Computing the R-Matrix of the Quantum Toroidal Algebra

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Abstract. We consider the problem of the R-matrix of the quantum toroidal algebra $U_{q,t}({\mathfrak{gl}}_1)$ in the Fock representation. Using the connection between the R-matrix $R(u)$ ($u$ being the spectral parameter) and the theory of Macdonald operators we obtain explicit formulas for $R(u)$ in the operator and matrix forms. These formulas are expressed in terms of the eigenvalues of a certain Macdonald operator which completely describe the functional dependence of $R(u)$ on the spectral parameter $u$. We then consider the geometric R-matrix (obtained from the theory of $K$-theoretic stable bases on moduli spaces of framed sheaves), which is expected to coincide with $R(u)$ and thus gives another approach to the study of the poles of the R-matrix as a function of $u$.

1. Introduction

The quantum toroidal algebra $U_{q,t}({\mathfrak{gl}}_1)$ is an important object in geometric representation theory, mathematical physics and algebraic combinatorics. In fact, arguably the most important appearance of this algebra lies at the intersection of these three fields, namely the action on Fock space:

$$U_{q,t}({\mathfrak{gl}}_1) \sim \mathcal{F}$$

Specifically, $\mathcal{F}$ can be thought of either as the $K$-theory group of the Hilbert scheme of points on $\mathbb{A}^2$, the Hilbert space of quantum mechanics for an arbitrary number of identical bosons, or the ring of symmetric functions in countably many variables; in each of these incarnations, the action encapsulates the symmetries of the Fock space.

As $U_{q,t}({\mathfrak{gl}}_1)$ is a Hopf algebra, one can use its (topological) coproduct to define actions:

$$U_{q,t}({\mathfrak{gl}}_1) \sim \mathcal{F} \otimes \mathcal{F}$$

(and more generally on tensor products of arbitrarily many Fock spaces). But as is often the case in quantum algebra, the order of the tensor factors matters, and simply swapping the factors in does not commute with the $U_{q,t}({\mathfrak{gl}}_1)$ action. To fix this, one constructs the universal $R$-matrix:

$$R \in U_{q,t}({\mathfrak{gl}}_1) \otimes U_{q,t}({\mathfrak{gl}}_1)$$

and uses it to obtain a $U_{q,t}({\mathfrak{gl}}_1)$-module intertwiner:

$$R(u) : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}(u)$$

The operators satisfy the quantum Yang–Baxter equation:

$$R_{1,2}(u_2/u_1)R_{1,3}(u_3/u_1)R_{2,3}(u_3/u_2) = R_{2,3}(u_3/u_2)R_{1,3}(u_3/u_1)R_{1,2}(u_2/u_1)$$

where $R_{i,j}(u_j/u_i)$ acts in the $i$-th and $j$-th spaces of $\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}$. Let $N(u)$ be the vacuum-vacuum matrix element of $R(u)$ (which can be found in formula) and set:

$$R(u) := N(u)^{-1} R(u)$$

Since the universal $R$-matrix is an infinite sum (closely related to the canonical tensor of the Hopf pairing between two halves of the quantum toroidal algebra $U_{q,t}({\mathfrak{gl}}_1)$), one can obtain formulas for $R(u)$ as a power series in $u$. While the coefficients of this power series do not admit closed formulas, there are many ways to compute them, both systematically and algorithmically. However, the main result of the present paper is that $R(u)$ is actually the expansion of a rational function with prescribed poles.

Theorem 1. (cf. Corollary) The operator is the power series expansion of an operator whose coefficients are rational functions in $u$, with simple poles at $\{u = q^i t^{-j}\}_{i,j \geq 1}$.

Moreover, we give formulas for $R(u)$ and $R(u)$ in terms of bosons and in the standard Heisenberg basis of $\mathcal{F}$, in Propositions and respectively.
Our techniques for proving the Theorem above are two-fold: on one hand, we heavily use the Macdonald polynomial basis of \( \mathcal{F} \), and express vertex operators in terms of this basis. On the other hand, we recall from \([1,17]\) the use of the shuffle algebra in dealing with such vertex operators. We prove certain shuffle algebra formulas for the R-matrix \( R(u) \), which will allow us to identify its functional dependence on the parameter \( u \) and prove Theorem 1. Although we believe certain key formulas that we need are new, the approach summarized in the present paragraph has been used in many works, such as \([2,5,7,8,17,20]\), and in the closely related problem of the instanton R-matrix \([12,13,25,28]\).

In the second half of the present paper, we present an alternative approach to proving the first paragraph of Theorem 1 via the K-theoretic stable basis construction (which originated in \([14]\) and was developed in \([1,22,24]\) and other works) \(^1\). The starting point is the observation that \( \mathcal{F} \) is isomorphic to the \((\text{algebraic, equivariant})\) K-theory groups of the Hilbert scheme of points on \( \mathbb{A}^2 \), a beautiful idea in geometric representation theory that arose from several important directions (notably the actions of the Heisenberg algebra on the cohomology of Hilbert schemes, due to Grojnowski and Nakajima, and the Bridgeland-King-Reid-Haiman equivalence of categories between modules over wreath products and sheaves on the Hilbert scheme). Taking this idea one step further, one identifies \( \mathcal{F} \otimes \mathcal{F} \) with the K-theory groups of moduli spaces of rank 2 sheaves on the plane. The latter are symplectic varieties, and so any Lagrangian correspondence involving the said moduli spaces will give rise to homomorphisms:

\[
\mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \left( \frac{u_2}{u_1} \right)
\]

where \( u_1, u_2 \) are simply equivariant parameters. The “correct” Lagrangian correspondences are the stable bases constructed in \([1,14,22,24]\), which were defined so that the operators \((7)\) satisfy the quantum Yang-Baxter equation. Though the details have not been written down (to the authors’ knowledge), it is overwhelmingly expected that the operators \((1)\) and \((7)\) match under the identification \( u = \frac{u_2}{u_1} \). If this is so, then the fact that \( R(u) \) is a rational function in \( u \) would follow almost immediately, as operators given by Lagrangian correspondences are rational functions in \( \frac{u_2}{u_1} \) for general reasons.

The structure of the present paper is the following:

- In Section 2, we review Macdonald polynomials, the Fock space and vertex operators
- In Section 3, we review the quantum toroidal algebra \( U_{q,t}(\mathfrak{gl}_1) \) and its shuffle algebra incarnation, and prove Theorem 1
- In Section 4, we recall stable bases for Hilbert schemes, and present the alternative approach to the R-matrix using the operators \((7)\)

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2. The Fock space and symmetric functions

In this Section we provide some basic information about symmetric functions, Fock space, Heisenberg algebra and Macdonald operators.

### 2.1. Partitions

Let \( \mathcal{P} \) be the set of all partitions \( \mu = (\mu_1, \mu_2, \ldots, \mu_j) \), \( \mu_j \geq \mu_{j+1} \). The length of a partition \( \mu \), denoted \( \ell(\mu) \), is equal to the number of non-zero parts in \( \mu \). The weight of a partition \( |\mu| \) is given by the sum of all parts \( |\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} \). The notation \( \mu \vdash j \) means that \( \mu \) is a partition of \( j \), \( |\mu| = j \). Let \( r \) be a positive integer and \( \mu \) a partition. Define the part multiplicity function \( m_r(\mu) \) which counts the number of parts in \( \mu \) equal to \( r \). The part multiplicity vector \( m(\mu) \) is defined as \( m(\mu) = (m_1(\mu), m_2(\mu), \ldots, m_{\ell(\mu)}(\mu), 0, \ldots) \). It is clear that we have a bijection between the set of all partitions \( \mathcal{P} \) and non-negative integer vectors of part multiplicities. Using this correspondence we can add partitions by component-wise addition of the corresponding part multiplicity vectors. We can also subtract partitions when the subtraction of the part multiplicities does not produce vectors with negative entries. So for three partitions \( \mu, \nu, \lambda \) the notation:

\[
\mu \pm \nu = \lambda \quad \text{means} \quad m(\mu) \pm m(\nu) = m(\lambda)
\]

---

\(^1\)We will take an expositional point of view rather than a purely rigorous one, so we do not claim an alternative proof due to certain details that we leave out for the sake of brevity. Some of these details are straightforward and some are less so, and we will inform the reader which is which. 

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This defines a partial order: \( \nu \subseteq \mu \), if \( m(\mu) - m(\nu) \in \mathbb{Z}_{\geq 0} \). The intersection of two partitions \( \mu \cap \nu \) is the partition \( \lambda = \mu \cap \nu \) such that \( \lambda \subseteq \mu \), \( \lambda \subseteq \nu \) and the differences \( \mu - \lambda \) and \( \nu - \lambda \) have no common parts. We will also use the dominance order on partitions \( \lambda \geq \mu \), which means that \( \lambda \) and \( \mu \) have the same weight and \( \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \) for all \( k > 0 \).

Let us introduce the short hand notations involving factorials of part multiplicities of a partition \( \mu \):

\[
\mu! := \prod_{a=1}^{\infty} m_a(\mu)! 
\]

We define an analogue of the binomial coefficient for arguments in \( \mathcal{P} \). Let \( \mu, \nu \in \mathcal{P} \), we write:

\[
\begin{bmatrix} \mu \\ \nu \end{bmatrix} := \begin{cases} \frac{\mu!}{(\mu - \nu)!}, & \text{if } \nu \subseteq \mu \\ 0, & \text{otherwise} \end{cases}
\]

The usual properties of binomial coefficients apply to (8), in particular the symmetry:

\[
\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \mu & \mu - \nu \end{bmatrix}
\]

The summ ands will typically involve binomial coefficients (8) and due to their vanishing condition the summands will have finite support.

2.2. Symmetric functions. Let \( \Lambda \) be the ring of symmetric functions in the alphabet \( x = (x_1, x_2, \ldots) \) over \( \mathbb{Z} \) and \( \Lambda_F = \Lambda \otimes \mathbb{F} \) where \( \mathbb{F} = \mathbb{Q}(q, t) \). The basic symmetric functions are the monomial symmetric function \( m_\lambda(x) \) and the power sum symmetric function \( p_\lambda(x) \), \( \lambda \in \mathcal{P} \), where:

\[
p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_{\lambda}(x)}, \quad p_r(x) = \sum_i x_i^r
\]

For an integer \( r \) and a partition \( \lambda \) we set:

\[
z_\lambda(q, t) := \lambda! \prod_{r \in \lambda} \frac{1 - q^r}{1 - t^r}
\]

The Macdonald scalar product on \( \Lambda_F \) is defined by:

\[
\langle p_\lambda | p_\mu \rangle_{q, t} = \delta_{\lambda, \mu} z_\lambda(q, t)
\]

Let us now recall the Macdonald polynomials \( P_\lambda(x; q, t) \) which are characterised by the following two conditions [13, Ch. VI]:

\[
P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda, \mu}(q, t) m_\mu(x)
\]

\[
\langle P_\lambda | P_\mu \rangle_{q, t} = 0, \quad \text{for } \lambda \neq \mu
\]

As shown in loc. cit., these conditions imply that:

\[
\langle P_\lambda | P_\mu \rangle_{q, t} = \frac{1}{b_\lambda(q, t)}, \quad \text{where } b_\lambda(q, t) = \prod_{\mu \in \lambda} \frac{1 - q^{\lambda_\mu} p_{\mu}(\lambda) + 1}{1 - q^{\lambda_\mu} + 1 p_{\mu}(\lambda)}
\]

Thus, the dual Macdonald polynomials \( Q_\lambda(x; q, t) \) are:

\[
Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(x; q, t)
\]

and the scalar product becomes:

\[
\langle P_\lambda | Q_\mu \rangle_{q, t} = \delta_{\lambda, \mu}
\]
We will effectively work with two bases of symmetric functions: the power sum basis and the Macdonald basis. Let us define the transition coefficients between these bases:

$$P_\lambda(x; q, t) = \sum_\mu g_{\lambda, \mu}(q, t) p_\mu(x)$$  \hspace{1cm} (15)

Since $P_\lambda(x; q, t)$ and $p_\mu(x)$ are both homogeneous functions, this means that $g_{\lambda, \mu}(q, t)$ are non-zero when $|\lambda| = |\mu|$, i.e.:

$$g_{\lambda, \mu}(q, t) = 0, \quad |\lambda| \neq |\mu|$$  \hspace{1cm} (16)

2.3. The Fock space. Let $\mathcal{H}_n$ be the Heisenberg algebra over $F$, generated by $\{a_n, a_{-n}\}$ for $n \in \mathbb{Z}_{>0}$ with the defining relations:

$$[a_r, a_s] = [a_{-r}, a_{-s}] = 0 \quad \text{and} \quad [a_r, a_{s}] = \delta_{r,s} \frac{1 - q^r}{1 - t^r}$$  \hspace{1cm} (17)

for all $r, s > 0$. By multiplying these generators we get:

$$a_\mu = \prod_{r \in \mu} a_r, \quad a_{-\mu} = \prod_{r \notin \mu} a_r, \quad \mu \in \mathcal{P}$$

The representation of this Heisenberg algebra is given on the Fock space $\mathcal{F}$ and the dual Fock space $\mathcal{F}^*$ by the formulas:

$$a_{-\lambda} |a_\mu\rangle = |a_{\mu+\lambda}\rangle, \quad a_\lambda |a_\mu\rangle = z_\lambda(q, t) \left[ \frac{\mu}{\lambda} \right] |a_{\mu-\lambda}\rangle$$

$$\langle a_\mu | a_\lambda \rangle = \delta_{\alpha\beta} z_\alpha(q, t)$$  \hspace{1cm} (18)

The scalar product reads:

$$\langle a_\lambda | a_\mu \rangle = \delta_{\lambda, \mu} z_\lambda(q, t)$$  \hspace{1cm} (19)

We denote by $T_{\alpha, \beta}^{\mu, \nu}$ the matrix elements of the generic basis element $a_{-\mu} a_\nu$ of the Heisenberg algebra:

$$T_{\alpha, \beta}^{\mu, \nu} := \langle a_\alpha | a_{-\mu} a_\nu | a_\beta \rangle$$  \hspace{1cm} (20)

We set $T$ to be the matrix of coefficients $T_{\alpha, \beta}^{\mu, \nu}$ for all partitions $\alpha, \beta, \mu, \nu$. The matrix elements of $T$ are computed using standard binomial identities:

$$T_{\alpha, \beta}^{\mu, \nu} = \delta_{\alpha-\mu, \beta-\nu} z_\alpha(q, t) z_\nu(q, t) \left[ \frac{\beta}{\nu} \right], \quad (T^{-1})_{\alpha, \beta}^{\mu, \nu} = \delta_{\mu-\alpha, \nu-\beta} \frac{(-1)^{\ell(\mu-\alpha)}}{z_\alpha(q, t) z_\nu(q, t)} \left[ \frac{\mu}{\alpha} \right]$$  \hspace{1cm} (21)

As graded vector spaces, the ring of symmetric functions and the Fock space $\mathcal{F}$ are isomorphic:

$$\iota : \mathcal{F} \to \Lambda F, \quad |a_\lambda\rangle \mapsto |p_\lambda\rangle$$

Using the isomorphism $\iota$ we can view Macdonald functions as vectors in the Fock space. They could be defined using the transition coefficients $g_{\lambda, \mu}(q, t)$ between Macdonald functions and the power sums $\{p_\mu(x)\}$:

$$\langle P_{\lambda} | = \sum_\mu g_{\lambda, \mu}(q, t) \langle a_\mu |, \quad |Q_{\lambda}\rangle = b_{\lambda}(q, t) \sum_\mu g_{\lambda, \mu}(q, t) |a_\mu\rangle$$  \hspace{1cm} (22)

These vectors give us an orthonormal basis. We also define the normalized vectors:

$$\langle \langle a_\lambda | := \frac{1}{z_\lambda(q, t)} \langle a_\lambda |, \quad |a_\lambda\rangle := |a_\lambda\rangle$$  \hspace{1cm} (23)

so that we have:

$$\langle \langle a_\lambda | a_\mu \rangle\rangle = \delta_{\lambda, \mu}$$  \hspace{1cm} (24)
2.4. Macdonald operators. Following [4, 27] we build the vertex operators:

\[
\eta(z) := \exp \left( \sum_{r=1}^{\infty} \frac{1 - t^{-r}}{r} a_{-r} z^r \right) \exp \left( - \sum_{r=1}^{\infty} \frac{1 - t^r}{r} a_r z^{-r} \right)
\]
\[
\xi(z) := \exp \left( - \sum_{r=1}^{\infty} \frac{1 - t^{-r}}{r} (t/q)^{r/2} a_{-r} z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{1 - t^r}{r} (t/q)^{r/2} a_r z^{-r} \right)
\]

The Fourier modes of these operators, denoted by \(\eta_n\) and \(\xi_n\):

\[
\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n}, \quad \xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n}
\]

have a well defined action on \(\mathcal{F}\). A product of Heisenberg operators is normally ordered if all the positive modes are on the right and the negative modes are on the left. We use the standard notation \(a_\mu a_{-\mu} := a_{-\mu} a_\mu\). As such, the products \(\eta(z)\eta(w)\) and \(\xi(z)\xi(w)\) can be ordered according to:

\[
\eta(z)\eta(w) = \zeta \left( \frac{z}{w} \right) \eta(z)\eta(w) : \\
\xi(z)\xi(w) = \zeta \left( \frac{w}{z} \right) \xi(z)\xi(w) :
\]

where:

\[
\zeta(x) = \frac{(x - 1)(x - qt^{-1})}{(x - q)(x - t^{-1})}
\]

In formulas (27) and (28), the rational function \(\zeta\) is expanded in non-negative powers of \(z/w\). Using the vertex operator one can write the free field realizations of Macdonald operators. The basic Macdonald operators are the operators \(\hat{E}_1\) and \(\hat{E}_1^*\) which act on the Macdonald functions as:

\[
\hat{E}_1 | P_\lambda \rangle = \sum_{i \geq 1} (q^{\lambda_i} - 1) t^{-i} | P_\lambda \rangle, \quad \hat{E}_1^* | P_\lambda \rangle = \sum_{i \geq 1} (q^{-\lambda_i} - 1) t^i | P_\lambda \rangle
\]

These operators are realized on the Fock space by the zero modes \(\eta_0\) and \(\xi_0\):

\[
\eta_0 = (t - 1) \hat{E}_1 + 1, \quad \xi_0 = (t^{-1} - 1) \hat{E}_1^* + 1
\]

For any partition \(\lambda \in \mathcal{P}\), consider the ring homomorphism \(\varepsilon_\lambda : \Lambda \rightarrow \mathbb{F}\) generated by:

\[
\varepsilon_\lambda(x_i) = q^{\lambda_i} t^{-i}, \quad i > 1
\]

The eigenvalue equations for \(\eta_0\) and \(\xi_0\) are:

\[
\frac{t}{1 - t} \eta_0 | P_\lambda \rangle = \varepsilon_\lambda(e_1(x)) | P_\lambda \rangle, \quad \frac{t}{1 - t} \xi_0 | P_\lambda \rangle = \varepsilon_\lambda(e_1(x^{-1})) | P_\lambda \rangle
\]

where the alphabet \((x^r)\) means \((x_1^r, x_2^r, \ldots)\) for any integer \(r\), and \(e_1(x)\) is the first elementary symmetric function \(e_1(x) = x_1 + x_2 + \ldots\). More generally [27] we define the operators:

\[
\hat{E}_n := \frac{1}{n! (1 - t^{-1})^{n+1/2}} \int \prod_{i=1}^{n} \frac{dz_i}{2\pi i z_i} \prod_{1 \leq i < j \leq n} \frac{(z_i - z_j)}{(z_i - t^{-1} z_j)} : \eta(z_1) \ldots \eta(z_n) :
\]
\[
\hat{E}_n^* := \frac{1}{n! (1 - t)^{n+1/2}} \int \prod_{i=1}^{n} \frac{dz_i}{2\pi i z_i} \prod_{1 \leq i < j \leq n} \frac{(z_i - z_j)}{(z_i - t z_j)} : \xi(z_1) \ldots \xi(z_n) :
\]

whose action in the Macdonald basis is:

\[
\hat{E}_n | P_\lambda \rangle = \varepsilon_\lambda(e_n(x)) | P_\lambda \rangle, \quad \hat{E}_n^* | P_\lambda \rangle = \varepsilon_\lambda(e_n(x^{-1})) | P_\lambda \rangle
\]

Since the operators \(\hat{E}_n\) are diagonal in the Macdonald basis, they commute with each other, and thus they generate a ring of operators which is isomorphic to the ring of symmetric functions (see [2]).

\footnote{Note that the rational function (29) is actually \(\zeta(x^{-1})\) in the notation of (29), but either choice gives rise to the same algebra structures.}

\footnote{These operators are denoted by \(E_{q,t}\) and \(E_{q^{-1},t^{-1}}\) in [13] Ch. VII respectively.}
3. The quantum toroidal and shuffle algebras

We will now discuss the algebraic structure that governs the Fock space, and encompasses both the vertex operators $\eta(z)$, $\zeta(z)$ and the Macdonald operators $E_n$ and $E_n^*$. We will focus on two isomorphic incarnations of this algebra:

$$U_{q,t} (\hat{g}_1) \cong \mathcal{D}S$$

where the left-hand side is the quantum toroidal algebra that was featured in the Introduction, and the right-hand side is the (double extended) shuffle algebra $S$. We will find the latter incarnation to be more convenient for our computations.

3.1. The triangular decomposition. The quantum toroidal algebra of type $\mathfrak{g}_1$ (sometimes called the Ding-Iohara-Miki algebra [6, 13]) can be presented by generators and relations, although we believe that it is more enlightening for our purposes to start from its triangular decomposition:

$$U_{q,t} (\hat{g}_1) = U_{q,t}^+ (\hat{g}_1) \otimes U_{q,t}^\circ (\hat{g}_1) \otimes U_{q,t}^- (\hat{g}_1)$$

The three tensor factors above can be referred to as the “positive nilpotent”, “Cartan” and “negative nilpotent” subalgebras, by analogy with the theory of finite-dimensional Lie algebras. Moreover, the first two and the latter two factors fit together in analogues of the “Borel subalgebras”:

$$U_{q,t}^\geq (\hat{g}_1) = U_{q,t}^+ (\hat{g}_1) \otimes U_{q,t}^\circ (\hat{g}_1)$$

$$U_{q,t}^\leq (\hat{g}_1) = U_{q,t}^- (\hat{g}_1) \otimes U_{q,t}^\circ (\hat{g}_1)$$

where $\mathbb{F} = \mathbb{Q}(q,t)$. Our reason for stating the facts above before actually defining $U_{q,t} (\hat{g}_1)$ is that one can reconstruct the algebra structure from the bialgebra structures on the halves (48) and (39), plus the bialgebra pairing:

$$U_{q,t}^\geq (\hat{g}_1) \otimes U_{q,t}^\leq (\hat{g}_1) \xrightarrow{(\cdot, \cdot)} \mathbb{F}$$

that we will recall in Subsection 3.3. Thus, we find it more convenient to work backwards.

3.2. The positive halves. Recall the rational function $\zeta(x)$ from [29].

Definition 1. Consider the algebras:

$$U_{q,t}^+ (\hat{g}_1) = \mathbb{F}[e_n]_{n \geq 0} / \left( \text{relation } (43) \right)$$

$$U_{q,t}^- (\hat{g}_1) = \mathbb{F}[f_n]_{n \geq 0} / \left( \text{relation } (44) \right)$$

where $c(z) = \sum_{n \geq 0} \frac{x^n}{n!}$, $f(z) = \sum_{n \geq 0} \frac{z^n}{n!}$, and:

$$e(z)e(w)\zeta\left(\frac{z}{w}\right) = e(w)e(z)\zeta\left(\frac{w}{z}\right)$$

$$f(z)f(w)\zeta\left(\frac{w}{z}\right) = f(w)f(z)\zeta\left(\frac{z}{w}\right)$$

To make sense of the relation above, one clears the denominators of the $\zeta$ functions and then identifies the coefficients of any $z^aw^b$ in the left and right-hand sides. Note that $U_{q,t}^\geq (\hat{g}_1) = U_{q,t}^\leq (\hat{g}_1)^{op}$.

One would like $U_{q,t}^+ (\hat{g}_1)$ and $U_{q,t}^- (\hat{g}_1)$ to admit bialgebra structures, but as in the theory of finite-dimensional quantum groups, before doing so one needs to enlarge these algebras. To do so, define:

$$U_{q,t}^{\geq \pm} (\hat{g}_1) = \mathbb{F}\left[c, \psi_0^+, \psi_1^+, \psi_2^+, \ldots\right] / c \text{ and } \psi_0^+ \text{ invertible}$$

Let us form the power series $\psi^\pm (z) = \sum_{r=0}^{\infty} \frac{z^r}{r!}$, and define the “Borel” subalgebras:

$$U_{q,t}^{\geq \pm} (\hat{g}_1) = U_{q,t}^\pm (\hat{g}_1) \otimes U_{q,t}^{\circ \pm} (\hat{g}_1)$$

$$U_{q,t}^{\leq \pm} (\hat{g}_1) = U_{q,t}^\pm (\hat{g}_1) \otimes U_{q,t}^{\circ \pm} (\hat{g}_1)$$

where the multiplication in the algebras above is governed by the relations:

$$\psi^+ (z)e(w)\zeta\left(\frac{z}{w}\right) = e(w)\psi^+ (z)\zeta\left(\frac{w}{z}\right)$$
\[
\psi^-(z)f(w)\zeta\left(\frac{w}{z}\right) = f(w)\psi^-(z)\zeta\left(\frac{z}{w}\right)
\]

(49)

as well as the fact that \(c\) is central. To make sense of the relations above, expand the rational functions in the right-hand sides as a power series in \(w^{-1}\) and \(w\), respectively.

### 3.3. The non-negative halves

The extended algebras \(U_{q,t}^\geq(gl_1)\) and \(U_{q,t}^\leq(gl_1)\) admit topological coproducts, completely determined by the following formulas:

\[
\Delta(c) = c \otimes c
\]
\[
\Delta(\psi^+(w)) = \psi^+(w) \otimes \psi^+(w/e^{(1)})
\]
\[
\Delta(\psi^-(w)) = \psi^-(w/e^{(2)}) \otimes \psi^-(w)
\]
\[
\Delta(e(z)) = e(z) \otimes 1 + \psi^+(z) \otimes e(z/e^{(1)})
\]
\[
\Delta(f(z)) = 1 \otimes f(z) + f(z/e^{(2)}) \otimes \psi^-(z)
\]

(50)–(54)

where \(e^{(1)} = e \otimes 1, e^{(2)} = 1 \otimes e\). The formulas above induce bialgebra structures on \(U_{q,t}^\geq(gl_1)\) and \(U_{q,t}^\leq(gl_1)\).

The counit \(\varepsilon\) annihilates all generators \(e_n, f_n\) and \(\psi^+_{r}\), for \(r \neq 0\), and sends \(c\) and \(\psi^+_{0}\) to 1.

**Remark 1.** Note that \(U_{q,t}^\geq(gl_1)\) and \(U_{q,t}^\leq(gl_1)\) are actually Hopf algebras, and it is easy to see how to write down the antipode maps:

\[
U_{q,t}^\geq(gl_1) \xrightarrow{\Delta} U_{q,t}^\geq(gl_1), \quad U_{q,t}^\leq(gl_1) \xrightarrow{\Delta} U_{q,t}^\leq(gl_1)
\]

(55)

from (51)–(53). As we will not need the antipode in the present paper, we leave this as an exercise to the interested reader. Indeed, it is well known that the antipode is not an extra structure on a bialgebra, but a property which is determined by the product and coproduct.

**Definition 2.** A bilinear pairing:

\[
\langle \cdot, \cdot \rangle : U_{q,t}^\geq(gl_1) \otimes U_{q,t}^\leq(gl_1) \rightarrow \mathbb{F}
\]

(56)

is called a bialgebra pairing if it satisfies the properties:

\[
\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle
\]
\[
\langle aa', b \rangle = \langle a \otimes a', \Delta^{op}(b) \rangle
\]

(57)–(58)

for all \(a, a' \in U_{q,t}^\geq(gl_1)\) and all \(b, b' \in U_{q,t}^\leq(gl_1)\).

Consider the assignments:

\[
\langle e(z), f(w) \rangle = \frac{1}{\alpha} \cdot \delta\left(\frac{z}{w}\right)
\]

(59)

where:

\[
\alpha := \frac{1-q t^{-1}}{(1-q)(1-t^{-1})}
\]

(60)

and:

\[
\langle \psi^+(z), \psi^-(w) \rangle = \frac{\zeta\left(\frac{w}{z}\right)}{\zeta\left(\frac{z}{w}\right)}
\]

(61)

(the right-hand side of (61) should be expanded as a power series in \(|z| \gg |w|\)) and \(\langle e, - \rangle = \langle -, e \rangle = 1\). All other pairings between the \(e\)'s, \(f\)'s and \(\psi\)'s vanish. It is straightforward to see that the assignments above determine a bialgebra pairing, in accordance with (57)–(58). Moreover, with respect to the \(\mathbb{Z} \times \mathbb{Z}\) grading of the quantum toroidal algebra given by:

\[
\deg e_n = (1, n), \quad \deg f_n = (-1, n), \quad \deg \psi^\pm_n = (0, \pm n), \quad \deg e = (0, 0)
\]

(62)

we have \(\langle a, b \rangle \neq 0\) only if \(\deg a + \deg b = 0\).
3.4. The full algebra. The main reason for introducing all the structure above is to give the following.

**Definition 3.** The Drinfeld double of $U_{q,t}^\triangleright\triangleright(\mathfrak{g}l_1)$ and $U_{q,t}^\triangleleft\triangleleft(\mathfrak{g}l_1)$ is defined as:

$$U_{q,t}^\triangleright\triangleleft(\mathfrak{g}l_1) = U_{q,t}^\triangleright\triangleright(\mathfrak{g}l_1) \otimes U_{q,t}^\triangleleft\triangleleft(\mathfrak{g}l_1) / \{ e \otimes 1 - 1 \otimes e, \psi^\triangleright_0 \otimes 1 - 1 \otimes (\psi^\triangleright_0)^{-1} \}$$  (63)

where the multiplication is controlled by the following relation:

$$a_1 b_1 \langle a_2, b_2 \rangle = \langle a_1, b_1 \rangle b_2 a_2$$  (64)

for all $a \in U_{q,t}^\triangleright\triangleright(\mathfrak{g}l_1) = U_{q,t}^\triangleright\triangleright(\mathfrak{g}l_1) \subseteq U_{q,t}(\mathfrak{g}l_1)$ and $b \in U_{q,t}^\triangleleft\triangleleft(\mathfrak{g}l_1) = 1 \otimes U_{q,t}^\triangleleft\triangleleft(\mathfrak{g}l_1) \subseteq U_{q,t}(\mathfrak{g}l_1)$. In the formula above, we use Sweedler notation $\Delta(a) = a_1 \otimes a_2$ and $\Delta(b) = b_1 \otimes b_2$ for the coproduct, meaning that there are implied summation signs in front of the tensors and in front of the LHS and RHS of (64).

We leave it as an exercise to the interested reader that formula (64) implies the relations:

$$\psi^-(cz)e(w)\zeta\left(\frac{z}{w}\right) = e(w)\psi^-(cz)\zeta\left(\frac{w}{z}\right)$$  (65)

$$\psi^+(cz)f(w)\zeta\left(\frac{w}{z}\right) = f(w)\psi^+(cz)\zeta\left(\frac{z}{w}\right)$$  (66)

$$[e(z), f(w)] = \frac{1}{2} \left( \delta_c(zw)\psi^-(w) - \delta_c(wz)\psi^+(z) \right)$$  (67)

The currents $\psi^\pm(z)$ can be viewed as exponential generating functions:

$$\psi^\pm(z) = \psi^\pm_0 \exp \left( \sum_{n>0} \frac{1}{n} (1-q^n)(1-t^{-n})(1-q^{-n}t^n)h_{\pm n}z^n \right)$$  (68)

Then the Drinfeld double relation implies that $\{h_{\pm n}\}_{n>0}$ generate a deformed Heisenberg algebra:

$$[h_n, h_m] = [h_{-n}, h_{-m}] = 0 \quad \text{and} \quad [h_n, h_{-m}] = \delta_{m,n} \frac{n(e^n - e^{-n})}{(1-q^n)(1-t^{-n})(q^{-n}t^n - 1)}$$  (69)

for all $m, n > 0$. Combining all the quadratic relations in the present Section, specifically (43, 44, 48), (45), (46), (47), (49), (50), (51) gives us the usual generators and relations of the quantum toroidal algebra.

**Remark 2.** In order for $U_{q,t}(\mathfrak{g}l_1)$ defined as in (63) to be an algebra, we need to make sense of products:

$$ab', a''b''' \ldots$$

for various $a, a', a'', \ldots \in U_{q,t}^\triangleright\triangleright(\mathfrak{g}l_1) \subseteq U_{q,t}(\mathfrak{g}l_1)$ and $b, b', b''$, $\ldots \in U_{q,t}^\triangleleft\triangleleft(\mathfrak{g}l_1) \subseteq U_{q,t}(\mathfrak{g}l_1)$. In order for such a product to unambiguously describe an element of (63), we need a way to convert products of the form $ab$ to linear combinations of products of the form $ba$, and vice versa. This is done by the following formula, which is equivalent to (64) by the properties of the antipode map (55):

$$ab = \langle a_1, b_1 \rangle b_2 a_2 \langle a_3, S(b_3) \rangle$$  (70)

where the right-hand side involves the Sweedler notation for the iterated coproduct: $\Delta^{(2)}(a) = a_1 \otimes a_2 \otimes a_3$ and $\Delta^{(2)}(b) = b_1 \otimes b_2 \otimes b_3$. The interested reader may check that there exists an antipode map (it is uniquely determined by the coproduct formulas) which satisfies all the Hopf algebra axioms. The reason we choose to not write down the antipode map explicitly is that formula (64) leads to more elegant formulas than the equivalent formulas (70) in the case of the quantum toroidal algebra.

3.5. The action I. Let $u \in \mathbb{C}^*$ be the spectral parameter. It is straightforward to check that there is an action $[6, 26]$:  

$$U_{q,t}(\mathfrak{g}l_1) \sim \Lambda^F(u)$$  (71)

generated by the assignments:

$$c \mapsto (q/t)^{1/2}$$  (72)

$$\psi^\triangleright_0 \mapsto 1$$  (73)

$$e(z) \mapsto u \xi(z)$$  (74)

$$f(z) \mapsto u^{-1} \eta(z)$$  (75)

$$\psi^+(z) \mapsto \exp \left( -\sum_{r=1}^{\infty} \frac{1}{r} (1-t^r)(1-q^{-r}t^r)(q/t)^{r/2}a_rz^{-r} \right)$$  (76)
\[ \psi^{-}(z) \mapsto \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} (1 - t^{-r})(1 - q^{-r} t^r) a_{-r} z^r \right) \]  

(77)

Indeed, all one needs to show is that the formulas above satisfy relations (43), (44), (48), (49), (65), (66), (67), (69), and we leave this as an exercise to the interested reader.

3.6. The universal \( R \)-matrix. Given a Hopf algebra \( A \), an element \( R \in A \otimes A \) which satisfies:

\[
\begin{align*}
\mathcal{R} \Delta (g) &= \Delta^{op}(g) \mathcal{R}, \quad \forall g \in A \\
(\Delta \otimes 1) \mathcal{R} &= \mathcal{R}_{1,3} \mathcal{R}_{2,3} \\
(1 \otimes \Delta) \mathcal{R} &= \mathcal{R}_{1,3} \mathcal{R}_{1,2}
\end{align*}
\]

(78) (79) (80)

is called a universal \( R \)-matrix. As a consequence of (66), (69) and (80) we have the Yang–Baxter equation:

\[ \mathcal{R}_{1,2} \mathcal{R}_{1,3} \mathcal{R}_{2,3} = \mathcal{R}_{2,3} \mathcal{R}_{1,3} \mathcal{R}_{1,2} \]  

(81)

Drinfeld doubles such as (63) admit universal \( R \)-matrices:

\[ \mathcal{R} \in U_{q,t}(\mathfrak{gl}_1) \otimes U_{q,t}(\mathfrak{gl}_1) \]  

(82)

(the completed tensor product is necessary because the coproduct (50)–(54) is topological, i.e. its values are infinite sums) which are none other than the canonical tensors of the pairing (56):

\[ \mathcal{R} = \sum_i a_i \otimes b^i \]  

(83)

where \( \{a_i\} \) and \( \{b^i\} \) are orthogonal bases of the subalgebras \( U_{q,t}^+(\mathfrak{gl}_1) \) and \( U_{q,t}^-(\mathfrak{gl}_1) \). If these subalgebras were finite-dimensional and the pairing were non-degenerate, then (83) would be completely well-defined. However, both of the aforementioned properties fail for the quantum toroidal algebra \( U_{q,t}(\mathfrak{gl}_1) \). The way to fix this failure is to note that the restriction of the pairing (59):

\[ \langle \cdot, \cdot \rangle : U_{q,t}^+(\mathfrak{gl}_1) \otimes U_{q,t}^-(\mathfrak{gl}_1) \longrightarrow \mathbb{F} \]  

(84)

is indeed non-degenerate, and that the failure of non-degeneracy is simply due to the central elements \( \psi_0 \) being in the kernel of the pairing. The way to fix this is to add central elements \( d \) and \( d^\perp \) to the quantum toroidal algebra, and extend the Hopf pairing in such a way that \( \{d, d^\perp\} \) are orthogonal to \( \{\log(c), \log(\psi_0)\} \). On this extended algebra, the pairing becomes non-degenerate and we have:

\[ \mathcal{R} = \tilde{\mathcal{R}} \mathcal{K} \]  

(85)

where we separate the “Cartan part” of the \( R \)-matrix:

\[ \mathcal{K} = \exp \left( \sum_{r=1}^{\infty} \frac{(1 - q^{-r})(1 - t^{-r})(1 - q^{-r} t^r)}{r} (b_r \otimes h_{-r}) \right) e^{\log(c) \otimes d + d \otimes \log(c) + \log(\psi_0^-) \otimes d^\perp + d^\perp \otimes \log(\psi_0^-)} \]  

(86)

from the canonical tensor of the restricted pairing (83), namely:

\[ \tilde{\mathcal{R}} = \sum_i a_i \otimes a^i = 1 \otimes 1 + \alpha \sum_{n \in \mathbb{Z}} e_n \otimes f_{-n} + \text{higher terms} \]  

(87)

where \( a_i \) and \( a^i \) run over dual bases of \( U_{q,t}^+(\mathfrak{gl}_1) \) and \( U_{q,t}^-(\mathfrak{gl}_1) \). In the formula above, we have for all \( i \):

\[ \deg a_i + \deg a^i = 0 \]  

(88)

where deg is the \( \mathbb{Z} \times \mathbb{Z} \) grading on \( U_{q,t}(\mathfrak{gl}_1) \) which is completely determined by \( \deg e_n = (1,n) \) and \( \deg f_n = (-1,n) \) and multiplicativity (see (62)). Relation (88) is due to the fact that the pairing (59), (61) only pairs non-trivially elements of opposite degrees. In the present paper, we will study the object (87), and slightly abusively refer to it as “the universal \( R \)-matrix of the quantum toroidal algebra”. 
3.7. The shuffle algebra. In the previous Subsections, we discussed the definition of the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_1)$, as well as its Hopf algebra structure and universal $R$-matrix. We saw that the quantum toroidal algebra acts on Fock space (via the vertex operators $\xi(z)$ and $\eta(z)$), but one thing which wasn’t clear is how to incorporate the Macdonald operators $\hat{E}_n$ and $\hat{E}_n^+$ into this algebra action. Moreover, while [17] gives a recipe for defining the universal $R$-matrix, it is by no means an explicit formula. To remedy these issues, we will introduce an algebraic structure called the shuffle algebra, which will be isomorphic to the positive half of the quantum toroidal algebra:

$$S \cong U_{q,t}^+(\mathfrak{gl}_1)$$

However, while $U_{q,t}^+(\mathfrak{gl}_1)$ is defined by generators and relations, $S$ is completely explicit, and also leads to explicit formulas. To define the shuffle algebra, start from the vector space of symmetric polynomials in arbitrarily many variables:

$$V = \bigoplus_{n=0}^{\infty} \mathbb{F}(z_1, \ldots, z_n)^\text{symmetric}$$

Let us make $V$ into an algebra via the so-called shuffle product:

$$F(z_1, \ldots, z_n) \ast F'(z_1, \ldots, z_{n'}) = \text{Sym} \left[ \frac{F(z_1, \ldots, z_n)F'(z_{n+1}, \ldots, z_n+n')}{n!n'!} \prod_{1 \leq i,j \leq n} \zeta \left( \frac{z_i}{z_j} \right) \right]$$

where Sym refers to the sum over all $(n + n')!$ permutations of the variables. It is a straightforward exercise to check that $V$ is an associative algebra with unit the function $1$ in zero variables.

**Definition 4.** The shuffle algebra is the subalgebra $S \subset V$ generated by:

$$\{z^n\}_{n \in \mathbb{Z}}$$

It was shown in [17] that $S$ coincides with the subset of $V$ of rational functions of the form:

$$F(z_1, \ldots, z_n) = f(z_1, \ldots, z_n) \prod_{1 \leq i,j \leq n} \frac{1 - \frac{z_i}{z_j}}{1 - \frac{z_i}{z_j} (1 - \frac{z_i}{z_j})}$$

where $f$ is a symmetric Laurent polynomial which satisfies the so-called wheel conditions of [4]:

$$f \left( \frac{xq}{t}, xq, x, z_4, \ldots, z_n \right) = f \left( \frac{xq}{t}, x, z_4, \ldots, z_n, \frac{xq}{t} \right) = 0$$

As such, it is very easy to note that the following explicit rational functions lie in $S$:

$$F_n = a^n \prod_{1 \leq i,j \leq n} \zeta \left( \frac{z_i}{z_j} \right) = a^n \prod_{1 \leq i,j \leq n} \left( \frac{z_i-z_j}{z_i} \right) \left( z_i-\frac{qt^{-1}z_j}{z_i} \right)$$

It was shown in [4] that the rational functions $F_n$ commute with each other, and thus generate a commutative subalgebra of the shuffle algebra:

$$\mathbb{F}[F_1, F_2, \ldots] \subset S$$

The connection between the shuffle algebra and the quantum toroidal algebra is the following result.

**Proposition 1.** The maps $f_n \mapsto z^n$ and $e_n \mapsto z_i^n$ extend to isomorphisms:

$$U_{q,t}^- (\mathfrak{sl}_1) \xrightarrow{\sim} S \quad \text{and} \quad U_{q,t}^+ (\mathfrak{gl}_1) \xrightarrow{\sim} S^\text{op}$$

respectively (above and henceforth, $S^\text{op}$ refers to $S$ with the opposite algebra structure).

3.8. The coproduct on the shuffle algebra. The Hopf algebra structure on $U_{q,t}(\mathfrak{gl}_1)$ can be presented in terms of the shuffle algebra (see [17], or [20] for a more recent review in a related language to ours). Explicitly, first consider the extended shuffle algebras:

$$S^\leq = S \otimes U_{q,t}^- (\mathfrak{gl}_1) \underbrace{\text{\,}/\text{relation}}_{(92)}$$

$$S^\geq = S^\text{op} \otimes U_{q,t}^+ (\mathfrak{gl}_1) \underbrace{\text{\,}/\text{relation}}_{(100)}$$

$$S^\text{op} = S^\text{op} \otimes U_{q,t}^+ (\mathfrak{gl}_1) \underbrace{\text{\,}/\text{relation}}_{(100)}$$

$$S^\leq = S \otimes U_{q,t}^- (\mathfrak{gl}_1) \underbrace{\text{\,}/\text{relation}}_{(92)}$$

$$S^\geq = S^\text{op} \otimes U_{q,t}^+ (\mathfrak{gl}_1) \underbrace{\text{\,}/\text{relation}}_{(100)}$$
the following coproduct formulas:

\[
F(z_1, \ldots, z_n)\psi^-(w) = \psi^-(w)F(z_1, \ldots, z_n)\prod_{i=1}^{n} \zeta\left(\frac{z_i}{w}\right)
\]

(99)

\[
G(z_1, \ldots, z_n)\psi^+(w) = \psi^+(w)G(z_1, \ldots, z_n)\prod_{i=1}^{n} \zeta\left(\frac{w}{z_i}\right)
\]

(100)

for all \(F \in \mathcal{S}\) and \(G \in \mathcal{S}^{op}\). The right-hand sides of the formulas above are expanded as power series in \(w\) in the same direction as \(\psi^\pm(w)\). By comparing \(\text{(100)} - \text{(99)}\) with \(\text{(46)} - \text{(47)}\), we can see that the isomorphisms of Proposition \(\text{I}\) extend to isomorphisms:

\[
\mathcal{S}^< \xrightarrow{\text{U}_{q,t}} U_{q,t}(\mathfrak{gl}_1) \quad \text{and} \quad \mathcal{S}^> \xrightarrow{\text{U}_{q,t}} U_{q,t}(\mathfrak{gl}_1)
\]

(101)

The reason for the extended algebras \(\mathcal{S}^>\) and \(\mathcal{S}^<\) is that they admit topological bialgebra structures via the following coproduct formulas:

\[
\Delta(c) = c \otimes c
\]

(102)

\[
\Delta(\psi^+(w)) = \psi^+(w) \otimes \psi^+(w/c(1))
\]

(103)

\[
\Delta(\psi^-(w)) = \psi^-(w/c(2)) \otimes \psi^-(w)
\]

(104)

\[
\Delta(F) = \sum_{k=0}^{n} F(z_1c(2), \ldots, z_k c(2) \otimes z_{k+1}, \ldots, z_n) \cdot \prod_{i=1}^{k} \psi^-(z_i) \prod_{i=k+1}^{n} \zeta\left(\frac{z_i}{z_j}\right)
\]

(105)

\[
\Delta(G) = \sum_{k=0}^{n} \prod_{j=k+1}^{n} \psi^+(z_j) \cdot G(z_1, \ldots, z_k \otimes z_{k+1} c(1), \ldots, z_n c(1)) \prod_{i=1}^{k} \zeta\left(\frac{z_i}{z_j}\right)
\]

(106)

for any \(F(z_1, \ldots, z_n) \in \mathcal{S}\) and any \(G(z_1, \ldots, z_n) \in \mathcal{S}^{op}\). To make sense of the right-hand side of \(\text{(105)}\) and \(\text{(106)}\), one expands the rational function as a power series in \(|z_i| < |z_j|\) (for all \(1 \leq i \leq k, k < j \leq n\)) and in every monomial of the resulting expression places all powers of \(z_1, \ldots, z_k\) to the left of the \(\otimes\) sign, and all powers of \(z_{k+1}, \ldots, z_n\) to the right of the \(\otimes\) sign. It is easy to see that the coproducts \(\text{(103)} - \text{(106)}\) match the coproducts \(\text{(50)} - \text{(54)}\) under the isomorphisms \(\text{(101)}\).

3.9. The action II. With Proposition \(\text{I}\) in mind, it should come to no surprise that one can present the action of the quantum toroidal algebra on Fock space in the language of the shuffle algebra (this was observed in \[4\]). Specifically, composing the isomorphisms \(\text{(96)}\) with the action \(\text{(71)}\) implies that:

\[
F \in \mathcal{S} \quad \text{acts on } \Lambda_F \text{ via } \frac{1}{n!} \int_{C_n} \prod_{i=1}^{n} \frac{dz_i}{2\pi i z_i} F(z_1, \ldots, z_n) : \eta(z_1) \cdots \eta(z_n) :
\]

(107)

\[
G \in \mathcal{S}^{op} \quad \text{acts on } \Lambda_F \text{ via } \frac{1}{n!} \int_{C_n} \prod_{i=1}^{n} \frac{dz_i}{2\pi i z_i} G(z_1, \ldots, z_n) : \xi(z_1) \cdots \xi(z_n) :
\]

(108)

where \(C_n\) is the contour given by \(|z_1| = \cdots = |z_n| = 1\). The formulas above allow one to motivate the shuffle algebra as the abstract structure which governs the composition of the vertex operators \(\eta(x)\) and \(\xi(x)\), respectively. When \(F = F_n\), the formulas above lead to a family of Macdonald operators similar to those in \(\text{(83)} - \text{(84)}\):

\[
\hat{F}_n := \frac{1}{n!} \int_{C_n} \prod_{i=1}^{[|q|<1]} \frac{dz_i}{2\pi i z_i} F_n(z_1, \ldots, z_n) : \eta(z_1) \cdots \eta(z_n) :
\]

(109)

\[
\hat{F}_n^* := \frac{1}{n!} \int_{C_n} \prod_{i=1}^{[|q|>1]} \frac{dz_i}{2\pi i z_i} F_n(z_1, \ldots, z_n) : \xi(z_1) \cdots \xi(z_n) :
\]

(110)

where the superscript on the integral sign denotes the assumption on the sizes of the parameters \(q\) and \(t\) that must be made in order to evaluate the contour integral. It is convenient to express the action of
these operators in the Macdonald basis using their generating functions:

\[ \hat{F}(v) := \sum_{n=0}^{\infty} v^{-n} \hat{F}_n, \quad \hat{F}^\ast(v) := \sum_{n=0}^{\infty} (-v)^n \hat{F}^\ast_n \]  

(111)

In addition we define two functions:

\[ N(v) = \exp \left( \sum_{r>0} \frac{1}{r} \frac{1 - q^r t^{-r}}{(1 - q^r)(1 - t^{-r})} v^{-r} \right) \]  

(112)

\[ N^\ast(v) = \exp \left( \sum_{r>0} \frac{1}{r} \frac{1 - q^r t^r}{(1 - q^r)(1 - t^r)} v^r \right) \]  

(113)

where we suppose \(|q|, |t^{-1}|, |v^{-1}| < 1\) in (112) and \(|q|, |t^{-1}|, |v^{-1}| > 1\) in (113).

**Lemma 1.** The action of \(\hat{F}(v)\) and \(\hat{F}^\ast(v)\) in the Macdonald basis is given by:

\[ \hat{F}(v) |P_\lambda\rangle = f_\lambda(v) |P_\lambda\rangle, \quad \hat{F}^\ast(v) |P_\lambda\rangle = f_\lambda^\ast(v) |P_\lambda\rangle \]  

(114)

and the eigenvalues \(f_\lambda(v), f_\lambda^\ast(v)\) are defined by:

\[ f_\lambda(v) := N(v) \prod_{(i,j) \in \Lambda} \frac{1 - q^{-t^{-i+1}q^{j-1}}}{1 - v^{-t^{-i}q^j}} \]  

(115)

\[ f_\lambda^\ast(v) := N^\ast(v) \prod_{(i,j) \in \Lambda} \frac{1 - v^{-t^{-i}q^{j+1}}}{1 - v^{-t^j}q^{-j}} \]  

(116)

**Proof.** We will prove the formula for \(\hat{F}(v)\), as the one for \(\hat{F}^\ast(v)\) is analogous. The eigenvalue \(f_\lambda(v)\) can be written in the exponential form:

\[ f_\lambda(v) = \exp \left( \sum_{r>0} \frac{1}{r} \frac{1 - q^r t^{-r}}{1 - q^r} v^{-r} t^r \varepsilon_\lambda(p_r(x)) \right) \]  

(117)

which can be shown with the help of the identity:

\[ \frac{1}{1 - q^r} \varepsilon_\lambda(p_r(x)) = -q^{-t^{-r}} \sum_{i=1}^{\ell(\lambda)} \lambda_i \sum_{j=1}^{\lambda_i} q^{ir} t^{-ir} + \frac{t^{-r}}{(1 - t^{-r})(1 - q^r)} \]  

(118)

Let us define:

\[ F(v) = \sum_{n=0}^{\infty} v^{-n} F_n \]  

(119)

\[ E(v) = \sum_{n=0}^{\infty} v^{-n} \frac{t^{-n(n+1)/2}}{(1 - t^{-1})^n} \prod_{1 \leq i \neq j \leq n} \frac{(z_i - z_j)}{(z_i - t^{-1}z_j)} \]  

(120)

as elements of \(\mathcal{S}[[v^{-1}]]\). It was proved in [18] Subsection 6.5 that there exist elements \(P_n \in \mathcal{S}\) such that:

\[ F(v) = \exp \left( \sum_{r=1}^{\infty} \frac{1 - q^r t^{-r}}{(1 - q^r)(1 - t^{-r})} P_r v^{-r} \right) \]  

(121)

\[ E(v) = \exp \left( \sum_{r=1}^{\infty} \frac{-1}{t^{-r} - 1} P_r v^{-r} \right) \]  

(122)

Since formula (117) yields an action, we conclude that there exist operators \(\hat{P}_n\) on \(\Lambda_T\) such that formulas (121) and (122) also hold with hats above \(F, E\) and \(P\), where:

\[ \hat{E}(v) = \sum_{r=0}^{\infty} v^{-r} \hat{E}_n \]  

However, formula (115) then implies:

\[ \hat{P}_n |P_\lambda\rangle = \varepsilon_\lambda ((t^n - 1)p_n(x)) |P_\lambda\rangle \]  

Combining this with the version of (121) with hats on top of \(F\) and \(P\) yields precisely (114).
3.10. **Shuffle algebra formulas for the R-matrix.** The shuffle algebra language yields a description of the universal \( R \)-matrix in the Fock representation. Specifically, it was shown in [20, Theorem 4.16] that the image of \( \mathcal{R} \) under the action map:

\[
U_{q,t}^{\pm} \hat{g}_l \otimes U_{q,t}^{-\pm} \hat{g}_l \longrightarrow \text{End}(\Lambda_T(u)) \otimes U_{q,t}^{-\pm} \hat{g}_l \cong \text{End}(\Lambda_T(u)) \otimes \mathcal{S}
\]

is given by the formula:

\[
\sum_{n=0}^{\infty} \frac{(u_0)^n}{n!} \int \prod_{i=1}^{n} \frac{dw_i}{2\pi i w_i} : \xi(w_1) \ldots \xi(w_n) : \otimes S(w_1, \ldots, w_n)
\]

where the formal series \( S(w_1, \ldots, w_n) \in S[[w_1^{\pm 1}, \ldots, w_n^{\pm 1}]] \) is defined by:

\[
S(w_1, \ldots, w_n) = \sum_{d_1, \ldots, d_n \in \mathbb{Z}} w_1^{d_1} \ldots w_n^{d_n} \text{Sym} \left[ z_1^{d_1} \ldots z_n^{d_n} \right] \prod_{1 \leq i \neq j \leq n} \zeta \left( \frac{z_i}{z_j} \right)
\]

We may then apply formula (107) to conclude that the formal series \( S(w_1, \ldots, w_n) \) acts on \( \Lambda_T(u) \) as:

\[
u^{-n} : \eta(w_1) \ldots \eta(w_n) : = \prod_{1 \leq i \neq j \leq n} \zeta \left( \frac{w_i}{w_j} \right)
\]

Combining this with (123) we can compute \( \mathcal{R} \) in a tensor product of Fock representations. Explicitly, let \( R(u_2/u_1) \) be the image of the universal \( R \)-matrix \( \mathcal{R} \) under the action map:

\[
U_{q,t}^{\pm} \hat{g}_l \otimes U_{q,t}^{-\pm} \hat{g}_l \longrightarrow \text{End}(\Lambda_T(u_1)) \otimes \text{End}(\Lambda_T(u_2)), \quad \mathcal{R} \sim R(u_2/u_1)
\]

We denote by \( R(u_2/u_1) \) and \( K \) the corresponding images of \( \mathcal{R} \) and \( K \). From now on we will assume \(|q| < 1 < |t|\) and \(|u| > 1\).

**Proposition 2.** The matrix \( R(u) \) is given by:

\[
R(u) = \hat{R}(u)K
\]

where \( \hat{R}(u) = \sum_{n=0}^{\infty} u^{-n} \hat{R}_n \) is given by the formula:

\[
\hat{R}_n = \frac{\alpha^n}{n!} \int \prod_{i=1}^{n} \frac{dw_i}{2\pi i w_i} : \xi(w_1) \ldots \xi(w_n) : \otimes \eta(w_1) \ldots \eta(w_n) : \prod_{1 \leq i \neq j \leq n} \zeta \left( \frac{w_i}{w_j} \right)
\]

and:

\[
K = \exp \left( \sum_{r>0} \frac{1}{r} \left( 1 - t^r \right) \left( 1 - q^r t^{-r} \right) \frac{(t/q)^{r/2} a_r \otimes a_{-r}}{1 - q^r} \right) (q/t)^{r(d \otimes 1 + 1 \otimes d)}
\]

We define a version of the path ordered exponential:

\[
\mathcal{P} \exp \left( \int \frac{dw}{2\pi i w} A(w)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{dw_i}{2\pi i w_i} A(w_1) \ldots A(w_n)
\]

**Remark 3.** \( \hat{R}(u) \) is a path ordered exponential:

\[
\hat{R}(u) = \mathcal{P} \exp \left( \frac{\alpha}{u} \int \frac{dw}{2\pi i w} \xi(w) \otimes \eta(w) \right)
\]

**Proof.** Using (24) and (25) we can remove the rational function from the integrand in \( \hat{R}_n \) in (123) together with the normal ordering:

\[
\hat{R}_n = \frac{\alpha^n}{n!} \int \prod_{i=1}^{n} \frac{dw_i}{2\pi i w_i} \xi(w_1) \ldots \xi(w_n) \otimes \eta(w_1) \ldots \eta(w_n)
\]

Then we use the definition of \( \mathcal{P} \exp \) and arrive at (131).

**Remark 4.** The partial vacuum-vacuum expectations and the vacuum-vacuum expectations of \( \hat{R}(u) \) read:

\[
\langle \langle \omega | \otimes \varnothing \rangle \rangle \hat{R}(u) \langle | \varnothing \rangle \otimes \varnothing \rangle = \hat{F}(u)
\]

\[
\langle \langle \omega | \otimes \langle \omega \rangle \rangle \rangle \hat{R}(u) \langle | \varnothing \rangle \otimes \langle \omega \rangle \rangle = N(u)
\]

which is an immediate consequence of (128). We should note that (132) was derived in [5] by using commutation relations of the quantum toroidal algebra.
3.11. The \( \hat{R} \)-matrix in terms of bosonic operators. We focus on the \( \hat{R}(u) \) and its expression in the Heisenberg basis \([18]\). We can write:

\[
\hat{R}(u) = \sum_{\mu,\nu,\rho} [\hat{R}(u)]_{\mu,\rho}^{\nu,\sigma} a_{-\mu} a_{\nu} \otimes a_{-\rho} a_{\sigma}
\]

(135)

and

\[
[\hat{R}(u)]_{\mu,\rho}^{\nu,\sigma}(u) = 0, \quad |\mu| + |\rho| \neq |\nu| + |\sigma|
\]

(136)

The vanishing condition (136) follows from (88). We define the normalized Heisenberg basis (18). We can write:

\[
\text{Definition 5. We define the normalized } R \text{-matrix:}
\]

\[
R(u) := \frac{1}{N(u)} R(u)
\]

(134)

Recalling the explicit form of \( \xi \) and \( \eta \) we have:

\[
\hat{F}_n = \frac{1}{n!} \sum_{c_1, \ldots, c_n} \int d(z_1, \ldots, z_n) \exp \left( \sum_{r>0} \frac{1-t^{-r}}{r} c_{-r} p_r(z) \right) \exp \left( \sum_{r>0} \frac{1-t^{-r}}{r} c_{r} p_r(z^{-1}) \right)
\]

(140)

Clearly the expansion coefficients of \( \hat{R}(u) \) in \( c_{-\mu} c_{\nu} \) coincide with the expansion coefficients of \( \hat{F}(u) \) in \( a_{-\mu} a_{\nu} \). If we denote these expansion coefficients by \( X_{\mu,\nu}(u) \), then we have:

\[
\hat{R}(u) = \sum_{\mu,\nu,\sigma} X_{\mu,\nu}(u) c_{-\mu} c_{\nu}, \quad \hat{F}(u) = \sum_{\mu,\nu,\sigma} X_{\mu,\nu}(u) a_{-\mu} a_{\nu}
\]

(142)

Clearly the expansion coefficients of \( \hat{R}(u) \) in \( c_{-\mu} c_{\nu} \) coincide with the expansion coefficients of \( \hat{F}(u) \) in \( a_{-\mu} a_{\nu} \). If we denote these expansion coefficients by \( X_{\mu,\nu}(u) \), then we have:

\[
\hat{R}(u) = \sum_{\mu,\nu,\sigma} X_{\mu,\nu}(u) c_{-\mu} c_{\nu}, \quad \hat{F}(u) = \sum_{\mu,\nu,\sigma} X_{\mu,\nu}(u) a_{-\mu} a_{\nu}
\]

(142)

Now we calculate \( X_{\mu,\nu}(u) \). Define the matrix elements \( f_{\alpha,\beta}(u) \) of \( \hat{F}(u) \) in the Heisenberg basis:

\[
f_{\alpha,\beta}(u) := \langle a_{\alpha} | \hat{F}(u) | a_{\beta} \rangle
\]

(143)

Recall the coefficients \( T_{\alpha,\beta}^{\mu,\nu} \) and \( (T^{-1})_{\alpha,\beta}^{\mu,\nu} \) from \([21]\). The connection between \( f_{\alpha,\beta}(u) \) and \( X_{\mu,\nu}(u) \) is:

\[
f_{\alpha,\beta}(u) = \sum_{\mu,\nu,\sigma} T_{\alpha,\beta}^{\mu,\nu} X_{\mu,\nu}(u), \quad X_{\mu,\nu}(u) = \sum_{\alpha,\beta} (T^{-1})_{\alpha,\beta}^{\mu,\nu} f_{\alpha,\beta}(u)
\]

(144)

The coefficients \( f_{\alpha,\beta}(u) \) are computed by writing \( \hat{F}(u) \) in its eigenbasis of Macdonald polynomials:

\[
\hat{F}(u) = \sum_{\lambda \in P} f_{\lambda}(u) |Q_{\lambda}\rangle \langle P_{\lambda}|
\]

(145)

Let us insert (145) into (143) and use the transition coefficients \( g_{\alpha,\beta}(q,t) \) from (15) and the scalar product (19). We obtain:

\[
f_{\alpha,\beta}(u) = \sum_{\lambda \in P} f_{\lambda}(u) \langle a_{\alpha} | Q_{\lambda} \rangle \langle P_{\lambda} | a_{\beta} \rangle = \sum_{\lambda \in P} \sum_{\mu,\nu} b_{\lambda}(q,t) f_{\lambda}(u) g_{\lambda,\mu}(q,t) g_{\lambda,\nu}(q,t) \langle a_{\alpha} | a_{\mu} \rangle \langle a_{\nu} | a_{\beta} \rangle
\]

(145)
Substituting the formula above into the second equation of (144), and using (21) to express \((T^{-1})_{\alpha,\beta}^{\mu,\nu}\), yields:

\[
X_{\mu,\nu}(u) = \sum_{\alpha,\beta,\gamma,\delta} \delta_{\mu,\alpha,\beta,\gamma,\delta} \left[ \frac{1}{\alpha} g_{\lambda,\alpha}(q, t) g_{\lambda,\beta}(q, t) z_{\alpha}(q, t) z_{\beta}(q, t) \right] \sum_{\lambda \in \mathcal{P}} b_{\lambda}(q, t) f_{\lambda}(u) \sum_{\sigma,\rho,\kappa,\nu} \left[ (\alpha + \beta - \sigma) \right] \left[ (\lambda + \gamma - \rho) \right] \left[ (\mu + \nu - \kappa) \right] \left[ (\delta - \kappa) \right] \left[ (\delta - \rho) \right] (146)
\]

Since the operator \(K\) in (127) does not depend on the spectral parameter \(u\), Proposition 3 shows that the dependence on \(u\) of \(R(u)\) is given by \(f_{\lambda}(u)\) which is written explicitly in (115). For the normalized matrix \(R(u)\) the spectral parameter dependence is given by the rational functions \(N(u)^{-1} f_{\lambda}(u)\) and we conclude:

**Corollary 1.** The operator \(R(u)\) only has simple poles, and they are located at \(u = q^i t^{-j} \). If we are interested in the restriction of \(R(u)\) to the degree \(\leq N\) part of \(\Lambda_2\), then the only poles we encounter are \(u = q^i t^{-j} \) with \((i, j)\) among the boxes of partitions of weight \(\leq N\).

### 3.12. Computing the matrix elements of \(R(u)\)

We derive an explicit formula for the matrix elements of the normalized matrix \(R(u)\). This formula uses Proposition 3 and therefore the final answer depends on the transition coefficients \(g_{\mu,\nu}(q, t)\) between the Macdonald functions and the power sums. We recall the normalized vectors of the Fock space (23) and define the matrix elements:

\[
R_{\alpha,\beta}^{\gamma,\delta}(u) := \langle \langle \alpha, \beta | R(u) | \gamma, \delta \rangle \rangle
\]

where \(\langle \langle \alpha, \beta | \otimes \langle \langle \beta | \rangle \rangle\). Similarly to (150) these matrix elements satisfy the vanishing condition:

\[
R_{\alpha,\beta}^{\gamma,\delta}(u) = 0, \quad |\alpha| + |\beta| \neq |\gamma| + |\delta|
\]

Due to (51) the matrix elements \(R_{\alpha,\beta}^{\gamma,\delta}(u)\) satisfy the Yang–Baxter equation:

\[
\sum_{a,b,c \in \mathcal{P}} R_{\alpha,\beta}^{a,b}(u_2/u_1) R_{\alpha,\gamma}^{b,c}(u_3/u_1) R_{\alpha,\delta}^{c,a}(u_2/u_1) = \sum_{a,b,c \in \mathcal{P}} R_{\alpha,\beta}^{a,b}(u_2/u_1) R_{\alpha,\gamma}^{b,c}(u_3/u_2) R_{\alpha,\delta}^{c,a}(u_2/u_1)
\]

for all fixed external indices \(\alpha, \alpha', \alpha''\) and \(\beta, \beta', \beta''\) (the summations over partitions \(a, b, c\) on both sides of (149) are finite due to (151)).

**Proposition 4.** Let \(\alpha, \beta, \gamma, \delta\) be partitions. The matrix elements of \(R(u)\) are given explicitly by:

\[
R_{\alpha,\beta}^{\gamma,\delta}(u) = \sum_{\lambda \in \mathcal{P}} b_{\lambda}(q, t) \prod_{(i,j) \in \lambda} \frac{1 - u^{i-1} q^{j+1}}{1 - u^{i} q^{-j}} \sum_{\sigma,\rho,\kappa,\nu} g_{\lambda,\alpha,\beta,\gamma,\delta}(q, t) C_{\alpha,\beta}^{\gamma,\delta}(\sigma)
\]

where:

\[
C_{\alpha,\beta}^{\gamma,\delta}(\sigma) = (-1)^{\ell(\alpha + \gamma + \delta)} (q/t)^{\beta + \gamma + \delta} z_{(\alpha + \gamma + \delta)}(q, t) \sum_{k, \kappa, \rho} (q/t)^{-|\kappa|} \left( \frac{\delta}{\sigma - \kappa} \right) \left( \frac{\beta - \sigma}{\beta - \rho} \right)
\]

**Proof.** From (131) and (127) we have \(R(u) = N(u)^{-1} \tilde{R}(u) K\). Denote the matrix elements of \(\tilde{R}(u)\) and \(K\) by:

\[
\tilde{R}_{\alpha,\beta}^{\gamma,\delta}(u) := \langle \langle \alpha, \beta | \tilde{R}(u) | \gamma, \delta \rangle \rangle, \quad K_{\alpha,\beta}^{\gamma,\delta} := \langle \langle \alpha, \beta | K | \gamma, \delta \rangle \rangle
\]

We compute \(\tilde{R}(u)\) starting from (122). Recall the definition of the operators \(c_{\pm \alpha}\) and then evaluate \(\langle \langle \alpha, \beta | c_{-\mu} c_{\nu} | \gamma, \delta \rangle \rangle\) using (23) and (21):

\[
\langle \langle \alpha, \beta | c_{-\mu} c_{\nu} | \gamma, \delta \rangle \rangle = \langle \langle \alpha, \beta | \prod_{\rho \in \mu} (1 \otimes a_{-r} - (q/t)^{-r/2} a_{-r} \otimes 1) \prod_{\rho \in \nu} (1 \otimes a_{r} - (q/t)^{-r/2} a_{r} \otimes 1) | \gamma, \delta \rangle \rangle
\]

15
After writing the coefficients $T$ explicitly (21) and using:

$$z_{\nu}(q, t) = z_{\nu - \rho}(q, t)z_{\rho}(q, t)$$

we get:

$$R_{\alpha, \beta}^\gamma(u) = \sum_{\mu, \nu \in \mathcal{P}} X_{\mu, \nu}(u) \sum_{\sigma \leq \mu, \rho \leq \nu} \frac{(-1)^{(\mu + \nu - (\sigma + \rho))}(q/t - \frac{1}{2}|\mu - \nu - \rho|)(\mu |\sigma| + \nu |\rho|)}{z_{\alpha}(q, t)z_{\beta}(q, t)} T_{\gamma, \sigma}^{\alpha - \nu, \rho} T_{\beta, \rho}^\sigma$$

$$= (-1)^{(\alpha + \gamma)} \sum_{\mu \in \alpha + \beta, \nu \in \gamma + \delta} z_{\mu}(q, t)X_{\mu, \nu}(u)$$

The Kronecker delta $\delta_{\beta - \sigma, \delta - \rho}$ allows us to fix $\rho = \delta - (\beta - \sigma)$ while the other Kronecker delta can be simplified:

$$R_{\alpha, \beta}^\gamma(u) = (-1)^{(\alpha + \gamma)}(q/t)^{\frac{1}{2}|(\alpha + \gamma)| - |\beta|}\sum_{\mu \in \alpha + \beta, \nu \in \gamma + \delta} z_{\mu}(q, t)X_{\mu, \nu}(u)$$

Next we “reverse” the summations over $\mu$ and $\nu$ by replacing everywhere $\mu \rightarrow (\alpha + \beta) - \mu$ and $\nu \rightarrow (\gamma + \delta) - \nu$. After that the Kronecker delta becomes $\delta_{\mu, \nu}$ and allows us to remove the summation over $\nu$:

$$R_{\alpha, \beta}^\gamma(u) = \sum_{\mu} z_{(\gamma + \delta) - \mu}(q, t)X_{(\alpha + \beta) - \mu, (\gamma + \delta) - \mu}(u)\tilde{C}_{\alpha, \beta}^\gamma(\mu)$$

where:

$$\tilde{C}_{\alpha, \beta}^\gamma(\mu) := (-1)^{(\alpha + \gamma)}(q/t)^{\frac{1}{2}|(\alpha + \gamma)| - |\beta|}\sum_{\sigma} (q/t)^{|\sigma - \mu|}(\alpha + \beta - \mu)(\mu - \gamma)(\delta - \beta)$$

The evaluation of $K$ produces:

$$K_{\alpha, \beta}^\gamma = \delta_{\alpha + \beta, \gamma + \delta}(q/t)^{|\alpha + \delta|/2}\prod_{r \in \gamma - \alpha} (1 - q^r t^{-r})$$

We need to compute $R_{\alpha, \beta}^\gamma(u) = N(u)^{-1} \sum_{\sigma, \rho} \tilde{C}_{\alpha, \beta}^\gamma(\mu)K_{\sigma, \rho}^\gamma$. After inserting the explicit form of the coefficient $X$ (149) and simplifying we find:

$$R_{\alpha, \beta}^\gamma(u) = \sum_{b_\lambda \in \mathcal{P}} b_\lambda(q, t) \prod_{(i, j) \in \lambda} \frac{1 - u^{j+1}q^{i+j+1}}{1 - u^{i+1}q^{j+1}} \sum_{\kappa \in (\alpha + \beta) \cap (\gamma + \delta)} g_{\lambda, (\alpha + \beta) - \kappa}(q, t)g_{\lambda, (\gamma + \delta) - \kappa}(q, t)$$

$$\sum_{\sigma, \rho} \frac{(-1)^{(\kappa - \mu)}z_{(\gamma + \delta) - \mu}(q, t)}{z_{(\gamma + \delta) - \mu}(q, t)} \tilde{C}_{\alpha, \beta}^\gamma(\mu)K_{\sigma, \rho}^\gamma$$

The first line of this expression matches with (150) up to the factor $C_{\alpha, \beta}^\gamma(\kappa)$. The remaining part of the proof consists of showing that:

$$\sum_{\mu \in \kappa} \frac{(-1)^{(\kappa - \mu)}z_{(\gamma + \delta) - \mu}(q, t)}{z_{(\gamma + \delta) - \mu}(q, t)} \tilde{C}_{\alpha, \beta}^\gamma(\mu)K_{\sigma, \rho}^\gamma = C_{\alpha, \beta}^\gamma(\kappa), \quad \forall \kappa \subseteq (\alpha + \beta) \cap (\gamma + \delta)$$

On the left hand side of (159) one needs to use (153) to rewrite $z_{(\gamma + \delta) - \mu}(q, t)$ in terms of $z_{(\gamma + \delta) - \kappa}, z_{(\gamma + \delta) - \mu}$ and a binomial coefficient. Dividing both sides of the resulting equation by $z_{(\gamma + \delta) - \kappa}$ we obtain an identity satisfied by two Laurent polynomials in $q/t$ (expand the product $\prod_{r \in \gamma - \alpha}(1 - (q/t)r)^{\rho}$ appearing in $K_{\sigma, \rho}^\gamma$ in powers of $q/t$). The rest is an exercise in computing summations on the left hand side of the resulting identity using standard manipulations with binomial coefficients. □
4. Stable bases and Hilbert schemes

We will now provide an alternative viewpoint on the $R$-matrix, through a geometric construction known as the stable basis (which originated in [14], and was developed in [1, 22–24] and other works). We will start by reviewing the general theory, focusing on the crucial aspects, all the while glossing over technical details. Thus the following presentation should not be taken as a completely rigorous discussion (however, we will provide appropriate references), but rather as an invitation to a more in-depth study that does not require prior knowledge of algebraic geometry. Specifically, we provide:

- in Subsections 4.1 - 4.6: a quick review of certain basic aspects of algebraic geometry, which would be contained in most introductory courses (such as the first three chapters of [10])
- in Subsections 4.7 - 4.10: an introduction to Hilbert schemes and moduli spaces of framed sheaves on the plane, following [16]
- in Subsections 4.11 - 4.13: a discussion of certain symplectic algebraic varieties equipped with torus actions, following [14]
- in Subsections 4.14 - 4.16: a definition (modulo technical details) of the $K$-theoretic stable basis ([1, 22, 23]) and its usefulness in constructing trigonometric $R$-matrices ([24])

4.1. Algebraic varieties. An affine algebraic variety is, by definition, the subset of $\mathbb{C}^m$ cut out by a certain collection of polynomial equations:

$$V = \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m, f_1(x_1, \ldots, x_m) = \cdots = f_k(x_1, \ldots, x_m) = 0 \right\}$$

(160)

A (Zariski) open subset of $V$ refers to the following subset, defined for any polynomials $g_1, \ldots, g_l$:

$$U = \left\{ (x_1, \ldots, x_m) \in V, g_1(x_1, \ldots, x_m) \neq 0 \text{ or } \cdots \text{ or } g_l(x_1, \ldots, x_m) \neq 0 \right\}$$

(161)

Our main objects of interest are algebraic varieties $X$ over $\mathbb{C}$, i.e. spaces covered by open subsets:

$$U \subset X$$

(162)

which are all isomorphic to open subsets of affine algebraic varieties, as in (161). Moreover, for any open subsets $U, U' \subset X$, the gluing maps between $U \cap U'$ viewed as a subset of $U$ and $U \cap U'$ viewed as a subset of $U'$ are given by rational functions. A subvariety $Z \subset X$ is given by imposing further polynomial equations on the open subsets (162), in a way which is compatible with the gluing maps between any two open subsets. The “smallest” subvarieties of $X$ are points:

$$p \in X$$

which are sometimes called closed points[4]. Many familiar geometric notions, such as dimension (always over $\mathbb{C}$) and tangent spaces, apply to algebraic varieties, see [10] for details.

The algebraic treatment of varieties starts by considering their coordinate rings, i.e. the rings of algebraic functions on these varieties. For instance, if $V \subset \mathbb{C}^m$ is the affine algebraic variety cut out by the polynomial equations (160), then the Nullstellensatz tells us that its coordinate ring is:

$$R_V = \mathbb{C}[x_1, \ldots, x_m] / \sqrt{(f_1, \ldots, f_k)}$$

Going one step further, if $U \subset V$ is the open subset defined by $g \neq 0$ for some polynomial $g$, then:

$$R_U = (R_V)_g = \left\{ \frac{a}{g^n} : a \in R_V, n \in \mathbb{N} \right\}$$

As for general algebraic varieties, they have local coordinate rings $R_U$ for every open subset (162). The gluing maps between two open subsets $U$ and $U'$ lead to ring homomorphisms:

$$R_U \to R_{U \cap U'}, R_U \leftarrow R_{U'}$$

(163)

If $Z \subset X$ is a subvariety, then the restriction maps $R_U \to R_{U \cap Z}$ are quotient maps of rings.

---

[4]This is because, in the language of scheme theory, the word “point” also refers to irreducible subvarieties of $X$ of positive dimension
4.2. Sheaves. Since the study of algebraic geometry is mostly centered around coordinate rings of varieties, and these rings are commutative, it makes sense to study module theory over such rings.

**Definition 6.** A (coherent) sheaf $\mathcal{F}$ on $X$ is an object which consists of $R_U$-modules $F_U$, as $U$ goes over all open subsets. When defining a sheaf, one needs to specify how to glue the various modules $F_U$.

We refer to [10, Chapter 2] for the complete set of axioms that defines a coherent sheaf, and simply work with the intuitive “definition” above. The basic example of a coherent sheaf is the structure sheaf of $X$, i.e. the one defined by:

$$F_U = R_U$$

for all open subsets $U$, with gluing maps provided by. The structure sheaf is usually denoted by $\mathcal{O}_X$. More generally, a coherent sheaf $\mathcal{F}$ for which the $R_U$-modules $F_U$ are free for all small enough open subsets $U$, is called locally free. Locally free sheaves can be identified with the familiar notion of vector bundles from topology. However, not all sheaves are locally free, as the following examples show.

**Example 2.** Let $X = \mathbb{A}^2$. Its ring of polynomial functions is $\mathbb{C}[x, y]$, so any module over this ring will correspond to a coherent sheaf over $X$. In particular, ideals:

$$I \subset \mathbb{C}[x, y]$$

are examples of such modules/sheaves. If we assume that the co-length:

$$n := \dim_{\mathbb{C}} \mathbb{C}[x, y]/I$$

is finite, then $I$ is not a free module, and so the corresponding coherent sheaf is not locally free.

**Example 3.** Given a point $p \in X$, the skyscraper sheaf $\mathcal{C}_p$ is defined by the property that:

$$(\mathcal{C}_p)_U = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

If $p \in U$, then the action of $f \in R_U$ on $(\mathcal{C}_p)_U$ is simply given by multiplication with the scalar $f(p) \in \mathbb{C}$.

If $Z \subset X$ is any subvariety, then any coherent sheaf $\mathcal{F}$ on $X$ can be restricted to $Z$, i.e. we can define the coherent sheaf:

$$\mathcal{F}|_Z$$

on $Z$ whose defining $R_{U \cap Z}$ modules are (for any open subset $U \subset X$):

$$F_U \otimes_{R_U} R_{U \cap Z}$$

endowed with the gluing maps inherited from those of $\mathcal{F}$. The extremal case of restriction is when $Z = p$ is a point, in which case $\mathcal{F}|_p$ is called the fiber of $\mathcal{F}$ at $p$.

4.3. Operations with sheaves. Given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, their direct sum and tensor product:

$$\mathcal{F} \oplus \mathcal{G} \quad \text{and} \quad \mathcal{F} \otimes \mathcal{G}$$

are defined by taking the direct sums and tensor products, respectively, of all the $R_U$-modules $F_U$ and $G_U$. A map $f : \mathcal{F} \to \mathcal{G}$ of coherent sheaves on $X$ consists of $R_U$-module maps $F_U \to G_U$ for all open subsets $U \subset X$, which are suitably compatible under gluing. The kernel and image of $f$ are also coherent sheaves, denoted by:

$$\text{Ker } f \quad \text{and} \quad \text{Im } f$$

If $\text{Ker } f = 0$ then $f$ is called injective, while if $\text{Im } f = \mathcal{G}$ then $f$ is called surjective.

**Definition 7.** A short exact sequence of coherent sheaves:

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

is one such that the morphism $f$ is injective, the morphism $g$ is surjective, and $\text{Im } f = \text{Ker } g$.  

---

5 We note that while the $R_U$-modules $(\text{Ker } f)_U$ are simply $\text{Ker } (F_U \to G_U)$, the $R_U$-modules $(\text{Im } f)_U$ are actually defined by a procedure known as sheafification applied to $\text{Im } (F_U \to G_U)$, see [10].
Definition 8. The (0-th) algebraic $K$-theory group of $X$ is the abelian group:

\[ K(X) \]

generated by symbols $[F]$ for all coherent sheaves $F$ on $X$, modulo the relations:

\[ [F] - [G] + [H] = 0 \] (167)

for all short exact sequences $F \rightarrow G \rightarrow H$.

When $X$ is smooth, $K(X)$ is a ring with multiplication defined by derived tensor product of sheaves. The fact that we need the word “derived” in the previous sentence stems from the fact that the usual tensor product $F \otimes G$ does not preserve the relations (167) unless one of $F$ or $G$ is locally free. However, if $X$ is smooth, any coherent sheaf $F$ admits a resolution by a finite complex of locally free sheaves $\cdots \rightarrow E_i \rightarrow E_{i-1} \rightarrow \cdots$, so we may define the derived tensor product:

\[ [F] \hat{\otimes} [G] = \sum_i (-1)^i [E_i \otimes G] \] (168)

As an exercise in commutative algebra, one may check that the definition above does not depend on any of the choices made, and endows $K(X)$ with a commutative ring structure.

Example 4. As the previous paragraph shows, it is very elegant to work with locally free sheaves (i.e. vector bundles), but one often needs to work with general coherent sheaves. For example, given any subvariety $Z \subset X$, its structure sheaf $\mathcal{O}_Z$ is a coherent sheaf on $X$ and thus an element in $K$-theory:

\[ [\mathcal{O}_Z] \in K(X) \]

In the extremal case, when $Z = \{ p \}$ is a point, this construction recovers skyscraper sheaves: $\mathcal{O}_p = \mathbb{C}_p$.

4.4. Torus actions. Throughout the present paper, an algebraic torus is $T = (\mathbb{C}^*)^k$ for various natural numbers $k$. An algebraic variety $X$ is called a $T$-variety if it is endowed with an action:

\[ T \curvearrowright X \]

by which we mean that general elements $(t_1, \ldots, t_k) \in T = (\mathbb{C}^*)^k$ act on $T$-invariant open subsets $U \subset X$ (and implicitly on the local coordinate rings $R_U$) by rational functions in $t_1, \ldots, t_k$.

Example 5. The standard action:

\[ \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{A}^2_\mathbb{C}, \quad (t_1, t_2) \cdot (x, y) = \left( \frac{x}{t_1}, \frac{y}{t_2} \right) \] (169)

corresponds to the following action on the ring of functions:

\[ \mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{C}[x, y], \quad (t_1, t_2) \cdot f(x, y) = f(t_1 x, t_2 y) \] (170)

Definition 9. Let $X$ be a $T$-variety. A coherent sheaf $\mathcal{F}$ on $X$ is called $T$-equivariant if the torus $T$ acts on the $R_U$-modules $F_U$ in a way which is compatible with the action of $T$ on the rings $R_U$:

\[ t \cdot (r f) = (t \cdot r)(t \cdot f), \quad \forall t \in T, r \in R_U, f \in F_U \]

for all $T$-invariant open subsets $U \subset X$.

Example 6. Combining Examples 4 with 5, it is not hard to see that a finite colength ideal $I_\lambda$ is $\mathbb{C}^* \times \mathbb{C}^*$ equivariant if and only if it is a monomial ideal:

\[ I_\lambda = (x_{\lambda_1}, x_{\lambda_2} y, \ldots, x_{\lambda_d} y^{d-1}, y^d) \] (171)

for some partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > 0$. The number $\lambda_1 + \cdots + \lambda_d$ is the colength of the ideal $I_\lambda$.

6This is because for any open subset $U \subset X$, the ring $R_{U \cap Z}$ is a $R_U$-module via the restriction morphism $R_U \rightarrow R_{U \cap Z}$. 19
4.5. **Equivariant K-theory.** If $X$ is a $T$-variety, then all the notions of Subsection 4.3 make sense for $T$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$. In particular, morphisms of sheaves $\mathcal{F} \to \mathcal{G}$ are called $T$-equivariant if the corresponding $R_{\mathcal{U}}$-module homomorphisms $F_{\mathcal{U}} \to G_{\mathcal{U}}$ respect the $T$-action. A short exact sequence is called $T$-equivariant if all the sheaves and morphisms in its definition are $T$-equivariant.

**Definition 10.** The $T$-equivariant $(0\text{-th})$ algebraic K-theory group of $X$, denoted by:

$$K^T(X)$$

is defined as in Definition 8 but allowing only $T$-equivariant coherent sheaves and short exact sequences.

Besides the structures discussed in Subsection 4.3 (including derived tensor product if $X$ is smooth), equivariant K-theory has the additional structure of a module over the representation ring of $T$:

$$\text{Rep}_T \simeq K^T(X)$$ (172)

where $\text{Rep}_T$ denotes the set of linear combinations of finite-dimensional (algebraic) $T$-representations, made into a ring via direct sum and tensor product; alternatively, one may think of $\text{Rep}_T$ as the ring of characters of such $T$-representations. Explicitly, if $V$ is a finite-dimensional $T$-representation and $\mathcal{F}$ is a $T$-equivariant coherent sheaf on $X$, then $V \otimes \mathcal{F}$ has a natural structure of a $T$-equivariant coherent sheaf on $X$. If $T = (\mathbb{C}^*)^k$, then:

$$\text{Rep}_T = \mathbb{C}[q_1^{\pm 1}, \ldots, q_k^{\pm 1}]$$ (173)

where $q_i$ denotes the one-dimensional $T$-representation corresponding to the character $(t_1, \ldots, t_k) \mapsto t_i$.

4.6. **The localization theorem.** Let $X$ be a smooth algebraic variety, which entails the existence of the tangent (respectively cotangent) locally free sheaf $\text{Tan} X$ (respectively $\text{Tan}^\vee X$). We further assume that $X$ has a $T$-action whose fixed point locus $X^T$ is finite. We will write:

$$K^T(X)_{\text{loc}} = K^T(X) \otimes_{\text{Rep}_T} \text{Frac}(\text{Rep}_T)$$

for the localized equivariant K-theory groups of $X$. This formally means that elements of $K^T(X)_{\text{loc}}$ can be multiplied not only with elements of the ring $\text{Rep}_T$ (i.e. Laurent polynomials in the elementary characters $q_1, \ldots, q_k : T \to \mathbb{C}^*$) but with elements of the field:

$$\mathbb{F} = \text{Frac}(\text{Rep}_T)$$

(i.e. rational functions in $q_1, \ldots, q_k$). Thus, $K^T(X)_{\text{loc}}$ is a $\mathbb{F}$-vector space.

**Theorem 7.** (229) If $X$ is a smooth $T$-variety, we have an isomorphism of $\mathbb{F}$-vector spaces:

$$K^T(X)_{\text{loc}} \cong \bigoplus_{p \in X^T} \mathbb{F} \cdot [p]$$

where $[p] = [\mathcal{C}_p]$ is the class of the skyscraper sheaf at the torus fixed point $p \in X^T$ (we assume that there are finitely many torus fixed points). The isomorphism above is explicitly given by:

$$[\mathcal{F}] \mapsto \sum_{p \in X^T} \mathcal{F}|^L_{\mathcal{F}} \wedge^\bullet (\text{Tan}^\vee_p X) \cdot [\mathcal{C}_p]$$ (174)

for all $T$-equivariant coherent sheaves $\mathcal{F}$, where the derived fiber (see 108) is defined as:

$$\mathcal{F}|^L_p = [\mathcal{F}] \wedge^\bullet (\text{Tan}^\vee_p X)$$

Thus, $\mathcal{F}|^L_p$ can be thought of as an alternating sum of $T$-equivariant vector spaces (supported at $p \in X$), as can the total exterior power $\wedge^\bullet (\text{Tan}^\vee_p X)$. The fraction in (174) is defined as the ratio of the alternating sums of the $T$-characters in said vector spaces, and this ratio is an element of $\mathbb{F}$.

The main substance of Theorem 7 lies in the fact that the classes of the skyscraper sheaves $\mathcal{C}_p$ form a $\mathbb{F}$-basis of $K^T(X)_{\text{loc}}$. Once one accepts this fact, formula (174) is a simple exercise which follows from:

$$\mathcal{C}_p|^L_{p'} = \begin{cases} \wedge^\bullet (\text{Tan}^\vee_p X) & \text{if } p = p' \\ 0 & \text{otherwise} \end{cases}$$ (175)

(= the equality above is one of characters of alternating sums of $T$-equivariant vector spaces).
4.7. Hilbert schemes. Let us now apply the notions above to a particularly interesting algebraic variety.

Definition 11. The Hilbert scheme of \( n \) points on the affine plane \( \mathbb{A}^2 \) is:

\[
\text{Hilb}_n = \{ \text{colength } n \text{ ideals } I \subset \mathbb{C}[x,y] \} \tag{176}
\]

To interpret \( \text{Hilb}_n \) as an algebraic variety, we consider the following alternative description. Given endomorphisms \( X',Y' \) of a vector space \( V \), a vector \( v \in V \) is called cyclic if \( \text{span}\{ P(X,Y)v \} = V \), as \( P \) goes over all non-commutative polynomials with coefficients in \( \mathbb{C} \). Then we have:

\[
\text{Hilb}_n = \{ (X,Y,v) \in \text{Mat}_{n \times n} \times \text{Mat}_{n \times n} \times \mathbb{C}^n \text{ such that } [X,Y] = 0 \text{ and } v \text{ is cyclic} \} / \text{GL}_n \tag{177}
\]

where \( \text{GL}_n \) acts on \( X,Y \) by conjugation and on \( v \) by left multiplication. To get from an ideal \( I \) as in (176) to a triple \( (X,Y,v) \) as in (177), we fix an identification \( \mathbb{C}^n = \mathbb{C}[x,y]/I \), and let \( X \) and \( Y \) be the operators of multiplication by \( x \) and \( y \). Meanwhile, \( v \) is simply \( 1 \mod I \). To get from a triple \( (X,Y,v) \) as in (177) to an ideal \( I \) as in (176) is also a straightforward exercise, one which we leave to the interested reader. The presentation (177) is manifestly an algebraic variety: the matrix entries of \( X,Y,v \) are in one-to-one correspondence with partitions of weight \( n \), so its presence is cosmetic at the moment, but will play a key role when we study moduli spaces of higher rank sheaves. In the language of (177), the action (178) is given by:

\[
(t_1,t_2,\xi) \cdot (X,Y,v) = (t_1X,t_2Y,\xi v)
\]

A point of \( \text{Hilb}_n \) is fixed by \( T \) precisely when the corresponding ideal is \( T \)-equivariant, which as we saw in Example 6 is precisely asking that the ideal be monomial. Therefore, we conclude that the fixed points of \( \text{Hilb}_n \) are in one-to-one correspondence with partitions of weight \( n \):

\[
\text{Hilb}_n^T = \{ I_\lambda, \lambda \vdash n \}
\]

Let \( q_1,q_2,u \) be the usual elementary characters of the three factors of the torus (178). If we write:

\[
\mathbb{C}_1 = \mathbb{C}(q_1,q_2,u)
\]

then the localization Theorem 7 gives us:

\[
K^T(\text{Hilb}_n)_{\text{loc}} \cong \bigoplus_{[\lambda]=n} \mathbb{C}_1 \cdot [\lambda]
\]

where \( [\lambda] = [C_{\lambda}] \). To get the Fock space, we simply need to let \( n \) run over all non-negative integers:

\[
\text{Hilb} = \bigcup_{n=0}^{\infty} \text{Hilb}_n, \quad K^T(\text{Hilb}) = \bigoplus_{n=0}^{\infty} K^T(\text{Hilb}_n)
\]

and we conclude that:

\[
K^T(\text{Hilb})_{\text{loc}} \cong \bigoplus_{\lambda} \mathbb{C}_1 \cdot [\lambda]
\]

By work of Haiman, it is natural to identify:

\[
K^T(\text{Hilb})_{\text{loc}} \cong \Lambda_{\mathcal{F}}(u) \tag{179}
\]

by sending \([\lambda]\) to the modified Macdonald polynomials \( \tilde{H}_\lambda \) (close relatives of \( P_\lambda \), and we refer to [8] for details). The notation \( \Lambda_{\mathcal{F}}(u) \) simply refers to \( \Lambda_{\mathcal{F}} \), but we insert the parameter \( u \) in the notation to keep track of the third \( \mathbb{C}^* \) action. This action is trivial on \( \text{Hilb}_n \), but it will play a role in the next Subsection.

---

Footnote: Care must be taken to properly formulate the operation of taking the \( \text{GL}_n \) quotient in the context of algebraic varieties. The appropriate language here is that of geometric invariant theory, which deals with the existence and properties of such quotients.
4.9. Moduli of higher rank sheaves. The Hilbert scheme is, perhaps after the Grassmannian, one of the most fundamental examples of a moduli space: an algebraic variety whose points parameterize algebro-geometric objects of a different nature (in the present case, ideals in \( \mathbb{C}[x, y] \)). As we have seen in Example 2, ideals can be interpreted as coherent sheaves on \( \mathbb{P}^2_\mathbb{C} \). As such, they have rank 1, meaning that for any open subset \( U \subset \mathbb{A}^2_\mathbb{C} \) which misses finitely many points, the ideal is a rank 1 free module over the local coordinate ring of \( U \). One can therefore ask if there exists a moduli space of rank \( r \) sheaves on \( \mathbb{A}^2_\mathbb{C} \), and the answer is sadly no. However, the following closely related object exists and behaves nicely.

**Definition 12.** Let \( \infty \subset \mathbb{P}^2_\mathbb{C} \) be the line at infinity, so \( \mathbb{A}^2_\mathbb{C} = \mathbb{P}^2_\mathbb{C} \setminus \infty \). There exists an algebraic variety \( \mathcal{M}(r) \), whose points are in one-to-one correspondence with pairs:

\[
(\mathcal{F}, \phi)
\]

where \( \mathcal{F} \) is a rank \( r \) coherent sheaf on \( \mathbb{P}^2_\mathbb{C} \), and \( \phi \) is an isomorphism:

\[
\phi : \mathcal{F}|_{\infty} \xrightarrow{\sim} \mathcal{O}_{\infty}^r
\]

The isomorphism \( \phi \) is called a framing of \( \mathcal{F} \), and a pair (180) is called a framed sheaf. We will write \( \mathcal{M}(r)_n \subset \mathcal{M}(r) \) for the connected component of framed sheaves whose second Chern class is \( n \).

The gist of the construction above is that a framed sheaf is trivial (i.e. free) in a neighborhood of \( \infty \subset \mathbb{P}^2_\mathbb{C} \), but it might have some non-trivial structure away from \( \infty \). In particular, for general algebraic reasons, a rank 1 framed sheaf embeds inside the structure sheaf:

\[
\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_\mathbb{C}}
\]

Because of the framing, the inclusion above is an equality near \( \infty \), but on the open subset \( \mathbb{A}^2_\mathbb{C} = \mathbb{P}^2_\mathbb{C} \setminus \infty \), it corresponds to an ideal:

\[
I \subset \mathbb{C}[x, y]
\]

hence \( \mathcal{M}(1) \cong \text{Hilb} \). With this example in mind, it should be of no surprise that \( \mathcal{M}(r)_n \) admits the following alternate presentation for any \( r \), by analogy with (177):

\[
\mathcal{M}(r)_n = \left\{ (X, Y, A, B) \in \text{Mat}_{n \times n} \times \text{Mat}_{n \times n} \times \text{Mat}_{n \times r} \times \text{Mat}_{r \times n} \right\} /	ext{GL}_n
\]

such that \([X, Y] = AB \) and \( A \) is cyclic (184)

where \( \text{GL}_n \) acts by conjugation on \( X \) and \( Y \) and by left (resp. right) multiplication on \( A \) (resp. \( B \)). We call the matrix \( A \) cyclic if span\{\( P(X, Y)A \cdot \mathbb{C}^* \)\} = \( V \), as \( P \) goes over all non-commutative polynomials with coefficients in \( \mathbb{C} \). We refer to [15] Chapter 2] for a detailed description on how to connect a framed sheaf with a quadruple \( (X, Y, A, B) \) as above.

4.10. The torus action on \( \mathcal{M}(r) \). There is an action of \( \text{T}_r = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r \) on \( \mathcal{M}(r) \), where the first two factors act on the sheaf \( \mathcal{F} \) (through their action on the projective plane \( \mathbb{P}^2_\mathbb{C} \)) while the last \( r \) factors act by multiplying the framing isomorphism (181) with diagonal matrices. In terms of the presentation (185), this action is given by:

\[
(t_1, t_2, \xi_1, \ldots, \xi_r) \cdot (X, Y, A, B) = (t_1X, t_2Y, AD, t_1t_2D^{-1}B), \quad \text{where } D = \text{diag}(\xi_1, \ldots, \xi_r)
\]

The fixed points of this action are all direct sums of monomial ideals, i.e. for various partitions \( \lambda^1, \ldots, \lambda^r \):

\[
\mathcal{F}_{\lambda^1, \ldots, \lambda^r} = \mathcal{I}_{\lambda^1} \oplus \cdots \oplus \mathcal{I}_{\lambda^r}
\]

where the inclusion \( \mathcal{I}_{\mu} \subset \mathcal{O}_{\mathbb{P}^2_\mathbb{C}} \) is defined as the identity near \( \infty \), and as the inclusion \( I_{\mu} \subset \mathbb{C}[x, y] \) on the open subset \( \mathbb{A}^2_\mathbb{C} = \mathbb{P}^2_\mathbb{C} \setminus \infty \). The fact that \( \mathcal{I}_{\mu} \subset \mathcal{O}_{\mathbb{P}^2_\mathbb{C}} \) in a neighborhood of \( \infty \) determines a canonical framing of \( \mathcal{F}_{\lambda^1, \ldots, \lambda^r} \), which makes (185) into a framed sheaf. More specifically, we have:

\[
\mathcal{M}(r)_n^{T_r} = \{ \mathcal{F}_{\lambda^1, \ldots, \lambda^r} \mid \lambda^1, \ldots, \lambda^r \in \mathcal{P} \text{ satisfy } |\lambda^1| + \cdots + |\lambda^r| = n \}
\]

Let \( q_1, q_2, u_1, \ldots, u_r \) be the usual elementary characters of the factors of \( T_r \). If:

\[
\mathcal{F}_r = \mathbb{C}(q_1, q_2, u_1, \ldots, u_r)
\]

then the localization Theorem [2] gives us:

\[
K_r^T(\mathcal{M}(r)_n)_{\text{loc}} \cong \bigoplus_{|\lambda^1| + \cdots + |\lambda^r| = n} \mathcal{F}_r \cdot [\lambda^1, \ldots, \lambda^r]
\]

22
The non-degeneracy of the form (188) implies that the dimension of $p$ between the tangent and cotangent spaces at any $T$ character $q$ write scheme of $n$ bundles (induced by (188)) is not $T^2$ inspired by [14]), which we will now recall, exists for a wide class of so-called conical symplectic resolutions that was considered in Subsection 4.10, and the decomposition

$$
T \dashrightarrow \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r
$$

In terms of connected components indexed by the second Chern class $n$, we have:

$$
\bigcup_{n_1 + n_2 = n} \lambda_1 \cdots \lambda_{n_2} \bigoplus \lambda_{n_2} \cdots \lambda_n \cong \lambda_1 \cdots \lambda_n
$$

Example 9. Instead of asking for the fixed point set of $\mathcal{M}(r)$ under the whole $T_r$ action, one may ask for its fixed point set under a subtorus. In particular, for any decomposition $r = r_1 + r_2$, we may consider the one-parameter subtorus:

$$
\phi_{r_1, r_2} : \mathbb{C}^* \to T_r, \quad t \mapsto (1, 1, 1, \ldots, 1, t, \ldots, t)
$$

The fixed point set with respect to this torus is simply given by those rank $r$ framed sheaves which split as a direct sum of two framed sheaves, of respective ranks $r_1$ and $r_2$:

$$
\mathcal{M}(r_1) \times \mathcal{M}(r_2) \cong \mathcal{M}(r)^{\phi_{1, r_2}}, \quad (\mathcal{F}_1, \mathcal{F}_2) \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2
$$

4.11. Symplectic forms and tangent spaces. The $K$-theoretic stable basis construction ( [11][22][23], inspired by [14]), which we will now recall, exists for a wide class of so-called conical symplectic resolutions $X$. Such algebraic varieties $X$ are smooth and equipped with a non-degenerate symplectic form:

$$
\omega : \text{Tan} X \otimes \text{Tan} X \to \mathcal{O}_X
$$

Non-degeneracy implies that (188) yields an isomorphism between the tangent and cotangent bundles. Moreover, we assume that there exists a torus $T$ acting on $X$, which naturally splits into two parts:

$$
T = \mathbb{C}^*_\omega \times A
$$

where the subtorus $\mathbb{C}^*_\omega$ scales the symplectic form with weight 1, while the subtorus $A$ preserves the symplectic form. This means that the aforementioned isomorphism between the tangent and cotangent bundles (induced by (188)) is not $T$-equivariant, but becomes $T$-equivariant upon twisting with the character $q : T \to \mathbb{C}^*$ that sends $(t, a) \mapsto t$ with respect to the decomposition (189):

$$
\text{Tan}^X X \cong q \otimes \text{Tan} X
$$

The non-degeneracy of the form (188) implies that the dimension of $X$ = the rank of Tan $X$ must be even. Moreover, the $T$-equivariant nature of the symplectic form of (190) implies that we have isomorphisms:

$$
\text{Tan}_p X \cong q \otimes \text{Tan}_p X
$$

between the tangent and cotangent spaces at any $p \in X_T$. Therefore, the weights that appear in the $T$-representation $\text{Tan}_p X$ can be paired up with respect to the symplectic form, and we conclude that:

$$
\chi_T(\text{Tan}_p X) = \sum_{i=1}^{\dim X} \left( \chi_i + \frac{1}{q \chi_i} \right)
$$

for various $T$-characters $\chi_i : T \to \mathbb{C}^*$. Here and below, if $V$ is any representation of the torus $T$, we will write $\chi_T(V) \in \text{Rep}_T$ for the character of $T$ in $V$. Explicitly, $\chi_T(V)$ will be a Laurent polynomial in the elementary characters of $T$ (these are denoted by $q_1, q_2, u_1, \ldots, u_r$ for the torus $T_r$ of Subsection 4.11).

Example 9. The discussion above applies to the moduli spaces of framed sheaves $X = \mathcal{M}(r)_n$, which are $2nr$ dimensional as algebraic varieties over $\mathbb{C}$ (this generalizes the well-known fact that the Hilbert scheme of $n$ points has dimension $2n$). We consider the action of the torus:

$$
T = T_r = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r
$$

that was considered in Subsection 4.11 and the decomposition (189) has:

$$
\mathbb{C}^*_\omega = \{(t, t, 1, \ldots, 1)\} \hookrightarrow T_r \twoheadrightarrow A = \{(t, t^{-1}, \xi_1, \ldots, \xi_r)\}
$$

---

This class contains all Nakajima quiver varieties, of which the moduli spaces $\mathcal{M}(r)_n$ are specific examples.
In particular, the weight of the symplectic form is \( q_1 q_2 \). As for the \( T_r \)-character in the tangent space to a fixed point of the form \( \mathbb{C}^* \), if we let \( q_1, q_2, u_1, \ldots, u_r \) denote the usual elementary characters of the factors of \( T_r \), we have:

\[
\chi_T \left( \Tan_{F_{\lambda_1, \ldots, \lambda_r}} \mathcal{M}(r) \right) = \sum_{i,j=1}^{r} \sum_{\lambda \in \mathcal{P}} \left( \frac{u_i}{u_j} q_1^{-a_\lambda(q_1)} q_2^{-l_\lambda(q_2)} + \frac{u_j}{u_i} q_1^{-a_\lambda(q_1)} q_2^{-l_\lambda(q_2)} \right)
\]  \hspace{1cm} (191)

where \( \circ \in \lambda \) means that square goes over all the boxes in the Young diagram of the partition \( \lambda \), and the arm (respectively leg) length \( a_\lambda(q) \) (respectively \( l_\lambda(q) \)) refers to the number of unit steps one must take to the right (respectively above) of the box \( \circ \) in order to reach the vertical (respectively horizontal) boundary of the Young diagram \( \lambda \). We remark that the arm (respectively leg) length can be negative if the aforementioned boundary is left (respectively below) of the box \( \circ \).

In the case \( r = 1 \), namely the Hilbert scheme of points, formula (191) reads:

\[
\chi_{\mathbb{C}^* \times \mathbb{C}^*} (\Tan_{\lambda} \Hilb) = \sum_{\lambda \in \mathcal{P}} \left( q_1^{1-a_\lambda(q_1)} q_2^{l_\lambda(q_2)} + q_1^{l_\lambda(q_2)} q_2^{1-a_\lambda(q_1)} \right)
\]

Then formula (175) implies the following restriction formula in \( K \)-theory:

\[
\mathcal{C}_{I_1 [I_1]} = \prod_{\lambda \in \mathcal{P}} \left( 1 - q_1^{1-a_\lambda(q_1)} q_2^{l_\lambda(q_2)} \right) \left( 1 - q_1^{-a_\lambda(q_1)} q_2^{-l_\lambda(q_2)} \right)
\]  \hspace{1cm} (192)

where the left-hand side is naturally interpreted as the torus character in a chain complex of \( \mathbb{C}^* \times \mathbb{C}^* \) representations. Note that the right-hand side of the expression above matches (12), which is supported where the left-hand side is naturally interpreted as the torus character in a chain complex of \( \mathbb{C}^* \times \mathbb{C}^* \) representations.

4.12. Attracting subvarieties. For a smooth algebraic variety \( X \) endowed with the action of an algebraic torus \( T \), we can perform the following construction. For any one-parameter torus \( \sigma : \mathbb{C}^* \to T \) and any connected component of the fixed locus:

\[
Z \subset X^\sigma
\]  \hspace{1cm} (193)

we may define its attracting set:

\[
\text{Attr}_\sigma(Z) = \{ x \in X \text{ s.t. } \lim_{t \to 0} \sigma(t) \cdot x \text{ exists and lies in } Z \}
\]  \hspace{1cm} (194)

The repelling set is defined analogously, but replacing \( \sigma \) with \( \sigma^{-1} \). One may define a partial order on the set of connected components of \( X^\sigma \) generated by:

\[
Z \succeq Z' \text{ if } \text{Attr}_\sigma(Z) \cap Z' \neq \emptyset
\]

and transitivity. This allows us to define the full attracting set of a connected component (193) as:

\[
\text{Attr}^f_\sigma(Z) = \bigcup_{Z \preceq Z'} \text{Attr}_\sigma(Z')
\]  \hspace{1cm} (195)

While attracting sets (194) need not be subvarieties of \( X \) in general (the reason is that they are only locally closed, instead of closed, subsets of \( X \) in the Zariski topology), the full attracting set (195) will be. This is the very point of the definition of full attracting subvarieties: if one defines the notion of a fixed point “flowing” into another fixed point when a small perturbation of the former literally flows into the latter under the action of the one-parameter subtorus \( \sigma \), then the full attracting subvariety of \( Z \) consists of all points of \( X \) which flow into points which flow into points which ... flow into points of \( Z \).

4.13. Normal bundles. Let us now apply the situation in the previous Subsection when \( X \) is a symplectic variety, and \( \sigma : \mathbb{C}^* \to A \) is a one-parameter subtorus that preserves the symplectic form. In this case, consider any connected component of the fixed point locus:

\[
Z \subset X^A
\]

The normal bundle to \( Z \) will split as a direct sum:

\[
N_Z X = N_\sigma^{\text{rep}} X \oplus N_\sigma^{\text{at}} X
\]  \hspace{1cm} (196)

where \( N_\sigma^{\text{rep}} X \) (respectively \( N_\sigma^{\text{at}} X \)) is the normal bundle of the repelling (respectively attracting) subvariety of \( Z \). In other words, the weights of the torus \( A \) acting in the fibers of \( N_\sigma^{\text{rep}} X \) (respectively \( N_\sigma^{\text{at}} X \)) are
precisely those weights which are positive (respectively negative) with respect to the cocharacter \( \sigma \). The symplectic form pairs \( N_Z^+ X \) and \( N_Z^- X \) non-trivially, and so we have the following analogue of (190):

\[
(N_Z^+ X) \cong q \otimes N_Z^- X
\]
as \( T \)-equivariant vector bundles on \( Z \). In particular, the dimensions of the attracting and repelling normal bundles in (190) are equal to each other, which can be thought of as saying that the attracting/repelling subvarieties are half-codimensional with respect to the subvariety \( Z \subset X \) (in particular, this implies that the dimension of \( Z \) must be even).

**Example 10.** Let us consider the situation of Example 2 when \( X = M(r)_n \), and the fixed point set \( X^A \) consists of finitely many points:

\[
p = \mathcal{F}_{\lambda_1, \ldots, \lambda_r}
\]

In this case, the normal bundle to such a fixed point is simply the tangent space at \( p \), whose \( T_r \)-character is given by formula (191). For any cocharacter \( \sigma : C^* \to A \subset T_r \), one of the two terms:

\[
\frac{u_i}{u_j} q_1^{a_1(\lambda^i) - 1} q_2^{b_2(\lambda^i)} \quad \text{and} \quad \frac{u_i}{u_j} q_1^{a_1(\lambda^j) - 1} q_2^{b_2(\lambda^j)}
\]

will be positive and the other will be negative with respect to \( \sigma \). Therefore, we have:

\[
\text{Tan}_p X = \text{Tan}_p^+ X \oplus \text{Tan}_p^- X
\]

where the \( T_r \)-character of \( \text{Tan}_p^+ X \) (respectively \( \text{Tan}_p^- X \)) is simply the sum of those terms among (198) which are positive (respectively negative) with respect to \( \sigma \). Special choices of \( \sigma \) will make it very obvious which is the positive term and which is the negative term. For example, if:

\[
\sigma_{d_1, \ldots, d_r}(t) = (t, t^{-1}, t^{d_3}, \ldots, t^{d_r})
\]

for integers \( d_1 \ll \cdots \ll d_r \) (the inequalities should be interpreted as saying that \( d_i \) is much smaller than \( d_{i+1} \) compared to the weight \( n \) of the partitions that make up the fixed point (197)), then the first term in (198) will be the positive one and the second term will be the negative one whenever \( i < j \).

4.14. The stable basis. We are ready to give the definition of the \( K \)-theoretic stable basis (1122 24), inspired by (14), at least modulo certain details. For simplicity, we will only deal with symplectic algebraic varieties \( X \) endowed with \( T \)-actions for which all the preceding discussion applies, and whose fixed point set \( X^T \) is finite. Our main example is \( X = M(r)_n \), which satisfies all of these properties.

**Definition 13.** For any generic cocharacter \( \sigma : C^* \to A \), there is a collection of elements:

\[
\{ s_p \}_{p \in X^T} \subset K^T(X)
\]

which are uniquely determined by the following properties for all \( p \in X^T \):

1. \( s_p \) is supported on the full attracting subvariety \( \text{Attr}^\sigma(p) \)
2. \( s^p_{\mid p} = \wedge(\text{Tan}_p^- X) \otimes \text{monomial}^p \)
3. for any fixed point \( p' < p \), the \( A \)-weights of \( s^p_{\mid p'} \) are contained (as elements of \( \text{Lie}_R(A^-) \)) in:

\[
\text{convex hull of weights appearing in } \wedge(\text{Tan}_p^- X) + \text{shift}_{p, p'} \subset \text{Lie}_R(A^-)
\]

where convex hull refers to the convex hull minus one of its vertices.

**Remark 5.** The (so-far undefined) quantities \( \text{monomial}^p \) and \( \text{shift}_{p, p'} \) are certain characters of \( T \) and \( A \), respectively. The definition of these quantities takes as input two more pieces of data that one must specify in order to give the full definition of the stable basis, namely:

- a polarization, i.e. the choice of a decomposition of the tangent spaces to \( X \) into halves
- a \( T \)-equivariant line bundle \( L \) on \( X \)

The interested reader may find a complete discussion of these issues in [21, Section 9]. We do not dwell on them, because these choices are quite natural in the case at hand: because the moduli space of sheaves \( X = M(r) \) is the cotangent bundle of a certain stack \( S \), \( \text{Tan} M(r) \) is locally isomorphic to \( \text{Tan} S \otimes \text{Tan} S^* \), and this decomposition into “halves of the tangent bundle” yields the requisite polarization. As for the line bundle \( L \), it will always be taken to be the structure sheaf \( \mathcal{O}_X \) in the constructions that follow.

\footnote{Given a cocharacter \( \sigma : C^* \to A \), a character \( \chi : A \to C^* \) is called positive (respectively negative) if \( \chi \circ \sigma(t) = t^n \) for \( n \) a positive (respectively negative) integer. The notion applies similarly if \( \chi : T \to C^* \) for a bigger torus \( T \supset A \).}
Properties (1) and (2) imply that the collection (200) is upper triangular in the fixed point basis $[p]$:

$$s_p = [p] \cdot \frac{\text{monomial}_p}{\lambda_1(\text{Tr}_p X)} + \sum_{p' < p} [p'] \cdot \text{coefficient}$$

where the numerator of the fraction is a character of $T$, i.e., a monomial in the representation ring $\text{Rep}_T$, and the quantities marked “coefficient” are elements of $\text{Frac}(\text{Rep}_T)$. Therefore, the elements (200) form a basis of $K^T(X)_{\text{loc}}$, which is called the stable basis. Property (3) implies the uniqueness of the stable basis, and also provides an algorithm for computing it via long division of Laurent polynomials in several variables: fix a total ordering of the fixed points $p_1, \ldots, p_N$ which refines the partial ordering $\prec$. Then we start from the collection $\{t_p = [\mathcal{O}_{\text{Attr}}(X)] \otimes \text{monomial}_p\}_{p \in \lambda^T} \subset K^T(X)$ and at the $i$-th step, we will modify:

$$t_{p_i} \rightsquigarrow s_{p_i} = t_{p_i} + \sum_{j=1}^{i-1} \text{coefficient} \cdot s_{p_j}$$

in such a way that property (3) holds for $p = p_i$. The fact that the already constructed $s_{p_1}, \ldots, s_{p_{i-1}}$ already satisfy properties (2) and (3) implies that the coefficients in the above formula are completely determined. The existence of the stable basis satisfying properties (1)-(3) was established in [1, 22, 23].

4.15. The geometric $R$-matrix. Let us now set $X = \mathcal{M}(r)_n$, and consider the one-parameter subgroup (199) for $d_1 \ll \cdots \ll d_r$. The construction of the previous Subsection gives rise to a stable basis:

$$\{s^\pm_{\lambda^1, \ldots, \lambda^r} \}_{\text{partitions } \lambda^1, \ldots, \lambda^r} \subset K^{T^r}(\mathcal{M}(r))$$

Similarly, the analogous construction for the inverse one-parameter subgroup $\sigma_{d_1, \ldots, d_r}^{-1}$ yields a basis:

$$\{s^{-\lambda^1, \ldots, -\lambda^r}_\lambda \}_{\text{partitions } \lambda^1, \ldots, \lambda^r} \subset K^{T^r}(\mathcal{M}(r))$$

In other words, the basis (201) is defined just like the basis (203), but switching the roles of attracting and repelling varieties. This implies that the bases $s^\pm_{\lambda^1, \ldots, \lambda^r}$ and $s^{-\lambda^1, \ldots, -\lambda^r}_\lambda$ are upper and lower triangular, respectively, in the basis of fixed points $[\lambda^1, \ldots, \lambda^r]$ with respect to the ordering $\prec$.

**Definition 14.** For any decomposition $r = r_1 + r_2$, we have stable basis maps:

$$K^{T_{r_1}}(\mathcal{M}(r_1)) \otimes K^{T_{r_2}}(\mathcal{M}(r_2)) \xrightarrow{\text{Stab}^\pm} K^{T_r}(\mathcal{M}(r))$$

given by:

$$\text{Stab}^\pm \left(s^\pm_{\lambda^1, \ldots, \lambda^r} \otimes s^{-\mu^1, \ldots, -\mu^r}_\mu\right) = s^\pm_{\lambda^1, \ldots, \lambda^r, \mu^1, \ldots, \mu^r}$$

for all partitions $\lambda^1, \ldots, \lambda^r, \mu^1, \ldots, \mu^r$. The maps (205) are isomorphisms after tensoring with $\mathbb{F}_r$.

In the usual stable basis framework of [14], it is natural to think of the maps (205) as the fundamental construction, instead of the elements (203–204). Indeed, Definition 15 can be upgraded to the language of Lagrangian correspondences, in which the subvariety $\mathcal{M}(r_1) \times \mathcal{M}(r_2)$ plays the role of individual torus fixed points of $\mathcal{M}(r)$ (and in fact, the aforementioned subvariety is the torus fixed locus with respect to the one-parameter subgroup (187)). An overview of the theory in this more general language can be found in [21, Section 9].

**Definition 15.** For any decomposition $r = r_1 + r_2$, the composition:

$$R_{r_1, r_2} : K^{T_{r_1}}(\mathcal{M}(r_1)) \otimes K^{T_{r_2}}(\mathcal{M}(r_2)) \xrightarrow{\text{Stab}^+} K^{T_r}(\mathcal{M}(r)) \xrightarrow{(\text{Stab}^{-1})^{-1}} K^{T_{r_1}}(\mathcal{M}(r_1)) \otimes K^{T_{r_2}}(\mathcal{M}(r_2))$$

is called a geometric $R$-matrix. It satisfies the quantum Yang-Baxter equation:

$$R_{r_1, r_2} R_{r_1, r_3} R_{r_2, r_3} = R_{r_2, r_3} R_{r_1, r_3} R_{r_1, r_2}$$

as endomorphisms of $K^{T_{r_1}}(\mathcal{M}(r_1)) \otimes K^{T_{r_2}}(\mathcal{M}(r_2)) \otimes K^{T_{r_3}}(\mathcal{M}(r_3))$, for all natural numbers $r_1, r_2, r_3$.\[\]\[10\] Implicit in the notation (207) is that $R_{r_1, r_3}$ acts as (206) as an endomorphism of $K(\mathcal{M}(r_1)) \otimes K(\mathcal{M}(r_3))$ tensor the identity on $K(\mathcal{M}(r_2))$, etc.
4.16. **Poles of the $R$-matrix.** As we have seen in (170) and (186), matching modified Macdonald polynomials with the skyscraper sheaves at the torus fixed points allows us to identify $K^{T_r}(\mathcal{M}(r))$ with a tensor product of Fock spaces. Under this isomorphism, the $R$-matrix (206) gives rise to an endomorphism:

$$R_{r_1,r_2} \in \text{End}_{F_r}(\Lambda_F(u_1) \otimes \cdots \otimes \Lambda_F(u_r))$$ (208)

where $r = r_1 + r_2$. Since the endomorphism above is “geometric” in nature (the rigorous term here is that it is given by a correspondence), all its matrix coefficients are rational functions in $q_1, q_2, u_1, \ldots, u_r$. However, the very nature of the stable basis allows us to say more.

**Proposition 5.** *The poles of the endomorphism $R_{r_1,r_2}$ are all of the form:*

$$u_j = u_j q_1^x q_2^y$$

*as $1 \leq i \leq r_1 < j \leq r$ and $x, y \in \mathbb{Z}$.*

**Proof.** As explained in Subsection 9.3.5 of [21], the endomorphism $R_{r_1,r_2}$ is a product of endomorphisms $R_{1,1}$ acting in various tensor products of the form $\Lambda_F(u_i) \otimes \Lambda_F(u_j)$ for $1 \leq i \leq r_1 < j \leq r$ (and the identity on the other tensor factors) and so it is enough to prove the Proposition in the case $r_1 = r_2 = 1$. As is clear from (205) and (207), when expressed in the basis $s^\lambda_\mu \otimes s^\mu_\lambda$, the endomorphism $R_{1,1}$ is the product of the following matrices:

- the change of basis from $s^\lambda_\mu$ to $s^\lambda_\mu$ as $\lambda$ and $\mu$ go over all partitions.
- the change of basis from $s^\mu_\lambda \otimes s^\mu_\lambda$ to $s^\lambda_\mu \otimes s^\lambda_\mu$ as $\lambda$ and $\mu$ go over all partitions.

The second bullet is simply the tensor product of a matrix acting in $\Lambda_F(u_1)$ and a matrix acting in $\Lambda_F(u_2)$, so it does not produce any poles involving $u_1$ and $u_2$. As for the first bullet, we can further sub-divide it into:

- the change of basis from $s^\lambda_\mu$ to $[\lambda, \mu]$ as $\lambda$ and $\mu$ go over all partitions.
- the change of basis from $[\lambda, \mu]$ to $s^\lambda_\mu$ as $\lambda$ and $\mu$ go over all partitions.

As shown by (202), the poles of the aforementioned changes of basis are among the linear factors in:

$$\wedge^*(\text{Tan}_{x,y}^{x,y} \mathcal{M}(2))$$

From formula (191), those linear factors which involve both $u_1$ and $u_2$ in the expression above are:

$$1 - \frac{u_1}{u_2} q_1^x q_2^y \quad \text{and} \quad 1 - \frac{u_2}{u_1} q_1^x q_2^y$$

for various $(x, y) \in \mathbb{Z}^2$ which arise as the weights of boxes in various partitions. 

A finer analysis of the poles of the $R$-matrix via shift operators (following the lines of [11] Subsections 3.4 and 3.5) allows one to prove that the poles of $R_{r_1,r_2}$ are simple and all of the form $u_j = u_j q_1^x q_2^y$ as $1 \leq i \leq r_1 < j \leq r$ and $x, y \in \mathbb{N}$. We thank Yakov Kononov and Andrey Smirnov for this remark.

4.17. **Connection between the $R$-matrices.** As the reader probably suspects at this stage, the endomorphisms (205) are expected to match the ones produced by the $R$-matrix of the quantum toroidal algebra.

**Conjecture 1.** *The endomorphism $R_{1,1} \in \text{End}_{F_2}(\Lambda_F(u_1) \otimes \Lambda_F(u_2))$ matches $R(u_2/u_1)$ of (126).*

As we already mentioned in Remark 5, part of the Conjecture above involves fixing the various choices that go into the Definition of the stable basis, with the goal of having it match $R(u_2/u_1)$ on the nose. In fact, these choices are quite strongly determined by the existence of an action:

$$U_{q_1, q_2} \bigg|_{c=1, \psi_\psi^{-1} = (q_1, q_2)^{-r}} \sim K^{T_r}(\mathcal{M}(r))_{\text{loc}}$$ (209)

which generalizes (71) under the identification (179). A way to reformulate the actions above (as $r$ runs over $\mathbb{N}$) is to frame them as an algebra homomorphism:

$$\Upsilon : U_{q_1, q_2} \bigg|_{c=1, \psi_\psi^{-1} = (q_1, q_2)^{-r}} \longrightarrow \prod_{r=1}^{\infty} \text{End}_{F_r}(K^{T_r}(\mathcal{M}(r))_{\text{loc}})$$ (210)

(implicit in the formula above is that “$r$” denotes the grading element with respect to the product in the right-hand side). On the other hand, the usual FRT formalism (explained in [14] as pertains to our
context) says that once one has $R$-matrices \[ \text{(206)} \] for all $r_1$ and $r_2$ that satisfy the quantum Yang-Baxter equation, taking arbitrary matrix coefficients of these $R$-matrices gives rise to a Hopf subalgebra:

\[ \mathcal{U} \subset \prod_{r=1}^{\infty} \text{End}_{\mathbb{C}}(K^{T_r}(\mathcal{M}(r)))_{\text{loc}} \quad (211) \]

Tautologically, $R_{r_1, r_2}$ are the images of the universal R-matrix of $\mathcal{U}$ in the representations $K^{T_r}(\mathcal{M}(r))_{\text{loc}}$.

**Conjecture 2.** The map $\mathcal{Y}$ of \[ \text{(210)} \] induces a Hopf algebra isomorphism:

\[ \mathcal{U}_{\mathfrak{g}_1, \mathfrak{g}_2} \left( \mathfrak{g}_1 \right) \bigg|_{c=1, \vec{\mu}_0 = 1, \vec{\mu}_0 = (q_1, q_2) = -r} \cong \mathcal{U} \quad (212) \]

There are several parts to proving Conjecture 2, and we will list them in increasing order of difficulty.

1. Show that the image of the map $\mathcal{Y}$ lands in the subalgebra $\mathcal{U}$ of \[ \text{(211)} \]. This is actually fairly easy, since the action \[ \text{(209)} \] is generated by tautological line bundles on the so-called simple Nakajima correspondences (see \[ \text{(20)} \]) and also the proof of Proposition \[ \text{(6)} \]; these correspondences are Lagrangian, and they fit in quite well within the framework of stable bases.

2. Show that \[ \text{(212)} \] is not just a map of algebras, but a map of bialgebras. To this end, one needs to show that the map \[ \text{(210)} \] intertwines the coproduct \[ \text{(50)} \]–\[ \text{(54)} \] on the left-hand side with the coproduct induced by stable basis maps (see \[ \text{(24)} \] Section 3.3) on the right-hand side.

3. Show that the map \[ \text{(212)} \] is a bijection: we will deal with injectivity in the following Proposition, but note that surjectivity is the more significant task.

**Proposition 6.** The map \[ \text{(210)} \] is injective.

**Proof.** There is a triangular decomposition which is orthogonal to \[ \text{(37)} \] (see \[ \text{(20)} \] for details):

\[ \mathcal{U}_{\mathfrak{g}_1, \mathfrak{g}_2} \left( \mathfrak{g}_1 \right) \bigg|_{c=1, \vec{\mu}_0 = 1, \vec{\mu}_0 = (q_1, q_2) = -r} = S \otimes \mathbb{F}[\psi^\pm_k]_{k \in \mathbb{N}} \otimes S^{op} \quad (213) \]

such that the three factors in the decomposition above act on:

\[ K^{T_r}(\mathcal{M}(r))_{\text{loc}} = \bigoplus_{n=0}^{\infty} K^{T_r}(\mathcal{M}(r)_n)_{\text{loc}} \]

by decreasing $n$, preserving $n$, and increasing $n$, respectively. More specifically:

\[ \psi^\pm(z) \cdot [\lambda^1, \ldots, \lambda^r] = [\lambda^1, \ldots, \lambda^r] \cdot \prod_{a=1}^r \left[ \frac{z - u_a q_1^{-1} q_2^{-1}}{z - u_a} \prod_{\omega = (i, j) \in \lambda^a} \xi \left( \frac{u_a q_1 q_2^{-1}}{z} \right) \right] \quad (214) \]

where $\psi^\pm(z) = \sum_{k=0}^{\infty} \psi^\pm_k z^{-k}$. In the formula above, $\xi$ denotes the rational function \[ \text{(29)} \] with $q \mapsto q_1$, $t \mapsto q_2$. Since $[\lambda^1, \ldots, \lambda^r]$ correspond to tensor products of modified Macdonald polynomials according to \[ \text{(68)} \], then the polynomial ring generated by the operators $\psi^\pm_k$ coincides with the polynomial ring generated by the (plethystically modified versions of the) Macdonald operators \[ \text{(33)} \] and \[ \text{(34)} \]. In particular, it is easy to deduce from \[ \text{(214)} \] that the elements \[ \{ h_{\pm k} \}_{k \in \mathbb{N}} \in U_{\mathfrak{g}_1, \mathfrak{g}_2} (\mathfrak{g}_1) \] of \[ \text{(68)} \] act by:

\[ h_{\pm k} \cdot [\lambda^1, \ldots, \lambda^r] = [\lambda^1, \ldots, \lambda^r] \cdot \sum_{a=1}^r \left[ \frac{u^\pm_k}{(1 - q_1^{\pm k})(1 - q_2^{\pm k})} \right] \quad (215) \]

It is sometimes convenient to replace the operators above by:

\[ h'_{\pm k} = -h_{\pm k} \pm \text{Id}_{K^{T_r}(\mathcal{M}(r))_{\text{loc}}} \cdot \sum_{a=1}^r \frac{u^\pm_k}{(1 - q_1^{\pm k})(1 - q_2^{\pm k})} \quad \Rightarrow \]

\[ h'_{\pm k} \cdot [\lambda^1, \ldots, \lambda^r] = [\lambda^1, \ldots, \lambda^r] \cdot \sum_{a=1}^r \frac{u^\pm_k}{(1 - q_1^{\pm k})(1 - q_2^{\pm k})} \cdot \sum_{\omega = (i, j) \in \lambda^a} \xi \left( \frac{u^\pm_k}{q_1^{\pm k} q_2^{\pm k}} \right) \]

Similarly, the tensor factors $S$ and $S^{op}$ act by the following formulas:

\[ F \cdot [\mu^1, \ldots, \mu^r] = \sum_{\{\mu^a \leq \lambda^a\}_{a=1}^r} [\lambda^1, \ldots, \lambda^r] \cdot F(\ldots, u_a q_1^{\pm 1} q_2^{\pm 1}, \ldots)_{c=1}^r \sum_{\omega = (i, j) \in \lambda^a \setminus \mu^a} \text{factor} \mu^1, \ldots, \mu^r \quad (216) \]
\[ G \cdot [\lambda^1, \ldots, \lambda^r] = \sum_{\{\mu^s \leq \lambda^1\}^r_{s=1}, \ldots, \sum_{i \in \mathbb{N}} |\lambda^s(\mu^s)|=n} \left[ \mu^1, \ldots, \mu^r \right] \cdot G(\ldots, u_\alpha q_1^{j_1} q_2^{j_2} \ldots)_{\beta=(i,j) \in \lambda^s(\mu^s)} \cdot \text{factor}_{\lambda^1, \ldots, \lambda^r} \]  

(217)

for any \( F(z_1, \ldots, z_n) \in \mathcal{S} \) and \( G(z_1, \ldots, z_n) \in \mathcal{S}^{op} \), where \( \mu \leq \lambda \) means that the Young diagram of the partition \( \mu \) is contained in the Young diagram of the partition \( \lambda \). In formulas (216)–(217), the terms denoted factor_{\mu^1, \ldots, \mu^r} and factor_{\lambda^1, \ldots, \lambda^r} are certain non-zero elements of \( \mathcal{F} \) that the interested reader may find in [19] Subsection 3.10; the important thing for now is that they do not depend on \( F \) and \( G \), respectively. We conclude from the formulas above that \( F(z_1, \ldots, z_n) \in \mathcal{S} \) (respectively \( G(z_1, \ldots, z_n) \in \mathcal{S}^{op} \)) acts on \( r \)-tuples of partitions by adding (respectively removing) \( n \) boxes; the respective matrix coefficient is proportional to \( F \) (respectively \( G \)) applied to the collection of weights \( \{u_\alpha q_1^{j_1} q_2^{j_2}\}_{\beta=(i,j)} \) of the added (respectively removed) boxes.

With this description of the action (209) in mind, let us prove the injectivity of the map (210). By (213), we need to show that for an arbitrary non-zero tensor:

\[ T = \sum_{i} F_i \otimes \chi_i \otimes G_i \]  

(218)

there exists a large enough \( r \) and collections of partitions \( \lambda^1, \ldots, \lambda^r, \mu^1, \ldots, \mu^r \) such that the coefficient:

\[ \left\langle [\mu^1, \ldots, \mu^r] | T | [\lambda^1, \ldots, \lambda^r] \right\rangle \]  

(219)

is non-zero. To this end, choose any \( n_1, n_2 \in \mathbb{N} \) and \( r = n_1 + n_2 \). Then we consider the following \( r \)-tuples of partitions, all of which either consist of a single box or are empty:

\[ (\lambda^1, \ldots, \lambda^r) = (\emptyset, \ldots, \emptyset, \underbrace{\emptyset, \ldots, \emptyset}_{n_1 \text{ terms}}, \underbrace{\underbrace{\emptyset, \ldots, \emptyset}_{n_2 \text{ terms}}, \ldots, \underbrace{\emptyset, \ldots, \emptyset}_{n_2 \text{ terms}}}_{r-n_1-n_2 \text{ terms}}) \]

Then let us choose \( n_1 \) and \( n_2 \) to be the maximal number of variables among the shuffle elements \( F_i \) and \( G_i \) (respectively) which appear in formula (218). The non-zero contributions to the matrix coefficient (219) all arise from having the \( G_i \)’s remove the last \( n_2 \) boxes from the \( r \)-tuple of partitions \( (\lambda^1, \ldots, \lambda^r) \) and have the \( F_i \)’s add the first \( n_1 \) boxes, in order to obtain the \( r \)-tuple of partitions \( (\mu^1, \ldots, \mu^r) \). With this in mind, the coefficient (219) equals:

\[ \sum_i F_i(u_1, \ldots, u_n) \cdot \chi_{i, h_{\lambda^1}^{\mu^1} \cdots u_{n_1+1}^{\lambda^1} \cdots u_{n_2}^{\mu^r} \cdots u_{r-n_1-n_2}^{\lambda^r}) \cdot \text{non-zero factor} \]

where the sum above only goes over those indices \( i \) such that \( F_i \) has \( n_1 \) variables and \( G_i \) has \( n_2 \) variables. Since the parameters \( u_1, \ldots, u_r \) are generic, the non-vanishing of the expression above follows from the non-vanishing of the tensor (218), and we are done.

Remark 6. Besides the algebraic approach outlined above, a more straightforward way to prove Conjecture 7 would involve comparing formulas for the two R-matrices involved, for example (139) with the trigonometric version of the product formula of [23, equation (130)] (we thank Andrey Smirnov for pointing out the latter formula to us).

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29
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