alpha \eta) K}} = \alpha \rho^*
\]

\( \square \)

Note that taking \( \alpha = \sqrt{\varepsilon} \) yields Theorem 4.1 as stated in the main article.

### D Proofs for Section 5

**Theorem D.1** (Theorem 5.1 part 1, precisely stated). Suppose that \( f(x) \) satisfies Assumptions 1–4 but that \( f(x) \) is only accessible through a comparison oracle with parameters \( \mu, \delta_0 = 0.5 \) and \( \kappa > 1 \). Select a target gradient accuracy, \( \eta \leq \nu / L \). Then, choose \( m, r \) and \( K \) according to:

\[
m = \frac{4C}{\mu^2 \eta^2} \left( \frac{10Ls}{\alpha^2 (1 + \rho^*)^2 \nu \frac{\nu}{L}} \right)^{2s-2} s \log(2d/s)
\]

\[
r = \frac{\alpha (1 + \rho^*) \nu \sqrt{d}}{Ls}
\]

\[
K = \frac{2 (\Delta_0 - \alpha \eta)^3}{(\alpha \rho^*)^2 (2\alpha \nu \frac{\nu}{L} - 2\alpha \eta)}
\]

Then SCOBO finds \( x_K \) such that:

\[
f(x_K) - f_* \leq \frac{L}{2} \alpha^2 (1 + \rho^*)^2
\]

using \( mK \) oracle queries, with probability at least:

\[
1 - 8K \exp \left( -c \eta^4 s \log(2d/s) \right)
\]

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Proof. As argued in the proof of Theorem C.5, if $\Delta_k \leq \alpha(1 + \rho^*)$ for any $0 \leq k < K - 1$ then we are done. So, assume that $\Delta_k > \alpha(1 + \rho^*)$ for all $k$. By restricted strong convexity (Assumption 3), this implies $\|g_k\| \geq \alpha(1 + \rho^*)$. Now appealing to Theorem B.6 for the prescribed choice of $m$, we obtain $\|\hat{g}_k - g_k\|_2 \leq \eta$ with probability at least $1 - 8 \exp\left(-c\eta^4 s \log(2d/s)\right)$. Use the union bound to conclude that

$$
\|\hat{g}_k - g_k\|_2 \leq \eta \quad \text{for} \quad 0 \leq k \leq K - 1
$$

with probability greater than $1 - 8K \exp\left(-c\eta^4 s \log(2d/s)\right)$. Conditional on this, we apply Theorem C.5 to obtain:

$$
f(x_K) - f_\star \leq \frac{L}{2} \alpha^2 (1 + \rho^*)^2
$$

as desired. \qed

**Theorem D.2** (Theorem 5.1 part 2, precisely stated). Suppose that $f(x)$ satisfies Assumptions 1–3 but now suppose that the oracle parameters are $\mu, \delta_0 < 0.5$ and $\kappa = 1$. For target gradient accuracy $\eta < \nu/L$, choose $m, r$ and $K$ according to:

$$
K = \frac{2(\Delta_0 - \alpha\eta)^3}{(\alpha \rho^*)^2 \left( \frac{2\alpha \nu}{L} - 2\alpha \eta \right)}
$$
$$
m = \frac{C}{\eta^2 \delta_0^2} s \log(2d/s)
$$
$$
r = \frac{\alpha(1 + \rho^*)}{L \sqrt{d}}
$$

Then SCOBO finds $x_K$ such that:

$$
f(x_K) - f_\star \leq \frac{L}{2} \alpha^2 (1 + \rho^*)^2
$$

using $mK$ oracle queries, with probability at least:

$$
1 - 8K \exp\left(-c\eta^4 s \log(2d/s)\right)
$$

The proof of Theorem D.2 is very similar to that of D.1, so we omit the details. In both cases, choosing $\alpha$ such that

$$
\varepsilon = \frac{L}{2} \alpha^2 (1 + \rho^*)^2
$$

then yields the results as stated in the main article.

Proof of Corollary 5.2. From Assumption 2, and the fact that $x_{k+1} = x_k - \alpha \hat{g}_k$, we get that:

$$
f(x_{k+1}) \leq f(x_k) - \alpha \|g_k\| \langle \hat{g}_k, g_k / \|g_k\| \rangle + \frac{L}{2} \alpha^2 \tag{a}
$$

$$
\leq f(x_k) - \alpha \|g_k\| \left(1 - \frac{\eta^2}{2}\right) + \frac{L\alpha^2}{2}
$$

where (a) follows from the fact that $\|\hat{g}_k - g_k / \|g_k\|\| \leq \eta$. Hence, we get that $f(x_{k+1}) \leq f(x_k)$ as long as:

$$
\|g_k\| \geq \frac{L\alpha}{2 - \eta^2} \tag{23}
$$
So, suppose that (23) does not hold. By smoothness (Assumption 2) and restricted strong convexity (Assumption 3) we obtain:

\[
\begin{align*}
    f(x_k) - f_* & \leq \frac{L}{2} \Delta_k^2 \\
    & \leq \frac{1}{\nu^2} \|g_k\|^2 \\
    & \leq L^2 \alpha^2 \\
    & \leq L^2 \alpha^2 \left( \frac{L^2}{\nu^2} \right) \left( \frac{1}{2 - \eta^2} \right)
\end{align*}
\]

Because \( \eta < 1 \) we have that \( \frac{1}{2 - \eta^2} \leq 1 \). Moreover, one can easily check that \( \rho^* \geq \rho^*|_{\eta=0} = L/\nu \).

Thus:

\[
\begin{align*}
    f(x_k) - f_* & \leq \frac{L}{2} \alpha^2 \left( 1 + \rho^* \right)^2 (a) = \varepsilon
\end{align*}
\]

where (a) follows from (22).

**Proof of Theorem 5.4.** Write \( C_f(x_{k+1}, x_k) = \xi_i \cdot \text{sign}(f(x_k) - f(x_{k+1})) \), where \( \xi_i \) is a binomial random variable:

\[
\xi_i = \begin{cases} 
+1 & \text{with prob. } 0.5 + \delta_0 \\
-1 & \text{with prob. } 0.5 - \delta_0 
\end{cases}
\]

Then \( MC^M(x_{k+1}, x_k) = X \text{sign}(f(x_k) - f(x_{k+1})) \) where \( X = \sum_{i=1}^{M} \xi_i \) is a (shifted) binomial random variable with parameters \( M, 0.5 + \delta_0 \). By the Chernoff bound:

\[
P[X < 0] \leq \exp \left( -\frac{\delta_0^2}{1 + 2\delta_0} M \right)
\]

If \( M = \frac{5 + 10\delta_0}{\delta_0^2} \) then one can easily verify that:

\[
\exp \left( -\frac{\delta_0^2}{1 + 2\delta_0} M \right) = \exp(-5) < 0.007
\]

Hence if \( C^M_f(x_{k+1}, x_k) < 0 \) we can conclude that \( \text{sign}(f(x_k) - f(x_{k+1})) = -1 \) with probability 0.993. By Lemma 5.2, with probability much greater than 0.997, this can only happen if \( f(x_{k+1}) \leq \varepsilon \). From the union bound we get that \( f(x_{k+1}) \leq \varepsilon \) with probability at least \( 1 - (0.007 + 0.003) = 0.99 \). 

**E Warm started inexact line search**

Extended from Section 6, we introduce a warm started inexact line search method. The vanilla inexact line search, *i.e.* Algorithm 5, starts its step size searching from \( \alpha = \alpha_{\text{def}} \) at every iteration of SCOBO. Although \( \alpha \) converges to the interval \( (\alpha_*/2, \alpha_*) \) exponentially, it may still waste unnecessary effort in the case that optimal step sizes do not change much between 2 iterations. Especially, when \( \alpha_*/\alpha_{\text{def}} \) is larger, a noticeable difference, in terms of the number of comparison oracles, can be observed if we do not restart the line search all over every iteration.

In the warm started inexact linear search, we use the estimated step size from last iteration of SCOBO as the warm started initialization for the new linear search. Since the warm started
Algorithm 6 Warm started inexact line search

1. **Input:** $x$: current point; $\hat{g}_k$: estimated gradient; $\alpha$: initial step size; $\alpha_{\text{def}}$: default step size; $M$: number of trials for comparison; $\omega$: confidence parameter; $\psi$: searching parameter.

2. if $C_f^M(x, x + \alpha \hat{g}_k) \leq -\omega$ then
   3. while $C_f^M(x + \alpha \hat{g}_k, x + \psi \alpha \hat{g}_k) \leq -\omega$ do
      4. $\alpha = \psi \alpha$
   5. end while
3. else if $C_f^M(x, x + \alpha \hat{g}_k) \geq \omega$ then
   4. while $C_f^M(x, x + \psi^{-1} \alpha \hat{g}_k) \geq \omega$ and $\alpha > \alpha_{\text{def}}$ do
      5. if $\psi^{-1} \alpha < \alpha_{\text{def}}$ then
         6. $\alpha = \alpha_{\text{def}}$
      7. else
         8. $\alpha = \psi^{-1} \alpha$
   9. end if
10. end while
11. end if
12. end if
13. end if
14. end if
15. **Output:** $\alpha$

Initialization can be larger than the optimal step size at the current iteration, we must also include the mechanism to reduce step size from the initial $\alpha$. We first use $M$-trial comparison oracles to determine if we want to extend or reduce the initial step size with confidence. We will keep the initial step size if the confidence is mediocre in both directions. Once decided, we keep extend/reduce the step size by a factor of $\psi$ until the stopping condition is satisfied.

Nevertheless, by Theorem 5.1, we can use a smaller minimum step size for targeting a better error bound, so the final convergence accuracy of SCOBO with warm started line search is better than the SCOBO with bigger fixed step size. Though SCOBO with vanilla line search (Alg. 5) can achieve similar accuracy by setting a very small default step size, but it can waste a lot queries on linear search if the default step size is too tiny. However, if the default step size is too large, then the accuracy of SCOBO become loose. Overall, we claim the warm started line search with small default step size has both good convergence performance and query efficiency. We summarize the warm started inexact line search as Algorithm 6.