Certainty Equivalent Quadratic Control for Markov Jump Systems

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Abstract—Real-world control applications often involve complex dynamics subject to abrupt changes or variations. Markov jump linear systems (MJS) provide a rich framework for modeling such dynamics. Despite an extensive history, theoretical understanding of parameter sensitivities of MJS control is somewhat lacking. Motivated by this, we investigate robustness aspects of certainty equivalent model-based optimal control for MJS with quadratic cost function. Given the uncertainty in the system matrices and in the Markov transition matrix is bounded by \(\epsilon\) and \(\eta\) respectively, robustness results are established for (i) the solution to coupled Riccati equations and (ii) the optimal cost, by providing explicit perturbation bounds which decay as \(O(\epsilon + \eta)\) and \(O((\epsilon + \eta)^2)\) respectively.

I. INTRODUCTION

The Linear Quadratic Regulator (LQR) is both theoretically well understood and commonly used in practice when the system dynamics are known. It also provides an interesting benchmark, when system dynamics are unknown, for reinforcement learning with continuous state and action spaces and for adaptive control [1], [2], [3], [4], [5], [6], [7].

A natural generalization of linear dynamical systems is Markov jump linear systems (MJS) that allow the dynamics of the underlying system to switch between multiple linear systems according to an underlying finite Markov chain. Similarly, a natural generalization of LQR problem to MJS is to use mode-dependent cost matrices, which allows to have different control goals under different modes. While the optimal control for MJS-LQR is well understood when one has perfect knowledge of the system dynamics [8], [9], in practice it may not be optimal due to the imperfect knowledge of the system dynamics and the transition matrix. For instance, one might use system identification techniques to learn an approximate model for the system. Designing optimal controllers for MJS-LQR with this approximate system dynamics and transition matrix in place of the true ones leads to so-called certainty equivalent (CE) control which is used extensively in practice. However, a theoretical understanding of the suboptimality of the CE control for MJS-LQR is lacking. The main challenge here is the hybrid nature of the problem that requires consideration of both the system dynamics uncertainty \(\epsilon\), and the underlying Markov transition matrix uncertainty \(\eta\).

The solution of infinite horizon MJS-LQR involves coupled algebraic Riccati equations. Our goal is to understand how sensitive the solution of these equations and the corresponding optimal cost are to the perturbations in system model. To this aim, we first develop explicit \(O(\epsilon + \eta)\) perturbation bound for the solution to coupled algebraic Riccati equations that arise in the context of MJS-LQR. This in turn is used to establish explicit \(O((\epsilon + \eta)^2)\) suboptimality bound. Finally, numerical experiments are provided to support our theoretical claims. Our proof strategy requires nontrivial advances over those of [4], [10]. Specifically, the coupled nature of Riccati equations requires novel perturbation arguments as these coupled equations lack some of the nice properties of the standard Riccati equations, like uniqueness of solutions under certain conditions or being amenable to matrix factorization based approaches.

II. RELATED WORK

The performance analysis of CE control for the classical LQR problem for linear time invariant (LTI) systems relies on the perturbation/sensitivity analysis of the underlying algebraic Riccati equations (ARE), i.e. how much the ARE solution changes when the parameters in the equation are perturbed. This problem is studied in many works [11]. Early results on ARE solution perturbation bound are presented in [12] (continuous-time) and [10] (discrete-time). Most literature, however, only discusses perturbed solutions within the vicinity of the ground-truth solution. The uniqueness of such a perturbed solution is not discussed until [13], which is further refined in [14] to provide explicit perturbation bounds and generalization to complex equations. Tighter bounds are obtained [15] when the parameters have a special structure like sparsity. Channelled by ARE perturbation results, the end-to-end CE LQR control suboptimality bound in terms of dynamics perturbation is established in [4]. The field of CE MJS-LQR control and corresponding coupled ARE (cARE) perturbation analysis, however, is very barren. To the best of our knowledge, there is no work on performance guarantee for CE MJS-LQR control, and the only two works [16], [17] for cARE perturbation analysis only consider continuous-time cARE that arises in robust control applications, which is not applicable in MJS-LQR setting. Our work is also related to robust control for MJS (see, e.g., [18], [9]), where the focus is to numerically compute a controller to achieve a guaranteed cost under a given uncertainty bound. Whereas, we aim to characterize how the degradation in performance depends on perturbations in different parameters when CE control is used. Therefore, our work contributes to the body of work in robust control and CE control of MJS from a different perspective, and also paves the way to
use these ideas in the context of learning-based control with performance guarantees.

III. PRELIMINARIES AND PROBLEM SETUP

We use boldface uppercase (lowercase) letters to denote matrices (vectors). For a matrix $V$, $\rho(V)$, $\sigma(V)$, and $\|V\|$ denote its spectral radius, smallest singular value, and spectral norm, respectively. We let $\|V\|_1 := \|V\| + 1$. The Kronecker product of two matrices $M$ and $N$ is denoted as $M \otimes N$. $V_{1:s}$ denotes a set of $s$ matrices $\{V_i\}_{i=1}^s$ of same dimensions. We use $\text{diag}(V_{1:s})$ to denote a block diagonal matrix whose $i$-th diagonal block is given by $V_i$. We define $[s] := \{1, 2, \ldots, s\}$, $g(V_{1:s}) := \min_{i \in [s]} g(V_i)$, $\|V_{1:s}\| := \max_{i \in [s]} \|V_i\|$, and $\|V_{1:s}\|_+ := \max_{i \in [s]} \|V_i\|_+$. We use $\alpha U_{1:s} + \beta V_{1:s}$ to denote $\{\alpha U_i + \beta V_i\}_{i=1}^s$. We use $\|z\|$ and $\|z\|_\infty$ for inequalities that hold up to a constant factor.

A. Markov Jump Systems

We consider the problem of optimally controlling MJS, which are governed by the state equation,

$$x_{t+1} = A_\omega(t)x_t + B_\omega(t)u_t + w_t,$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^p$ and $w_t \in \mathbb{R}^m$ denote the state, input (or action) and noise at time $t$ respectively. Throughout, we assume $x_0 \sim N(0, I_n)$ and $\{w_t\}_{t=0}^\infty \sim N(0, \sigma_w^2 I_m)$. There are $s$ modes in total, and the dynamics of the $i$-th mode is given by $A_i^\omega(t), B_i^\omega(t)$. The active mode at time $t$ is indexed by $\omega(t) \in [s]$. In MJS the mode sequence $\omega(t)_{t=0}^\infty$ follows an ergodic Markov chain with transition matrix $T^\omega \in \mathbb{R}^{s \times s}$ such that for all $t \geq 0$, the $i$-th element of $T^\omega$ denotes the conditional probability

$$[T^\omega]_{ij} := \mathbb{P}(\omega(t+1) = j | \omega(t) = i), \forall i, j \in [s].$$

Due to ergodicity, for any initial distribution $\pi_0 \in \mathbb{R}^s$ of $\omega(0)$, there exists a unique stationary distribution $\pi = \pi_T \in \mathbb{R}^s$ such that $(T^\omega)^T \pi_0 = \pi_T$, as $t \to \infty$. Throughout, we assume the initial state $x_0$, Markov chain $\{\omega(t)\}_{t=0}^\infty$, and noise $\{w_t\}_{t=0}^\infty$ are mutually independent.

For mode-dependent controller $K_{1:s}$ that yields inputs $u_t = K_{\omega(t)}x_t$, we use $L_i := A_i^\omega + B_i^\omega K_i$ to denote the closed-loop state matrix for mode $i$. We use $x_{t+1} = L_\omega(t)x_t$ to denote the noise-free autonomous MJS, either open-loop ($L_i = A_i^\omega$) or closed-loop ($L_i = A_i^\omega + B_i^\omega K_i$). Due to the randomness in $\{\omega(t)\}_{t=0}^\infty$, it is common to consider the stability of MJS in the mean-square sense which is defined as follows.

**Definition 1.** [9, Definitions 3.8, 3.40] (a) We say MJS in (1) with $u_0 = 0$ is mean square stable (MSS) if there exists $x_\infty, \Sigma_\infty$ such that for any initial state/mode $x_0$, $\omega(0)$, as $t \to \infty$, we have

$$\mathbb{E}[x_t] - x_\infty \to 0, \quad \mathbb{E}[x_t x_t^\top] - \Sigma_\infty \to 0.$$  

(b) We say MJS in (1) with $w_0 = 0$ is (mean square) stabilizable if there exists mode-dependent controller $K_{1:s}$ such that the closed-loop MJS $x_{t+1} = (A_\omega(t) + B_\omega(t)K_\omega(t))x_t$ is MSS. We call such $K_{1:s}$ a stabilizing controller.

One can check the stabilizability of an MJS via linear matrix inequalities [9, Proposition 3.42]. It is well-known that the stability of non-switching systems is related to the spectral radius of the state matrix. Similarly, the mean-square stability of an autonomous MJS $x_{t+1} = L_\omega(t)x_t$ is related to the spectral radius of the augmented state matrix: $L \in \mathbb{R}^{s \times s \times n^2}$ with $i$-th $n^2 \times n^2$ block given by

$$[L]_{ij} := [T^\omega]_{ij} L_j^T \otimes L_i^T, \forall i, j \in [s].$$

Before stating a lemma to relate the MSS with the spectral radius of $L$, we define the operator,

$$\varphi^*_i(V_{1:s}) := \sum_{j=1}^s [T^\omega]_{ij} V_j, \forall i \in [s].$$

**Lemma 2.** [9, Theorem 3.9] The following are equivalent: (a) MJS $x_{t+1} = L_\omega(t)x_t$ is MSS; (b) $\rho(L) < 1$; (c) there exists $V_{1:s}$ with $V_i > 0$, such that $V_i - L_i^T \varphi^*_i(V_{1:s})L_i > 0, \forall i \in [s]$.

These assertions reduce to the classical stability results regarding spectral radius and Lyapunov equation when $s = 1$. Moreover, it can be shown that the augmented matrix $L^T$ maps $\{\mathbb{E}[x_t x_t^\top l_{\omega(t)=i}]\}_{i=1}^s$ to $\{\mathbb{E}[x_{t+1} x_{t+1}^\top l_{\omega(t+1)=i}]\}_{i=1}^s$ [9, p.35], hence its spectral radius determines MSS.

B. Linear Quadratic Regulator

The optimal control problem we consider in this paper is the following Markov jump system infinite-horizon linear quadratic regulator (MJS-LQR) problem:

$$\inf_{\{u_0, u_1, \ldots\}} \mathbb{E}\left[ \sum_{t=0}^\infty x_t^\top Q(t)x_t + u_t^\top R(t)u_t \right]$$

s.t. $x_{t+1} = A_\omega(t)x_t + B_\omega(t)u_t + w_t$.

Here, we consider the long-term average quadratic cost

$$J(u_0, u_1, \ldots) := \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q(t)x_t + u_t^\top R(t)u_t \right]$$

where $Q(t)$ and $R(t)$ are mode-dependent cost matrices chosen by users, and the expectation is over the randomness of initial state $x_0$, noise $\{w_t\}_{t=0}^\infty$ and Markovian modes $\{\omega(t)\}_{t=0}^\infty$. Unlike classical LQR for LTI systems, where cost matrices are usually fixed throughout the time horizon, the mode-dependent cost matrices in MJS-LQR allows us to have different control goals under different modes. To guarantee MJS-LQR (6) is solvable, we assume the MJS in (1) and the cost matrices satisfy the following.

**Assumption 3.** (a) For all $i \in [s]$, the cost matrices $Q_i$ and $R_i$ are positive definite; and (b) the MJS in (1) with $w_0 = 0$ is stabilizable and each pair $(Q_i^\frac{1}{2}, A_i)$ is observable.

The following lemma characterizes some properties of the minimizer of (7).

**Lemma 4.** [9, Theorem 4.6 and Corollary A.21] Under Assumption 3 (a) and (b), for all $i \in [s]$, the coupled discrete-time algebraic Riccati equations (cDARE)

$$X_i = A_i^\top \varphi^*_i(X_{1:s})A_i + Q_i - A_i^\top \varphi^*_i(X_{1:s})B_i^\top (R_i + B_i^\top \varphi^*_i(X_{1:s})B_i)^{-1} B_i^\top \varphi^*_i(X_{1:s})A_i$$

where $X_i = \mathbb{E}[x_i x_i^\top]$ is an $n^2 \times n^2$ matrices.
have a unique solution $P_i^{1:s}$ among $\{X_{1:s} : X_i \succeq 0, \forall i\}$, and $P_i^* \succ 0$ for all $i \in [s]$. Moreover, the mode-dependent state feedback controller

$$K_i^* = -(R_i + B_i^T \varphi_i^*(P_i^{1:s})B_i)^{-1} B_i^T \varphi_i^*(P_i^{1:s})A_i^*$$  \hspace{1cm} (9)

stabilizes the MJS in (1) and minimizes the cost (7) with input $u_t = u_t^* := K_i^*(x_t)$ and optimal cost $J^* := J(u_0^*, u_1^*, \ldots) = \sigma_0^2 \sum_{i=1}^s \text{tr}(\sigma_i^T(i)P_i^*)$.

In Assumption 3, $R_i \succ 0$ means every component of control variable will be penalized. With $Q_i^2, A_i$ being observable, $Q_i \succ 0$ is not mandatory for Lemma 4 to hold. This condition is needed mainly for our subsequent theoretical developments.

In the remaining paper, we let cDARE($A_i^{1:s}, B_i^{1:s}, T^*$) denote equations with structure given by (8), where $T^*$ determines the operator $\varphi^*$. In practice, cDARE can be solved efficiently either with LMI or recursively [9].

C. Certainty Equivalent Controller

In this work we seek to control MJS with unknown dynamics ($A_i^{1:s}, B_i^{1:s}, T^*$) based on approximate parameters ($A_{1:s}, B_{1:s}, T$). The cost matrices ($Q_{1:s}, R_{1:s}$) are assumed known and the modes $\{\omega(t)\}_{t=0}^\tau$ are observed at runtime. We analyze the CE approach, that is, using approximate parameters ($A_{1:s}, B_{1:s}, T$), we solve the perturbed cDARE($A_{1:s}, B_{1:s}, T$),

$$X_i = \hat{A}_i^T \hat{\varphi}_i(X_{1:s}) \hat{A}_i + Q_i - \hat{A}_i^T \hat{\varphi}_i(X_{1:s}) \hat{B}_i,$$

$$R_i + \hat{B}_i^T \hat{\varphi}_i(X_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(X_{1:s}) \hat{A}_i,$$  \hspace{1cm} (10)

for all $i \in [s]$ and $X_i \succeq 0$, where the operator $\hat{\varphi}$ is defined as

$$\hat{\varphi}_i(V_{1:s}) := \sum_{j=1}^s |T|_{ij} V_j.$$  \hspace{1cm} (11)

Let $\hat{P}_{1:s}$ be the positive definite solution of (10), then the CE controller is given by

$$\hat{K}_i = -(R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i,$$  \hspace{1cm} (12)

for all $i \in [s]$. Lastly, we apply the input $u_t = \hat{K}_{\omega(t)}x_t$ to control the true MJS. Let $\hat{J} := J(\hat{u}_0, \hat{u}_1, \ldots)$ be the cost incurred by playing the CE controller on the true MJS. Throughout we assume that for all $i \in [s]$, the approximate parameters have the following accuracy levels:

$$\|A_i^* - \hat{A}_i\| \leq \epsilon, \|B_i^* - \hat{B}_i\| \leq \epsilon, \|T^* - \hat{T}\|_\infty \leq \eta.$$  \hspace{1cm} (13)

In the next section, we address the following questions: (a) When can the perturbed cDARE in (10) be guaranteed to have a unique positive semi-definite solution $\hat{P}_{1:s}$? (b) What is a tight upper bound on $\|\hat{P}_{1:s} - \hat{P}_{1:s}\|$? (c) When does $\hat{K}_{1:s}$ stabilize the true MJS? (d) How large is the suboptimality gap $\hat{J} - J^*$?

IV. Perturbation Analysis for MJS-LQR

Before we formally state our results, we introduce a few more concepts and assumptions. We use $L_i^* := A_i^* + B_i^* K_i$ to denote the closed-loop state matrix under the optimal MJS-LQR controller (9), and define the augmented state matrix $\hat{L}^*$ similar to (4) such that its $ij$-th block is given by

$$[\hat{L}^*]_{ij} := [T^*]_{ij} L_i^{2T} \otimes L_i^{2T}.$$  \hspace{1cm} (14)

From Lemma 4, we know the closed-loop MJS $x_{t+1} = \hat{L}_{\omega(t)}x_t$ is MSS, thus $\rho(\hat{L}^*) < 1$ by Lemma 2. Furthermore, we define the following to quantify the decay of $\hat{L}^*$.

Definition 5. For an arbitrary $\gamma \in [\rho(\hat{L}^*), 1)$, we define

$$\gamma(\hat{L}^*, \gamma) := \max \{\|\hat{L}^* - \gamma \hat{L}^*\|, \gamma \|\hat{L}^*\| \}.$$  \hspace{1cm} (15)

Note that $\gamma(\hat{L}^*, \gamma)$ is finite by Gelfand’s formula. It is easy to see that $\gamma(\hat{L}^*, \gamma)$ monotonically decreases with $\gamma$ and that $\gamma(\hat{L}^*, \gamma) \geq 1$. This quantity measures the transient response of a non-switching system with state matrix $\hat{L}^*$ and can be upper bounded by its $H_\infty$ norm [19].

Finally, for the ease of exposition, we define a few constants:

$$\xi := \min \{\|B_{1:s}\|_2 \|R_{1:s}\|_2^{-1} \|L_{1:s}\|_2^{-2}, \|g(P_{1:s})\|_2\},$$

$$C_\xi := \|A_{1:s}^*\|^2_2 \|B_{1:s}^*\|^2_2 + \|P_{1:s}^*\|^2_2 \|R_{1:s}\|_2^{-1},$$

$$C_\eta := \|A_{1:s}^*\|^2_2 \|B_{1:s}^*\|^2_2 + \|P_{1:s}^*\|^2_2 \|R_{1:s}\|_2^{-1},$$

$$C_\eta' := \|A_{1:s}^*\|^2_2 \|B_{1:s}^*\|^2_2 \|P_{1:s}^*\|^2_3 \|R_{1:s}\|_2^{-1},$$

$$\Gamma_* := \max \{\|A_{1:s}\|_2 + \|B_{1:s}\|_1 + \|P_{1:s}\|_1 + \|K_{1:s}\|_1\}.$$  \hspace{1cm} (16)

In the following, we will show that despite being coupled, cDARE for MJS-LQR satisfies nice properties. To be more precise, we show that if the approximate MJS is accurate enough, i.e., $\epsilon$ and $\eta$ are sufficiently small, we can guarantee that not only the positive definite solution $\hat{P}_{1:s}$ to the perturbed cDARE uniquely exists, but also $\hat{P}_{1:s}$ does not deviate much from $P_{1:s}$.

Theorem 6. Let $\epsilon, \eta \geq 0$ be as in (13). Under Assumptions 3, and as long as $\epsilon \leq \min \{C_\eta' \xi (1-\gamma)^2, 1, \|B_{1:s}\|_1, \|Q_{1:s}\|_1\}$, $\eta \leq C_\eta \xi (1-\gamma)^2$, the perturbed cDARE in (10) is guaranteed to have a unique solution $\hat{P}_{1:s}$ in $\{X_{1:s} : X_i \succeq 0, \forall i\}$ such that $\hat{P}_i \succ 0$ for all $i$ and

$$\|\hat{P}_{1:s} - P_{1:s}\| \leq \frac{C_{\text{perturbation}}(\hat{L}^*, \gamma)}{1-\gamma} (6C_\xi \epsilon + 2C_\eta \eta).$$  \hspace{1cm} (17)

In this theorem, the choice of parameter $\gamma$ balances the numerator and the denominator of $\gamma(\hat{L}^*, \gamma)$. From the constants, we see we would have milder requirement on $\epsilon$ and $\eta$ and tighter bound on $\|\hat{P}_{1:s} - P_{1:s}\|$ when (i) $\|A_{1:s}\|_2, \|B_{1:s}\|_2, \|L_{1:s}\|_2, \gamma(\hat{L}^*, \gamma)$, and (ii) $\|R_{1:s}\|_2$ are smaller. These translate to the cases when (i) the true MJS is easier to stabilize; (ii) the closed-loop MJS under the optimal controller is more stable; and (iii) the input dominates more in the cost function. The role of $\gamma(\hat{L}^*, \gamma)$
in this theorem is closely related to the damping property in ARE perturbation analysis [12]. Coefficients for \( \epsilon \) and \( \eta \) on the RHS of (17) are also known as condition numbers in algebraic Riccati equation sensitivities literature [14]. The uniqueness result in Theorem 6 guarantees the perturbation bound (17) indeed applies to the perturbed solution one would obtain in practice. Using similar proof techniques, the exact same theorem can be proved for cases when approximate \( \tilde{Q}_{1:s} \) is used in place of \( Q_{1:s} \) in the computations, which can be useful when the cost for \( x_i \) is in the form of \( \| y_i \|^2 \) where \( y_i = C_{w(i)} x_i \) represents observation, and we only have approximate parameter \( \hat{C}_{1:s} \).

In this case, \( Q_{1:s} = C_{1:s}^T C_{1:s} \), and we only have approximate parameter \( \hat{C}_{1:s} \).

Finally, using Theorem 6, we quantify the mismatch between the performance of the optimal controller \( \hat{K}_{1:s} \) and the certainty equivalent controller \( K_{1:s} \). We then quantify the suboptimality gap \( J - J^* \) in terms of the controller mismatch and derive an upper bound on this mismatch so that the certainty equivalent controller \( \hat{K}_{1:s} \) stabilizes the MJS in the mean-square sense. This leads to the following suboptimality result.

**Theorem 7.** Let \( \epsilon, \eta \geq 0 \) be as in (13). Suppose Assumptions 3 (a) and (b) hold and additionally the Markov chain \( \{\omega(t)\}_{t=0}^{\infty} \) is ergodic. Then, as long as \( \epsilon \) and \( \eta \) are such that the upper bounds in Theorem 6 are valid and \( C_{\epsilon} \epsilon + C_\eta \eta \geq \frac{1}{\sqrt{n}} \), the CE controller \( \hat{K}_{1:s} \) stabilizes the true MJS and we have
\[
J - J^* \leq \sigma_w \sigma_x n \min \{n, p\} \left( \| \tilde{R}_{1:s} \| + \Gamma_{\epsilon}^2 \right) \frac{\tau_\epsilon \gamma_\eta (\bar{\gamma}_\epsilon^2 \bar{\gamma}_\eta^2 + \Gamma_{\epsilon}^2) \Gamma_{\epsilon}^2}{(1 - \gamma)^2 (\bar{\gamma}_\epsilon^2 \bar{\gamma}_\eta^2 + \Gamma_{\epsilon}^2)} (6C_{\epsilon} \epsilon + 2C_\eta \eta)^2.
\] (18)

Our result states that the suboptimality decays as the square of the size of uncertainty. Also note that the CE controller \( \hat{K}_{1:s} \) stabilizes both the approximate MJS \( (\hat{A}_{1:s}, \hat{B}_{1:s}, \hat{T}) \) and MJS \( (A^*_{1:s}, B^*_{1:s}, T^*) \), hence the upper bounds on \( \epsilon \) and \( \eta \) also provide a stability margin type result for the optimal LQR controllers [20] for MJS.

To see the significance of our bound, suppose the uncertainty in the system dynamics and the transition matrix are due to the estimation error induced by a system identification procedure that uses \( T \) samples. Then, if the estimation error decays as \( O(1/\sqrt{T}) \), Theorem 7 states that the suboptimality decays as \( O(1/T) \) which is similar to what is known in the case of LDS [4] except that we suffer from an additional \( s \) and \( n \) dependence in the numerator because of multiple modes in MJS. More concretely, having estimates twice as good will reduce the suboptimality to one fourth, which can be achieved by doubling the data if a system identification algorithm with sample complexity \( O(1/\sqrt{T}) \) is used. If we look at the dependence of suboptimality gap on the system properties, our result states that more modes \( s \), larger system order \( n \) and larger noise variance \( \sigma_w \) adversely affect the gap.

**V. Numerical Experiments**

In this section, we present some numerical results to support our proposed theory. All of the synthesis and performance experiments are run in MATLAB.

Consider a system with \( n \) states, \( p \) inputs, and the number of modes \( s \). The entries of the true system matrices \( (A^*_{1:s}, B^*_1) \) were generated randomly from a standard normal distribution. We scaled each \( A^*_i \) to have spectral radius equal to 0.3 to obtain a mean square stable MJS. For the cost matrices \( (Q_{1:s}, R_{1:s}) \), and the approximate \( (\hat{A}_{1:s}, \hat{B}_{1:s}) \), we set
\[
\hat{Q}_i = \hat{Q}_i \hat{Q}_i^T, \quad \hat{R}_i = \hat{R}_i \hat{R}_i^T, \quad \hat{A}_i = A^*_i + \epsilon_A A^*_i, \quad \hat{B}_i = B^*_i + \epsilon_B B^*_i,
\]
where \( \hat{Q}_i \), \( \hat{R}_i \), \( A^*_i \), and \( B^*_i \) were generated using \texttt{randn} function; and \( \epsilon_A \) and \( \epsilon_B \) are some fixed scalars. The approximate \( \hat{T} \) was sampled from a Dirichlet Process \( \text{Dir}(s - 1, \cdot I_s + 1) \). To generate the true Markov matrix \( T^* \) for MJS, we let \( T^* = T + \epsilon \), where the perturbation \( \epsilon = \eta T (\text{Dir}(s - 1, \cdot I_s + 1) - T) \). We note that when \( \eta T = 0 \), there is no perturbation at all and \( T^* = T \); when \( \eta T = 1 \), \( T^* \) will preserve no information of \( T \). We also assume that we had equal probability of starting in any initial mode.

We next study how the system errors vary with \( \epsilon_A, \epsilon_B, \eta T \in \{0.01, 0.02, 0.05, 0.1, 0.2, 0.3\} \), and the number of modes \( s \in \{10, 20, 30, 40\} \). We set the number of states and inputs to \( n = 10 \) and \( p = 5 \), respectively. For each choice of \( \epsilon_A, \epsilon_B, \eta T \), we run 100 experiments, and record \( (P_{1:s}, \hat{P}_{1:s}) \) and the costs for these matrices.

Let \( \Delta_P \) denote the maximum of \( \| P_{1:s} - \hat{P}_{1:s} \| / \| P_{1:s} \| \) over the experiments and modes. We also use \( \Delta_J := (J - J^*)/J^* \) to denote the relative suboptimality gap for MJS, where \( J^* \) and \( J \) are the costs incurred by playing the optimal controller and the certainty equivalent controller on the true system, respectively. In Figures 1 and 2, we plot \( \Delta_P \) and \( \Delta_J \) versus \( \epsilon_A, \epsilon_B, \eta T \), and \( \epsilon \) increase, respectively. Each curve on the plot represents a fixed number of modes \( s \). These empirical results are all consistent with (17). In particular, Figure 1 (right) shows that given the uncertainty in the system matrices and in the Markov transition matrix is bounded by \( \epsilon \), the perturbation bound to coupled Riccati equations has the rate \( O(\epsilon) \) which degrades linearly as \( \epsilon \) increase. Further, it can be easily seen that the gaps indeed increase with the number of modes in the system.

Figure 2 demonstrates the relationship between the relative suboptimality \( \Delta_J \) and the five parameters \( \epsilon_A, \epsilon_B, \eta T, \epsilon \) and \( s \). As can be seen in Figure 2 (right), given the uncertainty in the system matrices and in the Markov transition matrix is bounded by \( \epsilon \), the perturbation bounds to the optimal cost decay quadratically \( (O(\epsilon^2)) \) which is consistent with (18).

**VI. Conclusions**

In this work, we provide a perturbation analysis for cDARE, which arise in the solution of MJS-LQR, and an end-to-end suboptimality guarantee for certainty equivalence control for MJS-LQR. Our results show the robustness of
the optimal policy to perturbations in system dynamics and establish the validity of the certainty equivalent control in a neighborhood of the original system. This work opens up multiple future directions. First, with proper system identification algorithms, we can analyze model-based online/adaptive algorithms where control policy is updated continuously over a single trajectory. Second, a natural extension would be to study MJS with output measurements where states are only partially observed, i.e., the LQG setting. This will require considering the dual coupled Riccati equations for filtering.

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A. Useful matrix facts

The following results on matrices will be used repeatedly throughout the proof and will be referred using the fact number.

**Fact 1.** Let $M, N$ be two symmetric and positive semidefinite matrices, then

\[
\|N(I + MN)^{-1}\| \leq \|N\|, \tag{19}
\]

\[
\|(I + MN)^{-1}\| \leq 1 + \|N\|\|M\|. \tag{20}
\]

**Fact 2.** Let $M, N$ be two arbitrary matrices, where $M$ and $M + N$ are invertible, then

\[
(M + N)^{-1} = M^{-1} - M^{-1}N(M + N)^{-1} = M^{-1} - (M + N)^{-1}NM^{-1}. \tag{21}
\]

**Fact 3.** For two arbitrary matrices $M$ and $N$ such that $I + M$ and $I + N$ are both invertible, we have

\[
(I + M)^{-1} - (I + N)^{-1} = (I + M)^{-1}(N - M)(I + N)^{-1}. \tag{22}
\]

Above, (19) is due to [4, Lemma 7] (in their supplement). To see (20), first note that $(I + MN)^{-1} = I - MN(I + MN)^{-1}$ by matrix inversion lemma, and then apply (19). (21) and (22) also follow from matrix inversion lemma.

**Fact 4.** Consider a block-diagonal matrix $X \in \mathbb{R}^{n \times sn}$ composed of $X_{1:s}$ such that the $i$th diagonal block of $X$ is given by $X_i \in \mathbb{R}^{n \times n}$. Let $\text{vec}()$ be the operator that vectorizes all diagonal blocks of $X$ into a vector, i.e. $\text{vec}(X) := (\text{vec}(X_1), \ldots, \text{vec}(X_s))$. Let $\text{vec}^{-1}$ denote the inverse of $\text{vec}$ such that $\text{vec}^{-1}(\text{vec}(X)) = X$. Then,

\[
\|\text{vec}\| = \sup_{X = \text{diag}(X_{1:s})} \|\text{vec}(X)\| = \sqrt{n} s \tag{23}
\]

\[
\|\text{vec}^{-1}\| = \sup_{\|x\| = 1} \|\text{vec}^{-1}(x)\| = 1. \tag{24}
\]

Fact 4 follows by noting that (i) achieves the supremum when $X_i = I_n$ for all $i$ and (ii) achieves the supremum when $x = (1, 0, \ldots, 0)$. The following fact is adapted from [4] and is useful in bounding the spectral radius of a perturbed matrix.

**Fact 5.** Let $M$ be an arbitrary matrix in $\mathbb{R}^{n \times n}$ and let $\rho(M) \leq \gamma$. Then for all $k \geq 1$ and real matrices $\Delta$ of appropriate dimensions, we have $\|\text{vec}(M + \Delta)^k\| \leq \tau(M, \gamma)(\tau(M, \gamma)\|\Delta\| + \gamma)^k$.

B. Proof of Theorem 6

We first provide a lemma, used in the proof of Theorem 6, that establishes that positive definite solutions of coupled algebraic Riccati equations are unique among the set of positive semidefinite matrices when they exist. Note that existence of such solutions guarantee stabilizability in mean square sense.

**Lemma 8.** ([9, Lemma A.14]) Consider $cDARE(A_{1:s}, B_{1:s}, T)$ for a generic MJS($A_{1:s}, B_{1:s}, T$) and LQR cost matrices $Q_{1:s}, R_{1:s}$. Assume $Q_i, R_i \succ 0$ for all $i \in [s]$. Then, if there exists a positive definite solution $P_{1:s}$ to $cDARE(A_{1:s}, B_{1:s}, T)$, then it is the unique solution among $\{X_{1:s} : X_i \succeq 0, \forall i \in [s]\}$.

**Main Proof for Theorem 6.**

The outline of the proof is as follows:

(a) We first construct an operator $\mathcal{K}(X_{1:s})$ using the difference between the true $cDARE(A_{1:s}, B_{1:s}, T^*)$ and the perturbed $cDARE(A_{1:s}, B_{1:s}, T)$, whose fixed point(s) $X_{1:s}^*$ (when exist) will guarantee $P_{1:s}^* + X_{1:s}^*$ is a solution to the perturbed $cDARE(A_{1:s}, B_{1:s}, T)$.

(b) Then we show when $\epsilon$ and $\eta$ are small enough, $\mathcal{K}(X_{1:s})$ will be a contraction mapping on a closed set $\mathcal{S}_\nu$ whose radius $\nu$ is a function of $\epsilon$ and $\eta$. Thus, there exists a unique fixed point $X_{1:s}^* \in \mathcal{S}_\nu$ by contraction mapping theorem, and $P_{1:s}^* + X_{1:s}^*$ is a solution to the perturbed $cDARE(A_{1:s}, B_{1:s}, T)$.

(c) Finally, given there exists a unique solution to the perturbed $cDARE(A_{1:s}, B_{1:s}, T)$ in the neighborhood $\mathcal{S}_\nu$ of $P_{1:s}^*$, we will show this solution is the only possible solution among all positive semi-definite matrices. To do this, we first show $P_{1:s}^* + X_{1:s}^* \succ 0$. By Lemma 8, we know the perturbed $cDARE$ (10) has a unique solution, given by $\hat{P}_{1:s} := P_{1:s}^* + X_{1:s}^*$, among $\{X_{1:s} : X_i \succeq 0, \forall i\}$. Furthermore, $\|\hat{P}_{1:s} - P_{1:s}^*\| = \|X_{1:s}^*\| \leq \nu(\epsilon, \eta)$.

**Step (a): Construct operator $\mathcal{K}$.**

First we define a few notations for the ease of exposition. For all $i \in [s]$, let $S_i^* := B_i^*R_i^{-1}B_i^*$ and $\hat{S}_i := \hat{B}_i R_i^{-1}\hat{B}_i^*$. Define block diagonal matrices $A_i^*, \hat{A}_i, B_i^*, \hat{B}_i, Q, R, P_i^*, K_i, L_i^*, S_i, \Phi(X), \hat{\Phi}(X)$ such that their $i$th diagonal blocks are given by $A_i^*, \hat{A}_i, B_i^*, \hat{B}_i, Q_i, R_i, P_i^*, K_i, L_i^*, S_i, \Phi(X_i), \hat{\Phi}(X_i)$ respectively. Note that $X_{1:s}$ represent arguments of matrix functions or unknown variables used in matrix equations, e.g. (8). We will see many equations that hold for each single block also hold for the diagonally concatenated notations.
We have $K^* = -(R + B^T\Phi^*(P^*)B^*)^{-1}B^T\Phi^*(P^*)A^*$ from (9), then using the matrix inversion lemma, we can get

$$L^* = A^* + B^*K^* = (I + S^*\Phi^*(P^*))^{-1}A^*.$$  

(25)

Furthermore, by diagonally concatenating cDARE (8) and then applying the matrix inversion lemma again, we have

$$X = A^*T\Phi^*(X)I + S^*\Phi^*(X))^{-1}A^* + Q.$$  

(26)

Then, we define the following Riccati difference function using the difference between LHS and RHS of (26), while keeping the RHS parameter dependent:

$$F(X; A^*, B^*, T^*):=X - A^*T\Phi^*(X)I + S^*\Phi^*(X))^{-1}A^* - Q.$$  

(27)

Though not explicitly listed, $\Phi^*$ and $S^*$ on the RHS of (27) depend on arguments $T^*$ and $B^*$ respectively. Since $P^*_{1:s}$ is the solution to the true cDARE($A_{1:s}^*, B_{1:s}^*, T^*$), we have $F(P^*; A^*, B^*, T^*) = 0$. Similarly, if there exists solution $P_{1:s}$ to the perturbed cDARE($A_{1:s}, B_{1:s}, T$), then $P := \text{diag}(P_{1:s})$ would satisfy $F(P; A, B, T) = 0$.

Now we consider the function $F(P^* + X; A^*, B^*, T^*)$. When $P^* + X \geq 0$, we have the following.

$$(i) \quad P^* + X - A^*T[\Phi^*(P^*) + \Phi^*(X)]I + S^*\Phi^*(P^*) + S^*\Phi^*(X))^{-1}A^* - Q$$

$$(ii) \quad P^* + X - A^*T[\Phi^*(P^*) + \Phi^*(X)] =: \Gamma$$

$$(iii) \quad P^* + X - A^*T[\Phi^*(P^*) + \Phi^*(X)]I - GS^*\Phi^*(X))^{-1}A^* - Q$$

$$(iv) \quad X - A^*T[\Phi^*(P^*) + \Phi^*(X)]I - GS^*\Phi^*(X))L^* + A^*T\Phi^*(P^*)L^*$$

$$(v) \quad X - L^*T[I + \Phi^*(P^*)S^*][\Phi^*(P^*) + \Phi^*(X)]I - GS^*\Phi^*(X)] - \Phi^*(P^*)]L^*$$

$$= X - L^*T[I + \Phi^*(P^*)S^*][-\Phi^*(P^*)\Gamma S^* + I - \Phi^*(X)\Gamma S^*] \Phi^*(X)L^*$$

$$=: \Lambda$$

where (i) follows from the definition in (27); (ii) follows from the identity $(M + N)^{-1} = M^{-1} - (M + N)^{-1}NM^{-1}$ and the fact $I + S^*\Phi^*(P^*) + S^*\Phi^*(X)$ is invertible due to the assumption that $P^* + X \geq 0$; (iii) and (v) follow from the fact $(I + S^*\Phi^*(P^*))^{-1}A^* = L^*$ in (25); (iv) follows from the fact $F(P^*; A^*, B^*, T^*) = P^* - A^*T\Phi^*(P^*)L^* - Q = 0$ in (27).

Note that

$$\Lambda = -\Phi^*(P^*)\Gamma S^* + I - \Phi^*(X)\Gamma S^* - \Phi^*(P^*)S^*\Phi^*(P^*)\Gamma S^* + \Phi^*(P^*)S^* - \Phi^*(P^*)S^*\Phi^*(X)\Gamma S^*$$

$$= -(\Phi^*(P^*) + \Phi^*(P^*)S^*\Phi^*(P^*) - \Phi^*(P^*) \Phi^*(P^*)S^*\Phi^*(X)] + \Phi^*(P^*)S^*\Phi^*(X)) \Gamma S^*$$

$$+ I - \Phi^*(X)\Gamma S^*$$

$$= I - \Phi^*(X)\Gamma S^*.$$  

(29)

Therefore

$$F(P^* + X; A^*, B^*, T^*) = X - L^*T\Phi^*(X)L^* + L^*T\Phi^*(X)(I + S^*\Phi^*(P^*) + S^*\Phi^*(X))^{-1}S^*\Phi^*(X)L^*.$$  

(30)

If we define

$$\mathcal{T}(X) := X - L^*T\Phi^*(X)L^*,$$

$$\mathcal{H}(X) := L^*T\Phi^*(X)(I + S^*\Phi^*(P^*) + S^*\Phi^*(X))^{-1}S^*\Phi^*(X)L^*,$$

we have

$$F(P^* + X; A^*, B^*, T^*) = \mathcal{T}(X) + \mathcal{H}(X).$$  

(33)

Let $Y_i := X_i - L_i^*T\phi_i^*(X_{1:s})L_i^*$, and $Y := \text{diag}(Y_{1:s})$, then linear operator $\mathcal{T}$ can be viewed as $\mathcal{T} : X \mapsto Y$. By vectorization, we see for every $i$

$$(I - [T^*]_{ii} \cdot L_i^* \otimes L_i^* T) \text{vec}(X_i) - \sum_{j \neq i} [T^*]_{ij} L_i^* \otimes L_i^* T \text{vec}(X_j) = \text{vec}(Y_i).$$  

(34)
Theorem 6.

Lemma 9. Assume combined with (41), makes the bounds in Lemma 9 applicable and we get:

\[ X = T^{-1}(Y) = \overset{\cdot}{v}(I - L^*)^{-1} \overset{\cdot}{v}(Y), \]

where \( \dot{v} \) denotes operator composition. Since \( T^{-1} \) is well defined, we can construct the following operator:

\[ K(X) = T^{-1}(F(P^* + X; A^*, B^*, T^*) - F(P^* + X; A, B, T) - \mathcal{H}(X)). \]

From (33), we see that if there exists a fixed point \( X^* \) for \( K \), then \( F(P^* + X^*; A, B, T) = 0 \), i.e. \( P_{1:s}^* + X_{1:s} \) is a solution to the perturbed cDARE\( (A_{1:s}, B_{1:s}, T) \).

**Step (b):** Show that \( K \) is a contraction to conclude existence of a perturbed solution.

We consider the set

\[ S_{\nu} := \{ X : \| X \| \leq \nu, X = \text{diag}(X_{1:s}), P^* + X \succeq 0 \}. \]

We will show when \( \nu \) is small enough, \( K \) maps \( S_{\nu} \) into itself and is a contraction mapping. Thus, \( K \) is guaranteed to have a fixed point in \( S_{\nu} \). To do this, we first present a lemma that bounds \( K \) when \( \epsilon \) and \( \nu \) are sufficiently small. Then, we provide a choice of \( \nu \) that makes this bound valid.

**Lemma 9.** Assume \( \epsilon \leq \min\{\| B^* \|, 1 \} \). Suppose \( X, X_1, X_2 \in S_{\nu} \) with \( \nu \leq \min\{1, \| S^* \|^{-1} \} \), then

\[
\| K(X) \| \leq \frac{\sqrt{\text{ns} \cdot (L^*, \gamma)}}{1 - \gamma} \left( \| L^* \| \| S^* \| \| R^{-1} \| \| \epsilon \| \right) + C \eta \| \| S^* \| \| R^{-1} \| \| \epsilon \| \| \eta \|
\]

Proving is given in Appendix D. To apply this lemma, let

\[ \nu = \frac{\sqrt{\text{ns} \cdot (L^*, \gamma)}}{1 - \gamma} \left( 6C \| \epsilon \| + 2C \| \eta \| \right). \]

Applying the upper bounds for \( \epsilon \) and \( \eta \) in the premises of Theorem 6 to (40), we have

\[
\| K(X) \| \leq \frac{1 - \gamma}{\sqrt{\text{ns} \cdot (L^*, \gamma)}} \left( \frac{1}{12} \| B^* \| \| P^* \| \| R^{-1} \| \| \epsilon \| \right) + \frac{1}{24} \min\{ \| B^* \| \| R^{-1} \| \| \epsilon \|, S \} \| \| \eta \|
\]

by cancelling off \( \epsilon \) and \( \eta \) in (38) using (40), and applying the third upper bound for \( \nu \) in (41). Since \( \nu \leq \sigma(P^*) \) in (41), we can see \( P^* + K(X) \succeq 0 \), thus \( K(X) \in S_{\nu} \), i.e. \( K \) maps \( S_{\nu} \) into itself. Furthermore, applying upper bounds for \( \epsilon \) and \( \eta \) in Theorem 6 and the third upper bound for \( \nu \) in (41) to (39) gives

\[
\| K(X_1) - K(X_2) \| \leq \frac{1}{4} + \frac{1}{4} \| L^* \| \| S^* \| \| R^{-1} \| \| \epsilon \| \| \eta \|
\]

We have shown \( K(X) \) not only maps closed set \( S_{\nu} \) into itself but also is a contraction mapping on \( \nu \), which means \( K(X) \) has a unique fixed point \( X^* \in S_{\nu} \). By definition of \( K(X) \) and the identity in (33), we see \( F(P^* + X^*; A, B, T) = 0 \), i.e. \( P^* + X^* \) is a solution to the perturbed cDARE\( (A_{1:s}, B_{1:s}, T) \).

**Step (a):** Show the uniqueness of the perturbed solution.

By definition of \( S_{\nu} \) and (41), we know \( \| X \| \leq \nu < \sigma(P^*) \), thus \( P^* + X^* \succeq 0 \). This implies \( P_i^* + X_i^* \succeq 0 \) for all \( i \).

By Lemma 8, we know \( P_{1:s} := P_{1:s}^* + X_{1:s}^* \) is the only possible solution to the perturbed cDARE\( (A_{1:s}, B_{1:s}, T) \) among \( \{ X_{1:s} ; X_i \geq 0, \forall i \} \).

Finally, note that \( \| \dot{P}_{1:s} - P_{1:s}^* \| = \| X_{1:s}^* \| = \| X^* \| \leq \nu \) where \( \nu \) is defined in (40), which concludes the proof for Theorem 6.
C. Proof of Theorem 7

We first outline our high-level proof strategy for Theorem 7:

- We first bound the mismatch between the optimal controller $K^*_i$ and the certainty equivalent controller $\hat{K}_i$ in terms of the upper bounds on the quantities $\|\hat{A}_i - A^*_i\|$, $\|\hat{B}_i - B^*_i\|$, $\|\hat{P}_i - P^*_i\|$ and $\|\hat{T} - T^*\|_\infty$.

- Next, assuming the certainty equivalent controller stabilizes the MJIS in the mean-square sense, we quantify the suboptimality gap $\hat{J} - J^*$ in terms of the controller mismatch $\|\hat{K}_i - K^*_i\|$.

- Lastly, using the matrix Fact 5, we derive an upper bound on the mismatch $\|\hat{K}_i - K^*_i\|$ so that the certainty equivalent controller $\hat{K}_i$ indeed stabilizes the MJIS in the mean-square sense. Using the derived upper bound on the mismatch $\hat{J} - J^*$, we get our final result.

Lemma 10 (Controller mismatch). Let $\epsilon, \eta > 0$ be fixed scalars. Suppose $\|T - T^*\|_\infty \leq \eta$, $\|\hat{A}_i - A^*_i\| \leq \epsilon$, $\|\hat{B}_i - B^*_i\| \leq \epsilon$ and $\|\hat{P}_i - P^*_i\| \leq f(\epsilon, \eta)$ for all $i \in [s]$ and for some function $f$ such that $\max\{\epsilon, \eta\} \leq f(\epsilon, \eta) \leq \Gamma_s$. Then, under Assumption 3, we have

$$\|\hat{K}_i - K^*_i\| \leq 28\Gamma_s^3 \frac{\sigma(R_i) + \Gamma_s^3}{\sigma(R_i)^2} f(\epsilon, \eta) \quad \text{for all} \quad i \in [s].$$

Proof. To begin, recall that given $P_{1:s}$ and $\hat{P}_{1:s}$, the optimal controller and the certainty equivalent controller is given by

$$K_i^* = -(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} B_i^T \varphi_i^*(P^*_{1:s}) A_i,$$

and

$$\hat{K}_i = -(R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i,$$

respectively for all $i \in [s]$. As an auxiliary step, we define $\hat{K}_i := -(R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i$. Then, we have $\|K_i^* - \hat{K}_i\| \leq \|K_i^* - K_i\| + \|K_i - \hat{K}_i\|$. We bound the first term on the right side of this inequality as follows,

$$\|K_i^* - \hat{K}_i\| = \|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} B_i^T \varphi_i^*(P^*_{1:s}) A_i - (R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i\|,$$

$$\leq \|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} \| B_i^T \varphi_i^*(P^*_{1:s}) A_i - (R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i\|$$

$$+ \|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} - (R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1}\| \|\hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i\|. \tag{43}$$

Note that, since we are assuming that the cost matrices $Q_i$ and $R_i$ are positive definite for all $i \in [s]$, therefore, we have

$$\|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1}\| = \frac{1}{\sigma(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)} \leq \frac{1}{\sigma(R_i)}. \tag{44}$$

Next, we bound the difference,

$$\|B_i^T \varphi_i^*(P^*_{1:s}) A_i - \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{A}_i\| = \|B_i^T \varphi_i^*(P^*_{1:s}) A_i - (B_i^* + \Delta B_i)^T (\varphi_i^*(P^*_{1:s}) + \varphi_i^*(\Delta P_i) A_i + \Delta A_i)\|,$$

$$\leq \|B_i^T \varphi_i^*(P^*_{1:s}) A_i - (B_i^* + \Delta B_i)^T (\varphi_i^*(P^*_{1:s}) A_i + \varphi_i^*(\Delta P_i) A_i + \varphi_i^*(\Delta P_i) A_i + \varphi_i^*(\Delta P_i) A_i)\|,$$

$$\leq \|B_i^T \varphi_i^*(P^*_{1:s}) A_i - (B_i^* + \Delta B_i)^T (\varphi_i^*(P^*_{1:s}) A_i + \varphi_i^*(\Delta P_i) A_i + \varphi_i^*(\Delta P_i) A_i + \varphi_i^*(\Delta P_i) A_i)\|,$$

$$\leq \|A_i^\ast\| + \|B_i^\ast\| \|\varphi_i^*(P^*_{1:s})\| \epsilon + \|\hat{B}_i^\ast\| \|f(\epsilon, \eta)\| \epsilon + \|\hat{A}_i^\ast\| \|\varphi_i^*(P^*_{1:s})\| \epsilon + \|\hat{A}_i^\ast\| \|f(\epsilon, \eta)\| \epsilon^2,$$

$$\leq 3\Gamma_s^2 f(\epsilon, \eta), \tag{45}$$

where the second inequality follows from the assumption that $f(\epsilon, \eta) \geq \max\{\epsilon, \eta\}$. To proceed, we use the matrix identity $X^{-1} - Y^{-1} = X^{-1}(Y - X)Y^{-1}$, to get the following norm bound,

$$\|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} - (R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1}\|,$$

$$= \|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} (\hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i - B_i^T \varphi_i^*(P^*_{1:s}) B_i) (R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i)^{-1}\|,$$

$$\leq \|(R_i + B_i^T \varphi_i^*(P^*_{1:s}) B_i)^{-1} \| \|B_i^T \varphi_i^*(P^*_{1:s}) B_i\| \|R_i + \hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i\|^{-1} \|\hat{B}_i^T \hat{\varphi}_i(\hat{P}_{1:s}) \hat{B}_i - B_i^T \varphi_i^*(P^*_{1:s}) B_i\|,$$

$$\leq \frac{1}{\sigma(R_i)^2} (\|B_i^T\|^2 + \|\hat{B}_i^T\| \|\varphi_i^*(P^*_{1:s})\| + \|B_i^T\| \|\varphi_i^*(P^*_{1:s})\| + \|B_i^T\| \epsilon + \|B_i^T\| \epsilon + \max_{i \in [s]} \|P_i\| \epsilon^2), \tag{46}$$
Using a similar idea, we get
\[
\| \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{A}_i \| = \| (B_i^* + \Delta B_i^*)^T (\varphi_i^* (P_{1:s}^*) + \varphi_i (\Delta P_{1:s}^*)) (A_i^* + \Delta A_i^*) \|
\]
\[
= \| B_i^T \varphi_i^* (P_{1:s}^*) A_i^* + B_i^T \varphi_i (\Delta P_{1:s}^*) A_i^* + B_i^T \varphi_i^* (P_{1:s}^*) \Delta A_i^* + B_i^T \varphi_i (\Delta P_{1:s}^*) \Delta A_i^* \\
+ \Delta B_i^T \varphi_i^* (P_{1:s}^*) A_i^* + \Delta B_i^T \varphi_i (\Delta P_{1:s}^*) A_i^* + \Delta B_i^T \varphi_i^* (P_{1:s}^*) \Delta A_i^* + \Delta B_i^T \varphi_i (\Delta P_{1:s}^*) \Delta A_i^* \|
\]
\[
\leq \left( \| A_i^* \| \| B_i^* \| + \| A_i^* \| (\max_{i \in [s]} \| P_i^* \|) + \| B_i^* \| (\max_{i \in [s]} \| P_i^* \|) + \| A_i^* \| \epsilon + \max_{i \in [s]} \| P_i^* \| \epsilon + \epsilon^2 \right) \\
f(\epsilon, \eta) + \| A_i^* \| \| B_i^* \| (\max_{i \in [s]} \| P_i^* \|),
\]
\[
\leq 3 \Gamma^2 f(\epsilon, \eta) + \Gamma^3.
\] (47)

Substituting (44), (45), (46), and (47) into (43), we obtain
\[
\| K_i^* - K_i \| \leq 12 \Gamma^2 \frac{\sigma(R_i) + \Gamma^2}{\sigma(R_i)^2} f(\epsilon, \eta), \quad \text{for all } i \in [s].
\] (48)

This gives the final bound on \( \| K_i^* - K_i \| \). Using similar proof techniques, we can also bound \( \| K_i - \hat{K}_i \| \) as follows,
\[
\| K_i - \hat{K}_i \| = \| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{A}_i - (R_i + \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{B}_i)^{-1} \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{A}_i \|
\]
\[
\leq \| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| \| \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \| \hat{A}_i - (R_i + \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| \| \hat{B}^i \varphi_i (\hat{P}_{1:s}) \| \hat{A}_i \|
\]
\[
+ \| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} - (R_i + \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| \| \hat{B}^i \varphi_i (\hat{P}_{1:s}) \| \hat{A}_i \|
\] (49)

In the following, we will bound each norm in the above expression separately to get a bound on \( \| K_i - \hat{K}_i \| \). First, we have
\[
\| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| = \frac{1}{\sigma (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)} \leq \frac{1}{\sigma (R_i)}.
\] (50)

Next, we bound the difference,
\[
\| \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{A}_i - \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{A}_i \|
\]
\[
\leq \| \hat{B}^i \| \| \varphi_i^* (\hat{P}_{1:s}) - \varphi_i (\hat{P}_{1:s}) \| \| \hat{A}_i \|
\]
\[
\leq \| A_i^* + \Delta A_i^* \| \| B_i^* + \Delta B_i^* \| \sum_{j \in [s]} \| (T^*_{ij} - [T]_{ij}) \| P_j \|
\]
\[
\leq \left( \| A_i^* \| + \epsilon \right) \left( \| B_i^* \| + \epsilon \right) \| \varphi_i^* (\hat{P}_{1:s}) - \varphi_i (\hat{P}_{1:s}) \| \| \hat{A}_i \|
\]
\[
\leq \eta \left( \| A_i^* \| \| B_i^* \| + \| A_i^* \| (\max_{i \in [s]} \| P_i^* \|) + \| B_i^* \| (\max_{i \in [s]} \| P_i^* \|) + \| A_i^* \| \epsilon + \| B_i^* \| \epsilon + \max_{i \in [s]} \| P_i^* \| \epsilon + \epsilon^2 \right) f(\epsilon, \eta)
\]
\[
+ \| A_i^* \| \| B_i^* \| (\max_{i \in [s]} \| P_i^* \|),
\]
\[
\leq \eta \left( \max_{i \in [s]} \| P_i^* \| \right).
\] (51)

To proceed, using matrix identity \( X^{-1} - Y^{-1} = X^{-1} (Y - X) X^{-1} \), we get the following norm bound,
\[
\| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} - (R_i + \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{B}_i)^{-1} \|
\]
\[
= \| (R_i + \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| \| \hat{B}^i \varphi_i^* (\hat{P}_{1:s}) \| \| \hat{A}_i \|
\]
\[
\leq \| (R_i + \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{B}_i)^{-1} \| \| \hat{B}^i \varphi_i (\hat{P}_{1:s}) \| \| \hat{A}_i \|
\]
\[
\leq \eta \left( \min_{\sigma_i} \left( \max_{i \in [s]} \| P_i^* \| \right) \right)
\]
\[
\leq \eta \left( \max_{i \in [s]} \| P_i^* \| \right).
\] (52)

Using similar idea, we get the following norm bound,
\[
\| \hat{B}^i \varphi_i (\hat{P}_{1:s}) \hat{A}_i \| = \| (B_i^* + \Delta B_i^*)^T (\varphi_i (P_{1:s}^*) + \varphi_i (\Delta P_{1:s}^*)) (A_i^* + \Delta A_i^*) \|
\]
\[ f(\epsilon, \eta) + \|A_i^*\|\|B_i^*\| (\max_{t \in [s]} \|P_t^*\|), \]
\[ \leq 3\Gamma_i^2 f(\epsilon, \eta) + \Gamma_i^3. \]  
(53)

Substituting (50), (51), (52) and (53) into (49), we get the following norm bound,
\[ \|\tilde{K}_i - K_i\| \leq 16\Gamma_i^2 \frac{(\sigma(R_i) + \Gamma_i^3)}{\sigma(R_i)^2} \eta. \]  
(54)

This gives the final bound on \(\|\tilde{K}_i - K_i\|\). Finally, we combine the two norm bounds (48) and (54), to obtain the statement of the theorem,
\[ \|K_i^* - \tilde{K}_i\| \leq \|K_i^* - \tilde{K}_i\| + \|\tilde{K}_i - K_i\|, \]
\[ \leq 12\Gamma_i^2 \frac{(\sigma(R_i) + \Gamma_i^3)}{\sigma(R_i)^2} f(\epsilon, \eta) + 16\Gamma_i^3 \frac{(\sigma(R_i) + \Gamma_i^3)}{\sigma(R_i)^2} \eta, \]
\[ \leq 28\Gamma_i^2 \sigma(R_i) + \Gamma_i^3 \leq f(\epsilon, \eta), \]  
(55)

for all \(i \in [s]\). This completes the proof. \(\square\)

**Lemma 11 (Suboptimality gap).** Let \(\epsilon, \eta > 0\) be fixed scalars. Suppose \(\|T - T^*\|_\infty \leq \eta, \|\tilde{A}_i - A_i^*\| \leq \epsilon, \|\tilde{B}_i - B_i^*\| \leq \epsilon\) and \(\|\tilde{P}_i - P_i^*\| \leq f(\epsilon, \eta)\) for all \(i \in [s]\) and for some function \(f\) such that \(\max_{\epsilon, \eta} f(\epsilon, \eta) \leq \gamma < 1\). Then, under Assumption 3, as long as \(f(\epsilon, \eta) \leq \frac{1}{1804\sigma^2(\max(\|R_i\|, \|P_i\|))} \sigma(R_i)^2\), the certainty equivalent controller \(u_t = \tilde{K}_i(t)x_t\) achieves
\[ \hat{j} - J^* \leq \sigma^2 \min\{n, p\} \|R_1^*\| + \Gamma_i^3 \|T^*\|_\infty. \]

**Proof.** To prove this lemma, we need to quantify the suboptimality gap \(\hat{j} - J^*\) in terms of the controller mismatch \(\|\tilde{K}_i^* - K_i\|\) and derive an upper bound on this mismatch so that the certainty equivalent controller \(\tilde{K}_i\) stabilizes the MJS in the mean-square sense. For this purpose, recall that the goal of infinite time horizon LQR problem is to solve the following optimization problem,
\[ \inf_{\{u_0, u_1, \ldots\}} \limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} x_t^T Q(t)x_t + u_t^T R(t)u_t \right] \text{ s.t. } x_{t+1} = A_i^* x_t + B_i^* u_t + w_t, \]  
(56)

where the expectation is taken over the initial state \(x_0 \sim \mathcal{N}(0, I_n)\), Markovian modes \(\{\omega(t)\}_{t=0}^{\infty}\), and the i.i.d. noise \(\{w_t\}_{t=0}^{\infty} \sim i.i.d. \mathcal{N}(0, \sigma^2 I_n)\). If the input is given by \(u_t = K_i(t)x_t\), then, setting \(L_i(t) = A_i^* + B_i^* K_i(t)\), the state updates as follows,
\[ x_{t+1} = L_i(t)x_t + w_t \implies x_t = \begin{cases} x_0 & \text{if } t = 0, \\ L_i(0)x_0 + w_0 & \text{if } t = 1, \\ \prod_{j=1}^{t-1} L_i(j)x_0 + \sum_{\tau=0}^{t-2} \prod_{j=\tau+1}^{t-1} L_i(j)w_\tau + w_{t-1} & \text{if } t \geq 2 \end{cases} \]  
(57)

where \(\prod_{i=0}^{t-1} L_i(t) = L_i(t-1)L_i(t-2) \cdots L_i(0)\). For ease of notation, we set \(C_i(t) = Q(t) + K_i(t)R(t)K_i(t)\). Then, using (57) and the independence of \(x_0, \{w_t\}_{t=0}^{\infty}\) and \(\{\omega(t)\}_{t=0}^{\infty}\), the MJS cost function (56) can be simplified as follows,
\[ J(A_i^*, B_i^*, K_i, L_i) = \limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} x_t^T Q(t)x_t + K_i(t)R(t)K_i(t)x_t \right], \]
\[ = \limsup_{T \to \infty} \frac{1}{T} \left[ \mathbb{E} \left[ x_0^T C_{\omega(0)} x_0 \right] + \mathbb{E} \left[ x_0^T L_{\omega(0)}^T C_{\omega(1)} L_{\omega(0)} x_0 \right] \right. \]
\[ + \mathbb{E} \left[ w_0^T C_{\omega(1)} w_0 \right] + \sum_{t=2}^{T} \left. \mathbb{E} \left[ x_0^T \left( \prod_{i=0}^{t-1} L_i(t) \right)^T C_{\omega(t)} \left( \prod_{i=0}^{t-1} L_i(t) \right) x_0 \right] \right] \]
Given a noiseless closed loop MJS, before that, we introduce a few more concepts and definitions that will be used in the remaining proof.

Towards the end of this proof, we will show that when LQR problem (56) reduces to finding the suboptimally gap for a corresponding noiseless problem as follows,

$$\hat{\rho} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right].$$

Recall that, by assumption we have $\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right] = 0,$ where the expectation is with respect to $x_0 \sim \mathcal{N}(0, I_n)$ and the Markovian modes $\{\omega(t)\}_{t=0}^{\infty} \sim \mathcal{N}(0, \sigma^2_{\omega(t)} I_n).$ Let $M_{\text{init}}$ denote the original Markov chain and $M_{\text{final}}$ denotes a new Markov chain with the same transition matrix $T^*$ but with the initial distribution as the stationary distribution of $M_{\text{init}}.$ Observe that, for the Markov chain $M_{\text{final}},$ we have $\omega(0) \sim \omega(1) \sim \cdots \sim \pi_{T^*}.$ Using these, we define the following two cost functions for noiseless Markov jump systems,

$$J(A_{\omega(t)}, B_{\omega(t)}, K_{1:s}) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right] + \sigma^2_{\omega} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right],$$

subject to $\frac{d}{dt} x_t = (A_{\omega(t)} + B_{\omega(t)} K_{\omega(t)}) x_t.$ To proceed, using the assumption that the original Markov chain is ergodic, in the limit when $T \to \infty,$ equation (58) becomes

$$J(A_{1:s}, B_{1:s}, K_{1:s}) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right] + \sigma^2_{\omega} \mathbb{E} \left[ \text{tr} \left( C_{\omega(t)} + \sum_{i=0}^{T-1} \prod_{j=r+1}^{T} L_{\omega(j)}^T C_{\omega(t)} \prod_{i=0}^{T-1} L_{\omega(i)} \right) \right].$$

Recall that, by assumption we have $\rho(L^*) \leq \gamma < 1$ for some $\gamma > 0.$ From Proposition 2, this is equivalent to saying that the optimal controller $K_{1:s}$ stabilizes the noiseless MJ S in the mean-square sense. If additionally, the certainty equivalent controller $K_{1:s}$ stabilizes the noiseless MJ S in the mean-square sense, the problem of finding the suboptimality gap for the LQR problem (56) reduces to finding the suboptimally gap for a corresponding noiseless problem as follows,

$$J - J^* = \sigma^2_{\omega} \left( J_{\text{MJS}}(A_{1:s}, B_{1:s}, K_{1:s}) - J_{\text{MJS}}(A_{1:s}, B_{1:s}, K_{1:s}) \right).$$

Towards the end of this proof, we will show that when $K_{1:s}$ stabilizes the noiseless MJS in the mean-square sense and the Riccati perturbations $f(\epsilon, \eta)$ are sufficiently small, then $K_{1:s}$ also stabilizes the noiseless MJS in the mean-square sense. Before that, we introduce a few more concepts and definitions that will be used in the remaining proof.

**Definition.** Given a noiseless closed loop MJS, $\bar{x}_{t+1} = (A_{\omega(t)} + B_{\omega(t)} K_{\omega(t)}) \bar{x}_t,$ we define the following quantities:

(a) We denote by $\Sigma[\bar{x}_t] = \sum_{i \in [s]} \Sigma_i[\bar{x}_t]$ the covariance matrix of the state $\bar{x}_t,$ where $\Sigma_i[\bar{x}_t] := \mathbb{E}[\bar{x}_t \bar{x}_t^T | \bar{x}_0 = i].$

(b) When $K_{1:s}$ stabilizes the noiseless MJS in the mean-square sense, we know $\sum_{i \in [s]} \Sigma_i[\bar{x}_t]$ exists. We denote this limit as $\Sigma^K$ and we have $\Sigma^K = \sum_{i=0}^{\infty} T^i (\Sigma[\bar{x}_0]),$ where $T(V_{1:s}) = (T_1(V_{1:s}), \ldots, T_s(V_{1:s}))$ and $T_j(V_{1:s}) = \sum_{i=1}^{s_j} [T^*]_{ij}(A_i + B_i K_i)^T V_j(A_i + B_i K_i).$

(c) We denote by $\{K_{1:s}^{X^K_i}\}_{i=1}^{s}$ the unique positive definite solution of the following coupled Lyapunov equations for MJS,

$$X_{i}^{K^K} = Q_i + K_i^T R_i K_i + (A_i + B_i K_i)^T \varphi_i^T(X_{1:s}^{K^K})(A_i + B_i K_i), \quad \text{for all } i \in [s].$$
(d) The optimal control problem for noiseless MJS is the following infinite time horizon linear quadratic regulator problem

\[
\inf_{\{u_0,u_1,\ldots\}} \mathbb{E} \left[ \sum_{t=0}^{\infty} \tilde{x}_t^\top Q_{\omega(t)} \tilde{x}_t + u_t^\top R_{\omega(t)} u_t \right] \quad \text{s.t.} \quad \tilde{x}_{t+1} = A_{\omega(t)}^* \tilde{x}_t + B_{\omega(t)}^* u_t,
\]

where the expectation is over the initial state \(x_0 \sim N(0, \Sigma_{\omega})\) and the Markovian modes \(\{\omega(t)\}_{t=0}^{\infty}\).

(e) We define \( \bar{J}(A_{1:s}^*, B_{1:s}^*, K_{1:s}) := \mathbb{E} \left[ \sum_{t=0}^{\infty} \tilde{x}_t^\top (Q_{\omega(t)} + K_{\omega(t)}^\top R_{\omega(t)} K_{\omega(t)}) \tilde{x}_t \right] \quad \text{s.t.} \quad \tilde{x}_{t+1} = (A_{\omega(t)}^* + B_{\omega(t)}^* K_{\omega(t)}) \tilde{x}_t. \)

We are now ready to state a lemma which bounds the suboptimality gap of a noiseless LQR problem in terms of the mismatch between the optimal controller \(K_{1:s}\) and the certainty equivalent controller \(K_{1:s}\).

**Lemma** (Lemma 3 of [21]). Suppose, \(K_{1:s}^*\) and \(K_{1:s}\) stabilize the noiseless system \(\tilde{x}_{t+1} = A_{\omega(t)}^* \tilde{x}_t + B_{\omega(t)}^* u_t\) in the mean-square sense. Then, the costs incurred by the optimal controller \(K_{1:s}^*\) and certainty equivalent controller \(K_{1:s}\) on the true system satisfy

\[
J(A_{1:s}^*, B_{1:s}^*, K_{1:s}) - \bar{J}(A_{1:s}^*, B_{1:s}^*, K_{1:s}) = \sum_{i=1}^{s} \text{tr}(\Sigma_i^{K}(K_i^* - \hat{K}_i)^\top (R_i + B_i^T \phi_i(P_{1:s}^K)B_i^*) (K_i^* - \hat{K}_i)).
\]

Recall that, by assumption \(K_{1:s}^*\) stabilizes the noiseless MJS in the mean-square sense. If we assume, the certainty equivalent controller \(\hat{K}_{1:s}\) also stabilizes the noiseless MJS in the mean-square sense, we can use the above Lemma in (62) to get the following suboptimality bound for the LQR problem (56),

\[
\bar{J} - J^* \leq \sigma_0^2 \sum_{i=1}^{s} \sigma_i^{\Sigma(K)} ||R_i + B_i^T \phi_i(P_{1:s}^K)B_i^*|| ||K_i^* - \hat{K}_i||^2.
\]

\[
\leq \sigma_0^2 \sum_{i=1}^{s} \min\{n,p\} ||\Sigma_i^{K}|| ||R_i|| + ||B_i^*||^2 \left( \prod_{i=1}^{s} ||B_i^*|| \right) ||K_i^* - \hat{K}_i||^2,
\]

\[
\leq 800\sigma_0^2 \min\{n,p\} s ||\Sigma_i^{K}|| ||R_{1:s}|| + \Gamma_0^2 \Gamma_0 (\sigma(R_{1:s}) + \Gamma_0^3)^2 f(\epsilon, \eta)^2. \tag{65}
\]

What remains is to show that \(K_{1:s}\) stabilizes the true system \(\tilde{x}_{t+1} = (A_{\omega(t)}^* + B_{\omega(t)}^* \hat{K}_{\omega(t)}) \tilde{x}_t\) in the mean square sense. For this purpose, we define \(\tilde{L}_i := A_i^* + B_i^* \hat{K}_i = L_i^* + B_i^* \Delta_{K_i}\) where \(\Delta_{K_i} = K_i - K_i^*\). Then, we have \(\hat{L}_i \circ \tilde{L}_i = L_i^* \circ L_i^* + L_i^* \circ \Delta_{K_i} B_i^* + \Delta_{K_i} B_i^* \circ L_i^* + \Delta_{K_i} \circ \Delta_{K_i} B_i^*\). Using these, we define the following matrix,

\[
\hat{L} := \begin{bmatrix}
[T^*]_{11} \hat{L}_1 \circ \tilde{L}_1 & [T^*]_{12} \hat{L}_1 \circ \tilde{L}_2 & \cdots & [T^*]_{1s} \hat{L}_1 \circ \tilde{L}_s \\
[T^*]_{21} \hat{L}_2 \circ \tilde{L}_1 & [T^*]_{22} \hat{L}_2 \circ \tilde{L}_2 & \cdots & [T^*]_{2s} \hat{L}_2 \circ \tilde{L}_s \\
\vdots & \vdots & \ddots & \vdots \\
[T^*]_{s1} \hat{L}_s \circ \tilde{L}_1 & [T^*]_{s2} \hat{L}_s \circ \tilde{L}_2 & \cdots & [T^*]_{ss} \hat{L}_s \circ \tilde{L}_s
\end{bmatrix}.
\]

\[
\tag{66}
\]

To show that \(\hat{K}_{1:s}\) stabilizes the system \(\tilde{x}_{t+1} = (A_{\omega(t)}^* + B_{\omega(t)}^* \hat{K}_{\omega(t)}) \tilde{x}_t\) in the mean square sense it suffices to show that \(\rho(\hat{L}) < 1\) due to Proposition 6. For this purpose, we use the Fact 5 and the observation that, for any block matrix \(M\) with blocks \(M_{ij}\), \(||M||^2 \leq \sum_{i,j} ||M_{ij}||^2\), to obtain

\[
||\hat{L}^k|| = ||(\hat{L}^* + \Delta)^k|| \leq \tau(\hat{L}^*, \gamma)(\tau(\hat{L}^*, \gamma)||\Delta|| + \gamma)^k,
\]

where

\[
||\Delta|| = \sum_{i=1}^{s} \left( 2 ||A_i^* + B_i^* K_i^*|| ||B_i^*|| ||K_i^* - \hat{K}_i|| + ||B_i^*||^2 ||K_i^* - \hat{K}_i||^2 \right),
\]

\[
\leq s \left( 2 \Gamma_0^3 \max_{i \in [s]} ||K_i^* - \hat{K}_i|| + \Gamma_0^2 \max_{i \in [s]} ||K_i^* - \hat{K}_i||^2 \right),
\]

\[
(a) \leq 3s \Gamma_0^3 \max_{i \in [s]} ||K_i^* - \hat{K}_i||,
\]

\[
(b) \leq 90s \Gamma_0 (\sigma(R_{1:s}) + \Gamma_0^3)^2 f(\epsilon, \eta),
\]

\[
(c) \leq \frac{1 - \gamma}{2\tau(\hat{L}^*, \gamma)},
\]

where, we get (b) from the derived bound on \(||K_i^* - \hat{K}_i||\), while (a) and (c) from the assumption that \(f(\epsilon, \eta) \leq \frac{1 - \gamma}{2\tau(\hat{L}^*, \gamma)}\). This implies ||\hat{L}^k|| \leq \tau(\hat{L}^*, \gamma)(1 + \gamma)^k, which implies \(\rho(\hat{L}) \leq (1 + \gamma)/2 < 1\). To summarize,
we showed that when the optimal controller $K^*_f$ stabilizes the noiseless system $\bar{x}_{t+1} = A^*_f(t)\bar{x}_t + B^*_f(t)u_t$ in the mean-square sense and $f(\epsilon, \eta)$ satisfies the above inequality then the certainty equivalent controller $K_{1,s}$ also stabilizes the true system in the mean-square sense. Lastly, we observe that $\|\Sigma^K\| \leq \sum_{i=0}^\infty \|E[\tilde{x}_i\tilde{x}_j]\| \leq \sum_{i=0}^\infty \|E[\tilde{x}_i^2]\| \leq \sum_{i=0}^\infty \tau(\tilde{L}^*, \gamma)\left(\frac{1+\tau \gamma}{2}\right)^k \|\tilde{x}_0\|^2 \leq \frac{2\tau(\tilde{L}^*, \gamma)}{1-\gamma}$ for some $\tau(\tilde{L}^*, \gamma) \geq 1$ and $\gamma < 1$. Combining this with (65), we get the advertised suboptimality bound. This completes the proof.

D. Proof of Lemma 9

Since $\mathcal{K}$ is defined using $T^{-1}$, given in (35), in order to bound $\mathcal{K}$, we start by bounding $\|T^{-1}\|$. Let us first bound one of the factors in $T^{-1}$:

$$\|I - \tilde{L}^*\|^{-1} = \left\| \sum_{k=0}^\infty (\tilde{L}^*)^k \right\| \leq \sum_{k=0}^\infty \|\tilde{L}^*\|^k \leq \sum_{k=0}^\infty \tau(\tilde{L}^*, \gamma)\gamma^k \leq \frac{\tau(\tilde{L}^*, \gamma)}{1-\gamma}$$

where (i) follows from Definition 5, and (ii) holds since $\rho(\tilde{L}^*) \leq \gamma < 1$. Therefore, using Fact 4, we have

$$\|T^{-1}\| \leq \|\text{vec}^{-1}\|(I - \tilde{L}^*)^{-1}\|\text{vec} \leq \sqrt{n}\tau(\tilde{L}^*, \gamma) \frac{1}{1-\gamma}.$$  (69)

Next, to simplify the notations, for $X, X_1, X_2 \in S_\nu$, we let $P_X := P^* + X$, $P_{X_1} := P^* + X_1$, $P_{X_2} := P^* + X_2$, $\Delta_A := \hat{A} - A^*$, $\Delta_B := \hat{B} - B^*$, and $\Delta_S := \hat{S} - S^*$. We now derive some basic relations that will be used frequently later.

Recall by definition $X = \text{diag}(X_{1:s})$ and $\Phi^*(X) = \text{diag}(\varphi^*_1(X_{1:s}))$, so

$$\|\Phi^*(X)\| = \max_{i \in [s]} \|\varphi^*_1(X_{1:s})\| = \max_{i \in [s]} \|T^*\|_{ij} X_j \| \leq \|X\|.$$  (70)

Similarly, we have $\|\hat{\Phi}(X)\| \leq \|X\|$. Furthermore,

$$\|\Phi^*(X) - \hat{\Phi}(X)\| = \max_{i \in [s]} \|\varphi^*_1(X_{1:s}) - \hat{\varphi}_1(X_{1:s})\|$$

$$= \max_{i \in [s]} \sum_{j \in [s]} \|T^*\|_{ij} - |T|_{ij} X_j \| \leq \eta\|X\|.$$  (71)

For any $X \in S_\nu$, we have

$$\|P_X\| \leq \|P^*\| + \nu \leq \|P^*\|_\infty,$$  (72)

where we used assumption $\nu \leq 1$ in the statement of Lemma 9. Combining (70) and (72), we have

$$\max \left\{ \left\|\Phi^*(P_X)\right\|, \left\|\Phi(P_X)\right\| \right\} \leq \|P^*\|_\infty.$$  (73)

Consider $\Delta_S$, since $\Delta_S = \hat{S} - S^* = \tilde{B}R^{-1}\tilde{B}^T - B^*R^{-1}B^T = \Delta_B R^{-1}B^T + B^*R^{-1}\Delta_B + \Delta_B R^{-1}\Delta_B$ and assumption $\|\Delta_B\| \leq \epsilon \leq \|B^*\|$ in the statement of Lemma 9, we have

$$\|S^*\| \leq \|B^*\|^2\|R^{-1}\|, \quad \|\Delta_S\| \leq 3\|B^*\||\|R^{-1}\|\epsilon, \quad \|\hat{S}\| \leq 4\|B^*\|^2\|R^{-1}\|.$$  (74)

Following Lemma 1, (73), and (74), we have

$$\max \left\{ \left\|\Phi^*(I + S^*\Phi^*(P_X))^{-1}\right\|, \left\|\Phi^*(I + S^*\hat{\Phi}(P_X))^{-1}\right\| \right\} \leq 1 + \|S^*\|\|P^*\|_\infty \leq \|B^*\|^2\|R^{-1}\| + \|P^*\|_\infty,$$  (75)

$$\max \left\{ \left\|\Phi^*(I + \tilde{S}\Phi^*(P_X))^{-1}\right\|, \left\|\Phi^*(I + \tilde{S}\hat{\Phi}(P_X))^{-1}\right\| \right\} \leq 1 + \|\tilde{S}\|\|P^*\|_\infty \leq 4\|B^*\|^2\|R^{-1}\| + \|P^*\|_\infty.$$  (76)

Finally, recall the definition of $F(X; A^*, B^*, T^*)$ in (27), and consider the following notation:

$$G_1(X) := F(P_X; \hat{A}^*, \hat{B}^*, \hat{T}) - F(P_X; \hat{A}, \hat{B}, \hat{T}),$$  (77)

$$G_2(X) := F(P_X; \hat{A}^*, \hat{B}^*, \hat{T}) - F(P_X; \hat{A}^*, \hat{B}^*, \hat{T}).$$  (78)

Now we are ready to start the main proof for Lemma 9. We will do this in two steps: (a) Proof of (38). (b) Proof of (39).

Step (a): Proof of the bound (38) in Lemma 9.

We can see from the definition of $K(X)$ in (36):

$$K(X) = T^{-1}(G_1(X) + G_2(X) - \mathcal{H}(X)),$$  (79)
We will upper bound \( \| H(X) \|, \| G_1(X) \|, \) and \( \| G_2(X) \| \) for any \( X \in S_\nu \), then combining these with the bound for \( \| T^{-1} \| \) in (69), we can conclude step (a).

Since \( H(X) = L^* \Phi^*(X)(I + S^* \Phi^*(P^*) + S^* \Phi^*(X))^{-1}S^* \Phi^*(X)L^* \) in (33), we have
\[
\| H(X) \| \leq \| L^* \|^2 \| S^* \| \| X \|^2 \leq \| L^* \|^2 \| S^* \| \nu^2,
\]
where (19) and (70) are used. Now consider \( G_1(X) \)
\[
G_1(X) = F(P_X^*; A^*, B^*, T) - F(P_X^*; A, B, T)
\]
\[
\overset{\text{(i)}}{=} - A^T \Phi(P_X^*)(I + S^* \Phi(P_X^*))^{-1} A^* + (A^* + \Delta_A)^T \Phi(P_X^*)(I + \tilde{\Phi}(P_X^*))^{-1} (A^* + \Delta_A)
\]
\[
\overset{\text{(ii)}}{=} - A^T \Phi(P_X^*)(I + S^* \Phi(P_X^*))^{-1} \Delta_S \Phi(P_X^*)(I + \tilde{\Phi}(P_X^*))^{-1} A^*
\]
\[
+ \Delta_A^T \Phi(P_X^*)(I + S^* \Phi(P_X^*))^{-1} A^* + A^T \Phi(P_X^*)(I + \tilde{\Phi}(P_X^*))^{-1} \Delta_A
\]
\[
+ \Delta_A^T \Phi(P_X^*)(I + S^* \Phi(P_X^*))^{-1} \Delta_A,
\]
where (i) follows from the definition of \( F \) in (27), and (ii) uses (22). Then,
\[
\| G_1(X) \| \leq \| A^* \|^2 \| \Phi(P_X^*) \|^2 \| S \| + 2 \| \Phi(P_X^*) \| \| A^* \| \| \epsilon \| + \| \tilde{\Phi}(P_X^*) \| \epsilon^2
\]
\[
\leq 3 \| A^* \|^2 \| P^* \| \| B^* \| \| R^{-1} \| \| \epsilon \| + 2 \| P^* \| + \| A^* \| \| \epsilon \| + \| B^* \| \| P^* \| + \| \epsilon \|
\]
\[
\leq 3 \| A^* \|^2 \| B^* \|^2 \| P^* \|^2 \| R^{-1} \|^2 \| \epsilon \|
\]
where (19), (73), (74), (71), (72), and (75) are used. Using upper bounds for \( \| T^{-1} \|, \| H(X) \|, \| G_1(X) \|, \) and \( \| G_2(X) \| \) in (69), (80), (82), and (84), then the relation in (79) gives
\[
\| \mathcal{K}(X) \| \leq \sqrt{\frac{n \sigma^2}{1 - \gamma}} \left( \| L^* \|^2 \| S^* \| \nu^2 + 3 \| A^* \|^2 \| B^* \|^2 \| R^{-1} \|^2 \| \epsilon \| + \| A^* \|^2 \| B^* \|^2 \| P^* \|^2 \| R^{-1} \|^2 \| \epsilon \| \right)
\]
which shows (38) in Lemma 9.  

**Step (b):** Proof of the bound (39) in Lemma 9. Following as in step (a) and invoking the linearity of \( T^{-1} \), we have
\[
\mathcal{K}(X_1) - \mathcal{K}(X_2) = T^{-1}(G_1(X_1) - G_1(X_2) + G_2(X_2) - G_2(X_2) - H(X_1) + H(X_2)).
\]
We will upper bound \( \| H(X_1) - H(X_2) \|, \| G_1(X_1) - G_1(X_2) \|, \) and \( \| G_2(X_1) - G_2(X_2) \| \) for any \( X_1, X_2 \in S_\nu \), then combining these with the bound for \( \| T^{-1} \| \) in (69), we can prove step (b).

First we consider \( H(X_1) - H(X_2) \)
\[
H(X_1) - H(X_2)
\]
\[
= L^* \Phi^*(X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_1)L^* - L^* \Phi^*(X_2)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2)L^*
\]
\[
= L^* \Phi^*(X_1) \left( (I + S^* \Phi^*(P_X^*))^{-1} - (I + S^* \Phi^*(P_X^*))^{-1} \right) S^* \Phi^*(X_1)L^*
\]
\[
- L^* \Phi^*(X_2) - (I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2)L^* - L^* \Phi^*(X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2 - X_1)L^*
\]
\[
= L^* \Phi^*(X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2 - X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_1)L^*
\]
\[
- L^* \Phi^*(X_2 - X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2)L^* - L^* \Phi^*(X_1)(I + S^* \Phi^*(P_X^*))^{-1}S^* \Phi^*(X_2 - X_1)L^*.
\]
Then,
\[
\|\mathcal{H}(X_1) - \mathcal{H}(X_2)\| \\
\leq l^*\|X_1\| + \|S^*\|\|X_2 - X_1\| + l^*\|X_2 - X_1\| + l^*\|X_2 - X_1\| + l^*\|X_2 - X_1\| + l^*\|X_2 - X_1\| \\
\leq 3l^*\|S^*\|\nu\|X_2 - X_1\|.
\] (88)

The first inequality follows from \(\|N(I + MN)^{-1}\| \leq \|N\|\) in (19), and \(\|\Phi^*(X)\| \leq \|X\|\) in (70). The last inequality can be obtained by recalling the assumption \(\nu \leq \|S^*\|^{-1}\) in the statement of Lemma 9.

For \(\mathcal{G}_1(X_1) - \mathcal{G}_1(X_2)\), we have
\[
\mathcal{G}_1(X_1) - \mathcal{G}_1(X_2) = F(P_{X_1}; A^*, B^*, \hat{T}) - F(P_{X_2}; A^*, B^*, \hat{T}) = F(P_{X_1}; A^*, B^*, \hat{T}) + F(P_{X_2}; A^*, B^*, \hat{T}) \\
= -A^*\Phi(P_{X_1})/(I + S^*\Phi(P_{X_1}))^{-1}A^* + (A^* + \Delta_A)^\top\Phi(P_{X_1})/(I + \hat{S}\Phi(P_{X_1}))^{-1}(A^* + \Delta_A) \\
+ A^*\Phi(P_{X_2})/(I + S^*\Phi(P_{X_2}))^{-1}A^* - (A^* + \Delta_A)^\top\Phi(P_{X_2})/(I + \hat{S}\Phi(P_{X_2}))^{-1}(A^* + \Delta_A) \\
\begin{align}
\leq & -A^*\Phi(P_{X_1})/(I + S^*\Phi(P_{X_1}))^{-1}A^* + \Delta_A\Phi(P_{X_1})/(I + \hat{S}\Phi(P_{X_1}))^{-1}\hat{A} - \Delta_A\Phi(P_{X_1})/(I + \hat{S}\Phi(P_{X_1}))^{-1}\hat{A} \\
&+ A^*\Phi(P_{X_2})/(I + S^*\Phi(P_{X_2}))^{-1}A^* - A^*\Phi(P_{X_2})/(I + \hat{S}\Phi(P_{X_2}))^{-1}\Delta_A \\
=: & M_1 \\
&+ M_2.
\end{align}
\] (89)

For \(M_1 - M_2\), we have
\[
M_1 - M_2 = \Phi(P_{X_1})/(I + S^*\Phi(P_{X_1}))^{-1} - \Phi(P_{X_1})/(I + S^*\Phi(P_{X_2}))^{-1} \\
\begin{align}
\leq & \Phi(P_{X_1})(I + S^*\Phi(P_{X_1}))^{-1}S^*\Phi(X_2 - X_1)(I + S^*\Phi(P_{X_2}))^{-1} + \Phi(X_1 - X_2)(I + S^*\Phi(P_{X_2}))^{-1} \\
=: & \Phi(P_{X_1})(I + S^*\Phi(P_{X_1}))^{-1}S^* - I + \Phi(X_1 - X_2)(I + S^*\Phi(P_{X_2}))^{-1}.
\end{align}
\] (90)

Then,
\[
\|M_1 - M_2\| \leq (\|P^*\| + \|B^*\|\|R^{-1}\| + 1)\|X_2 - X_1\| + \|B^*\|\|R^{-1}\| + \|P^*\| + \|B^*\|\|R^{-1}\| + \|X_2 - X_1\|.
\] (91)

The first line is obtained by invoking: (i) \(\|N(I + MN)^{-1}\| \leq \|N\|\) in (19), (ii) \(\|\Phi(X)\| \leq \|X\|\) in (70), (iii) \(\|\Phi(P_{X_1})\| \leq \|P^*\| + \|B^*\|\|R^{-1}\|\) in (73), (iv) \(\|S^*\| \leq \|B^*\|\|R^{-1}\|\) in (74), and (v) \(\|I + S^*\Phi(P_{X_1})^{-1}\| \leq \|B^*\|\|R^{-1}\| + \|P^*\|\) in (75). Now using (19), (73), (74), (70), and (76) similarly, we have
\[
\|M_1 - M_2\| \leq \|\Phi(P_{X_1})(I + S^*\Phi(P_{X_1}))^{-1}\hat{S} - I + \Phi(X_2 - X_1)(I + S^*\Phi(P_{X_2}))^{-1}\| \\
\begin{align}
\leq & (\|P^*\| + 4\|B^*\|\|R^{-1}\| + 1)\|X_2 - X_1\| + 4\|B^*\|\|R^{-1}\| + \|P^*\| + \|B^*\|\|R^{-1}\| + \|X_2 - X_1\| \\
\leq & 16\|B^*\|\|R^{-1}\| + \|P^*\|\|R^{-1}\| + \|X_2 - X_1\|.
\end{align}
\] (92)

Through (19) and (73), we can have
\[
\|M_1\| \leq \|P^*\| + \|M_2\| \leq \|P^*\| + \|P^*\|.
\] (93)

Plugging (91), (92), and (93) into (89) gives
\[
\|
\mathcal{G}_1(X_1) - \mathcal{G}_1(X_2)\| \leq 51\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| + 16\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| \\
\begin{align}
\leq & 51\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| + 16\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| \\
\leq & 51\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| + 16\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\| \leq \epsilon.
\end{align}
\] (94)

where we additionally use the fact \(\|\Delta_S\| \leq 3\|B^*\|\|R^{-1}\|\) in (74) and assumption \(\epsilon \leq 1\) in the statement of Lemma 9.

For \(\mathcal{G}_2(X_1) - \mathcal{G}_2(X_2)\), following the same strategy, we can obtain the following bound:
\[
\|
\mathcal{G}_2(X_1) - \mathcal{G}_2(X_2)\| \leq 2\|A^*\|\|B^*\|\|P^*\|\|R^{-1}\|\|X_2 - X_1\|\eta.
\] (95)
Using upper bounds for \( \|T^{-1}\|, \|\mathcal{H}(X_1) - \mathcal{H}(X_2)\|, \|G_1(X_1) - G_1(X_2)\|, \) and \( \|G_2(X_1) - G_2(X_2)\| \) in (69), (88), (94), and (95), then relation in (86) gives

\[
||K(X_1) - K(X_2)|| \leq \sqrt{n}r(L^*, \gamma) \left( \frac{n}{1 - \gamma} \right) \left( 3\|L^*\|^2\|S^*\|^2\|\nu\| + 51\|A^*\|\|B^*\|\|P^*\|^3\|R^{-1}\|\|\epsilon\| + 2\|A^*\|\|B^*\|\|P^*\|^3\|R^{-1}\|\|\eta\| \right) \|X_1 - X_2\|. \tag{96}
\]

which shows (39) in Lemma 9 and concludes the proof.