On infinite walls in deformation quantization

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Abstract

We examine the deformation quantization of a single particle moving in one dimension (i) in the presence of an infinite potential wall, (ii) confined by an infinite square well, and (iii) bound by a delta function potential energy. In deformation quantization, considered as an autonomous formulation of quantum mechanics, the Wigner function of stationary states must be found by solving the so-called $\ast$-genvalue (“stargenvalue”) equation for the Hamiltonian. For the cases considered here, this pseudo-differential equation is difficult to solve directly, without an ad hoc modification of the potential. Here we treat the infinite wall as the limit of a solvable exponential potential. Before the limit is taken, the corresponding $\ast$-genvalue equation involves the Wigner function at momenta translated by imaginary amounts. We show that it can be converted to a partial differential equation, however, with a well-defined limit. We demonstrate that the Wigner functions calculated from the standard Schrödinger wave functions satisfy the resulting new equation. Finally, we show how our results may be adapted to allow for the presence of another, non-singular part in the potential.
1 Introduction

Deformation quantization\(^1\) is touted as a completely autonomous method of doing quantum mechanics, that is especially suited to the study of the classical limit.

Some quantum systems that are simple to treat in the Schrödinger formulation, however, are difficult to analyze in deformation quantization. Even the treatment of a free particle is not completely straightforward.\(^2\) Other examples are a particle moving in one dimension in the presence of an infinite wall, and a particle confined by an infinite square well.\(^3\) Surely, if deformation quantization is to take its rightful place as one of the possible ways of doing quantum mechanics, such simple systems must be treatable in it.\(^3\)

For stationary, pure states, the \(*\)-genvalue (“stargenvalue”) equation must be solved to find the Wigner function. We will restrict to such considerations in this work. For the infinite wall(s) case, simply imposing the usual boundary conditions on its solutions does not lead to the expected Wigner functions. In \(^4\) it was found that if the infinite wall and infinite square well potentials were modified by an additional “boundary potential” in an ad hoc way, then the expected Weyl transform of the density operator was a solution. The authors of \(^4\) did also show that the added potential terms were consistent with the Schrödinger treatment of those systems. They did not, however, derive the terms from first principles.

Furthermore, the normal intimate relation of deformation quantization with classical mechanics is altered by their modification. In the deformation quantization of standard systems, quantum mechanics is treated as a deformation of classical mechanics, with deformation parameter \(\hbar\). The Hamiltonian is not modified by quantum corrections, and the deformation is encoded entirely in the \(*\)-product. On the other hand, the “boundary term” added to the potential in \(^4\) is of order \(\hbar^2\), taking the form \(-\frac{\hbar^2}{2m} \delta'(x-a)\), where \(x=a\) is the position of an infinite wall.

Here we study the one-dimensional infinite wall by treating it as the limit of a solvable exponential potential, \(\lim_{\alpha \to \infty} e^{2\alpha x}\). Our original hope was to derive the prescription of \(^4\) from first principles. We follow instead a different path,

1\(^1\)Deformation quantization is also known as the Weyl-Wigner(-Moyal) formalism, phase-space quantization, and by other names as well. For elementary introductions, see \(^1\). More advanced reviews are listed in \(^2\). The modern interpretation of the formalism as a deformation of classical mechanics was first reported, and developed extensively, in \(^3\).

2\(^2\)See our Appendix.

3\(^3\)MW thanks Brian Wynder for emphasizing this point to him.
however.

Before the limit of the exponential potential is taken, the \( \star \)-genvalue equation involves the Wigner function evaluated at momenta shifted by imaginary amounts. Finding the limit of this equation directly is problematic. We show, however, that it can be converted to a partial differential equation, with a well-behaved limit. The resulting new equation can be treated in the standard way: no modification of the potential is required, and the normal Dirichlet boundary conditions can be used. We demonstrate that the Wigner functions calculated from the standard Schrödinger wave functions satisfy the new equation. Finally, we adapt our results to allow for the presence of another, non-singular part in the potential.

We should point out here that even the Schrödinger (canonical, operator) quantization of infinite walls has some subtleties (see [5] and references therein). In their operator quantization of such one-dimensional potentials, the authors of [5] adopt an approach similar to ours: they resolve apparent paradoxes by “acknowledging the existence of the rest of the real line.” Their treatment has no need for additional boundary potential terms like those introduced in [4].

A quick review of deformation quantization is given in section 2. Section 3 is the meat of the paper, containing the treatments of various cases of infinite walls and wells described as limits of exponentials. In section 4, the results are adapted to include an additional, regular potential, besides those described in section 3. Section 5 is our conclusion, and the Appendix treats the pure deformation quantization of a free particle in one dimension.

2 Deformation quantization

This section is a quick review of deformation quantization [1, 2], setting our notation and providing the results we will need. We will restrict attention to pure states, described by a state \( |\psi\rangle \), or a density operator \( \hat{\rho} = |\psi\rangle \langle \psi| \).

In deformation quantization, observables are not represented by operators, but rather by functions on phase space. They are multiplied using a pseudo-differential \( \star \)-product (“star product”) that is associative but non-commutative. Included is the Wigner function describing the state of the system, the central object in deformation quantization. It obeys an evolution equation involving a \( \star \)-commutator. In the \( \hbar \to 0 \) limit, \( \star \)-commutators reduce to Poisson brackets, and the equations of motion of classical mechanics are recovered. In that sense, this autonomous method describes quantum mechanics as a deformation of classical
mechanics [3].

Deformation quantization can be understood as a transform of the standard way of doing quantum mechanics using the density operator (matrix). Let us restrict attention (throughout) to the case of a single particle moving on the $x$-axis, so that phase space has coordinates $(x,p)$.

In canonical quantization, the phase space coordinates $x$ and $p$ are promoted to the operators $\hat{x}$ and $\hat{p}$, obeying the Heisenberg commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Observables like $x^2 p$ are promoted according to an operator ordering scheme.

Choosing the Weyl ordering, our example becomes

$$W^{-1}(x^2 p) = \frac{1}{3} \left( \hat{x}^2 \hat{p} + \hat{x} \hat{p} \hat{x} + \hat{p} \hat{x}^2 \right). \quad (1)$$

For a function $f(x,p)$ on phase space, this generalizes to

$$W^{-1} f = f(\partial_a, \partial_b) e^{a \hat{x} + b \hat{p}} |_{a,b=0}, \quad (2)$$

where $\partial_a := \partial/\partial a$, etc. In particular, it follows that

$$W^{-1} e^{ax + bp} = e^{a \hat{x} + b \hat{p}}. \quad (3)$$

This gives rise to another expression

$$W^{-1} f = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp f(x,p) e^{i\tau (\hat{p} - p) + i\sigma (\hat{x} - x)}, \quad (4)$$

where (3) has been used in the usual Fourier formula for $f$.

The crucial property of this ordering is

$$(W^{-1} f) (W^{-1} g) = W^{-1} (f \ast g), \quad (5)$$

proved by Grönewold. The Grönewold-Moyal $\ast$-product (pronounced star-product) takes the form

$$f(x,p) \ast g(x,p) := f(x,p) \exp \left\{ \frac{i\hbar}{2} \left( \frac{\partial}{\partial x} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial x} \right) \right\} g(x,p). \quad (6)$$

Here the arrows indicate the directions in which the derivatives act, and they act only on $f$ and $g$, and not to the left or right of them. That is, eqn. (6) stands for

$$f(x,p) \ast g(x,p) = e^{\frac{i\hbar}{2} \left( \partial_{x'} \partial_p - \partial_p \partial_{x'} \right)} f(x',p') g(x,p) \big|_{x'=x, p'=p}. \quad (7)$$

Deformation quantization may be carried out in ways that correspond to many different operator orderings. Here, we will restrict to the Weyl ordering $W^{-1}$, resulting in the famous Grönewold-Moyal $\ast$-product. The use of the inverse notation ($W^{-1}$) is for later convenience.
Eqn. (5) shows that the $\star$-product is an associative, non-commutative product of phase space functions that mimics the product of operators. In deformation quantization, the operator product is replaced by this $\star$-product, so that operators can be represented by ordinary functions (and distributions) on phase space.

The exponent of the $\star$-product (6) indicates the most important property of deformation quantization: its intimate relation to classical physics. In classical mechanics, it is the Poisson bracket of functions on phase space,

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} = f (\frac{\partial}{\partial x} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial x}) g$$

that enters the dynamical equations. In the operator formulation of quantum mechanics, it is the commutator $[\hat{f}, \hat{g}]$ of operator observables $\hat{f}$ and $\hat{g}$ that is important. In deformation quantization, the $\star$-commutator

$$[f, g]_\star := f \star g - g \star f$$

of functions $f$ and $g$ takes its place. The equation

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} [f, g]_\star = \{f, g\}$$

encodes the correspondence relation between classical and quantum mechanics in deformation quantization.

The inverse $W$ of the Weyl map $W^{-1}$ is known as the Weyl transform. One formula for it,

$$W \hat{f} = \frac{1}{2\pi \hbar} \int d\zeta d\varphi \ Tr(\hat{f} e^{i[(\hat{p} - p)\zeta + (\hat{x} - x)\varphi]/\hbar})$$

(11)

treats momenta and coordinates in symmetric fashion. A useful variant can be derived from it:

$$W \hat{f} = \hbar \int dy e^{-ipy} \langle x + \frac{hy}{2} | \hat{f} | x - \frac{hy}{2} \rangle$$

(12)

as well as a similar formula involving momentum eigenstates. The Weyl transform $W \hat{f}$ of the operator $\hat{f}$ is also known as its Weyl symbol.

The object

$$\Delta(x, p; \hat{x}, \hat{p}) := e^{i[(\hat{p} - p)\zeta + (\hat{x} - x)\varphi]/\hbar}$$

(13)

is prominent in the fundamental equation (11). It is sometimes called the quantizer, and its importance was stressed in [6].

The Weyl transform obeys

$$W(\hat{f}) \star W(\hat{g}) = W(\hat{f} \hat{g})$$

(14)
an inverse analog of (5). This makes it possible to work exclusively with phase space functions, as long as they are multiplied with the $\star$-product. Deformation quantization is the realization of this idea - it is the Weyl transform of quantum mechanics done with the density operator.

The Weyl transform of the density matrix $\hat{\rho} = |\psi\rangle \langle \psi|$, 

$$\mathcal{W}(\hat{\rho}) = \hbar \int dy e^{-ipy} \psi^* (x - \frac{\hbar y}{2}) \psi (x + \frac{\hbar y}{2}), \tag{15}$$

is the central object in deformation quantization; it describes the quantum state of the system. After normalization, it is known as the Wigner function:

$$\rho := \frac{\mathcal{W}(\hat{\rho})}{2\pi\hbar}. \tag{16}$$

Combining the last two equations gives

$$\rho[\psi] := \frac{1}{2\pi} \int dy e^{-ipy} \psi^* (x - \frac{\hbar y}{2}) \psi (x + \frac{\hbar y}{2}). \tag{17}$$

We will use the notation $\rho[\psi]$ to emphasize that this is the Wigner function calculated from a known Schrödinger wave function $\psi$.

The Wigner function evolves according to

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho]. \tag{18}$$

For stationary states, $\frac{\partial \rho}{\partial t} = 0$, so that

$$[H, \rho] = 0. \tag{19}$$

For $\rho$ describing an energy eigenstate with eigenvalue $E$,

$$H \star \rho = \rho \star H = E \rho. \tag{20}$$

These simplified dynamical equations allow us to solve for the Wigner function of the stationary states of the system.

By (15), the Wigner function can always be calculated if the Schrödinger wave function is known. We are interested here, however, in considering deformation quantization as an autonomous method of doing quantum mechanics. That means we want to find the Wigner functions by solving either (18) or (20), and we will restrict to the latter in this paper. Of course, (16) can still be used as a very useful check of our results.

Solving (20) can lead to more general solutions than those of physical interest. Imposing the constraints

$$\rho \star \rho = \frac{1}{\hbar} \rho, \quad \rho^* = \rho, \tag{21}$$

6
is important in finding the correct solutions [3]. Clearly, the first relation is just the Weyl transform of the usual projection condition on the density operator. More generally, the requirement is

$$\rho_i \star \rho_j = \frac{1}{\hbar} \delta_{ij} \rho_j ,$$ (22)

for Wigner functions describing a discrete set of states, and

$$\rho_\alpha \star \rho_\beta = \frac{1}{\hbar} \delta(\alpha - \beta) \rho_\beta$$ (23)

for states with a continuous label.

All observable probabilities can be calculated using the Wigner function. First of all, by (15), the probability densities are

$$|\psi(x)|^2 = \int dp \rho(x, p) , \quad |\psi(p)|^2 = \int dx \rho(x, p) .$$ (24)

Clearly then, the Wigner function is normalized and real: \(\int dx dp \rho = 1\) and \(\rho^* = \rho\). The expectation value of an operator

$$\langle \hat{f} \rangle = \int dx dp \rho \star f ,$$ (25)

where \(f := W(\hat{f})\). Roughly, one can think of the integral over phase space as the analog of the trace, and as discussed above, the star product takes the place of the operator product. The important cyclic property of a trace is encoded in

$$\int dx dp f \star g = \int dx dp g \star f = \int dx dp fg .$$ (26)

### 3 Free particle except for infinite walls/wells

In this section, we will treat a particle moving freely on the \(x\)-axis, except for the presence of one or more infinite potential walls or wells. More precisely, the infinite barrier, the infinite square well, and the delta-function well will all be studied as limits of potentials built from exponentials.

We are interested in the “pure” deformation quantization of these systems. That is, we would like to derive their Wigner functions using the equations of deformation quantization only, without reference to the well known Schrodinger wave functions, for example. Here we restrict consideration to stationary states. That means, therefore, that we must examine the \(\star\)-genvalue equation [20] for the corresponding Hamiltonians.
As mentioned in the introduction, these problems are not straightforward, as we now explain [4]. Consider equation (20) more carefully, for the example of an infinite wall, i.e. for the potential energy

\[
V(x) = \begin{cases} 
  0, & x < 0; \\
  \infty, & x > 0.
\end{cases}
\] (27)

Following the Schrödinger treatment of this system, one would restrict to \( x < 0 \), and impose the boundary condition \( \rho(0, p) = 0 \). For \( x < 0 \), the \( \star \)-genvalue equation is that of a free particle, with real and imaginary parts given in the Appendix as equations (54) and (51), respectively. But the imaginary part, \( p \partial_x \rho = 0 \), does not lead to sensible results for this potential (see the Appendix).

To look for some guidance, we can study the expected solution, the Weyl transform of the known density matrix. The Schrödinger wave function is

\[
\psi(x) = \theta(-x) \left[ e^{i\sqrt{Ex}} - e^{-i\sqrt{Ex}} \right].
\] (28)

Using (17), the corresponding Wigner function is found to be

\[
\rho(\psi) = \theta(-x) \tilde{\rho}(x, p),
\] (29)

with

\[
\tilde{\rho}(x, p) = \frac{2 \sin[2x(p + \sqrt{E})]}{p + \sqrt{E}} + \frac{2 \sin[2x(p - \sqrt{E})]}{p - \sqrt{E}} \\
+ 2 \cos(2x \sqrt{E}) \frac{2 \sin(2xp)}{p}.
\] (30)

This \( \rho(x, p) \) does not satisfy the imaginary part (54) of the \( \star \)-genvalue equation \( H \star \rho = E \rho \).

Our goal in this article is to find a dynamical equation replacing the \( \star \)-genvalue equation that can be solved to find the correct Wigner functions for these problems.

### 3.1 Infinite wall

The Liouville Hamiltonian is

\[
H_\alpha = p^2 + e^{2\alpha x}.
\] (31)

Constants like the mass have been set so as to simplify considerations \((2m = 1, \text{ e.g.)}\). A pure deformation quantization of this system has already been carried

\[\text{In those equations and henceforth, we set } \hbar = 1.\]
out in \[7\]. Taking the $\alpha \to \infty$ limit of $V(x) = e^{2\alpha x}$ yields an infinite wall with $V = 0$ for $x < 0$, and $V = \infty$ for $x > 0$.

Therefore, the limit of the solution found in \[7\] yields the correct Wigner function. That is not what we wish to do, however. We hope instead to take the limit of the $\star$-genvalue equation, in order to find an equation that can be solved directly, leading to the physical Wigner function.

The $\star$-genvalue equation is easily found to be

$$H_\alpha \star \rho(x, p) = \left[ \left( p - \frac{i}{2} \partial_x \right)^2 + e^{2\alpha(x+i\star p)} \right] \rho(x, p) = E \rho(x, p) . \quad (32)$$

It separates into \Im (imaginary) part

$$\left[ -p \partial_x + e^{2\alpha x} \sin(\alpha \partial_p) \right] \rho(x, p) = 0 , \quad (33)$$

and \Re (real) part

$$\left[ p^2 - E - \frac{1}{4} \partial_x^2 + e^{2\alpha x} \cos(\alpha \partial_p) \right] \rho(x, p) = 0 . \quad (34)$$

Formally, these equations can be rewritten as

$$e^{-2\alpha x} \partial_x \rho(x, p) = -\frac{i}{2p} \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right] , \quad (35)$$

and

$$e^{-2\alpha x} \left( p^2 - E - \frac{1}{4} \partial_x^2 \right) \rho(x, p)$$

$$+ \frac{1}{2} \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] = 0 . \quad (36)$$

Using (35) to find $\partial_x^2 \rho(x, p)$, and substituting this into (36), we arrive at an equation without derivatives,

$$0 = \left( p^2 - E \right) \rho(x, p) + \frac{1}{p} \left( \frac{e^{2\alpha x}}{4} \right)^2 \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p+2i\alpha} + \frac{\rho(x, p - 2i\alpha) - \rho(x, p)}{p-2i\alpha} \right]$$

$$- \frac{e^{2\alpha x}}{4p} \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right]$$

$$+ \frac{e^{2\alpha x}}{2} \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] . \quad (37)$$

This can be considered a difference equation in the momentum variable. Only imaginary shifts of the momentum arguments are involved; that is, besides $\rho(x, p)$, the result involves the four quantities

$$\rho(x, p \pm i\alpha) , \quad \rho(x, p \pm 2i\alpha) . \quad (38)$$
The $\alpha \to \infty$ limit of (37) is problematic.

We can trade the four quantities of (38), however, for the derivatives

$$\partial_x^n \rho(x, p), \quad n = 1, 2, 3, 4.$$  \hspace{1cm} (39)

The resulting differential equation will have a well-defined limit as $\alpha \to \infty$, the result we are seeking.

Four equations relating the “variables” of (38) to those of (39) are required. Two are already provided: (35) and (36). The two additional equations can be derived by taking derivatives of (36):

$$0 = \partial_x^3 \rho(x, p) - 4(p^2 - E) \partial_x \rho(x, p)$$
$$\quad + 4\alpha e^{2\alpha x} [\rho(x, p + i\alpha) + \rho(x, p - i\alpha)]$$
$$\quad - 2e^{2\alpha x} [\partial_x \rho(x, p + i\alpha) + \partial_x \rho(x, p - i\alpha)], \quad (40)$$

and

$$0 = \partial_x^2 \rho(x, p) - 4(p^2 - E) \partial_x^2 \rho(x, p)$$
$$\quad - 8\alpha^2 e^{2\alpha x} [\rho(x, p + i\alpha) + \rho(x, p - i\alpha)]$$
$$\quad - 4\alpha i e^{4\alpha x} \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p + i\alpha} + \frac{\rho(x, p - 2i\alpha) - \rho(x, p)}{p - i\alpha} \right]$$
$$\quad + 2e^{2\alpha x} \left\{ [4(p + i\alpha)^2 - E] \rho(x, p + i\alpha)$$
$$\quad + 2e^{2\alpha x} [\rho(x, p + 2i\alpha) + \rho(x, p)]$$
$$\quad + [4(p - i\alpha)^2 - E] \rho(x, p - i\alpha)$$
$$\quad + 2e^{2\alpha x} [\rho(x, p - 2i\alpha) + \rho(x, p)] \right\}. \quad (41)$$

With the help of symbolic computation, we find a simple, differential equation results:

$$0 = \frac{1}{16} \partial_x^4 \rho(x, p) + \frac{1}{2} (p^2 + E) \partial_x^2 \rho(x, p)$$
$$\quad + (p^4 - 2Ep + E^2) \rho(x, p) - e^{4\alpha x} \rho(x, p). \quad (42)$$

Taking the limit $\alpha \to \infty$, we find the new equation

$$\frac{1}{16} \partial_x^4 \rho(x, p) + \frac{1}{2} (p^2 + E) \partial_x^2 \rho(x, p) + (p^4 - 2Ep + E^2) \rho(x, p) = 0, \quad (43)$$

\footnote{Hölder’s theorem (see [3], e.g.) states that no solution of the simple difference equation $y(x + 1) - y(x) = 1/x$ satisfies any algebraic differential equation. Replacing a difference equation with a differential equation may, therefore, omit interesting solutions. In our case, however, we will see that the physical Wigner function is a solution of our resulting differential equation.}
valid for $x < 0$.

It is simple to verify that the Wigner function $\tilde{\rho}(x, p)$ of (30) satisfies the new equation (43), for $x < 0$.

It is also interesting to note that eqn. (43) can be rewritten as

$$\left(p^2 \ast \rho \ast p^2 - E^2 \rho\right) - 2E \Re \left(p^2 \ast \rho - E \rho\right) = 0 .$$

The new equation is a linear combination of ones that follow from the $\star$-genvalue equation for a free particle.

### 3.2 Infinite square well

We will now consider the case of two infinite walls, as limits of two exponential potentials. More precisely, we will be taking the $\alpha \to \infty$ limit of the $\star$-genvalue equation following from the sinh-Gordon Hamiltonian

$$H_\alpha = p^2 + e^{-2\alpha(x+1)} + e^{2\alpha(x-1)} .$$

The method and the result for the two-wall potential will be remarkably similar to those for the single-wall case.

The $\star$-genvalue equation is

$$\left\{ \left( p - \frac{i}{2} \partial_x \right)^2 + 2e^{-2\alpha} \cosh \left[ 2\alpha \left( x + \frac{i}{2} \partial_p \right) \right] \right\} \rho(x, p) = H_\alpha \ast \rho(x, p) = E \rho(x, p) .$$

It has $\Im$ and $\Re$ parts

$$\partial_x \rho(x, p) = -\frac{i e^{-2\alpha}}{p} \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right] \sinh(2\alpha x) ,$$

and

$$\left( p^2 - E - \frac{1}{4} \partial_x^2 \right) \rho(x, p) + e^{-2\alpha} \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] \cosh(2\alpha x) = 0 .$$

Combining the previous two equations leads to

$$0 = (p^2 - E) \rho(x, p) + \frac{e^{-4\alpha}}{4p} \cosh^2(2\alpha x) \left[ \rho(x + 2i\alpha) - \rho(x, p) \frac{\rho(x, p) - \rho(x, p - 2i\alpha)}{p + i\alpha} \right]$$

$$-\frac{ie^{-2\alpha}}{2p} \cosh(2\alpha x) \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right]$$

$$- e^{-2\alpha} \cosh(2\alpha x) \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] .$$

(49)
This equation involves no derivatives, but only the quantities
\[ \rho(x, p \pm i\alpha), \quad \rho(x, p \pm 2i\alpha) \] (50)
as well as \( \rho(x, p) \). We again wish to eliminate these in favor of derivatives \( \partial_n^\alpha \rho \), \( n = 1, 2, 3, 4 \). To do so, we make use of the additional relations
\[
\begin{align*}
\partial_3^\alpha \rho(x, p) &= 4(p^2 - E) \partial_2 \rho(x, p) \\
&\quad + 2ae^{-2\alpha} \sinh(2\alpha x) \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] \\
&\quad - 4ie^{-4\alpha} \cosh(2\alpha x) \sinh(2\alpha x) \times \\
&\quad \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p + i\alpha} + \frac{\rho(x, p) - \rho(x, p - 2i\alpha)}{p - i\alpha} \right],
\end{align*}
\] (51)
and
\[
\begin{align*}
\partial_4^\alpha \rho(x, p) &= 4(p^2 - E) \partial_2^2 \rho(x, p) \\
&\quad + 16a^2 e^{-2\alpha} \cosh(2\alpha x) \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] \\
&\quad + 16ie^{-4\alpha} \sinh^2(2\alpha x) \times \\
&\quad \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p + i\alpha} + \frac{\rho(x, p) - \rho(x, p - 2i\alpha)}{p - i\alpha} \right] \\
&\quad + e^{-2\alpha} \cosh(2\alpha x) \left\{ [4(p + i\alpha)^2 - E] \rho(x, p + i\alpha) \\
&\quad - e^{-2\alpha} \cosh(2\alpha x) \left[ \rho(x, p + 2i\alpha) + \rho(x, p) \right] \\
&\quad + [4(p - i\alpha)^2 - E] \rho(x, p - i\alpha) \\
&\quad - e^{-2\alpha} \cosh(2\alpha x) \left[ \rho(x, p) + \rho(x, p - 2i\alpha) \right] \right\}. \quad (52)
\end{align*}
\]

The resulting new equation is quite complicated, and so we refrain from writing it here. It does, however, have a well-defined limit \( \alpha \to \infty \), for \( x \in [-1, 1] \):
\[ \frac{1}{16} \partial_4^\alpha \rho(x, p) + \frac{1}{2}(p^2 + E) \partial_2^2 \rho(x, p) + (p^4 - 2Ep + E^2) \rho(x, p) = 0. \] (53)
This is identical to the analogous result found for the one-wall case.

As for the one-wall case, the Wigner functions calculated from the Schrödinger wave functions
\[ \psi(x) = \theta(-x + 1) \theta(x + 1) \cos(\sqrt{E}x), \quad E = \frac{n^2 \pi^2}{4} \] (54)
satisfy this new equation, in the following sense. Using the \( \theta \) functions, we find \( \rho[\psi] = \theta(-x + 1) \theta(x + 1) \rho \), where
\[
\rho(x, p) = \frac{\sin[(2p + n\pi)(1 - |x|)]}{2p + n\pi} + \frac{\sin[(2p - n\pi)(1 - |x|)]}{2p - n\pi}
\]
or, equivalently,

\[ \bar{\rho}(x, p) = \frac{\sin \left[ 2\sqrt{E}(1 - |x|) \right]}{2(p + \sqrt{E})} + \frac{\sin \left[ 2\sqrt{E}(1 - |x|) \right]}{2(p - \sqrt{E})} + \frac{\cos(\sqrt{Ex}) \sin [2p(1 - |x|)]}{p} \]  

(56)

\( \bar{\rho} \) satisfies the new equation \( \Box \) As in the one-wall case, the expression for the Wigner function valid where \( V = 0 \) solves the derived equation.

### 3.3 Delta-function potential well

A \( \delta \)-function potential well can be studied as the \( \alpha \to \infty \) limit of the Hamiltonian

\[ H_\alpha = p^2 - 2\alpha e^{-2\alpha|x|} \]  

(57)

Restricting to \( x > 0 \), the \(*\)-genvalue equation is

\[ H_\alpha \ast \rho(x, p) = \left[ \left(p - \frac{i}{2} \partial_x \right)^2 - 2\alpha e^{-2\alpha(x + \frac{1}{2} \partial_x)} \right] \rho(x, p) \]

\[ = E \rho(x, p) \]  

(58)

Its \( \Im \) and \( \Re \) parts are

\[ e^{2\alpha x} \partial_x \rho(x, p) = -\frac{i\alpha}{p} \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right] \]  

(59)

and

\[ e^{2\alpha x} \left( p^2 - E - \frac{1}{4} \partial_x^2 \right) \rho(x, p) - \alpha \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] = 0 \]  

(60)

Combining these last two relations yields an equation involving no derivatives,

\[ 0 = 4(p^2 - E) \rho(x, p) - 4\alpha e^{-2\alpha x} \left[ \rho(x + i\alpha) + \rho(x, p - i\alpha) \right] \\
- \frac{4\alpha^2 e^{-4\alpha x}}{p} \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p + i\alpha} + \frac{\rho(x, p - 2i\alpha) - \rho(x, p)}{p - i\alpha} \right] \\
- 2\alpha^2 e^{-2\alpha x} \left[ \rho(x, p + i\alpha) - \rho(x, p - i\alpha) \right] \]  

(61)
but instead depending on $\rho(x,p)$ and

$$\rho(x,p \pm i\alpha), \quad \rho(x,p \pm 2i\alpha).$$  \hfill (62)

In the by-now familiar way, we trade the dependence on these for $x$-derivatives, with the help of the additional equations

$$\partial^3_x \rho(x,p) = 4(p^2 - E) \partial_x \rho(x,p)$$

$$+ 8\alpha^2 e^{-2\alpha x} \left[ \rho(x,p + i\alpha) + \rho(x,p - i\alpha) \right]$$

$$+ 4\alpha^2 i e^{-4\alpha x} \left[ \frac{\rho(x,p + 2i\alpha) - \rho(x,p)}{p + i\alpha} + \frac{\rho(x,p) - \rho(x,p - 2i\alpha)}{p - i\alpha} \right],$$  \hfill (63)

and

$$\partial^4_x \rho(x,p) = 4(p^2 - E) \partial^2_x \rho(x,p)$$

$$- 16\alpha^3 e^{-2\alpha x} \left[ \rho(x,p + i\alpha) + \rho(x,p - i\alpha) \right]$$

$$- 16\alpha^3 e^{-4\alpha x} \left[ \frac{\rho(x,p + 2i\alpha) - \rho(x,p)}{p + i\alpha} + \frac{\rho(x,p) - \rho(x,p - 2i\alpha)}{p - i\alpha} \right]$$

$$- 4\alpha e^{-2\alpha x} \left\{ -4(p + i\alpha)^2 + E \right\} \rho(x,p + i\alpha)$$

$$- 4\alpha e^{-2\alpha x} \left[ \rho(x,p + 2i\alpha) + \rho(x,p) \right] + \left[ -4(p - i\alpha)^2 + E \right] \rho(x,p - i\alpha)$$

$$- 4\alpha e^{-2\alpha x} \left[ \rho(x,p - 2i\alpha) + \rho(x,p) \right].$$  \hfill (64)

The resulting new differential equation

$$\frac{1}{16} \partial^4_x \rho(x,p) + \frac{(p^2 + E)}{2} \partial^2_x \rho(x,p)$$

$$+ \left[ p^4 - 2Ep + E^2 + 4\alpha^2 e^{-4\alpha x} \right] \rho(x,p) = 0$$

(65)

has a well-defined limit. Taking $\alpha \to \infty$, gives

$$\frac{1}{16} \partial^4_x \rho(x,p) + \frac{1}{2}(p^2 + E) \partial^2_x \rho(x,p) + (p^4 - 2Ep + E^2) \rho(x,p) = 0,$$  \hfill (66)

since $x > 0$. The result is identical for $x < 0$, and to the result found above for both the one-wall and the infinite square well cases.

The sole state bound by the delta-function well has wave function

$$\psi(x) = e^{-|x|}, \quad E = -1.$$  \hfill (67)

The corresponding Wigner function $\rho[\psi]$ is

$$\rho(x,p) = \frac{e^{-2x} \left[ \cos(2xp) + \frac{1}{p} \sin(2xp) \right]}{p^2 + 1}.$$  \hfill (68)

It satisfies the new differential equation, where $V = 0$.  

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4 Wall with an additional potential

So far, we have treated systems that are free except for infinite walls or wells. Describing such potentials as limits of exponential terms, we found in all cases that the same differential equation governs the Wigner functions where $V = 0$. It would be useful to generalize to systems involving additional, regular potential terms, in the presence of infinite walls/wells.

As a first step, we will consider here the one-wall case, with an unspecified potential added. That is, we’ll consider again the limit of the Liouville potential, with the additional, regular potential $V(x)$ present. After a general treatment, we will consider the simple, special case $V(x) = x^2$ as a check.

The Hamiltonian is

$$H_\alpha = p^2 + e^{2\alpha x} + V(x),$$

yielding the $*$-genvalue equation

$$H_\alpha * \rho(x,p) = E \rho(x,p),$$

$$= \left\{ \left( p - \frac{i}{2} \partial_x \right)^2 + e^{2\alpha (x + \frac{i}{2} \partial_p)} \right\} \rho(x,p) + V(x) * \rho(x,p).$$

Its $\Im$ and $\Re$ parts are

$$[- p \partial_x + e^{2\alpha x} \sin(\alpha \partial_p)] \rho(x,p) + \Im [V(x) * \rho(x,p)] = 0,$$

and

$$\left[ p^2 - E - \frac{1}{4} \partial_x^2 + e^{2\alpha x} \cos(\alpha \partial_p) \right] \rho(x,p) + \Re [V(x) * \rho(x,p)] = 0.$$

Combining these gives

$$0 = 4(p^2 - E) \rho(x,p) + \frac{e^{4\alpha x}}{4p} \left[ \frac{\rho(x + 2i\alpha) - \rho(x,p)}{p + i\alpha} + \frac{\rho(x - 2i\alpha) - \rho(x,p)}{p - i\alpha} \right]$$

$$+ \frac{i\alpha e^{2\alpha x}}{p} [\rho(x,p + i\alpha) - \rho(x,p - i\alpha)]$$

$$+ 2e^{2\alpha x} [\rho(x,p + i\alpha) + \rho(x,p - i\alpha)]$$

$$+ 4\Re [V(x) * \rho(x,p)] - \frac{1}{p} \partial_x \Im [V(x) * \rho(x,p)]$$

$$+ \frac{ie^{2\alpha x}}{2p} \left\{ \Im [V(x) * \rho(x,p + i\alpha)] - \frac{\Im [V(x) * \rho(p - i\alpha)]}{p - i\alpha} \right\}.$$
Following the procedure used above, we would like to eliminate the quantities \( \rho(x, p \pm i\alpha) \), \( \rho(x, p \pm 2i\alpha) \) in favor of derivatives of the Wigner function. As in the cases previously considered, two additional equations can be derived straightforwardly:

\[
0 = -\partial_x^3 \rho(x, p) + 4(p^2 - E) \partial_x \rho(x, p) + 4\alpha e^{2i\alpha} \left[ \frac{\rho(x, p + 2i\alpha) - \rho(x, p)}{p + i\alpha} - \frac{\rho(x, p) - \rho(x, p - 2i\alpha)}{p - i\alpha} \right] \\
- ie^{4\alpha} \left[ \Im \left[ V(x) \star \rho(x, p + i\alpha) \right] \frac{p + i\alpha}{p + i\alpha} + \Im \left[ V(x) \star \rho(x, p - i\alpha) \right] \frac{p - i\alpha}{p - i\alpha} \right] \\
+ 2e^{2\alpha} \left( 4 \left[ (p + i\alpha)^2 - E \right] \rho(x, p + i\alpha) + 2\alpha^2 e^{2i\alpha} \left[ \rho(x, p + 2i\alpha) + \rho(x, p) \right] \\
+ \left[ 4(p - i\alpha)^2 - E \right] \rho(x, p - i\alpha) + 2e^{2i\alpha} \left[ \rho(x, p - 2i\alpha) + \rho(x, p) \right] \right) \\
+ 8\alpha e^{2i\alpha} \left[ \Im \left[ V(x) \star \rho(x, p + i\alpha) \right] \frac{p + i\alpha}{p + i\alpha} + \Im \left[ V(x) \star \rho(x, p - i\alpha) \right] \frac{p - i\alpha}{p - i\alpha} \right] \\
+ 8e^{2\alpha} \left[ \Re \left[ V(x) \star \rho(x, p + i\alpha) \right] + \Re \left[ V(x) \star \rho(x, p - i\alpha) \right] \right] \\
+ 4\partial_x^2 \Re \left[ V(x) \star \rho(x, p) \right],
\]

and

\[
0 = -\partial_x^4 \rho(x, p) + 4(p^2 - E) \partial_x^2 \rho(x, p) + 8\alpha^2 e^{2i\alpha} \left[ \rho(x, p + 2i\alpha) + \rho(x, p - 2i\alpha) \right] \\
- 4ie^{4\alpha} \left[ \Re \left[ V(x) \star \rho(x, p + i\alpha) \right] \frac{p + i\alpha}{p + i\alpha} + \Re \left[ V(x) \star \rho(x, p - i\alpha) \right] \frac{p - i\alpha}{p - i\alpha} \right] \\
+ 2e^{2\alpha} \left( 4 \left[ (p + i\alpha)^2 - E \right] \rho(x, p + i\alpha) + 2\alpha^2 e^{2i\alpha} \left[ \rho(x, p + 2i\alpha) + \rho(x, p) \right] \\
+ \left[ 4(p - i\alpha)^2 - E \right] \rho(x, p - i\alpha) + 2e^{2i\alpha} \left[ \rho(x, p - 2i\alpha) + \rho(x, p) \right] \right) \\
+ 8\alpha e^{2i\alpha} \left[ \Re \left[ V(x) \star \rho(x, p + i\alpha) \right] \frac{p + i\alpha}{p + i\alpha} + \Re \left[ V(x) \star \rho(x, p - i\alpha) \right] \frac{p - i\alpha}{p - i\alpha} \right] \\
+ 8e^{2\alpha} \left[ \Re \left[ V(x) \star \rho(x, p + i\alpha) \right] + \Re \left[ V(x) \star \rho(x, p - i\alpha) \right] \right] \\
+ 4\partial_x^2 \Re \left[ V(x) \star \rho(x, p) \right].
\]

This is not sufficient in this more general case, however. We now have 4 further quantities

\[
\left( \frac{\Re}{3} \right) \left[ V(x) \star \rho(x, p \pm i\alpha) \right],
\]

(76)

to get rid of. Here \( \left( \frac{\Re}{3} \right) \) indicates the real part or the imaginary part.

With considerably more work, however, the required 4 equations can be derived. We find

\[
\left( \frac{\Re}{3} \right) \left[ V(x) \star \left( p^2 - E - \frac{1}{4} \partial_x^2 \rho(x, p) \right) \right]
\]

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\[
\begin{align*}
&= -\frac{e^{2\alpha}}{2} (\Re/\Im) \left[ V(x) \star \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] \right] \\
&- (\Re/\Im) \left[ V(x) \star \Re \left[ V(x) \star \rho(x, p) \right] \right] , \quad (77)
\end{align*}
\]

and
\[
\begin{align*}
&- (\Re/\Im) \left[ V(x) \star p \partial_x \rho(x, p) \right] \\
&- \frac{ie^{2\alpha}}{2} (\Re/\Im) \left\{ V(x) \star \left[ \rho(x, p + i\alpha) + \rho(x, p - i\alpha) \right] \right\} \\
&+ (\Re/\Im) \left\{ V(x) \star \Im \left[ V(x) \star \rho(x, p) \right] \right\} = 0 . \quad (78)
\end{align*}
\]

By symbolic computation, the resulting differential equation can be derived. In the limit, it goes to
\[
\begin{align*}
\frac{1}{16} \partial_x^4 \rho(x, p) + \frac{(p^2 + E)}{2} \partial_x^2 \rho(x, p) + (p^4 - 2Ep + E^2) \rho(x, p) \\
+ (p^2 - E) \Re \left[ V(x) \star \rho(x, p) \right] - p \partial_x \Im \left[ V(x) \star \rho(x, p) \right] \\
- \frac{1}{4} \partial_x^2 \Im \left[ V(x) \star \rho(x, p) \right] - \Im \left[ V(x) \star p \partial_x \rho(x, p) \right] \\
+ \Im \left\{ V(x) \star \Im \left[ V(x) \star \rho(x, p) \right] \right\} + \Re \left\{ V(x) \star \Re \left[ V(x) \star \rho(x, p) \right] \right\} \\
+ \Re \left\{ V(x) \star \left[ \left( p^2 - E - \frac{1}{4} \partial_x^2 \right) \rho(x, p) \right] \right\} = 0 , \quad (79)
\end{align*}
\]

for \( x < 0 \). Clearly, this equation reduces to the one found above when \( V = 0 \).

As a simple check of this result, consider the example of the simple harmonic potential \( V(x) = x^2 \). Only the odd-parity wave functions have \( \psi(0) = 0 \), and so only they survive the presence of the infinite potential wall at \( x = 0 \). The ground state wave function is therefore
\[
\psi(x) = \theta(-x) \ x \ e^{-x^2} , \quad (80)
\]

with energy \( E = 3 \). The corresponding Wigner function is \( \rho[\psi] = \theta(-x)\bar{\rho} \) with
\[
\begin{align*}
\bar{\rho}(x, p) &= x^2 \operatorname{erf}(x - ip) \pi e^{-p^2-x^2} \\
&- \frac{1}{2} \operatorname{erf}(x + ip) \pi e^{-p^2-x^2} + x^2 \operatorname{erf}(x + ip) \pi e^{-p^2-x^2} \\
&+ \sqrt{\pi} x e^{-2x(x-ip)} + i\sqrt{\pi} pe^{-2x(x-ip)} \\
&+ \operatorname{erf}(x - ip) p^2 \pi e^{-p^2-x^2} - \frac{1}{2} \operatorname{erf}(x + ip) \pi e^{-p^2-x^2} \\
&+ \sqrt{\pi} x e^{-2x(x+ip)} - i\sqrt{\pi} pe^{-2x(x+ip)} \\
&+ \operatorname{erf}(x - ip) p^2 \pi e^{-p^2-x^2} . \quad (81)
\end{align*}
\]
Here erf denotes the error function

\[ \text{erf}(x) = \int_0^x e^{-t^2} \, dt . \] (82)

We have verified that (81) does indeed satisfy the equation (79).

5 Conclusion

If deformation quantization is truly an autonomous formulation of quantum mechanics, problems that are solved simply using other methods must be treatable in it. First-principle deformation quantization of the systems should be possible, even if difficult. For stationary states, the Wigner functions should be derivable by solving the \( \star \)-genvalue equation for the Hamiltonian.

Here we have studied a single quantum particle travelling freely on the \( x \)-axis, except for the presence of one or two infinite walls. The simple Schrödinger wave functions of these systems are well known. The Wigner functions expected from the stationary-state wave functions can therefore be easily worked out (see \( \rho[\psi] \) in eqn (17)). Dias and Prata [4] point out, however, that they do not satisfy the corresponding \( \star \)-genvalue equations.

To cure this problem, they propose the addition of “boundary potentials” to the Hamiltonian. They further show that these additional terms are consistent with the Schrödinger quantization of the systems. They do not, however, derive the extra potentials from first principles.

In the hopes of filling this gap, we treated the infinite wall and infinite square well potentials as limits of solvable, exponential ones. The delta-function well is also considered, since it can be treated in a very similar way. Our results do not relate easily to the proposal of [4], however.

A well-defined limit was found for the corresponding \( \star \)-genvalue equations. Happily, this new differential equation is common to all the systems. Furthermore, the expected Wigner functions \( \rho[\psi] \) satisfy it away from the infinite walls. The revised hope is therefore that the new equation can be solved to find the correct Wigner functions directly, and boundary conditions on the Wigner functions can be imposed in the way familiar from Schrödinger quantization.

That hope has not been realized here. We have not (yet) shown that the new differential equations can be solved to find the correct, physical Wigner functions. We only verified that the \( \rho[\psi] \)’s satisfied the new equations, outside the \( V = \infty \) regions.
Nevertheless, finding a well-defined limit of the $\star$-genvalue equations for these systems is a significant first step. The new equation was also generalized to allow for a regular potential in addition to infinite walls/wells.

It might be interesting to use other solvable potentials instead of the exponential ones. It would at least (hopefully) verify that our results are independent of our particular choice of representation of the infinite potential wall as a limit. For example, \( \lim_{N \to \infty} x^N \) yields the infinite square well. Aspects of this limit have already been studied in [9].

Appendix: Free particle

In the reviews we have seen, a pure deformation quantization of a free particle moving in one dimension is conspicuously absent. Here we attempt such a treatment. If nothing else, this appendix should indicate why the free example is typically omitted.

Putting \( 2m = 1 \) to get the Hamiltonian \( H = p^2 \), the $\star$-genvalue equation becomes
\[
\left( p - \frac{ih}{2} \partial_x \right)^2 \rho = E\rho .
\] (83)

With \( \hbar = 1 \), the imaginary part of this equation is
\[
p \partial_x \rho = 0 ,
\] (84)
while
\[
\left( p^2 - \frac{1}{4} \partial_x^2 \right) \rho = E\rho .
\] (85)

is the real part.

The factor \( p \) in (84) is crucial: we can conclude that when \( p \neq 0 \), \( \partial_x \rho = 0 \), but not when \( p = 0 \). Substituting the ansatz
\[
\rho(x,p) = f(p) + \delta(p) g(x) ,
\] (86)
in (85), we find
\[
(p^2 - E)f(p) - \delta(p) \left( E + \frac{1}{4} \partial_x^2 \right) g(x) = 0 .
\] (87)

Considering \( p = 0 \) yields
\[
g(x) = b \exp \left( 2i\sqrt{E}x \right) + b^* \exp \left( -2i\sqrt{E}x \right) ,
\] (88)
where $\rho = \rho^*$ has been imposed. Then (87) reduces to $(p^2 - E)f(p) = 0$, solved by

$$f(p) = a_+ \delta(p - \sqrt{E}) + a_- \delta(p + \sqrt{E}),$$

(89)

with $a_{\pm}$ arbitrary real constants.

The terms of (89) correspond to plane waves of momentum $\pm \sqrt{E}$, as can be verified by solving

$$p \star \rho = \rho \star p = \sqrt{E} \rho.$$  

(90)

The expression

$$\rho = a_+ \delta(p - \sqrt{E}) + a_- \delta(p + \sqrt{E}),$$

(91)

is the Wigner function of a mixed state of the two momentum eigenstates. The terms of (88) are necessary for coherent superpositions of the two momentum eigenstates, and they represent interference between them. That they are required can be seen by considering the Wigner function of a simple harmonic oscillator in the long-period limit, or the large-width limit of a particle in an infinite square well.

The general result is

$$\rho = \delta(p) \left\{ b \exp(2i\sqrt{E}x) + b^* \exp(-2i\sqrt{E}x) \right\} + a_+ \delta(p - \sqrt{E}) + a_- \delta(p + \sqrt{E}).$$

(92)

To restrict to pure-state Wigner functions, we impose

$$\rho \star \rho \propto \delta(0) \rho,$$

(93)

valid for the Wigner functions corresponding to non-normalizable pure states. We find

$$\rho \star \rho = \delta(0) \left\{ (a_+^2 + |b|^2)\delta(p - \sqrt{E}) + (a_-^2 + |b|^2)\delta(p + \sqrt{E}) \right. \\
+ (a_+ + a_-)\delta(p) \left[ b \exp(2i\sqrt{E}x) + b^* \exp(-2i\sqrt{E}x) \right]\}.$$ 

(94)

The calculation is straightforward, except that we interpret $\delta(p + \sqrt{E})\delta(p + \sqrt{E})$ as $\delta(0)\delta(p + \sqrt{E})$, e.g., since they yield equivalent results as distributions (i.e. upon integration). Similarly, assuming $E > 0$, we put $\delta(p + \sqrt{E})\delta(p - \sqrt{E})$ to zero, e.g.

In (93), this result yields the constraint

$$|b|^2 = a_+ a_- \Rightarrow b = \sqrt{a_+ a_-} e^{i\phi}, \phi \in \mathbb{R}.$$ 

(95)
The general pure-state solution to the free-particle eigenvalue equation is therefore

\[
\rho = a_+ \delta(p - \sqrt{E}) + a_- \delta(p + \sqrt{E}) \\
+ 2\sqrt{a_+ a_-} \delta(p) \cos(2\sqrt{E}x + \phi) .
\]  

(96)

On the other hand, calculating

\[
\rho[\psi] := \frac{1}{2\pi} \int dy e^{-ipy} \psi^*(x - \frac{y}{2})\psi(x + \frac{y}{2}) ,
\]  

(97)

with the pure-state wave function

\[
\psi = \alpha_+ e^{i\sqrt{Ex}} + \alpha_- e^{-i\sqrt{Ex}}
\]  

(98)

yields

\[
\rho[\psi] = |\alpha_+|^2 \delta(p - \sqrt{E}) + |\alpha_-|^2 \delta(p + \sqrt{E}) \\
+ \delta(p) \left\{ \alpha_+^* \alpha_- e^{-2i\sqrt{Ex}} + \alpha_+ \alpha_-^* e^{2i\sqrt{Ex}} \right\} .
\]  

(99)

Comparing (99) with (96) reveals a one-to-one correspondence, however, given by the relations

\[
\alpha_\pm = \sqrt{a_\pm} e^{i\phi_\pm} , \quad \phi - \phi_+ + \phi_- = 0 .
\]  

(100)

As should be, only the relative phase \(\phi_+ - \phi_-\) of (98) is relevant to the Wigner function.

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