SASAKI-LIKE ALMOST CONTACT COMPLEX RIEMANNIAN MANIFOLDS

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Abstract. A Sasaki-like almost contact complex Riemannian manifold is defined as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. Explicit compact and non-compact examples are given. A canonical construction producing a Sasaki-like almost contact complex Riemannian manifold from a holomorphic complex Riemannian manifold is presented and called an $S^1$-solvable extension.

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1. Introduction

The almost contact complex Riemannian manifold is an odd-dimensional pseudo-Riemannian manifold equipped with a 1-form $\eta$ and a codimension one distribution $H = \text{Ker}(\eta)$ endowed with a complex Riemannian structure. More precisely, the $2n$-dimensional distribution $H$ is equipped with a pair consisting of an almost complex structure and a pseudo-Riemannian metric of signature $(n, n)$ compatible in the way

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that the almost complex structure acts as an anti-isometry on the metric. Almost contact complex Riemannian manifolds are investigated and studied in [7, 11, 12, 13, 14, 15, 16, 18].

The main goal of this note is to find a class of almost contact complex Riemannian manifolds resemble some basic properties of the well known Sasakian manifolds. We define the class of Sasaki-like spaces as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. We note that a holomorphic complex Riemannian manifold is a complex manifold endowed with a complex Riemannian metric whose local components in holomorphic coordinates are holomorphic functions (see [17]). We determine the Sasaki-like almost contact complex Riemannian structure with an explicit expression of the covariant derivative of the structure tensors (cf. Theorem 3.3) and construct explicit compact and non-compact examples. We also present a canonical construction producing a Sasaki-like almost contact complex Riemannian manifold from any holomorphic complex Riemannian manifold which we called an $S^{1}$-solvable extension (cf. Theorem 3.5). Studying the curvature of Sasaki-like spaces we show that it is completely determined by the curvature of the underlying holomorphic complex Riemannian manifold. We develop gauge transformations of Sasaki-like spaces, i.e. we find the class of contact conformal transformations of an almost contact complex Riemannian manifolds which preserve the Sasaki-like condition.

Convention 1.1. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact complex Riemannian manifold with a pseudo-Riemannian metric $g$ of signature $(n+1, n)$.

a) We shall use $x, y, z, u$ to denote smooth vector fields on $M$, i.e. $x, y, z, u \in \mathfrak{X}(M)$.

b) We shall use $X, Y, Z, U$ to denote smooth horizontal vector fields on $M$, i.e. $X, Y, Z, U \in H = \ker(\eta)$.

c) The $2n$-tuple $\{e_{1}, \ldots, e_{n}, e_{n+1} = \varphi e_{1}, \ldots, e_{2n} = \varphi e_{n}\}$ denotes a local orthonormal basis of the horizontal space $H$.

d) For an orthonormal basis $\{e_{0} = \xi, e_{1}, \ldots, e_{n}, e_{n+1} = \varphi e_{1}, \ldots, e_{2n} = \varphi e_{n}\}$ we denote $\varepsilon_{i} = \text{sign}(g(e_{i}, e_{i})) = \pm 1$, where $\varepsilon_{i} = 1$ for $i = 0, 1, \ldots, n$ and $\varepsilon_{i} = -1$ for $i = n+1, \ldots, 2n$.

2. Almost contact complex Riemannian manifolds

Let $(M, \varphi, \xi, \eta)$ be an almost contact manifold, i.e. $M$ is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(\varphi, \xi, \eta)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a vector field $\xi$ and its dual 1-form $\eta$ such that the following algebraic relations are satisfied:

\[ \varphi \xi = 0, \quad \varphi^{2} = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \]

An almost contact structure $(\varphi, \xi, \eta)$ on $M$ is called normal and respectively $(M, \varphi, \xi, \eta)$ is a normal almost contact manifold if the corresponding almost complex structure $\hat{J}$ on $M' = M \times \mathbb{R}$ defined by

\[ \hat{J}X = \varphi X, \quad \hat{J}\xi = \frac{d}{dr}, \quad \hat{J}\frac{d}{dr} = -\frac{1}{r}\xi \]
is integrable (i.e. $M'$ is a complex manifold) [22]. The almost contact structure is normal if and only if the Nijenhuis tensor of $(\varphi, \xi, \eta)$ is zero [1]. The Nijenhuis tensor $N$ of the almost contact structure is defined by

$$N = [\varphi, \varphi] + d\eta \otimes \xi,$$

and $[\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2 [x, y] - \varphi [\varphi x, y] - \varphi [x, \varphi y],$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$.

Let the almost contact manifold $(M, \varphi, \xi, \eta)$ be endowed with a pseudo-Riemannian metric $g$ of signature $(n + 1, n)$ which is compatible with the almost contact structure in the following way

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).$$

The associated metric $\tilde{g}$ of $g$ on $M$ is defined by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$. Both metrics $g$ and $\tilde{g}$ are necessarily of signature $(n + 1, n)$.

The manifold $(M, \varphi, \xi, \eta, g)$ is known as an almost contact manifold with B-metric or an almost contact $B$-metric manifold [7]. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold. We will call these manifolds almost contact complex Riemannian manifolds.

The structure group of the almost contact complex Riemannian manifolds is $O(n, \mathbb{C}) \times I_1 = (GL(n, \mathbb{C}) \cap O(n, n)) \times I_1$, i.e. it consists of real square matrices of order $2n + 1$ of the following type

$$\begin{pmatrix} A & B & \tilde{\vartheta}^T \\ -B & A & \tilde{\vartheta}^T \\ \tilde{\vartheta} & \tilde{\vartheta} & 1 \end{pmatrix},$$

$$A^T A - B^T B = I_n,$$

$$B^T A + A^T B = O_n,$$

$$A, B \in GL(n; \mathbb{R}),$$

where $\vartheta$ and its transpose $\vartheta^T$ are the zero row $n$-vector and the zero column $n$-vector; $I_n$ and $O_n$ are the unit matrix and the zero matrix of size $n$, respectively.

The covariant derivatives of $\varphi$, $\xi$, $\eta$ with respect to the Levi-Civita connection $\nabla$ play a fundamental role in the differential geometry on the almost contact manifolds. The structure tensor $F$ of type $(0,3)$ on $(M, \varphi, \xi, \eta, g)$ is defined by

$$F(x, y, z) = g((\nabla x \varphi)(y, z)).$$

It has the following properties [7]:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

The relations of $\nabla \xi$ and $\nabla \eta$ with $F$ are:

$$\nabla y = g((\nabla x \eta)(y, z)) = F(x, \varphi y, \xi).$$

The following 1-forms associated with $F$:

$$\theta(z) = \sum_{i=1}^{2n} \varepsilon_i F(e_i, e_i, z),$$

$$\theta^*(z) = \sum_{i=1}^{2n} \varepsilon_i F(e_i, \varphi e_i, z)$$

satisfy the obvious relation $\theta^* \circ \varphi = -\theta \circ \varphi^2$.

Besides the Nijenhuis tensor $N$, the following symmetric $(1,2)$-tensor $\hat{N}$ is defined in [16] as follows: consider the symmetric brackets $\{x, y\}$ given by

$$g(\{x, y\}, z) = g(\nabla_x y + \nabla_y x, z) = x g(y, z) + y g(x, z) - z g(x, y) - g([y, z], x) + g([z, x], y);$$
set
\[ \{\varphi, \varphi\}(x, y) = \{\varphi x, \varphi y\} + \varphi^2\{x, y\} - \varphi\{\varphi x, y\} - \varphi\{x, \varphi y\} \]
and define the symmetric tensor $\widehat{N}$ as follows [16]
\[ \widehat{N} = \{\varphi, \varphi\} + (\mathcal{L}_\xi g) \otimes \xi, \]
where $\mathcal{L}$ denotes the Lie derivative. The tensor $\widehat{N}$ is also called the associated Nijenhuis tensor.

We define the corresponding tensors of type $(0,3)$ by the same letter, $N(x, y, z) = g(N(x, y), z)$, $\widehat{N}(x, y, z) = g(\widehat{N}(x, y), z)$. Both tensors $N$ and $\widehat{N}$ can be expressed in terms of the fundamental tensor $F$ as follows [16]
\[ \begin{align*}
N(x, y, z) &= F(\varphi x, y, z) - F(\varphi y, x, z) - F(x, y, \varphi z) + F(y, x, \varphi z) \\
&\quad + \eta(z)[F(x, \varphi y, \xi) - F(y, \varphi x, \xi)],
\end{align*} \tag{6} \]
\[ \begin{align*}
\widehat{N}(x, y, z) &= F(\varphi x, y, z) + F(\varphi y, x, z) - F(x, y, \varphi z) - F(y, x, \varphi z) \\
&\quad + \eta(z)[F(x, \varphi y, \xi) + F(y, \varphi x, \xi)].
\end{align*} \tag{7} \]

2.1. Relation with holomorphic complex Riemannian manifolds. Let us remark that the $2n$-dimensional distribution $H = \text{ker}(\eta)$ is endowed with an almost complex structure $J = \varphi|_H$ and a metric $h = g|_H$, where $\varphi|_H, g|_H$ are the restrictions of $\varphi, g$ on $H$, respectively, and the metric $h$ is compatible with $J$ as follows
\[ h(JX, JY) = -h(X, Y), \quad \widehat{h}(X, Y) := h(X, JY). \tag{8} \]
The distribution $H$ can be considered as an $n$-dimensional complex Riemannian distribution with a complex Riemannian metric $g^C = h + i\widehat{h} = g|_H + ig|_H$.

We recall that a $2n$-dimensional almost complex manifold $(N, J, h)$ endowed with a pseudo-Riemannian metric of signature $(n, n)$ satisfying (8) is known as an almost complex manifold with Norden metric [19, 20, 25, 3, 23, 21], an almost complex manifold with B-metric [4, 6] or an almost complex manifold with complex Riemannian metric [10, 17, 5, 2, 9]. When the almost complex structure $J$ is parallel with respect to the Levi-Civita connection $\nabla^h$ of the metric $h$, $\nabla^h J = 0$, then the manifold is known as a Kähler-Norden manifold, a Kähler manifold with B-metric or a holomorphic complex Riemannian manifold. In this case the almost complex structure $J$ is integrable and the local components of the complex metric in holomorphic coordinate system are holomorphic functions. A four-dimensional example of a Kähler manifold with Norden metric has been given in [19], another approach to the Kähler manifolds with Norden metric has been used in [20] and in [25], there has been proved that the four-dimensional sphere of Kotel’nikov-Study carries a structure of a Kähler manifold with Norden metric.

2.2. The case of parallel structures. The simplest case of almost contact complex Riemannian manifolds is when the structures are $\nabla$-parallel, $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\widehat{g} = 0$, and it is determined by the condition $F(x, y, z) = 0$. In this case the distribution $H$ is involutive. The corresponding integral submanifold is a totally geodesic submanifold which inherits a holomorphic complex Riemannian
structure and the almost contact complex Riemannian manifold is locally a pseudo-Riemannian product of a holomorphic complex Riemannian manifold with a real interval.

3. Sasaki-like almost contact complex Riemannian manifolds

In this section we consider the complex Riemannian cone over an almost contact complex Riemannian manifold and determine a Sasaki-like almost contact complex Riemannian manifold with the condition that its complex Riemannian cone is a holomorphic complex Riemannian manifold.

3.1. Holomorphic complex Riemannian cone.

Let \((M, \varphi, \xi, \eta, g)\) be an almost contact Riemannian manifold of dimension \(2n + 1\). We consider the cone over \(M\) \(\mathcal{C}(M) = M \times \mathbb{R}^{-}\) with the almost complex structure determined in (2) and the complex Riemannian metric defined by

\[
\tilde{g}\left( (x, a \frac{d}{dr}), (y, b \frac{d}{dr}) \right) = r^2 g(x, y) + \eta(x)\eta(y) - ab,
\]

where \(r\) is the coordinate on \(\mathbb{R}^{-}\) and \(a, b\) are \(C^\infty\) functions on \(M \times \mathbb{R}^{-}\).

Using the general Koszul formula

\[
2g(\nabla xy, z) = xg(y, z) + yg(z, x) - zg(x, y)
\]

we calculate from (9) that the non-zero components of the Levi-Civita connection \(\nabla\) of the complex Riemannian metric \(g\) on \(\mathcal{C}(M)\) are given by

\[
\begin{align*}
\tilde{g} (\tilde{\nabla}_X Y, Z) &= r^2 g(\nabla_X Y, Z), \\
\tilde{g} (\tilde{\nabla}_X Y, \xi) &= r^2 g(\nabla_X Y, \xi) + \frac{1}{2} (r^2 - 1) d\eta(X, Y), \\
\tilde{g} (\tilde{\nabla}_X \xi, Z) &= r^2 g(\nabla_X \xi, Z) - \frac{1}{2} (r^2 - 1) d\eta(X, Z), \\
\tilde{g} (\tilde{\nabla}_\xi Y, Z) &= r^2 g(\nabla_\xi Y, Z) - \frac{1}{2} (r^2 - 1) d\eta(Y, Z), \\
\tilde{g} (\tilde{\nabla}_\xi \xi, Z) &= g(\nabla_\xi Y, \xi), \\
\tilde{g} (\tilde{\nabla}_Y \xi, Z) &= g(\nabla_\xi Y, Z), \\
\tilde{g} (\tilde{\nabla}_{\frac{d}{dr}} X, Z) &= rg(X, Z), \\
\tilde{g} (\tilde{\nabla}_{\frac{d}{dr}} Y, Z) &= rg(Y, Z).
\end{align*}
\]
Applying (2) we calculate from the formulas above that the non-zero components of the covariant derivative $\tilde{\nabla}\tilde{J}$ of the almost complex structure $\tilde{J}$ are given by
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) Y, Z) = r^2 g((\nabla_X\varphi) Y, Z), \]
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) Y, \xi) = r^2 \{ g ((\nabla_X\varphi) Y, \xi) + g(X, Y) \} + \frac{1}{2}(r^2 - 1) \, d\eta(X, \varphi Y), \]
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) Y, \frac{d}{dx}) = -r \{ g ((\nabla_X\xi, Y) + g(X, \varphi Y)) + \frac{1}{2}(r^2 - 1) \, d\eta(X, Y, Z), \]
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) \xi, Z) = -r \{ 2 \, d\eta(X, \xi, Z) - d\eta(X, \varphi Z), \]
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) \xi, \frac{d}{dx}) = -\frac{1}{2} \, g((\nabla_X\xi, \varphi Y), \tilde{g}((\tilde{\nabla}_X\tilde{J}) \xi, Z) = -g(\nabla_X\xi, Z) - \frac{1}{2}(r^2 - 1) \, d\eta(Y, \varphi Z)}{g((\tilde{\nabla}_X\tilde{J}) \xi, \varphi Y)}, \]
\[ \tilde{g}((\tilde{\nabla}_X\tilde{J}) \frac{d}{dx}, Z) = -\frac{1}{2} \, g((\nabla_X\xi, Z). \]

We have

**Proposition 3.1.** The complex Riemannian cone $\mathcal{C}(M)$ over an almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g)$ is a holomorphic complex Riemannian space if and only if the following conditions hold
\[ F(X, Y, Z) = F(\xi, Y, Z) = F(\xi, \xi, Z) = 0, \]
\[ F(X, Y, \xi) = -g(X, Y). \]

**Proof.** We obtain from the expressions above that the complex Riemannian cone $(\mathcal{C}(M), \tilde{J}, \tilde{g})$ is a holomorphic Riemannian manifold (a Kähler manifold with Norden metric), i.e. $\tilde{\nabla}\tilde{J} = 0$, if and only if the almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g)$ satisfies the following conditions
\[ F(X, Y, Z) = 0, \]
\[ F(X, Y, \xi) = -g(X, Y) - \frac{1}{2r} \, (r^2 - 1) \, d\eta(X, \varphi Y), \]
\[ F(\xi, Y, Z) = \frac{1}{2r} \, (r^2 - 1) \, \{d\eta(\varphi Y, Z) - d\eta(Y, \varphi Z) \}, \]
\[ F(\xi, \xi, Z) = 0, \quad \nabla_\xi \xi = 0. \]

The condition $\tilde{\nabla}\tilde{J} = 0$ implies the integrability of $\tilde{J}$, hence the structure is normal.

Further, according to (13), we get
\[ (\nabla_X\eta) Y = -g(X, \varphi Y) + \frac{1}{2r} \, (r^2 - 1) \, d\eta(X, Y), \]
yielding $d\eta(X, Y) = \frac{1}{2r} \, (r^2 - 1) \, d\eta(X, Y)$ since the metric $\tilde{g}$ is symmetric. The latter equality shows $d\eta(X, Y) = 0$ which together with (16) yields
\[ (\nabla_X\eta) Y = -g(X, \varphi Y). \]

From (15) we get $d\eta(\xi, X) = (\nabla_\xi\eta)(X) - (\nabla_X\eta)(\xi) = 0$. Hence, $d\eta = 0$. Substitute $d\eta = 0$ into (13)-(14) to complete the proof of the proposition. \hfill \Box

**Definition 3.1.** An almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g)$ is said to be Sasaki-like if the structure tensors $\varphi, \xi, \eta, g$ satisfy the equalities (11) and (12).
To characterize the Sasaki-like almost contact complex Riemannian manifold by the structure tensors, we need the next general result

**Theorem 3.2.** Let \((M, \varphi, \xi, \eta, g)\) be an almost contact complex Riemannian manifold. Then the covariant derivative of \(\varphi\) is given by the formula

\[
g(\nabla_x \varphi) y, z = -\frac{1}{4} \left[ N(\varphi x, y, z) + N(\varphi x, z, y) + \hat{N}(\varphi x, y, z) + \hat{N}(\varphi x, z, y) \right]
+ \frac{1}{2} \eta(x) \left[ N(\xi, y, \varphi z) + \hat{N}(\xi, y, \varphi z) + \eta(z) \hat{N}(\xi, \xi, \varphi y) \right].
\]

(18)

*Proof.* Taking the sum of (6) and (7), we obtain

\[
F(\varphi x, y, z) - F(x, y, \varphi z) = \frac{1}{2} \left[ N(x, y, z) + \hat{N}(x, y, z) \right] - \eta(z) F(x, \varphi y, \xi).
\]

The identities (4) together with (1) imply

\[
F(x, y, \varphi z) + F(x, z, \varphi y) = \eta(z) F(x, \varphi y, \xi) + \eta(y) F(x, \varphi z, \xi).
\]

A suitable combination of (19) and (20) yields

\[
F(\varphi x, y, z) = \frac{1}{4} \left[ N(x, y, z) + N(x, z, y) + \hat{N}(x, y, z) + \hat{N}(x, z, y) \right].
\]

(21)

Applying (1), we obtain from (21)

\[
F(x, y, z) = \eta(x) F(\xi, y, z)
- \frac{1}{4} \left[ N(\varphi x, y, z) + N(\varphi x, z, y) + \hat{N}(\varphi x, y, z) + \hat{N}(\varphi x, z, y) \right].
\]

(22)

Set \(x = \xi\) and \(z \to \varphi z\) into (19) and use (1) to get

\[
F(\xi, y, z) = \frac{1}{2} \left[ N(\xi, y, \varphi z) + \hat{N}(\xi, y, \varphi z) \right] + \eta(z) F(\xi, \xi, y).
\]

(23)

Finally, set \(y = \xi\) into (23) and use the general identities \(N(\xi, \xi) = F(\xi, \xi, \xi) = 0\) to obtain

\[
F(\xi, \xi, z) = \frac{1}{2} \hat{N}(\xi, \xi, \varphi z).
\]

(24)

Substitute (24) into (23) and the obtained identity insert into (22) to get (18). \(\square\)

The next result determines the Sasaki-like spaces by the structure tensors.

**Theorem 3.3.** Let \((M, \varphi, \xi, \eta, g)\) be an almost contact complex Riemannian manifold. The following conditions are equivalent:

a) The manifold \((M, \varphi, \xi, \eta, g)\) is a Sasaki-like almost contact complex Riemannian manifold;

b) The covariant derivative \(\nabla \varphi\) satisfies the equality

\[
(\nabla_x \varphi) y = -g(x, y) \xi - \eta(y) x + 2\eta(x) \eta(y) \xi;
\]

(25)

c) The Nijenhuis tensors \(N\) and \(\hat{N}\) satisfy the relations:

\[
N = 0, \quad \hat{N} = -4 (\bar{g} - \eta \otimes \eta) \otimes \xi.
\]

(26)
Proof. It is easy to check using (5) and (4) that (25) is equivalent to the system of the equations (11) and (12) which established the equivalence between a) and b) in view of Proposition 3.1.

Substitute (25) consequently into (6) and (7) to get (26) which gives the implication b) ⇒ c).

Now, suppose (26) holds. Consequently, we obtain \( \tilde{N}(\xi, y) = 0 \). Now, (25) follows with a substitution of the last equality together with (26) into (18) which completes the proof. \( \square \)

Corollary 3.4. Let \((M, \varphi, \xi, \eta, g)\) be a Sasaki-like almost contact complex Riemannian manifold. Then we have

a) the manifold \((M, \varphi, \xi, \eta, g)\) is normal, \(N = 0\), the fundamental 1-form \(\eta\) is closed, \(d\eta = 0\), and the integral curves of \(\xi\) are geodesics, \(\nabla\xi\xi = 0\);

b) the 1-forms \(\theta\) and \(\theta^*\) satisfy the equalities \(\theta = -2n\eta\) and \(\theta^* = 0\).

3.2. Examples. In this section we construct a number of examples of Sasaki-like almost contact complex Riemannian manifolds.

3.2.1. Example 1. Consider the solvable Lie group \(G\) of dimension \(2n + 1\) with a basis of left-invariant vector fields \(\{e_0, \ldots, e_{2n}\}\) defined by the commutators

\[
[e_0, e_1] = e_{n+1}, \ldots, [e_0, e_n] = e_{2n}, [e_0, e_{n+1}] = -e_1, \ldots, [e_0, e_{2n}] = -e_n.
\]

Define an invariant almost contact complex Riemannian structure on \(G\) by

\[
g(e_0, e_0) = g(e_1, e_1) = \cdots = g(e_n, e_n) = 1
\]

\[
g(e_{n+1}, e_{n+1}) = \cdots = g(e_{2n}, e_{2n}) = -1,
\]

\[
g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \ldots, 2n\}, i \neq j,
\]

\[
\xi = \varphi e_0, \quad \varphi e_1 = e_{n+1}, \ldots, \varphi e_n = e_{2n}.
\]

Using the Koszul formula (10) we check that (11) and (12) are fulfilled, i.e. this is a Sasaki-like almost contact complex Riemannian structure.

Let \(\epsilon^0 = \eta, \epsilon^1, \ldots, \epsilon^{2n}\) be the corresponding dual 1-forms, \(\epsilon^i(e_j) = \delta_j^i\). From (27) and the formula for an arbitrary 1-form \(\alpha\)

\[
d\alpha(A, B) = A\alpha(B) - B\alpha(A) - \alpha([A, B]),
\]

it follows that the structure equations of the group are

\[
de^0 = d\eta = 0, \quad \de^1 = \epsilon^0 \wedge e^{n+1}, \ldots, \de^n = \epsilon^0 \wedge e^{2n},
\]

\[
de^{n+1} = -\epsilon^0 \wedge e^1, \ldots, \de^{2n} = -\epsilon^0 \wedge e^n
\]

and the Sasaki-like almost contact complex Riemannian structure has the form

\[
g = \sum_{i=0}^{2n} \epsilon_i (\epsilon^i)^2, \quad \varphi e^1 = e^{n+1}, \ldots, \varphi e^n = e^{2n}.
\]

The group \(G\) is the following rank-1 solvable extension of the Abelian group \(\mathbb{R}^{2n}\)

\[
\epsilon^0 = dt, \quad \epsilon^1 = \cos t \, dx^1 + \sin t \, dx^{n+1}, \quad \epsilon^{n+1} = -\sin t \, dx^1 + \cos t \, dx^{n+1},
\]

\[
\cdots, \quad \epsilon^n = \cos t \, dx^n + \sin t \, dx^{2n}, \quad \epsilon^{2n} = -\sin t \, dx^n + \cos t \, dx^{2n}.
\]
Clearly, the 1-forms defined in (31) satisfy (29) and the Sasaki-like almost contact complex Riemannian metric has the form

\[(32) \quad g = dt^2 + \cos 2t \left( \sum_{i=1}^{2n} \varepsilon_i (dx^i)^2 \right) - \sin 2t \left( -2 \sum_{i=1}^{n} dx^i dx^{n+i} \right). \]

It is known that the solvable Lie group \(G\) admits a lattice \(\Gamma\) such that the quotient space \(G/\Gamma\) is compact (c.f. [24, Chapter 3]). The invariant Sasaki-like almost contact complex Riemannian structure \((\varphi, \xi, \eta, g)\) on \(G\) descends to \(G/\Gamma\) which supplies a compact Sasaki-like almost contact complex Riemannian manifold in any dimension.

It follows from (27), (30), (31) and (32) that the distribution \(H = \text{span}\{e_1, \ldots, e_{2n}\}\) is integrable and the corresponding integral submanifold can be considered as the holomorphic complex Riemannian flat space \(\mathbb{R}^{2n} = \text{span}\{dx^1, \ldots, dx^{2n}\}\) with the following holomorphic complex Riemannian structure

\[ Jdx^1 = dx^{n+1}, \ldots, Jdx^n = dx^{2n}; \quad h = \sum_{i=1}^{2n} \varepsilon_i (dx^i)^2, \quad \tilde{h} = -2 \sum_{i=1}^{n} dx^i dx^{n+i}. \]

3.2.2. \(S^1\)-solvable extension. Inspired by Example 1 we proposed the following more general construction. Let \((N^{2n}, J, h, \tilde{h})\) be a \(2n\)-dimensional holomorphic complex Riemannian manifold, i.e. the almost complex structure \(J\) acts as an anti-isometry on the neutral metric \(h\), \(h(JX, JY) = -h(X, Y)\) and it is parallel with respect to the Levi-Civita connection of \(h\). In particular, the almost complex structure \(J\) is integrable. The associated neutral metric \(\tilde{h}\) is defined by \(\tilde{h}(X, Y) = h(JX, Y)\) and it is also parallel with respect to the Levi-Civita connection of \(h\).

Consider the product manifold \(M^{2n+1} = \mathbb{R}^+ \times N^{2n}\). Let \(dt\) be the coordinate 1-form on \(\mathbb{R}^+\) and define an almost contact complex Riemannian structure on \(M^{2n+1}\) as follows

\[(33) \quad \eta = dt, \quad \varphi|_H = J, \quad \eta \circ \varphi = 0, \quad g = dt^2 + \cos 2t \ h - \sin 2t \ \tilde{h}. \]

**Theorem 3.5.** Let \((N^{2n}, J, h, \tilde{h})\) be a \(2n\)-dimensional holomorphic complex Riemannian manifold. Then the product manifold \(M^{2n+1} = \mathbb{R}^+ \times N^{2n}\) equipped with the almost contact complex Riemannian structure defined in (33) is a Sasaki-like almost contact complex Riemannian manifold.

If \(N^{2n}\) is compact then \(M^{2n+1} = S^1 \times N^{2n}\) with the structure (33) is a compact Sasaki-like almost contact complex Riemannian manifold.

**Proof.** It is easy to check using (10), (33) and the fact that the complex structure \(J\) is parallel with respect to the Levi-Civita connection of \(h\) that the structure defined in (33) satisfies (11) and (12) and thus \((M, \varphi, \xi, \eta, g)\) is a Sasaki-like almost contact complex Riemannian manifold.

Now, suppose \(N^{2n}\) is a compact holomorphic complex Riemannian manifold. The equations (33) imply that the metric \(g\) is periodic on \(\mathbb{R}\) and therefore it descends to the compact manifold \(M^{2n+1} = S^1 \times N^{2n}\). Thus we obtain a compact Sasaki-like almost contact complex Riemannian manifold. \(\square\)
We call the Sasaki-like almost contact complex Riemannian manifold constructed in Theorem 3.5 from a holomorphic complex Riemannian manifold an $S^1\text{-solvable extension of a holomorphic complex Riemannian manifold.}$

3.2.3. Example 2. Let us consider the Lie group $G^5$ of dimension 5 with a basis of left-invariant vector fields $\{e_0, \ldots, e_4\}$ defined by the commutators

\[
\begin{align*}
[e_0, e_1] &= \lambda e_2 + e_3 + \mu e_4, & [e_0, e_2] &= -\lambda e_1 - \mu e_3 + e_4, \\
[e_0, e_3] &= -e_1 - \mu e_2 + \lambda e_4, & [e_0, e_4] &= \mu e_1 - e_2 - \lambda e_3, & \lambda, \mu \in \mathbb{R}.
\end{align*}
\]

Let $G^5$ be equipped with an invariant almost contact complex Riemannian structure as in (28) for $n = 2$. We calculate using (10) that the non-zero connection 1-forms of the Levi-Civita connection are

\[
\begin{align*}
\nabla_{e_0}e_1 &= \lambda e_2 + \mu e_4, & \nabla_{e_1}e_0 &= -e_3, & \nabla_{e_0}e_2 &= -\lambda e_1 - \mu e_3, & \nabla_{e_3}e_0 &= -e_4, \\
\nabla_{e_0}e_3 &= -\mu e_2 + \lambda e_4, & \nabla_{e_3}e_0 &= e_1, & \nabla_{e_0}e_4 &= \mu e_1 - \lambda e_3, & \nabla_{e_4}e_0 &= e_2, \\
\nabla_{e_1}e_3 &= -\nabla_{e_2}e_4 = \nabla_{e_3}e_1 = \nabla_{e_4}e_2 &= -e_0.
\end{align*}
\]

Similarly as in Example 1 we verify that the constructed manifold $(G^5, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact complex Riemannian manifold.

Take $\mu = 0$ and $\lambda \neq 0$. Then the structure equations of the group become

\[
\begin{align*}
de^0 &= d\eta = 0, & de^1 &= e^0 \wedge e^3 + \lambda e^0 \wedge e^2, & de^2 &= e^0 \wedge e^4 - \lambda e^0 \wedge e^1, \\
de^3 &= -e^0 \wedge e^1 + \lambda e^0 \wedge e^4, & de^4 &= -e^0 \wedge e^2 - \lambda e^0 \wedge e^3.
\end{align*}
\]

A basis of 1-forms satisfying (34) is given by $e^0 = dt$ and

\[
\begin{align*}
e^1 &= \cos(1-\lambda) t \, dx^1 - \cos(1+\lambda) t \, dx^2 + \sin(1-\lambda) t \, dx^3 - \sin(1+\lambda) t \, dx^4, \\
e^2 &= \sin(1-\lambda) t \, dx^1 + \sin(1+\lambda) t \, dx^2 - \cos(1-\lambda) t \, dx^3 - \cos(1+\lambda) t \, dx^4, \\
e^3 &= -\sin(1-\lambda) t \, dx^1 + \sin(1+\lambda) t \, dx^2 + \cos(1-\lambda) t \, dx^3 - \cos(1+\lambda) t \, dx^4, \\
e^4 &= \cos(1-\lambda) t \, dx^1 + \cos(1+\lambda) t \, dx^2 + \sin(1-\lambda) t \, dx^3 + \sin(1+\lambda) t \, dx^4.
\end{align*}
\]

Then the Sasaki-like metric is of the form

\[
g = dt^2 - 4 \cos 2t \left( dx^1 dx^2 - dx^3 dx^4 \right) - 4 \sin 2t \left( dx^1 dx^4 + dx^2 dx^3 \right).
\]

From (34) it follows that the distribution $H = \text{span}\{e_1, \ldots, e_4\}$ is integrable and the corresponding integral submanifold can be considered as the holomorphic complex Riemannian flat space $\mathbb{R}^4 = \text{span}\{dx^1, \ldots, dx^4\}$ with the holomorphic complex Riemannian structure given by

\[
Jdx^1 = dx^3, \quad Jdx^2 = dx^4; \quad h = -4(dx^1 dx^2 - dx^3 dx^4), \quad \tilde{h} = 4(dx^1 dx^4 + dx^2 dx^3)
\]

and the Sasaki-like metric (35) takes the form

\[
g = dt^2 + \cos 2t \, h - \sin 2t \, \tilde{h}.
\]
4. Curvature properties

Let \((M, \varphi, \xi, \eta, g)\) be an almost contact complex Riemannian manifold. The curvature tensor of type \((1, 3)\) is defined by \(R = [\nabla, \nabla] - \nabla_\cdot\cdot\). We denote the curvature tensor of type \((0, 4)\) by the same letter, \(R(x, y, z, u) = g(R(x, y)z, u)\). The Ricci tensor \(\text{Ric}\), the scalar curvature \(\text{Scal}\) and the \(\ast\)-scalar curvature \(\text{Scal}^*\) are the usual traces of the curvature, \(\text{Ric}(x, y) = \sum_{i=0}^{2n} \varepsilon_i R(e_i, x, y, e_i), \text{Scal} = \sum_{i=0}^{2n} \varepsilon_i \text{Ric}(e_i, e_i), \text{Scal}^* = \sum_{i=0}^{2n} \varepsilon_i \text{Ric}(e_i, \varphi e_i)\).

**Proposition 4.1.** On a Sasaki-like almost contact complex Riemannian manifold \((M, \varphi, \xi, \eta, g)\) the next formula holds

\[
R(x, y, \varphi z, u) = R(x, y, z, \varphi u) - [g(y, z) - 2\eta(y)\eta(z)]g(x, \varphi u) + [g(y, u) - 2\eta(y)\eta(u)]g(x, \varphi z) - [g(x, z) - 2\eta(x)\eta(z)]g(y, \varphi u) - [g(x, u) - 2\eta(x)\eta(u)]g(y, \varphi z).
\]

In particular, we have

\[
R(x, y)\xi = \eta(y)x - \eta(x)y, \quad [X, \xi] \in H, \quad \nabla_\xi X = -\varphi X - [X, \xi] \in H; \\
R(\xi, X)\xi = -X, \quad \text{Ric}(y, \xi) = 2n\ \eta(y), \quad \text{Ric}(\xi, \xi) = 2n.
\]

**Proof.** The Ricci identity for \(\varphi\) reads

\[
R(x, y, \varphi z, u) - R(x, y, z, \varphi u) = g\left(\left(\nabla_x \nabla_y \varphi\right) z, u\right) - g\left(\left(\nabla_y \nabla_x \varphi\right) z, u\right).
\]

Applying (25) to the above equality and using (17) we obtain (36) by straightforward calculations. Set \(z = \xi\) into (36) and using (1) we get the first equality in (37). The rest follows from (17) and the condition \(d\eta = 0\). The equalities (38) follow directly from the first equality in (37).

4.1. The horizontal curvature. From \(d\eta = 0\) it follows locally \(\eta = dx\), \(H\) is integrable and the manifold is locally the product \(M^{2n+1} = N^{2n} \times \mathbb{R}\) with \(TN^{2n} = H\). The submanifold \((N^{2n}, J = \varphi|_H, h = g|_H)\) is a holomorphic complex Riemannian manifold. Indeed, we obtain from (11) that \(h(\nabla^h J)Y, Z) = F(X, Y, Z) = 0\), where \(\nabla^h\) is the Levi-Civita connection of \(h\).

We may consider \(N^{2n}\) as a hypersurface of \(M\) with the unit normal \(\xi = \frac{\partial}{\partial x}\). The equality (17) yields

\[
g(\nabla_x \xi, Y) = -g(\nabla_X Y, \xi) = -g(\varphi X, Y) = -\tilde{g}|_H(X, Y), \quad \nabla_\xi \xi = 0.
\]

This means that the second fundamental form is equal to \(\tilde{g}|_H = \tilde{h}\). The Gauss equation (see e.g. [8, Chapter VII, Proposition 4.1]) yields

\[
R(X, Y, Z, U) = R^h(X, Y, Z, U) + g(\varphi X, Z)g(\varphi Y, U) - g(\varphi Y, Z)g(\varphi X, U),
\]

where \(R^h\) is the curvature tensor of the holomorphic complex Riemannian manifold \((N^{2n}, J, h)\).
For the horizontal Ricci tensor we obtain from (39) and (38) that

\[
(40) \quad Ric(Y, Z) = \sum_{i=1}^{2n} \varepsilon_i R(e_i, Y, Z, e_i) + R(\xi, Y, Z, \xi)
\]
\[
= Ric^h(Y, Z) + g(\varphi Y, \varphi Z) + g(Y, Z) = Ric^h(Y, Z),
\]

where \(Ric^h\) is the Ricci tensor of \(h = g|_H\).

It follows from Proposition 4.1 that the curvature tensor in the direction of \(\xi\) on a Sasaki-like almost contact complex Riemannian manifold is completely determined by \(\eta, \varphi, g, \tilde{g}\). Indeed, using the properties of the Riemannian curvature, we derive from (37)

\[
R(x, y, \xi, z) = R(\xi, z, x, y) = \eta(y)g(x, z) - \eta(x)g(y, z).
\]

Now, the equation (39) implies that the Riemannian curvature of a Sasaki-like almost contact complex Riemannian manifold is completely determined by the curvature of the underlying holomorphic complex Riemannian manifold \((N^{2n}, TN^{2n} = H, J, h)\).

4.2. Example 3: \(S^1\)-solvable extension of the h-sphere. The next example illustrates Theorem 3.5. Consider \(\mathbb{R}^{2n+2}, n > 2\), as a flat holomorphic complex Riemannian manifold, i.e. \(\mathbb{R}^{2n+2}\) is equipped with the canonical complex structure \(J'\) and the canonical Norden metrics \(h'\) and \(\tilde{h}'\) defined by

\[
h'(x', y') = \sum_{i=1}^{n+1} (x'^iy' - x'^{n+i+1}y'^{n+i+1}), \quad \tilde{h}'(x', y') = -\sum_{i=1}^{n+1} (x'^{n+i+1} + x'^{n+i+1}y')
\]

for the vectors \(x' = (x^1, \ldots, x^{2n+2})\) and \(y' = (y^1, \ldots, y^{2n+2})\) in \(\mathbb{R}^{2n+2}\). Identifying the point \(z' = (z^1, \ldots, z^{2n+2})\) in \(\mathbb{R}^{2n+2}\) with the position vector \(Z\), we consider the complex hypersurface \(S^2_h(z'_0; a, b)\) defined by the equations

\[
h'(z' - z'_0, z' - z'_0) = a, \quad \tilde{h}'(z' - z'_0, z' - z'_0) = b,
\]

where \((0, 0) \neq (a, b) \in \mathbb{R}^2\). The co-dimension two submanifold \(S^2_h(z'_0; a, b)\) is \(J'\)-invariant and the restriction of \(h'\) on \(S^2_h(z'_0; a, b)\) has rank \(2n\) due to the condition \((0, 0) \neq (a, b)\). The holomorphic complex Riemannian structure on \(\mathbb{R}^{2n+2}\) inherits a holomorphic complex Riemannian structure \((J'|_{S^2_h}, h'|_{S^2_h})\) on the complex hypersurface \(S^2_h(z'_0; a, b)\). The holomorphic complex Riemannian manifold \((S^2_h(z'_0; a, b), J'|_{S^2_h}, h'|_{S^2_h})\) is sometimes called an h-sphere with center \(z'_0\) and a pair of parameters \((a, b)\). The h-sphere \(S^2_h(z'_0, 1, 0)\) is the sphere of Kotel’nikov-Study [25]. The curvature of an h-sphere is given by the formula [3]

\[
(41) \quad R'|_{S^2_h} = \frac{1}{a^2 + b^2} \{a(\pi_1 - \pi_2) - b\pi_3\},
\]

where \(\pi_1 = \frac{1}{2} h'|_{S^2_h} \otimes h'|_{S^2_h}, \pi_2 = \frac{1}{2} \tilde{h}'|_{S^2_h} \otimes \tilde{h}'|_{S^2_h}, \pi_3 = -h'|_{S^2_h} \otimes \tilde{h}'|_{S^2_h}\) and \(\otimes\) stands for the Kulkarni-Nomizu product of two \((0, 2)\)-tensors; for example,

\[
h' \otimes \tilde{h}'(X, Y, Z, U) = h'(Y, Z) \tilde{h}'(X, U) - h'(X, Z) \tilde{h}'(Y, U) + \tilde{h}'(Y, Z) h'(X, U) - \tilde{h}'(X, Z) h'(Y, U).
\]
Consequently, we have

\[
Ric'|_{S^2_h} = \frac{2(n-1)}{a^2 + b^2} (ah'|_{S^2_h} + bh'|_{S^2_h}), \quad Scal'|_{S^2_h} = \frac{4n(n-1)a}{a^2 + b^2}.
\]

The product manifold \(M^{2n+1} = \mathbb{R}^+ \times S^2_h(z_0'; a, b)\) equipped with the following almost contact complex Riemannian structure

\[
\eta = dt, \quad \varphi|_H = J'|_{S^2_h}, \quad \eta \circ \varphi = 0, \quad g = dt^2 + \cos 2t \ h'|_{S^2_h} - \sin 2t \ \tilde{h}'|_{S^2_h}
\]
is a Sasaki-like almost contact complex Riemannian manifold according to Theorem 3.5.

The horizontal metrics on \(M^{2n+1} = \mathbb{R}^+ \times S^2_h(z_0'; a, b)\) are

\[
h = g|_H = \cos 2t \ h'|_{S^2_h} - \sin 2t \ \tilde{h}'|_{S^2_h}, \quad \tilde{h} = \tilde{g}|_H = \sin 2t \ h'|_{S^2_h} + \cos 2t \ \tilde{h}'|_{S^2_h}.
\]

The Levi-Civita connection \(\nabla'\) of the metric \(h'|_{S^2_h}\) coincides with the Levi-Civita connection of \(\tilde{h}'|_{S^2_h}\) since \(\nabla'J' = 0\). Using this fact, the Koszul formula (10) together with (43) gives for the Levi-Civita connection \(\nabla^h\) of \(h\) the expression

\[
h(\nabla^h_X Y, Z) = \cos 2t \ h'|_{S^2_h} (\nabla'_X Y, Z) - \sin 2t \ h'|_{S^2_h} (\nabla'_X Y, JZ) = h (\nabla'_X Y, Z),
\]
which implies \(\nabla^h_X Y = \nabla'_X Y\). The latter equality together with (43) yields for the curvature of \(h\) the formula \(R^h = \cos 2t \ R'|_{S^2_h} - \sin 2t \ \tilde{R}'|_{S^2_h}\), where \(\tilde{R}'|_{S^2_h} := J' R'|_{S^2_h}\). The above equality together with (41) implies

\[
R^h = \frac{1}{a^2 + b^2} \left\{ \cos 2t [a(\pi_1 - \pi_2) - b\pi_3] - \sin 2t [-a\pi_3 - b(\pi_1 - \pi_2)] \right\}
= \frac{1}{a^2 + b^2} \left\{ (a \cos 2t + b \sin 2t)(\pi_1 - \pi_2) - (b \cos 2t - a \sin 2t)\pi_3 \right\}.
\]

We obtain from (39), (43) and (44) that the horizontal curvature \(R|_H\) of the Sasaki-like almost contact complex Riemannian manifold \(M^{2n+1} = \mathbb{R}^+ \times S^2_h(z_0'; a, b)\) is given by the formula

\[
R|_H = R^h - (\sin 2t)^2 \pi_1 - (\cos 2t)^2 \pi_2 + \sin 2t \cos 2t \pi_3
= \left\{ \frac{1}{a^2 + b^2} (a \cos 2t + b \sin 2t) - (\sin 2t)^2 \right\} \pi_1
- \left\{ \frac{1}{a^2 + b^2} (a \cos 2t + b \sin 2t) + (\cos 2t)^2 \right\} \pi_2
- \left\{ \frac{1}{a^2 + b^2} (b \cos 2t - a \sin 2t) - \sin 2t \cos 2t \right\} \pi_3.
\]
For the horizontal Ricci tensor we obtain from (40), (42) and (43) the formula

\[ \text{Ric}_H = \text{Ric}^h = \frac{2(n - 1)}{a^2 + b^2}(ah'|_{\mathcal{S}^n} + b\eta'|_{\mathcal{S}^n})\]

\[ = \frac{2(n - 1)}{a^2 + b^2} \left[ (a \cos 2t - b \sin 2t)h + (b \cos 2t + a \sin 2t)\eta \right].\]

5. Contact conformal (homothetic) transformations

In this section we investigate when the Sasaki-like condition is preserved under contact conformal transformations. We recall that a general contact conformal transformation of an almost contact complex Riemannian manifold \((M, \varphi, \xi, \eta, g)\) is defined by [11, 13, 14]

\[ \eta = e^{w} \eta, \quad \xi = e^{-w} \xi, \]

\[ \eta(x, y) = e^{2w} \cos 2v \, g(x, y) + e^{2w} \sin 2v \, g(x, \varphi y) + (e^{2w} - e^{2w} \cos 2v)\eta(x)\eta(y), \]

where \(u, v, w\) are smooth functions.

If the functions \(u, v, w\) are constant we have a contact homothetic transformation. The tensors \(F\) and \(F\) are connected by [11], see also [15, (22)],

\[ 2F(x, y, z) = 2e^{2w} \cos 2vF(x, y, z) + 2e^{2w}\eta(x)\left[\eta(y)dw(\varphi z) + \eta(z)dw(\varphi y)\right] \]

\[ + e^{2w} \sin 2v \left[F(\varphi y, z, x) - F(y, \varphi z, x) + F(x, \varphi y, \xi)\eta(z)\right] \]

\[ + e^{2w} \sin 2v \left[F(\varphi z, y, x) - F(z, \varphi y, x) + F(x, \varphi z, \xi)\eta(y)\right] \]

\[ + (e^{2w} - e^{2w} \cos 2v)\left[F(x, y, \xi) + F(\varphi y, \varphi x, \xi)\right]\eta(z) \]

\[ + (e^{2w} - e^{2w} \cos 2v)\left[F(x, z, \xi) + F(\varphi z, \varphi x, \xi)\right]\eta(y) \]

\[ + (e^{2w} - e^{2w} \cos 2v)\left[F(y, z, \xi) + F(\varphi z, \varphi y, \xi)\right]\eta(x) \]

\[ + (e^{2w} - e^{2w} \cos 2v)\left[F(z, y, \xi) + F(\varphi y, \varphi z, \xi)\right]\eta(x) \]

\[ - 2e^{2w} \cos 2v\left[du(\varphi z) + dv(z)\right] - 2e^{2w} \sin 2v\left[du(z) - dv(\varphi z)\right]g(\varphi x, \varphi y) \]

\[ - 2e^{2w} \cos 2v\left[du(\varphi y) + dv(y)\right] - 2e^{2w} \sin 2v\left[du(y) - dv(\varphi y)\right]g(\varphi x, \varphi z) \]

\[ - 2e^{2w} \cos 2v\left[du(z) - dv(\varphi z)\right] + 2e^{2w} \sin 2v\left[du(\varphi z) + dv(z)\right]g(x, \varphi y) \]

\[ - 2e^{2w} \cos 2v\left[du(y) - dv(\varphi y)\right] + 2e^{2w} \sin 2v\left[du(\varphi y) + dv(y)\right]g(x, \varphi z). \]

The Sasaki-like condition (25) also reads as

\[ F(x, y, z) = g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y). \]
We obtain the Sasaki-like condition for the metric $\mathcal{F}$ substituting (45) into (47) which yields

$$\mathcal{F}(x, y, z) = e^{w+2u} \left\{ \cos 2v [\eta(z)g(\varphi x, \varphi y) + \eta(y)g(\varphi x, \varphi z)] - \sin 2v [\eta(z)g(x, \varphi y) + \eta(y)g(x, \varphi z)] \right\}.$$

(48)

Substitute (47) into (46) to get

$$\mathcal{F}(x, y, z) = e^{2w} \eta(x) \{\eta(y)dw(\varphi z) + \eta(z)dw(\varphi y)\} + e^{2u} \left\{ \cos 2v [\eta(z)g(\varphi x, \varphi y) + \eta(y)g(\varphi x, \varphi z)] - \sin 2v [\eta(z)g(x, \varphi y) + \eta(y)g(x, \varphi z)] \right\}$$

$$- e^{2u} \left\{ \cos 2v [du(\varphi z) + dv(z)] + \sin 2v [du(z) - dv(\varphi z)] \right\} g(\varphi x, \varphi y)$$

$$+ \{\cos 2v [du(\varphi y) + dv(y)] + \sin 2v [du(y) - dv(\varphi y)]\} g(\varphi x, \varphi z)$$

$$+ \{\cos 2v [du(z) - dv(\varphi z)] - \sin 2v [du(\varphi z) + dv(z)]\} g(x, \varphi y)$$

$$+ \{\cos 2v [du(y) - dv(\varphi y)] - \sin 2v [du(\varphi y) + dv(y)]\} g(x, \varphi z) \right\}.$$

(49b)

The equalities (49) and (48) imply

$$(1 - e^w)e^{2u} \left\{ \cos 2v [\eta(z)g(\varphi x, \varphi y) + \eta(y)g(\varphi x, \varphi z)]$$

$$- \sin 2v [\eta(z)g(x, \varphi y) + \eta(y)g(x, \varphi z)] \right\}$$

$$- e^{2u} \left\{ \cos 2v [du(\varphi z) + dv(z)] + \sin 2v [du(z) - dv(\varphi z)] \right\} g(\varphi x, \varphi y)$$

$$+ \{\cos 2v [du(\varphi y) + dv(y)] + \sin 2v [du(y) - dv(\varphi y)]\} g(\varphi x, \varphi z)$$

$$+ \{\cos 2v [du(z) - dv(\varphi z)] - \sin 2v [du(\varphi z) + dv(z)]\} g(x, \varphi y)$$

$$+ \{\cos 2v [du(y) - dv(\varphi y)] - \sin 2v [du(\varphi y) + dv(y)]\} g(x, \varphi z) \right\}$$

$$+ e^{2w} \eta(x) \{\eta(y)dw(\varphi z) + \eta(z)dw(\varphi y)\} = 0.$$

(50)

Set $x = y = \xi$ into (50) to get

$$dw(\varphi z) = 0.$$

(51)

Now, using (51) we write (50) in the form

$$A(z)g(\varphi x, \varphi y) + B(z)g(x, \varphi y) + A(y)g(\varphi x, \varphi z) + B(y)g(x, \varphi z) = 0,$$

(52)
where the 1-forms $A$ and $B$ are defined by
\begin{align}
A(z) &= \cos 2v \left[ (e^w - 1)\eta(z) + d u(\varphi z) + d v(z) \right] + \sin 2v \left[ d u(z) - d v(\varphi z) \right], \\
B(z) &= \sin 2v \left[ (e^w - 1)\eta(z) + d u(\varphi z) + d v(z) \right] - \cos 2v \left[ d u(z) - d v(\varphi z) \right].
\end{align}
Taking the trace of (52) with respect to $x = e_i$, $z = e_i$ and $y = e_i$, $z = e_i$ to get
\begin{equation}
-(2n + 1)A(z) + \eta(z)A(\xi) + B(\varphi z) = 0, \quad A(z) - \eta(z)A(\xi) - B(\varphi z) = 0.
\end{equation}
We derive from (54) that $A(z) = 0$. Similarly, we obtain $B(z) = 0$. Now, (53) imply
\begin{align}
\cos 2v [d u(\varphi z) + d v(z)] + \sin 2v [d u(z) - d v(\varphi z)] &= (1 - e^w) \cos 2v \eta(z), \\
\sin 2v [d u(\varphi z) + d v(z)] - \cos 2v [d u(z) - d v(\varphi z)] &= (1 - e^w) \sin 2v \eta(z).
\end{align}
Comparing (47) and (46) we derive

**Proposition 5.1.** Let $(M, \varphi, \xi, \eta, g)$ be a Sasaki-like almost contact complex Riemannian manifold. Then the structure $(\varphi, \xi, \eta, g)$ defined by (45) is Sasaki-like if and only if the smooth functions $u, v, w$ satisfy the following conditions
\begin{equation}
d w \circ \varphi = 0, \quad d u - d v \circ \varphi = 0, \quad d u \circ \varphi + d v = (1 - e^w)\eta.
\end{equation}
In particular
\begin{equation}
d u(\xi) = 0, \quad d v(\xi) = 1 - e^w.
\end{equation}
In the case $w = 0$ the global smooth functions $u$ and $v$ does not depend on $\xi$ and are locally defined on the complex submanifold $N^{2n}$, $TN = H$, and the complex valued function $u + \sqrt{-1}v$ is a holomorphic function on $N^{2n}$.

**Proof.** Solve the linear system (55) to get the second and the third equality into (56). Now, (51) completes the proof of (56). □

5.1. **Contact homothetic transformations.** Let us consider contact homothetic transformations of an almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g)$. Since the functions $u, v, w$ are constant, it follows from (45) using the Koszul formula (10) that the Levi-Civita connections $\nabla$ and $\nabla$ of the metrics $\overline{g}$ and $g$, respectively, are connected by the formula
\begin{equation}
\nabla_x y = \nabla_x y + e^{2(u-w)} \sin 2v \left[ g(\varphi x, \varphi y)\xi - (e^{-2w} - e^{2(u-w)} \cos 2v) \right] g(x, \varphi y)\xi.
\end{equation}
For the corresponding curvature tensors $\overline{R}$ and $R$ we obtain from (57) that
\begin{align}
\overline{R}(x, y) z &= R(x, y) z \\
&+ e^{2(u-w)} \sin 2v \left\{ g(y, \varphi z)\eta(x)\xi - g(\varphi y, \varphi z)\varphi x \right. \\
&\left. - g(x, \varphi z)\eta(y)\xi + g(\varphi x, \varphi z)\varphi y \right\} \\
&+ (e^{-2w} - e^{2(u-w)} \cos 2v) \left\{ g(\varphi y, \varphi z)\eta(x)\xi + g(y, \varphi z)\varphi x \\
&\left. - g(\varphi x, \varphi z)\eta(y)\xi - g(x, \varphi z)\varphi y \right\}.
\end{align}
We have
Proposition 5.2. The Ricci tensor of an almost contact complex Riemannian manifold is invariant under a contact homothetic transformation,

\begin{equation}
\overline{\text{Ric}} = \text{Ric}.
\end{equation}

Consequently, we obtain

\begin{equation}
\overline{\text{Scal}} = e^{-2u} \cos 2v \text{ Scal} - e^{-2u} \sin 2v \text{ Scal}^* + (e^{-2u} - e^{-2u} \cos 2v) \text{ Ric}(\xi, \xi),
\end{equation}

\begin{equation}
\overline{\text{Scal}}^* = e^{-2u} \sin 2v \text{ Scal} + e^{-2u} \cos 2v \text{ Scal}^* - e^{-2u} \sin 2v \text{ Ric}(\xi, \xi).
\end{equation}

In particular, the scalar curvatures of a Sasaki-like almost contact complex Riemannian manifold changes under a contact homothetic transformation with $w = 0$ as follows

\begin{equation}
\overline{\text{Scal}} = e^{-2u} \cos 2v \text{ Scal} - e^{-2u} \sin 2v \text{ Scal}^* + 2n (1 - e^{-2u} \cos 2v),
\end{equation}

\begin{equation}
\overline{\text{Scal}}^* = e^{-2u} \sin 2v \text{ Scal} + e^{-2u} \cos 2v \text{ Scal}^* - 2n e^{-2u} \sin 2v.
\end{equation}

Proof. Taking the trace of (58) we get $\overline{\text{Ric}} = \text{Ric}$.

Consider the basis $\{e_0 = \xi, e_1, \ldots, e_n, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n\}$, where

\begin{equation}
e_i = e^{-u} \{\cos v e_i - \sin v \varphi e_i\}, \quad i = 1, \ldots, n.
\end{equation}

It is easy to check that this basis is orthonormal for the metric $\overline{g}$. Then (59) gives

\begin{equation}\text{Scal} = \sum_{i=0}^{2n} e_i \overline{\text{Ric}}(e_i, e_i) \quad \text{and} \quad \overline{\text{Scal}} = \sum_{i=0}^{2n} e_i \overline{\text{Ric}}(e_i, \varphi e_i)
\end{equation}

which yield the formulas for the scalar curvatures.

The formulas (61) follow from (60) and (38).

Consequently, we have

Proposition 5.3. A Sasaki-like almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g)$ is Einstein if and only if the underlying holomorphic complex Riemannian manifold $(N^{2n}, TN^{2n} = H, J, h)$ is an Einstein manifold with scalar curvature not depending on the vertical direction $\xi$.

Proof. Compare (38) with (40) to see that $(M, \varphi, \xi, \eta, g)$ is an Einstein manifold if and only if $N$ is an Einstein manifold with Einstein constant equal to $2n$, $\text{Ric}^h = 2n g$. Further, consider a contact homothetic transformation with $w = v = 0$ we get that $(M, \varphi, \xi, \eta, \overline{g} = e^{2u} g + (1 - e^{2u}) \eta \otimes \eta)$ is again Sasaki-like due to Proposition 5.1. Applying Proposition 5.2 and (40), we get the following sequence of equalities

\begin{equation}\overline{\text{Ric}}^h = \text{Ric}|_H = \text{Ric}|_H = \text{Ric}^h = \frac{\text{Scal}}{2n}g|_H = \frac{e^{-2u} \text{Scal}^h}{2n}g|_H,
\end{equation}

which yield $\overline{\text{Scal}}^h = e^{-2u} \text{Scal}^h = 4n^2$ by choosing the constant $u$ to be equal to $e^{-2u} = \frac{4n^2}{\text{Scal}^h}$, i.e. the Einstein constant of the complex holomorphic Einstein manifold $N$ can always be made equal to $4n^2$ which completes the proof.

Suppose we have a Sasaki-like almost contact complex Riemannian manifold which is Einstein, $\text{Ric} = 2n g$, and make a contact homothetic transformation

\begin{equation}\overline{\eta} = \eta, \quad \overline{\xi} = \xi, \quad \overline{g}(x, y) = c \ g(x, y) + d \ g(x, \varphi y) + (1 - c) \eta(x)\eta(y),
\end{equation}

where $c$ and $d$ are constant with $c + d = 1$. Then $\overline{\text{Ric}} = \text{Ric}$, and $\overline{\eta}$ is a contact form since $\overline{\eta}^2 = 0$. A Sasaki-like almost contact complex Riemannian manifold $(M, \varphi, \xi, \eta, g, \overline{g})$ is Einstein if and only if $\text{Scal}(\xi) = 2n$. Then $\overline{\text{Scal}} = 2n e^{-2u}$ which is constant.

Note: The above text contains mathematical content that is not fully transcribed or formatted correctly. It appears to be a continuation of a discussion on Sasaki-like almost contact complex Riemannian manifolds, focusing on properties of the Ricci tensor and scalar curvatures under contact homothetic transformations. The propositions and equations provided are part of a larger theorem or proof, suggesting a focus on complex Riemannian geometry with specific emphasis on metrics and curvature properties under transformation.
where $c$, $d$ are constants. According to Proposition 5.2 we obtain using (62) that
\[
\overline{\text{Ric}}(x, y) = \text{Ric}(x, y) = 2n g(x, y)
\]
\[
= \frac{2n}{c^2 + d^2} \left\{ c \overline{g}(x, y) - d \overline{g}(x, \varphi y) + (c^2 + d^2 - c) \eta(x) \eta(y) \right\}.
\]
(63)

We may call a Sasaki-like space whose Ricci tensor satisfies (63) an $\eta$-complex-Einstein Sasaki-like manifold and if the constant $d$ vanishes, $d = 0$, we have $\eta$-Einstein Sasaki-like space. Thus, we have shown

**Proposition 5.4.** Any $\eta$-complex-Einstein Sasaki-like space is contact homothetic to an Einstein Sasaki-like space.

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