Monopole-like Configuration from Quantized SU(3) Gauge Fields

V. Dzhunushaliev

Phys. Dept., Kyrgyz-Russian Slavic University, Bishkek, 720000, Kyrgyz Republic

D. Singleton

Physics Dept., CSU Fresno, 2345 East San Ramon Ave. M/S 37 Fresno, CA 93740-8031, USA

Monopole field configurations have been extensively studied in both Abelian and non-Abelian gauge theories. The question of the quantum corrections to these systems is a difficult one, since the classical monopoles have non-perturbatively large couplings, which makes the standard, perturbative methods for calculating quantum corrections suspect. Here we apply a modified version of Heisenberg’s quantization technique for strongly interacting, nonlinear fields to a classical solution of the SU(3) Yang-Mills field equations. This classical solution is not monopole-like and has an energy density which diverges as \( r \to \infty \). However, the quantized version of this solution has a monopole-like far field, and a non-divergent energy density as \( r \to \infty \). This may point to the conclusion that monopoles may arise not from quantizing classical monopole configurations, but from quantizing field configurations which at the classical level do not appear monopole-like.

I. INTRODUCTION

Monopoles have been studied within the context of both Abelian and non-Abelian gauge theories. Usually, the approach is to start with a classical monopole configuration (i.e., having a magnetic field which becomes Coulombic as \( r \to \infty \)) and then consider the quantum corrections to the system. One difficulty with this approach is that the coupling strength of monopole configurations is large due to the Dirac quantization condition which requires that there be an inverse relationship between electric and magnetic couplings. A perturbatively small electric coupling requires a non-perturbatively large magnetic coupling. The perturbative quantization techniques do not work well with monopoles for the same reason that they have trouble with QCD in the low energy regime: the couplings are non-perturbatively large.

Here we will use a modification of a non-perturbative quantization method originally used by Heisenberg to quantize nonlinear, strongly coupled spinor fields. However, rather than applying this quantization to an SU(3) field configuration which is monopole-like already at the classical level, we apply it to a non-monopole configuration, which has fields and an energy density which diverge at spatial infinity (r → ∞). After applying Heisenberg’s quantization technique to this solution we find that the fields and energy density become physically well behaved, and the asymptotic magnetic field becomes monopole-like. This result may indicate that monopoles are inherently quantum objects. Rather than quantizing a classical field theory configuration which appears monopole-like, real monopoles might arise as a consequence of quantizing non-monopole classical solutions.

II. CLASSICAL SU(3) SOLUTION

In this section we will briefly review the classical, spherically symmetric, SU(3) Yang-Mills theory solution to which we will apply the modified Heisenberg quantization method. We begin with the following ansatz for pure SU(3) Yang-Mills theory

\[
A_0 = \frac{1}{2} \lambda^a \left( \lambda^a_{ij} + \lambda^a_{ji} \right) \frac{x^i x^j}{r^2} w(r),
\]

\[
A_i^a = \left( \lambda^a_{ij} - \lambda^a_{ji} \right) \frac{x^j}{r^2} + \lambda^a_{jk} \left( \epsilon_{ilj} x^k + \epsilon_{ilk} x^j \right) \frac{x^l}{r^3} v(r),
\]

where \( \lambda^a \) are the Gell-Mann matrices; \( a = 1, 2, \ldots, 8 \) is a color index; the Latin indices \( i, j, k, l = 1, 2, 3 \) are space indices; \( i^2 = -1; r, \theta, \phi \) are the usual spherically coordinates. This is a simplified version of an ansatz considered by several groups.
Under this ansatz the Yang-Mills equations \((D^\mu F^a_{\mu\nu} = 0)\) become
\[
\begin{align*}
  r^2 v'' &= v^3 - v - v w^2, \\
  r^2 w'' &= 6 v w^2.
\end{align*}
\] (3), \(4\) approach the form
where the primes indicate differentiation with respect to \(r\). In the asymptotic limit \(r \to \infty\) the solutions to Eqs. \(3\), \(4\) lead to the field energy and action of this solution diverging as \(w\) in some phenomenological studies of quarkonia bound states [9]. It was also found that for a broad range of boundary conditions for the functions \(T\)
where
\[
\begin{align*}
  \gamma &= \text{the time ordering operator;} \\
  \psi &= \text{spinor self interaction term which involved three spinor fields and various combinations of } \gamma^\mu\text{'s and/or } \gamma^5\text{'s. The } \gamma^\mu\text{'s and/or } \gamma^5\text{'s. The}
\end{align*}
\] (4)

where \(\gamma^\mu\) are Dirac matrices; \(\hat{\psi}(x)\), \(\hat{\psi}(x)\) are the spinor field and its adjoint respectively; \(\Im[\hat{\psi}(\hat{\psi})]\) is the general nonlinear spinor self interaction term which involved three spinor fields and various combinations of \(\gamma^\mu\)’s and/or \(\gamma^5\)’s. The constant \(l\) has units of length, and sets the scale for the strength of the interaction. Next one defines \(\tau\) functions as
\[
\tau(x_1 x_2 \ldots y_1 y_2 \ldots) = \langle 0 | T[\hat{\psi}(x_1) \hat{\psi}(x_2) \ldots \hat{\psi}(y_1) \hat{\psi}(y_2) \ldots]|\Phi\rangle
\] (11)
where \(T\) is the time ordering operator; \(|\Phi\rangle\) is a state for the system described by Eq. \(10\). Applying Eq. \(10\) to \(11\) we obtain the following infinite system of equations for various \(\tau\)’s
\[
\begin{align*}
  l^{-2} \gamma^\mu \frac{\partial}{\partial x_1(r)} \left\{ \tau(x_1 \ldots x_n|y_1 \ldots y_n) \right\} &= \Im[\tau(x_1 \ldots x_n x_r|y_1 \ldots y_n y_r)] + \\
  \delta(x_r - y_1) \tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n|y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) + \\
  \delta(x_r - y_2) \tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n|y_1 y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) + ...
\end{align*}
\] (12)
Eq. (12) represents one of an infinite set of coupled equations which relate various order (given by the index \( n \)) of the \( \tau \) functions to one another. To make some head way toward solving this infinite set of equations Heisenberg employed the Tamm-Dankoff method whereby he only considered \( \tau \) functions up to a certain order. This effectively turned the infinite set of coupled equations into a finite set of coupled equations.

For the SU(3) Yang-Mills theory this idea leads to the following Yang-Mills equations for the quantized SU(3) gauge field

\[
D_\mu \tilde{F}^{\alpha \mu} = 0
\]  

(13)

here \( \tilde{F}^{\alpha \mu} \) is the field operator of the SU(3) gauge field.

One can show that Heisenberg’s method is equivalent to the Dyson-Schwinger system of equations for small coupling constants. One can also make a comparison between the Heisenberg method and the standard Feynman diagram technique. With the Feynman diagram method quantum corrections to physical quantities are given in terms of an infinite number of higher order, loop diagrams. In practice one takes only a finite number of diagrams into account when calculating the quantum correction to some physical quantity. This standard diagrammatic method requires a small expansion parameter (the coupling constant), and thus does not work for strongly coupled theories. The Heisenberg method was intended for strongly coupled, nonlinear theories, and we will apply a variation of this method to the classical solution discussed in the last section.

We will consider a variation of Heisenberg’s quantization method for the present non-Abelian equations by making the following assumptions \([11]\):

1. The physical degrees of freedom relevant for studying the above classical solution are given entirely by the two ansatz functions \( v, w \) appearing in Eqs. (3), (4). No other degrees of freedom will arise through the quantization process.

2. From Eqs. (5), (6) we see that one function \( w(r) \) is a smoothly varying function for large \( r \), while another function, \( v(r) \), is strongly oscillating. Thus we take \( w(r) \) to be an almost classical degree of freedom while \( v(r) \) is treated as a fully quantum mechanical degree of freedom. Naively one might expect that only the behavior of second function would change while first function stayed the same. However since both functions are interrelated through the nonlinear nature of the field equations we find that both functions are modified.

To begin we replace the ansatz functions by operators \( \hat{v}(x), \hat{w}(x) \).

\[
r^2 \hat{v}'' = \hat{v}^3 - \hat{v} - \hat{v}\hat{w}^2, \\
r^2 \hat{w}'' = 6\hat{w}\hat{v}^2
\]

(14)

(15)

These equations can be seen as an approximation of the quantized SU(3) Yang-Mills field equations \([13]\). Taking into account assumption (2) we let \( \hat{w} \to w \) become just a classical function again, and replace \( \hat{v}^2 \) in Eq. (15) by its expectation value to arrive at

\[
r^2 \hat{v}'' = \langle \hat{v}^3 \rangle - \langle \hat{v} \rangle - \langle \hat{v} \rangle \langle \hat{w}^2 \rangle, \\
r^2 \hat{w}'' = 6 \langle \hat{w} \langle \hat{v}^2 \rangle \rangle
\]

(16)

(17)

where the expectation value \( \langle \hat{v}^2 \rangle \) is taken with respect to some quantum state \( |q\rangle \): \( \langle \hat{v}^2 \rangle = \langle q|\hat{v}^2|q\rangle \). We can average Eq. (16) to get

\[
r^2 \langle \hat{v} \rangle'' = \langle \hat{v}^3 \rangle - \langle \hat{v} \rangle - \langle \hat{v} \rangle \langle \hat{w}^2 \rangle, \\
r^2 \langle \hat{w} \rangle'' = 6 \langle \hat{w} \langle \hat{v}^2 \rangle \rangle
\]

(18)

(19)

Eqs. (18), (19) are almost a closed system for determining \( \langle \hat{v} \rangle \) except for the \( \langle \hat{v} \rangle \) and \( \langle \hat{v}^3 \rangle \) terms. One can obtain differential equations for these expectation values by applying \( r^2 \partial / \partial r \) to \( \hat{v}^2 \) or \( \hat{v}^3 \) and using Eqs. (18) - (17). However the differential equations for \( \langle \hat{v}^2 \rangle \) or \( \langle \hat{v}^3 \rangle \) would involve yet higher powers of \( \hat{v} \) thus generating an infinite number of coupled differential equations for the various \( \langle \hat{v}^n \rangle \). In the next section we will use a path integral inspired method \([12]\) to cut this progression off at some finite number of differential equations.

**IV. PATH INTEGRATION OVER CLASSICAL SOLUTIONS**

Within the path integral method the expectation value of some field \( \Phi \) is given by

\[
\langle \Phi \rangle = \int \Phi e^{iS[\Phi]} D\Phi
\]

(20)
The classical solutions, $\Phi_{cl}$, give the dominate contribution to the path integral. For a single classical solution one can approximate the path integral as

$$\int e^{iS[\Phi]} D\Phi \approx A e^{iS[\Phi_{cl}]}$$

(21)

where $A$ is a normalization constant. Consequently the expectation of the field can be approximated by

$$\int \Phi e^{iS[\Phi]} D\Phi \approx \Phi_{cl}.$$  

(22)

We are interested in the case where $\Phi$ is the gauge potential $A^a_{\mu}$, in which case our approximation becomes

$$\langle A^a_{\mu} \rangle \approx \int \left( A^a_{\mu} \right)_{\phi_0} e^{iS[\tilde{A}^a_{\mu}]_{\phi_0}} D \left( \tilde{A}^a_{\mu} \right)_{\phi_0}$$

(23)

$$(\tilde{A}^a_{\mu})_{\phi_0}$$ are the classical solutions of the Yang - Mills equations labeled by a parameter $\phi_0$. In the present case the classical solution with the asymptotic form $[\tilde{A}^a_{\mu}]_{\phi_0}$ has an infinite energy and action. When one considers the Euclidean version of the path integral above the exponential factor in (23) becomes $\exp[-S[(\tilde{A}^a_{\mu}]_{\phi_0}]$ which for an infinite action would naively imply that this classical configuration would not contribute to the path integral at all. However, there examples where infinite action classical solutions have been hypothesized to play a significant role in the path integral. The most well known example of this is the meron solution $[13]$, which has an infinite action. Analytically the singularities of the meron solutions can be dealt with by replacing the regions that contain the singularities by instanton solutions. Since instanton solutions have finite action this patched together solution of meron plus instanton has finite action. However, the Yang-Mills field equations are not satisfied at the boundary where the meron and instanton solutions are sewn together. In addition recent lattice studies $[14]$ have indicated that merons (or the patched meron/instanton) do play a role in the path integral. Here we will treat this divergence in the action in an approximate way through a redefinition of the path integral integration measure. Since the divergence in the action comes from first term of the $w(r)$ ansatz function of $[\tilde{A}^a_{\mu}]_{\phi_0}$ which does not contain $\phi_0$ we will take the action for different $\phi_0$’s to be approximately the same: $S[(\tilde{A}^a_{\mu}]_{\phi_0}] \approx S_0 \rightarrow \infty$. Then changing the functional integration measure in the following way $-D(\tilde{A}^a_{\mu})_{\phi_0} \rightarrow e^{-iS_0} D(\tilde{A}^a_{\mu})_{\phi_0}$ – allows us to approximate the expectation value of $A^a_{\mu}$ as

$$\langle A^a_{\mu} \rangle \approx \sum_{\text{all classical solutions}} \left( \tilde{A}^a_{\mu} \right)_{\phi_0} p_{\phi_0}$$

(24)

where $p_{\phi_0}$ is the probability for a given classical solution. We will consider the classical solutions whose asymptotic behavior is given by $[\tilde{A}^a_{\mu}]_{\phi_0}$, and we will take the different solutions (as distinguished by different $\phi_0$’s) to have equal probability $p_{\phi_0} \approx \text{const}$. Therefore

$$\langle v \rangle \approx \frac{1}{2\pi} \int_0^{2\pi} v_{cl} d\phi_0 = \frac{A}{2\pi} \int_0^{2\pi} \sin(x^\alpha + \phi_0) d\phi_0 = 0,$$

(25)

$$\langle v^2 \rangle \approx \frac{1}{2\pi} \int_0^{2\pi} v_{cl}^2 d\phi_0 = \frac{A^2}{2\pi} \int_0^{2\pi} \sin^2(x^\alpha + \phi_0) d\phi_0 = \frac{A^2}{2},$$

(26)

$$\langle v^3 \rangle \approx \frac{1}{2\pi} \int_0^{2\pi} v_{cl}^3 d\phi_0 = \frac{A^4}{2\pi} \int_0^{2\pi} \sin^3(x^\alpha + \phi_0) d\phi_0 = 0,$$

(27)

$$\langle v^4 \rangle \approx \frac{1}{2\pi} \int_0^{2\pi} v_{cl}^4 d\phi_0 = \frac{A^4}{2\pi} \int_0^{2\pi} \sin^4(x^\alpha + \phi_0) d\phi_0 = \frac{3}{8} A^4$$

(28)

here $v_{cl}$ is the function from the Eq. The path integral inspired Eqns. (25) - (28) are the heart of the cutoff procedure that we wish to apply to Eqns. (18), (19). On substituting Eqns. (25), (26) into Eqns. (18), (19) we find that Eq.(18) is satisfied identically and Eq.(19) takes the form

$$r^2 w'' = 3A^2 w = \alpha(\alpha - 1) w, \quad \alpha > 1$$

(29)
which has the solutions

\[ w = w_0 r^\alpha, \]
\[ w = \frac{w_0}{r^{\alpha - 1}} \]  

(30)  

(31)

where \( w_0 \) is some constant. The first solution is simply the classically averaged singular solution \( \text{(3)} \) which still has the bad asymptotic divergence of the fields and energy density. The second solution, \( \text{(31)} \), is more physically relevant since it leads to asymptotic fields which are well behaved.

The solution of Eq. \( \text{(31)} \) implies the following important result: the quantum fluctuations of the strongly oscillating, nonlinear fields leads to an improvement of the bad asymptotic behavior of these nonlinear fields. This means that after quantization the monotonically growing and strongly oscillating components of the gauge potential become functions with good asymptotic behavior.

As \( r \rightarrow \infty \) we find the following SU(3) color fields

\[ \langle H^a_\rho \rangle \propto \left( \frac{\langle v^2 \rangle - 1}{r^2} \right) \approx \frac{Q}{r^2} \quad \text{with} \quad Q = \frac{1}{6} \alpha (\alpha - 1) - 1, \]  

(32)

\[ \langle H^a_{\rho, \theta} \rangle \propto \langle v^\prime \rangle \approx 0, \]  

(33)

\[ \langle E^a_\mu \rangle \propto \langle \frac{rw^\prime - w}{r^2} \rangle \approx - \frac{\alpha w_0}{r^{\alpha + 1}}, \]  

(34)

\[ \langle E^a_{\rho, \theta} \rangle \propto \langle v \rangle \frac{w}{r} = 0. \]  

(35)

We can see that as \( r \rightarrow \infty \) \( |\langle E^a_\mu \rangle| \ll |\langle H^a_\rho \rangle| \). In particular at infinity we find only a monopole “magnetic” field \( H^a_\rho = Q/r^2 \) with a “magnetic” charge \( Q \). This result can be summarized as: the approximate quantization of the SU(3) gauge field (by averaging over the classical singular solutions) gives a monopole-like configuration from an initial classical configuration which was not monopole-like. We will call this a “quantum monopole” to distinguish it from field configurations which are monopole-like already in the classical theory.

V. ENERGY DENSITY

The divergence of the fields of the classical solution given by Eqs. \( \text{(1)} - \text{(5)} \) leads to a diverging energy density for the solution, and thus an infinite total energy. The energy density \( \varepsilon \) of the quantized solution is

\[ \varepsilon \propto (E^a_\mu)^2 + (H^a_\rho)^2 \propto \left( \frac{rg' - g}{r^2} \right)^2 + \frac{2(f^2)g^2}{r^4} + \frac{2(f^2)}{r^4} + \frac{((f^2 - 1)^2)}{r^4} \]  

(36)

The first two terms on the right hand side of Eq. \( \text{(36)} \), which involve the “classical” ansatz function \( g(r) \), go to zero faster than \( 1/r^4 \) as \( r \rightarrow \infty \) due to the form of \( g(r) \) in Eq. \( \text{(31)} \). Thus the leading behavior of \( \varepsilon \) is given by the last two terms in Eq. \( \text{(36)} \) which have only the “quantum” ansatz function \( f(r) \). To calculate \( \langle f^2 \rangle \) let us consider

\[ r^2 \frac{d}{dr} \langle f'(r) f'(r) \rangle = \langle f'(r') f^3(r) \rangle - \langle f'(r') f(r) \rangle (1 + g^2(r)) \]  

(37)

In the limit \( r' \rightarrow r \) we have

\[ r^2 \langle f'(r) f''(r) \rangle = \frac{r^2}{2} \left \langle f^2(r) \right \rangle' = \frac{1}{3} \left \langle f^4(r) \right \rangle' - \frac{1}{2} \left \langle f^2(r) \right \rangle' (1 + g^2(r)). \]  

(38)

From Eqs. \( \text{(26)} \) and \( \text{(28)} \) we see that \( \langle f^2 \rangle' = 0 \) which implies \( \langle f^2 \rangle = \text{const} \). Thus the third term in \( \text{(36)} \) gives the leading asymptotic behavior as \( r \rightarrow \infty \) to be

\[ \varepsilon \approx \frac{\text{const}}{r^2} \]  

(39)

and the total energy of this “quantum monopole” (excitation) is infinite. This fact indicates that our approximation \( \text{(20)} \) is good only for \( \langle f^n(r) \rangle \) calculations but not for the derivative \( \langle f^2(r) \rangle \).
VI. CONCLUSIONS

Starting from an infinite energy, classical solution to the SU(3) Yang-Mills field equations we found that the bad asymptotic behavior of this solution was favorably modified by a variation of the quantization method proposed by Heisenberg to deal with strongly coupled, nonlinear field theories. In addition, although the original classical solution was not monopole-like, it was found that the quantized solution was monopole-like. This may imply that if real monopoles exist they may be inherently quantum mechanical objects *i.e.* that monopoles arise from the quantization of non-monopole classical solutions. This is to be contrasted with the standard idea that monopoles result from quantizing solutions which are already monopole-like at the classical level. One possible application of this is to the dual-superconductor picture of the QCD vacuum. In this picture one models the QCD vacuum as a stochastic gas of appearing/disappearing monopoles and antimonopoles as in Fig. 1. These monopole/antimonopole fluctuations can form pairs (analogous to Cooper pairs in real superconductors) which can Bose condense leading to a dual Meissner effect expelling color electric flux from the QCD vacuum, except in narrow flux tubes which connect and confine the quarks. Lattice calculations confirm such a model: monopoles appear to play a major role in the QCD lattice gauge path integral. Based on the results of the present paper it may be that the monopoles which are considered in the dual superconductor QCD vacuum picture should be the “quantum” monopoles discussed here rather than “classical” monopoles (*i.e.* monopoles which are already monopole-like at the classical level).

![Fig. 1: QCD vacuum ≈ stochastic gas of quantum monopoles/antimonopoles.](image)

VII. ACKNOWLEDGMENT

VD is grateful for Viktor Gurovich for the fruitful discussion.

[1] P.A.M. Dirac, Proc. Roy. Soc., A133, 60 (1931); P.A.M. Dirac, Phys. Rev. 74, 817 (1949)
[2] G. ’t Hooft, Nucl. Phys. B79, 276 (1974); A.M. Polyakov, JETP Lett. 20, 194 (1974)
[3] W. Heisenberg, Nachr. Akad. Wiss. Göttingen, N8, 111 (1953); W. Heisenberg, Nachr. Akad. Wiss. Göttingen; W. Heisenberg, Zs. Naturforsch., 9a, 292 (1954); W. Heisenberg, F. Kortel and H. Mütter, Zs. Naturforsch., 10a, 425 (1955); W. Heisenberg, Zs. für Phys., 144, 1 (1956); P. Askali and W. Heisenberg, Zs. Naturforsch., 12a, 177 (1957); W. Heisenberg, Nucl. Phys., 4, 532 (1957); W. Heisenberg, Rev. Mod. Phys., 29, 269 (1957)
[4] W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles.*, Max-Planck-Institut für Physik und Astrophysik, Interscience Publishers London, New York, Sydney, 1966
[5] V. Dzhunushaliev and D. Singleton “Confining solutions of SU(3) Yang-Mills theory”, “Contemporary Fundamental Physics”, by Nova Science Press, Edited by V.Dvoeglazov, [hep-th/9902076](http://arxiv.org/abs/hep-th/9902076)
[6] W.J. Marciano and H. Pagels, Phys. Rev. D12, 1093 (1975)
[7] Z. Horvath and L. Palla, Phys. Rev. D14, 1711 (1976)
[8] D.V. Gal’tsov and M.S. Volkov, Phys. Lett. B274, 173 (1992)
[9] E. Eichten, et. al., Phys. Rev. D17, 3090 (1978)
[10] D. Singleton and A. Yoshida, Int. J. Mod. Phys. A12, 4823 (1997)
[11] V. Dzhunushaliev and D. Singleton, Int. J. Theor. Phys, 38, 887(1999); [hep-th/9912194](http://arxiv.org/abs/hep-th/9912194)
[12] V. Dzhunushaliev, Phys. Lett. B498, 218 (2001); hep-th/0010185.

[13] C.G. Callan, R. Dashen and D.J. Gross, Phys. Lett. B66, 375 (1977); Phys Rev. D17, 2717 (1978); Phys. Rev. D19, 1826 (1979).

[14] J.V. Steele and J.W. Negele, Phys. Rev. Lett., 85, 4207 (2000).

[15] J.W. Negele, “Instanton and Meron Physics in Lattice QCD”, hep-lat/0007027; J.V. Steele, “Can Merons Describe Confinement?”, hep-lat/0007030.

[16] G. ’t Hooft, in Proc. Europ. Phys. Soc. Conf. on High Energy Physics (1975), p.1225.

[17] S. Mandelstam, Phys. Rev. D19, 2391 (1979).

[18] T. Suzuki and I. Yotsuyanagi, Phys. Rev. D42, 4257(1990); J.D. Stack, S.D. Neiman and R.J. Wensley, Phys. Rev. D50, 3399(1994); M.N. Chernodub and M.I. Polikarpov, “Abelian projections and monopoles”, hep-th/9710203.