SUBLINEAR EQUATIONS AND SCHUR’S TEST
FOR INTEGRAL OPERATORS

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ABSTRACT. We study weighted norm inequalities of (p, r)-type,
\[ \| G(f \, d\sigma) \|_{L^r(\Omega, \sigma)} \leq C \| f \|_{L^p(\Omega, \sigma)}, \]
for 0 < r < p and p > 1, where \( G(f \, d\sigma)(x) = \int_{\Omega} G(x, y) f(y) \, d\sigma(y) \) is an integral operator associated with a nonnegative kernel \( G(x, y) \) on \( \Omega \times \Omega \), and \( \sigma \) is a locally finite positive measure in \( \Omega \).

We show that this embedding holds if and only if
\[ \int_{\Omega} (G\sigma)^{\frac{p}{p-r}} \, d\sigma < +\infty, \]
provided \( G \) is a quasi-symmetric kernel which satisfies the weak maximum principle.

In the case \( p = \frac{r}{q} \), where \( 0 < q < 1 \), we prove that this condition characterizes the existence of a non-trivial solution (or supersolution) \( u \in L^r(\Omega, \sigma) \), for \( r > q \), to the sublinear integral equation
\[ u - G(u^q \, d\sigma) = 0, \quad u \geq 0. \]

We also give some counterexamples in the end-point case \( p = 1 \), which corresponds to solutions \( u \in L^q(\Omega, \sigma) \) of this integral equation, studied recently in [19], [20]. These problems appear in the investigation of weak solutions to the sublinear equation involving the (fractional) Laplacian,
\[ (-\Delta)^{\alpha} u - \sigma u^q = 0, \quad u \geq 0, \]
for \( 0 < q < 1 \) and \( 0 < \alpha < \frac{n}{2} \) in domains \( \Omega \subseteq \mathbb{R}^n \) with a positive Green function.

1. INTRODUCTION

Let \( \Omega \) be a locally compact, Hausdorff space. For a positive, lower semicontinuous kernel \( G : \Omega \times \Omega \rightarrow (0, +\infty) \), we denote by
\[ G(f \, d\sigma)(x) = \int_{\Omega} G(x, y) f(y) \, d\sigma(y), \quad x \in \Omega, \]
the corresponding integral operator, where \( \sigma \in \mathcal{M}^+(\Omega) \), the class of positive locally finite Radon measures in \( \Omega \).

We study the \((p, r)\)-weighted norm inequalities
\[ \| G(f \, d\sigma) \|_{L^r(\Omega, \sigma)} \leq C \| f \|_{L^p(\Omega, \sigma)}, \quad \forall f \in L^p(\Omega, \sigma), \]

2010 Mathematics Subject Classification. Primary 35J61, 42B37; Secondary 31B15, 42B25.
Key words and phrases. Weighted norm inequalities, sublinear elliptic equations, weak maximum principle, Green’s function, fractional Laplacian.
in the case $0 < r < p$ and $p \geq 1$, where $C$ is a positive constant which does not depend on $f$.

The main goal of this paper is to find explicit characterizations of (1.1) in terms of $G \sigma$ under certain assumptions on $G$. We also study connection of inequality (1.1) with $p = \frac{r}{q}$, where $0 < q < 1$, to the existence of a positive function $u \in L^r(\Omega, \sigma)$ such that

$$u \geq G(u^q \sigma) \ d\sigma - \text{a.e. in } \Omega,$$

in the case $r > q$. In other words, $u$ is a supersolution for the sublinear integral equation

$$(1.3) \quad u - G(u^q \sigma) = 0, \quad 0 < u < +\infty \ d\sigma - \text{a.e. in } \Omega,$$

where $0 < q < 1$.

In this paper, we assume that the kernel $G$ of the integral operator is quasi-symmetric, and satisfies a weak maximum principle (WMP); see Sec. 2. Such restrictions are satisfied by the Green kernel associated with many elliptic operators, including the fractional Laplacian $(-\Delta)\alpha$, as well as quasi-metric kernels, and radially symmetric, decreasing convolution kernels $G(x, y) = k(|x - y|)$ on $\mathbb{R}^n$ (see, e.g., [1], [2], [18], [19], [20] and the literature cited there).

If $G$ is Green’s kernel associated with the Laplacian in an open domain $\Omega \subseteq \mathbb{R}^n$, (1.3) is equivalent to the sublinear elliptic boundary value problem

$$\begin{cases} -\Delta u - \sigma u^q = 0, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $0 < q < 1$.

We observe that solutions $u \in L^r(\Omega, \sigma)$ to (1.4) in the case $r = 1 + q$ correspond to finite energy solutions $u \in L^{1+2}_0(\Omega)$ in the Dirichlet space, i.e.,

$$\int\Omega |\nabla u|^2 dx < +\infty,$$

where $u$ has zero boundary values (see [5]).

The more difficult end-point case $p = 1$ of (1.1), along with solutions $u \in L^q(\Omega, \sigma)$ in the case $r = q$, was studied recently in [19], [20]. After a certain modification, it leads to solutions $u \in L^q_{\text{loc}}(\Omega, \sigma)$, i.e., all solutions to (1.3), or (1.4) understood in a weak sense (see [16]). For Riesz kernels on $\Omega = \mathbb{R}^n$ such $(1, q)$-weighted norm inequalities, along with weak solutions to the sublinear problem

$$\begin{cases} (-\Delta)^\alpha u - \sigma u^q = 0, & u > 0 \text{ in } \mathbb{R}^n, \\ \liminf_{x \to \infty} u = 0, & u \in L^q_{\text{loc}}(\sigma), \end{cases}$$

for $0 < \alpha < \frac{4}{n}$, were treated earlier in [5], [6], [7].

Our main result is the following theorem.

**Theorem 1.1.** Let $\sigma \in \mathcal{M}^{+}(\Omega)$. Suppose $G$ is a positive, quasi-symmetric, lower semicontinuous kernel on $\Omega \times \Omega$ which satisfies the weak maximum principle.
(i) If $1 < p < +\infty$ and $0 < r < p$, then the $(p,r)$-weighted norm inequality (1.1) holds if and only if

$$\int_{\Omega} (G\sigma)^{\frac{p}{r}} d\sigma < +\infty. \tag{1.6}$$

(ii) If $0 < q < 1$ and $q < r < \infty$, then there exists a positive (super)solution $u \in L^r(\Omega,d\sigma)$ to (1.3) if and only if (1.1) holds with $p = \frac{r}{q}$, or equivalently,

$$\int_{\Omega} (G\sigma)^{\frac{1}{r-\frac{1}{q}}} d\sigma < +\infty. \tag{1.7}$$

Remark 1.2. We observe that the “if” parts of statements (i) and (ii) of Theorem 1.1 fail if $p = 1$, and $r = q$, respectively. The “only if” parts hold for all $0 < r < p$ in statement (i), and $r > 0$ in statement (ii).

Remark 1.3. It is known that inequality (1.1) with $p = \frac{r}{q} \geq 1$ in the case $0 < q < 1$ yields the existence of a positive supersolution $u \in L^r(\Omega,\sigma)$ for (1.2). This statement follows from a lemma due to Gagliardo [12], and does not require $G$ to be quasi-symmetric or to satisfy the WMP (see Sec. 3 below). However, the converse statement does not hold without the WMP (see [20] in the case $r = q$).

Remark 1.4. Without the assumption that $G$ satisfies the WMP, the “only if” parts of statement (i) (with $p = \frac{r}{q} \geq 1$) and statement (ii) (with $r \geq q$) hold only for $0 < r \leq 1 - q^2$ (see Lemma 3.1 below).

In particular, if there exists a positive (super)solution $u \in L^q(\Omega,\sigma)$, then (1.7) holds with $r = q$ for $0 < q \leq q_0$, where $q_0 = \frac{\sqrt{5} - 1}{2} = 0.61\ldots$ is the conjugate golden ratio. However, (1.7) with $r = q$ generally fails (even for symmetric kernels) in the case $q_0 < q < 1$; the cut-off $q = q_0$ here is sharp [20].

In Sec. 4 below, we discuss related results, and provide some counterexamples in the case $p = 1$.

2. Kernels and Potential Theory

Let $G : \Omega \times \Omega \to (0, +\infty]$ be a positive kernel. We will assume that $\Omega$ is a locally compact space Hausdorff space, and $G$ is lower semicontinuous, so that we can apply elements of the classical potential theory developed for such kernels (see [3], [11]). Most of our results hold for non-negative kernels $G(x,y) \geq 0$. In that case, some statements concerning the existence of positive solutions (rather than supersolutions) require the additional assumption that $G$ is non-degenerate; see [20].

By $\mathcal{M}^+(\Omega)$ we denote the class of all nonnegative, locally finite, Borel measures on $\Omega$. We use the notation supp$(\nu)$ for the support of $\nu \in \mathcal{M}^+(\Omega)$ and $||\nu|| = \nu(\Omega)$ if $\nu$ is a finite measure.

For $\nu \in \mathcal{M}^+(\Omega)$, the potential of $\nu$ is defined by

$$G\nu(x) := \int_{\Omega} G(x,y) d\nu(y), \quad \forall x \in \Omega,$$
and the potential with the adjoint kernel
\[ G^\star \nu(y) := \int_\Omega G(x, y) \, d\nu(x), \quad \forall y \in \Omega. \]

A positive kernel \( G \) on \( \Omega \times \Omega \) is said to satisfy the weak maximum principle (WMP) with constant \( h \geq 1 \) if, for any \( \nu \in \mathcal{M}^+(\Omega) \),
\[
\sup\left\{ G\nu(x) : x \in \text{supp}(\nu) \right\} \leq M = \implies \sup\left\{ G\nu(x) : x \in \Omega \right\} \leq hM,
\]
for any constant \( M > 0 \). When \( h = 1 \), \( G \) is said to satisfy the strong maximum principle. It holds for Green’s kernels associated with the classical Laplacian, or fractional Laplacian \((-\Delta)^\alpha\) in the case \( 0 < \alpha \leq 1 \), for all domains \( \Omega \) with positive Green’s function. The WMP holds for Riesz kernels on \( \mathbb{R}^n \) associated with \((-\Delta)^\alpha\) in the full range \( 0 < \alpha < \frac{n}{2} \), and more generally for all radially non-increasing kernels on \( \mathbb{R}^n \) (see \[1\]).

The WMP also holds for the so-called quasi-metric kernels (see \[8\], \[9\], \[15\], \[20\]). We say that \( d(x, y) : \Omega \times \Omega \to [0, +\infty) \) satisfies the quasimetric triangle inequality with quasimetric constant \( \kappa \) if
\[
d(x, y) \leq \kappa[d(x, z) + d(z, y)],
\]
for any \( x, y, z \in \Omega \). We say that \( G \) is a quasi-metric kernel (with quasimetric constant \( \kappa > 0 \)) if \( G \) is symmetric and \( d(x, y) = \frac{1}{G(x, y)} \) satisfies \( \kappa \).

A kernel \( G : \Omega \times \Omega \to (0, +\infty) \) is said to be quasi-symmetric if there exists a constant \( a \) such that
\[
a^{-1}G(y, x) \leq G(x, y) \leq aG(y, x), \quad \forall x, y \in \Omega.
\]
Many kernels associated with elliptic operators are quasi-symmetric and satisfy the WMP (see \[2\]).

For \( 0 < q < 1 \), and \( \sigma \in \mathcal{M}^+(\Omega) \), we are interested in positive solutions \( u \in L^r(\sigma) \) \((r > 0)\) to the integral equation
\[
u = G(u^q \sigma), \quad u > 0 \quad d\sigma - a.e.
\]
and positive supersolutions \( u \in L^r(\sigma) \) to the integral inequality
\[
u \geq G(u^q \sigma), \quad u > 0 \quad d\sigma - a.e.
\]

In \[20\], we characterized the existence of positive solutions \( u \in L^q(\Omega, \sigma) \) and \( u \in L^q_{\text{loc}}(\sigma) \). The latter correspond to the so-called “very weak” solutions to the sublinear boundary value problem \( \text{(1.4)} \) (see \[9\], \[16\]). It is easy to see that the condition \( u \in L^q_{\text{loc}}(\sigma) \) is necessary for the existence of any positive (super)solution, since otherwise \( u \equiv +\infty \) \( d\sigma \)-a.e. (see \[20\]).

For a measure \( \lambda \in \mathcal{M}^+(\Omega) \), the energy of \( \lambda \) is given by
\[
\mathcal{E}(\lambda) := \int_\Omega G\lambda \, d\lambda.
\]

The notion of energy is closely related to another major tool of potential theory, the capacity of a set, and the associated equilibrium measure.
For a kernel $G : \Omega \times \Omega \to (0, +\infty]$, we consider the Wiener capacity
\begin{equation}
\text{cap}(K) := \sup \left\{ \mu(K) : \begin{array}{l}
\lambda \cdot G \mu(y) \leq 1 \text{ on supp}(\mu), \\
\mu \in \mathcal{M}^+(K)
\end{array} \right\},
\end{equation}
defined for compact sets $K \subset \Omega$.

The extremal measure $\mu$ for which the supremum in (2.5) is attained is called the equilibrium measure. Alternatively, capacity can be defined as a solution to the following extremal problem involving energy:
\begin{equation}
\text{cap}(K) := \left[ \inf \left\{ \mathcal{E}(\mu) : \begin{array}{l}
\mu \in \mathcal{M}^+(K), \\
\mu(K) = 1
\end{array} \right\} \right]^{-1}.
\end{equation}

We say that a property holds nearly everywhere (or n.e.) on $K$ when the exceptional set $Z \subset K$ where this property fails has zero capacity, $\text{cap}(Z) = 0$.

We will use the following fundamental theorem [3], [11].

**Theorem 2.1.** Let $G$ be a positive symmetric kernel on $\Omega \times \Omega$, and let $K \subset \Omega$ a compact set. The two extremal problems
\begin{align*}
\max \left\{ \lambda(K) : \begin{array}{l}
G \lambda \leq 1 \text{ on supp}(\lambda), \\
\lambda \in \mathcal{M}^+(K)
\end{array} \right\}, \\
\max \left\{ 2\lambda(K) - \mathcal{E}(\lambda) : \lambda \in \mathcal{M}^+(K) \right\},
\end{align*}
always have solutions, which are precisely the same, and each maximum coincides with the Wiener capacity $\text{cap}K$. The class of all solutions consists of measures $\lambda \in \mathcal{M}^+(K)$ for which
\[\mathcal{E}(\lambda) = \lambda(\Omega) = \text{cap}(K).\]

The potential of any solution has the following properties:
\begin{enumerate}
\item $G \lambda(x) \geq 1$ n.e. in $K$,
\item $G \lambda(x) \leq 1$ on $\text{supp}(\lambda)$,
\item $G \lambda(x) = 1$ $d\lambda$-a.e. in $\Omega$.
\end{enumerate}

The extremal measure $\lambda$ in Theorem 2.1 is the equilibrium measure for the set $K$. We observe that since $G$ is a positive kernel, the capacity of all compact sets $K$ is finite. (This is true even for non-negative kernels if $G(x,x) > 0$ for all $x \in \Omega$; see [11]).

### 3. Weighted Norm Inequalities, Supersolutions, and Energy Estimates

We begin this section with a proof of Theorem 2.1. We remark that the “only if” part of statement (i) of Theorem 2.1 is proved without using the assumption that $G$ is quasi-symmetric. Furthermore, the proof of this part works in the case $p = 1$ as well.

**Proof of Theorem 2.1** We first prove statement (i). If the $(p,r)$-inequality holds for $0 < r < p$, where $p \geq 1$, then assuming that $f = (G \sigma)^{\frac{p}{r}} \in L^p(\Omega, \sigma)$ and using it as a test function, we deduce
\begin{equation}
\int_\Omega \left[ G \left( (G \sigma)^{\frac{p}{r}} d\sigma \right) \right]^r d\sigma \leq C^r \left[ \int_\Omega (G \sigma)^{\frac{p}{r}} d\sigma \right]^r.
\end{equation}
where $C$ is the embedding constant in (1.1). We now use the pointwise inequality
\[(3.1) \quad \left[ G\sigma(x) \right]^\frac{p}{p-r} \leq s h^{s-1} G \left( (G\sigma)^{s-1} d\sigma \right)(x), \quad x \in \Omega, \]
for all $s \geq 1$, established in [14] Lemma 2.5 and Remark 2.6 for non-negative kernels satisfying the WMP with constant $h \geq 1$. Applying (3.1) with $s = \frac{p}{p-r}$, we obtain
\[
\int_\Omega (G\sigma)^\frac{pm}{p-r} d\sigma \leq \left( \frac{p}{p-r} \right)^r h^\frac{s^2}{r} C^r \left[ \int_\Omega (G\sigma)^\frac{pm}{p-r} d\sigma \right]^\frac{r}{p}.
\]
Since $0 < r < p$, this estimate yields
\[
\int_\Omega (G\sigma)^\frac{pm}{p-r} d\sigma \leq \left( \frac{p}{p-r} \right) \frac{p}{p-r} h^\frac{2}{(p-r)^2} C^r.
\]
The extra assumption that $f = (G\sigma)^\frac{pm}{p-r} \in L^p(\Omega, \sigma)$ is easy to remove by using $\chi_K f$ in place of $f$, where $K$ is a compact subset of $\Omega$ on which $G\sigma(x) \leq n$, and then letting $n \to +\infty$ (see details in [20]).

In the opposite direction, suppose that (1.6) holds for $0 < r < p$ and $p > 1$. Without loss of generality we may assume that $f \geq 0$. By Hölder’s inequality,
\[
\int_\Omega |G(fd\sigma)|^r d\sigma = \int_\Omega \left[ \frac{G(fd\sigma)}{G\sigma} \right]^r (G\sigma)^r d\sigma \leq \left[ \int_\Omega \left( \frac{G(fd\sigma)}{G\sigma} \right)^p d\sigma \right]^{\frac{r}{p}} \left[ \int_\Omega (G\sigma)^\frac{pm}{p-r} d\sigma \right]^{1-\frac{r}{p}}.
\]

We next sketch a proof of a $(1,1)$-weak type estimate obtained in a more general context in [20] Lemma 5.10:
\[(3.2) \quad \left\| \frac{G(fd\sigma)}{G\sigma} \right\|_{L^{1\infty}(\Omega, d\sigma)} \leq c \|f\|_{L^1(\Omega, d\sigma)},\]
where $c = c(h, a)$ depends only on the constants $h \geq 1$ in the weak maximum principle, and $a > 0$ in the quasi-symmetry condition.

Since $G$ is quasi-symmetric, we can assume without loss of generality that it is symmetric by replacing $G$ with $\frac{1}{2}(G + G^*)$. Let $E_t = \{ x \in \Omega : \frac{G(fd\sigma)}{G\sigma}(x) > t \}$, where $t > 0$. For an arbitrary compact set $K \subset E_t$, we denote by $\mu \in \mathcal{M}_+(K)$ an equilibrium measure on $K$ (see Sec. 2 above) such that $G\mu \geq 1$ n.e. on $K$ and $G\mu \leq 1$ on supp($\mu$).

It is easy to see that in fact
\[(3.3) \quad G\mu \geq 1 \quad d\sigma - \text{a.e. on } K.
\]
Indeed, from (1.6) it follows that $G\sigma < +\infty \, d\sigma$-a.e. Since $G\mu \geq 1$ n.e. on $K$, the set $Z = \{ x \in K : G\mu(x) < 1 \}$ has zero capacity, and consequently,
\[
\sigma(Z) = \sigma(\{ x \in Z : G\sigma(x) < +\infty \}) \leq \sum_{n=1}^{+\infty} \sigma(\{ x \in Z : G\sigma(x) \leq n \})
\]
Thus, \( \sigma(Z) = 0 \), which proves (3.3).

Since \( G \mu \leq 1 \) on supp(\( \mu \)), it follows that \( G \mu \leq h \) on \( \Omega \) by the WMP. From this and (3.3), using Fubini’s theorem, we deduce

\[
\sigma(K) \leq \int_K G \mu \, d\sigma = \int_K G \sigma_K \, d\mu \\
\leq \int_K \frac{G(f \, d\sigma)}{t} \, d\mu = \frac{1}{t} \int_K G \mu \, f \, d\sigma \\
\leq \frac{1}{t} \int_{\Omega} h f \, d\sigma = \frac{h}{t} \|f\|_{L^1(\Omega, \sigma)}.
\]

Taking the supremum over all \( K \subset E_t \), we obtain

\[
\sigma(E_t) \leq \frac{h}{t} \|f\|_{L^1(\Omega, \sigma)},
\]

which proves (3.2).

The corresponding \( L^\infty \) estimate is obvious:

\[
\left\| \frac{G(f \, d\sigma)}{G \sigma} \right\|_{L^\infty(\Omega, d\sigma)} \leq \|f\|_{L^\infty(\Omega, d\sigma)}.
\]

Thus, for \( 1 < p < +\infty \), by the Marcinkiewicz interpolation theorem we obtain

\[
\left\| \frac{G(f \, d\sigma)}{G \sigma} \right\|_{L^p(\Omega, d\sigma)} \leq C \|f\|_{L^p(\Omega, d\sigma)},
\]

for all \( f \in L^p(\Omega, d\sigma) \). Hence, combining the preceding estimates, we deduce

\[
\int_{\Omega} [G(f \, d\sigma)]^r \, d\sigma \leq C \|f\|_{L^p(\Omega, \sigma)} \left[ \int_{\Omega} (G \sigma)^{\frac{pr}{r-q}} \, d\sigma \right]^{\frac{1}{1-q}}.
\]

This proves statement (i).

We now prove statement (ii). Let \( 0 < q < 1 \). Suppose there exists a positive supersolution \( u \in L^r(\Omega, \sigma) \) with \( r > q \). As shown in [14, Corollary 3.6], if \( G \) satisfies the WMP, then any nontrivial supersolution \( u \) satisfies the global pointwise bound

\[
(3.4) \quad u(x) \geq (1 - q)^{\frac{1}{r-q}} h^{-\frac{r}{r-q}} [G \sigma(x)]^{\frac{1}{1-q}} \, d\sigma - \text{a.e.}
\]

Thus, (1.7) holds.

Conversely, by statement (i), (1.7) with \( r > q \) implies the \((p,r)\)-inequality (1.1) with \( p = \frac{r}{q} \). Letting \( u_0 = c [G \sigma(x)]^{\frac{1}{1-q}} \), where \( c > 0 \) is a positive constant, we get a sequence of iterations

\[
u_{j+1} = G(u_j^q \, d\sigma), \quad j = 0, 1, \ldots,
\]

where by induction we see that \( u_{j+1} \geq u_j \), provided the constant \( c \) is small enough. Here the initial step \( u_1 \geq u_0 \) follows from (3.1) with \( s = \frac{1}{1-q} \), since

\[
u_1 = G(u_0^q \, d\sigma) = c^q G \left[ (G \sigma)^{\frac{q}{1-q}} \, d\sigma \right] \geq c [G \sigma(x)]^{\frac{1}{1-q}} = u_0,
\]
for an appropriate choice of \( c = c(q, h, a) \). By (1.1) with \( p = \frac{r}{q} \) and \( f = u_j \), we have by induction,

\[
\|u_{j+1}\|_{L^r(\Omega, \sigma)} = \|G(u_j^q d\sigma)\|_{L^r(\Omega, \sigma)} \leq C \|u_j\|_{L^r(\Omega, \sigma)}^q < +\infty.
\]

Since \( 0 < q < 1 \) and \( u_j \leq u_{j+1} \), it follows that

\[
\|u_{j+1}\|_{L^r(\Omega, \sigma)} \leq C(r, q, h, a), \quad j = 0, 1, \ldots
\]

Using the monotone convergence theorem, we obtain a positive solution

\[
u = \lim_{j \to \infty} u_j, \quad \nu \in L^r(\Omega, \sigma).
\]

Theorem 1.1 makes use of energy conditions of the type

\[
(3.5) \quad \int_{\Omega} (G\sigma)^{\gamma} d\sigma < \infty,
\]

for some \( s > 0 \). Note that when \( s = 1 \), this gives the energy \( \mathcal{E}(\sigma) \) introduced above.

In the next lemma, we deduce (3.5) for \( s = \frac{r}{1-q} \) provided there exists a positive supersolution \( \nu \in L^r(\Omega, \sigma) \) to (1.2), for non-negative, quasi-symmetric kernels \( G \), without assuming that (1.1) holds, or that \( G \) satisfies the WMP. In the special case \( r = q \) it was proved in [20, Lemma 5.1].

Lemma 3.1. Let \( \sigma \in \mathcal{M}^+(\Omega) \), and let \( 0 < q < 1 \). Suppose \( G \) is a non-negative quasi-symmetric kernel on \( \Omega \times \Omega \). Suppose there is a positive supersolution \( \nu \in L^r(\Omega, \sigma) \) \((r > 0)\), i.e., \( G(u^q d\sigma) \leq \nu d\sigma \)-a.e. Let \( 0 < q \leq 1 - r^2 \). Then

\[
(3.6) \quad \int_{\Omega} (G\sigma)^{\gamma} d\sigma \leq a^{\frac{n}{1-q}(1-r^2)} \int_{\Omega} \nu^r d\sigma < +\infty,
\]

where \( a \) is the quasi-symmetry constant of \( G \).

Proof. Suppose \( \nu \in L^r(\Omega, \sigma) \), where \( 0 < r < 1 \), is a positive supersolution. Let \( \gamma \geq 1 \). By Hölder’s inequality with exponents \( \gamma \) and \( \gamma' = \frac{r}{r-1} \), we estimate

\[
G\sigma(x) = \int_{\Omega} u^{\frac{\gamma}{r}} u^{-\frac{\gamma}{r}} G(x, y) d\sigma(y)
\leq \left[ G(u^q d\sigma(x))^\frac{\gamma}{r} \right]^\frac{1}{\gamma'} \left[ G(u^{-\frac{\gamma}{r}} d\sigma(x))^\frac{1}{\gamma'} \right]^\frac{\gamma}{r}
\leq \left[ u(x)^\frac{\gamma}{r} \right]^\frac{1}{\gamma'} \left[ G(u^{-\frac{\gamma}{r}} d\sigma(x))^\frac{1}{\gamma'} \right]^\frac{\gamma}{r}.
\]

Let \( \gamma = 1 + \frac{q}{1-r^2} \), where \( 0 < r \leq 1 - q^2 \). Then \( \frac{(1-q)r}{r} \geq 1 \). Using the preceding inequality, along with Hölder’s inequality with the conjugate exponents

\[
\frac{(1-q)(1-r+q)}{1-r-q^2} > 1 \quad \text{and} \quad \frac{(1-q)(1-r+q)}{rq} \geq 1,
\]

and Fubini’s theorem, we estimate
\[
\int_{\Omega} (G(\sigma))^\frac{r}{r-q} d\sigma \leq \int_{\Omega} u^{\frac{r}{1-q}} \left[ G(u^{-1} d\sigma) \right]^{\frac{r}{1-q}} d\sigma
\]
\[= \int_{\Omega} u^{\frac{r}{1-q}(1+r-q)} \left[ u^q G(u^{-1} d\sigma) \right]^{\frac{r}{1-q}} d\sigma\]
\[\leq \left[ \int_{\Omega} u^q d\sigma \right]^{\frac{r}{1-q}} \left[ \int_{\Omega} G(u^{-1} d\sigma) u^q d\sigma \right]^{\frac{r}{1-q}} \]
\[= \left[ \int_{\Omega} u^q d\sigma \right]^{\frac{r}{1-q}} \left[ \int_{\Omega} G^*(u^q d\sigma) u^{-1} d\sigma \right] \]
\[\leq a^{\frac{rq}{1-q}} \left[ \int_{\Omega} u^q d\sigma \right]^{\frac{r}{1-q}}.\]

In the last estimate we used the inequality \( G^*(u^q d\sigma) \leq au. \) Since \( 1 - r - q^2 + rq = (1 - q)(1 - r + q), \) this completes the proof of (3.6). \( \square \)

We next show that, for general non-negative kernels \( G, \) the \((p, r)\)-weighted norm inequality (1.1) with \( p = \frac{L}{q} \geq 1 \) yields the existence of a supersolution \( u \in L^r(\Omega, \sigma) \) to (1.2). This is deduced from Gagliardo’s lemma [12] (see also [22]), as in the special case \( r = q \) in [20].

We will apply this lemma to construct a measurable function \( \phi \) such that

\( 0 < [G(\phi d\sigma)]^q \leq \phi < +\infty \ d\sigma - \text{a.e.}, \)

for \( 0 < q < 1. \) Clearly, if \( \phi \) satisfies the above estimate, then \( u = \phi^{\frac{1}{q}} \) satisfies (1.2).

Moreover, \( u \in L^r(\Omega, \sigma) \) if \( \phi \in L^p(\Omega, \sigma), \) where \( p = \frac{L}{q} \geq 1. \)

We recall that a convex cone \( P \subset B \) is strictly convex at the origin if, for any \( \phi, \psi \in P, \alpha \phi + \beta \psi = 0 \) implies \( \phi = \psi = 0, \) for any \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1. \)

**Lemma 3.2** (Gagliardo [12]). Let \( B \) be a Banach space, and let \( P \subset B \) be a convex cone which is strictly convex at the origin and such that if \( (\phi_n) \subset P, \phi_{n+1} - \phi_n \in P, \) and \( \|\phi_n\| \leq M \) for all \( n = 1, 2, \ldots, \) then there exists \( \psi \in P \) so that \( \|\phi_n - \psi\| \to 0. \)

Let \( S: P \to P \) be a continuous mapping with the following properties:

1. For \( \phi, \psi \in P, \) such that \( \phi - \psi \in P, \) we have \( S\phi - S\psi \in P. \)
2. If \( \|\phi\| \leq 1 \) and \( \phi \in P, \) then \( \|Su\| \leq 1. \)

Then for every \( \lambda > 0 \) there exists \( \phi \in P \) so that \( (1 + \lambda)\phi - S\phi \in P \) and \( 0 < \|\phi\| \leq 1. \) Moreover, for every \( \psi \in P \) such that \( 0 < \|\psi\| \leq \frac{1}{1+\lambda}, \) \( \phi \) can be chosen so that \( \phi = \psi + \frac{1}{1+\lambda} S\phi. \)

We will apply this lemma to \( B = L^p(\sigma), \) \( p \geq 1, \) and the cone of non-negative functions \( P \) in \( B. \) In this case obviously one can ensure that \( \phi > 0 \ d\sigma-\text{a.e.} \)

**Lemma 3.3.** Let \( (\Omega, \sigma) \) be a sigma-finite measure space, and let \( G \) be a non-negative kernel on \( \Omega \times \Omega. \) Let \( 0 < r < +\infty \) and \( 0 < q < 1. \) Suppose (1.1) holds for
\( p = \frac{r}{q} \geq 1 \) with an embedding constant \( C = \kappa > 0 \). Then, for every \( \lambda > 0 \), there is a positive \( \phi \in L^p(\sigma) \) satisfying (3.7) so that
\[
\|\phi\|_{L^p(\sigma)} \leq (1 + \lambda)^{\frac{1}{1-q}} \kappa^{\frac{1}{q}}. 
\]

**Proof.** The supersolution \( \phi \) is constructed using Lemma 3.2. Define \( S : L^p(\sigma) \to L^p(\sigma) \) by
\[
S \phi := \left[ \frac{1}{\kappa^q} G(\phi d\sigma) \right]^q,
\]
for all \( \phi \in L^p(\sigma) \), \( \phi \geq 0 \). Inequality (1.1) gives that \( S \) is a bounded continuous operator. In fact, by (1.1) we see that if \( \|\phi\|_{L^p(\sigma)} \leq 1 \), then
\[
\|S(\phi)\|_{L^p(\sigma)}^p = \frac{1}{\kappa^q} \int_\Omega [G(\phi \sigma)]^p d\sigma
\]
\[
= \frac{1}{\kappa^q} \left( \int_\Omega \phi^p d\sigma \right)^q \leq 1.
\]
Therefore, by Lemma 3.2 there exists \( \phi \in L^p(\sigma) \) such that
\[
(1 + \lambda) \phi \geq \frac{1}{\kappa^q} [G(\phi \sigma)]^q,
\]
\( \|\phi\|_{L^p(\sigma)} \leq 1 \), and \( \phi > 0 \) \( d\sigma \)-a.e. Setting \( \phi_0 = c \phi \), where
\[
c = \left[ \frac{1}{(1 + \lambda) \kappa^q} \right]^{\frac{1}{1-q}},
\]
we deduce that \( \phi > 0 \) \( d\sigma \)-a.e., and
\[
\phi_0 \geq G(\phi_0 \sigma)^q, \quad \|\phi_0\|_{L^p(\sigma)} \leq (1 + \lambda)^{\frac{1}{1-q}} \kappa^{\frac{1}{q}}.
\]

\[\square\]

**Remark 1.3** follows immediately from Lemma 3.3.

**Remark 3.4.** For \( p = \frac{r}{q} \), a counterexample in [20] demonstrates that, without the WMP, the existence of a supersolution \( u \in L^r(\sigma, \sigma) \) to (1.2) in the case \( r = q \) does not imply the \((p, r)\)-weighted norm inequality (1.1), even for positive symmetric kernels \( G \). A slight modification of that counterexample shows that the same is true in the case \( r > q \) as well.

### 4. A Counterexample in the End-Point Case \( p = 1 \)

In the case \( p = 1, 0 < q < 1 \), the \((1, q)\)-weighted norm inequality (1.1) with \( r = q \) follows from a similar inequality for the space of measures \( \mathcal{M}^+(\Omega) \) in place of \( L^1(\Omega, \sigma) \),
\[
\|Gv\|_{L^q(\Omega, \sigma)} \leq C \|v\|, \quad \forall v \in \mathcal{M}^+(\Omega),
\]
where \( \|v\| = v(\Omega) \). This inequality was shown in [20] to be equivalent to the existence of a positive supersolution \( u \in L^q(\Omega, \sigma) \) to (1.2) for quasi-symmetric kernels \( G \) satisfying the WMP. In this case, (4.1) is equivalent to (1.1) with \( r = q \) and \( p = 1 \) in view of Lemma 3.3.
However, a characterization of (4.1), or (1.1) with \( r = q \) and \( p = 1 \), in terms of the energy estimate (1.7) with \( r = q \) is not available, contrary to the case \( r > q \): the condition

\[
\int_{\Omega} (G\sigma)^{\frac{1}{1-q}} d\sigma < +\infty
\]

is not sufficient for (4.1).

On the other hand, it is not difficult to see that (4.1) holds for all \( \nu \in \mathcal{M}(\Omega) \) if and only if it holds for all finite linear combinations of point masses, \( \nu = \sum_{j=1}^{n} a_j \delta_{x_j}, \ a_j > 0 \). It had been conjectured that, for \( 0 < q < 1 \), condition (4.2) combined with (4.1) for single point masses \( \nu = \delta_{x} \), i.e.,

\[
\int_{\Omega} G(x,y)^q d\sigma(y) \leq C < +\infty, \ \forall x \in \Omega,
\]

was not only necessary, but also sufficient for (4.1). (Notice that in the case \( q \geq 1 \) (4.3) is obviously necessary and sufficient for (4.1); see [20].)

In this section, we give a counterexample to this conjecture for Riesz potentials on \( \mathbb{R}^n \),

\[
I_{2\alpha} \nu (x) = \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-2\alpha}}, \ x \in \mathbb{R}^n,
\]

where \( \nu \in \mathcal{M}(\mathbb{R}^n) \), and \( 0 < 2\alpha < n \). Clearly, Riesz kernels \(|x-y|^{2\alpha-n}\) are symmetric, and satisfy the WMP.

Suppose \( 0 < q < 1, \ n \geq 1, \) and \( 0 < 2\alpha < n \). We construct \( \sigma \in \mathcal{M}(\mathbb{R}^n) \) such that

\[
\mathcal{E}(\sigma) = \mathcal{E}_{\alpha,q}(\sigma) := \int_{\mathbb{R}^n} \left( I_{2\alpha} \sigma \right)^{\frac{1}{1-q}} d\sigma < +\infty,
\]

and

\[
\mathcal{K}(\sigma) = \mathcal{K}_{\alpha,q}(\sigma) := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-2\alpha}q} < +\infty,
\]

but

\[
K(\sigma) = K_{\alpha,q}(\sigma) := \sup \left\{ \left. \left( I_{2\alpha} \nu \right)^{\frac{1}{1-q}} \right|_{\mathcal{M}(\mathbb{R}^n)} : \nu \in \mathcal{M}(\mathbb{R}^n), \ \nu \neq 0 \right\} = +\infty.
\]

In other words, we need to construct a measure \( \sigma \) such that \( \mathcal{E}(\sigma) < +\infty \) (in the special case \( q = \frac{1}{2} \) this means that \( \sigma \) has finite energy), and (4.6) holds for all \( \delta \)-functions \( \nu = \delta_{x} \ (x \in \mathbb{R}^n) \), but (4.6) fails for a linear combination of \( \delta \)-functions

\[
\nu = \sum_{j=1}^{\infty} a_j \delta_{x_j}, \ \text{where} \ \sum_{j=1}^{\infty} a_j < +\infty, \ a_j > 0.
\]

We will use a modification of the example considered in [6] for other purposes.

We will need the following lemma and its corollary in the radially symmetric case (see [6]).
Lemma 4.1. Let \(0 < q < 1\) and \(0 < 2\alpha < n\). If \(d\sigma = \sigma(|x|) \, dx\) is radially symmetric, then \(\kappa(\sigma) < +\infty\) if and only if \(\mathcal{K}(\sigma) < +\infty\). Moreover, there exists a constant \(c = c(q, \alpha, n) > 0\) such that \(\kappa(\sigma)\) satisfies
\[
\mathcal{K}(\sigma) \leq \kappa(\sigma)^q \leq c \mathcal{K}(\sigma),
\]
where in the this case
\[
\mathcal{K}(\sigma) = \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|y|^{(n-2\alpha)q}}.
\]

Remark 4.2. For radially symmetric \(\sigma\), condition \(\mathcal{K}(\sigma) < +\infty\) is equivalent to \(\sigma \in L^{1,\infty}(\mathbb{R}^n, \sigma)\), which is necessary and sufficient for \(\mathcal{K}(\sigma) < +\infty\) in this case; see [19], [20]. Here \(L^{1,\infty}(\mathbb{R}^n, \sigma)\) denotes the corresponding Lorentz space with respect to the measure \(\sigma\).

Corollary 4.3. Let \(\sigma_{R,\gamma} = \chi_{B(0,R)}|x|^{-\gamma}\), where \(0 \leq \gamma < n - q(n-2\alpha)\) and \(R > 0\). Then
\[
\mathcal{K}(\sigma) = \frac{\omega_n R^{n-\gamma-q(n-2\alpha)}}{n-\gamma-q(n-2\alpha)},
\]
and
\[
\frac{\omega_n}{n-\gamma-q(n-2\alpha)} \leq \frac{\kappa(\sigma_{R,\gamma})^q}{R^{n-\gamma-q(n-2\alpha)}} \leq \frac{c}{n-\gamma-q(n-2\alpha)},
\]
where \(c = c(q, \alpha, n)\), and \(\omega_n = |S^{n-1}|\) is the surface area of the unit sphere.

Let
\[
\sigma = \sum_{k=1}^{\infty} c_k \sigma_k,
\]
where
\[
\sigma_k = \sigma_{R_k,\gamma_k}(x+k), \quad R_k = |x_k| = k, \quad \gamma_k = n - q(n-2\alpha) - \epsilon_k,
\]
and the positive scalars \(c_k, \epsilon_k\) are picked so that \(\sum_{k=1}^{\infty} c_k < \infty\), \(\epsilon_k \to 0\), and \(0 < \gamma_k < n\). Notice that \(\gamma_k \to n - q(n-2\alpha)\) as \(k \to \infty\), which is a critical exponent for the inequality (4.17) (with \(\sigma_k\) in place of \(\sigma\)) discussed below.

More precisely, for \(0 < q < 1\) and \(0 < \delta < +\infty\), we set
\[
a_k = \frac{1}{k \log^\frac{1}{\delta}(k+1)}, \quad c_k = \frac{1}{k^{2-q+\delta}}, \quad \epsilon_k = \frac{1}{k^{1+\delta}}, \quad k = 1,2,\ldots,
\]
so that
\[
\sum_{k=1}^{\infty} a_k < +\infty, \quad \sup_{k \geq 1} \frac{c_k}{\epsilon_k} < +\infty, \quad \sum_{k=1}^{\infty} \frac{c_k}{\epsilon_k^{1-q}} < +\infty, \quad \text{but} \quad \sum_{k=1}^{\infty} \frac{c_k \epsilon_k^q}{\epsilon_k} = +\infty.
\]

We first verify condition (4.4). Notice that
\[
c_1 A \leq \left(\sigma_{\alpha,q}(\sigma)\right)^{1-q} \leq c_2 A,
\]
where $A$ is the least constant in the inequality (see [4]; [5], Lemma 3.3)

\[(4.17) \int_{\mathbb{R}^n} |I_{\alpha} f|^1 d\sigma \leq A \|f\|^1_{L^2(dx)}, \quad \text{for all } f \in L^2(\mathbb{R}^n, dx),\]

or, equivalently,

\[(4.18) \int_{\mathbb{R}^n} |I_{\alpha} (gd \sigma)|^1 d\sigma \leq A^2 \|g\|^1_{L^2(d\sigma)}, \quad \text{for all } g \in L^2(\mathbb{R}^n, \sigma),\]

where the constants of equivalence $c_1, c_2$ in (4.16) depend only on $\alpha, q,$ and $n$.

Consequently, $[E_{\alpha,q}(\sigma)]^1$ is equivalent to a norm on a subset of $\mathcal{H}^+ (\mathbb{R}^n)$, so that

\[(4.19) \left[ E_{\alpha,q} \left( \sum_k \sigma_k \right) \right]^1 \leq c \sum_k \left[ E_{\alpha,q} (\sigma_k) \right]^1, \]

where $c = c(\alpha, q, n)$ is a positive constant which depends only on $\alpha, q,$ and $n$.

We claim that,

\[(4.20) E_{\alpha,q} (\sigma_k) \leq \frac{C R_k^{\frac{\alpha}{1-q} \gamma}}{\varepsilon_k}, \quad k = 1, 2, \ldots,\]

where $C = C(\alpha, q, n)$.

Indeed, by the semigroup property of Riesz kernels,

\[
I_{2\alpha} \sigma_k (x) = c(\alpha, n) \int_{B(0, R_k)} \frac{dt}{|x - t|^n |x + x_k|^\alpha} \\
\leq c(\alpha, n) \int_{\mathbb{R}^n} \frac{dt}{|x - t|^n |x + x_k|^\alpha} = c |x + x_k|^{2\alpha - \gamma},
\]

where $c = c(n, 2\alpha + n - \gamma)$ remains bounded by a constant $C(\alpha, q, n)$ as $k \to +\infty$, since $\lim_{k \to +\infty} (2\alpha + n - \gamma) = 2\alpha + q(n - 2\alpha) < n$.

Notice that $(\gamma - 2\alpha) \frac{\alpha}{1-q} + \gamma = n - \frac{\alpha}{1-q}$. Hence, by the preceding estimate,

\[
E_{\alpha,q} (\sigma_k) = \int_{\mathbb{R}^n} \left( I_{2\alpha} \sigma_k \right)^{1+q} d\sigma_k \\
\leq c^{1+q} \int_{|x + x_k| < R_k} \frac{dx}{|x + x_k|^{n + \frac{\alpha}{1-q}}} \\
= c^{1+q} \omega_n \int_0^{R_k} \frac{r^{\frac{\alpha}{1-q}}}{r^\alpha} dr \\
\leq \frac{C(\alpha, q, n) R_k^{\frac{\alpha}{1-q}}}{\varepsilon_k},
\]

which proves (4.20).

It follows from (4.19) and the preceding estimate that, for $\sigma$ defined by (4.12),

\[(4.21) \left[ E_{\alpha,q}(\sigma) \right]^{1-q} \leq c(\alpha, q, n) \sum_k c_k \left[ E_{\alpha,q} (\sigma_k) \right]^{1-q} \\
\leq c(\alpha, q, n) C(\alpha, q, n)^{1-q} \sum_k \frac{c_k R_k^{\frac{\alpha}{1-q}}}{\varepsilon_k^{1-q}} < +\infty,
\]
Proof. Suppose first that $|x| > \frac{R}{2}$.

To prove (4.5), we will need the following lemma.

**Lemma 4.4.** Let $R > 0, 0 < \beta < n$, and $0 < \varepsilon < n - \beta$. For $\gamma = n - \beta - \varepsilon > 0$, we have

$$(4.22) \quad \phi_{R, \gamma}(x) := \int_{|t| < R} \frac{dt}{|x - t|^\beta |t|^\gamma} \approx \left\{ \begin{array}{ll} \frac{R^{n-\varepsilon}|x|^\gamma}{\varepsilon} & \text{if } |x| \leq \frac{R}{2}, \\ R^\beta \left( \frac{R}{|x|} \right)^\beta & \text{if } |x| > \frac{R}{2}, \end{array} \right.$$  

where the constants of equivalence depend only on $\beta$ and $n$.

**Proof.** Suppose first that $|x| > \frac{R}{2}$. Then

$$\phi_{R, \gamma}(x) = \int_{|t| < \frac{R}{2}} \frac{dt}{|x - t|^\beta |t|^\gamma} + \int_{\frac{R}{2} < |t| < R} \frac{dt}{|x - t|^\beta |t|^\gamma} := I + II.$$  

Clearly, in the first integral $\frac{|t|}{2} \leq |x - t| \leq \frac{3|t|}{2}$, and so $I$ is bounded above and below by

$$\frac{\omega_n c(\beta)}{|x|^\beta} \int_0^R r^{n-\gamma} dr = \frac{c(\beta, n) R^{n-\gamma}}{|x|^\beta}.$$  

To estimate the second term, notice that, for $|x| > 2R$ and $|t| < R$, we have $|x - t| > \frac{|t|}{2}$, so that

$$II \leq \frac{c(\beta, n)}{R^n \beta} \int_{\frac{R}{2} < |t| < R} dt = \frac{c(\beta, n) R^{n-\gamma}}{|x|^\beta}.$$  

For $\frac{R}{2} < |x| < 2R$ and $|t| < R$, we have $|x - t| < 3R$, and consequently

$$II \leq \frac{c(\beta, n)}{R^n} \int_{|x - t| < 3R} \frac{dt}{|x - t|^\beta} = \frac{\omega_n c(\beta, n)}{R^n} \int_0^{3R} r^{n-\beta} dr = C(\beta, n) R^{n-\beta-\gamma} \leq \frac{C(\beta, n) R^{n-\gamma}}{|x|^\beta}.$$  

Thus, $II \leq c(n, \beta) I$, which proves (4.22) in the case $|x| \geq \frac{R}{2}$.

Suppose now that $|x| \leq \frac{R}{2}$. Then

$$\phi_{R, \gamma}(x) = \int_{|t| < \frac{|x|}{2}} \frac{dt}{|x - t|^\beta |t|^\gamma} + \int_{\frac{|x|}{2} < |t| < 2|x|} \frac{dt}{|x - t|^\beta |t|^\gamma} + \int_{2|x| < |t| < R} \frac{dt}{|x - t|^\beta |t|^\gamma} := III + IV + V.$$  

Clearly, in the first integral $\frac{|t|}{2} < |x - t| < \frac{3|x|}{2}$, and so $III$ is bounded above and below by

$$\frac{c(\beta)}{|x|^\beta} \int_{|t| < \frac{|x|}{2}} \frac{dt}{|t|^\gamma} = \frac{\omega_n c(\beta)}{|x|^\beta} \int_0^{\frac{|x|}{2}} r^{n-\gamma} dr = \frac{c(\beta, n)}{(n-\gamma)2^{n-\gamma} |x|^\varepsilon}.$$
The second integral $IV$ is bounded above and below by

$$c(\gamma) \int_{\frac{1}{4}<|t|<2|x|} \frac{dt}{|x-t|^\gamma},$$

Clearly,

$$IV \leq c(\gamma) \int_{|x-t|<3|x|} \frac{dt}{|x-t|^\beta} = \omega_c c(\gamma) \int_0^{3|x|} t^{n-1-\beta} dt \leq \omega_n c(\gamma)|x|^\gamma,$$

so that $IV \leq c(\beta, n) III$.

Finally, the integral $V$ is bounded above and below by

$$c(\beta) \int_{2|x|-|t|<R} \frac{dt}{|t|^\gamma+\beta} = c(\beta) \int_{2|x|}^R t^{n-1-\gamma-\beta} dt = c(\beta) \frac{R^e-(2|x|)^e}{\varepsilon}.$$

Combining these estimates we complete the proof of (4.22). $\square$

By Lemma 4.4 with $\beta = (n-2\alpha)q$, $R = R_k$, $\varepsilon = \varepsilon_k$, and $\gamma = \gamma_k = n-\beta-\varepsilon_k$, we obtain, for $k = 2, 3, \ldots,$

$$\phi_{R_k,\gamma_k}(x-x_k) = \int_{|t+x_k|<R_k} \frac{dt}{|x-t|^{|n-2\alpha|q}|t+x_k|^\gamma}$$

\begin{align*}
\leq C(\alpha, q, n) \begin{cases}
R_k^{\varepsilon_k} & \text{if } |x-x_k| < 1, \\
R_k^{\varepsilon_k-1} & \text{if } 1 \leq |x-x_k| \leq \frac{R_k}{2}, \\
R_k^{\varepsilon_k} & \text{if } |x-x_k| > \frac{R_k}{2}.
\end{cases}
\end{align*}

(4.23)

In the case $k = 1$, we use the estimate $\phi_{R_1,\gamma_1}(x-x_1) \leq C(\alpha, q, n) \frac{R_1^{\varepsilon_1}}{\varepsilon_1}$ for all $x \in \mathbb{R}^n$. 

We next estimate

\[ K(\sigma) = \sup_{x \in \mathbb{R}^n} \sum_{k=1}^{\infty} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ \leq \sup_{x \in \mathbb{R}^n} \sum_{k=1}^{\infty} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ + \sup_{x \in \mathbb{R}^n} \sum_{1 < |x - x_k| < R_k/2} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ + \sum_{x \in \mathbb{R}^n} \sum_{|x - x_k| \geq R_k} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ =: I + II + III. \]

Suppose that \( j \leq |x| \leq j + 1 \) for some \( j = 0, 1, \ldots \). We first estimate \( I \). Since \( |x - x_k| \leq 1 \), and \(|x_k| = k\), it follows that

\[ k = |x_k| \leq 1 + |x| \leq 1 + |x - x_k| + |x_k| = k + 2. \]

Consequently, \( j - 1 \leq k \leq j + 2 \) if \( j \geq 2 \), and \( 1 \leq k \leq 3 \) if \( j = 0, 1, 2 \). Hence, the corresponding sum contains no more than four terms, and therefore

\[ I := \sup_{x \in \mathbb{R}^n} \sum_{1 \leq |x - x_k| \leq 1} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ \leq C(\alpha, q, n) \sup_{j \geq 0} \sum_{j - 1 \leq k \leq \max(j - 1, 1) \leq \max(j, 3)} c_k R_k^{R_k} \epsilon_k \]

\[ \leq C(\alpha, q, n), \]

since by (4.13) and (4.15),

\[ \sup_{k \geq 1} R_k^{R_k} < +\infty, \quad \text{and} \quad \sup_{k \geq 1} \frac{c_k}{\epsilon_k} < +\infty. \]

To estimate \( II \), notice that \( 0 < \epsilon_k \log R_k \leq C \), and consequently

\[ \frac{R_k^{R_k} - 1}{\epsilon_k} \leq C \log R_k. \]

Hence, by (4.23) and (4.14),

\[ II := \sup_{x \in \mathbb{R}^n} \sum_{1 < |x - x_k| < R_k/2} c_k \phi_{R_k, \eta}(x - x_k) \]

\[ \leq C(\alpha, q, n) \sup_{x \in \mathbb{R}^n} \sum_{1 < |x - x_k| < R_k/2} \frac{c_k (R_k^{R_k} - 1)}{\epsilon_k} \]

\[ \leq C(\alpha, q, n) \sum_{k=1}^{\infty} c_k \log R_k < +\infty. \]
Finally, we estimate \( III \) using (4.23) and (4.14). Since \( \sup_k R^k < +\infty \), we deduce

\[
III := \sup_{x \in \mathbb{R}^n} \sum_{|x-x_k| \geq \frac{\varepsilon}{2}} c_k \phi_{R_k}(x-x_k)
\]

\[
\leq C(\alpha, q, n) \sup_{x \in \mathbb{R}^n} \sum_{|x-x_k| \geq \frac{\varepsilon}{2}} c_k R^k
\]

\[
\leq C(\alpha, q, n) \sum_{k=1}^{+\infty} c_k C(\alpha, q, n).
\]

This proves (4.5).

It remains to verify (4.6) for \( \sigma = \sum_{k=1}^{+\infty} c_k \sigma_k \) and \( \nu = \sum_{j=1}^{+\infty} a_j \delta_{x_j} \) defined above. We estimate

\[
\|I_{2\alpha}v\|_{L^q(\sigma)}^q = \sum_{k=1}^{+\infty} c_k \int_{\mathbb{R}^n} \left( \sum_{j=1}^{+\infty} \frac{a_j}{|x+x_j|^{(n-2\alpha)}} \right)^q d\sigma_k
\]

\[
\geq \sum_{k=1}^{+\infty} c_k \int_{\mathbb{R}^n} \frac{a_k^q}{|x+x_k|^{(n-2\alpha)}} d\sigma_k
\]

\[
= \sum_{k=1}^{+\infty} c_k a_k^q \int_{|x+x_k|<R_k} \frac{dx}{|x+x_k|^{(n-2\alpha)q+\gamma_k}}.
\]

Since

\[
\int_{|x+x_k|<R_k} \frac{dx}{|x+x_k|^{(n-2\alpha)q+\gamma_k}} = \int_{|x|<R_k} \frac{dx}{|x|^{(n-2\alpha)q+\gamma_k}} = \omega_k \int_0^R r^{-1+\gamma_k} dr = \omega_k \frac{R^{\gamma_k}}{\varepsilon_k},
\]

and \( R^k_{\varepsilon} \geq 1 \), it follows by (4.14) that

\[
\|I_{2\alpha}v\|_{L^q(\sigma)}^q \geq \omega_k \sum_{k=1}^{+\infty} \frac{c_k a_k^q}{\varepsilon_k} = \omega_k \frac{1}{k\log(k+1)} = +\infty.
\]

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