GLOBAL EXISTENCE OF STRONG SOLUTION TO NON-ISOTHERMAL IDEAL GAS SYSTEM

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ABSTRACT. This paper aims to establish the global existence of strong solutions to a non-isothermal ideal gas model. We first show global well-posedness in the Sobolev space \(H^2(\mathbb{R}^3)\) by using energy estimates. We then prove the global well-posedness for small-data solutions in the critical Besov space by using Banach’s fixed point theorem.

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1. INTRODUCTION

Starting from a given free energy, Lai-Liu-Tarfulea [19] established a general framework for deriving non-isothermal fluid models by combining classical thermodynamic laws and the energetic variational approach(see [14, 17]). As an application, three full non-isothermal systems (the non-isothermal ideal gas, non-isothermal porous media, and non-isothermal generalized porous media equations) are established based on three specific free energies. What is more, under some appropriate assumption on the conductivity coefficient \(\kappa_3\), a maximum/minimum principle is developed for the first two models by adapting an idea originally from the work [23]. These maximum/minimum principles establish the positivity of the absolute temperature, which implies the thermodynamic consistency of the corresponding models.

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However, [19] does not address the long time behavior (existence and uniqueness) of the solution to the non-isothermal models mentioned, while it is the core theory for partial differential system. At present, there are many results on the existence and behavior of weak solutions to various non-isothermal fluid models; see [12, 8, 10, 13, 22] for the Navier-Stokes-Fourier system, which is a powerful generalization of the classical Navier-Stokes equations and is used to model thermodynamic fluid flow, [7] for the non-isothermal general Ericksen-Leslie system, [16] for the non-isothermal Poisson-Nernst-Planck-Fourier system, and [20] for the Brinkman-Fourier system with ideal gas equilibrium.

This paper aims to study the global well-posedness of the following non-isothermal ideal gas system in $\mathbb{R}^3$:

\[
\begin{aligned}
\partial_t \rho &= \kappa_1 \Delta (\rho \theta), \\
\kappa_2 (\rho \theta)_t - \kappa_1 (\kappa_1 + \kappa_2) \nabla \cdot (\theta \nabla (\rho \theta)) &= \nabla \cdot (\kappa_3 \nabla \theta).
\end{aligned}
\]

For the reader’s convenience, we briefly sketch the construction of (1.1). As can be seen in the model, the main unknown variables are:

1. a non-negative measurable function $\rho = \rho(t, x)$ which denotes the mass density;
2. a positive measurable function $\theta = \theta(t, x)$ representing the absolute temperature.

In addition, a vector field $u = u(t, x)$ denoting the velocity field of the fluid will be used as an intermediate variable.

For an ideal gas, we have the following definition of free energy

\[
\Psi(\rho, \theta) = \kappa_1 \theta \rho \ln \rho - \kappa_2 \rho \theta \ln \theta.
\]

Then the (specific) entropy of the system, denoted by $\eta$, and the (specific) internal energy, denoted by $e$, are connected to the free energy $\Psi$ by the standard Helmholtz relation (see formula (2.5.26) in the classical book [5])

\[
\begin{aligned}
\eta(\rho, \theta) &:= -\partial_\theta \Psi, \\
e(\rho, \theta) &:= \Psi - \partial_\theta \Psi \theta = \Psi + \eta \theta, \\
\eta_\theta &= \frac{\kappa_2 \rho}{\theta}.
\end{aligned}
\]

The total energy and total dissipation are then chosen to be

\[
E^{total} = \int_{\Omega_t} \Psi(\rho, \theta) dx, \quad D^{total} = \frac{1}{2} \int_{\Omega_t} \rho u^2 dx.
\]
Employing the energetic variational approach then establishes the following Darcy type diffusion law

\[
\begin{align*}
    p &= \Psi \rho - \rho = \kappa_1 \rho \theta, \\
    \nabla p &= -\rho u, \\
    \partial_\rho p &= \kappa_1 \rho.
\end{align*}
\]  

(1.4)

We remark that, according to [2, 21], the internal energy and pressure are both linearly proportional to the product of temperature and density. It is easy to verify this fact by combining (1.2), (1.3) and (1.4).

Now, we rewrite the internal energy function in terms of the new state variables \( \rho, \eta \), yielding

\[
(1.5) \quad e_1(\rho, \eta) = e(\rho, \theta(\rho, \eta)),
\]

which then implies

\[
(1.6) \quad \begin{cases} 
    e_{1\rho} = \theta, & e_{1\eta} = \Psi \rho, \\
    \nabla p = \rho \nabla e_{1\rho} + \eta \nabla e_{1\eta}.
\end{cases}
\]

Recalling the continuity equation for a closed system

\[
(1.7) \quad \rho_t + \nabla \cdot (\rho u) = 0,
\]

combine this with the two classical thermodynamic laws, the first of which relates the rate of change of the internal energy with dissipation and heat

\[
(1.8) \quad \frac{de}{dt} = \nabla \cdot W + \nabla \cdot q,
\]

where \( W \) denotes the amount of thermodynamic work done by the system on its surroundings and \( q \) denotes the quantity of energy supplied to the system as heat. The second thermodynamic law describes the evolution of the entropy

\[
(1.9) \quad \partial_t \eta + \nabla \cdot (\eta u) = \nabla \cdot \left( \frac{q}{\theta} \right) + \Delta,
\]

where \( \Delta \geq 0 \) denotes the rate of entropy production, and Fourier’s law yields

\[
(1.10) \quad q = \kappa_3 \nabla \theta,
\]
where \(\kappa_3\) denotes the material conductivity (which may depend on \(\rho\) and \(\theta\)). Combining (1.6), (1.7), (1.8), (1.9), and (1.10), we obtain

\[
\frac{de_1(\rho, \eta)}{dt} = e_{1\rho} \rho_t + e_{1\eta} \eta_t
\]

\[
= e_{1\rho} (\rho \cdot (\rho u)) + e_{1\eta} (\eta \cdot (\eta u)) + \nabla \cdot \left( \frac{q}{\theta} \right)
\]

(1.11)

\[
= - \nabla \cdot (e_{1\rho} \rho u + e_{1\eta} \eta u) + (\rho \nabla e_{1\rho} + \eta \nabla e_{1\eta}) \cdot u + \theta \nabla \cdot \left( \frac{q}{\theta} \right) + \theta \Delta
\]

\[
= \nabla \cdot W + \nabla p \cdot u + \nabla \cdot q - \frac{q}{\theta} \cdot \nabla \theta + \theta \Delta
\]

\[
= \nabla \cdot W - \rho u^2 + \nabla \cdot q - \frac{\kappa_3 |\nabla \theta|^2}{\theta} + \theta \Delta.
\]

Therefore

\[
\begin{cases}
W = -(e_{1\rho} \rho + e_{1\eta} \eta) u,
\Delta = \frac{1}{\theta} \left( \rho |u|^2 + \frac{\kappa_3 |\nabla \theta|^2}{\theta} \right),
\end{cases}
\]

(1.12)

which in turn gives

\[
\eta_t + \nabla \cdot (\eta u) = \eta_\theta (\theta_t + u \cdot \nabla \theta) + \eta_\rho (\rho_t + u \cdot \nabla \rho) + \eta \nabla \cdot u
\]

\[
= \eta_\theta (\theta_t + u \cdot \nabla \theta) + \eta_\rho (-\rho \nabla \cdot u) + \eta \nabla \cdot u
\]

\[
= \eta_\theta (\theta_t + u \cdot \nabla \theta) + (\eta - \eta_\rho \rho) \nabla \cdot u
\]

(1.13)

\[
= \eta_\theta (\theta_t + u \cdot \nabla \theta) + \eta_p \nabla \cdot u
\]

\[
= \nabla \cdot \left( \frac{q}{\theta} \right) + \Delta
\]

\[
= \nabla \cdot \left( \frac{q}{\theta} \right) + \frac{1}{\theta} \left( \rho |u|^2 + \frac{q \cdot \nabla \theta}{\theta} \right),
\]

which finally yields

(1.14)

\[
\eta_\theta (\theta_t + u \cdot \nabla \theta) + \eta_p \nabla \cdot u = \nabla \cdot \left( \frac{q}{\theta} \right) + \frac{1}{\theta} \left( \rho |u|^2 + \frac{q \cdot \nabla \theta}{\theta} \right).
\]

Combining (1.14), (1.3) and (1.4) allows us to conclude that

\[
\frac{\kappa_2 \rho}{\theta} (\theta_t + u \cdot \nabla \theta) + \kappa_1 \rho \nabla \cdot u
\]

(1.15)

\[
= \nabla \cdot \left( \frac{\kappa_3 \nabla \theta}{\theta} \right) + \frac{1}{\theta} \left( -\kappa_1 \nabla (\rho \theta) \cdot u + \frac{\kappa_3 |\nabla \theta|^2}{\theta} \right),
\]

so that

(1.16)

\[
\kappa_2 (\rho \theta)_t - \kappa_1 (\kappa_1 + \kappa_2) \nabla \cdot (\theta \nabla (\rho \theta)) = \nabla \cdot (\kappa_3 \nabla \theta),
\]

which completes the derivation of the non-isothermal ideal gas model (1.1).

Our main goal is to establish well-posedness for the system (1.1). Motivated by
similar works on the classical Navier-Stokes equations ([6, 9]), we first motivate our choice of working spaces. We observe that (1.1) is invariant under the transformation
\[ (\rho(t, x), \theta(t, x)) \rightarrow (\rho(\lambda^2 t, \lambda x), \theta(\lambda^2 t, \lambda x)), \]
\[ (\rho_0(x), \theta_0(x)) \rightarrow (\rho(\lambda x), \theta(\lambda x)). \]

**Definition 1.1.** A function space \( E \subset S'(\mathbb{R}^3) \times S'(\mathbb{R}^3) \) is called a critical space if the associated norm is invariant under the transformation (1.17).

Obviously \( \dot{H}^{3/2} \times \dot{H}^{3/2} \) is a critical space for the initial data, but \( \dot{H}^{3/2} \) is not included in \( L^\infty \). We cannot expect to get \( L^\infty \) control on the density and the temperature by taking \( (\rho_0 - 1, \theta_0 - 1) \in \dot{H}^{3/2} \times \dot{H}^{3/2} \). Moreover, the product between functions does not extend continuously from \( \dot{H}^{3/2} \times \dot{H}^{3/2} \) to \( \dot{H}^{3/2} \), so that we will run into difficulties when estimating the nonlinear terms. Similar to the Navier-Stokes system studied in [6], we could use homogeneous Besov spaces \( \dot{B}_2^{3/2}(\mathbb{R}^3) \) (defined in [1], Chapter 2). \( \dot{B}_2^{3/2} \) is an algebra embedded in \( L^\infty \) which allows us to control the density and temperature from above without requiring more regularity on derivatives of \( \rho_0 \) and \( \theta_0 \).

Our first result proves global well-posedness for (1.1) when the initial data is close to a stable equilibrium \( (\underline{\rho}, \underline{\theta}) \) in the subcritical space \( H^2 \times H^2 \). The working space \( X(T) \) is defined by the norm
\[ \|u\|_{X(T)} := \sup_{0 \leq \tau \leq T} \|u(\tau)\|_{H^2}^2 + \int_0^T \left( \|\nabla u\|_{H^2}^2 + \|\partial_t u\|_{H^1}^2 \right) d\tau, \]
for any distribution \( u \) and \( T > 0 \).

**Theorem 1.2.** Let \( \underline{\rho}, \underline{\theta} > 0 \) be fixed constants. There exist two positive constants \( c \) and \( M \) such that for all \( \rho_0 \) and \( \theta_0 \) where \( (\rho_0 - \underline{\rho}, \theta_0 - \underline{\theta}) \in H^2 \times H^2 \) and
\[ \|\rho_0 - \underline{\rho}\|_{H^2} + \|\theta_0 - \underline{\theta}\|_{H^2} \leq c, \]
the system (1.1) has a unique global solution \( (\rho, \theta) \) with \( (\rho - \underline{\rho}, \theta - \underline{\theta}) \in X(T) \) for all \( T > 0 \). Moreover, if we define \( \tilde{\rho} := \rho - \underline{\rho} \) and \( \tilde{\theta} := \theta - \underline{\theta} \), then
\[ \| (\tilde{\rho}, \tilde{\theta}) \|_{X(T)} \leq \frac{1}{2} c. \]

Based on the above scaling analysis, it will suffice to prove an \( H^2 \) energy estimate for \( \rho \) and \( \theta \).

Our second result then establishes the existence and uniqueness of a solution to the system (1.1) for initial data close to a stable equilibrium \( (\underline{\rho}, \underline{\theta}) \) in the critical space \( \dot{B}_2^{3/2} \times \dot{B}_2^{3/2} \). For convenience, we assume that \( \underline{\rho} = \underline{\theta} = 1 \). The working space \( E(t) \) is then defined by
\[ E(T) := \left\{ u \in C \left( [0, T], \dot{B}_2^{3/2} \right), \quad \nabla^2 u \in L^1 \left( 0, T; \dot{B}_2^{3/2} \right) \right\}, \quad T > 0. \]
Theorem 1.3. There exist two positive constants $c$ and $M$ such that for all $(a_0, \tilde{\theta}_0) \in \dot{B}^{3/2}_{2,1} \times \dot{B}^{3/2}_{2,1}$ with
\begin{equation}
\|a_0\|_{\dot{B}^{3/2}_{2,1}} + \|\tilde{\theta}_0\|_{\dot{B}^{3/2}_{2,1}} \leq \frac{c}{2M},
\end{equation}
the system (1.1) has a unique global solution $(\rho, \theta)$ with initial data $\theta_0 = \tilde{\theta}_0 + 1$ and $\rho_0 = 1/(1 + a_0)$. Moreover, if we define $\rho = 1/(1 + a)$ and $\tilde{\theta} = \theta - 1$, then for all $T > 0$
\begin{equation}
\|(a, \tilde{\theta})\|_{E(T)} \leq c.
\end{equation}

The rest of the paper unfolds as follows. Section 2 will present some basic tools in Fourier analysis: Littlewood-Paley decomposition and paraproduct calculus in Besov spaces. Section 3 will prove the global existence and uniqueness result in Sobolev spaces (Theorem 1.2). Section 4 will prove the global well-posedness result in the critical Besov space by using Banach’s fixed point Theorem.

2. Notation and preliminaries

For any $1 \leq p \leq \infty$ and measurable $f : \mathbb{R}^n \to \mathbb{R}$, we will use $\|f\|_{L^p(\mathbb{R}^n)}$, $\|f\|_{L^p}$ or simply $\|f\|_p$ to denote the usual $L^p$ norm. For a vector valued function $f = (f^1, \cdots, f^m)$, we still denote $\|f\|_p := \sum_{j=1}^m \|f^j\|_p$.

For any $0 < T < \infty$ and any Banach space $\mathcal{B}$ with norm $\| \cdot \|_{\mathcal{B}}$, we will use the notation $C([0, T], \mathcal{B})$ or $C^0_t \mathcal{B}$ to denote the space of continuous $\mathcal{B}$-valued functions endowed with the norm
\[ \|f\|_{C([0, T], \mathcal{B})} := \max_{0 \leq t \leq T} \|f(t)\|_{\mathcal{B}}. \]

Also for $1 \leq p \leq \infty$, we define
\[ \|f\|_{L^p_t(\mathcal{B})} := \| \|f(t)\|_{\mathcal{B}} \|_{L^p_t([0, T])}. \]

We shall adopt the following convention for the Fourier transform:
\[ \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} dx; \]
\[ f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\cdot\xi} d\xi. \]

For $s \in \mathbb{R}$, the fractional Laplacian $|\nabla|^s$ then corresponds to the Fourier multiplier $|\xi|^s$ defined as
\[ \langle \nabla \rangle^s \hat{f}(\xi) = |\xi|^s \hat{f}(\xi). \]
whenever it is well-defined. For $s \geq 0$, $1 \leq p < \infty$, we define the semi-norm and norms:

\[
\|f\|_{\dot{W}^{s,p}} = \|\nabla^s f\|_p, \\
\|f\|_{W^{s,p}} = \|\nabla^s f\|_p + \|f\|_p.
\]

When $p = 2$ we denote $\dot{H}^s = \dot{W}^{s,2}$ and $H^s = W^{s,2}$ in accordance with the usual notation.

For any two quantities $X$ and $Y$, we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant $C$ on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim Z_1, Z_2, \ldots, Z_k Y$ if $X \leq CY$ and the constant $C$ depends on the quantities $Z_1, \ldots, Z_k$.

For any two quantities $X$ and $Y$, we shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

We will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let $\varphi_0$ be a radial function in $C^\infty_c(\mathbb{R}^n)$ and satisfy

\[
0 \leq \varphi_0 \leq 1, \quad \varphi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \varphi_0(\xi) = 0 \text{ for } |\xi| \geq 7/6.
\]

Let $\varphi(\xi) := \varphi_0(\xi) - \varphi_0(2\xi)$ which is supported in $\frac{1}{2} \leq |\xi| \leq \frac{7}{6}$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

\[
\widehat{S_j f}(\xi) = \varphi_0(2^{-j}\xi) \hat{f}(\xi), \\
\widehat{\Delta_j f}(\xi) = \varphi(2^{-j}\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n.
\]

We will denote $P_{>j} = I - S_j$ ($I$ is the identity operator) and for any $-\infty < a < b < \infty$, denote $P_{[a,b]} = \sum_{a \leq j \leq b} \Delta_j$. Sometimes for simplicity of notation (and when there is no obvious confusion) we will write $f_j = \Delta_j f$ and $f_{a \leq j \leq b} = \sum_{j=a}^b f_j$. By using the support property of $\varphi$, we have $\Delta_j \Delta_{j'} = 0$ whenever $|j - j'| \geq 1$.

Thanks to the above Littlewood-Paley decomposition, a number of functional spaces can be characterized. Let us give the definition of homogeneous Besov spaces at first.

**Definition 2.1.** For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^n)$, we set

\[
\|u\|_{\dot{B}^{s,p}_r(\mathbb{R}^n)} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j u\|_{L^p}^r \right)^{\frac{1}{r}},
\]

with the usual modification if $r = \infty$. 
We then define the Besov space by $\dot{B}^s_{p,r} = \{ u \in \mathcal{S}'(\mathbb{R}^3), \| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} < \infty \}$. In the following, for convenience of notations, we always use $\dot{B}^s_{p,r}$ instead of $\dot{B}^s_{p,r}(\mathbb{R}^3)$ and similar notations for other norms. Let us now state some classical properties for the Besov spaces without proof.

**Proposition 2.2.** The following properties hold:
1) Derivatives: we have
$$\| \nabla u \|_{\dot{B}^{s-1}_{p,r}} \leq C \| u \|_{\dot{B}^s_{p,r}}.$$  

2) Sobolev embedding: If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^s_{p_2,r_2}.$

If $s_1 > s_2$ and $1 \leq p, r_1, r_2 \leq +\infty$, then $\dot{B}^s_{p,r_1} \hookrightarrow \dot{B}^s_{p,r_2}.$

3) Algebraic property: for $s > 0$, $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra.

4) Real interpolation: $\left( \dot{B}^{s_1}_{p_{1,r_1}}, \dot{B}^{s_2}_{p_{2,r_2}} \right)_{\theta,r'} = \dot{B}^{s_1+(1-\theta)s_2}_{p,r'},$

We recall some product laws in Besov spaces coming directly from the paradifferential calculus of J. M. Bony (see[4]).

**Proposition 2.3.** We have the following product laws:
$$\| uv \|_{\dot{B}^s_{p,r}} \lesssim \| u \|_{\dot{B}^s_{p,r}} \| v \|_{\dot{B}^s_{p,r}} + \| v \|_{L^\infty} \| u \|_{\dot{B}^s_{p,r}} \quad \text{if} \quad s > 0,$$
$$\| uv \|_{\dot{B}^s_{p,r}} \lesssim \| u \|_{\dot{B}^{s_1}_{p,r} \cap L^\infty} \| v \|_{\dot{B}^{s_2}_{p,r}} \quad \text{if} \quad s_1 \leq \frac{3}{2}, s_2 > \frac{3}{2} \quad \text{and} \quad s_1 + s_2 > 0,$$
$$\| uv \|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{p,r}} \lesssim \| u \|_{\dot{B}^{s_1}_{p,r}} \| v \|_{\dot{B}^{s_2}_{p,r}} \quad \text{if} \quad s_1, s_2 < \frac{3}{2} \quad \text{and} \quad s_1 + s_2 > 0,$$
$$\| uv \|_{\dot{B}^s_{p,r}} \lesssim \| u \|_{\dot{B}^s_{p,r}} \| v \|_{\dot{B}^{3/2}_{p,r} \cap L^\infty} \quad \text{if} \quad |s| < \frac{3}{2}.$$  

Moreover, if $r = 1$, the third inequality also holds for $s_1, s_2 \leq \frac{3}{2}$ and $s_1 + s_2 > 0$.

### 3. Global well-posedness in Sobolev spaces

The present section is dedicated to proving Theorem 1.2. Before starting, we assume that $\tilde{\rho} := \rho - \frac{1}{3} \bar{\theta} := \theta - \bar{\theta}$. Then we first rewrite (1.1) as

$$\begin{cases}
\partial_t \tilde{\rho} - \kappa_1 \bar{\theta} \Delta \tilde{\rho} - \kappa_1 \bar{\theta} \Delta \bar{\theta} = \kappa_1 \Delta (\tilde{\rho} \bar{\theta}), \\
\rho_k \kappa_2 \partial_t \tilde{\theta} = \kappa_1 \kappa_2 \nabla \bar{\theta} \cdot \nabla (\rho \theta) - \kappa_1^2 \nabla \cdot (\theta \nabla (\rho \theta)) = \nabla \cdot (\kappa_3 (\theta) \nabla \tilde{\theta}).
\end{cases}$$

For simplicity, here we assume that $\bar{\rho} = \bar{\theta} = 1$. And furthermore, we decompose the coefficients $\kappa_3 (\theta) = \tilde{\kappa}_3 + \bar{\kappa}_3 (\bar{\theta})$, which satisfies $\bar{\kappa}_3 (0) = 0$. We also assume that $\tilde{\kappa}_3'$ and $\bar{\kappa}_3''$ exist and are bounded. Then (3.1) can be written by

$$\begin{cases}
\partial_t \tilde{\rho} - \kappa_1 \Delta \tilde{\rho} - \kappa_1 \Delta \bar{\theta} = \kappa_1 \Delta (\tilde{\rho} \bar{\theta}), \\
\kappa_2 \partial_t \tilde{\theta} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \tilde{\theta} - \kappa_1^2 \Delta \bar{\theta} = \kappa_1 (\kappa_1 + \kappa_2) \left( \nabla \bar{\theta} \cdot \nabla \tilde{\rho} + \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} + \nabla \tilde{\theta} \cdot \nabla (\tilde{\rho} \bar{\theta}) \right) + \kappa_1^2 \Delta (\tilde{\rho} \bar{\theta}) + \nabla \cdot (\tilde{\kappa}_3 (\tilde{\theta}) \nabla \bar{\theta}) - \kappa_2 \rho_k \partial_t \bar{\theta}.
\end{cases}$$

3.1. \textbf{L}^2 \textbf{energy estimate.} Taking the L^2 inner product with \( \tilde{\rho} \) and \( \tilde{\theta} \) with respect to the first and second equations, one has

\begin{equation}
\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|^2_{L^2} + \kappa_1\|\nabla\tilde{\rho}\|^2_{L^2} = -\kappa_1\int_{\mathbb{R}^3} \nabla\tilde{\theta} \cdot \nabla\tilde{\rho} \, dx - \kappa_1\int_{\mathbb{R}^3} \nabla(\tilde{\rho}\tilde{\theta}) \cdot \nabla\tilde{\rho} \, dx
\end{equation}

(3.3)

\begin{align*}
\text{where} \\
& I_1 = -\kappa_1\int_{\mathbb{R}^3} \nabla\tilde{\theta} \cdot \nabla\tilde{\rho} \, dx, \quad I_2 = -\kappa_1\int_{\mathbb{R}^3} \nabla(\tilde{\rho}\tilde{\theta}) \cdot \nabla\tilde{\rho} \, dx, \\
& I_3 = -\kappa_1^2\int_{\mathbb{R}^3} \nabla\tilde{\theta} \cdot \nabla\tilde{\rho} \, dx, \quad I_4 = -\kappa_1^2\int_{\mathbb{R}^3} \nabla(\tilde{\rho}\tilde{\theta}) \cdot \nabla\tilde{\rho} \, dx, \\
& I_5 = \kappa_1(\kappa_1 + \kappa_2)\int_{\mathbb{R}^3} \left( \nabla\tilde{\rho} \cdot \nabla\tilde{\theta} + \nabla\tilde{\theta} \cdot \nabla\tilde{\rho} + \nabla(\tilde{\rho}\tilde{\theta}) \right) \tilde{\theta} \, dx, \\
& I_6 = -\int_{\mathbb{R}^3} \tilde{\kappa}_3(\tilde{\theta})\nabla\tilde{\theta} \cdot \nabla\tilde{\rho} \, dx, \quad I_7 = \kappa_2\int_{\mathbb{R}^3} \tilde{\rho}\partial_t\tilde{\theta} \, dx.
\end{align*}

(3.4)

Firstly, by Hölder and Cauchy inequalities, we have

\[ I_1 \leq \frac{1}{2}\kappa_1\|\nabla\tilde{\theta}\|^2_{L^2} + \frac{1}{2}\kappa_1\|\nabla\tilde{\rho}\|^2_{L^2}, \quad I_3 \leq \frac{1}{2}\kappa_1^2\|\nabla\tilde{\theta}\|^2_{L^2} + \frac{1}{2}\kappa_1^2\|\nabla\tilde{\rho}\|^2_{L^2}. \]

Then by linear combination of (3.3) and (3.4), one can get that

\begin{equation}
\frac{1}{2}\kappa_1(1 + \delta)\frac{d}{dt}\|\tilde{\rho}\|^2_{L^2} + \frac{1}{2}\kappa_2\frac{d}{dt}\|\tilde{\theta}\|^2_{L^2} + \frac{1}{2}\delta\kappa_1^2\|\nabla\tilde{\rho}\|^2_{L^2} + (\tilde{\kappa}_3 - \frac{1}{2}\delta\kappa_1^2)\|\nabla\tilde{\theta}\|^2_{L^2}
\end{equation}

\leq \kappa_1(1 + \delta)I_1 + I_2 + I_4 + I_5 + I_6 + I_7,

(3.5)

where \( \delta \) is a small positive constant that satisfies

\[ \delta\kappa_1^2 < 2\tilde{\kappa}_3. \]

To bound \( I_2 \), we decompose it into two parts and by Hölder inequality, we have

\begin{align*}
I_2 &= -\kappa_1\int_{\mathbb{R}^3} \tilde{\theta}\nabla\tilde{\rho} \cdot \nabla\tilde{\rho} \, dx - \kappa_1\int_{\mathbb{R}^3} \tilde{\rho}\nabla\tilde{\theta} \cdot \nabla\tilde{\rho} \, dx \\
&\leq \kappa_1\|\tilde{\theta}\|_{L^\infty}\|\nabla\tilde{\rho}\|^2_{L^2} + \kappa_1\|\tilde{\rho}\|_{L^\infty}\|\nabla\tilde{\rho}\|_{L^2}\|\nabla\tilde{\theta}\|_{L^2}.
\end{align*}

(3.6)
Similarly, $I_4$ and $I_5$ can be bounded by

$$I_4 \leq \kappa_1^3 \|\tilde{\rho}\|_{L^\infty} \|\nabla \tilde{\theta}\|_{L^2}^2 + \kappa_2^3 \|\tilde{\theta}\|_{L^\infty} \|\nabla \tilde{\rho}\|_{L^2} \|\nabla \tilde{\theta}\|_{L^2},$$

(3.7)  

$$I_5 \leq \kappa_1 (\kappa_1 + \kappa_2) \left( \|\tilde{\rho}\|_{L^\infty} + \|\tilde{\theta}\|_{L^\infty} + \|\tilde{\rho}\|_{L^2} \|\tilde{\theta}\|_{L^\infty} + \|\tilde{\theta}\|_{L^2}^2 \right)$$  

$$\times \left( \|\nabla \tilde{\rho}\|_{L^2} \|\nabla \tilde{\theta}\|_{L^2} + \|\nabla \tilde{\rho}\|_{L^2}^2 \right).$$

For $I_6$, notice that $\tilde{\kappa}_3(0) = 0$. Then we use Taylor formula and Hölder inequality to get that

$$I_6 \leq C \|\tilde{\theta}\|_{L^\infty} \|\nabla \tilde{\theta}\|_{L^2}^2.$$

Now we turn to the last term $I_7$. The Hölder inequality implies that

$$I_7 = \kappa_2 \int \tilde{\rho} \partial_\theta \tilde{\theta} \, dx \leq \kappa_2 \|\tilde{\rho}\|_{L^\infty} \|\partial_\theta \tilde{\theta}\|_{L^2} \|\tilde{\theta}\|_{L^6}$$

$$\leq C \|\rho\|_{H^1} \left( \|\partial_\theta \tilde{\theta}\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 \right).$$

Combining the estimates (3.6) to (3.9) with (3.5), we have

$$\frac{1}{2} \kappa_1 (1 + \delta) \frac{d}{dt} \|\tilde{\rho}\|_{L^2}^2 + \frac{1}{2} \kappa_2 \frac{d}{dt} \|\tilde{\theta}\|_{L^2}^2 + \frac{1}{2} \delta \kappa_1^2 \|\nabla \tilde{\rho}\|_{L^2}^2 + \frac{1}{2} \delta \kappa_2^2 \|\nabla \tilde{\theta}\|_{L^2}^2 + \tilde{\kappa}_3 - \frac{1}{2} \delta \kappa_1^2 \|\nabla \tilde{\rho}\|_{L^2}^2 \|\nabla \tilde{\theta}\|_{L^2}^2$$

(3.10)  

$$\leq C \left( \kappa_1^2 + \kappa_2^2 + 1 + \frac{\kappa_2}{\kappa_1} \right) \left( \|\tilde{\rho}\|_{H^2} + \|\tilde{\theta}\|_{H^2} + \|\tilde{\rho}\|_{H^2}^2 + \|\tilde{\theta}\|_{H^2}^2 \right)$$

$$\times \left( \|\partial_\theta \tilde{\theta}\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 + \|\nabla \tilde{\rho}\|_{H^2}^2 \right).$$

3.2. $H^2$ energy estimate. Due to the equivalence of $\|(\tilde{\rho}, \tilde{\theta})\|_{H^2}$ with $\|(\rho, \tilde{\theta})\|_{L^2} + \|(\tilde{\rho}, \tilde{\theta})\|_{H^2}$, it is sufficient to bound the homogeneous $H^2$ norm of $(\tilde{\rho}, \tilde{\theta})$. Applying $\partial_\theta^2$ for $i = 1, 2, 3$ to (3.2) and then taking the $L^2$ inner product with $(\partial_\theta^2 \tilde{\rho}, \partial_\theta^2 \tilde{\theta})$, respectively, we find that

$$\frac{1}{2} \frac{d}{dt} \|\partial_\theta^2 \tilde{\rho}\|_{L^2}^2 + \kappa_1 \|\nabla \partial_\theta^2 \tilde{\rho}\|_{L^2}^2$$

(3.11)  

$$= -\kappa_1 \int \nabla \partial_\theta^2 \tilde{\theta} \cdot \nabla \partial_\theta^2 \tilde{\rho} \, dx - \kappa_1 \int \nabla \partial_\theta (\tilde{\rho} \partial_\theta) \cdot \nabla \partial_\theta^2 \tilde{\rho} \, dx$$

$$\leq \frac{1}{2} \kappa_1 \|\nabla \partial_\theta \tilde{\theta}\|_{L^2}^2 + \frac{1}{2} \kappa_1 \|\nabla \partial_\theta \tilde{\rho}\|_{L^2}^2 - \kappa_1 \int \nabla \partial_\theta (\tilde{\rho} \partial_\theta) \cdot \nabla \partial_\theta^2 \tilde{\rho} \, dx,$$
\[ \frac{1}{2} \kappa_2 \rho \frac{d}{dt} \| \nabla^2 \bar{\theta} \|^2_{L^2} + (\kappa_1^2 + \tilde{\kappa}_3) \| \nabla \partial_t^2 \bar{\theta} \|^2_{L^2} \]

\[ = -k_1^2 \int_{\mathbb{R}^3} \nabla \partial_t^2 \bar{\rho} \cdot \nabla \partial_t^2 \bar{\theta} \, dx - k_1^2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \bar{\theta}) \cdot \nabla \partial_t^2 \bar{\theta} \, dx \]

\[ + \kappa_1 (\kappa_1 + \kappa_2) \int_{\mathbb{R}^3} \partial_t^2 \left( \nabla \bar{\rho} \cdot \nabla \bar{\theta} + \nabla \bar{\theta} \cdot \nabla \bar{\theta} + \nabla \tilde{\theta} \cdot \nabla \bar{\rho} \right) \partial_t^2 \bar{\theta} \, dx \]

\[ - \int_{\mathbb{R}^3} \partial_t^2 \left( \tilde{\kappa}_3 (\bar{\theta}) \nabla \bar{\theta} \right) \cdot \nabla \partial_t^2 \bar{\theta} \, dx + \kappa_2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \partial_t \bar{\theta}) \partial_t^2 \bar{\theta} \, dx \]

\[ \leq \frac{1}{2} \kappa_1^2 \| \nabla \partial_t^2 \bar{\theta} \|_{L^2}^2 + \frac{1}{2} \kappa_2^2 \| \nabla \partial_t^2 \bar{\rho} \|_{L^2}^2 - k_1^2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \bar{\theta}) \cdot \nabla \partial_t^2 \bar{\theta} \, dx \]

\[ + \kappa_1 (\kappa_1 + \kappa_2) \int_{\mathbb{R}^3} \partial_t^2 \left( \nabla \bar{\rho} \cdot \nabla \bar{\theta} + \nabla \bar{\theta} \cdot \nabla \bar{\theta} + \nabla \tilde{\theta} \cdot \nabla \bar{\rho} \right) \partial_t^2 \bar{\theta} \, dx \]

\[ - \int_{\mathbb{R}^3} \partial_t^2 \left( \tilde{\kappa}_3 (\bar{\theta}) \nabla \bar{\theta} \right) \cdot \nabla \partial_t^2 \bar{\theta} \, dx + \kappa_2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \partial_t \bar{\theta}) \partial_t^2 \bar{\theta} \, dx. \]

Again, we denote \( J_1 \) to \( J_5 \) by

\[ J_1 = -\kappa_1 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \bar{\theta}) \cdot \nabla \partial_t^2 \bar{\rho} \, dx, \quad J_2 = -k_1^2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \bar{\theta}) \cdot \nabla \partial_t^2 \bar{\theta} \, dx, \]

\[ J_3 = \kappa_1 (\kappa_1 + \kappa_2) \int_{\mathbb{R}^3} \partial_t^2 \left( \nabla \bar{\rho} \cdot \nabla \bar{\theta} + \nabla \bar{\theta} \cdot \nabla \bar{\theta} + \nabla \tilde{\theta} \cdot \nabla \bar{\rho} \right) \partial_t^2 \bar{\theta} \, dx, \]

\[ J_4 = -\int_{\mathbb{R}^3} \partial_t^2 \left( \tilde{\kappa}_3 (\bar{\theta}) \nabla \bar{\theta} \right) \cdot \nabla \partial_t^2 \bar{\theta} \, dx, \quad J_5 = \kappa_2 \int_{\mathbb{R}^3} \nabla \partial_t^2 (\bar{\rho} \partial_t \bar{\theta}) \partial_t^2 \bar{\theta} \, dx. \]

Then the linear combination of (3.11) and (3.12) implies that

\[ \frac{1}{2} \kappa_1 (1 + \delta) \frac{d}{dt} \| \partial_t^2 \bar{\rho} \|^2_{L^2} + \frac{1}{2} \kappa_2 \frac{d}{dt} \| \partial_t^2 \bar{\theta} \|^2_{L^2} + \frac{1}{2} \kappa_1^2 \| \nabla \partial_t^2 \bar{\rho} \|_{L^2}^2 -(\kappa_1^2 + \tilde{\kappa}_3) \| \nabla \partial_t^2 \bar{\theta} \|_{L^2}^2 \]

\[ \leq \kappa_1 (1 + \delta) J_1 + J_2 + J_3 + J_4 + J_5. \]

By the Hölder inequality, one has

\[ J_1 \leq C \kappa_1 \left( \| \bar{\theta} \|_{L^\infty} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} + \| \nabla \partial_t \bar{\rho} \|_{L^6} \| \partial_t \bar{\theta} \|_{L^3} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} \right) \]

\[ + \| \nabla \bar{\rho} \|_{L^5} \| \nabla \partial_t^2 \bar{\theta} \|_{L^6} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} + \| \bar{\rho} \|_{L^\infty} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} \| \nabla \partial_t^2 \bar{\theta} \|_{L^2} \]

\[ + \| \nabla \partial_t \bar{\theta} \|_{L^6} \| \partial_t \bar{\rho} \|_{L^3} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} + \| \bar{\theta} \|_{L^3} \| \partial_t^2 \bar{\rho} \|_{L^6} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} \]

\[ \leq C \kappa_1 \left( \| \bar{\theta} \|_{H^2} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} + \| \bar{\rho} \|_{H^2} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} \right). \]

Similarly, we have

\[ J_2 \leq C \kappa_2^2 \left( \| \bar{\theta} \|_{H^2} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} + \| \bar{\rho} \|_{H^2} \| \nabla \partial_t^2 \bar{\rho} \|_{L^2} \right). \]
and
\begin{align}
J_3 & \leq C(\kappa_1^2 + \kappa_2^2)\left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} \left( \left\| \nabla \tilde{\theta} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\rho} \right\|_{L^6} + \left\| \nabla \tilde{\rho} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\theta} \right\|_{L^6} \\
& \quad + \left\| \nabla \tilde{\theta} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\rho} \right\|_{L^6} + \left\| \tilde{\rho} \right\|_{L^\infty} \left\| \nabla \tilde{\theta} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\rho} \right\|_{L^6} \\
& \quad + \left\| \tilde{\theta} \right\|_{L^\infty} \left\| \nabla \tilde{\rho} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\theta} \right\|_{L^6} + \left\| \tilde{\theta} \right\|_{L^\infty} \left\| \nabla \tilde{\rho} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\rho} \right\|_{L^6}
\right) \\
& \quad + \left\| \nabla \rho \right\|_{L^\infty} \left\| \nabla \tilde{\theta} \right\|_{L^3} \left\| \nabla \tilde{\rho} \right\|_{L^6}
\right)
\leq C(\kappa_1^2 + \kappa_2^2)\left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} \left( \left\| \tilde{\theta} \right\|_{H^2} + \left\| \tilde{\rho} \right\|_{H^2} + \left\| \tilde{\theta} \right\|_{H^2}^2 + \left\| \tilde{\rho} \right\|_{H^2}^2 \right)
\times \left( \left\| \nabla \partial_t^2 \tilde{\rho} \right\|_{L^2} + \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} + \left\| \nabla^2 \tilde{\theta} \right\|_{L^2} \right),
\end{align}

where we have used the Sobolev embedding that for any distribution \( f \),
\[
\left\| f \right\|_{L^\infty} \leq C\left\| f \right\|_{H^2}, \quad \left\| f \right\|_{L^3} \leq C\left\| \nabla f \right\|_{H^1}.
\]

Similarly, for \( J_4 \), we have
\begin{align}
J_4 & \leq C\left\| \tilde{\theta} \right\|_{L^\infty} \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} + \left\| \partial_t \tilde{\theta} \right\|_{L^3} \left\| \nabla \partial_t \tilde{\theta} \right\|_{L^6} \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} \\
& \quad + \left\| \partial_t \tilde{\theta} \right\|_{L^\infty} \left\| \nabla \partial_t \tilde{\theta} \right\|_{L^6} \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2}
\leq C\left\| \tilde{\theta} \right\|_{H^2} \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2} + C\left\| \tilde{\theta} \right\|_{H^2} \left\| \nabla \partial_t \tilde{\theta} \right\|_{L^2} \left\| \nabla \partial_t^2 \tilde{\theta} \right\|_{L^2}.
\end{align}

As to the last term \( J_5 \), we decompose it into four parts
\[
J_5 = \kappa_2 \int_{\mathbb{R}^3} \partial_t^2 \tilde{\rho} \partial_t \tilde{\theta} \, dx
= \kappa_2 \left( \int_{\mathbb{R}^3} \partial_t^2 \tilde{\rho} \partial_t \tilde{\theta} \, dx + 2\kappa_2 \int_{\mathbb{R}^3} \partial_t \tilde{\rho} \partial_t \partial_t \tilde{\theta} \partial_t \tilde{\theta} \, dx - \frac{1}{2}\kappa_2 \int_{\mathbb{R}^3} \partial_t \tilde{\rho} \left( \partial_t^2 \tilde{\theta} \right)^2 \, dx + \frac{1}{2}\kappa_2 \int_{\mathbb{R}^3} \partial_t \tilde{\rho} \left( \partial_t \tilde{\rho} \partial_t \tilde{\theta} \right)^2 \, dx \right)
:= J_{51} + J_{52} + J_{53} + J_{54}.
\]

For \( J_{53} \), we can quickly by the equation of \( \tilde{\rho} \) get that
\begin{align}
J_{53} &= -\frac{1}{2}\kappa_2 \int_{\mathbb{R}^3} \partial_t \tilde{\rho} \left( \partial_t \tilde{\theta} \right)^2 \, dx \\
&= -\frac{1}{2}\kappa_1 \kappa_2 \int_{\mathbb{R}^3} \left( \Delta \tilde{\rho} + \Delta \tilde{\theta} + \Delta (\tilde{\rho} \tilde{\theta}) \right) \left( \partial_t \tilde{\theta} \right)^2 \, dx \\
&\leq C\kappa_1 \kappa_2 \left( \left\| \tilde{\rho} \right\|_{H^2} + \left\| \tilde{\theta} \right\|_{H^2} + \left\| \tilde{\rho} \right\|_{H^2}^2 + \left\| \tilde{\theta} \right\|_{H^2}^2 \right) \left( \left\| \nabla \tilde{\theta} \right\|_{H^2}^2 + \left\| \nabla \tilde{\rho} \right\|_{H^2}^2 \right).
\end{align}

To bound \( J_{51} \) and \( J_{52} \), we roughly estimate them as follows,
\begin{align}
J_{51} + J_{52} &\leq C(\kappa_1, \kappa_2) \left( \left\| \partial_t^2 \tilde{\rho} \right\|_{L^2}^2 + \left\| \partial_t \tilde{\rho} \right\|_{L^2}^2 + \left\| \partial_t \partial_t \tilde{\theta} \right\|_{L^2}^2 \right) + \frac{\kappa_2^2}{16 \kappa_1^2} \delta \left\| \partial_t \partial_t \tilde{\theta} \right\|_{L^2}^2.
\end{align}

Let us postpone the estimate of \( J_{54} \), since its bound can be deduced directly by taking time integral. We need to get the additional estimates on \( \partial_t \tilde{\theta} \) and \( \partial_t \partial_t \tilde{\theta} \).
For that, applying $\partial^k_t$ with $k = 0, 1, i = 1, 2, 3$ to the equation of $\tilde{\theta}$, taking $L^2$ inner product with $\partial_t \partial^k_t \tilde{\theta}$, one has

\begin{equation}
(3.20)
\end{equation}

\[\kappa_2 \| \partial_t \partial^k_t \tilde{\theta} \|^2_{L^2} + \frac{1}{2}(\kappa_1^2 + \tilde{\kappa}^2) \frac{d}{dt} \| \nabla \partial^k_t \tilde{\theta} \|^2_{L^2} = k_1^2 \int_{\mathbb{R}^3} \Delta \partial^k_t \tilde{\rho} \cdot \partial_t \partial^k_t \tilde{\theta} \, dx + k_1^2 \int_{\mathbb{R}^3} \Delta \partial^k_t (\tilde{\rho} \tilde{\theta}) \cdot \partial_t \partial^k_t \tilde{\theta} \, dx
\]

\[+ \kappa_1 (\kappa_1 + k_2) \int_{\mathbb{R}^3} \partial^k_t \left( \nabla \tilde{\rho} \cdot \nabla \tilde{\theta} + \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} + \nabla \tilde{\theta} \cdot \nabla (\tilde{\theta} \tilde{\rho}) \right) \partial_t \partial^k_t \tilde{\theta} \, dx
\]

\[+ \int_{\mathbb{R}^3} \partial^k_t \nabla \cdot (\tilde{\kappa}_2 (\tilde{\theta}) \nabla \tilde{\theta}) \cdot \partial_t \partial^k_t \tilde{\theta} \, dx + k_2 \int_{\mathbb{R}^3} \partial^k_t (\tilde{\rho} \partial_t \tilde{\theta}) \cdot \partial_t \partial^k_t \tilde{\theta} \, dx
\]

\[\leq C \frac{\kappa_1}{\kappa_2} \| \Delta \partial^k_t \tilde{\rho} \|^2_{L^2} + C \frac{\kappa_1}{\kappa_2} \| \Delta \partial^k_t (\tilde{\rho} \tilde{\theta}) \|^2_{L^2} + C \frac{\kappa_2^2 (\kappa_1 + k_2)^2}{\kappa_2} \| \partial^k_t (\nabla \tilde{\rho} \cdot \nabla \tilde{\theta}) \|^2_{L^2}
\]

\[+ C \frac{\kappa_2^2 (\kappa_1 + k_2)^2}{\kappa_2} \| \partial^k_t (\nabla \tilde{\theta} \cdot \nabla (\tilde{\theta} \tilde{\rho})) \|^2_{L^2}
\]

\[+ C \frac{1}{\kappa_2} \| \partial^k_t \nabla (\tilde{\kappa}_2 (\tilde{\theta}) \nabla \tilde{\theta}) \|^2_{L^2} + k_2 \| \partial^k_t (\tilde{\rho} \partial_t \tilde{\theta}) \|^2_{L^2} + \frac{1}{2} \kappa_2 \| \partial_t \partial^k_t \tilde{\theta} \|^2_{L^2}.
\]

which implies

\begin{equation}
(3.21)
\end{equation}

\[\frac{1}{2} \kappa_2 \| \partial_t \partial^k_t \tilde{\theta} \|^2_{L^2} + \frac{1}{2}(\kappa_1^2 + \tilde{\kappa}^2) \frac{d}{dt} \| \nabla \partial^k_t \tilde{\theta} \|^2_{L^2} \leq C \frac{\kappa_1^4}{\kappa_2} \| \Delta \partial^k_t \tilde{\rho} \|^2_{L^2} + C \frac{\kappa_1^4}{\kappa_2} \| \Delta \partial^k_t (\tilde{\rho} \tilde{\theta}) \|^2_{L^2} + C \frac{\kappa_2^2 (\kappa_1 + k_2)^2}{\kappa_2} \| \partial^k_t (\nabla \tilde{\rho} \cdot \nabla \tilde{\theta}) \|^2_{L^2}
\]

\[+ C \frac{\kappa_2^2 (\kappa_1 + k_2)^2}{\kappa_2} \| \partial^k_t (\nabla \tilde{\theta} \cdot \nabla (\tilde{\theta} \tilde{\rho})) \|^2_{L^2}
\]

\[+ C \frac{1}{\kappa_2} \| \partial^k_t \nabla (\tilde{\kappa}_2 (\tilde{\theta}) \nabla \tilde{\theta}) \|^2_{L^2} + k_2 \| \partial^k_t (\tilde{\rho} \partial_t \tilde{\theta}) \|^2_{L^2} \| \partial_t \partial^k_t \tilde{\theta} \|^2_{L^2}.
\]

Multiplying by $\frac{\kappa_2}{4C \kappa_1} \delta$ on both sides of (3.21) and combining the resulting inequality with (3.13)-(4.20), we get that

\begin{equation}
(3.22)
\end{equation}

\[\frac{1}{2} \kappa_1 (1 + \delta) \frac{d}{dt} \| \partial^2_t \tilde{\rho} \|^2_{L^2} + \frac{1}{2} \kappa_2 \frac{d}{dt} \| \partial^2_t \tilde{\theta} \|^2_{L^2} + \frac{\kappa_2}{8C} (1 + \frac{\tilde{\kappa}}{\kappa_1^2}) \frac{d}{dt} \| \nabla \partial^2_t \tilde{\theta} \|^2_{L^2}
\]

\[+ \frac{\delta \kappa_1^2}{4} \| \nabla \partial^2_t \tilde{\rho} \|^2_{L^2} + (\tilde{\kappa}_3 - \frac{1}{2} \delta \kappa_1^2) \| \nabla \partial^2_t \tilde{\theta} \|^2_{L^2} + \frac{\kappa_2^2}{16C \kappa_1^4} \delta \| \partial_t \partial^2_t \tilde{\theta} \|^2_{L^2}
\]

\[\leq C (\kappa_1, \kappa_2, \kappa_3) \left( \| \tilde{\rho} \|^2_{H^2} + \| \tilde{\rho} \|^2_{H^2} + \| \tilde{\theta} \|^2_{H^2} + \| \tilde{\theta} \|^2_{H^2} \right)
\]

\[\times \left( \| \nabla \tilde{\rho} \|^2_{H^2} + \| \nabla \tilde{\theta} \|^2_{H^2} + \| \partial_t \partial^2_t \tilde{\theta} \|^2_{L^2} \right) + J_{54}.
\]

Integrating (3.22) over $[0, t]$ with respect to time variable, denoting

\[X(t) = \sup_{0 \leq \tau \leq t} \left( \| \tilde{\rho} (\tau) \|^2_{H^2} + \| \tilde{\theta} (\tau) \|^2_{H^2} \right) + \int_0^t \left( \| \nabla \tilde{\rho} \|^2_{H^2} + \| \nabla \tilde{\theta} \|^2_{H^2} + \| \partial_t \tilde{\theta} (t) \|^2_{H^1} \right) \, dt,
\]

\[X(0) = \| \tilde{\rho}_0 \|^2_{H^2} + \| \tilde{\theta}_0 \|^2_{H^2},
\]
combining with
\[\int_0^t J_{54} \, dt \leq C(\kappa_2)(\|\tilde{\rho}\|_{H^2}^2 - \|\tilde{\rho}_0\|_{H^2}^2),\]
we can obtain that
\[X(t) \leq C_1(\kappa_1, \kappa_2, \bar{\kappa}_3) \left( X(0) + (X(0))^{\frac{3}{2}} \right) + C_2(\kappa_1, \kappa_2, \bar{\kappa}_3) \left( X(t)^{\frac{3}{2}} + X(t)^2 \right),\]
By using the standard continuity argument, we complete the proof of Theorem 1.2.

4. Global existence for small data with critical regularity

In this section, we will obtain the global existence of solutions to system (1.1) in Theorem 1.3. From now on, we define the density and the temperature by the form
\[a := \frac{1}{\rho} - 1, \quad \tilde{\theta} := \theta - 1.\]
Then system (1.1) can be rewritten as
\[(4.1)\begin{cases} \partial_t a - \kappa_1 \Delta a + \kappa_1 \Delta \tilde{\theta} = F, \\ \kappa_2 \partial_t \tilde{\theta} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \tilde{\theta} + \kappa_1^2 \Delta a = G, \end{cases}\]
where
\begin{align*}
F &= -2\kappa_1 \frac{\tilde{\theta} + 1}{1 + a} |\nabla a|^2 + 2\kappa_1 \nabla a \cdot \nabla \tilde{\theta} - \kappa_1 a \Delta \tilde{\theta} + \kappa_1 \tilde{\theta} \Delta a, \\
G &= 2\kappa_1^2 \frac{(\tilde{\theta} + 1)^2}{(1 + a)^2} |\nabla a|^2 - (3\kappa_1^2 + \kappa_1 \kappa_2) \frac{(\tilde{\theta} + 1)}{1 + a} \nabla a \cdot \nabla \tilde{\theta} - \kappa_1^2 \Delta a + \kappa_1 \kappa_2 \Delta \tilde{\theta} + \kappa_1 \tilde{\theta} \Delta \tilde{\theta} + (1 + a) \nabla \cdot (\bar{\kappa}_3 \tilde{\theta} \nabla \tilde{\theta}) + \kappa_1 (\kappa_1 + \kappa_2) |\nabla \tilde{\theta}|^2.
\end{align*}
Proving the global existence result is based on the following variant of Banach’s fixed point theorem. For the proof, we refer e.g. to [18].

**Lemma 4.1.** Let \(X\) be a reflexive Banach space or let \(X\) have a separable pre-dual. Let \(K\) be a convex, closed and bounded subset of \(X\) and assume that \(X\) is embedded into a Banach space \(Y\). Let \(\Phi : X \rightarrow X\) map \(K\) into \(K\) and assume there exists \(c < 1\) such that
\[\|\Phi(x) - \Phi(y)\|_Y \leq c \|x - y\|_Y, \quad x, y \in K.\]
Then there exists a unique fixed point of \(\Phi\) in \(K\).

Based on the natural scaling of system (1.1), we choose our working space to be
\[E(T) := \left\{ u \in C \left( [0, T], \dot{B}_{2,1}^{3/2} \right), \quad \nabla^2 u \in L^1 \left( [0, T], \dot{B}_{2,1}^{3/2} \right) \right\}, \quad T > 0,\]
with the norm
\[\|u\|_{E(T)} := \|u\|_{L^\infty(\dot{B}_{2,1}^{3/2})} + \|\nabla^2 u\|_{L^1(\dot{B}_{2,1}^{3/2})}.\]
The following proposition quantifies the smoothing effect of the linear system of (4.1).

**Proposition 4.2.** Let us consider the initial data \((a_0, \tilde{\theta}_0)\) in \(\dot{B}^s_{2,1}(\mathbb{R}^3)\) with regularity \(s \leq \frac{3}{2}\). Introducing a pair of forces \((F, G)\) in \(L^1_t(\dot{B}^s_{2,1}(\mathbb{R}^3))\), we denote by \((a, \tilde{\theta})\) the unique solution of the following linear parabolic system:

\[
\begin{aligned}
\partial_t a - \kappa_1 \Delta a + \kappa_1 \Delta \tilde{\theta} &= F, \\
\kappa_2 \partial_t \tilde{\theta} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \tilde{\theta} + \kappa_1^2 \Delta a &= G.
\end{aligned}
\]

Then \((a, \tilde{\theta})\) belongs to \(L^\infty_t(\dot{B}^s_{2,1}(\mathbb{R}^3))\) and the pairs \((\partial_t a, \partial_t \tilde{\theta})\) and \((\Delta a, \Delta \tilde{\theta})\) belong to \(L^1_t(\dot{B}^s_{2,1}(\mathbb{R}^3))\). Furthermore, there exists a positive constant \(C\) depending only on \(\kappa_1, \kappa_2\) and \(\bar{\kappa}_3\) such that

\[
\|(a, \tilde{\theta})\|_{L^\infty_t(\dot{B}^s_{2,1})} + \|\partial_t a, \partial_t \tilde{\theta}\|_{L^1_t(\dot{B}^s_{2,1})} + \|\Delta a, \Delta \tilde{\theta}\|_{L^1_t(\dot{B}^s_{2,1})} \\
\leq C(\kappa_1, \kappa_2, \bar{\kappa}_3) \left( \|(a_0, \tilde{\theta}_0)\|_{L^\infty_t(\dot{B}^s_{2,1})} + \|(F, G)\|_{L^1_t(\dot{B}^s_{2,1})} \right). 
\]

**Proof.** We first apply the homogeneous dyadic block \(\hat{\Delta}_q\) to system (4.3), and multiply both equations by \(\hat{\Delta}_q a\) and \(\hat{\Delta}_q \tilde{\theta}\), respectively, and integrate over \(\mathbb{R}^3\). We then get

\[
\frac{1}{2} \frac{d}{dt} \|\hat{\Delta}_q a\|^2_{L^2} + \kappa_1 \|\hat{\Delta}_q a\|^2_{L^2} = \kappa_1 \int_{\mathbb{R}^3} \hat{\Delta}_q \tilde{\theta} \cdot \nabla \hat{\Delta}_q a \, dx + \int_{\mathbb{R}^3} \hat{\Delta}_q F \, \hat{\Delta}_q a \, dx,
\]

and

\[
= \frac{1}{2} \kappa_1 \int_{\mathbb{R}^3} \hat{\Delta}_q \tilde{\theta} \cdot \nabla \hat{\Delta}_q a \, dx + \int_{\mathbb{R}^3} \hat{\Delta}_q G \, \hat{\Delta}_q \tilde{\theta} \, dx.
\]

By Hölder and Cauchy inequalities, the first term on the right side of (4.5) can be bounded by

\[
\kappa_1 \int_{\mathbb{R}^3} \nabla \hat{\Delta}_q \tilde{\theta} \cdot \nabla \hat{\Delta}_q a \, dx \leq \frac{1}{2} \kappa_1 \|\nabla \hat{\Delta}_q \tilde{\theta}\|_{L^2}^2 + \frac{1}{2} \kappa_1 \|\nabla \hat{\Delta}_q a\|_{L^2}^2.
\]

Plugging the above inequality into (4.5), and making the linear combination of the resulting inequality with (4.6), one has

\[
\frac{1}{2} \kappa_1 (1 + \delta) \frac{d}{dt} \|\hat{\Delta}_q a\|_{L^2}^2 + \frac{1}{2} \kappa_2 \frac{d}{dt} \|\hat{\Delta}_q \tilde{\theta}\|_{L^2}^2 + \delta \kappa_1^2 \|\nabla \hat{\Delta}_q a\|_{L^2}^2 + (\bar{\kappa}_3 - \frac{1}{2} \delta \kappa_1^2) \|\nabla \hat{\Delta}_q \tilde{\theta}\|_{L^2}^2
\]
\[
\leq \kappa_1 (1 + \delta) \|\hat{\Delta}_q F\|_{L^2} \|\hat{\Delta}_q a\|_{L^2} + \|\hat{\Delta}_q G\|_{L^2} \|\hat{\Delta}_q \tilde{\theta}\|_{L^2},
\]

where \(\delta\) is a small positive number. Setting

\[
f_q^2 = \kappa_1 (1 + \delta) \|\hat{\Delta}_q a\|_{L^2}^2 + \kappa_2 \|\hat{\Delta}_q \tilde{\theta}\|_{L^2}^2
\]
and $\kappa = \min\{\frac{\delta_1}{1 + \delta}, \frac{\delta_1 - \frac{1}{2}\delta_4^2}{\kappa_2}\}$, we then by Bernstein inequality have

\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \kappa 2^{2\delta} f_q^2 \leq C_{\kappa_1, \kappa_2} (\|\hat{\Delta}_q F\|_{L^2} + \|\hat{\Delta}_q G\|_{L^2}) f_q.
\]

To finish this, we multiply the above inequality by $2^{2\delta}$, and denote

$g_q = 2^{2\delta} \sqrt{\kappa_1 (1 + \delta)\|\hat{\Delta}_q a\|_{L^2}^2 + \kappa_2 \|\hat{\Delta}_q \bar{\theta}\|_{L^2}^2},$

we then get

\[
\frac{1}{2} \frac{d}{dt} g_q^2 + \kappa 2^{2\delta} g_q^2 \leq C(\kappa_1, \kappa_2) 2^{2\delta} (\|\hat{\Delta}_q F\|_{L^2} + \|\hat{\Delta}_q G\|_{L^2}) g_q.
\]

Using $h_q^2 = g_q^2 + \epsilon^2$, integrating over $[0, t]$ and then letting $\epsilon$ tend to 0, we infer

\[
g_q(t) + \kappa 2^{2\delta} \int_0^t g_q(\tau) d\tau \leq g_q(0) + C(\kappa_1, \kappa_2) 2^{2\delta} \int_0^t (\|\hat{\Delta}_q F\|_{L^2} + \|\hat{\Delta}_q G\|_{L^2}) d\tau.
\]

We finally conclude that

\[
\| (a, \bar{\theta}) \|_{L_t^\infty(B_{r_1}^2)} + \| (a, \bar{\theta}) \|_{L_t^1(B_{r_1}^2)}^2 \leq C(\kappa_1, \kappa_2) \left( \| (a_0, \bar{\theta}_0) \|_{L_t^\infty(B_{r_1}^2)} + \| (F, G) \|_{L_t^1(B_{r_1}^2)} \right).
\]

Combining (4.11) with the equations of $(a, \bar{\theta})$, we eventually finished the proof of this proposition.

Our construction of the global solution relies on a combination of Proposition 4.2 with Lemma 4.1. To this end, for any given $T > 0$, we define the set $K(T)$ by

$K(T) := \{ (b, \tau) \in E(T) \times E(T), \ b(0) = a_0, \ \tau(0) = \bar{\theta}_0 \text{ and } \| (b, \tau) \|_{E(T)} \leq c \}$

for some suitable small positive constants $c$, which will be determined shortly. Next, given $(b, \tau) \in K(T)$, we define the mapping

$\Phi(b, \tau) := (a, \bar{\theta}),$

where $(a, \bar{\theta})$ is defined as the unique solution of the corresponding linearized problem of (4.1)

\[
\begin{align*}
\frac{\partial_t a - \kappa_1 \Delta a + \kappa_1 \Delta \bar{\theta}}{\kappa_2} &= F(b, \tau), \\
\kappa_2 \partial_t \bar{\theta} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \bar{\theta} + \kappa_1^2 \Delta a &= G(b, \tau), \\
(a, \bar{\theta})|_{t=0} &= (a_0, \bar{\theta}_0),
\end{align*}
\]
where
\[ F(b, \tau) = -2\kappa_1 \frac{\tau + 1}{1 + b} |\nabla b|^2 + 2\kappa_1 \nabla b \cdot \nabla \tau - \kappa_1 b \Delta \tau + \kappa_1 \tau \Delta b, \]
\[ G(b, \tau) = 2\kappa_1 \frac{(\tau + 1)^2}{(1 + b)^2} |\nabla b|^2 - \left(3\kappa_1^2 + \kappa_1 \kappa_2\right) \frac{(\tau + 1)}{1 + b} \nabla b \cdot \nabla \tau \]
\[ - \kappa_1^2 \frac{\tau^2 + 2\tau}{1 + b} \Delta b + \kappa_1 \Delta b + \bar{\kappa}_3 \tau \Delta \tau 
+ (1 + b) \nabla \cdot (\bar{\kappa}_3(\tau) \nabla \tau) + \kappa_1 (\kappa_1 + \kappa_2)|\nabla \tau|^2. \]  
(4.13)

Following Propositions 4.2, we easily obtain that
\[ (4.14) \]
\[ \|\Phi(b, \tau)\|_{L^1(T)} \leq C \left(\|\alpha_0, \bar{\theta}_0\|_{\dot{B}^{3/2}_{2,1}} + \|F(b, \tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})} + \|G(b, \tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\right). \]

In order to prove that \( \Phi(K(T)) \subset K(T) \) under the smallness condition on \( \alpha_0 \) and \( \bar{\theta}_0 \), one needs to bound the right side of (4.14). We ignore \( \kappa_1, \kappa_2, \) and \( \bar{\kappa}_3 \), as they are fixed constants.

For the first term in (4.13), we rewrite it as
\[ \frac{\tau + 1}{1 + b} |\nabla b|^2 = m_1(b) |\nabla b|^2 (\tau + 1) + |\nabla b|^2 (\tau + 1), \]
where \( m_1(b) := \frac{1}{4\kappa_1} - 1 \) satisfying \( m_1(0) = 0 \). By Lemma 1.6 in [6], the continuity of the product in Besov spaces (Chapter 2 in [1]), we get
\[ \left\| \frac{\tau + 1}{1 + b} |\nabla b|^2 \right\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq C \left(1 + \|b\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\right) \left\| |\nabla b|^2 \right\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \left(\|\tau\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} + 1\right) \]
\[ \leq C (1 + c)^2 c^2. \]

Similarly, we have
\[ \|\nabla b \cdot \nabla \tau\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq C \|b\|^2, \]
\[ \|b \cdot \Delta \tau\|_{L^1_T(\dot{B}^{3/2}_{2,1})} + \|\tau \Delta b\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq C \|\tau\|^2. \]

Combining the above estimates, we find that
\[ \|F(b, \tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq C (1 + c)^2 c^2. \]

The terms in \( G \) can be bounded in essentially the same way. For the first term in \( G \), we rewrite it as
\[ \frac{(\tau + 1)^2}{(1 + b)^2} |\nabla b|^2 = m_2(b) |\nabla b|^2 (\tau + 1)^2 + |\nabla b|^2 (\tau + 1)^2, \]
where \( m_2(b) := \frac{1}{4\kappa_2} - 1 \) satisfying \( m_2(0) = 0 \). We then infer that
\[ \left\| \frac{(\tau + 1)^2}{(1 + b)^2} |\nabla b|^2 \right\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \leq C \left[1 + \|b\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\right] \left\| |\nabla b|^2 \right\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \left(\|\tau\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} + 1\right)^2 \]
\[ \leq C (1 + c)^3 c^2. \]
The term $\frac{\tau^2}{1+b} \Delta b \cdot \nabla \tau$ is handled the same as $\frac{\tau^1}{1+b} |\nabla b|^2$. The third term in $G$ can be rewritten as

$$\frac{\tau^2}{1+b} \Delta b = m_1(b) \tau \Delta b(\tau + 2) + \tau \Delta b(\tau + 2),$$

which is estimated in the same way. The fourth term of $G$ is in fact $-m_1(b) \Delta b$, so that

$$\left\| \frac{b}{1+b} \Delta b \right\|_{L^1_T(B^{3/2}_{2,1})} \leq C \|b\|_{L^\infty_T(B^{3/2}_{2,1})} \|\Delta b\|_{L^1_T(B^{3/2}_{2,1})} \leq C^2.$$

Likewise

$$\|b \Delta \tau\|_{L^1_T(B^{3/2}_{2,1})} + \|\tau \Delta \tau\|_{L^1_T(B^{3/2}_{2,1})} \leq C^2.$$

The seventh term of $G$ becomes $(1+b)\tilde{\kappa}_3(0) |\nabla \tau|^2 + (1+b)\tilde{\kappa}_3(\tau) \Delta \tau$, where $\tilde{\kappa}_3(0) = 0$ and $\tilde{\kappa}_3' \Delta \tau$ is handled the same as $\tilde{\kappa}_3(\tau) \Delta \tau$. The $L^1_T(B^{3/2}_{2,1})$-norm for this term is controlled by $C(1+c) c^2$. The last term in $G$ is similarly bounded by $C c^2$. Thus, we have

$$\|G(b, \tau)\|_{L^1_T(B^{3/2}_{2,1})} \leq C (1+c)^3 c^2.$$

Finally, combining the above estimate and the one for $F$ with (4.14), we obtain that

$$\|\Phi(b, \tau)\|_{E(T)} \leq C \|(a_0, \tilde{\theta}_0)\|_{B^{3/2}_{2,1}} + C(1+c)^3 c^2.$$

This implies $\Phi(K(T)) \subset K(T)$ provided that

$$c \leq \min\{1, \frac{1}{16C}\} \quad \text{and} \quad \|(a_0, \tilde{\theta}_0)\|_{B^{3/2}_{2,1}} \leq \frac{1}{2C} c.$$

Next, we will prove that for any $T > 0$, the map $\Phi(b, \tau)$ is contractive on $K(T)$. Indeed, for $(v_i, \tau_i) \in K(T)$, let $(a_i, \tilde{\theta}_i) = \Phi(v_i, \tau_i)$ for $i = 1, 2$. Moreover, we set $\bar{a} = a_1 - a_2$ and $\bar{\theta} = \tilde{\theta}_1 - \tilde{\theta}_2$. Then $(\bar{a}, \bar{\theta})$ satisfies the equation

$$\begin{cases}
\partial_t \bar{a} - \kappa_1 \Delta \bar{a} + \kappa_1 \Delta \bar{\theta} = \delta F, \\
\kappa_2 \partial_t \bar{\theta} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \bar{\theta} + \kappa_1^2 \Delta \bar{a} = \delta G,
\end{cases}
$$

and $$(\bar{a}, \bar{\theta})|_{t=0} = (0, 0),$$

where $\delta F = F(b_1, \tau_1) - F(b_2, \tau_2)$ and $\delta G = G(b_1, \tau_1) - F(b_2, \tau_2)$. Applying Proposition 4.2 yields:

$$(\bar{a}, \bar{\theta}) \in L^\infty_T(B^{3/2}_{2,1}) + (\Delta \bar{a}, \Delta \bar{\theta}) \leq C \left( \|\delta F\|_{L^1_T(B^{3/2}_{2,1})} + \|\delta G\|_{L^1_T(B^{3/2}_{2,1})} \right).$$
Now, $\delta F$ can be rewritten as follows

\begin{equation}
\delta F = -2\kappa_1 \left( \frac{1}{1 + b_1} - \frac{1}{1 + b_2} \right) |\nabla b_2|^2 (\tau_2 + 1) - 2\kappa_1 \frac{1}{1 + b_1} \nabla \delta b \cdot \nabla b_2 (\tau_2 + 1) \\
- 2\kappa_1 \frac{1}{1 + b_1} \nabla b_1 \cdot \nabla \delta b (\tau_2 + 1) - 2\kappa_1 \frac{1}{1 + b_1} |\nabla b_1|^2 \delta \tau + 2\kappa_1 \nabla \delta b \cdot \nabla \tau_2 \\
+ 2\kappa_1 \nabla b_1 \cdot \nabla \delta \tau - \kappa_1 \delta b \Delta \tau_2 - \kappa_1 b_1 \Delta \delta \tau + \kappa_1 \delta \tau \Delta b_2 + \kappa_1 \tau_1 \Delta \delta b,
\end{equation}

where $\delta b = b_1 - b_2$, $\delta \tau = \tau_1 - \tau_2$. Moreover, for the first term we also have

\begin{equation}
\frac{1}{1 + b_2} - \frac{1}{1 + b_1} = \frac{1}{(1 + b_2)(1 + b_1)} \delta b = (m_1(b_1) + 1)(m_1(b_2) + 1) \delta b.
\end{equation}

Then we can estimate the terms in $\delta F$ analogously to the terms in (4.13) to obtain

\begin{align*}
||\delta F||_{L_T^p(\bar{\Omega}^{3/2})} &\leq C(||b_1||_{L_T^p(\bar{\Omega}^{3/2})} + 1)(||b_2||_{L_T^p(\bar{\Omega}^{3/2})} + 1)||\delta b||_{L_T^p(\bar{\Omega}^{3/2})} ||\nabla b_2||_{L_T^p(\bar{\Omega}^{3/2})}^2 \\
& \quad \times (||\tau_2||_{L_T^p(\bar{\Omega}^{3/2})} + 1) \\
& \quad + C(||b_1||_{L_T^p(\bar{\Omega}^{3/2})} + 1)||\nabla \delta b||_{L_T^p(\bar{\Omega}^{3/2})} ||\nabla b_2||_{L_T^p(\bar{\Omega}^{3/2})} ||\nabla b_1||_{L_T^p(\bar{\Omega}^{3/2})} ||\tau_2||_{L_T^p(\bar{\Omega}^{3/2})} + 1) \\
& \quad + C(||b_1||_{L_T^p(\bar{\Omega}^{3/2})} + 1)||\nabla \delta b||_{L_T^p(\bar{\Omega}^{3/2})} ||\nabla \tau_2||_{L_T^p(\bar{\Omega}^{3/2})} + 1) \\
& \quad + C(||b_1||_{L_T^p(\bar{\Omega}^{3/2})} + 1)||\nabla \tau_1||_{L_T^p(\bar{\Omega}^{3/2})} ||\delta \tau||_{L_T^p(\bar{\Omega}^{3/2})} \\
& \leq C(1 + c)^3 c (||\delta b||_{E(T)} + ||\delta \tau||_{E(T)}).
\end{align*}

A similar methodology is applied for the terms in $\delta G$. We write

\begin{align*}
\delta G &= \kappa_1^2 J_1 - (3\kappa_1^2 + \kappa_1 \kappa_2) J_2 - \kappa_1^2 J_3 + \kappa_1^2 J_4 \\
& \quad + \kappa_1 \kappa_2 J_5 + \kappa_1^2 J_6 + J_7 + \kappa_1 (\kappa_1 + \kappa_2) J_8.
\end{align*}

Each of the $J_i$ terms correspond to the difference operator $\delta$ applied to each respective term in the expression for $G$ in (4.13). Specifically, we have

\begin{align*}
J_1 &= \left( \frac{1}{1 + b_1} - \frac{1}{1 + b_2} \right) \left( \frac{(\tau_2 + 1)^2}{1 + b_2} + \frac{(\tau_2 + 1)^2}{1 + b_1} \right) |\nabla b_2|^2 \\
& \quad + \frac{\delta \tau (\tau_1 + \tau_2 + 2)}{(1 + b_1)^2} |\nabla b_2|^2 + \frac{(\tau_1 + 1)^2}{(1 + b_1)^2} \nabla \delta b \cdot (\nabla b_1 + \nabla b_2),
\end{align*}

\begin{align*}
J_2 &= \left( \frac{1}{1 + b_1} - \frac{1}{1 + b_2} \right) \left( \frac{\tau_1}{1 + b_1} \right) |\nabla b_2| \cdot \nabla \tau_2 + \frac{\delta \tau}{1 + b_1} \nabla b_2 \cdot \nabla \tau_2 \\
& \quad + \frac{\tau_1 + 1}{1 + b_1} \nabla \delta b \cdot \nabla \tau_2 + \frac{\tau_1 + 1}{1 + b_1} \nabla b_1 \cdot \nabla \delta \tau_2,
\end{align*}

\begin{align*}
J_3 &= \left( \frac{1}{1 + b_1} - \frac{1}{1 + b_2} \right) \left( \frac{\tau_1}{1 + b_2} \right) \Delta b_2 + \frac{\delta \tau (\tau_1 + \tau_2 + 2)}{1 + b_1} \Delta b_2 + \frac{\tau_1^2 + 2 \tau_1}{1 + b_1} \Delta \delta b,
\end{align*}

\begin{align*}
J_4 &= \left( \frac{b_1}{1 + b_1} - \frac{b_2}{1 + b_2} \right) \Delta b_2 + \frac{b_1}{1 + b_1} \Delta \delta b,
\end{align*}

\begin{align*}
J_5 &= \delta b \Delta \tau_2 + b_1 \Delta \delta \tau.
\end{align*}
\[ J_6 = \delta \tau \Delta \tau_2 + \tau_1 \Delta \delta \tau, \]
\[ J_7 = \delta b(\tilde{\kappa}_3(\tau_2) \Delta \tau_2 + \tilde{\kappa}_3'(\tau)|\nabla \tau_2|^2) \]
\[ + (1 + b_1)(\tilde{\kappa}_3(\tau_1) - \tilde{\kappa}_3(\tau_2)) \Delta \tau_2 + (1 + b_1)\tilde{\kappa}_3'(\tau_1) \Delta \delta \tau \]
\[ + (1 + b_1)(\tilde{\kappa}_3' \tau_1) - \tilde{\kappa}_3'(\tau_2))|\nabla \tau_2|^2 + (1 + b_1)\tilde{\kappa}_3'(\tau_1) \nabla \delta \tau \cdot (\nabla \tau_1 + \nabla \tau_2), \]
\[ J_8 = \nabla \delta \tau \cdot (\nabla \tau_1 + \nabla \tau_2). \]

The estimate for each \( J_i \) is similar to those already established, keeping in mind the continuity of products in Besov spaces, Lemma 1.6 in [6], and the identity (4.20) to control several terms in \( J_1, J_2, J_3, \) and \( J_4. \) The only subtle point is that, to estimate \( J_7, \) we need to invoke the mean-value theorem to write

\[ |\tilde{\kappa}_3(\tau_1) - \tilde{\kappa}_3(\tau_2)| \leq C|\delta \tau| \quad |\tilde{\kappa}_3'(\tau_1) - \tilde{\kappa}_3'(\tau_2)| \leq C|\delta \tau|, \]

where \( C \) above depends on the upper bound for \( |\tilde{\kappa}_3'| \) and \( |\tilde{\kappa}_3''| \). Since this upper bound exists by assumption, we can infer that

\[ \|\delta G\|_{L^1_t(\dot{H}^{2/3}_x)} \leq C(1 + c)^3 \left( \|\delta b\|_{E(T)} + \|\delta \tau\|_{E(T)} \right). \]

Therefore,

\[ \|\tilde{\alpha}, \tilde{\theta}\|_{L^\infty_t(\dot{B}^{2/3}_{2,1})} + \|(\Delta \tilde{\alpha}, \Delta \tilde{\theta})\|_{L^1_t(\dot{B}^{2/3}_{2,1})} \leq C(1 + c)^3 \left( \|\delta b\|_{E(T)} + \|\delta \tau\|_{E(T)} \right). \]

If we additionally assume that \( c \leq \frac{1}{64C} \), we then have that for all \( T > 0 \)

\[ \|\tilde{\alpha}, \tilde{\theta}\|_{L^\infty_t(\dot{B}^{2/3}_{2,1})} + \|(\Delta \tilde{\alpha}, \Delta \tilde{\theta})\|_{L^1_t(\dot{B}^{2/3}_{2,1})} \leq \frac{1}{2} \left( \|\delta b\|_{E(T)} + \|\delta \tau\|_{E(T)} \right). \]

Thus, \( \Phi \) is contractive as a mapping from \( E(T) \) to \( E(T) \). The proof of Theorem 1.3 follows from Lemma 4.1.

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