CATEGORIFICATION OF DONALDSON-THOMAS INVARIANTS
VIA PERVERSE SHEAVES
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Abstract. We show that there is a perverse sheaf on a fine moduli space
of stable sheaves on a smooth projective Calabi-Yau 3-fold, which is locally
the perverse sheaf of vanishing cycles for a local Chern-Simons functional,
possibly after taking an étale Galois cover. This perverse sheaf lifts to a mixed
Hodge module and gives us a cohomology theory which enables us to define
the Gopakumar-Vafa invariants mathematically.

1. Introduction

The Donaldson-Thomas invariant (DT invariant, for short) is a virtual count
of stable sheaves on a smooth projective Calabi-Yau 3-fold \( Y \) over \( \mathbb{C} \) which was
defined as the degree of the virtual fundamental class of the moduli space \( \mathcal{X} \) of stable
sheaves ([35]). Using microlocal analysis, Behrend showed that the DT invariant is
in fact the Euler number of the moduli space, weighted by a constructible function
\( \nu_X \), called the Behrend function ([1]). Since the ordinary Euler number is the
alternating sum of Betti numbers of cohomology groups, it is reasonable to ask if
the DT invariant is in fact the Euler number of a cohomology theory on \( \mathcal{X} \). On the
other hand, it has been known that the moduli space is locally the critical locus
of a holomorphic function, called a local Chern-Simons functional ([12]). Given
a holomorphic function \( f \) on a complex manifold \( V \), one has the perverse sheaf
\( \phi_f(\mathbb{Q}[\dim V - 1]) \) of vanishing cycles supported on the critical locus and the Euler
number of this perverse sheaf at a point \( x \) equals \( \nu_X(x) \). This motivated Joyce and
Song to raise the following question ([12, Question 5.7]).

Let \( \mathcal{X} \) be the moduli space of simple coherent sheaves on \( Y \). Does there exist a
natural perverse sheaf \( P^\bullet \) on the underlying analytic variety \( X = \mathcal{X}_{\text{red}} \) which is
locally isomorphic to the sheaf \( \phi_f(\mathbb{Q}[\dim V - 1]) \) of vanishing cycles for \( f \), \( V \) above?

The purpose of this paper is to provide an affirmative answer.

Theorem 1.1. (Theorem 3.14 and Theorem 3.16)
Let \( \mathcal{X} \) be a quasi-projective moduli space of simple sheaves on a smooth projective
Calabi-Yau 3-fold \( Y \) with universal family \( \mathcal{E} \) and let \( X = \mathcal{X}_{\text{red}} \) be the reduced scheme
of \( \mathcal{X} \). Then there exist an étale Galois cover
\[
\rho : X^\dagger \to X = X^\dagger / G
\]
and a perverse sheaf \( P^\bullet \) on \( X^\dagger \), which is locally isomorphic to the perverse sheaf
\( \phi_f(\mathbb{Q}[\dim V - 1]) \) of vanishing cycles for the local Chern-Simons functional \( f \). In
fact, for any étale Galois cover $\rho: X^{\dagger} \to X$, there exists such a perverse sheaf $P^*$ if and only if the line bundle $\rho^* \det(\text{Ext}^k_\pi(E,E))$ admits a square root on $X^{\dagger}$ where $\pi: X \times Y \to X$ is the projection and $\text{Ext}^k_\pi(E,E) = R\pi_* R\mathcal{H}om(E,E)$.

We will also prove the same for mixed Hodge modules (Theorem 7.1), i.e. there is a mixed Hodge module $M^*$ on $X^{\dagger}$ whose underlying perverse sheaf is $\text{rat}(M^*) = P^*$.

As an application of Theorem 1.1, when $\det \text{Ext}^k_\pi(E,E)$ admits a square root, the hypercohomology $\mathbb{H}^i(X, P^*)$ gives us the DT (Laurent) polynomial

$$DT^Y_i(X) = \sum_i t^i \dim \mathbb{H}^i(X, P^*)$$

such that $DT^Y_{-1}(X)$ is the ordinary DT invariant by [1].

Another application is a mathematical theory of Gopakumar-Vafa invariants (GV for short) in [9]. Let $X$ be a moduli space of stable sheaves supported on curves of homology class $\beta \in H_2(Y, \mathbb{Z})$. The GV invariants are integers $n_h(\beta)$ for $h \in \mathbb{Z}_{\geq 0}$ defined by an $sl_2 \times sl_2$ action on some cohomology of $X$ such that $n_h(\beta)$ is the DT invariant of $X$ and that they give all genus Gromov-Witten invariants $N_g(\beta)$ of $Y$. By Theorem 1.1, when $\det \text{Ext}^k_\pi(E,E)$ admits a square root, there exists a perverse sheaf $P^*$ on $X$ which is locally the perverse sheaf of vanishing cycles. By the relative hard Lefschetz theorem for the morphism to the Chow scheme ([30]), we have an action of $sl_2 \times sl_2$ on $H^*(X, P^*)$ where $P^*$ is the graduation of $P^*$ by the filtration of $P^*$ which is the image of the weight filtration of the mixed Hodge module $M^*$ with $\text{rat}(M^*) = P^*$. This gives us a geometric theory of GV invariants which we conjecture to give all the GW invariants $N_g(\beta)$.

Our proof of Theorem 1.1 relies heavily on gauge theory. By the Seidel-Thomas twist ([12, Chapter 8]), it suffices to consider only vector bundles on $Y$. Let $B = A/G$ be the space of semiconnections on a hermitian vector bundle $E$ modulo the gauge group action and let $B_{si}$ be the open subset of simple points. Let $B: B \to \mathbb{C}$ be the (holomorphic) Chern-Simons functional. Let $X \subset B_{si}$ be a locally closed complex analytic subspace. We call a finite dimensional complex submanifold $V$ of $B_{si}$ a CS chart if the critical locus of $f = cs|_V$ is $V \cap X$ and is an open complex analytic subspace of $X$. By [12], at each $x \in X$, we have a CS chart $V$ with $T_xV = T_xX$, which we call the Joyce-Song chart (JS chart, for short). Thus we have a perverse sheaf $P^*_V$ on $V$ which is the dimensions of the JS charts $V$ vary from point to point. In this paper, we show that there are

1. a locally finite open cover $X = \cup_\alpha U_\alpha$;
2. a (continuous) family $V_\alpha \to U_\alpha$ of CS charts of constant dimension $r$, each of which contains the JS chart;
3. a homotopy $V_{\alpha \beta} \to U_{\alpha \beta} \times [0,1]$ from $V_\alpha|_{U_{\alpha \beta}} = V|_{t=0}$ to $V_\beta|_{U_{\alpha \beta}} = V|_{t=1}$ where $U_{\alpha \beta} = U_\alpha \cap U_\beta$. We call such a collection CS data. (See Proposition 3.12.)

From the CS data, we can extract perverse sheaves $P^*_\alpha$ on $U_\alpha$ for all $\alpha$ and gluing isomorphisms $\sigma_{\alpha \beta}: P^*_\alpha|_{U_{\alpha \beta}} \to P^*_\beta|_{U_{\alpha \beta}}$. (See Proposition 3.13.) The 2-cocycle obstruction for gluing $\{P^*_\alpha\}$ to a global perverse sheaf is shown to be

$$\sigma_{\alpha \beta \gamma} = \sigma_{\alpha \beta} \circ \sigma_{\beta \gamma} \circ \sigma_{\alpha \beta} = \pm 1 \in \mathbb{Z}_2.$$
which coincides with the 2-cocycle obstruction for gluing the determinant line bundles of the tangent bundles of CS charts. Since the perfect obstruction theory $\text{Ext}^\bullet_\pi(E, E)$ for $X$ is symmetric, the determinant of the tangent bundle is a square root of $\det \text{Ext}^\bullet_\pi(E, E)$. Therefore the local perverse sheaves $\{P^\bullet_\alpha\}$ glue to a global perverse sheaf if and only if there is a square root of $\det \text{Ext}^\bullet_\pi(E, E)$ in $\text{Pic}(X)$. (See Theorem 3.14.)

When $X$ is the moduli scheme of one dimensional stable sheaves on $Y$, we show that the torsion-free part of $\det \text{Ext}^\bullet_\pi(E, E)$ has a square root by Grothendieck-Riemann-Roch. More generally, Z. Hua ([11]) proved that it is true for all sheaves. In §9, we simplify his proof and generalize his result to the case of perfect complexes. By taking a spectral cover using a torsion line bundle, we obtain a finite étale Galois cover $\rho : X^\dagger \to X$ with a cyclic Galois group $G$ and a perverse sheaf $P^\bullet$ on $X^\dagger$ which is locally the perverse sheaf of vanishing cycles of a local CS functional. (See Theorem 3.16.)

The layout of this paper is as follows. In §2, we recall necessary facts about the perverse sheaves of vanishing cycles and their gluing. In §3, we collect the main results of this paper. In §4, we prove that there exists a structure which we call preorientation data on $X$. In §5, we show that preorientation data induce CS data mentioned above. In §6, we show that CS data induce local perverse sheaves, gluing isomorphisms and the obstruction class for gluing. In §7, we prove an analogue of Theorem 1.1 for mixed Hodge modules. In §8, we develop a theory of GV invariants. In §9, we discuss the existence of square root of $\det \text{Ext}^\bullet_\pi(E, E)$.

An incomplete version of this paper was posted in the arXiv on October 15, 2012 (1210.3910). Some related results were independently obtained by C. Brav, V. Bussi, D. Dupont, D. Joyce and B. Szendroi in [5]. We are grateful to Dominic Joyce for his comments and suggestions. We thank Martin Olsson for his comments, Zheng Hua for informing us of his paper [11] and Yan Soibelman for his comments. We also thank Takuro Mochizuki for answering questions on mixed Hodge modules.

**Notations.** A complex analytic space is a local ringed space which is covered by open sets, each of which is isomorphic to a ringed space defined by an ideal of holomorphic functions on an analytic open subset of $\mathbb{C}^n$ for some $n > 0$, and whose transition maps preserve the sheaves of holomorphic functions. A complex analytic variety is a reduced complex analytic space. We will denote the variety underlying a complex analytic space $X$ by $X$. We will use smooth functions to mean $C^\infty$ functions. In case the space is singular, with a stratification by smooth strata, smooth functions are continuous functions that are smooth along each stratum. We use analytic functions to mean continuous functions that locally have power series expansions in the real and imaginary parts of coordinate variables. We will work with analytic topology unless otherwise mentioned.

2. **Perverse sheaves of vanishing cycles**

In this section, we recall necessary facts about perverse sheaves of vanishing cycles. Let $X$ be a complex analytic variety and $D^b_c(X)$ the bounded derived category of constructible sheaves on $X$ over $\mathbb{Q}$. Perverse sheaves are sheaf complexes which behave like sheaves.
Definition 2.1. An object $P^\bullet \in D^b(X)$ is called a perverse sheaf (with respect to the middle perversity) if

1. $\dim \{ x \in X \mid H^i(\varepsilon^* P^\bullet) = H^i(B_x(x) ; P^\bullet) \neq 0 \} \leq -i$ for all $i$;
2. $\dim \{ x \in X \mid H^i(\varepsilon^* P^\bullet) = H^i(B_x(x) , B_x(x) - \{ x \} ; P^\bullet) \neq 0 \} \leq i$ for all $i$

where $\varepsilon^* : \{ x \} \hookrightarrow X$ is the inclusion and $B_x(x)$ is the open ball of radius $\varepsilon$ centered at $x$ for $\varepsilon$ small enough.

Perverse sheaves form an abelian category $\mathcal{Perv}(X)$ which is the core of a t-structure ([2, §2]). An example of perverse sheaf is the sheaf of vanishing cycles which is the focus of this paper.

Definition 2.2. Let $f : V \rightarrow \mathbb{C}$ be a continuous function on a pseudo-manifold $V$. We define

$$ A^r_f := \phi_f(\mathbb{Q}[-1]) = R\Gamma_{\{ \text{Re} f \leq 0 \}} \mathbb{Q}|_{f^{-1}(0)}. $$

When $V$ is a complex manifold of dimension $r$ and $f$ is holomorphic, $A^r_f|_{r}$ is a perverse sheaf on $f^{-1}(0)$. (See [13, Chapter 8].) Let $X_f$ be the critical set ($df = 0$) of $f$ in $V$. Since $A^r_f|_{r} = \phi_f(\mathbb{Q}[r - 1])$ is zero on the smooth manifold $f^{-1}(0) - X_f$, $A^r_f|_{r}$ is a perverse sheaf on $X_f$, called the perverse sheaf of vanishing cycles for $f$.

The stalk cohomology of $A^r_f|_{r}$ at $x \in f^{-1}(0)$ is the reduced cohomology $H^r(M_f)$ of the Milnor fiber

$$ M_f = f^{-1}(\delta) \cap B_r(x) \quad \text{for} \quad 0 < \delta \ll \varepsilon \ll 1. $$

Proposition 2.3. Let $f, f_0, f_1 : V \rightarrow \mathbb{C}$ be continuous functions on a pseudo-manifold $V$.

1. Let $Z$ be a subset of $f_0^{-1}(0) \cap f_1^{-1}(0)$. Suppose $\Phi : V \rightarrow V$ is a homeomorphism such that $\Phi|_Z = \text{id}_Z$ and $f_1 \circ \Phi = f_0$. Then $\Phi$ induces an isomorphism $\Phi^* : A^r_{f_1}|_Z \xrightarrow{\cong} A^r_{f_0}|_Z$ in $D^b(Z)$ by pulling back.

2. Suppose $\Phi_t : V \rightarrow V$, $t \in [0, 1]$, is a continuous family of homeomorphisms preserving $f$, i.e. $f \circ \Phi_t = f$ for all $t$, such that $\Phi_1|_Z = \text{id}_Z$ and $\Phi_0 = \text{id}_V$. Then the pullback isomorphism $\Phi_1^* : A^r_f|_Z \xrightarrow{\cong} A^r_f|_Z$ is the identity morphism.

Proof. By the definition of $A^r_f$, we have $A^r_{f_0} = A^r_{f_1 \circ \Phi} \cong \Phi^* A^r_{f_1}$. Let $\iota : Z \hookrightarrow V$ denote the inclusion. Since $\Phi \circ \iota = \iota$, we have the isomorphism

$$ \Phi^* : \iota^* A^r_{f_1} = \iota^* \Phi^* A^r_{f_1} \xrightarrow{\cong} \iota^* A^r_{f_0}. $$

Because $\Phi_t$ preserves the set $\{ \text{Re} f \leq 0 \}$, the isotopy $\{ \Phi_t \}$ induces a homotopy from the identity chain map $\Phi_0^* = \text{id}_A^r|_Z$ to $\Phi_1^*$ by choosing a flabby resolution $I^\bullet$ by the complex of singular cochains. Since homotopic chain maps are equal in the derived category, we find that the induced isomorphism $\Phi_1^* : A_f^r|_Z \xrightarrow{\cong} A_f^r|_Z$ is indeed the identity morphism.

Example 2.4. Let $q = \sum_{i=1}^{r} y_i^2$ on $\mathbb{C}^r$. The set $\{ \text{Re} q > 0 \} \subset \mathbb{C}^r$ is a disk bundle over $\mathbb{R}^{r-1}$ which is obviously homotopic to $S^{r-1}$. From the distinguished triangle

$$ R\Gamma_{\{ \text{Re} q \leq 0 \}} \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow R\iota_* \iota^* \mathbb{Q} $$

where $\iota : \{ \text{Re} q > 0 \} \hookrightarrow \mathbb{C}^r$ is the inclusion, we find that $A^r_q[1]$ is a sheaf complex supported at the origin satisfying $A^r_q[1] \cong \mathbb{Q}[-r + 1]$, i.e.

$$ A^r_q[1] \cong \mathbb{Q}. $$
Suppose $\Phi : \mathbb{C}^r \to \mathbb{C}^r$ is a homeomorphism such that $q \circ \Phi = q$. Since $A^\bullet_r \cong \mathbb{Q}$, the isomorphism $\Phi^* : A^\bullet_0 \to A^\bullet_0$ is either 1 or $-1$. The sign is determined by the change in the orientation of the sphere $S^{r-1}$ in the Milnor fiber. Since $q$ is preserved by $\Phi$, $d\Phi|_0 : T_0\mathbb{C}^r \to T_0\mathbb{C}^r$ is an orthogonal linear transformation with respect to $q$ whose determinant is either 1 or $-1$. It is easy to see that these two sign changes are identical, i.e.

$$\Phi^* = \det(d\Phi|_0) \cdot \text{id}.$$ 

The following fact about the sheaf $A^\bullet_f$ of vanishing cycles will be useful.

**Proposition 2.5.** (1) Let $g : W \to \mathbb{C}$ be a holomorphic function on a connected complex manifold $W$ of dimension $d$ and let $q = \sum_{i=1}^d y_i^2$. Let $V = W \times \mathbb{C}^r$ and $f : V \to \mathbb{C}$ be $f(z, y) = g(z) + q(y)$. Then the summation form of $f$ induces an isomorphism

$$A^\bullet_f[d + r] \cong pr_1^{-1}A^\bullet_g[d] \otimes pr_2^{-1}A^\bullet_q[r] \cong pr_1^{-1}A^\bullet_g[d] \otimes \mathbb{Q} \cong pr_1^{-1}A^\bullet_g[d]$$

of perverse sheaves on the critical set $X_f$ of $f$.

(2) Let $\Phi : V \to V$ be a biholomorphic map such that $f \circ \Phi = f$ and $\Phi|_W = \text{id}_{W \times \{0\}}$. Then $\Phi^* : A^\bullet_f \to A^\bullet_f$ is $\det(d\Phi|_{W \times \{0\}}) \cdot \text{id}_{A^\bullet_f}$ and $\det(d\Phi|_{W \times \{0\}}) = \pm 1$.

**Proof.** (1) is a result of D. Massey in [24, §2]; (2) is proved in [5, Theorem 3.1].

It is well known that perverse sheaves and isomorphisms glue.

**Proposition 2.6.** Let $X$ be a complex analytic space with an open covering $\{X_\alpha\}$.

(1) Suppose that for each $\alpha$ we have $P^\bullet_\alpha \in \text{Perv}(X_\alpha)$ and for each pair $\alpha, \beta$ we have isomorphisms

$$\sigma_{\alpha\beta} : P^\bullet_\alpha|_{X_\alpha \cap X_\beta} \cong P^\bullet_\beta|_{X_\alpha \cap X_\beta}$$

satisfying the cocycle condition $\sigma_{\beta\gamma} \circ \sigma_{\alpha\beta} = \sigma_{\alpha\gamma}$. Then $\{P^\bullet_\alpha\}$ glue to define a perverse sheaf $P^\bullet$ on $X$ such that $P^\bullet|_{X_\alpha} \cong P^\bullet_\alpha$ and that $\sigma_{\alpha\beta}$ is induced by the identity map of $P^\bullet|_{X_\alpha \cap X_\beta}$.

(2) Suppose $P^\bullet, Q^\bullet \in \text{Perv}(X)$ and $\sigma_\alpha : P^\bullet|_{X_\alpha} \cong Q^\bullet|_{X_\alpha}$ such that $\sigma_{\alpha|_{X_\alpha \cap X_\beta}} = \sigma_{\beta|_{X_\alpha \cap X_\beta}}$. Then there exists an isomorphism $\sigma : P^\bullet \to Q^\bullet$ such that $\sigma|_{X_\alpha} = \sigma_\alpha$ for all $\alpha$.

See [5, Theorem 2.5] for precise references for proofs of Proposition 2.6. One way to prove Proposition 2.6 is to use the elementary construction of perverse sheaves by MacPherson and Vilonen.

**Theorem 2.7.** [22, Theorem 4.5] Let $S \subset X$ be a closed stratum of complex codimension $c$. The category $\text{Perv}(X)$ is equivalent to the category of objects $(B^\bullet, C) \in \text{Perv}(X - S) \times \text{Sh}_{\mathbb{Q}}(S)$ together with a commutative triangle

$$R^{-c-1}\pi_\star \kappa^* B^\bullet \quad \xrightarrow{m} \quad R^{-c-1}\pi_\star \gamma^* B^\bullet$$

such that $\ker(m)$ and $\coker(m)$ are local systems on $S$, where $\kappa : K \hookrightarrow L$ and $\gamma : L - K \hookrightarrow L$ are inclusions of the perverse link bundle $K$ and its complement.
$L - K$ in the link bundle $\pi : L \to S$. The equivalence of categories is explicitly given by sending $P^\bullet \in \text{Perv}(X)$ to $B^\bullet = P^\bullet|_{X - S}$ together with the natural morphisms

$$
\begin{align*}
R^{-c-1}\pi_*\kappa_*\kappa!B & \to R^{-c}\pi_*\gamma!\gamma^*B^\bullet \\
\downarrow m & \downarrow n \\
R^{-c}\pi_*\varphi^*P^\bullet & \to \quad \quad \quad R^{-c}\pi_*\varphi^*P^\bullet
\end{align*}
$$

where $\varphi : D - K \hookrightarrow D$ is the inclusion into the normal slice bundle.

See [22, §4] for precise definitions of $K$, $L$ and $D$. Morally the above theorem says that an extension of a perverse sheaf on $X - S$ to $X$ is obtained by adding a sheaf on $S$. Since sheaves glue, we can glue perverse sheaves stratum by stratum.

**Proof of Proposition 2.6.** We stratify $X$ by complex manifolds and let $X^{(i)}$ denote the union of strata of codimension $\leq i$. On the smooth part $X^{(0)}$, $P^\bullet_\alpha$ are honest sheaves and hence they glue to a sheaf $P^\bullet|_{X^{(0)}}$. For $X^{(1)} = X^{(0)} \cup S$, we find that since $P^\bullet_\alpha$ are isomorphic on intersections $X_\alpha \cap X_\beta$, the sheaves $R^{-c}\pi_*\varphi^*P^\bullet_\alpha$ glue and so do the natural triangles

$$
\begin{align*}
R^{-2}\pi_*\kappa_*\kappa!\left(P^\bullet|_{X^{(0)}}\right) & \to R^{-1}\pi_*\gamma!\gamma^*\left(P^\bullet|_{X^{(0)}}\right) \\
\downarrow m & \downarrow n \\
R^{-1}\pi_*\varphi^*P^\bullet_\alpha & \to \quad \quad \quad R^{-1}\pi_*\varphi^*P^\bullet_\alpha
\end{align*}
$$

Hence we obtain a perverse sheaf $P^\bullet|_{X^{(0)}} \in \text{Perv}(X^{(0)})$. It is obvious that we can continue this way using Theorem 2.7 above until we obtain a perverse sheaf $P^\bullet$ on $X$ such that $P^\bullet|_{X^{(i)}} = P^\bullet_\alpha$.

The gluing of isomorphisms is similar. \hfill \Box

Another application of Theorem 2.7 is the following **rigidity property of perverse sheaves**.

**Lemma 2.8.** Let $P^\bullet$ be a perverse sheaf on an analytic variety $U$. Let $\pi : T \to U$ be a continuous map from a topological space $T$ with connected fibers and let $T'$ be a subspace of $T$ such that $\pi|_{T'}$ is surjective. Suppose an isomorphism $\mu : \pi^{-1}P^\bullet \cong \pi^{-1}P^\bullet$ satisfies $\mu|_{T'} = \text{id}_{(\pi^{-1}P^\bullet)|_{T'}}$. Then $\mu = \text{id}_{\pi^{-1}P^\bullet}$.

**Proof.** We first prove the simple case: if we let $C$ be a locally constant sheaf over $\mathbb{Q}$ of finite rank on $Z \subset U$ and $\tilde{\mu} : \pi^{-1}C \to \pi^{-1}C$ be a homomorphism such that $\tilde{\mu}|_{T' \cap \pi^{-1}(Z)} = \text{id}$, then $\tilde{\mu}$ is the identity morphism. Indeed, since the issue is local, we may assume that $Z$ is connected and that $C \cong \mathbb{Q}^r$ so that $\tilde{\mu} : \mathbb{Q}^r \to \mathbb{Q}^r$ is given by a continuous map $\pi^{-1}(Z) \to GL(r, \mathbb{Q})$. By connectedness, this obviously is a constant map which is 1 along $T' \cap \pi^{-1}(Z)$. We thus proved the lemma in the sheaf case.

For the general case, we use Theorem 2.7. As in the proof of Proposition 2.6 above, we stratify $U$ and let $U^{(i)}$ be the union of strata of codimension $\leq i$. Since $P^\bullet$ is a perverse sheaf, $P^\bullet|_{U^{(0)}}[\dim U]$ is isomorphic to a locally constant sheaf and hence $\mu|_{U^{(0)}}$ is the identity map. For $U^{(1)} = U^{(0)} \cup S$, using the notation of Theorem 2.7, $C = R^{-1}\pi_*\varphi^*P^\bullet$ is a locally constant sheaf and $\mu$ induces a homomorphism $\pi^{-1}C \to \pi^{-1}C$ which is identity on $T' \cap \pi^{-1}(S)$. Therefore $\mu$ induces the identity
morphism of the pullback of (2.2) by \( \pi \) to itself and hence \( \mu \) is the identity on \( U^{(1)} \).
Continuing in this fashion, we obtain Lemma 2.8.

### 3. Preorientation Data and Perverse Sheaves

In this section, we collect the main results of this paper. We first introduce the notion of preorientation data which induces a family of CS charts. This will give us a collection of local perverse sheaves and gluing isomorphisms. We identify the cocycle condition for the gluing isomorphisms as the existence of a square root of the determinant bundle of \( \text{Ext}^*_E(\mathcal{E}, \mathcal{E}) = R\pi_* R\text{Hom}(\mathcal{E}, \mathcal{E}) \) where \( \mathcal{E} \) denotes the universal bundle over \( X \times Y \to X \).

#### 3.1. Chern-Simons functionals on connection spaces

In this subsection we briefly recall the necessary gauge theoretic background. More details will be provided in later sections. Our presentation largely follows [12, Chapter 9].

Let \( Y \) be a smooth projective Calabi-Yau 3-fold over \( \mathbb{C} \), with a Hodge metric implicitly chosen. We fix a nowhere vanishing holomorphic \((3, 0)\)-form \( \Omega \) on \( Y \). Let \( E \) be a smooth complex vector bundle on \( Y \) with a smooth hermitian metric. In this paper, a smooth semiconnection is a differential operator \( \bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E) \) satisfying the \( \bar{\partial} \)-Leibniz rule. We denote by \( \Omega^{0, k}(E) \) the space of smooth \((0, k)\)-forms on \( Y \) taking values in \( E \). Following the notation in gauge theory, we denote \( adE = E^\vee \otimes E \); thus fixing a smooth semiconnection \( \bar{\partial}_0 \), all other semiconnections can be expressed as \( \bar{\partial}_0 + a \), with \( a \in \Omega^{0,1}(adE) \).

We fix a pair of integers \( s \geq 4 \) and \( \ell > 6 \), and form the completion \( \Omega^{0, k}(adE)_s \) of \( \Omega^{0, k}(adE) \) under the Sobolev norm \( L^s \). \( L^s \) is the sum of \( L^2 \)-norms of up to \( s \)-th partial derivatives.) We say \( \bar{\partial}_0 + a \) is \( L^s \) if \( a \) is \( L^s \), assuming \( \bar{\partial}_0 \) is smooth. We denote by \( \mathcal{G} \) the gauge group of \( L^\ell_{s+1} \)-sections of \( \text{Aut}(E) \) modulo \( \mathbb{C}^* \), which is the \( L^\ell_{s+1} \) completion of \( C^\infty(\text{Aut}(E))/\mathbb{C}^* \). We denote by \( A \) the space of \( L^\ell_{s+1} \)-semiconnections on \( E \). We have an isomorphism of affine spaces, after a choice of smooth \( \bar{\partial}_0 \in A \), via

\[
\bar{\partial}_0 + a : \Omega^{0,1}(adE)_s \to A, \quad a \mapsto \bar{\partial}_0 + a.
\]

The gauge group \( \mathcal{G} \) acts on \( A \) via \( g \cdot (\bar{\partial}_0 + a) = (g^{-1})^*(\bar{\partial}_0 + a) \). Let \( A_{s\iota} \) be the \( \mathcal{G} \)-invariant open subset of simple semiconnections, i.e. the automorphism groups are all \( \mathbb{C}^* \cdot \text{id}_E \). Let

\[
B_{s\iota} = A_{s\iota} / \mathcal{G} \subset A / \mathcal{G} := B.
\]

Then \( B_{s\iota} \) is a complex Banach manifold.

An element \( \partial \in A \) is called integrable if the curvature \( F^{0, 2}_{\partial} := (\partial)^2 \) vanishes. If \( \partial \) is integrable and \( a \in \Omega^{0,1}(adE) \), then \( \partial + a \) is \( F^{0, 2}_{\partial + a} = \partial a + a \wedge a \). By Sobolev inequality, \( F^{2, 2}_{\partial + a} \) is a continuous operator from \( \Omega^{0,1}(adE)_s \) to \( \Omega^{0,2}(adE)_{s-1} \), analytic in \( a \). An integrable smooth semiconnection \( \bar{\partial} \) defines a holomorphic vector bundle \((E, \bar{\partial})\) on \( Y \).

Picking \( a \) (reference) integrable \( \bar{\partial} \in A \), the holomorphic Chern-Simons functional is defined as

\[
CS : A \to \mathbb{C}, \quad CS(\bar{\partial} + a) = \frac{1}{4\pi^2} \int_Y \text{tr} \left( \frac{1}{2} (\bar{\partial}a) \wedge a + \frac{1}{3} a \wedge a \wedge a \right) \wedge \Omega.
\]
This is a cubic polynomial in $a$ whose quadratic part is

$$CS_2(\bar{\partial} + a) = \frac{1}{4\pi^2} \int_Y \text{tr}(\frac{1}{2}(\bar{\partial}a) \wedge a) \wedge \Omega.$$ 

Since the directional derivative of $CS$ at $\bar{\partial} + a$ in the direction of $b$ is

$$\delta CS(\bar{\partial} + a)(b) = \frac{1}{4\pi^2} \int_Y \text{tr}(b \wedge F^{0,2}_{\bar{\partial}+a}) \wedge \Omega,$$

$\delta CS(\bar{\partial} + a) = 0$ if and only if $\bar{\partial} + a$ is integrable. Thus the complex analytic subspace $A_{si}^{int}$ of simple integrable smooth semiconnections in $A_{si}$ is the critical locus (complex analytic subspace) of $CS$. Let $\Xi_{si} = A_{si}^{int}/G \subset B_{si}$. Since $CS$ is $G$-equivariant, it descends to

$$\Xi : B_{si} \to \mathbb{C}$$

whose critical locus (complex analytic subspace) in $B_{si}$ is $\Xi_{si}$.

3.2. The universal family over $X$. Let $X \subset \Xi_{si} \subset B$ be an open complex analytic subspace and denote by $X = X_{\text{red}}$ the $X_{\text{red}}$ endowed with the reduced scheme structure. We assume that $X$ is quasi-projective and that there is a universal family of holomorphic bundles $E$ over $X \times Y$ that induces the morphism $X \to B$. Our convention is that $E$ is with its holomorphic structure $\bar{\partial}_X$ implicitly understood. For $x \in X$, we denote by $E_x = E|_{x \times Y}$, the holomorphic bundle associated with $x \in X$. We denote by $\bar{\partial}_x$ the restriction of $\bar{\partial}_X$ to $E_x$. We use $adE_x$ to denote the smooth vector bundle $E_x^* \otimes E_x$.

We fix a hermitian metric $h$ on $E$ that is analytic in $X$ direction (see §4.3). For $x \in X$, we denote by $h_x$ the restriction of $h$ to $E_x$. Using $h_x$, we form $\Omega^{0,k}(adE_x)_s$, the space of $L^s_x$-valued $(0,k)$-forms. We form the space $A_x$ of $L^s_x$ semiconnections on $E_x$, which is isomorphic to $\Omega^{0,1}(adE_x)_s$ via $\bar{\partial}_x + \cdot$ (cf. (3.1)). We form the adjoint $\bar{\partial}^*_x$ of $\bar{\partial}_x$ using the hermitian metric $h_x$.

Since $\text{Aut}(E_x) = \mathbb{C}^*$, a standard argument in gauge theory shows that the tangent space of $B$ at $x$ is

$$T_xB \cong \Omega^{0,1}(adE_x)_s/\text{Im}(\bar{\partial}_x)_s \cong \ker(\bar{\partial}^*_x), \quad x \in X,$$

where the first isomorphism depends on a choice of smooth $E_x \cong E$; the second isomorphism is canonical using the Hodge theory of $(E_x, \bar{\partial}_x, h_x)$. (We use the subscript “$s$” to indicate that it is the image in $\Omega^{0,1}(adE_x)_s$.) For any open subset $U \subset X$, we denote $T_UB = TB|_U$. Since $X \subset B$ is a complex analytic subspace, it is a holomorphic Banach bundle over $U$.

For $x \in X$, we form the Laplacian

$$\square_x = \bar{\partial}_x \bar{\partial}^*_x + \bar{\partial}^*_x \bar{\partial}_x : \Omega^{0,1}(adE_x)_s \to \Omega^{0,1}(adE_x)_{s-2},$$

and its truncated eigenspace

$$\Theta_x(\epsilon) = \mathbb{C}\text{-span} \{a \in \Omega^{0,1}(adE_x)_s \mid \bar{\partial}^*_x a = 0, \quad \square_x a = \lambda a, \quad \lambda < \epsilon \} \subset \ker(\bar{\partial}^*_x).$$

Note that $T_xX \cong \ker(\square_x)^0_{s,1} \subset \Theta_x(\epsilon)$ for any $\epsilon > 0$; since $(E_x, \bar{\partial}_x, h_x)$ are smooth, $\Theta_x(\epsilon) \subset \Omega^{0,1}(adE_x)$ (i.e., consisting of smooth forms).
3.3. Preorientation data. We recall the quadratic form $cs_{2,x}$ on $T_xB$, $x \in X$, induced by the CS-function $cs : B \to \mathbb{C}$, defined explicitly by

$$cs_{2,x}(a_1, a_2) = \frac{1}{8\pi^2} \int \text{tr}(a_1 \wedge \overline{\partial}_x a_2) \wedge \Omega, \quad a_i \in \Omega^{0,1}(ad\mathcal{E}_x)_s/\ker(\overline{\partial}_x)_s.$$ 

Since $T_xX$ is the null subspace of $cs_{2,x}$, we have the induced non-degenerate quadratic form

$$(3.3) \quad Q_x : T_xB/T_xX \times T_xB/T_xX \rightarrow \mathbb{C}.$$ 

We introduce the notion of orientation bundles and their homotopies. Let $r$ be a positive integer. (The notion of analytic subbundles will be recalled in the next section.)

**Definition 3.1.** Let $U \subset X$ be open. A rank $r$ orientation bundle on $U$ is a rank $r$ analytic subbundle $\Xi \subset T_uB$ such that

1. there is an assignment $U \ni x \mapsto \epsilon_x \in (0, 1)$, continuous in $x \in U$, such that for every $x \in U$, $\Theta_x(\epsilon_x) \subset \Xi := \Xi|_x$;
2. at each $x \in U$, $Q_x := Q_x|_{\Xi_x/T_xX}$ is a non-degenerate quadratic form.

We also need a notion of homotopy between orientation bundles.

**Definition 3.2.** Let $\Xi_a$ and $\Xi_b$ be two orientation bundles over $U$. A homotopy from $\Xi_a$ to $\Xi_b$ is a family of orientation bundles $\Xi_t$ on $U$, such that

1. the family is analytic in $t$, and $\Xi_0 = \Xi_a$ and $\Xi_1 = \Xi_b$;
2. for all $t \in [0, 1]$, $\Xi_t$ satisfy (1) of Definition 3.1 for a single $\epsilon$.

The following is one of the key ingredients in our construction of perverse sheaves on moduli spaces.

**Definition 3.3.** We say $X$ is equipped with rank $r$ preorientation data if there are

1. a locally finite open cover $X = \cup_{\alpha} U_\alpha$,
2. a rank $r$ orientation bundle $\Xi_\alpha$ on $U_\alpha$ for each $\alpha$, and
3. a homotopy $\Xi_{\alpha\beta}$ from $\Xi_\alpha|_{U_{\alpha\beta}}$ to $\Xi_\beta|_{U_{\alpha\beta}}$ for each $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Since the open cover is locally finite, for each $x \in X$, we can find an open set $U_x$ which is contained in any $U_\alpha$ with $x$ inside. We can further choose $\epsilon_x > 0$ such that $\Xi_\alpha \supset \Theta_x(\epsilon_x)$ whenever $x \in U_\alpha$. To simplify the notation, from now on we will suppress $\epsilon_x$ and write $\Theta_x$ for the subbundle $\Theta_x(\epsilon_x)$.

In §4, we will prove the following.

**Proposition 3.4.** Every quasi-projective $X \subset \mathcal{V}_{si}$ with universal family admits preorientation data.

3.4. Families of CS charts and local trivializations. We introduce another key ingredient, called CS charts.

**Definition 3.5.** Let $f$ be a holomorphic function on a complex manifold $V$ such that $0$ is the only critical value of $f$. The reduced critical locus (also called the critical set) is the common zero set of the partial derivatives of $f$. The critical locus of $f$ is the complex analytic space $\mathfrak{X}_f$ defined by the ideal $(df)$ generated by the partial derivatives of $f$. 

Definition 3.6. An $r$-dimensional CS chart for $\mathfrak{X}$ is an $r$-dimensional complex submanifold $V$ of $A_{si}$, which embeds holomorphically into $B_{si}$ by the projection $A_{si} \to B_{si}$, such that, letting $ι : V \to B$ be the inclusion, the critical locus $\mathfrak{X}_{cs,0} \subset V$ is an open complex analytic subspace of $\mathfrak{X} \subset \mathfrak{V}_{si}$.

We say the chart $(V,ι)$ contains $x \in X$ if $x \in ι(\mathfrak{X}_{cs,0})$.

Example 3.7. By [12, Theorem 5.5], for any $x = \partial_0 \in \mathfrak{V}_{si}$,

$$V = \{ \bar{\partial}_0 = \partial_0 + a \mid \bar{\partial}_0^2 = 0 \} \subset A_{si}$$

is a CS chart of $\mathfrak{X}$ containing $x$ of dimension $\dim_T x$, for a sufficiently small $ε > 0$. This chart depends on $x$ and the choice of a hermitian metric on $E$ on which the adjoint $\bar{\partial}_0$ depends. In this paper, we call this chart the Joyce-Song chart at $x$. We remark that when $\bar{\partial}_0$ and the hermitian metric are smooth, all $\bar{\partial}_0 + a \in V$ are smooth.

Definition 3.8. Let $ρ : Z \to X$ be a continuous map from a topological space $Z$ to $X$. Let $r$ be a positive integer. A family of $r$-dimensional CS charts (for $\mathfrak{X}$) is a subspace $V \subset Z \times B$ that fits in a diagram

$$\begin{array}{ccc}
V & \longrightarrow & Z \times B \\
\pi & \downarrow & \downarrow \text{pr}_2 \\
& Z & \\
\end{array}$$

such that for each $x \in Z$ the fiber $V_x := π^{-1}(x) \subset B_{si}$ and is an $r$-dimensional CS chart of $\mathfrak{X}$ containing $ρ(x)$.

Given $V \subset Z \times B$ a family of CS charts over $ρ : Z \to X$, we define $Δ(Z) = \{(x, ρ(x)) \mid x \in Z\} \subset Z \times V$, where $ρ(x)$ is the unique point in $V_x$ whose image in $B$ is $ρ(x)$.

Definition 3.9. A local trivialization of the family $Z \leftarrow V \to Z \times B$ consists of an open $U_0 \subset Z$, an open neighborhood $\mathcal{U}$ of $Δ(U_0) \subset V_{U_0} := π^{-1}(U_0)$, and a continuous

$$(3.4) \quad U_0 \times V_{U_0} \supset \mathcal{U} \longrightarrow \Psi V_{U_0} \times U_0,$$

such that $Ψ$ commutes with the two tautological projections $U \subset U_0 \times V_{U_0} \to U_0 \times U_0$ and $V_{U_0} \times U_0 \to U_0 \times U_0$, and that

(1) letting $U_0 \to U_0 \times U_0$ be the diagonal, then $Ψ|_{U \times V_{U_0} \times U_0} = \text{id}$;

(2) for any $x,y \in U_0$, letting $U_{x,y} = U \cap (x \times V_y)$ and $Ψ_{x,y} := Ψ|_{U_{x,y}} : U_{x,y} \to V_x$, then $Ψ_{x,y}$ is biholomorphic onto its image; $Ψ_{x,y}$ restricted to $U_{x,y} \cap \mathfrak{X}_{f_x}$ is an open immersion into $\mathfrak{X}_{f_x} \subset V_x$, commuting with the tautological open immersions $\mathfrak{X}_{f_x}, \mathfrak{X}_{f_y} \subset \mathfrak{X}$.

We say that $V \subset Z \times B$ admits local trivializations if for any $x_0 \in Z$ there is a local trivialization over an open neighborhood $U_0$ of $x_0$ in $Z$.

Definition 3.10. Let $Z \subset \mathbb{R}^n$ be a (real) analytic subset defined by the vanishing of finitely many analytic functions. Let $V \subset Z \times B$ be a family of CS charts over $ρ : Z \to X$. A complexification of $V$ consists of a complexification $Z^C \subset \mathbb{C}^n$ of $Z$ (thus having $Z^C \cap \mathbb{R}^n = Z$), a holomorphic $ρ^C : Z^C \to X$ extending $ρ : Z \to X$,
and a holomorphic family of CS charts $\mathcal{V}^C \subset Z^C \times B$ over $Z^C$ (i.e. $\mathcal{V}^C$ is a complex analytic subspace of $Z^C \times B$) such that

$$\mathcal{V}^C \times Z^C = \mathcal{V} \subset Z \times B.$$ 

Let $U \subset X$ be an open subset and $\mathcal{V} \subset U \times B$ be a family of CS charts over $U$ which admits local trivializations. Let $U_0 \subset U$ be an open subset and $\Psi$ in (3.4) a local trivialization of $\mathcal{V}$ over $U_0$.

**Definition 3.11.** We say that the local trivialization $\Psi$ is complexifiable if for any $x \in U_0$, there is an open neighborhood $x \in O_x \subset U_0$ such that if we let $\mathcal{V}_{O_x} \subset O_x \times B$ and $\Psi_{O_x} : U_{O_x} \to V_{O_x} \times O_x$, where $U_{O_x} = U \times U \times V \times O_x$ and $\Psi_{O_x}$ is the pullback of $\Psi$, the following hold:

1. the family $\mathcal{V}_{O_x} \subset O_x \times B$ admits a complexification $\mathcal{V}_{O_x}^C \subset O_x^C \times B$ over a complexification $O_x^C$ of $O_x$;
2. there is a holomorphic local trivialization $\Psi_{O_x}^C : U_{O_x}^C \to V_{O_x}^C \times O_x^C$, i.e. $U_{O_x}^C \subset O_x^C \times V_{O_x}^C$ is open and contains the diagonal $\Delta(O_x^C)$, such that $\Psi_{O_x}^C$ is holomorphic,

$$U_{O_x}^C \times O_x^C \subset V_{O_x}$$

and $\Psi = \Psi_{O_x}^C|_U : U_{O_x} \to V_{O_x} \times O_x \subset V_{O_x}^C \times O_x^C$.

In §5, we will prove the following.

**Proposition 3.12.** Let $X \subset B_{si}$ be equipped with preorientation data $(\cup U_\alpha, \Xi_\alpha, \Xi_{\alpha\beta})$. Then there are

1. a family of $r$-dimensional CS charts $\mathcal{V}_\alpha \subset U_\alpha \times B$ with complexifiable local trivializations at all $x \in U_\alpha$;
2. an open neighborhood $U_x$ and a subfamily $W_x$ of CS charts in $\mathcal{V}_\alpha|_{U_x}$ for each $x \in U_\alpha$, i.e. a subbundle $W_x$ of $\mathcal{V}_\alpha|_{U_x}$ which admits compatible complexifiable local trivializations

$$U_x \times \mathcal{V}_\alpha|_{U_x} \supset U \xrightarrow{\Psi} \mathcal{V}_\alpha|_{U_x} \times U_x$$

$$U_x \times W_x|_{U_x} \supset U' \xrightarrow{\Psi} W_x|_{U_x} \times U_x$$

3. a family of CS charts $\mathcal{V}_{\alpha\beta}$ parameterized by $U_{\alpha\beta} \times [0,1]$ with $\mathcal{V}_{\alpha\beta}|_{U_{\alpha\beta} \times \{0\}} = \mathcal{V}_{\alpha}|_{U_{\alpha\beta}}$, $\mathcal{V}_{\alpha\beta}|_{U_{\alpha\beta} \times \{1\}} = \mathcal{V}_{\beta}|_{U_{\alpha\beta}}$, which has complexifiable local trivializations at all $(x,t) \in U_{\alpha\beta} \times [0,1]$, such that $\mathcal{V}_{\alpha\beta}|_{U_{\alpha\beta} \times [0,1]}$ contains the subfamily $W_x \times [0,1]$ of CS charts over $U_x \times [0,1]$.

We call the above $(\mathcal{V}_\alpha, W_x, \mathcal{V}_{\alpha\beta})$ CS data for $X$.

**3.5. Local perverse sheaves and gluing isomorphisms.** Given CS data, we can construct perverse sheaves $P_\alpha^*$ on each $U_\alpha$ and gluing isomorphisms $\sigma_{\alpha\beta} : P_\alpha^*|_{U_{\alpha\beta}} \to P_{\beta}^*|_{U_{\alpha\beta}}$.

We will prove the following in §6.

**Proposition 3.13.** (1) Let $\pi : \mathcal{V} \to U$ be a family of CS charts on $U \subset X \subset B_{si}$ with complexifiable local trivializations at every point $x \in U$. Then the perverse sheaves of vanishing cycles for

$$f_x : \mathcal{V}_x = \pi^{-1}(x) \subset B_{si} \xrightarrow{cs} \mathbb{C}$$

are...
glue to a perverse sheaf $P^\bullet$ on $U$, i.e. $P^\bullet$ is isomorphic to $A^*_f[x]$ in a neighborhood of $x$.

(2) Let $V_\alpha$ and $V_\beta$ be two families of CS charts on $U$ with complexifiable local trivializations. Let $P^\bullet_\alpha$ and $P^\bullet_\beta$ be the induced perverse sheaves on $U$. Let $V$ be a family of CS charts on $U \times [0,1]$ with complexifiable local trivializations such that $V|_{U \times \{0\}} = V_\alpha$ and $V|_{U \times \{1\}} = V_\beta$. Suppose for each $x \in U$, there are an open neighborhood $U_x \subset U$ and a subfamily $W$ of both $V_\alpha|_{U_x}$ and $V_\beta|_{U_x}$ such that $W \times [0,1]$ is a complexifiable subfamily of CS charts in $V|_{U_x \times [0,1]}$. Then there is an isomorphism $\sigma_{\alpha\beta} : P^\bullet_\alpha \cong P^\bullet_\beta$ of perverse sheaves.

(3) If there are three families $V_\alpha, V_\beta, V_\gamma$ with homotopies among them as in (2), then the isomorphisms $\sigma_{\alpha\beta}, \sigma_{\beta\gamma}, \sigma_{\gamma\alpha}$ satisfy

$$\sigma_{\alpha\beta\gamma} := \sigma_{\gamma\alpha} \circ \sigma_{\beta\gamma} \circ \sigma_{\alpha\beta} = \pm \text{id}.$$ 

In fact, the isomorphism in (2) is obtained by gluing pullback isomorphisms via biholomorphic maps $\chi_{\alpha\beta} : V_{\alpha,x} \rightarrow V_{\beta,x}$ at each $x$. The sign $\pm 1$ in (3) is the determinant of the composition

$$T_x V_{\alpha,x} \xrightarrow{dx_{\alpha\beta}} T_x V_{\beta,x} \xrightarrow{dx_{\beta\gamma}} T_x V_{\gamma,x} \xrightarrow{dx_{\gamma\alpha}} T_x V_{\alpha,x}$$

where $V_{\cdot,x}$ is the fiber of $V$ over $x$. By Serre duality, $\det T V_{\alpha,x}$ is a square root of $\det \text{Ext}^\bullet_{\text{om}}(E,E)$ and hence the 2-cocycle $\sigma_{\alpha\beta\gamma}$ defines an obstruction class in $H^2(X, \mathbb{Z}_2)$ for the existence of a square root of $\det \text{Ext}^\bullet_{\text{om}}(E,E)$. Combining Propositions 3.4, 3.12 and 3.13, we thus obtain the following.

**Theorem 3.14.** Let $X \subset X \subset \mathcal{X}_s$ be quasi-projective and equipped with a universal family $E$. Then there is a perverse sheaf $P^\bullet$ on $X$ which is locally a perverse sheaf of vanishing cycles if and only if there is a square root of the line bundle $\det \text{Ext}^\bullet_{\text{om}}(E,E)$ in $\text{Pic}(X)$.

It is obvious that the theorem holds for any étale cover of $X$.

### 3.6. Divisibility of the determinant line bundle

In this subsection, we show that for moduli spaces $\mathcal{X}$ of stable sheaves on $Y$, and for the universal sheaf $E$ on $X \times Y$, $\det \text{Ext}^\bullet_{\text{om}}(E,E)$ has a square root possibly after taking a Galois étale cover $X^\dagger \rightarrow X$. By Theorem 3.14, there is a globally defined perverse sheaf $P^\bullet$ on $X^\dagger$ which is locally the perverse sheaf of vanishing cycles.

To begin with, using the exponential sequence

$$H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z}),$$

we see that

$$\det \text{Ext}^\bullet_{\text{om}}(E,E) = \det R^\pi_* \text{RHom}(E,E)$$

admits a square if and only if its first Chern class in $H^2(X, \mathbb{Z})$ is even. We determine the torsion-free part of the first Chern class of (3.5) using the Grothendieck-Riemann-Roch theorem:

$$\text{ch}(\text{Ext}^\bullet_{\text{om}}(E,E)) = \pi_* \left( \text{ch}(R^\pi_* \text{RHom}(E,E)) \text{td}(Y) \right).$$

Since $E$ is flat over $X$, $\alpha_i := c_i(E) \in A^*(X \times Y)_{\mathbb{Q}}$. Let $r = \text{rank } E$. Then one has

$$\text{ch}(E) = r + \alpha_1 + \frac{1}{2}(\alpha_1^2 - 2\alpha_2) + \frac{1}{6}(\alpha_1^3 - 3\alpha_1\alpha_2 + 3\alpha_3) + \delta_4 + \cdots,$$

where $\delta_4 \in A^4(X \times Y)_{\mathbb{Q}}$, and $\cdots$ are elements in $A^{>4}(X \times Y)_{\mathbb{Q}}$. Thus

$$\text{ch}(R^\pi_* \text{RHom}(E,E)) = \text{ch}(E) \text{ch}(E^\vee)$$
admits a square root

L

Proof. Since the torsion-free part of L

line bundle

Note that the pullback of det Ext

pullback of

L
torsion line bundle

Lemma 3.15. Let X be a fine moduli scheme of simple sheaves on Y. There is a torsion line bundle L on X = X_red with L^⊗k ∼= O_X for some k > 0 such that the pullback of det Ext^∗_*(E, E) by the cyclic étale Galois cover X^1 = \{ s ∈ L | s^k = 1 \} → X admits a square root \( L \) on \( X^1 \).

Proof. Since the torsion-free part of \( c_1(\det \text{Ext}^*_{**}(E, E)) \) is even, there is a torsion line bundle L on X such that \( L \otimes \det \text{Ext}^*_{**}(E, E) \) is torsion-free and admits a square root. Note that the pullback of \( L \) to \( X^1 \) is trivial.

For a moduli space \( X \) of stable sheaves with universal bundle, we can apply the Seidel-Thomas twists ([12]) so that we can identify \( X \) as a complex analytic subspace of \( B_s \). By Theorem 3.14, we have the following.

Theorem 3.16. If \( X \) is a fine moduli scheme of stable sheaves on Y, there exist a cyclic étale Galois cover \( X^1 → X \) and a perverse sheaf \( P^* \) on \( X^1 \) which is locally the perverse sheaf of vanishing cycles.

4. Existence of preorientation data

In this section we prove Proposition 3.4. We construct orientation bundles, their homotopies, and their complexifications.
4.1. The semiconnection space. We continue to use the convention introduced at the beginning of Subsection 3.2. Thus, $\mathcal{E}$ is the universal family of locally free sheaves on $X \times Y$ and $(\mathcal{E}_x, \overline{\partial}_x)$ is the restriction $\mathcal{E}|_{x \times Y}$, and we use $\text{ad}_{\mathcal{E}}$ to denote the smooth vector bundle $\mathcal{E}^\vee_x \otimes \mathcal{E}_x$.

Since $X$ is quasi-projective, $X$ has a stratification according to the singularity types of points in $X$. We fix such a stratification. We say a continuous function (resp. a family) on an open $U \subset X$ is smooth if its restriction to each stratum is smooth.

We now choose a smooth hermitian metric $h$ on $\mathcal{E}$. Since $X$ is quasi-projective, by replacing $\mathcal{E}$ by its twist via a sufficiently negative line bundle from $X$, for a sufficiently ample $\mathcal{H}$ on $Y$, we can make $\mathcal{E}^\vee \otimes p_Y^* \mathcal{H}$ generated by global sections, where $p_Y : X \times Y \to Y$ is the projection. Thus for an integer $N$, we have a surjective homomorphism of vector bundles $\mathcal{O}^{\oplus N}_{X \times Y} \to \mathcal{E}^\vee \otimes p_Y^* \mathcal{H}$; dualizing and untwisting $\mathcal{H}^\vee$, we obtain a subvector bundle homomorphism $\mathcal{E} \subset p_Y^* \mathcal{H}^{\oplus N}$. We then endow $\mathcal{H}$ a smooth hermitian metric; endow $\mathcal{E} \subset p_Y^* \mathcal{H}^{\oplus N}$ the pullback metric. We define $h$ to be the induced hermitian metric on $\mathcal{E}$ via the holomorphic subbundle embedding $\mathcal{E} \subset p_Y^* \mathcal{H}^{\oplus N}$. For $x \in X$, we denote by $h_x$ the restriction of $h$ to $\mathcal{E}_x$.

For the integers $s$ and $\ell$ chosen before, we denote the $L^1$-completion of $\Omega^{0,k}(\text{ad}\mathcal{E}_x)$ by $\Omega^{0,k}(\text{ad}\mathcal{E}_x)_s$. We form the formal adjoint $\overline{\partial}_x^*$ of $\overline{\partial}_x$ using the hermitian metric $h_x$. Then $\overline{\partial}_x$ and $\overline{\partial}_x^*$ extend to differential operators

$$
\overline{\partial}_x (\text{resp. } \overline{\partial}_x^*) : \Omega^{0,k}(\text{ad}\mathcal{E}_x)_{s+1-k} \to \Omega^{0,k+1}(\text{ad}\mathcal{E}_x)_{s-k} \quad (\text{resp. } \Omega^{0,k-1}(\text{ad}\mathcal{E}_x)_{s+k}).
$$

We use $\ker(\overline{\partial}_x)_{s+1-k}$ to denote the kernel of $\overline{\partial}_x$ in $\Omega^{0,k}(\text{ad}\mathcal{E}_x)_{s+1-k}$; likewise, $\ker(\overline{\partial}_x^*)_{s+1-k}$.

We form the Laplacian

$$
\square_x = \overline{\partial}_x \overline{\partial}_x^* + \overline{\partial}_x^* \overline{\partial}_x : \Omega^{0,k}(\text{ad}\mathcal{E}_x)_{s+1-k} \to \Omega^{0,k}(\text{ad}\mathcal{E}_x)_{s-k}.
$$

We denote by $\square_x^{-1}(0)^{0,k}$ the set of harmonic forms (the kernel of $\square_x$) in $\Omega^{0,k}(\text{ad}\mathcal{E}_x)$.

It will be convenient to fix a local trivialization of $\mathcal{E}$. Given $x_0 \in X$, we realize an open neighborhood $U_0 \subset X$ of $x_0$ as $U_0 = X_{f_0}$, where $(V_0, f_0) := (V_{x_0}, f_{x_0})$ is the JS chart. We fix a smooth isomorphism $E \simeq E_{x_0}$. Since $V_0$ is a complex manifold, by shrinking $x_0 \in V_0$ if necessary, we assume that $V_0$ is biholomorphic to an open subset of $\mathbb{C}^n$ for some $n$. We let $z = (z_1, \cdots, z_n)$ be the induced coordinate variables on $V_0$. By abuse of notation, we also use $z$ to denote a general element in $V_0$.

Let $\overline{\partial}_0 + a_0(z)$ be the family of semiconnections on $E$ of the chart $(V_0, f_0)$. Because $a_0(z)$ satisfies the system in Example 3.7, $\overline{\partial}_x a_0(z) = 0$. Let $E_{V_0} = V_0 \times E$, as a vector bundle over $V_0 \times Y$. Let $\overline{\partial}_z$ be the $\overline{\partial}$-operator along the $z$ direction of the product bundle $E_{V_0} = V_0 \times E$. Then $\overline{\partial}_{V_0} := \overline{\partial}_0 + \overline{\partial}_z + a_0(z)$ is a semiconnection on $E_{V_0}$. It is known that (cf. [12, Chapter 9], [26])$^{1}$ $\overline{\partial}_0 f_0 = (\overline{\partial}_{0} a_0(z) = 0)$, which is the same as $((\overline{\partial}_{\mathcal{E}})^2 = 0)$ since $\overline{\partial}_x a_0(z) = 0$.

Using $U_0 = X_{f_0}$, the restriction $(E_{V_0}, \overline{\partial}_{V_0})|_{U_0}$ is a holomorphic bundle over $U_0 \times Y$. By construction, it is biholomorphic to $\mathcal{E}|_{U_0} := \mathcal{E}_{U_0}$. Let

$$
\zeta : (E_{U_0}, \overline{\partial}_{V_0}|_{U_0}) \to \mathcal{E}_{U_0}
$$

\hspace{1cm} (4.1)

\hspace{1cm} $^{1}$If we don’t put any subscript at $\Omega^{\partial,\cdot}(\cdot)$, it means the set of smooth sections.
be a biholomorphism. Since $E_{x_0}$ is simple, we can assume that $\zeta$ extends the smooth isomorphism $E \simeq E_{x_0}$ we begun with. In the following, we call $\zeta$ a framing of $E$ over $U_0$.

Via this framing, we identify $\Omega^{0,k}(\text{ad}E_x)$, with $\Omega^{0,k}(\text{ad}E)_s$; the connection form $\alpha_0(z)$ locally has convergent power series expansion in $z$ everywhere over $V_0$, with coefficients smooth $\text{ad}E$-valued $(0,1)$-forms.

Another application of this local framing is that it gives local trivializations of $T_X B$. Using the holomorphic family of semicunctions $\overline{\partial}_0 + \alpha_0(z)$, we embeds $V_0$ into $B$ as a complex submanifold. (Since $(V_0,f_0)$ is a CS chart, $V_0 \subset B_{st}$.) Using $B = A/G$, we have an induced surjective homomorphism of holomorphic Banach bundles

\begin{equation}
V_0 \times \Omega^{0,1}(\text{ad}E)_s \longrightarrow T_{V_0} B.
\end{equation}

By shrinking $0 \in V_0$ if necessary, the induced $V_0 \times \ker(\overline{\partial}_0^0)_{s,1} \rightarrow T_{V_0} B$ is an isomorphism of holomorphic Banach bundles. Since for $x \in X$, $\overline{\partial}_x^* : \Omega^{0,1}(\text{ad}E_x) \rightarrow \Omega^0(\text{ad}E_x)_{s-1}/C$ is surjective, $\ker(\overline{\partial}_x^*)_{s,1} := \bigcap_{x \in X} \ker(\overline{\partial}_x^*)_{s,1}$ is a smooth Banach bundle. The same holds for $\ker(\overline{\partial}_x^*)_{s,2} \subset \Omega^0(\text{ad}E)_{s-1}$.

4.2. Truncated eigenspaces. We form the (partially) truncated eigenspace

$$\Theta_x(\epsilon) := \mathbb{C}\text{-span} \{ a \in \Omega^{0,1}(\text{ad}E)_s \mid \overline{\partial}_x^* a = 0, \Box_x a = \lambda a, \lambda < \epsilon \} \subset \ker(\overline{\partial}_x^*)_{s,1}$$

and its $(0, 2)$-analogue

$$\Theta_x(\epsilon) : = \mathbb{C}\text{-span} \{ a \in \Omega^{0,2}(\text{ad}E)_s \mid \Box_x a = 0, \Box_x a = \lambda a, \lambda < \epsilon \} \subset \ker(\overline{\partial}_x^*)_{s,2}.$$

Let $U \subset X$ be an open subset. We form

$$\Theta_U(\epsilon) = \prod_{x \in U} \Theta_x(\epsilon) \subset \ker(\overline{\partial}_x^*)_{s,1}\big|_U$$

and

$$\Theta'_U(\epsilon) = \prod_{x \in U} \Theta'_x(\epsilon) \subset \ker(\overline{\partial}_x^*)_{s,2}\big|_U.$$
Since $X$ is quasi-projective, we can cover $X$ by a countable number of open subsets each of which has compact closure in $X$. Applying the proof to each of these open set, we conclude that a continuous $\epsilon(\cdot)$ exists making the first statement of the lemma hold. The proof of the part on $\ker(J^0_{x_0}f, 1)$ is parallel, using that for every $x \in X$, because $Y$ is a Calabi-Yau threefold, $\Box_{x}^{-1}(0)^{0,1} \cong \Box_{x}^{-1}(0)^{0} \cong \mathbb{C}$. This proves the lemma.

We will fix this function $\epsilon(\cdot)$ in the remainder of this section.

**Lemma 4.2.** Let $U \subset X$ be connected and open, and let $\epsilon_0 \in (0, \epsilon_x)$ separate the eigenvalues of $\Box_x$ for all $x \in U$. Then $\Theta_U(\epsilon_0)$ (resp. $\Theta^*_U(\epsilon_0)$) is a smooth subbundle of $\ker(J^0_{X})^{0,1}_{|U}$ (resp. $\ker(J^1_{X})^{0,2}_{|U}$).

**Proof:** Since $\epsilon_0$ is not an eigenvalue of $\Box_x$ for all $x \in U$, and since $U$ is connected, the family
\[
\Theta_x(\epsilon_0) = \text{span}\{a \in \Omega^{0,1}(adE_x) \mid \Box_x a = \lambda a, \lambda < \epsilon_0\}, \quad x \in U,
\]have identical dimensions and form a smooth (finite rank) vector bundle over $U$ (cf. [14]). Because $\epsilon_0 < \epsilon_x$ for all $x \in U$, $\Theta_x(\epsilon_0) \cap \text{Im}(\Theta_x) = \{0\}$. Therefore, $\Theta_x(\epsilon_0) = \Theta_x(\epsilon_0)$ for all $x \in U$. This proves that $\Theta_U(\epsilon_0)$ is a smooth bundle over $U$ and thus a smooth subbundle of $\ker(J^0_{X})^{0,1}_{|U}$. The proof for the other is identical. □

### 4.3. Complexification.
We recall the notion of analytic families. Let $D \subset \mathbb{C}^n$ be an open subset, with $z = (z_1, \cdots, z_n)$ its induced coordinate variables, and $\text{Re} z = (\text{Re} z_1, \cdots, \text{Re} z_n)$ and same for $\text{Im} z$. An analytic function on $D$ is a smooth $\mathbb{C}$-valued function on $D$ that locally has convergent power series expansions in $\text{Re} z$ and $\text{Im} z$. We denote by $\mathcal{O}^{\mathbb{C}}_D$ the sheaf of analytic functions on $D$. An analytic section of $\mathcal{O}^{\mathbb{C}}_{D}$ is a section of the sheaf $\mathcal{O}^{0} \oplus \mathcal{O}^{0,1} \oplus \mathcal{O}^{1} \oplus \mathcal{O}^{1,0}$. 

**Definition 4.3.** Let $F \to U$ be a holomorphic vector bundle over a reduced complex analytic subspace $U$. We say a continuous section $s \in C^0(U, F)$ is analytic if at every $x \in U$, there is an open neighborhood $U_0 \subset U$ of $x \in U$, a holomorphic trivialization $F|_{U_0} \cong \mathcal{O}^{0} \oplus \mathcal{O}^{0,1} \oplus \mathcal{O}^{1} \oplus \mathcal{O}^{1,0}$ and a closed holomorphic embedding $U_0 \subset D$ into a smooth complex manifold $D$ such that $s|_{U_0}$ is the restriction of an analytic section of $\mathcal{O}^{\mathbb{C}}_D$. We say a rank 1 complex subbundle $F' \subset F$ is analytic if locally $F'$ is spanned by 1 analytic sections of $F$. In case $F$ is a holomorphic Banach vector bundle over $U$, the same definition holds with $\mathcal{O}^{\mathbb{C}}_D$ replaced by the local holomorphic Banach bundle trivializations $F|_{U_0} \cong B \times U_0$, for Banach spaces $B$.

The purpose of this subsection is to prove

**Proposition 4.4.** Let the situation be as in Lemma 4.2. Then the bundle $\Theta_U(\epsilon)$ is an analytic subbundles of $T_U B$.

We will prove the proposition after we construct a complexification of the family $J^0_x$. Since this is a local study, for any $x_0 \in U$, we pick an open neighborhood $U_0 \subset U$ and fix an isomorphism $\zeta$ (cf. (4.1)) derived from realizing $U_0 = X_{f_0}$ for the JS chart $(V_0, f_0) = (V_{x_0}, f_{x_0})$.

Let $D \subset V_0$ be an open neighborhood of $0 \in V_0$. As in the discussion leading to (4.1), we denote by $J^0_{f_0} + a_0(z)$ the family of semiconnections on $E_{V_0} = E \times V_0$ over $V_0 \times Y$. The connection form $a_0(z)$ is a $\Omega^{0,1}(adE)$-valued holomorphic function on $D$, with $a_0(0) = 0$. 

Writing $z_k = u_k + iv_k$, we can view $D$ as an open subset of $\mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ is with the coordinate variable $(u_1, \ldots, u_n, v_1, \ldots, v_n)$. By allowing $u_k$ and $v_k$ to take complex values, we embed $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$, thus embed $D \subset \mathbb{C}^{2n}$ as a (totally real) analytic subset. We call an open $D^c \subset \mathbb{C}^{2n}$ a complexification of $D$ if $D^c \cap \mathbb{R}^{2n} = D$.

We use $w$ to denote the complex coordinate variables of $\mathbb{C}^{2n}$.

**Lemma 4.5.** We can find a complexification $D^c \supset D$ such that the function $a_0(z)$ extends to a holomorphic $a_0(\cdot)_C : D^c \to \Omega^{0,1}(adE)$.

**Proof.** The extension is standard. Since $a_0(z)$ is derived from the JS chart, it is holomorphic in $z$. Thus for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in D$, $a_0(z)$ equals to a convergent power series in $(z_k - \alpha_k)$ in a small disk centered at $\alpha$ with coefficients in $\Omega^{0,1}(adE)$. Letting $\alpha_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$, and writing $z_k = u_k + iv_k$, the power series becomes a power series in $(u_k - a_k)$ and $(v_k - b_k)$.

Because $u_k$ and $v_k$ are complex coordinate variables of $\mathbb{C}^{2n} \supset \mathbb{R}^{2n} \supset D$, $a_0(z)$ extends to a holomorphic $\Omega^{0,1}(adE)$-valued function in a small neighborhood of $\alpha$ in $\mathbb{C}^{2n}$. Because the extension of a function defined on an open subset of $\mathbb{R}^{2n}$ to a germ of holomorphic function on $\mathbb{C}^{2n}$ is unique, the various extensions of $a_0(z)$ using power series expansions at various $\alpha \in D$ give a single extension of $a_0(z)$ to a holomorphic $a_0(w)_C$ on some complexifications $D^c \supset D$. $\square$

For $D \subset V_0$ a neighborhood of $0 = x_0 \in V_0$, we denote

$$O_0 := D \cap X_{f_0} = D \cap X.$$  

For $x \in O_0$, we write $\overline{\partial}_x^* = \overline{\partial}_0^* + a_0(x)^!$. The extension problem for $a_0(x)^!$ is more delicate because it is not defined away from $O_0$.

**Lemma 4.6.** For any $y \in Y$, there is an open neighborhood $S \subset Y$ of $y \in Y$ and an open neighborhood $D \subset V_0$ of $0 \in V_0$ so that the hermitian metric $h|_{O_0 \times S}$ extends to an $L^1_{s+2}$ hermitian metric on $E_{D \times S} := E_D|_{D \times S}$, analytic in $z \in D$.

**Proof.** Let $S \subset Y$ be an open neighborhood of $y$ so that $S$ is biholomorphic to the unit ball in $\mathbb{C}^3$, and that $E|_{O_0 \times S} \cong \Omega^{0,1}_{O_0 \times S}$ and $H|_{S} \cong \Omega_{S}$.

We let $k_S$ be the hermitian norm of $1$ in $\Omega_S \cong H|_{S}$ of the hermitian metric of $H$ fixed earlier. Then $k_S$ is a smooth positive function on $S$. We let $s_1, \ldots, s_r$ be the standard basis of $\Omega^{0,1}_{O_0 \times S} \cong \Omega^{0,1}_{O_0 \times S}$.

Because $E \subset p_Y^* H^{\mathbb{P}^n}$ is a subvector bundle over $X \times Y$, using $H|_{S} \cong \Omega_{S}$, the image of $s_k$ in $p_Y^* H^{\mathbb{P}^n}|_{O_0 \times S}$ has the presentation $s_k = (s_{k,1}, \ldots, s_{k,n})$, where $s_{k,j} \in \Gamma(O_{O_0 \times S})$. Then the hermitian metric form of $h$ on $E|_{O_0 \times S}$ in the basis $s_1, \ldots, s_r$ takes the form

$$h(s_k, s_l) = k_S \cdot \sum_j s_{k,j} \overline{s_{l,j}}. \tag{4.3}$$

To extend this expression over $D \times S$, we will modify the semiconnection $\overline{\partial}_{D \times S} := \overline{\partial}_0^* + \overline{\partial}_z + a_0^3_{D \times S}$ to an integrable semiconnection $\overline{\partial}_{D \times S}$ and extend $s_1, \ldots, s_r$ to holomorphic sections of $(E_{D \times S}, \overline{\partial}_{D \times S})$.

Let $m \subset \mathcal{O}_D$ be the maximal ideal generated by $z_1, \ldots, z_n$, and let $I \subset \mathcal{O}_D$ be the ideal sheaf of $D \cap X \subset D$. Then $F_{\mathcal{O}_D}^{0,2} \equiv 0 \mod I$. We construct $\overline{\partial}_{D \times S}$ by power series expansion. We let $s' = s + 2$, and set $b_0(z) = 0$. Suppose we have found $b_0(z) \in \Omega^{0,1}(adE|_S)_{s'} \otimes \mathcal{O}_I$ such that

$$\overline{\partial}_0 b_0(z) \in \Omega^{0,1}(adE|_S)_{s'} \otimes \mathcal{O}_I \text{ and } F_{\mathcal{O}_D}^{0,2} \equiv 0 \mod m^2 \cap I, \tag{4.4}$$
where \( F_{\bar{\partial} + a_0 + b}^{0,2} := (\bar{\partial} a_0 + b k(z)) \). Then by the Bianchi identity, using \( a_0(0) = b k(0) = 0 \), we have
\[
(4.5) \quad \bar{\partial} F_{\bar{\partial} + a_0 + b}^{0,2} = (\bar{\partial} a_0 + b k(z)) F_{\bar{\partial} + a_0 + b}^{0,2} \equiv 0 \mod m^{k+1} \cap I.
\]
We let \( \phi_\alpha \in m^k \cap I, \alpha \in \Lambda_k \), be a \( C \)-basis of \((m^k \cap I)/(m^{k+1} \cap I)\); we write
\[
F_{\bar{\partial} + a_0 + b}^{0,2} \equiv \sum_{\alpha \in \Lambda_k} A_\alpha \phi_\alpha \mod m^{k+1} \cap I.
\]
Because \( a_0(z) \) is from the family on the JS chart, \( a_0(z) \in \Omega^{0,1}(adE) \otimes C I \) (i.e. is smooth). Thus using \((4.4)\) and \((4.5)\), we conclude that \( A_\alpha \in \Omega^{0,2}(adE|_S)_{K} \), and \( \bar{\partial} A_\alpha = 0 \). Since \( S \) is biholomorphic to the unit ball in \( \mathbb{C}^3 \), it is strictly pseudoconvex with smooth boundary. Applying a result of solutions of the \( \bar{\partial} \)-equation with \( L^2 \) estimate (cf. [16, Theorem 6.11]), there is a constant \( C \) depending only on \( S \) such that for each \( \alpha \in \Lambda_k \), we can find \( B_\alpha \in \Omega^{0,1}(adE|_S)_{K} \) such that
\[
(4.6) \quad \bar{\partial} B_\alpha = A_\alpha \quad \text{and} \quad \| B_\alpha \|_{L^2(S)} \leq C \| A_\alpha \|_{L^2(S)}.
\]
We define \( \delta_k(z) = \sum_{\alpha \in \Lambda_k} B_\alpha \phi_\alpha \), and let \( b_{k+1}(z) = b_k(z) + \delta_k(z) \). Then \((4.4)\) holds with \( k \) replaced by \( k + 1 \).

We consider the infinite sum \( \sum_{k=1}^\infty \delta_k(z) \). Using the estimate \((4.6)\), and the Morrey’s inequality \( \| u \|_{C^{0,1-\epsilon}(S)} \leq C' \| u \|_{L^1(S)} \) for the domain \( S \), a standard power series convergence argument (cf. [17, Section 5.3(c)]) shows that possibly after shrinking \( 0 \in D \) and \( y \in S \), \( \sum_{k=1}^\infty \delta_k|DS| \) converges to a \( b(z) \in \Omega^{0,1}(adE|_S)_{K} \otimes C I \). Then the semiconnection
\[
\tilde{\nabla}_{DS} := \bar{\partial} + \bar{\partial}_z + a_0(z) + b(z)
\]
on \( E_{DS} \) is integrable and is the desired modification.

We now extend the metric. Possibly after shrinking \( 0 \in D \) and \( y \in S \), we can assume that the subbundle homomorphism \( E|_{O_0 \times S} \to p_Y^* \mathcal{H}^{\mathbb{H}^N}|_{O_0 \times S} \) extends to a subbundle homomorphism
\[
g : (E_{DS}, \tilde{\nabla}_{DS}) \longrightarrow p_Y^* \mathcal{H}^{\mathbb{H}^N}|_{DS};
\]
the sections \( s_1, \cdots, s_r \) extend to holomorphic sections \( \tilde{s}_1, \cdots, \tilde{s}_r \) of \((E_{DS}, \tilde{\nabla}_{DS})\) that span the bundle \( E_{DS} \). Using \( \mathcal{H}|_S \cong \mathcal{O}_S \), and writing \( g(\tilde{s}_k) = (\tilde{s}_{k,1}, \cdots, \tilde{s}_{k,N}) \), we define
\[
h_S(\tilde{s}_k, \tilde{s}_l) := k_S \sum_{j=1}^N \bar{s}_{k,j} \bar{s}_{l,j},
\]
which defines a hermitian metric \( h_S \) of \( E_{DS} \), extending the metric \((4.3)\).

It remains to express the metric \( h_S \) in a basis constant along \( D \). We let \( e_k = s_k|_{D \times S}: e_1, \cdots, e_r \) form a smooth basis of \( E|_S \). We let \( \hat{e}_k \) be the pullback of \( e_k \) via the tautological projection \( E_{DS} \to E|_S \). Under this basis, \( a_0(z) + b(z) \) becomes an \( r \times r \)-matrix with entries \( \Omega^{0,1}(adE|_S)_{K} \)-valued holomorphic functions over \( D \). Let \( c_{kj} \) be functions so that \( \tilde{s}_k = \sum_j c_{kj} \hat{e}_j \). Because \( \tilde{s}_k \) are \( \tilde{\nabla}_{DS} \) holomorphic, using \( \bar{\partial}_z \hat{e}_k = \bar{\partial}_0 \hat{e}_k = 0 \), we have
\[
\bar{\partial}_z c_{kj} \in (a_0(z) + b(z))_{kt} c_{ij} + \bar{\partial}_0 c_{kj} = 0.
\]
Since the only the term \( \bar{\partial}_z c_{kj} \) takes value in \((0,1)\)-forms of \( D \), (others take value in \((0,1)\)-forms of \( S \)) we have \( \bar{\partial}_z c_{kj} = 0 \). Therefore, \( c_{kj} \) are holomorphic in \( z \).
This proves that the hermitian metric form of $h_S$ under the basis $\bar{e}_1, \ldots, \bar{e}_r$ is real analytic in $(\text{Re } z, \text{Im } z)$. Finally, we add that $\bar{s}_i$ and $\bar{e}_j$ are $L^0_s$, thus $c_{k_j}$ lie in $L^\ell_s = L^\ell_{s+2}$. This proves that the metric $h_S$ is $L^\ell_{s+2}$.

**Corollary 4.7.** There is an open neighborhood $0 \in D \subset V_0$ and an $L^\ell_{s+2}$ hermitian metric $\bar{h}_D$ on $E_D$ such that $\bar{h}_D$ is analytic in $z \in D$ and extends $h|_{O_0 \times Y}$.

**Proof.** By the previous Lemma, for any $y \in Y$, we can find an open neighborhood $D \times S \subset V_0 \times Y$ of $(0, y) \in V_0 \times Y$ and a hermitian metric $\bar{h}_S$ on $E_D \times S$ that extends $h|_{O_0 \times S}$, and is analytic in $z \in D$. Because $Y$ is compact, we can cover $Y$ by finitely many such opens $S_a, a = 1, \ldots, l$, paired with $0 \in D_a \subset V_0$. Let $D_0 = \cap D_a$. Then $0 \in D_0 \subset V_0$ is open and $\bar{h}_S_a$ are defined over $D_0 \times S_a$.

We then pick a smooth partition of unity $\sum_{a=1}^l \chi_a = 1$ with $\chi_a : Y \to [0, 1]$ such that the closure $\{(\chi_a > 0) \}$ lies in $S_a$. Then $\bar{h}_D = \sum_{a=1}^l \chi_a \cdot \bar{h}_S_a$ is an $L^\ell_{s+2}$ hermitian metric on $E_{D_0}$ that is analytic in $z \in D_0$, and extends $h|_{O_0 \times Y}$.

**Lemma 4.8.** Let the hermitian metric $\bar{h}_D$ on $E_D$ be given by Corollary 4.7, and let $\overline{\mathcal{D}}^*_z = \overline{\mathcal{D}}_0 + a_0(z)^t$, $z \in D$, be the formal adjoint of $\overline{\mathcal{D}}_z$ using the hermitian metric $\bar{h}_z := \bar{h}_{D_0 \times Y}$. Then we can find a complexification $D^C \supset D$ such that the function $a_0(z)^t$ extends to a holomorphic $a_0(z)^t : D^C \to \Omega^0(adE \otimes \mathbb{C} T^0_Y)^t, s$.

**Proof.** Using the explicit dependence of $\overline{\mathcal{D}}^*_z := (\overline{\mathcal{D}}_0 + a_0(z))^*$ on the metric $\bar{h}_z$, we see immediately that $\overline{\mathcal{D}}^*_z := \overline{\mathcal{D}}^*_z - \overline{\mathcal{D}}^*_0 \in \Omega^0(adE \otimes \mathbb{C} T^0_Y)^t, s$ is analytic in $(\text{Re } z, \text{Im } z)$. Following the proof of Lemma 4.5, there is a complexification $D^C \supset D$ such that $a_0(z)^t$ extends to $a_0(w)^t, s$ defined over $D^C$ and holomorphic in $w \in D^C$. Here we have used that $\bar{h}_D$ is $L^\ell_{s+2}$ to ensure that $a_0(w)^t$ are $L^\ell_s$.

In the remainder of this section, we fix a complexification $D^C \supset D$ so that both $a_0(z)$ and $a_0(z)^t$ extend holomorphically to $a_0(w)^t, s$ and $a_0(w)^t, s$ on $D^C$. We define

$$O^C_0 = \{F^0_{\overline{\mathcal{D}}_0 + a_0(w), s} = 0\}_s \subset D^C.$$  

For $w \in D^C$, we define $\overline{\mathcal{D}}^*_w = \overline{\mathcal{D}}_0 + a_0(w)^t$.

**Corollary 4.9.** We have $(\overline{\mathcal{D}}^*_w)^2_{O^C_0} = 0$.

**Proof.** Via the holomorphic map $\eta : D^C \to V_0$, $w = (u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto z = (u_1 + iv_1, \ldots, u_n + iv_n), \text{ we see that the pullback } a_0(\eta(w)) \text{ is a holomorphic extension of } a_0(z). \text{ Thus by the uniqueness of holomorphic extension, we have } a_0(w)^t, s = a_0(\eta(w)). \text{ Thus } O^C_0 = \eta^{-1}(O_0). \text{ In particular, every irreducible component } A \subset O_0 \text{ has its complexification } A_C = \eta(A), \text{ and vice versa (cf. [27, Proposition 5.3]).}$

Since $(\overline{\mathcal{D}}^*_w)^2$ is holomorphic and vanishes along $O_0$, by studying its vanishing near a general point of any irreducible component $A$ of $O_0$, and noticing that $O^C_0$ is with the reduced analytic subspace structure, we conclude that $(\overline{\mathcal{D}}^*_w)^2_{O^C_0} = 0$. 

We now complexify $\Theta_D(\ell)$ using the span of generalized eigenvectors of the “Laplacian” of $\overline{\mathcal{D}}_w$. We define

$$\square_w = \overline{\mathcal{D}}_w \overline{\mathcal{D}}^*_w + \overline{\mathcal{D}}^*_w \overline{\mathcal{D}}_w : \Omega^{0,J}(adE)_s \to \Omega^{0,J}(adE), s-2, \quad w \in D^C.$$  

This is a family of second order elliptic operators, holomorphic in $w \in D^C$, whose symbols are identical to that of $\square_{x_0}$.  


Lemma 4.10. Let the notation be as before. Suppose $\epsilon_0 > 0$ separates eigenvalues of $\partial_z$ and $\epsilon_0 < \epsilon(z)$ for all $z \in O_0$. Then we can choose $D^c \supset D$ such that $\Theta_{O_0}(\epsilon_0) \subset O \times \Omega^{0,1}(adE)_s$ extends to a holomorphic subbundle $\Theta_{O_0}(\epsilon_0) \subset O_0 \times \Omega^{0,1}(adE)_s$.

Proof. We extend $\Theta_D(\epsilon_0)$ to $D^c$ using the generalized eigenforms of $\square_w$. Since $\square_w$ is holomorphic in $w$, and since $1 + \square_z, z \in D$, are invertible, by [14, page 365] after shrinking $D^c \supset D$ if necessary, the family

$$(1 + \square_w)^{-1} : \Omega^{0,1}(adE)_s \rightarrow \Omega^{0,1}(adE)_s, \quad w \in D^c,$$

is a holomorphic family of bounded operators.

We now extend $\Theta_D(\epsilon_0)$. First, note that $\lambda$ is a spectrum of $\square_z$ if and only if $(1 + \lambda)^{-1}$ is a spectrum of $(1 + \square_z)^{-1}$, and they have identical associated spaces of generalized eigenforms. Since $\square_z, z \in D$, has discrete spectrum (eigenvalues) and $\epsilon$ is not its eigenvalue, for any $x \in D$, we can pick a small open neighborhood $D_x \subset D$ of $x \in D$ and a sufficiently small $\delta, \epsilon_0 \gg \delta > 0$, so that no eigenvalues of $(1 + \square_z)^{-1}$ for $z \in D_x$ lie in $\{\lambda | (1 + \epsilon_0)^{-1} \leq \delta\}$. Then by the continuity of the spectrum, we can find an open $D^c_x \subset D^c$, $D^c_x \cap D = D_x$, such that no $(1 + \square_w)^{-1}$, $w \in D^c_x$, contains spectrum in the region $\{\lambda | (1 + \epsilon_0)^{-1} \leq \delta/2\}$. Applying [14, Theorem VII-1.7], over $D^c_x$ we have decompositions $\Omega^{0,1}(adE)_s = E_{1,w} \oplus E_{2,w}$ such that $E_{1,w}$ and $E_{2,w}$ are holomorphic in $w$, invariant under $(1 + \square_w)^{-1}$, and $T_{i,w} := (1 + \square_w)^{-1}|_{E_{i,w}} : E_{i,w} \rightarrow E_{i,w}$ has its spectrum in $\{|\lambda| < (1 + \epsilon_0)^{-1}\}$ for $i = 1$ and in $\{|\lambda| > (1 + \epsilon_0)^{-1}\}$ for $i = 2$.

For us, the key property is that $E_{1,w} = \Theta_w(\epsilon_0)$ when $w \in D_x$. For $w \in D^c_x$, we define $\Theta_w(\epsilon_0) = E_{1,w}$. Then $\Theta_{D^c}(\epsilon_0) := \bigoplus_{w \in D^c_x} \Theta_w(\epsilon_0)$ extends $\Theta_D(\epsilon_0)$ holomorphically to $D^c$. By covering $D$ by open subsets like $D_x$, and using that the holomorphic extensions of $\Theta_D(\epsilon_0)$ are unique, when they exist, we conclude that for a complexification $D^c \supset D$,

$$\Theta_{D^c}(\epsilon_0) := \prod_{w \in D^c} \Theta_w(\epsilon_0) \subset D^c \times \Omega^{0,1}(adE)_s$$

extends $\Theta_D(\epsilon_0)$ and is a holomorphic bundle over $D^c$. \hfill \Box

Corollary 4.11. $\Theta_{U_0}(\epsilon_0)$ is an analytic subbundle of $T_{U_0}B$.

Proof. Applying the surjective homomorphism (4.2), and using the complexification constructed, the conclusion follows. \hfill \Box

4.4. The existence of orientation bundles. We prove Proposition 3.4. We begin with a rephrasing of the non-degeneracy condition under $cs_{2,x}$. We define a pairing

$$(\cdot, \cdot)_x : \Omega^{0,1}(adE_x)_s \times \Omega^{0,2}(adE_x)_{s-1} \rightarrow \mathbb{C}, \quad x \in X,$$

via $(a_1, a_2)_x = \frac{1}{\sqrt{\pi}} \int \tr(a_1 \wedge a_2) \wedge \Omega$. It relates to the quadratic form $cs_{2,x}$ via

$$(a, b)_{x} = (a, \partial_x b)_x, \quad a, b \in T_xB \cong \ker((\not\partial)^s_x)^{0,1}.$$

Given a subspace $W \subset T_xB \cong \ker((\not\partial)^s_x)^{0,1}$ that contains $T_xX \cong \square_x^{-1}(0)^{0,1}$, we define its companion spaces by

$$(4.11) \quad W' = \square_x^{-1}(0)^{0,2} \oplus \partial_x(W) \quad \text{and} \quad W'' := \square_x^{-1}(0)^{0,1} \oplus \square_x(W).$$

Recall that $Q_x$ is the descent of $cs_{2,x}$ to $T_xB/T_xX$. 
Lemma 4.12. Let $W \subset T_x \mathfrak{X}$ be a subspace containing $T_x \mathfrak{X}$, and let $W'$ be its companion space. Then $Q_{\varepsilon|W/T_x \mathfrak{X}}$ is non-degenerate if and only if the restricted pairing $(\cdot, \cdot)_x : W \times W' \to \mathbb{C}$ is a perfect pairing.

Proof. Since $Y$ is a Calabi-Yau threefold, by Serre duality, the pairing $(\cdot, \cdot)_x$ restricted to $\square_x^{-1}(0)^{0,1} \times \square_x^{-1}(0)^{0,2}$ is perfect. Let $e_1, \ldots, e_l \in W$ be so that $\square_x e_1, \ldots, \square_x e_l$ form a basis of $\square_x(W)$. By Hodge theory, $e_1, \ldots, e_l$ and $\square_x^{-1}(0)^{0,1}$ span $W$. By (4.10), and that $\square_x^{-1}(0)^{0,1}$ is orthogonal to $\text{Im}(\square_x)$ under $(\cdot, \cdot)_x$, $Q_x$ is non-degenerate on $W/T_x \mathfrak{X} = W/\square_x^{-1}(0)^{0,1}$ if and only if $(e_i, \square_x e_j)_x$ form an invertible $l \times l$ matrix, which is equivalent to that $(\cdot, \cdot)_x$ on $W \times W'$ is perfect. This proves the lemma.

Lemma 4.13. Let $x \in X$ and $r \geq d_x = \dim T_x \mathfrak{X}$ be an integer. Then we can find an open neighborhood $U \subset X$ of $x$ such that there exists a rank $r$ orientation bundle over $U$.

Proof. We first recall an easy fact. Let $q$ be a non-degenerate quadratic form on $\mathbb{C}^n$. Let $0 < l \leq n$ be an integer, and let $Gr(l, \mathbb{C}^n)$ be the Grassmannian of $l$ dimensional subspaces of $\mathbb{C}^n$. We introduce

\[(4.12) \quad Gr(l, \mathbb{C}^n)^0 = \{[S] \in Gr(l, \mathbb{C}^n) \mid q|_S \text{ is non-degenerate}\}.
\]

Since $q$ is non-degenerate, it is direct to check that $Gr(l, \mathbb{C}^n)^0$ is the complement of a divisor in $Gr(l, \mathbb{C}^n)$, and thus is connected and smooth.

We now prove the lemma. We first treat the case $r = d_x$. Since $\square_x$ has non-negative discrete eigenvalues, there is an $\epsilon > 0$ so that it has no eigenvalues in $(0, 2\epsilon)$. By Corollary 4.11, over an open neighborhood $U \subset X$ of $x$, $\Theta_U(\epsilon)$ is an analytic subbundle of $T_U \mathfrak{X}$. To show that it is an orientation bundle, we only need to verify that for any $y \in U$, $Q_y$ restricting to $\Theta_y(\epsilon)/\square_y^{-1}(0)^{0,1}$ is non-degenerate. By Lemma 4.12, this is equivalent to that $(\cdot, \cdot)_y : \Theta_y(\epsilon) \times \Theta_y(\epsilon)' \to \mathbb{C}$ is perfect. Because it is perfect at $y = x$, and because being perfect is an open condition, possibly after shrinking $x \in U$ if necessary, it is perfect for every $y \in U$. This proves the case $r = d_x$.

In case $l = r - d_x > 0$, applying the discussion at the beginning of this proof, we find an $l$-dimensional subspace $W \subset T_x \mathfrak{X}/T_x \mathfrak{X}$ so that $Q_{x|W}$ is non-degenerate. We let $\Xi_x$ be the preimage of $W$ under the quotient map $T_U \mathfrak{X} \to T_U \mathfrak{X}/T_x \mathfrak{X}$. To complete the proof, we extend $\Xi_x$ to a neighborhood of $x \in X$ and show that it is an orientation bundle.

We pick an open neighborhood $U \subset X$ of $x \in X$ such that the isomorphism $\zeta$ in (4.1) has been chosen; we pick a basis of $W$, say $u_1, \ldots, u_l \in \ker((\square_x/y)_s)^{0,1}/\square_x^{-1}(0)^{0,1} = \text{Im}(\square_x/y)_{0,1}^{0,1}$. We then extend $u_k$ to be the constant section of $U \times \Omega^{0,1}(adE)_s$, and let $\tilde{u}_k$ be its image sections in $T_U \mathfrak{X}$. It is a holomorphic extension of $u_k$. By shrinking $x \in U$ if necessary, $\Theta_U(\epsilon)$ and the sections $\tilde{u}_1, \ldots, \tilde{u}_l$ span a subbundle $\Xi$ of $T_U \mathfrak{X}$. Because $\Theta_U(\epsilon)$ is an analytic subbundle of $U \times \Omega^{0,1}(adE)_s$, and because $\tilde{u}_k$ are holomorphic, we see that $\Xi$ is an analytic subbundle of $T_U \mathfrak{X}$ and contains $\Theta_U(\epsilon)$.

It remains to check that for any $y \in U$, $Q_y$ restricted to $\Xi_y/T_y \mathfrak{X}$ is non-degenerate. By the previous lemma, this is equivalent to the fact that the pairing $(\cdot, \cdot)_y : \Xi_y \times \Xi_y' \to \mathbb{C}$ is perfect, where $\Xi_y'$ is the companion space of $\Xi_y$ (cf. (4.11)). Because this pairing is perfect when $y = x$, by shrinking $x \in U$ if necessary, we can make it perfect for all $y \in U$. Therefore, $\Xi$ is an orientation bundle over $U$. \qed
Lemma 4.14. Suppose $\Xi_\alpha$ and $\Xi_\beta$ are two rank $r$ orientation bundles over an open $U$. Then for any $x \in U$, there is an open neighborhood $U_0 \subset U$ of $x$ such that there is a homotopy from $\Xi_\alpha|_{U_0}$ to $\Xi_\beta|_{U_0}$.

Proof. We begin with an easy fact. For any finite dimensional subspace $W_0 \subset W := T_xB/T_xX$, there is a finite dimensional subspace $W \subset W$ containing $W_0$, such that $Q_x|W$ is non-degenerate. Indeed, let $N \subset W_0$ be the null-subspace of $Q_x|W_0$. Since $Q_x$ is non-degenerate, we can find a subspace $M \subset W$ such that $W_0 \cap M = 0$ and $Q_x : N \times M \to \mathbb{C}$ is perfect. Then the space $W = W_0 \oplus M \subset W$ is the desired subspace.

We now construct the desired homotopy. We first find a finite dimensional subspace $W \subset W$ so that it contains both $\Xi_\alpha|_x/T_xX$ and $\Xi_\beta|_x/T_xX$, and $Q_x$ is non-degenerate on $W$. Let $l = r - d_x$. We form the Grassmannian $Gr(l,W)$ and its Zariski open subset $Gr(l,W)^0$ as in (4.12) (with $Q_x$ in place of $q$). Then both $[\Xi_\alpha|_x/T_xX]$ and $[\Xi_\beta|_x/T_xX]$ are in $Gr(l,W)^0$. Because $Gr(l,W)^0$ is a smooth connected quasi-projective variety, we can find an analytic arc $[S_l] \in Gr(l,W)^0$, $t \in [0,1]$, such that $S_0 = \Xi_\alpha|_x/T_xX$ and $S_1 = \Xi_\beta|_x/T_xX$. We let $\Xi_{t,x}$ be the preimage of $S_l$ under the quotient homomorphism

$$\pi_x : T_xB \to T_xB/T_xX.$$ 

Then $[S_l]$ form an analytic family of subspaces in $T_xB$, interpolating between $\Xi_\alpha|_x$ and $\Xi_\beta|_x$.

We extend this to an analytic family of orientation bundles in a neighborhood of $x \in U$. As before, we realize an open neighborhood $U_0 \subset U$ of $x \in U$ as $U_0 = X_{f_0}$, $x = 0 \in V_0$, for the JS chart $(V_0,f_0)$. Then we have the isomorphism of holomorphic Banach bundles $T_{U_0}B \cong U_0 \times \ker((\overline{\partial})_x^0)^0.1$. By choosing $U_0$ sufficiently small, we can find an $\epsilon > 0$ so that $\Theta_{U_0}(\epsilon) \subset \Xi_\alpha|_{U_0}$, $\Theta_{U_0}(\epsilon) \subset \Xi_\beta|_{U_0}$, and $\Theta_x(\epsilon) = \overline{\partial}_{\frac{1}{1}}^0(0)^0.1$.

Next, for $i = \alpha$ and $\beta$, we find $l$ analytic sections $s^1_1, \cdots s^1_l$ of $T_{U_0}B$ such that $\Theta_{U_0}(\epsilon)$ and $s^1_1, \cdots s^1_l$ span $\Xi_{U_0}$. Because $s^0_1(x)$ and $s^0_1(x)$ all lie in $pi_x^{-1}(W)$, we can find arcs $\xi_k(t) \in \Xi_{x_i}^{-1}(W)$, analytic in $t \in [0,1]$, such that $\xi_k(0) = s^0_k(x)$ and $\xi_k(1) = s^0_k(x)$, and

$$\Xi_{t,x} = \Theta_x(\epsilon) \oplus \mathbb{C}-\text{span}(\xi_1(t), \cdots \xi_l(t)).$$

Using the isomorphism $T_{U_0}B \cong U_0 \times \ker((\overline{\partial})_x^0)^0.1$, we can view $\xi_k(t)$ as analytic arcs in $\ker((\overline{\partial})_x^0)^0.1 \subset \Omega^{0,1}(adE)_s$. We then define

$$s^1_k(y) = \xi_k(t) + (1 - t)(s^0_k(x) - s^0_k(x)) + t(s^0_k(y) - s^0_k(x)).$$

Clearly, they are analytic in $t$, and $s^1_k(y) = s^1_k(y)$ and $s^1_k(y) = s^1_k(y)$ for all $y \in U_0$. Therefore, by shrinking $x \in U_0$ if necessary, for every $t \in [0,1]$, the sections $s^1_1(y), \cdots s^1_l(y)$ and $\Theta_{U_0}(\epsilon)$ span a rank $r$ analytic subbundle $\Xi_{\mathbb{C}}^0 \subset U_0 \times \Omega^{0,1}(adE)_s$. Because the arcs $\xi_k(t)$ are analytic in $t$, the family $\Xi_{\mathbb{C}}^0$ is analytic in $t$. Finally, because $[S_l]$ all lie in $Gr(l,W)^0$, by shrinking $x \in U_0$ if necessary,

$$\Xi_{t,x} := \text{image of } \Xi_{\mathbb{C}}^0 \text{ under } U_0 \times \Omega^{0,1}(adE)_s \to T_{U_0}B$$

form an analytic family of orientation bundles providing the desired homotopy between $\Xi_\alpha|_{U_0}$ and $\Xi_\beta|_{U_0}$.

$\square$

Proof of Proposition 3.4. We first pick a locally finite cover $U_\alpha$ so that each $U_\alpha$ has an orientation bundle $\Xi_\alpha$. For each $x \in X$, we pick an open neighborhood $U_{\alpha}$ of $x \in X$ so that (1) $U_x \subset U_\alpha$ whenever $x \in U_\alpha$, and that (2) for every pair $\alpha, \beta$ with
We comment that a locally finite refinement of the covering as follows: For each $x \in X$, we fix any $\alpha(x)$ such that $U_x \subset U_{\alpha(x)}$. Since $X$ is quasi-projective, we have a metric $d(\cdot, \cdot)$ on $X$ induced from projective space. By shrinking $U_x$ if necessary, we may assume $U_x$ is the ball $B(x, 2\epsilon_x)$ of radius $2\epsilon_x > 0$ centered at $x$. Let $O_x = B(x, \epsilon_x)$ and $\Xi_x = \Xi_{\alpha(x)}|U_x$. Then $\{O_x\}$ is an open cover of $X$ and $\Xi_x$ is an orientation bundle on $O_x$. Suppose that $O_x \cap O_y \neq \emptyset$. Without loss of generality, we may assume $\epsilon_x \leq \epsilon_y$. Then $O_x \subset B(y, 2\epsilon_y) = U_y \subset U_{\alpha(y)}$. Also we have $x \in O_x \subset U_x \subset U_{\alpha(x)}$. Hence $O_x \subset U_{\alpha(x)} \cap U_{\alpha(y)}$ and thus we have a homotopy from $\Xi_x|O_x \cap O_y$ to $\Xi_y|O_x \cap O_y$ as desired. 

5. CS data from preorientation data

In this section we prove Proposition 3.12. We construct CS charts from orientation bundles, their local trivializations, and complexifications.

5.1. Constructing families of CS charts. Let $\Xi$ be an orientated bundle on $U$. We generalize Joyce-Song’s construction in [12] to form a $\Xi$-aligned family of CS charts.

Given $\Xi$, for any $x \in U$, we view $\Xi_x \subset \ker(\overline{\partial}_x)_{0,1}$ and denote its companion space $\Xi_x'' \subset \Omega^{0,1}(adE_x)$ be as defined in (4.11) with $W$ replaced by $\Xi_x$. Using condition (1) of Definition 3.1, one sees that $\Xi'' := \bigsqcup_{x \in U} \Xi''_x$ is an analytic subbundle of $\Omega^{0,1}(adE)_{s-2}|U$.

We define the quotient homomorphism of Banach bundles
\begin{equation}
P : \Omega^{0,1}(adE)_{s-2}|U \longrightarrow \Omega^{0,1}(adE)_{s-2}|U/\Xi'',
\end{equation}
whose restriction to $x \in U$ is denoted by $P_x : \Omega^{0,1}(adE_x)_{s-2} \rightarrow \Omega^{0,1}(adE_x)_{s-2}/\Xi''_x$. For $x \in U$, we form the elliptic operator
\begin{equation}
L_x : \Omega^{0,1}(adE_x)_{s} \longrightarrow \Omega^{0,1}(adE_x)_{s-2}/\Xi''_x, \quad L_x(a) = P_x(\Box_x a + \overline{\partial}_x a \wedge a).
\end{equation}

For a continuous $\varepsilon(\cdot) : U \rightarrow (0, 1)$ to be specified shortly, we define
\begin{equation}
V_x = \{a \in \Omega^{0,1}(adE_x)_s \mid L_x(a) = 0, \|a\|_s < \varepsilon(x)\}.
\end{equation}
(Here $\|\cdot\|_s$ is defined using $h_x$.) Letting $\Pi_x : \Omega^{0,1}(adE_x)_s \rightarrow B$ be the composite of the tautological isomorphism $\overline{\partial}_x + \cdots : \Omega^{0,1}(adE_x)_s \cong A_x$ (cf. (3.1)) with the tautological projection $A_x \rightarrow B$, we define
\begin{equation}
\mathcal{V}_x = \Pi_x(V_x).
\end{equation}
We comment that $\mathcal{V}_x$ only depends on $(\Xi_x, h_x, \varepsilon(x))$.

Let $f_x : V_x \rightarrow \mathbb{C}$ (or $f_x : V_x \rightarrow \mathbb{C}$) be the composite of $V_x \rightarrow B$ and $cs : B \rightarrow \mathbb{C}$.

**Proposition 5.1.** Let $U \subset X$ be open and $\Xi$ a rank $r$ orientation bundle on $U$. Then there is a continuous $\varepsilon(\cdot) : U \rightarrow (0, 1)$ such that the family $V_x$, $x \in U$, constructed via (5.3) using $\varepsilon(\cdot)$ is a smooth family of complex manifolds of dimension $r$, and such that all $(V_x, f_x)$ are CS charts of $X$.

**Proof.** We relate $V_x$ to the JS charts by first showing that $L_x(a) = 0$ if and only if
\begin{equation}
\overline{\partial}_x a = 0 \quad \text{and} \quad P_x \circ \overline{\partial}_x F^{0,2}_{\overline{\partial}_x + a} = 0.
\end{equation}
Indeed, it is immediate that (5.5) implies $L_x(a) = 0$. For the other direction, suppose $L_x(a) = 0$. Since $\Xi''_x \subset \ker(\overline{\partial}_x)_{s-2}$, applying $\overline{\partial}_x$ to $L_x(a) = 0$, we obtain...
\( \overline{\partial}_x a = 0 \), which forces \( \overline{\partial}_x^* a = 0 \). Having this, we obtain \( P_x \circ \overline{\partial}_x^* F^{0,2} = 0 \). This proves the equivalence.

By direct calculation, the linearization of \( L_x \) at \( a = 0 \) is
\[
\delta L_x = P_x \circ \square_x : \Omega^{0,1}(ad E_x)_s \longrightarrow \Omega^{0,1}(ad E_x)_{s-2}/\Xi_x,
\]
which is surjective with kernel \( \Xi_x \). Since the operators \( L_x \) depend smoothly on \( x \in U \), applying the implicit function theorem, for a continuous \( \varepsilon(\cdot) : U_0 \to (0,1) \) (taking sufficiently small values), the solution spaces \( V_x \), \( x \in U \), form a family of manifolds of real dimensions \( 2r \). Since the operators \( L_x \) are holomorphic in \( a \), each solution space \( V_x \) is a complex submanifold of \( \Omega^{0,1}(ad E_x)_s \) and their images in \( B \) lie in \( B_x \). Finally, because the family \( \square_x \) is smooth in \( x \in U \), the family \( V_x \) is a smooth family of complex manifolds. Using local isomorphism \( \zeta \) in (4.1), we see that the family \( V_x, x \in U \), form a smooth family of complex submanifolds of \( B_x \). This proves the first part of the Proposition.

For the second part, we first show that by choosing \( \varepsilon(x) \) small enough, \( \mathcal{V}_x \cap \mathcal{X} \) contains an open complex analytic subspace of \( \mathcal{X} \) containing \( x \). At individual \( x \in U \), this follows from that each \( V_x \) contains (an open neighborhood of \( x \) in the) JS chart \((V_x^{JS}, f_x^{JS})\). However, to prove that we can choose \( \varepsilon(x) \) continuously in \( x \), we argue directly.

As \( (E_x, \overline{\partial}_x) \) are simple, the tautological \( \Pi_x : \ker(\overline{\partial}_x^* f)^{0,1} \to B \) is biholomorphic near \( \Pi_x(0) = x \). We let
\[
\mathfrak{F}_x : \Omega^{0,1}(ad E_x)_s \to \Omega^{0,2}(ad E_x)_{s-1}, \quad \mathfrak{F}_x(a) = F^{0,2} \overline{\partial}_x a,
\]
be the curvature section. Then \( \ker(\overline{\partial}_x^* f)^{0,1} \cap (\mathfrak{F}_x = 0) \) contains an open complex analytic subspace of \( \mathcal{X} \) containing \( x \) (cf. [12, Chapter 9], [26]). Since \( \ker(\overline{\partial}_x^* f)^{0,1} \cap (\mathfrak{F}_x = 0) \) is contained in \( (L_x = 0) \), \( \mathcal{V}_x \cap \mathcal{X} \) contains an open neighborhood of \( x \in \mathcal{X} \).

We now prove that for \( \varepsilon(x) \) small, \( \mathcal{F}_{\mathcal{X}} = \Pi_x^{-1}(\mathcal{V}_x \cap \mathcal{X}) \). We follow the proof of [12, Prop. 9.12]. Let \( x \in U \). We define a subbundle
\[
R_x := \{ (\overline{\partial}_x + a, b) \mid P_x \circ \overline{\partial}_x b = \overline{\partial}_x (\overline{\partial}_x b - b \& a - a \& b = 0) \} \subset V_x \times \Omega^{0,2}(ad E_x)_s.
\]
Applying the implicit function theorem, because the two equations in the bracket are holomorphic in \( a \), for \( \varepsilon(x) \) small enough, \( R_x \) is a holomorphic subbundle of \( V_x \times \Omega^{0,2}(ad E_x)_s \) over \( V_x \). Then the Bianchi identity coupled with the equivalence relations (5.5) ensures that the restriction of the curvature section to \( V_x \), namely \( \mathfrak{F}_x|_{V_x} \), is a section of \( R_x \). Since \( \mathcal{X} \) locally near \( \overline{\partial}_x \) is defined by the vanishing of \( \mathfrak{F}_x \), we conclude
\[
\Pi_x^{-1}(\mathcal{V}_x \cap \mathcal{X}) = V_x \cap (\mathfrak{F}_x|_{V_x} = 0).
\]
It remains to show that \( (\mathfrak{F}_x|_{V_x} = 0) = (df_x = 0) \). We define a bundle map
\[
\varphi_x : R_x \to T^\mathcal{X} V_x, \quad (\overline{\partial}_x + a, b) \mapsto (\overline{\partial}_x + a, \alpha_b),
\]
where \( \alpha_b \in T^\mathcal{X}_{\overline{\partial}_x + a} V_x \) is \( \alpha_b(\cdot) = \frac{1}{\sqrt{\varepsilon}} \int tr(\cdot \& b) \& \Omega \). Clearly, \( \varphi_x \) is holomorphic and \( \varphi_x \circ \mathfrak{F}_x|_{V_x} = df_x \). We show that by choosing \( \varepsilon(x) \) small, we can make \( \varphi_x \) an isomorphism of vector bundles over \( V_x \). We first claim that restricting to \( x \) we have
\[
R_x|_x = \{ b \in \Omega^{0,2}(ad E_x)_{s-1} \mid P_x \circ \overline{\partial}_x^* b = \overline{\partial}_x^* \overline{\partial}_x b = 0 \} = \overline{\partial}_x (\Xi_x) \oplus \square_x^{-1}(0)^{0,2}.
\]
Indeed the first identity follows from the definition of \( R_x \). We prove the second identity. For any \( b \in R_x|_x \), since \( \overline{\partial}_x^* \overline{\partial}_x b = 0 \), we have \( \overline{\partial}_x b = 0 \). Thus we can write
\( b = b_0 + \overline{\partial}_x c \) with \( \Box_c b_0 = 0 \). Then \( P_x \circ \overline{\partial}_x b = 0 \) implies that we may take \( c \in \Xi_x \).
This proves (5.7).

Applying Lemma 4.12, we know that the pairing \((\cdot, \cdot)_x\) (cf. (4.9)) restricted to \( T_x V_x \times R_x|_x \) is perfect and thus \( \varphi_x|_x \) is an isomorphism. Therefore, by choosing \( \varepsilon(x) \) sufficiently small, \( \varphi_x \) is an isomorphism over \( V_x \) and thus \( (V_x, f_x) \) is a CS chart of \( X \). Note that the choice of \( \varepsilon(x) \) is to make sure that \( \varphi_x \) are isomorphisms over \( V_x \). Since this relies on the openness argument, we can choose \( \varepsilon(\cdot) : U \to (0,1) \) continuously to ensure this. This proves the proposition. \( \square \)

The family of CS charts are canonical.

**Lemma 5.2.** Let \( \Xi_a \) and \( \Xi_b \) be two orientation bundles over \( U \). Suppose \( \Xi_a \subset \Xi_b \) is a vector subbundle. Let \( \varepsilon(\cdot) : U \to (0,1) \) be the size function that produces a family of CS charts \( \mathcal{V}(\Xi_b) \subset U \times \mathcal{B} \) using \( \Xi_b \). Then the same \( \varepsilon(\cdot) \) produces a family of CS charts \( \mathcal{V}(\Xi_a) \subset U \times \mathcal{B} \), and \( \mathcal{V}(\Xi_a) \subset \mathcal{V}(\Xi_b) \subset U \times \mathcal{B} \).

**Proof.** This follows from the construction because the only data needed to construct \( V_x \) are \( \Xi_x \) and \( \varepsilon(x) \). \( \square \)

### 5.2. Local trivializations

In this subsection, we prove

**Proposition 5.3.** The family \( \mathcal{V} \subset U \times \mathcal{B} \) constructed in Proposition 5.1 is locally trivial everywhere.

The study is local. For any \( x_0 \in U \), we pick an open neighborhood \( U_0 \subset U \) of \( x_0 \in U \) so that \( U_0 = X_{f_0} \) for the JS chart \( (V_0, f_0) \), coupled with the isomorphism \( \zeta \) in (4.1). Using the induced \( \mathcal{E}_x \cong E \) for \( x \in U_0 \), the CS charts \( V_x \) become complex submanifolds of \( \Omega^{0,1}(adE)_s \). We define

\[
(V_{U_0}) = \bigcap_{x \in U_0} x \times V_x \subset U_0 \times \Omega^{0,1}(adE)_s.
\]

By shrinking \( x_0 \in U_0 \) if necessary, we assume that \( U_0 \) lies in \( X_{f_x} \subset X \) for all \( x \in U_0 \).

To keep the notation transparent, for \( x, y \in U_0 \), we denote by \( y_x \in X_{f_x} \subset V_x \) the point whose image in \( X \) is \( y \), i.e. the point in \( X_{f_x} \) associated to \( y \in U_0 \). We denote by \( \overline{\partial}_x + a_x(y_x) \) the connection form of \( y_x \in V_x \). Because \( U_0 \subset X_{f_y} \), for any \( x, y, z \in U_0 \), as \( (E, \overline{\partial}_x + a_x(z_x)) \cong (E, \overline{\partial}_y + a_y(z_y)) \), there is a unique \( g \in \mathcal{G} \) so that

\[
\overline{\partial}_y + a_y(z_y) = g(\overline{\partial}_x + a_x(z_x)) = \overline{\partial}_x - \overline{\partial}_x g \cdot g^{-1} + ga_x(z_x)g^{-1}.
\]

As before, for open \( \mathcal{U} \subset U_0 \times V_{U_0} \) containing \( \Delta(U_0) \subset U_0 \times V_{U_0} \), viewed as a subset of \( V_y \). We denote \( \Delta(U_0) = \{(x, x) \mid x \in U_0 \} \subset U_0 \times V_{U_0} \).

**Lemma 5.4.** We can find an open \( \mathcal{U} \subset U_0 \times V_{U_0} \) containing \( \Delta(U_0) \subset U_0 \times V_{U_0} \) and a smooth \( g : \mathcal{U} \to \mathcal{G} \) such that the following hold:

1. for any \( x, y \in U_0 \), \( g_{x,y}: g|_{\mathcal{U}_{x,y}} : \mathcal{U}_{x,y} \to \mathcal{G} \) is holomorphic and \( g_{x,x}(\cdot) = 1 \);
2. for \( x, y, z \in U_0 \), \( (x, z) \in \mathcal{U}_{x,y} \), and letting \( a(x, y, z) = g_{x,y}(z)(\overline{\partial}_y + a_y(z)) - \overline{\partial}_x \), we have \( \overline{\partial}_x a(x, y, z) = 0 \), and \( a(x, x, x) = 0 \).

**Proof.** For \( \alpha \in \mathcal{G} \), and for \( x, y \in U_0 \) and \( z \in V_y \), we form \( c_{x,y,z}(\alpha) := \alpha(\overline{\partial}_y + a_y(z)) - \overline{\partial}_x \), and define \( R_{x,y,z}(\cdot) = \overline{\partial}_x c_{x,y,z}(\cdot) : \mathcal{G}_{s+1} \to \Omega^{0}(adE)_{s-1}/\mathbb{C} \).

Note that \( R_{x,x,x}(1) = 0 \).
We calculate the linearization of the operator $R_{x,y,z}(\cdot)$ at $(x, x, x)$ and $\alpha = 1$:
\[ \delta R_{x,x,x}|_{\alpha=1} = -\overline{\Omega}_x \overline{\delta}_x : \Omega^0(adE)_{x+1}/C \longrightarrow \Omega^0(adE)_{x-1}/C. \]
Because $(E, \overline{\delta}_x)$ are simple, $\delta R_{x,x,x}|_{\alpha=1}$ are isomorphisms. Thus by the implicit function theorem, for an open $U \subset U_0 \times V_{U_0}$ containing the diagonal $\Delta(U_0) \subset U_0 \times V_{U_0}$, we can find a unique smooth
\[ g_\cdot(\cdot) : U \longrightarrow \mathcal{G} \]
such that $R_{x,y,z}(g_{x,y}(z)) = 0$, and $g_{x,x}(z) = 1$. Because the equation $R_{x,y,z}(\alpha)$ is holomorphic in $z$ and $\alpha$, $g_{x,y}(z)$ is holomorphic in $z \in U_{x,y} \subset V_y$. Thus both (1) and (2) of the lemma are satisfied.

We remark that by the uniqueness of the solution $g_{x,y}(z)$, by an extension of (5.9) (to the non-reduced case), we have $a(x, y, \cdot)|_{U_{x,y} \cap X_f_y} = a_y(\cdot)|_{U_{x,y} \cap X_f_y}$.

**Proof of Proposition 5.3.** We take the family $g_{x,y}(z)$ defined on $U \subset U_0 \times V_{U_0}$ constructed in the previous Lemma. To find a local trivialization $\Psi : U \rightarrow V_{U_0} \times U_0$, we solve $b(x, y, z)$ satisfying the system

\[ L_x (a(x, y, z) + b(x, y, z)) = 0, \quad (b(x, y, z), \nu)_x = 0, \quad \forall \nu \in \Xi'_{\nu}, \]

where $L_x$ is defined in (5.2). By the remark at the end of the previous proof, we see that restricting to $U_{x,y} \cap X_f_y$, $b(x, y, \cdot) = 0$ are solutions. Also, since $a(x, x, z) = a_x(z)$ and $L_x(a_x(z)) = 0$, $b(x, x, z) = 0$ are solutions.

We denote $(\Xi'_{\nu}) = \{ b \mid (b, \nu)_x = 0, \forall \nu \in \Xi'_{\nu} \} \subset \Omega^{0,1}(adE)_x$. We define
\[ M_{x,y,z}(\cdot) = L_x (a(x, y, z) + \cdot) : (\Xi'_{\nu}) \longrightarrow \Omega^{0,1}(adE)_{x-2}/\Xi'_{\nu}. \]

By our construction, the linearization $\delta M_{x,x,x}$ at 0 is
\[ \delta M_{x,x,x}|_0 = P_x \circ \square_x : (\Xi'_{\nu}) \longrightarrow \Omega^{0,1}(adE)_{x-2}/\Xi'_{\nu}, \]
which is an isomorphism. Thus applying the implicit function theorem, for an open $U' \subset U$ containing the diagonal $\Delta(U_0)$, we can solve the system $M_{x,y,z}(b(x, y, z)) = 0$ uniquely and smoothly in $(x, y, z)$. Further, since $g_{x,y}(z)$ is holomorphic in $z$, and the operator $M_{x,y,z}(\cdot)$ is a linear operator holomorphic in $z$, $b(x, y, z)$ is holomorphic in $z$.

We set $\Psi : U' \rightarrow U_0 \times V_{U_0}$ to be
\[ \Psi_{x,y}(\overline{\delta}_y + a_y(z)) = g_{x,y}(z)(\overline{\delta}_y + a_y(z)) + b(x, y, z) - \overline{\delta}_x. \]

Because $M_{x,y,z}(0)|_{U_{x,y} \cap X_f_y} = 0$, we conclude that $\Psi_{x,y}$ is an open embedding of $U_{x,y} \cap X_f_y$ into $X_f_y \subset V_x$. Therefore, $\Psi$ is the desired local trivialization.

The local trivializations are canonical.

**Lemma 5.5.** Let $\Xi_\alpha \subset \Xi_b$ be two orientation bundles over $U$, as in Lemma 5.2. Suppose $U_0 \subset U$ is an open subset and $\Psi_i : U_i \rightarrow V(\Xi_i)_{U_0} \times U_0$, $i = a, b$, be local trivializations constructed. Let $V(\Xi_\alpha)_{U_0} \subset V(\Xi_b)_{U_0}$ be the inclusion given by Lemma 5.2. Then
\[ \Psi_a|_{U_a \cap U_0} = \Psi_b|_{U_b \cap U_0} : U_a \cap U_0 \longrightarrow V(\Xi_\alpha)_{U_0} \times U_0 \subset V(\Xi_b)_{U_0} \times U_0. \]

**Proof.** This follows from that $(\Xi'_{\nu_a}) \subset (\Xi'_{\nu_b})$, and thus $\Psi_a$ is also the solution to the equations for $\Psi_b$ when restricted to $V_a \subset V_b$. By the uniqueness of the solution, we have the identity. \qed
Corollary 5.6. Let $\Xi$ (resp. $\Xi_t$) be an orientation bundle (resp. a homotopy of orientation bundles) on $U$. Then $\Xi$ (resp. $\Xi_t$) can be complexified locally everywhere.

Proof. The proof is straightforward, knowing that both $\Xi$ and $\Xi_t$ are analytic. □

5.3. Complexifications. We construct complexifications of a family of CS charts.

Proposition 5.7. The family of CS charts constructed in Proposition 5.1 and the local trivialization constructed in Proposition 5.3 can be complexified locally everywhere.

Proof. Given an orientation $\Xi$ on an open $U \subset X$, for any $x_0 \in U$, we pick an open neighborhood $U_0 \subset U$ of $x_0 \in U$ so that $U_0 = X_{f_0}$ for the JS chart $(V_0, f_0)$, where $x_0 = 0 \in V_0$. By Lemma 4.8, we pick an open $0 \in D \subset V_0$ so that $\overline{\partial}_z$ and $\overline{\partial}^z$ extend to holomorphic $\overline{\partial}_w = \overline{\partial}_0 + a_0(w)_C$ and $\overline{\partial}_w = \overline{\partial}_0 + a_0(w)_C$ over a complexification $D^C$ of $D$. By Corollary 5.6, we extend $\Theta_D(\epsilon)$ and $\Xi_D^\epsilon$ to holomorphic subbundles $\Theta_D^\epsilon(\epsilon)$ and $\Xi_D^\epsilon(\epsilon)$ of $T_D^{\epsilon}$. (Here we follow the convention that for any $Z \to B$, we denote $T_ZB = TB \times_B Z$.)

We form the projection $P_w$ as in (5.1), with $E$ (resp. $\Xi''$) replaced by $E$ (resp. $\Xi''$). We form $V_w$ using (5.3), with $\Sigma_w$ replaced by $E$ and subscript “$w$” replaced by the subscript “$w$”. Since the proof that the resulting family $V \subset U \times B$ is a smooth family of complex manifolds only uses the isomorphism property of the linearization of $L_w$ and the implicit function theorem, the same study extends to small perturbations of $L_w$. Thus possibly after shrinking $D^C \supset D$ if necessary, the family

$$V_{D^C} = \prod_{w \in D^C} w \times V_w \subset D^C \times \Omega^{0,1}(adE)_s$$

is a smooth family of complex manifolds. Because all $\overline{\partial}_w$, $\overline{\partial}_w^*$ and $\Xi''_D^w$ are holomorphic in $w \in D^C$, the system $L_w$ is holomorphic in $w$. Therefore, $V_{D^C}$ is a complex manifold and is a complex submanifold of $D^C \times \Omega^{0,1}(adE)_s$.

Next we study the issue of being CS charts. Let $\Pi_w : V_w \to B$ be defined by $\overline{\partial}_w + \cdot$, and define $f_w = \Pi_w \circ \psi : V_w \to C$. We show that $(V_w, f_w)$ are CS charts for $w \in O_0^C$ (cf. (4.8)) is the complexification of $O_0 = D \cap X$.

Going through the proof of Proposition 5.1, we first need to check that for $w \in O_0^C$, $a \in V_w$ satisfies the system (cf. (5.5))

$$(5.11) \quad \overline{\partial}_w^*a = 0 \quad \text{and} \quad P_w \circ \overline{\partial}_w^*(\overline{\partial}_w a + a \wedge a) = 0.$$ 

First because $\Xi_{O_0^C}$ is the complexification of $\Xi_{O_0}$, and $\Xi_w \subset \ker(\overline{\partial}_x^*)_{x=1}^{0,1}$ for $x \in O_0$, we have that $\Xi_w \subset \ker(\overline{\partial}_w^*)_{w=1}^{0,1}$ for $w \in O_0^C$. On the other hand, $a \in V_w$ means that $\overline{\partial}_w \overline{\partial}_w^*a + \overline{\partial}_w^* (\overline{\partial}_w a + a \wedge a) \equiv 0 \mod \Xi''_w$. Thus applying $\overline{\partial}_w^*$ to this relation, we obtain $\overline{\partial}_w^* \overline{\partial}_w^* \overline{\partial}_w a = 0$. Finally, since $\overline{\partial}_x^* \overline{\partial}_x : \Omega^{0}(adE)_s/C \to \Omega^{0}(adE)_{s-2}/C$ are isomorphisms for $x \in D$, by shrinking $D^C \supset D$ if necessary, $\overline{\partial}_w^* \overline{\partial}_w$ are isomorphisms for $w \in O_0^C$, thus $\overline{\partial}_w^*a = 0$. This proves that for all $w \in O_0^C$, all $a \in V_w$ satisfy (5.11).

After this, mimicking the proof of Proposition 5.1, we see that $(V_w, f_w)$ is a CS chart if for the similarly defined subbundle $R_w \subset V_w \times \Omega^{0,2}(adE)_s$, the similarly defined vector bundle homomorphism $\varphi_w : R_w \to T^*V_w$ is an isomorphism. But this follows from that $\varphi_w$ are isomorphism for $x \in O_0$, and that being isomorphism is
an open condition. Thus by shrinking $D^C \supset D$ if necessary, all $\varphi_x$ are isomorphisms for $w \in O_0^C$. This proves that the $V_{O_0^C}$ is a complexification of $V_{O_0^C}$.

The proof that the local trivialization $\Psi$ can be complexified is similar, using that all the operators and families used to construct $\Psi$ are holomorphic on $D^C$, and that the only tool used to construct $\Psi$ is the implicit function theorem. We skip the repetition here.

5.4. Homotopy of family of CS charts. Let $U \subset X$ be open and let $\Xi_t$, $t \in [0, 1]$, be a homotopy between the orientation bundles $\Xi_0$ and $\Xi_1$ on $U$. For any $x_0 \in U$, we pick an open neighborhood $x_0 \in U_0 \subset U$ such that $U_0 = X_{f_0}$ for the JS chart $(V_0, f_0)$, that $U_0$ has compact closure in $U$, and that there is an $\varepsilon > 0$ such that $\Theta_{U_0}(\varepsilon)$ is an orientation bundle contained in $\Xi_t$ for all $t \in [0, 1]$.

Because of the compactness of the closure of $U_0$ in $U$, we can find a sufficiently small $\varepsilon > 0$ (in place of $\varepsilon(\cdot) : U_0 \to (0, 1)$) such that for each $t \in [0, 1]$, we have the family of CS charts $V_t \subset U_0 \times \mathcal{B}$ by applying Proposition 5.1 using $\Xi_t|_{U_0}$ and the size function $\varepsilon(\cdot) = \varepsilon$. We denote $V_{[0, 1]} = \bigsqcup_{t \in [0, 1]} V_t$, and call it the homotopy of the families $V_0$ and $V_1$.

Applying Proposition 5.1 to the orientation bundle $\Theta_{U_0}(\varepsilon)$, using the same $\varepsilon(\cdot) = \varepsilon$, we obtain the family of CS charts $\mathcal{W} \subset U_0 \times \mathcal{B}$.

**Proposition 5.8.** We have tautological inclusion $[0, 1] \times \mathcal{W} \subset \mathcal{V}_{U_0}$ as subspace of $[0, 1] \times U_0 \times \mathcal{B}$. Further, this pair can be complexified locally everywhere.

**Proof.** The proof is similar to the complexification of the family of CS charts and their local trivializations studied in the previous subsection. We omit the repetitions.

## 6. Perverse sheaves from CS data

In this section we prove Proposition 3.13.

6.1. A perverse sheaf from a family of CS charts. In this subsection we prove (1) of Proposition 3.13. Let $\mathcal{V} \subset U \times \mathcal{B}_{s_i}$ be a family of CS charts over $U$ and $\Psi : U \times \mathcal{V} \supset \mathcal{U} \to \mathcal{V} \times U$ be a local trivialization. Let

$$f : \mathcal{V} \xrightarrow{\subset} U \times \mathcal{B}_{s_i} \xrightarrow{pr} \mathcal{B}_{s_i} \xrightarrow{cs} \mathbb{C}$$

and let $\pi_\mathcal{V}$ be the projection from $U \times \mathcal{V}$ or $\mathcal{V} \times U$ to $\mathcal{V}$. We need a technical result that there is a homeomorphism interpolating $(f \circ \pi_\mathcal{V}) \circ \Psi$ and $f \circ \pi_\mathcal{V}$.

**Proposition 6.1.** There is an open subset $\mathcal{U}' \subset \mathcal{U}$ containing the diagonal $\Delta(U) \subset \mathcal{U}$ and an injective local homeomorphism $\Phi : \mathcal{U}' \to \mathcal{U}$ preserving the projections to $U \times U$, such that

$$(f \circ \pi_\mathcal{V}) \circ \Psi \circ \Phi = f \circ \pi_\mathcal{V} : \mathcal{U}' \to \mathbb{C}.$$

**Proof of Proposition 3.13 (1).** For $x \in U$, we can choose a sufficiently small open neighborhood $O_x$ of $x$ in $U$ such that $O_x$ is contained in the critical set $X_{f_x} \cap \mathcal{U}' \subset \mathcal{V}_x \cap \mathcal{U}'$ for any $z \in O_x$. Let $\psi_x : O_x \times \mathcal{V}_x \to \mathcal{V}$ be the restriction of $\Psi$ to $O_x \times \mathcal{V}_x \subset O_x \times \mathcal{V}$. By shrinking $\mathcal{V}_x$ if necessary and restricting $\Phi$ to $O_x \times \mathcal{V}_x \subset O_x \times \mathcal{V}|_{O_x}$ over $O_x \times \{x\} \cong O_x$, we have a homeomorphism

$$\lambda_x : O_x \times \mathcal{V}_x \xrightarrow{\Phi} O_x \times \mathcal{V}_x \xrightarrow{\psi_x} \mathcal{V}$$

that pulls back $f$ to $f_x \circ \text{pr}_2$ where $\text{pr}_2$ denotes the projection onto the second factor and $f_x : \mathcal{V}_x \subset \mathcal{B}_{s_i} \xrightarrow{cs} \mathbb{C}$. If we further restrict it to $\{z\} \times \mathcal{V}_x$ and vary $z$
in \(O_x\), we obtain a continuous family of homeomorphisms \(\lambda_{x,z} : V_z \cong V_x\), defined by \(\lambda_{x,z} = \lambda_{x_0, z}\). That pulls back \(f_z\) to \(f_x\) and thus we have a continuous family of isomorphisms \(\lambda^*_{x,z} : A^*_{f_x} \cong A^*_{f_z}\) by Proposition 2.3.

We have to show that the perverse sheaves \(P^*_{x} := A^*_{f_x}[r]\) on \(O_x \subset V_x\) glue to give us a perverse sheaf \(P^*\) on \(U\). Let \(y \in U\) such that \(O_x \cap O_y \neq \emptyset\). For any \(z \in O_x \cap O_y\), we have isomorphisms

\[
P^*_{x} \xrightarrow{\lambda^*_{x,z}} P^*_y \xrightarrow{\lambda^*_{y,z}} P^*_{y}
\]

over a neighborhood of \(z\). For another \(z'\) in the connected component of \(O_x \cap O_y\) containing \(z\), we choose a path \(z_t\) for \(t \in [0, 1]\) with \(z_0 = z\), \(z_1 = z'\). Then \(\lambda^{-1}_{y,z} \circ \lambda_{x,z} : V_x \rightarrow V_y\) is a continuous family of homeomorphisms that pulls back \(f_y\) to \(f_x\). By Proposition 2.3, we have the equality

\[
\lambda^*_{y,z} \circ (\lambda^*_{x,z})^{-1} = (\lambda^*_{x,z})^{-1} \circ \lambda^*_{y,z} : P^*_x \cong P^*_y.
\]

By Proposition 2.6, \(\lambda^*_{x,z} \circ (\lambda^*_{x,z})^{-1}\) glue to give us an isomorphism \(\tau_{xy} : P^*_x \cong P^*_y\) over \(O_x \cap O_y\).

Now the cocycle condition for gluing \(\{P^*_z\}\) is obviously satisfied because \(\tau_{xz} \circ \tau_{yz} \circ \tau_{xy}\) at any \(z \in O_x \cap O_y \cap O_z\) is \(\lambda^*_{x,z} \circ (\lambda^*_{x,z})^{-1} \circ \lambda^*_{y,z} \circ (\lambda^*_{x,z})^{-1} \circ \lambda^*_{z,y} \circ (\lambda^*_{x,z})^{-1} \circ \lambda^*_{z,y} = 1\).

Therefore the perverse sheaves \(P^*_x\) glue to give us a perverse sheaf \(P^*\) on \(U\).

The proof of Proposition 6.1 requires techniques for complex analytic subspaces. For this, we use the complexification of the family of CS charts and its local trivializations. Let \(x \in U\). By assumption, we have an open neighborhood \(O_x\) of \(x\) in \(U\), a complex manifold \(D^C_x\) containing \(O_x\) as a closed real analytic subset and a holomorphic family \(\mathcal{V}_D^C \rightarrow D^C_x\) of CS charts such that \(\mathcal{V}_D^C|O_x \cong V \times U\) and that there is a holomorphic local trivialization

\[
D^C_x \times \mathcal{V}_D^C \supset U_{D^C_x} \xrightarrow{\Psi^C} \mathcal{V}_D^C \times D^C_x.
\]

By shrinking \(O_x\) and \(V\) around \(x\) if necessary, we may assume that \(U_{D^C_x} = D^C_x \times V_x\) and \(O_x \subset V_x\) for all \(x \in O_x\). If we restrict \(\Psi\) to \(D^C_x \times \{x\}\), we get a holomorphic map \(\psi_x\) fitting into the commutative diagram

\[
\begin{array}{ccc}
D^C_x \times V_x & \xrightarrow{\psi_x} & \mathcal{V}_D^C
\\ & \searrow \downarrow \nearrow & \\
& D^C_x & \\
\end{array}
\]

which is a fiberwise open embedding. (We use \(\simeq\) to mean injective holomorphic maps.) Let \(f : \mathcal{V}_D^C \rightarrow \mathbb{C}\) be similarly defined as in (6.1), and let \(f_x : V_x \subset B_{si}^{\mathbb{C}} \rightarrow \mathbb{C}\). We let

\[
f_0 = (f \circ \pi_V) \circ (\text{id}_{D^C_x} \times \psi_x) \quad \text{and} \quad f_1 = (f \circ \pi_V) \circ (\Psi^C \circ (\text{id}_{D^C_x} \times \psi_x));
\]

both are holomorphic functions \(D^C_x \times D^C_x \times V_x \rightarrow \mathbb{C}\).

**Proposition 6.2.** After shrinking \(D^C_x\) and \(V_x\) if necessary, there is an open subset \(U'^C_x \subset D^C_x \times D^C_x \times V_x\) containing the diagonal \(\Delta = \{(y, y, y) \mid y \in D^C_x\}\) and an injective local homeomorphism \(\Phi^C : U'^C_x \rightarrow D^C_x \times D^C_x \times V_x\) commuting with the projection \(pr_{12} : D^C_x \times D^C_x \times V_x \rightarrow D^C_x \times D^C_x\) such that \(f_1 \circ \Phi^C = f_0\).
By cancelling the isomorphism id_{D_C^x} \times \psi_x and restricting to the real part \( O_x \subset D_C^x \), Proposition 6.1 is an immediate consequence of Proposition 6.2.

We now prove Proposition 6.2. Let
\[ f_t = (1-t)f_0 + tf_1 \]
be holomorphic functions on \( D_C^x \times D_C^x \times \mathcal{V}_x \). Let \( (d_V f_t) \) be the ideal generated by the partial derivatives of \( f_t \) in the direction of \( \mathcal{V}_x \). We will find a homeomorphism \( \Phi \) in a neighborhood of \((x,x,x)\) such that \( f_t \circ \Phi = f_0 \) by introducing a vertical vector field along the fibers of \( pr_{12} : D_C^x \times D_C^x \times \mathcal{V}_x \to D_C^x \times D_C^x \). We need a few lemmas.

**Lemma 6.3.** Let \( V \) be a smooth complex manifold and \( S \) a complex analytic subspace. Suppose \( W_1 \subset W_2 \) are two closed complex analytic subspaces of \( V \times S \) such that the induced projection \( W_1 \subset V \times S \to S \) is flat, and that there is a closed complex analytic subspace \( S_0 \subset S \) such that \( W_1 \times_S S_0 = W_2 \times_S S_0 \). Then there is an open subset \( U_0 \subset V \times S \) that contains \( V \times S_0 \) such that \( W_1 \cap U_0 = W_2 \cap U_0 \) as complex analytic subspaces in \( U_0 \).

**Proof.** Let \( \mathcal{K} \) be the kernel of the surjection \( \mathcal{O}_{W_2} \to \mathcal{O}_{W_1} \). Since \( \mathcal{O}_{W_1} \) is flat over \( \mathcal{O}_S \), we have an exact sequence
\[ 0 \to \mathcal{K} \otimes \mathcal{O}_S \to \mathcal{O}_{W_2} \otimes \mathcal{O}_S \to \mathcal{O}_{W_1} \otimes \mathcal{O}_S \to 0. \]
Since \( W_1 \times_S S_0 = W_2 \times_S S_0 \), \( \phi \) is an isomorphism, thus \( \mathcal{K} \otimes \mathcal{O}_S \otimes S_0 = 0 \). As \( \mathcal{K} \) is a coherent sheaf of \( \mathcal{O}_{V \times S} \)-modules, there is an open \( U_0 \subset V \times S \), containing \( V \times S_0 \), such that \( \mathcal{K}|_{U_0} = 0 \). This proves that \( W_1 \cap U_0 = W_2 \cap U_0 \), as complex analytic subspaces of \( U_0 \). \( \Box \)

**Lemma 6.4.** Let \( Z_t \) be the complex analytic subspace of \( D_C^x \times D_C^x \times \mathcal{V}_x \) defined by the ideal \( (d_V f_t) \). Then \( Z_t \) is independent of \( t \) in an open neighborhood of \((x,x,x)\) \in \( D_C^x \times D_C^x \times \mathcal{V}_x \).

**Proof.** Let \( Z = D_C^x \times D_C^x \times X \), where \( X \) is the critical locus of \( f_x : \mathcal{V}_x \subset B_{r} \subset \mathbb{C} \) in \( \mathcal{V}_x \) defined by the ideal \( (d_f) \). We first show \( Z_0 = Z_1 = Z \). Since the critical locus of \( f_t |_{\{z \times \mathbb{C} \times \mathbb{C} \}} \) is \( X \), by the definition of local trivialization for \( i = 0,1 \), \( Z \subset Z_i \) for \( i = 0,1 \). By Lemma 6.3, \( Z = Z_i \) for \( i = 0,1 \).

Let \( F,G : \mathbb{A}^1 \times D_C^x \times D_C^x \times \mathcal{V}_x \to \mathbb{C} \) be functions defined by
\[ F(t,z,z',p) = (1-t)f_0(z,z',p) + tf_1(z,z',p) \quad \text{and} \quad G(t,z,z',p) = f_0(z,z',p). \]
Since \( f_t \) is a linear combination of \( f_0 \) and \( f_1 \), we have the inclusion
\[ X_F := (d_V F = 0) \supset X_G := (d_V G = 0) = \mathbb{A}^1 \times Z \]
of analytic subspaces. Also the restrictions of \( F \) and \( G \) to \( \Gamma = \mathbb{A}^1 \times \{ (z,z) | z \in D_C^x \} \) coincide and hence \( X_F |_{\Gamma} = X_G |_{\Gamma} \). By Lemma 6.3, \( X_F = X_G = \mathbb{A}^1 \times Z \) in an open neighborhood of \( \Gamma \). Restricting to \( t \in [0,1] \), we obtain the lemma. \( \Box \)

We need another lemma. Let \( V \) and \( S \) be as before. We assume \( V \subset \mathbb{C}^r \) is an open subset. We endow \( V \) the standard inner and hermitian product.

**Lemma 6.5.** Let the notation be as stated. Let \( Z \subset V \) be a closed complex analytic subspace. Suppose \( f : V \times S \to \mathbb{C} \) is a holomorphic function with only one fiberwise critical value \( 0 \) such that the vanishing locus (complex analytic subspace) of \( d_V f \)
is $Z \times S$. Then for any convergent $p_n \to p_0 \in V \times S$ such that $d_V f(p_n) \neq 0$ and $d_V f(p_n) \to 0$, we have

$$\lim_{n \to \infty} \frac{f(p_n)}{\|d_V f(p_n)\|} = 0.$$  

**Proof.** We prove the case $S = pt$. The general case is exactly the same. Let $\nu : \tilde{V} \to V$ be the resolution of the ideal sheaf of $Z$ so that $\nu^{-1}(Z)$ is a normal crossing divisor in $\tilde{V}$. Let $\tilde{p}_n$ be the unique lifting of $p_n$ in $\tilde{V} - \nu^{-1}(Z)$. Suppose $n_k$ is a subsequence so that $\tilde{p}_{n_k} \to q \in \nu^{-1}(Z)$. We investigate the limiting behavior of $|f(p_{n_k})|/\|df(p_{n_k})\|$. By our choice of resolution $\nu$, locally near $q \in \tilde{V}$, the pullback $\nu^*(df)$ of the ideal $(df)$ is a principal ideal sheaf generated by some monomial $\varphi = z_1^{k_1} \cdots z_m^{k_m}$ with $k_i > 0$ for holomorphic functions $z_1, \cdots, z_r$ on $V$ such that restricting to the fiber $\tilde{V}$ they give a local coordinates of $\tilde{V}$ centered at $q$. Let $w_1, \cdots, w_r$ be local coordinates of $V$ centered at $p_0$. Then $\varphi$ divides $\nu^* \frac{\partial^r f}{\partial w_i^r}$ for all $i$.

We claim that $\nu^* f$ is divisible by $z_1^{k_1+1} \cdots z_m^{k_m+1}$. Indeed if we expand near $z_1 = 0$, $\nu^* f = c_m z_1^{m_1} + c_{m_1+1} z_2^{m_1+1} + \cdots$, where $c_j$ are holomorphic in $z_2, \cdots, z_r$, and $c_{m_1} \neq 0$. Then $\varphi$ divides

$$\sum_i \frac{\partial f}{\partial w_i} \frac{\partial w_i}{\partial z_1} = \frac{\partial}{\partial z_1} \nu^* f = m_1 c_m z_1^{m_1-1} + (m_1 + 1)c_{m_1+1} z_1^{m_1} + \cdots.$$  

Hence $m_1 \geq k_1 + 1$ and $z_1^{k_1+1}$ divides $\nu^* f$. Likewise $z_i^{k_i+1}$ divides $\nu^* f$ for each $i$. Therefore $\nu^* f \subset \nu^*(df) \sqrt{\nu^*(df)}$ where $\sqrt{\nu^*(df)}$ is the radical of $\nu^*(df)$. We write $\nu^* f = \sum g_i \nu^* \frac{\partial f}{\partial w_i}$ with $g_i|_{\nu^{-1}(Z)} = 0$. Therefore

$$\frac{\nu^* |f(p_{n_k})|}{\|df(p_{n_k})\|} = \frac{\nu^* |f(\tilde{p}_{n_k})|}{\|df(\tilde{p}_{n_k})\|} \leq \left( \sum \left| g_i(\tilde{p}_{n_k}) \right|^2 \right)^{1/2} \to 0.$$  

Because $\tilde{V} \to V$ is proper, and because this convergence holds for all convergent subsequence $\tilde{p}_{n_k}$, we find that $\lim_{n \to \infty} \frac{f(p_n)}{\|df(p_n)\|} = 0$. \hfill $\square$

**Proof of Proposition 6.2.** We use the inner product and the hermitian metric on $V_x$ via embedding $V_x \cong V_x \subset \mathcal{O}^{b_1}(adE_x)_x$. We let $\nabla_V f_t$ be the relative gradient vector field of $f_t$ on $D_C^2 \times D_C^2 \times V_x$ as the metric dual of $d_V f_t$, the differential of $f_t$ in $V_x$ direction. Note that $\nabla_V f_t$ is a vertical vector field with respect to the projection $pr_{12} : D_C^2 \times D_C^2 \times V_x \to D_C^2 \times D_C^2$ which is differentiable on each fiber. We define a time dependent vector field on the complement of $Z = Z_t = (d_V f_t = 0)$ by

$$(6.3) \quad \xi_t = \frac{f_0 - f_t}{\|\nabla_V f_t\|^2} \nabla_V f_t.$$  

We claim that it extends to a well defined vector field on some neighborhood of $(x, x, x)$ in $D_C^2 \times D_C^2 \times V_x$. It suffices to show that

$$|\xi_t| = \left| \frac{f_0 - f_t}{\|\nabla_V f_t\|^2} \right| \frac{|f_0 - f_t|}{\|d_V f_t\|}$$  

approaches zero as a point approaches $Z$. Since $Z_t = Z_0 = Z_1$ by Lemma 6.4, we have an inclusion $(d_V f_t) \subset (d_V f_1)$ of ideals for $i = 0, 1$. Hence we can express the vertical partial derivatives of $f_0$ and $f_1$ as linear combinations of the vertical partial derivatives of $f_t$. Thus in a neighborhood of $(x, x, x)$, we have

$$\|d_V f_0\| \leq C\|d_V f_1\| \quad \text{and} \quad \|d_V f_1\| \leq C\|d_V f_1\|.$$
for some \( C > 0 \). By Lemma 6.5, we have
\[
|\xi_t| = \frac{|f_0 - f_1|}{|d_V f_1|} \leq C^{-1} \left( \frac{|f_0|}{|d_V f_0|} + \frac{|f_1|}{|d_V f_1|} \right) \to 0, \quad \text{as } d_V f_0, d_V f_1 \to 0.
\]
This proves that the vector field \( \xi_t \) is well defined in a neighborhood of \((x, x, x)\).

Let \( x_t \) for \( t \in [0, 1] \) be an integral curve of the vector field \( \xi_t \), so that
\[
dx_t = \xi_t(x_t).
\]
Since \( \xi_t \) is a vertical vector field for \( pr_{13} \), \( x_t \) lies in a fiber of \( pr_{13} \). Then \( f_t(x_t) \) is constant in \( t \) because
\[
\frac{d}{dt} f_t(x_t) = d_V f_t(x_t) + f_t - f_0 = \nabla_V f_t \cdot \dot{x}_t + f_t - f_0 = 0.
\]
Therefore the flow of the vector field \( \xi_t \) from \( t = 0 \) to \( t = 1 \) gives a homeomorphism \( \Phi \) of a neighborhood of \((x, x, x)\) into \( D^C_x \times D^C_x \times V_x \) such that \( f_1 \circ \Phi = f_0 \). Because \( \xi_t = 0 \) for all \( t \) over the diagonal \( \{(z, z)\} \subset D^C_x \times D^C_x \), \( \Phi \) is the identity map over the diagonal. \( \square \)

6.2. Gluing isomorphisms. In this subsection we prove (2) of Proposition 3.13.

We have a family \( V \) of CS charts on \( U \times [0, 1] \) with \( V|_{U \times (0)} = V_\alpha \) and \( V|_{U \times (1)} = V_\beta \). For \( x \in U \), there exist an open neighborhood \( O_x \) of \( x \) in \( U \) and a subfamily \( W \times [0, 1] \) of CS charts in \( V|_{O_x \times [0, 1]} \). Moreover, we have perverse sheaves \( P^*_{\alpha} \) and \( P^*_{\beta} \) on \( U \) given by \( V_\alpha \) and \( V_\beta \) respectively, whose restrictions to \( O_x \) are the perverse sheaves of vanishing cycles \( A^*_{j^\alpha_x} \) and \( A^*_{j^\beta_y} \) respectively where \( j^\alpha_x : V_\alpha|_{x} \subset B_{a_i} \xrightarrow{c^\alpha} \mathbb{C} \) and \( j^\beta_y : V_\beta|_{x} \subset B_{a_i} \xrightarrow{c^\beta} \mathbb{C} \).

Recall that \( P^*_{\alpha} \) was obtained by gluing \( A^*_{j^\alpha_x} \) by the isomorphisms
\[
A^*_{j^\alpha_x} \xrightarrow{(\lambda^\alpha_{x,z})^*} A^*_{j^\beta_y} \xrightarrow{(\lambda^\beta_{y,z})^*} A^*_{j^\beta_y}
\]
for \( x, y \in U \) and \( z \in O_x \cap O_y \).

**Lemma 6.6.** Suppose for each \( x \in U \) we have an isomorphism \( \chi^*_x : A^*_{j^\alpha_x} \xrightarrow{\cong} A^*_{j^\beta_y} \) such that for \( z \in O_x \) we have a commutative diagram of isomorphisms
\begin{align*}
\begin{array}{ccc}
A^*_{j^\alpha_x} & \xrightarrow{(\lambda^\alpha_{x,z})^*} & A^*_{j^\beta_y} \\
\downarrow{\chi_x^*} & & \downarrow{\chi_y^*} \\
A^*_{j^\alpha_x} & \xrightarrow{(\lambda^\beta_{y,z})^*} & A^*_{j^\beta_y}
\end{array}
\end{align*}

Then the isomorphisms \( \chi^*_z \) glue to give an isomorphism \( \sigma_{\alpha\beta} : P^*_{\alpha} \to P^*_{\beta} \).

**Proof.** If \( z \in O_x \cap O_y \), we have a commutative diagram
\begin{align*}
\begin{array}{ccc}
A^*_{j^\alpha_x} & \xrightarrow{(\lambda^\alpha_{x,z})^*} & A^*_{j^\beta_y} \\
\downarrow{\chi_x^*} & & \downarrow{\chi_y^*} \\
A^*_{j^\alpha_x} & \xrightarrow{(\lambda^\beta_{y,z})^*} & A^*_{j^\beta_y}
\end{array}
\end{align*}
By Proposition 2.6 (2), we have an isomorphism $\sigma_{\alpha\beta}$ that restricts to $\chi^*_x$.  

**Lemma 6.7.** Let $W \subset V$ be a complex submanifold in a complex manifold. Let $f : V \to \mathbb{C}$ be a holomorphic function and $g = f|_W$. Suppose the critical loci $\mathcal{X}_f$ and $\mathcal{X}_g$ are equal. Let $x$ be a closed point of $\mathcal{X}_f = \mathcal{X}_g$. Then there is a coordinate system $\{z_1, \cdots, z_r\}$ of $V$ centered at $x$ such that $W$ is defined by the vanishing of $z_1, \cdots, z_m$ and 

$$f = q(z_1, \cdots, z_m) + g(z_{m+1}, \cdots, z_r) \quad \text{where} \quad q(z) = \sum_{i=1}^m z_i^2, \quad \text{and} \quad g = f|_W.$$ 

**Proof.** We choose coordinates $\{y_1, \cdots, y_r\}$ of $V$ centered at $x$ such that $W$ is defined by the vanishing of $y_1, \cdots, y_m$. Let $I$ be the ideal generated by $y_1, \cdots, y_m$. Since $\mathcal{X}_f = \mathcal{X}_g$, i.e. $(df) = (dg) + I$, we have 

$$\frac{\partial f}{\partial y_i}|_W = \sum_{j=m+1}^r a_{ij} \frac{\partial g}{\partial y_j}, \quad i = 1, \cdots, m$$ 

for some functions $a_{ij}$ regular at $x$. By calculus, we have 

$$f = g(y_{m+1}, \cdots, y_r) + \sum_{i=1}^m \frac{\partial f}{\partial y_i}|_W \cdot y_i + I^2$$ 

$$= g(y_{m+1}, \cdots, y_r) + \sum_{j=m+1}^r \frac{\partial g}{\partial y_j} \sum_{i=1}^m a_{ij} y_i + I^2$$ 

$$= g(z_{m+1}, \cdots, z_r) + \sum_{i,k=1}^m b_{ik} y_i y_k$$ 

where $z_j = y_j + \sum_{i=1}^m a_{ij} y_i$ for $j \geq m + 1$ and $b_{ik}$ are some functions holomorphic near $x$. Since the kernel of the Hessian of $f$ at $x$ is the tangent space of $\mathcal{X}_f = \mathcal{X}_g \subset W$ at $x$, the quadratic form $q = \sum_{i,k=1}^m b_{ik} y_i y_k$ is nondegenerate near $x$. Hence we can diagonalize $q = \sum_{i=1}^m z_i^2$ by changing the coordinates $y_1, \cdots, y_m$ to new coordinates $z_1, \cdots, z_m$. It follows that $z_1, \cdots, z_r$ is the desired coordinate system. \hfill $\Box$

Let $\mathcal{V}^t = \mathcal{V}|_{U \times \{t\}}$ so that $\mathcal{V}^0 = \mathcal{V}_\alpha$ and $\mathcal{V}^1 = \mathcal{V}_\beta$. Let $\delta_0 : U \to \mathcal{V}^t$ be the tautologous section which sends $x \in U$ to $x$ in the fiber $\mathcal{V}^t_x \subset \mathcal{B}_{\alpha \beta}$ of $\mathcal{V}^t$ over $x$. Let $f_t : \mathcal{V}^t \subset U \times \mathcal{B}_{\alpha \beta} \to \mathbb{C}$ be the family CS functional.

**Lemma 6.8.** For $x \in U$, there exist an open neighborhood $U_0$ of $\delta_0(x)$ in $\mathcal{V}^0$ and a homeomorphism $\chi$ of $U_0 \times [0,1]$ into $\mathcal{V}$ such that 

1. the restriction $\chi_t$ (of $\chi$) to $U_0 \times \{t\}$ is a holomorphic map to an open set in $\mathcal{V}^t$; 
2. $\chi|_{U_0 \cap W}$ maps $U_0 \cap W$ into $W \subset \mathcal{V}_t$ and is the identity map; 
3. $\chi_0 : U_0 \to \mathcal{V}^0$ is the identity map; 
4. $f_t \circ \chi_t = f_0$.

**Proof.** Since $[0,1]$ is compact, it suffices to find such a $\chi$ over an interval $[t_0, t_0 + \epsilon]$ at each $t_0 \in [0,1]$ with $0 < \epsilon \ll 1$. For $t_0 \in [0,1]$, by choosing a complexification of the pair $W \times [0,1] \subset \mathcal{V}$ near $(x, t_0)$, we can find coordinate functions $y_1, \cdots, y_r$ of the fibers of $\mathcal{V} \to U \times [0,1]$ at $(x, t_0)$ such that $y_1, \cdots, y_m$ are holomorphic along fibers of $\mathcal{V} \to U \times [0,1]$, and $W \times [0,1]$ is defined by the vanishing of $y_1, \cdots, y_m$
(cf. Proposition 5.8). Let $t$ be the coordinate for $[0, 1]$. We can repeat the proof of Lemma 6.7 with

$$f : V \hookrightarrow U \times [0, 1] \times B_{si} \xrightarrow{pr_{3}} B_{si} \xrightarrow{cs} \mathbb{C}$$

and $g = f|_{W \times [0, 1]}$. By Lemma 6.3, $(d_V f) = (d_V g) + I$ where $I = (y_1, \ldots, y_m)$ and $(d_V g)$ (resp. $(d_V f)$) denotes the ideal generated by the partial derivatives in the fiber direction of $V \to U \times [0, 1]$ (resp. $W \times [0, 1] \to U \times [0, 1]$). Then we obtain a new coordinate system $\{z_i\}$ of $V$ over $U \times [0, 1]$ at $(x, t_0)$ such that

$$f = \sum_{i=1}^m z_i^2 + g(z_{m+1}, \ldots, z_r)$$

with $z_j|_{W \times [0, 1]} = y_j$ for $j \geq m + 1$. Then the coordinate change from $\{z_j|_{t_0}\}$ to $\{z_j\}$ defines the desired map $\chi$.

Let $f_x^* : V_x^i \subset B_{si} \xrightarrow{cs} \mathbb{C}$ be the CS functional on the CS chart $V_x^i$.

**Lemma 6.9.** The isomorphism $\chi_x^* : A_{f_x^*} \to A_{f_x^*}$ induced from $\chi_1$ is independent of the choice of $\chi$.

**Proof.** If $\chi'$ is another such homeomorphism, then $\chi^{-1}_1 \circ \chi'_1 : V_2^0 \to V_1^0$ is a homotopy of homeomorphisms which is id at $t = 0$. By Proposition 2.3, $(\chi^{-1}_1 \circ \chi'_1) : A_{f_x^*} \to A_{f_x^*}$ is the identity. This proves the lemma. □

**Lemma 6.10.** For each $x \in U$, we have an isomorphism $\chi_x^* : A_{f_x^*} \xrightarrow{\cong} A_{f_x^*}$ such that (6.4) holds.

**Proof.** By Proposition 6.2, we have a homeomorphism $\xi_\alpha : O_x \times V_\alpha|_{O_x} \to V_\alpha|_{O_x} \times O_x$ that gives the gluing isomorphism $(\lambda_\alpha^*|_{O_x})^* \circ \sigma_{\alpha\beta}$ over $(x, z)$. By construction, the restriction of $\xi_\alpha$ to $(O_x \times V_\alpha|_{O_x}) \times O_x \times O_x$ to the diagonal $O_x \subset O_x \times O_x$ is the identity map. The composition of homeomorphisms

$$O_x \times V_\alpha|_{O_x} \xrightarrow{id \times \chi} O_x \times V_\beta|_{O_x} \cong \chi^{-1} \circ \xi_\beta \circ \chi^{-1} \circ \xi_\alpha \circ \chi^{-1} \circ \xi_\alpha \circ \chi^{-1}$$

is the identity over the diagonal $O_x \subset O_x \times O_x$. Upon fixing a local trivialization $V_\alpha$ over $O_x \subset O_x \times O_x$, we have $A_{f_x^*} \cong pr_{-1} A_{f_x^*}$ because the analytic space $O_x$ is locally contractible. By Lemma 2.8 for $O_x \times V_x \subset O_x \times O_x \times V_x \to V_x$, we obtain the commutativity of the diagram of isomorphisms

$$pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*} \xrightarrow{\cong} pr_{-1} A_{f_x^*}$$

Restricting to the fiber over $(x, z)$, we obtain (6.4) possibly after shrinking $O_x$. □

By Lemma 6.6 and Lemma 6.10, we have the desired isomorphism $\sigma_{\alpha\beta} : P^* \to P_{\beta}^*$ over $U_\alpha \cap U_\beta$ and thus we proved (2) of Proposition 3.13.
6.3. Obstruction class. In this subsection we prove (3) of Proposition 3.13 and complete our proof of Theorem 3.14.

Let $x \in U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$. By our construction in §6.2, in a neighborhood of $x$, $\sigma_{\gamma \alpha} \circ \sigma_{\beta \gamma} \circ \sigma_{\alpha \beta}$ is given by a biholomorphic map $\varphi : V_{\alpha,x} \to V_{\alpha,x}$ preserving $f_x^\varphi$ whose restriction to $W$ is the identity. By Proposition 2.5, $\sigma_{\gamma \alpha} \circ \sigma_{\beta \gamma} \circ \sigma_{\alpha \beta}$ is $\det(d\varphi|_x) \cdot \id$ and $\det(d\varphi|_x) = \pm 1$. Thus we obtain a $\mathbb{Z}_2$-valued Čech 2-cocycle $\{\sigma_{\alpha \beta \gamma}\}$ of the covering $\{U_\alpha\}$. One checks directly that the cocycle is closed and its cohomology class $\sigma \in H^2(X, \mathbb{Z}_2)$ is the obstruction class for gluing the perverse sheaves $\{P^*_\alpha\}$.

**Proposition 6.11.** Given preorientation data $\{\Xi_\alpha\}$ on $X$, the perverse sheaves $\{P^*_\alpha\}$ in §6.1 glue to give a globally defined perverse sheaf $P^*$ on $X$ if and only if the obstruction class $\sigma \in H^2(X, \mathbb{Z}_2)$ vanishes.

The cocycle $\sigma_{\alpha \beta \gamma}$ is by definition the cocycle for gluing the determinant line bundles $\det(TV_{\alpha,x})$ by the isomorphisms induced from the biholomorphic maps $V_{\alpha,x} \cong V_{\beta,x}$. In particular, $\det(TV_{\alpha,x})$ glue to a globally defined line bundle on $X$ if and only if $\sigma \in H^2(X, \mathbb{Z}_2)$ is zero.

Since a neighborhood $O_x$ of $x$ in $X$ is the critical locus of the CS functional on $V_{\alpha,x}$, we have a symmetric obstruction theory

$$F := [T_{V_{\alpha,x}} \to \Omega_{V_{\alpha,x}}] \to \mathbb{L}_X|O_x.$$ 

Hence the determinant line bundle $\det F$ is the inverse square of the determinant line bundle of the tangent bundle $T_{V_{\alpha,x}}$. On the other hand, by [35], $F \cong \operatorname{Ext}^\bullet_\pi(E, E)[2]$ on $O_x$ where $\pi : X \times Y \to X$ is the projection and $E$ is the universal bundle. Hence the determinant bundle of $T_{V_{\alpha,x}}$ is a square root of the determinant bundle of $\operatorname{Ext}^\bullet_\pi(E, E)[1]$. Therefore if the obstruction class $\sigma \in H^2(X, \mathbb{Z}_2)$ is zero, then $\det \operatorname{Ext}^\bullet_\pi(E, E)$ has a square root.

Suppose $\det \operatorname{Ext}^\bullet_\pi(E, E)$ has a square root $L$. Then the local isomorphisms $\det T_{V_{\alpha,x}} \cong L|O_x$ induce gluing isomorphisms for $\{\det T_{V_{\alpha,x}}\}$. Therefore the obstruction class $\sigma$ to gluing them is zero. This implies the gluing of $\{P^*_\alpha\}$ by Proposition 6.11. This completes the proof of Theorem 3.14.

We add that if a square root $L$ of $\det \operatorname{Ext}^\bullet_\pi(E, E)$ exists, then any two such square roots differ by tensoring a 2-torsion line bundle (i.e. a $\mathbb{Z}_2$-local system) on $X$, and vice versa.

**Remark 6.12.** In [19], Kontsevich and Soibelman defined orientation data as choices of square roots of $\det \operatorname{Ext}^\bullet_\pi(E, E)$ satisfying a compatibility condition. See Definition 15 in [19]. For the existence of a perverse sheaf which is the goal of this paper, it suffices to have a square root. However for wall crossings in the derived category, the compatibility condition seems necessary. See §5 of [19] for further discussions.

7. Mixed Hodge modules

In this section, we prove that the perverse sheaf $P^*$ in Theorem 1.1 lifts to a mixed Hodge module (MHM for short) of Morihiko Saito ([31]).

A mixed Hodge module $M^*$ consists of a $\mathbb{Q}$-perverse sheaf $P^*$, a regular holonomic $D$-module $A^*$ with $DR(A^*) \cong P^* \otimes_{\mathbb{Q}} \mathbb{C}$, a $D$-module filtration $F$ and a weight filtration $W$, with polarizability and inductive construction. Like perverse sheaves,
mixed Hodge modules form an abelian category $\text{MHM}(X)$. There is a forgetful functor

$$\text{rat} : \text{MHM}(X) \rightarrow \text{Perv}(X)$$

which is faithful and exact. Moreover, if $f : V \rightarrow \mathbb{C}$ is a holomorphic function on a complex manifold $V$ of dimension $r$, there is a MHM $M_f^\bullet = \phi_f^\bullet (\mathbb{Q}[r-1])$ such that

$$\text{rat}(M_f^\bullet) = \phi_f(\mathbb{Q}[r-1]) = A_f^\bullet[r]$$

is the perverse sheaf of vanishing cycles of $f$. Further, if $\Phi : V \rightarrow V$ is a biholomorphic map and $g = f \circ \Phi$, then $\Phi$ induces an isomorphism $\Phi_\ast : M_f^\bullet \rightarrow M_g^\bullet$. We also note that the category $\text{MHM}(X)$ is a sheaf, i.e. gluing works ([33]).

The goal of this section is to prove the following.

**Theorem 7.1.** The perverse sheaf $P^\bullet$ in Theorem 1.1 lifts to a MHM $M^\bullet$. Namely, there exists a MHM $M^\bullet$ such that $\text{rat}(M^\bullet) = P^\bullet$.

Like Theorem 3.14, Theorem 7.1 is a direct consequence of the following analogue of Proposition 3.13 together with Propositions 3.4 and 3.12.

**Proposition 7.2.** (1) Let $\pi : V \rightarrow U$ be a family of CS charts on $U \subset X \subset B_{si}$ with complexifiable local trivializations at every point $x \in U$. Then the MHM of vanishing cycles for

$$f_x : V_x = \pi^{-1}(x) \subset B_{si} \xrightarrow{cs} \mathbb{C}$$

glue to a MHM $M^\bullet$ on $U$, i.e. $M^\bullet$ is isomorphic to $M^\bullet$ in a neighborhood of $x$.

(2) Let $V_\alpha$ and $V_\beta$ be two families of CS charts on $U$ with complexifiable local trivializations. Let $M^\bullet_\alpha$ and $M^\bullet_\beta$ be the induced MHMs on $U$. Let $V$ be a family of CS charts on $U \times [0, 1]$ with complexifiable local trivializations such that $V|_{U \times \{0\}} = V_\alpha$ and $V|_{U \times \{1\}} = V_\beta$. Suppose for each $x \in U$, there are an open $U_x \subset U$ and a subfamily $W$ of both $V_\alpha|_{U_x}$ and $V_\beta|_{U_x}$ such that $W \times [0, 1]$ is a complexifiable subfamily of CS charts in $V|_{U_x \times [0, 1]}$. Then there is an isomorphism $\sigma^m_{\alpha \beta} : M^\bullet_\alpha \cong M^\bullet_\beta$ of MHMs.

(3) If there are three families $V_\alpha, V_\beta, V_\gamma$ with homotopies among them as in (2), then the isomorphisms $\sigma^m_{\alpha \beta}, \sigma^m_{\beta \gamma}, \sigma^m_{\gamma \alpha}$ satisfy

$$\sigma^m_{\alpha \beta \gamma} := \sigma^m_{\alpha \beta} \circ \sigma^m_{\beta \gamma} \circ \sigma^m_{\gamma \alpha} = \pm \text{id}.$$

The proof of Proposition 7.2 is a line by line repetition of the proof of Proposition 3.13 in §6, with perverse sheaves replaced by MHMs if homeomorphisms are replaced by biholomorphic maps. Notice that the use of Propositions 2.3 and 2.5 are justified by the fact that $\text{rat}$ is a faithful functor. For example, Proposition 3.13 (3) immediately implies Proposition 7.2 (3) because $\text{rat}(\sigma_{\alpha \beta}^m) = \sigma_{\alpha \beta}$ and $\text{rat}(\pm \text{id}) = \pm \text{id}$.

Now the only non-holomorphic map used in §6 is the homeomorphism $\Phi$ in the proof of Proposition 6.2 which was defined by the integral flow of the vector field (6.3). Therefore we have a proof of Theorem 7.1 as soon as we can replace (6.3) by a holomorphic vector field $\xi_t$ which vanishes on $X$ and satisfies

\begin{equation}
\tag{7.1}
d_V f_t(\xi_t) = f_0 - f_1.
\end{equation}

in the notation of §6.1.

To see this, we first recall that by Lemma 6.4, the ideal $\mathcal{I} = (d_V f_t)$ is independent of $t$ and defines an analytic space $Z$.

**Lemma 7.3.** $f_1 - f_0 \in \mathcal{I}^2$. 
Proof. We let $\iota : D^C_\mathcal{I} \times V^C_\mathcal{I} \to A_s$ and $\iota' : V^C_\mathcal{I} \times D^C_\mathcal{I} \to A_s$ be the compositions of the projections to $V^C_\mathcal{I}$ with the tautological map $V^C_\mathcal{I} \to A_s$ constructed in §6.

Then $f_0 = cs \circ \iota \circ (id_{D^C_\mathcal{I}} \times \psi_x)$ and $f_1 = cs \circ \iota' \circ \Psi^C \circ (id_{D^C_\mathcal{I}} \times \psi_x)$. Since the problem is local, we can reduce the proof to the following case.

Let $\xi \in D^C_\mathcal{I} \times D^C_\mathcal{I} \times V^C_\mathcal{I}$ be any point in the subspace defined by the ideal $\mathcal{I}$. We pick an open neighborhood $\xi \in W \subset D^C_\mathcal{I} \times D^C_\mathcal{I} \times V^C_\mathcal{I}$, so that $W$ is endowed with holomorphic coordinates $z = (z_1, \ldots, z_m)$ with $\xi = (0, \ldots, 0) \in W$. Let $I = \mathcal{I} \otimes_{O_{D^C_\mathcal{I} \times D^C_\mathcal{I} \times V^C_\mathcal{I}}} \mathcal{O}_W$ denote the ideal of $W \cap Z$. Then it suffices to show that

$$f_0|_W - f_1|_W \in I^2.$$

We now describe the difference $f_0|_W - f_1|_W$. For simplicity, we abbreviate $(id_{D^C_\mathcal{I}} \times \psi_x)|_W$ to $\tilde{\psi}_x$. By the construction of $\Psi^C$, we know that there are holomorphic $g : W \to G$ and $\epsilon : W \to \Omega^{0,1}(adE)_s$ satisfying $\epsilon|_{W \cap \mathcal{I}} \equiv 0$ such that

$$\iota' \circ \Psi^C \circ (id_{D^C_\mathcal{I}} \times \psi_x)|_W = \iota' \circ \Psi^C \circ \tilde{\psi}_x = g \cdot (\iota \circ \tilde{\psi}_x) + \epsilon = g \cdot (\iota \circ \tilde{\psi}_x + \epsilon'),$$

where $g \cdot (-)$ denotes the gauge group action; $\cdot + \epsilon$ is via the affine structure $A_s \times \Omega^{0,1}(adE)_s \to A_s$, and $\epsilon' : W \to \Omega^{0,1}(adE)_s$ is the holomorphic map making the third identity hold, which satisfies $\epsilon'|_{W \cap \mathcal{I}} \equiv 0$. Since $cs$ is invariant under gauge transformations, (7.2) is equivalent to

$$cs \circ \iota \circ \tilde{\psi}_x - cs \circ (\iota \circ \tilde{\psi}_x + \epsilon') \in I^2.$$

We use finite dimensional approximation to reduce this to a familiar problem in several complex variables. First, since $\epsilon'$ takes values in $C^\infty$-forms, we can lift it to $\tilde{\epsilon} : W \to \Omega^{0,1}(adE)_{L^2}$ for a large $t$ so that $\Omega^{0,1}(adE)_{L^2} \subset \Omega^{0,1}(adE)_s$. Since $\Omega^{0,1}(adE)_{L^2}$ is a separable Hilbert space, we can approximate it by an increasing sequence of finite dimensional subspaces $R_k \subset \Omega^{0,1}(adE)_{L^2}$. Let $q_k : \Omega^{0,1}(adE)_{L^2} \to W_k \subset \Omega^{0,1}(adE)_s$ be the orthogonal projection. Then we have a convergence of holomorphic functions

$$\lim_{k \to \infty} cs \circ (\iota \circ \tilde{\psi}_x + q_k \circ \tilde{\epsilon}) = cs \circ (\iota \circ \tilde{\psi}_x + \epsilon')$$

uniformly on every compact subset of $W$. We claim that

$$cs \circ \iota \circ \tilde{\psi}_x - cs \circ (\iota \circ \tilde{\psi}_x + q_k \circ \tilde{\epsilon}) \in I^2.$$

Note that the claim and the uniform convergence imply (7.3).

We prove (7.4). For a fixed $k$, we pick a basis $e_1, \ldots, e_n$ of $R_k$; we introduce complex coordinates $w = (w_1, \ldots, w_n)$, and form a holomorphic function

$$F_k : W \times \mathbb{C}^n \longrightarrow \mathbb{C}; \quad F_k(z, w) = cs \circ (\iota \circ \tilde{\psi}_x + \sum_{j=1}^n w_j e_j).$$

If we write $q_k \circ \tilde{\epsilon} = \delta_1 e_1 + \cdots + \delta_n e_n : W \to R_k$, then all $\delta_j$ are holomorphic functions lying in $I$. Therefore,

$$(cs \circ (\iota \circ \tilde{\psi}_x + q_k \circ \tilde{\epsilon}))(z) = F(z, \delta_1(z), \cdots, \delta_n(z)).$$

Since $\delta_j \in I$, applying Taylor expansion along $(z, 0)$, we conclude that

$$F_k(z, \delta_1(z), \cdots, \delta_n(z)) \equiv F_k(z, 0) + \sum_{j=1}^n \frac{\partial F_k}{\partial w_j}(z, 0) \cdot \delta_j(z) \mod I^2.$$
Since $\frac{\partial F_k}{\partial w}(z,0)$ involve the partial derivatives of the \(c_s\) via the chain rule, we conclude that $\frac{\partial F_k}{\partial w}(z,0) \in I$. This proves that $F_k(z,\delta_1(z),\cdots,\delta_n(z)) - F_k(z,0) \in I^2$, which is (7.4). This proves the Lemma.

Since $f_0 - f_1 \in I^2$ and $I = (dv_\xi)$, we can always find a time dependent holomorphic vertical vector field $\xi_t$ which satisfies (7.1) and vanishes along $X$ (cf. [5]). This gives us the desired biholomorphic map $\Phi$. This completes our proof of Theorem 7.1. Note that we don’t need Lemma 6.5 for this choice of $\xi_t$.

Remark 7.4. (Gluing of polarization) To use the hard Lefschetz property of [30], we also need the gluing of polarization and monodromy for $M^\bullet$ and the graduation $\text{gr}^W M^\bullet$. Note that our gluing isomorphisms arise from continuous families of CS charts. Since a continuous variation of rational numbers is constant, the obvious polarizations for the constant sheaves $Q_V$ on $V$ is constant in a continuous family of CS charts. The induced polarizations on the perverse sheaves $A^\bullet_f[r] = \phi_f Q[r-1]$, defined in [30, §5.2], are functorial. Therefore the polarizations on $A^\bullet_f[r]$ glue to give us a polarization on $P^\bullet$. By the same argument, we have a polarization on $\text{gr}^W M^\bullet$ and the monodromy operators also glue.

8. GOPAKUMAR-VAFA INvariants

In this section, we provide a mathematical theory of the Gopakumar-Vafa invariant as an application of Theorem 3.16.

8.1. Intersection cohomology sheaf. As before, let $Y$ be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$. From string theory ([9, 15]), it is expected that

1. there are integers $n_h(\beta)$, called the Gopakumar-Vafa invariants (GV for short) which contain all the information about the Gromov-Witten invariants $N_g(\beta)$ of $Y$ in the sense that

\[
\sum_{g,\beta} N_g(\beta)q^{2g-2} = \sum_{k,h,\beta} n_h(\beta) \frac{1}{k} \left(2 \sin \left(\frac{k\lambda_2}{2}\right)\right)^{2h-2} q^{k\beta}
\]

where $\beta \in H_2(Y,\mathbb{Z})$, $q^\beta = \exp(-2\pi i \int_{\beta} c_1(\mathcal{O}_Y(1)))$;

2. $n_h(\beta)$ come from an $\text{sl}_2 \times \text{sl}_2$ action on some cohomology theory of the moduli space $X$ of one dimensional stable sheaves on $Y$;

3. $n_0(\beta)$ should be the Donaldson-Thomas invariant of the moduli space $X$.

By using the global perverse sheaf $P^\bullet$ constructed above and the method of [10], we can give a geometric theory for GV invariants.

We recall the following facts from [30, 31].

Theorem 8.1. (1) (Hard Lefschetz theorem) If $f : X \rightarrow Y$ is a projective morphism and $P^\bullet$ is a perverse sheaf on $X$ which underlies a pure (polarizable) MHM $M^\bullet$, then the cap product induces an isomorphism

\[
\omega^k : \text{pr}^{k-H-k} Rf_* P^\bullet \rightarrow \text{pr}^k Rf_* P^\bullet
\]

where $\omega$ is the first Chern class of a relative ample line bundle.
(2) (Decomposition theorem) If \( f : X \to Y \) is a proper morphism and \( P^* \) as above, then

\[
R_f_* P^* \cong \bigoplus_k \mathcal{H}^k R_f_* P^*[-k]
\]

and each summand \( \mathcal{H}^k R_f_* P^*[-k] \) is a perverse sheaf underlying a MHM which is again polarizable semisimple and pure.

Let \( \mathfrak{X} \) be the moduli space of stable one-dimensional sheaves \( E \) on \( Y \) with \( \chi (E) = 1 \) and \( [E] = \beta \in H_2 (Y, \mathbb{Z}) \). In particular, the rank of \( E \) is zero and \( c_1 (E) = 0 \). By [23, Theorem 6.11], there is a universal family \( E \) on \( \mathfrak{X} \times Y \). Let \( X = \mathfrak{X}_{red} \) be the reduced complex analytic subspace of \( \mathfrak{X} \).

Let \( \tilde{X} \) be the semi-normalization of \( X \) and let \( S \) be the image of the morphism \( \tilde{X} \to \text{Chow}(Y) \) to the Chow scheme of curves in \( Y \). By [18], the morphism \( \tilde{X} \to X \) is one-to-one and hence a homeomorphism because \( \tilde{X} \) is projective and \( X \) is separated. The natural morphism \( f : \tilde{X} \to S \) is projective and the intersection cohomology sheaf \( \text{IC}^* = IC_{\mathfrak{X}} (\mathbb{C}^*) \) underlies a pure simple MHM. In [10], S. Hosono, M.-H. Saito and A. Takahashi show that the hard Lefschetz theorem applied to \( f \) and \( c : S \to \text{pt} \) gives us an action of \( sl_2 \times sl_2 \) on the intersection cohomology \( IH^*(X) = \mathbb{H}^*(\tilde{X}, IC^*) \) as follows: The relative Lefschetz isomorphism

\[
\mathcal{H}^{-k} R_{f*}(IC^*) \to \mathcal{H}^k R_{f*}(IC^*)
\]

for \( f \) gives an action of \( sl_2 \), called the left action, via the isomorphisms

\[
\mathbb{H}^* (\tilde{X}, IC^*) \cong \mathbb{H}^* (S, R_{f*} IC^*) \cong \bigoplus_k \mathbb{H}^* (S, \mathcal{H}^k R_{f*}(IC^*)[-k])
\]

from the decomposition theorem. On the other hand, since \( \mathcal{H}^k R_{f*}(IC^*)[-k] \) underlies a MHM which is again semisimple and pure, \( \mathbb{H}^* (S, \mathcal{H}^k R_{f*}(IC^*)[-k]) \) is equipped with another action of \( sl_2 \), called the right action, by hard Lefschetz again. Therefore we obtain an action of \( sl_2 \times sl_2 \) on the intersection cohomology \( IH^*(\tilde{X}) \) of \( \tilde{X} \).

If \( C \in S \) is a smooth curve of genus \( h \), the fiber of \( f \) over \( C \) is expected to be the Jacobian of line bundles on \( C \) whose cohomology is an \( sl_2 \)-representation space

\[
\left( \frac{1}{2} \oplus 2(0) \right) \otimes h,
\]

where \( \left( \frac{1}{2} \right) \) denotes the 2-dimensional representation of \( sl_2 \) while \( (0) \) is the trivial 1-dimensional representation. In [10], the authors propose a theory of the Gopakumar-Vafa invariants by using the \( sl_2 \times sl_2 \) action on \( IH^*(\tilde{X}, \mathbb{C}) \) as follows: By the Clebsch-Gordan rule, it is easy to see that one can uniquely write the \( sl_2 \times sl_2 \)-representation space \( IH^*(\tilde{X}, \mathbb{C}) \) in the form

\[
IH^*(\tilde{X}, \mathbb{C}) = \bigoplus_k \left( \frac{1}{2} \right)_L \oplus 2(0)_L \otimes R_h,
\]

where \( \left( \frac{1}{2} \right)_L \) denotes the \( k + 1 \) dimensional irreducible representation of the left \( sl_2 \) action while \( R_h \) is a representation space of the right \( sl_2 \) action. Now the authors of [10] define the GV invariant as the Euler number \( Tr_{R_h} (-1)^H R \) of \( R_h \) where \( H_R \) is the diagonal matrix in \( sl_2 \) with entries 1, -1.

However it seems unlikely that the invariant \( n_h (\beta) \) defined using the intersection cohomology as in [10] will relate to the GW invariants of \( Y \) as proposed by Gopakumar-Vafa because intersection cohomology is unstable under deformation. We propose to use the perverse sheaf \( P^* \) on \( X \) constructed above instead of \( IC^* \).
8.2. **GV invariants from perverse sheaves.** In this subsection, we assume that \( \det \text{Ext}^*_E(E,E) \) admits a square root so that we have a perverse sheaf \( P^\bullet \) and a MHM \( \hat{M}^\bullet \) which are locally the perverse sheaf and MHM of vanishing cycles for a local CS functional.

**Remark 8.2.** In [11], it is proved that if the Calabi-Yau 3-fold \( Y \) is simply connected and \( H^*(Y, \mathbb{Z}) \) is torsion-free, then \( \det \text{Ext}^*_E(E,E) \) admits a square root. For instance, when \( Y \) is a quintic threefold, we have the desired perverse sheaf and MHM.

Since the semi-normalization \( \gamma : \hat{X} \to X \) is bijective, the pullback \( \hat{P}^\bullet \) of \( P^\bullet \) is a perverse sheaf and \( \gamma_\ast \hat{P}^\bullet \cong P^\bullet \). By Theorem 7.1, \( P^\bullet \) lifts to a MHM \( \hat{M}^\bullet \) and its pullback \( \hat{M}^\bullet \) satisfies \( \text{rat}(\hat{M}^\bullet) = \hat{P}^\bullet \) since \( \text{rat} \) preserves Grothendieck’s six functors ([31]). Let \( \hat{M}^\bullet = \text{gr}^W \hat{M}^\bullet \) be the graded object of \( \hat{M}^\bullet \) with respect to the weight filtration \( W \). Then \( \hat{M}^\bullet \) is a direct sum of polarizable Hodge modules ([32]). Let \( P^\bullet = \text{rat}(\hat{M}^\bullet) \) which is the graduation \( \text{gr}^W P^\bullet \) by the weight filtration of \( P^\bullet \) because \( \text{rat} \) is an exact functor ([31]).

By [30, §5], the hard Lefschetz theorem and the decomposition theorem hold for the semisimple polarizable MHM \( \hat{M}^\bullet \). Hence by applying the functor \( \text{rat} \), we obtain the hard Lefschetz theorem and the decomposition theorem for \( \hat{P}^\bullet \). Therefore, we can apply the argument in §8.1 to obtain an action of \( sl_2 \times sl_2 \) on the hypercohomology \( \mathbb{H}^\ast(\hat{X}, \hat{P}^\bullet) \) to write

\[
\mathbb{H}^\ast(\hat{X}, \hat{P}^\bullet) \cong \bigoplus_h \left( \left( \frac{1}{2} \right)_L \oplus 2(0)_L \right) \otimes h \otimes R_h.
\]

**Definition 8.3.** We define the Gopakumar-Vafa invariant as

\[
n_h(\beta) := \text{Tr}(R_h(-1)^H) = \sum (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet).
\]

The GV invariant \( n_h(\beta) \) is integer valued and defined by an \( sl_2 \times sl_2 \) representation space \( \mathbb{H}^\ast(\hat{X}, \hat{P}^\bullet) \) as expected from [9].

**Proposition 8.4.** The number \( n_0(\beta) \) is the Donaldson-Thomas invariant of \( \hat{x} \).

**Proof.** Recall that the DT invariant is the Euler number of \( X \) weighted by the Behrend function \( \nu_X \) on \( X \) and that \( \nu_X(x) \) for \( x \in X \) is the Euler number of the stalk cohomology of \( P^\bullet \) at \( x \). Therefore the DT invariant of \( \hat{x} \) is the Euler number of \( \mathbb{H}^\ast(X, P^\bullet) \).

Since the semi-normalization \( \gamma : \hat{X} \to X \) is a homeomorphism, \( \gamma_\ast \hat{P}^\bullet \cong P^\bullet \) and \( \mathbb{H}^\ast(X, P^\bullet) \cong \mathbb{H}^\ast(\gamma_\ast \hat{P}^\bullet) \cong \mathbb{H}^\ast(\hat{X}, \hat{P}^\bullet) \) so that

\[
\text{DT}(\hat{x}) = \sum_k (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet) = \sum_k (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet).
\]

Since \( \hat{P}^\bullet \) has a filtration \( W \) with \( \hat{P}^\bullet = \text{gr}^W \hat{P}^\bullet \), we have the equality of alternating sums

\[
\sum_k (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet) = \sum_k (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet).
\]

Since the Euler number of the torus part \( \left( \left( \frac{1}{2} \right)_L \oplus 2(0)_L \right) \otimes h \) is zero for \( h \neq 0 \),

\[
\sum_k (-1)^k \dim \mathbb{H}^k(\hat{X}, \hat{P}^\bullet) = \text{Tr}(R_0(-1)^H)_h = n_0(\beta).
\]

This proves the proposition. \( \square \)
Furthermore, we propose the following conjecture.

**Conjecture 8.5.** (1) The GV invariants $n_h(\beta)$ are invariant under deformation of the complex structure of $Y$.

(2) The GV invariants $n_h(\beta)$ depend only on $\beta$ and are independent of the constant term $\chi(E)$ of the Hilbert polynomial.

(3) The GV invariants $n_h(\beta)$ are independent of the choice of a polarization of $Y$.

(4) The identity (8.1) holds.

Note that for $h = 0$, (1) follows from Proposition 8.4 and [35]. Also by [12] and [35], (3) is known for $h = 0$. Of course, (1)-(3) are consequences of (4). Furthermore, establishing the identity (8.1) will equate Definition 8.3 with that introduced by Pandharipande-Thomas [29] for a large class of CY 3-folds (cf. [28]).

**8.3. K3-fibered CY 3-folds.** In this last subsection, we show that Conjecture 8.5 holds for a primitive fiber class of K3 fibered CY 3-folds.

We first consider the local case. We let $\Delta \subset \mathbb{C}$ be the unit disk, $t \in \Gamma(O_\Delta)$ the standard coordinate function, and let $p : Y \to \Delta$ be a smooth family of polarized K3 surfaces. We suppose the central fiber $Y_0$ contains a curve class $\beta_0 \in H^{1,1}(Y_0, \mathbb{R}) \cap H^2(Y_0, \mathbb{Z})$, not proportional to the polarization, such that $\beta_0$ ceases to be $(1, 1)$ in the first order deformation of $Y_0$ in $Y$, which means that if we let $\hat{\beta} \in \Gamma(\Delta, \mathbb{R}^2 p_* Z_Y)$ be the continuous extension of $\beta_0$ and let $\hat{\omega} \in \Gamma(\Delta, p_* \Omega^2_Y/\Delta)$ be a nowhere vanishing section of relative $(2, 0)$-form, then $p_*(\hat{\omega} \wedge \hat{\beta}) \notin t^2 O_\Delta$.

For $c \in \Delta$, we let $\iota_c : Y_c \to Y$ be the closed embedding. We let $\beta \in H^1(Y, \mathbb{Z})$ be such that $\beta_0 = \iota_0^* \beta_0$. Since $Y \to \Delta$ is a family of polarized K3 surfaces, the family of relative ample line bundle is ample on $Y$. We form the moduli $\mathcal{M}_Y(-\beta, 1)$ (resp. $\mathcal{M}_Y(\beta_0, 1)$) of one dimensional stable sheaves $E$ of $O_Y$-modules (resp. $O_{Y_0}$-modules) with $c_2(E) = -\beta$ (resp. $c_1(E) = \beta_0$) and $\chi(E) = 1$. Since $\beta$ is a fiber class, the moduli $\mathcal{M}_Y(-\beta, 1)$ is well defined and is a complex scheme.

Because the polarization of $Y$ restricts to the polarization of $Y_0$, we have a closed embedding

$$\mathcal{M}_{Y_0}(\beta_0, 1) \subset \mathcal{M}_Y(-\beta, 1).$$

**Lemma 8.6.** Suppose $\beta_0$ ceases to be $(1, 1)$ in the first order deformation of $Y_0$ in $Y$, and there are no $c \neq 0 \in \Delta$ such that $\iota_c^* \beta \in H^{1,1}(Y_c, \mathbb{R})$. Then the embedding (8.2) is an isomorphism of schemes.

**Proof.** We first claim that for any sheaf $[E] \in \mathcal{M}_Y(-\beta, 1)$, $E = \iota_0^* E'$ for a sheaf $[E'] \in \mathcal{M}_{Y_0}(\beta_0, 1)$. Indeed, let $\text{spt}(E)$ be the scheme-theoretic support of $E$. Since $E$ is stable, it is connected and proper, thus its underlying set is contained in a closed fiber $Y_c \subset Y$ for some closed $c \in \Delta$. Denoting by the same $t \in \Gamma(O_Y)$ the pullback of $t \in O_\Delta$, since $E$ is coherent, there is a positive integer $k$ so that $\text{spt}(E) \subset ((t - c)^k = 0)$. In particular, $E$ is annihilated by $(t - c)^k$.

Since $t - c \in \Gamma(O_Y)$, multiplying by $t - c$ defines a sheaf morphism $(t - c) : E \to E$, which has non-trivial kernel since $(t - c)^k$ annihilates $E$. Since $E$ is stable, this is possible only if $E$ is annihilated by $t - c$. Therefore, letting $\mathcal{E}' = E/(t - c) \cdot E$, which is a sheaf of $O_{Y'}$-modules, we have $E = \iota_c^* \mathcal{E}'$. It remains to show that $c = 0$. If not, then $c_1(E') = \iota_c^* \beta$ will be in $H^{1,1}(Y_c, \mathbb{R})$, a contradiction. This proves the claim.

We now prove that (8.2) is an isomorphism. Indeed, by the previous argument, we know that (8.2) is a homeomorphism. To prove that it is an isomorphism, we
need to show that for any local Artin ring $A$ with quotient field $\mathbb{C}$ and morphism $\varphi_A : \text{Spec } A \to \mathcal{M}_Y(-\beta, 1)$, $\varphi_A$ factors through $\mathcal{M}_{Y_0}(\beta_0, 1)$. By an induction on the length of $A$, we only need to consider the case where there is an ideal $I \subset A$ such that $\dim_C I = 1$ and the restriction $\varphi_{A/I} : \text{Spec } A/I \to \mathcal{M}_Y(-\beta, 1)$ already factors through $\mathcal{M}_{Y_0}(\beta_0, 1)$.

Let $\mathcal{E}$ be the sheaf of $A \times \mathcal{O}_Y$-modules that is the pullback of the universal family of $\mathcal{M}_Y(-\beta, 1)$ via $\varphi_A$. As $\varphi_{A/I}$ factors through $\mathcal{M}_{Y_0}(\beta_0, 1)$, $\mathcal{T} \subset \mathcal{E} \subset \mathcal{E}$. If $\mathcal{T} \cdot \mathcal{E} = 0$, then $\varphi_A$ factors, and we are done. Suppose not. Then since $\mathcal{E}_0 = \mathcal{E} \otimes_A \mathbb{C}$ is stable, there is a $c \in I$ so that $\mathcal{T} \cdot \mathcal{E} = c \cdot \mathcal{E} \subset \mathcal{E}$. Thus $(t - c) \cdot \mathcal{E} = 0$. We now let $\psi : \text{Spec } A \to \mathcal{E}$ be the morphism defined by $\psi^*(c) = c$, and let $Y_A = Y \times_{\mathcal{E}, \psi} \text{Spec } A$. Then $Y_A \to \text{Spec } A$ is a family of K3 surfaces with a tautological embedding $\iota_A : Y_A \to Y \times \text{Spec } A$. Then $(t - c) \cdot \mathcal{E} = 0$ means that there is an $A$-flat family of sheaves $\mathcal{E}'$ of $\mathcal{O}_{Y_\mathcal{E}}$-modules so that $\iota_A^* \mathcal{E}' = \mathcal{E}$.

Let $q$ be the projection and $\iota$ be the tautological morphism fitting into the Cartesian square

$$
\begin{array}{ccc}
Y_A & \xrightarrow{\iota} & Y \\
\downarrow q & & \downarrow p \\
\text{Spec } A & \xrightarrow{\psi} & \Delta
\end{array}
$$

Then $c_1(\mathcal{E}') = t^* \beta$. Thus for the relative $(2, 0)$-form $\tilde{\omega}$, we have

$$
0 = q_*(c_1(\mathcal{E}') \wedge t^* \tilde{\omega}) = q_*(\beta \wedge \tilde{\omega}) = \psi^*-p_*(\beta \wedge \tilde{\omega}).
$$

By the assumption that $p_*(\beta \wedge \tilde{\omega})$ is not divisible by $t^2$, the above vanishing implies that $\psi$ factors through $0 \in \Delta$. This proves that $c = 0$ and $\varphi_A$ factors through $\mathcal{M}_{Y_0}(\beta_0, 1)$. This proves the proposition. $\square$

By the above lemma, we find that the moduli scheme $X = X = \mathcal{M}_Y(-\beta, 1) = \mathcal{M}_{Y_0}(\beta_0, 1)$ is a smooth projective variety of dimension

$$
\dim X = \beta_0^2 + 2 = 2k,
$$

because the obstruction space $\text{Ext}^2(E, E)_0$ is trivial for any stable sheaf $E$ on a K3 surface $Y_0$. Hence we can set $P^* = \mathbb{Q}_X$ and thus $H^*(X, P^*) = H^*(X, \mathbb{Q})$. Let

$$
Y_0 = S \xrightarrow{\pi} \mathbb{P}^1
$$

be an elliptic K3 surface and $\beta_0 = C + kF$ where $C$ is a section and $F$ is a fiber. We can calculate the GV invariants in this case by the same calculation as in [10, Theorem 4.7]. Since the details are obvious modifications of those in [10, §4], we briefly outline the calculation. Indeed by Fourier-Mukai transform, $X$ is isomorphic to the Hilbert scheme $S^{[k]}$ of points on $S$ and the Chow scheme in this case is a complete linear system $\mathbb{P}^k$. The Hilbert-Chow morphism is

$$
S^{[k]} \xrightarrow{h} S^{(k)} \xrightarrow{\pi} (\mathbb{P}^1)^{(k)} \cong \mathbb{P}^k.
$$

It is easy to see that the cohomology $H^*(S, \mathbb{Q})$ of $S$ as an $sl_2 \times sl_2$ representation space by (relative) hard Lefschetz applied to $S \to \mathbb{P}^1$ is

$$
(\frac{1}{2})_L \otimes (\frac{1}{2})_R + 20 \cdot (0)_L \otimes (0)_R.
$$

If we denote by $t_L$ (resp. $t_R$) the weight of the action of the maximal torus for the left (resp. right) $sl_2$ action, we can write $H^*(S, \mathbb{Q})$ as $(t_L + t_L^{-1})(t_R + t_R^{-1}) + 20.$
Hence $H^\ast(S^{(k)}, \mathbb{Q})$ is the invariant part of

$$
\left(\left(\frac{1}{2}\right)_L \otimes \left(\frac{1}{2}\right)_R + 20 \cdot (0)_L \otimes (0)_R\right)^k
$$

by the symmetric group action. In terms of Poincaré series, we can write

$$
\sum_k P_{t_L, t_R}(S^{(k)})q^k = \frac{1}{(1-t_L t_R q)(1-t_L t_R^{-1} q)(1-t_L^{-1} t_R q)(1-t_L^{-1} t_R^{-1} q)(1-q)^{20}}.
$$

Applying the decomposition theorem ([2]) for the semismall map $S^{(k)} \to S^{(k)}$, we find that

$$
\sum_k P_{t_L, t_R}(S^{(k)})q^k = \prod_{m \geq 1} \frac{1}{(1-t_L q^m)(1-t_L^{-1} t_R q^m)(1-t_L^{-1} t_R^{-1} q^m)(1-t_L^{-1} t_R^{-1} q^m)(1-q)^{20}}
$$

which gives

$$
\sum_k P_{t_L, t_R}(S^{(k)})|_{t_R = -1} = \prod_{m \geq 1} \frac{1}{(1-t_L q^m)^2(1+t_L^{-1} q^m)^2(1-q)^{20}}.
$$

By definition, the GV invariants are defined by writing (8.3) as

$$
\sum_{h,k} q^k (t_L + t_L^{-1} + 2)^h \otimes R_h(S^{(k)})|_{t_R = -1} = \sum_{h,k} q^k n_h(k) (t_L + t_L^{-1} + 2)^h
$$

By equating (8.3) and (8.4) with $t_L = -y$, we obtain

$$
\sum_{h,k} (-1)^h n_h(k) (y^{1/2} - y^{-1/2})^{2h} q^{k-1} = \frac{1}{q \prod_{m \geq 1} (1-y q^m)^2(1-y^{-1} q^m)^2(1-q)^{20}}
$$

with $\beta_0^2 = 2k-2$. On the other hand, by [25, Theorem 1], we have

$$
\sum_{h,k} (-1)^h r_h(k) (y^{1/2} - y^{-1/2})^{2h} q^{k-1} = \frac{1}{q \prod_{m \geq 1} (1-y q^m)^2(1-y^{-1} q^m)^2(1-q)^{20}}
$$

where $r_h(k)$ are the BPS invariants from the Gromov-Witten theory for $Y \to \Delta$. Combining these two identities, we find that

$$
n_h(k) = r_h(k),
$$

which verifies Conjecture 8.5 for the local Calabi-Yau 3-fold $Y \to \Delta$. We thus obtain

**Proposition 8.7.** Let $Y \to \mathbb{P}^1$ be a K3 fibered projective CY threefold and let $\iota_0 : Y_0 \subset Y$ be a smooth fiber. Let $\beta_0 \in H_2(Y_0, \mathbb{Z})$ be a curve class so that its Poincare dual $\beta_0^\vee \in H^2(Y_0, \mathbb{Z})$ ceases to be $(1,1)$ type in the first order deformation of $Y_0$ in the family $Y_c, c \in \mathbb{P}^1$. Then $X_0 = X_0 := \mathcal{M}_{Y_0}(\beta_0^\vee, 1) \subset \mathcal{X} := \mathcal{M}_Y(-\iota_0, \beta_0^\vee, 1)$ is a (smooth) open and closed complex analytic subspace, and (8.1) holds for the GV invariants of the perverse sheaf $P^\ast$ (of $\mathcal{X}$) restricted to $X_0$ where $N_y(\beta_0)$ in (8.1) are the GW invariants contributed from the connected components of stable maps to $Y$ that lie in $Y_0$.

It will be interesting to extend the constructions in this paper to the setting of stable pairs. Then it may be possible to extend the theory of Gopakumar-Vafa invariants to the moduli scheme of stable pairs. Let $M$ be the moduli scheme of stable pairs $(F, s)$ with fixed topological type, where $F$ is a pure sheaf of one dimensional
Lemma 9.3. We may assume Riemann-Roch together with the Todd class.

Proof. we find that $c_1$ bundles; let $\pi, \rho$ support and $s$ 44 YOUNG-HOON KIEM AND JUN LI

Then we have $\alpha = \frac{r}{2}$.

Proof of Theorem 9.1. Let $\alpha_1 = c_1(\mathcal{E})$. By Lemma 9.3, we may assume $\alpha_1 = 0$. Then we have

$\text{ch}(\mathcal{E}) = r - \alpha_2 + \frac{\alpha_3}{2} + \frac{\alpha_2^2 - 2\alpha_4}{12}$

where $r$ is the rank of $\mathcal{E}$. Since $R\text{Hom}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$, we have

$\text{ch}(R\text{Hom}(\mathcal{E}, \mathcal{E})) = r^2 - 2r\alpha_2 + \alpha_3^2 + \frac{r}{6}(\alpha_2^3 - 2\alpha_4)$. By the Grothendieck-Riemann-Roch formulas $\text{ch}(\pi_! \mathcal{E}) = \int_Y \text{ch}(\mathcal{E}) \cdot Td_Y$ and

$\text{ch}(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E})) = \text{ch}(R\pi_! R\text{Hom}(\mathcal{E}, \mathcal{E})) = \int_Y \text{ch}(R\text{Hom}(\mathcal{E}, \mathcal{E})) \cdot Td_Y$.

9. Appendix: Square root of determinant line bundle

The purpose of this appendix is to give a direct proof of the following theorem of Hua [11], where it is stated only for sheaves. The argument presented below is a simplification of the proof in [11]. A byproduct of this simplification is that the proof now works for any perfect complexes, not just sheaves as in [11].

Theorem 9.1. [11, Theorem 3.1] Let $\mathcal{E} \to X \times Y$ be a perfect complex of vector bundles; let $\pi, \rho$ be the projections from $X \times Y$ to $X, Y$ respectively; let $\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E}) = R\pi_! R\text{Hom}(\mathcal{E}, \mathcal{E})$. Then the torsion-free part of $c_1(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E})) \in H^2(X, \mathbb{Z})$ is divisible by 2.

For a proof, we need two lemmas.

Lemma 9.2. Theorem 9.1 holds when $\mathcal{E}$ is a line bundle $\mathcal{L}$.

Proof. Since the Chern character of $R\text{Hom}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}$ is 1, by Grothendieck-Riemann-Roch together with the Todd class

$Td_Y = 1 + \frac{c_2(Y)}{12}$,

we find that $c_1(\text{Ext}_\pi^\bullet(\mathcal{L}, \mathcal{L})) = 0$. □

Lemma 9.3. We may assume $c_1(\mathcal{E}) = 0$.

Proof. Let $\mathcal{L} = (\det \mathcal{E})^{-1}$ and $\mathcal{F} = \mathcal{E} \oplus \mathcal{L}$. Then

$c_1(\mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{L}) = c_1(\mathcal{E}) - c_1(\mathcal{L}) = 0$.

Moreover we have

$c_1(\text{Ext}_\pi^\bullet(\mathcal{F}, \mathcal{F})) = c_1(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E})) + c_1(\text{Ext}_\pi^\bullet(\mathcal{L}, \mathcal{L})) + c_1(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{L})) + c_1(\text{Ext}_\pi^\bullet(\mathcal{L}, \mathcal{E}))$.

The last two terms cancel by Serre duality and the second term is zero by Lemma 9.2. Hence $c_1(\text{Ext}_\pi^\bullet(\mathcal{F}, \mathcal{F}))$ is divisible by 2 if and only if $c_1(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E}))$ is divisible by 2. □

Proof of Theorem 9.1. Let $\alpha_i = c_1(\mathcal{E})$. By Lemma 9.3, we may assume $\alpha_1 = 0$. Then we have

$\text{ch}(\mathcal{E}) = r - \alpha_2 + \frac{\alpha_3}{2} + \frac{\alpha_2^2 - 2\alpha_4}{12}$

where $r$ is the rank of $\mathcal{E}$. Since $R\text{Hom}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$, we have

$\text{ch}(R\text{Hom}(\mathcal{E}, \mathcal{E})) = r^2 - 2r\alpha_2 + \alpha_3^2 + \frac{r}{6}(\alpha_2^3 - 2\alpha_4)$. By the Grothendieck-Riemann-Roch formulas $\text{ch}(\pi_! \mathcal{E}) = \int_Y \text{ch}(\mathcal{E}) \cdot Td_Y$ and

$\text{ch}(\text{Ext}_\pi^\bullet(\mathcal{E}, \mathcal{E})) = \text{ch}(R\pi_! R\text{Hom}(\mathcal{E}, \mathcal{E})) = \int_Y \text{ch}(R\text{Hom}(\mathcal{E}, \mathcal{E})) \cdot Td_Y$. 

we have
\[
c_1(\pi^! E) = \int_Y \left( \frac{\alpha_2^2 - 2\alpha_4}{12} - \frac{c_2(Y)}{12} \right) \quad \text{and}
\]
\[
c_1(\text{Ext}^*(E, E)) = \int_Y \left( \frac{\alpha_2^2}{6}(\alpha_2^2 - 2\alpha_4) - 2r\alpha_2 \cdot \frac{c_2(Y)}{12} \right) = \int_Y \alpha_2^2 + 2r c_1(\pi^! E).
\]
So it suffices to show that \(\int_Y \alpha_2^2\) is divisible by 2. By the Künneth formula, we can write
\[
\alpha_2 = \pi^* A_2 + \pi^* A_1 \cdot \rho^* B_1 + \rho^* B_2
\]
modulo torsion, where \(A_i \in H^{2i}(Y, \mathbb{Z})\) and \(B_i \in H^{2i}(X, \mathbb{Z})\). Then we have
\[
\int_Y \alpha_2^2 = 2\left( \int_Y A_2 A_1 \right) \cdot B_1
\]
which is obviously divisible by 2. \(\square\)

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