Entanglement-Assisted Quantum Data Compression

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Abstract—Ask how the quantum compression of ensembles of pure states is affected by the availability of entanglement, and in settings where the encoder has access to side information. We find the optimal asymptotic quantum rate and the optimal tradeoff (rate region) of quantum and entanglement rates. It turns out that the rate can be described equivalently by the cq-state

$$\omega^{XAC} = \sum_{x \in X} p(x) |x\rangle A \otimes |\psi_x\rangle_{XAC}$$

(1)

Thus, the source is now an ensemble $E = \{ p(x), |\psi_x\rangle_{XAC} \}$ of product states, which can be described equivalently by the cq-state

$$\omega^{XAC} = \sum_{x \in X} p(x) |x\rangle A \otimes |\psi_x\rangle_{XAC}$$

(1)

yet another equivalent description is via the random variable $X \in X$, distributed according to $p, i.e. \Pr\{ X = x \} = p_x$; this also makes the pure states $|\psi_x\rangle$ random variables.

We will consider the information theoretic limit of many copies of $\omega$, i.e. $\omega^{X^n A^n C^n} = (\omega^{XAC})^{\otimes n}$

Using the notation

$$x^n = x_1 x_2 \ldots x_n \quad p(x^n) = p(x_1) p(x_2) \ldots p(x_n),$$

Further notation. Conditional entropy and conditional mutual information, $S(A|B)_{\omega}$ and $I(A:B|C)_{\omega}$, respectively, are defined in the same way as their classical counterparts:

$$S(A|B)_\omega = S(AB)_\omega - S(B)_\omega \quad \text{and} \quad I(A:B|C)_\omega = S(A|C)_\omega - S(A|BC)_\omega$$

The fidelity between two states $\omega$ and $\xi$ is defined as $F(\omega, \xi) = \| \sqrt{\omega} \sqrt{\xi} \|_1 = \text{Tr} \sqrt{\omega^2 \xi}$, with the trace norm $\| | X \|_1 = \text{Tr} | X | = \text{Tr} X^T X$.

II. COMPRESSION ASSISTED BY ENTANGLEMENT

We assume that the encoder, Alice, and the decoder, Bob, have initially a maximally entangled state $\Phi_K^{A_0 B_0}$ on registers $A_0$ and $B_0$ (both of dimension $K$). With probability $p(x^n)$, the source provides Alice with the state $|\psi^n_x\rangle = \sigma_x^{C^n}$. Then, Alice performs her encoding operation $C : A^n C^n A_0 \rightarrow C^n C$ on the systems $A^n$, $C^n$ and her part $A_0$ of the entanglement, which is a quantum channel, i.e. a completely positive and trace preserving (CPTP) map. Note that our notation is a slight abuse, which we maintain as it is simpler while it cannot lead to confusions, since channels really are maps between the trace class operators on the involved Hilbert spaces.) The dimension of the compressed system obviously has to be smaller than the original source, i.e. $|C_A| \leq |A|^n$. We call $Q = \frac{1}{2} \log |C_A|$.
and $E = \frac{1}{2} \log K$ the quantum and entanglement rates of the compression protocol, respectively. The system $C_A$ is then sent to Bob via a noiseless quantum channel, who performs a decoding operation $D : C_A B_0 \rightarrow \hat{A}^n$ on the system $C_A$ and his part of entanglement $B_0$.

According to Stinespring’s theorem [7], all these CPTP maps can be dilated to isometries $V_A : A^n C^n A_0 \rightarrow C^n C_A W_A$ and $V_B : C_A B_0 \rightarrow \hat{A}^n W_B$, where the new systems $W_A$ and $W_B$ are the environment systems of Alice and Bob, respectively.

We say the encoding-decoding scheme has fidelity $1 - \epsilon$, or error $\epsilon$, if

$$F := \text{F} \left( \omega^{X^n C^n \xi C^n} \right) := \sum_{x^n \in S_X^n} p(x^n) F \left( \| \psi_{x^n} \|_{x^n} \| A^n \|_{\sigma x^n} \| C^n \|_{\xi} \right) \geq 1 - \epsilon,$$

where $\xi^{X^n C^n \xi C^n} = \sum_{x^n} \rho(x^n) \| x^n \|_{x^n} \otimes \xi^{x C^n}$ and $\xi^{x C^n} = (D \otimes C) \psi_{x^n} \psi_{x^n}^{A^n} \otimes \sigma x^n \|_{\sigma x^n} \otimes \Phi_{A_B B_0}$. We say that $(E, Q)$ is an (asymptotically) achievable rate pair if for all $n$ there exist codes such that the fidelity converges to 1, and the entanglement and quantum rates converge to $E$ and $Q$, respectively. The rate region is the set of all achievable rate pairs, as a subset of $\mathbb{R} \times \mathbb{R}^+$.

Note that this means that we demand not only that Bob can reconstruct the source states $\psi_{x^n}$ with high fidelity on average, but that Alice retains the side information states $\sigma x^n$ as well with high fidelity.

There are two extreme cases of the side information that have been considered in the literature: If $C$ is a trivial system, or more generally if the states $\sigma x^n$ are all identical, then the aforementioned task is the entanglement-assisted version of blind Schumacher compression. If $C = X$, or more precisely $|\sigma x| = |x|$, then Alice has access to classical random variable $X$, and the task reduces to visible Schumacher compression with entanglement assistance. The blind-visible terminology is with high fidelity.

Remark 1: In the case of no entanglement being available, i.e. $E = 0$ ($K = 1$), the problem is fully understood: The asymptotic rate $Q = S(A)$ from [1], [2] is achievable without touching the side information, and it is optimal, even in the visible case (which includes all other side information), by the weak and strong converses of [4], [5] and [6].

### III. Optimal Quantum Rate

To formulate the minimum compression rate under unlimited entanglement assistance, we need the following concept.

**Definition 2:** An ensemble of pure states $\mathcal{E} = \{ p(x), |\psi_x \rangle \rangle \langle \psi_x | \langle x | \rangle \}_{x \in X}$ is called reducible if its states fall into two or more orthogonal subspaces. Otherwise the ensemble $\mathcal{E}$ is called irreducible. We apply the same terminology to the source cqg-state $\omega^{XAC}$.

Notice that a reducible ensemble can be written uniquely as a disjoint union of irreducible ensembles $\mathcal{E} = \bigcup_{y \in Y} \mathcal{E}_y$, with a partition $X = \bigcup_{y \in Y} X_y$ and irreducible ensembles $\mathcal{E}_y = \{ p(x|y), |\psi_x \rangle \rangle \langle \psi_x | \langle x | \rangle \}_{x \in X_y}$, where $q(y)p(x|y) = p(x)$ for $x \in X_y$ and $q(y) = \sum x \in X_y p(x)$. We define the subspace spanned by the vectors of each irreducible ensemble as $F_y := \text{span} \{|\psi_x \rangle : x \in X_y \}$. The irreducible ensembles $\mathcal{E}_y$ are pairwise orthogonal, i.e. $F_y \perp F_{y'}$ for all $y' \neq y$. We may thus introduce the random variable $Y = Y(X)$ taking values in the set $Y$ with probability distribution $q(y)$; namely, $Y$ is a deterministic function of $X$ such that $\text{Pr} \{ X \in X_y \} = 1$. We define the modified source $\omega^{XACY} = \sum_y p(y) \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y$ with side information systems $CY$. Because there is an isometry $V : AC \rightarrow ACY$ which acts as

$$V |\psi_x \rangle \rangle \langle \psi_x | \langle x | \rangle \|_{\sigma x} \| Y \|_{\xi(x)}^Y = |\psi_x \rangle \rangle \langle \psi_x | \langle x | \rangle \|_{\sigma x} \| Y \|_{\xi(x)}^Y = |\psi_x \rangle \rangle \langle \psi_x | \langle x | \rangle \|_{\sigma x} \| Y \|_{\xi(x)}^Y,$$

the extended source $\omega^{XACY}$ is equivalent to the original source and side information $\omega^{XAC}$ modulo a local operation of Alice.

We first present the optimal asymptotic compression rate in the following theorem and prove the achievability of it, but we leave the converse proof to the end of this section, as it requires introducing further machinery.

Theorem 3: For the given source $\omega^{XACY}$, the optimal asymptotic compression rate assisted by unlimited entanglement is $Q = \frac{1}{2} (S(A) + S(A|CY))$. \[ \text{Proof.} \]

We first show that this rate is achievable. Consider the following purification of $\omega^{XAC}$,

$$|\omega^{XAC} = \sum \sqrt{p(x)} \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y,$$

with side information systems $CY$. This is obtained from $|\omega^{XAC} = \sum \sqrt{p(x)} \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y$ by Alice applying the isometry $V$ from Eq. (3).

We apply quantum state redistribution (QSR) [9], [10] as a subprotocol, where the objective is for Alice to send to Bob $A^n$, using $C^n Y^n$ as side information, while $(XX')^n$ serves as reference system; the figure of merit is the fidelity with the original pure state $|\omega^{XAC} = \sum \sqrt{p(x)} \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y$, QRS gives us the first inequality of the following chain:

$$1 - o(1) \leq F \left( |\omega^{XAC} = \sum \sqrt{p(x)} \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y \right) \leq F \left( |\omega^{XAC} = \sum \sqrt{p(x)} \| x \|_X \| A \|_{\sigma x} \| Y \|_{\xi(x)}^Y \right),$$

where the second inequality follows from monotonicity of the fidelity under partial trace. Thus, the protocol satisfies our fidelity criterion (2).

The communication rate we obtain from QSR is $Q = \frac{1}{2} I(A : X'Y') = \frac{1}{2} (S(A) + S(A|CY))$. Furthermore, QSR guarantees entanglement consumption at the rate $E = \frac{1}{2} I(A : CY) = \frac{1}{2} (S(A) - S(A|CY)).$
Definition 4: For a source $\omega^{XAC}$ and $\epsilon \geq 0$, define

$$I_\epsilon(\omega) := \max_{V: AC \rightarrow \hat{AC}W \text{ isometry}} I(X: \hat{CW})$$

where $\xi^{X\hat{AC}W} = (I_X \otimes V)\omega^{XAC}(I_X \otimes V^\dagger) = \sum_x p(x)|x\rangle\langle x|\xi_x^{\hat{AC}W}$.

In this definition, the dimension of the environment is w.l.o.g. bounded as $|W| \leq |A|^2|C|^2$, hence, the optimisation is of a continuous function over a compact domain, so we have a maximum rather than a supremum.

Lemma 5: The function $I_\epsilon(\omega)$ has the following properties:

1. It is a non-decreasing function of $\epsilon$.
2. It is concave in $\epsilon$.
3. It is continuous for $\epsilon \geq 0$.
4. For any two states $\omega_1^{X_1A_1C_1}$ and $\omega_2^{X_2A_2C_2}$ and for $\epsilon \geq 0$,
   $$I_\epsilon(\omega_1 \otimes \omega_2) \leq I_\epsilon(\omega_1) + I_\epsilon(\omega_2).$$
5. For any state $\omega^{XAC}$, $I_0(\omega) \leq S(CY)$.

Proof. 1. The definition of $I_\epsilon(\omega)$ directly implies that it is a non-decreasing function of $\epsilon$.

2. To prove the concavity, let $V_1 : AC \rightarrow \hat{AC}W$ and $V_2 : AC \rightarrow \hat{AC}W$ be the isometries attaining the maximum for $\epsilon_1$ and $\epsilon_2$, respectively, which act as follows:
   $$V_1|\psi_1\rangle^A|\sigma_2\rangle^C = |\xi_1\rangle^{\hat{AC}W} \text{ and } V_2|\psi_2\rangle^A|\sigma_2\rangle^C = |\xi_2\rangle^{\hat{AC}W}.$$

For $0 \leq \lambda \leq 1$, define the isometry $U : AC \rightarrow \hat{AC}WR^\dagger$ by letting, for all $x$,

$$U|\psi_\lambda\rangle^A|\sigma\rangle^C = \sqrt{\lambda}|\xi_1\rangle^{\hat{AC}W}|00\rangle^{RR^\dagger} + \sqrt{1-\lambda}|\xi_2\rangle^{\hat{AC}W}|11\rangle^{RR^\dagger},$$

where systems $R$ and $R^\dagger$ are qubits. Then, the reduced state on the systems $X\hat{A}C$ is

$$\tau_{X\hat{AC}}^{\hat{A}C} = \sum_x p(x)|x\rangle\langle x|^{X\hat{AC}} = \lambda_1^{\hat{AC}} + (1-\lambda)\xi_x^C,$$

where the fidelity is bounded as follows:

$$F(\omega^{XAC}, \tau^{X\hat{AC}}) = \sum_x p(x) \sqrt{\langle \psi_x|\lambda_x^{\hat{AC}} + (1-\lambda)|\xi_x^C\rangle^2} \geq \lambda \sum_x p(x) \sqrt{\langle \psi_x|\lambda_x^{\hat{AC}} + (1-\lambda)|\xi_x^C\rangle^2} + (1-\lambda) \sum_x p(x) \sqrt{\langle \psi_x|\lambda_x^{\hat{AC}} + (1-\lambda)|\xi_x^C\rangle^2} \geq 1 - (\lambda \epsilon_1 + (1-\lambda)\epsilon_2),$$

where the second line follows from the concavity of the function $\sqrt{\lambda}$, and the last line follows by the definition of the isometries $V_1$ and $V_2$. Now, define $W^\dagger := WR^\dagger$ and let $\epsilon = \lambda \epsilon_1 + (1-\lambda)\epsilon_2$. According to Definition 4, we obtain

$$I_\epsilon(\omega) \geq I(X : \hat{CW}^\dagger) = I(X : \hat{C}W) = I(Y : \hat{C}W^\dagger) \geq S(Y) + I(X : \hat{C}W) \geq S(Y) + I(X : \hat{C}W) + S(C) \geq S(Y) + S(C),$$

where the first line follows because $Y$ is a function of $X$. The second and fourth line are due to the chain rule. The third line follows because for the classical system $Y$ the conditional entropy $S(Y|C)$ is non-negative. The penultimate line follows because for any $x$ the state on the system $C$ is pure. The last line is due to strong sub-additivity of the entropy. Furthermore, for every $y$, the ensemble $E_y$ is irreducible; hence, the conditional mutual information $I(X : W|Y) = 0$ which follows from the detailed discussion on page 2028 of [11].
Proof of the converse part of Theorem 3. We start by observing
\[ nQ + S(B_0) \geq S(C_A) + S(B_0) \geq S(C_A B_0) = S(\hat{A}^n W_B), \]
where the first inequality is due to subadditivity of the entropy, and the equality follows because the decoding isometry \( V_B \) does not change the entropy. Hence, we get
\[ nQ + S(B_0) \geq S(\hat{A}^n) + S(W_B | \hat{A}^n) \]
\[ \geq S(A^n) + S(W_B | \hat{A}^n X^n) \]
\[ \geq S(A^n) + S(W_B | \hat{A}^n X^n) - n \delta(n, \epsilon) \]
\[ = S(A^n) + S(\hat{C}^n W_A | X^n) - S(A^n | X^n) - n \delta(n, \epsilon) \]
\[ \geq S(A^n) + S(\hat{C}^n W_A | X^n) - 3n \delta(n, \epsilon), \]
where in the first and second line we use the chain rule and subadditivity of entropy. The inequality in the third line follows from the decodability of the system \( A^n \): the fidelity criterion (2) implies that the output state on systems \( \hat{A}^n \) is \( 2\sqrt{n} \epsilon \)-close to the original state \( A^n \) in trace norm; then apply the Fannes-Audenaert inequality [13, 14] where \( \delta(n, \epsilon) = \sqrt{2n} \log |A| + \frac{1}{n} h(\sqrt{2n}) \). The equalities in the fourth and the fifth line are due to the chain rule and the fact that for any \( x^n \) the overall state of \( A^n \hat{C}^n W_A W_B \) is pure. In the last line, we use the decodability of the systems \( X^n A^n \), that is the output state on systems \( X^n \hat{A}^n \) is \( 2\sqrt{n} \epsilon \)-close to the original states \( X^n A^n \) in trace norm, then we apply the Alicki-Fannes inequality [15, 16].

Moreover, we bound \( Q \) as follows:
\[ nQ \geq S(C_A) \geq S(C_A \hat{C}^n W_A) \]
\[ = S(A^n C^n A_0) - S(C^n W_A) \]
\[ = S(A^n C^n Y^n) + S(A_0) - S(\hat{C}^n W_A), \]
where the first equality follows because the encoding isometry \( V_A : A^n C^n A_0 \to C_A \hat{C}^n W_A \) does not change the entropy. Adding Eqs. (4) and (5), we thus obtain
\[ Q \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{1}{2n} I(\hat{C}^n W_A : X^n) - \frac{3}{2} \delta(n, \epsilon) \]
\[ \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{1}{2n} I(\hat{C}^n W_A W_B : X^n) - \frac{3}{2} \delta(n, \epsilon) \]
\[ \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{1}{2n} I(\epsilon \omega^n) - \frac{3}{2} \delta(n, \epsilon) \]
\[ \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{1}{2} I(\epsilon) - \frac{3}{2} \delta(n, \epsilon) \]
where the second line is due to data processing. The third line follows from Definition 4. The last line follows from point 4 of Lemma 5. In the limit of \( \epsilon \to 0 \) and \( n \to \infty \), the rate is bounded by
\[ Q \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{1}{2} I_0(\epsilon) \]
\[ \geq \frac{1}{2} (S(A) + S(ACY)) - \frac{\epsilon}{2} S(CY) \]
\[ = \frac{1}{2} (S(A) + S(A|CY)), \]
where the first line follows from point 3 of Lemma 5 stating that \( I_\epsilon(\omega) \) is continuous at \( \epsilon = 0 \). The second line is due to point 5 of Lemma 5.

IV. COMPLETE RATE REGION

In this section, we find the complete rate region of achievable rate pairs \((E, Q)\).

Theorem 6: For the source \( \omega^{XACY} \), all asymptotically achievable entanglement and quantum rate pairs \((E, Q)\) satisfy
\[ Q \geq \frac{1}{2} (S(A) + S(A|CY)), \]
\[ Q + E \geq S(A). \]
Conversely, all the rate pairs satisfying the above inequalities are achievable.

Proof. The first inequality comes from Theorem 3. For the second inequality, consider any code with quantum communication rate \( R \) and entanglement rate \( E \). By using an additional communication rate \( E \), Alice and Bob can distribute the entanglement first, and then apply the given code, converting it into one without preshared entanglement and communication rate \( Q + E \), having exactly the same fidelity. By Remark 1, \( Q + E \geq S(A) \).

As for the achievability, the corner point \((\frac{1}{2} I(A : CY), \frac{1}{2} (S(A) + S(A|CY)))\) is achievable, because QSR which is used as the achievability protocol in Theorem 3 uses \( \frac{1}{2} I(A : CY) \) ebits of entanglement between Alice and Bob. Furthermore, all the points on the line \( Q + E = S(A) \) for \( Q \geq \frac{1}{2} (S(A) + S(A|CY)) \) are achievable because one ebit can be distributed by sending a qubit. All other rate pairs are achievable by resource wasting. The rate region is depicted in Fig. 1.

![Figure 1. The optimal rate region of quantum and entanglement rates.](image)

V. DISCUSSION

First of all, let us look what our result tell us in the cases of blind and visible compression.

Corollary 7: In blind compression (i.e. if \( C \) is trivial, or more generally the states \( \sigma_x \) are all identical), the compression
of the source \( \omega^{XACY} \) reduces to the entanglement-assisted Schumacher compression for which Theorem 3 gives the optimal asymptotic quantum rate

\[
Q \geq \frac{1}{2} (S(A) + S(A|Y)) - S(A) - \frac{1}{2} S(Y).
\]

This implies that if the source is irreducible, then this rate is equal to the Schumacher limit \( S(A) \). In other words, the entanglement does not help the compression. Moreover, due to Theorem 6, a rate \( \frac{1}{2} S(Y) \) of entanglement is consumed in the compression, and \( E + Q \geq S(A) \) in general.

The blind compression of a source \( \omega^{XAY} \) is also considered in [11], but there instead of entanglement, a noiseless classical channel was assumed in addition to the quantum channel. It was shown that the optimal quantum rate assisted with free classical communication is equal to \( S(A) - S(Y) \), while a rate \( S(Y) \) of classical communication suffices. By sending the classical information using dense coding [18], spending \( \frac{1}{2} \) ebit and \( \frac{1}{2} \) qubit per cbit, we can recover the quantum and entanglement rates of Corollary 7. This means that our converse implies the optimality of the quantum rate from [11].

Thus we are motivated to look at a modified compression model where the resources used are classical communication and entanglement. Namely, we let Alice and Bob share entanglement at rate \( E \) and use classical communication at rate \( C \), but otherwise the objective is the same as in Section II; define the rate region as the set of all asymptotic achievable classical communication and entanglement rate pairs \( (C, E) \), such that the decoding fidelity asymptotically converges to \( 1 \).

**Theorem 8:** For a source \( \omega^{XAY} \), a rate pair \( (C, E) \) is achievable if and only if

\[
C \geq 2S(A) - S(Y), \quad E \geq S(A) - S(Y).
\]

**Proof.** We start with the converse. The first inequality follows from Theorem 3, because with unlimited entanglement shared between Alice and Bob, \( \frac{1}{2} (S(A) + S(A|Y)) = S(A) - \frac{1}{2} S(Y) \) qubits of quantum communication is equivalent to \( 2S(A) - S(Y) \) bits of classical communication due to teleportation [17] and dense coding [18]. The second inequality follows from [11], because with free classical communication, the quantum rate is lower bounded by \( S(A) - S(Y) \) which, due to super dense coding [18], is equivalent to sharing \( S(A) - S(Y) \) ebits when classical communication is for free.

The achievability of the corner point \( (2S(A) - S(Y), S(A) - S(Y)) \) follows from [11] because the compression protocol uses \( S(A) - S(Y) \) qubits and \( S(Y) \) bits of classical communication which is equivalent to using \( S(A) - S(Y) \) ebits of entanglement and \( 2S(A) - 2S(Y) + S(Y) \) bits of classical communication, due to dense coding [18]. Other rate pairs are achievable by resource wasting. The rate region is depicted in Fig. 2.

**Corollary 9:** In the visible case, our compression problem reduces to the visible version of Schumacher compression with entanglement assistance. In this case, according to Theorem 3 the optimal asymptotic quantum rate is \( Q = \frac{1}{2} S(A) \). Moreover, a rate \( E = \frac{1}{2} S(A) \) of entanglement is consumed in the compression scheme, and \( E + Q \geq S(A) \) in general.

We remark that the visible compression assisted by unlimited entanglement is also a special case of remote state preparation considered in [19], from which we know that the rate \( Q = \frac{1}{2} S(A) \) is achievable and optimal.

The visible analogue of [11], of compression using qubit and cbit resources, was treated in [20], where the achievable region was determined as the union of all pairs \( (C, Q) \) such that \( Q \geq S(A|Z) \) and \( C \geq I(X : Z) \), for any random variable \( Z \) forming a Markov chain \( Z \rightarrow X \rightarrow A \). Compare to the complicated boundary of this region the much simpler one of Corollary 9, which consists of two straight lines.

We close by discussing several open questions for future work: First, the final discussion of different pairs of resources to compress suggests that an interesting target would be the characterisation of the full triple resource tradeoff region for \( Q, C \) and \( E \) together.

Secondly, we recall that our definition of successful decoding included preservation of the side information \( \sigma^E_Z \) with high fidelity. What is the optimal compression rate \( \hat{Q} \) if the side information does not have to be preserved? For an example where this change has a dramatic effect on the optimal compression rate, consider the ensemble \( E \) consisting of the three two-qubit states \( |0\rangle^A |0\rangle^C, |1\rangle^A |0\rangle^C \) and \( |\rangle^A |\rangle^C \) (where \( |\rangle^E = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \)), with probabilities \( \frac{1}{2} - t, \frac{1}{2} - t \) and \( 2t \), respectively. Note that \( E \) is irreducible, hence \( t \approx 0 \), we get an optimal quantum rate of \( Q \approx 1 \), because \( S(A) \approx S(A|C) \approx 1 \).

However, by applying a CNOT unitary (with \( A \) as control and \( C \) as target), the ensemble is transformed into \( E' \) consisting of the states \( |0\rangle^A |0\rangle^C, |1\rangle^A |1\rangle^C \) and \( |\rangle^A |\rangle^C \). The state of \( A \) is not changed, only the side information, which is why we denote it \( C' \). Hence we can apply Theorem 3 to get a quantum rate \( Q \approx \frac{1}{2} \), because \( S(A) \approx 1, S(A|C) \approx 0 \).

Thirdly, note that the lower bound \( Q + E \geq S(A) \) in Theorem 6 holds with a strong converse (see the proof and [6]). But
does $Q \geq \frac{1}{2}(S(A) + S(A|CY))$ hold as a strong converse rate with unlimited entanglement? Likewise, in the setting of [11] with unlimited classical communication, is $Q \geq S(A) - S(Y)$ a strong converse bound for the quantum rate?

REFERENCES

[1] B. Schumacher, “Quantum coding,” Phys. Rev. A, vol. 51, no. 4, pp. 2738–2747, Apr 1995.
[2] R. Jozsa and B. Schumacher, “A new proof of the quantum noiseless coding theorem,” J. Mod. Optics, vol. 41, no. 12, pp. 2343–2349, 1994.
[3] M. Ohya and D. Petz, Quantum Entropy and Its Use. Springer Verlag, Berlin Heidelberg, 1993 (2nd edition 2004).
[4] H. Barnum, C. A. Fuchs, R. Jozsa, and B. Schumacher, “General fidelity limit for quantum channels,” Phys. Rev. A, vol. 54, no. 6, pp. 4767–4771, Dec 1996.
[5] M. Horodecki, “Limits for compression of quantum information carried by ensembles of mixed states,” Phys. Rev. A, vol. 57, no. 5, pp. 3364–3369, May 1998.
[6] A. Winter, “Coding Theorems of Quantum Information Theory,” Ph.D. dissertation, Universität Bielefeld, Department of Mathematics, Germany, July 1999, arXiv:quant-ph/9907077.
[7] W. F. Stinespring, “Positive Functions on $C^*$-Algebras,” Proc. Amer. Math. Society, vol. 6, no. 2, pp. 211–216, 1955.
[8] M. Horodecki, “Optimal compression for mixed signal states,” Phys. Rev. A, vol. 61, 052309, Apr 2000.
[9] I. Devetak and J. T. Yard, “Exact Cost of Redistributing Multipartite Quantum States,” Phys. Rev. Lett., vol. 100, 230501, Jun 2008.
[10] J. Oppenheim, “State redistribution as merging: introducing the coherent relay,” May 2008, arXiv[quant-ph]:0805.1065.
[11] H. Barnum, P. Hayden, R. Jozsa, and A. Winter, “On the reversible extraction of classical information from a quantum source,” Proc. Royal Soc. London Ser. A, vol. 457, no. 2012, pp. 2019–2039, Aug. 2001.
[12] R. T. Rockafeller, Convex Analysis. Princeton University Press, 1970.
[13] M. Fannes, “A continuity property of the entropy density for spin lattice systems,” Commun. Math. Phys., vol. 31, no. 4, pp. 291–294, Dec 1973.
[14] K. M. R. Audenaert, “A sharp continuity estimate for the von Neumann entropy,” J. Phys. A: Math. Theor., vol. 40, no. 28, pp. 8127–8136, 2007.
[15] R. Alicki and M. Fannes, “Continuity of quantum conditional information,” J. Phys. A: Math. Gen., vol. 37, no. 5, pp. L55–L57, 2004.
[16] A. Winter, “Tight Uniform Continuity Bounds for Quantum Entropies: Conditional Entropy, Relative Entropy Distance and Energy Constraints,” Commun. Math. Phys., vol. 347, no. 1, pp. 291–313, Oct 2016.
[17] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels,” Phys. Rev. Lett., vol. 70, no. 13, pp. 1895–1899, Mar 1993.
[18] C. H. Bennett and S. J. Wiesner, “Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states,” Phys. Rev. Lett., vol. 69, no. 20, pp. 2881–2884, Nov 1992.
[20] P. Hayden, R. Jozsa, and A. Winter, “Trading quantum for classical resources in quantum data compression,” J. Math. Phys., vol. 43, no. 9, pp. 4404–4444, Sept 2002.