Numerical solution to a multi-dimensional linear inverse heat conduction problem by a splitting-based conjugate gradient method

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Abstract. In this paper we consider a multi-dimensional inverse heat conduction problem with time-dependent coefficients in a box, which is well-known to be severely ill-posed, by a variational method. The gradient of the functional to be minimized is obtained by aids of an adjoint problem and the conjugate gradient method with a stopping rule is then applied to this ill-posed optimization problem. To enhance the stability and the accuracy of the numerical solution to the problem we apply this scheme to the discretized inverse problem rather than to the continuous one. The difficulties with large dimensions of discretized problems are overcome by a splitting method which only requires the solution of easy-to-solve one-dimensional problems. The numerical results provided by our method are very good and the techniques seem to be very promising.

1. Introduction
Inverse heat conduction problems (IHCPs), because of their important applications in many branches of technology, science, etc., have been extensively studied over the last 50 years or so. Although there exists a vast literature on one-dimensional problems, there are much fewer papers devoted to multi-dimensional cases, especially when the coefficients of the equations describing the heat transfer processes depend on time. For recent surveys on the subject we refer to [1, 2, 4]. The aim of this work is to suggest a fast and stable numerical method for a multi-dimensional IHCP with time-dependent coefficients in a parallelepiped. To our knowledge, our result is one of very few papers dealing with multi-dimensional IHCPs with time-dependent coefficients.

Let \( \Omega \) be the open parallelepiped \( (l_1, L_1) \times (l_2, L_2) \times \cdots \times (l_n, L_n) \) with \( l_1, l_2, \ldots, l_n, L_1, L_2, \ldots, L_n \) \( (n \geq 2) \) being given. Denote by \( \partial \Omega \) the boundary of \( \Omega \). For \( t \in (0, T] \), set \( Q_t := \Omega \times (0, t] \), \( S_t := \partial \Omega \times (0, t] \), \( S = S_T \). Suppose that \( \partial \Omega \) is split into three parts \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \), where \( \Gamma_i \cap \Gamma_j = \emptyset, i, j = 1, 2, 3, i \neq j \). We denote \( \Gamma_i \times (0, T] \) by \( S_i, i = 1, 2, 3 \). Consider the problem of
determining $\partial u/\partial N|_{S_3}$ and $u|_{t=0}$ from the system

$$u_t - \sum_{i=1}^{n} (a_i(x,t)u_{x_i})_{x_i} + a(x,t)u = f, \quad (x,t) \in QT,$$

$$u|_{S_1} = \varphi(\xi, t), \quad (\xi, t) \in S_1,$$

$$\frac{\partial u}{\partial N}|_{S_1 \cup S_2} = g(\xi, t), \quad (\xi, t) \in S_1 \cup S_2. \tag{3}$$

Here, $\varphi$ and $g$ are given functions, $\nu$ is the outer normal to $S$, and $\frac{\partial u}{\partial N}|_{S} := \sum_{i=1}^{n} a_i(x,t)u_{x_i}\cos(\nu, x_i)|_{S}$.

In this paper, we assume that the following conditions are satisfied

$$a_i, \ a \in C(\bar{QT}), a_i(x, t) \geq \lambda > 0, \forall i = 1, 2, \ldots, n, \ a(x, t) \geq 0, \forall (x, t) \in \bar{QT}, \tag{4}$$

$$f \in L_2(QT), \ \varphi \in L_2(S_1), \ g \in L_2(S_1 \cup S_2), \tag{5}$$

The problem (1)–(3) is severely ill-posed (see, e.g. [1, 2, 4]). In this paper we shall use the variational method suggested in [3] and the conjugate gradient method (CGM) to this IHCP. The idea is very simple: since the initial condition and the Neumann condition $\partial u/\partial N|_{S_3}$ are not known, we consider them as a control $v$ to minimize the defect functional $J_0(v) = \frac{1}{2}\|u|_{S_1} - \varphi\|_{L_2(S_1)}^2$. The gradient of the defect functional is found via the direct and adjoint problems. Since the optimization problem is still unstable, we have to use a regularization method for it. In fact, we shall use the CGM with a stopping rule proposed by Nemirovskii [7] which has been proved to have optimal order regularization properties. It then comes out that for evaluating the gradient one should first numerically solve the direct and adjoint problems and thus obtain an approximation to the gradient. However, it should be noted that when we discretize the variational problem we get a discretized functional and the gradient of this new one is not the same as that obtained by the above method. In fact, the last is only an approximation to it, and the approximation error becomes more and more significant in the iterative procedure. Besides, since direct discretization of the direct and adjoint problems leads to extremely large systems of algebraic equations, their numerical solutions are extremely expensive. To overcome these difficulties we suggest the following scheme: 1) discretize the direct problem and form a corresponding discretized functional, 2) introduce the discretized adjoint problem and 3) evaluate the gradient of the discretized functional, 4) use the CGM for the discretized variational problem. To avoid the large dimensions of the discretized problems we use a splitting method (see, e.g. [6, 8]) for this purpose. The technique only requires solving one-dimensional problems, the numerical solution of which is easily and directly calculated. The numerical results provided by our method are very good and the techniques seem to be very promising. Also, we make use of Tikhonov regularization to the problem in combination with CGM. It came out that numerical results with or without Tikhonov regularization are of the same quality.

2. Problem setting

In this section we summarize some results on the inverse problem (1)–(3) and its related direct problem. For more details, we refer the reader to [4].

2.1. The direct problem

This part is devoted to a non-homogeneous second boundary value problem for linear parabolic equations. This problem is referred to as the direct one.
Let the conditions (4)–(5) be satisfied. Consider the second boundary value problem for the parabolic equation

\[ u_t - \sum_{i=1}^{n} (a_i(x, t)u_{x_i})_{x_i} + a(x, t)u = f, \quad (x, t) \in Q_T, \]  
\[ u(x, 0) = v_0(x), \quad x \in \Omega, \]  
\[ \frac{\partial u}{\partial N}|_S = g(\xi, t), \quad (\xi, t) \in S, \quad \text{with } g = v_1 \text{ on } S_3. \]  

It was proved in [4] that, if \( v_0 \in L^2(\Omega), g \in L^2(S) \), then there is a unique solution in \( V^{1,0}(Q_T) \) of the problem (6)–(8).

### 2.2. Variational method for the inverse problem

Since the initial condition \( u|_{t=0} \) and the Neumann condition at \( S_3 \) in the problem (1)–(3) are not known, we consider them as a control. We thus reformulate (1)–(3) into the following optimization problem:

**Minimize the functional**

\[ J_0(v) = \frac{1}{2} \| u(v)|_{S_1} - \varphi \|_{L^2(S_1)}^2, \]  

**subject to** (6)–(8). Here we write \( u(v) := u(x, t; v) \) to emphasize the dependence of \( u \) on \( v = (v_0, v_1) \).

Let \( \psi(x, t) := \psi(x, t; v) \) be a solution in \( V^{1,0}(Q_T) \) of the problem adjoint to (6)–(8)

\[ \psi_t = -\sum_{i=1}^{n} (a_i(x, t)\psi_{x_i})_{x_i} - a(x, t)\psi, \quad (x, t) \in Q_T, \]  
\[ \psi(x, T) = 0, \quad x \in \Omega, \]  
\[ \frac{\partial \psi}{\partial N}|_{S_1} = u(v)|_{S_1} - \varphi, \quad \frac{\partial \psi}{\partial N}|_{S_2 \cup S_3} = 0. \]

**Theorem 1.** Let conditions (4)–(5) be satisfied. Then the functional \( J_0 \) is Fréchet differentiable and its gradient can be represented in the form:

\[ \nabla J_0(v) = \left\{ \begin{array}{l} \psi(v)|_{t=0} \\ \psi(v)|_{S_3} \end{array} \right\}. \]

### 3. Discretized inverse problem

To minimize the objective functional \( J_0(v) \) using the CGM, an usual way is to approximate it as well as its gradient by discretized formulas. Although this approach is rather simple for implementing, it can introduce significant approximation errors in evaluating the gradient of the objective functional. Indeed, the errors are caused by: 1) approximation of the direct problem (6)–(8) by a discretized one; 2) numerical integration of the objective functional (9); and 3) approximation of the adjoint problem (10)–(12) by a discretized one.

In order to diminish the approximation errors, we propose the following approach: we first discretize the direct problem and form the corresponding discretized objective functional; then the gradient of the discretized objective functional is formulated based on the discretized adjoint problem. The main advantage of this approach is that the gradient of the discretized objective functional can be exactly computed. The only approximation error is introduced in discretizing the direct problem. The formulation of the discretized problems is presented in the following subsections.
3.1. A splitting finite difference scheme for the direct problem

Subdivide the domain $\Omega$ into elementary cells by the rectangular uniform grid specified by $l_i = x^0_i < x^1_i = h_i < \cdots < x^{N_i}_i = L_i, \ i = 1, 2, \ldots, n$. Here $h_i = (L_i - l_i)/N_i$ is the grid size in the $x_i$-direction, $i = 1, \ldots, n$. The grid vertices are denoted by $x^k = (x^k_1, \ldots, x^k_n)$, where $k = (k_1, k_2, \ldots, k_n)$, $0 \leq k_i \leq N_i$. We also denote by $h = (h_1, \ldots, h_n)$ the vector of spatial grid sizes and $e_i$ the unit vector in the $x_i$-direction, $i = 1, \ldots, n$. In the following, we will make use of the set of the indices of all grid vertices in the domain $\Omega$ (i.e. including boundary vertices).

For the convenience of the presentation, we denote this set by $\Omega_h$, i.e. $\Omega_h = \{ k = (k_1, \ldots, k_n) : 0 \leq k_i \leq N_i, \ \forall i = 1, \ldots, n \}$. For an arbitrary function $u(x, t)$ defined in $Q_T$, we denote by $u^k(t)$ its approximate value at $(x^k, t)$. We also use the notation $u^{k+e_i/2}(t)$ to denote its approximate value at $(x^k + h_i e_i/2, t)$. In order to define grid functions on the boundaries of the domain $\Omega$, we use, for each $i$, the notation $k'_i = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$ to represent the index of a grid point on the plane $\{ x_i = t_i \}$ or $\{ x_i = L_i \}$. For a function $g(\xi)$ defined on one of these planes, we denote by $g^{k_i}$ its approximate value at $\xi = (x^k_1, \ldots, x^k_i, \ldots, x^k_n)$.

By approximating the spatial derivatives in (6) and (8) by finite difference quotients and replacing the initial condition (7) by a grid function, we obtain the following difference-differential problem

$$
\begin{align*}
\frac{dU}{dt} + \sum_{i=1}^{n} A_i(t) U(t) &= F(t), \\
U(t = 0) &= U_0,
\end{align*}
$$

(14)

where $U(t)$ is the grid function defined by $U(t) = \{ u^k(t), \ k \in \Omega_h \}$ and $A_i(t)$ is the finite-dimensional operator specified by $A_i(t) U(t) = \{ A_i(t) U(t) \}_{k \in \Omega_h}$ with

$$
\begin{cases}
A_i(t) U(t)_{k} = \frac{a^k(t) u^k(t)}{n} + \left\{
\begin{array}{ll}
a_i^{k+e_i/2}(t)[u^k(t)-u^{k-e_i}(t)] - a_i^{k+e_i/2}(t)[u^{k+e_i}(t)-u^k(t)], & 1 \leq k_i \leq N_i - 1, \\
0, & k_i = 0, \\
a_i^{k+e_i/2}(t)[u^{k+e_i}(t)-u^{k-e_i}(t)], & k \in \Omega_h : k_i = N_i,
\end{array}
\right.
\end{cases}
$$

(15)

The right hand side function $F(t) = \{ F^k(t), \ k \in \Omega_h \}$ is specified by

$$
F^k(t) = \left\{
\begin{array}{ll}
f^k(t), & k \in \Omega_h : 1 \leq k_i \leq N_i - 1, \ \forall i = 1, \ldots, n, \\
f^k(t) + g^{k_i}(t)/h_i, & k \in \Omega_h : k_i = 0 \ or \ k_i = N_i, \ i = 1, \ldots, n.
\end{array}
\right.
$$

(16)

Finally, the initial condition $U_0$ is defined by $U_0 = \{ u^0_k, \ k \in \Omega_h \}$.

It follows from (15) that for each value of $t$, the operators $A_i(t)$, $i = 1, \ldots, n$, are semi-positive definite. So we can apply splitting schemes for solving the problem (14). Consider a uniform time grid specified by $0 = t_0 < t_1 < \cdots < t_m < \cdots < t_{N_i} = T$. We denote by $A_i^m = A_i(t_{m} + \tau/2)$ and $F^m = F(t_{m} + \tau/2)$ for $m = 0, \ldots, N_i - 1$, where $\tau = T/N_i$. In this paper, we use a two-circle component-by-component splitting scheme. The idea of this scheme is that we replace (14) by a sequence of $2n$ one-dimensional linear systems associated with $A_i$, $i = 1, \ldots, n$ (the first $n$ systems are solved in forward order while the last $n$ ones are solved reversely). The scheme is written as (6), pp. 166-167:

$$
\begin{align*}
\left( E + \frac{\tau}{4} A_i^m \right) U^{m+i/(2n)} &= \left( E - \frac{\tau}{4} A_i^m \right) U^{m+(i-1)/(2n)}, & i = 1, \ldots, n - 1, \\
\left( E + \frac{\tau}{4} A_i^m \right) U^{m+i/2} - \frac{\tau}{2} F^m &= \left( E - \frac{\tau}{4} A_i^m \right) U^{m+(i-1)/2}, \\
\left( E + \frac{\tau}{4} A_i^m \right) U^{m+(i+1)/(2n)} &= \left( E - \frac{\tau}{4} A_i^m \right) \left( U^{m+1/2} + \frac{\tau}{2} F^m \right), \\
\left( E + \frac{\tau}{4} A_i^m \right) U^{m+i-1/(2n)} &= \left( E - \frac{\tau}{4} A_i^m \right) U^{m+i-1/(2n)}, & i = n - 1, n - 2, \ldots, 1
\end{align*}
$$

(17)
for \( m = 0, \ldots, N_t - 1 \), and
\[
U^0 = U_0. \tag{18}
\]

In (17), \( E \) represents the identity matrix associated with the coefficient matrices \( A^m_i \). We remark that the spatial index of the grid functions \( U(t) \) and \( F(t) \) has been ignored in (17) for simplifying the notation. The reader should be aware that \( U^m \) and \( F^m \) are vectors for each superscript \( m \).

Since in the following we only consider discretized problems and for clarity of the presentation, we denote by \( U = \{U^{k,m}, \ k \in \bar{\Omega}_h; \ m = 1, \ldots, N_t\} \) the solution of the splitting scheme (17). We also use the notation \( U = \{U^m, \ m = 1, \ldots, N_t\} \) when it is not necessary to mention the spatial index.

It was proved in [6] that this splitting scheme is absolutely stable and approximates the solution of problem (14) with second order accuracy in \( t \), i.e. it is of the same order as Crank-Nicholson’s scheme, provided that the coefficients and the right hand side function are smooth. The main advantage of the splitting scheme is that its equations can be solved as one-dimensional three-diagonal linear systems. Solutions of these systems can be exactly and directly calculated [8].

The splitting scheme (14)–(17) can be rewritten in the following compact form
\[
\begin{cases}
U^{m+1} = A^{m}U^{m} + \tau B^{m}F^{m}, \ m = 0, \ldots, N_t - 1, \\
U^0 = U_0,
\end{cases}
\tag{19}
\]
with \( A^{m} = \Lambda_0^m \Lambda_2^m \cdots \Lambda_i^m \Lambda_n^m \cdots \Lambda_2^m \Lambda_1^m, B^{m} = \Lambda_0^m \Lambda_2^m \cdots \Lambda_i^m \Lambda_n^m \), and \( \Lambda_i^m = (E + \frac{\tau}{2} A_i^m)^{-1} (E - \frac{\tau}{2} A_i^m) \), \( i = 1, \ldots, n \).

### 3.2. Discretized inverse problem

For convenience of the presentation, we denote by \( \Gamma_i^h \) the set of grid points at the corresponding boundary \( \Gamma_i \) of \( \Omega \). In this section, we consider the approximate values at the grid vertices of the functions \( v_0(x) = u(x, t = 0), v_1 = g(\xi, t)|_{S_3} \) as the unknown variables that need to be estimated. We denote these vectors respectively by \( V_0 = \{v_0^k, \ k \in \bar{\Omega}_h\} \) and \( V_1 = \{v_1^{k,m}, \ k \in \Gamma_i^h, \ m = 0, \ldots, N_t - 1\} \). We also denote by \( V = \{V_0, V_1\} \) the pair of the unknown variables.

The measured data \( \varphi(x, t), (x, t) \in S_1 \), in the continuous formulation is replaced by the discrete measured data \( \varphi = \{\varphi^{k,m}, \ k \in \Gamma_i^h, \ m = 1, \ldots, N_t\} \) at the grid points. Note that we do not take into account the measurement at the initial time instant in the discretized problem setup. In the discretized inverse problem, the unknown variable \( V \) is estimated by minimizing the following discretized objective function
\[
J^h_{\gamma,\tau}(V) = \frac{1}{2} \sum_{k \in \Gamma_i^h} \sum_{m=1}^{N_t} \left[ U^{k,m}(V) - \varphi^{k,m} \right]^2 + \frac{1}{2} \gamma^2 \|V\|^2, \tag{20}
\]
where \( U(V) \) is the solution to the splitting scheme (19), \( \|V\|^2 = \sum_{k \in \Omega_h} (V_k)^2 + \sum_{k \in \Omega_h} \sum_{m=0}^{N_t-1} (V_1^{k,m})^2 \).

### 3.3. Gradient of the discretized objective function

**Theorem 2.** The derivatives of the discretized objective function (20) subject to the splitting scheme (19) are given by the formulas
\[
\frac{\partial J^h_{\gamma,\tau}(V)}{\partial V_0^k} = \left( (A_0^k)^* \eta_0 \right)^k + \gamma^2 V_0^k, \ k \in \bar{\Omega}_h, \tag{21}
\]
\[
\frac{\partial J^h_{\gamma,\tau}(V)}{\partial V_1^{k,m}} = \frac{\tau}{\kappa^2} \left( (B^m)^* \eta_m \right)^k + \gamma^2 V_1^{k,m}, \ k \in \Gamma_i^h, \ m = 0, \ldots, N_t - 1, \tag{22}
\]
Here \((A^m)^*\) and \((B^m)^*\) are respectively the adjoint matrices of \(A^m\) and \(B^m\), \(h^3\) is the grid size on \(Γ_3\), and \(η\) is the solution to the adjoint problem

\[
\begin{align}
η^{N_t-1} & = ζ^N_t, \\
η^{m-1} & = (A^m)^*γ^m + ζ^m, m = N_t - 1, \ldots, 1,
\end{align}
\]

where \(ζ^m = \{ζ^k,m, k ∈ \mathbb{Ω}_h\}\) with

\[
ζ^k,m = \begin{cases} 
U^{k,m}(V) − φ^{k,m}_l, k ∈ Γ^h_t, \\
0, \text{ otherwise},
\end{cases}
\]

It can be proved that the solution of the adjoint problem (23) as well as the derivatives of the discretized objective function given in (21)–(22) are obtained by solving a sequence of one-dimensional systems of linear equations as in the splitting scheme (17).

### 3.4. Conjugate gradient algorithm for the discretized inverse problem

To apply the conjugate gradient algorithm [7] to the discretized inverse problem, we define the operator \(A_{l,h}\) by \(A_{l,h}V := \{U^{k,m}, k ∈ Γ^h_t, m = 1, \ldots, N_t\}\) where \(U\) is the solution of the splitting scheme (19) with \(U^0 = V_0\) and \(F = \{F^{k,m}(V_1)\}\) being given by

\[
F^{k,m}(V_1) = \begin{cases} 
\frac{1}{η}V^{k,m}, k ∈ Γ^h_t, m = 0, \ldots, N_t - 1, \\
0, \text{ otherwise},
\end{cases}
\]

It can be proved that \(A_{l,h}\) is a linear operator.

We note that the conjugate gradient algorithm [7] requires to solve two direct and one adjoint problems in each iteration. However, by using the residual of the objective function, we can reduce this to two problems as shown in the following algorithm for the discretized inverse problem (20) subject to (19):

1. **Initialization:** 1.1) Given an initial guess \(V^{(0)}\), 1.2) Calculate the residual \(r^{(0)} = U(V^{(0)})|Γ^h_t × \{1, \ldots, N_t\} − φ_l\) by solving the splitting scheme (17) with \(V = V^{(0)}\), 1.3) Calculate \(J^{h,τ}_l(V^{(0)}) = \frac{1}{2}∥r^{(0)}∥^2 + \frac{1}{2}γ^2∥V^{(0)}∥^2\), 1.4) Calculate the gradient \(r^{(0)} = \nabla J^{h,τ}_l(V^{(0)})\) by solving the adjoint problem (23) with \(ζ\) equal to \(r^{(0)}\) on \(Γ^h_t × \{0, \ldots, N_t\}\) and using the formulas (21) and (22), 1.5) Define the initial search direction \(d^{(0)} := \{d^{(0)}_0, d^{(0)}_1\} = −r^{(0)}\).

2. For \(i = 0, 1, 2, \ldots\)
   2.1) Solve the splitting scheme (19) with \(U_0 = d^{(i)}_0\) and \(F = F(d^{(i)}_1)\) in (24). Calculate \(α_k = ∥r^{(i)}∥^2 / (∥A_{l,h}d^{(i)}∥^2 + γ^2∥d^{(i)}∥^2)\)
   2.2) Update the solution \(V^{(i+1)} = V^{(i)} + α_id^{(i)}\)
   2.3) Calculate the residual \(r^{(i+1)} = r^{(i)} + α_iA_{l,h}d^{(i)}\)
   2.4) Calculate the gradient \(r^{(i+1)}\) by solving the adjoint problem (23) with \(ζ\) equal to \(r^{(i+1)}\) on \(Γ^h_t × \{0, \ldots, N_t\}\) and using the formulas (21) and (22).
   2.5) \(J^{h,τ}_l(V^{(i+1)}) = \frac{1}{2}∥r^{(i+1)}∥^2 + \frac{1}{2}γ^2∥V^{(i+1)}∥^2\), \(β_{i+1} = ∥r^{(i+1)}∥^2 / ∥r^{(i)}∥^2\)
   2.6) Update the search direction \(d^{(i+1)} = β_{i+1}d^{(i)} − r^{(i+1)}\)
   2.7) Stopping check: if the stopping rule is satisfied, stop; otherwise, return to step 2.1

End

The optimal iteration index is determined as the first \(i = i_{\text{max}}\) such that the stopping rule \(∥A_{l,h}V^{(i)} − φ_l∥ \leq γ_1ε\) (with \(γ_1\) slightly greater than 1) is fulfilled. In addition we stop if \(∥r^{(i)}∥ < ε\).
4. Numerical Examples

We have tested our method for several problems. However, due to lack of space, we illustrate the performance of the CGM for the discretized inverse problem for a two-dimensional numerical example with the initial condition being given. We take the domain $Q_T = \{(x_1, x_2, t) \in (0.75, 1) \times (0, 1) \times (0, 1)\}$. We consider the problem of estimating the Neumann boundary condition (heat flux) $\partial u / \partial N$ at the plane $S_3 = \{x_1 = 1\} \times \{0 < t \leq 1\}$ from the measured data at the opposite plane $S_1 = \{x_1 = 0.75\} \times \{0 < t \leq 1\}$. The discretized grid sizes used in this example are $h = (0.01, 0.01)$ and $\tau = 0.01$ resulting in $26 \times 101 \times 101$ grid points.

The algorithm is tested for two cases. In the first case, the Tikhonov regularization term is not taken into account. In the second case, we add the regularization term with the regularization parameter $\gamma = 10^{-5}$. This parameter is only empirically chosen. The algorithm is run for at most 40 iterations. The stopping parameter $\gamma_1$ is chosen to be 1.05 in the case without Tikhonov regularization. In the other case, the algorithm is stopped when the relative reduction of the objective function is less than $10^{-4}$ or its gradient is less than $10^{-4}$. Again, these parameters are chosen empirically. The algorithm is initiated at the initial guess $V^{(0)} = 0$.

In the example, the exact solution to the direct problem (6)–(8) is taken as

$$u(x, t) = 10(x_1^2 + 20x_1 + 300)x_2^2(1 - x_2)^2/(t + 1), \quad (x, t) \in Q_1.$$  (25)

The coefficients $a_i$ and $a$ are chosen as $a_1(x, t) = a_2(x, t) = 10^{-2}(10 + x_1 t)$, $a(x, t) = 0$. In this case we have $f(x, t) = -10(x_1^2 + 20x_1 + 300)x_2^2(1 - x_2)^2/(t + 1)^2 - 0.1(4x_1 t + 20 t + 20)x_2^2(1 - x_2)^2/(t + 1) - 0.1(10 + x_1 t)(x_1^2 + 20x_1 + 300)(12x_2^2 - 12x_2 + 2)/(t + 1)$ and $\mu_0(x) = 10(x_1^2 + 20x_1 + 300)x_2^2(1 - x_2)^2$.

Here, since we only work with the discretized problems, the data is calculated by adding a random noise of magnitude of 0.01 to the numerical solution of the direct problem. Note that our numerical calculations have indicated that the splitting scheme is about ten times faster than Crank-Nicholson’s scheme for the two-dimensional problem. This advantage is more clear in higher dimensions [8].

The algorithm is stopped after 13 iterations without Tikhonov regularization. When Tikhonov regularization was taken into account, the algorithm reached the maximum number of iterations allowed (Figure 1). The estimated values of $g|_{S_3}$ are depicted in Figure 1 while the difference between the estimated and the exact values are depicted in Figure 2. It is clear from Figure 2 that the estimation error $\|g^{estimated}(1, \cdot, t) - g^{exact}(1, \cdot, t)\|_{L^2(0, 1)}$ is small in approximately 90% of the considered time interval and it is large in the remaining interval. This phenomenon has also been reported in the literature [2, 4].

Comparing the results with and without Tikhonov regularization, we can see that the algorithm without Tikhonov regularization stops much faster. However, the accuracy of the two cases are almost the same. Testing the algorithm, we observe that the accuracy of the estimate depends on the distance between the measurement plane and the plane at which the heat flux is estimated. The closer the distance is, the better the estimate will be.

5. Conclusions

This paper proposed a conjugate gradient algorithm for a multi-dimensional IHCP with time-dependent coefficients, based on a splitting method for the direct problem. The problem is formulated as an optimization problem using the variational approach. To avoid approximation errors in evaluating the gradient of the objective functional, we formulated the discretized objective function whose gradient can be calculated in an exact and direct way.

The two-dimensional numerical results showed that the estimation is good except at some final time instants. This can be improved if we use the concept of ”future time” ([2, 4, 5]) by taking more Cauchy data in the time direction. However, to save the space of the paper
Comparing the CGM with and without Tikhonov regularization, we saw that the results in the two cases are almost the same. However, without Tikhonov regularization term the algorithm converges faster.

References
[1] Alifanov O M, Artiukhin E A and Rumiantsev S V 1995 Extreme Methods for Solving Ill-Posed Problems With Applications to Inverse Heat Transfer Problems (New York: Begell House Publishers).
[2] Beck J V, Blackwell B and Clair Jr C R St 1985 Inverse Heat Conduction: Ill-Posed Problems (New York: Wiley-Interscience).
[3] Dinh Nho Hao 1992 A noncharacteristic Cauchy problem for linear parabolic equations II : A variational method Numer. Funct. Anal. and Optimiz. 13(5&6) 541–564.
[4] Dinh Nho Hao 1998 Methods for Inverse Heat Conduction Problems (Frankfurt/Main: Peter Lang Verlag).
[5] Dinh Nho Hao and Reinhardt H J 1998 Gradient methods for inverse heat conduction problems Inverse Problems in Engineering 6(3) 177–211.
[6] Marchuk G I 1975 Methods of Numerical Mathematics (New York: Springer-Verlag).
[7] Nemirovskii A S 1986 The regularizing properties of the adjoint gradient method in ill-posed problems U. S. S. R. Comput. Maths. Math. Phys. 26(2) 7–16.
[8] Nguyen Trung Thành 2007 Infrared Thermography for the Detection and Characterization of Burried Objects, PhD Thesis, Vrije Universiteit Brussel.