Cosmology of the very early universe

E. Galindo Dellavalle\textsuperscript{b}, G. Germán\textsuperscript{a\*}, A. de la Macorra\textsuperscript{c†}

\textsuperscript{(a)} The Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, UK

\textsuperscript{(b)} Abacus College, Threeways House, George Street, Oxford, OX1 2BJ, UK

\textsuperscript{(c)} Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, 01000 México D.F. México

April 15, 2008

Abstract

We study cosmological solutions for the very early universe beginning at the Planck scale for a universe containing radiation, curvature and, as a simplification of a possible scalar field potential, a cosmological constant term. The solutions are the natural counterpart of the well known results for a post-inflationary universe of non-relativistic matter, curvature and a cosmological constant. Contrary to the common belief that inflation arises independently of the initial curvature we show that in the positive curvature case the universe collapses again into a Big Crunch without allowing the cosmological term to dominate and to produce inflation. There is a critical value for the cosmological constant which divides the regions where inflation is allowed from those where inflation cannot occur. One can also have loitering solutions where the scale factor remains almost constant growing to produce inflation (or decreasing to a Big Crunch) after a time which depends on the amount of energy above (or below) the critical energy. At the critical energy the solution approaches asymptotically a particular value for the scale factor (Einstein’s static pre-inflationary universe). The cases where the cosmological term vanishes or becomes negative are also studied providing a complete discussion of Friedmann models.

\* E-mail: g.german1@physics.ox.ac.uk

†E-mail: macorra@fisica.unam.mx
1 Introduction

Studies of the evolution of the universe typically deal with the radiation and matter dominated eras in the old cosmology and with inflation and quintessence in the modern standard cosmology scenario. However, the pre-inflationary epoch is clearly also of great interest since it may contain the link between more fundamental theories (strings, branes) and low energy physics. With this in mind we have made a simplified attempt to understand the evolution of the universe in a pre-inflationary era. We study the problem of the evolution of the very early universe in the presence of radiation, curvature and a cosmological constant. The cosmological term is a very natural simplification of the more realistic and difficult problem of dealing with the evolution of a scalar field. The problem we discuss below is based on an exact solution to Friedmann equations. This solution is closely related to the one originally obtained by Harrison [1]. However to the best of our knowledge no further study of such solution has been published perhaps because interest has centred mostly in inflationary and post-inflationary solutions. This solution is the natural counterpart of the classical post-inflationary problem which has been discussed since long ago [2] and which deals with a universe containing non-relativistic matter, curvature and a cosmological term.

2 A Solution to Friedmann equations

We begin by writing the equations of Friedmann and acceleration in a convenient way

\[ \frac{\ddot{R}^2}{R^2} = \frac{8\pi G}{3} (\rho_\gamma + \rho_\Lambda - \rho_k), \]

(1)

\[ \frac{\ddot{R}}{R} = \frac{8\pi G}{3} (-\rho_\gamma + \rho_\Lambda), \]

(2)

where

\[ \rho_\gamma = \rho_{\ast\gamma} \left( \frac{R_\ast}{R} \right)^4, \quad \rho_\Lambda = \frac{\Lambda}{8\pi G} \equiv \epsilon^2 G, \quad \rho_k = \frac{3kc^2}{8\pi GR^2}, \]

(3)

are the densities of radiation, cosmological constant and curvature, respectively. The obvious time dependence has been dropped for clarity of notation. As usual \( G \) denotes Newton’s constant and \( c \) is the velocity of light. Adding Eq.(1) and Eq.(2) and multiplying by \( R^2 \) we get

\[ R\ddot{R} + \dot{R}^2 = \frac{16\pi G}{3} \rho_\Lambda R^2 - kc^2. \]

(4)

We notice that the l.h.s of Eq.(1) can be written as

\[ R\ddot{R} + \dot{R}^2 = \frac{1}{2} \frac{d^2}{dt^2} R^2. \]

(5)

Introducing the new variable

\[ X = R^2 - 2kc^2 t_\Lambda^2, \]

(6)
where
\[ t_\Lambda = \sqrt{\frac{3}{32\pi G\rho_\Lambda}}. \]  
(7)

We see that Eq. (4) can be rewritten in the very simple form
\[ \ddot{X} = t_\Lambda^{-2}X. \]  
(8)

Thus the solution for \( R \) is given by
\[ R = \left( c_1 e^{-\frac{t}{t_\Lambda}} + \frac{1}{2} c_2 e^{\frac{t}{t_\Lambda}} + 2k c^2 t_\Lambda^2 \right)^{1/2}. \]  
(9)

The arbitrary constants \( c_1 \) and \( c_2 \) are, as usual, fixed by the initial conditions on \( R \) and on \( \dot{R} \) by using Friedmann equation. Requiring that \( R(t = t_*) = R_* \), where the asterisk signals some initial condition, gives
\[ c_1 = R_*^2 e^{\frac{t_*}{t_\Lambda}} - \frac{1}{2} c_2 e^{\frac{2t_*}{t_\Lambda}} - 2t_*^2 k c^2 e^{\frac{t_*}{t_\Lambda}}. \]  
(10)

From the Friedmann equation Eq. (1) evaluated at \( t = t_* \) we get the second constant
\[ c_2 = R_*^2 e^{-\frac{t_*}{t_\Lambda}} - 2t_*^2 k c^2 e^{-\frac{t_*}{t_\Lambda}} + \left( \frac{\rho_* \gamma}{\rho_\Lambda} - \frac{4t_*^2 k c^2}{R_*^2} + 1 \right)^{1/2} R_*^2 e^{-\frac{t_*}{t_\Lambda}}. \]  
(11)

Thus the solution is given by
\[ R = R_* \left( \cosh \tau + \frac{2t_*^2 k c^2}{R_*^2} (1 - \cosh \tau) + \left( \frac{\rho_* \gamma}{\rho_\Lambda} - \frac{4t_*^2 k c^2}{R_*^2} + 1 \right)^{1/2} R_*^2 e^{-\frac{t_*}{t_\Lambda}} \right) \left( \sinh \tau \right)^{1/2}, \]  
(12)

where we have introduced the new variable \( \tau \) defined in terms of \( t \) by
\[ \tau = \frac{t - t_*}{t_\Lambda}. \]  
(13)

We can simplify Eq. (12) by introducing the (constant) density parameters
\[ \Omega_{si} \equiv \frac{\rho_{si}}{\rho_{sc}} \equiv \frac{\rho_i(t = t_*)}{\rho_c(t = t_*)}, \]  
(14)

where \( \rho_{si} \) is the critical density at \( t = t_* \) and \( \rho_{si} \) the energy density of the substance \( i \) at \( t = t_* \). We also use the Friedmann equation written in terms of density parameters evaluated at \( t = t_* \)
\[ \Omega_{s\gamma} + \Omega_{s\Lambda} - \Omega_{sk} = 1. \]  
(15)

We then have the final expression for our solution which is valid for any \( k \) and \( \Lambda \)
\[ R = R_* \left( \frac{1}{2} \Omega_{s\Lambda} + \left( 1 - \frac{1}{2} \Omega_{s\Lambda} \right) \cosh \tau + \frac{1}{\sqrt{\Omega_{s\Lambda}}} \sinh \tau \right)^{1/2}. \]  
(16)
Closely related solutions were originally obtained by Harrison case by case. Here we concentrate in the cosmological consequences of such a solution. The Hubble parameter $H \equiv \frac{\dot{R}}{R}$ is

$$H = \frac{1}{2t_\Lambda} \left( \frac{R_*}{R} \right)^2 \left[ \left( 1 - \frac{1}{2} \frac{\Omega_{*k}}{\Omega_{*\Lambda}} \right) \sinh \tau + \frac{1}{\sqrt{\Omega_{*\Lambda}}} \cosh \tau \right].$$  \hspace{1cm} (17)

The velocity $\dot{R}$ is given by

$$\dot{R} = H R,$$

and the acceleration $\ddot{R}$ is then

$$\ddot{R} = \frac{1}{2t_\Lambda} \left( 1 - \frac{1}{2} \frac{\Omega_{*k}}{\Omega_{*\Lambda}} \left( \frac{R_*}{R} \right)^2 \right) - \frac{H}{2t_\Lambda} \left( \frac{R_*}{R} \right)^2.$$

The case when $\rho_\Lambda = 0$ follows from an expansion of Eq.(16) on $\tau$ and throwing away terms proportional to $\rho_\Lambda$ (contained in $t_\Lambda$, see Eq.(7)). We can also follow the previous procedure. In any case the solution is

$$R = R_* \left( 1 + 2H_* (t - t_*) - \Omega_{*k} H_*^2 (t - t*)^2 \right)^{1/2},$$ \hspace{1cm} (20)

where

$$H_*^2 = \frac{8\pi G}{3} (\rho_{*\gamma} - \rho_{*k}).$$ \hspace{1cm} (21)

Finally the solution for a negative cosmological constant also follows from Eq.(16)

$$R = R_* \left(-\frac{1}{2} \frac{\Omega_{*k}}{\Omega_{*\Lambda}} + \left( 1 + \frac{1}{2} \frac{\Omega_{*k}}{\Omega_{*\Lambda}} \right) \cos \tau + \frac{1}{\sqrt{\Omega_{*\Lambda}}} \sin \tau \right)^{1/2}.$$ \hspace{1cm} (22)

From now on we particularize to the case where the initial values correspond to the Planck era

$$(t_*, R_*, \rho_{*\gamma}) = (t_{pl}, l_{pl}, \rho_{pl}),$$ \hspace{1cm} (23)

where $t_{pl}, l_{pl}$ and $\rho_{pl}$ are Planck time, Planck length and Planck density, respectively. We further work in units where $\hbar = c = m_{pl} = 1$ so that

$$(t_*, R_*, \rho_{*\gamma}, G) = (t_{pl}, l_{pl}, \rho_{pl}, m_{pl}^{-2}) = (1, 1, 1, 1).$$ \hspace{1cm} (24)

Thus the densities are given by

$$(\rho_{*\gamma}, \rho_{*\Lambda}, \rho_{*k}, \rho_{*c}) = \left( 1, \varepsilon^2, \frac{3k}{8\pi}, 1 + \varepsilon^2 - \frac{3k}{8\pi} \right).$$ \hspace{1cm} (25)

We introduce the quantity $b(t)$ which by analogy with the usual $a(t)$ is here defined by

$$b(t) \equiv \frac{R}{R_*},$$ \hspace{1cm} (26)

where $b_* = b(t = t_*) = b_{pl} = 1$. The notation reminds us that $b(t)$ refers to an epoch "before" inflation while the usual $a(t)$ refers to the evolution of the scale factor "after" inflation.
Figure 1: The solution Eq.\((16)\) for a universe containing radiation, curvature and a positive cosmological term is here shown as a function of time. All quantities are in Planck units. We see that the positive curvature case leads to a Big Crunch shortly after the birth of the universe not allowing the possibility of inflation. Inflation can occur in this case only if the energy density of the cosmological term is bigger than a critical value, see Eq.\((30)\).

3 Friedmann models

A typical value for the inflationary scale is \(\mathcal{O}(10^{15}\text{GeV})\) thus, to illustrate the behaviour of \(b(t)\) and all the other quantities of interest we take \(\varepsilon = 10^{-8}\).

- **Positive cosmological term.** In Fig. 1 we show the behaviour of \(b(t)\) as given by Eq.\((16)\) for the three possible values of the curvature. The case \(k = 1\) (closed universe) presents a Big Crunch (BC) at a time very close to the Planck time not allowing the possibility of an inflationary epoch. For the open and flat universes \(k = -1, k = 0\), respectively inflation is unavoidable due to the dominance of the \(\rho_{\Lambda}\) term at large \(b(t)\). The case \(k = 1\) is particularly interesting. We can calculate, as a function of \(\Omega_{\ast\Lambda}\), the time at which the BC occurs. This is given by

\[
\tau = \cosh^{-1} \left( 1 + \frac{8\Omega_{\ast\Lambda}}{(\Omega_{\ast k} - 2\Omega_{\ast\Lambda})^2 - 4\Omega_{\ast\Lambda}} \right). \tag{27}
\]

From Eq.\((27)\) we see that the BC disappears when

\[
(\Omega_{\ast k} - 2\Omega_{\ast\Lambda})^2 - 4\Omega_{\ast\Lambda} = 0, \tag{28}
\]

or

\[
\Omega_{\ast\Lambda c} = \frac{1}{2} \left( 1 + \Omega_{\ast k} - \sqrt{1 + 2\Omega_{\ast k}} \right), \tag{29}
\]

where \(k = +1\) and the subindex \(c\) refers to a critical value. We can also solve in terms of \(\varepsilon\) with the result

\[
\rho_{\Lambda c} = \frac{1}{4} \frac{\rho_{\ast k=1}^2}{\rho_{\ast\gamma}} = \left( \frac{3}{16\pi} \right)^2 \varepsilon_c^2. \tag{30}
\]
Figure 2: At the critical density Eq.(30) the solution for a positive cosmological term Eq.(16) reduces to Eq.(31). We plot this solution as a function of time $t$. The value it approaches $b_E = 1/\sqrt{\varepsilon_c}$ corresponds to a static Einstein solution for a pre-inflationary universe. This mini universe is of the order of the Planck length.

Figure 3: Solutions where the energy density is close to the critical density Eq.(30) are here shown as functions of time. For an energy $\varepsilon < \varepsilon_c$ the solution collapses into a Big Crunch while the opposite case gives rise to an inflationary solution with the scale factor growing at an ever increasing velocity. By tuning the energy close to the critical value we can have loitering solutions where the scale factor remains almost constant during an arbitrary interval of time.

With this value of $\rho_{\Lambda c} = \varepsilon_c^2$ we find that the solution Eq.(16) reduces to

$$b_c(t) = \frac{1}{\sqrt{\varepsilon_c}} \left(1 + (\varepsilon_c - 1)e^{-\tau} \right)^{1/2}.$$  \hspace{1cm} (31)

This solution is illustrated in Fig.2, where we see that it approaches asymptotically the value $b_E = 1/\sqrt{\varepsilon_c}$. We can recognize $b_E$ as an Einstein solution for a static universe in the pre-inflationary era. The size of this mini universe is of the order of the Planck length. As discussed long ago for a post-inflationary universe, a slightly bigger value for the energy density than the critical one $\varepsilon_c$ gives rise to a universe which, although closed, expands with an ever increasing velocity due to the dominance of the cosmological...
term. On the other hand an energy scale less than $\varepsilon_c$ leads irremediably to a BC. This behaviour is illustrated in Fig.3. We see that we can keep the solution waiting close to $b_E$ as long as we wish (loitering solution) by tuning the energy density to values close to the critical value. Loitering solutions in standard cosmology for a closed FRW model with matter and a cosmological constant were originally studied by Lemaître [3]. Some other more general studies can be found in [4]. The behaviour of the Hubble parameter is peculiar in loitering solutions. This has been noted by Sahni in the context of loitering braneworld models [5]. There, he notices that loitering solutions are characterized by the fact that the Hubble parameter dips in value during loitering. We illustrate this behaviour in Fig.4 (long dashed curve). The acceleration in a loitering solution always becomes positive for large $b(t)$ when the cosmological constant term becomes dominating. The transition from a decelerating to an accelerating universe is signaled by the point at which the acceleration vanishes. For our solution Eq.(16) (with $k = 1$) this is given by

$$\tau_0 = \text{arc cosp} \left[ \frac{9 - 16\pi\varepsilon \left( 3 + \varepsilon \left( 3 - 16\pi\varepsilon + 2\sqrt{\frac{(3-16\pi\varepsilon)(3-8\pi(1+\epsilon^2))}{\varepsilon}} \right) \right)}{9 - 256\pi^2\varepsilon} \right]. \quad (32)$$

The behaviour of $\tau_0$ as a function of the energy $\varepsilon$ is shown in Fig.5.

We have also investigated the behaviour of the energy density parameters for the loitering solution. This is illustrated in Fig.6 where we see that it is the curvature term the one which dominates the energy of the universe during loitering, while the cosmological constant and radiation terms remain almost indistinguishable one from the other. This peculiar behaviour during loitering is also consistently shown in Fig.7.
Figure 5: The value of $\tau$ for which the acceleration of the scale factor vanishes is here shown as a function of the energy $\varepsilon$ where $\varepsilon_c < \varepsilon < 1$. For $\varepsilon = \varepsilon_c$ the value of $\tau_0$ grows without limit corresponding to the asymptotic solution Eq. (31) illustrated in Fig. 2. For $\varepsilon = 1$ (in Planck units), the scale factor accelerates right from the beginning ($\tau_0 = 0$). The closer to $\varepsilon_c$ we tune the energy the longer the loitering epoch. Thus the age of the universe can increase dramatically.

Figure 6: The energy density parameters for the loitering solution with $\varepsilon = \varepsilon_c + 10^{-6}$ and $k = 1$ are shown as functions of time. We can clearly see that during loitering (compare with Fig. 4) the curvature term dominates over radiation and the cosmological constant which are almost indistinguishable. For larger values of $t$ (not shown) the cosmological term dominates, going asymptotically to one.

where we plot the effective equation of state parameter $\omega_{\text{ef}}$ which takes a value around $-1/3$ during loitering. The $\omega_{\text{ef}}$ is given by

$$
\omega_{\text{ef}} = \frac{\frac{1}{3} \rho_\gamma - \rho_\Lambda - \frac{1}{3} \rho_{k=1}}{\rho_\gamma + \rho_\Lambda + \rho_{k=1}}
$$

(33)

- **Vanishing cosmological term.** In this case the solution is given by Eq. (20) for the three values of $k$. This solution is illustrated in Fig. 8. Note that this figure is almost identical to Fig. 1. The reason is that for small $t$ the cosmological term present in Eq. (16) is irrelevant. Thus solutions Eq. (16) and Eq. (20) become different for larger
Figure 7: The effective equation of state parameter $\omega_{ef}$ as given by Eq.(33) is shown as a function of time for the loitering solution. We see that during loitering $\omega_{ef} \simeq -1/3$ which, consistently with Fig.6, corresponds to curvature domination. For larger values $\omega_{ef}$ goes to $-1$ which is the value corresponding to a cosmological constant term as it should be.

Figure 8: The solution Eq.(20) for a universe containing radiation, curvature and a vanishing cosmological term is here shown as a function of time. The figure is almost identical to Fig.1 because the cosmological term is only relevant at large $b(t)$, see Fig.9.

values of $t$. This is shown in Fig.9 for the cases $k = -1$ and $k = 0$ only since the $k = +1$ case is essentially the same in both solutions whenever $\varepsilon < \varepsilon_c$.

• **Negative cosmological term.** In this case the solution is given by Eq.(22) for the three values of $k$. This solution is illustrated in Fig.10.

## 4 Conclusions

We have studied a model of the very early universe containing radiation, curvature and a cosmological constant. We find that in the cases $k = -1$, $k = 0$ and $\Lambda > 0$ inflation occurs unavoidably while in the case $k = +1$ there is no inflation unless the energy density is bigger than a certain critical value. This critical energy signals
Figure 9: Here we show the solution Eq. (20) for the vanishing cosmological case when $k = -1$ and $k = 0$. We compare with Fig. 1 for large $b(t)$ where the cosmological term begins to dominate the expansion of the universe. The case $k = +1$ for $\varepsilon < \varepsilon_c$ is almost identical in both cases.

Figure 10: We show the solution Eq. (22) for a universe containing radiation, curvature and a negative cosmological term as a function of time. In all cases the universe collapses into a Big Crunch oscillating in different ways for the negative and vanishing curvature cases while collapsing once for the positive curvature.
the limit between contracting solutions which end in a Big Crunch from those which expand forever. At the critical value, the solution approaches asymptotically an Einstein solution for a static universe in the pre-inflationary regime. The mini universe defined by this solution is of the order of the Planck length. Particular emphasis is placed in the closed universe case because there loitering solutions can occur. These solutions have somewhat peculiar properties: the Hubble parameter dips in value during loitering and the energy density parameter associated with curvature dominates over radiation and the cosmological constant. We also solve and briefly discuss the cases with a vanishing and negative cosmological constant thus providing a complete study of the Friedmann models. All of these results have also been obtained in the well studied problem of a universe of matter, curvature and a cosmological constant. They are due to the competition between curvature and the matter terms. Thus, technically, replacing matter by radiation which decays even faster should not change any of these features. However it is most gratifying that in this case one can find closed form analytical solutions to the problem.

5 Acknowledgements

This work was supported by the project PAPIIT IN114903-3 and CONACYT: 42096, 45718. G.G. is grateful to Prof. G.G. Ross and Prof. S. Sarkar for the hospitality extended to him at the RPCTP during a sabbatical leave from Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México. Support from DGAPA, UNAM and the RPCTP, Oxford is gratefully acknowledged. This work is part of the Instituto Avanzado de Cosmología (IAC) collaboration.

References

[1] E.R. Harrison, Mon. Not. R. Astr. Soc. (1967)137, 69-79. See also Stephani H., Kramer D., Maccallum M., Hoenselaers C., Herlt E., *Exact Solutions of Einstein Field Equations*, Chapter 14, Cambridge U. Press, 2nd Ed. 2003.

[2] Some general references are:
  S. Weinberg, *Cosmology*, Oxford University Press, 2008.
  S. Dodelson, *Modern Cosmology*, Academic Press, 2003.
  J.A. Peacock, *Cosmological Physics*, Cambridge U. Press, 1999.
  P.J.E. Peebles, *Principles of Physical Cosmology*, Princeton U. Press, 1993.
  S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, 1972.

[3] A.G. Lemaître, Mon. Not. R. Astr. Soc. (1937)91, 438

[4] V. Sahni and Yu. Shtanov, [astro-ph/0410221](https://arxiv.org/abs/astro-ph/0410221).
  V. Sahni, H. Feldman and A. Stebbins, Astrophys. J. 385(1992)1.
  S. Alenxander, R. Brandenberger, D. Easson, Phys. Rev. D62, 103509(2000).
  R. Branderberger, D. Easson, D. Kimberley, Nucl. Phys. B623(2002)421.
[5] V. Sahni, Proceedings of the 14th Workshop on General Relativity and Gravitation (JGRG14), Kyoto University, Japan. W. Hikida, M. Sasaki, T. Tanaka and T. Nakamura Eds., p 95-115. astro-ph/0502032