\textbf{sl}_q(2) \text{ Realizations for Kepler and Oscillator Potentials and q-Canonical Transformations}

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\textbf{Abstract}

The realizations of the Lie algebra corresponding to the dynamical symmetry group \(SO(2,1)\) of the Kepler and oscillator potentials are q-deformed. The q-canonical transformation connecting two realizations is given and a general definition for q-canonical transformation is deduced. q-Schrödinger equation for a Kepler like potential is obtained from the q-oscillator Schrödinger equation. Energy spectrum and the ground state wave function are calculated.

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1 Introduction

There are mathematical and physical aspects of q-deformations\[1\]. From the mathematical point of view, one usually demands that the q-deformed algebra to be a Hopf algebra. The physical point of view is somehow, less restrictive: obtaining the underlying undeformed picture in $q \to 1$ limit is the basic condition. Hence, q-deformation of a physical system is not unique. For example, the harmonic oscillator which is the most extensively studied system, has several q-deformed descriptions\[2\]. q-deformation of physical systems other than the oscillator are not well studied. On the other hand, most of the concepts of classical and quantum mechanics become obscure after q-deformations. For example, q-deformed change of phase space variables leaving basic q-commutation relations invariant is presented\[3\], and a canonical transformation connecting q-oscillators is studied\[4\]; but, q-canonical transformations establishing relationships between different potentials are not known.

The purpose of this work is to present a q-canonical transformation and to define a q-deformed Kepler like potential in a consistent manner with the q-oscillator. The possession of the same dynamical symmetry group $SO(2,1)$ by the harmonic oscillator and the Kepler problems will guide us.

In general, the phase space realizations of the Lie algebras corresponding to a dynamical symmetry group which are relevant to different physical systems are connected by canonical transformations. We generalize this connection to define q-canonical transformations. We hope that the procedure may also help to define new q-deformed potentials from the known ones.

In section 2 we review the known relation between the undeformed Kepler and oscillator problems.

In section 3 we present q-deformations of two realizations of $sl(2)$ (which is the Lie algebra of $SO(2,1)$) relevant to the Kepler and oscillator potentials. We define the q-canonical transformation connecting two realizations. We then give a general definition of q-canonical transformation.

In section 4 we define a q-Schrödinger equation for a Kepler like potential from the q-Schrödinger equation of the q-oscillator by a coordinate change.

Finally, we fix the energy spectrum of the q-oscillator, and find the ground state wave function; then, obtain energy spectrum and ground state wave function of the q-deformed Kepler problem.
2 Review of the Relations between Kepler and Oscillator Potentials

It is well known that $SO(2, 1)$ is the dynamical symmetry group of the radial parts of the Schrödinger equations of the Kepler and the harmonic oscillator potentials.

In one (space) dimension the phase space realizations of the corresponding Lie algebra $sl(2)$ relevant to the Kepler and the harmonic oscillator problems are given by

$$H = 2px,$$
$$X_+ = -\sqrt{2}x,$$
$$X_- = -\frac{1}{\sqrt{2}}p^2x,$$

and

$$L_0 = up_u + \frac{i}{2},$$
$$L_+ = -\sqrt{2}u^2,$$
$$L_- = -\frac{1}{4\sqrt{2}}p_u^2,$$

with

$$px - xp = i, \quad pu_u - up_u = i.$$

The above generators satisfy the usual commutation relations

$$[H, X_\pm] = \pm 2iX_\pm, \quad [X_+, X_-] = -iH,$$  \hspace{1cm} (3)

$$[L_0, L_\pm] = \pm 2iL_\pm, \quad [L_+, L_-] = -iL_0.$$  \hspace{1cm} (4)

The eigenvalue equation for the Kepler Hamiltonian

$$H_K\Psi \equiv \left(\frac{p^2}{2\mu} + \frac{\beta^2}{x}\right)\Psi = E\Psi,$$  \hspace{1cm} (5)

$^3$See for example [1].
which is equivalent to
\[
\left( \frac{p^2}{2\mu} - E \right) x \Phi = \beta^2 \Phi,
\]
is solved by diagonalizing the operator
\[
\frac{1}{\sqrt{2}} \left( \frac{1}{\mu} X_- + EX_+ \right).
\]

On the other hand solution of the oscillator problem is simply obtained by diagonalizing the operator
\[
-\frac{1}{\sqrt{2}} \left( \frac{1}{\mu} L_- + \frac{1}{2}\mu \omega^2 L_+ \right).
\]

Classically (i.e. before the $\hbar$-deformation) the Kepler and the oscillator phase space variables are connected by the canonical transformation
\[
x = u^2, \quad p = \frac{p_u}{2u}.
\]

The canonical transformations of the above type are also employed for solving the H-atom path integral. In fact, since the path integrations make use of the classical dynamical variables, the canonical point transformations are routinely used to transform the path integral of a given potential into a solvable form.

Relation between the Schrödinger equations corresponding to the one-dimensional oscillator and Kepler type potentials is the following.

Schrödinger equation of the one-dimensional oscillator
\[
\left( -\frac{1}{2\mu} \frac{d^2}{du^2} + \frac{1}{2}\mu \omega^2 u^2 \right) \psi = E\psi,
\]
is transformed by the coordinate change suggested by (5)
\[
u = \sqrt{x},
\]
into
\[
\left( -\frac{1}{2\mu} \frac{d^2}{dx^2} + \frac{E}{8x} - \frac{3/32\mu}{x^2} \right) \phi = -\frac{\mu \omega^2}{8} \phi,
\]
with
\[ \psi = \frac{1}{\sqrt{x}} \phi. \]  

(9)

The energy \( E \) and the frequency \( \omega^2 \) of the oscillator play the role of the coupling constant \( \beta^2 \) and the energy \( E_K \) of the Kepler problem:

\[ \frac{E}{8} = \frac{\omega}{4}(2n + 1) = -\beta^2, \]  
\[ E_K = -\mu \omega^2/8. \]  

(10)  
\[ (11) \]

\( (8) \) is equivalent to the one-dimensional Kepler problem with an extra potential barrier \(-(3/32\mu)/x^2\) or to the two-dimensional Kepler problem with “angular momentum” \( p^2_{\theta} = -3/16 \).

If we solve \( \omega \) from (10) as

\[ \omega(\beta) = -\frac{4\beta^2}{2n + 1}, \]  

(12)

and insert into (11), we obtain the Kepler energy

\[ E_K = -\frac{2\mu \beta^4}{(2n + 1)^2}. \]  

(13)

3  q-canonical Transformation between the Kepler and Oscillator Realizations of \( sl_q(2) \)

3.1  q-deformation of the Kepler Realization

To q-deform the algebra of the generators (1), we prefer to q-deform the commutation relation between \( p \) and \( x \) (we use the same notation for the q-deformed and undeformed objects),

\[ xp - qp = -i\sqrt{q}, \]  

(14)

but keep the functional forms of \( H, X_{\pm} \) same as the ones given in (1). The q-deformed commutation relations are then given by [7]
\[ HX_- - qX_- H = -2i\sqrt{q}X_- \]
\[ HX_+ - \frac{1}{q}X_+ H = 2i\frac{1}{\sqrt{q}}X_+ \]  \tag{15}
\[ X_+X_- - q^2X_-X_+ = \frac{-i}{2}\sqrt{q}(1+q)H. \]

Note that after rescaling the above generators
\[ X_\pm \to \frac{i\sqrt{q}(1+q)}{\sqrt{q}}X_\pm, \quad H \to 2iH, \]
and setting \( q = r^2 \), one arrives at
\[ HX_- - r^2X_- H = -rX_- \]
\[ r^2HX_+ - X_+ H = rX_+ \]
\[ X_+X_- - r^4X_-X_+ = r^2H, \]  \tag{16}
which is the Witten’s second deformation of \( sl(2) \).

### 3.2 \( q \)-deformation of the Oscillator Realization

We like to \( q \)-deform the generators given in (2), which are relevant to the oscillator problem, in a consistent manner with the deformation of the Kepler realization (1).

We define the \( q \)-deformed commutation relation of \( p_u \) and \( u \), as
\[ up_u - \sqrt{q}puu = ib(q), \]  \tag{17}
and fix \( b(q) \) by requiring the commutation relations of the \( q \)-deformed algebra of the generators (2) to be the same as (15). We rescale \( L_0, L_\pm \),
\[ L_0 \to \frac{a}{2\sqrt{q}}L_0, \quad L_\pm \to \left[ a(1+\sqrt{q})^3 \right]^{1/2} L_\pm, \]
with
\[ a = \frac{(1+\sqrt{q})(1+q)}{2\sqrt{q}}, \]
and fix \( b(q) \) as
\[
b = -\frac{1}{2}(1 + \frac{1}{\sqrt{q}}).
\]
The q-deformed algebra, then becomes
\[
\begin{align*}
L_0L_+ - qL_-L_0 &= -2i\sqrt{q}L_- \\
L_0L_+ - \frac{1}{q}L_+L_0 &= 2i\frac{1}{\sqrt{q}}L_+ \\
L_+L_- - q^2L_-L_+ &= -\frac{i}{2}\sqrt{q}(1 + q)L_0,
\end{align*}
\]
which is the same as the \( sl_q(2) \) algebra of the Kepler problem (13).

Note that before q-deformation, \( sl(2) \) algebra (4) admits three different choices for \( L_0 \)
\[
L_0 = up_u + \frac{i}{2}, \quad L_0 = p_uu - \frac{i}{2}, \quad L_0 = \frac{1}{2}(up_u + p_uu).
\]
In the q-deformed case however, if we like to have the generators to be independent of \( q \) (except an overall factor), the ordering degeneracy in (19) is removed, that is \( L_0 \) can only take the form given in (2).

### 3.3 q-canonical Transformation

Let us introduce a transformation similar to (5):
\[
x = \left(\frac{u}{b}\right)^2 = \left(\frac{2\sqrt{q}}{1 + \sqrt{q}}\right)^2 u^2, \quad p = \frac{1}{2}u^{-1}p_u.
\]
Then, the q-commutation relation (14) yields
\[
\frac{q}{1 + \sqrt{q}}(up_u - qu^{-1}p_uu^2) = -i\sqrt{q},
\]
which is consistent with (17). Indeed, by the virtue of (17), the above commutation relation becomes
\[
\frac{q}{1 + \sqrt{q}}(up_u - \sqrt{q}p_uu + ib\sqrt{q}) = -i\sqrt{q}.
\]
which is again equal to (17).

Now, we are ready to define q-canonical transformation.

**Definition.** We like to keep the phase space realizations of the q-deformed generators to be formally the same as the undeformed generators of the dynamical symmetry group. We then define the transformation \( x,p \rightarrow u,p_u \) to be the q-deformed canonical transformation if

i) algebras generated by the realizations \( X_i(x,p) \) and \( L_i(u,p_u) \) are the same and,

ii) the q-commutation relations between \( p \) and \( x \), and \( p_u \) and \( u \) are preserved.

In accordance with the above definition, we conclude that (20) is a q-canonical transformation.

By rescaling the q-canonical variables \( p, x \) and \( p_u, u \) as

\[
(x, p) \rightarrow q^{-1/4}(x, p), \quad (u, p_u) \rightarrow \sqrt{|b|/\sqrt{q}} (u, p_u),
\]

and setting

\[ q \rightarrow q^{-2}, \]

the q-commutators (14) and (21) become

\[
px - q^2 xp = i, \quad (22)
\]
\[
p_u u - qu p_u = i. \quad (23)
\]

In the rest of the paper, these q-commutation relations will be used.

There is another definition of q-deformed canonical transformation[4]: phase space coordinates are transformed under the condition that the q-commutators remain invariant. On the other hand, our condition in the definition of q-canonical transformation is to obtain in a suitable limit, the undeformed mappings connecting different potentials which possess the same dynamical symmetry. Thus our definition of q-canonical transformation is dynamics dependent, i.e. the basic q-commutators are potential dependent (22-23).
4 q-Canonical Transformation from q-Oscillator Schrödinger Equation to q-Kepler Problem

Introduce the q-deformed derivative $D_q(u)$

$$D_q(u)f(u) \equiv \frac{f(u) - f(qu)}{(1-q)u}. \quad (24)$$

In terms of this definition one can show that

$$D_q(u)\{f(u)g(u)\} = D_q(u)f(u)g(u) + f(qu)D_q(u)g(u). \quad (25)$$

Since the q-deformed derivative $D_q(u)$ satisfies

$$D_q(u)u - quD_q(u) = 1,$$

we can set

$$p_u = iD_q(u),$$

which is consistent with (23).

In terms of this q-differential realization one can obtain the q-deformed Schrödinger equation for the q-oscillator

$$(-\frac{1}{2\mu}D^2_q(u) + \frac{\mu}{2}\omega^2_q c^2_q(u) - \frac{1}{2}E_q)\psi_q(u) = 0, \quad (26)$$

where

$$c_q(u) = \sqrt{qu}, \quad \omega_q = [\omega]_q \equiv \frac{1-q\omega}{1-q}. \quad (27)$$

Obviously, the choice (27) is not unique. The conditions to be satisfied are

$$\lim_{q \to 1} c_q(u) = u, \lim_{q \to 1} \omega_q = \omega.$$

We adopt the change of variable suggested by (20)

$$u = \sqrt{x}. \quad (28)$$

The q-derivative $D_q(u)$ transforms as

$$D_q(u) = (1+q)\sqrt{x}D_{q^2}(x). \quad (29)$$

\(^4\)For example see [10] and the references given therein.
\( D_{q^2}(x) \) satisfies
\[
D_{q^2}(x) - q^2 x D_{q^2}(x) = 1,
\] (30)
hence in accordance with (22) it can be identified with \(-ip\). Therefore, (28) and (29) are equivalent to the q-canonical transformation (20).

The q-Schrödinger equation (26) then becomes
\[
\left[ -\frac{1}{2\mu} (1+q)^2 x D_{q^2}^2(x) - \frac{1}{2\mu} (1+q) D_{q^2}^2(x) + \frac{1}{2\mu} [\omega_q]^2 q x - \frac{1}{2} E_q \right] \phi_q(x) = 0,
\] (31)
with
\[
\phi_q(x) = \psi_q(\sqrt{x}).
\] (32)
To get rid of the term linear in \( D_q(x) \) in (31), we introduce the ansatz
\[
\phi_q(x) = x^\alpha \varphi_q(x).
\] (33)
Choosing
\[
\alpha = \frac{\ln((3-q)/2)}{2\ln q},
\]
and by multiplying (31) from left by \(1/(1+q)^2 x\), we obtain
\[
\left[ -\frac{1}{2\mu} D_{q^2}^2(x) - \frac{2q^2 - 2q - 3}{8\mu q^2 (1+q)^2} \frac{E_q}{2(1+q)^2} \right] \varphi_q(x)
\]
\[
= \frac{\mu [\omega]^2 q}{2(1+q)^2} \varphi_q(x),
\] (34)
which is the q-deformed Schrödinger equation of the Kepler potential with an extra potential barrier.

The q-oscillator energy \( E_q \) is dependent on \( \omega \) and \( q \). From the identification of the coupling constant
\[
-\beta^2 = E_q,
\] (35)
we can solve \( \omega \) in terms of \( \beta \) and \( q \), as \( \omega(\beta, q) \). Hence in terms of the solutions of the q-Schrödinger equation for q-oscillator (26) we can obtain the solutions

\[\text{[For a study of q-deformed H-atom in an unrelated manner to the q-oscillator see [1].]}\]
of
\[
\left[ \frac{-1}{2\mu} D_q^2(x) + \frac{\beta^2/2(1+q)^2}{x} + \frac{(2q^2 - 2q - 3)/8\mu q^2(1+q)^2}{x^2} \right] \varphi_q(x) = E_K \varphi_q(x), \tag{36}
\]
where
\[
E_K = \frac{q\mu[\omega(\beta,q)]^2}{2(1+q)^2}, \tag{37}
\]
is the q-deformed analog of the energy spectrum of the Kepler problem.

5 Energy Spectrum and Ground State Wave Functions

General solution of the q-Schrödinger equation of the q-deformed oscillator \cite{26} is not known. We fix the energy spectrum to be of the conventional form\cite{2}
\[
E_{qn} = [\omega(2n+1)]_q = \frac{1 - q^{\omega(2n+1)}}{1 - q}. \tag{38}
\]
Inserting the above energy spectrum into (35) we obtain
\[
\omega_{qn}(\beta,q) = \frac{1 - [(1 - q)(1 + \beta^2)]^{\text{ln}q/2(2n+1)}}{1 - q}. \tag{39}
\]
The energy spectrum (37) of the q-Kepler problem then becomes
\[
E_{Kn} = \frac{q\mu}{2(1 - q^2)^2} \{1 - [(1 - q)(1 + \beta^2)]^{\text{ln}q/2(2n+1)}\}^2. \tag{40}
\]
We now like to build the ground state wave function of the q-oscillator. For this aim, we introduce
\[
e_q(z^2) = 1 + \sum_{n=1}^\infty \left( \prod_{k=1}^n \frac{2(1-q)}{1-q^{2k}} \right) z^{2n}, \tag{41}
\]
which is defined to satisfy
\[
D_q(z)e_q(z^2) = 2ze_q(z^2).
\]
Hence, the equation for the ground state of the $q$-oscillator

$$
\left(-\frac{1}{2\mu}D_q^2(u) + \frac{q}{2}\mu[\omega_q^2 u^2]\right)\psi_0^q(u) = \frac{1}{2}[\omega_q]q\psi_0^q(u),
$$

possesses the solution

$$
\psi_0^q(u) = e_q \left(-\frac{\mu}{2}\frac{1 - q^\omega}{1 - q} u^2 \right).
$$

By introducing the above definition into (32) and using (33) we obtain the ground state wave function of the $q$-Kepler problem

$$
\varphi_0^q(x) = x^{-\alpha}\psi_0^q(\sqrt{x}) = x^{-\alpha}e_q \left(-\frac{\mu}{2}\frac{1 - q^\omega}{1 - q} x \right),
$$

which corresponds to the energy $E_{K_0}$. 

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