A Hopf-Lax formula for Hamilton-Jacobi equations with Caputo time derivative

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Abstract

We prove a representation formula of Hopf-Lax type for the solution of a Hamilton-Jacobi equation involving Caputo time-fractional derivative. Equations of these type are associated with optimal control problems where the controlled dynamics is replaced by a time-changed stochastic process describing the trajectory of a particle subject to random trapping effects.

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1 Introduction

In the recent times, several classical parabolic equations have been revisited by replacing the standard derivative with fractional ones [1, 3, 5, 8, 10, 16]. Fractional time derivatives are given by convolution integral of the time-derivative with power-law kernels. They arise in several phenomena in connection with anomalous diffusion and are typical for memory effects in complex systems (see [14] for a review). The probabilistic interpretation of the corresponding physical models leads to the study of subdiffusive or, more in general, non markovian processes. From a mathematical point of view, the presence of nonlocal terms with respect to the time variable poses several technical difficulties.

Aim of this paper is to study the connection between Hamilton-Jacobi equations and anomalous diffusions, recovering a subordinated version of the Hopf-Lax formula. Consider the Cauchy problem

\[
\begin{cases}
\partial_t u + H(Du) = 0 & (x,t) \in Q, \\
u(x,0) = g(x) & x \in \mathbb{R}^d,
\end{cases}
\]

where the Hamiltonian $H$ is convex and superlinear and $Q = \mathbb{R}^d \times (0, \infty)$. Then the classical Hopf-Lax formula

\[
u(x,t) = \min_{y \in \mathbb{R}^d} \left\{ tL \left( \frac{x-y}{t} \right) + g(y) \right\},
\]

where $L$ is the Legendre transform of $H$, gives the unique viscosity solution of (1.1). Moreover, if $g$ is Lipschitz continuous, then $u$ is also Lipschitz continuous and it is the maximal almost everywhere (a.e.) subsolution of...
(1.1). Formula (1.2) is derived from the optimal control interpretation of the Cauchy problem. Indeed, the Hamilton-Jacobi equation in (1.1) can be interpreted as the dynamic programming equation satisfied by the value function of a control problem with dynamics

\[
\begin{cases}
  \dot{x}(s) = a(s) & s \in (0, t), \\
x(t) = x,
\end{cases}
\]

where \(a: (0, \infty) \to \mathbb{R}^d\) is the control variable, and cost functional

\[
J(x, t, a) = \int_0^t L(a(s))ds + g(x(0)).
\]

Since \(L\) is independent of \((x, t)\), straight lines are proved to be the minimizing trajectories in (1.4) and (1.2) is so obtained (see [4, 7] for details). In this paper we consider the Cauchy problem

\[
\begin{cases}
  \partial_{(0, q)}^\beta u + H(Du) = 0 & (x, t) \in Q, \\
u(x, 0) = g(x) & x \in \mathbb{R}^d,
\end{cases}
\]

where

\[
\partial_{(0, q)}^\beta u(x, t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \partial_\tau u(x, \tau) \frac{d\tau}{(t - \tau)^\beta}
\]

is the Caputo time-fractional derivative of order \(\beta \in (0, 1)\) of \(u\). To deduce a Hopf-Lax formula for (1.5) we rely, as in the classical case, on the optimal control interpretation of the problem. Let \(E_t\) be a continuous, nondecreasing stochastic process defined as the inverse of a \(\beta\)-stable subordinator \(D_t\), i.e. \(E_t := \inf\{\tau > 0 : D_\tau > t\}\) for \(t \geq 0\). The stochastic process \(X(t) = x(E(t))\), where \(x(t)\) is given by (1.3), solves the stochastic differential equation

\[
\begin{cases}
  dX(t) = \bar{a}(t) dE_t, & t \in (0, \infty) \\
  X(t) = x,
\end{cases}
\]

where \(\bar{a}(t) = a(E(t))\) for \(a \in \mathcal{A}\). The subordinator \(E_t\) can be interpreted as a change of the time-scale which introduces trapping events in the evolution of the process \(X(t)\), whereas, when not trapped, the particle moves according to the standard dynamics \(x(t)\). Define the cost functional

\[
J_\beta(x, t, a) = \mathbb{E}_{x,t} \left\{ \int_0^t L(a(s))dE_s + g(X(0)) \right\}.
\]

It is clear that straight lines are still the optimal trajectories minimizing (1.7), but traveled at a velocity which depends on the time scale \(E_t\). We prove that the value function \(u_\beta\) associated to the time-changed control problem is given by the Hopf-Lax formula

\[
u_\beta(x, t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x - y}{E_t} \right) + g(y) \right\} \right].
\]

The previous formula is similar to (1.2), but it takes into account the average speed at which the straight trajectories are traveled. We also prove that \(u_\beta\) is the maximal subsolution and an a.e. solution of problem (1.5), but we are not able to prove that it is a viscosity solution in the sense of the definition introduced in [5, 16] (see Remark 3.4 for more details). We can rewrite formula (1.2) as the convolution of the solution of (1.1) with a kernel given by the probability density function (PDF) of the process \(E_t\), i.e.

\[
u_\beta(x, t) = \int_0^\infty u(x, s)\mathcal{E}_\beta(s, t)ds.
\]
Employing a standard numerical solver for (1.1) to compute \( u \), we use the previous formula to illustrate with some numerical examples the effect of the Caputo derivative on control problems and fronts propagation.

The paper is organized as follows. In Section 2, we briefly recall some properties of the subordinator process and we introduce the Hopf-Lax formula. Section 3 is devoted to the time-fractional Hamilton-Jacobi equation. Finally, in Section 4, some numerical examples are discussed in order to stress the main differences with the classical theory.

2 The subordinator process and the Hopf-Lax formula

Let \( \{D_t\}_{t \geq 0} \) be a stable subordinator of order \( \beta \in (0, 1) \), i.e. a one-dimensional, non-decreasing Lévy process whose PDF \( g(s, \tau) \) has Laplace transform equal to \( e^{-\tau s^\beta} \). The inverse stable process \( \{E_t\}_{t \geq 0} \), defined as the first passage time of the process \( D_T \) over the level \( t \), i.e.

\[
E_t = \inf\{\tau > 0 : D_\tau > t\},
\]

has sample paths which are continuous, non-decreasing and such that \( E_0 = 0 \), \( E_t \to \infty \) as \( t \to \infty \). It is worthwhile to observe that \( E_t \) does not have stationary or independent increments. The process \( E_t \) can be used to model systems with two time scales: a deterministic one given by the standard time \( t \), referred to the external observer, and a stochastic one given by \( E_t \), internal to the physical process (see [9, 12, 13]). We recall some basic properties of the process \( E_t \) which we will exploit in the following

**Proposition 2.1.** For \( t > 0 \), it holds:

- For any \( \alpha > 0 \), there exists a constant \( C(\alpha, \beta) > 0 \) such that
  \[
  \mathbb{E}[E_t^\alpha] = C(\alpha, \beta)t^{\alpha\beta}.
  \]

- The process \( E_t \) has PDF
  \[
  \mathcal{E}_\beta(s, t) = \frac{t}{\beta} s^{-1 - \frac{1}{\beta}} g(s, t).
  \]

For the proof of the following result we refer to [13]

**Proposition 2.2.** The function \( \mathcal{E}_\beta(\cdot, t) \) is a weak solution of

\[
\partial_{(0,t)}^\beta \mathcal{E}_\beta(r, t) = -\partial_r \mathcal{E}_\beta(r, t), \quad r \in (0, \infty).
\]

In the following, we assume that

\[
H : \mathbb{R}^d \to \mathbb{R} \text{ is convex and } \lim_{|p| \to \infty} \frac{H(p)}{p} = +\infty;
\]

\[
g : \mathbb{R}^d \to \mathbb{R} \text{ is Lipschitz continuous, bounded.}
\]

The Legendre transform \( L \) of \( H \), defined by \( L(q) = \sup\{pq - H(p)\} \), is well defined, convex and superlinear. Let \( u_\beta \) be the value function of the stochastic control problem (1.6)-(1.7), i.e.

\[
u_\beta(x, t) = \inf_{\alpha \in \mathcal{A}} J_\beta(x, t, \alpha),
\]

where \( \mathcal{A} := \{\alpha : (0, \infty) \to \mathbb{R}^d : \alpha \text{ is a progressively measurable process}\} \).
Proposition 2.3. The value function $u_\beta$ defined in (2.6) is given by the Hopf-Lax formula

$$u_\beta(x, t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right].$$ (2.7)

Proof. Fix $(x, t) \in \mathbb{R}^d \times (0, +\infty)$. Let $Y : \Omega \to \mathbb{R}^d$ be a r.v. such that

$$\min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} = E_t L \left( \frac{x-Y}{E_t} \right) + g(Y).$$ (2.8)

Note that $Y$ is well defined since, by (2.4) and (2.5), the minimum in the LHS of (2.8) is achieved for any $\omega \in \Omega$. For the control law $\bar{a}(s) = (x-Y)/E_t$, consider the solution $X(s)$ of (1.6). Then

$$X(s) = Y + \frac{x-Y}{E_t} E_s$$

and therefore $X(0) = Y$ (recall that $E_0 = 0$) and $X(t) = x$. Hence

$$u_\beta(x, t) \leq \mathbb{E}_{x,t} \left\{ \int_0^t L \left( \frac{x-Y}{E_t} \right) dE_s + g(X(0)) \right\} = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right].$$

We prove the reverse inequality. Given a control $\alpha \in \mathcal{A}$, by the convexity of $L$ we have

$$L \left( \frac{1}{E_t} \int_0^t \alpha(s) dE_s \right) \leq \frac{1}{E_t} \int_0^t L(\alpha(s)) dE_s.$$

If $X(t)$ is the solution of (1.6), since $\int_0^t \alpha(s) dE_s = x - X(0)$, by the previous inequality we get

$$\mathbb{E}_{x,t} \left\{ \int_0^t L(\alpha(s)) dE_s + g(X(0)) \right\} \geq \mathbb{E}_{x,t} \left\{ E_t L \left( \frac{x-X(0)}{E_t} \right) + g(X(0)) \right\}$$

$$\geq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right]$$

and, for the arbitrariness of $a \in \mathcal{A}$,

$$u_\beta(x, t) \geq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right].$$

In order to prove some regularity properties of the function $u_\beta$, we need a preliminary result.

Lemma 2.4. For $s \in [0, t)$, we have

$$u_\beta(x, t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_s) L \left( \frac{x-y}{E_t - E_s} \right) + u_\beta(y, s) \right\} \right].$$ (2.9)

Proof. For $y \in \mathbb{R}^d$, let $Z : \Omega \to \mathbb{R}^d$ be a r.v. such that

$$u_\beta(y, s) = \mathbb{E}_{x,t} \left[ E_s L \left( \frac{y-Z}{E_s} \right) + g(Z) \right]$$

By the identity

$$\frac{x-Z}{E_t} = \left( 1 - \frac{E_s}{E_t} \right) \frac{x-y}{E_t - E_s} + \frac{E_s}{E_t} \frac{y-Z}{E_s}$$

we have

$$u_\beta(x, t) = \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x-z}{E_t} \right) + g(z) \right]$$

for any $z \in \mathbb{R}^d$. Then, using the convexity of $L$, we get

$$u_\beta(x, t) \leq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_s) L \left( \frac{x-y}{E_t - E_s} \right) + u_\beta(y, s) \right\} \right].$$
and by the convexity of \( L \), we get
\[
L \left( \frac{x-Z}{E_t} \right) \leq \left( 1 - \frac{E_s}{E_t} \right) L \left( \frac{x-y}{E_t - E_s} \right) + \frac{E_s}{E_t} L \left( \frac{y-Z}{E_s} \right).
\]

Therefore
\[
\begin{align*}
    u_\beta(x,t) &= \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right] \\
    &\leq \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x-Z}{E_t} \right) + g(Z) \right] \\
    &\leq \mathbb{E}_{x,t} \left[ (E_t - E_s)L \left( \frac{x-y}{E_t - E_s} \right) + E_s L \left( \frac{y-Z}{E_s} \right) + g(Z) \right] \\
    &= \mathbb{E}_{x,t} \left[ (E_t - E_s)L \left( \frac{x-y}{E_t - E_s} \right) + u_\beta(y,s) \right].
\end{align*}
\]

Since the previous inequality holds for any \( y \in \mathbb{R}^d \), we get
\[
u_\beta(x,t) \leq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_s)L \left( \frac{x-y}{E_t - E_s} \right) + u_\beta(y,s) \right\} \right].
\]

To prove the reverse inequality, let \( W \) be a r.v. such that
\[
u_\beta(x,t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x-y}{E_t} \right) + g(y) \right\} \right] = \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x-W}{E_t} \right) + g(W) \right].
\]

If \( Y : \Omega \to \mathbb{R}^d \) is a r.v., since
\[
\mathbb{E}_{x,t} [u_\beta(Y,s)] \leq \mathbb{E}_{x,t} \left[ E_s L \left( \frac{Y-W}{E_s} \right) + g(W) \right],
\]

it follows that
\[
\nu_\beta(x,t) \geq \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x-W}{E_t} \right) - E_s L \left( \frac{Y-W}{E_s} \right) + u_\beta(Y,s) \right]. \quad (2.10)
\]

Set \( Y = \frac{E_s}{E_t} x + (1 - \frac{E_s}{E_t}) W \). Then \( \frac{x-y}{E_t} = \frac{E_s}{E_t} = \frac{y-W}{E_s} \) and by (2.10)
\[
\nu_\beta(x,t) \geq \mathbb{E}_{x,t} \left[ (E_t - E_s) L \left( \frac{x-Y}{E_t - E_s} \right) + u_\beta(Y,s) \right] \\
\geq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_s)L \left( \frac{x-y}{E_t - E_s} \right) + u_\beta(y,s) \right\} \right].
\]

\[
\square
\]

**Remark 2.5.** Arguing as in Lemma 2.4, it is also possible to prove that if \( \tau : \Omega \to [0,t) \) is a stopping time, then
\[
u_\beta(x,t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_\tau)L \left( \frac{x-y}{E_t - E_\tau} \right) + u_\beta(y,\tau) \right\} \right]. \quad (2.11)
\]

**Proposition 2.6.** We have
\[
\begin{align*}
    (i) \text{ For all } x, \tau \in \mathbb{R}^d, \ t \in (0,\infty) & \quad |\nu_\beta(x,t) - \nu_\beta(\tau,t)| \leq L_g|x-\tau|, \\
    \text{where } L_g \text{ is the Lipschitz constant of the initial datum } g. \quad (2.12)
\end{align*}
\]

\[
\begin{align*}
    (ii) \text{ There exists a constant } C \text{ such that for all } x \in \mathbb{R}^d, \ t \in (0,\infty) & \quad |\nu_\beta(x,t) - g(x)| \leq Ct^\beta. \quad (2.13)
\end{align*}
\]
(iii) There exists a constant $C$ such that for all $x \in \mathbb{R}^d$, $t, \bar{t} \in (0, \infty)$, $\bar{t} < t$,
\[ |u_\beta(x, t) - u_\beta(x, \bar{t})| \leq C(t - \bar{t})^\beta. \] (2.14)

Proof. Fixed $t > 0$, $x, \bar{x} \in \mathbb{R}^d$, let $Z : \Omega \to \mathbb{R}^d$ be a r.v. such that
\[ E_t L \left( \frac{x - Z}{E_t} \right) + g(Z) = \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x - y}{E_t} \right) + g(y) \right\}. \]

Then
\[ u_\beta(x, t) - u_\beta(x, t) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x - y}{E_t} \right) + g(y) \right\} - \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x - Z}{E_t} \right) + g(Z) \right] \right] \]
\[ \leq \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x - Z}{E_t} \right) + g(\bar{x} - x + Z) \right] - \mathbb{E}_{x,t} \left[ E_t L \left( \frac{x - Z}{E_t} \right) + g(Z) \right] \]
\[ \leq \mathbb{E}_{x,t} \left[ g(\bar{x} - x + Z) - g(Z) \right] \leq L_g |\bar{x} - x|. \]

Exchanging the role of $x$, $\bar{x}$, we get (2.12).

Fix $x \in \mathbb{R}^d$ and $t > 0$. We recall that for any $t, \gamma > 0$, the $\gamma$-moment of $E_t$ is given by Then, setting $y = x$ in the RHS of (2.7) and recalling (2.1), we get
\[ u_\beta(x, t) \leq \mathbb{E}_{x,t} [ E_t L(0) + g(x) ] = L(0) \mathbb{E}_{x,t}[E_t] + g(x) = L(0) c(\beta, 1)t^\beta + g(x). \] (2.15)

To get the other inequality in (2.13), we observe that
\[ u_\beta(x, t) - g(x) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x - y}{E_t} \right) + g(y) - g(x) \right\} \right] \]
\[ \geq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ E_t L \left( \frac{x - y}{E_t} \right) - L_g |x - y| \right\} \right] = - \mathbb{E}_{x,t} \left[ E_t \max_{z \in \mathbb{R}^d} \{ -L(z) + L_g |z| \} \right] \]
\[ = \mathbb{E}_{x,t} \left[ -E_t \max_{|w| \leq L_g} \{|w| - L(w)\} \right] = - \max_{|w| \leq L_g} \{ H(w)\} \mathbb{E}_{x,t}[E_t]. \] (2.16)

By (2.1) and (2.15), we get (2.13).

To prove (2.14), fix $x \in \mathbb{R}^d$ and $0 < \bar{t} < t$. Setting $y = x$, $s = \bar{t}$ in the RHS of (2.9) and recalling (2.1), we get
\[ u_\beta(x, t) \leq \mathbb{E}_{x,t} \left[ (E_t - E_{\bar{t}}) L(0) + u_\beta(x, \bar{t}) \right] = L(0) \mathbb{E}_{x,t}[E_t - E_{\bar{t}}] + u_\beta(x, \bar{t}) \]
\[ = L(0) c(\beta, 1)(t^{\bar{t}} - \bar{t})^{\beta} + u_\beta(x, \bar{t}). \]

On the other side, by (2.9) with $s = \bar{t}$ and (2.12), arguing as in (2.16), we have
\[ u_\beta(x, t) - u_\beta(x, \bar{t}) = \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_{\bar{t}}) L \left( \frac{x - y}{E_t - E_{\bar{t}}} \right) + u_\beta(y, \bar{t}) - u_\beta(x, \bar{t}) \right\} \right] \]
\[ \geq \mathbb{E}_{x,t} \left[ \min_{y \in \mathbb{R}^d} \left\{ (E_t - E_{\bar{t}}) L \left( \frac{x - y}{E_t - E_{\bar{t}}} \right) - L_g |x - y| \right\} \right] \]
\[ = \mathbb{E}_{x,t} \left[ -(E_t - E_{\bar{t}}) \max_{|w| \leq L_g} \{|w| - L(w)\} \right] \]
\[ = - \max_{|w| \leq L_g} \{ H(w)\} \mathbb{E}_{x,t}[E_t - E_{\bar{t}}] \geq - \max_{|w| \leq L_g} \{ H(w)\} c(\beta, 1)(t - \bar{t})^{\beta}. \]

\[ \square \]
3 The fractional Hamilton-Jacobi equation

We exploit the results of the previous section to show that the function \( u_\beta \) given by (2.7) is a solution of the Cauchy problem (1.5).

**Definition 3.1.** A function \( v \in C^0(\bar{Q}) \) is said to be a.e. subsolution of (1.5) is \( Du \in L^\infty(Q), \partial_t v(\cdot,x) \in L^1_{\text{loc}}(0,\infty) \) for all \( x \in \mathbb{R}^d \) and

\[
\partial_{(0,t)}^\beta v + H(Dv) \leq 0 \quad \text{a.e. in } Q, \\
v(x,0) \leq g(x) \quad x \in \mathbb{R}^d.
\]

(3.1)

(3.2)

Observe that, since \( \partial_t v(\cdot,x) \in L^1_{\text{loc}}(0,\infty) \) for all \( x \in \mathbb{R}^d \), the fractional derivative \( \partial_{(0,t)}^\beta v(x,t) \) is well defined for any \( (x,t) \in Q \).

**Proposition 3.2.** The function \( u_\beta \) is the maximal a.e. subsolution of (1.5). In addition, \( u_\beta \) is a.e. solution of (1.5) in \( Q \).

**Proof.** By Proposition 2.6, we have that \( Du_\beta \in L^\infty(Q), \partial_t u_\beta(\cdot,x) \in L^1_{\text{loc}}(0,\infty) \) for all \( x \in \mathbb{R}^d \). Given \( (x,t) \in \mathbb{R}^d \times (0,\infty) \) such that \( Du(x,t) \) exists, we prove that for any \( q \in \mathbb{R}^d \),

\[
\partial_{(0,t)}^\beta u_\beta(x,t) + Du_\beta(x,t) \cdot q - L(q) \leq 0.
\]

(3.3)

Indeed, fix \( q \in \mathbb{R}^d \) and \( h > 0 \). Consider the control law \( \alpha(s) \equiv q \). Then the solution \( X(s) \) of (1.6) is given by \( X(s) = x - (E_t - E_s)q \). Define the stopping time

\[
\tau_h = \sup\{s \in (t - h, t) : |X(s) - x| = h\}.
\]

By (2.11), for \( \tau = \tau_h \) and \( y = X(\tau_h) \), we have

\[
E_{x,t}[u_\beta(X(t),t) - u_\beta(X(\tau_h),\tau_h)] \leq L(q) E_{x,t}[E_t - E_{\tau_h}]
\]

(3.4)

By Ito's formula [6], we also have

\[
E_{x,t}[u_\beta(X(t),t) - u_\beta(X(\tau_h),\tau_h)] = E_{x,t}\left[ \int_{\tau_h}^t \partial_s u_\beta(X(s),s)ds + \int_{\tau_h}^t Du_\beta(X(s),s)dX(s) \right]
\]

(3.5)

Therefore, recalling (3.4),

\[
E_{x,t}\left[ \int_{\tau_h}^t \partial_s u_\beta(X(s),s)ds + \int_{\tau_h}^t (Du_\beta(X(s),s) \cdot q - L(q))dE_s \right] = E_{x,t}\left[ \int_{\tau_h}^t \partial_s u_\beta(X(s),s)ds + \int_{\tau_h}^{+\infty} \left( \int_0^r (Du_\beta(Y(s),D_s) \cdot q - L(q))ds \right)(E_\beta(r,t) - E_\beta(r,\tau_h))dr \right] \leq 0
\]
where $D_s$ is the inverse of $E_s$, i.e. $E_{D_s} = s$. Dividing the previous inequality by $h$ and passing to the limit for $h \to 0^+$, by the Dominated Convergence Theorem we get

$$
\partial_t u_\beta(x,t) + \mathbb{E}_{x,t} \left[ \int_0^{t - \infty} \left( \int_0^r (Du_\beta(Y(s), D_s) \cdot q - L(q)) ds \right) \partial_t \mathcal{E}_\beta(r,t) dr \right] \leq 0.
$$

(3.6)

Set $ \Phi(r) = \int_0^r (Du_\beta(Y(s), D_s) \cdot q - L(q)) ds$. Since $\mathcal{E}_\beta$ is a solution of (2.3), then we have (see [3, Lemma 4.2])

$$
\partial_t \mathcal{E}_\beta(r,t) = -D_{(0,t]}^{1-\beta} \partial_\beta \mathcal{E}_\beta (r,t) = -\delta_0(r) \delta_0(t), \quad r \in (0, \infty).
$$

where $D_{(0,t]}^{1-\beta}$ is the Riemann-Liouville derivative of order $1 - \beta$, which is defined for a continuous function $f : [0, t] \to \mathbb{R}$ by

$$
D_{(0,t]}^{1-\beta} f(t) := \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t f(\tau) \frac{1}{(t-\tau)^{1-\beta}} d\tau.
$$

Therefore

$$
\mathbb{E}_{x,t} \left[ \int_0^{t - \infty} \Phi(r) \partial_t \mathcal{E}_\beta(r,t) dr \right] = -\mathbb{E}_{x,t} \left[ \int_0^{t - \infty} \Phi(r) D_{(0,t]}^{1-\beta} \partial_\beta \mathcal{E}_\beta(r,t) dr \right]
$$

$$
- \Phi(0) \delta_0(t) = -\mathbb{E}_{x,t} \left[ D_{(0,t]}^{1-\beta} \left( \int_0^{t - \infty} \Phi(r) \partial_\beta \mathcal{E}_\beta(r,t) dr \right) \right]
$$

$$
= -\mathbb{E}_{x,t} \left[ D_{(0,t]}^{1-\beta} \left( [\Phi(r) \mathcal{E}_\beta(r,t)]_{0}^{t - \infty} - \int_0^{t - \infty} \partial_\beta \Phi(r) \mathcal{E}_\beta(r,t) dr \right) \right]
$$

(3.7)

Since $\lim_{r \to +\infty} \mathcal{E}_\beta(r,t) = 0$ and $\partial_r \Phi(r) = Du_\beta(Y(r), D_r) \cdot q - L(q)$, we have

$$
\mathbb{E}_{x,t} \left[ \int_0^{t - \infty} \Phi(r) \partial_t \mathcal{E}_\beta(r,t) dr \right] = \mathbb{E}_{x,t} \left[ D_{(0,t]}^{1-\beta} \left( \int_0^{t - \infty} \mathcal{E}_\beta(r,t) (Du_\beta(Y(r), D_r) \cdot q - L(q)) dr \right) \right]
$$

$$
= \mathbb{E}_{x,t} \left[ D_{(0,t]}^{1-\beta} (Du_\beta(X(t), t) \cdot q - L(q)) \right] = D_{(0,t]}^{1-\beta} (Du_\beta(x,t) \cdot q - L(q)).
$$

(3.8)

Replacing (3.8) in (3.6), we get

$$
\partial_t u_\beta(x,t) + D_{(0,t]}^{1-\beta} (Du_\beta(x,t) \cdot q - L(q)) \leq 0.
$$

(3.9)

Applying the fractional integral $I_{(0,t]}^{1-\beta} = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{-\beta} d\tau$ to the previous equation, we finally get

$$
\partial_\beta u_\beta(x,t) + Du_\beta(x,t) \cdot q - L(q) \leq 0,
$$

(3.10)

hence the claim (3.3). It follows that $u_\beta$ is an a.e. subsolution of (1.5).

We now prove that $u_\beta$ is the maximal a.e. subsolution. Assume by contradiction that there exist $(x,t) \in Q$, $\varepsilon > 0$ and a a.e. subsolution $v$ of (1.5) such that

$$
u_\beta(x,t) \leq v(x,t) - 2\varepsilon.
$$

(3.11)

It is not restrictive to assume that that $v \in C^1(Q)$ (see Lemma 3.3 at the end of the proof). Let $x$ be a $\varepsilon$-optimal control for $u_\beta$, i.e.

$$u_\beta(x,t) \geq \mathbb{E}_{x,t} \left\{ \int_0^t L(\alpha(s)) dE_s + g(X(0)) \right\} - \varepsilon,
$$
where $X(s)$ is given by the solution of (1.6) corresponding to $\alpha$. By Ito’s formula

$$v(x, t) = E_{x,t} [v(X(t), t)] = E_{x,t} \left[ v(X(0), 0) + \int_0^t dv(X(s), s) \right]$$

$$E_{x,t} \left[ v(X(0), 0) + \int_0^t \partial_x v(X(s), s)ds + \int_0^t \Phi(\cdot) \right]$$

$$\leq E_{x,t} \left[ g(X(0)) + \int_0^t \partial_x v(X(s), s)ds + \int_0^t (L(\alpha(s)) + H(Dv(X(s), s))) \right]$$

$$\leq u_\beta(x, t) + \varepsilon + E_{x,t} \left[ \int_0^t \partial_x v(X(s), s)ds + \int_0^t H(Dv(X(s), s))) \right]$$

$$= u_\beta(x, t) + \varepsilon + E_{x,t} \left[ \int_0^t \partial_x v(X(s), s)ds + \int_0^\infty \int_0^t H(Dv(Y(s), D_s)) (\varepsilon_\beta(r, t) - \varepsilon_\beta(r, 0))dr \right]$$

where $\Phi(r) = \int_0^\infty H(Dv(Y(s), D_s)) ds$. Integrating (2.3) and observing that $\varepsilon_\beta(r, 0) = 0$ for all $r \in [0, +\infty)$, we have

$$\varepsilon_\beta(r, t) = -I_{(0,t]}(\partial_t \varepsilon_\beta(r, \cdot)) + \delta_0(r) H(r),$$

where $\delta_0$ and $H$ are the Dirac function at 0 and the Heaviside function. Performing a computation similar to (3.7), we have that

$$E_{x,t} \left[ \int_0^\infty \Phi(r) \varepsilon_\beta(r, t) dr \right] = -E_{x,t} \left[ I_{(0,t]}(\int_0^\infty \Phi(r) \partial_t \varepsilon_\beta(r, \cdot) dr) \right]$$

$$= E_{x,t} \left[ I_{(0,t]}(\int_0^\infty \partial_t \Phi(r) \varepsilon_\beta(r, \cdot) dr) \right] = E_{x,t} \left[ I_{(0,t]}(\int_0^\infty H(Dv(Y(r), D_r)) \varepsilon_\beta(r, \cdot) dr) \right]$$

Replacing the previous identity in (3.5), we finally get that

$$v(x, t) \leq u_\beta(x, t) + \varepsilon + E_{x,t} \left[ \int_0^t \partial_x v(X(s), s)ds + I_{(0,t]}^\beta [H(Dv(X(s), \cdot))] \right].$$

Since $v$ is a $C^1$ subsolution of (1.5), by applying the operator $I_{(0,t]}^\beta$ to the equation satisfied by $v$ we get

$$\int_0^t \partial_x v(x, s)ds + I_{(0,t]}^\beta [H(Dv(x, \cdot))] \leq 0 \quad \forall (x, t) \in Q.$$

Replacing the previous inequality in (3.12), we get a contradiction to (3.11).

We finally prove that $u_\beta$ satisfies (1.5) a.e. in $Q$. Assume by contradiction that there exists $(x_0, t_0) \in Q$ and $\delta, \varepsilon$ positive such that, defined $U = (x_0 - \delta, x_0 + \delta) \times (t_0 - \varepsilon, t_0 + \varepsilon)$, we have

$$\partial_{(0,t]}^{\beta} u_\beta + H(Du_\beta) \leq -2\delta < 0 \quad \text{a.e. in } U.$$  

Define the function $\phi(t) = (t - (t_0 - \varepsilon)) \chi_{(t_0 - \varepsilon, t_0)} + ((t_0 + \varepsilon) - t) \chi_{(t_0, t_0 + \varepsilon)}$ for $t \in \mathbb{R}$, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$. Set $C_\beta = \varepsilon^\beta / \Gamma(2 - \beta)$ and observe that $\partial_{(0,t]}^{\beta} \phi(t) = (t - t_0 + \varepsilon)^\beta / \Gamma(2 - \beta)$ for $t \in (t_0 - \varepsilon, t_0)$, $\partial_{(0,t]}^{\beta} \phi(t) = C_\beta - (t - t_0)^\beta / \Gamma(2 - \beta)$ for $t \in (t_0, t_0 + \varepsilon)$ and $\partial_{(0,t]}^{\beta} \phi(t) = 0$ otherwise. Hence $\partial_{(0,t]}^{\beta} \phi(t) \leq C_\beta$ for all $t \in \mathbb{R}$. Defined $\bar{u}(x, t) = u_\beta(x, t) + \frac{\phi(t)}{C_\beta}$ for $(x, t) \in Q$, by (13.13) we have

$$\partial_{(0,t]}^{\beta} \bar{u} + H(D\bar{u}) \leq \partial_{(0,t]}^{\beta} u_\beta + H(Du_\beta) + \delta \leq -\delta \quad \text{a.e. in } U.$$
Moreover, if ε is small enough in such a way that \( t_0 - \varepsilon > 0 \), it follows that \( \bar{u}(x,0) = u_\beta(x,0) = g(x) \) and therefore \( \bar{u} \) is an a.e. subsolution of (1.5). Since \( \bar{u}(x_0,t_0) = u_\beta(x_0,t_0) + \frac{\delta}{C_\beta} \), we get a contradiction to the maximality of \( u_\beta \) among the subsolutions of (1.5).

**Lemma 3.3.** Let \( v \) be a subsolution of (1.5). Then there exists a sequence of subsolutions \( v_\delta \in C^1(Q) \) such that \( v_\delta \) tends to \( v \) locally uniformly for \( \delta \to 0 \).

**Proof.** Given a subsolution \( v \), we define \( v(x,t) = v(x,0) \) per \( t \in (-\infty,0) \), hence we can write

\[
\partial_{(0,t)}^\beta v(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial_\tau v(x,t-\tau)}{\tau^\beta} d\tau = \frac{1}{\Gamma(1-\beta)} \int_0^\infty \frac{\partial_\tau v(x,t-\tau)}{\tau^\beta} d\tau.
\]

Let \( v_\delta = v * \rho_\delta(x,t) \) where \( \rho_\delta \) is a standard mollifier in \( \mathbb{R}^{d+1} \), i.e. \( \rho_\delta(\cdot) = \frac{1}{\delta^{d+1}} \rho \left( \frac{\cdot}{\delta} \right) \) with \( \rho \) a smooth function such that \( \text{supp} \rho \subset \{|x| < 1, |t| < 1\} \) and \( \int_{\mathbb{R}^{d+1}} \rho dx dt = 1 \). Then \( v_\delta \to v \) locally uniformly for \( \delta \to 0 \) and by convexity

\[
H(Dv_\delta(x,t)) \leq (H(Dv) * \rho_\delta)(x,t) \quad \forall (x,t) \in Q.
\]

Moreover

\[
(\partial_{(0,t)}^\beta v(x,t) * \rho_\delta)(x,t) = \int_{\mathbb{R}^{d+1}} \frac{1}{\Gamma(1-\beta)} \left( \int_0^\infty \frac{\partial_\tau v(y,s-\tau)}{\tau^\beta} d\tau \right) \rho_\delta(y,s-t) ds dy
\]

\[
= \frac{1}{\Gamma(1-\beta)} \int_0^\infty \left( \int_{\mathbb{R}^{d+1}} \frac{\partial_\tau v(y,s-\tau)}{\tau^\beta} \right) \rho_\delta(y,s-t) ds dy \frac{1}{\tau^\beta} d\tau
\]

\[
= \frac{1}{\Gamma(1-\beta)} \int_0^\infty \left( \int_{\mathbb{R}^{d+1}} \frac{\partial_\tau v(y,r)}{\tau^\beta} \right) \rho_\delta(y,r-t-\tau) dr dy \frac{1}{\tau^\beta} d\tau
\]

\[
= \frac{1}{\Gamma(1-\beta)} \int_0^\infty (\partial_\tau v * \rho_\delta)(x,t-\tau) \frac{1}{\tau^\beta} d\tau = \partial_{(0,t)}^\beta v_\delta(x,t).
\]

Replacing the previous identity and (3.14) in (3.1), we get

\[
\partial_{(0,t)}^\beta v_\delta(x,t) + H(Dv_\delta(x,t)) \leq 0 \quad \forall (x,t) \in Q.
\]

Since \( \|v - v_\delta\|_\infty \leq C\delta \), with \( C \) depending on \( \|Dv\|_\infty \), by subtracting \( C\delta \) to \( v_\delta \), we have that \( v_\delta \) also satisfies (3.2).

**Remark 3.4.** It is well known that Hamilton-Jacobi equations such as (1.1) in general do not admit classical solutions and the correct notion of weak solution is the one of viscosity solution ([2]). A theory of viscosity solutions for a general class of Hamilton-Jacobi equations with Caputo time derivative have been recently developed in [5, 16]. However, in these papers, the connection between Hamilton-Jacobi equations and the corresponding optimal control theory has not been pursued. In Theorem 3.2, we establish this connection for a.e. (sub-)solutions, but we are not able to show the corresponding property for viscosity solutions. Indeed, in the proof of the subsolution and supersolution conditions, applying the Ito’s formula as in the classical viscosity solution argument, we get an equation involving Riemann-Liouville time derivative, see for example (3.9). The delicate point is that, for passing from (3.9) to (3.10), we perform a fractional integration and therefore we need that the equation is satisfied globally, while the notion of viscosity solution is only local.
4 Integral formula and numerical examples

We propose some examples in order to show a comparison between $u$, the solution of the classical Hamilton-Jacobi equation (1.1), and $u_\beta$, the solution of the time-fractional Hamilton-Jacobi equation (1.5). We start rewriting formula (2.7) as

$$u_\beta(x,t) = \int_0^\infty \min_{y \in \mathbb{R}^d} \left\{ rL \left( \frac{x - y}{r} \right) + g(y) \right\} \mathcal{E}_\beta(r,t) dr = \int_0^\infty u(x,r) \mathcal{E}_\beta(r,t) dr$$

(4.1)

where $u$ is given by the formula (1.2) and $\mathcal{E}_\beta(\cdot,t)$ is the PDF of $E_t$. Recalling (2.2), (4.1) can be also rewritten as

$$u_\beta(x,t) = \frac{t}{\beta} \int_0^\infty u(x,s) g_\beta(ts^{-1/\beta}) \frac{ds}{s^{1+\frac{1}{\beta}}}.$$  

(4.2)

We will use formula (4.2) to compute the function $u_\beta$. We assume that the function $u$ is known in order to avoid additional numerical errors due to its approximation which could further affect $u_\beta$ and hide some important properties. Moreover we approximate the integral by a quadrature formula and we employ the Matlab toolbox Stable Distribution [11] to compute $g_\beta(s)$. The toolbox requires 4 parameters ($\alpha$, $\beta$, $\gamma$, $\delta$) in order to compute a stable distribution (see [15]). For the distribution $g_\beta$ corresponding to the value $\beta = 0.4, 0.5, 0.6, 0.8$ used in the tests, we consider the following parameters

| $\beta$ | $\beta_0$ | $\gamma$ | $\delta$ |
|---------|-----------|----------|----------|
| 0.4     | 0.4       | $\gamma_c$ | $\gamma_c - 0.15$ |
| 0.5     | 0.5       | $\gamma_c$ | $\gamma_c$ |
| 0.6     | 0.6       | $\gamma_c$ | $\gamma_c + 0.15$ |
| 0.8     | 0.8       | $\gamma_c$ | $\gamma_c + 0.5$ |

having set $\gamma_c = \frac{1}{2}$. We to briefly describe the algorithm:

**Algorithm 1** Computation of $u_\beta$

1: Define a uniform grid of size $\Delta x$ in space and $\Delta t$ in time;
2: Approximate $\int_0^\infty$ in (4.2) with $\int_{0}^{M}$ for a given parameter $M$;
3: Define a partition of $[0,M]$ of size $\Delta s = \frac{M}{N_{int}}$ given by the points $s_k = k\Delta s, k = 0, \ldots, N_{int}$;
4: Compute the matrix $g_\beta(s_k,t_j)$ by the toolbox Stable Distribution;
5: Compute $\mathcal{E}(s_k,t_j)$ by (2.2);
6: Define $U(i,k) := u(x_i, s_k)$ where $u$ is the solution of (1.1);
7: Compute the integral in (4.2) by midpoint rule;
8: return $u_\beta(x_i, t_j)$.

4.1 Test 1

Consider the Hamilton-Jacobi equation

$$\partial_t u + |Du|^2 = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty),$$

(4.3)

with the initial datum $g(x) = -|x|^2$. Then, the solution of the problem is given by

$$u(x,t) = -(|x| + t)^2 \quad (x,t) \in \mathbb{R} \times (0,\infty).$$
For $\beta \in (0, 1)$, the solution of
\[ \partial_{\alpha, \eta}^\beta u + |Du|^2 = 0 \] (4.4)
with the same initial datum is given by
\[ u_\beta(x, t) = -\int_0^\infty (|x| + r)^2 E_\beta(r, t) dr. \] (4.5)

In order to highlight the impact of $\beta$ on the solution of the time-fractional Hamilton-Jacobi equation, we consider $x = 0$ in (4.5) and study the evolution of $u(0, t)$ for $t \in (0, 2]$ for different values of $\beta$. By (2.1) and (4.5), we have
\[ u_\beta(0, t) = \mathbb{E}[E_\beta^2] = C(2, \beta) t^{2\beta}. \]
Comparing the solutions of (4.3) and (4.4), see figure 1, we see that the effect of the Caputo derivative is to induce a faster evolution for small time, while a slower one as the time increases, a typical effect of the polynomial decay at infinity of the distribution of the subordinator.

![Figure 1: $u_\beta(0, t)$ for $t \in (0, 2]$ and $\beta = 0.4, 0.5, 0.6, 0.8, 1.$](image)

### 4.2 Test 2

We consider equation (4.3) with the initial condition
\[ g(x) = \max\{0, x^2 - 1\}. \] (4.6)

In this case, the solution of (4.3) is given by
\[ u(x, t) = \max\left\{0, \frac{x^2}{1 + 2t} - 1\right\}. \] (4.7)

As before we compute $u_\beta$ by means of formula (4.2). Comparing the behavior of $u_\beta$ and $u$ in figure 2, we
can see that for small times the evolution of \( u_\beta \) is faster than the one of \( u \), since \( u_\beta(x,t) \leq u(x,t) \) and \( \text{supp}(u(t)) \subset \text{supp}(u_\beta(t)) \) for \( t \leq 0.5 \). While the time increases, the evolution of \( u_\beta \) slows down with respect to the one of \( u \). It is also interesting to observe the more regular behavior of \( u_\beta \) in the space variable. Indeed the initial edge of \( g \) is instantaneously smoothed for the fractional equation, while it persists for (4.3) (see Figure 3). We also observe that \( u_\beta \) is not \( C^2 \) in space and a “memory” of the initial edge of \( g \) is preserved in the second derivative.

Figure 3: Space derivative of \( u \) (left) and \( u_\beta \) with \( \beta = \frac{1}{2} \) (right), at different times.
4.3 Test 3

The last test refers to the Hamilton-Jacobi equation

\[ \partial_t u + |Du| = 0 \quad (x, t) \in \mathbb{R}^2 \times (0, \infty), \quad (4.8) \]

which represents the motion at a constant speed of a level curve of the viscosity solution. Even if the Hamiltonian \( H(p) = |p| \) does not satisfy assumptions (2.4), it is well known that formula (1.2) is still valid and it simplifies in \( u(x, t) = \min\{g(y) : |x - y| \leq t\} \). We consider the corresponding time-fractional equation

\[ \partial^\beta_{(0,t)} u + |Du| = 0 \quad (x, t) \in \mathbb{R}^2 \times (0, \infty), \quad (4.9) \]

whose solution if given by \( u_\beta(x, t) = \mathbb{E}_{x,t}[\min\{g(y) : |x - y| \leq E_t\}] \). In the first example, see Figure 4, we compare the evolution of a unitary circle for (4.8) and for (4.9) with \( \beta = \frac{1}{2} \). Given the initial datum \( g(x) = |x|^2 - 1 \), we observe that also in the fractional case its evolution is given by circles of increasing radius, but the propagation speed is not uniform and tends to slows down after some times. A similar property it is also observed in the case of a initial front given by two circles, see Figure 5.

![Figure 4: Evolution of the 0-level sets of \( u \) (left) and \( u_\beta \) (right) with \( \beta = \frac{1}{2} \), for \( t \in [0,15] \).](image)

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Figure 5: Evolution of the 0-level sets of \( u \) (left) and \( u_\beta \) (right) with \( \beta = \frac{1}{2} \) for \( t \in [0, 9] \).

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