Structural Damage Identification Based on Sensitivity Analysis and Sparse Constraint

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Abstract. In the problem of structural damage identification, only a few elements or substructures are damaged, which means the damage of the structure has sparsity. Hence, this paper proposed a structural damage identification method based on sensitivity analysis and sparse constraint. This method established the sensitivity equations through the first-order sensitivity analysis of the structure. Then a damaged elements compression method was used to reduce the number of damaged elements. And the original problem was transformed into a constrained optimization problem with the L1 norm of damage coefficients as the objective function to describe the sparsity. Finally, the results of a numerical example showed that the method can effectively identify single-damage and multi-damage conditions with great robustness.

1. Introduction
During the service of civil engineering structures, the internal damage of structures will inevitably occur due to long-term load, fatigue effect, environmental corrosion, material aging, and mutation effect [1-2]. When the damage accumulates to a certain extent, it may even lead to serious accidents such as collapse [1-2]. Therefore, structural health monitoring has become one of the key and hot issues in civil engineering [3].

As the core part of structural health monitoring, structural damage identification is particularly important. And the model updating method based on dynamic characteristics is currently the most concerned structural damage identification method [4-6]. Sensitivity analysis is a widely used model updating method with simple construction and specific physical meaning.

Esfandiar[i] [7] used the mode shapes of intact structures and the modal frequencies of damaged structures to construct the sensitivity matrix of the mode shapes of damaged structures to the structural damage coefficient and successfully used it to identify the structural damage of plane trusses and frame structures. Grip et al. [8] based on the sensitivity method, used a convex function Huber function to simulate the regularization constraints, but the parameter selection of convex function Huber function had a great impact on the recognition results. Mansourabadi et al. [9] combined the wavelet change with the sensitivity analysis of frequency response function, used the sensitivity analysis of wavelet transform coefficient to modify the model to achieve the effect of damage identification, and finally carried out verification on the truss and frame structure. Krishnanunni et al. [10] analysed the sensitivity of frequency and static displacement and used the cuckoo algorithm to solve the problem, and the results showed that the value of the weight coefficient of frequency and static displacement had a great influence on the identification. Although there have been many research results in sensitivity analysis, construction of the objective function and constraint conditions still have important significance.
In real damage conditions, there are only a few damaged elements, so it is necessary to study the application of sparsity in structural damage identification. In this paper, a damage identification based on sensitivity and sparsity constraint is proposed. This method introduces the L1 norm as the objective function to represent the sparsity, adopts a compression method to reduce the number of damaged elements during the iterative process, and takes the normalized sensitivity equations as constraints to identify the damaged elements. A numerical example is used to verify the effectiveness of the method, and the results show that the method has high accuracy and good noise resistance.

2. Theory

2.1. Sensitivity analysis
Without considering damping, for a structure with \( n_e \) elements and \( n_{DOF} \) degrees of freedom, its dynamic characteristic equation is

\[
K\phi - \lambda_i M\phi_i = 0
\]

(1)

where \( K \) and \( M \) are the global stiffness and mass matrix of the structure. Additionally, \( \lambda_i \) and \( \phi_i \) denotes the \( i \)th modal eigenvalue and eigenvector of the structure.

The structural characteristic sensitivity coefficient is defined as the rate of change of modal parameters to structural parameters, which can characterize the degree of influence of changes in structural parameters on changes in modal parameters. Fox et al \[11\] deduced the expression of the first-order characteristic sensitivity, the expression of the sensitivity coefficient of the \( i \)th frequency to the \( n \)th structural parameter is given by

\[
S_{\lambda_n} = \phi_i^T \left( \frac{\partial K}{\partial p_n} - \lambda_i \frac{\partial M}{\partial p_n} \right) \phi_i
\]

(2)

Similarly, the sensitivity coefficient expression of the \( i \)th eigenvector to the \( n \)th structural parameter is

\[
S_{\phi_n} = \sum_{r \neq i} \frac{\phi_i^T \left( \frac{\partial K}{\partial p_n} - \lambda_i \frac{\partial M}{\partial p_n} \right) \phi_r}{\lambda_r - \lambda_i} - \frac{1}{2} \phi_i^T \frac{\partial M}{\partial p_n} \phi_i
\]

(3)

It is generally believed that structural damage will cause the degradation of structural stiffness without affecting the structural mass. Therefore, this paper defines structural damage as the degradation coefficient of the element stiffness matrix. The element stiffness matrix of the damaged structure is

\[
K^d = \sum_{n=1}^{N} K_n^d = \sum_{n=1}^{N} K_n (1 + \alpha_n) = K + \sum_{n=1}^{N} K_n \alpha_n
\]

(4)

where \( \alpha_n \) denotes the damage coefficient of \( n \)th element. And \( d \) represents the damaged structure.

Substitute equation (4) into equation (2) and equation (3), then

\[
S_{\lambda_n} = \phi_i^T K_n^d \phi_i
\]

\[
S_{\phi_n} = \sum_{r \neq i} \frac{\phi_i^T K_n \phi_r}{\lambda_r - \lambda_i} \phi_i
\]

(5)

According to equation (5), the first-order characteristic sensitivity equations of the structure can be constructed as

\[
S\alpha = \Delta \theta
\]

(6)
where \( S = [S_x, S_y]^T \), \( \Delta \theta = [\Delta \lambda, \Delta \phi]^T \).

And in the traditional method, the above equations generally use the least square method to calculate the damage coefficient, namely \( \alpha = (S^T S)^{-1} S^T \Delta \theta \).

2.2. Damage identification

2.2.1. Modal incomplete. Due to the limitation of the number of sensors and the layout conditions, it is difficult to test all the degrees of freedom of the structure, which will cause the number of degrees of freedom collected to be smaller than that of the finite element model. This paper introduces a 0-1 operator matrix to solve this problem. Assuming that the measured degree of freedom of the structure is \( m \), the processed mode change and mode sensitivity coefficient can be given by

\[
\Delta \phi = \text{diag}(l_1, \ldots, l_m) \cdot \Delta \phi \\
\tilde{S}_\phi = \text{diag}(l_1, \ldots, l_m) \cdot S_\phi
\]

where \( l_m = 1 \) when \( m \in m \), and \( l_m = 0 \) when \( m \notin m \).

2.2.2. Sparse constraint. Due to the different magnitudes of the sensitivity coefficients of the eigenvalue and the eigenvector, the sensitivity matrix \( S \) is ill-conditioned. Therefore, this paper standardizes each row of equation (6) with the maximum value to solve this problem, namely

\[
\tilde{S}_\alpha = W[S_x, \tilde{S}_\phi]^T \alpha = W[\Delta \lambda, \Delta \phi] = \Delta \tilde{\theta}
\]

where \( W \) is the weight matrix. \( W = \text{diag}(\ldots, w_j, \ldots) \), \( w_j = \max(|\tilde{S}_j|, |\Delta \tilde{\theta}_j|) \).

Considering that the actual damage usually has a certain degree of sparsity, and the L1 norm can well represent the sparsity of the coefficients. This paper takes the L1 norm of the damage coefficient as the objective function, and the standardized sensitivity equation set as the constraint condition. The problem is converted to a constrained optimization problem as follow:

\[
\min f = \| \alpha \|_1 \\
\text{s.t.} \quad \tilde{S}_\alpha = \Delta \tilde{\theta}
\]

The SQP algorithm based on trust region is used to solve the above constrained optimization problem.

2.2.3. Damaged elements compression. Considering that the sensitivity coefficients of different elements may be sharply different, so the elements with small coefficients may be annihilated in the recognition process due to noise and other factors. Aiming at this problem, this paper proposed a compression method to compress the number of damaged elements.

From equation (6), it can be seen that if the sensitivity coefficient of a damaged element is large, it is easy to be identified. Hence, in each iteration, delete the undamaged elements which have sensitivity coefficients larger than the mean of the sensitivity matrix.

2.2.4. Iterative process. Considering that the sensitivity analysis is based on the approximate method of Taylor expansion, this paper uses an iterative method to solve the problem. The exit condition of iteration is usually determined by the relative value of the residual or the stability of iteration. In this paper, iterative stability is used as the exit condition. And the iterative process is as follows:

Step 1: Set the measuring point and the modal extraction order, and input the initial finite element model and the modal parameters and physical parameters of the damaged structure.

Step 2: Initialize \( k = 1 \) and \( \alpha^{(0)} = [0, \ldots, 0]^T \). Assume that all elements to be damaged elements.
Step 3: Calculate the sensitivity coefficients of modal parameters and construct the constrained optimization problem according to equation (9), and solve it to obtain $\alpha^{(k)}$.

Step 4: If $\|\alpha^{(1)} - \alpha^{(k-1)}\|_2 \leq \varepsilon$, go to step 7.

Step 5: Delete the elements which's damage coefficients are less than 0.05% and sensitivity coefficients are larger than the mean of all elements.

Step 6: Use $\alpha^{(k)}$ to update the finite element model and $k = k + 1$, go to step 3, when $k > k_{\text{max}}$ go to step 7.

Step 7: Output the final damage coefficient $\alpha = (1 + \alpha^{(1)})(1 + \alpha^{(2)}) \cdots (1 + \alpha^{(k)}) - 1$.

2.2.5. Noise. In actual measurement, the data will inevitably be interfered by noise, so it is necessary to consider the influence of noise on the modal test data. Therefore, this paper used numerical simulation to impose noise on the mode shapes as follow:

$$\phi_i = \phi_i (1 + \eta R_{i,\text{max}})$$  \hspace{1cm} (10)

where $\phi_i$ and $\phi_i$ are the $i$th mode value before and after imposing noise. And $\eta$ denotes the disturbance level, $R$ is the matrix of Gaussian random numbers with mean value 0 and variance of 1.

3. Numerical example

In order to verify the effectiveness of the method proposed in this paper, a numerical simulation of damage identification was carried out on a 45-element two-span steel truss shown in figure 1. The area of the truss members is $A = 0.02m^2$, the modulus of elasticity is $E = 206GPa$, and the density is $\rho = 7850kg \cdot m^3$.

Figure 1. Two-span steel truss.

It is assumed that the first 3 modal information after structural damage can be obtained. The sensor placement method adopts the sensor placement method proposed by Zeng [12], which are respectively arranged at nodes 2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 16, 17, 18, 19, 20 in the $y$ direction and node 14 in the $x$ direction.

Three damage conditions are set up (see Table 1), and the effects of three noise levels of 0%, 3% and 5% on the results of structural damage identification are considered.

Table 1. Damage cases.

| Damage cases | Damaged elements and ratios |
|--------------|----------------------------|
| 1            | The damaged element is 25, the damage ratio is 25%. |
| 2            | The damaged elements are 7, 15, and 36, and the damage ratios are 30%, 20%, and 30%. |
| 3            | The damaged elements are 1, 19, 33, and 35, and the damage ratios are 25%, 20%, 20%, and 30%. |

To study the effect of sparse constraint for sensitivity analysis, this paper used the traditional sensitivity analysis without sparse constraint and the method proposed in this paper for structural damage identification respectively. Figure 2, figure 4 and figure 6 indicate the damage identification results of traditional sensitivity analysis for different cases with considering different noise levels. And
Figure 3, Figure 5 and Figure 7 indicate the damage identification results of sensitivity analysis considering sparse constraint for different cases with considering different noise levels.

From Figure 2, Figure 4 and Figure 6, it can be seen that the traditional sensitivity analysis can identify the damaged elements accurately when the noise level is 0%. However, when the noise level becomes larger, the number of misidentifications of the detection result will increase. For example, for single damage in Figure 2, elements 23, 24, 27, and 29 are identified as damaged elements when the noise level is 5%. And it can be seen in Figure 4 and Figure 6 that the situation mentioned above also exists in case 2 and case 3. In a word, the traditional sensitivity analysis without sparse constraint could not identify the damaged elements correctly when there exists noise.

Figure 2. Result of sensitivity analysis without sparse constraint for case 1.

Figure 3. Result of the method proposed in this paper for case 1.

Figure 4. Result of sensitivity analysis without sparse constraint for case 2.

Figure 5. Result of the method proposed in this paper for case 2.

Figure 6. Result of sensitivity analysis without sparse constraint for case 3.

Figure 7. Result of the method proposed in this paper for case 3.

As a comparison, figure 3, figure 5 and figure 7 show the results of the method proposed in this paper for different cases under different noise levels. From figure 3, it can be seen that for a single damage case, the method can identify the damaged element 25 with only a few mistakes under the 5% noise level. Additionally, for multiple-damage cases in figure 5 and figure 7, the method can also identify damage accurately under different noise levels which means that the method proposed by this paper has strong robustness. So it can be concluded that sparse constraint and damaged elements compression can improve the identification result’s accuracy of sensitivity analysis apparently.

The results of the numerical example show that the iterative method proposed in this paper can identify single and multiple damage cases accurately under modal incomplete and 5% noise level.
4. Conclusions
A structural damage identification method based on sensitivity analysis and sparse constraint is demonstrated in this paper. Based on the analysis and discussion done in this paper, the main conclusions are as follows:

(1) Due to the influence of incomplete modal information, the sensitivity matrix is an ill-conditioned matrix that is difficult to solve. This paper standardizes and simplifies the sensitivity matrix to reduce the difficulty of solving.

(2) This paper introduces the L1 norm as a sparse constraint to describe the sparseness of the damage and uses damaged elements compression to reduce the effect of noise. The results show that the sparse constraint and damaged elements compression can improve the accuracy of sensitivity analysis.

(3) The method proposed in this paper has good anti-noise performance, and can still identify damaged elements accurately when the noise level is no more than 5%.

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