Generation of Magnetic Field in the Pre-recombination Era

Rajesh Gopal\textsuperscript{1} and Shiv K. Sethi\textsuperscript{1}
\textsuperscript{1}Raman Research Institute, Bangalore 560080, India
emails: rajesh@rri.res.in, sethi@rri.res.in

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ABSTRACT
We study the possibility of generating magnetic fields during the evolution of electron, proton, and photon plasma in the pre-recombination era. We show that a small magnetic field can be generated in the second order of perturbation theory for scalar modes with adiabatic initial conditions. The amplitude of the field is $\lesssim 10^{-30}$ G at the present epoch for scales from sub-kpc to $\gtrsim 100$ Mpc.

1 INTRODUCTION

Magnetic fields are ubiquitous in the universe and presumably play an important role in most objects in the universe. Their origin however is not well understood (see e.g. Parker 1979; Zeldovich, Ruzmaikin & Sokoloff 1983). Galactic fields of micro-Gauss strength could have arisen from dynamo amplification of seed fields $\simeq 10^{-20}$ G (see e.g. Ruzmaikin, Shukurov & Sokoloff 1988; Beck et al 1996; Shukurov 2004; Brandenburg & Subramanian 2004). On the other hand, large scale magnetic fields could have originated from primordial magnetic fields $\simeq 10^{-9}$ G generated during inflationary epoch in the early universe (Turner & Widrow 1988, Ratra 1992, see Grasso & Rubenstein 2001; Giovannini 2004 for reviews).

Small seed fields could have arisen from various astrophysical processes during the later stages of evolution of the universe ($z \lesssim 20$) (e.g. Subramanian, Narasimha and Chitre 1995; Kulsrud et al 1997; Grasso & Rubenstein 2001 and Widrow 2002 for reviews). Harrison (1970) considered a scenario in which a small seed field $\simeq 10^{-25}$ G is generated owing to the vorticity in the photon-baryon fluid in the pre-recombination plasma. More recently, Hogan (2000) and Berezhiani & Dolgov (2003) also considered the generation of magnetic fields from photon pressure in the pre-recombination epoch. From astrophysical processes, during the post-recombination universe, the typical scales of the seed field are $\simeq$ a few Mpc (see e.g. Subramanian, Narasimha and Chitre 1995; Kulsrud et al 1997). On the other hand, magnetic fields can be generated on much larger scales in the pre-recombination universe.

In the pre-recombination universe, photons, electrons, and protons can be treated as tightly coupled fluids. Photons however preferentially exert pressure on electrons (the pressure on protons is suppressed by a factor $(m_e/m_p)^2$). During the evolution of the photon-baryon plasma a difference in velocity fields of electrons and protons can therefore be generated, and this holds the promise of generating magnetic fields from this induced current. In addition the approximation that photons and baryons are tightly coupled, and therefore can be treated as one fluid, breaks down for scales up to several Mpc by the time of recombination (during the process of recombination this scale exceeds the horizon scale). At smaller scales the photons free-stream and this in principle can lead to additional contribution to induced currents that might generate magnetic fields.

In this paper we study the coupled electron, proton, and photon plasma in first and second order in perturbation theory to understand the generation of magnetic fields during the evolution of the plasma.

In the next section we describe the relevant equations for our study. In §3 we discuss the evolution of magnetic fields and its sources in the first and second order in perturbation theory. In §4 discuss our results and give concluding remarks. Throughout the paper, numerical values of different quantities are given for the spatially flat FRW model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ (Spergel et al. 2003, Reiss et al. 2004, Tonry et al. 2003, Perlmutter et al. 1999, Riess et al. 1998) with $\Omega_b h^2 = 0.02$ (Spergel et al. 2003, Tytler et al. 2000) and $h = 0.7$ (Freedman et al. 2001).

2 PRE-RECOMBINATION PLASMA

The primary components of the plasma in the radiation era which lasts between neutrino decoupling and recombination are photons, free electrons and protons. Observations of the Cosmic Microwave Background Radiation (CMBR) which is the relic of the radiation existing in this era show that the plasma is almost homogeneous and in thermal equilibrium at the time
of recombination (see e.g. Peebles 1993). However, anisotropies observed in the CMBR also indicate that there are spatial fluctuations superimposed on this uniform background density. The initial condition for each mode of fluctuation is set before it enters the horizon. In the present analysis we assume that the fluctuations are adiabatic which means that the entropy per fluid particle is conserved. Recent WMAP observations favour this initial condition (Peiris et al. 2003). The electrons interact with each other and with protons through Coulomb scattering. The mean free paths for e-e, e-p and p-p collisions are the same in this thermal plasma (see e.g. Shu 1992) and are much smaller than the astrophysically relevant scales ($\simeq 1$ Mpc). Hence a continuum description treating them as fluids can be used. In such a macroscopic description the effect of scattering between different species is taken into account by including a momentum exchange term in the Euler equation. For photons however the dominant interaction is Thomson scattering off free electrons with mean free path (comoving), $l_{\text{ec}}$, at $z \simeq 1000$ for a fully ionized universe being, $l_{\text{ec}} = 1/(a\sigma_T n_e) \simeq 3$ Mpc. Here, $\sigma_T$ is the Thomson cross section for e-\gamma scattering, $n_e$ is the electron number density and $a$ is the scale factor. This is comparable to the length scales in consideration and hence a Boltzmann particle description is essential.

### 2.2 Fluid equations for electrons and protons

As discussed earlier, since the mean free paths of electrons and protons are very small compared to astrophysical scales, we can describe their evolution accurately using continuity and Euler equations for an ideal fluid. In linear theory, the density field $\rho_{e,p}(x, \eta)$ is expanded as $\rho_{e,p}(x, \eta) = \bar{\rho}_{e,p}(\eta)(1 + \delta_{e,p}(x, \eta))$, where $\bar{\rho}$ is the unperturbed background density and $\delta$ is the fractional perturbation. In what follows quantities denoted with a bar on top are background unperturbed quantities. The continuity equations for each of the above species is given as (e.g. Ma & Bertschinger 1995):

$$\delta_{e,p} + \nabla \cdot \mathbf{v}_{e,p} - 3\Phi = 0$$  \hfill (5)
The corresponding Euler equations are:

\[
\dot{v}_e = -\frac{\nabla P_e}{\rho_e} - \nabla \Psi - \frac{ae}{m_e} E - \frac{ae}{m_e} v_e \times B + \left(\frac{v_e - v_p}{\tau_{\gamma e}}\right) R + \frac{v_p - v_e}{\tau_{ep}} \tag{6}
\]

\[
\dot{v}_p = -\frac{\nabla P_p}{\rho_p} - \nabla \Psi + \frac{ae}{m_p} E + \frac{ae}{m_p} v_p \times B + \frac{v_e - v_p}{\tau_{ep}} \left(\frac{m_e}{m_p}\right) \tag{7}
\]

Here, \(\tau_{ep}\) is the (co-moving) electron-proton collision time scale; \(\tau_{\gamma e} = 1/(n_e \sigma_{\gamma e} a)\) is the photon-electron Thompson scattering time scale. \(E, B\) are the physical electric and magnetic fields, \(R \equiv 4\rho_e/3\rho_e\), and \(P_{e,p}\) are the pressures which, for adiabatic fluids, can be written as \(P \equiv P(p)\). By taking curl of the Euler equations and using Maxwell’s equations, we can get the evolution equation for the vorticities \((\Omega_{e,p} \equiv \nabla \times \mathbf{v}_{e,p})\) of the fluids as:

\[
\dot{\Omega}_e + \frac{\dot{a}}{a} \Omega_e = -\frac{e}{m_e a} \frac{d}{d\eta} (a^2 B) - \frac{ae}{m_e} \nabla \times (v_e \times B) + \left(\frac{\Omega_e - \Omega_p}{\tau_{\gamma e}}\right) R - \frac{\nabla^2 B}{4\pi n_e c \tau_{ep}} \tag{8}
\]

3 EVOLUTION EQUATION FOR THE MAGNETIC FIELD

To arrive at the equation governing the evolution of magnetic field, we use the Euler equations for the charged fluids and Maxwell’s equations (Appendix B). Subtracting Eq. (6) from Eq. (7) and using Maxwell’s equations we first obtain the evolution of the current \(J\):

\[
\frac{m_e}{e^2} \frac{\partial}{\partial \eta} (\mathbf{J} / n_e) + \frac{\dot{a}}{a} \frac{m_e}{e^2 n_e} \mathbf{J} = \frac{1}{n_e} \nabla P_e + \frac{ae}{m_e} \mathbf{V} + a (\mathbf{V} \times B) - \frac{\nabla}{\tau_{\gamma e} (n_e)} R \frac{m_e}{e} - \frac{m_e J}{n_e e^2 \tau_{ep}} \tag{9}
\]

In the above equation we have neglected forces on the proton fluid due to pressure gradient and electric field since they are smaller than that for the electron fluid by the factor \(m_e/m_p\). Taking curl of equation (9) and using Maxwell’s equations, we get the equation for the generation of magnetic fields:

\[
\frac{1}{a} \frac{\partial}{\partial \eta} (a^2 B) = \frac{m_e}{e^2} \nabla \times \left(\frac{\mathbf{J}}{n_e}\right) + \frac{m_e}{e} \nabla \times \left(\frac{\nabla P_e}{\rho_e}\right) - \nabla \times (\mathbf{v}_e \times B) + \frac{m_e}{e} \nabla \times \left(\frac{\mathbf{J}}{n_e \tau_{ep}}\right) + \frac{m_e}{e} \nabla \times \left(\frac{R (v_e - v_p)}{\tau_{\gamma e}}\right) - \frac{\dot{a} m_e}{a e^2} \nabla \times \left(\frac{\mathbf{J}}{n_e}\right) \tag{10}
\]

For studying the generation of magnetic fields in the early universe most of the terms in the above equation can be dropped. The first term on the right hand side can be shown to be negligible as compared to the term on the left hand side (see e.g. Widrow 2002). Similarly all the terms proportional to \(B\) can be neglected if one wishes to study the generation of magnetic fields from zero magnetic field initial conditions. These terms can back-react once the magnetic field is generated. We show later that the back-reaction terms are negligible for the magnitude of the generated magnetic field. These considerations simplify the above equation to:

\[
\frac{1}{a} \frac{\partial}{\partial \eta} (a^2 B) = \mathbf{S}(\mathbf{x}, \eta) \tag{11}
\]

with

\[
\mathbf{S}(\mathbf{x}, \eta) = \frac{m_e}{e} \nabla \times \left(\frac{\nabla P_e}{\rho_e}\right) + \frac{m_e}{e} \nabla \times \left(\frac{R (v_e - v_p)}{\tau_{\gamma e}}\right) \tag{12}
\]

We now discuss the nature of these source terms of magnetic field generation in first and second order in perturbation theory.

3.1 Evaluation of the source term: Linear theory

The source term for any Fourier mode \(\mathbf{S}(k, \eta)\) can be simplified for the linear case. In this case, \(R = 4\rho_e/(3\rho_e)\), and \(\tau_{\gamma e} = 1/(n_e \sigma_{\gamma e})\), are unperturbed quantities and hence don’t carry any spatial dependence. The first term of the right hand side of Eq. (12) identically vanishes in this case. The source term can then be written as:

\[
\mathbf{S}(k, \eta) = \frac{m_e R}{e \tau_{\gamma e}} (\Omega_e - \Omega_p) \tag{13}
\]

Thus, we see that the source for the magnetic field in the plasma is the differential vorticity between electrons and photons. The vorticity equation for photons is essentially the Boltzmann moment equation for \(l = m = 1\) (Eq. (A8)). We note that the source of photon vorticity is \(4\omega_{l1}/\tau_{\gamma e} \propto \Omega_e\). This implies that the only source which can excite any \(l\)-moment for \(m = 1\) is the electron fluid vorticity. The evolution equation for the electron fluid vorticity (Eq. (5)) shows that the only sources of vorticity are the magnetic field and the photon vorticity. This means that if the vorticities were zero in the initial condition,
as is the case with initial zero-vorticity conditions we consider here, none of these quantities can be generated for any scale in the linear regime. In particular we can conclude that no magnetic field is generated in linear order for scalar perturbations. It should be noted that using Eq. 13 and Eq. 14 allows us to follow modes at which photons are free-streaming at any given epoch. Therefore the above conclusion holds for all scales larger than the scales at which electrons and protons can be treated as fluids.

3.2 Source term in the second order

There are various terms which have to be included in going to second order in perturbation theory. Second order terms can arise from treating metric perturbations to second order (Martinez-Gonzalez, Sanz & Silk 1992) or by including the second order terms in the electron-photon scattering (Vishniac 1987, Jubas & Dodelson 1995, Hu, Scott, & Silk 1994). Vishniac (1987) adopted the simple procedure of including the spatial dependence of densities to include the second order effects. Detailed analyses (Jubas & Dodelson 1995, Hu, Scott, & Silk 1994) showed that Vishniac’s procedure gives the most important second order effect in the electron-proton scattering for sub-horizon scales. This allows us to study scales smaller than the horizon at the last scattering surface, \( H^{-1} \approx 100 \, \text{Mpc} \). At larger scales other second order effects from electron-photon scattering and the second order metric perturbations might be comparable or dominate. We adopt Vishniac’s procedure here and obtain the second order term from treating the spatial dependence of densities i.e. in \( R, \tau_e, \rho_e \) in the source term for magnetic field generation (Eq. 12). To get estimates of the generated magnetic field we solve for the difference in photon and baryonic bulk velocity in the tight-coupling approximation. We argue below that the main contribution to the source of the magnetic field at

\[ \tau_e(\eta) \]

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4 CONCLUSION AND DISCUSSION

We have studied the possibility of generating magnetic fields during the evolution of the photon-baryon plasma in the pre-recombination universe. For scalar perturbation in linear theory magnetic field is not generated at any scale; this includes scales at which the photon-baryon coupling approximation breaks down. We show that in the second order in perturbation theory a small magnetic field is generated. The strength of the generated magnetic field is \( \lesssim 10^{-30} \) for scales from \( \simeq 100 \) Mpc to sub-kpc at the present epoch.

In Eq. (10), we have neglected several terms which could back-react on the generated magnetic field. It can be readily noticed from Eq (A9), that the source field. And therefore we were justified in neglecting those terms for studying the generation of magnetic field. As discussed above magnetic fields at small scales are frozen in the plasma from epochs \( \simeq \omega^{-1} \). It could be asked whether the radiative viscosity prior to the recombination can damp these fields. The maximum length scale damped by pre-recombination radiative viscosity is \( \propto B \) (Jedamzik, Katalinić, & Olinto 1998, Subramanian & Barrow 1998). For the small magnetic fields we obtain, the maximum scale of dissipation can be shown to be much smaller than any relevant length scales.

APPENDIX A: PHOTON BOLTZMAN EQUATION AND INITIAL CONDITIONS

In this appendix we discuss the complete spherical harmonic moment expansion of the Boltzmann equation without assuming azimuthal symmetry of \( \Delta \). The assumption of azimuthal symmetry precludes the existence of vortical modes for the velocity fields of photons and electrons. Hence if we are looking for the generation of magnetic fields in the linear order, we have to relax this assumption since a possible presence of magnetic fields violates azimuthal symmetry. The notations used in this section are self contained.

The Fourier transformed Boltzmann equation can be written as:

\[
\dot{\Delta} + ik\mu + i k \mu \Psi + \dot{\Phi} = \frac{1}{\tau_e} \left( \Delta_0 - \Delta + 4 \nu_e \cdot \hat{n} + 3 \frac{n_i n_j \Pi_{ij}}{2} \right) \tag{A1}
\]

The photon brightness function \( \Delta \) can be expanded in terms of the scalar spherical harmonics \( Y_{\ell m} \) as:

\[
\Delta(\hat{n}) = \sum \sqrt{\frac{4\pi}{2\ell + 1}} \Delta_{\ell m} Y_{\ell m}(\hat{n}) \tag{A2}
\]

where, the coefficients \( \Delta_{\ell m} \) are given by the inverse relation,

\[
\Delta_{\ell m} = \sqrt{\frac{2\ell + 1}{4\pi}} \int d\Omega Y_{\ell m}^*(\hat{n}) \Delta(\hat{n}) \tag{A3}
\]

The photon fluid variables like over-density \( \delta_\gamma \) and velocity \( \nu_\gamma \) are then given by:

\begin{align*}
\delta_\gamma & = \int \frac{d\Omega}{4\pi} \Delta = \Delta_0 \\
\dot{k} \cdot \nu_\gamma & = \frac{\Delta_{10}}{4} \\
\Omega_\gamma & = |\hat{k} \times \nu_\gamma| = \frac{\Delta_{11}}{4}
\end{align*}

(A4) (A5) (A6)

By substituting the expansion for \( \Delta \) in Eq. (A1) and using the familiar properties of spherical harmonics we arrive at the following hierarchy of equations for the evolution of the moments \( \Delta_{\ell m} \). For details of such an expansion we refer to the paper (Hu and White 1997).

\[
\dot{\Delta}_{\ell m} + i k \frac{A_{\ell,m}}{(2\ell - 1)} \Delta_{\ell - 1,m} + i k \frac{A_{\ell + 1,m}}{(2\ell + 3)} \Delta_{\ell + 1,m} + \frac{\Delta_{\ell m}}{\tau_e} = S_{\ell m} \tag{A8}
\]

Here, \( A_{\ell m} = \sqrt{\ell^2 - m^2} \). The source \( S_{\ell m} \) is given as:

\[
S_{\ell m} = \delta_{\ell 1} \frac{\Delta_{\ell m}}{\tau_e} + 4 \Phi \delta_{\ell 0} \delta_{m 0} + \left( \frac{4 \nu_{\ell m}}{\tau_e} - k \nu_\gamma \delta_{m 0} \right) \delta_{\ell 1} + \frac{1}{10} \Delta_{\ell m} \delta_{\ell 2} \tag{A9}
\]

In the above equations, the coefficients \( \nu_{\ell m} \) are the coefficients in the multipole expansion of the \( \nu_e \cdot \hat{n} \) term such that \( \nu_e \cdot \hat{n} = \sum \nu_{\ell m} Y_{\ell m} \delta_{\ell 1} \). The vorticity of the photon fluid is tracked by the evolution of the \( \ell = 1, m = 1 \) moment \( \Delta_{11} \). We notice from Eq (A10) that the source \( S_{11} = 4 \nu_{11}/\tau_e \). This is the only source for the evolution of the \( m = 1 \) moment. The first two moment equations in the hierarchy give the familiar continuity and Euler equations for photons:

\[
\dot{\delta}_\gamma + \frac{4i}{3} k \cdot \nu_\gamma - 4 \Phi = 0 \tag{A10}
\]
\[ \dot{\Psi} + i k \frac{\delta_0}{4} + \Pi - i k \Psi = \frac{\nu_e - \nu_\gamma}{\tau_{\nu e}} \]  

(A11)

In the above equation, \( \Pi = \frac{3}{2} i k_3 \Pi_{ij} \).

### A1 Initial Conditions: Tight-coupling approximation

The electron-proton plasma recombines at a redshift \( z_{\text{rec}} \approx 10^4 \) (see e.g. Peebles 1993). At any epoch in the universe before recombination, \( \eta \approx z_{\text{rec}} \approx 2 H_0^{-1}(1 + z_{\text{rec}})^{-1/2}/\Omega_m^{1/2} \), there are roughly five physically relevant length scales: (a) Super-horizon scale, \( k \gtrsim \eta^{-1} \) (b) scales that are sub-horizon but larger than the sound Horizon, \( \eta/\sqrt{3} \lesssim k \lesssim \eta^{-1} \). At these scales the evolution of velocity fields is determined by gravitational potentials. (c) Scales smaller than the sound horizon scale but larger than the Silk damping scale, \( \eta/\sqrt{3} \lesssim k \lesssim k_{\text{damp}} \). At these scales the baryon velocity evolution is determined by both gravitational potentials and the photon pressure, (d) scales that are in the damping regime but larger than the scales at which photon free-stream, \( k_{\text{damp}} \lesssim k \lesssim k_{\text{fs}}, k_{\text{fs}} \approx (2 \text{Mpc})^{-1}(10^3/(1+z))^{-2} \). The densities and velocities of baryons decay exponentially in this regime (see e.g. Peebles 1980) and (e) \( k \gtrsim k_{\text{fs}} \), at these scales photons are free-streaming and therefore photons and baryons cannot be treated as coupled fluids. During the evolution in the expanding universe before recombination, the electron velocity and density perturbations at most scales first pass through the Silk damping regime before reaching this phase. Therefore during this phase \( \delta_e, \nu_e \approx 0 \). The only exception to this occurs around the epoch of recombination when the free-streaming length increases very rapidly. As the sources of magnetic field generation are nearly zero in this regime, the dynamics of plasma at these scales play an unimportant role for our study. In the evolution in linear theory all scales undergo either some or all of these phases of evolution.

Initial condition for each mode is set outside the horizon. Up to phase (c) discussed above, \( k \ll k_{\text{fs}} \). During this phase the photons are tightly coupled to the baryons and this greatly simplifies the problem (Peebles & Yu 1970, Peebles 1980, Hu & Sugiyama 1995). In this approximation, to zeroth order in \( \tau_{\nu e} \): \( \nu_e = \nu_\gamma \); and \( \Pi^T = 0 \). Also to this order in \( \tau_{\nu e} \): \( \delta_e = 3/4 \delta_t \); this can be readily obtained by subtracting the electron continuity equation from the photon continuity equation (Eqs. A10 and A11). To solve for the difference between electron and photon bulk velocity we need to expand to the first order in \( \tau_{\nu e} \). To this order, from Eq. A11 (Peebles & Yu 1970, Peebles 1980):

\[ \nu_e - \nu_\gamma = \tau_{\nu e} \left( \frac{\partial \nu_e}{\partial \eta} + \frac{1}{4} \delta_e \right) \]  

(A12)

Here the quantities in the bracket on the right hand side are to be evaluated to the zeroth order in \( \tau_{\nu e} \). Eq. A12 along with the evolution of electron velocity (Eq. A17) and the difference of electron and proton velocities (Eq. A19) can be used to give the following expression for the electric field in the tight coupling approximation:

\[ aE(x, t) = \frac{m_e}{c} \left( \frac{a}{a} R \nu_e - \frac{\nabla p_e}{\rho_e} + \frac{1}{4} R \nabla \delta_e \right) \]  

(A13)

In deriving Eq. A13 all terms proportional to the magnetic field were dropped as they cannot act as sources for generating magnetic field. Taking the curl of this equation and using Maxwell’s equation (Eq. A18) one obtains the equation for magnetic field generation:

\[ \frac{1}{a} \frac{\partial}{\partial \eta} (a^2 B) = \frac{m_e}{c} \nabla \times \left( \frac{a}{a} R \nu_e + \frac{\nabla p_e}{\rho_e} + \frac{1}{4} R \nabla \delta_e \right) \]  

(A14)

This equation verifies the discussion above that the source of magnetic field generation is electron vorticity in the linear theory. With non-vortical initial conditions, Eq. A14 shows that all the sources of magnetic field generation are zero in the linear perturbation theory. We wish to consider the second order effect by considering the spatial dependence of densities. This gives: \( R = \tilde{R}(\delta_e - \delta_t) \approx -1/3 \tilde{R} \delta_t \) in the tight coupling approximation, as \( \delta_e - 3/4 \delta_t \), during adiabatic expansion (see e.g. Peebles 1980). In second order, the second term on the right hand side is the Biermann battery term. In the adiabatic initial condition we consider here, \( p = \rho \gamma \) with \( \gamma = 5/3 \). And the source term \( \propto \nabla \delta_e \times \nabla \delta_t = 0 \) and therefore in this limit the Biermann battery term doesn’t contribute. It should be noted that the plasma evolves adiabatically only for scales that are not affected by Silk damping (see below). However as the densities and velocities damp in this regime one doesn’t expect much contribution from these scales. Biermann battery term can also contribute for initial conditions different from the adiabatic initial conditions. The third term on the right hand side also vanishes even in the second order in the tightly-coupled regime. Therefore the only source of magnetic field generation is the first term on the right hand side of Eq. A14. Eq. A14 can therefore be simplified to:

\[ \frac{1}{a} \frac{\partial}{\partial \eta} (a^2 B) = \tilde{R} \frac{m_e}{3c} \nabla \times \left( \frac{a}{a} \delta_e \nu_e \right) \]  

(A15)

This equation can be used to get an order-of-magnitude estimate of the generated magnetic field.

Eq. A12 can be used to calculate, to the zeroth order, the evolution equation of electron velocity field (Peebles & Yu
1970). This equation along with the continuity equation (Eq. 6) and \( \nabla \cdot \mathbf{E} = 0 \), Eq. (26), and dropping all terms proportional to \( \mathbf{B} \), gives:

\[
\ddot{\delta}_e = -\frac{\dot{a}}{a} \frac{\dot{\delta}_e}{(1 + R)} - \frac{k^2 p_e}{\rho_e(1 + R)} - k^2 \Psi - \frac{R}{4(1 + R)} k^2 \delta_e + \frac{3\dot{a}}{a} \frac{\dot{\Psi}}{(1 + R)} + 3\dot{\Phi} \tag{A16}
\]

This equation can be solved along with the evolution equation of \( \delta_e \) by WKB approximation and these solutions can be matched to large scale solutions (Hu & Sugiyama 1995). We discuss here approximate solutions at different epochs. First we discuss solutions during phase (c) of the evolution. We note that all the terms on the right hand side except for the \( \delta_e \) term are smaller as compared to this term for scales smaller than the sound horizon scale \( \sim 1/\sqrt{3}\eta \). The electron pressure is always negligible as compared to the photon pressure in the pre-recombination universe. With these simplification and bearing in mind that \( 1/R \ll 1 \) during the evolution, Eq. (A16) is solved to give:

\[
\delta_e(k, \eta) = A(k) \cos \left( \int_0^\eta \omega_e d\eta' \right) \tag{A17}
\]

Here we have only retained the solution compatible with adiabatic initial conditions (see e.g. Hu & Sugiyama 1995) and

\[
\omega_\phi = \frac{k}{\sqrt{3(1 + 1/R)}} \tag{A18}
\]

In phase (d) of the evolution of the plasma, the tight-coupling approximation breaks down and the photon-slip which damps perturbations (Silk damping) must be taken into account. The solution including the Silk damping is (see e.g. Peebles 1980):

\[
\delta_e(k, \eta) = A(k) \cos(\omega_e \eta) \exp(-\omega_e \eta) \tag{A19}
\]

Here, \( \omega_d \simeq \frac{2k^2 \tau_e}{15} \) (A20)

The silk damping scale, at any epoch, can be obtained from this expression: \( k_{\text{silk}} \simeq (15/(2\tau_e \eta))^{1/2} \simeq (4 \text{Mpc})^{-1}((1 + z)/10^3)^{-5/4} \) in the matter-dominated era. The velocity field in the linear evolution remains non-vortical, and hence can be found from the continuity equation (Eq. 6). It should be noted that solutions for the baryon density and velocity fields differ from the corresponding quantities for the electrons only by replacing \( R \) defined here as \( R' = 4\rho_e/(3p_e) \) (see e.g. Hu & Sugiyama 1995). As \( R' \gg 1 \) for the evolution of the plasma in the pre-recombination universe, for baryonic densities compatible with primordial nucleosynthesis, the baryon and electron quantities can be used interchangeably in evaluating the second order expression above.

The evolution of electron density and velocity in the oscillatory regime and the super-horizon solutions, prior to the epoch of recombination, can be summarized as (for solutions at super-horizon scales in this conformal-Newton gauge see e.g. Ma & Bertschinger 1995):

| \( \delta_e \) | Oscillator \( \Psi = \text{const} \) | for \( k \lesssim \eta^{-1} \) (RD and MD) |
| \( \delta_e \) | Oscillator \( k \lesssim \left( \frac{\eta}{\sqrt{3}} \right)^{-1} \) (RD and MD) | and for \( k \gtrsim \eta^{-1} \) (RD) |
| \( \delta_e \) | \( \eta^2 \) for \( \eta^{-1} \lesssim k \lesssim \left( \frac{\eta}{\sqrt{3}} \right)^{-1} \) (MD) |
| \( v_e \) | \( k\Psi \eta \) for \( k \lesssim \eta^{-1} \) (RD and MD) | and for \( \left( \frac{\eta}{\sqrt{3}} \right)^{-1} \gtrsim k \gtrsim \eta^{-1} \) (MD) |
| \( v_e \) | Oscillator \( k \lesssim \left( \frac{\eta}{\sqrt{3}} \right)^{-1} \) (RD and MD) |

Here RD and MD correspond to radiation and matter dominated epochs, respectively.

**APPENDIX B: MAXWELL’S EQUATIONS FOR FRW BACKGROUND**

The Maxwell’s equations for the FRW metric in terms of physical fields, \( \mathbf{E}, \mathbf{B}, \mathbf{J} \) are as follows:

\[
\nabla \times (a^2 \mathbf{B}) = 4\pi a^3 \mathbf{J} + \frac{\partial(a^2 \mathbf{E})}{\partial \eta} \tag{B1}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{B2}
\]

\[
\nabla \times (a^2 \mathbf{E}) = -\frac{\partial(a^2 \mathbf{B})}{\partial \tau} \tag{B3}
\]
\[ \nabla \cdot (a^2 \mathbf{E}) = 4 \pi a^3 e (n_p - n_e) \]  
(B4)

The current \( \mathbf{J} \) is written in terms of fluid quantities as:

\[ \mathbf{J} = e (n_p \mathbf{v}_p - n_e \mathbf{v}_e) \]  
(B5)

Here, \( n_{e,p} \) are the electronic and protonic number densities which are assumed to be equal to the lowest order i.e \( \bar{n}_e = \bar{n}_p = n \). From Eq. (B1) it follows that \( \nabla \cdot \mathbf{J} = 0 \) if the second term can be neglected, which is the case here (see e.g. Parker 1979).

Eq. (B6) along with Eq. (B5) then shows that:

\[ \nabla \cdot \mathbf{E} = 0 \]  
(B6)

in the linear theory.

APPENDIX C: ACKNOWLEDGEMENT

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