Equidistribution of Elements of Norm 1 in Cyclic Extensions

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Abstract

Upon quotienting by units, the elements of norm 1 in a number field $K$ form a countable subset of a torus of dimension $r_1 + r_2 - 1$ where $r_1$ and $r_2$ are the numbers of real and pairs of complex embeddings. When $K$ is Galois with cyclic Galois group we demonstrate that this countable set is equidistributed in a finite cover of this torus with respect to a natural partial ordering induced by Hilbert’s Theorem 90.

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1 Introduction

Our main theorem is the following equidistribution result.

Main Theorem. The elements of norm 1 in a cyclic number field $K$ are contained in a torus under a natural quotient. In this quotient, these elements are equidistributed with respect to a so-called visible height function $h$ induced by Hilbert’s Theorem 90.

This is stated precisely in Theorem 1.4 in Section 1.3 and generalizes previous work on the case of quadratic extensions [4].

Let $K$ be a fixed embedding of a number field of degree $d$ over $\mathbb{Q}$. We define $\mathcal{O}$ to be the ring of integers of $K$ and $\mathcal{U}$ the group of units. Let $\mathcal{N}$ be the subset of elements in $K$ with norm $(\mathcal{N} = \mathcal{N}_{K/\mathbb{Q}})$ equal to 1. By Hilbert’s Theorem 90, if $K$ is cyclic with Galois group generated by (a preferred generator) $\sigma$, for $\beta \in \mathcal{N}$, there exists $\alpha \in \mathcal{O}$ such that $\beta = \alpha/\sigma(\alpha)$. We define

$$\pi : K^\times \rightarrow \mathcal{N} \quad \text{by} \quad \pi(\alpha) = \alpha/\sigma(\alpha).$$

The map $\pi$ is a surjective homomorphism with kernel $\mathbb{Q}^\times$, so that the induced map $\tilde{\pi} : K^\times/\mathbb{Q}^\times \rightarrow \mathcal{N}$ is an isomorphism. We will say $\alpha \in \mathcal{O}$ is a visible point for $\beta$ if $\pi(\alpha) = \beta$ and $|N(\alpha)|$ is minimal for all integers with this property. In this case $\tilde{\pi}^{-1}(\beta)$ is the coset $\alpha \mathbb{Q}^\times$. We denote the set of visible points by $\mathcal{V}$. We then define a visible height on $\mathcal{N}$ by $h(\beta) = |N(\alpha)|$ for $\alpha \in \pi^{-1}(\beta)$ a visible point.

The quotient $K^\times/\mathbb{Q}^\times$ decomposes as $\text{Tor}(K^\times/\mathbb{Q}^\times) \oplus F_{\infty}$ where the first factor is finite and the second is the free group with countably infinite rank. By the isomorphism $\tilde{\pi}$ we conclude that the group structure of $\mathcal{N}$ is the product of the group of roots of unity in $K$ and a free group.
We consider a natural quotient of $N$ into a compact group to formulate our equidistribution question using Weyl’s Criterion. Specifically, with $\log$ denoting the regulator map (see Equation 1.1) we consider equidistribution of $N = \log N + \log \pi(U)$ where $U$ is the group of units in $\mathcal{O}$. The set $\pi(U)$ is a finite index subgroup of $U$ (as proven in Proposition 1.1), and from Dirichlet’s Unit Theorem it follows that $N$ is contained in a torus. In Proposition 1.3 we show that $h$ induces a well-defined height, which we also call $h$, on $N$.

To prove that $N$ is equidistributed with respect to $h$, we use Weyl’s criterion and the Wiener-Ikehara Tauberian Theorem to reduce the problem to an analysis of $L$-functions. After corresponding our natural $L$-series to a partial Hecke $L$-series, we use known results about the analytic properties of partial Hecke $L$-series to demonstrate that Weyl’s Criterion is satisfied.

1.1 Equidistribution and Weyl’s Criterion

Given a probability space $(X, \mathcal{F}, \nu)$ and countable set $C \subset X$ we call $h : C \to [0, \infty)$ a height on $X$ if

$$C(r) := h^{-1}[0, r] = \{x \in C : h(x) \leq r\}$$

is finite for every $r > 0$. We then say that $C$ is equidistributed (or uniformly distributed) in $(X, \mathcal{F}, \nu)$ with respect to the height $h$ if for any measurable set $B \in \mathcal{F}$ with positive measure,

$$\lim_{r \to \infty} \frac{|B \cap C(r)|}{|C(r)|} = \nu(B).$$

In words, $C$ is equidistributed with respect to $h$ if, for every positive $B \in \mathcal{F}$ the proportion of points of bounded height which lie in $B$ converges to the measure of $B$ as the height bound increases.

When $X$ is a compact abelian group with Borel $\sigma$-algebra and Haar probability measure—we say simply that $C$ is equidistributed in $X$ with respect to $h$. In this case, we may use Weyl’s criterion to establish equidistribution. Specifically, $C$ is equidistributed in a compact abelian group $X$ with respect to $h$ if, for any character $\chi : X \to \mathbb{T},$

$$\lim_{n \to \infty} \frac{1}{|C(r)|} \sum_{x \in C(r)} \chi(x) = \begin{cases} 1 & \text{if } \chi \text{ is trivial;} \\ 0 & \text{otherwise.} \end{cases}$$

We now precisely define the compact abelian group and height to which we will apply Weyl’s Criterion.

1.2 A Hilbert-Dirichlet Torus

For a number field $K$, the set of archimedean places is denoted $S_\infty$, and has cardinality $r_1 + r_2$ where $r_1$ is the number of real places and $r_2$ is the number of complex places. For each place $v \in S_\infty$ we denote by $\| \cdot \|_v$ either the usual absolute value if $v$ is real, or the usual complex absolute value squared if $v$ is complex.

We define the regulator map $\log : K^\times \to \mathbb{R}^{r_1 + r_2}$ by

$$\alpha \to (\log \|\alpha\|_v)_{v \in S_\infty}. \quad (1.1)$$

We also define $\Sigma : \mathbb{R}^{r_1 + r_2} \to \mathbb{R}$ by $x \mapsto x_1 + x_2 + \cdots + x_{r_1 + r_2}$. Then, by the proof of Dirichlet’s Unit Theorem, $\log U$ is a lattice in $\ker \Sigma$ and the kernel of $\log$ restricted to $U$ is the group of roots of unity in $K$. This is usually stated as

$$U \cong \text{Tor}(U) \times \mathbb{Z}^{r_1 + r_2 - 1}.$$
The quotient
\[ T = \ker \Sigma / \log U \]
is isomorphic to the torus \( T^{r_1 + r_2 - 1} \). Our normalization of \( \| \cdot \|_v \) together with the fact that \( N \) consists of elements of norm 1 implies that \( \log N \) lies in \( \ker \Sigma \). From here forward we will assume that \( K \) is Galois over \( \mathbb{Q} \) with cyclic Galois group \( G \) generated by \( \sigma \).

**Proposition 1.1.** The set \( S := \ker \Sigma / \log \pi(U) \) is a torus and is a finite cover of \( T = \ker \Sigma / \log U \).

**Proof.** This will follow from the fact that \( \pi(U) \) is a finite index subgroup of \( U \), which we will now establish. We will show that for all \( u \in U \cap N \), \( u^e \in \pi(U) \) where \( e \) is the least common multiple of the ramification indices of all prime ideals in \( \mathcal{O} \).

Since \( u \in N \), there is a visible point \( \alpha \) such that \( u = \pi(\alpha) \). For each prime ideal \( p \), there is a non-negative integer \( n_p(\alpha) \) such that
\[
\alpha \mathcal{O} = \prod_p p^{n_p(\alpha)} = \sigma(\alpha) \mathcal{O},
\]
where the second equation follows because \( \alpha/\sigma(\alpha) = u \) is a unit. If \( \alpha \) is a unit then every \( n_p \) is zero. If \( \alpha \) is not a unit, at least one \( n_p > 0 \) and, in any case, \( n_p = 0 \) for all but finitely many primes \( p \).

The Galois group acts transitively on the primes above a given rational prime \( p \). This implies that if \( p, q \) are primes above \( p \), then \( n_p(\alpha) = n_q(\alpha) \) and in this case we can write \( n_p(\alpha) \) for this common integer. That is,
\[
\alpha \mathcal{O} = \prod_p p^{n_p(\alpha)}.
\]

We come to the central observation: if \( n_p(\alpha) \geq e_p \), the ramification index of \( p \), then \( p \mathcal{O} \) divides \( \alpha \mathcal{O} \) and hence \( \alpha = p \beta \) for some \( \beta \in \mathcal{O} \). In this situation \( \pi(\alpha) = \pi(\beta) \) and \( p^d | N(\beta) | = | N(\alpha) | \), contradicting the visibility of \( \alpha \). In particular \( n_p(\alpha) = 0 \) for all non-ramified primes \( p \).

Consider \( u^e = \pi(\alpha^e) \). For all primes \( p \) we write \( e = e_p g_p \), for some positive integers \( g_p \). We have that
\[
\alpha^e \mathcal{O} = \prod_p p^{e_p(\alpha)} = \prod_p \prod_{p \mid |} p^{g_p n_p(\alpha)} = \prod_p p^{g_p n_p(\alpha)}.\]

As such, \( \alpha^e \mathcal{O} \) is generated by a rational integer. It follows that \( \alpha^e = n w \) for some \( n \in \mathbb{N} \) and \( w \in \mathcal{U} \). We have \( u^e = \pi(\alpha^e) = \pi(n w) = \pi(w) \) as needed.

\[ \square \]

### 1.3 A Visible Height From Hilbert’s Theorem 90

We now prove a key lemma from which it follows that the visible height is well-defined on \( N \).

**Lemma 1.2.** Assume that \( \alpha \in \mathcal{V} \) is a visible point for \( \beta \in N \), and \( \gamma \in \mathcal{O} \). Then \( \beta = \pi(\gamma) \) if and only if there exists a non-zero rational integer \( n \) such that \( \gamma = n \alpha \). Such an \( n \) and \( \alpha \) are unique up to multiplication by \(-1\).

In particular, \( \alpha \) is a visible point for \( \beta \) if and only if \(-\alpha \) is.

\[ ^1 \text{Note that this implies that } K \text{ is either totally real, or totally imaginary, but we will still write, for instance, } U \cong \text{Tor}(U) \times \mathbb{Z}^{r_1 + r_2 - 1} \text{ with the understanding that one of } r_1 \text{ and } r_2 \text{ is } 0. \]
Proof. If $\gamma = n\alpha$ and $\alpha$ is a visible point for $\beta$ we have

$$\beta = \pi(\alpha) = \frac{\alpha}{\sigma(\alpha)} = \frac{n\alpha}{n\sigma(\alpha)} = \frac{n\alpha}{\sigma(n\alpha)} = \pi(n\alpha) = \pi(\gamma).$$

Conversely, assume that $\pi(\alpha) = \pi(\gamma)$ then

$$\frac{\alpha}{\gamma} = \frac{\sigma(\alpha)}{\sigma(\gamma)} = \sigma\left(\frac{\alpha}{\gamma}\right),$$

and hence $\alpha/\gamma$ is fixed by $G$. It follows that there exist relatively prime rational integers $m$ and $n$ such that $\alpha/\gamma = m/n$, and by the minimality of $|N(\alpha)|$, $|m| \leq |n|$. Thus both $\alpha$ and $\gamma = n\alpha/m$ are algebraic integers which map onto $\beta$.

If we assume $n/m > 0$, then $n/m \geq 1$ since $|m| \leq |n|$ and for $j = |n/m|$ we have that $\gamma - j\alpha$ is an algebraic integer. If $n/m \neq j$ then it is non-zero with

$$\pi(\gamma - j\alpha) = \pi\left(\left(\frac{n}{m} - j\right)\alpha\right) = \frac{(n/m - j)\alpha}{\sigma((n/m - j)\alpha)} = \pi(\alpha).$$

Since $n/m - j \in \mathbb{Z} - \{0\}$,

$$|N(\gamma - j\alpha)| = |N\left(\left(\frac{n}{m} - j\right)\alpha\right)| = \left(\frac{n}{m} - j\right)^d |N(\alpha)| < |N(\alpha)|.$$

This contradicts the minimality of $|N(\alpha)|$. Therefore, $n/m = j$ so that $n/m$ is an integer and $\gamma = n\alpha$. The case when $n/m < 0$ is similar.

To show uniqueness, by the above if both $\alpha$ and $\alpha'$ are visible points for $\beta$ then $\pi(\alpha) = \pi(\alpha')$ and we have $\alpha = n'\alpha'$ and also $\alpha' = n\alpha$ for $n, n' \in \mathbb{Z}$. But then $n, n' = \pm 1$. \(\square\)

Hilbert’s Theorem 90 implies that for every $\beta \in \mathcal{N}$ there is a visible point $\alpha \in \mathcal{V}$, and in this situation, we define $h(\beta) = |N(\alpha)|$.

For any $n > 0$ there are finitely many integers in $\mathcal{O}$ with absolute norm bounded by $n$. It follows from this fact and Lemma 1.2 that $h$ is a well-defined height on $\mathcal{N}$. A height on $\mathcal{N}$ does not necessarily induce a well defined on $\mathcal{N}$. We observe that for each $\beta \in \mathcal{N}$ and $u \in \pi(\mathcal{U})$, $h(u\beta) = h(\beta)$. By the uniqueness statement in Lemma 1.2 we arrive at the following.

Proposition 1.3. The height $h$ induces a well defined height on $\overline{\mathcal{N}} := \log \mathcal{N} + \log \pi(\mathcal{U})$.

The set $\mathcal{N}$ of norm 1 elements is contained in $\ker \Sigma$ and so $\overline{\mathcal{N}} = \log \mathcal{N} + \log \pi(\mathcal{U})$ is contained in $\ker \Sigma/\log \pi(\mathcal{U})$. Propositions 1.1 and 1.3 imply that $\overline{\mathcal{N}}$ is a countable set in a compact abelian group with a natural height, and therefore we can state our Main Theorem precisely.

Theorem 1.4. $\overline{\mathcal{N}} = \log \mathcal{N} + \log \pi(\mathcal{U})$ is equidistributed in $S = \ker \Sigma/\log \pi(\mathcal{U})$ with respect to $h$.

1.4 Hecke $L$-Series and the Wiener-Ikehara Tauberian Theorem

Let $\overline{\mathcal{N}}(r) = \{x \in \overline{\mathcal{N}} : h(x) \leq r\}$. The function

$$r \mapsto \sum_{x \in \overline{\mathcal{N}}(r)} \chi(x)$$
is the summatory function of the $L$-series

$$L(\chi; s) = \sum_{x \in \mathbb{N}} \frac{\chi(x)}{h(x)^s}.$$ 

This observation is useful since asymptotics of the summatory function (as a function of $r$) follow from analytic properties of $L(\chi; s)$ using the Wiener-Ikehara Tauberian theorems. Specifically, if we can show that, for the trivial character $\chi = 1$, $L(s; 1)$ has a pole at $s = \sigma_0$ and there exists $\epsilon > 0$ such that for all other characters, $L(\chi; s)$ is analytic for $\Re(s) > \sigma_0 - \epsilon$, then there exists a non-zero constant $C$ such that as $r \to \infty$,

$$\#(N(r)) \sim Cr^{\sigma_0} \quad \text{and} \quad \sum_{x \in \mathbb{N}(r)} \chi(x) = o(r^{\sigma_0}),$$ 

and Weyl’s criterion will be satisfied. See [2, Ch.VII §3] or [3, Ch. XV §3.5].

Let $\mathcal{O}$ be the set of non-zero principal integral ideals in $K$ and let $\mathbb{N}$ denote the ideal norm. Let $\iota : K^\times \to (\mathbb{R}^\times)^r_1 \times (\mathbb{C}^\times)^r_2$ be a preferred Minkowski embedding of $K^\times$. Given a character, we denote the complex conjugate character with a bar.

**Theorem 1.5.** Each character $\chi$ on $S$ lifts to a continuous character $\chi_\infty$ on $(\mathbb{R}^\times)^r_1 \times (\mathbb{C}^\times)^r_2$. Define $\chi_\iota$ to be the character on the the group of principal fractional ideals $\mathcal{O}$ given by

$$\chi_\iota(\alpha \mathcal{O}) = \chi_\infty(\iota(\alpha)).$$

Then, in the half-plane $\Re(s) > 1$ the function $L(\chi; s)$ converges absolutely and

$$L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{\alpha \in \mathcal{O}} \overline{\chi_\iota(\alpha)} \frac{\overline{\chi_\infty(\alpha)}}{\overline{N(\alpha)}},$$

where $\zeta(s)$ is the Riemann zeta function. Moreover, $\chi_\iota$ is trivial if and only if $\chi$ is trivial. For $\chi$ non-trivial, $L(\chi; s)$ has an analytic continuation to the half-plane $\Re(s) > \frac{1}{d}$, and $L(\chi; 1)$ has a meromorphic continuation to this half-plane with a simple pole at $s = 1$.

Our notation is suggestive as $\chi_\infty$ extends to the $\infty$-type of the unramified Hecke character $\chi_\infty \iota$ on the idèle class group of $K$. It will then follow that

$$\zeta_K(\chi_\iota; s) := \sum_{\alpha \in \mathcal{O}} \overline{\chi_\iota(\alpha)} \frac{\overline{\chi_\infty(\alpha)}}{\overline{N(\alpha)}},$$

is a partial Hecke $L$-series (here partial means summed over the class of principal ideals). If $K$ happens to have class number 1, then $\zeta_K(\chi_\iota; s)$ is a (complete) Hecke $L$-series.

In [1] it is proved that $\zeta_K(\chi_\iota; s)$ has an analytic continuation (as a function of $s$) to the entire plane, except when $\chi_\iota = 1$ is trivial, in which case it has a meromorphic continuation to a function with a single simple pole at $s = 1$. It follows that, if $\chi$ is nontrivial, then so is $\chi_\iota$, and $L(\chi; s) = \zeta_K(\chi_\iota; s)/\zeta(ds)$ has an analytic continuation to the half-plane $\Re(s) > 1/d$. If $\chi = 1$ is trivial, then $L(\chi; s)$ has a meromorphic continuation to the same half-plane, but with a simple pole at $s = 1$. Weyl’s criteria now follows from the Wiener-Ikehara Tauberian Theorem. Therefore, Theorem 1.4 follows from Theorem 1.5.
2 The Proof of Theorem 1.5

Starting with a continuous character \( \chi \) on \( S \) we note that \( \overline{N} = \log N + \log \pi(U) \) is dense in \( S \). Given \( \beta \in \mathbb{N} \),

\[
\pi^{-1}(\beta) = \{ x \in K^{\times} : x/\sigma(x) = \beta \},
\]

and \( K^{\times} \) is the disjoint union

\[
K^{\times} = \bigsqcup_{\beta \in \mathbb{N}} \pi^{-1}(\beta).
\]

We then define \( \chi_K : K^{\times} \to \mathbb{T} \) by \( \chi_K(x) = \chi(\log \beta + \log \pi(U)) \) whenever \( x \in \pi^{-1}(\beta) \). With Lemma 1.2 then \( \chi_K(r) = 1 \) for all \( r \in \mathbb{Q} \), which corresponds to the fact that if \( \pi(\alpha) = \beta \) then \( \pi^{-1}(\beta) = \alpha \mathbb{Q}^{\times} \). Note that if \( u \) is a unit, \( \chi_K(u) = \chi(\log \pi(U)) = 1 \).

For all \( x \in K^{\times} \) we define \( \chi_\infty((x)) = \chi_K(x) \). Because \( \ell(K^{\times}) \) is dense in \( (\mathbb{R}^{\times})^r \times (\mathbb{C}^{\times})^{r^2} \),

the continuity of \( \log \) and \( \pi \) mean that we may uniquely define

\[
\chi_\infty : (\mathbb{R}^{\times})^r \times (\mathbb{C}^{\times})^{r^2} \to \mathbb{T}
\]

by considering points in \( (\mathbb{R}^{\times})^r \times (\mathbb{C}^{\times})^{r^2} \) as limits of elements of \( K^{\times} \) under the (multiplicative) Minkowski embedding \( K^{\times} \hookrightarrow (\mathbb{R}^{\times})^r \times (\mathbb{C}^{\times})^{r^2} \).

It remains to show that \( \chi_\infty \) is a character. This will follow again by density and the fact that \( \chi_K \) is a multiplicative homomorphism. To see this, suppose \( x, y \in K^{\times} \). Since \( \pi \) is multiplicative \( \pi(xy) = \pi(x)\pi(y) \), and \( \chi \) is a character on \( S \) we have that

\[
\begin{align*}
\chi_K(xy) &= \chi(\log \pi(xy) + \log \pi(U)) \\
&= \chi(\log \pi(x) + \log \pi(y) + \log \pi(U)) \\
&= \chi(\log \pi(x) + \log \pi(U))\chi(\log \pi(y) + \log \pi(U)) \\
&= \chi_K(x)\chi_K(y).
\end{align*}
\]

Clearly, \( \chi_K \) is trivial if and only if our original character \( \chi \) is trivial.

We wish to relate \( L(\chi; s) \) to a Hecke \( L \)-function in order to establish the necessary analytic properties to verify Weyl’s criterion. Let \( \mathfrak{U} \) be the set of principal ideals generated by visible points. If \( v \in \mathfrak{U} \) then there is a visible point \( \alpha \) such that \( v = \alpha \mathfrak{O} \). If \( \alpha' \) is another visible point such that \( v = \alpha' \mathfrak{O} \) then \( \alpha' = u\alpha \) for some \( u \in \mathfrak{U} \). Then,

\[
\log(\pi(\alpha')) = \log(\pi(\alpha)) + \log(\pi(u)), \quad \text{and hence} \quad \overline{\pi(\alpha)} = \overline{\pi(\alpha')}
\]

in \( \overline{N} \) and hence \( \log \circ \pi \) induces a bijection from \( \mathfrak{U} \) to \( \overline{N} \). We write \( \overline{\mathfrak{U}} \) for the image of \( \mathfrak{U} \) under this map. This induces a map from \( \mathfrak{U} \to \overline{N} \) by defining for \( a = \gamma \mathfrak{O} \) with \( \gamma \in K^{\times} \) that \( \overline{a} = \overline{\pi(\gamma)} \). As such,

\[
\overline{\chi_3(a)} = \chi(\overline{a}) = \chi(\overline{\pi(\gamma)}).
\]

For \( v \in \mathfrak{U} \), as above \( h(\overline{v}) = |N(\alpha)| = Nv \) and we have

\[
L(\chi; s) = \sum_{x \in \overline{N}} \frac{\chi(x)}{h(x)^s} = \sum_{v \in \mathfrak{U}} \frac{\overline{\chi_3(v)}}{Nv^s}.
\]

Multiplying and dividing by the scaled Riemann zeta function \( \zeta(ds) \), up to reordering,

\[
L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{n=1}^\infty \sum_{v \in \mathfrak{U}} \frac{\overline{\chi_3(v)}}{n^s Nv^s} = \frac{1}{\zeta(ds)} \sum_{n=1}^\infty \sum_{v \in \mathfrak{U}} \frac{\overline{\chi_3(v)}}{N(nv)^s}.
\]
By Lemma 1.2, each non-zero algebraic integer $\gamma$ can be written uniquely as $n\alpha$ for visible point $\alpha$ and positive integer $n$. The principal ideal $a = \gamma O$ can be written uniquely as $nv = \alpha O$ and we have $\pi(\gamma) = \pi(\alpha)$ and therefore $\pi = \pi(\gamma) = \pi(\gamma) = \pi(\alpha)$. That is, up to reordering,

$$L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{a \in \mathcal{D}} \frac{\chi(a)}{N(a s)} = \frac{1}{\zeta(ds)} \zeta_K(\chi; s).$$

The right-hand side of this equation can be written as an absolutely convergent series in the half-plane $\text{Re}(s) > 1$ because of the analytic properties of both $\zeta(s)$ and $\zeta_K(\chi; s)$. Because this series can be reordered as $L(\chi; s)$ it follows that $L(\chi; s)$ is also an absolutely convergent series in the half-plane $\text{Re}(s) > 1$. The stated analytic properties follow from the analytic properties of $\zeta(s)$ and $\zeta_K(\chi; s)$.

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