Formal Estimation of Collision Risks for Autonomous Vehicles: A Compositional Data-Driven Approach* 

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Abstract. In this work, we propose a compositional data-driven approach for the formal estimation of collision risks for autonomous vehicles (AVs) with black-box dynamics acting in a stochastic multi-agent framework. The proposed approach is based on the construction of sub-barrier certificates for each stochastic agent via a set of data collected from its trajectories while providing a-priori guaranteed confidence on the data-driven estimation. In our proposed setting, we first cast the original collision risk problem for each agent as a robust optimization program (ROP). Solving the acquired ROP is not tractable due to an unknown model that appears in one of its constraints. To tackle this difficulty, we collect finite numbers of data from trajectories of each agent and provide a scenario optimization program (SOP) corresponding to the original ROP. We then establish a probabilistic bridge between the optimal value of SOP and that of ROP, and accordingly, we formally construct the sub-barrier certificate for each unknown agent based on the number of data and a required level of confidence. We then propose a compositional technique based on small-gain reasoning to quantify the collision risk for multi-agent AVs with some desirable confidence based on sub-barrier certificates of individual agents constructed from data. For the case that the proposed compositionality conditions are not satisfied, we provide a relaxed version of compositional results without requiring any compositionality conditions but at the cost of providing a potentially conservative collision risk. Eventually, we develop our approaches for non-stochastic multi-agent AVs. We demonstrate the effectiveness of our proposed results by applying them to a vehicle platooning consisting of 100 vehicles with 1 leader and 99 followers. We formally estimate the collision risk for the whole network by collecting sampled data from trajectories of each agent.

1. INTRODUCTION

In the near future, we expect to see fully autonomous vehicles (AVs), aircrafts, and robots, all of which should be able to make their own decisions without direct human involvement. Although this technology theme for AVs provides many potential advantages, e.g., fewer traffic collisions, reduced traffic congestion, increased roadway capacity, relief of vehicle occupants from driving, etc., there have been at least two critical challenges that need to be considered. First and foremost, closed-form mathematical models for many complex and heterogeneous physical systems, including AVs, are either not available or equally complex to be of practical use.

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Accordingly, one cannot employ model-based techniques to analyze this type of complex unknown systems. Although there are some model identification techniques available in the relevant literature to learn the model from data, e.g., [1, 2, 3, 4, 5] to name a few, acquiring an accurate model for complex systems is always complicated, time-consuming and expensive. As the second difficulty, providing safety certification and guaranteeing correctness of the design of such autonomous vehicles in a formal as well as time- and cost-effective way have always been the major obstacles in their successful deployment. These main challenges motivated us to develop a compositional data-driven approach to bypass the system identification phase and directly evaluate the AVs performance from data collected from their trajectories.

Over the past few years, a number of advances has been made in developing various techniques for evaluating AVs performance. Such techniques have mainly relied on methods from artificial intelligence (AI), machine learning, control theory, and optimization. In this regard, formal verification of heuristic intersection management (HAIM) using statistical model checking is studied in [6]. A statistical model checking approach to validate the collision risk assessment generated by a probabilistic perception system is proposed in [7]. Validation of perception and decision-making systems for autonomous driving via statistical model checking is investigated in [8]. A machine learning-based approach for uncertainty modeling and runtime verification of autonomous vehicles driving control is proposed in [9]. Statistical model checking for the safety validation of autonomous vehicles is investigated in [10].

Another promising approach for the safety verification of complex dynamical systems is to employ barrier certificates, initially introduced in [11, 12]. This approach has received significant attentions in the past decade, as a discretization-free technique, for formal verification and synthesis of non-stochastic [13, 14, 15], and stochastic dynamical systems [16, 17, 18, 19, 20, 21, 22, 23], to name a few. In particular, barrier certificates are Lyapunov-like functions defined over the state space of the system to enforce a set of inequalities on both the function itself and its one-step transition. An appropriate level set of a barrier certificate separates an unsafe region, we call it here collision set, from all system trajectories starting from a given set of initial states. Consequently, the existence of such a function provides a formal probabilistic certificate for the safety of the system. A barrier certificate for dynamical systems is schematically depicted in Figure 1.

Our main contribution in this work is to propose, for the first time, a compositional data-driven scheme for the formal estimation of collision risks for stochastic multi-agent AVs with black-box dynamics. Our proposed approach is based on the construction of sub-barrier certificates for each stochastic agent via a set of data collected from its trajectories. To do so, we first reformulate the original collision risk problem as a robust optimization program (ROP). Since the proposed ROP is not tractable due to an unknown model appearing in its constraints, we collect finite numbers of data from trajectories of each agent and provide a scenario optimization program (SOP) corresponding to the original ROP. We then build a probabilistic
relation between the optimal value of SOP and that of ROP, and as a result, we formally construct the sub-barrier certificate for each agent based on the number of data and a required level of confidence. We propose a compositional technique based on small-gain conditions to quantify the collision risk for the multi-agent AV based on constructed sub-barrier certificates of individual agents from data. We also propose a relaxed version of compositional results without requiring any compositionality condition but at the cost of providing a potentially conservative collision risk. Finally, we present our proposed approaches for non-stochastic multi-agent AVs. We demonstrate the effectiveness of our proposed techniques by applying them to a vehicle platooning consisting of 100 vehicles with 1 leader and 99 followers.

Data-driven safety verification of stochastic systems via barrier certificates is also studied in [24]. Our proposed approach here differs from the one in [24] in three main directions. First and foremost, we propose here a compositional data-driven scheme based on small-gain reasoning for the formal estimation of collision risks for multi-agent AVs, whereas the results in [24] only deal with monolithic systems, and accordingly, they are not practical in the case of facing high-dimensional systems. Moreover, we propose here a relaxed compositional approach in the case that our compositional conditions are not fulfilled. Second, we propose our approaches for both stochastic and non-stochastic large-scale systems, while the results of [24] are only tailored to stochastic systems. As the last main contribution, in order to propose our compositional results, we deal here with a non-convex class of robust and scenario optimization programs, whereas the proposed results in [24] are only applicable to convex optimization problems.

The rest of the paper is organized as follows. Section 2 gives mathematical preliminaries and notation, and formal definitions of stochastic (multi)agent AVs. Section 3 provides notions of (sub)barrier certificates. Section 4 is dedicated to data-driven construction of sub-barrier certificates. In Section 5, we propose our solution to establish a formal relation between the optimal value of SOP and that of ROP. Section 6 contains
the main compositionality results for multi-agent AVs based on small-gain reasoning. In Section 7, we propose a relaxed version of compositional results in which the compositionality conditions are no longer required. We develop our proposed results for non-stochastic multi-agent AVs in Section 8. Section 9 finally verifies our approaches over a vehicle platooning.

2. Problem Description

2.1. Preliminaries and Notation. We consider a probability space \((\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)\), where \(\Omega\) is the sample space, \(\mathcal{F}_\Omega\) is a sigma-algebra on \(\Omega\) comprising subsets of \(\Omega\) as events, and \(\mathbb{P}_\Omega\) is a probability measure assigning probabilities to events. Given the probability space \((\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)\), we denote the \(N\)-Cartesian product set of \(\Omega\) by \(\Omega^N\), and its corresponding product measure by \(\mathbb{P}^N\). A topological space \(S\) is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Any Borel space \(S\) is assumed to be endowed with a Borel sigma-algebra, which is denoted by \(\mathcal{B}(S)\).

We denote the set of real, positive and non-negative real numbers by \(\mathbb{R}, \mathbb{R}^+, \) and \(\mathbb{R}_0^+\), respectively. \(\mathbb{N} := \{0, 1, 2, \ldots\}\) represents the set of non-negative integers and \(\mathbb{N}_{\geq 1} := \{1, 2, \ldots\}\) is the set of positive integers. Given \(M\) vectors \(x_i \in \mathbb{R}^n\), \(x = [x_1; \ldots; x_M]\) denotes the corresponding column vector of dimension \(\sum_i n_i\). Given any \(a \in \mathbb{R}\), \(|a|\) denotes the absolute value of \(a\). Given a vector \(x \in \mathbb{R}^n\), \(\|x\|\) denotes the Euclidean norm of \(x\). We also denote by \(\|A\|_F := \sqrt{\text{trace}(A^T A)}\) the Frobenius norm of any matrix \(A \in \mathbb{R}^{n \times n}\). We denote the indicator function of a subset \(A\) of a set \(X\) by \(1_A : X \rightarrow \{0, 1\}\), where \(1_A(x) = 1\) if and only if \(x \in A\), and \(0\) otherwise. Symbol \(1_n\) denotes a column vector in \(\mathbb{R}^{n \times 1}\) with all elements equal to one. Given a symmetric matrix \(A\), \(\lambda_{\text{max}}(A)\) denotes the maximum eigenvalue of \(A\). If a system \(A\) satisfies a property \(\varphi\), we denote it by \(A \models \varphi\). We also use \(\models\) in this work to show the feasibility of a solution for an optimization problem.

Given functions \(f_i : X_i \rightarrow Y_i\), for any \(i \in \{1, \ldots, M\}\), their Cartesian product \(\prod_{i=1}^M f_i : \prod_{i=1}^M X_i \rightarrow \prod_{i=1}^M Y_i\) is defined by \((\prod_{i=1}^M f_i)(x_1, \ldots, x_M) = [f_1(x_1); \ldots; f_M(x_M)]\). Given a measurable function \(f : \mathbb{N} \rightarrow \mathbb{R}^n\), the (essential) supremum of \(f\) is denoted by \(\|f\|_{\infty} := (\text{ess}\sup\{\|f(k)\|, k \geq 0\})\). A function \(\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\) is said to be a class \(\mathcal{K}\) function if it is continuous, strictly increasing, and \(\phi(0) = 0\). A class \(\mathcal{K}\) function \(\phi(\cdot)\) belongs to class \(\mathcal{K}_\infty\) if \(\phi(s) \rightarrow \infty\) as \(s \rightarrow \infty\).

2.2. Stochastic (Multi)Agent AVs. In this work, we consider dynamics of each agent as a discrete-time stochastic system, as formalized in the following definition.

Definition 2.1. Each agent of AVs is a tuple

\[ A_i = (X_i, W_i, s_i, f_i), \]  

where:
• $X_i \subseteq \mathbb{R}^{n_i}$ is a Borel space as the state set of the agent;
• $W_i \subseteq \mathbb{R}^{p_i}$ is a Borel space as the interaction set of the agent;
• $\varsigma_i$ is a sequence of independent-and-identically distributed (i.i.d.) random variables from a sample space $\Omega_i$ to the set $\mathcal{V}_{\varsigma_i}$, namely $\varsigma_i := \{\varsigma_i(k) : \Omega_i \rightarrow \mathcal{V}_{\varsigma_i}, k \in \mathbb{N}\}$;
• $f_i : X_i \times W_i \times \mathcal{V}_{\varsigma_i} \rightarrow X_i$ is a measurable function characterizing the state evolution of $A_i$.

The evolution of the state of $A_i$ for a given initial state $x_{0i} = x_i(0) \in X_i$, and an interaction sequence $\{w_i(k) : \Omega_i \rightarrow W_i, k \in \mathbb{N}\}$ is described as

$$A_i : x_i(k + 1) = f_i(x_i(k), w_i(k), \varsigma_i(k)), \quad k \in \mathbb{N}. \quad (2.2)$$

We call the random sequence $x_{x_{0i},w_i} : \Omega_i \times \mathbb{N} \rightarrow X_i$ satisfying (2.2) the solution process of $A_i$ under an interaction $w_i$ and an initial state $x_{0i}$.

Since the main goal in this work is to formally estimate the collision risks for AVs, we assume that the controller for each agent is already designed and deployed to the system. We model the effect of other agents, influencing the state of the current agent, via the interaction $w_i$ which can be also captured by the deployed closed-loop controller inside AVs. In the following, we provide a formal definition of multi-agent AVs without interactions $w_i$ that is considered as a composition of several agents with interactions.

**Definition 2.2.** Consider $M \in \mathbb{N}_{\geq 1}$ agents $A_i = (X_i, W_i, \varsigma_i, f_i), i \in \{1, \ldots, M\}$, with their interactions partitioned as

$$w_i = [w_{i1}; \ldots; w_{i(i-1)}; w_{i(i+1)}; \ldots; w_{iM}]. \quad (2.3)$$

A multi-agent AV is defined as $\mathcal{A} = (X, \varsigma, f)$, denoted by $\mathcal{I}(A_1, \ldots, A_M)$, such that $X := \prod_{i=1}^{M} X_i$, $\varsigma = [\varsigma_1; \ldots; \varsigma_M]$, and $f := \prod_{i=1}^{M} f_i$ subjected to the following interconnection constraint:

$$\forall i, j \in \{1, \ldots, M\}, i \neq j: w_{ij} = x_j, \quad X_j \subseteq W_{ij}. \quad (2.4)$$

Such a multi-agent AV can be represented by

$$\mathcal{A} : x(k + 1) = f(x(k), \varsigma(k)), \quad k \in \mathbb{N}, \quad \text{with } f : X \times \mathcal{V}_\varsigma \rightarrow X. \quad (2.5)$$

An example of a multi-agent AV $\mathcal{A}$ with two agents $A_1$ and $A_2$ is illustrated in Figure 2.

**Remark 2.3.** Note that in multi-agent systems, the underlying dynamics should not be necessarily interconnected. In particular, the main goal in multi-agent systems is to design a controller to satisfy a global objective in which the interconnection comes with the desired specification. In our work, we call the overall system “multi-agent AVs” since we assume that the controller is already designed and the interaction between agents, captured by the deployed closed-loop controller, results in coupled dynamics.
In the next section, in order to estimate the collision risk for multi-agent AVs in finite time horizons, we present notations of barrier and sub-barrier certificates for, respectively, multi-agent AVs $A$ (without interactions) and individual agents $A_i$ (with interactions).

3. (Sub-)Barrier Certificates

In this section, we first present the notation of barrier certificates for multi-agent AVs $A$ without interactions, as the following definition.

**Definition 3.1.** Given the multi-agent AV $A = (X, \varsigma, f)$, a non-negative function $B : X \to \mathbb{R}^+_0$ is called a barrier certificate (BC) for $A$ if there exist $\gamma, \lambda \in \mathbb{R}^+$, with $\lambda > \gamma$, $0 < \kappa < 1$, and $\psi \in \mathbb{R}^+_0$, such that:

\begin{align*}
B(x) &\leq \gamma, \quad \forall x \in X_0, \\
B(x) &\geq \lambda, \quad \forall x \in X_c, \\
E\left[B(f(x, \varsigma)) \mid x\right] &\leq \kappa B(x) + \psi, \quad \forall x \in X,
\end{align*}

where $X_0, X_c \subseteq X$ are initial and collision sets, respectively.

Given a collision event $\varphi = (X_0, X_c, T)$, the multi-agent AV $A$ may have a collision, denoted by $A \models_T \varphi$, if a trajectory of $A$ starting from the initial set $X_0$ reaches the collision set $X_c$ within the time horizon $T$. Since trajectories of the multi-agent AV are probabilistic, we are interested in computing the collision risk $P\{A \models_T \varphi\}$. Now we employ Definition 3.1 and quantify the collision risk for the multi-agent AVs in (2.5) via the next theorem, which is borrowed from [25].
Theorem 3.2. Consider the multi-agent AV $A$ defined in Definition 2.2 and a finite time horizon $T \in \mathbb{N}_0$. Suppose there exists a non-negative barrier certificate $B$ satisfying conditions (3.1)-(3.3). Then the collision risk for the multi-agent AV $A$ is bounded by

$$\Pr\{A = \varphi | T\} \leq \begin{cases} 1 - (1 - \gamma)(1 - \psi)^T, & \text{if } \lambda \geq \frac{\psi}{1 - \kappa}, \\ \left(\frac{\psi}{1 - \kappa}\lambda\right)^T + \left(\frac{\psi}{1 - \kappa}\lambda\right)(1 - \kappa)^T, & \text{if } \lambda < \frac{\psi}{1 - \kappa}. \end{cases} \quad (3.4)$$

Proof: According to condition (3.2), one has $X_c \subseteq \{x \in X | B(x) \geq \lambda\}$. Then we have

$$\Pr\{x_0(k) \in X_c \text{ for } 0 \leq k < T \mid x_0 = x(0)\} \leq \Pr\left\{\sup_{0 \leq k < T} B(x(k)) \geq \lambda \mid x_0 = x(0)\right\}. \quad (3.5)$$

The quantified collision risk in (3.4) follows directly by applying [25, Theorem 3, Chapter III] to (3.5) and employing, respectively, conditions (3.3) and (3.1) in Definition 3.1. \hfill \blacksquare

In general, searching for barrier certificates for multi-agent AVs as in Definition 3.1 is computationally very expensive, even if the model is known, mainly due to the high-dimension of the system. Accordingly, we present in the following a definition of sub-barrier certificates (SBC) for individual agents $A_i$ and propose in Section 6 a compositional approach based on small-gain reasoning to construct a BC of multi-agent AVs based on SBC of individual agents.

Definition 3.3. Given an agent $A_i = (X_i, W_i, \varsigma_i, f_i)$, a non-negative function $B_i : X_i \to \mathbb{R}_0^+$ is called a sub-barrier certificate (SBC) for $A_i$ if there exist $\gamma_i, \lambda_i \in \mathbb{R}^+$, $\alpha_i, \rho_i, \psi_i \in \mathbb{R}_0^+$, and $0 < \kappa_i < 1$, such that:

$$B_i(x_i) \geq \alpha_i \|x_i\|^2, \quad \forall x_i \in X_i, \quad (3.6)$$

$$B_i(x_i) \leq \gamma_i, \quad \forall x_i \in X_{0i}, \quad (3.7)$$

$$B_i(x_i) \geq \lambda_i, \quad \forall x_i \in X_{ci}, \quad (3.8)$$

$$\mathbb{E}\left[B_i(f_i(x_i, w_i, \varsigma_i)) \mid x_i, w_i\right] \leq \kappa_i B_i(x_i) + \rho_i \|w_i\|^2 + \psi_i, \quad \forall x_i \in X_i, \forall w_i \in W_i, \quad (3.9)$$

where $X_{0i}, X_{ci} \subseteq X_i$ are initial and collision sets of agents, respectively.

Remark 3.4. Note that $\alpha_i$ and $\rho_i$ in Definition 3.3 could be, without loss of generality, $\mathcal{K}_\infty$ functions as proposed in [22]. We consider them in this work as linear functions for the sake of an easier presentation.

In the next section, we provide a data-driven scheme for the construction of SBC for each agent.

4. Data-Driven Construction of Sub-BARRIER Certificates

In this section, we assume that the transition map $f_i$ and the distribution of $\varsigma_i$ in (2.2) are both unknown, and we employ the term black-box models to refer to this type of systems. We fix the structure of SBC as
\[ \mathcal{B}_i(q_i, x_i) = \sum_{j=1}^r q_i \tilde{p}_{ij}(x_i) \] with some user-defined (possibly nonlinear) basis functions \( \tilde{p}_{ij}(x_i) \) and unknown coefficients \( q_i = [q_i, \ldots, q_r] \in \mathbb{R}^r \). In the case of having polynomial-type barrier certificates, basis functions \( \tilde{p}_{ij}(x_i) \) are monomials over \( x_i \).

In order to enforce conditions (3.6)-(3.9) in Definition 3.3 we first cast our collision risk problem for each agent as the following robust optimization program (ROP), for any \( i \in \{1, \ldots, M\} \):

\[
\text{ROP} : \begin{cases}
\min_{\Theta, \eta_i} & \eta_i \\
\text{s.t.} & \max \left\{ g_{i_1}(x_i, \Theta_i), g_{i_2}(x_i, w_i, \Theta_i) \right\} \leq \eta_i, x \in \{1, \ldots, 4\}, \forall x_i \in X_i, \forall w_i \in W_i, \\
& \Theta_i = [\gamma_i; \lambda_i; \psi_i; \alpha_i; \rho_i; q_1; \ldots; q_r], \\
& \eta_i \in \mathbb{R}, \gamma_i, \lambda_i \in \mathbb{R}^+, \alpha_i, \rho_i, \psi_i \in \mathbb{R}_0^+, \\
\end{cases}
\tag{4.1}
\]

where:

\[
g_{i_1} = -\mathcal{B}_i(q_i, x_i), \quad g_{i_2} = \alpha_i \|x_i\|^2 - \mathcal{B}_i(q_i, x_i), \\
g_{i_3} = (\mathcal{B}_i(q_i, x_i) - \gamma_i) \mathbf{1}_{X_{i_1}}(x_i), \quad g_{i_4} = (\lambda_i - \mathcal{B}_i(q_i, x_i)) \mathbf{1}_{X_{i_2}}(x_i), \\
g_{i_5} = \mathbb{E} \left[ \mathcal{B}_i(q_i, f_i(x_i, w_i, \varsigma_i)) \right| x_i, w_i] - \kappa_i \mathcal{B}_i(q_i, x_i) - \rho_i w_i^2 - \psi_i, \tag{4.2}
\]

with \( \mathbf{1}_{X_{i_1}}(x_i) \) and \( \mathbf{1}_{X_{i_2}}(x_i) \) being indicator functions acting on initial and collision sets, respectively. We denote the optimal value of ROP by \( \eta^*_R \). If \( \eta^*_R \leq 0 \), a solution to the ROP implies the satisfaction of conditions (3.6)-(3.9) in Definition 3.3.

We modified conditions (3.6)-(3.9) of Definition 3.3 by artificially adding \( \eta_i \) to its right-hand side and defining it as the objective function of the ROP. In the sequel, we provide our solution to find an SBC for each unknown agent by establishing a probabilistic relation between the optimal value of ROP (i.e., \( \eta^*_R \)) and that of its corresponding scenario optimization program (SOP).

To solve the proposed ROP in (4.1), we face two major difficulties. First, the proposed ROP in (4.1) has infinitely-many constraints since the state and interaction of \( A_i \) live in continuous sets (i.e., \( x_i \in X_i, w_i \in W_i \)). In addition and more importantly, one needs to know the precise map \( f_i \) in \( g_{i_5} \) in order to solve the optimization problem in (4.1). To tackle these two difficulties, we propose a scenario optimization program corresponding to the original ROP. Suppose \( (\hat{x}_i, \hat{w}_i)_{i=1}^{N_i} \) denote \( N_i \) independent-and-identically distributed (i.i.d.) sampled data from \( X_i \times W_i \). In our setting, we take two consecutive sampled data-points from trajectories of \( A_i \) as the pair of \( (x_i(k), x_i(k+1)) \) and denote it by \((\hat{x}_i, f_i(\hat{x}_i))\). We now propose the following scenario optimization program (SOP), for any \( i \in \{1, \ldots, M\} \):

\[
\text{SOP} : \begin{cases}
\min_{\Theta, \eta_i} & \eta_i \\
\text{s.t.} & \max \left\{ g_{i_1}(\hat{x}_i, \Theta_i), g_{i_2}(\hat{x}_i, \hat{w}_i, \Theta_i) \right\} \leq \eta_i, x \in \{1, \ldots, 4\}, \forall x_i \in X_i, \forall w_i \in W_i, \\
& \Theta_i = [\gamma_i; \lambda_i; \psi_i; \alpha_i; \rho_i; q_1; \ldots; q_r], \\
& \eta_i \in \mathbb{R}, \gamma_i, \lambda_i \in \mathbb{R}^+, \alpha_i, \rho_i, \psi_i \in \mathbb{R}_0^+, \\
\end{cases}
\tag{4.3}
\]

where:

\[
g_{i_1} = -\mathcal{B}_i(q_i, \hat{x}_i), \quad g_{i_2} = \alpha_i \|\hat{x}_i\|^2 - \mathcal{B}_i(q_i, \hat{x}_i), \\
g_{i_3} = (\mathcal{B}_i(q_i, \hat{x}_i) - \gamma_i) \mathbf{1}_{X_{i_1}}(\hat{x}_i), \quad g_{i_4} = (\lambda_i - \mathcal{B}_i(q_i, \hat{x}_i)) \mathbf{1}_{X_{i_2}}(\hat{x}_i), \\
g_{i_5} = \mathbb{E} \left[ \mathcal{B}_i(q_i, f_i(\hat{x}_i, \hat{w}_i, \varsigma_i)) \right| x_i, w_i] - \kappa_i \mathcal{B}_i(q_i, \hat{x}_i) - \rho_i \hat{w}_i^2 - \psi_i, \tag{4.4}
\]

with \( \mathbf{1}_{X_{i_1}}(\hat{x}_i) \) and \( \mathbf{1}_{X_{i_2}}(\hat{x}_i) \) being indicator functions acting on initial and collision sets, respectively. We denote the optimal value of SOP by \( \eta^*_S \). If \( \eta^*_S \leq 0 \), a solution to the SOP implies the satisfaction of conditions (3.6)-(3.9) in Definition 3.3.
where $g_i-\tilde{g}_{i_5}$ are the functions as defined in (4.2). We denote the optimal value of SOP by $\eta_{N_5}^*$. Although the problems of infinitely-many constraints and unknown map $f_i$ are resolved here in SOP$_N$, there is still no closed-form solution for the expected value in $g_{i_5}$. Hence, we employ an empirical approximation of the expected value and propose a new scenario optimization problem, denoted by SOP$_\xi$, as the following, for any $i \in \{1, \ldots, M\}$:

$$\begin{align}
\text{SOP}_\xi : \quad & \min_{[\Theta_i, \eta_i]} \eta_i \\
\text{s.t.} \quad & \max \left\{ g_{i_5}(\hat{x}_{i_5}, \Theta_i), \tilde{g}_{i_5}(\hat{x}_{i_5}, \hat{w}_{i_5}, \Theta_i) \right\} \leq \eta_i, \forall z \in \{1, \ldots, 4\}, \forall \hat{x}_{i_5} \in X, \forall \hat{w}_{i_5} \in W_i, \\
& \forall l \in \{1, \ldots, N_i\}, \Theta_i = [\gamma_i; \lambda_i; \psi_i; \alpha_i; \rho_i; q_i; \ldots; q_i], \\
& \eta_i \in \mathbb{R}, \gamma_i, \lambda_i \in \mathbb{R}^+, \alpha_i, \rho_i, \psi_i \in \mathbb{R}_0^+, \\
\end{align}$$

(4.4)

with

$$\tilde{g}_{i_5}(\hat{x}_{i_5}, \hat{w}_{i_5}, \Theta_i) = \frac{1}{N_i} \sum_{j=1}^{N_i} \mathcal{B}_i(q_i, f_i(\hat{x}_{i_5}, \hat{w}_{i_5}, \xi_j)) - \kappa_i \mathcal{B}_i(q_i, \hat{x}_{i_5}) - \rho_i \|\hat{w}_{i_5}\|^2 - \psi_i + \mu_i,$$

(4.5)

where $N_i \in \mathbb{N}$ and $\mu_i \in \mathbb{R}_0^+$ are, respectively, the number of samples required for the empirical approximation and the corresponding error introduced by this approximation. We denote the optimal value of the objective function SOP$_\xi$ by $\eta_{\xi}^*$.

**Remark 4.1.** Note that condition $\tilde{g}_{i_5}$ is not convex due to a bilinearity between decision variables $q_i$ and the unknown variable $\kappa_i$. Although a mild convexity can be observed since both $\kappa_i$ and $q_i$ are scalar, the SOP$_\xi$ in (4.4) is in principle non-convex. To tackle this issue, we consider $\kappa_i$ in a known finite set with a cardinality $m_i$, while solving SOP$_\xi$ in (4.4). Accordingly, we propose our data-driven approach in Section 5 by incorporating the effect of this bilinearity in computing the minimum number of data required for solving SOP$_\xi$ (4.4) (cf. Theorem 5.1).
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**Lemma 4.2.** Suppose $\mathcal{B}_i(q_i, x_i)$ is a solution for SOP$\Rightarrow$ in (4.4). For a-priori approximation error $\mu_i \in \mathbb{R}_{0}^+$, a desired confidence $\beta_{1i}$, $(0, 1]$, and an upper bound $Q_i$ on the variance of the sub-barrier certificate applied on $f_{i}$, i.e., $\text{Var}[\mathcal{B}_i(q_i, f_i(x_i, w_i, t_i))] \leq Q_i \in \mathbb{R}^+$, $\forall x_i \in X_i, w_i \in W_i$, one has

$$\mathbb{P}\{\mathcal{B}_i(q_i, x_i) \Rightarrow \text{SOP}_N\} \geq 1 - \beta_{1i}, \quad \text{with } \hat{N}_i \geq \frac{Q_i}{\beta_{1i} \mu_i^2},$$

for any $i \in \{1, \ldots, M\}$.

5. Construction of SBC for Unknown Agents

In this section, we aim at establishing a formal relation between the optimal value of SOP$\Rightarrow$ in (4.4) and that of ROP in (4.1), inspired by the fundamental results proposed in [27]. Accordingly, we formally quantify the SBC of unknown agents based on the number of data and a required level of confidence. To do so, we first raise the following assumption.

**Assumption 1.** Functions $g_{i_1} - g_{i_4}$ are all Lipschitz continuous with respect to $x_i$, and $g_{i_5}$ is Lipschitz continuous with respect to $x_i, w_i$, with Lipschitz constants $L_{g_{i_1}} - L_{g_{i_4}}, L_{g_{i_5}}$, respectively.

Under Assumption 1, we propose the main result of this section.

**Theorem 5.1.** Consider an unknown agent $A_i$ as in (2.1), and initial and collision regions $X_{0i}$ and $X_{ci}$, respectively. Let Assumption 1 hold. Consider the corresponding SOP$\Rightarrow$ in (4.4) with its associated optimal value $\eta_i^*$, and the solution $\Theta_i^* = [\gamma^*_i; \lambda^*_i; \eta^*_i; \alpha_i^*; \rho_i^*; q_i^*; \ldots; q_i^*]$, with $N_i(\xi_{2i}, \beta_{2i})$, $\xi_{2i} := (\xi_{2i_1}, \ldots, \xi_{2i_m})$, where

$$N_i(\xi_{2i}, \beta_{2i}) := \min \left\{ N_i \in \mathbb{N} \left| \sum_{k=1}^{m_i} \sum_{j=0}^{c_{i1}-1} \left( N_i \right)^{\beta_{2i} - j} (1 - \xi_{2i_k})^{N_i - j} \leq \beta_{2i} \right. \right\}, \quad (5.1)$$

$\beta_{2i} \in [0, 1]$, $\xi_{2i_k} = \left( \frac{\xi_{2i_k}}{\xi_{2i_k}} \right)^{n_i + p_i}$, $\xi_{1i} \in [0, 1] \leq L_{g_{i_k}} = \max \{ L_{g_{i_1}}, L_{g_{i_2}}, L_{g_{i_3}}, L_{g_{i_4}}, L_{g_{i_5}} \}, k \in \{1, \ldots, m_i\}$, with $n_i, p_i, c_i, m_i$ being, respectively, dimensions of state and interaction sets, the number of decision variables in SOP$\Rightarrow$, and the cardinality of a finite set that $\kappa_i \in (0, 1)$ takes value from it. Then if $\eta_i^* + \xi_{1i} \leq 0$, the solution $\Theta_i^*$ is a feasible solution for ROP (4.1), i.e., $\Theta^* \Rightarrow \text{ROP}$, with a confidence of at least $1 - \beta_{1i} - \beta_{2i}$.  

**Proof:** Based on [27] Theorem 4.1 and Remark 4.2, the probabilistic distance between optimal values of ROP and SOP$\Rightarrow$ can be formally lower bounded by

$$\mathbb{P}^{N_i}\left\{ 0 \leq \eta_{R_i} - \eta_{N_i}^* \leq \xi_{1i} \right\} \geq 1 - \beta_{2i},$$

1One can readily verify that $\eta_{R_i}^*$ is always bigger than or equal to $\eta_{N_i}^*$ because $\eta_{R_i}^*$ is computed for infinitely-many constraints, whereas $\eta_{N_i}^*$ is computed only for finitely many of them.
provided that
\[ N_i \geq N_i(g_{ik}(\frac{\varepsilon_1}{\text{LSP},s_{g_{ik}}}), \beta_2), \]
where \( g_{ik} : [0, 1] \rightarrow [0, 1] \) is given by
\[ g_{ik}(s) = s^{n_i + \mu_i}, \quad \forall s \in [0, 1], \]
and \( \text{LSP} \) is a Slater constant as defined in [27, equation (5)]. Since the original ROP in (4.1) can be cast as a min-max optimization problem, the Slater constant \( \text{LSP} \) can be selected as 1 [27, Remark 3.5].

One can readily conclude that \( \eta_i^* \leq \eta_i^* + \varepsilon_1 \), with a confidence of at least \( 1 - \beta_2 \). From Lemma 4.2 we have \( \eta_i^* \leq \eta_i^* \), with a confidence of at least \( 1 - \beta_1 \). Let us now define events \( \mathcal{E}_1 := \{ \eta_i^* \leq \eta_i^* \} \) and \( \mathcal{E}_2 := \{ \eta_i^* \leq \eta_i^* + \varepsilon_1 \} \), where \( \mathbb{P}\{\mathcal{E}_1\} \geq 1 - \beta_1 \) and \( \mathbb{P}\{\mathcal{E}_2\} \geq 1 - \beta_2 \). From the above derivations, one has \( \eta_i^* \leq \eta_i^* + \varepsilon_1 \leq \eta_i^* + \varepsilon_1 \). Since \( \eta_i^* + \varepsilon_1 \leq 0 \), it implies that \( \eta_i^* \leq 0 \). We are now computing the concurrent occurrence of events \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), namely \( \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} \):

\[ \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} = 1 - \mathbb{P}\{\tilde{\mathcal{E}}_1 \cup \tilde{\mathcal{E}}_2\}, \]

where \( \tilde{\mathcal{E}}_1 \) and \( \tilde{\mathcal{E}}_2 \) are the complement of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively. Since

\[ \mathbb{P}\{\tilde{\mathcal{E}}_1 \cup \tilde{\mathcal{E}}_2\} \leq \mathbb{P}\{\tilde{\mathcal{E}}_1\} + \mathbb{P}\{\tilde{\mathcal{E}}_2\}, \]

and by leveraging (5.2), one can readily conclude that

\[ \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} \geq 1 - \mathbb{P}\{\tilde{\mathcal{E}}_1\} - \mathbb{P}\{\tilde{\mathcal{E}}_2\} \geq 1 - \beta_1 - \beta_2. \]

Therefore, the solution \( \Theta_i^* \) via solving SOP in (4.4) is a feasible solution for ROP (4.1) with a confidence of at least \( 1 - \beta_1 - \beta_2 \), which completes the proof.

We summarize the results of Theorem 5.1 in Algorithm 1 to describe the required procedure.

**Algorithm 1** Data-driven construction of SBC for unknown agents \( A_i \)

**Require:** \( \beta_1 \in (0, 1], \beta_2 \in [0, 1], \mu_i \in \mathbb{R}_0^+ \), \( s_{g_{ik}} \in \mathbb{R}^+ \), \( Q_i \in \mathbb{R}^+ \), and degree of SBC

1. Choose \( \varepsilon_1 \in [0, 1] \) such that \( \varepsilon_1, \leq s_{g_{ik}} \)
2. Compute \( \hat{N}_i \) required for the empirical approximation according to Lemma 4.2
3. Compute the minimum number of samples \( N_i \) according to (5.1)
4. Solve the SOP in (4.4) with acquired \( \hat{N}_i \) and \( N_i \), and obtain \( \eta_i^* \) and \( \Theta_i^* \)

**Ensure:** if \( \eta_i^* + \varepsilon_1 \leq 0 \), then \( \Theta_i^* \) is a feasible solution for ROP with a confidence of at least \( 1 - \beta_1 - \beta_2 \).

In order to compute the required number of samples in Theorem 5.1 one needs to first compute \( s_{g_{ik}} \). In the next lemmas, we propose a systematic approach to compute \( s_{g_{ik}} \) for both linear and nonlinear agents.
Lemma 5.2. Consider a linear operator \( A_i : x_i(k+1) = A_i x_i(k) + D_i w_i(k) + R_i \zeta_i(k), \) with \( A_i, R_i \in \mathbb{R}^{n_i \times n_i}, D_i \in \mathbb{R}^{p_i \times n_i}, \) and \( \zeta_i(\cdot) \sim \mathcal{N}(0, I_n). \) Let \( \|A_i\|_F \leq \mathcal{L}_{A_i}, \|D_i\|_F \leq \mathcal{L}_{D_i}, \) where \( \mathcal{L}_{A_i}, \mathcal{L}_{D_i} \in \mathbb{R}_+^+. \) Then \( \mathcal{L}_{g_{i,k}} \) for a quadratic SBC in the form of \( x_i^T P_i x_i, \) with a positive-definite matrix \( P_i \in \mathbb{R}^{n_i \times n_i}, \) is computed as

\[
\mathcal{L}_{g_{i,k}} = \max \left\{ \mathcal{L}_{g_{i,1}}, \mathcal{L}_{g_{i,2}}, \mathcal{L}_{g_{i,3}}, \mathcal{L}_{g_{i,4}}, \mathcal{L}_{g_{i,5}} \right\} = \max \left\{ 2s_i (\lambda_{\max}(P_i) + \mathcal{L}_{A_i}), (\mathcal{L}_{g_{i,s_{ik}}}^2 + \mathcal{L}_{g_{i,w}}^2) \right\},
\]

with

\[
\mathcal{L}_{g_{i,s_{ik}}} = 2\lambda_{\max}(P_i)(s_i(\mathcal{L}_{A_i}^2 + \kappa_i) + s_i' \mathcal{L}_{A_i} \mathcal{L}_{D_i}), \quad \mathcal{L}_{g_{i,w}} = 2\lambda_{\max}(P_i)(s_i' \mathcal{L}_{D_i}^2 + s_i \mathcal{L}_{A_i} \mathcal{L}_{D_i}) + 2 \mathcal{L}_{\rho_i} s_i',
\]

where \( \mathcal{L}_{\alpha_i}, \mathcal{L}_{\rho_i}, s_i, s_i' \) are, respectively, upper bounds on \( \alpha_i, \rho_i, \) and the norm of \( x_i, w_i, \) i.e., \( \alpha_i \leq \mathcal{L}_{\alpha_i} \in \mathbb{R}_0^+, \rho_i \leq \mathcal{L}_{\rho_i} \in \mathbb{R}_0^+, \|x_i\| \leq s_i \in \mathbb{R}_0^+, \forall x_i \in X_i, \) and \( \|w_i\| \leq s_i' \in \mathbb{R}_0^+, \forall w_i \in W_i. \)

**Proof:** We first compute Lipschitz constants of \( \mathcal{L}_{g_{i,1}}, \mathcal{L}_{g_{i,4}} \) with respect to \( x \) and \( \mathcal{L}_{g_{i,5}} \) with respect to \( x_i \) and \( w_i, \) and then take the maximum among them. By defining

\[
\mathcal{L}_{g_{i,s_{ik}}}: \begin{cases} \max_{x_i \in X_i} \| \frac{\partial g_{i,s_{ik}}}{\partial x_i} \|, \\ \text{s.t.} \quad \|x_i\| \leq s_i, \|w_i\| \leq s_i', \end{cases}
\]

one has

\[
\mathcal{L}_{g_{i,s_{ik}}} = \max_{x_i \in X_i} \| 2(A_i^T P_i A_i - \kappa_i P_i) x_i + 2A_i^T P_i D_i w_i \|
\]

\[
\leq \max_{x_i \in X_i} \| 2(A_i^T P_i A_i - \kappa_i P_i) \|_F \|x_i\| + \|2A_i^T P_i D_i\|_F \|w_i\|
\]

\[
\leq 2s_i (\|A_i^T P_i A_i\|_F + \kappa_i \|P_i\|_F) + 2s'_i \|A_i\|_F \|P_i\|_F \|D_i\|_F
\]

\[
\leq 2s_i (\|P_i\|_F \|A_i\|_F^2 + \kappa_i \|P_i\|_F) + 2s'_i \|A_i\|_F \|P_i\|_F \|D_i\|_F
\]

\[
\leq 2\lambda_{\max}(P_i)(s_i(\mathcal{L}_{A_i}^2 + \kappa_i) + s_i' \mathcal{L}_{A_i} \mathcal{L}_{D_i}).
\]

Similarly, defining

\[
\mathcal{L}_{g_{i,w}}: \begin{cases} \max_{w_i \in W_i} \| \frac{\partial g_{i,w}}{\partial w_i} \|, \\ \text{s.t.} \quad \|x_i\| \leq s_i, \|w_i\| \leq s_i', \end{cases}
\]

one has

\[
\mathcal{L}_{g_{i,w}} = \max_{w_i \in W_i} \| 2D_i^T P_i D_i w_i + 2D_i^T P_i A_i x_i - 2\rho_i w_i \|
\]

\[
\leq \max_{w_i \in W_i} \| 2D_i^T P_i D_i \|_F \|w_i\| + 2\|D_i^T P_i A_i\|_F \|x_i\| + 2\rho_i \|w_i\|
\]

\[
\leq 2s'_i \|D_i^T P_i D_i\|_F + 2s_i \|D_i^T P_i A_i\|_F \|x_i\| + 2\mathcal{L}_{\rho_i} s_i'
\]

\[
\leq 2\lambda_{\max}(P_i)(s'_i \mathcal{L}_{D_i}^2 + s_i \mathcal{L}_{A_i} \mathcal{L}_{D_i}) + 2 \mathcal{L}_{\rho_i} s_i'.
\]
Then $L_{g_{i,k}} = (L_{g_{i,kx}}^2 + L_{g_{i,sw}}^2)^{\frac{1}{2}}$. Similarly for $g_{i_1} - g_{i_4}$, we have $L_{g_{i_1}} = L_{g_{i_3}} = L_{g_{i_4}} \leq 2s_i\lambda_{\text{max}}(P_i)$, $L_{g_{i_2}} \leq 2s_i(\lambda_{\text{max}}(P_i) + \omega_i)$. Then $L_{g_i} = \max \{L_{g_{i_1}}, L_{g_{i_2}}, L_{g_{i_3}}, L_{g_{i_4}}, L_{g_{i,k}}\} = \max \{2s_i(\lambda_{\text{max}}(P_i) + \omega_i), (L_{g_{i,kx}}^2 + L_{g_{i,sw}}^2)^{\frac{1}{2}}\}$, which completes the proof.

**Remark 5.3.** One needs to know upper bounds for $\lambda_{\text{max}}(P_i), \omega_i, \rho_i$ in order to compute $L_{g_{i,k}}$, and accordingly, the required number of samples as Step 3 in Algorithm 1. The pre-assumed upper bounds should be then enforced as some additional conditions during solving the SOPs as mentioned in Step 4 of Algorithm 1.

Lemma 5.2 provides a systematic approach for computing $L_{g_{i,k}}$ tailored to quadratic barrier certificates in the form of $x_i^T P_i x_i$. If one is interested in polynomial-type SBC, one can still leverage the results of Lemma 5.2 since, without loss of generality, any polynomial function can be reformulated as a quadratic function of monomials. Similarly, we provide another lemma for the computation of $L_{g_{i,k}}$ but for nonlinear agents $A_i$.

**Lemma 5.4.** Consider nonlinear agents $A_i$: $x_i(k + 1) = f_i(x_i(k), w_i(k)) + R_i s_i(k)$ with $s_i(\cdot) \sim \mathcal{N}(0, \Sigma_i)$. Let $\|f_i(x_i, w_i)\| \leq L_{f_i}, \|\partial_{x_i} f_i(x_i, w_i)\|_F = \|\partial f_i(x_i, w_i) / \partial x_i\|_F \leq L_{x_i}, \|\partial_{w_i} f_i(x_i, w_i)\|_F = \|\partial f_i(x_i, w_i) / \partial w_i\|_F \leq L_{w_i}$, where $L_{f_i}, L_{x_i}, L_{w_i} \in \mathbb{R}^+$. Then $L_{g_i}$, for a quadratic SBC in the form of $x_i^T P_i x_i$, a positive-definite matrix $P_i \in \mathbb{R}^{n_i \times n_i}$, is computed as

$$L_{g_i} = \max \{L_{g_{i,1}}, L_{g_{i,2}}, L_{g_{i,3}}, L_{g_{i,4}}, L_{g_{i,k}}\} = \max \{2s_i(\lambda_{\text{max}}(P_i) + \omega_i), (L_{g_{i,kx}}^2 + L_{g_{i,sw}}^2)^{\frac{1}{2}}\},$$

with

$$L_{g_{i,kx}} = 2\lambda_{\text{max}}(P_i)(L_{f_i} L_{x_i} + \kappa_i s_i), \quad L_{g_{i,sw}} = 2\lambda_{\text{max}}(P_i) L_{f_i} L_{w_i} + 2L_{\rho_i} s_i',$$

where $L_{\omega_i}, L_{\rho_i}, s_i, s_i'$ are, respectively, upper bounds on $\alpha_i, \rho_i$, and the norm of $x_i, w_i$, i.e., $\alpha_i \leq L_{\omega_i} \in \mathbb{R}^+, \rho_i \leq L_{\rho_i} \in \mathbb{R}^+, \|x_i\| \leq s_i \in \mathbb{R}^+, \forall x_i \in X_i$, and $\|w_i\| \leq s_i' \in \mathbb{R}^+, \forall w_i \in W_i$.

**Proof:** By defining

$$L_{g_{i,kx}} : \begin{cases} \max_{x_i \in X_i} \|\partial g_{i,kx} / \partial x_i\|, \\
\text{s.t.} \quad \|x_i\| \leq s_i, \|w_i\| \leq s_i', \quad L_{g_{i,sw}} : \begin{cases} \max_{w_i \in W_i} \|\partial g_{i,sw} / \partial w_i\|, \\
\text{s.t.} \quad \|x_i\| \leq s_i, \|w_i\| \leq s_i', \end{cases}
\end{cases}$$

one has $L_{g_{i,kx}} \leq 2\lambda_{\text{max}}(P_i)(L_{f_i} L_{x_i} + \kappa_i s_i), L_{g_{i,sw}} \leq 2\lambda_{\text{max}}(P_i) L_{f_i} L_{w_i} + 2L_{\rho_i} s_i'$. Then $L_{g_{i,k}} = (L_{g_{i,kx}}^2 + L_{g_{i,sw}}^2)^{\frac{1}{2}}$. Similarly for $g_{i_1} - g_{i_4}$, we have $L_{g_{i_1}} = L_{g_{i_3}} = L_{g_{i_4}} \leq 2s_i\lambda_{\text{max}}(P_i), L_{g_{i_2}} \leq 2s_i(\lambda_{\text{max}}(P_i) + \omega_i)$. Then $L_{g_i} = \max \{L_{g_{i_1}}, L_{g_{i_2}}, L_{g_{i_3}}, L_{g_{i_4}}, L_{g_{i,k}}\} = \max \{2s_i(\lambda_{\text{max}}(P_i) + \omega_i), (L_{g_{i,kx}}^2 + L_{g_{i,sw}}^2)^{\frac{1}{2}}\}$, which completes the proof.

In the next section, we consider multi-agent AV $\mathcal{A}$ as in Definition 2.2 and provide a compositional framework for the construction of BC for $\mathcal{A}$ using SBC of $A_i$. 
6. Compositionality Results for Multi-Agent AVs

In this section, we analyze multi-agent AV $A = \mathcal{A}(A_1, \ldots, A_M)$ by driving a small-gain type compositional condition and discussing how to construct a barrier certificate for the multi-agent AV based on sub-barrier certificates of its individual agents. The construed BC is useful to estimate the collision risk for multi-agent AVs according to Theorem 3.2. Before presenting the main compositionality result of the work, we define

$$\Lambda := \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_M)$$

with $\hat{\lambda}_i = 1 - \kappa_i$, and

$$\Delta := \{\hat{\delta}_{ij}\}$$

with $\hat{\delta}_{ii} = 0$, $\forall i \in \{1, \ldots, M\}$.

In the next theorem, we show how one can construct a BC for the multi-agent AV $A$ using SBC of $A_i$.

**Theorem 6.1.** Consider the multi-agent AV $A = \mathcal{A}(A_1, \ldots, A_M)$ induced by $M \in \mathbb{N}_{\geq 1}$ agents $A_i$. Suppose that each $A_i$ admits an SBC $B_i$ with a confidence of at least $1 - \beta_1^i - \beta_2^i$, as proposed in Theorem 5.1. If

$$\sum_{i=1}^M \lambda_i > \sum_{i=1}^M \gamma_i,$$  \hspace{1cm} (6.1)

$$1^T_M (-\Lambda + \Delta) := [\pi_1; \ldots; \pi_M]^T < 0,$$  

equivalently, $\pi_i < 0$, $\forall i \in \{1, \ldots, M\}$, \hspace{1cm} (6.2)

then

$$B(q, x) := \sum_{i=1}^M B_i(q_i, x_i)$$

is a BC for the multi-agent AV $A$ with a confidence of at least $1 - \sum_{i=1}^M \beta_1^i - \sum_{i=1}^M \beta_2^i$, and with

$$\gamma := \sum_{i=1}^M \gamma_i, \hspace{0.5cm} \lambda := \sum_{i=1}^M \lambda_i, \hspace{0.5cm} \psi := \sum_{i=1}^M \psi_i, \hspace{0.5cm} \kappa := 1 + \pi, \hspace{0.5cm} \text{with } \max_{1 \leq i \leq M} \pi_i < \pi < 0, \hspace{0.5cm} \text{and } \pi \in (-1, 0).$$

**Proof:** We first show that conditions (6.1), (6.2) hold. For any $x := [x_1; \ldots; x_M] \in X_0 = \prod_{i=1}^M X_{0i}$, we have

$$B(q, x) = \sum_{i=1}^M B_i(q_i, x_i) \leq \sum_{i=1}^M \gamma_i = \gamma,$$

and similarly for any $x := [x_1; \ldots; x_N] \in X_c = \prod_{i=1}^M X_{ci}$, one has

$$B(q, x) = \sum_{i=1}^M B_i(q_i, x_i) \geq \sum_{i=1}^M \lambda_i = \lambda,$$

satisfying conditions (6.1), (6.2) with $\gamma = \sum_{i=1}^M \gamma_i$ and $\lambda = \sum_{i=1}^M \lambda_i$. Note that $\lambda > \gamma$ according to (6.1).

Now we show that condition (6.3) holds, as well. By employing condition (6.0) and compositionality condition $1^T_M (-\Lambda + \Delta) < 0$, one can obtain the chain of inequalities in (6.3). By defining

$$\kappa_s := \max \left\{ s + 1^T_M (-\Lambda + \Delta) B(q, x) \middle| 1_M^T B(q, x) = s \right\}, \hspace{0.5cm} \psi := \sum_{i=1}^M \psi_i,$$  \hspace{1cm} (6.4)
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\[
\begin{align*}
\mathbb{E}\left[ \mathcal{B}(q, f(x, \varsigma)) \mid x \right] &= \mathbb{E}\left[ \sum_{i=1}^{M} \mathcal{B}_i(q_i, f_i(x_i, w_i, \varsigma_i)) \mid x, w \right] = \sum_{i=1}^{M} \mathbb{E}\left[ \mathcal{B}_i(q_i, f_i(x_i, w_i, \varsigma_i)) \mid x, w \right] \\
&= \sum_{i=1}^{M} \mathbb{E}\left[ \mathcal{B}_i(q_i, f_i(x_i, w_i, \varsigma_i)) \mid x_i, w_i \right] \leq \sum_{i=1}^{M} (\kappa_i \mathcal{B}_i(q_i, x_i) + \rho_i \|w_i\|^2 + \psi_i) \\
&\leq \sum_{i=1}^{M} (\kappa_i \mathcal{B}_i(q_i, x_i) + \rho_i \|w_i\|^2 + \psi_i) = \sum_{i=1}^{M} (\kappa_i \mathcal{B}_i(q_i, x_i) + \rho_i \sum_{j=1, i \neq j}^{M} \|x_j\|^2 + \psi_i) \\
&= \sum_{i=1}^{M} (\kappa_i \mathcal{B}_i(q_i, x_i) + \sum_{j=1, i \neq j}^{M} \lambda j \mathcal{B}_j(q_j, x_j) + \psi_i) \\
&= \mathcal{B}(q, x) + \mathbf{1}_M^\top (-\Lambda + \Delta) \left[ \mathcal{B}_1(q_1, x_1); \ldots; \mathcal{B}_M(q_M, x_M) \right] + \sum_{i=1}^{M} \psi_i \\
&\leq (1 + \pi) \mathcal{B}(q, x) + \psi.
\end{align*}
\]

where \( \mathcal{B}(q, x) = [\mathcal{B}_1(q_1, x_1); \ldots; \mathcal{B}_M(q_M, x_M)] \), condition (6.3) is also satisfied. We now show that \( \kappa = 1 + \pi \)
and \( 0 < \kappa < 1 \). Since \( \mathbf{1}_M^\top (-\Lambda + \Delta) := [\pi_1; \ldots; \pi_M]^\top < 0 \), and \( \max_{1 \leq i \leq M} \pi_i < \pi < 0 \) with \( \pi \in (-1, 0) \), one has

\[
\begin{align*}
\mathbf{s} + \mathbf{1}_M^\top (-\Lambda + \Delta) \mathcal{B}(q, x) &= \mathbf{s} + [\pi_1; \ldots; \pi_M]^\top \left[ \mathcal{B}_1(q_1, x_1); \ldots; \mathcal{B}_M(q_M, x_M) \right] \\
&= \mathbf{s} + \pi_1 \mathcal{B}_1(q_1, x_1) + \cdots + \pi_M \mathcal{B}_M(q_M, x_M) \leq \mathbf{s} + \pi \left( \mathcal{B}_1(q_1, x_1) + \cdots + \mathcal{B}_M(q_M, x_M) \right) \\
&= \mathbf{s} + \pi \mathbf{s} = (1 + \pi) \mathbf{s}.
\end{align*}
\]

Then \( \kappa \mathbf{s} \leq (1 + \pi) \mathbf{s} \), and accordingly \( \kappa \leq 1 + \pi \). Since \( \max_{1 \leq i \leq M} \pi_i < \pi < 0 \) with \( \pi \in (-1, 0) \), then \( 0 < \kappa = 1 + \pi < 1 \), which completes the proof. Note that since each agent \( A_i \) admits its SBC \( \mathcal{B}_i \) with a confidence of at least \( 1 - \beta_{1_i} - \beta_{2_i} \), one can readily conclude that \( \mathcal{B} \) is a BC for the multi-agent AV \( A \) with a confidence of at least \( 1 - \sum_{i=1}^{M} \beta_{1_i} - \sum_{i=1}^{M} \beta_{2_i} \).

**Remark 6.2.** If one is interested in enforcing the compositionality conditions during solving SOP\( \varsigma \) in (4.4), enforcing \( \mathbf{1}_M^\top (-\Lambda + \Delta) < 0 \) is not directly possible due to having a bilinearity between decision variables \( \rho_i \) and \( \alpha_j \). As a potential solution, one can a-priori fix either \( \rho_i \) or \( \alpha_j \) in order to resolve the bilinearity \( \frac{\rho_i}{\alpha_j} \), and then enforce compositionality conditions (6.1), (6.1) as two additional conditions while solving SOP\( \varsigma \) in (4.4).
In case the compositionality conditions (6.1), (6.2) are not satisfied, we propose, in the next section, an alternative compositional approach that does not require any compositionality conditions but at the cost of providing a potentially conservative collision risk for the multi-agent AVs.

7. Relaxed Compositional Approach

In this section, we propose an alternative compositional approach which is relax compared to the small-gain reasoning in the sense that (i) condition (3.6) is no longer needed, (ii) compositionality conditions (6.1), (6.2) are no more required, (iii) $\kappa_i$ in condition (3.9) is equal to one, and (iv) required number of data in Theorem 5.1 is reduced since $\kappa_i = 1$, $\forall i \in \{1, \ldots, M\}$, and accordingly $m_i = 1$. On the downside, the provided collision risk for the multi-agent AV is potentially conservative. In our proposed alternative setting, we omit $g_{i2}$ and modify some other conditions of ROP in (4.1) as

$$g_{i5} = \mathbb{E}[B_i(q_i, f_i(x_i, w_i, \varsigma_i)) \mid x_i, w_i] - B_i(q_i, x_i) - \rho_i \|w_i\|^2 - \psi_i, \quad g_{i6} = \gamma_i - \lambda_i - \psi_i,$$

(7.1)

for some $\psi_i < 0$. We now compute the collision risk for each individual agent $A_i$ via the following theorem.

**Theorem 7.1.** Consider an agent $A_i$ defined in Definition 2.1 and a finite time horizon $T \in \mathbb{N}_0$. Suppose that there exists a non-negative SBC $B_i$ satisfying conditions (3.6) - (3.9) with $\lambda_i > \gamma_i$. Then the collision risk for each agent $A_i$ is computed as

$$P\{A_i \mid T \varphi_i\} \leq \delta_i, \quad \text{with} \quad \delta_i = \frac{\gamma_i + \hat{\psi}_i T}{\lambda_i},$$

(7.2)

where $\hat{\psi}_i = \rho_i \|w_i\|_\infty^2 + \psi_i$.

**Proof:** The first part of the proof is similar to that of Theorem 5.2. Since

$$\mathbb{E}[B_i(f_i(x_i, w_i, \varsigma_i)) \mid x_i, w_i] \leq \kappa_i B_i(x_i) + \hat{\psi}_i, \quad \text{with} \quad \hat{\psi}_i = \rho_i \|w_i\|_\infty^2 + \psi_i,$$

(7.3)

the collision risk estimation for each agent $A_i$ in (7.2) follows directly by applying [25] Corolary 2-1, Chapter III] to (3.5) and employing, respectively, conditions (7.3) and (3.7).

In comparison with the small-gain compositional approach, one needs here to solve SOP $\varsigma$ in (4.4) without $g_{i2}$ and with $g_{i5}, g_{i6}$ as in (7.1). Moreover, the interconnection of $A_i, \forall i \in \{1, \ldots, M\}$, is defined here by an interconnection map $h : \prod_{i=1}^M X_i \to \prod_{i=1}^M W_i$, and accordingly, the interconnection constraint in (2.4) can be generalized to $[w_1; \ldots; w_M] = h(x_1, \ldots, x_M)$. Similar to Theorem 5.1 one can relate the optimal value of SOP $\varsigma$ to that of ROP in (4.1) with the new conditions in (7.1), and consequently, formally quantify collision risks of unknown agents $A_i$ based on the number of data as in (5.1) with $m_i = 1$, and a required level of confidence. We now propose the relaxed compositionality results of this section.
Theorem 7.2. Consider the multi-agent AV $A = (X, \zeta, f)$ composed of $M$ agents $A_i = (X_i, W_i, \varsigma_i, f_i)$, with a collision event $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_M$. Let the collision risk for each $A_i$ be at most $\delta_i$ (in the sense of Theorem 7.1) with a confidence of at least $1 - \beta_i$, with $\beta_i = \beta_1 + \beta_2, \forall i \in \{1, \ldots, M\}, i.e.,$

$$P \left\{ P \{ A_i \models \varphi_i \} \leq \delta_i \right\} \geq 1 - \beta_i. $$

Then the collision risk for the multi-agent AV $A$ is at most $\sum_{i=1}^{M} \delta_i$ with a confidence of at least $1 - \sum_{i=1}^{M} \beta_i$, i.e.,

$$P \left\{ P \{ A_1 \models \varphi_1 \cup \cdots \cup A_M \models \varphi_M \} \leq \sum_{i=1}^{M} \delta_i \right\} \geq 1 - \sum_{i=1}^{M} \beta_i. $$

Proof: Since

$$P \{ A_1 \models \varphi_1 \cup \cdots \cup A_M \models \varphi_M \} \leq P \{ A_1 \models \varphi_1 \} + \cdots + P \{ A_M \models \varphi_M \},$$

we have

$$P \{ A_1 \models \varphi_1 \cup \cdots \cup A_M \models \varphi_M \} \leq \delta_1 + \cdots + \delta_M = \sum_{i=1}^{M} \delta_i.$$ By defining collision events $E_i$ and $E$ as

$$E_i = \left\{ P \{ A_i \models \varphi_i \} \leq \delta_i \right\}, \forall i \in \{1, \ldots, M\},$$

$$E = \left\{ P \{ A_1 \models \varphi_1 \cup \cdots \cup A_M \models \varphi_M \} \leq \sum_{i=1}^{M} \delta_i \right\},$$

one has $P \{ E_i \} \geq 1 - \beta_i, \forall i \in \{1, \ldots, M\}$. We now aim at computing a lower bound for $P \{ E \}$. We have

$$P \{ \bar{E}_1 \cup \cdots \cup \bar{E}_M \} \leq P \{ \bar{E}_1 \} + \cdots + P \{ \bar{E}_M \} \leq \beta_1 + \cdots + \beta_M = \sum_{i=1}^{M} \beta_i,$$

with $\bar{E}_i$ being complements of $E_i, \forall i \in \{1, \ldots, M\}$. Since $E \subseteq \bar{E}_1 \cup \cdots \cup \bar{E}_M$, one has $P \{ E \} \leq P \{ \bar{E}_1 \cup \cdots \cup \bar{E}_M \}$. By employing (7.4), one has $P \{ \bar{E} \} \leq \sum_{i=1}^{M} \beta_i$. Subsequently, $P \{ E \} \geq 1 - \sum_{i=1}^{M} \beta_i$, which completes the proof.

As it can be observed, the confidence level for the both small-gain and relaxed compositional approaches is the same, i.e., $1 - \sum_{i=1}^{M} \beta_1, - \sum_{i=1}^{M} \beta_2$. In the relaxed compositional approach, the collision risk for the multi-agent AV is computed based on the linear combination of estimated collision risks for individual agents, i.e., $\sum_{i=1}^{M} \delta_i$. Whereas using the small-gain compositional approach, we first construct the overall BC based on SBC of agents and then estimate the collision risk of the multi-agent AV via results of Theorem 3.2. Accordingly, the estimated collision risk via the small-gain compositional approach is potentially less conservative but at the cost of satisfying some additional conditions.

In the next section, we develop our data-driven approaches for deterministic multi-agents AVs without any stochasticity $\varsigma_i$. 
8. Deterministic (Multi)Agent AVs

In this section, we consider dynamics of each agent as a discrete-time deterministic system given by

$$A_i: x_i(k + 1) = f_i(x_i(k), w_i(k)), \quad k \in \mathbb{N}, \quad (8.1)$$

with $f_i: X_i \times W_i \to X_i$, and represent it with $A_i = (X_i, W_i, f_i)$. The multi-agent AV $A$ without interactions $w_i$, constructed as a composition of several agents $A_i$ with interactions, can be represented by $A = (X, f)$ with $f : X \to X$, and $A: x(k + 1) = f(x(k))$. We now define barrier certificates for deterministic multi-agent AVs as the next definition.

Definition 8.1. Given the deterministic multi-agent AV $A = (X, f)$ with initial and collision sets $X_0, X_c \subseteq X$, a function $B: X \to \mathbb{R}$ is called a barrier certificate (BC) for $A$ if there exist $\gamma, \lambda \in \mathbb{R}$, with $\lambda > \gamma$, and $0 < \kappa < 1$, such that conditions (8.1), (8.2) are satisfied, and

$$B(f(x)) \leq \kappa B(x), \quad \forall x \in X. \quad (8.2)$$

Now we employ Definition 8.1 and quantify the collision risk for multi-agent AVs via the next theorem.

Theorem 8.2. Consider a deterministic multi-agent AV $A = (X, f)$. Suppose $B$ is a BC for $A$ as in Definition 8.1. Then the collision risk is zero for the multi-agent AV within an infinite time horizon, i.e., $x_{x_0}(k) \cap X_c = \emptyset$ for any $x_0 \in X_0$ and any $k \in \mathbb{N}$.

Proof: We show the statement based on contradiction. Let $x_{x_0}$ of $A$ start at some $x_0 \in X_0$. Suppose $x_{x_0}$ reaches inside $X_c$. Based on conditions (8.1), (8.2), one has $B(x(0)) \leq \gamma$ and $B(x(k)) \geq \lambda$ for some $k \in \mathbb{N}$. Since $B(x(k))$ is a BC and by employing condition (8.2), one can conclude that $\lambda \leq B(x(k)) \leq B(x(0)) \leq \gamma$. This contradicts condition $\lambda > \gamma$, which completes the proof.

Since searching for the BC as in Definition 8.1 is computationally expensive, we now define sub-barrier certificates for deterministic agents $A_i$ as the following definition.

Definition 8.3. Consider a deterministic agent $A_i = (X_i, W_i, f_i)$, and $X_{0i}, X_{ci} \subseteq X_i$ as its initial and collision sets, respectively. A function $B_i: X_i \to \mathbb{R}$ is called a sub-barrier certificate (SBC) for $A_i$ if there exist $\gamma_i, \lambda_i \in \mathbb{R}$, $\alpha_i, \rho_i \in \mathbb{R}_0^+$, and $0 < \kappa_i < 1$, such that conditions (8.6)-(8.8) are satisfied, and $\forall x_i \in X_i, \forall w_i \in W_i$,

$$B_i(f_i(x_i, w_i)) \leq \kappa_i B_i(x_i) + \rho_i \|w_i\|^2. \quad (8.3)$$

Here, we assume that the transition map $f_i$ in (8.1) is unknown. We similarly recast the conditions of SBC as the proposed ROP in (4.1) without $g_{i1}$, and with

$$g_{i5}(x_i, \Theta_i) = B_i(q_i, f_i(x_i, w_i)) - \kappa_i B_i(q_i, x_i) - \rho_i \|w_i\|^2. \quad (8.4)$$
Since the proposed ROP in (4.1) has infinitely many constraints and a precise transition map \( f_i \) is needed for solving the problem, we employ the proposed SOP_N in (4.3) instead of solving the ROP in (4.1). Note that we do not need to employ SOP_\( \varsigma \) in (4.4) since there is no stochasticity inside the model. We denote the optimal value of the objective function SOP_N by \( \eta_N^* \). Similar to Theorem 5.1 we propose the next theorem to relate the optimal value of SOP_N in (4.3) to that of ROP in (4.1), and accordingly, formally quantify the SBC of unknown agents \( A_i \) based on the number of data and a required level of confidence.

**Theorem 8.4.** Consider an unknown deterministic agent \( A_i \) as in (8.1), and initial and collision regions \( X_0 \) and \( X_{c,i} \), respectively. Let \( g_{i_2}, g_{i_4} \) be Lipschitz continuous with respect to \( x_i \), and \( g_{i_5} \) as in (8.4) be Lipschitz continuous with respect to \( x_i, w_i \), with Lipschitz constants \( \mathcal{L}_{g_{i_2}}, \mathcal{L}_{g_{i_4}}, \mathcal{L}_{g_{i_5}} \), respectively. Consider the corresponding SOP_N in (4.3) with its associated optimal value \( \eta_N^* \) and solution \( \Theta_i^* = [\gamma_i^*; \lambda_i^*; \rho_i^*; q_i^*; \cdots; q_i^*] \), with \( N_i \geq N_i(\varepsilon_2, \beta_2) \), as in (5.1) with \( \varepsilon_1, \beta_2 \in [0, 1] \) where \( \varepsilon_1, \beta_2 \leq \mathcal{L}_{g_{i_5}} = \max \{\mathcal{L}_{g_{i_2}}, \mathcal{L}_{g_{i_3}}, \mathcal{L}_{g_{i_4}}, \mathcal{L}_{g_{i_5}}\} \). If \( \eta_N^* + \varepsilon_1 \leq 0 \), then the solution \( \Theta_i^* \) is a feasible solution for ROP in (4.1) with a confidence of at least \( 1 - \beta_2 \).

Proof of Theorem 8.4 is similar to that of Theorem 5.1 and is omitted here.

### 8.1. Small-Gain Compositional Approach.

Given a deterministic multi-agent AV \( A = \{A_1, \ldots, A_M\} \), we show in the next theorem that how to construct a BC for the multi-agent AV using SBC of \( A_i \).

**Theorem 8.5.** Consider the deterministic multi-agent AV \( A = \{A_1, \ldots, A_M\} \) induced by \( M \in \mathbb{N}_{\geq 1} \) agents \( A_i \) as in (8.1). Suppose that each \( A_i \) admits an SBC \( B_i \) with a confidence of at least \( 1 - \beta_2 \), as proposed in Theorem 8.4. If the compositional conditions (6.1), (6.2) are satisfied, then

\[
B(q, x) := \sum_{i=1}^{M} B_i(q_i, x_i)
\]

is a BC for the multi-agent AV \( A \) with a confidence of at least \( 1 - \sum_{i=1}^{M} \beta_2 \), and with

\[
\gamma := \sum_{i=1}^{M} \gamma_i, \quad \lambda := \sum_{i=1}^{M} \lambda_i, \quad \kappa := 1 + \pi, \quad \text{with} \quad \max_{1 \leq i \leq M} \pi_i < \pi < 0, \quad \text{and} \quad \pi \in (-1, 0).
\]

Proof of Theorem 8.5 is similar to that of Theorem 6.1 and is omitted here.

### 8.2. Relaxed Compositional Approach.

In the case that the compositional conditions are not satisfied, we propose a relax compositional approach, in which (i) condition (3.6) is no longer needed, (ii) compositional conditions (6.1), (6.2) are no more required, (iii) \( \kappa_i \) in condition (8.3) is equal to one, and (iv) required number of data in Theorem 8.4 is reduced, but at the cost of providing the collision risk estimation for multi-agent AVs in finite time horizons. To do so, we first propose a new definition of SBC for deterministic agents \( A_i \) as the following.
Definition 8.6. Consider a deterministic agent $A_i = (X_i, W_i, f_i)$, and $X_{0i}, X_{ci} \subseteq X_i$ as its initial and collision sets, respectively. A function $B_i : X_i \rightarrow \mathbb{R}$ is called a sub-barrier certificate (SBC) for $A_i$ if there exist $\alpha_i, \beta_i \in \mathbb{R}_0^+$, $\gamma_i, \lambda_i \in \mathbb{R}$ with $\gamma_i + \beta_i \| w_i \|_{\infty} \mathcal{T}_i < \lambda_i$, for a finite time horizon $T_i \in \mathbb{N}_0$, such that conditions (3.7), (3.8) are satisfied, and $\forall x_i \in X_i, \forall w_i \in W_i$,

$$B_i(f_i(x_i, w_i)) \leq B_i(x_i) + \beta_i \| w_i \|^2. \quad (8.5)$$

Now we employ Definition 8.6 and estimate the collision risk for each agent $A_i$ in a finite time horizon via the next theorem.

Theorem 8.7. Consider a deterministic agent $A_i$ in (8.1) and a finite time horizon $T_i \in \mathbb{N}_0$. Suppose $B_i$ is an SBC for $A_i$ as in Definition 8.6. Then, the collision risk is zero for each agent $A_i$ within the finite time horizon $T_i = \frac{\lambda_i - \gamma_i}{\beta_i \| w_i \|_{\infty}^2}$, i.e., $x_{0i}(k) \cap X_{ci} = \emptyset$ for any $x_{0i} \in X_{0i}$ and any $k \in [0, T_i)$, with $T_i = \frac{\lambda_i - \gamma_i}{\beta_i \| w_i \|_{\infty}^2}$.

Proof: According to (8.5), since $B_i(x_i(k))/B_i(x_i(0)) \leq \beta_i \| w_i \|^2 \mathcal{T}_i$, one can recursively infer that $B_i(x_i(k)) - B_i(x_i(0)) \leq \beta_i \| w_i \|^2 k$. By employing condition (3.7), we have $B_i(x_i(k)) \leq \gamma_i + \beta_i \| w_i \|^2 k$. Now since $\gamma_i + \beta_i \| w_i \|^2 \mathcal{T}_i < \lambda_i$, one can readily conclude that $B_i(x_i(k)) < \lambda_i$. From condition (3.8), one gets $x_i(k) \notin X_{ci}$ for any $k \in [0, T_i)$. This implies that the collision risk is zero for each agent $A_i$ for the finite time horizon $T_i = \frac{\lambda_i - \gamma_i}{\beta_i \| w_i \|_{\infty}^2}$, which completes the proof.

We similarly recast the conditions of SBC as the proposed ROP in (4.1) without $g_{i1}, g_{i2}$, and with

$$g_{i5}(x_i, \Theta_i) = B_i(q_i, f_i(x_i, w_i)) - B_i(q_i, x_i) - \beta_i \| w_i \|^2, \quad g_{i6}(x_i, \Theta_i) = \gamma_i + \beta_i \| w_i \|^2 \mathcal{T} - \lambda_i - \gamma_i.$$

for some $g_i < 0$. Similar to Theorem 8.3, one can relate the optimal value of SOPN in (4.3) to that of ROP in (4.1), and accordingly, quantify the SBC of unknown agents $A_i$ based on the number of data as in (5.1) with $m_i = 1$, and a required level of confidence. We now propose our relaxed compositional approach for multi-agent AVs $A$ with unknown deterministic dynamics via the next theorem.

Theorem 8.8. Consider the deterministic multi-agent AV $A = (X, f)$ composed of $M$ agents $A_i = (X_i, W_i, f_i)$, $i \in \{1, \ldots, M\}$, with a collision event $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_M$. Let the collision risk for each $A_i$ be zero (in the sense of Theorem 8.7) for a finite time horizon $T_i = \frac{\lambda_i - \gamma_i}{\beta_i \| w_i \|_{\infty}^2}$ with a confidence of at least $1 - \beta_2$, $\forall i \in \{1, \ldots, M\}$, i.e., $P\left\{A_i \neq \varphi_i \right\} \geq 1 - \beta_2$. Then the collision risk for the multi-agent AV $A$ is zero for the finite time horizon $T = \min_{i \in \{1, \ldots, M\}} T_i$, with a confidence of at least $1 - \sum_{i=1}^M \beta_2$, i.e.,

$$P\left\{A_1 \neq \varphi_1 \cap \cdots \cap A_M \neq \varphi_M \right\} \geq 1 - \sum_{i=1}^M \beta_2.$$

Proof of Theorem 8.8 is similar to that of Theorem 7.2 and is omitted here.
The confidence level for multi-agent AVs with unknown deterministic dynamics in the both small-gain and relaxed compositional approaches is the same, i.e., $1 - \sum_{i=1}^{M} \beta_i$. However, the collision risk in the relaxed compositional approach is estimated in finite time horizons, whereas small-gain compositional approach estimates the collision risk in infinite time horizons but at the cost of fulfilling some additional conditions.

9. Case Study: Vehicle Platooning

In this section, we illustrate our proposed results by applying them to a vehicle platooning in a network of 100 vehicles with 1 leader and 99 followers (see Figure 3). The model of this case study is adapted from [28]. The evolution of states can be described by the following multi-agent AV

$$
A: x(k + 1) = Ax(k) + u(k) + R\varsigma(k),
$$

where $A$ is a block matrix with diagonal blocks $A_x$, and off-diagonal blocks $A_{i(i-1)} = A_w, i \in \{2, \ldots, M\}$, where

$$
A_x = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix},
A_w = \begin{bmatrix}
0 & \varpi \\
0 & 0
\end{bmatrix},
$$

with $\varpi = 0.01$ being the interconnection degree, and all other off-diagonal blocks being zero matrices of appropriate dimensions. Moreover, $R$ is a partitioned matrix with main diagonal blocks $\bar{R} = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.06 \end{bmatrix}$, and all other off-diagonal blocks being zero matrices of appropriate dimensions. Furthermore, $x(k) = [x_1(k); \ldots; x_M(k)]$, $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_M(k)]$, and $u(k) = [u_1(k); \ldots; u_M(k)]$, where $u_i(k)$ is the controller input for each agent.

By considering each individual agent $A_i$ described as

$$
A_i: x_i(k + 1) = A_xx_i(k) + u_i(k) + A_wu_i(k) + \bar{R}\varsigma_i(k),
$$

one can readily verify that $A = \mathcal{I}(A_1, \ldots, A_M)$, where $w_i(k) = [0; w_{i(i-1)}(k)], i \in \{1, \ldots, M\}$, (with $w_{i(i-1)} = x_{i-1}, w_{1,0} = 0$). The state of the $i$-th vehicle is defined as $x_i = [d_i; \nu_i]$, for any $i \in \{1, \ldots, M\}$, where $d_i$ denotes the relative distance between the vehicle $i$ and its proceeding vehicle $i-1$ (the 0-th vehicle represents

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vehicle_platooning_diagram}
\caption{Vehicle platooning in a network of 100 vehicles.}
\end{figure}
the leader), and \( \nu_i \) is its velocity in the leader’s frame. The overall control objective in vehicle platooning is for each vehicle to adjust its speed in order to maintain a safe distance from the vehicle ahead. We assume that the controller for each agent is already designed and deployed as \( u_i = [-0.2d_i + 1.1\nu_i + 0.1; 0.01d_i - 0.9\nu_i] \), for any \( i \in \{1, \ldots, M\} \).

Regions of interest for each vehicle are given as \( X_i = [0, 0.7] \times [-3.55, 0.2], X_{0i} = [0.35, 0.7] \times [-0.9, 0.2] \), and \( X_{c_i} = [0, 0.3] \times [-3.55, -2.9] \), for all \( i \in \{1, \ldots, M\} \). We assume the model of each agent and the distribution of the noise are both unknown. The main goal is to compositionally construct a BC for the multi-agent AV based on SBCs of individual agents via data collected from trajectories of agents and solving SCP\(\varsigma \) (4.3).

Accordingly, we formally estimate the collision risk for the multi-agent AV within a finite time horizon with some desirable confidence.

We first fix the structure of our sub-barrier certificates as \( B_i(q_i, x_i) = q_i, d_i^2 + q_i d_i^3 + q_i d_i \nu_i + q_i \nu_i^2 + q_i \nu_i^3 \), for all \( i \in \{1, \ldots, 100\} \). According to Algorithm 11 we fix the threshold \( \varepsilon_{1i} = 0.08 \) and the confidence \( \beta_{2i} = 10^{-4} \), a-priori. In order to reduce the number of decision variables, and accordingly, reduce the number of data in (5.1) required for solving the SOP\(\varsigma \) (4.3), we a-priori fix \( \lambda_i = 10, \psi_i = 10^{-2}, \alpha_i = 10^{-4}, \rho_i = 9 \times 10^{-7} \), for all \( i \in \{1, \ldots, 100\} \). Now we need to compute \( L_{g} \) which is required for computing the minimum number of data. We compute \( s_i = 3.8148, s'_i = 3.15 \), and construct matrix \( P_i \) based on coefficients of the SBC as

\[
P_i = \begin{bmatrix}
q_{i1} & 0 & q_{i4} & 0 \\
0 & q_{i2} & 0 & 0 \\
\frac{q_{i3}}{2} & 0 & q_{i6} & 0 \\
0 & 0 & 0 & q_{i4}
\end{bmatrix}.
\]

By enforcing \(-0.001 \leq q_{i1}, q_{i2}, q_{i3}, q_{i5} \leq 0.001, -0.14 \leq q_{i4} \leq 0.14 \), we ensure that \( \lambda_{\text{max}}(P_i) \leq 0.14 \) as discussed in Remark 5.3. We assume that \( \kappa_i \in \{0.9, 0.99\} \) with the cardinality \( m_i = 2 \). Then according to Lemma 5.2 we compute \( L_{g_{ik}} = 1.7804 \), and accordingly, \( \varepsilon_{2i_k} = (\frac{\varepsilon_{1i}}{2q_{ik}})^{\nu_i + p_i} = (\frac{0.08}{1.7804})^3 = 9.0723 \times 10^{-5} \). Now we have all the required ingredients to compute \( N_i \). The minimum number of data required for solving SOP\(\varsigma \) in (4.4) is computed as:

\[
N_i(\varepsilon_{2i_k}, \beta_{2i_k}) \geq \min \left\{ N_i \in \mathbb{N} \mid \sum_{k=1}^{2} \sum_{j=0}^{6} \left( \begin{array}{c} N_i \\ j \end{array} \right) \varepsilon_{2i_k} (1 - \varepsilon_{2i_k})^{N_i - j} \leq 10^{-4} \right\} = 244993.
\]

Now we need to compute \( \tilde{N}_i \) which is required for solving the SOP\(\varsigma \) (4.4) \(\text{i.e., condition } \tilde{g}_{i_k} \). According to Lemma 4.2 we fix \( \mu_i = 0.08 \) with a-priori confidence \( \beta_{1i} = 10^{-4} \) and compute \( \tilde{N}_i = 11 \). We refer the interested reader to [24] for more details on the quantification of \( \tilde{N}_i \). Leveraging the computed parameters, the SOP\(\varsigma \)
in (4.4) is solved with $\kappa_i = 0.99$ and the following decision variables:

$$B_i(q_i, x_i) = -0.0008d_i^2 + 0.001d_i + 0.0001d_i\nu_i + 0.14\nu_i^4 - 0.0006\nu_i^2, \quad \eta_i^* = -0.085, \quad \gamma_i^* = 0.1. \quad (9.1)$$

Since $\eta_i^* + \varepsilon_i = -0.005 \leq 0$, according to Theorem 5.1 one can guarantee that the constructed SBC via collected data together with other decision variables in (9.1) are valid for the original ROP (4.1) with a confidence of at least $1 - \beta_1 - \beta_2 = 1 - 10^{-4} - 10^{-4}$. Satisfaction of conditions (3.6)-(3.8) via constructed SBC from data is illustrated in Figures 4, 5.

**Figure 4.** Satisfaction of condition (3.6). As observed, this condition is negative for all ranges of $d_i \in [0, 0.7]$ and $\nu_i \in [-3.55, 0.2]$.

**Figure 5.** Satisfaction of conditions (3.7), (3.8) based on SBC of each unknown vehicle. Pink and purple boxes are initial and collision sets, respectively. As seen, the initial set is inside the $\gamma_i$-level set of the SBC (i.e., $B_i(q_i, x_i) = \gamma_i$) and the collision set is outside the $\lambda_i$-level set of the SBC (i.e., $B_i(q_i, x_i) = \lambda_i$).

We now proceed with Theorem 6.1 to construct a BC for the multi-agent AV with a level of confidence using SBCs of individual agents constructed from data. By constructing $\Lambda := \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_M)$ with $\hat{\lambda}_i = 0.01$, and
\[ \Delta := \{ \hat{\delta}_{ij} \} \text{ with } \hat{\delta}_{ij} = \frac{\mathbf{M}_{ij}}{\alpha_j} = 9 \times 10^{-3}, \] one can readily verify that the compositionality condition 
\begin{equation}
I_M^T (-\Lambda + \Delta) < 0
\end{equation}
is satisfied. Moreover, the compositionality condition (6.1) is also met since
\[ \lambda_i > \gamma_i, \forall i \in \{1, \ldots, 100\}. \]
Then by employing the results of Theorem 6.1 one can conclude that
\[ B(q, x) = \sum_{i=1}^{100} (-0.0008d_i^2 + 0.001d_i^4 + 0.0001d_i \nu_i + 0.14\nu_i^4 - 0.0006\nu_i^6) \]
is a BC for the multi-agent AV with \( \gamma = 10, \lambda = 1000, \kappa = 0.99 \) and \( \psi = 0.01 \), with a confidence of at least
\[ 1 - \sum_{i=1}^{100} \beta_1 = 98\%. \]
By employing Theorem 3.2, we guarantee that the collision risk for the multi-agent AV is at most 1% with a confidence of at least 98% during the time horizon \( T = 100 \), \text{i.e.,}
\begin{equation}
P^{N} \{ \mathcal{P} \{ \mathcal{A} \models_{100} \varphi \} \leq 0.01 \} \geq 0.98. \tag{9.2} \end{equation}
In order to verify our results, we assume that we have access to the model of agents and plot the closed-loop state (relative distance and velocity) trajectories of a representative vehicle with 10 different noise realizations as in Figure 6. As it can be observed, none of 10 state trajectories violates the safety specification, which is in accordance with our collision risk guarantee in (9.2).

![Figure 6. Closed-loop (relative distance and velocity) trajectories of a representative vehicle with 10 different noise realizations in a network of 100 vehicles.](image)
10. Discussion

In this work, we proposed a compositional data-driven scheme for formal estimation of collision risks for stochastic multi-agent AVs while providing a-priori guaranteed confidence on our estimation. We first reformulated the original collision risk problem as a robust optimization program (ROP), and then provided a scenario optimization program (SOP) corresponding to the original ROP by collecting finite numbers of data from trajectories of each agent. We then built a probabilistic relation between the optimal value of SOP and that of ROP, and accordingly, constructed the sub-barrier certificate for each unknown agent based on the number of data and a required level of confidence. By leveraging a compositional technique based on small-gain reasoning, we quantified the collision risk for the multi-agent AVs based on constructed sub-barrier certificates of individual agents. In the case that the compositionality condition is not satisfied, we proposed a relaxed-version of compositional results without requiring any compositionality conditions. We finally developed our techniques for non-stochastic multi-agent AVs. We demonstrated the effectiveness of our proposed results by applying them to a vehicle platooning in a network of 100 vehicles.

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