ORE’S THEOREM ON SUBFACTOR PLANAR ALGEBRAS

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Abstract. This article proves that an irreducible subfactor planar algebra with a distributive biprojection lattice admits a minimal 2-box projection generating the identity biprojection. It is a generalization (conjectured in 2013) of a theorem of Øystein Ore on distributive intervals of finite groups (1938), and a corollary of a natural subfactor extension of a conjecture of Kenneth S. Brown in algebraic combinatorics (2000). We deduce a link between combinatorics and representations in finite group theory.

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1. Introduction

Any finite group $G$ acts outerly on the hyperfinite II$_1$ factor $R$, and the group subfactor $(R \subseteq R \rtimes G)$, of index $|G|$, remembers the group [11]. Jones proved in [12] that the set of possible values for the index $|M : N|$ of a subfactor $(N \subseteq M)$ is

$$\{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\} \cup [4, \infty].$$

By Galois correspondence [19], the lattice of intermediate subfactors of $(R \subseteq R \rtimes G)$ is isomorphic to the subgroup lattice of $G$. Moreover, Watatani [28] extended the finiteness of the subgroup lattice to any irreducible finite index subfactor. Then, the subfactor theory can be seen as an augmentation of the finite group theory, where the indices are not necessarily integers. The notion of subfactor planar algebra [14] is a diagrammatic axiomatization of the standard invariant of a finite index II$_1$ subfactor [13]. Bisch [4] proved that the intermediate subfactors are given by the biprojections (see Definition 4.2) in the 2-box space of the corresponding planar

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algebra. The recent results of Liu [15] on the biprojections are also crucial for this article (see Section 3).

Oystein Ore proved in 1938 that a finite group is cyclic if and only if its subgroup lattice is distributive, and he extended one way as follows:

**Theorem 1.1** ([20], Theorem 7). Let $[H, G]$ be a distributive interval of finite groups. Then there is $g \in G$ such that $(Hg) = G$.

This article generalizes Ore’s Theorem 1.1 to planar algebras as follows:

**Theorem 1.2.** Let $\mathcal{P}$ be an irreducible subfactor planar algebra with a distributive biprojection lattice. Then there is a minimal 2-box projection generating the identity biprojection (it is called w-cyclic).

In general, we deduce a non-trivial upper bound for the minimal number of minimal projections generating the identity biprojection. Note that Theorem 1.2 was conjectured for the first time in a conference of the author in 2013 [21, Conjecture 5.11]. The following application is a dual version of Theorem 1.1. See Definition 6.3 for the notations.

**Theorem 1.3.** Let $[H, G]$ be a distributive interval of finite groups. Then $\exists$ $V$ irreducible complex representation of $G$ such that $G(\text{V}H) = H$.

Next, we deduce a non-trivial upper bound for the minimal number of irreducible components for a faithful complex representation of $G$, involving the subgroup lattice only. This is a new link between combinatorics and representations in finite group theory.

Finally, the appendix proves that in the irreducible depth 2 case, the coproduct of two minimal central projections is given by the fusion rule of the corresponding irreducible complex representations.

This article generalizes results from finite group theory to subfactor theory (as for [2,9,21,23,28–30]), applying back to new results in finite group theory (which is quite rare). An expert in group theory suggested the author to write a group theoretic translation of the proof of these applications [22]. Otherwise, the author investigated (with Mamta Balodi) another approach for a direct proof of these applications, related to a problem in algebraic and geometric combinatorics, “essentially” due to K.S. Brown, asking whether the Möbius invariant of the bounded coset poset $P$ of a finite group (which is equal to the reduced Euler characteristic of the order complex of the proper part of $P$) is nonzero ([20 page 760] and [5 Question 4]). These investigations gave rise to [3]. In fact, these applications are a consequence of a relative version of Brown’s problem. Shareshian and Woodroofe proved in [26] another consequence of Brown’s problem. In [23 Section 6], the author extended Brown’s problem to any irreducible subfactor planar algebra and explained in details how it implies Theorem 1.2.

For the convenience of the reader and because this article proves the optimal version of Ore’s theorem on irreducible subfactor planar algebras, we will reproduce some preliminaries of [21,23], for being quite self-contained.

2. **Basics on lattice theory**

A lattice $(L, \wedge, \vee)$ is a poset $L$ in which every two elements $a, b$ have a unique supremum (or join) $a \vee b$ and a unique infimum (or meet) $a \wedge b$. Let $G$ be a finite

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group. The set of subgroups $K \subseteq G$ forms a lattice, denoted by $\mathcal{L}(G)$, ordered by $\subseteq$, with $K_1 \vee K_2 = \langle K_1, K_2 \rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$. A sublattice of $(L, \wedge, \vee)$ is a subset $L' \subseteq L$ such that $(L', \wedge, \vee)$ is also a lattice. If $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$. Any finite lattice is bounded, i.e. admits a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$. The atoms are the minima of $L \setminus \{\hat{0}\}$. The coatoms are the maxima of $L \setminus \{\hat{1}\}$. Consider a finite lattice, $b$ the join of its atoms and $t$ the meet of its coatoms, then let call $[\hat{0}, b]$ and $[t, \hat{1}]$ its bottom and top intervals. A lattice is distributive if the join and meet operations distribute over each other. A distributive bounded lattice is called Boolean if any element $a$ admits a unique complement $a^\complement$ (i.e. $a \wedge a^\complement = \hat{0}$ and $a \vee a^\complement = \hat{1}$).

**Lemma 2.1.** Let $a$ and $b$ be two elements of a Boolean lattice. If $a \vee b = \hat{1}$ then $b \geq a^\complement$. In particular, if $a$ is an atom then $b \in \{a^\complement, \hat{1}\}$.

**Proof.** It is immediate after the following computation:

$$a^\complement = a^\complement \wedge \hat{1} = a^\complement \wedge (a \vee b) = (a^\complement \wedge a) \vee (a^\complement \wedge b) = a^\complement \wedge b.$$

The subset lattice of $\{1, 2, \ldots, n\}$, with union and intersection, is called the Boolean lattice $\mathcal{B}_n$ of rank $n$. Any finite Boolean lattice is isomorphic to some $\mathcal{B}_n$. A lattice is called top (resp. bottom) Boolean if its top (resp. bottom) interval is Boolean. We refer to [27] for more details.

**Proposition 2.2.** A finite distributive lattice is top and bottom Boolean.

**Proof.** See [27] items a-i p254-255 which uses Birkhoff’s representation theorem (a finite lattice is distributive if and only if it embeds into some $\mathcal{B}_n$).

3. **Ore’s theorem on intervals of finite groups**

We will give our short alternative proof of Theorem [11] by extending it to any top Boolean interval (see Proposition 2.2). The proof of the second claim below is different from that of [21] Theorem 2.5], for being a correct translation of the proof of Theorem 5.9. This single variation reveals how an extension of [21] was possible.

**Definition 3.1.** An interval of finite groups $[H, G]$ is called $H$-cyclic if there is $g \in G$ such that $\langle Hg \rangle = G$. Note that $\langle Hg \rangle = \langle H, g \rangle$.

**Theorem 3.2.** A top Boolean interval $[H, G]$ is $H$-cyclic.

**Proof.** The proof follows from the claims below.

**Claim:** Let $M$ be a maximal subgroup of $G$. Then $[M, G]$ is $M$-cyclic.

**Proof:** For $g \in G$ with $g \notin M$, we have $\langle M, g \rangle = G$ by maximality.

**Claim:** A Boolean interval $[H, G]$ is $H$-cyclic.

**Proof:** Let $K$ be an atom in $[H, G]$. By induction on the rank of the Boolean lattice (initiated by the previous claim), we can assume $[K, G]$ to be $K$-cyclic, i.e. there is $g \in G$ such that $\langle K, g \rangle = G$. Now, for all $g' \in Kg$ we have

$$\langle K, g \rangle = \langle Kg \rangle = \langle Kg' \rangle = \langle K, Hg' \rangle.$$ 

But $Kg$ decomposes into a finite partition of $H$-cosets $Hg_i$ with $i = 1, \ldots, |K : H|$. It follows that for all $i$, we have $K \vee \langle Hg_i \rangle = G$, and so $\langle Hg_i \rangle \in \{K^\complement, G\}$ by Lemma
If for all $i$ we have $\langle H g_i \rangle = K^C$, then

$$G = \langle Kg \rangle = \bigcup_i H g_i = \bigvee_i \langle H g_i \rangle = \bigvee_i K^C = K^C,$$

which is a contradiction. So there is $i$ such that $\langle H g_i \rangle = G$. The result follows. □

Claim: $[H, G]$ is $H$-cyclic if its top interval $[K, G]$ is $K$-cyclic.

Proof: Consider $g \in G$ with $\langle K, g \rangle = G$. For any coatom $M \in [H, G]$, we have $K \subseteq M$ by definition, and so $g \notin M$, then a fortiori $\langle H, g \rangle \notin M$. It follows that $\langle H, g \rangle = G$. □

The converse is false because $\langle S_2, (1234) \rangle = S_4$ whereas $[S_2, S_4]$ is not top Boolean.

4. Biprojections and basic results

For the notions of subfactor, subfactor planar algebra and basic properties, we refer to [13–15]. See also [24] Section 3 for a short introduction. Let $(N \subseteq M)$ be a finite index irreducible subfactor. The $n$-box spaces $P_n,+$ and $P_n,-$ of the planar algebra $P = P(N \subseteq M)$, are $N' \cap M_{n-1}$ and $M' \cap M_n$. A projection is an operator $p$ such that $p = p^2 = p^*$. Let $N \subseteq K \subseteq M$ be an intermediate subfactor. Then, the Jones projection $e_K^M : L^2(M) \to L^2(K)$ is an element of $P_{2,+}$. Consider $e_1 := eN^M$ and $id := eM^M$ the identity. Note that $tr(id) = 1$ and $tr(e_1) = |M : N|^{-1} = \delta^{-2}$. Let $\langle a | b \rangle := \langle b^* a \rangle$ be the inner product of $a$ and $b \in P_{2,+}$. Let $F : P_{2,+} \to P_{2,+}$ be the Fourier transform, and let $a \ast b$ be the coproduct of $a$ and $b$. Then $a \ast b = F(F^{-1}(a)F^{-1}(b))$. Note that $a \ast e_1 = e_1 \ast a = \delta^{-1} a$ and $a \ast id = id \ast a = \delta tr(a) id$. Let $\overline{\tau} := F(F(a))$ be the contragredient of $a$. Let $R(a)$ be the range projection of $a$. We define the relations $a \preceq b$ and $a \sim b$ by $R(a) \subseteq R(b)$ and $R(a) = R(b)$, respectively.

Lemma 4.1. Let $p, q \in P_{2,+}$ be projections. Then

$$e_1 \preceq p \ast \overline{\tau} \iff pq \neq 0.$$

Proof. The result follows by irreducibility (i.e. $P_{1,+} = \mathbb{C}$).

Note that if $p \in P_{2,+}$ is a projection then $\overline{\tau}$ is also a projection.

Definition 4.2 ([13,17,18]). A biprojection is a projection $b \in P_{2,+} \setminus \{0\}$ with $F(b)$ a multiple of a projection.

Note that $e_1 = eM^N$ and $id = eM^M$ are biprojections.

Theorem 4.3 ([4] p212). A projection $b \in P_{2,+}$ is a biprojection if and only if it is the Jones projection $eK^M$ of an intermediate subfactor $N \subseteq K \subseteq M$.

Then, the set of biprojections is a finite lattice [28], of the form $[e_1, id]$.

Theorem 4.4. An operator $b \in P_{2,+}$ is a biprojection if and only if $e_1 \preceq b = b^2 = b^* = \overline{b} \sim b \ast b$.

Moreover, $b \ast b = \delta tr(b)b$.

Proof. See [17] items 0-3 p191 and [18] Theorem 4.12. □
Lemma 4.5. Consider $a_1, a_2, b \in \mathcal{P}_{2,+}$ with $b$ a biprojection, then
\[(b \cdot a_1 \cdot b) \ast (b \cdot a_2 \cdot b) = b \cdot (a_1 \ast (b \cdot a_2 \cdot b)) \ast b = b \cdot ((b \cdot a_1 \cdot b) \ast a_2) \cdot b\]
\[(b \ast a_1 \ast b) \cdot (b \ast a_2 \ast b) = b \ast (a_1 \cdot (b \ast a_2 \ast b)) \ast b = b \ast ((b \ast a_1 \ast b) \cdot a_2) \ast b\]

Proof. By exchange relations \[\text{(17)}\] on $b$ and $\mathcal{F}(b)$. \[\square\]

Now, we define the biprojection generated by a positive operator.

Definition 4.6. Consider $a \in \mathcal{P}_{2,+}$ positive, and let $p_n$ be the range projection of $\sum_{k=1}^n a^{*k}$. By finiteness there exists $N$ such that for all $m \geq N$, $p_m = p_N$, which is a biprojection \[\text{(18)}\] Lemma 4.14], denoted $\langle a \rangle$, called the biprojection generated by $a$. It is the smallest biprojection $b \geq a$. For $S$ a finite set of elements in $\mathcal{P}_{2,+}$, let $\langle S \rangle$ be $\langle \sum_{s \in S} R(s) \rangle$.

Lemma 4.7. Let $a, b, c, d$ be positive operators of $\mathcal{P}_{2,+}$. Then
\[
\begin{align*}
(1) & \quad a \ast b \text{ is also positive,} \\
(2) & \quad [a \preceq b \text{ and } c \preceq d] \rightarrow a \ast c \preceq b \ast d, \\
(3) & \quad a \preceq b \Rightarrow \langle a \rangle \leq \langle b \rangle, \\
(4) & \quad a \sim b \Rightarrow \langle a \rangle = \langle b \rangle.
\end{align*}
\]

Proof. It’s precisely \[\text{(18) Theorem 4.1 and Lemma 4.8} \] for (1) and (2). Next, if $a \preceq b$ then by (2), for any integer $k$, $a^{*k} \preceq b^{*k}$, so for any $n$,
\[
\sum_{k=1}^n a^{*k} \preceq \sum_{k=1}^n b^{*k},
\]
then $\langle a \rangle \leq \langle b \rangle$ by Definition \[\text{(4.0)}\]. Finally, (4) is immediate from (3). \[\square\]

Let $N \subseteq K \subseteq M$ be an intermediate subfactor. The planar algebras $\mathcal{P}(N \subseteq K)$ and $\mathcal{P}(K \subseteq M)$ can be derived from $\mathcal{P}(N \subseteq M)$, see \[\text{(14)}\].

Theorem 4.8. Consider the intermediate subfactors $N \subseteq P \subseteq K \subseteq Q \subseteq M$. Then there are two isomorphisms of von Neumann algebras
\[
\begin{align*}
l_K : \mathcal{P}_{2,+}(N \subseteq K) & \rightarrow e_K^M \mathcal{P}_{2,+}(N \subseteq M) e_K^M, \\
r_K : \mathcal{P}_{2,+}(K \subseteq M) & \rightarrow e_K^M \ast \mathcal{P}_{2,+}(N \subseteq M) \ast e_K^M,
\end{align*}
\]
for usual $\ast$, $\times$ and $()^{*}$, such that
\[
l_K(e_P^M) = e_P^M \text{ and } r_K(e_Q^M) = e_Q^M.
\]
Moreover, the coproduct $\ast$ is also preserved by these maps, but up to a multiplicative constant, $|M : K|^{1/2}$ for $l_K$ and $|K : N|^{-1/2}$ for $r_K$. Then, $\forall m \in \{l_K^{1/2}, r_K^{1/2}\}$, $\forall a_1 > 0$ in the domain of $m$, $m(a_1) > 0$ and $\langle m(a_1) \rangle = m((a_1, \ldots, a_n))$.

Proof. Immediate from \[\text{(11) or (16)}\], using Lemma \[\text{(15)}\]. We can compute the multiplicative constant for $l_K$ on the coproduct, directly as follows. Let $\alpha$ be the constant such that for any $a, b \in \mathcal{P}_{2,+}(N \subseteq K),$
\[
\begin{align*}
l_K(a \ast b) = \alpha l_K(a) \ast l_K(b).
\end{align*}
\]
Note that $l_K(e_K^M) = e_K^M$, $e_K^M \ast e_K^M = |M : N|^{1/2} |M : K|^{-1} e_K^M$ and
\[
\begin{align*}
l_K(e_K^M) & = |K : N|^{1/2} e_K^M. \]
\]
So,

$$\alpha = |M : K||K : N|^{1/2}/|M : N|^{1/2} = |M : K|^{1/2}.$$

We can compute similarly the constant for $r_K$. \hfill \Box

**Notations 4.9.** Let $b_1 \leq b \leq b_2$ be the biprojections $e_i^M \leq e_i^N \leq e_i^Q$. We define $\sum_{b} := l_K$ and $r_b := r_K$; also $\mathcal{P}(b_1, b_2) := \mathcal{P}(P \subseteq Q)$ and $\bigl|b_2 : b_1\bigr| := \text{tr}(b_2)/\text{tr}(b_1) = |Q : P|.

5. **Ore’s theorem on subfactor planar algebras**

We will generalize Theorem 3.2 to any irreducible subfactor planar algebra $\mathcal{P}$. The proof (organized in lemmas and propositions) is inspired by the proof of [25, Theorem 4.9] and Clifford theory.

**Proposition 5.1.** Let $p \in \mathcal{P}_{2,+}$ be a minimal central projection. Then, there exists $u \leq p$ minimal projection such that $\langle u \rangle = \langle p \rangle$.

**Proof.** If $p$ is a minimal projection, then it’s ok. Else, let $b_1, \ldots, b_n$ be the coatoms of $[e_1, \langle p \rangle]$ ($n$ is finite by [28]). If $p \not\leq \sum_{i=1}^{n} b_i$ then $\exists u \leq p$ minimal projection such that $u \not\leq b_i \forall i$, so that $\langle u \rangle = \langle p \rangle$. Else $p \leq \sum_{i=1}^{n} b_i$ (with $n > 1$, otherwise $p \leq b_1$ and $\langle p \rangle \leq b_1$, contradiction). Consider $E_i = \text{im}(b_i)$ and $F = \text{im}(p)$, then $F = \sum_i E_i \cap F$ (because $p$ is a minimal central projection) with $1 < n < \infty$ and $E_i \cap F \subseteq F \forall i$ (otherwise $\exists i$ with $p \leq b_i$, contradiction), so $\dim(E_i \cap F) < \dim(F)$ and there exists $U \subseteq F$ one-dimensional subspace such that $U \not\subseteq E_i \cap F \forall i$, and so a fortiori $U \not\subseteq E_i \forall i$. It follows that $u = p_U \leq p$ is a minimal projection such that $\langle u \rangle = \langle p \rangle$. \hfill \Box

Thanks to Proposition 5.1, we can give the following definition:

**Definition 5.2.** A planar algebra $\mathcal{P}$ is weakly cyclic (or w-cyclic) if it satisfies one of the following equivalent assertions:

- $\exists u \in \mathcal{P}_{2,+}$ minimal projection such that $\langle u \rangle = \text{id}$,
- $\exists p \in \mathcal{P}_{2,+}$ minimal central projection such that $\langle p \rangle = \text{id}$.

Moreover, $(N \subseteq M)$ is called w-cyclic if its planar algebra is w-cyclic.

Let $\mathcal{P} = \mathcal{P}(N \subseteq M)$ be an irreducible subfactor planar algebra. Take an intermediate subfactor $N \subseteq K \subseteq M$ and its biprojection $b = e_i^M$.

**Proposition 5.3.** The planar algebra $\mathcal{P}(e_1, b)$ is w-cyclic if and only if there is a minimal projection $u \in \mathcal{P}_{2,+}$ such that $\langle u \rangle = b$.

**Proof.** The planar algebra $\mathcal{P}(N \subseteq K)$ is w-cyclic if and only if there is a minimal projection $x \in \mathcal{P}_{2,+}(N \subseteq K)$ such that $\langle x \rangle = e_i^K$, if and only if $l_K(\langle x \rangle) = l_K(e_i^K)$, if and only if $\langle u \rangle = e_i^M$ (by Theorem 4.8), with $u = l_K(x)$ a minimal projection in $e_i^M \mathcal{P}_{2,+} e_i^K$ and in $\mathcal{P}_{2,+}$. \hfill \Box

**Proposition 5.4.** The planar algebra $\mathcal{P}(b, \text{id})$ is w-cyclic if and only if there is a minimal projection $v \in \mathcal{P}_{2,+}$ such that $\langle b, v \rangle = \text{id}$ and $r_b^{-1}(b \ast v \ast b)$ is a positive multiple of a minimal projection.

**Proof.** The planar algebra $\mathcal{P}(K \subseteq M)$ is w-cyclic if and only if there is a minimal projection $x \in \mathcal{P}_{2,+}(K \subseteq M)$ such that $\langle x \rangle = e_i^M$, if and only if $r_K(\langle x \rangle) = r_K(e_i^M)$, if and only if $\langle r_K(x) \rangle = e_i^M$ by Theorem 4.8. The results follows by Lemmas 5.6 and 5.7 below. \hfill \Box
Lemma 5.5. Let $A$ be a $*$-subalgebra of $P_{2,+}$. Then, any element $x \in A$ is positive in $A$ if and only if it is positive in $P_{2,+}$.

Proof. If $x$ is positive in $A$, then it is of the form $aa^*$, with $a \in A$, but $a \in P_{2,+}$ also, so $x$ is positive in $P_{2,+}$. Conversely, if $x$ is positive in $P_{2,+}$ then $\langle xy|y \rangle = \text{tr}(y^*xy) \geq 0$, for any $y \in P_{2,+}$, so in particular, for any $y \in A$, which means that $x$ is positive in $A$. \qed

Note that Lemma 5.5 will be applied to $A = bP_{2,+}b$ or $b*P_{2,+}b$.

Lemma 5.6. For any minimal projection $x \in P_{2,+}(b, \text{id})$, $r_b(x)$ is positive and for any minimal projection $v \preceq r_b(x)$, there is $\lambda > 0$ such that $b \ast v \ast b = \lambda r_b(x)$.

Proof. First $x$ is positive, so by Theorem 4.8, $r_b(x)$ is also positive. For any minimal projection $v \preceq r_b(x)$, we have $b \ast v \ast b \preceq r_b(x)$, because
\[ b \ast v \ast b \preceq b \ast r_b(x) \ast b = b \ast b \ast u \ast b \ast b \sim b \ast u \ast b = r_b(x), \]
by Lemma 4.7(2) and with $u \in P_{2,+}$. Now by Lemma 4.7(1), $b \ast v \ast b > 0$, so $r_b^{-1}(b \ast v \ast b) > 0$ also, and by Theorem 4.8
\[ r_b^{-1}(b \ast v \ast b) \preceq x. \]
But $x$ is a minimal projection, so by positivity, $\exists \lambda > 0$ such that
\[ r_b^{-1}(b \ast v \ast b) = \lambda x. \]
It follows that $b \ast v \ast b = \lambda r_b(x)$. \qed

Lemma 5.7. For $v \in P_{2,+}$ positive, $\langle b \ast v \ast b \rangle = \langle b, v \rangle$.

Proof. First, by Definition 4.4, $b \ast v \ast b \leq \langle b, v \rangle$, so by Lemma 4.7(3), $\langle b \ast v \ast b \rangle \leq \langle b, v \rangle$. Next $e_1 \leq b$ and $x \ast e_1 = e_1 \ast x = \delta^{-1}x$, so
\[ v = \delta^2e_1 \ast v \ast e_1 \leq b \ast v \ast b. \]
Moreover by Theorem 4.4, $\tau \preceq \langle b \ast v \ast b \rangle$, but by Lemma 4.1
\[ \tau \ast b \ast v \ast b \preceq \tau \ast e_1 \ast v \ast b \sim \tau \ast v \ast b \geq e_1 \ast b \sim b. \]
Then $b, v \preceq \langle b \ast v \ast b \rangle$, so we also have $\langle b, v \rangle \leq \langle b \ast v \ast b \rangle$. \qed

Proposition 5.8. Let $\mathcal{P}$ be an irreducible subfactor planar algebra and $[e_1, \text{id}]$ its biprojection lattice. Let $[t, \text{id}]$ be the top interval of $[e_1, \text{id}]$. Then, $\mathcal{P}$ is $w$-cyclic if $\mathcal{P}(t, \text{id})$ is so.

Proof. Let $b_1, \ldots, b_n$ be the coatoms of $[e_1, \text{id}]$ and $t = \bigwedge_{i=1}^n b_i$. By assumption and Proposition 5.4 there is a minimal projection $v \in P_{2,+}$ with $\langle t, v \rangle = \text{id}$. If $\exists i$ such that $v \leq b_i$, then $\langle t, v \rangle \leq b_i$, contradiction. So $\forall i, v \not\leq b_i$ and then $\langle v \rangle = \text{id}$. \qed

Theorem 5.9. An irreducible subfactor planar algebra $\mathcal{P}$ with a top Boolean biprojection lattice $[e_1, \text{id}]$ is $w$-cyclic.

Proof. By Proposition 5.8 we can assume $[e_1, \text{id}]$ Boolean.
We will make a proof by induction on the rank of the Boolean lattice.
If $[e_1, \text{id}]$ is of rank 1, then for any minimal projection $u \neq e_1$, $\langle u \rangle = \text{id}$. Now suppose $[e_1, \text{id}]$ Boolean of rank $n > 1$, and assume the result true for any rank $< n$. Let $b$ be an atom of $[e_1, \text{id}]$. Then $[b, \text{id}]$ is Boolean of rank $n - 1$, so by assumption
By Theorem 4.8, Lemmas 4.7, 5.6 and 5.7, for any minimal projection \( v \in \mathcal{P}_{2,+}(b, \text{id}) \) with \( \langle x \rangle = \text{id} \).

Thus \( \langle v \rangle \in \{ b, \text{id} \} \) by Lemma 2.1. Assume that for every minimal projection \( v \preceq r_b(x) \) we have \( \langle v \rangle = b \); because \( r_b(x) > 0 \), by the spectral theorem, there is an integer \( m \) and minimal projections \( v_1, \ldots, v_m \) such that \( r_b(x) \sim \sum_{i=1}^m v_i \), so

\[
\text{id} = \langle r_b(x) \rangle = \sum_{i=1}^m \langle v_i \rangle \leq \bigvee_{i=1}^m \langle v_i \rangle = \bigvee_{i=1}^m b^i = b^m,
\]

thus \( \text{id} \preceq b^m \), contradiction. So there is a minimal projection \( v \preceq r_b(x) \) such that \( \langle v \rangle = \text{id} \), and the result follows.

The proof of Theorem 1.2 follows by Proposition 2.2. In general, we deduce the following non-trivial upper bound:

**Corollary 5.10.** The minimal number \( r \) of minimal projections generating the identity biprojection of \( \mathcal{P} \) (i.e. \( \langle u_1, \ldots, u_r \rangle = \text{id} \)) is at most the minimal length \( \ell \) for an ordered chain of biprojections

\[
e_1 = b_0 < b_1 < \cdots < b_\ell = \text{id}
\]

such that \([b_i, b_{i+1}]\) is top Boolean.

**Proof.** Immediate from Theorem 5.9 and Lemma 5.12. \( \square \)

**Remark 5.11.** Let \((N \subset M)\) be an irreducible finite index subfactor. Then Corollary 5.10 reformulates as a non-trivial upper bound for the minimal number of (algebraic) irreducible sub-\(N\)-\(N\)-bimodules of \(M\), generating \(M\) as von Neumann algebra.

**Lemma 5.12.** Let \( b' < b \) be biprojections. If \( \mathcal{P}(b', b) \) is w-cyclic, then there is a minimal projection \( u \in \mathcal{P}_{2,+} \) such that \( \langle b', u \rangle = b \).

**Proof.** Take the von Neumann algebras isomorphisms (Theorem 4.8)

\[
l_b : \mathcal{P}_{2,+}(e_1, b) \rightarrow b\mathcal{P}_{2,+}b
\]

and, with \( a = l_b^{-1}(b') \),

\[
r_a : \mathcal{P}_{2,+}(b', b) \rightarrow a \ast \mathcal{P}_{2,+}(e_1, b) \ast a.
\]

Then, by assumption, the planar algebra \( \mathcal{P}(b', b) \) is w-cyclic, so by Proposition 5.4

\[
\exists u' \in \mathcal{P}_{2,+}(e_1, b) \text{ minimal projection such that } \langle a, u' \rangle = l_b^{-1}(b).
\]

Then by applying the map \( l_b \) and Theorem 4.8 we get

\[
b = \langle l_b(a), l_b(u') \rangle = \langle b', u \rangle
\]

with \( u = l_b(u') \) a minimal projection of \( b\mathcal{P}_{2,+}b \), so of \( \mathcal{P}_{2,+} \). \( \square \)

We can idem assume \( r_a^{-1}(a \ast l_b^{-1}(u) \ast a) \) minimal projection of \( \mathcal{P}_{2,+}(b', b) \).
6. Applications to finite group theory

We will give several group theoretic translations of Theorem 5.9 and Corollary 5.10, giving a new link between combinatorics and representations in finite group theory. Let $G$ be a finite group acting outerly on the hyperfinite II_1 factor $R$. Note that the subfactor $(R \subseteq R \rtimes G)$ is w-cyclic if and only if $G$ is cyclic. More generally, for $H$ a subgroup of $G$:

**Theorem 6.1.** The subfactor $(R \rtimes H \subseteq R \rtimes G)$ is w-cyclic if and only if $[H, G]$ is $H$-cyclic.

**Proof.** By Proposition 5.4, $(R \rtimes H \subseteq R \rtimes G)$ is w-cyclic if and only if $\exists u \in \mathcal{P}_{2,+}(R \subseteq R \rtimes G) \simeq \bigoplus_{g \in G} C_{e_g} \simeq C^G$ minimal projection such that $(b, u) = \text{id}$ with $b = e^{R \rtimes G}_{R \rtimes H}$ and $r_b^{-1}(b * u * b)$ minimal projection, if and only if $\exists g \in G$ such that $\langle H, g \rangle = G$, because $u$ is of the form $e_g$ and $\forall g' \in HgH$, $Hg'H = HgH$. □

Theorem 3.2 is a translation of Theorem 5.9 using Theorem 6.1.

**Corollary 6.2.** The minimal cardinal for a generating set of $G$ is at most the minimal length $\ell$ for an ordered chain of subgroups

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

such that $[H_i, H_{i+1}]$ is top Boolean.

**Proof.** Immediate from Corollary 5.10 and Theorem 6.1 □

**Definition 6.3.** Let $W$ be a representation of a group $G$, $K$ a subgroup of $G$, and $X$ a subspace of $W$. Let the fixed-point subspace be

$$W^K := \{w \in W \mid kw = w, \forall k \in K\}$$

and the pointwise stabilizer subgroup

$$G(X) := \{g \in G \mid gx = x, \forall x \in X\}$$

**Lemma 6.4.** Let $p \in \mathcal{P}_{2,+}(R^G \subseteq R)$ be the projection on a space $X$ and $b$ the biprojection of a subgroup $H$ of $G$. Then

$$p \leq b \iff H \subseteq G(X).$$

It follows that the biprojection $(p)$ corresponds to the subgroup $G(X)$.

**Proof.** First, $p \leq b$ if and only if $\forall x \in X, bx = x$. Now, there is $\lambda > 0$ such that

$$\mathcal{F}^{-1}(b) = \lambda \sum_{h \in H} e_h.$$ 

So, if $bx = x$, then $\mathcal{F}(e_h)x = \mathcal{F}(e_h)(bx) = (\mathcal{F}(e_h)b)x = bx = x$. Thus,

$$p \leq b \iff \forall h \in H, \forall x \in X, \mathcal{F}(e_h)x = x \iff \forall h \in H, h \in G(X).$$

The result follows. □

We deduce an amusing alternative proof of a well-known result [6, §226]:

**Corollary 6.5.** A complex representation $V$ of a finite group $G$ is faithful if and only if for any irreducible complex representation $W$ there is an integer $n$ such that $W \preceq V^\otimes n$. 
Proof. Let $V$ be a complex representation of $G$, and let $V_1, \ldots, V_s$ be (equivalence class representatives of) its irreducible components. Then
\[
\ker(\pi_V) = \bigcap_{i=1}^s \ker(\pi_{V_i}) = \bigcap_{i=1}^s G(V_i).
\]
Now, $V$ is faithful if and only if $\ker(\pi_V) = \{e\}$, if and only if (by Lemma 6.4) $\bigvee_{i=1}^s \langle p_i \rangle = \id$, with $p_i \in P_{2,+}(R^G \subseteq R) \simeq CG$, the minimal central projection on $V_i$. But by Definition 4.6, $\langle p_1, \ldots, p_s \rangle$ is the range projection of $N \sum_{n=1}^N (p_1 + \cdots + p_s)^* n$ for $N$ large enough. The result follows by Corollary 7.5. □

Definition 6.6. The group $G$ is called linearly primitive if it admits an irreducible complex representation $V$ which is faithful, i.e. $G(V) = \{e\}$.

Definition 6.7. The interval $[H, G]$ is called linearly primitive if there is an irreducible complex representation $V$ of $G$ such that $G(V H) = H$.

Theorem 6.8. The subfactor $(R^G \subseteq R^H)$ is w-cyclic if and only if $[H, G]$ is linearly primitive.

Proof. By Proposition 5.3, $(R^G \subseteq R^H)$ is w-cyclic if and only if
\[
\exists u \in P_{2,+}(R^G \subseteq R) \simeq CG
\]
minimal projection such that $\langle u \rangle = e_{R^H}^R$, if and only if, by Lemma 6.4 $H = G(U)$ with $U = \text{im}(u)$. Let $p$ be the central support of $u$, and $V$ its range (irreducible). Then $H \subset G(V U) \subset G(U)$, so $H = G(V U)$. □

Corollary 6.9. The subfactor $(R^G \subseteq R)$ is w-cyclic if and only if $G$ is linearly primitive.

We will prove the dual versions of Theorem 3.2 and Corollary 6.2 giving the link between combinatorics and representations theory.

Corollary 6.10. Let $[H, G]$ be a bottom Boolean interval of finite groups. Then \(\exists V\) irreducible complex representation of $G$ such that $G(V H) = H$.

Proof. It is the group theoretic reformulation of Theorem 5.9 for $P(R^G \subseteq R^H)$, thanks to Theorem 6.8. □

The proof of Theorem 1.3 follows by Proposition 2.2.

Corollary 6.11. The minimal number of irreducible components for a faithful complex representation of $G$ is at most the minimal length $\ell$ for an ordered chain of subgroups
\[
\{e\} = H_0 < H_1 < \cdots < H_\ell = G
\]
such that $[H_i, H_{i+1}]$ is bottom Boolean.

Proof. It’s a reformulation of Corollary 5.10 for $P = P(R^G \subseteq R)$, using Definition 1.6, Proposition 5.1 and the fact that the coproduct of two minimal central projections of $P_{2,+}$ is given by the tensor product of the associated irreducible representations of $G$, by Corollary 7.5. □
Definition 6.12. A subgroup \( H \) of a group \( G \) is called core-free if any normal subgroup of \( G \) contained in \( H \) is trivial.

Note that Corollary 6.11 can be improved by taking for \( H_0 \) any core-free subgroup of \( H_1 \), thanks to the following lemma.

Lemma 6.13. For \( H \) core-free, \( G \) is linearly primitive if \([H,G]\) is so.

Proof. Take \( V \) as above. Now, \( V^H \subset V \) so \( G_{(V)} \subset G_{(V^H)} \), but \( \ker(\pi_V) = G_{(V)} \), it follows that \( \ker(\pi_V) \subset H \); but \( H \) is a core-free subgroup of \( G \), and \( \ker(\pi_V) \) a normal subgroup of \( G \), so \( \ker(\pi_V) = \{e\} \), which means that \( V \) is faithful on \( G \), i.e. \( G \) is linearly primitive. \( \square \)

Remark 6.14. We get as well a non-trivial upper bound for the minimal number of irreducible components for a faithful (co)representation of a finite dimensional Kac algebra, involving the lattice of left coideal \( \star \)-subalgebras, by Galois correspondence [8, Theorem 4.4].

7. Appendix

In the irreducible depth 2 case, the relation between coproduct and fusion rules is well-known to experts, but we did not find a proof in the literature. Because we need it in the proof of Corollary 6.11 for the convenience of the reader, we will prove this relation in this appendix.

Let \( \mathcal{P} \) be a subfactor planar algebra which is irreducible and depth 2, i.e. \( \mathcal{P}_{1,+} = \mathbb{C} \) and \( \mathcal{P}_{3,+} \) is a factor. By [7, Section 3], \( \mathcal{P} = \mathcal{P}(R^\mathbb{A} \subset R) \), with \( \mathbb{A} \) a Kac algebra equal to \( \mathcal{P}_{2,+} \) and acting outerly on the hyperfinite II_1 factor \( R \).

Theorem 7.1 (Splitting, [10] p39). Any element \( x \in \mathbb{A} \) splits as follows:

\[
\begin{align*}
x &= \varepsilon(1) x^{(1)} \otimes x^{(2)} \quad \text{and} \quad x &= \varepsilon(1) x^{(1)} x^{(2)} \\
\text{with } \Delta(x) &= x^{(1)} \otimes x^{(2)} \text{ the sumless Sweedler notation of the comultiplication.}
\end{align*}
\]

Corollary 7.2. If \( a, b \in \mathbb{A} \) are central, then so is \( a \star b \).

Proof. This diagrammatic proof by splitting is due to Vijay Kodiyalam.

\[
(a \star b) \cdot x = \begin{array}{c}
\begin{array}{c}
\varepsilon(1) x^{(1)} x^{(2)} \\
\varepsilon(1) x^{(1)} x^{(2)} \\
\varepsilon(1) x^{(1)} x^{(2)}
\end{array}
\end{array} \quad \delta (a \otimes b) \Delta(x) = x \cdot (a \star b) \quad \square
\]

Proposition 7.3. Consider \( a, b, x \in \mathbb{A} \). Then \( \langle a \star b | x \rangle = \delta \langle a \otimes b | \Delta(x) \rangle \).

Proof. By irreducibility, \( \text{tr}(x \star y) = \delta \text{tr}(x) \text{tr}(y) \). Then, by Theorem 7.1

\[
\text{tr}(x^*(a \star b)) = \text{tr}((x^{(1)*}_1 a) \star (x^{(2)*}_2 b)) = \delta \text{tr}(x^{(1)*}_1 a) \text{tr}(x^{(2)*}_2 b).
\]

The result follows by definition, \( \langle x | y \rangle := \text{tr}(y^* x) \) and \( \langle a \otimes b | c \otimes d \rangle := \langle a | c \rangle \langle b | d \rangle \). \( \square \)
Note that as a finite dimensional von Neumann algebra,
\[ \mathcal{A} \simeq \bigoplus_i \text{End}(H_i). \]
As a Kac algebra, it acts on a tensor product \( V \otimes W \) as follows:
\[ \Delta(x)(v \otimes w) = (x^{(1)}v) \otimes (x^{(2)}w), \]
and \( H_i \otimes H_j \) decomposes into irreducible representations
\[ H_i \otimes H_j = \bigoplus_k M_{ij}^k \otimes H_k \]
with \( M_{ij}^k \) the multiplicity space. It follows that
\[ n_in_j = \sum_k n_{ij}^kn_k \]
with \( n_k = \dim(H_k) \) and \( n_{ij}^k = \dim(M_{ij}^k) \). The following proposition gives the relation between comultiplication and fusion rules \( (n_{ij}^k) \).

**Proposition 7.4.** The inclusion matrix of the unital inclusion of finite dimensional von Neumann algebras \( \Delta(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A} \) is \( \Lambda = (n_{ij}^k) \).

**Proof.** The irreducible representations of \( \mathcal{A} \otimes \mathcal{A} \) are \( (H_i \otimes H_j)_{i,j} \), so by the double commutant theorem and the Schur’s lemma, we get that
\[ \pi_{H_i \otimes H_j}(\mathcal{A} \otimes \mathcal{A}) = \pi_{H_i \otimes H_j}(\mathcal{A} \otimes \mathcal{A})'' = \text{End}(H_i \otimes H_j) \simeq M_{n_in_j}(\mathbb{C}). \]
Moreover, by the fusion rules
\[ \pi_{H_i \otimes H_j}(\Delta(\mathcal{A})) \simeq \bigoplus_k M_{ij}^k \otimes \pi_{H_k}(\mathcal{A}) \simeq \bigoplus_k M_{ij}^k \otimes M_{n_k}(\mathbb{C}). \]
So, the inclusion matrix of the following inclusion is \( (n_{ij}^k) \).
\[ \pi_{H_i \otimes H_j}(\Delta(\mathcal{A})) \subseteq \pi_{H_i \otimes H_j}(\mathcal{A} \otimes \mathcal{A}). \]
Take \( V = \bigoplus_{i,j} H_i \otimes H_j \). Then, we have the isomorphism of inclusions:
\[ [\Delta(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}] \simeq [\pi_V(\Delta(\mathcal{A})) \subseteq \pi_V(\mathcal{A} \otimes \mathcal{A})]. \]
But \( \pi_V = \bigoplus_{i,j} \pi_{H_i \otimes H_j} \), so the result follows. \( \square \)

**Corollary 7.5.** Let \( p_i \in \mathcal{A} \) be the minimal central projection on \( H_i \). The relation between coproduct and fusion rules is the following:
\[ p_i \ast p_j = \delta \sum_k n_{ij}^kp_k. \]

**Proof.** By Corollary 7.2, there is \( c_{ij}^k \geq 0 \) such that \( p_i \ast p_j = \sum_k c_{ij}^kp_k \). So, \( \langle p_i \ast p_j | p_k \rangle = c_{ij}^k \text{tr}(p_k) \). But, by Propositions 7.3 and 7.4
\[ \langle p_i \ast p_j | p_k \rangle = \delta \langle p_i \otimes p_j | \Delta(p_k) \rangle = \delta n_{ij}^k \text{tr}(p_k). \]
The result follows. \( \square \)
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