A NOTE ON GEOGRAPHY OF BILINEARIZED LEGENDRIAN CONTACT HOMOLOGY FOR DISCONNECTED LEGENDRIAN SUBMANIFOLDS

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Abstract. In this short note, we provide a criterion for DGA-homotopy of augmentations of Chekanov-Eliashberg algebra of disconnected Legendrian submanifolds. We apply the criterion to obtain the extension of geography results of Bourgeois and Galant concerning bilinearized Legendrian contact homology to the case of disconnected Legendrian submanifolds.

1. Introduction

For $M$ an $n$-dimensional smooth manifold we denote by $J^1(M) = T^*M \times \mathbb{R}$ its one-jet bundle. We endow it with a canonical contact structure given by the kernel of the co-oriented one-form $dz - \eta$, where $\eta$ is the Liouville one form on $T^*M$. To study the Legendrian isotopy classes of such Legendrians we can define the Chekanov-Eliashberg algebra $(\mathcal{A}(\Lambda), \partial)$ for generic closed Legendrian submanifold $\Lambda$ of $J^1(M)$ (see [1], [2]). We will consider Legendrian submanifolds whose Maslov class vanishes to obtain $\mathbb{Z}$-graded differential graded algebra. Where differential $\partial$ counts rigid pseudoholomorphic disks in the symplectization of $J^1(M)$.

The homology of $\mathcal{A}(\Lambda)$ is hard to work with and so we (bi)linearize the differential using augmentations $\varepsilon : (\mathcal{A}, \partial) \to (\mathbb{Z}_2, 0)$ (see [1], [2] for linearization and [3] for bilinearization). We denote by $LCH^\varepsilon(\Lambda)$ and $LCH_1^{\varepsilon_1, \varepsilon_2}(\Lambda)$ the linearized and bilinearized Legendrian contact homology respectively.

In this context, the question of the DGA-homotopy of augmentations of the Chekanov-Eliashberg algebra naturally appears. In [3] Bourgeois and Chantraine proved that the cardinality of the set

$$\mathcal{E}(\Lambda) = \{[\varepsilon] \sim | \varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0) \text{ is an augmentation of } \mathcal{A}(\Lambda)\}$$

of DGA-homotopy classes of augmentations is a Legendrian isotopy invariant. It is not a simple task to decide whether two given augmentations $\varepsilon_1$ and $\varepsilon_2$ belong to the same the DGA-homotopy class. Therefore, it is surprising that there exist criteria that can be used to distinguish those classes.

The first hints of the existence of such a criterion can be tracked to work on duality long exact sequence of Ekholm, Etnyre and Sabloff (see [4]). Combining those with results concerning bilinearized Legendrian contact homology of Bourgeois and Chantraine (see [3]) leads to a necessary condition that reads:

If $\varepsilon_1$ and $\varepsilon_2$ are DGA-homotopic, then $\tau_0$ vanishes.

Here $\tau_0 : LCH_0^{\varepsilon_1, \varepsilon_2}(\Lambda) \to H_0(\Lambda)$ is the map from the duality long exact sequence relating bilinearized Legendrian contact homology and the Morse homology of a connected Legendrian submanifold. This condition was found to be also sufficient by Bourgeois and Galant in [5] for connected Legendrian submanifolds.

For disconnected Legendrian submanifolds this condition fails. One can easily find a Legendrian link with non-vanishing $\tau_0$ arising from the duality sequence for two DGA-homotopic augmentations. This happens because the condition does not pass to the chain level anymore.

However, in the connected case, one can show using the duality that $\tau_0$ vanishes if and only if $\tau_n : LCH_n^{\varepsilon_1, \varepsilon_2}(\Lambda) \to H_n(\Lambda)$ is surjective, in particular, if $\tau_n$ hits the fundamental class of the Legendrian since it is connected. Therefore, we restate and prove the condition so that it holds even in the disconnected case:
**Theorem 1.1.** Let $M$ be an $n$-dimensional smooth manifold and $\Lambda$ for be a closed Legendrian submanifold of $(J^1(M), dz - \eta)$ with vanishing Maslov class. Denote by $[\Lambda]$ its fundamental class. Let $\varepsilon_1, \varepsilon_2$ be two augmentations of the Chekanov-Eliashberg algebra $A(\Lambda)$ over $\mathbb{Z}_2$. Then the following holds:

\[(1.1) \quad \varepsilon_1 \text{ and } \varepsilon_2 \text{ are DGA homotopic } \iff [\Lambda] \text{ is an element of the image of } \tau_n.\]

There is one possible interpretation of our result in the context of exact Lagrangian fillings of Legendrian submanifolds due to Ekholm, Honda, Kálmán and Karlsson

**Theorem (6, 7).** An exact Lagrangian filling $L$ of a closed Legendrian submanifold $\Lambda$ induces an augmentation $\varepsilon_L : (A(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$. If $L_1$ and $L_2$ are two exact Lagrangian fillings of $\Lambda$ that are isotopic through exact Lagrangian fillings, then $\varepsilon_{L_1} \sim \varepsilon_{L_2}$.

Therefore, we immediately obtain the following necessary condition.

**Corollary 1.2.** If $L_1$ and $L_2$ are two exact Lagrangian fillings of $\Lambda$ that are isotopic through exact Lagrangian fillings, then $[\Lambda]$ is an element of the image of $\tau_{-n}$.

Nevertheless, the main application of our result will lie in the geography of bilinearized Legendrian contact homology for disconnected Legendrian submanifolds. In other words, we are asking about what polynomials can be attained as Poincaré polynomials of bilinearized Legendrian contact homology, that is

$$ P_{\Lambda, \varepsilon_1, \varepsilon_2}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} LCH_k^{\varepsilon_1, \varepsilon_2}(\Lambda) \ t^k, $$

and vice versa, which Legendrian submanifolds realize a particular admissible polynomial. This question was fully answered for linearized Legendrian contact homology by Bourgeois, Sabloff and Traynor in [8] as it was observed in [3]. And because of the results of Bourgeois and Chantraine [3] it is enough to describe the geography when $\varepsilon_1 \not\sim \varepsilon_2$. That was done in [3] by Bourgeois and Galant for the connected case. In this note, we extend the results to the disconnected case. More specifically, we define a version of $\mathrm{bLCH}$-admissible polynomials for disconnected Legendrian submanifolds (links in dimension three) called $\mathrm{lbLCH}$-admissible polynomials (see Definition 4.1) and we prove:

**Theorem 1.3.** Let $\Lambda$ be a Legendrian submanifold of $J^1(M)$, $\dim M = n$, with vanishing Maslov class, that consists of $r$ $n$-dimensional components for any natural number $r$, and $\varepsilon_1, \varepsilon_2$ be two DGA-non-homotopic augmentations. Then $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ is a $\mathrm{lbLCH}$-admissible polynomial.

and to complete the geography also the other direction:

**Theorem 1.4.** Let $M$ be a smooth manifold of dimension $n$. If $P \in \mathbb{N}_0[t, t^{-1}]$ is any $\mathrm{lbLCH}$-admissible polynomial, then there is a Legendrian submanifold of $J^1(M)$ whose connected components are connected Legendrian submanifolds and two DGA-non-homotopic augmentations $\varepsilon_1, \varepsilon_2$ of its Chekanov-Eliashberg algebra so that $P_{\Lambda, \varepsilon_1, \varepsilon_2} = P$.

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2. Background

2.1. (Bi)linearization.

Definition 2.1. Let \( \varepsilon_1, \varepsilon_2 \) be augmentations of \( \mathcal{A}(\Lambda) \). A linear map \( K : \mathcal{A}(\Lambda) \to \mathbb{Z}_2 \) satisfying \( K(ab) = \varepsilon_1(a)K(b) + K(a)\varepsilon_2(b) \) for all \( a, b \in \mathcal{A}(\Lambda) \) is called a \((\varepsilon_1, \varepsilon_2)\)-antiderivation. If it exists and \( \varepsilon_1 - \varepsilon_2 = K \circ \partial \), then the augmentations are said to be DGA-homotopic, notation \( \varepsilon_1 \sim \varepsilon_2 \).

Note that the relation \( \sim \) above is an equivalence and that each augmentation is uniquely determined by its values on Reeb chords of \( \Lambda \) and so for compact Legendrian \( \Lambda \) we have a finite set

\[
\mathcal{E}(\Lambda) = \{[\varepsilon]_\sim \mid \varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0) \text{ is an augmentation of } \mathcal{A}(\Lambda)\}.
\]

Theorem 2.2 (Theorem 1.3. in [3]). Let \( \{\Lambda_t\}_{t \in [0,1]} \) is a Legendrian isotopy, then we have a bijection of \( \mathcal{E}(\Lambda_0) \) and \( \mathcal{E}(\Lambda_1) \).

Theorem 2.3 (Theorem 1.4, [3]). Let \( \Lambda \) be a compact generic Legendrian submanifold of \( J^1(M) \) with vanishing Maslov number, then if \( \varepsilon_1, \varepsilon_2 \) are two DGA-homotopic augmentations of \( \mathcal{A}(\Lambda) \), then \( LCH^{\varepsilon_1}(\Lambda) \cong LCH^{\varepsilon_2}(\Lambda) \).

Therefore, the cardinality of the set \( \mathcal{E}(\Lambda) \) is a Legendrian isotopy invariant. However, for \( \{\Lambda_t\}_{t \in [0,1]} \) a Legendrian isotopy and \( f : \mathcal{E}(\Lambda_0) \to \mathcal{E}(\Lambda_1) \) be the bijection from Theorem 2.2 then it is true that \( LCH^{\varepsilon_1}(\Lambda_0) \cong LCH^{f(\varepsilon)}(\Lambda_1) \).

Theorem 2.4 (Theorem 1.2, [3]). Let \( \Lambda \) be a compact generic Legendrian submanifold of \( J^1(M) \). Consider the set

\[
\mathcal{H}(\Lambda) = \bigcup_{[\varepsilon]_\sim \in \mathcal{E}(\Lambda)} \{LCH^\varepsilon(\Lambda)\}.
\]

Let \( \{\Lambda_t\}_{t \in [0,1]} \) is a Legendrian isotopy, then the sets \( \mathcal{H}(\Lambda_0) \) and \( \mathcal{H}(\Lambda_1) \) coincide.

Analogously to linearized Legendrian contact homology we have the following.

Theorem 2.5 (Theorem 1.2, [3]). Let \( \Lambda \) be a compact generic Legendrian submanifold of \( J^1(M) \) with vanishing Maslov number, then if \( \varepsilon_1, \varepsilon_2, \varepsilon \) are augmentations of \( \mathcal{A}(\Lambda) \), where \( \varepsilon_1 \sim \varepsilon_2 \), then \( LCH^{\varepsilon_1, \varepsilon_2}(\Lambda) \cong LCH^{\varepsilon_2, \varepsilon_1}(\Lambda) \) and \( LCH^{\varepsilon, \varepsilon_2}(\Lambda) \cong LCH^{\varepsilon_2, \varepsilon}(\Lambda) \).

In particular,

Corollary 2.6. Let \( \Lambda \) be a compact generic Legendrian submanifold of \( J^1(M) \) with vanishing Maslov number, then if \( \varepsilon_1, \varepsilon_2 \) are augmentations of \( \mathcal{A}(\Lambda) \), where \( \varepsilon_1 \sim \varepsilon_2 \), then \( LCH^{\varepsilon_1, \varepsilon_2}(\Lambda) \cong LCH^{\varepsilon_2, \varepsilon_1}(\Lambda) = LCH^{\varepsilon_2, \varepsilon}(\Lambda) \).

Theorem 2.7 (Theorem 1.2, [3]). Let \( \Lambda \) be a compact generic Legendrian submanifold of \( J^1(M) \). Consider the set

\[
\mathcal{H}^b(\Lambda) = \bigcup_{([\varepsilon_1]_\sim, [\varepsilon_2]_\sim) \in \mathcal{E}(\Lambda) \times \mathcal{E}(\Lambda)} \{LCH^{\varepsilon_1, \varepsilon_2}(\Lambda)\}.
\]

Let \( \{\Lambda_t\}_{t \in [0,1]} \) is a Legendrian isotopy, then the sets \( \mathcal{H}^b(\Lambda_0) \) and \( \mathcal{H}^b(\Lambda_1) \) coincide.

In view of Theorems 2.4 and Corollary 2.6 we see that \( \mathcal{H}(\Lambda) \subset \mathcal{H}^b(\Lambda) \). And so bilinearized Legendrian contact homology is a stronger invariant of Legendrian isotopy which encodes the non-commutativity of the Chekanov-Eliashberg algebra that is lost in the process of linearization.

In general, it is not an easy task to determine the DGA-homotopy class of a augmentation of \( \mathcal{A}(\Lambda) \) computationally and thus determining the cardinality of \( \mathcal{E}(\Lambda) \) is an interesting problem. In the standard 3-dimensional space the cardinality of \( \mathcal{E}(\Lambda) \) was studied by Ng, Rutherford, Shende, Sivek in [9].

It is quite surprising that there is a criterion for DGA-homotopy of augmentations in any dimension. This first appeared in work of Bourgeois and Galant (see [5]).
Theorem 2.8 (Proposition 3.3. in [5]). Let $\Lambda$ be a connected compact generic Legendrian submanifold of $J^1(M)$ has dimension with vanishing Maslov number, where $M$ has dimension $n$. Then if $\varepsilon_1, \varepsilon_2$ are augmentations of $A(\Lambda)$ it holds that

$$\dim_{\mathbb{Z}_2} LCH^{\varepsilon_2, \varepsilon_1}_{n}(\Lambda) - \dim_{\mathbb{Z}_2} LCH^{\varepsilon_1, \varepsilon_2}_{-1}(\Lambda) = \begin{cases} 0, & \varepsilon_1 \not\sim \varepsilon_2, \\ 1, & \varepsilon_1 \sim \varepsilon_2. \end{cases}$$

Equivalently,

$$\varepsilon_1 \sim \varepsilon_2 \iff \tau_0 = 0.$$
And so

\[ \sigma_k(q) = \sum_{\dim M(\epsilon c^{-1})/\mathbb{R} = \text{index}_j(q) - 1} \#_2 \mathcal{M}_{c,\epsilon, q} \sum_{j=1}^{k} \epsilon_1(c_{i_1}) \ldots \epsilon_1(c_{i_{n-1}}) c \epsilon_2(c_{i_n}) \ldots \epsilon_2(c_{i_k}). \]

The main difference between the linearized and bilinearized case is that the one has to pay attention to the ordering of augmentations in the duality formula from which the name of the sequence stems. More precisely, consider \( \epsilon_1, \epsilon_2 \) two augmentations of \( \mathcal{A}(\Lambda) \) of some \( \Lambda = \prod_{j=1}^{r} \Lambda_j \) as above. We have two dual sequences: first for the ordering \( (\epsilon_1, \epsilon_2) \), that we call positive:

\[ \cdots \rightarrow \text{LCH}^{n-k-1}_{\epsilon_1, \epsilon_2}(\Lambda) \rightarrow \text{LCH}^{\epsilon_1, \epsilon_2}_k(\Lambda) \overset{\tau_{+,-}}{\rightarrow} H_k(\Lambda) \overset{\sigma_{+,n,-}}{\rightarrow} \text{LCH}^{n-k}_\epsilon(\Lambda_{\epsilon_1}, \epsilon_2) \rightarrow \cdots \]

second for the ordering \( (\epsilon_2, \epsilon_1) \), that we call negative:

\[ \cdots \rightarrow \text{LCH}^{n-k-1}_{\epsilon_1, \epsilon_2}(\Lambda) \rightarrow \text{LCH}^{\epsilon_1, \epsilon_2}_k(\Lambda) \overset{\tau_{-,-}}{\rightarrow} H_k(\Lambda) \overset{\sigma_{-,n,-}}{\rightarrow} \text{LCH}^{n-k}_{\epsilon_2, \epsilon_1}(\Lambda) \rightarrow \cdots \]

Recall that since all components \( \Lambda_j \) for \( j = 1, \ldots, r \) of our Legendrian \( \Lambda \) are closed we have an intersection pairing \( \bullet : H_k(\Lambda_j) \otimes H_{n-k}(\Lambda_j) \rightarrow \mathbb{Z}_2 \) for each \( k \in \mathbb{Z} \). Now define the intersection pairing on \( \Lambda \) for \( k \in \mathbb{Z} \) and for \( c = (c_1, \ldots, c_r) \in H_k(\Lambda) = H_k(\Lambda_1) \oplus \cdots \oplus H_k(\Lambda_r) \) and \( d = (d_1, \ldots, d_r) \in H_{n-k}(\Lambda) = H_{n-k}(\Lambda_1) \oplus \cdots \oplus H_{n-k}(\Lambda_r) \) to be

\[ c \bullet d = \sum_{j=1}^{r} c_j \bullet d_j. \]

For precise definition of the pairing \( \bullet \) on the components of the disconnected Legendrian submanifold \( \Lambda \) see Section 3.3.3. of [3].

Define a pairing \( \langle \cdot, \cdot \rangle : C(\mathcal{R}(\Lambda))^* \otimes \mathbb{Z}_2 C(\mathcal{R}(\Lambda)) \rightarrow \mathbb{Z}_2 \) on generators \( \{c_i : i \in \{1, \ldots, \#(\mathcal{R}(\Lambda))\} \} \) as follows

\[ \langle c_i^*, c_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \]

Note that this precisely corresponds to evaluation \( \langle c_i^*, c_j \rangle = c_j^*(c_i) \). In particular, if \( |c_i^*| \neq |c_j| \), then \( \langle c_i^*, c_j \rangle = 0 \).

Consider \( \partial^{\epsilon_1, \epsilon_2} \) the differential on \( C(\mathcal{R}(\Lambda)) \) and \( \mu^{1}_{\epsilon_2, \epsilon_1} \) on \( C(\mathcal{R}(\Lambda))^* \) the dual differential to \( \partial^{\epsilon_2, \epsilon_1} \), that is

\[ \langle \mu^{1}_{\epsilon_2, \epsilon_1}(a), b \rangle = \langle a, \partial(b) \rangle, \]

for \( a \in C(\mathcal{R}(\Lambda)) \) and \( b \in \mu^{1}_{\epsilon_2, \epsilon_1} \). Moreover, consider the tensor product \( W = C(\mathcal{R}(\Lambda))^* \otimes \mathbb{Z}_2 C(\mathcal{R}(\Lambda)) \) endowed with the standard differential

\[ d_W(c^* \otimes d) = \mu^{1}_{\epsilon_2, \epsilon_1}(c^*) \otimes d + c^* \otimes \partial^{\epsilon_2, \epsilon_1}(d) \]

for \( c^* \in C(\mathcal{R}(\Lambda))^* \) and \( d \in C(\mathcal{R}(\Lambda)) \). Using the generalized Künneth formula we obtain that

\[ H_\bullet(W, d_W) \cong \bigoplus_{k+l=\bullet} H_k(\mathcal{R}(\Lambda))^* \otimes \mathbb{Z}_2 H_l(\mathcal{R}(\Lambda), \partial^{\epsilon_2, \epsilon_1}), \]

where torsion does not occur for we work over the field \( \mathbb{Z}_2 \).

Define \( F : W \rightarrow \mathbb{Z}_2 \) by \( F(a) = \sum_i \langle c_i, d_i \rangle \) for \( a = \sum c_i \otimes d_i \in W \), then \( F \circ d_W = 0 \) since the differentials are dual to each other. Therefore \( F \) descends to homology of \( (W, d_W) \) and so does the pairing.

We will denote the pairing that we have just constructed as

\[ \langle \cdot, \cdot \rangle_+ : \text{LCH}^{\epsilon_2, \epsilon_1}(\Lambda) \otimes \mathbb{Z}_2 \text{LCH}^{\epsilon_1, \epsilon_2}(\Lambda) \rightarrow \mathbb{Z}_2, \]

and for the opposite order of augmentations we will write

\[ \langle \cdot, \cdot \rangle_- : \text{LCH}^{\epsilon_1, \epsilon_2}(\Lambda) \otimes \mathbb{Z}_2 \text{LCH}^{\epsilon_2, \epsilon_1}(\Lambda) \rightarrow \mathbb{Z}_2. \]

**Proposition 2.10.** Then for any non-zero class \( [a] \in \text{LCH}^{\epsilon_1, \epsilon_2} \), there is a Reeb chord \( c \in \mathcal{R}(\Lambda) \) so that \( \langle [a], [c] \rangle_- \neq 0 \), where the pairing

\[ \langle \cdot, \cdot \rangle_- : \text{LCH}^{k}_{\epsilon_1, \epsilon_2}(\Lambda) \otimes \mathbb{Z}_2 \text{LCH}^{\epsilon_2, \epsilon_1}(\Lambda) \rightarrow \mathbb{Z}_2 \]

is as above.
Proof. Let \( a = \sum_{i=1}^{m} c_i^* + db \) be a representative of the class \([a]\), where \( c_i \in \mathcal{R}(\Lambda) \) are all distinct, non-exact, and \( |c_i^*| = k \), and moreover, \( b \in C_{k+1}^*(\Lambda) \). Now because the class \([a]\) is non-zero then \( m > 0 \). If we had that \( \langle [a], [c] \rangle = 0 \) for all \( c \in \mathcal{R}(\Lambda) \) of grading \( k \), then

\[
0 = \langle [a], [c] \rangle = \langle a, c \rangle = \left( \sum_{i=1}^{m} c_i^* + db, c \right) = \left( \sum_{i=1}^{m} c_i^*, c \right) = \sum_{i=1}^{m} c_i^*(c),
\]

therefore, \( m = 0 \) which is a contradiction. \( \qed \)

We have an analogue of Proposition 3.9 in [4].

**Proposition 2.11.** The pairs of maps \( \tau_{+,k} \) and \( \sigma_{-,k} \), and \( \tau_{-,k} \) and \( \sigma_{+,k} \) are adjoint in the following sense:

Let us have \( c \in H_k(\Lambda) \) and a chord \( q \) of grading \( n - k \):

\[
\langle \sigma_{-,n-k}(c), [q] \rangle_+ = c \cdot \tau_{+,n-k}([q]),
\]

\[
\langle \sigma_{+,n-k}(c), [q] \rangle_- = c \cdot \tau_{-,n-k}([q]),
\]

where \( \bullet : H_k(\Lambda) \otimes H_{n-k}(L) \to \mathbb{Z}_2 \) is the intersection pairing, and pairings

\[
\langle \cdot, \cdot \rangle_- : \text{LCH}^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \otimes_{\mathbb{Z}_2} \text{LCH}^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to \mathbb{Z}_2,
\]

\[
\langle \cdot, \cdot \rangle_+ : \text{LCH}^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \otimes_{\mathbb{Z}_2} \text{LCH}^{n-k}_{\varepsilon_2,\varepsilon_1}(\Lambda) \to \mathbb{Z}_2
\]

are as above.

**Proof.** Let us prove the first equation the other case is analogous.

The right handed size counts holomorphic disks with \( q \) mixed positive puncture and \( c \) negative Morse puncture and with possibly other augmented negative punctures between \( q \) and \( c \) with \( \varepsilon_1 \) and between \( c \) and \( q \) with \( \varepsilon_2 \). Now the bijective correspondence from Theorem 3.6 form [4] implies that this disks corresponds to the lifted generalized disk with \( \gamma \) a negative gradient flow line of the perturbing function \( f \) ending at \( c \) and connecting it to the boundary of the disk.

To pass to the right side that is from homology to cohomology we change the sign of the perturbing function, that is \( f = -f \) on the left side. Now the orientation of \( \gamma \) is reversed and so \( c \) is a positive Morse puncture and \( q \) is a negative mixed puncture. Since the order of augmentations is reversed , the disk contributes to \( \sigma_{-,n-k} \). \( \qed \)

**Figure 2.** Effect of passing from homology to cohomology on a generalized lifted disk.

### 2.3. The map \( \tau_n \)

Let us focus on the \( n \)-th level of the duality long exact sequence. Consider a Reeb chord \( a \) with grading \( n \), than the differential \( \partial \) counts \( \langle u, \gamma \rangle \) lifted generalized disks where \( u \) is a pseudo-holomorphic curve and \( \gamma \) is the gradient flow line from a generic point of the boundary to \( m_j \) a maximum of the Morse function \( f \) on \( \Lambda_j \) the corresponding component of \( \Lambda \), this in particular implies that the beginning point of \( \gamma \) on the boundary must be the maximum \( m_j \). For \( l \in \{1, \ldots, m+1\} \) denote this moduli space by

\[
\mathcal{M}_{a; b, p}(a; b_1, \ldots, b_{l-1}, p_l, b_{l}, \ldots, b_m),
\]
where $p_i$ lies on the boundary component of the punctured disk which was mapped to $\Lambda_j$ where $j = j_i$ assuming $b_i$ is a Reeb chord from the connected component $\Lambda_{j_i}$ to the connected component $\Lambda_{j_l}$ of the disconnected Legendrian submanifold $L$, for all $l \in \{1, \ldots, m\}$ if $l - 1 = m$ then $j = j_m$.

We denote by $[\Lambda_j]$ the homology class which $m_j$ represents. And so by the dimension formula (\[10\], Section 3.3.1)

$$0 = \dim(u, \gamma) = \dim \mathcal{M}(\alpha; b_1, \ldots, b_m) + 1 - \text{Index}_f(p_l) = |\alpha| - |\beta| - 1 + 1 - n$$

and the fact that $|\alpha| = n$ we get that $|\beta| = 0$.

![Figure 3. Pointed disc.](image)

This gives us the description of action the map $\tau_n$ on $a$ that is

\begin{equation}
\tau_n(a) = \sum_{|b|=0} \#_2 \mathcal{M}_{a,b,p} \sum_{j=1}^m \varepsilon_1(b_1) \cdots \varepsilon_1(b_{j-1})[\Lambda_{i_{j-1}}] \varepsilon_2(b_j) \cdots \varepsilon_2(b_m)
\end{equation}

2.4. Effect of Legendrian ambient 0-surgery. Let $r \in \mathbb{N}$. Consider disconnected Legendrian submanifold $\Lambda = \bigsqcup_{j=1}^r \Lambda_j$. Denote by $\Lambda_{S,1}$ the submanifold resulting from the Legendrian ambient 0-surgery by connecting $\Lambda_1$ and $\Lambda_2$. Now inductively $\Lambda_{S,k}$ denotes the submanifold resulting from the Legendrian ambient 0-surgery by connecting $\Lambda_{S,k-1}$ and $\Lambda_k$ for $k = 3, \ldots, r$.

Now we will restrict the setting to the first iteration of the 0-surgery for simplicity. The effect of the ambient Legendrian 0-surgery using the embedded sphere $S^0$ into $\Lambda_1$ and $\Lambda_j$ for some $i, j \in \{1, \ldots, r\}$ on the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda)$ was described by Dimitroglou Rizell in Section 1 of [10], more specifically, for embedded spheres of all dimensions from 0 to $n - 1$. Denote by $\Lambda_S$ the Legendrian submanifold obtained by performing the surgery. The situation is as follows:

The algebra $\mathcal{A}(\Lambda_S)$ is isomorphic to the algebra $\mathcal{A}(\Lambda, S)$ defined as the free product of the algebra $\mathcal{A}(\Lambda)$ and the line $\mathbb{Z}_2(s)$ of a formal generator $s$ which corresponds to $c_S$ a new Reeb chord of $\Lambda_S$ that is of degree $|c_S| = n - 1$. Note that this means that for any $\varepsilon$ augmentation of $\mathcal{A}(\Lambda_S)$ we have that $\varepsilon(c_S) = 0$. The differential on $\mathcal{A}(\Lambda, S)$ is to be denoted by $\partial_S$ and it decomposes into $\partial_S = \partial + h$ on generators. Here $\partial$ is the differential of $\mathcal{A}(\Lambda)$ and

\begin{equation}
h(a) = \sum_{|a|-|b|-|s|=1} |\mathcal{M}_{a:b,w}(a; b_1, \ldots, b_m)| s^{w_1} b_1 \cdots b_m s^{w_{m+1}}
\end{equation}

counts number of holomorphic disks with boundary on $L$ and with $w_i$ marked points on the corresponding part of boundary of the disk that is mapped to one of the points that are in the image of $S^0$. Here $|s| = (w_1 + \ldots + w_{m+1})(n - 1)$. For more details see ([10], Section 6).

Consider $\varepsilon_1, \varepsilon_2$ two augmentations of $\mathcal{A}(\Lambda)$ then $\varepsilon_1^{\Gamma}, \varepsilon_2^{\Gamma}$ are two augmentations of $\mathcal{A}(\Lambda, S)$ induced in the as the pull-back. More specifically, they both vanish on the element $s$ and coincide with the original augmentations of original Reeb chords. Now $\partial_S^{\varepsilon_1, \varepsilon_2}$ the bilinearized differential decomposes $\partial_S^{\varepsilon_1, \varepsilon_2} = \partial^{\varepsilon_1, \varepsilon_2} + h^{\varepsilon_1, \varepsilon_2}$ on generators.

If one of $w_j > 1$ or if there are two $j \neq j'$ such that $w_j \neq 0$ and $w_{j'} \neq 0$, then the corresponding disk contributes by zero to the bilinearized differential. It must hold that there is exactly one
\[ j \in \{1 \ldots, m + 1\} \text{ such that } w_j = 1 \text{ for a disk to contribute to } h^{ε_1,ε_2}. \text{ Otherwise, it has already contributed to the usual bilinearized differential } \partial^{ε_1,ε_2}. \text{ Let us denote by } \rho_ε = h^{ε_1,ε_2}. \]

\[
\rho_ε = \sum_{|a| - |b| = n} |M_{a; b, w}(a; b_1, \ldots, b_m)| \sum_{i=1}^{m} \varepsilon_1(b_1) \ldots \varepsilon_1(b_{i-1}) s \varepsilon_2(b_i) \ldots \varepsilon_2(b_m). \]

Let \(ε_1, ε_2\) be two augmentations of \(A(Λ)\) over \(\mathbb{Z}_2\). Denote by \(ε_1^S, ε_2^S\) the induced augmentations of \(A(Λ, S)\).

The inclusion of the line \(\mathbb{Z}_2\langle s \rangle_{n-1}\) into \(C_*(Λ, S)\) makes it into a subcomplex and the fact that \(C_*(Λ, S)/\mathbb{Z}_2\langle s \rangle\) makes it into a subcomplex and the fact that \(C_*(Λ, S)/\mathbb{Z}_2\langle s \rangle \cong C_*(Λ)\) now yields a short exact sequence of complexes

\[
0 \rightarrow (\mathbb{Z}_2\langle s \rangle, s, \partial_1^{ε_1,ε_2}) \rightarrow (C_*(Λ, S), \partial_1^{ε_1,ε_2}) \rightarrow (C_*(Λ), \partial^{ε_1,ε_2}) \rightarrow 0
\]

this induces the following long exact sequence in homology

\[
\ldots \rightarrow LCH_0^{ε_1,ε_2}(Λ) \rightarrow LCH_1^{ε_1,ε_2}(Λ) \rightarrow \mathbb{Z}_2\langle s \rangle_{n-1} \rightarrow LCH_1^{ε_1,ε_2}(Λ) \rightarrow \ldots
\]

Since \(\mathbb{Z}_2\langle s \rangle_{n-1} = 0\) if \(i \neq n\) we obtain the isomorphism:

\[
0 \rightarrow LCH_0^{ε_1,ε_2}(Λ) \rightarrow LCH_1^{ε_1,ε_2}(Λ) \rightarrow 0.
\]

3. Proof of Theorem 1.1

First, let us say why we formulate Theorem 1.1 using the map \(τ_n\) and not a map \(τ_0\) like it is done in Theorem 2.8, that is,

\[
(3.1) \quad \varepsilon_1 \sim ε_2 \iff τ_0 = 0.
\]

To prove (3.1) one has to show that it holds in homology that

\[
ε_1(·) - ε_2(·) = τ_0(·) \bullet [Λ],
\]

where the pairing with the fundamental class \(· \bullet [Λ] : H_0(Λ) \rightarrow \mathbb{Z}_2\) is an vector space automorphism of \(\mathbb{Z}_2\). The isomorphism is purely formal character in the connected case, however, in the disconnected case, this is no longer and isomorphism and formulation (3.1) fails even in the following basic example of a disconnected Legendrian submanifold.

As in [5] let us denote by \(Λ^{(2)}\) the standard \(n\)-dimensional Legendrian Hopf link in \(J^1(\mathbb{R}^n)\) so that the Maslov potential on the upper component is the Maslov potential of the lower component enlarged by 1.

The Corollary 4.7 in [5] then implies that there are two augmentations \(ε_L\) and \(ε_R\) of Chekanov-Eliashberg algebra of \(Λ^{(2)}\) so that \(ε_L(m_{12}) = 1\) and \(ε_R(m_{12}) = 0\) for a chord \(m_{12}\) and they vanish otherwise.

Therefore, posing \(ε_1 = ε_R, ε_2 = ε_R,\) the map \(τ_0\) sends each Reeb chord to zero except for the chord \(m_{12}\) which is send to the diagonal of \(H_0(Λ) \oplus H_0(Λ) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\). And so, in this case \(τ_0 \neq 0\) even though \(ε_1 \sim ε_2\).

From now on, Λ denotes a disconnected Legendrian submanifold with \(r\) connected closed components and \(Λ_S\) a connected Legendrian submanifold obtained by performing \(r - 1\) Legendrian ambient surgeries on Λ. And by \(ε_1^S, ε_2^S\) denote the augmentations induced by this surgery so that
they vanish on the formal generators added to the Chekanov-Eliashberg algebra and coincide with $\varepsilon_1$ and $\varepsilon_2$ otherwise. This implies that on chords of $\Lambda$ are decorated with the same number by both $\varepsilon_i^S$ and $\varepsilon_i$ for both $i = 1, 2$. Considering Lemma 3.1 from [5] we obtain the following proposition

**Proposition 3.1.** $\varepsilon_1 \sim \varepsilon_2$ if and only if $\varepsilon_1^S \sim \varepsilon_2^S$

**Lemma 3.2.** Consider the following diagram:

$$
\begin{array}{ccc}
LCH_0^{\varepsilon_1, \varepsilon_2^S} (\Lambda_S) & \xrightarrow{\pi^+} & LCH_0^{\varepsilon_1, \varepsilon_2^S} (\Lambda) \\
\end{array}
\begin{array}{ccc}
\uparrow^{\gamma} & & \uparrow^{\tau_{+0}} \\
H_0(\Lambda_S) & & H_0(\Lambda) \\
\end{array}
\begin{array}{ccc}
\downarrow^{\alpha} & & \downarrow^{\alpha} \\
\mathbb{Z}_2 & & \mathbb{Z}_2 \\
\end{array}
$$

(3.2)

where $\alpha : H_0(\Lambda) \to H_0(\Lambda)$ is defined as follows. Denote by $[\ast_{\Lambda_j}]$ for $j = 1, \ldots, r$ the classes of points in $H_0(\Lambda)$ that represent distinct components $\Lambda_j$ of $\Lambda$, and $[\ast_{\Lambda_S}] \in H_0(\Lambda_S)$ a class of point in $\Lambda_S$, that is connected. Moreover, consider the map $\gamma : H_0(\Lambda_S) \to \mathbb{Z}_2$ defined as $a[\ast_{\Lambda_S}] \to a$ for any $a \in \mathbb{Z}_2$, and let us denote by $\alpha : H_0(\Lambda) \to \mathbb{Z}_2$ the composition $\gamma \circ \bar{\alpha}$, i.e. $\bar{\alpha}(\bigoplus_{j=1}^r a_j[\ast_{\Lambda_j}]) \mapsto \left(\sum_{j=1}^r a_j\right)[\ast_{\Lambda_S}]$ for any $a_j \in \mathbb{Z}_2$. The map $\pi^+$ denotes the composition of isomorphisms in level zero of the surgery long exact sequence. Then the diagram above commutes.

**Proof.** Let $n_j$ be minimum of the perturbing Morse function on the component $\Lambda_j$ if $q$ is a chord of degree 0 that starts on $\Lambda_i$ and ends on $\Lambda_j$, thanks to the rigidity of the formal disk the starting point of a generalized lifted disk must map either to the starting point of $q$ or to the ending point of it. In the first case, the disk contributes with $\varepsilon_2(q)[\ast_{\Lambda_j}]$ to $\tau_{+0}(q)$. In the second case, the disk contributes with $\varepsilon_1(q)[\ast_{\Lambda_j}]$ to $\tau_{+0}(q)$. And so $\tau_{+0}(q) = \varepsilon_1(q)[\ast_{\Lambda_j}] + \varepsilon_2(q)[\ast_{\Lambda_j}]$. By the proof of Proposition 3.2 in [5] we have that $\tau_{+0}^S = \varepsilon_1 + \varepsilon_2$. Now it is clear that for each $q$ the diagram commutes and so it commutes. \hfill $\square$

**Lemma 3.3.** For each $c \in H_0(\Lambda)$ and the map $\alpha : H_0(\Lambda) \to \mathbb{Z}_2$ from the statement of Lemma 3.2 it holds that

$$
\alpha(c) = c \cdot [\Lambda].
$$

(3.3)

**Proof.** The map $\alpha$ is an element of $(H_0(\Lambda))^*$ of the dual of $H_0(\Lambda)$. The intersection pairing $\bullet : H_0(\Lambda) \otimes \mathbb{Z}_2, H_n(\Lambda) \to \mathbb{Z}_2$ defines $\Theta : (H_0(\Lambda))^* \to H_n(\Lambda)$ an isomorphism that sends $\delta \in (H_0(\Lambda))^*$ to a class $\Theta(\delta) \in H_n(\Lambda)$ so that $p \bullet \Theta(\delta) = \delta(p)$ for each $p \in H_0(\Lambda)$.

Let us claim that $\Theta(\alpha) = [\Lambda]$. Then our claim is equivalent to the statement that for every $p \in H_0(\Lambda)$ it holds that $\alpha(p) = p \bullet [\Lambda]$. Since elements $e_j \in H_0(\Lambda)$ with only non-zero component equal to $[\ast_{\Lambda_j}]$ generate the space $H_0(\Lambda)$ and clearly $\alpha(e_j) = 1$. By the definition of the pairing $\bullet$ for the disconnected Legendrian submanifold $\Lambda$ we have that

$$
e_j \bullet [\Lambda] = [\ast_{\Lambda_j}] \bullet [\Lambda] + \sum_{i=1, i \neq j}^r 0 \bullet [\Lambda_j] = [\ast_{\Lambda_j}] \bullet [\Lambda] = 1$$

(3.4)
where the last component is by the Poincaré duality for closed component $\Lambda$ and so $([*_{\Lambda_1}] \oplus [\Lambda_2]) = 1$. The reasoning for the other generating classes $e_j$ is analogous.

**Proof of Theorem** \[\text{I.A.}\] First, consider $\varepsilon_1 \not\sim \varepsilon_2$. Then $\varepsilon_1^S \not\sim \varepsilon_2^S$ by Proposition 3.1. Thanks to Proposition 3.2 in \[\text{I.B.}\] we have that $\tau_{\varepsilon_1^S,0} \neq 0$. For the sake of contradiction suppose that $[\Lambda] \in \text{im}_Z \tau_{-,n}$, then by the exactness of the duality sequence for the negative order of augmentations we obtain that $\sigma_{-,0}([\Lambda]) = 0$. And so by Proposition 2.11 we obtain that

$$0 = \langle \sigma_{-,0}([\Lambda]), q \rangle_\pi = \tau_{+,0}(q) \bullet [\Lambda]$$

for every Reeb chord that gives rise to a generator of $LCH_{n_2}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ and a generator of $LCH_{n_1}^{\varepsilon_1, \varepsilon_2}(\Lambda)$. However, using the commutativity from Lemma 3.2 and the form of the map $\alpha$ from Lemma 3.3 we obtain that for every Reeb chord $q$

$$\gamma \circ \tau_{\alpha_0}^S \circ (\pi_0^+)^{-1}(q) = \alpha \circ \tau_{+,0}(q) = \tau_{+,0}(q) \bullet [\Lambda] = 0$$

which yields that $\tau_{\varepsilon_1^S,0} = 0$ because both $\gamma$ and $\tau_{+,0}^+$ are isomorphisms. This is the contradiction with $\tau_{\varepsilon_1^S,0} \neq 0$.

On the other hand, assume that $\varepsilon_1 \sim \varepsilon_2$, then by Proposition 3.1 we have that $\varepsilon_1^S \sim \varepsilon_2^S$, which yields $\tau_{\varepsilon_1^S,0} = 0$ as above. Suppose that $[\Lambda] \not\in \text{im}_Z \tau_{-,n}$, then $\sigma_{-,0}([\Lambda]) \neq 0$, and thanks to Proposition 2.11 there exists a chord $q$ so that $\langle \sigma_{-,0}([\Lambda]), q \rangle_\pi \neq 0$.

If $\tau_{+,0} = 0$, then $0 \neq \langle \sigma_{-,0}([\Lambda]), q \rangle_\pi = \tau_{+,0}(q) \bullet [\Lambda] = 0$ which is a contradiction.

If $\tau_{+,0} \neq 0$, then the commutativity of the diagram in Lemma 3.2 Lemma 3.3 and the fact that $\tau_{\varepsilon_1^S,0} = 0$ imply that

$$0 = \gamma \circ \tau_{\alpha_0}^S \circ (\pi_0^+)^{-1}(q) = \alpha \circ \tau_{+,0}(q) = \tau_{+,0}(q) \bullet [\Lambda] = \sigma_{-,0}([\Lambda]), q \rangle_\pi \neq 0$$

which is a contradiction. This completes the proof.

Consider a disconnected Legendrian submanifold $\Lambda$ as above. We want to show that there are no further obstructions regarding the dimension of the image of the map $\tau_{-,n}$ or map $\tau_{+,n}$. In the next section, we will use this result to prove that there is not any other obstruction on the DGA-homotopy of the given augmentation of Chekanov-Eliashberg algebras of Legendrian submanifolds having $n$-spheres as its connected components, since this amounts to the discussion of the geography of bilinearized Legendrian contact homology for such submanifolds.

**Proposition 3.4.** For any integer $r \geq 2$ and any non-negative integer $m < r$ there exists a disconnected Legendrian submanifold $\Lambda'$ and augmentations $\varepsilon_{\Lambda'}^L, \varepsilon_{\Lambda'}^R$ of its Chekanov-Eliashberg algebra so that $\dim_\mathbb{Z}_2 \text{im} \tau_{+,n} = m$.

**Proof.** Perform a Legendrian ambient surgery on those two components of $\Lambda^{(2)}$ producing a Legendrian $\Lambda'$.

![Figure 6. The construction of the Legendrian submanifold $\Lambda'$ when $n = 1$.](image)

Pull-back the augmentations $\varepsilon_L$ and $\varepsilon_R$ onto the algebra $\mathcal{A}(\Lambda')$. Consequently Proposition 3.2 from \[\text{I.B.}\] implies that those two pull-backed augmentations $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ are not DGA-homotopic since $\partial(m_{12}) = 0$ by Proposition 4.5 and $\tau_{+,0}(m_{12}) = \tilde{\varepsilon}_L(m_{12}) - \tilde{\varepsilon}_R(m_{12}) = 1 \neq 0$. The fact $1$ yields that $\dim_\mathbb{Z}_2 \text{im} \tau_{+,n} = 0$. 

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The disconnected Legendrian submanifold $\Lambda^r = \bigsqcup_{j=1}^{r} \Lambda_j$ is defined as $r$ unlinked horizontally displaced copies $\Lambda_j$ of $\Lambda$. That means that the bilinearized complex splits into $r$ copies because there are no Reeb chords among distinct components. In particular,  

$$\tau^\Lambda_{\tau,n} = \bigoplus_{j=1}^{r} \tau^\Lambda_{j,n}$$

that is for the following augmentations the rank of the resulting map $\tau^\Lambda_{\tau,n} : LCH^L_{r,n} \rightarrow H_n(\Lambda^r) = \bigoplus_{j=1}^{r} H_n(\Lambda_j)$ is the sum of the ranks of the maps $\tau^\Lambda_{j,n} : LCH^L_{r,n} \rightarrow H_n(\Lambda_j)$ playing the same role as $\tau^\Lambda_{\tau,n}$ but in the duality exact sequence of the corresponding component. Fix some $\tau^\Lambda_{j,n}$ displaced copies $\Lambda_j$ because otherwise we could factor the corresponding $(\tau^m_L, \tau^m_R)$-derivative through the chords of $r$-th component $\Lambda^r$, which is impossible, and thus

$$\dim_{\mathbb{Z}_2} \ker \tau^\Lambda_{\tau,n} = \sum_{j=1}^{r} \dim_{\mathbb{Z}_2} \ker \tau^\Lambda_{j,n} = m \cdot 1 + (r - m) \cdot 0 = m$$

as we desired. 

\[ \square \]

4. Proof of Theorem 1.3 and Theorem 1.4

For a Legendrian submanifold $\Lambda$ and two non-homotopic augmentations $\varepsilon_1, \varepsilon_2$ let us denote

\[(4.1) \quad P^+ = P_{\Lambda, \varepsilon_1, \varepsilon_2}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} LCH^L_{k,\varepsilon_1,\varepsilon_2}(\Lambda) t^k.\]

and similarly $P^-$ for the opposite ordering of augmentations. Those Laurent polynomial split as $P^\pm = p^\pm + q^\pm$, where $q^\pm(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \ker \tau^\pm_{\pm,k}$ and $p^\pm(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \ker \tau^\pm_{\pm,k}$.

Similarly to [3] we can define the following.

**Definition 4.1.** Let $n$ be natural numbers and $P \in \mathbb{N}_0[t, t^{-1}]$ be a Laurent polynomial so that $P = q + p$. We say that $P$ is lbLCH-admissible if:

(i) $q \in \mathbb{N}_0[t]$ is a polynomial, $\deg(q) \leq n$, and $q(0) \geq 1$,

(ii) if $n$ is odd, then $p(-1)$ is even, and if $n$ is even, then $p(-1) = 0$.

**Proof of Theorem 1.3** To prove (i), $\im \tau_{\pm,k} \subset H_k(\Lambda)$ and so $q^\pm \in \mathbb{Z}[t]$ and $\deg(q^\pm) \leq n$. Theorem 1.1 and the fact that $\varepsilon_1 \neq \varepsilon_2$ imply that the class $[L]$ is not in the image of $\tau^\pm_{\pm,n}$. Therefore, $q^\pm = \dim_{\mathbb{Z}_2} \im \tau^\pm_{\pm,n} < r$. By adjoiningness, we know that $\dim_{\mathbb{Z}_2} \im \tau^\pm_{\pm,0} = r - \dim_{\mathbb{Z}_2} \im \tau^\pm_{\pm,n} \geq 1$.

The proof of (ii) coincides with the proof of Proposition 4.2 in [3]. Therefore, both $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ and $P_{\Lambda, \varepsilon_2, \varepsilon_1}$ are lbLCH-admissible.

\[ \square \]
Let us have arbitrary $P = q + p$ lbLCH-admissible Poincaré polynomial.

For the Hopf link $\Lambda^{(2)}$ we have that $P_{\Lambda^{(2)}}(1, \varepsilon_L, \varepsilon_R) = 1 + t^n$. And because by Proposition 3.5 in [5] for non-homotopic augmentations $\varepsilon_L, \varepsilon_R$ the connected sum acts as subtraction of the term $t^n$ we obtain that for the disconnected Legendrian submanifold $\Lambda^m$ from Proposition 3.4 for $m = q(0) - 1$ its Poincaré polynomial is the following constant

\begin{equation}
(4.2)
\quad P_{\Lambda^m, \varepsilon_L', \varepsilon_R'} = m.
\end{equation}

Note that if a link consists of multiple components so that their projections into $T^*M$ does not intersect, then the Poincaré polynomial is given by the sum of Poincaré polynomials of corresponding components.

Now choose the Laurent polynomials $\tilde{p}, \tilde{q}$ with non-negative coefficients so that

\begin{equation}
(4.3)
\quad p = \tilde{p} \text{ and } q - q(0) - q_n t^n = \tilde{q} - 1,
\end{equation}

and moreover, let $\Psi_{\tilde{q}, \tilde{p}} = \Lambda^{(2N)}_{\tilde{p}}$ be the connected Legendrian submanifold in $J^1(M)$ and $\varepsilon_L, \varepsilon_R$ the augmentations of its Eliashberg-Chekanov algebra so that $P_{\Psi_{\tilde{q}, \tilde{p}}, \varepsilon_L, \varepsilon_R} = \tilde{q} + \tilde{p}$ that is due to Bourgeois and Galant (see [5]).

Consider the disconnected Legendrian submanifold $\Lambda$ that consists of the submanifold $\Psi_{q, p}$, $q_n$ horizontally displaced copies of $\Lambda_W$, the $n$-dimensional Legendrian lift of the Whitney immersion with the single augmentation $\varepsilon_w$, and polynomial $P_{\Lambda_W, \varepsilon_w} = t^n$, and the disconnected Legendrian submanifold $\Lambda^m$ with the following augmentations:

\begin{equation}
(4.4)
\quad \varepsilon_1 = \begin{cases} 
\varepsilon_L^m; & \text{on chords of } \Lambda^m, \\
\varepsilon_L; & \text{on chords of } \Psi_{\tilde{q}, \tilde{p}}, \\
\varepsilon_w; & \text{on chords of } \Lambda_W,
\end{cases}
\end{equation}

and for $\varepsilon_2$ analogously with $R$ and $L$ exchanged. Observe that since the Chekanov-Eliashberg algebra of the Legendrian $\Lambda$ splits, then the DGA-homotopy descends to the components, but on the copy corresponding to $\Psi_{q, p}$ the augmentations $\varepsilon_1$ and $\varepsilon_2$ are not DGA-homotopic. The claim that the disconnected Legendrian submanifold $\Lambda$ has the Poincaré polynomial equal to $P$ easily follows

\begin{equation}
(4.5)
\quad P_{\Lambda, \varepsilon_1, \varepsilon_2} = m + \tilde{p} + \tilde{q} + q_n t^n = q(0) - 1 + \tilde{p} + \tilde{q} + q_n t^n = p + q = P.
\end{equation}

\[\square\]

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