Matching Conditions and Gravitational Collapse in Two-Dimensional Gravity

R.B. Mann and S.F. Ross
Department of Physics
University of Waterloo
Waterloo, Ontario
N2L 3G1

May 27, 1992
WATPHYS TH-92/01
Abstract

The general theory of matching conditions is developed for gravitational theories in two spacetime dimensions. Models inspired from general relativity and from string theory are considered. These conditions are used to study collapsing dust solutions in spacetimes with non-zero cosmological constant, demonstrating how two-dimensional black holes can arise as the endpoint of such collapse processes.
1 Introduction

Although the study of matching conditions (phenomena at the boundary between regions of space-time covered by different metrics) was first considered around 1920, it has been a topic of considerable interest in general relativity only since the influential paper by Israel in 1966 [1, 2, 3, 4, 5]. The objective is to describe the form of the surface separating the two regions, that is, to give the spatial position of the surface as a function of its proper time. In general relativity, this is accomplished by imposing matching conditions on the metrics at the surface, where the metric on either side and the stress-energy of the space-time are known [1]. This study has several important applications in particle physics, astrophysics and cosmology [4], as well as for gravitational collapse [5, 6].

Interest in the study of such conditions in two spacetime dimensions arises from recent work on theories of gravitation in this context [7, 8, 9, 10, 11, 12, 13]. Such theories have a rich structure which can give rise to spacetimes with black holes, thereby making them an interesting arena for the study of quantum gravitational effects as their mathematical complexity is significantly reduced from that of (3 + 1) dimensional general relativity. The formation of two-dimensional black hole spacetimes from collapsing distributions of matter is of particular interest in this regard. Some [14, 15] (but not all [8]) of these static black hole spacetimes result from the coupling of a dilaton field to the spacetime metric. The presence of the dilaton necessitates a consideration of appropriate matching conditions for the collapsing matter [14].

In this paper we formulate the (1+1) dimensional analogue of the matching conditions and apply them to both static spacetimes and to collapsing matter in a variety of situations. Specifically we consider theories of gravitation in which a generic stress-energy tensor generates the spacetime curvature, analogous to (3 + 1)-dimensional general relativity. This may be done via auxiliary fields [16], leading to a theory in which the Ricci scalar is set equal to the trace of the conserved stress-energy tensor [1], or by considering ‘string-inspired’ models, in which spacetime curvature arises due to a metric-dilaton coupling as well as due to a generic stress-energy tensor [9, 14]. We show in each case that the formation of black holes from collapsing dust proceeds in a similar manner to the analogous collapse in general relativity.
2 Two-Dimensional Gravity

Before discussing the field equations, it is worthwhile to establish the equivalent of the Gauss-Codazzi relations in two dimensions. As the boundary $\Sigma$ between any two regions of $(1 + 1)$ dimensional spacetime is in fact one-dimensional (a line), there is no intrinsic curvature, and the extrinsic curvature may be given by the scalar

$$K \equiv u^\alpha u^\beta \nabla_\alpha n_\beta$$

where $u^\alpha$ is the tangent to $\Sigma$, and $n^\alpha$ is the normal. By applying

$$R_{\alpha\beta\gamma\delta} = R g_{\alpha[\gamma} g_{\beta]\delta),$$

which is only true in two dimensions, and the fact that $u^\alpha$ and $n^\alpha$ are orthogonal unit vectors, it is possible to derive the equivalent of the Gauss-Codazzi relations,

$$R = 2(n^\alpha \nabla_\alpha K + \epsilon K^2),$$

where $\epsilon = -1$ if $\Sigma$ is timelike, $+1$ if it is spacelike. This relates the only remaining degree of freedom in the curvature of the spacetime, the Ricci scalar, to the extrinsic curvature of the surface.

Two-dimensional gravity must be founded on a different set of field equations, as the result (2) implies that Einstein’s tensor vanishes identically in two space-time dimensions,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv 0.$$  

One proposed alternative is the theory

$$R - \Lambda = 8\pi GT,$$

which is the simplest non-trivial analogue of general relativity that may be constructed in two dimensions \[18\]. This equation derived from the action \[16\]

$$S = \int d^2x \sqrt{-g} g^{\mu\nu} \left( \frac{1}{2} \partial_\mu \Psi \partial_\nu \Psi + \Psi R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Lambda \right) + L_M,$$

which also implies conservation of energy,

$$T^{\mu\nu}_{;\nu} = 0,$$  

2
if we define $\delta \mathcal{L}_M \equiv -8\pi G \sqrt{-g} \Gamma_{\mu\nu} \delta g^{\mu\nu}$. For reasonable two-dimensional stress-energy tensors, the field equation (5) qualitatively reproduces many of the solutions of Einstein’s equations, including gravitational collapse, a cosmological solution, a post-Newtonian expansion, and black holes [3, 4], plus a number of other similarities in the semiclassical regime [3, 13, 17]. We shall refer to this as the “R=T” theory.

The study of matching conditions in this theory leads to some results similar to general relativity, although we do see differences, particularly in the physical interpretation. The analogue of the Lanczos equations for this field equation is [19]

$$4\pi GS = [K],$$

(8)

and integration of the field equation (5) using (3) gives

$$S = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{0} T dn,$$

(9)

that is, $S$ is the integral of the trace of the stress-energy through the surface which is analogous to the relation

$$S_{ij} = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{0} T_{ij} dn,$$

(10)

in general relativity [4].

Another two-dimensional theory of recent interest has been developed from a non-critical string theory in two target-space dimensions [3, 10], although the most interesting solution, a metric having the form of an asymmetric black hole,

$$ds^2 = -(1 - ae^{-Qx})dt^2 + \frac{dx^2}{1 - ae^{-Qx}}$$

(11)

was first found as the solution to a scale-invariant higher-derivative theory of gravity [11], and has also been found to be a solution to $c = 1$ Liouville gravity in two dimensions [12]. The effective target space action for the string theory is [3]

$$S = \int d^2xe^{-2\Phi} \sqrt{-g}(R - 4(\nabla \Phi)^2 + c).$$

(12)

From a gravitational point of view, the asymmetry of the solution (11) about the origin is somewhat objectionable, as it is difficult to understand how
such a solution could arise from gravitational collapse of a distribution of matter confined to some finite spatial region (for an alternative viewpoint see [15]). It is possible [14] to derive a symmetric solution locally identical to (11) by assuming the presence of an appropriate source of stress-energy centered about the origin. This may be done by introducing a corresponding matter term $L_M$ in the action (12) which may be thought of as modelling some unknown higher-order effects. It may also be shown that the action with matter term is equivalent to that of a massive scalar field $\psi = e^{-\Phi}$ non-minimally coupled to curvature [14], allowing a less ambiguous interpretation of the matter term.

We therefore propose the revised action

$$S = \int d^2x \sqrt{-g} \left\{ e^{-2\Phi} (R - 4(\nabla \Phi)^2 + c) + L_M \right\}. \quad (13)$$

This leads to the field equations

$$R_{\mu \nu} - 2 \nabla_\mu \nabla_\nu \Phi = 8\pi G e^{2\Phi} T_{\mu \nu}, \quad (14)$$

and

$$R - 4(\nabla \Phi)^2 + 4 \nabla^2 \Phi + c = 0. \quad (15)$$

This system is also quite rich in structure, having a cosmological solution, a post-Newtonian expansion, the symmetric black hole solution

$$ds^2 = -(1 - ae^{-Q|x|}) dt^2 + \frac{dx^2}{1 - ae^{-Q|x|}}. \quad (16)$$

and gravitational collapse. These have been studied in detail in [14, 20]. Although these field equations sacrifice some of the simplicity of the field equation (5), they remain substantially simpler in practical terms than general relativity. The analogues of the Lanczos equations for these field equations are

$$8\pi GS_{00} = \epsilon[K], \quad (17)$$

$$4\pi GS_{01} = [u^\alpha \nabla_\alpha \Phi], \quad (18)$$

$$8\pi GS_{11} = -\epsilon[K] + 2[n^\alpha \nabla_\alpha \Phi], \quad (19)$$

in natural coordinates [1] where $x^0$ and $x^1$ are the geodesic distances along $u^\alpha$ and $n^\alpha$ respectively. These equations introduce the possibility of non-zero
elements of $S_{ij}$ in the normal direction, in contrast to the situation in general relativity.

Integration of the field equations (14,15) using the relation between the Ricci scalar and extrinsic curvature (3) gives

$$S_{ij} = \lim_{\varepsilon \to 0} \int_{0}^{\varepsilon} e^{2\Phi} T_{ij} dn,$$

(20)

that is, the surface stress-energy is the integral of the stress-energy multiplied by a weighting factor through the surface in natural coordinates which is the analogue of (10) in this case.

3 Matching: The Static Case

In the R=T theory, the metric

$$ds^2 = -f(|x|)dt^2 + f(|x|)^{-1}dx^2$$

(21)

where

$$f(|x|) = -\frac{\Lambda}{2} x^2 + 2M|x| - C$$

(22)

is a solution of the field equation (3) for $8\pi GT = 4M\delta(x)$, that is, it is the general vacuum solution with a point source of matter and a cosmological constant. It is easy to see that this metric describes black hole solutions and cosmological event horizons, as the coordinate singularities of the metric are given by the condition $f(|x|) = 0$, giving pairs of horizons on either side of the point singularity. We therefore say that this metric represents the analogue of the Schwarzschild-De Sitter solution in this two dimensional theory. Note however that the Cosmic Censorship principle does not apply, as, for given $\Lambda$ and $M$, there is some $C$ such that $f(|x|) = 0$ has no positive roots, and thus there can be uncloaked singularities.

As any two-dimensional space-time metric has only one degree of freedom, any symmetric static metric can be written in the form (21). It is therefore very useful to consider the problem analogous to the boundary between two Schwarzschild-De Sitter metrics [4],

$$ds^2_\pm = -f_\pm dt^2_\pm + \frac{dx^2_\pm}{f_\pm},$$

(23)
where the physical example we consider is the boundary between two solutions of the form

\[ f_{\pm} = -\frac{1}{2} \Lambda_{\pm} x^2 + 2M_{\pm} |x| - C. \] (24)

The metric intrinsic to \( \Sigma \) is

\[ ds^2_{\Sigma} = -d\tau^2 \] (25)

in all two-dimensional systems. Matching the metrics at the surface \( \Sigma \) described by \(|x| = \mathcal{R}(\tau)\) gives

\[ f_{\pm}^{-1} \dot{\mathcal{R}}^2 - f_{\pm} i^2_{\pm} = -1, \] (26)

where the overdot denotes the derivative with respect to proper time, as before. Assuming the surface is timelike, the tangent vector will be a timelike unit vector,

\[ u^\mu u^\nu g_{\mu\nu} = -1 \] (27)

implying by (26) that

\[ u^\mu = (i_{\pm}, \dot{\mathcal{R}}). \] (28)

Similarly, by using the orthogonality of the tangent and normal vectors, \( u^\mu n_\mu = 0 \), and the fact that \( n_\mu \) is also a unit vector,

\[ n_\mu n_\nu g^{\mu\nu} = 1, \] (29)

we find

\[ n_\mu = (-\dot{\mathcal{R}}, i_{\pm}). \] (30)

Applying the condition (26) and the two forms (28,30) to the definition [1] of the extrinsic curvature yields

\[ K_{\pm} = -\frac{\dot{\mathcal{R}} + \frac{1}{2} f'_{\pm}}{\sqrt{\dot{\mathcal{R}}^2 + f_{\pm}}}, \] (31)

which may be simplified to

\[ K_{\pm} = -\frac{d}{d\mathcal{R}} \sqrt{\dot{\mathcal{R}}^2 + f_{\pm}}. \] (32)

Substituting this form of \( K \) in the Lanczos equation \([K] = 8\pi GS\) gives

\[ \dot{F}_+ - \dot{F}_- = -8\pi GS \dot{\mathcal{R}}, \] (33)
\[ F_\pm = \sqrt{\mathcal{R}_\pm^2 + f_\pm} \] as before, analogous to the situation in general relativity \[2\]. However, we do not have the additional information (which comes from the angular components in higher dimensions) that enabled us to obtain the first integral previously. If we define \( \sigma \) by

\[ \frac{d(\sigma \mathcal{R})}{d\tau} = 2S\dot{\mathcal{R}} \quad (34) \]

then we obtain

\[ F_+ - F_- = -4\pi G\sigma \mathcal{R} - c, \quad (35) \]

with \( c \) a constant of integration. This equation is identical in form to that which occurs in general relativity \[2\]. However the physical interpretation of \( \sigma \) is unclear in this case, and (in contrast to general relativity \[2\]) we cannot eliminate the constant of integration \( c \). This constant means two initial conditions are required to give a unique solution, where only one was necessary in general relativity. In the special case of the thin shell of dust in a vacuum, \( S \) is a constant, the surface density, and we get \( \sigma = 2S \) from the definition \( (34) \). In general relativity, \( \sigma \) is defined to be the surface density, so this result is encouraging.

Turning now to the string-motivated theory described by \( (13) \), if we consider the case of the symmetric static metric

\[ ds^2 = -f_\pm dt^2 + \frac{dx^2}{f_\pm}, \quad (36) \]

the development could be followed through in much the same way as it was above for the R=T theory. However, in the case of \( (14,15) \), we have another important matching condition. We must require that the dilaton field \( \Phi \) be continuous, as otherwise we would need an infinite surface density of the associated charge on the surface (this is similar to the requirement that the scalar potential \( V \) be continuous in classical electromagnetism). We have two important static metrics in this theory, which are (written in the form \( (36) \))

\[ f_+ = 1 - ae^{-Q_+ |x|}, \quad (37) \]

the vacuum solution developed in \( (14) \), for which we find

\[ \Phi_+ = \frac{-Q_+ |x|}{2}, \quad (38) \]
and

\[ f_- = 1 - \frac{k}{1 + \cosh Q_- x} \]  \hspace{1cm} (39)

which is actually a proposed vacuum solution of the string theory with a non-zero tachyon field \([21]\), with the associated dilaton field

\[ \Phi_- = \ln \left[ Q_- \sqrt{2/k} \sinh \left( \frac{Q_-}{2} |x| \right) \sqrt{\sinh^2 \left( \frac{Q_-}{2} |x| \right) + 1} \right] . \]  \hspace{1cm} (40)

Thus, if we regard these two solutions as the outside and inside metrics for some surface layer, requiring continuity of the dilaton field \( \Phi_+ (\mathcal{R}) = \Phi_- (\mathcal{R}) \) gives the equation

\[ e^{-Q_+ \mathcal{R}} = \frac{2Q_+^2}{k} \sinh^2 \left( \frac{Q_-}{2} \mathcal{R} \right) \left( \sinh^2 \left( \frac{Q_-}{2} \mathcal{R} \right) + 1 \right), \]  \hspace{1cm} (41)

which has only the trivial solution \( \mathcal{R} (\tau) = C \), where \( C \) is a constant determined in terms of the parameters on both sides. It may be the case that there is no solution, as (41) may not have a positive real root. The constant solution does not imply that there is no stress-energy on the surface; instead

\[ K_\pm = -\frac{d\sqrt{f_\pm}}{d\mathcal{R}}, \]  \hspace{1cm} (42)

as \( \mathcal{R} = 0 \), and thus the first “Lanczos” equation (17) implies

\[ 8\pi GS_{00} = -\frac{d}{d\mathcal{R}} (\sqrt{f_+} - \sqrt{f_-}). \]  \hspace{1cm} (43)

We may make the obvious generalization that for any matching problem in which the dilaton field is static on both sides of the surface, requiring continuity of the dilaton field will give at most the trivial solution \( \mathcal{R} (\tau) = C \), and some constant stress-energy on the surface. A complete solution would require matching of the tachyon field as well, but we shall not consider this problem any further.

### 4 Matching: The Collapsing Case

Consider a non-static metric written in comoving coordinates

\[ ds^2 = -dt^2 + a^2(t)dy^2, \]  \hspace{1cm} (44)
where the cosmic scale factor $a(t)$ is an arbitrary function representing the one degree of freedom of the metric. We take the position of the surface $\Sigma$ to be $|y| = r(\tau)$ in this metric. This metric represents the cosmological solution, as the length scale will vary in time; in two dimensions open and closed universes do not yield distinct metrics [8]. In the two-dimensional theories described above, the exact solutions to date can be written either in this form or in the static form (21); in particular, the interior metric of collapsing dust is written in this form with the spatial coordinate taken to be co-moving, implying $\dot{r} = 0$.

If the metric on one side (say $V_-$) of a surface has the form (44), matching it to the intrinsic metric of the surface gives the condition

$$-\dot{t}^2_+ + a^2 \dot{r}^2 = -1. \quad (45)$$

Assuming the surface is timelike, the expressions (28) and (30) for $u^\alpha$ and $n_\alpha$ are valid, and we can explicitly evaluate the extrinsic curvature, giving

$$K_- = -\frac{\dot{r}}{t_-} - 2\frac{a'}{a} \dot{r}. \quad (46)$$

As this does not appear to be the derivative of a function, that is $K_- \neq df/dr$ for any $f$, we cannot integrate the Lanczos equation as we did previously. Therefore, this is as far as we can carry the general argument, and further discussion requires that we specify $a(t)$.

We therefore turn to the special case of a collapsing “ball” of dust in one spatial dimension. The study of a ball of collapsing dust in general relativity provides an idealized model of the physical collapse of a star to form a black hole. Here we consider the analogue of this problem in two-spacetime dimensions to see whether or not the 2D black holes studied previously [7, 9, 13] can arise as the endpoint of such a collapse.

Although a ball of collapsing dust in a spacetime with vanishing cosmological constant was studied for these reasons early in the development of the R=T theory [3], it has been recently suggested [22] that as the space-time outside this dust is flat, it represents a very special case, and is not a reliable model for the corresponding interactions in general relativity. We therefore consider collapsing dust in a spacetime with non-zero cosmological constant. As we shall see, the qualitative properties of the collapse are the same as in ref. [8], and all the results reduce to the previous ones when $\Lambda \to 0$. 

9
The dust is falling freely under the influence of purely gravitational forces, so we may make it the basis of a comoving coordinate system. We then have a Robertson-Walker metric on the interior region,

\[ ds^2 = -d\tau^2 + a^2(\tau)dy^2, \]  

(47)

where \( y \) is the comoving spatial coordinate, \( \tau \) is the proper time of the dust, and \( a \) is the scale factor. The trace of the stress-energy in the interior region is \( T = -\rho \), where \( \rho(\tau) \) is the density of the dust. The conservation law (7) thus implies that \( \rho a = \rho_0 a_0 \), where \( \rho_0 \) is the initial density of the dust, and \( a_0 \) is the initial scale factor [8]. The field equation then becomes

\[ \ddot{a} = -4\pi G\rho_0 a_0 + \frac{\Lambda}{2}a, \]  

(48)

where the overdot denotes \( d/d\tau \), and we impose the initial conditions

\[ a_0 = a(0) = 1, \dot{a}(0) = 0, \]  

(49)

that is, we take the dust to be initially at rest with \( a \) at its maximum, and scale \( a \) to its initial value. The solution for \( a \) with these initial conditions can be easily seen to be

\[ a(\tau) = (1 - \frac{8\pi G\rho_0}{\Lambda}) \cosh(\sqrt{\frac{\Lambda}{2}}\tau) + \frac{8\pi G\rho_0}{\Lambda} \]  

(50)

if \( \Lambda \) is positive, and

\[ a(\tau) = (1 - \frac{8\pi G\rho_0}{\Lambda}) \cos(\sqrt{\frac{\Lambda}{2}}\tau) + \frac{8\pi G\rho_0}{\Lambda} \]  

(51)

if \( \Lambda \) is negative. As \( \Lambda \rightarrow 0 \)

\[ a(\tau) \rightarrow 1 - 2\pi G\rho_0 \tau^2, \]  

(52)

which is the solution found in [8].

To make the form of the solution more obvious, we write the interior metric as

\[ ds^2 = -d\tau^2 + ((1 - b) \cosh(\sqrt{\frac{\Lambda}{2}}\tau) + b)^2dy^2 \]  

(53)
if $\Lambda$ is positive, and

$$ds^2 = -d\tau^2 + ((1 + b) \cos(\sqrt{\Lambda'/2} \tau) - b)^2 dy^2$$

(54)

if $\Lambda$ is negative, where we have defined $\Lambda' = |\Lambda|$, and $b = 8\pi G \rho_0 / \Lambda'$. For a collapsing dust, we must have $a(\tau) = 0$ at some finite time $\tau$. For $\Lambda$ negative,

$$\tau = \sqrt{\frac{2}{\Lambda'}} \cos^{-1}\left(\frac{b}{b+1}\right),$$

(55)

but when $\Lambda$ is positive, we must have

$$b > 1 \Rightarrow \rho_0 > \frac{\Lambda}{8\pi G}$$

(56)

for (53) to correspond to a collapsing dust. If this condition is not satisfied, the gravitational collapse of the dust will be overcome by the general expansion of the space-time caused by the cosmological constant. When (56) is satisfied, $a(\tau) = 0$ at

$$\tau = \sqrt{\frac{2}{\Lambda'}} \cosh^{-1}\left(\frac{b}{b-1}\right).$$

(57)

Note that $\tau \to (2\pi G \rho_0)^{-1/2}$ as $\Lambda \to 0$ in both cases, reproducing the previous result [3].

In the exterior coordinates, where the stress-energy trace $T = 0$, the metric is the two-dimensional Schwarzschild-De Sitter analogue (21). In the matching, we treat the mass $M$ and constant $C$ as adjustable parameters.

We consider next the problem of extending the outside co-ordinates to the inside of the dust. In the integrating factor technique, we attempt to construct a continuous transformation from the interior to the exterior coordinates over the whole of the interior dust-filled region [3]. That is to say, we attempt to write the interior metric in the form

$$ds^2 = -B(t, x)dt^2 + A(t, x)dx^2$$

(58)

where $x$ and $t$ are the same as before, and $B$ and $A$ must satisfy the boundary conditions

$$B(t, \mathcal{R}) = f(\mathcal{R}), \quad A(t, \mathcal{R}) = \frac{1}{f(\mathcal{R})}$$

(59)
at the dust edge, whose position is $|y| = r$ in the interior coordinates and $|x| = R$ in the exterior coordinates. We assume that
\[ dt = \eta(\dot{S}d\tau + S'dy), x = a(\tau)y \]
where $\eta$ is an integrating factor. Requiring that the forms (58) and (47) of the interior metric be equivalent then gives
\[ B\eta^2\dot{S}\dot{S}' = Aya, \]
\[ B\eta^2\dot{S}^2 - Ay^2\dot{a}^2 = 1, \]
\[ B\eta^2S'^2 - Aa^2 = -a^2, \]
and (62) and (63) imply
\[ A = \frac{1}{1 - y^2\dot{a}^2}. \]

If the cosmological constant vanishes, it is possible to solve for $S$ and $\eta$, and explicit forms of $A$ and $B$ are given in [8]. In the more general case, we were able to find $S$ by assuming $S = f(y)g(a(\tau))$, but we were unable to find a suitable expression for $\eta$. As a transformation of this kind was found in the case $\Lambda = 0$, we expect that it is still in principle possible here, since there should be no significant change in the behavior of the dust edge. We therefore assume
\[ f(R) = \frac{1}{A} = 1 - r^2\dot{a}^2, \]
which is necessary for the transformation to be continuous at the dust edge, and $R = ra(\tau)$, that is, the position of the dust edge in the exterior coordinates is the proper distance from the origin to the dust edge. Note that with the latter assumption, we can interpret the boundary conditions (49) as representing a ball of dust with initial radius $R_0 = r$ initially at rest in the exterior coordinates, and we see that $a = 0$ corresponds to the collapse of the dust to $R = 0$.

We next show that the above conditions imply that the dust edge is in fact a boundary surface in two dimensions. The extrinsic curvature in the interior region is given by (30), and as the interior coordinates are co-moving, $\dot{r} = 0$, and thus $K_\perp = 0$. The extrinsic curvature in the exterior coordinates is given by (32),
\[ K_+ = -\frac{d}{dR}\sqrt{R^2 + f}. \]
If we now set $R = ra(\tau)$, $f(R) = 1 - r^2a^2$, we see that $\dot{R}^2 + f(R) = 1$, and thus $K_+ = 0$. Thus, if these conditions are satisfied, the dust edge will be a boundary surface. This is important, as the dust edge in general relativity is a boundary surface, and thus a collapse that required a surface shell of matter would not be a good model for the collapsing dust in four dimensions. Finally, we should note that (65) is not a necessary condition for the dust edge to be a boundary surface, as we only need $\dot{R}^2 + f(R) = c$, $c$ some arbitrary constant, to obtain $K_+ = 0$.

Applying the conditions (65) and $R = ra(t)$ to the interior metrics (53) and (54), and the exterior metric (21), we find

$$M = 2\rho_0 r, \quad C = 1 - r^2(8\pi G\rho_0 - \frac{\Lambda}{2}).$$

(67)

This mass identification is hardly surprising, as it simply means that a line of dust of density $\rho_0$ and length $2r$ is equivalent to a point source of mass $2\rho_0 r$. The identification of the constant of integration is relatively unimportant, as it has no physical significance. Note that only the mass identification is required for the dust edge to be a boundary surface, that is, it is equivalent to the necessary condition $\dot{R}^2 + f(R) = c$.

As positive and negative cosmological constants give slightly different results we treat them separately. For positive $\Lambda$, the matching conditions are satisfied if (53) is matched to (21) with

$$f(x) = \Lambda' br|x| - \frac{\Lambda'}{2} x^2 + 1 - \frac{\Lambda'}{2} r^2(2b - 1).$$

(68)

There is no Birkhoff theorem in two dimensions \[17\], so we must ask under what conditions the collapse of the dust leads to a black hole. The positions of the event horizons are given by the roots of (68), as these are where the signature of the metric (21) changes sign, so

$$|x_h| = br \pm \sqrt{(b - 1)^2r^2 + \frac{2}{\Lambda'}}.$$ 

(69)

There is always at least one root, but we interpret this larger root as the cosmological event horizon due to the expansion of space-time caused by the positive cosmological constant. Thus, an event horizon will form around the collapsing dust only if the smaller root is positive, that is, if

$$b > \frac{1}{\Lambda' r^2} + \frac{1}{2} \Rightarrow \rho_0 > \frac{1}{8\pi Gr^2} + \frac{\Lambda'}{16\pi G}. $$

(70)
This condition reduces to the one in [8] if Λ = 0. Note that it is possible to satisfy this condition on the initial density, but not the condition (56). This, however, is physically meaningless, as in this case the dust would not collapse, and the Schwarzschild radius given by (69) would never lie within the range of the exterior coordinates. Alternatively, if (56) is satisfied, but (70) is not, collapse occurs to a naked point source. Both (56) and (70) must be satisfied for a black hole to form.

If an event horizon does form, we would now like to know when it forms, that is, we would like to find the comoving time \( \tau_h \) at which the event horizon and the dust edge coincide. This is found by substituting \( x_h = ra(\tau_h) \) in (69), which gives

\[
\tau_h = \sqrt{2 \Lambda} \sinh^{-1} \left( \frac{1}{\sqrt{2 \Lambda r(b-1)}} \right) .
\]

(71)

Note that the comoving time at which the horizon forms is finite, and \( \tau_h \to \frac{1}{4\pi G \rho_0 r} \) as \( \Lambda \to 0 \), so we recover the previous result [8].

A light signal emitted from the surface at time \( t \) obeys the null condition

\[
\frac{dx}{dt} = \Lambda' br|x| - \frac{\Lambda'}{2} x^2 + 1 - \frac{\Lambda'}{2} r^2(2b-1)
\]

(72)

and arrives at a point \( \tilde{x} \) at time

\[
\tilde{t} = t + \int_{ra(\tau)}^{\tilde{x}} \frac{dt}{dx} dx = t + \frac{2}{\Lambda' \sqrt{(b-1)^2 r^2 + 2/\Lambda'}} \tanh^{-1} \left[ \frac{|x| - br}{\sqrt{(b-1)^2 r^2 + 2/\Lambda'}} \right]_{ra(\tau)}^{\tilde{x}},
\]

(73)

and thus \( \tilde{t} \to \infty \) as \( ra(\tau) \to x_h \) given by (69), that is, as \( \tau \to \tau_h \), so the collapse to the Schwarzschild radius appears to take an infinite amount of time, and the collapse to \( R = 0 \) is unobservable from outside, as in [8]. The form of \( \tilde{t} \) also reduces to the previous one as \( \Lambda \to 0 \).

The comoving time interval \( d\tau \) between emissions of wave crests is equal to the natural wavelength \( \lambda \) that would be emitted in the absence of gravitation, and the interval \( d\tilde{t} \) between arrivals of wave crests is the observed wavelength \( \tilde{\lambda} \). Thus the red shift of light from the dust edge is

\[
z = \frac{d\tilde{t}}{d\tau} - 1 = \frac{1}{1 + r\dot{a}(\tau)} - 1,
\]

(74)
and
\[ r\dot{a}(\tau_h) = r(1 - b)\sqrt{\frac{\Lambda'}{2}} \sinh \left( \sqrt{\frac{\Lambda'}{2}} \tau_h \right) = -1, \quad (75) \]
therefore \( z \to \infty \) as \( \tau \to \tau_h \), as before [8]. Thus, the collapsing fluid will fade from sight, as the red shift of light from its surface diverges.

For negative \( \Lambda \), the matching conditions are satisfied if (54) is matched to (21) with
\[ f(x) = \Lambda' br|x| + \frac{\Lambda'}{2} x^2 + 1 - \frac{\Lambda'}{2} r^2(2b + 1). \quad (76) \]
The position of the event horizon is now given by the root of (76), which gives
\[ |x_h| = -br \pm \sqrt{(b + 1)^2r^2 - \frac{2}{\Lambda'}}, \quad (77) \]
and we see that there is at most one positive root of (76). This is not surprising, as with \( \Lambda \) negative, the space-time is contracting, and there is no cosmological horizon. This Schwarzschild radius is positive if
\[ b > \frac{1}{\Lambda' r^2} - \frac{1}{2} \Rightarrow \rho_0 > \frac{1}{8\pi Gr^2} - \frac{\Lambda'}{16\pi G}. \quad (78) \]
This also reduces to the condition in [8] if \( \Lambda = 0 \).

The comoving time at which the horizon forms may be found by substituting \( x_h = ra(\tau_h) \) in (77), which gives
\[ \tau_h = \sqrt{\frac{2}{\Lambda'}} \sin^{-1} \left( \sqrt{\frac{2}{\Lambda'} \frac{1}{r(b + 1)}} \right). \quad (79) \]
As before, \( \tau_h \) is finite, with limit \( \tau_h \to 1/4\pi G\rho_0 r \) as \( \Lambda \to 0 \), so we recover the previous result [8]. The null condition for a light signal emitted from the dust edge is
\[ \frac{dx}{dt} = \Lambda' br|x| + \frac{\Lambda'}{2} x^2 + 1 - \frac{\Lambda'}{2} r^2(2b + 1) \quad (80) \]
so light emitted at \((x, t)\) arrives at \( \tilde{x} \) at time
\[ \tilde{t} = t + \int_{ra(\tau)}^{\tilde{x}} \frac{dt}{dx}dx \]
\[ = t - \frac{2}{\Lambda' \sqrt{(b + 1)^2r^2 - 2/\Lambda'}} \tanh^{-1} \left[ \frac{|x| + br}{\sqrt{(b + 1)^2r^2 - 2/\Lambda'}} \right]_{ra(\tau)}. \quad (81) \]
and thus $\bar{t} \to \infty$ as $ra(\tau) \to x_h$ given by (77), that is, as $\tau \to \tau_h$. The collapse to the Schwarzschild radius appears to take an infinite amount of time, as before. The form of $\bar{t}$ also reduces to the previous one as $\Lambda \to 0$.

The red shift of light from the horizon is still given by (74), and we have

$$r\dot{a}(\tau_h) = -r(1 + b)\sqrt{\frac{2}{\Lambda'}} \sin \left(\sqrt{\frac{2}{\Lambda'}\tau_h}\right) = -1, \quad (82)$$

so $z \to \infty$ as $\tau \to \tau_h$, the red shift of light from the surface diverges as the horizon forms. We should note finally that (74) reduces to the expression in [8] for the red shift when the cosmological constant vanishes.

The case of collapsing dust in the string-motivated theory was studied in [14]. We include the results here in the interests of completeness, and because they provide another example of the effects of requiring continuity of the dilaton field. The interior metric for the dust is of the form

$$ds^2 = -d\tau^2 + a^2(\tau)dy^2, \quad (83)$$

where the $y$ coordinate is comoving with the dust. The interior dilaton field is

$$\Phi_-(\tau) = \Phi_0 - \ln \left(\cos\left(\frac{Q}{2}\tau\right)\right). \quad (84)$$

The exterior metric is the vacuum solution (16),

$$ds^2_+ = -(1 - ae^{-Q|x|})dt^2 + \frac{dx^2}{1 - ae^{-Q|x|}}, \quad (85)$$

and the exterior dilaton field is

$$\Phi_+ = -\frac{Q}{2}x. \quad (86)$$

We require continuity of the dilaton field through the surface, which gives the equation

$$\mathcal{R}(\tau) = x_0 + \frac{2}{Q} \ln \left(\cos\left(\frac{Q}{2}\tau\right)\right) \quad (87)$$

in the exterior coordinates. There will in general be a non-zero stress-energy on the surface of the dust in the string theory,

$$8\pi G_{S_{ij}} = \begin{pmatrix} -[K] & 0 \\ 0 & -[K] \end{pmatrix}, \quad (88)$$

and there is a surface dilaton charge, a feature that differs from both the R=T case and that of general relativity.
5 Conclusions

We have shown that the endpoint of dust collapse in both the R=T and the string-motivated two dimensional theories of gravity considered here yield the static black holes studied previously [7, 9]. The general results qualitatively parallel the four-dimensional case, although the technical details are quite different. In particular, we have (3), which has the same content as the Gauss-Codazzi equations. The “Lanczos” equations in the two cases take almost the same form as in general relativity, and give the same result that $S_{ij}$, referred to as a surface stress-energy, is the integral of the volume stress-energy through the surface in natural coordinates, although this is multiplied by an exponential of the dilaton field in the string-motivated case. It is worth remarking that in the latter theory, the normal components of $S_{ij}$ are non-zero, unlike general relativity and the linear theory.

At the boundary between two static metrics in the R=T theory, we find that the equation for $R(\tau)$ is the same as it was in general relativity, with the problem being reduced to solving the differential equation (35). However, the interpretation of this result is quite different: $\sigma$ does not have a simple physical interpretation, and there is an additional arbitrary constant in the solution. On the other hand the string motivated system of equations (14, 15) has a physically significant auxiliary field, the dilaton field $\Phi$, and we find that requiring that $\Phi$ be continuous gives us the function $R(\tau)$ describing the surface in both cases of interest. As the dilaton field represents a physical potential in the string theory, we must require that it be continuous if we wish to have only finite effects on the surface.

For the R=T theory we have extended the collapsing dust scenario of ref. 8 to include spacetimes with a non-zero cosmological constant. The solution with $\Lambda$ positive will collapse only if the initial density $\rho_0$ exceeds a critical value given by (53), since the gravitational attraction of the dust must overcome the cosmological expansion of the space-time. The condition (53) is sufficient to make the dust edge a boundary surface, although it is not necessary. It is equivalent to the mass and constant identifications (67). This is the mass we would expect for the dust, and it is equivalent to the necessary condition for the dust edge to be a boundary surface. Horizons will form as the dust collapses only if the conditions (71) and (78) on the initial density of the dust are satisfied. We also find that when the horizon forms, it does so in finite comoving time and infinite coordinate time, and that the
red shift of light from the dust edge diverges as the horizon forms.

As the qualitative results of the study of the collapsing dust are the same as those of the previous study [8], we conclude that black holes may be successfully modelled in two dimensions. Of greater interest, of course, is the inclusion of quantum effects in these processes. The simplicity of the theories described here make them ideal testing grounds for further work in this area.

6 Acknowledgments

This work was supported in part by the Natural Sciences and Engineering Research Council. We would like to thank K. Lake for helpful discussions.

References

[1] W. Israel, Nuovo Cimento B44 (1966) 1.

[2] J.E. Chase, Nuovo Cimento B67 (1970) 136.

[3] K. Lake, Phys. Rev. D19 (1979) 2487.

[4] A. Aurilia, R.S. Kissack, R. Mann and E. Spallucci, Phys. Rev. D35 (1987) 2961.

[5] K. Lake, “Some notes on the Propagation of Discontinuities in Solutions to the Einstein Equations”, Kingston, Queen’s University.

[6] S. Weinberg, Gravitation and Cosmology (John Wiley & sons, New York 1972).

[7] R.B. Mann, A. Shiekh, and L. Tarasov, Nucl. Phys. B341 (1990) 134.

[8] A.E. Sikkema and R.B. Mann, Class. Quantum Grav. 8 (1991) 219.

[9] G. Mandal, A.M. Sengupta, and S.R. Wadia, Mod. Phys. Lett. B 6 (1991) 1685.
[10] E. Witten, Phys. Rev. D44 (1991) 314.

[11] H.-J. Schmidt, J. Math. Phys. 32 (1991) 1562.

[12] A.R. Lugo and J.G. Russo, Stanford University preprint, SU-ITP-896.

[13] R.B. Mann, Gen. Rel. Grav. 24 (1992) 433.

[14] R.B. Mann, M.S. Morris and S.F. Ross, Waterloo preprint WATPHYS TH-91/04.

[15] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. D45 (1992) R1005.

[16] R.B. Mann, S.M. Morsink, A.E. Sikkema and T.G. Steele, Phys. Rev. D43 (1991) 3948.

[17] S.M. Morsink and R.B. Mann, Class. Quant. Grav. 8 (1991) 2257; R.B. Mann and T.G. Steele, Class. Quant. Grav. 9 (1992) 475.

[18] R.B. Mann, Found. Phys. Lett. 4 (1991) 425.

[19] J.D. Brown, “Lower Dimensional Gravity”, Singapore, World Scientific (1988).

[20] R.B. Mann and S.F. Ross, WATPHYS-TH92/02.

[21] R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B371 (1992) 269.

[22] M. Kriele, private communication.