On the genericity of loxodromic actions

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One way of picking a “generic” element of a finitely generated group is to pick a random element with uniform probability in a large ball centered on 1 in the Cayley graph. If the group acts on a $\delta$-hyperbolic space, with at least one element acting loxodromically, then it is plausible that generic elements should act loxodromically with high probability. In this paper we prove that the probability of acting loxodromically is bounded away from 0, provided the group satisfies a very weak automaticity condition, and provided a certain compatibility condition linking the automatic with the $\delta$-hyperbolic structure is satisfied.

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1 Introduction and Motivation

For several decades, it was an unproven, but widely believed slogan that “generic elements of the mapping class groups of surfaces are pseudo-Anosov”. Here the word “generic” admits several different interpretations. For one of the possible interpretations, this slogan has recently been proven in a very satisfactory way. Indeed, A. Sisto [18], inspired by previous work of J. Maher [12], proved that the element obtained by a long random walk in the Cayley graph of the mapping class group of a surface tends to be pseudo-Anosov, with a probability which tends to 1 exponentially quickly as the length of the walk tends to infinity. (See also [17] for a completely different approach.) Actually, Sisto works in a far more general context: if a finitely generated, non-elementary group acts on a $\delta$-hyperbolic space, then the probability that the element obtained by a random walk in the group acts non-loxodromically (or even just in a non-“contracting” manner) decays exponentially quickly with the length of the walk. In the special case of the mapping class group acting on the curve complex of the surface, this implies the original slogan.

In the present paper, we use a different notion of genericity. For us, a generic element is not obtained by a long random walk, but it is picked randomly with uniform probability among the elements in a large “ball” centered on the identity element in the Cayley graph of the group. Almost nothing was known about the genericity (in this sense) of pseudo-Anosov elements until the paper [6] by S. Caruso.
Caruso’s result states that in the braid group $B_n$, equipped with Garside’s generators, the proportion of elements in the ball of radius $l > 1$ in the Cayley graph which are rigid and pseudo-Anosov is bounded below by a strictly positive constant (independently of $l$). The word “rigid” will be defined below. For the moment we just point out that the proportion of rigid elements does definitely not tend to 1 as $l$ tends to infinity. Caruso’s proof uses as a key ingredient a result of González-Meneses–Wiest [11, Theorem 5.16] which states that if a rigid braid is reducible, then its reducibility is obvious (there is a round or “almost round” reducing curve).

In the present paper we establish an analogue of Caruso’s result in the framework of a finitely generated group satisfying a very weak automaticity condition, acting on a $\delta$-hyperbolic space. We prove that a positive proportion of elements is rigid and acts loxodromically. In the proof, we replace the use of González-Meneses–Wiest’s lemma by a natural geometric hypothesis on the compatibility between the automatic and the hyperbolic structure.

We are hoping to show in a later paper how to use the general theorem presented here in order to deduce results about the frequency of pseudo-Anosov elements in mapping class groups. It would, of course, be interesting to prove true genericity (the proportion tending to 1, not just being bounded away from 0) of pseudo-Anosov elements. In the case of braid groups, this was achieved by Caruso–Wiest [7], and it might be possible to adapt the techniques of [7] to the present context.

2 Statement of the result

Throughout this paper, we consider a finitely generated monoid $G$ (for instance a group), equipped with a fixed finite generating set $S$. We always suppose that each element of $G$ has a preferred representative word in the generators, called the normal form of the element, and the language of normal forms is recognized by a deterministic finite state automaton (FSA). We recall briefly what this means: we have a finite graph, whose vertices are called the states of the automaton, and whose edges are oriented and labelled by generators. From each state there is exactly one exiting edge per generator. There is a unique start state, and a unique fail state – all edges exiting the fail state immediately loop back to it. In practice, we shall omit the fail state and the arrows leading to it, for simplicity. Some of the states (possibly all except the fail state) are labelled as accept states. A word is recognized or accepted by the automaton if the path in the automaton beginning at the start state and following the edge labels of the
word ends in an accept state. For an introduction to the theory of finite state automata, the reader can consult [10, Chapter 1] or [16].

We say a word in the generators $S$ is rigid if the word $w^n$ (the word repeated $n$ times) is accepted by the automaton for all positive integers $n$. For instance, this is the case if the words $w$ and $w^2$ are both accepted, and if the paths traced out in the automaton by the two paths end in the same state. We say an element $g$ of $G$ is rigid if the normal form word representing it is rigid, i.e. if the normal form of $g^n$ is equal to the normal form of $g$, repeated $n$ times. In the past, rigid elements have only been defined for Garside groups [3], but the preceding definition is just an obvious generalization.

If $A$ is the transition matrix of the automaton, then the $(i,j)$th entry of $A^l$ counts the number of words of length $l$ starting at the $i$th and ending at the $j$th state. This makes it possible to use Perron-Frobenius theory to asymptotically count words [19].

We say a finite state automaton is recurrent if there is some integer $l$ such that there is a path of length exactly $l$ from from any state (other than the failure state) to itself and to any other state. In terms of matrices, this means that $A^l$ is a matrix with all entries strictly positive – this property is very useful in Perron-Frobenius theory. The automata which we are interested in in practice (related to mapping class groups, for instance) are unfortunately often not recurrent; nevertheless the reader should keep the recurrent case in mind, because it is simpler while already containing the key of the argument.

For the purposes of this paper we introduce another notation (which is unnecessary if the automaton under consideration is recurrent). We consider the (possibly empty) set of states which can be reached from all non-fail states of the automaton. This set of states, together with the edges connecting them, will be called the accessible sub-automaton, and its states the accessible states. Note that a path can enter, but never leave, the accessible sub-automaton. We shall say a FSA is dominated by its accessible sub-automaton if the sub-automaton is non-empty and recurrent, and if the exponential growth rate of the number of words accepted by its accessible sub-automaton is strictly larger than that of the original automaton, with the sub-automaton removed.

**Definition 2.1** We say that $G$ satisfies the automatic normal form hypothesis if every element of $G$ has a unique preferred word in the generating set representing it, and if the language of preferred representatives (or “normal forms”) is recognized by a deterministic finite state automaton (FSA) with the following properties:

- the automaton must be dominated by its accessible sub-automaton, and
• There is at least one word \( w_{\text{rigid}} \) such that both \( w_{\text{rigid}} \) and \( w_{\text{rigid}}^2 \) are accepted by the automaton, where the paths traced out in the automaton by the two words both end in the same accept state \( E \). Moreover, this state \( E \) has to lie in the accessible sub-automaton.

Note that the automatic normal form hypothesis does not require any kind of the fellow traveller conditions familiar from automatic or combable groups. It is really a fairly mild hypothesis. For instance, it is satisfied if the language of normal forms is recognized by a FSA which is recurrent and where the start state is also an accept state – in this case the trivial word can play the role of the word \( w_{\text{rigid}} \).

**Example 2.2** If \( G = B_n^+ \) is the positive braid monoid, and \( S \) is Garside’s generating set, then the FSA has one start state, one fail state, and \( n! \) other states, which are labelled by the \( n! \) generators, and which are all accept states. In this automaton, edges labelled \( g \in S \) either terminate in the state labelled \( g \) or in the fail state. The accessible sub-automaton of this automaton contains these \( n! \) states, but not the start state. As proven in [6], it is recurrent with \( l = 5 \). The normal form of any rigid braid can play the role of the word \( w_{\text{rigid}} \).

**Notation 2.3** For an integer \( l \), we denote \( B(l) \) be the set of elements of \( G \) whose normal form is of length at most \( l \). Warning: since the normal form words need not be geodesics in the Cayley graph, this set does not necessarily coincide with the ball of radius \( l \) and center \( 1_G \) in the Cayley graph of \( G \).

Now we suppose that \( G \) acts on the left by isometries on a \( \delta \)-hyperbolic complex \( X \) (which need not be proper), and that this action on \( X \) is determined by the action of the generators \( g \in S \). Let \( P \) be a point of \( X \) – we think of it as a base point, and it is fixed once and for all. If we define \( c = \max_{g \in S} d_X(P, g.P) \), then for any word \( x_1x_2x_3 \ldots \) in the generators of \( G \), any two successive points in the sequence \( P, x_1.P, x_1x_2.P, x_1x_2x_3.P, \ldots \) in \( X \) are at distance at most \( c \). (It is philosophically satisfying, but of no mathematical importance, to choose \( P \) so as to make \( c \) as small as possible.)

We shall need a compatibility condition between the automatic and the hyperbolic structures:

**Definition 2.4** Suppose a monoid \( G \) satisfying the automatic normal form hypothesis is acting on a \( \delta \)-hyperbolic complex \( (X, P) \). We say that the **geodesic words hypothesis**
is satisfied if there exists a constant $R \geq 0$ such that for every normal form word $x_1x_2x_3\ldots$, and for every $i, j \in \mathbb{N}$ with $0 \leq i < j$, the sequence of points

\[
x_1x_2\ldots x_iP, \ x_1x_2\ldots x_{i+1}P, \ \ldots, x_1x_2\ldots x_jP
\]

stays in the $R$-neighbourhood of every geodesic connecting its endpoints.

**Remark 2.5** The geodesic words hypothesis is actually equivalent to the condition that the family of paths

\[
\{(P, x_1.P, x_1x_2.P, x_1x_2x_3.P, \ldots) | \text{a word recognized by the FSA}\}
\]

forms a uniform family of unparameterized quasi-geodesics in $X$. However, we shall not use this equivalent point of view.

We recall the classification of isometries of $\delta$-hyperbolic spaces: an element $g$ of the monoid $G$ acts *elliptically* if the orbit $P, g.P, gg.P, \ldots$ in $X$ is bounded. It acts *parabolically* if $\lim_{n \to \infty} \frac{1}{n} \cdot d_X(P, g^n.P) = 0$. In this case, the sequence of points $P, g.P, gg.P, \ldots$ of $X$, is not a quasi-geodesic in $X$: the point $g^n.P$ gets arbitrarily far away from any geodesic connecting $P$ and $g^{2n}.P$; moreover, the isometry of $X$ given by the action of $g$ has a unique fixed point on $\partial X$. Finally, $g$ acts *loxodromically* (or *hyperbolically*) if $\lim_{n \to \infty} \frac{1}{n} \cdot d_X(P, g^n.P)$ exists and is positive. In this case, the orbit of $P$ under $\langle g \rangle$ is a quasi-geodesic, and the isometry of $X$ given by the action of $g$ has two fixed points on $\partial X$. For a proof that any element belongs to exactly one of these three classes see [8, Section 9] (which requires $X$ to be proper, but the proof does not use this hypothesis). See also [5, Section 3].

**Notation 2.6** Suppose a monoid $G$ satisfying the automatic normal form hypothesis acts on a $\delta$-hyperbolic space. We say that there are *interesting loxodromic actions* if there exists a loop in the accessible sub-automaton such that the products of generators of $G$ read along this loop acts loxodromically on $X$. This is for instance the case if there are rigid elements which act loxodromically.

We are now ready to state our main result.

**Theorem 2.7** Let $G$ be a finitely generated monoid acting on a $\delta$-hyperbolic complex $X$, and satisfying the automatic normal form hypothesis and the geodesic words hypothesis. Suppose further that there are interesting loxodromic actions. Then

\[
\liminf_{l \to \infty} \frac{\# \{ \text{elements of } B(l) \text{ which are rigid and act loxodromically on } X \}}{\# B(l)} > 0.
\]
Remark 2.8  Unfortunately, the geodesic words hypothesis is often difficult to verify in practice. This is the reason why the current paper does not contain any non-trivial examples.

Open Problems 2.9  (1) It should be possible to use Theorem 2.7 in order to study the proportion of pseudo-Anosov elements in mapping class groups. We recall that a mapping class is pseudo-Anosov if and only if it acts loxodromically on the curve complex of the surface. It seems likely that any mapping class group, equipped with any reasonable normal form recognized by a FSA, satisfies the geodesic words hypothesis. In fact, it is even conceivable that any uniform family of quasi-geodesics in a mapping class group projects to a uniform family of unparameterized quasi-geodesics in the curve complex.

(2) One tool for checking the geodesic words hypothesis could be the results on train track splitting sequences of H. Masur, L. Mosher and S. Schleimer [15]. For instance, all that’s needed for reproving that pseudo-Anosov braids are generic [6, 7] without using [11, Theorem 5.16] is proving that Garside normal forms are train track splitting sequences – that would be interesting in its own right.

(3) Apply the theorem in other contexts, for instance of $\text{Out}(F_n)$ acting on the free factor complex $\mathcal{FF}_n$. We recall that in this setup, an element acts loxodromically if and only if it is iwip. Another possible framework is that of (strongly) relatively hyperbolic groups acting on the coned-off Cayley graph. This framework might be easier to deal with because here quasi-geodesics in the group project to unparameterized quasi-geodesics in the $\delta$-hyperbolic space (see [9, Section 8]). Thus the geodesic words hypothesis should not pose any difficulty in this case.

(4) Prove the stronger result that the proportion of elements of $G$ which act loxodromically actually tends to 1 (we have only proven that its lim inf is positive). One possible strategy for doing so involves constructing a “blocking element” in $G$, analogue to the blocking braids of [7].

(5) Strengthen the statements by proving the genericity not only of loxodromic, but even of contracting actions, in the sense of Sisto [18].

(6) Prove that the translation distance in $X$ of the action of generic elements with normal forms of length $l$ is bounded below by a linear function in $l$ (c.f [13]).

3  Proof of Theorem 2.7

Throughout the proof, our strategy for counting elements of $G$ with various properties will be to count normal form words representing elements with the same properties.
The first step of the argument is to prove that the proportion of elements which are rigid is bounded away from zero.

**Lemma 3.1**

\[
\liminf_{l \to \infty} \frac{\# \{ \text{rigid words of length } l \text{ recognized by the FSA} \}}{\# \{ \text{words of length } l \text{ recognized by the FSA} \}} > 0.
\]

**Proof** We recall that there is, by hypothesis, a path in the automaton, starting at the start state and ending at some accept state \(E\) in the accessible sub-automaton, tracing out a word \(w_{\text{rigid}}\) in the automaton such that the path along the word \(w_{\text{rigid}}^2\) ends in the same state \(E\). In particular, the word \(w_{\text{rigid}}\) is rigid. Since the automaton is dominated by the accessible sub-automaton, we have

\[
\liminf_{l \to \infty} \frac{\# \{ \text{words of length } l \text{ with prefix } w_{\text{rigid}}, \text{ recognized by the FSA} \}}{\# \{ \text{words of length } l \text{ recognized by the FSA} \}} > 0.
\]

Next we observe that, for any word \(w\) read out along a loop starting and ending in the state \(E\), the word \(w_{\text{rigid}}w\) is also rigid. Let us denote the by \(k\) the length of the word \(w_{\text{rigid}}\). Now, among the paths of length \(l - k\) in the automaton (in fact in the accessible sub-automaton) which start at the state \(E\), the proportion of those which also end at \(E\) tends to some positive number as \(l\) tends to \(\infty\) – for a proof of this fact, see for instance [6, Lemma 3.5(i) and (ii)].

In summary, among the words of length \(l\), there is a positive proportion which starts with the word \(w_{\text{rigid}}\), and among those, there is a positive proportion of words which end at the state \(E\). Since all these words are rigid, the lemma is proven. \(\Box\)

We now turn to the second step of the proof of Theorem 2.7, namely proving that among the rigid elements constructed in the previous step, the proportion of loxodromically acting ones tends to 1 exponentially quickly.

Let \(w\) be a word in normal form which is rigid – this means that the word \(w^n\) is also in normal form for all positive integers \(n\), and thus that the geodesic words hypothesis applies to all \(w^n\). We will prove that this imposes serious restrictions on how the element of \(G\) represented by \(w\) can act on \(X\).

**Lemma 3.2** Let \(w\) be a rigid word in normal form. Then \(w\) cannot act parabolically; moreover, if it acts elliptically, then it does not contain any subword whose action moves the base point \(P\) by more than \(5R\). (Here \(R\) is the constant appearing in the geodesic words hypothesis.)
Proof. The rigid element represented by $w$ cannot act parabolically on $X$, because if it did, then the infinite sequence of points $P, w.P, ww.P, www.P, \ldots$ would not lie uniformly close to a geodesic ray, as required by the geodesic words hypothesis.

Next, we look at the case where the rigid element $w$ acts elliptically. We denote $D = d_X(P, w.P)$. Since $w$ acts by isometries, we have $d_X(w^i.P, w^{i+1}.P) = D$ for all integers $k$.

Claim A. We claim that $D \leq 3R$. (This claim is our analogue of the result of González-Meneses–Wiest [11, Theorem 5.16].)

In order to prove Claim A, we suppose, for a contradiction, that $D = 3R + \epsilon$, for some positive integer $\epsilon$. Since $w$ acts elliptically, the orbit $P, w.P, ww.P, www.P, \ldots$ is bounded – let $\rho$ be its diameter. Now we choose an integer $N$ so large that $\frac{\rho}{N} < \epsilon$. By the geodesic words hypothesis, the geodesic from $P$ to $w^N.P$ contains all points $w^i.P$, for $0 \leq i \leq N$, in its $R$-neighbourhood. In particular, for every point $w^i.P$ there is a point $v_i$ on this geodesic with $d_X(w^i.P, v_i) \leq R$. We now have the $N$ points $v_1, \ldots, v_N$ on a geodesic of length at most $\rho$, so there is a pair of indices $(i, j)$ with $0 \leq i < j \leq N$ such that $d_X(v_i, v_j) \leq \frac{\rho}{N} < \epsilon$. This implies that $d_X(w^i.P, w^j.P) < 2R + \epsilon$. Hence the point $w^{i+1}.P$, which lies at distance exactly $3R + \epsilon$ from $w^i.P$, cannot lie in the $R$-neighbourhood of a geodesic segment from $w^i.P$ to $w^j.P$. This contradicts the geodesic words hypothesis, and the claim is proven.

We have just seen that a rigid element $w$ which acts elliptically cannot move the basepoint very far: $d_X(P, w.P) \leq 3R$. In fact, neither can any subword of $w$:

Claim B. If $w = x_1x_2x_3 \ldots$ is rigid, we claim that any subword $x_ix_{i+1} \ldots x_{j-1}x_j$ satisfies

$$d_X(P, x_ix_{i+1} \ldots x_{j-1}x_j.P) \leq 5R$$

(Claim B is our analogue of the theorem of Bernadete-Gutierrez-Nitecki [2, 4]). In order to prove this claim, we observe that $x_1 \ldots x_{i-1}.P$ and $x_1 \ldots x_{j-1}.P$ each lie $R$-close to some point of a geodesic from $P$ to $w.P$. Since this geodesic is only of length at most $3R$, we obtain

$$d_X(x_1 \ldots x_{i-1}.P, x_1 \ldots x_{j-1}.P) \leq R + 3R + R = 5R$$

Since $x_1 \ldots x_{j-1}$ acts by an isometry on $X$, we deduce that $d_X(P, x_i \ldots x_j.P) \leq 5R$, also. The proof of Claim B and of Lemma 3.2 is complete.

We observe that words whose action displaces the base point by more than $5R$ do actually exist (by the hypothesis that there are loops in the accessible sub-automaton which act loxodromically). Let $w_{\text{far}}$ be one such word.
Now we recall that we have a distinguished rigid word \( w_{\text{rigid}} \) recognized by the automaton, whose length we denote \( k \) and which is read in the automaton along a path which ends in an accessible accept state which we denote \( E \). As seen in the proof of Lemma 3.1, there is a non-negligible proportion of elements of length \( l \) which are recognized by the automaton and which are of the form \( w_{\text{rigid}}w \), where the word \( w \) is read along a loop of length \( l - k \) based at the state \( E \). In particular, these words \( w_{\text{rigid}}w \) are rigid. Now, we have seen in Lemma 3.2 that these words are guaranteed to act loxodromically if they contain a subword \( w_{\text{far}} \) which moves the base point by more than \( 5R \).

Among the words of length \( l - k \) which can be read along loops based at \( E \) in the accessible sub-automaton, the proportion of those that do not contain the subword \( w_{\text{far}} \) tends to 0 as \( l \) tends to infinity. For a proof of this statement, see Lemma 3.5(iii) of [6]. (In fact, the convergence is exponentially fast, but we won’t use this fact.)

In summary, the proportion of rigid, loxodromically acting words among all recognized words of length equal to \( l \) is bounded away from 0. The very last step is to deduce that the same holds among all recognized words of length less than or equal to \( l \). The proof of Theorem 2.7 is now complete.

### 4 An elementary example: \( \text{SL}(2, \mathbb{Z}) \)

In this last section, we study one elementary example explicitly, namely the case of the group \( \text{SL}(2, \mathbb{Z}) \), which is the mapping class group of the once-punctured torus, acting on the Farey graph, which is the curve complex of the once-punctured torus – see [14, Section 1.5]. In this case, Theorem 2.7 does not contain any new information, since the full genericity (in the sense of this paper) of pseudo-Anosov elements has been known for a long time. A proof of this fact is in [17, Theorem 18], and the closely related case of the 3-strand braid group is treated very explicitly in [1].

As generators of \( \text{SL}(2, \mathbb{Z}) \) we choose the matrices

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\]

and their inverses. Then we can use the rewriting \( ABAA \rightarrow \overline{AB}, \ ABAB \rightarrow \overline{BA}, \ ABA \rightarrow \overline{AB}, \ BABA \rightarrow \overline{BA} \) and \( \overline{BABA} \rightarrow BA \) – thus we can choose that words in our language should not contain subword \( \overline{ABA} \) (except for the element \( \overline{ABA} \) itself). Similarly, subwords \( \overline{BAB} \) and also \( \overline{ABA} \) and \( \overline{BAB} \) are forbidden, except for the element \( \overline{ABA} \) itself. Due to the rewritings \( \overline{ABA} \rightarrow \overline{BAB}, \ \overline{BABA} \rightarrow \overline{ABA}, \ \overline{ABA} \rightarrow \overline{BABB} \)
and $\overline{B}AB \rightarrow A\overline{B}A$, we can also rule out the subwords $AB\overline{A}$, $BA\overline{B}$, $AB\overline{A}$ and $\overline{B}AB$. Now any element of $\text{SL}(2, \mathbb{Z})$ except $ABA$ and $\overline{A}\overline{B}A$ has a unique representative word not containing any of these subwords. The automaton describing this language is given in Figure 1. We see that several words can play the role of the word $w_{\text{rigid}}$, for instance $w_{\text{rigid}} = A\overline{B}$, but we shall use $w_{\text{rigid}} = B$.

Figure 1: A finite state automaton for $\text{SL}(2, \mathbb{Z})$. All states (except the start state) are accept states. The states are named after the last two letters read. The symbol $X_{\sim A}$ means “any letter but $A$, or no letter”. The bottom part of the picture shows the accessible sub-automaton. The top part shows (shaded grey) the non-accessible states, and how they are connected to the accessible states. Not shown here is the fail state.

The adjacency matrix $M$ of the accessible sub-automaton is of size $8 \times 8$ – it is given in Figure 4. Here the entries 0 have been suppressed, for better legibility. Now we recall that

- the number of normal form words of length $l$ starting with the letter $B$ is equal to the sum of the terms in column number 4 (in blue) of the matrix $M^{l-1}$
On the genericity of loxodromic actions

|       | $X_{BA}$ | $BA$  | $AB$  | $X_{-AB}$ | $BA$  | $AB$  | $X_{-BA}$ |
|-------|---------|-------|-------|----------|-------|-------|----------|
| $X_{BA}$ | 1       | 1     | 1     |          |       |       |          |
| $BA$   |         |       |       | 1        | 1     | 1     |          |
| $AB$   |         |       |       |          |       |       | 1        |
| $X_{-AB}$ | 1       | 1     | 1     |          |       |       |          |
| $BA$   |         |       |       | 1        |       |       |          |
| $AB$   |         |       |       |          | 1     |       |          |
| $X_{-BA}$ |       |       |       |          |       |       | 1        |

Figure 2: The adjacency matrix of the accessible sub-automaton

- the number of normal form words of length $l$ of the form $w_{\text{rigid}} = Bw$, where $w$ is a word read along a loop of length $l - 1$ based in the state $X_{-AB}$ is equal to the term of the matrix $M^{l-1}$ in the position marked by a red star * (the position (4,4)). Recall that all these words are rigid.

Perron-Frobenius theory tells us that there is a unique eigenvalue $\lambda$ of maximal modulus, which is real and has a one-dimensional eigenspace. In the present case, $\lambda = \sqrt{2} + 1$ (the "silver ratio"). We shall denote $\kappa = \frac{1}{\lambda} = \sqrt{2} - 1$. When $l$ tends to infinity, the matrix $M^l$ multiplied by an appropriate (very small) number gets arbitrarily close to the matrix

\[
\begin{pmatrix}
\lambda & 1 & 1 & \lambda & 1 & 1 & \lambda \\
1 & \kappa & \kappa & 1 & 1 & \kappa & 1 \\
1 & \kappa & \kappa & 1 & 1 & \kappa & 1 \\
\lambda & 1 & 1 & \lambda & 1 & 1 & \lambda \\
1 & \kappa & \kappa & 1 & 1 & \kappa & 1 \\
1 & \kappa & \kappa & 1 & 1 & \kappa & 1 \\
\lambda & 1 & 1 & \lambda & 1 & 1 & \lambda
\end{pmatrix}
\]

Among the normal form words of length $l$, the proportion of those that start with the letter $B$ is bounded away from zero (independently of $l$). Among those, in turn, the proportion of ones that end in the state $X_{-AB}$, and thus are rigid, tends to $\frac{\lambda}{4\lambda+1} = \frac{\sqrt{2}+1}{4\sqrt{2}+8} = 0.17677\ldots$ This yields a lower bound on the proportion of rigid elements. (In fact, using a more careful counting, it is possible to prove that the proportion of rigid elements among all elements in the $l$-ball tends to $\frac{8\lambda+4}{16\lambda+16} = \frac{2+\sqrt{2}}{8} = 0.42677\ldots\}$
In order to study the words representing pseudo-Anosov elements among the rigid ones, we equip the Farey graph with the base point $P = 1 = \frac{1}{1}$. Then the geodesic words hypothesis is satisfied with $R = 1$. By Claim B, the normal form of any rigid element which does not act loxodromically cannot contain any subword that moves the base point by more than five. The proportion, among rigid (or indeed any) words of length $l$, of words which do not contain any such subword tends to zero as $l$ tends to infinity.

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