On the strict comparison theorem for
G-expectations

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Abstract. This paper investigates the strict comparison theorem under the framework of G-expectation, i.e., let \(X \leq Y\) q.s., if \(X, Y\) satisfy some additional conditions, then \(\hat{E}[X] < \hat{E}[Y]\).

1 Introduction

In 2006, to deal with the model uncertainty problem in finance, Peng establish the sublinear expectation theory and introduce a new sublinear expectation, called G-expectation, which have many well properties as classical linear expectation except linearity (see Peng [10, 11, 12, 13, 14]). Unlike the well-known g-expectation, G-expectation was introduced via fully nonlinear parabolic partial differential equations.

In this paper, we consider the following problem: for a linear expectation \(E_P\), we know that if \(X \leq Y\) then \(E_P[X] \leq E_P[Y]\), furthermore, if \(X \leq Y\) and \(P(X < Y) > 0\), then \(E_P[X] < E_P[Y]\). If we replace the linear expectation \(E_P\) by Peng’s G-expectation \(\hat{E}\), the former holds obviously, we interest that when the latter holds. We give three forms of strict comparison theorem for G-expectation and some interesting examples.

This paper is organized as follows: in Section 2, we recall some basic notions and results of G-expectation. In Section 3, we prove strict comparison theorems for G-expectation and give some interesting examples.

2 Preliminaries

We present some preliminaries in the theory of G-expectations. More details can be found in Peng [10, 11, 12, 13, 14].

Let \(\Omega\) be a given set and let \(\mathcal{H}\) be a linear space of real valued functions defined on \(\Omega\) satisfying: if \(X_i \in \mathcal{H}\), \(i = 1, \cdots, d\), then

\[\varphi(X_1, \cdots, X_d) \in \mathcal{H}, \ \forall \varphi \in C_{t,x}\text{Lip}(\mathbb{R}^d),\]
where \( C_{L, L_p}(\mathbb{R}^d) \) is the space of all real continuous functions defined on \( \mathbb{R} \) such that
\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in \mathbb{R}^d, \text{ } k \text{ depends on } \varphi.
\]

A sublinear expectation \( \mathbb{E} \) on \( \mathcal{H} \) is a functional \( \mathbb{E} : \mathcal{H} \to \mathbb{R} \) satisfying the following properties:

1. Monotonicity: If \( X \geq Y \) then \( \mathbb{E}[X] \geq \mathbb{E}[Y] \).
2. Constant preserving: \( \mathbb{E}[c] = c \).
3. Sub-additivity: \( \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \).
4. Positive homogeneity: \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0 \).

**Definition 2.1** In a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), a random variable \( Y \) is said to be independent from another random variable \( X \) if
\[
\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, y)|x = X]], \forall \varphi \in C_{L, L_p}(\mathbb{R} \times \mathbb{R}).
\]

Two random variables \( X_1 \) and \( X_2 \) are called identically distributed, denoted by \( X_1 \sim X_2 \), if
\[
\mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2)], \forall \varphi \in C_{L, L_p}(\mathbb{R}).
\]

If \( \bar{X} \) is identically distributed with \( X \) and independent from \( X \), then \( \bar{X} \) is said to be an independent copy of \( X \).

**Definition 2.2** In a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), a random variable \( X \in \mathcal{H} \) is said to be \( G \)-normal distributed with \( \sigma^2 = \mathbb{E}[X^2] \) and \( \sigma^2 = -\mathbb{E}[-X^2] \), denoted by \( X \sim N(0, [\sigma^2, \sigma^2]) \), if
\[
aX + b\bar{X} \sim \sqrt{a^2 + b^2}X, \forall a, b \geq 0,
\]
where \( \bar{X} \) is an independent copy of \( X \).

We can give a characterization of \( G \)-normal distribution by fully nonlinear parabolic partial differential equation (see Peng \[10, 11, 12, 13, 14\]).

**Proposition 2.3** Let \( X \sim N(0, [\sigma^2, \sigma^2]) \), then for each \( \varphi \in C_{L, L_p}(\mathbb{R}) \), the function \( u \) defined by
\[
u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], t \geq 0, x \in \mathbb{R},
\]
is the unique viscosity solution of the following \( G \)-heat equation:
\[
\begin{cases}
\partial_t u - G(\partial_{xx} u) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}, \\
u_{t=0} = \varphi,
\end{cases}
\]
where \( G(\alpha) = (\sigma^2 \alpha^+ - \sigma^2 \alpha^-)/2 \).
We will give the notion of $G$-expectations. Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$. We consider the canonic process: $B_t(\omega) = \omega_t, t \in [0, \infty), \omega \in \Omega$. For each fixed $T > 0$, we set

$$Lip(\Omega_T) := \{ \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) : \forall n \geq 1, t_1, \cdots, t_n \in [0, T], \forall \varphi \in C_{l, Lip}(\mathbb{R}^n) \}.$$ 

and

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

Then we can construct a consistent sublinear expectation called $G$-expectation $\hat{E}[-]$ on $Lip(\Omega)$, such that $B_1$ is $G$-normal distributed under $\hat{E}$ and for each $s, t \geq 0$ and $t_1, \cdots, t_N \in [0, t]$, we have

$$\hat{E}[\varphi(B_{t_1}, \cdots, B_{t_2}, B_{t+s} - B_t)] = \hat{E}[\psi(B_{t_1}, \cdots, B_{t_n})],$$

where $\psi(x_1, \cdots, x_N) = \hat{E}[\varphi(x_1, \cdots, x_N, \sqrt{2}B_t)]$. Under $G$-expectation $\hat{E}[-]$, the canonic process $\{B_t : t \geq 0\}$ is called $G$-Brownian motion.

The completion of $Lip(\Omega)$ under the Banach norm $\hat{E}[\cdot]$ is denoted by $L_{1G}(\Omega)$. $\hat{E}[-]$ can be extended uniquely to a sublinear expectation on $L_{1G}(\Omega)$ (see Peng [12, 13, 14]).

We denote by $\mathcal{B}(\Omega)$ the Borel $\sigma$-algebra of $\Omega$. It was proved in Hu and Peng [5] (see also Denis, Hu and Peng [3]) that there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \ \forall X \in L_{1G}(\Omega).$$

For such weakly compact family $\mathcal{P}$, we can introduce the natural Choquet capacity

$$v(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

**Definition 2.4** A set $A \subset \Omega$ is polar if $v(A) = 0$. A property holds quasi-surely (q.s.) if it holds outside a polar set.

**Definition 2.5** A real function $X$ on $\Omega$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O$ with $v(O) < \varepsilon$ such that $X|_O$ is continuous.

The following proposition can be found in Denis, Hu and Peng [3].

**Proposition 2.6** For each $X \in L_{1G}(\Omega)$, there exists $Y$ such that $Y = X$ q.s. and $Y$ is quasi-continuous.

### 3 Main Theorem

In this section, we consider strict comparison theorem for Peng’s $G$-expectation $\hat{E}$, where $\hat{E}$ defined on $(\Omega, L_{1G}(\Omega))$ and there exists a weakly compact family $\mathcal{P}$ such that $\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X]$.  

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Theorem 3.1 Let $X,Y \in L^1_G(\Omega)$ and $X \leq Y$ q.s. If
\[ \inf_{P \in P} P(X < Y) > 0, \]
then \( \hat{E}[X] < \hat{E}[Y]. \)

\textbf{Proof.} Since \( \hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y], \) we only consider the case of \( X \leq 0 \) q.s. and \( \inf_{P \in P} P(X < 0) > 0. \)

For such \( X, \) we can choose \( \varepsilon \) such that \( 0 < \varepsilon < \inf_{P \in P} P(X < 0). \) Since \( X \in L^1_G(\Omega), \) by Proposition 2.6 there exists \( Y \) such that \( X = Y \) q.s. and \( Y \) is quasi-continuous. Noting that \( X = Y \) q.s. implies \( \hat{E}[X] = \hat{E}[Y], \) and if \( \hat{E}[Y] < 0 \) we also have \( \hat{E}[X] < 0. \) Without loss of generality we can assume that \( X \) is quasi-continuous. By Definition 2.5 there exists an open set \( O \) such that \( v(O) < \inf_{P \in P} P(X < 0) - \varepsilon \) and \( X|_O \) is continuous.

Set \( A = \{ \omega : X(\omega) \geq 0 \}, \) \( A_n = \{ \omega : X(\omega) \geq -1/n \}, \) \( F = A \cap O^c \) and \( F_n = A_n \cap O^c, \) then we can check that \( F_n \) is closed and \( F_n \downarrow F. \) Since \( P \) is weakly compact, we have \( v(F_n) \downarrow v(F) \) (see Huber and Strassen [3]). There exists \( n_0 \in \mathbb{N} \) such that \( v(F_{n_0}) \leq v(F) + \varepsilon. \)

We have
\[
\begin{align*}
v(X &\geq -1/n_0) - 1 = v((A_{n_0} \cap O^c) \cup (A_{n_0} \cap O)) - 1 \\
&\leq v(F_{n_0}) + v(A_{n_0} \cap O) - 1 \\
&\leq v(F) + \varepsilon + v(O) - 1 \\
&\leq v(A) + \varepsilon + v(O) - 1 \\
&= -\left( \inf_{P \in P} P(X < 0) - v(O) - \varepsilon \right) < 0
\end{align*}
\]
Since \( X \leq 0 \) q.s., we have \( v(X > t) = 0 \) for \( t \geq 0. \) Finally, we get
\[
\hat{E}[X] = \sup_{P \in P} E_P[X] = \sup_{P \in P} \left( \int_0^\infty P(X > t)\,dt + \int_{-\infty}^0 (P(X > t) - 1)\,dt \right)
\leq \int_0^\infty \sup_{P \in P} P(X > t)\,dt + \int_{-\infty}^0 (\sup_{P \in P} P(X > t) - 1)\,dt
\leq \int_0^\infty v(X > t)\,dt + \int_{-\infty}^0 (v(X \geq t) - 1)\,dt = \int_{-\infty}^0 (v(X \geq t) - 1)\,dt
\leq \int_{-1/n_0}^0 (v(X \geq t) - 1)\,dt \leq (v(X \geq -1/n_0) - 1)/n_0 < 0.
\]

\square

In fact, the condition \( \inf_{P \in P} P(X < Y) > 0 \) is not easy to verify. But for \( X,Y \in Lip(\Omega), \) we can represent them by \( X = \varphi(B_{t_1},B_{t_2} - B_{t_1},\cdots,B_{t_n} - B_{t_{n-1}}) \) and \( Y = \psi(B_{t_1},B_{t_2} - B_{t_1},\cdots,B_{t_n} - B_{t_{n-1}}), \) where \( \varphi,\psi \in C_l Lip(\mathbb{R}^n) \) and \( B \) is the \( G \)-Brownian motion on \( (\Omega,\mathcal{L}_1^G(\Omega),\mathcal{E}) \). When \( \varphi > 0, \) we have the following results.
Lemma 3.2 Suppose $\sigma > 0$. Let $X \in \text{Lip}(\Omega)$ with the form $X = \varphi(B_1)$ and $\varphi(x) \leq 0, \forall x \in \mathbb{R}$. If there exists $x_0 \in \mathbb{R}$ such that $\varphi(x_0) < 0$, then $\mathbb{E}[X] < 0$.

The proof of this lemma depends on some deep estimates of fully nonlinear parabolic partial differential equations, which initially obtained by Krylov and Safonov [8], we will prove it in Appendix.

Theorem 3.3 Let $\sigma > 0$ and $X, Y \in \text{Lip}(\Omega)$ with the forms $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ and $Y = \psi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$, where $\varphi(x) \leq \psi(x), \forall x \in \mathbb{R}^n$. Then $\mathbb{E}[X] < \mathbb{E}[Y]$ if and only if there exists $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) < \psi(x_0)$.

Proof. The necessity is obviously, we only need to prove the sufficiency.

We first consider the case of $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1})$ and $Y = \psi(B_{t_1}, B_{t_2} - B_{t_1})$ with $\varphi \leq \psi$ and there exists $(x_1, x_2) \in \mathbb{R}^2$ such that $\varphi(x_1, x_2) < \psi(x_1, x_2)$. By Lemma 3.2 we have $\mathbb{E}[\varphi(x, B_{t_2} - B_{t_1})] \leq \mathbb{E}[\psi(x, B_{t_2} - B_{t_1})]$ for each $x \in \mathbb{R}$ and $\mathbb{E}[\varphi(x_1, B_{t_2} - B_{t_1})] < \mathbb{E}[\psi(x_1, B_{t_2} - B_{t_1})]$. Let us use Lemma 3.2 again, we conclude $\mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1})] < \mathbb{E}[\psi(B_{t_1}, B_{t_2} - B_{t_1})]$.

For $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ and $Y = \psi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$, we repeat above procedure, thus $\mathbb{E}[X] < \mathbb{E}[Y]$. $\square$

The condition of $\sigma > 0$ is necessary. When $\sigma = 0$, the above strict comparison theorem does not hold.

Example 3.4 Let $\sigma = 0$ and $X = B_1 \wedge 0$, then we can check that $u(t, x) = x \wedge 0$ is the unique viscosity solution of $G$-heat equation: $\partial_t u = \sigma^2 (\partial^2_x u)^{+}/2$ with initial condition $u(0, x) = x \wedge 0$. But we have $\mathbb{E}[X] = u(1, 0) = 0$. The strict comparison theorem does not hold.

Corollary 3.5 Let $B_t$ be a $G$-Brownian motion with $\sigma^2 = \mathbb{E}[B_t^2] \geq -\mathbb{E}[-B_t^2] = \sigma^2 > 0$. Then we have

$$\inf_{P \in \mathcal{P}} P(a \leq B_t \leq b) > 0,$$

where $-\infty < a < b < \infty$ and $t > 0$.

Remark 3.6 Suppose $\sigma > 0$ and $X, Y \in \text{Lip}(\Omega)$ satisfy the same conditions in Theorem 3.3. By Corollary 3.5, we can prove that $\inf_{P \in \mathcal{P}} P(X < Y) > 0$, so by Theorem 3.7, we also have $\mathbb{E}[X] < \mathbb{E}[Y]$. In the end, we give another form of strict comparison theorem.

Theorem 3.7 Let $X, Y \in L^1_G(\Omega)$ and $X \leq Y$ q.s.. If $\nu(X < Y) > 0$ and $X$ has mean certainty, i.e., $\mathbb{E}[X] = -\mathbb{E}[-X]$, then $\mathbb{E}[X] < \mathbb{E}[Y]$.

Proof. Since $X \leq Y$ q.s. and $\nu(X < Y) > 0$, there exists $P \in \mathcal{P}$ such that $P(X \leq Y) = 1$ and $P(X < Y) > 0$. Therefore $E_P[X] < E_P[Y]$. Since $\mathbb{E}[X] = -\mathbb{E}[-X]$, we have

$$\mathbb{E}[X] = E_P[X] < E_P[Y] \leq \mathbb{E}[Y].$$
But unfortunately, in general, if \( X \leq Y \) and \( v(X < Y) > 0 \), it does not imply \( \hat{E}[X] < \hat{E}[Y] \). We give the following counterexample, in which we will use the notion of quadratic variation process of \( G \)-Brownian motion \( \langle B \rangle_t \). More details about this process can be found in Peng [10, 11, 12, 13, 14].

**Example 3.8** Let \( B_t \) be a \( G \)-Brownian motion with \( \sigma^2 = \hat{E}[B_t^2] > \sigma^2 = -\hat{E}[\sigma B_t^2] > 0 \). We consider \( \langle B \rangle_t \) and \( \sigma^2 t \), then we have \( \langle B \rangle_t \leq \sigma^2 t \) q.s., and we can choose \( P \in \mathcal{P} \) such that \( B_t \) becomes the classical Brownian motion with \( E_P[B_t^2] = \sigma^2 \). Under this \( P \), the quadratic variation process \( \langle B \rangle_t \) equals \( \sigma^2 t \) \( P \)-a.s., so we have \( P(\langle B \rangle_t < \sigma^2 t) = 1 \). It is easy to get \( v(\langle B \rangle_t < \sigma^2 t) = 1 \), but \( \hat{E}[\langle B \rangle_t] = \sigma^2 t = \hat{E}[\sigma^2 t] \).

### A Proof of Lemma 3.2

In this section, we complete the proof of Lemma 3.2. We always suppose \( \sigma > 0 \).

The following lemma is a special case of Theorem 1.1 in Krylov and Safonov [8]. We denote \( B_R(x^0) := \{ x : |x - x^0| < R \} \), \( B_R := B_R(0) \) and \( Q_{\theta,R} := (0, \theta R^2) \times B_R \).

**Lemma A.1** Let \( \theta > 1 \) and \( R \leq 2 \), \( u \in C^{1,2}(Q_{\theta,R}) \), \( u \geq 0 \) be such that

\[
\partial_t u - a(t,x)\partial_{xx}u = 0, \quad \text{in } Q_{\theta,R},
\]

where \( a \in L^{\infty}((0, \infty) \times \mathbb{R}) \) and for some \( \lambda > 0 \), \( \lambda^{-1} \leq a(t,x) \leq \lambda \), \( \forall (t,x) \in (0, \infty) \times \mathbb{R} \).

Then there is a constant \( C \) depending only on \( \lambda, \theta \) such that

\[
u(\theta R^2, x) \geq Cu(R^2, 0), \quad \forall x \in B_{R/2}.
\]

We now give the proof of Lemma 3.2.

**Proof of Lemma 3.2** We first consider the case of \( \varphi \in C_{b, Lip}(\mathbb{R}) \) (bounded and Lipschitzian continuous). Then the \( G \)-heat equation \( \mathcal{H} \) has the unique classical solution, i.e., \( u(t, x) \in C^{1,2}((0, 2) \times \mathbb{R}) \) (see Krylov [7]). Since \( \varphi(x) \leq 0 \) for all \( x \in \mathbb{R} \), by the well-known maximal principle, we have \( u(t, x) \leq 0 \). Since \( \varphi(x_0) < 0 \), by the continuity of \( u(t, x) \), there exists \( \varepsilon > 0 \) and \( (t_\varepsilon, x_\varepsilon) \in (0, 1/2) \times \mathbb{R} \), such that \( u(t_\varepsilon, x_\varepsilon) < -\varepsilon \).

We set \( v(t, x) = -u(t, x_\varepsilon - Mx) \), where \( M > 2|x_\varepsilon|/\sqrt{t_\varepsilon} \). It is easy to verify that \( v(t, x) \) is the unique solution of the following PDE:

\[
\begin{cases}
\partial_t v - a(t,x)\partial_{xx}v = 0, \quad (t, x) \in (0, 2) \times \mathbb{R}, \\
v|_{t=0} = -\varphi(x_\varepsilon - Mx), \quad x \in \mathbb{R},
\end{cases}
\]

where

\[
a(t, x) = \begin{cases}
\sigma^2/2M^2 & \text{for } (t, x) \text{ such that } \partial_{xx}u(t, x_\varepsilon - Mx) \geq 0, \\
\sigma^2/2M^2 & \text{otherwise}.
\end{cases}
\]
Then \( v(t,x) \) satisfies all the condition of above Lemma. We get \( v(1, x_x/M) \geq C u(t_x, 0) = -C u(t_x, x_x) > 0 \), where \( C > 0 \) depending on \( \tau, \varphi, x, t_x \). So we have

\[ \hat{E}[\varphi(B_1)] = u(1,0) = -v(1, x_x/M) < 0. \]

For each \( \varphi \in C_{b,Lip}(\mathbb{R}) \) with \( \varphi(x) \leq 0 \) and \( \varphi(x_0) < 0 \), we can choose \( \varphi' \in C_{b,Lip}(\mathbb{R}) \) such that \( \varphi(x) \leq \varphi'(x) \leq 0 \) and \( \varphi'(x_0) = \varphi(x_0) < 0 \), then we have

\[ \hat{E}[\varphi(B_1)] \leq \hat{E}[\varphi'(B_1)] < 0. \]

\( \square \)

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