Abstract

A quantum compiler is a software program for decomposing (“compiling”) an arbitrary unitary matrix into a sequence of elementary operations (EO). Coppersmith showed that the $N_B$-bit Discrete Fourier Transform matrix $U_{FT}$ can be decomposed in a very efficient way, as a sequence of order($N_B^2$) elementary operations. Can a quantum compiler that doesn’t know a priori about Coppersmith’s decomposition nevertheless decompose $U_{FT}$ as a sequence of order($N_B^2$) elementary operations? In other words, can it rediscover Coppersmith’s decomposition by following a much more general algorithm? Yes it can, if that more general algorithm is the recursive application of the Cosine-Sine Decomposition (CSD).
1 Introduction

In quantum computing, elementary operations are operations that act on only a few (usually one or two) qubits. For example, CNOTs and one-qubit rotations are elementary operations. A quantum compiling algorithm is an algorithm for decomposing (“compiling”) an arbitrary unitary matrix into a sequence of elementary operations (SEO). A quantum compiler is a software program that implements a quantum compiling algorithm.

Henceforth, we will refer to Ref. [1] as Tuc99. Tuc99 gives a quantum compiling algorithm, implemented in a software program called Qubiter. The Tuc99 algorithm uses a matrix decomposition called the Cosine-Sine Decomposition (CSD) which is well known in the field of Computational Linear Algebra[2]. Tuc99 uses CSD in a recursive manner. Henceforth we will refer to the recursive application of CSD as re-CSD or reap-CSD.

A modest desideratum for a quantum compiler is that it should recognize when a matrix is a tensor product of one-bit operators, and decompose such a matrix into a tensor product of one-bit operators. Qubiter does this for the $N_B$-bit Hadamard matrix $H_{N_B}$.

In Ref. [3], Coppersmith showed how to express the $N_B$-bit Discrete Fourier Transform matrix $U_{FT(N_B)}$ in a very efficient way, as a sequence of order($N_B^2$) elementary operations. His decomposition will henceforth be called the quantum Fast Fourier Transform (qFFT). Another more difficult desideratum for a quantum compiler is that it should decompose $U_{FT(N_B)}$ into a sequence of order($N_B^2$) elementary operations. Qubiter does this too.

Numerical evidence that Qubiter can compile $H_{N_B}$ and $U_{FT(N_B)}$ for $N_B = 1, 2, 3, 4$ in this ideal way was reported in Tuc99. The goal of this paper is to explain analytically why Qubiter behaves in this ideal way. Qubiter does not behave this way because it is hardwired to recognize $H_{N_B}$ and $U_{FT(N_B)}$. Such a highly specialized approach would be of limited scope. Instead, the reason it behaves this way is because efficient expansions of both $H_{N_B}$ and $U_{FT(N_B)}$ can both be viewed as special cases of re-CSD, and re-CSD is Qubiter’s specialty. This is a promising result. It hints that re-CSD is a door to compiling efficiently a large class of unitary matrices that includes: $H_{N_B}, U_{FT(N_B)},$ and an infinitude of other matrices.

2 Notation

In this section, we will introduce some notation that is used throughout this paper. For additional information about notation, the reader is referred to Ref. [4]. Ref. [4] is a review paper by the author of this paper that uses the same notational conventions as this paper.

For integers $a, b$ such that $a \leq b$, let $Z_{a,b} = \{a, a+1, \ldots, b-1, b\}$. We will often use $N_B$ to denote the number of bits in a quantum register, and $N_S = 2^{N_B}$ to
denote the corresponding number of states.

First, let us introduce the members of our cast of characters that are 2d matrices. The Pauli matrices are
\[ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1} \]

This drama will also feature the 2d identity matrix and the one-bit Hadamard matrix:
\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2} \]

Note that \( H \) is related to \( e^{-i\frac{\pi}{4} \sigma_y} \), a \( \pi/2 \) rotation about the Y axis, as follows:
\[ e^{-i\frac{\pi}{4} \sigma_y} = \cos(\frac{\pi}{4}) - i\sigma_y \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = H \sigma_z. \tag{3} \]

Of course, matrices of dimension greater than 2 will also make an appearance in this play. Some will be built by using the tensor product and direct sum of matrices. In particular, for any matrix \( A \), we can tensor-multiply or direct-sum several copies of \( A \). For any positive integer \( r \), let
\[ A^{\otimes r} = A \otimes A \otimes \ldots \otimes A, \tag{4} \]
and
\[ A^{\oplus r} = A \oplus A \oplus \ldots \oplus A. \tag{5} \]

For example, \( I^{\otimes r} \otimes A = A^{\otimes 2r} \). A fact that will be useful later on is that for any two matrices \( A, B \), the transpose operation distributes over \( \oplus \) and \( \otimes \): \( (A \oplus B)^T = A^T \oplus B^T \) and \( (A \otimes B)^T = A^T \otimes B^T \).

For any positive integer \( r \), let \( D_r \):
\[ D_r = (H \sigma_z) \otimes I^{\otimes r-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I^{\otimes r-1} & -I^{\otimes r-1} \\ I^{\otimes r-1} & I^{\otimes r-1} \end{bmatrix}, \tag{6} \]
where \( I^{\otimes 0} = 1 \). Note that \( D_r \) is \( 2^r \) dimensional.

The main theme of this drama is the recursive application of the CSD (re-CSD or reap-CSD). The CSD (in the form used here) is defined as follows. Given a unitary matrix \( U \) of even dimension \( N \), \( U \) can be expressed as
\[ U = (L_0 \oplus L_1) D (R_0 \oplus R_1), \tag{7} \]
with
\[ D = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}, \tag{8} \]
and

\[ C^2 + S^2 = 1, \]  

where \( L_0, L_1, R_0, R_1 \) are unitary matrices of dimension \( N/2 \), and where \( C \) and \( S \) are real diagonal matrices. Eq.(7) is represented diagrammatically in Fig.1. Matrix \( D \) is assigned to the node, matrix \( U \) is assigned to the incoming arrow, matrices \( L_0 \oplus L_1 \) and \( R_0 \oplus R_1 \) are each assigned to an outgoing arrow. When the CSD is used recursively, then the various applications of CSD can each be represent as in Fig.1 and connected to form a CSD binary tree. We will refer to the matrix assigned to the arrow entering the root node of the CSD tree as the initial matrix, \( U_{in} \), of the tree. Fig.2 shows another convention for CSD trees that will be used here. Namely, a black-filled node will represent a node that is assigned the same matrix that is assigned to the node’s single incoming arrow.

In this paper, we are mostly concerned with “degenerate” CSD trees that have been pruned so that they contain only the leftmost or rightmost branches (See Fig.3). As discussed in most books about programming algorithms, there are several algorithms for traversing all the nodes of a tree. The algorithm that will be used in this paper (also used by Qubiter) is one of the most common, and is called the in-order tree transversal strategy. In this strategy, one visits (1) the left sub-tree, (2)the root node, (3) the right sub-tree, in a recursive manner. If one lists, in accordance with the in-order strategy, the node labels of either of the two trees in Fig.3 one obtains for both trees:

\[ U_{in} = X_1X_2X_3 \ldots X_9. \]  

(10)
Henceforth we will refer to the left and right hand side trees of Fig.\textit{3} as the \textbf{uphill} and \textbf{downhill trees}, respectively.

### 3 Hadamard Matrices

In this section, we will consider re-CSD with initial matrix equal to the $N_B$-bit Hadamard matrix $H^\otimes N_B$. This problem is closely related to the one considered in the next section, re-CSD with initial matrix equal to the $N_B$-bit Discrete Fourier Transform matrix. For simplicity, we will assume that $N_B = 4$. How to generalize our results from $N_B = 4$ to arbitrary $N_B$ will be obvious.

Fig\textit{4} shows the CSD tree that is produced by Qubiter when the initial matrix is $H^\otimes 4$. We will spend the remainder of this section explaining Fig\textit{4} (As discussed in Tuc99, the CSD is not unique. Due to this non-uniqueness, there are many possible CSD trees that can be produced from compiling the same initial matrix $H^\otimes 4$. Fig\textit{4} is just one of these possibilities. Tuc99 discusses what choices must be made in order to steer Qubiter towards producing this particular tree.)

We can express the initial matrix $H^\otimes 4$ as a product of one-qubit Hadamard matrices:

$$H^\otimes 4 = H(3)H(2)H(1)H(0).$$

Eq.\textit{(11)} can be expressed recursively as

$$H^\otimes 4 = \Gamma(3210),$$

(12a)
Figure 4: re-CSD with initial matrix equal to the 4-bit Hadamard matrix.

\[ \Gamma(3210) = (H\sigma_z)(3)\sigma_z(3)\Gamma(210), \]  
\[ \Gamma(210) = (H\sigma_z)(2)\sigma_z(2)\Gamma(10), \]  
\[ \Gamma(10) = (H\sigma_z)(1)\sigma_z(1)\Gamma(0), \]  
and

\[ \Gamma(0) = (H\sigma_z)(0)\sigma_z(0). \]  

It is convenient to translate the various bit-labelled operators in Eqs. (12) into matrices. Define \( \Gamma_r \) for \( r \in Z_{0,4} \) by

\[ \Gamma(3210) = \Gamma_4, \]  
\[ \Gamma(210) = I \otimes \Gamma_3, \]
\[ \Gamma(10) = I^\otimes 2 \otimes \Gamma_2 , \quad (13c) \]

\[ \Gamma(0) = I^\otimes 3 \otimes \Gamma_1 , \quad (13d) \]

and

\[ \Gamma_0 = 1 . \quad (13e) \]

Note that for \( r \in \mathbb{Z}_{0,4} \), \( \Gamma_r \) is a matrix of dimension \( 2^r \). In fact, \( \Gamma_r = H^\otimes r \). The bit-labelled operators \((H\sigma_z)(\alpha)\) can be expressed in terms of the \( D_r \) matrices defined by Eq.(6):

\[ (H\sigma_z)(3) = D_4 , \quad (14a) \]

\[ (H\sigma_z)(2) = I \otimes D_3 , \quad (14b) \]

\[ (H\sigma_z)(1) = I^\otimes 2 \otimes D_2 , \quad (14c) \]

and

\[ (H\sigma_z)(0) = I^\otimes 3 \otimes D_1 \quad (14d) \]

Likewise, the bit-labelled operators \( \sigma_z(\alpha) \) can be expressed as matrices:

\[ \sigma_z(3) = \begin{bmatrix} I^\otimes 3 & 0 \\ 0 & -I^\otimes 3 \end{bmatrix} , \quad (15a) \]

\[ \sigma_z(2) = I \otimes \begin{bmatrix} I^\otimes 2 & 0 \\ 0 & -I^\otimes 2 \end{bmatrix} , \quad (15b) \]

\[ \sigma_z(1) = I^\otimes 2 \otimes \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} , \quad (15c) \]

and

\[ \sigma_z(0) = I^\otimes 3 \otimes \sigma_z . \quad (15d) \]

After replacing bit-labelled operators by their matrix equivalents via Eqs.(13), (14) and (15), the recursion relation defined by Eqs.(12) becomes simply:

\[ \Gamma_{r+1} = D_{r+1} \begin{bmatrix} \Gamma_r & 0 \\ 0 & -\Gamma_r \end{bmatrix} , \quad (16) \]

for \( r \in \mathbb{Z}_{0,3} \).
The downhill tree of Fig.4 was obtained using re-CSD in combination with Eq.(16) and the following identities:

\[
\sigma_z \otimes H^\otimes 3 = H^\otimes 3 \otimes (-H^\otimes 3), \tag{17a}
\]

\[
\sigma_z \otimes H^\otimes 2 = H^\otimes 2 \otimes (-H^\otimes 2) \otimes (-H^\otimes 2) \otimes H^\otimes 2, \tag{17b}
\]

\[
\sigma_z \otimes H = H \otimes (-H) \otimes (-H) \otimes H \otimes (-H) \otimes H \otimes (-H), \tag{17c}
\]

and

\[
\sigma_z \otimes 4 = \sigma_z \otimes (-\sigma_z) \otimes (-\sigma_z) \otimes (\sigma_z \otimes (\sigma_z \otimes (\sigma_z \otimes (\sigma_z)))). \tag{17d}
\]

Listing the node matrices of the downhill tree given by Fig.4 (listing them in the order visited by an in-order tree transversal) gives:

\[
H^\otimes 4 = D_4 D_3 \oplus 2 D_2 \oplus 4 D_1 \oplus 8 \sigma_z \otimes 4. \tag{18}
\]

Using re-CSD, Qubiter expresses \(H^\otimes 4\) in the form given by the right hand side of Eq.(18). Then it recognizes that: (1) the matrices of the form \(D_r \oplus s\) are one-qubit Y-axis rotations, and (2) the matrix \(\sigma_z \otimes 4\) is a \(\sigma_z\) matrix applied separately to each qubit.

Note that Eq.(11) listed the mutually commuting operators \(\{H(\alpha) : \alpha \in Z_{0,3}\}\) in one of 4! equivalent orders. If we take the transpose of both sides of Eq.(11), we reverse the order of the \(H(\alpha)\) operators:

\[
H^\otimes 4 = H(0) H(1) H(2) H(3). \tag{19}
\]

In fact, we can take the transpose of all equations between and including Eqs.(11) to Eqs.(18). In particular, we get

\[
\Gamma_{r+1}^T = \begin{bmatrix} \Gamma_r^T & 0 \\ 0 & -\Gamma_r^T \end{bmatrix} D_{r+1}^T
\]

for \(r \in Z_{0,3}\). Also,

\[
H^\otimes 4 = \sigma_z \otimes 4 (D_1^T \oplus 8 (D_2^T \oplus 4 (D_3^T \oplus 2 D_4^T). \tag{21}
\]

Eq.(21) also follows if we list the node matrices of the uphill tree given by Fig.4 (listing them in the order visited by an in-order tree transversal).
4 Discrete Fourier Transform Matrices

In this section, we will consider re-CSD with initial matrix equal to the $N_B$-bit Discrete Fourier Transform matrix, defined by $(U_{FT})_{x,y} = \frac{1}{\sqrt{N_S}} e^{\frac{2\pi i x y}{N_S}}$, where $x, y \in \mathbb{Z}_{0,N_S-1}$. For simplicity, we will assume that $N_B = 4$.

![Diagram](image)

Figure 5: re-CSD with initial matrix equal to the 4-bit Discrete Fourier Transform matrix.

Fig. 5 shows the CSD tree that is produced by Qubiter when the initial matrix is $U_{FT}$ for $N_B = 4$. We will spend the remainder of this section explaining Fig. 5.

In Ref. [3], Coppersmith showed how to express $U_{FT}$ as a sequence of order $(N_B^2)$ elementary operations. We will call his decomposition the quantum Fast Fourier Transform (qFFT). For a pedagogical discussion of the qFFT circuit and related matters, see Ref. [4]. Ref. [4] has the virtue that it uses the same notation as this paper.
Define the root of unity $\omega$ and the 2d matrix $\Omega$ by

$$\omega = \exp(i \frac{2\pi}{N_S}), \quad \Omega = \text{diag}(1, \omega) = \omega^n,$$

(22)

where $n$ is the number operator.

As in Ref.[4], for any 2 distinct bits $\alpha, \beta \in Z_{0,N_B-1}$, let us define an operator $V(\alpha, \beta)$ by

$$V(\alpha, \beta) = \exp[i\pi \frac{n(\alpha)n(\beta)}{2^{n-\beta}}].$$

(23)

As in Ref.[4], $R$ will denote the bit reversal matrix. For $N_B = 4$, it maps bits $0 \to 3, 1 \to 2, 2 \to 1$, and $3 \to 0$. It can be expressed in terms of exchange operators, which in turn can be expressed in terms of CNOTs.

As shown in Ref.[4], the qFFT for $N_B = 4$ is

$$U_{FT} = H(3)V(3, 2)V(3, 1)V(3, 0)H(2)V(2, 1)V(2, 0)H(1)V(1, 0)H(0)R. \quad (24)$$

A diagrammatical way of saying the same thing is:

$$U_{FT} = \begin{array}{c}
\vdots \\
H \\
\vdots \\
H \\
H \\
R
\end{array}. \quad (25)$$

Note that Eq.(24) for $U_{FT}$ and Eq.(11) for $H^{\otimes 4}$ are very similar. They differ in that the expression for $U_{FT}$ contains, in addition to the one-bit Hadamard matrices, the bit reversal matrix $R$, and diagonal matrices inserted between the one-bit Hadamard matrices. It is convenient at this point to lump together the diagonal operators that occur between the one-bit Hadamard matrices. Define diagonal operators $\Delta(\cdot)$ by

$$\Delta(3210) = \sigma_z(3)V(3, 2)V(3, 1)V(3, 0), \quad (26a)$$

$$\Delta(210) = \sigma_z(2)V(2, 1)V(2, 0), \quad (26b)$$

$$\Delta(10) = \sigma_z(1)V(1, 0), \quad (26c)$$

and

$$\Delta(0) = \sigma_z(0). \quad (26d)$$
Next, let us consider re-CSD with the initial matrix $U_{FT}R$. Eq. (24) can be expressed recursively as

\[ U_{FT}R = \Gamma(3210) , \quad (27a) \]

\[ \Gamma(3210) = (H\sigma_z)(3)\Delta(3210)\Gamma(210) , \quad (27b) \]

\[ \Gamma(210) = (H\sigma_z)(2)\Delta(210)\Gamma(10) , \quad (27c) \]

\[ \Gamma(10) = (H\sigma_z)(1)\Delta(10)\Gamma(0) , \quad (27d) \]

and

\[ \Gamma(0) = (H\sigma_z)(0)\Delta(0) . \quad (27e) \]

It is convenient to translate the various bit-labelled operators in Eqs. (27) into matrices. Define $\Gamma_r$ for $r \in \mathbb{Z}_0^4$ by

\[ \Gamma(3210) = \Gamma_4 , \quad (28a) \]

\[ \Gamma(210) = I \otimes \Gamma_3 , \quad (28b) \]

\[ \Gamma(10) = I^{\otimes 2} \otimes \Gamma_2 , \quad (28c) \]

\[ \Gamma(0) = I^{\otimes 3} \otimes \Gamma_1 , \quad (28d) \]

and

\[ \Gamma_0 = 1 . \quad (28e) \]

Note that for $r \in \mathbb{Z}_0^4$, $\Gamma_r$ is a matrix of dimension $2^r$. The bit-labelled operators $(H\sigma_z)(\alpha)$ can be expressed in terms of the $D_r$ matrices defined by Eq. (6):

\[ (H\sigma_z)(3) = D_4 , \quad (29a) \]

\[ (H\sigma_z)(2) = I \otimes D_3 , \quad (29b) \]

\[ (H\sigma_z)(1) = I^{\otimes 2} \otimes D_2 , \quad (29c) \]

and

\[ (H\sigma_z)(0) = I^{\otimes 3} \otimes D_1 . \quad (29d) \]
The bit-labelled operators $\Delta(\cdot)$ can be expressed in terms of the $\Omega$ matrix defined by Eq.(22):

\[
\Delta(3210) = \sigma_z(3)e^{i\pi[n(2) + n(1) + n(0)]n(3)} = \sigma_z(3)e^{i\pi[4n(2)+2n(1)+n(0)]n(3)} \\
= \begin{bmatrix}
I^\otimes 3 & 0 \\
0 & -\Omega^4 \otimes \Omega^2 \otimes \Omega
\end{bmatrix} = A_3 \oplus B_3 , \tag{30a}
\]

\[
\Delta(210) = \sigma_z(2)e^{i\pi[n(1) + n(0)]n(2)} = \sigma_z(2)e^{i\pi[4n(1)+2n(0)]n(2)} \\
= \begin{bmatrix}
I^\otimes 2 & 0 \\
0 & -\Omega^4 \otimes \Omega^2
\end{bmatrix} = I \otimes (A_2 \oplus B_2) , \tag{30b}
\]

\[
\Delta(10) = \sigma_z(1)e^{i\pi n(0)n(1)} = \sigma_z(1)e^{i\pi[4n(0)]n(1)} \\
= \begin{bmatrix}
I^\otimes 2 & 0 \\
0 & -\Omega^4
\end{bmatrix} = I^\otimes 2 \otimes (A_1 \oplus B_1) , \tag{30c}
\]

and

\[
\Delta(0) = \sigma_z(0) = I^\otimes 3 \otimes \sigma_z = I^\otimes 3 \otimes (A_0 \oplus B_0) . \tag{30d}
\]

The matrices $A_r$ and $B_r$, where $r \in Z_{0,3}$, are first mentioned in Eqs.(30). They are implicitly defined by those equations. Note that for $r \in Z_{0,3}$, $A_r$ and $B_r$ are diagonal unitary matrices of dimension $2^r$. After replacing bit-labelled operators by their matrix equivalents via Eqs.(28), (29) and (30), the recursion relation defined by Eqs.(27) becomes simply:

\[
\Gamma_{r+1} = D_{r+1} \begin{bmatrix}
A_r \Gamma_r & 0 \\
0 & B_r \Gamma_r
\end{bmatrix} , \tag{31}
\]

for $r \in Z_{0,3}$. 

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The downhill tree of Fig. 5 was obtained using re-CSD in combination with Eq. (31).

Listing the node matrices of the downhill tree given by Fig. 5 (listing them in the order visited by an in-order tree transversal) gives:

\[
U_{FT} \Gamma_4 = D_4(A_3 \oplus B_3)D_3^{\otimes 2}(A_2 \oplus B_2)D_2^{\otimes 4}(A_1 \oplus B_1) \oplus 4 \sigma_z \oplus 8 . 
\] (32)

Qubiter first finds the necessary permutation \( R \) using a strategy to be discussed in the next section. Then Qubiter uses re-CSD to express \( U_{FT} \Gamma_4 \) in the form given by the right hand side of Eq. (32). Then it recognizes that: (1) the matrices of the form \( D_r^{\otimes s} \) are one-qubit Y-axis rotations, (2) the diagonal matrices \( \sigma_z^{\otimes 8} \) and \( (A_r \oplus B_r)^{\otimes s} \) are expressible as products of doubly controlled phase factors (such as \( V \)).

If we take the transpose of both sides of Eq. (24), we reverse the order of all the operators on the right hand side:

\[
U_{FT} = RH(0)V(1, 0)H(1)V(2, 0)V(2, 1)H(2)V(3, 0)V(3, 1)V(3, 2)H(3) .
\] (33)

Diagrammatically,

In fact, we can take the transpose of all equations between and including Eqs. (24) to Eqs. (32). In particular, we get

\[
\Gamma_{r+1}^T = \begin{bmatrix} \Gamma_r^T & 0 \\ 0 & \Gamma_r^T \end{bmatrix} D^{\otimes 4}_{r+1}
\] (35)

for \( r \in \mathbb{Z}_{0,3} \). Also,

\[
RU_{FT} = \Gamma_4^T = \sigma_z^{\otimes 8}(A_1 \oplus B_1)^{\otimes 4}(D_2^{\otimes 4}(A_2 \oplus B_2)^{\otimes 2}(D_3^{\otimes 2}(A_3 \oplus B_3)D_4^{\otimes 2} .
\] (36)

Eq. (36) also follows if we list the node matrices of the uphill tree given by Fig. 5 (listing them in the order visited by an in-order tree transversal).

\section*{5 Bit Permutations Before Each CSD}

In decomposing \( U_{FT} \) via re-CSD, we first pre or post multiplied \( U_{FT} \) by the bit reversal matrix \( R \). This example makes it clear that in using re-CSD, before each application
of the CSD, it is helpful to permute either the rows or the columns (or both) of the input matrix $U$ in Fig. 1. Only a certain type of permutation will work; it must also be a bit permutation, so that it can be expressed as a product of bit exchange operators, which in turn can be expressed as a product of CNOTs. But what is a good strategy for choosing such a permutation?

Decompositions such as the ones given here for $H^\otimes 4$ and $U_{FT}$ come from CSD trees that contain only a single branch. To promote the growth of such degenerate trees, we want to stunt the growth of either the left matrices $L_0, L_1$ or the right matrices $R_0, R_1$ in Fig. 1. The left matrices are “stunted” if $L_0$ and $L_1$ are diagonal matrices. Likewise, the right matrices are “stunted” if $R_0$ and $R_1$ are diagonal.

The general CSD is

$$
\begin{bmatrix}
U_{00} & U_{01} \\
U_{10} & U_{11}
\end{bmatrix}
= \begin{bmatrix}
L_0 & 0 \\
0 & L_1
\end{bmatrix}
\begin{bmatrix}
C & S \\
-S & C
\end{bmatrix}
\begin{bmatrix}
R_0 & 0 \\
0 & R_1
\end{bmatrix}
= \begin{bmatrix}
L_0C R_0 & L_0SR_1 \\
L_1(-S)R_0 & L_1CR_1
\end{bmatrix}.
$$

(37)

When the left matrices $L_0, L_1$ are diagonal, $U_{00}U_{10}^\dagger$ and $U_{01}U_{11}^\dagger$ are diagonal matrices. Analogously, when the right matrices $R_0, R_1$ are diagonal, $U_{00}U_{01}^\dagger$ and $U_{10}U_{11}^\dagger$ are diagonal matrices. So probably a good strategy for selecting a row and/or column permutation to perform before each CSD is to find a bit permutation that minimizes the absolute value of the off-diagonal elements of $U_{00}U_{10}^\dagger \oplus U_{01}U_{11}^\dagger$ and/or $U_{00}U_{01}^\dagger \oplus U_{10}U_{11}^\dagger$. The idea is to coax the growth of sparse trees that have only a few branches (rightmost and leftmost branches might or might not be included in such a tree.)

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