Greedy bi-criteria approximations for $k$-medians and $k$-means

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Abstract

This paper investigates the following natural greedy procedure for clustering in the bi-criterion setting: iteratively grow a set of centers, in each round adding the center from a candidate set that maximally decreases clustering cost. In the case of $k$-medians and $k$-means, the key results are as follows.

- When the method considers all data points as candidate centers, then selecting $O(k \log(1/\varepsilon))$ centers achieves cost at most $2 + \varepsilon$ times the optimal cost with $k$ centers.
- Alternatively, the same guarantees hold if each round samples $O(k/\varepsilon^5)$ candidate centers proportionally to their cluster cost (as with $k\text{means}++$, but holding centers fixed).
- In the case of $k$-means, considering an augmented set of $n^{1/\varepsilon^2}$ candidate centers gives $1 + \varepsilon$ approximation with $O(k \log(1/\varepsilon))$ centers, the entire algorithm taking $O(dk \log(1/\varepsilon)n^{1 + 1/\varepsilon^2})$ time, where $n$ is the number of data points in $\mathbb{R}^d$.
- In the case of Euclidean $k$-medians, generating a candidate set via $n^{O(1/\varepsilon^2)}$ executions of stochastic gradient descent with adaptively determined constraint sets will once again give approximation $1 + \varepsilon$ with $O(k \log(1/\varepsilon))$ centers in $dk \log(1/\varepsilon)n^{O(1/\varepsilon^2)}$ time.

Ancillary results include: guarantees for cluster costs based on powers of metrics; a brief, favorable empirical evaluation against $k\text{means}++$; data-dependent bounds allowing $1 + \varepsilon$ in the first two bullets above, for example with $k$-medians over finite metric spaces.
1 Introduction

Consider the task of covering or clustering a set of $n$ points $X$ using centers from a set $Y$. A solution $C \subseteq Y$ must balance two competing criteria: its size $|C|$ should be small, as should its cost

$$\phi_X(C) := \sum_{x \in X} \min_{y \in C} \Delta(x, y),$$

where $\Delta : X \times Y \to \mathbb{R}_+$ is a non-negative function defined on $Y$ and a superset $X \supseteq Y$.

Amongst many conventions for balancing these two criteria, perhaps the most prevalent is to fix a reference solution $A$ with $k := |A|$ centers, and to seek a solution $C$ which minimizes approximation ratio $\phi_X(C) / \phi_X(A)$ while constraining $|C| = k$. Problems of this type are generally NP-hard: for example, the $k$-means problem, where $\Delta(x, y) := ||x - y||^2$ in Euclidean space, and the metric $k$-medians problem, where $\Delta(x, y) := D(x, y)$ for some metric $D$ over finite $X = Y$, are each NP-hard to approximate within some constant factor larger than one (Jain et al. 2002; Awasthi et al. 2015).

On the other hand, if $|C|$ is allowed to slightly exceed $|A|$, the task of approximation becomes much easier. Returning to the example of $k$-means, for any $\varepsilon > 0$, increasing the center budget to $|C| \leq k \ln(1/\varepsilon)$ grants the existence of algorithms with approximation factor $1 + \varepsilon$ while taking time polynomial in the size of the input $X$ (Makarychev et al. 2016).

The classical problem of set cover is similar: it is NP-hard, but its natural greedy algorithm finds a cover of size $|C| = \lceil k \ln(n) \rceil$ whenever one of size $k$ exists (Johnson 1974). The analogous greedy iterative procedure for $\phi_X$ — which incrementally adds elements from $Y$ to maximally decrease $\phi_X$ — is the basis of this paper and all its algorithms, but with one twist: the set of centers in each round, $Y_i$, is adaptively chosen by a routine called select. Instantiating select in various ways yields the following results.

Results for $k$-means. This problem takes $X = Y = \mathbb{R}^d$ and $\Delta(x, y) = ||x - y||^2$.

- When select returns all of $X$, then $O(k \log(1/\varepsilon))$ centers suffice to achieve approximation factor $(1 + \varepsilon)(1 + \kappa_1)$, where $\kappa_1 \in [0, 1]$ is a problem-dependent constant (cf. Theorem 3.1). By contrast, the main competing method kmeans++ currently achieves approximation factor $2 + \varepsilon$ with $O(k/\varepsilon^2)$ centers (Aggarwal et al. 2009; Wgj 2016). (A lower bound on the number of centers in this regime is not known; when exactly $k$ centers are used, the approximation factor is below $\ln(k)$ only with exponentially small probability (Brunscheid and Rögl 2013).)

- When select returns $O(k/\varepsilon^3)$ points from $X$ subsampled similarly to kmeans++ (Arthur and Vassilvitskii 2007), once again $O(k \log(1/\varepsilon))$ centers suffice but with a slightly worse approximation factor $(1 + \varepsilon)(1 + \kappa_2)$, where $\kappa_2 \in [\kappa_1, 1 + \varepsilon]$ is another problem-dependent constant (cf. Theorem 3.2).

- When select returns the means of all subsets of $X$ of size $\lceil 1/\varepsilon \rceil$, then $O(k \log(1/\varepsilon))$ centers suffice for approximation factor $(1 + \varepsilon)^2$ (cf. Theorem 3.3). Thus the method requires time $O(k d \log(1/\varepsilon) n^{1+1/\varepsilon})$, improving upon a running time $O(\log(1/\varepsilon)/\varepsilon^2)$ with $O(k \log(1/\varepsilon))$ centers, due to Makarychev et al. (2016), whose algorithm randomly projects to $O(\log(n)/\varepsilon^2)$ dimensions, then constructs extra candidate centers via a gridding argument (Matousek 2000), then rounds an LP solution, and finally lifts the resulting partition back to $\mathbb{R}^d$. The local search method analyzed by Bandyapadhyay and Varadarajan (2016) returns a solution with approximation factor $1 + \varepsilon$ using $(1 + \varepsilon)k$ centers, but its running time is exponential in $(1/\varepsilon)^d$. Lastly, assuming the instance satisfies a certain separation condition with parameter $\kappa > 0$, a running time of $O(n^3)(k \log(n))^{\text{poly}(1/\varepsilon, 1/\kappa)}$ is also possible (Awasthi et al. 2010).
Results for generalized \( k \)-medians. Variants of the preceding results hold in the following generalized setting, versions of which appear elsewhere in the literature (e.g., [Arthur and Vassilvitskii, 2007]): \( \Delta(x, y) = D(x, y)^p \) for metric \( D \) on space \( \mathcal{X} = \mathcal{Y} \) (not necessarily infinite), and \( p \geq 1 \).

- The earlier Theorems 3.1 and 3.2 go through in this setting still with \( O(k \log(1/\varepsilon)) \) centers, but respectively granting approximation ratios \((1 + \kappa_1)^p\) and \((1 + \kappa_2)^p\) (cf. Theorems 3.1 and 3.2). A notable improvement in this regime is the case of \( p = 1 \) with finite metrics, where \( \kappa_1 = 0 \). An approximation factor of \( 1 + \varepsilon \) was obtained in prior work for finite metrics with exactly \( k \) centers, however requiring separation with a parameter \( \kappa > 0 \), and with an algorithm whose running time is \((nk)^{\text{poly}(1/\varepsilon, 1/\kappa)}\) (Awasthi et al., 2010).

- Achieving approximation ratio \((1 + \varepsilon)\) with \( O(k \log(1/\varepsilon)) \) centers is again possible when \( D \) is induced by a norm in \( \mathcal{X} = \mathcal{Y} = \mathbb{R}^d \). If the norm is Euclidean and \( p = 1 \), it suffices to generate candidate centers with \( n^3[1/\varepsilon^2] \) executions of projected stochastic gradient descent with adaptively determined constraint sets (cf. Theorem 3.4); for other norms or exponents \( p \), \( O(n^3\varepsilon^{-d}) \) candidate centers need to be sampled (cf. Theorem 3.5). Existing approximation schemes that only use \( k \) centers (for either Euclidean \( k \)-medians and \( k \)-means) have complexity that is either exponential in \( k \) ([Kumar et al., 2004, 2005; Feldman et al., 2007]) or more than exponential in \( d \) ([Kolliopoulos and Rao, 1999; Cohen-Addad et al., 2016; Friggstad et al., 2016]).

Related works. Analysis of greedy methods is prominently studied in the context of maximizing submodular functions ([Nemhauser et al., 1978]), and the recent literature offers many techniques for efficient implementation (e.g., [Badanidiyuru and Vondrak, 2014; Buchbinder et al., 2015]). It is most natural to view the objective function to be minimized in the present work as a supermodular function, as opposed to viewing its negation as a submodular function. These different viewpoints lead to different approximation results, even for the same greedy scheme. Moreover, the specific objective function considered in this work has additional structure that permits computational speedups (cf. Section 3.1) not generally available for other supermodular objectives. A more detailed discussion is presented in Appendix B.

In the context of Euclidean \( k \)-medians and \( k \)-means problems (where \( \mathcal{X} = \mathcal{Y} = \mathbb{R}^d \) and \( \Delta(x, y) = \|x - y\|^p_2 \) for \( p \in \{1, 2\} \)), the present work is related to the study of bi-criteria approximation algorithms, which find a solution using \( \beta k \) centers whose cost is at most \( \alpha \) times the cost of the best solution using \( k \) centers. For any \( \varepsilon \in (0, 1) \), the factors \( \alpha = 2 + \varepsilon \) and \( \beta = \text{poly}(1/\varepsilon) \) are achievable for both the \( k \)-medians problem ([Lin and Vitter, 1992]) and the \( k \)-means problems ([Aggarwal et al., 2009; Wei, 2016]) by \( \text{poly}(n, d, k, 1/\varepsilon) \)-time algorithms that only select centers from among the data points \( \mathcal{X} \). The bi-criteria approximation methods of [Makarychev et al., 2016] and [Bandyapadhyay and Varadarajan, 2016] for \( k \)-means are already discussed above, as are proper approximation schemes for \( k \)-medians and \( k \)-means ([Kumar et al., 2004, 2005; Feldman et al., 2007; Kolliopoulos and Rao, 1999; Cohen-Addad et al., 2016; Friggstad et al., 2016]).

Organization. The generic greedy scheme is presented in Section 2. Generalized \( k \)-medians problems are discussed in Section 3. Lastly, experiments with \( k \)-means constitute Section 4.

Notation. The set of positive integers \( \{1, 2, \ldots, N\} \) is denoted by \([N]\), and \( [x]_+ := \max\{0, x\} \) for \( x \in \mathbb{R} \). The reference solution \( \mathcal{A} := \{a_1, a_2, \ldots, a_k\} \) of cardinality \( k \) is treated as fixed in each discussion, however only in some circumstances it is optimal. This solution \( \mathcal{A} \) partitions \( \mathcal{X} \) into \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \), where \( \mathcal{A}_j := \{x \in \mathcal{X} : \text{arg min}_{j \in [k]} \Delta(x, a_j) = j\} \), breaking arg min ties using any fixed, deterministic rule. Observe that \( \phi_\mathcal{X}(C) = \sum_{j=1}^k \phi_{\mathcal{A}_j}(C) \) for any \( C \subseteq \mathcal{Y} \), and \( \phi_\mathcal{X}(\mathcal{A}) = \sum_{j=1}^k \sum_{x \in \mathcal{A}_j} \Delta(x, a_j) \). The mean of a finite subset \( \mathcal{A} \subseteq \mathbb{R}^d \) is denoted by \( \mu(\mathcal{A}) := \sum_{x \in \mathcal{A}} x/|\mathcal{A}| \).
Algorithm greedy

**Input:** input points $X \subseteq \mathcal{X}$, initial centers $C_0 \subseteq \mathcal{Y}$, number of iterations $t$, candidate selection procedure select, tolerance $\tau \geq 0$.

For $i = 1, 2, \ldots, t$:
- Choose candidate centers $Y_i := \text{select}(X, C_{i-1})$.
- Set $C_i := C_{i-1} \cup \{c_i\}$ for any $c_i \in Y_i$ that satisfies
  $$\phi_X(C_i) \leq (1 + \tau) \cdot \min_{c \in Y_i} \phi_X(C_{i-1} \cup \{c\})$$.

**Output:** $C_t$.

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Figure 1: Greedy algorithm for general $k$-medians problems.

## 2 Greedy method

The greedy scheme is presented in Figure 1. As discussed before, it greedily adds a new center in each round so as to maximally decrease cost. A routine select provides the candidate centers in each round, and the minimization over these candidates need only be solved to accuracy $1 + \tau$.

The bounds on greedy will depend on one of two conditions being satisfied on $(C_{i-1}, Y_i)$ in each round, at least with some probability. These conditions are parameterized by an approximation factor $\gamma$. In the sequel (e.g., generalized $k$-medians problems), the proofs will proceed by establishing one of these conditions, and then directly invoke the guarantees on greedy.

**Condition 1.** For each $j \in [k]$, there exists $c \in Y_i$ such that $\phi_{A_j}(\{c\}) \leq \gamma \cdot \phi_{A_j}(\{a_j\})$.

**Condition 2.** There exists $c \in Y_i$ such that
$$\max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \min_{c \in Y_i} \phi_{A_j}(\{c\}) \right] \geq \max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \gamma \cdot \phi_{A_j}(\{a_j\}) \right].$$

Note that Condition 1 implies Condition 2.

**Theorem 2.1.** Let $\varepsilon > 0$ and $\alpha \geq \gamma \geq 1$ be given, along with initial clustering $C_0$ satisfying $\phi_X(C_0) \leq \alpha \cdot \phi_X(A)$, and lastly greedy chooses $c_i \in Y_i$ with $\tau = 0$. If either

1. Condition 1 or Condition 2 hold for $(C_{i-1}, Y_i, \gamma)$ in each round, and $t \geq k \ln((\alpha - \gamma)/(\gamma \varepsilon))$; or
2. Condition 1 or Condition 2 hold for $(C_{i-1}, Y_i, \gamma)$ conditionally independently with probability at least $\rho > 0$ in each round, and $t \geq \max\{k \ln((\alpha - \gamma)/(\gamma \varepsilon))/(2\rho), 2 \ln(1/\delta)/\rho^2\}$ for some $\delta > 0$;

then $\phi_X(C_t) \leq \gamma \cdot (1 + \varepsilon) \cdot \phi_X(A)$ holds unconditionally under the assumptions (1) above, and with probability at least $1 - \delta$ under assumptions (2).

The proof is an immediate consequence of the following more general lemma.
Lemma 2.1. If Condition 1 or Condition 2 are satisfied with some $\gamma \geq 1$ for $(C_{i-1}, Y_i)$ in each round and $\tau < 1/(k-1)$, then the set of representatives $C_t$ returned by greedy satisfies

$$\phi_X(C_t) \leq \left(1 - \frac{1}{k}\right)^s \cdot (1 + \tau)^s \cdot \phi_X(C_0) + \left(1 - \frac{1}{k}\right)^s \cdot (1 + \tau)^s \cdot \frac{1 + \tau}{1 - (k-1)\tau} \cdot \phi_X(A)$$

with $s = t$. If instead Condition 1 or Condition 2 holds with probability at least $\rho$ conditionally independently across rounds, then this bound on $\phi_X(C_t)$ holds with probability at least $1 - \delta$ with $s = \lceil t\rho - \sqrt{t \ln(1/\delta)}/2 \rceil$.

Proof. First consider any pair $(C_{i-1}, Y_i)$ satisfying Condition 1 or Condition 2 which simply means Condition 2 holds. Then

$$\phi_X(C_{i-1}) - \phi_X(C_t) \geq \phi_X(C_{i-1}) - \min_{c \in Y_i} \phi_X(C_{i-1} \cup \{c\}) \quad \text{(definition of $C_t$)}$$

$$\geq \max_{c \in Y_i} \max_{j \in [k]} \phi_{A_j}(C_{i-1}) - \phi_{A_j}(C_{i-1} \cup \{c\})$$

$$\geq \max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \min_{c \in Y_i} \phi_{A_j} \{c\} \right]_{+}$$

$$\geq \max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \gamma \cdot \phi_{A_j} \{a_j\} \right]_{+} \quad \text{(Condition 2)}$$

$$\geq \frac{1}{k} \sum_{j=1}^{k} \left[ \phi_{A_j}(C_{i-1}) - \gamma \cdot \phi_{A_j} \{a_j\} \right]_{+}$$

$$\geq \frac{1}{k} \left( \phi_X(C_{i-1}) - \gamma \cdot \phi_X(A) \right).$$

Rearranging the inequality gives the recurrence inequality

$$\phi_X(C_i) \leq \left(1 - \frac{1}{k}\right) \cdot (1 + \tau) \cdot \phi_X(C_{i-1}) + \frac{\gamma}{k} \cdot (1 + \tau) \cdot \phi_X(A). \quad (1)$$

Now let $(B_0, \ldots, B_{s-1})$ denote the subsequence of $(C_0, \ldots, C_{t-1})$ where the corresponding pairs $(C_{i-1}, Y_i)$ satisfy Condition 1 or Condition 2. Since $\phi_X$ can not increase on rounds where neither condition holds, it still follows that

$$\phi_X(B_i) \leq \left(1 - \frac{1}{k}\right) \cdot (1 + \tau) \cdot \phi_X(B_{i-1}) + \frac{\gamma}{k} \cdot (1 + \tau) \cdot \phi_X(A),$$

and therefore

$$\phi_X(B_t) \leq \left(1 - \frac{1}{k}\right)^s \cdot (1 + \tau)^s \cdot \phi_X(B_0) + \sum_{i=0}^{s-1} \left(1 - \frac{1}{k}\right)^i \cdot (1 + \tau)^i \cdot \frac{\gamma}{k} \cdot \phi_X(A).$$

If the conditions hold for every round, then $s = t$ and the proof is done since $\tau < 1/(k-1)$. Otherwise, it remains to bound $s$; but since the conditions hold on a given round with probability at least $\rho$ conditionally independently of previous rounds, it follows by Azuma’s inequality that $\Pr \{ s \leq t\rho - \sqrt{t \ln(1/\delta)}/2 \} \leq \exp \left(-2t(\ln(1/\delta)/(2t)) \right) \leq \delta$ as desired. \qed

4
Connection to supermodular and submodular optimization. Theorem 2.1 recovers the
analysis of the standard greedy method for set cover (Johnson [1974]): if \( X \) is a set of points and \( \mathcal{Y} \) is a family of subsets of \( X \), then the choices \( \Delta(x, S) := 1 + 1[x \not\in S] \) and \( Y_i = \mathcal{Y} = \text{select()} \) satisfy Condition 1 with \( \gamma = 1 \). A valid cover has cost \( n \); \text{greedy} (with \( C_0 = \emptyset, \varepsilon = 1/n \)) finds a valid cover with cardinality \( \leq k \ln(n) \) when one of cardinality \( k \) exists.

More generally, the behavior of \text{greedy} on the objective \( \phi_X \) — when \( Y_i = \mathcal{Y} \) for each \( i \) — is well understood, because \( \phi_X \) is a monotone (non-increasing) supermodular function. Indeed, the results of Nemhauser et al. [1978] show that monotonicity and supermodularity of \( \phi_X \), together, imply the key recurrence eq. (1) in the proof of Lemma 2.1. Typically, this \text{greedy} algorithm is analyzed for the clustering problem by regarding the function \( f(S) := \phi_X(\{c_0\}) - \phi_X(S \cup \{c_0\}) \) as a submodular objective to be maximized (e.g., Mirzasoleiman et al. [2013]); here \( c_0 \in \mathcal{Y} \) is some distinguished center fixed \text{a priori}. The results from this form of analysis are generally incomparable to those obtained in the present work. More details are given in Appendix B.

3 Generalized \( k \)-medians problems

The results of this section will specialize \( \mathcal{X}, \mathcal{Y}, \mathcal{A}, \) and \( \Delta \) in the following two ways.

The first setting, \textit{generalized} \( k \)-\textit{medians}, is as follows. There is a single ambient space \( \mathcal{X} = \mathcal{Y} \) for data points and centers, and every data point is a possible center — when \( \mathcal{Y} = \{0, 1\} \). The first setting, \( \Delta(x, y) := 1 + 1[y \not\in \mathcal{Y}] \) and the triangle inequality \( D(x, z) \leq D(x, y) + D(y, z) \). Lastly define \( \Delta(x, y) := D(x, y)^p \) for some real number exponent \( p \geq 1 \).

Secondly, the distinguished sub-cases of \textit{Euclidean} \( k \)-\textit{medians} and \( k \)-\textit{means} are as follows. The ambient space \( \mathcal{X} = \mathcal{Y} = \mathbb{R}^d \) is \( d \)-dimensional Euclidean space. For Euclidean \( k \)-\textit{medians}, \( \Delta(x, y) = \|x - y\|_2^p \); for \( k \)-\textit{means}, \( \Delta(x, y) = \|x - y\|_2 \). Moreover, for \( k \)-\textit{means}, it is assumed (without loss of generality) that the reference solution \( \mathcal{A} \) satisfies \( a_j = \mu(A_j) \) for each \( j \in [k] \).

Associated with each generalized \( k \)-\textit{medians} instance is a real number \( q \geq 1 \); for \( k \)-\textit{means} instances, \( q = 1 \), but otherwise \( q = p \). Additionally, define the normalized cost

\[
\psi_A(C) := (\phi_A(C)/|A|)^{1/q}.
\]

This normalization is convenient in the proofs, but is not used in the main theorems. Lastly, all invocations of \text{greedy} in this section set the tolerance parameter as \( \tau = 0 \).

All results in this section will assume an \( \alpha \)-approximate initialization \( C_0 \). There exist easy methods attaining this approximation guarantee for \( \alpha = \mathcal{O}(1) \) with \( |C_0| = \mathcal{O}(k) \), for instance \text{kmeans++} (cf. Theorem 2.1).

The basic approximation guarantee will make use of the following data-dependent quantity:

\[
\kappa_1 := \max \left\{ \min_{x \in A_j} \frac{\psi_{x}^{-1}(\{a_j\})}{\psi_{A_j}^{-1}(\{a_j\})} : j \in [k], |A_j| > 0 \right\}.
\]

Note that \( \kappa_1 \leq 1 \) in general, but \( \kappa_1 \) can easily be smaller; for instance \( \kappa_1 = 0 \) with finite metrics.

**Theorem 3.1.** Consider an instance of the generalized \( k \)-\textit{medians} problem in \((\mathcal{X}, \mathcal{D})\) with exponent \( p \). Let \( \varepsilon > 0 \) be given, along with \( C_0 \) with \( \phi_X(C_0) \leq \alpha(1 + \kappa_1)^q \phi_X(\mathcal{A}) \) for some \( \alpha \geq 1 \). Suppose \text{greedy} is run for \( t \leq k \ln((\alpha - 1)/\varepsilon) \) rounds with \( X = Y_i = \text{select()} \) in each round. Then the resulting centers \( C_t \) satisfy

\[
\phi_X(C_t) \leq (1 + \kappa_1)^q (1 + \varepsilon) \phi_X(\mathcal{A})
\]

where \( \kappa_1 \in [0, 1] \).
The key to the proof is the following property of generalized $k$-medians problems, which implies Condition \(1 \) holds in every round of \textbf{greedy}. This inequality generalizes the usual bias-variance equality for $k$-means problems; a similar inequality appeared without $\kappa_1$ for a slightly less general setting in [Arthur and Vassilvitskii, 2007].

\textbf{Lemma 3.1} (See also [Arthur and Vassilvitskii, 2007 Lemmas 3.1 and 5.1]). Let a generalized $k$-medians problem be given. Then for any $j \in [k]$ and $y \in Y$,

\[ \psi_{A_j}(\{y\}) \leq \psi_{A_j}(\{a_j\}) + \psi_{\{y\}}(\{a_j\}), \]

and moreover

\[ \min_{y \in A_j} \psi_{A_j}(\{y\}) \leq (1 + \kappa_1)\psi_{A_j}(\{a_j\}) \leq 2\psi_{A_j}(\{a_j\}). \]

\textit{Proof.} The second bound is implied by the first, since the choice $z := \arg\min_{y \in A_j} \psi_{\{y\}}(a_j)$ satisfies $\psi_{\{z\}}(a_j) = \kappa_1 \psi_{A_j}(\{a_j\})$ where $\kappa_1 \leq 1$, and so

\[ \min_{y \in A_j} \psi_{A_j}(\{y\}) = \psi_{A_j}(\{z\}) \leq \psi_{A_j}(\{a_j\}) + \psi_{\{z\}}(\{a_j\}) \leq (1 + \kappa_1)\psi_{A_j}(\{a_j\}) \leq 2\psi_{A_j}(\{a_j\}). \]

For the first bound, the special case of $k$-means follows from the standard bias-variance equality (cf. Lemma \[A.1\]). Otherwise, $q = p \geq 1$, and the triangle inequality for $D$ together with Minkowski’s inequality (applied in $\mathbb{R}^{|A_j|}$ with counting measure) implies

\[ \phi_{A_j}(\{y\})^{1/p} \leq \left( \sum_{x \in A_j} (D(x, a_j) + D(a_j, y))^p \right)^{1/p} \leq \left( \sum_{x \in A_j} D(x, a_j)^p \right)^{1/p} + \left( \sum_{x \in A_j} D(a_j, y)^p \right)^{1/p}. \]

Dividing both sides by $|A_j|^{1/p}$ gives the bound. \hfill \Box

\textit{Proof of Theorem 3.1.} By Lemma 3.1, every round of \textbf{greedy} satisfies Condition \[1 \] with $\gamma \leq (1 + \kappa_1)^q$. The result now follows by Theorem 2.1. \hfill \Box

### 3.1 Reducing computational cost via random sampling

One drawback of setting $Y_i = X$ as in Theorem 3.1 is computational cost: \textbf{greedy} must compute $\phi_X(C_{i-1} \cup \{c\})$ for each $c \in X$. One way to improve the running time, not pursued here, is to speed up $\phi_X$ via subsampling and other approximate distance computations (Andoni and Indyk, 2008; Feldman and Langberg, 2011). Separately, and this approach comprises this subsection: the size of $Y_i$ can be made independent of $|X|$. This is achieved via a random sampling scheme similar to \textbf{kmeans++} (Arthur and Vassilvitskii, 2007), but repeatedly sampling many new centers given the same fixed set of prior centers.

This random sampling scheme also has a data-depandant quantity. Unfortunately $\kappa_1$ is unsuitably small in general, as it only guarantees the existence of one good center in $X$: instead, there needs to be a reasonable number of needles in the hay. To this end, given $\varepsilon > 0$, define

\[ \text{core}(A_j; \kappa) := \left\{ x \in A_j : \psi_{\{x\}}(\{a_j\}) \leq \kappa \psi_{A_j}(\{a_j\}) \right\}, \]

\[ \kappa_2 := \inf \left\{ \kappa \geq 0 : \forall j \in [k], |\text{core}(A_j; \kappa)| \geq \varepsilon |A_j|/(1 + \varepsilon) \right\}, \]

\[ A_j := \text{core}(A_j; \kappa_2). \]

The quantity $\kappa_2$ will capture problem adaptivity in the main bound below. By Lemma 3.2, $\kappa_1 \leq \kappa_2 \leq (1 + \varepsilon)^{1/q}$. 

\[ 6 \]
Theorem 3.2. Let $\varepsilon > 0$ and $\delta > 0$ be given, along with $C_0$ with $\phi_X(C_0) \leq \alpha (1+\kappa_2)^q (1+\varepsilon)^{q-1} \phi_X(A)$ for some $\alpha \geq 1$. Suppose greedy is run for $t \geq 4k \ln((\alpha - 1)/\varepsilon) + 8 \ln(1/\delta)$ rounds where $Y_i$ is chosen by the following scheme:

**select++**: return $4k(1 + \varepsilon)/(\varepsilon + 4)$ samples according to $\Pr[x] \propto \Delta(x, C_{i-1})$. Then with probability at least $1 - \delta$, the resulting centers $C_t$ satisfy

$$\phi_X(C_t) \leq (1 + \kappa_2)^q (1 + \varepsilon)^q \phi_X(A)$$

where $\kappa_2 \in [\kappa_1, (1 + \varepsilon)^{1/q}]$.

The key to the proof is Lemma 3.3, showing $(C_{i-1}, Y_i)$ satisfies Condition 2 with high probability. In order to prove this, the following tools are adapted from a high probability analysis of kmeans++ due to [Aggarwal et al. (2009); the full proof of Lemma 3.2 can be found in Appendix C.

Lemma 3.2. Consider any iteration $i$.

1. $\kappa_1 \leq \kappa_2 \leq (1 + \varepsilon)^{1/q}$
2. If $y \in \tilde{A}_j$, then $\psi_{A_j}(C_{i-1} \cup \{y\}) \leq (1 + \kappa_2) \psi_{A_j}(A)$.
3. If $\psi_{A_j}(C_{i-1}) > (1 + \varepsilon)(1 + \kappa_2) \psi_{A_j}(A)$, then every $y \in Y_i$ satisfies $\Pr[y \in \tilde{A}_j | y \in A_j] \geq (\varepsilon/(1 + \varepsilon))^{q+3}/4$.

Lemma 3.3. Suppose $\phi_X(C_{i-1}) > \gamma (1 + \varepsilon) \phi_X(A)$ where $\gamma := (1 + \varepsilon)^{q-1}(1 + \kappa_2)^q$. Then every element $c \in Y_i$ as chosen by select++ in Theorem 3.2 satisfies Condition 2 with constant $\gamma$ with probability at least $(\varepsilon/(1 + \varepsilon))^{q+4}/4k$.

Proof. Fix any cluster $A_m$ satisfying

$$m := \arg \max_{j \in [k]} \phi_{A_j}(C_{i-1}) - \gamma \phi_{A_j}(A).$$

This $A_m$ must satisfy $\phi_{A_m}(C_{i-1}) > \gamma (1 + \varepsilon) \phi_{A_m}(A)$ and thus $\psi_{A_m}(C_{i-1}) > (1 + \varepsilon)(1 + \kappa_2) \phi_{A_m}(A)$, since otherwise

$$\phi_X(C_{i-1}) - \gamma \phi_X(A) = \sum_j \phi_{A_j}(C_{i-1}) - \gamma \phi_{A_j}(A) \leq 0,$$

a contradiction.

Observe that the probability of sampling a center $c$ from $A_m$ is

$$\Pr[c \in A_m] = \frac{\phi_{A_m}(C_{i-1}) - \gamma \phi_{A_m}(A) + \gamma \phi_{A_m}(A)}{\phi_X(C_{i-1})}$$

$$= \frac{\max_j (\phi_{A_j}(C_{i-1}) - \gamma \phi_{A_j}(A)) + \gamma \phi_{A_m}(A)}{\phi_X(C_{i-1})}$$

$$\geq \frac{k^{-1} \sum_j (\phi_{A_j}(C_{i-1}) - \gamma \phi_{A_j}(A)) + \gamma \phi_{A_m}(A)}{\phi_X(C_{i-1})}$$

$$\geq \frac{1}{k} \left(1 - \frac{\gamma \phi_X(A)}{\phi_X(C_{i-1})}\right) > \frac{1}{k} \left(1 - \frac{1}{1 + \varepsilon}\right) = \frac{\varepsilon}{k(1 + \varepsilon)}.$$
Consider an instance of Theorem 3.3. The previous settings only achieved approximation ratio $1 + \varepsilon$ with repeated sampling, and thereafter invoking Theorem 2.1.

If $c \in A_m$ implies

$$\phi_{A_m}(C_{i-1} \cup \{c\}) \leq (1 + \kappa_2)^q \phi_{A_m}(A) \leq \gamma \phi_{A_m}(A),$$

and Condition 2 holds with probability at least $p_0$ since

$$\max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \min_{c \in Y_i} \phi_{A_j}(\{c\}) \right]_+ \geq \left[ \phi_{A_m}(C_{i-1}) - \min_{c \in Y_i} \phi_{A_m}(\{c\}) \right]_+ \quad \text{(since } m \in [k])$$

$$\geq \left[ \phi_{A_m}(C_{i-1}) - \gamma \phi_{A_m}(\{a_m\}) \right]_+ \quad \text{(by choice of } Y_i)$$

$$= \max_{j \in [k]} \left[ \phi_{A_j}(C_{i-1}) - \gamma \phi_{A_j}(\{a_j\}) \right]_+ \quad \text{(by choice of } m). \quad \square$$

The proof of Theorem 3.2 concludes by noting the success probability of Condition 2 is boosted with repeated sampling, and thereafter invoking Theorem 2.1.

**Lemma 3.4.** If a single sample from distribution $D$ satisfies Condition 3 with probability at least $\rho > 0$, then sampling $\lceil 1/\rho \rceil$ points iid from $D$ satisfies Condition 2 with probability at least $1 - 1/\rho$.

**Proof of Theorem 3.2.** If $\phi_X(C_i) \leq \gamma (1 + \varepsilon) \phi_X(A)$ for any $i$, then it holds for all $j \geq i$. Thus suppose $\phi_X(C_i) \leq \gamma (1 + \varepsilon) \phi_X(A)$ for all $i$; by Lemma 3.3 and Lemma 3.4 and since Condition 1 implies Condition 2, then Condition 2 holds in every iteration each with probability at least $1/2$, and the result follows by Theorem 2.1. \[\square\]

### 3.2 Approximation ratios close to one

The previous settings only achieved approximation ratio $1 + \varepsilon$ when $X$ and $A$ allowed it (e.g., when $\kappa_1$ and $\kappa_2$ were small). This subsection will cover three settings, each with three corresponding choices for select giving $1 + \varepsilon$ in general. The first method is for k-means $\Delta(x, y) = \|x - y\|_2$, the second for vanilla Euclidean $k$-medians $\Delta(x, y) = \|x - y\|_2$, and the last for $\Delta(x, y) = \|x - y\|^p$ for any norm and $p \geq 1$. Throughout this subsection, it is required that $A$ denotes an optimal solution.

First, in the special case of $k$-means, it suffices for select to return a single non-adaptive set of size $n^{1/\varepsilon}$ in each round.

**Theorem 3.3.** Consider an instance of the $k$-means problem. Let $\varepsilon \in (0, 1)$ be given, along with $C_0$ with $\phi_X(C_0) \leq \alpha (1 + \varepsilon) \phi_X(A)$ for some $\alpha \geq 1$. Suppose greedy is run for $t \geq k \ln((\alpha - 1)/\varepsilon)$ rounds using a select routine that always returns the same subset $\bar{Y}$ defined by $\bar{Y} := \{\sum_{i=1}^{\lceil 1/\varepsilon \rceil} x_i/\lceil 1/\varepsilon \rceil : x_1, x_2, \ldots, x_{\lceil 1/\varepsilon \rceil} \in X\}$. Then the resulting centers $C_t$ satisfy

$$\phi_X(C_t) \leq (1 + \varepsilon)^2 \cdot \phi_X(A).$$

**Proof.** Lemma A.2 implies that Condition 1 is satisfied with the set $\bar{Y}$ and $\gamma = 1 + \varepsilon$. The theorem therefore follows from Theorem 2.1. \[\square\]
The construction of the set $\bar{Y}$ from Theorem \ref{thm:3.3} crucially relies on the bias-variance decomposition available for squared Euclidean distance (cf. Lemma \ref{lem:A.1}).

Next consider the Euclidean $k$-medians case $\Delta(x,y) := \|x - y\|_2$. Since the mean (as used in Theorem \ref{thm:3.3}) minimizes $z \mapsto \phi_A\{\{z\}\}$ for $k$-means, it is natural to replace this mean selection with a more generic optimization procedure. Additionally, by using a stochastic online procedure, there is hope of using poly$(1/\varepsilon)$ data points as in Theorem \ref{thm:3.3}.

There is one catch — the standard convergence time for stochastic gradient descent (henceforth sgd) depends polynomially on the diameter of the space being searched. In order to obtain multiplicative optimality as in Condition \ref{cond:1} the diameter of the space must be related to the cost of an optimal cluster $\phi_A(A)$. Fortunately, this quantity can be guessed with $n^{-3}$ trials.

**Lemma 3.5.** Define the procedure

**guess-ball:** uniformly sample center $y \in X$ and sizes $b, m \in [n]$, let $B$ denote the $b$ points closest to $y$, and return the triple $(y, m, B)$.

For any subset $A \subseteq X$ with mean $c := \mu(A)$, with probability at least $n^{-3}$, simultaneously: $\phi_A\{\{c\}\} \leq \phi_B\{\{y\}\} \leq (1 + 2^q)\phi_A\{\{c\}\}$, and $m = |A|$, and $\Delta(y, c) \leq \phi_A\{\{c\}\}/|A|$.

The proof of Lemma \ref{lem:3.5} will be given momentarily, but with that concern out of the way, now note the sgd-based select.

**Theorem 3.4.** Consider the case of Euclidean $k$-medians, meaning $\Delta(x,y) = \|x - y\|_2$. Define a procedure select-sgd which generates $2n^{3+\lfloor1/\varepsilon^2\rfloor}$ iid samples as follows:

Set $r := \phi_B\{\{y\}\}/m$ where $(y, m, B)$ are from guess-ball. Perform $s := \lceil 1/\varepsilon^2 \rceil$ iterations of sgd (cf. Theorem \ref{thm:A.2}) starting from $y$, on objective function $w \mapsto E_x\|x - w\|_2$ constrained to the Euclidean ball of radius $r$ around $y$, and using step size $\eta := 2r/\sqrt{s}$. Return the unweighted average of the sgd iterates.

If greedy is run with initial clusters $C_0$ with $\phi_X(C_0) \leq \alpha(1 + 16\varepsilon)\phi_X(A)$ for some $\alpha \geq 1$, the above select-sgd routine, and $t \geq 4k(\ln((\alpha - 1)/\varepsilon) + 8\ln(1/\delta))$, then with probability at least $1 - \delta$, the output centers $C_t$ satisfy $\phi_X(C_t) \leq (1 + \varepsilon)(1 + 16\varepsilon)\phi_X(A)$.

The full proof is in Appendix \ref{app:D} but can be sketched as follows. For $k$-medians, subgradients have norm 1, and the guarantees on guess-ball give $\Delta(y, a_j) \leq \phi_A\{\{a_j\}\}/|A_j|$, so $\lceil 1/\varepsilon^2 \rceil$ sgd iterations suffice (cf. Theorem \ref{thm:A.2}), if somehow the random data points were drawn directly from $A_j$. But $|A_j| \geq 1$, so all data points are drawn from it with probability at least $n^{-\lceil1/\varepsilon^2\rceil}$. Note that this proof grants the existence of a good sequence of $1/\varepsilon^2$ examples together with a good triple $(y, m, B)$, thus another approach, mirroring non-adaptive scheme in Theorem \ref{thm:3.3} is to enumerate these $n^{3+\lfloor1/\varepsilon^2\rfloor}$ possibilities, but process them with sgd rather than the uniform averaging in Theorem \ref{thm:3.3}.

**Proof of Lemma 3.5.** Fix any cluster $A$. This proof establishes the existence of a point $y \in X$ and a set $B \subseteq X$ of closest points which satisfies all required properties together with $m := |A|$. Since one such triple exists, then the probability of sampling one uniformly at random is at least $n^{-3}$.

Choose $y \in X$ with $\Delta(y, c) = \min_{x \in A} \Delta(x, c) \leq \phi_A\{\{c\}\}/|A|$. Let $B$ denote the $|A|$ points closest to $y$ in $X$ (ties broken arbitrarily). By Lemma \ref{lem:3.1},

$$\phi_B\{\{y\}\} \leq \phi_A\{\{y\}\} \leq 2^q \phi_A\{\{c\}\}.$$  

If it also holds that $\phi_B\{\{y\}\} \geq \phi_A\{\{c\}\}$, the proof is complete.
Otherwise suppose $\phi_B(\{y\}) < \phi_A(\{c\})$, which also implies $|A| \geq 1$, since otherwise $|X| = 0 = \phi_A(\{c\}) = \phi_A(\{y\})$. Consider the process of iteratively adding to $B$ those points in $X \setminus B$ which are closest to $y$, stopping this process at the first time when $\phi_B(\{y\}) \geq \phi_A(\{c\})$; it is claimed that this final $B$ also satisfies $\phi_B(\{y\}) \leq (1 + 2^q) \phi_A(\{c\})$.

To this end, note that the penultimate $B'$ did not satisfy $B' \supsetneq A$, since that would mean $\phi_B'(\{y\}) \geq \phi_A(\{y\}) \geq \phi_A(\{c\}) > \phi_B'(\{y\})$.

As such, the final added point $v$ can be no further from $y$ than the furthest element of $A \setminus B'$, meaning by Lemma 3.1

$$\phi_{\{v\}}(\{y\}) \leq \max_{u \in A \setminus B'} \phi_{\{u\}}(\{y\}) \leq \phi_A(\{y\}) = |A| (\psi_A(\{y\}))^q \leq |A| (2\psi_A(\{c\}))^q \leq 2^q \phi_A(\{c\}),$$

thus

$$\phi_B(\{y\}) = \phi_B'(\{y\}) + \phi_{\{v\}}(\{y\}) < \phi_A(\{c\}) + 2^q \phi_A(\{c\}). \quad \square$$

To close, note how to get $1 + \varepsilon$ with $\Delta(x, y) = \|x - y\|^p$ for a general norm and exponent $p$. Unfortunately, the number of samples used here is exponential in the dimension.

**Theorem 3.5.** Let a generalized $k$-medians problem with $\Delta(x, y) = \|x - y\|^p$ for some norm $\| \cdot \|$ and exponent $p \geq 1$ be given, along with $\varepsilon \in (0, 1)$ and $\delta > 0$, and initial clusters $C_0$ with $\phi_X(C_0) \leq \alpha(1 + \varepsilon) \phi_X(A)$ for some $\alpha \geq 1$. Suppose greedy is run for $t \geq 4k \ln((\alpha - 1)/\varepsilon) + 8 \ln(1/\delta)$ rounds where $Y_i$ is chosen by select-ball, a procedure which returns $O(n^3 \varepsilon^{-qd/p})$ iid samples, each generated as follows:

Obtain $(y, m, B)$ from guess-ball, and output a uniform random sample from the $D$-ball of radius $2(\phi_B(\{y\}))/m^{1/p}$ centered at $y$.

Then with probability at least $1 - \delta$, the resulting centers $C_t$ satisfy $\phi_X(C_t) \leq (1 + \varepsilon)^2 \phi_X(A)$.

The full proof appears in Appendix D but is easy to sketch. By Lemma 3.5 not only is a point $y \in A_j$ with $\Delta(y, a_j) \leq \phi_{A_j}(\{a_j\})/|A_j|$ in hand, but additionally an accurate estimate on $\Delta(y, a_j)$. The chosen sampling radius is large enough to include points around $a_j$ which are all $(1 + \varepsilon)$ accurate, and the probability of sampling this smaller ball via the larger is just the ratio of their volumes. The result follows by boosting the probability via Lemma 3.4 and applying Theorem 2.1.

## 4 Experiments with $k$-means

Experimental results appear in Figure 2. The table in Figure 2a summarizes the improvement over kmeans++ by greedy with kmeans++-inspired select++ (cf. Section 3.1): for each of 5 UCI datasets with 1000-20000 points, kmeans++ and greedy/select++ were run 10 times for $t \in \{10, 50\}$, and then a ratio of median and minimum performance was recorded. Of course, while the experiment is favorable, it is somewhat unfair as kmeans++ requires less computation. On the other hand, greedy is more amenable to various speedups, for instance it is trivially parallelized.

The curves in Figure 2b plot the (median) cost on aba as a function of the number of centers. These plots indicate an area where the analysis in the present work may be improved. Namely, the results of Section 3 require good initialization for the best bounds (e.g., a naive analysis with $C_0 = \emptyset$ introduces logarithmic dependencies on interpoint distance ratios). According to Figure 2b, this is potentially an artifact of the analysis: the dashed line uses kmeans++ for the first 25 centers and greedy for the remaining 25, and it does not outperform full greedy. Another possibility as that there are other data-dependent quantities which remove the need for good initialization.
|       | med(gr)/med(++) | min(gr)/min(++) |
|-------|----------------|----------------|
|       | k = 10         | k = 50         |
| aba   | 0.747          | 0.843          |
|       | 0.662          | 0.693          |
| car   | 0.794          | 0.837          |
|       | 0.915          | 0.924          |
| eeg   | 0.723          | 0.767          |
|       | 0.775          | 0.823          |
| let   | 0.746          | 0.855          |
|       | 0.787          | 0.804          |
| mag   | 0.729          | 0.824          |
|       | 0.788          | 0.811          |

(a) Cost ratio on five datasets.

Figure 2: \(k\)-means comparisons between \texttt{kmeans++} (++), \texttt{greedy} with subsampling (gr), and greedy initialized with \texttt{kmeans++} (g+).

(b) \texttt{kmeans++} initialization does not help.

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A Technical tools

Lemma A.1 (Bias-Variance). For any finite subset $A \subseteq \mathbb{R}^d$ and any $z \in \mathbb{R}^d$,

$$\sum_{x \in A} \|x - z\|^2_2 = \sum_{x \in A} \left( \|x - \mu(A)\|^2_2 + \|\mu(A) - z\|^2_2 \right).$$
Then with probability at least $1$.

Theorem A.1

Theorem A.2

This is a simple consequence of the first moment method and Lemma A.1.

Proof. For convenience, define $Y_i = \left( \sum_{x \in A} \|x - z\|_2^2 \right) = \sum_{x \in A} \|x - \mu(A) + \mu(A) - z\|_2^2$

$$= \sum_{x \in A} \left( \|x - \mu(A)\|_2^2 + \|\mu(A) - z\|_2^2 \right) + 2 \left( \mu(A) - z, \sum_{x \in A} (x - \mu(A)) \right).$$

Lemma A.2 (Inaba et al. 1994). For any finite subset $A \subseteq \mathbb{R}^d$ and any $\epsilon > 0$, there exists $x_1, x_2, \ldots, x_m \in A$ with $m = \lceil 1/\epsilon \rceil$ such that $\mu_m := \sum_{i=1}^m x_i / m$ satisfies

$$\sum_{x \in A} \|x - \mu_m\|_2 \leq (1 + \epsilon) \sum_{x \in A} \|x - \mu(A)\|_2.$$ 

Proof. This is a simple consequence of the first moment method and Lemma A.1.

Lemma A.3. If $x \in [0, 1]$ and $p \geq 1$, then $(1 + x)^{1/p} - 1 \geq x(2^{1/q} - 1)$.

Proof. For convenience, define $f(x) := (1 + x)^{1/p} - 1$. $f$ is concave, thus along $[0, 1]$ is lower bounded by its secant, which passes between $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, 2^{1/p} - 1)$.

Theorem A.1 (Bernstein’s inequality for martingales). Let $(Y_i)_{i=1}^n$ be a bounded martingale difference sequence with respect to the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$. Assume that for some $b, v > 0$, $|Y_i| \leq b$ for all $i$ and $\sum_{i=1}^n \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}) \leq v$ almost surely. For all $\delta \in (0, 1)$,

$$\Pr \left[ \sum_{i=1}^n Y_i > \sqrt{2v \ln(1/\delta)} + b \ln(1/\delta)/3 \right] \leq \delta.$$ 

Theorem A.2 (Convergence analysis for stochastic gradient descent). Consider the standard setup of stochastic gradient descent (sgd).

- Let convex function $f : \mathbb{R}^d \to \mathbb{R}$, reference point $\bar{w}$, and convex compact set $S$ be given.

- Let a random subgradient oracle be given, which for any $w \in S$ returns (random) $\hat{g}$ with $\mathbb{E}(\hat{g}) \in \partial f(w)$.

- Let $L \geq 0$ be given which bounds the norms on full and stochastic gradients almost surely, meaning $\sup_{w \in S} \sup_{\hat{g} \in \partial f(w)} \|\hat{g}\|_2 \leq L$ and $\|\hat{g}\|_2 \leq L$ almost surely for any $\hat{g}$ returned by the oracle at any $w \in S$.

- Let $B \geq 0$ be given so that $\sup_{w \in S} \|w - \bar{w}\|_2 \leq B$.

- Let $w_1 \in S$ and total number of iterations $t$ be given, and suppose $w_{i+1} := \Pi_S(w_i - \eta \hat{g}_i)$ where $\eta := BL/\sqrt{t}$ and $\hat{g}_i$ is a stochastic gradient given by the oracle at $w_i$, and $\Pi_S$ is orthogonal projection onto $S$.

Then with probability at least $1 - 1/e$, $f(\bar{w}_t) \leq f(\bar{w}) + 4BL/\sqrt{t}$. 

B Analysis based on supermodularity

The behavior of the standard greedy algorithm (i.e., greedy where $Y_i = Y = \text{select}()$ is well-understood in the context of minimizing monotone supermodular set functions, and indeed, the objective function $\phi_X$ fits this bill. To see that $\phi_X$ is supermodular, consider any $C \subseteq C' \subseteq Y$ and $c \in Y \setminus C'$. Then

$$\phi_X(C) - \phi_X(C \cup \{c\}) = \sum_{x \in X} \left[ \min_{c' \in C} \Delta(x, c') - \Delta(x, c) \right]_+$$

$$\geq \sum_{x \in X} \left[ \min_{c' \in C'} \Delta(x, c') - \Delta(x, c) \right]_+ = \phi_X(C') - \phi_X(C' \cup \{c\}) .$$

Monotonicity holds since $C \subseteq C' \subseteq Y$ implies $\min_{c \in C} \Delta(x, c) \geq \min_{c' \in C'} \Delta(x, c')$ for every $x \in X$.

B.1 Standard analysis of greedy algorithm

[Nemhauser et al. (1978)] show that supermodularity of an arbitrary set function $f: 2^Y \to \mathbb{R}$ is equivalent to the following property: for any $S \subseteq T \subseteq Y$,

$$\sum_{y \in T \setminus S} (f(S) - f(S \cup \{y\})) \geq f(S) - f(T) . \tag{2}$$

If $f$ is also monotone, then for any $S, S^* \subseteq Y$ (where $S$ denotes a current solution and $S^*$ denotes an arbitrary reference solution),

$$\max_{y \in S^* \setminus S} f(S) - f(S \cup \{y\}) \geq \frac{1}{|S^* \setminus S|} \sum_{y \in S^* \setminus S} (f(S) - f(S \cup \{y\}))$$

$$\geq \frac{1}{|S^* \setminus S|} (f(S) - f(S^*))$$

$$\geq \frac{1}{|S^*|} (f(S) - f(S^*)) , \tag{3}$$

where eq. (3) follows from eq. (2) with $T = S \cup S^*$. This shows that a greedy choice of $y \in Y$ to minimize $f(S \cup \{y\})$ yields a reduction in objective value at least as large as $(f(S) - f(S^*)) / |S^*|$. Using $f = \phi_X, S = C_{t-1}, S^* = A$, and the fact that $\phi_X(C_i) \leq (1 + \tau) \cdot \min_{c \in Y} \phi_X (C_{i-1} \cup \{c\})$ (as $Y_i = Y = \text{select}()$), the above inequality implies

$$\phi_X(C_i) \leq \left( 1 - \frac{1}{k} \right) \cdot (1 + \tau) \cdot \phi_X(C_{i-1}) + \frac{1}{k} \cdot (1 + \tau) \cdot \phi_X(A) , \tag{4}$$

which exactly matches the key recurrence eq. (1) in the proof of Lemma 2.1 in the case $\gamma = 1$.

This clustering problem is more commonly viewed in the literature as a submodular maximization problem (e.g., [Mirzasoleiman et al. 2013]) with objective $f(S) := \phi_X(\{c_0\}) - \phi_X(S \cup \{c_0\})$, where $c_0 \in Y$ is some distinguished center fixed a priori. There, the same analysis of the greedy algorithm, after $t$ rounds starting with $C_0 = \{c_0\}$, yields a guarantee of the form

$$\phi_X(\{c_0\}) - \phi_X(C_t \cup \{c_0\}) \geq \left( 1 - \frac{1}{e} \right) \max_{C \subseteq Y: |C| \leq t} \left( \phi_X(\{c_0\}) - \phi_X(C \cup \{c_0\}) \right) ,$$

which can be rewritten as

$$\phi_X(C_t \cup \{c_0\}) \leq \frac{1}{e} \phi_X(\{c_0\}) + \left( 1 - \frac{1}{e} \right) \min_{C \subseteq Y: |C| \leq t} \phi_X(\{c_0\}) .$$

This is generally incomparable to Theorem 2.1
B.2 Reducing computational cost via uniform random sampling

For general monotone supermodular objectives \( f \), random sampling can reduce the computational cost of the standard greedy algorithm, although the savings are modest compared to what can be achieved in the special case of \( f = \phi_X \) where additional structure is exploited. This subsection describes such a “folklore” result (see, e.g., Buchbinder et al., 2015, Theorem 1.3).

Consider a greedy choice of \( y \in \bar{Y} \) to minimize \( f(S \cup \{y\}) \), where \( \bar{Y} \), a multiset of size \( m \), is formed by independently drawing centers from \( Y \) uniformly at random. Again, let \( S^* \subseteq Y \) be an arbitrary reference solution. Then

\[
\Pr[\bar{Y} \cap (S^* \setminus S) = \emptyset] = \left(1 - \frac{|S^* \setminus S|}{|Y|}\right)^m,
\]

and hence

\[
\mathbb{E}\left[f(S) - \min_{y \in Y} f(S \cup \{y\})\right] \geq \left(1 - \left(1 - \frac{|S^* \setminus S|}{|Y|}\right)^m\right) \frac{1}{|S^* \setminus S|} \sum_{y \in S^* \setminus S} (f(S) - f(S \cup \{y\}))
\]

\[
\geq \left(1 - \left(1 - \frac{|S^* \setminus S|}{|Y|}\right)^m\right) \frac{1}{|S^* \setminus S|} (f(S) - f(S \cup S^*)) \quad \text{(by eq. (3))}
\]

\[
\geq \left(1 - \exp\left(-\frac{|S^* \setminus S|}{|Y|} \cdot m\right)\right) \frac{1}{|S^* \setminus S|} (f(S) - f(S^*)) \quad \text{(by convexity)}.
\]

If \( m \geq (|Y|/|S^*|) \ln(1/\delta) \), then the above inequality implies

\[
\mathbb{E}\left[f(S) - \min_{y \in Y} f(S \cup \{y\})\right] \geq \frac{1 - \delta}{|S^*|} (f(S) - f(S^*)).
\]

This leads to the same key recurrence as in eq. (4) (in expectation) when \( Y_i \) is formed by drawing \( m \geq (|Y|/k) \ln(1/\delta) \) centers independently and uniformly at random from \( Y \).

A standard way to apply this technique to the generalized \( k \)-medians problem described in Section 3 (where typically one takes \( Y = \mathbb{R}^d \)) is to form each \( Y_i \) by drawing \( m \) centers independently and uniformly at random from \( X \). The number of centers considered in each round of this greedy implementation is linear in \( n = |X| \) (and this is also true of other methods studied by Buchbinder et al., 2015). In contrast, the number of centers considered using the random sampling technique from Section 3.1 is independent of \( n \).

C kmeans++ tools extracted from Aggarwal et al. (2009)

This appendix proves Lemma 3.2, the main tool in the results of Section 3.1. These tools are then used to prove adaptive (depending on \( \kappa_2 \)) for vanilla kmeans++ in Theorem C.1. The analysis is based on a high probability analysis of kmeans++ due to Aggarwal et al., 2009, merely simplified and adjusted to the setting here.

Throughout this section, the following additional notation will be convenient.
• $C_{i-1}(z) := \arg\min_{y \in C_{i-1}} \Delta(z, y)$.

• $\kappa_3 := (1 + \varepsilon)(1 + \kappa_2)$.

• $\kappa_4 := (1 + \varepsilon)\kappa_3$.

• Split the optimal clusters into “good” and “bad” clusters, depending on how well they’re “covered” by the centers in $C_{i-1}$.

$$\text{Good}_i := \left\{ j \in [k] : \psi_{A_j}(C_{i-1}) \leq \kappa_3\psi_{A_j}(\{a_j\}) \right\},$$

$$\text{Bad}_i := \left\{ j \in [k] : \psi_{A_j}(C_{i-1}) > \kappa_3\psi_{A_j}(\{a_j\}) \right\}.$$

### C.1 Proof of Lemma 3.2

**Proof of part 1 of Lemma 3.2.** For the lower bound, given any $\kappa_0 < \kappa_1$, then there must exist $j \in [k]$ with $|A_k| > 0$ and $\min_{x \in A_j} \psi_x(\{a_j\}) > \kappa_0\psi_{A_j}(\{a_j\})$, thus $|\text{core}(A_j; b)| = 0$, and $\kappa_2 > \kappa_0$ by definition of $\kappa_2$. Since this holds for every $\kappa_0 < \kappa_1$, then $\kappa_2 \geq \kappa_1$.

For the upper bound, Set $\kappa_0 := (1 + \varepsilon)^{1/q}$, and consider any $j \in [k]$, setting $A_j := \text{core}(A_j; \kappa_0)$ for convenience. The goal is to show $|A_j| \geq \varepsilon|A_j|/(1 + \varepsilon)$, and thus $\kappa_2 \leq \kappa_0 = (1 + \varepsilon)^{1/q}$ by definition of $\kappa_2$.

The claim is trivial if $A_j = A_j$. If $A_j \subseteq A_j$, then

$$\psi_{A_j}(\{a_j\}) \geq \sum_{x \in A_j \setminus A_j} \Delta(x, a_j) > \left(|A_j| - |\tilde{A}_j|\right) \kappa_0\psi_{A_j}(\{a_j\})/|A_j|.$$  

Rearranging, $|\tilde{A}_j| > |A_j|/(1 - 1/\kappa_0^q) = \varepsilon|A_j|/(1 + \varepsilon)$.

**Proof of part 2 of Lemma 3.2.** By Lemma 3.1, every $y \in \tilde{A}_j$ satisfies

$$\psi_{A_j}(C_{i-1} \cup \{y\}) \leq \psi_{A_j}(\{y\}) \leq \psi_{A_j}(\{a_j\}) + \psi_{\{\{y\}\}}(\{a_j\}) \leq (1 + \kappa_2)(\psi_{A_j}(\{a_j\})).$$

The proof of part 3 of Lemma 3.2 will use the following lemma.

**Lemma C.1.** For any $j \in \text{Bad}_i$, and any $\hat{c} \in C_{i-1}$,

$$\psi_{\{\hat{c}\}}(\{a_j\}) \geq (\kappa_3 - 1)\psi_{A_j}(\{a_j\}).$$

**Proof.** Take any $\hat{c} \in C_{i-1}$. Then, using the fact that $j \in \text{Bad}_i$ and Lemma 3.1

$$\kappa_3(\psi_{A_j}(\{a_j\})) \leq \psi_{A_j}(\{\hat{c}\}) \leq \psi_{A_j}(\{a_j\}) + \psi_{\{\hat{c}\}}(\{a_j\}).$$

Rearranging gives the bound.

Part 3 of Lemma 3.2 is a consequence of the following more detailed bound.

**Lemma C.2.** For any $j \in \text{Bad}_i$,

$$\Pr[\hat{c}_i \in \tilde{A}_j \mid \hat{c}_i \in A_j] \geq \frac{|\tilde{A}_j| \left(1 - (\kappa_2/(\kappa_3 - 1))^{q/p}\right)^p}{|A_j| \left(1 + \frac{1}{\kappa_3 - 1}\right)^q} \geq \frac{1}{4} \left(\frac{\varepsilon}{1 + \varepsilon}\right)^{q+3}.$$
Proof. To start,
\[
\Pr \left[ \hat{c}_i \in \tilde{A}_j \mid \hat{c}_i \in A_j \right] = \frac{\phi_{\tilde{A}_j}(C_{i-1})}{\phi_{A_j}(C_{i-1})} = \frac{\left| \tilde{A}_j \right|}{\left| A_j \right|} \left( \frac{\psi_{\tilde{A}_j}(C_{i-1})}{\psi_{A_j}(C_{i-1})} \right)^q.
\]

The proof proceeds by bounding the numerator and denominator separately.

For the numerator, fix a particular \( \tilde{x} \in \tilde{A}_j \), and observe
\[
\psi_{\{\tilde{x}\}}(C_{i-1}(\tilde{x}))^{q/p} \\
\geq D(a_j, C_{i-1}(\tilde{x})) - D(a_j, \tilde{x}) \quad \text{(triangle inequality of }D) \\
\geq D(a_j, C_{i-1}(a_j)) - D(a_j, \tilde{x}) \quad \text{(since } \tilde{x} \in \tilde{A}_j) \\
\geq D(a_j, C_{i-1}(a_j)) - \left( \kappa_2 \psi_{A_j}(\{a_j\}) \right)^{q/p} \quad \text{(Lemma }C.1 \text{ symmetry of }D) \\
= \psi_{\{a_j\}}(C_{i-1}(a_j))^{q/p} \left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right)^{q/p} \right),
\]

therefore
\[
\psi_{\tilde{A}_j}(C_{i-1}) \geq \min_{\tilde{x} \in \tilde{A}_j} \psi_{\{\tilde{x}\}}(C_{i-1}) \geq \psi_{\{a_j\}}(C_{i-1}(a_j)) \left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right)^{q/p} \right)^{p/q}.
\]

For the denominator
\[
\psi_{A_j}(C_{i-1}) \leq \psi_{A_j}(C_{i-1}(\{a_j\})) \\
\leq \psi_{A_j}(a_j) + \psi_{\{a_j\}}(C_{i-1}(a_j)) \quad \text{(Lemma }3.1 \text{ symmetry of }D) \\
\leq \left( 1 + \frac{1}{\kappa_3 - 1} \right) \psi_{\{a_j\}}(C_{i-1}(a_j)) \quad \text{(Lemma }C.1 \text{)}.
\]

Combining the numerator and denominator bounds,
\[
\Pr \left[ \hat{c}_i \in \tilde{A}_j \mid \hat{c}_i \in A_j \right] \geq \frac{\left| \tilde{A}_j \right|}{\left| A_j \right|} \left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right)^{q/p} \right)^{p/q}.
\]

Lastly, to simplify the inequalities, first note \( |\tilde{A}_j|/|A_j| \geq \varepsilon/(1 + \varepsilon) \) by definition of \( \kappa_2 \). Next consider the case of \( k \)-means, implying \( q = 1 \neq 2 = p \). Then
\[
\Pr \left[ \hat{c}_i \in \tilde{A}_j \mid \hat{c}_i \in A_j \right] \geq \frac{\varepsilon}{1 + \varepsilon} \left( \frac{\kappa_3 - 1}{\kappa_3} \right) \left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right)^{1/2} \right)^2.
\]

To control these terms, note since \( \kappa_2 \geq 0 \) that
\[
\frac{\kappa_3 - 1}{\kappa_3} = 1 - \frac{1}{\kappa_3} = 1 - \frac{1}{(1 + \varepsilon)(1 + \kappa_2)} \geq 1 - \frac{1}{1 + \varepsilon} = \frac{\varepsilon}{1 + \varepsilon}.
\]

For the second term, since \( \sqrt{\cdot} \) is concave, the tangent bound \( \sqrt{a} \leq 1 + (a - 1)/2 \) holds, thus
\[
\left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right)^{1/2} \right)^2 \geq \frac{1}{4} \left( 1 - \left( \kappa_2/(\kappa_3 - 1) \right) \right)^2 = \left( \frac{\varepsilon}{2(1 + \varepsilon)} \right)^2.
\]

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When the instance is not \( k \)-means, \( q = p \geq 1 \), and so

\[
\Pr \left[ \hat{c}_i \in \tilde{A}_j \mid \hat{c}_i \in A_j \right] \geq \frac{|\tilde{A}_j|}{A_j} \left( \frac{\kappa_3 - 1}{\kappa_3} \right)^p \left( 1 - \frac{\kappa_2}{(\kappa_3 - 1)} \right)^p
\]

\[
= \frac{|\tilde{A}_j|}{A_j} \left( \frac{\kappa_3 - 1 - \kappa_2}{\kappa_3} \right)^p = \left( \frac{\epsilon}{1 + \epsilon} \right)^{p+1}.
\]

\[ \square \]

C.2 \textit{kmeans++} guarantee for generalized \( k \)-medians problems

\textbf{Theorem C.1.} Let \( C_t \) denote the centers output by \textit{greedy} when run with \( C_0 = \emptyset \) and \textit{select} as in \textbf{Theorem 3.2}, except \(|Y_i| = 1\) (e.g., only a single sample, as with \textit{kmeans++}). With probability at least \( 1 - \delta \), if \( t \geq 8(k + \ln(1/\delta))((1 + \epsilon)/\epsilon)^{q+4} \), then \( \phi_X(C_t) \leq (1 + \epsilon)^{2q}(1 + \kappa_2)^q \phi_X(A) \).

The proof uses the following lemma.

\textbf{Lemma C.3.} In step \( i \), at least one of the following is true.

\begin{itemize}
  \item \( \psi_X(S_{i-1}) \leq \kappa_4 \psi_X(\{A\}) \),
  \item \( \Pr \left[ \hat{c}_i \in \bigcup_{j \in \text{Bad}_i} A_j \right] \geq 1 - \kappa_3/\kappa_4 \geq \epsilon/(1 + \epsilon) \).
\end{itemize}

\textbf{Proof.} Suppose \( \phi_X(C_{i-1}) > \kappa_4 \phi_X(\{A\}) \). Then

\[
\Pr \left[ \hat{c}_i \in \bigcup_{j \in \text{Bad}_i} A_j \right] = \frac{\sum_{j \in \text{Bad}_i} \phi_{A_j}(C_{i-1})}{\phi_X(S)}
\]

\[
= 1 - \frac{\sum_{j \in \text{Good}_i} \phi_{A_j}(C_{i-1})}{\phi(C_{i-1})}
\]

\[
\geq 1 - \frac{\kappa_3 \sum_{j \in \text{Good}_i} \phi_{A_j}(\{a_j\})}{\kappa_4 \phi_X(A)} \geq 1 - \frac{\kappa_3^q}{\kappa_4^q} = \frac{\epsilon}{1 + \epsilon}.
\]

\[ \square \]

\textbf{Proof of Theorem C.1} Consider the success events

\[ \mathcal{E}_i := \{ \phi_X(C_{i-1}) \leq \kappa_4 \phi_X(A) \} \lor \{ |\text{Bad}_i| = 0 \} \lor \{ |\text{Bad}_{i+1}| < |\text{Bad}_i| \} \]

which states that at least one of the following statements is true upon choosing \( c_i \in Y_i \) (note \( \{c_i\} = Y_i \):

1. The approximation ratio \( \phi_X(C_{i-1})/\phi_X(A) \) before choosing \( c_i \) is already at most \( \kappa_4^q \).
2. \( \text{Bad}_i \) is empty, \( \phi_X(C_{i-1}) \leq \kappa_3^q \phi_X(A) \).
3. The choice \( c_i \) causes one bad set for \( C_{i-1} \) to become good for \( C_i \).

After \( k \) successes, \( \text{Bad}_i \) is empty, thus it remains to control the number of stages before \( k \) successes. By \textbf{Lemma 3.2} and since \( \kappa_3 \geq 1 + \kappa_2 \), if \( c \in \tilde{A}_j \) for some \( j \in \text{Bad}_i \), then \( j \notin \text{Bad}_{i+1} \). Therefore, by
1. Combining these, both stochastic and full gradients of $f$ at the only point of non-differentiability, $n$.

The result is a consequence of the following claim: any single center provided

Moreover, for any $w$ around $y$ and any $\bar{A}_j$, namely the constraint set used by sgd within $select-sgd$, the corresponding stochastic gradient has norm 1 since

At the only point of non-differentiability, $x = w$, it suffices to control the magnitude of every directional derivative (Rockafellar, 1970, Theorem 23.2), but by a similar calculation these are also 1. Combining these, both stochastic and full gradients of $f$ have norm 1, thus by Theorem A.2 with probability at least $1 - 1/e$, the output $\bar{w}$ satisfies

Since $A_j$ was arbitrary, Condition 1 is satisfied with $\gamma = 1 + 16\varepsilon$ with probability at least $(1 - 1/e)n^{3+\lceil 1/\varepsilon^2 \rceil}$. 

D  Deferred proofs from Section 3.2

First, the guarantee for $select-sgd$.

Proof of Theorem 3.4. The result is a consequence of the following claim: any single center provided by $select-sgd$ satisfies Condition 1 with probability $n^{3+\lceil 1/\varepsilon^2 \rceil}/2$. Indeed, this suffices as with the proof of Theorem 3.2, this probability can be boosted (e.g., via Lemma 3.4), and then Theorem 2.1 completes the proof.

To establish Condition 1, fix any optimal cluster $A_j$, and any $\bar{w}$ output by $select-sgd$. By Lemma 3.5, with probability at least $n^{-3}$, the estimate $\phi_B(\{y\})/m$ satisfies $\|y - a_j\|_2 \leq \phi_{A_j}(A_j)/|A_j| \leq \phi_B(\{y\})/m \leq 3\phi_{A_j}(A_j)/|A_j|$. Moreover, with probability at least $(|A_j|/n)^{1/\varepsilon^2} \geq n^{-1/\varepsilon^2}$, every data point randomly sampled during sgd was drawn from $A_j$. Consequently, this sample is equivalent to one drawn directly from $A_j$ itself. Under this iid sampling, the function $f(c) := E_x(\Delta(x, c)) = \phi_{A_j}(\{c\})/|A_j|$ is convex with optimum $a_j$.

To apply the bounds for sgd in Theorem A.2, the norms of iterates and subgradients (stochastic and full subgradients of $f$) must be controlled. Letting $S$ denote the ball of radius $r = \phi_B(\{y\})/m$ around $y$, namely the constraint set used by sgd within $select-sgd$, every $w \in S$ satisfies

Moreover, for any $w \in S$ and any random $x \in A_j$ with $w \neq x$, the corresponding stochastic gradient has norm 1 since

$$\left\| \frac{\partial}{\partial w} \left\| x-w \right\|_2 \right\|_2 = \left\| \frac{\partial}{\partial w} \left\| x-w \right\|_2 \right\|_2 = \frac{2\|x-w\|_2}{2\|x-w\|_2} = 1.$$
Lastly, the guarantees for select-ball.

**Lemma D.1.** Given current centers $S$, suppose a single new center $\hat{c}$ is sampled according to the distribution in select-ball where $\varepsilon \leq 1$. Then $S \cup \{\hat{c}\}$ satisfies Condition 1 with $\gamma = 1 + \varepsilon$ with probability $\Omega(n^{-3}\varepsilon^{q/d/p})$.

**Proof.** Fix any reference cluster $A_j$. First consider the ball $B_r$ of radius (measured by $D$) $r := (\gamma^{1/q} - 1)^{q/p}\psi_{A_j}(\{a_j\})^{q/p}$ around $a_j$; by Lemma 3.1, every $z \in B_r$ satisfies

$$\psi_{A_j}(\{z\}) \leq \psi_{A_j}(\{a_j\}) + D(z, a_j)^{p/q} \leq \gamma^{1/q}\psi_{A_j}(\{a_j\})$$

and in particular $\phi_{A_j}(\{z\}) \leq \gamma\phi_{A_j}(\{a_j\})$, meaning Condition 1 holds for this $j \in [k]$ with $\gamma$ for every $z \in B_r$; but $j \in [k]$ was arbitrary, so Condition 1 holds with probability exceeding the probability of the chosen point $\hat{c}$ falling within $B_r$. Furthermore, note by concavity of $(\cdot)^{1/q}$ the resulting tangent bound $\gamma^{1/q} - 1 \leq (\gamma - 1)/q = \varepsilon/q$, thus $r \leq \psi_{A_j}(\{a_j\})^{q/p}$.

Now consider the triple $(y, m, B)$ returned by guess-ball, and moreover the ball $B_R$ of radius $R := 2\left(\phi_B(\{y\})/m\right)^{1/p}$ centered at $y$. By Lemma 3.5 and the above upper bound $r \leq \psi_{A_j}(\{a_j\})^{q/p}$, every $z \in B_r$ satisfies

$$D(z, y) \leq D(z, a_j) + D(a_j, y) \leq r + \left(\phi_{A_j}(\{a_j\})/|A_j|\right)^{1/p} \leq \psi_{A_j}(\{a_j\})^{q/p} + \psi_{A_j}(\{a_j\})^{q/p} \leq R,$$

meaning $B_r \subseteq B_R$. As such, the probability of hitting a point in $B_r$ with a uniform sample from $B_R$ is the volume ratio of the two balls, which by Lemma 3.5 and the secant lower bound $\gamma^{1/q} - 1 = (1 + \varepsilon)^{1/q} - 1 \geq \varepsilon(2^{1/q} - 1)$ for $\varepsilon \in [0, 1]$ (cf. Lemma A.3) satisfies

$$\Pr[z \in B_r|z \in B_R] = \left(\frac{r}{R}\right)^d \geq \left(\frac{(\gamma^{1/q} - 1)^{q/p}\psi_{A_j}(\{\hat{c}\})^{q/p}}{2(1 + 2\varepsilon)^{1/p}\psi_{A_j}(\{\hat{c}\})^{q/p}}\right)^d \geq \left(\frac{\varepsilon^{q/p}}{2 \cdot 3^{q/p}}\right)^d.$$

The result now follows by multiplying this success probability with the $n^{-3}$ success probability for guess-ball (cf. Lemma 3.5).

**Proof of Theorem 3.5.** As with the proof of Theorem 3.2, the proof follows by combining Lemma D.1 with Lemma 3.4 and Theorem 2.1. 

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