A MASS TRANSPORT APPROACH TO THE OPTIMIZATION OF ADAPTED COUPLINGS OF REAL VALUED RANDOM VARIABLES.

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Abstract. In this work, we investigate an optimization problem over adapted couplings between pairs of real valued random variables, possibly describing random times. We relate those couplings to a specific class of causal transport plans between probabilities on the set of real numbers endowed with a filtration, for which their provide a specific representation, which is motivated by a precise characterization of the corresponding deterministic transport plans. From this, under mild hypothesis, the existence of a solution to the problem investigated here is obtained. Furthermore, several examples are provided, within this explicit framework.

Stochastic analysis, optimal transport, optimization, filtrations.

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1. Introduction

For the sake of clarity, in this work, the set $\mathbb{T}$ which within models encountered here may label times, will be the whole real line $\mathbb{R}$. We investigate the connections between adapted mass transport problems and optimization over fixed marginals couplings $(X,Y,(\Omega,A,P))$ of the specific form :

Definition 1.1. Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two random variables defined on a same $\mathbb{P}$– complete probability space $(\Omega,A,P)$. We say that $(X,Y,(\Omega,A,P))$ is an adapted coupling, if and only if, there exist two random variables $\tau : \Omega \to \mathbb{R}$, and $Z : \Omega \to \mathbb{R}$, such that

$$Y = (X + Z)1_{\{X \leq \tau\}} + \tau 1_{\{X > \tau\}}, \quad \mathbb{P} - a.s.,$$

holds, $\forall A, B \in \mathcal{B}(\mathbb{R})$ BOREL sets of $\mathbb{R}$, such that $A \subset ]-\infty, t[$ and $B \subset [t, +\infty[$.

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Given \( \eta, \nu \in M_1(\mathbb{R}) \), two Borel probability measures on \( \mathbb{R} \), we denote by \( \text{Cpl}_a(\eta, \nu) \) the set of such adapted couplings \((X,Y,(\Omega,\mathcal{A},\mathbb{P}))\) which further satisfy \( p_X = \eta \) and \( p_Y = \nu \), where \( p_X \) (resp. \( p_Y \)) denotes the probability law of the random variable \( X \) (resp. \( Y \)) on the probability space \((\Omega,\mathcal{A},\mathbb{P})\).

In models, we may interpret \( X \) as a time where a signal, for instance a postal letter, or a phone call, or any physical signal within experiments, is received, and \( Y \) as the time where a response signal is emitted, while we interpret \( \tau \) as a time waiting for the signal actually received at \( X \), and \( Z \) as a delay time of answer; if the entry signal has not been received before the waiting time \( \tau \) has been exhausted, it results into a signal sent at \( \tau \); otherwise, a signal is emitted with a delay \( Z \) after the entry signal has been received, before \( \tau \), at a time \( X \). Thus, we interpret (1) as a delay hypothesis, while we interpret (2) as a waiting time hypothesis. Within this perspective, notice that a sufficient condition for (2) is that \( \tau \) is \( \mathbb{P}- \) independent to \( X \), while another sufficient condition is obtained by taking \( \tau = X \), \( \mathbb{P}-a.s. \), whenever \( Y \geq X \) holds \( \mathbb{P}- \) almost surely. Therefore, taking now for granted the existence of such couplings, let \( X, \tau, Z \) be three random variables defined on a same \( \mathbb{P}- \) complete probability space \((\Omega,\mathcal{A},\mathbb{P})\), where \( Z \) is non-negative, where \( \tau \) satisfies (2), and define \( Y \) by (1.1) ; then \((X,Y,(\Omega,\mathcal{A},\mathbb{P}))\) is an adapted coupling according to Definition 1.1. In particular, if \( Z = 0 \), then in the previous example, we obtain \( Y = \min(X,\tau) \) : thus \( \tau \) suggests a particular case of so-called censoring times (see [16] p.124), which might involve further interpretations for \( \tau \). Cases where \( X \leq \tau \), \( \mathbb{P}-a.s. \) are of particular interest with respect to the recent literature : in this case \( Y \geq X \), \( \mathbb{P}-a.s. \), and therefore, whenever both \( X \) and \( Y \) are \( \mathbb{P}- \) integrable, it results that

\[
E_{\mathbb{P}}[Y|\sigma(X)] \geq X, \quad \mathbb{P}-a.s.,
\]

the left hand term denoting a conditional expectation with respect to the \( \sigma- \) field generated by the random variable \( X \). Otherwise stated, some of those couplings are also submartingale couplings, while \((-X,-Y,(\Omega,\mathcal{A},\mathbb{P}))\) is therefore a particular case of so-called supermartingale couplings, which have been recently investigated within a canonical form, for instance see [14]. However, submartingale couplings don’t exhaust the class of adapted couplings within the acceptation of Definition 1.1. Indeed, at the inverse, we may find random variables \( X \) and \( \tau \), \( \mathbb{P}- \) independent from one to each other such that \( X > \tau \), \( \mathbb{P}-a.s. \), and take \( Z = 0 \) in which case \( X \) and \( Y = \tau \) are \( \mathbb{P}- \) independent, without necessarily being neither martingale nor submartingale couplings ; for a recall on definitions of martingale couplings, for instance see [9], while for other among attractive approaches using martingales together with optimal
transport, for instance see [3]. Further, as noticed earlier, Definition 1.1 doesn’t require \( \tau \) to be necessarily independent to \( X \) in (2), but rather states a specific conditional independence which motivated the interpretation adopted above of \( \tau \) as a waiting time for \( X \).

Let \( c : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a non-negative BOREL measurable map, where for given \( x, y \in \mathbb{R} \), \( c(x, y) \) is interpreted within this model as the cost to send a signal at time \( y \) when an entry signal is received at time \( x \); for instance, when applying this model to a post office gestion, for \( x < y \), the number \( c(x, y) \geq 0 \) may coincide with the cost, which is expressed in euros, to stock a letter or a package from the time described by \( x \), where it has been received at the post office, until the time described by \( y \) where it is delivered out to the target recipient. In this work, given two BOREL probability measures \( \eta \) and \( \nu \) on \( \mathbb{R} \), we are interested in the following problem

\[
\inf E_\mathbb{P}[c(X,Y)],
\]

where the infimum is taken on the set of all the adapted couplings \((X, Y, (\Omega, \mathcal{A}, \mathbb{P}))\) of \( \eta \) to \( \nu \).

Here, we address this problem through the connections between those adapted couplings and so-called causal (or adapted) transport plans. Indeed, as stated below, it turns out that adapted couplings defined according to Definition 1.1 actually provide a specific representation for causal transport plans between probabilities on the set of real numbers endowed with suitable probabilistic filtrations, within the precise definition adopted in [11], which will be the ariane thread of this note. This motivates the structure of this work, which is the following :

In section 2 we fix the notation with a particular emphasize on the filtration which we take on \( \mathbb{R} \), and on the transport kernel which is used to define the filtration generated by a transport plan adopted in [11]; within models, recall that this filtration encapsulates the information flow which is required to perform the plan as time evolves. In section 3 Definition 3.1 provides a recall of the latter, within this specific framework, which is used to state the definition of causal transport plans which we adopt here. Then, Lemma 3.1 mentions a characterization for such plans between BOREL probability measures on \( \mathbb{R} \), which is then used to shorten subsequent proofs : it stands on specific properties of a conditional probability cumulative distribution function

\[
F_\gamma : (x, y) \in \mathbb{R}^2 \to F_\gamma^x(y) \in \mathbb{R},
\]

which is BOREL measurable (respectively increasing and c\textsuperscript{à}ud-l\textsuperscript{à}g) along the first (resp. second) variable ; the french acronym c\textsuperscript{à}ud-l\textsuperscript{à}g refers to right-continuous functions with left limits. To obtain a better understanding of those latter plans within the
Figure 1.1. These figures, which will be described more accurately below, illustrate Lemma 3.1 and Theorem 3.1. The image on the left represents the graph of the restriction of a conditional cumulative distribution function

\[ F_\gamma : (x, y) \in \mathbb{R}^2 \to \mathbb{P}(\{Y \leq y\}|X = x) \in [0, 1] \]

which is associated to the joint law \( p_{X,Y} \) of an adapted coupling \((X, Y, (\Omega, \mathcal{A}, \mathbb{P}))\) according to Definition 1.1: as it will be shown by Theorem 1.1, it turns out to coincide with some causal transport plan \( \gamma \), when \( \mathbb{R} \) is endowed with a suitable probabilistic filtration; above, the ”\( X = x \)” refers to a desintegration of measure. The image on the right represents the graph of a function \( T : \mathbb{R} \to \mathbb{R} \) such that \((X, T(X), (\Omega, \mathcal{A}, \mathbb{P}))\) is an adapted coupling for any real-valued random variable \( X \): as it will be pointed out below, the joint law \( p_{X,T(X)} \) actually boils down to a deterministic causal transport plan. Within models, in both cases illustrated above, the upper half-plane (resp. the lower half-plane) above (resp. below) the first diagonal corresponds to cases where a response signal was sent at a time \( y \) relatively \textbf{after} (resp. \textbf{before}) the time \( x \) where the entry signal has been received. Depending on those two regions of the plane, curves may exhibit quite distinct aspects.

particular circumstances which we encounter in this context, we then state an accurate characterization to identify the deterministic ones, in Theorem 3.1, while Lemma 3.2 provides details to clearly grasp its origin. As a byproduct, this theorem yields a strong motivation for the representation provided by the adapted couplings adopted in Definition 1.1. A proof that this representation actually holds in the general case is provided in Theorem 4.1 of section 4, where, under mild hypothesis, it is then
applied to obtain, in Corollary 4.1, the existence of a solution to the problem which is stated above, from adapted tools of mass transport. It may be remarked that some arguments adopted from optimal transport can facilitate the identification of the optima. Furthermore, several examples are provided, notably from the *jump times* of a simple Poisson process, which are aimed to provide some explicit illustrations of specificities of the related causal transport plans, within this framework.

2. Preliminaries and notation

In this whole work, for the sake of clarity, we set $T = \mathbb{R}$, while $\mathbb{R}$ is endowed with its usual topology, whose associated Borel $\sigma-$ field is denoted by $\mathcal{B}(\mathbb{R})$. The set of Borel probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is denoted by $M_1(\mathbb{R})$. Whenever $\mathcal{G} \subset \mathcal{B}(\mathbb{R})$ denotes a sigma-field, and $\nu \in M_1(\mathbb{R})$, the set of $\nu-$ negligible subsets is defined by $\mathcal{N}^\nu := \{N \subset \mathbb{R} : \exists A \in \mathcal{B}(\mathbb{R}), N \subset A, \text{ and, } \nu(A) = 0\}$. Moreover $\mathcal{G}^\nu$, which we still call here the $\nu-$ completion of $\mathcal{G}$, denotes the smallest $\sigma-$ fields on $\mathbb{R}$ which contains both all the elements of $\mathcal{G}$ and all the elements of $\mathcal{N}^\nu$; notice that the negligible sets are taken with respect to the whole $\mathcal{B}(\mathbb{R})$, while stating the $\nu-$ completion of a sub $\sigma-$ field $\mathcal{G}$. For much details on probabilities and on specific tools as we use it subsequently, we refer to [1]. For $\nu \in M_1(\mathbb{R})$, the unique extension of $\nu$ to $\mathcal{B}(\mathbb{R})$ is still denoted by $\nu$, which doesn’t seem to yield any confusion below. We endow $\mathbb{R}$ with the filtration $(\mathcal{B}_t(\mathbb{R}))_{t \in T}$, i.e. $\mathcal{B}_s(\mathbb{R}) \subset \mathcal{B}_t(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$, $\forall s,t \in T$ such that $s \leq t$, which is defined by

$$\mathcal{B}_t(\mathbb{R}) = \{A \in \mathcal{B}(\mathbb{R}) : A \subset ]-\infty, t[ \text{ or } [t, +\infty[ \subset A\},$$

so that $\mathcal{B}_t(\mathbb{R})$ is equivalently defined by $\mathcal{B}_t(\mathbb{R}) = \sigma(\mathcal{C}_t)$, the smallest $\sigma-$ field on $\mathbb{R}$ which contains all the subsets which constitute

$$\mathcal{C}_t = \{]-\infty, a], a < t\},$$

$\forall t \in T$. The cartesian product $\mathbb{R} \times \mathbb{R}$ is endowed with its usual product topology which turns it into a Polish space, while the associated Borel sigma field is denoted by $\mathcal{B}(\mathbb{R} \times \mathbb{R})$; $M_1(\mathbb{R} \times \mathbb{R})$ denotes the set of Borel probability measures defined on the measurable space $(\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R} \times \mathbb{R}))$, while we adapt conventional notation analogous to the case of $\mathbb{R}$ for completions. Given $\eta, \nu \in M_1(\mathbb{R})$, in optimal transport ([10], [13], [15], [18]), a probability $\gamma \in M_1(\mathbb{R} \times \mathbb{R})$ is said to be a transport plan from $\eta$ to $\nu$, if and only if, its first marginal is $\eta$ (i.e. $p_1^* \gamma = \eta$) and its second marginal is $\nu$ (i.e. $p_2^* \gamma = \nu$), $p_1 : (x, y) \in \mathbb{R} \times \mathbb{R} \rightarrow x \in \mathbb{R}$ (resp. $p_2 : (x, y) \in \mathbb{R} \times \mathbb{R} \rightarrow y \in \mathbb{R}$) denoting the canonical projections functions, while $p_1^* \gamma$ (resp. $p_2^* \gamma$) denotes the pushforward
(see [1]) of $\gamma$ by $p_1$ (resp. by $p_2$); $p_1, \gamma(A) = \gamma(A \times \mathbb{R})$ (resp. $p_2, \gamma(B) = \gamma(\mathbb{R} \times B)$), $\forall A \in \mathcal{B}(\mathbb{R})$ (resp. $\forall B \in \mathcal{B}(\mathbb{R})$). Then,

$$\Pi(\eta, \nu) = \{ \gamma \in M_1(\mathbb{R} \times \mathbb{R}) : p_1, \gamma = \eta, p_2, \gamma = \nu \}$$

denotes the set of transport plans from $\eta$ to $\nu$.

To describe accurately cases where the mass is split over the transport, recall that for any $\gamma \in \Pi(\eta, \nu)$, there exists a function $x \in \mathbb{R} \rightarrow Q^x_\gamma \in M_1(\mathbb{R})$ (see [6], [7]), such that the function

$$\phi_B : x \in \mathbb{R} \rightarrow Q^x_\gamma(B) \in \mathbb{R}$$

is Borel measurable, $\forall B \in \mathcal{B}(\mathbb{R})$, and

$$\gamma(A \times B) = \int_A Q^x_\gamma(B) \eta(dx), \forall A, B \in \mathcal{B}(\mathbb{R}),$$

the latter denoting a Lebesgue integral with respect to the Borel probability measure $\eta$, which is well defined from the previous hypothesis on \((2.3)\); within mass transport models, $Q^x_\gamma(B)$ may be interpreted as the proportion of the mass located at $x \in \mathbb{R}$ which is carried to $B \in \mathcal{B}(\mathbb{R})$, since $\gamma$ coincides with a desintegration kernel \(\gamma\) with respect to the canonical projection $p_1$ (see [5]), outside an $\eta-$ negligible set ; we have, for any $A \in \mathcal{B}(\mathbb{R})$, $Q^x_\gamma(A) = \gamma(\{p_2 \in A\} | p_1 = x)$, $\eta$ – a.s., where ”$p_1 = x$” refers here to a desintegration of measure. Subsequently, any such function $Q_\gamma$ will be called a transport kernel associated to $\gamma$ ; it is unique up to an $\eta-$ negligible set, which will justify to take $\eta-$ completions below. Further, for $x \in \mathbb{R}$, $F^x_\gamma$ denotes the cumulative distribution function of $Q^x_\gamma$ ; $F^x_\gamma(a) = Q^x_\gamma([-\infty, a])$, $\forall a \in \mathbb{R}$, and we set

$$F^\gamma : (x, y) \in \mathbb{R}^2 \rightarrow F^y_\gamma(y) \in \mathbb{R},$$

to which we refer to as a conditional cumulative distribution function associated to $\gamma$, as it may depend on the choice of $Q_\gamma$. Furthermore, given a real-valued random variable $X$ which is defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that is, a function $X : \Omega \rightarrow \mathbb{R}$ whose level sets $\{\omega \in \Omega : X(\omega) \leq l\} \in \mathcal{A}$ are all contained in $\mathcal{A}$, $\forall l \in \mathbb{R}$, for short and once given $A \subset \mathbb{R}$, we may use the notation $\{X \in A\}$ to denote the inverse image $X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\}$, while we use the notation $E_\mathbb{P}[X]$ to denote the mathematical expectation of $X$ under the probability $\mathbb{P}$ ; $E_\mathbb{P}[X] := \int_\Omega X(\omega) d\mathbb{P}(d\omega)$ denotes a Lebesgue integral with respect to the probability measure $\mathbb{P}$ whenever $X$ is $\mathbb{P}-integrable$ (i.e. $E_\mathbb{P}[|X|] < +\infty$), in which case $E_\mathbb{P}[X] \in \mathbb{R}$, while we set $E_\mathbb{P}[X] = +\infty$ whenever $X \geq 0$, $\mathbb{P}-a.s.$, and $X$ is not $\mathbb{P}-integrable$. The law of the random variable $X$ is denoted by $p_X \in M_1(\mathbb{R})$, where $p_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{X \in A\}), \forall A \in \mathcal{B}(\mathbb{R})$, while further assuming that $Y : \Omega \rightarrow \mathbb{R}$ denotes a random variable defined on the same probability space as $X$, $p_{X,Y} \in M_1(\mathbb{R} \times \mathbb{R})$ denotes the joint law.
of $X$ and $Y$, that is, the pushforward $p_{X,Y} := (X,Y)_*\mathbb{P}$ of the probability measure $\mathbb{P}$ by the measurable function $(X,Y) : \omega \in \Omega \to (X(\omega), Y(\omega)) \in \mathbb{R} \times \mathbb{R}$, so that $p_{X,Y}(A) = \mathbb{P}((X,Y) \in A), \forall A \in \mathcal{B}(\mathbb{R} \times \mathbb{R})$; whenever $X$ is $\mathbb{P}$– independent to $Y$ we denote by $p_X \otimes p_Y$ their joint law $p_{X,Y}$. Finally, given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, if $C \in \mathcal{A}$ is such that $\mathbb{P}(C) > 0$, we apply the standard notation $\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)}$, $\forall A \in \mathcal{A}$, as it is used in Definition 1.1 with $C = \{X \geq t\}, t \in \mathbb{T}$, while for any $A \in \mathcal{A}$, $1_A : \Omega \to \mathbb{R}$ denotes here the indicator function of $A$; $1_A(\omega) = 1$ if $\omega \in A$, while $1_A(\omega) = 0$ if $\omega \notin A$, $\forall \omega \in \Omega$.

3. Causal mass transport between Borel probabilities on the set of real numbers.

3.1. The filtration generated by a transport plan, a characterization. We recall some conventional terminology adopted in [11], which will be required subsequently.

**Definition 3.1.** Let $\eta, \nu \in M_1(\mathbb{R})$, and $\gamma \in \Pi(\eta, \nu)$. The filtration $(\mathcal{G}_t(\gamma))_{t \in \mathbb{T}}$ generated by $\gamma$ is defined, for any $t \in \mathbb{T}$, by

$$
\mathcal{G}_t(\gamma) = \sigma(\phi_B : B \in \mathcal{B}_t(\mathbb{R}), \nu(\partial B) = 0)^\eta;
$$

the $\eta$– completion of the smallest sigma field which turns into measurable functions all the functions $\phi_B$ as in (2.3) such that $B \in \mathcal{B}_t(\mathbb{R})$ and $\nu(\partial B) = 0$. The set $\Pi_c(\eta, \nu)$ of causal (or adapted) transport plans $\gamma$ from $\eta$ to $\nu$ is defined by

$$
\Pi_c(\eta, \nu) = \{\gamma \in \Pi(\eta, \nu) : \mathcal{G}_t(\gamma) \subset \mathcal{B}_t(\mathbb{R})^\eta, \forall t \in \mathbb{T}\}.
$$

**Lemma 3.1.** Let $\eta, \nu \in M_1(\mathbb{R})$, $\mathcal{G} \in \Pi(\eta, \nu)$, and further denote by $Q_\gamma : x \in \mathbb{R} \to Q_\gamma^x \in M_1(\mathbb{R})$ a transport kernel associated to $\gamma$ by (2.4). Then the following assertions are equivalent :

(i) $\gamma \in \Pi_c(\eta, \nu)$

(ii) For any $t \in \mathbb{T}$ such that $\eta([t, +\infty[) > 0, \forall A \in \mathcal{B}_t(\mathbb{R}$, and for all $x \geq t$ outside an $\eta$– negligible set, we have

$$
Q_\gamma^x(A) = \gamma([p_2 \in A]|\{p_1 \in [t, +\infty[\});
$$

conditional probability, with respect to an event.

(iii) For any $t \in \mathbb{T}$ such that $\eta([t, +\infty[) > 0$, and for any $a < t$, $\exists c_{a,t} \in \mathbb{R}$ such that we have

$$
F_\gamma^x(a) = c_{a,t}, \forall x \geq t, \text{ outside an } \eta – \text{ negligible set},
$$

$F_\gamma^x : \mathbb{R} \to [0, 1]$ denoting the cumulative distribution function of $Q_\gamma^x$, $\forall x \in \mathbb{R}$.
Proof: Since the definitions yield
\[
\mathcal{B}_t(\mathbb{R}) = \rho^{-1}_t(\mathcal{B}(\mathbb{R})) = \{x \in \mathbb{R} : \rho_t(x) \in A\} : A \in \mathcal{B}(\mathbb{R}),
\]
where the function \(\rho_t : s \in \mathbb{R} \to \min(s, t) \in \mathbb{R}\) is continuous, \(\forall t \in \mathbb{T}\), and \(\rho_t \circ \rho_u = \rho_{\min(t, u)}\), \(\forall t, u \in \mathbb{T}\), \(\circ\) denoting the usual pullback of functions, from [11] we first deduce that, for any \(t \in \mathbb{T}\), we have
\[
\mathcal{G}_t(\gamma) = \sigma(\phi_A : A \in \mathcal{B}_t(\mathbb{R}))^\eta;
\]
(3.5) since \(\mathcal{B}_t(\mathbb{R})\) is also the \(\sigma\)-field generated by \(\{\rho^{-1}_t(F) : F \text{ closed subset of } \mathbb{R}\}\), the latter equation (3.5) can also be checked readily by a monotone class argument, once noticed that, since \(\rho_t\) is continuous, for any closed set \(F \subset \mathbb{R}\), the holding condition \(\lambda(\{y > 0 : \nu(\partial(\rho^{-1}_t(d^{-1}_F([0, y]))) > 0\}) = 0\), where \(\lambda\) denotes the LEBESGUE mesure, and where \(d_F : x \in \mathbb{R} \to \inf_{y \in F}|x - y| \in \mathbb{R}\), yields the existence of a decreasing sequence \((B_n)_{n \in \mathbb{N}}\) of closed subsets of the form \(\rho^{-1}_t(d^{-1}_F([-\infty, t_n]))\), for some \(t_n > 0\), which further satisfy \(\nu(\partial B_n) = 0\), \(B_n \in \mathcal{B}_t(\mathbb{R})\), \(\forall n \in \mathbb{N}\) while \(\rho^{-1}_t(F) = \bigcap_{n \in \mathbb{N}} B_n\), so that \(\phi_{\rho^{-1}_t(F)} = \lim_{n \to \infty} \phi_{B_n}\), \(\eta\) - a.s., and therefore \(\phi_{\rho^{-1}_t(F)}\) is \(\mathcal{G}_t(\gamma)\)-measurable. From (3.5), we first notice that \(\gamma \in \Pi(\eta, \nu)\), if an only if,
\[
Q_\gamma(A) = E_\eta[Q_\gamma(A)|\mathcal{B}_t(\mathbb{R})^\eta], \quad \eta - \text{a.s.}
\]
(3.6) for all \(A \in \mathcal{B}_t(\mathbb{R})\), \(\forall t \in \mathbb{T}\); the right-hand term of (3.6) denotes a conditional expectation with respect to the \(\sigma\)-field \(\mathcal{B}_t(\mathbb{R})^\eta\), where \(t \in \mathbb{T}\). We now turn to the main part of the proof. First assume (i), and let \(t \in \mathbb{R}\). For \(A \in \mathcal{B}_t(\mathbb{R})\), from the very definition of the conditional expectation with respect to a sigma field (for instance see [3], [7]), and from the properties of the filtration \((\mathcal{B}_t(\mathbb{R}))\) just recalled above, it directly follows that
\[
E_\eta[Q_\gamma(A)|\mathcal{B}_t(\mathbb{R})^\eta] = Q_\gamma(A)1_{-\infty,t[} + \gamma(\{p_2 \in A\}|\{p_1 \in [t, +\infty[\})1_{[t, +\infty[}, \quad \eta - \text{a.s.,}
\]
if \(\eta([t, +\infty[) > 0\), while (3.6) already holds as soon as \(\eta([t, +\infty[) = 0\). As a consequence, (3.6) holds for all \(A \in \mathcal{B}_t(\mathbb{R})\), and \(\forall t \in \mathbb{T}\), if and only if,
\[
\gamma(\{p_2 \in A\}|\{p_1 \in [t, +\infty[\})1_{[t, +\infty[} = Q_\gamma(A)1_{[t, +\infty[}, \quad \eta - \text{a.s.,}
\]
holds, for all \(A \in \mathcal{B}_t(\mathbb{R})\), and \(\forall t \in \mathbb{T}\) such that \(\eta([t, +\infty[) > 0\), which is (ii) : we have proved that (i) is equivalent to (ii). On the other hand, since for \(t \in \mathbb{T}\) and \(a < t\), ]\(-\infty, a]\) \(\subset \mathcal{B}_t(\mathbb{R})\), (ii) yields (iii) as a particular case, by taking
\[
c_{a,t} = \gamma(\{p_2 \in ]-\infty, a]\}|\{p_1 \in [t, +\infty[\}) \in [0, 1].
\]
(3.7)
Finally, assuming that (iii) holds, for \(t \in \mathbb{T}\) such that \(\eta([t, +\infty[) > 0\), and \(a < t\), we notice that we still obtain (3.7), so that we have (ii) for all \(A \in \mathcal{B}_t(\mathbb{R})\) of the form
Figure 3.2. This figure illustrates Lemma 3.1. Representation of the restriction to the square $[0, 2000] \times [0, 2000]$ of a conditional cumulative distribution function $F_\gamma$, which is associated to the causal transport plan

$$
\gamma = \frac{1}{2} p_{X,x+z} + \frac{1}{4} p_X \otimes \delta_{t_0} + \frac{1}{4} p_X \otimes p_X,
$$

where $t_0 = 700$, and where $X$, and $Z$ denote two $\mathbb{P}$-independent real valued random variables defined on a same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, whose laws are given by $p_X = \mathcal{E} \left( \frac{1}{100} \right)$, and by $p_Z = \mathcal{E} \left( \frac{1}{200} \right)$; $\mathcal{E}(\beta) \in M_1(\mathbb{R})$ denotes the so-called exponential law of parameter $\beta \in \mathbb{R}_+^*$, that is, the Borel probability measure absolutely continuous with respect to the Lebesgue measure, whose density $f_\beta : \mathbb{R} \to \mathbb{R}$ is given by

$$
f_\beta(x) = \beta \exp(-\beta x) \mathbb{1}_{[0, +\infty[}(x), \ \forall x \in \mathbb{R}.
$$

\[ -\infty, a], a < t. \] Since $B_t(\mathbb{R}) = \sigma(\mathcal{C}_t)$, where $\mathcal{C}_t \subset B(\mathbb{R})$ is given by $[2.2]$, (ii) follows from a monotone class argument (see [3]).

3.2. Deterministic causal transport plans. Let $\eta \in M_1(\mathbb{R})$, let $T : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, and define $\gamma_T = (I_\mathbb{R}, T)_* \eta$, the deterministic transport plan of $\eta$ by $T$; we can set $Q_{\gamma_T}^x = \delta_{T(x)}$, $\forall x \in \mathbb{R}$, $\delta_y$ denoting the Dirac mass (see [1]) centered on $y \in \mathbb{R}$, $\forall y \in \mathbb{R}$, while $\gamma_T$ coincides with the pushforward measure of
η by the measurable function
\[(I_\mathbb{R}, T) : x \in \mathbb{R} \rightarrow (x, T(x)) \in \mathbb{R} \times \mathbb{R},\]
so that \((I_\mathbb{R}, T) \ast \eta(C) = \eta(\{x \in \mathbb{R} : (x, T(x)) \in C\}), \forall C \in \mathcal{B}(\mathbb{R} \times \mathbb{R}).\) Further recall that, within those specific hypothesis, from (3.5) we obtain
\[\mathcal{G}_t(\gamma T) = \mathcal{G}_t^T, \tag{3.8}\]
where
\[\mathcal{G}_t^T = \{T^{-1}(A) : A \in \mathcal{B}_t(\mathbb{R})\}^n, \forall t \in \mathbb{T}; T^{-1}(A) = \{x \in \mathbb{R} : T(x) \in A\}\] still denotes the inverse image of \(A \in \mathcal{B}(\mathbb{R})\) by the function \(T\). Finally, for the sake of consistency with [11] and for short, given \(\eta \in M_1(\mathbb{R})\), subsequently \(T\) will be said to be adapted (or causal), if and only if,
\[\gamma_T \in \Pi_c(\eta, \nu),\]
where \(\nu = T \ast \eta\), if and only if,
\[\mathcal{G}_t^T \subset \mathcal{B}_t(\mathbb{R})^n, \forall t \in \mathbb{T},\]
where the last equivalence follows from (3.8); again, \(T \ast \eta \in M_1(\mathbb{R})\) denotes the push-forward or direct image of \(\eta \in M_1(\mathbb{R})\) by the BOREL measurable function \(T : \mathbb{R} \rightarrow \mathbb{R}\), that is \(T \ast \eta(A) = \eta(\{x \in \mathbb{R} : T(x) \in A\}), \forall A \in \mathcal{B}(\mathbb{R}).\)

**Lemma 3.2.** Let \(\eta \in M_1(\mathbb{R})\), and let \(T : \mathbb{R} \rightarrow \mathbb{R}\) be a BOREL measurable function such that \(T\) is adapted, i.e. \(T^{-1}(\mathcal{B}_t(\mathbb{R})) \subset \mathcal{B}_t(\mathbb{R})^n, \forall t \in \mathbb{T}\). Further assume that we have
\[\eta(\{x \in \mathbb{R} : T(x) < x\}) > 0. \tag{3.9}\]
Then, the following assertions are satisfied:

(i) \(\forall t \in \mathbb{T}, \forall A \in \mathcal{B}(\mathbb{R})\) such that either \(A \subset ]-\infty, t[\) or \([t, +\infty[ \subset A\), the following implication holds:
\[\eta(\{x \geq t : T(x) \in A\}) > 0 \implies \eta(\{x \in \mathbb{R} : T(x) \in A\}[t, +\infty[) = 1.\]

(ii) Let
\[D = \{t \in \mathbb{R} : \eta(\{x \geq t : T(x) < t\}) > 0\}.\]
Then, \(D \neq \emptyset\), and there exists a unique \(t_0 \in \mathbb{R}\) which is such that \(t_0 < t, \forall t \in D,\) and
\[\eta(\{x \in \mathbb{R} : T(x) = t_0\}[t, +\infty[) = 1, \forall t \in D.\]

(iii) \(\inf D \in \mathbb{R}\) is not attained and we have
\[\eta([[ \inf D, +\infty[)) > 0\]
and
\[\eta(\{x \in \mathbb{R} : T(x) = t_0\}) \inf D, +\infty[) = 1.\]
(iv) The following hold:

\[
\begin{align*}
\eta\left(\{x \leq t_0 : T(x) \geq x\}\right) &= \eta([-\infty, t_0]) \\
\eta([t_0, +\infty[) &= 0 \\
\eta(\{x \in \mathbb{R} : T(x) = t_0\}|[t_0, +\infty[) &= 1
\end{align*}
\]

**Proof:** Since \(\gamma_T = (I_{\mathbb{R}}, T)_*\eta \in \Pi_c(\eta, \nu)\), where \(\nu = T_*\eta\), Lemma 3.1 applies. Hence, assume that \(t \in \mathbb{T}\) and \(A \in \mathcal{B}(\mathbb{R})\) are such that \(\eta(\{x \geq t : T(x) \in A\}) > 0\); in particular, \(\eta([t, +\infty[) > 0\). As we can state

\[
\mathbb{Q}_{\gamma_T}^x(A) = \mathbb{I}_{T(x)}(A) = \mathbb{1}_A(T(x)) = \mathbb{1}_{T^{-1}(A)}(x), \quad \forall x \in \mathbb{R},
\]

\(\mathbb{Q}_{\gamma_T}\) denoting a transport kernel for \(\gamma_T\) recalled above, from Lemma 3.1 we obtain

\[
\mathbb{1}_{T^{-1}(A)}\mathbb{1}_{[t, +\infty[} = \gamma_T(\{p_2 \in A\}|\{p_1 \in [t, +\infty[\}) \mathbb{1}_{[t, +\infty[}, \quad \eta - a.s.,
\]

so that \(\eta(\{x \geq t : T(x) \in A\}) > 0\) implies \(\gamma_T(\{p_2 \in A\}|\{p_1 \in [t, +\infty[\}) = 1\), from which the implication (i) follows. Further noticing that

\[
\eta(\{x \in \mathbb{R} : T(x) < x\}) = \eta\left(\bigcup_{q \in \mathbb{Q}} \{x \geq q : T(x) < q\}\right),
\]

and that a countable union of negligible sets is again negligible, we obtain \(D \neq \emptyset\) from (3.9). Then, we choose any \(t \in D\), and define

\[
t_0 = \inf(\{y \in \mathbb{R} : \eta(\{x \geq t : T(x) < y\}) \geq \eta(\{x \geq t : T(x) < t\})\}).
\]

Together with (i), from the definitions, it is then an easy exercise to check that \(t_0 \in \mathbb{R}\) doesn’t depend on the choice of \(t \in D\), while it meets the required hypothesis. Finally, the rest of the statement results by following the successive steps, while applying (i) as far as necessary. \(\Box\)

**Theorem 3.1.** Let \(\eta \in M_1(\mathbb{R})\), let \(T : \mathbb{R} \to \mathbb{R}\) be a Borel measurable function, set \(\nu = T_*\mu\), and still define \(\gamma_T = (I_{\mathbb{R}}, T)_*\eta\) to be the deterministic transport plan of \(\eta\) by \(T\). Then, \(\gamma_T \in \Pi_c(\eta, \nu)\), if and only if, \(T\) satisfies one among the two hypothesis below:

1. \(T(x) \geq x, \eta - a.s.\)
2. \(\exists t_0 \in \mathbb{R}\) which is such that, \(\forall x \leq t_0\) outside an \(\eta-\)negligible set, we have \(T(x) \geq x\), and for any \(x > t_0\) outside an \(\eta-\)negligible set, we have \(T(x) = t_0\).
This figures illustrate Theorem 3.1. In red, the first diagonal of the plane. In blue, the representation of the restriction of the graph of the Borel measurable function $T : \mathbb{R} \to \mathbb{R}$, given by, from the image on the left to the image on the right, $T(x) = c, \forall x \in \mathbb{R}$, with $c = 700$, by $T(x) = (x + z_0) \mathbbm{1}_{[-\infty,t_0]}(x) + t_0 \mathbbm{1}_{[t_0,\infty[}(x), \forall x \in \mathbb{R}$, with $t_0 = 30$ and $z_0 = 200$, by $T : x \in \mathbb{R} \to (ax^2 + bx + c) \mathbbm{1}_{[-\infty,t_0]} + t_0 \mathbbm{1}_{[t_0,\infty[}(x) \in \mathbb{R}$, with $t_0 = 250$, $a = \frac{1}{1500}$, $b = 1$, and $c = 300$, and finally by $T : x \in \mathbb{R} \to x + a \cos(bx) + c \in \mathbb{R}$, where $a = 10$, $b = \frac{1}{5}$, and $c = 220$. For any $\eta \in M_1(\mathbb{R})$, those functions $T$ induce a deterministic causal transport plan $\gamma_T = (I_{\mathbb{R}},T) \ast \eta \in \Pi_c(\eta,\nu)$ of $\eta$ to $\nu = T_{\ast} \eta$.

Proof: Assuming that one among (1) or (2) is satisfied, it is enough to notice that $B_t(\mathbb{R}) = \sigma(C_t)$, where $C_t$ is given by (2.2), so that we easily check that $\gamma_T \in \Pi_c(\eta,\nu)$, as $G_{t}^{T} \subset B_t(\mathbb{R})^\nu, \forall t \in \mathbb{T}$; recall that (3.8) holds. Conversely, assuming that $\gamma_T \in \Pi_c(\eta,\nu)$, the result follows from Lemma 3.2.

Remark: (2) above is clearly equivalent to : \( \exists t_0 \in \mathbb{R} \) and a non-negative Borel measurable function $g_T : \mathbb{R} \to [0, +\infty[$, such that

\[ T(x) = (x + g_T(x)) \mathbbm{1}_{[-\infty,t_0]}(x) + t_0 \mathbbm{1}_{[t_0,\infty[}(x), \eta - a.e.. \]

As a first practical consequence of Theorem 3.1, we get:

**Corollary 3.1.** Let $T : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, $\eta \in M_1(\mathbb{R})$, and assume that $\nu = T_{\ast} \eta$ and $\eta$ are diffuse (i.e. $\eta(\{x\}) = \nu(\{x\}) = 0, \forall x \in \mathbb{R}$), and that $\eta([t, +\infty[) > 0, \forall t \in \mathbb{T};$ for instance, $\eta$ has a cumulative distribution function which
is strictly increasing on \( ]y, +\infty[ \) for some \( y \in \mathbb{R} \). Then, \( T \) is adapted, if and only if,

\[
T(x) \geq x, \, \eta - a.s.
\]

\( \square \)

**Example 3.1.** Let \( a, b \in \mathbb{R} \), and define \( T : x \in \mathbb{R} \to ax + b \in \mathbb{R} \); recall again that within those hypothesis, given \( \eta \in M_1(\mathbb{R}) \), \( T \) is said to be adapted, if and only if, \( \gamma = (I_\mathbb{R}, T)_*\eta \in \Pi_c(\eta, \nu) \), where \( \nu = T_*\eta \), if and only if, \( T^{-1}(\mathcal{B}_t(\mathbb{R})) \subset \mathcal{B}_t(\mathbb{R})^\eta \), \( \forall t \in T \).

- Assuming that \( \eta = \mathcal{N}(0, 1) \) (the Gauss law), \( T \) is adapted, if and only if, either \( a = 0 \), or both \( a = 1 \) and \( b \geq 0 \).
- If \( \eta = \mathcal{E}(1) \) (exponential law) then \( T \) is adapted, if and only if, either \( a = 0 \), or both \( a \geq 1 \) and \( b \geq 0 \).

4. **Characterization of adapted couplings and probabilistic optimization.**

The following Theorem 4.1 will allow to address the problem stated in the introduction by using tools of causal optimal transport, in Corollary 4.1.

**Theorem 4.1.**

1. Let \( \eta, \nu \in M_1(\mathbb{R}) \). Then, for any \( \gamma \in \Pi_c(\eta, \nu) \), there exists an adapted coupling \((X, Y, (\Omega, \mathcal{A}, \mathbb{P})) \in \text{Cpl}_a(\eta, \nu)\) such that \( p_{X,Y} = \gamma \), where \( p_{X,Y} \in M_1(\mathbb{R} \times \mathbb{R}) \) denotes the joint law of the random variables \( X : \Omega \to \mathbb{R} \) and \( Y : \Omega \to \mathbb{R} \) on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\), and where \( \text{Cpl}_a(\eta, \nu) \) (resp. \( \Pi_c(\eta, \nu) \)) is given by Definition 1.1 (resp. by Definition 3.1).

2. For any adapted coupling \((X, Y, (\Omega, \mathcal{A}, \mathbb{P})) \in \text{Cpl}_a(p_X, p_Y)\), we have

\[
p_{X,Y} \in \Pi_c(p_X, p_Y),
\]

\( p_X \) (resp. \( p_Y \)) denoting the probability law of \( X \) (resp. of \( Y \)) on \((\Omega, \mathcal{A}, \mathbb{P})\).

**Proof:** First assume that \( \gamma \in \Pi_c(\eta, \nu) \), and define

\[
\begin{cases}
(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R} \times \mathbb{R})^\gamma, \gamma) \\
X = p_1 \\
Y = p_2 \\
\tau = \min(p_1, p_2) \\
Z = (Y - X)1_{\{x \leq \tau\}}
\end{cases}
\]
Figure 4.4. This figure illustrates Example 4.1 for $0 < a < b$. From classical results (see 2/ in 1.7. of [8]), for $\gamma = pr_a r_b$ it follows that we can choose

$$F_\gamma^x(y) = 1_{[0, +\infty]}(y - x) \left(1 - \text{erf} \left(\frac{b - a}{\sqrt{2(y - x)}}\right)\right),$$

for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$, where $\text{erf}$ denotes the so-called error function, which is given by $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt$, $\forall z \in \mathbb{R}_+$; $\mathbb{R}_+$ is the support of $p_{r_a}$. The figure represents the restriction of the graph of $F_\gamma$, for $a = 6$ and $b = 11$.

$p_1$ (resp. $p_2$) still denotes the canonical projection on the first (resp. second) component of the product space $\mathbb{R} \times \mathbb{R}$. Then, it is enough to check that $(X, Y, (\Omega, A, \mathbb{P}))$ satisfies the axioms of Definition 1.1 to obtain (1), which follows from Lemma 3.1. Then, (2) follows by applying again Lemma 3.1 (iii), together with the definitions.

Example 4.1. Let $(B_t)_{t \in [0, +\infty[}$ be a standard real valued Brownian motion, starting from 0, which is defined on a $\mathbb{P}$-complete probability space $(\Omega, A, \mathbb{P})$ (see [16]); this is the continuous Lévy process associated to the Gauss law $\mathcal{N}(0, 1)$ which is an infinitely divisible distribution, while the law of this continuous stochastic process is the so-called Wiener measure, see [12], [16], [17], [19]. In particular, $(B_t)_{t \in [0, +\infty[}$ is a family of gaussian random variables labeled by $\mathbb{R}_+$, such that on the one hand the function $t \in [0, +\infty[ \to B_t(\omega) \in \mathbb{R}$ is continuous for any $\omega \in \Omega$ outside a $\mathbb{P}$-negligible set, while on the other hand, for any $s, t \in [0, +\infty[$ such that $s < t$, there exists two independent random variables $G$ and $\tilde{G}$, depending respectively on $s$ and on both $s$ and $t$, defined on a same probability space, and with a same Gauss law $p_G = p_{\tilde{G}} = \mathcal{N}(0, 1) \in M_1(\mathbb{R})$, such that the joint law $p_{B_s, B_t}$ of the two random variables $B_s : \Omega \to \mathbb{R}$ and $B_t : \Omega \to \mathbb{R}$ on $(\Omega, A, \mathbb{P})$ coincide with the joint law of the
random variables $\sqrt{sG}$ and $\sqrt{sG} + \sqrt{t - sG}$, that is

$$p_{B_s, B_t} = p_{\sqrt{sG}, \sqrt{sG} + \sqrt{t - sG}}.$$ 

For $c \in \mathbb{R}$, define the, $\mathbb{P}-$ almost surely finite, first passage time random variable

$$T_c := \inf\{t \geq 0 : B_t = c\},$$

which is a so-called stopping time with respect to the filtration generated by $(B_t)_{t \in [0, +\infty]}$ (see [8]). Assuming that $0 < a < b$, from Theorem 4.1, we obtain

$$p_{T_a, T_b} \in \Pi_c(p_{T_a}, p_{T_b}),$$

$p_{T_a}$ (resp. $p_{T_b}$) denoting the probability law of $T_a$ (resp. of $T_b$), while $p_{T_a, T_b} = (T_a, T_b), \mathbb{P} \in M_1(\mathbb{R} \times \mathbb{R})$ denotes their joint law, on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. 

**Corollary 4.1.** Assume that $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a non-negative and lower semi-continuous function, and let $\eta, \nu \in M_1(\mathbb{R})$. Then, there exists an adapted coupling $(X, Y, (\Omega, \mathcal{A}, \mathbb{P})) \in Cpl_a(\eta, \nu)$ which attains

$$\inf\{E_\mathbb{P}[c(X, Y)] : (X, Y, (\Omega, \mathcal{A}, \mathbb{P})) \in Cpl_a(\eta, \nu)\}, \quad (4.10)$$

where $Cpl_a(\eta, \nu)$ denotes the set defined in Definition 1.1. Moreover, this problem is equivalent to the causal (or adapted) Monge-Kantorovich problem

$$\inf_{\gamma \in \Pi_c(\eta, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y)\gamma(dx, dy), \quad (4.11)$$

where $\Pi_c(\eta, \nu)$ is given by Definition 3.1. That is, $\gamma \in \Pi_c(\eta, \nu)$ attains the infimum of $\{4.11\}$, if and only if, there exists some adapted coupling $(X, Y, (\Omega, \mathcal{A}, \mathbb{P})) \in Cpl_a(\eta, \nu)$ which attains the infimum of $\{4.10\}$, and $\gamma = p_{X,Y}$, where $p_{X,Y}$ denotes the joint law of the pair of random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, on the $\mathbb{P}-$ complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. 

**Proof:** From Theorem 4.1 this optimization problem is equivalent to the causal Monge-Kantorovich problem (4.11). Hence, the result follows from [11]. 

**Example 4.2.** For $m \in \mathbb{N}$, $\lambda > 0$, denote by $\Gamma \left( m, \frac{1}{\lambda} \right) \in M_1(\mathbb{R})$ the Borel probability measure absolutely continuous with respect to the Lebesgue measure, whose density $\rho_{m, \frac{1}{\lambda}} : \mathbb{R} \to \mathbb{R}$ is given by 

$$\rho_{m, \frac{1}{\lambda}}(x) = \lambda \exp(-\lambda x) \frac{(\lambda x)^{m-1}}{(m-1)!} \mathbb{1}_{[0, +\infty]}(x), \ \forall x \in \mathbb{R};$$
those are particular cases of the so-called gamma distributions. For \( n \in \mathbb{N} \), and \( p \in \mathbb{N} \), define
\[
\eta = \Gamma \left( n, \frac{1}{\lambda} \right)
\]
and
\[
\nu = \Gamma \left( n + p, \frac{1}{\lambda} \right),
\]
for some \( \lambda \in [0, +\infty[ \). Further, let \((N_t)_{t \in \mathbb{R}_+} \) be a simple Poisson process with parameter \( \lambda \), which is defined on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) (for a definition, see [3], [16] or [17]), and set
\[
T_m = \inf \{ t \in \mathbb{R} : N_t \geq m \},
\]
which is a random variable on \((\Omega, \mathcal{A}, \mathbb{P})\), which describes the time of the \( m \)–th jump of the process, \( \forall m \in \mathbb{N} \). Then, denoting by \( p_{T_n, T_{n+p}} \) the joint law of the jump times \( T_n \) and \( T_{n+p} \) on \((\Omega, \mathcal{A}, \mathbb{P})\), from Theorem 4.1 we obtain
\[
p_{T_n, T_{n+p}} \in \Pi_c(\eta, \nu),
\]
so that Jensen’s inequality ensures that it also attains the infimum of
\[
\inf_{\gamma \in \Pi_c(\eta, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y| \gamma(dx, dy).
\]

**Remark:** (On the identification of the optima, and the \( c \)-cyclical monotonicity.) Let \( \eta \in M_1(\mathbb{R}) \), \( \nu \in M_1(\mathbb{R}) \), further assume that \( c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is non-negative, continuous, and that there exists \( u \in L^1(\eta) \), \( v \in L^1(\nu) \), such that \( c(x, y) \leq u(x) + v(y) \), \( \forall (x, y) \in \mathbb{R} \times \mathbb{R} \). Notice that, from the proofs that the \( c \)-cyclical monotonicity is a necessary condition for optima of usual Monge-Kantorovich problem (see the fundamental theorem of optimal transport in [2]), for \( \gamma \in \Pi_c(\eta, \nu) \) to be an optimum of the causal mass transport problem \((4.11)\), a necessary condition is that for any \( n \in \mathbb{N} \), and for any \( \{(x_i, y_i) : i \in \{1, \ldots, n\}\} \) contained in the support of \( \gamma \), such that
\[
\max_{i \in \{1, \ldots, n\}} x_i < \min_{i \in \{1, \ldots, n\}} y_i, \tag{4.12}
\]
and for any permutation \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \), we necessarily have
\[
\sum_{i=1}^n c(x_i, y_i) \leq \sum_{i=1}^n c(x_i, y_{\sigma(i)});
\]
indeed, assuming \((4.12)\), \((iii)\) of Lemma 3.1 ensures that, following the proof by absurd which is proposed for the classical problem in [2], as soon as \( \gamma \in \Pi_c(\eta, \nu) \), the plan \( \tilde{\gamma} \) of lower cost than \( \gamma \) which is built in [2] can be chosen to be an element of \( \Pi_c(\eta, \nu) \), as far as, given \( t_1 \in \max_{i \in \{1, \ldots, n\}} x_i, \min_{i \in \{1, \ldots, n\}} y_i \), the neighbourhoods \((U_i)_{i \in \{1, \ldots, n\}}\) (resp. \((V_i)_{i \in \{1, \ldots, n\}}\)) of the respective \((x_i)_{i \in \{1, \ldots, n\}}\) (resp. \((y_i)_{i \in \{1, \ldots, n\}}\)), where
the mass may be modified, can be chosen to be such that $\bigcup_{i=1}^{n} U_i \subset \mathbb{R}[−\infty,t_1]$ and $\bigcup_{i=1}^{n} V_i \subset [t_1, +\infty[.$

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