Delayed state synchronization of continuous-time multi-agent systems in the presence of unknown communication delays

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Abstract: This paper studies delayed synchronization of continuous-time multi-agent systems (MAS) in the presence of unknown nonuniform communication delays. A delay-free transformation is developed based on a communication network which is a directed spanning tree, which can transform the original MAS to a new one without delays. By using this transformation, we design a static protocol for full-state coupling and a dynamic protocol for delayed state synchronization for homogeneous MAS via full- and partial-state coupling. Meanwhile, the delayed output synchronization is also studied for heterogeneous MAS, which is achieved by using a low-gain and output regulation based dynamic protocol design via the delay-free transformation.

Key Words: Multi-agent systems, Delayed state synchronization, Continuous-time, Communication delays

1 Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1, 7, 9, 11, 13, 17, 24] and references therein).

Recently synchronization in a network with time delay has attracted a great deal of interest. As clarified in [3], we can identify two kinds of delay. Firstly there is communication delay, which results from limitations on the communication between agents. Secondly we have input delay, which is due to computational limitations of an individual agent. Many works have focused on dealing with input delay (see e.g. [10, 12, 16, 20, 22, 23, 26, 27]), but communication delay is much less understood at this moment. In the case of communication delay, only for a constant synchronization trajectory do we preserve the diffusive nature of the network. This diffusive nature is an intrinsic part of the currently available design techniques and hence only this case has been studied. Some works in this area can be seen in [2, 8, 21, 22].

References [4] and [5] solved the synchronization problem for nonlinear heterogeneous MAS with unknown non-uniform constant communication delays. Some other results for non-uniform communication delays can also be found in [8, 14, 15, 21, 25].

On the other hand, the concept of delayed synchronization was introduced in [4] and [5]. Compared with general synchronization problem, delayed synchronization allows a fixed signal lag from parents node to their son node when constant communication delay is considered. It means that there exists a fixed distance (or some other physical quantities) between two agents to keep moving. But, due to the restriction of strongly connected network, the synchronized trajectory must converge to a constant.

In this paper, we study delayed synchronization problems of MAS in the presence of unknown communication delays. The communication network is assumed to be a directed spanning tree (i.e., it has one root node and the other non-root nodes have indegree one). The contribution of this paper is threefold:

- A delay-free transformation is established to remove the effect from unknown communication delays, and obtain a transformed MAS without communication delays.
- We develop the delayed state synchronization results of homogeneous MAS based on the delay-free transformation, and obtain a dynamic synchronized trajectory. Static and dynamic protocol designs are provided for both full- and partial-state coupling cases respectively.
• We also develop the delayed output synchronization results of heterogeneous MAS. A low-gain and output regulation based dynamic protocol design is provided via the delay-free transformation.

We will see that, compared to earlier work, our approach is limited to a graph which is a directed spanning tree. However, the intrinsic advantage is that the synchronized trajectory is not limited to a constant but will follow the trajectory of the root agent.

**Notations and definitions:** Given a matrix $A \in \mathbb{R}^{m \times n}$, $A^T$ and $A^*$ denote the transpose and conjugate transpose of $A$, respectively while $\|A\|$ denotes the induced 2-norm of $A$. A square matrix $A$ is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between $A$ and $B$. $I_n$ denotes the $n$-dimensional identity matrix and $0_n$ denotes $n \times n$ zero matrix; we will use $I$ or $0$ if the dimension is clear from the context.

A weighted directed graph $G$ is defined by a triple $(V, E, \mathcal{A})$ where $V = \{1, \ldots, N\}$ is a node set, $E$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighting matrix, and $a_{ij} > 0$ if $(i, j) \in E$. Each pair in $E$ is called an edge. A path from node $i_k$ to $i_l$ is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k - 1$.

A directed tree is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the root, which has no parent node. In this case, the root has a directed path to every other node in the tree. A directed spanning tree is a subgraph which is a directed tree containing all the nodes of the original graph. An agent is called a root agent if it is the root of some directed spanning tree of the associated graph. Let $\Pi_G$ denote the set of all root agents for a graph. For a weighted graph $G$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the Laplacian matrix associated with the graph $G$.

In the case where $G$ has non-negative weights, $L$ has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $1$.

### 2 Problem formulation

We will study a MAS consisting of $N$ identical agents:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^p$ are the state, input and the output, respectively, of agent $i$ for $i = 1, \ldots, N$.

**Assumption 1** We assume that

• $(A, B, C)$ is stabilizable and detectable.

• All eigenvalues of $A$ are in the closed left half complex plane.

The communication network provides agent $i$ with the following information,

$$z_i(t) = \sum_{j=1}^{N} a_{ij} [y_j(t) - y_j(t - \tau_{ij})]$$

(3)

where $a_{ij} \geq 0$ and $a_{ii} = 0$. This communication topology of the network can be described by a weighted graph $\mathcal{G}$ with weighting matrix $\mathcal{A} = [a_{ij}]$. We can obtain the associated Laplacian matrix $L$ via $[1]$. Here $\tau_{ij} \in \mathbb{R}^+$ represents an unknown constant communication delays from agent $j$ to agent $i$. This communication delay implies that it takes $\tau_{ij}$ seconds for agent $j$ to transfer its state information to agent $i$. Furthermore, we assume Agent 1 is root of graph in this paper.

**Definition 1** For any $\beta > 0$, let $\mathcal{G}_\beta^N$ denote the set of directed graphs with $N$ nodes which are equal to a directed spanning tree for which the corresponding Laplacian matrix $L$ is lower triangular with the top row identical to zero which has the property that $\ell_{ii} \geq \beta$ for $i = 2, \ldots, N$ while agent 1 is the root agent. Similarly for any $\alpha > \beta > 0$, let $\mathcal{G}_\alpha^N$ denote the set of directed graphs with $N$ nodes which are equal to a directed spanning tree for which the corresponding Laplacian matrix $L$ is lower triangular with the first row equal to zero with the property that $\beta \leq \ell_{ii} \leq \alpha$ for $i = 2, \ldots, N$.

**Remark 1** Note that any graph which is a directed spanning tree will have a lower triangular Laplacian matrix after a possible reordering of the agents.

For the graph defined by Definition[1] we know the Laplacian matrix $L$ is lower diagonal with the first row identical to zero. Therefore, we have

$$L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \ell_{N1} & \cdots & \ell_{N,N-2} & \ell_{N,N-1} & \ell_{N,N} \end{pmatrix}.

$$

Since the graph is equal to a directed spanning tree, there are in every row (except the first one) exactly two elements unequal to 0.

Our goal is to achieve delayed state synchronization among agents in a MAS, that is

$$\lim_{t \to \infty} \left[ x_i(t) - x_j(t - \tau_{ij}) \right] = 0,$$

(4)

for all $i, j \in \{1, \ldots, N\}$.

For the MAS (2), we formulate delayed state synchronization problems as follows.

**Problem 1** Consider a MAS described by agents (2) and (3) associated with a directed graph $G \in \mathcal{G}_\beta^N$ is equal to a spanning tree, where $\mathcal{G}_\beta^N$ is defined in Definition[1]. The delayed
state synchronization problem with a set of graphs $G^N_\alpha$ in the presence of unknown, nonuniform, arbitrarily large communication delays is to find a distributed static protocol of the type,

$$u_i(t) = F^\top(t), \quad (i = 1, \ldots, N)$$

(5)

for each agent such that (4) is satisfied for all $i, j \in \{1, \ldots, N\}$, for any directed graph $G \in G^N_\alpha$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$. 

**Problem 2** Consider a MAS described by agents (2) and (3) associated with a directed graph $G \in G^N_\alpha$ is equal to a spanning tree, where $G^N_\alpha$ is defined in Definition 1. The delayed state synchronization problem with a set of graphs $G^N_\alpha$ in the presence of unknown, nonuniform, arbitrarily large communication delays is to find a distributed dynamic protocol of the type,

$$\begin{align*}
\tilde{x}_i(t) &= A^\top x_i(t) + B^\top_c \zeta_i(t), \\
u_i(t) &= C^\top x_i(t) + D^\top_c \zeta_i(t),
\end{align*}$$

(6)

for each agent such that (4) is satisfied for all $i, j \in \{1, \ldots, N\}$, for any directed graph $G \in G^N_\alpha$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$. 

### 3 Delayed state synchronization for homogeneous MAS with communication delays

In this section, we will give delayed state synchronization results based on algebraic Riccati equation for full- and partial-state coupling.

#### 3.1 Full-state coupling

Firstly, we consider full-state coupling (i.e., $C = I$). We define

$$\tilde{x}_i(t) = x_i(t + \bar{\tau}_{i,1})$$

where $\bar{\tau}_{i,1}$ denotes the sum of delays from agent $i$ to the root (agent 1) based on its path, and $\tau_{ij} = \bar{\tau}_{i,1} - \bar{\tau}_{j,1}$. Then, we have

$$\begin{align*}
\zeta_i(t) &= \zeta_i(t + \bar{\tau}_{i,1}) = \sum_{j=1}^N a_{ij} \left[ x_i(t + \bar{\tau}_{i,1}) - x_j(t + \bar{\tau}_{i,1} - \tau_{ij}) \right] \\
&= \sum_{j=1}^N a_{ij} (\tilde{x}_i(t) - \tilde{x}_j(t))
\end{align*}$$

(7)

and $\tilde{u}_i(t) = u_i(t + \bar{\tau}_{i,1})$. Especially, we have $\bar{\tau}_{i,1} = 0$, i.e., $\tilde{x}_i(t) = x_i(t)$ and $\tilde{u}_i(t) = 0$ (since $\zeta_i(t) = \zeta_i(t) = 0$). Thus, (2), (3) and (5) yield:

$$\dot{\tilde{x}}_i(t) = A \tilde{x}_i(t) + \sum_{j=1}^N \ell_{ij} BF \tilde{x}_j(t)$$

(8)

Thus, our synchronization objective can be expressed as

$$\lim_{t \to \infty} \left[ \tilde{x}_i(t) - \tilde{x}_j(t) \right] = 0.$$

We define

$$\tilde{x}(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \vdots \\ \tilde{x}_N(t) \end{pmatrix}.$$ 

We have

$$\dot{\tilde{x}}(t) = (I \otimes A + L \otimes (BF)) \tilde{x}(t).$$

(9)

This is referred to as our delay-free transformation. Further, let

$$\eta_1(t) = \tilde{x}_1(t), \quad \text{and} \quad \eta_i(t) = \tilde{x}_i(t) - \tilde{x}_1(t) \quad \text{with} \quad i = 2, \ldots, N,$

we have

$$\eta(t) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix} = (T \otimes I) \tilde{x}(t).$$

(10)

where

$$L_Q = TL_T^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & L_Q \end{pmatrix} \begin{pmatrix} 0 & \ell_{21} & \cdots & \ell_{N1} \\ \ell_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{N2} & \cdots & \ell_{N,N-1} & 0 \end{pmatrix}.$$ 

(11)

and $L_Q$ is a positive-definite lower triangular matrix. Obviously, due to the structure of $L_Q$, the synchronization of (9) is equivalent to the asymptotic stability of the following $N - 1$ subsystems,

$$\dot{\eta}_i(t) = (A + \ell_{ii} BF) \eta_i(t), \quad i = 2, \ldots, N.$$ 

(12)

If (12) is globally asymptotically stable for $i = 2, \ldots, N$, we see from the above that $\eta_i(t) \to 0$ for $i = 2, \ldots, N$. This implies that

$$\tilde{x}(t) - (T^{-1} \otimes I) \begin{pmatrix} \eta_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to 0.$$

Note that the first column of $T^{-1}$ is equal to the vector $1$ and therefore

$$\tilde{x}_i(t) - \eta_i(t) \to 0$$

for $i = 1, \ldots, N$. This implies that we achieve state synchronization.
Conversely, suppose that the network (12) reaches state synchronization. In this case, we shall have
\[ \ddot{x}(t) = 0 \]
for all initial conditions. Then \( \eta(t) - (T1) \otimes \ddot{x}_1(t) \rightarrow 0 \). Since 1 is the first column of \( T \), we have
\[
T1 = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]
Therefore, \( \eta(t) - (T1) \otimes \ddot{x}_1(t) \rightarrow 0 \) implies that \( \eta_i(t) - \ddot{x}_1(t) \rightarrow 0 \) and \( \eta_i(t) \rightarrow 0 \) for \( i = 2, \ldots, N \) for all initial conditions. Thus, we obtain the following lemma.

**Lemma 1** The MAS (2) achieves state synchronization if and only if the system (12) is globally asymptotically stable for \( i = 2, 3, \ldots, N \). The synchronized trajectory converges to the trajectory of the root agent.

We let
\[
w = (1 \ 0 \ \cdots \ 0)
\]
be the normalized left eigenvector associated with the zero eigenvalue of \( L \). Then, from Lemma 1 we have
\[
\eta_i(t) = A\eta_i(0), \quad \eta_i(0) = \ddot{x}_1(0),
\]
In other words, \( \eta_i(0) \) is the initial condition of the root agent. Therefore, the synchronized trajectory given by (13) yields that the synchronized trajectory is given by
\[
x_i(t) = e^{At} \ddot{x}_1(0) = e^{At} x_1(0),
\]
which is the trajectory of the root agent and delay-free. Therefore, the root agent is sometimes referred to as the leader.

**Protocol design:** For full-state coupling, we design a parameterized static protocol of the form:
\[
u_i(t) = -\rho B^T P \dot{\epsilon}_i(t),
\]
where \( P > 0 \) is the unique solution of the continuous-time algebraic Riccati equation,
\[
A^T P + PA - PBB^T P + Q = 0,
\]
with \( Q > 0 \) and \( \rho \geq \frac{1}{P} \) with \( \ell_{ii} \geq \beta > 0 \).

Using an algebraic Riccati equation we can design a suitable protocol provided \( (A, B) \) is stabilizable. The synchronization based on protocol (15) is as follows.

**Theorem 1** Consider a MAS described by (2) and (3). Let any \( \beta > 0 \) be given, and consider the set of network graphs \( G_n^\beta \) with \( \ell_{ii} \geq \beta \).

If \( (A, B) \) is stabilizable, then the state synchronization problem stated in Problem 1 with \( G = G_n^\beta \) is solvable. In particular, the protocol (15) solves the state synchronization problem for any graph \( G \in G_n^\beta \) and \( \tau_{ij} \in \mathbb{R}^+ \). Moreover, the synchronized trajectory is given by (14).

**Proof:** For protocol (15), we can obtain
\[
\ddot{u}_i(t) = -\rho B^T P \dot{\epsilon}_i(t).
\]
By using the delay-free transformation, it means that we only need to prove that the system
\[
\dot{z}(t) = (A - \ell_{ii} \rho BB^T P)z(t)
\]
is asymptotically stable for any \( \ell_{ii} \) that satisfies \( \ell_{ii} \geq \beta \). We observe that
\[
(A - \ell_{ii} \rho BB^T P)^T P + P(A - \ell_{ii} \rho BB^T P)
\]
\[
= -Q - (2\ell_{ii} - 1) PBB^T P
\]
\[
\leq -Q.
\]
Therefore, \( (A - \ell_{ii} \rho BB^T P) \) is Hurwitz stable for any \( \ell_{ii} \geq \beta > 0 \). Based on Lemma 1 the delayed state synchronization result can be proved.

### 3.2 Partial-state coupling

In the following, we give the transformation for partial state coupling. Similar to the case of full-state coupling, we have the following expression for (2) and (3) by using our delay-free transformation,
\[
\begin{align}
\dot{x}_i(t) &= A\tilde{x}_i(t) + B\bar{u}_i(t) \\
\bar{z}_i(t) &= \sum_{j=1}^{N} a_{ij}(\tilde{y}_i(t) - \bar{y}_j(t)),
\end{align}
\]
with \( \tilde{x}_i(t) = x_i(t - \bar{\tau}_{ii}), \) \( \bar{y}_j(t) = y_j(t - \bar{\tau}_{ii}) \).

The MAS described by (2) and (3) after implementing the dynamic protocol (6) is described by
\[
\begin{align}
\dot{\tilde{x}}_i(t) &= A\tilde{x}_i(t) + B\bar{u}_i(t) \\
\bar{z}_i(t) &= \sum_{j=1}^{N} a_{ij}(\tilde{y}_i(t) - \bar{y}_j(t)),
\end{align}
\]
for \( i = 1, \ldots, N \), where
\[
\tilde{x}_i(t), \quad \bar{z}(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_N(t) \end{pmatrix}
\]
Define
\[
\tilde{A} = \begin{pmatrix} A & BC_c \\ 0 & A_c \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} BD_c \\ B_c \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}.
\]
Then, the overall dynamics of the \( N \) agents can be written as
\[
\dot{\bar{z}}(t) = (I_N \otimes \tilde{A} + L \otimes \tilde{BC})\bar{z}(t).
\]
So, this is the delay-free system obtained after our transformation for MAS with unknown communication delays via partial state coupling.
The synchronization of (20) is equivalent to the asymptotic stability of the following $N - 1$ subsystems,
\[
\dot{\eta}_i = (\bar{A} + \ell_{ii} B\bar{C})\eta_i, \quad i = 2, \ldots, N.
\] (21)

Similar to Lemma 1, we obtain the following lemma for partial state coupling.

**Lemma 2** The MAS (20) achieves state synchronization if and only if the system (21) is globally asymptotically stable for $i = 2, 3, \ldots, N$. The synchronized trajectory converges to the trajectory of the root agent of (20).

Meanwhile, we have
\[
\dot{\eta}_1(t) = A\dot{\eta}_1(t), \quad \dot{\eta}_1(0) = \bar{x}_1(0).
\] (22)

Therefore, the synchronized trajectory given by (22) yields that the synchronized trajectory is given by
\[
x_i(t) = (I \ 0)e^{A t}\bar{x}_1(0) = e^{A t}x_1(0).
\] (23)

**Protocol design:** For partial-state coupling, we design a parameterized dynamic protocol of the form:
\[
\begin{align*}
\dot{x}_i(t) &= (A + KC)x_i(t) - K\zeta_i(t), \\
u_i(t) &= -\beta^{-1}B^TP_\delta x_i(t),
\end{align*}
\] (24)

where $K$ is a matrix such that $A + KC$ is Hurwitz stable, $P_\delta > 0$ is the unique solution of the continuous-time algebraic Riccati equation,
\[
A^TP_\delta + P_\delta A - P_\delta BB^TP_\delta + \delta I = 0.
\] (25)

with $\ell_{ii} \geq \beta > 0$.

**Theorem 2** Consider a MAS described by (2) and (3). Let any $\alpha > \beta > 0$ be given, and consider the set of network graphs $\mathbb{G}_\alpha^N$ with $\alpha \geq \ell_{ii} \geq \beta$.

If $(A, B)$ is stabilizable and $(A, C)$ is observable, then the state synchronization problem stated in Problem 2 with $G = \mathbb{G}_\alpha^N$ is solvable. In particular, there exists a $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$, the dynamic protocol (24) solves the state synchronization problem for any graph $G \in \mathbb{G}_\alpha^N$ and $\tau_{ij} \in \mathbb{R}^+$. Moreover, the synchronized trajectory is given by (23).

**Proof:** For dynamic protocol (24), we have
\[
\begin{align*}
\dot{x}_i(t) &= (A + KC)x_i(t) - K\zeta_i(t), \\
u_i(t) &= -\beta^{-1}B^TP_\delta x_i(t),
\end{align*}
\] (24)

by using our delay-free transformation.

From (2), (3), and protocol (24), it means that we only need to prove that the system
\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) - \ell_{ii}\beta^{-1}BB^TP_\delta x_i(t), \\
\dot{x}(t) &= (A + KC)x(t) - KC\bar{x}(t),
\end{align*}
\] (26)

is asymptotically stable for $\alpha \geq \ell_{ii} \geq \beta$.

Define $e(t) = \bar{x}(t) - x(t)$. The system (26) can be rewritten in terms of $x$ and $e$ as
\[
\begin{align*}
\dot{x}(t) &= (A - \ell_{ii}\beta^{-1}BB^TP_\delta)x(t) + \ell_{ii}\beta^{-1}BB^PE(t), \\
\dot{e}(t) &= (A + KC + \ell_{ii}\beta^{-1}BB^TP_\delta)e(t) - \ell_{ii}\beta^{-1}BB^P\bar{x}(t).
\end{align*}
\] (27)

Since $\ell_{ii} \geq \beta$, we have
\[
(A - \ell_{ii}\beta^{-1}BB^TP_\delta)^2P_\delta + P_\delta(A - \ell_{ii}\beta^{-1}BB^TP_\delta)
\leq -\delta I - P_\delta BB^TP_\delta.
\]

Define $V_1 = \bar{x}(t)P_\delta x(t)$ and $v = -B^TP_\delta x(t)$. We can derive that
\[
\dot{V}_1 \leq -\delta\|\bar{x}(t)\|^2 - \|v\|^2 + \theta(\delta)\|e(t)\||v|,
\]

where
\[
\theta(\delta) = \ell_{ii}\beta^{-1}\|BB^TP_\delta\|.
\]

Clearly, $\theta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Let $Q$ be the positive definite solution of the Lyapunov equation,
\[
(A + KC)^TQ + Q(A + KC) = -2I.
\]

Since $P_\delta \rightarrow 0$ and $\ell_{ii}$ is bounded (we have $\ell_{ii} < \alpha$), there exists a $\delta_1$ such that for all $\delta \in (0, \delta_1)$,
\[
(A + KC + \ell_{ii}\beta^{-1}BB^TP_\delta)^TQ + Q(A + KC + \ell_{ii}\beta^{-1}BB^TP_\delta) \leq -I.
\]

Define $V_2 = e(t)^TQe(t)$. We get
\[
\dot{V}_2 \leq -\|e(t)\|^2 + M\|e(t)\||v|
\]

where
\[
M = 2\ell_{ii}\beta^{-1}\|QB\|.
\]

Define $V = 4M^2V_1 + 2V_2$. Then
\[
\dot{V} \leq -4M^2\delta\|\bar{x}(t)\|^2 - 2\|e(t)\|^2 - 4M^2\|v\|^2
+ (4M^2\theta(\delta) + 2M)\|e(t)\||v|.
\]

There exists a $\delta^* \leq \delta_1$ such that $4M^2\theta(\delta) \leq 2M$ for all $\delta \in (0, \delta^*)$. Hence for a $\delta \in (0, \delta^*)$,
\[
\dot{V} \leq -4M^2\delta\|\bar{x}(t)\|^2 - \|e(t)\|^2 - (\|e(t)\| - 2M\|v\|)|v|^2.
\]

We conclude that the system (26) is asymptotically stable for $\alpha \geq \ell_{ii} \geq \beta$. Based on Lemma 2, the delayed state synchronization result can be proved.

**4 Delayed output synchronization for heterogeneous MAS with communication delays**

In this section, we consider the following heterogeneous MAS,
\[
\begin{align*}
\dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \\
y_i(t) &= C_i x_i(t),
\end{align*}
\] (28)

where $x_i(t) \in \mathbb{R}^{m_i}$, $u_i(t) \in \mathbb{R}^{m_u}$, and $y_i(t) \in \mathbb{R}^m$ are the state, input and the output, respectively, of agent $i$ for $i = 1, \ldots, N$. 


Meanwhile, the communication network provides agent $i$ with form of $23$, including time delay $\tau_{ij}$. Similarly, we can obtain a delay-free transformation for $28$ by letting $\tilde{x}_i(t) = x_i(t + \tilde{\tau}_{i1})$, $\tilde{y}_i(t) = y_i(t + \tilde{\tau}_{i1})$, $\tilde{u}_i(t) = u_i(t + \tilde{\tau}_{i1})$, and $\tilde{z}_i(t) = z_i(t + \tilde{\tau}_{i1})$.

Meanwhile, MAS $28$ satisfies the following assumption.

**Assumption 2** We assume that

- $(A_i, B_i, C_i)$ is stabilizable and detectable.
- All eigenvalues of $A_i$ are in the closed left half complex plane.
- $(A_i, B_i, C_i, 0)$ is right-invertible.
- $(A_i, B_i, C_i, 0)$ has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of $A_1$ (the system matrix of the root agent).

Thus, we can transform $2$, $3$, and $5$ as

\[
\begin{align*}
\dot{\tilde{x}}_i(t) &= A_i \tilde{x}_i(t) + B_i \tilde{u}_i(t) \\
\tilde{y}_i(t) &= C_i \tilde{x}_i(t) \\
\tilde{u}_i(t) &= \sum_{j=1}^{n_i} \ell_{ij} F_i \tilde{x}_j(t).
\end{align*}
\]

(29)

Given the model (29) and graph which is a directed spanning tree, all earlier approaches can also be applied to a heterogeneous MAS. Here we will give design scheme to obtain the delayed state synchronization results based on $6$.

Since the graph is equal to a directed spanning tree, it only has a single root which is Agent 1. Moreover, $u_1 = 0$. In this section, our goal is achieve delayed output synchronization

\[
\lim_{t \to \infty} \left[ y_i(t) - y_j(t - \tau_{ij}) \right] = 0.
\]

(30)

For the heterogeneous MAS $28$, we formulate delayed output synchronization problem as follows.

**Problem 3** Consider a MAS described by agents $28$ and $3$ associated with a directed graph $G \in \mathbb{G}^N$ is equal to a spanning tree where $\mathbb{G}_{p}^N$ is defined in Definition $7$. The delayed output synchronization problem given the set of graph $\mathbb{G}_p^N$ in the presence of unknown, nonuniform, arbitrarily large communication delays is to find a distributed dynamic protocol of the type $6$, for each agent such that $30$ is satisfied for all $i \in \{1, \ldots, N\}$, for any directed graph $G \in \mathbb{G}_p^N$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$. Then, we let

\[
e_i(t) = \tilde{y}_i(t) - \tilde{y}_1(t),
\]

and (29) can be rewritten as

\[
\begin{align*}
\begin{pmatrix}
\dot{\tilde{x}}_i(t) \\
\tilde{y}_i(t) \\
\tilde{u}_i(t)
\end{pmatrix}
&= \begin{pmatrix}
A_i & 0 & 0 \\
0 & A_i & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_i(t) \\
\tilde{y}_i(t) \\
\tilde{u}_i(t)
\end{pmatrix}
+ \begin{pmatrix}
B_i \\
0 \\
0
\end{pmatrix}
\tilde{u}_i(t)
\end{align*}
\]

(31)

Then, let $O_i$ be the observability matrix,

\[
O_i = \begin{pmatrix}
C_i & -t & C_i \\
C_i A_i & -t & C_i A_i \\
\vdots & \vdots & \vdots \\
C_i A_i^{n_i-1} & -t & C_i A_i^{n_i-1}
\end{pmatrix}
\]

Let $q_i$ denote the dimension of the null space of $O_i$, and define $k_i = n_i - q_i$. Next, define $\Lambda^u_i \in \mathbb{R}^{n_i \times q_i}$ and $\Phi^u_i \in \mathbb{R}^{n_i \times q_i}$ such that

\[
O_i \left( \begin{pmatrix}
\Lambda^u_i \\
\Phi^u_i
\end{pmatrix}
\right) = 0, \quad \text{rank} \left( \begin{pmatrix}
\Lambda^u_i \\
\Phi^u_i
\end{pmatrix}
\right) = q_i.
\]

Because $(A_i, C_i)$ is observable, $\Lambda^u_i$ and $\Phi^u_i$ have full column rank. Next, we define $\Lambda^c_i$ and $\Phi^c_i$ such that $A_i = [\Lambda^u_i \Lambda^c_i] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i = [\Phi^u_i \Phi^c_i] \in \mathbb{R}^{n_i \times n_i}$ are nonsingular. Thus, we define a new state $\tilde{x}_i(t) \in \mathbb{R}^{n_i+q_i}$ as

\[
\begin{pmatrix}
\tilde{x}_i(t) \\
\tilde{y}_i(t)
\end{pmatrix}
= \begin{pmatrix}
\tilde{x}_i(t) - A_i M_i \Phi^{-1} \tilde{x}_i(t) \\
-N_i \Phi^{-1} \tilde{x}_i(t)
\end{pmatrix}
\]

(32)

where

\[
M_i = \begin{pmatrix}
I_{q_i} & 0 \\
0 & 0
\end{pmatrix} \in \mathbb{R}^{n_i \times n_i}, \quad N_i = \begin{pmatrix}
0 & I_{k_i}
\end{pmatrix} \in \mathbb{R}^{k_i \times n_i}.
\]

Based on this new state variable $\tilde{x}_i(t)$, we can transform (31) as

\[
\begin{align*}
\dot{\tilde{x}}_i(t) &= \tilde{A}_i \tilde{x}_i(t) + \tilde{B}_i \tilde{u}_i(t) \\
\tilde{e}_i(t) &= \tilde{C}_i \tilde{x}_i(t) + \left( \tilde{C}_i - \tilde{C}_i \tilde{A}_i^{\tilde{N}_i} \right) \tilde{x}_i(t)
\end{align*}
\]

Further, we define $\phi_i(t) = \Xi_i \tilde{x}_i(t)$ with

\[
\Xi_i = \begin{pmatrix}
\tilde{C}_i \\
\tilde{C}_i \tilde{A}_i \\
\vdots \\
\tilde{C}_i \tilde{A}_i^{\tilde{N}_i-1}
\end{pmatrix}
\]

(33)

where $\tilde{N} \geq n_i + 1$. Note that $\Xi_i$ is not necessarily a square matrix; however, due to observability of $(\tilde{A}_i, \tilde{C}_i)$, $\Xi_i$ is injective, which implies that $\Xi_i^\top \Xi_i$ is nonsingular. Meanwhile, we obtain a new expression of (32),

\[
\begin{align*}
\dot{\phi}_i(t) &= \left( A_o + K_o \right) \phi_i(t) + B_o \tilde{u}_i(t) \\
\tilde{e}_i(t) &= C_o \phi_i(t)
\end{align*}
\]

(34)

where

\[
A_o = \begin{pmatrix}
0 & I_{p(n-1)} \\
0 & 0
\end{pmatrix} \in \mathbb{R}^{p \times p \tilde{n}}, \quad C_o = \begin{pmatrix}
I_p & 0
\end{pmatrix} \in \mathbb{R}^{p \times p \tilde{n}},
\]

$B_o = \Xi_i \tilde{B}_i$, and $K_o^i = \begin{pmatrix}
0 \\
G_i
\end{pmatrix}$ with $G_i = \tilde{C}_i \tilde{A}_i^{\tilde{N}_i}(\Xi_i^\top \Xi_i)^{-1} \Xi_i^\top$.

Thus, we will design a dynamic protocol of the form

\[
\begin{align*}
\dot{\phi}_i(t) &= \left( A_o + K_o \right) \phi_i(t) + B_o \tilde{u}_i(t) + H_x Q_x C_o(\tilde{z}_i(t) - \sum_{j=1}^{N_i} \ell_{ij} C_o \phi_j(t)) \\
\tilde{e}_i(t) &= F_i \tilde{e}_i(t) \\
\tilde{u}_i(t) &= G_i \tilde{e}_i(t)
\end{align*}
\]
Then, we set \( \xi = \{ K_i, \Gamma_i - K_i \Pi_i \} \). \( K_i \) is chosen such that \( A_i + B_i K_i \) is Hurwitz stable. \( \Gamma_i \) and \( \Pi_i \) satisfy the following regulator equations,

\[
\Pi_i \tilde{A}^{\ell}_{22} = A_i \Pi_i + \tilde{A}^{\ell}_{12} + B_i \Gamma_i \quad C_i \Pi_i = \bar{c}^i_2
\]

\( H_\varepsilon = \text{diag}(I_p e^{-\varepsilon}, I_p e^{-2 \varepsilon}, \ldots, I_p e^{-\varepsilon \hat{n}}) \in \mathbb{R}^{p \times p \hat{n}}. Q_\varepsilon > 0 \) is the unique solution of the following algebraic Riccati equation

\[
(A_o + K_e)Q_\varepsilon + Q_\varepsilon (A_o + K_e)^\top - 2(\beta Q_\varepsilon C_o^\top C_o Q_\varepsilon + I_p e^{-\varepsilon}) \geq 0
\]

with \( 0 < \beta \leq \varepsilon_{ii} \) for \( i = 2, \ldots, N \),

\[
K_e = \left( \begin{array}{c} 0 \\ \varepsilon^{-n+1} KH_\varepsilon \end{array} \right) \in \mathbb{R}^{p \times p \hat{n}},
\]

\( K \in \mathbb{R}^{p \times p \hat{n}} \) is chosen matrix. And \( (A_o + K_e, C_o) \) is always observable. Meanwhile, \( Q_\varepsilon \to 0 \) when \( \varepsilon \to 0 \). By designing the dynamic protocol \( \Xi \), the following theorem can be obtained.

**Theorem 3** Consider a MAS described by agents (28) and (3) satisfying Assumption 2.

The delayed output synchronization problem stated in Problem 3 is solvable for the set of graphs \( \mathcal{G}_\pi \). In particular, protocol (35) solves the state synchronization problem for any graph \( G \in \mathcal{G}_\pi \) and any \( \tau_{ij} \in \mathbb{R}^+ \).

**Proof:** To achieve output regulation our design includes two steps: observer design and state feedback design.

Step 1) Observer design.

Let \( L_\xi \) be the matrix obtained by removing the first row and column of \( L \) as already used in (11). Clearly, \( L_\xi \) is a lower triangular matrix and all its eigenvalues are greater than \( \beta \), i.e., \( \varepsilon_{ii} \geq \beta \).

Let \( \hat{\phi}_i(t) = \phi_i(t) - \bar{\phi}_i(t) \) for \( i = 2, \ldots, N \), then we have

\[
\dot{\hat{\phi}}_i(t) = (A_o + K_i^o)\hat{\phi}_i(t) - H_\varepsilon Q_\varepsilon C_o^\top (F_c^\top \bar{c}^i_2 - \sum_{j=2}^N \varepsilon_{ij} \bar{c}^i_j \hat{\phi}_j(t))
\]

\[
= (A_o + K_o)\hat{\phi}_i(t) - (K_o - K_i^o)\hat{\phi}_i(t)
\]

\[
- \sum_{j=2}^N \varepsilon_{ij} H_\varepsilon Q_\varepsilon C_o^\top C_o \hat{\phi}_j(t)
\]

with \( K_o = [0, K_i^o]^\top \).

Then, we set \( \zeta_i(t) = \varepsilon^{-1} H_\varepsilon^{-1} \hat{\phi}_i(t) \), we have

\[
\varepsilon \dot{\zeta}_i(t) = (A_o + K_e)\zeta_i(t) - K_e^\top \zeta_i(t) - \sum_{j=2}^N \varepsilon_{ij} H_\varepsilon Q_\varepsilon C_o^\top C_o \zeta_j(t)
\]

where

\[
K'_e = \left( \begin{array}{c} 0 \\ \varepsilon^{-n+1} (K_o - K_i^o) H_\varepsilon \end{array} \right).
\]

Since \( \tilde{L} \) is a positive lower triangular matrix and the graph is diverge, we have

\[
\varepsilon \dot{\xi}_2(t) = (A_o + K_e - \ell_{22} Q_\varepsilon C_o^\top C_o) \xi_2(t) - K_e^\top \xi_2(t)
\]

\[
\varepsilon \dot{\xi}_j(t) = (A_o + K_e - \ell_{jj} Q_\varepsilon C_o^\top C_o) \xi_j(t) - K_e^\top \xi_j(t)
\]

\[
- \ell_{jj} Q_\varepsilon C_o^\top C_o \xi_j(t)
\]

with \( i < j \). That means we just need to prove \( A_o + K_e - \ell_{ii} Q_\varepsilon C_o^\top C_o - K_e^\top \) is asymptotically stable.

Thus, based on (36) and \( \varepsilon_{ii} \geq \beta \), we have

\[
Q_\varepsilon^{-1} (A_o + K_e - \ell_{ii} Q_\varepsilon C_o^\top C_o - K_e^\top)
\]

\[
+ (A_o + K_e - \ell_{ii} Q_\varepsilon C_o^\top C_o - K_e^\top) \varepsilon^{-1} Q_\varepsilon^{-1}
\]

\[
= Q_\varepsilon^{-1} [-I_p e^{-\varepsilon} - 2(\varepsilon_{ii} - \beta) Q_\varepsilon C_o^\top C_o Q_\varepsilon]
\]

\[
- K_e^\top Q_\varepsilon C_o^\top C_o Q_\varepsilon
\]

\[
\leq Q_\varepsilon^{-1} [-I_p e^{-\varepsilon} - K_e^\top Q_\varepsilon C_o^\top C_o Q_\varepsilon]
\]

(38)

Since \( K_e^\top \) is a high-order term of \( \varepsilon \), there exist a \( \varepsilon^* \) such that

\[
2\|K_e^\top Q_\varepsilon\| \leq 1.
\]

It means we can obtain \( A_o + K_e - \ell_{ii} Q_\varepsilon C_o^\top C_o - K_e^\top \) is asymptotically stable for any \( \varepsilon < \varepsilon^* \).

Thus, we obtain \( \xi_j(t) \to 0 \) as \( t \to \infty \). That is to say, we have \( \hat{\phi}_i(t) \to \hat{\phi}_i(t) \). Finally, we have

\[
\hat{\hat{\xi}}_i(t) \to \hat{\hat{\xi}}_i(t), \text{ as } t \to \infty.
\]

Step 2) State feedback design to solve output regulation.

The observer designing implies \( \hat{u}_i(t) = F_i \hat{\xi}_i(t) \) for \( t \to \infty \), i.e., control signal converges to a state feedback protocol.

Meanwhile, we need to achieve \( \lim_{t \to \infty} e_i(t) = 0 \), it can be considered as an output regulation problem. From (32), we define \( \tilde{\xi}(t) = [(\tilde{\xi}_1(t))^\top, (\tilde{\xi}_2(t))^\top]^\top \). Thus, we have exosystem

\[
\tilde{\xi}_2(t) = \tilde{A}_{22} \tilde{\xi}_2(t)
\]

and the regulation system

\[
\tilde{\xi}_i(t) = A_i \tilde{\xi}_i(t) + \tilde{A}_{12} \tilde{\xi}_2(t) + B_i \hat{u}_i(t).
\]

Since \( (A_i, B_i) \) is stabilizable and the eigenvalues of \( \tilde{A}_{22} \) are in closed right-half plane, we know protocol \( \hat{u}_i(t) = F_i \hat{\bar{\xi}}_i \) can solve this regulation problem based on [19] Theorem 2.3.1, if the regulator equation (35) is solvable. From [19] Corollary 2.5.1, the regulator equations are solvable if, for each \( \lambda \) that is an eigenvalue of \( \tilde{A}_{22} \), the rank of Rosenbrock system matrix

\[
\begin{pmatrix}
A_i - \lambda C_i \\
0
\end{pmatrix}
\]

is \( n_i + p \). The Rosenbrock system matrix has normal rank \( n_i + p \) due to rightinvertibility of the quadruple \( (A_i, B_i, C_i, 0) \) (see [18] Property 3.1.6)). Since this quadruple has no invariant zeros coinciding with eigenvalues of \( A_1 \) and the eigenvalues of \( \tilde{A}_{22} \) are a subset of the eigenvalues of \( A_1 \), it follows that the rank of the Rosenbrock system matrix is equal to the normal rank for each \( \lambda \) that is an eigenvalue of \( \tilde{A}_{22} \).

Thus, we can achieve the delayed output synchronization by \( \lim_{t \to \infty} e_i(t) = 0 \), i.e., achieve (30).
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