Entropy, Critical Exponent and Immersed Surfaces in Hyperbolic 3-Manifolds

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Abstract

We consider a \( \pi_1 \)-injective immersion \( f : \Sigma \rightarrow M \) from a compact surface \( \Sigma \) to a hyperbolic 3–manifold \( M \). Let \( \Gamma \) denote the copy of \( \pi_1 \Sigma \) in \( \text{Isom}(\mathbb{H}^3) \) induced by the immersion and \( \delta(\Gamma) \) be the critical exponent. Suppose \( \Gamma \) is convex cocompact and \( \Sigma \) is negatively curved, we prove that there are two geometric constants \( C_1(\Sigma, M) \) and \( C_2(\Sigma, M) \) not bigger than 1 such that \( C_1(\Sigma, M) \cdot \delta(\Gamma) \leq h(\Sigma) \leq C_2(\Sigma, M) \cdot \delta(\Gamma) \), where \( h(\Sigma) \) is the topological entropy of the geodesic flow on. When \( f \) is an embedding, we show that \( C_1(\Sigma, M) \) and \( C_2(\Sigma, M) \) are exactly the geodesic stretches (a.k.a. Thurston’s intersection number) with respect to certain Gibbs measures. Moreover, we prove the rigidity phenomenon arising from this inequality. Lastly, as an application, we discuss immersed minimal surfaces in hyperbolic 3–manifolds and these discussions lead us to results similar to A. Sanders’ work [San14] on the moduli space of \( \Sigma \) introduced by C. Taubes [Tau04].

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1 Introduction

1.1 Main results

We consider a $\pi_1$-injective immersion $f : \Sigma \to \text{M}$ from a compact surface $\Sigma$ to a hyperbolic 3-manifold $\text{M}$. Let $\Gamma$ denote the copy of $\pi_1\Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion $f$, and we endow $\Sigma$ with the induced metric $g$ from the given hyperbolic metric $h$ on $\text{M}$. The topological entropy $h(\Sigma)$ of the geodesic flow on $T^1\Sigma$, and the critical exponent $\delta_\Gamma$ of $\Gamma$ on $\mathbb{H}^3$ are two natural geometric quantities associated to this setting. Recall that when $(\Sigma, g)$ is a negatively curved manifold, then each closed geodesic on $\Sigma$ corresponds to a unique conjugacy class $[\gamma] \in \pi_1\Sigma$, and vice versa. We can write the topological entropy of the geodesic flow on $T^1\Sigma$ as

$$h(\Sigma) = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1\Sigma]; l_g(\gamma) \leq T \},$$

where $l_g(\gamma)$ is the length of the closed geodesic $[\gamma]$ with respect to the metric $g$. Moreover, the critical exponent $\delta_\Gamma$ can be understood by lengths of closed geodesics as well. By Sullivan’s theorem [Sul79], when $\Gamma$ is convex cocompact, we have

$$\delta_\Gamma = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1\Sigma]; l_h(\gamma) \leq T \},$$

where $l_h(\gamma)$ is the length of $[\gamma]$ using the hyperbolic metric $h$.

The main tool used in the note will be the Thermodynamic Formalism. Specifically, the reparametrization method introduced by Ledrappier [Led95] and Sambarino [Sam14]. Through the reparametrization method, we can link two different Anosov flows on $\mathbb{H}^3$ by a Hölder continuous function. Thus, we can compare periods of closed orbits associating with different Anosov flows, therefore, their topological entropies.

Our main theorem shows that we can relate the two geometric quantities $h(\Sigma)$ and $\delta_\Gamma$ by an inequality. Moreover, the equality cases exhibit rigidity features.

Theorem (Theorem 3.1). Let $f : \Sigma \to \text{M}$ be a $\pi_1$-injective immersion from a compact surface $\Sigma$ to a hyperbolic 3-manifold $\text{M}$, and $\Gamma$ be the copy of $\pi_1\Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion $f$. Suppose $\Gamma$ is convex cocompact and $(\Sigma, f^*h)$ is negatively curved, then

$$C_1(\Sigma, \text{M}) \cdot \delta_\Gamma \leq h(\Sigma) \leq C_2(\Sigma, \text{M}) \cdot \delta_\Gamma,$$

where $C_1(\Sigma, \text{M})$ and $C_2(\Sigma, \text{M})$ are two geometric constants not bigger than 1. Moreover, each equality holds if and only if the marked length spectrum of $\Sigma$ is proportional to the marked length spectrum of $\text{M}$, and the proportion is the ratio $\frac{\delta_\Gamma}{h(\Sigma)}$.

Remark 1.1.: 

1. By Sullivan’s theorem [Sul84], one can replace the critical exponent $\delta_\Gamma$ in (1) by the Hausdorff dimension $\dim_H \Lambda(\Gamma)$ of the limit set $\Lambda(\Gamma)$.

2. This result should be compared with Theorem 1 [Bur93], Theorem 1.2 [Kni95], and Theorem A [Sam13]. All these results possess a similar flavor of comparing entropies.

3. In Glorieux’s thesis [Glo15], he follows Knieper’s method and deduces an upper bound of $h(\Sigma)$ in the case that $\Sigma$ is embedded in a quasi-Fuchsian manifold $\text{M}$. We will prove that the upper bound in Glorieux’s thesis is exactly the same as the one in Theorem 3.1.

Next two theorems depict geometric meanings of $C_1(\Sigma, \text{M})$ and $C_2(\Sigma, \text{M})$ mentioned in Theorem 3.1. These two constants could be regarded as averages of lengths of closed geodesics with respect to different metrics $g$ and $h$. 
Theorem (Theorem 4.1). Let $f : \Sigma \to M$ be a $\pi_1$–injective immersion from a compact surface $\Sigma$ to a hyperbolic 3–manifold $M$, and $\Gamma$ be the copy of $\pi_1 \Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion $f$. Suppose $\Gamma$ is convex cocompact and $(\Sigma, f^* h)$ is negatively curved, then

$$C_2(\Sigma, M) = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g)} l_h(\gamma)}{\sum_{[\gamma] \in R_T(g)} l_g(\gamma)}; \quad C_1(\Sigma, M) = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(h)} l_h(\gamma)}{\sum_{[\gamma] \in R_T(h)} l_g(\gamma)}$$

where

$$R_T(g) := \{ [\gamma] \in [\pi_1 \Sigma] : l_g(\gamma) \leq T \}, \text{ and } R_T(h) := \{ [\gamma] \in [\pi_1 \Sigma] : l_h(\gamma) \leq T \}.$$ 

In additional, when $f : \Sigma \to M$ is an embedding, we have another geometrical interpretation of $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$. The following theorem shows that $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$ are the geodesic stretches of $\Sigma$ relative to $M$ with respect to certain Gibbs measures.

Theorem (Theorem 4.2). Let $f : \Sigma \to M$ be a $\pi_1$–injective embedding from a compact surface $\Sigma$ to a hyperbolic 3–manifold $M$, and $\Gamma$ be the copy of $\pi_1 \Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the embedding $f$. Suppose $\Gamma$ is convex cocompact and $(\Sigma, f^* h)$ is negatively curved, then

$$C_1(\Sigma, M) = I_\mu(\Sigma, M), \quad C_2(\Sigma, M) = I_{\mu BM}(\Sigma, M).$$

Here, $I_\mu(\Sigma, M)$ and $I_{\mu BM}(\Sigma, M)$ are the geodesic stretches with respect to a Gibbs measure $\mu$ and the Bowen-Margulis measure $\mu_{BM}$ of the geodesic flow $\phi$ on $T^1 \Sigma$.

Remark 1.2. Our definition of the geometric stretch $I_\mu(\Sigma, M)$ in Section 4 is inspired by Knieper [Kni95]. In the general setting, the geodesic stretch was introduced in the paper [CF90] of Croke and Fathi is known as the Thurston’s intersection number.

1.2 Applications

By the Gauss equation, immersed minimal surfaces in hyperbolic 3–manifolds are negatively curved. Minimal surfaces in hyperbolic 3–manifolds is a very rich subject and have drawn a lot of attention, with important contributions by Uhlenbeck [Uhl83] and Taubes [Tau04]. In this note, we take a glance at this rich subject from a dynamical system point of view.

The following corollary is a consequence of the main theorem.

Corollary (Corollary 5.1). Let $f : \Sigma \to M$ be a $\pi_1$–injective minimal immersion from a compact surface $\Sigma$ to a hyperbolic 3–manifold $M$, and $\Gamma$ be the copy of $\pi_1 \Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion. Suppose $\Gamma$ is convex cocompact, then there are explicit constants $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$ not bigger than 1 such that

$$C_1(\Sigma, M) \cdot \delta_1 \leq h(\Sigma) \leq C_2(\Sigma, M) \cdot \delta_1$$

Moreover, each equality holds if and only if the marked length spectrum of $\Sigma$ is proportional to the marked length spectrum of $M$, and the proportion is the ration $\frac{\delta_1}{\mu(\Sigma)}$.

Remark 1.3.:

From [Uhl83], we learn that the $\pi_1$–injectivity is guaranteed if we put some curvature conditions on $\Sigma$. Namely, all principal curvatures are between $-1$ and 1. Furthermore, in such cases, immersed minimal surfaces are indeed embedded. Therefore, we can interpret the constants $C_i(\Sigma, M)$ of such pairs $(\Sigma, M)$ as geodesic stretches.
In the last part of this note, we change gear to the Taubes’ moduli space of $\Sigma$. Taubes [Tau04] constructs the space of minimal hyperbolic germs $H$ which is a deformation space for the set whose archetypal elements is a pair that consists of a Riemannian metric $g$ and the second fundamental form $B$ from a closed, oriented, negative Euler characteristic minimal surfaces $\Sigma$ in some hyperbolic 3–manifold $M$.

Uhlenbeck [Uhl83] proved that there exists a representation $\rho : \pi_1(\Sigma) \to \text{Isom} (\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ leaving this minimal immersion invariant. In other words, there is a map

$$\Phi : H \to R(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C})),$$

where $R(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))$ is the space of conjugacy classes of representations of $\pi_1(\Sigma)$ into $\text{PSL}(2, \mathbb{C})$.

The following corollary gives an upper and a lower bound of the topological entropy $h(g,B)$ of the geodesic flow on $T^1\Sigma$ provided the data $(g,B) \in H$.

**Corollary** (Corollary 5.2). Let $\rho \in R(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))$ be a discrete, faithful and convex cocompact representation. Suppose $(g,B) \in \Phi^{-1}(\rho)$, then there are explicit constants $C_1(g,B)$ and $C_2(g,B)$ not bigger than 1 such that

$$C_1(g,B) \cdot \delta_{\rho(\pi_1\Sigma)} \leq h(g,B) \leq C_2(g,B) \cdot \delta_{\rho(\pi_1\Sigma)} \leq \delta_{\rho(\pi_1\Sigma)}$$

with the last equality if and only if $B$ is identically zero which holds if and only if $\rho$ is Fuchsian.

**Remark 1.4**:  
1. When $\rho$ is quasi-Fuchsian, the above upper bound of $h(g,B)$ is a special case treated in Sanders’ paper [San14]. However, the inequality in Sanders’ work has no information about the constant $C_2(g,B)$.
2. The lower bound given in Sanders’ paper [San14] is derived by Manning’s formula [Man81], which gives a lower bound of the topological entropy $h(g,B)$ in terms of the curvature of $(\Sigma, g)$. Whereas, our method doesn’t see the curvature directly. It will be interesting to compare our lower bound with Sanders’ lower bound of $h(g,B)$.

From above corollary, we recover the Bowen’s rigidity theorem [Bow79].

**Corollary** (Bowen’s rigidity [Bow79]). A quasi-Fuchsian representation $\rho \in \mathcal{QF}$ is Fuchsian if and only if $\dim H \Lambda(\Gamma) = 1$.

Lastly we focus on a two special subsets of the minimal hyperbolic germs: the Fuchsian space $\mathcal{F}$ and the almost-Fuchsian space $\mathcal{AF}$. The Fuchsian space is the space of all hyperbolic metrics on $\Sigma$, i.e.

$$\mathcal{F} = \{(m,0) \in H; m \text{ is a hyperbolic metric on } \Sigma\}$$

and the almost-Fuchsian space $\mathcal{AF}$ is defined by the condition that $\|B\|_g^2 < 2$. By Uhlenbeck’s theorem in [Uhl83] we know that if $(g,B) \in \mathcal{AF}$ then there exists a unique quasi-Fuchsian 3–manifold $M$, up to isometry, such that $\Sigma$ is an embedded minimal surface in $M$ with the induced metric $g$ and the second fundamental form $B$.

The following theorem was first proved in [San14], and we recover this theorem by the reparametrization method.

**Corollary** (Theorem 5.1, Theorem 3.5 [San14]). Consider the entropy function restricting on the almost-Fuchsian space $h : \mathcal{AF} \to \mathbb{R}$, then

1. the entropy function $h$ realizes its minimum at the Fuchsian space $\mathcal{F}$, and
2. for \((m, 0) \in \mathcal{F}\), \(h\) is monotone increasing along the ray \(r(t) = (g_t, tB)\) provided \(\|tB\|_{g_t} < 2\), i.e. \(r(t) \subset \mathcal{AF}\), where \(g_t = e^{2\pi t} m\).

In the end this note, we give another proof of the following theorem given in [San14]. Similar to Bridgeman’s method in [Bri10], the pressure form is used in showing that the Hessian of the entropy defines a metric on \(\mathcal{F}\), which is bounded below by the Weil-Petersson metric.

**Theorem** (Theorem 5.2, Theorem 3.8 [San14]). One can define a Riemannian metric on the Fuchsian space \(\mathcal{F}\) by using the Hessian of \(h\). Moreover, this metric is bounded below by \(2\pi\) times the Weil-Petersson metric on \(\mathcal{F}\).

It is natural to ask if this metric is different from the Weil-Petersson metric. It will be interesting to learn more relations between this metric coming from the Hessian of the entropy and the Weil-Petersson metric or pressure metric on the quasi-Fuchsian spaces.

**Acknowledgments**

The author is extremely grateful to his Ph.D advisor Dr. François Ledrappier. This work would have never been possible without François’ support, guidance, patience, and sharing of his insightful ideas. The author also would like to thank Olivier Glorieux and Andy Sanders for useful discussions on their works.

## 2 Preliminaries

### 2.1 Thermodynamic formalism

For general knowledges of the Thermodynamic Formalism, a great reference is the book written by Parry and Pollicott [PP90]. The reparametrization method is discussed in detail in Sambarino’s work [San14].

#### 2.1.1 Flows and reparametrization

Let \(X\) be a compact metric space with a continuous flow \(\phi = \{\phi_t\}_{t \in \mathbb{R}}\) on \(X\) without any fixed point, and \(\mu\) is a \(\phi\)-invariant probability measure on \(X\). Consider a positive continuous function \(F : X \rightarrow \mathbb{R}_{>0}\) and define

\[
\kappa(x, t) := \int_0^t F(\phi_s(x)) ds.
\]

The function \(\kappa\) satisfies the cocycle property \(\kappa(x, t + s) = \kappa(x, t) + \kappa(\phi_t x, s)\) for all \(x, t \in \mathbb{R}\) and \(x \in X\).

Since \(F > 0\) and \(X\) is compact, \(F\) has a positive minimum and \(\kappa(x, \cdot)\) is an increasing homeomorphism of \(\mathbb{R}\). We then have a map \(\alpha : X \times \mathbb{R} \rightarrow \mathbb{R}\) such that

\[
\alpha(x, \kappa(x, t)) = \kappa(x, \alpha(x, t)) = t.
\]

for all \((x, t) \in X \times \mathbb{R}\).

**Definition 2.1.** Let \(F : X \rightarrow \mathbb{R}\) be a positive continuous function. The **reparametrization** of the flow \(\phi\) by \(F\) is the flow \(\phi^F = \{\phi^F_t\}_{t \in \mathbb{R}}\) defined by \(\phi^F_t(x) = \phi_{\alpha(x, t)}(x)\).

**Definition 2.2.** Two continuous functions \(F, G : X \rightarrow \mathbb{R}\) are **Livšic cohomologous** if there exists a continuous function \(V : X \rightarrow \mathbb{R}\) which is \(C^1\) in the flow direction such that

\[
F(x) - G(x) = \frac{\partial}{\partial t} \bigg|_{t=0} V(\phi_t(x)),
\]

and we denote this relation by \(F \sim G\).
2.1.2 Periods and measures

Let \( O \) be the set of closed orbits of \( \phi \). For \( \tau \in O \), let \( l(\tau) \) be the period of \( \tau \) with respect to \( \phi \), then the period of \( \tau \) with respect to the reparametrized flow \( \phi^F \) is

\[
\int_0^{l(\tau)} F(\phi_s(x)) \, ds,
\]

where \( x \) is any point on \( \tau \). Let \( \delta_\tau \) be the Lebesgue measure supported by the orbit \( \tau \), and we denote

\[
\langle \delta_\tau, F \rangle = \int_0^{l(\tau)} F(\phi_s(x)) \, ds.
\]

If \( \mu \) is a \( \phi \)-invariant probability measure on \( X \) and \( F : X \to \mathbb{R} \) is a continuous function, and let \( \phi^F \) be the reparametrization of \( \phi \) by \( F \). We define \( \widehat{F} \cdot \mu \) to be the probability measure: for any continuous function \( G \) on \( X \)

\[
\widehat{F} \cdot \mu(G) = \frac{\int_X G \cdot F \, d\mu}{\int_X F \, d\mu}.
\]

Then \( \widehat{F} \cdot \mu \) is a \( \phi^F \)-invariant probability measure.

2.1.3 Entropy, pressure and equilibrium states

We denote by \( h_\phi(\mu) \) the measure theoretic entropy of \( \phi \) with respect to a \( \phi \)-invariant probability measure \( \mu \) (cf. [PP90] for a precise definition). Let \( \mathcal{M}^\phi \) denote the set of \( \phi \)-invariant probability measures, and \( C(X) \) denote the set of continuous functions on \( X \). The pressure of a function \( F : X \to \mathbb{R} \) is defined as

\[
P_\phi(F) := \sup_{m \in \mathcal{M}^\phi} \left( h_\phi(m) + \int_X F \, dm \right).
\]

We define the topological entropy of the flow \( \phi \) by

\[
h_\phi = P_\phi(0).
\]

If there is no ambiguity on which flow is referred, for example \( \phi \), then we might drop the subscript \( \phi \) and use \( h \) to denote the topological entropy, and \( h(\mu) \) to denote the measure theoretic entropy of \( \phi \) with respect to \( \mu \).

For a continuous function \( F \), if there exists a measure \( m \in \mathcal{M}^\phi \) on \( X \) such that

\[
P_\phi(F) = h_\phi(m) + \int_X F \, dm,
\]

then \( m \) is called an equilibrium state of \( F \), and denoted it by \( m = \bar{m}_F \). An equilibrium state of the function \( F \equiv 0 \) is called a measure of maximum entropy.

Remark 2.1. From the definition of the pressure, we list two immediate properties:

1. \( h_\phi = \sup_{m \in \mathcal{M}^\phi} h(m) \).

2. \( P_\phi \) is monotone, in the sense that if \( F \geq G \) then \( P_\phi(F) \geq P_\phi(G) \).

The following Abramov formula relates the measure theoretic entropies of the flow \( \phi \) and its reparametrization \( \phi^F \).

Theorem (Abramov formula, [Abr59]). Suppose \( \phi \) is a continuous flow on \( X \) and \( \phi^F \) is the reparametrization of
by a positive continuous function $F$, then for all $\mu \in \mathcal{M}^\phi$

$$h_{\phi^F}(\hat{F};\mu) = \frac{h_{\phi}(\mu)}{\int_X Fd\mu}.$$ 

The following Bowen’s formula links the topological entropy of the reparametrized flow $\phi^F$ and the reparametrization function $F$.

**Theorem 2.1** (Bowen’s formula, Sambarino [Sam14]). If $\phi$ is a continuous flow on a compact metric space $X$ and $F : X \to \mathbb{R}$ is a positive continuous function, then

$$P_{\phi}(-hF) = 0$$

if and only if $h = h_{\phi^F}$. Moreover, if $h = h_{\phi^F}$ and $m$ is an equilibrium state of $-hF$, then $Fm$ is a measure of maximal entropy of the reparametrized flow $\phi^F$.

### 2.1.4 Anosov flow

A $C^{1+\alpha}$ flow $\phi_t : X \to X$ on a compact manifold $X$ is called Anosov if there is a continuous splitting of the unit tangent bundle $T^1X = E^0 \oplus E^s \oplus E^u$, where $E^0$ is the one-dimensional bundle tangent to the flow direction, and there exists $C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$ for $t \geq 0$. We say that the flow is transitive if there is a dense orbit.

**Example.** Let $M$ be a compact Riemannian manifold with negative sectional curvature and $\phi_t : T^1M \to T^1M$ is the geodesic flow on the unit tangent bundle of $M$. Then $\phi_t : T^1M \to T^1M$ is a transitive Anosov flow.

Recall that a function $F : X \to \mathbb{R}$ is called $\alpha$–Hölder continuous if there exists $C > 0$ and $\alpha \in (0, 1]$ such that for all $x, y \in X$ we have $|F(x) - F(y)| \leq C \cdot d_X(x, y)^\alpha$. In most cases, we will abbreviate $\alpha$–Hölder continuous to Hölder continuous.

If $\phi_t$ is a transitive Anosov flow on a compact manifold $X$, we know more about the pressure and equilibrium states.

**Theorem 2.2** (Bowen-Ruelle [BR75]). If $\phi_t$ is a transitive Anosov flow on a compact manifold $X$, then for each $F : X \to \mathbb{R}$ Hölder continuous function, there exists a unique equilibrium state $m_F$ of $F$, which is also known as the Gibbs measure of $F$. Moreover if $F$ and $G$ are Hölder continuous functions such that $m_F = m_G$, then $F - G$ is Livšic cohomologous to a constant.

**Remark.** Because the equilibrium state $m_F$ of $F$ is unique, we know that $m_F$ is ergodic. i.e., the Gibbs measure of $F$ is ergodic.

Recall that $O$ is the set of period orbits of $\phi$. For a continuous function $F : X \to \mathbb{R}_{\geq 0}$ and $T \in \mathbb{R}$, we define

$$R_T(F) = \{\tau \in O : \langle \delta_{\tau}, F \rangle \leq T\}.$$ 

**Proposition 2.1** (Bowen [Bow72]). The topological entropy of a transitive Anosov flow $\phi$ is finite and positive. Moreover,

$$h_\phi = \lim_{T \to \infty} \frac{1}{T} \log \# \{\tau \in O : l(\tau) \leq T\}.$$ 

If $F : X \to \mathbb{R}$ is a positive Hölder continuous function, then

$$h_F := h_{\phi^F} = \lim_{T \to \infty} \log \# R_T(F),$$
is finite and positive.

**Theorem 2.3** (Equidistribution, Bowen [Bow72], Parry-Pollicott [PP90]). Suppose \( \phi \) is a transitive Anosov flow on a compact manifold \( X \). Then there exists a unique probability measure of maximum entropy \( \mu_\phi \). Moreover, for all continuous function \( G \) on \( X \), we know

\[
    \int_X G d\mu_\phi = \mu_\phi(G) = \lim_{T \to \infty} \frac{1}{\# R_T(1)} \sum_{\tau \in R_T(1)} \langle \delta_\tau, G \rangle = \lim_{T \to \infty} \sum_{\tau \in R_T(1)} \langle \delta_\tau, G \rangle.
\]

The probability measure \( \mu_\phi \) is called the ** Bowen-Margulis measure** of the flow \( \phi \).

### 2.1.5 Livšic type theorems

Recall that two continuous functions \( F, G : X \to \mathbb{R} \) are **Livšic cohomologous** if there exists a continuous function \( V : X \to \mathbb{R} \) which is \( C^1 \) in the flow direction such that

\[
    F(x) - G(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\phi_t(x)).
\]

**Remark 2.2.** By definition, the following properties are immediate:

1. If \( F \) and \( G \) are Livšic cohomologous then they have the same integral over any \( \phi \)-invariant measure.

2. The pressure \( P_\phi(F) \) only depends on the Livšic cohomology class of \( F \).

3. \( R_T(F) \) only depends on the Livšic cohomology class of \( F \).

In the rest of this note, we will only discuss the Livšic cohomology of Hölder continuous functions on \( X \). Specifically, two Hölder continuous \( F, G : X \to \mathbb{R} \) are called Livšic cohomologous if there exists a Hölder continuous \( V : X \to \mathbb{R} \) which is \( C^1 \) in the flow direction such that

\[
    F(x) - G(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\phi_t(x)).
\]

**Theorem 2.4** (Livšic Theorem [Liv07]). Let \( \phi_t : X \to X \) be a transitive Anosov flow. Let \( F : X \to \mathbb{R} \) be a Hölder continuous function such that \( \langle \delta_\tau, F \rangle = \int_0^{l(\tau)} F \circ \phi_t(x) dt = 0 \) for each \( \phi \)-closed orbit \( \tau \) for any \( x_\tau \in \tau \), then \( F \) is cohomologous to 0.

**Theorem 2.5** (Positive Livšic Theorem [Sam14]). Let \( \phi_t : X \to X \) be a transitive Anosov flow. Let \( F : X \to \mathbb{R} \) be a Hölder continuous function such that \( \langle \delta_\tau, F \rangle > 0 \) for each \( \phi \)-closed orbit \( \tau \), then \( F \) is cohomologous to a Hölder continuous function \( G(x) \) such that \( G(x) > 0 \), \( \forall x \in X \).

For the geodesic flow on the unit tangent bundle of negative curved manifolds, we have a similar theorem as the following:

**Theorem 2.6** (Nonnegative Livšic Theorem [LT05]). Suppose \( \Sigma \) is a compact Riemannian manifold with negative sectional curvature. Let \( \phi_t : T^1 \Sigma \to T^1 \Sigma \) be the geodesic flow on \( T^1 \Sigma \). Let \( F : T^1 \Sigma \to \mathbb{R} \) be a Hölder continuous function such that \( \langle \delta_\tau, F \rangle \geq 0 \) for each \( \phi \)-closed orbit \( \tau \), then \( F \) is cohomologous to a Hölder continuous function \( G(x) \) such that \( G(x) \geq 0 \), \( \forall x \in T^1 \Sigma \).

Here we recall the Anosov closing lemma [Ano67]:

\[
    \bigcup_{t \in \mathbb{R}} \overline{\{ \phi_t(x) \mid x \in X \}} = X.
\]
Theorem 2.7 (Anosov Closing Lemma). Let $\phi_t : X \to X$ be a transitive Anosov flow. Then for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, d_X) > 0$ such that if for $v \in X$, $t > 0$ satisfying

$$d_X(v, \phi_t v) < \delta,$$

then there exists a closed orbit $\tau_w = \{\phi_s w\}_{s=0}^{t'}$ of period $t'$, where $|t - t'| < \varepsilon$, such that

$$d_X(\phi_s v, \phi_s w) < \varepsilon \text{ for } 0 \leq s \leq t.$$

2.1.6 Variance and derivatives of the pressure

In this subsection we shall recall the definition and basic properties of the variance. Let $\phi_t : X \to X$ be a transitive Anosov flow on a compact metric space $X$, and $C^\alpha(X)$ be the set of $\alpha$-Hölder continuous function on $X$.

Definition 2.3. Suppose $F \in C^\alpha(X)$ and $m_F$ is the equilibrium state of $F$. For any $G \in C^\alpha(X)$ we define the variance of $G$ with respect to $m_F$ by

$$\text{Var}(G, m_F) := \lim_{T \to \infty} \frac{1}{T} \int_X \left( \int_0^T G(\phi_t(x)) dt - T \int G dm_F \right)^2 dm_F(x).$$

The following properties give us some handy formulas of the derivatives of the pressure.

Proposition 2.2 (Parry-Pollicott, Prop 4.10, 4.11 [PP90]). Suppose that $\phi_t : X \to X$ is a transitive Anosov flow on a compact metric space $X$, and $F,G \in C^\alpha(X)$. If $m_F$ is the equilibrium state of $F$, then

1. The function $t \mapsto P_{\phi}(F + tG)$ is analytic,

2. The first derivative is given by

$$\left. \frac{dP_{\phi}(F + tG)}{dt} \right|_{t=0} = \int_X G dm_F,$$

3. If $\int G dm_F = 0$, then the second derivative could be formulated as

$$\left. \frac{d^2 P_{\phi}(F + tG)}{dt^2} \right|_{t=0} = \text{Var}(G, m_F),$$

4. If $\text{Var}(G, m_F) = 0$, then $G$ is Livšic cohomologous to zero.

2.1.7 Pressure metric

Here we keep the same setting as in previous subsection that $\phi_t : X \to X$ is a transitive Anosov flow on a compact metric space $X$. We consider the space $\mathcal{P}(X)$ of Livšic cohomology classes of pressure zero Hölder continuous functions on $X$, i.e.

$$\mathcal{P}(X) := \{ F; F \in C^\alpha(X) \text{ for some } \alpha \text{ and } P_{\phi}(F) = 0 \} / \sim.$$

The tangent space of $\mathcal{P}(X)$ at $F$ is

$$T_F \mathcal{P}(X) = \ker ((DP_{\phi})(F)) = \{ G; G \in C^\alpha(X) \text{ for some } \alpha \text{ and } \int G dm_F = 0 \} / \sim,$$

where $m_F$ is the equilibrium state of $F$. 
Since the variance vanishes only on functions \( G \) that are cohomologous to 0,\
\[
\|G\|_p^2 := \frac{\text{Var}(G, m_F)}{-\int F dm_F}
\]
is positive definite on equivalence classes of functions cohomologous to 0, and thus a metric on \( T_F \mathcal{P}(X) \). We call this metric \( \| \cdot \|_p \) the pressure metric on \( T_F \mathcal{P}(X) \).

**Proposition 2.3.** If \( \{c_t\}_{t \in (-1, 1)} \) is a smooth one parameter family contained in \( \mathcal{P}(X) \), then
\[
\|\dot{c}_0\|_p^2 = \int \dddot{c}_0 dm_{c_0} - \int c_0 dm_{c_0}
\]
where \( \dot{c}_0 = \frac{d}{dt}c_t \bigg|_{t=0} \) and \( \dddot{c}_0 = \frac{d^2}{dt^2}c_t \bigg|_{t=0} \).

**Proof.** This follows the direction computation of the (Gâteaux) second derivative of \( P_\phi(c_t) \):
\[
\frac{d^2}{dt^2} P_\phi(c_t) \bigg|_{t=0} = (D^2 P_\phi)(c_0)(\dot{c}_0, \dot{c}_0) + (DP_\phi)(c_0)(\dddot{c}_0)
\]
\[= \text{Var}(\dot{c}_0, m_{c_0}) + \int \dddot{c}_0 dm_{c_0} .
\]
Since \( P_\phi(c_t) = 0 \), we have
\[
\|\dot{c}_0\|_p^2 = \frac{\text{Var}(\dot{c}_0, m_{c_0})}{\int c_0 dm_{c_0}} = \int \dddot{c}_0 dm_{c_0} - \int c_0 dm_{c_0}.
\]

\[\square\]

### 2.2 Negatively curved manifolds and the group of isometries

In this subsection, we survey several facts of \( \delta \)-hyperbolic spaces and their group of isometries. A good reference is the book [GdlH90] edited by Ghys and de la Harpe.

#### 2.2.1 \( \delta \)-hyperbolic spaces

A metric space \((X, d)\) is said to be geodesic if any two points \(x, y \in X\) can be joined be a geodesic segment \([x, y]\) that is a naturally parametrized path from \(x\) to \(y\) whose length is equal to \(d(x, y)\), and is called proper if all closed balls are compact.

**Definition 2.4.** A geodesic metric space \((X, d)\) is called \( \delta \)-hyperbolic (where \( \delta \geq 0 \) is some real number) if for any geodesic triangle in \(X\) each side of the triangle is contained in the \( \delta \)-neighborhood of the union of two other sides. A metric space \((X, d)\) is called hyperbolic if it is \( \delta \)-hyperbolic for some \( \delta \geq 0 \).

In the following, we lists two types of hyperbolic spaces appearing in this note.

**Example.**

1. **Pinched Hadamard manifold** \((\tilde{M}, d_\tilde{g})\): a complete and simply connected Riemannian manifold \((\tilde{M}, \tilde{g})\) whose sectional curvature is bounded by two negative numbers. The metric on \(\tilde{M}\) is the distance function \(d_\tilde{g}\) induced by the Riemannian metric \(\tilde{g}\).

2. The **Cayley graph** \(C(G, S)\) and its word metric \(w\): given a finitely generated group \(G\) and a finite generating set \(S\) of \(G\), \(C(G, S)\) is a graph whose vertices are elements of \(G\). Two vertices \(g, h \in G\) are connected by an edge if and only if there is a generator \(s \in S\) such that \(h = gs\). The word metric \(w\) on \(C(G, S)\) is defined by
assuming that each edge has unit length, and \( w(g, h) \) is the minimum of the length of all paths connecting \( g \) and \( h \).

**Remark 2.3.** A group \( G \) is called *hyperbolic* if for one (and for all) finite generating set the Cayley graph is hyperbolic. For example, finitely generated free groups and surface groups for surfaces with genus > 1.

We say that two geodesics ray \( \tau_1 : [0, \infty) \to X \) and \( \tau_2 : [0, \infty) \to X \) are *equivalent* and write \( \tau_1 \sim \tau_2 \) if there is a \( K > 0 \) such that for all \( t > 0 \)

\[
d(\tau_1(t), \tau_2(t)) < K.
\]

It is easy to see that \( \sim \) is indeed an equivalence relation on the set of geodesic rays. We then define the *geometric boundary* \( \partial_\infty X \) of \( X \) by

\[
\partial_\infty X := \{ [\tau] : \tau \text{ is a geodesic ray in } X \}.
\]

Moreover, we know that when \( X \) is proper, \( \partial_\infty X \) is metrizable. More precisely, the following *visual metric*, given by Gromov, defines a metric on \( \partial_\infty X \).

**Definition 2.5.** Let \((X, d)\) be a \( \delta \)-hyperbolic proper metric space. Let \( a > 1 \) and let \( o \in X \) be a basepoint. We say that a metric \( d_a \) on \( \partial_\infty X \) is a *visual metric* with respect to the basepoint \( o \) and the visual parameter \( a \) if there exists \( C > 0 \) such that for any two distinct point \( \xi, \eta \in \partial_\infty X \), and for any biinfinite geodesic \( \tau \) connecting \( \xi \) to \( \eta \) in \( X \) we have

\[
\frac{1}{C}a^{-d(o, \tau)} \leq d_a(\xi, \eta) \leq C a^{-d(o, \tau)}.
\]

**Theorem** (Gromov, cf. Theorem 1.5.2 [Bou95]). There is \( a_0 > 1 \) such that for any basepoint \( o \in X \) and any \( a \in (1, a_0) \) the boundary \( \partial_\infty X \) admits a visual metric \( d_a \) with respect to \( o \).

**Remark 2.4 (cf. Ch.11 [CDP90], Remark 1.5.3 [Bou95]).**

1. For a pinched Hadamard manifold \( M \) whose sectional curvature is not larger than \(-\alpha^2\), then the boundary \( \partial_\infty M \) admits a visual metric \( d_a \) for all \( a \in (1, e^b] \).

2. Let \( d_a \) and \( d_{a'} \) be two different visual metrics with respect to a fixed basepoint \( o \in X \) and different visual parameters \( a \) and \( a' \), then \( d_a \) and \( d_{a'} \) are Hölder equivalent. i.e., there exists \( C \geq 1 \) and \( \alpha = \frac{\log a'}{\log a} \) such that for \( \xi, \eta \in \partial_\infty X \)

\[
\frac{1}{C} \cdot (d_a(\xi, \eta))^{\alpha} \leq d_{a'}(\xi, \eta) \leq C \cdot (d_a(\xi, \eta))^{\alpha}.
\]

3. Let \( d_a \) and \( d'_a \) be two different visual metrics with respect to a fixed visual parameter \( a \) and different basepoints \( o, o' \in X \), then the metric \( d_a \) and \( d'_a \) are Lipchitz. i.e., there exists \( C \geq 1 \) such that for all \( \xi, \eta \in \partial_\infty X \)

\[
\frac{1}{C} \cdot d_a(\xi, \eta) \leq d'_a(\xi, \eta) \leq C \cdot d_a(\xi, \eta).
\]

### 2.2.2 Quasi-isometry

**Definition 2.6.** A function \( q : X \to Y \) from a metric space \((X, d_X)\) to a metric space \((Y, d_Y)\) is called a \((C, L)\)-*quasi-isometry embedding* if there is \( C, L > 0 \) such that:

For any \( x, x' \in X \), we have

\[
\frac{1}{C} d_X(x, x') - L \leq d_Y(q(x), q(x')) \leq C \cdot d_X(x, x') + L.
\]

If, in addition, there exists an *approximate inverse* map \( \bar{q} : Y \to X \) that is a \((C, L)\)-quasi-isometric embedding such that for all \( x \in X \) and \( y \in Y \)

\[
d_X(\bar{q} q(x), x) \leq L, \quad d_Y(q \bar{q}(y), y) \leq L,
\]
then we call \( q \) a \((C, L)\)-quasi-isometry. \((X, d_X)\) and \((Y, d_Y)\) are called quasi-isometric.

In most cases, the quasi-isometry constants \( C \) and \( L \) do not matter, so we shall use the words quasi-isometry and quasi-isometry embedding without specifying constants.

**Theorem 2.8** (Bourdon, Theorem 1.6.4 [Bou95]). Let \((X, d_X)\) and \((Y, d_Y)\) be hyperbolic spaces. Suppose the boundaries equip with visual metrics. Then

1. Any quasi-isometry embedding \( q : X \to X' \) extends to a bi-Hölder embedding \( q : \partial_\infty X \to \partial_\infty Y \) with respect to the corresponding visual metrics.

2. Any quasi-isometry \( q : X \to X' \) extends to a bi-Hölder homeomorphism \( q : \partial_\infty X \to \partial_\infty Y \) with respect to the corresponding visual metrics.

**Definition 2.7.** A \((C, L)\)-quasi-geodesic is a \((C, L)\)-quasi-isometry embedding \( q : \mathbb{R} \to X \).

**Theorem 2.9** (Morse Lemma, cf. Ch.5, Theorem 6 [GdlH90]). Suppose \( X \) and \( Y \) are hyperbolic spaces, and \( q : X \to Y \) is a \((C, L)\)-quasi-isometry. Then every geodesic \( \gamma \subset X \) its image \( q(\gamma) \) is a quasi-geodesic on \( Y \) and is within a bounded distance \( R \) from a geodesic on \( Y \). Moreover, this constant \( R \) is only depending on \( X, Y \) and the quasi-isometry constants \( C \) and \( L \).

**Remark 2.5.** When the space \( Y \) is a pinched Hadamard manifold, we have a stronger result of the above theorem. Specifically, every geodesic \( \gamma \subset X \) its image \( q(\gamma) \) is a quasi-geodesic on \( Y \) and is within a bounded distance \( R \) from a unique geodesic on \( Y \).

Let \( X \) be a hyperbolic space, we denote its group of isometries by \( \text{Isom}(X) \). The following lemma connects some subgroups of \( \text{Isom}(X) \) and the hyperbolic space \( X \).

**Theorem 2.10** (Švarc-Milnor lemma, cf. Lemma 3.37 [Kap09]). Let \( X \) be a proper geodesic metric space. Let \( G \) be a subgroup of \( \text{Isom}(X) \) acting properly discontinuously and cocompactly on \( X \). Pick a point \( o \in X \). Then the group \( G \) is finitely generated; for some choice of finitely generating set \( S \) of \( G \), the map \( q : G \to X \), given by \( q(\gamma) = \gamma(o) \), is a quasi-isometry. Here \( G \) is given the word metric induced from \( C(G, S) \).

### 2.2.3 Negatively curved manifolds and the group of isometries

Let \((X, g)\) be a negatively curved compact Riemannian manifold. The universal covering \((\tilde{X}, \tilde{g})\) of \((X, g)\) is a pinched Hadamard manifold, and \(\pi_1 X\) is finitely generated and acting canonically on \(\tilde{X}\). Let \(\Gamma\) denote the group of deck transformations of the covering \(\tilde{X}\). We know that \(\Gamma \subset \text{Isom}(\tilde{X})\), \(\Gamma \cong \pi_1 X\), and \(X\) is isometric to \(\Gamma \backslash \tilde{X}\). More precisely, using generators there exists a nature isomorphism \(i_X : \pi_1 X \to \Gamma\), given by \(\gamma_X := i_X(\gamma), \forall \gamma \in \pi_1 X\). Thus, using this isomorphism we can define a \(\pi_1 X\)-action on \(\tilde{X}\) by \(\gamma \cdot x = (\gamma_X)(x)\). It’s clear that this \(\pi_1 X\)-action is nothing different from the \(\Gamma\)-action on \(\tilde{X}\).

Because \((X, g)\) is negatively curved, every \(\gamma \in \Gamma\) corresponds to a unique geodesic \(\tau_\gamma^X\) on \(X\). Besides, each conjugacy class \([\gamma]\) \(\in [\Gamma]\) corresponds to a unique closed geodesic geodesic \(\tau_\gamma^X\) on \(X\) and vice versa. Moreover, the length of the closed geodesic \(\tau_\gamma^X\) is exactly the translation distance of \(\gamma \in \pi_1 X\), i.e. \(l_g(\tau_\gamma^X) = l_g(\gamma) := d_g(x, \gamma \cdot x) = d_g(x, \gamma_X(x))\).

**Definition/Theorem** (Margulis [Mar70]). Let \((X, g)\) be a compact negatively curved Riemannian manifold and \(\Gamma\) be the group of deck transformations of \(\tilde{X}\), then the topological entropy \(h(X)\) geodesic flow on \(T^1 X\) is given by

\[
h(X) = \lim_{T \to \infty} \frac{1}{T} \log \# \{[\gamma] \in [\pi_1 X]; l_g(\gamma) \leq T\}.
\]

Now let’s consider a compact 3–manifold \(M\) equipped with a hyperbolic metric \(h\). Then there exists a discrete and faithful representation \(\rho : \pi_1 M \to \text{Isom}(\mathbb{H}^3)\) such that \(M \cong \rho(\pi_1 M) \backslash \mathbb{H}^3\) where \((\mathbb{H}^3, \tilde{h})\) is the universal covering of \((M, h)\). For the sake of brevity, in what follows we will denote the lifted metric of \(\tilde{h}\) on \(\mathbb{H}^3\) by \(h\).
Definition 2.8. Let $\Gamma$ be a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$, the limit set $\Lambda(\Gamma)$ is the set of limit points $\Gamma x$ for any $x \in \mathbb{H}^3$.

Definition 2.9. The critical exponent $\delta_\Gamma$ is defined as following:

$$\delta_\Gamma := \inf \{ s; \sum_{\gamma \in \Gamma} e^{-sd_h(x,\gamma x)} < \infty \},$$

for any point $x \in \mathbb{H}^3$ and $d_h$ is the hyperbolic distance on $\mathbb{H}^3$.

Definition 2.10. A discrete subgroup $\Gamma$ of $\text{Isom}(\mathbb{H}^3)$ is called convex cocompact if $\Gamma$ acts cocompactly on the convex hull of the limit set of $\Gamma$, i.e., $\Gamma \setminus \text{Conv}(\Lambda(\Gamma))$ is compact.

Theorem 2.11 (Sullivan [Sul79]). Suppose $\Gamma$ is a non-elementary, convex cocompact, and discrete subgroup of $\text{Isom}(\mathbb{H}^3)$, then

$$\delta_\Gamma = \lim_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\Gamma] ; l_h(\gamma) \leq T \},$$

where $l_h(\gamma) = d_h(o,\gamma o)$, $o$ is the origin of $\mathbb{H}^3$.

2.3 Hölder cocycles

Let $(X, g)$ be a compact negatively curved manifold, $\tilde{X}$ be its universal covering, and $\Gamma$ be the group of deck transformations of the covering $\tilde{X}$. Recall that the $\pi_1 X$-action on $\tilde{X}$ is defined by $\gamma \cdot x = i_X(\gamma)(x)$, where $i$ is the isomorphism $i_X : \pi_1 \Sigma \to \Gamma$.

Definition 2.11. A Hölder cocycle is a function $c : \pi_1 X \times \partial_\infty \tilde{X} \to \mathbb{R}$ such that

$$c(\gamma_0 \gamma_1, x) = c(\gamma_0, \gamma_1 \cdot x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \pi_1 X$ and $x \in \partial_\infty \tilde{X}$, and $c(\gamma, \cdot)$ is Hölder continuous for every $\gamma \in \pi_1 X$.

Given a Hölder cocycle $c$ we define the periods of $c$ to be the number

$$l_c(\gamma) := c(\gamma, \gamma_X^+)$$

where $\gamma_X^+$ is the attracting fixed point of $\gamma \in \pi_1 X \setminus \{ e \}$ on $\partial_\infty \tilde{X}$.

Remark 2.6. The cocycle property implies that the period of an element $\gamma$ only depends on its conjugacy class $[\gamma] \in [\pi_1 X]$.

Two cocycles $c$ and $c'$ are said to be cohomologous if there exists a Hölder continuous function $U : \partial_\infty \tilde{X} \to \mathbb{R}$ such that for all $\gamma \in \pi_1 X$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma \cdot x) - U(x).$$

One easily deduces from the definition that the set of periods of a cocycle is a cohomological invariant.

Definition 2.12. The exponential growth rate for a Hölder cocycle $c$ is defined as:

$$h_c := \limsup_{T \to \infty} \frac{1}{T} \log \# \{ [\gamma] \in [\pi_1 X] : l_c(\gamma) \leq T \}.$$
2.3.1 From cocycle cohomology to Livšic cohomology

Theorem 2.12 (Ledrappier [Led95]). For each Hölder cocycle $c : \pi_1 X \times \partial_\infty \tilde{X} \to \mathbb{R}$, there exists a Hölder continuous function $F_c : T^1 X \to \mathbb{R}$, such that for all $\gamma \in \pi_1 X - \{e\}$, one has

$$l_c(\gamma) = \int_{[\gamma]} F_c.$$  

The map $c \mapsto F_c$ induces a bijection between the set of cohomology classes of $\mathbb{R}$–valued Hölder cocycles, and the set of Livšic cohomology classes of Hölder continuous functions from $T^1 X$ to $\mathbb{R}$.

Using the above Theorem 2.12, Sambarino give the following reparametrization theorem in [Sam14].

Theorem 2.13 (Sambarino [Sam14]). Let $c$ be a Hölder cocycle with positive periods such that $h_c$ is finite. Then the action of $\Gamma$ on $(\partial_\infty \tilde{X} \times \partial_\infty \tilde{X} - \text{diagonal}) \times \mathbb{R}$ via $c$, that is,

$$\gamma(x, y, s) = (\gamma x, \gamma y, s - c(\gamma, y)),$$

is proper and compact. Moreover, the flow $\psi$ on $\pi_1 X \setminus (\partial_\infty \tilde{X} \times \partial_\infty \tilde{X} - \text{diagonal}) \times \mathbb{R}$, defined by

$$\psi_t \Gamma(x, y, s) = \Gamma(x, y, s - t),$$

is conjugated to $\phi^{F_c} : T^1 X \to T^1 X$ which is the reparametrization of the geodesic flow $\phi$ on $T^1 X$ by a Hölder continuous function $F_c$ provided $l_c(\gamma) = \int_{[\gamma]} F_c$ for all $[\gamma] \in [\pi_1 X]$. Furthermore, the conjugating map is also Hölder continuous, and the topological entropy of $\psi$ is $h_c$.

2.4 Immersed surfaces in hyperbolic 3–manifolds

In this subsection, we review some well-known facts about immersed surfaces in hyperbolic 3–manifolds. Let $\Sigma$ be a differentiable 2-manifold and $M$ be a 3-manifold, we say a differentiable mapping $f : \Sigma \to M$ is an immersion if $df_p : T_p \Sigma \to T_{f(p)} M$ is injective for all $p \in S$. If, in addition, $f$ is a homeomorphism onto $f(\Sigma) \subset M$, where $f(\Sigma)$ has the subspace topology induced from $M$, we say that $f$ is an embedding. Moreover, if the induce homomorphism $f_* : \pi_1 \Sigma \to \pi_1 M$ is injective, then we call $f$ is a $\pi_1$–injective.

Throughout, we consider that $M$ is a hyperbolic 3–manifold equipped with a hyperbolic metric $h$ and $\Sigma$ is a compact, 2–dimension manifolds with negative Euler characteristic. Before moving further, we recall several terminologies in differential geometry. Given an immersion $f : \Sigma \to M$, let $g = f^* g$ be the induced Riemannian metric on $\Sigma$, $\nabla$ be the Levi-Civita connection on $(M, h)$, $N$ be the unit outward normal vector field to the surface $f(\Sigma) \subset M$, and $\partial_1$ and $\partial_2$ be the coordinate fields of $T \Sigma$.

The second fundamental form $B : T \Sigma \times T \Sigma \to \mathbb{R}$ of $f$ is the symmetric 2-tensor on $\Sigma$ defined by, locally,

$$B(\partial_i, \partial_j) = \langle \partial_i, -\nabla_{\partial_j} N \rangle_h,$$

where $\langle, \rangle_h$ is the hyperbolic metric $h$ on $M$.

The shape operator $S_g : T \Sigma \to T \Sigma$ is the symmetric self-adjoint endomorphism defined by raising one index of the second fundamental form $B$ with respect to the metric $g$.

The mean curvature $H$ of the immersion $f : \Sigma \to M$ (or, of the immersed surface $(\Sigma, g)$) is the trace of the shape operator. We call an immersion $f : \Sigma \to M$ minimal if the mean curvature $H$ vanishes identically.

Moreover, we can relate the induced Riemannian metric $g$ and shape operator $S_g$ by the famous Gauss-Codazzi
equations:

\[ K_g = -1 + \det S_g, \quad \text{(Gauss eq.)} \]  
\[ \nabla_d f(X)(S_g(Y)) - \nabla_d f(Y)S_g(X) = S_g([X,Y]). \quad \text{(Codazzi eq.)} \]

where \( X, Y \in T\Sigma \) and \([, ,]\) is the Lie bracket on \( T\Sigma \).

**Remark 2.7.**

1. We call real eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( S_g \) the *principal curvatures* of the immersion \( f : \Sigma \to M \).

2. The shape operator \( S_g \) and the second fundamental form \( B \) are linked by

\[ B(X,Y) = \langle X, S_g(Y) \rangle_g, \forall X, Y \in T\Sigma. \]

3. If \( f \) is a minimal immersion the Gauss-Codazzi equations could be expressed in terms of \( B \) by

\[ K_g = -1 - \frac{1}{2} \|B\|_g^2, \quad (\nabla_{\partial_i}B)_{jk} = (\nabla_{\partial_j}B)_{ik}, \]

where \( \| \cdot \|_g \) is the tensor norm w.r.t. metric \( g \) and \( \partial_1 \) and \( \partial_2 \) are coordinate fields of \( TM \). Moreover, in this case the Gauss equation implies \( K_g \leq -1 \). i.e., \((\Sigma, g)\) is a negatively curved surface.

### 2.4.1 Minimal hyperbolic germs

In the context of minimal hyperbolic germs, the surface \( \Sigma \) is always assumed to be closed. Let \((g, B)\) be a pair consisting of a Riemannian metric \( g \) and a symmetric 2-tensor \( B \) on \( \Sigma \).

**Definition 2.13.** A pair \((g, B)\) is called a *minimal hyperbolic germ* if it satisfies the following equations:

\[
\begin{align*}
K_g &= -1 - \frac{1}{2} \|B\|_g^2, \\
(\nabla\partial_i B)_{jk} &= (\nabla\partial_j B)_{ik}, \\
H &= 0.
\end{align*}
\]

Recall that \( \text{Diff}_0(\Sigma) \) is the space of orientation preserving diffeomorphisms of \( \Sigma \) isotopic to the identity. There is a natural \( \text{Diff}_0(\Sigma) \) action (i.e. by pullback) on the space of minimal hyperbolic germs, and we are mostly interested in the following quotient space.

**Definition 2.14.** The space \( \mathcal{H} \) of minimal hyperbolic germs is the quotient:

\[ \mathcal{H} = \{ \text{minimal hyperbolic germs} \}/\text{Diff}_0(\Sigma). \]

Taubes shows that \( \mathcal{H} \) is a smooth manifold of dimension \( 12g - 12 \) where \( g \) is the genus of \( \Sigma \). The fundamental theorem of surface theory ensures that each \((g, B) \in \mathcal{H}\) can be integrated to an immersed minimal surface in a hyperbolic 3-manifold with the Riemannian metric \( g \) and the second fundamental form \( B \).

Moreover, \( \mathcal{H} \) is closely related with the Teichmüller space. To be more precise, we first recall facts in the Teichmüller theory. The *Teichmüller space* \( \mathcal{T} \) of \( \Sigma \) is the space of isotopic classes of complex structures on \( \Sigma \). Alternatively, by the uniformization theorem, \( \mathcal{T} \) can also be identified with the space of isotopic classes of conformal structures on \( \Sigma \), i.e. conformal classes of Riemannian metrics with curvature \(-1\). It is clear that we can identify \( \mathcal{T} \) with a subspace \( \mathcal{F} \) of \( \mathcal{H} \).
Namely, the Fuchsian space:
\[ F = \{(m, 0) \in \mathcal{H}; m \text{ is a Riemannian metric of constant curvature } -1\}. \]

Let \([g]\) be the conformal class of a Riemannian metric \(g\) on \(S\) and \(X = (S, [g])\) be the Riemann surface associated with \(g\). It is well-known that \(T^*_X T\) the fiber of the holomorphic cotangent bundle over \(X\) can be identified with \(Q(X)\) the space of holomorphic quadratic differentials on \(X\).

The following theorem of Hopf [Hop51] helps us see the relation between \(\mathcal{H}\) and \(Q(X)\).

**Theorem 2.14** (Hopf [Hop51]). If \((g, B) \in \mathcal{H}\), then \(B\) is the real part of a (unique) holomorphic quadratic differential \(\alpha \in Q(X)\). More precisely, if \((x_1, x_2) = x_1 + ix_2 = z\) is a local isothermal coordinate of \(X\) and \(B = B_{11}dx_1^2 + B_{22}dx_2^2 + 2B_{12}dx_1dx_2\), then

\[ \alpha(g, B) = (B_{11} - iB_{12})(x_1, x_2)dz. \]

**Remark.** In fact \(B_{11} = -B_{22}\) because \((\Sigma, g)\) is minimal, and it is not hard to see \(||\alpha||_g = ||B||_g||.\)

Moreover, the space \(\mathcal{H}\) admits a smooth map to \(T^*\mathcal{T}\) given by

\[ \Psi : \mathcal{H} \rightarrow T^*\mathcal{T} \]

\[ (g, B) \mapsto ([g], \alpha(g, B)). \]

For any two holomorphic quadratic differentials \(\alpha\) and \(\beta\) in \(Q(X)\), the **Weil-Petersson pairing** is given by

\[ \langle \alpha, \beta \rangle_{WP} = \int_{\Sigma} \frac{\alpha \beta}{m}, \]

where \(m\) is the hyperbolic metric on \(\Sigma\) conformal to \(g\). It’s also well-known that this pairing defines a Kähler metric, the **Weil-Petersson metric**, on the Teichmüller space whose geometry has been intensely studied. In the last section, we will discuss several applications of our results related with the Weil-Petersson metric on \(F\).

We now change gear to the Gauss equation. Since every Riemannian metric \(g\) on \(\Sigma\) is conformal to a unique hyperbolic metric \(m\), we can write \(g = e^{2u}m\) where \(e^{2u}\) is the conformal factor. Therefore, we can rewrite the Gauss equation as the following.

**Theorem 2.15** (Gauss equation, Theorem 4.2 [Uhl83]). The Gauss equation may be written, in terms of \(m\),

\[ -1 - \frac{1}{2} \|B\|^2_m = K_g = e^{-2u}(-\Delta_m u - 1), \]

where \(K_g\) is the Gaussian curvature of \((\Sigma, g)\).

From another point of view, using Uhlenbeck’s result in [Uhl83] we can relate the space of minimal hyperbolic germs \(\mathcal{H}\) with the character variety \(\mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))\), where \(\mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))\) is the space of conjugacy classes of representations of \(\pi_1(S)\) into \(\text{PSL}(2, \mathbb{C})\). More precisely, Uhlenbeck [Uhl83] proves that for each data \((g, B) \in \mathcal{H}\) there exists a representation \(\rho : \pi_1(\Sigma) \rightarrow \text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})\) leaving this minimal immersion invariant. i.e., there is a map

\[ \Phi : \mathcal{H} \rightarrow \mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C})). \]
2.4.2 Almost-Fuchsian germs

**Definition 2.15.** The space of almost-Fuchsian germs $\mathcal{AF}$ is defined by

$$\mathcal{AF} = \{(g, B) \in H; \|B\|_{g}^2 < 2\}.$$ 

**Remark 2.8.** $\|B\|_{g}^2 < 2$ is equivalent to that the principal curvatures $\lambda_1$ and $\lambda_2$ are between $-1$ and 1.

This definition is motivated by the observation of Uhlenbeck [Uhl83] that when $(g, B) \in \mathcal{AF}$ there exists a unique quasi-Fuchsian manifold $M$, up to isometry, such that $\Sigma$ is an embedded minimal surface in $M$ with the induced metric $g$ and the second fundamental form $B$. In other words, almost-Fuchsian germs are more than quasi-Fuchsian, but not Fuchsian.

In the following, we discuss a ray in $\mathcal{AF}$, which could be considered as a path in the Teichmüller space $\mathcal{T}$. Specifically, for a hyperbolic metric $m \in \mathcal{F}$, and a holomorphic quadratic differential $\alpha \in Q((\Sigma, [m]))$, we consider the ray

$$r(t) = (g_t, tB) \subset \mathcal{AF},$$

where $g_t$ and $B = \text{Re}(\alpha)$ satisfying $\|tB\|_{g_t}^2 < 2$. Notice that $g_t$ is conformal to the hyperbolic metric $m$, so we can write $g_t = e^{2u_t}m$ where the conformal factor $e^{2u_t}$ is a $C^2$ function on $\Sigma$. By studying the Gauss equation, Uhlenbeck proved $u_t$ is smooth on $t$, hence $r(t)$ is smooth when $t$ is small. We state that result in the below.

**Theorem 2.16 (Uhlenbeck, Theorem 4.4 [Uhl83]).** Consider the maps $F : W^{2,2}(\Sigma) \times [0, \infty) \to L^2(\Sigma)$,

$$F(u, t) = \Delta_m u + 1 - e^{2u} - \frac{1}{2} \|tB\|_{m}^2 e^{-2u},$$

where $W^{2,2}(\Sigma)$ is the classical Sobolev space. Then there exists $\tau_0 > 0$ and a smooth solution curve

$$c : [0, \tau_0] \to W^{2,2}(\Sigma) \times [0, \infty)$$

$$t \mapsto \left(u(t), t\right)$$

such that $c(0) = (0, 0)$ and $F(c(t)) = 0$ for all $t \in [0, \tau_0]$.

3 Proof of the main result

Throughout this section, $\Sigma$ denotes a compact 2–dimensional manifold with negative Euler characteristic. Let $f : \Sigma \to M$ be a $\pi_1$–injective immersion from $\Sigma$ to a hyperbolic 3-manifold $M$ and $\Gamma$ be the copy of $\pi_1 \Sigma$ in $\text{Isom}(\mathbb{H}^3)$ induced by the immersion $f$. More precisely, let $\rho : \pi_1 M \to \text{Isom}(\mathbb{H}^3)$ be the discrete and faithful representation, up to conjugacy, corresponding to $M$, i.e. $M = \rho(\pi_1 M) \setminus \mathbb{H}^3$. Then $\Gamma = \rho(f_\ast(\pi_1 \Sigma))$ where $f_\ast$ is the induced homomorphism of $f : \Sigma \to M$.

The hypotheses throughout here are: $\Gamma$ is a convex cocompact and $(\Sigma, g)$ is negatively curved where $g = f^* h$ and $h$ is the given hyperbolic metric on $M$.

Notice that because $(\Sigma, g)$ is a compact negatively curved surface, its universal covering $(\tilde{\Sigma}, \tilde{g})$ is a pinched Hadamard manifold. Let $\Gamma_\Sigma$ denote the group of deck transformations of the covering $\tilde{\Sigma}$. Then we know $\Gamma_\Sigma \cong \pi_1 \Sigma$ and $\Gamma_\Sigma \subset \text{Isom}(\tilde{\Sigma})$.

**Lemma 3.1.** There exists a quasi-isometry $q : \tilde{\Sigma} \to \text{Conv}(\Lambda(\Gamma))$, where $\text{Conv}(\Lambda(\Gamma))$ is the convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}^3$. Moreover, $q$ extends to a bi-Hölder and $\Gamma$–equivariant map between the boundaries.
Proof. Because \((\bar{\Sigma}, \bar{g})\) and \((\text{Conv}(\Lambda(\Gamma)), h)\) are both pinched Hadamard manifold, and \(\Gamma_\Sigma \in \text{Isom}(\bar{\Sigma})\) acts cocompactly on \(\bar{\Sigma}\) and \(\Gamma\) acts convex cocompactly on \(\mathbb{H}^3\), by Theorem 2.10 (Švarc-Milnor lemma), we know that there are quasi-isometries \(q_1: \bar{\Sigma} \to C(\Gamma_\Sigma, S')\) and \(q_2 : C(\Gamma, S) \to \text{Conv}(\Lambda(\Gamma))\), where \(C(\Gamma, S)\) is the Cayley graph of \(\Gamma = \langle S \rangle\).

Because \(\Gamma\) and \(\Gamma_\Sigma\) are both isomorphic to \(\pi_1 \Sigma\), by Švarc-Milnor lemma the identity map \(i: C(\Gamma_\Sigma, S') \to C(\Gamma, S)\) is a quasi-isometry. So, \(q_2q_1: \bar{\Sigma} \to \text{Conv}(\Lambda(\Gamma))\) is the desired quasi-isometry. The second assertion is a consequence of Proposition 2.8, because the quasi-isometry \(q = q_2q_1\) extends to a bi-Hölder map \(q : \partial_\infty \bar{\Sigma} \to \Lambda(\Gamma) = \partial_\infty \text{Conv}(\Lambda(\Gamma))\).

Lastly, by the construction of \(q\), it is easy to see that \(q\) is \(\Gamma\)-equivariant. \(\square\)

Now, we have two different objects \(\Gamma_\Sigma \subset \text{Isom}(\bar{\Sigma})\) and \(\Gamma \subset \text{Isom}(\mathbb{H}^3)\). Nevertheless, they are the same as a group. Because \(\pi_1 \Sigma\) is finitely generated, there are canonical isomorphisms between \(\pi_1 \Sigma, \Gamma_\Sigma\) and \(\Gamma\), by sending generators to generators. Namely, \(i_{\Sigma}: \pi_1 \Sigma \to \Gamma_\Sigma\) and \(i_{\Sigma}: \pi_1 \Sigma \to \Gamma\). For brevity, we denote elements in \(\pi_1 \Sigma, \Gamma_\Sigma\) and \(\Gamma\) by \(\gamma, \gamma_\Sigma\) and \(\gamma_M\), respectively, where \(i_{\Sigma}(\gamma) = \gamma_\Sigma\) and \(i_{\Sigma}(\gamma) = \gamma_M\).

**Lemma 3.2.** The above quasi-isometry \(q : \partial_\infty \bar{\Sigma} \to \Lambda(\Gamma)\) sends the attracting (repelling) limit point \(\gamma_\Sigma^+ (\gamma_\Sigma^-)\) of the hyperbolic element \(\gamma_\Sigma \in \Gamma_\Sigma \subset \text{Isom}(\bar{\Sigma})\) to the attracting (repelling) limit point \(\gamma_M^+ (\gamma_M^-)\) of \(\gamma_M \in \Gamma \subset \text{Isom}(\mathbb{H}^3)\).

**Proof.** Notice that the boundary map \(q : \partial_\infty \bar{\Sigma} \to \Lambda(\Gamma)\) is an equivariant homeomorphism, and having an attracting (repelling) point is a topological feature. Therefore, \(q\) maps the attracting point of \(\gamma_\Sigma \in \Gamma_\Sigma\) to the attracting point of \(\gamma_M \in \Gamma\). \(\square\)

Now we are ready to state and prove the main theorem.

**Theorem 3.1 (Main theorem).** Let \(f : \Sigma \to M\) be a \(\pi_1\)-injective immersion from a compact surface \(\Sigma\) to a hyperbolic 3-manifold \(M\), and \(\Gamma\) be the copy of \(\pi_1 \Sigma\) in \(\text{Isom}(\mathbb{H}^3)\) induced by the immersion \(f\). Suppose \(\Gamma\) is convex cocompact and \((\Sigma, f^* h)\) is negatively curved, then

\[
C_1(\Sigma, M) \cdot \delta_\Gamma \leq h(\Sigma) \leq C_2(\Sigma, M) \cdot \delta_\Gamma.
\]

Here \(h(\Sigma)\) is the topological entropy of the geodesic flow on \(T^1 \Sigma\), \(\delta_\Gamma\) is the critical exponent, and \(C_1(\Sigma, M), C_2(\Sigma, M)\) are two geometric constants smaller or equal to 1. Moreover, each equality holds if and only if the marked length spectrum of \(\Sigma\) is proportional to the marked length spectrum of \(M\), and the proportion is the ratio \(\frac{\delta_\Gamma}{h(\Sigma)}\).

We will discuss the geometric meaning of \(C_1(\Sigma, M)\) and \(C_2(\Sigma, M)\) in detail in the next section. In this section we shall focus on the proof and the rigidity phenomena coming from the equality cases.

**Proof of the main theorem.** Let \(\phi\) denote the geodesic flow on the unit tangent bundle of \((\Sigma, g)\), i.e. \(\phi : T^1 \Sigma \to T^1 \Sigma\).

**The first step** is to construct a Hölder reparametrization function \(F : T^1 \Sigma \to \mathbb{R}_{>0}\) such that the topological entropy \(h_F\) of the reparametrized flow \(\phi^F\) is the critical exponent \(\delta_\Gamma\) of \(\Gamma\) in \(\mathbb{H}^3\).

Recall the Busemann function \(B^h_\eta(x, y) : \partial_\infty \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}\), for \(\eta \in \partial_\infty \mathbb{H}^3\) and \(x, y \in \mathbb{H}^3\) is given by

\[
B^h_\eta(x, y) := \lim_{z \to \eta} d_h(x, z) - d_h(y, z).
\]

Using the quasi-isometry \(q\) defined in Lemma 3.1, we define a map \(c : \pi_1 \Sigma \times \partial_\infty \bar{\Sigma} \to \mathbb{R}\) by

\[
c : \pi_1 \Sigma \times \partial_\infty \bar{\Sigma} \to \mathbb{R}
\]

\[
(\gamma, \xi) \mapsto B^h_{q(\xi)}(f(o), \gamma^{-1} \cdot f(o)),
\]

for \(o \in \bar{\Sigma}\).

**Claim:** \(c\) is a Hölder cocycle.
pf.
\[
c(\gamma_1 \gamma_2, \xi) = B^h_{\phi(\xi)}(f(o), (\gamma_1 \gamma_2)^{-1} \cdot f(o)) \\
= B^h_{\phi(\xi)}(f(o), (\gamma_2^{-1} \gamma_1^{-1}) \cdot f(o)) \\
= B^h_{\phi(\xi)}(f(o), (\gamma_2^{-1} \cdot f(o)) + B^h_{\phi(\xi)}(\gamma_2^{-1} \cdot f(o), (\gamma_2^{-1} \gamma_1^{-1}) \cdot f(o)) \\
= c(\gamma_2, \xi) + B^h_{\phi(\xi)}(f(o), \gamma_1^{-1} \cdot f(o)) \\
= c(\gamma_2, \xi) + B^h_{\phi(\gamma_2 \xi)}(f(o), \gamma_1^{-1} \cdot f(o))
\]
by Lemma 3.2

Therefore, \(c\) is a cocycle. To see \(c\) is Hölder, we first notice that the boundary map \(q : \partial_\infty \tilde{\Sigma} \to \Lambda(\Gamma) \subset \partial_\infty \mathbb{H}^3\) is bi-Hölder as we have discussed in the beginning of this section. Moreover, we know that \(\Lambda(\Gamma)\) embeds in \(\partial_\infty \mathbb{H}^3\) and \(B^h_\phi(x, y)\) is smooth on \(\partial_\infty \mathbb{H}^3\). Therefore, \(c(\gamma, \cdot)\) is Hölder continuous on \(\partial_\infty \Sigma\), and we finish the proof of this claim.

Notice that the period \(c(\gamma, \gamma_+^+) = B^h_{\phi(\gamma_1)}(f(o), \gamma^{-1} f(o)) = l_h(\gamma) > 0\) for all \([\gamma] \in [\pi_1 \Sigma]\). Thus, \(l_c(\gamma) = l_h(\gamma)\) for all \([\gamma] \in [\pi_1 \Sigma]\), and we can easily see that

\[
h_c = \delta_T = \lim_{T \to \infty} \frac{1}{T} \log \# \{[\gamma] \in [\pi_1 \Sigma]; l_h(\gamma) \leq T\} < \infty.
\]

Thus, by Theorem 2.13, there exists a positive Hölder continuous maps \(F_c\) on \(T^1 \Sigma\) such that the translation flow defined by the Hölder cocycle \(c\) is conjugated to the reparametrization \(\phi^{F_c}\) of the geodesic flow \(\phi_t : T^1 \Sigma \to T^1 \Sigma\) by \(F_c\). In particular, for all \([\gamma] \in [\pi_1 \Sigma]\)

\[
c(\gamma, \gamma_+^+) = \int_{[\gamma]} F_c = l_h(\gamma),
\]
and the topological entropy of the flow \(\phi^{F_c}\) is exactly the exponential growth rate of \(c\), i.e. \(h_{F_c} = h_c\).

Notice that for the constant function \(1\) on \(T^1 \Sigma\), we have \(l_g(\gamma) = \int_{[\gamma]} 1\) for all \([\gamma] \in [\pi_1 \Sigma]\). Therefore, we have the pressure of the function \(-h_1 \cdot 1\) is zero, i.e. \(P(-h_1 \cdot 1) = 0\), where

\[
h_1 = \lim_{T \to \infty} \frac{1}{T} \log \# \{[\gamma] \in [\pi_1 \Sigma]; l_g(\gamma) \leq T\}
\]
is the topological entropy of the geodesic flow \(\phi\) on \(T^1 \Sigma\).

From now on we denote \(F_c\) by \(F\).

The second step is to show that

\[
h(\Sigma) \leq \int F d\mu_{BM} \cdot h_F,
\]
where \(\mu_{BM}\) the Bowen-Margulis measure of the geodesic flow \(\phi : T^1 \Sigma \to T^1 \Sigma\).

Since

\[
P(-h_F \cdot F) = 0 = h(\mu_{-h_F F}) - h_F \int F d\mu_{-h_F F}
\]

\[
P(-h(\Sigma) \cdot 1) = 0 = h(\mu_{BM}) - h(\Sigma) \cdot \int 1 d\mu_{BM} = h(\mu_{BM}) - h(\Sigma).
\]

where \(\mu_{-h_F F}\) is the equilibrium state of \(-h_F F\). Since \(\mu_{BM} \in \mathcal{M}^\phi\), by the variational principle we have

\[
P(-h_F \cdot F) = 0 \geq h(\mu_{BM}) - h_F \int F d\mu_{BM}.
\]
Furthermore,
\[ h_F \int Fd\mu_{BM} \geq h(\mu_{BM}) = h(\Sigma). \]

The **third step** is to show the inequality
\[ \int_{c_1(\Sigma, M)} Fd\mu_{-Fh_F} : h_F \leq h(\Sigma). \]

By Remark 2.1, we have
\[ h(\Sigma) \geq h(\mu_{-Fh_F}) \]
\[ \iff h(\Sigma) - h_F \int Fd\mu_{-Fh_F} \geq h(\mu_{-Fh_F}) - h_F \int Fd\mu_{-Fh_F} = 0 \]
\[ \iff h(\Sigma) \geq h_F \cdot \int Fd\mu_{-Fh_F}. \]

The **fourth step** is to show that 0 ≤ \( C_1(\Sigma, M) \) ≤ 1 and 0 ≤ \( C_2(\Sigma, M) \) ≤ 1.

Because \( C_1(\Sigma, M) = \int Fd\mu_{-Fh_F} \), \( C_2(\Sigma, M) = \int Fd\mu_{BM} \) and \( F \) is positive, it is enough to show that \( F \) could be chosen to be smaller or equal than 1.

**Claim:** \( F \leq 1 \).

This is a consequence of Theorem 2.6. For each conjugacy class \([\gamma] \in [\pi_1 \Sigma]\) there exists a unique closed geodesic \( \tau^\Sigma_\gamma \) on \( \Sigma \) such that \( l_g(\gamma) = l_g(\tau^\Sigma_\gamma) \). Because \( f \) is \( \pi_1 \)-injective, \( f \) maps \( \tau^\Sigma_\gamma \) to a closed curve \( f(\tau^\Sigma_\gamma) \) on \( M \) which is in the same free homotopy class generated by \([\gamma]\). More precisely, let \( \tau^M_\gamma \) denote the closed geodesic on \( M \) in the conjugacy class \([\gamma]\), then we know that \( f(\tau^\Sigma_\gamma) \) and \( \tau^M_\gamma \) are in the same free homotopy class. Moreover, because \( g \) is the induced metric \( f^* h \), we know that \((\Sigma, g)\) is Riemannian isometric to \((f(\Sigma), h)\). Thus, \( l_g(\tau^\Sigma_\gamma) = l_h(f(\tau^\Sigma_\gamma)) \).

Therefore, \( \forall [\gamma] \in [\pi_1 \Sigma], \)
\[ l_g(\gamma) = l_g(\tau^\Sigma_\gamma) = l_h(f(\tau^\Sigma_\gamma)) \geq l_h(\tau^M_\gamma) = l_h(\gamma). \]

Therefore, for all \([\gamma] \in [\pi_1 \Sigma] \)
\[ \int [\gamma] 1 = l_g(\gamma) \geq l_h(\gamma) = \int [\gamma] F. \]

By Theorem 2.6, we have \( 1 - F \) is cohomologous to a nonnegative Hölder continuous function \( H \), and \( H \) is unique up to cohomology. Thus, we have that \( F \sim 1 - H \) and \( 1 - H \leq 1 \). By choosing \( F \) to be \( 1 - H \), we now finish the proof of this claim.

The **fifth step** is to examine the equality cases.

If \( h(\Sigma) = h_F \int Fd\mu_{-Fh_F} \), then \( h(\Sigma) = h(\mu_{-Fh_F}) \). i.e., \( \mu_{-Fh_F} \) is the equilibrium state of the constant function \(-h(\Sigma) \cdot 1\). By the uniqueness part of Theorem 2.2, we have that \( Fh_F \) is cohomologous to the constant \( h(\Sigma) \), i.e. \( F \sim \frac{h(\Sigma)}{F} \). Similarly, if \( h(\Sigma) = h_F \cdot \int Fd\mu_{BM} \), then \( \mu_{BM} = \mu_{-h_FF} \). Hence, again, \( h(\Sigma) \sim F \cdot h_F \), i.e. \( F \sim \frac{h(\Sigma)}{h_F} \).

\( \square \)

**Corollary 3.1.** If \( h(\Sigma) = \delta_\Gamma \), then \( \Sigma \) is a totally geodesic submanifold in \( M \).

**Proof.** Notice \( h(\Sigma) = \delta_\Gamma \) implies \( F = 1 \). This means that the length of each closed geodesic on \( \Sigma \) has the same length with the corresponding closed geodesic on \( M \). Furthermore, we know that the closed geodesics in \( \Sigma \) are dense. In the sense that, for any point \( p \in \Sigma \), the set of tangent vectors \( v \in T_p \Sigma \) such that the exponential map \( \exp_p tv \) gives a closed geodesic is dense in \( T_p \Sigma \). Therefore, the shape operator \( S_g \) is zero when evaluating on this
dense subset of vectors on $T_p\Sigma$. By the continuity of the shape operator $S_g$, we have $S_g \equiv 0$. Therefore $\Sigma$ is totally geodesic in $M$.

4 Geometric meaning of $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$

Throughout this section we keep the same setting as in the last section: $f : \Sigma \to M$ is a $\pi_1$--injective immersion from a compact surface $\Sigma$ to a hyperbolic 3-manifold $M$, $\pi_1\Sigma \cong \Gamma$ is the subgroup of $\text{Isom}(\mathbb{H}^3)$ induced by $f$ and $h$ is the given hyperbolic metric on $M$, and assuming that $(\Sigma, f^* h)$ is negatively curved and $\Gamma$ is convex cocompact. In this section we will discuss the geometric meaning of these two constants. When $f$ is an immersion, these two constants could be regarded as averages of lengths of closed geodesics; moreover when $f$ is an embedding, we show that these two constants are exactly the geodesic stretches defined imitating [Kni95].

4.1 Immersion

First, using the equidistribution property of Gibbs measures, we can understand $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$ by averages of the length of closed geodesics with respect to different metrics.

**Theorem 4.1.** Let $\mu_{BM}$ be the Bowen-Margulis measure of the geodesic flow $\phi : T^1\Sigma \to T^1\Sigma$ and $\mu_{-h_F}F$ be the Gibbs measure for $-h_F F$ defined in Theorem 3.1. Then

$$C_2(\Sigma, M) = \int F d\mu_{BM} = \lim_{T \to \infty} \frac{1}{\# R_T(g)} \sum_{[\gamma] \in R_T(g)} \frac{l_h(\gamma)}{l_g(\gamma)} = \lim_{T \to \infty} \frac{1}{\# R_T(h)} \sum_{[\gamma] \in R_T(h)} \frac{l_h(\gamma)}{l_g(\gamma)}$$

and

$$C_1(\Sigma, M) = \int F d\mu_{-h_F}F = \left( \lim_{T \to \infty} \frac{1}{\# R_T(h)} \sum_{[\gamma] \in R_T(h)} \frac{l_g(\gamma)}{l_h(\gamma)} \right)^{-1} = \lim_{T \to \infty} \frac{1}{\# R_T(g)} \sum_{[\gamma] \in R_T(g)} \frac{l_h(\gamma)}{l_g(\gamma)}$$

where

$$R_T(g) := \{ [\gamma] \in [\pi_1\Sigma] : l_g(\gamma) \leq T \} \text{ and } R_T(h) := \{ [\gamma] \in [\pi_1\Sigma] : l_h(\gamma) \leq T \}.$$  

**Proof.** This is a consequence of the equidistribution theorem (Theorem 2.3).

By Theorem 2.3, we have

$$C_2(\Sigma, M) = \int F d\mu_{BM} = \lim_{T \to \infty} \frac{1}{\# R_T(1)} \sum_{\tau \in R_T(1)} \frac{\langle \delta_\tau, F \rangle}{\langle \delta_\tau, 1 \rangle} = \lim_{T \to \infty} \frac{1}{\# R_T(1)} \sum_{\tau \in R_T(1)} \frac{\langle \delta_\tau, F \rangle}{\langle \delta_\tau, 1 \rangle}.$$  

Notice that every closed orbit $\tau$ of the geodesic flow $\phi$ on $T^1\Sigma$ corresponds to a unique conjugacy class $[\gamma^\tau]$ of $\pi_1\Sigma$, and vice versa. Moreover, the period of $\tau$ is the length of $\gamma^\tau$ on $\Sigma$, i.e.

$$l_h(\gamma^\tau) = \langle \delta_\tau, F \rangle, \quad l_g(\gamma^\tau) = \langle \delta_\tau, 1 \rangle.$$  

Since there is an one-to-one correspondence between $R_T(1)$ and $R_T(g)$, we can rewrite the equation above by
\[ C_2(\Sigma, M) = \int Fd\mu_{BM} = \lim_{T \to \infty} \frac{1}{\# R_T(g)} \sum_{[\gamma] \in R_T(g)} l_h(\gamma) = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(g)} l_h(\gamma)}{\sum_{[\gamma] \in R_T(g)} l_h(\gamma)}. \]

For the other equation, by Theorem 2.1, we know that \( \mu_{\phi^F} = F_{\mu_{-Fh_F}}. \) Therefore
\[
\mu_{\phi^F}(\frac{1}{F}) = F_{\mu_{-Fh_F}}(\frac{1}{F}) = \frac{\int (\frac{1}{F}) \cdot Fd\mu_{-Fh_F}}{\int Fd\mu_{-Fh_F}} = \frac{1}{\int Fd\mu_{-Fh_F}}.
\]

By Theorem 2.3, we have
\[
\mu_{\phi^F}(\frac{1}{F}) = \lim_{T \to \infty} \frac{1}{\# R_T(F)} \sum_{\tau' \in R_T(F)} \frac{\langle \delta_{\tau'}, \frac{1}{F} \rangle}{\langle \delta_{\tau'}, 1 \rangle} = \lim_{T \to \infty} \frac{\sum_{\tau' \in R_T(F)} \langle \delta_{\tau'}, \frac{1}{F} \rangle}{\sum_{\tau' \in R_T(F)} \langle \delta_{\tau'}, 1 \rangle}.
\]

Notice that for a closed geodesic \( \tau' \) of the geodesic flow \( \phi : T^1 \Sigma \to T^1 \Sigma, \langle \delta_{\tau'}, \frac{1}{F} \rangle = \int_0^1 (\tau') \frac{1}{\phi(t)} : F(\phi_t)dt = l_h(\tau') \) and similarly \( \langle \delta_{\tau'}, F \rangle = \int_0^1 l_h(t) : F(\phi_t)dt = l_h(\tau') \). By the one-to-one correspondence between closed orbit \( \tau' \) and conjugacy class \([\gamma, \tau']\), we have an one-to-one correspondence between \( R_T(F) \) and \( R_T(h) \).

Hence, we have the following equation:
\[
C_1(\Sigma, M) = \int Fd\mu_{-Fh_F} = \left( \mu_{\phi^F}(\frac{1}{F}) \right)^{-1} = \left( \lim_{T \to \infty} \frac{1}{\# R_T(h)} \sum_{[\gamma] \in R_T(h)} \frac{l_h(\gamma)}{l_h(\gamma)} \right)^{-1} = \lim_{T \to \infty} \frac{\sum_{[\gamma] \in R_T(h)} l_h(\gamma)}{\sum_{[\gamma] \in R_T(h)} l_h(\gamma)}.
\]

\[\square\]

Remark 4.1.: 

1. From this result, although we don’t understand the measure \( \mu_{-Fh_F} \) much, we still see that the integral \( \int Fd\mu_{-Fh_F} \) is exactly \( \int Gd\mu_{BM} \) where \( G \) is the reparametrization function that we get when we reparametrize the geodesic flow \( \psi \) on \( T^1 M \) to conjugate the geodesic flow \( \phi_t \) on \( T^1 \Sigma \), and \( \mu_{BM} \) is the Bowen-Margulis measure of \( \psi \).

2. From above expression of \( C_1(\Sigma, M) \) and \( C_2(\Sigma, M) \), we can also see \( C_1(\Sigma, M) \leq 1 \) and \( C_2(\Sigma, M) \leq 1 \). It is because for each \([\gamma] \in [\pi_1 \Sigma]\), we know \( l_g(\gamma) \geq l_h(\gamma) \) (cf. step 4 in the proof of Theorem 3.1).

4.2 Embedding

In this subsection, we will assume that \( f : \Sigma \to M \) is an embedding. To state our results more precisely and to put it in context, we first introduce the geodesic stretch and discuss the relation between the geodesic stretch, \( C_1(\Sigma, M) \) and \( C_2(\Sigma, M) \).

Notice that, we can lift \( f : \Sigma \to M \) to an embedding between their universal coverings, i.e. \( \tilde{f} : \tilde{\Sigma} \to \tilde{M} = \mathbb{H}^3 \). Moreover, one can easily check that this lifting is \( \pi_1 \Sigma \)-equivariant. Specifically, for each \( \gamma \in \pi_1 \Sigma \), let \( \gamma \Sigma \in \Gamma \Sigma \) and \( \gamma M \in \Gamma \) be the corresponding element of \( \gamma \) in the deck transformation groups \( \Gamma \Sigma \subset \text{Isom}^{+} \Sigma \) and \( \Gamma \subset \text{Isom}^{+} (\mathbb{H}^3) \), respectively.
respectively. Then for each $\bar{x} \in \tilde{\Sigma}$ we have

$$\tilde{f}(\gamma \cdot \bar{x}) := \tilde{f}(\gamma \Sigma(\bar{x})) = \gamma_M(\tilde{f}(\bar{x})) =: \gamma \cdot \tilde{f}(\bar{x}).$$

Using this embedding $\tilde{f} : \tilde{\Sigma} \to \mathbb{H}^3$ we can define a tangent map $f : T^1\tilde{\Sigma} \to T^1\mathbb{H}^3$ by

$$f : (\bar{x}_0, w) \mapsto (\tilde{f}(\bar{x}_0), d_{\tilde{f}\bar{x}_0}(w))$$

where $\bar{x}_0 \in \tilde{\Sigma}$ and $w$ is a unit vector on the tangent plane $T_{\bar{x}_0}\tilde{\Sigma}$. Notice that $\pi_1\Sigma$ acts on $T^1\tilde{\Sigma}$ and $T^1\mathbb{H}^3$ in an obvious way. Thus $f$ is also $\pi_1\Sigma$-equivariant. More precisely, $\gamma \cdot f(\bar{x}_0, w) = (\gamma \cdot f(\bar{x}_0), d_{\gamma f\bar{x}_0}(w)) = (f(\gamma \cdot \bar{x}_0), d_{\gamma f\bar{x}_0}(w)) = f(\gamma \cdot (\bar{x}_0, w))$.

The following lemma depicts a key feature of the embedding $f : \Sigma \to M$.

**Lemma 4.1.** $(\tilde{\Sigma}, d_\Sigma)$ is quasi-isometric to $(\tilde{f}(\Sigma), d_h) \subset (\mathbb{H}^3, d_h)$ where $d_h$ is the distance on $\tilde{\Sigma}$ induced by $g$ and $d_h$ is the hyperbolic distance on $\mathbb{H}^3$.

**Proof.** Because $\tilde{f}$ is an embedding and $\pi_1\Sigma$-equivariant, we know that $(\tilde{f}(\Sigma), d_h)$ is a proper geodesic space and $\Gamma \in \text{Isom}(\tilde{f}(\Sigma)) \subset \text{Isom}(\mathbb{H}^3)$ acts properly discontinuously and compactly on $\tilde{f}(\Sigma)$. Hence, by Theorem 2.10 (Švarc-Milnor lemma), $(\tilde{\Sigma}, d_\Sigma)$ is quasi-isometric $(\tilde{f}(\Sigma), d_h)$. (Because $(\tilde{\Sigma}, d_\Sigma)$ and $(\tilde{f}(\Sigma), d_h)$ are both quasi-isometric to the Cayley graph of $\pi_1\Sigma$ with a word metric.)

**Definition 4.1.** For all $v \in T^1\tilde{\Sigma}$ and $t > 0$, we define

$$a(v, t) := d_h(\pi \circ f(v), \pi \circ f \circ \tilde{\phi}_t(v)),$$

where $\pi : T^1\tilde{\Sigma} \to \tilde{\Sigma}$ is the natural projection and $\tilde{\phi}$ is the lift of $\phi$.

**Remark 4.2.** $a(v, t)$ is $\pi_1\Sigma$-invariant, because $f$ is $\pi_1\Sigma$-equivariant and $\pi_1\Sigma$ is acting on $\tilde{\Sigma}$ via $\Gamma_\Sigma \subset \text{Isom}(\tilde{\Sigma})$.

**Lemma 4.2.** For all $v \in T^1\tilde{\Sigma}$ and $t_1, t_2 > 0$,

$$a(v, t_1 + t_2) \leq a(v, t_1) + a(\tilde{\phi}_{t_1}(v), t_2).$$

**Proof.** It’s easy consequence of the triangle inequality of $d_h$.

The following corollary is a consequence of Kingman’s sub-additive ergodic theorem [Kin73].

**Corollary 4.1.** Let $\mu$ be a $\phi_t$-invariant probability measure on $T^1\Sigma$. Then for $\mu$ - a.e. $v \in T^1\Sigma$

$$I_{\mu}(\Sigma, M, v) := \lim_{t \to \infty} \frac{a(v, t)}{t},$$

and defines a $\mu$-integrable function on $T^1\Sigma$, invariant under the geodesic flow $\phi_t$.

**Proof.** Kingman’s original theorem works on measure preserving transformations; nevertheless, it works on flow as well. More precisely, for flows, we consider the time one map to be the measure preserving transformation. Thus, the only condition that we need is $\sup\{a(v, t); v \in T^1\tilde{\Sigma}, 0 \leq t \leq 1\} \in L^1(\mu)$. Notice that we can always translate $v \in T^1\Sigma$ to a fixed copy of $T^1\Sigma$ in $T^1\tilde{\Sigma}$ by a deck transformation. Because $T^1\Sigma$ is compact and $a(v, t)$ is $\pi_1\Sigma$-invariant, we have that $\sup\{a(v, t); v \in T^1\tilde{\Sigma}, 0 \leq t \leq 1\}$ is bounded.

From the above corollary, we can define the geodesic stretch as the following.
Definition 4.2. The geodesic stretch \( I_\mu(\Sigma, M) \) of \( \Sigma \) relative to \( M \) and a \( \phi_t \)-invariant probability measure \( \mu \), i.e. \( \mu \in \mathcal{M}^0 \), is defined as

\[
I_\mu(\Sigma, M) := \int_{T^1 \Sigma} I_\mu(\Sigma, M, v) d\mu.
\]

Remark 4.3. If \( \mu \in \mathcal{M}^0 \) is ergodic, then \( I_\mu(\Sigma, M) = \lim_{t \to \infty} \frac{a(v, t)}{t} \) for \( \mu \)-a.e. \( v \in T^1 \Sigma \).

Since \( f : (\Sigma, d_g) \to (f(\Sigma), d_h) \) is a quasi-isometry, by Theorem 2.8 we know that \( f \) extends to a bi-Hölder map between \( \partial_\infty \Sigma \) and \( \partial_\infty f(\Sigma) = \Lambda(\Gamma) \). By the same discussion as in Lemma 3.2, we know that \( f \) maps the attracting (repelling) fixed point \( \gamma^+_\Sigma (\gamma^-_\Sigma) \) of \( \gamma_\Sigma \in \Gamma \Sigma \) to the corresponding attracting (repelling) fixed point \( \gamma^+_M (\gamma^-_M) \) of \( \gamma_M \in \Gamma \).

Moreover, each conjugacy class \([\gamma] \in [\pi_1 \Sigma]\) corresponds to a unique closed geodesic \( \tau^\Sigma_M \) on \( \Sigma \) and \( \tau^M_\gamma \) on \( M \), and \( \tau^\Sigma_\gamma \) also corresponds to the unique geodesic \( \tau^\Sigma_{\gamma^+_\Sigma} \) connecting \( \gamma^+_\Sigma \) and \( \gamma^+_M \) on \( \partial_\infty \Sigma \). Notice that \( \tilde{f}(\gamma^+_\Sigma) = \gamma^+_M \) and \( \tilde{f}(\gamma^-_\Sigma) = \gamma^-_M \) on \( \partial_\infty \Sigma \). So \( f(\tau^\Sigma_M) \) is a quasi-geodesic on \( \mathbb{H}^3 \) within a bounded Hausdorff distance from the geodesic \( \tau^M_{\gamma} \) on \( \mathbb{H}^3 \), where \( \tau^M_\gamma \) is the geodesic on \( \text{Conv}(\Lambda(\Gamma)) \subset \mathbb{H}^3 \) connecting \( \gamma^-_M \) and \( \gamma^+_M \) on \( \Lambda(\Gamma) \).

Lemma 4.3. If \( \mu \in \mathcal{M}^0 \) and ergodic, then there exists a sequence of conjugacy classes \([\gamma_n]\) \( \subset [\pi_1 \Sigma] \), i.e. closed geodesics, such that

\[
\int F d\mu = \lim_{n \to \infty} \frac{l_b(\gamma_n)}{l_g(\gamma_n)},
\]

where \( F \) is the reparametrization function defined in Theorem 3.1.

Proof. First, by the sub-additive ergodic theorem we know that for \( \mu \)-a.e. \( v \in T^1 \Sigma \)

\[
\lim_{t \to \infty} \frac{a(v, t)}{t} = I_\mu(\Sigma, M).
\]

By the Birkhoff ergodic theorem we have for \( \mu \)-a.e. \( v \in T^1 \Sigma \)

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t F(\phi_s v) ds = \int F d\mu.
\]

We define two sets

\[
A := \{ v \in T^1 \Sigma : v \text{ satisfies (5)} \},
\]

\[
B := \{ v \in T^1 \Sigma : v \text{ satisfies (6)} \}.
\]

Since \( A \) and \( B \) are both full \( \mu \)-measure, we have \( A \cap B \neq \emptyset \).

Pick \( v \in A \cap B \), and \( \varepsilon_n \searrow 0 \) as \( n \to \infty \). By the Anosov Closing Lemma (Theorem 2.7), for each \( \varepsilon_n \), there exists \( \delta_n = \delta_n(\varepsilon_n) \) such that for \( v \in T^1 \Sigma \) and \( T_n = T_n(\delta_n) > 0 \) satisfying \( D_\theta(\phi_{T_n} v, v) < \varepsilon_n \), then there exists \( w_n \in T^1 \Sigma \) generates a periodic orbit \( \tau^\Sigma_n \) on \( \Sigma \) of period \( l_g(\tau^\Sigma_n) = T' \) such that \( |T_n - T'| < \varepsilon_n \) and \( D_\theta(\phi_{T' n} v, \phi_{T' n} w) < \varepsilon_n \) for all \( s \in [0, T_n] \).

Furthermore, because the geodesic flow \( \phi_t \) on \( T^1 \Sigma \) is a transitive Anosov flow and \( T^1 \Sigma \) is compact, by the Poincaré recurrent theorem, for each \( \delta_n \) given as above, we can pick \( T_n \) to be the \( n \)-th return time of the flow \( \phi_t \) to the set \( B_{\delta_n}(v) \), i.e. \( D_\theta(\phi_{T_n} v, v) < \delta_n \) for each \( n \).

Suppose \( \tau^\Sigma_n \) corresponds to \([\gamma_n] \in [\pi_1 \Sigma]\), then since \( \mu \) is ergodic, by the Birkhoff ergodic theorem we have

\[
\int_{T^1 \Sigma} F d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(\phi_t v) dt.
\]

Claim: \( \int F d\mu = \lim_{n \to \infty} \frac{\int_{\gamma_n} F}{l_g(\gamma_n)} \).
pf. Notice that
\[
\frac{1}{l_g(\gamma_n) + \varepsilon_n} \int_0^{l_g(\gamma_n) - \varepsilon_n} F(\phi_t v) \leq \frac{1}{t_n} \int_0^{t_n} F(\phi_t v) \leq \frac{1}{l_g(\gamma_n) - \varepsilon_n} \int_0^{l_g(\gamma_n) + \varepsilon_n} F(\phi_t v).
\]
Because \( F \) is Hölder, we know that \( |F(\phi_t v) - F(\phi_t w_n)| \leq C \cdot D_g(\phi_t v, \phi_t w_n)^\alpha \leq C \cdot \varepsilon_n^\alpha \).
When \( n \) is big enough such that \( l_g(\gamma_n) > 2\varepsilon_n \) (notice that \( \varepsilon_n \to 0 \) and \( l_g(\gamma_n) \to \infty \)), we have
\[
\left| \frac{1}{t_n} \int_0^{t_n} F(\phi_t v) - \frac{1}{l_g(\gamma_n)} \int_0^{l_g(\gamma_n)} F(\phi_t w_n) \right| \leq \frac{l_g(\gamma_n)}{l_g(\gamma_n) - \varepsilon_n} \left( l_g(\gamma_n) \cdot (l_g(\gamma_n) - \varepsilon_n) \right) \\
\leq \frac{1}{l_g(\gamma_n) - \varepsilon_n} \left( l_g(\gamma_n) \cdot C \cdot \varepsilon_n^\alpha + 2\varepsilon_n \cdot \|F\|_{\infty} \right) \\
\leq 2C \cdot \varepsilon_n^\alpha + \frac{2\varepsilon_n}{l_g(\gamma_n) - \varepsilon_n} \cdot \|F\|_{\infty}.
\]
So, we finish the proof of this claim.
Moreover, from the construction of \( F \), \( \forall \gamma_n \in [\pi_1 \Sigma] \) we have
\[
\int_{[\gamma_n]} F = l_h(\gamma_n).
\]
Therefore,
\[
\int F \, d\mu = \lim_{n \to \infty} \frac{\int_{\gamma_n} F}{l_g(\gamma_n)} = \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)}.
\]
\( \square \)

Lemma 4.4. There exists a sequence of conjugacy classes \( \{[\gamma_n]\} \subset [\pi_1 \Sigma] \), i.e. closed geodesics, such that
\[
\lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)} = I_{\mu}(\Sigma, M).
\]

Proof. Choose the \( \{\gamma_n\} \in [\pi_1 \Sigma] \) to be the sequence \( \{[\gamma_n]\} \) we found in Lemma 4.3.

Claim:
\[
\lim_{n \to \infty} \frac{a(w_n, l_g(\gamma_n))}{l_g(\gamma_n)} = \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)}
\]

pf. By definition,
\[
a(w_n, l_g(\gamma_n)) := d_h(\pi \circ f \circ w_n, \pi \circ f \circ \phi_{l_g(\gamma_n)} w_n).
\]
For such \( \{\gamma_n\} \in [\pi_1 \Sigma] \), let \( \tau^\Sigma_n \) and \( \tau^M_n \) denote the corresponding closed geodesics on \( \Sigma \) and \( M \), and \( \tilde{\tau}_n^\Sigma \) and \( \tilde{\tau}_n^M \) denote their lifting on \( \Sigma \) and \( \text{Conv}(\Lambda(\Gamma)) \), respectively. Then we know that \( \tilde{\tau}_n^\Sigma \) and \( \tilde{\tau}_n^M \) are at most Hausdorff distance \( R \) from each other. Therefore we can choose \( x_n \in \tau^M_n \) such that \( d_h(\pi w_n, x_n) < R \). Because \( d_h \) is \( \Gamma \)-invariant, \( \tilde{\tau}_n^\Sigma \to \H^3 \) is an embedding, and \( \pi \circ f \circ w_n \) and \( \pi \circ f \circ \phi_{l_g(\gamma_n)} w_n \) project to the same point on \( \Sigma \), we have
\[
d_h(\gamma_n \cdot x_n, \pi \circ f \circ \phi_{l_g(\gamma_n)} w_n) = d_h(\pi \circ f \circ w_n, x_n) < R.
\]
Hence, by the triangle inequality
\[
\left| d_h(\pi \circ f \circ w_n, \pi \circ f \circ \phi_{l_g(\gamma_n)} w_n) - d_h(x_n, \gamma_n \cdot x_n) \right| \leq \left| d_h(\pi \circ f \circ w_n, x_n) \right| + \left| d_h(\gamma_n \cdot x_n, \pi \circ f \circ \phi_{l_g(\gamma_n)} w_n) \right| \leq R.
\]
Therefore,
\[
\lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)} = \lim_{n \to \infty} \frac{l_h(\gamma_n) - 2R}{l_g(\gamma_n)} \leq \lim_{n \to \infty} \frac{a(w_n, l_g(\gamma_n))}{l_g(\gamma_n)} \leq \lim_{n \to \infty} \frac{l_h(\gamma_n) + 2R}{l_g(\gamma_n)} = \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)}
\]
and we finish the proof of this claim.

Claim:
\[
I_\mu(\Sigma, M) = \lim_{t \to \infty} \frac{a(v, t)}{t} = \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)}.
\]

pf. Pick the \( t_n \) as we mentioned in the first paragraph. Then
\[
|a(v, t_n) - a(w_n, l_g(\gamma_n))| \leq \left| d_h(\pi \circ f \circ v, \pi \circ f \circ \phi_{t_n} v) - d_h(\pi \circ f \circ w_n, \pi \circ f \circ \phi_{t_n} w_n) \right|
\leq d_h(\pi \circ f \circ v, \pi \circ f \circ w_n) + d_h(\pi \circ f \circ \phi_{t_n}(\gamma_n) w_n, \pi \circ f \circ \phi_{t_n} v)
\leq C \cdot \left( d_g(\pi \circ v, \pi \circ w_n) + d_g(\pi \circ \phi_{t_n}(\gamma_n) w_n, \pi \circ \phi_{t_n} v) \right) + 2L
\leq C \cdot (\delta_2 + \varepsilon) + 2L,
\]
where \( C \) and \( L \) are the quasi-isometry constants only depending on the embedding \( f : \Sigma \to M \).

Therefore,
\[
\lim_{t_n \to \infty} \frac{a(v, t_n)}{t_n} = \lim_{n \to \infty} \frac{a(w_n, l_g(\gamma_n))}{l_g(\gamma_n)} = \lim_{n \to \infty} \frac{l_h(\gamma_n)}{l_g(\gamma_n)}.
\]

\[
\square
\]

**Theorem 4.2.** Suppose that \( \Sigma \) is a compact and negatively curved surface embedded in a hyperbolic 3-manifold \( M \) as in Theorem 3.1. Assuming that \( \Gamma \) is convex cocompact, then
\[
C_1(\Sigma, M) = I_\mu(\Sigma, M),
C_2(\Sigma, M) = I_{\mu_{BM}}(\Sigma, M),
\]
where \( \mu \) is a \( \phi \)-invariant Gibbs measure and \( \mu_{BM} \) is the Bowen-Margulis measure of the geodesic flow \( \phi_t \) on \( T^1 \Sigma \).

**Proof.** It’s a consequence of the above two lemmas, because \( \mu_{BM} \) and \( \mu = \mu_{-HF} \) are Gibbs measures, which are, in particular, ergodic measures of the flow \( \phi_t \).

\[
\square
\]

**Remark 4.4.** Theorem 4.2 also indicates that \( C_1(\Sigma, M), C_2(\Sigma, M) \leq 1 \), because \( a(v, t) \leq t \) for all \( t > 0 \) and \( v \in T^1 \Sigma \).

5 Applications to immersed minimal surfaces in hyperbolic 3–manifolds

5.1 Immersed minimal surfaces in hyperbolic 3–manifolds

In what follows, \( M \) denotes a hyperbolic 3–manifold equipped with a hyperbolic metric \( h \) and \( \Sigma \) is a compact, 2–dimensional manifold with negative Euler characteristic. Recall that \( f : \Sigma \to M \) is called a minimal immersion if \( f \) is an immersion and the its mean curvature \( H \) vanishes identically.

Let \( g = f^* h \) denote the induced metric on \( \Sigma \) via the immersion \( f \). By the Gauss equation, when \( f : \Sigma \to M \) is a minimal immersion, the Gaussian curvature \( K_o \leq -1 \).

So, applying the Theorem 3.1 to this case, we have the following corollary.
Corollary 5.1. Let $f : \Sigma \to M$ be a $\pi_1$-injective minimal immersion from a compact surface $\Sigma$ to a hyperbolic 3–manifold $M$, and $\Gamma$ be the copy of $\pi_1 \Sigma$ in Isom($\mathbb{H}^3$) induced by the immersion $f$. Suppose $\Gamma$ is convex cocompact, then there are explicit constants $C_1(\Sigma, M)$ and $C_2(\Sigma, M)$ not bigger than 1 such that

$$C_1(\Sigma, M) \cdot \delta_\Gamma \leq h(\Sigma) \leq C_2(\Sigma, M) \cdot \delta_\Gamma.$$ 

Moreover, each equality holds if and only if the marked length spectrum of $\Sigma$ is proportional to the marked length spectrum of $M$, and the proportion is the ratio $\frac{\delta_\Gamma}{h(\Sigma)}$.

5.2 Minimal hyperbolic germs

5.2.1 Minimal hyperbolic germs

In the next three subsections, following Uhlenbeck’s assumptions in [Uhl83], we shall assume $\Sigma$ is a closed surface.

Recall that $\mathcal{H}$ is the set of the isotopy classes of pairs consisting of a Riemann metric $g$ and a symmetric 2-tensor $B$ on $\Sigma$ such that the trace of $B$ w.r.t. $g$ is zero and $(g, B)$ satisfies the Gauss-Codazzi equations (cf. Remark 2.7). Such a pair $(g, B) \in \mathcal{H}$ can be integrated to an immersed minimal surfaces of a hyperbolic 3-manifold with the induced metric $g$ and second fundamental form $B$. Moreover, we know that for each data $(g, B) \in \mathcal{H}$ there exists a representation $\rho : \pi_1(\Sigma) \to \text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ leaving this minimal immersion invariant. Thus, we have a map

$$\Phi : \mathcal{H} \to \mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C})), \quad (g, B) \mapsto \rho,$$

where $\mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))$ is the space of conjugacy classes of representations of $\pi_1(\Sigma)$ into $\text{PSL}(2, \mathbb{C})$.

The following corollary is an obvious consequence of Theorem 3.1. Recall that $h(g, B)$ denotes the topological entropy of the geodesic flow for the immersed surface $(\Sigma, g)$ with second fundamental form $B$.

Corollary 5.2. Let $\rho \in \mathcal{R}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{C}))$ be a discrete, convex cocompact representation and suppose $(g, B) \in \Phi^{-1}(\rho) \neq \emptyset$. Then there are explicit constants $C_1(g, B)$ and $C_2(g, B)$ not bigger than 1 such that

$$C_1(g, B) \cdot \delta_{\rho(\pi_1 \Sigma)} \leq h(g, B) \leq C_2(g, B) \cdot \delta_{\rho(\pi_1 \Sigma)} \leq \delta_{\rho(\pi_1 \Sigma)}$$

with the last equality if and only if $B$ is identically zero which holds if and only if $\rho$ is Fuchsian.

Proof. $(g, B) \in \Phi^{-1}(\rho)$ means that there exists a $\pi_1$–injective immersion $f : \Sigma \to \rho(\pi_1 \Sigma) \backslash \mathbb{H}^3 = M$ such that the induced metric is $g$ and the second fundamental form is $B$, where $(M, h)$ is a convex cocompact hyperbolic 3–manifold. Therefore, by Theorem 3.1 we have

$$C_1(\Sigma, M) \cdot \delta_{\rho(\pi_1 \Sigma)} \leq h(\Sigma) \leq C_2(\Sigma, M) \cdot \delta_{\rho(\pi_1 \Sigma)}.$$ 

Then we pick $C_1(g, B) = C_1(\Sigma, M)$ and $C_2(g, B) = C_2(\Sigma, M)$. The rightmost inequality is because $C_2(g, B) = C_2(\Sigma, M) \leq 1$, and the rigidity is the consequence of Corollary 3.1. \hfill \square

Remark 5.1. By Sullivan’s theorem, we know that $\delta_{\rho(\pi_1 \Sigma)} = \dim H \Lambda(\rho(\pi_1 \Sigma))$. Thus, we can replace the critical exponent by the Hausdorff dimension in the above corollary.

5.2.2 Quasi-Fuchsian Spaces

We call a discrete faithful representation $\rho : \pi_1(\Sigma) \to \text{Isom}(\mathbb{H}^3)$ quasi-Fuchsian if and only if the limit set $\Lambda(\rho(\pi_1 \Sigma))$ of $\rho(\pi_1 \Sigma)$ is a Jordan curve and the domain of discontinuity $\partial_{\infty} \mathbb{H}^3 \backslash \Lambda(\rho(\pi_1 \Sigma))$ is composed by two invariant, connected, simply-connected components. $\mathcal{QF}$ denotes the set of quasi-Fuchsian representations.
We notice that if $\rho \in \mathcal{QF}$, elements in $\Phi^{-1}(\rho)$ are $\pi_1(\Sigma)$–injective minimal immersions from $\Sigma$ to $\rho(\pi_1(\Sigma)) \setminus \mathbb{H}^3$. Moreover, Uhlenbeck in [Uhl83] points out that for $\rho \in \mathcal{QF}$, $\Phi^{-1}(\rho)$ is always a non-empty set.

**Corollary 5.3.** Let $\rho \in \mathcal{QF}$ be a quasi-Fuchsian representation and $(g, B) \in \Phi^{-1}(\rho)$. Then there are explicit constants $C_1(g, B)$ and $C_2(g, B)$ are not bigger than 1 such that

$$C_1(g, B) \cdot \delta_{\rho(\pi_1(\Sigma))} \leq h(g, B) \leq C_2(g, B) \cdot \delta_{\rho(\pi_1(\Sigma))} \leq \delta_{\rho(\pi_1(\Sigma))}$$

with the last equality if and only if $B$ is identically zero which holds if and only if $\rho$ is Fuchsian.

Using the above corollary, we can give another proof the famous Bowen’s rigidity theorem.

**Corollary 5.4** (Bowen’s rigidity [Bow79]). A quasi-Fuchsian representation $\rho \in \mathcal{QF}$ is Fuchsian if and only if $\dim_H \Lambda_{\Gamma} = 1$.

**Proof.** For any $(g, B) \in \Phi^{-1}(\rho)$, we have $\Sigma$ is an immersed minimal surface in a quasi-Fuchsian manifold $M = \rho(\pi_1(\Sigma)) \setminus \mathbb{H}^3$ with the induced metric $g$ and the second fundamental form $B$. Let $K(\Sigma)$ denote the Gaussian curvature of $\Sigma$ in $M$, then by the Gauss-Codazzi equation $K(\Sigma) \leq -1$. Therefore using the Theorem B in [Kat82], we have

$$h(g, B) \geq \left( \frac{- \int K(\Sigma) dA}{\text{Area}(\Sigma)} \right)^{\frac{1}{2}} \geq 1.$$

Hence the result is derived by the above lower bound of $h(\Sigma)$ plus the above corollary. \qed

### 5.2.3 Almost-Fuchsian spaces

Recall that the *almost-Fuchsian* space $\mathcal{AF}$ is the space of minimal hyperbolic germs $(g, B) \in \mathcal{H}$ such that $\|B\|_g < 2$. Given a hyperbolic metric $m \in \mathcal{F}$ and a holomorphic quadratic differential $\alpha \in Q([m])$, we discuss an informative smooth path

$$r(t) = (g_t, tB) \subset \mathcal{AF},$$

where $g_t = e^{2u_t} m$ and $B = \text{Re}(\alpha)$ satisfying $\|tB\|_{g_t} < 2$. Notice that $u_t : \Sigma \to \mathbb{R}$ is well-defined and smooth on $t$ (cf. Theorem 2.16).

Through studying this path, we can learn many geometric features of the Fuchsian space $\mathcal{F}$. First, we will see how the entropy behaves while we change the data along the ray $r(t)$ in $\mathcal{AF}$. To be more precise, in the following we denote the unit tangent bundle of $\Sigma$ associated with the data $r(t)$ by $T^1\Sigma_{r(t)}$.

We recover the following theorem by employing the reparametrization method.

**Theorem 5.1** (Sanders, Theorem 3.5 [San14]). Consider the entropy function restricting on the almost-Fuchsian space $h : \mathcal{AF} \to \mathbb{R}$, then

1. the entropy function $h$ realizes its minimum at the Fuchsian space $\mathcal{F}$, and
2. for $(m, 0) \in \mathcal{F}$, $h$ is monotone increasing along the ray $r(t) = (g_t, tB)$ provided $\|tB\|_{g_t} < 2$, i.e. $r(t) \subset \mathcal{AF}$, where $g_t = e^{2u_t} m$.

Fixed $t_0 > 0$, $r(t_0) = (e^{2u_{t_0}} m, t_0 B)$ defines the geodesic flow $\phi_{t_0} : T^1\Sigma_{r(t_0)} \to T^1\Sigma_{r(t_0)}$. For any $t > t_0$, we want to show $h(t_0, g_0) \leq h(t, g)$ and equality holds if and only if $B = 0$. Since $(\Sigma, g_t)$ is negatively curved, using the distance function $d_{g_t}$, we can construct the Busemann cocycle $B_{\xi}^g(x, y)$ like we did in the proof of Theorem 3.1. Then by Theorem 2.13, we can reparametrize the geodesic flow $\phi_{t_0}$ induced by the data $r(t_0) = (e^{2u_{t_0}} m, t_0 B)$ on $T^1\Sigma_{r(t_0)}$.
by a Hölder function \( F_t \) on \( T^1\Sigma_{(t_0, g_0)} \) such that \( \phi^t : T^1\Sigma_{(t)} \to T^1\Sigma_{(t)} \) is conjugated to \((\phi^{t_0})^{F_t} : T^1\Sigma_{(t_0)} \to T^1\Sigma_{(t_0)} \). We consider the pressure \( P_{\phi^{t_0}} : C(T^1\Sigma_{(t_0)}) \to \mathbb{R} \), and we have

\[
P_{\phi^{t_0}}(-h_{F_t} \cdot F_t) = 0 = P_{\phi^{t_0}}(-h(g_t, tB) \cdot F_t)
\]

\[
P_{\phi^{t_0}}(-h_{F_{t_0}} \cdot 1) = 0 = P_{\phi^{t_0}}(-h(g_{t_0}, t_0B) \cdot 1).
\]

Remark 5.2. Without using Theorem 2.13, when \( t_0 \) and \( t \) are small, the structure stability of Anosov flows also gives us the reparametrizing function \( F_t \). We will see more details about this perspective in the next subsection.

**Proof of the theorem 5.1.** Because almost-Fuchsian is quasi-Fuchsian, the first assertion is a consequence of Corollary 5.3 and the fact that \( h(g, B) \geq 1 \) (cf. the proof of Corollary 5.4). The remaining part of this proof is devoted to the second assertion.

It’s enough to show \( l_{g_t}^{(\gamma)}(\gamma) \leq l_{g_0}^{(\gamma)}(\gamma) \), for \( t > t_0 \) and \( \forall [\gamma] \in [\pi_1\Sigma] \). Because if \( l_{g_t}^{(\gamma)}(\gamma) \leq l_{g_0}^{(\gamma)}(\gamma) \) for all \([\gamma] \in [\pi_1\Sigma]\), then we have \( F_t \) is cohomologous to a function which is bigger then \( F_{t_0}(= 1) \), abusing the notation, we denote this function by \( F_t \) again, i.e. \( l_{g_t}^{(\gamma)}(\gamma) \leq l_{g_0}^{(\gamma)}(\gamma) \implies F_t \leq F_{t_0} = 1 \).

**Claim:** \( F_t \leq F_{t_0} = 1 \implies h_t \geq h_{t_0} \) for \( t \leq t_0 \) and the equality holds if and only if \( F_t \sim 1 \).

pf. we repeat the same trick used in the proof of Theorem 3.1. Since the pressure is monotone (see Remark 2.1), we have

\[
0 \leq F_t \leq F_{t_0} \implies 0 \geq -h_t F_t \geq -h_{t_0} F_{t_0} \implies P_{\phi^{t_0}}(-h_t F_t) \geq P(-h_{t_0} F_{t_0}).
\]

Thus,

\[
0 = P_{\phi^{t_0}}(-h_{t_0}) = P_{\phi^{t_0}}(-h_{t_0} F_{t_0}) = P_{\phi^{t_0}}(-h_t F_t) \geq P_{\phi^{t_0}}(-h_{t_0} F_{t_0}) = P_{\phi^{t_0}}(-h_t).
\]

By definition,

\[
0 = P_{\phi^{t_0}}(-h_{t_0}) = \sup_{\nu \in \mathcal{M}^{\phi^{t_0}}} h(\nu) + \int (-h_{t_0}) d\nu = \sup_{\nu \in \mathcal{M}^{\phi^{t_0}}} h(\nu) = h(\mu_{h_{t_0}}) = h_{t_0}
\]

where \( \mu_{h_{t_0}} \) is the equilibrium state of the function \(-h_{t_0} \cdot 1 \) and

\[
0 \geq P_{\phi^{t_0}}(-h_t) = \sup_{\nu \in \mathcal{M}^{\phi^{t_0}}} h(\nu) + \int (-h_t) d\nu = h_{t_0} - h_t.
\]

To see the equality case, we notice that \( h_{t_0} = h_t \) implies that \( h(\mu_{h_{t_0} F_t}) = h_t = h_{t_0} \), i.e.

\[
0 = P_{\phi^{t_0}}(-h_{t_0} \cdot 1) = h(\mu_{h_{t_0} F_t}) - \int h_{t_0} \cdot 1 d\mu_{h_{t_0} F_t}.
\]

In other words, \( \mu_{F_t, h_t} \) is the equilibrium state of the constant function \(-h_{t_0} \cdot 1 \). Hence, by the uniqueness of equilibrium state (cf. Theorem 2.2) we have \( \mu_{h_{t_0} F_t} = \mu_{h_{t_0} \cdot 1} \) and which implies \( h_{t_0} F_t \sim h_{t_0} \cdot 1 \), i.e. \( F_t \sim 1 \). We now complete the proof of this claim.

By the above claim, we know the equality holds if and only if \( F_t \sim 1 \), i.e. \( l_{g_t}^{(\gamma)}(\gamma) = l_{g_0}^{(\gamma)}(\gamma) \) for all \([\gamma] \in [\pi_1\Sigma]\). By the marked length spectrum theorem [Ota90], this means that \( g_0 = g_1 \). In other words, \( h(g_t, tB) \) is strictly increasing when \( u_t \neq 0 \).

To prove \( l_{g_t}^{(\gamma)} \leq l_{g_0}^{(\gamma)} \) for \( t > t_0 \), it is enough to show that \( g_t = e^{2u_t} m \) is decreasing, i.e. \( \frac{d}{dt} u_t < 0 \) for all \( t > 0 \).

Because fixing a free homology class of a closed curve \( \tau \) and let \( c_t \) denote the closed geodesic in this class under
the metric $g_t$, assuming that $g_t$ is decreasing, then we have $||v||_{g_t} \leq ||v||_{g_0}$ for $t > t_0$. Thus,

$$l_{g_t}(c_t) \leq l_{g_0}(c_{t_0}) = \int_{c_{t_0}} ||v||_{g_0} \leq \int_{c_{t_0}} ||v||_{g_t} = l_{g_t}(c_{t_0}),$$

where $v$ is the unit tangent vector of $c_{t_0}$ for the metric $g_0$, i.e. $v(s) := \frac{d}{ds}(c_{t_0}(s))/||\frac{d}{ds}(c_{t_0}(s))||_{g_0}$.

**Claim:** $g_t = e^{2u_t} m$ is decreasing, i.e. $\frac{d}{dt} u_t < 0$ for all $t > 0$.

pf. by Theorem 2.15 (the Gauss equation), we have

$$K_{g_t} = -1 - \frac{1}{2} t^2 e^{-4u_t} ||B||^2_m = e^{-2u_t} (-\Delta_m u_t - 1). \quad (8)$$

Taking the derivative of equation 8 w.r.t. $t$ and evaluating at $t_0$ provided $||t_0 B||^2_{g_0} < 2$, then

$$-\Delta_m \dot{u}_{t_0} = e^{2u_{t_0}} \cdot \dot{u}_{t_0} (||t_0 B||^2_{g_0} - 2) - t_0 e^{-2u_{t_0}} ||B||^2_m.$$

Because for the fixed $t_0$, at a maximum of $u_{t_0}$ we have $-\Delta_m \dot{u}_{t_0} \geq 0$. Thus

$$e^{2u_{t_0}} \cdot \dot{u}_{t_0} (||t_0 B||^2_{g_0} - 2) - t_0 e^{-2u_{t_0}} ||B||^2_m \geq 0.$$

The hypothesis $||t_0 B||^2_{g_0} < 2$ implies that $\dot{u}_{t_0} \leq 0$ at each maximum; hence $\dot{u}_{t_0} \leq 0$ for all points on $\Sigma$. Moreover, if $\dot{u}_{t_0} = 0$ for some $t_0 > 0$, then we have $B = 0$. This means $u_t \equiv 0$ for all $t$.

$\square$

**Remark 5.3.** The main difference between our proof and Sanders’ proof in [San14] is that in [San14] he used a sophisticated formula of the derivative of the topological entropy whereas in our reasoning we examine the length changing along the path directly.

### 5.2.4 Another metric on $\mathcal{F}$

Following the previous theorem, Sanders proves that we can define a metric on the Fuchsian space $\mathcal{F} \subset \mathcal{H}$ by taking the Hessian of the topological entropy along the path $r(s) = (e^{2u_s} m, sB)$. The striking point is that this metric is bounded below by the Weil-Petersson metric on $\mathcal{F}$.

Recall that the fiber of the cotangent bundle of $m \in \mathcal{F}$ is identified with the space of holomorphic quadratic differentials on the Riemann surface $(\Sigma, m)$. Thus, in order to connect the Hessian of the entropy with the Weil-Petersson metric, we will prove that the Hessian of the topological entropy along the given path $r(s)$ gives us a way to measure holomorphic quadratic differentials on $(\Sigma, m)$. It is because $r(s)$ is defined by the data $(m, B)$, where $B$ is given by a holomorphic quadratic differential $\alpha$ such that $B = \text{Re}(\alpha)$.

In the following we give another proof of Sanders’ theorem by using the pressure metric we introduced in the beginning of this note.

Before we start proving this result, we recall several important concepts of Anosov flows. We first notice that by the structure stability of the Anosov flow (cf. Prop. 1 in [Pol94] or [dLMM86]), when $s$ is small, the geodesic flows $\phi^s : T^1 \Sigma_{r(s)} \rightarrow T^1 \Sigma_{r(s)}$ conjugates to the reparametrized flow $\phi^F_s : T^1 \Sigma_{r(0)} \rightarrow T^1 \Sigma_{r(0)}$, where $\phi : T^1 \Sigma_{r(0)} \rightarrow T^1 \Sigma_{r(0)}$ is the geodesic flow on $T^1 \Sigma_{r(0)}$ and $F_s$ the is the reparametrizing H"older continuous function. Since the path $r(s)$ is a smooth one parameter family in $\mathcal{A} \mathcal{F}$, the structure stability theorem also indicates that $\{ F_s \}$ form a smooth one parameter family of H"older continuous functions on $T^1 \Sigma_{r(0)}$.

Since we shall only be interested in metrics $g_s$ close to $g_0(= m)$, it suffices to consider one parameter families given
by perturbation series: for $s$ small,

$$g_s = g_0 + s \cdot \dot{g}_0 + \frac{s^2}{2} \ddot{g}_0 + ..., \text{ and } F_s = F_0 + s \cdot \dot{F}_0 + \frac{s^2}{2} \ddot{F}_0 + ...,$$

where $\dot{g}_0, \ddot{g}_0, ...$ are symmetric 2-tensors on $T^1 \Sigma_{r(0)}$ and $\dot{F}_0, \ddot{F}_0, ...$ are Hölder continuous functions on $T^1 \Sigma_m$.

The following lemma connects the derivatives of $g_s$ with $F_s$.

**Lemma 5.1** (Pollicott, Lemma 5 [Pol94]).

\[ \int_{T^1 \Sigma} \dot{g}_0(v,v) \, d\mu_0 = \int_{T^1 \Sigma} \dot{g}_0(v,v) \, d\mu_0, \quad (9) \]

and

\[ \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0 \leq \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0. \quad (10) \]

**Remark.** The proof of above lemma is a straightforward computation. However, in the sake of brevity we omit the proof.

The following lemma reveals the relation between Weil-Petersson metric and the second derivative of the metric $g_s$.

**Lemma 5.2.**

\[ \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0 = \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0 = -2\pi \int_{\Sigma} \| \alpha \|^2_m \, dV_m. \]

**Proof.** Claim:

\[ \int \ddot{u}_0 \, d\mu_0 = -2\pi \int_{\Sigma} \| \alpha \|^2_m \, dV_m. \]

pf. for $m \in \mathcal{F}$ and $\alpha \in Q([m])$, we notice that the Gauss equation (Theorem 2.15) for this given data $r(s) = (e^{2u} \cdot m, s \cdot \text{Re} \alpha)$ is that

$$\Delta_m u_s + 1 - e^{-2u_s} - s e^{-2u_s} \| \alpha \|^2_m = 0,$$

where $\| \alpha \|_m$ is the norm of $\alpha$ with respect to the hyperbolic metric $m$.

Taking the derivative with respect to $s$, we have

$$-\Delta_m \ddot{u}_s = e^{2u_s} \cdot \ddot{u}_s (\| s \alpha \|^2_m - 2) - s e^{-2u_s} \| \alpha \|^2_m.$$

The maximum principle implies that $\ddot{u}_0 = 0$. We differentiate the above equation again and evaluate at $s = 0$, then we get

$$-\Delta_m \ddot{u}_0 = -2\ddot{u}_0 - 2 \| \alpha \|^2_m. \quad (11)$$

By integrating the equation (11) with respect to the volume form of $m$, and because $\Sigma$ is has no boundary we have

$$0 = -2 \int_{\Sigma} \ddot{u}_0 \, dV_m - 2 \int_{\Sigma} \| \alpha \|^2_m \, dV_m.$$

Because in this case the Bowen-Margulis measure $\mu_0$ of the geodesic flow $\phi : T^1 \Sigma_m \rightarrow T^1 \Sigma_m$ is the Liouville measure, we have

$$2\pi \int_{\Sigma} \ddot{u}_0 \, dV_m = \int_{T^1 \Sigma} \ddot{u}_0 \, d\mu_0.$$

**Claim:**

\[ \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0 = \int_{T^1 \Sigma} \ddot{g}_0(v,v) \, d\mu_0. \]

pf. It is straightforward, because $\ddot{u}_0 = 0$ and $\ddot{g}_0 = 2\ddot{u}_0 m$. \qed
Now we are ready to state and prove the main result of this subsection.

**Theorem 5.2** (Sanders, Theorem 3.8 [San14]). One can define a Riemannian metric on the Fuchsian space \( F \) by using the Hessian of \( h \). Moreover, this metric is bounded below by \( 2\pi \) times the Weil-Petersson metric on \( F \).

**Proof of the Theorem 5.2.** Here we consider \( c_t := -h(g_t \cdot tB) \cdot F_t = -h_t F_t \). Since \( P_\rho(c_t) = 0 \), we know that \( \int c_0 d\mu_0 = 0 \) where \( \mu_0 \) is the Bowen-Margulis measure of the flow \( \phi : T^1 \Sigma(0) \to T^1 \Sigma(0) \), i.e. \( m_{c_0} = \mu_0 \) and \( c_0 \in T_{c_0} P(T^1 \Sigma) \). Therefore, by Proposition 2.3, the pressure metric of \( \dot{c}_0 \) is

\[
\|\dot{c}_0\|^2_p = \frac{\text{Var}(\dot{c}_0, \mu_0)}{\int \dot{c}_0 d\mu_0} = \frac{\int \dot{c}_0 d\mu_0}{\int \dot{c}_0 d\mu_0}.
\]

Notice that \( h_0 = 1 \), \( F_0 = 1 \), and \( \dot{u}_0 = 0 \), so by Lemma 5.1

\[
\int \dot{F} d\mu_0 = \int \dot{g}_0 d\mu_0 = \int 2\dot{u}_0 m(v, v) d\mu_0 = 0,
\]

and hence

\[
0 \leq \|\dot{c}_0\|^2_p = \tilde{h}_0 + 2\dot{h}_0 \int \dot{F}_0 d\mu_0 = \tilde{h}_0 + \int \dot{\tilde{F}}_0 d\mu_0.
\]

Therefore,

\[
\tilde{h}_0 \geq -\int_{T^1 \Sigma} \dot{\tilde{F}}_0 d\mu_0 \\
\geq -\int_{T^1 \Sigma} \frac{\tilde{g}_0(v, v)}{2} d\mu_0 \quad \text{Lemma 5.1}
\]

\[
= -\int_{T^1 \Sigma} \tilde{u}_0 m(v, v) d\mu_0 \quad \text{Lemma 5.2}
\]

\[
= -\int_{T^1 \Sigma} \tilde{u}_0 d\mu_0
\]

\[
= 2\pi \int_{\Sigma} \|\alpha\|^2_m dV_m \quad \text{Lemma 5.2}
\]

\[
= 2\pi \|\alpha\|^2_{WP}.
\]

\( \square \)

**Remark 5.4.** Comparing with Bridgeman’s results in [Bri10], we suspect:

1. \( \int_{T^1 \Sigma} \dot{\tilde{F}}_0 d\mu_0 = \int_{T^1 \Sigma} \frac{\tilde{g}_0(v, v)}{2} d\mu_0 \), and

2. \( \|\dot{c}_0\|_p = 0 \).

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