FAITHFULNESS OF DIRECTED COMPLETE POSETS BASED ON
SCOTT CLOSED SET LATTICES

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Abstract. By Thron, a topological space \( X \) has the property that \( C(X) \) isomorphic to \( C(Y) \) implies \( X \) is homeomorphic to \( Y \) iff \( X \) is sober and \( T_D \), where \( C(X) \) and \( C(Y) \) denote the lattices of closed sets of \( X \) and \( T_0 \) space \( Y \), respectively. When we consider dcpos (directed complete posets) equipped their Scott topologies, a similar question arises: which dcpos \( P \) have the property that for any dcpo \( Q \), \( C_\sigma(P) \) isomorphic to \( C_\sigma(Q) \) implies \( P \) is isomorphic to \( Q \) (such a dcpo \( P \) will be called Scott closed set lattice faithful, or SCL-faithful in short)? Here \( C_\sigma(P) \) and \( C_\sigma(Q) \) denote the lattices of Scott closed sets of \( P \) and \( Q \), respectively. Following a characterization of continuous (quasicontinuous) dcpos in terms of \( C_\sigma(P) \), one easily deduces that every continuous (quasicontinuous) dcpo is SCL-faithful. Note that the Scott space of every continuous (quasicontinuous) dcpo is sober. Compared with Thron’s result, one naturally asks whether every SCL-faithful dcpo is sober (with the Scott topology). In this paper we shall prove that some classes of dcpos are SCL-faithful, these classes contain some dcpos whose Scott topologies are not bounded sober. These results will help to obtain a complete characterization of SCL-faithful dcpos in the future.

1. Introduction

In \cite{12}, Thron proved the interesting result: a topological space \( X \) has the property that \( C(X) \) isomorphic to \( C(Y) \) implies \( X \) is homeomorphic to \( Y \) iff \( X \) is sober and \( T_D \), where \( C(X) \) and \( C(Y) \) denote the lattices of closed sets of \( X \) and \( T_0 \) space \( Y \), respectively. A directed complete poset (dcpo, for short) \( P \) will be called Scott closed set lattice faithful, or SCL-faithful in short if for any dcpo \( Q \), \( P \) is isomorphic to \( Q \) whenever the Scott-closed-set lattice \( C_\sigma(P) \) of \( P \) and \( C_\sigma(Q) \) of \( Q \) are isomorphic. One of the classic result in domain theory is that a dcpo \( P \) is continuous iff the lattice \( C_\sigma(P) \) is a completely distributive lattice (Theorem II-1.14 of \cite{3}). From this it follows that every continuous dcpo is SCL-faithful. In a similar way, one deduces that every quasicontinuous dcpo is SCL-faithful. Compared with Thron’s result, one naturally asks whether every SCL-faithful dcpo is sober in their Scott topology.

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2. Preliminaries

For any subset $A$ of a poset $P$, let $\uparrow A = \{x \in P : y \leq x \text{ for some } y \in A\}$ and $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$. A subset $A$ is called an upper set if $A = \uparrow A$, and a lower set if $A = \downarrow A$. A subset $U$ of a poset $P$ is Scott open if (i) $U = \uparrow U$ and (ii) for any directed subset $D$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, whenever $\bigvee D$ exists. All Scott open sets of a poset $P$ form a topology on $P$, denoted by $\sigma(P)$ and called the Scott topology on $P$. The complements of Scott open sets are called Scott closed sets. Clearly, a subset $A$ is Scott closed iff (i) $A = \downarrow A$ and (ii) for any directed subset $D \subseteq A$, $\bigvee D \in A$ whenever $\bigvee D$ exists. The set of all Scott closed sets of $P$ will be denoted by $C_\sigma(P)$. The space $(P, \sigma(P))$ is denoted by $\Sigma P$, called the Scott space of $P$ (See [3] for more about Scott spaces).

A poset $P$ is directed complete if its every directed subset has a supremum. A directed complete poset is briefly called a dcpo.

A subset $A$ of a topological space is irreducible if $A \subseteq F_1 \cup F_2$ with $F_1$ and $F_2$ closed, then $A \subseteq F_1$ or $A \subseteq F_2$ holds. The set of all nonempty irreducible closed subsets of a space $X$ will be denoted by $\text{Irr}(X)$.

For any $T_0$ topological space $(X, \tau)$, the specialization order $\leq_\tau$ on $X$ is defined by $x \leq_\tau y$ iff $x \in \overline{\{y\}}$ where “$\overline{\cdot}$” means taking closure.

**Remark 2.1.** (1) For any topological space $X$, $(\text{Irr}(X), \subseteq)$ is a dcpo. If $D$ is a directed subset of $\text{Irr}(X)$, the supremum of $D$ in $(\text{Irr}(X), \subseteq)$ equals $\overline{\bigcup D}$ (the closure of $\bigcup D$), which is the same as the supremum of $D$ in the complete lattice of all closed sets of $X$.

(2) For any $x \in X$, $\overline{\{x\}} \in \text{Irr}(X)$. A $T_0$ space $X$ is called sober if $\text{Irr}(X) = \{\overline{\{x\}} : x \in X\}$, that is, every nonempty irreducible closed set is the closure of a point.

(3) If $(X, \tau)$ and $(Y, \eta)$ are topological spaces such that the open set lattices $(\tau, \subseteq)$ and $(\eta, \subseteq)$ of $X$ and $Y$ are isomorphic, then the posets $\text{Irr}(X)$ and $\text{Irr}(Y)$ are isomorphic.

For a $T_0$ space $X$, a sobrification of $X$ is a sober space $Y$ together with a continuous mapping $\eta_X : X \to Y$, such that for any continuous mapping $f : X \to Z$ with $Z$ sober, there is a unique continuous mapping $\hat{f} : Y \to Z$ such that $f = \hat{f} \circ \eta_X$. The sobrification of a $T_0$ space is unique up to homeomorphism.

**Remark 2.2.** The following facts on sobrifications are well-known.

(1) If $Y$ is a sober space, then $Y$ is a sobrification of a $T_0$ space $X$ iff the closed set lattice $C(X)$ of $X$ is isomorphic to the closed set lattice $C(Y)$ of $Y$ (Equivalently, the open set lattice of $Y$ is isomorphic to that of $X$).

(2) The set $\text{Irr}(X)$ of all nonempty closed irreducible sets of a $T_0$ space $X$ equipped the hull-kernel topology is a sobrification of $X$, where the mapping $\eta_X : X \to \text{Irr}(X)$ is defined by $\eta_X(x) = \overline{\{x\}}$ for all $x \in X$. The closed sets of the hull-kernel topology consists of all sets of the form $h(A) = \{F \in \text{Irr}(X) : F \subseteq A\}$ ($A$ is a closed set of $X$).
A $T_0$ space will be called Scott sobrificable if there is a dcpo $P$ such that $\Sigma P$ is the sobrification of $X$. Also for a $T_0$ space $(X, \tau)$, $(X, \tau)$ is homeomorphic to $\Sigma P$ for some poset $P$ iff $(X, \tau)$ is homeomorphic to the Scott space $\Sigma(X, \leq)$.  

**Lemma 2.3.** A $T_0$ space $(X, \tau)$ is Scott sobrificable iff for any Scott closed set $F$ of the dcpo $\text{Irr}(X)$, there is a closed set $A$ of $X$ such that $F = h(A)$.  

**Proof.** Note that $\text{Irr}(X)$ equipped with the hull-kernel topology is a sobrification of $X$. Also every $h(A) = \{ F \in \text{Irr}(X) : F \subseteq A \}$ is a Scott closed set of the dcpo $(\text{Irr}(X), \subseteq)$, where $A$ is a closed set of $X$. Thus the hull kernel topology on $\text{Irr}(X)$ is contained in the Scott topology of $\text{Irr}(X)$. Thus $(X, \tau)$ is Scott sobrificable iff there is a dcpo $P$ such that $\text{Irr}(X)$ is homeomorphic to $\Sigma P$. This is then equivalent to that the hull kernel topology on $\text{Irr}(X)$ coincides with the Scott topology on $(\text{Irr}(X), \subseteq)$, hence every Scott closed set of $(\text{Irr}(X), \subseteq)$ equals $h(A)$ for some closed set $A$ of $(X, \tau)$.  

A topological space $(X, \tau)$ is called a d-space (or monotone convergence space) if (i) $X$ is $T_0$, (ii) the poset $(X, \leq)$ is a dcpo, and (iii) for any directed subset $D \subseteq X$, $D$ converges (as a net) to $\bigvee D$. If $(X, \tau)$ is a d-space, then every closed set $F$ of $X$ is a Scott closed set of the dcpo $(X, \leq)$.  

**Remark 2.4.** (1) Every sober space is a d-space.  
(2) Every Scott space of a dcpo is a d-space.  

**Lemma 2.5.** Let $(X, \tau)$ be a d-space. If $\{ x_i : i \in I \}$ is a directed subset of $(X, \leq)$, then the supremum $\sup \{ \text{cl}(\{x_i\}) : i \in I \}$ of $\{ \text{cl}(\{x_i\}) : i \in I \}$ in $\text{Irr}(X)$ equals $\text{cl}(\{x\})$, where $x = \bigvee \{ x_i : i \in I \}$.  

3. **Main results**  

In this section, we establish some classes of SCL-faithful dcpos, using irreducible sets, quasicontinuous elements and M property, respectively.  

A $T_0$ space is called bounded-sober if every nonempty upper bounded (with respect to the specialization order on $X$) closed irreducible subset of the space is the closure of a point [13]. Every sober space is bounded-sober, the converse implication is not true.  

If $X$ is a $T_0$ space such that every irreducible closed proper subset is the closure of an element, then $X$ is bounded-sober.  

In the following, a dcpo whose Scott space is sober (bounded-sober) will be simply called a sober (bounded-sober) dcpo.  

**Lemma 3.1.** For a bounded-sober dcpo $P$, $\Sigma P$ is Scott sobrificable if and only if $P$ is sober.  

**Proof.** We only need to check that if $\Sigma P$ is not sober, then it is not Scott sobrificable.  

Since $\Sigma P$ is not sober, there is a nonempty irreducible closed set $F$ such that $F$ is not the closure of any point. By that $\Sigma P$ is bounded-sober, one can verify that the set $\mathcal{F} = \downarrow_{\text{Irr}(\Sigma P)} \{ \text{cl}(\{x\}) : x \in F \}$ is a Scott closed set of $\text{Irr}(\Sigma P)$. But any closed set $B$ of $\Sigma P$ containing all $\text{cl}(\{x\})(x \in F)$ must contain $F$, thus $h(B) \neq \mathcal{F}$. By Lemma 2.3, $\Sigma P$ is not Scott sobrificable.
In the following, we shall write $P \cong Q$ if the two posets $P$ and $Q$ are isomorphic.

**Theorem 3.2.** Let $P$ be a sober dcpo. For any bounded-sober dcpo $Q$, if $C_\sigma(P) \cong C_\sigma(Q)$ then $P \cong Q$.

**Proof.** Let $Q$ be a bounded-sober dcpo such that $C_\sigma(P) \cong C_\sigma(Q)$. Then $\Sigma P$ is a sobrification of $\Sigma Q$. By Lemma 3.1, $\Sigma Q$ is sober, therefore $\Sigma P$ and $\Sigma Q$ are homeomorphic, which implies $P \cong Q$. \hfill \square

**Definition 3.3.** An element $a$ of a poset $P$ is called down-linear if the subposet $\downarrow a = \{x \in P : x \leq a\}$ is a chain (for any $x_1, x_2 \in \downarrow a$, it holds that either $x_1 \leq x_2$ or $x_2 \leq x_1$).

The image of a down-linear element under an order isomorphism is clearly down-linear.

**Lemma 3.4.** Let $X$ be a d-space. If $F \in \text{Irr}(X)$ is a down-linear element of the poset $\text{Irr}(X)$, then there exists a unique $x \in X$ such that $F = \text{cl}\{\{x\}\}$.

**Proof.** First, the set $\{\text{cl}\{\{x\}\} : x \in F\}$ is a subset of $\downarrow F$ in $\text{Irr}(X)$, so it is a chain. Thus $\{x : x \in F\}$ is a chain of $(X, \leq)$. Let $x = \sup\{x : x \in F\}$. Then noticing that $F$ is closed, we have $\text{cl}\{\{x\}\} = F$. \hfill \square

In the following, for a dcpo $P$, we shall use $\text{Irr}_\sigma(P)$ to denote the dcpo of all nonempty irreducible Scott closed subsets of $P$. Without specification, irreducible sets of a poset mean the irreducible sets with respect to the Scott topology.

**Theorem 3.5.** Let $P$ be a dcpo satisfying the following conditions:

(1) Let $\Sigma P$ be a nonempty irreducible Scott closed proper set $F$, $F$ is either a down-linear element of $\text{Irr}_\sigma(P)$ or it is the supremum of a directed set of down-linear irreducible closed sets.

Then $P$ is SCL-faithful.

**Proof.** Let dcpo $P$ satisfy the above condition (DL-sup) and $Q$ be a dcpo such that $C_\sigma(P) \cong C_\sigma(Q)$.

(1) Let $F \in \text{Irr}_\sigma(P)$ and $F \neq P$. If $F$ is down-linear, then by Lemma 3.4 $F$ is the closure of a unique point. If $F$ is the supremum of a directed set of down-linear irreducible closed sets, then by Lemma 3.4 $F$ can be represented as the directed supremum of $\text{cl}\{\{x_i\}\}(i \in J)$. Thus, $\{x_i : i \in J\}$ is a directed set of $P$. Let $x = \sup\{x_i : i \in J\}$. Then noticing that $F$ is closed, we have that $\text{cl}\{\{x\}\} = F$ is also a closure of a point.

(2) Since $C_\sigma(P) \cong C_\sigma(Q)$, $Q$ also satisfies condition (DL-sup). As the proof of (1) only make use of condition (DL-sup), so every nonempty closed irreducible proper subset of $\Sigma Q$ is the closure of a point.

By the definition of the bounded-sobriety, we see that (1) and (2) imply that $\Sigma P$ and $\Sigma Q$ are all bounded-sober.

(3) If either $\Sigma P$ or $\Sigma Q$ is sober, then by Theorem 3.2 $P \cong Q$. If neither $\Sigma P$ nor $\Sigma Q$ is sober, then $P$ and $Q$ are irreducible sets and are not the closure of any point. Thus $Q \cong \{\text{cl}\{\{y\}\} : y \in Q\} \cong \text{Irr}_\sigma(Q) - \{Q\} \cong \text{Irr}_\sigma(P) - \{P\} \cong \{\text{cl}\{\{x\}\} : x \in P\} \cong P$, as desired. \hfill \square

**Example 3.6.** In [9], Johnstone constructed the first non-sober dcpo as $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with partial order defined by

$(m, n) \leq (m', n') \iff$ either $m = m'$ and $n \leq n'$ or $n' = \infty$ and $n \leq m'$.

Then
\( (a) \) \((X, \leq)\) is a dcpo, \(X\) is irreducible and \(X \neq \text{cl}(\{x\})\) for any \(x \in X\).

\( (b) \) If \(F\) is a proper irreducible Scott closed set of \(X\), then \(F = \downarrow (m,n)\) for some \((m,n) \in X\).

\( (c) \) If \(m \neq \infty\), \(\downarrow (m,n)\) is a down-linear element of \(\text{Irr}_\sigma(X)\). If \(m = \infty\), then \(\downarrow (m,n)\) is the supremum of the chain \(\{\downarrow (m,k) : k \neq \infty\}\) whose members are down-linear.

Hence by Theorem 3.9, we deduce that dcpo \(X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})\) is SCL-faithful. Thus an SCL-faithful dcpo need not be sober.

Next, we give another class of SCL-faithful dcpos.

**Remark 3.7.** (cf. [11]) Let \(A\) be a nonempty Scott closed set of a dcpo \(P\). Then

(i) \(A\) is a dcpo.

(ii) For any subset \(B \subseteq A\), \(B\) is a Scott closed set of dcpo \(A\) iff it is a Scott closed set of \(P\). Thus \(C_\sigma(A) = \downarrow_{C_\sigma(P)} A = \{B \in C_\sigma(P) : B \subseteq A\}\).

A finite subset \(F\) of a dcpo \(P\) is way-below an element \(a \in P\), denoted by \(F \ll a\), if for any directed subset \(D \subseteq P\), \(a \leq \bigvee D\) implies \(D \cap F \neq \emptyset\). A dcpo \(P\) is quasicontinuous if for any \(x \in P\), the family

\[
\text{fin}(x) = \{F : F \text{ is finite and } F \ll x\}
\]

is a directed family (for any \(F_1, F_2 \in \text{fin}(x)\) there is \(F \in \text{fin}(x)\) such that \(F \subseteq \downarrow F_1 \cap \uparrow F_2\)) and for any \(y \not\ll x\) there is \(F \in \text{fin}(x)\) satisfying \(y \not\ll F\) (see Definition III-3.2 of [3]). Every continuous dcpo is quasicontinuous.

Every quasicontinuous dcpo is sober (Proposition III-3.7 of [3]). A dcpo \(P\) is quasicontinuous iff the Scott open set lattice of \(P\) is hypercontinuous (Theorem VII-3.9 of [3]). From this and Remark 4, we have the following.

**Lemma 3.8.** Every quasicontinuous dcpo is SCL-faithful.

An element \(x\) of a dcpo \(P\) is called a quasicontinuous element if the sub-dcpo \(\downarrow x\) is a quasicontinuous dcpo.

**Theorem 3.9.** Let \(P\) be a dcpo. Then \(P\) is SCL-faithful if it satisfies the following two conditions:

1. \(\Sigma P\) is bounded sober;
2. every element of \(P\) is the supremum of a directed set of quasicontinuous elements.

**Proof.** Assume that \(P\) is a dcpo satisfying the two conditions. Let \(Q\) be a dcpo and \(F : C_\sigma(P) \to C_\sigma(Q)\) be an isomorphism. Then \(F\) restricts to an isomorphism \(F : \text{Irr}_\sigma(P) \to \text{Irr}_\sigma(Q)\).

1. Let \(x \in P\) be a quasicontinuous element. Then \(F(\downarrow x)\) is in \(C_\sigma(Q)\) and \(\downarrow_{C_\sigma(P)} (\downarrow x) = \{B \in C_\sigma(P) : B \subseteq \downarrow x\}\) is isomorphic via \(F\) to \(\downarrow_{C_\sigma(Q)} F(\downarrow x) = \{E \in C_\sigma(Q) : E \subseteq F(\downarrow x)\}\) (all Scott closed sets of \(F(\downarrow x)\)). Since \(\downarrow x\) is quasicontinuous, it is SCL-faithful. Hence \(\downarrow x\) is isomorphic to \(F(\downarrow x)\), implying that there is a largest element in \(F(\downarrow x)\), denoted by \(f(x)\). It is easily observable that the mapping \(f\) is well defined on the set of quasicontinuous elements of \(P\), and for any two quasicontinuous elements \(x_1, x_2 \in P\), \(f(x_1) \leq f(x_2)\) iff \(x_1 \leq x_2\).

2. If \(x \in P\) is the supremum of a directed set \(\{x_i : i \in I\}\) of quasicontinuous elements \(x_i\), then \(F(\downarrow x) = \text{sup}_{\text{Irr}_\sigma(Q)} \{F(\downarrow x_i) : i \in I\} = \text{sup}_{\text{Irr}_\sigma(Q)} \{\downarrow f(x_i) : i \in I\} = \downarrow y_x\), where \(y_x = \text{sup}_Q \{f(x_i) : i \in I\}\) and \(f(x_i)\) is the element in \(Q\) defined for quasicontinuous elements \(x_i\) in (1). Let \(f(x) = y_x\) again.
Thus we have an monotone mapping \( f : P \rightarrow Q \). Following that \( F \) is an isomorphism, we have that \( f(x_1) \geq f(x_2) \) iff \( x_1 \geq x_2 \).

It remains to show that \( f \) is surjective.

(3) If \( y \in \downarrow f(P) \), then \( \downarrow y \subseteq F(\downarrow x) \) for some \( x \in P \). Since \( F \) restricts to an isomorphism between the dcpos \( \text{Irr}_\sigma(P) \) and \( \text{Irr}_\sigma(Q) \), there is \( H \in \text{Irr}_\sigma(P) \) such that \( H \subseteq \downarrow x \) and \( F(H) = \downarrow y \). But \( P \) is bounded-sober, so \( H = \downarrow x' \) for some \( x' \in P \). It follows that \( y = f(x') \), implying \( y \in f(P) \). Therefore \( f(P) \) is a down set of \( Q \). Also clearly \( f(P) \) is closed under sups of directed sets, so it is a Scott closed subset of \( Q \).

(4) Since \( F \) is an isomorphism between the lattices \( C_\sigma(P) \) and \( C_\sigma(Q) \), \( Q = F(P) = F(\text{sup}_{C_\sigma(P)}(\{x : x \in P\})) = \text{sup}_{C_\sigma(Q)}(F(\{x : x \in P\})) = \text{sup}_{C_\sigma(Q)}(\{f(x) : x \in P\}) \).

For each \( x \in P \), \( \downarrow f(x) \subseteq f(P) \) and \( f(P) \) is a Scott closed set of \( Q \), it holds then that \( \text{sup}_{C_\sigma(Q)}(\{f(x) : x \in P\}) \subseteq f(P) \). Therefore \( Q \subseteq f(P) \), which implies \( Q = f(P) \). Hence \( f \) is also surjective. The proof is thus completed.

If \( x \in P \) is a down-linear element of a dcpo \( P \), then \( \downarrow x \) is a chain, so it is continuous (hence quasicontinuous).

**Corollary 3.10.** If \( P \) is a dcpo satisfying the following conditions, then \( P \) is SCL-faithful:

1. \( P \) is bounded-sober.
2. every element \( a \in P \) is the supremum of a directed set of down-linear elements.

**Example 3.11.** In order to answer the question whether every well-filtered dcpo is sober posed by Heckmann [5], Kou [10] constructed another non-sober dcpo \( P \) as follows:

Let \( X = \{x \in \mathbb{R} : 0 < x \leq 1\} \), \( P_0 = \{(k,a,b) \in \mathbb{R} : 0 < k < 1, 0 < b \leq a \leq 1\} \) and \( P = X \cup P_0 \).

Define the partial order \( \sqsubseteq \) on \( P \) as follows:

(i) for \( x_1, x_2 \in X \), \( x_1 \sqsubseteq x_2 \) iff \( x_1 = x_2 \);
(ii) \((k_1, a_1, b_1) \sqsubseteq (k_2, a_2, b_2) \) iff \( k_1 \leq k_2, a_1 = a_2 \) and \( b_1 = b_2 \).
(iii) \( (k, a, b) \sqsubseteq x \) iff \( a = x \) or \( kb \leq x < b \).

If \( u = (h, a, b) \in P_0 \), then \( \downarrow u = \{(k, a, b) : k \leq h\} \) is a chain. If \( u = x \in P_0 \), then \( u = \{\{(k, x, x) : 0 < k < 1\}, \) where each \((k, x, x) \) is a down-linear element and \( \{(k, x, x) : 0 < k < 1\} \) is a chain. Thus \( P \) satisfies (2) of Corollary 3.10.

Let \( F \) be an irreducible nonempty Scott closed set of \( P \) with an upper bound \( v \). If \( v = (h, a, b) \in P_0 \), then \( F \subseteq (h, a, b) = \{(k, a, b) : k \leq h\} \). Take \( m = \bigvee \{ k : (k, a, b) \in F \} \). Then \( F = (m, a, b) \), is the closure of point \((m, a, b) \).

Now assume that \( F \) does not have an upper bound in \( P_0 \), then \( v = x \) for some \( x \in P_0 \). If \( v \not\in F \), then due to the irreducibility of \( F \), there exist \( a, b \) such that \( F \subseteq \{(k, a, b) : 0 < k < 1\} \), which will imply that \( F \) has an upper bound of the form \((m, a, b)\), contradicting the assumption. Therefore \( v \in F \), implying that \( F = (m, a, b) \) is a lower set is the closure of point \( v \).

It thus follows that \( P \) satisfies (1) as well. By Corollary 3.10, \( P \) is SCL-faithful.

In [7], Ho and Zhao introduced the following notions.

**Definition 3.12.** Let \( L \) be a poset and \( x, y \in L \). The element \( x \) is beneath \( y \), denoted by \( x \prec y \), if for every nonempty Scott-closed set \( S \subseteq L \) with \( \bigvee S \) existing, \( y \leq \bigvee S \) implies \( x \in S \). An element \( x \) of \( L \) is called C-compact if \( x \prec x \). Let \( \kappa(L) \) denote the set of all the C-compact elements of \( L \).
Let $P$ be a poset, $A \subseteq P$ finite. The set $\text{mub}(A)$ of the minimal upper bounds of $A$ is complete, if for any upper bound $x$ of $A$, there exists $y \in \text{mub}(A)$ such that $y \leq x$.

A poset $P$ is said to satisfy property $m$, if for all finite set $A \subseteq P$, $\text{mub}(A)$ is complete.

A poset $P$ is said to satisfy property $M$, if $P$ satisfies property $m$ and for all finite set $A \subseteq P$, $\text{mub}(A)$ is finite.

**Remark 3.13.** Let $L$ be a complete lattice and $a \in L$ be a $C$-compact element. If $x, y \in L$ such that $a \leq x \lor y$, then $a \leq \bigvee(\downarrow x \lor \downarrow y)$ and $\downarrow x \lor \downarrow y$ is Scott closed, so $a \in \downarrow x \lor \downarrow y$, implying $a \leq x$ or $a \leq y$. Thus $a$ is $\lor$-irreducible.

**Corollary 3.14.** For any dcpo $P$, $\kappa(C_\sigma(P)) \subseteq \text{Irr}_\sigma(P)$. That is $C$-compact closet sets are all irreducible.

**Lemma 3.15.** Let $P$ be a dcpo. Then

(1) For all $x \in P$, $\downarrow x \in \kappa(C_\sigma(P))$.

(2) If $P$ satisfies property $M$, then $A \in \kappa(C_\sigma(P))$ iff $A = \downarrow x$ for some $x \in P$.

The following theorem gives the third class of SCL-faithful dcpos using property $M$.

**Theorem 3.16.** If $P$ is a dcpo satisfying property $M$ and the condition (2) in Theorem 3.9 then $P$ is SCL-faithful.

**Proof.** Let $P$ be a dcpo satisfying condition (2) in Theorem 3.9 and property $M$, and $Q$ be a dcpo with an order isomorphism $F : C_\sigma(P) \to C_\sigma(Q)$.

Then the restrictions $F : \kappa(C_\sigma(P)) \to \kappa(C_\sigma(Q))$ and $F : \text{Irr}_\sigma(P) \to \text{Irr}_\sigma(Q)$ are all order isomorphisms.

For each $q \in Q$, by Lemma 3.15(1), $\downarrow q \in \kappa(C_\sigma(Q))$, then $F^{-1}(\downarrow q) = \downarrow x_q$ for a unique $x_q \in P$ by Lemma 3.15(2). Now define a map $g : Q \to P$ such that $g(q) = x_q$ iff $F^{-1}(\downarrow q) = \downarrow x_q$. The mapping $g$ satisfies the condition that $g(q_1) \leq g(q_2)$ iff $q_1 \leq q_2$ since $F^{-1}$ is an isomorphism. Note that $\kappa(C_\sigma(Q)) \cong \kappa(C_\sigma(P)) \cong P$ is a dcpo.

Since $P$ satisfies the condition (2) in Theorem 3.9 by the proof of Theorem 3.9 there is a monotone mapping $f : P \to Q$ such that $F(\downarrow x) = \downarrow f(x)$ holds for every $x \in P$ (note that parts (1) and (2) of proof of Theorem 3.9 do not need the condition that $P$ is bounded sober).

Then for any $x \in P$, $\downarrow x = F^{-1}(\downarrow f(x))$, so $x = g(f(x))$. Thus $g : Q \to P$ is also a surjective mapping, therefore an isomorphism between $P$ and $Q$, as desired.

Note that Kou’s and Johnstone’s examples of dcpos are bounded sober but do not have property $M$.

4. Remarks and some possible further work

We close the paper with some extra remarks and problems for further exploration.

**Remark 4.1.** (1) Recently, Ho, Jung and Xi [6] constructed a pair of non-isomorphic dcpos having isomorphic Scott topologies, showing the existence of non-SCL-faithful dcpos. Their counterexample also reveals that sobriety is not a sufficient condition for a dcpo to be SCL-faithful.

(2) If $P$ is an SCL-faithful dcpo and $P^*$ is the dcpo obtained by adding a top element to $P$, then one can show that $P^*$ is also SCL-faithful. Let $X$ be the dcpo of Johnstone. Then $X^*$ is SCL-faithful, but $X^*$ is not bounded sober ($X$ is an irreducible Scott closed set of $X^*$.
which is not the closure of any point of \(X^*\). Thus a SCL-faithful dcpo need not be bounded sober. So, bounded sobriety is not a necessary condition for a dcpo to be SCL-faithful.

(3) The bounded sobriety in Theorem 3.9 might be further weakened. We can try other conditions which are weaker than this, like the "dominatedness" used in [6].

(4) Given a class \(\mathcal{M}\) of dcpos, define
\[
\mathcal{M}^\flat = \{P : P \text{ is a dcpo and for any } Q \in \mathcal{M}, C_\sigma(P) \cong C_\sigma(Q) \text{ implies } P \cong Q\}.
\]

A class \(\mathcal{M}\) of dcpos is called reflexive if \(\mathcal{M}^{\flat \flat} = \mathcal{M}\).

The class \(S\) of all SCL-faithful dcpos and the class DCPO of all dcpos are reflexive. Do we have other reflexive classes of dcpos other than these two?

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References

[1] Drake, D. and Thron, W. J.; On the representation of of an abstract lattice as the family of closed sets of a topological space, Trans. Amer. Math. Soc. 120(1965), 57 - 71.
[2] Engelking, R.: General Topology, Vol.6, Sigma Series in Pure Mathematics, 1989.
[3] Gierz, G. et al.: Continuous lattices and Domains, Encyclopedia of Mathematics and Its Applications, Vol.93, Cambridge University Press, 2003.
[4] He, Q. Y. and Xu, L. S.: The C-algebraicity of Scott-closed lattices and its applications, Applied Mathematics: A Journal of Chinese Universities(Series A) 369-374. (in Chinese)
[5] Heckmann, R.: An upper power domain construction in terms of strongly compact sets, Lecture Notes in Computer Science 598, Springer-Verlag, 1992, 272-293.
[6] Ho, W. K. and Jung, A. and Xi, X. Y.: The Ho-Zhao problem, 2016 (priprint).
[7] Ho, W. K and Zhao, D. S.: Lattices of Scott-closed sets, Comment. Math. Univ. Carolin. 50 (2009) 297-314.
[8] Isbell, J. R.: Completion of a construction of Johnstone, Proc. Amer. Math. Soc., 85(1982), 333-334.
[9] Johnstone, P.: Scott is not always sober, In: Continuous Lattices, Lecture Notes in Math. 871, Springer-Verlag, (1981), 282-283.
[10] Kou, H.: well-filtered DCPOS Need not be Sober, In: Domains and Processes, Semantic Structures in Computation, 1(2001), 41-50.
[11] Lawson, J. D. and Xu, L. S.: Posets Having Continuous Intervals, Theoretical Computer Science, 316(2004), 89-103.
[12] Thron, W. J.: Lattice-equivalence of topological spaces, Duke Math. J. 29 (1962), 671-679.
[13] Zhao, D. S. and Fan, T. H.: dcpo-completion of posets, Theoretical Computer Science 411(2010), 2167-2173.