Confinement and gluon propagator in Coulomb gauge QCD

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We consider the effects of the Faddeev-Popov determinant in the Coulomb gauge on the confinement properties of the QCD vacuum. We show that the determinant is needed to regularize the otherwise divergent functional integrals near the Gribov horizon but still enables large field configurations to generate IR enhanced running coupling. The physical gluon propagator is found to be strongly suppressed in the IR consistent with expectations from lattice gauge calculations.

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\[ \text{I. INTRODUCTION} \]

Quantitative understanding of confinement and more generally of the dynamics of gluons at low energies remains as the major challenge in QCD. In the past few years lattice simulations and phenomenological studies have provided new insights into the nature of the low energy behavior of the gluon propagator and role of gluons in forming the hadronic spectrum \[1, 2\]. Since gluons can only participate in strong interactions, spectroscopy of hadrons with excited gluonic modes is of crucial importance for investigations of confinement. It has recently been shown that hybrid mesons with excited quark and gluon modes should have properties similar to that of ordinary hadronic resonances and thus gluonic excitations may appear in the meson spectrum \[3, 4, 5, 6\]. Searches for exotic mesons have produced a few tantalizing candidates \[7, 8, 9, 10, 11, 12, 13\] and new experiments planned for JLab and GSI in light and charm meson spectroscopy, respectively, are expected to produce a map of gluonic excitations. In this paper we address gluon propagation in the QCD vacuum. This investigation was prompted by recent lattice results indicating that in both covariant and Coulomb gauges the low momentum gluons do not propagate \[14, 15, 16, 17, 18\]. This is precisely what one expects for physical degrees of freedom e.g. two transverse gluons in a physical gauge \[19, 20, 21, 22\]. In the four-dimensional Euclidean formulation of covariant gauge QCD, however, the lack of IR enhancement in the gluon propagator contradicts the naive expectation that the color confining force could be simply related to the gluon propagator. A popular, phenomenological approach to gluon (and quark) low energy dynamics is based on a truncation of the self-consistent set of Dyson-Schwinger equations. In many such approaches the gluon propagator plays a central role in providing the effective interaction between quarks, for example, it is used to generate dynamical chiral symmetry breaking \[23\]. A soft gluon propagator implies that confinement has to be described by other means. For example it has been argued that in covariant gauges the Kugo-Ojima confinement criterion for absence of colored non-singlets in the physical spectrum can be satisfied with a soft gluon propagator and an enhanced ghost propagator \[24, 25, 26, 27\]. We will show that this also seems to be the case in Coulomb gauge formulation.

In a covariant formulation one sacrifices positivity constraints and the Fock space representation and introduces additional (ghost) non-physical degrees of freedom. Alternatively, by relaxing the requirement of manifest Lorentz covariance it is possible to eliminate all non-physical components and study confinement and other low energy phenomena within the framework of quantum mechanical wave functions. Such an approach has obvious, important implications for quark model based phenomenology. Furthermore, at finite density it allows for well established, diagrammatic, many-body techniques to be used. A many-body approach has proven to be successful in treating a variety of low energy phenomena in QCD. For example the random phase approximation, which is typically \(e.g.\) for electron gas) relevant at high-densities, in QCD is also applicable at low-densities and may result in a self-consistent realization of confinement \[22, 28, 29\]. Due to the long-range nature of the confining interaction, at low-densities quasiparticle excitations have infinite energy which eliminates colored states from the physical spectrum. Due to the presence of bare quark-antiquark pairs near the Fermi-Dirac surface, the quasiparticle vacuum breaks chiral symmetry and leads to a non-vanishing scalar quark density. The collective excitations of this quark-antiquark plasma correspond to the Goldstone bosons \[30, 31, 32, 33, 34\].

The picture described above relies on the existence of a long range, confined quark-quark interaction. Such an interaction is expected to arise from the Coulomb operator which in the Coulomb gauge Hamiltonian describes direct interactions between (color) charge densities. Unlike QED, where this interaction is simply determined by the distance between sources, in QCD it is a complicated function of the transverse gluon field and can not be thought of as a simple potential \(e.g.\) of the Cornell type \[35\]. The conjecture that the Coulomb operator is related to the confining interaction is based on the observation that it is positive definite and vanishes at the Gribov horizon. The Gribov horizon defines the boundary of the gluon field domain. A number of approxima-
tions have been developed to calculate the expectation value of the Coulomb operator and to verify this conjecture \[22, 28, 31, 32\]. In the process it has been realized that the Gribov region still contains physically equivalent field configurations. To what extent the necessary identification of the wave functional at these gauge-equivalent points modifies the expectation value of the Coulomb operator remains an open issue \[32\].

The standard approach to a many body system is to introduce a physically motivated ansatz for the wave functional and to define approximations for the evaluation of expectation values. We have followed this approach in Ref. \[22\] where we generated the confining interaction but a specific assumption on the behavior of the gluon propagator at low energies had to be imposed. In particular, the gluon dispersion relation which follows from minimizing the vacuum expectation value was solved self-consistently together with the confining interaction but a particular boundary condition was chosen for the so-called Fundamental Modular Region, \( \Lambda \) as shown in Fig. 1. The boundary of \( \Lambda \) is a set of gauge fields which lead to degenerate absolute minima. The fundamental region resides inside the so-called Gribov region, \( \Omega \), corresponding to all minima of \( I[\Lambda,g] \) and thus also satisfying the transversality condition. The boundary of \( \Omega \) defines a set of configurations for which the gauge mapping is singular and \( \mathcal{J}[A] = 0 \). In what follows we will primarily study the

\[ H_0 = \frac{1}{2} \int d^3x \left[ \mathcal{J}^{-1} \Pi^a(x) \mathcal{J} \Pi^a(x) + (\nabla \cdot A^a(x))^2 \right], \]

with

\[ \mathcal{J} \equiv \text{Det} (1 - \lambda) = e^{T r \log(1 - \lambda)}, \] being the Faddeev-Popov (FP) determinant. The matrix \( \lambda \), in the FP operator \((1 - \lambda)^{-1}\), is given by,

\[ \lambda_{x,a,y,b} = \left( -\frac{1}{\nabla^2} \right) x_y g f_{ac,b} A^c(y) \nabla_y, \]

and in Eq. (1) the trace is over the spatial: \( x, y \) and color: \( a, b, c \) indices. The FP determinant is the Jacobian of coordinate transformation from the canonical coordinates of the Weyl gauge, \( A^{\mu,a}(x) = (0, \mathbf{V}^a(x)) \) with kinetic energy given by, \( 1/2 \int d^3x (-i \delta/\delta \mathbf{V}^a(x))^2 \), to the Coulomb gauge fields, \( A^a(x) \) defined through a gauge map,

\[ \mathbf{V}^a(x) \rightarrow (A^a(x), \vec{\phi}(x)) = u(\vec{\phi}) A^a u(\vec{\phi}) + u^{-1}(\vec{\phi}) \nabla u(\vec{\phi}). \]
role of configurations near singular boundary points on ∂Ω. Since there exist configurations for which ∂Λ and ∂Ω overlap, the point $A^a(x) = 0$ lies in Λ and both Λ and Ω are convex, some fluctuations around the null field reach the coordinate singularity, ∂Ω without leaving Λ. Furthermore it has been pointed out [41] that it is the common boundary points which dominate functional integrals. Thus even if field configurations outside of FMR are included these may not lead to substantial errors.

In the limit $\mathcal{J} = 1$, the kinetic term describes a set of coupled harmonic oscillators and its (unnormalized) ground state $|0\rangle$ is given by

$$\langle A|\omega_0\rangle = \exp\left(-\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_0(k) A^a(k) A^a(-k)\right) \tag{10}$$

with $\omega_0(k) = k = |k|$ being the free gluon energy and,

$$A^a(k) \equiv \int dx e^{-ik \cdot x} A^a(x), \tag{11}$$

represents the normal modes. The FP determinant leads to a suppression of the ground state wave functional near the Gribov horizon. This can be illustrated using an analogy between the Weyl and the Coulomb gauge kinetic terms and a harmonic oscillator in cartesian and spherical coordinates, respectively. In N-dimensions, the S-wave harmonic oscillator radial wave function satisfies,

$$\frac{1}{2} \sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] R(r) = \frac{1}{2} \left[ \mathcal{J} \frac{\partial}{\partial r} \mathcal{J} \frac{\partial}{\partial r} + \omega^2 r^2 \right] R(r). \tag{12}$$

Here the jacobian is given by $\mathcal{J} = r^{N-1} \sim \exp(-N \log r)$ and it vanishes at the boundary $r \to 0$ of the domain of $r$. The ground state wave function, $R(r)$ is finite at that boundary, $R(r) = \exp(-r^2 \omega^2/2)$ but the radial wave function defined by $u(r) = J^{1/2} R(r)$, vanishes as $r \to 0$. The Hamiltonian can be redefined to absorb the jacobian,

$$H \rightarrow \tilde{H} = \mathcal{J}^{-1/2} H J^{1/2} = \frac{1}{2} \left( p_r^2 + \omega^2 r^2 \right) + V_C, \tag{13}$$

were, $p_r = -i\partial/\partial r$ and the additional potential is given by $V_C = \mathcal{J}^{-2}[p_r, \mathcal{J}]^2/4 - \mathcal{J}^{-1}[p_r, \mathcal{J}]/2$. The Hamiltonian $\tilde{H}$ is hermitian with respect to a flat measure in the radial direction,

$$\frac{\int Dx R(x) H R(x)}{\int d\Omega} = \int dr \mathcal{J} R(r) H R(r) = \int dr u(r) \tilde{H} u(r). \tag{14}$$

In terms of $\tilde{H}$ and $u(r)$ one effectively recovers the simple harmonic motion in one dimension (modulo $V_C$), except for a boundary condition, $u(r) \to 0$ at a point $r = 0$ corresponding to the singularity of the coordinate transformation.

The behavior of the wave function near the Gribov horizon is strongly correlated with confinement. In the Coulomb gauge, at the Gribov horizon, $\mathcal{J} = 0$ the Coulomb kernel diverges and this can be interpreted as a manifestation of confinement [22]. However, since the radial wave functional vanishes at singular points, δΩ, of the coordinate transformation, functional integrals over color singlet states are expected to be finite. In Ref. [22] we have explored this scenario, but we have not accounted for the boundary condition on the ground state wave functional. In effect we used a gaussian ansatz for $u$ and not for $R$, i.e. the wave functional was finite at the Gribov horizon. To ensure that functional integrals over the FP operator converge we had to choose a particular condition on the parameters of the ground state wave functional. Summarizing, the FD determinant is a crucial element of the QCD Coulomb gauge dynamics as it suppresses the integrands in functional integrals near the Gribov horizon.

As long as gauge fields are within the Gribov region, Ω, the Coulomb potential is positive and it is possible to use a variational approach. The simplest variational ansatz for which diagrammatic expansion is possible is given by the functional generalization of the harmonic oscillator ground state which is equivalent to the quasiparticle approximation. This approximation can be systematically improved through a cluster expansion and excited states can also be studied [23]. In this context one often uses the formalism of second quantization which is natural when dealing with gaussian integrals over polynomials (Wick theorem). In our case, however, before one gets to this stage one has to deal with non-polynomial operators, e.g. $\mathcal{J}[A]$ or $K[A]$, and thus it is simpler to proceed with the Schrödinger representation.

### III. THE QUASIPARTICLE SPECTRUM

In the non-interacting case, $H = H_0$ the perturbative vacuum of Eq. [10] minimizes the energy density of the system, i.e.

$$0 = \frac{\partial}{\partial \omega(k)} \langle \omega | H_0 | \omega \rangle / V \bigg|_{\omega = \omega_0}$$
Here $\mathcal{V}$ is the total number of gluon degrees of freedom, $\mathcal{V} = \delta_{ab} \delta_{ij} \int d^3x = (N_c^2 - 1) \times 2 \times \text{volume}$. To describe the quasiparticle spectrum we will use the same gaussian variational ansatz. The vev of the Hamiltonian becomes,

$$E(\omega) = \langle \omega | H \omega \rangle / \langle \omega | \omega \rangle \equiv E_K(\omega) + E_C(\omega),$$

where

$$E_K = \frac{1}{2 \langle \omega | \omega \rangle} \int DA^a \int dxe^{-\int d\omega(k) A^a(k) A^a(-k)} \left[ \Pi^a(x) \Pi^a(x) + (B^a(x))^2 \right] e^{-\int d\omega(k) A^a(k) A^a(-k)},$$

and

$$E_C = \frac{1}{2 \langle \omega | \omega \rangle} \int DA^a \int dxdye^{-\int d\omega(k) A^a(k) A^a(-k)} \left[ \rho^a(x) \mathcal{J} K[A]_{\omega;0,y} \rho^b(y) \right] e^{-\int d\omega(k) A^a(k) A^a(-k)}.$$

The ground state normalization is given by,

$$\langle \omega | \omega \rangle = \int DA^a \mathcal{J}[A] e^{-\int d\omega(k) A^a(k) A^a(-k)} \equiv \langle \mathcal{J}[A] \rangle$$

We use the $\langle \cdot \cdot \cdot \rangle$ to represent functional integrals over the ground state ansatz functional with the flat measure. After integrating by parts the kinetic and Coulomb kernel contributions can be written as ($dk \equiv d\omega(k)/(2\pi)^3$),

$$E_K = \frac{1}{2 \langle \omega | \omega \rangle} \int DA^a \mathcal{J}[A] e^{-\int d\omega(k) A^a(k) A^a(-k)} \times \left[ \int [d\omega(k) A^a(k) A^a(-k)] \right],$$

and

$$E_C = \frac{1}{2 \langle \omega | \omega \rangle} \int DA^a \int \Pi^a_{\omega}[dk] \mathcal{J}[A] e^{-\int d\omega(k) A^a(k) A^a(-k)} \rho^a(k_1, k_2) K[A]_{\omega;0,k_1,k_2} \rho^b(k_3, k_4),$$

respectively. The charge density is now given by,

$$\rho^a(k_1, k_2) = f_{abc} \omega(k_2) A^b(k_1) \cdot A^c(k_2),$$

and the Coulomb kernel by,

$$K[A]_{\omega;0,k} = \int dxdy e^{i(k_1 + k_2) \cdot x} K[A]_{\omega;0,y} e^{i(k_1 + k_2) \cdot y}.$$

Even though details of the boundary of the functional integrals are not known, the partial integration is presumably justified since the integrand vanishes as $A \rightarrow \infty$ and at the boundary of the Gribov region, $\mathcal{J} \rightarrow 0$. Compared to the harmonic oscillator example discussed earlier, the partial integration combines contributions from $V_C$ and $p^2$ in Eq. 10 to the vev of the Hamiltonian and express them as coordinate space integrals over the gaussian wave function. In our case the complication in evaluating functional integrals over $DA$ is due to the nonlinear dependence of $\mathcal{J}[A]$ and $K[A]$ on the FP operator, $(1 - \lambda)^{-1} = (1 - \lambda[A])^{-1}$. These integrals are performed by expanding such functionals in powers of $A$, performing Gaussian integrals over polynomials of $A$ and approximating them by products of two-point correlations. This is will be illustrated in particular cases below. To simplify the notation, the triplet of indices representing momentum, color and spin will be denoted by greek letters, e.g. $a = (k, \alpha, i)$ and a doublet containing a momentum and a color index by $\alpha = (k, a)$. The summation convention will be used with upper and lower indices differing by a replacement $k \rightarrow -k$, e.g. $A^a \equiv A^{ia}(k)$,

$$\sum_{a} A^a A^a = A^a A^a = \sum_{a} \sum_{ij} [dk] A^a(k) \delta_{ij} (k) A^a(-k).$$

In this notation $\lambda$ from the FP operator can be written as,

$$\lambda_{\alpha \beta} = \lambda(p, \alpha)(q, \beta) \equiv \lambda_{\alpha \beta}(p, q) A^\gamma,$$

$$= g f_{abc} \int [dk] (2\pi)^3 \delta^3(p - k - q) \frac{A^c(k) \cdot iq}{p^2} \equiv \lambda_{\alpha \beta} A^\gamma,$$
with
\[
\lambda^\alpha_{\gamma\beta} = \lambda^{(p,a)}_{(k,c,k),(q,b)}(2\pi)^3 \delta^3(p-k-q) \frac{i g [\delta_T(k)] q^k}{p^2} f_{acb}.
\]

Evaluation of functional integrals is simplified by introducing corresponding generating functionals,
\[
\langle A^{\alpha_1} \cdots A^{\alpha_n} F[A] \rangle = \int \Pi_\beta dA^\beta F[A] A^{\alpha_1} \cdots A^{\alpha_n} e^{-\sum_\gamma \omega_\gamma A^{\gamma A}},
\]
and since,
\[
\int \Pi_\beta dA^\beta F[A] = e^{\sum_\gamma \frac{\omega_\gamma J_\gamma}{2\omega_\gamma}} \langle F[A + J/2\omega] \rangle
\]
we obtain,
\[
\langle A^{\alpha_1} \cdots A^{\alpha_2} F[A] \rangle =
\]
with the first term representing propagator the absence of the FP determinant (J = 1) and the second term
\[
\Pi^\alpha_{\beta} = \frac{\delta}{\delta J_\alpha} \frac{\delta}{\delta J^\beta} \sum_{J=0} \langle J [A + J/2\omega] \rangle / \langle J[A] \rangle = \frac{1}{2\omega_\alpha} \left[ \delta^\alpha_\beta + \omega_\alpha \frac{\delta}{\delta J_\alpha} \frac{\delta}{\delta J^\beta} \sum_{J=0} \langle J [A + J/2\omega] \rangle / \langle J[A] \rangle \right],
\]
where \( [\lambda^\alpha]^\gamma_{\beta} \equiv \partial \lambda^\gamma_{\beta} / \partial A_\alpha = \lambda^\gamma_{\beta} \). This relation can be represented through an infinite set of coupled integral Dyson equations containing all dressed vertices. As argued in Ref. [22, 23], however, vertex corrections give a finite and small modification and will be ignored. The dominant contributions in both the IR and the UV regions of the loop momentum integrals over instantaneous propagators come from diagrams with a maximal number of soft Coulomb lines and a maximal number of primitive self energy loops, respectively. The primitive self energy is shown in Fig. 2 and is given by
\[
I^0_{\alpha \beta} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\omega(k)(q-k)^2},
\]
and explicitly,
\[
I^0_{\alpha \beta} = I^0(q) = g^2 N_C \int [dk] \frac{1 - (\bar{q} \cdot k)^2}{2\omega(|k|)(q-k)^2}.
\]
This self-energy is UV divergent and has to be renormalized. We will discuss renormalization in detail in the following section. To proceed we need to introduce the expectation value of the FP operator,
\[
d^\alpha_{\beta} = \langle [(1 - \lambda)^{-1}]_{\beta} J[A] / \langle J[A] \rangle \rangle = \delta^\alpha_{\beta} d_{\alpha}.
\]
A few lowest order diagram contributions to this vev are shown in Fig. 3. At the two-loop order the first and second diagram in the second line dominate in the IR and UV, respectively and are retained. In higher orders the
dominant contribution comes from a series of rainbow-ladder diagrams obtained by summing the class of diagrams generated by these two lowest order loop diagrams and results in the following approximation to the Dyson series,

\[ d^{\alpha \beta} = \delta^{\alpha \beta} d_{\alpha} = \delta^{\alpha \beta} + \left\{ \left[ \lambda(1 - \lambda) \right] \delta^{\alpha \beta} J \right\} / \langle J \rangle \]

\[ = \delta^{\alpha \beta} + \sum_{\gamma} \frac{1}{2\Omega_{\gamma}} \left[ \lambda_{\gamma} d_{\lambda} \delta^{\alpha \beta} \right] = \delta^{\alpha \beta} \left[ 1 + I_0 d_{\alpha} \right], \]  

(36)

where

\[ \sum_{\gamma} \frac{1}{2\Omega_{\gamma}} \lambda_{\gamma} d_{\lambda} \delta^{\alpha \beta} = \delta^{\alpha \beta} I_0, \]  

(37)

and explicitly, with \( d_{\alpha} = d(q) \), \( I_0 = I(q) \), is given by

\[ d(q) = \frac{g}{1 - g I(q)}, \]

\[ I(q) = N_C \int [dk] (1 - (\hat{q} \cdot \hat{k})^2) \left( \frac{q}{2\Omega(|k|)} \right)^2 d(|q - k|). \]  

(38)

Using the same approximation (of ignoring vertex corrections), the Dyson equation for the instantaneous propagator becomes,

\[ \frac{\delta}{\delta F^\alpha \delta J_{\beta j = 0}} \langle J \left[ A + \frac{J}{2\omega} \right] \rangle / \langle J [A] \rangle = - \left[ \frac{\lambda^\alpha}{2\omega_\alpha} \left( \frac{\lambda^\beta}{2\omega_\beta} \right) \right] \left( \delta_{\alpha \beta} d^\gamma \right) \delta^\gamma_{\lambda} - \sum_{\rho} \left[ \frac{\lambda^\alpha}{2\omega_\alpha} \left( \frac{\lambda^\beta}{2\omega_\beta} \right) \right] \delta^\gamma_{\lambda} d^\gamma_{\rho} \frac{1}{2\Omega_{\rho}} \left( \lambda^\rho \right) \left( \delta_{\alpha \beta} d^\gamma \right) \delta^\gamma_{\sigma}. \]  

(39)

and is shown in Fig. 4. Since neutrality of the vacuum implies

\[ \left[ \lambda^\alpha d_{\lambda} d^\gamma \right] \delta^\gamma_{\lambda} = \sum_{\gamma} \lambda^\gamma_{\alpha} d_{\sigma} \lambda^\sigma_{\beta} \delta^\gamma_{\lambda} = 2\delta^\alpha_{\beta} F_\alpha, \]  

(40)

we finally obtain,

\[ \Omega_\alpha = \omega_\alpha + F_\alpha, \]  

(41)

where \( F_\alpha = F(q) \), is explicitly given by

\[ F(q) = \frac{N_C}{2} \int [dk] \left( \frac{1 - (\hat{k} \cdot \hat{q})^2}{(q - k)^2} \right) d(|k|) d(|q - k|). \]  

(42)

We can now return to the calculation of the vacuum expectation of the full Hamiltonian. By minimizing with respect to \( \omega \) this determines \( \omega \), and from Eq. (41), the kinetic propagator, \( 1/2\Omega \). In terms of this propagator, the kinetic vacuum expectation value, \( E_K \) is given by

\[ E_K = \frac{1}{2} \sum_{\alpha} \left( \omega_\alpha^2 + p_\alpha^2 \right) \Pi^\alpha_{\alpha} + \]

\[ + \frac{1}{2} V_{\alpha \beta} V_{\gamma \delta} \frac{\partial}{\partial J_\alpha} \frac{\partial}{\partial J_\beta} \frac{\partial}{\partial J_\gamma} \frac{\partial}{\partial J_\delta} \sum_{\rho} \frac{\delta^\rho_{\alpha \beta}}{2\omega_\rho} \langle J \left[ A + \frac{J}{2\omega} \right] \rangle. \]  

(43)

The second term originates from the square of the magnetic field, \( B'^2 = \nabla \times A^2 \gamma V + V_{\alpha \beta} A^\alpha A^\beta \), and it is shown in Fig. 5,

\[ E_K / V = \frac{1}{2} \int [dq] \left( \omega^2 (q) + \frac{q^2}{2\Omega(q)} \right) \]

\[ + \frac{\delta^2 N_C}{32} \int [dq] [dk] \left( 3 - (\hat{k} \cdot \hat{q})^2 \right) \frac{1}{\Omega(|k|) \Omega(|q|)}. \]  

(44)
The 4-point function contribution comes from the equation for the transverse gluon propagator and is given by,

\[
\frac{1}{2 \Omega_\gamma} = \frac{1}{2 \omega_\gamma} - \frac{1}{2 \omega_\beta} + \frac{1}{2 \omega_\alpha}
\]

FIG. 4: The rainbow-ladder approximation to the Dyson equation for the transverse gluon propagator

\[
E_k = \frac{\Omega_k}{2} + \frac{1}{2 \Omega_\alpha} + \frac{1}{2 \Omega_\beta}
\]

FIG. 5: The vacuum expectation value of the kinetic energy. The 4-point function contribution comes from the \(B^2\) term

\[
\Omega^2(q) - F^2(q) - q^2 = \frac{N_C g^2}{4} \int [d\mathbf{k}] \frac{(3 - (\mathbf{q} \cdot \mathbf{k})^2)}{\Omega(k)}
\]

This magnetic contribution involves a transverse gluon 4-point function, which as discussed earlier, is approximated by the product of two 2-point functions, i.e. the gluon-gluon scattering amplitude shown in Figs. 6 is not computed. The Coulomb energy vev is shown in Fig. 8 and is given by,

\[
E_C = \frac{1}{2 \omega J_\alpha \omega J_\beta \omega J_\gamma \omega J_\delta} \left[ e \sum_{\rho} \frac{\rho \rho_\rho}{\rho \rho} \right] \rho \gamma
\]

\[
\times \rho \alpha \beta ((1 - \lambda)^{-2}(-\nabla^2) \mathcal{J} \left[ A + J \right] ) \rho \delta
\]

(45)

\[
\frac{E_C}{\mathcal{V}} = \frac{N_C}{32} \int [d\mathbf{k}] [d\mathbf{q}] \frac{K(|\mathbf{q} - \mathbf{k}|)}{(\mathbf{q} - \mathbf{k})^2} \left( 1 + (\hat{k} \cdot \mathbf{q}) \right)^2
\]

\[
\times \frac{(\Omega(|\mathbf{k}|) - F(|\mathbf{k}|) - \Omega(|\mathbf{q}|) + F(|\mathbf{q}|))^2}{\Omega(|\mathbf{k}|)\Omega(|\mathbf{q}|)}
\]

(46)

Finally the gap equation follows from, \(0 = \partial(E_K + E_C)/\partial \omega_\alpha\),

\[
\Omega^2(q) - F^2(q) - q^2 = \frac{N_C g^2}{4} \int [d\mathbf{k}] \frac{(3 - (\mathbf{q} \cdot \mathbf{k})^2)}{\Omega(k)}
\]

with \(N_C\) \(g\) \(f(q)d^2(q)\),

(47)

and \(f\) satisfying

\[
f(q) = 1 + N_C \int [d\mathbf{k}] \frac{1 - (\mathbf{q} \cdot \mathbf{k})^2}{2\Omega(|\mathbf{k}|)|\mathbf{q} - \mathbf{k}|^2} f(|\mathbf{q} - \mathbf{k}|).
\]

(49)
It is also instructive to analyze the single quasiparticle dispersion relation,
\[
E_\alpha \delta^0_\beta = \langle JA^\alpha HA_\beta \rangle / \langle J \rangle. 
\] (50)

\[
E(q) = \frac{1}{2\Omega(q)} \left[ \Omega^2 + F^2(q) + q^2 + \frac{N_C g^2}{4} \int [dk]\frac{(3 - (\mathbf{q} \cdot \mathbf{k})^2)}{\Omega(k)} + \frac{N_C}{4} \int [dk](1 + (\mathbf{q} \cdot \mathbf{k})^2) \frac{K(q - k)}{\Omega(k)(q - k)^2} \right],
\] (51)

After combining with the gap equation one obtains,
\[
E(q) = \Omega(q) \left[ 1 + \frac{N_C}{4} \int [dk](1 + (\mathbf{q} \cdot \mathbf{k})^2) \frac{K(q - k)}{\Omega(k)(q - k)^2} \right],
\] (52)

which is identical to the expression found in Ref. [22] modulo replacement \( \omega \to \Omega \).

**IV. RENORMALIZATION**

So far we have been ignoring potential UV divergences. These divergences should be removed by renormalizing appropriate operators and the coupling constant, \( g \). It turns out that all four equations of interest, Eqs. (38), (41), (47), (49), require renormalization. These equations have to be regularized first and this can be done by cutting-off the momentum integrals, \( \int [dk] \to \int^\Lambda [dk] \). The physical, renormalized solutions, \( d(q), f(q), \omega(q) \) and \( \Omega(q) \) should be \( \Lambda \) independent. We will first discuss renormalization of the expectation value of the FP operator, \( d(q) \). Assuming that a renormalized solution for \( \Omega(q) \) has been found, the equation for \( d(k) \) is renormalized by adjusting the bare coupling, \( g \to g(\Lambda) \), i.e. the renormalized vev of the FP operator, \( d(k) \) will play the role of the running coupling. The \( \Lambda \) dependence of \( g(\Lambda) \) is determined by the UV behavior of Eq. (58),
\[
\frac{dg(\Lambda)}{d\Lambda} = - \frac{\beta}{(4\pi)^2} g^2(\Lambda) \frac{d\Omega(\Lambda)}{\Omega(\Lambda)}, \] (53)

with \( \beta = 8N_C/3 \). In this and all other renormalization group equations we keep only relevant and marginal contributions, i.e. no power corrections, \( O(p^n/A^n) \) with \( n > 0 \) are included since they do not require renormalization. Since \( \beta > 0 \) and physical solutions require \( d(k), \Omega(k) > 0 \) the solution of Eq. (53) vanishes in the limit \( \Lambda \to \infty \). In the limit \( k \to \infty \) the integral \( I(k = \Lambda, \Lambda) \) given by Eq. (58), with the second argument referring to the upper limit of integration, is finite. Thus in leading logarithmic approximation, \( d(\Lambda) \to g(\Lambda) \) as \( \Lambda \to \infty \). Furthermore, from Eqs. (11) it follows that for large \( q, q \sim \Lambda \),
\[
\frac{d\Omega(q)}{dq} \to \frac{d\omega(q)}{dq} + O(d^2(q)) \to \frac{d\omega(q)}{dq} + O(g^2). \] (54)

Similarly from Eq. (47), to leading logarithmic approximation, we find \( d\omega(q)/dq = 1 + O(g^2) \). Thus finally,
\[
\Lambda \frac{dg(\Lambda)}{d\Lambda} = - \frac{\beta}{(4\pi)^2} g^2(\Lambda) + O(g^5(\Lambda)), \] (55)

which, ignoring the terms \( O(g^5) \) has a solution given by
\[
g(\Lambda) = \frac{g(\mu)}{\left(1 + \frac{\beta}{(4\pi)^2} g^2(\mu) \log(\Lambda^2/\mu^2)^{1/2} \right)}. \] (56)

The asymptotic behavior as \( \Lambda \to \infty \) is therefore given by,
\[
g(\Lambda) = \frac{4\pi}{\beta^{1/2} \log^{1/2}(\Lambda^2)}. \] (57)

The renormalized equation for \( d(k) \) is completely specified once \( g(\mu) \), the value of the coupling at an arbitrarily chosen renormalization scale, \( \mu \) is fixed. It should be stressed, however, that this solution is valid only to within terms of the order of \( 1/\log^{3/2}(\Lambda^2/\mu^2) \). In practical applications we will be renormalizing at a low energy scale, \( \mu \) e.g. related to the string tension or the glue-ball mass and thus such corrections become unimportant as \( \Lambda \to \infty \). For relevant operators, however, as we will see below, such logarithmic corrections are multiplied by positive powers of \( \Lambda \) and thus cannot be neglected.

For practical (numerical) applications we have found a different, momentum subtraction renormalization (MSR) scheme to be more practical. In this scheme the renormalized equation for \( d(q) \) is obtained by subtracting Eq. (58) at \( q = \mu \),
\[
\frac{1}{d(q)} - \frac{1}{d(\mu)} = -\frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \hat{q}) \int_{0}^{\Lambda} d\mathbf{k} k^2 \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \hat{q})^2)}{\Omega(k)} \frac{d\mathbf{q} - k}{(q - k)^2} + (q \to \mu).
\] (58)

In this renormalization scheme the coupling constant is therefore given by

\[
\frac{1}{g_{\text{MSR}}(\Lambda)} = \frac{1}{d(\mu)} + \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \hat{q}) \int_{0}^{\Lambda} d\mathbf{k} k^2 \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \hat{q})^2)}{\Omega(k)} (\mu - k)^2.
\] (59)

and \(g(\Lambda) = g_{\text{MSR}}(\Lambda)\) to within corrections of the order \(O(\mu/\Lambda)\) i.e. they agree asymptotically. From now on we will drop the MSR subscript. Finally for \(k \sim \Lambda \to \infty\)

\[
d(k) = g(\Lambda) \left[1 + O(g^2(\Lambda) \log(\Lambda^2/k^2))\right],
\] (60)

which is expected, as discussed above.

We now proceed to discuss renormalization of the equation for \(f(k)\). Physically \(f(k)\) represents additional contributions to the \(\text{vev}\) of the square of the FP operator, \(\langle (1 - \lambda)^{-2} \rangle \sim d^2 f\) not present in the expectation value, \(\langle (1 - \lambda)^{-2} \rangle \sim d^2\). Any UV divergent contribution to \(f\) should therefore be renormalized by renormalizing the operator \(g^2/(1 - \lambda)^2\), since the operator \(g/(1 - \lambda)\), has already being renormalized. This is done by a renormalization constant, \(Z_K(\Lambda)\) introduced by replacing the composite Coulomb operator, \(K[A] \to Z_K(\Lambda)K[A]\). Using the renormalized Coulomb operator the equation Eq. (59) for \(f(k)\) becomes,

\[
f(k) = Z_K(\Lambda) + \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \hat{q}) \int_{0}^{\Lambda} d\mathbf{k} k^2 \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \hat{q})^2)}{\Omega(k)} \frac{d^2(\mathbf{q} - k)f(\mathbf{q} - k)}{(q - k)^2},
\] (61)

and in the limit \(\Lambda \to \infty\) one obtains,

\[
\Lambda \frac{dZ_K(\Lambda)}{d\Lambda} = -\frac{\beta}{(4\pi)^2} \frac{d^2 f(\Lambda)}{\Omega(\Lambda)} Z_K(\Lambda),
\] (62)

which in the leading logarithmic approximation has a solution given by

\[
Z_K(\Lambda) = \frac{Z_K(\mu)}{\log^{1/2}(\Lambda^2/\mu^2)}.
\] (63)

resulting, in the MSR scheme in \(Z_K(\Lambda)\) given by,

\[
Z_K(\Lambda) = f(\mu) - \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \hat{q}) \int_{0}^{\Lambda} d\mathbf{k} k^2 \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \hat{q})^2)}{\Omega(k)} \frac{d^2(\mathbf{q} - k)f(\mathbf{q} - k)}{(q - k)^2} - (q \to \mu),
\] (64)

\[
f(k) - f(\mu) = \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \hat{q}) \int_{0}^{\Lambda} d\mathbf{k} k^2 \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \hat{q})^2)}{\Omega(k)} \frac{d^2(\mathbf{q} - k)f(\mathbf{q} - k)}{(q - k)^2} - (q \to \mu).\]

As expected, for UV values of \(k \sim \Lambda \to \infty\) we find

\[
f(k) = Z_K(\Lambda) \left[1 + O(g^2(\Lambda) \log(\Lambda^2/k^2))\right].
\] (66)

Much of the discussion on renormalization of \(d\) and \(f\) has
already being given in Ref. [22]. It should be noted that if for large \( k, d^2(k)/\Omega(k) < 1/\log(k)^n \) with \( n > 1 \) there is no renormalization for \( f(k) \). Our analysis suggests that \( n = 1 \) thus the unrenormalized equation for \( f \) has a sub-leading \( \log(\log(k)) \). We suspect this is an artifact of the rainbow-ladder truncation.

As long as one works with the leading logarithmic approximation, \( \Omega(k) = \omega(k) = k \), and there is no effect of the FP determinant on \( d \) or \( f \). The inclusion of the FP determinant has an effect on the low momentum behavior of \( d \) and \( f \), but it also introduces a new divergent integral, \( F(q) \) in Eq. (12). Since the origin of \( F \) is the FP determinant \( \mathcal{J} \), it is the FP determinant that has to be renormalized in order to make \( \Omega(q) \) finite. The renormalized FP determinant should be chosen as,

\[
\mathcal{J} \to \left[ \mathcal{J} e^{\sum_n \delta \omega_n A^n A_n + \cdots} \right]_\Lambda. \tag{67}
\]

Here \( \cdots \) stands for higher powers of the field operators, however, since including the FP determinant within a gaussian approximation to the functional integrals only the quadratic term needs to be retained. It can be easily verified that replacing \( \mathcal{J} \) by Eq. (67) leads to the replacement,

\[
F(q) \to F(q, \Lambda) + \delta \omega(\Lambda), \tag{68}
\]

where \( F(q, \Lambda) \) stands for the integral in Eq. (12) with the upper limit set to \( \Lambda \). The counterterm \( \delta \omega(\Lambda) \) will be chosen to make \( \Omega(q) \) UV finite. Since \( F(q) \) has mass dimension of one, in general one expects two counterterms would be needed, one proportional to \( \Lambda \) and the other to one power of the momentum. From the UV behavior of the integrand in Eq. (12) it follows, however, that only the first is needed and we obtain,

\[
\frac{d \delta \omega(\Lambda)}{d \Lambda} = -\frac{\beta}{(4\pi)^2} d^2(\Lambda), \tag{69}
\]

whose solution is given by

\[
\delta \omega(\Lambda) = \delta \omega(\mu) - \frac{\beta}{(4\pi)^2} \int_\mu^\Lambda dk d^2(k). \tag{70}
\]

We note here that corrections to the leading asymptotic behavior \( d(k) \sim g(k) \) cannot be neglected here since for \( F(q) \) they result in terms of \( O(\Lambda) \). Thus it is necessary to keep \( d(k) \) rather then \( g \) in the renormalized expression for \( F \). As in the case of \( d \) and \( f \) in the following we will use the MSR scheme for \( \Omega(q) \) which gives,

\[
\Omega(q) - \Omega(\mu) - \omega(q) + \omega(\mu) = \frac{\beta}{(4\pi)^2} \int_{-1}^1 (\mathbf{k} \cdot \mathbf{q}) \int_0^\Lambda dk d^2(k) \left[ \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \mathbf{q})^2)}{(\mathbf{q} - \mathbf{k})^2} d(\mathbf{k}) d(\mathbf{q} - \mathbf{k}) - (q \to \mu), \tag{71}\right]
\]

with asymptotic behavior, \( k \sim \Lambda \to \infty \) given by,

\[
\Omega(k) = \omega(k) + \frac{\beta}{(4\pi)^2} g^2(\Lambda) \Lambda \left[ 1 + O \left( g^2(\Lambda) \log(\Lambda^2/k^2) \right) \right], \tag{72}
\]

and with \( \delta \omega(\Lambda) \) in MSR given by,

\[
\delta \omega(\Lambda) = \Omega(\mu) - \omega(\mu) - \frac{\beta}{(4\pi)^2} \int_{-1}^1 (\mathbf{k} \cdot \mathbf{q}) \int_0^\Lambda dk d^2(k) \left[ \frac{3}{4} \frac{(1 - (\mathbf{k} \cdot \mathbf{q})^2)}{(\mathbf{q} - \mathbf{k})^2} d(\mathbf{k}) d(\mathbf{q} - \mathbf{k}) \right]. \tag{73}
\]

The gap equation is the one which cannot be renormalized in a simple way like the previous equations. This is due to inconsistencies in the approximation used. Specifically, the gap equation is derived by taking the functional derivative of the energy expectation value with respect to \( \omega \). In the second integral in Eq. (17) we have retained only the derivative of the gluon lines and not of the Coulomb kernel. The former leads to terms in the integrand proportional to \( F \) thus formally of \( O(g^4) \). Similarly derivatives of the Coulomb operator \( d^2 f \) lead to terms proportional to \( d^4 f / \Omega^2 \) i.e. also of \( O(g^4) \). Thus if terms proportional to the difference \( \Omega - \omega \) are kept in the numerator of the gap equation it would be necessary to include derivatives of the Coulomb kernel. However all these \( O(g^4) \) terms involve two-loop integrals and for simplicity will be neglected. The simplified gap equation then reads,

\[
\omega^2(q) = q^2 + 2 F(q) \omega(q) + \frac{N_c g^2}{4} \int [dk] \frac{1}{\omega(k)} \left[ K(q - k) \frac{\omega^2(k) - \omega^2(q)}{(q - k)^2} \right]. \tag{74}
\]

i.e. is the same as in Ref. [22] except for the term involving \( F \) in the \( r.h.s \) and all terms are of \( O(g^2) \). This
simplification is justifiable since our goal is to study the
effect of the FP determinant on the low momentum prop-
erties and thus possible modifications of UV behavior are
largely irrelevant.

In a covariant formulation the renormalized theory has
the same operator structure as the bare one. This is not
the case in the Hamiltonian approach. Renormalization
introduces non-canonical operators. The strength of such
operators is determined by the cutoff. The gap equation
has a quadratic divergence which is to be renormalized
by a gluon "mass" counter-term in the Hamiltonian,

$$\delta H(\Lambda) = \frac{1}{2} m^2(\Lambda) A_\alpha A_\alpha. \quad (75)$$

This is the only relevant operator e.g. of dimension two.
The constant $m^2(\Lambda)$ is fixed by requiring that the gap
equation leads to a $\Lambda$-independent solution. Thus we in-
sist that $\omega(k)$ is $\Lambda$-independent and this guarantees that

$$\omega(q) = \omega(\mu) - q^2 - F(q) \omega(q) + F(\mu) \omega(\mu) + \mu^2 =$$

$$+ \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \mathbf{q}) \int_{0}^{\infty} dk k^2 \left[ \omega^2(k) - \omega^2(q) \right] \frac{3 (1 + (\mathbf{k} \cdot \mathbf{q})^2) K(q - k)}{8 \omega(k)} \frac{\text{d}q}{q(k - \mu)^2} - (q \to \mu), \quad (78)$$

with the asymptotic behavior for $k \sim \Lambda \to \infty$ given by,

$$\omega^2(k) = q^2 \left[ 1 + O \left( g^2(\Lambda) \log(\Lambda^2/k^2) \right) \right]$$

$$+ \frac{\beta}{(4\pi)^2} g^2(\Lambda) Z_K(\Lambda) \Lambda^2, \quad (79)$$

any divergence of an operator matrix element calculated
with respect to the state $|\omega\rangle$ will be associated with the
operator itself and not the state. The counter-term $\delta H(\Lambda)$
contributes to the r.h.s of Eq. (74) with $m^2(\Lambda)$ and from
the UV behavior of Eq. (74) we find,

$$\frac{dm^2(\Lambda)}{d\Lambda} = - \frac{\beta}{(4\pi)^2} \left[ 2 g^2(\Lambda) \Lambda^2 \omega(\Lambda) + d^2(\Lambda) f(\Lambda) \omega(\Lambda) \right], \quad (76)$$

whose solution is

$$m^2(\Lambda) - m^2(\mu) =$$

$$= - \frac{\beta}{(4\pi)^2} \int_{\mu}^{\Lambda} \frac{dk}{\omega(k)} \left[ 2 g^2(\Lambda) k^2 + d^2(k) f(k) \omega^2(k) \right]. \quad (77)$$

In the MSR scheme the gap equation then becomes,

$$m^2(\Lambda) = \omega^2(\mu) + \mu^2$$

$$- 2 \frac{\beta}{(4\pi)^2} g^2(\Lambda) \int_{0}^{\Lambda} dk k^2 \frac{1}{\omega(k)} - \frac{\beta}{(4\pi)^2} \int_{-1}^{1} (\mathbf{k} \cdot \mathbf{\mu}) \int_{0}^{\Lambda} dk k^2 \left[ \omega^2(k) - \omega^2(\mu) \right] \frac{3 (1 + (\mathbf{k} \cdot \mathbf{\mu})^2) K(\mu - k)}{8 \omega(k)} \frac{\text{d}q}{(\mu - k)^2}. \quad (80)$$

The renormalized equations for the vev of the FP opera-
tor, the corrections to $d^2$ needed to obtain the Coulomb
potential, the gluon propagator and the ground state
wave function are given by Eqs. (82), (83), (77), and (84),
respectively. These equations depend on four parame-
ters, the renormalization constants, $d(\mu)$, $f(\mu)$, $\Omega(\mu)$
and $\omega(\mu)$. In the following section we will study the solutions
of these equations and their physical interpretation.

\section{V. RESULTS}

As discussed above there are four constants which need
to be fixed. This can be done, for example, by comparing
the Coulomb potential in position space,

$$V_{eff}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d^2(k) f(k)}{k^2} \quad (81)$$

with the lattice, static quark-antiquark potential. This
procedure was used in Ref. [22]. Unfortunately, the de-
pendence of $V_{eff}$ on the renormalization constants is
complicated thus fitting the lattice potential will not
necessarily provide much physical insight. Furthermore
it has recently been shown that there are differences
between the lattice Coulomb potential and the Wilson
loop potential [42]. We will thus proceed by simplifying
the resulting equations and imposing constraints on the
renormalization constants. The main difference between
the present analysis and what was done in Ref. [22] has to do
with inclusion of the Faddeev-Popov determinant. Our
goal here is to investigate the role of the FP determinant
trough the behavior of \( \Omega(k) \) at low momenta. If the
determinant is omitted, one has \( \Omega(k) = \omega(k) \) and in this
case the remaining three equations, for \( d(q) \), Eq. (82),
\( f(q) \), Eq. (83) and \( \omega(q) \), Eq. (85), were analyzed in
Ref. [22]. These equations have solutions provided \( \omega(k) \)
is finite as \( k \to 0 \). If \( \omega(k) \to 0 \) then the equation for
\( d(k) \) will develop a pole at a finite, positive value of mo-
mentum and if \( \omega(k) \to \infty \) as \( k \to 0 \) then for a confining
potential, \( K(k) \to 1/k^\alpha \), with \( \alpha > 2 \) the gap equation
has no solution. A renormalization condition at \( \mu = 0 \),
\( \omega(0) = m_g \) was therefore imposed with \( m_g \) fixed by the
Wilson loop string tension. A simplified set of equations
can be obtained by making an angular approximation,

\[
|q-k| \to q\theta(q-k) + k\theta(k-q)
\]

(82)

where \( q = |q| \) and \( k = |k| \). In Ref. [22] it was shown that
the angular approximation lead to results which are
very close to the exact numerical solutions. We will thus
follow this approximation here since it allows us to con-
siderably simply the numerical analysis. Using the an-
gular approximation the equation for the FP operator, \( d(k) \)
becomes,

\[
1 \frac{d(q)}{d(\mu)} - 1 \frac{d(\mu)}{d(q)} = -\beta \frac{1}{(4\pi)^2} \int_0^q dk \frac{k^2 d(q)}{q^2} \Omega(k) + \frac{\beta}{(4\pi)^2} \int_\mu^q dk \frac{d(k)}{\Omega(k)} + \frac{\beta}{(4\pi)^2} \int_0^\mu dk \frac{k^2 d(\mu)}{\mu^2 \Omega(k)},
\]

(83)

the gluon propagator function, \( \Omega \) is given by

\[
\Omega(q) = \Omega(\mu) + \omega(q) - \omega(\mu) + \frac{\beta}{(4\pi)^2} \int_0^q dk \frac{k^2 d(k)d(q)}{q^2} + \frac{\beta}{(4\pi)^2} \int_\mu^q dk d^2(k) - \frac{\beta}{(4\pi)^2} \int_0^\mu dk \frac{k^2 d(\mu)}{\mu^2 d(k)d(\mu)},
\]

(84)

and the gap equation for \( \omega(q) \) becomes,

\[
\omega^2(q) - \omega^2(\mu) - q^2 - F(q)\omega(q) + F(\mu)\omega(\mu) + \mu^2 =
\]

\[
+ \frac{\beta}{(4\pi)^2} \int_0^q dk \frac{k^2 K(q)}{q^2} \frac{\omega^2(k) - \omega^2(q)}{\omega(k)} + \int_q^\infty dk K(k) \frac{\omega^2(k) - \omega^2(q)}{\omega(k)} - (q \to \mu).
\]

(85)

As discussed above if one ignores the FP determinant it is
not possible to choose an arbitrary renormalization con-
tion for \( \omega(0) \). Furthermore in this case there is a critical
(maximum) value of \( d(\mu) = d_c(\mu) = 4\pi \sqrt{3} \omega(0)/5\mu \) for
which Eq. (83) has a solution. Appearance of such a criti-
cal coupling is an artifact of the approximation (rainbow-
ladder truncation) used in evaluation of the expectation
value of the Hamiltonian. The problem can be illustrated
by considering the following integral, as a schematic rep-
resentation of the function integrals representing the \( vev \)
of the FP operator,

\[
I(g) = \int dx \mathcal{J}(x) \frac{e^{-\omega x^2}}{1 - g x}.
\]

(86)

Here \( x \) represents the gauge potential, \( J(x) \sim e^{log(1-x)} \)
plays the role of the FP determinant, and \( 1/(1 - g x) \) the
FP operator. It one sets \( \mathcal{J} = 1 \) the integral becomes
divergent at \( x = 1 \) unless \( g = 0 \). The rainbow-latter
approximation can be though off as a procedure for evaluating the integral by expanding $1/(1-gx)$ in a power series in $gx$ and then integrating term by term keeping only a subset of contributions. In particular retaining the two-point correlations means the following approximation,

$$
\int dxx^2\omega^{-\omega x^2} \rightarrow \left[ \int dxx^2\omega^{-\omega x^2} \right]^n = \frac{1}{\omega^{3/2}}. \tag{87}
$$

This results in the following result for $I(g)$,

$$
I(g)/I(0) \sim \frac{1 - g^2/\omega}{1 - g^2/\omega}, \tag{88}
$$

which has a critical coupling $g = g_c = \sqrt{\omega}$. The FP determinant, however, makes the integral well defined for all values of $g$ and thus no critical coupling is expected in this case. This is also what happens if Eqs. (24) is taken into account. As long as $\Omega(0) > 0$ the $q = 0$ value of $\omega(q)$ does not play a role in determining the position of the pole in the FP operator, and we can for simplicity assume $\omega(q) = 0$. Then from Eq. (23) approximation $\Omega(q) = \Omega(0)$ for $q < \mu$, we obtain,

$$
d(q) = \frac{d(0)}{\left(1 + \frac{5}{3}\beta/(4\pi)^2d^2(0)q/\Omega(0)\right)^{1/2}}. \tag{89}
$$

Finally from Eq. (24) we can derive a relation between $\Omega(0)$ and $d(\mu)$,

$$
z_0\Omega(0) = \mu/2(4\pi)^2\beta d^2(0), \tag{90}
$$

where $z_0 \approx 4$ is a root of the nonlinear equation shown in Fig. 8. Thus as $d(\mu)$ increases so does $\Omega(0)$ but there is no upper limit on $d(\mu)$. This is due to the FP determinant which regularizes functional integrals near the Gribov horizon. As $\Omega(0)$ increases, the transverse-gluon

2-point correlation function decreases at low momentum and the ghost correlator function, $d(k)$ increases. This is precisely what was found in other gauges using Dyson-Schwinger methods and in other approximations to the Coulomb gauge.

In Figs. 9-11 we plot results of numerical solutions to the set of coupled equations. These should be compared with Figs. 4-6 from Ref. 22. In Fig. 12 we plot the inverse of $\Omega(k)$ which is representing the transverse-gluon correlation function. As expected it is suppressed at low momenta and approaches the perturbative limit as $k \rightarrow \infty$. 

FIG. 8: Plot of the function $f(z) \equiv 1 - (r.h.s. \text{ of Eq. (24)})/\Omega(0) = -3\ln(1 + 5z/3)/5 + 31/25 - 25/125z + 144(1 - 1/\sqrt{1 + 5z/3})/625z^2$, with $z \equiv d^2(\mu)\mu/\beta\Omega(0)$.

FIG. 9: A plot of the function $f(z) \equiv 1 - (r.h.s. \text{ of Eq. (24)})/\Omega(0) = -3\ln(1 + 5z/3)/5 + 31/25 - 25/125z + 144(1 - 1/\sqrt{1 + 5z/3})/625z^2$, with $z \equiv d^2(\mu)\mu/\beta\Omega(0)$.

FIG. 10: Numerical solution for the vacuum expectation value of the FP operator, $d(k)$.

The three curves correspond do $d(k = \mu) = 38.4$ (solid), $d(k = \mu) = 3.8$ (dashed) and $d(k = \mu) = 1.1$ (dashed-dotted) respectively, indicating that solution may exist for arbitrary choice of $d(\mu)$ i.e. no critical coupling.

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FIG. 10: Numerical solution for the vacuum expectation value of the FP operator, $d(k)$. The three curves correspond do $d(k = \mu) = 38.4$ (solid), $d(k = \mu) = 3.8$ (dashed) and $d(k = \mu) = 1.1$ (dashed-dotted) respectively, indicating that solution may exist for arbitrary choice of $d(\mu)$ i.e. no critical coupling.
VI. SUMMARY

In this paper we have studied the role of the Faddeev-Popov determinant in the Coulomb gauge. The FP determinant specifies the measure in the functional integrals over gauge field configurations and has so far been ignored in most calculations of QCD matrix elements in the Coulomb gauge. The FP determinant vanishes at the boundary of the Gribov region nevertheless it still allows for large field configurations near the boundary to enhance matrix elements. In particular we have shown that the FP operator, corresponding to the running coupling and the ghost propagator is strongly enhanced in the IR, but at the same time no artificial critical coupling exists. The same is true for the Coulomb kernel which specifies the static, temporal Wilson loop. Finally, the instantaneous part of the transverse gluon propagator is found to be suppressed as is found in other gauges.

VII. ACKNOWLEDGMENT

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