QUASI-CLASSICAL ASYMPTOTICS FOR PSEUDO-DIFFERENTIAL OPERATORS WITH DISCONTINUOUS SYMBOLS: WIDOM’S CONJECTURE

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Abstract. Relying on the known two-term quasiclassical asymptotic formula for the trace of the function \( f(A) \) of a Wiener-Hopf type operator \( A \) in dimension one, in 1982 H. Widom conjectured a multi-dimensional generalization of that formula for a pseudo-differential operator \( A \) with a symbol \( a(x, \xi) \) having jump discontinuities in both variables. In 1990 he proved the conjecture for the special case when the jump in any of the two variables occurs on a hyperplane. The present paper gives a proof of Widom’s Conjecture under the assumption that the symbol has jumps in both variables on arbitrary smooth bounded surfaces.

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1. INTRODUCTION

For two domains $Λ, Ω \subset \mathbb{R}^d$, $d \geq 1$ consider in $L^2(\mathbb{R}^d)$ the operator defined by the formula

\begin{equation}
(\tilde{T}_\alpha(a)u)(x) = \left(\frac{\alpha}{2\pi}\right)^d \chi_\Lambda(x) \int_\Omega \int_{\mathbb{R}^d} e^{i\alpha \xi \cdot (x-y)} a(x, \xi) \chi_\Lambda(y) u(y) dy d\xi, \quad \alpha > 0,
\end{equation}

for any Schwartz class function $u$, where $\chi_\Lambda(\cdot)$ denotes the characteristic function of $\Lambda$, and $a(\cdot, \cdot)$ is a smooth function, with an appropriate decay in both variables. Clearly, $\tilde{T}_\alpha$ is a pseudo-differential operator with a symbol having discontinuities in both variables. We are interested in the asymptotics of the trace $\text{tr} g(\tilde{T}_\alpha)$ as $\alpha \to \infty$ with a smooth function $g$ such that $g(0) = 0$. In 1982 H. Widom in [39] conjectured the asymptotic formula

\begin{equation}
\text{tr} g(\tilde{T}_\alpha(a)) = \alpha^d \mathfrak{W}_0(\mathfrak{A}(g); \Lambda, \Omega) + \alpha^{d-1} \log \alpha \mathfrak{W}_1(\mathfrak{A}(g); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),
\end{equation}

where $\mathfrak{W}_0$ and $\mathfrak{W}_1$ are certain trace-class functionals.
with the following coefficients. For any symbol \( b = b(x, \xi) \), any domains \( \Lambda, \Omega \) and any \( C^1 \)-surfaces \( S, P \), let

\[
W_0(b) = W_0(b; \Lambda, \Omega) = \frac{1}{(2\pi)^d} \int_{\Lambda} \int_{\Omega} b(x, \xi) d\xi dx,
\]

\[
W_1(b) = W_1(b; S, P) = \frac{1}{(2\pi)^{d-1}} \int_{S} \int_{P} b(x, \xi) |n_S(x) \cdot n_P(\xi)| dS dS_x,
\]

where \( n_S(x) \) and \( n_P(\xi) \) denote the exterior unit normals to \( S \) and \( P \) at the points \( x \) and \( \xi \) respectively, and

\[
A(g; b) = \frac{1}{(2\pi)^2} \int_0^1 \frac{g(bt) - tg(b)}{t(1-t)} dt, \quad A(g) := A(g; 1).
\]

The main objective of the paper is to prove the formula (1.2) for a large class of functions \( g \) and bounded domains \( \Lambda, \Omega \).

The interest in the pseudo-differential operators with discontinuous symbols goes back to the classical Szegö formula for the determinant of a Toeplitz matrix, see [35] and [15]. There exists a vast body of literature devoted to various non-trivial generalizations of the Szegö formula in dimension \( d = 1 \), and it is not our intention to review them here. Instead, we refer to the monographs by A. Böttcher-B. Silbermann [5], and by N.K. Nikolski [25] for the background reading, T. Ehrhardt’s paper [9] for a review of the pre-2001 results, and the recent paper by P. Deift, A. Its, I. Krasovsky [8], for the latest results and references. A multidimensional generalization of the continuous variant of the Szegö formula was obtained by I.J. Linnik [24] and H. Widom [37], [38]. In fact, paper [38] addressed a more general problem: instead of the determinant, suitable analytic functions of the operator were considered, and instead of the scalar symbol matrix-valued symbols were allowed: for \( \Omega = \mathbb{R}^d \) and \( a(x, \xi) = a(\xi) \) it was shown that

\[
\text{tr} g(\tilde{T}_a(a)) = \alpha^d V_0 + \alpha^{d-1} V_1 + o(\alpha^{d-1}),
\]

with some explicitly computable coefficients \( V_0, V_1 \), such that \( V_0 = \mathcal{W}_0(g(a)) \) for the scalar case. Under some mild extra smoothness assumptions on the boundary \( \partial \Lambda \), R. Roccabrote (see [26]) found the term of order \( \alpha^{d-2} \) in the above asymptotics of \( \text{tr} g(\tilde{T}_a(a)) \).

The situation changes if we assume that \( \Lambda \neq \mathbb{R}^d \) and \( \Omega \neq \mathbb{R}^d \), i.e. that the symbol has jump discontinuities in both variables, \( x \) and \( \xi \). As conjectured by H. Widom, in this case the second term should be of order \( \alpha^{d-1} \log \alpha \), see formula (1.2). For \( d = 1 \) this formula was proved by H. Landau-H. Widom [20] and H. Widom [39]. For higher dimensions, the asymptotics (1.2) was proved in [40] under the assumptions that one of the domains is a half-space, and that \( g \) is analytic in a disk of a sufficiently large radius. After this paper there have been just a few publications with partial results. Using an abstract version of the Szegö formula with a remainder estimate, found by A. Laptev and Yu. Safarov (see [21], [22]), D. Gioev (see [11, 12]) established a sharp bound

\[
\text{tr} g(\tilde{T}_a(a)) - \alpha^d \mathcal{W}_0(g(a)) = O(\alpha^{d-1} \log \alpha).
\]
In [13] D. Gioev and I.Klich observed a connection between the formula (1.2) and the behaviour of the entanglement entropy for free Fermions in the ground state. As explained in [13], the studied entropy is obtained as \( \text{tr} \, h(\tilde{T}_\alpha) \) with some bounded domains \( \Lambda, \Omega \), the symbol \( a(x, \xi) = 1 \), and the function \( (1.7) \)

\[
h(t) = -t \log t - (1 - t) \log(1 - t), \quad t \in (0, 1).
\]

Since \( h(0) = h(1) = 0 \), the leading term, i.e. \( \mathcal{W}_0(h(1)) \), vanishes, and the conjecture (1.2) gives the \( \alpha^{d-1} \log \alpha \)-asymptotics of the trace, which coincides with the expected quasi-classical behaviour of the entropy. However, the formula (1.2) is not justified for non-smooth functions, and in particular for the function (1.7). Instead, in the recent paper [16] R. Helling, H. Leschke and W. Spitzer proved (1.2) for a quadratic \( g \). With \( g(t) = t - t^2 \) this gives the asymptotics of the particle number variance, which provides a lower bound of correct order for the entanglement entropy.

The operators of the form (1.1) also play a role in Signal Processing. Although the main object there is band-limited functions of one variable, in [33] D. Slepian considered some multi-dimensional generalizations. In particular, he derived asymptotic formulas for the eigenvalues and eigenfunctions of \( \tilde{T}_\alpha(1) \) for the special case when both \( \Lambda \) and \( \Omega \) are balls in \( \mathbb{R}^d \). Some of those results are used in [34]. These results, however, do not allow to study the trace \( \text{tr} \, g(\tilde{T}_\alpha(1)) \).

The main results of the present paper are Theorems 2.3 and 2.4. They establish asymptotic formulas of the type (1.2) for the operator \( T(a) = T_\alpha(a) \), defined in (2.4), which is slightly different from \( \tilde{T}_\alpha(a) \), but as we shall see later, the difference does not affect the first two terms of the asymptotics (1.2). Theorem 2.3 proves formula (1.2) for functions \( g \) analytic in a disk of sufficiently large radius. Theorem 2.4 proves (1.2) for the real part of \( T(a) \) with an arbitrary \( C^\infty \)-function \( g \).

The proof comprises the following main ingredients:

1. Trace class estimates for pseudo-differential operators with discontinuous symbols,
2. Analysis of the problem for \( d = 1 \),
3. A geometrical estimate,
4. Reduction of the initial problem to the case \( d = 1 \).

The most important and difficult is Step 4. Here we divide the domain \( \Lambda \) into a boundary layer, which contributes to the first and second terms in (1.2), and the inner part, which matters only for the first term. Then we construct a suitable partition of unity subordinate to this covering, which is given by the functions \( q^\downarrow = q^\downarrow(x) \) and \( q^\uparrow = q^\uparrow(x) \) respectively. An interesting feature of the problem is that the thickness of the boundary layer does not depend on \( \alpha \). The asymptotics of \( \text{tr}(q^\downarrow g(T_\alpha)) \), \( \alpha \to \infty \), do not feel the boundary, and relatively standard quasi-classical considerations lead to the formula \( (1.8) \)

\[
\text{tr}(q^\uparrow g(T_\alpha)) = \alpha^d \mathcal{W}_0(q^\uparrow g(a)) + O(\alpha^{d-1}), \quad \alpha \to \infty.
\]
To handle the trace $\text{tr}(q^\downarrow g(T_\alpha))$ we construct an appropriate covering of the boundary layer by open sets of a specific shape. For each of these sets the boundary $\partial \Lambda$ is approximated by a hyperplane, which makes it possible to view the operator $T_\alpha$ as a PDO on the boundary hyperplane, whose symbol is an operator of the same type, but in dimension one. This reduction brings us to Step 2 of the plan. The 1-dim situation was studied in H. Widom’s paper [39]. Although its results are not directly applicable, the method developed there allows us to get the required asymptotics. These ensure that

$$\text{tr}(q^\downarrow g(T_\alpha)) = \alpha^d \mathfrak{W}_0(g^\downarrow g(a)) + \alpha^{d-1} \log \alpha \mathfrak{W}_1(g; \alpha) + o(\alpha^{d-1} \log \alpha), \alpha \to \infty.$$  

Adding up (1.8) and (1.9), gives (1.2).

In the reduction to the 1-dim case an important role is played by a result of geometrical nature, which is listed above as the third ingredient. It is loosely described as follows. Representing $\xi \in \mathbb{R}^d$ as $\xi = (\hat{\xi}, t)$, with $\hat{\xi} = (\xi_1, \xi_2, \ldots, \xi_{d-1})$, define for each $\hat{\xi} \in \mathbb{R}^{d-1}$ the set

$$\Omega(\hat{\xi}) = \{t : (\hat{\xi}, t) \in \Omega\} \subset \mathbb{R}.$$  

If it is non-empty, then it is at most countable union of disjoint open intervals in $\mathbb{R}$ whose length we denote by $\tilde{\rho}_j, j = 1, 2, \ldots$. The important observation is that under appropriate restrictions on the smoothness of the boundary $\partial \Omega$, the function

$$\tilde{m}_\delta(\hat{\xi}) = \sum_j \tilde{\rho}_j^{-\delta},$$  

belongs to $L^1(\mathbb{R}^{d-1})$ for all $\delta \in (0, 2)$. The precise formulation of this result is given in Appendix 1.

From the technical viewpoint Steps 2 and 4 are based on the trace class estimates derived at Step 1. In order to work with discontinuous symbols we also establish convenient estimates for smooth ones. The emphasis is on the estimates which allow one to control explicitly the dependence on the parameter $\alpha$, and on the scaling properties of the symbols.

At this point it is appropriate to compare our proof with H. Widom’s paper [40], where (1.2) was justified for the case when $\Lambda$ (or $\Omega$) was a half-space. In fact, our four main steps are the same as in [40]. However, in [40] the relative weight of these ingredients was different. If $\Lambda$ is a half-space, then the reduction to the 1-dim case (i.e. Step 4) is almost immediate whereas in the present paper, for general $\Lambda$ such a reduction is a major issue. In [40] at Step 3 it was sufficient to have the geometric estimate for $\delta = 1$. In the present paper it is crucial to have such an estimate for $\delta > 1$. Moreover, trace class estimates were derived in [40] under the assumption that $\Lambda$ was a half-space, which is clearly insufficient for our purposes. As far as the 1-dim asymptotics are concerned (i.e. Step 2), our estimates are perhaps somewhat more detailed, since apart from the parameter $\alpha$, they allow one to monitor the dependence on the scaling parameters as well.

The detailed structure of the paper is described at the end of Section 2.
Some notational conventions. We conclude the Introduction by fixing some basic notations which will be used throughout the paper. For \( x \in \mathbb{R}^d \) we denote \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \). Very often we split \( x = (x_1, x_2, \ldots, x_d) \) in its components as follows:

\[
\begin{align*}
\mathbf{x} &= (\mathbf{x}, x_d), \\
\mathbf{\tilde{x}} &= (x_1, x_2, \ldots, x_{d-1}), \\
\mathbf{\circ} &= (x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_d).
\end{align*}
\]

and for some \( l = 1, 2, \ldots, d \),

\[
\mathbf{\circ} = (x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_d).
\]

The notation \( B(x, R) \) is used for the open ball in \( \mathbb{R}^d \) of radius \( R > 0 \), centered at \( x \in \mathbb{R}^d \). For some \( \rho > 0 \) let

\[
(1.11) \quad \mathcal{Q}^{(n)}_\rho = (-2\rho, 2\rho)^n
\]

be the \( n \)-dimensional open cube.

The characteristic function of the domain \( \Lambda \subset \mathbb{R}^d, d \geq 1 \), is denoted by \( \chi_\Lambda = \chi_\Lambda(x) \).

To avoid cumbersome notation we write \( \chi_{\mathbf{x}, \ell}(\mathbf{x}) := \chi_{B(\mathbf{x}, \ell)}(\mathbf{x}) \).

For a function \( u = u(x), x \in \mathbb{R}^d \) its Fourier transform is defined as follows:

\[
\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int e^{-i\mathbf{x} \cdot \mathbf{\xi}} u(x) dx.
\]

The integrals without indication of the domain of integration are taken over the entire Euclidean space \( \mathbb{R}^d \).

The notation \( \mathcal{S}_p, 0 < p \leq \infty \) is used for the standard Schatten-von Neumann classes of compact operators in a separable Hilbert space, see e.g. [3], [32]. In particular, \( \mathcal{S}_1 \) is the trace class, and \( \mathcal{S}_2 \) is the Hilbert-Schmidt class. Unless otherwise stated the underlying Hilbert space is assumed to be \( L^2(\mathbb{R}^d) \).

By \( C, c \) (with or without indices) we denote various positive constants whose precise value is of no importance.

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2. Main result

2.1. Definitions and main results. In this paper we need several types of pseudo-differential operators, depending on the parameter $\alpha > 0$. For the symbol $a = a(x, \xi)$, amplitude $p = p(x, y, \xi)$, and any function $u$ from the Schwartz class on $\mathbb{R}^d$ we define

$$\begin{align*}
(\text{Op}_\alpha^a p)u(x) &= \left(\frac{\alpha}{2\pi}\right)^d \int \int e^{i\alpha(x-y)\xi} p(x, y; \xi) u(y)d\xi dy, \\
(\text{Op}_\alpha^l a)u(x) &= \left(\frac{\alpha}{2\pi}\right)^d \int \int e^{i\alpha(x-y)\xi} a(x, \xi) u(y)d\xi dy, \\
(\text{Op}_\alpha^r a)u(x) &= \left(\frac{\alpha}{2\pi}\right)^d \int \int e^{i\alpha(x-y)\xi} a(y, \xi) u(y)d\xi dy.
\end{align*}$$

If the function $a$ depends only on $\xi$, then the operators $\text{Op}_\alpha^l(a)$, $\text{Op}_\alpha^r(a)$ and $\text{Op}_\alpha^a(a)$ coincide with each other, and we simply write $\text{Op}_\alpha(a)$. Later we formulate conditions on $a$ and $p$ which ensure boundedness of the above operators uniformly in the parameter $\alpha \geq 1$. Let $\Lambda, \Omega$ be two domains in $\mathbb{R}^d$, and let $\chi_\Lambda(x)$, $\chi_\Omega(\xi)$ be their characteristic functions. We always use the notation

$$P_{\Omega, \alpha} = \text{Op}_\alpha(\chi_\Omega).$$

We study the operator

$$T_\alpha(a) = T_\alpha(a; \Lambda, \Omega) = \chi_\Lambda P_{\Omega, \alpha} \text{Op}_\alpha^l(a) P_{\Omega, \alpha} \chi_\Lambda,$$

and its symmetrized version:

$$S_\alpha(a) = S_\alpha(a; \Lambda, \Omega) = \chi_\Lambda P_{\Omega, \alpha} \text{Re Op}_\alpha^l(a) P_{\Omega, \alpha} \chi_\Lambda.$$ 

Note that $T_\alpha(a)$ differs from the operator (1.1) by the presence of an extra projection $P_{\Omega, \alpha}$ on the left of $\text{Op}_\alpha^l(a)$. As we shall see later in Section 4, this difference does not affect the first two terms of the asymptotics (1.2).

Let us now specify the class of symbols and amplitudes used throughout the paper. We denote by $S^{(n_1, n_2, m)}$ the set of all (complex-valued) functions $p = p(x, y, \xi)$, which are bounded together with their partial derivatives up to order $n_1$ w.r.t. $x$, $n_2$ w.r.t. $y$ and $m$ w.r.t. $\xi$. It is convenient to define the norm in this class in the following way. For arbitrary numbers $\ell > 0$ and $\rho > 0$ define

$$N^{(n_1, n_2, m)}(p; \ell, \rho) = \max_{0 \leq n \leq n_1} \sup_{x, y, \xi} \ell^{n+k} \rho^r |\nabla_x^n \nabla_y^k \nabla_\xi^r p(x, y, \xi)|.$$

Here we use the notation

$$|\nabla^t f(t)|^2 = \sum_{j_1, j_2, \ldots, j_l = 1}^r |\partial_{j_1} \partial_{j_2} \cdots \partial_{j_l} f(t)|^2.$$
for a function $f$ of the variable $t \in \mathbb{R}$. The presence of the parameters $\ell, \rho$ allows one
to consider amplitudes with different scaling properties.

In the same way we introduce the classes $\mathcal{S}^{(n,m)}$ (resp. $\mathcal{S}^{(m)}$) of all (complex-valued)
functions $a = a(x, \xi)$ (resp. $a = a(\xi)$), which are bounded together with their partial
derivatives up to order $n$ w.r.t. $x$, and $m$ w.r.t. $\xi$. The norm $N^{(n,m)}(a; \ell, \rho)$ (resp.
$N^{(m)}(a; \rho)$) is defined in a way similar to (2.5). Note the straightforward inequality: if
$a \in \mathcal{S}^{(n,m)}, b \in \mathcal{S}^{(n,m)}$, then $ab \in \mathcal{S}^{(n,m)}$ and
\[
N^{(m,n)}(ab; \ell, \rho) \leq C_{m,n} N^{(m,n)}(a; \ell, \rho) N^{(m,n)}(b; \ell, \rho).
\]
As we show later, if the amplitude $p$ and/or symbol $a$ belong to an appropriate class $S$
then the PDO's (2.1)–(2.3) are bounded.

Let $M$ be a non-degenerate linear transformation, and let $k, k_1 \in \mathbb{R}^d$ be some vectors.
By $A = (M, k)$ we denote the affine transformation $Ax = Mx + k$. A special role
is played by the Euclidean isometries, i.e. by the affine transformations of the form
$E = (O, k)$, where $O$ is an orthogonal transformation. The set of all Euclidean isometries
on $\mathbb{R}^d$ is denoted by $E(d)$. Let us point out some useful unitary equivalence for the
operators (2.1) - (2.3). For the affine transformation $(M, k)$ define the unitary operator
$U = U_{M,k} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by
\[
(U_{M,k} u)(x) = \sqrt{|\det M|} u(Mx + k), u \in L^2(\mathbb{R}^d).
\]
It is straightforward to check that for an arbitrary $k_1 \in \mathbb{R}^d$,
\[
(U_{M,k} e^{-i\alpha k_1} Op_\alpha(b) e^{i\alpha k_1} U_{M,k}^{-1})(b_{M,k,k_1}) = Op_\alpha(b_{M,k,k_1}), \tag{2.6}
\]
and
\[
\begin{align*}
U_{M,k} \chi \Lambda U_{M,k}^{-1} &= \chi \Lambda_{M,k} U_{M,k} e^{i\alpha k_1} P_{\Omega,\alpha} e^{-i\alpha k_1} U_{M,k}^{-1} = P_{\Omega,\alpha} \Lambda_{M,k}, \tag{2.7}
\Lambda_{M,k} &= M^{-1}(\Lambda - k), \quad \Omega_{M,k_1}^T = M^T(\Omega - k_1).
\end{align*}
\]
For the operator $T(a)$ this implies that
\[
(U_{M,k} e^{-i\alpha k_1} T(a; \Lambda, \Omega) e^{i\alpha k_1} U_{M,k}^{-1})(a_{M,k,k_1}) = T(a_{M,k,k_1}; \Lambda_{M,k}, \Omega_{M,k_1}^T). \tag{2.8}
\]
Note that the asymptotic coefficients $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are invariant with respect to affine
transformations, i.e.
\[
\begin{align*}
\mathfrak{M}_0(b; \Lambda, \Omega) &= \mathfrak{M}_0(b_{M,k,k_1}; \Lambda_{M,k}, \Omega_{M,k_1}^T), \tag{2.9}
\mathfrak{M}_1(b; \partial \Lambda, \partial \Omega) &= \mathfrak{M}_1(b_{M,k,k_1}; \partial \Lambda_{M,k}, \partial \Omega_{M,k_1}^T).
\end{align*}
\]
The first of the relations (2.9) is immediately checked by changing variables under the
integral (1.3). The second one is proved in Appendix 4, Lemma 16.1.
As one particular type of a linear transformation, it is useful to single out the case when \( M = \ell I \) for some \( \ell > 0 \), and \( k = 0 \), i.e. \( M \) is a scaling transformation. In this situation we denote
\[
(W_\ell u)(x) = (U_{M,0} u)(x) = \ell^2 u(\ell x).
\]

Then a straightforward calculation gives for any \( \rho > 0 \):
\begin{equation}
W_\ell \text{Op}_a^\alpha(b) W_\ell^{-1} = \text{Op}_b^\beta(b_\ell,\rho), \quad b_\ell,\rho(x,y,\xi) = b(\ell x, \ell y, \rho \xi), \quad \beta = \alpha \ell \rho.
\end{equation}

In particular,
\begin{equation}
W_\ell \chi_\Lambda W_\ell^{-1} = \chi_{\Lambda_\ell}, \quad \Lambda_\ell = \ell^{-1} \Lambda,
\end{equation}

It is important that the norm (2.5) is invariant under certain linear transformations. Note first of all that
\begin{equation}
N^{(n_1,n_2,m)}(p;\ell,\rho) = N^{(n_1,n_2,m)}(p_\ell \rho_1, \ell \rho_1^{-1}),
\end{equation}
for arbitrary positive \( \ell, \ell_1, \rho, \rho_1 \). Moreover, the norm is also invariant under Euclidian isometries:
\begin{equation}
N^{(n_1,n_2,m)}(p;\ell,\rho) = N^{(n_1,n_2,m)}(p_{\mathbf{O},k,k_1}; \ell, \rho).
\end{equation}

Sometimes we refer to \( \ell \) and \( \rho \) as scaling parameters.

Now we can specify the classes of domains which we study. We always assume that \( \Lambda \) and \( \Omega \) are domains with smooth boundaries in the standard sense. However, for the reference convenience and to specify the precise conditions on the objects involved, we state our assumptions explicitly.

**Definition 2.1.** We say that a domain \( \Gamma \subset \mathbb{R}^d, d \geq 2 \), is a \( C^m \)-graph-type domain, with some \( m \geq 1 \), if one can find a real-valued function \( \Phi \in C^m(\mathbb{R}^{d-1}) \), with the properties
\begin{equation}
\begin{cases}
\Phi(\hat{0}) = 0, \\
\nabla \Phi \text{ is uniformly bounded on } \mathbb{R}^{d-1}, \\
\nabla \Phi \text{ is uniformly continuous on } \mathbb{R}^{d-1},
\end{cases}
\end{equation}

and some transformation \( \mathbf{E} = (\mathbf{O}, \mathbf{k}) \in E(d) \) such that
\[
\mathbf{E}^{-1} \Gamma = \{ x : x_d > \Phi(\hat{x}) \}, \quad \hat{x} = (x_1, x_2, \ldots, x_{d-1}).
\]

In this case we write \( \Gamma = \Gamma(\Phi; \mathbf{O}, \mathbf{k}) \) or \( \Gamma = \Gamma(\Phi) \), if the omission of the dependence on \( \mathbf{E} \) does not lead to confusion.

We often use the notation
\begin{equation}
M_\Phi = \| \nabla \Phi \|_\infty.
\end{equation}

It is clear that \( \Gamma(\Phi; \mathbf{O}, \mathbf{k}) = \mathbf{E} \Gamma(\Phi; \mathbf{I}, 0) \) with \( \mathbf{E} = (\mathbf{O}, \mathbf{k}) \). The point \( \mathbf{k} \) is on the boundary of the domain \( \Gamma(\Phi; \mathbf{O}, \mathbf{k}) \).

If \( \Lambda = \Gamma(\Phi; \mathbf{O}, \mathbf{k}) \), then the domain \( \Lambda_\ell \) (see definition (2.11)) has the form
\begin{equation}
\Lambda_\ell = \Gamma(\Phi_\ell; \mathbf{O}, \ell^{-1} \mathbf{k}), \quad \Phi_\ell(\hat{x}) = \ell^{-1} \Phi(\ell \hat{x}).
\end{equation}
Note that the value (2.15) is invariant under scaling:

\[(2.17) \quad \|\nabla \Phi\|_{L^\infty} = \|\nabla \Phi_\ell\|_{L^\infty}.\]

In what follows we extensively use the relations (2.6), (2.7) and (2.8) in order to reduce the domains or symbols to a more convenient form. For these purposes, let us make a note of the following elementary property for the domain \(\Lambda = \Gamma(\Phi; I, 0)\). According to (2.7), for any \(k \in \partial \Lambda\) we have

\[(2.18) \quad \Lambda_{I,k} = \Gamma(\Phi_k; I, 0), \quad \Phi_k(\hat{x}) = \Phi(\hat{x} + \hat{k}) - k_d.\]

Clearly, \(\Phi_k(0) = 0\) and \(\|\nabla \Phi\|_{L^\infty} = \|\nabla \Phi_k\|_{L^\infty}\).

In the next definition we introduce general \(C^m\)-domains. Let \(B(x, r) = \{v \in \mathbb{R}^d : |x - v| < r\}\) be the ball of radius \(r > 0\) centered at \(x\).

**Definition 2.2.** Let \(\Lambda \subset \mathbb{R}^d, d \geq 2\) be a domain, \(w \in \mathbb{R}^d\) be a vector and \(R > 0\) be a number.

1. For a \(w \in \partial \Lambda\) we say that in the ball \(B(w, R)\) the domain \(\Lambda\) is represented by the \(C^m\)-graph-type domain \(\Gamma = \Gamma(\Phi; O, w), m \geq 1\), if there is a number \(R = R_w > 0\), such that

\[(2.19) \quad \Lambda \cap B(w, R) = \Gamma \cap B(w, R).\]

2. For a \(w \in \Lambda\) we say that in the ball \(B(w, R)\) the domain \(\Lambda\) is represented by \(\mathbb{R}^d\), if there is a number \(R = R_w > 0\), such that

\[(2.20) \quad \Lambda \cap B(w, R) = B(w, R).\]

3. The domain \(\Lambda\) is said to be \(C^m, m \geq 1\), if for each point \(w \in \partial \Lambda\) there is a number \(R = R_w > 0\), such that in the ball \(B(w, R)\) the domain \(\Lambda\) is represented by a \(C^m\)-graph-type domain \(\Gamma(\Phi; O, w)\) with some \(C^m\)-function \(\Phi = \Phi_w : \mathbb{R}^{d-1} \to \mathbb{R}\), satisfying (2.14), and some orthogonal transformation \(O = O_w\). In this case we also say that the boundary \(\partial \Lambda\) is a \(C^m\)-surface.

The next two theorems represent the main results of the paper:

**Theorem 2.3.** Let \(\Lambda, \Omega \subset \mathbb{R}^d, d \geq 2\) be bounded domains in \(\mathbb{R}^d\) such that \(\Lambda\) is \(C^1\) and \(\Omega\) is \(C^3\). Let \(a = a(x, \xi)\) be a symbol with the property

\[(2.20) \quad \max_{0 \leq n \leq d+2} \sup_{0 \leq m \leq d+2} |\nabla_x^n \nabla_\xi^m a(x, \xi)| < \infty,\]

supported on the set \(B(z, \ell) \times B(\mu, \rho)\) with some \(z, \mu \in \mathbb{R}^d\) and \(\ell, \rho > 0\). Let \(g\) be a function analytic in the disk of radius \(R = C_1 N^{(d+2)(a; \ell, \rho)}\), such that \(g(0) = 0\). If the constant \(C_1\) is sufficiently large, then

\[(2.21) \quad \text{tr} g(T(a)) = \alpha^d \mathfrak{M}_0(g(a); \Lambda, \Omega) + \alpha^{d-1} \log \alpha \mathfrak{M}_1(g(a); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),\]

as \(\alpha \to \infty\).
For the self-adjoint operator $S(a)$ we have a wider choice of functions $g$:

**Theorem 2.4.** Let $\Lambda, \Omega \subset \mathbb{R}^d$, $d \geq 2$ be bounded domains in $\mathbb{R}^d$ such that $\Lambda$ is $C^1$ and $\Omega$ is $C^3$. Let $a = a(x, \xi)$ be a symbol satisfying (2.20) with a compact support in both variables. Then for any function $g \in C^\infty(\mathbb{R})$, such that $g(0) = 0$, one has

$$
\text{tr} g(S(a)) = \alpha^d \mathfrak{M}_0(g(\text{Re}a); \Lambda, \Omega)
$$

(2.22)

$$
+ \alpha^{d-1} \log \alpha \, \mathfrak{M}_1(\mathfrak{A}(g; \text{Re}a); \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),
$$

as $\alpha \to \infty$.

**Remark 2.6.** It would be natural to expect that the variables $x$ and $\xi$ in the operator $T(a)$ have “equal rights”. Indeed, it was shown in [40], p. 173, by an elementary calculation, that the roles of $x, \xi$ are interchangeable. On the other hand, the conditions on $\Lambda$ and $\Omega$ in the main theorems above, are clearly asymmetric. At present it is not clear how to rectify this drawback.

**Remark 2.7.** Denote by $n(\lambda_1, \lambda_2; \alpha)$ with $\lambda_1\lambda_2 > 0, \lambda_1 < \lambda_2$ the number of eigenvalues of the operator $S(a)$ which are greater than $\lambda_1$ and less than $\lambda_2$. In other words,

$$
n(\lambda_1, \lambda_2; \alpha) = \text{tr} \chi_I(S(a)), \ I = (\lambda_1, \lambda_2).
$$

Since the interval $I$ does not contain the point 0, this quantity is finite. Theorem 2.4 can be used to find the leading term of the asymptotics of the counting function $n(\lambda_1, \lambda_2; \alpha)$, by approximating the characteristic function $\chi_I$ with smooth functions $g$. Suppose for instance that $a_- < a(x, \xi) < a_+$, $x \in \Lambda, \xi \in \Omega$, with some positive constants $a_-, a_+$, and that $[a_-, a_+] \subset I$. Then it follows from Theorem 2.4 that

$$
n(\lambda_1, \lambda_2; \alpha) = \left( \frac{\alpha}{2\pi} \right)^d |\Lambda| |\Omega| + \alpha^{d-1} \log \alpha \, \mathfrak{M}_1(\mathfrak{A}(\chi_I; a)) + o(\alpha^{d-1} \log \alpha).
$$

A straightforward calculation shows that

$$
\mathfrak{A}(\chi_I; a(x, \xi)) = \frac{1}{(2\pi)^2} \log \left( \frac{a(x, \xi)}{\lambda_1} - 1 \right).
$$

Another interesting case is when $[\lambda_1, \lambda_2] \subset (0, a_-)$. This guarantees that $\mathfrak{M}_0(\chi_I(a)) = 0$, and the asymptotics of $n(\lambda_1, \lambda_2; \alpha)$ are described by the second term in (2.22):

$$
n(\lambda_1, \lambda_2; \alpha) = \alpha^{d-1} \log \alpha \, \mathfrak{M}_1(\mathfrak{A}(\chi_I; a)) + o(\alpha^{d-1} \log \alpha), \ \alpha \to \infty.
$$
An elementary calculation gives:
\[
\mathcal{A}(\chi_I; a(x, \xi)) = \frac{1}{(2\pi)^2} \log \frac{\lambda_2(a(x, \xi) - \lambda_1)}{\lambda_1(a(x, \xi) - \lambda_2)}.
\]

The proof of Theorems 2.3 and 2.4 splits in two unequal parts. The crucial and the most difficult part is to justify the asymptotics for a polynomial \( g \):

**Theorem 2.8.** Let \( \Lambda, \Omega \subset \mathbb{R}^d, d \geq 2 \) be bounded domains in \( \mathbb{R}^d \) such that \( \Lambda \) is \( C^1 \) and \( \Omega \) is \( C^3 \). Let \( a = a(x, \xi) \) be a symbol satisfying (2.20) with a compact support in both variables. Then for \( g_p(t) = t^p, p = 1, 2, \ldots, \)
\[
\text{tr} g_p(T(a)) = \alpha^d \mathcal{W}_0(g_p(a); \Lambda, \Omega)
\]
\[
+ \alpha^{d-1} \log \alpha \, \mathcal{W}_1(a; \partial \Lambda, \partial \Omega) + o(\alpha^{d-1} \log \alpha),
\]
as \( \alpha \to \infty \). If \( T(a) \) is replaced with \( S(a) \), then the same formula holds with the symbol \( a \) replaced by \( \text{Re} a \) on the right-hand side.

Once this theorem is proved, the asymptotics can be closed with the help of the sharp bounds (12.11) and (12.17), which were derived in [11], [12] using the abstract version of the Szegö formula with a remainder estimate obtained in [21] (see also [22]).

**2.2. Asymptotic coefficient \( \mathcal{A}(g; b) \).** Here we provide some simple estimates for the coefficient \( \mathcal{A} \) defined in (1.5).

**Lemma 2.9.** Suppose that the analytic function \( g \) is given by the series
\[
g(z) = \sum_{m=0}^{\infty} \omega_m z^m
\]
with a radius of convergence \( R > 0 \). Let
\[
g^{(1)}(t) = \sum_{m=2}^{\infty} (m-1) |\omega_m| t^{m-1}, \quad |t| \leq R.
\]
Then for any \( b, |b| < R \) the following estimate holds:
\[
|\mathcal{A}(g; b)| \leq \frac{1}{(2\pi)^2 |b| g^{(1)}(|b|)}.
\]

**Proof.** Consider first \( g_m(z) = z^m, m \geq 2 \), so that
\[
(2\pi)^2 |\mathcal{A}(g_m; b)| \leq |b|^m \int_0^1 \frac{t - t^m}{t(1-t)} dt \leq (m-1)|b|^m.
\]
Thus
\[
|\mathcal{A}(g; b)| \leq \sum_{m=2}^{\infty} |\omega_m| |\mathcal{A}(g_m; b)| \leq \frac{1}{(2\pi)^2} \sum_{m=2}^{\infty} (m-1) |\omega_m| |b|^m,
\]
which leads to (2.24). \( \square \)
Lemma 2.10. Suppose that $g \in C^1(\mathbb{R})$ and $g(0) = 0$. Then for any $b \in \mathbb{R}$ the following estimate holds:

$$|\mathcal{A}(g; b)| \leq \frac{1}{\pi^2} |b| \|g'\|_{L^\infty(-|b|, |b|)}.$$  

Proof. Denote $w = \|g'\|_{L^\infty(-|b|, |b|)}$. Since $|g(t)| \leq w|t|$ for all $|t| \leq |b|$, we have

$$\left| \int_0^{\frac{1}{2}} \frac{g(bt) - tg(b)}{t(1-t)} dt \right| \leq 2w|b| \int_0^{\frac{1}{2}} \frac{1}{1-t} dt \leq 2w|b|.$$

Similarly,

$$\left| \int_{\frac{1}{2}}^1 \frac{g(bt) - tg(b)}{t(1-t)} dt \right| \leq 2w|b| \int_{\frac{1}{2}}^1 \frac{1}{t} dt \leq 2w|b|.$$

Adding up the two estimates, one gets (2.25).

2.3. Plan of the paper. We begin with estimates for norms and trace norms of the pseudo-differential operators with smooth symbols, see Sect. 3. Information about various classes of compact operators can be found in [14], [3], [32]. The first trace norm estimate for PDO’s was obtained in [30], and later reproduced in [31], Proposition 27.3, and [28], Theorem II-49. The fundamental paper [2] contains estimates in various compact operators classes for integral operators in terms of smoothness of their kernels. There are also publications focused on conditions on the symbol which guarantee that a PDO belongs to an appropriate Neumann-Schatten ideal, see e.g. [27], [1], [36], [7] and references therein. In spite of a relatively large number of available literature, these results are not sufficient for our purposes. We need somewhat more detailed information about trace norms. In particular, we derive an estimate for the trace norm of a PDO with weights, i.e. of $h_1 \text{Op}_\alpha(a) h_2$, where the supports of $h_1$ and $h_2$ are disjoint. The most useful was paper [29], which served as a basis for our approach. Although our estimates are quite elementary, and, probably not optimal, they provide bounds of correct orders in $\alpha$ and the scaling parameters.

Sections 4, 5 are devoted to trace-class estimates for PDO’s with various jump discontinuities. The most basic estimates are those for the commutators $[\text{Op}_\alpha(a), \chi_\Lambda]$, $[\text{Op}_\alpha(a), P_{\Omega,\alpha}]$ where $\Lambda$ and $\Omega$ are graph-type domains, see Lemmas 4.3 and 4.5. Similar bounds were derived in [40] for the case when one of the domains $\Lambda, \Omega$ is a half-space. The new estimates which play a decisive role in the proof, are collected in Sect. 5. Here the focus is on the PDO’s with discontinuous symbols sandwiched between weights having disjoint supports. As in the case of smooth symbols in Sect. 3, it is important for us to control the dependence of trace norms on the distance between the supports. The simplest result in Sect. 5, illustrating this dependence is Lemma 5.1. The estimates culminate in Lemma 5.5 which bounds the error incurred when replacing $\Lambda$ by a half-space in the operator $\overline{T}(a; \Lambda, \Omega)$.

The estimate (6.1) obtained in Sect. 6 is used only for closing the asymptotics in Sect. 12. In contrast to the results obtained in Sections 4 and 5, which estimate norms
3. Estimates for PDO’s with smooth symbols

In this technical section we establish estimates for the norms and trace norms of the operators (2.1) – (2.3).

In Subsections 3.1 and 3.2 we assume that $\alpha = 1$, which does not restrict generality in view of the scaling invariance (2.10). On the other hand, in Subsection 3.4 it is more convenient to allow arbitrary values $\alpha > 0$.

3.1. Boundedness. To find convenient estimates for the norms we use the idea of [19], Theorem 2 (see also [23], Lemma 2.3.2 for a somewhat simplified version).

**Theorem 3.1.** Let $\omega \geq 0$ be a number, and let $p(x,y,\xi)$ be an amplitude such that for some $\omega \geq 0$, the function

$$\langle x - y \rangle^{-\omega} |\nabla_x^{n_1} \nabla_y^{n_2} \nabla_{\xi}^m p(x,y,\xi)|$$
is a bounded for all

\[ n_1, n_2 \leq r := \left[ \frac{d}{2} \right] + 1, \quad \text{and} \quad m \leq s := [d + \omega] + 1. \]

Then \( \text{Op}_1^a(p) \) is a bounded operator and

\[ \| \text{Op}_1^a(p) \| \leq C \max_{n_1, n_2 \leq r} \sup_{x, y, \xi} \frac{|\nabla_{x}^{n_1} \nabla_{y}^{n_2} \nabla_{\xi}^m p(x, y, \xi)|}{|x - y|^{-\omega}}, \]

with a constant \( C \) depending only on \( d \) and \( \omega \).

Proof. Denote \( \text{Op}_1^a(p) = \text{Op}(p) \). Let us estimate

\[
\begin{align*}
(\text{Op}(p)u, v) &= \frac{1}{(2\pi)^d} \int \int \int e^{i(x-y)} p(x, y, \xi) u(y) \overline{v(x)} dy d\xi dx \\
&= \frac{1}{(2\pi)^2d} \int \int \int \int e^{i(x-y)} e^{i(t-\xi)\cdot y} p(x, y, \xi) \hat{u}(t) \overline{\hat{v}(\eta)} dy d\xi dx dt d\eta
\end{align*}
\]

with arbitrary \( u, v \in C_0^\infty(\mathbb{R}^d) \). Thus we may assume that \( p \) is in the Schwarz class on \( \mathbb{R}^d \). Let \( P_x(\xi) \) be the operator

\[
\frac{1 - i\xi \cdot \nabla_x}{\langle \xi \rangle^2}.
\]

Since \( P_x(\xi) e^{i\xi \cdot x} = e^{i\xi \cdot x} \), after integration by parts \( k \) times in \( x \) and \( y \), we get

\[
(2\pi)^{2d} (\text{Op}(p)u, v)
\]

\[
= \int \int \int \int \int e^{i(x-y)} e^{i(t-\xi)\cdot y} \left( P_x^{(\xi-\eta)} \right)^k \left( P_y^{(t-\xi)} \right)^k p(x, y, \xi) \hat{u}(t) \overline{\hat{v}(\eta)} dy d\xi dx dt d\eta.
\]

Since \( P_x(\xi) e^{i\xi \cdot x} = e^{i\xi \cdot x} \), we can also integrate \( s \) times by parts in \( \xi \):

\[
(2\pi)^{2d} (\text{Op}(p)u, v) = \int \int \int \int \int \int e^{i(x-y)} e^{i(t-\xi)\cdot y} \left( P_x^{(\xi-\eta)} \right)^s \left( P_y^{(t-\xi)} \right)^k p(x, y, \xi) \hat{u}(t) \overline{\hat{v}(\eta)} dy d\xi dx dt d\eta.
\]

This integral is a finite sum of terms of the form

\[
\int \int \int \int e^{i(x-y)} e^{i(t-\xi)\cdot y} g_{sm}(x - y) \phi_{kn_1}(\xi - \eta) \psi_{kn_2}(t - \xi)
\]

\[
p_{n_1n_2m}(x, y, \xi) \hat{u}(t) \overline{\hat{v}(\eta)} dy d\xi dx dt d\eta,
\]

where \( m \leq s, n_1, n_2 \leq k, |p_{n_1n_2m}(x, y, \xi)| \leq |\nabla_x^{n_1} \nabla_y^{n_2} \nabla_{\xi}^m p(x, y, \xi)| \), and

\[
|\phi_{kn_1}(\mu)| + |\psi_{kn_2}(\mu)| \leq C(\mu)^{-k}, \quad |g_{sm}(z)| \leq C(z)^{-s}.
\]
Rewrite the above integral in the form
\[
I := (2\pi)^d \int \int \int e^{i\xi \cdot (x-y)} p_{n_1n_2m}(x, y, \xi) g_{sm}(x - y) \overline{\Phi(x, \xi)} \overline{\Psi(y, \xi)} \, dx \, dy \, d\xi,
\]
with
\[
\Phi(x, \xi) = \frac{1}{(2\pi)^d} \int e^{i\eta \cdot x} \hat{\phi}_{kn_1}(\xi - \eta) \hat{v}(\eta) \, d\eta,
\]
\[
\Psi(y, \xi) = \frac{1}{(2\pi)^d} \int e^{it \cdot y} \hat{\psi}_{kn_2}(t - \xi) \hat{u}(t) \, dt.
\]
Both functions \(\Phi\) and \(\Psi\) are \(L^2(\mathbb{R}^2)\) and
\[
\|\Psi\|_{L^2}^2 \leq C\|u\|_{L^2}^2, \quad \|\Phi\|_{L^2}^2 \leq C\|v\|_{L^2}^2.
\]
Indeed, by Parseval’s identity,
\[
\|\Phi\|_{L^2}^2 = \int \int |\phi_{kn_1}(\xi - \eta)|^2 |\hat{v}(\eta)|^2 \, d\eta \, d\xi = \|\phi_{kn_1}\|_{L^2}^2 \|v\|_{L^2}^2,
\]
and the norm of \(\phi_{kn_1}\) is finite if we choose \(k > d/2\). Similarly for \(\Psi\). Thus the integral \(I\) defined above can be estimated as follows:
\[
|I| \leq C \sup_{x, y, \xi} \langle x - y \rangle^{-\omega} |p_{n_1n_2m}(x, y, \xi)|
\]
\[
\left[ \int \int (x - y)^{-s+\omega} \|\Phi(x, \xi)\|^2 \, dx \, dy \, d\xi \right]^{\frac{1}{2}} \left[ \int \int (x - y)^{-s+\omega} \|\Phi(y, \xi)\|^2 \, dx \, dy \, d\xi \right]^{\frac{1}{2}}.
\]
Here we have used Hölder’s inequality. The product of the last two integrals equals
\[
\|\Phi\|_{L^2} \|\Psi\|_{L^2} \int \langle z \rangle^{-s+\omega} \, dz,
\]
and the integral of \(\langle z \rangle^{-s+\omega}\) is finite if we choose \(s > \omega + d\). \(\square\)

One should say that there is a simpler looking test of boundedness for PDO’s of the type \(\text{Op}^a_{1}(a)\) which requires no smoothness w.r.t. \(\xi\), but imposes certain decay condition at infinity in the variable \(x\), see [18], Theorem 18.11’. A similar result can be also obtained for the operator \(\text{Op}^a_{p}(p)\), but it would require from \(p(x, y, \xi)\) a decay in both \(x\) and \(y\), which is not convenient for us.

3.2. Trace class estimates. Again, we assume that \(\alpha = 1\). We are going to use the ideas from [29]. In fact, our estimates are nothing but more precise quantitative variants of Proposition 3.2 and Theorem 3.5 from [29].

Assuming that \(p \in L^1(\mathbb{R}^d)\), introduce the “double” Fourier transform:
\[
\hat{p}(\eta, \mu, \xi) = \frac{1}{(2\pi)^d} \int \int e^{-ix \cdot \eta - iy \cdot \mu} p(x, y, \xi) \, dx \, dy.
\]
Lemma 3.2. Suppose that \( \hat{p} \in L^1(\mathbb{R}^d), p \in L^1(\mathbb{R}^d) \). Then for any \( h_1, h_2 \in L^2(\mathbb{R}^d) \) the operator \( h_1 \text{Op}^a_1(p)h_2 \) is trace class and

\[
\|h_1 \text{Op}^a_1(p)h_2\|_{\mathcal{E}_1} \leq (2\pi)^{-d} \|h_1\|_{L^2} \|h_2\|_{L^2} \int \int \int |\hat{p}(\eta, \mu, \xi)|d\eta d\mu d\xi.
\]

Proof. Represent the amplitude \( p \) as follows:

\[
p(x, y, \xi) = \frac{1}{(2\pi)^d} \int \int e^{ix\cdot\eta + iy\cdot\mu} \hat{p}(\eta, \mu, \xi) d\eta d\mu.
\]

Let \( g_j, m_j \) be two orthonormal sequences in \( L^2(\mathbb{R}^d) \). Then

\[
(h_1 \text{Op}^a_1(p)h_2g_j, m_j) = \left( \frac{1}{2\pi} \right)^{2d} \int \int \left[ \int e^{ix\cdot\eta + iy\cdot\mu} h_1(x)m_j(x) dx \right. \\
\times \left. \int e^{-iy\cdot\xi + ix\cdot\mu} h_2(y) g_j(y) dy \right] \hat{p}(\eta, \mu, \xi) d\xi d\eta d\mu,
\]

and by the Bessel inequality,

\[
\sum_j |(h_1 \text{Op}^a_1(p)h_2g_j, m_j)| \leq \left( \frac{1}{2\pi} \right)^{2d} \|h_1\|_{L^2} \|h_2\|_{L^2} \int \int \int |\hat{p}(\eta, \mu, \xi)|d\eta d\mu d\xi.
\]

By Theorems 11.2.3,4 from [3], page 246, the operator \( h_1 \text{Op}^a_1(p)h_2 \) is trace class and its trace norm satisfies the required bound. \( \square \)

It is usually more convenient to write these estimates in terms of the amplitudes themselves, and not their Fourier transforms. In all the statements below we always assume that the amplitudes (symbols) have the required partial derivatives and that the integrals involved, are finite.

Corollary 3.3. Let \( h_1, h_2 \) be arbitrary \( L^2 \)-functions. Then

\[
\|h_1 \text{Op}^a_1(p)h_2\|_{\mathcal{E}_1} \leq C \|h_1\|_{L^2} \|h_2\|_{L^2} \sum_{n_1, n_2 = 0}^{d+1} \int \int |\nabla_x^{n_1} \nabla_y^{n_2} p(x, y, \xi)| dxdy d\xi.
\]

Proof. Integrating by parts, we get:

\[
|\hat{p}(\eta, \mu, \xi)| \leq C(1 + |\eta|)^{-d-1}(1 + |\mu|)^{-d-1} \sum_{n_1, n_2 = 0}^{d+1} \int \int |\nabla_x^{n_1} \nabla_y^{n_2} p(x, y, \xi)| dxdy.
\]

Substituting this estimate in (3.1), we get the required result. \( \square \)

It is also useful to have bounds for operators with \( h_1 \) and \( h_2 \) having disjoint supports. Below we denote by \( \zeta \in C^\infty(\mathbb{R}) \) a function such that

\[
(3.2) \quad \zeta(t) = 1 \text{ if } |t| \geq 2, \quad \text{and} \quad \zeta(t) = 0 \text{ if } |t| \leq 1.
\]

In all the subsequent estimates constants may depend on \( \zeta \) and its derivatives, but it is unimportant for our purposes.
Corollary 3.4. Let $h_1, h_2$ be two $L^2$-functions. Suppose that $p(x, y, \xi) = 0$ if $|x - y| \leq R$ with some $R \geq c$. Then

$$\|h_1 \text{Op}_1^a(p) h_2\|_{\mathcal{S}_1} \leq C_m \|h_1\|_{L^2} \|h_2\|_{L^2} \sum_{n_1, n_2=0}^{d+1} \int \int_{|x-y| \geq R} \frac{\|\nabla_{x}^{n_1} \nabla_{y}^{n_2} p(x, y, \xi)\|}{|x - y|^m} dxdy d\xi,$$

for any $m = 0, 1, \ldots$

In particular, if $p(x, y, \xi)$ depends only on $x$ and $\xi$, i.e. $p(x, y, \xi) = a(x, \xi)$, and the essential supports of $h_1$ and $h_2$ are separated by a distance $R$, then

$$\|h_1 \text{Op}_1^a(a) h_2\|_{\mathcal{S}_1} + \|h_1 \text{Op}_1^a(a) h_2\|_{\mathcal{S}_1} \leq C_m \|h_1\|_{L^2} \|h_2\|_{L^2} \sum_{n_0=0}^{d+1} \int \int |\nabla_{x}^{n_0} a(x, \xi)| dxd\xi,$$

for any $m \geq d + 1$.

Proof. If the essential supports of $h_1$ and $h_2$ are separated by a distance $R$, then $h_1 \text{Op}_1^a(p) h_2$ can be rewritten as

$$h_1 \text{Op}_1^a(\tilde{p}) h_2, \tilde{p}(x, y, \xi) = p(x, y, \xi) \zeta(2|x - y| R^{-1}),$$

with the function $\zeta \in C^\infty(\mathbb{R})$ defined in (3.2). Thus the bound (3.4) immediately follows from (3.3).

Proof of (3.3). Let $P = -i|x - y|^{-2}(x - y) \cdot \nabla_{\xi}$. Clearly, $Pe^{i\xi(x-y)} = e^{i\xi(x-y)}$, so, integrating by parts $m$ times, we get the following formula for the kernel of the operator $\text{Op}_1^a(p)$:

$$\frac{1}{(2\pi)^d} \int e^{i\xi(x-y)} q(x, y, \xi) d\xi,$$

with

$$q(x, y, \xi) = i^m \frac{1}{|x - y|^{2m}} (x - y) \cdot \nabla_{\xi} p(x, y, \xi).$$

It is straightforward to see that

$$\sum_{n_1, n_2=0}^{d+1} |\nabla_{x}^{n_1} \nabla_{y}^{n_2} q(x, y, \xi)| \leq C \frac{1}{|x - y|^m} \sum_{n_1, n_2=0}^{d+1} |\nabla_{x}^{n_1} \nabla_{y}^{n_2} \nabla_{\xi}^m p(x, y, \xi)|.$$

By Corollary 3.3 this implies the proclaimed result. \qed
Lemma 3.5. Let $h_1, h_2$ be arbitrary $L^2$-functions. Then
\[
\|h_1 \text{Op}_1^a(p) h_2\|_{\mathcal{E}_1} \leq C_Q \|h_1\|_{L^2} \|h_2\|_{L^2}
\]
\begin{equation}
\times \sum_{n_1, n_2=0}^{d+1} \sum_{m=0}^{Q} \int \int \int \frac{|\nabla_x^{n_1} \nabla_y^{n_2} \nabla_{\xi}^m p(x,y,\xi)|}{1 + |x-y|^Q} dxdyd\xi,
\end{equation}
for any $Q = 0, 1, \ldots$.

Proof. Let a function $\zeta$ be as defined in (3.2). Denote
\[
p_1(x,y,\xi) = p(x,y,\xi)(1 - \zeta(x-y)), \quad p_2(x,y,\xi) = p(x,y,\xi)\zeta(x-y).
\]
Estimate separately the trace norms of the operators $h_1 \text{Op}_1^a(p_1) h_2$ and $h_1 \text{Op}_1^a(p_2) h_2$. Since $|x-y| \leq 2$ on the support of $p_1$, by Corollary 3.3, the trace norm of $h_1 \text{Op}_1^a(p_1) h_2$ does not exceed the right hand side of (3.5). For $p_2$ one uses Corollary 3.4, which also gives the required bound. \(\square\)

In the next Theorem we replace the $L^2$-norms of functions $h_1, h_2$ by much weaker ones. Let $C = [0,1]^d$ be the unit cube, and let $C_z = C + z, z \in \mathbb{R}^d$. For $s \in (0, \infty]$ and any function $h \in L_{\text{loc}}^2$ introduce the following quasi-norm:
\begin{equation}
\left\| h \right\|_s = \left[ \sum_{x \in \mathbb{Z}^d} \left( \int_{C_x} |h(x)|^2 dx \right)^s \right]^{\frac{1}{2}}, \quad 0 < s < \infty,
\end{equation}
\begin{equation}
\left\| h \right\|_{\infty} = \sup_{x \in \mathbb{R}^d} \left( \int_{C_x} |h(x)|^2 dx \right)^s, \quad s = \infty.
\end{equation}

Theorem 3.6. Let $h_1, h_2$ be arbitrary $L_{\text{loc}}^2$-functions. Then
\[
\|h_1 \text{Op}_1^a(p) h_2\|_{\mathcal{E}_1} \leq C_Q \left\| h_1 \right\|_{L^\infty} \left\| h_2 \right\|_{L^\infty} \sum_{n_1, n_2=0}^{d+1} \sum_{m=0}^{Q} \int \int \int \frac{|\nabla_x^{n_1} \nabla_y^{n_2} \nabla_{\xi}^m p(x,y,\xi)|}{1 + |x-y|^Q} dxdyd\xi,
\]
for any $Q = 0, 1, \ldots$.
In particular, if $p(x,y,\xi)$ depends only on $x$ and $\xi$, i.e. $p(x,y,\xi) = a(x,\xi)$, then
\[
\|h_1 \text{Op}_1^a(h_2)\|_{\mathcal{E}_1} + \|h_1 \text{Op}_1^a(h_2)\|_{\mathcal{E}_1}
\]
\begin{equation}
\leq C \left\| h_1 \right\|_{L^\infty} \left\| h_2 \right\|_{L^\infty} \sum_{n=0}^{d+1} \sum_{m=0}^{d+1} \int \int \nabla_x^n \nabla_{\xi}^m a(x,\xi) dxd\xi.
\end{equation}

Proof. The estimate (3.8) follows immediately from (3.7) with $Q = d + 1$. Let us prove (3.7).
Let \( \zeta_j \in C^\infty_0(\mathbb{R}^d) \), \( j \in \mathbb{Z} \), be a partition of unity subordinate to a covering of \( \mathbb{R}^d \) by unit cubes, such that the number of intersecting cubes is uniformly bounded, and

\[
|\nabla^n_x \zeta_j(x)| \leq C_n, \quad n = 1, 2, \ldots,
\]

for all \( x \in \mathbb{R}^d \) uniformly in \( j \). By \( \chi_j \) we denote the characteristic function of the cube labeled \( j \). In view of (3.9), for the amplitude

\[
q_{j,s}(x,y,\xi) = \zeta_j(x)\zeta_s(y)p(x,y,\xi),
\]

we obtain from Lemma 3.5 that

\[
\|h_1 \chi_j \operatorname{Op}^a_1(q_{j,s}) h_2 \chi_s\|_{S^1} \leq C \|h_1 \chi_j\|_{L^2} \|h_2 \chi_s\|_{L^2} \sum_{n_1,n_2=0}^{d+1} \sum_{m=0}^{Q} \int \int \int \chi_j(x)\chi_s(y) \frac{|\nabla^{n_1}_x \nabla^{n_2}_y \nabla^m_\xi p(x,y,\xi)|}{1 + |x-y|^Q} dx dy d\xi,
\]

The \( L^2 \)-norms of \( h_1 \chi_j \) and \( h_2 \chi_s \) are estimated by \( \|h_1\|_\infty \) and \( \|h_2\|_\infty \) respectively. Thus, remembering that the number of intersecting cubes is uniformly bounded, we obtain from the bound

\[
\|h_1 \operatorname{Op}^a_1(p) h_2\|_{S^1} \leq \sum_{j,s} \|h_1 \operatorname{Op}^a_1(q_{j,s}) h_2\|_{S^1},
\]

the required estimate (3.7). \( \square \)

In the same way one proves the “local” variant of Corollary 3.4:

**Theorem 3.7.** Let \( h_1, h_2 \) be two \( L^2 \)-functions. Suppose that \( p(x,y,\xi) = 0 \) if \( |x-y| \leq R \) with some \( R \geq c \). Then

\[
\|h_1 \operatorname{Op}^a_1(p) h_2\|_{S^1} \leq C \|h_1\|_\infty \|h_2\|_\infty \sum_{n_1,n_2=0}^{d+1} \int \int |\nabla^{n_1}_x \nabla^{n_2}_y \nabla^m_\xi p(x,y,\xi)| |x-y|^m dx dy d\xi,
\]

for any \( m = 0, 1, \ldots \).

In particular, if \( p(x,y,\xi) \) depends only on \( x \) and \( \xi \), i.e. \( p(x,y,\xi) = a(x,\xi) \), and the essential supports of \( h_1 \) and \( h_2 \) are separated by a distance \( R \), then

\[
\|h_1 \operatorname{Op}^a_1(a) h_2\|_{S^1} + \|h_1 \operatorname{Op}^r_1(a) h_2\|_{S^1} \leq C \|h_1\|_\infty \|h_2\|_\infty R^{d-m} \sum_{n=0}^{d+1} \int \int |\nabla^{n}_x \nabla^m_\xi a(x,\xi)| dx d\xi,
\]

for any \( m \geq d+1 \).
3.3. Operators of a special form. In addition to the general PDO’s we often work with operators of the form $h \text{Op}_1(a), a = a(\xi)$, which have been studied quite extensively. We need the following estimate which can be found in [2], Theorem 11.1 (see also [4], Section 5.8), and for $s \in [1, 2]$ in [32], Theorem 4.5.

**Proposition 3.8.** Suppose that $h \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $a \in L^2_{\text{loc}}(\mathbb{R}^d)$ are functions such that $|h|_s, |a|_s < \infty$ with some $s \in (0, 2)$. Then $h \text{Op}_1(a) \in \mathcal{S}_s$ and

$$
\|h \text{Op}_1(a)\|_{\mathcal{S}_s} \leq C \|h\|_s \|a\|_s.
$$

3.4. Amplitudes from classes $\mathcal{S}^{(n_1,n_2,m)}$: semi-classical estimates. Here we apply the trace class estimates obtained so far to symbols and amplitudes from the classes $\mathcal{S}^{(n_1,n_2,m)}$. Now we are concerned with estimates for $\alpha$-PDO, with an explicit control of dependence on $\alpha$. Moreover, we shall explicitly monitor the dependence on scaling parameters in terms of norms $N^{(n_1,n_2,m)}(p; \ell, \rho)$.

**Lemma 3.9.** Assume that $p \in \mathcal{S}^{(k,k,d+1)}$ with

$$
k = \left\lceil \frac{d}{2} \right\rceil + 1.
$$

Let $\ell > 0, \rho > 0$ be two parameters such that $\alpha \ell \rho \geq c$. Then $\text{Op}_1^\alpha(p)$ is a bounded operator and

$$
\| \text{Op}_1^\alpha(p)\| \leq C N^{(k,k,d+1)}(p; \ell, \rho).
$$

**Proof.** Using (2.12) with $\ell_1 = (\alpha \rho)^{-1}, \rho_1 = \rho$, and the unitary equivalence (2.10), we conclude that it suffices to prove the sought inequalities for $\alpha = \rho = 1$ and arbitrary $\ell \geq c$. Without loss of generality suppose also that $N^{(k,k,d+1)}(p; \ell, 1) = 1$, so that

$$
|\nabla_x^{n_1}\nabla_y^{n_2}\nabla_\xi^m p(x, y, \xi)| \leq \ell^{-n_1-n_2} \leq C,
$$

for all $n_1, n_2 \leq k, m \leq d + 1$. Now the required bound follows from Theorem 3.1 with $\omega = 0$. \hfill \Box

**Lemma 3.10.** Suppose that $p \in \mathcal{S}^{(k,k,d+2)}$ with $k$ defined in (3.11), and that $\alpha \ell \rho \geq c$. Denote $a(x, \xi) = p(x, x, \xi)$. Then $a \in \mathcal{S}^{(k,d+2)}$, and

$$
\| \text{Op}_1^\alpha(p) - \text{Op}_1^\alpha(a)\| \leq C(\alpha \ell \rho)^{-1}N^{(k,k,d+2)}(p; \ell, \rho).
$$

Moreover, for any symbol $a \in \mathcal{S}^{(k,d+2)}$, we have

$$
\| \text{Op}_1^\alpha(a) - \text{Op}_1^\alpha(a)\| \leq C(\alpha \ell \rho)^{-1}N^{(k,d+2)}(a; \ell, \rho).
$$

**Proof.** The bound (3.14) follows from (3.13) with $p(x, y, \xi) = a(y, \xi)$, so that $p \in \mathcal{S}^{(k,k,d+2)}$ and $N^{(n_1,n_2,m)}(p; \ell, \rho) = N^{(n_2,m)}(a; \ell, \rho)$ for $n_1, n_2 \leq k, m \leq d + 2$. As in the proof of Lemma 3.9, in view of (2.12) and (2.10) we may assume that $\alpha = 1, \rho = 1$ and $\ell \geq c$. In order to apply Theorem 3.1 note that the amplitude $b(x, y, \xi) = p(x, y, \xi) - p(x, x, \xi)$ satisfies the bounds

$$
|\nabla_x^{n_1}\nabla_y^{n_2}\nabla_\xi^m b(x, y, \xi)| \leq C \ell^{-n_1-n_2}N^{(k,k,d+2)}(p; \ell, 1),
$$
for any $n_1, n_2 \leq k, m \leq d + 2$, and
\[
|\nabla^m_\xi b(x, y, \xi)| \leq \ell^{-1}N^{(0,1,d+2)}(p; \ell, 1)|x - y|,
\]
for any $m \leq d + 2$. Therefore,
\[
\langle x - y \rangle^{-1}|\nabla^m_x \nabla^m_y \nabla^m_\xi b(x, y, \xi)| \leq C\ell^{-1}N^{(k,k,d+2)}(p; \ell, 1), \quad n_1, n_2 \leq k, m \leq d + 2.
\]
Now by Theorem 3.1 with $\omega = 1$ we get
\[
\|\text{Op}_\alpha^a(b)\| \leq C\ell^{-1}N^{(k,k,d+2)}(p; \ell, 1),
\]
which leads to (3.13). \hfill \Box

Now we obtain appropriate trace class bounds. All bounds will be derived under one of the following conditions. For the operator $\text{Op}_\alpha^a(p)$ we assume that

the support of the amplitude $p = p(x, y, \xi)$ is contained

\begin{equation}
\text{(3.15)} \quad \text{either in } B(z, \ell) \times \mathbb{R}^d \times B(\mu, \rho)
\end{equation}

\begin{equation}
\text{(3.16)} \quad \text{or in } \mathbb{R}^d \times B(z, \ell) \times B(\mu, \rho),
\end{equation}

with some $z, \mu \in \mathbb{R}^d$ and some $\ell > 0, \rho > 0$. For the operators $\text{Op}_\alpha^l(a), \text{Op}_\alpha^r(a)$ we assume that

\begin{equation}
\text{(3.17)} \quad \text{the support of the symbol is contained in } B(z, \ell) \times B(\mu, \rho).
\end{equation}

The constants in the obtained estimates will be independent of $z, \mu$ and $\ell, \rho$.

**Lemma 3.11.** Let $p \in S^{(d+1,d+1,d+1)}$ be an amplitude satisfying either the condition (3.15) or (3.16), and let $a \in S^{(d+1,d+1)}$ be a symbol satisfying the condition (3.17).

1. If $\alpha\ell\rho \geq c$, then $\text{Op}_\alpha^a(p) \in \mathcal{G}_1$ and $\text{Op}_\alpha^l(a) \in \mathcal{G}_1, \text{Op}_\alpha^r(a) \in \mathcal{G}_1$, and

\begin{equation}
\|\text{Op}_\alpha^a(p)\|_{\mathcal{G}_1} \leq C(\alpha\ell\rho)^dN^{(d+1,d+1,d+1)}(p; \ell, \rho),
\end{equation}

\begin{equation}
\|\text{Op}_\alpha^l(a)\|_{\mathcal{G}_1} \leq C(\alpha\ell\rho)^dN^{(d+1,d+1,d+1)}(a; \ell, \rho).
\end{equation}

2. If $a \in S^{(d+1,m)}$ with some $m \geq d + 1$, and $h_1, h_2$ are $L^2_{\text{loc}}$-functions, whose supports are separated by a distance $R > 0$. Suppose that $\alpha\rho R \geq c$ and $\alpha\ell\rho \geq c$. Then $h_1\text{Op}_\alpha^l(a)h_2 \in \mathcal{G}_1$ and

\[
\|h_1\text{Op}_\alpha^l(a)h_2\|_{\mathcal{G}_1} + \|h_1\text{Op}_\alpha^r(a)h_2\|_{\mathcal{G}_1}
\leq C_m(\alpha\ell\rho)^d(\alpha R\rho)^{-m+d}\|h_1\|_{\infty}\|h_2\|_{\infty}N^{(d+1,m)}(a; \ell, \rho),
\]

where the norm $\| \cdot \|_s$ is defined in (3.6).
Proof. The estimate (3.19) is a special case of (3.18).

In view of (2.6) for both (3.18) and (3.20) we may assume that \( z = \mu = 0 \). Furthermore, using (2.12) and (2.10) with \( \ell_1 = (\alpha \rho)^{-1}, \rho_1 = \rho \), we see that it suffices to prove the sought inequalities for \( \alpha = 1, \rho = 1 \) and arbitrary \( \ell \geq c \), and, in the case of (3.20), arbitrary \( R > c \). Assume without loss of generality that \( N^{(d+1,d+1,d+1)}(p; \ell, 1) = 1 \) and \( N^{(d+1,m)}(a; \ell, 1) = 1 \).

Proof of (3.20). Suppose for definiteness that the support of the amplitude \( p \) satisfies (3.15). Since

\[
|\nabla_x^n \nabla_y^n \nabla_\xi^m p(x, y, \xi)| \leq \ell^{-n_1-n_2} \chi_0.1(x) \chi_{0,1}(\xi) \leq C \chi_0.1(x) \chi_{0,1}(\xi),
\]

for all \( n_1, n_2 \leq d + 1 \) and \( m \leq d + 1 \), the bound (3.7) with \( Q = d + 1 \) and \( h_1 = h_2 = 1 \) gives that

\[
\| \text{Op}^1_1(p)\|_{\mathfrak{S}_1} \leq C \int_{|\xi| \leq 1} \int_{\mathbb{R}^d} \int_{|x| \leq \ell} \frac{1}{1 + |x - y|^{d+1}} \, dxdy|\xi| \leq C \ell^d,
\]

which leads to (3.18).

Proof of (3.20). Since

\[
|\nabla_x^n \nabla_\xi^s a(x, \xi)| \leq \ell^{-n} \chi_{0.1}(x) \chi_{0,1}(\xi) \leq C \chi_{0,1}(x) \chi_{0,1}(\xi),
\]

for all \( n \leq d + 1 \) and \( s \leq m \), the bound (3.10) with \( h_1 = h_2 = 1 \) gives that

\[
\| h_1 \text{Op}^1_1(a) h_2 \|_{\mathfrak{S}_1} + \| h_1 \text{Op}^r_1(a) h_2 \|_{\mathfrak{S}_1} \leq C \left| h_1 \right|_{\mathfrak{S}_\infty} \left| h_2 \right|_{\mathfrak{S}_\infty} \int_{|\xi| \leq 1} \int_{|x| \leq \ell} dxd|\xi|
\]

\[
\quad \leq C \left| h_1 \right|_{\mathfrak{S}_\infty} \left| h_2 \right|_{\mathfrak{S}_\infty} R^{d-m} \ell^d,
\]

which leads to (3.20). \( \square \)

**Lemma 3.12.** Let \( p \in S^{(d+1,d+1,d+2)} \) be an amplitude satisfying either the condition (3.15) or (3.16), and let \( a \in S^{(d+1,d+2)} \) be a symbol satisfying the condition (3.17). Suppose that \( \alpha \ell \rho \geq c \). Denote \( b(x, \xi) = p(x, x, \xi) \). Then \( b \in S^{(d+1,d+2)} \), it satisfies (3.17), and

\[
(3.21) \quad \| \text{Op}_{\alpha}^a(p) - \text{Op}_{\alpha}^r(a)\|_{\mathfrak{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+1,d+1,d+2)}(p; \ell, \rho).
\]

Moreover,

\[
(3.22) \quad \| \text{Op}_{\alpha}^a(a) - \text{Op}_{\alpha}^r(a)\|_{\mathfrak{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+1,d+1,d+2)}(a; \ell, \rho).
\]

Proof. The bound (3.22) follows from (3.21) with \( p(x, y, \xi) = a(y, \xi) \), so that \( p \in S^{(d+1,d+1,d+2)} \) and \( N^{(d+1,d+1,d+2)}(p; \ell, \rho) = N^{(d+1,d+2)}(a; \ell, \rho) \).

Proof of (3.21). As in the proof of Lemma 3.11, in view of (2.6), (2.12) and (2.10) we may assume that \( z = \mu = 0 \) and \( \alpha = 1, \rho = 1 \) and \( \ell \geq c \). For definiteness suppose that the condition (3.15) is satisfied. Without loss of generality assume that
\( \mathcal{N}^{(d+1,d+1,d+2)}(p;\ell,1) = 1 \). In order to apply Theorem 3.6 note that the amplitude
\[ g(x,y,\xi) = p(x,y,\xi) - p(x,x,\xi) \]
satisfies the bounds
\[ |\nabla^n_x \nabla^{n_2}_y \nabla^{n_3}_\xi g(x,y,\xi)| \leq C\ell^{-n_1-n_2} \chi_0,\ell(x)\chi_0,1(\xi), \]
\[ |\nabla^m_\xi g(x,y,\xi)| \leq \ell^{-1}|x-y|\chi_0,\ell(x)\chi_0,1(\xi), \]
for any \( n_1, n_2 \leq d+1 \) and \( m \leq d+2 \), so that
\[ |\nabla^n_x \nabla^{n_2}_y \nabla^{n_3}_z g(x,y,\xi)| \leq C\ell^{-1}(x-y)\chi_0,\ell(x)\chi_0,1(\xi), \quad n_1, n_2 \leq d+1, \quad m \leq d+2. \]

Thus by Theorem 3.6 with \( Q = d+2 \), we have
\[ \|\text{Op}_\alpha^s(g)\|_{\mathcal{E}_1} \leq C\ell^{-1} \int \int \int \frac{1}{1+|x-y|^{d+1}} dy dx d\xi \leq C\ell^{d-1}, \]
which is the required bound. \( \square \)

**Corollary 3.13.** Let \( a, b \in \mathcal{S}^{(d+1,d+2)} \) be two symbols such that either \( a \) or \( b \) satisfies (3.17). Suppose that \( \alpha \ell \rho \geq c \). Then
\[ (3.23) \quad \|\text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b) - \text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b)\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} \mathcal{N}^{(d+1,d+2)}(a;\ell,\rho) \mathcal{N}^{(d+1,d+2)}(b;\ell,\rho). \]

**Proof.** Suppose that \( a \) satisfies (3.17). By (3.19) and (3.14),
\[ \|\text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b) - \text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b)\|_{\mathcal{E}_1} \leq \|\text{Op}_\alpha^1(a)\|_{\mathcal{E}_1} \|\text{Op}_\alpha^1(b) - \text{Op}_\alpha^1(b)\| \]
\[ \leq C(\alpha \ell \rho)^{d-1} \mathcal{N}^{(d+1,d+1)}(a;\ell,\rho) \mathcal{N}^{(k,d+2)}(b;\ell,\rho), \]
where \( k \) is defined in (3.11). The operator \( \text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b) \) has the form \( \text{Op}_\alpha^2(p) \) with \( p(x,y,\xi) = a(x,\xi)b(y,\xi) \). Clearly, \( p \) satisfies the condition (3.15), \( p \in \mathcal{S}^{(d+1,d+1,d+2)} \) and
\[ \mathcal{N}^{(d+1,d+1,d+2)}(p;\ell,\rho) \leq C \mathcal{N}^{(d+1,d+2)}(a;\ell,\rho) \mathcal{N}^{(d+1,d+2)}(b;\ell,\rho). \]
The symbol of \( \text{Op}_\alpha^1(ab) \) is \( p(x,x,\xi) \). Thus by (3.21),
\[ \|\text{Op}_\alpha^1(a) \text{Op}_\alpha^1(b) - \text{Op}_\alpha^1(ab)\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} \mathcal{N}^{(d+1,d+2)}(a;\ell,\rho) \mathcal{N}^{(d+1,d+2)}(b,\ell,\rho). \]
Together with (3.24) this gives (3.23).

The case when \( b \) satisfies (3.17) is done in a similar way, and we omit the proof. \( \square \)

We need also an estimate for the trace of the difference \( \text{Op}_\alpha^1(a)^m - \text{Op}_\alpha^1(a^m) \) with an explicit control of the dependence on \( m \).

**Lemma 3.14.** Let \( a \in \mathcal{S}^{(d+2,d+2)} \) satisfy (3.17). Suppose that \( \alpha \ell \rho \geq 1 \). Then
\[ (3.25) \quad \|(\text{Op}_\alpha^1(a))^m - \text{Op}_\alpha^1(a^m)\|_{\mathcal{E}_1} \leq D^{m-1}(m-1)^{2(d+2)}(\alpha \ell \rho)^{-1}(\mathcal{N}^{(d+1,d+2)}(a;\ell,\rho))^m, \]
for any \( m = 1, 2, \ldots, \) with the constant
\[ D = D_{tr} + D_{\text{norm}}, \]
where \( D_{\text{norm}}, D_{tr} \) are the constants in the bounds (3.12) and (3.23) respectively under the condition \( \alpha \ell \rho \geq 1 \).
Proof. Suppose without loss of generality that \( \ell = \rho = 1 \) and \( N^{(d+1,d+2)}(a; 1, 1) = 1 \), so that by Lemma 3.9,
\[
\| \text{Op}_\alpha(a) \| \leq D_{\text{norm}}, \text{ if } \alpha \geq 1.
\]

Note that
\[
N^{(d+1,d+2)}(a^p; 1, 1) \leq p^{2(d+2)},
\]
for any \( p = 1, 2, \ldots \).

We prove the estimate (3.25) by induction. For \( m = 2 \) it follows from (3.23) that
\[
\| (\text{Op}_\alpha(a))^2 \| \leq D_{\text{tr}} \bigl( \alpha + 1 \bigr) \leq D \alpha + 1.
\]
Suppose that the sought estimate holds for \( m = p \), i.e.
\[
\| (\text{Op}_\alpha(a))^p \| \leq D^{p-1}(p-1)^{2(d+2)} \alpha^{d-1},
\]
and let us show that it holds for \( m = p + 1 \). Rewrite:
\[
(\text{Op}_\alpha(a))^{p+1} - \text{Op}_\alpha(a^{p+1}) = [(\text{Op}_\alpha(a))^p - \text{Op}_\alpha(a^p)] \text{Op}_\alpha(a)
\]
\[
+ [\text{Op}_\alpha(a^p) \text{Op}_\alpha(a) - \text{Op}_\alpha(a^{p+1})].
\]
By (3.27), (3.26) and (3.23),
\[
\| (\text{Op}_\alpha(a))^{p+1} - \text{Op}_\alpha(a^{p+1}) \| \leq D^{p-1}(p-1)^{2(d+2)} D_{\text{norm}} \alpha + D_{\text{tr}} p^{2(d+2)} D^{d-1}
\leq p^{2(d+2)} \alpha^{d-1} (D_{\text{norm}} + D_{\text{tr}}).
\]
Since \( D_{\text{norm}} \geq 1 \), the right-hand side of the above bound does not exceed \( D^{p} p^{2(d+2)} \alpha^{d-1} \), as required.

4. Trace-class estimates for operators with non-smooth symbols

Here we obtain trace class estimates for operators with symbols having jump discontinuities. More precisely, we consider symbols containing characteristic functions \( \chi_\Lambda \) and/or \( P_{\Omega,\alpha} \). For \( d \geq 2 \) both domains \( \Lambda \) and \( \Omega \) are supposed to be \( C^1 \)-graph-type domain. We intend to consider the cases \( d \geq 2 \) and \( d = 1 \) simultaneously. For \( d = 1 \) we assume as a rule that \( \Lambda \) and \( \Omega \) are half-infinite open intervals. For the reference convenience we state these conditions explicitly:

**Condition 4.1.** If \( d = 1 \), then \( \Lambda = \{ x \in \mathbb{R} : x \geq x_0 \} \) and \( \Omega = \{ \xi \in \mathbb{R} : \xi \geq \xi_0 \} \) with some \( x_0, \xi_0 \in \mathbb{R} \).

If \( d \geq 2 \), then both \( \Lambda \) and \( \Omega \) are \( C^1 \)-graph-type domains in the sense of Definition 2.1, i.e. \( \Lambda = \Gamma(\Phi; O_{\Lambda}, k_\Lambda) \), \( \Omega = \Gamma(\Psi; O_{\Omega}, k_\Omega) \) with some Euclidean isometries \( (O_{\Lambda}, k_\Lambda) \), \( (O_{\Omega}, k_\Omega) \) and some \( C^1 \)-functions \( \Phi, \Psi \), satisfying the conditions (2.14).
The precise values of the constants $M_\Phi = \| \nabla \Phi \|_{L^\infty}$ and $M_\Psi = \| \nabla \Psi \|_{L^\infty}$ do not play any role in this section. In fact, our results will be uniform in the functions $\Phi$, $\Psi$, satisfying the condition
\begin{equation}
\max(M_\Phi, M_\Psi) \leq M,
\end{equation}
with some constant $M$. Referring to the unitary equivalence (2.7), we often assume that either $O_\Lambda = I$, $k_\Lambda = 0$, or $O_\Omega = I$, $k_\Omega = 0$.

4.1. A partition of unity. For any $C^1$-graph-type domain $\Lambda$ we construct a specific partition of unity on $\mathbb{R}^d$. The following remark on the domain $\Lambda = \Gamma(\Phi; I, 0)$ will be useful for this construction. In view of the condition (4.1),
\begin{equation}
|x_d - \Phi(\hat{x}) - (y_d - \Phi(\hat{y}))| \leq \langle M \rangle |x - y|, \quad \langle M \rangle := \sqrt{1 + M^2}
\end{equation}
for all $x, y \in \mathbb{R}^d$. In particular, if $x \in \Lambda$ and $y \notin \Lambda$, i.e. if $x_d > \Phi(\hat{x})$ and $y_d \leq \Phi(\hat{y})$, we have $x_d - \Phi(\hat{x}) \leq \langle M \rangle |x - y|$, and hence
\begin{equation}
\text{dist}\{x, \mathcal{C}\Lambda\} \geq \frac{1}{\langle M \rangle} (x_d - \Phi(\hat{x})), \quad x \in \Lambda = \Gamma(\Phi; I, 0),
\end{equation}
where $\mathcal{C}\Lambda$ is the standard notation for the complement of $\Lambda$.

Lemma 4.2. Let $\Lambda = \Gamma(\Phi; O, k)$ be a graph type domain with $\Phi$ satisfying the condition (4.1), and let
\begin{equation}
\Lambda(t) = \Gamma(\Phi + t; O, k), \quad t \in \mathbb{R}.
\end{equation}
Then for any $\delta > 0$ there exist two non-negative functions $\zeta_1 = \zeta_1^{(\delta)}$, $\zeta_2 = \zeta_2^{(\delta)} \in C^\infty(\mathbb{R}^d)$ such that $\zeta_1 + \zeta_2 = 1$,
\begin{equation}
\zeta_1(x) = \begin{cases} 0, & \text{if } x \notin \Lambda^{(\delta)}, \\ 1, & \text{if } x \in \Lambda^{(3\delta(M))}, \end{cases}
\end{equation}
and
\begin{equation}
|\nabla^s \zeta_1(x)| + |\nabla^s \zeta_2(x)| \leq C_s \delta^{-s}, \quad s = 1, 2, \ldots,
\end{equation}
uniformly in $x$, with constants $C_s$, which may depend on $M$, but are independent of the function $\Phi$, satisfying (4.1), on the transformation $O$ or on the vector $k$.

Proof. Since the left-hand side of (4.4) is invariant with respect to the orthogonal transformations and translations, without loss of generality we may assume that $O = I$ and $k = 0$. Let $\psi_j \in C^\infty_0, j = 1, 2, \ldots$, be a partition of unity of $\mathbb{R}^d$ subordinate to the covering of $\mathbb{R}^d$ by balls of radius 2 centred at the points of the integer lattice $\mathbb{Z}^d$. Clearly,
\begin{equation}
|\nabla^s \psi_j(x)| \leq C_s,
\end{equation}
uniformly in $j$. Denote
\begin{equation}
\phi_j(x) = \psi_j(x \delta^{-1}),
\end{equation}
Consequently \( \zeta \) covering of \( \square \) satisfied in view of (4.5). This completes the proof.

\[ |\nabla^s \phi_j(x)| \leq C_s \delta^{-s}, \]

uniformly in \( k \). Define

\[ \zeta_1(x) = \sum_j \phi_j(x), \]

where the summation is taken over all indices \( j \) such that

\[ \phi_j(x)\chi_{\Lambda^{(s)}}(x) = \phi_j(x), \]

so that \( \zeta_1(x) = 0 \) for \( x \notin \Lambda^{(s)} \). By (4.3) the distance from any point \( x \in \Lambda^{(s)}, \epsilon > \delta \) to \( \mathcal{C}\Lambda^{(s)} \) is bounded from below by \( \langle M \rangle^{-1}(\epsilon - \delta) \), so we conclude that

\[ \text{dist}\{\Lambda(R), \mathcal{C}\Lambda^{(s)}\} \geq 2\delta, \ R = 3\delta(M). \]

Consequently \( \zeta_1(x) = 1 \) for all \( x \in \Lambda(R) \), as required. Define \( \zeta_2 = 1 - \zeta_1 \). Then (4.4) is satisfied in view of (4.5). This completes the proof. \( \square \)

For the functions \( \zeta_1, \zeta_2 \) constructed in the above lemma we write sometimes \( \zeta_1^{(s)}(x; \Lambda) \) and \( \zeta_2^{(s)}(x; \Lambda) \).

4.2. Trace class estimates. Assume as before, that Condition 4.1 is satisfied. If \( d \geq 2 \), the in all subsequent estimates the constants are independent of the transformations \( (O_\Lambda, k_\Lambda) \) and \( (O_\Omega, k_\Omega) \), and are uniform in the functions \( \Phi \) and \( \Psi \) satisfying (4.1), but may depend on the constant \( M \) in (4.1). If \( d = 1 \), then the estimates are uniform in the numbers \( x_0, \xi_0 \), which enter the definitions of \( \Lambda \) and \( \Omega \).

As in the previous section we assume as a rule that the symbols are compactly supported and satisfy the condition (3.17). The constants in the obtained estimates will be independent of \( z, \mu, \ell, \rho \).

Lemma 4.3. Suppose that the symbol \( a \in S^{(d+1,d+2)} \) satisfies (3.17) if \( d \geq 2 \), and that it is supported on \( \mathbb{R} \times B(\mu, \rho) \) if \( d = 1 \), with some \( \mu \in \mathbb{R} \). Assume that \( \alpha \ell \rho \geq c \). Let \( \text{Op}_\alpha(a) \) denotes any of the operators \( \text{Op}_\alpha^l(a) \) or \( \text{Op}_\alpha^r(a) \). Then

\[ \|\chi_{\Lambda} \text{Op}_\alpha(a)(1 - \chi_{\Lambda})\|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+1,d+2)}(a; \ell, \rho), \]

and

\[ \|[\text{Op}_\alpha(a), \chi_{\Lambda}]\|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+1,d+2)}(a; \ell, \rho). \]

Proof. For definiteness we prove the above estimates for the operator \( \text{Op}_\alpha^l(a) \).

The estimate (4.7) follows from from (4.6) due to the identity

\[ [\text{Op}_\alpha^l(a), \chi_{\Lambda}] = (1 - \chi_{\Lambda}) \text{Op}_\alpha^l(a)\chi_{\Lambda} - \chi_{\Lambda} \text{Op}_\alpha^l(a)(1 - \chi_{\Lambda}). \]

Let us prove (4.6). In view of (2.6), (2.13) and (2.7) we may assume that \( O_\Lambda = I, k = 0 \), i.e. \( \Lambda = \Gamma(\Phi; I, 0) \), if \( d \geq 2 \), and \( \Lambda = (0, \infty) \), if \( d = 1 \).
Assume that \( d \geq 2 \). Using (2.12), (2.11) and (2.10) with \( \ell_1 = (\alpha \rho)^{-1}, \rho_1 = \rho \), one reduces the estimate (4.6) to
\[
\| \chi_{\Lambda_{\ell_1}} \text{Op}_{a_1}(a_{\ell_1}, \rho_1)(1 - \chi_{\Lambda_{\ell_1}}) \|_{S_1} \leq C(\alpha \rho \rho_1)^{d-1} N^d(a_{\ell_1}, \rho_1; \alpha \rho \rho_1, 1),
\]
where we have used the notation \( \Lambda_{\ell_1} = \Gamma(\Phi_{\ell_1}; I, 0), \Phi_{\ell_1}(\hat{x}) = \ell_1^{-1} \Phi(\ell_1 \hat{x}) \). Thus in view of (2.17) it suffices to prove (4.6) for \( \alpha = \rho = 1 \), and arbitrary \( \ell \geq c \).

The next step is to replace the characteristic functions with their smoothed-out versions. Let \( \zeta_1 \) and \( \zeta_2 \) be the functions constructed in Lemma (4.2) for \( \delta = 1 \). It is clear that
\[
\chi_{\Lambda}(x) \leq \zeta_1(x; \Lambda_{-3(M)}),
\]
\[
1 - \chi_{\Lambda}(x) \leq \zeta_2(x; \Lambda),
\]
Therefore it suffices to estimate the trace-norm of the operator
\[
\zeta_1 \text{Op}_{a_1}(a) \zeta_2.
\]

Denote (4.8)
\[
p(x, y, \xi) = \zeta_1(x; \Lambda_{-3(M)}) a(x, \xi) \zeta_2(y; \Lambda),
\]
so that by (4.4),
\[
|\nabla_x^{n_1} \nabla_y^{n_2} \nabla_\xi^m p(x, y, \xi)| \leq CN^{n_1, m}(a; \ell, 1) \chi_{x, \ell}(x) \chi_{y, \ell}(y),
\]
for all \( n_1 \leq d + 1, m \leq d + 2 \) and all \( n_2 \). Here we have used the fact that \( \ell \geq c \). For the sake of brevity assume, without loss of generality, that
\[
N^{d+1, d+2}(a; \ell, 1) = 1.
\]

By Theorem 3.6 with \( Q = d + 2 \),
\[
\| \text{Op}_a^\alpha(p) \|_{S_1} \leq C \int_{x_d \geq \Phi(x) - 3(M)} \int_{y_d \leq \Phi(y) + 3(M)} \frac{\chi_{x, \ell}(x) \chi_{y, \ell}(y)}{1 + |x - y|^{d+2}} d\xi dy dx
\]
\[
\leq C(I_1(\ell) + I_2(\ell)),
\]
where
\[
I_1(\ell) = \int_{x_d \geq \Phi(x) + 3(M)} \int_{y_d \leq \Phi(y) + 3(M)} \frac{\chi_{x, \ell}(x)}{1 + |x - y|^{d+2}} dy dx,
\]
\[
I_2(\ell) = \int_{x_d - \Phi(x) \leq 3(M)} \int \frac{\chi_{x, \ell}(x)}{1 + |x - y|^{d+2}} dy dx.
\]

By (4.3), in the integral \( I_1(\ell) \) we have
\[
|x - y| \geq \frac{1}{\langle M \rangle} (x_d - \Phi(x) - 3(M)).
\]
Thus we get
\[
I_1(\ell) \leq C \int_{x \in B(\hat{z}, \ell)} \int_{x_d \geq \Phi(\hat{x}) + 3\langle M \rangle} (1 + |x_d - \Phi(\hat{x}) - 3\langle M \rangle|^2)^{-1} dx_d d\hat{x}
\]
\[
\leq \tilde{C} \int_{\hat{x} \in B(\hat{z}, \ell)} d\hat{x} \leq C' \ell^{d-1}.
\]
The integral \(I_2(\ell)\) is estimated in a more straightforward way:
\[
I_2(\ell) \leq C \int_{x \in B(\hat{z}, \ell)} \int_{|x_d - \Phi(\hat{x})| \leq 3\langle M \rangle} dx_d d\hat{x} \leq \tilde{C} \int_{\hat{x} \in B(\hat{z}, \ell)} d\hat{x} \leq C' \ell^{d-1},
\]
so that \(I_1(\ell) + I_2(\ell) \leq C' \ell^{d-1}\), which entails (4.6).

Suppose now that \(d = 1\). As in the case \(d \geq 2\) it suffices to prove the estimate for \(\alpha = \rho = 1\) and arbitrary \(\ell \geq c\). Let \(p(x, y, \xi) = \zeta_1(x)a(x, \xi)\zeta_2(y)\) with \(\zeta_1, \zeta_2 \in C^\infty\) such that
\[
\zeta_1(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 1, \end{cases} \quad \zeta_2(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 1. \end{cases}
\]
Then
\[
|\partial_x^{n_1}\partial_y^{n_2}\partial_\xi^m p(x, y, \xi)| \leq C N^{(n_1, m)}(a; \ell, 1) \chi_{\mu, 1}(\xi).
\]
Therefore Theorem 3.6 with \(Q = 3\) gives
\[
\|\text{Op}_1^a(p)\|_{\mathfrak{S}_1} \leq C N^{(2, 3)}(a; \ell, 1) \int_{x \geq -1} \int_{y \leq +1} \int \frac{\chi_{\mu, 1}(\xi)}{1 + |x - y|^3} d\xi dxdy \leq C' N^{(2, 3)}(a; \ell, 1),
\]
which leads to (4.6)

**Corollary 4.4.** Let \(d = 1\), and let \(a \in S^{(3)}\) be a function supported on \((\mu - \rho, \mu + \rho)\) with some \(\mu \in \mathbb{R}, \rho > 0\). Suppose that \(\Lambda = (x_0, \infty)\) with some \(x_0 \in \mathbb{R}\). Then
\[
(4.9) \quad \|\chi_\Lambda \text{Op}_\alpha(a)(1 - \chi_\Lambda)\|_{\mathfrak{S}_1} \leq C N^{(3)}(a; \rho),
\]
and
\[
(4.10) \quad \|[\text{Op}_\alpha(a), \chi_\Lambda]\|_{\mathfrak{S}_1} \leq C N^{(3)}(a; \rho),
\]
for all \(\alpha > 0\) and \(\rho > 0\), uniformly in \(x_0, \mu \in \mathbb{R}\).

**Proof.** Consider \(a = a(\xi)\) as a function of two variables \(x, \xi\), so that
\[
N^{(2, 3)}(a; \ell, \rho) = N^{(3)}(a; \rho)
\]
for any \(\ell > 0\). Thus, using (4.6) and (4.7) with \(\ell = (\alpha \rho)^{-1}\) we get (4.9) and (4.10) for arbitrary \(\alpha > 0\) and \(\rho > 0\).

We need a similar result for the operator \(P_{31, \alpha}\) instead of \(\chi_\Lambda\):
Lemma 4.5. Suppose that the symbol $a \in S^{(d+2,d+1)}$ satisfies (3.17), and that it is supported on $(z - \ell, z + \ell) \times \mathbb{R}$ with some $z \in \mathbb{R}$ if $d = 1$. Assume that $\alpha \ell \rho \geq c$. Let $\text{Op}_a(a)$ denote any of the operators $\text{Op}_a^l(a)$ or $\text{Op}_a^r(a)$. Then

\begin{equation}
\|P_{\Omega,\alpha} \text{Op}_a(a)(1 - P_{\Omega,\alpha})\|_{\mathfrak{S}_1} \leq C(\alpha \ell \rho)^{d-1}\mathbf{N}^{(d+2,d+1)}(a; \ell, \rho),
\end{equation}

and

\begin{equation}
\|[\text{Op}_a(a), P_{\Omega,\alpha}]\|_{\mathfrak{S}_1} \leq C(\alpha \ell \rho)^{d-1}\mathbf{N}^{(d+2,d+1)}(a; \ell, \rho).
\end{equation}

Proof. Inverting the roles of the variables $x$ and $\xi$, one obtains the above estimates directly from Lemma 4.3.

Let us prove the analogue of Corollary 3.13 for the operators of the form $T(a; \Lambda, \Omega)$.

Corollary 4.6. Let $d \geq 1$ and $\alpha \ell \rho \geq c$. Let $a, b \in S^{(d+2,d+2)}$ be symbols, such that either $a$ or $b$ satisfies the condition (3.17). Then

\begin{equation}
\|T(a)T(b) - T(1)T(ab)\|_{\mathfrak{S}_1} + \|T(b)T(a) - T(ab)T(1)\|_{\mathfrak{S}_1} + \|[T(a), T(b)]\|_{\mathfrak{S}_1},
\end{equation}

\begin{equation}
\leq C(\alpha \ell \rho)^{d-1}\mathbf{N}^{(d+2,d+2)}(a; \ell, \rho)\mathbf{N}^{(d+2,d+2)}(b; \ell, \rho).
\end{equation}

Proof. For definiteness assume that $a$ satisfies (3.17). For brevity we write $\text{Op}$ instead of $\text{Op}_a$. Then

\begin{equation}
\|T(a)T(b) - T(1)T(ab)\|_{\mathfrak{S}_1} \leq 2\|[\text{Op}(a), P_{\Omega,\alpha}]\|_{\mathfrak{S}_1}\|\text{Op}(b)\|
\end{equation}

\begin{equation}
+ \|[\text{Op}(a), \chi_\Lambda]\|_{\mathfrak{S}_1}\|\text{Op}(b)\| + \|\text{Op}(a)\text{Op}(b) - \text{Op}(ab)\|_{\mathfrak{S}_1}.
\end{equation}

The claimed estimate for $T(a)T(b) - T(1)T(ab)$ follows from (4.7), (4.12) Lemma 3.9 and Corollary 3.13. In the same way one proves the required bound for the trace norm of $T(b)T(a) - T(ab)T(1)$.

Using any of these two bounds for the symbols $a' = ab$ and $b' = 1$ we get the same estimate for the trace norm of $T(ab)T(1) - T(1)T(ab)$, which leads to (4.13) for the commutator $[T(a), T(b)]$.

We need a version of the above lemma for the one-dimensional case with a specific choice of $a$ and $b$:

Lemma 4.7. Let $d = 1$, and let $a = a(\xi), b = b(\xi)$ be functions from $S^{(3)}$ such that either $a$ or $b$ is supported on $(\mu - \rho, \mu + \rho)$ with some $\mu \in \mathbb{R}$ and $\rho > 0$. Assume that $\Lambda = (x_0, \infty)$ with some $x_0 \in \mathbb{R}$, and let $\Omega$ be an arbitrary open subset of $\mathbb{R}$. Then for all $\alpha > 0$, $\rho > 0$ we have

\begin{equation}
\|T(a)T(b) - T(ab)T(1)\|_{\mathfrak{S}_1} + \|T(a)T(b) - T(1)T(ab)\|_{\mathfrak{S}_1}
\end{equation}

\begin{equation}
+ \|[T(a), T(b)]\|_{\mathfrak{S}_1} \leq C\mathbf{N}^{(3)}(a; \rho)\mathbf{N}^{(3)}(b; \rho),
\end{equation}

with a constant independent of $x_0, \mu$ and $\Omega$. 

Proof. For definiteness assume that \( b \) is supported on \((\mu - \rho, \mu + \rho)\). For brevity we write Op instead of \( \text{Op}_{\alpha} \). Since \( [\text{Op}(b), P_{\Omega, \alpha}] = 0 \) and \( \text{Op}(a) \text{Op}(b) = \text{Op}(ab) \), we can write
\[
\|T(a)T(b) - T(ab)T(1)\|_{\varepsilon_1} + \|T(b)T(a) - T(1)T(ab)\|_{\varepsilon_1} \\
\leq 2\|\text{Op}(b), \chi_{\Lambda}\|_{\varepsilon_1} \|\text{Op}(a)\| \leq C\|a\|_N \mathcal{N}^{(3)}(b; \rho).
\]
At the last step we have used (4.10). Using either of the above estimates for \( b' = ab, a' = 1 \), we get the estimate (4.14) for the trace norm of \( T(ab)T(1) - T(1)T(ab) \). This leads to the same bound for the commutator \([T(a), T(b)]\). \( \square \)

5. Further trace-class estimates for operators with non-smooth symbols

5.1. Weights with disjoint supports. Here we obtain more special bounds for trace norms of PDO’s with discontinuous symbols.

As in Section 4, we always assume that both domains \( \Lambda \) and \( \Omega \) satisfy Condition 4.1. To avoid cumbersome proofs in this section we always assume that \( d \geq 2 \), although many of the estimates easily generalize to \( d = 1 \). As before, in all subsequent estimates the constants are independent of the functions \( \Phi \) and \( \Psi \) defining \( \Lambda \) and \( \Omega \), but may depend on the constant \( M \) in (4.1).

It is technically convenient to introduce smoothed-out versions of the characteristic functions of the balls. Let \( h, \eta \in C_0^\infty(\mathbb{R}^d) \) be two non-negative functions such that \( 0 \leq h \leq 1, 0 \leq \eta \leq 1 \), \( h(x) = 1 \) for \( |x| \leq 1 \), and \( h(x) = 0 \) for \( |x| \geq 5/4 \); \( \eta(\xi) = 1 \) for \( |\xi| \leq 1 \), and \( \eta(\xi) = 0 \) for \( |\xi| \geq 5/4 \). Denote
\[
\begin{align*}
&h_{z, \ell}(x) = h((x - z)\ell^{-1}); \\
&\eta_{\mu, \rho}(\xi) = \eta((\xi - \mu)\rho^{-1}), \ \Xi_{\mu, \rho} = \text{Op}_{\alpha}(\eta_{\mu, \rho}).
\end{align*}
\]

Constants in all estimates will be independent of \( z, \mu \) and \( \rho, \ell \).

Lemma 5.1. Let \( \alpha \ell \rho \geq c \). Then for any \( r \geq 8/5 \),
\[
\|h_{z, \ell} \Xi_{\mu, \rho} P_{\Omega, \alpha}(1 - h_{z, \ell})\|_{\varepsilon_1} + \| (1 - h_{z, \ell}) \Xi_{\mu, \rho} P_{\Omega, \alpha} h_{z, \ell} \|_{\varepsilon_1} \leq C(\alpha \ell \rho)^{d-1} r^{-\omega},
\]
with arbitrary \( \omega < 1/2 \), uniformly in \( z, \mu \in \mathbb{R}^d \). The constant \( C \) is independent of the parameters \( \alpha, \ell, \rho \).

Proof. For brevity we write \( \text{Op}_{\alpha} \) instead of \( \text{Op}_{\alpha}' \). Furthermore, we make some assumptions which do not restrict generality.

1. By virtue of (2.6) and (2.7) we may assume that the coordinates are chosen in such a way that \( \Omega = \Gamma(\Psi; \mathbf{I}, \mathbf{0}) \);
2. Using (2.12), (2.17) and (2.10) with \( \ell_1 = \ell \) and \( \rho_1 = (\alpha \ell)^{-1} \), we conclude that it suffices to prove the estimate for \( \ell = \alpha = 1 \) and \( \rho \geq c \).
It is sufficient to establish the sought bound for the first term on the left-hand side of (5.2), since the estimated operators are mutually adjoint.

Split the symbol $\eta_{\mu,\rho}\chi_\Omega$ into two parts: smooth and non-smooth in the following way. Let $\zeta^{(\delta)}_1(\xi) = \zeta^{(\delta)}_1(\xi; \Omega)$ and $\zeta^{(\delta)}_2(\xi) = \zeta^{(\delta)}_2(\xi; \Omega)$ be the functions constructed in Lemma 4.2 for $\delta > 0$. Define

$$\eta_{\mu,\rho}\chi_\Omega = \psi_1 + \psi_2\chi_\Omega, \; \psi_j = \eta_{\mu,\rho}\zeta^{(\delta)}_j, \; j = 1, 2.$$  

To handle $\psi_1$ note that in view of (4.4),

$$|\nabla^m \xi \psi_1(\xi)| \leq \begin{cases} C\rho^{-m}, & \xi_d - \Psi(\hat{\xi}) \geq 3\langle M \rangle \delta, \\ C\delta^{-m}, & |\xi_d - \Psi(\hat{\xi})| \leq 3\langle M \rangle \delta, \end{cases}$$

for all $\xi \in B(\mu, 5\rho/4)$, and all $m = 0, 1, \ldots$. As $r \geq 8/5$, the distance between the supports of $h = h_{z,1}$ and $\tilde{h} = 1 - h_{z,r}$ is at least $7r/32$. Thus, to estimate the trace-class norm, we can use Theorem 3.7. Precisely, using (3.10), one obtains:

$$\|h \operatorname{Op}_1(\psi_1) \tilde{h}\|_{\mathcal{E}_1} \leq C r^{d-m} \left[ \rho^{-m} \int_{|\xi - \mu| \leq 5\rho/4} d\xi \right] \left[ \int_{|\xi_d - \Psi(\hat{\xi})| \geq 3\langle M \rangle \delta} d\xi + \int_{|\xi_d - \Psi(\hat{\xi})| \leq 3\langle M \rangle \delta} d\xi \right]$$

(5.4) 

$$\leq C \left( (\rho r)^{d-m} + \rho^{d-1} r^{d-m} \delta^{1-m} \right).$$

To handle $\operatorname{Op}_1(\psi_2\chi_\Omega)$ we recall, that by definition of $\zeta^{(\delta)}_2$, the symbol $\psi_2\chi_\Omega$ is supported in

$$\{ \xi : |\xi - \mu| \leq 5\rho/4, |\xi_d - \Psi(\hat{\xi})| \leq 3\langle M \rangle \delta \},$$

so that $|\psi_2\chi_\Omega| \leq C\delta^{1/2} \rho^{d-1}$, see (3.6) for definition of $| \cdot |_s$. Therefore, by Proposition 3.8,

$$\|h \operatorname{Op}_1(\psi_2) P_{\Omega,\alpha} \tilde{h}\|_{\mathcal{E}_1} \leq \|h \operatorname{Op}_1(\psi_2\chi_\Omega)\|_{\mathcal{E}_1} \leq C\delta^{1/2} \rho^{d-1}.$$  

Together with (5.4) this gives

$$\|h \Xi_{\mu,\rho} P_{\Omega,\alpha} \tilde{h}\|_{\mathcal{E}_1} \leq C \left[ (\rho r)^{d-m} + \rho^{d-1} (r^{d-m} \delta^{1-m} + \delta^{1/2}) \right].$$

Taking $\delta = r^{-2\omega}$ with $\omega < 1/2$, and making $m$ sufficiently large, we estimate the right hand side by

$$C \rho^{d-1} r^{-\omega},$$

which leads to the required estimate (5.2). □

**Lemma 5.2.** Let $\alpha \ell \rho \geq c$ and

$$\kappa_1 = \frac{1}{2(2d - 1)}.$$

(5.5)
Then for any \( \kappa \in (0, \kappa_1) \) there exists a number \( r_1 > 0 \), depending only on \( \kappa \) and the dimension \( d \), such that

\[
(5.6) \quad \| \Xi_{\mu, r} h_{z, t} \chi_\omega P_{\Omega, \alpha}(1 - h_{z, r}) \|_{\mathcal{E}_1} + \| \Xi_{\mu, r} h_{z, t} \chi_\omega P_{\Omega, \alpha}(1 - \Xi_{\mu, r}) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} r^{-\kappa},
\]

for all \( r \geq r_1 \), uniformly in \( z, \mu \in \mathbb{R}^d \). The constant \( C \) is independent of the parameters \( \alpha, \ell, \rho, r \).

**Proof.** Denote

\[
R = \Xi_{\mu, r} h_{z, t} \chi_\omega P_{\Omega, \alpha} G,
\]

where

\[
G = 1 - h_{z, r} \ell \quad \text{or} \quad G = 1 - \Xi_{\mu, r} \rho.
\]

Represent:

\[
R = X_1 X_2 + Y_1 Y_2,
\]

\[
X_1 = \Xi_{\mu, r} h_{z, t} \chi_\omega (1 - \Xi_{\mu, r}), \quad X_2 = P_{\Omega, \alpha} G,
\]

\[
Y_1 = \Xi_{\mu, r} \chi_\omega, \quad Y_2 = h_{z, t} \Xi_{\mu, r} P_{\Omega, \alpha} G.
\]

Clearly,

\[
\| X_2 \|, \| Y_1 \| \leq 1,
\]

so it remains to estimate the trace norms of \( X_1, Y_2 \). Inverting the roles of the variables \( x, \xi \), we get from Lemma 5.1 that

\[
\| X_1 \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} t^{-\omega},
\]

for all \( t \geq 8/5 \) and arbitrary \( \omega < 1/2 \). As far as \( Y_2 \) is concerned, if \( G = 1 - \Xi_{\mu, r} \rho \), then by choosing \( t = 4r/5 \) we guarantee that \( \Xi_{\mu, r} \rho G = 0 \), so that

\[
\| R \|_{\mathcal{E}_1} \leq \| X_1 \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} r^{-\omega},
\]

for all \( r \geq 2 \). Since \( \kappa_1 \leq 1/2 \), this leads to (5.6) for all \( r \geq 2 \).

For \( G = 1 - h_{z, r} \ell \), Lemma 5.1 gives:

\[
\| Y_2 \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} r^{-\omega},
\]

for all \( r \geq 8/5 \) and \( \omega > 1/2 \). Thus,

\[
(5.7) \quad \| R \|_{\mathcal{E}_1} \leq \| X_1 \|_{\mathcal{E}_1} + \| Y_2 \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} (t^{d-1} r^{-\omega} + t^{-\omega}), \quad t \geq 8/5, r \geq 8/5.
\]

The minimum of the right hand side is attained at

\[
(5.8) \quad t = r^\frac{\omega}{d-1+\omega} = r^{\frac{\omega}{2}} = \frac{\omega}{d-1+\omega}, \quad \kappa = \frac{\omega^2}{d-1+\omega}.
\]

For sufficiently large \( r_1 \), under the condition \( r \geq r_1 \) we have \( t \geq 2 \), so that (5.7) is applicable, and hence

\[
\| R \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{-1} r^{-\kappa}.
\]

The formula (5.8) maps \( \omega \in (0, 1/2) \) into \( \kappa \in (0, \kappa_1) \). This implies (5.6). \( \square \)

Now we obtain a more elaborate version of Lemma 5.2.
Lemma 5.3. Let $\alpha \ell \rho \geq c$. Let $\kappa_j, j = 1, 2, \ldots$, be the sequence of positive numbers such that $\kappa_1$ is defined by (5.5) and
\begin{equation}
\kappa_{j+1} = \frac{\kappa_j \kappa_1}{2(d-1) + \kappa_1 + \kappa_j}, \quad j = 1, 2, \ldots.
\end{equation}
Then for any $p = 1, 2, \ldots$, and any $\kappa \in (0, \kappa_p)$ there exists a number $r_p = r_p(\kappa, d) > 0$, such that
\begin{equation}
\|\Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^p(1 - h_{z, r \ell})\|_{\mathcal{E}_1} + \|\Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^p(1 - \Xi_{\mu, r \rho})\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} r^{-\kappa},
\end{equation}
and
\begin{equation}
\|\Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^p(1 - h_{z, r \ell})\|_{\mathcal{E}_1} + \|\Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^p(1 - \Xi_{\mu, r \rho})\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} r^{-\kappa},
\end{equation}
for all $r \geq r_p$, uniformly in $z, \mu \in \mathbb{R}^d$. The constant $C$ is independent of the parameters $\alpha, \ell, \rho, r$.

Proof. We begin by proving (5.10). Denote
\begin{align*}
R_1^{(s)}(\ell, \rho; r) &= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^s(1 - h_{z, r \ell}) , \\
R_2^{(s)}(\ell, \rho; r) &= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^s(1 - \Xi_{\mu, r \rho}).
\end{align*}
If $p = 1$, then the sought bound holds due to Lemma 5.2. For $p \geq 2$ we proceed by induction. Suppose that for some $s < p, s \geq 1$, we have
\begin{equation}
\|R_1^{(s)}\|_{\mathcal{E}_1} + \|R_2^{(s)}\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} r^{-\sigma},
\end{equation}
for arbitrary $\sigma < \kappa_s$ and all $r \geq r_s$. Let us show that this implies the same bound for $p = s + 1$, arbitrary $\sigma < \kappa_{s+1}$, and $r \geq r_{s+1}$ with some $r_{s+1}$. To this end rewrite:
\begin{align*}
R_1^{(s+1)}(\ell, \rho; r) &= S_1 + S_2 + S_3, \\
S_1 &= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^s(1 - h_{z, r \ell})\chi_{\Omega P_{\Omega, \alpha}}(1 - h_{z, r \ell}) \\
&= R_1^{(s)}(\ell, \rho; t)\chi_{\Omega P_{\Omega, \alpha}}(1 - h_{z, r \ell}), \\
S_2 &= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^s(1 - \Xi_{\mu, r \rho})h_{z, t \ell}\chi_{\Omega P_{\Omega, \alpha}}(1 - h_{z, r \ell}) \\
&= R_2^{(s)}(\ell, \rho; t)h_{z, t \ell}\chi_{\Omega P_{\Omega, \alpha}}(1 - h_{z, r \ell}), \\
S_3 &= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^s\Xi_{\mu, r \rho}h_{z, t \ell}\chi_{\Omega P_{\Omega, \alpha}}(1 - h_{z, r \ell}) \\
&= \Xi_{\mu, \ell} h_{z, \ell}(\chi_{\Omega P_{\Omega, \alpha}})^sR_1^{(1)}(t \ell, t \rho; r t^{-1}),
\end{align*}
with an arbitrary \( t > 0 \). It follows from (5.12) and from Lemma 5.2 that
\[
\|S_1\|_{\mathcal{E}_1} + \|S_2\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} t^{-\sigma_s},
\]
\[
\|S_3\|_{\mathcal{E}_1} \leq C(\alpha \ell^2 \rho)^{d-1} (rt^{-1})^{-\sigma_1},
\]
for any \( \sigma_s < \kappa_s, \sigma_1 < \kappa_1 \), and all \( t \geq r_s \) and \( rt^{-1} \geq r_1 \). Thus
\[
\|R^{(q+1)}(\ell, \rho; r)\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} (t^{2(d-1)+\sigma_1 r^{-\sigma_1} + t^{-\sigma_s}}).
\]
The right hand side is minimized when
\[
(5.13) \quad t^{-\sigma_s} = t^{2(d-1)+\sigma_1 r^{-\sigma_1}}, \quad \text{i.e.} \quad t^{\sigma_s} = r^{\sigma_{s+1}}, \quad \sigma_{s+1} = \frac{\sigma_1 \sigma_s}{2(d - 1) + \sigma_1 + \sigma_s}.
\]
This choice of \( t \) satisfies \( t \geq r_s \) if \( r \geq r_{s+1} \) with a sufficiently large \( r_{s+1} \). In addition, since \( \sigma_{s+1} < \sigma_s \), increasing \( r_{s+1} \) if necessary, one guarantees that \( rt^{-1} \geq r_1 \). Moreover, by definition (5.9), the formula (5.13) maps \( \sigma_s \in (0, \kappa_s) \) to \( \sigma_{s+1} \in (0, \kappa_{s+1}) \). This completes the proof of (5.12) for \( p = s + 1 \), and hence, by induction leads to (5.10).

The bounds (5.11) follow directly from (5.10). For example, the first trace norm on the left hand side of (5.11) equals the trace norm of the adjoint operator, i.e.
\[
\|h_{z, \ell, \Xi_{\mu, \rho}}(P_{\Omega, \alpha} \chi_{\Lambda})^p (1 - h_{z, r, \ell})\|_{\mathcal{E}_1}.
\]
Now, we get the required bound for this trace norm after exchanging the roles of the variables \( \mathbf{x} \) and \( \xi \) and using (5.10). Similarly for the second term on the left hand side of (5.11). \( \square \)

Let \( \Upsilon_{\delta}(\mathbf{z}, \mathbf{e}), \delta > 0, \mathbf{z} \in \mathbb{R}^d, \mathbf{e} \in \mathbb{S}^{d-1} \), be the “layer” defined by
\[
(5.14) \quad \Upsilon_{\delta}(\mathbf{z}, \mathbf{e}) = \{ \mathbf{x} \in \mathbb{R}^d : |(\mathbf{x} - \mathbf{z}) \cdot \mathbf{e}| < \delta \}.
\]

Lemma 5.4. Suppose that \( \delta \rho \alpha \geq c \). Let the sequence \( \kappa_j, j = 1, 2, \ldots \) be as defined in (5.9), and let \( r_p = r_p(\kappa, \mu, \rho, d) > 0 \) be the numbers found in Lemma 5.3 for arbitrary \( \kappa \in (0, \kappa_p), p = 1, 2, \ldots \). Then
\[
\|\Xi_{\mu, \rho} h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p (1 - h_{z, r, \ell}) \Upsilon_{\kappa}(\mathbf{w}, \mathbf{e}) (1 - \chi_{B(\mathbf{z}, 4\ell)}(1 - h_{z, r, \ell})\|_{\mathcal{E}_1}
\]
\[
\leq C(\alpha \ell \rho)^{d-1} \left[ r^{-\kappa} + r^{2(d-1)} (\ell \delta^{-1})^{-\kappa} \right],
\]
for all \( r \geq r_p, \ell \geq r_p \delta \), uniformly in \( \mathbf{w}, \mathbf{z}, \mu \in \mathbb{R}^d \). The constant \( C \) is independent of the parameters \( \alpha, \ell, \rho, r, \delta \).

Proof. Let us fix a \( \kappa \in (0, \kappa_p) \). Replace the operator on the left-hand side by
\[
\Xi_{\mu, \rho} h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p \Xi_{\mu, r, \rho} \Upsilon_{\kappa}(\mathbf{w}, \mathbf{e}) (1 - \chi_{B(\mathbf{z}, 4\ell)}) h_{z, r, \ell}
\]
with \( r \geq r_p \). By Lemma 5.3 this operation may change the trace norm at most by
\[
C(\alpha \ell \rho)^{d-1} r^{-\kappa}.
\]
Let \{B(x_l, \delta)\}, l = 1, 2, \ldots, N be a collection of balls covering \(\Upsilon_\delta(w, e) \cap B(z, r\ell)\) such that \(B(x_l, \delta) \subset \Upsilon_{2\delta}(w, e)\). Then
\[
\|h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p \Xi_{\mu, r\rho} (1 - \chi_{B(z, 4\ell)}) \chi_{B(x_l, \delta)}\| \leq \|(1 - h_{x_l, \ell}) (\chi_{\Lambda} P_{\Omega, \alpha})^p \Xi_{\mu, r\rho} h_{x_l, \delta}\| \leq 1.
\]
Since \(\ell \geq r\rho\delta\), it follows from (5.11) that the right hand side does not exceed
\[
C(\alpha \delta r\rho)^d(\ell \delta^{-1})^{-\kappa},
\]
uniformly in \(l = 1, 2, \ldots, N\). Thus
\[
\|\Xi_{\mu, r\rho} h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p \Xi_{\mu, r\rho} \chi_{\Upsilon_\delta(w, e)} (1 - \chi_{B(z, 4\ell)}) \chi_{B(z, r\ell)}\| \leq 1.
\]
Choosing the covering in such a way that the number of balls \(N\) does not exceed \(C(r\ell \delta^{-1})^{d-1}\), and hence
\[
\|\Xi_{\mu, r\rho} h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p \Xi_{\mu, r\rho} \chi_{\Upsilon_\delta(w, e)} (1 - \chi_{B(z, 4\ell)}) \chi_{B(z, r\ell)}\| \leq C(\alpha \ell r^2 \rho)^{d-1}(\ell \delta^{-1})^{-\kappa}.
\]
Returning to the initial operator, we obtain:
\[
\|\Xi_{\mu, r\rho} h_{z, \ell} (\chi_{\Lambda} P_{\Omega, \alpha})^p \chi_{\Upsilon_\delta(w, e)} (1 - \chi_{B(z, 4\ell)}) \chi_{B(z, r\ell)}\| \leq C(\alpha \ell \rho)^{d-1}[r^{-\kappa} + r^{2(d-1)}(\ell \delta^{-1})^{-\kappa}],
\]
as claimed. \(\square\)

5.2. **Reduction to the flat boundary.** The above lemmas provide useful tools for the study of the operator \(T(1; \Lambda, \Omega)\). Our objective in this subsection is to show that under suitable conditions one can replace \(\Lambda\) by a half-space. Since the domain \(\Omega\) remains unchanged, we omit \(\Omega\) from the notation of \(T\) and simply write \(T(1; \Lambda)\).

As before we assume that \(\Lambda, \Omega\) satisfy Condition 4.1. In addition, we assume that \(\Lambda = \Gamma(\Phi; I, 0)\), i.e. the domain is given by
\[
\Lambda = \{x : x_d > \Phi(\hat{x})\}.
\]
Define
\[
(5.16) \quad \Pi = \{x : x_d > \nabla \Phi(\hat{0}) \cdot \hat{x}\}.
\]
Since \(\nabla \Phi\) is uniformly continuous, we have
\[
(5.17) \quad \max_{|x - \hat{x}| \leq s} |\nabla \Phi(\hat{x}) - \nabla \Phi(\hat{0})| =: \varepsilon(s) \to 0, \ s \to 0,
\]
so that
\[
(5.18) \quad \max_{|\hat{x}| \leq s} |\Phi(\hat{x}) - \Phi(\hat{0}) - \nabla \Phi(\hat{0}) \cdot \hat{x}| \leq \varepsilon(s)s.
\]
In the next lemma we use the parameters \( \kappa_k \) defined in (5.9), and also the notation
\[
\tilde{r}_p(\kappa, d) = \max_{1 \leq k \leq p} r_k(\kappa, d), \quad 0 < \kappa < \kappa_p,
\]
where \( r_k(\kappa, d) \) are the numbers introduced for \( \kappa \in (0, \kappa_k) \) in Lemma 5.3. Since \( \kappa_k \) is a decreasing sequence, the numbers \( \tilde{r}_p(\kappa, d) \) are well-defined.

**Lemma 5.5.** Let \( \Lambda = \Gamma(\Phi; I, 0) \) and \( \Pi \) be as defined above. Let the point \( z \in \mathbb{R}^d \) and parameters \( \ell, t > 0 \) be such that
\[
B(z, 4\ell) \subset \Lambda \cap \Pi \cap B(0, t).
\]
Suppose that \( \alpha \ell \rho \geq c \). Let the sequence \( \kappa_j, j = 1, 2, \ldots \) be as defined in (5.9), and let \( \tilde{r}_p = \tilde{r}_p(\kappa, d) > 0 \) be the numbers defined in (5.19). Then for any \( p = 1, 2, \ldots \), and any \( \kappa \in (0, \kappa_p) \), under the conditions
\[
\tilde{r}_p \ell \leq t, \quad 2\tilde{r}_p(4t \varepsilon(4t)) \leq \ell,
\]
one has
\[
\| \Xi_{\mu, \rho}h_{z, \ell}(g_p(T(1; \Lambda)) - g_p(T(1; \Pi))) \|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} R_{\kappa}(\alpha; \ell, \rho, t),
\]
with
\[
R_{\kappa}(\alpha; \ell, \rho, t) = (t\ell^{-1})^{-\kappa} + (t\ell^{-1})^{2(d-1)}(\alpha \ell \rho)^{-\kappa} + (t\ell^{-1})^{2(d-1)}(t\ell^{-1}\varepsilon(4t))^\kappa,
\]
and a constant \( C \) independent of \( z, \mu \).

**Proof.** Rewrite the difference of the operators on the left-hand side as
\[
g_p(T(1; \Lambda)) - g_p(T(1; \Pi)) = \sum_{k=0}^{p-1} g_k(T(1; \Pi))(T(1; \Lambda) - T(1; \Pi))g_{p-k}(T(1; \Lambda)),
\]
and estimate every term in this sum individually. Since
\[
T(1; \Lambda) - T(1; \Pi) = (\chi_\Lambda - \chi_\Pi)P_{\Omega, \alpha} \chi_\Lambda + \chi_\Pi P_{\Omega, \alpha}(\chi_\Lambda - \chi_\Pi),
\]
the \( k \)th term in the sum takes the form
\[
g_k(T(1; \Pi))(\chi_\Lambda - \chi_\Pi)P_{\Omega, \alpha} g_{p-k}(T(1; \Lambda))
\]
\[
+ g_k(T(1; \Pi))P_{\Omega, \alpha}(\chi_\Lambda - \chi_\Pi)g_{p-k}(T(1; \Lambda))
\]
\[
= X_1 X_2 + Y_1 Y_2,
\]
with
\[
X_1 = (\chi_\Pi P_{\Omega, \alpha})^k(\chi_\Lambda - \chi_\Pi), \quad X_2 = \chi_\Pi P_{\Omega, \alpha} g_{p-k}(T(1; \Lambda)) - \chi_\Pi P_{\Omega, \alpha} g_{p-k}(T(1; \Lambda)),
\]
\[
Y_1 = (\chi_\Pi P_{\Omega, \alpha})^{k+1}(\chi_\Lambda - \chi_\Pi), \quad Y_2 = g_{p-k}(T(1; \Lambda)).
\]
Clearly, the norms of $X_2$ and $Y_2$ do not exceed 1. Let us estimate the trace norms of $X_1$ and $Y_1$. Represent $X_1$ as

$$X_1 = X_{11} + X_{12},$$

$$X_{11} = (\chi_\Pi P_{\Omega,\alpha})^k(\chi_\Lambda - \chi_\Pi)h_{0,3t},$$

$$X_{12} = (\chi_\Pi P_{\Omega,\alpha})^k(\chi_\Lambda - \chi_\Pi)(1 - h_{0,3t}).$$

Due to the condition (5.20), $B(z, 5t/4) \subset B(0, 3t)$, so that

$$1 - h_{0,3t} = (1 - h_{0,3t})(1 - h_{z,t}).$$

Pick an arbitrary $\varkappa < \kappa$, and assume that

$$t\ell^{-1} \geq \max_{1 \leq k \leq p} r_k(\varkappa, d),$$

where $r_k$ are the numbers from Lemma 5.3. Thus it follows from Lemma 5.3 that

$$\|\Xi_{\mu,\rho}h_{z,\ell}X_{11}\|_{\mathcal{S}_1} \leq \|\Xi_{\mu,\rho}h_{z,\ell}(\chi_\Pi P_{\Omega,\alpha})^k(1 - h_{z,t})\|_{\mathcal{S}_1} \leq C(\alpha\ell\rho)^{d-1}(t\ell^{-1})^{-\varkappa}.$$ (5.24)

In order to estimate $\|X_{11}\|_{\mathcal{S}_1}$, we note that the difference $\chi_\Lambda - \chi_\Pi$ is supported on the set $\Lambda \triangle \Pi$, i.e. on

$$\{x : \Phi(\hat{x}) < x_d < \nabla\Phi(\hat{0}) \cdot \hat{x}\} \cup \{x : \nabla\Phi(\hat{0}) \cdot \hat{x} < x_d < \Phi(\hat{x})\}.$$ By (5.18),

$$\max_{x,|\hat{x}| \leq 4t} |\Phi(\hat{x}) - \nabla\Phi(\hat{0}) \cdot \hat{x}| \leq 4t \varepsilon(4t).$$

Thus

$$\Lambda \triangle \Pi \cap B(0, 4t) \subset \Upsilon_{\nu}(0, e), \nu = \varepsilon(4t)4t,$$

(see (5.14) for definition of $\Upsilon_{\nu}$), where

$$e = \frac{(-\nabla\Phi(\hat{0}), 1)}{\sqrt{|\nabla\Phi(\hat{0})|^2 + 1}}$$

is the unit normal to the hyperplane $\Pi$. In order to use Lemma 5.4 we must ensure that the thickness $\nu$ satisfies the condition $\alpha\nu\rho \geq c$, and hence it is more convenient to consider a thicker layer $\Upsilon_\delta(0, e)$ with

$$\delta = c_0(\alpha\rho)^{-1} + 4t\varepsilon(4t),$$

with some $c_0 > 0$. Note that due to (5.20), $h_{0,3t} = h_{0,3t} \chi_{B(z,8t)}$ and

$$\chi_{\Lambda \triangle \Pi} = \chi_{\Lambda \triangle \Pi}(1 - \chi_{B(z,4t)}).$$

Therefore

$$\|\Xi_{\mu,\rho}h_{z,\ell}X_{11}\|_{\mathcal{S}_1} \leq \|\Xi_{\mu,\rho}h_{z,\ell}(\chi_\Pi P_{\Omega,\alpha})^k\chi_{\Upsilon_\delta(0, e)}(1 - \chi_{B(z,4t)})\chi_{B(z,8t)}\|_{\mathcal{S}_1}. (5.25)$$
Furthermore, in view of the first condition in (5.21), we have
from (5.21) and the condition $\alpha \ell \rho \geq c$ for a sufficiently small $c_0$ we get $r_p \delta \leq \ell$. Furthermore, in view of the first condition in (5.21), we have

$$r := \frac{8t}{\ell} \geq 8r_p.$$

Applying Lemma 5.4 to the right hand side of (5.25), we get

$$\|\Xi_{\mu,\rho}h_{z,\ell} X_{11}\|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1} \left[ r^{-\kappa} + r^{2(\ell^d - 1)/d - \kappa} \right] \leq C(\alpha \ell \rho)^{d-1} \left[ (\ell^{-1})^{1 - \kappa} + (\ell^{-1})^{2d(\ell^d - 1)/d} (\ell^{-1})^{(\ell^d - 1)/d} (4/\ell)^{1 - \kappa} \right].$$

Together with (5.24), this bound leads to the estimate of the form (5.22) for the operator $X_1 X_2$. Similarly, one obtains an estimate for the trace norm of $Y_1$, which leads to the required bound for $\|Y_1 Y_2\|_{\mathfrak{e}_1}$. Together, they ensure (5.22).

In the next lemma we replace the product of the two test functions $h_{z,\ell}(x)$ and $\eta_{\mu,\rho}(\xi)$ with an arbitrary compactly supported symbol $b(x, \xi)$.

Lemma 5.6. Let $\Lambda$ and $\Pi$ be as in Lemma 5.5 and let $\alpha \ell \rho \geq c$. Assume that for some point $z \in \mathbb{R}^d$, some $t > 0$, and some $\kappa \in (0, \kappa_p)$ the conditions (5.20) and (5.21) are satisfied. Let $b \in S^{(d+1, d+2)}$ be a symbol supported in $B(z, \ell) \times B(\mu, \rho)$ with some $\mu \in \mathbb{R}^d$. Then

$$\| \text{Op}^I_{\alpha}(b)(g_p(T(1, \Lambda)) - g_p(T(1, \Pi))) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1} \mathcal{R}_\kappa(\alpha; \ell, \rho, t) N^{(d+1, d+2)}(b; \ell, \rho),$$

with the factor $\mathcal{R}_\kappa(\alpha; \ell, \rho, t)$ defined in (5.23).

Proof. We write Op instead of $\text{Op}^I_{\alpha}$. Rewrite Op(b) as follows:

$$\text{Op}(b) = \text{Op}(b) \Xi_{\mu,\rho} h_{z,\ell} + \text{Op}(b)(1 - h_{z,\ell}).$$

In view of the bound

$$\| \text{Op}^I_{\alpha}(b) \Xi_{\mu,\rho} h_{z,\ell} (\g_p(1, \Lambda)) - g_p(T(1, \Pi)) \|_{\mathfrak{e}_1} \leq \| \text{Op}^I_{\alpha}(b) \| \Xi_{\mu,\rho} h_{z,\ell} (\g_p(1, \Lambda)) - g_p(T(1, \Pi)) \|_{\mathfrak{e}_1},$$

the bound (5.26) for the first term on the right hand side of (5.27) follows from Lemmas 5.5 and 3.9. To estimate the second term assume that $h(x) = 1$ for all $x \in B(0, 9/8)$, so that the supports of $b(\cdot, \xi)$ and $1 - h_{z,\ell}$ are separated by the distance of at least $\ell/8$. Thus the bound (3.20) with $Q = d + 2$ gives

$$\| \text{Op}(b)(1 - h_{z,\ell}) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-2} N^{(d+1, d+2)}(b; \ell, \rho).$$

Since $\kappa < \kappa_p < 1$, together with the estimate for the first term this bound produces (5.26).
6. A Hilbert-Schmidt class estimate

Although we mostly use trace class estimates, in the proof of the asymptotics (2.22) for arbitrary smooth functions we need one estimate in the Hilbert-Schmidt class. As in Sect. 5 we assume that the domains Λ and Ω satisfy Condition 4.1 and that \( d \geq 2 \).

The constants in all subsequent estimates are independent on the functions \( \Phi, \Psi \), or the parameters \( O, k \), but may depend on the constant \( M \) in (4.1).

**Lemma 6.1.** Let \( \Lambda \) and \( \Omega \) be two domains satisfying Condition 4.1. Let \( a = a(x, \xi) \) be a symbol from \( S^{(d+2, d+2)} \), supported in \( B(z, \ell) \times B(\mu, \rho) \) with some \( z, \mu \in \mathbb{R}^d \) and \( \ell, \rho > 0 \). Let \( \operatorname{Op}_a(a) \) denote any of the operators \( \operatorname{Op}_l^\alpha(a) \) or \( \operatorname{Op}_r^\alpha(a) \), and let \( \alpha \ell \rho \geq 2 \). Then

\[
\| \chi_\Lambda \operatorname{P}_{\alpha, \alpha}^\Lambda \operatorname{Op}_a(a) \operatorname{P}_{\alpha, \alpha}^\Omega (1 - \chi_\Lambda) \| _{S^2} \leq C(\alpha \ell \rho)^{d-1} \log(\alpha \ell \rho) \left( N^{(d+2, d+2)}(a; \ell, \rho) \right)^2,
\]

uniformly in \( z, \mu \in \mathbb{R}^d \).

This lemma was first proved in [11] (see Theorem 3.2.2, p. 129) and can be also found in [12], Theorem 2.1. Our proof is a minor variation of that from [11] and [12]. It begins with a lemma which appeared in [6], Lemma 2.10, and [11], Lemma 3.4.1. For the sake of completeness we provide our proof of this lemma, which is somewhat more elementary than that of [11].

**Lemma 6.2.** Let \( u \in L^2(\mathbb{R}^d) \) be a function satisfying the bound

\[
\int_{\mathbb{R}^d} |u(x + h) - u(x)|^2 dx \leq \kappa |h|^\beta,
\]

for any \( h \in \mathbb{R}^d \), with some \( \beta \geq 0 \) and some constant \( \kappa > 0 \). Then for all \( r > 0 \),

\[
\int_{|\xi| \geq r} |\hat{u}(\xi)|^2 d\xi \leq C \kappa r^{-\beta},
\]

with a universal constant \( C \).

**Proof.** By Parseval’s identity,

\[
\kappa |h|^\beta \geq \int_{\mathbb{R}^d} |u(x + h) - u(x)|^2 dx = \int_{\mathbb{R}^d} |e^{ih \cdot \xi} - 1|^2 |\hat{u}(\xi)|^2 d\xi \geq \int_{|\xi| \geq r} |e^{ih \cdot \xi} - 1|^2 |\hat{u}(\xi)|^2 d\xi.
\]

Integrating in \( h \) over the ball \( |h| \leq h_0 \), we get

\[
\int_{|h| \leq h_0} |e^{ih \cdot \xi} - 1|^2 dh = \int_{|h| \leq h_0} (2 - 2 \cos(h \cdot \xi)) dh = 2 w_d h_0^d - 2(2\pi)^{\frac{d}{2}} |\xi|^{-\frac{d}{2}} h_0^d J_{\frac{d}{2}}(h_0 |\xi|),
\]
where \( w_d \) is the volume of the unit ball in \( \mathbb{R}^d \), and \( J_{d/2} \) is the standard Bessel function of order \( d/2 \). Assume that \( h_0 r \geq 1 \) so that

\[
J_{d/2}(h_0|\xi|) \leq \frac{C}{\sqrt{h_0|\xi|}}, \quad |\xi| \geq r.
\]

Therefore

\[
\int_{|h| \leq h_0} |e^{ih\cdot\xi} - 1|^2 dh \geq 2h_d \left( w_d - C \frac{h_0 r}{(h_0 r)^{d+1}} \right).
\]

Taking \( h_0 = C_1 r^{-1} \) with a sufficiently large \( C_1 \), we ensure that the right-hand side is bounded from below by \( 2h_d w_d \), and hence (6.2) leads to the bound

\[
\int_{|\xi| \geq r} |\hat{u}(\xi)|^2 d\xi \leq C r \beta h_0^\beta = C r^{-\beta}, \quad \text{as claimed.}
\]

Now we use this lemma for a specific choice of the function \( u \):

**Lemma 6.3.** Let \( \Omega = \Gamma(\Psi; O, k) \subset \mathbb{R}^d \) be a \( C^1 \)-graph-type-domain, and let \( \eta_{\mu, \rho}, \mu \in \mathbb{R}^d, \rho > 0 \), be as defined in (5.1). Then

\[
\int_{\mathbb{R}^d} |\eta_{\mu, \rho}(\xi + h)\chi_{\Omega}(\xi + h) - \eta_{\mu, \rho}(\xi)\chi_{\Omega}(\xi)|^2 d\xi \leq C \rho^{d-1} (M_{\Psi}) |h|,
\]

for all \( h \in \mathbb{R}^d \), uniformly in \( \mu \in \mathbb{R}^d \) and \( \rho > 0 \).

**Proof.** Without loss of generality assume that \( O = I, k = 0 \). Also, in view of (2.17) we may assume that \( \rho = 1 \).

It suffices to prove the estimate (6.3) for \( |h| < 1 \). Estimate the integrand by

\[
|\eta_{\mu,1}(\xi + h) - \eta_{\mu,1}(\xi)| + |\eta_{\mu,1}(\xi)|^2 |\chi_{\Omega}(\xi + h) - \chi_{\Omega}(\xi)|^2.
\]

Thus the integral on the right-hand side of (6.3) up to a constant does not exceed \( I_1 + I_2 \) with

\[
I_1 = \int_{\mathbb{R}^d} |\eta_{\mu,1}(\xi + h) - \eta_{\mu,1}(\xi)| d\xi, \quad I_2 = \int \eta_{\mu,1}^2(\xi) |\chi_{\Omega}(\xi + h) - \chi_{\Omega}(\xi)|^2 d\xi.
\]

It is straightforward to see that

\[
I_1 \leq C|h| \int_{|\mu - \xi| \leq 3} d\xi \leq C|h|.
\]

In \( I_2 \) the integration is restricted to the set \( (\Omega - h) \Delta \Omega \), i.e. to the set

\[
\{ \xi : \Psi(\hat{\xi} + \hat{h}) - h_d \leq \xi_d \leq \Psi(\hat{\xi}) \} \cup \{ \xi : \Psi(\hat{\xi}) \leq \xi_d \leq \Psi(\hat{\xi} + \hat{h}) - h_d \}.
\]
On this set we have
\[ |\xi_d - \Psi(\hat{\xi})| \leq |\Psi(\hat{\xi} + \hat{h}) - \Psi(\hat{\xi})| + |\hat{h}| \leq M_\Psi |\hat{h}| + |\hat{h}| \leq 2\langle M_\Psi \rangle |\hat{h}|. \]

Therefore,
\[ I_2 \leq \int_{|\xi - \hat{\mu}| \leq 2} \int_{|\xi_d - \Psi(\hat{\xi})| \leq 2\langle M_\Psi \rangle |\hat{h}|} d\xi_d d\hat{\xi} \leq C\langle M_\Psi \rangle |\hat{h}|. \]

This leads to (6.3). \( \square \)

**Lemma 6.4.** Let \( \Lambda \) and \( \Omega \) be two domains satisfying Condition 4.1. Let \( h_{z,\ell}, \eta_{\mu,\rho} \) be functions defined in (5.1). Suppose that \( \alpha \ell \rho \geq 2 \). Then
\[ \| \chi_{\Lambda} h_{z,\ell} \Xi_{\mu,\rho} P_{\Omega,\alpha}(1 - \chi_{\Lambda}) \|^2_{L_2} \leq C(\alpha \ell \rho)^{d-1} \log(\alpha \ell \rho), \]
uniformly in \( \mu, z \in \mathbb{R}^d \).

**Proof.** Due to (2.10), (2.11) and (2.17) we may assume that \( \ell = \rho = 1 \), and \( \alpha \geq 2 \). Denote \( h = h_{z,1}, \eta = \eta_{\mu,1} \). Denote \( b(\xi) = \eta(\xi)\chi_{\Omega}(\xi) \), and
\[ \hat{b}(t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{it\xi} b(\xi) d\xi. \]
Thus
\[ \| \chi_{\Lambda} h \text{Op}_{\gamma}(\eta) P_{\Omega,\alpha}(1 - \chi_{\Lambda}) \|^2_{L_2} = \frac{\alpha^{2d}}{(2\pi)^d} \int_{\mathbb{R}^d} |h(x)|^2 (1 - \chi_{\Lambda}(y)) \chi_{\Lambda}(x) |\hat{b}(\alpha(x - y))|^2 dxdy. \]

Split the integral in two: for \( |x_d - \Phi(\hat{x})| \leq \alpha^{-1} \), which we denote by \( I_1 \), and for \( |x_d - \Phi(\hat{x})| > \alpha^{-1} \), which we denote by \( I_2 \). To handle \( I_1 \) it suffices to use the Parceval’s identity:
\[ \int_{\mathbb{R}^d} |\hat{b}(\alpha(x - y))|^2 dy = \alpha^{-d} \int_{\mathbb{R}^d} |b(\xi)|^2 d\xi \leq C \alpha^{-d}, \]
so that
\[ I_1 \leq C \alpha^d \int_{|x_d - \Phi(\hat{x})| < \alpha^{-1}} |h(x)|^2 dx \leq C \alpha^{d-1}. \]

For \( I_2 \) observe that according to (4.3),
\[ |x - y| \geq \frac{1}{\sqrt{\langle M_\Psi \rangle}} |x_d - \Phi(\hat{x})|. \]

By Lemma 6.2,
\[ \int_{|x - y| \geq \frac{1}{\sqrt{\langle M_\Psi \rangle}} |x_d - \Phi(\hat{x})|} |\hat{b}(\alpha(x - y))|^2 dy \leq C \alpha^{-1-d} \langle M_\Psi \rangle \sqrt{\langle M_\Psi \rangle} |x_d - \Phi(\hat{x})|^{-1}. \]
Consequently,
\[ I_2 \leq C \alpha^{d-1} \int_{|x_d - \Phi(\hat{x})| \geq \alpha^{-1}} |h(x)|^2 |x_d - \Phi(\hat{x})|^{-1} dx \]
\[ \leq C \alpha^{d-1} \int_{|\hat{x} - \hat{z}| \leq 1/4} \left[ \int_{\alpha^{-1}}^{\alpha} t^{-1} dt + \alpha^{-1} \int_{|t - \hat{z}| \leq 1/4} dt \right] d\hat{x} = C \alpha^{d-1} \log \alpha. \]

The proof is complete. \[\square\]

**Proof of Lemma 6.1.** For definiteness we prove the above estimate for Op\(\alpha(a) = Op_I(a)\). Furthermore, it suffices to prove the sought estimate for \(\ell = 1, \alpha \geq 2\) and \(N^{(d+2,d+2)}(a; 1, 1) = 1\). We write \(G_1 \approx G_2\) with two operators \(G_1, G_2\), if
\[ \|G_1 - G_2\|_{\mathcal{E}_1} \leq C \alpha^{d-1}. \]

By (4.12),
\[ P_{\Omega, \alpha} Op_I(a) P_{\Omega, \alpha} \approx Op_I(a) P_{\Omega, \alpha}. \]

Denote \(h = h_{\tau, 1}, \eta = \eta_{\mu, 1}\). Since \(a(x, \xi) = a(x, \xi) h(x) \eta(\xi)\), according to (3.23),
\[ Op_\alpha(a) \approx Op_\alpha(a) Op_\alpha(h\eta). \]

Moreover, in view of (4.7),
\[ \chi_\Lambda Op_\alpha(a) \approx Op_\alpha(a) \chi_\Lambda. \]

Thus
\[ \chi_\Lambda Op_\alpha(a) P_{\Omega, \alpha}(1 - \chi_\Lambda) \approx Op_\alpha(a) \chi_\Lambda h Op_\alpha(\eta) P_{\Omega, \alpha}(1 - \chi_\Lambda). \]

Since for any trace class operator, \(\|S\|_{\mathcal{E}_2}^2 \leq \|S\|_{\mathcal{E}_1} \|S\|\), and \(Op_\alpha(a)\) is bounded uniformly in \(\alpha\) (see Lemma 3.9), we have
\[ \|\chi_\Lambda P_{\Omega, \alpha} Op_\alpha(a) P_{\Omega, \alpha}(1 - \chi_\Lambda)\|_{\mathcal{E}_2}^2 \leq C \|\chi_\Lambda h Op_\alpha(\eta) P_{\Omega, \alpha}(1 - \chi_\Lambda)\|_{\mathcal{E}_2}^2 + C \alpha^{d-1}. \]

Now the required estimate follows from Lemma 6.4. \[\square\]

7. LOCALISATION

Now it is time to replace the global assumptions on the domain by the local ones. Now we do not need to assume that the domains \(\Lambda\) and \(\Omega\) are of graph type, see Definition 2.1. Instead we assume that inside a ball of fixed radius, both \(\Lambda\) and \(\Omega\) are representable either by \(C^1\)-graph-type domains or by \(\mathbb{R}^d\), in the sense of Definition 2.2. For the reference convenience we state this assumption explicitly:

**Condition 7.1.** The domain \(\Lambda\) (resp. \(\Omega\)) satisfies one of the following two conditions:

1. If \(d = 1\), then for some numbers \(x_0 \in \mathbb{R}\) (resp. \(\xi_0 \in \mathbb{R}\)) and \(R > 0\), we have \(\Lambda \cap (x_0 - R, x_0 + R) = (x_0, x_0 + R)\) (resp. \(\Omega \cap (\xi_0 - R, \xi_0 + R) = (\xi_0, \xi_0 + R)\)).

   If \(d \geq 2\), then for some point \(w \in \partial \Lambda\) (resp. \(\eta \in \partial \Omega\)) and some number \(R > 0\), in the ball \(B(w, R)\) (resp. \(B(\eta, R)\)) the domain \(\Lambda\) (resp. \(\Omega\)) is represented by
a graph-type domain $\Gamma(\Phi; O_\Lambda, w)$ (resp. $\Gamma(\Psi; O_\Omega, \eta)$) with some $C^1$-function $\Phi$ (resp. $\Psi$), satisfying (2.14), and an orthogonal transformation $O_\Lambda$ (resp. $O_\Omega$).

(2) If $d = 1$, then for some numbers $w \in \mathbb{R}$ (resp. $\eta \in \mathbb{R}$) and $R > 0$, we have $\Lambda \cap (w - R, w + R) = (w - R, w + R)$ (resp. $\Omega \cap (\eta - R, \eta + R) = (\eta - R, \eta + R)$).

If $d \geq 2$, then for some point $w \in \Lambda$ (resp. $\eta \in \Omega$) and some number $R > 0$, in the ball $B(w, R)$ (resp. $B(\eta, R)$) the domain $\Lambda$ (resp. $\Omega$) is represented by the entire Euclidean space $\mathbb{R}^d$.

As before, our estimates will be uniform in the functions $\Phi, \Psi$ satisfying the bound (4.1) with some constant $M$, but may depend on the value of $M$.

For $d = 1$ we use the notation $\Lambda_0 := (x_0, \infty)$ (resp. $\Omega_0 := (\xi_0, \infty)$) if $\Lambda$ (resp. $\Omega$) satisfies Condition 7.1(1) and $\Lambda_0 = \mathbb{R}$ (resp. $\Omega_0 = \mathbb{R}$) if $\Lambda$ (resp. $\Omega$) satisfies Condition 7.1(2).

For $d \geq 2$ we use the notation $\Lambda_0 := \Gamma(\Phi)$ (resp. $\Omega_0 := \Gamma(\Psi)$) if $\Lambda$ (resp. $\Omega$) satisfies Condition 7.1(1) and $\Lambda_0 = \mathbb{R}^d$ (resp. $\Omega_0 = \mathbb{R}^d$) if $\Lambda$ (resp. $\Omega$) satisfies Condition 7.1(2).

For brevity we also denote

$$T(a) = T(a; \Lambda, \Omega), \quad T^{(0)}(a) = T(a; \Lambda_0, \Omega_0).$$

In this section we study the trace norms of the operators of the type

$$\text{Op}_\alpha(b) g_p(T(a; \Lambda, \Omega)),$$

where $g_p(t) = t^p, p = 1, 2, \ldots$, and the symbols $a = a(x, \xi), b = b(x, \xi)$ satisfy the following conditions:

(7.1) $a, b \in S^{(d+2,d+2)}$, and $b$ is supported in $B(z, \ell) \times B(\mu, \rho) \subset B(w, R) \times B(\eta, R)$,

with some $z, \mu \in \mathbb{R}^d$ and $\ell, \rho > 0$. Sometimes in the proofs for brevity we use the notation $\text{Op}(b)$ instead of $\text{Op}_\alpha(b)$. If $\Lambda = \Lambda_0, \Omega = \Omega_0$, then one can take $R = \infty$.

**Lemma 7.2.** Let $\Lambda, \Omega \subset \mathbb{R}^d$ and $\Lambda_0, \Omega_0 \subset \mathbb{R}^d, d \geq 1$, be a specified above. Suppose that the symbol $b$ satisfies (7.1). Denote by $\text{Op}_\alpha(b)$ any of the operators $\text{Op}_\alpha^l(b)$ or $\text{Op}_\alpha^r(b)$. Then

(7.2) $\| \text{Op}_\alpha(b) P_{\Omega, \alpha} \chi_{\Lambda} - P_{\Omega_0, \alpha} \chi_{\Lambda_0} \text{Op}_\alpha(b) \|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2,d+2)}(b; \ell, \rho),$

and

(7.3) $\| \text{Op}_\alpha(b) \chi_\Lambda P_{\Omega, \alpha} - \chi_{\Lambda_0} P_{\Omega_0, \alpha} \text{Op}_\alpha(b) \|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2,d+2)}(b; \ell, \rho).$

**Proof.** Without loss of generality assume

$$N^{(d+2,d+2)}(b; \ell, \rho) = 1.$$

In view of (3.22) any of the above inequalities for $\text{Op}_\alpha^l(b)$ immediately implies the same inequality for $\text{Op}_\alpha^r(b)$. We prove (7.2) for the operator $\text{Op}_\alpha^l(b)$. The inequality (7.3) is
proved in the same way. Write:

\[ \text{Op}^l_{\alpha}(b) P_{\Omega,\alpha \chi_{\Lambda}} = \text{Op}^t_{\alpha}(b) P_{\Omega,\alpha \chi_{\Lambda}} \]

\[ = [\text{Op}^l_{\alpha}(b), P_{\Omega,\alpha}] \chi_{\Lambda} + P_{\Omega,\alpha}(\text{Op}^t_{\alpha}(b) - \text{Op}^r_{\alpha}(b)) \chi_{\Lambda} + P_{\Omega,\alpha}[\text{Op}^r_{\alpha}(b), \chi_{\Lambda}] \]

\[ + P_{\Omega,\alpha \chi_{\Lambda}}(\text{Op}^r_{\alpha}(b) - \text{Op}^l_{\alpha}(b)) + P_{\Omega,\alpha \chi_{\Lambda}} \text{Op}^l_{\alpha}(b). \]

Now (7.2) follows by virtue of Lemmas 3.12, 4.3 and 4.5. Similarly one proves (7.3). □

**Lemma 7.3.** Let \( T(a) \) and \( T^{(0)}(a) \) be as described above. Let \( d \geq 1 \), and let \( a, b \) be some symbols satisfying condition (7.1). Then under the assumption \( \alpha \ell \rho \geq c \) one has

\[ \| T(b) - T^{(0)}(b) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)}(b; \ell, \rho), \]

\[ \| \text{Op}^l_{\alpha}(b) T(a) - T^{(0)}(ab) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)}(b; \ell, \rho) N^{(d+2, d+2)}(a; \ell, \rho), \]

and

\[ \| \text{Op}^l_{\alpha}(b)[T(a) - T^{(0)}(a)] \|_{\mathfrak{e}_1} + \| T^{(0)}(b)[T(a) - T^{(0)}(a)] \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)}(b; \ell, \rho) N^{(d+2, d+2)}(a; \ell, \rho). \]

**Proof.** Without loss of generality assume that

\[ \text{Op}(b) = \text{Op}^l_{\alpha}(b). \]

Note that (7.2) immediately leads to

\[ \| \text{Op}(b)(P_{\Omega,\alpha \chi_{\Lambda}} - P_{\Omega_{\ell,\rho} \alpha \chi_{\Lambda}}) \|_{\mathfrak{e}_1} + \| (\chi_{\Lambda} P_{\Omega,\alpha} - \chi_{\Lambda} P_{\Omega,\alpha}) \text{Op}(b) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1}. \]

To prove (7.4), write

\[ T(b) - T^{(0)}(1) \text{Op}(b) = (\chi_{\Lambda} P_{\Omega,\alpha} - \chi_{\Lambda} P_{\Omega_{\ell,\rho} \alpha}) \text{Op}(b) P_{\Omega} \chi_{\Lambda} \]

\[ + \chi_{\Lambda} P_{\Omega_{\ell,\rho} \alpha}(\text{Op}(b) P_{\Omega,\alpha} \chi_{\Lambda} - P_{\Omega_{\ell,\rho} \alpha} \chi_{\Lambda} \text{Op}(b)), \]

so that by (7.8) and (7.2),

\[ \| T(b) - T^{(0)}(1) \text{Op}(b) \|_{\mathfrak{e}_1} \leq \| (\chi_{\Lambda} P_{\Omega,\alpha} - \chi_{\Lambda} P_{\Omega_{\ell,\rho} \alpha}) \text{Op}(b) \|_{\mathfrak{e}_1} \]

\[ + \| \text{Op}(b) P_{\Omega,\alpha} \chi_{\Lambda} - P_{\Omega_{\ell,\rho} \alpha} \chi_{\Lambda} \text{Op}(b) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1}, \]

which leads to

\[ \| T(b) - T^{(0)}(1) \text{Op}^l_{\alpha}(b) \|_{\mathfrak{e}_1} + \| T^{(0)}(b) - T^{(0)}(1) \text{Op}^t_{\alpha}(b) \|_{\mathfrak{e}_1} \leq C(\alpha \ell \rho)^{d-1}. \]

Thus (7.4) follows.
In order to prove (7.5) rewrite:
\[
\text{Op}(b) T(a) = (\text{Op}(b) \chi_{\Lambda} P_{\Omega} - \chi_{\Lambda_0} P_{\Omega_0,\alpha} \text{Op}(b)) \text{Op}(a) P_{\Omega,\alpha} \chi_{\Lambda} \\
+ \chi_{\Lambda_0} P_{\Omega_0,\alpha} (\text{Op}(b) \text{Op}(a) - \text{Op}(ab)) P_{\Omega,\alpha} \chi_{\Lambda} \\
+ \chi_{\Lambda_0} P_{\Omega_0,\alpha} \text{Op}(ab) (P_{\Omega,\alpha} \chi_{\Lambda} - P_{\Omega_0,\alpha} \chi_{\Lambda_0}) + T^{(0)}(ab).
\]

Now (7.5) follows from (3.12), (7.3), (3.23) and (7.8).

The inequality (7.5) immediately implies (7.6) for the first trace norm in (7.6). Moreover, it follows from (7.9) that \( T^{(0)}(b) \) in the second term in (7.6) can be replaced by \( T^{(0)}(1) \text{Op}_\alpha(b) \). Now the required bound for the second term follows from the bound for the first trace norm in (7.6).

Lemma 7.4. Let \( d \geq 1 \), and let \( a, b \) be some symbols satisfying condition (7.1). Assume that \( \alpha \ell \rho \geq c \). Then
\[
\| \text{Op}_\alpha^I(b) g_p(T(a)) - g_p(T^{(0)}(a)) \|_1 \leq C(\alpha \ell \rho)^{d-1} \mathbf{N}^{(d+2,d+2)}(b; \ell, \rho) \mathbf{N}^{(d+2,d+2)}(a; \ell, \rho)^p.
\]

\[
\| \text{Op}_\alpha^I(b^p) g_p(T(a)) - g_p(T^{(0)}(ab)) \|_1 \leq C(\alpha \ell \rho)^{d-1} \mathbf{N}^{(d+2,d+2)}(b; \ell, \rho)^p \mathbf{N}^{(d+2,d+2)}(a; \ell, \rho)^p.
\]

Proof. We write for brevity Op instead of \( \text{Op}_\alpha^I \), and assume without loss of generality that (7.7) is satisfied.

Step 1. Let us show first that
\[
\| \text{Op}(b) g_p(T(a)) - g_{p-1}(T^{(0)}(a)) T^{(0)}(ab) \|_{\mathbf{S}_1} \leq C(\alpha \ell \rho)^{d-1}.
\]

We do it by induction. If \( p = 1 \), then (7.12) is exactly (7.5). Suppose (7.12) holds for some \( p = m \), and let us deduce (7.12) for \( p = m + 1 \). Write:
\[
\text{Op}(b) g_{m+1}(T(a)) - g_{m}(T^{(0)}(a)) T^{(0)}(ab)
\]
\[
= \left( \text{Op}(b) g_m(T(a)) - g_{m-1}(T^{(0)}(a)) T^{(0)}(ab) \right) T(a) \\
+ g_{m-1}(T^{(0)}(a)) \left( T^{(0)}(ab) T(a) - T^{(0)}(a) T^{(0)}(ab) \right).
\]

The claimed term for the first term on the right-hand side follows from (7.12) for \( p = m \), and Lemma 3.9.

By (7.6), in the second term we can replace \( T(a) \) with \( T^{(0)}(a) \). It remains to use Corollary 4.6 and Lemma 3.9. Thus by induction (7.12) holds for all \( p = 1, 2, \ldots \). The estimate (7.10) is an immediate consequence of (7.12).
Step 2. Let us show that

\[(7.13) \quad \|(Op(b)^p g_p(T(a)) - g_p(T^{(0)}(ab))\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1}.\]

Again we use induction. For \(p = 1\) this inequality repeats (7.12). Suppose it holds for some \(p = m\). In order to prove it for \(p = m + 1\) write

\[
(\text{Op}(b))^{m+1} g_{m+1}(T(a)) - g_{m+1}(T^{(0)}(ab)) = (\text{Op}(b))^m (\text{Op}(b) g_{m+1}(T(a)) - g_m(T^{(0)}(a)) T^{(0)}(ab)) + (\text{Op}(b))^m (g_{m+1}(T(a)) - g_m(T^{(0)}(ab))) T^{(0)}(ab).
\]

Now we use (7.13) for \(p = m\), (7.12) for \(p = m + 1\), and Lemma 3.9.

Step 3. It follows from Corollary 3.13 that

\[\|(Op(b)^p) - (Op(b))^p\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1},\]

and hence

\[\|(Op(b)^p - (Op(b))^p) g_p(T(a))\|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1},\]

in view of Lemma 3.9. To complete the proof put together Steps 1, 2 and 3. \(\square\)

For the one-dimensional case we need a more involved version of the above lemma:

**Lemma 7.5.** Let \(d = 1\), and let \(a = a(\xi), b = b(\xi)\) be functions from \(\mathbf{S}^{(3)}\), such that \(b\) is supported on the interval \((\mu - \rho, \mu + \rho)\) with some \(\mu \in \mathbb{R}\) and \(\rho > 0\). Assume that \(\Lambda = (x_0, \infty)\) with some \(x_0 \in \mathbb{R}\) and let \(\Omega\) be an arbitrary subset of \(\mathbb{R}\). Then for all \(\alpha > 0, \rho > 0\), we have

\[\|Op_\alpha(b^p) g_p(T(a)) - g_p(T(ab))\|_{\mathcal{E}_1} \leq C(N^{(3)}(a; \rho))^{p} (N^{(3)}(b; \rho))^p.\]

**Proof.** Lemma 7.4 is not directly applicable, although the proof is. Assume without loss of generality that

\[N^{(3)}(b; \rho) = N^{(3)}(a; \rho) = 1.\]

Step 1. Let us show first that

\[(7.14) \quad \|Op(b) g_p(T(a)) - g_{p-1}(T(a)) T(ab)\|_{\mathcal{E}_1} \leq C.\]

We do it by induction. If \(p = 1\), then

\[Op(b)T(a) - T(ab) = [Op(b), \chi_\Lambda] Op(a) P_{\Omega, \alpha} \chi_\Lambda,\]
and (7.14) follows from (4.10) and the bound \( \| \text{Op}(a) \| \leq \| a \|_{L^\infty} \). Suppose (7.14) holds for some \( p = m \), and let us deduce (7.14) for \( p = m + 1 \). Write:

\[
\text{Op}(b)g_{m+1}(T(a)) - g_m(T(a))T(ab) = \left( \text{Op}(b)g_m(T(a)) - g_{m-1}(T(a))T(ab) \right)T(a) \\
+ g_{m-1}(T(a)) \left( T(ab)T(a) - T(a)T(ab) \right).
\]

The claimed estimate follows from (7.14) for \( p = m \), and Lemma 4.7, and more precisely, from the bound

\[ \| T(ab)T(a) - T(a)T(ab) \|_{S^1} \leq C. \]

**Step 2.** Let us show that

\[ (7.15) \quad \| (\text{Op}(b)^pg_p(T(a)) - g_p(T(ab)) \|_{S^1} \leq C. \]

Again we use induction. For \( p = 1 \) this inequality repeats (7.14). Suppose it holds for some \( p = m \). In order to prove it for \( p = m + 1 \) write

\[
(\text{Op}(b))^{m+1}g_{m+1}(T(a)) - g_{m+1}(T(ab)) = (\text{Op}(b))^m \left( \text{Op}(b)g_{m+1}(T(a)) - g_m(T(a))T(ab) \right) \\
+ \left( (\text{Op}(b))^m g_m(T(a)) - g_m(T(ab)) \right) T(ab).
\]

Now one uses (7.14) for \( p = m + 1 \), and (7.15) for \( p = m \) to show that the right hand side does not exceed

\[ C \| b \|_{L^\infty}^m + C \| a \|_{L^\infty} \| b \|_{L^\infty} \leq C'. \]

The proof is complete.

**Lemma 7.6.** Let \( d \geq 1 \), and let \( a, b \) be some symbols satisfying condition (7.1). Suppose that \( \alpha \ell \rho \geq c \). Then

\[
\| \text{Op}_{\alpha}^I((ba)^p)g_p(T^0(1)) \|_{S^1} \leq C(\alpha \ell \rho)^{d-1} (N^{d+2,d+2}(b; \ell, \rho))^p (N^{d+2,d+2}(a; \ell, \rho))^p.
\]

**Proof.** Estimate:

\[
\| \text{Op}_{\alpha}^I((ba)^p)g_p(T^0(1)) \|_{S^1} \leq \| g_p(T^0(ab)) \|_{S^1} - \text{Op}_{\alpha}^I((ba)^p)g_p(T^0(1)) \|_{S^1}.
\]

It remains to use Lemma 7.4 twice. 

\[ \square \]
Lemma 7.7. Let $d \geq 1$, and let $a, b$ be some symbols satisfying condition (7.1). Then under the assumption $\alpha \ell \rho \geq c$, one has
\[
\| \operatorname{Op}_{\alpha}(b) g_T(T(a)) - \operatorname{Op}_{\alpha}(b a) g_T(T(0)(1)) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)} (b; \ell, \rho) \left( N^{(d+2, d+2)} (a; \ell, \rho) \right)^p.
\]

Proof. Assume without loss of generality that (7.7) is satisfied. Also, due to (7.10) we may assume that for $\alpha \leq R$ that
\[
\| \operatorname{Op} (b) g_T(T(0)(a)) - \operatorname{Op} (b a) g_T(T(0)(1)) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1},
\]
Furthermore, by Lemma 7.6 used with $R = \infty$,
\[
\| Z_1 \|_{\mathcal{E}_1} \leq \| \operatorname{Op}_{\alpha}(b) - \operatorname{Op}_{\alpha}(b a) \| \| \operatorname{Op}(a) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1},
\]
By Corollary 3.13 and Lemma 3.9,
\[
\| Z_3 \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1}.
\]
Put together, these bounds produce (7.16).

Theorem 7.8. Let $d \geq 1$, and let $a, b$ be some symbols satisfying condition (7.1). Suppose that $\alpha \ell \rho \geq c$. In addition assume that either $\Lambda$ or $\Omega$ satisfies Condition 7.1(2), i.e. either $\Lambda_0 = \mathbb{R}^d$ or $\Omega_0 = \mathbb{R}^d$. Then
\[
\| \operatorname{tr} \left( \operatorname{Op}_{\alpha}(b) g_T(T(a)) - \alpha^d \mathbf{1}(a; \Lambda, \Omega) \right) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)} (b; \ell, \rho) \left( N^{(d+2, d+2)} (a; \ell, \rho) \right)^p.
\]

For $a = 1$ and the function $g(t) = t - t^p$ one has
\[
\| \operatorname{Op}_{\alpha}(b) g_T(T(1)) \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1} N^{(d+2, d+2)} (b; \ell, \rho).
\]

Proof. Without loss of generality assume (7.7). If $\Lambda_0 = \mathbb{R}^d$, then $T(0)(1) = P_{\Lambda_0, \alpha}$, and it follows from Lemma 7.7 that
\[
\| \operatorname{Op} (b) g_T(T(a)) - \operatorname{Op} (b a) P_{\Lambda_0, \alpha} \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1}.
\]
Take the trace:

\[
\text{tr}(\text{Op}(b\alpha^p)P_{\Omega_0,\alpha}) = \left(\frac{\alpha}{2\pi}\right)^d \int_{\mathbb{R}^d} \int_{\Omega_0} b(x, \xi)g_p(a(x, \xi))d\xi dx = \alpha^d \mathcal{W}_0(bg_p(a); \Lambda, \Omega).
\]

This gives (7.17). Similarly for \(\Omega_0 = \mathbb{R}^d\).

To obtain (7.18) we use (7.19) with \(a = 1\).

\[
\square
\]

8. Model problem in dimension one

One of the pivotal points of the proof is the reduction to a model operator for \(d = 1\). This section is entirely devoted to the study of this problem.

8.1. Model problem: reduction to multiplication. The pair of the model operators on \(L^2(\mathbb{R}_+)\) is defined as follows:

\[
T_{\pm} := T_\alpha(1; \mathbb{R}_+, \mathbb{R}_\pm) = T_1(1; \mathbb{R}_+, \mathbb{R}_\pm).
\]

To simplify simultaneous considerations of \(T_+\) and \(T_-\), we change \(\xi \to -\xi\) in \(T_-\) we arrive at

\[
(T_{\pm}u)(x) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty e^{\pm i\xi(x-y)}u(y)dyd\xi, \quad u \in S(\mathbb{R}).
\]

As in [39], using the Mellin transform \(\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})\):

\[
\hat{u}(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-\frac{1}{2}+is}u(x)dx,
\]

one can easily show that the operator \(T_{\pm}\) is unitarily equivalent to the multiplication by the function

\[
\frac{1}{1 + e^{\pm 2\pi^2s}}
\]

in \(L^2(\mathbb{R})\). Thus \(\mathcal{M}g(T_{\pm})\mathcal{M}^*\) is also multiplication by a function. If \(g\) is \(C^1\), and \(g(0) = g(1) = 0\), this function is integrable, and hence one can write the kernel of \(g(T_{\pm})\):

\[
K_{\pm}(x, y) = \frac{1}{2\pi} \left(\frac{xy}{2}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} \left(\frac{y}{x}\right)^{\pm is} g\left(1 + e^{2\pi s}\right) ds.
\]

Note that the function \(K_{\pm}(x, y)\) is homogeneous of degree \(-1\):

\[
K_{\pm}(tx, ty) = t^{-1}K(x, y),
\]

for any \(t > 0\). By a straightforward change of variables, one sees that

\[
K_{\pm}(1, 1) = \frac{1}{2\pi} \int g\left(1 + e^{2\pi s}\right) ds = \mathfrak{A}(g),
\]

where \(\mathfrak{A}(g)\) is defined in (1.5).
8.2. Model problem: asymptotics. We are computing the asymptotics of the trace

\[ I_{\pm}(\alpha) = \text{tr}(\text{Op}_{\alpha}(b)g(T_{\pm})), \]

with a suitable symbol \( b \), for arbitrary function \( g \), satisfying the condition

\[ g \in C^1(\mathbb{R}), \quad g(0) = g(1) = 0. \]

Rewrite this trace as follows:

\[ I_{\pm}(\alpha; b) = \frac{\alpha}{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}} b(z, \xi) e^{i\alpha \xi(z-x)} K_{\pm}(x, z) d\xi dz dx. \]

The symbol \( b = b(x, \xi) \) is assumed to be of the form

\[ b(x, \xi) = \psi(x)a(x, \xi), \]

where \( \psi \in C_0^\infty(\mathbb{R}) \) is a non-negative function such that \( \psi(x) \leq 1 \) for all \( x \in \mathbb{R} \) and

\[ \psi(x) = \begin{cases} 0, & x \notin (1, 4r), \\ 1, & x \in (4, r), \end{cases} \]

with a parameter \( r > 4 \).

We begin with some estimates:

**Lemma 8.1.** Let \( \Lambda \) and \( \Omega \) be arbitrary open subsets of \( \mathbb{R} \). Let the symbol \( a \) and function \( \psi \) be as above. Then for any continuous function \( g \) any \( L \geq r \) and \( \alpha r \rho \geq c \), we have

\[ \| \text{Op}_{\alpha}(\psi a)g(T_{\pm}(1)) \|_{\mathcal{E}_1} \leq C\alpha \rho r \mathcal{N}(2, 2)(a; L, \rho). \]

**Proof.** Let \( \tilde{h} := h_{0, 2r}(x) \) be the function defined in (5.1). The estimated trace norm does not exceed \( C\| \text{Op}_{\alpha}(\tilde{h}a)\|_{\mathcal{E}_1} \). By Lemma 3.11,

\[ \| \text{Op}_{\alpha}(\tilde{h}a)\|_{\mathcal{E}_1} \leq C\alpha \rho r \mathcal{N}(2, 2)(a; r, \rho). \]

Since the right-hand side is monotone increasing in \( r \) and \( L \geq r \), the claimed estimate follows. \( \square \)

Now we compute the asymptotics of \( I_{\pm}(\alpha; b) \) as \( \alpha \to \infty \).

**Theorem 8.2.** Let the symbol \( b \) be as in (8.4), and let the function \( g \) be as in (8.3). Then for any \( \delta > 0 \), \( L > 0 \) and \( \rho > 0 \), such that \( \alpha r \rho \geq c \), we have

\[ |I_{\pm}(\alpha; b) - \mathcal{A}(g)a(0, 0) \log r| \leq C[1 + rL^{-1} + (\alpha \rho)^{-\delta} \log r] \mathcal{N}(1, 2)(a; L, \rho), \]

uniformly in \( r \geq 5 \), with a constant \( C = C(\delta) \).

This lemma is a modified version of a similar result from [39] which gives the asymptotics of the trace \( I_{\pm}(\alpha; b) \) as \( \alpha \to \infty \), when \( \psi(x) = 1 \), and the parameters \( L, \rho \) are fixed. On the contrary, the formula (8.7) explicitly depends on four parameters \( L, r, \rho, \alpha \). Later we use (8.7) with large \( r \) and such \( \rho \) and \( L \) that the right-hand side is uniformly bounded.
Proof. Without loss of generality we may assume that $\rho = 1$ and $N^{(1,2)}(a; L, 1) = 1$, so that
\begin{equation}
|\partial^s_x a(x, \xi)| \leq 1, \ s = 0, 1, 2, \ |\partial_x a(x, \xi)| \leq L^{-1},
\end{equation}
for all $x, \xi$.

Step I. We conduct the proof for the sign “+” only. The case of “−” is done in the
same way. For brevity the subscript “+” is omitted from the notation.

Let $\phi \in C_0^\infty(\mathbb{R})$ and consider the integral
\[ J'(\alpha) = \frac{\alpha}{2\pi} \int_0^\infty \int_0^\infty \int_{-1}^1 K(x, z)(1 - \phi(xz^{-1}))e^{i\alpha z(x - z)}b(z, \xi)d\xi dz dx. \]

Integrating by parts in $\xi$ twice, by (8.8) we get
\[ \left| \int_{-1}^1 e^{i\alpha z(x - z)}b(z, \xi)d\xi \right| \leq \frac{C}{(1 + \alpha |z - x|)^2}. \]

In view of (8.1) we can estimate:
\begin{align*}
|J'(\alpha)| & \leq C\alpha \int_0^\infty \int_0^\infty \frac{|K(x, z)||1 - \phi(xz^{-1})|}{(1 + \alpha |x - z|)^2} dx dz \\
& = C\alpha \int_0^\infty \int_0^\infty \frac{|K(x, 1)||1 - \phi(x)|}{(1 + \alpha z|x - 1|)^2} dx dz = C \int_0^\infty \frac{|K(x, 1)||1 - \phi(x)|}{|x - 1|} dx.
\end{align*}

This integral converges at infinity since $K(x, 1) = O(x^{-1/2})$. If $K(1, 1) = 0$, then, then
it converges also at $x = 1$ for any function $\phi \in C_0^\infty$, and in particular for $\phi = 0$. In this
case the integral $J'(\alpha) = I(\alpha)$ is uniformly bounded, so (8.7) is proved.

If $K(1, 1) \neq 0$, then the above integral converges under the assumption $\phi(1) = 1$. Thus it remains to find the asymptotics of the integral
\[ J(\alpha) = \frac{\alpha}{2\pi} \int_0^\infty \int_0^\infty \int_\mathbb{R} K(x, z)\phi(xz^{-1})e^{i\alpha z(x - z)}b(z, \xi)d\xi dz dx. \]

To choose a suitable function $\phi$, let $\nu > 0$ be a number such that $K(x, 1)$ is separated
from zero for all $x \in [1 - \nu, 1 + \nu]$. Take a function $\omega \in C_0^\infty(\mathbb{R})$ supported on $[1 - \nu, 1 + \nu]$, such that
\[ \omega(x) = 1, \ 1 - \nu/2 \leq x \leq 1 + \nu/2, \]
and define
\[ \phi(x) = \omega(x) \frac{K(1, 1)}{K(x, 1)}, \]
so that
\[ K(x, z)\phi(xz^{-1}) = z^{-1}K(1, 1)\omega(xz^{-1}). \]
Now $J(\alpha)$ can be rewritten as follows:

$$J(\alpha) = \alpha \frac{K(1, 1)}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \omega(xz^{-1}) e^{i\alpha \xi z (1-xz^{-1})} dx \right] b(z, \xi) d\xi dz$$

$$= \alpha \frac{K(1, 1)}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}} e^{i\alpha \xi z} \tilde{\omega}(\alpha \xi z) b(z, \xi) d\xi dz$$

Rewriting it again, using (8.2), we get

$$J(\alpha) = \alpha A(g) \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{|\xi| < 1} e^{i\alpha \xi \tilde{\omega}(\alpha \xi)} \tau(\xi) d\xi,$$

with

$$\tau(\xi) = \int_0^\infty b(z, \xi z^{-1}) \frac{dz}{z} = \int_1^{2r} b(z, \xi z^{-1}) \frac{dz}{z}.$$

Split $J(\alpha)$ in two integrals:

$$J(\alpha) = J_1(\alpha) + J_2(\alpha),$$

$$J_1(\alpha) = \alpha \frac{A(g)}{\sqrt{2\pi}} \int_{|\xi| < 1} e^{i\alpha \xi \tilde{\omega}(\alpha \xi)} \tau(\xi) d\xi,$$

$$J_2(\alpha) = \alpha \frac{A(g)}{\sqrt{2\pi}} \int_{|\xi| \geq 1} e^{i\alpha \xi \tilde{\omega}(\alpha \xi)} \tau(\xi) d\xi.$$

Since $|\tilde{\omega}(\eta)| \leq C_\delta |\eta|^{-\delta - 1}$ for any $\delta > 0$, we have

$$\alpha \int_{|\xi| \geq 1} |\tilde{\omega}(\alpha \xi)| d\xi \leq C\alpha^{-\delta}.$$

Thus, estimating

$$|\tau(\xi)| \leq \int_1^{2r} z^{-1} dz = \log(2r),$$

we obtain

$$J_2(\alpha) \leq C\alpha^{-\delta} \log r.$$

The contribution of $J_1(\alpha)$ requires more careful calculations:

**Step II. Study of $J_1(\alpha)$**. We compare (8.9) with the number

$$\tau_0 = a(0, 0) \int_1^{4r} \frac{dz}{z} = a(0, 0)(\log r + \log 4).$$

Estimate the difference:

$$|\psi(z)a(z, \xi z^{-1}) - a(0, 0)| \leq |\psi(z)(a(z, \xi z^{-1}) - a(0, \xi z^{-1}))|$$

$$+ |\psi(z)(a(0, \xi z^{-1}) - a(0, 0))| + |\psi(z) - 1||a(0, 0)|.$$
Thus
\[ |\tau(\xi) - \tau_0| \leq \max_{z,t} |\partial_z a(z,t)| \int_1^{4r} dz + \max_{z,t} |\partial_t a(z,t)||\xi| \int_1^{4r} z^{-2} dz \]
\[ + |a(0,0)| \left( \int_1^4 \frac{dz}{z} + \int_1^{4r} \frac{dz}{z} \right) \]
\[ \leq C(\max_{z,t} |\partial_z a(z,t)| r + \max_{z,t} |\partial_t a(z,t)| + |a(0,0)|) \]
\[ \leq C(1 + rL^{-1}), \]
for all \(|\xi| < 1\), where we have used (8.8). In view of (8.12) this entails the bound
\[ \left| J_1(\alpha) - \alpha A(g)a(0,0) \log r \right| \leq C \alpha (1 + rL^{-1} \int_{|\xi|<1} |\hat{\omega}(\alpha\xi)| d\xi \right) \leq C' \alpha (1 + rL^{-1}). \]
By definition of the function \( \omega \) and by (8.10),
\[ \frac{\alpha}{\sqrt{2\pi}} \int_{|\xi|<1} e^{i\alpha\xi} \hat{\omega}(\alpha\xi) d\xi = 1 + O(\alpha^{-\delta}), \]
and hence, from the boundedness of \( J'(\alpha) \) and from (8.11) it follows that
\[ |I(\alpha; b) - A(g)a(0,0) \log r| \leq |J'(\alpha)| + |J_2(\alpha)| + |J_1(\alpha) - A(g)a(0,0) \log r| \]
\[ \leq C'(1 + rL^{-1} + \alpha^{-\delta} \log r). \]
This leads to (8.7). \( \square \)

8.3. Multiple intervals. Here instead of the model operator \( T_\pm \) we consider the operator with a more general set \( \Omega \). We begin with introducing some quantities related to finite subsets of the real line.

Let \( \rho > 0 \), and let \( X \subset (-2\rho, 2\rho) \) be a finite set. Now we introduce a number describing the spacing of the elements of \( X \). If \( X = \emptyset \), then for any \( \delta \geq 0 \) we define
\[ m_\delta(X) = (4\rho)^{-\delta}, X = \emptyset. \]
If \#\( X = N \geq 1 \), then we label the points \( \xi \in X \) in increasing order: \( \xi_1, \xi_2, \ldots, \xi_N \), and define
\[ m_\delta(X) = \sum_{j=1}^{N} \rho_j^{-\delta}, \quad \begin{cases} \rho_j = \text{dist}\{\xi_j, X \setminus \{\xi_j\}\}, & N \geq 2, \\ \rho_1 = 4\rho, & N = 1. \end{cases} \]
Note that
\[ \#X \leq 2^\delta \rho^\delta m_\delta(X), \] for any \( \delta \geq 0 \),
and, more generally,
\[ m_\nu(X) \leq (2\rho)^{\delta-\nu} m_\delta(X), \] for any \( 0 \leq \nu \leq \delta \).
Note also the natural “monotonicity” of \( m_\delta \) in the set \( X \): if \( X_1 \subset X_2 \subset (-2\rho, 2\rho) \) are two finite sets, then

\[
m_\delta(X_1) \leq m_\delta(X_2),
\]

for any \( \delta \geq 0 \).

Now let \( \Omega \) be a subset of real line such that for some \( \rho > 0 \)

\[
\Omega \cap (-2\rho, 2\rho) = \left( \bigcup_j I_j \right) \cap (-2\rho, 2\rho),
\]

where \( \{I_j\}, j = 1, 2, \ldots \) is a finite collection open intervals such that \( I_j \cap I_s = \emptyset \) for \( j \neq s \). Let \( X \) be the set of their endpoints inside \((-2\rho, 2\rho)\). We are interested in the asymptotics of the trace

\[
\text{tr} \left( \text{Op}_\alpha^l(b) g(T(1; \mathbb{R}^+, \Omega)) \right), \quad g(t) = t^p - t,
\]

with some \( p \in \mathbb{N} \),

where \( b = b(x, \xi) \) is as in (8.4).

We begin with the simplest case, \( X = \emptyset \). Let \( \zeta \in C^\infty_0(\mathbb{R}) \) be a non-negative function, such that

\[
\zeta(x) = \begin{cases} 
0, & |x| \geq 1/4, \\
1, & |x| < 1/8.
\end{cases}
\]

**Lemma 8.3.** Let \( g \) be as in (8.19), and let the symbol \( a \) be as in (8.4). If \( X = \emptyset \), then under the condition \( \alpha L \rho \geq c \) we have

\[
\| \text{Op}_\alpha^l(a) g(T(1; \mathbb{R}^+, \Omega)) \|_{\mathcal{E}_1} \leq C N^{(1,2)}(a; L, \rho).
\]

**Proof.** Also, without loss of generality assume that \( N^{(1,2)}(a; L, \rho) = 1 \), so that

\[
\| \text{Op}_\alpha^l(a) \| \leq C,
\]

by Lemma 3.9.

Since \( a(x, \xi) = 0 \) for \( |\xi| \geq \rho \), we have

\[
a(x, \xi) = a(x, \xi)(\zeta(\xi))^p, \quad \zeta(\xi) = \zeta(\xi(8\rho)^{-1}),
\]

for any \( p \in \mathbb{N} \). By Lemma 7.5,

\[
\| \text{Op}_\alpha^l(\zeta^p) g_p(T(1; \mathbb{R}^+, \Omega)) - g_p(T(\zeta; \mathbb{R}^+, \Omega)) \|_{\mathcal{E}_1} \leq C' \left( N^{(3)}(\zeta; \rho) \right)^p \leq C,
\]

for all \( \alpha > 0 \). Since \( \zeta(\xi) = 0 \) for \( |\xi| \geq 2\rho \), we have either \( \zeta\chi = 0 \) or \( \zeta\chi = \zeta \). If \( \zeta\chi = 0 \), then \( T(\zeta; R^+, \Omega) = 0 \), and the above estimate together with (8.22) imply that

\[
\| \text{Op}_\alpha^l(a) g(T(1; \mathbb{R}^+, \Omega)) \|_{\mathcal{E}_1} \leq \| \text{Op}_\alpha^l(a) \| \left( \| \text{Op}_\alpha^l(\zeta^p) g_p(T(1; \mathbb{R}^+, \Omega)) \|_{\mathcal{E}_1} + \| \text{Op}_\alpha^l(\zeta) g_1(T(1; \mathbb{R}^+, \Omega)) \|_{\mathcal{E}_1} \right) \leq C,
\]

as claimed.
If $\bar{\zeta} \chi_{\Omega} = \bar{\zeta}$, then $T(\bar{\zeta}; \mathbb{R}^+, \Omega) = T(\bar{\zeta}; \mathbb{R}^+, \mathbb{R})$, and using Lemma 7.5 backwards, we get
\[
\| \text{Op}_\alpha(\tilde{\zeta})^p(g_p(T(1; \mathbb{R}^+, \Omega)) - g_p(\chi_{\mathbb{R}^+})) \|_{S_1} \leq C,
\]
for any $p = 1, 2, \ldots$, which leads to
\[
\| \text{Op}_\alpha^l(a)(g_p(T(1; \mathbb{R}^+, \Omega)) - g_p(\chi_{\mathbb{R}^+})) \|_{S_1} \leq C,
\]
in view of (8.22). Therefore
\[
\| \text{Op}_\alpha(a)(g(T(1; \mathbb{R}^+, \Omega)) - g(\chi_{\mathbb{R}^+})) \|_{S_1} \leq C.
\]
Since $g(\chi_{\mathbb{R}^+}) = 0$, this entails (8.21). The proof is complete.

If $\mathcal{X} \neq \emptyset$, then we find the asymptotics of the trace (8.19) using Theorem 8.2. To this end we need to build a suitable partition of unity, which will reduce the problem to the case $\Omega = \mathbb{R}^+$ or $\Omega = \mathbb{R}^-$. We attach to each point $\xi_j \in \mathcal{X}$, $j = 1, 2, \ldots, N := \#\mathcal{X}$, the cut-off function
\[
u_j(\xi) = \zeta\left(\frac{\xi - \xi_j}{\rho_j}\right),
\]
where $\rho_j, \zeta$ are defined in (8.14) and (8.20) respectively. Clearly, $\nu_j \in S^{(\infty)}$ and
\[
N^{(m)}(\nu_j; \rho_j) < C_m,
\]
with a constant $C_m$ depending only on $m$. By construction the supports of the functions $\nu_j$ associated with distinct points of the set $\mathcal{X}$, do not overlap. To complete our definition of the partition of unity we set
\[
u(\xi) = 1 - \sum_{1 \leq j \leq N} \nu_j(\xi).
\]
Insert now this partition of unity into the trace (8.19) and establish the asymptotics of the constituent traces individually.

**Lemma 8.4.** Let the symbols $a$ and $b$ be as in (8.4), and let $g$ be as in (8.19). Suppose that $N := \#\mathcal{X} \geq 1$. Let $\nu_j \in C_0^{\infty}(\mathbb{R})$, $j = 1, 2, \ldots, N$, be as above. Assume that
\[
\alpha L \rho \geq c, \quad \alpha \rho_j \geq c.
\]
Then for any $\delta > 0$ we have
\[
|\text{tr}(\text{Op}_\alpha^l(b \nu_j)g(T(1; \mathbb{R}^+, \Omega))) - \mathfrak{A}(g)a(0, \xi_j) \log r| \leq C(1 + rL^{-1} + (\alpha \rho_j)^{-\delta} \log r)N^{(1,2)}(a; L, \rho),
\]
with a constant $C$ independent of $j$.

**Proof.** Translating by $\xi_j$, we may assume that $\xi_j = 0$. Without loss of generality assume $N^{(1,2)}(a; L, \rho) = 1$. Denote $\tilde{u}_j(\xi) = u_j(\xi/2)$, so that $\tilde{u}_j u_j = u_j$. By Lemma 7.5 we have
\[
\| \text{Op}_\alpha(\tilde{u}_j^p)g_p(T(1; \mathbb{R}^+, \Omega)) - g_p(T(\tilde{u}_j; \mathbb{R}^+, \Omega)) \|_{S_1} \leq C,
\]
for all $\alpha > 0$. If $\xi_j$ is the left endpoint of one of the intervals $\{I_j\}$ in (8.18), then $\tilde{u}_j \chi_\Omega = \tilde{u}_j \chi_{\mathbb{R}_+}$. If $\xi_j$ is the right endpoint, then $\tilde{u}_j \chi_\Omega = \tilde{u}_j \chi_{\mathbb{R}_-}$. Thus by Lemma 7.5 again,
\begin{equation}
\| \text{Op}_\alpha(u_j)g(T(1; \mathbb{R}_+, \Omega)) - \text{Op}_\alpha(u_j)g(T_\pm) \|_{\mathcal{E}_1} \leq C,
\end{equation}
where the sign “+” (resp. “−”) is used for the left (resp. right) endpoint. As $\alpha L \rho \geq c$, Lemma 3.9 gives
\[ \| \text{Op}_\alpha(\psi a) \| \leq C. \]
Thus in combination with (8.23) we obtain
\[ \| \text{Op}_\alpha(\psi a)g(T(1; \mathbb{R}_+, \Omega)) - g(T_\pm) \|_{\mathcal{E}_1} \leq C, \]
for $\alpha L \rho \geq c$. To find the trace of $\text{Op}_\alpha(\psi a)g(T_\pm)$ note that $au_j \in \mathcal{S}(2,2)$, $u_j(\xi) = 0$ for $|\xi| \geq \rho_j/4$, and
\[ N^{(1,2)}(au_j; L, \rho_j) \leq CN^{(1,2)}(a; L, \rho), \]
since $\rho \geq \rho_j$. Since $\alpha \rho_j \geq c$, now the result follows from Theorem 8.2.

\section*{Lemma 5.8.}
Let the symbol $a$ be as in (8.4), and let $g$ be as in (8.19). Suppose that $N := \# X \geq 1$. Then under the condition $\alpha L \rho \geq c$ we have for any $\delta \geq 1$:
\[ \| \text{Op}_\alpha(au)g(T(1; \mathbb{R}_+, \Omega)) \|_{\mathcal{E}_1} \leq C \delta \rho^\delta m_3(X) N^{(1,2)}(a; L, \rho). \]

\textbf{Proof.} Again, without loss of generality assume that $N^{(1,2)}(a; L, \rho) = 1$.

By definition of $u_j$ and $u$, we have $u(\xi) = 0$ whenever $|\xi - \xi_j| \leq \rho_j/8$ for any $\xi_j \in X$. Let $l$ be a number such that $\rho_l = \min_j \rho_j$. Without loss of generality assume that $l = 1$, so that $u \in \mathcal{S}(\infty)$ with
\[ N^{(m)}(u; \rho_1) \leq C_m. \]
Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that
\[ \varphi(t) = \begin{cases} 1, |t| \leq 1/4; \\
0, |t| \geq 3/4, \end{cases} \]
and that the collection $\varphi(t - n), n \in \mathbb{Z}$, forms a partition of unity on $\mathbb{R}$. Denote
\[ \phi_n(\xi) = \zeta(\xi (8 \rho)^{-1}) u(\xi) \varphi(12 \xi \rho_1^{-1} - n), n \in \mathbb{Z}, \]
where $\zeta$ is defined in (8.20), so that $\phi_n(\xi) = 0$ for $|\xi| \geq 2 \rho$. Denote by $J \leq C \rho \rho_1^{-1}$ the smallest natural number such that
\[ \sum_{|n| \leq J} \phi_n(\xi) = \zeta(\xi (8 \rho)^{-1}) u(\xi). \]

Let us show that
\begin{equation}
\| \text{Op}_\alpha(\phi_n)g(T(1; \mathbb{R}_+, \Omega)) \|_{\mathcal{E}_1} \leq C,
\end{equation}
for all $\alpha > 0$ uniformly in $n, |n| \leq J$. Indeed, let
\[ \tilde{\phi}_n(\xi) = \varphi(4 \rho_1^{-1}(\xi - n \rho_1/12)), \]
so that \( \phi_n = \phi_n \tilde{\varphi}_n \). Consider only those \( n \), for which \( \phi_n \neq 0 \). Since \( u(\xi) = 0 \) for all \( \xi \in (\xi_j - \rho_j, \xi_j + \rho_j) \), \( j = 1, 2, \ldots, N \), the latter requirement implies that

\[
\text{either } \tilde{\varphi}_n \chi_\Omega = 0 \text{ or } \tilde{\varphi}_n \chi_\Omega = \tilde{\varphi}_n.
\]

By Lemma 7.5,

\[
\| Op_\alpha(\tilde{\varphi}_n \rho) g_p(T(1; \mathbb{R}_+; \Omega)) - g_p(T(\tilde{\varphi}_n; \mathbb{R}_+; \Omega)) \|_{\mathcal{E}_1} \leq C' (N^{(3)}(\tilde{\varphi}_n; \rho_1))^p \leq C,
\]

for all \( \alpha > 0 \). In the case \( \tilde{\varphi}_n \chi_\Omega = 0 \), this immediately implies (8.24) with \( g(t) = t^p - t \), since \( \phi_n = \phi_n \tilde{\varphi}_n \rho \) for any \( p = 1, 2, \ldots \). Suppose now that \( \tilde{\varphi}_n \chi_\Omega = \tilde{\varphi}_n \), so that

\[
g_p(T(\tilde{\varphi}_n; \mathbb{R}_+; \Omega)) = g_p(T(\tilde{\varphi}_n; \mathbb{R}_+; \mathbb{R})).
\]

Using Lemma 7.5 "backwards", we get

\[
\| Op_\alpha(\tilde{\varphi}_n \rho) (g_p(T(1; \mathbb{R}_+; \Omega)) - g_p(\chi_{\mathbb{R}_+})) \|_{\mathcal{E}_1} \leq C,
\]

which leads to

\[
\| Op_\alpha(\phi_n) (g(T(1; \mathbb{R}_+; \Omega)) - g(\chi_{\mathbb{R}_+})) \|_{\mathcal{E}_1} \leq C.
\]

Since \( g(\chi_{\mathbb{R}_+}) = 0 \), this entails (8.24). Summing over all \( n \)'s, we get

\[
\| Op_\alpha^t (au) g(T(1; \mathbb{R}_+; \Omega)) \|_{\mathcal{E}_1} \leq \| Op_\alpha(a) \| \sum_{|n| \leq J} \| Op_\alpha(\phi_n) g(T(1; \mathbb{R}_+; \Omega)) \|_{\mathcal{E}_1}
\]

\[
\leq C' J \leq C \rho \rho_1^{-1} \leq C \rho m_1(X),
\]

where we have used Lemma 3.9 to estimate the norm of \( Op_\alpha^t (a) \) under the condition \( \alpha L \rho \geq c \), and the bound

\[
\rho_1^{-1} \leq m_1(X),
\]

which follows from the definition (8.14). Using (8.16) we complete the proof. \( \square \)

**Theorem 8.6.** Let \( g \) be as in (8.19). Let the symbols \( a, b \) and the function \( \psi \) be as in (8.4). Assume that \( \alpha \rho \geq c \), \( L \geq r \geq 1 \). Then

\[
\left| \text{tr} \left( Op_\alpha^t (\psi a) g(T(1; \mathbb{R}_+; \Omega)) \right) - \mathcal{A}(g) \sum_{\xi \in X} a(0, \xi) \log r \right|
\]

\[
\leq C_\delta m_\delta(X) [\rho^\delta + \alpha^{1-\delta} \rho r + \alpha^{-\delta} \log r] N^{(2)}(a; L, \rho),
\]

for any \( \delta \geq 1 \), where the sum on the left hand side equals zero if \( X = \emptyset \).

**Proof.** Assume that \( N^{(2)}(a; L, \rho) = 1 \). If \( X = \emptyset \), then the estimate immediately follows from Lemma 8.3 and the definition (8.13).

Suppose that \( X \) is non-empty. Represent the trace

\[
\mathcal{T} = \text{tr} \left( Op_\alpha^t (\psi a) g(T(1; \mathbb{R}_+; \Omega)) \right)
\]
on the left-hand side of (8.25) as the sum
\[ \mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \]
\[ \mathcal{I}_1 = \sum_{1 \leq j \leq N; \alpha \rho_j \geq 1} \text{tr}(\text{Op}_\alpha^j(\psi au_j)g(T(1; \mathbb{R}_+, \Omega))), \]
\[ \mathcal{I}_2 = \sum_{1 \leq j \leq N; \alpha \rho_j < 1} \text{tr}(\text{Op}_\alpha^j(\psi au_j)g(T(1; \mathbb{R}_+, \Omega))), \]
\[ \mathcal{I}_3 = \text{tr}(\text{Op}_\alpha^j(\psi au)g(T(1; \mathbb{R}_+, \Omega))). \]
To every term with \( \alpha \rho_j \geq 1 \) we apply Lemma 8.4, so that
\[ |\mathcal{I}_1 - \mathcal{A}(g) \sum_{\xi_j \in X; \alpha \rho_j \geq 1} a(0, \xi_j) \log r| \leq C \left[ N(1 + rL^{-1}) + \log r \alpha^{-\delta} \sum_{j=1}^N \rho_j^{-\delta} \right], \]
(8.26)
where we have used (8.15), and the inequality \( rL^{-1} \leq 1 \). If \( \alpha \rho_j < 1 \) we use the estimate (8.6), which we rewrite as follows:
\[ |\text{tr}(\text{Op}_\alpha^j(\psi au_j)g(T(1; \mathbb{R}_+, \Omega))) - \log r \mathcal{A}(g)a(0, \xi_j)| \leq C \alpha \rho r, \]
which leads to
\[ |\mathcal{I}_2 - \mathcal{A}(g) \sum_{\xi_j \in X; \alpha \rho_j < 1} a(0, \xi_j) \log r| \leq C \tilde{N} \alpha \rho r, \]
where \( \tilde{N} \) is the number of points \( \xi_j \) such that \( \alpha \rho_j < 1 \). Similarly to (8.15) one can estimate:
\[ \tilde{N} \leq \alpha^{-\delta} \sum_{j: \alpha \rho_j < 1} \rho_j^{-\delta} \leq \alpha^{-\delta} m_\delta(X), \]
for any \( \delta \geq 0 \), so that
(8.27)
\[ |\mathcal{I}_2 - \mathcal{A}(g) \sum_{\xi_j \in X; \alpha \rho_j < 1} a(0, \xi_j) \log r| \leq C \alpha^{1-\delta} m_\delta(X) \rho r. \]

To estimate \( \mathcal{I}_3 \) use Lemma 8.5: \( |\mathcal{I}_3| \leq C_\delta \rho^\delta m_\delta(X) \), for any \( \delta \geq 1 \). Together with (8.26) and (8.27) this leads to (8.25). 

9. Partitions of unity, and a reduction to the flat boundary

Now we are in a position to evaluate the asymptotics of the trace
\[ \mathcal{I}_\omega(b; \Lambda, \Omega, g) := \text{tr}(\text{Op}_\omega(b)g(T(1; \Lambda, \Omega))) \]
for a compactly supported symbol \( b \), with the function \( g(t) = t^p - t, p = 1, 2, \ldots \). Theorem 7.8 gives the required asymptotics for the case when \( b \) is supported on a ball contained inside \( \Lambda \), and the answer in this case does not include any information about the
boundary of \( \Lambda \). Now we are ready to tackle the more difficult case, when the support of \( b \) has a non-trivial intersection with the boundary \( \partial \Lambda \). We assume that the domains \( \Lambda \) and \( \Omega \) satisfy Condition 4.1, i.e., \( \Lambda = \Gamma(\Phi; O_\Lambda, k_\Lambda), \Omega = \Gamma(\Psi; O_\Omega, k_\Omega) \), with some orthogonal transformations \( O_\Lambda, O_\Omega \), some \( k_\Lambda, k_\Omega \in \mathbb{R}^d \), and some functions \( \Phi, \Psi \in C^1(\mathbb{R}^{d-1}) \), which satisfy (4.1). Without loss of generality we suppose that \( O_\Lambda = I, k_\Lambda = 0 \), i.e.

\[
\Lambda = \Gamma(\Phi; I, 0) = \{x \in \mathbb{R}^d : x_d > \Phi(\hat{x})\}.
\]

About the symbol \( b \) we assume that either

\[
(9.2) \quad b \in \mathcal{S}(d+2,d+2), \text{ supp } b \subset \mathbb{R}^d \times B(0,\rho),
\]

or

\[
(9.3) \quad b \in \mathcal{S}(d+2,d+2), \text{ supp } b \subset B(0,1) \times B(0,\rho),
\]

with some \( \rho > 0 \). For some intermediate results we need weaker smoothness assumptions on \( b \). Sometimes the dependence of the trace (9.1) on some of the arguments is omitted from the notation and then we write \( \mathfrak{T}_\alpha(b; \Lambda; g) \) or \( \mathfrak{T}_\alpha(b; \Lambda) \) etc.

9.1. **Two partitions of unity.** Our strategy is to “approximate” \( \Lambda \) by a half-space and then use the approach developed in [40]. The first step is to divide the domain \( \Lambda \) into two subsets: a “boundary layer”, which will eventually contribute to the second term of the asymptotics (2.23), and the “inner part”, which affects only the first term of the asymptotics. The study of these two domains require two different partitions of unity. We begin with the partition of unity for the boundary layer.

Let \( v_1, v_2 \in C^\infty(\mathbb{R}) \) be two non-negative functions such that \( v_1(t) + v_2(t) = 1 \) for all \( t \in \mathbb{R} \) and

\[
(9.4) \quad v_1(t) = \begin{cases} 0, & t \leq 1, \\ 1, & t \geq 2. \end{cases}
\]

Define a partition of unity subordinate to the covering of the half-axis by the intervals

\[
\Delta_{-1} = (-2, 3), \Delta_k = (2r^k, 3r^{k+1}), \quad k = 0, 1, \ldots,
\]

where \( r > 2 \). Denote

\[
v_1^{(k)}(t) = v_1\left(\frac{t - r^k}{r^k}\right), \quad v_2^{(k)}(t) = v_2\left(\frac{t - r^{k+1}}{r^{k+1}}\right), \quad k = 0, 1, \ldots,
\]

and define

\[
\zeta_{-1}(t) = v_2(t - 1)v_1(t + 2), \quad \zeta_k(t) = v_1^{(k)}(t)v_2^{(k)}(t), \quad k = 0, 1, 2, \ldots.
\]

It is clear that \( \zeta_k(t) = 0 \) if \( t \notin \Delta_k, \quad k = -1, 0, \ldots \). It follows from the definition of \( v_1, v_2 \) that for \( r > 2 \) and any \( K \),

\[
\sum_{k=-1}^{K} \zeta_k(t) = v_2^{(K)}(t), \quad \sum_{k=-1}^{\infty} \zeta_k(t) = 1, \quad t \geq 0.
\]
Define two cut-off functions on $\mathbb{R}^d$:

\[(9.5) \quad q^\downarrow(x) = v_2^K(\alpha(x_d - \Phi(\hat{x}))), \quad q^\uparrow(x) = v_1^K(\alpha(x_d - \Phi(\hat{x}))).\]

To find the trace asymptotics of $\Sigma_\alpha(b)$ we study the traces $\Sigma_\alpha(q^\downarrow b)$ and $\Sigma_\alpha(q^\uparrow b)$ for the following value of the parameter $K$:

\[(9.6) \quad K = K(\alpha; r, A) = \left\lceil \frac{\log \alpha - A}{\log r} \right\rceil,\]

with some number $A > 0$. Here $\lceil \cdots \rceil$ denotes the integer part. This value of $K$ is chosen thus to ensure that $q^\downarrow$ is supported in a thin “boundary layer” whose width does not depend on $\alpha$, whereas $q^\uparrow$ is supported “well inside” the domain $\Lambda$. More precisely,

\[(9.7) \quad \begin{cases} q^\downarrow(x) = 0, & x_d - \Phi(\hat{x}) \geq 3re^{-A}, \\ q^\uparrow(x) = 0, & x_d - \Phi(\hat{x}) \leq 2r^{-1}e^{-A}. \end{cases}\]

As we show later on, the asymptotics of the trace $\Sigma_\alpha(q^\downarrow b)$ (see (9.1)) “feels” the boundary $\partial\Lambda$, whereas $\Sigma_\alpha(q^\uparrow b)$ can be handled as if $\Lambda$ were the entire space $\mathbb{R}^d$. The trace with $q^\downarrow$ requires a more careful analysis. In particular, we need a further partition of unity in variable $\hat{x}$. Cover $\mathbb{R}^{d-1}$ by cubes of the form $Q_m = Q_0 + m$, $m \in \mathbb{Z}^{d-1}$, where

$Q_0 = (-1,1)^{d-1}$.

Let $\sigma_m = \sigma_m(\hat{x})$ be a partition of unity, associated with this covering, such that

\[(9.8) \quad \sigma_m(\hat{x}) = \sigma_0(\hat{x} - m),\]

which guarantees that

$|\nabla_j^2 \sigma_m(\hat{x})| \leq C_j,$

for all $j$ uniformly in $m \in \mathbb{Z}^{d-1}$. For each $k = 0, 1, \ldots$ we use the partition of unity

\[(9.9) \quad \sigma_{k,m}(\hat{x}) = \sigma_m(\alpha \hat{x}r^{-k-1}), \quad m \in \mathbb{Z}^{d-1}.\]

Now define for all $x \in \mathbb{R}^d$

\[(9.10) \quad \begin{cases} q_{-1}(x) = \zeta_{-1}(\alpha(x_d - \Phi(\hat{x}))), \\ q_{k,m}(x) = \zeta_k(\alpha(x_d - \Phi(\hat{x})))\sigma_{k,m}(\alpha \hat{x}r^{-k-1}), & k = 0, 1, \ldots, m \in \mathbb{Z}^{d-1}, \end{cases}\]

so that

\[(9.11) \quad q^\uparrow(x) = q_{-1}(x) + \sum_{k=0}^{K} \sum_{m \in \mathbb{Z}^{d-1}} q_{k,m}(x).\]

The contributions of the boundary layer and the inner region are found in Lemmas 10.6 and 10.7 respectively. For Lemma 10.7 we need another, more standard partition of unity. To define it we follow the scheme described in [17], Ch.1. Let us state the required result in the form convenient for our purposes.
Proposition 9.1. Let $\ell \in C^1(\mathbb{R}^d)$ be a function such that

$$
|\ell(x) - \ell(y)| \leq \rho |x - y|,
$$

for all $x, y \in \mathbb{R}^d$ with some $\rho \in [0,1)$. Then there exists a set $x_j \in \mathbb{R}^d$, $j \in \mathbb{N}$ such that the balls $B(x_j, \ell(x_j))$ form a covering of $\mathbb{R}^d$ with the finite intersection property, i.e. each ball intersects no more than $N = N(\rho) < \infty$ other balls. Furthermore, there exist non-negative functions $\psi_j \in C_0^\infty(\mathbb{R}^d)$, $j \in \mathbb{N}$, supported in $B(x_j, \ell(x_j))$ such that

$$
\sum_j \psi_j(x) = 1,
$$

and

$$
|\nabla^m \psi_j(x)| \leq C_m \ell(x)^{-m},
$$

for all $m$ uniformly in $j$.

For our purposes the convenient choice of $\ell(x)$ for all $x \in \mathbb{R}^d$ is

$$
\ell(x) = \begin{cases} 
\frac{1}{32\langle M \rangle} (x_d - \Phi(\hat{x})), & x_d > \Phi(\hat{x}) + \alpha^{-1}, \\
\frac{1}{32\alpha \langle M \rangle}, & x_d \leq \Phi(\hat{x}) + \alpha^{-1}. 
\end{cases}
$$

Here $M > 0$ is the constant from condition $(4.1)$. Since $|\nabla \ell| \leq 1/32$ a.e., the condition $(9.12)$ is satisfied with $\rho = 1/32$.

Lemma 9.2. If $x_d - \Phi(\hat{x}) > \alpha^{-1}$, then $B(x, 32\ell) \subset \Lambda$.

Proof. By $(4.2)$, under the condition $x_d - \Phi(\hat{x}) > \alpha^{-1}$ we have for any $y \in B(x, 32\ell)$:

$$
y_d - \Phi(\hat{y}) \geq x_d - \Phi(\hat{x}) - \langle M \rangle |x - y| = 32\langle M \rangle \ell(x) - \langle M \rangle |x - y| > 0.
$$

This proves that $y \in \Lambda$, as claimed. \qed

Incidentally, the partition of unity just introduced is also useful for the boundary layer. Indeed, our study of the traces $T_\alpha(q_k,m;b)$, $k = 0, 2, \ldots, m \in \mathbb{Z}^{d-1}$, is based on the estimates obtained in the previous sections for symbols from the classes $S^{(m,n)}$. Observe, however, that those estimates are not directly applicable to symbols of the type $q_k,m$, since these functions do not have isotropic scaling properties in-built in the norms $(2.5)$. Thus in Lemma 9.4 below we study individually operators with $\psi_j q_k,m$ and then sum up in $j$.

9.2. Reduction to the flat boundary. The first step in finding the asymptotics of $T_\alpha(q_k,m;b;\Lambda)$ is to replace $\Lambda$ by a half-space, as in Lemma 5.26. We emphasize that the choice of this half-space depends on $k = 0, 1, \ldots$ and $m \in \mathbb{Z}^{d-1}$.

For a fixed $\hat{z} \in \mathbb{R}^{d-1}$ define

$$
\Phi_0(\hat{y}) = \Phi_0(y;\hat{z}) = \Phi(\hat{z}) + \nabla \Phi(\hat{z}) \cdot (\hat{y} - \hat{z}).
$$
Then by (5.17),

\[
\sup_{\tilde{z} \in \mathbb{R}^{d-1}} \max_{\tilde{y}, |\tilde{z} - \tilde{y}| \leq s} |\Phi(\tilde{y}) - \Phi_0(\tilde{y}, \tilde{z})| \leq s\varepsilon(s),
\]

For each \(k = 0, 1, \ldots, m \in \mathbb{Z}^{d-1}\) define the approximating half-space by

\[
\Pi_{k,m} = \{x : x_d > \Phi_0(\hat{x}; \hat{x}_{k,m})\}, \quad \hat{x}_{k,m} = \alpha^{-1}r^{k+1}m.
\]

For our analysis of various asymptotics it is convenient to introduce the notion of a \(W\)-sequence.

**Definition 9.3.** Let \(w_k = w_k(r, A), k \geq 0,\) be a sequence of non-negative numbers, depending on the parameters \(r, A > 0,\) and let \(K = K(\alpha; r, A)\) be as defined in (9.6). We say that \(w_k\) is a \(W\)-sequence if

\[
\lim_{r \to \infty} \lim_{A \to \infty} \limsup_{\alpha \to \infty} \frac{1}{\log \alpha} \sum_{k \leq K(\alpha; r, A)} w_k(r, A) = 0.
\]

If a \(W\)-sequence depends on some other fixed parameter, for instance \(M\) from (4.1), then it is not reflected in the notation and we still write simply \(w_k(r, A).\)

**Lemma 9.4.** Let \(\Lambda\) and \(\Omega\) be some domains satisfying Condition 4.1 with \(O_\Lambda = I, k_\Lambda = 0.\) Suppose that the symbol \(b\) satisfies (9.2). Then for any \(k = 0, 1, \ldots, K = K(\alpha; r, A),\)

\[
\|q_{k,m} \text{Op}_\alpha(b)(g_p(T(1; \Lambda)) - g_p(T(1; \Pi_{k,m})))\|_{S_1} \leq \rho^{(k+1)d-1}w_k(r, A)N^{d+1,d+2}(b; 1, \rho),
\]

with some \(W\)-sequence \(w_k(r, A)\) independent of the symbol \(b,\) uniformly in \(m \in \mathbb{Z}^{d-1}, \rho \geq c.\) Moreover, \(w_k(r, A)\) does not depend on the functions \(\Phi, \Psi,\) and on the transformation \((O_\Omega, k_\Omega),\) but may depend on the parameter \(M.\)

**Proof.** Denote \(\Pi = \Pi_{k,m}.\) Using (2.18) we may assume that \(m = \hat{0}\) and \(\Phi(\hat{0}) = 0,\) so that \(\Phi_0(\hat{x}) = \nabla \Phi(\hat{0}) \cdot \hat{x}.\) For \(\mu > 0\) denote

\[
D_k(\mu) = \{x : 2\mu^{-1}\alpha^{-1}r^k < x_d - \Phi(\hat{x}) < 3\mu^{-1}\alpha^{-1}r^{k+1}, |\hat{x}| < \mu\alpha^{-1}r^{k+1}\}.
\]

It is clear that \(\text{supp} \ q_{k,0} \subset D_k(1).\) Also, since \(k \geq 0,\) we have \(x_d - \Phi(\hat{x}) > \alpha^{-1}\) for \(x \in D_k(2).\)

**Step 1:** Let us show first that

\[
D_k(2) \subset \Lambda \cap \Pi \cap B(\hat{0}, 13(M)\alpha^{-1}r^{k+1}),
\]
for sufficiently large $A$. By construction $D_k(2) \subset \Lambda$. For an arbitrary $x \in D_k(2)$ write, using (9.14):

$$x_d - \Phi_0(\check{x}) = x_d - \Phi(\check{x}) + (\Phi(\check{x}) - \Phi_0(\check{x}))$$

$$\geq x_d - \Phi(\check{x}) - 2\alpha^{-1}r^{k+1}\varepsilon(2\alpha^{-1}r^{k+1})$$

$$\geq \alpha^{-1}r^k - 2\alpha^{-1}r^k r\varepsilon(2\alpha^{-1}r^{k+1})$$

$$= \alpha^{-1}r^k [1 - 2r\varepsilon(2\alpha^{-1}r^{k+1})].$$

Estimating

$$\alpha^{-1}r^{k+1} \leq re^{-A}$$

for all $k \leq K$, and taking a sufficiently large $A$, we can guarantee that the right hand side of (9.19) is positive which proves that $D_k(2) \subset \Pi$. Furthermore, since $\Phi(0) = 0$ and $|\nabla\Phi(x)| \leq M$, for any $x \in D_k(2)$ we have

$$|x_d| \leq |x_d - \Phi(\check{x})| + |\Phi(\check{x})| \leq 6\alpha^{-1}r^{k+1} + M|\check{x}| \leq 12(M)\alpha^{-1}r^{k+1},$$

so that

$$|x| \leq \sqrt{x_d^2 + |\check{x}|^2} \leq 13(M)\alpha^{-1}r^{k+1},$$

and hence $x \in B(0, 13(M)\alpha^{-1}r^{k+1})$, as claimed. Thus (9.18) is proved.

Step 2. Let $B(x_j, \ell_j), \ell_j := \ell(x_j)$ be the balls from Proposition 9.1, which form a covering of $\mathbb{R}^d$ with the finite intersection property. Let $J \subset \mathbb{Z}$ be the set of all indices $j$ such that $B(x_j, 4\ell_j) \subset D_k(2)$. Let us prove that

$$D_k(1) \subset \bigcup_{j \in J} B(x_j, \ell_j).$$

(9.20)

It suffices to show for arbitrary $x \in D_k(2)$, that if $w \in B(x, 4\ell(x))$, but $w \notin D_k(2)$, then $B(x, \ell(x)) \cap D_k(1) = \emptyset$. In view of (9.13), $\ell(x) = (32\langle M \rangle)^{-1}(x_d - \Phi(\check{x}))$.

We consider separately three cases. Case 1. Suppose that $w \in B(x, 4\ell(x))$ and $w_d - \Phi(\check{w}) < \alpha^{-1}r^k$. Then by (4.2),

$$x_d - \Phi(\check{x}) \leq w_d - \Phi(\check{w}) + \langle M \rangle|w - x|$$

$$< \alpha^{-1}r^k + \frac{1}{8}(x_d - \Phi(x_d)).$$

Thus $x_d - \Phi(\check{x}) < 8(7\alpha)^{-1}r^k$, so that for any $y \in B(x, \ell(x))$ we have by (4.2) again that

$$y_d - \Phi(\check{y}) \leq x_d - \Phi(\check{x}) + \langle M \rangle|x - y| \leq \frac{33}{32}(x_d - \Phi(\check{x})) < 2\alpha^{-1}r^k,$$

and hence $B(x, \ell(x)) \cap D_k(1) = \emptyset$.

Case 2. Suppose that $w \in B(x, 4\ell(x))$ and $w_d - \Phi(\check{w}) > 6\alpha^{-1}r^{k+1}$. Then by (4.2),

$$x_d - \Phi(\check{x}) \geq w_d - \Phi(\check{w}) - \langle M \rangle|w - x|$$

$$> 6\alpha^{-1}r^{k+1} + \frac{1}{8}(x_d - \Phi(x_d)).$$

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Thus $x_d - \Phi(\hat{x}) > 48(9\alpha)^{-1}r^{k+1}$, so that for any $y \in B(x, \ell(x))$ we have

$$y_d - \Phi(\hat{y}) \geq x_d - \Phi(\hat{x}) - \langle M \rangle |x - y| > \frac{31}{32}(x_d - \Phi(\hat{x})) > 3\alpha^{-1}r^{k+1},$$

and hence $B(x, \ell(x)) \cap D_k(1) = \emptyset$ again.

Case 3. Suppose now that $w \in B(x, 4\ell(x))$, but $|\hat{w}| \geq 2\alpha^{-1}r^{k+1}$. By virtue of Case 2 above, we may assume that $x_d - \Phi(\hat{x}) \leq 6\alpha^{-1}r^{k+1}$. Consequently,

$$|\hat{x}| \geq |\hat{w}| - |x - w| \geq 2\alpha^{-1}r^{k+1} - \frac{1}{8\langle M \rangle}(x_d - \Phi(\hat{x})) \geq \frac{5}{4}\alpha^{-1}r^{k+1}.$$ 

Thus for any $y \in B(x, \ell(x))$,

$$|\hat{y}| \geq |\hat{x}| - |x - y| \geq |\hat{x}| - \frac{1}{32\langle M \rangle}(x_d - \Phi(\hat{x})) \geq \frac{5}{4}\alpha^{-1}r^{k+1} - \frac{1}{5\langle M \rangle}\alpha^{-1}r^{k+1} > \alpha^{-1}r^{k+1},$$

so that $B(x, \ell(x)) \cap D_k(1) = \emptyset$. This proves (9.20).

Step 3. Let $\psi_j$ be the partition of unity from Proposition 9.1. According to (9.20)

$$q_{k,0}(x) = \sum_{j \in J} \phi_j(x), \quad \phi_j(x) = q_{k,0}(x)\psi_j(x).$$

Thus the trace norm

$$Z = \|q_{k,0} \cdot \text{Op}_\alpha^l(b)(g_p(T(1; \Lambda)) - g_p(T(1; \Pi)))\|_{s_1},$$

is estimated as follows:

(9.21) $Z \leq \sum_{j \in J} Z_j, \quad Z_j = \|\text{Op}_\alpha^l(\phi_j b)(g_p(T(1; \Lambda)) - g_p(T(1; \Pi)))\|_{s_1}.$

For each $Z_j$ we use Lemma 5.6 with some $\varkappa \in (0, \varkappa_p)$. Since $x_j \in D_k(2)$, we have

(9.22) $\frac{1}{32\langle M \rangle}\alpha^{-1}r^{k} \leq \ell_j \leq \frac{6}{32\langle M \rangle}\alpha^{-1}r^{k+1},$

and as $k \leq K$, this means that $\ell_j \leq re^{-A}$. For sufficiently large $A$, we have $\ell_j \leq 1$, so that

$$N^{(d+1,d+2)}(b\phi_j; \ell_j, \rho) \leq CN^{(d+1,d+2)}(b; 1, \rho).$$

By (9.18) and by definition of the set $J$, the ball $B(x_j, 4\ell_j), j \in J$ satisfies the condition (5.20) with

$$t = \alpha^{-1}r^{k+2}.$$

for all $r \geq 13\langle M \rangle$. Furthermore, in view of (9.22),

$$5\langle M \rangle r < tl_j^{-1} < 32\langle M \rangle r^2, \quad j \in J.$$

Thus for sufficiently large $r$ the first condition in (5.21) is satisfied uniformly in $k$. Now estimate for $k \leq K$:

$$tl_j^{-1}\varepsilon(4t) < 32\langle M \rangle r^2 \varepsilon(r^2 e^{-A}).$$
For sufficiently large $A$ this quantity is arbitrarily small, and hence the second condition in (5.21) is also satisfied. Finally, $\alpha \ell_j \rho \geq cr^k \rho \geq c'$. Thus by Lemma 5.6,

$$Z_j \leq C(\alpha \ell_j \rho)^{d-1} R_{x}(\alpha; \ell_j, \rho, t) N^{(d+1,d+2)}(b; 1, \rho),$$

uniformly in $j \in J$. By definition (5.23),

$$R_{x}(\alpha; \ell_j, \rho, t) \leq C [r^{-\varepsilon} + r^{4(d-1)} r^{-\varepsilon} + r^{4(d-1)} (r^{2} e^{-A})^\varepsilon] =: \tilde{w}_k(r, A),$$

for all $\rho \geq c$, uniformly in $j \in J$. By (9.21),

$$Z \leq \sum_{j \in J} Z_j \leq C(\alpha \rho)^{d-1} N^{(d+1,d+2)}(b; 1, \rho) \tilde{w}_k(r, A) \sum_{j \in J} (\ell(x_j))^{-1}. $$

Due to the finite intersection property (see Proposition 9.1), and to the property $B(x_j, 4\ell(x_j)) \subset D_k(2)$ for all $j \in J$, the sum on the right-hand side is estimated by the integral

$$\int_{D_k(2)} f(x)^{-1} d\mathbf{x} = \int_{|\hat{x}| \leq 2} \int_{\alpha^{-1} r^{k+1}} t^{-1} dt d\hat{x} \leq C \alpha^{1-d} r^{(k+1)(d-1)} \log r,$$

and hence

$$Z \leq C r^{(k+1)(d-1)} \rho^{d-1} w_k(r, A) N^{(d+1,d+2)}(b; 1, \rho), \quad w_k(r, A) = \log r \ \tilde{w}_k(r, A).$$

Step 4. To complete the proof of (9.4) it remains to show that $w_k(r, A)$ is a $W$-sequence, i.e. it satisfies the property (9.16). To this end consider each term in the definition of $w_k = \log r \ \tilde{w}_k$ (see (9.23)) separately. Clearly,

$$\sup_A \sup_\alpha \frac{\log r}{\log \alpha} \sum_{k \leq K} \frac{r^{-\varepsilon}}{r^{-\varepsilon}} \leq r^{-\varepsilon} \to 0, \ r \to \infty,$$

so this sum satisfies (9.16). Consider now

$$\frac{\log r}{\log \alpha} \sum_{k \leq K} r^{4(d-1)} r^{-\varepsilon} \leq \frac{\log r}{\log \alpha} r^{d(d-1)} \frac{1}{1 - r^{-\varepsilon}} \to 0, \ \alpha \to \infty,$$

so this sum also satisfies (9.16). For the last term in the definition (9.23), note that

$$r^{k+2} \alpha^{-1} \leq r^2 e^{-A},$$

and hence

$$\sup_\alpha \frac{\log r}{\log \alpha} \sum_{k \leq K} r^{4(d-1)} (r^{2} e^{-A})^\varepsilon \leq r^{4(d-1)} (r^{2} e^{-A})^\varepsilon \to 0, \ A \to \infty,$$

which follows from the fact that $\varepsilon(s) \to 0$ as $s \to 0$. Therefore $w_k$ satisfies (9.16), and the proof of the Lemma is now complete. \qed
10. Asymptotics of the trace (9.1)

Now we compute the asymptotics of the trace $\mathcal{T}_\alpha(q_{k,m}; \Pi_{k,m}; \Omega; g)$ (see (9.1) for definition of $\mathcal{T}_\alpha$, and (9.15) for definition of $\Pi_{k,m}$), where $k = 0, 1, \ldots, m \in \mathbb{Z}^{d-1}$,

$$
(10.1) \quad g(t) = t^p - t, \quad \text{with some } p \in \mathbb{N}.
$$

As before we assume that $\Lambda$ and $\Omega$ are graph-type domains, but we also need stronger smoothness conditions on $\Psi$, and some specific restrictions on the orthogonal transformations $O_\Lambda, O_\Omega$.

**Condition 10.1.** Let $\Phi \in C^1(\mathbb{R}^{d-1})$, $\Psi \in C^3(\mathbb{R}^{d-1})$ be some real-valued functions satisfying (4.1) and

$$
(10.2) \quad \|\nabla^2 \Psi\|_{L^\infty} + \|\nabla^3 \Psi\|_{L^\infty} \leq \tilde{M},
$$

with some $\tilde{M} > 0$. Assume that $\Phi(\hat{0}) = \Psi(\hat{0}) = 0$. The domain $\Lambda$ is defined as $\Lambda = \Gamma(\Phi; I, 0)$, and there is an index $l = 1, 2, \ldots, d$ such that the domain $\Omega$ is defined by

$$
(10.3) \quad \Omega = \begin{cases} 
\Omega^+ = \{\xi : \xi_l > \Psi(\hat{\xi})\}, & \hat{\xi} = (\xi_1, \ldots, \xi_{l-1}, \xi_{l+1}, \ldots, \xi_d), \\
\text{or } \Omega^- = \{\xi : \xi_l < \Psi(\hat{\xi})\}, & \hat{\xi} = (\xi_1, \ldots, \xi_{l-1}, \xi_{l+1}, \ldots, \xi_d).
\end{cases}
$$

One can easily see that the above definition of the domain $\Omega$ in fact describes $\Omega$ as a graph-type domain $\Gamma(\hat{\Psi}; O; 0)$ with some easily identifiable orthogonal transformation $O$ and function $\hat{\Psi}$. For example, if $l = d - 1$, then for the domain $\Omega^-$ the entries $O_{js}$ of the matrix $O$ are given by

$$
O_{js} = \begin{cases}
\delta_{j,s}, & 1 \leq j \leq d - 2, \\
-\delta_{j+1,s}, & j = d - 1, \\
\delta_{j-1,s}, & j = d,
\end{cases}
$$

and the function $\hat{\Psi}$ by $\hat{\Psi}(\hat{\xi}) = -\Psi(\hat{\xi})$.

In what follows one of the main players will be the set

$$
(10.4) \quad \Omega^{(\pm)}(\hat{\xi}) = \{t \in \mathbb{R} : \langle \hat{\xi}, t \rangle \in \Omega^{(\pm)}\} = \{\xi_d : \pm \xi_l > \pm \Psi(\hat{\xi})\}.
$$

We are interested in the structure of the set $\Omega^{(\pm)}(\hat{\xi}) \cap (-2\rho, 2\rho)$ for $\hat{\xi} \in \mathbb{S}^{d-1}$ with an appropriate $\rho > 0$, see (1.11) for the definition of the cube $\mathbb{S}_\rho$. The set $\Omega^{(\pm)}(\hat{\xi}) \cap (-2\rho, 2\rho)$ is either empty, or it is an open set, i.e. it is a countable union of disjoint open intervals whose endpoints either coincide with $\pm 2\rho$, or lie inside of the interval $(-2\rho, 2\rho)$. Denote by $X^{(\pm)}(\hat{\xi})$ the set of those endpoints which lie strictly inside $(-2\rho, 2\rho)$. Define also

$$
X(\hat{\xi}) = \{\xi_d \in (-2\rho, 2\rho) : \Psi(\hat{\xi}) = \xi_l\}.
$$

Obviously, $X^{(\pm)}(\hat{\xi}) \subset X(\hat{\xi})$. In case the set $X(\hat{\xi})$ is finite we use the quantity $m_d(X(\hat{\xi}))$ introduced in (8.14). The next lemma is of primary importance:
Let $\Psi \in C^3(\mathbb{R}^{d-1})$ be a function satisfying Condition 10.1, and let the sets $X^{(\pm)}(\xi), X(\xi) \subset (-2\rho, 2\rho)$ be as defined above. Then

(1) For almost all $\hat{\xi} \in \mathcal{C}_{\rho}^{(d-1)}$:
   - (a) the set $X(\hat{\xi})$ is finite,
   - (b) $X(\hat{\xi}) = X^{(\pm)}(\hat{\xi})$,
   - (c) the disjoint open intervals forming $\Omega^{\pm}(\hat{\xi})$ have distinct endpoints.

(2) For any $\delta \in (0, 2)$ the function $m_\delta(X(\hat{\xi}))$ satisfies the bound

$$
\int_{\xi \in \mathcal{C}_{\rho}^{(d-1)}} m_\delta(X(\hat{\xi})) d\hat{\xi} \leq C \rho^{-1-\delta}(1 + (\rho + \rho^2) M).
$$

The constant $C$ depends only on $\delta$ and dimension $d$.

This lemma follows immediately from Theorem 13.1.

First we establish the asymptotics for the case $\nabla \Phi(\hat{x}_{k,m}) = 0$. Throughout this section all $\mathcal{W}$-sequences $w_k(r, A)$ do not depend on the functions $\Phi \in C^1, \Psi \in C^3$, but may depend on the constants $M$ in (4.1) and $M$ in (10.2).

Lemma 10.3. Assume that

(1) $g$ is given by (10.1),
(2) $\Lambda, \Omega$ are two graph-type domains satisfying Condition 10.1,
(3) $0 \leq k \leq K$ with $K$ defined in (9.6),
(4) $b$ is a symbol satisfying (9.2),
(5) $\nabla \Phi(\hat{x}_{k,m}) = 0$.

Then

$$
\left| \Theta_\alpha(q_k, m; \Pi_{k,m}, \Omega, g) - \alpha^{d-1} \log r \mathfrak{A}(g) \mathfrak{M}_1(\sigma_{k,m} b; \partial \Lambda, \partial \Omega) \right| 
\leq r^{(k+1)(d-1)} w_k(r, A) N^{(2,2)}(b, 1, \rho),
$$

with some $\mathcal{W}$-sequence $w_k(r, A)$ independent of the symbol $b$ and of the point $m \in \mathbb{Z}^{d-1}$, uniformly in $\rho \in [c, C]$ with arbitrary positive constants $c, C$ such that $c < C$.

Proof. Using (2.18) we may assume that $\Phi(\hat{x}_{k,m}) = 0$. We also assume throughout that $\alpha \geq 2, r \geq 5, A \geq 2$, and $c \leq \rho \leq C$.

Step 1: a reduction to the one-dimensional case. Let $\Omega^{(\pm)}(\hat{\xi})$ be as defined in (10.4).

Since $\Pi_{k,m} = \{ x : x_d > 0 \}$ (see (9.15)), a straightforward calculation shows that $g_p(T(1; \Pi_{k,m}, \Omega))$ is a PDO in $L^2(\mathbb{R}^{d-1}, \mathcal{H}), \mathcal{H} = L^2(\mathbb{R})$, with the operator-valued symbol

$$
g_p(T(1; \mathbb{R}^+, \Omega^{(\pm)}(\hat{\xi}))).
$$

Therefore the operator $X = q_{k,m} \mathcal{O}_p(\alpha)b\left(T(1; \Pi_{k,m}, \Omega)\right)$ can be viewed as a PDO with the operator-valued symbol

$$
X_\alpha(\hat{x}, \hat{\xi}) = q_{k,m}(\hat{x}, \cdot) \mathcal{O}_p(\alpha)(b(\hat{x}, \cdot; \hat{\xi}, \cdot)) g\left(T(1; \mathbb{R}^+, \Omega^{(\pm)}(\hat{\xi}))\right),
$$
Recall that with the operator-valued symbol $A$, i.e., $X_{\alpha} \hat{x}, \hat{\xi}$, the asymptotics of the trace of the operator $X(\hat{x}, \hat{\xi})$. By definition (9.10),

$$X_{\alpha}(\hat{x}, \hat{\xi}) = \sigma_{k,m}(\hat{x}) \tilde{y}_{\alpha}(\hat{x}, \hat{\xi}),$$

with the operator-valued symbol

$$\tilde{y}_{\alpha}(\hat{x}, \hat{\xi}) = \tilde{\psi} \text{Op}_{\alpha}(b(\hat{x}, \cdot; \hat{\xi}, \cdot)) g(T(1; \mathbb{R}^+, \Omega^{(\pm)}(\hat{\xi}))), \quad \tilde{\psi}(x_d) = \zeta_k(x_d - \Phi(\hat{x})).$$

After the change of the variable $x_d = r^k \alpha^{-1} t$,

the operator $\tilde{y}_{\alpha}(\hat{x}, \hat{\xi})$ becomes unitarily equivalent to

$$(10.6) \quad y_{\beta}(\hat{x}, \hat{\xi}) = \psi \text{Op}_{\beta}(a) g(T(1; \mathbb{R}^+, \Omega^{(\pm)}(\hat{\xi}))), \quad \beta = r^k,$$

where

$$\psi(t) = \zeta_k(r^k t - \alpha \Phi(\hat{x}))) = v_1(t - 1 - \alpha r^{-k} \Phi(\hat{x})) v_2(t r^{-1} - 1 - \alpha r^{-k-1} \Phi(\hat{x})), $$

$$a(t, \xi) = b(\hat{x}, r^k \alpha^{-1} t; \hat{\xi}, \xi).$$

The asymptotics of $\text{tr} y_{\beta}(\hat{x}, \hat{\xi})$ is found with the help of Theorem 8.6. Let us check that the symbol $a$, the function $\psi$, and the set $\Omega^{(\pm)}(\hat{\xi})$ satisfy the conditions of this Theorem.

Note that $a \in S^{(2,2)}$ and

$$(10.7) \quad N^{(2,2)}(a; L, \rho) \leq N^{(2,2)}(b; 1, \rho), \quad L := \alpha r^{-k} \geq e^A,$$

where we have used that $k \leq K$ (see (9.6)). Moreover, $a(t, \xi) = 0$ for $|\xi| \geq \rho$.

Recall that $\alpha r^{-(k+1)} \hat{x} \in Q_m$, so that in view of (5.17) and (9.6),

$$(10.8) \quad |\Phi(\hat{x})| = |\Phi(\hat{x}) - \Phi(\hat{x}_m)| \leq \varepsilon(r^{k+1} \alpha^{-1}) r^{k+1} \alpha^{-1} \leq \varepsilon(r e^{-A}) \alpha^{-1} r^{k+1} e^{-A},$$

and hence

$$\alpha r^{-k} |\Phi(\hat{x})| \leq r \varepsilon(r e^{-A}).$$

For sufficiently large $A$, we can guarantee that $r \varepsilon(r e^{-A}) \leq 1/4$. Thus by the definition (9.4), the function $\psi(t)$ satisfies the requirements (8.5).

From now on we always assume that $\hat{\xi}$ is in the subset of full measure in $C^{(d-1)}$ such that the properties (1)(a)-(c) from lemma 10.2 hold. This guarantees that $\Omega^{(\pm)}(\hat{\xi})$ is of the form required in Theorem 8.6.

Step 2: asymptotics of $\text{tr} y_{\beta}(\hat{x}, \hat{\xi})$. Recall that $r \geq 2$, and note also that for sufficiently large $A$ we have $L \geq r$, see (10.7) for definition of $L$. Moreover, $\beta \rho = r^k \rho \geq c$, see (10.6)
for definition of $\beta$. Thus by Theorem 8.6,
\[
\left| \text{tr } Y_{\beta}(\hat{x}, \hat{\xi}) - \mathfrak{A}(g) \log r \sum_{\xi \in X(\hat{\xi})} a(0, \xi) \right|
\leq C m_\delta(X(\hat{\xi}))(1 + \beta^{1-\delta} r) N^{(2,2)}(a; L, \rho),
\]
for any $\delta \geq 1$. Rewrite, remembering the definition of $a(x, \xi)$:
\[
\left| \text{tr } Y_{\beta}(\hat{x}, \hat{\xi}) - \mathfrak{A}(g) \log r \sum_{\xi \in X(\hat{\xi})} b(\hat{x}, 0; \hat{\xi}, \Psi(\hat{\xi})) \right|
\leq C m_\delta(X(\hat{\xi}))[1 + \beta^{1-\delta} r] N^{(2,2)}(a; L, \rho),
\]
Analyze the asymptotic term on the left-hand side. By virtue of (10.8) for any $\xi$ we have
\[
|b(\hat{x}, 0; \xi) - b(\hat{x}, \Phi(\hat{x}); \xi)| \leq N^{(1,0)}(b; 1, \rho) r e^{-A} \varepsilon(r e^{-A}),
\]
so that by (8.15),
\[
\left| \sum_{\xi \in X(\hat{\xi})} (b(\hat{x}, 0; \xi) - b(\hat{x}, \Phi(\hat{x}); \xi)) \right| \leq \#X(\hat{\xi}) N^{(1,0)}(b; 1, \rho) r e^{-A} \varepsilon(r e^{-A})
\leq (2\rho)^\delta m_\delta(X(\hat{\xi})) r e^{-A} \varepsilon(r e^{-A}) N^{(1,0)}(b; 1, \rho).
\]
Assume that $A$ is so large that $r \log r e^{-A} \varepsilon(r e^{-A}) \leq 1$. Thus
\[
(10.9) \quad \begin{cases}
\left| \text{tr } Y_{\beta}(\hat{x}, \hat{\xi}) - \mathfrak{A}(g) \log r \sum_{\xi \in X(\hat{\xi})} b(\hat{x}, \Phi(\hat{x}); \hat{\xi}, \Psi(\hat{\xi})) \right| \leq C \mathcal{L}_k(\hat{\xi}; r), \\
\mathcal{L}_k(\hat{\xi}; r) = m_\delta(X(\hat{\xi}))(1 + r^{(1-\delta)k+1}) N^{(2,2)}(b; 1, \rho),
\end{cases}
\]
where we have replaced $\beta$ with its value, i.e. $r^k$.

Step 3: asymptotics of $\text{tr } X_\alpha$. By Lemma 15.1,
\[
\text{tr } X_\alpha = \left( \frac{\alpha}{2\pi} \right)^{d-1} \int \sigma_{k,m}(\hat{x}) \text{tr } Y_{\beta}(\hat{x}, \hat{\xi}) d\hat{x} d\hat{\xi},
\]
and hence, by (10.9),
\[
(10.10) \quad \left| \text{tr } X_\alpha - \left( \frac{\alpha}{2\pi} \right)^{d-1} \log r \mathfrak{A}(g) \mathfrak{B}_k(\alpha, m; r) \right| \leq C \mathfrak{R}_k(\alpha, m; r, A),
\]
where the leading term is
\[
(10.11) \quad \mathfrak{B}_k(\alpha, m; r) = \int \int \sigma_{k,m}(\hat{x}) \sum_{\xi \in X(\hat{\xi})} b(\hat{x}, \Phi(\hat{x}); \hat{\xi}, \Psi(\hat{\xi})) d\hat{x} d\hat{\xi},
\]
and the remainder $R_k(\alpha, m; r)$ is

$$
R_k(\alpha, m; r) = \alpha^{d-1} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \sigma_m(\alpha x r^{k-1}) L_k(\xi; r) d\hat{x} d\hat{\xi}
$$

(10.12)

$$
\leq C r^{(k+1)(d-1)} \int_{\mathbb{R}^{d-1}} L_k(\xi; r) d\hat{\xi}.
$$

Here we have used the definition (9.9). Let us calculate the leading term first. According to Lemma 14.6,

$$
B_k(\alpha, m; r) = \int_{\partial \Omega} \int_{\partial \Lambda} \sigma_{k,m}(\hat{x}, \xi) \frac{|n_{\partial \Omega}(\xi) \cdot e_d|}{\sqrt{1 + |\nabla \Phi(\hat{x})|^2}} dS_x dS_\xi,
$$

where

$$
n_{\partial \Omega}(\xi) := \left( -\partial_{\xi_1} \Psi(\hat{\xi}), \ldots, -\partial_{\xi_{l-1}} \Psi(\hat{\xi}), 1, -\partial_{\xi_{l+1}} \Psi(\hat{\xi}), \ldots, -\partial_{\xi_d} \Psi(\hat{\xi}) \right),
$$

is the unit normal to the surface $\xi_l = \Psi(\hat{\xi})$ at the point $\xi = (\xi_1, \ldots, \xi_{l-1}, \Psi(\hat{\xi}), \xi_{l+1}, \ldots, \xi_d)$. On the other hand, the unit normal to $\partial \Lambda$ at the point $x = (\hat{x}, \Phi(\hat{x}))$ is given by the vector

$$
n_{\partial \Lambda}(x) = \frac{(-\nabla \Phi(\hat{x}), 1)}{\sqrt{1 + |\nabla \Phi(\hat{x})|^2}}.
$$

Thus

$$
\left| n_{\partial \Omega}(\xi) \cdot n_{\partial \Lambda}(x) - \frac{n_{\partial \Omega}(\xi) \cdot e_d}{\sqrt{1 + |\nabla \Phi(\hat{x})|^2}} \right| \leq |\nabla \Phi(\hat{x}) - \nabla \Phi(\hat{0})|.
$$

By (5.17) the right hand side does not exceed

$$
\varepsilon(r^{k+1}\alpha^{-1}) \leq \varepsilon(re^{-A}),
$$

so the main term of the asymptotics satisfies the bound

$$
|\mathfrak{B}_k(\alpha, m; r) - (2\pi)^{d-1} \mathfrak{W}_1(\sigma_k,m b; \partial \Lambda, \partial \Omega)|
\leq C \varepsilon(re^{-A}) \max |b(\mathbf{x}, \xi)| \alpha^{1-d} r^{(k+1)(d-1)},
$$

uniformly in $m \in \mathbb{Z}^{d-1}$. Since $\varepsilon(re^{-A}) \to 0$ as $A \to \infty$, the above estimate can be rewritten as follows:

$$
|\mathfrak{B}_k(\alpha, m; r) - (2\pi)^{d-1} \mathfrak{W}_1(\sigma_k,m b; \partial \Lambda, \partial \Omega)|
\leq \alpha^{1-d} r^{(k+1)(d-1)} w_k^{(1)}(r, A) N_k^{(2,2)}(b; 1, \rho),
$$

(10.13)

with some $\mathfrak{W}$-sequence $w_k^{(1)}(r, A)$. 
Step 4: estimating the remainder (10.12). Let us show that the remainder (10.12) defines a $W$-sequence. By (10.12) and (10.9),

$$R_k(\alpha, m; r) \leq Cr^{k+1}(d-1)(1 + r^{(1-\delta)k+1}) \int_{|\xi| \leq \rho} m_\delta(X(\hat{\xi})) d\hat{\xi} N^{(2,2)}(b; 1, \rho),$$

for any $\delta \geq 1$, where $\beta = r^k$. By (10.5),

$$\int_{|\xi| \leq \rho} m_\delta(X(\hat{\xi})) d\hat{\xi} \leq C r^{d-1-\delta}(1 + (\rho + \rho^2)M),$$

for all $\delta \in [1, 2)$. Thus $R_k(\alpha, m; r)$ satisfies the bound

$$R_k(\alpha, m; r) \leq Cr^{k+1}(d-1)w_k^{(2)}(r; 1, \rho),$$

(10.15)

for $w_k^{(2)}(r, A) = 1 + r^{1+(1-\delta)k}$.

Assume that $\delta > 1$. In view of (9.6),

$$\frac{1}{\log \alpha} \sum_{k=0}^{K} (1 + r^{k(1-\delta)+1}) \leq \frac{1}{\log r} + \frac{1}{\log \alpha} \frac{r}{1 - r^{1-\delta}}.$$

Thus the triple limit $\lim_{r \to \infty} \sup_{A \to \infty} \limsup_{\alpha \to \infty}$ of the above expression equals zero. This proves that $w_k^{(2)}$ is a $W$-sequence.

Step 5: end of the proof. According to (10.10), (10.13) and (10.15), for each $k = 0, 1, \ldots, K$ we have

$$|\text{tr } X_\alpha - \left(\frac{\alpha}{2\pi}\right)^{d-1} A(g) \log r \mathfrak{M}_1(\sigma_k, m b; \partial \Lambda, \partial \Omega)|$$

$$\leq r^{(k+1)(d-1)}w_k(r, A)N^{(2,2)}(b; 1, \rho),$$

with some $W$-sequence $w_k(r, A)$. The proof is complete. \hfill $\Box$

Now we can remove the condition $\nabla \Phi(\hat{x}_k, m) = 0$. Recall the notation (2.15).

Lemma 10.4. Suppose that Conditions (1)-(4) of Lemma 10.3 are satisfied. We assume in addition that

$$M_\phi \leq \frac{1}{2}, \quad M_\phi M_\phi \leq \frac{1}{2}, \quad \text{if } l = d.$$  

Then

$$\left| \Sigma_\alpha(q_k, m b; \Pi_k, m, \Omega, g) - \alpha^{d-1} \log r \mathfrak{A}(g) \mathfrak{M}_1(\sigma_k, m b, \partial \Lambda, \partial \Omega) \right|$$

$$\leq r^{(k+1)(d-1)}w_k(r, A)N^{(2,2)}(b, 1, \rho),$$

with some $W$-sequence $w_k(r, A)$ independent of the symbol $b$ and of the point $m \in \mathbb{Z}^{d-1}$, uniformly in $\rho \in [c, C]$ for arbitrary constants $0 < c < C$. 
Proof. We reduce the problem to the one considered in Lemma 10.3 using a suitable linear transformation. Recall the definition of $\Pi_m$:

$$\Pi_{k,m} = \{x : x_d > \Phi(\tilde{x}_{k,m}) + \hat{b} \cdot (\tilde{x} - \tilde{x}_{k,m})\}, \quad \hat{b} := \nabla \Phi(\tilde{x}_{k,m}).$$

Now we use (2.6) and (2.7) with $k = (\tilde{x}_{k,m}, \Phi(\tilde{x}_{k,m})), k_1 = 0$, and the non-degenerate transformation defined by

$$Mx = (\tilde{x}, x_d + \hat{b} \cdot \tilde{x}), \quad \text{so that} \quad M^T x = (\hat{x} + \hat{b} x_d, x_d).$$

Due to the unitary equivalence (2.6) and (2.7),

$$\Sigma_\alpha(q_{k,m} b; \Pi_{k,m}, \Omega; g) = \Sigma_\alpha(\tilde{q}_k b_{M,k}; \Pi, \Omega^T_M; g),$$

where $\Pi = \{x : x_d > 0\}$, $\Omega^T_M = M^T \Omega$,

$$b_{M,k}(x, \xi) = b(Mx + k, (M^T)^{-1} \xi),$$

and

$$\tilde{q}_k(\tilde{x}, x_d) = \sigma_{k,0}(\tilde{x}) \zeta_k(\alpha(x_d - \Phi_1(\tilde{x}))).$$

The function $\Phi_1(\tilde{x})$ satisfies the conditions of Lemma 10.3, and in particular, the condition $\nabla \Phi_1(\tilde{0}) = 0$. Furthermore, according to (10.3), for $l \neq d$,

$$(10.17) \quad \Omega^T_M = \{\xi : \xi_l > \Psi_1^\circ(\xi)\} \quad \text{or} \quad \Omega^T_M = \{\xi : \xi_l < \Psi_1^\circ(\xi)\},$$

with

$$\Psi_1^\circ(\xi) = \Psi(\tilde{\xi} - \hat{b} \xi_d, \xi_d) + b_l \xi_d, \quad \tilde{\xi} = (\xi_1, \ldots, \xi_{l-1}, \xi_{l+1}, \ldots, \xi_{d-1}).$$

Note that all partial derivatives of $\Psi_1$ up to order 3 are uniformly bounded by a constant depending only on the parameters $M$ in (4.1) and $M$ in (10.2). If $l = d$, then in view of (10.3),

$$(M^T)^{-1} \Omega^T_M = \{\xi : \xi_d > \Psi(\tilde{\xi})\} \quad \text{or} \quad (M^T)^{-1} \Omega^T_M = \{\xi : \xi_d < \Psi(\tilde{\xi})\}.$$

In view of (10.16), we can use Lemma 14.5 with $B = (M^T)^{-1}$, which ensures that $\Omega^T_M$ is again given by (10.17) with a $C^3(\mathbb{R}^{d-1})$-function $\Psi_1$, whose partial derivatives up to order 3 are uniformly bounded by a constant depending only on the parameters $M$ in (4.1) and $M$ in (10.2).

Thus for both $l < d$ and $l = d$ the domain $\Omega^T_M$ satisfies (10.3) with some function $\Psi_1$. Finally, due to the condition $M^T \leq 1/2$, the new symbol $b_{M}(x, \xi)$ vanishes if $|\xi| \geq 2\rho$.

Thus one can apply Lemma 10.3 with $m = 0$ and with $2\rho$ instead of $\rho$. This leads to the estimate

$$\left|\Sigma_\alpha(\tilde{q}_k b_{M,k}; \Pi, \Omega^T_M; g) - \alpha^{d-1} \log r \mathfrak{A}(g) \mathfrak{W}_1(\sigma_{k,0} b_{M,M}; \partial M, k, \partial \Omega^T_M)\right| \leq r^{(k+1)(d-1)} w_k(r, A) N^{(2,2)}(b_{M}, 1, \rho),$$

where $w_k$ and $N^{(2,2)}$ are defined by

$$w_k(r, A) = \int_{\mathbb{R}^d} \frac{1}{|\xi|} \left|\hat{q}_k(\xi, x_d) \hat{b} \xi_d - \alpha \sigma_{k,0}(\xi) \zeta_k^{-1}(\xi) (\sigma_{k,0}(\xi) \hat{b} \xi_d - \alpha \zeta_k^{-1}(\xi))\right| d\xi, \quad N^{(2,2)}(b_{M}, 1, \rho) = \sup_{|\xi| < \rho} \left|\frac{1}{\xi} \hat{b}_{M}(\xi, x_d)\right|.$$
with some $\mathcal{W}$-sequence $w_k$, and with $\Lambda_{M,k} = M^{-1}(\Lambda - k)$. According to (2.9),

$$\mathcal{W}_1(\sigma_{k,0}b_{M,k}; \partial \Lambda_{M,k}, \partial \Omega_M^T) = \mathcal{W}_1(\sigma_{k,m}b; \partial \Lambda, \partial \Omega).$$

Furthermore,

$$N^{(2,2)}(b_{M,k}; 1, \rho) \leq CN^{(2,2)}(b; 1, \rho),$$

with a universal constant $C$. This leads to the proclaimed bound. \hfill \square

**Corollary 10.5.** Suppose that the conditions (1)-(4) of Lemma 10.3 are satisfied, and that (10.16) holds. Then

$$\left| \mathcal{F}_{\alpha}(g_{k,m}b; \Lambda, \Omega; g) - \alpha^{d-1}\log r \mathcal{A}(g) \mathcal{W}_1(\sigma_{k,m}b; \partial \Lambda, \partial \Omega) \right|$$

(10.18)

$$\leq Cr^{(k+1)(d-1)}w_k(r, A)N^{(d+1,d+2)}(b, 1, \rho),$$

with some $\mathcal{W}$-sequence $w_k(r, A)$, independent of the symbol $b$ and of the point $m \in \mathbb{Z}^{d-1}$, uniformly in $\rho \in [c, C]$ for arbitrary constants $0 < c < C$.

**Proof.** The estimate follows directly from Lemmas 9.4 and 10.4. \hfill \square

The next step is to obtain the appropriate asymptotic formulas for $q^\downarrow$ and $q^\uparrow$ (see (9.5)) instead of $q_{k,m}$.

**Lemma 10.6.** Suppose that $g$ is as in (10.1), that the domains $\Lambda, \Omega$ satisfy Condition 10.1, and that (10.16) holds. Suppose also that $b$ is a symbol satisfying (9.3). Then

$$\lim_{r \to \infty} \limsup_{A \to \infty} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log \alpha} \mathcal{F}_{\alpha}(q^\downarrow b; \Lambda, \Omega; g) = \mathcal{A}(g)\mathcal{W}_1(b; \partial \Lambda, \partial \Omega),$$

(10.19)

uniformly in $\rho \in [c, C]$ with arbitrary constants $0 < c < C$.

**Proof.** Represent $q^\downarrow(x)$ in accordance with (9.11):

$$q^\downarrow(x) = \zeta^{-1}(\alpha(x_d - \Phi(\hat{x}))) + \sum_{k=0}^{K} \sum_{m \in \mathbb{Z}^{d-1}} q_{k,m}(x),$$

(10.20)

and calculate the contribution of each term to the sought trace.

First we consider $\zeta^{-1}$. Since the support of the symbol $b$ in the $x$-variable is the ball $B(0, 1)$, the support of $\zeta^{-1}$ is contained in the set

$$\{x : |x_d - \Phi(\hat{x})| \leq C\alpha^{-1}, |\hat{x}| < 1\}.$$ 

Cover this set with open balls of radius $\alpha^{-1}$, and denote by $\phi_k$, $k = 1, 2, \ldots, N$ the partition of unity associated with this covering such that $|\nabla^l\phi_k| \leq C_l\alpha^l$, for all $l = 1, 2, \ldots$, uniformly in $k$. Clearly,

$$N^{(n_1,n_2)}(\phi_k b; \alpha^{-1}, \rho) \leq CN^{(n_1,n_2)}(b; 1, \rho),$$

with some $\mathcal{W}$-sequence $w_k$. This leads to the proclaimed bound.
for any $n_1, n_2$, uniformly in $k$. Now we can estimate, using (3.19):

$$\|\zeta - 1\|_{s_1} \lesssim \sum_{k=1}^{N} \|O_p(\phi_k b)\|_{s_1} \leq C N^{d+1,d+1}(b; 1, \rho).$$

Clearly, the covering can be chosen in such a way that $N \leq C\alpha^{d-1}$, which implies that

$$\|\zeta - 1\|_{s_1} \lesssim C\alpha^{d-1} N^{d+1,d+1}(b; 1, \rho).$$

Consider now the sum on the right-hand side of (10.20). For each trace $T_{\alpha}(q_k \cdot m b; \Lambda, \Omega; g)$ we use Corollary 10.5, and then sum up the obtained inequalities over $m \in \mathbb{Z}^{d-1}$ and $k \geq 0$. Let us handle the asymptotic coefficient first:

$$Y(\alpha, r, A) := K \sum_{k=0}^{K} \sum_{|m| \leq Cr^{-(k+1)}} \alpha^{d-1} \log r \ A(g) W_1(\sigma_{k,m} b ; \partial \Lambda, \partial \Omega)$$

where we have used that fact that $\sum_{m} \sigma_{k,m} = 1$ for any $k = 0, 1, \ldots$. Since

$$\sum_{k=0}^{K} 1 = \frac{\log \alpha - A}{\log r} + O(1),$$

we have

$$\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} Y(\alpha, r, A) = A(g) W_1(b; \partial \Lambda, \partial \Omega),$$

for any $A \in \mathbb{R}$ and $r > 0$.

Let us consider the remainder. To estimate the sum up the right-hand sides of (10.18) over different values of $k$ and $m$, we observe that the summation over $m$ for each value of $k$, is restricted to $|m| \leq C r^{-(k+1)}$, since the support of the symbol $b$ in the $x$-variable is contained in the unit ball. Thus

$$Z(\alpha, r, A) := \sum_{k=0}^{K} \sum_{|m| \leq C r^{-(k+1)}} w_k(r, A) r^{(k+1)(d-1)} \leq C \alpha^{d-1} \sum_{k=0}^{K} w_k(r, A).$$

By definition of the $W$-sequence (see (9.16)),

$$\lim_{r \to \infty} \limsup_{A \to \infty} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} Z(\alpha, r, A) = 0.$$

Together with (10.21) this leads to (10.19).

Lemma 10.7. Suppose that the conditions of Lemma 10.6 are satisfied. Then

$$\lim_{r \to \infty} \limsup_{A \to \infty} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \zeta_\alpha(g b; \Lambda, \Omega; g) = 0,$$

uniformly in $\rho \in [c, C]$ with arbitrary constants $0 < c < C$. 

\qed
Proof. We use the partition of unity from Proposition 9.1 associated with the slowly-varying function defined in (9.13). Let \( x_j \) be the sequence constructed in Proposition 9.1, and let \( \ell_j = \ell(x_j) \). Denote by \( J \) the set of indices \( j \) such that \( \psi_j q^\dagger \chi_{B(0,1)} \neq 0 \). By (9.7) we have \( x_d - \Phi(\hat{\mathbf{x}}) \geq 2r^K \alpha^{-1} \geq 2r^{-1} e^{-A} \) for all \( \mathbf{x} \) in the support of \( q^\dagger \). Thus \( \ell_j \geq cr^{-1} e^{-A} \) for all indices \( j \in J \), and hence by Lemma 9.2, \( B(x_j, 32 \ell_j) \subset \Lambda \), and in particular, the support of \( \psi_j \) is strictly inside \( \Lambda \). Since \( N^{(m)}(\psi_j; \ell_j) \leq C_m \) for all \( m \), it follows from Theorem 7.8 that

\[
\| \text{Op}_\alpha(\psi_j q^\dagger b) g(T(1; \Lambda, \Omega)) \|_{\mathcal{E}_1} \leq \| \text{Op}_\alpha(\psi_j b) g(T(1; \Lambda, \Omega)) \|_{\mathcal{E}_1} \leq C(\alpha \ell_j \rho)^{d-1} N^{d+2,d+2}(b \psi_j; \ell_j, \rho) \leq C(\alpha \ell_j \rho)^{d-1} N^{d+2,d+2}(b; 1, \rho),
\]

for all \( j \in J \). Consequently,

\[
\| \text{Op}_\alpha(q^\dagger b) g(T(1; \Lambda, \Omega)) \|_{\mathcal{E}_1} \leq C \alpha^{d-1} N^{d+2,d+2}(b; 1, \rho) \sum_{j \in J} \ell_j^{d-1},
\]

where we have estimated \( \rho \leq C \). Using the finite intersection property, we can estimate:

\[
\sum_{j \in J} \ell_j^{d-1} \leq C \int_{cr^{-1}e^{-A} < x_d - \Phi(\hat{\mathbf{x}}) < C} \ell(\mathbf{x})^{-1} d\mathbf{x} \leq C \int_{|\mathbf{x}| < C} \int_{cr^{-1}e^{-A} < t < C} t^{-1} dt d\mathbf{x} \leq C'(A + \log r).
\]

Therefore

\[
\| (\text{Op}_\alpha(q^\dagger b) g(T(1; \Lambda, \Omega)) \|_{\mathcal{E}_1} \leq C(A + \log r) \alpha^{d-1} N^{d+2,d+2}(b; 1, \rho).
\]

Thus the triple limit in (10.22) equals zero, as claimed. \( \square \)

Let us now combine Lemmas 10.6 and 10.7.

Corollary 10.8. Suppose that the conditions of Lemma 10.6 are satisfied. Then

\[
(10.23) \quad \lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \text{tr \text{Op}_\alpha(b) g(T(1; \Lambda, \Omega))} = \mathfrak{A}(g) \mathfrak{M}_1(b; \partial \Lambda, \partial \Omega).
\]

Proof. To avoid cumbersome formulae, throughout the proof we write \( G_1 \approx G_2 \) for any two trace-class operators depending on \( \alpha \), such that

\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \| G_1 - G_2 \|_{\mathcal{E}_1} = 0.
\]

For brevity write \( T := T(1; \Lambda, \Omega) \). By (4.6),

\[
\text{Op}_\alpha^l(b) g(T) \approx \chi_A \text{Op}_\alpha^l(b) g(T).
\]

Rewrite the symbol \( \chi_A b \) in the form

\[
\chi_A b = q^\dagger \chi_A b + q^\dagger \chi_A b.
\]
Again by (4.6),
\[ \chi_{\Lambda} q^\dagger \text{Op}_\alpha^l(b) g(T) \approx q^\dagger \text{Op}_\alpha^l(b) g(T), \]
and \( q^\dagger \chi_{\Lambda} = q^\dagger \) by definition of \( q^\dagger \). Thus
\[ \text{Op}_\alpha^l(b) g(T) \approx q^\dagger \text{Op}_\alpha^l(b) g(T) + q^\dagger \text{Op}_\alpha^l(b) g(T). \]
Now apply (10.19) and (10.22). Since the left-hand side of (10.23) does not depend on \( A \) or \( r \), the claimed result follows. \( \square \)

Next, from the formula (10.23) containing \( T(1; \Lambda, \Omega) \) we deduce a similar asymptotics for the operator \( T(a; \Lambda, \Omega) \).

**Theorem 10.9.** Assume that

1. \( \Lambda, \Omega \) are two graph-type domains satisfying Condition 10.1,
2. (10.16) is satisfied,
3. \( a \in S^{(d+2,d+2)} \),
4. \( b \) is a symbol satisfying (9.3).

Then for any \( p = 1, 2, \ldots \),
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr} \left( \text{Op}_\alpha^l(b) g_p(T(a; \Lambda, \Omega)) \right) - \alpha^d \mathfrak{W}_0(bg_p(a); \Lambda, \Omega) ight.
\]
\[ - \alpha^{d-1} \log \alpha \mathfrak{W}_1(b\mathfrak{A}(gp;a); \partial \Lambda, \partial \Omega) \right) = 0. \tag{10.24} \]

**Proof.** As in the proof of Corollary 10.8 we use the notation \( G_1 \approx G_2 \). For brevity we omit \( \Lambda, \Omega, \partial \Lambda, \partial \Omega \) from notation. Due to Lemma 7.7,
\[ \text{Op}(b) g_p(T(a)) \approx \text{Op}(ba^p) g_p(T(1)). \tag{10.25} \]
Furthermore, by Lemma 7.7 again, with the notation \( g(t) = t^p - t \), we have:
\[ \text{Op}(ba^p) g_p(T(1)) = \text{Op}(ba^p) g(T(1)) + \text{Op}(ba^p) T(1) \approx \text{Op}(ba^p) g(T(1)) + T(ba^p). \]
By Corollary 10.8,
\[ \lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \text{tr} \text{Op}(ba^p) g(T(1)) = \mathfrak{W}_1(b\mathfrak{A}(g;p;a)), \]
where we have taken into account that \( \mathfrak{A}(g_p - g_1; a) = \mathfrak{A}(g;p,a) \). By (4.11),
\[ T(ba^p) \approx \chi_{\Lambda} \text{Op}(ba^p) P_{\Omega, \alpha} \chi_{\Lambda}. \]
The trace of the operator on the right-hand side equals \( \alpha^d \mathfrak{W}_0(bg_p(a); \Lambda, \Omega) \). Thus
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr} \left( \text{Op}_\alpha^l(ba^p) g_p(T(1)) \right) - \alpha^d \mathfrak{W}_0(bg_p(a)) 
\]
\[ - \alpha^{d-1} \log \alpha \mathfrak{W}_1(b\mathfrak{A}(g;p;a)) \right) = 0. \]
The reference to (10.25) completes the proof of (10.24). \( \square \)
11. Proof of Theorem 2.8

From now on we assume that the domains $\Lambda, \Omega$ satisfy the conditions of Theorem 2.8. We begin the proof of Theorem 2.8 with removing conditions 1 and 2 from Theorem 10.9.

Theorem 11.1. Assume that $\Lambda$ and $\Omega$ are as in Theorem 2.8. Suppose also that $a, b \in S^{(d+2,d+2)}$ and the symbol $b$ is compactly supported. Then the asymptotic formula (10.24) holds.

Proof. Our first step is to construct suitable coverings of $\Lambda$ and $\Omega$ by open balls, and associated partitions of unity.

By Definition 2.2, we can cover $\Lambda$ by finitely many open balls $B(w_j, R), j = 1, 2, \ldots$ of some radius $R$ in such a way that if $w_j \in \partial \Lambda$, then $B(w_j, R) \subset \Lambda$. Using Corollary 14.2, and making $R$ smaller if necessary, one can always assume that for each $w_j \in \partial \Lambda$, in the ball $B(w_j, R)$ the domain $\Lambda$ is represented by a $C^1$-graph-type domain $\Gamma(\Phi; O, w)$ with a function, satisfying (2.14), and the bound

\[ M_{\Phi} \leq \frac{1}{8\sqrt{d}}. \]  

Denote by $\phi_j \in C_0^\infty(R^d), j = 1, 2, \ldots$, a partition of unity subordinate to the above covering of $\Lambda$. Using the unitary equivalence (2.10) we may assume that $R = 1$. For each ball $B(w_j, 1)$ further steps depend on the location of the point $w_j$.

Case 1: $w_j \in \partial \Lambda$. Translating $w_j$ to the point zero, and applying an appropriate orthogonal transformation, we may assume that (2.19) holds with $O = I, k = 0$, i.e.

\[ \Lambda \cap B(0, 1) = \{ x : x_d > \Phi(\hat{x}) \} \cap B(0, 1). \]

Now we construct an appropriate partition of unity for the domain $\Omega$. Just as for $\Lambda$ above, by Definition 2.2 one can cover $\Omega$ by finitely many open balls $B(\xi_l, \rho), l = 1, 2, \ldots$ of some radius $\rho > 0$ in such a way that if $\xi_l \notin \partial \Omega$, then $B(\xi_l, \rho) \subset \Omega$. Using Corollary 14.4, and making $\rho$ smaller if necessary, one can always assume that for each $\xi_l \in \partial \Omega$, in the ball $B(\xi_l, \rho)$ the domain $\Omega$ is represented by a graph-type domain described in (14.8) with a $C^3$-function $G$, satisfying the bound $\| \nabla G \|_{L^\infty} \leq 4\sqrt{d}$. Redenoting $G$ by $\Psi$, and recalling (11.1) we conclude that the condition (10.16) is satisfied. Without loss of generality we may assume that $\nabla^2 G$ and $\nabla^3 G$ are uniformly bounded on $\mathbb{R}^{d-1}$. Denote by $\eta_j, j = 1, 2, \ldots$, the partition of unity subordinate to the constructed covering.

Sub-case 1.1: $\xi_l \in \partial \Omega$. Applying an appropriate translation, we may assume that $\xi_l = 0$. Thus by the above construction, in the balls $B(0, 1)$ and $B(0, \rho)$ resp. the domains $\Lambda$ and $\Omega$ resp. are represented by the graph-type domains satisfying Condition 10.1.

The symbol $\tilde{b}(x, \xi) = b(x, \xi)\phi_j(x)\eta_l(\xi)$ is supported on the domain $B(0, 1) \times B(0, \rho)$, so that by Lemma 7.4 we may assume that $\Lambda = \Gamma(\Phi; I, 0)$ and $\Omega$ is as defined by (10.3).
Thus by Theorem 10.9,
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j} \eta_{l}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} \eta_{l} g_{p}(a)) \right)
\]
\[
(11.2)
\]
\[
- \alpha^{d-1} \log \alpha \mathcal{M}_{1}(b\phi_{j} \eta_{l} \mathcal{A}(g_{p}; a)) = 0.
\]

Here and below for brevity we omit \( \Lambda, \Omega, \partial \Lambda, \partial \Omega \) from notation.

Sub-case 1.2: \( \xi_{l} \in \Omega \). Since the domain \( \Omega \) satisfies Condition 7.1 (2), we can use Theorem 7.8, which gives the formula
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j} \eta_{l}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} \eta_{l} g_{p}(a)) \right) = 0.
\]

Summing up over all values of the index \( l \), from (11.2) and (11.3) we get
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j} \eta_{l}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} g_{p}(a)) \right)
\]
\[
(11.4)
\]
\[
- \alpha^{d-1} \log \alpha \mathcal{M}_{1}(b\phi_{j} \mathcal{A}(g_{p}; a)) = 0,
\]
where we have denoted \( \eta = \sum_{l} \eta_{l} \). Since \( 1 - \eta \) is supported outside \( \Omega \), we have
\[
\text{Op}_{\alpha}^{l}(b\phi_{j}(1 - \eta)) g_{p}(T(a)) = -\phi_{j}(\text{Op}_{\alpha}^{l}(\eta), \chi_{\Lambda}) P_{\Omega, \alpha} \text{Op}_{\alpha}^{l}(a) P_{\Omega, \alpha} \chi_{\Lambda} g_{p-1}(T(a)).
\]

Due to (4.7), adding this term to the left-hand side of (11.4), does not change the asymptotics. Thus
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} g_{p}(a)) \right)
\]
\[
(11.5)
\]
\[
- \alpha^{d-1} \log \alpha \mathcal{M}_{1}(b\phi_{j} \mathcal{A}(g_{p}; a)) = 0,
\]

Case 2: \( w_{j} \in \Lambda \). By Definition 2.2 we can cover \( \overline{\Omega} \) by finitely many open balls \( B(\xi_{l}, \rho), l = 1, 2, \ldots \) of some radius \( \rho \) in such a way that if \( \xi_{l} \notin \partial \Omega \), then \( B(\xi_{l}, \rho) \subset \Omega \). Denote by \( \eta_{l}, l = 1, 2, \ldots \), the partition of unity subordinate to this covering. Since \( B(w_{j}, 1) \subset \Lambda \), the domain \( \Lambda \) satisfies Condition 7.1(2), and hence, by Theorem 7.8,
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j} \eta_{l}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} \eta_{l} g_{p}(a)) \right) = 0.
\]

Summing over \( l \), we get
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr}(\text{Op}_{\alpha}^{l}(b\phi_{j} \eta_{l}) g_{p}(T(a))) - \alpha^{d} \mathcal{M}_{0}(b\phi_{j} g_{p}(a)) \right) = 0,
\]
with \( \eta = \sum_i \eta_i \). As in the previous case, we can remove \( \eta \) from this formula, so that

\[
(11.6) \quad \lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr} \left( \text{Op}_\alpha^I (b\phi_j) g_p(T(a)) \right) - \alpha^d \mathcal{M}_0 (b\phi_j g_p(a)) \right) = 0.
\]

End of the proof. Summing up over all values of \( j \), it follows from (11.5) and (11.6) that

\[
(11.7) \quad \lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left( \text{tr} \left( \text{Op}_\alpha^I (b\phi) g_p(T(a)) \right) - \alpha^d \mathcal{M}_0 (g_p(a)) \right)
\]

\[
- \alpha^{d-1} \log \alpha \mathcal{M}_1 (\mathcal{A} (g_p; a)) = 0,
\]

where we have denoted \( \phi = \sum_j \phi_j \). By Corollary 3.13,

\[
\| \text{Op}_\alpha^I (b\phi) - \text{Op}_\alpha^I (b\phi) \|_{\mathcal{E}_1} \leq C \alpha^{d-1}.
\]

Furthermore, \( \phi \chi = \chi \), and hence \( \phi g_p(T(a)) = g_p(T(a)) \). Thus one can remove \( \phi \) from the asymptotics (11.7) altogether. This leads to (10.24), as required. □

Proof of Theorem 2.8. Let us deduce (2.23) from Theorem 11.1. Let \( R > 0 \) be a number such that \( a(x, \xi) = 0 \) for \( |x|^2 + |\xi|^2 \geq R^2 \). Let \( b \in C^\infty_0 (\mathbb{R}^d) \) be a function such that

\[
b(x, \xi) = 1 \quad \text{for} \quad |x|^2 + |\xi|^2 \leq 2R^2 \quad \text{and} \quad b(x, \xi) = 0 \quad \text{for} \quad |x|^2 + |\xi|^2 \geq 4R^2.
\]

Write:

\[
(11.8) \quad \text{tr} \ g_p(T(a)) = \text{tr} \left( \text{Op}_\alpha^I (b) g_p(T(a)) \right) + \text{tr} \left( \text{Op}_\alpha^I (1 - b) g_p(T(a)) \right).
\]

Note that

\[
\text{Op}_\alpha^I (1 - b) T(a) = \text{Op}_\alpha^I (1 - b) \text{Op}_\alpha^I (a) \chi \Lambda P_{\Omega,a} \chi \Lambda - \text{Op}_\alpha^I (1 - b) [\text{Op}_\alpha^I (a), \chi \Lambda P_{\Omega,a}] P_{\Omega,a} \chi \Lambda.
\]

By Lemmas 4.3 and 4.5,

\[
\| [\text{Op}_\alpha^I (a), \chi \Lambda P_{\Omega,a}] \|_{\mathcal{E}_1} \leq C \alpha^{d-1}.
\]

Furthermore, by Corollary 3.13,

\[
\| \text{Op}_\alpha^I (1 - b) \text{Op}_\alpha^I (a) \|_{\mathcal{E}_1} = \| \text{Op}_\alpha^I (1 - b) \text{Op}_\alpha^I (a) - \text{Op}_\alpha^I (a(1 - b)) \|_{\mathcal{E}_1} \leq C \alpha^{d-1}.
\]

Therefore the second term in (11.8) gives a contribution of order \( O(\alpha^{d-1}) \). Applying Theorem 11.1 to the first term in (11.8), we obtain (2.23).

It remains to justify the formula (2.23) for the operator \( S(a) \) with \( a \) replaced by \( \text{Re} \ a \) on the right-hand side. Let us show that

\[
(11.9) \quad \| g_p(T(\text{Re} \ a)) - g_p(\text{Re} \ T(a)) \|_{\mathcal{E}_1} \leq C \alpha^{d-1}.
\]

Rewrite the difference on the left-hand side as

\[
g_p(T(\text{Re} a)) - g_p(\text{Re} T(a)) = \sum_{k=0}^{p-1} g_k(\text{Re} T(a)) (T(\text{Re} a) - \text{Re} T(a)) g_{p-1-k}(T(\text{Re} a)),
\]
so that
\[
\|g_p(T(\text{Re} a)) - g_p(\text{Re} T(a))\|_{\mathcal{E}_1} \\
\leq p(\|\text{Op}(a)\|^p - 1 + \|\text{Op}(\text{Re} a)\|^{p-1}) \|\text{Op}(\text{Re} a) - \text{Re} \text{Op}(a)\|_{\mathcal{E}_1}.
\]

The operator \(\text{Re} \text{Op}(a)\) is nothing but \(\frac{\text{Op}(a) + \text{Op}^r(\pi)}{2}\), and hence, by (3.22),
\[
\|\text{Op}(\text{Re} a) - \text{Re} \text{Op}(a)\|_{\mathcal{E}_1} = \frac{1}{2} \|\text{Op}(\pi) - \text{Op}^r(\pi)\|_{\mathcal{E}_1} \leq C \alpha^{d-1}.
\]

Using Lemma 3.9, we obtain (11.9) from (11.10). Using (2.23) for \(T(\text{Re} a)\), we now obtain (2.23) for the operator \(S(a)\).

The proof of Theorem 2.8 is complete. \(\square\)

12. Closing the asymptotics: proof of Theorems 2.3 and 2.4

The crucial point of the proof of Theorems 2.3 and 2.4 is the sharp estimates (12.11) and (12.17). These estimates were essentially derived in [11], Chapter 3, see also [12]. We obtain them for more general symbols, which require some additional functional calculus considerations. For the sake of completeness we provide proofs even for the results borrowed from [11], [12].

12.1. Non-self-adjoint case. The inequality (12.5) in the next lemma was proved in [12].

**Lemma 12.1.** Let \(A\) be a trace class operator in a separable Hilbert space \(\mathcal{H}\), and let \(P\) be an orthogonal projection in \(\mathcal{H}\). Suppose that
\[
g(z) = \sum_{m=1}^{\infty} \omega_m z^m,
\]
is a function analytic in a disk of radius \(R > \|A\|\). Denote
\[
g^{(1)}(t) = \sum_{m=2}^{\infty} (m-1) |\omega_m| t^{m-1},
\]
\[
g^{(2)}(t) = \sum_{m=2}^{\infty} m(m-1) |\omega_m| t^{m-2}.
\]

Then
\[
\|g(PAP) - Pg(A)P\|_{\mathcal{E}_1} \leq g^{(1)}(\|A\|) \|PA(I - P)\|_{\mathcal{E}_1},
\]
and
\[
\|g(PAP) - Pg(A)P\|_{\mathcal{E}_1} \leq \frac{g^{(2)}(\|A\|)}{2} \|PA(I - P)\|_{\mathcal{E}_2} \|(I - P)AP\|_{\mathcal{E}_2}.
\]
Proof. Denote $Q = I - P$. Then for $m \geq 2$ we write

$$PA^m P = PA(P + Q)A^{m-1}P = PA(PA)A^{m-1}P + (PA)QA^{m-1}P = P(AP)A(P + Q)A^{m-2}P + (PA)QA^{m-1}P = P(AP)^2 A^{m-2}P + (PA)^2 QA^{m-2}P + (PA)QA^{m-1}P = P(2^{m-1}) + \sum_{j=1}^{m-1} (PA)^{m-j}QA^j P = (PA)^m + \sum_{j=1}^{m-1} (PA)^{m-j}QA^j P.$$  

(12.6)

Estimate the term under the sum:

$$\| (PA)^{m-j}QA^j P \|_{\mathcal{S}_1} \leq \| (PA)^{m-1-j} \| \| PAQ \|_{\mathcal{S}_1} \| A^j P \| \leq \| A \|^{m-1} \| PAQ \|_{\mathcal{S}_1},$$

so that

$$\| PA^m P - (PA)^m \|_{\mathcal{S}_1} \leq (m - 1) \| A \|^{m-1} \| PAQ \|_{\mathcal{S}_1}.$$ 

Now (12.4) follows.

On the other hand, for $j \geq 2$,

$$A^j P = A^{j-1}(P + Q)AP = A^{j-1}PAP + A^{j-1}QAP = A^{j-2}(P + Q)APAP + A^{j-1}QAP = A^{j-2}(PA)^2P + A^{j-2}Q(AP)^2 + A^{j-1}QAP = A^{j-1}P + \sum_{k=1}^{j-1} A^k Q(AP)^{j-k} P.$$  

(12.7)

Substituting (12.7) in (12.6) we obtain

$$PA^m P - (PA)^m = \sum_{j=1}^{m-1} (PA)^{m-j}QA^j P = \sum_{j=1}^{m-1} (PA)^{m-j}QA(PA)^{j-1}P + \sum_{j=2}^{m-1} (PA)^{m-j}Q \sum_{k=1}^{j-1} A^k Q(AP)^{j-k} P.$$
This entails the bound
\[
\| PA^m P - (PAP)^m \|_{\mathfrak{e}_1} \leq (m - 1)\| A \|^{m-2} \| PAP \|_{\mathfrak{e}_1} \\
+ \sum_{j=2}^{m-1} \| PA \|^{m-j-1} \| PAP \|_{\mathfrak{e}_2} \sum_{k=1}^{j-1} \| A \|^k \| QAP \|_{\mathfrak{e}_2} \| AP \|^{j-k-1} \\
\leq (m - 1)\| A \|^{m-2} \| PAP \|_{\mathfrak{e}_1} + \| A \|^{m-2} \| PAP \|_{\mathfrak{e}_2} \| QAP \|_{\mathfrak{e}_2} \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} 1^j
\leq \frac{m(m - 1)}{2} \| A \|^{m-2} \| PAP \|_{\mathfrak{e}_2} \| QAP \|_{\mathfrak{e}_2}.
\]
This leads to (12.5). \qed

Thus, if \( P, P_1 \) are two orthogonal projections, then
\[
\| g(PP_1AP_1P) - PP_1g(A)P_1P \|_{\mathfrak{e}_1} \\
\leq \| g(PP_1AP_1P) - P_1g(A)P_1P \|_{\mathfrak{e}_1} + \| g(PP_1AP_1P) - P_1g(A)P_1P \|_{\mathfrak{e}_1} \\
\leq \frac{g(2(\|A\|))}{2} \| PP_1AP_1(I - P) \|_{\mathfrak{e}_2} \| (I - P)P_1AP_1P \|_{\mathfrak{e}_2} \\
+ g(1(\|A\|)) \| P_1A(I - P_1) \|_{\mathfrak{e}_1}.
\]
Now we apply the above estimates to pseudo-differential operators. Let \( A = \text{Op}_\alpha^l(a) \) with a compactly supported symbol \( a \), and let \( D \) be the constant introduced in Lemma 3.14. In particular, by Lemma 3.9,
\[
\| \text{Op}_\alpha^l(a) \| \leq DN^{(d+2,d+2)}(a; \ell, \rho) =: t_0, \quad \text{and} \quad \| T(a) \| \leq t_0,
\]
under the condition \( \alpha \ell \rho \geq 1 \).

**Lemma 12.2.** Suppose that \( a \in S^{(d+2,d+2)} \) is a symbol such that
\[
\text{supp } a \subset B(z, \ell) \times B(\mu, \rho),
\]
and that \( \alpha \ell \rho \geq 1 \). Let \( g \) be given by the series (12.1) with the radius of convergence \( R > t_0 \). Denote
\[
\tilde{g}(t) = \sum_{m=2}^{\infty} (m - 1)^2(\omega_m)^{(d+2)}t^{m-1}.
\]
Then
\[
\| g(A) - \text{Op}_\alpha^l(g(a)) \|_{\mathfrak{e}_1} \leq (\alpha \ell \rho)^{d-1}N^{(d+2,d+2)}(a; \ell, \rho)\tilde{g}(t_0).
\]

**Proof.** By (3.25),
\[
\| (\text{Op}_\alpha^l(a))^m - \text{Op}_\alpha^l(a^m) \|_{\mathfrak{e}_1} \leq (m - 1)^{2(d+2)}(\alpha \ell \rho)^{d-1}N^{(d+2,d+2)}(a; \ell, \rho)t_0^{m-1},
\]
which leads to (12.10). \qed
In the next lemma we still need to remember the dependence of the parameter \( t_0 \) on
the symbol \( a \), but we do not specify the dependence of the constants in the estimates
on the symbol \( a \) or the parameters \( \ell, \rho \).

**Lemma 12.3.** Let the symbol \( a \) be as in Lemma 12.2. Let \( g \) be given by the series (12.1)
with the radius of convergence \( R > t_0 \). Then

\[
\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \operatorname{tr} g(T(a)) - \mathfrak{M}_0(g(a); \Lambda, \Omega) \right| \leq C g^{(2)}(t_0).
\]

**Proof.** Using an appropriate partition of unity, Lemma 7.2 and estimates (4.6), (4.11),
we can easily derive from the estimates (4.11) and (6.1) that

\[
\| P_{\Omega, \alpha} \mathcal{O}_\alpha(a)(I - P_{\Omega, \alpha}) \| \lesssim \alpha^{d-1},
\]

\[
\| \chi \Lambda P_{\Omega, \alpha} \mathcal{O}_\alpha(a)P_{\Omega, \alpha}(1 - \chi \Lambda) \| \lesssim \alpha^{d-1} \log \alpha,
\]

where \( \mathcal{O}_\alpha(a) \) denotes any of the operators \( \mathcal{O}_\alpha' \), \( \mathcal{O}_\alpha^* \). Moreover, by (12.9), \( \| \mathcal{O}_\alpha'(a) \| \leq t_0 \). Therefore it follows from (12.8) that

\[
\| g(T(a)) - \chi \Lambda P_{\Omega, \alpha}g(\mathcal{O}_\alpha'(a)) P_{\Omega, \alpha} \chi \Lambda \| \lesssim \alpha^{d-1} \log \alpha \left[ \log \alpha \ g^{(2)}(t_0) + g^{(1)}(t_0) \right].
\]

Together with (12.10) this gives

\[
\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \| g(T(a)) - T(g(a)) \| \lesssim C g^{(2)}(t_0).
\]

In order to find the trace of \( T(g(a)) \) we note that in view of (12.12),

\[
\| (I - P_{\Omega, \alpha}) \mathcal{O}_\alpha'(g(a)) P_{\Omega, \alpha} \| \lesssim \alpha^{d-1},
\]

so that

\[
\operatorname{tr} T(g(a)) = \operatorname{tr} \chi \Lambda P_{\Omega, \alpha}g(\mathcal{O}_\alpha'(a)) P_{\Omega, \alpha} \chi \Lambda + O(\alpha^{d-1}).
\]

Writing out the kernel of the operator on the right-hand side and integrating, we find
that its trace equals \( \alpha^d \mathfrak{M}_0(g(a); \Lambda, \Omega) \). The proof is complete. \( \square \)

12.2. **Proof of Theorem 2.3.** Represent the function \( g \) in the form

\[
g(z) = g_p(z) + r_p(z), \quad g_p(z) = \sum_{m=1}^{p} \omega_m z^m, \quad r_p(z) = \sum_{m=p+1}^{\infty} \omega_m z^m,
\]

so that

\[
\operatorname{tr} g(T(a)) = \operatorname{tr} g_p(T(a)) + \operatorname{tr} r_p(T(a)).
\]

For any \( \varepsilon > 0 \) one can find a number \( p \) such that

\[
r_p^{(1)}(t_0) + r_p^{(2)}(t_0) < \varepsilon,
\]

see definitions (12.2) and (12.3). By (2.24),

\[
| \mathfrak{A}(r_p; b) | \leq |b| r_p^{(1)}(|b|),
\]
for any number $b \in \mathbb{C}$, $|b| < R$. Thus by definition of $\mathfrak{W}_1$ and $\mathfrak{A}$,

$$|\mathfrak{W}_1(\mathfrak{A}(r_p, a); \partial \Lambda, \partial \Omega)| \leq C r_p^{(1)}(t_0) \leq C \varepsilon,$$

so that according to Lemma 12.3,

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} r_p(T(a)) - \alpha^d \mathfrak{W}_0(r_p(a); \Lambda, \Omega) \right| - \alpha^{d-1} \log \alpha \mathfrak{W}_1(\mathfrak{A}(r_p, a); \partial \Lambda, \partial \Omega) \leq C(1 + r_p^{(2)}(t_0)) < C \varepsilon.$$

Adding this formula and (2.23) for the trace $\text{tr} g_p(T(a))$, we obtain

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} g(T(a)) - \alpha^d \mathfrak{W}_0(g(a); \Lambda, \Omega) \right| - \alpha^{d-1} \log \alpha \mathfrak{W}_1(\mathfrak{A}(g; a); \partial \Lambda, \partial \Omega) \leq C \varepsilon.$$

Since the parameter $\varepsilon$ is arbitrary, this proves (2.21). □

12.3. Self-adjoint case. A central role is played by the following abstract result, established in [21] (see also [22]), which we have slightly rephrased.

**Proposition 12.4.** Let $A$ be a self-adjoint bounded operator in a separable Hilbert space $H$, and let $P$ be an orthogonal projection in $H$ such that $PA \in S^2$. Then for any function $\psi \in C^2(I)$, $I = [-\|A\|, \|A\|]$, such that $\psi(0) = 0$, the operators $P\psi(A)P$ and $\psi(PA)$ are trace class, and

$$\left| \text{tr}(P\psi(A)P - \psi(PA)) \right| \leq \frac{1}{2} \|\psi''\|_{L^\infty(I)} \|PA(1 - P)\|_{S^1}^2.$$

We also need a less elegant, but still useful estimate. The conditions on the operators in the next lemma are certainly not optimal, but they suffice for our purposes.

**Lemma 12.5.** Suppose that $A$ is a self-adjoint trace class operator in a Hilbert space $H$, and let $P$ be an orthogonal projection in $H$. Then for any function $\psi$ such that $t\psi \in L^1(\mathbb{R})$, we have

$$\|P\psi(A)P - P\psi(PA)P\|_{S^1} \leq \frac{1}{\sqrt{2\pi}} \|PA(I - P)\|_{S^1} \int |t| |\dot{\psi}(t)|dt. \quad (12.13)$$

**Proof.** For an arbitrary self-adjoint operator $B$ denote

$$U(t) = U(t; B) = e^{iBt}, t \in \mathbb{R},$$

so that

$$i\partial_t U(t) + BU(t) = 0, \quad U(0) = I.$$

Compare the operators

$$W_1(t) = PU(t; PA)P \quad \text{and} \quad W_2(t) = PU(t; A)P.$$
It is clear, that
\[ i\partial_t W_1(t) + PAPW_1(t) = 0, \]
\[ i\partial_t W_2(t) + PAPW_2(t) + PA(I - P)U(t; A)P = 0. \]
Therefore \( W(t) := W_2(t) - W_1(t) \) satisfies the equation
\[ i\partial_t W(t) + PAPW(t) = -PA(I - P)U(t; A)P. \]
Thus
\[ W(t) = i\int_0^t U(t - s; PAP)PA(I - P)U(s; A)Pds, \]
and hence
\[ \| P(U(t, A) - U(t; PAP))P \|_{\mathcal{E}_1} \leq |t|\| PA(I - P) \|_{\mathcal{E}_1}, \quad t \in \mathbb{R}. \]
For any self-adjoint \( B \) we have
\[ \psi(B) = \frac{1}{\sqrt{2\pi}} \int U(t; B)\hat{\psi}(t)dt, \]
where \( \hat{\psi} \) is the Fourier transform of \( \psi \). Thus in view of (12.14),
\[ \| P\psi(A)P - P\psi(PAP)P \|_{\mathcal{E}_1} \leq \frac{1}{\sqrt{2\pi}}\| PA(I - P) \|_{\mathcal{E}_1} \int |t| |\hat{\psi}(t)|dt, \]
which is the required bound. \( \square \)

Below we recall an elementary result of functional calculus for pseudo-differential operators. For a function \( \psi : \mathbb{R} \to \mathbb{C} \) we denote
\[ a_\psi(x, y, \xi) = \psi\left( \frac{a(x, \xi) + a(y, \xi)}{2} \right), \quad A_\psi = \text{Op}_a^\alpha(a_\psi), \]
so that
\[ a_1(x, y, \xi) = \frac{a(x, \xi) + a(y, \xi)}{2}, \quad A_1 = \text{Re Op}_a^\alpha(a). \]

**Lemma 12.6.** Suppose that \( a \in S^{\alpha, 2, \alpha + 2} \) be a symbol such that
\[ \text{supp } a \subset B(\mathbf{z}, \ell) \times B(\mathbf{\mu}, \rho). \]
Let \( \psi \in C_0^\infty(\mathbb{R}) \). Then
\[ \| \psi(A_1) - A_\psi \|_{\mathcal{E}_1} \leq C(\alpha \ell \rho)^{d-1}\left( N^{\alpha, 2, \alpha + 2}(a; \ell, \rho) \right)^{2d+3}, \]
with a constant depending only on the function \( \psi \).
Proof. Let
\[ w_t = w_t(x, y, \xi) = e^{itA_1(x, y, \xi)}, \quad W(t) = \text{Op}_{a}(w_t), \]
and \( U(t) = U(t; A_1) = e^{itA_1} \). Let us show that
\begin{equation}
\| U(t) - W(t) \|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} t^{2d+5} \left( N^{(d+2, d+2)}(a; \ell, \rho) \right)^{2d+3}.
\end{equation}

The operator \( W(t) \) satisfies the equation
\[ i\partial_t W(t) + \text{Op}_{a}(a_1 w_t) = 0, \]
so that
\[ i\partial_t W(t) + \text{Op}_{a}(a_1 W(t)) = M(t), \quad M(t) = \text{Op}_{a}(a_1 W(t) - \text{Op}_{a}(a_1 w_t)). \]
Since \( A_1 = \text{Op}_{a}(a_1) \), the difference \( E(t) := W(t) - U(t) \) satisfies the equation
\[ i\partial_t E(t) + A_1 E(t) = M(t), \quad E(0) = 0. \]
Thus
\[ E(t) = -i \int_0^t U(t - s; A_1) M(t) ds. \]
By Lemma 3.12 and Corollary 3.13,
\[ \| M(t) \|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} t^{2d+4} \left( N^{(d+2, d+2)}(a; \ell, \rho) \right)^{2d+3}, \]
so that (12.16) holds.

Since
\[ \psi(A_1) = \frac{1}{\sqrt{2\pi}} \int U(t; A_1) \hat{\psi}(t) dt, \]
it follows from (12.16) that
\[ \left\| \psi(A_1) - \frac{1}{\sqrt{2\pi}} \int W(t) \hat{\psi}(t) dt \right\|_{\mathcal{S}_1} \leq C(\alpha \ell \rho)^{d-1} \int |t|^{2d+5} |\hat{\psi}(t)| dt \left( N^{(d+2, d+2)}(a; \ell, \rho) \right)^{2d+3}. \]
By definition of \( w_t \),
\[ \frac{1}{\sqrt{2\pi}} \int W(t) \hat{\psi}(t) dt = \text{Op}_{a}(a_\psi). \]
This completes the proof. \(\square\)

Let us apply the estimates in Proposition 12.4, and Lemmas 12.5 and 12.6 to the operator \( S(a) = \chi_{\Lambda} P_{1, \alpha} \text{Re Op}_{a}^{l}(a) P_{1, \alpha} \chi_{\Lambda} \). By Lemma 3.9 the operator \( A_1 = \text{Re Op}_{a}^{l}(a) \) is bounded by some constant \( t_0 > 0 \) uniformly in \( \alpha \geq 1 \), so that \( \| S(a) \| \leq t_0 \) for all \( \alpha \geq 1 \) as well. In the lemma below the dependence on the symbol \( a \) is included in the constants.
Lemma 12.7. Let $\Lambda$ and $\Omega$ be bounded $C^1$-domains, and let $a = a(x, \xi)$ be a compactly supported symbol from $S^{(d+2,d+2)}$. If $\psi \in C^\infty$ and $\psi(0) = 0$, then

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} \psi \left( S(a) \right) - \alpha^d \mathcal{M}_0(\psi(\text{Re } a); \Lambda, \Omega) \right| \leq C \| \psi'' \|_{L^\infty(-t_0,t_0)}.$$

Proof. As in the previous lemma, we use the notation $A_1 = \text{Re Op}_{\alpha}^l(a)$. Using an appropriate partition of unity, Lemma 7.2 and estimates (4.11), we can easily derive from the estimates (4.6), (4.11), we can easily derive

$$\| P_{\Omega,\alpha} A_1 (I - P_{\Omega,\alpha}) \|_{\mathcal{S}} \leq C \alpha^{d-1},$$

$$\| \chi P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} (1 - \chi) \|_{L^2}^2 \leq C \alpha^{d-1} \log \alpha.$$

Remembering that $\| S(a) \| \leq t_0$, together with Proposition 12.4 the second estimate gives:

$$\left| \text{tr} \psi \left( \chi P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \chi \right) - \text{tr} \left( \chi \Lambda \psi \left( P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \right) \chi \right) \right| \leq C \| \psi'' \|_{L^\infty(-t_0,t_0)} \alpha^{d-1} \log \alpha.$$

Now, according to Lemma 12.6 and Lemma 12.5 with $A = A_1 = \text{Re Op}_{\alpha}^l(a)$, $P = P_{\Omega,\alpha}$, we have

$$\| \chi \Lambda \psi \left( P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \right) \chi \chi \Lambda P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \chi \|_{\mathcal{S}} \leq C \alpha^{d-1}, A\psi = \text{Op}_{\alpha}^a(a\psi).$$

The last two estimates together imply that

$$\left| \text{tr} \psi \left( \chi \Lambda P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \chi \right) - \text{tr} \chi \Lambda P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \chi \right| \leq C \| \psi'' \|_{L^\infty(-t_0,t_0)} \alpha^{d-1} \log \alpha + C \alpha^{d-1}.$$

Arguing as in the proof of Lemma 12.3, we find that the trace of the operator

$$\chi \Lambda P_{\Omega,\alpha} A_1 P_{\Omega,\alpha} \chi \chi \Lambda,$$

equals $\alpha^d \mathcal{M}_0(\psi(\text{Re } a); \Lambda, \Omega) + O(\alpha^{d-1})$. Now (12.17) follows. \qed

12.4. Proof of Theorem 2.4. Represent the function $g$ in the form

$$g(t) = \beta t + \psi(t), \beta = g'(0),$$

with a function $\psi \in C^\infty$ such that $\psi(0) = \psi'(0) = 0$. Therefore

$$g(t) = \beta t + \int_0^t \int_0^\tau \psi''(s) ds d\tau.$$

For any $\varepsilon > 0$ one can find a polynomial $z = z(t)$ of a sufficiently large degree $p - 2$ such that

$$\max_{|t| \leq t_0} |z(t) - \psi''(t)| \leq \varepsilon.$$

Therefore,

$$g(t) = g_p(t) + \phi(t),$$

where

$$g_p(t) = \beta t + \int_0^t \int_0^\tau z(s) ds d\tau.$$
is a polynomial of degree $p$, and
\[
\phi(t) = \int_0^t \int_0^\tau (\psi''(s) - z(s)) ds d\tau.
\]
Thus
\[
\text{tr} g(S(a)) = \text{tr} g_p(S(a)) + \text{tr} \phi(S(a)).
\]
By construction, $\phi(0) = \phi'(0) = 0$ and
\[
\max_{|t| \leq t_0} |\phi''(t)| < \varepsilon, \quad \max_{|t| \leq t_0} |\phi'(t)| < \varepsilon t_0, \quad \max_{|t| \leq t_0} |\phi(t)| < \varepsilon t_0^2.
\]
By \(2.25\),
\[
|\mathfrak{A}(\phi; b)| \leq 4|b| \max_{|t| \leq |b|} |\phi'(t)|,
\]
for any number $b \in \mathbb{R}$. Thus by definition of $\mathfrak{W}_1$ and $\mathfrak{A}$,
\[
|\mathfrak{W}_1(\mathfrak{A}(\phi, \text{Re } a); \partial \Lambda, \partial \Omega)| \leq C \varepsilon,
\]
so that according to Lemma 12.7,
\[
\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} \phi(S(a)) - \alpha^d \mathfrak{W}_0(\phi(\text{Re } a); \Lambda, \Omega) \right|
\]
\[
- \alpha^{d-1} \log \alpha \mathfrak{W}_1(\mathfrak{A}(\phi; \text{Re } a); \partial \Lambda, \partial \Omega) \leq C \varepsilon.
\]
Adding this formula and \(2.23\) for the trace $\text{tr} g_p(S(a))$, we obtain
\[
\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} \left| \text{tr} g(S(a)) - \alpha^d \mathfrak{W}_0(g(\text{Re } a); \Lambda, \Omega) \right|
\]
\[
- \alpha^{d-1} \log \alpha \mathfrak{W}_1(\mathfrak{A}(g; \text{Re } a); \partial \Lambda, \partial \Omega) \leq C \varepsilon.
\]
Since the parameter $\varepsilon$ is arbitrary, this proves \(2.22\). □

13. Appendix 1: A Lemma by H. Widom

In this Appendix we establish a variant of Lemma A from H. Widom’s paper [40].

13.1. Function $m_d$. In this section we use the notation \((8.13)\) and \((8.14)\) introduced in Subsect. 8.3 for an arbitrary finite set $X \subset (-2\rho, 2\rho)$, where $\rho > 0$ is a fixed number. Let $C^n_\rho$ be the $n$-dimensional cube defined in \((1.11)\). Let $\Phi \in C^3(\overline{C^n_\rho})$ be a real-valued function. For each $\hat{x} \in \overline{C^n_\rho}$ and each $l = 1, 2, \ldots, d$ define the sets $X_l, X_l^{(+)}$, $X_l^{(-)}$ in the following way. First, we set
\[
(13.1) \quad X_l(\hat{x}) = X_l(\hat{x}; \Phi) = \{x_d \in (-2\rho, 2\rho) : \Phi(\hat{x}) = x_l\}.
\]
Using the Area Formula (see [10], Theorem 1, Section 3.3.2), one can show that for almost all $\hat{x}$ the set $X_l(\hat{x})$ is finite. Thus the function $m_d(X_l(\hat{x}))$ (see \((8.14)\)) is well-defined a.a. $\hat{x}$. Denote
\[
\Lambda_l^{(\pm)}(\hat{x}) = \{x_d \in (-2\rho, 2\rho) : \pm x_l > \pm \Phi(\hat{x})\}.
\]
For each $\hat{x} \in \mathbb{R}^{d-1}$ this set is either empty or it is a countable union of open intervals. Denote by $X_l^{(\pm)}(\hat{x}) = X_l^{(\pm)}(\hat{x}; \Phi)$ the set of all endpoints of these intervals which are strictly inside $(-2\rho, 2\rho)$. Clearly, $X_l^{(\pm)}(\hat{x}) \subset X_l(\hat{x})$.

**Theorem 13.1.** Let $\Phi \in C^3([-2\rho, 2\rho])$ and $\rho > 0$. Then for any $l = 1, 2, \ldots, d$ the following statements hold:

1. For almost all $\hat{x} \in \mathbb{R}^{d-1}$:
   a. The set $X_l(\hat{x})$ is finite,
   b. $X_l(\hat{x}) = X_l^{(\pm)}(\hat{x})$,
   c. The disjoint open intervals forming $\Lambda_l^{(\pm)}(\hat{x})$ have distinct endpoints.

2. For any $\delta \in (0, 2)$ the function $\mu_\delta(X_l(\hat{x}))$ satisfies the bound
   \[
   \int_{\hat{x} \in \mathbb{R}^{d-1}} \mu_\delta(X_l(\hat{x})) d\hat{x} \leq C\rho^{d-1-\delta} (1 + \rho_\rho \|\nabla^2 \Phi\|_\infty + \rho_2 \|\nabla^3 \Phi\|_\infty).
   \]
   The constant $C$ depends only on $\delta$ and dimension $d$.

**Remark 13.2.** As indicated above, we can immediately see that the set $X_l^{(\pm)}(\hat{x})$ is finite a.e. $\hat{x}$. Indeed, if $l = d$, then obviously this set consists of one point. Suppose that $l < d$. Denote by $\Xi : \mathbb{R}^d_{d-1} \rightarrow \mathbb{R}^d_{d-1}$ the mapping
\[
\Xi(\hat{x}) = (x_1, \ldots, x_l-1, \Phi(\hat{x}), x_{l+1}, \ldots, x_{d-1}).
\]
Then by the area formula (see, e.g. Theorem 1, p.96 in [10]),
\[
\int_{\mathbb{R}^d_{d-1}} J_\Xi(\hat{x}) d\hat{x} = \int \sum_{\hat{x} \in \mathbb{R}^d_{d-1}} 1 d\hat{x},
\]
where $J_\Xi$ is the Jacobian of the map $\Xi$, i.e. $|\partial_\hat{x} \Phi|$. The integral on the left-hand side is finite, and hence the integrand on the right-hand side is finite a.e. $\hat{x}$, which implies that $X_l(\hat{x})$ is finite a.e. $\hat{x}$.

In the remaining proof of Theorem 13.1 we rely on H. Widom’s paper [40], where the above theorem was proved for $\delta = 1$. We begin with the study of the 2-dimensional case.

13.2. **Special case** $d = 2$. Let $\phi \in C^3([-2\rho, 2\rho])$. The definition of $X_l(s; \phi)$ takes the form
\[
X_1(s) = X_1(s; \phi) = \{t \in (-2\rho, 2\rho) : \phi(t) = s\};
\]
\[
X_2(s) = X_2(s; \phi) = \{t \in (-2\rho, 2\rho) : \phi(s) = t\}.
\]
The lemma below is a variant of Sublemma 1 from [40].

**Lemma 13.3.** Let $\phi$ be as defined above. Then for $l = 1, 2$ the following statements hold:

1. For almost all $s \in (-2\rho, 2\rho)$:
Proof. For \((\star)\) equality follows immediately. Let \(m = \Lambda\) be the set of non-critical points of \(\phi\), i.e. if \(\phi(x) = s\), then \(\phi'(x) \neq 0\). For any such \(s\) the sets \(X_1^+(s), X_1^-(s)\) trivially coincide with \(X_1(s)\). Also, if the set \(\Lambda_1^\pm(s)\) is not empty, then the constituent open intervals do not have common endpoints. By Sard’s Theorem the set of non-critical points is a set of full measure, so that in combination with Remark 13.2 this proves Part (1) of the Lemma.

**Proof of Part (2).** Decompose the open set

\[
E = \{ x \in (-2\rho, 2\rho) : \phi'(x) \neq 0 \},
\]

into the union of disjoint open intervals. Let \(s \in (-2\rho, 2\rho)\) be a non-critical point of \(\phi\), and denote for brevity \(m(s) = m_\delta(X_1(s))\). Denote by \(L_\pm(s)\) the subset of those open intervals forming \(\Lambda_1^\pm(s)\) which do not have \(-2\rho\) or \(2\rho\) as their endpoints, and denote

\[
m^\pm(s) = \begin{cases} (2\rho)^{-\delta}, & \text{if } L_\pm(s) = \emptyset; \\ \sum_{J \in L_\pm(s)} |J|^{-\delta}, & \text{otherwise.} \end{cases}
\]

Then it is clear that

\[
m(s) \leq m^+(s) + m^-(s).
\]

Estimate separately \(m^+(s)\) and \(m^-(s)\).

Consider an interval \(J \in L_+(s)\). Since \(s\) is a non-critical value of \(\phi\), the left endpoint of \(J\) falls in some interval \(K \subset E\), on which \(\phi' < 0\), and the right endpoint falls in some interval \(I \subset E\), on which \(\phi' > 0\). Thus \(J = (x_K(s), x_I(s))\), where \(x_K(s)\) and \(x_I(s)\) are unique solutions of the equation \(\phi(x) = s\) on the intervals \(K\) and \(I\) respectively. Writing

\[
I = (\beta_I^-, \beta_I^+),
\]

we immediately conclude that

\[
|J| \geq x_I(s) - \beta_I^-.
\]
and hence

\[(13.4) \quad m^{(+)}(s) \leq \sum_I (x_I(s) - \beta_I^{(-)})^{-\delta},\]

where the summation is taken over all intervals \(I \subset E\) such that \(\phi'(x) > 0, x \in I,\) and \(s \in \phi(I)\). Remembering the set of critical points of \(\phi\) has measure zero, we can use (13.4) to estimate:

\[
\int_{-2\rho}^{2\rho} m^{(+)}(s) ds \leq C \rho^{1-\delta} + \sum_I \int_{\phi(I)} (x_I(s) - \beta_I^{(-)})^{-\delta} ds,
\]

where the summation is taken over all intervals \(I \subset E\) on which \(\phi'(x) > 0\). Estimate the integral for each \(I\) individually, denoting for brevity \(\beta^{(\pm)} = \beta^{(\pm)}_I\).

**Case 1:** \(0 < \delta < 1\). Write:

\[
\int_{\phi(I)} (x_I(s) - \beta^{(-)})^{-\delta} ds = \int_{\beta^{(-)}}^{\beta^{(+)}} \phi'(x)(x - \beta^{(-)})^{-\delta} dx.
\]

Since \(\phi'(\beta^{(-)}) = 0\), we have

\[
|\phi'(x)| \leq \|\phi''\|_{L^\infty} |I|, \quad x \in I,
\]

and hence the integral is bounded from above by

\[
\|\phi''\|_{L^\infty} |I|^{2-\delta} \int_0^1 t^{-\delta} dt \leq C \|\phi''\|_{L^\infty} |I|^{2-\delta}.
\]

Therefore

\[
\int_{-2\rho}^{2\rho} m^{(+)}(s) ds \leq C \rho^{1-\delta} + C \|\phi''\|_{L^\infty} \sum_{I \subset E} |I|^{2-\delta}
\]

\[(13.5) \quad \leq C \rho^{1-\delta}(1 + \rho \|\phi''\|_{L^\infty}),\]

where we have used that \(\sum_{I \subset E} |I| \leq 4\rho\).

**Case 2:** \(\delta = 1\). Write:

\[
\int_{\phi(I)} (x_I(s) - \beta^{(-)})^{-1} ds = \int_{\beta^{(-)}}^{\beta^{(+)}} \phi'(x)(x - \beta^{(-)})^{-1} dx = \int_{\beta^{(-)}}^{\beta^{(+)}} \phi''(x) \log \frac{\beta^{(+)}}{x} - \beta^{(-)} dx,
\]

where we have used the fact that \(\phi'(\beta^{(-)}) = 0\). The last integral does not exceed

\[
\|\phi''\|_{L^\infty} (\beta^{(+)}) - \beta^{(-)}) \int_1^1 \log \frac{1}{t} dt \leq C \|\phi''\|_{L^\infty} |I|,
\]

so that

\[(13.6) \quad \int_{-2\rho}^{2\rho} m^{(+)}(s) ds \leq C + C \|\phi''\|_{L^\infty} \sum_{I \subset E} |I| \leq C(1 + \rho \|\phi''\|_{L^\infty}).\]
Case 3: $1 < \delta < 2$. Write:

$$\int_{\phi(I)} (x_I(s) - \beta(-))^{-\delta} ds = \int_{\beta(-)}^{\beta(+)} \phi'(x)(x - \beta(-))^{-\delta} dx$$

(13.7) \[ = \frac{1}{\delta - 1} \int_{\beta(-)}^{\beta(+)} \phi''(x) [(x - \beta(-))^{1-\delta} - (\beta(+)-\beta(-))^{1-\delta}] dx. \]

Here we have used again the fact that $\phi'(\beta(-)) = 0$. The interval $I$ can be of one of the following two types:

- $\beta(+) = 2\rho$, in which case $E$ contains only one interval of this type;
- $\beta(+) = \alpha$ is a critical point of $\phi$, i.e. $\phi'(\beta(+)) = 0$.

If $\beta(+) = 2\rho$, then the integral (13.7) does not exceed

$$C_\delta \|\phi''\|_{L^\infty}(\beta(+) - \beta(-))^{2-\delta} \int_0^1 (t^{1-\delta} - 1) dt \leq C_\delta \|\phi''\|_{L^\infty}\rho^{2-\delta}.$$ 

If $\phi'(\beta(+)) = 0$, then there exists a point $x_0 \in I$ such that $\phi''(x_0) = 0$, so that

$$|\phi''(x)| \leq \|\phi''\|_{L^\infty}|I|, \ x \in I.$$ 

Thus the integral (13.7) does not exceed

$$C \|\phi''\|_{L^\infty}|I| \int_{\beta(-)}^{\beta(+)} (x - \beta(-))^{1-\delta} dx \leq C_\delta \|\phi''\|_{L^\infty}|I|^{3-\delta}.$$ 

Therefore

$$\int_{-2\rho}^{2\rho} m^+(s) ds \leq C \rho^{1-\delta} + C\rho^{2-\delta}\|\phi''\|_{L^\infty} + C \|\phi''\|_{L^\infty} \sum_{I \subset E} |I|^{3-\delta}$$

(13.8) \[ \leq C \rho^{1-\delta}(1 + \rho\|\phi''\|_{L^\infty} + \rho^2\|\phi''\|). \]

Estimates (13.5), (13.6) and (13.8) for $m^-(s)$ are proved in the same way. A reference to (13.3) completes the proof. \(\square\)

13.3. **Proof of Theorem 13.1.** If $l = d$, then the Theorem follows trivially. In particular, $\#X_d(\hat{x}) \leq 1$ for all $\hat{x} \in C_{\rho}^{(d-1)}$, so that $m_\delta(X_d(\hat{x})) \leq \rho^{-\delta}$, and hence the required inequality (13.2) follows immediately.

Suppose that $l \neq d$. In this case for each $\hat{x} = (x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{d-1})$ define the auxiliary function

$$\phi(t) = \phi(\hat{x})(t) = \Phi(\hat{x})\big|_{x_d=t},$$
so that \( X_l(\tilde{x}; \Phi) = X_1(x_l; \phi_{\tilde{x}}) \), and \( \tilde{X}^{(\pm)}(\tilde{x}; \Phi) = \tilde{X}_1^{(\pm)}(x_l; \phi_{\tilde{x}}) \). Therefore Part (1) of the Theorem follows from Part (1) of Lemma 13.3. Moreover,

\[
\int_{C} m_\delta(X_l(\hat{x}; \Phi)) d\hat{x} = \int_{C} \int_{-2\rho}^{2\rho} m_\delta(X_1(s; \phi_{\hat{x}})) ds d\hat{x}.
\]

The estimate (13.2) follows now from Part (2) of Lemma 13.3. \( \square \)

14. Appendix 2: change of variables

In this section we provide some elementary, but useful information from the multi-variable calculus.

14.1. Change of variables: surfaces. First we prove a few lemmas describing surfaces in different coordinates.

Lemma 14.1. Let \( S \subset \mathbb{R}^d \) be a set. Suppose that in a neighbourhood of a point \( w \in S \), the set \( S \) is locally described by the function \( \Phi \), i.e.

\[
S \cap B(w, R) = \{ x : x_d = \Phi(\hat{x}) \} \cap B(w, R),
\]

for some \( R > 0 \), where \( \hat{x} = (x_1, x_2, \ldots, x_{d-1}) \), and \( \Phi \in C^m(\mathbb{R}^{d-1}), m \geq 1. \) Then there exists a number \( R_1 \), an orthogonal transformation \( O \), and a function \( F \in C^m(\mathbb{R}^{d-1}) \), such that for \( E = (O, w) \in E(d) \),

\[
E^{-1}S \cap B(0, R_1) = \{ x : x_d = F(\hat{x}) \} \cap B(0, R_1),
\]

\[
F(\hat{0}) = 0, \nabla F(\hat{0}) = 0.
\]

Proof. Since the set \( S_1 = S - w \) satisfies the condition (14.1) with \( w = 0 \), it suffices to prove the lemma for \( w = 0 \). Thus we may assume that \( \Phi(\hat{0}) = 0 \). Rewrite the equation \( x_d = \Phi(\hat{x}) \) as follows:

\[
x_d = \hat{b} \cdot \hat{x} + \hat{\Phi}(\hat{x}), \quad \hat{b} = \nabla \Phi(\hat{0}), \quad \hat{\Phi}(\hat{x}) = \Phi(\hat{x}) - \hat{b} \cdot \hat{x}.
\]

Since \( \Phi \in C^m, m \geq 1, \) we have

\[
\max_{|\hat{x}| \leq s} |\nabla \hat{\Phi}(\hat{x})| =: \delta(s) \to 0, \quad s \to 0,
\]

and hence

\[
\max_{|\hat{x}| \leq s} |\hat{\Phi}(\hat{x})| \leq \delta(s)s.
\]

Denote \( b = (-\hat{b}, 1) \) and rewrite the equation again:

\[
n \cdot x = |b|^{-1} \hat{\Phi}(\hat{x}), \quad n = b|b|^{-1}.
\]

Let \( O \) be an orthogonal matrix in which the last column equals \( n \), and denote \( y = O^T x \), so that \( n \cdot x = y_d \), and the above equation takes the form

\[
y_d = W(y_d; \hat{y}), \quad W(y_d; \hat{y}) := |b|^{-1} \hat{\Phi}(\hat{O}\hat{y}).
\]
By construction, for all \( y \in B(0, s) \) we have

\[
y_d \in I(s) := [-\delta(s)s, \delta(s)s].
\]

From now on we assume that \( s \leq R \) is so small that \( \delta(s) \leq 1/4 \). Thus the cylinder

\[
C = \{ \hat{y} \in \mathbb{R}^{d-1} : |\hat{y}| < s/4 \} \times I(s)
\]

belongs to the ball \( B(0, s) \subset B(0, R) \). Therefore for all \( \hat{y} : |\hat{y}| < s/4 \), the function \( W(\cdot; \hat{y}) \) maps the interval \( I(s) \) into itself. Moreover, in view of the condition \( \delta(s) \leq 1/4 \),

\[
|\partial_{y_d} W(y_d, \hat{y})| \leq \max_{|\hat{x}| \leq s} |\nabla \hat{\Phi}(\hat{x})| \leq \delta(s) \leq \frac{1}{4},
\]

for all \( y \in C \), so that by the Contraction Mapping Theorem, for each \( \hat{y} \) the equation (14.4) has a unique solution \( y_d \in I(s) \). Denote this solution by \( F(\hat{y}) \). Clearly, \( F(0) = 0 \). Using the Implicit Function Theorem one shows that this solution is a \( C^m \)-function of \( \hat{y} \). Moreover,

\[
\nabla_\hat{y} F(\hat{y}) = \frac{\nabla_\hat{y} W(t, \hat{y})}{1 - \partial_t W(t, \hat{y})} \bigg|_{t = F(\hat{y})},
\]

so that \( \nabla F(\hat{0}) = 0 \) as required. Now extend \( F \) to the entire space \( \mathbb{R}^{d-1} \) as a \( C^m \)-function, and take \( R_1 = s/8 \).

This lemma immediately yields a useful transformation of domains:

**Corollary 14.2.** Let \( \Lambda \subset \mathbb{R}^d \) be a domain and let \( w \in \partial \Lambda \). Suppose that in the ball \( B(w, R) \) the domain \( \Lambda \) is represented by a \( C^m \)-graph-type domain \( \Gamma = \Gamma(\Phi; O, w) \), where \( m \geq 1 \). Then there exists a number \( R_1 \), an orthogonal transformation \( \tilde{O} \), and a function \( F \in C^m(\mathbb{R}^{d-1}) \), satisfying (2.14) and \( \nabla F(\hat{0}) = 0 \), such that in the ball \( B(w, R_1) \) the domain \( \Lambda \) is represented by the graph-type domain \( \Gamma(F; \tilde{O}, w) \).

Now we establish the legitimacy of assumption (10.3). More precisely, we show that any \( C^m \)-domain can be made to satisfy the condition (10.3) locally.

**Lemma 14.3.** Let \( S \) be a set such that for some Euclidean isometry \( E = (O, k) \),

\[
E^{-1} S \cap B(0, R) = \{x : x_d = \Phi(\hat{x})\} \cap B(0, R), \tag{14.5}
\]

for some \( R > 0 \), and with some function \( \Phi \in C^m(\mathbb{R}^{d-1}) \), such that \( \Phi(0) = 0 \). Then there exists a number \( R_1 > 0 \) such that one can find a number \( l = 1, 2, \ldots, d \) and a function \( G \in C^m(\mathbb{R}^{d-1}) \) such that \( G(\hat{0}) = 0 \),

\[
S \cap B(k, R_1) = \{x : x_l = k_l + G(\hat{x} - \hat{k})\} \cap B(k, R_1), \tag{14.6}
\]

\[
\hat{x} = (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_d),
\]

and \( \|\nabla G\|_{L^\infty} \leq 4\sqrt{d} \).
Proof. Since the set $S_1 = S - k$ satisfies the condition (14.5) with $E = (O, 0)$, it suffices to prove the lemma for $k = 0$.

In view of Lemma 14.1 we may assume without loss of generality that the transformation $E$ is chosen in such a way that $\nabla \Phi(0) = 0$. Denote, as in the proof of Lemma 14.1,

$$\delta(s) = \max_{|x| \leq s} |\nabla \Phi(\hat{x})|, \ s \leq R.$$ 

Rewrite the equation $x_d = \Phi(\hat{x})$ for the variable $t : x = E^{-1}t = O^Tt$:

$$(O^Tt)_d - \Phi(O^Tt) = 0,$$

so that

$$n \cdot t - \Phi(O^Tt) = 0,$$

with some unit vector $n \in \mathbb{R}^d$. Let $l$ be such that $|n_l| = \max_j |n_j| \geq d^{-1/2}$. Rewrite the equation in the form:

(14.7) $$t_l = W(t_l; \hat{t}), \ W(t_l; \hat{t}) = -\frac{1}{n_l} \hat{n} \cdot \hat{t} + \frac{1}{n_l} \Phi(O^Tt).$$

Since $|\Phi(\hat{x})| \leq \delta(s)s$ for all $|\hat{x}| \leq s$, the equation (14.7) implies that for all $t \in B(0, s)$ we have

$$t_l \in I(\hat{t}; s) := \left[ -\frac{1}{n_l} \hat{n} \cdot \hat{t} - \sqrt{d}\delta(s)s, -\frac{1}{n_l} \hat{n} \cdot \hat{t} + \sqrt{d}\delta(s)s \right].$$

From now on we assume that $s \leq R$ is so small that $\sqrt{d}\delta(s) \leq 1/4$. Thus the set

$$C(s) := \{ t : |t| \leq (4\sqrt{d})^{-1}s, \ t_l \in I(\hat{t}; s) \},$$

is contained in $B(0, s) \subset B(0, R)$. Therefore for all $\hat{t} : |\hat{t}| \leq (4\sqrt{d})^{-1}s$, the function $W(\cdot, \hat{t})$ maps the interval $I(\hat{t}; s)$ into itself. Moreover, in view of the condition $\sqrt{d}\delta(s) \leq 1/4$, we have

$$|\partial_t W(t_l; \hat{t})| \leq \sqrt{d}\delta(s) \leq \frac{1}{4},$$

for all $t \in C(s)$, so that by the Contraction Mapping Theorem, for each $\hat{t}$ the equation (14.7) has a unique solution $t_l \in I(\hat{t}; s)$. Denote this solution by $G(\hat{t})$. Clearly, $G(0) = 0$.

By the Implicit Function Theorem, $G$ is a $C^m$-function on $\{ \hat{t} : |\hat{t}| \leq s(4\sqrt{d})^{-1} \}$. In particular,

$$\nabla_{\hat{t}} G(\hat{t}) = \left. \nabla_{\hat{t}} W(t, \hat{t}) \right|_{t = G(\hat{t})},$$

so that

$$\sup_{|\hat{t}| < s(4\sqrt{d})^{-1}} |\nabla_{\hat{t}} G(\hat{t})| \leq \frac{4\sqrt{d}}{3} \left( 1 + \max_{|x| \leq s} |\nabla \Phi(\hat{x})| \right) \leq \frac{4\sqrt{d}}{3} (1 + \delta(s)) \leq \frac{4\sqrt{d} + 1}{3}.$$
Now take \( R_1 = s(12\sqrt{d})^{-1} \) and extend \( G \) to \( \mathbb{R}^{d-1} \) in such a way that \( \|\nabla G\|_{L^\infty} \leq 4\sqrt{d} \).

**Corollary 14.4.** Let \( \Lambda \subset \mathbb{R}^d \) be a domain and let \( w \in \partial \Lambda \). Suppose that in the ball \( B(w, R) \) the domain \( \Lambda \) is represented by the \( C^m \)-graph-type domain \( \Gamma = \Gamma(\Phi; O, w) \), \( m \geq 1 \). Then there exist a real number \( R_1 > 0 \) and an integer \( l = 1, 2, \ldots, d \), a a function \( G \in C^m(\mathbb{R}^{d-1}) \), such that

\[
\|\nabla G\|_{L^\infty} \leq 4\sqrt{d},
\]

and

\[
\Lambda \cap B(w, R_1) = \begin{cases} 
\{ x : x_l > w_l + G(\hat{x} - \hat{w}) \} \cap B(w, R_1), \\
\text{or } \{ x : x_l < w_l + G(\hat{x} - \hat{w}) \} \cap B(w, R_1).
\end{cases}
\]

**Proof.** Denote \( E = (O, k) \). By Definition 2.1 (1),

\[
E^{-1}\Lambda \cap B(0, R) = \{ x : x_d > \Phi(\hat{x}) \} \cap B(0, R).
\]

Thus the boundary \( S := \partial \Lambda \) satisfies (14.5). Due to Lemma 14.3 the boundary \( S \) also satisfies (14.6) with a function \( G \), which satisfies all required properties. Thus the domain \( \Lambda \) is given by (14.8). \( \square \)

Finally, we need one more technical result in which we use a linear transformation of a very specific form.

**Lemma 14.5.** Let \( B \) be the linear transformation

\[
Bx = (\hat{x} + \hat{b}x_d, x_d),
\]

with some vector \( \hat{b} \in \mathbb{R}^{d-1} \). Suppose that the surface \( S \) is described by

\[
BS = \{ x : x_d = \Phi(\hat{x}) \},
\]

with some function \( \Phi \in C^m(\mathbb{R}^{d-1}), m \geq 1 \), such that

\[
\|\nabla \Phi\|_{L^\infty} \leq M, \quad \sum_{k=1}^m \|\nabla^k \Phi\|_{L^\infty} \leq C,
\]

with some number \( M > 0 \). Suppose that \( |\hat{b}| \leq (2M)^{-1} \). Then there is a function \( F \in C^m(\mathbb{R}^{d-1}) \) such that

\[
S = \{ x : x_d = F(\hat{x}) \},
\]

and

\[
\sum_{k=1}^m \|\nabla^k F\|_{L^\infty} \leq C,
\]

uniformly in \( \Phi \), satisfying (14.10).
Proof. Since the surface $BS - \Phi(\hat{0})e_d$ satisfies (14.9) with the function $\Phi(\check{x}) - \Phi(\hat{0})$, it suffices to prove the lemma assuming that $\Phi(\hat{0}) = 0$. By definition each $t \in S$ satisfies the equation

$$\tag{14.11} (Bt)_d = \Phi(\hat{Bt}), \quad \text{i.e.} \quad t_d = W(t_d; \hat{t}) := \Phi(\hat{t} + \hat{b}t_d).$$

Denote $b := |\hat{b}| \leq (2M)^{-1}$. Since $\Phi(\hat{0}) = 0$ and $Mb \leq 2^{-1}$, we have

$$\tag{14.12} |\Phi(\hat{t} + \hat{b}t_d)| \leq M(|\hat{t}| + b|t_d|) \leq M|\hat{t}| + \frac{1}{2}|t_d|$$

for all $t$. Therefore a solution of the equation (14.11) satisfies

$$|t_d| \leq 2M|\hat{t}|.$$ 

Thus for each $\hat{t} \in \mathbb{R}^{d-1}$ the function $W(\cdot; \hat{t})$ maps the interval

$$I(\hat{t}) = [-2M|\hat{t}|, 2M|\hat{t}|]$$

into itself. Furthermore, in view of the condition $bM \leq 1/2$, we have

$$\partial_{t_d} W(t_d; \hat{t}) \leq Mb \leq \frac{1}{2}.$$ 

Thus by the Contraction Mapping Theorem, for each $\hat{t} \in \mathbb{R}^{d-1}$ the equation (14.5) has a unique solution $t_d \in I(\hat{t})$. We denote this solution by $F(\hat{t})$. Moreover, by the Implicit Function Theorem, $F$ is a $C^m$-function, with derivatives bounded uniformly in $\Phi$, satisfying (14.10). $\square$

14.2. Change of variables: integration. We need a simple version of the change of variables in the area formula (see, e.g. Theorem 1, p.96 in [10]). We use again the notation

$$\check{x} = (x_1, x_2, \ldots, x_{d-1}), \hat{x} = (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_d),$$

for a fixed $l = 1, 2, \ldots, d$. Let $F \in C^1(\mathbb{R}^{d-1})$. For each $\check{x}$ let $X(\check{x}) \subset \mathbb{R}$ be the set

$$X(\check{x}) = \{x_d \in \mathbb{R} : F(\hat{x}) = x_1\}.$$ 

Recall also that the vector

$$\tag{14.13} n_S(\check{x}) := \left(\frac{-\partial_{x_1} F(\hat{x}), \ldots, -\partial_{x_{l-1}} F(\hat{x}), 1, -\partial_{x_{l+1}} F(\hat{x}), \ldots, -\partial_{x_d} F(\hat{x})}{\sqrt{1 + |\nabla F(\hat{x})|^2}}\right),$$

defines the (continuous) unit normal to the surface

$$\tag{14.14} S = \{x : x_l = F(\hat{x})\},$$

at the point $x = (x_1, \ldots, x_{l-1}, F(\hat{x}), x_{l+1}, \ldots, x_d)$.

Below for brevity we use the notation $(\check{x}, F(\hat{x}))$ for the vector $x$ with $x_l = F(\hat{x})$. 
Lemma 14.6. Let \( f \in C_0^\infty(\mathbb{R}^d), F \in C^1(\mathbb{R}^{d-1}) \) be some functions, and let \( S \subset \mathbb{R}^d \) be the surface (14.14). Then
\[
\int_{\mathbb{R}^{d-1}} \sum_{x_d \in X(\mathbf{x})} f(\hat{x}, F(\hat{x})) d\mathbf{x} = \int_S |\mathbf{n}_S(\mathbf{x}) \cdot \mathbf{e}_d| f(\mathbf{x}) dS_\mathbf{x}.
\]

Proof. If \( l = d \), then \( X(\hat{x}) = \{ F(\hat{x}) \} \), so that the integral equals
\[
\int_{\mathbb{R}^{d-1}} f(\hat{x}, F(\hat{x})) d\mathbf{x} = \int_{\mathbb{R}^{d-1}} \frac{f(\hat{x}, F(\hat{x}))}{\sqrt{1 + |\nabla F(\hat{x})|^2}} \sqrt{1 + |\nabla F(\hat{x})|^2} d\mathbf{x} = \int_S |\mathbf{n}_S(\mathbf{x}) \cdot \mathbf{e}_d| f(\mathbf{x}) dS_\mathbf{x},
\]
as required.

Suppose that \( l < d \). Denote by \( \Xi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1} \) the mapping
\[
\Xi(\hat{x}) = (x_1, \ldots, x_{l-1}, F(\hat{x}), x_{l+1}, \ldots, x_d).
\]
Then by the Change of Variables formula (see, e.g. Theorem 2, p.99 in [10]),
\[
\int_{\mathbb{R}^{d-1}} \sum_{x_d \in X(\mathbf{x})} f(\hat{x}, F(\hat{x})) d\mathbf{x} = \int_{\mathbb{R}^{d-1}} \sum_{\hat{x} \in \Xi^{-1}(\mathbf{x})} f(\hat{x}, F(\hat{x})) d\hat{x} = \int_{\mathbb{R}^{d-1}} J_\Xi(\hat{x}) f(\hat{x}, F(\hat{x})) d\hat{x},
\]
where \( J_\Xi \) is the Jacobian of the map \( \Xi \). A straightforward computation shows that \( J_\Xi = |\partial_{x_d} F| \). Now, in view of (14.13), the last integral can be rewritten as
\[
\int_{\mathbb{R}^{d-1}} \frac{|\partial_{x_d} F(\hat{x})|}{\sqrt{1 + |\nabla F(\hat{x})|^2}} f(\hat{x}, F(\hat{x})) \sqrt{1 + |\nabla F(\hat{x})|^2} d\hat{x} = \int_S |\mathbf{n}_S(\mathbf{x}) \cdot \mathbf{e}_d| f(\mathbf{x}) dS_\mathbf{x},
\]
as required. \( \square \)

15. Appendix 3: A trace-class formula

Let \( \mathcal{H} \) be a separable Hilbert space. Consider in \( L^2(\mathbb{R}^n, \mathcal{H}) \), \( n \geq 1 \) the pseudodifferential operator \( T \) with an \( \mathcal{H} \)-valued symbol \( t(\mathbf{x}, \xi) \), i.e.
\[
(T\phi)(\mathbf{x}) = (\text{Op}_1(t)\phi)(\mathbf{x}) = \frac{1}{(2\pi)^n} \int \int e^{i(\mathbf{x}-\mathbf{y})\xi} t(\mathbf{x}, \xi)\phi(\mathbf{y}) d\mathbf{y} d\xi,
\]
for any \( \phi \in L^2(\mathbb{R}^n, \mathcal{H}) \). Our objective is to give a proof of the standard formula for the trace of \( T \) under the assumption that both \( t \) and \( T \) are trace-class. Note that we do not provide conditions which ensure that \( T \in \mathcal{S}_1 \).

Lemma 15.1. Suppose that the operator \( t(\mathbf{x}, \xi) \) is trace class a.a. \( \mathbf{x} \) and \( \xi \), that \( T = \text{Op}_1(t) \) is trace-class, and that
\[
(15.1) \quad \|t(\cdot, \cdot)\|_{L^1} \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n).
\]
Then
\[
\text{tr} \ T = \frac{1}{(2\pi)^n} \int \text{tr} t(\mathbf{x}, \xi) d\mathbf{x} d\xi.
\]
Proof. Since $T \in S_1$, for any family of bounded operators $K_s$ in $L^2(\mathbb{R}^n, \mathfrak{H})$, strongly converging to $I$ as $s \to \infty$, we have

$$\|T - TK_s\|_{S_1} \to 0, \ s \to \infty.$$  

Thus it suffices to consider instead of $T$ the operator $B = \text{Op}_a^1(b)$ with the amplitude $b(x, y, \xi) = t(x, \xi)\eta(\xi)\psi(y)$, where $\eta, \psi \in C_0^\infty(\mathbb{R}^n)$, and prove that

$$\text{tr } B = \frac{1}{(2\pi)^n} \int \int \psi(x)\eta(\xi) \text{tr } t(x, \xi) dyd\xi. \quad (15.2)$$

Since $B$ is trace-class, for any orthonormal basis (o.n.b.) $F_j$ in $L^2(\mathbb{R}^n, \mathfrak{H})$ we have

$$\text{tr } B = \sum_j \langle BF_j, F_j \rangle,$$

and the series converges absolutely, see [3], Ch. 11, Theorem 2.7. We choose the o.n.b. labeled by two indices. Let $f_j, j \in \mathbb{N}$, be an o.n.b. of $\mathfrak{H}$, and let $g_s, s \in \mathbb{N}$, be an o.n.b. of $L^2(\mathbb{R}^n)$, so that $f_j \otimes g_s$ is an o.n.b. of $L^2(\mathbb{R}^n, \mathfrak{H})$. Consider the finite sum

$$S_{N,M} = \sum_{s=1}^N \sum_{j=1}^M \langle Bf_j \otimes g_s, f_j \otimes g_s \rangle$$

$$= \frac{1}{(2\pi)^n} \sum_{s=1}^N \int e^{i\xi(x-y)}\eta(\xi)T_M(x, \xi)g_s(y)\overline{\psi(y)}g_s(x)dyd\xi dx,$$

with

$$T_M(x, \xi) = \sum_{j=1}^M \langle t(x, \xi)f_j, f_j \rangle_{\mathfrak{H}}.$$

Since

$$|T_M(x, \xi)| \leq \|t(x, \xi)\|_{S_1},$$

and $T_M(x, \xi) \to \text{tr } t(x, \xi) =: T(x, \xi), M \to \infty$ pointwise, in view of (15.1), by the dominated convergence theorem we conclude that the double sum $S_{N,M}$ converges as $M \to \infty$, to

$$S_N = \frac{1}{(2\pi)^n} \int e^{-i\xi y} \sum_{s=1}^N \psi(y)\eta(\xi)\hat{T}^{(s)}(\xi)g_s(y)dyd\xi,$$

$$\hat{T}^{(s)}(\xi) = \int e^{i\xi x}T(x, \xi)g_s(x)dx.$$  

As $N \to \infty$, the integral

$$I_N(\xi) = \int e^{-i\xi y} \psi(y) \sum_{s=1}^N \hat{T}^{(s)}(\xi)g_s(y)dy.$$
for almost all $\xi$ converges to
\[ \int T(y, \xi) \psi(y) dy \]
by Plancherel’s Theorem. By Bessel inequality $I_N(\xi)$ is bounded from above by
\[ \left( \int |\psi(y)|^2 dy \right)^{\frac{1}{2}} \left( \sum_{s=1}^{N} |\hat{T}^{(s)}(\xi)|^2 \right)^{\frac{1}{2}} \leq \|\psi\|_{L^2} \left( \int |T(x, \xi)|^2 dx \right)^{\frac{1}{2}} \]
uniformly in $N$. By (15.1) the right hand side is an $L^2$-function and $\eta$ has compact support, so that by the Dominated Convergence Theorem, in (15.3) one can pass to the limit as $N \to \infty$:
\[ \lim_{N \to \infty} S_N = \frac{1}{(2\pi)^n} \int T(y, \xi) \eta(\xi) \phi(y) dy d\xi. \]
This coincides with (15.2), which completes the proof of the Lemma. \hfill \square

16. Appendix 4: Invariance with respect to the affine change of variables

Our aim is to show that the coefficient (1.4) does not change under the affine change of variables. Let $M$ be a non-degenerate linear transformation of $\mathbb{R}^d$, let $k, k_1$ be vectors in $\mathbb{R}^d$, and let $a = a(x, \xi)$ be a continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ with a compact support in both variables. As in (2.6) denote
\[ a_{M,k,k_1}(x, \xi) = a(Mx + k, (M^T)^{-1}\xi + k_1). \]
For a set $S \subset \mathbb{R}^d$ denote $S_{M,k} = M^{-1}(S - k), S_{M,k_1}^T = M^T(S - k)$.

**Lemma 16.1.** Let $F, G \in C^1(\mathbb{R}^d)$ be two real-valued functions such that $\nabla G(x) \neq 0$, $\nabla G(\xi) \neq 0$, and let $S, P$ be two surfaces, defined by the equations $F(x) = 0$ and $G(\xi) = 0$ respectively. Suppose that the function $a$ is as defined above. Then for any non-degenerate linear transformation $M$ and any pair of vectors $k, k_1 \in \mathbb{R}^d$, one has
\[ \mathfrak{M}_1(a_{M,k,k_1}; S_{M,k}, P_{M,k_1}^T) = \mathfrak{M}_1(a; S, P). \]

**Proof.** The surfaces $S_{M,k}$ and $P_{M,k_1}^T$ are defined by the equations
\[ F_{M,k}(x) = 0 \quad \text{and} \quad G_{M,k_1}(\xi) = 0 \]
respectively. Define
\[ Z(t, s; M, k, k_1) = \int_{F_{M,k}(x) > t} \int_{G_{M,k_1}(\xi) > s} a_{M,k,k_1}(x, \xi) |\nabla F_{M,k}(x) \cdot \nabla G_{M,k_1}(\xi)| d\xi dx. \]
A straightforward change of variables gives the equality
\[ Z(t, s; M, k, k_1) = Z(t, s; I, 0, 0). \]
By Proposition 3, Ch 3.4, [10],

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} Z(t, s; M, k, k_1) \bigg|_{t=s=0} = \int_{F_{M,k}(x)=0} \int_{G_{M,k_1}(\xi)=0} a_{M,k,k_1}(x, \xi) \left| \frac{\nabla F_{M,k}(x) \cdot \nabla G_{M,k_1}(\xi)}{\left| \nabla F_{M,k}(x) \right| \left| \nabla G_{M,k_1}(\xi) \right|} \right| dS_\xi dS_x
\]

\[
= \mathcal{W}_1(a_{M,k,k_1}; S_{M,k}, P^T_{M,k_1})
\]

where we have used that the vectors

\[
\frac{\nabla F_{M,k}(x)}{\left| \nabla F_{M,k}(x) \right|} \quad \text{and} \quad \frac{\nabla G_{M,k_1}(\xi)}{\left| \nabla G_{M,k_1}(\xi) \right|}
\]

define unit normals to the surfaces \( S_{M,k} \) and \( P^T_{M,k_1} \) at the points \( x \) and \( \xi \) respectively. Now the proclaimed equality follows from (16.1). \( \square \)

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