THE EULER PRODUCT EXPRESSIONS OF THE ABSOLUTE TENSOR PRODUCTS OF THE DIRICHLET L-FUNCTIONS

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Abstract. In this paper, we calculate the absolute tensor square of the Dirichlet L-functions and show that it is expressed as an Euler product over pairs of primes. The method is to construct an equation to link primes to a series which has the factors of the absolute tensor product of the Dirichlet L-functions. This study is a generalization of Akatsuka’s theorem on the Riemann zeta function, and gives a proof of Kurokawa’s prediction proposed in 1992.

1. Introduction

In 1992 Kurokawa [1] defined the absolute tensor products (Kurokawa tensor products). The definition is given by

$$(Z_1 \otimes \cdots \otimes Z_r)(s) := \prod_{\rho_1, \cdots, \rho_r \in \mathbb{C}} ((s - \rho_1 - \cdots - \rho_r))^\mu(\rho_1, \cdots, \rho_r)$$

for some zeta functions $Z_j(s)$ ($j = 1, \cdots, r$), where the symbol $\prod$, which was introduced by Deninger [2], represents the zeta regularized product (see below) and the integer $\mu(\rho_1, \cdots, \rho_r)$ is defined by

$$\mu(\rho_1, \cdots, \rho_r) := \mu_1(\rho_1) \cdots \mu_r(\rho_r) \times \begin{cases} 
1 & (\Im(\rho_1), \cdots, \Im(\rho_r) \geq 0), \\
(-1)^{r-1} & (\Im(\rho_1), \cdots, \Im(\rho_r) < 0), \\
0 & \text{(otherwise),}
\end{cases}$$

where $\mu_j(\rho)$ denotes the order of $\rho$ which is a zero of $Z_j(s)$; now, we regard the poles of $Z_j(s)$ as the zeros with negative orders in this paper. Here the zeta regularized products are defined by

$$\prod_{n=1}^{\infty} ((s - a_n))^{b_n} := \exp \left( -\lim_{w \to 0} \frac{Z_{a,b}(w, s)}{w^2} \right)$$

where $a := \{a_n\}_{n=1}^{\infty}$ and $b := \{b_n\}_{n=1}^{\infty}$ are complex sequences such that $Z_{a,b}(w, s) := \sum_{n=1}^{\infty} b_n (s - a_n)^{-w}$ converges locally, uniformly and absolutely in some $s$-region included in $\mathbb{C} - a$ for $\Re(w) > C$ with some constant $C \in \mathbb{R} > 0$ and is a meromorphic function of $w$ at $w = 0$. If $b \subset \mathbb{Z}$ then $\prod_{n=1}^{\infty} ((s - a_n))^{b_n}$ is a meromorphic function of $s$ in the whole $\mathbb{C}$ and has zeros only at $s = a_n$. The integer $b_n$ contributes to the order of $a_n$. See [3] for more details concerning the zeta regularized products. The factors of the zeta regularized products are derived from the summands of $Z_{a,b}(w, s)$, so we call $Z_{a,b}(w, s)$ the “factors series” in this paper.

In [1] Kurokawa also predicted that the absolute tensor product of $r$ arithmetic zeta functions which have the expression by the Euler product over primes would have the Euler product over $r$-tuples $(p_1, \cdots, p_r)$ of primes. The validity of Kurokawa’s prediction has been confirmed in some cases, for example, the cases of the Hasse zeta functions of finite fields by Koyama and Kurokawa [4] for $r = 2$, by
Akatsuka [5] for $r = 3$ and by Kurokawa and Wakayama [6] for general $r$. Also, the case of the Riemann zeta function for $r = 2$ was first proved by Koyama and Kurokawa [4], and then by Akatsuka [7] in a different way.

In this paper, according to Akatsuka’s method in [7], we will reach the Euler product expression of the absolute tensor product $(L_{\chi_1} \otimes L_{\chi_2})(s)$, where $L_{\chi_1}(s) := L(s, \chi_j) (j \in \mathbb{Z}_{>0})$ denotes the Dirichlet $L$-function corresponding to a primitive Dirichlet character $\chi_j$ to the modulus $N_j$ with $N_j \in \mathbb{Z}_{>2}$. The key item which leads to our goal is an equation which links the “factors series” of $(L_{\chi_1} \otimes \cdots \otimes L_{\chi_r})(s)$ to $r$-tuples of prime numbers (see Theorem 4.1 below). We name such equation the “key equation”. Letting $r = 1$, where $r$ is a parameter in the “key equation”, we obtain the zeta regularized product expression of $L(s, \chi_1)$:

**Theorem 1.1.** Let $\rho_{\chi_1}$ denote the imaginary zeros of $L(s, \chi_1)$ counted with multiplicity, and let $\tau_{\chi_1}^{(0)}$ be a possible real number with $0 < \tau_{\chi_1}^{(0)} < \frac{1}{2}$ and $L \left( \frac{1}{2} \pm \tau_{\chi_1}^{(0)} \right) = 0$. Then $L(s, \chi_1)$ has the following expression:

$$
L(s, \chi_1) = \prod_{\Im(\rho_{\chi_1}) \neq 0} \left( (s - \rho_{\chi_1}) \prod_{n=1}^{\infty} \left( s + 2n - \frac{3 + \chi_1(-1)}{2} \right) \right) 
\times \left( s - \frac{1}{2} - \tau_{\chi_1}^{(0)} \right)^{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})} \left( s - \frac{1}{2} + \tau_{\chi_1}^{(0)} \right)^{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})} \right),
$$

where $\mu_{\chi_1}(\tau_{\chi_1}^{(0)})$ and $\mu_{\chi_1}(0)$ denote the order of $\frac{1}{2} \pm \tau_{\chi_1}^{(0)}$ and the order of $\frac{1}{2}$ respectively.

**Remark 1.2.** As Theorem 1.1, define $\tau_{\chi_j}^{(0)}$ by $0 < \tau_{\chi_j}^{(0)} < \frac{1}{2}$ and $L \left( \frac{1}{2} \pm \tau_{\chi_j}^{(0)} \right) = 0$ for $j \in \mathbb{Z}_{>0}$. It is well known that the orders of $\frac{1}{2} \pm \tau_{\chi_j}^{(0)}$ are equal and at most one. For convenience, if $\mu_{\chi_j}(\tau_{\chi_j}^{(0)}) = 0$ then we define $\tau_{\chi_j}^{(0)} := \frac{1}{2}$.

Let $\rho_{\chi_j}$ denote the imaginary zeros of $L(s, \chi_j)$. From (1.1) and the definition of the absolute tensor products, we find that $(L_{\chi_1} \otimes L_{\chi_2})(s)$ has the following expression:

$$
(L_{\chi_1} \otimes L_{\chi_2})(s)
= \prod_{\Im(\rho_{\chi_1}), \Im(\rho_{\chi_2}) < 0} \left( s - \rho_{\chi_1} - \rho_{\chi_2} \right)^{-1} \prod_{\Im(\rho_{\chi_1}), \Im(\rho_{\chi_2}) > 0} \left( s - \rho_{\chi_1} - \rho_{\chi_2} \right)
\times \prod_{(a,b) \in \{1,2\} \times \{1,2\}} \left( s - \rho_{\chi_a} + 2n - \frac{3 + \chi_b(-1)}{2} \right)^{\mu_{\chi_b}(\tau_{\chi_b}^{(0)})}
\times \prod_{\Im(\rho_{\chi_a}) > 0} \left( s - \rho_{\chi_a} - \frac{1}{2} + \tau_{\chi_a}^{(0)} \right)^{\mu_{\chi_a}(\tau_{\chi_a}^{(0)})}
\times \prod_{\Im(\rho_{\chi_a}) > 0} \left( s - \rho_{\chi_a} - \frac{1}{2} - \tau_{\chi_a}^{(0)} \right)^{\mu_{\chi_a}(\tau_{\chi_a}^{(0)})}
\times \prod_{\Im(\rho_{\chi_a}) > 0} \left( s - \rho_{\chi_a} - \frac{1}{2} \right)^{\mu_{\chi_a}(0)}
$$
Now, let $p, q$ be primes and $j, m, n$ be positive integers, and let $\alpha$ be any fixed number with $0 < \alpha < 1$. For the complex numbers $\tau_{x_j}$ with $\rho_{x_j} = \frac{1}{2} + i\tau_{x_j}$, we define $\tau_{x_j}^{(1)} := \min\{\Re(\tau_{x_j}) > 0\}$; we fix $\epsilon_j$ arbitrarily with $0 < \epsilon_j < \min\{\tau_{x_j}^{(1)}, \tau_{x_j}^{(1)}\}$. Also, we define $\epsilon^{(r)} := \min_{j \in \{1, \ldots, r\}} \{\epsilon_j\}$. Define that

\begin{align*}
E_1(w, s, \{\chi_j\}) &:= \frac{i}{2\pi} \sum_p \sum_{m=1}^{\infty} \chi_1(p^m)\chi_2(p^m)p^{-ms}(m \log p)^{w-2} \log p^2 + \frac{i(s-2)}{2\pi} \sum_p \sum_{m=1}^{\infty} \chi_1(p^m)\chi_2(p^m)p^{-ms}(m \log p)^{w-1} \log p^2,

E_2(w, s, \{\chi_j\}) &:= \frac{i}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p, m, q, n, p^m \neq q^n} \chi_a(p^m)\chi_b(q^n)p^{-m(s-1)q^{-n}(m \log p)^w \log p} \frac{\chi(qn)p^{-m(s+\alpha)q^{-n(1+\alpha)}}}{n(m \log p + n \log q)} \log p,

E_3(w, s, \{\chi_j\}) &:= \frac{1}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p, m, q, n} \chi_a(p^m)\chi_b(q^n)p^{-m(s+\alpha)q^{-n(1+\alpha)}}n(m \log p + n \log q)^w \log p,

E_4(w, s, \{\chi_j\}) &:= -\frac{1}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p, m, n} \chi_a(-1)\chi_b(p^m)p^{-m(s+\alpha)}q^{-n(1+\alpha)}n(m \log p + n \log q)^w \log p,

E_5(w, s, \{\chi_j\}) &:= \frac{i}{2} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p, m} \chi_a(p^m)p^{-m(1-\chi_b(-1))} \sin(im \log p) \log p^w \log p,

E_6(w, s, \{\chi_j\}) &:= \frac{i}{2} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p, m} \chi_a(p^m)p^{-m(1-\chi_b(-1))} \sin(im \log p) \log p^{w-1} \log p,
\end{align*}
Then, letting \( r = 2 \) in the “key equation”, we can deduce the Euler product expression of \((L_{\chi_1} \otimes L_{\chi_2})(s)\) as follows:

**Theorem 1.3.** In \( \Re(s) > 2 \) we have

\[
(L_{\chi_1} \otimes L_{\chi_2})(s) = \exp \left( \sum_{k=1}^{10} E_k(s, \{\chi_j\}^2_{j=1}) \right),
\]
where \( E_k(s, \{ \chi_j \}^2_{j=1}) := E_k(0, s, \{ \chi_j \}^2_{j=1}) \), that is,

\[
E_1(s, \{ \chi_j \}^2_{j=1}) := -\frac{i}{2\pi} \sum_{p,m} \chi_1(p^m) \chi_2(p^m) p^{-ms} + \frac{i(s - 2)}{2\pi} \sum_{p,m} \chi_1(p^m) \chi_2(p^m) p^{-ms} \log p,
\]

\[
E_2(s, \{ \chi_j \}^2_{j=1}) := \frac{i}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p,m,q,n} \chi_a(p^m) \chi_b(q^n) p^{-m(s-1)} q^{-n} \log p, \quad n(n \log p - n \log q),
\]

\[
E_3(s, \{ \chi_j \}^2_{j=1}) := \frac{1}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p,m,q,n} \chi_a(p^m) \chi_b(q^n) p^{-m(s+\alpha)} q^{-n(1+\alpha)} \log q, \quad n(m \log p + n \log q),
\]

\[
E_4(s, \{ \chi_j \}^2_{j=1}) := \frac{i}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p,m,n} \chi_a(-1) \chi_b(p^m) p^{-m(s+\alpha)} e^{-i\alpha \pi} m^2 m(\log p - n\pi) \log p,
\]

\[
E_5(s, \{ \chi_j \}^2_{j=1}) := \frac{i}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \sin(\log p),
\]

\[
E_6(s, \{ \chi_j \}^2_{j=1}) := \frac{i}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \int_{S^1} \sum_{p,m} p^{-m(s-u)} (\log p) \chi_a(p^m) \log L(u, \chi_b) du,
\]

\[
E_7(s, \{ \chi_j \}^2_{j=1}) := \frac{1}{2\pi} \sum_{(a,b) \in \{(1,2),(2,1)\}} \left( \log \left( \frac{\chi_a(-1) \Gamma(1+\alpha) N^a G(\chi_a)}{(2\pi)^{1+\alpha}} \right) + \gamma + \log \left( \frac{2\pi}{N^a} + \frac{\pi i}{2} \right) \right) \times \sum_{p,m} \chi(b(p^m) p^{-m(s+\alpha)} m^2 \log p
\]

\[
+ \sum_{a=1}^2 \left( -\frac{1+\alpha}{4} \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \right) m
\]

\[
+ \frac{i}{2\pi} \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \int_0^\infty \frac{1}{e^u - 1} \frac{u + m(\log p)(1 - e^{-\alpha \pi})}{u + m \log p} du\right),
\]

\[
E_8(s, \{ \chi_j \}^2_{j=1}) := \sum_{(a,b) \in \{(1,2),(2,1)\}} \mu_{\chi_a} (\chi_b) \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \frac{1}{m},
\]

\[
E_9(s, \{ \chi_j \}^2_{j=1}) := \sum_{(a,b) \in \{(1,2),(2,1)\}} \mu_{\chi_a} (\chi_b) \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \frac{1}{m},
\]

\[
E_{10}(s, \{ \chi_j \}^2_{j=1}) := \sum_{(a,b) \in \{(1,2),(2,1)\}} \mu_{\chi_a} (0) \sum_{p,m} \chi_a(p^m) p^{-m(s+\alpha)} \frac{1}{m}.
\]

The proofs of Theorem 1.1 and Theorem 1.3 are given in Section 5 and Section 6 respectively. The contents of the other sections are as follows. In Section 2 some lemmas are proved which are made use of in Section 3 or later. In Section 3 a series
is introduced which includes information on the zeros of the Dirichlet L-functions and some properties of the series is shown. In Section 4 the “key equation” is deduced.

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2. Lemmas

In this section, we prove some lemmas which are used later.

Lemma 2.1. Let $c \in \mathbb{C} - \{0\}$ and $\delta \in \mathbb{R}_{>0}$ be any fixed numbers.

(i) Suppose that $f(u)$ satisfies $f(u) = O(1) (u \to 0)$, $O(u^{-\delta}) (u \to \infty)$ and is holomorphic on $\mathbb{C} - \{0\}$. Define

$$F_1(z) := \int_0^\infty \frac{f(u)}{u - cz} du \quad (\Im(cz) < 0).$$

Then $F_1(z) + f(cz) \log z$ is a single-valued meromorphic function of $z$ on $\mathbb{C} - \{0\}$.

(ii) Suppose that $f(u)$ satisfies $f(u) = O(1) (u \to 0)$, $O(u^{1-\delta}) (u \to \infty)$ and is holomorphic on $\mathbb{C} - \{0\}$. Define

$$F_2(z) := \int_0^\infty \frac{f(u)}{u^2 - (cz)^2} du \quad (\Im(cz) < 0).$$

Then $F_2(z) + \frac{f(cz) - f(-cz)}{2cz} \log z$ is a single-valued meromorphic function of $z$ on $\mathbb{C} - \{0\}$.

Proof of Lemma 2.1. (i) If $cz$ is in the fourth quadrant, then by Cauchy’s theorem we have

$$\lim_{X \to \infty} \int_{P_1 \cup P_2} \frac{f(u)}{u - cz} du = 0, \quad (2.1)$$

where

$$P_1 := \{u \in \mathbb{R} \mid 0 \leq u \leq X\},$$

$$P_2 := \left\{ X e^{i\phi} \mid 0 \leq \phi \leq \frac{3\pi}{2} \right\} \cup \{u \in \mathbb{C} \mid \Re(u) = 0, -X \leq \Im(u) \leq 0\}$$

for $X \in \mathbb{R}_{>0}$ and we go around the integral path in the counterclockwise direction. It follows from (2.1) that

$$F_1(z) = -\lim_{X \to \infty} \int_{P_1 \cup P_2} \frac{f(u)}{u - cz} du. \quad (2.2)$$

Since the integral path in the right-hand side of (2.2) doesn’t include the positive real axis, (2.2) remains holomorphic while $cz$ moving from the fourth quadrant into the first one across that axis. Therefore, (2.2) gives the analytic continuation of $F_1(z)$ with $cz$ in the first quadrant. On the other hand, when $cz$ is in the first quadrant, by Cauchy’s theorem we have

$$\lim_{X \to \infty} \int_{P_1 \cup P_2} \frac{f(u)}{u - cz} du = 2\pi i f(cz).$$

From this and (2.2) it follows that

$$F_1(z) = -2\pi i f(cz) + \int_0^\infty \frac{f(u)}{u - cz} du. \quad (2.3)$$
Rem 2.2. If \( \Im(\text{cz}) > 0 \) then we have
\[
F_2(z) = -\pi i \frac{h(\text{cz})}{\text{cz}} + \int_0^\infty \frac{h(u)}{u^2 - (\text{cz})^2} du.
\]
We use this in the proof of Lemma 2.6.

Define that
\[
H(t) := \frac{1}{t} \int_0^\infty \frac{1}{e^u - 1} \frac{u - it(1 - e^{-in})}{u - it} du \quad (\Re(t) < 0), \tag{2.4}
\]
\[
I_j(t) := \frac{1}{t} \int_0^\infty \frac{u^{2j} e^{(1+2j)u}}{(e^u - 1)(a^2 + 4t^2)} du \quad (\Re(t) > 0), \tag{2.5}
\]
\[
J_j(t) := I_j(t) + \frac{\log t}{4\sin^2 \frac{u}{2}} \tag{2.6}
\]
for \( j \in \mathbb{Z}_{>0} \). For these functions, we show the following two lemmas: Lemma 2.4 and Lemma 2.6.

Rem 2.3. In the following, it is found that \( H(t) \), \( I_j(t) \) and \( J_j(t) \) has the analytic continuations, and let the same symbols denote those continuations respectively.

Lemma 2.4. (i) \( H(t) \) has the following asymptotic behavior at \( t = 0 \) :
\[
H(t) = -\frac{e^{-i(\alpha + \frac{1}{2})t}}{2\sin \frac{u}{2}} \log t + O(1).
\]
(ii) \( H(t) + \frac{e^{-i(\alpha + \frac{1}{2})t}}{2\sin \frac{u}{2}} \log t \) is a single-valued meromorphic function on \( t \in \mathbb{C} \).
(iii) \( H(t) \) has the simple pole at \( t = 2n\pi \) (\( n \in \mathbb{Z} - \{0\} \)) with residue
\[
\omega_n e^{-\omega_n 2\alpha \pi i} \left( \arg t - \frac{1 - \omega_n}{2} \right),
\]
where \( \omega_n := \frac{n}{|m|} \).

Rem 2.5. If \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \) and the argument lies in \( \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) \), it follows from Lemma 2.4 (ii) that \( H(t) \) is a meromorphic function because \( \frac{e^{-i(\alpha + \frac{1}{2})t}}{2\sin \frac{u}{2}} \log t \) is such one.

Proof of Lemma 2.4. (i) (2.4) is equivalent to
\[
tH(t) = \left( \int_0^1 + \int_1^\infty \right) \frac{1}{e^u - 1} \frac{u - it(1 - e^{-in})}{u - it} du \quad (\Re(t) < 0), \tag{2.7}
\]
The second integral is holomorphic on \( t \in \mathbb{C} - i\mathbb{R}_{\leq 1} \) and particularly at \( t = 0 \) becomes
\[
\int_1^\infty \frac{1}{e^u - 1} du. \tag{2.8}
\]
Next, we consider the first integral of the right-hand side of (2.7). For $|u| < 2\pi$, we have

$$\frac{u - it(1 - e^{-\alpha u})}{e^u - 1} = \sum_{n=0}^{\infty} a_n(t)u^n$$

(2.9)

and then by the binomial theorem the right-hand side of (2.9) becomes

$$\sum_{n=0}^{\infty} a_n(t)(u - it)^n$$

$$= \sum_{n=0}^{\infty} a_n(t)(it)^n + \sum_{n=1}^{\infty} a_n(t)(u - it)^n + \sum_{n=2}^{\infty} a_n(t) \sum_{k=1}^{n-1} \binom{n}{k} (u - it)^{n-k}(it)^k$$

so

(the first integral of the right-hand side of (2.7))

$$= \sum_{n=0}^{\infty} a_n(t)(it)^n \int_0^1 \frac{1}{u - it} du + \sum_{n=1}^{\infty} a_n(t) \int_0^1 (u - it)^{n-1} du$$

$$+ \sum_{n=2}^{\infty} a_n(t) \sum_{k=1}^{n-1} \binom{n}{k} (it)^k \int_0^1 (u - it)^{n-k-1} du.$$  

(2.10)

The third term of (2.10) is holomorphic for $|t| < 1$ and vanishes at $t = 0$. The second term of (2.10) is equal to

$$\int_0^1 \sum_{n=1}^{\infty} a_n(0)u^{n-1} du = \int_0^1 \left( \frac{1}{e^u - 1} - \frac{1}{u} \right) du$$  

(2.11)

at $t = 0$, where we use $a_0(0) = 1$ because we have

$$\frac{u}{e^u - 1} = \sum_{n=0}^{\infty} a_n(0)u^n$$

from (2.9). Then, the first term of (2.10) is equal to

$$\sum_{n=0}^{\infty} a_n(t)(it)^n(\log(1 - it) - \log(-it)) = -\frac{ite^{-i\alpha t}}{e^{it} - 1} \log t + h_1(t)$$

for $|t| < 1$, where $h_1(t)$ is a power series which converges for $|t| < 1$. Noting that by Cramér [8, p. 117, (20)] it was shown that

$$(\text{the right-hand side of (2.11)}) + (2.8) = 0,$$

it follows that

$$H(t) = -\frac{ie^{-i\alpha t}}{e^{it} - 1} \log t + h_2(t) = -\frac{e^{-i(\alpha + \frac{3\pi}{2})t}}{2\sin \frac{3\pi}{2}} \log t + h_2(t),$$  

(2.12)

where $h_2(t)$ is a power series which converges in $|t| < 1$. The proof of (i) is complete.

(ii) We find that $H(t) + \frac{e^{-i(\alpha + \frac{3\pi}{2})t}}{2\sin \frac{3\pi}{2}} \log t$ is holomorphic for $|t| < 1$ from (2.12) and is a single-valued function on $t \in \mathbb{C} - \{0\}$ from Lemma 2.1 (i). The proof of (ii) is complete.

(iii) By (2.4) it is easily found that $H(t)$ is holomorphic if $\arg t \in \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right)$. From this and Lemma 2.4 (ii) we can obtain the desired result. □
Lemma 2.6. \( J_j(t) \) has the following properties:
(i) \( J_j(t) \) is a single-valued meromorphic function on \( t \in \mathbb{C} - \{0\} \).
(ii) \( J_j(t) \) satisfies that for \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \)
\[
J_j(t) + J_j(-t) = \begin{cases} 
-\pi i e^{-\frac{\chi_{j(-1)}i}{2}t} - \frac{i\pi}{2\sin t} & (\Re(t) < 0), \\
\pi i e^{\frac{\chi_{j(-1)}i}{2}t} + \frac{i\pi}{2\sin t} & (\Re(t) > 0), 
\end{cases}
\]
where the argument lies in \((-\frac{\pi}{2}, \frac{3\pi}{2})\).

Proof of Lemma 2.6. (i) We should use Lemma 2.1 (ii) as
\[
c = -2t, \; z = t \quad \text{and} \quad f(u) = \frac{ue^{\frac{1+\chi_j(-1)}{4}u} (\frac{u}{2} \cos \frac{u}{2} - t \sin \frac{u}{2})}{t(e^u - 1)}.
\]
(ii) Let \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \) and the argument lie in \((-\frac{\pi}{2}, \frac{3\pi}{2})\). Then, since \( \frac{\log t}{4\sin \frac{t}{2}} \) is meromorphic, by Lemma 2.6 (i) we find that \( I_j(t) \) is as well.

Now, by Remark 2.2, for \( \Re(t) < 0 \) we have
\[
I_j(t) = -\pi i \frac{e^{-\frac{\chi_{j(-1)}i}{2}t}}{\sin t} + \frac{1}{t} \int_0^\infty \frac{ue^{\frac{1+\chi_j(-1)}{4}u} (\frac{u}{2} \cos \frac{u}{2} - t \sin \frac{u}{2})}{(e^u - 1)(u^2 + 4t^2)} du.
\]
Adding \( \frac{\log t}{4\sin \frac{t}{2}} \) to the both sides of (2.14), we obtain
\[
J_j(t) = (\text{the right-hand side of (2.14)}) + \frac{\log t}{4\sin \frac{t}{2}} \quad (\Re(t) < 0). \tag{2.15}
\]
On the other hand, from (2.5) and (2.6) we can obtain the following equations: for \( \Re(t) < 0 \)
\[
J_j(-t) = I_j(-t) - \frac{\log(-t)}{4\sin \frac{t}{2}}
= -\frac{1}{t} \int_0^\infty \frac{ue^{\frac{1+\chi_j(-1)}{4}u} (\frac{u}{2} \cos \frac{u}{2} - t \sin \frac{u}{2})}{(e^u - 1)(u^2 + 4t^2)} du - \frac{\log t}{4\sin \frac{t}{2}} - \frac{i\pi}{4\sin \frac{t}{2}}. \tag{2.16}
\]
From (2.15) and (2.16) it is follows that
\[
J_j(t) + J_j(-t) = -\pi i \frac{e^{\frac{1+\chi_{j(-1)}i}{4}t}}{\sin t} - \frac{i\pi}{4\sin \frac{t}{2}} \quad (\Re(t) < 0), \tag{2.17}
\]
which is (2.13) with \( \Re(t) < 0 \). By replacing \( t \) with \(-t\) in (2.17), we obtain (2.13) with \( \Re(t) > 0 \).

\( \square \)

Lemma 2.7 was proved by Akatsuka [7].

Lemma 2.7. (i) [7, Lemma 2.5] For any \( X, Y \in \mathbb{R}_{>0} \) satisfying \( X < Y \)
\[
\log Y - \log X \geq \frac{Y - X}{Y}.
\]
(ii) [7, Remark 2.1] \( \sum_p \sum_{m=1}^\infty \frac{p^{-m}}{m^2 \log p} < \infty. \)
(iii) [7, p639, (4.4)] For any fixed \( \delta \in \mathbb{R}_{>0} \) and any \( A \in \mathbb{R} \)
\[
\sum_p \sum_{m=1}^\infty p^{-m(1+\delta)}(m \log p)^A \log p < \infty.
\]

We prove a formula for the gamma function in the following lemma.
Lemma 2.8. Let any fixed $\psi \in \mathbb{R}$ satisfy $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$ and let $\arg \nu \in (-\psi - \frac{\pi}{2}, -\psi + \frac{\pi}{2})$ and $\Re(w) > 0$. Then, we have

$$\frac{\Gamma(w)}{\nu^w} = \int_0^{\infty} e^{\nu t} \frac{dt}{t}.$$  

Proof of Lemma 2.8. For any fixed $\psi \in \mathbb{R}$ satisfying $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$, let $\arg \nu = -\psi$. Then, we have

$$\frac{\Gamma(w)}{\nu^w} = \int_0^{\infty} e^{-\nu t} \left( \frac{t}{\nu} \right)^w \frac{dt}{t} = \int_0^{\infty} e^{-\nu t} \frac{dt}{t} = \int_0^{\infty} e^{-\nu t} \frac{dt}{t}. $$

When $w$ is fixed in $\Re(w) > 0$, the both sides are holomorphic in

$$\{ \nu \in \mathbb{C} \mid \Re(\nu e^{i\psi}) > 0 \} = \{ \nu \in \mathbb{C} \mid -\psi < \arg \nu < \frac{\pi}{2} - \psi \}.$$ 

This completes the proof. □

3. Properties of a series concerning the zeros of the Dirichlet L-functions

For a series $\theta(t) := \sum \Re(\tau) e^{-\tau t} (\Re(t) > 0)$ where $\tau \in \mathbb{C}$ with $\rho = \frac{1}{2} + it$ for the imaginary zeros $\rho$ of the Riemann zeta function, Cramér [8] deduced the explicit formula and then Guinand [9] obtained the meromorphic continuation and the poles by proving the functional equation and derived the approximate behavior. Akatsuka [7] introduced $\theta^*(t) := \theta(t) - e^{-t} (t \in \mathbb{C} - i\mathbb{R}_{\leq 0})$ and proved the properties on the basis of the results of Cramér and Guinand.

We define a following series :

$$l_{\chi_j}(t) := \sum_{\Re(\tau_{\chi_j}) > 0} e^{-\tau_{\chi_j} t} (\Re(t) > 0).$$

for $j \in \mathbb{Z}_{>0}$. With reference to the methods of the above three mathematicians we research in this series.

We define the complete Dirichlet L-function $\hat{L}(s, \chi_j)$ by

$$\hat{L}(s, \chi_j) := \left( \frac{\pi}{N_j} \right)^{\left( \frac{s}{2} + \frac{1}{4} - \frac{\chi_j}{2} \right)} \Gamma \left( \frac{s}{2} + \frac{1 - \chi_j}{4} \right) L(s, \chi_j)$$

and define that

$$\xi(s, \chi_j) := \hat{L} \left( s + \frac{1}{2}, \chi_j \right).$$

It is well known that $\frac{\xi'}{\xi}(s, \chi_j)$ satisfies the functional equation :

$$\frac{\xi'}{\xi}(-s, \chi_j) = -\frac{\xi'}{\xi}(s, \chi_j).$$

First, we prove the meromorphic and the functional equation of $l_{\chi_j}(t)$.

Theorem 3.1. $l_{\chi_j}(t)$ has a meromorphic continuation to $\mathbb{C} - i\mathbb{R}_{\leq 0}$ for which

$l_{\chi_j}(t) + l_{\chi_j}(-t)$ 

$$= \begin{cases} 
\frac{ie^{-\chi_j(-1)i\frac{t}{2}}}{2 \sin t} - \mu_{\chi_j}(\tau_{\chi_j}^{(0)}) (e^{i\chi_j(0)t} + e^{-i\chi_j(0)t}) - \mu_{\chi_j}(0) & (\Re(t) < 0), \\
\frac{ie^{-\chi_j(-1)i\frac{t}{2}}}{2 \sin t} - \mu_{\chi_j}(\tau_{\chi_j}^{(0)}) (e^{i\chi_j(0)t} + e^{-i\chi_j(0)t}) - \mu_{\chi_j}(0) & (\Re(t) > 0), 
\end{cases}$$

where the argument lies in $(-\frac{\pi}{2}, \frac{3\pi}{2})$. 

10 H. TANAKA
Proof of Theorem 3.1. If \( \Re(t) > 0 \), then by Cauchy’s theorem we have

\[
I_{\chi_j}(t) = \frac{1}{2\pi i} \int_{C_1 \cup C_{2,j} \cup C_3} e^{ist} \frac{\zeta'(s)}{\zeta(s)} \zeta(s, \chi_j) ds
\]

where

\[
C_1 := \{ s \in \mathbb{C} \mid \Re(s) = -\frac{1}{2}, \Im(s) \geq 0 \}
\]

\[
C_{2,j} := \left\{ \frac{1}{2} \cos \varphi + i \varepsilon_j \sin \varphi \mid 0 \leq \varphi \leq \pi \right\}
\]

\[
C_3 := \{ s \in \mathbb{C} \mid \Re(s) = \frac{1}{2}, \Im(s) \geq 0 \}
\]

and we go around the integral path in the counterclockwise direction.

First, we consider the integral of the path \( C_1 \). It becomes

\[
\int_{C_1} = \int_{-\infty}^{0} e^{i(\frac{1}{2} + iy)} \frac{\zeta'(s)}{\zeta(s)} \left(-\frac{1}{2} + iy, \chi_j\right) dy
\]

\[
= -ie^{-\frac{\pi i}{4}} \int_{0}^{\infty} e^{-yt} \frac{\zeta'(s)}{\zeta(s)} \left(\frac{1}{2} - iy, \chi_j\right) dy
\]

\[
= ie^{-\frac{\pi i}{4}} \int_{0}^{\infty} e^{-yt} \left(-\frac{1}{2} \log \left(\frac{\pi}{N_j}\right) + \frac{1}{2} \Gamma' \left(\frac{3 - \chi_j(-1)}{4} - \frac{iy}{2}\right) + \frac{L'}{L}(1 - iy, \chi_j)\right) dy.
\]

The term concerning (3.4) becomes

\[
-\frac{i}{2} e^{-\frac{\pi i}{4}} \log \left(\frac{\pi}{N_j}\right) \int_{0}^{\infty} e^{-yt} dy = -\frac{i}{2t} e^{-\frac{\pi i}{4}} \log \left(\frac{\pi}{N_j}\right).
\]

Concerning (3.5), since

\[
\frac{\Gamma'}{\Gamma}(s) = \int_{0}^{\infty} \left(\frac{e^{-u} - e^{-(s-1)u}}{e^u - 1}\right) du \quad (\Re(s) > 0),
\]

it follows that

\[
\frac{i}{2} \int_{0}^{\infty} e^{-yt} \int_{0}^{\infty} \left(\frac{e^{-u}}{u} - \frac{e^{-u(s-1) + \frac{iy}{2}}}{e^u - 1}\right) du dy
\]

\[
= \frac{i}{2} e^{-\frac{\pi i}{4}} \int_{0}^{\infty} \left(\frac{e^{-u} - e^{-(s-1)u}}{e^u - 1}\right) \int_{0}^{\infty} e^{-yt} dy du
\]

\[
= \frac{i}{2} e^{-\frac{\pi i}{4}} \int_{0}^{\infty} \left(\frac{e^{-u}}{u} - \frac{1 + \chi_j(-1)}{t(t - \frac{iy}{2})}\right) du.
\]

Concerning (3.6), since the Euler product \( \prod_p (1 - \chi_j(p)p^{-s})^{-1} = L(s, \chi_j) \) converges uniformly on \( \Re(s) = 1 \), we have

\[
ie^{-\frac{\pi i}{4}} \int_{0}^{\infty} e^{-yt} \left(-\sum_{p} \sum_{m=1}^{\infty} \chi_j(p^m) \log p \frac{p^{-m(1-iy)}}{p^{-m(1-iy)}}\right) dy
\]
\[= -ie^{-i\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t - im \log p} \int_0^\infty e^{(-t + im \log p)y} dy\]
\[= -ie^{-i\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t - im \log p}.\]

Similarly, we can calculate the integral of the path \(C_3\) in (3.3) and obtain the following result:

\[\int_{C_3} = -\frac{ie^{-i\frac{\pi}{4}}}{2t} \log \left(\frac{\pi}{N}\right)\]

\[+ \frac{i}{2} \int_0^\infty \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{i}u} \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{t + im \log p}} \right) \right) du \]  
\[= -ie^{-i\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t + im \log p}.\]

Noting that

\[(3.7) + (3.8)\]

\[= \frac{i}{2} \left( \frac{2 \cos \frac{\pi}{4}}{t} \int_0^\infty e^{-u} du - \int_0^\infty e^{\frac{1 + \xi_j(-1)}{i}u} \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{t + im \log p}} \right) du \right)\]

\[= \frac{i}{2} \left( \frac{2 \cos \frac{\pi}{4}}{t} \int_0^\infty \left( e^{-u} - \int_0^\infty e^{\frac{1 + \xi_j(-1)}{i}u} e^{\frac{1 + \xi_j(-1)}{t + im \log p}} du \right) du \right)\]

\[= \frac{i}{2} \left( \frac{2 \cos \frac{\pi}{4}}{t} \int_0^\infty \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{i}u} - e^{\frac{1 + \xi_j(-1)}{t + im \log p}} \right) du \right)\]

we can deduce that for \(\Re(t) > 0\)

\[l_{x_j}(t) = -\frac{\cos \frac{\pi}{4}}{2\pi t} \log \left(\frac{\pi}{N_{x_j}}\right) + \frac{\cos \frac{\pi}{4}}{2\pi t} \int_0^\infty \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{i}u} \right) du + \frac{1}{\pi} I_j(t)\]

\[- \frac{1}{2\pi} e^{-\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t - im \log p} - \frac{1}{2\pi} e^{-\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t + im \log p}\]

\[+ \frac{1}{2\pi} \int_{C_{2,x}(\pi \to 0)} e^{im\xi(s, x_j)} ds.\]

Adding \(\log t\) to the both sides, we have

\[l_{x_j}(t) + \frac{\log t}{4\pi \sin \frac{\pi}{4}}\]

\[= -\frac{\cos \frac{\pi}{4}}{2\pi t} \log \left(\frac{\pi}{N_{x_j}}\right) + \frac{\cos \frac{\pi}{4}}{2\pi t} \int_0^\infty \left( e^{-u} - e^{\frac{1 + \xi_j(-1)}{i}u} \right) du + \frac{1}{\pi} I_j(t)\]

\[- \frac{1}{2\pi} e^{-\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t - im \log p} - \frac{1}{2\pi} e^{-\frac{\pi}{4}} \sum_0^{\infty} \sum_{m=1}^{\infty} \chi_j(p^m)\frac{p^{-m}}{t + im \log p}\]

\[+ \frac{1}{2\pi} \int_{C_{2,x}(\pi \to 0)} e^{im\xi(s, x_j)} ds.\]

(3.9)
Theorem 3.2. Define $S_j := \{ \frac{1}{2\pi} \cos \varphi + i \frac{1}{2} \sin \varphi + \frac{1}{2} \varphi \mid 0 \leq \varphi \leq \pi \}$. (i) $l_{\chi_j}(t)$ has the following expression for $\Re(t) > 0$:

$$l_{\chi_j}(t)$$
where

\[ l_{\chi_j}(t) = \frac{it}{2\pi} e^{\pm \frac{1}{2} \sum_{m=1}^{\infty} \chi_j(p^m)p^{-m}} e^{-i(\alpha + \frac{1}{4})t} \left( \int \sum_{m=1}^{\infty} \frac{\chi_j(p^m)p^{-m(1+\alpha)}}{m(t + im \log p)} + i \log \left( \frac{\chi_j(-1)\Gamma(1+\alpha)N^\alpha jG(\chi_j)}{(2\pi)^{1+\alpha}} \right) - \frac{(1+\alpha)\pi}{2} \right) \left\{ \begin{array}{c} \chi_j(-1)e^{-\alpha \pi m} \frac{1}{m(t-m\pi)} + i \log \left( \frac{\chi_j(-1)\Gamma(1+\alpha)N^\alpha jG(\chi_j)}{(2\pi)^{1+\alpha}} \right) - \frac{(1+\alpha)\pi}{2} \\
+ \frac{1}{t} \left( \gamma + \log \left( \frac{2\pi}{N_j} \right) + \frac{\pi i}{2} \right) - \frac{t}{2\pi} e^{-\frac{1}{2} \int_{S_j(\pi \to 0)} e^{-ist} \log L(s, \chi_j)ds} \\
- \frac{1}{2\sin t} \mu_{\chi_j}(\tau^{(0)}_{\chi_j})(e^{\nu_{\chi_j}(\tau^{(0)}_{\chi_j})t} + e^{-\nu_{\chi_j}(\tau^{(0)}_{\chi_j})t}) - \mu_{\chi_j}(0) \end{array} \right. \right\} \right. \tag{3.12} \]

(ii) \( l_{\chi_j}(t) \) has the following expression for \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \):

\[ l_{\chi_j}(t) = \left. \begin{array}{c} \frac{\log t}{2\pi t} - \frac{1}{2\pi t} \left( \log \left( \frac{2\pi}{N_j} \right) + \gamma + \frac{3\pi i}{2} \right) + O(1) \end{array} \right. \]

(\( \gamma \)) - (iv), the argument lies in \( (-\frac{\pi}{2}, \frac{3\pi}{2}) \).

**Proof of Theorem 3.2** (i) If \( \Re(t) > 0 \), then by Cauchy’s theorem

\[ l_{\chi_j}(t) = \frac{1}{2\pi i} \int_{R_1 \cup S_j \cup R_2} e^{(s+\frac{1}{2})t} \frac{L'}{L}(s, \chi_j)ds, \tag{3.13} \]

where

\[ R_1 := \{ s \in \mathbb{C} \mid \Re(s) = -\alpha, \ \Im(s) \geq 0 \}, \]

\[ R_2 := \{ s \in \mathbb{C} \mid \Re(s) = 1, \ \Im(s) \geq 0 \}, \]

and we go around the integral path in the counterclockwise direction. By the partial integration, (3.13) becomes

\[ l_{\chi_j}(t) = -\frac{t}{2\pi} e^{-\frac{1}{2} t} \left( \int_{R_1} + \int_{S_j} + \int_{R_2} \right) e^{ist} \log L(s, \chi_j)ds. \tag{3.14} \]

By using the functional equation

\[ L(s, \chi_j) = \frac{N^s_j}{(2\pi)^{1-s}} G(\chi_j) \Gamma(1-s)(e^{-\frac{\pi i}{2}(1-s)} + \chi_j(-1)e^{\frac{\pi i}{2}(1-s)})L(1-s, \chi_j), \]
the integral of the path $R_1$ becomes
\[
\int_{R_1} = \int_{0}^{\infty} e^{i(-\alpha + iy)t} \log L(-\alpha + iy, \chi_j) idy
\]
\[
= ie^{-i\alpha t} \int_{0}^{\infty} e^{-yt} \left( \log \left( \frac{N^\alpha G(\chi_j)}{(2\pi)^{1+\alpha}} \right) + iy \log \left( \frac{2\pi}{N_j} \right) + \log \Gamma(1 + \alpha - iy) + \log(e^{-\frac{\pi}{2}(1+\alpha-iy)} + \chi_j(-1)e^{\frac{\pi}{2}(1+\alpha-iy)}) + \log L(1 + \alpha - iy, \bar{\chi}_j) \right) dy. \tag{3.15}
\]

The integrals concerning (3.15) and (3.16) become
\[
\int_{0}^{\infty} e^{-yt} \log \left( \frac{N^\alpha G(\chi_j)}{(2\pi)^{1+\alpha}} \right) dy = \frac{-1}{t} \log \left( \frac{N^\alpha G(\chi_j)}{(2\pi)^{1+\alpha}} \right) \tag{3.20}
\]
and
\[
i \left( \log \left( \frac{2\pi}{N_j} \right) \right) \int_{0}^{\infty} ye^{-yt} dy = -\frac{i}{t} \log \left( \frac{2\pi}{N_j} \right). \tag{3.21}
\]

respectively. By the partial integration the integral concerning (3.17) is equal to
\[
\int_{0}^{\infty} e^{-yt} \log \Gamma(1 + \alpha - iy) dy
\]
\[
= \left[ -\frac{1}{t} e^{-yt} \log \Gamma(1 + \alpha - iy) \right]_{0}^{\infty} - \frac{i}{t} \int_{0}^{\infty} e^{-yt} \frac{\Gamma'(s)}{\Gamma(s)}(1 + \alpha - iy) dy
\]
\[
= -\frac{1}{t} \log \Gamma(1 + \alpha) + \frac{i\gamma}{t} \int_{0}^{\infty} e^{-yt} dy - \frac{i}{t} \int_{0}^{\infty} e^{-yt} \int_{0}^{\infty} \frac{1 - e^{(-\alpha + iy)u}}{e^u - 1} du dy \tag{3.22}
\]
because
\[
\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma + \int_{0}^{\infty} \frac{1 - e^{(1-s)u}}{e^u - 1} du \quad (\Re(s) > 0).
\]
The integral in the third term of (3.22) becomes
\[
\int_{0}^{\infty} \frac{1}{e^u - 1} \int_{0}^{\infty} (e^{-yt} - e^{-\alpha u + (-t+i)uy}) dy du
\]
\[
= \int_{0}^{\infty} \frac{1}{e^u - 1} \left( \frac{1}{t} + \frac{e^{-\alpha u}}{t - iu} \right) du
\]
\[
= -\frac{1}{t} \int_{0}^{\infty} \frac{1}{e^u - 1} \cdot \frac{u + it(1 - e^{-\alpha u})}{u + it} du.
\]
Hence,
\[
(3.22) = -\frac{1}{t} \log \Gamma(1 + \alpha) - \frac{i\gamma}{t^2} + \frac{i}{t^2} \int_{0}^{\infty} \frac{1}{e^u - 1} \cdot \frac{u + it(1 - e^{-\alpha u})}{u + it} du. \tag{3.23}
\]
The integral concerning (3.18) is equal to
\[
\int_{0}^{\infty} e^{-yt} \log(e^{-\frac{\pi}{2}(1+\alpha-iy)} + \chi_j(-1)e^{\frac{\pi}{2}(1+\alpha-iy)}) dy
\]
\[ = \int_{\infty}^{0} e^{-yt} \log \chi_j(-1) dy + \int_{\infty}^{0} \frac{\pi i}{2} (1 + \alpha - iy) e^{-yt} dy + \int_{\infty}^{0} e^{-yt} \log(1 + \chi_j(-1) e^{-\pi i(1 + \alpha - iy)}) dy. \] 

(3.24)

Since the third term of (3.24) becomes

\[ \sum_{m=1}^{\infty} \left( \frac{-1}{m} \right)^{m-1} \chi_j(-1)^m e^{-im\pi(1 + \alpha)} \int_{\infty}^{0} e^{-(t + m\pi)} dy = \sum_{m=1}^{\infty} \frac{\chi_j(-1)^m e^{-im\pi}}{m(t + m\pi)}, \]

we have

\[ (3.24) = -\frac{1}{t} \log \chi_j(-1) - \frac{(1 + \alpha) \pi i}{2t} - \frac{\pi}{t^2} + \sum_{m=1}^{\infty} \frac{\chi_j(-1)^m e^{-im\alpha}}{m(t + m\pi)}. \] 

(3.25)

The integral concerning (3.19) becomes

\[ \int_{\infty}^{0} e^{-yt} \log L(1 + \alpha - iy, \chi_j) dy = \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m)}{m} p^{-m(1 + \alpha)} \int_{\infty}^{0} e^{-(t + im \log p)y} dy \]

\[ = -\sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m) p^{-m(1 + \alpha)}}{m(t - im \log p)}. \] 

(3.26)

The integral of the path \( R_2 \) of (3.14) becomes

\[ \int_{0}^{\infty} e^{i(1 + \gamma)t} \log L(1 + iy, \chi_j) dy = ie^{it} \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m)}{m} p^{-m} \int_{0}^{\infty} e^{-(t + im \log p)y} dy \]

\[ = ie^{it} \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m) p^{-m}}{m(t + im \log p)}. \] 

(3.27)

Applying (3.20), (3.21), (3.23), (3.25), (3.26) and (3.27) to (3.14), we obtain the desired result.

(ii) \( \text{By Theorem 3.2 (i), we find that for } \Re(t) < 0 \)

\[ l_{\chi_j}(-t) \]

\[ = -\frac{it}{2\pi} e^{-\frac{\pi i}{2}} \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m) p^{-m}}{m(t - im \log p)} + \frac{e^{i(\alpha + \frac{1}{2})t}}{2\pi} \left( \frac{i}{t} \sum_{p} \sum_{m=1}^{\infty} \frac{\chi_j(p^m) p^{-m(1 + \alpha)}}{m(t + im \log p)} \right) \]

\[ - it \sum_{m=1}^{\infty} \frac{\chi_j(-1)^m e^{im\pi}}{m(t - m\pi)} + i \log \left( \frac{\chi_j(-1) \Gamma(1 + \alpha) N_j^\alpha G(\chi_j)}{(2\pi)^{1+\alpha}} \right) - \frac{(1 + \alpha)\pi}{2} \]

\[ + \frac{1}{t} \left( \gamma + \log \left( \frac{2\pi}{N_j} \right) + \frac{\pi i}{2} \right) - H(t) + \frac{t}{2\pi} e^{it} \int_{S(\pi \to 0)} e^{-ist} \log L(s, \chi_j) ds. \]

By using the equation for \( l_{\chi_j}(t) \) deduced in Theorem 3.1, we obtain (3.12) for \( \Re(t) < 0 \). Since the right-hand side of (3.12) is meromorphic for \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \) if the argument lies in \( \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) \), the proof of (ii) is completed.

In the following, let \( t \in \mathbb{C} - i\mathbb{R}_{\leq 0} \) and the argument lie in \( \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) \).

(iii) \( \text{By Theorem 3.2 (ii) and Lemma 2.4 (i), we find that} \)

\[ l_{\chi_j}(t) = -\frac{1}{2\pi t} \left( \gamma + \log \left( \frac{2\pi}{N_j} \right) + \frac{\pi i}{2} \right) - \frac{log t}{4\pi \sin \frac{t}{2}} - \frac{ie^{-i\frac{\pi}{2}t}}{2\sin t} + O(1) \quad (t \to 0) \]

\[ = -\log t \frac{1}{2\pi t} \left( \gamma + \log \left( \frac{2\pi}{N_j} \right) + \frac{3\pi i}{2} \right) + O(1) \quad (t \to 0). \]

(iv) \( \text{By (3.1), we find trivially that } l_{\chi_j}(t) \text{ is holomorphic for } \Re(t) > 0. \) From this and the expression obtained in Theorem 3.2 (ii), the desired result follows. \( \square \)
We consider the bounds of \( l_{x, j}(t) \) which is needed later.

**Lemma 3.3.** (i) For \( \Re(t) \geq 1 \)

\[
l_{x, j}(t) = O(e^{-\varepsilon_j \Re(t) + \frac{1}{2} |\Im(t)|}).
\]

(ii) For \( \Re(t) \leq -1 \)

\[
l_{x, j}(t) = \frac{e^{-\varepsilon_j \Re(t)}}{e^{-t} - e^{-it}} + O(e^{\varepsilon_j \Re(t) + \frac{1}{2} |\Im(t)|}) + e^{\varepsilon_j |\Im(t)|}.
\]

(iii) If \( t = \sigma + iU \) with \( U \geq 2 \) and \( -U \leq \sigma \leq U \), then

\[
l_{x, j}(t) = \frac{it}{2\pi} e^{-\frac{it}{2}} \sum_{p, m < e^{|U|}} \frac{\chi_j(p^m) p^{-m}}{m(t - im \log p)} + O(U e^{\varepsilon_j + \frac{1}{2} U}).
\]

**Proof of Lemma 3.3.** (i) If \( \Re(t) \geq 1 \), then we have

\[
l_{x, j}(t) = \sum_{\Re(\tau_j) > \varepsilon_j} e^{-\tau_j t}
\]

from (3.1). Since

\[
\left| \sum_{\Re(\tau_j) > \varepsilon_j} e^{-\tau_j t} \right| \leq \sum_{\Re(\tau_j) > \varepsilon_j} |e^{-\tau_j t}| = \sum_{\Re(\tau_j) > \varepsilon_j} e^{-\Re(\tau_j) \Re(t) + \Im(\tau_j) \Im(t)}
\]

\[
\leq e^{\frac{1}{2} |\Im(t)|} \sum_{\Re(\tau_j) > \varepsilon_j} e^{-\Re(\tau_j) \Re(t)}
\]

\[
= e^{\frac{1}{2} |\Im(t)|} e^{-\varepsilon_j \Re(t)} \sum_{\Re(\tau_j) > \varepsilon_j} e^{-\Re(\tau_j) - \varepsilon_j} \Re(t)
\]

\[
= O(e^{-\varepsilon_j \Re(t) + \frac{1}{2} |\Im(t)|}),
\]

we obtain the desired result.

(ii) For \( \Re(t) \leq -1 \), we have

\[
l_{x, j}(t) = -l_{x, j}(-t) - \frac{it}{2\sin t} - \mu_{x, j}(\tau_j(0)) (e^{i\tau_j(0)t} + e^{-i\tau_j(0)t}) - \mu_{x, j}(0)
\]

\[
= - \sum_{\Re(\tau_j) > \varepsilon_j} e^{\tau_j t} + \frac{it}{2\sin t} - \mu_{x, j}(\tau_j(0)) (e^{i\tau_j(0)t} + e^{-i\tau_j(0)t}) - \mu_{x, j}(0)
\]

(3.28)

by Theorem 3.1. Concerning the first and third term of the right-hand side of (3.28), we have

\[
\left| \sum_{\Re(\tau_j) > \varepsilon_j} e^{\tau_j t} \right| \leq \sum_{\Re(\tau_j) > \varepsilon_j} |e^{\tau_j t}| = \sum_{\Re(\tau_j) > \varepsilon_j} e^{\Re(\tau_j) \Re(t) - \Im(\tau_j) \Im(t)}
\]

\[
\leq e^{\frac{1}{2} |\Im(t)|} \sum_{\Re(\tau_j) > \varepsilon_j} e^{\Re(\tau_j) \Re(t)}
\]

\[
= e^{\frac{1}{2} |\Im(t)|} e^{-\varepsilon_j \Re(t)} \sum_{\Re(\tau_j) > \varepsilon_j} e^{\Re(\tau_j) - \varepsilon_j} \Re(t)
\]

\[
= O(e^{-\varepsilon_j \Re(t) + \frac{1}{2} |\Im(t)|}).
\]
and
\[ |e^{ix_j(t)} + e^{-ix_j(t)}| \leq e^{-\epsilon x_j(\sigma - 1)} + e^{x_j(\sigma - 1)} = O(e^{x_j(\sigma - 1)}) \]
respectively. Hence, we obtain the desired result.

(iii) When \( t = \sigma + iU \) with \( U \geq 2 \) and \( -U \leq \sigma \leq U \), we have
\[
l_{x_j}(t) = \frac{it}{2\pi} e^{-\frac{\pi}{2}U} \sum_{p,m} \frac{\chi_j(p^m)p^{-m}}{m(mU + im\log p)} + O(Ue^{(\frac{1}{2} + \epsilon)U})
\]
by estimating trivially each term of the right-hand side of (3.12) in Theorem 3.2 except the first term.

If \( p^m \geq e^{2U} \), then \( U \leq \frac{\log p}{2} \), so \( m \log p - U \geq \frac{m \log p}{2} \). Therefore, we have
\[
|\chi_j(p^m)p^{-m}| \leq \frac{\sum_{p,m} p^{-m}}{m(m \log p - U)} \leq \frac{4Ue^{\frac{U}{2}}}{m^2 \log p} \sum_{p,m} \frac{p^{-m}}{m^2 \log p}
\]
\[
\leq 4Ue^{\frac{U}{2}} \sum_{p,m} \frac{p^{-m}}{m^2 \log p} = O(Ue^{\frac{U}{2}}).
\]
In the last equation, we use Lemma 2.7 (ii). This completes the proof. \( \square \)

Now, we fix \( \theta_j \) arbitrarily with \( 0 < \theta_j < \frac{\pi}{4} \) and \( \tan \theta_j < \epsilon_j \).

**Corollary 3.4.** (i) For \( u \geq \frac{1}{\cos \theta_j} \)
\[
l_{x_j}(ue^{-i\theta_j}) = O(e^{-\frac{\pi}{4} \sin \theta_j}),
\]
\[
l_{x_j}(ue^{i(\pi - \theta_j)}) = O(e^{x_j(\sin \theta_j)}).
\]
(ii) If \( R \geq 1 \) and \( -R \tan \theta_j \leq y \leq R \tan \theta_j \) then
\[
l_{x_j}(R + iy) = O(e^{\frac{\pi}{2}}).
\]
(iii) If \( \sigma \in \mathbb{R}, M \in \mathbb{Z} \geq 100\) and \( U := \log (M + \frac{1}{2}) \) then
\[
l_{x_j}(\sigma + iU) = \begin{cases} O(e^{\frac{\pi}{2}}) & (\sigma \geq 1), \\ O(U^2 e^{(\epsilon_j + \frac{1}{2})U}) & (-1 \leq \sigma \leq 1), \\ O(e^{\epsilon_j \sigma + \frac{\pi}{2}} + e^{(\epsilon_j)U}) & (\sigma \leq -1). \end{cases}
\]

**Proof of Corollary 3.4.** (i) First, by Lemma 3.3 (i) we find
\[
l_{x_j}(ue^{-i\theta_j}) = O(e^{-\epsilon_j \Re(ue^{-i\theta_j}) + \frac{1}{2} \Re(ue^{-i\theta_j})})
\]
\[
= O(e^{-\epsilon_j (\cos \theta_j - \frac{1}{2} \sin \theta_j)})
\]
\[
= O(e^{-\frac{1}{2} \epsilon_j \sin \theta_j}).
\]
In the last equation we use the fact that \( \frac{1}{2} \sin \theta_j < \epsilon_j \cos \theta_j - \frac{1}{2} \sin \theta_j \) because \( \tan \theta_j < \epsilon_j \). Hence, (3.30) has been proved.

Next, by Lemma 3.3 (ii) we have
\[
l_{x_j}(ue^{i(\pi - \theta_j)}) = \frac{e^{-\frac{x_j(\epsilon_j)}{2} \Re(ue^{i(\pi - \theta_j)})}}{e^{x_j(\epsilon_j) \Re(ue^{i(\pi - \theta_j)})}} - e^{-iue^{i(\pi - \theta_j)}}
\]
\[
+ O(e^{\epsilon_j \Re(ue^{i(\pi - \theta_j)}) + \frac{1}{2} \Re(ue^{i(\pi - \theta_j)})} + e^{x_j(\epsilon_j) \Re(ue^{i(\pi - \theta_j)})} + e^{x_j(\epsilon_j) \Re(ue^{i(\pi - \theta_j)})}).
\]
Now,

\[ |(the \ first \ term \ of \ the \ right-hand \ side \ of \ (3.32))| \leq \frac{e^{\frac{1}{2}u \sin \theta_j}}{e^{u \sin \theta_j} - e^{-u \sin \theta_j}} = O(e^{-\frac{1}{2}u \sin \theta_j}). \]

Hence, we can deduce

\[ l_{\chi_j}(ue^{i(\pi - \theta_j)}) = O(e^{-\frac{1}{2}u \sin \theta_j} + e^{-u(\varepsilon_j \cos \theta_j - \frac{1}{2} \sin \theta_j)} + e^{\tau_{ij}u \sin \theta_j}), \]

where in the last equation we use the fact that \( \varepsilon_j \cos \theta_j - \frac{1}{2} \sin \theta_j > 0 \) because \( \tan \theta_j < \varepsilon_j \). Hence, (3.31) holds.

(ii) From Lemma 3.3 (i) and \( \tan \theta_j < \varepsilon_j \), we can easily deduce the desired result.

(iii) If \( \sigma \geq 1 \) (respectively \( \sigma \leq -1 \)) then we can trivially deduce the desired result from Lemma 3.3 (i) (respectively Lemma 3.3 (ii)).

If \( -1 \leq \sigma \leq 1 \) then we can derive

\[ l_{\chi_j}(\sigma + iU) = \frac{i(\sigma + iU)}{2\pi}e^{-\frac{1}{2}(\sigma + iU)} \sum_{p^m < e^{2U}} \bar{X}_j(p^{2m}p^{-m}) \sum_{p^m < e^{2U}} \frac{p^{-m}}{m(\sigma + iU - im \log p)} + O(Ue^{(\varepsilon_j + \frac{1}{2})U}) \]

from Lemma 3.3 (iii). Concerning the first term of the right-hand side, we find that

\[ \left| (\sigma + iU)e^{-\frac{1}{2}(\sigma + iU)} \right| = O(Ue^{\frac{1}{2}}), \]

\[ \sum_{p^m < e^{2U}} \bar{X}_j(p^{2m}p^{-m}) \sum_{m(\sigma + iU - im \log p)} \left| \frac{p^{-m}}{U - m \log p} \right| \leq \sum_{p^m < e^{2U}} \frac{p^{-m}}{U - m \log p} \]

\[ = \sum_{p^m < M + \frac{1}{2}} \frac{p^{-m}}{U - m \log p} + \sum_{M + \frac{1}{2} \leq p^m < (M + \frac{1}{2})^2} \frac{p^{-m}}{m \log p - U} \]

and that by Lemma 2.7 (i)

\[ (the \ first \ term \ of \ (3.33)) \leq \left( M + \frac{1}{2} \right) \sum_{p^m < M + \frac{1}{2}} \frac{p^{-m}}{M + \frac{1}{2} - p^m} \]

\[ \leq \left( M + \frac{1}{2} \right) \sum_{n = 0}^{M} \frac{1}{n \left( M + \frac{1}{2} - n \right)} \]

\[ = \sum_{n = 0}^{M} \frac{1}{n} + \sum_{n = 0}^{M} \frac{1}{M + \frac{1}{2} - n} \]

\[ \ll \log M \ll U, \]

\[ (the \ second \ term \ of \ (3.33)) \leq \sum_{p^m < (M + \frac{1}{2})^2} \frac{p^{-m}}{m \log p - U} \]

\[ \leq \sum_{p^m < (M + \frac{1}{2})^2} \frac{1}{p^m - (M + \frac{1}{2})}. \]
Hence, we can obtain
\[ l_x(r + iU) = O(U^2e^{\frac{r}{2}} + Ue^{(r^2 + \frac{1}{2})U}) = O(U^2e^{(r^2 + \frac{1}{2})U}). \]

This completes the proof. \( \square \)

4. The “key equation”

In this section, we prove an equation which links the "factors series" of \((L_{x_j} \otimes \cdots \otimes L_{x_j})(s)\) to \(r\)-tuples of prime numbers \((r \in \mathbb{Z}_{\geq 1})\).

Define that
\[
\vartheta^{(r)} := \min_{j \in \{1, \ldots, r\}} \{ \theta_j \}, \\
\tau^{(0)}_r := \max_{j \in \{1, \ldots, r\}} \{ \tau^{(0)}_x \}, \\
D_{\vartheta^{(r)}, \tau^{(0)}_r} := \left\{ (w, z) \in \mathbb{C}^2 \mid -\frac{r}{2}\sin\vartheta^{(r)} < \Re(ze^{-i\vartheta^{(r)}}) < -r\tau^{(0)}_r\sin\vartheta^{(r)} \right\}, \\
= \left\{ (w, z) \in \mathbb{C}^2 \mid -\frac{r}{2}\tan\vartheta^{(r)} < \Re(z) + \Im(z)\tan\vartheta^{(r)} < -r\tau^{(0)}_r\tan\vartheta^{(r)} \right\},
\]

where \(L^{(1)}_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}) := \frac{1}{\Gamma(w)} \int_0^{\infty e^{-is^{(r)}}} e^{zt} \prod_{j=1}^{r} l_{x_j}(t)t^{w-1}dt, \)

\(L^{(2)}_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}) := (-1)^{r-1} \frac{e^{\pi i w}}{\Gamma(w)} \int_0^{\infty e^{-is^{(r)}}} e^{zt} \prod_{j=1}^{r} \left( l_{x_j}(t) + \sum_{n=1}^{\infty} e^{-(2n - 1 - \chi^{(r)}_j)t} \right) \bigg|_{t^{w-1}dt}, \)

\(R_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}) := \frac{2\pi i}{\Gamma(w)} \lim_{N \to \infty} \sum_{p, m < N + 1/2} \text{Res}_{t=\text{ir}m\log p} e^{-zt} \prod_{j=1}^{r} l_{x_j}(t)t^{w-1}. \)

Then, we show

**Theorem 4.1 (The "key equation").** Let \((w, z) \in D_{\vartheta^{(r)}, \tau^{(0)}_r}\) satisfy \(\Im(z) < -\left(\frac{1}{2} + \varepsilon^{(r)}\right)r\) and \(\text{Re}(w) > r\). Then,

\[ L^{(1)}_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}) + L^{(2)}_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}) = R_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}). \]  

**Proof of Theorem 4.1.** Let \(\lambda\) be any fixed real number with \(0 < \lambda < \log 2\) and we define

\[ F_{\vartheta^{(r)}}(w, z, \{x_j\}^{r}_{j=1}: \lambda) := \frac{1}{\Gamma(w)} \int_{V_{\lambda, \vartheta^{(r)}}} e^{zt} \prod_{j=1}^{r} l_{x_j}(t)t^{w-1}dt, \]

where \(V_{\lambda, \vartheta^{(r)}}\) is the union of \(V_1(\infty \to \lambda), V_2(\pi - \vartheta^{(r)} \to -\vartheta^{(r)})\) and \(V_3(\lambda \to \infty)\) when

\[ V_1 := \{ \nu e^{i(\pi - \vartheta^{(r)})} \mid \nu \geq \lambda \}, \]

\[ V_2 := \{ \lambda e^{i\varphi} \mid -\vartheta^{(r)} \leq \varphi \leq \pi - \vartheta^{(r)} \}. \]
Therefore, \( F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r): \) converges absolutely and uniformly on any compact subset of \( D_{\theta^{(r)}, \pi^{(r)}} \).

Now, when \((w, z) \in D_{\theta^{(r)}, \pi^{(r)}}\) and \(0 < \eta < \lambda\), we have

\[
F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r; \lambda) = F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r; \eta)
\]

by Theorem 3.2 (iv) and Cauchy's theorem, where

\[
W_{\eta, \lambda, \theta^{(r)}} := \{\lambda e^{it} \mid \theta^{(r)} \leq \varphi \leq \pi - \theta^{(r)}\} \cup \{\nu e^{-j\theta^{(r)}} \mid \nu \leq \lambda\}
\]

and we go around the integral path in the counterclockwise direction. If \(\Re(w) > r\), then by Theorem 3.2 (iii) we have

\[
F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r; \lambda) = \lim_{\eta \downarrow 0} F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r; \eta)
\]

By replacing \(t\) with \(-t\), using Theorem 3.1 and taking note of

\[
\frac{ie^{x-it}}{2 \sin t} = -\frac{e^{(2x-1)-it}}{1 - e^{-2it}} = -\sum_{n=1}^{\infty} e^{-2(n-1)-it}
\]

we find that the first term of (4.4) is equal to

\[
(-1)^{r-1} \frac{e^{x-it}}{\Gamma(w)} \int_{0}^{\infty} e^{-j\theta^{(r)}} \prod_{j=1}^{r} \left( t_{\chi_j} + \sum_{n=1}^{\infty} e^{-2n-1 - \frac{1}{2} \lambda^{(r)}} t_{\chi_j} \right) t^{w-1} dt.
\]

Hence, we have

\[
F_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^r; \lambda) = L_{\theta^{(r)}}^{(1)}(w, z, \{\chi_j\}_{j=1}^r) + L_{\theta^{(r)}}^{(2)}(w, z, \{\chi_j\}_{j=1}^r).
\]

Next, we define that \( U := \log |M + \frac{1}{2} X| \) for \( M \in \mathbb{Z}_{\geq 100} \) and let \((w, z) \in D_{\theta^{(r)}, \pi^{(r)}}\) with \(\Im(z) < -\frac{1}{2} + \varepsilon^{(r)} r\) and \(R \in \mathbb{R}\) with \(R \tan \theta^{(r)} \geq U\). By Theorem 3.2 (iv) and the residue theorem, we have

\[
\int_{P_{1} \cup P_{2} \cup P_{3}} e^{-zt} \prod_{j=1}^{r} l_{\chi_j}(t) t^{w-1} dt = 2\pi i \sum_{m, p, m} \operatorname{Res}_{t = \im \log p} e^{-zt} \prod_{j=1}^{r} l_{\chi_j}(t) t^{w-1}.
\]
where
\[
P_1 := \left\{ -u + iu \tan \theta^{(r)} \left| \frac{U}{\tan \theta^{(r)}} \leq u \leq \lambda \cos \theta^{(r)} \right. \right\}
\cup \left\{ \lambda e^{i\varphi} \mid -\theta^{(r)} \leq \varphi \leq \pi - \theta^{(r)} \right\}
\cup \left\{ u - iu \tan \theta^{(r)} \mid \lambda \cos \theta^{(r)} \leq u \leq R \right\},
\]
\[
P_2 := \{ R + iy \mid -R \tan \theta^{(r)} \leq y \leq U \},
\]
\[
P_3 := \left\{ \sigma + iU \left| -\frac{U}{\tan \theta^{(r)}} \leq \sigma \leq R \right. \right\}
\]
and we go around the integral path in the counterclockwise direction. First, we consider the limit of (4.5) as \( R \to \infty \). Concerning the integral of the path \( P_2 \), we have
\[
\left| \int_{-R \tan \theta^{(r)}}^{-R \tan \theta^{(r)}} e^{-z(R+iy)} \prod_{j=1}^{r} l_{\lambda_j}(R + iy)(R + iy)^{-w-1} dy \right| 
\leq R^{(w)-1} e^{-\Re(z)R} \int_{-R \tan \theta^{(r)}}^{R \tan \theta^{(r)}} e^{(\Im(z)+\frac{\pi}{2})y} dy
\leq R^{(w)-1} e^{-\Re(z)R} \frac{\Im(z)}{\Im(z) + \frac{\pi}{2}}
\leq \frac{R^{(w)-1}}{\Im(z) + \frac{\pi}{2}} e^{-\Re(z)\Im(\theta^{(r)}) + \frac{\pi}{2} \tan \theta^{(r)}R}, \tag{4.6}
\]
where in the last inequality we use the fact that \( \Im(z) + \frac{\pi}{2} < 0 \). From \( \Re(z) + \Im(z) \tan \theta + \frac{\pi}{2} \tan \theta > 0 \) because \( (w, z) \in D_{\theta^{(r)}, r^{(0)}} \), it follows that (4.6) vanishes as \( R \to \infty \). Hence, we have
\[
\int_{P_1 \cup P_3} e^{-z} \prod_{j=1}^{r} l_{\lambda_j}(t)t^{-w-1} dt = 2\pi i \sum_{p = 0 \text{ mod } M} \text{Res}_{t = im \log p} e^{-z} \prod_{j=1}^{r} l_{\lambda_j}(t)t^{-w-1}, \tag{4.7}
\]
where
\[
P_3 := \left\{ -u + iu \tan \theta^{(r)} \left| \frac{U}{\tan \theta^{(r)}} \leq u \leq \lambda \cos \theta^{(r)} \right. \right\}
\cup \left\{ \lambda e^{i\varphi} \mid -\theta^{(r)} \leq \varphi \leq \pi - \theta^{(r)} \right\}
\cup \left\{ u - iu \tan \theta^{(r)} \mid u \geq \lambda \cos \theta^{(r)} \right\},
\]
and we go around the integral path in the counterclockwise direction. Next, we consider the limit of (4.7) as \( M \to \infty \). Concerning the integral of the path \( P_3 \), we have
\[
\left| \int_{P_3} \right| = \left| \int_{\Re(z) = 0}^{\Re(z) = \infty} e^{-z} \prod_{j=1}^{r} l_{\lambda_j}(\sigma + iU)(\sigma + iU)^{-w-1} d\sigma \right|
\leq \int_{\Re(z) = 0}^{\Re(z) = \infty} e^{-\Re(z)\sigma + \Im(z)U} \prod_{j=1}^{r} l_{\lambda_j}(\sigma + iU) \left| \max \{ |\sigma|, U \} \right|^{\Re(z)-1} d\sigma
\]
we have

\begin{equation}
(4.8)
\end{equation}

About the first term of (4.8), by using Corollary 3.4 (iii) we can deduce

\begin{align*}
\int_{-1}^{1} U_{\tan \theta(c)} e^{-U(z)\sigma + \Im(z)U}(e^{\epsilon(z)r} + e^{\epsilon(z)\theta(c)r})\sigma & \lesssim e^{\Im(z)U} \left( \frac{U}{\tan \theta(c)} \right)^{\Im(Uw)} d\sigma \\
& + \int_{-1}^{1} e^{-U(z)\sigma + \Im(z)U}(e^{\epsilon(z)r} + e^{\epsilon(z)\theta(c)r})U_{\Im(Uw)} d\sigma \\
& \leq e^{\Im(z)U} \left( \frac{U}{\tan \theta(c)} \right)^{\Im(Uw)} \\
& \quad \times \int_{-1}^{1} (e^{\epsilon(z)r} + e^{-U(z)\sigma + \Im(z)U}) d\sigma \\
& + e^{\Im(z)U} \left( \frac{U}{\tan \theta(c)} \right)^{\Im(Uw)} \int_{-1}^{1} (e^{\epsilon(z)r} + e^{-U(z)\sigma + \Im(z)U}) d\sigma \\
& = e^{\Im(z)U} \left( \frac{U}{\tan \theta(c)} \right)^{\Im(Uw)} \int_{-1}^{1} (e^{\epsilon(z)r} + e^{-U(z)\sigma + \Im(z)U}) d\sigma.
\end{align*}

(4.9)

Since

\begin{equation}
\int_{-1}^{1} e^{A\sigma} d\sigma \ll A \begin{cases} 
1 & (A > 0), \\
\frac{U}{\tan \theta(c)} & (A = 0), \\
e^{-\frac{U}{\tan \theta(c)}} & (A < 0)
\end{cases} \ll \frac{U}{\tan \theta(c)} \left( 1 + e^{-\frac{U}{\tan \theta(c)}} \right),
\end{equation}

we have

\begin{align*}
(4.9) & \ll \left( \frac{U}{\tan \theta(c)} \right)^{\Im(Uw)} \left( \frac{U}{\tan \theta(c)} \right) \\
& \quad \times \left( e^{\Im(z)+\frac{U}{\tan \theta(c)}} + e^{\Im(z)+\Im(z)\tan \theta(c)+\frac{U}{\tan \theta(c)}} \right) \\
& \quad + e^{\Im(z)\tan \theta(c)} \left( \Im(z) + \Im(z)\tan \theta(c) + \frac{U}{\tan \theta(c)} \right) \\
& \to 0 \quad (M \to \infty),
\end{align*}

where in the last limit we use the fact that

\begin{equation}
\Im(z) + \Im(z)\tan \theta(c) + \frac{r}{2} < 0 \quad (4.10)
\end{equation}
and
\[
\begin{aligned}
\Re(z) + \Im(z) \tan \theta(r) + \frac{r}{2} \tan \theta(r) - \varepsilon(r) r < 0, \\
\Re(z) + \Im(z) \tan \theta(r) + r r_t^{(0)} \tan \theta(r) < 0
\end{aligned}
\]

because \((w, z) \in D_{\theta(r), r_t^{(0)}}\) and \(\tan \theta(r) < \varepsilon(r)\). About the second term of (4.8), by using Corollary 3.4 (iii) we have
\[
\begin{aligned}
\int_{-1}^{1} \ll_r \int_{-1}^{1} e^{-\Re(z) \sigma + \Im(z) U} (U^{2} e^{(\varepsilon(r) + \frac{r}{4})} U)^{r} U^{\Re(w)-1} d\sigma \\
\ll_z U^{\Re(w)+2r-1} e^{(\Im(z) + \frac{1}{2} + \varepsilon(r)) U} \\
\rightarrow 0 \ (M \rightarrow \infty),
\end{aligned}
\]

where in the last limit we use \(\Im(z) + (\frac{1}{2} + \varepsilon(r)) r < 0\). About the third term of (4.8), by Corollary 3.4 (iii) we have
\[
\begin{aligned}
\int_{1}^{\infty} \ll_r \int_{-1}^{1} e^{-\Re(z) \sigma + \Im(z) U} e^{\frac{r}{r_t^{(0)}} \max \{|\sigma|, U\}^{\Re(w)-1}} d\sigma \\
= e^{(\Im(z) + \frac{1}{2}) U} \left( \int_{1}^{U} e^{-\Re(z) \sigma} U^{\Re(w)-1} d\sigma + \int_{U}^{\infty} e^{-\Re(z) \sigma} U^{\Re(w)-1} d\sigma \right) \ (4.11)
\end{aligned}
\]

\[
\ll_z U^{\Re(w)-1} + 1 \ (4.12)
\]

\(\rightarrow 0 \ (M \rightarrow \infty),\)

where in transforming (4.11) into (4.12) we use \(\Re(z) > -(\Im(z) + \frac{1}{2}) \tan \theta(r) > 0\) because \((w, z) \in D_{\theta(r), r_t^{(0)}}\), and in the last limit we use (4.10). Hence, we obtain
\[
F_{\theta(r)}(w, z, \{\chi_j\}_{j=1}^{r}; \lambda) = R_{\theta(r)}(w, z, \{\chi_j\}_{j=1}^{r}).
\]

This completes the proof. 

In the following sections, it is necessary that the left-hand side of (4.3) be a meromorphic function of \(w = w_0\). To obtain the property we show a lemma. It is the generalization of the lemma proved by Hirano, Kurokawa and Wakayama [10, Lemma 1].

Let \(\psi \in (-\pi, \pi]\) be any fixed real number and \(f(t)\) be a locally integrable function on \(\{re^{i\psi} \mid r \in (0, \infty)\}\). We define
\[
M_{\phi}[f : w] := \int_{0}^{\infty} e^{i\psi} f(t) t_{w-1} dt.
\]

Now, assume that \(f(t)\) satisfies
\[
f(t) = \begin{cases} 
O(t^{-a+\varepsilon}) & (t \rightarrow 0), \\
O(t^{-b-\varepsilon}) & (t \rightarrow \infty e^{i\psi})
\end{cases}
\]

for \(a, b \in \mathbb{R}\) with \(a < b\) and \(M_{\phi}[f : w]\) converges absolutely, so is an analytic function, in \(a < \Re(w) < b\). Then, the following lemma holds.

Lemma 4.2. Suppose that \(f(t)\) has the following approximate behaviors as \(t \rightarrow 0\) and \(t \rightarrow \infty e^{i\psi}\) :
\[
f(t) \sim \begin{cases} 
\sum_{k=0}^{\infty} \sum_{n=0}^{N_1(k)} A_{1}(n, k)(\log t)^{n} t^{a_1(k)} & (t \rightarrow 0), \\
\sum_{k=0}^{\infty} \sum_{n=0}^{N_2(k)} A_{2}(n, k)(\log t)^{n} t^{a_2(k)} & (t \rightarrow \infty e^{i\psi}),
\end{cases}
\]

(4.13)
where $N_1(k)$ are non-negative and finite integers for each $k$ and $a_1(k)$ and $a_2(k)$ are complex sequences with $\Re(a_1(k))$ and $\Re(a_2(k))$ monotonically increasing. Then $M_\omega[f : w]$ has a meromorphic continuation into $w \in \mathbb{C}$ with poles at $w = -a_1(k)$ and $w = -a_2(k)$ for each $k$. Especially the poles at $s = -a_i(k)$ are simple if $N_i(k) = 0$.

**Proof of Lemma 4.2.** First we define $f_m(t)$ as

$$f_m(t) := f(t) - \sum_{n=0}^{m} a_1(n,k)(\log t)^n t^{a_1(k)}.$$ 

Then, in $a < \Re(w) < b$, we have

$$M_\omega[f : w] = \int_{0}^{e^{i\omega}} f_m(t) t^{w-1} dt + \int_{0}^{e^{i\omega}} \sum_{n=0}^{m} a_1(n,k)(\log t)^n t^{a_1(k)+w-1} dt$$

$$+ \int_{e^{i\omega}}^{\infty} f(t) t^{w-1} dt. \tag{4.14}$$

The first and third terms of the right-hand side of (4.14) are analytic function of $w$ in $-\Re(a_1(m+1)) < \Re(w)$ and in $\Re(w) < b$ respectively. The second term becomes

$$\sum_{n=0}^{m} a_1(n,k) \int_{0}^{e^{i\omega}} (\log t)^n t^{a_1(k)+w-1} dt,$$

and then by partial integration we can transform it into

$$\sum_{n=0}^{m} \sum_{r=0}^{N_1(k)n} a_1(n,k) \frac{(-1)^r n(n-1) \cdots (n-r+1)(i\omega)^{n-r} e^{i\omega(w+1)}}{w + a_1(k)r+1}.$$

Hence, we see that $M_\omega[f : w]$ is a meromorphic function of $w$ with having poles at $w = -a_1(k)$ in $-\Re(a_1(m+1)) < \Re(w) < b$, especially the orders of which at $s = -a_1(k)$ are simple if $N_1(k) = 0$. Since $\Re(a_1(m+1)) \to \infty (m \to \infty)$, it is shown that the meromorphy of $M_\omega[f : w]$ in the left half plane $\Re(w) < b$.

In a similar way, we can obtain a meromorphic continuation into the right half plane $b \leq \Re(w)$. \hfill \Box

From Lemma 4.2 the meromorphy of the left-hand side of (4.3) follows.

**Corollary 4.3.** If $\left(\frac{1}{2} + r^{(0)}\right) r \tan \theta^{(r)} < \Re(s) \tan \theta^{(r)} - \Im(s) < r \tan \theta^{(r)}$ and $\Re(s) > r(1 + \varepsilon^{(r)})$, then $L^{(1)}_{\theta^{(r)}}(w, -i(s - \frac{1}{2}), \{\chi_j\}_{j=1}^{r})$ and $L^{(2)}_{\theta^{(r)}}(w, -i(s - \frac{1}{2}), \{\chi_j\}_{j=1}^{r})$ are meromorphic functions of $w$ on the whole $\mathbb{C}$.

**Proof of Corollary 4.3.** By the consideration about $F^{(r)}_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^{r}; \lambda)$ in the proof of Theorem 4.1, $L^{(1)}_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^{r})$ and $L^{(2)}_{\theta^{(r)}}(w, z, \{\chi_j\}_{j=1}^{r})$ are holomorphic functions of $w$ under the assumption that

$$(w, z) \in D_{\theta^{(r)}, r^{(0)}}, \Re(z) < -\left(\frac{1}{2} + \varepsilon^{(r)}\right) r \text{ and } \Re(w) > r.$$ 

We can remove $\Re(w) > r$ because it follows from Theorem 3.2 (iii) that

$$e^{-zt} \prod_{j=1}^{r} f_{\chi_j}(t)$$
and
\[
e^{zt} \prod_{j=1}^{r} \left( l(x_j(t)) + \sum_{n=1}^{\infty} e^{-\left(2n-1-\frac{s}{2}\right) t} + \mu(x_j)(\tau(0)) e^{i\theta(0) t} + \mu(x_j)(0) \right)
\]

which appear in \( L_{\theta(1)}^{(i)}(w, z, \{ \tau \}) \) \((i = 1, 2)\) satisfy the condition concerning \( t \to 0 \) in (4.13). By putting \( z = -i \left( s - \frac{1}{2} \right) \) we obtain the desired results. \( \square \)

5. The zeta regularized product expression of \( L(s, \chi_1) \)

Our goal in this section is to prove Theorem 1.1. We obtain an equation which links the “factors series” of \( L(s, \chi_1) \) to prime numbers by calculating the both sides of (4.3) with \( r = 1 \) and then prove Theorem 1.1.

5.1. The “key equation” for \( r = 1 \).

**Lemma 5.1.** Let \((w, z) \in D_{\theta(1), \tau(0)}\) satisfy \( \Im(z) < -\left( \frac{1}{2} + \varepsilon(1) \right) \) and \( \Re(w) > 1 \). Then,
\[
L_{\theta(1)}^{(1)}(w, z, \chi_1) = \sum_{\Re(\tau(1)) > 0} \frac{1}{(z + \tau(1)) \Re(w)},
\]
\[
L_{\theta(1)}^{(2)}(w, z, \chi_1) = e^{\pi i w} \left( \sum_{\Re(\tau(1)) > 0} \frac{1}{(\tau(1) - z) \Re(w)} \sum_{n=1}^{\infty} \frac{1}{(z + \left(2n - 1 - \frac{\tau(1)}{2}\right) \Re(w))} \right) + \frac{\mu(\tau(0))}{(-z - i \tau(0)) \Re(w)} + \frac{\mu(\tau(0))}{(-z + i \tau(0)) \Re(w)}.
\]

**Proof of Lemma 5.1.** Since \((w, z) \in D_{\theta(1), \tau(0)}\) and \( \Re(\tau(1)) > \varepsilon(1) > \tan \theta(1) \), we have
\[
\Re(z + \tau(1)) + \Im(z + \tau(1)) \tan \theta(1) > \varepsilon(1) - \tan \theta(1) > 0,
\]
and from this we find \( \arg(z + \tau(1)) \in \left( \theta(1) - \frac{\pi}{2}, \theta(1) + \frac{\pi}{2} \right) \). Therefore, by using Lemma 2.8 as \( \psi = -\theta(1) \) we obtain
\[
L_{\theta(1)}^{(1)}(w, z, \chi_1) = \frac{1}{\Gamma(w)} \sum_{\Re(\tau(1)) > 0} \int_{0}^{\infty} e^{-\left(z + \tau(1)\right) t \Re(w) - t} dt
\]
\[
= \sum_{\Re(\tau(1)) > 0} \frac{1}{(z + \tau(1)) \Re(w)}.
\]

In a similar way as \( L_{\theta(1)}^{(2)}(w, z, \chi_1) \) we can reach the desired result concerning \( L_{\theta(1)}^{(2)}(w, z, \chi_1) \).

**Lemma 5.2.** If \( \left( \frac{1}{2} + \tau(0) \right) \tan \theta(1) < \Re(s) \tan \theta(1) - \Im(s) < \tan \theta(1) \), \( \Re(s) > 1 + \varepsilon(1) \) and \( \Re(w) > 1 \) then we have
\[
L_{\theta(1)}^{(1)} \left( w, -i \left( s - \frac{1}{2} \right), \chi_1 \right) = e^{\pi i w} \sum_{\Re(\tau(1)) < 0} \frac{1}{(s - \rho(1)) \Re(w)}.
\]

(5.1)
Proof of Lemma 5.2. Putting \( z = -i(s - \frac{1}{2}) \) in Lemma 5.1, we obtain the conditions concerning \((w, s)\) and have

\[
L_{\theta(1)}^{(1)} \left( w, -i \left( s - \frac{1}{2} \right), \chi_1 \right) = \sum_{\Re(\tau_{\chi_1}) > 0} \frac{1}{(-i(s - \frac{1}{2}) + \tau_{\chi_1})^w} + \sum_{n=1}^{\infty} \frac{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}{(s + 2n - \Re(\tau_{\chi_1})(-1))^w}
\]

where the argument lies in \((-\frac{\pi}{2}, \frac{\pi}{2})\). The serieses in (5.1) and (5.2) converge absolutely, locally and uniformly in the given \((w, s)\)-region above.

Proof of Lemma 5.3. By Theorem 3.2 (ii) and (iv), we find that the residue in

\[
R_{\theta(1)} \left( w, -i \left( s - \frac{1}{2} \right), \chi_1 \right) = \sum_{p,m} \chi_1(p^m)p^{-ms}(m \log p)^{w-1} \log p.
\]

The series converges absolutely and uniformly on any compact subset of \( \{ (w, s) \in \mathbb{C}^2 \mid \Re(s) > 1 \} \).
Theorem 5.4. If \( \frac{1}{2} + \tau_1^{(0)} \) tan \( \theta^{(1)} \) < Re(s) tan \( \theta^{(1)} - \Im(s) < \tan \theta^{(1)} \), Re(s) > 1 + \varepsilon^{(1)} \) and Re(w) > 1, we have

\[
\begin{align*}
\sum_{\Im(p_{\chi_1}) \neq 0} \frac{1}{(s - p_{\chi_1})^w} &+ \sum_{n=1}^{\infty} \frac{1}{(s + 2n - \frac{3 + \chi_1(-1)}{2})^w} \\
+ \frac{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}{(s - \frac{1}{2} - \tau_{\chi_1}^{(0)})^w} + \frac{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}{(s - \frac{1}{2} + \tau_{\chi_1}^{(0)})^w} + \frac{\mu_{\chi_1}(0)}{(s - \frac{1}{2})^w} \\
= -\frac{1}{\Gamma(w)} \chi_1(p^m)p^{-ms}(m \log p)^{w-1} \log p.
\end{align*}
\]

Proof of Theorem 5.4. We put \( r = 1 \) and \( z = -i \left( 1 \right) \) in Theorem 4.1 and then by applying Lemma 5.2 and 5.3 we have

\[
\begin{align*}
\sum_{\Im(p_{\chi_1}) \neq 0} \frac{1}{(s - p_{\chi_1})^w} &+ \sum_{n=1}^{\infty} \frac{1}{(s + 2n - \frac{3 + \chi_1(-1)}{2})^w} \\
+ \frac{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}{(s - \frac{1}{2} - \tau_{\chi_1}^{(0)})^w} + \frac{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}{(s - \frac{1}{2} + \tau_{\chi_1}^{(0)})^w} + \frac{\mu_{\chi_1}(0)}{(s - \frac{1}{2})^w} \\
= -\frac{1}{\Gamma(w)} \chi_1(p^m)p^{-ms}(m \log p)^{w-1} \log p,
\end{align*}
\]

under the conditions that

\[
\left( \tau_{\chi_1}^{(0)} + \frac{1}{2} \right) \tan \theta^{(1)} < \Re(s) \tan \theta^{(1)} - \Im(s) < \tan \theta^{(1)},
\]

\( \Re(s) > 1 + \varepsilon^{(1)} \) and \( \Re(w) > 1 \).

Then, replacing \( \chi_1 \) with \( \chi_1 \) in (5.5), we obtain (5.4). \( \square \)

5.2. Proof of Theorem 1.1.

Proof. The left-hand side of (5.4) is a meromorphic function of \( w \) on the whole \( \mathbb{C} \) by Corollary 4.3. Hence, by using the definition of the zeta regularized product we have

\[
\exp \left( -\text{Res}_{w=0} \left( \frac{\text{the left-hand side of (5.4)}}{w^2} \right) \right) = \prod_{\Im(p_{\chi_1}) \neq 0} (s - p_{\chi_1}) \prod_{n=1}^{\infty} \left( s + 2n - \frac{3 + \chi_1(-1)}{2} \right)^{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}(s - \frac{1}{2} - \tau_{\chi_1}^{(0)})^{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}(s - \frac{1}{2} + \tau_{\chi_1}^{(0)})^{\mu_{\chi_1}(\tau_{\chi_1}^{(0)})}(s - \frac{1}{2})^{\mu_{\chi_1}(0)},
\]

(5.6)
On the other hand, since \( \frac{1}{1(w)} = w + O(w^2) \) \( (w \to 0) \), we have

\[
\exp \left( -\operatorname{Res}_{w=0} \left( \frac{\text{the right-hand side of (5.4)}}{w^2} \right) \right) = \exp \left( \sum_{p,m} \frac{\chi_1(p^m)p^{-ms}}{m} \right) = \prod_p (1 - \chi_1(p)p^{-s})^{-1}.
\]

By the property of the zeta regularized products, (5.6) is a meromorphic function on the whole \( \mathbb{C} \). Hence (1.1) holds. \( \square \)

6. The Euler product expression of \((L_{X_1} \otimes_{F_1} L_{X_2})(s)\)

In a similar way as section 5, we show Theorem 1.3.

6.1. The “key equation” for \( r = 2 \).

**Lemma 6.1.** If \((2z_2^{(0)} + 1) \tan \theta^{(2)} < \Re(s) \tan \theta^{(2)} - \Im(s) < 2 \tan \theta^{(2)}, \Re(s) > 2(1 + \varepsilon^{(2)}) \) and \( \Re(w) > 2 \) then we have

\[
L^{(1)}_{\theta^{(2)}}(w, -i(s - 1), \{ \chi_j \}_{j=1}^2) = e^{\frac{\pi i}{24}} \sum_{\chi(\rho_{x_1}), \chi(\rho_{x_2}) < 0} \frac{1}{(s - \rho_{x_1} - \rho_{x_2})^w}.
\]

\[
L^{(2)}_{\theta^{(2)}}(w, -i(s - 1), \{ \chi_j \}_{j=1}^2)
= -e^{\frac{\pi i}{24}} \sum_{\chi(\rho_{x_1}), \chi(\rho_{x_2}) > 0} \frac{1}{(s - \rho_{x_1} - \rho_{x_2})^w}
+ \sum_{(a,b) \in \{(1,2), (2,1)\}} \left( \sum_{\chi(\rho_{x_1}) > 0} \sum_{n=1}^{\infty} \frac{\mu_{x_1}(\rho_{x_1})}{(s - \rho_{x_1} - \frac{1}{2} - \tau_{x_1})^w} \right)
+ \sum_{\chi(\rho_{x_1}) > 0} \left( \sum_{n=1}^{\infty} \frac{\mu_{x_1}(\rho_{x_1})}{(s + 2n - 2 - \frac{\chi(1)}{2} + \tau_{x_1})^w} \right)
+ \sum_{n=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{\mu_{x_1}(\rho_{x_1})}{(s + 2n - 2 - \frac{\chi(1)}{2} - \tau_{x_1})^w} \right)
+ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{(s + 2n_1 + 2n_2 - 3 - \frac{\chi(1) + \chi(2)}{2})^w}\]

\[
+ \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 - \tau_{x_1} - \tau_{x_2})^w} + \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 + \tau_{x_1} + \tau_{x_2})^w} + \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 - \tau_{x_1} + \tau_{x_2})^w} + \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 + \tau_{x_1} - \tau_{x_2})^w} + \sum_{(a,b) \in \{(1,2), (2,1)\}} \left( \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 - \tau_{x_1})^w} + \frac{\mu_{x_1}(\rho_{x_1}) \mu_{x_2}(\rho_{x_2})}{(s - 1 + \tau_{x_1})^w} \right).
\]

The serieses which appear here converge absolutely, locally and uniformly in the given \((w,s)\)-region above.
Proof of Lemma 6.1. In a similar way as Lemma 5.1 and 5.2 we can prove these.

Lemma 6.2. If \((2\varepsilon_2^{(0)} + 1) \tan \theta^{(2)} < \Re(s) \tan \theta^{(2)} - \Im(s) < 2 \tan \theta^{(2)}, \Re(s) > 2(1 + \varepsilon^{(2)})\) and \(\Re(w) > 2\) then we have

\[
R_{\theta^{(2)}}(w, -i(s - 1), \{\chi_j\}_{j=1}^{2}) = \frac{e^{\frac{\pi i w}{2}}}{\Gamma(w)} \sum_{k=1}^{10} E_k(w, s, \{\chi_j\}_{j=1}^{2}). \tag{6.1}
\]

Proof of Lemma 6.2. Let \(p\) and \(m\) be any fixed prime number and positive integer respectively. By Theorem 3.2 (ii) we have

\[
l_{\chi_1}(t) \cdot l_{\chi_2}(t) = \tilde{\chi}_1(p^m) \tilde{\chi}_2(p^m) \left( \frac{ite^{-\frac{it}{2}p^{-m}}}{2\pi m(t - im \log p)} \right)^2 + \sum_{(a,b) \in \{(1,2),(2,1)\}} \frac{ite^{-\frac{it}{2}a(p^m)b^{-m}}}{2\pi m(t - im \log p)} \left( -\frac{it}{2\pi} \sum_{p^m \neq q^m} \frac{\tilde{\chi}_1(q^n)q^{-n}}{n(t - in \log q)} \right.
\]

\[
- \frac{e^{i(\alpha + \frac{1}{2})\pi \gamma}}{2\pi} \left( \frac{it}{2} \sum_{q,n} \frac{\tilde{\chi}_1(q^n)q^{-n(1+\alpha)}}{n(t + in \log q)} - \frac{it}{2} \sum_{n=1}^{\infty} \frac{\tilde{\chi}_1(-1)e^{-i(\alpha \pi)}}{n(t - n\pi)} \right.
\]

\[
+ i \log \left( \frac{\chi_b(-1)\Gamma(1 + \alpha)N_b^2 G(\chi_b)}{(2\pi)^{1+\alpha}} \right) - \frac{(1 + \alpha)\pi}{2} + \frac{1}{t} \left( \gamma + \log \left( \frac{2\pi}{N_b} \right) + \frac{\pi i}{2} \right) - \frac{1}{t} \int_{0}^{\infty} \frac{1}{e^u - 1} \cdot \frac{u - it(1 - e^{-\alpha \pi})}{u - it} \, du \right)
\]

\[
- \frac{t}{2\pi} \frac{e^{\frac{it}{2}}}{{\sin t}} \int_{S(\varepsilon^{(2)} \to 0)} e^{-ist} \log L(s, \chi_b) \, ds - \frac{ie^{\frac{\pi i (\alpha - 1) \gamma}{2\sin t}}}{{\sin t}} \left( \mu_{\chi_b}((e^{i\alpha \pi}))L(s, \chi_b) - e^{-i\alpha \pi}(\mu_{\chi_b}(t)) \right)
\]

+ (the holomorphic parts at \(t = im \log p\)).

Applying this to

\[
R_{\theta^{(2)}}(w, -i(s - 1), \{\chi_j\}_{j=1}^{2}) = \frac{2\pi i}{\Gamma(w)} \sum_{p,m} \text{Res}_{t=im \log p} \left( e^{\frac{i(s-1)t}{2}}l_{\chi_1}(t) \cdot l_{\chi_2}(t)t^{w-1} \right)
\]

leads to (6.1).
Lemma 6.3. For \( k \in \{1, 2, \cdots, 10\} \), \( E_k(w, s, \{\chi_j\}_{j=1}^2) \) converges absolutely and uniformly on any compact subset of \( \{(w, z) \in \mathbb{C}^2 \mid \Re(s) > \beta_k\} \), where

\[
\beta_k = \begin{cases} 
1 & (k = 1), \\
2 & (k = 2, 6), \\
1 - \alpha & (k = 3, 4, 7), \\
\max\left\{ \frac{1 - \chi_2(-1)}{2}, \frac{1 - \chi_2(-1)}{2} \right\} & (k = 5), \\
\frac{\delta}{2} + \frac{\epsilon(0)}{\tau^2} & (k = 8), \\
\frac{\delta}{2} - \frac{\epsilon(0)}{\tau^2} & (k = 9), \\
\frac{\delta}{2} & (k = 10).
\end{cases}
\]

Proof of Lemma 6.3. The desired results follow from Lemma 2.7 (iii) immediately except for \( E_2(w, s, \{\chi_j\}_{j=1}^2) \), \( E_3(w, s, \{\chi_j\}_{j=1}^2) \) and \( E_4(w, s, \{\chi_j\}_{j=1}^2) \).

Concerning \( E_4(w, s, \{\chi_j\}_{j=1}^2) \), we can easily prove its absolute and locally uniform convergence by Lemma 2.7 (iii).

We consider \( E_3(w, s, \{\chi_j\}_{j=1}^2) \). Let \( (w, s) \in \mathbb{C}^2 \) satisfy \( \Re(s) > 1 - \alpha + \delta \) and \( A \leq \Re(w) \leq B \) for any fixed real numbers \( \varepsilon, A \) and \( B \) with \( \delta > 0 \) and \( A < B \).

Then, for any prime numbers \( p, q \) and any \( m, n \in \mathbb{Z}_{\geq 1} \) we have

\[
\frac{\chi_a(p^m)\chi_b(q^n)p^{-m(s+\alpha)}q^{-n(m \log p + \log q)}}{n(m \log p + n \log q)} \leq \begin{cases} 
\frac{2^{-(1+\delta)}q^{-n(\log 2)^{A+1}}}{n^2 \log q} & (p = 2, m = 1), \\
\frac{p^{-m(1+\delta)}q^{-n(m \log p + \log q)}}{n^2 \log q} & (otherwise),
\end{cases}
\]

where \( (a, b) \in \{(2, 1), (2, 1)\} \). From Lemma 2.7 (ii) we have

\[
\sum_{q, n} \frac{2^{-(1+\delta)}q^{-n(\log 2)^{A+1}}}{n^2 \log q} < \infty.
\]

From Lemma 2.7 (ii), (iii) we have

\[
\sum_{p, m, q, n} p^{-m(1+\epsilon)}q^{-n(m \log p + \log q)} \leq \left( \sum_{p, m} p^{-m(1+\epsilon)}(m \log p) \log p \right) \left( \sum_{q, n} \frac{q^{-n}}{n^2 \log q} \right) < \infty.
\]

Hence, we find that \( E_3(w, s, \{\chi_j\}_{j=1}^2) \) converges absolutely and uniformly on any compact subset of \( \{(w, s) \in \mathbb{C}^2 \mid \Re(s) > 1 - \alpha\} \).

We consider \( E_2(w, s, \{\chi_j\}_{j=1}^2) \). Let \( (w, s) \in \mathbb{C}^2 \) satisfy \( \Re(s) > 2 + \delta \) and \( A \leq \Re(w) \leq B \) for any fixed real numbers \( \delta, A \) and \( B \) with \( \delta > 0 \) and \( A < B \). Then, for any prime numbers \( p, q \) and any \( m, n \in \mathbb{Z}_{\geq 1} \) we have

\[
\frac{\chi_a(p^m)\chi_b(q^n)p^{-m(s-1)}q^{-n(m \log p + \log q)}}{n(m \log p - n \log q)} \leq \begin{cases} 
\frac{2^{-(1+\delta)}q^{-n(\log 2)^{A+1}}}{n^2 \log q} & (p = 2, m = 1), \\
\frac{p^{-m(1+\delta)}q^{-n(m \log p + \log q)}}{n^2 \log q} & (otherwise),
\end{cases}
\]

where \( (a, b) \in \{(2, 1), (2, 1)\} \). From Lemma 2.7 (ii) we have

\[
\sum_{q, n} \frac{2^{-(1+\delta)}q^{-n(\log 2)^{A+1}}}{n^2 \log q} < \infty.
\]

From Lemma 2.7 (ii), (iii) we have

\[
\sum_{p, m, q, n} p^{-m(1+\epsilon)}q^{-n(m \log p + \log q)} \leq \left( \sum_{p, m} p^{-m(1+\epsilon)}(m \log p) \log p \right) \left( \sum_{q, n} \frac{q^{-n}}{n^2 \log q} \right) < \infty.
\]

Hence, we find that \( E_2(w, s, \{\chi_j\}_{j=1}^2) \) converges absolutely and uniformly on any compact subset of \( \{(w, s) \in \mathbb{C}^2 \mid \Re(s) > 1 - \alpha\} \).
where \((a, b) \in \{(1, 2), (2, 1)\}\). In the case of \((p, m) = (2, 1)\), from \(\log x - \log 2 \geq (1 - \frac{\log 2}{\log 3}) \log x\) for any \(x \in \mathbb{R}_{\geq 3}\) and Lemma 2.7 (ii) it follows that

\[
\sum_{q^n \geq 3} \frac{2^{-(1+\delta)} q^{-n} (\log 2)^{A+1}}{n(n \log q - \log 2)} \leq \frac{2^{-(1+\delta)} (\log 2)^{A+1}}{1 - \frac{\log 2}{\log 3}} \sum_{q,n} q^{-n} \log q < \infty.
\]

In the case of \((p, m) \neq (2, 1)\), we have

\[
\sum_{p,m,q,n} \frac{p^{-m(1+\delta)} q^{-n} (m \log p)^B \log p}{n|m \log p - n \log q|} \leq \sum_{p,m,q,n} \frac{p^{-m(1+\delta)} q^{-n} (m \log p)^B \log p}{n|m \log p - n \log q|}
\]

\[
= \sum_{p,m,q,n} \left( \sum_{p^m < q^n < p^{2m}} + \sum_{p^m < q^n < p^{2m}} + \sum_{p^m < q^n \geq p^{2m}} \right) \quad (6.2)
\]

Concerning the third term of (6.2), we have \(n \log q - m \log p \geq \frac{n \log 2}{2}\) because \(2m \log p \leq n \log q\). Therefore, from Lemma 2.7 (ii), (iii) we have

\[
\text{(the third term of (6.2))} \leq 2 \left( \sum_{p,m} p^{-m(1+\delta)} (m \log p)^B \log p \right) \left( \sum_{q,n} q^{-n} \right) \frac{\log q}{n^2 \log q} < \infty.
\]

Concerning the second term of (6.2), from Lemma 2.7 (i) we have

\[
\sum_{q^n < p^{2m}} \frac{q^{-n}}{n|m \log p - n \log q|} \leq \sum_{q^n < p^{2m}} \frac{q^{-n}}{q^n - p^n} \leq \sum_{l=1}^{p^{2m} - p^n - 1} \frac{1}{(p^m + l) - p^m} \ll m \log p.
\]

Hence, from Lemma 2.7 (iii) we find

\[
\text{(the second term of (6.2))} \ll \sum_{p,m} p^{-m(1+\delta)} (m \log p)^{B+1} \log p < \infty. \quad (6.4)
\]

Concerning the first term of (6.2), from Lemma 2.7 (i) we have

\[
\sum_{q^n < p^m} \frac{q^{-n}}{n|m \log p - n \log q|} \leq \sum_{q^n < p^m} \frac{q^{-n}}{p^m - q^n} \leq p^m \sum_{l=1}^{p^m - 1} \frac{1}{l(p^m - l)} = \sum_{l=1}^{p^m - 1} \frac{1}{l} + \sum_{l=1}^{p^m - 1} \frac{1}{p^m - l} \ll m \log p.
\]
Hence, from Lemma 2.7 (iii) we find
\[
(\text{the first term of } 6.2) \ll \sum_{p \leq m} p^{-m(1+\varepsilon)} (m \log p)^{B+1} \log p < \infty. \quad (6.5)
\]

From (6.3), (6.4) and (6.5) it follows that (6.2) converges. This completes the proof. 

From Lemma 6.1, Lemma 6.2 and Lemma 6.3 we derive the “key equation” for \( r = 2 \).

**Theorem 6.4.** If \((2\tau_2^{(0)} + 1) \tan \theta(2) < \Re(s) \tan \theta(2) - \Im(s) < 2 \tan \theta(2), \Re(s) > 2(1 + \varepsilon(2)) \) and \( \Re(w) > 2 \) then the following equation holds:

\[
- \sum_{\chi \neq \chi_0} \sum_{(a,b) \in \{(1,2),(2,1)\}} \left( \sum_{\chi = \chi_0} \mu_{\chi_0} \left( \frac{\theta(0)}{s} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( s - \frac{\tau_1 + \tau_2}{2} \right)^n \right)
+ \sum_{\chi \neq \chi_0} \sum_{(a,b) \in \{(1,2),(2,1)\}} \left( \sum_{\chi = \chi_0} \mu_{\chi_0} \left( \frac{\theta(0)}{s} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( s - \frac{\tau_1 + \tau_2}{2} \right)^n \right)
+ \sum_{\chi \neq \chi_0} \sum_{(a,b) \in \{(1,2),(2,1)\}} \left( \sum_{\chi = \chi_0} \mu_{\chi_0} \left( \frac{\theta(0)}{s} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( s - \frac{\tau_1 + \tau_2}{2} \right)^n \right)
+ \sum_{\chi \neq \chi_0} \sum_{(a,b) \in \{(1,2),(2,1)\}} \left( \sum_{\chi = \chi_0} \mu_{\chi_0} \left( \frac{\theta(0)}{s} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( s - \frac{\tau_1 + \tau_2}{2} \right)^n \right)
= -\frac{1}{\Gamma(w)} \sum_{k=1}^{10} E_k(w, s, \{\chi_j\}_{j=1}^2).
\]

**Proof of Theorem 6.4.** We put \( r = 2 \) and \( z = -i(s - 1) \) in Theorem 4.1 and then by applying Lemma 6.1 and Lemma 6.2 and replacing \( \chi \) with \( \chi \) we obtain the desired result. 

6.2. **Proof of Theorem 1.3.**

**Proof.** The left-hand side of the formula in Theorem 6.4 is a meromorphic function of \( w \) on the whole \( \mathbb{C} \) by Corollary 4.3. Hence, by using the definition of zeta
regularized products we have

\[
\exp \left( - \operatorname{Res}_{w=0} \left( \frac{\text{the left-hand side of the formula in Theorem 6.4}}{w^2} \right) \right) \\
= \prod_{\mathfrak{A}(\rho_{x_1}) > 0} \left( \left( s - \rho_{x_1} - \rho_{x_2} \right)^{-1} \prod_{\mathfrak{A}(\rho_{x_1}) > 0} \left( s - \rho_{x_1} - \rho_{x_2} \right) \right) \\
\times \prod_{(a,b) \in \{(1,2),(2,1)\}} \left( \prod_{\mathfrak{A}(\rho_{x_1}) > 0, n \geq 1} \left( s - \rho_{x_a} + 2n - \frac{3 + \chi_b(-1)}{2} \right) \right) \\
\times \prod_{\mathfrak{A}(\rho_{x_a}) > 0} \left( s - \rho_{x_a} - \frac{1}{2} - \tau_{x_a}^{(0)} \right) \mu_{x_a}(\tau_{x_a}^{(0)}) \\
\times \prod_{\mathfrak{A}(\rho_{x_a}) > 0} \left( s - \rho_{x_a} - \frac{1}{2} + \tau_{x_a}^{(0)} \right) \mu_{x_a}(\tau_{x_a}^{(0)}) \\
\times \prod_{n \geq 1} \left( s + 2n + \frac{3 + \chi_b(-1)}{2} \right) \mu_{x_a}(\tau_{x_a}^{(0)}) \\
\times \prod_{n \geq 1} \left( s + 2n + \frac{3 + \chi_b(-1)}{2} \right) \mu_{x_a}(\tau_{x_a}^{(0)}) \\
\times \prod_{n_{1, n_2} \geq 1} \left( s + 2n_{1} + 2n_{2} - \chi_1(-1) + \chi_2(-1) \right) \\
\times \left( s - 1 - \tau_{x_1}^{(0)} - \tau_{x_2}^{(0)} \right) \left( s - 1 - \tau_{x_1}^{(0)} + \tau_{x_2}^{(0)} \right) \\
\times \left( s - 1 + \tau_{x_1}^{(0)} - \tau_{x_2}^{(0)} \right) \left( s - 1 + \tau_{x_1}^{(0)} + \tau_{x_2}^{(0)} \right) \mu_{x_1}(\tau_{x_1}^{(0)}) \mu_{x_2}(\tau_{x_2}^{(0)}) \\
\times \prod_{(a,b) \in \{(1,2),(2,1)\}} \left( s - 1 - \tau_{x_1}^{(0)} - \tau_{x_2}^{(0)} \right) \left( s - 1 + \tau_{x_1}^{(0)} + \tau_{x_2}^{(0)} \right) \mu_{x_a}(\tau_{x_a}^{(0)}) \mu_{x_b}(\tau_{x_b}^{(0)}) \\
\times (s - 1) \mu_{x_1}(0) \mu_{x_2}(0) \\
= (L_{x_1} \otimes L_{x_2})(s).
\]

On the other hand, by Theorem 6.4 and noting that \( \frac{1}{w} = w + O(w^2) \ (w \to 0) \), we have

\[
\exp \left( - \operatorname{Res}_{w=0} \left( \frac{\text{the right-hand side of the formula in Theorem 6.4}}{w^2} \right) \right) \\
= \exp \left( \sum_{k=1}^{10} E_k(0, s, \{\chi_j\}_{j=1}^{2}) \right)
\]
for \( \Re(s) > 2 \). This completes the proof. \( \square \)
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