Waves of complicated structure in nonlinear dispersive media

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Abstract. A semi-analytical asymptotic model for large amplitude solitary internal waves in shallow fluids is presented. These solitary waves only exist for large amplitudes beyond the limit of applicability of the KdV/mKdV equations. The theory describes complicated structures including multi humped waves and waves with algebraic asymptotes. All these effects lead to drastically different wave structures even for almost identical density profiles. Trapped core inside the wave prevents appearance of such multiple scales within the core area. The structural stability of waves of large amplitude is investigated using the approach presented earlier in the literature. Multiscaled waves without vortex core are shown to be structurally unstable. It is anticipated that complicated multiscaling phenomena could exist for solitary waves in various geophysical situations.

1. Introduction
Propagation velocities and horizontal scales are important characteristics of a plane disturbance in a dispersive medium. In an ideal density stratified shallow fluid, a wave of small, but finite amplitude has one typical scale resulting from the local balance between nonlinearity and dispersion in the framework of Korteweg-de Vries (KdV) equation as discussed in [1]. Solitary waves of permanent form for which capillary dispersion has the same order as the gravitational one may have oscillatory outskirts as predicted in [2]. When viscosity is accounted, transient effects leading to various length scales are discussed for the modified KdV type equation, for example, in [3]. Various length scales also appear due to external factors such as complicated boundary conditions including bottom topography, this situation was discussed in [4] and [5] among others. In the present work it is shown that for the gravitational differential dispersion, ignoring all other earlier mentioned external or internal effects, there are solitary waves with various scales and even various (not only exponential) asymptotes. These solutions exist only for disturbances of finite amplitude exceeding the range of applicability of the modified KdV model, which incorporates both quadratic and cubic nonlinearities. Higher nonlinearity in the existing small amplitude KdV or mKdV models leads to the correction of the wave length scale without generation of multiscaling. For appearance of multiscaling, the various competitive nonlinearities should be of the same order and that order needs to be higher then the cubic one as analytically shown below. This effect was initially noticed in [6]. The authors of [7] presented numerical procedure that provides fast calculations for gravitational waves between rigid lids. This model is able to work with fine density stratifications. In [7] a two-humped and usual one-humped solitary internal waves solutions for nearly identical density profiles in a
two-pycnocline density stratification were calculated. The authors of [8] have numerically shown that in some stratifications with two pycnoclines three conjugate flow solutions leading to two-humped solitary waves were present. In [9] a continuous stratification in order to characterize the role of vertical structure of the fluid density in the context of waves close to the limiting amplitude was theoretically examined. In the current paper, an asymptotic model presented in earlier papers by the author addresses multiscaling phenomena for internal solitary waves under free surface in the frame of Dubreil-Jacotin-Long equation (DJL) [10]. Special attention is given to the case of complicated nonlinearity involving both quadratic, cubic and quartic nonlinear terms for the case of continuous stratification with a single pycnocline. Solitary waves of permanent form, their existence, structural stability and asymptotic behaviour are discussed. It is expectable that such multiscaling solitary waves could exist in other physical systems where complicated competitive nonlinearities are balanced by dispersion.

2. Asymptotic model for internal waves in shallow fluids

Let us consider the two-dimensional steady motion of an ideal stratified fluid in a frame of reference moving with phase speed of wave \( c \). The approach is asymptotic being based on the DJL equation for waves without a priori limitation on amplitude. This approach has started from the pioneering work [11]. Let us consider the stratification in the form

\[
\rho_0(z) = \rho_{00}(1 - \sigma(z + \delta f(z))), \quad \delta \ll 1, \quad \sigma \ll 1, \quad f \sim 1,
\]

where \( \sigma \) denotes Boussinesq parameter. In [12] it was shown that for this case the dimensionless (primed) stream function \( \psi' = -\psi/cH \) of a solitary disturbance obeys the equation

\[
\psi_{zz} + \mu^2 \psi_{xx} + \lambda(\psi - z) - \frac{\sigma}{2}(\psi^2 - 1 - 2\psi \lambda(\psi - z)) + \delta \lambda z f_\psi(\psi) = o(\sigma, \delta, \mu^2),
\]

where \( \mu \) is the aspect ratio \( H/L \) and \( \lambda = \frac{2gH}{c^2} \).

In equation (2) \( z \) denotes the vertical axis, taken positive upwards and \( x \) corresponds to the horizontal axis; \( z \) and \( x \) are scaled with \( H \) and \( L \), the given vertical and horizontal scales respectively. The primes in equation (2) are dropped. It is decided to locate the bottom and the surface at the dimensionless heights \( z = -0.5 \) and \( z = 0.5 + \eta(x) \), respectively, where \( \eta(x) \) denotes surface displacement. The boundary conditions at the bottom and surface are

\[
\psi_x = 0 \quad \text{on} \quad z = -0.5,
\]

\[
\sigma(\psi_x \psi_z \psi_{zz} - \psi_z^2 \psi_{xx}) + \lambda \psi_x = o(\sigma) \quad \text{on} \quad z = 0.5 + \eta(x),
\]

\[
\psi_x = -\eta_x \psi_z.
\]

The solution of equations (2-5) is sought in the form

\[
\psi = \psi^{(0)} + \mu^2 \psi^{(1)} + \ldots, \quad \lambda = \lambda^{(0)} + \mu^2 \lambda^{(1)} + \ldots, \quad \eta = \eta^{(0)} + \mu^2 \eta^{(1)} + \ldots,
\]

where zeroth order variables are of order unity. Below we shall provide solution for the first mode, which is most frequently observed in nature. The analysis for the higher modes is similar.

In the zeroth order

\[
\psi^{(0)} = z + A(x) \cos(\pi z), \quad \lambda^{(0)} = \pi^2, \quad \eta^{(0)} = 0,
\]
where the amplitude function \( A(x) \) is to be determined at a higher order. For the solution to the first order equation to exist the solvability condition (Fredholm alternative) demands

\[
A_{xxx} + \lambda^{(1)} A_x - \frac{\sigma}{\mu^2} (2A_x - 8\pi AA_x + 2\pi^2 A^2 A_x) + \frac{2\delta}{\mu^2} Q_x(A) = 0,
\]

\[
Q(A) = A \int_{-0.5}^{0.5} \cos^2(\pi z) f_\psi(\psi = \psi^{(0)}) dz.
\]

In order to (locally) balance nonlinearity and dispersion we have to require \( \max(\sigma/\mu^2, \delta/\mu^2) \sim 1 \) thus determining \( L \). In [11] it was suggested to consider the nonlinear terms as power series in the Boussinesq parameter instead of the small amplitude parameter. The paper [12] extended this idea to account a more general undisturbed flow state including sheared flows in deep fluids.

The Weierstrass approximation theorem states that every continuous function defined on a closed interval can be uniformly approximated as closely as desired by a polynomial function. Thus, the integral below can be represented with the help of some \( N \) interval can be uniformly approximates as closely as desired by a polynomial function. Thus, the integral below can be represented with the help of some \( N \)th-order polynomial according to the Weierstrass approximation theorem. In the current study only polynomial formula for stratification is considered, thus it directly leads to nonlinearities in the polynomial form:

\[
\int_0^A Q(A') dA' = A^2 P_N(A).
\]

For the wave of amplitude \( A_0 \) equation (10) yields

\[
A_0^2 A^{-2} = (A_0 - A) \Phi(A, A_0), \quad \Phi(A, A_0) = 2\frac{\delta}{\mu^2} \frac{P_N(A_0) - P_N(A)}{A_0 - A} + \frac{\sigma}{\mu^2} \left( \frac{8}{3} - \frac{A_0 - A}{A} \right),
\]

\[
\lambda^{(1)} = \frac{\sigma}{\mu^2} \left( \frac{8}{3} + 2 + \frac{4}{3} \right) - 2\frac{\delta}{\mu^2} P_N(A_0).
\]

Equations (12,13) determine completely both profile and phase velocity of a solitary wave with amplitude \( A_0 \).

3. Various scales in waves of finite amplitudes
The function \( f \) in the form of a \( M \)th-order polynomial generates \( P_N \) with the index \( N = M - 1 \). The power index of \( \Phi \) is thus \( \max(1, M - 2) \). The condition for equation (8) to possess a multiscaled solution reduces to the condition that \( \Phi(A, A_0) \) must be sign-defined with several extrema within \([0, A_0]\). Thus it must have more than two imaginary roots on that interval. It determines \( M \geq 4 \), i.e. for a stratification in the form of cubic polynomial or if wave amplitude is small enough to neglect \( A^4 \) and higher order nonlinearities, multiscaled solitary waves do not exist because \( f \) has no imaginary roots for this case. This is why classical KdV or mKdV can not provide multiscaled solitary waves over flat bottom. Let us consider wave structures for the density stratification in the form,

\[
\rho_0(z) = \rho(1 - \sigma z + 0.5\sigma^2 z^2 + \alpha\sigma^2 z^4),
\]

which produces quadratic, cubic and quartic terms in equation (8). Thus equation (12) for this case of stratification becomes

\[
\Phi(A, A_0) = \frac{\sigma}{\mu^2} \left[-\frac{8\pi}{3} \left( \frac{1}{3} + 2\alpha - \frac{160\alpha}{9\pi^2} \right) + \frac{\pi^2}{3} (A + A_0) + \frac{128\alpha\pi^2}{75} (A^2 + A_0^2 + AA_0) \right].
\]
Two-humped solitary wave with amplitude for the particular stratification profile equation (14) with $\alpha = -1.39$ and $\sigma = 0.01$ is shown in figure 1. Indeed, the maximum derivative on $x$ in the dimensionless coordinates is of order unity. However, the wave has a pronounced two-scale structure with typical length scales, which are much larger than the length $L$ used to scale the derivative.

Generally, one can expect at most $M/2$ different scales for a stratification in the form of polynomial with even power index $M$, and $(M - 1)/2$ otherwise.

Further, we wish to examine the structure of solitary waves of permanent form for the stratification given by equation (14) and consider the case $\alpha = -1.39$ and $\alpha = -1.3856$ with focus on the waves of permanent form under free surface along with their limiting forms and structural stability. Other values of $\alpha$ lead to more extensive consideration with a number of particular cases. Such study is beyond the scope of the present paper. First, for $\alpha = -1.39$ there exist only permanent waves with positive amplitudes. Wave phase velocity is defined by the following expression

$$c^{(1)}(A_0) = \frac{c - c(A_0 = 0)}{\mu^2} = \frac{4A_0}{3\pi} \left( 2\alpha + \frac{1}{3} - \frac{160\alpha}{9\pi^2} \right) - \frac{A_0^2}{6} + \frac{64\alpha A_0^3}{75\pi}. \tag{16}$$

The phase velocity is an increasing function for $0 < A_0 < A_2$ and $A_0 > A_1$. For $A_1 < A_0 < A_2$ the phase velocity decreases with amplitude and there are no steady solitary wave solutions. When $0 < A_0 < A_2$ solitary waves are widening as amplitude increases with a table top limiting shape with a local maximum for the wave velocity. Amplitude functions for such waves are shown in figure 2. Such waves are structurally stable according to [14] as both the wave energy

![Figure 1. Amplitude function and streamline pattern for a two-humped solitary wave, $\alpha = -1.39$.](image1)

![Figure 2. Profiles of stable solitary wave are shown by solid lines. Dashed lines correspond to $A_2 = 0.1311, A_1 = 0.1793$. Limiting amplitude reaches when $A_0 = A_2$.](image2)
\[ E = \int_{-\infty}^{\infty} A^2 dx \] and the wave velocity increase as amplitude increases.

For \( A_0 > A_1 \) wave profiles are shown in figure 3. Waves change from the table top solution via multiscaled structures to solitary waves with a single scale.

\[
\frac{25\pi^2}{1152\alpha} (x - x_0) = \frac{2}{\sqrt{D} + 1} + \ln \left( \frac{\sqrt{D} + 1 - 1}{\sqrt{D} + 1 + 1} \right), \quad D = \frac{384\alpha A}{25\pi}, \tag{17}
\]

where \( x_0 \) is an arbitrary constant. For the particular stratification considered here waves

quite interesting case corresponds to \( \alpha = -1.3856 \). In this case \( A_2 = A_1 \) and the limiting form of a bore like solution has one exponential and one algebraic \( \frac{1}{x^2} \) asymptotes. Permanent waves with amplitudes above the amplitude of a bore solution are unstable while the waves below this amplitude are stable.

\[ x_0 \quad \text{is an arbitrary constant.} \]

For the particular stratification considered here waves

are structurally unstable \[14\] since wave energy decreases as shown in figure 5 but the wave velocity increases with the increasing of wave amplitude. Interesting observation is these waves
of sufficiently large amplitude could be stable as the energy is eventually increased as shown in figure 5. For the stratification considered here, it does not matter because solution with vortex core appears at lower amplitude when energy is still the decreasing function of amplitude. However it leads to the interesting phenomenon - waves with vortex core could stabilize the wave. The idea is that the vortex core leads to widening of wave [15] and consequently to the increase of its energy, thus the structural stability criterion will be satisfied. For the considered particular stratification waves with vortex core are initially unstable as increase of energy due to the vortex core and associated widening does not compensate the decrease of energy in the wave outside the vortex core. Nonetheless, above some amplitude waves become structurally stable. When wave amplitude further increases the permanent wave of limiting amplitude becomes infinitely wide as shown by the authors of [15]. The theory described above is valid for wave amplitudes below $A_\ast$, a certain amplitude at which a vortex core started to appear inside the wave. For nearly linear density profile $A_\ast = 1/\pi$. [15] have shown that

$$B_x^2 \sim R(A_\ast)(1 - B) - \frac{8\nu}{15}(1 - B^{5/2}),$$

(18)

where $\nu$ is the supercriticality parameter defined such that $B$ varies from zero to one as wave amplitude does from $A_\ast$ to the maximum value allowed predicted there. $R(A_\ast)$ depends on stratification profile and is fixed. It is straightforward to notice that $B(x)$ is monotonic and therefore multiscaling in the vortex core area does not exist when $A > A_\ast$.

Multiscaling effects similar to the discussed above, could be observed in various physical media. The authors of [16] reported that solitary Rossby waves in channels obey the same KdV type equation with complicated nonlinearity due to the mean shear variations. Coriolis force for Rossby waves plays the same role as gravitational force for the internal gravity waves.

4. Conclusion
For a particular case of a nonlinear dispersive medium like a shallow stratified fluid embedded in the gravity field, we have addressed multiscaled solitary waves which are predicted when there exists competition of several different types of nonlinearity. The mechanism leading to these solutions differs from the mechanism of multiscaling due to the competition of different types of dispersion or effects due to the dissipation. We have shown that the length used to scale the $x$-derivative does not simply coincide with the typical length scale of the wave, as for KdV. Moreover, multiscaled (multi humped) disturbances exist for sufficiently large amplitudes, at least terms in forth order of waves amplitude should be accounted. Multiscaling (multi humped) phenomenon exists or does not exist for almost identical density profiles, two pycnoclines cases studied earlier in [7] and [8] are not necessary for the existence of multiscaling.
Continuous stratification given by equation (14) was studied in more detail. The structure of permanent solitary waves and how multiscaling appeared were presented. It is shown that the examined solitary waves could have exponential and power asymptotics. It is also noticed that the conjugate flow states could lead to quite significant streamline displacements as shown in figure 4. Structural stability was examined using the criterion proposed in [14]. It was shown that both stable and unstable solutions of the KdV type equation with quadratic, cubic and quartic nonlinearities are available. Multiscaled waves without trapped core belong to the unstable solutions. Trapped core inside the wave prevents appearance of such multiple scales within the core area. However, trapped core could stabilize the multiscaled solution in the sense of structural stability. The case when trapped core and multiscaling are combined together is beyond the scope of the present study and will be presented elsewhere. It is noted that multiscaling phenomena could exist for solitary waves in various physical environments, for example, for Rossby waves on a shear flow [16] and inertial waves in swirling flows [17].

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