Abstract: Like many routing protocols, the Tor anonymity network has decentralized path selection, in clients locally and independently choose paths. As a result, network resources may be left idle, leaving the system in a suboptimal state. This is referred to as the price of anarchy, where agents acting in their own self interest can make poor decisions when viewed in a global context. In this paper we explore the cost of anarchy in Tor by examining the potential performance increases that can be gained by centrally optimizing circuit and relay selection using global knowledge. In experiments with both offline and online algorithms, we show that centrally coordinated clients can achieve up to 75% higher bandwidth compared to traditional Tor. Drawing on these findings, we design and evaluate a decentralized version of our online algorithm, in which relays locally distribute information enabling clients to make smarter decisions locally and perform downloads 10-60% faster. Finally, we perform a privacy analysis of the decentralized algorithm against a passive and active adversary trying to reduce anonymity of clients and increase their view of the Tor network. We conclude that this decentralized algorithm does not enable new attacks, while providing significantly higher performance.

1 Introduction

Tor [7] is one of the most popular and widely used low-latency anonymity systems, with 2 million daily clients supported by a network of over 6000 volunteer relays. Tor clients tunnel traffic through a sequence of 3 relays called a circuit. Traffic sent over the circuit is encrypted so that no single relay can learn the identity of both the client and destination. With a limited amount of resources available from the volunteer relays, it is important to distribute traffic to utilize these resources as efficiently as possible. Doing so ensures high performance to latency sensitive applications and attracts potential clients to use the network, increasing anonymity for all users.

Similar to ordinary routing, clients make decisions locally based on their view of the network. In Tor, directory authorities are centralized servers that keep track of all relays and their information. Clients download this information, stored in consensus files and server descriptors, every few hours from the directory authorities. With this information clients create 10 circuits to have available, selecting relays for the circuits at random weighted by the relays bandwidth. When the client starts using Tor to send traffic, it then selects the circuit in best standing to use for the next 10 minutes. Since it can take the directory authority up to 24 hours to properly update relay information and distribute it to clients, much research has focused on how clients can make better decisions, such as considering latency [2] or relay congestion [37] when selecting relays and circuits.

The main issue with leaving network decisions to clients is that local incentives can lead clients to make decisions that they believe will maximize their performance, when actually these decisions result in a global sub-optimal state with overall client performance suffering. In these situations it is common to compare what happens when decisions are made with global knowledge. This can be achieved by centrally processing requests as they are made, or performing decisions offline when the entire set of requests is known a priori. The performance gap that results from using decentralized compared to centralized decision making is called the price of anarchy.

While Tor clients are not necessarily selfish, they are making local decisions without context of the state of the global network. In this paper we analyze the price of anarchy in the Tor network. We develop both offline and online centralized algorithms that optimize relay and circuit selection decisions made by clients. Using the global view of all active circuits in the Tor network, the algorithms are able to more intelligently select which circuits clients should use for each download they perform, resulting in significant performance improvements to clients. While these are not necessarily protocols that should be run in practice, it allows us to bound the actual price of anarchy in Tor. This demonstrates the potential improvements that better resource allocation in Tor could achieve.
In this paper we make the following major contributions:

- Using a modified client model, we create a genetic algorithm to compute the optimal offline circuit selection for a given sequence of download requests. Since the algorithm is given access to the full sequence of requests before processing, this allows us to establish the optimal performance for a given network and load. This serves as a baseline for the competitive analysis of other algorithms.

- In addition, we develop an online centralized algorithm that processes download requests serially. Based on a delay-weighted capacity routing algorithm the centralized circuit selection tries to avoid low bandwidth bottlenecks when assigning circuits to requests.

- Using techniques from the centralized algorithm, we develop a decentralized algorithm that results in similar resource allocations, with relays releasing a small amount of information to allow clients to make smarter circuit selection choices.

- We perform privacy analysis on the decentralized circuit selection algorithm that considers both information that can be learned from a passive adversary, and examines how an active adversary could attempt to abuse the algorithm to increase their view of the network.

- All algorithms are analyzed in a large scale simulated environment using the Shadow simulator [15]. This allows us to precisely compare the effects and performance gains achieved across the different circuit selection algorithms.

2 Background

In this section we discuss some of the Tor architecture, basics on how the clients operate and send data through the network, and related work on increasing performance in Tor.

2.1 Tor Architecture

When a client first joins the Tor network, it downloads a list of relays with their relevant information from a directory authority. The client then creates 10 circuits through a guard, middle, and exit relay. The relays used for each circuit are selected at random weighted by bandwidth. For each TCP connection the client wishes to make, Tor creates a stream which is then assigned to a circuit; each circuit will typically handle multiple streams concurrently. The Tor client will find a viable circuit that can be used for the stream, which will be then be used for the next 10 minutes or until the circuit becomes unusable.

Internal to the overlay network, relay pairs establish a single TLS connection for all communication between them. To send traffic through a circuit, data is packed into 512-byte onion-crypted cells using secret keys negotiated with each relay during the circuit building process. Once a relay has received a cell, it peels off its layer of encryption, finds the next relay on the circuit to send to, and places it on a circuit queue where it waits to be sent. After a cell has traversed the entire circuit, the exit recovers the initial data sent by the client and is forwarded to the end destination.

2.2 Related Work

One of the major keys to increasing anonymity for Tor users is ensure a large anonymity set, that is, a large user base. To do so Tor needs to offer low latency to the clients; bad performance in the form of slow web browsing can lead to fewer users using the system overall. To this end, there has been a plethora of research looking to address ways to increase performance in Tor. These roughly fall into 4 areas: scheduling, selection, transports, and incentives.

Scheduling: Internally Tor is constantly making scheduling decisions on what to send and how much should be processed. These decisions happen on all levels, between streams, circuits, and connections. On the circuit level, much work has been done in an attempt to classify latency sensitive circuits, either prioritizing them [3, 35] or outright throttling noisy ones [18]. With regards to deciding how much should be sent on a circuit, there has been work comparing Tor’s end-to-end window algorithm with an ATM-style link-based algorithm [4]. Additional work [14] has been done on scheduling across circuits from all connections, limiting how much is written to a connection at a time so Tor has more control over what gets sent, opposed to the kernel.

Selection: When creating circuits, Tor selects from relays at random weighted by their bandwidth. Determining the bandwidth is a non-trivial issue, and much work [33, 34] has been done looking at a range of methods, from using self-reported values, central nodes making
direct measurements, and peer-to-peer methods relying on existing relays. There also has been a lot of research in improving the relay selection for circuit creation. These range from incorporating latency and congestion measurements, using a virtual coordinate system, to adjusting the weighting strategy, all in an attempt improve overall client performance.

**Transport:** One of the noted performance issues in Tor is the fact that the single TLS connection between relays can cause unnecessary blocking, where circuits could keep sending data but TCP mechanisms prevent it [8]. Reardon [26] attempted to address this by implementing TCP-over-DTLS allowing a connection to be dedicated to each circuit. In a similar vein, there has been numerous work [5, 9, 10] looking into increasing the number of TCP connections between relays and scheduling circuits between them in an attempt to avoid unnecessary blocking. Nowlan et al. introduce uTCP and uTLS [23, 24], which allows for out-of-order delivery in Tor so it can process cells from circuits that are not blocking.

**Incentives:** While the previous lines of research involved improving efficiency in how Tor handles traffic, another set looked at potential ways to incentive clients to also contribute bandwidth as a relay, increasing the overall network resources available to clients. This was first explored by Ngan, Dingledine, and Wallach [22], where they would prioritize traffic from relays providing high quality service in the Tor network. Jansen, Hopper, and Kim [16] extend this idea, allowing relays to earn credits which can be redeemed for higher prioritized traffic. Building on this, Jansen, Johnson, and Syverson [17] introduce a more lightweight solution that allows for the same kind of prioritization of traffic without as much overhead.

### 3 Experimental Setup

To maximize our understanding of performance in the Tor network, we want to run large scale experiments that accurately represent the actual operations of Tor clients and relays. To this end we use Shadow [1, 15], a discrete event network simulator with the capability to run actual Tor code in a simulated network environment. Shadow allows us to setup a large scale network configuration of clients, relays, and servers, which can all be simulated on a single machine. This lets us run experiments privately without operating on the actual Tor network, avoiding potential privacy concerns of dealing with real users. Additionally, it lets us have a global view and precise control over every aspect of the network, giving us insights and capabilities into Tor that would be near impossible to achieve in a live network. Most importantly, Shadow performs deterministic runs, allowing for reproducible results and letting us isolate exactly what we want to test for performance effects.

Our experiments are configured to use Shadow v1.9.2 and Tor v0.2.5.10. We use the large network configuration deployed with Shadow which consists of 500 relays, 1350 web clients, 150 bulk clients, 300 performance clients, and 500 servers. The default client model in Shadow [15] has clients download a file of a specific size, and after the download is finished it chooses how long to pause until it starts the next download. Web clients download 320 KB files and will randomly pause between 1 and 60,000 milliseconds until it starts the next download. Bulk clients download 5 MB files with no break in between downloads. The performance clients are split into three groups, downloading 50 KB, 1 MB, and 5 MB files, then pausing 1 minute between each download it performs. Along with the default client model we also consider a new type of client model called the fixed download model. Instead of clients having a single start time and inter-download pause times, they have a list of downloads with each download having a unique start time and end time. To create a fixed download experiment we extract the start and stop times for all client downloads from a default client run and use those as the base for the fixed download experiment.

For measuring performance between experiments we examine different metrics depending on which client model is being used. For the default client model we look at the time to first byte and download times of web and bulk clients along with the total read bandwidth across all clients in the network. When using the new fixed download model, download times lose meaning as they have fixed start and end times. The major metric that will change between experiments is the amount of data being pushed through the Tor network, so total client read bandwidth is the only metric we consider when using the fixed download model.

### 4 Price of Anarchy

The price of anarchy [29] refers to situations where decentralized decision making can lead a system into a sub-optimal configuration due to the selfish nature of agents participating in the system. In networking
this problem is specifically referred to as selfish routing [28, 30], where users select routes based on a local optimization criteria (latency, bandwidth, etc.) resulting in a sub-optimal global solution due to potential conflicting interests. While clients in Tor are not necessarily selfish, as they select relays at random weighted by their bandwidth, they are oblivious with respect to relay and network conditions. In that sense, localized decision making with respect to relays and circuits has the potential to lead to a sub-optimal result, where network resources are left idle which could be used to increase client performance.

To fully explore the price of anarchy in Tor we want to examine just how much of a performance increase can be achieved if we have some centralized authority making decisions rather than the clients themselves. For this we consider both an offline algorithm, which is able to see the entire set of download requests when operating, and an online algorithm which needs to process inputs serially and is unaware of future requests. In this section we detail how each of these algorithms work and look at any potential performance benefits.

4.1 Offline Algorithm

An offline algorithm needs access to the entire input set while operating, in this case that means knowing the times of when each client is active and performing a download. So instead of using the default client model in Shadow, we use the fixed download model which gives us a list of all download start and stop times in the experiment. From the experimental setup we extract all downloads $d_i$ with their corresponding start and end times $(s_i, e_i)$, along with the list of relays $r_j$ and their bandwidth $bw_j$. These parameters are then passed into the offline algorithm which returns a mapping $d_i \rightarrow c_i(r_{i1}, r_{i2}, r_{i3})$ of downloads and a circuit consisting of three relays which should be used by the client performing the download. Note that we still use the constraints from Tor that all relays in the circuit must be unique and that $r_{i3}$ must be an exit relay, but we do not force the algorithm to select $r_{i1}$ to be a guard. This is done since it is impossible to use a non-exit as an exit relay, however there is no mechanism to enforce the guard in a circuit actually has the guard flag set.

When using the fixed download client model the offline circuit selection problem starts to strongly resemble job scheduling [11]. For example, we have a set of downloads (jobs) that have to be processed on three relays (machines). Each download has a start time (release date), and can either have an end time (due date) or a file of specific size to download (number of operations). The relays can process downloads simultaneously (jobs can be preempted) and the goal is to download each file as fast as possible (minimize complete time or lateness). There are some complications in how exactly to define the machine environment for job scheduling, since the relays are heterogeneous and the fact that the amount of processing done on a relay is determined by the bottleneck in the circuit and not necessarily the bandwidth capacity of the relay. We would definitely need an algorithm which handled job splitting across machines, as we need 3 relays processing a download, and these problems seem to be often NP-hard.

Due to these complications we instead develop a genetic algorithm in order to compute a lower-bound of an optimal offline solution. Our genetic algorithm will have a population of solutions with a single solution consisting of a complete mapping of downloads to circuits. We can breed two solutions by iterating through each download and randomly choosing a circuit from either parent. However, the most important part of a genetic algorithm is a fitness function allowing us to score each solution. Since the metric we are most concerned with in the fixed download client model is the amount of bandwidth being pushed through the network, we want a method for estimating the total amount of network bandwidth used given a mapping of downloads to circuits.

In order to calculate the total bandwidth across an entire set of downloads, we first need a method for calculating the circuit bandwidth across a set of active circuits at a single point of time. To do this we make two
Algorithm 1 Estimate bandwidth of active circuits

1: function CalcCircuitBW(activeCircuits)
2:     activeRelays ← GetRelays(activeCircuits)
3:     while not activeCircuits.empty() do
4:         r ← GetBottleneckRelay(activeRelays)
5:         circuits ← GetCircuits(activeCircuits, r)
6:         circuitBW ← r.bw / circuits.len
7:         for c ∈ circuits do
8:             c.bw ← circuitBW
9:             for circRelay ∈ c.relays do
10:                if circRelay.bw = 0 then
11:                   activeRelays.remove(circRelay)
12:               end if
13:         end for
14:     end for
15:     activeCircuits.remove(c)
16: end while
17: end function

Algorithm 2 Compute total bandwidth for downloads

1: function CalcTotalBW(downloads)
2:     start, end ← GetTimeInterval(downloads)
3:     bandwidth ← 0
4:     time ← start
5:     while time ≤ end do
6:         circuits ← GetActiveCircs(downloads, time)
7:         CalcCircuitBW(circuits)
8:         for circuit ∈ circuits do
9:             bandwidth += circuit.bw
10:         end for
11:         time ← time + tick
12:     end while
13:     return bandwidth
14: end function

observations: (1) bottleneck relays will determine the bandwidth of the circuit, and (2) that a relay will split its bandwidth equally among all circuits it is the bottleneck on. The first observation comes from the end-to-end window based congestion control algorithm used in Tor, ensuring clients and exit relays only send as much as a circuit can handle. The second observation isn’t trivially true, since with the priority circuit scheduler some circuits will be sending more than others on a small enough time interval. In aggregate though all circuits will be given roughly the same amount of bandwidth.

The pseudocode for calculating the circuit bandwidth across a set of active circuits is shown in Algorithm 1 and works as follows. First the algorithm identifies the relay \( r \) with the lowest bandwidth per circuit (line 4). We know this relay is the bottleneck on all circuits that \( r \) appears on, as by definition every other relay on the circuit will have a higher available circuit bandwidth. Next the algorithm iterates through each circuit \( c_i \) that \( r \) appears on and assigns the circuit bandwidth as \( r \)'s per circuit bandwidth (lines 5-8). In addition, for each \( c_i \) the algorithm also iterates over each relay \( r_j \) on circuit \( c_i \), decrementing the bandwidth of \( r_j \) by the circuit bandwidth assigned to circuit \( c_i \) (line 10). While iterating over the relays, if any of them runs out of available bandwidth, the relay is removed from the list of active relays (lines 11-13). After we have iterated over the relays in circuit \( c_i \), it is removed from the list of active circuits (line 15). Note that during this process relay \( r \) will always end up with a bandwidth of 0, and no circuits remaining in the list of active circuits will contain \( r \). Furthermore, any other relay that reached a bandwidth value of 0 will also not be contained on any circuit remaining in the active circuit list. This means that we can never have a situation where a circuit has a relay that is no longer active with a bandwidth of 0, resulting in a circuit bandwidth of 0. The algorithm repeats this process, extracting the bottleneck relay and updating relay and circuit bandwidths, until no active circuits remain. An example of this this algorithm is shown in Figure 1 highlighting which relay is selected and how the circuit bandwidth is assigned and relay bandwidth updated.

With an algorithm to compute the bandwidth across all active circuits at a single point in time, we can now compute the overall bandwidth consumption across an entire set of downloads with assigned circuits, which is outlined in Algorithm 2. First the algorithm retrieves the earliest start time and latest end time across all downloads (line 2). With this we then iterate over every “tick” between the start and end time (lines 5-12), calculating the bandwidth across all active circuits during that tick (lines 6-7). Then for each active circuit we simply add the circuit bandwidth to a total bandwidth counter (lines 8-10). This gives us a total bandwidth value for a circuit to download mapping that can be used as the fitness score for solutions in the genetic algorithm. With this fitness score we can now run the genetic algorithm across populations of circuit to download mappings. The main parameters for the genetic

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1 The proof for this is shown in Appendix A
algorithm are the breed percentile \( b \), indicating the top \( b\% \) of solutions to pick from when breeding, the elite top \( e\% \) of solutions that get copied over into the next generation, and the mutation probability \( m \). For our purposes a mutation can occur when we are selecting a circuit for a single download, and we simply select a relay in the circuit to be replaced when another randomly chosen relay.

### 4.2 Online Algorithm

The offline genetic algorithm has access to the entire set of downloads when making circuit selection decisions. We want to see how it compares to an online algorithm, one that has to make circuit selection decisions serially. This way it has global knowledge of which active downloads are using which circuits, but has no idea how long downloads will last and does not know when future downloads start. While at first glance the online problem might seem similar to traditional routing, there are some problems using routing algorithm with respect to circuit selection in Tor. Typically routing algorithms depict the network as a graph in order to run efficient algorithms in order to extract which paths to select. Since relays, not links, are the resource constrained entities we can invert the graph, with each relay consisting of and in and out vertex and the edge between them representing the bandwidth of the relay. Additionally, since in Tor there is a link between every relay, for each relay we add an edge with infinite bandwidth connecting its out vertex with every other relays in vertex. But now if we want to run any traditional algorithms (e.g. max flow), there is not way to impose the requirement that all circuits must contain exactly three hops. In order to do this we would have to create a graph of all potential circuits, where each relay would have a separate vertex for when it can be a guard, middle, and exit relay. But not only does this explode our state space, there is now no way to make sure that when a relay is used as a guard, the available bandwidth on the middle and exit vertex decreases correspondingly.

So while we cannot use traditional routing algorithm for online circuit selection, we can borrow some techniques from routing algorithms that share similar goals. Specifically, for our online algorithm we borrow techniques used in the delay-weighted capacity (DWC) routing algorithm\[2\]. The goal of the DWC algorithm is similar to what we wish to accomplish for circuit selection, that is to find the path with the highest potential capacity. There are quite a few differences in the original DWC algorithm and what we need to accomplish, mainly that we are given a set of circuits to choose from and do not need to extract paths from a graph. The key insight we can use from the DWC algorithm is bottleneck identification and weight assignment. The algorithm first extracts a set of least delay paths, then for each path it identifies the bottleneck link, incrementing the links weight by the inverse bandwidth of the link. Then the DWC algorithm simply selects the path with the lowest sum of weights across all links in the path. By doing this it avoids the most critical links, those that are the bottlenecks providing the lowest amount of bandwidth.

To adopt the DWC routing algorithm for an online circuit selection algorithm, we need a method for identifying which relays are bottlenecks on active circuits. Algorithm\[3\] which estimates the bandwidth for all active circuits, can very easily be adapted for this purpose. Note that when we get the relay with the lowest per circuit bandwidth (line 4), we know that this relay is the bottleneck on each active circuit it appears on. So when we iterate through those circuits (line 7) and assign them the per circuit bandwidth value (line 8) we can also update the weight for the relay. By simply adding

\[
r.text \leftarrow r.text + \frac{1}{\text{circuitBW}}
\]

after the circuit bandwidth assignment (between lines 8 and 9). Now when the function is done each circuit will have a bandwidth and each relay will have a corresponding DWC weight. So to run the DWC circuit selection algorithm, we first iterate through the downloads in order based on the download start times. For each new download we get the set of current active downloads and their assigned circuits. We then run the updated algorithm that calculates active circuit bandwidth and relay DWC weights. Then, for each circuit we are considering for the new download, we calculate the circuit weight as the sum of relay weights in the circuit. Then the new download is assigned the circuit with the lowest calculated weight. If there are multiple circuits with the lowest weight we select the circuit with the highest available bandwidth, defined as the minimum across all relay available bandwidth values.

\[\text{2 Our experiments used parameters } b = 0.2 \text{ and } e = 0.1. \text{ Mutations were only allowed during some of the experiments and was set at } m = 0.01.\]
4.3 Circuit Sets

In both the offline and online circuit selection algorithms, each download has a set of potential circuits that can be selected from. For our experiments we consider three potential circuit sets. The first is the original 10 circuits that were available during the initial vanilla run. Along with extracting the fixed download start and end times from a completed vanilla experiment, we also record which circuits were available on the client when each download started. This allows us to specifically focus on circuit selection, keeping relay selection identical across all experiments. The second circuit set we consider is the full set of all potential valid circuits. We consider a circuit \( r_1, r_2, r_3 \) valid if \( r_3 \) is an actual exit relay, and if \( r_1 \neq r_2, r_2 \neq r_3, \) and \( r_1 \neq r_3 \). Note we do not require that \( r_1 \) actually has the guard flag set. This is because while there are mechanisms internally in Tor that prevent any use of a non-exit relay being used as an exit relay, there is nothing stopping a client from using a non-guard relay as their guard in a circuit. We also remove any “duplicate” circuits that contain the same relays, just in a different order. This is done since neither algorithm considers inter-relay latency, only relay bandwidth, so there is no need to consider circuits with identical relays.

The final circuit set we consider is the pruned circuit set. The motivation behind the pruned set can be seen in Figure 2. It shows that we can add an active circuit and actually reduce the amount of bandwidth being pushed through the Tor network. In the example shown it will always be better to use one of the already active circuits shown on the left instead of the new circuit seen on the right side of the graphic. There are two main ideas behind building the pruned circuit set: (1) when building circuits always use non-exit relays for the guard and middle if possible and (2) select relays with the highest bandwidth. To build a circuit to add to the pruned set, the algorithm first finds the exit relay with the highest bandwidth. If none exist, the algorithm can stop as no new circuits can be built. Once it has an exit, the algorithm then searches for the non-exit relay with the highest bandwidth, and this relay is used for the middle relay in the circuit. If one is not found it searches the remaining exit relays for the highest bandwidth relay to use for the middle in the circuit. If it still cannot find a relay, the algorithm stops as there are not enough relays left to build another circuit. The search process for the middle is replicated for the guard, again stopping if it cannot find a suitable relay. Now that the circuit has a guard, middle, and exit relay, the circuit is added to the pruned circuit set and the algorithm calculates the circuit bandwidth as the minimum bandwidth across all relays in the circuit. Each relay then decrements its relay bandwidth by the circuit bandwidth. Any relay that now has a bandwidth of 0 is permanently removed from the relay list. This is repeated until the algorithm can no longer build valid circuits.

4.4 Results

In this section we explore the performance of our offline genetic and online DWC circuit selection algorithms. To start we first configured and ran a large scale experiment running vanilla Tor with no modifications, while using the default client model. The results of this run were then used to generate an experiment using the fixed download model, where each download had the same start and end times that was seen in the vanilla experiment. We also extracted the original circuit sets for each download. These download times are then fed into both algorithms, along with which circuit sets to use, which then output a circuit selection for each download.

We ran the offline genetic and online DWC algorithms once for each circuit set, resulting in mappings of fixed downloads to circuits. Large scale experiments were run using each of the outputted circuit selections, and the results were compared to the original vanilla Tor experimental run. Recall since we are using the fixed download client model, the only metric we are concerned with is network usage, in this case the total client download bandwidth during the experiment. Figures 3a and 3b shows the total client bandwidth achieved with the genetic and DWC algorithms respec-

3 The pseudocode for this algorithm is shown in Appendix B.
Fig. 3. Total client bandwidth when using the offline and online circuit selection while changing the set of available circuits, along with the relay bandwidth capacity of the best results from both algorithms.

tively. The first thing to note is that both algorithms were able to produce relay and circuit selections that improved bandwidth by 72%, with median client bandwidth jumping from 86 MBps up to 147 MBps. To see why we can look at relay capacity seen in Figure 3c. This looks at the percent of bandwidth being used on a relay compared to their configured BandwidthRate. Note that this can go higher than 100% because Tor also has a BandwidthBurstRate that allows the relay to temporarily send more than the BandwidthRate over short periods of time. This shows us that in the best performing DWC and genetic algorithms, half of the time relays are using 90% or more of their configured bandwidth, which was only happening about 20% of the time in vanilla Tor. This shows that the algorithms were able to take advantage of resources that were otherwise left idle in vanilla Tor.

Interesting while both algorithms were able to achieve near identical maximum results, these came from different circuit sets. The genetic algorithm achieved its best results when using the pruned set while the DWC algorithm performed best with the full set of valid circuits. Furthermore, when the genetic algorithm used the full set and the DWC algorithm used the pruned set, they both actually performed worse when compared to vanilla Tor. We suspect that when using the full circuit set, the genetic algorithm gets stuck in a local maximum that it is unable to escape, and using the pruned circuit set prevents the algorithm from entering one of these suboptimal local maximums. When the algorithms were restricted to the original circuit sets both algorithms were able to improve performance compared to vanilla Tor. The offline genetic algorithm was able to increase client bandwidth to 111 MBps, while the online DWC algorithm saw performance results close to what was achieved with the full circuit set, with median client bandwidth at 138 MBps compared to 147 MBps with the full circuit set.

While the genetic algorithm serves as a lower bound on an optimal offline solution, given the near identical results seen when using the pruned circuit set in the genetic algorithm compared with the full circuit set in the DWC algorithm, we believe the online DWC algorithm is operating fairly close to an optimal solution, resulting in a much better allocation of resources. Most likely to see even more increased performance we would need to either increase the bandwidth resources available (e.g. add more relays or increase the bandwidth of existing relays), or add more clients to the network that can consume any remaining idle resources.

5 Decentralized Circuit Selection

In the previous section we saw that when making circuit selection in a centralized fashion, with a global view of the network, we were able to see large performance increases with resources being used much more efficiently. In this section we look at a decentralized solution that borrows techniques from the online DWC algorithm, allowing for clients to make smarter circuit selections locally. The main insight from the online DWC algorithm outlined in Section 4.2 is that we want to avoid bottleneck relays, particularly if they are low-bandwidth bottlenecks. The obvious advantage from using a centralized approach is we have a global view of active circuits and can accurately estimate where the bottlenecks are and how much bandwidth they are providing per circuit. In a decentralized setting no one entity has this global view that can determine who is and is not a bottleneck.
on each circuit. So instead we want a method for relays themselves to estimate which circuits they are a bottleneck on. With estimates of bottleneck circuits, they can locally compute their DWC weight and leak it to clients, allowing clients to make more intelligent circuit selection decisions.

5.1 Local Weight Computation

The main challenge in having relays locally compute their DWC weight is having the relays estimate which circuits they are the bottleneck on. To calculate bottleneck circuits, we make three observations: (1) to be a bottleneck on any circuit the relay’s bandwidth should be fully consumed (2) all bottleneck circuits should be sending more cells than non-bottleneck circuits, and (3) the number of cells being sent on bottleneck circuits should be fairly uniform. The last point is due to congestion control built into Tor, where the sending rate on a circuit should converge to the capacity of the bottleneck. So the two main things we need is a method for calculating the bandwidth of a circuit and a way to classify circuits as bottleneck or non-bottleneck based on their bandwidth value.

While calculating circuit bandwidth might seem easy, we need to be careful to treat bulk and web circuits equally. Bulk clients continuously send data through the circuit, so those should always be at or close to capacity. Web clients are more bursty, sending traffic in short periods and resting for an unknown amount of time. In order to assign a circuit a bandwidth we consider two parameters. First is the bandwidth window, in which we keep track of all traffic sent on a circuit for the past $w$ seconds. The other parameters is the bandwidth granularity, a sliding window of $g$ that we scan over the $w$ second bandwidth window to find the maximum number of cells transferred during any $g$ period. That maximum value is then assigned as the circuit bandwidth. With circuits assigned a bandwidth value we need a way to cluster the circuits into bottleneck and non-bottleneck groups. For this we look at three different clustering methods.

Jenks Natural Breaks: The Jenks natural breaks algorithm attempts to cluster one-dimensional data into classes that minimizes the in-class variance. The function takes in a one-dimensional array of data and the $n$ classes that the data should be clustered into. It returns the breaks $[b_1, b_2], [b_2, b_3], ..., [b_n, b_{n+1}]$ and a goodness of variance fit (GVF) $\text{var} \in [0, 1]$, where higher variances indicate better fits. For clustering circuits we can use the Jenks algorithm in two different ways. First is to cluster the data into 2 classes, with circuits in the $[b_1, b_2]$ range classified as non-bottlenecks and those in $[b_2, b_3]$ classified as bottlenecks. Second is we can keep incrementing the number of classes we cluster the circuits into until the GVF value passes some threshold $\tau$. Once the threshold is passed we then classify all circuits in the $[b_n, b_{n+1}]$ range as bottlenecks and everyone else as non-bottlenecks.

Head/Tail: The head/tail clustering algorithm [20] is useful when the underlying data has a long tail, which could be useful for bottleneck identification, as we expect to have a tight clustering around the bottlenecks with other circuits randomly distributed amongst the lower bandwidth values. The algorithm first splits the data into two sets, the tail set containing all values less than the arithmetic mean, and everything greater to or equal to the mean is put in the head set. If the percent of values that ended up in the head set is less than some threshold, the process is repeated using the head set as the new data set. Once the threshold is passed the function returns the very last head set as the head cluster of the data. The pseudocode for this algorithm is shown in Algorithm 3. For bottleneck identification we simply pass in the circuit bandwidth data and the head cluster returned contains all the bottleneck circuits.

Kernel Density Estimator: The idea behind the kernel density estimator [25, 27] is we are going to try and fit a multimodal distribution based on a Gaussian kernel to the circuits. Instead of using the bandwidth estimate for each circuit, the estimator takes as input the entire bandwidth history seen across all circuits, giving the estimator more data points to build a more accurate density estimate. For the kernel bandwidth we initially use the square root of the mean of all values. Once we have a density we compute the set of local minima $\{m_1, \ldots, m_n\}$ and classify every circuit with bandwidth above $m_n$ as a bottleneck. If the resulting density estimate is unimodal and we do not have any local minima, we repeat.

### Algorithm 3 Head/Tail clustering algorithm

1: function HeadTail(data, threshold)  
2: $m \leftarrow \text{sum(data)}/\text{len(data)}$  
3: head ← $\{d \in \text{data} | d \geq m\}$  
4: if $\text{len(head)}/\text{len(data)} < \text{threshold}$ then  
5: head ← $\text{HeadTail(head, threshold)}$  
6: end if  
7: return head  
8: end function
this process, halving the kernel bandwidth until we get a multimodal distribution.

With a method to calculate circuit bandwidth and identify bottleneck circuits, the relay can calculate their weight in the same way described in Section 4.2. We iterate over each bottleneck circuit and add the inverse of the bandwidth to the local weight calculation. This information can be leaked to clients, allowing them to select the circuit with the lowest weight summed across all relays in the circuit.

5.2 Implementation

Since the decentralized algorithm needs real-time bandwidth information, we implement the algorithm directly in Tor with clients making circuit selection decisions as downloads start. So while we had to use the fixed download client model in Section 4 in order to precompute circuit selection, for the decentralized algorithm we can use the default client model where clients have a single start time and web clients pause randomly between downloads they perform. While this means they algorithms need to run in real time, we can now allows include the time to first byte and file download time in our evaluation metrics.

To incorporate the decentralized algorithm in Tor we first implemented the local weight calculation method described in Section 4.1. As discussed previously the method has three parameters, bandwidth granularity $g$, bandwidth window $w$, and which clustering method to use. For each circuit relays keep a count of how many cells were read and written in each $100$ millisecond interval over the past $w$ seconds. Then to determine the circuit bandwidth it iterates through the circuit history to determine the maximum number of cells sent over a $g$ millisecond period. The circuit bandwidth values are then clustered into bottleneck circuits, with the weight calculated as the sum of inverse circuit bandwidth across all bottlenecks. Now that a relay can compute their own DWC weight, they need a way to leak this information to clients. For this we implement a new cell called a gossip cell. Each relay records its weight in a gossip cell and appends it on the downstream queue to the client for each circuit it has. Since the cell is encrypted no other relay will be able to modify the weight. To prevent gossip cells from causing congestion, relays send the gossip cells on circuits every $5$ seconds based on the circuit ID; specifically when $\text{now}() \equiv \text{circID} \mod 5$. For clients we modified the circuit_get_best function to select circuits with the lowest weight, using circuit bandwidth as the tie breaker. While compiling a list of “acceptable” circuits that Tor normally selected from, we additionally keep track of the minimal circuit weight across the acceptable circuits, and additionally filter out circuits with larger weight values. If there are multiple circuits remaining after this, we iterate through the remaining circuits selecting circuits with the highest circuit bandwidth, defined as the minimum advertised relay bandwidth across all relays in the circuit. If after this step we still have more than one circuit available we default to Tor’s circuit selection among the remaining circuits.

For the centralized DWC algorithm we can no longer precompute circuit selection since we are using the default client model, so we need a way to run the centralized algorithm during the experiment as clients start downloads. To accomplish this we introduce a new central authority node which clients will need to query for circuit selection decisions. When a client starts the central authority connects to the client using the Tor control protocol. It disables the client from making circuit selections decisions locally, and then listens on the CIRC and STREAM events. With the CIRC events the central authority know what circuits are available on every client in the network. The STREAM events then notify the central authority when a client begins a download, at which time the central authority can select which circuit the client has that they should use for the new download. Through the STREAM events the central authority is also notified when the stream has completed so the central authority can remove the circuit assigned to the stream from the active circuit list. With this the central authority obtains a global view of the network with knowledge of every current active circuit. So it can use the Algorithm 1 discussed in Section 4.2 to compute which relays are bottlenecks on each circuit, update the weight of relays accordingly, and select the lowest weighted circuit that clients have available.

5.3 Results

With the various parameters and clustering methods available to the decentralized algorithm, the first thing we are interested in is which parameters results in the most accurate bottleneck estimation. Since we are comparing the decentralized algorithm against the centralized DWC algorithm, we will use the central authorities bottleneck estimation as ground truth. To make the comparison we configured an experiment to run with the central authority making circuit selections. Every
Table 1. The bottleneck clustering methods mean square error across varying bandwidth granularity and window parameters, with red values indicating scores less than weighted random estimator.

![Graphs showing different download times and client bandwidth comparisons](image)

Fig. 4. Download times and client bandwidth compared across circuit selection in vanilla Tor, decentralized, and centralized algorithms.

time the central authority selects a circuit for a client it outputs the bottleneck estimation for every relay. Additionally, during the run all relays periodically output the entire 10 second bandwidth history for every circuit it is on. With this information we can compute which circuits every relay would have classified as a bottleneck depending on the parameters used. For comparing estimates, every time the central authority outputs their bottleneck figures, every relay that outputs their circuit history within 10 simulated milliseconds will have their local bottleneck estimate compared to the estimate produced by the central authority.

We are interested in the combination of parameters and clustering methods that minimizes the mean-squared error between the central authority and local relay bottleneck estimations. Table 1 shows the results of the different clustering methods with bandwidth granularity set at either 100 millisecond or 1 second, and with the bandwidth window varying between 1, 2, 5 and 10 seconds. For comparison we created a weighted random estimator that simply randomly selects from the entire set of estimates produced by the central authority. The weighted random estimator produced a mean-squared error of 1.33, which was better than almost half of all configurations. For example the Jenks two class clustering method consistently produced results worse than the weighted random estimator no matter what parameters were used. Across all clustering methods larger bandwidth windows resulted in higher mean-squared errors. This indicates that circuits that were inactive (and thus excluded from the central authorities calculation) were still being assigned a high bandwidth value and clustered into the bottleneck group. Interestingly, while the kernel density estimator produced the best results on average, the overall best parameter configuration was using the head/tail estimator with bandwidth granularity $g = 100\text{ms}$ and bandwidth window $w = 1s$.

Using these parameters we ran an experiment with the decentralized DWC algorithm, along with vanilla Tor and centralized DWC. Note that every experiment was configured to have relays send gossip cells every 5 seconds in addition with including a central authority connected to every client. In non-centralized experiments, when a download starts the central authority tells the client to select a circuit themselves instead of making the selection for them, and we had vanilla Tor ignore relay weights when making circuit selection. This is done to ensure that any latency and bandwidth overhead is identical across experiments and the only difference is the circuit selection algorithm being used. The results of all three experiments are shown in Figure 4. The centralized experiments show across the board improvements in every metric compared to vanilla Tor, with clients experiencing almost 20% higher bandwidth. While the decentralized experiment produced results slightly under the centralized, we still see improvements compared to vanilla Tor. Figure 4d shows client bandwidth still increasing 8% with the decentralized algorithm. Bulk clients in both the centralized and decentralized experiments achieved between 5% and 30%
faster downloads. While all web clients saw faster download times in the centralized experiment, the results from the decentralized experiment are slightly more mixed. 70% of web client downloads performed just as fast as they did in the centralized experiment, with 15% falling in between centralized and vanilla Tor. The final 15% of web client downloads experienced just slightly longer download times, with 1-2% increases compared to vanilla Tor. Time to first byte shown in Figure 4a shows the worst performance in the decentralized experiments, with roughly 25% of downloads experiencing an extra 1-2 seconds to receive their first byte. This could be caused by the fact that there is a delay in a relay updating its weight and all clients receiving it, causing clients to select relays that are temporarily congested resulting in poor times to first byte.

Along with the discussed parameters for the local weight computation, both the centralized and decentralized have an *implicit* parameter, the number of active circuits client should attempt to maintain. By default Tor clients keep around 10 active circuits it can select from. This is to make sure that there is always some circuit that can be used for downloads, and the client doesn’t have to incur the 5-10 second overhead of circuit creation when a download starts. For both the centralized and decentralized algorithms, having more circuits available to select from could increase performance. As was seen in Section 4.2 when we had the full set of circuits to select from, the centralized DWC algorithm produced the best results. To test this we configured the experiments to have clients maintain a set of 10, 25, and 50 available circuits. Results for these experiments are shown in Figures 5 and 6. Across almost every metric the decentralized algorithm saw worse results when using more circuits. This is particularly evident looking at client bandwidth in Figure 6d. When selecting from 50 circuits only about 40% of the time clients achieved a higher bandwidth, and using 25 circuits always resulted in worse performing clients. This is most likely due to the fact that increasing the number of circuits that clients create is interfering with the bottleneck classification algorithm, resulting in worse performance. For the centralized algorithm all download times remained fairly identical no matter how many circuits were available. The only metric that saw any difference was client bandwidth shown in Figure 6d. Here we see increasing the number of available circuits to 25 and 50 achieves an increase of 8% and 19% in client bandwidth compared to using only 10 circuits. Considering that no other metrics improved when increasing this is most likely due to overhead from maintaining circuits that is not being controlled for.
Fig. 7. (a) weight difference on relays compared to the difference in number of bottleneck circuits on the relay (b) CDF of the difference in percent of circuits a relay sees when a relay lies about their weight (c) percent of circuits a relay is on for the top 20 relays going from vanilla Tor to decentralized DWC (blue), in addition with the gains achieved when relays lie about their weight (green)

5.4 Privacy Analysis

With the addition of gossip cells and the new decentralized circuit selection protocol, there are some avenues that could be abused in an attempt to reduce client anonymity. The first major concern is that since the gossip cells purposefully leak information about the internal state of the relays, that this could open up a new side channel where an adversary can correlate client activity to change in information leaked to determine which relays are used by a client. The second issue is the increase in percent of circuits that relays end up being selected on, both from the change in selection algorithm and also due to the fact that an adversarial relay could abuse the protocol. This can be done by lying about their own weight or artificially inflating other relays weight, thereby increasing the chance they will be selected allowing them to observe a higher portion of the Tor network. In this section we examine the impact of some of these privacy issues.

5.4.1 Information Leakage

The first issue is that relays advertising their local weight calculation could leak some information to an adversary. Mittal et al. [21] showed how an adversary could use throughput measurements to identify bottleneck relays in circuits. Relay weight values could be used similarly, where an adversary could attempt to correlate start and stop times of connections with the weight values of potential bottleneck relays. To examine how much information is leaked by the weight values, every time a relay sent a GOSSIP cell we recorded the weight of the relay and the number of active circuits using the relay. Then for every two consecutive GOSSIP cells sent by a relay we then recorded the difference in weight and number of active circuits, which should give us a good idea of how well the change in these two values are correlated over a short period of time. Figure 7a shows the distribution of weight differences across various changes in clients actively using the relay. Since we are interested in times when the relay is a bottleneck on new circuits, we excluded times when the weight difference was 0 as this is indicative that the relay was not a bottleneck on any of the new circuits. This shows an almost nonexistent correlation between the two metrics, where we are just as likely to see a rise or drop in weight after more bottleneck circuits start using a relay. In the situation similar to the one outlined in [21], where an adversary is attempting to identify bottleneck relays used in a circuit, we are particularly interested in the situation where the number of active circuits using the relay as a bottleneck increases by 1. If there were large (maybe temporary) changes noticeable to an adversary they could identify the bottleneck relay. But as we can see in Figure 7a, the distribution of weight changes when client difference is 1 is very large, meaning it would be almost impossible to identify the bottleneck relay by just monitoring relay weights.

5.4.2 Relay Usage Changes

Since relays are still selected at random weighted by their bandwidth, over the long term relay usage should remain the same in vanilla Tor and when using the decentralized DWC circuit selection algorithm. Over the short term this might change though, since circuits are selected based on network information that could cause...
peeks in relay usage. In addition to short term fluctuations, an active adversarial relay could lie about their DWC weight, consistently telling clients they have a weight of 0, increasing the chances clients select circuits that the adversaries relay lies on. These changes could allow a relay to increase their view of the network without having to actually provide more bandwidth, potentially reducing anonymity of clients using Tor.

To determine the effect that the decentralized algorithm has on relay usage, we extracted all circuits selected by clients during each experiment. In addition, for the experiment running the decentralized circuit select algorithm we had clients output every circuit it considered, along with each relays weight. This allows us to perform a static analysis of which circuits would have been selected if a single relay was lying about their weight. In each situation we are interested in the change in the percent of circuits selected containing a specific relay, showing how much more of the network an adversary might be able to see. Figure 7b shows the CDF of the change in percent of circuits experienced by each relay. It shows the difference when going from vanilla Tor to regular decentralized DWC, in addition to going from vanilla Tor to decentralized DWC when relays lie about their weight. While the increases are generally small, the largest changes come from simply using the new decentralized algorithm, with 80% of relays seeing a small drop in the percent of circuits selected they end up on. Even the relays that see an increase almost never end up seeing more than 1% additional circuits. Furthermore there is very little gained when relays start lying about their weight, with most increases being an extra 0.1-0.2%. Figure 7c shows the 20 relays with the largest gains. These are the highest bandwidth relays in the network, providing 39% of the total available bandwidth. Here we see in the most extreme case, the very highest bandwidth relay could view 9% more circuits than they would have seen in vanilla Tor over the short term. Over the long term however, the only real gain coming from the relay lying about their weight would be capped at about 3-4%.

5.4.3 Denial of Service

While adversarial relays are limited in the number of extra circuits they can be placed on by lying about their weight, they still could have the ability to reduce the chances that other relays are selected. To achieve this they would need to artificially inflate the weight of other relays in the network, preventing clients from selecting circuits that the target relays appear on. Recall that the local weight calculation is based on how many bottleneck circuits the relay estimates they are on. This means that an adversary cannot simply just create inactive circuits through the relay to inflate their weight, those circuits would never be labeled as bottleneck. So to actually cause the weight to increase the adversary needs to actually send data through the circuits. To test the effectiveness, we configured an experiment to create 250 one-hop circuits through the target relay. After 15 minutes the one-hop circuits were activated, downloading as much data as they could. Note that we want as many circuits through the relay as possible to make their weight as large as possible. The relay weight is summed across all bottleneck circuits, $\sum bw(c_i)^{-1}$. If we have $n$ one-hop circuits through a relay of bandwidth $bw$, each circuit will have a bandwidth of roughly $\frac{bw}{n}$, so the weight on the relay will be $\sum_i\frac{bw}{n}^{-1} = \frac{n}{bw}$. 

Fig. 8. (a) attackers bandwidth and targets weight when an adversary is running the denial of service attack (b) number of clients using the target for a circuit (c) number of downloads completed by clients using the target relay before and after attack is started, shown both for vanilla Tor and using decentralized circuit selection.
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6 Conclusion

In this paper we explore the price of anarchy in Tor by examining how using offline and online algorithms for circuit selection is able to significantly increase performance to end clients. These algorithms are able to more efficiently utilize network resources, resulting in almost twice as much bandwidth being consumed. Furthermore, since the offline and online produced almost identical results, it suggests that that network utilization was near or at capacity. In addition we adapt the online DWC algorithm to create a decentralized version, where relays themselves are able to estimate calculations made by a central authority. While the results are not quite as dramatic, the decentralized algorithm was still able to achieve 20-25% higher bandwidth consumption when compared to vanilla Tor. Finally we analyze the potential loss in anonymity when using the decentralized algorithm, both against a passive and active adversary.

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A Bandwidth Algorithm Proof

Let $R$ be the relay selected with $B$ bandwidth and $C$ circuits. Let $R'$ be a different relay with $B'$ bandwidth and $C'$ circuits. By definition $R$ is selected such that $\frac{B}{C} \leq \frac{B'}{C'}$. When iterating through the $C$ circuits let $n$ be the number that $R'$ is on. Note that this means that $n \leq C'$. After the circuits have been iterated through and operations performed, $R'$ will have $B' - \frac{B}{C} \cdot n$ bandwidth left with $C' - n$ circuits.

Assume that $R'$ has 0 bandwidth afterwards, so $B' - \frac{B}{C} \cdot n = 0$. We want to show this means that $R'$ is on no more circuits so that $C' = n = 0$. We have

$$B' - \frac{B}{C} \cdot n = 0 \Rightarrow B' = \frac{B}{C} \cdot n \Rightarrow n = \frac{B' \cdot C}{B}$$

So that means that the number of circuits $R'$ is left on is

$$C' - n = C' - C - \frac{B' \cdot C}{B} = \frac{B \cdot C' - B' \cdot C}{B} = \frac{B \cdot C' - B' \cdot C}{B}$$

However, $R$ was picked such that

$$\frac{B}{C} \leq \frac{B'}{C'} \Rightarrow B \cdot C' \leq B' \cdot C \Rightarrow B \cdot C' - B' \cdot C \leq 0$$

This gives us

$$C' - n = \frac{B \cdot C' - B' \cdot C}{B} \leq 0$$

since we know $B > 0$ and

$$n \leq C' \Rightarrow 0 \leq C' - n$$

which implies that $0 \leq C' - n \leq 0 \Rightarrow C' - n = 0$. 

B Circuit Pruning Algorithm

Algorithm 4 Generate pruned circuit set

1: function BuildPrunedSet(relays)
2:     circuits ← List()
3:     while TRUE do
4:         if relays.len() > 3 then
5:             break
6:         end if
7:         if relays.numExits() > 0 then
8:             break
9:         end if
10:        relays.sortByBW()
11:        exit ← relays.getFirstExit()
12:        middle ← relays.getFirstNonExit()
13:        guard ← relays.getFirstNonExit()
14:        if middle == Null then
15:            middle ← relays.getFirstExit()
16:        end if
17:        if guard == Null then
18:            guard ← relays.getFirstExit()
19:        end if
20:        circuits.append(guard, middle, exit)
21:        bw ← mini(guard.bw, middle.bw, exit.bw)
22:        guard.bw ← guard.bw − bw
23:        middle.bw ← middle.bw − bw
24:        exit.bw ← exit.bw − bw
25:        if guard.bw == 0 then
26:            relays.remove(guard)
27:        end if
28:        if middle.bw == 0 then
29:            relays.remove(middle)
30:        end if
31:        if exit.bw == 0 then
32:            relays.remove(exit)
33:        end if
34:     end while
35:     return circuits
36: end function