We analyze the superfield equations of the 4-dimensional N=4 SYM-theory using light-cone gauge conditions and the harmonic-superspace approach. The harmonic superfield equations of motion are drastically simplified in this gauge, in particular, the basic harmonic-superfield matrices and the corresponding harmonic analytic gauge connections become nilpotent on-shell.

1. Introduction

It is known that the superfield constraints of the N = 3 and N = 4 super-Yang-Mills (SYM) theories \[4\] are equivalent to the corresponding equations of motion. Moreover, the component fields of these theories coincide on mass shell. In the harmonic approach to the N = 3 SYM-theory \[2\], the SU(3)/U(1) × U(1) harmonics have been used for the covariant reduction of the spinor coordinates and derivatives and for the off-shell description of this theory in terms of the corresponding G-analytic superfields. As it has been shown in Ref. \[5\] the light-cone version of the N = 3 harmonic superspace simplifies drastically superfield equations of motion. Moreover, the component constraints of chirality or different types of harmonic analyticities \[6\]. It will be shown in Ref. \[7\], in particular, it has been shown that the self-duality condition for the N = 4 superfield strength corresponds to the special reality condition in the harmonic superspace. Stress that the N = 4 harmonic superspace describes the on-shell superfields only in contrast to analogous harmonic formalisms for N = 2 and 3.

The short on-shell harmonic superfields in the Abelian N = 4 SYM-theory satisfy the constraints of chirality or different types of harmonic and Grassmann analyticities \[8\]. It will be shown that these analyticities are also useful in the N = 4 non-Abelian SYM-theory.

We shall analyze the classical solutions of the harmonic-superspace equations using the convenient light-cone gauge conditions for superfield connections (see, e.g. \[8\] \[9\]). These gauge conditions yield the nilpotent superfield matrices in the bridge representation of the N = 4 SYM-theory.

2. Harmonic-superspace formulation of N = 4 SYM equations

The covariant coordinates of the D = 4, N = 4 superspace are

\[ z^M = (x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) , \]

where \( \alpha, \dot{\alpha} \) are the SL(2,C) indices and \( i \) are indices of the fundamental representations of the group SU(4).

We shall study solutions of the SYM-equations using the non-covariant notation

\[ x^\pm \equiv x^1 = t + x^3 , \quad x^\mp \equiv x^{2} = t - x^3 , \]
\[ y \equiv x^{12} = x^1 + i x^2 , \quad \bar{y} \equiv x^{21} = x^1 - i x^2 , \]
\[ (\theta^+_i, \theta^-_i) \equiv \theta^\alpha , \quad (\bar{\theta}^+_{\dot{i}}, \bar{\theta}^-_{\dot{i}}) \equiv \bar{\theta}^{\dot{\alpha}} . \]

suitable when the Lorentz symmetry is reduced to SO(1,1). The general N = 4 superspace has the odd dimension (8,8) in this notation.

The D = 4, N = 4 SYM-constraints \[10\] have the following reduced-symmetry form:

\[ \{ \nabla_+^k, \nabla_+^l \} = 0 , \quad \{ \nabla_+^k, \nabla_+^{l+} \} = 0 , \]
\[ \{ \nabla_+^k, \nabla_+^{l+} \} = 2i \delta^k_l \nabla_+^y , \]
\[ \{ \nabla_+^k, \nabla_-^l \} = W^{kl} , \quad \{ \nabla_+^k, \nabla_-^l \} = 2i \delta^k_l \nabla_y . \]
\( \{ \nabla^k, \nabla_{l+} \} = 2i\delta^k_\ell \nabla_y, \quad \{ \nabla_{k+}, \nabla_{l-} \} = W_{kl}, \quad (5) \)
\( \{ \nabla^k, \nabla^l \} = 0, \quad \nabla_{k+}, \nabla_{l-} = 0, \)
\( \{ \nabla^k, \nabla_{l-} \} = 2i\delta^k_\ell \nabla = \) (6)

where \( \nabla \) are the covariant derivatives in the \( (4,8,8) \)-dimensional superspace, \( W_{kl} \) and \( W^k_l \) are the gauge-covariant superfields constructed from the gauge connections. These superfields satisfy the subsidiary conditions

\[ W^{ik} \equiv W_{ik} = -\frac{1}{2} \epsilon^{ikjl} W_{jl}. \] (7)

The equations of motion for the superfield strengths follow from the Bianchi identities

\[ \nabla_{\pm}^i W^{k l} + \nabla^k W^{i l} = 0, \]
\[ \nabla_{\pm}^i W^{k l} = \frac{1}{2} \delta^k_\ell \nabla_{j+} W^{j l} - \delta^k_\ell \nabla_{j-} W^{j l}. \] (8)

Let us consider the light-cone gauge conditions

\[ A^k_+ = 0, \quad \bar{A}_k + = 0, \quad A_k = 0, \] (9)

then the constrains (3) are solved explicitly.

The harmonic superspaces for the \( N = 4 \) SYM-theory have been discussed in Refs.\( ^3 \). It has been shown that the G- and H-analytic Abelian on-shell superfield strength lives in the harmonic superspace with \( (4+4) \) Grassmann coordinates. We shall use the analogy with the HSS description of the \( N = 3 \) SYM-equations \( ^3 \) and consider the gauge invariance and geometric structure of the superfield \( N = 4 \) equations. Stress that the variety of different G-analytic superspaces for \( N = 4 \) is more rich than for the case \( N = 2 \) and 3, however, we do not know the off-shell superfield structure of the \( N = 4 \) SYM-theory.

We shall use the \( SU(4)/U(1)^3 \) harmonics \( ^3 \) for the HSS interpretation of the non-Abelian \( N = 4 \) constraints \( ^3 \) by analogy with the Abelian case. These harmonics parametrize the corresponding coset space. They form an \( SU(4) \) matrix and are defined modulo \( U(1) \times U(1) \times U(1) \) transformations

\[ u^1_i = u_i^{(1,0,1)}, \quad u^2_i = u_i^{(-1,0,1)}, \]
\[ u^3_i = u_i^{(0,1,-1)}, \quad u^4_i = u_i^{(0,-1,1)} \] (10)

where \( i \) is the index of the quartet representation of \( SU(4) \).

The complex conjugated harmonics have opposite \( U(1) \) charges

\[ u^1_i = u_i^{(-1,0,1)}, \quad u^2_i = u_i^{(1,0,1)}, \]
\[ u^3_i = u_i^{(0,0,1)}, \quad u^4_i = u_i^{(0,0,-1)}. \] (11)

Note that we use indices \( I, J = 1, 2, 3, 4 \) for the projected components of the harmonic matrix which do not transform with respect to the ‘ordinary’ \( SU(4) \) transformations. The authors of Ref.\( ^3 \) prefer to use the \( SU(4)/S(U(2) \times U(2)) \) harmonics for the \( N = 4 \) theory.

The corresponding harmonic derivatives \( \partial^I_f \) act on these harmonics and satisfy the \( SU(4) \) algebra.

The special conjugation of the \( SU(4) \) harmonics has the following form:

\[ u^1_i \leftrightarrow u^4_i, \quad u^2_i \leftrightarrow u^3_i, \]
\[ u^3_i \leftrightarrow u^2_i, \quad u^4_i \leftrightarrow u^1_i \] (12)

and the conjugation of the harmonic derivatives is

\[ \partial^1_f f \leftrightarrow -\overline{\partial^1_f f}, \quad \partial^4_f f \leftrightarrow -\overline{\partial^4_f f}. \] (13)

where \( f(u) \) is an arbitrary harmonic function.

The analytic coordinates in the \( N = 4 \) superspace \( H(4,12|6,6) \) are

\[ \zeta = (X^+, X^-, Y, Y, \theta^\pm_2, \theta^\pm_3, \theta^\pm_4, \bar{\theta}^\pm_2, \bar{\theta}^\pm_3, \bar{\theta}^\pm_4). \] (14)

\[ X^+ = x^+ + i(\theta^+_2 \bar{\theta}^{1+} - \theta^+_1 \bar{\theta}^{2+}), \]
\[ X^- = x^- + i(\theta^-_2 \bar{\theta}^{-1+} - \theta^-_1 \bar{\theta}^{-2+}), \]
\[ Y = y + i(\theta^+_2 \bar{\theta}^{-1-} - \theta^-_1 \bar{\theta}^{-2-}), \]
\[ \bar{Y} = \bar{y} + i(\theta^-_2 \bar{\theta}^{1+} - \theta^+_1 \bar{\theta}^{2+}), \]
\[ \theta^+_I = \theta^+_I u_i^+, \quad \bar{\theta}^+_I = \bar{\theta}^+_I u_i^+. \] (15)

The spinor derivatives have the following simple form in these coordinates:

\[ D^1_\pm = \partial^1_\pm, \quad D^4_\pm = \partial^4_\pm. \] (16)
\[ D^2_+ = \partial^2_+ + i\theta^2_+ \partial_+ + i\theta^2_- \partial_-; \]
\[ D^-_+ = \partial^-_+ + i\theta^-_+ \partial_+ + i\theta^-_- \partial_-; \] (17)
\[ D^1_+ = \bar{\partial}_1_+ + 2\bar{\theta}^+_I \partial_+ + 2\bar{\theta}^-_I \partial_-, \]
\[ D^-_+ = \bar{\partial}_1_- + 2i\bar{\theta}^+_I \partial_+ + 2i\bar{\theta}^-_I \partial_- \] (18)

The corresponding harmonic derivatives are

\[ D^1_\pm = \partial^1_\pm + i\theta^1_+ \partial_+ + i\theta^1_- \partial_-; \]
\[ + i\theta^2_+ \partial^1_+ \partial_+ + i\theta^2_- \partial^1_- - \theta^2_- \partial_+ \]
\[ - \theta^2_+ \partial^1_+ + \theta^1_+ \partial^1_+ \partial_+ + \theta^-_+ \partial^1_- \partial_-. \] (19)
\[ D_4^3 = \partial_3^2 + i\theta_4^+ \partial_4^3 - \partial_4^3 \partial_+ + i\theta_4^+ \partial_4^3 - \partial_4^3 \partial_+ \]
\[ + i\theta_4^- \partial_4^3 - \partial_4^3 - \theta_4^+ \partial_4^3 \partial_+ + \partial_4^3 \partial_+ . \]
\[ (22) \]

Other projections of the Grassmann and harmonic derivatives can be constructed analogously.

Let us consider the harmonic projections of the CB covariant derivatives and the corresponding connections

\[ \nabla_{I+} = u_k^I \nabla_k^I = D_{I+} , \]
\[ \nabla_{I-} = u_k^I \nabla_k^I = D_{I-} , \]
\[ \nabla_{I+} = u_k^I \nabla_k^I = D_{I+} + A_I^L , \]
\[ \nabla_{I-} = u_k^I \nabla_k^I = D_{I-} + \bar{A}_{I-} . \]
\[ (23-26) \]

Taking into account these relations we can transform the CB-constraints (3-6) to the equivalent (2,2)-dimensional set of the G-integrability relations:

\[ \{ \nabla_{I+} , \nabla_{I-} \} = \{ \nabla_{I+} , \nabla_{I\pm} \} = \{ \nabla_{I\pm} , \nabla_{I\pm} \} = 0 . \]
\[ (27) \]

Thus, the N = 4 SYM-geometry preserves the Grassmann (6,6) analyticity. It can be shown that the harmonic projection of the superfield strength \( u_k^I W_{ij} \) follows from the basic (6,6)-analyticity in the HSS geometric formalism.

Now we shall discuss the solution of the G-integrability relations

\[ A_{I\pm}(v) = e^{-v} (D_{I\pm} e^v) , \]
\[ \bar{A}_{I\pm}(v) = e^{-v} (D_{I\pm} e^v) , \]
\[ (28-29) \]

where \( v(z, u) \) is the superfield bridge matrix.

The gauge transformations of the bridge

\[ e^v \rightarrow e^\lambda e^v e^{-\tau} , \]
\[ (30) \]

contain the (6,6)-analytic AB-gauge parameters \( \lambda \)

\[ (D_{I\pm}, \bar{D}_{I\pm}) \lambda = 0 \]
\[ (31) \]

and the harmonic-independent constrained CB-gauge parameters \( \tau \).

Matrix \( e^v \) determines a transform of the CB-gauge superfields to the analytic basis (AB). The analytic gauge group acts on the harmonic connections in AB

\[ \nabla_{K}^I = e^{v} D_{K}^I e^{-v} = D_{K}^I + V_{K}^I (v) , \]
\[ \delta V_{K}^I = D_{K}^I \lambda + [V_{K}^I , \lambda] . \]
\[ (32-33) \]

Our gauge choice \( A_{I\pm} = \bar{A}_{I\pm} = 0 \) corresponds to the following partial gauge conditions for the bridge:

\[ (D_{I\pm}^I, \bar{D}_{I\pm}) \lambda = 0 . \]
\[ (34) \]

We treat bridge \( v \) as the basic on-shell superfield, so the SYM-equations of this approach are formulated for this superfield

\[ [D_{K}^I, e^{-v} D_{I}^J e^v] = [D_{K}^I, e^{-v} \bar{D}_{J} e^v] = 0 \]
\[ (35) \]

where \( I < K \).

The subsidiary condition (34) is equivalent to the reality condition for the harmonic projection of the superfield strength \( u_k^I W_{ij} \) and corresponds to the following equation in the bridge representation:

\[ - D_{I\pm}^2 (e^{-v} D_{I\pm}^I e^v) = D_{I\pm}^2 (e^{-v} \bar{D}_{I\pm} e^v) . \]
\[ (36) \]

By analogy with the N = 3 formalism one can choose the following light-cone gauge for the N = 4 bridge:

\[ v = \theta_1^+ b^1 + \tilde{\theta}_4^- b_4 + \theta_1^- \tilde{\theta}_4^- d_4^1 , \]
\[ (37) \]

where the fermionic matrices \( b^1, \tilde{b}_4 \) and the bosonic matrix \( d_4^1 \) are the (6,6) analytic superfields. This bridge is nilpotent

\[ v^2 = \theta_1^- \tilde{\theta}_4^- [b_4, b^1] , \quad v^3 = 0 , \]
\[ e^{-v} = I - v + \frac{1}{2} v^2 = I - \theta_1^- b^1 - \tilde{\theta}_4^- b_4 \]
\[ + \theta_1^- \tilde{\theta}_4^- (1/2) [b_4, b^1] - d_4^1 . \]
\[ (38-39) \]

In the gauge group \( SU(n) \), our superfields satisfy the conditions

\[ (b^1)^\dagger = b_4 , \quad (d_4^1)^\dagger = -d_4^1 , \]
\[ \text{Tr} b^1 = \text{Tr} d_4^1 = 0 . \]
\[ (40-41) \]

Consider the parametrization of the basic spinor connections in our gauge

\[ A_{I\pm}(v) = b^1 - \theta_1^- (b^1)^2 + \tilde{\theta}_4^- f_4^1 \]
\[ + \theta_1^- \tilde{\theta}_4^- [b^1, f_4^1] , \]
\[ \bar{A}_{I\pm}(v) = \tilde{b}_4 - \tilde{\theta}_4^- (\tilde{b}_4)^2 + \theta_1^- \tilde{f}_4^1 \]
\[ - \theta_1^- \tilde{\theta}_4^- [\tilde{b}_4, \tilde{f}_4^1] , \]
\[ (42-43) \]
where the following auxiliary superfields are introduced:
\[ f_1^I = d_1^I - \frac{1}{2} \{ b^1, \bar{b}_4 \}, \]
\[ \bar{f}_1^I = d_1^I - \frac{1}{2} \{ b^1, \bar{b}_4 \}. \] (44)

The H-analyticity equations
\[ (D_2^1, D_3^1, D_4^1, D_5^1) \cdot A_1^1(v) = 0 \] (45)
are equivalent to the following (6,6)-analytic equations:
\[ (D_2^1, D_3^1) b^1 = - (\theta_2^1, \theta_3^1) (b^1)^2, \] (46)
\[ (D_2^1, D_3^1) f_1^I = - (\bar{\theta}_2^1, \bar{\theta}_3^1) f_1^I, \] (47)
\[ (D_2^1, D_3^1) f_1^I = (\theta_2^1, \theta_3^1) [f_1^I, b^1], \] (48)
\[ (D_2^1, D_3^1) f_1^I = 0. \] (49)

We shall discuss below the relations between the matrices \( b^1 \) and \( \bar{b}_4 \) which arise from the transform of the CB-representation of the gauge group before the gauge fixing:
\[ e^{\nu} \nabla_1^\pm e^{-\nu} = D_1^\pm, \]
\[ e^{\nu} \nabla_4^\pm e^{-\nu} = D_4^\pm. \] (50)

The harmonic connections in the bridge representations \( V_1^I(v) \) satisfy automatically the harmonic zero-curvature equations
\[ D_1^I V_1^I - D_1^I V_1^I + [V_2^I, V_4^I] = \delta_1^I V_1^I - \delta_1^I V_1^I. \] (51)

Basic SYM-equations are equivalent to the dynamical G-analyticity conditions
\[ (D_2^{\pm}, D_4^{\pm}) V_1^I(v) = 0, \quad I < K. \] (52)

In gauge \( [57] \), these equations give us the following relations:
\[ V_2^I(v) = \theta_2^I b^1, \] (53)
\[ V_3^I(v) = \theta_3^I b^1, \] (54)
\[ V_4^I = (V_2^I)^1 = - \bar{\theta}_2^I \bar{b}_4, \] (55)
\[ V_4^I = (V_2^I)^1 = - \bar{\theta}_3^I \bar{b}_4, \] (56)
where all connections are nilpotent. Similar relations have been considered in the harmonic formalism of the \( N = 3 \) SYM-theory \([3]\).

One can also construct the non-analytic harmonic connections
\[ e^{\nu} D_1^I e^{-\nu} = V_1^2 = - \theta_1^I D_1^I b^1. \] (57)

The conjugated harmonic connection depend, respectively, on matrix \( b_4 \) only
\[ V_3^4 = (V_2^4)^1 = - \bar{\theta}_4^I D_3^I b_4. \] (58)

It is not difficult to show that the harmonic AB-connections \( V_1^2 \) satisfies the partial (8,6)-analyticity condition
\[ D_4^2 V_1^2 = 0 \] (59)
and the conjugated connection possesses the (6,8)-analyticity
\[ D_4^{\pm} V_3^4 = 0. \] (60)

The basic AB-superfield strengths can be constructed in terms of the harmonic connections by analogy with the \( N = 2 \) SYM-theory \([1]\)
\[ W^{12} = - D_4^1 D_4^1 V_1^2 = - D_4^2 b^1, \] (61)
\[ W_{34} = - D_4^{+} D_4^{-} V_3^4 = - D_3^{+} c_4. \] (62)
They satisfy the non-Abelian G- and H-analyticity conditions which generalize the shortness conditions for the corresponding Abelian superfields \([7]\).

The reality condition
\[ W^{12} = - W_{34} \] (63)
is equivalent to the single linear differential relation between the matrices \( b^1 \) and \( \bar{b}_4 \) which can be easily solved via the following representation with the anti-Hermitian (6,6)-analytic bosonic matrix \( A^{13} \)
\[ b^1 = \bar{D}_{3+} A^{13}, \] (64)
\[ \bar{b}_4 = D_{3+}^{2} A^{13}, \] (65)
\[ W^{12} \equiv - W_{34} = - D_3^{+} \bar{D}_{3+} A^{13}. \] (66)

Consider the evident relation
\[ (b^1)^2 = \frac{1}{2} D_{3+} [A^{13}, \bar{D}_{3+} A^{13}] . \] (67)

Equations \([67]\) generate the following relations for \( A^{13} \)
\[ (D_2^1, D_3^1) A^{13} = \frac{1}{2} (\theta_2^1, \theta_3^1) [A^{13}, \bar{D}_{3+} A^{13}]. \] (68)
Thus, the harmonic-superspace representation and light-cone gauge conditions simplify significantly the analysis of the $N = 4$ SYM-equations. We hope that this representation allows us to construct the interesting solutions of these equations.

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