Solution to a Soft Fuzzy Group Decision-Making Problem Involving a Soft Fuzzy Number Valued Information System

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ABSTRACT
In this paper, we introduce an operation union on the collection of soft fuzzy numbers [1] related to multi-parameter set and elucidated with a hypothetical example. For a given soft fuzzy number valued information system (IS), we define a strict partial ordering and a fuzzy number valued utility function on the initial universal set relating to each attribute, which in turn yields a utility soft information corresponding to each entity. We also define a finite collection of soft fuzzy number valued information systems, soft fuzzy number valued hierarchical information systems and their corresponding soft unions. A group decision-making problem with individual attribute set for decision makers wherein the perceptions are expressed using soft fuzzy numbers is modelled involving the collection of soft fuzzy number valued information systems. Such a problem is called soft fuzzy group decision-making problem. A new procedure to solve the problem of finding importance (weights) of the decision makers in such a situation is also proposed in which utility soft information plays a major role. An algorithm is developed to solve the same. Validation of the methodology is shown with an illustration of real life situation.

1. Introduction
In real life situations, we come across problems that comprises of exact, imprecise or uncertain, simple or complex information that needs to be analysed for various requirements. We begin with the collection of facts that are available to us and end at a stage where we are equipped with models and methodologies applicable to handle the existing scenario. The search for new models to interpret the knowledge acquired and methods to handle any situations in various fields of subjects is an ongoing research process. To process and analyse any aspect of an entity or a collection of entities, there is a need to find an appropriate model which will enable us to capture all aspect of its nature without loss of information. In such situations, the information expressed in terms of multi-parameter sets is of great importance. A collection of soft fuzzy number valued information systems is one such model introduced in this paper which is used as tools to model several characteristics,
uncertainty, impreciseness, etc. in complex situations leading to the development of the soft fuzzy group decision-making problem (SFGDMP).

1.1. Literature Survey

In many real life situations, more than one individual or decision makers or experts were involved in a decision-making process. To handle various kinds of problems that arose in group decision-making (GDM), different approaches were developed and studied by several researchers. In most of the GDM problems agreed attribute set were considered by the decision makers. Hwang and Lin [2] in their book (Part III) had presented an overall view of methods and techniques in group participation analysis till 1987 wherein some sections had been dedicated to GDM problems in classical (non-fuzzy) setup that involved individual attribute set corresponding to decision makers. In the literature of GDM, problems involving individual attribute set were further studied by few researchers (to cite [3–5]).

In 1965, Zadeh [6] formulated fuzzy sets, which in an imprecise environment captured the inexactness present in a system. Decision-making in fuzzy environment was initiated by Bellman and Zadeh [7], which paved the way to the development of several methods to solve multi-attribute decision-making problems. Zadeh [8] elucidated the concept of linguistic variables to handle situations that involved less preciseness in humanistic systems which was further studied by several researchers using appropriate kind of fuzzy numbers. Jean and Andrew [9] in 1973 applied social preferences as fuzzy binary relation in GDM problems and by the end of the decade researchers [10–12] dealt with multiple-aspect decision-making in the presence of uncertainty wherein weights and ratings were represented as fuzzy variables. GDM in the fuzzy environment was further studied in several directions and it was recorded in the collection of papers [13] edited by Kacprzyk and Fedrizzi, published in 1990. Evaluation or selection of alternatives under multiple attribute set is one type of problem in GDM. Over the years, many researchers developed various methods in solving GDM problems (to cite a few [14–16]).

A detailed literature survey on multiple attribute GDM problems in both classical and fuzzy environment due to Kabak and Ervural [17] was recorded in the year 2017.

On the other hand, the concept of a soft set as a mathematical tool for dealing uncertainty was introduced by Molodtsov [18] in 1999 to be a parameterised family of subsets of some universal set $U$. Combination of soft sets with fuzzy sets was studied to capture the nature of entities in the problem in hand. In 2001, Maji and Roy [19] defined a fuzzy soft set to be a soft set in which the set of all subsets of $U$ were replaced by the collection of fuzzy sets on $U$. With the on set of the new millennium, Biswas et al. [20] had applied soft sets in decision-making problems. GDM involving fuzzy soft set theory was studied by few researchers (to cite a few [21,22]).

In 2012 and 2013, Samantha and Das [23] defined the soft real set in which the initial universal set was considered to be $\mathbb{R}$, the set of real numbers, where they studied the properties in-depth and applied it to decision-making problems. Beaula and Raja introduced fuzzy soft numbers [24] as a fuzzy set over soft real number [25–29]. In 2018, the authors [30] had defined a real measure on soft real set for comparison purposes and applied in a multi-attribute decision-making problems. In 2019, the concept of soft fuzzy numbers
(combining fuzzy numbers and soft set), fuzzy number valued measure on soft fuzzy numbers and soft fuzzy number valued information systems ($\tilde{\mathcal{I}}\tilde{\mathcal{S}}$) was introduced and studied by the authors [1] wherein a decision-making problem to handle in-depth information was considered.

1.2. Motivation

Down the century, many researchers had considered the importance of decision makers in GDM problems and the problem of determining the same led to new research avenues. Determination of the objective weights of decision makers was considered as one such avenue and in 2019, an overview of various methods was reviewed by Kabak and Koksalmis [31].

The objective methods so far developed were applicable only for certain types of problems in which the decision matrix of each decision maker was considered with agreed attribute set. A procedure for determining the importance of decision makers for each alternative in a GDM problem with individual attribute set had not been considered yet.

In this paper, for the first time, we record a new formulation involving collection of soft fuzzy number valued information systems and a new methodology to solve the problem of determining the importance of decision makers in such a situation. Using these SFGDMP involving collection of soft fuzzy number valued information systems is discussed.

1.3. Outline of the Paper

The paper is systematised as follows: in Section 2, we provide needed prerequisites and some results needed for further study. A hypothetical example is introduced to study the different situations involved in the paper. In Section 3, we discuss the union of soft fuzzy numbers related to multi-parameter sets and property of the fuzzy number valued measure on the same with respect to the weights of the parameters are studied. In Section 4, we define a finite collection of soft fuzzy number valued information systems and is discussed in detail. Soft union on the collection of soft fuzzy number valued information systems and soft fuzzy number valued hierarchical information systems are defined. In Section 5, we deal with the mathematical formulation for a SFGDMP problem and an algorithm to solve the same. In Section 6, the methodology proposed is applied and discussed as a case study based on a real life situation, using a secondary data collected from websites. Also conclusion of the paper are recorded.

2. Preliminaries

In this section, we consider fuzzy numbers $\mathcal{F}(\mathbb{R})$, soft fuzzy number valued information system $\tilde{\mathcal{I}}\tilde{\mathcal{S}}$ and its related properties.

**Definition 2.1:** [32] A fuzzy number denoted by $\tilde{A}$ was defined as a mapping $\tilde{A} : \mathbb{R} \to [0, 1]$ that satisfies

1. Normality, i.e. there exists $t_0 \in \mathbb{R}$, such that $\tilde{A}(t_0) = 1$.
2. Convexity, i.e. for $s \leq t \leq r$, $\tilde{A}(t) \geq \tilde{A}(s) \land \tilde{A}(r) \equiv \min(\tilde{A}(s), \tilde{A}(r))$. 


Definition 2.5: [34] The scalar multiplication of any fuzzy number $\bar{A}$ was given by $[\bar{A}]_\alpha = \{t : \mu_{\bar{A}}(t) \geq \alpha\}$ for $0 < \alpha \leq 1$ and shown to a closed interval $[a^\alpha, b^\alpha]$ (say).

The collection of such fuzzy numbers is denoted as $\mathcal{F}(\mathbb{R})$.

The $\alpha$ cut of a fuzzy number $\bar{A}$ was given by $[\bar{A}]_\alpha = \{t : \mu_{\bar{A}}(t) \geq \alpha\}$ for $0 < \alpha \leq 1$ and shown to a closed interval $[a^\alpha, b^\alpha]$ (say).

Definition 2.2: [32] For $\bar{A}_1, \bar{A}_2 \in \mathcal{F}(\mathbb{R})$ with $[\bar{A}_1]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[\bar{A}_2]_\alpha = [a_2^\alpha, b_2^\alpha]$, a partial ordering $\leq$ on $\mathcal{F}(\mathbb{R})$ was defined by

$$\bar{A}_1 \leq \bar{A}_2 \text{ if and only if } a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha \forall \alpha \in (0, 1) .$$

(1)

Definition 2.3: [32] A fuzzy number $\bar{A}$ is said to be non-negative if $\bar{A}(t) = 0$ for $t < 0$. The collection of all non-negative fuzzy numbers is denoted as $\mathcal{F}^+(\mathbb{R})$.

For rest of the paper, we consider only $\mathcal{F}^+(\mathbb{R})$.

Definition 2.4: [33] For any two fuzzy numbers $\bar{A}_1, \bar{A}_2$ in $\mathcal{F}^+(\mathbb{R})$ with $[\bar{A}_1]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[\bar{A}_2]_\alpha = [a_2^\alpha, b_2^\alpha]$, the arithmetic operations $\oplus$ on collection of fuzzy numbers $\mathcal{F}^+(\mathbb{R})$ were expressed using resolution identity due to Ref. [8] as follows:

$$\bar{A}_1 \oplus \bar{A}_2 = \cup_{\alpha \in [0, 1]} [\bar{A}_1 \oplus \bar{A}_2]_\alpha \text{ where } [\bar{A}_1 \oplus \bar{A}_2]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha]$$

$$\bar{A}_1 \oplus \bar{A}_2 \in \mathcal{F}^+(\mathbb{R}).$$

Definition 2.5: [34] The scalar multiplication of any fuzzy number $\bar{A} \in \mathcal{F}^+(\mathbb{R})$ by a non-negative real number $\lambda$ was defined as $\lambda(\bar{A}) = \cup_{\alpha \in [0, 1]} [\lambda(\bar{A})]_\alpha$ with

$$[\lambda(\bar{A})]_\alpha = [\lambda a^\alpha, \lambda b^\alpha] \text{ and } \lambda(\bar{A}) \in \mathcal{F}^+(\mathbb{R}).$$

Definition 2.6: [35] The distance $d$ between any two fuzzy numbers $\bar{A}_1, \bar{A}_2 \in \mathcal{F}^+(\mathbb{R})$ was defined by $d(\bar{A}_1, \bar{A}_2) = \sup_{\alpha \in [0, 1]} \delta([\bar{A}_1]_\alpha, [\bar{A}_2]_\alpha)$, where

$$\delta([\bar{A}_1]_\alpha, [\bar{A}_2]_\alpha) = \max(|a_1^\alpha - a_2^\alpha|, |b_1^\alpha - b_2^\alpha|),$$

whenever

$$[\bar{A}_1]_\alpha = [a_1^\alpha, b_1^\alpha] \text{ and } [\bar{A}_2]_\alpha = [a_2^\alpha, b_2^\alpha].$$

Fuzzy numbers that were very often used in various real life applications were triangular and trapezoid fuzzy numbers. Recently, in 2013, the concept of linear octagonal fuzzy number was introduced by Malini and Kennedy [36], which was found to be more useful for solving real life problems.

Definition 2.7: [36] A fuzzy number $\bar{A}$ was said to be a linear octagonal fuzzy number denoted by $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8$. Forrest of the paper, we consider only $\mathcal{F}^+(\mathbb{R})$. Recently, in 2013, the concept of linear octagonal fuzzy number was introduced by Malini and Kennedy [36], which was found to be more useful for solving real life problems.
$a_8 \in \mathbb{R}$ with membership function $\tilde{A}(x)$ given by

$$
\tilde{A}(x) = \begin{cases} 
  k \left( \frac{x - a_1}{a_2 - a_1} \right), & a_1 \leq x \leq a_2, \\
  k, & a_2 \leq x \leq a_3, \\
  k + (1 - k) \left( \frac{x - a_3}{a_4 - a_3} \right), & a_3 \leq x \leq a_4, \\
  1, & a_4 \leq x \leq a_5, \\
  k + (1 - k) \left( \frac{a_6 - x}{a_6 - a_5} \right), & a_5 \leq x \leq a_6, \\
  k, & a_6 \leq x \leq a_7, \\
  k \left( \frac{a_8 - x}{a_8 - a_7} \right), & a_7 \leq x \leq a_8, \\
  0, & \text{otherwise},
\end{cases}
$$

where $0 \leq k \leq 1$.

**Remark 2.1:** A linear octagonal fuzzy number would look like Figure 1.

The $\alpha$ - cut of a linear octagonal fuzzy number was computed as follows:

$$
[\tilde{A}_\alpha] = \begin{cases} 
  \left[ a_1 + \frac{\alpha}{k}(a_2 - a_1), a_8 - \frac{\alpha}{k}(a_8 - a_7) \right], & \alpha \in [0, k], \\
  \left[ a_8 + \frac{\alpha - k}{1 - k}(a_4 - a_3), a_5 - \frac{\alpha - 1}{k - 1}(a_6 - a_5) \right], & \alpha \in (k, 1].
\end{cases}
$$

**Definition 2.8:** [36] Let $\tilde{A}$ be an octagonal fuzzy number. The measure on $\tilde{A}$ was defined by $M_{\text{Oct}}(\tilde{A}) = (1/4) [(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1 - k)]$.

**Remark 2.2:** [36] Any two linear octagonal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ could be compared using the following:
Remark 2.3: Linear octagonal fuzzy numbers yield better results for the choices of $k < 0.5$ [15,37] in solving transportation and decision-making problems. For our study, we consider linear octagonal fuzzy numbers with $k = 0.3$.

Definition 2.9: [18] Let $U$ be a universal set and $E$ be a set of parameters. A soft set is defined as a mapping $F$ from $E$ to the set of all subsets of $U$ denoted by $(F, E)$.

Definition 2.10: [23] A soft real set $(F, E)$ was defined as a mapping $F : E \rightarrow \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is the collection of bounded subsets of $\mathbb{R}$.

A soft real number denoted $(F, E)$ was defined as a particular soft real set which is a singleton soft real set that has been identified with the corresponding soft element.

Definition 2.11: [30] For a soft real set $(F, E)$, the measure denoted by $\tilde{M}^*[(F, E)]$ is defined by $\tilde{M}^*[(F, E)] = \prod_{e \in E} m^*(F(e))$, where $m^*$ stands for the Lebesgue outer measure.

Note that for each $e \in E$, we have $F(e) \subset \mathbb{R}$ and the Lebesgue outer measure of $F(e)$ was given by $m^*(F(e)) = \inf \sum_i l(I^e_i)$ where the infimum is taken over all countable collections of intervals $\{I^e_i\}$ such that $F(e) \subset \bigcup_i I^e_i$.

Remark 2.4: $\tilde{M}^*[(F, E)] = 0$ if $F(e)$ is a singleton set.

For comparing any two soft real numbers, we define the following:

Definition 2.12: A real measure on soft real number $(F, E)$, with $E = \{e_j\}_{j=1}^l$ denoted by $\tilde{M}[(F, E)]$, is defined by $\tilde{M}[(F, E)] = \sum_{j=1}^l w_j F(e)$, where $w_j$ are the weights assigned to the parameters $e_j$ such that $\sum_{j=1}^l w_j = 1$.

Remark 2.5: The measure defined in Definition 2.12 is applied in the example cited in the Remark 4.1.

Definition 2.13: [1] A soft fuzzy number was defined as a mapping $\tilde{f} : E \rightarrow \mathcal{F}^* (\mathbb{R})$, where $E$ is the parameter set. The collection of soft fuzzy numbers was denoted as $\mathcal{F}^* (\mathbb{R})(E)$.

Remark 2.6: The soft fuzzy number considered in Definition 2.13 is used in Example 2.1.

Remark 2.7: If the fuzzy number associated with the parameters is linear octagonal fuzzy numbers then the corresponding soft fuzzy numbers are called soft linear octagonal fuzzy number.

To understand the concept of soft fuzzy numbers and various concepts introduced in Sections 3, 4 and 5, we consider the following hypothetical example.
Table 1. Assessment: single contestant.

| Judges | Phrase 1 | Phrase 2 | Phrase 3 |
|--------|----------|----------|----------|
| I      | EX       | G        | EX       |
| II     | G        | G        | G        |
| III    | VG       | EX       | G        |

Table 2. Linguistic terms–fuzzy numbers.

| Linguistic terms | Octagonal fuzzy numbers |
|------------------|-------------------------|
| Excellent (EX)   | (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) |
| Very good (VG)  | (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) |
| Good (G)         | (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) |

Example 2.1: In a beauty contest, suppose there are three judges involved to evaluate individual contestant based on the two aspects ‘Elegance’ and ‘Intelligence’. The judges analyse the ‘elegance’ of the contestants by considering phrases such as ‘stylish dress up’ (phrase 1), ‘graceful walk’ (phrase 2) and ‘Presentation’ (phrase 3).

To test and evaluate the ‘intelligence’ of individuals, the judges may post different questions. Suppose the questions such as

1. ‘What qualities should miss world embody?’ (Q₁).
2. ‘What do you think is the greatest threat your generation right now is facing?’ (Q₂).
3. ‘Who is the most inspiring woman for you in the world?’ (Q₃).
4. ‘Do you think there should be a change in the world, if so What is the change you expect?’ (Q₄).
5. ‘If you can choose one famous figure from the past or present to solve one of your global problems, Whom would it be and why?’ (Q₅).
6. ‘What is the most significant change you have seen in the world over the last 10 years?’ (Q₆).
7. ‘Why should you be Miss world? Give me a compelling reason.’ (Q₇).

Let the verbal assessment of the three judges be recorded using the linguistic terms ‘Excellent (EX)’, ‘Very good (VG)’ and ‘Good (G)’.

We consider a situation that records the result regarding elegance of one of the contestants given by the three judges based on various related ‘phrases’ as listed in Table 1.

We express the assessment of the three judges as soft fuzzy numbers \((\tilde{f}, E), (\tilde{g}, E)\) and \((\tilde{h}, E)\) with the ‘phrases’ as parameter set \(E = \{e₁, e₂, e₃\}\), where \(e₁ =\) phrase 1, \(e₂ =\) phrase 2 and \(e₃ =\) phrase 3.

The various linguistic terms considered are represented by linear octagonal fuzzy numbers (see Definition 2.7) given in Table 2.

The soft linear octagonal fuzzy numbers are

\[
\tilde{f} = \{\tilde{f}(e₁) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}(e₂) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \\
\tilde{f}(e₃) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3)\} \]
\[
\tilde{g}(e) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \\
\tilde{g}(e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \\
\tilde{g}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)
\]

\[
(\tilde{h}, E) = (\tilde{h}(e_1)) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \\
\tilde{h}(e_2) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \\
\tilde{h}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)
\]

**Definition 2.14:** [1] The sum \( \oplus : \tilde{F}^+(\mathbb{R})(E) \times \tilde{F}^+(\mathbb{R})(E) \to \tilde{F}^+(\mathbb{R})(E) \) was defined as follows: for any two soft fuzzy numbers \((\tilde{f}, E), (\tilde{g}, E) \in \tilde{F}^+(\mathbb{R})(E), \)

\[
(\tilde{f}, E) \oplus (\tilde{g}, E) = (\tilde{f}(e) + \tilde{g}(e) \text{ for } e \in E).
\]

The scalar multiplication was defined by \( \lambda (\tilde{f}, E) = (\lambda \tilde{f}(e) \text{ for } e \in E) \) any non-negative real number \( \lambda \).

**Definition 2.15:** [1] Let \( (\tilde{f}, E) \in \tilde{F}^+(\mathbb{R})(E) \) with \( E = \{e_j\}_{j=1}^l \). A fuzzy number valued measure on \( \tilde{f}, E \) was defined by

\[
\tilde{M}((\tilde{f}, E)] = \oplus \sum_{j=1}^l [w_j \tilde{f}(e_j)],
\]

where \( w_j \geq 0 \) are weights of the parameters in \( E \) with \( \sum_{j=1}^l w_j = 1 \). Here note that \( \oplus \sum \) represents sum of fuzzy numbers.

**Definition 2.16:** [1] Let \( M : \tilde{F}(\mathbb{R}) \to \mathbb{R} \), \( M(\tilde{A}) \) denote the defuzzified value of a fuzzy number \( \tilde{A} \in \tilde{F}(\mathbb{R}) \) based on any suitable defuzzification method under consideration. Then any two soft fuzzy numbers \((\tilde{f}, E), (\tilde{g}, E) \in \tilde{F}^+(\mathbb{R})(E)) \) are related by the relation ‘<’ given by

\[
(\tilde{f}, E) \prec (\tilde{g}, E),
\]

if \( \tilde{M}((\tilde{f}, E)] < (\sim, \leq \text{ or } >) \tilde{M}((\tilde{g}, E)] \) and \( \tilde{M}((\tilde{f}, E)]) < (\approx, \leq \text{ or } >) \tilde{M}((\tilde{g}, E)]) \).

Note that \( (\tilde{f}, E) \prec (\tilde{g}, E) \) only if

\[
M(\tilde{M}((\tilde{f}, E)]) < (\approx, \leq \text{ or } >) M(\tilde{M}((\tilde{g}, E)]),
\]

\((\tilde{F}^+(\mathbb{R}), \leq)\) is a partially ordered set.

Note that though we have considered element \((\tilde{f}, E), \) we do not distinguish between two elements \((\tilde{f}_1, E), (\tilde{f}_2, E) \) which have the same measure \( \tilde{M}, \) rather we consider instead of individual element’s \((\tilde{f}, E) \) equivalence classes of elements of same measure \( \tilde{M} \) without explicit mention.
Definition 2.17: [1] A soft fuzzy number valued information system is a quadruple

\[ \tilde{S} = (U, A, \tilde{F}^* (\mathbb{R})(\mathcal{E}), \tilde{l}) \]

where \( U = \{ u_i \}_{i=1}^m \) is the set of objects under consideration, \( A = \{ a_j \}_{j=1}^n \) is the attribute set, \( \mathcal{E} = \{ E_1, E_2, \ldots, E_n \} \) and \( E_j = \{ e_{kj} \}_{k=1}^{I_j} \) is the parameter set associated with attribute \( a_j \), \( l_j \) representing the number of parameters in \( E_j \) and if \( \tilde{l} : U \times A \to \tilde{F}^* (\mathbb{R})(\mathcal{E}) \) is a mapping such that \( \tilde{l}(u_i, a_j) = (\tilde{f}_{ij}, E_j) \in \tilde{F}^* (\mathbb{R})(\mathcal{E}) \) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) where \( (\tilde{f}_{ij}, E_j) \) is a soft fuzzy number.

We shall define order on collection of alternatives based on soft information as follows:

Remark 2.8: From Definition 2.17, we obtain soft fuzzy number valued function on \( U \) for each \( a \in A \) and denote collection of such function by \( \tilde{U}(\tilde{F}^* (\mathbb{R})(\mathcal{E})). \) \( \tilde{S} \) can be viewed as mapping \( \tilde{F} : A \to \tilde{U}(\tilde{F}^* (\mathbb{R})(\mathcal{E})). \) Such a mapping defines a soft fuzzy number soft set denoted \( (\tilde{F}, A) \).

Definition 2.18: For a given \( \tilde{S} = (U, A, \tilde{F}^* (\mathbb{R})(\mathcal{E}), \tilde{l}) \), for each attribute \( a \in A \), we define an order relation \( \lesssim_a \) on \( U \) given by \( u \lesssim_a v \) iff \( \tilde{l}(u, a) \lesssim \tilde{l}(v, a) \) for \( u, v \in U \) where \( \lesssim \) is defined in Definition 2.16.

Proposition 2.1: \((U, \lesssim_a)\) is a partially ordered set with \( \lesssim_a \) being a partially ordered relation on \( U \).

Proof: (1) For \( u, v, w \in U \) and \( \tilde{l}(u, a), \tilde{l}(v, a), \tilde{l}(w, a) \in \tilde{F}^* (\mathbb{R})(\mathcal{E}) \), we prove the following:

(a) To prove \( \lesssim_a \) is reflexive. Since \( M(\tilde{M}[\tilde{l}(u, a)]) \leq M(\tilde{M}[\tilde{l}(u, a)]) \) holds good, we have from Definition 2.16, \( \tilde{l}(u, a) \lesssim \tilde{l}(u, a) \), i.e. \( u \lesssim_a u \) for all \( u \in U \).

(b) To prove \( \lesssim_a \) is antisymmetric. If \( M(\tilde{M}[\tilde{l}(u, a)]) \leq M(\tilde{M}[\tilde{l}(v, a)]) \) and \( M(\tilde{M}[\tilde{l}(v, a)]) \geq M(\tilde{M}[\tilde{l}(u, a)]) \), then \( M(\tilde{M}[\tilde{l}(u, a)]) = M(\tilde{M}[\tilde{l}(v, a)]) \) which in turn implies that if \( \tilde{l}(u, a) \lesssim \tilde{l}(v, a) \) and \( \tilde{l}(v, a) \lesssim \tilde{l}(u, a) \) then \( \tilde{l}(u, a) \approx \tilde{l}(v, a) \). I.e. \( u \lesssim_v v \) and \( v \lesssim_v u \) then \( u \approx_v v \). Hence \( \lesssim_a \) is antisymmetric.

(c) Again from Definition 2.16, we have that if \( M(\tilde{M}[\tilde{l}(u, a)]) \leq M(\tilde{M}[\tilde{l}(v, a)]) \) and \( M(\tilde{M}[\tilde{l}(v, a)]) \leq M(\tilde{M}[\tilde{l}(w, a)]) \) and \( M(\tilde{M}[\tilde{l}(w, a)]) \leq M(\tilde{M}[\tilde{l}(u, a)]) \) which in turn implies that if \( \tilde{l}(u, a) \lesssim \tilde{l}(v, a) \) if \( \tilde{l}(v, a) \lesssim \tilde{l}(w, a) \) then \( \tilde{l}(u, a) \lesssim \tilde{l}(w, a) \). I.e. if \( u \lesssim_v v \) and if \( v \lesssim_w w \), then \( u \lesssim_w w \). Hence \( \lesssim_a \) is transitive.

Definition 2.19: For a given \( \tilde{S} = (U, A, \tilde{F}^* (\mathbb{R})(\mathcal{E}), \tilde{l}) \), for each \( a \in A \), we define the fuzzy number valued utility function \( \tilde{U}_a : (U, \lesssim_a) \to (\tilde{F}^* (\mathbb{R}), \lesssim) \) given by

\[ \tilde{U}_a(u) = \bigoplus_{s \in U | s \lesssim_u v} \tilde{M}[\tilde{l}(s, a)]. \]
Proposition 2.2: \( \tilde{U}_a \) is an order preserving function.

Proof: To prove \( \tilde{U}_a \) is order preserving, we need to prove the condition

\[
\tilde{U}_a(u) \preceq \tilde{U}_a(v) \text{ whenever } u \preceq_a v \tag{2.1}
\]

for all \( u, v \in U \).

For \( v \in U \),

\[
\tilde{U}_a(v) = \bigoplus \sum_{t \in U | t \preceq_a v} \tilde{M}(\tilde{l}(t, a)).
\]

Choose an arbitrary \( u \prec_a v \) and fix it. For all \( s \preceq_a u \) and \( t \preceq_a v \), we have

\[
\tilde{U}_a(v) = \bigoplus \sum_{s \in U | s \preceq_a v} \tilde{M}(\tilde{l}(s, a)) \bigoplus \bigoplus \sum_{t \in U | t \preceq_a v} \tilde{M}(\tilde{l}(t, a)) \succ \bigoplus \sum_{s \in U | s \preceq_a v} \tilde{M}(\tilde{l}(s, a))
\]

\[
= \tilde{U}_a(u).
\]

Hence condition (2.1) holds

Definition 2.20: For a soft fuzzy number valued information system

\( \tilde{IS} = (U, A, \mathcal{F}^*(\mathbb{R})(\mathcal{E}), \tilde{l}) \) for each \( u \in U \), define a mapping \( \tilde{f}^u : A \to \mathcal{F}^*(\mathbb{R}) \) such that

\[
\tilde{f}^u(a) = \bigoplus \sum_{s \in U | s \preceq_a v} \tilde{M}(\tilde{l}(s, a)), a \in A.
\]

Then \( \tilde{f}^u \) will be a soft fuzzy number called soft utility information of \( u \) in \( \tilde{IS} \).

Definition 2.21: [1] A soft fuzzy number valued hierarchical information system is a quintuple \( \tilde{SH} = (U, A, H_A, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_A), \tilde{l}_H) \) where \( H_A = \{H_{a_j} | a_j \in A \} \), where \( H_{a_j} \) denote the concept hierarchy tree of attribute \( a_j \) for \( j = 1, 2, \ldots, n \). \( \mathcal{E}_A = \{E_{a_j} \}_{j=1}^n \) with \( E_{a_j} = \{E_{js} \}_{s=1}^{l_j} \) the collection of parameter sets associated with \( l_j \) leaf nodes of the concept hierarchy tree and \( \tilde{l}_{a_j} : U \times A \to \mathcal{F}^*(\mathbb{R})(\mathcal{E}_A) \) is a function such that \( \tilde{l}_{a_j}(u, a_j) \) consists of corresponding collection of soft fuzzy numbers in all levels of concept hierarchy tree.
Table 3. Evaluation of three judges.

| Judge I | Judge II | Judge III |
|---------|----------|-----------|
| e₁₁     | e₁₂      | e₂₁       |
| G       | VG       | EX        |

3. Properties of Soft Fuzzy Numbers Related to Multi-parameter Sets

In this section, we define union on collection of soft fuzzy numbers related to multi-parameter sets and discuss it with an example. Also study the behaviour of fuzzy number valued measure over the operation introduced.

Let $E = \{E_1, \ldots, E_n\}$ be the collection of parameter sets under consideration with $E_1 = \{e_{1,k}\}_{k=1}^{l_1}, \ldots, E_n = \{e_{n,k}\}_{k=1}^{l_n}$, where $l_1, \ldots, l_n$ denote the corresponding number of parameters in $E_1, \ldots, E_n$ and $\tilde{F}^\ast (\mathbb{R})(E)$ be the collection of soft fuzzy numbers related to multi-parameter set $E$.

Definition 3.1: The soft union denoted $\tilde{\sqcup}$ of any finite collection of soft fuzzy numbers $\{(\tilde{f}_j, E_j)\}_{j=1}^{n} \in \tilde{F}^\ast (\mathbb{R})(E)$ is defined as a function $\tilde{\tilde{f}}_{\tilde{\sqcup}} : H \rightarrow \tilde{F}^\ast (\mathbb{R})$ where $H = \bigcup_{j} E_j$ is given by

$$\tilde{\tilde{f}}_{\tilde{\sqcup}}(e) = \left\{ \begin{array}{ll}
\tilde{f}_j(e), & e \in E, e \neq E_i, i \neq j, \\
\frac{1}{t} \oplus \sum_{j=1}^{t} (\tilde{f}_j e)_{j=1}^{t}, & e \in \bigcap_{j=1}^{t} E_j, t \neq n
\end{array} \right.$$ 

and $(\tilde{\tilde{f}}_{\tilde{\sqcup}}, H)$ is a soft fuzzy number related to parameter set $H$.

Example 3.1: In this example, we consider the evaluations of the judges of one contestant to test her intelligence based on the questions given in Example 2.1. Let $E_1 = \{e_{11} = Q_1, e_{12} = Q_2\}, E_2 = \{e_{21} = Q_3, e_{22} = Q_4\}$ and $E_3 = \{e_{31} = Q_5, e_{32} = Q_7\}$ be the set of questions posed by the judges I, II and III, respectively. This gives raise to multi-parameter sets (say) $E_1, E_2, E_3$. Suppose the judges have commented upon the reply of the contestant as linguistic terms given in Table 3, then the evaluation of the judges I, II, and III are represented as soft linear octagonal fuzzy numbers $(\tilde{f}_1, E_1)$, $(\tilde{f}_2, E_2)$ and $(\tilde{f}_3, E_3)$, respectively. Using the operation defined in Definition 3.1, the combined evaluation is obtained as a soft linear octagonal fuzzy number $(\tilde{\tilde{f}}_{\tilde{\sqcup}}, H)$, with $H = E_1 \cup E_2 \cup E_3$, given by

$$\tilde{\tilde{f}}_{\tilde{\sqcup}}(e) = \left\{ \begin{array}{ll}
\tilde{f}_j(e), & e \in E, e \neq E_i, i \neq j, \\
\frac{1}{t} \oplus \sum_{j=1}^{t} (\tilde{f}_j e)_{j=1}^{t}, & e \in \bigcap_{j=1}^{t} E_j, t \neq n
\end{array} \right.$$ 

and $(\tilde{\tilde{f}}_{\tilde{\sqcup}}, H)$ is a soft fuzzy number related to parameter set $H$.

Remark 3.1: The evaluation of judge I (say) about the contestant based on both the aspects 'elegance' and 'intelligence' gives rise to soft linear octagonal fuzzy numbers
related to multi-parameter sets $E$ and $E_1$. Thus, $(\tilde{f}, E)$ and $(\tilde{f_1}, E_1)$ from Examples 2.1 and 3.1 are soft linear octagonal fuzzy numbers related to multi-parameter sets under consideration.

Properties of $\tilde{M}$ related to multi-parameters sets involving soft union are studied in the following proposition:

**Proposition 3.1:** For any two soft fuzzy numbers $(\tilde{f_1}, E_1), (\tilde{f_2}, E_2) \in \mathcal{F}^*(\mathbb{R})(\mathcal{E})$ related to $\mathcal{E} = \{E_1, E_2\}$, $E_1 = \{e_{1,k}\}_{k=1}^{l_1}$, $E_2 = \{e_{2,k}\}_{k=1}^{l_2}$ with $\{w_{1,k}\}_{k=1}^{l_1}$ and $\{w_{2,k}\}_{k=1}^{l_2}$ the associated weights of the parameters, we have

\[
\tilde{M}[(\tilde{f_1}, E_1) \sqcup (\tilde{f_2}, E_2)] \geq \frac{1}{2^2} (\tilde{M}[(\tilde{f_1}, E_1) \oplus (\tilde{f_2}, E_2)]).
\]

**Proof:** Considering Definition 3.1 for $j = 1, 2$, the soft union of $(\tilde{f_1}, E_1)$ and $(\tilde{f_2}, E_1)$ is the soft fuzzy number $(\tilde{f_\sqcup}, H)$ with $H = E_1 \cup E_2$ and

\[
\tilde{f_\sqcup}(e) = \begin{cases} 
\tilde{f_1}(e), & e \in E_1, \\
\tilde{f_2}(e), & e \in E_2, \\
\frac{\tilde{f_1}(e) \oplus \tilde{f_2}(e)}{2}, & e \in E_1 \cap E_2.
\end{cases}
\]

Case 1: $E_1 \cap E_2 = \emptyset$

From Definition 2.15, we have

\[
\tilde{M}[(\tilde{f_1}, E_1) \sqcup (\tilde{f_2}, E_2)] = \tilde{M}[(\tilde{f_\sqcup}, H)]
\]

\[
= \bigoplus_{e \in H} \sum_{e \in H} w_e \tilde{f_\sqcup}(e), \quad \text{such that} \sum_{e \in H} w_e = 1
\]

\[
= \bigoplus_{e \in E_1} \frac{1}{2} \tilde{f_1}(e) + \bigoplus_{e \in E_2} \frac{1}{2} \tilde{f_2}(e)
\]

\[
= \bigoplus_{k=1}^{l_1} \frac{w_{1,k}}{2} \tilde{f_\sqcup}(e_{1,k}) + \bigoplus_{k=1}^{l_2} \frac{w_{2,k}}{2} \tilde{f_\sqcup}(e_{2,k}).
\]

Let $[\tilde{f_1}(e_{1,k})]_\alpha = [a_{f_{1,k}}^\alpha, b_{f_{1,k}}^\alpha]$ for $k = 1, \ldots, l_1$, then

\[
\left[ \frac{w_{1,k}}{2} \tilde{f_\sqcup}(e_{1,k}) \right]_\alpha = \left[ \frac{w_{1,k}}{2} \tilde{f_1}(e_{1,k}) \right]_\alpha
\]

\[
= \left[ \frac{w_{1,k}}{2} a_{f_{1,k}}^\alpha, \frac{w_{1,k}}{2} b_{f_{1,k}}^\alpha \right].
\]
Therefore,
\[ \sum_{k=1}^{l_1} \frac{w_{1,k}}{2} \tilde{f}_1(e_{1,k}) = \frac{1}{2} \left[ \sum_{k=1}^{l_1} \frac{w_{1,k}a_{f_1}^\alpha}{2} f_{1,k} \right] + \frac{1}{2} \left[ \sum_{k=1}^{l_1} \frac{w_{1,k}b_{f_1}^\alpha}{2} f_{1,k} \right] \]
which is a \( \alpha \) - cut of the fuzzy number
\[ \bigoplus \frac{1}{2} \sum_{k=1}^{l_1} \frac{w_{1,k}}{2} \tilde{f}_1(e_{1,k}) = \frac{1}{2} \tilde{M}([\tilde{f}_1, E_1]). \]

Along the lines for \([\tilde{f}_2(e_{2,k})]_\alpha = [a_{f_2}^\alpha, b_{f_2}^\alpha], k = 1, \ldots, l_2, \) we have
\[ \sum_{k=1}^{l_2} \frac{w_{2,k}}{2} \tilde{f}_2(e_{2,k}) = \frac{1}{2} \tilde{M}([\tilde{f}_2, E_2]). \]

Hence
\[ \sum_{k=1}^{l_1} \frac{w_{1,k}}{2} \tilde{f}_1(e_{1,k}) + \sum_{k=1}^{l_2} \frac{w_{2,k}}{2} \tilde{f}_2(e_{2,k}) = \frac{1}{2} \tilde{M}([\tilde{f}_1, E_1] \sqcup [\tilde{f}_2, E_2]). \]

which yields
\[ \tilde{M}([\tilde{f}_1, E_1] \sqcup [\tilde{f}_2, E_2]) \approx \frac{1}{2} \tilde{M}([\tilde{f}_1, E_1]) \oplus \tilde{M}([\tilde{f}_2, E_2]). \]

**Case 2:** \( E_1 \cap E_2 \neq \emptyset \)

Without loss of generality, we suppose that \( E_1 \cap E_2 \) consist of one parameter (say), \( e = e_{1,l_1} = e_{2,l_2}, \) then the weights associated with the parameters in \( H_\sqcup \) are \([w_{1,k})/2 |_{k=1}^{l_1}, \]
\([w_{2,k})/2 |_{k=1}^{l_2-1} \) and \((w_{1,l_1} + w_{2,l_2})/2), \) so that \( \sum_{k=1}^{l_1-1} (w_{1,k})/2 + \sum_{k=1}^{l_2-1} (w_{2,k})/2 + ((w_{1,1} + w_{2,2})/2) = 1. \)

Call the \( \alpha \)-cut \([\tilde{f}_1(e)]_\alpha = [a_{f_1}^\alpha, b_{f_1}^\alpha] \) and \([\tilde{f}_2(e)]_\alpha = [a_{f_2}^\alpha, b_{f_2}^\alpha] \) for each \( j. \)

Then using operations on fuzzy numbers, for each \( e, \) we have
\[ [\tilde{f}_1(e) \oplus \tilde{f}_2(e)]_\alpha = [a_{f_1}^\alpha + a_{f_2}^\alpha, b_{f_1}^\alpha + b_{f_2}^\alpha] \]
and
\[ \frac{1}{2} \left[ \frac{w_{1,l_1} + w_{2,l_2}}{2} \tilde{f}_1(e) \oplus \tilde{f}_2(e) \right] = \frac{1}{2} \left[ \frac{w_{1,l_1} + w_{2,l_2}}{2} a_{f_1}^\alpha + a_{f_2}^\alpha, \frac{w_{1,l_1} + w_{2,l_2}}{2} b_{f_1}^\alpha + b_{f_2}^\alpha \right] \]
A collection of soft fuzzy number valued information system \( (\widetilde{S}_p)_{p=1}^q \), for some \( q \in \mathbb{Z}^+ \), can be obtained from Definition 2.17 and is given by Definition 4.1.

**Definition 4.1:** A collection of soft fuzzy number valued information system \( (\widetilde{S}_p)_{p=1}^q \), for some \( q \in \mathbb{Z}^+ \), are defined as the quadruple \( \widetilde{S}_p = (U, A_p, \mathcal{F}^* (\mathbb{R}) (E_p), \widetilde{I}_p) \), where \( U = \{ u_i \}_{i=1}^m \)
Table 4. Assessment by Judge I: five contestants.

| Contestants | Phrase 1 | Phrase 2 | Phrase 3 | Q1 | Q2 |
|-------------|----------|----------|----------|----|----|
| 1           | EX       | G        | EX       | G  | VG |
| 2           | G        | EX       | G        | VG | G  |
| 3           | EX       | G        | EX       | G  | G  |
| 4           | VG       | VG       | EX       | VG | G  |
| 5           | EX       | EX       | VG       | EX | EX |

Table 5. Assessment by Judge II: five contestants.

| Contestants | Phrase 1 | Phrase 2 | Phrase 3 | Q1 | Q2 |
|-------------|----------|----------|----------|----|----|
| 1           | G        | G        | VG       | EX | G  |
| 2           | G        | G        | EX       | VG | VG |
| 3           | EX       | G        | EX       | VG | G  |
| 4           | G        | VG       | VG       | EX | G  |
| 5           | VG       | VG       | G        | VG | G  |

Table 6. Assessment by Judge III: five contestants.

| Contestants | Phrase 1 | Phrase 2 | Phrase 3 | Q1 | Q2 |
|-------------|----------|----------|----------|----|----|
| 1           | VG       | EX       | G        | VG | G  |
| 2           | VG       | EX       | G        | G  | G  |
| 3           | VG       | G        | VG       | G  | VG |
| 4           | G        | G        | VG       | VG | G  |
| 5           | EX       | VG       | G        | G  | EX |

is the set of objects under consideration, \( A_p = \{a_j^p\}_{j=1}^{n_p} \), the attribute set, \( E_p = \{E_j^p\}_{j=1}^{n_p} \) the collection of parameter sets associated with attribute \( a_j^p \) and \( E_j^p = \{e_{jk}^p\}_{k=1}^{l_j} \), where \( l_j \) represents the number of parameters in \( E_j^p \) and \( \tilde{I}_p(u, a_j^p) = (\tilde{I}_{ij}, E_j^p) \in \mathcal{F}^*(\mathbb{R})(E_p) \) is a mapping such that \( \tilde{I}_p(u, a_j^p) \) is a mapping such that \( \tilde{I}_p(u, a_j^p) = (\tilde{I}_{ij}, E_j^p) \in \mathcal{F}^*(\mathbb{R})(E_p) \), the collection of soft fuzzy numbers related to multi-parameter set for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n_p \).

The situation considered in Example 2.1 can be better explained using the soft fuzzy number valued information system where the evaluations of the three judges and the information corresponding to them are formulated as collection of soft linear octagonal fuzzy number valued information systems.

**Example 4.1:** Evaluations in the final round. Here consider the case when there are five contestants evaluated by the three judges based on both the aspects ‘elegance’ and ‘intelligence’ and the linguistic assessments are given in Tables 4–6. Then the assessments are modelled as \( \tilde{I}_p = (U, A_p, \mathcal{F}^*(\mathbb{R})(E_p), \tilde{I}_p) \), where \( p = 1, 2, 3 \) refers to the three judges. \( U = \{PT_1, PT_2, PT_3, PT_4, PT_5\} \).

Here, the attribute set is common for all the judges, i.e. \( A_1 = A_2 = A_3 = A \).

\( A = \{a_1, a_2\} \), where \( a_1 = \text{‘elegance’} \) and \( a_2 = \text{‘intelligence’} \). The parameter set associated with \( a_1 \) is same for all the judges (from Example 2.1) while \( a_2 \) is evaluated through personalisation questions and is different for each judge (from Example 3.2). Therefore, we have \( E_1 = \{E, E_1\} \), \( E_2 = \{E, E_2\} \) and \( E_3 = \{E, E_3\} \).
The evaluations of the three judges corresponding to the attributes are given by $\tilde{f}_p$ and are represented as corresponding soft linear octagonal fuzzy number valued information systems given in Table 7.

The assessment of Judge $I$:

$\tilde{f}_{11}^1 (E) = \{ \tilde{f}_{11}^1 (e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{11}^1 (e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{11}^1 (e_3) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{21}^1 (E) = \{ \tilde{f}_{21}^1 (e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}$,

$\tilde{f}_{21}^1 (e_2) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{21}^1 (e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}$,

$\tilde{f}_{31}^1 (E) = \{ \tilde{f}_{31}^1 (e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{31}^1 (e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{31}^1 (e_3) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{41}^1 (E) = \{ \tilde{f}_{41}^1 (e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{41}^1 (e_2) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{41}^1 (e_3) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{51}^1 (E) = \{ \tilde{f}_{51}^1 (e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}$,

$\tilde{f}_{51}^1 (e_2) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{51}^1 (e_3) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \}$,
Table 7. Collection of soft linear octagonal fuzzy number valued information systems.

|   |   |   |
|---|---|---|
| $\tilde{I}_2 = (U, A, \mathcal{F}^+(\mathbb{R}) (\mathcal{E}_2), \tilde{h}_2)$ | $\tilde{I}_1 = (U, A, \mathcal{F}^+(\mathbb{R}) (\mathcal{E}_1), \tilde{h}_1)$ | $\tilde{I}_3 = (U, A, \mathcal{F}^+(\mathbb{R}) (\mathcal{E}_3), \tilde{h}_3)$ |
| $\tilde{f}_{21} (E_1) = \{\tilde{f}_{21}(e_{11}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{21}(e_{12}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3)\}$ | $\tilde{f}_{11} (E_1) = \{\tilde{f}_{11}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{11}(e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{11}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)\}$ | $\tilde{f}_{21} (E_1) = \{\tilde{f}_{21}(e_{11}) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{21}(e_{12}) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3)\}$ |

The assessment of Judge II:

- $\tilde{f}_{21} (E_1) = \{\tilde{f}_{21}(e_{11}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{21}(e_{12}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{21}(e_{13}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)\}$

- $\tilde{f}_{21} (E_2) = \{\tilde{f}_{21}(e_{21}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{21}(e_{22}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{21}(e_{23}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)\}$

- $\tilde{f}_{21} (E_3) = \{\tilde{f}_{21}(e_{31}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{21}(e_{32}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{21}(e_{33}) = (0.7, 0.8, 0.9, 1, 1, 1, 1, 1; 0.3)\}$
\[ \tilde{f}_{41}(e_2) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{41}(e_3) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ (\tilde{f}_{51}, E) = (\tilde{f}_{51}(e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{51}(e_2) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{51}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{12}, E_2) = (\tilde{f}_{12}(e_{11}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{12}(e_{12}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{22}, E_2) = (\tilde{f}_{22}(e_{11}) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{22}(e_{12}) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ (\tilde{f}_{32}, E_2) = (\tilde{f}_{32}(e_{11}) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{32}(e_{12}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{42}, E_2) = (\tilde{f}_{42}(e_{11}) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{42}(e_{12}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{52}, E_2) = (\tilde{f}_{52}(e_{11}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ \tilde{f}_{52}(e_{12}) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3). \]

The assessment of judge III:

\[ (\tilde{f}_{11}, E) = (\tilde{f}_{11}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{11}(e_2) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \tilde{f}_{11}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{21}, E) = (\tilde{f}_{21}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{21}(e_2) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{21}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \]

\[ (\tilde{f}_{31}, E) = (\tilde{f}_{31}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]

\[ \tilde{f}_{31}(e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{31}(e_3) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \]
\( \tilde{f}_{41}(E) = \{ \tilde{f}_{41}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}, \)
\( \tilde{f}_{41}(e_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3), \tilde{f}_{41}(e_3) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \},
\( \tilde{f}_{51}(E) = \{ \tilde{f}_{51}(e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}, \)
\( \tilde{f}_{51}(e_2) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3), \tilde{f}_{51}(e_3) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \},
\( \tilde{f}_{12}(E) = \{ \tilde{f}_{12}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \}, \)
\( \tilde{f}_{12}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \},
\( \tilde{f}_{22}(E) = \{ \tilde{f}_{22}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}, \)
\( \tilde{f}_{22}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \},
\( \tilde{f}_{32}(E) = \{ \tilde{f}_{32}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}, \)
\( \tilde{f}_{32}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \},
\( \tilde{f}_{42}(E) = \{ \tilde{f}_{42}(e_1) = (0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3) \}, \)
\( \tilde{f}_{42}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \},
\( \tilde{f}_{52}(E) = \{ \tilde{f}_{52}(e_1) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) \}, \)
\( \tilde{f}_{52}(e_1) = (0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3) \}).

**Remark 4.1:** Need for a soft fuzzy scenario in an information system.

Information systems are best suited to model complex situations involving attributes. In the fuzzy information system, the objects related to qualitative attributes are imprecise in nature and quantification of these using linguistic variables is evaluated with linguistic values (fuzzy numbers) describing fuzziness in such a system.

In Example 2.1, suppose the information considered is based on only fuzziness (not including the parameters), such a system gives only peripheral information and does not yield a foolproof model. I.e. suppose evaluation by the three judges \( (p = 1, 2, 3) \) corresponding to the attributes \( A = \{ a_1, a_2 \} \), where \( a_1 = \) ’Elegance’ and \( a_2 = \) ’Intelligence’, are recorded in Table 8 as information systems \( S_p = (U, A, V, \rho_p) \), where \( \rho_p : U \times A \rightarrow V \), \( V \) the set of linear octagonal fuzzy numbers that are used to describe the linguistic variables ‘EX’, ‘VG’ and ‘G’ as in Table 2.

The problem of ranking the contestants by the individual judges for this fuzzy information (fuzzy decision matrix) is solved using a fuzzy simple additive weighting method.
Table 8. Information systems involving linguistic variables.

|    |  $A$ | $a_1$ | $a_2$ |
|----|------|-------|-------|
| $S_1 = (U, A, V, \rho_1)$ |   | VG    | VG    |
| $PT_1$ |   | VG    | VG    |
| $PT_2$ |   | G     | G     |
| $PT_3$ |   | EX    | EX    |
| $PT_4$ |   | G     | G     |
| $PT_5$ |   | EX    | EX    |
| $S_2 = (U, A, V, \rho_2)$ |   | VG    | VG    |
| $PT_1$ |   | EX    | VG    |
| $PT_2$ |   | G     | EX    |
| $PT_3$ |   | VG    | VG    |
| $PT_4$ |   | G     | G     |
| $PT_5$ |   | EX    | G     |
| $S_3 = (U, A, V, \rho_3)$ |   | VG    | VG    |
| $PT_1$ |   | VG    | VG    |
| $PT_2$ |   | G     | G     |
| $PT_3$ |   | EX    | EX    |
| $PT_4$ |   | G     | G     |
| $PT_5$ |   | EX    | EX    |

Table 9. Individual ranking based on defuzzification measure.

|    | Judge I | Judge II | Judge III |
|----|---------|----------|-----------|
| $PT_1$ | 3       | 2        | 3         |
| $PT_2$ | 4       | 1        | 3         |
| $PT_3$ | 2       | 4        | 2         |
| $PT_4$ | 4       | 2        | 3         |
| $PT_5$ | 1       | 3        | 1         |

Table 10. Linguistic terms – five-point scale.

| Linguistic terms         | Points |
|--------------------------|--------|
| Excellent (EX)           | 5      |
| Very good (VG)           | 4      |
| Good (G)                 | 3      |
| Fair (FA)                | 2      |
| Poor (P)                 | 1      |

and the ranking order (Definition 2.8) using the defuzzification measure of linear octagonal fuzzy numbers is shown in Table 9.

Again in Example 2.1, suppose we consider the evaluation by the three judges on a five-point scale (crisp), incorporating parameters as recorded in Table 10.

The assessment by the three judges (given in Tables 4–6) are expressed as soft real number valued information systems shown in Table 11, where $I_p : U \times A \to \mathbb{R}(\mathcal{E}_p)$, $p = 1, 2, 3$ with the collection of parameter sets $\mathcal{E}_p = \{E, E_p\}$ and $\mathbb{R}(\mathcal{E}_p)$ the collection of soft real numbers related to $e_p E = \{e_1, e_2, e_3\}; E_1 = \{e_{1,1}, e_{1,2}\}; E_2 = \{e_{2,1}, e_{2,2}\}$ and $E_3 = \{e_{3,1}, e_{3,2}\}$.
Similarly, the evaluations for judge I are expressing the qualitative information are not captured appropriately.

A crisp simple additive weighting method is shown in Table 13.

Wherein we infer from columns 2, 4 and 6 that the contestants are ranked uniquely.

The ranking order of the contestants corresponding to individual judge based on the attribute corresponding to attribute a₁ are

\[(F_{11}^1, E) = \{ F_{11}^1(e_1) = 5, F_{11}^1(e_2) = 3, F_{11}^1(e_3) = 5 \},\]
\[(F_{21}^1, E) = \{ F_{21}^1(e_1) = 3, F_{21}^1(e_2) = 5, F_{21}^1(e_3) = 3 \},\]
\[(F_{31}^1, E) = \{ F_{31}^1(e_1) = 5, F_{31}^1(e_2) = 3, F_{31}^1(e_3) = 5 \},\]
\[(F_{41}^1, E) = \{ F_{41}^1(e_1) = 4, F_{41}^1(e_2) = 4, F_{41}^1(e_3) = 5 \},\]
\[(F_{51}^1, E) = \{ F_{51}^1(e_1) = 5, F_{51}^1(e_2) = 3, F_{51}^1(e_3) = 5 \}.

Similarly, the evaluations for judge I corresponding to attribute a₂ can be evaluated. Along lines, the evaluation of other judges for the attributes a₁, a₂ can be evaluated which are soft real numbers.

On computing the real measure (see Definition 2.12) on soft real numbers in the soft real number valued information system, we have crisp decision matrices corresponding to the three judges as in Table 12.

The ranking order of the contestants corresponding to individual judge based on the crisp simple additive weighting method is shown in Table 13.

In this case, repetition of the ranks has occurred due to the fact that the intricacies in expressing the qualitative information are not captured appropriately.

Note that the contestants are not ranked in the same way and uniquely in Tables 9 and 13. Hence, there is a need for a new model incorporating intricate points such as attributes, collection of sub-attributes, parameters, collection of sub-parameters and impreciseness which occur in the natural scenario. The answer to this is exhibited in this paper having the soft fuzzy number valued information system to model such a scenario (see Table 18) wherein we infer from columns 2, 4 and 6 that the contestants are ranked uniquely.

|   |   |   |
|---|---|---|
|   |   |   |
|   |   |   |
|   |   |   |
|   |   |   |

**Table 11. Soft real number information systems.**

The evaluation of Judge I (say) for the five contestants corresponding to the attribute a₁ are

\[(F_{11}^1, E) = \{ F_{11}^1(e_1) = 5, F_{11}^1(e_2) = 3, F_{11}^1(e_3) = 5 \},\]
\[(F_{21}^1, E) = \{ F_{21}^1(e_1) = 3, F_{21}^1(e_2) = 5, F_{21}^1(e_3) = 3 \},\]
\[(F_{31}^1, E) = \{ F_{31}^1(e_1) = 5, F_{31}^1(e_2) = 3, F_{31}^1(e_3) = 5 \},\]
\[(F_{41}^1, E) = \{ F_{41}^1(e_1) = 4, F_{41}^1(e_2) = 4, F_{41}^1(e_3) = 5 \},\]
\[(F_{51}^1, E) = \{ F_{51}^1(e_1) = 5, F_{51}^1(e_2) = 3, F_{51}^1(e_3) = 5 \}.

Similarly, the evaluations for judge I corresponding to attribute a₂ can be evaluated. Along lines, the evaluation of other judges for the attributes a₁, a₂ can be evaluated which are soft real numbers.

On computing the real measure (see Definition 2.12) on soft real numbers in the soft real number valued information system, we have crisp decision matrices corresponding to the three judges as in Table 12.

The ranking order of the contestants corresponding to individual judge based on the crisp simple additive weighting method is shown in Table 13.

In this case, repetition of the ranks has occurred due to the fact that the intricacies in expressing the qualitative information are not captured appropriately.

Note that the contestants are not ranked in the same way and uniquely in Tables 9 and 13. Hence, there is a need for a new model incorporating intricate points such as attributes, collection of sub-attributes, parameters, collection of sub-parameters and impreciseness which occur in the natural scenario. The answer to this is exhibited in this paper having the soft fuzzy number valued information system to model such a scenario (see Table 18) wherein we infer from columns 2, 4 and 6 that the contestants are ranked uniquely.
Table 12. Measure on soft real number in the soft real number valued information system.

| U  | A | a_1 | a_2 |
|----|---|-----|-----|
| Judge I  | PT_1 | 1.65 | 3.5 |
|       | PT_2 | 0.99 | 3.5 |
|       | PT_3 | 1.65 | 3.0 |
|       | PT_4 | 1.32 | 3.5 |
|       | PT_5 | 1.65 | 5.0 |
| Judge II | PT_1 | 3.30 | 4.0 |
|        | PT_2 | 3.63 | 4.0 |
|        | PT_3 | 4.29 | 3.5 |
|        | PT_4 | 3.63 | 4.0 |
|        | PT_5 | 3.63 | 3.0 |
| Judge III | PT_1 | 3.96 | 3.5 |
|        | PT_2 | 3.96 | 3.0 |
|        | PT_3 | 3.63 | 3.5 |
|        | PT_4 | 3.30 | 3.5 |
|        | PT_5 | 3.96 | 4.0 |

Table 13. Individual ranking – beauty contest – soft crisp information.

| U  | Judge I | Judge II | Judge III |
|----|---------|----------|-----------|
| PT_1 | 2       | 3        | 2         |
| PT_2 | 3       | 1        | 4         |
| PT_3 | 4       | 2        | 2         |
| PT_4 | 2       | 1        | 3         |
| PT_5 | 1       | 4        | 1         |

Remark 4.1: Consideration of only fuzzy information, the attribute set is an agreed set. The problem of determining the importance of the judges and then their combined evaluation to select the best contestant in such a situation could be done by any available methods (see Section 1.1). But these methods cannot be applied to the situation wherein parameters are considered to capture the in-depth information. Also through Remark 4.1, we have insisted the need of soft fuzzy information systems in choosing the beat contestant. Hence, new methodology is needed to handle a SFGDMP.

Remark 4.2: Using this information in the process of selecting the best pageant by combined evaluation by all the three judges is cited in Section 5.

Definition 4.2: Let \( \{\tilde{S}_p\}_{p=1}^{q} \) be a finite collection of the soft fuzzy number valued information system (where \( \tilde{S}_p = (U, A_p, F^*(\mathbb{R})(E_p), \tilde{I}_p) \), see Definition 4.1) for some \( q \in \mathbb{Z}^+ \).

Call \( \mathcal{A} = \bigcup_{p=1}^{q} A_p = \{a_i\}_{i=1}^{s} \) for \( s \leq n_1 + n_2 + \cdots + n_p \) and \( \mathcal{E} = \{\mathcal{E}_a\}_1 \) where

\[
E_{a_i}^p = \begin{cases} 
E_i^p, & a_1 \in A_p & \& a_i \notin A_q, \text{for any } q, E_i^p \text{ the associated parameter set in } \mathcal{E}_p, \\
\cup_{p=1}^{q_1} E_i^p, & a_1 \in \bigcap_{p=1}^{q_1} A_p, q_1 \leq q, \{E_i^p\}_{p=1}^{q_1} \text{ the associated parameter set in } \{\mathcal{E}_p\}_{p=1}^{q_1}.
\end{cases}
\]
and $\mathcal{F}^*(\mathbb{R})(\mathcal{E})$ the collection of soft fuzzy numbers associated with $\mathcal{E}$.

We define a mapping $\tilde{l}_1 : U \times A \rightarrow \mathcal{F}^*(\mathbb{R})(\mathcal{E})$ given by

$$
\tilde{l}_1(u_i, a_j) = \begin{cases} 
\tilde{l}(u_i, a_j), & a_1 \in A_p, a_1 \notin A_q, p \neq q, \text{ for any } q, \\
\tilde{\cup}(\tilde{l}(u_i, a_j))_{p=1}^{q_1}, & a_1 \in \cap_{p=1}^{q_1} A_p, \text{ for some } q_1 \leq q
\end{cases}
$$

and the quadruple $(U, A, \mathcal{F}^*(\mathbb{R})(\mathcal{E}), \tilde{l}_1)$ or in short $\tilde{S}_1$ is the soft union of soft fuzzy number valued information systems. Note that $\tilde{S}_1$ is a soft fuzzy number valued information system.

**Definition 4.3:** For two hierarchical soft fuzzy number valued information systems $\tilde{S}_1 = (U, A_1, H_{A_1}, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_1}), \tilde{l}_{A_1})$ and $\tilde{S}_2 = (U, A_2, H_{A_2}, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_2}), \tilde{l}_{A_2})$, we define

(i) $a_j \in \text{Att} A_1 \cap A_2$ if $H_{a_j} = H_{a_j}$ and, i.e. common only at nodes.

(ii) $a_j \in A_1 \cap A_2$ if $H_{a_j} = H_{a_j}$ and $\tilde{l}_{A_1}(u_i, a_j) = \tilde{l}_{A_2}(u_i, a_j)$, i.e. common at both nodes and leaf values.

**Definition 4.4:** $\tilde{S}_p = (U, A_p, H_{A_p}, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_p}), \tilde{l}_{A_p})$ for $p = 2$ be two hierarchical soft fuzzy number valued information systems, where $H_{A_p} = \{H_{a_p} / a_p \in A_p\}$, $H_{a_p}$ denotes the concept hierarchy tree of attribute $a_p$ for $j = 1, 2, \ldots, n_p$. $\mathcal{E}_{A_p} = \{\mathcal{E}_{A_p}\}_{j=1}^{n}$ with $\mathcal{E}_{a_p}^{j} = \{E_{a_p}^{j}\}_{p=1}^{P}$ is the collection of parameter sets associated with $a_j$ leaf nodes of the concept hierarchy tree and $\tilde{\cup}_{a_p} : U \times A \rightarrow \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_p})$ is a function such that $\tilde{\cup}_{a_p} (u_i, a_j)$ consists of corresponding collection of soft fuzzy numbers in all levels of concept hierarchy tree.

Call $\mathcal{A} = \bigcup_{i=1}^{p=1} A_p = \{a_i\}_{i=1}^s$ for $s \leq n_1 + n_2 + \cdots + n_p$, $H_{A_i} = \{H_{a_i} / a_i \in A_i\}$ with $H_{a_i} = H_{a_i}, a_i \in A_p$ and $\mathcal{E}_{A_i} = \{\mathcal{E}_{a_i}\}$, where

$$
E_{a_i} = \begin{cases} 
\{E_{a_i}^1\}_{s=1}^{1}, a_1 \in A_1 \& a_i \notin A_2, & E_{a_i}^1 \text{ parameter sets associated with leaf nodes in } \mathcal{E}_{a_i}, \\
\{E_{a_i}^2\}_{s=1}^{2}, a_1 \in A_2 \& a_i \notin A_1, & E_{a_i}^2 \text{ parameter sets associated with leaf nodes in } \mathcal{E}_{a_i}, \\
\bigcup_{p=1}^{2} E_{a_i}^{p}\}_{s=1}^{P}, a_i \in \text{Att } A_1 \cap A_2, & E_{a_i}^{p}\ \text{ parameter sets associated with leaf nodes in } \mathcal{E}_{a_i}, \\
\{E_{a_i}^{p}\}_{p=1}^{2}, \bigcup_{s=1}^{P}, & a_i \in A_1 \cap A_2
\end{cases}
$$

and $\mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_i})$ is the collection of soft fuzzy numbers associated with $\mathcal{E}_{A_i}$.

We define a mapping $\tilde{l}_{A_{ij}} : U \times A \rightarrow \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_i})$ given by

$$
\tilde{l}_{A_{ij}}(u_i, a_j) = \begin{cases} 
\tilde{l}_{h_j}(u_i, a_i), a_i \in A_1, & a_i \notin A_2, \\
\tilde{l}_{h_2}(u_i, a_i), a_i \in A_2, & a_i \notin A_1, \\
\tilde{\cup}(\tilde{l}_{h_p}(u_i, a_i))_{p=1}^{q_1}, a_i \in \text{Att } A_1 \cap A_2
\end{cases}
$$

and the quintuple $(U, A, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_{A_i}), \tilde{l}_{A_{ij}})$ or in short $\tilde{S}_{A_{ij}}$ is the soft union of soft fuzzy number valued hierarchical information systems. Note that $\tilde{S}_{A_{ij}}$ is a soft fuzzy number valued hierarchical information system.
5. SFGDMP

In multi-attribute GDM problems, importance (weights) to decision makers play a major role, wherein combined evaluation of decision makers are involved for selecting the optimal entity (alternative). Assigning or determining importance of decision makers is problem context. Subjective importance to the decision makers are assigned based on the expertise level (knowledge, experience, etc.). Difficulty of considering subjective importance may arise in many situations (expertise level not known, to avoid partiality in assigning weights, etc.). On the other hand, deriving objective importance of decision makers using the data provided (in the form assessments, evaluations, perception, etc.) is modelled using appropriate mathematical models.

In this section, we introduce SFGDMP as a situation in decision-making wherein the perception of each decision maker describing some aspect $Q$ of collection the entities based on corresponding individual attribute sets associated with underlying parameters. In such a situation, to handle the problem of determining the importance to decision makers, we have developed a new mathematical formulation which in turn is used to rank the entities. An algorithm is proposed to solve the same and validated with a real life situation.

The problem in hand is to

1. frame a suitable mathematical model for SFGDMP,
2. find a methodology to compute appropriate importance to decision makers for each,
3. alternative as the attributes are different for different decision makers,
4. measure the aspect by combining the evaluations of the decision makers and
5. choose the best entity.

5.1. Problem Description

Consider the SFGDMP involving a finite number of decision makers evaluating a finite collection of entities based on different characteristics features with associated parameter sets to choose the best entity. The problem is mathematically formulated as the collection of $\{\tilde{I}_{Sp}\}_{p=1}^{q}$ (or) $\{\tilde{IS}_{Hp}\}_{p=1}^{q}$ (see Definitions 4.1 and 4.5), where $U = \{u_1, u_2, \ldots, u_m\}$ be a finite collection of entities under consideration and $\{D_p\}_{p=1}^{q}$ be $q$ decision makers involved in evaluation of the entities.

$\tilde{I}_p : U \times A_p \rightarrow \mathcal{F}^*(\mathbb{R})(E_p)$ is a mapping such that $\tilde{I}_p(u_i, a^p_j) = (\overline{f}^p_{ij}, \overline{E}^p_j) \in \mathcal{F}^*(\mathbb{R})(E_p)$, the collection of soft fuzzy numbers related to multi-parameter set, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n_p$ and $p = 1, \ldots, q$, represents the perception of $q$ decision makers about the entities in relation to different characteristics (individual attribute set $A_p$) features describing the aspect $Q$ with associated parameter sets $\{E_p\}$.

Let $\lambda^p_i$ for $p = 1, \ldots, q$ and $i = 1, \ldots, m$ be the weights of the $q$ decision makers corresponding to the $i$th entity and $w^p_j$ denote the weights of the corresponding attributes in $A_p$ for $p = 1, \ldots, q$ and $w^p_{j_k}$ be the weights of the parameters in $E^p_j$ for $k = 1, \ldots, l_j$ and $p = 1, \ldots, q$ in the collection $\{\tilde{I}_{Sp}\}_{p=1}^{q}$.

Suppose the nature of the aspect involved in the problem are described by class of sub-characteristic features of characteristic features; weights of the parameters in parameter sets at leaf nodes can be assigned appropriately in the collection $\{\tilde{IS}_{Hp}\}_{p=1}^{q}$.
The problem is

1. to determine $\lambda_i^p$ such that $\sum_{p=1}^q \lambda_i^p = 1$,
2. to compute the aspect value of the $i$th entity $Q_i^p$ and combined aspect value $Q_i^C$
   incorporating $\lambda_i^p$ for $p = 1, \ldots, q$ and
3. to choose the optimum entity based on $Q_i^C$.

5.2. Methodology

As the attribute set is different for each of the decision maker, a new procedure is adapted
wherein objective importance to the decision makers are determined as real numbers for
each entity. Here, fuzzy number valued measure on utility soft information is obtained as
fuzzy numbers corresponding to the various entities in individual and combined decision
makers information system. We consider the closeness of each decision maker’s decision to
that of the combined decision in a fuzzy setup to derive the weights.

Applying this methodology, we propose an algorithm to solve the problem.

5.3. Procedure

**Step 1:** For each $p$, compute the corresponding fuzzy number valued measure $\tilde{U}_p(u_i)$ of
utility soft information of each entity $u_i$ by performing the following:

**Step 1 i:** If the information consists of $\tilde{I}_S^p$, then consider the following steps or if the
information consists of $\tilde{I}_H^p$, go to Step 1 v.

**Step 1 ii.:** For $i = 1, \ldots, m$ entity, input the soft fuzzy number $(\tilde{f}_j^p, E_j^p), a_j^p \in A_p, j =
1, 2, \ldots, n_p$, where $n_p$ are the number of attributes in $A_p$.

**Step 1 iii.:** Determine $\tilde{M}(\tilde{f}_j^p, E_j^p, a_j^p)$ and we reach $\tilde{I}_S^p, E_p \equiv A_p, (\tilde{f}_j, A_p)$, where $\tilde{f}_j (a_j^p) =
\tilde{M}_j^p$.

**Step 1 iv.:** Go to Step 1 x.

**Step 1 v:** For $\tilde{I}_H^p$, perform the following:

**Step 1 vi.:** Input the soft fuzzy number for each $i = 1, \ldots, m$ corresponding to each
sub-characteristic feature at the leaf nodes.

**Step 1 vii.:** Obtain the fuzzy number valued measure for the inputs.

**Step 1 viii.:** Compute Step I.ii.ii recursively back tracking till we reach $\tilde{I}_S^p, E_p \equiv A_p,$
where $\tilde{I}_p(u_i) = (\tilde{f}_i^p, A_p)$.

**Step 1 ix.:** Go to Step 1 x.

**Step 1 x:** For each $p$, compute the following to obtain fuzzy numbers $\tilde{U}_p(u_1), \ldots,$
$\tilde{U}_p(u_m)$.

**Step 1 xi.:** For each $u_i$, construct utility soft information $(\tilde{f}_i^p, A_p)$.

**Step 1 xii.:** Determine $\tilde{U}_p(u_i) = \tilde{M}(\tilde{f}_i^p, A_p)$.

**Step 1 xiii.:** Go to Step 2.

**Step 2:** To construct utility soft information in equally combined $\tilde{I}_S$ or $\tilde{I}_H$ and compute
fuzzy number valued measure $\tilde{U}(u_i)$ on it, perform the following:

**Step 2 i.:** If the information is recorded as $\tilde{I}_S^p$, do the following steps or if $\tilde{I}_H^p$ go to
Step 2 ii.
**Step 2 ii.** Obtain for each \( p, (1/q)(\tilde{l}_p) \).

**Step 2 iii.** Evaluate soft union of soft fuzzy number valued information systems obtained in previous step \( \tilde{S}_i = (U, \mathcal{A}, \mathcal{F}^*(\mathbb{R}), \tilde{l}_i) \).

**Step 2 iv.** Perform **Step 1 ii.** and **Step 1 iii.** for \( \tilde{S}_i = (U, \mathcal{A}, \mathcal{F}^*(\mathbb{R}), \mathcal{A}, \tilde{l}_i) \).

**Step 2 v.** Go to **Step 2 x.**

**Step 2 vi.** If the information is recorded as \( \mathcal{I}_{Sh}^q \), do the following steps:

**Step 2 vii.** Obtain for each \( p, (1/q)(\tilde{l}_{Hp}) \).

**Step 2 viii.** Compute the soft union \( \tilde{I}_{H} = (U, \mathcal{A}, \mathcal{F}^*(\mathbb{R}), (\mathcal{A}), \tilde{l}_{H}) \).

**Step 2 ix.** Perform **Step 1 vi.** to **Step 1 ix.** for the obtained \( \tilde{I}_{H} \) in the pervious step.

**Step 2 x.** Go to **Step 2 xi.**

**Step 2 xi.** Compute the following to obtain fuzzy numbers \( \tilde{u}_i(U_1), \ldots, \tilde{u}_i(U_m) \).

**Step 2 xii.** For each \( u_i \in U_i \), construct utility soft information \( \tilde{f} \), \( \mathcal{A} \).

**Step 2 xiii.** For each \( u_i \in U_i \), compute \( \tilde{u}(u_i) = \tilde{M}(\tilde{f}, \mathcal{A}) \) \( \in \mathcal{F}^*(\mathbb{R}) \).

**Step 2 xiv.** Using suitable defuzzification method, compute equally combined value of the aspect \( Q_i^c = \tilde{M}(\tilde{u}(u_i)) \) for each \( i \), and \( Q_i^p = \tilde{M}(\tilde{u}^p(u_i)) \), go to **Step 3.**

**Step 3:** Compute \( d((\tilde{t}_p^p \tilde{u}^p(u_i), t_i \tilde{u}(u_i))) \) and call it \( \lambda_i^p \) for each \( i \).

**Step 4:** Compute \( \lambda_{H}^i = (1/(1 + \lambda_i^p)) \) and the importance of the \( p \)th decision maker \( \lambda_i^p = \frac{\lambda_{H}^i}{(\sum_{p=1}^{q} \lambda_{H}^i)} \) for \( p = 1, \ldots, q \) such that \( \sum_{p=1}^{q} \lambda_i^p = 1 \).

**Step 5:** If \( \lambda_i^p = (1/q) \) for all \( p \) and each \( i \), then choose \( u_i \) for which \( Q_i^c \) is optimum, otherwise go to **Step 6.**

**Step 6:** Perform **Step 2** to **Step 2 iv.** or **Step 2 ix.** accordingly \( \tilde{I}_S \) or \( \tilde{I}_{Sh} \) with incorporating \( \lambda_i^p \) obtain the corresponding soft union. Go to **Step 6 i.**

**Step 6 i.** For each \( u_i \), construct utility soft information \( \tilde{g}^i \), \( \mathcal{A} \).

**Step 6 ii.** For each \( i \), determine \( \tilde{M}(\tilde{g}^i, \mathcal{A}) \) and call it \( \tilde{u}_{diff}(u_i) \).

**Step 7:** Using suitable defuzzification method, compute the combined value of the aspect \( Q_i^c = \tilde{M}(\tilde{u}_{diff}(u_i)) \) for each \( i \).

**Step 8:** Choose \( u_i \) for which \( Q_i^c \) is optimum.

In the above procedure, the following are computed by straight forward calculation using definitions discussed in Sections 1, 2 and 3.

**Remark 5.1:** In **Step 1 iii.**, \( \tilde{M}_p \) is computed using Definition 2.15.

**Remark 5.2:** **Step 2 ii.** and **Step 2 vii.** are obtained using scalar multiplication on soft fuzzy number (Definition 2.14) for the inputs.

**Remark 5.3:** **Step 2 ii** and **Step 2 viii** are computed using soft union (Definitions 4.3 and 4.5, respectively).

**Remark 5.4:** In **Step 3**, \( d((\tilde{t}_p^p \tilde{u}^p(u_i), t_i \tilde{u}(u_i))) \) refers to the closeness of decision of the \( p \)th Decision maker to that of the combined decision for each entity, where \( \{\tilde{u}^p(u_i)\}_{i=1}^{m} \) and \( \{\tilde{u}(u_i)\}_{i=1}^{m} \) are the collection of fuzzy number valued measure on utility soft information corresponding to each \( u_i \) in individual and combined information system, respectively. \( t_i^p \)
and \( t_i \) are the assigned weights based on the measure \( M(\bigcup^P(u_i)) \) and \( M(\bigcup(u_i)) \) such that \( \sum_{i=1}^{m} t_i^p = 1 \) and \( \sum_{i=1}^{m} t_i = 1 \).

**Remark 5.5:** In Step 1 xii, for each \( \tilde{I}_p \), a utility soft information \( (\tilde{f}^{(i)}_p, A_p) \) associated with \( A_p \) corresponding to each entity \( u_i \) is obtained using Definition 2.21 and the computation of the same involves a comparison among the entities using measure of the fuzzy numbers \( \tilde{M}^p_{ij} \).

Algorithm: Construct utility soft information

```plaintext
Require: \( \tilde{M}^p_{ij} (i = 1, \ldots, m, p = 1, \ldots, q, j = 1, \ldots, n) \) * Obtained from Step 1 iii.
1. for \( p \leftarrow 1, q \) do
2. for \( i \leftarrow 1, m \) do
3. for \( j \leftarrow 1, n \) do
4. \( M(\tilde{M}^p_{ij}) \)
5. end for
6. end for for \( i \leftarrow 1, m \) do
7. end for
8. for \( p \leftarrow 1, q \) do
9. for \( r \leftarrow 1, m \) do
10. for \( s \leftarrow 1, n \) do
11. \( M(\tilde{M}^p_{rs}) = M(\tilde{M}^p_{ij}) \)
12. sum(\( \tilde{f}^{(i)}_p (a^p_{ij}) \)) \leftarrow 0
13. for \( i \leftarrow 1, m \) do
14. if \( M(\tilde{M}^p_{rs}) \leq M(\tilde{M}^p_{ij}) \) then
15. sum(\( \tilde{f}^{(i)}_p (a^p_{ij}) \)) \leftarrow sum(\( \tilde{f}^{(i)}_p (a^p_{ij}) \)) + \( \tilde{M}^p_{ij} \) * Based on Definition 2.21
16. end if
17. end for
18. end for
19. end for
20. end for
```

This algorithm is used in Step 2 xiii. and Step 6 i. to obtain the corresponding utility soft information.

**Remark 5.6:** We note that the weights of the attributes and their associated parameters are problem dependent. If the attribute set has one characteristic feature and if only one decision maker involved, SFGDMP reduces to a decision-making problem considered in the earlier paper [1].

**Remark 5.7:** We consider the evaluations of three judges based on the aspect beauty for each contestant from the hypothetical Example 4.2. Suppose the judge’s expertise level is not known (some of the judges may be external or may be judging based on public perception), then the problem of determining the objective importance of the judges and the ranking of the contestant based on the combined evaluation is solved using the algorithm in the following:

Here, \( q = 3 \), i.e. the number of decision makers are three judges. Equal weights are assigned to the parameters involved in the problem. The weights to the attributes by all the three judges are shown in Table 14.
**Table 14.** Fuzzy number valued measure.

| PT  | $\tilde{f}(a_1)$, 0.4 | $\tilde{f}(a_2)$, 0.6 |
|-----|-----------------------|-----------------------|
| $P_{T_1}$ | (0.528, 0.627, 0.726, 0.825, 0.858, 0.891, 0.924, 0.957; 0.3) | (0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95, 0.95; 0.3) |
| $P_{T_2}$ | (0.363, 0.462, 0.561, 0.66, 0.726, 0.792, 0.858, 0.924; 0.3) | (0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95, 0.95; 0.3) |
| $P_{T_3}$ | (0.528, 0.627, 0.726, 0.825, 0.858, 0.891, 0.924, 0.957; 0.3) | (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3) |
| $P_{T_4}$ | (0.561, 0.66, 0.759, 0.858, 0.924, 0.99, 0.99, 0.99; 0.3) | (0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95, 0.95; 0.3) |
| $P_{T_5}$ | (0.627, 0.726, 0.825, 0.924, 0.957, 0.99, 0.99, 0.99; 0.3) | (0.7, 0.8, 0.9, 1, 1, 1, 1, 1; 0.3) |

**Step 1:** For $p = 1, 2, 3$, fuzzy number valued measure of utility soft information is computed as follows:

**Step 1 i.** Since the information for $p = 1$ is in the form $\tilde{I}_1$, perform the following steps:

**Step 1 ii.** Input the soft fuzzy numbers $\{\tilde{f}(i_1, E_1)\}$ and $\{\tilde{f}(i_2, E_1)\}$ corresponding to $a_1$ and $a_2$ from Table 7.

**Step 1 iii.** $\tilde{M}(\tilde{f}(i_1, E_1))$ and $\tilde{M}(\tilde{f}(i_2, E_1))$ are computed for each $i = 1, \ldots, 5$ and are shown in Table 14.

**Step 1 iv.** Go to **Step 1 xi**.

**Step 1 v.** Fuzzy numbers $\tilde{U}_1(P_{T_1})$, $\tilde{U}_1(P_{T_2})$, $\tilde{U}_1(P_{T_3})$, $\tilde{U}_1(P_{T_4})$, $\tilde{U}_1(P_{T_5})$ are obtained by performing the following steps:

**Step 1 vi.** Utility soft information is constructed using Definition 2.19 for $P_{T_i}$, $i = 1, \ldots, 5$, given by

$$\tilde{f}^{(1)}(a_1) = (1.419, 1.716, 2.013, 2.31, 2.442, 2.574, 2.706, 2.838; 0.3),$$

$$\tilde{f}^{(1)}(a_2) = (1.25, 1.65, 2.05, 2.45, 2.85, 3.25, 3.5, 3.75; 0.3),$$

$$\tilde{f}^{(2)}(a_1) = (0.363, 0.462, 0.561, 0.66, 0.726, 0.792, 0.858, 0.924; 0.3),$$

$$\tilde{f}^{(2)}(a_2) = (1.25, 1.65, 2.05, 2.45, 2.85, 3.25, 3.5, 3.75; 0.3),$$

$$\tilde{f}^{(3)}(a_1) = (1.419, 1.716, 2.013, 2.31, 2.442, 2.574, 2.706, 2.838; 0.3),$$

$$\tilde{f}^{(3)}(a_2) = (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3),$$

$$\tilde{f}^{(4)}(a_1) = (0.98, 2.376, 2.772, 3.168, 3.366, 3.564, 3.696, 3.828; 0.3),$$

$$\tilde{f}^{(4)}(a_2) = (1.25, 1.65, 2.05, 2.45, 2.85, 3.25, 3.5, 3.75; 0.3),$$

$$\tilde{f}^{(5)}(a_1) = (0.607, 3.102, 3.597, 4.092, 4.323, 4.554, 4.686, 4.818; 0.3),$$

$$\tilde{f}^{(5)}(a_2) = (1.95, 2.45, 2.95, 3.45, 3.85, 4.25, 4.5, 4.75; 0.3)).$$
Table 15. $\tilde{M}(\tilde{f}^{(i)}, A)$.

| $U$ | $\tilde{U}^1(PT_i)$ | $\tilde{U}^2(PT_i)$ | $\tilde{U}^3(PT_i)$ |
|-----|---------------------|---------------------|---------------------|
| $PT_1$ | (1.318, 1.676, 2.035, 2.394, 2.687, 2.980, 3.182, 3.385; 0.3) | (0.949, 1.229, 1.508, 1.788, 2.008, 2.227, 2.417, 2.606; 0.3) | (1.582, 2.020, 2.458, 2.896, 3.294, 3.692, 3.922, 4.151; 0.3) |
| $PT_2$ | (0.895, 1.175, 1.454, 1.734, 2.000, 2.267, 2.443, 2.620; 0.3) | (1.368, 1.747, 2.126, 2.506, 2.825, 3.144, 3.360, 3.576; 0.3) | (0.952, 1.210, 1.468, 1.726, 1.944, 2.162, 2.302, 2.441; 0.3) |
| $PT_3$ | (0.688, 0.886, 1.045, 1.224, 1.337, 1.450, 1.562, 1.675; 0.3) | (1.056, 1.374, 1.692, 2.010, 2.302, 2.593, 2.789, 2.984; 0.3) | (1.027, 1.346, 1.666, 1.985, 2.304, 2.623, 2.813, 3.002; 0.3) |
| $PT_4$ | (1.542, 1.940, 2.339, 2.737, 3.056, 3.376, 3.578, 3.781; 0.3) | (1.385, 1.783, 2.182, 2.580, 2.918, 3.257, 3.499, 3.741; 0.3) | (0.869, 1.148, 1.428, 1.708, 1.987, 2.267, 2.443, 2.620; 0.3) |
| $PT_5$ | (2.213, 2.711, 3.209, 3.707, 4.039, 4.372, 4.547, 4.777; 0.3) | (0.6385, 0.853, 1.072, 1.290, 1.508, 1.727, 1.879, 2.032; 0.3) | (1.852, 2.350, 2.848, 3.346, 3.774, 4.202, 4.462, 4.721; 0.3) |

**Step 1 vii.** The fuzzy number valued measure on $(\tilde{f}^{(i)}, A)$ for each $i$ is computed and is shown in Table 15.

Since for $p = 2, 3$, the information are recorded as $\tilde{I}_S^2$ and $\tilde{I}_S^3$ as given in Table 7; by performing **Step 1 i.** to **Step 1 xii.,** $\tilde{M}(\tilde{f}^{(i)}, A)$ and $\tilde{M}(\tilde{f}^{(i)}, A)$ are computed and tabulated in columns 3 and 4 of Table 15. Go to Step 2.

**Step 2:** The utility soft information for the equally combined soft fuzzy number valued information system is obtained by performing the following steps:

**Step 2 i.** For $p = 1, 2, 3, (1/3)\tilde{I}_S^p$ are obtained.

**Step 2 ii.** Using Definition 4.3, the soft union of $\{(1/3)\tilde{I}_S^p\}_{p=1}^3$ are computed and given by $\tilde{I}_S = (U, A, \mathcal{F}^*(\mathbb{R})(E), \tilde{I}_{(i)})$, where $A = \bigcup_{p=1}^3 A_p = A = \{a_1, a_2\}$ and $E = \{E_i\}_{i=1}^3$ with

$$E_{a_1} = \left\{ E \cup E \cup E, a_1 \in A_1 \cap A_2 \cap A_3, E \text{ the associated parameter sets in } E_1, E_2 \text{ and } E_3, \right\}$$

and the mapping $\tilde{I}_S : U \times A \rightarrow \mathcal{F}^*(\mathbb{R})(E)$ is given by

$$\tilde{I}_S(PT_i, a_l) = \left\lceil \frac{1}{3} \tilde{I}_1(PT_i, a_l) \cup \frac{1}{3} \tilde{I}_2(PT_i, a_l) \cup \frac{1}{3} \tilde{I}_3(PT_i, a_l) \right\rceil , a_l \in A,$$

where $\tilde{I}_S(u_i, a_1) = [(1/3)(\tilde{f}_{1\cup}, E)], \tilde{I}_S(u_i, a_2) = [(1/3)(\tilde{f}_{2\cup}, H)], H = E_1 \cup E_2 \cup E_3$ (see Example 3.2 for one contestant).

**Step 2 iii.** Performing **Step 2 ii.** and **Step 1 iii.** for $\tilde{I}_S = (U, A, \mathcal{F}^*(\mathbb{R})(A), \tilde{I}_{(i)}), [\tilde{M}(1/3)\tilde{f}_{1\cup}, E]$ and $[\tilde{M}(1/3)\tilde{f}_{2\cup}, H]$ are obtained corresponding to $a_1$ and $a_2$ as shown in Table 16.

**Step 2 xiii.** Fuzzy numbers $\tilde{U}(PT_1), \ldots, \tilde{U}(PT_5)$ are computed by performing **Step 2 xii.,** **Step 2 xiv.** wherein the utility soft information $(\tilde{f}^{(i)}, A)$ is constructed for each $PT_i \in U$, $\tilde{M}(\tilde{f}^{(i)}, A) \in \mathcal{F}^*(\mathbb{R})$ and the equally combined value of the aspect for each contestant $Q_{(i)}^{ce}$ are computed and given in Table 8.

We go to the following step:
Table 16. Fuzzy number valued measure \( [\tilde{\mathbf{M}}(1/3)(\tilde{f}_{\varphi_1}, E)] \) and \( [\tilde{\mathbf{M}}(1/3)(\tilde{f}_{\varphi_2}, H)] \).

| U         | \( a_1 \)                       | \( a_2 \)                       |
|-----------|---------------------------------|---------------------------------|
| \( PT_1 \) | 0.142, 0.174, 0.207, 0.240, 0.261, 0.283, 0.298, 0.312; 0.3 | 0.127, 0.160, 0.193, 0.226, 0.253, 0.281, 0.297, 0.314; 0.3 |
| \( PT_2 \) | 0.131, 0.163, 0.196, 0.229, 0.250, 0.272, 0.290, 0.309; 0.3 | 0.116, 0.149, 0.182, 0.215, 0.248, 0.281, 0.297, 0.314; 0.3 |
| \( PT_3 \) | 0.160, 0.192, 0.225, 0.258, 0.276, 0.294, 0.305, 0.316; 0.3 | 0.100, 0.133, 0.166, 0.199, 0.232, 0.265, 0.286, 0.308; 0.3 |
| \( PT_4 \) | 0.138, 0.171, 0.203, 0.236, 0.265, 0.294, 0.305, 0.316; 0.3 | 0.127, 0.160, 0.193, 0.226, 0.253, 0.281, 0.297, 0.314; 0.3 |
| \( PT_5 \) | 0.163, 0.196, 0.229, 0.261, 0.283, 0.305, 0.312, 0.319; 0.3 | 0.147, 0.180, 0.213, 0.246, 0.263, 0.280, 0.296, 0.313; 0.3 |

Table 17. \( \tilde{\mathbf{M}}(\tilde{\nu}^{(i)}, \mathcal{A}). \)

| U         | \( \tilde{\nu}(PT_i) \) | \( Q_i^{\infty} \) | \( t_i \) |
|-----------|--------------------------|--------------------|----------|
| \( PT_1 \) | 0.390, 0.495, 0.601, 0.706, 0.796, 0.886, 0.942, 0.998; 0.3 | 0.805 | 0.2 |
| \( PT_2 \) | 0.181, 0.234, 0.287, 0.339, 0.388, 0.436, 0.466, 0.496; 0.3 | 0.392 | 0.05 |
| \( PT_3 \) | 0.288, 0.360, 0.432, 0.504, 0.561, 0.616, 0.651, 0.686; 0.3 | 0.568 | 0.1 |
| \( PT_4 \) | 0.445, 0.564, 0.682, 0.800, 0.902, 1.003, 1.064, 1.124; 0.3 | 0.828 | 0.3 |
| \( PT_5 \) | 0.663, 0.827, 0.991, 1.156, 1.283, 1.411, 1.488, 1.566; 0.3 | 1.301 | 0.35 |

Table 18. \( \mathbf{M}(\tilde{\nu}^{(i)}, PT_i) \) and assigned weights.

| U         | \( \tilde{\nu}(PT_i) \) | \( t_i \) | \( \tilde{\nu}^{(i)}(PT_i) \) | \( \tilde{\nu}^{(i)}(PT_i) \) |
|-----------|--------------------------|----------|--------------------------|--------------------------|
| \( PT_1 \) | 2.484 | 0.2 | 0.1858 | 0.05 |
| \( PT_2 \) | 1.840 | 0.05 | 2.609 | 0.3 |
| \( PT_3 \) | 1.244 | 0.1 | 2.120 | 0.2 |
| \( PT_4 \) | 2.827 | 0.3 | 2.695 | 0.35 |
| \( PT_5 \) | 3.753 | 0.35 | 1.384 | 0.1 |

**Step 3:** Using Definition 2.6, the closeness of each judge’s decision to that combined is calculated, wherein the weights \( t_i \) and \( t_i^P \) are assigned based on the measure given in Tables 17 and 18 and given by

\[
\lambda_1^{\frac{1}{1}} = 0.677, \quad \lambda_2^{\frac{1}{1}} = 0.808, \quad \lambda_3^{\frac{1}{1}} = 0.985, \quad \lambda_4^{\frac{1}{1}} = 0.55, \quad \lambda_5^{\frac{1}{1}} = 0.471, \\
\lambda_1^{\frac{2}{2}} = 0.942, \quad \lambda_2^{\frac{2}{2}} = 0.488, \quad \lambda_3^{\frac{2}{2}} = 0.654, \quad \lambda_4^{\frac{2}{2}} = 0.507, \quad \lambda_5^{\frac{2}{2}} = 0.691, \\
\lambda_1^{\frac{3}{3}} = 0.823, \quad \lambda_2^{\frac{3}{3}} = 0.911, \quad \lambda_3^{\frac{3}{3}} = 0.653, \quad \lambda_4^{\frac{3}{3}} = 0.690, \quad \lambda_5^{\frac{3}{3}} = 0.475.
\]

**Step 4:** The normalised importance are calculated and given by

\[
\lambda_1^{\text{I}} = 0.677, \quad \lambda_2^{\text{I}} = 0.808, \quad \lambda_3^{\text{I}} = 0.985, \quad \lambda_4^{\text{I}} = 0.55, \quad \lambda_5^{\text{I}} = 0.471, \\
\lambda_1^{\text{II}} = 0.942, \quad \lambda_2^{\text{II}} = 0.488, \quad \lambda_3^{\text{II}} = 0.654, \quad \lambda_4^{\text{II}} = 0.507, \quad \lambda_5^{\text{II}} = 0.691, \\
\lambda_1^{\text{III}} = 0.823, \quad \lambda_2^{\text{III}} = 0.911, \quad \lambda_3^{\text{III}} = 0.653, \quad \lambda_4^{\text{III}} = 0.690, \quad \lambda_5^{\text{III}} = 0.475.
\]

**Step 5:** Using these \( \lambda_1^{\text{I}}, \lambda_2^{\text{II}}, \lambda_3^{\text{III}} \) values performing **Step 2 to Step 2 iv.**, we obtain \( [\tilde{l}_p(\tilde{f}(PT_i, a_l))] \) for \( p = 1, 2, 3, l = 1, 2 \) and \( \tilde{L}_{\mathcal{A}} = (U, \mathcal{A}, \mathcal{F}^*(\mathbb{R})(\mathcal{E}), \tilde{l}_{\mathcal{A}}) \), where

\[
\tilde{l}_{\mathcal{A}}(PT_i, a_l) = [\lambda_1^{\text{I}} \tilde{l}_1(PT_i, a_l)] \tilde{l}_2(PT_i, a_l) \tilde{l}_3(PT_i, a_l)] \tilde{l}_4(PT_i, a_l)] \tilde{l}_5(PT_i, a_l)], a_l \in \mathcal{A}, \ l = 1, 2.
\]
The utility soft information \(\tilde{\mu}((g_i), A)\) is constructed for each \(PT_i\) in the soft fuzzy number valued information \(\tilde{S}_I\) obtained in the previous step and the fuzzy number valued measure is computed by performing **Step 6 i.** and **Step 6 ii.**, respectively, and given in Table 19.

**Step 6:** For each \(i\), the combined value of the judges \(Q_c^i = M(\tilde{\mu}((g_i), A))\) are computed and shown in last column of Table 19.

**Step 7:** The combined evaluations of the judges yield the contestant \(PT_5\) to be ranked first.

### 6. Choosing the Best Car Brand Based on Safety Aspect: A SFGDMP Model

In this section, we consider the problem of choosing a car based on the aspect safety by combining the evaluations provided from websites [www.nhtsa.gov](http://www.nhtsa.gov) and [www.iihs.org](http://www.iihs.org) for the list of cars brands \(\{C_1, C_2, C_3, C_4, C_5\}\) (see Ref. [1]).

We quantify the star ratings, where an individual assessment based on the stars, i.e. highest number of stars accounting to better safety, is analysed as linguistic terms good, acceptable, marginal, poor with suitable comparative quantification. The linear octagonal fuzzy numbers representing the linguistic terms and the quantified ratings of the car brands are considered as given in Table 20.

The websites are considered to be the two experts and we have modelled their required information as collection of hierarchical soft fuzzy number valued information systems \((\tilde{I}_{S_{H_p}})_{p=1}^2\) wherein the safety features are considered as attribute sets and the corresponding tests conducted as parameter sets. The problem of determining the importance of experts and incorporating the same for combining their evaluations to choose the car brand with optimum level of safety are modelled as SFGDMP and solved by performing the Algorithm proposed in Section 5.

### 6.1. Problem Description and Solution

We consider a situation where two people in a family (say) \(P_1\) and \(P_2\) want to buy a car for a common usage. They are concerned about the safety measures, so \(P_1\) and \(P_2\) gather information from websites [www.iihs.org](http://www.iihs.org) and [www.nhtsa.gov](http://www.nhtsa.gov), respectively, to make a combined decision in the choice of a car.

#### Table 20. Fuzzy number representing linguistic terms.

| Star rating | Linguistic term | Octagonal fuzzy number          |
|-------------|-----------------|---------------------------------|
| 2           | Poor \((P)\)    | \((0.0, 0, 0.1, 0.2, 0.3, 0.4, 0.5; 0.3)\) |
| 3           | Marginal \((M)\)| \((0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9; 0.3)\) |
| 4           | Acceptable \((A)\)| \((0.5, 0.6, 0.7, 0.8, 0.9, 1, 1, 1; 0.3)\) |
| 5           | Good \((G)\)    | \((0.7, 0.8, 0.9, 1, 1, 1, 1, 1; 0.3)\) |
Suppose the person $P_1$ gathered information based on the problem (Case 2) worked in the earlier paper [1], then the required information is formulated as hierarchical soft fuzzy number valued information system $\widetilde{\mathcal{I}}_{H_1} = (U, A_1, H_{A_1}, \mathcal{F}^*(\mathbb{R})(\tilde{\mathcal{E}}^2_{A_1}), \tilde{\mathcal{I}}_{H_{A_1}})$.

The in-depth information gathered by person $P_2$ about ‘safety’(basic characteristic under consideration) of the car brands were available as star ratings based on various characteristic features $\{a_s^2\}_{s=1}^3$, $a_1^2 = \text{Frontal crash}$, $a_2^2 = \text{Side crash}$, $a_3^2 = \text{Rollover and sub-characteristic features}$ $\{a_s^2\}_{s=1}^3$ of $a_1^2$, $a_{21}^2 = \text{combined side barrier and sidepole}$, $a_{22}^2 = \text{Side barrier}$ and $a_{23}^2$ = side pole as shown in Figure 2.

The gathered information is formulated as the hierarchical soft fuzzy number valued information system $\widetilde{\mathcal{I}}_{H_2} = (U, A_2, H_{A_2}, \mathcal{F}^*(\mathbb{R})(\tilde{\mathcal{E}}^2_{A_2}), \tilde{\mathcal{I}}_{H_{A_2}})$, where $A_2 = \{a_1^2, a_2^2, a_3^2\}$ are the attributes, $H_{A_2} = \{H_1^2, H_2^2, H_3^2\}$ is associated hierarchical tree, $\mathcal{E}^2_{A_2} = \{\mathcal{E}^2_{a_1^2}, \mathcal{E}^2_{a_2^2}, \mathcal{E}^2_{a_3^2}\}$ is the collection of parameter sets associated with the leaf nodes $\mathcal{E}^2_{a_1^2} = \{E_{11}^2, E_{12}^2\}$ where $e_{11}^2$ and $e_{12}^2$ are the frontal barrier test corresponding to front driver side and passenger side, respectively, $\mathcal{E}^2_{a_2^2} = \{E_{21}^2 = \{e_{21}^2, e_{212}^2\}, E_{22}^2 = \{e_{22}^2, e_{222}^2\}, E_{23}^2 = \{e_{23}^2, e_{231}^2\}\}$, where $e_{21}^2$, $e_{212}^2$, $e_{22}^2$, $e_{222}^2$ and $e_{231}^2$ are the combined barrier test at front seat, seat barrier test at driver seat and rear passenger seat and side pole barrier test, respectively, and $\mathcal{E}^2_{a_3^2} = \{E_{31}^2 = \{e_{31}^2, e_{311}^2\}\}$ where $e_{31}^2$ is rollover resistance level test.

The various tests are considered as the set of parameters and the ratings of the car brands are recorded as soft octagonal fuzzy numbers for $i = 1, 2, \ldots, 5$ as follows:

$$\tilde{I}^2_{H_{a_1}}(C_i, a_{1i}^2) = (\tilde{f}_{i1}^2, E_{i1}^2)$$

$$\tilde{I}^2_{H_{a_2}}(C_i, a_{2i}^2) = (\tilde{f}_{i2}^2, E_{i2}^2)$$

$$\tilde{I}^2_{H_{a_3}}(C_i, a_{3i}^2) = (\tilde{f}_{i3}^2, E_{i3}^2)$$

and

The quantified ratings corresponding to expert 2 is shown in Table 21.
Table 21. Ratings of the car brand.

| Car brands | $e_{11}^2$ | $e_{12}^2$ | $e_{21}^2$ | $e_{22}^2$ | $e_{31}^2$ | $e_{32}^2$ |
|------------|------------|------------|------------|------------|------------|------------|
| C₁         | A          | G          | G          | G          | G          | A          |
| C₂         | G          | A          | G          | G          | G          | G          |
| C₃         | G          | A          | G          | G          | G          | G          |
| C₄         | G          | G          | G          | G          | G          | G          |
| C₅         | G          | G          | A          | G          | A          | G          |

Soft linear octagonal fuzzy numbers describing the rating of the five cars based on the test say the frontal crash test is given by

$$\tilde{f}_{11}^2(E_{11}) = \tilde{f}_{11}^2(e_{111}) = (0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1; 0.3),$$

$$\tilde{f}_{11}^2(e_{112}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{21}^2(E_{11}) = \tilde{f}_{21}^2(e_{111}) = (0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1; 0.3),$$

$$\tilde{f}_{21}^2(e_{112}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{31}^2(E_{11}) = \tilde{f}_{31}^2(e_{111}) = (0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1; 0.3),$$

$$\tilde{f}_{31}^2(e_{112}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{41}^2(E_{11}) = \tilde{f}_{41}^2(e_{111}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{41}^2(e_{112}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{51}^2(E_{11}) = \tilde{f}_{51}^2(e_{111}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3),$$

$$\tilde{f}_{51}^2(e_{112}) = (0.6, 0.7, 0.8, 0.9, 1, 1, 1, 1; 0.3).$$

Similarly, the ratings of the cars corresponding to other testing methods can be expressed as soft linear octagonal fuzzy numbers for the choice of $k = 0.3$.

6.2. Solution

The solution to this problem is obtained by executing the algorithm.

Here $p = 2$. Since the information provided by both the experts are modelled as the hierarchical soft fuzzy number valued information system, we start the algorithm from Step 1 ii.

The computations are done using MATLAB 2016a programme developed for the algorithm.

For each car brand $C_i$, the obtained fuzzy number valued measure on utility soft information $\tilde{U}^1(C_i)$ for expert 1, on utility soft information $\tilde{U}^2(C_i)$ for expert 2, on combined utility soft information $\tilde{U}(C_i)$ and $Q^x_i$ are recorded in columns 1, 2, 3 and 4 of Table 22, respectively.
Table 22. Utility measure.

| Car brands | $\tilde{U}^1(C_i)$ | $\tilde{U}^2(C_i)$ | $\tilde{U}(C_i)$ | $Q^c_i$ | $\tilde{U}_{\text{diff}}(C_i)$ | $Q^c$ |
|------------|-------------------|-------------------|------------------|--------|---------------------|--------|
| $C_1$      | (1.642, 1.892)    | (0.977, 1.147)    | (1.31, 1.52, 1.73) | 1.879  | (0.851, 0.994)     | 1.255  |
|            | 2.142, 2.392      | 0.317, 1.487      | 1.94, 2.00, 2.067 | 2.078  | 2.089; 0.3         | 1.342  |
|            | 2.435, 2.478      | 1.571, 1.655      | 2.078, 2.089; 0.3 | 1.342  | 1.402               |
|            | 2.485, 2.493; 0.3 | 1.67, 1.685; 0.3  |                  | 1.413  | 1.424; 0.3         |
| $C_2$      | (1.305, 1.501)    | (0.827, 0.967)    | (1.066, 1.234, 1.512 | (0.770, 0.893, 1.098 |
|            | 1.697, 1.893      | 1.107, 1.247      | 1.402, 1.57        | 1.016  | 1.139               |
|            | 1.916, 1.94       | 1.301, 1.355      | 1.609, 1.647       | 1.169  | 1.198               |
|            | 1.947, 1.953; 0.3 | 1.37, 1.385; 0.3  | 1.658, 1.669; 0.3  | 1.029  | 1.219; 0.3         |
| $C_3$      | (1.031, 1.192)    | (1.187, 1.387)    | (1.109, 1.29, 1.470 | 1.603  | (0.950, 1.105, 1.375 |
|            | 1.353, 1.514      | 1.587, 1.787      | 1.651, 1.711       | 1.260  | 1.416               |
|            | 1.552, 1.59, 1.56 | 1.871, 1.955      | 1.772, 1.783       | 1.467  | 1.518               |
|            | 1.603; 0.3        | 1.97, 1.985; 0.3  | 1.794; 0.3         | 1.529  | 1.541; 0.3         |
| $C_4$      | (1.404, 1.619)    | (1.327, 1.547)    | (1.366, 1.583, 1.952 | (0.915, 1.064, 1.322 |
|            | 1.834, 2.049      | 1.767, 1.987      | 1.801, 2.018       | 1.213  | 1.363               |
|            | 2.089, 2.128      | 2.071, 2.155      | 2.08, 2.142        | 1.410  | 1.458               |
|            | 2.135, 2.143; 0.3 | 2.17, 2.185; 0.3  | 2.153, 2.164; 0.3  | 1.470  | 1.481; 0.3         |
| $C_5$      | (1.492, 1.718)    | (1.257, 1.467)    | (1.375, 1.592, 1.960 | (1.062, 1.231, 1.517 |
|            | 1.943, 2.169      | 1.677, 1.887      | 1.810, 2.0279      | 1.401  | 1.570               |
|            | 2.204, 2.239      | 1.971, 2.055      | 2.087, 2.147       | 1.616  | 1.662               |
|            | 2.244, 2.25; 0.3  | 2.07, 2.085; 0.3  | 2.157, 2.167; 0.3  | 1.673  | 1.684; 0.3         |

Using the values from Table 22, the closeness of the decision of each expert to that of the combined was computed and the importance of the experts are obtained:

$$
\lambda_1^1 = 0.462, \quad \lambda_2^1 = 0.478, \quad \lambda_3^1 = 0.525, \quad \lambda_4^1 = 0.477, \quad \lambda_5^1 = 0.513;
\lambda_1^2 = 0.538, \quad \lambda_2^2 = 0.522, \quad \lambda_3^2 = 0.475, \quad \lambda_4^2 = 0.522, \quad \lambda_5^2 = 0.486;
$$

Incorporating the obtained weights, the combined evaluations of the experts, the corresponding fuzzy number valued measure on utility information $\tilde{U}_{\text{diff}}(C_i)$ and combined value of safety $Q^c_i$ are computed and tabulated in the last two columns of Table 22.

Based on the procedure, it was found that $C_5$ is the best brand based on the safety measure among the five car brands of choice.

Remark 6.1: In this real life situation, equal weights are assigned to the parameters involved. Weights to the various safety features are assigned according to the needs of persons $P_1$, $P_2$ and their individual perspectives. From columns 4 and 6 of Table 22, we note that the change in ranking of car brands is due to implementation of derived objective importance to the experts for each car brands, the weights assigned for attributes and parameters.

7. Conclusion and Future Study

In this paper, theoretical development of soft fuzzy numbers and soft fuzzy number valued information systems are studied in detail. The knowledge pertaining to collection of entities involved, based on $E$ are mathematically modeled as collection of $\tilde{IS}$ or $\tilde{ISH}$ and is used as a tool to solve SFGDMP using an algorithm developed. It is established that this type of problems cannot be effectively solved, but for the usage of soft fuzzy number valued information systems. In Section 6, we have limited our work for 5 midsize car brands based on information from two websites only wherein the qualitative information is quantified using the above developed model.
In the problems for each of the aspects there will be different attributes and parameters (defining some or all of the attributes). To cite a few:

| Aspects          | Attributes (sub-attributes)                                      | Parameters                                           |
|------------------|------------------------------------------------------------------|------------------------------------------------------|
| Technology       | Autonomous Emergency Braking                                     | Working of radar system                              |
|                  |                                                                   | Working of eyesight system                           |
| Interior Dimensions | • seating capacity                                             |                                                      |
| Exterior Dimensions | • fuel capacity                                               |                                                      |
| Comfort          | Power stereeing                                                 | tests for performance of motor                       |
|                  | Accessory Power Outlet                                         | module                                               |
| Performance      | Emission Norm Compliance                                        | working capacity of sensor                           |
|                  | Engine power                                                    | capability of the socket                             |

It is possible to consider similar problems using the model developed and we are continuing to work on some of them. It is also possible to develop models involving fuzzy analogue of soft fuzzy number valued information systems in different scenario.

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