Gradient expansion of the non-Abelian gauge-covariant Moyal star-product

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(Dated: May 2015 - July 2017)
Abstract

Motivated by the recent developments of gauge-covariant methods in the phase-space, a systematic method is presented aiming at the generalisation of the Moyal star-product to a non-Abelian gauge covariant one at any order. Such an expansion contains some dressing of the bare particle model by the gauge-fields explicitly, and might serve as a drastically simplifying tool for the elaborations of gauge-covariant quantum transport models. In addition, it might be of fundamental importance for the mathematical elaborations of gauge theory using the strict or deformation quantisation principles. A few already known examples of quantum kinetic theories are recovered without effort as an illustration of the power of this tool. A gauge-covariant formulation taking into account possible geometrical connections in both the position and momentum spaces is also constructed at leading orders, with applications to the generation of gauge-covariant effective theories in the phase-space. This paper is devoted to the pedestrian elaboration of the gradient expansions. Their numerous consequences will be explored in subsequent works.

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Keywords: Moyal star product; non-Abelian gauge theory; effective theory; kinetic theory; quantum transport; semi-classic and quasi-classic approximation; Wigner transform; gradient expansion; phase-space quantum mechanics; strict quantisation; deformation quantisation; topological field theory;
I. INTRODUCTION

There have been considerable renewal and important developments in the quasi-classic methods recently. They are all coming from many temptatives to retain some gauge structures in the gradient expansion. For instance, the prediction of a quantized transconductance in 2D systems under magnetic field (Niu et al., 1985; Thouless et al., 1982) – the quantum Hall effect phenomenology – can be recovered in a quite simple way using a gradient expansion (Mera, 2017; Zubkov, 2016). Gradient expansion methods are also in use for the understanding of topological phenomenologies in condensed-matter systems under interactions (Gurarie, 2011) and their bulk-boundary correspondance (Essin and Gurarie, 2011). In superfluids, the gradient expansion naturally leads to a low-energy effective action of topological nature as well (Volovik, 1987, 1988b; Volovik and Yakovenko, 1989).

The quasi-classic methods take their roots in different attempts to introduce quantum mechanics in the phase-space (Wigner, 1932) using the tools of statistical mechanics (Moyal, 1947) or the mathematical structure behind quantization of the position and momentum coordinates (Groenewold, 1946; Weyl, 1931), reviewed in (Curtright et al., 2014; Hillery et al., 1984; Polkovnikov, 2010; Zachos et al., 2005). Instead of using either the momentum $O (p_1, p_2) = \langle p_1 | \hat{O} | p_2 \rangle$ or the position $O (x_1, x_2) = \langle x_1 | \hat{O} | x_2 \rangle$ representations of a quantum operator $\hat{O}$ related via a Fourier transform in both variables, Wigner introduced the so-called mixed-coordinate Fourier transform, or Wigner transform, defined as

$$O (p, x) = \int d\tau \left[ e^{-ip\tau} O \left( x + \frac{\tau}{2}, x - \frac{\tau}{2} \right) \right] = \int \frac{dp}{2\pi} \left[ e^{ip\cdot x} O \left( p + \frac{p}{2}, p - \frac{p}{2} \right) \right]$$

(1.1)

in term of the center of mass $x = (x_1 + x_2) / 2$ and quasi-momentum $p = (p_1 + p_2) / 2$. There is no change of symbol between the Fourier representations $O (x_1, x_2)$ or $O (p_1, p_2)$ and the Wigner transformed $O (p, x)$. When the correlation encoded in the relative coordinate $\tau = x_1 - x_2$ or the momentum $p = p_1 - p_2$ are supposed to be weak, one may proceed to a perturbative expansion of the Wigner transform. At zero-th order the expansion would give back the pure classical results, and higher orders in the form of an expansion in powers of derivatives (the so-called gradient expansion) then give the quantum corrections to the statistical mechanics (Wigner, 1932).

It should not be surprising that the quasi-classic methods have profound applications in statistical mechanics, and condensed matter problems in particular. In fact, the later describes the phenomenology in the real space of periodic materials having inherent band structures. Also, many properties of a statistical system close to its equilibrium state present generally long wavelength scaling, to be compared with the atomistically ranged Fermi wavelength. Models taking into account both the space and momentum variations are thus of great interest there, and the gradient expansion is a powerful tool to compute the properties of a system close to its equilibrium. Methods of phase-space quantum mechanics are in fact exploited in one way or another in many problems of quantum transport (Abrikosov et al., 1963; Haug and Jauho, 2010; Kadanoff and Baym, 1962; Kopnin, 2001; Langenber and Larkin, 1986; Rammer and Smith, 1986; Rammer, 2008). Moreover, the quasi-classic methods in the phase-space (Agarwal and Wolf, 1970a,b,c) are also prominent in the understanding of the quantum-to-classical transition and quantum optics (Gardiner and Zoller, 2004; Walls and Milburn, 1994; Zachos et al., 2005).

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The success of the Wigner transform certainly lies in the complete understanding of its algebra. Indeed, once the Wigner transforms of two operators $O_{1,2}(x_1, x_2)$ are known in the form of $O_{1,2}(p, x)$, one can take the product of these two operators in the phase-space following the general recipe of the Moyal-Groenewold product

$$\int dx \int dy \left[ e^{-ip \cdot y} O_1 \left( x + \frac{p}{2}, y \right) O_2 \left( y, x - \frac{p}{2} \right) \right] = O_1(p, x) \star_0 O_2(p, x) \tag{1.2}$$

where the star $\star_0$-product (with 0-index meaning there is no gauge-potential associated to this product) has an explicit form known to all orders (Groenewold, 1946; Hillery et al., 1984)

$$\star_0 = \exp \left[ \frac{i}{2} \left( \partial_{\omega} \partial_p - \partial^p \partial_{\omega} \right) \right] \tag{1.3}$$

with the notation convention that $O \partial^{\dagger}_{\omega} \equiv \partial_{\omega} O$, i.e. the $\dagger$-derivative applies to the left. A gradient expansion is thus the calculation of the different orders of the exponential series of the $\star_0$ operation. This simplification tool allows to quickly calculate many universal properties of effective theories (Volovik, 2003), in addition to calculate kinetic models for statistical systems (Haug and Jauho, 2010; Rammert, 2008). For a recent overview on the subject including many relevant references, see (Curtright et al., 2014).

There have been recent trends to try to generalise the gradient expansion towards modern aspects of condensed matter problems. On the one hand, a few attempts have been done towards a systematic expansion including Berry phase effects of bands structure (Gosselin et al., 2007, 2008a,b, 2006; Gosselin and Mohrbach, 2009; Shindou and Balents, 2006, 2008; Shindou and Imura, 2005; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011) following the seminal works in (Chang and Niu, 1995, 1996; Sundaram and Niu, 1999), see reviews (Nagaosa et al., 2010; Xiao et al., 2009). Weyl semimetals also provide an important platform to use such models (Chernodub and Zubkov, 2017; Son and Spivak, 2013; Son and Yamamoto, 2012, 2013; Stephanov and Yin, 2012; Stone and Dwivedi, 2013). On the other hand, spin-orbit effects can be handle as a non-Abelian gauge theory (Berche and Medina, 2013; Fröhlich and Studer, 1993; Leurs et al., 2008), which serves as a guiding principle to the establishment of a kinetic theory when charge and spin are treated in a gauge-covariant way (Bergeret and Tokatly, 2014; Gorini et al., 2010; Konschelle, 2014). This later approach uses the so-called gauge-covariant Wigner transform, defined as (to be justified further in section III)

$$O(p, x) = \int dx \left[ e^{-ip \cdot x} U \left( x, x + \frac{p}{2} \right) O \left( x + \frac{p}{2}, x - \frac{p}{2} \right) U \left( x - \frac{p}{2}, x \right) \right] \tag{1.4}$$

where the parallel transport operator of the gauge potential $A_\alpha$ (the greek indices run through the four space-time coordinates, and the scalar product $p \cdot x = p^\alpha x_\alpha = p \cdot x - \omega t$ can be eventually taken Lorentz invariant, with $p$ the momentum vector in space, $x$ the space position vector, $\omega$ the frequency and $t$ the time coordinates)

$$U(b, a) = \text{Pexp} \left[ i \int_a^b dz \cdot A(z) \right] = \text{Pexp} \left[ i \int_0^1 ds \left[ (b - a) \cdot A(\tau_s) \right] \right] \tag{1.5}$$

along the path-ordered straight line $\tau_s = a + (b - a)s$ from $a$ to $b$ collapses the possible gauge transformations of the Wigner transform $O(p, x)$ to its position variable, i.e. the
function $O(p, x)$ transforms as $O'(p, x) = R(x) O(p, x) R^{-1}(x)$ under the gauge transformation $A'_\alpha(x) = R(x) A_\alpha(x) R^{-1}(x) - iU \partial_\alpha U^{-1}$ of representation $R(x)$. Because (1.4) reduces to the usual Wigner transform (1.1) in the limit of vanishing gauge fields $A \to 0$, one did not change the notation for the Wigner transform $O(p, x)$ of the operator $O(x_1, x_2)$. When not otherwise stated, a Wigner transform $O(p, x)$ refers to the general definition (1.4).

When the gauge theory involves only Abelian groups, one can drastically simplify the gauge-covariant Wigner transform (1.4) since in that case the two parallel transport operators (1.5) collapses to a single one

$$U_{\text{Abel.}} \left( x - \frac{r}{2}, x + \frac{r}{2} \right) = \exp \left[ -i \int_{x - r/2}^{x + r/2} A(r) \cdot dr \right] \quad (1.6)$$

such that the Wigner transform (1.4) reduces to the simpler expression

$$O(p, x) = \int dx \left[ e^{-i p \cdot r} U_{\text{Abel.}} \left( x - \frac{r}{2}, x + \frac{r}{2} \right) O \left( x + \frac{r}{2}, x - \frac{r}{2} \right) \right] \quad (1.7)$$

and some covariant formulations of the physical quantities in the phase-space follow without much efforts, see (Altshuler and Ioffe, 1992; Best et al., 1993; Bialynicki-Birula, 1977; Bialynicki-Birula et al., 1991; Javanainen et al., 1987; Kelly, 1964; Kopnin, 1994; Kubo, 1964; Langreth, 1966; Levanda and Fleurov, 1994, 2001; Luttinger, 1951; Serimaa et al., 1986; Stratonovich, 1956; Swieciecki and Sipe, 2013; Zachos and Curtright, 1999) for instance. A gauge-covariant Moyal expansion can even be established in the Abelian case (Iftimie et al., 2009; Karasev and Osborn, 2002, 2004, 2005; Mantoiu and Purice, 2004, 2005; Mueller, 1998), with many applications in statistical physics problems (Lein, 2010; Mueller, 1998).

In the contrary, calculations using the non-Abelian version of the gauge-covariant Wigner transform (1.4) are usually laborious and cumbersome. For some examples, one can refer to (Bergeret and Tokatly, 2014; Gorini et al., 2010; Konschelle, 2014) and the recent pedagogical review (Raimondi et al., 2016) for applications in condensed matter, and (Elze et al., 1986a,b; Winter, 1984, 1985) for the original works developing a transport theory of the quark-gluon plasma, or (Elze and Heinz, 1989; Weigert and Heinz, 1991) for early reviews. A crucial step to the understanding of its fundamental properties, and to the simplification of its numerous modern uses, would be to establish the non-Abelian gauge-covariant generalisation of the Moyal $\star$-product. Unfortunately, it is already clear from the expression of the Abelian gauge-covariant Moyal product, see e.g. (Mueller, 1998), that the $\star$-product can not be written in closed form if one wants to preserve the gauge structure, unless defining extra mathematical structures for which the exponential form of the Moyal product can be obtained in a symbolic fashion, like in (Bordemann et al., 1998, 1999; Iftimie et al., 2009; Karasev and Osborn, 2002, 2004, 2005; Mantoiu and Purice, 2004, 2005) where some involved mathematical structures are introduced. So we would be happy to find a systematic method to generate the Moyal expansion series order by order in a non-Abelian gauge-covariant way. Even if this construction looks cumbersome at first, it will help simplifying ulterior calculations, and understanding the gauge structure of the models.

This study is dedicated to such a construction. I will present in the following sections a method allowing to establish the first few terms of the non-Abelian gauge-covariant Moyal
\[
\int dx \int dy \left[ e^{-ip \cdot \frac{x}{2}} U(x, x + \frac{r}{2}) O_1 \left(x + \frac{r}{2}, y\right) O_2 \left(y, x - \frac{r}{2}\right) U \left(x - \frac{r}{2}, x\right) \right] = \\
= O_1 (p, x) \ast O_2 (p, x) \tag{1.8}
\]

in the form
\[
O_1 (p, x) \ast O_2 (p, x) = O_1 (p, x) O_2 (p, x) + \frac{i}{2} \left( \frac{\partial O_1}{\partial x^\alpha} \frac{\partial O_2}{\partial p_\beta} - \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial x^\alpha} \right) + \\
- \frac{i}{8} \left(F_{\alpha\beta} (x) \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial p_\beta} + 2 \frac{\partial O_1}{\partial p_\alpha} F_{\alpha\beta} (x) \frac{\partial O_2}{\partial p_\beta} + \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial p_\beta} F_{\alpha\beta} (x) \right) + \\
+ \frac{1}{12} \left( \frac{\partial O_1}{\partial p_\gamma} \frac{\partial F_{\alpha\beta}}{\partial x^\alpha} \frac{\partial O_2}{\partial p_\beta} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \frac{\partial F_{\alpha\beta}}{\partial p_\gamma} \frac{\partial O_2}{\partial p_\beta} \right) + \\
+ \frac{1}{24} \left[ \frac{\partial F_{\alpha\beta}}{\partial x^\alpha}, \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \frac{\partial O_2}{\partial p_\gamma} - \frac{\partial O_1}{\partial p_\gamma} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} \right] + O (\partial^4) \tag{1.9}
\]

up to the fourth order in gradient, and where we define the gauge field
\[
F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i [A_\alpha, A_\beta] \tag{1.10}
\]

and the covariant derivative
\[
\frac{\delta O}{\partial x^\alpha} = \frac{\partial O}{\partial x^\alpha} - i [A_\alpha (x), O (p, x)] \tag{1.11}
\]

applied to matrix. The fourth order term is also given in (3.44).

This result allows to drastically simplify the constructions of the different versions of the gauge-covariant kinetic theories. Indeed, one can compare the length of the demonstration of (1.9) given below with the relative simplicity of its utilisation in section II. It might also be of importance in the understanding of the so-called deformation quantisation (Bayen et al., 1977, 1978a,b; Kontsevich, 2003) or strict quantisation (Landsman, 1998, 2005; Rieffel, 1993, 1994), since the gauge-covariant Moyal expansion (1.9) will be shown to dress the bare theory with a gauge structure in section II.A. Indeed, one way of applying the gauge-covariant Moyal product is to start from the action of a bare particle, for instance the Dyson’s propagator
\[
G^{-1} (x_1, x_2) = \left( i \partial_t + A_0 + \frac{(\partial_x - i A)^2}{2m} + \mu \right) \delta (x_1 - x_2) \tag{1.12}
\]

for a Galilean relativistic free particle in a gas of chemical potential \( \mu \), and to get the gauge-covariant Wigner transform of it, which is nothing but
\[
G^{-1} (p, x) = \omega - \frac{p^2}{2m} + \mu \tag{1.13}
\]
i.e. the bare classical action. Then the gauge-covariant Moyal expansion (1.9) applied to the Dyson’s equation restaures the known transport theories in normal metals (Gorini et al., 2010) and superconductors (Bergeret and Tokatly, 2014; Konschelle, 2014) dressed with Abelian and/or non-Abelian gauge fields. This will be the subject of section II.B, where a
few already known gauge-covariant transport equations will be derive using (1.9) directly instead of dealing with the gauge-covariant Wigner transform (1.4) in a cumbersome way.

The gradient expansion (1.9) also leads to direct applications in deriving universal properties of statistical systems. In section II.C we derive the Chern-number quantisation of the transconductance in a non-interacting system, and its generalisation to interacting system in the form of a topological invariant for the Green’s functions. This well known result (Niu et al., 1985; Thouless et al., 1982) has been recovered recently using a gauge-flat gradient expansion (Mera, 2017; Zubkov, 2016) which, by properties of the Abelian electromagnetic gauge field, can be restated in a gauge-covariant fashion quite easily. We will show that the gradient expansion (1.9) directly gives the result without effort. The appearance of Pontryagin number in superfluids, first established in a series of paper by Volovik and Yakovenko (Volovik, 1987, 1988b; Volovik and Yakovenko, 1989), is also reviewed in section II.C in a quite straightforward way.

We then turn to the explicit demonstration of the gauge-covariant gradient expansion (1.9) up to the fourth order in section III. The demonstration uses a trick introduced in (Weigert and Heinz, 1991) of translating the gauge-covariant Wigner transform towards the Fock-Schwinger gauge (also called relativistic Poincaré, radial, quasi-canonical or fixed-point gauge fixing). In this gauge, the product of two operators can be recast in the form of the flat-space Moyal expansion (1.2). Then one has to transport back this equation to the gauge-covariant formulation of phase-space functions. That is, one has to understand how the different orders of space derivatives in the Fock-Schwinger gauge translate to the gauge-covariant Wigner transform. This is the work accomplished in section III.

Since there have been proposals to get some gauge-covariant expressions in the full phase-space when both \( D_x = \partial_x - iA_x (x) \) and \( D_p = \partial_p - iA_p (p) \) covariant derivatives pop out of the non-covariant phase-space method of effective diagonalisation of the band structure (Gosselin et al., 2007, 2008a,b, 2006; Gosselin and Mohrbach, 2009; Shindou and Balents, 2006, 2008; Shindou and Imura, 2005; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011), a few words about the possible realisations of the gradient expansion retaining the full geometry of the phase-space might be welcome. In fact, such an approach nicely generalises the demonstration done in section III. Section IV is dedicated to such a construction, generalising and unifying a few previous works.

A. Conventions

Before turning to the application of the gauge-covariant Moyal expansion (1.9), a few words about mathematical rigor are in order. Indeed, there will be no attempt toward rigorously define the conditions under which the expansion converges, nor to the rigorous definition of the gauge-covariant Wigner transform and its associated Weyl mapping from the classical to the quantum world. Reader interested in more rigorous construction might consult e.g. (Alvarez-Gaumé and Wadia, 2001; Estrada et al., 1989; Landsman, 1993; Robson, 1996; Szabo, 2003; Wallet, 2009) for general construction, and especially (Bordemann et al., 1998, 1999) for the construction of a covariant star-product, and references therein. (Sugimoto et al., 2006; Sugimoto and Nagaosa, 2012) also claim to have used a non-Abelian gauge-covariant method to extract the conservation laws of the spin degree of freedom, but I found nothing to compare with the approach presented below. In contrary, a covariant method have been proposed to compute the effective action in the high-energy sector (Banin and Pletnev, 2001; Pletnev and Banim, 1999; Salcedo, 2007) following similar recipes than
I. Nevertheless, these studies detail only the situation when the gauge field has effect in the position space, and there is no explicit construction of the star product as I do below. In addition, a simpler examination of the method might be welcome.

In the following, I propose a pedestrian approach to the establishment of the first few terms of the Moyal expansion up to the fourth order, and the results of this study can be seen as a quasi-empirical construction, hopefully useful for tackling modern problems of condensed matter, more generally for any problem when quantisation of the gauge-fields is not required. The aim is to provide a demonstration of (1.9) that a graduate student having basic knowledge on gauge theory and Moyal product (1.2) can follow.

A few words on the notation convention: The space-momentum Wigner correspondence is done using the definition

\[ O(p, x) = \int dx \left[ e^{-ip\cdot x} O(x + \frac{x}{2}, x - \frac{x}{2}) \right] = \int dp \left[ e^{ip\cdot x} O(p + \frac{p}{2}, p - \frac{p}{2}) \right] \]  

(1.14)

and the time-frequency correspondence using the Wigner transform follows from the convention

\[ O(\omega, t) = \int dt \left[ e^{i\omega t} O(t + \frac{t}{2}, t - \frac{t}{2}) \right] = \int d\omega \left[ e^{-i\omega t} O(\omega + \frac{\omega}{2}, \omega - \frac{\omega}{2}) \right] \]  

(1.15)

where the integrated variables, corresponding to relative coordinates, are noted with German alphabet. The covariant derivatives are written as

\[
\begin{align*}
\frac{Df}{\partial x^\alpha} &= \frac{\partial f}{\partial x^\alpha} - iA_\alpha(x) f \\
\frac{Df}{\partial t} &= \frac{\partial f}{\partial t} - iA_0(x) f \\
\frac{Df}{\partial \omega} &= \frac{\partial f}{\partial \omega} - iA(x) f
\end{align*}
\]

(1.16)

for a function \( f(x,t) \) and space-time dependent gauge potential \( A_\alpha(x,t) \). The correspondence is the same for

\[
\frac{\mathcal{D}F}{\partial x^\alpha} = \frac{\partial F}{\partial x^\alpha} - i[A_\alpha(x), F(p,x)]
\]

(1.17)

for a matrix (this covariant derivative appears only in the phase space). As a consequence, the Moyal \( \diamond_0 \) product reads

\[
\diamond_0 = \exp \left[ \frac{i}{2} \left( \frac{\partial^i}{\partial x} - \frac{\partial^i}{\partial p} \frac{\partial}{\partial x} - \frac{\partial^i}{\partial \omega} \frac{\partial}{\partial \omega} - \frac{\partial^i}{\partial t} \frac{\partial}{\partial t} \right) \right]
\]

(1.18)

where we restate the dimension by multiplying every \( p \)-derivative with a \( \hbar \). The context and the notations make clear whether we use the complete relativistic Wigner representation \( O(p, x, \omega, t) \), the space-momentum one \( O(p,x) \) (also written \( O(p,x,t_1,t_2) \) or \( O(p,x,\omega_1,\omega_2) \) for time-dependent problems) or the time-frequency one \( O(\omega,t) \) (also written \( O(\omega,t,x_1,x_2) \) or \( O(\omega,t,p_1,p_2) \) for space-dependent problems).

I put \( \hbar = 1 \) everywhere. To restate the dimension, one can add a \( \hbar \) term in front of all \( \partial_p \)-derivative. The gauge structure has natural dimension of inverse of length.

The notations adapt themselves to the situation of a \( p \)-dependent gauge potential \( A_\alpha(p) \) and covariant derivative \( \mathcal{D}^\alpha F = \partial^\alpha F - i[A_\alpha(p), F(p,x)] \), with the convention \( \partial^\alpha f = \partial f / \partial p_\alpha \) for the momentum derivative. See section IV.A for more details.

There is no try to discuss the symplectic structure of the phase-space, but positions of the indices are obvious in section IV.B.
B. Emergence of Gauge structures

Before entering the heart of the calculations, I would like to introduce some gauge structures naturally emerging in condensed matter systems. The reason is first of all pedagogical. I will discuss essentially classical systems in the following, and the star product (1.9) will be shown to dress the classical theory with some gauge structures. That is, the covariant star product exhibits some gauge connections and gauge fields, but their origin and exact expression in terms of microscopic degrees of freedom are hidden in the writing of $A_\alpha$ (in the derivatives $\mathfrak{D}_\alpha$) and $F_{\alpha\beta}$. It might then appear to the reader that this dressing comes from pure black magic, or even worse, that the proposed construction is of pure scholar interest without any physical ground. To explain why this is not the case is the purpose of this section, aiming at introducing the origin of the gauge structure from microscopic considerations.

Indeed, the characteristics of the gauge structures of interest for semi-classic arguments (e.g. which group is relevant for the description of a given phenomenology) inherited from (1.9) are in practise emerging from microscopic considerations. In a way, the semi-classic expansion I propose below inherits the geometry of the quantum state, but the method I will develop is blind with respect to this microscopic origin once the quantities $A_\alpha$, $\mathfrak{D}_\alpha$ and $F_{\alpha\beta}$ are known. As a matter of fact, physical results can be expressed in terms of the gauge connections and gauge fields only, without any mention of their origin, so to write everything in terms of $A_\alpha$, $\mathfrak{D}_\alpha$ and $F_{\alpha\beta}$ should not be such surprising. In this section, I introduce quickly the associated microscopic considerations, and refer to the relevant studies for more details. Then in the sequels of this paper, I will suppose that one already knows the specificity of the gauge structure, i.e. I will suppose one knows $A_\alpha$, $\mathfrak{D}_\alpha$ and $F_{\alpha\beta}$ from the outset. I will then show that a covariant gradient expansion exists, and built it in terms of the gauge connection and gauge fields without any more mention of their microscopic origin.

As a first example, nothing better than the celebrated electromagnetism. In that situation, it comes as a natural reflex to dress the bare particle via the minimal substitution $p_i \to p_i - A_i^{\text{e.m.}}$, with $A_i^{\text{e.m.}} = eA_i$ the so-called vector gauge potential, with gauge charge $e$. In that case, the associated gauge field is Abelian and is usually called the magnetic field in the space sector $F_{ij} = \varepsilon^{ijk}B_k$, when latin indices run along the three space dimensions, $i \in \{1,2,3\}$ and $\varepsilon^{ijk}$ is the completely antisymmetric symbol, and the electric field $F_0^i = E^i$ in the time sector having 0 for index. When putting $\mathcal{A}_\alpha^{\text{e.m.}}$ instead of $A_\alpha$ in (1.9)-(1.11), the Moyal product will dress the bare classical system with the electromagnetic interaction. Many simplifications follows from the Abelian nature of electromagnetism.

More involved gauge structures, namely non-Abelian ones, naturally show up in the realm of high energy physics. A pedagogical introduction to such physics can be find in (Aitchison and Hey, 2004). Once more time, substitution of the gauge structure in (1.9)-(1.11) will dress the bare theory with the adapted geometry. I now come back to condensed matter problems, and introduce (perhaps) less trivial gauge structures in the real and momentum space.

What appeared recently as an interesting change of paradigm in condensed matter problems was the possibility to treat complicated spin couplings using the language of gauge theory (Berche and Medina, 2013; Fröhlich and Studer, 1993; Leurs et al., 2008). For instance, the spin-orbit interaction naturally appears in a free electron gas of effective mass $m$, momentum $p$ and chemical potential $\mu$ without inversion symmetry following the Hamil-
tonian description
\[
H_{\text{s.o.}} = \frac{p^2}{2m} - \mu - \alpha_i^a \left( x \right) \frac{\sigma^a p_i}{2} = \left( p_i - m \alpha_i^a \left( x \right) \sigma^a / 2 \right)^2 - \mu_0 \tag{1.19}
\]
where repeated indices are summed, and \( \sigma^a \) are the Pauli matrices which span the spin algebra. The spin-orbit interaction is encoded in the \( p \)-independent tensor \( \alpha_i^a \). In the second expression \( \mu_0 = \mu - m \left( \alpha_i^a \right)^2 / 4 \) is a rescaling of the chemical potential which has no physical consequence. Now, we can formally define \( \mathcal{A}^{\text{s.o.}}_i = m \alpha_i^a \sigma^a / 2 \) as a gauge potential. The associated gauge structure corresponds to the geometry of rotations, here SU (2) for electronic spin, hence \( \mathcal{A}^{\text{s.o.}}_i \) is non-Abelian. When \( A_i \) is replaced with \( \mathcal{A}^{\text{s.o.}}_i = m \alpha_i^a \sigma^a / 2 \) in the Moyal expansion (1.9), the gradient expansion will describe the phenomenology of the spin interaction in condensed matter systems. Note that the usual shift \( H \rightarrow H - \mathcal{A}^{\text{s.o.}}_0 \) with \( \mathcal{A}^{\text{s.o.}}_0 = B^a \sigma^a / 2 \) can describe the usual Zeeman coupling between the spin and the electromagnetic field. Such a situation can also be described following the present study: it is sufficient to either deal with quasi-static situations when \( \mathcal{A}^{\text{s.o.}}_0 \) is time-independent, or to extend the gradient expansion to deal with adiabatic contributions, when one uses both the position-momentum and the time-frequency Wigner transformations, for instance (1.18) when there is no gauge-field. The notations introduced in section I.A explain how to deal with these situations.

The main merit of the gauge theory is then to be able to deal, e.g., with charge and spin degrees of freedom using the substitution \( A_i = \mathcal{A}^{\text{s.m.}}_i + \mathcal{A}^{\text{s.o.}}_i \) for instance. This is the strategy followed in order to deal with spintronics applications either in semi-conducting systems (Gorini et al., 2010; Raimondi et al., 2016) or superconducting systems (Bergeret and Tokatly, 2014; Konschelle, 2014).

As the last example of this short review, I would like to introduce the gauge structure in the momentum space, which attracted much interest in the later years, especially in the field of topological materials (Bérard and Mohrbach, 2004; Son and Yamamoto, 2012; Stephanov and Yin, 2012; Sundaram and Niu, 1999), see also (Bamler, 2016; Nagaosa et al., 2010; Xiao et al., 2009) for reviews. To understand how a gauge structure emerges from the momentum space, let us discuss a simplified model, with Hamiltonian
\[
H = H_0 \left( p \right) + V_0 x \tag{1.20}
\]
where \( H_0 \left( p \right) \) can be thought as the Hamiltonian representation of a band structure, and \( V_0 \) some perturbative potential. The position dependency of the perturbation is supposed to be linear, so \( V_0 \) is a constant matrix in (1.20). The model (1.20) is related to physical systems of interest: the perturbation \( V_0 x \) mimics a constant electrostatic contribution applied to a solid state or an atom, for instance, or some strain applied to the atom lattice.

Suppose next that the unitary transformation \( R \left( p \right) \) diagonalizes the band structure, i.e. \( R \left( p \right) H_0 \left( p \right) R^\dagger \left( p \right) = \mathcal{E} \left( p \right) \) is a diagonal matrix. Then one has
\[
U H U^\dagger = \mathcal{E} \left( p \right) + V_0 \left( p \right) R \left( p \right) x R^\dagger \left( p \right) \tag{1.21}
\]
where the perturbative matrix \( V_0 \left( p \right) = R \left( p \right) V_0 R^\dagger \left( p \right) \) might have acquired a momentum dependency under the transformation (since we are in the interaction representation now). Since \( x \) and \( p \) do not commute in quantum mechanics, one has (a similar argument has been used in (Gosselin et al., 2006))
\[
R \left( p \right) x R^\dagger \left( p \right) = x + R \left( p \right) [x, R^\dagger \left( p \right)] = x + iR \frac{\partial R^\dagger \left( p \right)}{\partial p} \tag{1.22}
\]
using the formal relation \( [x, f(p)] = i \partial_p f \) for any analytic function \( f \) of the momentum, and with \( [x, p] = i \). Then under a change of representation of the model (1.20), the system acquires a shift \( \hat{V}_0 R \partial_p R \dagger \) with origin in the momentum space. This is not yet a genuine gauge structure, since the shift \( R \partial_p R \dagger \) is curvature-free and has no physical effect (in the language of gauge theory, such a contribution is called a pure gauge). Nevertheless, if one restricts the total Hilbert space span by the model (1.20) in a way or an other using the projection \( \mathcal{P} \), usually in the low energy sector of an effective theory, the projected contribution \( \mathcal{A}^p = \mathcal{P} [R \partial_p R \dagger] \) to this restriction might become a true connection, and might have influences on the dynamics of the system. This is how the gauge structure emerges in the momentum space. One can deal also with the time-dependent Schrödinger equation and the adiabatic theorem (instead of the effective theory in the band structure, see (Moore, 2017) for a recent short review), in which case a frequency-like gauge potential appears, in addition to the momentum-like gauge structure. Once the gauge-potentials \( \mathcal{A}^p = \mathcal{P} [R \partial_p R \dagger] \) is inserted into the covariant gradient expansion of section IV.A (where we will deal with momentum covariant structures), one no more need the reference to the microscopic model in (1.22), that is, there is no more need to deal with non-commuting phase-space variables as these later ones will be naturally encapsulated in the Moyal expansion.

Note in passing that the shift (1.22) might not exist for interactions of polynomial higher orders in position, since the commutator (see (Transtrum and Van Huele, 2005) for the proof)

\[
[f(p,x), g(p,x)] = \sum_{k=1}^{\infty} \frac{(-i)^k}{k!} \left( \frac{\partial^k g}{\partial x^k} \frac{\partial^k f}{\partial p^k} - \frac{\partial^k f}{\partial x^k} \frac{\partial^k g}{\partial p^k} \right)
\]

of two analytic functions of the momentum and the position gets a clear gauge structure (once projected to some effective sub-spaces) only at linear order in space and/or momentum. Nevertheless, the apparent similarity between (1.23) and the Moyal commutator \([f, g]_\star = f \star g - g \star f\) using the \(\star\)-product (1.2) tends to show how phase-space gauge structures might emerge at the leading order of a semi-classic expansion of effective models, as discussed in (Gosselin et al., 2007, 2008a,b, 2006; Gosselin and Mohrbach, 2009; Shindou and Balents, 2006, 2008; Shindou and Imura, 2005; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011) in different contexts, usually not using covariant Wigner transform in condensed matter contexts.

We have thus seen that linear-in-momentum interaction naturally leads to an emergent position-dependent gauge structure (the example (1.19) of spin-orbit coupling), whereas a linear-in-position interaction naturally leads to an emergent momentum-dependent gauge structure in effective models (the example (1.22) is a prototype of such models). In the following, we will neglect the origin of the gauge structure. Instead, we will suppose there is a gauge structure at the microscopic level, and we will learn how to include this gauge structure at the semi-classic level in terms of \( A_\alpha \) and \( F_{\alpha\beta} \) without worrying anymore about their origin. Section IV.B introduces a covariant gradient expansion valid in the entire phase-space when both a position dependent and a momentum dependent gauge potentials are present.

II. THE COVARIANT MOYAL PRODUCT IN ACTION

In this section, a few examples of the uses of the Moyal expansion (1.9) are presented, in order to show how powerful this tool is. These examples are all already known in the
In the following, we mainly review the situation in condensed matter problems, where the gauge-covariant Wigner transform proved to be a convenient tool to unify the treatment of charge and spin degrees of freedom in terms of gauge principles applied to quantum kinetic equations (Bergeret and Tokatly, 2014; Gorini et al., 2010; Konschelle, 2014; Raimondi et al., 2016), see section II.B. In addition, we easily generalises the phenomenology of the quantum Hall effect to non-Abelian structure (namely for spin current) in section II.C. We start by explaining one important property of the Moyal product (1.9): it dresses the classical theory with quantum corrections coming from the gauge structure in section II.A.

A. Non-commutative dressing by the gauge structure

Let us start with the expansion of the momentum operator, namely, the calculation of the star product (1.9)

$$p_\alpha \star O_2 (p, x) = p_\alpha O_2 (p, x) - \frac{i}{2} \frac{\partial O_2}{\partial x^\alpha} + \frac{i}{8} \left( 3 F_{\alpha\beta} \frac{\partial O_2}{\partial p_\beta} + \frac{\partial O_2}{\partial p_\beta} F_{\alpha\beta} \right) + \cdots$$

up to the first order for convenience. We realise that this is nothing but the gauge-covariant quasi-classic expansion of the covariant derivative obtained after long algebra in e.g. (Elze and Heinz, 1989; Konschelle, 2014) (up to differences in conventions). So we have just shown that,

$$\int dt \left[ e^{-ip_\alpha U} \left( x, x + \frac{r}{2} \right) D_\alpha \left( x + \frac{r}{2}, x - \frac{r}{2} \right) O \left( x + \frac{r}{2}, x - \frac{r}{2} \right) U \left( x - \frac{r}{2}, x \right) \right] = p_\alpha \star O (p, x)$$

namely, the gauge-covariant gradient expansion of the covariant derivative $D_\alpha (x) O (x, y) = \partial O / \partial x^\alpha - i A_\alpha (x) O (x, y)$ equals the $\star$-product between $p_\alpha$ and the Wigner transform $O (p, x)$ of $O (x_1, x_2)$. In particular, one notes that, formally

$$\int dt \left[ e^{-ip_\alpha U} \left( x, x + \frac{r}{2} \right) D_\alpha \left( x + \frac{r}{2}, x - \frac{r}{2} \right) U \left( x - \frac{r}{2}, x \right) \right] = p_\alpha$$

in the Moyal product (1.9). Hence the $\star$-product dresses the theory with covariant contributions $D_\alpha$ and $F_{\alpha\beta}$ coming from the gauge structure of the microscopic theory, even if one starts with a bare situation, i.e. the $\star$ operation in (2.1) takes the bare classical $p_\alpha$ and $O (p, x)$ as inputs and outputs a covariant expression.

This is perhaps the most important property of (1.9): it maps two gauge-less classical quantities to their semi-classic counterparts and provides the dressing by the internal gauge structure of the microscopic theory.

In the same way, one has

$$\int dt \left[ e^{-ip_\alpha U} \left( x, x + \frac{r}{2} \right) O_1 \left( x + \frac{r}{2}, x - \frac{r}{2} \right) D_\alpha^\dagger \left( x - \frac{r}{2}, x \right) U \left( x - \frac{r}{2}, x \right) \right] =$$

$$= O_1 (p, x) \star p_\alpha = p_\alpha O_1 (p, x) + \frac{i}{2} \frac{\partial O_1}{\partial x^\alpha} + \frac{i}{8} \left( 3 F_{\alpha\beta} \frac{\partial O_1}{\partial p_\beta} + \frac{\partial O_1}{\partial p_\beta} F_{\alpha\beta} \right) + \cdots$$

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with $D^\dagger_{\alpha}(x) = \partial^\dagger/\partial x^\alpha + iA_{\alpha}(x)$. Note the change in sign in front of the gauge-field term due to its antisymmetry, and the balance between the prefactors of $\partial_p O_1$ and $\partial_p O_2$ in expressions (2.1) and (2.4).

To see the emergence of the non-commutative geometry associated with (1.9), we can also calculate the following Moyal commutation relations

\[
\begin{align*}
[p_{\alpha}, p_{\beta}]_* &= p_\alpha \ast p_\beta - p_\beta \ast p_\alpha = -iF_{\alpha \beta}(x) \\
[x^\alpha, p_\beta]_* &= x^\alpha \ast p_\beta - p_\beta \ast x^\alpha = i\delta^\alpha_\beta \\
[x^\alpha, x^\beta]_* &= x^\alpha \ast x^\beta - x^\beta \ast x^\alpha = 0
\end{align*}
\]

(2.5)

where the gauge field appears in front of the momentum derivative. One more time, such a commutation relation is usual in the quantum world since $[D_{\alpha}(x), D_{\beta}(x)] = -iF_{\alpha \beta}$, inducing a non-commutative geometry in the phase space. Such possibilities have been well explored in the case of the quantum Hall effect, see e.g. (Bellissard et al., 1994) and references therein. Here the gauge field can be non-Abelian as well. Of course (1.9) is truncated at some order in the derivatives, but it is clear that (2.5) is correct, since these higher orders will never appear in higher derivatives of $p_\alpha$ or $x^\alpha$. Many mathematical constructions start in practice with supposing commutations relations of the form (2.5). Here they are the consequences of the Moyal product (1.9).

**B. Quantum kinetic theory**

We now turn to the establishment of quantum kinetic theory using (1.9). The starting point of such constructions are the Dyson’s equations

\[
\int dy \int d\tau \left[ G^{-1} (x_1, t_1; y, \tau) G (y, \tau; x_2, t_2) \right] = \delta (x_1 - x_2) \delta (t_1 - t_2)
\]

(2.6)

and

\[
\int dy \int d\tau \left[ G(x_1, t_1; y, \tau) G^{-1} (y, \tau; x_2, t_2) \right] = \delta (x_1 - x_2) \delta (t_1 - t_2)
\]

(2.7)

from which one takes the sum and difference of the Wigner transforms. When the inverse Green’s function $G^{-1}$ presents a gauge structure, it is natural to use the covariant Wigner transform (1.4) instead of the bare one (1.1). Thus one would like to apply the covariant Moyal expansion to the classic quantity $G^{-1} (p, x)$.

We first suppose time-independent problems, in which case it is natural to write the quasi-classic Green’s function (we aim at recovering the results obtained in (Bergeret and Tokatly, 2014) for superconducting transport, in a simpler way)

\[
G^{-1} (p, x, t_1, t_2) = \tau_3 \left( i \frac{\partial}{\partial t_1} + A_0 (x) \right) - \left( \frac{p^2}{2m} - \mu \right) + \Delta (x)
\]

(2.8)

with the time-sector gauge-potential $A_0$ responsible for the Zeeman splitting, and supposed to be space-dependent only (see section I.B). The term $\Delta (x)$ can be either thought as a potential, or as the superconducting gap in the mean-field approximation. The $\tau_3$-Pauli matrix is responsible for the particle-hole redundancy in superconductors ($\Delta \propto \tau_3 g_i (x)$ in
that case), and can be taken as $\tau_3 \to 1$ in case of non-superconducting systems. Then the Dyson’s equation (2.6) transforms to

$$G^{-1}(p, x) \ast G(p, x) = G^{-1}(p, x) G(p, x) + \frac{i}{2} v_i \frac{\mathcal{D} G}{\partial x^i} + \frac{i}{2} \left( \frac{\mathcal{D} \Delta}{\partial x^i} + \tau_3 F_{i0} \right) \frac{\partial G}{\partial p_i}$$

$$+ \frac{i}{8} \left( 3 v_{ij} F_{ij} \frac{\partial G}{\partial p_j} + \frac{\partial G}{\partial p_j} v_i F_{ij} \right) + \cdots$$  (2.9)

where $\mathcal{D}_i A_0 = \partial_i A_0 - i[A_i, A_0] \equiv F_{i0}(x)$ corresponds to the electric-like gauge-field in the pure static limit. Doing the same manipulations for the right-applied Dyson’s equation (2.7) with the propagator

$$G^{-1}(p, x, t_1, t_2) = \tau_3 \left( -i \frac{\partial}{\partial t_2} + A_0(x) \right) - \left( \frac{p^2}{2m} - \mu \right) + \Delta(x)$$  (2.10)

and taking the difference between the two obtained gradient expansions finally leads to the so-called transport equation

$$i \left( \tau_3 \frac{\partial G}{\partial t_1} + \frac{\partial G}{\partial t_2} \tau_3 \right) + i v_i \frac{\mathcal{D} G}{\partial x^i} + [\tau_3 A_0(x) + \Delta(x), G(p, x, t_1, t_2)]$$

$$+ \frac{i}{2} \left\{ \tau_3 F_{0i} + v_k F_{ki}, \frac{\partial G}{\partial p_i} \right\} + \frac{i}{2} \left\{ \frac{\mathcal{D} \Delta}{\partial x^i}, \frac{\partial G}{\partial p_i} \right\} = 0$$  (2.11)

as obtained in (Bergeret and Tokatly, 2014) after long algebra. In (Bergeret and Tokatly, 2014), the term $\mathcal{D}_i \Delta$ is absent as being irrelevant for physical reasons. Also, a self-energy term due to the possible impurities was included in (Bergeret and Tokatly, 2014) and has been discarded here, for simplicity. We thus see that the Moyal expansion (1.9) allows to find quite easily the quasi-classic transport equation of a classical action $G^{-1}(p, x)$ in a much simpler way than the direct evaluation of the covariant Wigner transform (1.4).

In (2.11), the time dependency was not transformed into the time-frequency representation of the Wigner transform. One can nevertheless suppose some quadri-vectors $p_\alpha \equiv (\omega, p_i)$ and $x^\alpha = (-t, x^i)$ such that

$$\frac{\partial O_1}{\partial x^\alpha} \frac{\partial O_2}{\partial p_\alpha} = \frac{\partial O_1}{\partial x^i} \frac{\partial O_2}{\partial p_i} - \frac{\partial O_1}{\partial t} \frac{\partial O_2}{\partial \omega}$$  (2.12)

in (1.9), see section I.A for more details. Then the gauge-covariant Wigner transform of the time-derivative $D_0 = \partial / \partial t - i A_0$ corresponds to $\omega$, according to the recipe of section II.A. Starting then from

$$G^{-1}(p, \omega, x, t) = \tau_3 \omega - \left( \frac{p^2}{2m} - \mu \right) + \Delta(x, t)$$  (2.13)

and applying the Moyal rule (1.9) to the associated Dyson equations (2.6) applied to the left and (2.7) applied to the right (the propagator (2.13) is the same in both positions in the classic representation), then taking the difference between the two obtained gradient
expansions, one gets

\[
\frac{i}{2} \left\{ \tau_3, \frac{\partial G}{\partial t} \right\} + iv_i \frac{\partial G}{\partial x^i} + [\tau_3 \omega + \Delta (x, t), G(p, x, \omega, t)]
\]

\[
- \frac{i}{8} \left( F_{0j} \left( 3\tau_3 \frac{\partial G}{\partial p_j} + \frac{\partial G}{\partial \tau_3} \right) + \left( \tau_3 \frac{\partial G}{\partial p_j} + 3 \frac{\partial G}{\partial p_j} \tau_3 \right) F_{0j} \right)
\]

\[
+ \frac{i}{2} \left\{ v_i F_{ij}, \frac{\partial G}{\partial p_j} \right\} + \frac{i}{2} \left\{ v_i F_{i0}, \frac{\partial G}{\partial \omega} \right\} + \frac{i}{2} \left\{ \frac{\partial \Delta}{\partial x^i}, \frac{\partial G}{\partial p_i} \right\} - \frac{i}{2} \left\{ \frac{\partial \Delta}{\partial t}, \frac{\partial G}{\partial \omega} \right\} = 0
\]

(2.14)

for the transport equation of a singlet superconductor with spin-orbit (formally written as a gauge-potential \( A_i \)) and spin-splitting (a Zeeman term in the form of a time-sector gauge-potential \( A_0 \)) couplings (see section I.B). This transport equation was first obtained in (Konschelle, 2014), and we can appreciate the concision of the present approach using the Moyal expansion (1.9) with the cumbersome derivation proposed in (Konschelle, 2014). The limits of a normal metal in the presence of non-Abelian gauge fields (when \( \tau_3 \to 1 \) and \( \Delta \to 0 \), see (Gorini et al., 2010)) and of a superconductor with electromagnetic coupling (i.e. an Abelian gauge-field, see (Kopnin, 1994)) can be obtained from such an equation. An other equation, given by the sum of the two Dyson’s equations once Wigner transformed, is also given in (Konschelle, 2014), and can be obtained as well from the Moyal expansion (1.9) instead of the long derivation done in (Konschelle, 2014).

We just seen in this section that the Moyal expansion (1.9) provides a convenient way to overcome the complicated calculations done in the establishments and justifications of gauge-covariant transport equations. We here reviewed the condensed matter situations of such equations, where the above equations (in one form or another) have been used to establish many results, especially in the field of magneto-electric effects for low-energy spin manipulation using the macroscopic coherence of superconductivity, the so-called super-spintronics (Aprili and Quay, 2017; Eschrig, 2015; Linder and Robinson, 2015). Reader interested in the recent results might consult the review (Raimondi et al., 2016) for normal metallic systems and (Bergeret and Tokatly, 2015, 2016; Espedal et al., 2016; Jacobsen et al., 2015; Konschelle et al., 2016a, 2015, 2016b; Mal’shukov, 2016; Reeg and Maslov, 2015, 2017) for some descriptions of super-spintronics effects. Similar calculations can be done for the quark-gluon plasma, using eventually the Wigner function (i.e. the Wigner transform of the density operator) instead of the Green’s function (Elze and Heinz, 1989; Weigert and Heinz, 1991; Wong, 1996). There as well, to know the Moyal expansion (1.9) would have drastically simplified the obtention of the transport equations. In particular, it would simplify the search for effective actions at the one-loop order, see e.g. (Banin and Pletnev, 2001; Pletnev and Banin, 1999; Salcedo, 2007) and references therein.

C. Topology in the momentum space

In addition to the establishment of transport equations reviewed in section II, the gradient expansion (1.9) allows to evaluate perturbatively the Green’s function of a given system. Then it becomes possible, without much assumptions (like linear-response theory for systems slightly out of equilibrium for instance), to establish generic properties of matter using quasi-classic methods. This is what we review quickly in this section. As a simple yet interesting example, the momentum-space topology introduced by Volovik (Volovik, 2003), is reviewed.
In the way, we will generalise the results previously known for Abelian gauge-theory to non-Abelian situations at no cost. Indeed, the situations described below are usually identified using the flat-space Moyal expansion (1.2), whereas we establish these results using the gauge-covariant Moyal expansion (1.9) instead.

The basic idea of the obtention of topological quantities is to use the Dyson’s equation (2.6) in the phase-space,

\[ G^{-1}(p, x) \ast G(p, x) = 1 \]

and to expand the solution for \( G(p, x) \) in the form \( G = G_0 (1 + G_2 + \cdots) \) in power of gradient, once replacing the star product with (1.9). One obtains in that way

\[ G_0(p, x) = \left[ G^{-1}(p, x) \right]^{-1} \]

at zero-th order in gradient. Thus the zero-th order Green’s function is just the classical Green’s function. There is no first order term, and the second one reads

\[ G_2(p, x) = -\frac{i}{2} \left( \frac{\partial G_0^{-1}}{\partial p_\alpha} \frac{\partial G_0}{\partial p_\alpha} - \frac{\partial G_0}{\partial p_\alpha} \frac{\partial G_0^{-1}}{\partial x^\alpha} \right) + \frac{i}{8} \left( F_{\alpha\beta}(x) \frac{\partial G_0^{-1}}{\partial p_\alpha} \frac{\partial G_0}{\partial p_\beta} + 2 \frac{\partial G_0^{-1}}{\partial p_\alpha} F_{\alpha\beta}(x) \frac{\partial G_0}{\partial p_\beta} + \frac{\partial G_0^{-1}}{\partial p_\alpha} \frac{\partial G_0}{\partial p_\beta} F_{\alpha\beta}(x) \right) \]

which, to the best of our knowledge, is a novel result. We stop at this order to recover known results in the case of normal metal (where the Chern number appears) and superfluids (where the Pontryagin number appears). Higher order terms will be inspected in subsequent studies.

The strategy is for the moment to inject the perturbative Green’s function into the definition of the current (see (Zubkov, 2016) for a justification – the notations are relativistic with \( j^\alpha \equiv (\rho, j^i) \) a notation for the density of charge and the current vector, and \( p_\alpha \equiv (\omega, p) \) in the reciprocal space-time of frequency and momentum)

\[ j^\alpha = \frac{i}{\deg} \frac{e}{4} \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi} \text{Tr} \left\{ \frac{\partial G_0^{-1}}{\partial p_\alpha} G(p, x) \right\} \]

(2.18)

to get \( j^\alpha = j^{\alpha}_\ott{1} + j^{\alpha}_\ott{2} + \cdots \) with

\[ j^{\alpha}_\ott{1} = \frac{1}{\deg} \frac{e}{8} \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi} \text{Tr} \left\{ \frac{\partial G_0^{-1}}{\partial x^\beta} G_0 \left[ \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0, \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \right] \right\} \]

(2.19)

\[ j^{\alpha}_\ott{2} = \frac{1}{\deg} \frac{e}{8} \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi} \text{Tr} \left\{ F_{\beta\gamma}(x) \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0 \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \frac{\partial G_0^{-1}}{\partial p_\gamma} G_0 \right\} \]

(2.20)

up to higher order in the gradients. We supposed that \( [F_{\alpha\beta}, G_0] = 0 \) since usually the Green’s function is the bare one, as we discuss in section \( \Pi \), see especially (2.13) for the example of a superconductor. In (2.18), we defined the degeneracy number \( \deg \) as the number of redundant degrees of freedom of the Green’s function introduced in order to account for the symmetry of the model. It is essentially in use when dealing with both normal and superconducting systems. In this later case, \( \deg = 2 \) accounts for the particle-hole symmetry of the BCS model, whereas \( \deg = 1 \) in normal systems.

Let us have a look at \( j^{\alpha}_\ott{1} \). When the covariant derivative is replaced by the bare one \( \Box \rightarrow \partial \), this term is the only contribution coming from the \( \ast_0 \)-expansion (1.2) in the flat
space. In particular \( j_{(2)} = 0 \) in that case. There is a trick to get the linear-response current in the case of Abelian fields, even when one starts from the \(*_0\)-product instead of the covariant \(*\)-one (Mera, 2017; Zubkov, 2016). If one supposes that the position dependency of the classical Green’s function reads \( G_0 (p - A (x)) \) – the minimal substitution recipe – the expansions \( G_0 (p - A) = G_0 (p) - A_\alpha (x) \partial G_0 / \partial p_\alpha + \cdots \) and a similar one for \( G_0^{-1} \) are allowed and one gets

\[
\begin{align*}
  j_{(1), \text{min}}^\alpha &= j_{(1)}^\alpha (\mathcal{D} \to \partial) + \frac{e}{8 \deg} \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi} \text{Tr} \left\{ F_{\alpha\beta\gamma}^\text{Ab} (x) \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0 \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \frac{\partial G_0^{-1}}{\partial p_\gamma} G_0 \right\} \quad (2.21)
\end{align*}
\]

with the Abelian gauge field \( F_{\alpha\beta}^\text{Ab} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \), see details in (Mera, 2017; Zubkov, 2016). The extra contribution is exactly \( j_{(2)} \) whenever the gauge fields are Abelian. Despite giving similar contributions at the leading order, the strategy outlined in this paragraph is not well justified, because the star product \(*_0\) does not guarantee to obtain only covariant terms. In addition, the above heuristic minimal substitution to get \( j_{(1), \text{min}} \) seems to be hardly justified in the case of non-Abelian fields, whereas our strategy here to dress the bare Green’s function \( G_0 (p, x) \) by the gauge field using the covariant \(*\)-product expansion (1.9) gives a straightforward evaluation of the current in the linear-response framework, either in the Abelian or the non-Abelian cases. In addition, we impose no hypothesis about the structure of \( G_0 (p, x) \) when we derived (2.20).

We come back to the covariant calculation, and pursue our interpretation of the term \( j_{(2)} \) obtained in (2.20). Due to the antisymmetry of the gauge-field, and the cyclic property of the trace, one can define the symbol

\[
R^{\alpha\beta\gamma} = \frac{e}{8} \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi} \text{Tr} \left\{ \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0 \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \frac{\partial G_0^{-1}}{\partial p_\gamma} G_0 \right\} \quad (2.22)
\]

as being totally antisymmetric in its three indices. In the case of Abelian gauge field, one can represent (2.20), with \( \text{dim} \) the space dimension of the problem, as (notations taken from (Zubkov, 2016))

\[
\begin{align*}
  j_{(2)}^\alpha (x) &= R^{\alpha\beta\gamma} (x) F_{\beta\gamma} (x) = \frac{1}{\text{deg}} \frac{e}{4} \left\{ \begin{array}{l}
    \frac{\varepsilon^{\alpha\beta\gamma}}{4\pi} F_{\beta\gamma} \mathcal{M}, \quad \text{dim} = 2 \\
    \frac{\varepsilon^{\alpha\beta\gamma}}{4\pi^2} F_{\beta\gamma} N_\delta, \quad \text{dim} = 3
  \end{array} \right. \quad (2.23)
\end{align*}
\]

where we define the quantities

\[
\begin{align*}
  \mathcal{M} &= -\frac{\varepsilon^{\alpha\beta\gamma}}{3!4\pi^2} \iiint d\omega dp_x dp_y \text{Tr} \left\{ \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0 \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \frac{\partial G_0^{-1}}{\partial p_\gamma} G_0 \right\} \quad (2.24) \\
  N_\delta &= -\frac{\varepsilon^{\alpha\beta\gamma}}{3!8\pi^2} \iiint d\omega dp_x dp_y dp_z \text{Tr} \left\{ \frac{\partial G_0^{-1}}{\partial p_\alpha} G_0 \frac{\partial G_0^{-1}}{\partial p_\beta} G_0 \frac{\partial G_0^{-1}}{\partial p_\gamma} G_0 \right\} \quad (2.25)
\end{align*}
\]

and one can further show that (2.24) reduces to some topological numbers in peculiar circumstances. For instance, in the case of a normal metal without interactions at zero temperature, \( \text{deg} = 1 \) and the Green’s function reduces to

\[
G_0 (p, x) = \sum_n \frac{|n, p \rangle \langle n, p|}{i\omega - E_n (p)} \quad (2.26)
\]
where the electronic states are represented by the wave-function $|n, p\rangle$, with $n$ labeling the eigenenergies $E_n$. Injecting (2.26) in (2.24), and performing the contour integration over the imaginary Matsubara frequencies $i\omega$ at zero temperature, one can evaluate (see details in (Mera, 2017; Zubkov, 2016)),

$$
\mathcal{M} = \frac{1}{4\pi} \int \int d\rho_x d\rho_y [\mathcal{F}_{xy}] \tag{2.27}
$$

where $A_i = -i \langle n | \partial_i | n \rangle$ and $\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i$ corresponds to the Berry connection and curvature, respectively. This last expression is nothing but the celebrated quantisation formula for the transconductance in 2D systems under magnetic field with $\mathcal{M}$ the Chern number (Niu et al., 1985; Simon, 1983; Thouless et al., 1982).

We obtained this topological result in the case of a clean metal without interaction, represented by the expression (2.26). Nevertheless, variation of $G_0 \to G_0 + \delta G$ in the definition (2.24) indicates that this quantity is in fact invariant under small perturbation of the Green’s function, hence it gets some topological significations irrespective of the explicit representation of the Green’s function (Grinevich and Volovik, 1988; Volovik, 2003), and might continue to carry some topological significations even in the presence of interactions (Essin and Gurarie, 2011; Gurarie, 2011).

Another important example of the use of the gradient expansion is given by the quantisation of particle transconductance in the case of bi-dimensionnal Helium superfluids. In that later situation, $\deg = 2$ and the Green’s function reads

$$
G_0^{-1}(p, x) = \omega - \tau \cdot m(p) \tag{2.28}
$$

with $\tau_i$ the Pauli matrices which span the particle-hole algebra of the Cooper condensation in the superfluid. The vector $m$ encodes the symmetries of the superfluid gap. One then gets

$$
\mathcal{M} = -\frac{1}{16\pi} \int \int d\rho_x d\rho_y \left[ \hat{m} \cdot \left( \frac{\partial \hat{m}}{\partial \rho_x} \times \frac{\partial \hat{m}}{\partial \rho_y} \right) \right] \tag{2.29}
$$

which is nothing but the Pontryagin number. This quantity has been derived in (Volovik, 1988a; Volovik and Yakovenko, 1989), showing how the superfluids phenomenology can be similar to the quantum Hall effect, exhibiting transconductance quantisation, though of different origin. Indeed, in the quantum Hall case, the quantisation is due to the non-trivial electronic band structure under a magnetic field, whereas in superfluids, the non-trivial topology originates from the particle-hole space and the symmetries of the superfluid gap, the Pauli-$\tau$ matrices in (2.28).

One more time, note that the results (2.27) and (2.29) were previously obtained using a non-covariant gradient expansion. They have then been established using the trick explained around (2.21), using some minimal substitution $G_0(p - A)$ instead of the bare Green’s functions. In contrary, we established them using the covariant expansion of the $\star$-product, and (2.20) is therefore more general than the tricks used in the existing literature (Mera, 2017; Volovik, 1988a; Volovik and Yakovenko, 1989; Zubkov, 2016). At least one can show that our approach is gauge covariant to start with.

The above review helps us to understand the structure of the gauge-covariant Moyal expansion (1.9). Indeed, none of the topological numbers $M$ in (2.27) or (2.29) would have been obtained for a pure classic action (2.26) or (2.28). It is because the Moyal expansion (1.9) dresses the bare actions (2.26) and (2.28) by some gauge fields that the topological
obstructions (2.27) and (2.29) can be extracted from it. Though the gauge structure in this section explicitly comes from the Abelian electromagnetic structure, its origin in the general case of (1.9) might appear unclear at first sight. In fact, we never impose any restriction on the gauge field, beyond its generic definition (1.10). So, prior to any gradient expansion, the gauge structure of the theory must be extracted from extra considerations, like e.g. microscopic theories. In fact, one usually starts from a microscopic theory, from which the gauge structure can be established on the basis of symmetry construction. Once the gauge structure is established in the form of the minimal substitution $p \rightarrow p - A$, one can apply the gauge-covariant Wigner transform (1.4) to the equation of motion (e.g. the Schrödinger or the Dyson equation) on the bare theory, such that the gauge-covariant gradient expansion (1.9) will dress the perturbative theory by gauge interactions. At this level the gauge fields are classical fields, and the expansion (1.9) can be seen as a perturbative expansion in both gradients and gauge fields (higher orders in the gauge field will naturally appear in the higher orders, see section III for more details). A few strategies to understand how the gauge structure emerges in microscopic theories have been given in section I.B.

To conclude this section, let us calculate the spin current in 2D (the definition follows straightforwardly from arguments in (Zubkov, 2016) and (Konschelle, 2014), see also (Konschelle et al., 2016b) for a similar definition)

$$J^a_\alpha (x) = \frac{i}{4 \text{deg}} \int \frac{d\omega}{2\pi} \int \frac{dp_x}{2\pi} \int \frac{dp_y}{2\pi} \text{Tr} \left\{ \sigma^a \frac{\partial G^{-1}_0}{\partial p_\alpha} G_0 (p, x) \right\}$$

$$= \frac{1}{8 \text{deg}} \int \frac{d\omega}{2\pi} \int \frac{dp_x}{2\pi} \int \frac{dp_y}{2\pi} \text{Tr} \left\{ \sigma^a F_{\beta \gamma} (x) \frac{\partial G^{-1}_0}{\partial p_\alpha} G_0 \frac{\partial G^{-1}_0}{\partial p_\beta} G_0 \frac{\partial G^{-1}_0}{\partial p_\gamma} G_0 \right\}$$

$$= \frac{1}{4 \text{deg}} \frac{\varepsilon^{\alpha \beta \gamma}}{4\pi} F^a_{\beta \gamma} (x) \mathcal{M}$$

(2.30)

where we supposed the Green’s function $G_0$ to be independent of the spin, and we expand the non-Abelian gauge field $F_{\alpha \beta} \equiv F_{\alpha \beta}^a \sigma^a/2$ when it describes the spin degree of freedom with Pauli algebra span by spin matrices $\sigma^a$, $a \in \{1, 2, 3\}$. Then the trace applies on the spin degree of freedom only, and project the gauge field toward the Pauli matrix $\sigma^a$ present in the definition of the spin current.

The spin-current in (2.30) can then be generated by pure spin-orbit interactions (like Rashba or Dresselhaus ones, see (Berche and Medina, 2013; Winkler, 2003)), and might be quantised in the limit of zero temperature, following the same arguments giving (2.27) in normal metal or (2.29) in superfluid/superconducting systems. Nevertheless, the spin current is impossible to measure, since it contains 5 dimensions (three for the spin $a \in \{1, 2, 3\}$ and two for space $\alpha \in \{1, 2\}$) in our three dimensional experimental world. An other way to understand why the spin current can not be measured is to realise that it is not conserved in the usual sense (namely as $\partial_\alpha J^\alpha = 0$), but in a non-Abelian gauge covariant way (namely $\mathcal{D}_\alpha J^\alpha = \partial_\alpha J^\alpha - i [A_\alpha, J^\alpha] = 0$; the part $i [A_\alpha, J^\alpha]$ is usually considered as a spin-torque), see e.g. (Jin et al., 2006; Tokatly, 2017) and references therein.

Many other consequences of the gauge-covariant gradient expansion (1.9) are still to come, and will be the subject of subsequent works. At the moment, we turn to the presentation of the method leading to the expansion (1.9).
III. GAUGE-COVARIANT MOYAL STAR-PRODUCT

We now attack the heart section of this study. We will prove the gradient expansion (1.9), and add the next order term to it, namely the term of fourth order in gradient. A generic method will be presented, based in the difference between the gauge-covariant Wigner transform (1.4) and a gauge-fixed one in the Fock-Schwinger gauge (also called Poincaré-relativistic, or axial or radial gauge) proposed in (Weigert and Heinz, 1991). Section III.A provides a self-contained justification of the gauge-covariant and the gauge-fixed Wigner transforms. The method consists in realizing that in the Fock-Schwinger gauge, the gauge-covariant transform (1.4) looks formally like the Wigner transform (1.1) in the flat space. Then one can apply the well-known result (1.2) to it. The mapping from the gauge-flat to gauge-covariant formulation can be done by promoting the successive $x$-derivatives to some covariant derivatives, including gauge fields corrections. This is the task accomplished in section III.B. Unfortunately, I am not aware of any trick allowing the substitution to be done at any order, and so the substitution is done order by order. This is where the pedestrian approach appears. Section III.B is quite technical, and might be skipped in first reading since it does not participate in the understanding of the rest of the presentation. The gauge-covariant gradient expansion is then given in section III.C. The two limiting cases of Abelian gauge fields and pure gauge problems are given in section III.D.

A. Two different Wigner transforms

We begin by a short review of the gauge-covariant Wigner transform proposed in (Elze et al., 1986a,b; Winter, 1984) and the gauge-fixed Wigner transform proposed in (Weigert and Heinz, 1991).

The story starts with the following problem. Given the Wigner transform (1.1), and the possibility for the operators $O(x_1, x_2)$ to change under a gauge transformation to $O'(x_1, x_2) = R(x_1)O(x_1, x_2)R^{-1}(x_2)$, how may we construct a Wigner transform $O(p, x)$ which would be gauge-covariant? Namely we would like $O(p, x)$ to transform as $O'(p, x) = R(x)O(p, x)R^{-1}(x)$ under a gauge transformation. The solution to this problem, first proposed in (Luttinger, 1951; Stratonovich, 1956), is to realise that (1.1) can be written in the formal representation

$$O(p, x) = \int dx e^{-ip\cdot x} \left( e^{x^2/4}O(x_1, x_2)e^{-x^2/4} \right)_{x_1=x_2=x} \tag{3.1}$$

where the dagger-derivative $O^{\dagger}_x \equiv \partial_x O$ applies to the left. The exponentiated derivative shifts the argument of any function of its argument, $e^{x^2/4}f(x)e^{-x^2/4} = f(x + \xi)$. In effect, the formal writing (3.1) is strictly equivalent to the previous definition (1.1).

Under the representation (3.1), it is straightforward to generalise the Wigner transform to a covariant representation. It is sufficient to replace the derivative by its covariant version $\partial_x \to D_x = \partial_x - iA_x(x)$ and $\partial^2_x \to D^2_x = \partial_x^2 + iA_x(x)$. One obtains the representation (1.4) once we realise that

$$e^{x^2/4}f(x) = U(x, x + \xi)f(x + \xi) \tag{3.2}$$

with the parallel transport operator given in (1.5). We can drop the index $x_1 = x_2 = x$ from now on in (3.1). This operator has the property of a Dyson’s operator: $U(x, x) = 1$, $U(x_1, y)U(y, x_2) = U(x_1, x_2)$ and so $U(x_1, x_2)$ is the inverse of $U(x_2, x_1)$. Details of the
calculation done in (Elze et al., 1986b; Winter, 1984), partially reproduced in (Konschelle, 2014), show that the path connecting the points $x$ and $x + \mathbf{r}$ using the operator $e^{iD^s}$ must be a straight line.

For the following it is important to realise that the derivatives of $U$ are not trivial. In fact, one has

$$
\frac{DU(b, a)}{\partial b^\alpha} U(a, b) = i(b - a)^\beta \int_0^1 ds \left[ sU(b, \tau_s) F_{\beta \alpha}(\tau_s) U(\tau_s, b) \right]
$$

(3.3)

where the gauge field is defined as $F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i[A_\alpha, A_\beta]$ (with the convention $\partial_\alpha A_\beta \equiv \partial A_\beta / \partial x^\alpha$) in the non-Abelian case, and $\tau_s = a + (b - a)s$ represents the straight line between $b$ and $a$. Using the properties of the parallel transport operation, one gets similarly

$$
U(a, b) \frac{U(b, a) D^\dagger}{\partial a^\alpha} = i(b - a)^\beta \int_0^1 ds \left[(1 - s) U(a, \tau_s) F_{\beta \alpha}(\tau_s) U(\tau_s, a) \right]
$$

(3.4)

for the derivative applied to the second variable. The presence of $(1 - s)$ ensures that the path is taken in the opposite way.

The Wigner transform (1.4) is covariant with respect to the gauge transformation: $O'(p, x) = R(x) O(p, x) R^{-1}(x)$, as required. One can in particular applies the transformation

$$
O^{(Z)}(p, x) = U(Z, x) O(p, x) U(x, Z)
$$

(3.5)

$$
= \int d\mathbf{r} \left[ e^{-i\mathbf{r}/\hbar} U \left(Z, x + \frac{\mathbf{r}}{2} \right) O \left(x + \frac{\mathbf{r}}{2}, x - \frac{\mathbf{r}}{2} \right) U \left(x - \frac{\mathbf{r}}{2}, Z \right) \right]
$$

where the external point $Z$ is arbitrary for the moment. Nevertheless, it is the same parallel transport operator (1.5) which has been used in the definition of $O^{(Z)}(p, x)$. In particular, $U(Z, x)$ verifies (3.3). Suppose then that one applies the same gauge transformation to the gauge-potential, defining

$$
A^{(Z)}_\alpha(x) = U(Z, x) A_\alpha(x) U(x, Z) + iU(Z, x) \frac{\partial U(x, Z)}{\partial x^\alpha}
$$

(3.6)

$$
= (Z - x)^\beta \int_0^1 ds \left[ sU(Z, \tau_s) F_{\beta \alpha}(\tau_s) U(\tau_s, Z) \right]
$$

with $\tau_s = Z + (x - Z)s$ for the path between $x$ and $Z$, using (3.3) to kill $A_\alpha$. In particular, we have, in this new gauge

$$
(x - Z)^\alpha A^{(Z)}_\alpha(x) = 0
$$

(3.7)

since the gauge field is an antisymmetric quantity $F_{\alpha \beta} = -F_{\beta \alpha}$ by definition. This is nothing but the definition of the radial gauge, hence the transformation (3.5) is called the radial-gauge Wigner transform, or gauge-fixed Wigner transform. In addition, one has $A^{(Z)}_\alpha(x = Z) = 0$ and so the covariant derivatives reduce to the trivial ones at the point $x = Z$. This property allows easy manipulations. Indeed, taking $x = Z$ in the definition of the Wigner transform (1.4) but keeping $x_{1,2} = x \pm \mathbf{r}/2$ as independent variables reduces $O(p, x)$ to $O^{(Z)}(p, x)$ in (3.5). So the equation of motion for $O^{(Z)}(p, x)$ presents only flat-space derivatives $\partial_x$, and in particular, the Moyal product for two operators $O^{(Z)}_1(p, x)$ and
\(O_2^{(Z)}(p, x)\) is the well known one (1.2) since there is vanishing connection in this gauge when \(x = Z\). We thus have

\[
\int dx \int dy \left[ e^{-i\psi/h} U \left( Z, x + \frac{r}{2}, y \right) O_1 \left( y, x - \frac{r}{2} \right) U \left( x - \frac{r}{2}, Z \right) \right] = \nonumber
\]

\[
= O_1^{(Z)}(p, x) \ast_0 O_2^{(Z)}(p, x) = O_1^{(Z)} O_2^{(Z)} + \frac{i}{2} \left( \frac{\partial O_1^{(Z)}}{\partial p_\alpha} \frac{\partial O_2^{(Z)}}{\partial x^\alpha} - \frac{\partial O_1^{(Z)}}{\partial x^\alpha} \frac{\partial O_2^{(Z)}}{\partial p_\alpha} \right) + \cdots \quad (3.8)
\]

at first order of the star product (1.2). We no more need to precise the variables \((p, x)\) for \(O_{1,2}^{(Z)}\) since the gauge exponents specifies that we work in the phase space, allowing simpler notations.

We have thus established the Moyal product in the radial gauge: it is just the usual Moyal \(*_o\)-product (1.2). This is nevertheless not gauge covariant. To know the gauge-covariant gradient expansion, one must perform the parallel transport in the reverse sense, i.e. from the point \(Z\) back to \(x\). We thus take the limit \(Z \rightarrow x\) in the gradient expansion (3.8). It consists in replacing \(O_{1,2}^{(Z)} \rightarrow O_{1,2}(p, x)\) naively. Nevertheless, the space derivatives in (3.8) must be taken with care. One indeed realises that (Weigert and Heinz, 1991)

\[
\frac{\partial O(p, x)}{\partial x^\alpha} = \left( \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha} \right) O^{(z)} \bigg|_{Z=x} \quad (3.9)
\]

by a direct comparison of the definitions of the Wigner transforms in the radial gauge (3.5) and in the covariant representation (1.4). We finally have to explicitly calculate the limit

\[
O_1(p, x) \ast O_2(p, x) = \lim_{Z \rightarrow x} \left[ O_1^{(Z)}(p, x) \ast_0 O_2^{(Z)}(p, x) \right] \quad (3.10)
\]

using the substitution (3.9), e.g.

\[
O_1(p, x) \ast O_2(p, x) = O_1(p, x) O_2(p, x) + \frac{i}{2} \left( \frac{\partial O_1^{(Z)}}{\partial x^\alpha} \bigg|_{Z=x} \frac{\partial O_2}{\partial p_\alpha} - \frac{\partial O_1}{\partial x^\alpha} \bigg|_{Z=x} \frac{\partial O_2^{(Z)}}{\partial p_\alpha} \right) + \cdots \quad (3.11)
\]

at the leading order. The momentum derivative do not make any trouble, and we replace them straightforwardly in the above expression. The \(x\)-derivative of \(O^{(Z)}\) naturally appears in the \(*_o\)-product, so we just have to evaluate \(\partial Z O^{(Z)}\) and take its limit \(Z \rightarrow x\). This can be done exactly using the property (3.3) of the parallel transport, which let the gauge field appearing.

This is all for the general method, we must now turn to tedious evaluations of the substitution rules (3.10) from the radial gauge to the covariant representation for the sucessives derivatives, using (3.9) and its higher orders.

**B. Substitutions for the derivatives**

This section establishes the substitution rules between the gauge-fixed \(O^{(Z)}(p, x)\) and the gauge-covariant \(O(p, x)\) Wigner transforms. Namely, it gives, order by order how the mapping \(\partial_x^n O^{(Z)}|_{Z=x}\) works for \(n = 1, 2\) to all order in the covariant derivative of the gauge
field and for \( n = 3 \) up to the fourth order in gradient only. Details are cumbersome, and can be skip at first reading.

Our first goal is to evaluate \( \frac{\partial Z}{\partial \eta^a} O^Z \big|_{Z=x} \) in the expression (3.9) in order to establish the link between the gauge-fixed derivative \( \frac{\partial}{\partial n} O^Z \) and the gauge-covariant one. The pivotal formula is (3.3) which we rewrite as

\[
\frac{D U (b, a)}{\partial b^a} U (a, b) = i \mathbb{F}^{(b)}_\alpha (b, a) \tag{3.12}
\]

introducing the notation

\[
\mathbb{F}^{(c)}_\alpha (b, a) = (b - a)^3 \int_0^1 ds \left[ s U (c, \tau_s) F_{\beta \alpha} (\tau_s) U (\tau_s, c) \right] ; \quad \tau_s = a + (b - a) s \tag{3.13}
\]

for convenience. The exponent notation is fixed by the property \( \mathbb{F}^{(q)}_\alpha (b, a) = U (y, x) \mathbb{F}^{(x)}_\alpha (b, a) U (x, y) \) under a gauge transformation.

From the definition (3.5) for the radial gauge Wigner transform, one gets

\[
\frac{\partial O^Z}{\partial Z^a} = \int dx \left[ e^{-ip \cdot x} \frac{\partial U (b, a)}{\partial b^a} U (a, b) U \left( Z, x + \frac{y}{2} \right) O \left( x + \frac{y}{2}, x - \frac{y}{2} \right) U \left( x - \frac{y}{2}, Z \right) \right]_{a=x+y/2}^{b=Z}
\]

\[
\quad + \int dx \left[ e^{-ip \cdot x} U \left( Z, x + \frac{y}{2} \right) O \left( x + \frac{y}{2}, x - \frac{y}{2} \right) U \left( x - \frac{y}{2}, Z \right) U (b, a) \frac{\partial U (a, b)}{\partial b^x} \right]_{a=x-y/2}^{b=Z} \tag{3.14}
\]

using the property \( U (a, b) U (b, a) = 1 \) of the parallel transport. The terms written as \( U (b, a) \) and its derivatives can be extracted from the integral using the formal property

\[
\int dx \left[ e^{-ip \cdot x} f (x) U \left( Z, x + \frac{y}{2} \right) O \left( x + \frac{y}{2}, x - \frac{y}{2} \right) U \left( x - \frac{y}{2}, Z \right) \right] = f (i \partial_b) O^Z (p, x) \tag{3.15}
\]

valid for any analytical function \( f \). The same works for the function \( f (x) \) applied on the right, with \( \partial_b \)-derivatives instead of \( \partial_b \). Next, the unitarity of the parallel transport gives \( \partial_b U (a, b) U (b, a) = - U (a, b) \partial_b U (b, a) \), and so one gets

\[
\frac{\partial O^Z}{\partial Z^a} = i \left[ A_\alpha (Z) + \mathbb{F}^{(Z)}_\alpha \left( Z, x + \frac{i}{2} (\partial_b - \partial_b^i) \right) , O^Z (p, x) \right] \tag{3.16}
\]

where the limit \( Z \to x \) can be now taken straightforwardly, since there is no more \( x \)-derivatives in the right-hand side of (3.16). In particular, one can take the argument \( \partial_b - \partial_b^i \) in the expression (3.16) since \( \mathbb{F} \) is independent of the momentum. Relation (3.9) finally reads

\[
\frac{\partial O^Z}{\partial x^a} \bigg|_{Z=x} = \frac{\partial O (p, x)}{\partial x^a} - i \left[ \mathbb{F}^{(x)}_\alpha \left( x, x + \frac{i}{2} (\partial_b - \partial_b^i) \right) , O (p, x) \right] \tag{3.17}
\]

where the right-hand side is gauge-covariant, as required. The relation (3.17) between the radial gauged and the covariant representation of \( O (p, x) \) can be injected in the Moyal
expansion to get the non-Abelian gauge-covariant gradient expansion at first order, following the mapping (3.10). To do that one should expand $F^{(x)}_\alpha$ in power of $\partial_p - \partial_p^\dagger$. Using the substitution $U(x, x + z) F(x + z) U(x + z, x) = e^{z \mathcal{D}_z} F(x)$ proved in (Elze et al., 1986b), one formally has

$$F^{(x)}_\alpha (x, x + z) = \int_0^1 ds \left[(1 - s) e^{s(z \mathcal{D}_z)} F_{\alpha \beta} (x)\right] z^\beta = \sum_{n=0}^\infty \frac{(z \cdot \mathcal{D}_z)^n}{(n + 2)!} F_{\alpha \beta} (x) z^\beta$$

(3.18)

as a covariant expansion in power of the gradient of the gauge field. We used the formula

$$\int_0^1 ds \left[s^m e^{sx}\right] = \frac{\partial^m}{\partial x^m} \int_0^1 ds \left[e^{sx}\right] = \sum_{n=0}^\infty \frac{x^n}{n! (n + m + 1)}$$

(3.19)

to evaluate all integrals of the type of $F$ and its derivatives (yet to come).

We thus reuse in providing an explicit method to get a completely gauge-covariant Moyal expansion by substituting the usual gradient expansion with covariant contributions according to the general recipe (3.10) with

$$\frac{\partial O^{(Z)}}{\partial x^\alpha} \bigg|_{Z=x} = \mathcal{D} O(p, x) + \left[\sum_{n=0}^\infty i^n (z \cdot \mathcal{D}_z)^n F_{\alpha \beta} (x) z^\beta, O(p, x)\right]_{z=\partial_p - \partial_p^\dagger}$$

(3.20)

for the first order. The $\mathcal{D}_x$-derivative in the commutator applies only on the gauge field $F_{\alpha \beta}$, whereas the $\partial_p$-derivatives apply only on $O(p, x)$ inside the commutator. Nevertheless, even though the first-order derivative (3.17) can be formally written in term of a formal covariant derivative when $F^{(x)}_\alpha$ is interpreted as a pseudo-gauge potential, higher order derivatives $\partial_x^n O^{(Z)} \big|_{Z=x}$ cannot be straightforwardly replaced by higher derivatives $\mathcal{D}^n O(p, x)$.

To see that one just has to calculate the next order, namely we will use (3.9) to get

$$\frac{\partial^2 O(p, x)}{\partial x^\alpha \partial x^\beta} = \left(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha}\right) \left(\frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial Z^\beta}\right) \left.O^{(Z)}(p, x)\right|_{Z=x} =$$

$$= \left(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha}\right) \left.\frac{\partial O^{(Z)}}{\partial x^\beta} + i \left[A_\beta (Z) + F^{(Z)}_{\alpha \beta} \left(Z, x + \frac{i}{2} (\partial_p - \partial_p^\dagger)\right), O^{(Z)}\right]\right|_{Z=x}$$

(3.21)

where we used (3.16) in the second line. The relation (3.16) can be used again to evaluate $\partial_Z \partial_x O^{(Z)} = \partial_x \partial_Z O^{(Z)}$. From the property (3.9), one has

$$\left.\left(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha}\right) \left[A_\beta (Z) + F^{(Z)}_{\alpha \beta} \left(Z, x + \frac{i}{2} (\partial_p - \partial_p^\dagger)\right), O^{(Z)}\right]\right|_{Z=x} =$$

$$= \frac{\partial}{\partial x^\alpha} \left[A_\beta (x) + F^{(x)}_{\alpha \beta} \left(x, x + \frac{i}{2} (\partial_p - \partial_p^\dagger)\right), O(p, x)\right]$$

(3.22)

and so

$$\frac{\partial}{\partial x^\alpha} \left(O(p, x) - i \left[A_\beta (x) + F^{(x)}_{\alpha \beta} \left(x, x + \frac{i}{2} (\partial_p - \partial_p^\dagger)\right), O(p, x)\right]\right) =$$

$$= \left(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha}\right) \frac{\partial O^{(Z)}}{\partial x^\beta} \bigg|_{Z=x} =$$

$$= \frac{\partial^2 O^{(Z)}}{\partial x^\alpha \partial x^\beta} \bigg|_{Z=x} + i \frac{\partial}{\partial x^\beta} \left[A_\alpha (Z) + F^{(Z)}_{\alpha \beta} \left(Z, x + \frac{i}{2} (\partial_p - \partial_p^\dagger)\right), O^{(Z)}(p, x)\right] \bigg|_{Z=x}$$

(3.23)
now expanding the derivative \( \partial_\beta \) on the right-hand side, and because \( \partial_x A ( Z ) = 0 \), and finally using (3.17) to evaluate \( \partial_\beta O ( Z ) \big|_{Z = x} \), one gets the substitution rule for the second order derivative

\[
\frac{\partial^2 O ( Z )}{\partial x^\alpha \partial x^\beta} \big|_{Z = x} = \mathbb{D}^2 O ( p, x ) - i \left[ \frac{\partial}{\partial x^\beta} \mathbb{F}_\alpha ( Z, x + z ) \right] \left[ O ( Z ) ( p, x ) \right] z^{i ( \delta_\beta - \delta_\alpha ) / 2} \quad (3.24)
\]

which thus misses to be a simple replacement of the \( \partial^n O ( Z ) \big|_{Z = x} \) by the formal covariant derivative \( \mathbb{D}_x^n O ( p, x ) \). However, a rapid calculation gives

\[
\frac{\partial}{\partial x^\beta} \mathbb{F}_\alpha ( Z, x + z ) \big|_{z = x} = \int_0^1 ds \left[ s ( 1 - s ) U ( b, a ) \left( \frac{\partial F_{\alpha \gamma} ( a )}{\partial x^\alpha} - i \left[ \mathbb{F}_\beta ( a, b ), F_{\alpha \gamma} ( a ) \right] \right) U ( a, b ) \right]_{b = x}^{a = x + zs} z^\gamma \quad (3.25)
\]

and thus shows that all terms in (3.24) are gauge covariant as required. A term proporcional to \( F_{\alpha \beta} \) has been discarded in (3.25), since it will not appear in the Moyal expansion with contracted indices, i.e. the interchange \( \alpha \leftrightarrow \beta \) is allowed at any stage of the computation to simplify it. Though we just missed to give a formal resummaion of all the covariant contributions appearing in the Moyal expansion, we clearly succeed in establishing a gauge covariant method to get the Moyal expansion order by order using the mapping (3.10).

Using the same method we used to get (3.18), one can evaluate

\[
\mathbb{F}_\beta ( x + zs, z ) = U ( x + zs, x ) \left[ \sum_{n=0}^{\infty} \frac{n + 1}{( n + 2 )!} ( sz \cdot \mathbb{D}_z )^n F_{\delta \beta} ( x ) z^\delta \right] U ( x + zs, x ) \quad (3.26)
\]

using \( U ( a, b ) U ( b, c ) = U ( a, c ) \) to evaluate the term into brackets. The above contribution nevertheless does not contribute to the covariant Moyal expansion, because both pairs of indices \( ( \delta, \gamma ) \) and \( ( \alpha, \beta ) \) are dummy indices in the Moyal expansion, whereas there will be some commutators of the form \( [ F_{\alpha \gamma}, F_{\beta \delta} ] \) and their covariant derivatives in (3.25). Thus (3.25) reads

\[
\frac{\partial}{\partial x^\beta} \mathbb{F}_\alpha ( Z, x + z ) \big|_{Z = x} = \sum_{n=0}^{\infty} \frac{n + 1}{( n + 3 )!} ( z \cdot \mathbb{D}_z )^n \frac{\partial F_{\alpha \gamma} ( x )}{\partial x^\alpha} z^\gamma + \cdots \quad (3.27)
\]

being understood that the covariant derivatives \( \mathbb{D}_z \) apply only on the gauge-field next to them, and the forgotten terms in \( \cdots \) do not participate in the Moyal expansion. The expression (3.24) can thus be rewritten as

\[
\frac{\partial^2 O ( Z )}{\partial x^\alpha \partial x^\beta} \big|_{Z = x} = \mathbb{D}^2 O ( p, x ) + \sum_{n=0}^{\infty} \left[ \left( \frac{i}{2} \right)^n \frac{n + 2}{( n + 3 )!} \left( z^\delta \frac{\partial}{\partial x^\delta} \right)^n \frac{\partial F_{\beta \gamma} ( x )}{\partial x^\alpha} z^\gamma, O ( p, x ) \right] \bigg|_{z = \delta_\beta - \delta_\alpha} + \sum_{n=0}^{\infty} \left[ \left( \frac{i}{2} \right)^n \frac{n + 2}{( n + 2 )!} \left( z^\delta \frac{\partial}{\partial x^\delta} \right)^n F_{\beta \gamma} ( x ) z^\gamma, \frac{\partial O ( p, x )}{\partial x^\alpha} \right] \bigg|_{z = \delta_\beta - \delta_\alpha} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{4}{( n + 2 )! ( m + 2 )!} \left[ z^\delta \frac{\partial}{\partial x^\delta} \right]^n \frac{\partial F_{\beta \delta} ( x )}{\partial x^\alpha} z^\gamma, \left( \frac{z^\delta \partial}{\partial x^\epsilon} \right)^m O_{\epsilon \gamma} ( p, x ) \right] \bigg|_{z = \delta_\beta - \delta_\alpha} \quad (3.28)
\]
valid at all order in the covariant derivatives of the gauge fields, once injected in the covariant Moyal expansion following the recipe (3.10). One more time, let me repeat that the above expansion neglects terms odd in the permutation $\alpha \leftrightarrow \beta$ and/or $\gamma \leftrightarrow \delta$, since these contributions would vanish in the Moyal expansion. This is what I call the 'symmetrized equality', $\text{sym}$. There was no term forgotten in the first order calculation, and expression (3.20) was exact.

The above procedure can be performed up to any order. Higher order terms are nevertheless more and more cumbersome. Unfortunately, to truncate the expansion at order four in gradient, we need the next order term as well. To understand why, have a look at (3.27). Especially the term of the form $\mathcal{D} F \cdot z$, which gives a contribution $\mathcal{D} F \cdot \partial_p O$ in the substitution (3.28) for the second order derivative $\partial_p^2 O^{(Z)}|_{Z=x}$. In fact, at any order in the evaluation of $\partial_p^x O^{(Z)}|_{Z=x}$, a term of the form $\mathcal{D}^{n-1} F \cdot \partial_p O$ will be generated. This is clearly of order $n$ in gradient, provided we count the gradient of the gauge field as well. In general, one prefers to count the powers of the derivatives of the operator $O$, and so to take the correct contribution at order four in the gradients of $O$, one must include a contribution $\mathcal{D}^2 F \cdot \partial_p O$ from the third order derivative $\partial_p^3 O^{(Z)}|_{Z=x}$.

This will be the unique contribution we need to evaluate. Indeed, using schematic representation, one has, from the substitution rules from the Moyal expansion in the radial gauge to its gauge covariant representation

$$
\partial_x O_1^{(Z)} \cdot \partial_p O_2^{(Z)} \to \partial_x O_1 \cdot \partial_p O_2 + F \cdot \partial_p O_1 \cdot \partial_p O_2 + \mathcal{D} F \cdot \partial_p^2 O_1 \cdot \partial_p O_2 + \cdots \quad (3.29)
$$

up to higher order in gradients. This term is of first order in the $\ast_0$-expansion, but second order in gradient, since we count every gradient applied on either $O_1$ or $O_2$ in the later case. The term $\partial_p O_1^{(Z)} \cdot \partial_x O_2^{(Z)}$ looks schematically the same, and so we disregard it. At the second order in the $\ast_0$-expansion, one has terms of the form of fourth order gradient, e.g.

$$
\partial_x^2 O_1^{(Z)} \cdot \partial_p^2 O_2^{(Z)} \to \mathcal{D} F \cdot \partial_p O_1 \cdot \partial_p^2 O_2 + \mathcal{D}^2 F \cdot \partial_p^2 O_1 \cdot \partial_p^2 O_2 + \mathcal{D}^2 O_1 \cdot \partial_p^2 O_2 + \cdots \quad (3.30)
$$

and terms of the form

$$
\partial_x \partial_p O_1^{(Z)} \cdot \partial_x \partial_p O_2^{(Z)} \to \mathcal{D} \partial_p O_1 \cdot \mathcal{D} \partial_p O_2 + F \cdot \partial_p^2 O_1 \cdot \mathcal{D} \partial_p O_2 + \cdots \quad (3.31)
$$

so there is a term $\mathcal{D} F \cdot \partial_p O_1 \cdot \partial_p^2 O_2$ of the third order in gradient appearing when taking the mapping (3.10) of the second order of the $\ast_0$-expansion (and fourth order in gradient). This is an anomalous term which avoids to resum all the expansion in a convenient way using the method presented here.

At the next order (third in $\ast_0$, and sixth in gradient), one has terms of the form

$$
\partial_x^3 O_1^{(Z)} \cdot \partial_p^3 O_2^{(Z)} \to \mathcal{D}^2 F \cdot \partial_p O_1 \cdot \partial_p^3 O_2 + \mathcal{D}^3 F \cdot \partial_p^2 O_1 \cdot \partial_p^2 O_2 + \cdots + \mathcal{D}^2 O_1 \cdot \mathcal{D}^3 O_2 + \cdots \quad (3.32)
$$

where a fourth order term in gradient thus appear. The other contributions in the $\ast_0$ expansion, e.g.

$$
\partial_x^2 \partial_p O_1^{(Z)} \cdot \partial_x \partial_p O_2^{(Z)} \to \mathcal{D} F \cdot \partial_p^2 O_1 \cdot \mathcal{D} \partial_p^2 O_1 + \cdots \quad (3.33)
$$

will be of higher power in gradient (here we neglect terms of order 5 in gradient).

Thus, the price to pay for the use of the mapping (3.10) allowing the transformation from the flat Moyal product $\ast_0$ to the covariant one $\ast$ is a cumbersome counting of the order in a
gradient expansion. Since we want to stop our calculation to the fourth order in gradients, we will not give the full calculation of the evaluation of the third order in the $\star_0$-expansion

$$\star_0 = 1 + \frac{i}{2} \left( \partial_x^2 \partial_p - \partial_p^2 \partial_x \right) + \frac{1}{2} \left( \partial_x^2 \right)^2 \left( \partial_p^2 \right)^2 + \frac{1}{6} \left( \partial_x^2 \right)^3 \left( \partial_p^2 \partial_x - \partial_p^2 \partial_x \right)^2 + \cdots \quad (3.34)$$

namely of the term $\partial^3O(Z)|_{Z=x}$ but give the term of the form $\hat{D}^2F \cdot \partial_p O$ only. The third order derivative looks like

$$\frac{\partial^3 O(p, x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} = \left( \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha} \right) \left( \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial Z^\beta} \right) \left( \frac{\partial}{\partial x^\gamma} + \frac{\partial}{\partial Z^\gamma} \right) O^{(Z)}(p, x) \bigg|_{Z=x}$$

$$= \left( \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial Z^\alpha} \right) \left( \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial Z^\beta} \right) \left( \partial O^{(Z)} + i \left[ F^{(Z)}_\gamma (Z, x + z), O^{(Z)} \right] \right) \bigg|_{Z=x} + O \left( \partial^2 \right) \quad (3.35)$$

where we can neglect the terms in $A_\alpha$ since they will be of higher power in gradient (they would generate some covariant contribution $\hat{D}^3O$ when they are all taken into account properly since the expansion is covariant – we disregard these terms as being of higher order in gradient), as well as the terms $\partial_z F$ since they do not generate any term of the form $\hat{D} F$ and will thus be of higher power in $\partial_p$ as well. Now, we use the definition (3.16) to substitute $\partial_z O^{(Z)}$ with some $\left[ F^{(Z)}_\gamma (Z, x), O^{(Z)} \right]$, and we obtain

$$\frac{\partial^2 O^{(Z)}(p, x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \bigg|_{Z=x} = -3i \left[ \frac{\partial^2}{\partial x^\alpha \partial x^\beta} F^{(Z)}_\gamma (Z, x + z), O^{(Z)} \right] \bigg|_{Z=x} + O \left( \partial^2 \right) \quad (3.36)$$

at the leading order in gradient. We used that the Moyal expansion at third order gives a contribution $\partial_x^3 O_1 \partial_p^3 O_2$ and therefore the indices $\alpha, \beta$ and $\gamma$ are all contracted in the final result. There are thus three terms of the form $\partial_x^2 F$ since we discuss the third order derivatives. There were in fact two terms of the form (3.25) in the second order substitution (3.24), one being hidden in the expression of $\hat{D}^2 O(p, x)$ as one can check by expansion of the second order of (3.17).

Evaluating the second derivative of $F$ before taking the limit $Z \to x$ gives, at the leading order

$$\frac{\partial^3 O^{(Z)}(p, x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \bigg|_{Z=x} = \frac{3}{2} \int_0^1 ds \left( 1 - s \right)^2 \left[ \frac{\hat{D}^2 F^{(Z)}_\gamma (x)}{\partial x^\alpha \partial x^\beta} \left( \partial_p - \partial_p^1 \right)_\delta, O(p, x) \right] + O \left( \partial^2 \right)$$

$$= \frac{1}{8} \left\{ \frac{\hat{D}^2 F^{(Z)}_\gamma (x)}{\partial x^\alpha \partial x^\beta}, \frac{\partial O(p, x)}{\partial p_\delta} \right\} + O \left( \partial^2 \right) \quad (3.37)$$

which should be injected in the gradient expansion when one wants to retain fourth order terms in gradient.

We are now ready to give the expansion of the non-Abelian gauge covariant Moyal product at fourth order in gradient.

### C. Gauge-covariant gradient expansion

To get the gradient expansion at the desired order, we simply expand the different contributions (3.17) and (3.24) up to the fourth order in gradient, and add the anomalous contribution (3.37) coming from the higher order.
The expansion of the mapping (3.10) at first order in derivatives is already explicitly given in (3.20). One has

\[
\frac{\partial O(Z)}{\partial x^\alpha} \bigg|_{Z=x} = \mathfrak{D} O(p, x) + \frac{1}{4} \left\{ F_{\alpha\delta} (x), \frac{\partial O}{\partial p_\delta} \right\} + \left( \frac{i}{24} \mathfrak{D} F_{\alpha\delta}, \frac{\partial^2 O(p, x)}{\partial p_\gamma \partial p_\delta} \right) - \frac{1}{192} \left\{ \mathfrak{D}^2 F_{\alpha\delta}, \frac{\partial^3 O(p, x)}{\partial p_\gamma \partial p_\delta \partial p_\delta} \right\} + O (\partial^4) \tag{3.38}
\]

where the alternance of anti-commutators and commutators comes from the structure of (3.20) in power of \( (\partial_p - \partial_p^\dagger) \). We need the third order in gradient, since in the expansion of \( \star_0 \), the term \( \partial_x O(Z) \big|_{Z=x} \) appears in combination with a first order derivative \( \partial_p O(p, x) \) which is not affected by the mapping.

For the second order derivative, one uses (3.28) to get

\[
\frac{\partial^2 O(Z)}{\partial x^\alpha \partial x^\beta} \bigg|_{Z=x} = \frac{\partial^2 O(p, x)}{\partial x^\alpha \partial x^\beta} + \frac{1}{4} \left\{ F_{\beta\gamma} (x), \frac{\mathfrak{D} \partial O(p, x)}{\partial x^\alpha \partial p_\gamma} \right\} + \frac{1}{3} \left\{ \mathfrak{D} F_{\beta\gamma}, \frac{\partial O}{\partial p_\gamma} \right\} + \left( \frac{i}{16} \mathfrak{D}^2 F_{\beta\gamma}, \frac{\partial^2 O(p, x)}{\partial p_\delta \partial p_\delta} \right) + \frac{1}{16} \left\{ F_{\alpha\delta} (x), \left\{ F_{\beta\gamma} (x), \frac{\partial^2 O(p, x)}{\partial p_\delta \partial p_\delta} \right\} \right\} + O (\partial^3) \tag{3.39}
\]

with the anomalous term \( \mathfrak{D} F \cdot \partial_p O \) discussed in section III.B. Note that the covariant derivative \( \mathfrak{D} \) and the momentum derivative \( \partial_p \) commute.

Finally, the anomalous term at the third order (3.37) should be taken in order to get the complete gauge-covariant Moyal expansion up to the fourth order. The covariant Moyal product then reads

\[
O_1(p, x) \star O_2(p, x) = O_1(p, x) O_2(p, x) + \frac{i}{2} \left( \frac{\partial O_1}{\partial x^\alpha} \bigg|_{Z=x} \frac{\partial O_2}{\partial p_\alpha} - \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial x^\alpha} \bigg|_{Z=x} \right) + \frac{1}{2} \left( \frac{i}{2} \right)^2 \left( \frac{\partial^2 O_1}{\partial x^\alpha \partial x^\beta} \bigg|_{Z=x} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} - 2 \frac{\partial}{\partial p_\alpha} \frac{\partial O_2}{\partial x^\beta} \bigg|_{Z=x} \right) + \frac{i}{6} \left( \frac{i}{2} \right)^3 \left( \frac{\partial^3 O_1}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \bigg|_{Z=x} \frac{\partial^3 O_2}{\partial p_\alpha \partial p_\beta \partial p_\gamma} - 2 \frac{\partial}{\partial p_\alpha} \frac{\partial^2 O_2}{\partial x^\beta \partial x^\gamma} \bigg|_{Z=x} \right) + O (\partial^5) \tag{3.40}
\]

where the other terms of the expansion \( \left( \partial_x^i \partial_p - \partial_p^i \partial_x \right) \) do not contribute to the fourth order in gradient, as explained after (3.28). We just have to substitute (3.38), (3.39) and (3.37) in the above expansion to get the covariant Moyal product.

After a few algebra, and classifying the terms order by order in gradients, one finally has

\[
O_1(p, x) \star O_2(p, x) = O_1(p, x) O_2(p, x) + \mathcal{O}_\star (\partial^2) + \mathcal{O}_\star (\partial^3) + \mathcal{O}_\star (\partial^4) + O (\partial^5) \tag{3.41}
\]

where the first order term

\[
\mathcal{O}_\star (\partial^2) = \frac{i}{2} \left( \frac{\mathfrak{D} O_1 \partial O_2}{\partial x^\alpha} \bigg|_{Z=x} - \frac{\partial O_1}{\partial p_\alpha} \frac{\mathfrak{D} O_2}{\partial x^\alpha} \right) - \frac{i}{8} \left( F_{\alpha\beta} (x) \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial p_\beta} + 2 \frac{\partial O_1}{\partial p_\alpha} F_{\alpha\beta} (x) \frac{\partial O_2}{\partial p_\beta} + \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial p_\beta} F_{\alpha\beta} (x) \right) \tag{3.42}
\]
and the second order term

\[ \mathcal{O}_2 (\partial^2) = \frac{1}{12} \left( \frac{\partial O_1}{\partial p_\gamma} \frac{\partial F_{\gamma\beta}}{\partial x^\alpha} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \frac{\partial F_{\gamma\beta}}{\partial x^\alpha} \partial p_\gamma \right) + \]

\[ + \frac{1}{24} \left[ \frac{\partial F_{\gamma\beta}}{\partial x^\alpha} \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \partial p_\gamma - \frac{\partial O_1}{\partial p_\gamma} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} \right] \]  

(3.43)

were already written in (1.9). We here add the fourth order term

\[ \mathcal{O}_4 (\partial^4) = -\frac{1}{8} O_1 (p, x) \left( \mathcal{D}_x^\dagger \partial_x - \partial_x \mathcal{D}_x \right)^2 O_2 (p, x) + \]

\[ + \frac{1}{32} \left( \left\{ F_{\gamma\beta}, \frac{\partial \mathcal{D} O_1}{\partial x^\alpha \partial p_\gamma} \right\} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + 2 \frac{\partial \mathcal{D} O_1}{\partial x^\alpha \partial p_\gamma} \left\{ F_{\gamma\beta}, \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} \right\} \right) + \]

\[ + \frac{1}{32} \left( 2 \left\{ F_{\gamma\beta}, \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \right\} \frac{\partial \mathcal{D} O_2}{\partial x^\alpha \partial p_\gamma} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \left\{ F_{\gamma\beta}, \frac{\partial \mathcal{D} O_2}{\partial x^\alpha \partial p_\gamma} \right\} \right) + \]

\[ + \frac{i}{128} \left( \left\{ \frac{\partial^2 F_{\gamma\delta}}{\partial x^\alpha \partial x^\beta}, \frac{\partial \mathcal{D} O_1}{\partial p_\delta} \right\} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \left\{ \frac{\partial^2 F_{\gamma\delta}}{\partial x^\alpha \partial x^\beta}, \frac{\partial \mathcal{D} O_2}{\partial p_\delta} \right\} \right) + \]

\[ + \frac{i}{32^2} \left( \left\{ \frac{\partial^3 O_1}{\partial p_\alpha \partial p_\beta \partial p_\gamma} \frac{\partial^2 F_{\delta\mu}}{\partial x^\alpha \partial x^\beta}, \frac{\partial O_2}{\partial p_\delta} \right\} \frac{\partial O_2}{\partial p_\mu} - (\partial^2 \leftrightarrow \partial_p) \right) + \]

\[ - \frac{1}{128} \left( \left\{ F_{\alpha\delta}, \left\{ F_{\beta\gamma}, \frac{\partial^2 O_1}{\partial p_\delta \partial p_\gamma} \right\} \right\} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + \left\{ \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \left\{ F_{\alpha\delta}, \frac{\partial^2 O_2}{\partial p_\delta \partial p_\gamma} \right\} \right\} \right) - \]

\[ - \frac{1}{64} \left( \left\{ F_{\beta\gamma}, \frac{\partial^2 O_1}{\partial p_\delta \partial p_\gamma} \right\} \left\{ F_{\alpha\delta}, \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} \right\} \right) \]  

(3.44)

for the gauge-covariant Moyal expansion up to the four order in gradient of the operator. We define \( \mathcal{D}_x^\dagger = \partial_x O + i [O, A] = \mathcal{D}_x O \). The symbol \( (\partial^2 \leftrightarrow \partial_p) \) signifies that one must permute the triple and single derivatives, including their indices, without changing the indices of the covariant derivative of the gauge field, see (3.43) which can be noted

\[ \mathcal{O}_2 (\partial^2) = \frac{1}{12} \left( \frac{\partial O_1}{\partial p_\gamma} \frac{\partial F_{\gamma\beta}}{\partial x^\alpha} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + (\partial^2 \leftrightarrow \partial_p) \right) + \frac{1}{24} \left[ \frac{\partial F_{\gamma\beta}}{\partial x^\alpha} \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \partial p_\gamma - (\partial^2 \leftrightarrow \partial_p) \right] \]  

(3.45)

as well.

Expressions (3.41)-(3.44) constitute the first important result of this paper. It validates the method explained in section III.A aiming at calculating a gauge-covariant Moyal product at any order.

**D. Abelian and pure gauge cases**

In this section, we review some easier situations than the non-Abelian covariant case.

We first discuss the case of a pure gauge, characterised by a vanishing gauge field \( F_{\alpha\beta} = 0 \) but still having non-trivial gauge potential. This case exists only in the non-Abelian situation.

In this case one has

\[ \frac{D U (b, a)}{D b^a} U (a, b) = 0 \]  

(3.46)
and so $F = 0$ in all the calculations done in section III.B. From the expansion of the $\star$-product in (3.41), one might expect that

$$\lim_{F \to 0} \star = \exp \left[ \frac{i\hbar}{2} \left( \frac{\mathcal{D}^\dagger}{\partial x^\alpha} \frac{\partial}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha} \frac{\mathcal{D}^\dagger}{\partial x^\alpha} \right) \right]$$

(3.47)

with $O(p, x) \mathcal{D}^\dagger = O(p, x) \left[ \partial^\dagger_\alpha + i [, A_\alpha] \right] = \partial_\alpha O - i [A_\alpha, O]$. To show that, we simply write, from the definition (3.5) of $O^{(Z)}$

$$O^{(Z)}(p, x) = U(Z, x) O(p, x) U(x, Z) \Rightarrow \frac{\partial O^{(Z)}}{\partial x^\alpha} = U(Z, x) \frac{\mathcal{D} O(p, x)}{\partial x^\alpha} U(Z, x)$$

(3.48)

since $\partial_\alpha U(x, Z) U(Z, x) = iA_\alpha(x)$ in the limit of vanishing gauge field, see (3.46). There is thus no more $Z$-dependency in the covariant derivative $\mathcal{D} O(p, x)$. We thus have generically

$$\frac{\partial^n O^{(Z)}}{\partial x^n} = U(Z, x) \frac{\mathcal{D}^n O(p, x)}{\partial x^n} U(Z, x) \Rightarrow \frac{\partial^n O^{(Z)}}{\partial x^n} \bigg|_{Z=x} = \frac{\mathcal{D}^n O(p, x)}{\partial x^n}$$

(3.49)

and the limit $Z \to x$ is taken immediately with $U(x, x) = 1$, proving the Moyal product (3.47) in the pure gauge case. The pure gauge case is in particular useful to describe 1D situations in the quasi-static limit. In that case, only the gauge potentials $A_x$ and $A_0$ might survive, and the electric field is generated by the covariant derivative of $A_0$: $\mathcal{D}_x A_0 = \partial_x A_0 - i [A_x, A_0]$ corresponds to the definition of the quasi-static electric field.

In the case of Abelian field, one simply assumes that all gauge fields commute with the operators $O_{1,2}$, and that these later commute among themselves. In addition, the covariant derivatives $\mathcal{D}_x$ collapse to the usual ones $\partial_x$. So one has, at leading order

$$O_1(p, x) \star O_2(p, x) = O_1(p, x) O_2(p, x) + \frac{i}{2} \left( \frac{\partial O_1}{\partial x^\alpha} \frac{\partial O_2}{\partial p_\alpha} - \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial x^\alpha} - F_{\alpha \beta}(x) \frac{\partial O_1}{\partial p_\alpha} \frac{\partial O_2}{\partial p_\beta} \right) +$$

$$+ \frac{1}{12} \frac{\partial F_{\gamma \beta}}{\partial x^\alpha} \left( \frac{\partial O_1}{\partial p_\gamma} \frac{\partial^2 O_2}{\partial p_\alpha \partial p_\beta} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \frac{\partial O_2}{\partial p_\gamma} + \frac{\partial^2 O_1}{\partial p_\alpha \partial p_\beta} \frac{\partial O_2}{\partial p_\gamma} \right) + \mathcal{O}(\partial^4)$$

(3.50)

as first obtained in (Lein, 2010; Mueller, 1998). Note nevertheless that, in some cases, the terms are classified in different ways. For instance, the term $\partial F \cdot \partial^2 O_1 \cdot \partial p O_2$ is sometimes thought as a fourth order term in gradient, and thus comes in regards of terms like $\partial^2 O_1 \cdot \partial^2 O_2$, e.g. in (Karasev and Osborn, 2004). We choose here to expand in powers of the gradients of the operators $O_{1,2}$, rendering the second line of (3.50) a third order term.

We no more elaborate on the limit of Abelian gauge fields, since it has been investigated in many situations, see e.g. (Altshuler and Ioffe, 1992; Best et al., 1993; Bialynicki-Birula, 1977; Bialynicki-Birula et al., 1991; Javanainen et al., 1987; Kelly, 1964; Kubo, 1964; Langreth, 1966; Levanda and Fleurov, 1994, 2001; Luttinger, 1951; Serimaa et al., 1986; Stratonovich, 1956; Swiecicki and Sipe, 2013; Zachos and Curtright, 1999) and references therein.

IV. GAUGE-COVARIANT GRADIENT EXPANSION IN THE PHASE-SPACE

In all the previous sections, we discussed the situation when the gauge fields were position dependent. With the upraising of topological matter, it becomes of utmost importance to understand the possible non-trivial topologies of the band structure, see e.g. (Bamler, 2016;
Hasan and Kane, 2010; Nagaosa et al., 2010; Xiao et al., 2009) for reviews. Such non-trivial geometry comes from some gauge structure in the momentum space, as introduced quickly in section I.B. Despite the possible difficulties to deal with periodic systems in the Brillouin zone, the topology of the band structure might naively be thought as a momentum space topology in the phase space. In fact, naive interpretation along these lines seems to correctly describe the phenomenology of the already known materials, see e.g. (Bamler, 2016; Nagaosa et al., 2010; Xiao et al., 2009) and references therein. In this section, we discuss how to deal with such structure, when gauge field appears in the form of

\[ F^{\alpha\beta}(p) = \partial^\alpha A^\beta(p) - \partial^\beta A^\alpha(p) - i [A^\alpha(p), A^\beta(p)] \]  

(4.1)

instead of (1.10). We will note all the p-derivative with upper indices as in \( \partial^\alpha f \equiv \partial f / \partial p_\alpha \), to distinguish easily \( F_{\alpha\beta}(x) \) from \( F^{\alpha\beta}(p) \).

A. Momentum-like gauge-covariant gradient expansion

In this section, we discuss the properties of the gauge-covariant Moyal expansion in the reciprocal space. Namely, we define another Wigner transform being covariant in the momentum space instead of the position space as (1.4) was. Since the associated Moyal expansion can be obtained from (3.41) with minor replacements (essentially \( x \leftrightarrow p \)), we will be quick with the presentation.

As discussed in section III, the gauge-covariant Wigner transform can be constructed heuristically from the definition (3.1) where we substitute the flat derivative by covariant ones \( \partial_x \rightarrow D_x = \partial_x - i A_x(x) \), ultimately giving the parallel transport operators \( U(x_1, x_2) \) in the definition of a covariant Wigner transform (1.4). For the sake of pedagogy, we slightly change the notation here, and introduce the notation

\[ O^x(p, x) = \int dx \left[ e^{-ip \cdot x} \left[ e^{ip_1 / 2} O(x_1, x_2) e^{-ip_2 / 2} \right]_{x_1 = x_2 = x} \right] \]  

(4.2)

for the covariant Wigner transform of the operator \( O(x_1, x_2) \) that we discussed all along the previous sections.

When dealing with band structure calculations, one might prefer to start from the operator \( O(p_1, p_2) \) in the momentum representation. Since the momentum representation \( O(p_1, p_2) \) is obtained from the space representation \( O(x_1, x_2) \) via Fourier transformation, one can define as well the Wigner transform in the alternative form, see (1.1),

\[ O(p, x) = \int \frac{dp}{2\pi} \left[ e^{ip \cdot x} \left[ e^{p_1 / 2} O(p_1, p_2) e^{-p_2 / 2} \right]_{p_1 = p_2 = p} \right] \]

\[ \Rightarrow O^p(p, x) = \int \frac{dp}{2\pi} \left[ e^{ip \cdot x} \left[ e^{p_1 / 2} O(p_1, p_2) e^{-p_2 / 2} \right]_{p_1 = p_2 = p} \right] \]  

(4.3)

with \( D_p = \partial_p - i A(p) \). We note the gauge potential in the momentum space using the same symbols as the gauge potential \( A(x) \) in the real space, their variables make explicit which one we are dealing with. Despite the two representations of the flat-space Wigner transforms in (1.1), starting either from \( O(x_1, x_2) \) or \( O(p_1, p_2) \) and taking the partial Fourier transform with respect to \( x = x_1 - x_2 \) or \( p = p_1 - p_2 \), are strictly equivalent, their covariant extensions
$O^x$ and $O^p$ are not related by any Fourier transform. Clearly, $O^x$ in (4.2) is well adapted to treat problems with gauge fields in the position space. By extension, it becomes clear that $O^p$ in (4.3) would be of interest for problems with non-trivial geometry in the momentum space. So in this section we discuss briefly the phase-space gradient extension using (4.3) instead of (4.2) as the gauge-covariant Wigner transform.

Nevertheless, discussion can become quite short when realizing that the property (3.2) works in the same way in either the $x$ or $p$ spaces. So whenever we had

$$O^x (p, x) = \int d\mathbf{x} \left[ e^{-i p \cdot \mathbf{x}} U \left( \mathbf{x}, \mathbf{x} + \frac{\mathbf{r}}{2} \right) O \left( \mathbf{x} + \frac{\mathbf{r}}{2}, \mathbf{x} - \frac{\mathbf{r}}{2} \right) U \left( \mathbf{x} - \frac{\mathbf{r}}{2}, \mathbf{x} \right) \right]$$  (4.4)

in section III, we will now have

$$O^p (p, x) = \int \frac{d\mathbf{p}}{2\pi \hbar} \left[ e^{i p \cdot \mathbf{x}} U \left( \mathbf{p}, \mathbf{p} + \frac{\mathbf{r}}{2} \right) O \left( \mathbf{p} + \frac{\mathbf{r}}{2}, \mathbf{p} - \frac{\mathbf{r}}{2} \right) U \left( \mathbf{p} - \frac{\mathbf{r}}{2}, \mathbf{p} \right) \right]$$  (4.5)

in the present section. In particular, one can write symbolically

$$O^x (p, x) = U \left( \mathbf{x}, \mathbf{x} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}} \right) \int d\mathbf{x} \left[ e^{-i p \cdot \mathbf{x}} O \left( \mathbf{x} + \frac{\mathbf{r}}{2}, \mathbf{x} - \frac{\mathbf{r}}{2} \right) U \left( \mathbf{x} - \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, \mathbf{x} \right) \right]$$  (4.6)

using the usual Bopp substitution (Polkovnikov, 2010), see also (3.15). We recognise the usual flat-space Wigner transform inside the formal $U$’s operators. We write thus in symbolic form

$$O^x (p, x) = U \left( \mathbf{x}, \mathbf{x} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}} \right) O (p, x) U \left( \mathbf{x} - \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, \mathbf{x} \right)$$  (4.7)

and, accordingly, one has

$$O^p (p, x) = U \left( \mathbf{p}, \mathbf{p} - \frac{i}{2} \frac{\partial}{\partial \mathbf{x}} \right) O (p, x) U \left( \mathbf{p} + \frac{i}{2} \frac{\partial}{\partial \mathbf{x}}, \mathbf{p} \right)$$  (4.8)

for the covariant Wigner transform with momentum non-trivial geometric structures.

Next, whenever we wanted to construct the covariant Moyal product, we introduced the mapping (3.10) allowing to transform the $x$-derivatives to some covariant ones. This was insured by the definition of the $O^{(x)} (p, x)$ Wigner transform (3.5) which formally looks like the Wigner transform in the flat space $O (p, x)$ (in the notation of this section, $O (p, x)$ is the non-covariant Wigner transform (1.1)). So the mapping (3.10) can be formally represented as the transformation from the flat space to the covariant representations, and in the notations of this section, one would have

$$O^x_1 (p, x) \star O^x_2 (p, x) = U \left( \mathbf{x}, \mathbf{x} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}} \right) [O_1 (p, x) \star_0 O_2 (p, x)] U \left( \mathbf{x} - \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, \mathbf{x} \right)$$  (4.9)

instead of the transformation from $O^{(x)}$ to $O^x$. Said differently, the mappings (3.10) and (4.9) are equivalent. Finally, the reciprocal Moyal product would be defined as

$$O^p_1 (p, x) \star O^p_2 (p, x) = U \left( \mathbf{p}, \mathbf{p} - \frac{i}{2} \frac{\partial}{\partial \mathbf{x}} \right) [O_1 (p, x) \star_0 O_2 (p, x)] U \left( \mathbf{p} + \frac{i}{2} \frac{\partial}{\partial \mathbf{x}}, \mathbf{p} \right)$$  (4.10)
allowing to use all the tricks developed in section III.B with only a few adaptations. For instance, instead of (3.20), we will have now

\[
U \left( p, p - \frac{i}{2} \frac{\partial}{\partial x} \right) \frac{\partial O}{\partial p_\alpha} U \left( p + \frac{i}{2} \frac{\partial}{\partial x}, p \right) =
\]

\[
= \frac{\partial O (p, x)}{\partial p_\alpha} + \sum_{n=0}^{\infty} \frac{i^n (z \cdot \mathcal{D}_z)^n}{2^n (n + 2)!} F_{\alpha\beta} (p) z_\beta, O (p, x) \mid_{z=\partial_x - \partial^*_x} \tag{4.11}
\]

for the replacement of the first order \( p \)-derivative, with

\[
\frac{\partial O}{\partial p_\alpha} = \frac{\partial O}{\partial p_\alpha} - i \left[ A^{\alpha} (p), O (p, x) \right] ; F_{\alpha\beta} (p) = \frac{\partial A^\beta}{\partial p_\alpha} - \frac{\partial A^\alpha}{\partial p_\beta} - i \left[ A^{\alpha} (p), A^{\beta} (p) \right] \tag{4.12}
\]

so we essentially replace the covariant derivative by contravariant ones, and raise up the indices, keeping in mind the usual symplectic structure of the phase space.

We do not give the gradient expansion of the Moyal product, since it is mutadis mutandis the one given in (3.41): instead of replacing the \( x \)-derivatives in the \( \star_0 \)-product and keeping the \( p \)-derivatives in section III, we now replace the \( p \)-derivatives in the \( \star_0 \)-product, and let the \( x \)-derivative untouched. For the first derivative the substitution reads (4.11).

We conclude this section with a few remarks about the gauge structure of the momentum space. Firstly, its relation to the periodic band structure is not yet established, a task kept for later studies. A few first steps were done in (Zubkov, 2016), and it seems at first to be not much problematic, at least from a physicist point of view. Its rigorous mathematical establishment might be more complex, though. Secondly, we did not discuss the origin of the gauge potential \( A (p) \) and its covariant generalization to the gauge field \( F_{\alpha\beta} (p) \). As in the case of the \( x \)-space non-trivial geometry, the operators appearing in the Moyal product (3.41) would be the bare ones. So, an understanding of the non-trivial geometry in the momentum space must be done at an other, perhaps more microscopic, level. Reader interested in the topology of the band structure might consult (Bohm et al., 2003; Chruściński and Jamiolkowski, 2004; Shapere and Wilczek, 1989) and references therein, as well as (Bamberger, 2016; Hasan and Kane, 2010; Nagaosa et al., 2010; Xiao et al., 2009). Recent studies (Hidaka et al., 2016; Son and Spivak, 2013; Son and Yamamoto, 2012, 2013; Stephanov and Yin, 2012) discovered how the Berry connection associated to the band structure of Weyl semi-metal appears at the quasi-classical level using non-covariant gradient expansions of a topological action, see also (Duval et al., 2006a,b; Stone and Dwivedi, 2013) and references therein for a bit more mathematically oriented approach. Other approaches use the deformed wave-packet dynamics and its adiabatic corrections to get such phenomenologies, see (Xiao et al., 2009) for a review.

\section*{B. Phase-space gradient expansion in a covariant manner}

Section III was devoted to the construction of a covariant method to generalise the Moyal product to non-Abelian gauge structure in the position space, and section IV.A showed how it is possible to adapt such formalism to the situation when the system present non-trivial geometry in the momentum space. None of the gradient expansions developed so far present a complete symmetry of position and momentum, since the gauge structure was non-trivial only in either the position or the momentum space, not in the entire phase space. I would like
to bring a few arguments in favor of a generalisation of the two above approaches towards a
phase-space gradient expansion, in a fully covariant manner at all stages of the calculation.
To explore such a possibility, we start from the Dyson’s equation
\[ \int dy \left[ G^{-1}(x_1, y) G(y, x_2) \right] = \delta(x_1 - x_2) \] (4.13)
which transforms under the transformation (4.2) and the mapping (4.9) as
\[ U_x \left[ G^{-1}(p, x) \star_0 G(p, x) \right] U_x = 1 \]
with \( U_x = U \left( x, x + \frac{i}{2} \frac{\partial}{\partial p} \right) \); \( \bar{U}_x = U \left( x - \frac{i}{2} \frac{\partial}{\partial p}, x \right) \) (4.14)
with \( G(p, x) \) the Wigner transform (1.1) of the Green’s function. We have similarly
\[ U_p \left[ G^{-1}(p, x) \star_0 G(p, x) \right] U_p = 1 \]
with \( U_p = U \left( p, p - \frac{i}{2} \frac{\partial}{\partial x} \right) \); \( \bar{U}_p = U \left( p + \frac{i}{2} \frac{\partial}{\partial x}, p \right) \) (4.15)
using the transformation (4.3) and the mapping (4.10) being covariant in the momentum
space. Because we choose to transform the Dyson’s equation, the right-hand-side is always
the identity matrix, and so nothing forbids to apply the two mappings (4.9) and (4.10)
successively, i.e. to define
\[ U_p U_x \left[ G^{-1}(p, x) \star_0 G(p, x) \right] U_x U_p = 1 \] (4.16)
as the phase-space representation of the Dyson’s equation. At first order in the gradient, one has
\[ \frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial}{\partial x^\alpha} U_x \frac{\partial}{\partial x^\alpha} U_x = \frac{\partial}{\partial x^\alpha} + \frac{1}{4} \left\{ F_{\alpha\beta}(x), \frac{\partial}{\partial p_\beta} \right\} + \cdots \] (4.17)
\[ \frac{\partial}{\partial p_\alpha} \rightarrow \frac{\partial}{\partial p_\alpha} U_p \frac{\partial}{\partial p_\alpha} U_p = \frac{\partial}{\partial p_\alpha} + \frac{1}{4} \left\{ F_{\alpha\beta}(p), \frac{\partial}{\partial x^\beta} \right\} + \cdots \] (4.18)
whereas \( \partial_x \) is an invariant of the \( U_p \) mapping and \( \partial_p \) is invariant under the \( U_x \) mapping.
So applying only one time \( U_{x,p} \) on the Dyson’s equation does not generate fully gauge
covariant expressions in the full phase-space. To remedy this problem, we apply as many
\( U_{x,p} \) operations than one needs. For instance, at first order in gradient, and second order in
fields, one would have
\[ \frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial}{\partial x^\alpha} + \frac{1}{4} \left\{ F_{\alpha\beta}(x), \frac{\partial}{\partial p_\beta} \right\} + \frac{1}{16} \left\{ F_{\alpha\beta}(x), \left\{ F_{\beta\gamma}(p), \frac{\partial}{\partial p_\gamma} \right\} \right\} + \cdots \]
\[ \frac{\partial}{\partial p_\alpha} \rightarrow \frac{\partial}{\partial p_\alpha} + \frac{1}{4} \left\{ F_{\alpha\beta}(p), \frac{\partial}{\partial x^\beta} \right\} + \frac{1}{16} \left\{ F_{\alpha\beta}(p), \left\{ F_{\beta\gamma}(x), \frac{\partial}{\partial p_\gamma} \right\} \right\} + \cdots \] (4.19)
and we note this mapping with the special arrow \( \rightarrow \) for commodity. We confess the mapping
is clearly not well defined that way, but we give heuristic arguments for its construction.
Clearly, the mapping (4.19) requires an extra scale than the usual gradient expansion to be
truncated at the desired order. Here, we again used heuristic arguments, truncating it at
first order in gradient and second order in gauge fields in an arbitrary manner. Nevertheless, the mapping (4.19) is clearly covariant in the entire phase-space, as required.

Using the heuristic mapping (4.19), one obtains for the gradient expansion of the Dyson equation, which we note with a big star

\[
G^{-1}(p, x) \star G(p, x) = G^{-1}(p, x) G(p, x) + \frac{i}{2} \left( \frac{\mathcal{D}G^{-1}}{\partial p_\alpha} \frac{\partial}{\partial p_\alpha} - \frac{\partial}{\partial x_\alpha} \right) - \frac{i}{8} \left( F_{\alpha \beta}(x) \frac{\partial G^{-1}}{\partial p_\alpha} \frac{\partial G}{\partial p_\beta} + 2 \frac{\partial G^{-1}}{\partial p_\alpha} F_{\alpha \beta}(x) \frac{\partial G}{\partial p_\beta} + \frac{\partial G^{-1}}{\partial p_\alpha} \frac{\partial G}{\partial p_\beta} F_{\alpha \beta}(x) - \left( \frac{\mathcal{D}}{p_\alpha} \rightarrow \mathcal{D}_x \right) \right) - \frac{i}{32} \left( \left\{ \frac{\mathcal{D}G^{-1}}{\partial p_\alpha} \right\} \left\{ F^{\beta \gamma}, \frac{\mathcal{D}G}{\partial x_\gamma} \right\} + \left\{ F^{\beta \gamma}, \left\{ F^{\alpha \beta}, \frac{\mathcal{D}G^{-1}}{\partial p_\alpha} \right\} \right\} \frac{\mathcal{D}G}{\partial x_\gamma} + \frac{\partial G^{-1}}{\partial p_\alpha} \left\{ F^{\beta \gamma}, \frac{\mathcal{D}G}{\partial x_\gamma} \right\} \right) + \frac{i}{32} \left( \left\{ \frac{\mathcal{D}}{p_\alpha} \rightarrow \mathcal{D}_x \right\} \left\{ F^{\alpha \beta}(x) \rightarrow F^{\alpha \beta}(p) \right\} \right) + \cdots = 1 \quad (4.20)
\]

where the indices of the derivatives are conserved under the substitution \( \mathcal{D}_p \leftrightarrow \mathcal{D}_x \).

We have been able to justify the multiple applications of the transformations \( U_{x,p} \) on the derivatives because the right-hand-side of the Dyson’s equation is the unit matrix, which is an invariant of the \( U_{x,p} \)'s. Nevertheless, one sees easily that (4.20) reduces to the covariant \( \star \)-product established in section III when \( F^{\alpha \beta}(p) \rightarrow 0 \), and incidentally it reduces to the \( \star \)-product established in section IV.A in the limit \( F^{\alpha \beta}(x) \rightarrow 0 \). We thus promote – without more justification – the \( \star \)-product (4.20) as an eligible Moyal product in the phase-space. We now study a few immediate consequences of this Moyal algebra, and see that it completes some previously calculated star products appearing in the study of effective theories.

From (4.20), replacing \( G^{-1} \) and/or \( G \) with some variables \( x^\alpha \) and/or \( p_\beta \), one gets

\[
[x^\alpha, p_\beta]_\star = x^\alpha \star p_\beta - p_\beta \star x^\alpha = i \epsilon^\alpha_\beta + \frac{3i}{8} \left\{ F^{\alpha \gamma}(p), F_{\gamma \beta}(x) \right\}
\]

\[
[x^\alpha, x^\beta]_\star = x^\alpha \star x^\beta - x^\beta \star x^\alpha = i F^{\alpha \beta}(p)
\]

\[
[p_\alpha, p_\beta]_\star = p_\alpha \star p_\beta - p_\beta \star p_\alpha = -i F_{\alpha \beta}(x)
\]

for the non-commutative algebra of the phase-space. Previous works essentially discussed either the momentum space covariant construction (when \( F_{\alpha \beta} = 0 \)) being non trivial when monopoles are present in the band structures (Béard and Mohrbach, 2004; Duval and Horváthy, 2001; Sundaram and Niu, 1999) (see also (Volovik, 2003; Xiao et al., 2009) for reviews) or the position space covariant construction (when \( F^{\alpha \beta} = 0 \)) being non trivial under magnetic field – this is the famous Landau problem important for the quantum Hall effect (Bellissard et al., 1994). The price to pay to associate both effects is the higher order term \( \left\{ F^{\alpha \gamma}(p), F_{\gamma \beta}(x) \right\} \) appearing in the dynamics. This term has been identified in the past in the Abelian limit (Bliokh, 2006; Duval et al., 2006a,b; Xiao et al., 2005), and it is responsible for some characteristic phenomenologies of the Weyl semimetals, like the anomalous Hall effect (see (Nagaosa et al., 2010) for a review on related phenomena), the chiral anomaly (Son and Yamamoto, 2012, 2013; Stephanov and Yin, 2012) and the negative magneto-resistance (Son and Spivak, 2013).
One can as well calculate the equations of motion

$$\frac{1}{\hbar} \frac{\partial f}{\partial t} = [f \left( p, x \right), H]_\star = f \star H - H \star f$$

for $x^\alpha$ and $p_\alpha$. One gets

$$\frac{\partial x^\alpha}{\partial t} = \mathfrak{D}H + \frac{1}{2} \left\{ F^{\alpha\beta} (p), \frac{\partial H}{\partial x^\beta} \right\}$$

$$+ \frac{1}{16} \left( 2 \left\{ F^{\alpha\beta} (p), \left\{ F_{\beta\gamma} (x), \frac{\partial H}{\partial p_\gamma} \right\} \right\} + \left\{ \left\{ F^{\alpha\beta} (p), F_{\beta\gamma} (x) \right\}, \frac{\partial H}{\partial p_\gamma} \right\} \right)$$

(4.23)

and

$$\frac{\partial p_\alpha}{\partial t} = -\frac{\partial H}{\partial x^\alpha} - \frac{1}{2} \left\{ F_{\alpha\beta} (x), \frac{\partial H}{\partial p_\beta} \right\}$$

$$- \frac{1}{16} \left( 2 \left\{ F_{\alpha\beta} (x), \left\{ F^{\beta\gamma} (p), \frac{\partial H}{\partial x^\gamma} \right\} \right\} + \left\{ \left\{ F_{\alpha\beta} (x), F^{\beta\gamma} (p) \right\}, \frac{\partial H}{\partial x^\gamma} \right\} \right)$$

(4.24)

which reduces to the usual Hamilton relations whenever $A_\alpha (x) = A^\alpha (p) = 0$. In the quasi-static limit, one can replace $H \to \mathcal{E} - A_0 (p) - A_0 (x)$ with some time-sector gauge potential $A_0$ and energymatrix $\mathcal{E}$. In that case, the covariant derivatives $\mathfrak{D}_x H$ and $\mathfrak{D}_p H$ in (4.23) and (4.24) generate some quasi-static electric-like fields $F_{0\alpha} (x)$ and/or $F^{\alpha\alpha} (p)$.

When the second lines of (4.23) and (4.24) are neglected and the gauge fields are supposed Abelian, these equations of motion have been thoroughly studied (Gosselin et al., 2008a, 2006; Gosselin and Mohrbach, 2009; Shindou and Balents, 2008; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011), and often noted as $F^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma} \Omega_\gamma$ in terms of the Berry curvature $\Omega_\gamma$, whereas $F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} B^\gamma$ with the magnetic field $B^\gamma$. Nevertheless, some studies found an extra contribution (Shindou and Imura, 2005; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011), which could be written as $F (p, x)$ in the notation of this paper. I have no idea how such contributions might come out from the method presented in this paper. However, it is clear that the method used in (Shindou and Imura, 2005; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011) is intrinsically non-covariant, hence its elaboration is by far more intricate than the above methodology. Further elaborations will be presented in later studies.

Here I gave a(n almost) clear justification of the $x$ and $p$ gauge structure, and explained how one can elaborate a covariant semiclassical method, starting from a classical expression which is later dressed by the gradient expansion $\star (4.20)$.

V. CONCLUSION

Though it might appear as being not rigorously grounded, the present study enlighten a few characteristics of the non-commutative geometry in the phase space. In particular, a general method has been developed to generate a gauge-covariant gradient expansion order by order in section III. It is based on a formal mapping between the flat gauge Moyal product and the
gauge covariant one introduced in section III.A. In particular, the mapping between position derivatives in the two representations has been established, at least for the first few orders, in section III.B. When dealing with momentum space gauge structures (see section I.B for a quick introduction), the mapping applies instead on the momentum derivatives, see section IV.A. It has been shown that these mappings generate only covariant contributions to the Moyal star-product, unlike the usual approach intensively used in the physics literature.

Such star products automatically dress the classical theory with the gauge structure, in addition to provide the usual quantum corrections to the classical statistics in the form of the gradient expansion. Moreover, one can calculate many useful properties of statistical systems using the covariant \( \star \)-product, like e.g. the linear response theory, which eventually turn out to be of topological origin, see section II.C. It is also useful to extract low energy effective theories in a covariant way, in addition to construct some covariant quantum transport equations in a quite easy fashion, see section II.B. These kinetic theories are able to deal with any internal degree of freedom, like charge, spin, color, ...

A heuristic generalisation of these star products (covariant either in the momentum or in the position spaces) has been proposed in section IV.B, when the gauge structure is supposed to emerge in both position and momentum spaces. The former one might come from e.g. internal degree of freedom redundancy, whereas the gauge structure in the momentum space can emerge from non-trivial band structure in solid-state systems. Terms coupling both the \( x \) and \( p \) gauge fields naturally appear in this way. This approach might be useful for dealing with the emerging topological condensed matter systems. Many progresses were done in the past few years, using non-covariant methods. One can hope that the covariant method proposed here will be able to unite many (if not all) of the emerging phenomenologies with the old established ones. This might be a crucial point towards the complete understanding of the novel states of matter, because the proposed method present both versatility and easiness of utilisation.

Only part of the phenomenology associated to the generic method presented in this study has been studied so far (see sections II and II.C for short reviews), and many novel effects will be studied in the sequels of this study. As a matter of fact, I generalised many studies to the non-Abelian structure. In addition, I constructed a method being covariant at any level, whereas previous studies made non-covariant constructions and find bona-fide covariant expansion. Immediate consequences of the presented star products will be discussed in term of linear-response theory and covariant construction of effective models in the future.

I hope this study might be of interest for future more rigorous demonstrations of the gradient expansion in the phase-space, in addition to attract the attention on a convenient tool to study either high-energy or condensed matter modern problems.

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