TWO-LOOP SUPERSTRINGS VII

Cohomology of Chiral Amplitudes

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Abstract

The relation between superholomorphicity and holomorphicity of chiral superstring $N$-point amplitudes for NS bosons on a genus 2 Riemann surface is shown to be encoded in a hybrid cohomology theory, incorporating elements of both de Rham and Dolbeault cohomologies. A constructive algorithm is provided which shows that, for arbitrary $N$ and for each fixed even spin structure, the hybrid cohomology classes of the chiral amplitudes of the $N$-point function on a surface of genus 2 always admit a holomorphic representative. Three key ingredients in the derivation are a classification of all kinematic invariants for the $N$-point function, a new type of 3-point Green’s function, and a recursive construction by monodromies of certain sections of vector bundles over the moduli space of Riemann surfaces, holomorphic in all but exactly one or two insertion points.

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1 Introduction

A basic feature of superstrings in the Ramond-Neveu-Schwarz (RNS) formulation [1], is that the space-time theory arises from a two-dimensional $\mathcal{N} = 1$ supergravity theory on the string worldsheet [2]. The fields of this supergravity are the worldsheet metric $g_{mn}$ and gravitino field $\chi_{ma}$. Each equivalence class under all local symmetries - reparametrizations, local supersymmetry, Weyl, and super-Weyl invariance in the critical space-time dimension 10 - defines a supergeometry, and hence a super-complex structure on the worldsheet. The space of equivalence classes of supergeometries is supermoduli space, which is itself endowed with a supercomplex structure [3, 4]. The Chiral Splitting Theorem [5] guarantees that superstring amplitudes arise from a pairing of superholomorphic (right movers) and anti-superholomorphic (left movers) chiral amplitudes.

The notion of superholomorphicity on a worldsheet and its associated supermoduli space, and the notion of holomorphicity on the underlying bosonic worldsheet and its associated moduli space, are two different things, which do not coincide. Key physical properties of superstring amplitudes, such as the absence of unphysical singularities in the $S$-matrix, are intimately tied in with the holomorphic structure on an underlying bosonic worldsheet. As a consequence, the inherent holomorphic structure of strings has to be recovered from the distinct superholomorphic structure defined by the two-dimensional supergeometry. This latter notion (see [3, 4] and references therein) is still relatively unexplored, and the problem of recovering all the required holomorphic information from the superholomorphic structure has yet to be fully resolved.

Integrating out the Grassmann odd supermoduli [6], which are encoded in $\chi_{ma}$, provides a projection from a supergeometric structure to a purely bosonic geometry on the worldsheet. If this projection $(g_{mn}, \chi_{ma}) \to g_{mn}$ is carried out naively, by simply integrating over $\chi_{ma}$ at fixed $g_{mn}$, then the resulting holomorphic structure will fail to be invariant under local supersymmetry, and will thus fail to be defined intrinsically [7, 8]. The difficulties resulting from the naive projection $(g_{mn}, \chi_{ma}) \to g_{mn}$ were explored from many points of view in early studies of superstring perturbation theory [9, 10].

In [7, 8], it was proposed instead to fix a homology basis on the worldsheet, and to project the supergeometry $(g_{mn}, \chi_{ma})$ onto a bosonic geometry $\hat{g}_{mn}$, whose complex structure is defined by the super period matrix $\hat{\Omega}_{IJ}$. Since $\hat{\Omega}_{IJ}$ is invariant under local supersymmetry, this projection is well-defined for worldsheets of genus 2 throughout moduli space, and for arbitrary genus $h \geq 3$, away from a lower-dimensional subvariety. The super period matrix projection provides a consistent way of identifying the correct holomorphic structure defined by a supergeometry. This projection was applied successfully to the genus 2 superstring measure at fixed even spin structures in [11, 12, 13, 14, 15] producing the measure as a modular form on which the physical properties of proper factorization and finiteness could be checked explicitly.
Consistent projection of the measure should be viewed, however, only as a first step. Superstring amplitudes must similarly be projected onto holomorphic blocks with respect to $\hat{\Omega}_{IJ}$, and the major problem is to extract these from the superholomorphic amplitudes of the two-dimensional supergeometries. In [16, 17], this was solved for the the $N$-point function for massless NS bosons, for $0 \leq N \leq 4$, by exploiting the crucial simplification provided by the Gliozzi-Scherk-Olive (GSO) projection [18]. The GSO projection allows the extraction process to be carried out after averaging over spin structures, which resulted in many cancellations for $0 \leq N \leq 4$, and gave proofs of various non-renormalization theorems at two-loop order [17] and [19], whose validity had been conjectured earlier on general grounds [20, 21], and on the basis of string dualities, such as in [22, 23, 24].

The purpose of the present paper is to develop a concrete and systematic theory of the relation between the superholomorphic structures of chiral amplitudes defined by supergeometry data $(g_{mn}, \chi_{ma})$ and the holomorphic structures of the chiral amplitudes defined by the super period matrix $\hat{\Omega}_{IJ}$. Recall from [5] that the chiral $N$-point amplitude $\mathcal{F}[\delta]$ is a superholomorphic form in each of its vertex operator insertion points $z_r = (z_r, \theta_r)$ and in supermoduli. However, as was shown in [16], it incorporates forms of type

\[(0, 1)_r \otimes (0, 1)_s \quad (1.1)\]

in up to two insertion points $z_r$ and $z_s$. Thus the notion of $\partial_{z_r}$ is not well-defined on $\mathcal{F}[\delta]$, and it does not even make sense to speak of the holomorphicity of $\mathcal{F}[\delta]$. Nevertheless, we shall show in the present paper that $\mathcal{F}[\delta]$ admits the following decomposition,

\[\mathcal{F}[\delta] = \mathcal{Z}[\delta] + \mathcal{D}[\delta] \quad (1.2)\]

modulo Dirac $\delta$ functions at coincident points $^1$. Here, schematically, $\mathcal{Z}[\delta]$ is a holomorphic $(1, 0)$ form in each vertex insertion point, and $\mathcal{D}[\delta]$ is exact in one or two vertex insertion points, and holomorphic in all other vertex insertion points. The blocks $\mathcal{F}[\delta]$, $\mathcal{Z}[\delta]$, and $\mathcal{D}[\delta]$ are subject to certain monodromy properties, already familiar from [16], and which will be explained in detail in this paper. The exact piece $\mathcal{D}[\delta]$ will be immaterial in any physical superstring amplitude, and may be ignored. In this sense, the chiral amplitudes $\mathcal{F}[\delta]$ can be identified with the holomorphic amplitudes $\mathcal{Z}[\delta]$, and the holomorphic structure of superstrings has now been recovered.

As was already apparent in [17], the key to the extraction of the holomorphic blocks $\mathcal{Z}[\delta]$ is a new cohomology theory, in which certain cohomology classes would admit holomorphic representatives. We shall refer to it as the hybrid cohomology of chiral blocks, as the objects of interest are the chiral blocks of correlation functions in two-dimensional supergravity,

\[^1\text{By the “cancelled propagator argument”, in presence of the factor } E(z_i, z_j)^{k_i, k_j}, \text{ such Dirac } \delta \text{ functions do not contribute to the physical amplitudes. Mathematically, this follows from analytic continuation in the Mandelstam variables } s_{ij}, \text{ e.g. as was carried out explicitly for one-loop amplitudes in [25].}\]
and as the cohomology theory incorporates elements of both de Rham and Dolbeault cohomologies.

In this paper, we shall show that, for genus \( h = 2 \), the chiral amplitudes \( F[\delta] \) do admit holomorphic representatives, for arbitrary number \( N \) of external massless NS strings, and for each individual even spin structure \( \delta \). We shall show that, although intermediate calculations involve complicated combinatorics, the final result is remarkably simple. In fact, the structure of the various holomorphic blocks that enter may be schematically understood in terms of just a single fundamental building block. In the present paper, we develop an algorithm for the calculation of these blocks, and work out in detail the cases of \( D[\delta] \), which allows us to develop a constructive proof that each chiral amplitude does indeed admit a holomorphic representative with the correct monodromy properties. In the subsequent paper, we shall apply the same algorithm for computing explicitly also the holomorphic representatives \( Z[\delta] \). We turn next to a more precise description.

### 1.1 Hybrid cohomology of chiral blocks

The essentials of the hybrid cohomology theory which we need can be defined as follows.

Let \( \Sigma \) be a Riemann surface and let \( \hat{\Sigma} \) be the quotient of its universal cover by the commutator subgroup of the fundamental group \( \pi_1(\Sigma) \) of \( \Sigma \). For each fixed integer \( N \), let \( \hat{\Sigma}^N \) be the product of \( N \) copies of \( \hat{\Sigma} \), and \( \Gamma(\hat{\Sigma}^N, \Lambda^k) \), be the space of \( k \)-forms on \( \hat{\Sigma}^k \), which are smooth on \( \hat{\Sigma}^N \) away from the diagonal \( z_r = z_s \). The de Rham exterior differential \( d_r = dz_r \partial z_r + d\bar{z}_r \partial \bar{z}_r \) operates on \( \Gamma(\hat{\Sigma}^N, \Lambda^k) \),

\[
d_r : \Gamma(\hat{\Sigma}^N, \Lambda^k) \rightarrow \Gamma(\hat{\Sigma}^N, \Lambda^{k+1})
\]

A form \( F[\delta] \) in this cohomology theory is said to be **closed** if

\[
F[\delta] \in \bigcap_{r=1}^N \text{Ker} \, d_r \quad (1.4)
\]

A form \( D[\delta] \) in this cohomology theory is said to be **exact** if it is a closed form expressible as, modulo Dirac \( \delta \) functions supported only on the diagonal \( z_r = z_s \),

\[
D[\delta] = \sum_{r_1 < \ldots < r_\ell} d_{r_1} \cdots d_{r_\ell} S_{r_1 \ldots r_\ell}(z_1, \ldots, z_N), \quad \ell \geq 1
\]

with \( S_{r_1 \ldots r_\ell} \in \bigcap_{r \not\in \{r_1, \ldots, r_\ell\}} \text{Ker} \, d_r \). \hspace{1cm} (1.5)

The cohomology class of a closed form \( F[\delta] \) is its equivalence class modulo exact forms.

Given a complex structure on \( \Sigma \) and a closed form \( F[\delta] \), the main question is then whether the equivalence class of \( F[\delta] \), in terms of this cohomology theory, admits a representative of pure type \( (1,0) \) in each insertion point \( z_r \). Since a closed \((1,0)\) form on a
Riemann surface is automatically holomorphic, the fact that $F[\delta]$ is a closed form implies then that this representative must be holomorphic in all insertion points, away from the diagonal $z_r = z_s$. In practice, given a closed form $F[\delta]$, the existence of a pure $(1,0)$ representative implies that the leading $(0,1)_{r_1} \otimes \cdots \otimes (0,1)_{r_\ell}$ obstruction in $F[\delta]$ is of the form $\bar{\partial}_{r_1} \cdots \bar{\partial}_{r_\ell} S_{r_1 \cdots r_\ell}[\delta]$. Replacing $F[\delta]$ by $F[\delta] - d_{r_1} \cdots d_{r_\ell} S_{r_1 \cdots r_\ell}[\delta]$, we can iterate the process and recognize the problem as equivalent to all the $(0,1)$ obstructions being successively in the range of a product of $\bar{\partial}_r$ operators. In this sense, the problem of finding a holomorphic representative in the present cohomology theory is a hybrid mixture of de Rham and Dolbeault cohomologies.

Our main result is that the answer is affirmative for the chiral amplitudes $F[\delta]$ of the $N$-point function, when the complex structure of $\Sigma$ is defined by the super period matrix $\hat{\Omega}_{IJ}$. In fact, both the exact differential $D[\delta]$ and the resulting holomorphic representative $Z[\delta]$ exhibit a rich structure that can be summarized in the following theorem:

**Main Theorem**

Let the worldsheet $\Sigma$ be of genus $h = 2$, equipped with a supergeometry $(g_{mn}, \chi_{m\alpha})$. For any fixed even spin structure $\delta$ on $\Sigma$, and any positive integer $N$, let $F[\delta]$ be the chiral amplitude of the $N$-point function for scalar superfields, as given in (1.12) in the section below. Let the complex structure on $\Sigma$ be defined by the super period matrix $\hat{\Omega}_{IJ}$, as given explicitly in (2.17). Then there exist forms $S_r[\delta] \in \bigcap_{t \neq r} \text{Ker} \, d_t$, $S_{rs}[\delta] \in \bigcap_{t \neq r, s} \text{Ker} \, d_t$, \hspace{1cm} (1.6)

smooth away from the diagonal $z_r = z_s$, such that, modulo Dirac $\delta$ functions supported only on the diagonal, we have

$$F[\delta] = Z[\delta] + \sum_r d_r S_r[\delta] + \sum_{rs} d_r d_s S_{rs}[\delta], \hspace{1cm} (1.7)$$

where $Z[\delta]$ is a holomorphic form of pure type $(1,0)$ in each insertion point $z_r$. More precisely, the forms $S_r[\delta]$ and $S_{rs}[\delta]$ are linear combinations of certain basic forms $\Pi^{(n+2)}$, $\Pi_f^{(n+1)}$, $\Pi_\pm^{(m+1)(\ell+1)}$, \hspace{1cm} (1.8)

with coefficients given by certain basic kinematic invariants $K^\mu_{[t_1 \cdots t_n]}$, $K^{(m+1)(\ell+1)}_\pm$, \hspace{1cm} (1.9)

(see sections §4 and §11 below). In particular, the forms $S_r[\delta]$ and $S_{rs}[\delta]$ are actually $(1,0)$ forms, and hence holomorphic, in all insertion points $z_t$ for $t$ different from $r$ and $r, s$ respectively.

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1.2 Differential structure of the chiral amplitudes $\mathcal{F}[\delta]$

In this subsection, we shall review the key results of [16]. The $N$-point chiral amplitudes $\mathcal{F}[\delta]$ are naturally differential forms on $\Sigma^N$, and more precisely they are a 1-form in each insertion point $z_i$. This may be traced back to the fact that the expression for the un-integrated vertex operator [26] for a massless NS-NS string of momentum $k^\mu$ and polarization tensor $\epsilon^{ik}e^\nu$ is given in terms of the matter scalar superfield $X^\mu = x^\mu + \theta\psi_+^\mu + \bar{\theta}\psi_-^\mu$ as follows,

$$V(z, \bar{z}, \theta, \bar{\theta}, \epsilon, k) = \epsilon^\mu\epsilon^\nu|dzd\theta|^2(s\text{det} E_M^A)D_+X^\mu D_-X^\nu e^{ik \cdot X}$$

(1.10)

Chiral splitting of this vertex must be carried out with care, since $(s\text{det} E_M^A)$ is non-trivial on a genus 2 worldsheet, and depends on the underlying supergeometry $(g_{mn}, \chi_{ma})$. The resulting chiral vertex is the sum of three contributions,

$$\mathcal{V}^{(0)}(z; \epsilon, k) = \epsilon^\mu dz(\partial_z x_+^\mu - ik^\nu \psi_+^{\mu \nu}(z)) e^{ik \cdot x_+(z)}$$

$$\mathcal{V}^{(1)}(z; \epsilon, k) = -\frac{1}{2}\epsilon^\mu d\bar{z}\chi_+^{+ \mu}(z) e^{ik \cdot x_+(z)}$$

$$\mathcal{V}^{(2)}(z; \epsilon, k) = -\epsilon^\mu \hat{\mu} \hat{z} d\bar{z}(\partial_z x_+^\mu - ik^\nu \psi_+^{\mu \nu}(z)) e^{ik \cdot x_+(z)}$$

(1.11)

Here, $\mathcal{V}^{(0)}$ is the familiar vertex operator ($x_+(z)$ and $\psi_+(z)$ denote respectively the effective chiral boson and the chiral fermion components of the superfield $X^\mu$, as used, for example, in [16]), and $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ are corrections which depend both on the gravitino slice $\chi_+^+$ and the Beltrami differential $\hat{\mu} \hat{z}$ for the passage from period matrix to super period matrix. In all these expressions, $z$ is a holomorphic coordinate for the complex structure defined by the super period matrix $\Omega_{IJ}$, and not for the complex structure defined by the original bosonic metric $g_{mn}$ in the supergeometry $(g_{mn}, \chi_{ma})$.

We stress that $\mathcal{V}^{(0)}$ is a $(1, 0)$ form, but that $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ are $(0, 1)$ forms, and hence the full chiral vertex is a differential 1-form in the vertex insertion point $z$, containing forms of both types $(1, 0) \oplus (0, 1)$. Although $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ are the source of complications, their omission would certainly lead to unacceptable gauge-dependent results for the final superstring amplitudes.

The chiral amplitudes $\mathcal{F}[\delta]$ are correlators of the full chiral vertex operators, and thus receive contributions from 3 different types of differential forms,\(^2\)

$$\mathcal{F}[\delta] = \left\langle Q(p_I) \left( \frac{1}{8\pi^2} \int \chi S_m \int \chi S_m + \frac{1}{2\pi} \int \hat{\mu} T_m \right) \prod_{l=1}^N \mathcal{V}^{(0)}_l \right\rangle(c)$$

\(^2\)The correlators on the first line of $\mathcal{F}[\delta]$ are connected, as indicated by the subscript $(\cdots)_c$, in the following sense. To be excluded are all self-contractions of $T_m$, as well as the contributions in which both $x_+$ and $\psi_+$ are contracted between the two supercurrents $S_m S_m$. To be included are all the contractions of only a single field between the two $S_m$-operators, with the remaining operators contracted elsewhere.
\begin{align}
+ \sum_{r=1}^{N} \left< Q(p_I) \left( \frac{1}{2\pi} \int \chi S_m \nu_r^{(1)} + \nu_r^{(2)} \right) \prod_{t \neq r}^{N} \nu_t^{(0)} \right>
+ \frac{1}{2} \sum_{r \neq s}^{N} \left< Q(p_I) \nu_r^{(1)} \nu_s^{(1)} \prod_{t \neq r, s}^{N} \nu_t^{(0)} \right> \tag{1.12}
\end{align}

Here, \( S_m \) is the worldsheet supercurrent, and \( T_m \) is the stress tensor for the matter fields \( x_+ \) and \( \psi_+ \). The chiral amplitude is evaluated at fixed even spin structure \( \delta \) and fixed internal loop momenta \( p_I \), as guaranteed by the insertion of the operator

\[ Q(p_I) = \exp \left\{ ip_I^{\mu} \oint_{B_I} dz \partial_{\bar{z}} x_+^\mu(z) \right\}. \tag{1.13} \]

All quantities above are expressed with respect to the super periods \( \hat{\Omega}_{IJ} \). We recall the relations between the period matrix \( \Omega_{IJ} \), the super period matrix \( \hat{\Omega}_{IJ} \) and the Beltrami differential \( \hat{\mu}_{\bar{z}}z \) which provides the complex structure deformation between \( \Omega_{IJ} \) and \( \hat{\Omega}_{IJ} \),

\[ \Omega_{IJ} - \hat{\Omega}_{IJ} = i \int d^2 z \hat{\mu}_{\bar{z}}z \omega_I(z) \omega_J(z) \tag{1.14} \]

The full chiral amplitude \( \mathcal{B}[\delta] \) for \( N \) massless NS bosons, incorporating the chiral measure, ghost, and superghost contributions was derived in [16], and is given by\(^3\)

\[ \mathcal{B}[\delta](z; \epsilon, k, p_I) = d\mu_2[\delta] \left< Q(p_I) \prod_{t=1}^{N} \nu_t^{(0)} \right> + d\mu_0[\delta] \int_\zeta \mathcal{F}[\delta] \tag{1.15} \]

The integration \( \int_\zeta \) is over the odd super-moduli \( \zeta^\alpha, \alpha = 1, 2 \); the prefactors \( d\mu_0[\delta] \) and \( d\mu_2[\delta] \) are components of the chiral measure, defined and evaluated in [12]; their explicit form will not be needed here. The chiral blocks \( \mathcal{F}[\delta] \) and \( \mathcal{B}[\delta] \) are 1-forms (including both \((1, 0)\) and \((0, 1)\) components) in each vertex point \( z_r \) with the following monodromy,

\[ \mathcal{F}[\delta](z_r + \delta_{rs} A_K; \epsilon_r, k_r, p_I) = \mathcal{F}[\delta](z_r; \epsilon_r, k_r, p_I) \]
\[ \mathcal{F}[\delta](z_r + \delta_{rs} B_K; \epsilon_r, k_r, p_I) = \mathcal{F}[\delta](z_r; \epsilon_r, k_r, p_I - 2\pi\delta_{IK}k_s) \tag{1.16} \]

Thus they should be viewed as sections of a flat vector bundle over the moduli space of Riemann surfaces with \( N \)-punctures.

The following results on the differential structure of the chiral amplitudes \( \mathcal{F}[\delta] \) and \( \mathcal{B}[\delta] \) were proven in [16],

\textbf{(a) Closedness:} The forms \( \mathcal{F}[\delta] \) and \( \mathcal{B}[\delta] \) are closed in each variable \( z_j \);

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\(^3\)Strictly speaking, of the integral of \( \mathcal{F}[\delta] \) with respect to all \( \theta_r \) variables. To lighten the terminology, we shall not insist on this distinction when there is no possibility of confusion.
(b) **Slice-change:** Under infinitesimal changes of either the gravitino slice $\chi$ or the Beltrami differential $\hat{\mu}$, the forms $F[\delta]$ and $B[\delta]$ change by terms which are de Rham $d$-exact in one variable and de Rham $d$-closed in all other variables,

$$B[\delta](z; \epsilon, k, p_I) \to B[\delta](z; \epsilon, k, p_I) + \sum_{r=1}^{N} d_r \mathcal{R}_r[\delta](z; \epsilon, k, p_I)$$ (1.17)

Specifically, $\mathcal{R}_r[\delta]$ is a form of weight $(0, 0)$ in $z_r$, and a form of weight $(1, 0) \oplus (0, 1)$, which is de Rham closed, in each $z_s$ for $s \neq r$; Finally, $\mathcal{R}_r[\delta]$ has the same monodromy as $B[\delta]$.

Henceforth, we concentrate on the problem of finding a holomorphic representative within the cohomology classes of $B[\delta]$ and $F[\delta]$ in the sense of section §1.1. Since the first expression on the right hand side of the equation (1.15) is manifestly a holomorphic $(1, 0)$-form in each insertion point (away from the diagonal), $B[\delta]$ and $F[\delta]$ have the same non-holomorphic terms, and we can speak interchangeably of the existence of a holomorphic representative in either class. In [17], this problem was solved for $0 \leq N \leq 4$, and after summation over the even spin structures $\delta$. Here we shall treat the case of general $N$, and for each fixed spin structure $\delta$ separately.

### 1.3 The main $(0, 1)_r \otimes (0, 1)_s$ obstruction

The central result of the present paper is the development of an explicit algorithm for the construction of the differential blocks $d_r \mathcal{S}_r[\delta]$ and $d_r d_s \mathcal{S}_{rs}[\delta]$ for a general chiral $N$-point amplitude $F[\delta]$, so that the difference

$$F[\delta] - \sum_r d_r \mathcal{S}_r[\delta] - \sum_{[rs]} d_r d_s \mathcal{S}_{rs}[\delta]$$ (1.18)

is a pure $(1, 0)$ form in all insertion points. A crucial requirement of the algorithm is to make sure that $\mathcal{S}_r[\delta]$ and $\mathcal{S}_{rs}[\delta]$ are closed forms in any insertion point $z_t$ for $t$ different from $r$ and $r, s$ respectively. It is as a consequence of this requirement that the difference (1.18) is closed and a pure $(1, 0)$-form, and thus automatically holomorphic in all insertion points.

We establish in this manner the existence of the holomorphic representative $Z[\delta]$ for the blocks $F[\delta]$, and $\mathcal{H}[\delta]$ for the blocks $B[\delta]$. These holomorphic representatives have themselves a very rich structure, the derivation of which is postponed until a subsequent paper.

Although the intermediate calculations to isolate the blocks $d_r \mathcal{S}_r[\delta]$, $d_r d_s \mathcal{S}_{rs}[\delta]$, and $\mathcal{H}[\delta]$ will be quite involved, the final results are remarkably simple, and may all be related to one fundamental block.

To see how this comes about, we start from the chiral amplitude $F[\delta]$ in (1.12). Its first line is a $(1, 0)$ form in each insertion point, and these terms will not contribute to
blocks of the type $d_r S_r[\delta]$ and $d_r d_s S_{rs}[\delta]$, since they have no $(0,1)$ components. Its second line is a sum of forms which is of type $(1,0)$ in all but one insertion point, while the third line is a sum of forms of type $(1,0)$ in all but two insertion points. The last term is the top obstruction term, in that it exhibits the highest degree in $(0,1)$ forms. This term will determine $d_r d_s S_{rs}[\delta]$. It is natural to start by examining this term, and investigate how its $(0,1)_r \otimes (0,1)_s$ component can be recast in the form of an exact differential. The correlator in question is

$$\left\langle Q_{(1)} \mathcal{V}_r(1) \mathcal{V}_s(1) \prod_{l \neq r,s} \mathcal{V}_l(0) \right\rangle \quad (1.19)$$

The proportionality of $\mathcal{V}_r(1)$ to $\chi(r) \psi_+(r)$ forces this correlator to always contain a linear chain of RNS fermion $\psi_+$ contractions. Therefore, the above correlator will always contain a linear chain of Szegő kernels $S$, arranged as follows,

$$\frac{1}{4} d\bar{r} \chi(r) S(r,t_1) S(t_1,t_2) \cdots S(t_{n-1},t_n) S(t_n,s) \chi(s) d\bar{s} \quad (1.20)$$

Here, the points $t_1, t_2, \cdots, t_{n-1}, t_n$ are all distinct from one another and distinct from $r$ and $s$, as is guaranteed by the structure of the Wick contractions for the free field $\psi_+$. The remainder of the correlator which multiplies each such linear chain is manifestly holomorphic in all vertex insertion points.

This form in $(0,1)_r \otimes (0,1)_s$ can be recast in terms of a double exact differential in $r$ and $s$ provided there exists a function $\Pi^{(n+2)}(r,t_1,\cdots,t_n; s)$, which is a $(0,0)$ form in $r$ and $s$, and a $(1,0)$ form in $t_1, \cdots, t_n$, and is such that

$$\partial_r \partial_s \Pi^{(n+2)}(r,t_1,\cdots,t_n; s) = \frac{1}{4} \chi(r) S(r,t_1) S(t_1,t_2) \cdots S(t_{n-1},t_n) S(t_n,s) \chi(s) \quad (1.21)$$

The blocks $\Pi^{(n+2)}(r,t_1,\cdots,t_n; s)$ must be holomorphic in each of the points $t_1, \cdots, t_n$, away from coincident points $t_i = t_j$ for $i \neq j$, and away from the points $r$ and $s$. Finally, the blocks must have the same mirror symmetry as the correlator, so that we require

$$\Pi^{(n+2)}(r,t_1,\cdots,t_n; s) = (-)^n \Pi^{(n+2)}(s,t_n,\cdots,t_1; r) \quad (1.22)$$

Although one might initially have hoped that blocks $\Pi^{(n+2)}$ would exist without monodromy in $r, s$, it turns out to be impossible to achieve without introducing new singularities at extraneous points. Instead, we may require that its monodromy in $r$ is independent of $r$, and the monodromy in $s$ is independent of $s$. This non-trivial monodromy of $\Pi^{(n+2)}$, instead of causing a problem, will combine precisely with the monodromies of the $(0,1)$ form obstructions and help lift those as well.

The fundamental blocks $\Pi^{(n+2)}(r,t_1,\cdots,t_n; s)$ have the connectivity of a linear chain, and will be constructed explicitly in terms of bosonic and fermionic worldsheet Green functions in the body of the paper.
1.4 Recursive relations through monodromy

Under monodromy in their endpoints $r, s$, the fundamental blocks $\Pi^{(n+2)}(r; t_1, \cdots, t_n; s)$ produce new linear chain blocks $\Pi_I^{(n+1)}$ and $\Pi_{IJ}^{(n)}$, whose roles will be as follows,

$$\Pi^{(n+2)}(r; t_1, \cdots, t_n; s) \quad \text{contributes to } \mathcal{S}_{rs}[\delta]$$

$$r \rightarrow r + B_I \text{ monodromy}$$

$$\Pi_I^{(n+1)}(t_1; \cdots, t_n; s) \quad \text{contributes to } \mathcal{S}_s[\delta]$$

$$s \rightarrow s + B_J \text{ monodromy}$$

$$\Pi_{IJ}^{(n)}(t_1; \cdots, t_n) \quad \text{contributes to } \mathcal{H}[\delta] \quad (1.23)$$

These linear chain blocks are schematically depicted in Figure 1 below.

The fundamental blocks $\Pi^{(n+2)}(r; t_1, \cdots, t_n; s)$ may also be linked, by letting one endpoint, say $r$, coincide with the other endpoint $s$, or with one of the midpoints $t_1, \cdots, t_n$ of the linear chain. This produces singly linked chains, consisting of a single loop with $\ell + 1$ points $r, u_1, \cdots, u_\ell$ connected, at the point $r$, to a linear chain with $m + 2$ points $s, t_1, \cdots, t_m, r$.

By themselves, the singly linked blocks obtained this way will not be holomorphic in the point $r$, but we shall show that suitable “counter-terms” may be added to restore
holomorphicity at $r$, while maintaining holomorphicity in all $u$- and $t$-variables. It will be natural to “symmetrize” the resulting blocks, so that

$$\Pi_{\pm}^{(m+1|\ell+1)}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) = \pm (-)^{s} \Pi_{\pm}^{(m+1|\ell+1)}(s; t_1, \ldots, t_m, r, u_\ell, \ldots, u_1) \quad (1.24)$$

From the singly linked blocks $\Pi_{\pm}^{(m+1|\ell+1)}$, we obtain new holomorphic blocks by taking the monodromy in $s$. This process may be schematically represented as follows,

$$\Pi^{(m+\ell+3)}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell; v) \quad \text{linear chain}$$

$$s \rightarrow s + B_I \quad \text{monodromy}$$

$$\Pi_{\pm}^{(m|\ell+1)}(t_1; \ldots, t_m, r, u_1, \ldots, u_\ell) \quad \text{contributes to } \mathcal{H}[\delta] \quad (1.25)$$

The connectivity of the resulting singly linked chains is depicted in Figure 2 below.

![Diagram](image)

Figure 2: Connectivity of the singly linked chain blocks $\Pi_{\pm}^{(m+1|\ell+1)}$, $\Pi_{\pm}^{(m+1|\ell+1)}$; they are holomorphic at all points, except at the points represented by red squares.

Finally, the fundamental blocks $\Pi^{(n+2)}(r; t_1, \ldots, t_n; s)$ may also be linked by letting both end points coincide with distinct midpoints. These blocks only contribute to $\mathcal{H}[\delta]$ and will be calculated and studied in the subsequent paper.
1.5 Kinematic invariants

A fundamental component of the algorithm for finding $S_\epsilon[\delta]$ and $S_{rs}[\delta]$ is the identification of all kinematic invariants arising in the $N$-point function. These are discussed in detail in section §4, but we would like to single out some particularly important features in this Introduction.

The key quantity is the following odd Grassmann valued differential form of even degree, which is vector valued in Minkowski space-time,

$$K_\mu^r \equiv \epsilon_\mu^r \wedge dz_r + ik_\mu \theta_r = \epsilon^\mu d\theta_r \wedge dz_r + ik_\mu \theta_r$$  \hspace{1cm} (1.26)

Here $\epsilon^\mu_r$ is the physical polarization vector associated with external state $r$, and $k^\mu_r$ is the momentum of that state. In the body of the paper, we shall often use the composite $\epsilon^\mu_r = \epsilon^\mu d\theta_r$. As before, the coordinates of the vertex insertion point $r$ are given by $(z_r, \theta_r)$ and its complex conjugate.

The bilinears in $K_r$ give rise immediately to the gauge-invariant field strengths,

$$K_\mu^r K_\nu^r = i(\epsilon_\mu^r k_\nu^r - \epsilon_\nu^r k_\mu^r) dz_r \theta_r = i(\epsilon_\mu^r k_\nu^r - \epsilon_\nu^r k_\mu^r) d\theta_r \wedge dz_r \theta_r.$$  \hspace{1cm} (1.27)

Then all kinematic invariants are given either by linear chains in $K_\mu^r$ of the form

$$K_{[t_1 t_2 \ldots t_n]}^{\mu\nu} = K_{t_1}^{\mu} \left( \prod_{k=1}^{n-1} K_{t_k}^{\rho k} K_{t_{k+1}}^{\rho k} \right) K_{t_n}^{\nu}$$  \hspace{1cm} (1.28)

(contractured with either $p_\mu^r$, $ik_\mu^r$, or $\epsilon_\mu^r \theta_r$), or singly linked chains, of which there are two types $K_{(m+1)(\ell+1)}$, symmetric and anti-symmetric,

$$K_{(m+1)(\ell+1)}^+ = \frac{1}{2} \epsilon_\mu \theta_r \left( K_{[t_1 \ldots t_m]}^{\mu \rho} + (-)^\ell K_{[t_1 \ldots t_m]}^{\rho \mu} \right) ik_\rho$$
$$K_{(m+1)(\ell+1)}^- = \epsilon^\mu_\theta \epsilon^\rho_\theta \left( K_{[t_1 \ldots t_m]}^{\mu \rho} + (-)^\ell K_{[t_1 \ldots t_m]}^{\rho \mu} \right) ik_\rho$$  \hspace{1cm} (1.29)

Thus, the full chiral amplitude becomes a sum of such kinematic factors, with coefficients which are the holomorphic or differential blocks $\Pi$, $\Pi_I$ or $\Pi_\pm$.

This decomposition into sums of kinematic factors times basic Lorentz scalar amplitudes seems reminiscent, to some degree, of the color and helicity decomposition of scattering amplitudes in massless Yang-Mills theory (see e.g. [27], as well as the developments in [28, 29], and [30] where the subject is reviewed. While the Yang-Mills amplitudes, stripped of their color and helicity dependence, are found to obey certain analyticity and recursion relations in momentum space, our two-loop chiral amplitudes, stripped of their polarization tensor kinematic factors, are found to obey certain holomorphicity and recursion relations on the string worldsheet. It is tempting to imagine that this similarity has a real, and perhaps practical, significance.
1.6 Organization of the paper

The remainder of the paper is organized as follows.

Section §2 is devoted to a brief summary of two-dimensional supergeometry, the chiral splitting theorem, and the function theory which we need, including super Abelian differentials, the super period matrix, and the super prime form.

In section §3, we discuss simple examples of how the cohomology class of a superholomorphic object can contain a holomorphic representative. The most basic case is that of a superholomorphic Abelian differential, which already played a major role in the derivation in [11, 12, 13, 14] of the superstring measure. More complicated examples are the super prime form and its derivatives.

In section §4, all the kinematic invariants are derived.

In section §5, we provide a first splitting of the chiral amplitudes $\mathcal{F}[\delta]$ into differentials and remaining terms. The underlying combinatorics are explained in some detail. At this time, the differentials are not holomorphic in the remaining insertion variables, and are not yet the terms $d_r\mathcal{S}_r[\delta]$ and $d_rd_s\mathcal{S}_{rs}[\delta]$ that we seek. In order to eliminate the $(0, 1)$ terms, we shall need to solve $\bar{\partial}$ equations, with right hand side having non-trivial monodromies.

In section §6, we discuss such equations systematically. It turns out that all the equations we encounter can be solved using a unique function $Q_0(s; x, y)$, which can be viewed as a kind of Green’s function in 3 variables $s, x,$ and $y$. In preparation for the construction of the differentials $\mathcal{S}_r[\delta]$ and $\mathcal{S}_{rs}[\delta]$, we construct all the differential blocks that we need: first, the linear chain blocks in section §7, and then the singly linked chain blocks in section §8. As explained previously, the key starting block is $\Pi$, from which all other blocks descend by recursive monodromy relations.

In sections §9 and §10, the differentials obtained in the preliminary decomposition of section §5 can then be re-assembled, after solving suitable $\bar{\partial}$ equations, into terms of the form $d_r\mathcal{S}_r[\delta]$ and $d_rd_s\mathcal{S}_{rs}[\delta]$, this time with forms which are $(1, 0)$ and holomorphic in any insertion point $z_t$ with $t$ different from $r$ and $r, s$ respectively. Thus these are the hybrid exact differentials that we seek.

In section §11, we present a summary of the resulting formulas for $\mathcal{S}_r[\delta]$ and $\mathcal{S}_{rs}[\delta]$.

Finally, we should note that there is a very extensive literature on superstring perturbation theory. Some representative papers are the early works in [31, 32, 33], with the case of genus 2 considered in e.g. [34], and more recently in [35, 36, 37, 38, 39]. The pure spinor approach to multiloop superstring amplitudes is developed in [40]. Recent proposals for the superstring measure and scattering amplitudes beyond two-loop order may be found in [41], [42], and [43]. We refer to these papers, as well as to the earlier papers [12, 16, 17] of this series, for a fuller list of references on the subject.
2 Chiral $N$-point Amplitudes for Massless NS Strings

In this section, we present the basic facts about two-dimensional supergeometries and the correlators of chiral scalar superfields. The corresponding results were originally derived in [5]. Our goal here is to provide a succinct summary, in a form convenient for the extensive study of the chiral $N$-point amplitudes that will follow in the sequel of this paper. The main point to be brought forward in this section is that a supergeometry defines a notion of superholomorphicity, and that the $N$-point chiral amplitudes for massless NS strings are superholomorphic and expressible in terms of superholomorphic quantities which are analogous to standard objects from the theory of Riemann surfaces such as period matrices, Abelian differentials, and prime forms.

2.1 Function theory on a Riemann surface

The starting point is an orientable compact Riemann surface $\Sigma$ of genus 2 and with fixed even spin structure $\delta$. Let $(A_I, B_I), I = 1, 2,$ be a fixed choice of canonical homology basis for $\Sigma$, with intersection numbers $\#(A_I \cap A_J) = \#(B_I \cap B_J) = 0, \#(A_I \cap B_J) = \delta_{IJ}$. The worldsheet metric $g_{mn}$ defines a complex structure on $\Sigma$. The key ingredients of the associated complex function theory are the Abelian differentials $\omega_I(z)$, the period matrix $\Omega_{IJ}$, the Szegő kernel $S_\delta(z, w)$, the prime form $E(z, w)$, and the Green’s function $G(z, w)$.

The Abelian differentials $\omega_I, I = 1, 2,$ are the basis of holomorphic $(1,0)$-forms with respect to the metric $g_{mn}$ dual to the cycles $A_I$, and the period matrix $\Omega_{IJ}$ is the matrix of their periods around the $B_J$ cycles,

$$\int_{A_J} \omega_I = \delta_{IJ}, \quad \int_{B_J} \omega_I = \Omega_{IJ} \quad (2.1)$$

The three periods $\Omega_{IJ} = \Omega_{JI}$ are local coordinates on the genus two moduli space $\mathcal{M}_2$ of compact Riemann surfaces.

The Szegő kernel $S_\delta(z, w)$ is the Green’s function for the $\bar{\partial}$ operator on spinors of weight $(1/2, 0)$, and spin structure $\delta$, and obeys,

$$\partial_z S_\delta(z, w) = 2\pi \delta(z, w) \quad (2.2)$$

Throughout this paper, the even spin structure is fixed to be $\delta$, and we shall drop the subscript $\delta$ from the Szegő kernel, and simply denote it by $S(z, w) \equiv S_\delta(z, w)$.

The prime form $E(z, w)$ is a holomorphic function of weight $(-1/2, 0)$ in each variable $z$ and $w$ on the universal cover of $\Sigma$, which behaves as $E(z, w) \sim z - w$ for $z$ close to $w$, and has the following monodromy,

$$E(z + A_K, w) = E(z, w)$$

$$E(z + B_K, w) = E(z, w) \exp \left(-i\pi - \pi i \Omega_{KK} - 2\pi i \int_w^z \omega_K \right) \quad (2.3)$$
The Green function $G(z, w)$ is a meromorphic $(1, 0)$ form in $z$, which may be defined by

$$G(z, w) = \partial_z \ln \left( \frac{E(z, w)}{E(z, w_0)} \right)$$

(2.4)

It has simples poles at $w$ and $w_0$ with residues $\pm 1$, so that

$$\partial_z G(z, w) = +2\pi \delta(z, w) - 2\pi \delta(z, w_0)$$

$$\partial_{\bar{w}} G(z, w) = -2\pi \delta(z, w)$$

(2.5)

The point $w_0$ will be considered fixed throughout, and its dependence will not be exhibited. $G(z, w)$ is single-valued in $z$; it coincides with the third Abelian differential with vanishing $A$-periods in $z$; but has non-trivial monodromy in $w$, given by

$$G(z, w + A_K) = G(z, w)$$

$$G(z, w + B_K) = G(z, w) + 2\pi i \omega_K(z)$$

(2.6)

Note for later use that $\partial_{\bar{w}} G(z, w) = \partial_z G(w, z) = \partial_z \partial_{\bar{w}} \ln E(z, w)$ is symmetric in $z$ and $w$.

### 2.2 Two-dimensional supergeometry

The worldsheet $\Sigma$ of a supergeometry is equipped with a worldsheet metric and gravitino field $(g_{mn}, \chi_m^a)$, which in the superfield formalism may be parametrized by a superframe $E_A = dz^M E^A_M$ and a superconnection $\Omega = dz^M \Omega_M$. Here, $z^M = (z, \bar{z}, \theta, \bar{\theta})$ denote local coordinates for the super Riemann surface associated with $\Sigma$, and $A = (a, \alpha)$ labels the corresponding frame index. The torsion constraints on $E^A$, and $\Omega$ allow one to relate the superfields to the component fields by

$$E_m^a = e_m^a + \theta \gamma^a \chi_m,$$

$$g_{mn} = e_m^a e_n^b \delta_{ab}.\quad (2.7)$$

Supergeometry in the superfield formalism is invariant under super diffeomorphisms, and super Weyl transformations. Upon decomposition into components, these symmetries reduce to the customary diffeomorphism and Weyl invariance, as well as local supersymmetries. In local complex coordinates $z, \bar{z}$ on $\Sigma$, the bosonic frame indices $a$ and Einstein indices $m$ may be identified, and we use the frame labels $A = (z, \bar{z}, +, -)$. Infinitesimal diffeomorphisms $\delta_v$ and supersymmetries $\delta_\xi$ are generated by vector field $v^z$ of weight $(-1, 0)$ and the field $\xi^+$ of weight $(-1/2, 0)$ respectively and their action is given by

$$e_{\bar{z}}^m \delta_v e_m^z = \partial_{\bar{z}} v^z$$

$$e_{\bar{z}}^m \delta_\xi e_m^z = \partial_{\bar{z}} \xi^+$$

$$e_{\bar{z}}^m \delta_\xi \chi_m^+ = -2\partial_{\bar{z}} \xi^+$$

$$e_{\bar{z}}^m \delta_\xi \chi_m^z = \chi_m^+ \partial_{\bar{z}}$$

(2.8)

4 Throughout, auxiliary fields will play no role and will not be exhibited. Their explicit dependence may be found in [4, 7].
Super Weyl invariance of the critical superstring allows us to restrict to vanishing even Weyl changes $\epsilon^m \delta \epsilon_m = 0$, and vanishing odd Weyl changes $\chi_\bar{z}^+ = \chi_z^- = 0$. The action of local supersymmetry on $\chi_\bar{z}^+$ leaves two odd supermoduli, which we denote by $\zeta_1, \zeta_2$, and which may be used to parametrize the worldsheet gravitino field as follows,

$$\chi_\bar{z}^+ = \chi(z) = \zeta_1 \chi_1(z, \bar{z}) + \zeta_2 \chi_2(z, \bar{z})$$  \hspace{1cm} (2.9)

Henceforth, we shall use the notation $\chi(z)$ to denote this two-dimensional parametrization. To simplify notation, we indicate the $z$ and $\bar{z}$ dependence of $\chi$ only by $z$, but it must be kept in mind that the slice functions $\chi_{1,2}(z, \bar{z})$, and thus also $\chi(z)$ are generally allowed to be arbitrary smooth functions of $z$ and $\bar{z}$.

The super covariant derivative $\mathcal{D}^{(n)}_{-}$ on a superfield $V(z, \bar{z}, \theta, \bar{\theta}) = V_0 + \theta V_+ + \bar{\theta} V_-$ of weight $(n, 0)$ is given by

$$\mathcal{D}^{(n)}_{-} V = V_- + \bar{\theta} \left( \partial_{\bar{z}} V_0 + \frac{1}{2} \chi V_+ \right) - \theta \bar{\theta} \left( \partial_z V_+ + \frac{1}{2} \chi \partial_z V_0 + n \partial_z \chi V_0 - \frac{1}{4} \chi V_- \right)$$  \hspace{1cm} (2.10)

where $V_0, V_+$ and $\chi$ are all functions of $z, \bar{z}$. A superfield $V$ of weight $(n, 0)$ is said to be superholomorphic if

$$\mathcal{D}^{(n)}_{-} V = 0.$$  \hspace{1cm} (2.11)

We consider only superfields with a definite $\mathbb{Z}_2$-grading, which is defined to be the grading of the lowest component $V_0$. The components of a super holomorphic form $V$ are then given by $V_- = 0$, and $V_0, V_+$ obeying the equations,

$$\partial_z V_0 + \frac{1}{2} \chi(z) V_+ = 0 \quad \partial_z V_+ + \frac{1}{2} \chi(z) \partial_z V_0 + n \partial_z \chi(z) V_0 = 0$$  \hspace{1cm} (2.12)

Solutions to these equations may be viewed as $\chi$-deformations of holomorphic $(n, 0)$-forms $V_0$ which lead to even-grading (or simply “even”) superholomorphic $V$, or $\chi$-deformations of holomorphic $(n + 1/2, 0)$-forms $V_+$ which lead to odd-grading (or simply “odd”) superholomorphic $V$.

2.3 Super-Abelian differentials and super prime forms

The notions of holomorphic Abelian differentials generalize to supergeometry. Superholomorphic Abelian differentials $\hat{\omega} = \hat{\omega}_0 + \theta \hat{\omega}_+$ are forms of weight $(1/2, 0)$ satisfying

$$\mathcal{D}^{(1/2)}_{-} \hat{\omega} = 0.$$  \hspace{1cm} (2.13)

On a genus 2 super Riemann surface with even spin structure $\delta$, there exist no even superholomorphic Abelian differentials, since there exist no holomorphic $(1/2, 0)$ forms.
The space of odd superholomorphic Abelian differentials is 2-dimensional, and admits a basis \( \hat{\omega}_I(z) = \hat{\omega}_I(z) + \theta \hat{\omega}_I(z) \), given by

\[
\hat{\omega}_I(z) = \omega_I(z) - \frac{1}{16\pi^2} \int_x \int_y \partial_x \partial_y \ln E(z, x) \chi(x) S(x, y) \chi(y) \omega_I(y)
\]
\[
\hat{\omega}_I(0) = - \frac{1}{4\pi} \int_x S(z, x) \chi(x) \omega_I(x)
\]

(2.14)

Super holomorphic \((1/2, 0)\) forms \( \hat{\omega} = \hat{\omega}_0 + \theta \hat{\omega}_+ \) support the notion of a line integral, which may be defined by

\[
\int_{(w, \theta w)} \hat{\omega} = \int_w^z \left( dz \hat{\omega}_+ - \frac{1}{2} d\bar{z} \chi(z) \hat{\omega}_0 \right) + \theta_z \hat{\omega}_0(z) - \theta_w \hat{\omega}_0(w).
\]

(2.15)

The basis \( \hat{\omega}_I \) is dual to the basis \( A_I \) of homology cycles in the sense of line integrals (2.15), and we define its \( B_J \) periods to be the super period matrix \( \hat{\Omega}_{IJ} \),

\[
\int_{A_J} \hat{\omega}_I = \delta_{IJ}, \quad \int_{B_J} \hat{\omega}_I = \hat{\Omega}_{IJ}
\]

(2.16)

By construction, both \( \Omega_{IJ} \) and \( \hat{\Omega}_{IJ} \) are invariant under diffeomorphisms, but while \( \hat{\Omega}_{IJ} \) is also invariant under local supersymmetry transformations, \( \Omega_{IJ} \) is not. Thus \( \hat{\Omega}_{IJ} \) can be viewed as the supersymmetric completion of \( \Omega_{IJ} \). From (2.14) for \( \hat{\omega}_I \), it follows that

\[
\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_x \int_y \omega_I(x) \chi(x) S(x, y) \chi(y) \omega_I(y)
\]

(2.17)

Finally, we can also construct a super prime form \( \mathcal{E}_\delta(z, w) \) by

\[
\ln \mathcal{E}_\delta(z, w) = - F_0(z, w) - \theta_z F(z, w) - \theta_w F(w, z) - \theta_z \theta_w F_1(z, w)
\]

(2.18)

with the functions \( F_0, F_+, \) and \( F_1 \) defined by

\[
F_0(z, w) = - \ln E(z, w) + \frac{1}{16\pi^2} \int_x \int_y G(x, z) \chi(x) S(x, y) \chi(y) G(y, w)
\]
\[
F_+(z, w) = \frac{1}{4\pi} \int_x S(z, x) \chi(x) G(x, w)
\]
\[
F_1(z, w) = S(z, w) - \frac{i}{16\pi^2} \int_x \int_y S(z, x) \chi(x) \partial_x \partial_y \ln E(x, y) \chi(y) S(y, w)
\]

The super prime form \( \mathcal{E}_\delta(z, w) \) has properties similar to those of the customary prime form \( E(z, w) \): it is superholomorphic in \( z \) and \( w \), and \( \mathcal{E}_\delta(z, w) = \mathcal{D}_+^z \mathcal{E}_\delta(z, w) = 0 \) if and only if the superpoints coincide \( z = w \) [5].

---

5Throughout this paper, conditionally convergent integrals of the form \( \int_C \frac{f(z)}{z-w} d^2z \) are regulated as \( \partial_w \int_C \frac{f(z)}{z-w} d^2z \), for \( f \) smooth function with compact support.
2.4 The chiral splitting theorem

We consider a theory of scalar multiplets \((x^\mu, \psi^\mu, \psi^\dagger)\), \(0 \leq \mu \leq 9\), on the surface \(\Sigma\) with a supergeometry \((\epsilon_m^a, \chi_m^a)\). In the superfield formalism, the multiplet \((x^\mu, \psi^\mu, \psi^\dagger)\) corresponds to a scalar superfield \(X^\mu(z, \theta, \bar{\theta}) = x^\mu(z) + \theta \psi^\mu + \bar{\theta} \psi^\dagger\) whose action is

\[
I_m(E_M^A, X^\mu) = \frac{1}{4\pi} \int d^{12}z \, \text{sdet } E_M^A \, D_+ X^\mu D_- X^\mu
\]  

(2.20)

where \(d^{12}z = d^2z d\theta d\bar{\theta}\), and \(D_\pm = D_0\) are the covariant derivatives with respect to the supergeometry \((g_{mn}, \chi_m^a)\). The generating vertex for massless NS strings is given by [26]

\[
V(z, \bar{z}; \epsilon_o^\mu, \bar{\epsilon}_o^\mu, k^\mu) = \exp\left(ik^\mu X^\mu(z, \bar{z}) + \epsilon_o^\mu D_+ X^\mu + \bar{\epsilon}_o^\mu D_- X^\mu\right)
\]  

(2.21)

where \(k^2 = k \cdot \epsilon_o = k \cdot \bar{\epsilon}_o = 0\). The coefficients \(\epsilon_o\) and \(\bar{\epsilon}_o\) must be Grassmann odd (as will be indicated by the subscript \(o\)) in order to obtain a superfield vertex \(V\) that has definite (even) grading. In the next subsection, we shall explain more in detail how \(\epsilon_o\) and \(\bar{\epsilon}_o\) are determined. In particular, the physical massless NS-NS string vertex operators will correspond to retaining precisely the term linear in \(\epsilon_o\) and linear in \(\bar{\epsilon}_o\) in the expansion of \(V\) and discarding all other terms.

The matter part of the string amplitudes for the scattering of \(N\) massless NS superstrings is derived from the correlator of \(N\) vertex operators \(V\) for various polarizations \(\epsilon_{or}\), momenta \(k_r\) and vertex insertion points \(z_r\), with \(r = 1, \cdots, N\). Up to a factor of the vacuum amplitude (which is independent of \(\epsilon_{or}, k_r, z_r\)), this correlator is given by the functional integral over the superfield \(X\),

\[
\left\langle \prod_{r=1}^N V(z_r, \bar{z}_r; \epsilon_{or}^\mu, \bar{\epsilon}_{or}^\mu, k_r^\mu) \right\rangle \sim \int DX^\mu e^{-I_m(E_M^A, X^\mu)} \prod_{r=1}^N V(z_r, \bar{z}_r; \epsilon_{or}^\mu, \bar{\epsilon}_{or}^\mu, k_r^\mu)
\]  

(2.22)

The Chiral Splitting Theorem of [5] states that this correlator is chirally split at fixed internal loop momenta \(p_I, I = 1, 2\). More precisely, it was shown in [5] that we have

\[
\left\langle \prod_{r=1}^N V(z_r, \bar{z}_r; \epsilon_{or}^\mu, \bar{\epsilon}_{or}^\mu, k_r^\mu) \right\rangle \sim \int dp_I^\mu \left|\mathcal{F}[\delta](z_r; \epsilon_{or}^\mu, k_r^\mu; p_I^\mu)\right|^2
\]  

(2.23)

where \(p_I^\mu\) are parameters which can be interpreted as internal momenta circulating in the loop \(B_I\). (Such parameters had been introduced in [44] for the bosonic string.) The chiral blocks \(\mathcal{F}[\delta](z_r; \epsilon_{or}^\mu, k_r^\mu; p_I^\mu)\) are given explicitly by

\[
\mathcal{F}[\delta] = \exp\left\{\frac{i}{4\pi} p_I^\mu \Omega_{IJ} p_J^\mu + p_I^\mu \sum_{r=1}^N \left(\epsilon_{or}^\mu \hat{\omega}_I(z_r) + i k_r^\mu \int_{z_0}^{z_r} \hat{\omega}_I\right)\right\} \times
\]

\[^6\text{Throughout, a factor of } 2\pi \text{ in the internal loop momenta will be scaled out as compared to the conventions of [5]. Also, the dependence on super moduli of the chiral blocks } \mathcal{F}[\delta] \text{ will not be exhibited.}\]
\[
\exp \left( \frac{1}{2} \sum_{[r,s]}^{N} \left[ k_r^\mu k_s^\nu \ln \mathcal{E}_\delta(z_r, z_s) + \epsilon_{or}^\mu \epsilon_{os}^\nu \partial_r^\mu \partial_s^\nu \ln \mathcal{E}_\delta(z_r, z_s) \right. \right. \\
\left. \left. - i k_r^\mu \epsilon_{os}^\nu \partial_r^\nu \ln \mathcal{E}_\delta(z_r, z_s) - i k_s^\mu \epsilon_{or}^\nu \partial_s^\nu \ln \mathcal{E}_\delta(z_s, z_r) \right] \right) (2.24)
\]

where \( \partial_+ = \partial_0 + \theta \partial_z \), and the super period matrix \( \hat{\Omega} \), the super Abelian differentials \( \hat{\omega}_I \), and the super prime form \( \mathcal{E}_\delta \) have been defined and calculated earlier in this section.\(^7\)

The monodromy properties of \( \mathcal{F}[\delta] \) are as follows,
\[
\mathcal{F}[\delta](z_r + \delta_{rs} A_K, \epsilon_{or}^\mu; k_r^\mu; p_I^\mu) = \mathcal{F}[\delta](z_r, \epsilon_{or}^\mu; k_r^\mu; p_I^\mu) \\
\mathcal{F}[\delta](z_r + \delta_{rs} B_K, \epsilon_{or}^\mu; k_r^\mu; p_I^\mu) = \mathcal{F}[\delta](z_r, \epsilon_{or}^\mu; k_r^\mu; p_I^\mu - 2\pi \delta_{IK} k_s^\mu) (2.25)
\]

### 2.5 The use of Grassmann odd polarization vectors

The Grassmann odd nature of the coefficients \( \epsilon_o \) and \( \bar{\epsilon}_o \) was used originally in [5] only as a formal device to recover the physical vertex operators for massless NS states via a generating function. Actually, it is possible to trace the odd Grassmann nature of \( \epsilon_o, \bar{\epsilon}_o \) back to the structure of the integration measure that occurs in the physical vertex operator, and then use this structure beyond the formal level to our advantage.

The key observation is that the unintegrated vertex operator in the superfield formalism should naturally be viewed as a differential form on the super Riemann surface \( \Sigma \), generalizing the differential form structure of the component formulation of the vertex operators of [16] to superspace. The unintegrated physical vertex operator for massless NS strings is given by
\[
d^2 z (\text{sdet } E_M^A) \epsilon^\mu \bar{\epsilon}^\nu D_+ X^\mu D_- X^\nu \exp(ik \cdot X) (2.26)
\]

Here, \( k \) is again the momentum of the string, but \( \epsilon \) and \( \bar{\epsilon} \) are now Grassmann even, or simply, complex numbers-valued polarization vectors, satisfying \( k^2 = k \cdot \epsilon = k \cdot \bar{\epsilon} = 0 \). Additional insight is gained by using an observation made in [16], namely that the super volume form admits a natural chiral splitting,
\[
d^2 z (\text{sdet } E_M^A) = e^z \wedge e^\bar{z} \wedge d\theta \wedge d\bar{\theta} (2.27)
\]

with the chiral frame \( e^z \) given explicitly in Wess-Zumino gauge, and local complex coordinates \( z \) and \( \bar{z} \), by
\[
e^z = dz - \frac{1}{2} \theta \chi(z) d\bar{z}, \quad e^\bar{z} = d\bar{z} - \frac{1}{2} \bar{\theta} \bar{\chi}(z) dz (2.28)
\]

\(^7\)We shall use the summation subscript \([r,s]\) for a summation over both \( r \) and \( s \) with \( r \neq s \), while the subscript \( r \neq s \) will be reserved to denote a summation over \( r \) only for fixed \( s \).
Therefore, it is clearly natural to rearrange the unintegrated vertex operator of (2.26) by grouping together the chiral polarization vector $\epsilon$ and the chiral differentials $d\theta$ and $e^z$ (and similarly for $\bar{\epsilon}, d\bar{\theta}$ and $e^{\bar{z}}$),

\[
\left(\epsilon^\mu d\theta \wedge e^z \mathcal{D}_+ X^\mu\right) \wedge \left(\bar{\epsilon}^\mu d\bar{\theta} \wedge e^{\bar{z}} \mathcal{D}_- X^\mu\right) \exp(ik \cdot X) \tag{2.29}
\]

The generating function from which these vertex operators may be derived is given by

\[
\exp\left(ik^\mu X^\mu(z, \bar{z}) + \epsilon^\mu d\theta \wedge e^z \mathcal{D}_+ X^\mu + \bar{\epsilon}^\mu d\bar{\theta} \wedge e^{\bar{z}} \mathcal{D}_- X^\mu\right) \tag{2.30}
\]

which coincides with (2.21) provided we make the identifications,

\[
\begin{align*}
\epsilon^\mu_o & = \epsilon^\mu d\theta \wedge e^z = \epsilon^\mu d\theta \wedge \left(dz - \frac{1}{2} \theta \chi(z) d\bar{z}\right) \\
\bar{\epsilon}^\mu_o & = \bar{\epsilon}^\mu d\bar{\theta} \wedge e^{\bar{z}} = \bar{\epsilon}^\mu d\bar{\theta} \wedge \left(d\bar{z} - \frac{1}{2} \bar{\theta} \bar{\chi}(z) dz\right) \tag{2.31}
\end{align*}
\]

Given that the polarization vectors $\epsilon^\mu, \bar{\epsilon}^\mu$ are complex numbers, we see that $\epsilon^\mu_o$ and $\bar{\epsilon}^\mu_o$ are Grassmann odd, in view of the fact that $\theta, \chi, \bar{\theta}, \bar{\chi},$ and $d\theta, d\bar{\theta}$ are all Grassmann odd. Moreover, $\epsilon^\mu_o$ and $\bar{\epsilon}^\mu_o$ are differential forms of even degree, since $d\theta \wedge dz, d\theta \wedge d\bar{z}, d\bar{\theta} \wedge dz,$ and $d\bar{\theta} \wedge d\bar{z}$ are of even degree. The above set-up will be very useful in the sequel. In particular, $\mathcal{F}[\delta]$ should be viewed as a differential form in each of the vertex insertion points $z_r$. 

20
3 Holomorphicity and Super-Holomorphicity

The key to the approach to superstring perturbation theory developed in [11, 12, 13, 14, 16, 17] is a deformation of the complex structure on the world sheet defined by the metric $g_{mn}$ to the complex structure defined by the super period matrix $\hat{\Omega}_{IJ}$.

The deformation from $g_{mn}$ to $\hat{\Omega}_{IJ}$ was required in order to preserve local supersymmetry on the worldsheet, and produce a well-defined projection from supermoduli space to moduli space. But it turned out also that fundamental superholomorphic objects, such as the super Abelian differentials $\hat{\omega}_I(z)$, were related to the complex structure defined by $\hat{\Omega}_{IJ}$ rather than the one defined by the original metric $g_{mn}$ ([12], §5.1).

This relation between superholomorphicity with respect to $(g_{mn}, \chi^\alpha_m)$ and holomorphicity with respect to $\hat{\Omega}_{IJ}$ is the main theme of this paper. The purpose of this section is to present the simplest examples where this correspondence can be established, as a warm-up before the considerably more complicated treatment for the general case.

3.1 Deformations of complex structures and metrics

From the point of view of moduli, a deformation of complex structures can just be viewed as a deformation of period matrices (after a choice of canonical homology basis)

$$\Omega_{IJ} \rightarrow \hat{\Omega}_{IJ}$$

However, the objects of interest to us are tensors on the worldsheet, and we need to realize this deformation of complex structures as a deformation of metrics

$$g_{mn} \rightarrow \hat{g}_{mn}$$

As in [12], we proceed as follows. Let $\hat{g}_{mn}$ be a metric on the worldsheet $\Sigma$ whose period matrix in $\hat{\Omega}_{IJ}$. In genus $h = 2$, the dimension of odd supermoduli is 2, and thus we may assume that $\hat{g}_{mn} = g_{mn} + \mathcal{O}(\zeta^1 \zeta^2)$, where $\zeta^1$ and $\zeta^2$ are two odd supermoduli parameters. Let $z$ denote now complex coordinates\(^8\) for the metric $d\hat{s}^2$ associated with $\hat{g}_{mn}$. The deformation from the metric $ds^2$ to the metric $d\hat{s}^2$

$$d\hat{s}^2 = 2\hat{g}_{z\bar{z}}dzd\bar{z},$$

$$ds^2 = 2g_{z\bar{z}}dzd\bar{z} + \delta g_{z\bar{z}}dz^2 + \delta g_{z\bar{z}}d\bar{z}^2$$

is characterized by the Beltrami differential

$$\mu_{z\bar{z}} = \mu(z) = -\frac{1}{2} \hat{g}^{z\bar{z}} \delta g_{z\bar{z}}.$$
The condition that the period matrix for the metric $\hat{g}_{mn}$ is the super period matrix $\hat{\Omega}_{IJ}$ is then equivalent to

$$i \int_{\Sigma} \omega_I(z) \omega_J(z) \mu(z) = \Omega_{IJ} - \hat{\Omega}_{IJ}. \quad (3.5)$$

This equation determines the Beltrami differential $\mu(z)$ only up to a reparametrization $\mu(z) \rightarrow \mu(z) + \bar{\partial}v(z)$, where $v(z)$ is a smooth vector field on the surface $\Sigma$. Thus we can view the equivalence class of $\mu(z)$ as the intrinsic object, and a choice of $\mu(z)$ within this equivalence class as just a gauge choice. Obviously, this gauge choice did not enter the starting formulas for the superstring amplitudes, and it should cancel out at the end.

### 3.2 Super-holomorphic $\frac{1}{2}$-forms and holomorphic $(1, 0)$-forms

The relation between superholomorphic differentials of weight $(1/2, 0)$ and holomorphic differentials (with respect to $\hat{\Omega}_{IJ}$) of weight $(1, 0)$ was discovered in [12]. If $\hat{\omega}(z) = \hat{\omega}_0(z) + \theta \hat{\omega}_+(z)$ is a superholomorphic differential of weight $(1/2, 0)$, then there is a smooth, single-valued scalar function $\lambda(z)$ and a 1-form $\omega_z(z)dz$, holomorphic with respect to $\hat{g}_{mn}$, so that the integral in $\theta$ alone yields,

$$\int d\theta \wedge \hat{e}^z \hat{\omega}(z) = \omega_z(z)dz + d\lambda(z) \quad (3.6)$$

We re-derive this result here, in a formalism suitable for later extensions.

A first important observation [16] is that the superholomorphicity of $\hat{\omega}$ implies that the above left-hand side is a closed 1-form. Closedness is a property independent of the complex structure, and can be established using the isothermal coordinates $z$ defined by the metric $g_{mn}$. Thus $e^z$ is given by the expression (2.28), and we have

$$\int d\theta \wedge e^z \hat{\omega}(z) = dz \hat{\omega}_+(z) - \frac{1}{2} d\bar{z} \chi(z) \hat{\omega}_0(z) \quad (3.7)$$

In particular,

$$d \left( \int d\theta \wedge e^z \hat{\omega} \right) = - \left( \partial_z \hat{\omega}_+ + \frac{1}{2} \chi(z) \hat{\omega}_0 \right) dz \wedge d\bar{z} = 0 \quad (3.8)$$

In view of the superholomorphicity of $\hat{\omega}$ and hence of (2.12).

The next observation is that the $d\bar{z}$ component of the left-hand side of (3.6), with respect to the complex structure defined by $\hat{g}_{mn}$, is a $\bar{\partial}$-exact $(0, 1)$-form. It suffices to show this for each element $\hat{\omega}_I$ of the basis of superholomorphic differentials given by (2.14). Under the deformation of metrics $g_{mn} \rightarrow \hat{g}_{mn}$, the forms $dz$ and $d\bar{z}$ get deformed as,

$$dz \rightarrow dz - \mu(z)dz \quad d\bar{z} \rightarrow d\bar{z} - \bar{\mu}(z)dz, \quad (3.9)$$
where the $z$ coordinates on the left hand side of the arrows are holomorphic coordinates with respect to $g_{mn}$, while on the right hand side, they are holomorphic coordinates with respect to $\hat{g}_{mn}$. Thus, with $z$ holomorphic coordinates for $\hat{g}_{mn}$, we have

\[
\hat{e}^z = dz - \left( \mu(z) + \frac{1}{2} \theta \chi(z) \right) d\bar{z},
\]

\[
\hat{e}^{\bar{z}} = d\bar{z} - \left( \bar{\mu}(z) + \frac{1}{2} \bar{\theta} \chi(z) \right) dz
\] (3.10)

and hence

\[
\int d\theta \wedge \hat{e}^z \hat{\omega}_I(z) = dz \hat{\omega}_{I+} - d\bar{z} \left( \mu(z) \hat{\omega}_{I+}(z) + \frac{1}{2} \chi(z) \hat{\omega}_{I0}(z) \right)
\] (3.11)

We claim that the $d\bar{z}$ component is orthogonal to any holomorphic differential. Indeed, since it is already $O(\zeta_1 \zeta_2)$ in its dependence on supermoduli, in the pairing with an arbitrary holomorphic differential $\omega_J(z)$, we can ignore the distinction between holomorphic forms with respect to $g_{mn}$ and $\hat{g}_{mn}$. Thus we have,

\[
\int_{\Sigma} d^2z \omega_J \left( \mu \hat{\omega}_{I+} + \frac{1}{2} \chi \hat{\omega}_{I0} \right) = \int_{\Sigma} d^2z \omega_J \omega_{I+} + \frac{1}{2} \int_{\Sigma} d^2z \omega_J \chi \hat{\omega}_{I0} = 0
\] (3.12)

since the first term on the right hand side equals $\Omega_{IJ} - \hat{\Omega}_{IJ}$ by the definition of $\mu$, and the second term equals $\hat{\Omega}_{IJ} - \Omega_{IJ}$ in view of the formula (2.17) for $\hat{\Omega}_{IJ}$. Thus there exists a smooth function $\lambda_I(z)$, defined uniquely up to an additive constant, so that

\[
\mu(z) \omega_I(z) + \frac{1}{2} \chi(z) \hat{\omega}_{I0}(z) = -\partial_z \lambda_I(z).
\] (3.13)

We can write then

\[
\int d\theta \wedge \hat{e}^z \hat{\omega}_I(z) = \left( \int d\theta \wedge \hat{e}^z \hat{\omega}_I(z) - d\lambda_I(z) \right) + d\lambda_I(z)
\] (3.14)

The term between parentheses on the right hand side is by construction a $(1, 0)$-form, which is closed, since both $d\lambda_I$ and the left-hand side are closed. Thus it must be holomorphic. By examining its periods, we can easily recognize it as $dz \omega_I(z)$, and thus we obtain the desired formula,

\[
\int d\theta \wedge \hat{e}^z \hat{\omega}_I(z) = dz \omega_I(z) + d\lambda_I(z).
\] (3.15)

We observe that the function $\lambda_I$ depends on the choice of metric $\hat{g}_{mn}$ within all metrics with period matrix $\hat{\Omega}_{IJ}$. Under a gauge change $\mu(z) \rightarrow \mu(z) + \partial \nu(z)$, the function $\lambda_I$ changes by $\lambda_I(z) \rightarrow \lambda_I(z) - \nu(z) \omega_I(z)$, up to a $z$-independent additive term. The form of this additive term will be determined by the definition of the arbitrary additive constant in $\lambda_I(z)$, and we shall take this to be $\lambda_I(w_0) = 0$, where $w_0$ is the point on $\Sigma$ where the Green function $G$ is defined to vanish, $G(z, w_0) = 0$ for all $z \in \Sigma$.
3.3 The super prime form in $\hat{g}_{mn}$ conformal coordinates

The next illustrative example, closer to the full chiral blocks, is that of the super prime form $E_{\delta}(z, w)$. The super prime form $E_{\delta}(z, w)$ was defined by (2.18) and (2.19). It is important to recall that, in those formulas, the coordinates $z$ and $w$ are holomorphic coordinates with respect to the metric $g_{mn}$. As in the previous example, to discuss holomorphicity, we need to express $E_{\delta}(z, w)$ with respect to holomorphic coordinates $z$ and $w$ with respect to $\hat{g}_{mn}$. It is easy to see that the prime form $E$ and the Szegö kernel $S$ with respect to $g_{mn}$ can be expressed as

$$\ln E(z, w) \rightarrow \ln E(z, w) + \frac{1}{2\pi} \int \mu(x)G(x, z)G(x, w)$$

$$S(z, w) \rightarrow S(z, w) + \frac{1}{4\pi} \int \mu(x)\left(S(x, z)\partial_x S(x, w) - S(x, w)\partial_x S(x, z)\right)$$

where all expressions on the right hand side are defined with respect to the metric $\hat{g}_{mn}$. Taking into account the fact that $\mu = O(\zeta^1 \zeta^2)$, we can rewrite the super prime form as

$$\ln E_{\delta}(z, w) = -\hat{F}_0(z, w) - \theta_z \hat{F}_+(z, w) - \theta_w \hat{F}_+(w, z) - \theta_z \theta_w \hat{F}_1(z, w)$$

with the component functions defined by

$$\hat{F}_0(z, w) = -\ln E(z, w) + P(z, w)$$
$$\hat{F}_+(z, w) = F(z, w)$$
$$\hat{F}_1(z, w) = S(z, w) + \Psi(z, w)$$

where we use the following abbreviations,

$$F(z, w) = \frac{1}{4\pi} \int_x S(z, x)\chi(x)G(x, w)$$
$$P(z, w) = \frac{1}{16\pi^2} \int_x \int_y G(x, z)\chi(x)S(x, y)\chi(y)G(y, w) - \frac{1}{2\pi} \int_x \mu(x)G(x, z)G(x, w)$$
$$\Psi(z, w) = -\frac{1}{16\pi^2} \int_x \int_y S(z, x)\chi(x)\partial_x \partial_y \ln E(x, y)\chi(y)S(y, w)$$
$$+ \frac{1}{4\pi} \int \mu(x)\left(S(x, z)\partial_x S(x, w) - S(x, w)\partial_x S(x, z)\right)$$

The functions $F, P, \Psi$ as well as $\hat{\omega}_{I0}$ and $\lambda_I$ are the basic constituents of the holomorphic blocks that will be obtained in this paper.

3.4 Basic properties of the functions $\hat{\omega}_{I0}, \lambda_I, F, P$ and $\Psi$

Here, we shall collect all the basic properties of the functions $\hat{\omega}_{I0}, \lambda_I, F, P$ and $\Psi$ that will be required in the sequel. They are differentiation, monodromy and variation formulas.
under changes of slice for $\chi$ and $\mu$. Note that we have the following symmetry properties,
\[ P(z, w) = +P(w, z) \]
\[ \Psi(z, w) = -\Psi(w, z) \] (3.20)

3.4.1 Differential formulas
They are all written in holomorphic coordinates for $\hat{g}_{mn}$.
\[ \partial_{\bar{z}}\hat{\omega}_{I0}(z) = -\frac{1}{2}\chi(z)\omega_I(z) \] (3.21)
\[ \partial_{\bar{z}}\lambda_I(z) = -\mu(z)\omega_I(z) - \frac{1}{2}\chi(z)\hat{\omega}_{I0}(z) \]
\[ \partial_{\bar{z}}F(z, w) = +\frac{1}{2}\chi(z)G(z, w) \]
\[ \partial_{\bar{w}}F(z, w) = -\frac{1}{2}S(z, w)\chi(w) \]
\[ \partial_{\bar{z}}P(z, w) = -\frac{1}{2}\chi(z)F(z, w) + \mu(z)G(z, w) \]
\[ \partial_{\bar{z}}\Psi(z, w) = -\frac{1}{2}\partial_{\bar{z}}F(w, z)\chi(z) - \mu(z)\partial_{\bar{z}}S(z, w) - \frac{1}{2}\partial_{\bar{z}}\mu(z)S(z, w) \]

3.4.2 Monodromies
All monodromies on $A_K$-cycles vanish, while the monodromies on $B_K$-cycles are given as follows for $z' = z + B_K$,
\[ \hat{\omega}_{I0}(z') = \hat{\omega}_{I0}(z) \]
\[ \lambda_I(z') = \lambda_I(z) \]
\[ F(z', w) = F(z, w) \]
\[ F(z, w') = F(z, w) - 2\pi i\hat{\omega}_{K0}(z) \]
\[ P(z', w) = P(z, w) - 2\pi i\lambda_K(w) \]
\[ \Psi(z', w) = \Psi(z, w) \] (3.22)

3.4.3 Variations of $\mu$-Slice
The basic transformation is under a vector field $v^z = v(z)$ which is second order in $\zeta$,
\[ \delta_v\chi(z) = 0 \] (3.23)
\[ \delta_v\mu(z) = \partial_{\bar{z}}v(z) \]
\[ \delta_v\hat{\omega}_{I0}(z) = 0 \]
\[ \delta_v\lambda_I(z) = -v(z)\omega_I(z) \]
\[ \delta_v F(z, w) = 0 \]
\[ \delta_v P(z, w) = v(z)G(z, w) + v(w)G(w, z) + \partial_{w_0} v(w_0) \]
\[ \delta_v \Psi(z, w) = -v(z)\partial_z S(z, w) - \frac{1}{2}\partial_z v(z)S(z, w) - (z \leftrightarrow w) \]  

(3.24)

All other quantities, \( \omega_I, S, \ln E(z, w) \) are untransformed as long as they are evaluated with respect to the metric \( \hat{g}_{mn} \). (For simplicity, we assume that \( v(w_0) = 0 \).)

### 3.4.4 Variations of \( \chi \) and \( \mu \) by local supersymmetry

The basic transformation is under a spinor field \( \xi = \xi(z) \) which is first order in \( \zeta \),

\[ \delta_\xi \chi(z) = -2\partial_\zeta \xi(z) \]
\[ \delta_\xi \mu(z) = \xi(z)\chi(z) \]
\[ \delta_\xi \omega_I(z) = \xi(z)\omega_I(z) \]
\[ \delta_\xi \lambda_I(z) = \xi(z)\omega_I(z) \]
\[ \delta_\xi F(z, w) = -\xi(z)G(z, w) + \xi(w)S(z, w) \]
\[ \delta_\xi P(z, w) = +\xi(z)F(z, w) + \xi(w)F(w, z) \]
\[ \delta_\xi \Psi(z, w) = -\xi(z)\partial_z F(w, z) - \partial_w F(z, w)\xi(w) \]  

(3.25)

(For simplicity, we have assumed that \( \xi(w_0) = 0 \).)

### 3.4.5 Dependence on \( w_0 \)

The Green function \( G(r, s) \) is defined so that \( G(r, w_0) = 0 \) from which it follows that \( \lambda_I(w_0) = P(r, w_0) = P(w_0, s) = F(r, w_0) = 0 \). Denoting the corresponding quantities defined with respect to a new point \( w_0' \) by \( G'(r, s) \), we have the following relations

\[ G'(r, s) = G(r, s) - G(r, w_0') \]
\[ \lambda_I'(w) = \lambda_I(w) - \lambda_I(w_0') \]
\[ P'(r, s) = P(r, s) - P(r, w_0') - P(w_0', s) + P(w_0', w_0') \]
\[ F'(r, s) = F(r, s) - F(r, w_0') \]  

(3.26)

### 3.5 Holomorphic blocks from \( \partial_+^z \ln \mathcal{E}_\delta \) and \( \partial_+^w \partial_+^z \ln \mathcal{E}_\delta \)

The superholomorphic chiral amplitudes \( \mathcal{F}[\delta] \) of (2.24) involve, besides the differentials \( \omega_I \) and the super prime form \( \mathcal{E}_\delta \) also first and second derivatives of \( \ln \mathcal{E}_\delta \), multiplied by certain factors involving the frame \( e^z \). We shall now show that these superholomorphic objects also produce holomorphic forms on \( \Sigma \), up to exact differentials. All our considerations are modulo \( \delta \)-functions supported at coincident vertex insertion points; these terms do
not contribute in a suitable definition of the full physical amplitude, obtained by analytic
continuation in the external momenta. In the old dual model language, this fact is often
referred to as the cancelled propagator argument.

Using the component form of \( \ln \mathcal{E}_\delta \) of (3.17), the derivatives (with respect to complex
coordinates \( z \) and \( w \) on \( \Sigma \) associated with the complex structure of \( \hat{\Omega}_{IJ} \)) are given by

\[
\begin{align*}
\partial_+^z \ln \mathcal{E}_\delta(z, w) &= -\hat{F}_+(z, w) - \theta_w \hat{F}_1(z, w) - \theta_z \left( \partial_z \hat{F}_0(z, w) + \theta_w \partial_z \hat{F}_+(w, z) \right) \\
\partial_+^w \partial_+^z \ln \mathcal{E}_\delta(z, w) &= -\hat{F}_1(z, w) + \theta_z \partial_z \hat{F}_+(w, z) - \theta_w \left( \partial_w \hat{F}_+(z, w) + \theta_z \partial_w \partial_z \hat{F}_0(z, w) \right)
\end{align*}
\]

(3.27)

where all the partial derivatives on the right hand side can be taken to be partial derivatives
with respect to the holomorphic coordinates \( z \) of the metric \( \hat{g}_{mn} \), the difference between
these and the original derivatives with respect to \( g_{mn} \) having cancelled because \( F \) and \( P \)
are respectively of orders 1 and 2 in \( \zeta^\alpha \), and \( \ln E(z, w) \) is holomorphic. We shall now prove
the following formulas, in which the integrations are understood to be carried out over the
\( \theta_z \) and \( \theta_w \) variables only,

\[
\begin{align*}
\int d\theta_z e^z \partial_+^z \ln \mathcal{E}_\delta(z, w) &= dz \partial_z \ln E(z, w) - d_z P(z, w) - \theta_w d_z F(w, z) \\
\int d\theta_w e^w d\theta_z e^z \partial_+^w \partial_+^z \ln \mathcal{E}_\delta(z, w) &= dz dw \partial_w \partial_z \ln E(z, w) + d_w d_z P(z, w)
\end{align*}
\]

(3.28)

On the right hand side of the first line above, we have omitted a term \(-d\bar{z}\mu(z)\partial_z \ln E(z, \theta_0)\),
which is independent of \( w \), and will cancel out of the chiral amplitude thanks to momentum
conservation. The forms \( dz \partial_z \ln E(z, w) \) and \( dz dw \partial_w \partial_z \ln E(z, w) \) are holomorphic forms
on \( \Sigma \) with respect to the complex structure of \( \hat{\Omega}_{IJ} \). The remaining terms on the right hand
side of both formulas are exact differentials either in \( z \), in \( w \) or in both.

The proof of both formulas in (3.28) is obtained by first carrying out explicitly these
\( \theta_z \) and \( \theta_w \) integrations. We begin with the first formula,

\[
\begin{align*}
\int d\theta_z e^z \partial_+^z \ln \mathcal{E}_\delta(z, w) &= -dz \partial_z \hat{F}_0(z, w) - \theta_w dz \partial_z \hat{F}_+(w, z) - d\bar{z}\mu(z)\partial_z \ln E(z, w) \\
&\quad + \frac{1}{2} d\bar{z} \left( \chi(z) \hat{F}_+(z, w) + \theta_w \hat{F}_1(z, w) \chi(z) \right)
\end{align*}
\]

(3.29)

Using the differential relations (3.21), we can essentially absorb the \( d\bar{z} \) terms in an exact
de Rham differential, and recover the first line of (3.28). As mentioned above, we omit an
irrelevant term of the form \(-d\bar{z}\mu(z)\partial_z \ln E(z, \theta_0)\).

To prove the second line in (3.28), we apply the operation \( \int d\theta_w \wedge e^w \partial_+^w \) to the first line
of (3.28), and obtain,

\[
\begin{align*}
\int d\theta_w e^w d\theta_z e^z \partial_+^w \partial_+^z \ln \mathcal{E}_\delta(z, w) &= dz dw \partial_w \partial_z \ln E(z, w) + dw \partial_w dz P(z, w) \\
&\quad + dw dz \mu(w) \partial_w \partial_z \ln E(z, w) \\
&\quad - \frac{1}{2} dw \chi(w) dz F(w, z).
\end{align*}
\]

(3.30)
Here we note that the shift of $\partial_z$, from holomorphic coordinates $w$ with respect to $g_{mn}$ to holomorphic coordinates $\hat{w}$ with respect to $\hat{g}_{mn}$, results only in additional $\delta(z, \hat{w})$ terms, when applied to $\partial_z \ln E(z, \hat{w})$. In view again of the relations (3.21), we can write

$$dw \partial_w d_z P(z, \hat{w}) = d_w d_z P(z, \hat{w}) - d\hat{w} d_z \partial_{\hat{w}} P(z, \hat{w})$$

(3.31)

$$= d_w d_z P(z, \hat{w}) + \frac{1}{2} d\hat{w} \chi(\hat{w}) d_z F(w, z) - d\hat{w} d_z \mu(\hat{w}) \partial_{\hat{w}} \partial_z \ln E(z, \hat{w})$$

modulo $\delta$ functions along coincident points $z = \hat{w}$. This proves the desired relation.

The relation (3.28) shows that, just as in the case of super holomorphic differentials, the super prime form and its super-derivatives reduce to the ordinary prime form and its derivatives with respect to the complex structure of $\hat{g}_{mn}$, modulo exact differentials and $\delta$ functions. There is a subtlety, however: the function $P(z, \hat{w})$ has monodromy in both $z$ and $\hat{w}$, and the differential $d_w d_z P(z, \hat{w})$ is not exact in both variables $z$ and $\hat{w}$. We shall see later that the monodromy of $P$ is precisely what will be needed to supply the resulting holomorphic chiral blocks with the correct monodromy transformation properties.
4 Kinematic Invariants

The combinatorial structure of the chiral blocks of the $N$-point amplitudes is inherently quite complicated, and it will be very useful to decompose the blocks according to suitable kinematic invariants. In fact, identifying these kinematic invariants is a fundamental aspect of the holomorphic chiral block problem.

4.1 Grassmann parity of the polarization vectors

We make systematic use of the differential forms in superspace, and their commutation relations. Let $z_r$ be bosonic and $\theta_r$ be fermionic coordinates, with corresponding differentials $dz_r$ and $d\theta_r$. Their commutation rules are

\[
\begin{align*}
    z_r d\theta_s &= +d\theta_s z_r & dz_r \wedge dz_s &= -dz_s \wedge dz_r \\
    dz_r \theta_s &= +\theta_s dz_r & dz_r \wedge d\theta_s &= -d\theta_s \wedge dz_r \\
    \theta_r d\theta_s &= -d\theta_s \theta_r & d\theta_r \wedge d\theta_s &= +d\theta_s \wedge d\theta_r
\end{align*}
\]  

(4.1)

In particular, the chiral volume form $dz \wedge d\theta$ is even as a form but odd Grassmann valued.

With respect to local complex coordinates $z, \bar{z}$ associated with the complex structure induced by the metric $\hat{g}_{mn}$, the polarization differential forms, introduced in section 2.5 take the form,

\[
\begin{align*}
    \epsilon^\mu_o &= \epsilon^\mu \wedge \hat{\epsilon}^\bar{z} & \tilde{\epsilon}^\mu &= d\theta - \left( \mu(z) + \frac{1}{2} \theta \chi(z) \right) d\bar{z} \\
    \tilde{\epsilon}^\mu_o &= \tilde{\epsilon}^\mu \wedge \hat{\epsilon}^\bar{z} & \tilde{\epsilon}^\bar{\mu} &= d\bar{\theta} - \left( \bar{\mu}(z) + \frac{1}{2} \bar{\theta} \bar{\chi}(z) \right) dz
\end{align*}
\]  

(4.2)

The polarization vectors $\epsilon^\mu, \tilde{\epsilon}^\mu$ are ordinary complex numbers, so that $\epsilon^\mu_o, \tilde{\epsilon}^\mu_o$ and $\epsilon^\mu, \tilde{\epsilon}^\mu$ are Grassmann odd; $\epsilon^\mu_o, \tilde{\epsilon}^\mu_o$ are differential 2-forms, while $\epsilon^\mu, \tilde{\epsilon}^\mu$ are differential 1-forms. In particular, the chiral amplitude $\mathcal{F}[\delta]$ will be viewed as a differential form in each of the vertex insertion points $z_r$. Finally, we shall use the following notations throughout,

\[
\begin{align*}
    d_r &= \partial_r + \bar{\partial}_r & \partial_r &= dz_r \partial_{z_r} \\
    \bar{\partial}_r &= d\bar{z}_r \partial_{\bar{z}_r}
\end{align*}
\]  

(4.3)

where $d_r$ is the total differential in $r$, while $\partial_r$ and $\bar{\partial}_r$ are its $(1, 0)$ and $(0, 1)$ form components respectively.

4.2 The quantity $K^\mu_r$

We introduce the following key quantity

\[
K^\mu_r \equiv \epsilon^\mu_r \wedge dz_r + ik^\mu_r \theta_r = \epsilon^\mu d\theta_r \wedge dz_r + ik^\mu r \theta_r
\]  

(4.4)
which combines the momentum $k^\mu_r$ and the polarization vector $\varepsilon^\mu_r = \epsilon^\mu_r d\theta_r$. In view of the properties of $\varepsilon^\mu_r$, the quantity $K^\mu_r$ is odd Grassmann valued, as well as a linear combination of a differential form of degree 0 and a differential form of degree 2.

The combination $K^\mu_r$ plays a central role in the organization of the space-time structure of the chiral amplitudes. This may be seen, for example, by concentrating on those terms occurring in the chiral amplitude $F[\delta]$ of (2.24) which involve the Szegö kernel $S(z_r, z_s)$ only. Using the explicit form of the super prime form in (3.17) and (3.18), as well as its derivatives in (3.27), we find that the Szegö kernel $S(z_r, z_s)$ in the argument of the exponential in (2.24) has the following multiplicative coefficient,

$$-k^\mu_r k^\mu_s + \epsilon^\mu_{or} \epsilon^\mu_{os} - i k^\mu_r \epsilon^\mu_{os} \theta_r + i k^\mu_s \epsilon^\mu_{or} \theta_s = \tilde{K}^\mu_r \tilde{K}^\mu_s$$

where $\tilde{K}^\mu_r$ is defined in analogy with $K^\mu_r$ by

$$\tilde{K}^\mu_r \equiv \epsilon^\mu_{or} + i k^\mu_r \theta_r \quad \quad \epsilon^\mu_{or} = \epsilon^\mu_r d\theta_r \wedge \hat{e}^z_r$$

The quantity $\tilde{K}^\mu_r$ reduces to $K^\mu_r$ up to terms linear in $\mu$ and $\chi$, since $\epsilon^\mu_{or}$ reduces to $\varepsilon^\mu_r \wedge dz_r$. Next, we list some of the key properties of $K$ and $\tilde{K}$,

$$\begin{align*}
(1) & \quad \{ K^\mu_r, \theta_s \} = \{ K^\mu_r, d\theta_s \} = \{ K^\mu_r, \varepsilon^\nu_s \} = [K^\mu_r, d z_s] = [K^\mu_r, \varepsilon] = 0 \\
(2) & \quad \{ \tilde{K}^\mu_r, \theta_s \} = \{ \tilde{K}^\mu_r, d\theta_s \} = \{ \tilde{K}^\mu_r, \varepsilon^\nu_s \} = [\tilde{K}^\mu_r, d z_s] = [\tilde{K}^\mu_r, \varepsilon] = 0 \\
(3) & \quad K^\mu_r \tilde{K}^\nu_r = i(\varepsilon^\mu_r k^\nu_r - \varepsilon^\nu_r k^\mu_r)dz_r \theta_r = i(\epsilon^\mu_r k^\nu_r - \epsilon^\nu_r k^\mu_r)d\theta_r \wedge dz_r \theta_r \\
(4) & \quad \tilde{K}^\mu_r \tilde{K}^\nu_r = i(\epsilon^\mu_{or} k^\nu_r - \epsilon^\nu_{or} k^\mu_r)\theta_r = i(\epsilon^\mu_r k^\nu_r - \epsilon^\nu_r k^\mu_r)d\theta_r \wedge \hat{e}^z_r \\
(5) & \quad k^\mu_r K^\mu_r = k^\mu_r \tilde{K}^\mu_r = \varepsilon^\mu_r \theta_r K^\nu_r = \varepsilon^\mu_r \theta_r \tilde{K}^\nu_r = 0
\end{align*}$$

The quadratic character of the worldsheet fermions $\psi_+$, both in the action and in the vertex operators, requires that the number of Szegö kernels at any vertex insertion point be either 0 or 2. The presence of two Szegö kernels at an insertion point $s$, for example, then produces two factors of $K_s$, as well. Two factors of $K_r$ automatically produce the field strength combination $f^\mu_{\nu r} = \epsilon^\mu_r k^\nu_r - \epsilon^\nu_r k^\mu_r$ for the polarization vector $\epsilon^\mu_r$ and momentum $k^\mu_r$. Specifically, we have

$$\left( S(r, s) K^\mu_r K^\nu_s \right) \left( S(s, t) K^\mu_r K^\nu_t \right) = K^\mu_r K^\nu_s \left( S(r, s) S(s, t) K^\mu_r K^\nu_t \right)$$

with $K^\mu_r K^\nu_r$ given in terms of $f^\mu_{\nu r}$ by (3) above. The tremendous advantage is that the field strengths combination $f^\mu_{\nu r} = \epsilon^\mu_r k^\nu_r - \epsilon^\nu_r k^\mu_r$ are automatically invariant under space-time gauge transformations $\epsilon^\mu_r \rightarrow \epsilon^\mu_r + ck^\mu_r$, for any boost $c$.

Remarkably, it is not only the worldsheet fermion contributions, produced by the Szegö kernel, which will admit an organization in terms of the quantities $K^\mu_r$, but also the worldsheet boson contributions. The latter is not seen directly, but will be established starting in the subsequent section.
We note that the Bianchi identity on $f_{\mu}^{\nu}$ can be expressed simply as the following identity on the quantity $K_r$ and the associated momenta $k_r$,

$$k_{r}^{\mu}K_{r}^{\nu}K_{r}^{\rho} + k_{r}^{\nu}K_{r}^{\rho}K_{r}^{\mu} + k_{r}^{\rho}K_{r}^{\mu}K_{r}^{\nu} = 0 \quad (4.9)$$

In fact, all the kinematic invariants which we shall encounter can be expressed simply in terms of $K_r^\mu$, and we can describe them now.

### 4.3 Kinematic factors for linear chains

We shall denote all kinematic-type invariants by the same generic letter $K$, and distinguish them one from another only by their indices. The first type of kinematic factor arises from contracting indices along a linear chain with open ends. These open ends will be further contracted with internal loop momenta $p_{\mu}^i$, external momenta $k_{\mu}^r$ or polarization forms $\varepsilon_{\mu}^r$.

The key ingredient for the linear chain structure is given by

$$K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\nu} \equiv K_{t_1}^{\mu}K_{t_2}^{\rho_1}K_{t_3}^{\rho_1}K_{t_2}^{\rho_2}\cdots K_{t_{n-1}}^{\rho_{n-2}}K_{t_{n-1}}^{\rho_{n-1}}K_{t_n}^{\nu}$$

$$= K_{t_2}^{\mu} \prod_{k=1}^{n-1} K_{t_k}^{\rho_k} K_{t_{k+1}}^{\rho_k} K_{t_n}^{\nu} \quad (4.10)$$

By construction, these quantities involve only the gauge invariant field strengths $f_{\mu}^{\nu}$. They have the following reflection and cyclic permutation symmetry properties,

$$K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\nu} = (\varepsilon^n)^{[t_n t_{n-1} \cdots t_2 t_1]} K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\nu}$$

$$K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\mu} = (\varepsilon^n)^{[t_1 t_2 \cdots t_{n-1}t_n]} K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\mu}$$

$$K_{[t_1t_2\cdots t_{n-1}t_n]}^{\mu\mu} = K_{[t_{23}\cdots t_{n-1} t_1]}^{\mu\mu} \quad (4.11)$$

and satisfy various identities resulting from contractions,

$$k_{s}^{\mu}K_{[rs]}^{\mu\rho} = \frac{1}{2} K_{[rs]}^{\mu\mu} k_{r}^{\rho}$$

$$\frac{1}{2} K_{s}^{\nu} K_{s}^{\mu} K_{[rt_1\cdots t_n]}^{\mu\mu} = -i\varepsilon_{s}^{\mu} K_{s}^{\nu} K_{[rt_1\cdots t_n]}^{\mu\rho} K_{t_n}^{\nu} K_{t_n}^{\rho} + i\varepsilon_{s}^{\mu} K_{s}^{\nu} K_{[rt_1\cdots t_n]}^{\mu\rho} \quad (4.12)$$

These identities may be proven with the help of the Bianchi identity.

### 4.4 Kinematic factors for singly linked chains

The basic structure of a singly linked kinematic factor is obtained by taking a linear chain and linking one end to one of the midpoints of the linear chain with an extra factor of momentum evaluated at that midpoint; concretely, it is of the form,

$$K_{[t_1 t_2 \cdots t_{m} r u_1 \cdots u_{k}]}^{\mu\rho} k_{r}^{\rho} \quad (4.13)$$
Such factors arise from a bosonic field contraction at the end of the linear chain; the extra factor of momentum arises from a bosonic contraction with the exponential \( \exp \{ ik^\rho x_+^\rho(r) \} \).

Using the Bianchi identity, the kinematic combination that will occur later in the bosonic contractions can be written as,

\[
K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r = K^{\mu\nu}_{[t_1 \ldots t_m]} k^\rho_r K^\sigma_r K^{\sigma\rho}_{[u_1 \ldots u_\ell]}
\]

\[
= -K^{\mu\nu}_{[t_1 \ldots t_m]} k^\nu_r K^\sigma_r K^{\rho\sigma}_{[u_1 \ldots u_\ell]} - K^{\mu\nu}_{[t_1 \ldots t_m]} k^\sigma_r K^\rho_r K^{\sigma\rho}_{[u_1 \ldots u_\ell]}
\]

\[
= K^{\mu\nu}_{[t_1 \ldots t_m]} k^\nu_r K^{\sigma\sigma}_{[r u_1 \ldots u_\ell]} - K^{\mu\nu}_{[t_1 \ldots t_m]} k^\sigma_r K^{\rho\nu}_{[u_1 \ldots u_\ell]}
\]

\[
= K^{\mu\nu}_{[t_1 \ldots t_m]} k^\nu_r K^{\sigma\sigma}_{[r u_1 \ldots u_\ell]} + \epsilon^\ell_k (-) k^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} k^\rho_r
\]

Equivalently, we have the following anti-symmetrization formula,

\[
K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r - \epsilon^\ell_k K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} k^\rho_r = K^{\mu\nu}_{[t_1 \ldots t_m]} k^\rho_r K^{\sigma\sigma}_{[r u_1 \ldots u_\ell]}
\]

where the right hand side is the same form as we shall find in the fermionic loop contractions. Thus, it is natural to decompose the kinematic factor as follows,

\[
K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r = \frac{1}{2} \left( K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r - \epsilon^\ell_k K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} k^\rho_r \right)
\]

\[
+ \frac{1}{2} \left( K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r + \epsilon^\ell_k K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} k^\rho_r \right)
\]

or using the above identity,

\[
K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r = \frac{1}{2} \left( K^{\mu\nu}_{[t_1 \ldots t_m]} \ k^\rho_r K^{\sigma\sigma}_{[r u_1 \ldots u_\ell]} + K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \ k^\rho_r \right)
\]

where we have defined the “symmetrized” combination \( K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \) by

\[
K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \equiv \frac{1}{2} \left( K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} + \epsilon^\ell_k K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} \right)
\]

We now define the kinematic invariants \( K^{(m+1)\ell+1}_+ \) by

\[
K^{(m+1)\ell+1}_+ = \epsilon^\mu_s \theta^\sigma_s K^{\mu\nu}_{[t_1 \ldots t_m r u_1 \ldots u_\ell]} i k^\rho_r
\]

\[
K^{(m+1)\ell+1}_- = \epsilon^\mu_s \theta^\sigma_s K^{\mu\nu}_{[t_1 \ldots t_m]} i k^\rho_r K^{\sigma\sigma}_{[r u_1 \ldots u_\ell]}
\]
5 Combinatorial Organization

In this section, we shall start from the superspace formulation of chiral blocks as superholomorphic forms in the vertex insertion points, as well as on super moduli space, and decompose the blocks with two goals in mind.

- Separate the terms which are a total derivative in at least one vertex insertion point, and thus contain a non-vanishing \((0,1)\) form in that vertex insertion point; we generally denote these terms by \(D[\delta]\). The remainder will be all the terms which are \((1,0)\) forms in each vertex insertion point; we generally denote these terms by \(Z[\delta]\); the full chiral amplitude will then be given by the sum

\[
F[\delta] = Z[\delta] + D[\delta] \tag{5.1}
\]

- Bring to the fore the kinematic dependence solely in the form of the space-time gauge invariant combinations \(K^\mu_r\), introduced in the preceding section.

It should be kept in mind, however, that the \(D[\delta]\)-terms will not be closed in all insertion points, and as such, do not yet qualify to be exact terms in the sense of hybrid cohomology. A subsequent regrouping of terms producing such hybrid exact differentials, will be carried out in section §9.

5.1 Decomposition of chiral blocks

The starting point is the superspace representation for the chiral amplitudes, which was derived in (2.24) as a differential form in each of the vertex insertion points, and which we recall here for convenience,

\[
F[\delta] = \exp\left\{ \frac{i}{4\pi} p_I^\mu \hat{\Omega}_{IJ} p_J^\mu + p_I^\mu \sum_{r=1}^{N} \left( c_{or}^\mu \hat{\omega}_I(z_r) + i k_r^\mu \int_{z_0}^{z_r} \hat{\omega}_I \right) \right\} \times \\
\exp \frac{1}{2} \sum_{[r,s]} \left[ k_r^\mu k_s^\mu \ln \mathcal{E}_\delta(z_r,z_s) + c_{or}^\mu c_{os}^\mu \partial_r^\mu \partial_s^\mu \ln \mathcal{E}_\delta(z_r,z_s) \right. \\
- i k_r^\mu c_{or}^\mu \partial_r^\mu \ln \mathcal{E}_\delta(z_r,z_s) - i k_s^\mu c_{or}^\mu \partial_s^\mu \ln \mathcal{E}_\delta(z_s,z_r) \right] \tag{5.2}
\]

We use (3.17), (3.18) as well as (3.27) to recast the super Abelian differentials and the superprime form in terms of \(\hat{\omega}_{I}, \lambda_{I}, F, P\), and \(\Psi\), as well as the Szegö kernel and the prime form \(E\). With the help of the expressions derived in Section 3.5, all \(\mu(r)\) and \(\chi(r)\) dependence encountered in the polarization forms \(c_{or}\) is recombined in such a manner that all \((0,1)\) forms that occur are converted into total differentials applied to \(\lambda_{I}, F,\) or \(P\). All dependence of the Szegö kernel and the prime form evaluated between vertex insertion...
points is regrouped into an overall factor which will be denoted by \( Z_0 \); it is independent of \( \mu(z) \) and \( \chi(z) \), and thus independent of odd supermoduli. The final result is

\[
\mathcal{F}[\delta] = Z_0 \exp \left\{ p_\mu^I \sum_r \left( \varepsilon^\mu_r \theta_\lambda \lambda_\lambda I(r) + i k_\mu^I \lambda_\lambda \lambda_\lambda f_0(r) \right) \right\} \\
\times \exp \left\{ \sum_{[r,s]} \left( -\frac{1}{2} k_\mu^I k_s^I P(r, s) + i k_\mu^I \varepsilon^\mu_r \theta_\lambda \lambda_\lambda s d_s P(r, s) + \frac{1}{2} \varepsilon^\mu_r \varepsilon^\mu_s \theta_\lambda \lambda_\lambda s d_s d_s P(r, s) \right. \\
+ \frac{1}{2} \left( \bar{K}_r^\mu \bar{K}_s^\mu - K_r^\mu K_s^\mu \right) S(r, s) + \frac{1}{2} K_r^\mu K_s^\mu \Psi(r, s) \\
\left. + i K_r^\mu k_s^I F(r, s) + K_r^\mu \varepsilon^\mu_s d_s F(r, s) \right) \right\} \\
(5.3)
\]

### 5.2 The factor \( Z_0 \) and its generalizations

The factor \( Z_0 \) occurring in the formula for \( \mathcal{F}[\delta] \) is given by

\[
Z_0 = \exp \left\{ \frac{i}{4\pi} p_\mu^I \hat{\Omega}_{IJ} p_I^J + p_\mu^I \sum_{r=1}^N \left( \varepsilon^\mu_r \omega I(r) + i k_\mu^I \int^r \omega I \right) \right\} \\
\times \exp \left\{ \frac{1}{2} \sum_{[r,s]} \left[ K_r^\mu K_s^\mu S(r, s) + k_\mu^I k_s^I \ln E(r, s) + \varepsilon^\mu_r \varepsilon^\mu_s \theta_\lambda \lambda_\lambda r \theta_\lambda \lambda_\lambda s \ln E(r, s) \right. \\
\left. - i k_\mu^I \varepsilon^\mu_s \theta_\lambda \lambda_\lambda s \ln E(r, s) - i k_\mu^I \varepsilon^\mu_r \theta_\lambda \lambda_\lambda r \ln E(s, r) \right] \right\} \\
(5.4)
\]

The rearrangement of the kinematic factor multiplying \( S(r, s) \) has already been explained in section 4.2. It will turn out to be very convenient to reformulate the bosonic contributions in \( Z_0 \) in terms of correlators of the chiral worldsheet boson field \( x_+^\mu \), but leave Wick contractions undone until needed. This reformulation will drastically simplify the combinatorics, and facilitate the extraction of the kinematic combinations \( \bar{K}_r^\mu \bar{K}_s^\mu \) also for the terms involving the worldsheet bosons. Specifically, we shall introduce the field \( x^\mu_+(z) \) with two-point function,

\[
\langle x_+^\mu(z)x_+^\nu(w) \rangle = - \ln E(z, w) \\
(5.5)
\]

Furthermore, we shall use the combination below in order to deal efficiently with the internal momentum dependence,

\[
\bar{x}_+^\mu(z) = x_+^\mu(z) + p_\mu^I \int^z \omega_I \\
(5.6)
\]

The \( Z_0 \) factor is then given by\(^9\)

\[
Z_0 \equiv \exp \left\{ \frac{i}{4\pi} p_\mu^I \hat{\Omega}_{IJ} p_I^J + \frac{1}{2} \sum_{[r,s]} K_r^\mu K_s^\mu S(r, s) + \sum_r i k_\mu^I \bar{x}_+^\mu(r) + \sum_r \varepsilon^\mu_r \theta_\lambda \lambda_\lambda r \bar{x}_+^\mu(r) \right\} \\
(5.7)
\]

\(^9\)For further notational simplification, we shall not exhibit the vacuum expectation signs \( (\ ) \) on \( Z_0 \).
In $Z_0$, one could have left also the worldsheet fermion field uncontracted in the form $K_\mu r \psi_+^\alpha(r)$, instead of incorporating the Wick contracted Szegö kernel in $Z_0$. It does not appear, however, that much will be gained by doing so, and we shall not do so here.

To facilitate the organization of the combinatorics, it will be extremely useful to introduce a generalization of $Z_0$, which may be defined by removing the polarization vector contribution for the worldsheet bosons on a sequence of points $r_1, \cdots, r_n$. The reason this will be a helpful tool is that certain – but not all – boson vertex contractions in $Z_0$ will need to be carried out, and combined with certain combinations of the functions $\omega_{10}, \lambda_I, F, P$ in order to exhibit their kinematic dependence entirely in the form of the gauge invariant combinations $K_\mu r K_\nu s$. The required generalization is defined by

$$Z_0^{r_1 \cdots r_n} \equiv \exp \left\{ \frac{i}{4\pi} p^\mu_I \hat{\Omega}_{IJ} p^\mu_J + \frac{1}{2} \sum_{r,s} K^\mu_s K_\mu^r S(r,s) + \sum_r ik^\mu_+ \bar{x}_+^\mu(r) + \sum_{r \neq r_1, \cdots, r_n} \varepsilon^\mu_+ \theta_r \partial_r \bar{x}_+^\mu(r) \right\} (5.8)$$

Here it is assumed that all $r_i, i = 1, \cdots, n$ are distinct. Each $Z_0^{r_1 \cdots r_n}$ is strictly invariant under each of the monodromy transformations. Finally, we shall also use the following abbreviation,

$$X_\mu^+ \equiv ik^\mu_+ + K_\mu r K_\mu^r (dr)^{-1} \partial_r \bar{x}_+^\mu(r) \quad (5.9)$$

With these conventions and notations at our disposal, we are now ready to reformulate the chiral block and decompose them according to $F[\delta] = Z[\delta] + D[\delta]$, as explained earlier on in this section.

### 5.3 Summary of blocks of type $Z[\delta]$

It will be convenient to first present the result of the calculation explained above first, and then give a detailed derivation of each of the terms that contributes. We shall organize the contributions according to the number of vertex insertion points at which the contribution is not manifestly holomorphic. Since the sources of non-holomorphicity are a single occurrence of $\mu(z)$, which couples quadratically to the fields $x_+$ and $\psi_+$, and a double occurrence of $\chi(z)$, each of which couples bilinearly to $x_+$ and $\psi_+$, there can be at most 4 vertex insertion points at which any given contribution is not manifestly holomorphic. The result for the blocks $Z$ are as follows,

$$Z[\delta] = Z_{1a} + Z_{1b} + Z_{2a} + Z_{2b} + Z_{2c} + Z_{2d} + Z_{2e} + Z_{2f} + Z_{3a} + Z_{3b} + Z_{3c} + Z_{3d} + Z_{3e} + Z_4 \quad (5.10)$$

The first entry in the subscript stands for the number of non-holomorphic insertion points, while the supplementary index is used to distinguish different contributions to that order.
The corresponding individual contributions are given by

\[
\begin{align*}
Z_{1a} &= + \sum_r Z^r_0 p^\mu_t X^\mu_r \lambda_t (r) \\
Z_{1b} &= - \sum_r Z^r_0 \frac{1}{2} P^\mu_t s^\nu_r K^\mu_r K^\nu_r p^\mu_j \hat{\omega}_0 (r) \hat{\omega}_0 (r) \\
Z_{2a} &= - \sum_{[rs]} Z^{rs}_0 \frac{1}{2} p^\mu_t K^\mu_r K^\nu_s p^\nu_j \hat{\omega}_0 (r) \hat{\omega}_0 (s) \\
Z_{2b} &= + \sum_{[rs]} Z^{rs}_0 \frac{1}{2} P(r, s) X^\mu_r X^\mu_s \\
Z_{2c} &= + \sum_{[rs]} Z^{rs}_0 \frac{1}{2} K^\mu_r K^\mu_s \Psi (r, s) \\
Z_{2d} &= + \sum_{[rs]} Z^{rs}_0 K^\mu_r X^\mu_s F(r, s) p^\mu_t \left( K^\nu_r \hat{\omega}_0 (r) + K^\nu_s \hat{\omega}_0 (s) \right) \\
Z_{2e} &= + \sum_{[rs]} Z^{rs}_0 \frac{1}{2} K^\mu_r K^\nu_r K^\mu_s K^\nu_s (ds)^{-1} \left( \Phi_B (s; r, r) - ik^\sigma_s \partial_s \bar{x}_+^\sigma (s) Q_B (s; r, r) \right) \\
Z_{2f} &= - \sum_{[rs]} Z^{rs}_0 \frac{1}{2} K^\mu_r k^\nu_s k^\nu_r F(r, s) F(s, r) \\
Z_{3a} &= + \sum_{[rst]} Z^{rst}_0 K^\mu_r X^\mu_s F(r, s) p^\mu_t K^\nu_t \hat{\omega}_0 (t) \\
Z_{3b} &= + \sum_{[rst]} Z^{rst}_0 (-) \frac{i}{2} K^\mu_r K^\nu_r K^\mu_s k^\nu_s F(r, s) F(t, s) \\
Z_{3c} &= + \sum_{[rst]} Z^{rst}_0 \frac{1}{2} K^\mu_r K^\nu_r K^\nu_s K^\mu_s (ds)^{-1} \left( \Phi_B (s; r, t) - ik^\sigma_s \partial_s \bar{x}_+^\sigma (s) Q_B (s; r, t) \right) \\
Z_{3d} &= - \sum_{[rst]} Z^{rst}_0 K^\mu_r X^\mu_s K^\nu_t i k^\nu_r F(r, s) F(t, r) \\
Z_{3e} &= - \sum_{[rst]} Z^{rst}_0 \frac{1}{2} K^\mu_r X^\mu_s K^\nu_r X^\nu_r F(r, s) F(t, r) \\
Z_4 &= - \sum_{[rstu]} Z^{rstu}_0 \frac{1}{2} K^\mu_r X^\mu_s K^\nu_r X^\nu_u F(r, s) F(t, u) \\
\end{align*}
\]

All ingredients have been defined in earlier sections, except for the three point functions \( Q_B (s; r, t) \) and \( \Phi_B (s; r, t) \), which will be defined during the construction of these blocks to follow, where also the presence of the inverse differential \((ds)^{-1}\) will be explained. The functions \( Q_B (s; r, t) \) and \( \Phi_B (s; r, t) \) will be constructed explicitly in Section 6.
5.4 Summary of the blocks of type $D$

Similarly, all the blocks of type $D$ with an exact differentials in at least one of the vertex insertion points are as follows. Using the same nomenclature as for the $Z$ blocks, we have

$$D[\delta] = D_1 + D_{2a} + D_{2b} + D_{2c} + D_{2d} + D_{3a} + D_{3b} + D_{3c} + D_{3d} + D_{3f} + D_{4a} + D_{4b}$$

(5.12)

The corresponding individual contributions are given by,

$$D_1 = -\sum_r d_r \left( Z^r_0 \partial^\mu r \theta_r \lambda^r(r) \right)$$

$$D_{2a} = +\sum_{[rs]} d_r d_s \left( \frac{1}{2} Z^r_0 \partial^\mu r \theta_r \theta_s P(r, s) \right)$$

$$D_{2b} = -\sum_{[rs]} d_s \left( Z^r_0 \partial^\mu s \theta_s R^\mu(r) \right)$$

$$D_{2c} = -\sum_{[rs]} d_s \left( Z^r_0 \partial^\mu s \theta_s F(r, s) \right)$$

$$D_{2d} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3a} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3b} = +\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3c} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3d} = +\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3e} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{3f} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{4a} = +\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

$$D_{4b} = -\sum_{[rst]} d_s \left( Z^r_{rst} \frac{1}{2} K^\mu r \partial^\mu r K^\nu s K^\mu r (ds)^{-1} Q_B(s; r, r) \right)$$

(5.13)

We stress that each contribution is exact in one or two insertion points only, but not necessarily closed in any of the others. Note that the number of $\varepsilon$'s occurring in each
the number of total derivatives in each contribution.

To derive the above results for $Z$ and $D$, we proceed by expanding (5.3) in powers of $\dot{\omega}_{10}$, $\lambda_I$, $F$, $P$, $\Psi$, $Q_B$ and $\Phi_B$, inductively on the number of non-holomorphic vertex insertion points. Since $\lambda_I$, $P$, $\Psi$, $Q_B$, $\Phi_B$ are of order $O(\zeta^1 \zeta^2)$, and $\dot{\omega}_{10}$ and $F$ are of order $O(\zeta)$ in the supermoduli, only terms proportional to $\lambda_I$, $P$, $\Psi$, $Q_B$, $\Phi_B$, and the bilinears $\dot{\omega}_{10}\dot{\omega}_{10}$, $\dot{\omega}_{10} F$, and $FF$ must be retained. All others would be at least trilinear in supermoduli $\zeta$ and must thus vanish.

### 5.5 Contributions in $\lambda_I$

The contributions in $\lambda_I$ can only occur to first order, and arise from two sources in (5.3): one in $d_r \lambda_I$, and another in $\lambda_I$. Collecting both contributions yields all the dependence of $\mathcal{F}[\delta]$ on $\lambda_I$, and gives,

$$\mathcal{F}_\lambda = Z_0 \sum_r p^\mu_r \left( \varepsilon^\mu_r d_r \lambda_I(r) + i k^\mu_r \lambda_I(r) \right) \tag{5.14}$$

The term in $Z_0 d_r \lambda_I$ is not a total differential in $r$, because $Z_0$ itself has non-trivial $r$-dependence. We wish to isolate a total differential which includes also $Z_0$. Furthermore, $Z_0$ contains a contribution from $\varepsilon^\mu_r \partial_r \bar{x}^\mu_+(r)$, which provides with a further differential form contribution of type $(1,0)$. It will be advantageous to exhibit this contribution, and then partially Wick contract it. To this end, we use

$$Z_0 = Z_0^r \left( 1 + \varepsilon^\mu_r \theta_r \partial_r \bar{x}^\mu_+(r) \right) \tag{5.15}$$

This relation is an exact Taylor expansion in view of the fact that the term $\varepsilon^\mu_r \theta_r \partial_r \bar{x}^\mu_+(r)$ is linear in $\theta$, so that its square vanishes. Using this identity in (5.14), we obtain,

$$\mathcal{F}_\lambda = \sum_r Z_0^r p^\mu_r \left( \varepsilon^\mu_r d_r \lambda_I(r) + i k^\mu_r \lambda_I(r) + \varepsilon^\mu_r \partial_r \bar{x}^\mu_+(r) i k^\mu_r \lambda_I(r) \right)$$

$$= - \sum_r d_r \left( Z_0^r p^\mu_r \varepsilon^\mu_r \theta_r \lambda_I(r) \right)$$

$$+ \sum_r Z_0^r p^\mu_r \left( i k^\nu_r \partial_r \bar{x}^\nu_+(r) \varepsilon^\mu_r \theta_r + i k^\mu_r + \varepsilon^\nu_r \theta_r \partial_r \bar{x}^\nu_+(r) i k^\mu_r \right) \lambda_I(r)$$

Note that care is required in properly ordering standard differential 1-forms such as $d_r$ and $\partial_r$, with respect to $\varepsilon^\mu_r$, which is itself a differential 1-form. This ordering is responsible for the minus sign multiplying the first term on the right hand side of (5.16). Finally, in differentiating $Z_0$, we have used the relation

$$d_r \left( Z_0^r \varepsilon^\mu_r \theta_r \right) = Z_0^r \left( i k^\nu_r \partial_r \bar{x}^\nu_+(r) \varepsilon^\mu_r \theta_r \right) \tag{5.16}$$
Note that the contributions from the Szegö kernel in $Z_0$ cancel out in this differentiation, in view of the presence of the factor $\varepsilon^\mu r \theta_r$, and the fact that $(\theta_r)^2 = 0$, as well as the fact that the polarization vectors $\varepsilon^\mu r \theta_r$ cannot occur to an order higher than first. Finally, making use of the definition of $K^\mu r$, we may rearrange the terms in $F_\lambda$ as follows,

$$ik^\nu_r \partial_r \bar{x}^\nu_+(r) \varepsilon^\mu r \theta_r + ik^\mu r + \varepsilon^\nu r \theta_r \partial_r \bar{x}^\nu_+(r) ik^\mu r = ik^\mu r - i(k^\nu_r \varepsilon^\mu r - \varepsilon^\nu r k^\mu r) \theta_r \partial_r \bar{x}^\nu_+(r)$$

$$= ik^\mu r - K^\mu r K^\nu r (dr)^{-1} \partial_r \bar{x}^\nu_+(r)$$

$$= X^\mu r$$  \hspace{1cm} (5.17)

As a result, we have

$$F_\lambda = - \sum_r d_r \left( Z_0^\mu I^\mu r \theta_r \lambda I(r) \right) + \sum_r Z_0^\mu P^\mu r X^\mu r \lambda I(r)$$  \hspace{1cm} (5.18)

The first term on the right hand side produces $D_1$ of (5.13), while the second term produces $Z_{1a}$ of (5.11).

### 5.6 Contributions bilinear in $\dot{\omega}_{I0}$

By expanding the exponential of the terms in $\dot{\omega}_{I0}$ to second order, one obtains bilinears in $\dot{\omega}_{I0}$, which may be grouped into contributions at coincident points $r$, which yield $Z_{1b}$, and contributions at distinct points $r$ and $s$, which yield $Z_{2a}$. No $D$-terms appear.

### 5.7 Contributions in $P$

The contributions in $P(r, s)$ can occur to first order only, and arise in (5.3) from 3 sources: one term in $P$, one term in $d_r P$ and one term in $d_s P$,

$$F_P = Z_0 \sum_{[r, s]} \left( -\frac{1}{2} k^\mu r k^\mu s P(r, s) + ik^\mu r \varepsilon^\mu s d_s P(r, s) + \frac{1}{2} \varepsilon^\mu r \varepsilon^\mu s \theta_r \theta_s d_r d_s P(r, s) \right)$$  \hspace{1cm} (5.19)

We proceed as in the derivation of the contributions in $\lambda I$, except that this time we need to work in both variables $r$ and $s$. The generalization of (5.15) is given by

$$Z_0 = Z_0^{rs} \left( 1 + \varepsilon^\mu r \theta_r \partial_r \bar{x}^\mu_+(r) \right) \left( 1 + \varepsilon^\nu s \theta_s \partial_s \bar{x}^\nu_+(s) \right)$$  \hspace{1cm} (5.20)

As a result, we have

$$F_P = \sum_{[r, s]} Z_0^{rs} \left( 1 + \varepsilon^\mu r \theta_r \partial_r \bar{x}^\mu_+(r) \right) \left( 1 + \varepsilon^\nu s \theta_s \partial_s \bar{x}^\nu_+(s) \right) \times \left( -\frac{1}{2} k^\mu r k^\mu s P(r, s) + ik^\mu r \varepsilon^\mu s d_s P(r, s) + \frac{1}{2} \varepsilon^\mu r \varepsilon^\mu s \theta_r \theta_s d_r d_s P(r, s) \right)$$  \hspace{1cm} (5.21)
The single derivative term is handled as \( \lambda \) was,
\[
Z_{0s}^{rs} ik^\mu \varepsilon^\nu_s \theta_s d_s P(r, s) = -d_s \left( Z_{0s}^{rs} ik^\mu \varepsilon^\nu_s \theta_s P(r, s) \right) + ik^\mu_s \partial_s \bar{x}^\mu_s (s) i k^\mu_r \varepsilon^\nu_r \theta_r P(r, s) Z_{0s}^{rs}
\]
while the double derivative must be dealt with twice, and produces
\[
Z_{0s}^{rs} \frac{1}{2} \varepsilon^\mu_r \varepsilon^\nu_s \theta_r \theta_s d_r d_s P(r, s) = +d_r \left( Z_{0s}^{rs} \frac{1}{2} \varepsilon^\mu_r \varepsilon^\nu_s \theta_r \theta_s d_s P(r, s) \right)
+ d_s \left( Z_{0s}^{rs} \frac{1}{2} i k^\nu_r \partial_r \bar{x}^\nu_r (r) \varepsilon^\mu_r \varepsilon^\nu_r \theta_r P(r, s) \right)
- Z_{0s}^{rs} \frac{1}{2} i k^\sigma_s \partial_s \bar{x}^\sigma_s (s) i k^\nu_r \partial_r \bar{x}^\nu_r (r) \varepsilon^\mu_r \varepsilon^\nu_r \theta_r P(r, s)
\]

The same technique is now reiterated on the first term on the right hand side to bring out the differential \( d_s \) as well. Putting together all the non-derivative terms, and re-expressing the results in terms of \( K \) factors, we obtain
\[
\sum_{[rs]} Z_{0s}^{rs} \left( -\frac{1}{2} k^\mu_r k^\mu_s - i K^\mu_r K^\nu_s k^\mu_r \partial_s \bar{x}^\nu_r (s) + \frac{1}{2} K^\mu_r K^\nu_s K^\sigma_s K^\nu_r \partial_r \bar{x}^\sigma_r (r) \partial_s \bar{x}^\nu_r (s) \right) P(r, s)
\] (5.22)

Expressing the terms in parentheses in terms of \( X^\mu_r X^\mu_s \), produces the contribution \( Z_{2b} \). Collecting the exact differential terms, we get
\[
- \sum_{[rs]} d_s \left( Z_{0s}^{rs} \varepsilon^\mu_s \theta_s X^\mu_r P(r, s) \right) + \sum_{[rs]} d_r d_s \left( \frac{1}{2} Z_{0s}^{rs} \varepsilon^\mu_r \varepsilon^\nu_r \theta_r \theta_s P(r, s) \right)
\] (5.23)
which produces the terms \( D_{2b} + D_{2a} \).

**5.8 Contributions in \( \Psi \)**

There is only a single source for this contribution in (5.3), and its expansion to first order readily gives \( Z_{2c} \).

**5.9 Contributions in \( \hat{\omega}_{I0} F \)**

Terms in \( F \) may arise to first order, in which case they must ultimately be multiplied by one power of \( \hat{\omega}_{I0} \), or to second order. Here, we work out the first order case, leaving the case of second order to the next subsection. The double expansion, to first order in \( \hat{\omega}_{I0} \) and to first order in \( F \) produces the following contributions,
\[
\mathcal{F}_{\hat{\omega}F} = \sum_{[rs]} \sum_{t} Z_0 K^\mu_t \left( ik^\mu_r F(r, s) + \varepsilon^\mu_r \theta_s d_s F(r, s) \right) p^\mu_t K^\nu_t \hat{\omega}_{I0}(t)
\] (5.24)
Since $\varepsilon^\mu_\nu \theta_s K^\mu_r = 0$, a simplification occurs in the use of formula (5.20), and we have

$$Z_0 K^\mu_r = Z_0^{rs} K^\mu_r \left(1 + \varepsilon^\nu_\theta_s \partial_s \bar{x}^\nu_+ (s)\right)$$

which yields

$$\mathcal{F}_{\omega F} = \sum_{[rs]} \sum_t Z_0^{rs} K^\mu_r \left(i k^\mu_s F(r, s) + \varepsilon^\mu_\theta_s d_s F(r, s) + i k^\mu_s F(r, s) \varepsilon^\nu_\theta_s \partial_s \bar{x}^\nu_+ (s)\right) p^\nu_t K^\nu_t \omega_{t0}(t)$$

Using the following rearrangement of the derivative term,

$$Z_0^{rs} K^\mu_r \varepsilon^\mu_\theta_s d_s F(r, s) = -d_s \left(Z_0^{rs} K^\mu_r \varepsilon^\mu_\theta_s F(r, s)\right) + i k^\mu_s \partial_s \bar{x}^\nu_+ (s) Z_0^{rs} K^\mu_r \varepsilon^\mu_\theta_s F(r, s)$$

we isolate an exact differential as well as the non-derivative term as follows,

$$\mathcal{F}_{\omega F}^{(1)} = -\sum_{[r s]} \sum_t d_s \left(Z_0^{rs} p^\nu_t K^\mu_r \varepsilon^\mu_\theta_s F(r, s) K^\nu_t \omega_{t0}(t)\right)$$

$$\mathcal{F}_{\omega F}^{(2)} = +\sum_{[r s]} \sum_t Z_0^{rs} K^\mu_r X^\mu_s F(r, s) p^\nu_t K^\nu_t \omega_{t0}(t)$$

(5.26)

In the sums defining $\mathcal{F}_{\omega F}^{(1)}$, the point $t$ can never coincide with the point $s$, because the corresponding kinematic factor $\varepsilon^\mu_\theta_s K^\mu_r$ vanishes. But $t$ is allowed to coincide with the point $r$, and this contribution yields the term $\mathcal{D}_{2c}$. The remaining contributions are when $t$ is different from $r$ and $s$, and they produce the term $\mathcal{D}_{3a}$ of (5.13).

In the sums defining $\mathcal{F}_{\omega F}^{(2)}$, the point $t$ may coincide with the point $r$ or with the point $s$, yielding the two different terms in $\mathcal{Z}_{2d}$. The remaining contributions are when $t$ is different from both $r$ and $s$, and they produce the term $\mathcal{Z}_{3a}$ of (5.11).

### 5.10 Contributions in $FF$ and $\tilde{K}\tilde{K}$

The contributions bilinear in $F$ are by far the most involved kinematically, and they will also force us to introduce new basic functions $Q_B$ and $\Phi_B$ in addition to the building blocks $\tilde{\omega}_{t0}$, $\lambda_I$, $F$, $P$, and $\Psi$. A specific part of the contributions bilinear in $F$ will naturally combine with the contribution from the $(\tilde{K}\tilde{K} - KK)$ term in (5.3). The total contribution of the terms bilinear in $F$ is given by,

$$\mathcal{F}_{FF} = \frac{1}{2} Z_0 \sum_{[rs]} \left(i K^\mu_r k^\mu_s F(r, s) + K^\mu_r \varepsilon^\mu_\theta_s d_s F(r, s)\right)$$

$$\times \sum_{[u]} \left(i K^\nu_u k^\nu_u F(t, u) + K^\nu_u \varepsilon^\nu_\theta_u d_u F(t, u)\right)$$

$$41$$
We split this quadruple sum into three parts,

\[ \mathcal{F}_{FF} = \mathcal{F}_{FF}^{(1)} + \mathcal{F}_{FF}^{(2S)} + \mathcal{F}_{FF}^{(2A)} \] (5.27)

Here, \( \mathcal{F}_{FF}^{(1)} \) is the part of \( \mathcal{F}_{FF} \) for which \( s \neq u \), while \( \mathcal{F}_{FF}^{(2S)} + \mathcal{F}_{FF}^{(2A)} \) is the part of \( \mathcal{F}_{FF} \) for which \( s = u \). The last contribution is further separated into \( \mathcal{F}_{FF}^{(2S)} \) which is symmetric under interchange of \( r \) and \( t \), and \( \mathcal{F}_{FF}^{(2A)} \) which is anti-symmetric under interchange of \( r \) and \( t \). The reason for separating the \( u \neq s \) from the \( u = s \) parts is that in \( \mathcal{F}_{FF}^{(1)} \), the differentials in \( s \) and in \( u \) can be pulled out without hitting \( t \) or \( r \) respectively, while they cannot in \( \mathcal{F}_{FF}^{(2S)} \) or \( \mathcal{F}_{FF}^{(2A)} \). The reason for separating into symmetric and anti-symmetric parts under \( r \leftrightarrow t \) will be explained later.

### 5.10.1 Calculation of \( \mathcal{F}_{FF}^{(1)} \)

The contributions in \( \mathcal{F}_{FF}^{(1)} \) are naturally gotten by iterating the splitting off the exact differential twice for the result gotten for the terms linear in \( F \),

\[
\mathcal{F}_{FF}^{(1)} = + \frac{1}{2} \sum_{rs} \sum_{tu, s \neq u} Z_{0}^{rs} K_{r}^{\mu} X_{s}^{\mu} F(r, s) K_{t}^{\nu} X_{u}^{\nu} F(t, u) \\
\quad + \frac{1}{2} \sum_{rs} \sum_{tu, s \neq u} d_{u} d_{s} \left( Z_{0}^{rstu} K_{r}^{\mu} e_{s}^{\mu} s_{s} F(r, s) K_{t}^{\nu} e_{u}^{\nu} F(t, u) \right) \\
\quad - \sum_{rs} \sum_{tu, s \neq u} d_{s} \left( Z_{0}^{rst} K_{r}^{\mu} e_{s}^{\mu} s_{s} F(r, s) K_{t}^{\nu} X_{u}^{\nu} F(t, u) \right)
\] (5.28)

In the first term of (5.28), the summation instructions for the points \( r, s, t, u \) leave the following options.

- Either \( u = r \), for which there are two possibilities. Either \( t = s \) as well, which produces the term \( Z_{2f} \) of (5.13); or \( t \neq s \), which produces instead the term

\[
- \sum_{rst} Z_{0}^{rst} \frac{1}{2} K_{r}^{\mu} X_{s}^{\mu} K_{t}^{\nu} X_{u}^{\nu} F(r, s) F(t, r)
\] (5.29)

After a further simplification of this term, in view of the fact that \( K_{r}^{\mu} X_{r}^{\nu} = K_{r}^{\mu} k_{r}^{\nu} \), half of the contribution \( Z_{3d} \) of (5.11) is produced.

- Or \( u \neq r \), for which there are three possibilities. The first is \( t = s \), which produces the other half of the contribution \( Z_{3d} \) of (5.11). The second is \( t = r \), which produces \( Z_{3e} \). The third is \( t \neq r, s \), which produces \( Z_{4} \).

In the second term in (5.28), the summation instructions for the points \( r, s, t, u \) leave only two possibilities. In view of the structure of the kinematic factors, we are forced to
have \( u \neq r \) and \( t \neq s \). Thus we can have that either all 4 points are pairwise distinct, yielding \( D_{4a} \), or \( t = r \), yielding \( D_{3d} \).

In the third term in (5.28), the summation instructions for the points \( r, s, t, u \) leave only three possibilities. In view of the kinematic factors, we are forced to have \( s \neq t \). The first possibility is to have all 4 points are pairwise distinct, yielding \( D_{4b} \); the second possibility is \( t = r \), yielding \( D_{3e} \); and the third possibility is to have \( u = r \) yielding \( D_{3f} \).

5.10.2 Calculation of \( \mathcal{F}_{FF}^{(2A)} \)

Next, we calculate \( \mathcal{F}_{FF}^{(2A)} \), which may be rewritten as

\[
\mathcal{F}_{FF}^{(2A)} = \frac{1}{2} \sum_{[rst]} Z_{0}^{rst} K_{t}^{\nu} K_{r}^{\mu} \left( ik_{s}^{\nu} F(r, s) + \varepsilon_{s}^{\nu} \theta_{s} d_{s} F(r, s) \right) \left( ik_{s}^{\mu} F(t, s) + \varepsilon_{s}^{\mu} \theta_{s} d_{s} F(t, s) \right) \quad (5.30)
\]

Splitting off a single exact differential term, we obtain,

\[
\mathcal{F}_{FF}^{(2A)} = -\frac{i}{2} \sum_{[rst]} Z_{0}^{rst} K_{r}^{\nu} K_{t}^{\mu} X_{s}^{\mu} F(r, s) F(t, s)
+ \sum_{[rst]} d_{s} \left( \frac{1}{2} Z_{0}^{rst} K_{t}^{\mu} K_{r}^{\nu} ik_{s}^{\mu} \varepsilon_{s}^{\nu} \theta_{s} d_{s} F(r, s) F(t, s) \right) \quad (5.31)
\]

The first and second terms produce \( Z_{3b} \) and \( D_{3b} \) respectively. In both terms, anti-symmetry under exchange of \( r \) and \( t \) is guaranteed by the anti-symmetry of \( F(r, s) F(t, s) \); it renders the tensor structure symmetric under interchange of \( \mu \) and \( \nu \).

5.10.3 Contributions from \( (\tilde{K} \tilde{K} - KK) \)

Before turning to the contributions of \( \mathcal{F}_{FF}^{(2S)} \), we work out the contribution \( \mathcal{F}_{\tilde{K}\tilde{K}} \) arising from the differences between \( (\tilde{K}_{s}^{\mu} \tilde{K}_{s}^{\mu} - K_{s}^{\mu} K_{s}^{\mu}) S(r, s) \). This is because exact differentials can only be extracted after these terms are recombined with the terms from \( \mathcal{F}_{FF}^{(2S)} \). This contribution is given by,

\[
\mathcal{F}_{\tilde{K}\tilde{K}} = -\sum_{[rs]} Z_{0}^{rs} K_{r}^{\mu} \varepsilon_{s}^{\mu} \mu(s) d\bar{s} S(r, s) \quad (5.32)
\]

At the point \( s \), there must be brought down an additional \( K_{s} \) factor from a fermion propagator term. (One cannot bring down a factor of \( \hat{\omega}_{I0} \) because this would be third order in \( \chi \) and thus vanish.) Doing this in all possible ways gives rise to

\[
\mathcal{F}_{\tilde{K}\tilde{K}} = -\sum_{[rs] t \neq s} Z_{0}^{rs} K_{t}^{\nu} \varepsilon_{s}^{\mu} \mu(s) d\bar{s} S(r, s) K_{t}^{\nu} S(t, s) \quad (5.33)
\]
This term is a \((0,1)\) form in \(s\), and may be rearranged using \(K_s^\nu \varepsilon_s^\mu = -i \varepsilon_s^\mu \theta_s k_s^\nu\). Upon symmetrization of \(r\) and \(t\), \(\mu\) and \(\nu\) are anti-symmetrized and we get

\[
\mathcal{F}_{KK} = \sum_{[rs]} \sum_{t \neq s} \frac{1}{2} Z_{0}^{rs} K_t^\mu K_s^\nu K_s^\mu K_t^\nu \mu(s) \frac{d\tilde{s}}{d\bar{s}} S(r, s) S(t, s)
\]

\[
= \sum_{[rs]} \frac{1}{2} Z_{0}^{rs} K_t^\mu K_s^\nu K_s^\mu K_t^\nu \mu(s) \frac{d\tilde{s}}{d\bar{s}} S(r, s) S(r, s)
\]

\[
+ \sum_{[rst]} \frac{1}{2} Z_{0}^{rst} K_t^\mu K_s^\nu K_s^\mu K_t^\nu \mu(s) \frac{d\tilde{s}}{d\bar{s}} S(r, s) S(t, s)
\]

(5.34)

where we have separated the contributions from two and three points. Both contributions will be combined with those of \(\mathcal{F}_{FF}^{(2S)}\) in the next subsection.

**5.10.4 Contributions from \(\mathcal{F}_{FF}^{(2S)}\)**

We can return now to the term \(\mathcal{F}_{FF}^{(2S)}\). We shall extract an exact differential by recombining it with the preceding contributions from the differences \((\tilde{K}_t^\mu \tilde{K}_s^\nu - K_t^\mu K_s^\nu) S(r, s)\).

\[
\mathcal{F}_{FF}^{(2S)} = \frac{1}{2} \sum_{[rs]} \sum_{t} Z_{0}^{rs} K_t^{[\nu} K_r^{\mu]} \left(ik_s^{\mu} F(r, s) + \varepsilon_s^{\mu} \partial_s d_s F(r, s)\right)
\]

\[
\times \left(ik_r^{\nu} F(t, s) + \varepsilon_r^{\nu} \partial_d d_s F(t, s)\right)
\]

(5.35)

In the product, the contribution in \(F(r, s) F(t, s)\) cancels out since \(K_t^{[\nu} K_r^{\mu]} k_t^\nu k_r^\mu = 0\), while the contribution in \(d_s F(r, s) d_s F(t, s)\) cancels in view of \((\theta_s)^2 = 0\). As a result, we are left with the following contributions,

\[
\mathcal{F}_{FF}^{(2S)} = \frac{1}{4} \sum_{[rs]} \sum_{t} Z_{0}^{rst} K_t^\mu K_r^\nu K_s^\mu K_s^\nu (ds)^{-1} \left(F(r, s) d_s F(t, s) - d_s F(r, s) F(t, s)\right)
\]

(5.36)

The terms in parentheses cannot be recast as a total differential in terms of the basic building blocks \(F\) and \(P\) only. To extract an exact differential requires introducing an intrinsic three-point function.

In the next section, it will be shown that there exists an intrinsic three point function \(Q_B(s; r, t)\) which satisfies the following \(\bar{\partial}\) equation,

\[
-2 \partial_s Q_B(s; r, t) = F(r, s) \partial_s F(t, s) - \partial_s F(r, s) F(t, s) + 2 \mu(s) S(r, s) S(t, s)
\]

(5.37)

Thanks to the presence of the last term in \(\mu(s)\), it will turn out that the function \(Q_B(s; r, t)\) is well-defined in \(s, r\) and \(t\). This will be shown in the next section, where an explicit form for \(Q_B(s; r, t)\) will also be calculated. Note that the \(\mu(s)\) term needed in the definition of \(Q_B\) is precisely provided by the terms \(\mathcal{F}_{KK}\) computed in the preceding subsection.
In view of the definition of $Q_B(s;r,t)$, the term (5.34) and the $d\bar{s}$ terms of $\mathcal{F}^{(2S)}_{FF}$ combine into a single $\bar{\partial}_s$ differential of type $(0,1)$ in $s$, 

$$\frac{1}{2} \bar{\partial}_s \sum_{[rst]} \left( Z_{0}^{rs} K_{r}^{\mu} K_{t}^{\nu} K_{s}^{\nu} (ds)^{-1} Q_B(s; r, t) \right)$$  \hspace{1cm} (5.38)$$

We can express $\bar{\partial}_s$ as $\bar{\partial}_s = ds - \partial_s$, and use the $ds$ part to isolate a total differential in $s$. The remaining terms are pure $(1,0)$ in $s$. They may be regrouped by the use of another intrinsic three-point functions $\Phi_B(s; r, t)$, defined by,

$$\Phi_B(s; r, t) \equiv \frac{1}{2} \partial_s F(t, s) F(r, s) + \frac{1}{2} \partial_s F(r, s) F(t, s) - \partial_s Q_B(s; r, t)$$  \hspace{1cm} (5.39)$$

Combining all contributions of $\mathcal{F}^{(2S)}_{FF}$ and $\mathcal{F}_{\bar{K}K}$, using $Q_B$ and $\Phi_B$, we find

$$\mathcal{F}^{(2S)}_{FF} + \mathcal{F}_{\bar{K}K} = Z_{2c} + Z_{3c} + D_{2d} + D_{3c}$$  \hspace{1cm} (5.40)$$

where we have separated out the contributions with two and three points. This completes the proof of the decomposition of chiral blocks $\mathcal{F}[^\delta]$ into the blocks $Z[^\delta]$ and $D[^\delta]$, listed in (5.11) and (5.13).
6 Solving $\bar{\partial}$ Equations with Monodromy

In the derivation of a holomorphic representative for the cohomology class of the chiral blocks of the $N$-point function, we shall need several key quantities which are solutions of $\bar{\partial}$ equations where the right hand side has monodromy. A basic example is the quantity $Q_B(s; r, t)$ defined by the following equation

$$\partial \bar{s} Q_B(s; r, t) = \frac{1}{2} \partial \bar{s} F(r, s) F(t, s) + \frac{1}{2} \partial \bar{s} F(t, s) F(r, s) - \mu(s) S(r, s) S(t, s)$$

subject to the requirement that $Q_B(s; r, t)$ have no monodromy in $r$ and in $t$. The main difficulty stems from the fact that the right-hand-side of this equation has non-trivial monodromy in $s$ since $F(r, s)$ and $F(t, s)$ do. The existence and properties of quantities such as $Q_B(s; r, t)$ are novel features in the function theory of Riemann surfaces, and have to be treated with some care.

In fact, the monodromy on the right-hand-side may be traced back to the monodromy of the scalar Green function in the definition of $F$. This allows the problem to be reduced to the study of a more basic quantity $Q_0(s; x, y)$, which is defined to obey

$$\partial \bar{s} Q_0(s; x, y) = \frac{1}{2} \partial \bar{s} G(x, s) G(y, s) - \frac{1}{2} \partial \bar{s} G(y, s) G(x, s)$$

The solution $Q_0(s; x, y)$ to this equation is a scalar function in $s$ with monodromy, which remains to be specified. In the variables $x$ and $y$, $Q_0(s; x, y)$ must be a $(1, 0)_x \otimes (1, 0)_y$ form with vanishing monodromy (so that the integrals over $\Sigma$, needed to obtain $F$ in (6.1), will be well-defined). Naturally, we also require that it be odd under interchange of $x$ and $y$, namely $Q_0(s; y, x) = -Q_0(s; x, y)$. The solution of this equation then immediately allows us to solve for $Q_B$ in (6.1), and we have

$$Q_B(s; r, t) = \frac{1}{16\pi^2} \int_x \int_y S(r, x) S(t, y) \chi(x) \chi(y) Q_0(s; x, y)$$

$$+ \frac{1}{2\pi} \int_x G(x, s) \mu(x) S(r, x) S(t, x).$$

All other cases needed here of $\bar{\partial}$ equations whose right-hand-side has monodromy can be reduced and solved with the help of the function $Q_0(s; x, y)$. Before setting out to solve for $Q_0(s; x, y)$, some simple considerations on monodromy will be reviewed first.

6.1 The simplest $\bar{\partial}$ equation

Let $\varpi = \varpi d\bar{s}$ be a single-valued $(0, 1)$-form on the compact Riemann surface $\Sigma$. A basic fact of function theory on Riemann surfaces is that the equation

$$\bar{s} f = \varpi$$

(6.4)
admits a single-valued smooth solution $f$ on $\Sigma$ (unique up to an additive constant) if and only if $\varpi$ satisfies the orthogonality conditions
\[ \int_\Sigma \omega_I \wedge \varpi = 0 \quad (6.5) \]
for any holomorphic Abelian differential $\omega_I$ on $\Sigma$. The equation may then be solved explicitly in terms of the scalar Green function $G$,
\[ f(s) = -\frac{1}{2\pi} \int_\Sigma d^2 x G(x, s) \varpi_x \quad (6.6) \]
When $\varpi$ is single-valued, but the orthogonality condition $(6.5)$ is violated, the function $f$ defined in $(6.6)$ still solves $(6.4)$, but $f$ now has non-trivial monodromy in $s$,
\[ \begin{align*}
    f(s + A_K) &= f(s) \\
    f(s + B_K) &= f(s) + \int_\Sigma \omega_K \wedge \varpi 
\end{align*} \quad (6.7) \]
As a result, the solution $f$ should be viewed as defined on the universal covering space of $\Sigma$ (in fact, on its quotient $\hat{\Sigma}$ by the subgroup of $\pi_1(\Sigma)$ generated by commutators).

It should be stressed that, if functions $f$ with monodromy are allowed, the solution $f$ is no longer unique. Since one can add to any solution $f$ of $(6.4)$ an arbitrary holomorphic Abelian integral on $\hat{\Sigma}$ and obtain a new solution, we cannot have uniqueness without some a priori constraints on the monodromy of $f$. Clearly, solutions $f$ will be unique (up to constants) if their monodromies on both $A_K$ and $B_K$ cycles are fixed. A useful observation is that, under the assumption that the monodromies of $f$ are all constant, the $A$-monodromies determine the $B$-monodromies. Indeed, by subtracting a suitable linear combination of Abelian integrals, we may assume that the $A$-monodromies of $f$ are all 0. Then the $B$-monodromies of $f$ are determined by
\[ \int_\Sigma \omega_L \wedge \varpi = \sum_K \oint_{A_K} \omega_L \left(f(z + B_K) - f(z)\right) = f(z + B_L) - f(z), \quad (6.8) \]
which are indeed the monodromies exhibited by the solution $(6.7)$. Thus, once a canonical homology basis $(A_K, B_K)$ has been chosen, we may view the solution with vanishing $A$-monodromies as the canonical choice among solutions with constant monodromies.

Finally, the case of interest here is when $\varpi$ itself has monodromy, so that it becomes more appropriate to view it as a differential form on $\hat{\Sigma}$. The solution $(6.6)$ given above in terms of surface integrals does not extend to the case where $\varpi$ has monodromy, since the resulting $f$ would then depend upon the fundamental domain chosen to represent $\Sigma$ in its covering space. But it is possible to solve $(6.4)$ in terms of line integrals on the cover space $\hat{\Sigma}$. One proceeds by completing the $\bar{\partial}$ equation of $(6.4)$ into a total differential equation for $f$,
\[ d_s f = \varpi + \rho \quad (6.9) \]
where \( \rho = \rho_s ds \) is required to be a pure \((1, 0)\) form. Clearly, the \((0, 1)\) component of (6.9) coincides with \( \partial_s f = \varpi \), and the \((1, 0)\) component simply determines \( \rho \). Since \( \varpi \) has monodromy on \( \Sigma \), and is properly defined only on \( \hat{\Sigma} \), the entire equation (6.9) is viewed on \( \hat{\Sigma} \). Given \( \varpi \) and a differential form \( \rho \) which satisfies the integrability condition,

\[
0 = d_s \varpi + d_s \rho \quad \Leftrightarrow \quad 0 = \partial_s \varpi - \partial_s \rho_s
\]

the unique solution \( f \) on \( \hat{\Sigma} \) is obtained in terms of a line integral,

\[
f(s) = f(s_0) + \int_{s_0}^{s} (\varpi + \rho) \quad (6.11)
\]

The form \( \rho \) is, of course, not unique, since the addition to \( \rho \) of any holomorphic differential on \( \hat{\Sigma} \) will produce again a solution to the integrability condition.

The non-trivial monodromy of \( \varpi \) results in non-trivial \( s \)-dependence of the monodromy of \( f \). To see this, we compute the monodromy of \( f \) around any homology cycle \( C \),

\[
f(s + C) - f(s) = \int_{s_0}^{s+C} (\varpi + \rho)(t) - \int_{s_0}^{s} (\varpi + \rho)(t)
\]

\[
= \int_{s_0+C}^{s+C} (\varpi + \rho)(t) - \int_{s_0}^{s+C} (\varpi + \rho)(t) + \int_{s_0}^{s+C} (\varpi + \rho)(t)
\]

\[
= \int_{s_0}^{s} ((\varpi + \rho)(t+C) - (\varpi + \rho)(t)) + \int_{s_0}^{s+C} (\varpi + \rho)(t)
\]

Thus, the \( s \)-dependence of the monodromy of \( f \) is given by the line integral of the monodromy of the form \( \varpi + \rho \). The \( s \)-independent part of the monodromy, given by the last integral, is generally more complicated to compute explicitly, but it will fortunately not be needed here. In the case of the function \( Q_0(s; x, y) \), the form \( \rho \) will in fact be monodromy-free.

### 6.2 Solving the integrability equation for \( Q_0(s; x, y) \)

The methods outlined in the preceding subsection will now be applied to the calculation of the function \( Q_0(s; x, y) \), which is required to be single-valued on \( \Sigma \) in \( x \) and \( y \). The defining equation (6.2) may be simplified by carrying out the \( \bar{\partial}_s \) derivatives on the Green function, and we obtain,

\[
\partial_s Q_0(s; x, y) = -\pi \delta(s, x) G(y, s) + \pi \delta(s, y) G(x, s)
\]

(6.13)

The presence of only \( \delta \)-functions on the right hand side reveals that \( Q_0(s; x, y) \) is holomorphic in \( s \) away from \( x \) and \( y \), making it natural to seek solutions for \( Q_0 \) by meromorphic line integrals. The first step is to complete the \( \bar{\partial}_s \) equation (6.2) into a \( d_s \) equation,

\[
d_s Q_0(s; x, y) = \frac{1}{2} d_s G(x, s) G(y, s) - \frac{1}{2} d_s G(y, s) G(x, s) + \rho(s; x, y)
\]

(6.14)
where \( \rho(s; x, y) \) is required to be a pure \((1, 0)\) form in \( s \), and a single-valued \((1, 0)\) form in \( x \) and \( y \). The integrability condition on \( Q_0 \) is,

\[
0 = d_s G(x, s) \wedge d_s G(y, s) + ds \wedge d\bar{s} \partial_s \rho_s(s; x, y)
\]  
(6.15)

It is a special property of the problem at hand that the first term has vanishing monodromy and vanishing \( s \)-integral over \( \Sigma \). As a result, the \( \partial_s \rho_s \) equation admits a solution \( \rho_s \) with vanishing monodromy in \( s \),

\[
\rho_s(s; x, y) = -\frac{1}{2\pi} \int_w G(s, w) \left( \partial_w G(x, w) \partial_w G(y, w) - \partial_w G(y, w) \partial_w G(x, w) \right) + \sum t c_K(x, y) \omega_I(s)
\]  
(6.16)

where \( c_I(x, y) \) are two as yet arbitrary functions of \( x \) and \( y \) only, subject to the condition \( c_I(y, x) = -c_I(x, y) \). The \( w \)-integral may be carried out, and we obtain,

\[
\rho_s(s; x, y) = \left( G(s, y) - G(s, x) \right) \partial_x \partial_y \ln E(x, y) + \sum t c_I(x, y) \omega_I(s)
\]  
(6.17)

where we have used the identity \( \partial_s G(x, y) = \partial_s G(y, x) = \partial_x \partial_y \ln E(x, y) \). To summarize, we have the following set of two integrable equations for \( Q_0 \),

\[
\partial_s Q_0(s; x, y) = \frac{1}{2} \partial_s G(x, s) G(y, s) - \frac{1}{2} \partial_s G(y, s) G(x, s) + \left( G(s, y) - G(s, x) \right) \partial_x \partial_y \ln E(x, y) + \sum t c_I(x, y) \omega_I(s)
\]

\[
\partial_s Q_0(s; x, y) = -\pi \delta(s, x) G(y, s) + \pi \delta(s, y) G(x, s)
\]  
(6.18)

An essential property of \( \partial_s Q_0 \) is its behavior as \( s \rightarrow x \), which may be read off from the \( \partial_s Q_0 \) equation above, and is given by,

\[
\partial_s Q_0(s; x, y) = + \frac{1/2}{(s-x)^2} G(y, x) + O(1)
\]  
(6.19)

(and similarly as \( s \rightarrow y \)). The fact that \( \partial_s Q_0(s; x, y) \) has only double poles (no residues) at \( s = x \) and \( s = y \) is what allows \( \partial_s Q_0 \) to be integrated unambiguously along any path which avoids these singularities,

\[
Q_0(s; x, y) = \int^s \left[ \frac{1}{2} d_s G(x, s) G(y, s) - \frac{1}{2} d_s G(y, s) G(x, s) - \left( G(s, x) - G(s, y) \right) \partial_x \partial_y \ln E(x, y) \right] + \sum t c_I(x, y) \int^s \omega_I(s)
\]  
(6.20)

In particular, we have

\[
Q_0(s; x, y) = -\frac{G(y, x)}{2(s-x)} + O(1)
\]  
(6.21)

as \( s \rightarrow x \) (and similarly as \( s \rightarrow y \)). The resulting \( Q_0(s; x, y) \) is meromorphic in \( s \) with only poles and no logarithmic branch cuts.
6.3 Solving for $Q_0(s; x, y)$

Single-valuedness on $\Sigma$ of $Q_0(s; x, y)$ in $x$ and $y$ requires that the right hand side of (6.18) be single-valued in $x$ and $y$. This is manifest for the first and third lines in (6.18), but single-valuedness of the second line imposes conditions on $c_I(x, y)$: it must have vanishing monodromy around $A$-cycles, while around a cycle $B_K$, its monodromy must be,

$$c_I(x + B_K, y) = c_I(x, y) + 2\pi i \delta_{IK} \partial_x \partial_y \ln E(x, y)$$  \hspace{2cm} (6.22)

The monodromy equation for $c_I$, combined with its symmetry $c_I(y, x) = -c_K(x, y)$, allow for the following general solution in terms the scalar function $\varphi_I(w)$.

The function $\varphi_I(w)$ is meromorphic and multiple-valued on $\Sigma$. It may be defined by its pairing relation with holomorphic 1-forms $\omega_I$,

$$\int w \omega_J(\overline{w}) \partial_{\overline{w}} \varphi_I(w) = \delta_{IJ}$$  \hspace{2cm} (6.23)

the requirement $\varphi_I(w_0) = 0$, and its monodromy

$$\varphi_I(w + A_I) = \varphi_I(w)$$
$$\varphi_I(w + B_K) = \varphi_I(w) - 2\pi i \delta_{IJ}$$  \hspace{2cm} (6.24)

The associated Green’s function $G^\varphi(z, w)$, obtained by setting

$$G^\varphi(z, w) = G(z, w) + \sum_I \omega_I(z) \varphi_I(w)$$  \hspace{2cm} (6.25)

is single-valued in both $z$ and $w$, and satisfies the following equations

$$\partial_z G^\varphi(z, w) = +2\pi \left( \delta(z, w) - \delta(z, w_0) \right)$$
$$\partial_w G^\varphi(z, w) = -2\pi \delta(z, w) + \sum_I \omega_I(z) \partial_w \varphi_I(w)$$  \hspace{2cm} (6.26)

In terms of the function $\varphi_I(w)$, we may now easily write down the solution for $c_I(x, y)$,

$$c_I(x, y) = c_I^{(0)}(x, y) - \left( \varphi_I(x) - \varphi_I(y) \right) \partial_x \partial_y \ln E(x, y)$$  \hspace{2cm} (6.27)

where $c_I^{(0)}(x, y)$ is an arbitrary $(1, 0)_x \otimes (1, 0)_y$ form which is single-valued in $x$ and $y$ on $\Sigma$. In terms of the single-valued Green function $G^\varphi(s, x)$, the result may be expressed equivalently as,

$$\partial_s Q_0(s; x, y) = \frac{1}{2} \partial_s G(x, s) G(y, s) - \frac{1}{2} \partial_s G(y, s) G(x, s)$$
$$+ \left( G^\varphi(s, y) - G^\varphi(s, x) \right) \partial_x \partial_y \ln E(x, y) + \sum_I c_I^{(0)}(x, y) \omega_I(s)$$  \hspace{2cm} (6.28)
Although \( \varphi_I \) appears to introduce new data, the dependence on \( \varphi_I \) will be absorbed by the functions \( c^{(0)}_I \). To see this, we perform an infinitesimal variation \( \delta \varphi_I \) on \( \varphi_I \), under which we have

\[
\partial_s \delta Q_0(s; x, y) = \sum_I \left( \left( \delta \varphi_I(y) - \delta \varphi_I(x) \right) \partial_x \partial_y \ln E(x, y) + \delta c^{(0)}_I(x, y) \right) \omega_I(s) \tag{6.29}
\]

All dependence on \( \varphi_I \) may be eliminated by requiring that the differential \( \partial_s Q_0 \) have vanishing \( A \)-periods in \( s \). Since the differential \( \partial_s Q_0 \) has vanishing monodromy on \( A \)-cycles, this condition is equivalent to

\[
c^{(0)}_I(x, y) = -\frac{1}{2} \oint_{A_I} ds \left( \partial_s G(x, s)G(y, s) - \partial_s G(y, s)G(x, s) \right) + 2 \left( G^\varphi(s, y) - G^\varphi(s, x) \right) \partial_x \partial_y \ln E(x, y) \tag{6.30}
\]

Since the integrand on the right hand side is manifestly single-valued on \( \Sigma \) in \( x \) and \( y \), and all the poles in \( s \) have vanishing residue, the form \( c^{(0)}_I(x, y) \) defined this way is automatically single-valued in \( x \) and \( y \) as well. It is easy to check that \( Q_0 \) defined this way is invariant under changes of \( \varphi_I \), and is thus intrinsically defined.

In practice, it will often be convenient to calculate with the help of the function \( Q^\varphi_0(s; x, y) \) which is defined to obey

\[
\partial_s Q^\varphi_0(s; x, y) = \frac{1}{2} \partial_s G(x, s)G(y, s) - \frac{1}{2} \partial_s G(y, s)G(x, s) + \left( G^\varphi(s, y) - G^\varphi(s, x) \right) \partial_x \partial_y \ln E(x, y) \tag{6.31}
\]

and then obtain the function \( Q_0(s; x, y) \) by subtracting the \( A \)-periods in \( s \) of \( Q^\varphi_0 \),

\[
Q_0(s; x, y) = Q^\varphi_0(s; x, y) - \sum_I \left( \int_{w_0}^s \omega_I \right) \left( \oint_{A_I} dt \partial_t Q^\varphi_0(t; x, y) \right) \tag{6.32}
\]

The two constructions of \( Q_0(s; x, y) \) are equivalent, as may be checked by calculation.

### 6.4 Definition of \( Q_B \), \( Q_F \), and \( Q_P \)

During the course of our construction of holomorphic chiral blocks and analysis of their cohomology, we shall need solutions to the following \( \bar{\partial} \) equations which have right hand sides with non-zero monodromy in \( s \),

\[
\begin{align*}
\partial_s Q_B(s; r, t) &= \frac{1}{2} \partial_s F(r, s)F(t, s) + \frac{1}{2} \partial_s F(t, s)F(r, s) - \mu(s) S(r, s) S(t, s) \\
\partial_s Q_F(s; r, t) &= \frac{1}{2} \partial_s F(t, s)G(r, s) - \frac{1}{2} \partial_s G(r, s)F(t, s) \\
\partial_s Q_P(s; r, t) &= \frac{1}{2} \partial_s P(r, s)G(t, s) - \frac{1}{2} P(r, s) \partial_s G(t, s) \tag{6.33}
\end{align*}
\]
Each function is a scalar in $s$. $Q_B$ is a spinor in $r,t$, and has vanishing monodromy in these variables; by construction, we have $Q_B(s; r, t) = Q_B(s; t, r)$. $Q_F$ is a spinor in $t$, and a $(1,0)$ form in $r$, and has vanishing monodromy in these variables. $Q_P$ is a 1-form in $t$ in which it has no monodromy, and a scalar in $r$ in which it does have monodromy. All three equations are readily solved using the function $Q_0(s; x, y)$ constructed earlier;

$$Q_B(s; r, t) = \frac{1}{(4\pi)^2} \int_x \int_y S(r, x)S(t, y)\chi(x)\chi(y)Q_0(s; x, y)$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad + \frac{1}{2\pi} \int_x G(x, s)\mu(x)S(r, x)S(t, x)$$
$$Q_F(s; r, t) = -\frac{1}{4\pi} \int_x S(t, x)\chi(x)Q_0(s; r, x)$$
$$Q_P(s; r, t) = \frac{1}{16\pi^2} \int_x \int_y G(x, r)\chi(x)S(x, y)\chi(y)Q_0(s; y, t)$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad - \frac{1}{2\pi} \int_x G(x, r)\mu(x)Q_0(s; x, t).$$

All integrals in $x$ and $y$ are well-defined on $\Sigma$ since $Q_0(s; x, y)$ was constructed to be single-valued in $x$ and $y$.

### 6.5 Monodromies of $Q_0$, $Q_B$, $Q_F$, and $Q_P$

We shall now evaluate the monodromies in $s$ of the quantities $Q_0(s; x, y)$, $Q_B(s; x, y)$, $Q_F(s; x, y)$, and $Q_P(s; x, y)$. By construction, the $A$-cycle monodromies in $s$ of $Q_0(s; x, y)$, and hence of $Q_B(s; x, y)$, $Q_F(s; x, y)$, and $Q_P(s; x, y)$ all vanish. Thus, we need to evaluate only the $B$-cycle monodromies. Actually, we shall require only the $B$-cycle monodromies up to additive terms which are independent of $s$, and the formulas given below will hold only up to such terms.

From the form of $d_sQ_0(s; x, y)$ in (6.18), we can read off the monodromy of $d_sQ_0$ under $s \rightarrow s' = s + B_K$. Similarly, the monodromies of $d_sQ_B$, $d_sQ_F$, and $d_sQ_P$ follow from the definitions of $Q_B$, $Q_F$, and $Q_P$,

$$d_sQ_0(s'; x, y) - d_sQ_0(s, x, y) = -i\pi d_sG(y, s)\omega_K(x) + i\pi d_sG(x, s)\omega_K(y)$$
$$d_sQ_B(s'; r, t) - d_sQ_B(s, r, t) = -i\pi d_sF(t, s)\hat{\omega}_{K0}(r) - i\pi d_sF(r, s)\hat{\omega}_{K0}(t)$$
$$d_sQ_F(s'; r, t) - d_sQ_F(s, r, t) = +i\pi d_sF(t, s)\omega_K(r) - i\pi d_sF(r, s)\omega_K(t)$$
$$d_sQ_P(s'; r, t) - d_sQ_P(s, r, t) = +i\pi d_sG(t, s)\lambda_K(r) + i\pi d_sP(r, s)\omega_K(t).$$

(6.35)

Applying now the general formula (6.12), we find the monodromy of $Q_0(s; x, y)$. The monodromies of $Q_0(s; x, y)$, $Q_B(s; r, t)$, $Q_F(s; r, t)$, and $Q_P(s; r, t)$ follow at once,

$$Q_0(s'; x, y) - Q_0(s, x, y) = -i\pi G(y, s)\omega_K(x) + i\pi G(x, s)\omega_K(y)$$

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\[ Q_B(s'; r, t) - Q_B(s; r, t) = -i\pi F(t, s)\hat{\omega}_{K0}(r) - i\pi F(r, s)\hat{\omega}_{K0}(t) \]
\[ Q_F(s'; r, t) - Q_F(s; r, t) = +i\pi F(t, s)\omega_K(r) - i\pi G(r, s)\hat{\omega}_{K0}(t) \]
\[ Q_P(s'; r, t) - Q_P(s; r, t) = +i\pi G(t, s)\lambda_K(r) + i\pi P(r, s)\omega_K(t) \] (6.36)

up to an additive contribution which is independent of \( s \). The \( s \)-independent integral term arising in the monodromy of \( Q_0 \), for example, is given by,

\[ \int_{w_0}^{w_0 + B_K} d_t Q_0(t; x, y) \] (6.37)

Its form is akin to the integral occurring in the construction of the Riemann vector \( \Delta_I \), but its explicit expressions will not be needed here.

### 6.6 Derivatives of \( Q_0, Q_B, Q_F, \) and \( Q_P \)

It will be convenient to group together in this section all the derivatives of the quantities \( Q_B, Q_F, \) and \( Q_P \) that we will need. The \( \partial_s \) derivatives are given by their defining equations, while the other derivatives are obtained by differentiating their integral formulas, as well as the following result,

\[ \partial_s Q_0(s; x, y) = \pi G(x, s)\left(2\delta(y, x) - \delta(y, s) - \delta(y, w_0)\right) \] (6.38)

We find for \( Q_B(r, s; t) \)

\[ \partial_s Q_B(s; r, t) = -\frac{1}{4}\chi(s)\left(S(t, s)F(r, s) + S(r, s)F(t, s)\right) - \mu(s)S(r, s)S(t, s) \] (6.39)

as well as the derivatives,

\[ \partial_r Q_B(s; r, t) = -\frac{1}{2}\chi(r)Q_F(s; r, t) + G(r, s)\mu(r)S(t, r) \]
\[ \partial_r Q_B(s; r, r) = -\chi(r)Q_F(s; r, r) - \partial_r\left(\mu(r)G(r, s)\right) \] (6.40)

For \( Q_F \), we find,

\[ \partial_s Q_F(s; r, t) = -\frac{1}{4}\chi(s)S(t, s)G(r, s) + \pi\delta(r, s)F(t, r) \]
\[ \partial_t Q_F(s; r, t) = \frac{1}{2}\chi(r)S(t, r)G(r, s) - \pi\left(\delta(r, s) + \delta(r, w_0)\right) F(t, s) \]
\[ \partial_t Q_F(s; r, t) = -\frac{1}{2}\chi(t)Q_0(s; r, t) \] (6.41)
For $Q_P$, we find,
\[
\partial_s Q_P(s; r, t) = -\frac{1}{4} \chi(s) F(s, r) G(t, s) + \frac{1}{2} \mu(s) G(s, r) G(t, s) + \pi P(r, s) \delta(t, s)
\]
\[
\partial_t Q_P(s; r, t) = -\frac{1}{2} \chi(r) Q_F(s; t, r) + \mu(r) Q_0(s; r, t)
\]
\[
\partial_t Q_P(s; r, t) = \frac{1}{2} \chi(t) F(t, r) G(t, s) - \mu(t) G(t, r) G(t, s) - \pi \left( \delta(t, s) + \delta(t, w_0) \right) P(r, s)
\]

### 6.7 Transformations under changes of slice

The changes under slice $\mu$ are given by (up to contributions due to $v(w_0)$ and $\xi(w_0)$.)
\[
\delta_s Q_B(s; r, t) = -v(s) S(r, s) S(t, s) + v(r) G(r, s) S(t, r) + v(t) G(t, s) S(r, t)
\]
\[
\delta_s Q_F(s; r, t) = 0
\]
\[
\delta_s Q_P(s; r, t) = v(r) Q_0(s; r, t) - v(t) G(t, s) G(t, r) + \frac{1}{2} v(s) G(t, s) G(s, r)
\]

The changes under slice $\chi$ are given by
\[
\delta_s Q_B(s; r, t) = \frac{1}{2} \xi(s) F(t, s) S(r, s) + \frac{1}{2} \xi(s) F(r, s) S(t, s) + \xi(r) Q_F(s; r, t) + \xi(t) Q_F(s; t, r)
\]
\[
\delta_s Q_F(s; r, t) = \frac{1}{2} \xi(s) G(r, s) S(t, s) - \xi(r) G(r, s) S(t, r) + \xi(t) Q_0(s; r, t)
\]
\[
\delta_s Q_P(s; r, t) = \xi(r) Q_F(s; t, r) - \xi(t) G(t, s) F(t, r) + \frac{1}{2} \xi(s) G(t, s) F(s, r)
\]

### 6.8 Definition and properties of $\Phi_B$, $\Phi_F$, and $\Phi_P$

We define the $(1, 0)$ forms in $s$, denoted by $\Phi_B(s; r, t)$, $\Phi_F(s; r, t)$, and $\Phi_P(s; r, t)$ as follows,
\[
\Phi_B(s; r, t) \equiv \frac{1}{2} \partial_s F(t, s) F(r, s) + \frac{1}{2} \partial_t F(t, s) F(t, s) - \partial_s Q_B(s; r, t)
\]
\[
\Phi_F(s; r, t) \equiv \frac{1}{2} \partial_s F(t, s) G(r, s) - \frac{1}{2} \partial_t G(r, s) F(t, s) - \partial_s Q_F(s; r, t)
\]
\[
\Phi_P(s; r, t) \equiv \frac{1}{2} \partial_s P(r, s) G(t, s) - \frac{1}{2} \partial_t G(t, s) P(t, s) - \partial_s Q_P(s; r, t)
\]

Each term separately is monodromy-free in $r$ and $t$, but has monodromy in $s$. The quantities $\Phi_B(s; r, t)$ and $\Phi_P(s; r, t)$ satisfy the symmetry properties, $\Phi_B(s; r, t) = \Phi_B(s; t, r)$, and
\( \Phi_{P}(s; r, t) = -\Phi_{P}(s; t, r) \). Using the explicit expressions for \( Q_{B}, Q_{F}, \) and \( Q_{P} \), one shows that they are monodromy-free in \( s \). In fact, these quantities are essentially the analogs of the \( \rho \) differentials which entered into our general considerations in section 4.2, which also confirms why they should be monodromy-free in \( s \).

The derivatives of these quantities are readily obtained from their definitions,

\[
\begin{align*}
\partial_{s} \Phi_{B}(s; r, t) &= + \frac{1}{2} \chi(s) \left( \partial_{s} F(t, s) S(r, s) + \partial_{s} F(r, s) S(t, s) \right) \\
&+ \partial_{s} \left( \mu(s) S(r, s) S(t, s) \right) \\
\partial_{r} \Phi_{B}(s; r, t) &= - \frac{1}{2} \chi(r) \Phi_{F}(s; r; t) - \mu(r) \partial_{s} G(r, s) S(t, r) \\
\end{align*}
\]

as well as,

\[
\begin{align*}
\partial_{s} \Phi_{F}(s; r; t) &= \frac{1}{2} \chi(s) S(t, s) \partial_{s} G(r, s) - 2\pi \delta(r, s) \partial_{s} F(t, s) \\
\partial_{r} \Phi_{F}(s; r; t) &= - \frac{1}{2} \chi(r) S(t, r) \partial_{s} G(r, s) + 2\pi \delta(r, s) \partial_{s} F(t, s) \\
\partial_{t} \Phi_{F}(s; r; t) &= - \frac{1}{2} \chi(t) \left( G(s, r) - G(s, t) \right) \partial_{r} \partial_{t} \ln E(r, t) \\
\end{align*}
\]

### 6.9 Definition and derivatives of \( \Psi(r, s) \)

Finally, we also require the following composites, whose definitions are straightforward, but which we also include in this section for easy reference:

\[
\begin{align*}
\Psi(r, s) &= - \frac{1}{16\pi^{2}} \int_{x} \int_{y} S(r, x) \chi(x) \partial_{x} \partial_{y} \ln E(x, y) \chi(y) S(y, s) \\
&+ \frac{1}{4\pi} \int_{x} \mu(x) \left( S(r, x) \partial_{x} S(x, s) - S(x, s) \partial_{x} S(x, r) \right) \\
\end{align*}
\]

From this definition, we have \( \Psi(s, r) = -\Psi(r, s) \). The derivatives are

\[
\begin{align*}
\partial_{r} \Psi(r, s) &= - \frac{1}{2} \partial_{r} F(s, r) \chi(r) - \frac{1}{2} \mu(r) \partial_{s} S(r, s) - \frac{1}{2} \partial_{r} \left( \mu(r) S(r, s) \right) \\
\end{align*}
\]
7 Linear Chain Blocks

In this section and the next, we construct the blocks which will ultimately be extracted from the decomposition of chiral blocks given in the previous section. The extraction process itself will be given in subsequent sections. Since it is combinatorially involved, it is helpful to know in advance what the blocks are. The blocks are of several types: holomorphic in all insertion points, holomorphic in all but one insertion point, holomorphic in all but two insertion points (all away from the diagonal \( z_r = z_s; \) henceforth, this is to be understood and we shall not mention it explicitly.) The corresponding differentials will always include a full exterior differential \( d_r \) with respect to each insertion point \( z_r \) if the block is not holomorphic in \( z_r \). Furthermore, the blocks can be classified by a corresponding kinematic invariant.

This section is devoted to the definition and construction of the holomorphic blocks which have the connectivity of a linear chain. The most basic such block is the object \( \Pi^{(n+2)} \) which plays a key role in lifting the obstruction to holomorphicity caused by the presence of \( (0,1) \otimes (0,1) \) forms in the chiral amplitudes. It is in this role that they were already identified in the Introduction. The monodromy and derivative operations on the blocks \( \Pi^{(n+2)} \) produce new blocks \( \Pi^{(n+2)}_I, \Pi^{(n+2)}_{IJ}, \) and \( \Pi^{(n+2)}_F \) whose role in the cohomology construction of chiral amplitudes will be discussed in later sections.

7.1 Notation

It is convenient to introduce the following notation

\[
G^n(t_1, \ldots, t_{n+1}) \equiv G(t_1, t_2)G(t_2, t_3) \cdots G(t_n, t_{n+1}), \quad G^{(0)}(t, t) = 1
\]

\[
S^n(t_1, \ldots, t_{n+1}) \equiv S(t_1, t_2)S(t_2, t_3) \cdots S(t_n, t_{n+1}), \quad S^{(0)}(t, t) = 1 \quad (7.1)
\]

As with kinematic invariants, we will denote all blocks by the same generic letter, in this case \( \Pi \), and distinguish them from one another only by their indices.

7.2 Motivation and definition of the blocks \( \Pi \)

A natural starting point for the definition and construction of the blocks \( \Pi^{(n+2)} \) is found in their role in lifting the obstruction to holomorphicity caused by the presence of \( (0,1) \otimes (0,1) \) forms in the chiral amplitude. The contribution of such a form arises from the third line in the component form of the chiral amplitude given in (1.12), and is given by

\[
\left\langle Q(p_I) \mathcal{V}_r^{(1)} \mathcal{V}_s^{(1)} \prod_{l \neq r, s}^N \mathcal{V}_l^{(0)} \right\rangle \quad (7.2)
\]

The proportionality of \( \mathcal{V}_r^{(1)} \) to \( \chi(r) \psi_+ (r) \) forces this correlator to always contain a linear strand of RNS fermion \( \psi_+ \) contractions. Therefore, the above correlator will always contain
a linear strand of Szegö kernels, arranged as follows,

\[ d\bar{r}\chi(r)S(r, t_1)S(t_1, t_2)\cdots S(t_{n-1}, t_n)S(t_n, s)\chi(s)d\bar{s} \quad (7.3) \]

Here, the points \( t_1, t_2, \ldots, t_{n-1}, t_n \) are all distinct from one another and from \( r \) and \( s \), as is guaranteed by the structure of the Wick contractions for the free field \( \psi_+ \). The proposition to be proven here is that all such terms in the chiral amplitudes must in fact be exact differentials of well-defined functions. Lifting the holomorphicity obstruction just for this term thus requires that there exist a function \( \Pi^{(n+2)}(r; t_1, t_2, \ldots, t_{n-1}, t_n; s) \), which is a \((0, 0)\) form in \( r \) and \( s \), and a \((1, 0)\) form in \( t_1, \ldots, t_n \), and is such that\(^{10}\)

\[ \partial_r \partial_s \Pi^{(n+2)}(r; t_1, t_2, \ldots, t_{n-1}, t_n; s) = \frac{1}{4} \chi(r)S(r, t_1, t_2, \ldots, t_{n-1}, t_n, s)\chi(s) \quad (7.4) \]

The blocks \( \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) \) must be holomorphic in each of the points \( t_1, \ldots, t_n \), away from coincident points \( t_i = t_j \) for \( i \neq j \), and away from the points \( r \) and \( s \). Finally, the blocks must have the same mirror symmetry as the correlator, so that we require

\[ \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) = (-)^n \Pi^{(n+2)}(s; t_n, \ldots, t_1; r) \quad (7.5) \]

Finally, we require that its monodromy in \( r \) is independent of \( r \), and the monodromy in \( s \) is independent of \( s \). Although one might initially have hoped that blocks \( \Pi^{(n+2)} \) would exist without monodromy in \( r, s \), it turns out to be impossible to achieve without introducing new singularities at extraneous points. Instead of being a problem, the nontrivial monodromies of \( \Pi^{(n+2)} \) will combine precisely with the monodromies of the \((0, 1)\) form obstructions and help lift those as well.

### 7.3 Construction of the blocks \( \Pi \)

The simplest case arises for \( n = 0 \), where the block must satisfy

\[ \partial_r \partial_s \Pi^{(2)}(r; s) = \frac{1}{4} \chi(r)S(r, s)\chi(s) \quad (7.6) \]

The solution to this equation may be gathered from the derivative formulas of the basic functions \( P \) and \( F \) given in (3.21), and we find that \( \Pi^{(2)}(r, s) = P(r, s) \). The next few lowest order blocks may be obtained by combining \( P \), \( F \) and the bosonic Green function \( G \) and Szegö kernel \( S \), and we find for \( 0 \leq n \leq 2 \),

\[ \Pi^{(2)}(r, s) = P(r, s) \]

\[ \Pi^{(3)}(r; t; s) = P(r, t)G(t, s) - G(t, r)P(t, s) - F(t, r)F(t, s) \]

\[ \Pi^{(4)}(r; t_1, t_2; s) = +P(r, t_1)G(t_1, t_2)G(t_2, s) - G(t_1, r)P(t_1, t_2)G(t_2, s) + G(t_1, r)G(t_2, t_1)P(t_2, s) - F(t_1, r)F(t_1, t_2)G(t_2, s) - F(t_1, r)S(t_1, t_2)F(t_2, s) + G(t_1, r)F(t_2, t_1)F(t_2, s) \]

\(^{10}\)The factor \( 1/4 \) is introduced as a convenient normalization; henceforth we use the notation of (7.1).
A clear pattern for all values of \( n \) is recognized, and takes the following form,

\[
\Pi^{(n+2)} = \sum_{\alpha=0}^{n} (-)^{\alpha} G^{\alpha} P G^{n-\alpha} - \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-1-\alpha} (-)^{\alpha} G^{\alpha} F S^{\beta} F G^{n-1-\alpha-\beta}
\]  \( (7.8) \)

It is useful to write out the dependence on the points of \( \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) \),

\[
\Pi^{(n+2)} = \sum_{\alpha=0}^{n} (-)^{\alpha} G^{\alpha}(t_\alpha, \ldots, t_1, r) P(t_\alpha, t_{\alpha+1}) G^{n-\alpha}(t_{\alpha+1}, \ldots, t_n, s)
\]

\[
\quad - \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-1-\alpha} (-)^{\alpha} G^{\alpha}(t_\alpha, \ldots, t_1, r) F(t_{\alpha+1}, t_\alpha) S^{\beta}(t_{\alpha+1}, \ldots, t_{\alpha+\beta+1})
\]

\[
\times F(t_{\alpha+\beta+1}, t_{\alpha+\beta+2}) G^{n-1-\alpha-\beta}(t_{\alpha+\beta+2}, \ldots, t_n, s)
\]  \( (7.9) \)

By construction, the blocks \( \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) \) are \((0,0)\) forms in \( r \) and \( s \), and \((1,0)\) forms in \( t_1, \ldots, t_n \), and obey the symmetry relation (1.22). The expressions for \( n = 0, 1, 2 \) manifestly reproduce those found in (7.7). We shall later verify that \( \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) \) is indeed holomorphic in \( t_1, \ldots, t_n \), away from coincident points \( t_i = t_j \) for \( i \neq j \) and \( t_i \) away from the points \( r \) and \( s \).

### 7.4 Monodromy of \( \Pi \) produces new blocks \( \Pi_I \)

Monodromy of \( \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) \) produces new blocks. The \( A \)-cycle monodromy is absent in any of the arguments of \( \Pi^{(n+2)} \), but the blocks in general have \( B \)-cycle monodromy. We shall define the following \( B \)-cycle monodromy operator

\[
\Delta^{(z)}_K f(z) \equiv \frac{1}{2\pi i} \left( f(z + B_K) - f(z) \right)
\]  \( (7.10) \)

while leaving any possible other independent variables in \( f \) unchanged. By using the monodromy properties of \( P \), \( F \), and \( G \), we find the monodromy of \( \Pi^{(n+2)} \). The monodromies at the end points are given by,

\[
\Delta^{(r)}_K \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) = -\Pi^{(n+1)}_K(t_1; \ldots, t_n; s)
\]

\[
\Delta^{(s)}_K \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) = -(-)^n \Pi^{(n+1)}_K(t_n; t_{n-1}, \ldots, t_1; r)
\]  \( (7.11) \)

where the blocks \( \Pi^{(n+1)}_K \) are defined below. The monodromies at the midpoints \( t_k, k = 1, \ldots, n \), are given by,

\[
\Delta^{(t_k)}_K \Pi^{(n+2)}(r; t_1, \ldots, t_n; s) = -(-)^k G^k \Pi^{(n+1-k)}_K(t_{k+1}; t_{k+2}, \ldots, t_n; s)
\]

\[
+ \Pi^{(k)}_K(r; t_1, \ldots, t_{k-2}; t_{k-1}) G^{n+1-k}
\]  \( (7.12) \)

where we use the abbreviations \( G^k = G^k(t_k; \ldots, t_1, r) \) and \( G^{n+1-k} = G^{n+1-k}(t_k; \ldots, t_n, s) \).
The new blocks \( \Pi_{I}^{(n+2)}(r; t_1, \ldots, t_n; s) \), defined for \( n \geq -1 \), may be computed for the first few lowest orders,

\[
\begin{align*}
\Pi_{I}^{(1)}(s) &= \lambda_{I}(s) \\
\Pi_{I}^{(2)}(r, s) &= \lambda_{I}(r)G(r, s) + \omega_{I}(r)P(r, s) - \hat{\omega}_{I0}(r)F(r, s) \\
\Pi_{I}^{(3)}(r; t; s) &= \lambda_{I}(r)G(r, t)G(t, s) + \omega_{I}(r)P(r, t)G(t, s) \\
&\quad - \hat{\omega}_{I0}(r)F(r, t)G(t, s) + \hat{\omega}_{I0}(r)S(r, t)F(t, s) \\
&\quad + \omega_{I}(r)F(t, r)F(t, s) + \omega_{I}(r)G(t, r)P(s, t) \\
&\quad = \lambda_{I}(r)G^{n+1}(r, t_1, \ldots, t_n, s) \\
&\quad + \omega_{I}(r)\Pi_{I}^{(n+2)}(r; t_1, \ldots, t_n; s) \\
&\quad - \hat{\omega}_{I0}(r)\Pi_{F}^{(n+2)}(r; t_1, \ldots, t_n; s) \\
\end{align*}
\] (7.13)

The expressions for the blocks for general \( n \) are given as follows,

\[
\Pi_{I}^{(n+2)}(r; t_1, \ldots, t_n; s) = \sum_{\alpha=0}^{n} S^{\alpha}(r, t_1, \ldots, t_{\alpha})F(t_{\alpha}, t_{\alpha+1})G^{n-\alpha}(t_{\alpha+1}, \ldots, t_n, s) \\
\] (7.14)

Here, we have introduced yet another block \( \Pi_{F}^{(n+2)}(r; t_1, \ldots, t_n; s) \) which is Grassmann odd. Its explicit form will be presented in the next subsection.

### 7.5 The fermionic blocks \( \Pi_{F} \)

The monodromy calculation of the blocks \( \Pi \) yields the blocks \( \Pi_{I} \), which in turn are directly expressed in terms of the fermionic blocks \( \Pi_{F} \). The latter take the following explicit form in terms of the building blocks \( P, F, S \) and \( G \),

\[
\Pi_{F}^{(n+2)}(r; t_1, \ldots, t_n; s) = \sum_{\alpha=0}^{n} S^{\alpha}(r, t_1, \ldots, t_{\alpha})F(t_{\alpha}, t_{\alpha+1})G^{n-\alpha}(t_{\alpha+1}, \ldots, t_n, s) \\
\] (7.15)

The lowest order expressions for these blocks are

\[
\begin{align*}
\Pi_{F}^{(2)}(r, s) &= F(r, s) \\
\Pi_{F}^{(3)}(r; t; s) &= F(r, t)G(t, s) + S(r, t)F(t, s) \\
\Pi_{F}^{(4)}(r; t_1, t_2; s) &= F(r, t_1)G(t_1, t_2)G(t_2, s) + S(r, t_1)F(t_1, t_2)G(t_2, s) \\
&\quad + S(r, t_1)S(t_1, t_2)F(t_2, s) \\
&\quad = \Pi_{F}^{(m+2)}(r; t_1, \ldots, t_m, w)G^{\ell+1}(w, u_1, \ldots, u_{\ell}, s) \\
&\quad + S(r, t_1, \ldots, t_m, w)\Pi_{F}^{(\ell+2)}(w, u_1 \cdots, u_{\ell}, s) \\
\end{align*}
\] (7.16)

The blocks \( \Pi_{F}^{(n+2)} \) satisfy a generalized recursion relation, which allows one to split this block at any midpoint. This relation takes the form,

\[
\begin{align*}
\Pi_{F}^{(m+\ell+3)}(r; t_1, \ldots, t_m, w, u_1 \cdots, u_{\ell}; s) &= \Pi_{F}^{(m+2)}(r; t_1, \ldots, t_m, w)G^{\ell+1}(w, u_1, \ldots, u_{\ell}, s) \\
&\quad + S(r, t_1, \ldots, t_m, w)\Pi_{F}^{(\ell+2)}(w, u_1 \cdots, u_{\ell}, s) \\
\end{align*}
\] (7.17)
The use of $\Pi_F$ also allows one to obtain an analogous generalized recursion relation for the block $\Pi$ itself,

$$
\Pi^{(m+\ell+3)}(r; t_1, \ldots, t_m, w, u_1, \ldots, u_\ell; s) = 
\Pi^{(m+2)}(r; t_1, \ldots, t_m, w) G^{\ell+1}(w, u_1, \ldots, u_\ell, s) 
- (-)^m G^{m+1}(w, t_m, \ldots, t_1, r) \Pi^{(\ell+2)}(w, u_1, \ldots, u_\ell; s) 
- (-)^m \Pi^{(m+2)}_F(w; t_m, \ldots, t_1; r) \Pi^{(\ell+2)}_F(w, u_1, \ldots, u_\ell; s)
$$

(7.18)

The monodromies of the blocks $\Pi_F^{(n+2)}$ do not produce any new blocks, but it will nonetheless be useful to record their form,

$$
\Delta^{(r)}_n \Pi_F^{(n+2)}(r; t_1, \ldots, t_n; s) = 0
$$

(7.19)

$$
\Delta^{(s)}_n \Pi_F^{(n+2)}(r; t_1, \ldots, t_n; s) = +\Pi_F^{(n+1)}(r; t_1, \ldots, t_{n-1}; t_n) \omega_K(t_n) 
- S^n(r; t_1, \ldots, t_n) \omega_J(0; t_n)
$$

$$
\Delta^{(tk)}_n \Pi_F^{(n+2)}(r; t_1, \ldots, t_n; s) = +\Pi_F^{(k)}(r; t_1, \ldots, t_{k-2}; t_{k-1}) \omega_K(t_{k-1}) G^{n+1-k} 
- S^{k-1}(r; t_1, \ldots, t_{k-1}) \omega_J(k; t_{k-1}) G^{n+1-k}
$$

(7.20)

where we use again the abbreviation $G^{n+1-k} = G^{n+1-k}(t_k, \ldots, t_n, s)$.

### 7.6 Monodromy of $\Pi_I$ produces new blocks $\Pi_{IJ}$

The monodromy in the end points is given by

$$
\Delta^{(r)}_n \Pi_I^{(n+2)}(r; t_1, \ldots, t_n; s) = -\omega_I(r) \Pi_I^{(n+1)}(t_1; t_2, \ldots, t_n; s)
$$

$$
\Delta^{(s)}_n \Pi_I^{(n+2)}(r; t_1, \ldots, t_n; s) = +\Pi_I^{(n+1)}(r; t_1, \ldots, t_{n-1}; t_n)
$$

(7.21)

while the monodromies at the midpoints are given as follows,

$$
\Delta^{(tk)}_n \Pi_I^{(n+2)}(r; t_1, \ldots, t_n; s) = -(-)^k \omega_I(r) G^k \Pi_I^{(n+1-k)}(t_{k+1}; t_{k+2}, \ldots, t_n; s) 
+ \Pi_I^{(k)}(r; t_1, \ldots, t_{k-2}; t_{k-1}) G^{n+1-k}
$$

(7.22)

where we use again $G^k = G^k(t_k, \ldots, t_1, r)$ and $G^{n+1-k} = G^{n+1-k}(t_k, \ldots, t_n, s)$.

The blocks $\Pi_{IJ}^{(n+2)}(r; t_1, \ldots, t_n; s)$ are defined for all $n \geq -1$, and are new. They may be evaluated from the second relation in (7.20), and the monodromies of $G$, $\Pi$ and $\Pi_F$. This monodromy variation will produce one term involving $\Pi_F$ and one term involving $\lambda_I$. Both may be eliminated using (7.14), and we obtain the expression,

$$
\Pi_{IJ}^{(n+2)}(r; t_1, \ldots, t_n; s) = +\Pi_{IJ}^{(n+2)}(r; t_1, \ldots, t_n; s) \omega_I(s) 
+ (-)^n \omega_I(r) \Pi_{IJ}^{(n+2)}(s; t_n, \ldots, t_1; r) 
- \omega_I(r) \Pi_{IJ}^{(n+2)}(r; t_1, \ldots, t_n; s) \omega_I(s) 
+ \omega_J(0; r) S^{n+1}(r, t_1, \ldots, t_n, s) \omega_J(0; s)
$$

(7.23)
which satisfies the following symmetry relation,
\[ \Pi^{(n+2)}_{I^J}(r; t_1, \ldots, t_n; s) = (-)^n \Pi^{(n+2)}_{I^J}(s; t_n, \ldots, t_1; r) \] (7.23)

The lowest order blocks are given by
\[
\begin{align*}
\Pi^{(1)}_{I^J}(r) &= +\omega_I(r)\omega_J(r) - \omega_I(r)\lambda_J(r) + \lambda_I(r)\omega_J(r) \\
\Pi^{(2)}_{I^J}(r, s) &= +\omega_I(r)S(r, s)\omega_J(s) + \omega_I(r)F(r, s)\omega_J(s) - \omega_I(r)\omega_J(s) \\
&+ \omega_I(r)G(s, r)\lambda_J(s) + \lambda_I(r)G(r, s)\omega_J(s) + \lambda_I(r)P(r, s)\omega_J(s) \\
\Pi^{(3)}_{I^J}(r; t; s) &= +\omega_I(r)\Pi^{(n+1)}_{I^J}(r; t, s) + \omega_I(r)F(r, t, s)\omega_J(s) \\
&- \omega_I(r)\omega_J(s) - \omega_I(r)G(t, r)\omega_J(s) \\
&- \omega_I(r)G(r, t)\lambda_J(s) + \lambda_I(r)G(t, r)\omega_J(s) \\
&+ \lambda_I(r)G(r, t)\omega_J(s) + \omega_I(r)P(r, t)G(t, s)\omega_J(s) \\
&- \omega_I(r)\omega_J(s) - \omega_I(r)F(t, r)F(t, s)\omega_J(s) \\
&\quad - \omega_I(r)\omega_J(s)
\end{align*}
\] (7.24)

The monodromies of the blocks \( \Pi^{(n+2)}_{I^J}(r; t_1, \ldots, t_n; s) \) are readily evaluated from their expression in (7.22), and we find,
\[
\begin{align*}
\Delta^{(r)}_{K} \Pi^{(n+2)}_{I^J}(r; t_1, \ldots, t_n; s) &= -\omega_I(r)\Pi^{(n+1)}_{K^J}(t_1; t_2, \ldots, t_n; s) \\
\Delta^{(s)}_{K} \Pi^{(n+2)}_{I^J}(r; t_1, \ldots, t_n; s) &= +\Pi^{(n+1)}_{I^K}(r; t_1, \ldots; t_n)\omega_J(s)
\end{align*}
\] (7.25)

and for the midpoints, for \( 1 \leq k \leq n, \)
\[
\Delta^{(t_k)}_{K} \Pi^{(n+2)}_{I^J}(r; t_1, \ldots, t_n; s) = -(-)^k \omega_I(r)G^k\Pi^{(n+1-k)}_{K^J}(t_{k+1}; t_{k+2}, \ldots, t_n; s) \\
+ \Pi^{(t_k)}_{I^K}(r; t_1, \ldots, t_{k-2}; t_{k-1})G^{n+1-k}\omega_J(s)
\] (7.26)

where we use again \( G^k = G^k(t_k, \ldots, t_1, r) \) and \( G^{n+1-k} = G^{n+1-k}(t_k, \ldots, t_n, s). \) These relations show that no further new blocks are being produced by these final monodromies.

### 7.7 Derivatives of the blocks \( \Pi, \Pi_I, \Pi_{IJ} \) and \( \Pi_F \)

We make use of a series of very convenient differentiation formulas, whose role is to simplify the combinatorics of the derivative calculations. It is easy to prove the following identities,
\[
\begin{align*}
\partial_t \left( S(u, t)F(t, v) + F(u, t)G(t, v) \right) &= 2\pi \left( \delta(t, v) - \delta(t, u) \right)F(u, v) \\
\partial_t \left( F(t, u)S(t, v) - G(t, u)F(v, t) \right) &= 2\pi \left( \delta(t, v) - \delta(t, u) \right)F(v, u) \\
\partial_t \left( P(u, t)G(t, v) - G(t, u)P(t, v) - F(t, u)F(v, t) \right) &= 2\pi \left( \delta(t, v) - \delta(t, u) \right)P(u, v) \\
\partial_t \left( \lambda_I(r)G(r, t) + \omega_I(r)P(r, t) - \omega_I(t)F(r, t) \right) &= 2\pi \delta(r, t)\lambda_I(t) \\
\partial_t \left( \omega_I(r)S(r, t) + \omega_I(r)F(t, r) \right) &= 2\pi \delta(r, t)\omega_I(t)
\end{align*}
\] (7.27)
Effectively, the differentiations are governed just by the poles in the propagators $G$ and $S$.

- The derivatives of the blocks $\Pi^{(n+2)}(r; t_1, \cdots, t_n; s)$ at the midpoints are given by
  \[
  \partial_k \Pi^{(n+2)}(r; t_1, \cdots, t_n; s)
  = 2\pi \left( \delta(t_k, t_{k+1}) - \delta(t_k, t_{k-1}) \right) \Pi^{(n+1)}(r; t_1, \cdots, \hat{t}_k, \cdots, t_n; s)
  \]
  (7.28)
  where the caret $\hat{}$ denotes that this variable is to be omitted. In particular, this formula shows that $\Pi^{(n+2)}(r; t_1, \cdots, t_n; s)$ is holomorphic in the vertex insertion points $t_1, \cdots, t_n$, with simple poles. The derivatives at the end points are readily computed and we find,
  \[
  \partial_r \Pi^{(n+2)}(r; t_1, \cdots, t_n; s) = 2\pi \delta(r, t_1) \Pi^{(n+1)}(t_1; t_2, \cdots, t_n; s)
  - \frac{1}{2} \chi(r) \Pi^{(n+2)}_F(r; t_1, \cdots, t_n; s)
  + \mu(r) G^{n+1}(r, t_1, \cdots, t_n, s)
  \]
  (7.29)
  where $\Pi_F$ is the fermionic block given by (7.15).

- The derivatives of the blocks $\Pi^{(n+2)}_F(r; t_1, \cdots, t_n; s)$, may be obtained from the definition and generalized recursion relation for $\Pi_F$, and we find,
  \[
  \partial_r \Pi^{(n+2)}_F(r; t_1, \cdots, t_n; s) = \frac{1}{2} \chi(r) G^{n+1}(r, t_1, \cdots, t_n, s)
  \]
  \[
  \partial_s \Pi^{(n+2)}_F(r; t_1, \cdots, t_n; s) = -\frac{1}{2} S(r, t_1, \cdots, t_n, s) \chi(s)
  \]
  \[
  \partial_k \Pi^{(n+2)}_F(r; t_1, \cdots, t_n; s) = 0
  \]
  (7.30)

- The derivatives of the blocks $\Pi^{(n+2)}_I(r; t_1, \cdots, t_n; s)$ may be obtained from the expression for $\Pi_I$ in (7.14), and the derivatives of $\Pi$ and $\Pi_F$, and we find,
  \[
  \partial_r \Pi^{(n+2)}_I(r; t_1, \cdots, t_n; s) = +2\pi \delta(r, t_1) \Pi^{(n+1)}_I(t_1; t_2, \cdots, t_n; s)
  \]
  (7.31)
  \[
  \partial_s \Pi^{(n+2)}_I(r; t_1, \cdots, t_n; s) = \frac{1}{2} \omega_I(r) S^{n+1}(r; t_1, \cdots, t_n, s) \chi(s)
  \]
  while the derivatives at the midpoints are given by
  \[
  \partial_k \Pi^{(n+2)}_I(r; t_1, \cdots, t_n; s)
  = +2\pi \left( \delta(t_k, t_{k+1}) - \delta(t_k, t_{k-1}) \right) \Pi^{(n+1)}_I(r; t_1, \cdots, \hat{t}_k, \cdots, t_n; s)
  \]
  (7.32)
  Note that $\Pi^{(n+2)}_I(r; t_1, \cdots, t_n; s)$ is holomorphic in $r$ and in $t_1, \cdots, t_n$, but not in $s$.

- The derivatives of the blocks $\Pi^{(n+2)}_{IJ}(r; t_1, \cdots, t_n; s)$ are obtained from the expression for $\Pi_{IJ}$ in (7.22), and the derivatives of $\Pi$, $\Pi_I$, and $\Pi_F$, already computed earlier. Derivatives at the end points yield,
  \[
  \partial_r \Pi^{(n+2)}_{IJ}(r; t_1, t_2, \cdots, t_n; s) = +2\pi \delta(r, t_1) \Pi^{(n+1)}_{IJ}(t_1; t_2, \cdots, t_n; s)
  \]
  \[
  \partial_s \Pi^{(n+2)}_{IJ}(r; t_1, \cdots, t_{n-1}, t_n; s) = -2\pi \delta(s, t_n) \Pi^{(n+1)}_{IJ}(r; t_1, \cdots, t_{n-1}; t_n)
  \]
  (7.33)
Derivatives at the midpoints gives,

\[
\frac{\partial}{\partial t_k} \Pi_{I,J}^{(n+2)}(r; t_1, \cdots, t_n; s) = +2\pi \left( \delta(t_k, t_{k+1}) - \delta(t_k, t_{k-1}) \right) \Pi_{I,J}^{(n+1)}(r; t_1, \cdots, \hat{t_k}, \cdots, t_{n-1}; s)
\] (7.34)

where the caret \(^\wedge\) denotes that this variable is to be omitted. These identities show in particular that \(\Pi_{I,J}^{(n+2)}(r; t_1, \cdots, t_n; s)\) is holomorphic in all insertion points \(r, t_1, \cdots, t_n, s\).
8 Singly Linked Chain Blocks

In addition to the linear chain blocks, constructed and studied in the preceding section, further blocks occur in the chiral amplitudes corresponding to a chain with one link or with two links. The linking of the chain arises when a bosonic field $\partial x_+(r)$ in the supercurrent at the end of a linear chain is being contracted onto the $\exp\{ik_\nu \cdot x_+(t_\nu)\}$ operator of a vertex operator occurring in the linear chain. Since there are only two supercurrents, the linking can occur either once or twice. The basic building blocks of the linked chains are again the blocks $\Pi$, with either one or two end points linked to a midpoint,

\[
\begin{align*}
\text{singly linked} & \quad \Pi^{(n+2)}(t_i; t_1, \cdots, t_n; s) \delta_{r,t_i} \\
\text{doubly linked} & \quad \Pi^{(n+2)}(t_i; t_1, \cdots, t_n; t_j) \delta_{r,t_i} \delta_{s,t_j} 
\end{align*}
\]

(8.1)

The doubly linked chains will occur only in the construction of the holomorphic chiral blocks $Z$, but not for the differential chiral blocks $D$. Since the focus in this paper is on the differential chiral blocks, we shall postpone the study of the doubly linked chain blocks to a sequel paper in which the holomorphic chiral blocks $Z$ will be explicitly constructed, and deal here only with the simply linked chain blocks.

The single linking of a linear chain creates a closed loop. The linked chain block function is multiplied in the chiral amplitude by a kinematic factor. The kinematic factor, and thus the associated block function, is naturally decomposed into parts with definite symmetry properties under reflection of the linked chain. As a result, we define two singly linked block functions, $\Pi_+$ and $\Pi_-$, whose basic building blocks will be

\[
\begin{align*}
\Pi^{(m+1)[\ell+1]}_+ & \sim \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell; r) \\
& \quad + (-)^\ell \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_\ell, \cdots, u_1; r) \\
\Pi^{(m+1)[\ell+1]}_- & \sim \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell; r) \\
& \quad - (-)^\ell \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_\ell, \cdots, u_1; r)
\end{align*}
\]

(8.2)

In both blocks, the loop in the chain contains $\ell + 1$ points, $u_1, \cdots, u_\ell$ and $r$. This loop is connected at the point $r$ to a linear chain containing $m + 2$ points $s, t_1, \cdots, t_m, r$. By construction, the right hand sides of (8.2) is holomorphic in $t_1, \cdots, t_m$, and in $u_1, \cdots, u_\ell$ with simple poles occurring only between neighboring points, and between $t_1$ and $s$. The blocks are not holomorphic at the end point $s$ of the linear chain, just as the original block $\Pi$ is not holomorphic in its endpoints.

The key difficulty in completing the construction of the blocks $\Pi_{\pm}$ that are holomorphic in all variables, but the end point $s$, is to complete the blocks so that holomorphicity in $r$ is achieved. This will now be done separately for the blocks $\Pi_+$ and $\Pi_-$. The completion of the blocks $\Pi_+$ is straightforward and is achieved by further use of the basic block $\Pi$. The completion of the block $\Pi_-$ is considerably more involved and will necessitate the use of the functions $Q_0, Q_B, Q_F$, and $Q_P$. 

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8.1 The blocks \( \Pi^\ell_{+}(m+1|\ell+1) \)

The completion of the singly linked block \( \Pi^\ell_{+}(m+1|\ell+1) \) from (8.2) is straightforward, and is achieved through the addition of a closed loop \( \Pi^{(\ell+2)}(r; u_1, \cdots, u_\ell; r) \) multiplied by a linear chain of bosonic Green functions \( G \), as follows,

\[
2\Pi^\ell_{+}(m+1|\ell+1) (s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell) = \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell, r) + (-)^{\ell} \Pi^{(m+\ell+3)}(s; t_1, \cdots, t_m, r, u_\ell, \cdots, u_1) + (-)^m G^{m+1}(r, t_1, \cdots, t_1, s) \Pi^{(\ell+2)}(r; u_1, \cdots, u_\ell; r)
\]

The construction holds for \( \ell \geq 1 \), and \( m \geq 0 \); these singly linked chain blocks \( \Pi^\ell_{+} \) correspond to a loop with \( \ell + 1 \) points, attached at the point \( r \) to a linear chain with \( m + 2 \) points, including the point \( r \). When \( m = 0 \), the points \( t_k \) are absent, and the linear chain has only the points \( r \) and \( s \). These blocks are \((1,0)\)-forms in the insertion points \( t_1, \cdots, u_1, \cdots, u_\ell \), and \( r \), and scalars in \( s \), and clearly holomorphic in \( t_1, \cdots, t_m \) and \( u_1, \cdots, u_\ell \). They satisfy the following reflection symmetry property,

\[
\Pi^\ell_{+}(m+1|\ell+1) (s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell) = (-)^{\ell} \Pi^\ell_{+}(m+1|\ell+1) (s; t_1, \cdots, t_m, r, u_\ell, \cdots, u_1)
\]

We now show that they are also holomorphic in \( r \), thanks to the addition of the last term.

To carry out this check, we consider the sum of the first and second terms on the right hand side of (8.3), and denote this sum by \( 2\Pi^\ell_{+}(m+1|\ell+1) (s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell) \). We begin by computing its \( \partial_r \) derivative, using the formulas for the derivatives of \( \Pi^{(n+2)} \) evaluated earlier for the case \( n = m + \ell + 1 \). These derivatives are readily expressed in terms of the fermionic blocks \( \Pi^\ell_{F}(m+2) \), and we shall omit the \( \delta \)-function contributions at coincident vertex operator points. We find,

\[
2\partial_r \Pi^\ell_{+}(m+1|\ell+1) (s; t_1, \cdots, t_m, r, u_1, \cdots, u_\ell) = (-)^{m+\ell+1} \mu(r) G^{m+\ell+2}(r, u_\ell, \cdots, u_1, r, t_m, \cdots, t_1, s) + (-)^m G^{m+1}(r, t_1, \cdots, t_1, s) \Pi^{(\ell+2)}(r; u_1, \cdots, u_\ell; r)
\]

We use the recursion relation (7.17) for \( \Pi^\ell_{F} \), with \( w = r \), in the third and fourth terms on the right hand side of (8.5). The resulting expression consists of two terms involving \( \Pi^{(m+2)}_{F} \) and two terms involving \( \Pi^{(\ell+2)}_{F} \). The sum of the terms involving \( \Pi^{(m+2)}_{F} \) is given by,

\[
\frac{1}{2} (-)^{m+\ell} \chi(r) S(r, u_\ell) \cdots S(u_1, r) \Pi^{(m+2)}_{F}(r; t_m, \cdots, t_1; s) + \frac{1}{2} (-)^m \chi(r) S(r, u_1) \cdots S(u_\ell, r) \Pi^{(m+2)}_{F}(r; t_m, \cdots, t_1; s)
\]

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Using the reflection symmetry of the product of \( \ell + 1 \) Szegö kernels,
\[
S(r, u_\ell) \cdots S(u_1, r) = (-)^{\ell+1} S(r, u_1) \cdots S(u_\ell, r) \tag{8.7}
\]
it is clear that these terms cancel in \( \partial_r \Pi_+^{(m+1|\ell+1)} \). The remaining terms are
\[
2\partial_r \Pi_+^{(m+1|\ell+1)}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) \\
= (-)^{m+\ell+1} G^{m+1} \left[ \mu(r) G^{\ell+1}(r, u_\ell, \ldots, u_1; r) - \frac{1}{2} \chi(r) \Pi_F^{(\ell+2)}(r; u_1, \ldots, u_\ell; r) \right] \\
+ (-)^{m+1} G^{m+1} \left[ \mu(r) G^{\ell+1}(r, u_1, \ldots, u_\ell; r) - \frac{1}{2} \chi(r) \Pi_F^{(\ell+2)}(r; u_1, \ldots, u_\ell; r) \right] \tag{8.8}
\]
where we use the abbreviation \( G^{m+1} = G^{m+1}(r, t_m, \ldots, t_1, s) \). We recognize the coefficient of \( (-)^{m+1} G^{m+1} \) in the above expression as the \( \partial_r \) derivative of \( \Pi_F^{(\ell+2)}(r; u_1, \ldots, u_\ell; r) = (-)^{\ell} \Pi_F^{(\ell+2)}(r; u_\ell, \ldots, u_1; r) \). As a result, we have shown that,
\[
\partial_r \Pi_+^{(m+1|\ell+1)}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) = 0 \tag{8.9}
\]
and the desired assertion follows.

An alternative formula for the blocks \( \Pi_+ \) may be obtained using the recursion relation (7.18) for \( \Pi_+ \) itself. One finds,
\[
2\Pi_+^{(m+1|\ell+1)}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell; r) \\
= \Pi^{(m+2)}(s; t_1, \ldots, t_m; r) \left[ G^{\ell+1}(r, u_1, \ldots, u_\ell; r) + (-)^\ell G^{\ell+1}(r, u_\ell, \ldots, u_1; r) \right] \\
- (-)^m \Pi_F^{(m+2)}(r; t_m, \ldots, t_1; s) \left[ \Pi^{(\ell+2)}(r; u_1, \ldots, u_\ell; r) + (-)^\ell \Pi^{(\ell+2)}_F(r; u_\ell, \ldots, u_1; r) \right] \\
- (-)^m G^{m+1}(r, t_m, \ldots, t_1, s) \Pi^{(\ell+2)}(r; u_1, \ldots, u_\ell; r) \tag{8.10}
\]
Notice the sign change in the last term compared to the last term in (8.3).

### 8.2 The blocks \( \Pi_+^{(\ell+1)} \)

There is a natural extension of the connectivity of the blocks \( \Pi_+^{(m+1|\ell+1)} \) to the case where formally \( m = -1 \); we shall denote these blocks by \( \Pi_+^{(\ell+1)} \). They may be simply defined by
\[
\Pi_+^{(\ell+1)}(s; u_1, \ldots, u_\ell) = \Pi^{(\ell+2)}(s; u_1, \ldots, u_\ell; s) \tag{8.11}
\]
They are \((0, 0)\) forms in \( s \), and holomorphic \((1, 0)\) forms in \( u_1, \ldots, u_\ell \). Because of the symmetry of the block \( \Pi_+ \) itself, \( \Pi^{(\ell+2)}(s; u_1, \ldots, u_\ell; s) = (-)^\ell \Pi^{(\ell+2)}(s; u_\ell, \ldots, u_1; s) \), it is manifest that \( \Pi_+^{(\ell+1)}(s; u_1, \ldots, u_\ell) = 0 \) for \( \ell \) odd.
8.3 The blocks $\Pi^{(m+1|\ell+1)}$

The construction of the blocks $\Pi^{(m+1|\ell+1)}$ originates with the symmetrization of the block $\Pi^{(m+1|\ell+3)}$ given in (8.2). The blocks $\Pi^{(m+1|\ell+1)}_-$ have the following reflection symmetry,

$$\Pi^{(m+1|\ell+1)}_-(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) = -(-)^m \Pi^{(m+1|\ell+1)}_+(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell)$$ (8.12)

Using the generalized recursion relation (7.18), the symmetrized combination on the right hand side of $\Pi^{(m+1|\ell+1)}_-$ in (8.2) may alternatively be expressed as

$$-\Pi^{(m+2)}(s; t_1, \ldots, t_m; r) \left[ G^{\ell+1}(r, u_1, \ldots, u_\ell, r) + (-)^{\ell}G^{\ell+1}(r, u_\ell, \ldots, u_1, r) \right]$$

$$+(-)^m \Pi^{(m+2)}_F(r; t_m, \ldots, t_1; s) \left[ \Pi^{(\ell+2)}_F(r; u_1, \ldots, u_\ell; r) - (-)^{\ell} \Pi^{(\ell+2)}_F(r; u_\ell, \ldots, u_1; r) \right]$$

Notice that the terms of the type $G^{m+1} \Pi^{(\ell+2)}(r; u_1, \ldots, u_\ell; r)$, which occurred for $\Pi_+$ in (8.3), do not survive the symmetrization needed for $\Pi_-$. There is, however, another term, of the form $\Pi^{(m+2)} S^{\ell+1}$, which naturally arises in the chiral blocks and which does respect the symmetry of $\Pi_-$. Including this term, we shall use the following intermediate expression $\bar{\Pi}_-$ to complete the symmetrized form of (8.13) into a holomorphic form in $r$,

$$2\bar{\Pi}^{(m+1|\ell+1)}_-(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell)$$

$$= -\Pi^{(m+2)}(s; t_1, \ldots, t_m; r) \left[ G^{\ell+1}(r, u_1, \ldots, u_\ell, r) - (-)^{\ell}G^{\ell+1}(r, u_\ell, \ldots, u_1, r) \right]$$

$$+(-)^m \Pi^{(m+2)}_F(r; t_m, \ldots, t_1; s) \left[ \Pi^{(\ell+2)}_F(r; u_1, \ldots, u_\ell; r) - (-)^{\ell} \Pi^{(\ell+2)}_F(r; u_\ell, \ldots, u_1; r) \right]$$

$$+2\Pi^{(m+2)}(s; t_1, \ldots, t_m; r) S(r, u_1) S(u_1, u_2) \cdots S(u_{\ell-1}, u_\ell) S(u_\ell, r)$$

This expression is defined for $\ell \geq 1$ and $m \geq 0$; in the case $m = 0$, this singly linked chain consists of a loop containing the $\ell + 1$ points $u_1, \ldots, u_\ell$ and $r$, attached to a linear chain between $r$ and $s$. $\bar{\Pi}^{(m+1|\ell+1)}_-$ satisfies the same symmetry relation as $\Pi^{(m+1|\ell+1)}_+$ does in (8.12). It is holomorphic in $t_1, \ldots, t_m$ and $u_1, \ldots, u_\ell$, but fails to be holomorphic in $r$. Its $\partial_r$ derivative is readily calculated from the derivatives of $\Pi$ and $\Pi_F$. Neglecting contact terms between coincident points, we have

$$\partial_r \bar{\Pi}^{(m+1|\ell+1)}_-(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) = (-)^m G^{m+1} B^{(\ell+1)}_-(r; u_1, \ldots, u_\ell)$$

where we have used the abbreviation $G^{m+1} = G^{m+1}(r, t_m, \ldots, t_1, s)$. The function $B^{(\ell+1)}_-$ may be calculated explicitly, and takes the form,

$$B^{(\ell+1)}_-(r; u_1, \ldots, u_\ell) = \mu(r) S(r, u_1) \cdots S(u_\ell, r)$$

$$- \frac{1}{2} \mu(r) \left[ G^{\ell+1}(r, u_1, \ldots, u_\ell, r) - (-)^{\ell}G^{\ell+1}(r, u_\ell, \ldots, u_1, r) \right]$$

$$+ \frac{1}{4} \chi(r) \left[ \Pi^{(\ell+2)}_F(r; u_1, \ldots, u_\ell; r) - (-)^{\ell} \Pi^{(\ell+2)}_F(r; u_\ell, \ldots, u_1; r) \right]$$

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Remarkably, the function $B_{-}^{(\ell+1)}$ is independent of $m$. In part, this is thanks to the addition of the term $\Pi^{(m+2)}S^{\ell+1}$ in (8.14). In the subsequent subsection, we shall solve the equation that formally corresponds to the case $m = -1$, in terms of a new block $\Pi^{(\ell+1)}_{-}$, defined by

\begin{align*}
\partial_t \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) &= B^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) \\
\Pi^{(\ell+1)}_{-}(r; u_\ell, \ldots, u_1) &= (-)^{\ell+1}\Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell)
\end{align*}

(8.17)

The block $\Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell)$ will be holomorphic in $u_1, \ldots, u_\ell$ with simple poles only at coincident vertex insertion points. Given this block, it is now straightforward to construct the desired block $\Pi^{(m+1)\ell+1}_{-}$, and it is given by

\begin{align*}
\Pi^{(m+1)\ell+1}_{-}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) &= \Pi^{(m+1)\ell+1}_{-}(s; t_1, \ldots, t_m, r, u_1, \ldots, u_\ell) \\
&\quad - (-)^m G^{m+1}(r, t_m, \ldots, t_1) \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell)
\end{align*}

(8.18)

### 8.4 The blocks $\Pi^{(\ell+1)}_{-}$

To construct the blocks $\Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell)$, we start from its defining equation (8.17), and render all $r$-dependence explicit. To so so, we use the generalized recursion relation (7.17) for $m = 0$ and $\ell \to \ell - 1$, to obtain,

\begin{align*}
\Pi^{(\ell+2)}_{F}(r; u_1, \ldots, u_\ell; r) &= S(r, u_1)G(u_\ell, r)\Pi^{(\ell)}_{F}(u_1; u_2, \ldots, u_{\ell-1}; u_\ell) \\
&\quad + F(r, u_1)G(u_\ell, r)G^{\ell-1}(u_1, \ldots, u_\ell) \\
&\quad + S(r, u_1)F(u_\ell, r)S^{\ell-1}(u_1, \ldots, u_\ell)
\end{align*}

(8.19)

Next, we substitute this expressions (and its mirror with $u_i \to u_{\ell-i+1}$) for $\Pi^{(\ell+1)}_{F}$ on the right hand side of (8.17). The result for $\partial_t \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell)$ is then given by,

\begin{align*}
\partial_t \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) &= +\mu(r)S^{\ell+1}(r, u_1, \ldots, u_\ell, r) \\
&\quad - \frac{1}{2}\mu(r)G^{\ell+1}(r, u_1, \ldots, u_\ell, r) \\
&\quad + \frac{1}{2}(-)^{\ell}\mu(r)G^{\ell+1}(r, u_\ell, \ldots, u_1, r) \\
&\quad + \frac{1}{4}\chi(r)S(r, u_1)G(u_\ell, r)\Pi^{(\ell)}_{F}(u_1; u_2, \ldots, u_{\ell-1}; u_\ell) \\
&\quad + \frac{1}{4}\chi(r)F(r, u_1)G^{\ell}(u_1, \ldots, u_\ell, r) \\
&\quad + \frac{1}{4}\chi(r)F(u_\ell, r)S^{\ell}(r, u_1, \ldots, u_\ell) \\
&\quad - \frac{1}{4}(-)^{\ell}\chi(r)S(r, u_\ell)G(u_1, r)\Pi^{(\ell)}_{F}(u_\ell; u_{\ell-1}, \ldots, u_2; u_1)
\end{align*}

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\[-\frac{1}{4}(-)\ell \chi(r) F(r, u_\ell) G^{\ell}(u_1, \ldots, u_1, r) \]
\[-\frac{1}{4}(-)\ell \chi(r) F(u_1, r) S^{\ell}(r, u_\ell, \ldots, u_1) \]  \hspace{2cm} (8.20)

The dependence on \( r \) on the right hand side of (8.20) is typical of that of the derivatives of the three-point functions \( Q_0, Q_B, Q_F \) and \( Q_P \). We recall their expressions, neglecting any contact terms at coincident vertex insertion points,

\[
\begin{align*}
\partial_\ell Q_0(r; u_1, u_\ell) &= 0 \\
\partial_\ell Q_B(r; u_1, u_\ell) &= +\mu(r) S(r, u_1) S(u_\ell, r) \\
&\quad -\frac{1}{4} \chi(r) S(u_\ell, r) F(u_1, r) - \frac{1}{4} \chi(r) S(u_1, r) F(u_\ell, r) \\
\partial_\ell Q_F(r; u_1, u_\ell) &= -\frac{1}{4} \chi(r) S(u_\ell, r) G(u_1, r) \\
\partial_\ell Q_P(r; u_1, u_\ell) &= +\frac{1}{2} \mu(r) G(r, u_1) G(u_\ell, r) - \frac{1}{4} \chi(r) F(r, u_1) G(u_\ell, r) \hspace{2cm} (8.21)
\end{align*}
\]

Comparison of the right hand sides of (8.20) and (8.21) allows us to construct a particular solution \( \Pi^{(\ell+1)}_{-} \) to (8.20), given by

\[
\begin{align*}
\Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) &= +Q_B(r; u_1, u_\ell) S^{\ell-1}(u_1, \ldots, u_1) \\
&\quad -Q_P(r; u_1, u_\ell) G^{\ell-1}(u_1, \ldots, u_1) \\
&\quad +(-)^\ell Q_P(r; u_\ell, u_1) G^{\ell-1}(u_\ell, \ldots, u_1) \\
&\quad +Q_F(r; u_\ell, u_1) \Pi_{F}^{(\ell)}(u_1; u_2, \ldots, u_{\ell-1}; u_\ell) \\
&\quad -(-)^\ell Q_F(r; u_1, u_\ell) \Pi_{F}^{(\ell)}(u_\ell; u_{\ell-1}, \ldots, u_2; u_1) \\
&\quad -\frac{1}{4} \chi(r) F(r, u_1) G(u_\ell, r) \hspace{2cm} (8.22)
\end{align*}
\]

As a result, the general solution for (8.20) is of the form,

\[
\Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) = \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) + \Pi^{(\ell+1)}_{+}(r; u_1, \ldots, u_\ell) \hspace{2cm} (8.23)
\]

where \( \Pi^{(\ell+1)}_{+}(r; u_1, \ldots, u_\ell) \) is holomorphic in \( s \), apart from contact terms at coincident vertex insertion points, and satisfies the symmetry property of (8.20).

### 8.4.1 Holomorphicity in \( u_1 \) and \( u_\ell \)

The homogeneous solution \( \Pi^{(\ell+1)}_{+}(r; u_1, \ldots, u_\ell) \) is not arbitrary, however, but is further constrained by the requirement that the full blocks \( \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) \) be holomorphic in all the points \( u_1, \ldots, u_\ell \). This requirement is automatic for the points \( u_2, u_3, \ldots, u_{\ell-2}, u_{\ell-1} \), but not for the points \( u_1, u_\ell \). Thus, it will suffice to seek a term \( \Pi^{(\ell+1)}_{+}(r; u_1, \ldots, u_\ell) \) which is holomorphic in \( s \), and \( u_2, u_3, \ldots, u_{\ell-2}, u_{\ell-1} \), has the symmetry properties of (8.20), and compensates for the non-holomorphicity of \( \Pi^{(\ell+1)}_{-}(r; u_1, \ldots, u_\ell) \) in \( u_1 \). Given the symmetry
property (8.20), the resulting $\Pi_{-}^{(\ell+1)}(r; u_1, \ldots, u_\ell)$ will then be automatically holomorphic in $u_\ell$ as well. We begin by computing the $u_1$-derivative of $\tilde{\Pi}_{-}^{(\ell+1)}(r; u_1, \ldots, u_\ell)$,

\[
\partial_{u_1}\tilde{\Pi}_{-}^{(\ell+1)} = Q_0(r; u_1, u_\ell) \left[ \frac{1}{2} \chi(u_1)\Pi_F^{(\ell)}(u_1; \ldots; u_\ell) - \mu(u_1)G^{\ell-1}(u_1, \ldots, u_\ell) \right] \\
+ G(u_1, r) \left[ \mu(u_1)S^{\ell}(u_1, u_2, \ldots, u_\ell, u_1) - (-)^\ell \mu(u_1)G^{\ell}(u_1, u_2, \ldots, u_\ell, u_1) \right] \\
+ \frac{1}{2} (-)^\ell \chi(u_1)\Pi_F^{(\ell+1)}(u_1; u_\ell, u_{\ell-1}, \ldots, u_2; u_1)
\]

(8.24)

where we have used the recursion relation (7.17) in the form

\[
F(u_1, u_\ell)G^{\ell-1}(u_\ell, \ldots, u_1) + S(u_1, u_\ell)\Pi_F^{(\ell)}(u_\ell; \ldots; u_1) = \Pi_F^{(\ell+1)}(u_1; u_\ell \ldots; u_1)
\]

(8.25)

to combine two of the terms.

The observation that the $r$-dependence of the right hand side of (8.24) is holomorphic makes it possible to complete the block $\Pi_{-}^{(\ell+1)}$ into a block with the desired holomorphicity and symmetry properties.

The term proportional to $Q_0(r; u_1, u_\ell)$ in (8.24) may be recast as a $\partial_{u_1}$-derivative,

\[
Q_0(r; u_1, u_\ell) \left[ \frac{1}{2} \chi(u_1)\Pi_F^{(\ell)}(u_1; \ldots; u_\ell) - \mu(u_1)G^{\ell-1}(u_1, \ldots, u_\ell) \right] \\
= \partial_{u_1} \left( - Q_0(r; u_1, u_\ell)\Pi_F^{(\ell)}(u_1; u_2, \ldots, u_{\ell-1}; u_\ell) \right)
\]

(8.26)

Note that $Q_0$ is not holomorphic in $u_1$, even up to contact terms in coincident vertex insertion points, because there is a residual $\delta$-function at $w_0$,

\[
\partial_{u_1}Q_0(r; u_1, u_\ell) = -G(u_\ell, r) \left( 2\delta(u_1, u_\ell) - \delta(u_1, r) - \delta(u_1, w_0) \right)
\]

(8.27)

Here, $w_0$ is the point on $\Sigma$ where $G(z, w_0) = 0$ for all $z$, so that also $P(z, w_0) = F(z, w_0) = 0$ and thus also $\Pi_F^{(\ell)}(w_0; u_2, \ldots, u_{\ell-1}; u_\ell) = 0$. In view of the last equality, the pole in $u_1$ at $w_0$ which occurs in $Q_0(r; u_1, u_\ell)$ is cancelled by a corresponding zero in $\Pi_F^{(\ell)}(u_1; u_2, \ldots, u_{\ell-1}; u_\ell)$, thereby yielding (8.26) through differentiation of $\Pi_F^{(\ell)}$ only.

To identify the term proportional to $G(u_1, r)$ in (8.24), we relate it to the block $\tilde{\Pi}_{-}^{(\ell)}(u_1; u_2, \ldots, u_\ell)$. Specifically, the bracket on the second and third lines of the right hand side of (8.24) equals,

\[
\left[ \ldots \right] = B_{-}^{(\ell)}(u_1; u_2, \ldots, u_\ell) + \frac{1}{2} \partial_{u_1}\Pi^{(n+1)}(u_1; u_2, \ldots, u_\ell; u_1)
\]

(8.28)

as may be verified by explicit calculation.
8.4.2 General recursive formula for $\Pi^{(\ell+1)}$

Putting all together, using the fact that $B^{(\ell)}(u_1; u_2, \cdots, u_\ell) = \partial_{u_1} \tilde{\Pi}^{(\ell)}(u_1; u_2, \cdots, u_\ell)$, and the symmetry of (8.24), we find the following candidate expression for $\Pi^{(\ell+1)}(r; u_1, \cdots, u_\ell)$,

$$\tilde{\Pi}^{(\ell+1)}(r; u_1, \cdots, u_\ell) + Q_0(r; u_1, u_\ell)\Pi^{(\ell)}(u_1; u_2, \cdots, u_{\ell-1}; u_\ell)$$

$$-G(u_1, r)\Pi^{(\ell)}(u_1; u_2, \cdots, u_\ell) + (-)^{\ell}G(u_\ell, r)\Pi^{(\ell)}(u_\ell; u_{\ell-1}, \cdots, u_1)$$

$$-\frac{1}{2}G(u_1, r)\Pi^{(\ell+1)}(u_1; u_2, \cdots, u_\ell; u_1) + \frac{1}{2}(-)^{\ell}G(u_\ell, r)\Pi^{(\ell+1)}(u_\ell; u_{\ell-1}, \cdots, u_1; u_\ell)$$

This result is now holomorphic in $r$, $u_1$ and $u_\ell$ by construction.

The presence of the terms in $\tilde{\Pi}^{(\ell)}$ spoils the holomorphicity in the points $u_2$ and $u_{\ell-1}$ so that the above formula cannot yet be the complete expression for $\Pi^{(\ell+1)}(r; u_1, \cdots, u_\ell)$.

But it is now manifest how the full expression should be obtained: one should recursively treat the blocks $\tilde{\Pi}^{(\ell)}$, just as one did with the block $\Pi^{(\ell+1)}$, resulting in a linear recursion relation directly for the blocks,

$$\Pi^{(\ell+1)}(r; u_1, \cdots, u_\ell) = \tilde{\Pi}^{(\ell+1)}(r; u_1, \cdots, u_\ell) + Q_0(r; u_1, u_\ell)\Pi^{(\ell)}(u_1; u_2, \cdots, u_{\ell-1}; u_\ell)$$

$$-G(u_1, r)\Pi^{(\ell)}(u_1; u_2, \cdots, u_\ell) + (-)^{\ell}G(u_\ell, r)\Pi^{(\ell)}(u_\ell; u_{\ell-1}, \cdots, u_1)$$

$$-\frac{1}{2}G(u_1, r)\Pi^{(\ell+1)}(u_1; u_2, \cdots, u_\ell; u_1) + \frac{1}{2}(-)^{\ell}G(u_\ell, r)\Pi^{(\ell+1)}(u_\ell; u_{\ell-1}, \cdots, u_1; u_\ell)$$

The recursive argument proceeds as follows. Assuming that $\Pi^{(\ell)}(r; u_1, \cdots, u_{\ell-1})$ is holomorphic in $u_1, \cdots, u_{\ell-1}$, and that

$$\partial_r \Pi^{(\ell)}(r; u_1, \cdots, u_{\ell-1}) = B^{(\ell)}(r; u_1, \cdots, u_{\ell-1})$$

$$\Pi^{(\ell)}(r; u_{\ell-1}, \cdots, u_1) = (-)^{\ell}\Pi^{(\ell)}(r; u_1, \cdots, u_{\ell-1})$$

then the block $\Pi^{(\ell+1)}(r; u_1, \cdots, u_\ell)$ constructed by (8.30) obeys (8.17), and is holomorphic in all points $u_1, \cdots, u_\ell$, up to contact terms between coincident vertex insertion points. It remains to verify that these properties also hold on the lowest block $\Pi^{(2)}(r; u)$, which we do next.

8.4.3 The blocks $\Pi^{(2)}$ and $\Pi^{(3)}$

Following the above prescriptions for the blocks $\Pi^{(\ell+1)}$, we compute the two simplest cases. By their very definition, the first block vanishes,

$$\Pi^{(1)}(r) = 0$$
\[ \Pi^{(2)}_-(r; u) = Q_B(r; u, u) - 2Q_P(r; u, u) \]
\[ \Pi^{(3)}_-(r; u, v) = Q_B(r; u, v)S(u, v) - Q_P(r; u, v)G(u, v) + Q_P(r; v, u)G(v, u) + Q_F(r; v, u)F(u, v) - Q_F(r; u, v)F(v, u) \] (8.32)

Use of the recursion relation (8.30) then gives us the expressions for the full blocks,

\[ \Pi^{(2)}_+(r; u) = Q_B(r; u, u) - 2Q_P(r; u, u) - G(u, r)P(u, u) \]
\[ \Pi^{(3)}_+(r; u, v) = Q_B(r; u, v)S(u, v) + Q_0(r; u, v)P(u, v) - Q_P(r; u, v)G(u, v) + Q_P(r; v, u)G(v, u) + Q_F(r; v, u)F(u, v) - Q_F(r; u, v)F(v, u) - G(u, r)\Pi^{(2)}_+(u; v) + G(v, r)\Pi^{(2)}_-(v; u) \] (8.33)

Here, we have used the fact that \( \Pi^{(3)}_+(u; v; u) = 0 \) by symmetry.

Holomorphicity of \( \Pi^{(2)}_+(r; u) \) is slightly complicated by the fact that functions occur which are being evaluated at coincident points. Their derivatives are readily worked out and we have \( \partial_\bar{u}\Pi^{(2)}_+(r; u) = 0 \), in view of,

\[ \partial_\bar{u}P(u, u) = -\chi(u)F(u, u) + \partial_\bar{u}\mu(u) - 2\mu(u)\partial_\bar{u}\ln E(u, w_0) \] (8.34)
\[ \partial_\bar{u}Q_B(r; u, u) = -\chi(u)Q_F(r; u, u) - \partial_\bar{u}\left(\mu(u)G(u, r)\right) \]
\[ \partial_\bar{u}Q_P(r; u, u) = -\frac{1}{2}\chi(u)Q_F(r; u, u) + \frac{1}{2}\chi(u)F(u, u)G(u, r) - \partial_\bar{u}\mu(u)G(u, r) - \frac{1}{2}\mu(u)\partial_\bar{u}G(u, r) + \mu(u)G(u, r)\partial_\bar{u}\ln E(u, w_0) \]

Holomorphicity in \( u \) and \( v \) of \( \Pi^{(3)}_+(r; u, v) \) may be verified directly.
9 Extraction of Exact Differentials

Part I Linear Chain blocks

We return now to the task of showing that the cohomology class of the chiral block contains a pure, holomorphic, \( \otimes_{r=1}^{N}(1,0) \) representative, modulo chirally exact terms, that is, terms which are de Rham exact in at least one and possibly two insertion points, and are pure (1,0) and holomorphic type in all the others. Now, from [16], we know that the chiral block \( \mathcal{F}[\delta] \) is always a closed form in each \( z_r \). Thus, if we can show that there exist chirally exact forms \( d_r S_r[\delta] \) and \( d_r d_s S_{rs}[\delta] \) so that

\[
\mathcal{F}[\delta] - \left( \sum_r d_r S_r[\delta] + \sum_{[rs]} d_r d_s S_{rs}[\delta] \right) \in \bigotimes_{r=1}^{N}(1,0)_r
\]

it will automatically follow that the left-hand side is holomorphic in all \( z_r \), and is the holomorphic representative that we seek. This is the strategy which we shall adopt.

The discussion of the extractions of the chirally exact differentials is divided into two parts. In the first part, the necessary Wick contractions of both bosonic and fermionic fields will be carried out. Some simple cases with small number of vertex insertion points will be derived in detail, and the general contribution of linear chain differential blocks will be proven. In the second part, the remaining blocks will be used to extract the final differential blocks involving \( \Pi_{\pm} \).

9.1 A lemma for bosonic Wick contractions

Throughout, we shall need to perform partial contractions of the bosonic field \( x_+ \), through the composite \( X_\mu^s \), which is defined by

\[
X_\mu^s \equiv ik^\mu_s + K_\nu^s K_\mu^s \omega_I(s) + K_\nu^s K_\mu^s \partial_s x_+^\nu(s)
\]

To organize the combinatorics of these contractions, we first prove the following,

Lemma 1 The contractions of the composite \( X_\mu^s \) are given by

\[
Z_{0}^{r_1 \cdots r_n} X_\mu^s = Z_{0}^{r_1 \cdots r_n} \bar{X}_\mu^s + \sum_{t \not\in \{r_1 \cdots r_n\}} Z_{0}^{r_1 \cdots r_n st} K_\nu^s K_\mu^s X_t^\nu G(s,t) + \sum_{t \not\in \{r_1 \cdots r_n\}} \partial_t \left( Z_{0}^{r_1 \cdots r_n st} \varepsilon_t^\nu \partial_t K_\nu^s K_\mu^s G(s,t) \right)
\]

where \( \bar{X}_\mu^s \) stands for the instruction that its \( x_+ \) field is to be contracted only with the exponentials at \( r_1 \cdots r_n \).
To prove Lemma 1, it suffices to contract the $x_+$ field in $X_\mu^s$ with $Z_0^{r_1\cdots r_n s}$ at all insertion points $t$ which do not belong to the set \{r₁ \cdots rₙs\}. This contribution is given by

$$Z_0^{r_1\cdots r_n s} \big( X_\mu^s - \bar X_\mu^s \big) = Z_0^{r_1\cdots r_n st} K^\nu_\mu K^\mu_\nu \partial_s x^\nu_+ (s)$$

with the above instruction on $\partial_s x^\nu_+ (s)$. It is readily evaluated, and we get

$$Z_0^{r_1\cdots r_n s} \big( X_\mu^s - \bar X_\mu^s \big) = \sum_{t \not\in \{r_1 \cdots r_n s\}} Z_0^{r_1\cdots r_n st} K^\nu_\mu \left[ -i k^\mu_t G(s, t) \left( 1 + \varepsilon^\sigma_t \theta_t \partial_t \bar x^\sigma_+ (t) \right) - \varepsilon^\nu_t \theta_t \partial_t G(s, t) \right]$$

(9.5)

Here, the first term in the brackets [ ] arises from the contractions with the exponential, and from the expansion of the un-contracted derivative contribution in $Z_0^{r_1\cdots r_n s}$ at $t$, while the second term arises from the contraction of the derivative contribution in $Z_0^{r_1\cdots r_n s}$ at $t$. Pulling out the differential in $t$, we have

$$Z_0^{r_1\cdots r_n s} \big( X_\mu^s - \bar X_\mu^s \big) = \sum_{t \not\in \{r_1 \cdots r_n s\}} \partial_t \left( Z_0^{r_1\cdots r_n st} K^\nu_\mu \varepsilon^\nu_t \theta_t G(s, t) \right)$$

(9.6)

$$- \sum_{t \not\in \{r_1 \cdots r_n s\}} Z_0^{r_1\cdots r_n st} K^\nu_\mu \left[ i k^\nu_t + i(k^\nu_t \varepsilon^\sigma_t \theta_t - k^\sigma_t \varepsilon^\nu_t \theta_t) \partial_t \bar x^\sigma_+ (t) \right] G(s, t)$$

The object inside the brackets [ ] equals $X^\nu_t$, whence the result of Lemma 1.

The remaining contractions resulting from $\bar X_\mu^s$ are simple to express, and we have,

$$Z_0^{r_1\cdots r_n s} \bar X_\mu^s = Z_0^{r_1\cdots r_n s} \left( i k^\mu_s + K^\nu_\mu p^\nu_I \omega_I (s) - i K^\nu_\mu p^\nu_j \sum_{j=1}^n k^\nu_j G(s, r_j) \right)$$

(9.7)

It will be convenient to introduce a further notation $\check X_\mu^s$, defined by

$$\check X_\mu^s = \bar X_\mu^s + K^\nu_\mu p^\nu_I \omega_I (s)$$

(9.8)

It will allow us to deal directly with the non-trivial $p$-dependence of the blocks.

### 9.2 Extracting the simplest blocks

Before embarking on the extraction of the general blocks $\Pi$, $\Pi_I$, and $\Pi_\pm$ from the differential terms $D$ in (5.13), it will be helpful to understand the extraction mechanism on the simplest blocks first. This will also help in clarifying the ultimate fate of the total differential terms that arise in the second line of Lemma 1 formula (9.3).
(1) The simplest block is $D_1$. As it stands in (5.13), this block is already manifestly an exact differential in the point $r$, and holomorphic in all other insertion points. It readily produces a first differential block,

$$D_1 = \sum_r d_r S^{(1)}_r$$

$$S^{(1)}_r = -Z_0^{\mu} \varepsilon^\mu_r \theta_r \Pi^{(1)}_I(r)$$

(9.9)

where we have used the result $\Pi^{(1)}_I(r) = \lambda_I(r)$ of (7.13).

(2) Another simple block is $D_{2a}$, which is manifestly an exact differential in both the points $r$ and $s$, and holomorphic in all other insertion points. It readily produces a second differential block,

$$D_{2a} = \sum_{[rs]} d_r d_s S^{(2)}_{rs}$$

$$S^{(2)}_{rs} = \frac{1}{2} Z_0^{rs} \varepsilon^\mu_r \varepsilon^\mu_s \theta_r \theta_s \Pi^{(2)}(r, s)$$

(9.10)

where we have used the result $\Pi^{(2)}(r, s) = P(r, s)$ of (7.7). The remaining blocks in (5.13) are more complicated and will require some degree of recombination of various contributions in (5.13) to produce further suitable blocks.

(3) A simple recombination of two blocks is obtained by regrouping the differential block $D_{2c}$ with that part of block $D_{2b}$ which is produced by the $p$-dependence of the composite $X^{\mu}_r$ in $D_{2b}$; we shall denote this contribution by $D_{2bp}$. (The remainder $D_{2b} - D_{2bp}$ will be dealt with later.) Combining $D_{2bp}$ with $D_{2c}$ gives,

$$D_{2bp} + D_{2c} = -\sum_{[rs]} d_s \left( Z_0^{rs} p^\mu_r \varepsilon^\mu_s \theta_r \theta_s \Pi^{(2)}(r, s) \right)$$

(9.11)

Using the definition of the block in (7.13),

$$\Pi^{(2)}_I(r, s) = \lambda_I(r) G(r, s) + \omega_I(r) P(r, s)$$

(9.12)

the above combination of blocks may be recast as follows,

$$D_{2bp} + D_{2c} = \sum_s d_s S^{(2)}_s + \sum_{[rs]} \partial_s \left( Z_0^{rs} p^\mu_r \varepsilon^\mu_s \theta_r \theta_s \Pi^{(2)}_I(r, s) \right)$$

(9.13)

The block $d_s S^{(2)}_s$ is exact in $s$ and holomorphic in $r$ and is of the type we seek. The additional term with a $\partial_s$ derivative which arises on the first line of (9.13) is holomorphic.
in $s$, but not holomorphic in $r$. At first sight, the appearance of this term looks worrisome. Actually, it is a term which is of type $(1,0)$ in every insertion point, and it must be recombined with the $Z$ terms, which are all of this type. It will turn out that this purely $(1,0)$ derivative term, and others that will arise from later recombinations, cancel similar terms that arise from the contractions of terms in $Z$. This mechanism will be illustrated in section 10 for these simple cases, and will be instrumental in obtained general systematic formulas for holomorphic blocks, to be worked out in the next paper.

(4) The remainder of $\mathcal{D}_{2b} - \mathcal{D}_{2bp}$ is given by partial contraction using the Lemma 1,

$$
\mathcal{D}_{2b} - \mathcal{D}_{2bp} = -\sum_{rs} d_s \left( Z_0^{rs} P(r, s) \varepsilon_s^\mu \theta_r \hat{X}_r^\mu \right) 
- \sum_{rst} d_s \left( Z_0^{rst} \varepsilon_s^\mu \theta_s P(r, s) K_t^\mu K_t^\nu X_t^\nu G(r, t) \right) 
- \sum_{rst} d_s \partial_r \left( Z_0^{rst} \varepsilon_s^\mu \theta_r \varepsilon_r^\nu \theta_s P(t, s) K_t^\nu K_t^\rho G(t, r) \right)
$$

(9.14)

In the last line, we have interchanged $r$ and $t$. The derivative term on the last line combines with a contribution from $\mathcal{D}_{3d}$. To see this, we use the structure of the block $\Pi^{(3)}(s; t; r)$,

$$
\Pi^{(3)}(s; t; r) = -F(t, s)F(t, r) + P(t, s)G(t, r) - P(r, t)G(t, s)
$$

(9.15)

to recast $\mathcal{D}_{3d}$ in the following form,

$$
\mathcal{D}_{3d} = \sum_{rs} d_r d_s S^{(3)}_{rs} + \mathcal{D}'_{3d}
$$

$$
S^{(3)}_{rs} = \sum_{t \neq r, s} Z_0^{rst} \frac{1}{2} \varepsilon_r^\mu \theta_r \varepsilon_r^\nu \theta_r K_t^\mu K_t^\nu \Pi^{(3)}(s; t; r)
$$

$$
\mathcal{D}'_{3d} = -\sum_{rst} d_r d_s \left( Z_0^{rst} \varepsilon_s^\mu \theta_s \varepsilon_r^\nu \theta_r K_t^\nu K_t^\rho P(t, s)G(t, r) \right)
$$

(9.16)

We see that the block $d_r d_s S^{(3)}_{rs}$ is exact in $r$ and $s$, and holomorphic in $t$ because $\Pi^{(3)}(s; t; r)$ is holomorphic in $t$. In the $\mathcal{D}'_{3d}$ term, the term inside the parentheses is actually holomorphic in $r$, so that the total differential $d_r$ acts as a holomorphic $\partial_r$ differential. Using the relation $d_s \partial_r = -\partial_r d_s$ on these differential forms, we see that the last term of $\mathcal{D}_{2b} - \mathcal{D}_{2bp}$ is cancelled precisely by $\mathcal{D}'_{3d}$. As a result, we have

$$
\mathcal{D}_{2b} - \mathcal{D}_{2bp} + \mathcal{D}_{3d} = \sum_{rs} d_r d_s S^{(3)}_{rs} - \sum_{rs} d_s \left( Z_0^{rs} P(r, s) \varepsilon_s^\mu \theta_r \hat{X}_r^\mu \right) 
- \sum_{rst} d_s \left( Z_0^{rst} \varepsilon_s^\mu \theta_s P(r, s) K_r^\mu K_r^\nu X_r^\nu G(r, t) \right)
$$

(9.17)
(5) Next, the terms $D_{3e} - D_{2b} + D_{2o} + D_{3d}$ combine with $D_{3e}$ as follows. In $D_{3e}$, we use again equation (9.15) to eliminate the $FF$ term, so that

$$D_{3e} = D_{3e}' + \sum_{[rst]} d_s \left(Z_{0}^{rst} K_t^\mu K_r^\nu \varepsilon_s^\mu \theta_s \left[P(r, s)G(r, t) - P(r, t)G(r, s)\right]X_t^\nu\right)$$

$$D_{3e}' = - \sum_{[rst]} d_s \left(Z_{0}^{rst} K_t^\mu K_r^\nu \varepsilon_s^\mu \theta_s \Pi^{(3)}(s; r; t)X_t^\nu\right)$$

(9.18)

The block $D_{3e}'$ is holomorphic in $r$, but not in $t$. It will need to be further expanded and combined with other blocks to be rendered holomorphic also in $t$. As a result, the terms in $P(r, s)$ cancel between the second line of (9.17) and $D_{3e} - D_{3e}'$, yielding

$$D_{2b} - D_{2o} + D_{3d} + D_{3e} = \sum_{[rst]} d_s d_s S_{s}^{(3)} + D_{3e}' - \sum_{[rst]} d_s \left(Z_{0}^{rs} P(r, s)\varepsilon_s^\mu \theta_s \dot{X}_t^\nu\right)$$

$$- \sum_{[rst]} \partial_s \left(Z_{0}^{rst} K_t^\mu K_r^\nu \varepsilon_s^\mu \theta_s P(r, t)G(r, s)X_t^\nu\right)$$

(9.19)

The $\partial_s$-derivative term on the last line is a purely $(1, 0)$ form in all insertion points, and it must be recombined with the $Z$ terms, which are all of this type.

(6) We shall now show how the block involving $\Pi^{(3)}$ arises by combining some of these results. It is the block involving one internal momentum. One contribution arises from the fermionic completion of $D_{3a}$ with a single Szegö kernel $K_r^\sigma K_t^\nu S(r, t)$, and yields

$$D_{3a}' = - \sum_{[rst]} d_s \left(Z_{0}^{rst} p_t^\nu K_{[r]}^\mu \varepsilon_s^\mu \theta_s F(r, s)S(r, t)\dot{\omega}_{10}(t)\right)$$

(9.20)

Combining this with the $p$-dependent term of $D_{3e}'$, we obtain

$$D_{3a}' + D_{3e}' = \sum_{[rst]} d_s \left(Z_{0}^{rst} p_t^\nu K_{[r]}^\mu \varepsilon_s^\mu \theta_s \left[F(r, s)S(r, t)\dot{\omega}_{10}(t) + \Pi^{(3)}(s; r; t)\omega_{10}(t)\right]\right)$$

(9.21)

Using the definitions of $\Pi^{(3)}$ and $\Pi^{(3)}$, we deduce the following expression for the term in the square brackets,

$$F(r, s)S(r, t)\dot{\omega}_{10}(t) + \Pi^{(3)}(s; r; t)\omega_{10}(t)$$

$$= -\Pi^{(3)}(t; r; s) + \lambda_1(t)G(t, r)G(r, s) - \dot{\omega}_{10}(t)F(t, r)G(r, s)$$

(9.22)

As a result, we have

$$D_{3a}' + D_{3e}' = \sum_{s} d_s S_{s}^{(3)} + \sum_{[rst]} d_s \left(Z_{0}^{rst} p_t^\nu K_{[r]}^\mu \varepsilon_s^\mu \theta_s \left[\lambda_1(t)G(t, r) - \dot{\omega}_{10}(t)F(t, r)\right]G(r, s)\right)$$

$$S_{s}^{(3)} = - \sum_{s \neq r} \sum_{t \neq r, s} d_s \left(Z_{0}^{rst} p_t^\nu K_{[r]}^\mu \varepsilon_s^\mu \theta_s \Pi^{(3)}(t; r; s)\right)$$

(9.23)
The block $d_sS_s^{(3)}$ is exact in $s$. It is holomorphic in $r$ and $t$, since $\Pi_I^{(3)}(t;r;s)$ is. The first term inside the bracket of the first line produces a $\partial_s$ derivative term which is a pure $(1,0)$ form in all insertion points. It will combine with terms in $\mathcal{Z}$.

9.3 General pattern for linear chain blocks

The results obtained from the few simple cases above clearly suggest a pattern for the structure of the linear chain blocks in $\mathcal{D}$. The contributions $\mathcal{D}_{dd}$ to $\mathcal{D}$ with two exact differentials are given by the sum over all $n \geq 0$ of

$$
\mathcal{D}_{dd} = \sum_{n \geq 0} \sum_{[rs]} d_r d_s S_s^{(n+2)}
$$

$$(9.24)$$

$$
S_s^{(n+2)} = - \sum_{r,s \notin [t_1,\ldots,t_n]} Z_{r,s}^{st_1\cdots t_n} \frac{1}{2} \varepsilon_i^r \varepsilon_i^s \theta_r \theta_s K_{[t_1\cdots t_n]}^{\mu\nu} \Pi^{(n+2)}(r; t_1, \ldots, t_n; s)
$$

The contributions $\mathcal{D}_{dp}$ to $\mathcal{D}$ with one exact differential and one factor linear in the internal momenta $p$ is given by,

$$
\mathcal{D}_{dp} = \sum_{n \geq 0} \sum_{[rs]} d_r S_s^{(n+1)}
$$

$$(9.25)$$

$$
S_s^{(n+1)} = - \sum_{s \notin [t_1,\ldots,t_n]} Z_{s}^{st_1\cdots t_n} \rho^\mu K_{[t_1\cdots t_n]}^{\mu\nu} \varepsilon_i^s \theta_s \Pi^{(n+1)}(t_1; t_2, \ldots, t_n; s)
$$

We shall prove these formulas systematically in the next subsection, and complete them to include all differential blocks $\mathcal{D}$.

9.4 Partial bosonic Wick contractions using Lemma 1

To deal with the structure of the blocks and the various rearrangements into holomorphic blocks in a systematic way, we need to deal with the contractions of the $x_+^s$ field inside the object $X_{x_+}^s$. Partial contractions may be effected with the help of Lemma 1. We shall organize the contractions according to the number of iterations using Lemma 1 that have been performed. The differential blocks that involve these contractions are $\mathcal{D}_{2b}$, $\mathcal{D}_{3e}$ and $\mathcal{D}_{4b}$, and we decompose the result of their contractions as follows,

$$
\mathcal{D}_{2b} = \sum_{n=0}^{N-2} \mathcal{D}_{2b}^{(n+2)} + \sum_{n=1}^{N-2} \mathcal{D}_{2b\partial}^{(n+2)}
$$

$$
\mathcal{D}_{3e} = \sum_{n=0}^{N-3} \mathcal{D}_{3e}^{(n+3)} + \sum_{n=1}^{N-3} \mathcal{D}_{3e\partial}^{(n+3)}
$$

$$
\mathcal{D}_{4b} = \sum_{n=0}^{N-4} \mathcal{D}_{4b}^{(n+4)} + \sum_{n=1}^{N-4} \mathcal{D}_{4b\partial}^{(n+4)}
$$

(9.26)
where we define the contributions with a bar as those arising from \( \bar{X} \), and the contributions labeled by \( \partial \) as those arising from the derivative terms in Lemma 1.

In the notation (4.10) for kinematic invariants and (7.1) for iterated products of Green’s functions, the \( \mathcal{D} \) contributions are given by

\[
\mathcal{D}^{(n+2)}_{2b} = - \sum_{[rst_1 \cdots t_n]} d_s \left( Z^{rs_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \rho}_{[t_{r_{1 \cdots t_{n-1}}}]} P(r, s) G^n(r, t_1, \cdots, t_n) \bar{X}^\rho_{t_n} \right)
\]

\[
\mathcal{D}^{(n+3)}_{3c} = \sum_{[rsu_{t_1 \cdots t_n}]} d_s \left( Z^{rsu_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \nu}_{[u_{t_{1 \cdots t_{n-1}}}]} F(r, s) \right.
\]
\[
\times F(r, u) G^n(u, t_1, \cdots, t_n) \bar{X}^\rho_{t_n} \right)
\]

\[
\mathcal{D}^{(n+4)}_{4b} = \sum_{[rsuv_{t_1 \cdots t_n}]} d_s \left( Z^{rsuv_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \nu}_{[u_{t_{1 \cdots t_{n-1}}}]} F(r, s) \right.
\]
\[
\times F(u, v) G^n(v, t_1, \cdots, t_n) \bar{X}^\rho_{t_n} \right) \quad (9.27)
\]

The differential contributions are given by

\[
\mathcal{D}^{(n+2)}_{2b} = - \sum_{[rst_1 \cdots t_n]} d_s \partial_{t_n} \left( Z^{rs_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \rho}_{[t_{r_{1 \cdots t_{n-1}}}]} \varepsilon^\rho \theta_{t_n} P(r, s) G^n(r, t_1, \cdots, t_n) \right)
\]

\[
\mathcal{D}^{(n+3)}_{3c} = \sum_{[rsu_{t_1 \cdots t_n}]} d_s \partial_{t_n} \left( Z^{rsu_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \nu}_{[u_{t_{1 \cdots t_{n-1}}}]} \varepsilon^\rho \theta_{t_n} F(r, s) \right.
\]
\[
\times F(r, u) G^n(u, t_1, \cdots, t_n) \right)
\]

\[
\mathcal{D}^{(n+4)}_{4b} = \sum_{[rsuv_{t_1 \cdots t_n}]} d_s \partial_{t_n} \left( Z^{rsuv_{t_1 \cdots t_n}} \varepsilon^\mu \theta_s K^{\mu \nu}_{[u_{t_{1 \cdots t_{n-1}}}]} \varepsilon^\rho \theta_{t_n} \right.
\]
\[
\times F(r, s) F(u, v) G^n(v, t_1, \cdots, t_n) \right) \quad (9.28)
\]

Here we have systematically used the notations of (7.1) for linear chains of bosonic and fermionic Green functions, as well as the notations of (4.10) for the associated kinematic factors.

### 9.5 Partial fermionic Wick contractions in \( \mathcal{D}_{4b} \)

It will be useful to render the fermionic contractions partially explicit with a string of Szegő kernels between the points \( r \) and \( u \) in the blocks \( \mathcal{D}_{4b} \). By an abuse of notation, we shall now denote by \( \mathcal{D}_{4b}^{(n+4)} \) the derivative block with a total of \( n + 4 \) points, including the fermionic contraction points.
The blocks \( \mathcal{D}_{4b}^{(n+4)} \) are given by,

\[
\mathcal{D}_{4b}^{(n+4)} = \sum_{\beta=1}^{n} \sum_{[rsuv1 \cdots t_n]} d_s \partial_{t_n} \left( Z_0^{rsuv1 \cdots t_n} \varepsilon_s^\mu \theta_s K_{[rt_1 \cdots t_{\beta-1}uv\ldots t_n]}^\mu \varepsilon_t^\rho \theta_t F(r, s) \times S^\beta(r, t_1, \ldots, t_{\beta-1}, u) F(u, v) G^{n-\beta+1}(v, t_{\beta}, \ldots, t_n) \right)
\]

(9.29)

It is important to recall the correct instructions that are to be imposed on the operator \( \bar{X}^\rho_{t_n} \) in the first \( n \) terms of \( \mathcal{D}_{4b}^{(n+4)} \) and on the operator \( \bar{X}^\rho_v \) in the last line of \( \mathcal{D}_{4b}^{(n+4)} \).

### 9.5.1 Avoiding the danger of double counting

In particular, there is a danger of double counting contributions that must be addressed.

1. The \( x_+ \) field in \( \bar{X}^\rho_{t_n} \) is to be contracted with \( x_+ \) in the exponential factor of \( Z_0 \) at the points \( r, s, u, v, t_{\beta-1} \), and only at those points. No contractions at the points \( t_1, \ldots, t_{\beta-1} \) are to be included. From the point of view of \( x_+ \)-contractions, the points \( t_1, \ldots, t_{\beta-1} \) are new points, and these contributions are already included in the terms with \( \beta \to \beta - 1 \).

2. The term \( i k_{t_n}^\rho \) in \( \bar{X}^\rho_{t_n} \) must be evaluated at all points except \( r, s, u \), including when \( t_n \) coincides with the fermionic insertion points \( t_1, \ldots, t_{\beta-1} \).

3. The \( x_+ \) field in \( X^\rho_v \) is to be contracted with \( x_+ \) in the exponential of \( Z_0 \) at the points \( r, s, u, v \), and only at those points. No contractions at the points \( t_1, \ldots, t_n \) are to be included. From the point of view of \( x_+ \)-contractions, the points \( t_1, \ldots, t_n \) are new points, and these contributions are already included in the terms with \( n \to n + 1 \).

4. The term \( i k_v^\rho \) in \( \bar{X}^\rho_v \) must be evaluated at all points different from \( r, s, t \), including when \( v \) coincides with the fermionic insertion points \( t_1, \ldots, t_n \).

It will be convenient to reorganize these instructions according to the following pattern.
The reorganized instructions for the blocks \( \tilde{D}_{4b}^{(n+2)} \), are as follows,

\[
\tilde{D}_{4b}^{(n+2)} = \sum_{\beta=1}^{n-2} \sum_{[rst1\ldotstn]} d_s \left( Z_0^{rst1\ldotstn} \varepsilon_s^\mu \theta_s K_{\mu [r[t_1\ldots t_{n-1}]}^\mu \varepsilon_{t_n}^\nu \theta_{t_n} K_{\nu t_n]}^\nu \right) \tilde{X}_{tn}^\rho F(r,s) 
\times S^0(t_1, t_2, \ldots, t_\beta, t_\beta+1) G^{n-\beta-1}(t_{\beta+1}, t_{\beta+2}, \ldots, t_n) + \\
\sum_{[rst1\ldots tn]} d_s \left( Z_0^{rst1\ldotstn} \varepsilon_s^\mu \theta_s K_{\mu [r[t_1\ldots t_{n-1}]}^\mu \varepsilon_{t_n}^\nu \theta_{t_n} K_{\nu t_n]}^\nu \right) \tilde{X}_{tn}^\rho F(r,s) S^{n-1}(r, t_1, \ldots, t_{n-1}) F(t_{n-1}, t_n) 
\times S^{n-1}(r, t_1, \ldots, t_{n-1}) F(t_{n-1}, t_n)
\]

with the following instructions,

(I) The point \( t_n \) is genuinely distinct from all points \( rst1 \ldots t_{n-1} \). (Thus no \( ik^\rho \) terms arise for \( t_n \) coincident with those points.)

(II) The \( x_+ \) field in \( \tilde{X}_{tn}^\rho \), in both lines, must be contracted with the \( x_+ \) field in the exponential of \( Z_0 \) for all points \( r, s, t_1, \ldots, t_{n-1} \), namely including all the fermion contraction points.

The point \( t_n \) is exactly of the form of the \( ik^\rho \) term with \( v = r \).

\[
\tilde{D}_{4b}^{(n+2)} = \sum_{[rst1\ldots tn]} d_s \left( Z_0^{rst1\ldotstn} \varepsilon_s^\mu \theta_s K_{\mu [r[t_1\ldots t_{n-1}]}^\mu \varepsilon_{t_n}^\nu \theta_{t_n} K_{\nu t_n]}^\nu \right) \tilde{X}_{tn}^\rho F(r,s) 
\times S^0(t_1, t_2, \ldots, t_\beta, t_\beta+1) G^{n-\beta-1}(t_{\beta+1}, t_{\beta+2}, \ldots, t_n) + \\
\sum_{[rst1\ldots tn]} d_s \left( Z_0^{rst1\ldotstn} \varepsilon_s^\mu \theta_s K_{\mu [r[t_1\ldots t_{n-1}]}^\mu \varepsilon_{t_n}^\nu \theta_{t_n} K_{\nu t_n]}^\nu \right) \tilde{X}_{tn}^\rho F(r,s) S^{n-1}(r, t_1, \ldots, t_{n-1}) F(t_{n-1}, t_n) 
\times S^{n-1}(r, t_1, \ldots, t_{n-1}) F(t_{n-1}, t_n)
\]
Finally, the second summation may be safely lumped with the first summation as \( \beta = n - 1 \), so that our final formula is

\[
\bar{D}_{4b}^{(n+2)} = \sum_{\beta=1}^{n-1} \sum_{[r_{st1} \cdots t_n]} d_s \left( Z_0^{r_{st1} \cdots t_n} \varepsilon_s^\mu \theta_s K_{[r_{t1} \cdots t_{n-1}]}^{t} \bar{X}_t^\rho F(r, s) \right. \\
\times S^\beta(r, t_1, \cdots, t_\beta) F(\beta, t_{\beta+1}) G^{n-\beta-1}(t_{\beta+1}, t_{\beta+2}, \cdots, t_n) \\
+ \sum_{[r_{st1} \cdots t_n]} d_s \left( Z_0^{r_{st1} \cdots t_n} \varepsilon_s^\mu \theta_s K_{[r_{t1} \cdots t_{n-2}]}^{t} \sum_{a=1}^{n-2} i k_{ta}^\rho \delta_{ta, t_n} \\
\times F(r, s) S^{n-1}(r, t_1, \cdots, t_{n-1}) F(t_{n-1}, t_n) \right)
\]

subject to the same instructions (I) and (II). Note that these blocks are defined for \( n \geq 2 \), and that the second sum only contributes starting at \( n = 3 \).

### 9.6 Partial fermionic Wick contractions in the remaining blocks

In order to properly group together the various components of the blocks \( \Pi, \Pi_f, \) and \( \Pi_\pm \), we need to carry out the Wick contractions of the worldsheet fermion fields in order to complete the chain blocks, including their linking. We shall organize these contributions according to the number \( n \) of Szegö kernels that need to be inserted to complete the chain.

To organize this procedure effectively, it will be useful to recognize first that certain blocks \( D \) naturally fit in the same sequence of chains. For example, the block \( D_{2c} \) may really be viewed as an extension of the block \( D_{3a} \) in which we allow the point \( t \) to coincide with \( r \); similarly the block \( D_{2d} \) is an extension of \( D_{3c} \) with \( t = r \), and \( D_{3d} \) is an extension of \( D_{4a} \) with \( u = t \) and \( r \) and \( t \) interchanged. Thus, we have the following expansions in the number \( n \) of Szegö kernel insertions,

\[
D_{2c} + D_{3a} = \sum_{n=0}^N D_{3a}^{(n+2)} \\
D_{3b} = \sum_{n=1}^N D_{3b}^{(n+2)} \\
D_{2d} + D_{3c} = \sum_{n=0}^N D_{3c}^{(n+2)} \\
D_{3f} = \sum_{n=1}^N D_{3f}^{(n+2)} \\
D_{3d} + D_{4a} = \sum_{n=1}^N D_{4a}^{(n+2)}
\]

with

\[
D_{3a}^{(n+2)} = \sum_{[r_{st1} \cdots t_n]} d_s \left( Z_0^{r_{st1} \cdots t_n} \varepsilon_s^\mu \theta_s K_{[r_{t1} \cdots t_{n-1}]}^{t} p^\rho F(r, s) S^n(r, t_1 \cdots, t_n) \hat{w}_{t10}(t_n) \right) \quad (9.34)
\]

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To obtain the last line, we have taken \( \mathcal{D}_{4a} \) from (5.13), changed variables from \((r, s, u)\) to \((u, r, s)\) and recast the resulting formula for \(n + 3 \to n + 2\) vertex insertion points. We use the convention whereby \( \mathcal{D}_{3e}^{(n+2)} \) and \( \mathcal{D}_{4a}^{(n+2)} \) vanish for \(n = 0\), and \( \mathcal{D}_{4b}^{(n+2)} \) vanishes for both values \(n = 0, 1\).

### 9.7 Contributions to \( \varepsilon^\mu_r \theta_t K^{\mu
u}_{[r_1 \cdots t_n]} \varepsilon^\nu_s \theta_s \)

These contributions precisely coincide with the terms that have two differentials \(d_r d_s\); they combine as follows,

\[
\mathcal{D}_{2b}^{(n+2)} + \mathcal{D}_{3e}^{(n+2)} + \mathcal{D}_{4b}^{(n+2)} + \mathcal{D}_{4a}^{(n+2)} = \sum_{[r s]} d_r d_s S_{r s}^{(n+2)}
\]  

(9.35)

where \(S_{r s}^{(n+2)}\) was given explicitly in (9.24). This result is up to terms with two exact \((1, 0)\) differentials \(\partial_r \partial_s\), which will combine with similar terms in \(Z\).

### 9.8 Contributions to \( \varepsilon^\mu_s \theta_t K^{\mu
u}_{[r_1 \cdots t_{n+1}]} \hat{P}_I^\nu \)

These contributions precisely coincide with the terms that have a single differential and are linear in \(p\) (we do not count the \(p\)-dependence inside \(Z_0\) here). The dependence of the terms linear in \(p\) arises from the blocks \(\mathcal{D}_{1}, \mathcal{D}_{2c}, \mathcal{D}_{3a}\) as well as from the Wick contraction terms \(\mathcal{D}_{2b}^{(n+2)}, \mathcal{D}_{3e}^{(n+2)}, \) and \(\mathcal{D}_{4b}^{(n+2)}\). To make the \(p\)-dependence explicit, we split each of the contraction terms as follows,

\[
\hat{\mathcal{D}}_{2b}^{(n+2)} = \hat{\mathcal{D}}_{2b}^{(n+2)} + \mathcal{D}_{2b}^{(n+2)} \quad n \geq 0
\]

\[
\hat{\mathcal{D}}_{3e}^{(n+2)} = \hat{\mathcal{D}}_{3e}^{(n+2)} + \mathcal{D}_{3e}^{(n+2)} \quad n \geq 1
\]

\[
\hat{\mathcal{D}}_{4b}^{(n+2)} = \hat{\mathcal{D}}_{4b}^{(n+2)} + \mathcal{D}_{4b}^{(n+2)} \quad n \geq 2
\]

(9.36)

Here, the blocks denoted by \(\hat{\mathcal{D}}\) are given by the same formulas as the blocks \(\mathcal{D}\), but now with \(X\) replaced by \(\hat{X}\). The explicit expressions for the \(p\)-dependent terms are easily

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derived, and we have,

\[
D^{(n+2)}_{2bp} = \sum_{rst1\cdots t_n} d_s \left( Z^{rst1\cdots t_n} \varepsilon_s^\mu \theta_s^\nu K^{\mu \nu}_{[r_t1\cdots t_n]} \rho \hat{P}(r, s) G^n(r, t_1, \cdots, t_n) \omega_I(t_n) \right)
\]

\[
D^{(n+2)}_{3a} = \sum_{rst1\cdots t_n} d_s \left( Z^{rst1\cdots t_n} \varepsilon_s^\mu \theta_s^\nu K^{\mu \nu}_{[r_t1\cdots t_n]} \rho \hat{F}(r, s) S^n(r, t_1, \cdots, t_n) \omega(t_n) \right)
\]

\[
D^{(n+2)}_{3ep} = -\sum_{rst1\cdots t_n} d_s \left( Z^{rst1\cdots t_n} \varepsilon_s^\mu \theta_s^\nu K^{\mu \nu}_{[r_t1\cdots t_n]} \rho \hat{F}(r, s) F(r, t_1) G^n(t_1, \cdots, t_n) \omega_I(t_n) \right)
\]

\[
D^{(n+2)}_{4bp} = -\sum_{rst1\cdots t_n} d_s \left( Z^{rst1\cdots t_n} \varepsilon_s^\mu \theta_s^\nu K^{\mu \nu}_{[r_t1\cdots t_n]} \rho \hat{F} \sum_{\beta=1}^{n-1} F(r, s) S^\beta(r, t_1, \cdots, t_\beta) \times F(t_\beta, t_{\beta+1}) G^{n-\beta-1}(t_{\beta+1}, \cdots, t_n) \omega_I(t_n) \right)
\]

(9.37)

Therefore, the sum of all contributions linear in \( p \) is given by,

\[
\delta_{n,0} D_1 + D^{(n+2)}_{2bp} + D^{(n+2)}_{2c} + D^{(n+2)}_{3a} + D^{(n+2)}_{3ep} + D^{(n+2)}_{4bp} = \sum_s d_s S_s^{(n+1)}
\]

(9.38)

where \( S_s^{(n+1)} \) was given explicitly in (9.25). This result is up to terms with one exact \((1,0)\) differential \( \partial_s \), which will combine with similar terms in \( Z \).
10 Extraction of Exact Differentials

Part II Singly linked Chain blocks

In the preceding section, we have carried out the bosonic and fermionic Wick contractions needed to identify the chiral blocks, and we have explicitly accounted for the chiral differential blocks that involve II and II. The entire block functions \( D_1, D_{2a}, D_{2b}, D_{2c}, D_{3a}, D_{3b}, D_{3c}, D_{3d}, D_{4a}, D_{4b}, \) and \( D_{4b} \) were consumed in this process. In the present section, we shall collect all the remaining contributions to \( D \), which are \( D_{2b}, D_{2d} \) (included in \( D_{3c} \)), \( D_{3b}, D_{3c}, D_{3f} \) and \( D_{4b} \), and derive their representation in terms of the singly linked chain blocks \( \Pi_\pm \).

10.1 Remaining differential blocks

We begin by exhibiting the explicit forms of the remaining differentials,

\[
\hat{D}^{(n+2)}_{2b} = - \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n-1}]}^{\mu \rho} P(r, s) G^n(r, t_1, \cdots, t_n) \hat{X}^\rho_{t_n} \right) \quad (10.1)
\]

\[
\hat{D}^{(n+2)}_{3b} = + \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n}]}^{\mu \rho} i k_s^\rho F(r, s) S^n(r, t_1, \cdots, t_n) F(t_n, s) \right)
\]

\[
\hat{D}^{(n+2)}_{3c} = - \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n}]}^{\mu \rho} K_s^{\rho \mu}(ds)^{-1} K_{[rt_{1}\cdots t_{n}]}^{\mu \rho} S^n(r, t_1, \cdots, t_n) Q_B(s; r, t_n) \right)
\]

\[
\hat{D}^{(n+2)}_{3f} = + \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n-1}]}^{\mu \rho} F(r, s) F(r, t_1) G^n(t_1, \cdots, t_n) \hat{X}^\rho_{t_n} \right)
\]

and finally the most involved block,

\[
\hat{D}^{(n+2)}_{4b} = \sum_{\beta=1}^{n-1} \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n-1}]}^{\mu \rho} \hat{X}^\rho_{t_n} F(r, s) \right.
\]

\[
\times S^\beta(r, t_1, \cdots, t_\beta) F(t_\beta, t_{\beta+1}) G^{n-\beta-1}(t_{\beta+1}, t_{\beta+2}, \cdots, t_n) \Bigg) + \sum_{[rst\cdots t_n]} d_s \left( Z_{0}^{r_{st_{1}}\cdots t_{n}} z_s^\mu \theta_s K_{[rt_{1}\cdots t_{n}]}^{\mu \rho} \sum_{a=1}^{n-2} i k_{t_a}^\rho \delta_{t_a, t_n} \right.
\]

\[
\times F(r, s) S^{n-1}(r, t_1, \cdots, t_{n-1}) F(t_{n-1}, t_n) \Bigg) \quad (10.2)
\]

Note that, upon the identification \( t_0 \equiv r \), the block \( D_{3f} \) corresponds to the term \( a = 0 \) in the sum over \( a \) on the third line of the above expression for \( \hat{D}^{(n+2)}_{4b} \). Henceforth, we shall assume that this reformulation has been carried out.
The instructions on the contractions of the composite $\hat{X}^\rho_{t_n}$ are the same as the ones given in section 9.5; we repeat them here for completeness.

(I) The point $t_n$ is genuinely distinct from all points $rst_1 \cdots t_{n-1}$. (Thus no $ik^\rho$ terms arise for $t_n$ coincident with those points.)

(II) The $x_+$ field in $\hat{X}^\rho_{t_n}$, must be contracted with the $x_+$ field in the exponential of $Z_0$ for all points $r, s, t_1, \ldots, t_{n-1}$, namely including all the fermion contraction points.

The strategy adopted here will be to first isolate the terms associated with the bosonic Wick contraction between the points $t_n$ and $s$. These contractions produce two distinctive kinematical factors, namely $K^{\mu\nu}_{[rst_1 \cdots t_n]}$, from anti-symmetrization in $\varepsilon^\rho_\nu \theta_s k^\nu_s$, or the factor $\varepsilon^{[\mu s} \theta_s k^{\nu]}_{[rst_1 \cdots t_n]}$ from symmetrization. They will produce the blocks $\Pi_{+}^{(\ell+1)}$, and will completely consume the term $D_{3b}^{(n+2)}$ above. The remaining blocks are then organized according to the symmetry properties of $\Pi_{+}^{(m+1)(\ell+1)}$ or $\Pi_{-}^{(m+1)(\ell+1)}$ respectively.

### 10.2 Symmetric contraction between $t_n$ with $s$

The symmetric bosonic Wick contractions have the following characteristic kinematical factor, $\varepsilon^{[\mu s} \theta s k^{\nu]}_{[rst_1 \cdots t_n]}$, contributions to which we shall now isolate, and indicate with a subscript $+s$. From assembling the $s+$ parts of $\hat{D}_{2b}, \hat{D}_{3b}, \hat{D}_{4b}$, with the entire contribution of $\hat{D}_{3e}$, we find,

$$
\sum_s d_s S_{+s}^{(n+2)} = \hat{D}_{2b+s}^{(n+2)} + D_{3b}^{(n+2)} + D_{3e+s}^{(n+2)} + \hat{D}_{4b+s}^{(n+2)} \quad (10.3)
$$

$$
S_{+s}^{(n+2)} = - \sum_{s \not\in [r \cdots t_n]} Z_0^{rst_1 \cdots t_n} \frac{1}{2} \varepsilon^{[\mu s} \theta s k^{\nu]}_{[rst_1 \cdots t_n]} K^{\mu\nu}_{[rst_1 \cdots t_n]} \Pi_{+}^{(n+2)}(s; r, t_1, \ldots, t_n)
$$

The contributions in the block $\Pi_{+}^{(n+2)}(s; r, t_1, \ldots, t_n)$ derived directly from $\hat{D}_{2b}^{(n+2)}, D_{3b}^{(n+2)}, D_{4b}^{(n+2)}$, and $\hat{D}_{3e}^{(n+2)}$ are given by

$$
-2P(r, s)G^{n+1}(r, t_1, \ldots, t_n, s) + F(r, s)S^{n}(r, t_1, \ldots, t_n)F(t_n, s)
$$

$$
+2 \sum_{\beta=0}^{n-1} F(r, s)S^{n}(r, t_1, \ldots, t_\beta)F(t_\beta, t_{\beta+1}, \ldots, t_n)G^{n-\beta}(t_{\beta+1}, \ldots, t_n, s) \quad (10.4)
$$

where $t_0 \equiv r$, and $S(t_{-1}, t_0) \equiv 1$ in the above sum. Actually, in view of the symmetry properties of the kinematic factor, we may symmetrize the above contributions to $\Pi_{+}^{(n+2)}$, without loss of generality. The symmetry operation will divide the above contributions by a factor of 2, and add $(-)^{n+1}$ times those terms with $[rt_1t_2 \cdots t_{n-1}t_n] \rightarrow [tnt_{n-1} \cdots t_2t_1r]$ interchanged. One checks that

$$
F(r, s)S^{n}(r, t_1, \ldots, t_n)F(t_n, s) = (-)^{n+1}F(t_n, s)S^{n}(t_n, \ldots, t_1, r)F(r, s) \quad (10.5)
$$
As a result, we obtain the symmetrized block $\Pi_{+}^{(n+2)}(s; r, t_1, \ldots, t_n)$, given by

$$
-P(r, s)G^{n+1}(r, t_1, \ldots, t_n, s) + (-)^n G^{n+1}(t_n, \ldots, t_1, r)P(t_n, s) \\
+ F(r, s)S^n(r, t_1, \ldots, t_n)F(t_n, s) \\
+ \sum_{\beta=0}^{n-1} F(r, s)S^\beta(r, t_1, \ldots, t_\beta)F(t_\beta, t_{\beta+1})G^{n-\beta}(t_{\beta+1}, \ldots, t_n, s) \\
+ (-)^{n+1} \sum_{\beta=0}^{n-1} F(t_n, s)S(t_n, t_{n-1}) \cdots S(t_{n+1-\beta}, t_n-\beta) \\
\times F(t_{n-\beta}, t_{n-1-\beta})G^{n-\beta}(t_{n-\beta-1}, \ldots, r, s)
$$

(10.6)

This expression coincides with $-\Pi^{(n+3)}(s; r, t_1, \ldots, t_n; s)$ up to terms which are exact holomorphic differentials in $s$, and which will combine with similar contributions arising from contractions of $X$ in $Z$. Using now also the definition of the block $\Pi_{+}^{(n+2)}$ in terms of $\Pi^{(n+3)}$, we obtain the result announced in (10.3).

### 10.3 Anti-Symmetric contraction between $t_n$ with $s$

The anti-symmetric bosonic Wick contractions have the following characteristic kinematical factor, $K^{\mu\nu}_{[r_1\ldots t_n, s]}$, contributions to which we shall now isolate, and indicate with a subscript $-s$. From assembling the $-s$ parts of $\hat{D}_{2b}$, $\hat{D}_{3c}$, $\hat{D}_{4b}$, with the entire contribution of $\hat{D}_{3c}$, we find,

$$
\sum_s d_s S^{(n+2)}_{-s} = \hat{D}_{2b-s}^{(n+2)} + \hat{D}_{3c}^{(n+2)} + \hat{D}_{4b-s}^{(n+2)}
$$

(10.7)

The contributions in the block $\Pi_{-}^{(n+2)}(s; r, t_1, \ldots, t_n)$ derived directly from $\hat{D}_{2b}^{(n+2)}$, $\hat{D}_{3c}^{(n+2)}$, $\hat{D}_{4b}^{(n+2)}$, and $\hat{D}_{3c}^{(n+2)}$ are given by

$$
P(r, s)G^{n+1}(r, t_1, \ldots, t_n, s) - Q_B(s; r, t_n)S^n(r, t_1, \ldots, t_n) \\
- \sum_{\beta=0}^{n-1} F(r, s)S^{\beta}(r, t_1, \ldots, t_\beta)F(t_\beta, t_{\beta+1})G^{n-\beta}(t_{\beta+1}, \ldots, t_n, s)
$$

(10.8)

The kinematical factor has the following reflection and cyclic permutation symmetries,

$$
K^{\mu\nu}_{[r_1\ldots t_n, s]} = (-)^n K^{\mu\nu}_{[st_n\ldots t_1 r]} \\
K^{\mu\nu}_{[s t_n\ldots t_1 r]} = K^{\mu\nu}_{[t_n\ldots t_1 r s]}
$$

(10.9)

which combine to yield the mixed reflection and cyclic permutation symmetry,

$$
K^{\mu\nu}_{[r_1\ldots t_n]} = (-)^n K^{\mu\nu}_{[t_n\ldots t_1 r s]}
$$

(10.10)
It is under this symmetry that we can “symmetrize” the contributions of (10.8) by averaging (10.8) with \((-\)^n times the expression with \([rt_1 \cdots t_n s] \to [t_n \cdots t_1 s]r s\). The resulting symmetrized contributions to \(\Pi^{(n+2)}_\varepsilon(s;r_1,\cdots,t_n)\) will be denoted \(B^{(n+2)}_\varepsilon(s;r_1,\cdots,t_n)\), and are given as follows,

\[
B^{(n+2)}_\varepsilon = -Q_B(s;r_1,\cdots,t_n)S^n(r_1,\cdots,t_n) + \frac{1}{2}P(r,s)G^{n+1}(r_1,\cdots,t_n,s) \tag{10.11}
\]

\[
+ \frac{1}{2}(-)^nP(t_n,s)G^{n+1}(t_n,t_{n-1},\cdots,t_1,r,s)
\]

\[
- \frac{1}{2}\sum_{\beta=0}^{n-1} F(r,s)S^\beta(r_1,\cdots,t_\beta)F(t_\beta,t_{\beta+1})G^{n-\beta}(t_{\beta+1},\cdots,t_n,s)
\]

\[
- \frac{1}{2}(-)^n\sum_{\alpha=1}^{n} F(t_n,s)S^{n-\alpha}(t_n,\cdots,t_\alpha)F(t_\alpha,t_{\alpha-1})G^\alpha(t_{\alpha-1},\cdots,t_1,r,s)
\]

In view of the symmetrization, the blocks automatically have the symmetry,

\[
B^{(n+2)}_\varepsilon(s;r_1,\cdots,t_n) = (-)^nB^{(n+2)}_\varepsilon(s;t_n,t_{n-1},\cdots,t_1) \tag{10.12}
\]

This definition applies to all \(n \geq 1\). When \(n = 0\), we can still define the block \(B^{(2)}_\varepsilon(s;r)\) by the above expression by setting \(t_n = r\) and \(n = 0\). Thus the lowest order blocks are

\[
B^{(2)}_\varepsilon(s;r) = -Q_B(s;r,r) + P(r,s)G(r,s)
\]

\[
B^{(3)}_\varepsilon(s;r,t) = -Q_B(s;r,t)S(r,t)
\]

\[
+ \frac{1}{2}P(r,s)G(r,t)G(t,s) - \frac{1}{2}P(t,s)G(t,r)G(r,s)
\]

\[
+ \frac{1}{2}F(t,s)F(t,r)G(r,s) - \frac{1}{2}F(r,s)F(r,t)G(t,s) \tag{10.13}
\]

The presence of the cubic vertex function \(Q_B\) alerts us to the possibility that \(B^{(n+2)}_\varepsilon\) is related to the blocks \(\Pi^{(n+2)}_\varepsilon\) constructed in section 8. We recall here the lowest cases, adapted to the present notation,

\[
\Pi^{(2)}_\varepsilon(s;r) = Q_B(s;r,r) - 2Q_P(s;r,r) - G(r,s)P(r,r)
\]

\[
\Pi^{(3)}_\varepsilon(s;r,t) = Q_B(s;r,t)S(r,t) + Q_0(s;r,t)P(r,t)
\]

\[
- Q_P(s;r,t)G(r,t) + Q_P(s;t,r)G(t,r)
\]

\[
+ Q_F(s;t,r)F(r,t) - Q_F(s;r,t)F(t,r)
\]

\[
- G(r,s)\Pi^{(2)}_\varepsilon(r;t) + G(t,s)\Pi^{(2)}_\varepsilon(t;r) \tag{10.14}
\]

To compare, we compute

\[
\partial_s \left( B^{(2)}(s;r) + \Pi^{(2)}(s;r) \right) = \partial_s \left( P(r,s)G(r,s) - 2Q_P(s;r,r) - G(r,s)P(r,r) \right)
\]

\[
= \partial_s G(r,s) \left( P(r,s) - P(r,r) \right) = 0 \tag{10.15}
\]
Here, the term $Q_B(s; r, r)$, which arises in the expressions of both $B^{(2)}_-$ and $\Pi^{(2)}_-$ cancels out. The final result of (10.15) vanishes, from which we conclude that $B^{(2)}_-$ equals $\Pi^{(2)}_-$, up to terms holomorphic in $s$. As usual, in the full chiral amplitude, such exact holomorphic differentials in $s$ will combine with similar terms in $Z$.

The case of $B^{(3)}_-(s; r, t)$ is similar, and we calculate

$$\partial_s \left( B^{(3)}_-(s; r, t) + \Pi^{(3)}_-(s; r, t) \right) = \partial_s \left( \frac{1}{2} P(r, s) G(r, t) G(t, s) - \frac{1}{2} P(t, s) G(t, r) G(r, s) \right) + \frac{1}{2} F(t, s) F(t, r) G(r, s) - \frac{1}{2} F(r, s) F(r, t) G(t, s) - Q_P(s; r, t) G(r, t) + Q_P(s; t, r) G(t, r) + Q_F(s; t, r) F(r, t) - Q_F(s; r, t) F(t, r) \right) \quad (10.16)$$

Here, we have readily omitted the terms that produce $\delta$-functions at coincident vertex insertion points only, such as the term in $Q_0$, and the terms in $\Pi^{(2)}_-$ that arise in $\Pi^{(3)}_-$.

Using on the right hand side of (10.16) the defining $\partial_s P$ and $\partial_s F$, neglecting $\delta$-functions at coincident points immediately shows that (10.16) vanishes. The proof to all orders is readily obtained by generalizing the above case to all $n$, and we obtain,

**Lemma 2** The blocks $B^{(n+2)}_-(s; r, t_1, \ldots, t_n)$ and $-\Pi^{(n+2)}_-(s; r, t_1, \ldots, t_n)$ differ only by terms which are holomorphic in $s$,

$$\partial_s \left( B^{(n+2)}_-(s; r, t_1, \ldots, t_n) + \Pi^{(n+2)}_-(s; r, t_1, \ldots, t_n) \right) = 0. \quad (10.17)$$

This completes the proof of (10.7).

### 10.4 Blocks remaining after contractions of $t_n$ with $s$

Having identified and extracted the differential blocks $d_s S^{(n+2)}_+ S^{(n+2)}_-$ and $d_s S^{(n+2)}_+ S^{(n+2)}_-$, we are left with only the following differential blocks,

$$\tilde{D}^{(n+2)}_{2b} = - \sum_{[rst_1 \ldots t_n]} d_s \left( Z^{rst_1 \ldots t_n} s \theta_s K^{\mu \rho}_{[rst_1 \ldots t_n]} P(r, s) G^m(r, t_1, \ldots, t_n) \tilde{X}^\rho_{t_n} \right)$$

$$\tilde{D}^{(n+2)}_{4b} = \sum_{\beta=0}^{n-1} \sum_{[rst_1 \ldots t_n]} d_s \left( Z^{rst_1 \ldots t_n} s \theta_s K^{\mu \rho}_{[rst_1 \ldots t_n]} \tilde{X}^\rho_{t_n} F(r, s) \right.$$

$$\left. \times S^{\beta}(r, t_1, \ldots, t_\beta) F(t_\beta, t_{\beta+1}) G^{n-\beta-1}(t_{\beta+1}, \ldots, t_n) \right)$$

$$+ \sum_{[rst_1 \ldots t_n]} d_s \left( Z^{rst_1 \ldots t_n} s \theta_s K^{\mu \rho}_{[rst_1 \ldots t_n]} \sum_{a=0}^{n-2} i k^\rho_{t_a t_n} \delta_{t_a t_n} F(r, s) \right.$$

$$\left. \times S^{n-1}(r, t_1, \ldots, t_{n-1}) F(t_{n-1}, t_n) \right) \quad (10.18)$$

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Here, we have denoted by $\tilde{X}_{t_n}^{\rho}$ the instruction that $\tilde{X}_{t_n}^{\rho}$ is not to be contracted with $s$ any more (since those contractions were already isolated in sections 10.2 and 10.3), and the corresponding blocks also by $\tilde{D}$. In $\tilde{D}_{4b}^{(n+2)}$, we have now also included $\tilde{D}_{3b}^{(n+2)}$ as the contribution with $\beta = 0$ in the first sum, as well as $\tilde{D}_{3f}^{(n+2)}$ as the contribution with $a = 0$ in the second sum (where $t_0 \equiv r$). For later use, we shall relabel $r \rightarrow t_1$ and $t_i \rightarrow t_{i+1}$ when $i = 1, \ldots, n - 1$, and finally $t_{n+1} \rightarrow r$, so that

\[
\tilde{D}_{b}^{(n+2)} = - \sum_{[s t_1 \ldots t_n r]} d_s \left( Z_{0}^{st_1 \ldots t_n r} \varepsilon_{\mu} \theta_{s} K_{[t_1 \ldots t_n]}^{\mu \rho} P(s, t_1) G^{n}(t_1, \ldots, t_n, r) \tilde{X}_{r}^{\rho} \right) D\]

\[
\tilde{D}_{4b}^{(n+2)} = \sum_{s=1}^{m} \sum_{[s t_1 \ldots t_n r]} d_s \left( Z_{0}^{s t_1 \ldots t_n r} \varepsilon_{\mu} \theta_{s} K_{[t_1 \ldots t_n]}^{\mu \rho} \tilde{X}_{r}^{\rho} F(t_1, s) \right) \]

\[
\times S^{\beta-1}(t_1, \ldots, t_{\beta}) F(t_\beta, t_{\beta+1}) G^{n-\beta}(t_{\beta+1}, \ldots, t_n, r) \]
Each of these contributions has the connectivity of a closed loop with $\ell + 1$ points, attached at the point $r$ to a linear chain. We shall label the points on the closed loop by $r, u_1, \ldots, u_\ell$ and the points on the linear chain by $s, t_1, \ldots, t_m, r$.

- The fermion loop contributions may be expressed as follows\(^{11}\)

\[
D^{(m+1)(\ell+1)}_{(2b\bar{\psi})} = + \sum_{[s,t,r,u]} ds \left( Z_0^{st,ru} \varepsilon_s^\mu \theta_s K_{[t_1 \ldots t_m]}^{\mu \rho} i k^\rho_r K_{[u_1 \ldots u_\ell]}^{* \sigma} P(s,t_1) \right. \\
	imes G^m(t_1, \ldots, t_m, r) S^{\ell+1}(r, u_1, \ldots, u_\ell, r) \\
- \sum_{[s,t,r,u]} ds \left( Z_0^{st,ru} \varepsilon_s^\mu \theta_s K_{[t_1 \ldots t_m]}^{\mu \rho} i k^\rho_r K_{[u_1 \ldots u_\ell]}^{* \sigma} \\
\sum_{\beta=1}^m F(t_1, s) S^{\beta-1}(t_1, \ldots, t_\beta) F(t_\beta, t_{\beta+1}) \\
\times G^{m-\beta}(t_{\beta+1}, \ldots, t_m, r) S^{\ell}(r, u_1, \ldots, u_\ell, r) \right)
\]

Here and below, we use the notation $t_s$ as an abbreviation for $t_1 \cdots t_m$, and $u_s$ as an abbreviation for $u_1 \cdots u_\ell$.

- The bosonic contributions $D^{(m+1)(\ell+1)}_{(2b)}$ may be expressed as follows,

\[
D^{(m+1)(\ell+1)}_{(2b)} = - \sum_{[s,t,r,u]} ds \left( Z_0^{st,ru} \varepsilon_s^\mu \theta_s K_{[t_1 \ldots t_m u_1 \ldots u_\ell]}^{\mu \rho} \right. \\
	imes G^{m+\ell+1}(t_1, \ldots, t_m, r, u_1, \ldots, u_\ell, r) \left. \right)
\]

The bosonic contributions $D^{(m+1)(\ell+1)}_{(4b)}$ are given by

\[
D^{(m+1)(\ell+1)}_{(4b)} = \sum_{[s,t,r,u]} ds \left( Z_0^{st,ru} \varepsilon_s^\mu \theta_s K_{[t_1 \ldots t_m u_1 \ldots u_\ell]}^{\mu \rho} \right. \\
\times F(t_{\beta}, t_{\beta+1}) G^{m-\beta+1}(t_{\beta+1}, \ldots, t_{n+1}, r) \left. \right) \delta_{r,t_{m+1}}
\]

In the formula above, we have used the notations,

\[
\begin{align*}
n &= m + \ell \\
r &= t_{m+1} \\
u_i &= t_{m+i+1} & i &= 1, \ldots, \ell
\end{align*}
\]

\(^{11}\)Note that an extra $-$ sign has been included for the closed fermion loop. In our formalism, this comes about because $K_\mu^r S(r, u_1) K_{\mu_1}^u \cdots K_{\mu_\ell}^u S(u_\ell, r) K_\nu^r = -K_\mu_{u_1} K_{\mu_1}^u \cdots K_{\mu_\ell}^u K_\nu^r S(r, u_1) \cdots S(u_\ell, r)$. 

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and we continue to use this notation below. The contributions \( D_{4f}^{(m+1|\ell+1)} \) are given by

\[
D_{4f}^{(m+1|\ell+1)} = \sum_{[st_ru_s]} d_s \left( Z_{0}^{st_ru_s} \varepsilon^\mu \theta_s K_{[t_1\cdots t_m ru_1 \cdots u_4]}^{\mu \rho} i^{k_{\rho}} \right.
\times F(t_1, s) S^n(t_1, \cdots, t_{n+1}) F(t_{n+1}, r) \delta_{r, t_{m+1}}
\]

Note that \( D_{4f}^{(m+1|\ell+1)} \) coincides with the \( \beta = n + 1 \) contribution of \( D_{4b_{x}}^{(m+1|\ell+1)} \), so that, more succinctly,

\[
D_{4f}^{(m+1|\ell+1)} + D_{4b_{x}}^{(m+1|\ell+1)}
= \sum_{[st_ru_s]} d_s \left( Z_{0}^{st_ru_s} \varepsilon^\mu \theta_s K_{[t_1\cdots t_m ru_1 \cdots u_4]}^{\mu \rho} i^{k_{\rho}} \sum_{\beta=1}^{n+1} F(t_1, s) S^\beta(t_1, \cdots, t_{\beta})
\times F(t_\beta, t_{\beta+1}) G_{m-\beta+1}(t_{\beta+1}, \cdots, t_{n+1}, r) \delta_{r, t_{m+1}}. \right)
\]

We are now ready to isolate the contributions of the blocks \( \Pi_{+}^{(m+1|\ell+1)} \) that arise from these terms. To do so, we separate the + and − cases, according to the structure of the kinematic factors.

### 10.6 Contributions with kinematic factor \( K_{[t_1\cdots t_m ru_1 \cdots u_\ell]}^{\mu \rho} \)

The blocks \( D_{2b_{0}}^{(m+1|\ell+1)} \) and \( D_{4b_{\bar{w}}}^{(m+1|\ell+1)} \) do not contribute to this kinematic factor. The contribution to this kinematic factor from the remaining blocks will be denoted with the subscript +, and is given by

\[
\sum_s d_s S_{+s}^{(m+1|\ell+1)} = D_{2b_{c}\ell+}^{(m+1|\ell+1)} + D_{4b_{x}\ell+}^{(m+1|\ell+1)} + D_{4f+}^{(m+1|\ell+1)}
\]

(10.27)

\[
S_{+s}^{(m+1|\ell+1)} = - \sum_{s \not\in \{t, ru_s\}} Z_0^{st_ru_s} \varepsilon^\mu \theta_s K_{[t_1\cdots t_m ru_1 \cdots u_\ell]}^{\mu \rho} i^{k_{\rho}} \Pi_{+}^{(m+1|\ell+1)}(s; t_s, r, u_s)
\]

The contributions to \( \Pi_{+}^{(m+1|\ell+1)} \) arising directly from \( D_{2b_{c}\ell+}^{(m+1|\ell+1)} \), \( D_{4b_{x}\ell+}^{(m+1|\ell+1)} \), and \( D_{4f+}^{(m+1|\ell+1)} \) are given as follows by \( B_{+}^{(m+1|\ell+1)}(s; t_s, r, u_s) \),

\[
B_{+}^{(m+1|\ell+1)} = - \frac{1}{2} P(s, t_1) G_{m+1}(t_1, \cdots, t_m, r, u_1, \cdots, u_\ell, r)
+ \frac{1}{2} \sum_{\beta=1}^{n+1} F(t_1, s) S^\beta(t_1, \cdots, t_{\beta}) F(t_\beta, t_{\beta+1}) G_{m-\beta+1}(t_{\beta+1}, \cdots, t_{n+1}, r) \delta_{r, t_{m+1}}
+ (-)^\ell \left( [u_1 \cdots u_\ell] \rightarrow [u_\ell \cdots u_1] \right)
\]

(10.28)
with the identification
\[
\begin{align*}
n &= m + \ell \\
r &= t_{m+1} \\
u_i &= t_{m+i+1} & i = 1, \ldots, \ell
\end{align*}
\] (10.29)

It is straightforward to see that, up to terms which are holomorphic in \(s\), \(B_{\pm}^{(m+1|\ell+1)}\) coincides with \(-\Pi_{\pm}^{(m+1|\ell+1)}\). More precisely, we have

**Lemma 3** The blocks \(B_{\pm}^{(m+1|\ell+1)}\) and \(-\Pi_{\pm}^{(m+1|\ell+1)}\) differ only by terms which are holomorphic in \(s\),

\[
\partial_s \left( B_{\pm}^{(m+1|\ell+1)}(s; t_s, r, u_s) + \Pi_{\pm}^{(m+1|\ell+1)}(s; t_s, r, u_s) \right) = 0
\] (10.30)

As usual, we omit \(\delta\)-function contributions at coincident vertex insertion points for such considerations. We have continued to use the notations \(t_s\) and \(u_s\) as an abbreviation for \(t_1 \cdots t_m\), and \(u_1 \cdots u_\ell\) respectively.

### 10.7 Contributions with kinematic factor \(K_{[t_1 \cdots t_m]}^{\mu \rho} K_{[r u_1 \cdots u_\ell]}^{\sigma \sigma}\)

This part occurs in all blocks. This contribution will be denoted with the subscript \(-\), and is given by

\[
\begin{align*}
\sum_s d_s \mathcal{S}_s^{(m+1|\ell+1)} &= D_{2b}\mathcal{S}_s^{(m+1|\ell+1)} + D_{4b}\mathcal{S}_s^{(m+1|\ell+1)} + D_{4f}\mathcal{S}_s^{(m+1|\ell+1)} + D_{20}\mathcal{S}_s^{(m+1|\ell+1)} + D_{40}\mathcal{S}_s^{(m+1|\ell+1)} \\
\mathcal{S}_s^{(m+1|\ell+1)} &= - \sum_{s \in [t_s, r, u_s]} D_0^{t_s, r, u_s} \eta_s \mathcal{K}_{[t_1 \cdots t_m]}^{\mu \rho} k_{r}^\rho K_{[r u_1 \cdots u_\ell]}^{\sigma \sigma} \Pi_{\pm}^{(m+1|\ell+1)}(s; t_s, r, u_s)
\end{align*}
\] (10.31)

The contributions to \(\Pi_{\pm}^{(m+1|\ell+1)}\) arising directly from \(D_{2b}\mathcal{S}_s^{(m+1|\ell+1)}\), \(D_{4b}\mathcal{S}_s^{(m+1|\ell+1)}\), \(D_{4f}\mathcal{S}_s^{(m+1|\ell+1)}\), \(D_{20}\mathcal{S}_s^{(m+1|\ell+1)}\), and \(D_{40}\mathcal{S}_s^{(m+1|\ell+1)}\) are given as follows by \(B_{\pm}^{(m+1|\ell+1)}(s; t_s, r, u_s)\),

\[
\begin{align*}
B_{\pm}^{(m+1|\ell+1)} &= + \frac{1}{2} P(s, t_1) G^{m}(t_1, \cdots, t_m, r) S^{\ell+1}(r, u_1, \cdots, u_\ell, r) \\
&\quad - \frac{1}{2} P(s, t_1) G^{m+\ell+1}(t_1, \cdots, t_m, r, u_1, \cdots, u_\ell, r) \\
&\quad - \frac{1}{2} \sum_{\beta = 1}^m F(t_1, s) S^{\beta-1}(t_1, \cdots, t_\beta) F(t_\beta, t_{\beta+1}) \\
&\quad \times G^{m-\beta}(t_{\beta+1}, \cdots, t_m, r) S^{\ell+1}(r, u_1, \cdots, u_\ell, r) \\
&\quad + \frac{1}{2} \sum_{\beta = 1}^{n+1} F(t_1, s) S^{\beta-1}(t_1, \cdots, t_\beta) F(t_\beta, t_{\beta+1}) G^{m-\beta+1}(t_{\beta+1}, \cdots, t_{n+1}, r) \delta_{r, t_{m+1}} \\
&\quad - (-)^{\ell} \left( [u_1 \cdots u_\ell] \to [u_\ell \cdots u_1] \right)
\end{align*}
\] (10.32)
with the identification
\[
\begin{align*}
  n &= m + \ell \\
  r &= t_{m+1} \\
  u_i &= t_{m+i+1}, \quad i = 1, \ldots, \ell
\end{align*}
\] (10.33)

One may show by inspection that, up to terms which are holomorphic in \( s \), \( B_{(m+1)(\ell+1)} \) coincides with \( \Pi_{(m+1)(\ell+1)} \).

Some of the crucial steps in this identification are as follows. The first and third terms of \( B_{(m+1)(\ell+1)} \) in (10.32) sum to produce a term proportional to \( \Pi_{(m+2)}(s; t_1, \ldots, t_m; r) \) up to terms that are holomorphic in \( s \). The remaining sum may be split up into the parts \( 1 \leq \beta \leq m \) and the part \( m+1 \leq \beta \leq m+\ell+1 \). The first sum combines with the second term in \( B_{(m+1)(\ell+1)} \) to produce another term proportional to \( \Pi_{(m+2)}(s; t_1, \ldots; r) \) up to terms that are holomorphic in \( s \). The remaining sum may be recast in the following form, (changing notation for the summation variables, \( \beta = \alpha + m + 1 \)),

\[
F(t_1, s)S^m(t_1, \ldots, t_m, r) \sum_{\alpha=0}^{\ell} S^\alpha(r, u_1, \ldots, u_{\alpha-1}, t_\alpha) F(u_\alpha, u_{\alpha+1}) G^{\ell-\alpha}(u_{\alpha+1}, \ldots, u_\ell, r)
\]

The sum over \( \alpha \) may be recognized as \( \Pi_{(\ell+2)}(r; u_1, \ldots, u_\ell; r) \), while the prefactor equals the expression \((-)^m \Pi_{(m+2)}(r; t_m, \ldots, t_1; s) \), up to terms holomorphic in \( s \). Assembling all pieces, we have our final lemma,

**Lemma 4** The blocks \( B_{(m+1)(\ell+1)} \) and \( -\Pi_{(m+1)(\ell+1)} \) differ only by terms which are holomorphic in \( s \),

\[
\partial_s \left( B_{(m+1)(\ell+1)}(s; t_s, r, u_s) + \Pi_{(m+1)(\ell+1)}(s; t_s, r, u_s) \right) = 0 \quad (10.34)
\]

which completes the proof of (10.31). As usual, we omit \( \delta \)-function contributions at coincident vertex insertion points for such considerations. We have continued to use the notations \( t_s \) and \( u_s \) as an abbreviation for \( t_1 \cdots t_m \), and \( u_1 \cdots u_\ell \) respectively.
11 Summary and Additional Remarks

In this last section, we shall present a brief summary of the results obtained in this paper for the structure of the differential blocks \( D \). We shall also give a brief discussion of the fate of the various blocks, which are holomorphic \( \partial_s \) differentials purely of type \((1, 0)\) in all insertion points, and which will combine with similar terms in \( Z \), and which we have systematically omitted in the previous sections. This recombination in simple examples will set the stage for the next companion paper in which the complete structure of the holomorphic blocks \( Z \) will be derived and studied. We close with an analysis of the monodromy structure of the blocks \( D \) and \( Z \).

11.1 Summary of Differential blocks

The various results in this paper on the structure of the differential blocks may be summarized as follows,

\[
D[\delta] = \sum_{n=0}^{\infty} \sum_{[r_s]} d_r d_s S_r^{(n+2)} + \sum_{n=0}^{\infty} \sum_{r} d_s S_s^{(n+1)} + \sum_{m=-1}^{\infty} \sum_{s=1}^{\infty} d_s \left( S_s^{(m+1)(\ell+1)} + S_{-s}^{(m+1)(\ell+1)} \right)
\]

(11.1)

up to terms which are exact holomorphic \( \partial_s \) differentials, purely of type \((1, 0)\) in all insertion points. Here, the linear chain blocks \( S_r^{(n+2)} \) and \( S_s^{(n+1)} \) are given by

\[
S_r^{(n+2)} = - \sum_{r, s \notin \{t_1, \ldots, t_n\}} Z_{0}^{st_1 \cdots t_n} \frac{1}{2} \varepsilon_s^{\mu} \theta_{r \times s} K_{[t_1 \cdots t_n]}^{\mu \nu} \Pi^{(n+2)}(r; t_1, \ldots, t_n; s)
\]

\[
S_s^{(n+1)} = - \sum_{s \notin \{t_1, \ldots, t_n\}} Z_{0}^{st_1 \cdots t_n} P_{r[a}^{\mu} K_{[t_1 \cdots t_n]}^{\mu \nu} \varepsilon_s^{\nu} \theta_s^{[a} \Pi_{r]}^{(n+1)}(t_1; t_2, \ldots, t_n; s)
\]

(11.2)

while the singly linked blocks \( S_{\pm s}^{(m+1)(\ell+1)} \) are given by

\[
S_{\pm s}^{(m+1)(\ell+1)} = - \sum_{s \notin \{t, u\}} Z_{0}^{st_1 \cdots t_m \cdot u} \varepsilon_s^{\mu} \theta_{s}^{[a} K_{[t_1 \cdots t_m] u_{[a}}^{\mu \nu} i k_{r}^{\nu} \Pi_{\pm}^{(m+1)(\ell+1)}(s; t, r, u) \]

(11.3)

We use the notations \( t_s \) and \( u_s \) as abbreviations for \( t_1 \cdots t_m \) and \( u_1 \cdots u_\ell \) respectively.

We note that the contributions from \( m = -1 \) correspond to the contributions from the blocks \( \Pi_{\pm}^{(0)(\ell+1)}(s; t, r, u) = \Pi_{\pm}^{(0)(\ell+1)}(s; t_1, \ldots, u_\ell) \), in which no variables \( t_i \) occur, and in which we set \( r = s \).
### 11.2 Monodromy of $\mathcal{D}[\delta]$

All blocks have trivial monodromy under $A$-cycles. Using the monodromy operator around $B$-cycles, defined earlier by $\Delta^{(s)}_K p^\mu_t = -ik^\mu_t \delta_{IK}$, and the monodromy properties of the elementary functions $\lambda_I$, $\hat{\omega}^I_0$, $F$, $P$, and $Q_B$, we readily compute the variations under $B_K$ monodromy of the differential blocks,

$$\Delta^{(s)}_K \mathcal{D}[\delta] = \varepsilon^\mu_s \theta_s \partial_s \mathcal{D}^{(s)}_K [\delta]$$  \hspace{1cm} (11.4)

where the variation $\mathcal{D}^{(s)}_K [\delta]$ is given by,

$$\mathcal{D}^{(s)}_K [\delta] = - \sum_{\tau \neq s} Z^r_{0s} r^s \left( X^\mu_r \lambda_K (r) - p^\mu_r K^\nu_r \hat{\omega}^I_0 (r) \hat{\omega}^0_K (r) \right)$$

$$+ \sum_{s \neq [rt]} Z^{rst}_{0r} K^\mu_r K^\nu_t \left( p^\mu_t \hat{\omega}^I_0 (t) \hat{\omega}^0_K (t) + \hat{\omega}^I_0 (r) ik^\mu_t F (t, r) \right)$$

$$- \sum_{s \neq [rt]} Z^{rst}_{0r} K^\mu_r K^\nu_t X^\mu_t \hat{\omega}^0_K (r) F (r, t)$$

$$- \sum_{s \neq [rtu]} Z^{rstu}_{0r} K^\mu_r K^\nu_t \hat{\omega}^0_K (t) F (r, u)$$

$$+ \pi \sum_{s \neq [rt]} Z^{rstu}_{0r} K^\mu_r K^\nu_t k^\mu_s \hat{\omega}^0_K (r) \hat{\omega}^0_K (t)$$ \hspace{1cm} (11.5)

The key result is that $\Delta^{(s)}_K \mathcal{D}[\delta]$ turns out to be a holomorphic exact differential block. This result, together with the associated monodromy transformation laws of the $Z$ blocks will be studied in detail in the next paper.

### 11.3 Non-triviality of the differential blocks

Under changes of gauge slice, the blocks $\mathcal{F}[\delta]$ change by the addition of an exact differential, as was already shown in [16], Thus, the question naturally arises as to whether it is possible to choose a gauge in which the sum of all the differential blocks $\mathcal{D}[\delta]$ vanishes. We shall argue here that there exists no such gauge choice.

A natural candidate for such a choice might be the split gauge, introduced in [14] as a calculational tool to evaluate the superstring chiral measure in terms of $\vartheta$-functions. Split gauge is defined by taking the worldsheet gravitino $\chi$ to be supported at two points $q_1$ and $q_2$, such that $\chi$ is parametrized by the two odd moduli $\zeta^1, \zeta^2$ in the following manner,

$$\chi (z) = \zeta^1 \delta (z, q_1) + \zeta^2 \delta (z, q_2)$$ \hspace{1cm} (11.7)

and the points $q_1$ and $q_2$ are related to one another by a spin structure $\delta$-dependent equation,

$$S_\delta (q_1, q_2) = 0$$ \hspace{1cm} (11.8)
In split gauge, the super period matrix coincides with the bosonic period matrix \( \hat{\Omega}_{IJ} = \Omega_{IJ} \), and we may choose \( \hat{\mu} = 0 \). In split gauge, one also has

\[
\lambda_I(s) = P(r, s) = 0
\]  

(11.9)

As a result, the blocks \( \Pi^{(1)}_I(s) = \lambda_I(s) \) and \( \Pi^{(2)}(r, s) = P(r, s) \) vanish identically, so that \( S^{(1)}_s = S^{(2)}_{rs} = 0 \). This cancellation is only a property of the low order at which we consider the question, and higher blocks will not vanish. For example, in split gauge, we have the following simplified expression for the block \( \Pi^{(3)}(r; s, t) = -F(t, r)F(t, s) \), but this quantity is non-vanishing.

More generally, the blocks \( \Pi^{(n+2)} \) can never vanish identically as soon as \( n > 0 \). This may be seen directly by considering its double \((0, 1)\) derivatives at both end points, already mentioned in the introduction,

\[
\partial_r \partial_s \Pi^{(n+2)}(r; t_1, \cdots, t_n; s) = \frac{1}{4} \chi(r) S^{n+1}(r, t_1, \cdots, t_n, s) \chi(s) \]  

(11.10)

From inspection of the expression on the right hand side, it is clear that for \( n > 0 \), it cannot vanish for all \( t_1, \cdots, t_n \), and thus the block \( \Pi^{(n+2)} \) itself can also not vanish.

### 11.4 Recombination of holomorphic \( \partial_s \) differentials and \( \mathcal{Z} \)

We illustrate here, for the simplest blocks treated in section 9.2, how the exact holomorphic \( \partial_s \) differentials that arise at intermediate stages from the bosonic Wick contractions by Lemma 1, recombine with similar terms in \( \mathcal{Z} \). We follow closely the notations of section 9.2, and refer to the various cases with the numerals used there. In (1) and (2), no exact holomorphic \( \partial_s \) differentials arise for these blocks. In (3), combining \( \mathcal{D}_{2b} \) with \( \mathcal{D}_{2c} \) in (9.13), produces the exact holomorphic differential,

\[
\sum_{[rs]} \partial_s \left( Z_0^{rs} p_I^\epsilon_s \theta_s K_r^\nu K_r^\mu \lambda_I(r) G(r, s) \right) \]  

(11.11)

This contribution compensates an identical and opposite term arising from the contractions of certain \( \mathcal{Z} \)-blocks. To see how this works, we use Lemma 1 to carry out partial bosonic Wick contractions on \( \mathcal{Z}_{1a} \), as follows,

\[
\mathcal{Z}_{1a} = \sum_r Z_0^r p_I^\mu \left( i k_r^\mu + K_r^\mu K_r^\nu p_J J(r) \right) \lambda_I(r) + \sum_{[rs]} Z_0^{rs} p_I^\mu K_r^\nu X_s^\nu G(r, s) \lambda_I(r) + \sum_{[rs]} \partial_s \left( Z_0^{rs} p_I^\epsilon_s \theta_s K_r^\nu K_r^\mu G(r, s) \lambda_I(r) \right) \]  

(11.12)

The total \( \partial_s \) derivative terms in (9.13) and (11.12) precisely cancel one another in view of the anti-symmetry of \( K_r^\nu K_r^\mu \) under interchange in \( \mu \) and \( \nu \). Using also the definition of the
block $\Pi_{IJ}^{(1)}(r)$, we may assemble these simplest blocks as follows,

\[ \mathcal{Z}_{1a} + \mathcal{Z}_{1b} + \mathcal{D}_{2b} + \mathcal{D}_{2c} = \mathcal{H}_{pp}^{(1)} + d_r S_r^{(2)} + \sum_r Z_0^r p^r \imath k^r \lambda_I(r) + \sum_r Z_0^r p^r \imath k^r \lambda_I(r) + \mathcal{Z}_{1a} \]

\[ \mathcal{Z}_{1a} = \sum_{[rs]} Z_0^{rs} p_1^r k_1^r k_1^s X_s^r G(r, s) \lambda_I(r) \]

\[ \mathcal{H}_{pp}^{(1)} \equiv - \sum_r Z_0^r \frac{1}{2} p_1^r k_1^r k_1^s p^s \Pi_{IJ}^{(1)}(r) \]  

(11.13)

The $p$-parts in $X_s^r$ of $\mathcal{Z}_{1a}^{(2)}$ and $\mathcal{Z}_{2d}$ combine with the the 2-point fermionic contraction of $\mathcal{Z}_{2a}$ and the double $p$-part of $\mathcal{Z}_{2b}$ to produce the holomorphic block

\[ \mathcal{H}_{pp}^{(2)} \equiv - \sum_{[rs]} Z_0^{rs} \frac{1}{2} p_1^r k_1^r k_1^s \Pi_{IJ}^{(2)}(r, s) \]  

(11.14)

From the above pattern, we may deduce all the linear chain blocks which are bilinear in the internal loop momenta. We shall prove these formulas systematically in the next paper.

\[ \mathcal{H}_{pp}^{(n)} \equiv - \sum_{[t_1\cdots t_n]} Z_0^{t_1\cdots t_n} \frac{1}{2} p_1^\mu K_1^{\mu\nu} K_1^{\nu} \cdots K_{t_n-1}^{\mu\nu} K_{t_n-1}^{\nu} p_j^{(n)}(t_1, t_2, \cdots, t_{n-1}, t_n) \]  

(11.15)

The appearance of the blocks $\Pi_{IJ}^{(n+2)}$ in the $\mathcal{Z}$ blocks confirms the general structure announced in the Introduction, and will be studied in detail in the next paper.

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