CONTEXTUALITY, COHOMOLOGY AND PARADOX

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Abstract. Contextuality is a key feature of quantum mechanics that provides an important non-classical resource for quantum information and computation. Abramsky and Brandenburger used sheaf theory to give a general treatment of contextuality in quantum theory [New Journal of Physics 13 (2011) 113036]. However, contextual phenomena are found in other fields as well, for example database theory. In this paper, we shall develop this unified view of contextuality. We provide two main contributions: firstly, we expose a remarkable connection between contextuality and logical paradoxes; secondly, we show that an important class of contextuality arguments has a topological origin. More specifically, we show that “All-vs-Nothing” proofs of contextuality are witnessed by cohomological obstructions.

1. Introduction

Contextuality is one of the key characteristic features of quantum mechanics. It has been argued that it provides the “magic” ingredient enabling quantum computation [25]. There have been a number of recent experimental verifications that Nature does indeed exhibit this highly non-classical form of behaviour [8, 26, 52, 53].

The study of quantum contextuality has largely been carried out in a concrete, example-driven fashion, which makes it appear highly specific to quantum mechanics. Recent work by the present authors [2, 36] and others [14, 17, 19] has exposed the general mathematical structure of contextuality, enabling more general and systematic results. It has also made apparent that contextuality is a general and indeed pervasive phenomenon, which can be found in many areas of classical computation, such as databases [1] and constraints [3]. The work in [2] makes extensive use of methods developed within the logic and semantics of computation.

The key idea from [2] is to understand contextuality as arising where we have a family of data which is locally consistent, but globally inconsistent. This can be understood, and very effectively visualised (see Fig. 6) in topological terms: we have a base space of contexts (typically sets of variables which can be jointly measured or observed), a space of data or observations fibred over this space, and a family of local sections (typically valuations of the variables in the context) in these fibres. This data is consistent locally, but not globally: there is no global section defined on all the variables which reconciles all the local data. In topological language, we can say that the space is “twisted”, and hence provides an obstruction to forming a global section.

This provides a unifying description of a number of phenomena which at first sight seem very different:

- **Quantum contextuality.** The local data arises from performing measurements on compatible sets of observables. The fact that there is no global section corresponds to a no-go result for a hidden-variable theory to explain the observable data.

- **Databases.** The local data are the relation tables of the database. The fact that there is no global section corresponds to the failure in general of the universal relation assumption [18, 28, 33, 63].

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• Constraint satisfaction. The local data corresponds to the constraints, defined on subsets of the variables. The fact that there is no global section corresponds to the non-existence of a solution for the CSP.

In the present paper, we shall develop this unified viewpoint to give a logical perspective on contextuality. In particular, we shall look at contextuality in relation to logical paradoxes:

• We find a direct connection between the structure of quantum contextuality and classic semantic paradoxes such as “Liar cycles” [16,30,50,51]. There are also close connections to the study of circuits with feedback [11].

• Conversely, contextuality offers a novel perspective on these paradoxes. Contradictory cycles of references give rise to exactly the form of local consistency and global inconsistency we find in contextuality.

Mathematical structure. Sheaf theory [32] provides the natural mathematical setting for our analysis, since it is directly concerned with the passage from local to global. In this setting, it is furthermore natural to use sheaf cohomology to characterise contextuality. Cohomology is one of the major tools of modern mathematics, which has until now been conspicuous by its absence, both in theoretical computer science, and in quantum information. The use of cohomology to characterise contextuality was initiated in [5]. In the present paper, we take the cohomological approach considerably further, taking advantage of situations in which the outcomes of observations have an algebraic structure. This applies, for example, in the case of the standard Pauli spin observables, which have eigenvalues in \( \mathbb{Z}_2 \).

We study a strong form of contextuality arising from so-called “All-vs-Nothing” arguments [37]. We give a much more general formulation of such arguments than has appeared previously, in terms of local consistency and global inconsistency of systems of linear equations. We also show how an extensive class of examples of such arguments arises in the stabiliser fragment of quantum mechanics, which plays an important role in quantum error correction [42] and measurement-based quantum computation [47].

We then show how all such All-vs-Nothing arguments are witnessed by the cohomological obstruction to the extension of local sections to global ones previously studied in [5]. This obstruction is characterised in more abstract terms than previously, using the connecting homomorphism of the long exact sequence. Our main theorem establishes a hierarchy of properties of probability models, relating their algebraic, logical and topological structures.

Outline. The further structure of the paper is as follows. In §2 we show how our topological approach provides a unified description of many apparently different phenomena, linking contextuality, paradoxes and the “logical twisting” of spaces, leading to locally consistent data with obstructions to global consistency. This approach is then formalised in §3. In §4 we introduce All-vs-Nothing arguments and show how they arise concretely in stabiliser states. We then give our general formulation of these arguments as global logical inconsistencies. In §5 we develop the formal language of sheaf cohomology and apply it to provide cohomological witnesses of contextuality. Finally, in §6 we prove our main result, which relates the non-vanishing of the cohomology invariant to global logical inconsistency.

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2. The many faces of contextuality

We begin with the following scenario, depicted in Fig. 1. Alice and Bob are agents positioned at nodes of a network. Alice can access local bit registers \( a_1 \) and \( a_2 \), while Bob can access local bit
registers $b_1, b_2$. Alice can load one of her bit registers into a processing unit, and test whether it is 0 or 1. Bob can perform the same operations with respect to his bit registers. They send the outcomes of these operations to a common target, which keeps a record of the joint outcomes.

We now suppose that Alice and Bob perform repeated rounds of these operations. On different rounds, they may make different choices of which bit registers to access, and they may observe different outcomes for a given choice of register. The target can compile statistics for this series of data, and infer probability distributions on the outcomes.

The probability table in Fig. 2 (ignore the highlights for the moment) records the outcome of such a process. Each row of the table specifies a probability distribution on the possible joint outcomes, conditioned on the indicated choice of bit registers by Alice and Bob. Consider for example the cell at row 2, column 3 of the table. This corresponds to:

• Alice loads register $a_1$ and observes the value 0.
• Bob loads register $b_2$ and observes the value 1.

This event has the probability $1/8$, conditioned on Alice’s choice of $a_1$ and Bob’s choice of $b_2$.

We can now ask: *How can such an observational scenario be realised?* The obvious classical mechanism we can propose to explain these observations is depicted in Fig. 3. We postulate a source which on each round chooses values for each of the registers $a_1, a_2, b_1, b_2$, and loads each register with the chosen value. Alice and Bob will then observe the values which have been loaded by the source. We can suppose that this source is itself randomised, and chooses the values for the registers according to some probability distribution $P$ on the set of $2^4$ possible assignments. Then the question becomes: is there any distribution $P$ which would give rise to the table specified in Fig. 2?
A key observation is that, in order for this question to be non-trivial, we must assume that the choices of bit registers made by Alice and Bob are independent of the source. If the source could determine which registers are to be loaded on each round, as well as their values, then it becomes a trivial matter to achieve any given probability distribution on the joint outcomes.

Under this assumption of independence, it becomes natural to think of this scenario as a kind of correlation game. The aim of the source is to achieve as high a degree of correlation between the outcomes of Alice and Bob as possible, regardless of their choices on each round.

2.1. Logic rings a Bell. We shall now make an apparently innocuous deduction in elementary logic and probability theory.

Suppose we have propositional formulas $\varphi_1, \ldots, \varphi_N$, and that we can assign a probability $p_i$ to each $\varphi_i$. In particular, we have in the mind the situation where the boolean variables appearing in $\varphi_i$ correspond to empirically testable quantities, such as the values of bit registers in our scenario; $\varphi_i$ then expresses a condition on the outcomes of an experiment involving these quantities. The probabilities $p_i$ are obtained from the statistics of these experiments.

Now suppose that these formulas are not simultaneously satisfiable. Then (e.g.)

$$\bigwedge_{i=1}^{N-1} \varphi_i \rightarrow \neg \varphi_N \; , \; \text{or equivalently} \; \varphi_N \rightarrow \bigvee_{i=1}^{N-1} \neg \varphi_i .$$

Using elementary probability theory, we can calculate:

$$p_N \leq \text{Prob}\left(\bigvee_{i=1}^{N-1} \neg \varphi_i \right) \leq \sum_{i=1}^{N-1} \text{Prob}(\neg \varphi_i) = \sum_{i=1}^{N-1} (1 - p_i) .$$

The first inequality is the monotonicity of probability, and the second is sub-additivity. Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \leq N - 1 .$$

We shall refer to this as a logical Bell inequality. Note that it hinges on a purely logical consistency condition.

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1. This translates formally into a conditional independence assumption, which we shall not spell out here; see e.g. [10][12].
2.2. Logical analysis of the Bell table. We return to the probability table from Fig. 2. If we read 0 as true and 1 as false, the highlighted events (sets of assignments) in each row of the table are represented by the following propositions:

\[ \varphi_1 = (a_1 \land b_1) \lor (\neg a_1 \land \neg b_1) = a_1 \leftrightarrow b_1 \]
\[ \varphi_2 = (a_1 \land b_2) \lor (\neg a_1 \land \neg b_2) = a_1 \leftrightarrow b_2 \]
\[ \varphi_3 = (a_2 \land b_1) \lor (\neg a_2 \land \neg b_1) = a_2 \leftrightarrow b_1 \]
\[ \varphi_4 = (\neg a_2 \land b_2) \lor (a_2 \land \neg b_2) = a_2 \leftrightarrow \neg b_2 . \]

The events on the first three rows are the correlated outcomes; the fourth is anticorrelated. These propositions are easily seen to be jointly unsatisfiable. It follows that our logical Bell inequality should apply, yielding the inequality \( \sum_{i=1}^{4} p_i \leq 3 \). However, we see from the table that \( p_1 = 1 \) and \( p_i = 6/8 \) for \( i = 2, 3, 4 \). Hence the table yields a violation of the Bell inequality by \( 1/4 \).

This rules out the possibility of giving an explanation for the observational behaviour described by the table in terms of a classical source. We might then conclude that such behaviour simply cannot be realised. However, in the presence of quantum resources, and in particular entangled quantum states \[42\], this is no longer the case. More specifically, if we use the Bell state \((|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}\), with Alice and Bob performing 1-qubit local measurements corresponding to directions in the \( XY \)-plane of the Bloch sphere, at relative angle \( \pi/3 \), then this behaviour is physically realisable according to the predictions of quantum mechanics—our most highly-confirmed physical theory\[4\].

More broadly, we can say that this shows that quantum mechanics predicts correlations which exceed those which can be achieved by any classical mechanism. This is the content of Bell’s theorem \[9\], a famous result in the foundations of quantum mechanics, and in many ways the starting point for the whole field of quantum information. Moreover, these predictions have been confirmed by many experiments \[6, 7\].

The logical Bell inequality we used to derive this result is taken from \[4\]. Bell inequalities are a central technique in quantum information. In \[4\] it is shown that every Bell inequality (i.e. every inequality satisfied by the “local polytope”) is equivalent to a logical Bell inequality, based on purely logical consistency conditions.

2.3. Logical forms of contextuality. We shall now see how the same phenomena manifest themselves in a stronger form, which highlights a direct connection with logic.

Consider the table in Fig. 4 which depicts the same kind of scenario we considered previously. However, the entries are now either 0 or 1. The idea is that a 1 entry represents a positive probability. Thus we are distinguishing only between possible (positive probability) and impossible (zero probability). In other words, the rows correspond to the supports of some (otherwise unspecified) probability distributions. Note that only four entries of the table are filled in. Our claim is that

\[
\begin{array}{c|cccc}
A & B & (0,0) & (1,0) & (0,1) & (1,1) \\
\hline
a_1 & b_1 & 1 & & & \\
a_1 & b_2 & 0 & & & \\
a_2 & b_1 & 0 & & & \\
a_2 & b_2 & 0 & & & \\
\end{array}
\]

Figure 4. The Hardy paradox

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2Here, the choice of measurement directions by Alice and Bob plays the rôle of the “choice of local bit registers to load” in our original description.
just from these entries, referring only to the supports, we can deduce that there is no classical explanation for the behaviour recorded in the table. Moreover, this behaviour can be realised in quantum mechanics [24,41], yielding a stronger form of Bell’s theorem, due to Hardy [24].

What do “observables” observe? Classically, we would take the view that physical observables directly reflect properties of the physical system we are observing. These are objective properties of the system, which are independent of our choice of which measurements to perform, i.e. of our measurement context. More precisely, this would say that for each possible state of the system, there is a function \( \lambda \) which for each measurement \( m \) specifies an outcome \( \lambda(m) \), independently of which other measurements may be performed. This point of view is called non-contextuality, and may seem self-evident. However, this view is impossible to sustain in the light of our actual observations of (micro)-physical reality.

Consider once again the Hardy table depicted in Fig. 4. Suppose there is a function \( \lambda \) which accounts for the possibility of Alice observing value 0 for \( a_1 \) and Bob observing 0 for \( b_1 \), as asserted by the entry in the top left position in the table. Then this function \( \lambda \) must satisfy

\[
\lambda :: a_1 \mapsto 0, \quad b_1 \mapsto 0 .
\]

Now consider the value of \( \lambda \) at \( b_2 \). If \( \lambda(b_2) = 0 \), then this would imply that the event that \( a_1 \) has value 0 and \( b_2 \) has value 0 is possible. However, this is precluded by the 0 entry in the table for this event. The only other possibility is that \( \lambda(b_2) = 1 \). Reasoning similarly with respect to the joint values of \( a_2 \) and \( b_2 \), we conclude, using the bottom right entry in the table, that we must have \( \lambda(a_2) = 0 \). Thus the only possibility for \( \lambda \) consistent with these entries is

\[
\lambda :: a_1 \mapsto 0, \quad a_2 \mapsto 0, \quad b_1 \mapsto 0, \quad b_2 \mapsto 1 .
\]

But this would require the outcome \((0,0)\) for measurements \((a_2,b_1)\) to be possible, and this is precluded by the table.

It is easily seen that this argument amounts to showing that the following formulas are not jointly satisfiable:

\[
a_1 \land b_1, \quad \neg(a_1 \land b_2), \quad \neg(a_1 \land b_1), \quad a_2 \lor b_2 .
\]

We are thus forced to conclude that the Hardy models are contextual. Moreover, we can say that they are contextual in a logical sense, stronger than the probabilistic form we saw with the Bell tables, since we only needed information about possibilities to infer the contextuality of this behaviour.

**Strong contextuality.** Logical contextuality as exhibited by the Hardy paradox can be expressed in the following form: there is a local assignment (in the Hardy case, the assignment \( a_1 \mapsto 0, b_1 \mapsto 0 \)) which is in the support, but which cannot be extended to a global assignment which is compatible with the support. This says that the support cannot be covered by the projections of global assignments. A stronger form of contextuality is when no global assignments are consistent with the support at all. Note that this stronger form does not hold for the Hardy paradox.

Several much-studied constructions from the quantum information literature exemplify strong contextuality. An important example is the Popescu–Rohrlich (PR) box [46] shown in Fig. 5:

This is a behaviour which satisfies the no-signalling principle [20], meaning that the probability of Alice observing a particular outcome for her choice of measurement (e.g. \( a_1 = 0 \)), is independent of whether Bob chooses measurement \( b_1 \) or \( b_2 \); and vice versa. That is, Alice and Bob cannot signal to one another—the importance of this principle is that it enforces compatibility with relativistic constraints. However, despite satisfying the no-signalling principle, the PR box does not admit a quantum realisation. Note that the full support of this model correspond to the propositions used in showing the contextuality of the Bell table from Fig. 2, and hence the fact that these propositions are not simultaneously satisfiable shows the strong contextuality of the model.
Figure 5. The PR Box

| A | B | (0,0) | (1,0) | (0,1) | (1,1) |
|---|---|-------|-------|-------|-------|
| $a_1$ | $b_1$ | 1     | 0     | 0     | 1     |
| $a_1$ | $b_2$ | 1     | 0     | 0     | 1     |
| $a_2$ | $b_1$ | 1     | 0     | 0     | 1     |
| $a_2$ | $b_2$ | 0     | 1     | 1     | 0     |

Figure 6. The Hardy table and the PR box as bundles

In fact, there is provably no bipartite quantum-realisable behaviour of this kind which is strongly contextual [29, 35]. However, as soon as we go to three or more parties, strong contextuality does arise from entangled quantum states, as we shall see in §4.

Visualizing contextuality. The tables which have appeared in our examples can be displayed in a visually appealing way which makes the fibred topological structure apparent, and forms an intuitive bridge to the formal development of the sheaf-theoretic ideas in the next section.

Firstly, we look at the Hardy table from Fig. 4, displayed as a “bundle diagram” on the left of Fig. 6. Note that all unspecified entries of the Hardy table are set to 1.

What we see in this representation is the base space of the variables $a_1$, $a_2$, $b_1$, $b_2$. There is an edge between two variables when they can be measured together. The pairs of co-measurable variables correspond to the rows of the table. In terms of quantum theory, these correspond to pairs of compatible observables. Above each vertex is a fibre of those values which can be assigned to the variable—in this example, 0 and 1 in each fibre. There is an edge between values in adjacent fibres precisely when the corresponding joint outcome is possible, i.e. has a 1 entry in the table. Thus there are three edges for each of the pairs $\{a_1, b_2\}$, $\{a_2, b_1\}$ and $\{a_2, b_2\}$.

A global assignment corresponds to a closed path traversing all the fibres exactly once. We call such a path univocal since it assigns a unique value to each variable. Note that there is such a path, marked in blue; thus the Hardy model is not strongly contextual. However, there is no such path which includes the edge displayed in red. This shows the logical contextuality of the model.

Next, we consider the PR box displayed as a bundle on the right of Fig. 6. In this case, the model is strongly contextual, and accordingly there is no univocal closed path.
$S_1 : S_2$ is true,
$S_2 : S_3$ is true,
\vdots
$S_{N-1} : S_N$ is true,
$S_N : S_1$ is false.

Figure 7. The Liar Cycle

**Contextuality, logic and paradoxes.** The arguments for quantum contextuality we have discussed may be said to skirt the borders of paradox, but they do not cross those borders. The information we can gather from observing the co-measurable variables is locally consistent, but it cannot in general be pieced together into a globally consistent assignment of values to all the variables simultaneously. Thus we must give up the idea that physically observable variables have objective, “real” values independent of the measurement context being considered. This is very disturbing for our understanding of the nature of physical reality, but there is no direct contradiction between logic and experience. We shall now show that a similar analysis can be applied to some of the fundamental logical paradoxes.

**Liar cycles.** A Liar cycle of length $N$ is a sequence of statements as shown in Fig. 7.

For $N = 1$, this is the classic Liar sentence

$S : S$ is false.

These sentences contain two features which go beyond standard logic: references to other sentences, and a truth predicate. While it would be possible to make a more refined analysis directly modelling these features, we will not pursue this here, noting that it has been argued extensively and rather compellingly in much of the recent literature on the paradoxes that the essential content is preserved by replacing statements with these features by boolean equations [16, 30, 50, 51]. For the Liar cycles, we introduce boolean variables $x_1, \ldots, x_n$, and consider the following equations:

$$
x_1 = x_2, \quad \ldots, \quad x_{n-1} = x_n, \quad x_n = \neg x_1 .
$$

The “paradoxical” nature of the original statements is now captured by the inconsistency of these equations.

Note that we can regard each of these equations as fibered over the set of variables which occur in it:

$$
\{x_1, x_2\} : x_1 = x_2 \\
\{x_2, x_3\} : x_2 = x_3 \\
\vdots \\
\{x_{n-1}, x_n\} : x_{n-1} = x_n \\
\{x_n, x_1\} : x_n = \neg x_1
$$

Any subset of up to $n - 1$ of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, the Liar cycle of length 4 corresponds exactly to the PR box. The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram on the right of Fig. 6. To relate the notations, we make the
following correspondences between the variables of Fig. 6 and those of the boolean equations:

\[ x_1 \sim a_2, \ x_2 \sim b_1, \ x_3 \sim a_1, \ x_4 \sim b_2. \]

Thus we can read the equation \( x_1 = x_2 \) as \( "a_2 \text{ is correlated with } b_1" \), and \( x_4 = \neg x_1 \) as \( "a_2 \text{ is anti-correlated with } b_2" \).

Now suppose that we try to set \( a_2 \) to 1. Following the path in Fig. 6 on the right leads to the following local propagation of values:

\[
\begin{align*}
 a_2 = 1 &\Rightarrow b_1 = 1 \Rightarrow a_1 = 1 \Rightarrow b_2 = 1 \Rightarrow a_2 = 0 \\
 a_2 = 0 &\Rightarrow b_1 = 0 \Rightarrow a_1 = 0 \Rightarrow b_2 = 0 \Rightarrow a_2 = 1
\end{align*}
\]

The first half of the path corresponds to the usual derivation of a contradiction from the assumption that \( S_1 \) is true, and the second half to deriving a contradiction from the assumption that \( S_1 \) is false.

We have discussed a specific case here, but the analysis can be generalised to a large class of examples along the lines of [16,50]. The tools from sheaf cohomology which we will develop in the remainder of the paper can be applied to these examples. We plan to give an extended treatment of these ideas in future work.

*The Robinson Consistency Theorem.* As a final remark on the connections between contextuality and logic, we consider a classic result, the Robinson Joint Consistency Theorem [48]. It is usually formulated in a first-order setting. The version we will use has the following form:

**Theorem 2.1** (Robinson Joint Consistency Theorem). Let \( T_i \) be a theory over the language \( L_i \), \( i = 1, 2 \). If there is no sentence \( \phi \) in \( L_1 \cap L_2 \) with \( T_1 \vdash \phi \) and \( T_2 \vdash \neg \phi \), then \( T_1 \cup T_2 \) is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency. Note, however, that an extension of the theorem beyond the binary case fails. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently. A minimal counter-example is provided at the propositional level by the following “triangle”:

\[
T_1 = \{ x_1 \leftrightarrow \neg x_2 \}, \quad T_2 = \{ x_2 \leftrightarrow \neg x_3 \}, \quad T_3 = \{ x_3 \leftrightarrow \neg x_1 \}.
\]

This example is well-known in the quantum contextuality literature as the *Specker triangle* [31]. Although not quantum realizable, it serves as a basic template for more complex examples which are. See [2,13,27] for further details.

### 3. Sheaf formulation of contextuality

In this section we summarise the main ideas of the sheaf-theoretic formalism for contextuality [2].

In [2,3] we saw that logical contextuality can be expressed in terms of a bundle of outcomes over a base space of measurements and contexts. This idea can be formalized by regarding the bundle as a sheaf.

For our purposes, it will be sufficient to view the base space as the discrete space on a finite set \( X \) of variables. In the quantum case, these variables will be labels for measurements. The measurement contexts will be represented by a family \( \mathcal{M} = \{ C_i \}_{i \in I} \) of subsets of \( X \). These are the sets of variables which can be measured together—in quantum terms, the compatible families of observables. We assume that \( \mathcal{M} \) covers \( X \), i.e. \( \bigcup \mathcal{M} = X \); hence we call \( \mathcal{M} \) a *measurement cover.* We shall also assume that \( \mathcal{M} \) forms an antichain, so these are the *maximal contexts.* We also assume that all the variables have the same fibre, \( O \), of values or outcomes that can be assigned to them. Such a triple \( \langle X, \mathcal{M}, O \rangle \) is called a *measurement scenario.* We define a presheaf of sets over \( \mathcal{P}(X) \), namely \( \mathcal{E} : U \mapsto O^U \) with restriction

\[
\mathcal{E}(U \subseteq U') : \mathcal{E}(U') \rightarrow \mathcal{E}(U) : s \mapsto s|_U.
\]
showed the GHZ state to be strongly contextual using a kind of argument he dubbed ‘all versus quantum strong contextuality (though not in the general terms that we do) was Mermin [37], who uses sheaf cohomology, as we will see in §5.

Contextuality can be characterized as above. This formulation of contextuality makes it natural to support of the distributions $M(E_1), (E_2), \text{and } (E_3)$.

By an empirical model $\langle X, M, O \rangle$, we mean a sub-presheaf $S$ of $E$ satisfying (E1), (E2), and (E3).

In [2], we used the term “empirical model” for the probability table $\{pc\}_{C \in M}$. In the present paper, we shall only work with the associated support presheaf $S$, and so it is more convenient to refer to this as the model.

We can use this formalisation to characterize contextuality as follows.

Definition 3.2. For any empirical model $S$:

- For all $C \in M$ and $s \in S(C)$, $S$ is logically contextual at $s$, written $LC(S, s)$, if $s$ is not a member of any compatible family. $S$ is logically contextual, written $LC(S)$, if $LC(S, s)$ for some $s$.

- $S$ is strongly contextual, written $SC(S)$, if $LC(S, s)$ for all $s$. Equivalently, if it has no global section, i.e. if $S(X) = \emptyset$.

Note that, for every probability table $\{pc\}_{C \in M}$ that satisfies the no-signalling principle, the supports of the distributions $pc$ induce an empirical model $S$, and therefore logical or strong contextuality can be characterized as above. This formulation of contextuality makes it natural to use sheaf cohomology, as we will see in [5].

4. All-vs-Nothing Arguments

Quantum theory provides many instances of strong contextuality. Among the first to observe quantum strong contextuality (though not in the general terms that we do) was Mermin [37], who showed the GHZ state to be strongly contextual using a kind of argument he dubbed ‘all versus
nothing’. We show in [4.1] that arguments of this type can in fact be used to prove strong contextuality for a large class of states in quantum theory, particularly in stabiliser quantum mechanics, which plays a crucial role in quantum computation. Moreover, in [4.2] we give a much more general formulation of this type of argument that can be used to show strong contextuality for a much larger class of models.

4.1. All-vs-Nothing for quantum theory. The GHZ state is a tripartite state of qubits, defined as $(|↑↑↑⟩ + |↓↓↓⟩)/\sqrt{2}$. We assume that each party $i = 1, 2, 3$ can perform Pauli measurements in $\{X_i, Y_i\}$, and each measurement has outcomes in $O = Z_2 = \{0, 1\}$. Then, following Mermin’s argument, the possible joint outcomes satisfy these parity equations:

$$X_1 \oplus Y_2 \oplus Y_3 = 0 \quad Y_1 \oplus X_2 \oplus Y_3 = 0$$

$$Y_1 \oplus Y_2 \oplus X_3 = 0 \quad X_1 \oplus X_2 \oplus X_3 = 1$$

This system of equations is inconsistent because, regardless of the outcomes assigned to the observables $X_1, \ldots, X_3$, the left-hand sides sum to 0 (since each variable occurs twice) whereas the right-hand sides sum to 1. This shows that the model is strongly contextual, as there is no global assignment of outcomes to observables consistent with the observed local assignments.

The essence of the argument is that the possible local assignments satisfy systems of parity equations that admit no global solution. We call this an All-vs-Nothing argument.

In fact, such arguments arise naturally from a much larger class of states in stabiliser quantum theory [42]. Consider the Pauli $n$-group $P_n$, whose elements are $n$-tuples of Pauli operators (from $\{X, Y, Z, I\}$) with a global phase from $\{\pm 1, \pm i\}$.

**Definition 4.1.** An AvN triple in $P_n$ is a triple $(e, f, g)$ of elements of $P_n$ with global phases +1, which pairwise commute, and which satisfy the following conditions:

(A1) For each $i = 1, \ldots, n$, at least two of $e_i, f_i, g_i$ are equal.  
(A2) The number of $i$ such that $e_i = g_i \neq f_i$, all distinct from $I_i$, is odd.

Mermin’s argument, and the other All-vs-Nothing arguments which have appeared in the literature, can be seen to come down to exhibiting AvN triples.

**Theorem 4.2.** Let $S$ be the subgroup of $P_n$ generated by an AvN triple, and $V_S$ the subspace stabilised by $S$. For every state $|\psi⟩$ in $V_S$, the empirical model realised by $|\psi⟩$ under the Pauli measurements admits an All-vs-Nothing argument.

**Proof.** Firstly, we recall the quantum mechanics formula for the expected value of an observable $A$ on a state $v$:

$$\langle A \rangle_v = \langle v | A | v \rangle.$$

Note that

$$\langle v | A | v \rangle = 1 \iff A | v \rangle = | v \rangle.$$

Thus $A$ stabilises the state $v$ iff the expected value is 1. Suppose that $A$ is a dichotomic observable, with eigenvalues +1, −1 (see footnote 3), and $v$ is a state it stabilises. The support of the distribution on joint outcomes obtained by measuring $A$ on $v$ must contain only outcomes of even parity; while if $-A$ stabilises $v$, then the support will contain only outcomes of odd parity. If $A = P_1 \cdots P_n$ in $P_n$, the former case translates into the equation

$$x_1 \oplus \cdots \oplus x_n = 0$$

3 Although the eigenvalues of the Pauli matrices are +1 and −1, we relabel +1, −1, × as 0, 1, ⊕, respectively. The eigenvalues of a joint measurements $A_1 \otimes A_2 \otimes A_3$ are the products of the eigenvalues of the measurements at each site, so they are also ±1. As such, in the usual representation, these joint measurements are still dichotomic and only distinguish joint outcomes up to parity. Mermin’s argument shows that this information is sufficient to derive strong contextuality.
where we associate the variable $x_i$ with $P_i$; while in the latter case, it corresponds to the equation

$$x_1 \oplus \cdots \oplus x_n = 1.$$ 

If the set of equations satisfied by a state $v$ stabilised by a subgroup $S$ of $\mathcal{P}_n$ is inconsistent, we say that $v$ admits an All-vs-Nothing argument with respect to the measurements $h$ with global phase $+1$ such that either $h$ or $-h$ is in $S$.

We now show that any state $v$ in the subspace $V_S$ stabilised by the subgroup $S$ generated by an AvN triple $\langle e, f, g \rangle$ admits an All-vs-Nothing argument. Firstly, by the algebra of the Pauli matrices, we see from (A1) that if $\{e_i, f_i, g_i\} = \{P, Q\}$, with at least two equal to $P$, the componentwise product $e_i f_i g_i$ will, disregarding global phase, be $Q$. By (A2), we see that the product $efg = -h$, an element of $\mathcal{P}_n$ with global phase $-1$, which translates into a condition of odd parity on the support of any state stabilised by these operators for the measurement $h$. On the other hand, condition (A1) implies that under any global assignment to the variables, we can cancel the repeated items in each column, and deduce an even parity for $h$. \hfill \Box

If $e, f, g$ have linearly independent check vectors, they generate a subgroup $S$ such that $V_S$ has dimension $2^{n-3}$ \footnote{All rings considered in this paper will be commutative and with unit.}. Thus we obtain a large class of states admitting All-vs-Nothing arguments.

### 4.2. Generalized All-vs-Nothing arguments

Despite their established use in the quantum literature, All-vs-Nothing arguments as considered above do not exist for all strongly contextual models. Nonetheless, a natural generalization applies to more models. For instance, the model called ‘box 25’ in \cite{AVN-gh} admits no such (parity) argument, but it still satisfies the following equations, in which the coefficients of each variable on the left-hand sides add up to a multiple of 3, whereas the right-hand sides do not:

$$
\begin{align*}
a_0 + 2b_0 &\equiv 0 \mod 3 \\
0 + b_1 + c_0 &\equiv 2 \mod 3 \\
0 + b_0 + c_1 &\equiv 2 \mod 3
\end{align*}
$$

This example suggests using a general $\mathbb{Z}_n$ instead of just $\mathbb{Z}_2$. But once we realize that it is the ring structure of $\mathbb{Z}_n$ which plays the key rôle, we can obtain an even more general version.

Fix a ring $R$, and a measurement scenario $\langle X, \mathcal{M}, R \rangle$.

**Definition 4.3.** An $R$-linear equation is a triple $\phi = \langle C, a, b \rangle$ with $C \in \mathcal{M}$, $a : C \to R$ and $b \in R$. Write $V_\phi := C$. An assignment $s \in \mathcal{E}(C)$ satisfies $\phi$, written $s \models \phi$, if

$$
\sum_{m \in C} a(m)s(m) = b.
$$

This lifts to the level of systems of equations, or theories, and sets of assignments, or “models”:

- A system of equations $\Gamma$ has a set of satisfying assignments, $\mathbb{M}(\Gamma) := \{s \in \mathcal{E}(C) \mid \forall \phi \in \Gamma. \ s \models \phi\}$.

- A set of assignments $S \subseteq \mathcal{E}(C)$ determines an $R$-linear theory, $\mathbb{T}_{R}(S) := \{\phi \mid \forall s \in S. \ s \models \phi\}$.

**Definition 4.4.** Given an empirical model $\mathcal{S}$, define its $R$-linear theory to be

$$
\mathbb{T}_{R}(\mathcal{S}) := \bigcup_{C \in \mathcal{M}} \mathbb{T}_{R}(\mathcal{S}(C)) = \{\phi \mid \forall s \in \mathcal{S}(V_\phi). \ s \models \phi\}.
$$

We say that $\mathcal{S}$ is AvN$_R$, written AvN$_R(\mathcal{S})$, if $\mathbb{T}_{R}(\mathcal{S})$ is inconsistent, meaning that there is no global assignment $g : X \to R$ such that $\forall \phi \in \mathbb{T}_{R}(\mathcal{S}). \ g|_{V_\phi} \models \phi$.

**Proposition 4.5.** An AvN$_R$ model is strongly contextual.

**Proof.** Suppose $\mathcal{S}$ is not strongly contextual, i.e. that there is some $g \in \mathcal{S}(X)$. Then, for each $\phi \in \mathbb{T}_{R}(\mathcal{S}). \ g|_{V_\phi} \in \mathcal{S}(V_\phi)$, hence $g|_{V_\phi} \models \phi$. Thus, $\mathbb{T}_{R}(\mathcal{S})$ is consistent. \hfill \Box
4.3. Affine closures. We now consider the relationship between $R$-linear theories and empirical models more closely. We shall relate the existence of an $AvN_R$ argument to the strong contextuality of a derived model.

First, we focus on a single context, or set of variables, $U \subseteq X$. The maps between theories and models,

$$
\mathcal{P}\mathcal{E}(U) \xrightarrow{T} \text{Theories},
$$

form a Galois connection, corresponding to the lifting of the satisfaction relation to the powersets:

$$
S \subseteq M(\Gamma) \iff T(S) \supseteq \Gamma.
$$

We consider the closure operator $M \circ T$, which gives the largest set of assignments whose theory is still the same. First, note that $E(U) = R^U$ is a (free) $R$-module (and, when $R$ is a field, a vector space over $R$). Given solutions $s_1, \ldots, s_t$ to a linear equation, an affine combination of them,

$$
c_1 s_1 + \cdots + c_t s_t \quad \text{such that} \quad c_1 + \cdots + c_t = 1,
$$

is again a solution. In other words, the set of solutions $M(\Gamma)$ to a system of equations $\Gamma$ is an affine submodule of $E(U)$. This means that

$$(1) \quad \text{aff} \leq M \circ T,
$$

where $\text{aff} S$ stands for the affine closure of a set $S \subseteq E(U)$:

$$
\text{aff} S := \left\{ \sum_{i=1}^t c_i s_i \middle| s_i \in S, c_i \in R, \sum_{i=1}^t c_i = 1 \right\}.
$$

In the particular case of vector spaces (i.e. when $R$ is a field), (1) is in fact an equality: hence, affine subspaces are exactly the possible solution sets of a theory, and there cannot exist two different affine subspaces with the same theory, as may happen for general rings $R$.

We now lift this discussion to the level of empirical models. The natural thing to do is to take the affine closure at each context $C \in M$. However, one must be careful to ensure that this yields a well-defined empirical model. Firstly, note that the affine closure operation above is natural on $U$: it gives a natural transformation $\text{aff} : \mathcal{P} \circ \mathcal{E} \rightarrow \mathcal{P} \circ \mathcal{E}$, meaning that

$$(2) \quad (\text{aff } S)_{|U'} = \text{aff } (S_{|U'}) .$$

This follows easily from the fact that the module operations on $E(U)$ are defined coordinatwise.

**Definition 4.6.** Let $S$ be an empirical model on the scenario $(X, M, R)$. We define its affine closure, $\text{Aff } S$, as the empirical model given, at each $C \in M$, by

$$
(\text{Aff } S)(C) := \text{aff } (S(C)).
$$

The property in equation (2) guarantees that $\text{Aff } S$ can be consistently defined to be flasque below the cover as $(\text{Aff } S)(U) = \text{aff } (S(U))$. Note, however, that this equality does not hold for $U$ above the cover. In particular, it may happen that $S(X) = \emptyset$ ($S$ strongly contextual), but $(\text{Aff } S)(X) \neq \emptyset$.

Since $T_R(S)$ is given as the union of the theories at each maximal context, the Galois connection above lifts to the level of empirical models. Moreover, we also have

$$(3) \quad \text{Aff} \leq M \circ T,
$$

with equality when $R$ is a field.

One can therefore relate the notion of $S$ being $AvN_R$ to the strong contextuality of the affine closure of $S$.

---

5 Affineness is required because the equations may be inhomogeneous.
Proposition 4.7. Let $S$ be an empirical model on $(X, \mathcal{M}, R)$. Then, 
\[ \text{AvN}(S) \Rightarrow \text{SC}(	ext{Aff } S). \]

If $R$ is a field, the converse also holds.

Proof. From inequality (22), $T_R(S) = T_R(\text{Aff } S)$. Hence, if $S$ is $\text{AvN}_R$, then so is $\text{Aff } S$, implying by Proposition 4.6 that it is strongly contextual.

For the converse in the case that $R$ is a field, suppose that $T_R(S)$ is consistent. This means that there is a global assignment $g : X \rightarrow R$ satisfying all the equations in $T_R(S)$. But since for fields $MT_R(S) = \text{Aff } S$, we have that $g \in (\text{Aff } S)(X)$, hence the model $\text{Aff } S$ is not strongly contextual. \hfill $\square$

5. Cohomology witnesses contextuality

The logical forms of contextuality are characterised by the existence of obstructions to the extension of local sections to global compatible families. Thus, it seems natural to apply the tools of sheaf cohomology, which are well-suited to identifying obstructions of this kind, in order to provide cohomological witnesses for contextuality. This idea was put forward in previous work by the authors [5], the main points of which we now summarise. In the next section, we shall prove that such cohomological witnesses of contextuality exist for the whole class of $\text{AvN}_R$ models.

5.1. Čech cohomology. Let $X$ be a topological space, $\mathcal{M}$ be an open cover of $X$, and let $\mathcal{F} : \mathcal{O}(X)^{\mathcal{M}} \rightarrow \text{AbGrp}$ be a presheaf of abelian groups on $X$. We shall be particularly concerned with the case where $X$ is a set of measurements (a discrete space), and $\mathcal{M}$ is the cover of maximal contexts of a measurement scenario.

Definition 5.1. A $q$-simplex of the nerve of $\mathcal{M}$, is a tuple $\sigma = (C_0, \ldots, C_q)$ of elements of $\mathcal{M}$ with non-empty intersection, $|\sigma| := \cap_{i=0}^q C_i \neq \emptyset$. We write $N(\mathcal{M})^q$ for the set of $q$-simplices.

The 0-simplices are simply the elements of the cover $\mathcal{M}$, and the 1-simplices are the pairs $(C_i, C_j)$ of intersecting elements of the cover. Given a $q + 1$-simplex $\sigma = (C_0, \ldots, C_{q+1})$, we can obtain $q$-simplices
\[ \partial_j(\sigma) := (C_0, \ldots, \widehat{C_j}, \ldots, C_{q+1}) \quad 0 \leq j \leq q + 1 \]
by omitting one of the elements. Note that $|\sigma| \subseteq |\partial_j(\sigma)|$, and so the presheaf $\mathcal{F}$ has a restriction map
\[ \rho^{[\partial_j(\sigma)]}_| \mathcal{F}(|\partial_j(\sigma)|) \rightarrow \mathcal{F}(|\sigma|). \]

We now define the Čech cochain complex.

Definition 5.2. For each $q \geq 0$, the abelian group of $q$-cochains is defined by:
\[ C^q(\mathcal{M}, \mathcal{F}) := \prod_{\sigma \in N(\mathcal{M})^q} \mathcal{F}(|\sigma|). \]

The $q$-coboundary map,
\[ \delta^q : C^q(\mathcal{M}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{M}, \mathcal{F}), \]
is defined as follows: for each $\omega = (\omega(\tau))_{\tau \in N(\mathcal{M})^q} \in C^q(\mathcal{M}, \mathcal{F})$, and $\sigma \in N(\mathcal{M})^{q+1}$,
\[ \delta^q(\omega)(\sigma) := \sum_{j=0}^{q+1} (-1)^j \rho^{[\partial_j(\sigma)]}_| \omega(\partial_j \sigma). \]

The augmented Čech cochain complex is the sequence
\[ 0 \rightarrow C^0(\mathcal{M}, \mathcal{F}) \rightarrow C^1(\mathcal{M}, \mathcal{F}) \rightarrow \cdots. \]

Proposition 5.3. For each $q$, $\delta^q$ is a group homomorphism, and we have $\delta^{q+1} \circ \delta^q = 0$. 

Definition 5.4. For each $q \geq 0$, we define:

- the $q$-cocycles $Z^q(M, F) := \ker \delta^q$;
- the $q$-co-boundaries $B^q(M, F) := \text{im} \delta^{q-1}$.

By Proposition 5.3, these are subgroups of $C^q(M, F)$ with $B^q(M, F) \subseteq Z^q(M, F)$.

Definition 5.5. $\check{H}^q(M, F)$, the $q$-th Čech cohomology group, is defined as the quotient $Z^q(M, F)/B^q(M, F)$.

Note that $B^0(M, F) = 0$, and hence $\check{H}^0(M, F) \cong Z^0(M, F)$. A 0-cochain $\omega$ is a family $\{r_C \in F(C)\}_{C \in M}$. Since, for each 1-simplex $\sigma = (C, C')$,

$$\delta^0(\omega)(\sigma) = r_C |_{C \cap C'} - r_{C'} |_{C \cap C'},$$

$\omega$ is a cocycle (i.e. satisfies $\delta^0(\epsilon) = 0$) if and only if

$$r_C |_{C \cap C'} = r_{C'} |_{C \cap C'},$$

for all maximal contexts $C, C' \in M$ with non-empty intersection.

This almost states that $r$ is a compatible family, except that it does not require compatibility over restrictions to the empty context. For our present purposes, we are only interested in connected covers (since one can always reduce the analysis of a scenario to its connected components), in which case the exception is irrelevant. This is because, given any two contexts $C$ and $C'$ with empty intersection, there exists a sequence of contexts

$$C = C_0, C_1, \ldots, C_n = C'$$

such that $C_i \cap C_{i+1} \neq \emptyset$ for all $i$. Then, for any $i$, we have

$$r_{C_i} |_{\emptyset} = r_{C_i} |_{C_i \cap C_{i+1} \cap \emptyset} = r_{C_{i+1}} |_{C_i \cap C_{i+1} \cap \emptyset} = r_{C_{i+1}} |_{\emptyset}.$$

Consequently, $r_C |_{C \cap C'} = r_{C'} |_{\emptyset} = r_{C'} |_{C \cap C'}$, and so the family is compatible.

5.2. Relative cohomology and obstructions. In order to solve the problem of extending a local section to a global compatible family, we need to consider the relative cohomology of $F$ with respect to an open subset $U \subseteq X$. We will assume that the presheaf is flasque beneath the cover (as is the case with $\mathcal{S}$).

We define two auxiliary presheaves related to $F$. Firstly, $F|_U$ is defined by

$$F|_U(V) := F(U \cap V).$$

There is an evident presheaf map $p : F \rightarrow F|_U$ given as

$$p_V : F(V) \rightarrow F(U \cap V) : r \mapsto r|_{U \cap V}.$$

Secondly, $F_U$ is defined by $F_U(V) := \ker (p_V)$. Thus, we have an exact sequence of presheaves

$$0 \rightarrow F_U \rightarrow F \xrightarrow{p} F|_U.$$

The relative cohomology of $F$ with respect to $U$ is defined to be the cohomology of the presheaf $F_U$.

We now see how this can be used to identify cohomological obstructions to the extension of a local section. First, recall that in light of Proposition 5.3 the image of $\delta^0$, $B^1(M, F)$, is contained in $Z^1(M, F)$. Therefore, the map $\delta^0$ can be corestricted to a map $\delta^0 : C^0(M, F) \rightarrow Z^1(M, F)$, whose kernel is $Z^0(M, F) \cong \check{H}^0(M, F)$ and whose cokernel is $Z^1(M, F)/B^1(M, F) \cong \check{H}^1(M, F)$.

In summary, we have:

$$\check{H}^0(M, F) \xrightarrow{\ker \delta^0} C^0(M, F) \xrightarrow{\delta^0} Z^1(M, F) \xrightarrow{\text{coker} \delta^0} \check{H}^1(M, F).$$
We now lift the exact sequence of presheaves \[ \mathcal{M}, \mathcal{F} \] considered above to the level of cochains. The map \( C^0(\mathcal{M}, \mathcal{F}) \to C^0(\mathcal{M}, \mathcal{F}_{U}) \) is surjective due to flaccidity beneath the cover. Putting this together with the previous observation, we obtain the diagram below:

\[
\begin{array}{c}
0 \to C^0(\mathcal{M}, \mathcal{F}_{U}) \to C^0(\mathcal{M}, \mathcal{F}) \to C^0(\mathcal{M}, \mathcal{F}|_U) \to 0 \\
\quad \gamma^0 \\
0 \to Z^1(\mathcal{M}, \mathcal{F}_{U}) \to Z^1(\mathcal{M}, \mathcal{F}) \to Z^1(\mathcal{M}, \mathcal{F}|_U) \to 0
\end{array}
\]

whose two rows are short exact sequences. The snake lemma of homological algebra says that there exists a connecting homomorphism turning the kernels of the first row followed by the cokernels of the second into a long exact sequence, as shown in the following diagram.

\[
\begin{array}{c}
\hat{H}^0(\mathcal{M}, \mathcal{F}_{U}) \to \hat{H}^0(\mathcal{M}, \mathcal{F}) \to \hat{H}^0(\mathcal{M}, \mathcal{F}|_U) \\
0 \to C^0(\mathcal{M}, \mathcal{F}_{U}) \to C^0(\mathcal{M}, \mathcal{F}) \to C^0(\mathcal{M}, \mathcal{F}|_U) \to 0 \\
0 \to Z^1(\mathcal{M}, \mathcal{F}_{U}) \to Z^1(\mathcal{M}, \mathcal{F}) \to Z^1(\mathcal{M}, \mathcal{F}|_U) \to 0 \\
\hat{H}^1(\mathcal{M}, \mathcal{F}_{U}) \to \hat{H}^1(\mathcal{M}, \mathcal{F}) \to \hat{H}^1(\mathcal{M}, \mathcal{F}|_U)
\end{array}
\]

We are interested in the case where \( U \) is an element \( C_0 \) of the cover \( \mathcal{M} \). Then, it is clear that \( \hat{H}^0(\mathcal{M}, \mathcal{F}|_{C_0}) \) is isomorphic to \( \mathcal{F}(C_0) \), meaning that its elements are the local sections at \( C_0 \).

**Definition 5.6.** Let \( C_0 \) be an element of the cover \( \mathcal{M} \) and \( r_0 \in \mathcal{F}(C_0) \). Then, the cohomological obstruction of \( r_0 \) is the element \( \gamma(r_0) \) of \( \hat{H}^1(\mathcal{M}, \mathcal{F}_{C_0}) \), where \( \gamma \) is the connecting homomorphism.

The following justifies regarding these as obstructions.

**Proposition 5.7.** Let the cover \( \mathcal{M} \) be connected, \( C_0 \in \mathcal{M} \), and \( r_0 \in \mathcal{F}(C_0) \). Then, \( \gamma(r_0) = 0 \) if and only if there is a compatible family \( \{r_C \in \mathcal{F}(C)\}_{C \in \mathcal{M}} \) such that \( r_{C_0} = r_0 \).

**Proof.** The proof follows from the fact that \( \gamma: \hat{H}^0(\mathcal{M}, \mathcal{F}|_{C_0}) \to \hat{H}^1(\mathcal{M}, \mathcal{F}_{C_0}) \) is in a long exact sequence. Hence, the kernel of \( \gamma \) is equal to the image of the map \( \hat{H}^0(\mathcal{M}, \mathcal{F}) \to \hat{H}^0(\mathcal{M}, \mathcal{F}|_{C_0}) \). Recall that, since the cover is connected, the elements of \( \hat{H}^0(\mathcal{M}, \mathcal{F}) \) are the compatible families, and observe that this map takes such a family \( \{r_C \in \mathcal{F}(C)\}_{C \in \mathcal{M}} \) to the element \( r_{C_0} \in \mathcal{F}(C_0) \). Hence, its image is exactly those local sections at \( C_0 \) that belong to such a compatible family. \( \square \)

### 5.3. Witnessing contextuality

We now apply these tools to analyse the possibilistic structure of empirical models. The cohomological obstructions of Definition 5.6 would appear to be ideally suited to the problem of identifying contextuality. The caveat is that, in order to apply those tools, it is necessary to work over a presheaf of abelian groups, whereas we are concerned with \( \mathcal{S} \), which is merely a presheaf of sets.

We firstly consider how to build an abelian group from a set.

**Definition 5.8.** Given a ring \( R \), we define a functor \( F_R: \text{Set} \to R\text{-Mod} \) to the category of \( R \)-modules (and thus, in particular, to the category of abelian groups). For each set \( X \), \( F_R(X) \) is the
set of functions $\phi: X \to R$ of finite support. Given a function $f: X \to Y$, we define:

$$F_R f: F_R X \to F_R Y :: \phi \mapsto \lambda y, \sum_{f(x)=y} \phi(x) \cdot x.$$ 

This assignment is easily seen to be functorial. We regard a function $\phi \in F_R(X)$ as a formal $R$-linear combination of elements of $X$:

$$\sum_{x \in X} \phi(x) \cdot x .$$

There is a natural embedding $x \mapsto 1 \cdot x$ of $X$ into $F_R(X)$, which we shall use implicitly throughout. In fact, $F_R(X)$ is the free $R$-module generated by $X$; and in particular, $F_\mathbb{Z}(X)$ is the free abelian group generated by $X$.

Given an empirical model $S$ defined on the measurement scenario $(X, \mathcal{M}, O)$, we shall work with the (relative) Čech cohomology for the abelian presheaf $F_R S$ for some ring $R$.

**Definition 5.9.** With each local section, $s \in S(C)$, in the support of an empirical model, we associate the cohomological obstruction $\gamma_{F_R S}(s)$.

- If there exists some local section $s_0 \in S(C_0)$ such that $\gamma_{F_R S}(s_0) \neq 0$, we say that $S$ is cohomologically logically contextual, or $\text{CLC}_R(S)$. We also use the more specific notation $\text{CLC}_R(S, s_0)$.
- If $\gamma_{F_R S}(s) \neq 0$ for all local sections, we say that $e$ is cohomologically strongly contextual, or $\text{CSC}_R$.

The following proposition justifies considering cohomological obstructions as witnessing contextuality.

**Proposition 5.10** (**[5, Proposition 4.3]**).

- $\text{CLC}_R$ implies $\text{LC}$.
- $\text{CSC}_R$ implies $\text{SC}$.

**Proof.** Suppose an empirical model $e$ is not logically contextual. Then for every maximal context $C_0 \in \mathcal{M}$ and every $s_0 \in S(C_0)$, there is a compatible family $\{s_C \in S(C)\}_{C \in \mathcal{M}}$ with $s_{C_0} = s_0$. As $S(C)$ embeds into $F_R S(C)$, $\{s_C\}$ is also a compatible family in $F_R S$. Hence, by Proposition 5.7 we conclude that $\gamma(s) = 0$. The same argument can be applied to a single section witnessing the failure of strong contextuality.

Thus we have a sufficient condition for contextuality in the existence of a cohomological obstruction. Unfortunately, this condition is not, in general, necessary. It is possible that “false positives” arise in the form of families $\{r_C \in F_R S(C)\}_{C \in \mathcal{M}}$ which are not bona fide global sections in $S(X)$ in which genuine global sections do not exist.

Several examples are discussed in detail in [5]. An example for which a false positive arises is the Hardy model [23][24]. However, cohomological obstructions over $\mathbb{Z}$ (in fact, $\mathbb{Z}_2$ is enough) provide witnesses of strong contextuality for a number of well-studied models, including: the GHZ model [21][22][38], the Peres–Mermin “magic” square [39][40][44], and the 18-vector Kochen–Specker model [13], the PR box [46], and the Specker triangle [31][39]. In fact, the Kochen–Specker model and the Specker triangle belong to a large class of models, known as $\neg \text{GCD}$. In [6], it is shown all the models in this class admit cohomological witnesses for their strong contextuality. This result will be greatly generalised in [6].

The coefficients for cohomology can be taken from any commutative ring $R$. We now consider how the cohomological obstructions obtained with different rings relate to each other.

**Proposition 5.11.** Let $h: R' \to R$ be a ring homomorphism. Then, for any $C \in \mathcal{M}$ and $s \in S(C)$, $\gamma_{F_{R'} S}(s) = 0$ implies $\gamma_{F_R S}(s) = 0$, and so $\text{CSC}_R \Rightarrow \text{CSC}_{R'}$ and $\text{CLC}_R \Rightarrow \text{CLC}_{R'}$. 

Proof. If $h: R \rightarrow R'$ is a ring homomorphism, then for any set $X$ there is a map $F_h: F_R X \rightarrow F_{R'} X : r \mapsto h \circ r$.

which is a group homomorphism. Moreover, this assignment is natural in $X$. Hence, this determines a presheaf map $F_R S \rightarrow F_{R'} S$, and compatible families on the former are mapped to compatible families on the latter. Since the map $F_h$ above leaves the elements of the generating set fixed, the condition that the family agrees with $s_0$ at context $C_0$ is preserved. □

We conclude this section with a remark. If $\{r_C \in F_R S(C)\}_{C \in \mathcal{M}}$ is a compatible family, then the sum of the coefficients of the formal linear combinations $r_C$ is the same for all $C$. This holds because $S(\emptyset) = \mathcal{E}(\emptyset) = \{\ast\}$; so that for any $C \in \mathcal{M}$, we have

$$r_C|_\emptyset(\ast) = \sum_{s \in S(C)} r_C(s);$$

i.e. compatibility forces all these restrictions to the empty context to be the same. Therefore, when the obstruction of a section $s_0 \in S(C_0)$ (more precisely, of the linear combination $1 \cdot s_0 \in F_R S(C_0)$) vanishes, the corresponding family of linear combinations $\{r_C \in F_R S(C)\}_{C \in \mathcal{M}}$ must in fact contain only affine combinations—those whose coefficients sum to one.

6. Cohomology and Flaw arguments

The aim of this section is to show that if an empirical model is $\mathsf{AvN}_R$, then the cohomological obstructions witness its strong contextuality. Moreover, it is enough to consider cohomology with coefficients in the ring $R$ itself.

The result is stated as follows.

Theorem 6.1. Let $S$ be an empirical model on $(X, \mathcal{M}, R)$. Then:

$$\mathsf{AvN}_R(S) \Rightarrow \mathcal{E}(\mathsf{Aff} \mathcal{S}) \Rightarrow \mathcal{CSC}_R(S) \Rightarrow \mathcal{CSC}_Z(S) \Rightarrow \mathcal{SC}(S).$$

The first of these implications was already established in Proposition 4.7, the third in Proposition 5.10, and the fourth in Proposition 5.11.

In order to prove the second, we use the properties of the functor $F_R: \mathsf{Set} \rightarrow R\text{-}\mathsf{Mod}$ that constructs the $R$-module of formal $R$-linear combinations of elements of a set $X$. As already mentioned, $F_R X$ is the free $R$-module generated by $X$. This means that it is the left adjoint of the forgetful functor $U: R\text{-}\mathsf{Mod} \rightarrow \mathsf{Set}$.

$$\begin{array}{cc}
\mathsf{Set} & \perp \\
\downarrow & \\
R\text{-}\mathsf{Mod} & \\
U & \\
\end{array}$$

The unit $\eta$ of this adjunction is the obvious embedding, which we have been using, taking an element $x \in X$ to the formal linear combination $1 \cdot x$. The counit is the natural transformation $\epsilon: F_R \circ U \Rightarrow \text{Id}_{R\text{-}\mathsf{Mod}}$ given, for each $R$-module $M$, by the evaluation map $\epsilon_M: F_R U(M) \rightarrow M : r \mapsto \sum_{x \in M} r(x)x$.

We are interested in taking formal linear combinations of subsets of elements. Let us fix a module $M$ and a subset $S \subseteq U(M)$. Then the map $\epsilon_M$, by virtue of being an $R$-module homomorphism, maps the formal linear combinations of elements of $S$, $F_R(S)$, which coincide with the linear span in $F_R U(M)$ of $\eta[S] = \{1 \cdot s \mid s \in S\}$, to the linear span of $S$ in $M$, $\mathsf{span}_M S$. Moreover, it maps the formal affine combinations $F_R^{\mathsf{aff}}(S) = \mathsf{aff}_{F_R U(M)} \eta[S]$ to the affine closure $\mathsf{aff}_M S$.

Recall that we are dealing with measurement scenarios whose outcomes are identified with a ring $R$, hence where $\mathcal{E}(U)$ are themselves $R$-modules, i.e. $\mathcal{E}: \mathcal{P}(X)^{op} \rightarrow R\text{-}\mathsf{Mod}$. As such, the
count can be horizontally composed to yield a natural transformation, or map of presheaves, $\text{id}_E \ast e: F_R \circ U \circ E \rightarrow E$, given at each context $U \subseteq X$ by $\varepsilon_{E(U)}: F_R U \varepsilon(U) \rightarrow E(U)$. Now, given an empirical model $S$, we can apply the observation regarding subsets of the module at each context. But, since $\text{aff}_{E(U)} S(U) = (\text{Aff} S)(U)$ by definition for $U$ beneath the cover, and since containment still holds above it, we conclude that the presheaf map restricts as follows:

$$
\begin{align*}
F_R^{\text{aff}} U S & \longrightarrow F_R U S \longrightarrow F_R U E \\
\text{Aff} S & \longrightarrow \text{Span} S \longrightarrow E
\end{align*}
$$

We can now prove the theorem.

**Proof of Theorem** [6.7]. We show the contrapositive. Suppose that $S$ is not CSC, i.e., that $\gamma_{F_R S}(s_0) = 0$ for some $s_0 \in S(C_0)$. Then, by Proposition [6.7] this is equivalent to the existence of a compatible family $\{r_C \in F_R S(C)\}_{C \in M}$ with $r_{C_0} = s_0$. As observed at the end of [5.3] all these $r_C$ must be formal affine combinations of elements in $S(C)$. But then the presheaf map $F_R^{\text{aff}} U S \rightarrow \text{Aff} S$ above pushes this compatible family to a compatible family of $\text{Aff} S$, implying that the model $\text{Aff} S$ is not strongly contextual.

Essentially the same strategy can be used to prove an analogous result for logical contextuality.

The notion of inconsistent theory has to be adapted: instead of asking whether there is a global assignment satisfying all the equations in the theory, we can ask, given a partial assignment $s_0 \in E(C_0)$ whether there is such a global assignment with the additional requirement that it restricts to $s_0$. This can be seen as a generalisation of the notion of robust constraint satisfaction studied in [3] from the complexity perspective. We write $\text{AvN}_R(e, s_0)$ if the theory of $S$ has no solution extending $s_0$. Then we have:

$$
\text{AvN}_R(e, s_0) \Rightarrow \text{LC}(\text{Aff} S, s_0) \Rightarrow \text{CLC}_R(S, s_0) \Rightarrow \text{CLC}_{\mathbb{Z}}(S, s_0) \Rightarrow \text{LC}(S, s_0).
$$

**Discussion.** We have shown that for a large class of models, their logical or strong contextuality is witnessed by cohomology. This subsumes and greatly generalises the results in [5]. Moreover, these models include a large class of concrete constructions arising from stabiliser quantum mechanics, going well beyond existing results of this kind in the quantum information literature. It remains an objective for future work to achieve a precise characterisation of what cohomology detects, and more generally full equivalences between the various ways of expressing contextuality. Note that, as already mentioned, the first implication in Theorem [6.1] can be reversed under the assumption that $R$ is a field. If we use a more abstract notion of equational consistency, in terms of quotient modules rather than equations expressed in a “coordinatized” form, then it can be reversed even for general rings. The point of taking the ground ring to be a field is exactly that it allows coordinatization.

We also remark that the cohomological methods we have developed can be applied to an elaborated version of the treatment of logical paradoxes we gave in [22] following the lines of [10,50]. We aim to give a detailed treatment of this “cohomology of paradox” in future work. We also note the intriguing resemblances, on the conceptual level at least, to the work of Roger Penrose in [43].

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