A note on the Capelli identities for symmetric pairs of Hermitian type

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Abstract

We get several identities of differential operators in determinantal form. These identities are non-commutative versions of the formula of Cauchy-Binet or Laplace expansions of determinants, and if we take principal symbols, they are reduced to such classical formulas. These identities are naturally arising from the generators of the rings of invariant differential operators over symmetric spaces, and have strong resemblance to the classical Capelli identities. Thus we call those identities the Capelli identities for symmetric pairs.

1 Introduction

In this article we give several identities of differential operators associated to see-saw pairs of reductive Lie groups. These identities look similar to the classical Capelli identities ([1, 2]):

\[ \det(E_{i,j} + (n-j)\delta_{i,j}) = \det(x_{i,j}) \det(\partial_{i,j}) = \det(E'_{i,j} + (j-1)\delta_{i,j}), \]

where \( E_{i,j} = \sum_{k=1}^{n} x_{k,i} \partial_{k,j}, \quad E'_{i,j} = \sum_{k=1}^{n} x_{j,k} \partial_{i,k}, \quad \partial_{i,j} = \partial / \partial x_{i,j}. \)

The Capelli identities give two different determinantal expressions of the same differential operator, which can be interpreted as the image of the centers of universal enveloping algebras corresponding to a dual pair of reductive Lie groups (see [2, 4, 7]). In this setting, the original Capelli's identity corresponds to the dual pair \( GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \), and recently, there appear many kinds of generalizations in addition to the pioneering work of [2, 4]. See, for example, the references [13, 14, 5, 6, 7, 11, 9, 15, 16].

Our identities here also have a similar interpretation. In fact, they can be considered as different expressions of the same differential operator as the image of invariant differential operators on two symmetric spaces associated to a see-saw pair. Thus we call our identities the Capelli identities for symmetric pairs.
pairs. Note that our differential operators do not necessarily come from the center of the universal enveloping algebras, and yet they have a determinantal expressions. This is a remarkable difference of our identities from the classical Capelli identities.

To explain our identities more precisely, we need some notation. Let \( \mathfrak{g}_0 = \mathfrak{sp}_{2N}(\mathbb{R}) \) be a real symplectic Lie algebra, and let \( \mathfrak{g}_0, \mathfrak{t}_0, \mathfrak{m}_0 \) and \( \mathfrak{h}_0 \) be real reductive Lie subalgebras of \( \mathfrak{g}_0 \). We assume that they form a see-saw pair:

\[
\begin{array}{c}
\mathfrak{g}_0 \\
\cup \\
\mathfrak{t}_0 \\
\cup \\
\mathfrak{h}_0 \\
\times \\
\mathfrak{m}_0
\end{array}
\]

where \((\mathfrak{g}_0, \mathfrak{t}_0)\) and \((\mathfrak{m}_0, \mathfrak{h}_0)\) are symmetric pairs, \(\mathfrak{g}_0 \leftrightarrow \mathfrak{h}_0\) and \(\mathfrak{m}_0 \leftrightarrow \mathfrak{t}_0\) are dual pairs in \(\mathfrak{g}_0\).

Here \(\mathfrak{g}_0 \leftrightarrow \mathfrak{h}_0\) is called a dual pair if they are mutual commutants in the symplectic Lie algebra \(\mathfrak{g}_0\), and \((\mathfrak{g}_0, \mathfrak{t}_0)\) is called a symmetric pair if \(\mathfrak{t}_0\) is a fixed-point subalgebra of \(\mathfrak{g}_0\) under a non-trivial involution.

Recall the symplectic Lie algebra \(\mathfrak{g}_0\) has the Weil representation \(\omega\) (or also called the oscillator representation) acting on the polynomial ring \(\mathbb{C}[V]\) over a Lagrangian vector space \(V\) (see [3], [4]). In fact, this is a representation of the two-fold double cover of the symplectic group \(S_{2N}(\mathbb{R})\), which is called a metaplectic group. But we only consider the infinitesimal version of this, realized as a Harish-Chandra module. Thus the complexifications of Lie algebras in see-saw pairs act on \(\mathbb{C}[V]\) through the (differential of) the Weil representation \(\omega\).

We denote the complexification of \(\mathfrak{g}_0\) by \(\mathfrak{g}\) etc., and let \(\mathcal{PD}(V)\) be the ring of differential operators with polynomial coefficients on a vector space \(V\). Then we have the following picture:

\[
U(\mathfrak{g}) \xrightarrow{\omega} \mathcal{PD}(V) \xleftarrow{\omega} U(\mathfrak{m}).
\]

Let \(K\) and \(H\) be the complex Lie groups corresponding to \(\mathfrak{t}\) and \(\mathfrak{h}\) respectively. Thanks to the very definition of the dual pair \((\mathfrak{g}_0, \mathfrak{h}_0)\), the image of \(U(\mathfrak{g})\) is \(H\)-invariant. Similarly the image of \(U(\mathfrak{m})\) is \(K\)-invariant. Thus we have the following picture by restricting to invariant subalgebras:

\[
U(\mathfrak{g})^K \xrightarrow{\omega} \mathcal{PD}(V)^{K \times H} \xleftarrow{\omega} U(\mathfrak{m})^H. \tag{1}
\]

Due to the result of Howe [3], both \(U(\mathfrak{g})^K\) and \(U(\mathfrak{m})^H\) are mapped onto \(\mathcal{PD}(V)^{K \times H}\). In particular, we have \(\omega(U(\mathfrak{g})^K) = \omega(U(\mathfrak{m})^H)\).

In our previous paper [10] we studied the case where \((\mathfrak{g}_0, \mathfrak{t}_0)\) is a Hermitian symmetric pair, and \(\mathfrak{m}_0\) is a compact Lie algebra. Let \(\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0\) be a Cartan decomposition of \(\mathfrak{g}_0\). The algebra of \(K\)-invariants \(S(\mathfrak{p})^K\) of the symmetric algebra \(S(\mathfrak{p})\) is finitely generated. We take \(K\)-invariant elements \(X_d \in U(\mathfrak{g})^K\) \((d = 1, 2, \ldots)\) whose principal symbols are the generators of \(S(\mathfrak{p})^K\). Then \(\omega(X_d)\) is in \(\mathcal{PD}(V)^{K \times H}\), and there exists an inverse image in \(U(\mathfrak{m})^H\) thanks to \(\omega(U(\mathfrak{g})^K) = \omega(U(\mathfrak{m})^H)\). In [10], we determined an inverse image \(C_d \in U(\mathfrak{m})^H\) satisfying

\[
\omega(X_d) = \omega(C_d) \quad (X_d \in U(\mathfrak{g})^K, \ C_d \in U(\mathfrak{m})^H). \tag{2}
\]
We call this formula a *Capelli identity for a symmetric pair*, and $X_d$ and $C_d$ are *Capelli elements*.

In this article, we flip the role of symmetric pairs $(\mathfrak{g}_0, \mathfrak{t}_0)$ and $(\mathfrak{m}_0, \mathfrak{h}_0)$, and establish the similar identities.

Namely, let $(\mathfrak{m}_0, \mathfrak{h}_0)$ be a Hermitian symmetric pair, and assume $\mathfrak{g}_0$ to be compact. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the ($\pm 1$)-eigenspace decomposition with respect to the involution. We then take $K$-invariant elements $X_d \in U(\mathfrak{g})_K$ whose principal symbols are generators of $S(\mathfrak{p})_K$, and try to find $H$-invariant elements $C_d \in U(\mathfrak{m})_H$ satisfying (2). There are three see-saw pairs which fit into our setting:

| See-saw pairs with $\mathfrak{m}_0$ Hermitian type, $\mathfrak{g}_0$ compact | $\mathfrak{s}_0$ | $\mathfrak{g}_0$ | $\mathfrak{t}_0$ | $\mathfrak{m}_0$ | $\mathfrak{h}_0$ |
|---|---|---|---|---|---|
| Case $\mathbb{R}$ | $\mathfrak{sp}_{2m,n}(\mathbb{R})$ | $\mathfrak{u}_m$ | $\mathfrak{o}_m(\mathbb{R})$ | $\mathfrak{sp}_{2n}(\mathbb{R})$ | $\mathfrak{u}_n$ |
| Case $\mathbb{C}$ | $\mathfrak{sp}_{2m(p+q)}(\mathbb{R})$ | $\mathfrak{u}_m \oplus \mathfrak{u}_m$ | $\mathfrak{u}_m$ | $\mathfrak{u}_{p,q}$ | $\mathfrak{u}_p \oplus \mathfrak{u}_q$ |
| Case $\mathbb{H}$ | $\mathfrak{sp}_{4m}(\mathbb{R})$ | $\mathfrak{u}_{2m}$ | $\mathfrak{usp}_m$ | $\mathfrak{o}_{2n}^*$ | $\mathfrak{u}_n$ |

We get the Capelli identities for symmetric pairs in a complete form only for Case $\mathbb{C}$ in Table (3). For Cases $\mathbb{R}$ and $\mathbb{H}$, we only have explicit expressions of $\omega(X_d) \in PD(V)$, and do not get $C_d \in U(\mathfrak{m})_H$ up to now.

In Section 2 we prove the Capelli identities for Case $\mathbb{C}$, and in Sections 3 and 4 we give the equations up to $PD(V)$ for Cases $\mathbb{R}$ and $\mathbb{H}$, respectively. In addition, we show a formula for coefficients appearing in the Capelli identities in Appendix A.

## 2 Case $\mathbb{C}$

Here we establish two kinds of the Capelli identities for symmetric pairs for Case $\mathbb{C}$ in Table (3). The first identity in Subsection 2.2 has a simple expression as differential operators in $PD(V)$, but their inverse images (Capelli elements) $X_d \in U(\mathfrak{g})_K$ are more complicated than the second one in Subsection 2.3. The second identity has a simple Capelli element $X_d \in U(\mathfrak{g})_K$, for which it is easy to see the relation to generators of $S(\mathfrak{p})_K$. These two types of the Capelli identities become equal when taking principal symbols. Therefore they should be translated to each other by $\mathbb{C}$-linear combinations.

### 2.1 Formulas for the Weil representation

In this subsection, we give explicit formulas for the Weil representation. Let us recall our see-saw pair:

$$\mathfrak{g}_0 = \mathfrak{u}_m \oplus \mathfrak{u}_m \quad \mathfrak{t}_0 = \mathfrak{u}_m \quad \mathfrak{m}_0 = \mathfrak{u}_{p,q} \quad \mathfrak{h}_0 = \mathfrak{u}_p \oplus \mathfrak{u}_q$$

Here $\mathfrak{h}_0 = \mathfrak{u}_p \oplus \mathfrak{u}_q$ is diagonally embedded into the Lie algebra $\mathfrak{m}_0 = \mathfrak{u}_{p,q}$ of the indefinite unitary group (note that $\mathfrak{h}_0$ is the Lie algebra of a maximal compact
subgroup of $U(p, q)$; and $g_0$ decomposes into $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as follows:

$$g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{k}_0 = \{(A, -tA) \in g_0\}, \quad \mathfrak{p}_0 = \{(A, tA) \in g_0\}.$$ 

The real Lie algebra $\mathfrak{m}_0$ is embedded into $\mathfrak{sp}_{2m(p+q)}(\mathbb{R})$ as follows.

$$\mathfrak{m}_0 = \mathfrak{u}_{p,q} \hookrightarrow \mathfrak{sp}_{2m(p+q)}(\mathbb{R})$$

$$A + \sqrt{-1} B \mapsto \left( \begin{array}{ccc} A^{\oplus m} & (-B I_{p,q})^{\oplus m} \\ (I_{p,q} B I_{p,q})^{\oplus m} & (I_{p,q} A I_{p,q})^{\oplus m} \end{array} \right) (A, B \in \text{Mat}(p + q; \mathbb{R})), $$

where $I_{p,q}$ and $A^{\oplus m}$ is defined by

$$I_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}, \quad A^{\oplus m} = 1_m \ast A = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{pmatrix}$$

($m$ copies of $A$ on the diagonal). The real Lie algebra $g_0$ is embedded into $\mathfrak{sp}_{2m(p+q)}(\mathbb{R})$ as follows.

$$g_0 = \mathfrak{u}_m \oplus \mathfrak{u}_m \ni (A + \sqrt{-1} B, A' + \sqrt{-1} B') \longmapsto \left( \begin{array}{ccc} A * I_{p,q} + A' * I_{p,q} & -B * I_{p,q} - B' * I_{p,q} \\ B * I_{p,q} + B' * I_{p,q} & A * I_{p,q} + A' * I_{p,q} \end{array} \right) \in \mathfrak{sp}_{2m(p+q)}(\mathbb{R}),$$

where $A, B, A', B'$ are real matrices; and $I_{p,q}, I_{p,q}$ are defined by

$$I_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & 0_q \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} 0_p & 0 \\ 0 & 1_q \end{pmatrix} \in \text{Mat}(p + q; \mathbb{R}).$$

The notation $A * X$ means the Kronecker product

$$A * X = \begin{pmatrix} a_{11} X & a_{12} X & \cdots & a_{1m} X \\ a_{21} X & a_{22} X & \cdots & a_{2m} X \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} X & a_{m2} X & \cdots & a_{mm} X \end{pmatrix}.$$

Let $V = \text{Mat}(m, p; \mathbb{C}) \oplus \text{Mat}(m, q; \mathbb{C})$, and denote the Weil representation of $\mathfrak{sp}_{2m(p+q)}(\mathbb{C})$ on the polynomial ring $\mathbb{C}[V]$ by $\omega$. Through the embeddings into $\mathfrak{sp}_{2m(p+q)}(\mathbb{C})$, the complexified Lie algebras $g$ and $\mathfrak{m}$ act on $\mathbb{C}[V]$. We denote these representations also by $\omega$. Let $x_{s,i} (1 \leq s \leq m, 1 \leq i \leq p)$ and $y_{s,i} (1 \leq s \leq m, 1 \leq i \leq q)$ be the natural linear coordinate system of $V$, and define the following matrices.

$$X = (x_{s,i})_{1 \leq s \leq m, 1 \leq i \leq p}, \quad \partial^X = (\partial/\partial x_{s,i})_{1 \leq s \leq m, 1 \leq i \leq p} \in \text{Mat}(m, p; \mathcal{P}D(V)),$$

$$Y = (y_{s,i})_{1 \leq s \leq m, 1 \leq i \leq q}, \quad \partial^Y = (\partial/\partial y_{s,i})_{1 \leq s \leq m, 1 \leq i \leq q} \in \text{Mat}(m, q; \mathcal{P}D(V)),$$

$$P = (X, \partial^Y), \quad Q = (\partial^X, Y) \in \text{Mat}(m, p + q; \mathcal{P}D(V)).$$
The actions of basis elements of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C})$ is given by

$$
\omega((E_{s,t}, 0)) = \sum_{i=1}^{p} x_{s,i} \partial_{t,i}^X + \frac{p}{2} \delta_{s,t}, \quad \omega((0, E_{s,t})) = \sum_{i=1}^{q} y_{s,i} \partial_{t,i}^Y + \frac{q}{2} \delta_{s,t}
$$

$(1 \leq s, t \leq m)$, where $E_{s,t} \in \mathfrak{gl}_m(\mathbb{C})$ denotes the matrix unit with $1$ at the $(s, t)$-entry, and $\delta_{s,t}$ denotes Kronecker’s delta. For this, see (4.5) of [12], and $x_{s,i}$ in this article corresponds to $x_{(p+q)(s-1)+i}$ in [12], and $y_{s,i}$ in this article corresponds to $x_{(p+q)(s-1)+p+i}$ in [12]. The action of $m = \mathfrak{gl}_{p+q}(\mathbb{C})$ is given by

$$
\omega\left(\begin{pmatrix} E_{i,j} & 0 \\ 0 & 0 \end{pmatrix}\right) = \sum_{s=1}^{m} x_{s,i} \partial_{s,j}^X + \frac{m}{2} \delta_{i,j}, \quad \omega\left(\begin{pmatrix} 0 & E_{i,j} \\ 0 & 0 \end{pmatrix}\right) = \sqrt{-1} \sum_{s=1}^{m} x_{s,i} y_{s,j}, \quad \omega\left(\begin{pmatrix} 0 & E_{i,j} \\ 0 & 0 \end{pmatrix}\right) = \sqrt{-1} \sum_{s=1}^{m} y_{s,i} \partial_{s,j}^Y,
$$

where $E_{i,j}$’s should be interpreted in the suitable sizes (namely $p \times p$, $q \times q$, $p \times q$ or $q \times p$). Note that we used the normalization $x_{s,i} \mapsto \sqrt{2}x_{s,i}$ and $y_{s,i} \mapsto -\sqrt{2}y_{s,i}$ comparing to (4.5) of [12].

It is convenient to express the formulas of the Weil representation in a matrix form. We define matrices by arranging basis elements of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C})$ or $m = \mathfrak{gl}_{p+q}(\mathbb{C})$. Set

$$
E^X = \{(E_{s,t}, 0)\}_{1 \leq s, t \leq m}, \quad E^Y = \{(0, E_{s,t})\}_{1 \leq s, t \leq m},
$$

$$
B' = (E_{i,j})_{1 \leq i, j \leq p+q}, \quad B = \begin{pmatrix} 1_p & 0 \\ 0 & -\sqrt{-1} I_q \end{pmatrix}, \quad B' = \begin{pmatrix} 1_p & 0 \\ 0 & -\sqrt{-1} I_q \end{pmatrix},
$$

$$(E^X, E^Y) \in \text{Mat}(m; U(\mathfrak{g})), \quad B', B \in \text{Mat}(p+q; U(m)).$$

Then we can write the formulas above in a matrix form:

$$
\omega(E^X) = X^t \partial^X + \frac{p}{2} 1_m, \quad \omega(E^Y) = \partial^Y t Y - \frac{q}{2} 1_m, \quad \omega(E^X + E^Y) = P^t Q + \frac{p-q}{2} 1_m, \quad \omega(B) = \frac{1}{2} P Q + \frac{p-q}{2} I_{p,q},
$$

For example, $\omega(E^X)$ is by definition the $m \times m$ matrix whose $(s, t)$-entry is $\omega(E^X_{s,t})$, and it is equal to the $(s, t)$-entry of the $m \times m$ matrix $X^t \partial^X + (p/2) 1_m$.

Let $T^m_d = \{S \subset \{1, 2, \ldots, m\} ; \# S = d\}$, and $A_{S,T}$ denotes the submatrix of an $m \times m$ matrix $A$ with its rows and columns chosen from $S,T \subset T^m_d$. It is not so difficult to see that

$$
\sum_{S \in T^m_d} \det(E^X + E^Y)_{S,S} \in S(p)^K \quad (d = 1, 2, \ldots, m)
$$

is a generating set of $S(p)^K$, where $E^X$ and $E^Y$ are considered as matrices with entries $E_{s,t}$ in $S(\mathfrak{g})$. 

5
2.2 Capelli identity for Case $\mathbb{C}$ (1)

Here we give the first form of the Capelli identities for Case $\mathbb{C}$. We prove the identities in Subsection 2.3. This form of the identities has a simple expression as differential operators, but their inverse images (Capelli elements) $X_d \in U(\mathfrak{g})^K$ are more complicated than the second one given in Subsection 2.3.

Let us recall the picture of the Capelli identities for Case $\mathbb{C}$:

$$
\begin{align*}
U(\mathfrak{g})^K & \xrightarrow{\omega} \mathcal{PD}(V)^{K \times H} \xleftarrow{\omega} U(\mathfrak{m})^H \\
U(\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C}))(GL_m(\mathbb{C})) & \xrightarrow{\omega} U(\mathfrak{gl}_{p+d}(\mathbb{C}))(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))
\end{align*}
$$

We first give the formula which corresponds to $U(\mathfrak{g})^K \rightarrow \mathcal{PD}(V)^{K \times H}$ in Proposition 2.1. We often write the elements of $S \in \mathcal{I}_d^m$ by $S(1), S(2), \ldots, S(d)$ or $s_1, s_2, \ldots, s_d$ in increasing order. For two disjoint index sets $S' \in \mathcal{I}_d^m$ and $S'' \in \mathcal{I}_{m-d}^n$, let $l(S', S'')$ denotes the inversion number of the concatenated sequence $(S', S'')$.

**Definition 2.1.** We define the column-determinant with diagonal parameters $u = (u_1, u_2, \ldots, u_l)$ by

$$
\det(A_{S', T'}; u) = \sum_{\sigma \in S} \text{sgn}(\sigma) (A_{S'(\sigma(1)), T'(1)} + u_1 \delta_{S'(\sigma(1)), T'(1)}) \cdots
$$

$$
\cdots (A_{S'(\sigma(d)), T'(d)} + u_d \delta_{S'(\sigma(d)), T'(d)})
$$

$$
\text{det}(A_{S', T'} + 1_{S', T'},
\begin{pmatrix}
  u_1 & 0 & \cdots & 0 \\
  0 & u_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_d
\end{pmatrix},
)\right)
$$

for an $m \times m$ matrix $A$ and $S', T' \in \mathcal{I}_d^m$. Here $1_{S', T'}$ denotes the $(S', T')$-submatrix of the identity matrix.

**Proposition 2.1.** For $S, T \in \mathcal{I}_d^m$, the invariant differential operator

$$
\sum_{J \in \mathcal{I}_d^{m-l}} \det P_{S, J} \det Q_{T, J} \in \mathcal{PD}(V)
$$

can be expressed as an image of $U(\mathfrak{g}) = U(\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C}))$ under the Weil representation $\omega$ as follows.

$$
\sum_{J \in \mathcal{I}_d^{m-l}} \det P_{S, J} \det Q_{T, J} = \sum_{l=0}^{d} \sum_{S', T', S'', T''} (-1)^{(l', l'') + l(T', T'')} \det(\omega(E^{S', T'}; \alpha) \det(\omega(E^{T', T''}; \beta),
$$

where the second summation is taken over $S', T' \in \mathcal{I}_d^m$, $S'', T'' \in \mathcal{I}_{d-l}^m$ such that $S' \cap S'' = S$ and $T' \cap T'' = T$; and $\alpha$ and $\beta$ denote

$$
\alpha = (l - 1 - \frac{d}{2}, l - 2 - \frac{d}{2}, \ldots, -\frac{d}{2}),
$$

$$
\beta = -(d - l - 1) + \frac{d}{2}, -(d - l - 2) + \frac{d}{2}, \ldots, \frac{d}{2}).
$$

The proof is given in §2.4.1.
Definition 2.2. For an $n \times n$ matrix $A$, we define

$$\det(A) = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma) \text{sgn}(\tau) A_{\sigma(1), \tau(1)} A_{\sigma(2), \tau(2)} \cdots A_{\sigma(n), \tau(n)},$$

which is called the symmetrized determinant. Also we define the symmetrized determinant with diagonal parameters $u = (u_1, u_2, \ldots, u_n)$ by

$$\det(A; u) = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma) \text{sgn}(\tau) (A_{\sigma(1), \tau(1)} + u_1 \delta_{\sigma(1), \tau(1)}) \cdots (A_{\sigma(n), \tau(n)} + u_n \delta_{\sigma(n), \tau(n)}).$$

Next, we define a minor of the symmetrized determinant with uneven diagonal shift. Let $p, q$ be non-negative integers and $n = p + q$. For an $n \times n$ matrix $B$, $I, J \in \mathcal{T}_d$, and diagonal parameters $u = (u_1, u_2, \ldots, u_d)$, we define

$$\det_{p,q}(B_{I,J}; u) = \frac{1}{d!} \sum_{\sigma, \tau \in S_d} \text{sgn}(\sigma) \text{sgn}(\tau) (B_{\sigma(1), \tau(1)} - u_1 \varepsilon_{\sigma(1), \tau(1)}) \cdots (B_{\sigma(d), \tau(d)} - u_d \varepsilon_{\sigma(d), \tau(d)}),$$

where $I = \{i_1, i_2, \ldots, i_d\}$, $J = \{j_1, j_2, \ldots, j_d\}$ and $\varepsilon_{i,j}$ is a variant of Kronecker's delta defined by

$$\varepsilon = -I_{p,q} = \begin{pmatrix} -1 & 0 \\ 0 & 1_q \end{pmatrix}.$$

Proposition 2.2. For $I, J \in \mathcal{T}_d$ and $n = p + q$, the differential operator $\sum_{S \in \mathcal{T}_d} \det S_{I,J} \in \mathcal{PD}(V)$ can be expressed as an image of $U(m)$ under the Weil representation $\omega$ as follows.

$$\sum_{S \in \mathcal{T}_d} \det S_{I,J} = \text{Det}_{p,q}((\omega(B) - \frac{n}{2} I_{p,q})_{I,J}, d-1, d-2, \ldots, 0).$$
The proof of this proposition is given in §2.4.2. For \( d \geq 1 \), we define \( X_d \in U(\mathfrak{g})^K = U(\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C}))^{GL_m(\mathbb{C})} \) by

\[
X_d = \sum_{S \in I^m_d} \sum_{l=0}^d \sum_{S',S''} (-1)^{(l'(s'),l''(t'))} \det(E^X_{S';T';\alpha}) \det((E^Y_{S'';T'';\beta})),
\]

where \( \alpha = (l - 1 - p/2, l - 2 - p/2, \ldots, -p/2) \), \( \beta = (-(d - l - 1) + q/2, -(d - l - 2) + q/2, \ldots, q/2) \), and the third summation is taken over \( S', T' \in I^m_l \) and \( S'', T'' \in I^m_{d-l} \) which satisfy \( S = S' \cup S'' = T' \cup T'' \).

Similarly, we define \( C_d \in U(\mathfrak{m})^H = U(\mathfrak{gl}_{p+q}(\mathbb{C}))^{GL_p(\mathbb{C}) \times GL_q(\mathbb{C})} \) by

\[
C_d = \sum_{J \in I^{p+q}_d} \det_\mathbb{H}(\{B - \frac{\pi}{2} I_{p,q}\})_{J,J; d-1, d-2, \ldots, 0}.
\]

Finally, we combine Propositions 2.1 and 2.2 and obtain the Capelli identity for Case \( C \).

**Theorem 2.1** (Capelli identity for Case \( C \) (1)). Under the above notation, we have the Capelli identity

\[
\omega(X_d) = \sum_{S \in I^m_d, J \in I^{p+q}_d} \det P_{S,J} \det Q_{S,J} = \omega(C_d).
\]

We call \( X_d \) and \( C_d \) Capelli elements.

**Remark 2.2.** As already noted in Remark [2.1], we can prove that the principal symbol of \( X_d \) is

\[
\sum_{S \in I^m_d} \det(E^X + tE^Y)_{S,S} \in S(p)^K \simeq S(\text{Mat}(m; \mathbb{C}))^{GL_m(\mathbb{C})},
\]

and these invariant elements generate \( S(p)^K \). Thus the left-hand side of the theorem can be considered as the image under the Weil representation of invariant elements corresponding to the generators of \( S(p)^K \). These invariant differential operators are expressed by the elements of \( U(\mathfrak{m})^H = U(\mathfrak{gl}_{p+q})^{GL_p(\mathbb{C}) \times GL_q(\mathbb{C})} \) on the right-hand side.

**2.3 Capelli identity for Case \( C \) (2)**

Here we give the second form of the Capelli identities for Case \( C \). This form of the identities has simpler Capelli elements and it is easy to see the relation to generators of \( S(p)^K \). Let us recall the picture \( \mathfrak{C} \) for Case \( C \). We first give the formula which corresponds to \( U(\mathfrak{g})^K \rightarrow \mathcal{P}D(V)^{K \times H} \).

**Proposition 2.3.** For \( d \geq 1 \), we define \( X_d \in U(\mathfrak{g})^K \) by

\[
X_d = \sum_{S \in I^m_d} \det((E^X + tE^Y - \frac{p-2}{2}1_m)_{S,S}),
\]

and these invariant elements generate \( S(p)^K \). Thus the left-hand side of the theorem can be considered as the image under the Weil representation of invariant elements corresponding to the generators of \( S(p)^K \). These invariant differential operators are expressed by the elements of \( U(\mathfrak{m})^H = U(\mathfrak{gl}_{p+q})^{GL_p(\mathbb{C}) \times GL_q(\mathbb{C})} \) on the right-hand side.
where $E^X$ and $E^Y$ are given in Equation (4). Then its image under the Weil representation $\omega$ is given by

$$\omega(X_d) = \sum_{l=0}^{d} \frac{(m-l)!}{d!(m-d)!} \sum_{J \in I_1^{p+q}} c^d_J \sum_{S \in \mathcal{I}_m} \det P_{S,J} \det Q_{S,J}. $$

In the above formula, $c^d_J$ is an integer defined by

$$c^d_J = c^d_{\alpha, \beta} = \sum_{k \in \mathbb{Z}} \binom{\alpha + \beta}{k} (\beta - k)^d (-1)^k \frac{(\alpha - k)}{d!} (\alpha - k)^l,$$

for a non-negative integer $d$ and $J \in I_1^{p+q}$. Note that $d = \alpha + \beta$.

The proof is given in § 2.5.

We next give the formula corresponding to $PD(V^K \times H \leftarrow U(m)H)$.

**Proposition 2.4.** For $d \geq 1$, define $C_d \in U(m)H$ by

$$C_d = \sum_{l=0}^{d} \frac{(m-l)!}{d!(m-d)!} \sum_{J \in I_1^{p+q}} c^d_J \det_{p,q}(B - \frac{m}{d} I_{p,q}, J,J; l-1, l-2, \ldots, 0),$$

where $c^d_J$ is defined in Equation (10) and $B \in \text{Mat}(p+q; U(m))$ in Equation (5). Then its image under the Weil representation $\omega$ is given by

$$\omega(C_d) = \sum_{l=0}^{d} \frac{(m-l)!}{d!(m-d)!} \sum_{J \in I_1^{p+q}} c^d_J \sum_{S \in \mathcal{I}_m} \det P_{S,J} \det Q_{S,J}. $$

**Proof.** The formula is a linear combination over the elements in Proposition 2.2.

Thus we get the following theorem.

**Theorem 2.2** (Capelli identity for Case $C_2$). For $d \geq 1$, we define $X_d \in U(g)^K = U(\mathfrak{gl}_m(C) \oplus \mathfrak{gl}_n(C))^GL_m(C)$ and $C_d \in U(m)H = U(\mathfrak{gl}_{p+q}(C))^GL_p(C) \times GL_q(C)$ as in Propositions 2.3 and 2.4, respectively. Then we have

$$\omega(X_d) = \sum_{l=0}^{d} \frac{(m-l)!}{d!(m-d)!} \sum_{J \in I_1^{p+q}} c^d_J \sum_{S \in \mathcal{I}_m} \det P_{S,J} \det Q_{S,J} = \omega(C_d). $$

**2.4 Proof of the Capelli identity for Case $C_1$**

Here we give the proof of Propositions 2.1 and 2.2 in § 2.2.
2.4.1 Proof of Proposition 2.1

Let $e_s$ and $e_s^*$ be the elements in standard basis of $\mathbb{C}^m$ and its dual $(\mathbb{C}^m)^*$, respectively. We consider the exterior algebra $\Lambda(\mathbb{C}^m \oplus (\mathbb{C}^m)^*)$ and the ring $\mathcal{PD}(\mathcal{V})$ of differential operators with polynomial coefficients on $\mathcal{V}$. We define several elements in the algebra $\Lambda(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \otimes \mathcal{PD}(\mathcal{V})$:

$$\alpha_j = \sum_{s=1}^{m} e_s P_{s,j}, \quad \beta_j = \sum_{s=1}^{m} e_s^* Q_{s,j} \quad (1 \leq j \leq p+q); \quad \tau = \sum_{s=1}^{m} e_s e_s^*,$$

$$\Xi_X = \sum_{j=1}^{\tau} \alpha_j \beta_j = \sum_{s,t=1}^{m} e_s e_s^* (X^t \partial X)_{(s,t)},$$

$$\Xi_Y = \sum_{j=p+1}^{m} \alpha_j \beta_j = \sum_{s,t=1}^{m} e_s e_s^* (\partial Y^t Y)_{(s,t)}.$$

Here $(X^t \partial X)_{(s,t)}$ denotes the $(s,t)$-entry of the matrix $X^t \partial X$, and we omit $\wedge$ for the multiplication in the exterior algebra. For an index set $J$, we put $\alpha_J = \alpha_{j_1} \cdots \alpha_{j_r}$. We also define $\beta_J, e_S$ or $e_T^*$ in the same way. Then $\alpha_J$ and $\beta_J$ are written in terms of column-determinants as follows:

$$\alpha_J = \sum_{S \in I^m_p} e_S \det P_{S,J}, \quad \beta_J = \sum_{T \in I^m_p} e_T^* \det Q_{T,J}. \quad (12)$$

Recall the matrix $\varepsilon = -I_{p,q} \in \text{Mat}(p+q; \mathbb{C})$. We have the following relations for the elements defined above.

**Lemma 2.1.** (1) The element $\tau$ is central in the algebra $\Lambda(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \otimes \mathcal{PD}(\mathcal{V})$.

(2) We have the following commutation relations:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad \beta_i \beta_j + \beta_j \beta_i = 0 \quad (1 \leq i, j \leq p+q), \quad (13)$$

$$[P_{s,i}, Q_{t,j}] = \varepsilon_{i,j} \delta_{s,t} \quad (1 \leq i, j \leq p+q, 1 \leq s, t \leq m), \quad (14)$$

$$\alpha_i \beta_j + \beta_j \alpha_i = \varepsilon_{i,j} \tau \quad (1 \leq i, j \leq p+q), \quad (15)$$

$$[\Xi_X, \Xi_Y] = 0, \quad (16)$$

$$\alpha_{j} \Xi_X = (\Xi_X + \tau) \alpha_j \quad (1 \leq j \leq p), \quad (17)$$

$$\alpha_{j} \Xi_Y = (\Xi_Y - \tau) \alpha_j \quad (p+1 \leq j \leq p+q). \quad (18)$$

**Proof.** (1) $2m$ elements $e_s$ ($1 \leq s \leq m$) and $e_s^*$ ($1 \leq s \leq m$) are anti-commutative, and $\tau$ is a sum of $e_s e_s^*$, which is commutative with $e_i$ and $e_i^*$. Therefore $\tau$ is central in $\Lambda(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \otimes \mathcal{PD}(\mathcal{V})$.

(2)-Equation (13): The entries of $P = (X, \partial X)$ commute with each other. Hence $\alpha_i$ and $\alpha_j$ are anti-commutative. Similarly $\beta_i$ and $\beta_j$ are also anti-commutative.

(2)-Equation (14): $[P_{s,i}, Q_{t,j}]$ is nonzero only if $s = t$ and $i = j$. If $1 \leq i \leq p$, then $[P_{s,i}, Q_{s,i}] = [x_{s,i}, \partial X^s] = -1$. If $p+1 \leq i \leq p+q$, then $[P_{s,i}, Q_{s,i}] = [\partial X_{i-p}, y_{s,i-p}] = 1$. Hence we have $[P_{s,i}, Q_{t,j}] = \varepsilon_{i,j} \delta_{s,t}$.

(2)-Equation (15): We use Equation (14), and we get

$$\alpha_i \beta_j + \beta_j \alpha_i = \sum_{s,t=1}^{m} e_s e_s^* [P_{s,i}, Q_{t,j}] = \sum_{s,t=1}^{m} e_s e_s^* \varepsilon_{i,j} \delta_{s,t} = \varepsilon_{i,j} \tau.$$
Hence we have

\[
\alpha_j \Xi_X = \sum_{i=1}^{p} \alpha_j \alpha_i \beta_i = -\sum_{i=1}^{p} \alpha_i \alpha_j \beta_i = -\sum_{i=1}^{p} \alpha_i (-\beta_i \alpha_j + \varepsilon_{ij} \tau) = \Xi_X \alpha_j + \alpha_j \tau,
\]

(19)

where we used Equations (13) and (15). Equation (18) is proved similarly to Equation (17).

We have the following relations of $\Xi_X$ and $\Xi_Y$ with the symmetrized determinants.

**Lemma 2.2.** For indeterminate $z$, set $\Xi_X(z) = \Xi_X + z \tau$ and similarly $\Xi_Y(z) = \Xi_Y + z \tau$. Then, for the variables $u = (u_1, u_2, \ldots, u_d)$, we have

\[
\Xi_X(u_1) \Xi_X(u_2) \cdots \Xi_X(u_d) = d!(-1)^{\frac{d(d-1)}{2}} \sum_{S,T \in I_d^m} e_S e_T^\tau \det((X^t \partial_X) s, T; u),
\]

\[
\Xi_Y(u_1) \Xi_Y(u_2) \cdots \Xi_Y(u_d) = d!(-1)^{\frac{d(d-1)}{2}} \sum_{S,T \in I_d^m} e_S e_T^\tau \det((\partial Y^t Y) s, T; u).
\]

**Proof.** Let $\Xi = \sum_{s,t=1}^{m} e_s e_t^*(X^t \partial X)_{s,t}$ or $\Xi = \sum_{s,t=1}^{m} e_s e_t^*(\partial Y^t Y)_{s,t}$. Then we have

\[
\Xi(u) = \Xi + u \tau = \sum_{s,t=1}^{m} e_s e_t^* A_{s,t} + u \sum_{s=1}^{m} e_s e_s^* = \sum_{s,t=1}^{m} e_s e_t^* (A_{s,t} + u \delta_{s,t}).
\]

Hence we have

\[
\Xi(u_1) \cdot \Xi(u_2) \cdots \Xi(u_d) = \sum_{1 \leq s_1, \ldots, s_d \leq m} \prod_{1 \leq t_1, \ldots, t_d \leq m} e_{s_1} e_{t_1}^* \cdots e_{s_d} e_{t_d}^*(A_{s_1, t_1} + u \delta_{s_1, t_1}) \cdots (A_{s_d, t_d} + u \delta_{s_d, t_d})
\]

\[
= \sum_{S,T \in I_d^m} \sum_{\sigma, \tau \in E_d} \text{sgn}(\sigma) \text{sgn}(\tau) (-1)^{\frac{d(d-1)}{2}} e_S e_T^\tau
\]

\[
\times (A_{s(1), t(1)} + u_1 \delta_{s(1), t(1)}) \cdots (A_{s(d), t(d)} + u_d \delta_{s(d), t(d)})
\]

\[
= d!(-1)^{\frac{d(d-1)}{2}} \sum_{S,T \in I_d^m} e_S e_T^\tau \det(A_{s,t}; u_1, u_2, \ldots, u_d).
\]



**Proposition 2.5.** For $S, T \in I_d^m$, we have

\[
\sum_{J \in I_d^m} \det P_{S,J} \det Q_{T,J} = d \sum_{l=0}^{d} \sum_{S', T', S'', T'' \in I_d^m} (-1)^{(l+1)^2} \det((X^t \partial_X)_{S', T'}; l-1, l-2, \ldots, 0)
\]

\[
\times \det((\partial Y^t Y)_{S'', T''}; -(d-l-1), \ldots, -1, 0),
\]

where the second summation is taken over $S', T' \in I_d^m$, $S'', T'' \in I_{d-l}^m$ such that $S' \parallel S'' = S$ and $T' \parallel T'' = T$.
Proof. We show the equality by computing $\sum_{J \in \mathcal{T}_d^{p+q}} \alpha_J \beta_J$ in two different ways. First, we observe

$$\sum_{J \in \mathcal{T}_d^{p+q}} \alpha_J \beta_J = \sum_{J \in \mathcal{T}_d^{p+q}} \sum_{S, T \in \mathcal{T}_d^m} e_{S T}^* \det P_{S, J} \det Q_{T, J},$$

thanks to Equation (12). The coefficient of $e_{S T}^*$ in this expression is equal to the left-hand side of the desired formula.

Second, we compute $\sum_{J \in \mathcal{T}_d^{p+q}} \alpha_J \beta_J$ as follows:

$$\sum_{J \in \mathcal{T}_d^{p+q}} \alpha_J \beta_J = \sum_{l=0}^d \sum_{J \in \mathcal{T}_d^l} \sum_{K \in \mathcal{P} + \mathcal{T}_d^{p+q}} \alpha_J \alpha_K \beta_J \beta_K$$

$$= \sum_{l=0}^d \sum_{J \in \mathcal{T}_d^l} \sum_{K \in \mathcal{P} + \mathcal{T}_d^{p+q}} (-1)^{(d-l)} \alpha_J \beta_J \alpha_K \beta_K, \quad (20)$$

where $p + \mathcal{T}_d^{q-l} = \{p + k_1, p + k_2, \ldots, p + k_{d-l}\}; K \in \mathcal{T}_d^{p+q}$, and we used the anti-commutativity of factors $\alpha_k$ of $\alpha_K$ and $\beta_j$ of $\beta_J$. We compute the factor $\sum_{J \in \mathcal{T}_d^l} \alpha_J \beta_J$ in Equation (20) as follows:

$$\sum_{J \in \mathcal{T}_d^l} \alpha_J \beta_J = \frac{1}{l!} \sum_{j_1, \ldots, j_l=1}^p \alpha_{j_1} \alpha_{j_2} \cdots (\alpha_{j_l} \beta_{j_l}) \beta_{j_2} \cdots \beta_{j_{l-1}} \cdot (-1)^{l-1}$$

$$= \frac{(-1)^{l-1}}{l!} \sum_{j_1, \ldots, j_{l-1}=1}^p \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_{l-1}} \Xi_X \beta_{j_1} \beta_{j_2} \cdots \beta_{j_{l-1}}$$

$$= \frac{(-1)^{l-1}}{l!} \sum_{j'=1}^p \Xi_X (l-1) \cdot \alpha_{j'} \beta_{j'}, \quad (21)$$

where we used Equation (17) in order to move $\Xi_X$ to the left. By repeating this operation, we have

$$\Xi_X (l-1) \cdot \Xi_X (l-2) \cdots \Xi_X (0).$$

Similarly, for $r = d - l$, we have

$$\sum_{K \in \mathcal{P} + \mathcal{T}_d^q} \alpha_K \beta_K = \frac{(-1)^{r(r-1)/2}}{r!} \Xi_Y (-r + 1) \Xi_Y (-r + 2) \cdots \Xi_Y (0),$$

where we used Equation (18). Thus Expression (20) is computed as follows:

$$= \sum_{l=0}^d (-1)^{(d-l)} \frac{(-1)^{(d-l)/2}}{l!} \Xi_X (l-1) \cdot \Xi_X (l-2) \cdots \Xi_X (0)$$

$$\times \frac{(-1)^{(d-l)(d-l-1)/2}}{(d-l)!} \Xi_Y (-d + l + 1) \cdot \Xi_Y (-d + l + 2) \cdots \Xi_Y (0)$$

$$= \sum_{l=0}^d (-1)^{(d-l)} \sum_{S'T \in \mathcal{T}_d^l} e_{S'T}^* \det (X^i \partial_i X)_{S'T}; l-1, \ldots, 0$$

$$\times \sum_{S'', T'' \in \mathcal{T}_d^{d-l}} e_{S''T''}^* \det ((\partial^i Y)^{S''T''}; -d + l + 1, \ldots, 0), \quad (22)$$

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where we used Lemma 2.2. Now we look at the coefficient of \( e_S e_T^* \) in Equation (22). A summand of (22) has \( e_S e_T^* \) only if \( S' \parallel S'' = S \) and \( T' \parallel T'' = T \). Even in that case, we have to translate \( e_S e_T^* \) to \( e_S e_T^* \) in order to determine the sign of the summand. First, \((-1)^{\ell(d-\ell)}\) occurs by moving \( e_S \) to the left of \( e_T^* \). Second, \((-1)^{(s',s'')+(t',t'')}\) occurs by sorting \( e_S e_T^* \) and \( e_T^* e_S \), respectively. Thus it turns out that the coefficient of \( e_S e_T^* \) in Equation (22) is equal to the right-hand side of the desired formula of the proposition.

**Lemma 2.3.** Let \( E_{i,j} \) (\( 1 \leq i, j \leq n \)) be the matrix units, and \( E = (E_{i,j})_{1 \leq i,j \leq n} \in \text{Mat}(n; U(gl_n(\mathbb{C}))) \). Then the following relation between column-determinants and symmetrized determinants holds in \( U(gl_n(\mathbb{C})) \).

\[
\det(E_{i,j}; u-1, u-2, \ldots, u-d) = \det(E_{i,j}; u-1, u-2, \ldots, u-d) \\
= \det((t'E)_{i,j}; u-d, u-d+1, \ldots, u-1) \\
= \det((t'E)_{j,i}; u-d, u-d+1, \ldots, u-1).
\]

**Proof.** The proof is by exterior calculus, and we omit it.

**Proof of Proposition 2.4.** By Equation (4), we have

\[
X^t \partial X = \omega(E^X) - \frac{p}{2} 1_m, \quad \partial Y^t Y = \omega((t'E)^Y) + \frac{q}{2} 1_m, \tag{23}
\]

and hence entries of \( X^t \partial X \) and \( \partial Y^t Y \) have the same commutation relations as those of \( E^X \) and \( (t'E)^Y \), respectively. Therefore symmetrized determinants on the right-hand side of Proposition 2.2 can be changed to column-determinants thanks to Lemma 2.3. To the resulting formula, we again apply Equation (23), and we have proved Proposition 2.4.

**2.4.2 Proof of Proposition 2.2**

Let \( f_i \) and \( f_i^* \) be the elements in the standard basis of \( \mathbb{C}^{p+q} \) and \( (\mathbb{C}^{p+q})^* \) respectively. As before, we define some special elements in the algebra \( \bigwedge(\mathbb{C}^{p+q} \oplus (\mathbb{C}^{p+q})^*) \otimes \mathcal{P}(V) \).

\[
\eta_s = \sum_{i=1}^{p+q} f_i P_{s,i}, \quad \zeta_s = \sum_{i=1}^{p+q} f_i^* Q_{s,i} \quad (1 \leq s \leq m), \quad \Lambda = \sum_{i,j=1}^{p+q} f_i f_j^* (t' P Q)_{(i,j)} = \sum_{s=1}^{m} \eta_s \zeta_s, \quad \sigma = \sum_{i,j=1}^{p+q} \varepsilon_{s,i} f_i f_j^*.
\]

In this case, we obtain the row-determinants of \( P \) or \( Q \) when we make the products of \( \eta_s \)'s or \( \zeta_s \)'s. But the entries of \( P \) or \( Q \) are commutative with each other, so in fact, there is no difference between row and column-determinants. Thus we have

\[
\eta_1 \eta_2 \cdots \eta_m = \sum_{l \in \mathcal{T}_{p+q}} f_l \det P_{S,l}, \quad \zeta_1 \zeta_2 \cdots \zeta_m = \sum_{l \in \mathcal{T}_{p+q}} f_l^* \det Q_{S,l}.
\]

**Lemma 2.4.** (1) \( \eta_1, \eta_2, \ldots, \eta_m \) are anti-commutative (i.e. \( \eta_k \eta_l + \eta_l \eta_k = 0 \)).

(2) \( \zeta_1, \zeta_2, \ldots, \zeta_m \) are anti-commutative.
(3) \( \eta_t \zeta_t + \zeta_t \eta_s = \delta_{s,t} \sigma \) for \( 1 \leq s, t \leq m \).
(4) \([\Lambda, \eta_t] = \eta_t \sigma \) for \( 1 \leq t \leq m \).

Proof. (1) The entries of \( P = (X, \partial Y) \) commute with each other, and \( f_i \) and \( f_j \) are anti-commutative. Hence \( \eta_i \) and \( \eta_j \) are anti-commutative. The statement (2) can be proved similarly.

(3) By Equation (13), we get
\[
\eta_t \zeta_t + \zeta_t \eta_s = \sum_{i,j=1}^{p+q} f_i f_j^* [P_{i,j}, Q_{j,t}] = \sum_{i,j} f_i f_j^* \delta_{i,j, t} \varepsilon_{i,j} = \delta_{s,t} \sigma.
\]

(4) Note that \( \Lambda \eta_t = \sum_{s=1}^{m} \eta_s \zeta_t \eta_s = \sum_{s} \eta_s (-\eta_t \zeta_s + \delta_{s,t} \sigma) = \eta_t \Lambda + \eta_t \sigma \). Hence, \([\Lambda, \eta_t] = \eta_t \sigma \). \( \square \)

Lemma 2.5. Put \( B = tPQ \) and denote \( \Lambda(u) = \Lambda - u \sigma \). Then \( \Lambda = \sum_{i,j=1}^{p+q} f_i f_j^* B_{i,j} \).
We have the following relation between \( \Lambda \) and \( \text{Det}_{p,q} \).
\[
\Lambda(u_1) \Lambda(u_2) \cdots \Lambda(u_d) = d!(1 - \frac{d(d-1)}{2}) \sum_{I,J \in T_d^{p+q}} \text{Det}_{p,q}(B_{I,J}; u_1, u_2, \ldots, u_d).
\]

Proof. We can compute as follows.
\[
\Lambda(u_1) \Lambda(u_2) \cdots \Lambda(u_d)
\]
\[
= \sum_{i_1, \ldots, i_d=1}^{p+q} \prod_{i=1}^{d} f_{i_1} f_{j_1}^* \cdots f_{i_d} f_{j_d}^* (B_{i_1,j_1} - u_1 \varepsilon_{i_1,j_1}) \cdots (B_{i_d,j_d} - u_d \varepsilon_{i_d,j_d})
\]
\[
= \sum_{I,J \in T_d^{p+q}} \prod_{\sigma, \tau \in I,d} \text{sgn}(\sigma) \text{sgn}(\tau) (-1)^{d(d-1)/2} f_{I} f_{J}^*
\]
\[
\times (B_{\iota(1), \j(1)} - u_1 \varepsilon_{i(1), j(1)}) \cdots (B_{\iota(d), \j(d)} - u_d \varepsilon_{i(d), j(d)})
\]
\[
= d!(1 - \frac{d(d-1)}{2}) \sum_{I,J \in T_d^{p+q}} \text{Det}_{p,q}(B_{I,J}; u_1, \ldots, u_d).
\]
\( \square \)

Proof of Proposition 2.2. For \( I, J \in T_d^{p+q} \), we show that
\[
\sum_{S \in T_d^m} \text{det} P_{S,I} \text{det} Q_{S,J} = \text{Det}_{p,q}((tPQ)_1,J_d; d-1, d-2, \ldots, 0).
\]
(24)
Then Proposition 2.2 follows from Equation (24). To prove it, we compute \( \sum_{S \in T_d^m} \eta_S \zeta_S \) in two different ways. First, we compute as follows.
\[
\sum_{S \in T_d^m} \eta_S \zeta_S = \sum_{S \in T_d^m} \sum_{I,J \in T_d^{p+q}} \text{Det}_{p,q}(B_{I,J}; u_1, \ldots, u_d) \text{det} Q_{S,J}.
\]
The coefficient of \( f_{I} f_{J}^* \) is equal to the left-hand side of Equation (24).
Second we compute using Lemma 2.4 (4) as follows:
\[
\sum_{S \in T_d^m} \eta_S \zeta_S = \frac{1}{d!} \sum_{s_1, \ldots, s_d=1}^{m} \eta_{s_1} \cdots \eta_{s_d} \cdot \zeta_{s_1} \cdots \zeta_{s_d}
\]
\[
= (-1)^{d-1} \frac{1}{d!} \sum_{s_1, \ldots, s_d=1}^{m} \eta_{s_1} \cdots \eta_{s_d-1} \cdot \Lambda \cdot \zeta_{s_1} \cdots \zeta_{s_d-1}
\]
\[
= (-1)^{d-1} \frac{1}{d!} \sum_{s_1, \ldots, s_d=1}^{m} \Lambda(d-1) \cdot \eta_{s_1} \cdots \eta_{s_d-1} \cdot \zeta_{s_1} \cdots \zeta_{s_d-1}.
\]
In this way we repeat producing $\Lambda$ from $\eta_s\zeta_s$ and moving it to the left, and we have

$$\sum_{s \in I^m} \eta_s \zeta_s = \frac{(-1)^{d(d-1)/2}}{d!} \Lambda(d-1) \Lambda(d-2) \cdots \Lambda(0)$$

$$= \sum_{i,j \in I^m} f_i f_j \det_{p,q}(iPQ)_{l,j}; d-1, d-2, \ldots, 0,$$

by using Lemma 2.5. The coefficient of $f_i f_j$ is equal to the right-hand side of Equation (24). Thus we have proved Equation (24), and hence Proposition 2.2.

2.5 Proof of the Capelli identity for Case $\mathbb{C}$ (2)

Here we shall prove Proposition 2.6. We have already given the proof of Proposition 2.4 and Theorem 2.2 is a direct consequence of these propositions. We use the same setting as in § 2.4, Let $e_s$ and $e^*_s$ be the elements in the standard basis of $\mathbb{C}^m$ and $(\mathbb{C}^m)^*$ respectively. We again define several elements in $\wedge(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \otimes \mathcal{P} \mathcal{D}(V)$. The elements $\varepsilon = -I_{p,q}, \alpha_j, \beta_j$ and $\tau$ are the same as before, and newly defined elements are denoted with tilde.

$$\alpha_j = \sum_{s=1}^m e_s P_{s,j}, \quad \beta_j = \sum_{s=1}^m e_s^* Q_{s,j} \quad (1 \leq j \leq p + q); \quad \tau = \sum_{s=1}^m e_s e^*_s,$$

$$\tilde{\alpha}_j = \varepsilon_{j,j} \alpha_j, \quad \tilde{\beta}_j = \varepsilon_{j,j} \beta_j \quad (1 \leq j \leq p + q),$$

$$\Xi = \sum_{s,t=1}^m e_s e^*_t (P^t)(s,t) = \sum_{j=1}^{p+q} \alpha_j \beta_j = \sum_{j=1}^{p+q} \tilde{\alpha}_j \tilde{\beta}_j, \quad \tilde{\Xi} = \sum_{j=1}^{p+q} \tilde{\alpha}_j \beta_j = \sum_{j=1}^{p+q} \alpha_j \tilde{\beta}_j.$$

Lemma 2.6. We have the following relations for the elements above.

(1) $2(p + q)$ elements $\{\alpha_i, \tilde{\alpha}_i; 1 \leq i \leq p + q\}$ are anti-commutative.

(2) Similarly, $\{\beta_i, \tilde{\beta}_i; 1 \leq i \leq p + q\}$ are anti-commutative.

(3) We have the following commutation relations:

$$[\Xi, \tilde{\Xi}] = 0, \quad [\Xi, \alpha_i] = \tilde{\alpha}_i \tau, \quad [\Xi, \beta_i] = -\tilde{\beta}_i \tau \quad (1 \leq i \leq p + q).$$

Proof. The statements (1) and (2) follow from Equation (13) and the definition of $\tilde{\alpha}_j$ and $\tilde{\beta}_j$.

(3)-Equation (25): Since $\Xi = \Xi_X + \Xi_Y$ and $\tilde{\Xi} = -\Xi_X + \Xi_Y$, the assertion follows from Equation (16).

(3)-Equation (26): Again, since $\Xi = \Xi_X + \Xi_Y$ and $\tilde{\Xi} = -\Xi_X + \Xi_Y$, the first assertion follows from Equations (17) and (18). The second assertion can be proved similarly.

For integers $u$ and $v$, let us define one more element $\gamma(u,v)$ in $\wedge(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \otimes \mathcal{P} \mathcal{D}(V)$.

$$\gamma(u,v) = \sum_{1 \leq j_1, \ldots, j_{u+v} \leq p+q} \alpha_{j_1} \cdots \alpha_{j_u} \cdot \tilde{\alpha}_{j_{u+1}} \cdots \tilde{\alpha}_{j_{u+v}} \cdot \beta_{j_{u+v}} \cdots \beta_{j_{2}},$$

$$\gamma(0,0) = 1, \quad \gamma(u,v) = 0 \quad (u < 0 \text{ or } v < 0).$$
Lemma 2.7. We have the following relations of $\gamma(u, v)$ with $\Xi$.

(1) For non-negative integers $u$ and $v$, we have

$$\gamma(u, v)\Xi = \gamma(u + 1, v) + u\tau\gamma(u - 1, v + 1) + v\tau\gamma(u + 1, v - 1).$$

(2) For $d \geq 0$, $\Xi^d$ can be expressed as an integral linear combination of $\gamma(u, v)\tau^{d-u-v}$, where $u, v$ are non-negative integers and $d - u - v \geq 0$.

(3) $\Xi, \tilde{\Xi}, \tau$ and $\gamma(u, v)$ commute with each other.

(4) $\gamma(u, v)$ and $\gamma(u', v')$ commute with each other.

Proof. (1) We have

$$\begin{align*}
\gamma(u, v)\Xi &= \sum_{1 \leq j_1, \ldots, j_{u+v} \leq p+q} \alpha_{j_1} \cdots \alpha_{j_u} \cdot \tilde{\alpha}_{j_{u+1}} \cdots \tilde{\alpha}_{j_{u+v}} \cdot \beta_{j_{u+v}} \cdot \beta_{j_1} \cdot \Xi \\
&= \sum_{j_1, \ldots, j_{u+v}} \alpha_{j_1} \cdots \alpha_{j_u} \cdot \tilde{\alpha}_{j_{u+1}} \cdots \tilde{\alpha}_{j_{u+v}} \cdot \Xi \cdot \beta_{j_{u+v}} \cdots \beta_{j_1} \\
&\quad + \sum_{j_1, \ldots, j_{u+v}} \sum_{l=1}^{u+v} \alpha_{j_1} \cdots \alpha_{j_u} \cdot \tilde{\alpha}_{j_{u+1}} \cdots \tilde{\alpha}_{j_{u+v}} \cdot \beta_{j_{u+v}} \cdots [\beta_{j_l}, \Xi] \cdots \beta_{j_1}. \quad (27)
\end{align*}$$

Since $\Xi = \sum_i \alpha_i / \beta_i$, the first term of Expression (27) is equal to $\gamma(u + 1, v)$. For each summand of the second term in Expression (27), there appears $[\beta_{j_l}, \Xi] = \tilde{\beta}_{j_l} \tau$ (cf. Equation (26)), and this tilde over $\beta_{j_l}$ can be moved to $\alpha_{j_l}$ ($1 \leq l \leq u$), or cancel with $\tilde{\alpha}_{j_l}$ ($u + 1 \leq l \leq u + v$). Therefore summands of the second term of (27) is equal to $\gamma(u - 1, v + 1)\tau$ for $1 \leq l \leq u$, and equal to $\gamma(u + 1, v - 1)\tau$ for $u + 1 \leq l \leq u + v$. Thus, after summarizing the first term and the second term, we have

$$\gamma(u, v)\Xi = \gamma(u + 1, v) + u\tau\gamma(u - 1, v + 1) + v\tau\gamma(u + 1, v - 1).$$

(2) Starting with $\Xi = \gamma(1, 0)$, we repeat multiplying $\Xi$ from the right to this equation, and use (1) of this lemma. Then it is shown inductively that $\Xi^d$ can be expressed as a linear combination of $\gamma(u, v)\tau^{d-u-v}$ with integral coefficients.

(3) Note that $\tau$ is central, and that $\Xi$ commutes with $\tilde{\Xi}$ by Equation (26). We prove that $\gamma(u, v)$ can be expressed by $\Xi, \tilde{\Xi}$ and $\tau$, which will complete the proof. It follows from (1) of this lemma that

$$\gamma(u + 1, v) = \gamma(u, v)\Xi - u\tau\gamma(u - 1, v + 1) - v\tau\gamma(u + 1, v - 1).$$

Therefore $\gamma(u, v)$ can be written by using $\Xi, \tau$ and $\gamma(u', v')$ ($u' + v' < u + v$). Repeat this operation, then it turns out that $\gamma(u, v)$ has an expression of $\Xi = \gamma(1, 0)$, $\tilde{\Xi} = \gamma(0, 1)$ and $\tau$.

(4) Since $\gamma(u, v)$ is expressed by $\Xi, \tilde{\Xi}$ and $\tau$ by the proof of (3) above, $\gamma(u, v)$ and $\gamma(u', v')$ are commutative.
Definition 2.3. Thanks to Lemma 2.7 (2), we can define integers $b_{u,v}^d$ by

$$
\Xi^d = \sum_{0 \leq u + v \leq d} b_{u,v}^d \gamma(u, v) \tau^{d-u-v} \quad (d \geq 0).
$$

For non-negative integers $p, q, u, v$ and $J \in \mathcal{I}_w^{p+q}$, we set

$$
\varepsilon(J; u, v) = \varepsilon(\alpha, \beta; u, v) = \sum_{\sigma \in \mathcal{S}_{w+u+v}} \varepsilon_{J_{\sigma(u+1)}J_{\sigma(u+1)}J_{\sigma(u+1)}J_{\sigma(u+1)}} \varepsilon_{J_{\sigma(u+1)}J_{\sigma(u+1)}J_{\sigma(u+1)}J_{\sigma(u+1)}},
$$

$$
\varepsilon(0; 0, 0) = \varepsilon(0, 0; 0, 0) = 1,
$$

where $\alpha$ and $\beta$ are defined in Equation (11). When $u$ or $v$ is negative, we put $b_{u,v}^d = \varepsilon(J; u, v) = 0$.

Lemma 2.8. For non-negative integers $p, q, u, v$ and $w = u + v$, we have

$$
\gamma(u, v) = (-1)^{\frac{u(u-1)}{2}} \sum_{J \in \mathcal{I}_w^{p+q}} \varepsilon(J; u, v) \sum_{S,T \in \mathcal{I}_w} e_S e_T^* \det P_{S,J} \det Q_{T,J}.
$$

Proof. By the definition of $\gamma(u, v)$, we have

$$
\gamma(u, v) = \sum_{1 \leq j_1, \ldots, j_u \leq p+q} \alpha_{j_1} \cdots \alpha_{j_u} \cdot \alpha_{j_{u+1}} \cdots \alpha_{j_w} \cdot \beta_{j_w} \cdots \beta_{j_1},
$$

$$
= \sum_{1 \leq j_1, \ldots, j_w \leq p+q} \varepsilon_{j_{u+1}j_{u+1}} \cdots \varepsilon_{j_{u+1}j_{u+1}} \varepsilon_{j_{u+1}j_{u+1}} \cdots \varepsilon_{j_{u+1}j_{u+1}} \cdot \varepsilon_{\sigma} \cdot \varepsilon_{\sigma} \cdots \varepsilon_{\sigma} \varepsilon_{\sigma} \varepsilon_{\sigma} \cdots \varepsilon_{\sigma} \varepsilon_{\sigma} \cdots \varepsilon_{\sigma} \varepsilon_{\sigma} \cdots \varepsilon_{\sigma}.
$$

where $w = u + v$. In this expression, the sum over $\sigma \in \mathcal{S}_w$ of the products of $\varepsilon_{j,j}$ is equal to $\varepsilon(J; u, v)$, and there appears $(-1)^{w(w-1)/2}$ by sorting $\beta_{j_{u+1}}$’s into increasing order in $a$. Thus it follows from Equation (12) that the expression gives our desired formula.

Proof of Proposition 2.8. We prove Proposition 2.8 by computing $\Xi^d$ in two different ways, and make a contraction using $e_S e_T^* \mapsto \delta_{S,T}$, where $\delta_{S,T}$ denotes Kronecker’s delta. First, we get the following formula,

$$
\Xi^d = \sum_{S,T \in \mathcal{I}_w^d} (-1)^{\frac{d(d-1)}{2}} e_S e_T^* \det ((P^t Q)_{S,T}),
$$

by a similar computation to Lemma 2.2. We apply the contraction $e_S e_T^* \mapsto \delta_{S,T}$ to the right-hand side of Equation (28), and we have

$$
(-1)^{d(d-1)/2} \sum_{S,T \in \mathcal{I}_w^d} \det ((P^t Q)_{S,T}),
$$

which is equal to $\omega(X_d)$ multiplied by $(-1)^{d(d-1)/2}d!$ in view of Equation (17). Second, we compute in the following way. It follows from Definition 2.3 and Lemma 2.8 that

$$
\Xi^d = \sum_{l=0}^{d} \sum_{u+v=l} b_{u,v}^d (-1)^{\frac{d(d-1)}{2}} \sum_{J \in \mathcal{I}_w^{p+q}} \varepsilon(J; u, v) \cdot \Delta J \cdot \tau^{d-l},
$$
where we temporarily put

\[ \Delta J = \sum_{S,T \in I^n} e_S e_T^* \det Q_{S,T} \]

Since \( \tau^k = (\sum_{s=1}^m e_se_s^*)^k = k! \sum_{U \in I_k^n} (-1)^{k(k-1)/2} e_U e_U^* \), the expression above is equal to

\[
\sum_{l=0}^d \sum_{u+v=l} \sum_{J \in I^{l+q}_{l+q}} \frac{b_{u,v}^d}{s(T)} (-1)^{l(l+1)} \varepsilon(J; u, v) e_S e_T^* \det Q_{S,T} = (d-l)! \frac{1}{2^{(d-l)(d-l-1)}} e_U e_U^* \det Q_{S,T}.
\]

When applying the contraction \( e_S e_T^* \mapsto \delta_{S,T} \) to this expression, only the summands with \( S = T \) do not vanish. If we fix \( S(= T) \), then the number of choices of \( U \) is \( \binom{m-1}{d-1} \), since \( U \) should not intersect \( S \). Thus we contract the expression above, and obtain

\[
(\frac{(d-1)!}{d!})^2 \sum_{l=0}^d \sum_{u+v=l} \sum_{J \in I^{l+q}_{l+q}} \frac{b_{u,v}^d}{s(T)} (-1)^{l(l+1)} \varepsilon(J; u, v) (d-l)! \frac{(m-l)!}{d!} \det Q_{S,T} \]

which is equal to the right-hand side of Proposition 2.3 multiplied by \( (-1)^{d(d-1)/2} \) except that \( c^d_J \) is replaced with \( \sum_{u+v=l} b_{u,v}^d e(J; u, v) \).

We prove \( c^d_J = \sum_{u+v=l} b_{u,v}^d e(J; u, v) \) in Appendix A and this completes the proof of Proposition 2.3.

\[ \square \]

3 Case \( \mathbb{R} \)

Here we give the Capelli identities for symmetric pairs for Case \( \mathbb{R} \) in Table (3) without proof. There are two different types of the Capelli identities as remarked at the beginning of Section 2. In Theorem 3.2 we give the first form, which is simpler as differential operators; in Theorem 3.3 we give the second form, which has a simpler Capelli element \( X_d \in U(\mathfrak{g})^K \). In contrast to Case \( \mathbb{C} \), we only have the identities expressing \( \omega(\chi_d) \in \mathcal{P} \mathcal{D}(V)^{K \times H} \) in explicit differential operators for Capelli elements \( X_d \in U(\mathfrak{g})^K \). In other words, we only have identities corresponding to the left half of the picture below.

\[
\begin{align*}
U(\mathfrak{g})^K & \xrightarrow{\omega} \mathcal{P} \mathcal{D}(V)^{K \times H} & \xrightarrow{\omega} U(\mathfrak{m})^H \\
U(\mathfrak{gl}_m(\mathbb{C}))^{O_m(\mathbb{C})} & | | \quad U(\mathfrak{sp}_{2n}(\mathbb{C}))^{GL_n(\mathbb{C})}
\end{align*}
\]
3.1 Formulas for the Weil representation

Recall our see-saw pair:

\[ \mathfrak{g}_0 = \{ u_m \} \uplus \mathfrak{sp}_{2n}(\mathbb{R}) = \mathfrak{m}_0 \]
\[ \mathfrak{t}_0 = \{ \sigma_m(\mathbb{R}) \} \uplus \{ u_n \} = \mathfrak{h}_0 \]

Here \( \mathfrak{g}_0 = u_m \) is realized as the set of the \( m \times m \) skew Hermitian matrices, and \( \mathfrak{t}_0 = \sigma_m(\mathbb{R}) \) is the set of the alternating matrices. Let \( \mathfrak{p}_0 = \sqrt{-1} \text{Sym}_m(\mathbb{R}) \subset u_m \), and we have the direct sum decomposition \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0 \). The Lie algebra \( \mathfrak{m}_0 = \mathfrak{sp}_{2n}(\mathbb{R}) \) is defined as

\[ \mathfrak{sp}_{2n}(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -^tA \end{pmatrix} : A \in \mathfrak{gl}_n(\mathbb{R}), B, C \in \text{Sym}_n(\mathbb{R}) \right\}, \]

and \( \mathfrak{h}_0 = u_n \) is the Lie algebra of a maximal compact subgroup, which is embedded into \( \mathfrak{sp}_{2m}(\mathbb{R}) \) just as \( \mathfrak{g}_0 \) below.

\[ \mathfrak{g}_0 = \{ u_m \} \hookrightarrow \mathfrak{sp}_{2m}(\mathbb{R}) \]
\[ A + \sqrt{-1}B \mapsto \begin{pmatrix} A \ast 1_n & -B \ast 1_n \\ B \ast 1_n & A \ast 1_n \end{pmatrix} \quad (A,B : \text{real matrices}), \]

\[ \mathfrak{m}_0 = \mathfrak{sp}_{2n}(\mathbb{R}) \hookrightarrow \mathfrak{sp}_{2m}(\mathbb{R}) \]
\[ \begin{pmatrix} A & B \\ C & -^tA \end{pmatrix} \mapsto \begin{pmatrix} A \oplus m & B \oplus m \\ C \oplus m & -^tA \oplus m \end{pmatrix}. \]

Let \( V = \text{Mat}(m,n;\mathbb{C}) \), and \( x_{s,i} \) its linear coordinate system, and put \( \partial_{x_{s,i}} = \partial/\partial x_{s,i} \). Denote the Weil representation of \( \mathfrak{sp}_{2m}(\mathbb{C}) \) on the polynomial ring \( \mathbb{C}[V] \) by \( \omega \). Through the embeddings into \( \mathfrak{sp}_{2m}(\mathbb{C}) \), the complexified Lie algebras \( \mathfrak{g} = (\mathfrak{g}_0) \otimes_{\mathbb{R}} \mathbb{C} \) and \( \mathfrak{m} = (\mathfrak{m}_0) \otimes_{\mathbb{R}} \mathbb{C} \) act on \( \mathbb{C}[V] \). We denote these representations also by \( \omega \). Let \( E_{s,i} \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \) be the matrix unit. Then its representation through \( \omega \) is given by (see (4.5) of [12], e.g.)

\[ \omega(E_{s,i}) = \sum_{i=1}^n x_{s,i} \partial_{x_{s,i}} + \frac{n}{2} \delta_{s,t}. \]

Note that \( x_{s,i} \) of this article corresponds to \( x_{n(s-1)+i} \) in [12]. To give the representation of \( \mathfrak{m} = \mathfrak{sp}_{2n}(\mathbb{C}) \), we define the following elements in \( \mathfrak{m} \).

\[ X_{\varepsilon_i - \varepsilon_j} = \frac{1}{2} \begin{pmatrix} F_{i,j} & \sqrt{-1}G_{i,j} \\ -\sqrt{-1}G_{i,j} & F_{i,j} \end{pmatrix} \quad (1 \leq i, j \leq n), \]

\[ X_{\pm(\varepsilon_i + \varepsilon_j)} = \frac{1}{2} \begin{pmatrix} G_{i,j} & \mp \sqrt{-1}G_{i,j} \\ \pm \sqrt{-1}G_{i,j} & -G_{i,j} \end{pmatrix} \quad (1 \leq i, j \leq n), \]

where \( F_{i,j} = E_{i,j} - E_{j,i} \) and \( G_{i,j} = E_{i,j} + E_{j,i} \) are elements in \( \text{Mat}(n;\mathbb{C}) \). Note that we do not write \( X_0 \), but \( X_{\varepsilon_i - \varepsilon_j} \) to distinguish \( X_{\varepsilon_i - \varepsilon_j} \) from \( X_{\varepsilon_i - \varepsilon_j} \). Then \( X_{\varepsilon_i - \varepsilon_j} \) \( (1 \leq i \leq n) \) form a basis of a Cartan subalgebra of \( \mathfrak{m} = \mathfrak{sp}_{2n}(\mathbb{C}) \), and the
other elements defined above are the root vectors with respect to this Cartan subalgebra. The actions of these elements through $\omega$ are given as follows:

$$\omega(X_{s,i-1}) = \sum_{s=1}^{m} x_{s,i} \partial s,j + \frac{m}{2}\delta_{s,j} \quad (1 \leq i, j \leq n),$$

$$\omega(X_{s,i+1}) = \sum_{s=1}^{m} x_{s,i} x_{s,j}, \quad \omega(X_{s,i-1}) = \sum_{s=1}^{m} \partial s,i \partial s,j \quad (1 \leq i, j \leq n).$$

Note that we used the normalization $x_{s,i} \mapsto \sqrt{2} x_{s,i}$, which is slightly different from (4.5) of [12]. We define matrices of differential operators by

$$X = (x_{s,i})_{1 \leq s \leq m, 1 \leq i \leq n},$$

$$\partial = (\partial / \partial x_{s,i})_{1 \leq s \leq m, 1 \leq i \leq n} \in \text{Mat}(m, n; \mathcal{P}(V)),$$

$$P = (X, \partial),$$

$$Q = (\partial, X) \in \text{Mat}(m, 2n; \mathcal{P}(V)),$$

$$E = (E_{s,t})_{1 \leq s, t \leq m} \in \text{Mat}(m; \mathfrak{U}(g)),$$

and we use the notation like $\omega(E)$ of a matrix form as used in Case C. For example, we write $\omega(E + \bar{\omega}E) = P^tQ$. It is known that

$$\sum_{S \in T_d} \det(E + \bar{\omega}E)S,S \in S(p)^K \quad (d = 1, 2, \ldots, m)$$

is a generating set of $S(p)^K$, where $E$ is considered as a matrix with entries $E_{s,t}$ in $S(g)$.

### 3.2 Capelli identity for Case $\mathbb{R}$

We have two different types of the Capelli identities for Case $\mathbb{R}$.

**Theorem 3.1.** For $d \geq 1$, define $X_d \in U(g)^K = U(\mathfrak{gl}_m(\mathbb{C}))^{O_m(\mathbb{C})}$ by

$$X_d = \sum_{S \in T_d} \sum_{l=0}^{d} \sum_{S', T'} \sum_{S'', T''} (\pm 1) \det(E_{S', T'; l - 1 - \frac{n}{2}, l - 2 - \frac{n}{2}, \ldots, -\frac{n}{2}}) \times \det((\bar{\omega}E)_{S'', T''}; -(d - l - 1) + \frac{n}{2}, -(d - l - 2) + \frac{n}{2}, \ldots, \frac{n}{2}),$$

where the third summation is taken over $S', T' \in T_{d-l}^m$ and $S'', T'' \in T_l$ such that $S = S' \Pi S'' = T' \Pi T''$, and the signature is given by $(\pm 1) = (-1)^{((S', S'') + (T', T''))}$. Then we have the Capelli identity

$$\omega(X_d) = \sum_{S \in T_d} \sum_{J \in T_{d-n}^m} \det P_{S,J} \det Q_{S,J}.$$

**Theorem 3.2.** For $d \geq 1$, define $X_d \in U(g)^K = U(\mathfrak{gl}_m(\mathbb{C}))^{O_m(\mathbb{C})}$ by

$$X_d = \sum_{S \in T_d} \text{Det}(E + \bar{\omega}E)_{S,S},$$

where $\text{Det}$ denotes the symmetrized determinant. Then we have the Capelli identity

$$\omega(X_d) = \sum_{l=0}^{d} \frac{(m - l)!}{d!(m - d)!} \sum_{J \in T_{d-n}^m} c_d^J \sum_{S \in T_d} \det P_{S,J} \det Q_{S,J},$$

where $c_d^J$ is defined in Equation (11) with both $p$ and $q$ replaced by $n$.  

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4 Case $\mathbb{H}$

In this section, we give the Capelli identities for symmetric pairs for Case $\mathbb{H}$ in Table (3) without proof. As in the case of $\mathbb{R}$, for Capelli elements $X_d \in U(g)^K$, we only have the identities expressing $\omega(X_d) \in PD(V)^{K \times H}$ by explicit differential operators. In other words, we only consider the left half of the picture below.

\[
\begin{array}{c}
U(g)^K \xrightarrow{\omega} PD(V)^{K \times H} \xrightarrow{\omega} U(m)^H \\
\| \| \\
U(u_{2m})^{U_{Sp_m}} \xleftarrow{\|} U(o_{2n}^{U_n})
\end{array}
\]

4.1 Formulas for the Weil representation

Our see-saw pair in this section is:

\[
\begin{align*}
g_0 &= u_{2m} \\
\kappa_0 &= u_{Sp_m} \\
m_0 &= o_{2n}^{U_n} \\
h_0 &= u_n
\end{align*}
\]

Here $g_0, \kappa_0, m_0$ and $h_0$ are realized as follows.

\[
g_0 = u_{2m} = \{2m \times 2m \text{ skew Hermitian matrices}\},
\]

\[
\kappa_0 = usp_m = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} ; \begin{array}{l} X : m \times m \text{ skew Hermitian,} \\ Y \in Sym_m(\mathbb{C}) \end{array} \right\} \subset g_0,
\]

\[
m_0 = o_n^* = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} ; \begin{array}{l} X : n \times n \text{ skew Hermitian,} \\ Y \in Alt_n(\mathbb{C}) \end{array} \right\},
\]

\[
h_0 = u_n = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} ; X : n \times n \text{ skew Hermitian} \right\} \subset m_0.
\]

Define $p_0$ by

\[
p_0 = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} ; \begin{array}{l} X : m \times m \text{ skew Hermitian,} \\ Y \in Alt_m(\mathbb{C}) \end{array} \right\} \subset g_0,
\]

and we have the direct sum decomposition $g_0 = \kappa_0 \oplus p_0$. The real Lie algebras $g_0$ and $m_0$ are embedded into $sp_{4mn}(\mathbb{R})$ as follows:

\[
\begin{align*}
g_0 &= u_{2m} \hookrightarrow sp_{4mn}(\mathbb{R}) \\
A + \sqrt{-1}B &\mapsto \begin{pmatrix} A_{1} & -B_{1} \\ B_{1} & A_{1} \end{pmatrix} (A, B : \text{ real matrices}),
\end{align*}
\]

\[
\begin{align*}
m_0 &= o_{2n}^* \hookrightarrow sp_{4mn}(\mathbb{R}) \\
\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} &\mapsto \begin{pmatrix} X_1 - Y_1 & X_1 \\ Y_1 & X_1 \end{pmatrix}_{\oplus m} - \begin{pmatrix} X_2 - Y_2 & X_2 \\ Y_2 & X_2 \end{pmatrix}_{\oplus m} + \begin{pmatrix} X_2 - Y_2 & X_2 \\ Y_2 & X_2 \end{pmatrix}_{\oplus m} - \begin{pmatrix} X_1 - Y_1 & X_1 \\ Y_1 & X_1 \end{pmatrix}_{\oplus m}.
\end{align*}
\]
where we write \( X = X_1 + \sqrt{1}X_2 \) and \( Y = Y_1 + \sqrt{1}Y_2 \) with real matrices \( X_i, Y_i \) \((i = 1, 2)\).

Let \( V = \text{Mat}(2m, n; \mathbb{C}) \), and \( x_{s,i} \) \((1 \leq s \leq 2m, 1 \leq i \leq n)\) its linear coordinates, and put \( \partial_{s,i} = \partial/\partial x_{s,i} \). Denote the Weil representation of \( \mathfrak{sp}_{4nm}(\mathbb{C}) \) on the polynomial ring \( \mathbb{C}[V] \) by \( \omega \). The action of \( \omega \) for the basis element \( E_{s,t} \in \mathfrak{g} = \mathfrak{gl}_{2m}(\mathbb{C}) \) is given by (see (4.5) of \cite{12}, e.g.)
\[
\omega(E_{s,t}) = \sum_{i=1}^{n} x_{s,i} \partial_{s,i} + \frac{\sqrt{2}}{2} \delta_{s,t} \quad (1 \leq s, t \leq 2m),
\]
while the action of \( m = \mathfrak{o}_{2n}(\mathbb{C}) \) is
\[
\omega \left( \begin{pmatrix} E_{i,j} & 0 \\ 0 & -E_{j,i} \end{pmatrix} \right) = \sum_{s=1}^{2m} x_{s,i} \partial_{s,j} + m \delta_{i,j},
\]
\[
\omega \left( \begin{pmatrix} 0 & E_{i,j} - E_{j,i} \\ 0 & 0 \end{pmatrix} \right) = \sqrt{-1} \sum_{s=1}^{m} (x_{s,i}x_{s,j} - x_{s,j}x_{s,i}),
\]
\[
\omega \left( \begin{pmatrix} E_{i,j} - E_{j,i} & 0 \\ 0 & 0 \end{pmatrix} \right) = \sqrt{-1} \sum_{s=1}^{m} (\partial_{s,i} \partial_{s,j} - \partial_{s,j} \partial_{s,i}),
\]
where \( \pi = s + m \), and \( E_{i,j} \) or \( E_{j,i} \) denotes the matrix unit in \( \text{Mat}(n; \mathbb{C}) \). Note that we used the normalization \( x_{s,i} \mapsto \sqrt{2}x_{s,i} \) and \( x_{\pi,j} \mapsto -\sqrt{2}x_{\pi,j} \) \((1 \leq s \leq 2m, 1 \leq i \leq n)\) to (4.5) of \cite{12}. Note also that \( x_{s,i} \) of this article corresponds to \( x_{2m(s-1)+i} \) in \cite{12}, and \( x_{\pi,j} \) of this article corresponds to \( x_{(2n+1)(s-1)+i} \) in \cite{12} for \( 1 \leq s \leq m \). We set
\[
J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}
\]
and define the following matrices:
\[
X = (x_{s,i})_{1 \leq s \leq 2m, 1 \leq i \leq n} \quad \partial = (\partial/\partial x_{s,i})_{1 \leq s \leq 2m, 1 \leq i \leq n} \quad \in \text{Mat}(2m, n; \mathcal{P}D(V)),
\]
\[
P = (X, J\partial) \quad Q = (\partial, JX) \quad \in \text{Mat}(2m, 2n; \mathcal{P}D(V)),
\]
\[
E = (E_{s,t})_{1 \leq s, t \leq 2m} \quad \in \text{Mat}(2m; U(\mathfrak{gl}_{2m}(\mathbb{C}))),
\]
and we use the notation like \( \omega(E) \) of matrix form. For example, we write
\[
\omega(E + J^{t}EJ^{-1}) = P^{t}Q.
\]

### 4.2 Capelli identity for Case \( \mathbb{H} \)

Recall that the set \( \mathcal{I}^m_d \) \((1 \leq d \leq m)\) of strictly increasing indices. Here we define \( \mathcal{I}^n_d \) \((m \geq 1, d \geq 1)\) as the set of weakly increasing indices:
\[
\mathcal{I}^n_d = \left\{ S = \{s_1, s_2, \ldots, s_d\} : S \text{ is a multi-set with } 1 \leq s_1 \leq s_2 \leq \cdots \leq s_d \leq m \right\}.
\]

For \( S \in \mathcal{I}^n_d \), we define an integer \( S! \) by
\[
S! = t_1!t_2! \cdots t_m!,
\]

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where $t_j$ denotes the number of $j$ occurring in the multi-set $S$ ($\{t_j\}$ are not the members of $S$). For an $m \times m$ matrix $A$ and $I, J \in \overline{T}_m^n$, we write $A_{I,J} = (A_{i_a,j_b})_{1 \leq a,b \leq d}$. Note that $A_{I,J}$ is not necessarily a submatrix of $A$ in this case, since $I$ and $J$ might have duplicated indices.

We need the permanent (column-permanent) and the symmetrized permanent.

**Definition 4.1.** For an $n \times n$ matrix $A$, the permanent of $A$ is defined by

$$\text{per } A = \sum_{\sigma \in \mathfrak{S}_n} A_{\sigma(1),1}A_{\sigma(2),2} \cdots A_{\sigma(n),n}.$$ 

We define the permanent with diagonal parameters $u = (u_1, u_2, \ldots, u_d)$, and the symmetrized permanent by

$$\text{per}(A_{I,J}; u) = \sum_{\sigma \in \mathfrak{S}_d} (A_{i_{\sigma(t_1)},j_1} + u_1 \delta_{i_{\sigma(t_1)},j_1}) \cdots (A_{i_{\sigma(t_d)},j_d} + u_d \delta_{i_{\sigma(t_d)},j_d})$$

$$= \text{per}(A_{I,J} + 1_{I,J} \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_d \end{pmatrix})$$

where the third summation is taken over $S, T \in \overline{T}_m^n$ which satisfies $S = S' \cup S'' = T' \cup T''$. Then we have the Capelli identity:

$$\omega(X_d) = \sum_{S \in \overline{T}_m^n} \sum_{J \in \overline{T}_m^n} \frac{1}{S! J!} \text{per } P_{S,J} \text{ per } Q_{S,J}.$$
Theorem 4.2. For $d \geq 1$, define $X_d \in U(g)^K = U(gl_{2m}(\mathbb{C}))/U_{Sp}$ by

$$X_d = \sum_{S \in \mathcal{X}_d} \frac{1}{S!} \text{Per}(E + J\epsilon E^{-1})_{S,S}.$$ 

Then we have the Capelli identity:

$$\omega(X_d) = \sum_{l=0}^{d} \frac{(2m+d-1)!}{d!(2m+l-1)!} \sum_{J \in \mathcal{I}^2} \frac{c_{\beta,\alpha}^d}{J!} \sum_{S \in \mathcal{I}^m} \frac{1}{S!} \text{per} P_{S,J} \text{per} Q_{S,J},$$

where $c_{\beta,\alpha}^d$ is an integer defined in Equation (10) with both $p$ and $q$ replaced by $n$, and $\alpha$ and $\beta$ are determined from $J$ in the same way as in Equation (11).

Acknowledgment

We thank Yasuhide Numata for proving Lemmas A.5 and A.6 below. We also thank Hiroshi Oda for fruitful discussions on the identities. The first author is partially supported by JSPS Grant-in-Aid for Scientific Research (B) #17340037.

A Formula for $c_{\beta,\alpha}^d$

In this section we prove (Lemma A.6)

$$c_{\alpha,\beta}^d = \sum_{u+v=\alpha+\beta} b_{u,v}^d \epsilon(\alpha,\beta; u, v),$$

where $c_{\alpha,\beta}^d$, $b_{u,v}^d$ and $\epsilon(\alpha,\beta; u, v)$ are defined in Equation (10), Definition (2.3) and Lemma (2.8) respectively. We start with a simplification of $\epsilon(\alpha,\beta; u, v)$.

Lemma A.1. For non-negative integers $p, q, u, v$ and $J \in \mathcal{I}^{p+q}$, the integer $\epsilon(\alpha,\beta; u, v)$ is defined in Lemma (2.8)

$$\epsilon(J; u, v) = \epsilon(\alpha,\beta; u, v) = \sum_{\sigma \in \mathcal{S}_{u+v}} \epsilon_{j_{\sigma(u+1):j_{\sigma(u+1)}}} \cdots \epsilon_{j_{\sigma(u+v):j_{\sigma(u+v)}}} \epsilon(\alpha,\beta; u, v),$$

where $\alpha$ and $\beta$ are defined by Equation (11). We have

$$\epsilon(\alpha,\beta; u, v) = u!v! \sum_{a+b=v} (-1)^a \binom{\alpha}{a} \binom{\beta}{b}.$$ 

Proof. For a summand $\epsilon_{j_{\sigma(u+1):j_{\sigma(u+1)}}} \cdots \epsilon_{j_{\sigma(u+v):j_{\sigma(u+v)}}}$ of $\epsilon(\alpha,\beta; u, v)$, let $a$ be the number of $j_{\sigma(u+i)}$ $(1 \leq i \leq v)$ which is less than or equal to $p$. Similarly let $b$ be the number of $j_{\sigma(u+i)}$ $(1 \leq i \leq v)$ which is greater than or equal to $p+1$. In particular $a+b=v$, and the value of the summand is equal to $(-1)^a$.

We count the number of $\sigma$'s in $\mathcal{G}_{u+v}$ such that they give the same $a$ and $b$. There are $\binom{a}{u}$ choices of subset $\{\sigma(u+1), \sigma(u+2), \ldots, \sigma(u+v)\}$ of $\{1,2,\ldots,u+v\}$. By considering the order of this subset and the order of the complement $\{\sigma(1), \sigma(2), \ldots, \sigma(u)\}$, the number is multiplied by $u!v!$. Thus the number of such $\sigma$ is equal to $\binom{a}{u} \binom{\beta}{b} u!v!$. 

\[\square\]
Next we show a recurrence formula for $b_{u,v}^d$ in Lemma A.2 and two closed formulas in Lemmas A.4 and A.5.

**Lemma A.2.** We have a recurrence formula for $b_{u,v}^d$:

$$b_{0,0}^0 = 1, \quad b_{u,v}^{d+1} = b_{u-1,v}^d + (u + 1)b_{u,v-1}^d + (v + 1)b_{u-1,v+1}^d.$$  

**Proof.** By the definition of $b_{u,v}^d$, we have

$$\Xi_{d+1} = \sum_{0 \leq u + v \leq d} b_{u,v}^d \gamma(u,v) \tau^{d-u-v-\Xi}.$$  

It follows from Lemma 2.7 (1) that the expression above is equal to

$$\sum_{0 \leq u + v \leq d} b_{u,v}^d \tau^{d-u-v} (\gamma(u + 1, v + 1) + u \tau \gamma(u - 1, v + 1) + v \tau \gamma(u + 1, v - 1)).$$  

Hence we have $b_{u,v}^{d+1} = b_{u-1,v}^d + (u + 1)b_{u,v-1}^d + (v + 1)b_{u-1,v+1}^d$.  

**Lemma A.3.** We define the generating function $f_d$ of $b_{u,v}^d$ by

$$f_d(x,y) = \sum_{u,v \in \mathbb{Z}} b_{u,v}^d x^u y^v.$$  

Then it satisfies $f_{d+1}(x,y) = x f_d(x,y) + y \frac{\partial}{\partial x} f_d(x,y) + x \frac{\partial}{\partial y} f_d(x,y).$

**Proof.** By multiplying $x^u y^v$ to the formula of Lemma A.2, and take a sum over $u,v \in \mathbb{Z}$, and we have

$$f_{d+1} = \sum_{u,v \in \mathbb{Z}} b_{u-1,v}^d x^u y^v + \sum_{u,v \in \mathbb{Z}} (u + 1)b_{u,v-1}^d x^u y^v + \sum_{u,v \in \mathbb{Z}} (v + 1)b_{u-1,v+1}^d x^u y^v$$

$$= x f_d + y \frac{\partial}{\partial x} f_d + x \frac{\partial}{\partial y} f_d.$$  

**Lemma A.4.** We have a closed formula for $b_{u,v}^d$:

$$b_{u,v}^d = \sum_{m \geq 0, \mu, \nu \in \mathbb{Z}} \frac{(u - v + m + 2\mu - 2\nu)^d (-1)^{m+\nu}}{2^{u+v-m} u! \nu!} \binom{u}{\nu} \binom{v}{m} \binom{v-m}{\mu}.$$  

**Proof.** Set $\varphi_d = e^y f_d$. Since we have an identity $\frac{\partial}{\partial y} e^{-y} = e^{-y} \frac{\partial}{\partial y} - e^{-y}$ as differential operators, it follows from Lemma A.3 that

$$e^{-y} \varphi_{d+1} = e^{-y} x \varphi_d + e^{-y} y \frac{\partial}{\partial x} \varphi_d + x \left( e^{-y} \frac{\partial}{\partial y} - e^{-y} \right) \varphi_d,$$

and hence

$$\varphi_{d+1} = y \frac{\partial}{\partial x} \varphi_d + x \frac{\partial}{\partial y} \varphi_d.$$  

(29)
Put \( x = a - b, \) \( y = a + b. \) Then we have
\[
y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = a \frac{\partial}{\partial a} - b \frac{\partial}{\partial b} = \theta_a - \theta_b, \quad \text{where} \quad \theta_a = a \frac{\partial}{\partial a}.
\]
If we make the change of variables and put \( \psi_d(a, b) = \varphi_d(a - b, a + b), \) Equation 29 gives
\[
\psi_d = (\theta_a - \theta_b)\psi_{d-1} = (\theta_a - \theta_b)^d \psi_0 = (\theta_a - \theta_b)^d e^{a+b} \quad (\therefore \psi_0 = e^{a+b}).
\]
Since \( (\theta_a - \theta_b)(a^k b^l) = (k-l)a^k b^l, \) we have
\[
\psi_d = (\theta_a - \theta_b)^d \left( \sum_{k,l \geq 0} \frac{a^k b^l}{k! l!} \right) = \sum_{k,l \geq 0} \frac{(k-l)^d}{k! l!} a^k b^l.
\]
Therefore
\[
f_d = e^{-y} \psi_d = e^{-y} \sum_{k,l \geq 0} \frac{(k-l)^d}{k! l!} \left( \frac{x+y}{2} \right)^k \left( \frac{y-x}{2} \right)^l
\]
\[
= \sum_{k,l,m \geq 0} \frac{(-y)^m}{m!} \frac{(k-l)^d}{k! l!} \sum_{\mu,\nu \in \mathbb{Z}} \left( k \atop \mu \atop \nu \right) x^{k-m} y^\mu \frac{l!}{2^k} \frac{y^{l-\nu}(-x)^\nu}{2^l}
\]
\[
= \sum_{k,l,m \geq 0} \frac{(k-l)^d(-1)^{m+\nu}}{2^{k+l} k! l! m!} \left( k \atop \mu \atop \nu \right) x^{k-m+\nu} y^{\mu+l+\nu},
\]
(let \( u = k - \mu + \nu \) and \( v = m + \mu + l - \nu \))
\[
= \sum_{u,v,m \geq 0} \frac{(u-v+m+2\mu-2\nu)^d(-1)^{m+\nu}}{2^{u+v-m} m!} \frac{x^u y^v}{(u-v)! \mu! (v-m-\mu)! \nu!}
\]
\[
= \sum_{u,v,m \geq 0} \frac{(u-v+m+2\mu-2\nu)^d(-1)^{m+\nu}}{2^{u+v-m} m!} \left( u \atop \nu \atop \mu \atop m \right) (v-m) v^u y^v u! v!.
\]

Lemma A.5. We have another closed formula
\[
b_{u,v}^d = \frac{1}{2^{u+v} u! v!} \sum_{k,l \in \mathbb{Z}} \left( \begin{array}{c} 2v \\ k \end{array} \right) \frac{u^k}{l} (-1)^{k+l}(u+v-k-2l)^d.
\]

Proof. Put
\[
F_{u,v}^d(T) = \frac{1}{2^{u+v} u! v!} T^{u-v}(-2T + T^2 + 1)^v (1 - T^{-2})^u.
\]

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By the multinomial expansion, we have

\[ F_{u,v}(T) = \frac{1}{2^u v! u!} \sum_{m, \mu, \nu \in \mathbb{Z}} \binom{v}{m, \mu} \binom{u}{\nu} (-1)^{m+\nu} 2^m T^{u-v+m+2\mu-2\nu}. \]

Let \( \theta = T(\partial/\partial T) \), and apply \( \theta^d \) to the expression above and then substitute \( T = 1 \). Then we have

\[ \theta^d(F_{u,v}) \bigg|_{T=1} = \frac{1}{2^u v! u!} \sum_{m, \mu, \nu \in \mathbb{Z}} \binom{v}{m, \mu} \binom{u}{\nu} (-1)^{m+\nu} 2^m (u - v + m + 2\mu - 2\nu)^d, \]

which is equal to \( b_{u,v}^d \) by Lemma A.4. On the other hand

\[ F_{u,v}(T) = \frac{T^{u-v}}{2^u v! u!} (T-1)^{2v} (1-T^{-2})^u = \frac{T^{-u-v}}{2^u v! v!} (T-1)^{2v} (T^2 - 1)^u \]

\[ = \frac{1}{2^u v! v!} \sum_{k, l \in \mathbb{Z}} \binom{2v}{k} \binom{u}{l} (-1)^{k+l} T^{u+v-k-2l}. \]

Similar computation as above, i.e., applying \( \theta^d \) to this expression and substituting \( T = 1 \), gives us the same value as

\[ \theta_d(F_{u,v}) \bigg|_{T=1} = \frac{1}{2^u v! v!} \sum_{k, l \in \mathbb{Z}} \binom{2v}{k} \binom{u}{l} (-1)^{k+l} (u + v - k - 2l)^d. \]

\[ \square \]

By Lemmas A.5 and A.1, we have

\[ \sum_{u+v=\alpha+\beta} b_{u,v}^d(\alpha, \beta; u, v) \]

\[ = \sum_{u+v=\alpha+\beta} \frac{1}{2^u v! v!} \sum_{k, l \in \mathbb{Z}} \binom{2v}{k} \binom{u}{l} (-1)^{k+l} (u + v - k - 2l)^d \]

\[ \times \sum_{a+b=v} a! v! (-1)^a \binom{a}{a} \binom{b}{b} \]

\[ = \frac{1}{2^{\alpha+\beta}} \sum_{k, l, a, b \in \mathbb{Z}} \binom{\alpha}{a} \binom{\beta}{b} \binom{2a+2b}{k} \binom{\alpha + \beta - a - b}{l} \]

\[ \times (-1)^{k+l+a} (\alpha + \beta - k - 2l)^d. \quad (30) \]

Now we are ready to show the following lemma which is the goal of this section.

**Lemma A.6.** We have

\[ c_{\alpha, \beta}^d = \sum_{u+v=\alpha+\beta} b_{u,v}^d(\alpha, \beta; u, v). \]
Proof. Set

\[ F_{\alpha,\beta}(T) = \left( \frac{T^2 - 1}{2T} \right)^{\alpha+\beta} \left( \frac{-(T-1)^2}{(T^2 - 1)} + 1 \right)^{\alpha} \left( \frac{(T-1)^2}{(T^2 - 1)} + 1 \right)^{\beta} . \]

First, by the binomial expansion, we have

\[ F_{\alpha,\beta}(T) = \left( \frac{T^2 - 1}{2T} \right)^{\alpha+\beta} \sum_{a \in \mathbb{Z}} \binom{\alpha}{a} (-1)^a \left( \frac{T-1}{T^2 - 1} \right)^a \sum_{b \in \mathbb{Z}} \binom{\beta}{b} \left( \frac{T-1}{T^2 - 1} \right)^b \]

\[ \times \sum_{a,b \in \mathbb{Z}} \binom{\alpha + \beta - a - b}{l} \left( \frac{2a + 2b}{k} \right) T^{2a + 2b - k} (-1)^k \]

\[ = \frac{1}{2^{\alpha+\beta}} \sum_{a,b,k,l \in \mathbb{Z}} \binom{\alpha}{a} \binom{\beta}{b} \left( \frac{2a + 2b}{k} \right) \binom{\alpha + \beta - a - b}{l} \times (-1)^{a+k+l} \frac{1}{T^{\alpha + \beta - k - 2l}} . \]

We apply \( \theta_d \) to this expression, and let \( T = 1 \). Then we have

\[ \theta_d(F_{u,v}^d) \bigg|_{T=1} = \sum_{u+v=\alpha+\beta} b_{u,v}^d(\alpha, \beta; u, v), \]

by Equation (30). Second, we proceed as

\[ F_{\alpha,\beta}(T) = \frac{1}{(2T)^{\alpha+\beta}} \left( -(T-1)^2 + (T^2 - 1) \right)^{\alpha} \left( (T-1)^2 + (T^2 - 1) \right)^{\beta} \]

\[ = \frac{1}{(2T)^{\alpha+\beta}} (2T - 2)^{\alpha} (2T^2 - 2T)^{\beta} \]

\[ = \frac{(T - 1)^{\alpha+\beta}}{T^\alpha} \sum_{k \in \mathbb{Z}} \binom{\alpha + \beta}{k} T^{\alpha + \beta - k} (-1)^k \cdot \frac{1}{T^k} . \]

We apply \( \theta_d \) to this expression, and let \( T = 1 \). Then we have

\[ \theta_d(F_{u,v}^d) \bigg|_{T=1} = c_{\alpha,\beta}^d, \]

by the definition of \( c_{\alpha,\beta}^d \) (Equation (10)). \( \square \)

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