On the Asymptotic Performance of Matching Mechanisms with Affirmative Actions

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Abstract

Affirmative action policies, albeit controversial, attempt giving disadvantaged social groups preferential treatments to close the racial, ethnic, or socioeconomic gaps among different groups in our societies. This paper studies the asymptotic performance of two conventional matching mechanisms, the top trading cycles mechanism (TTCM) and the immediate acceptance mechanism (IAM), in the context of school choice markets with affirmative actions. We show that there exists no clear welfare dominance relationship between the quota-based affirmative action policy and its reserve-based counterpart for minority students under the TTCM, in the sense that these two affirmative actions induce different matching outcomes with non-negligible probability even in a sequence of random markets under relatively restricted regularity conditions. Given the possible preference manipulations under the IAM, we further characterize the asymptotically equivalent sets of Nash equilibrium outcomes of the IAM with these two affirmative actions when the market becomes sufficiently large. As the transition from one affirmative action policy to the other could possibly evoke substantial socioeconomic costs to local communities, we conclude that the IAM is more cost-effective compared to the TTCM in large school choice markets with affirmative actions, in the sense that it is unnecessary to identify the different welfare effects of these two affirmative actions under the IAM if the policymaker can assure a sufficient supply of popular schools.

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1 Introduction

Game theoretic approach has been widely applied in various marketplaces to evaluate, or to improve, the effectiveness of resource allocation mechanisms in these marketplaces. Auctions have been employed to allocate radio spectrum, electricity, natural gas and treasury bills in centralized markets which involve hundreds of billions of dollars worldwide (Milgrom, 2004, 2021). Compared to open market auctions, matching mechanisms are more

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prevalent in the marketplaces where price is incapable to signal agents’ preferences (Roth and Sotomayor, 1990; Roth, 2018). For instance, in public school choice and labor markets, price (i.e., tuition or wages) cannot decide who goes to which school or works at which company, candidates also need to be matched with (i.e., admitted or hired by) the particular colleges or employers they applied to.

Affirmative action policies, albeit controversial, attempt giving disadvantaged social groups preferential treatments to improve their socioeconomic status and representation in the societies. In the context of public school choice, many school districts in the United States and around the world often impose affirmative action policies to maintain the racial, ethnic and socioeconomic diversity at schools. The quota-based affirmative action (*majority quota*, henceforth) and the reserve-based affirmative action (*minority reserve*, henceforth) are two common policy designs in practice. Abdulkadiroğlu and Sönmez (2003) formalize the majority quota policy in school choice, which sets a maximum number less than the school’s capacity to students from socioeconomically advantaged groups (i.e., *majority students*) and leaves the difference to the policy-targeted student groups (i.e., *minority students* from racial and ethnic minority groups, or low-income families). The minority reserve policy proposed by Hafalir et al. (2013), on the other hand, gives higher priority to minority students up to the point that all reserved seats have been assigned to minorities.

The *top trading cycles mechanism* (TTCM, henceforth) and the *immediate acceptance mechanism* (IAM, henceforth; also known as the Boston mechanism) are two celebrated matching mechanisms for school choice, which are currently being used in many school districts in the U.S. and in other countries (Musset, 2012; Calsamiglia and Güell, 2018; Artemov, 2021). There have been numerous efforts to reconcile affirmative action policies with these two mechanisms in *finite* school choice markets (i.e., with a small number of students and schools as participants; see our discussions of related literature in Section 2). However, many real world matching markets do contain a large number of participants, for instance: the school choice system in Chile needs to match more than 270,000 students with 6,400 schools each year (Correa et al., 2022); India has been implementing a large-scale and comprehensive affirmative action program since 1950 in both publicly funded educational institutions and government sponsored jobs (Aygün and Turhan, 2020). This paper extends the performance comparison of the majority quota policy and its minority reserve counterpart under the TTCM and the IAM in *large* school choice markets (i.e., a sequence of random markets of different sizes). Our Theorem 1 shows that these two competing affirmative actions produce different matching outcomes under the TTCM with non-negligible probability, even if the number of reserved seats grows at a slower rate of $O(n^a)$ in a sequence of random markets, where $0 \leq a < 1/2$ and $n$ is the number of schools in a random market (see Condition (4) of Definition 1). As the purpose of imposing affirmative actions in school choice is to improve the matching outcomes (i.e., welfare) of minority students, the outcome non-equivalence between these two affirmative actions under the TTCM essentially results in an ambiguous Pareto dominance relationship for minorities in large matching markets (Corollary 1).

We then study the asymptotic performance of the majority quota and its minority reserve counterpart under the IAM. Our Theorem 2 implies that although the IAM is open to preference manipulations, these two affirmative actions are most likely to produce the same set of Nash equilibrium outcomes under the IAM when the market becomes sufficiently large. As transiting from one affirmative action policy to the other could possibly evoke substantial political, administrative and cognitive costs to local communities, an immediate policy implication of our results is that the IAM is more cost-effective compared to the TTCM in large
school choice markets with affirmative actions, in the sense that it is unnecessary to identify
the different welfare effects of these two affirmative actions under the IAM if the policymaker
can assure a sufficient supply of popular schools to the matching markets.

The rest of the paper proceeds as follows. Section 2 reviews the related literature. Section
3 sets up the large school choice model with affirmative actions. Section 4 introduces the
TTCM with affirmative actions algorithm and discusses its performance in large markets. Section
5 analyzes the corresponding IAM with affirmative actions in large markets. Section
6 concludes the paper.

2 Literature Review

The theory and design of matching markets was pioneered by Gale and Shapley (1962)
and Shapley and Scarf (1974), who introduce the deferred acceptance algorithm and the
TTCM in the one-to-one matching markets such as marriage and house allocations. Abdulkadiroğlu and Sönmez
(2003) first formalize the student optimal stable mechanism (SOSM, henceforth) based on
Gale and Shapley (1962)’s deferred acceptance algorithm and the TTCM in the public school
choice markets as a many-to-one matching problem, and compare its performance with the
IAM used in many school districts in the U.S. (e.g., Boston, Columbus, Minneapolis, and
Seattle); they also introduce the majority quota as a type-specific constraint to incorpo-
rate diversity concerns in the school choice problems. Kojima (2012) presents examples
to show that the minority students who purported to be the beneficiaries might instead
be made worse off under the TTCM with majority quota policies. Echenique and Yenmez
(2015) axiomatically characterize a class of substitute priority rules that allow schools to ex-
press preferences for diversity. Kominers and Sönmez (2016) treat affirmative action policies
from the perspective of designing priorities as a choice function for the schools. Dur et al.
(2018) and Çelebi and Flynn (2022) analyze and redesign assignment systems with prece-
dence orders and distance-based priorities in the context of Chicago and Boston public
school choice programs. Aygün and Turhan (2020) propose a cumulative offer mechanism
in which each school uses a dynamic reserves choice function, and compare its performance
with the existing engineering school admission programs in India. Other papers study real-
world implementations of affirmative action policies include the German university admis-
sions system where students are allowed to submit indifferent (instead of strictly ordered)
preferences (Westkamp, 2013), the Japan residency matching program with regional caps
(Kamada and Kojima, 2015), and the Brazilian public federal universities with multi dimen-
sional reserves (Aygün and Bö, 2021), among others.

Analyzing the asymptotic performance of various matching mechanisms in large mar-
kets scenarios has been receiving increasing attention in the literature. Our large market
setting is mostly close to Liu (2022), who analyzes the asymptotic performance of the SOSM
with these two competing affirmative actions. While this approach relies on a number of
regularity conditions, it has been used in several recent analyses on the asymptotic proper-
ties of matching mechanisms in different contexts. Among others, Immorlica and Mahdian
(2005) generalize the algorithm of Knuth (1976); Knuth et al. (1990) to the case of incom-
plete preference orders (i.e., a bounded preference order with limited length), and prove
that strategy-proofness is an approximate Nash equilibrium of stable matching mechanisms
in the two-sided one-to-one markets. Under an additional assumption of sufficient thick-
ness (which is similar to our Condition (5) of Definition 1), Kojima and Pathak (2009) extend
Immorlica and Mahdian (2005)’s results to the SOSM when schools are allowed to strate-
gically submit their priorities. Kojima et al. (2013) and Ashlagi et al. (2014) prove the existence of asymptotically stable matching mechanism in the U.S. National Resident Matching Program with two types of doctors (single and married). Azevedo and Leshno (2016) and Che et al. (2019) alternatively treat large matching markets as a continuum economy and establish the existence of a stable matching with diversity considerations; other large matching literature using the continuum economy modelling approach include Che and Kojima (2010); Kojima and Manea (2010); Lee (2016), among others.

As detailed in the preceding literature, the large market approach can often overcome impossibility results in finite matching markets by establishing the corresponding approximate properties in various matching markets with a large number of participants. To our knowledge, only a few studies have demonstrated that the large market approach does not eliminate all the distinct properties of different matching mechanisms: Kojima and Pathak (2009) present an example to illustrate that students still have incentives to manipulate the IAM in large school choice markets; Hatfield et al. (2016) show that neither the TTCM nor the IAM approximately respect improvements of school quality (i.e., a school matches with a set of more desirable students if it becomes more preferred by students); Che and Tercieux (2019) suggest that the inefficiency of the SOSM and instability of the TTCM remain significant in large markets when students have correlated preferences. Given the fading of many impossibility results in large markets, some researchers have criticized using approximation properties in matching and market design problems, in the sense that the asymptotic analysis of large matching markets may be too “permissive” to make market design irrelevant (Kojima, 2015). The current paper thus also supports the validity of the large market analytic approach, as it still enables us to capture the subtle difference between the matching mechanisms that can asymptotically satisfy some desirable properties from those that cannot.

3 Model

3.1 School Choice with Affirmative Actions

Let $S$ and $C$ be two finite sets of students and schools, $|S| \geq 2$. There are two types of students, majority and minority. $S$ is partitioned into two subsets of students based on their types. Denote $S^M$ the set of majority students, and $S^m$ the set of minority students, $S = S^M \cup S^m$ and $S^M \cap S^m = \emptyset$. Each student $s \in S$ has a strict preference order $P_s$ over the set of schools and being unmatched (denoted by $s$). All students prefer to be matched with some school instead of herself, $c P_s s$, for all $s \in S$. Each school $c \in C$ has a total capacity of $q_c$ seats, $q_c \geq 1$, and a strict priority order $> \in C$ over the set of students which is complete, transitive, and antisymmetric. Student $s$ is unacceptable by a school if $e >_c s$, where $e$ represents an empty seat in school $c$.

For each school $c$ with majority quota affirmative action policy, it cannot admit more majority students than its type-specific majority quota $q^M_c \leq q_c$, for all $c \in C$. Accordingly, the minority reserve policy gives priority to the minority applicants of school $c$ up to its minority reserve $r^M_c \leq q_c$, $\forall c \in C$, and allows $c$ to accept majority students up to its capacity $q_c$ if there are not enough minority applicants to fill the reserves.

A school choice market with affirmative actions is a tuple $\Gamma = (S, C, P, (q^M, r^M))$, where $P = (P_s)_{s \in S} > = (>)_{c \in C}$. When comparing the effects of a majority quota policy with its minority reserve counterpart in a market $\Gamma$, we assume $\Gamma$ is either with only majority quota or with only minority reserve, such that $r^M_c + q^M_c = q_c$, $\forall c \in C$, with $q^M =$ $(q^M_c)_{c \in C}$, $r^M =$ $(r^M_c)_{c \in C}$, and $q =$ $(q_c)_{c \in C}$. 


A matching $\mu$ is a mapping from $S \cup C$ to the subsets of $S \cup C$ in market $\Gamma$ such that, for all $s \in S$ and $c \in C$:

- $\mu(s) \in C \cup \{s\}$;
- $\mu(s) = c$ if and only if $s \in \mu(c)$;
- $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$;
- $|\mu(c) \cap S^M| \leq q^M_c$.

That is, a matching specifies the school where each student is assigned to or matched with herself, and the set of students assigned to each school; no school admits more students than its capacity, and no school admits more majority students than its majority quota.

A matching $\mu$ is blocked by a pair of student $s$ and school $c$ with majority quota, if $cP_s\mu(s)$ and either $|\mu(c)| < q_c$ and $s$ is acceptable to $c$, or:

- $s \in S^m$, $s >_c s'$, for some $s' \in \mu(c)$;
- $s \in S^M$ and $|\mu(c) \cap S^M| < q^M_c$, $s >_c s'$, for some $s' \in \mu(c)$;
- $s \in S^M$ and $|\mu(c) \cap S^M| = q^M_c$, $s >_c s'$, for some $s' \in \mu(c) \cap S^M$.

A matching $\mu$ is Q-stable, if $\mu(s) P_s\mu(s)$ for all $s \in S$, and has no blocking pair in $\Gamma$ with majority quota.

Accordingly, a matching $\mu$ is blocked by a pair of student $s$ and school $c$ with minority reserve, if $cP_s\mu(s)$ and either $|\mu(c)| < q_c$ and $s$ is acceptable to $c$, or:

- $s \in S^m$, $s >_c s'$, for some $s' \in \mu(c)$;
- $s \in S^M$ and $|\mu(c) \cap S^M| > r^m_c$, $s >_c s'$, for some $s' \in \mu(c)$;
- $s \in S^M$ and $|\mu(c) \cap S^M| \leq r^m_c$, $s >_c s'$, for some $s' \in \mu(c) \cap S^M$.

A matching $\mu$ is R-stable, if $\mu(s) P_s\mu(s)$ for all $s \in S$, and has no blocking pair in $\Gamma$ with minority reserve.

As the purpose of imposing affirmative actions in school choice markets is to improve the matching outcomes (i.e., welfare) of minority students, we need some type-specific criteria to evaluate the welfare effects of affirmative actions on minority students. Given two matchings $\mu$ and $\mu'$, $\mu$ Pareto dominates $\mu'$ for minorities if (i) $\mu(s) P_s \mu'(s)$ for at least one $s \in S^m$, and (ii) $\mu(s) R_s \mu'(s)$ for all $s \in S^m$, where $R_s$ represents two matched outcomes that are equally good for $s$.

A matching mechanism $f$ is a function that produces a matching $f(\Gamma)$ for each market $\Gamma$. A mechanism $f$ is strategy-proof if for each student $s \in S$ and for any $P$, there exists no $P'$ such that $\mu(P', P_{-s}) P \mu(P)$, where $P_{-s} = (P_i)_{i \in S \setminus s}$, that is, if a mechanism is strategy-proof, each student finds it optimal to report her preferences truthfully regardless of the preferences of other students. Finally, given two mechanisms $f$ and $f'$, we say $f$ Pareto dominates $f'$ for minorities if for all $\Gamma$, either $f'(\Gamma) = f(\Gamma)$ for all minorities or $f'(\Gamma)$ Pareto dominates $f(\Gamma)$ for minorities.
3.2 Large markets

A random market is a tuple $\tilde{\Gamma} = ((S^M, S^m), C, \succ, (q^M, r^m), k, (\mathcal{A}, \mathcal{B}))$, where $k$ is a positive integer, $\mathcal{A} = (\alpha_c)_{c \in C}$ and $\mathcal{B} = (\beta_c)_{c \in C}$ are the respective probability distributions on $C$, with $\alpha_c, \beta_c > 0$ for each $c \in C$. We assume that $\mathcal{A}$ for majorities to be different from $\mathcal{B}$ for minorities to reflect their distinct favors for schools.

A sequence of random markets is denoted by $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)$, where $\tilde{\Gamma}^n = ((S^{M,n}, S^{m,n}), C^n, \succ^n, (q^{M,n}, r^{m,n}), k^n, (\mathcal{A}^n, \mathcal{B}^n))$ is a random market of size $n$, with $|C^n| = n$ as the number of schools, $|r^{m,n}|$ the number of seats reserved for minorities, and $|S^n| = |S^{M,n}| + |S^{m,n}|$ as the number of students in market $\tilde{\Gamma}^n$.

Each random market induces a market by randomly generated preference orders of each student $s$ according to the following procedure introduced by Immorlica and Mahdian (2005):

**Step 1:** Select a school independently from the distribution $\mathcal{A}$ (resp. $\mathcal{B}$). List this school as the top ranked school of a majority student $s \in S^M$ (resp. minority student $s \in S^m$).

\[
\vdots
\]

**Step $l \leq k$:** Select a school independently from $\mathcal{A}$ (resp. $\mathcal{B}$) which has not been drawn from steps 1 to step $l - 1$. List this school as the $l^{th}$ most preferred school of a majority student $s \in S^M$ (resp. minority student $s \in S^m$).

Each major (resp. minority) student finds these $k$ schools acceptable, and only lists these $k$ schools in her preference order. Let $\bar{P}_s^n$ be the (truthful) preference order of student $s$ generated according to the preceding procedure, and $\bar{P}^n = (\bar{P}_s^n)_{s \in S^n}$ be the profile of truthful preferences. We introduce the following regularity conditions to guarantee the convergence of the random markets sequence.

**Definition 1.** Consider majority quotas $q^{M,n}$ and minority reserves $r^{m,n}$ such that $r^{m,n} + q^{M,n} = q^n$. A sequence of random markets $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)$ is regular, if there exist $a \in [0, \frac{1}{2})$, $\lambda, \kappa, \theta > 0$, $r \geq 1$, and positive integers $k$ and $\tilde{q}$, such that for all $n$:

1. $k^n \leq k$;
2. $q^n_c \leq \tilde{q}$ for all $c \in C^n$;
3. $|S^n| \leq \lambda n$, $\sum_{c \in C} q_c - |S^n| \geq \kappa n$;
4. $|r^{m,n}| \leq \theta n^a$;
5. $\frac{a_c}{a_{c'}} \in [\frac{1}{r}, r]$, $\frac{\beta_c}{\beta_{c'}} \in [\frac{1}{r}, r]$, for all $c, c' \in C^n$;
6. $\alpha_c = 0$, for all $c \in C^n$ with $q^n_{c, M} = 0$.

Condition (1) and (2) assume that the length of students’ preferences and the capacity of each school are bounded across schools and markets. Condition (3) requires that the number of students does not grow much faster than the number of schools, while there is an excess supply of school capacities to accommodate all students. Note that we do not distinguish the growth rate between majority and minority students, as minority students are generically treated as the intended beneficial student groups from affirmative action policies rather than race or other single socioeconomic status; in other words, the number of minority students is not necessarily less than majorities. Condition (4) requires that the number
of seats reserved for minority students grows at a slower rate of $O(n^a)$, where $a \in [0, \frac{1}{2})$. Condition (5) requires that the popularity of different schools, as measured by the probability of being selected by students from $\mathcal{A}$ for majorities and $\mathcal{B}$ for minorities, does not vary too much. Condition (6) requires that a majority student will not select a school that can only accept minority students (i.e., with majority quota $q_{c}^{M,n} = 0$), as these two affirmative actions trivially induce disparate matching outcomes in any arbitrarily large markets when a majority student applies to a school with zero majority quota. Most of these regularity conditions are common in the literature of large matching markets; see, for example, Kojima et al. (2013); Hatfield et al. (2016); Liu (2022) for more detailed illustrations.

We formally define the asymptotic outcome equivalence condition of these two affirmative actions in a sequence of random markets of different sizes as follows.

**Definition 2.** For any random market $\tilde{\Gamma}$, let $\eta_c(\tilde{\Gamma}; f, f')$ be probability that school $c \in C^n$ matched with different sets of students which induces $f(\tilde{\Gamma}) \neq f'(\tilde{\Gamma})$. We say two matching mechanisms are outcome equivalent in large markets, if for any sequence of random markets $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots)$ that is regular, $\max_{c \in C^n} \eta_c(\tilde{\Gamma}_n; f, f') \to 0$, as $n \to \infty$; that is, for any $\varepsilon > 0$, there exists an integer $m$ such that for any random market $\tilde{\Gamma}_n$ in the sequence with $n > m$ and any $c \in C^n$, we have $\max_{c \in C^n} \eta_c(\tilde{\Gamma}_n; f, f') < \varepsilon$.

### 4 Asymptotic Outcome Non-equivalence under the TTCM

For each market $\Gamma = (S, C, P, >, (q^M, r^m))$, the top trading cycles mechanism (TTCM) with affirmative actions runs as follows:

**Step 1:** Start with a matching in which no student is matched. For each school $c$, set its capacity counter at $q_c$. If $c$ has a majority quota, set its quota counter at its majority quota $q_{c}^{M}$; if $c$ has a corresponding minority reserve, set its reserve counter at its minority reserve $r_{c}^{m}$. If the reserve counter of school $c$ is positive, then it points to its most preferred minority student; otherwise it points to its most preferred student. Each student $s$ points to her most preferred acceptable school that still has a seat for her, and otherwise points to herself; that is, an acceptable school $c$ whose capacity counter is strictly positive and, if $s \in S^M$, its quota counter is strictly positive. There exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity counter of each school in a cycle is reduced by one and, if: (i) the assigned student $s$ is a majority student and the school matched to $s$ has a majority quota, then reduces the quota counter of the matched school by one; (ii) the assigned student $s$ is a minority student and the school matched to $s$ has a minority reserve, then reduces the reserve counter of the matched school by one. If no student remains, terminate. Otherwise, proceed to the next step.

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**Step 1:** Start with the matching and counter profile reached at the end of Step $l - 1$. For each remaining school $c$, if its reserve counter is positive, then $c$ points to its most preferred minority student among all remaining minority students; otherwise it points to its most preferred student among all remaining students. Each remaining student $s$ points to her most preferred acceptable school that still has a seat for her, and otherwise points to herself; that is, an acceptable school $c$ whose capacity counter is strictly positive and,
if $s \in S^M$, its quota counter is strictly positive. There exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity counter of each school in a cycle is reduced by one and, if: (i) the assigned student $s$ is a majority student and the school matched to $s$ has a majority quota, then reduces the quota counter of the matched school by one; (ii) the assigned student $s$ is a minority student and the school matched to $s$ has a minority reserve, then reduces the reserve counter of the matched school by one. If no student remains, terminate. Otherwise, proceed to the next step.

The algorithm terminates in a finite number of steps since there is at least one student matched and removed in any step of the algorithm. For a market $\Gamma$, if $r^m_c = 0$, $\forall c \in C$, i.e., a market with only majority quota, then the above algorithm reduces to the top trading cycles mechanism with majority quota (TTCM-Q, henceforth) proposed by Abdulkadiroğlu and Sönmez (2003); accordingly, if $q^M_c = q_c$, $\forall c \in C$, i.e., a market with only minority reserve, then the above algorithm reduces to the top trading cycles mechanism with minority reserve (TTCM-R, henceforth) proposed by Hafalir et al. (2013).

We introduce the following example to illustrate how the TTCM-Q and its TTCM-R counterpart assign students to schools in a finite school choice market.

Example 1. Consider the following market $\Gamma$ with two schools $C = \{c_1, c_2\}$, and three students $S = \{s_1, s_2, s_3\}$ where $S^M = \{s_1, s_2\}$ and $S^m = \{s_3\}$. $q_{c_1} = 2$ and $q_{c_2} = 1$. Schools and students have the following priority and preference orders:

|          | $\succ c_1$ | $\succ c_2$ | $P_{s_1}$ | $P_{s_i, i=2,3}$ |
|----------|-------------|-------------|-----------|------------------|
| $s_2$    | $s_1$       | $c_1$       | $c_1$     | $c_2$            |
| $s_3$    | $s_2$       |             |           |                  |
| $s_1$    | $s_3$       |             |           |                  |

Suppose that $\Gamma$ has the following majority quota and its corresponding minority reserve: $(q^M_{c_1}, q^M_{c_2}) = (1, 1)$, or correspondingly, $(r^m_{c_1}, r^m_{c_2}) = (1, 0)$ (i.e., $c_1$ can accept only one majority student under the TTCM-Q, while reserves a seat for minority students under the corresponding TTCM-R; $c_2$ has no affirmative actions).

At step 1 of the TTCM-Q, we will have the following the top trading cycle:

$$c_1 \rightarrow s_2 \rightarrow c_2 \rightarrow s_1 \rightarrow c_1.$$

That is, school $c_1$ first points to its most preferred student $s_2$, as it still has a positive quota counter ($q^M_{c_1} = 1$) to accept the majority student $s_2$. $s_2$ then points to her most preferred acceptable school $c_2$. $c_2$ also points to its most preferred student $s_1$, and $s_1$ points to her most preferred acceptable school $c_1$. As in the top trading cycle each student is assigned to the school she is pointing to, we have $s_2$ is assigned to $c_2$ and $s_1$ is assigned to $c_1$. $c_2$ is also removed from the algorithm as its capacity counter turns to zero ($q_{c_2} = 1$).

At step 2 of the TTCM-Q, since $c_2$ is the only acceptable school for the remaining student $s_3$, she points to herself instead of the remaining school $c_1$ and is left unmatched.

The matching outcome of the TTCM-Q is

$$f^{TTCM-Q}(\Gamma) = \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix}.$$
At step 1 of the TTCM-R, we will instead have the following the top trading cycle:

\[ c_1 \rightarrow s_3 \rightarrow c_2 \rightarrow s_1 \rightarrow c_1. \]

The only difference comes from the positive reserve counter \( r_c^m = 1 \) of school \( c_1 \), which makes \( c_1 \) point to its highest ranked minority student \( s_3 \) first. Thus, \( s_3 \) is assigned to \( c_2 \) while \( s_1 \) is assigned to \( c_1 \). \( c_2 \) is removed from the algorithm as its capacity counter turns to zero.

At step 2 of the TTCM-R, since \( c_2 \) is the only acceptable school for the remaining student \( s_2 \), she points to herself instead of the remaining school \( c_1 \) and is left unmatched.

The matching outcome of the TTCM-R is

\[ f^{TTCM-R}(\Gamma) = \left( \begin{array}{cc} c_1 & c_2 \\ s_1 & s_3 \end{array} \right) \]

Clearly, these two affirmative actions induce different matching outcomes under the TTCM in finite markets (i.e., with a small number of schools and students). Also, since all students point to their most preferred schools at each step of the TTCM-Q and the corresponding TTCM-R, both of these two mechanisms are strategy-proof (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013), i.e., no student has incentives to deviate from reporting her truthful preference order.

We next argue that such distinct performance of these two affirmative actions under the TTCM will not vanish even in arbitrarily large markets with sufficiently many schools and a relatively slow growth of reserved seats.

**Theorem 1.** The TTCM-Q and its corresponding TTCM-R are not outcome equivalent in large markets.

**Proof.** Consider a sequence of random markets \((\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)\), where there are \( n \) schools and \( \lambda n \) students, \( \lambda \geq 1 \), in each random market \( \tilde{\Gamma}^n \). Assume that the preferences of all students are generated according to the preference generation procedure defined in Section 3.2, with uniform distribution over all schools and preference length \( k = 1 \). Also, assume that school priorities are drawn identically and independently from the uniform distribution over students such that all students are acceptable. For each random market \( \tilde{\Gamma}^n \), denote \( t_n \in (0, 1) \) the portion of minority students, while \( 1 - t_n \) the corresponding portion of majority students. Also, assume that \( q_c = 1 \) or \( 2 \) for every school \( c \) in \( \tilde{\Gamma}^n \), denote \( \delta_n \in (0, 1) \) the portion of schools with two seats, while \( 1 - \delta_n \) the corresponding portion of schools with one seat. The preceding assumptions guarantee that the regularity conditions of Definition 1 are satisfied.

Let \( p_n \) be the probability that the two affirmative actions produce different outcomes under the TTCM in market \( \tilde{\Gamma}^n \). We will construct examples to show that the probability that \( p_n \) is strictly bounded away from zero in a sequence of random markets of different sizes \((\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)\).

\( p_1 > 0 \) is trivially satisfied when \( \tilde{\Gamma}^1 \) contains one majority student \( s_1 \) and one minority student \( s_2 \), while the exact school \( c_1 \) has one seat, \( \delta_1 \in (0, 1) \), and \( s_1 > c_1 s_2 \). For \( 2 \leq n < 4 \), it is a positive probability event that apart from other participants in \( \tilde{\Gamma}^n \), there are two schools \( c_1 \) and \( c_2 \), and three students \( s_1, s_2 \in S^{M,n}, s_3 \in S^{m,n} \), with the following priority and preference orders (recall Example 1):

Assume \( q_{c_1} = 2 \) and \( q_{c_2} = 1 \), with the following majority quota and its corresponding minority reserve: \((q_{c_1}^{M}, q_{c_2}^{M}) = (1, 1)\) or correspondingly, \((r_{c_1}^{m}, r_{c_2}^{m}) = (1, 0)\). Thus, as illustrated in Example 1, the majority student \( s_2 \) is matched with \( c_2 \) when \( c_1 \) has a majority quota \( q_{c_1}^{M} = 1 \), while the minority student \( s_3 \) is matched with \( c_2 \) when \( c_1 \) has the corresponding minority reserve \( r_{c_1}^{m} = 1 \), which gives \( p_n > 0 \), for each \( n \geq 2 \).
For $n \geq 4$. Let $\lambda = 1$. Denote $c_1$ an arbitrary school with no affirmative actions, $q_{c_1} = 1$. Let Event 1 be the event that there are exactly two minority students, denoted by $s_1$ and $s_2$ respectively, ranks $c_1$ first, $s_1, s_2 \in S^{m,n}$. The probability of Event 1 is

$$\left(\frac{n t_n}{1}\right) \times \left(\frac{n t_n - 1}{1}\right) \times \frac{1}{n^2} \times \left(1 - \frac{1}{n}\right)^{n-2},$$

where $t_n \in (0, 1)$ for any arbitrarily large $n \geq 4$. We can derive its limit when $n$ approaches $\infty$ as

$$\lim_{n \to \infty} \frac{n t_n (n t_n - 1)}{n^2} \left(1 - \frac{1}{n}\right)^{n-2} = \lim_{n \to \infty} \left(t_n\right)^2 \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{-2}$$

$$= \left(t_n\right)^2 \times \frac{1}{e} \times 1 = \frac{(t_n)^2}{e}.$$

Thus, for any sufficiently large $n$, the probability of Event 1 is at least, say, $\frac{(t_n)^2}{2e} > 0$.

Given Event 1, consider Event 2 such that except school $c_1$, there is exactly one school (denoted by $c_2$), $q_{c_2} = 2$ with either a majority quota $q_{c_2}^M = 1$ or its corresponding minority reserve policy $r_{c_2}^m = 1$, lists $s_1$ over all the rest students in its priority order; also, there is exactly one school (denoted by $c_3$) lists $s_2$ first. The conditional probability of Event 2 is given by

$$\left(\frac{n \delta_n}{1}\right) \times \left(\frac{n - 1}{1}\right) \times \frac{1}{n^2} \times \left(1 - \frac{2}{n}\right)^{n-3},$$

where $\delta_n \in (0, 1)$ for any arbitrarily large $n \geq 4$. The limit of the above expression is

$$\lim_{n \to \infty} \frac{n \delta_n (n - 1)}{n^2} \left(1 - \frac{2}{n}\right)^{n-3} = \lim_{n \to \infty} \delta_n \times \left(1 - \frac{2}{n}\right)^n \times \left(1 - \frac{2}{n}\right)^{-3}$$

$$= \delta_n \times \frac{1}{e^2} \times 1 = \frac{\delta_n}{e^2},$$

as $n$ approaches $\infty$. Thus, for any sufficiently large $n$, the conditional probability of Event 2 given Event 1 is at least, say, $\frac{\delta_n}{2e^2} > 0$.

Given Event 1 and 2, consider Event 3 such that except the two minority students $s_1$ and $s_2$, there is exactly one student (denoted by $s_3$) ranks $c_2$ first and exactly one student (denoted by $s_4$) ranks $c_3$ first, where $s_3, s_4 \in S^n$. The conditional probability of Event 3 is

$$\left(\frac{n - 2}{1}\right) \times \left(\frac{n - 3}{1}\right) \times \frac{1}{(n-1)^2} \times \left(1 - \frac{2}{n-1}\right)^{n-4}.$$

Similarly, we can derive the limit of this expression as $\frac{1}{e^3}$, when $n \to \infty$. For any sufficiently large $n$, the conditional probability of Event 3 given Event 1 and 2 is at least, say, $\frac{1}{2e^3}$.

Given Events 1, 2, and 3, let Event 4 be the event that apart from other students in $\Gamma^n$, $c_1$ ranks $s_3$ and $s_4$ higher than both $s_1$ and $s_2$. Since Events 1-3 do not impose any restrictions
on the rankings of these four students in $c_1$’s priority order, the conditional probability of Event 4 is $\frac{1}{5}$. Note that given Events 1-4 and the assumption that $k = 1$ and $q_{c_1} = 1$, the event that school $c_1$ is matched with $s_1$ or $s_2$ (under either the majority quota or its corresponding minority reserve) while being contained in a cycle involving another participant other than $c_1$, $s_1$, and $s_2$ occurs with conditional probability 1.

Given Events 1-4, let $\pi_1 > 0$ be the conditional probability that school $c_1$ is matched with $s_1$ when $c_2$ has the majority quota $q_{c_2}^M = 1$. The unconditional probability that $c_1$ is matched with $s_1$ when $q_{c_2}^M = 1$, is thus at least $\frac{\pi_1(r_0^2/\delta_n)}{48e^2} > 0$. Accordingly, let $\pi_2 > 0$ be the conditional probability (given events 1-4) that school $c_1$ is matched with $s_2$ when $c_2$ has $r_{c_2}^m = 1$. The unconditional probability that $c_1$ is matched with $s_2$ when $r_{c_2}^m = 1$, is thus at least $\frac{\pi_2(r_0^2/\delta_n)}{48e^2} > 0$.

Therefore, for any sufficiently large $n$, we cannot eliminate the probability that these two affirmative actions generate different matching outcomes under the TTCM in market $\Gamma^n$; i.e., there is an $\tilde{n}$ such that $p_n > 0$, for any $n \geq \tilde{n}$. This completes the proof.

**Remark 1.** Under the same regularity conditions as our Definition 1, Liu (2022) proves that the asymptotic outcome equivalence of the majority quota policy and its minority reserve counterpart under the SOSM. Such distinct asymptotic performance between the TTCM and the SOSM essentially comes from the priority trade nature of the TTCM. As presented in the preceding proof, blocking possible priority trades under the TTCM with affirmative actions requires that it is very unlikely for any two different students to list the same school without reserved seats in a sequence of random markets of arbitrary sizes (i.e., school $c_1$ in Event 1 when $n \geq 4$, or school $c_2$ in the $2 \leq n < 4$ case). This cannot be satisfied even under our relatively restricted regularity conditions of Definition 1. By contrast, the convergence process under the SOSM only demands that no two different students (either majority or minority) will list the same school with nonzero reserved seats with a high probability (see Lemma 1 in Section 5).

As the purpose of imposing affirmative actions in school choice is to improve the matching outcomes (i.e., welfare) of minority students, the asymptotically non-equivalent TTCM-Q and its corresponding TTCM-R also induce an ambiguous Pareto dominance relationship for minorities in large matching markets. To see this, recall the two minority students $s_1$ and $s_2$ in the $n \geq 4$ case of the preceding proof, the welfare improvement of $s_1$ (when $s_1$ is assigned to $c_1$, with conditional probability $\pi_1 > 0$) under the the TTCM-Q is clearly at the expense of $s_2$ (when $s_2$ is assigned to $c_1$, with conditional probability $\pi_2 > 0$) under the TTCM-R, and vice versa. We thus have the following corollary of Theorem 1.

**Corollary 1.** There exists no Pareto dominance relationship for minorities between the TTCM-Q and its corresponding TTCM-R in the sequence of random markets $(\Gamma^1, \Gamma^2, \ldots)$.

## 5 Asymptotic Equilibrium Outcomes Equivalence under the IAM

For each market $\Gamma = (S, C, P_\succ, (q^M, r^m))$, Afacan and Salman (2016) adapt the immediate acceptance mechanism (IAM) to school choice with affirmative actions. The **immediate acceptance mechanism with affirmative actions** algorithm runs as follows:

**Step 1:** Each student applies to her most preferred acceptable school (call it school $c$). The school $c$ first considers minority applicants and permanently accepts them up to its
minority reserve \( r^m_c \) one at a time following its priority order, if \( r^m_c > 0 \). School \( c \) then considers all the applicants who are yet to be accepted, and one at a time following its priority order. It permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than \( q^M_c \). The rest (if any) are rejected.

**Step 1:** Each student \( s \) who was rejected at Step \((l-1)\) applies to her next preferred acceptable choice (call it school \( c \), if any). If school \( c \) still has an available seat, it first considers minority applicants and permanently accepts them up to its remaining minority reserve one at a time following its priority order, if \( r^m_c > 0 \). School \( c \) then considers all the applicants who are yet to be accepted, and one at a time following its priority order. It permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than its remaining majority quota. The rest (if any) are rejected.

The algorithm terminates either when every student is matched to a school or every unmatched student has been rejected by all acceptable schools, which always terminates in a finite number of steps. For a market \( \Gamma \), if \( r^m_c = 0, \forall c \in C \), i.e., a market with only minority quota, then the above algorithm reduces to the IAM with majority quota (IAM-Q, henceforth). Also, if \( q^M_c = q_c, \forall c \in C \), i.e., a market with only minority reserve, then the above algorithm reduces to the IAM with minority reserve (IAM-R, henceforth).

We use the following example to illustrate how the IAM-Q and its IAM-R counterpart work in a finite school choice market, as well as students’ incentives to misreport their preferences.

**Example 2.** Consider the following market \( \Gamma \) with two schools \( C = \{c_1, c_2\} \), and four students \( S = \{s_1, s_2, s_3, s_4\} \) where \( S^M = \{s_1, s_2\} \) and \( S^m = \{s_3, s_4\} \). \( q_{c_1} = 2 \) and \( q_{c_2} = 1 \). Suppose that \( \Gamma \) has the following majority quota and its corresponding minority reserve: \((q^M_{c_1}, q^M_{c_2}) = (1, 1)\), or correspondingly, \((r^m_{c_1}, r^m_{c_2}) = (1, 0)\). Schools and students have the following priority and preference orders:

| \( >_{c_k}, k=1,2 \) | \( P_{s_i}, i=1,...,4 \) | \( P'_{s_4} \) |
|----------------|------------------|----------|
| \( s_1 \)     | \( c_1 \)         | \( c_2 \) |
| \( s_2 \)     | \( c_2 \)         | \( c_2 \) |
| \( s_3 \)     |                  | \( c_2 \) |
| \( s_4 \)     |                  |          |

That is, students \( s_1, s_2 \) and \( s_3 \) are non-strategic players who always truthfully report their own preferences \( P_{s_i} \), \( i = 1, 2, 3 \). On the other hand, although student \( s_4 \) holds \( P_{s_4} \) as her truthful preference order, she can strategically submit \( P'_{s_4} \) which instead lists \( c_2 \) as her only acceptable school.

When all students report their truthful preferences, all four students will apply to \( c_1 \) at Step 1 of either the IAM-Q or its IAM-R counterpart. Given \( q^M_{c_1} = 1 \) under the IAM-Q, \( c_1 \) will permanently accept its highest ranked majority student \( s_1 \) and highest ranked minority student \( s_3 \), and rejects the rest two students \( s_2 \) and \( s_4 \). On the other hand, given \( r^m_{c_1} = 1 \) under the IAM-R, \( c_1 \) will first permanently accept its highest ranked minority student \( s_3 \), and then accepts its highest ranked majority student \( s_1 \). \( c_1 \) then rejects the rest two students \( s_2 \) and \( s_4 \).

The remaining two students \( s_2 \) and \( s_4 \) apply to \( c_2 \) with an unfilled seat at Step 2 of either the IAM-Q or its IAM-R counterpart. Since \( c_2 \) has no reserved seat for minorities, it will
accept its most preferred applicant \( s_2 \), and rejects \( s_4 \) under either of these two matching mechanisms.

The IAM-Q and the IAM-R produce the same matching outcome as

\[
f^{IAM-Q}(\Gamma) = f^{IAM-R}(\Gamma) = \begin{pmatrix} c_1 & c_2 \\ s_1, s_3 \\ s_2 & s_4 \end{pmatrix}
\]

which leave \( s_4 \) unmatched.

Nevertheless, different from the strategy-proof TTCM-Q and its TTCM-R counterpart, students have incentives to manipulate their reported preference orders under the IAM with either of these two affirmative actions. To see this, let the unmatched student \( s_4 \) report \( P'_{s_4} \) instead, i.e., \( s_4 \) applies to \( c_2 \) at Step 1 of either the IAM-Q or its IAM-R counterpart. Since all students are permanently accepted under the IAM with these two affirmative actions, \( s_2 \) will be left unmatched at Step 2 instead. The IAM-Q and the IAM-R thus produce

\[
f^{IAM-Q}(\Gamma) = f^{IAM-R}(\Gamma) = \begin{pmatrix} c_1 & c_2 \\ s_1, s_3 \\ s_2 & s_4 \end{pmatrix}
\]

which unilaterally improve the matching outcome of the strategic student \( s_4 \).

To analyze the asymptotic performance of the majority quota and its minority reserve counterpart under the manipulable IAM, we first define the IAM-Q and its corresponding IAM-R as a preference revelation game in a sequence of random markets. Formally, given a regular random market \( \hat{\Gamma}^n \) of size \( n, n = 1, 2, \ldots \), a mechanism \( f \) and students’ corresponding truthful preference profile \( \hat{P}^n = (\hat{P}^n_s)_{s \in S^n} \), denote \( G_f(\hat{\Gamma}^n) = (\mathcal{P}^n, \hat{P}^n, f) \) the preference revelation game induced by \( f \), where \( \mathcal{P}^n \) is the strategy space of each student (i.e., all the possible stated preferences over schools), and \( f \) is the outcome function in which each student evaluates her assignments according to \( \hat{P}^n \).

**Definition 3.** A strategy profile \( P^* \in \Pi_{s \in S^n} \mathcal{P}^n \) is a Nash equilibrium of \( G_f(\hat{\Gamma}^n) \), if for each \( s \in S^n \), there is no strategy \( P'_s \in \mathcal{P}^n \) such that \( f_s(P'_s, P'^{-s}) \hat{P}^n f_s(P^*) \), where \( P'^{-s} = (P'^*_i)_{i \in S^n \setminus s} \).

Also, given a Nash equilibrium \( P^* \), its corresponding Nash equilibrium outcome is \( f(P^*) \).

Under the same preference generation procedure and the regularity conditions defined in Section 3.2, Liu (2022) writes his result to state that it is very unlikely for any two different students to list the same school with nonzero reserved seats under the SOSM with either of these two affirmative actions when the market becomes large. Since the underlying matching mechanisms will not affect the probability of listing any particular school (from either of the two distributions \( \mathcal{A} \) or \( \mathcal{B} \) in students’ preference orders, we omit the proof of Lemma 1 for brevity (see Expression (A.3) and the succeeding arguments in the Proof of Proposition 2 of Liu (2022) for details).

**Lemma 1.** (Liu, 2022) The probability that no two distinct students (either majority or minority) will list the same school \( c \in C^n \) with nonzero reserved seats in their preference orders converges to one, as \( n \to \infty \).

For any regular random market \( \hat{\Gamma}^n \) in the sequence of markets \( (\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots) \), let \( \hat{P}^n \) be the truthful preference order of student \( s \) generated according to the procedure defined in Section 3.2, and \( \xi^Q(\hat{P}^n) \) (resp. \( \xi^R(\hat{P}^n) \)) be the set of Q-stable (resp. R-stable) matchings under the profile of truthful preferences \( \hat{P}^n \) with majority quota (resp. minority reserve), where \( \hat{P}^n = (\hat{P}^n_s)_{s \in S^n} \).
Lemma 2. The probability that $\xi^q(\tilde{P}^n) = \xi^r(\tilde{P}^n)$ converges to one, as $n \to \infty$.

Proof. (i) $\xi^r(\tilde{P}^n) \subseteq \xi^q(\tilde{P}^n)$. Given a regular random market $\tilde{\Gamma}^n$, let $\mu \notin \xi^q(\tilde{P}^n)$ be a non R-stable matching in it, i.e., $\mu$ is blocked by a pair of student and school $(s, c) \in (S^n, C^n)$ when $\tilde{\Gamma}^n$ has the majority quota $q^{M,n}$. By the definition of R-stability, $(s, c)$ also blocks $\mu$ in $\tilde{\Gamma}^n$ with the corresponding minority reserve $r^{m,n} = q^n - q^{M,n}$. Thus, we have $\mu \notin \xi^r(\tilde{P}^n)$ in $\tilde{\Gamma}^n$.

(ii) $\xi^q(\tilde{P}^n) \subseteq \xi^r(\tilde{P}^n)$. Let $\mu \in \xi^q(\tilde{P}^n)$ be a Q-stable matching in a given regular random market $\tilde{\Gamma}^n$ of size $n$. We demonstrate that $\mu$ is asymptotically R-stable when the size of the market grows up; that is, the probability that $\mu \in \xi^r(\tilde{P}^n)$ converges to one in the sequence of markets $(\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)$, as $n \to \infty$.

As a majority student will never list a school $c$ with $q^M_c = 0$ (Condition (6) of Definition 1), a Q-stable matching $\mu$ in $\tilde{\Gamma}^n$ with majority quota $q^{M,n}$ is blocked by a pair of $(s, c)$ in $\tilde{\Gamma}^n$ with the corresponding minority reserve $r^{m,n}$, can only occur when $c \tilde{P}^n_s \mu(s)$, $|\mu(c) \cap S^M| = q^M_c$, $|\mu(c)| < q_c$, and $s' > s$, for all $s' \in \mu(c) \cap S^{M,n}$ and $s \in S^{M,n} \setminus \mu(c)$; that is, school $c$ has excessive majority applicants and insufficient number of minority applicants in $\tilde{\Gamma}^n$. By Lemma 1, we know that it is very unlikely for any two distinct students (i.e., $s$ and $s'$ here) to list the same school $c$ with nonzero reserved seats in their preference orders when $n$ becomes sufficiently large. This implies that the probability for any pair $(s, c) \in (S^n, C^n)$ forming a blocking pair in $\tilde{\Gamma}^n$ with $r^{m,n}$ but not in $\tilde{\Gamma}^n$ with the corresponding $q^{M,n}$ converges to zero, as $n \to \infty$. Thus, we have the probability that $\mu \in \xi^r(\tilde{P}^n)$ converges to one, as $n \to \infty$.

Lemma 3. (1.) The set of Nash equilibrium outcomes of the IAM-Q is equal to the set of Q-stable matchings under the truthful preferences $\tilde{P}^n$ in each $\tilde{\Gamma}^n$, $n = 1, 2, \ldots$.

(2.) The set of Nash equilibrium outcomes of the IAM-R is equal to the set of R-stable matchings under the truthful preferences $\tilde{P}^n$ in each $\tilde{\Gamma}^n$, $n = 1, 2, \ldots$.

Proof. (1.) The market-wise equivalence between the set of Nash equilibrium outcomes of the IAM-Q and the set of Q-stable matchings under the truthful preferences in each random market of size $n$, has been given by Theorem 3 of Ergin and Sönmez (2006). Thus, we only need to prove the second part.

(2.1) Given a regular random market $\tilde{\Gamma}^n$ of size $n$, $n = 1, 2, \ldots$, and the corresponding preference revelation game of IAM-R, let $P'$ be an arbitrary strategy profile and matching $\mu$ be its associated outcome. Suppose that $\mu$ is not R-stable under the truth preference profile $\tilde{P}^n$, we can thus find a pair of student and school $(s, c) \in (S^n, C^n)$ such that $c \tilde{P}^n_s \mu(s)$, and either $s > c$ or $s' > s$ for some $s' \in \mu(c)$, or $|\mu(c)| < q_c$ and $s$ is acceptable to $c$. This implies that $c$ is not at the top in $P'_s$, because otherwise student $s$ would have been assigned to school $c$. Let $P''_s$ be an alternative preference order of $s$ in which $c$ is positioned as her first choice. Clearly, $s$ will be assigned to $c$ under the strategy profile $(P''_s, P'_{-s})$, where $P'_{-s} = (P'_i)_{i \in S^n \setminus s}$. Thus, $P'$ is not a Nash equilibrium, as $P'_s$ offers a profitable deviation for student $s$ at $P'$ given that $c \tilde{P}^n_s \mu(s)$. Also, since $P'$ is arbitrarily chosen, the non R-stable matching $\mu$ cannot be obtained in the set of Nash equilibrium outcomes.

(2.ii) Let $\mu$ be a R-stable matching under $\tilde{P}^n$ in the regular random market $\tilde{\Gamma}^n$. We show that there exists a Nash equilibrium $P^*$, such that its associated outcome is $\mu$. For each student $s \in S^n$, let $P'_s$ be the preference order of student $s$ such that school $\mu(s)$ is positioned at the top, i.e., $\mu(s) P'_s c'$, $\forall c' \in C^n \setminus \mu(s)$. Thus, at $P^*$ the IAM-R will terminate at Step 1 and assign each student $s$ to $\mu(s)$. To show that $P^*$ is a Nash equilibrium, consider a pair of student and school $(s, c)$ such that $c \tilde{P}'_s \mu(s)$. As $\mu$ is R-stable, we know that $|\mu(c)| = q_c$, and each student who is matched with school $c$ under $\mu$ is more preferred to $s$; also, for each $s' \in \mu(c)$, $\mu(c)$ is her top ranked school at $P^*$. Thus, $s$ cannot be matched to $c$ by misreporting her
preferences. Since $s$ is arbitrarily chosen, the preceding argument suffices the non-existence of profitable deviations at $P^*$. We conclude that $P^*$ is a Nash equilibrium with the R-stable matching $\mu$ as its associated Nash equilibrium outcome.

We are now ready to present our main argument on the asymptotic performance of the IAM with affirmative actions.

**Theorem 2.** The sets of Nash equilibrium outcomes of the IAM-Q and its corresponding IAM-R are outcome equivalent in large markets.

As the transition from one affirmative action policy to the other could possibly evoke substantial political, administrative and cognitive costs to local communities, an immediate policy implication of Theorem 2 is that the IAM is more cost-effective compared to the asymptotically non-equivalent TTCM in large school choice markets with affirmative actions, in the sense that it is unnecessary to identify the different welfare effects of these two affirmative actions under the IAM if the policymaker can assure a sufficient supply of popular schools to the matching markets.

### 6 Conclusions

This paper studies the asymptotic performance of two celebrated matching mechanisms, the top trading cycles mechanism (TTCM) and the immediate acceptance mechanism (IAM), in the context of school choice markets with affirmative actions. We show that the majority quota policy and its minority reserve counterpart will induce different matching outcomes with non-negligible probability under the TTCM, even in arbitrarily large markets with sufficiently many schools and a relatively slow growth of reserved seats. Given the possible preference manipulations under the IAM, we further characterize the asymptotically equivalent set of Nash equilibrium outcomes of the IAM-Q and its corresponding IAM-R when the market becomes sufficiently large.

Our contribution to the literature is two-fold. From theoretical perspective, we extend the performance comparison of the majority quota policy and its minority reserve counterpart to the large matching market scenario; different from most extant studies on large matching markets, our results make a clear distinction on the asymptotic performance of the TTCM and the IAM with affirmative actions. From practical perspective, our results provide guidance to policymakers regarding the cost-effectiveness of the IAM over its TTCM counterpart in large school choice markets with affirmative actions.

Last, since we can treat affirmative actions as a generic type-specific constraint which is not limited in the context of school choice, future research can also work on identifying the asymptotic performance of different type-specific constraints in other matching markets with a large number of participants and type-specific preferential treatments (e.g., elite engineering college admissions, Covid-19 vaccine allocations, refugee resettlement, among others), and designing new matching mechanisms to improve the resource allocation effectiveness in these markets.

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