NON-TRIVIAL 3-WISE INTERSECTING UNIFORM FAMILIES

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ABSTRACT. A family of $k$-element subsets of an $n$-element set is called 3-wise intersecting if any three members in the family have non-empty intersection. We determine the maximum size of such families exactly or asymptotically. One of our results shows that for every $\epsilon > 0$ there exists $n_0$ such that if $n > n_0$ and $\frac{k}{n} + \epsilon < \frac{k}{n} < \frac{1}{2} - \epsilon$ then the maximum size is $4\left(\binom{n-4}{k-4}\right) + \left(\binom{n-4}{k-4}\right)$.

1. INTRODUCTION

In this paper we study the maximum size of non-trivial 3-wise intersecting $k$-uniform families on $n$ vertices. We start with some definitions. Let $n, k, r$ and $t$ be positive integers with $n > k > t$ and $r \geq 2$. Let $[n] = \{1, 2, \ldots, n\}$, and let $2^n$ and $\binom{n}{k}$ denote the power set of $[n]$ and the set of all $k$-element subsets of $[n]$, respectively. A family of subset $G \subset 2^n$ is called $r$-wise $t$-intersecting if $|G_1 \cap G_2 \cap \cdots \cap G_r| \geq t$ for all $G_1, G_2, \ldots, G_r \in G$. This family is called non-trivial $r$-wise $t$-intersecting, if $G$ moreover satisfies $|\cap_{G \in G} G| < t$. For simplicity we often omit $t$ if $t = 1$, e.g., an $r$-wise intersecting family means an $r$-wise 1-intersecting family. Let $M_r^t(n, k)$ denote the maximum size of a family $F \subset \binom{[n]}{k}$ which is non-trivial $r$-wise $t$-intersecting. Again for simplicity we write $M_r^t(n, k)$ for $M_r^t(n, k)$.

Hilton and Milner [7] determined $M_2(n, k)$, and Ahlswede and Khachatrian [1] determined $M_3^2(n, k)$ for all $t \geq 1$. Recently O’Neill and Verstr"{a}ete [11], and Balogh and Linz [2] studied $M_3(n, k)$ mainly for the cases $n \gg k$. Some other related results can be found in [5, 8, 9, 10, 15]. Our goal is to determine $M_3(n, k)$ exactly or asymptotically for most of $n, k$ with $0 < \frac{k}{n} < 1$.

This problem is closely related to determine the maximum $p$-measure of non-trivial $r$-wise $t$-intersecting families, see [12, 13]. For a real number $p$ with $0 < p < 1$ and a family $G \subset \binom{n}{k}$ we define the $p$-measure $\mu_p(G)$ of $G$ by

$$\mu_p(G) = \sum_{G \in G} p^{|G|} (1 - p)^{n - |G|}.$$ 

Let $W_r^t(n, p)$ (and $W_r(n, p)$ for $t = 1$) denote the maximum $p$-measure $\mu_p(G)$ of a family $G \subset \binom{n}{k}$ which is non-trivial $r$-wise $t$-intersecting. Brace and Daykin [3] determined $W_r^t(n, \frac{1}{2})$.

The well-known Erdős–Ko–Rado theorem [4] and its $p$-measure version state the following: if $\frac{k}{n} \leq \frac{1}{2}$ and $F \subset \binom{n}{k}$ is a 2-wise intersecting family then $|F|/\binom{n}{k} \leq \frac{k}{n}$, and

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if $p \leq \frac{1}{2}$ and $\mathcal{G} \subset 2^{[n]}$ is a 2-wise intersecting family then $\mu_p(\mathcal{G}) \leq p$. There are other such examples e.g., in [12], which suggest the following working hypothesis: if $\frac{k}{n} \approx p$ then $M_p^k(n, k)/\binom{n}{k} \approx W^r_p(n, p)$. This is indeed the case if $r = 3$ and $t = 1$ as we will describe shortly, and our result is corresponding to the following result on $W_3(n, p)$.

**Theorem 1 ([14]).** For $0 < p < 1$ and $q = 1 - p$ we have

$$
\lim_{n \to \infty} W_3(n, p) = \begin{cases} 
p^2 & \text{if } p \leq \frac{1}{3}, \\
4p^3q + p^4 & \text{if } \frac{1}{3} \leq p \leq \frac{1}{2}, \\
p & \text{if } \frac{1}{2} < p \leq \frac{2}{3}, \\
1 & \text{if } \frac{2}{3} < p < 1. 
\end{cases}
$$

To state our result we need to provide some large non-trivial 3-wise intersecting families that are potential candidates for $M_3(n, k)$. If $\frac{2}{3} < \frac{k}{n} < 1$ then $\binom{n}{k}$ is non-trivial 3-wise intersecting, and $M_3(n, k) = \binom{n}{k}$. From now on we assume that $\frac{k}{n} \leq \frac{2}{3}$. For integers $a < b$, let $[a, b] = \{a, a + 1, \ldots, b\}$. Define non-trivial 3-wise intersecting families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ by

$$
\mathcal{A}(n, k) = \left\{ F \in \binom{[n]}{k} : [2] \subset F, F \cap [3, k+1] \neq \emptyset \right\} \cup \{k+1 \setminus \{1\}, [k+1] \setminus \{2\}\},
$$

$$
\mathcal{B}(n, k) = \left\{ F \in \binom{[n]}{k} : |F \cap [4]| \geq 3 \right\},
$$

$$
\mathcal{C}(n, 2l) = \left\{ F \in \binom{[n]}{2l} : 1 \in F, |F \cap [2, 2l]| \geq l \right\} \cup \{[2, 2l] \cup \{i\} : i \in [2l+1, n]\},
$$

$$
\mathcal{C}(n, 2l + 1) = \left\{ F \in \binom{[n]}{2l+1} : 1 \in F, |F \cap [2, 2l+2]| \geq l + 1 \right\} \cup \{[2, 2l+2]\}.
$$

Then it follows that

$$
|\mathcal{A}(n, k)| = \binom{n-2}{k-2} - \binom{n-k-1}{k-2} + 2,
$$

$$
|\mathcal{B}(n, k)| = 4 \left( \binom{n-4}{k-3} + \binom{n-4}{k-4} \right),
$$

$$
|\mathcal{C}(n, 2l)| = \sum_{j=l}^{2l-1} \binom{2l-1}{j} \binom{n-2l}{2l-j-1} + (n-2l),
$$

$$
|\mathcal{C}(n, 2l + 1)| = \sum_{j=l+1}^{2l+1} \binom{2l+1}{j} \binom{n-2l-2}{2l-j} + 1.
$$

If $p = \frac{k}{n}$ is fixed and $n \to \infty$ then $|\mathcal{A}(n, k)|/\binom{n}{k} \to p^2$, and $|\mathcal{B}(n, k)|/\binom{n}{k} \to 4p^3q + p^4$. If, moreover, $\frac{1}{2} < p \leq \frac{2}{3}$ then $|\mathcal{C}(n, k)|/\binom{n}{k} \to p$ (see the proof of Theorem 4). Thus, by the working hypothesis with Theorem 1, we may expect that if $p = \frac{k}{n}$ is fixed and $n$ is sufficiently large, then

$$
M_3(n, k) \approx \begin{cases} 
|\mathcal{A}(n, k)| & \text{if } p \approx \frac{1}{3}, \\
|\mathcal{B}(n, k)| & \text{if } \frac{1}{3} \approx p \approx \frac{1}{2}, \\
|\mathcal{C}(n, k)| & \text{if } \frac{1}{2} \approx p \approx \frac{2}{3}, \\
\binom{n}{k} & \text{if } \frac{2}{3} < p < 1.
\end{cases}
$$
We will see that the aforementioned formula is roughly correct, but the actual statement we prove is a little more complicated. If \( k = 3 \) (and \( n \geq 4 \)) then we have \( A(n, 3) = B(n, 3) = C(n, 3) = \binom{n}{3} \) and it is easy to see that \( M_3(n, 3) = 4 \). From now on we assume that \( k \geq 4 \).

**Theorem 2.** We have
\[
M_3(n, k) = \begin{cases} 
|A(n, k)| & \text{if } k \geq 9 \text{ and } n \geq 3k - 2, \\
|B(n, k)| & \text{if } k \geq 4 \text{ and } \frac{5}{2}(k - 1) \leq n \leq 3(k - 1).
\end{cases}
\]

We record some results on \( M_3(n, k) \) for \( 4 \leq k \leq 8 \), which are not necessarily included in Theorem 2.

\[
\begin{align*}
M_3(n, 4) &= |B(n, 4)| & \text{if } n \geq 7, \\
M_3(n, 5) &= |B(n, 5)| & \text{if } n \geq 9, \\
M_3(n, 6) &= \begin{cases} 
|A(n, k)| & \text{if } n \geq 21, \\
|B(n, k)| & \text{if } 11 \leq n \leq 20,
\end{cases} \\
M_3(n, 7) &= \begin{cases} 
|A(n, k)| & \text{if } n \geq 21, \\
|B(n, k)| & \text{if } 12 \leq n \leq 20,
\end{cases} \\
M_3(n, 8) &= \begin{cases} 
|A(n, k)| & \text{if } n \geq 23, \\
|B(n, k)| & \text{if } 15 \leq n \leq 22.
\end{cases}
\]

**Theorem 3.** For every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all integers \( n \) and \( k \) with \( n > n_0 \) and \( \frac{5}{2} + \epsilon < \frac{k}{n} < \frac{1}{2} - \epsilon \), we have
\[
M_3(n, k) = |B(n, k)|.
\]

**Theorem 4.** For every \( \epsilon > 0 \) and every \( \delta > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for all integers \( n \) and \( k \) with \( n > n_0 \) and \( \frac{1}{2} + \epsilon < \frac{k}{n} < \frac{2}{3} - \epsilon \), we have
\[
(1 - \delta) \binom{n - 1}{k - 1} < M_3(n, k) < \binom{n - 1}{k - 1}.
\]

Indeed we prove \( (1 - \delta) \binom{n - 1}{k - 1} < |C(n, k)| \) under the assumptions in Theorem 4, from which it follows that
\[
|C(n, k)| \leq M_3(n, k) < \frac{1}{1 - \delta} |C(n, k)|.
\]

In section 2 we deduce Theorem 2 from the Ahlswede–Khacatrian theorem (Theorem 5) concerning non-trivial 2-wise 2-intersecting families. Then in section 3 we show a stronger stability version (Theorem 6) of Theorem 3 based on the corresponding stability result (Theorem 7) on \( W_3(n, p) \) from [14]. Finally in section 4 we prove Theorem 4 by estimating the size of the family \( C(n, k) \) using a variant of the de Moivre–Laplace theorem (Lemma 4).

2. PROOF OF THEOREM 2

We extend the construction of \( B(n, k) \) and define
\[
\mathcal{B}_i(n, k) = \{ F \in \binom{[n]}{k} : |F \cap [2 + 2i]| \geq t + i \}.
\]
Note that $B_i(n, k) = B(n, k)$. Note also that $B_i(n, k)$ is non-trivial 2-wise 2-intersecting for all $i \geq 1$, but non-trivial 3-wise intersecting only if $i = 1$. Ahlswede and Khachatrian determined the maximum size of non-trivial 2-wise $t$-intersecting families in $\binom{[n]}{k}$ completely. Here we include their result for the case $t = 2$.

**Theorem 5** (Ahlswede–Khachatrian [1]). The maximize size $M^*_2(n, k)$ of non-trivial 2-wise 2-intersecting families in $\binom{[n]}{k}$ is given by the following.

(i) If $2(k - 1) < n \leq 3(k - 1)$ then $M^*_2(n, k) = \max_{i \geq 1} |B_i(n, k)|$. If moreover $M^*_2(n, k) = |B_i(n, k)|$ then $(2 + \frac{1}{n+1})(k - 1) \leq n \leq (2 + \frac{1}{7})(k - 1)$.

(ii) If $3(k - 1) < n$ and $k \leq 5$ then $M^*_2(n, k) = |B(n, k)|$.

(iii) If $3(k - 1) < n$ and $k \geq 6$ then $M^*_2(n, k) = \max\{|A(n, k), |B(n, k)|\}$.

Moreover, if $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial 2-wise 2-intersecting family with $|\mathcal{F}| = M^*_2(n, k)$, then $\bigcap_{F \in \mathcal{F}} F = \emptyset$.

**Lemma 1.** If $k \geq 4$ and $2k \leq n \leq 3(k - 1)$ then $|A(n, k)| < |B(n, k)|$.

**Proof.** By (1) and (2), the inequality $|A(n, k)| < |B(n, k)|$ is equivalent to

$$\binom{n-2}{k-2} - \binom{n-k-1}{k-2} + 2 < 4 \binom{n-4}{k-3} + \binom{n-4}{k-4}.$$  

Using $\binom{n-2}{k-2} = \binom{n-4}{k-2} + 2 \binom{n-4}{k-3} + \binom{n-4}{k-4}$ and $\binom{n-4}{k-3} = \frac{k-2}{n-k-1} \binom{n-4}{k-2}$, the inequality stated above is equivalent to

$$\binom{n-4}{k-2} + 2 < \binom{n-k-1}{k-2}.$$  

Since $n \leq 3(k - 1)$ the LHS is at most 2, while the RHS satisfies

$$\binom{n-k-1}{k-2} \geq \binom{k-1}{k-2} = k - 1 \geq 3,$$

because $n \geq 2k$ and $k \geq 4$. \qed

**Lemma 2.** If $k \geq 9$ and $n \geq 3k - 2$ then $|A(n, k)| > |B(n, k)|$.

**Proof.** We need to show that

$$\binom{n-4}{k-2} \frac{n-3(k-1)}{n-k-1} + 2 > \binom{n-k-1}{k-2}. $$

For $k \geq 18$ we claim a stronger inequality $\binom{n-4}{k-2} \binom{n-3(k-1)}{n-k-1} \geq \binom{n-k-1}{k-2}$, that is,

$$g(n, k) := \frac{(n-4)(n-5) \cdots (n-k)(n-3k+3)}{(n-k-1)(n-k-2) \cdots (n-2k+2)} \geq 1.$$  

First we consider the case $n = 3k - 2$, and let

$$f(k) := g(3k - 2, k) = \frac{(3k-6) \cdots (2k-2)}{(2k-3) \cdots k}.$$  

Then we have

$$\frac{f(k)}{f(k+1)} = \frac{(2k-1)^2(2k-2)^2}{(3k-3)(3k-4)(3k-5)} < 1$$

There is a mistake in the proof. The LHS should be $\frac{n-k-1}{k-2}$ instead of $\frac{n-k-1}{k-2}$, but the inequality is still valid.

Therefore, we have

$$f(k) < f(k+1)$$

for $k \geq 9$ and $n \geq 3k - 2$. \qed
for $k \geq 4$. Since $f(9) > 1$ we have $1 < f(9) < f(10) < \cdots$, and so
\begin{equation}
(7) \quad g(3k - 2, k) > 1
\end{equation}
for $k \geq 9$.

Next fix $k \geq 18$ and let
\begin{equation}
(8) \quad h(n) := g(n + 1, k) - g(n, k)
\end{equation}
where
\[ c_n := \frac{(n - 4) \cdots (n - k + 1)}{(n - k - 1) \cdots (n - 2k + 3)} \cdot \frac{1}{(n - k)(n - 2k + 2)} > 0. \]
Then $h(n)$ is decreasing in $n$, and $h(3k + 1) = c_{3k+1}(k - 2)(17 - k) < 0$, so we have $h(n) < 0$ for $n \geq 3k + 1$. Thus it follows that
\[ g(3k + 1, k) > g(3k + 2, k) > \cdots \]
for $k \geq 18$. We also have $g(n, k) \rightarrow 1$ for $k$ fixed and $n \rightarrow \infty$, because both the denominator and numerator of (6) consist of $k - 2$ linear terms in $n$. Thus it follows that $g(n, k) \geq 1$ for $n \geq 3k + 1$. For $n = 3k - 2, 3k - 1, 3k$, we can verify $h(n) > 0$ by (8). This together with (7) yields
\[ 1 < g(3k - 2, k) < g(3k - 1, k) < g(3k, k) < g(3k + 1, k). \]
Therefore we have $g(n, k) \geq 1$ for all $n \geq 3k - 2$ and $k \geq 18$.

For the remaining cases $9 \leq k \leq 17$ we can verify (5) directly. \qed

Now we prove Theorem 2. Observe that if $\mathcal{F} \subset \binom{[n]}{k}$ is non-trivial 3-wise intersecting, then it is necessarily non-trivial 2-wise 2-intersecting. This yields that $M_3(n, k) \leq M_2^2(n, k)$.

First suppose that $k \geq 9$ and $n \geq 3k - 2$. In this case, by (iii) of Theorem 5 with Lemma 2, we have
\[ M_3(n, k) \leq M_2^2(n, k) = \max\{|\mathcal{A}(n, k)|, |\mathcal{B}(n, k)|\} = |\mathcal{A}(n, k)|. \]
Since $\mathcal{A}(n, k)$ is non-trivial 3-wise intersecting, we have $M_3(n, k) = |\mathcal{A}(n, k)|$.

Next suppose that $k \geq 4$ and $\frac{k}{2}(k - 1) \leq n \leq 3k - 3$. In this case, by (i) of Theorem 5, we have
\[ M_3(n, k) \leq M_2^2(n, k) = |\mathcal{B}_1(n, k)| = |\mathcal{B}(n, k)|. \]
Since $\mathcal{B}(n, k)$ is non-trivial 3-wise intersecting, we have $M_3(n, k) = |\mathcal{B}(n, k)|$. This completes the proof of Theorem 2. \qed

The results on $M_3(n, k)$ for $4 \leq k \leq 8$ follows from Theorem 5 and comparing $|\mathcal{A}(n, k)|$ and $|\mathcal{B}(n, k)|$ using (5).
3. Proof of Theorem 3

A family \(\mathcal{F} \subset 2^{[n]}\) is called shifted if \(F \in \mathcal{F}\) and \(F \cap \{i, j\} = \{j\}\), then \((F \setminus \{j\}) \cup \{i\} \in \mathcal{F}\) for all \(1 \leq i < j \leq n\). Theorem 3 is an immediate consequence from the following stronger stability result.

**Theorem 6.** For every \(\epsilon > 0\) there exist \(\gamma > 0\) and \(n_0 \in \mathbb{N}\) such that for all positive integers \(n\) and \(k\) with \(n > n_0\) and \(\frac{2}{3} + \epsilon < \frac{k}{n} < \frac{1}{2} - \epsilon\) the following holds: if \(\mathcal{F} \subset \binom{[n]}{k}\) is a shifted non-trivial 3-wise intersecting family with \(\mathcal{F} \not\subset \mathcal{B}(n, k)\), then \(|\mathcal{F}| < (1 - \gamma)|\mathcal{B}(n, k)|\).

We deduce Theorem 6 from the next technical lemma.

**Lemma 3.** For every \(\epsilon > 0\) and every \(p\) with \(\frac{2}{3} + \epsilon < p < \frac{1}{2} - \epsilon\) there exist \(\gamma > 0\) and \(n_0 \in \mathbb{N}\) such that for all positive integers \(n\) and \(k\) with \(n > n_0\) and \(|\frac{k}{n} - p| < \frac{\epsilon}{2}\) the following holds: if \(\mathcal{F} \subset \binom{[n]}{k}\) is a shifted non-trivial 3-wise intersecting family with \(\mathcal{F} \not\subset \mathcal{B}(n, k)\), then \(|\mathcal{F}| < (1 - \gamma)|\mathcal{B}(n, k)|\).

For real numbers \(\alpha > \beta > 0\), we write \(\alpha \pm \beta\) to mean the open interval \((\alpha - \beta, \alpha + \beta)\).

**Proof of Theorem 6.** Here we prove Theorem 6 assuming Lemma 3. Let \(\epsilon > 0\) be given.

Let \(I = (\frac{2}{3} + \epsilon, \frac{1}{2} - \epsilon)\). Noting that \(\frac{1}{2} - \frac{\epsilon}{2} = \frac{1}{3}\) we can divide \(I\) into at most \(\frac{1}{10\epsilon}\) small intervals \(I_p := p \pm \frac{\epsilon}{2}\). More precisely, by choosing real numbers \(\frac{2}{5} + \frac{3\epsilon}{2} \leq p_1 < p_2 < \cdots < p_N \leq 1 - \frac{3\epsilon}{2}\) appropriately, where \(N \leq \frac{1}{10\epsilon}\), we can cover \(I\) by \(\bigcup_{1 \leq i \leq N} I_{p_i}\). For every \(p \in I\) there exists \(i\) such that \(p \in I_i\). Apply Lemma 3 with this \(p\), then it provides \(\gamma = \gamma_i\) and \(n_0 = n_0(i)\). Finally let \(\gamma = \min_{1 \leq i \leq N} \gamma_i\) and \(n_0 = \max_{1 \leq i \leq N} n_0(i)\). Then Theorem 6 follows from Lemma 3.

Now we prove Lemma 3. For the proof we need the \(p\)-measure counterpart of Lemma 3. To state the result we define a non-trivial 3-wise intersecting family \(\mathcal{B}(n) \subset 2^{[n]}\) by

\[\mathcal{B}(n) = \{F \in 2^{[n]} : |F \cap [4]| \geq 3\}\]

Then \(\mu_p(\mathcal{B}(n)) = 4p^3q + p^4 =: f(p)\), where \(q = 1 - p\).

**Theorem 7** ([14]). Let \(\frac{2}{5} \leq p \leq \frac{1}{2}\), and let \(\mathcal{G} \subset 2^{[n]}\) be a shifted non-trivial 3-wise intersecting family. If \(\mathcal{G} \not\subset \mathcal{B}(n)\) then

\[\mu_p(\mathcal{G}) < \mu_p(\mathcal{B}(n)) - 0.0018 \leq (1 - 0.00576)f(p)\]

We deduce Lemma 3 from Theorem 7. To this end we assume the negation of Lemma 3, and derive a contradiction to Theorem 7. The main idea is that if we have a counterexample family \(\mathcal{F} \subset \binom{[n]}{k}\) to Lemma 3, then the family \(\mathcal{G} \subset 2^{[n]}\) consisting of all superset of \(F \in \mathcal{F}\) satisfies the assumptions of Theorem 7, but its \(p\)-measure is too large.

Suppose that Lemma 3 fails at some \(\epsilon_c > 0\) and some \(p_c\) with \(\frac{2}{5} + \epsilon_c < p_c < \frac{1}{2} - \epsilon_c\), where ‘c’ stands for counterexample. Fix these \(\epsilon_c\) and \(p_c\). Set

\[\gamma_c = \frac{0.00576}{4}, \quad I_c = p_c \pm \frac{\epsilon_c}{2}\]

Since \(f(p)\) is continuous in \(p\) we can find \(\epsilon_1\) with \(0 < \epsilon_1 \leq \frac{\epsilon_c}{4}\) such that

\[(1 - 3\gamma_c)f(p) > (1 - 4\gamma_c)f(p + \delta)\]
for all $0 < \delta \leq \epsilon_1$ and all $p \in I_c$. Define an open interval

$$J_{n,p} = ((p - \epsilon_1)n, (p + \epsilon_1)n) \cap \mathbb{N}.$$ 

By the concentration of the binomial distribution we can choose $n_1$ so that

$$\sum_{j \in J_{n,p}} \binom{n}{j} p^j q^{n-j} > \frac{1 - 3\gamma_c}{1 - 2\gamma_c}$$

for all $n > n_1$ and all $p \in I_0 := p_c + \frac{3}{4} \epsilon_c$. As $|B(n, k)|/\binom{n}{k} \to f(p)$ if $p = \frac{k}{n}$ is fixed and $n \to \infty$, we can find $n_2$ such that

$$\sum_{j \in J_{n_2,p}} \binom{n}{j} p^j q^{n-j} > (1 - 2\gamma_c) f \left( \frac{k}{n} \right)$$

for all $n > n_2$ and all $k$ with $\frac{k}{n} \in I_c$. Let $n_0 = \max\{n_1, n_2\}$.

With these $\epsilon_c, p_c, \gamma_c, n_0$ we can choose $n_c$, $k_c$ and $\mathcal{F}_c$ with $n_c > n_0$ and $\frac{k_c}{n_c} \in I_c$ such that $\mathcal{F}_c \subset \binom{[p_c]}{k_c}$ is a counterexample to Lemma 3, that is, $\mathcal{F}_c$ is shifted, non-trivial 3-wise intersecting, but

$$|\mathcal{F}_c| \geq (1 - \gamma_c)|\mathcal{B}(n, k)|.$$ 

Fix these $n_c, k_c$ and $\mathcal{F}_c$. By (13) and (12) we have

$$|\mathcal{F}_c| > (1 - 2\gamma_c) f \left( \frac{k_c}{n_c} \right) \binom{n_c}{k_c}.$$ 

Let $p = \frac{k_c}{n_c} + \epsilon_1$. Since

$$p_c - \frac{3}{4} \epsilon_c < \left( p_c - \frac{\epsilon_c}{2} \right) + \epsilon_1 < p < \left( p_c + \frac{\epsilon_c}{2} \right) + \epsilon_1 \leq p_c + \frac{3}{4} \epsilon_c,$$

we have $p \in I_0$. Also it follows from $k_c = (p - \epsilon_1)n_c$ that $J_{n_c,p} \subset [k_c, n_c]$. Define a shifted non-trivial 3-wise intersecting family $\mathcal{G} \subset 2^{[n]}$ by

$$\mathcal{G} = \bigcup_{j = k_c}^{n_c} \nabla_j(\mathcal{F}_c),$$

where $\nabla_j$ denotes the $j$-th upper shadow, that is, $\nabla_j(\mathcal{F}_c) = \{ H \in \binom{[n]}{j} : H \supset \exists F \in \mathcal{F}_c \}$. It follows from (14) and the Kruskal–Katona theorem that

$$|\nabla_j(\mathcal{F}_c)| \geq (1 - 2\gamma_c) f \left( \frac{k_c}{n_c} \right) \binom{n_c}{j}.$$
for \( j \in J_{n,c,p} \), see Claim 6 in [13] for a detailed proof. Then we have

\[
\mu_p(\mathcal{G}) \geq \sum_{j \in J_{n,c,p}} |\nabla_j(\mathcal{F})| p^j q^{n-j}.
\]

\[
\geq (1 - 2\gamma_c) f \left( \frac{k_c}{n_c} \right) \sum_{j \in J_{n,c,p}} \binom{n_c}{j} p^j q^{n-j} \quad \text{by (15)}
\]

\[
> (1 - 2\gamma_c) f \left( \frac{k_c}{n_c} \right) \frac{1}{1 - 3\gamma_c} \quad \text{by (11)}
\]

\[
= (1 - 3\gamma_c) f \left( \frac{k_c}{n_c} \right) \quad \text{by (10)}
\]

\[
= (1 - 4\gamma_c) f(p) \quad \text{by (9)}
\]

which contradicts Theorem 7. This completes the proof of Lemma 3, and so Theorem 6 (and Theorem 3).

\[\square\]

4. PROOF OF THEOREM 4

For the proof we prepare some technical estimations. Let \( \frac{1}{2} < p < \frac{2}{3} \) and \( q = 1 - p \). Let \( k = pn \). Fix a constant \( c > 0 \), and let

\[ J_{n,c} = \{ j \in \mathbb{N} : |j - p^2n| \leq c\sqrt{n}\}. \]

For \( j \in J_{n,c} \) we estimate

\[ \theta_j(n, p) = \frac{\binom{pn}{j} \binom{n-pn}{pn-j}}{\binom{n}{pn}} = \frac{\binom{k}{j} \binom{n-k}{n-j}}{\binom{n}{k}}. \]

**Lemma 4.** We have

\[ \theta_j(n, p) = \frac{1}{pq\sqrt{2\pi n}} \exp \left( -\frac{1}{2p^4q^4n^2} (j - p^2n)^2 \right) \left( p^2q^2n - (1 - 2p)^2(j - p^2n) + r_{n,p}(j) \right), \]

where \( \max_{j \in J_{n,c}} |r_{n,p}(j)| \to 0 \) as \( n \to \infty \).

**Proof.** This is a variant of the de Moivre–Laplace theorem, and it follows from a routine but tedious calculus. Here we include the outline. By Stirling’s formula we have

\[
\binom{a}{b} \sim \sqrt{\frac{a}{2\pi b(a-b)}} \frac{a^a}{b^b(a-b)^{a-b}},
\]

and so

\[
\theta_j(n, p) \sim \frac{n}{2\pi p q k(n-k)} \binom{pk}{n-k+1} \binom{qn-k-j}{n-k-j-1} (qn-k-j+1)\binom{p}{n-j}^{-p^2q^2n} (k-j)^{-k+j+1}.
\]
Noting that \( k + 1 = (j + \frac{1}{2}) + (k - j + \frac{1}{2}) \) and \( n - k + 1 = (qn - k + j + \frac{1}{2}) + (pm - j + \frac{1}{2}) \) we can rewrite \( \theta_j(n, p) \) as
\[
\theta_j(n, p) \sim \frac{1}{pq\sqrt{2\pi n}} \exp(-A),
\]
where
\[
A = (j + \frac{1}{2}) \log \frac{j}{pk} + (k - j + \frac{1}{2}) \log \frac{k-j}{pk} + (qn - k + j + \frac{1}{2}) \log \frac{qn-k+j}{q(n-k)} + (qn - j + \frac{1}{2}) \log \frac{qn-j}{q(n-k)}.
\]
Now, to use \( \log(1 + x) = x - \frac{x^2}{2} + O(x^3) \), we recall \( j \sim pk \) and we write
\[
\begin{align*}
\log \frac{j}{pk} &= \log(1 + \frac{j-pk}{pk}), & \log \frac{k-j}{pk} &= \log \frac{q}{p} + \log(1 - \frac{j-pk}{pk}), \\
\log \frac{qn-k+j}{q(n-k)} &= \log(1 + \frac{j-pk}{q(n-k)}), & \log \frac{qn-j}{q(n-k)} &= \log \frac{q}{q} + \log(1 - \frac{j-pk}{pk}).
\end{align*}
\]
Thus we have e.g., \( \log \frac{j}{pk} \sim (\frac{j-pk}{pk}) - \frac{1}{2}(\frac{j-pk}{pk})^2 \). Substituting these approximations into \( A \) and rearranging, we obtain the desired estimation. \( \square \)

Let \( \text{erf}(z) \) denote the error function, that is,
\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) \, dx.
\]

**Lemma 5.** We have
\[
\lim_{n \to \infty} \sum_{j \in J_{n,p}} \theta_j(n, p) = \text{erf} \left( \frac{3c}{\sqrt{2p}} \right).
\]

**Proof.** Let \( z = j - p^2 n \). Then, by Lemma 4, we can write \( \theta_j(n, p) \) as in (16), where
\[
A = \frac{1}{2p^4q^4n^2} z^2(p^2q^2n - (1 - 2p)^2 z).
\]
Thus it follows that
\[
\sum_{j \in J_{n,p}} \theta_j(n, p) \sim \int_{-c\sqrt{n}}^{c\sqrt{n}} \frac{1}{pq\sqrt{2\pi n}} \exp \left( \frac{1}{2p^4q^4n^2} z^2(p^2q^2n - (1 - 2p)^2 z) \right) \, dz.
\]
Finally, by changing \( z = \sqrt{\frac{n}{2}} x \), we get
\[
\lim_{n \to \infty} \sum_{j \in J_{n,p}} \theta_j(n, p) = \lim_{n \to \infty} \int_{-c}^{c} \frac{1}{pq\sqrt{2\pi}} \exp \left( -\frac{x^2}{2p^2q^2} + \frac{(1 - 2p)x^2}{2p^4q^4\sqrt{n}} \right) \, dx = \text{erf} \left( \frac{3c}{\sqrt{2p}} \right). \, \square
\]

Now we prove Theorem 4. Frankl [6] proved that if \( \frac{k}{n} \leq \frac{2}{3} \), then the maximum size of (not necessarily non-trivial) 3-wise intersecting families \( \mathcal{F} \subset \binom{[n]}{k} \) is \( \binom{n-1}{k-1} \), and moreover this bound is attained only if the family \( \mathcal{F} \) is trivial, that is, \( \bigcap_{F \in \mathcal{F}} F \neq \emptyset \). This means that \( M_3(n, k) < \binom{n-1}{k-1} \). Thus the remaining part of the proof below is to give a lower bound for \( M_3(n, k) \).
Let $\epsilon > 0$ and $\delta > 0$ be given. We need some constants to define $n_0$. First choose $c$ so that
\[
\min_{p \in [\frac{1}{2}, \frac{2}{3}]} \text{erf} \left( \frac{3c}{\sqrt{2p}} \right) = \text{erf} \left( \frac{9c}{2\sqrt{2}} \right) > 1 - \frac{\delta}{2}.
\]
Then, by Lemma 5, we can choose $n_1$ so that if $n > n_1$ then
\[
(17) \quad \sum_{j \in J_{n,p}} \theta_j(n, p) > \sqrt{1 - \delta}
\]
for all $p$ with $\frac{1}{2} \leq p \leq \frac{2}{3}$. Next choose $n_2$ so that if $\frac{1}{2} + \epsilon < p < \frac{2}{3} - \epsilon$, $n > n_2$, and $k = pn$, then $\frac{k}{2} < pk - c\sqrt{n}$ and $pk + c\sqrt{n} < k - 1$, or equivalently,
\[
\left( p - \frac{1}{2} \right) p\sqrt{n} > c \text{ and } c\sqrt{n} < pqn - 1.
\]
We need this condition to guarantee $J_{n,p} \subset \left[ \frac{k}{2}, k - 1 \right]$. Finally let $\delta_1, \delta_2 > 0$ be sufficiently small constants such that $(1 - \delta_1)^2/(1 + \delta_2) > \sqrt{1 - \delta}$, and choose $n_3$ so that if $n > n_3$ then
\[
p - \frac{j}{n} \geq p - p^2 - \frac{c}{\sqrt{n}} > pq(1 - \delta_1),
\]
and
\[
1 - 2p + \frac{j}{n} + \frac{1}{n} \leq 1 - 2p + p^2 + \frac{c}{\sqrt{n}} + \frac{1}{n} < q^2(1 + \delta_2).
\]
Then let $n_0 := \max\{n_1, n_2, n_3\}$.

Now let $n$ and $k$ be given, and let $p = \frac{k}{n}$. By the assumptions of the theorem we have $n > n_0$ and $\frac{1}{2} + \epsilon < p < \frac{2}{3} - \epsilon$.

First suppose that $k$ is even. In this case, by (3), we have
\[
|C(n, k)| > \sum_{j = k/2}^{k-1} \binom{k-1}{j} \binom{n-k}{k-j-1}.
\]
The summands in the RHS is
\[
\binom{k-1}{j} \binom{n-k}{k-j-1} = \frac{k-j}{k} \cdot \frac{k-j}{n-k-j} \cdot \frac{n-k}{k-j+1} \cdot \frac{n-k-j}{k-j} = \frac{p - \frac{j}{n}}{p} \binom{k}{j} \cdot \frac{p - \frac{j}{n}}{1 - 2p + \frac{j}{n} + \frac{1}{n}} \binom{n-k}{k-j}.
\]
For $j \in J_{n,p}$ we have
\[
\frac{p - \frac{j}{n}}{p} \cdot \frac{p - \frac{j}{n}}{1 - 2p + \frac{j}{n} + \frac{1}{n}} > \frac{(pq(1 - \delta_1))^2}{pq^2(1 + \delta_2)} = \frac{(1 - \delta_1)^2}{1 + \delta_2} > p\sqrt{1 - \delta}.
\]
Thus
\[
\binom{k-1}{j} \binom{n-k}{k-j-1} > p\sqrt{1 - \delta} \binom{k}{j} \binom{n-k}{k-j}.
\]
Consequently we have
\[
|C(n, k)| > \sum_{j \in J_{n,p}} \binom{k-1}{j} \binom{n-k}{k-j-1}
\]
\[
> p\sqrt{1 - \delta} \sum_{j \in J_{n,p}} \binom{k}{j} \binom{n-k}{k-j}
\]
\[
= p\sqrt{1 - \delta} \binom{n}{k} \sum_{j \in J_{n,p}} \theta_j(n, p)
\]
\[
> (1 - \delta) \binom{n-1}{k-1},
\]
where we use (17) for the last inequality.

Next suppose that \(k\) is odd. In this case, by (4), we have
\[
|C(n, k)| > \sum_{j=(k+1)/2}^{k} \binom{k}{j} \binom{n-k-1}{k-j-1}.
\]
The summands in the RHS is \(\binom{k}{j} \binom{n-k-1}{k-j-1} = \binom{k+j-1}{j} \binom{n-k}{k-j}\). For \(j \in J_{n,p}\) we have
\[
\frac{k-j}{n-k} \frac{p - \frac{j}{n}}{1 - p} > \frac{pq(1 - \delta_i)}{p^2} = p(1 - \delta_i) > p\sqrt{1 - \delta},
\]
and
\[
\binom{k}{j} \binom{n-k-1}{k-j-1} > p\sqrt{1 - \delta} \binom{k}{j} \binom{n-k}{k-j}.
\]
Thus we have \(|C(n, k)| > (1 - \delta) \binom{n-1}{k-1}\) just as in the previous case. This completes the proof of Theorem 4.

\[\square\]

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