The evidence for a spatially flat Universe

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ABSTRACT
We revisit the observational constraints on spatial curvature following recent claims that the Planck data favour a closed Universe. We use a new and statistically powerful Planck likelihood to show that the Planck temperature and polarization spectra are consistent with a spatially flat Universe, though because of a geometrical degeneracy cosmic microwave background anisotropy spectra on their own do not lead to tight constraints on the curvature density parameter $\Omega_K$. When combined with other astrophysical data, particularly geometrical measurements of baryon acoustic oscillations, the Universe is constrained to be spatially flat to extremely high precision, with $\Omega_K = 0.0004 \pm 0.0018$ in agreement with the 2018 results of the Planck team. In the context of inflationary cosmology, the observations offer strong support for models of inflation with a large number of e-foldings and disfavour models of incomplete inflation.

Key words: cosmology: cosmological parameters, large-scale structure of Universe, cosmic background radiation, observations

1 INTRODUCTION
One of the strongest motivations for the theory of inflation comes from the observation that our Universe is very nearly spatially flat. As Guth and others (Guth 1981; Linde 1990) have pointed out, exponential expansion during an inflationary phase provides an elegant solution to the ‘flatness’ problem. As a consequence of exponential expansion, $\Omega_K = 0$ is a powerful late time attractor.

The inflationary prediction of a spatially flat Universe is strongly supported by observations. The 2018 cosmic microwave background (CMB) results reported by the Planck team (combining Planck temperature and polarization data with Planck lensing and baryon acoustic oscillation measurements) give

$$\Omega_K = 0.0007 \pm 0.0019, \quad (1)$$

(Planck Collaboration et al. 2018a, hereafter PCP18) suggesting that our Universe is spatially flat to a 1σ accuracy of 0.2%. In addition, the Planck power spectra are extremely well fit by a nearly scale invariant adiabatic fluctuation spectrum. Taken together, these two results offer strong support for the inflationary model.

Recently, however, three papers (Park & Ratra 2019; Di Valentino et al. 2019; Handley 2019) have presented a different interpretation. These papers point out that the Planck temperature and polarization data show a preference for a closed universe (as noted in PCP18). Di Valentino et al. (2019) conclude that the Planck results favour positive curvature at the 3.4σ level (i.e. a probability to exceed (pte) of 0.034%). They interpret this high significance level as evidence for either undetected systematics in the Planck data, new physics, or an unusual statistical fluctuation (or some combination of all three). Park & Ratra (2019) and Handley (2019) reach qualitatively similar conclusions. Since the limit on spatial curvature is of such fundamental importance to cosmology, we revisit this problem in this paper.

2 CURVATURE AND CHOICE OF PRIOR
Let us assume that the spatial curvature of the Universe, $\Omega_K = K/((aH)^2)$, is of order unity at the start of inflation (where $a$ is the scale factor and $H$ is the Hubble parameter). If the Universe undergoes $N$ e-foldings of inflation (ending at $a = a_I$) the curvature parameter at the present day ($a = a_0$) will be

$$|\Omega_K| = e^{-2N} \left( \frac{a_I H_I}{a_0 H_0} \right)^2. \quad (2)$$

The term in square brackets depends on the duration of the reheating phase at the end of inflation and the energy scale of inflation and so is uncertain. For plausible parameters, with an energy scale of inflation of order $V_I \sim 10^{16}$ GeV,
(a_1 H_1)^2 / (a_0 H_0)^2 = e^{2N_*} \text{ with } N_* \approx 60^4 \text{ (e.g. Liddle & Leach 2003). A solution of the flatness and horizon problems requires } N > N_* \text{. For many models of inflation, e.g. } \alpha\text{-attractors (Carrasco et al. 2015), the number of e-foldings can be much greater than } N_* \text{, in which case our Universe is expected to be spatially flat to high accuracy. If, however, the number of e-foldings is comparable to } N_* \text{, spatial curvature may be detectable today. In any model involving a small number of e-foldings it is essential that fluctuations on the curvature scale remain small. This is quite natural in models of open inflation invoking a Coleman-de Luccia instanton (Coleman & de Luccia 1980). Models of this type have been discussed by many authors in the past (e.g. Gott 1982; Linde 1995; Bucher et al. 1995; Linde et al. 1999) and they have received renewed interest in the context of false vacuum decay within a string landscape (e.g. Freivogel et al. 2006; Yamauchi et al. 2011). It is also possible, with moderate fine tunings, to construct models with positive spatial curvature (e.g. Gratton et al. 2002; Linde 2003). The exact mechanism is unimportant for our purposes and neither is the curvature scale (e.g. Gratton et al. 2002; Linde 2003). The exact mechanism is unimportant for our purposes and neither is the choice of measure. We will simply assume that inflation generates a finite number of e-foldings with } N > N_* \text{, skewed to low values:}

\[ p(N)dN \propto N^{-\alpha}dN, \quad N > N_*, \quad (3) \]

with } \alpha > 1 \text{. Assuming } [K]/(aH)^2 = 1 \text{ at the start of inflation, the distribution of spatial curvatures at the present day is}

\[ p(\Omega K)d\Omega K = \frac{(\alpha - 1)}{4} \frac{N_*^{(\alpha - 1)}}{(N_* - \frac{1}{2} \ln |\Omega K|^\alpha)} \frac{d\Omega K}{\Omega K}, \quad (4) \]

This function is peaked at } \Omega K = 0 \text{, but has tails extending to non-zero values of } \Omega K \text{. In fact, for the distribution (4) the probability of finding } |\Omega K| > |\Omega K| \text{ is}

\[ P(|\Omega K| > |\Omega K|) \approx \frac{(\alpha - 1)}{2} \ln |\Omega K|, \quad (5) \]

and is non-negligible even though the most probable value is } |\Omega K| < |\Omega K| \text{. A specific model of this type of incomplete inflation has been discussed by Freivogel et al. (2006), though these authors used anthropic arguments in place of } N_* \text{, to cut off the distribution (3). We can therefore view experimental bounds on } \Omega K \text{ as constraints on incomplete inflation. The more accurately we can constrain the Universe to be spatially flat, the stronger the evidence for an inflationary attractor with a large number of e-foldings.}

Models have been suggested that skew inflation even more strongly to small numbers of e-foldings (Hawking & Turok 1998). However, the main purpose of this example, is to emphasise that there is no good physical justification to adopt a uniform prior in } \Omega K \text{ when analysing cosmological data. Since } \Omega K \text{ is poorly determined from CMB power spectra alone (with a non-Gaussian tail extending to large negative values), it is dangerous to interpret Bayesian posterior distributions in } \Omega K \text{ as probability distributions unless one can justify the choice of prior}^2 \text{. As a result, perceived tensions on the value of } \Omega K \text{ between } Planck \text{ and other cosmological data are on a very different footing to tensions in the value of, for example, the Hubble constant } H_0 \text{. As is well known, late time measurements of } H_0 \text{ differ from the base } \Lambda CDM \text{ value determined from } Planck \text{ by about } 4.3\sigma \text{ (e.g. Riess et al. 2019). However, the Hubble constant is so well determined by } Planck (H_0 = 67.44 \pm 0.58 \text{ km s}^{-1} \text{ Mpc}^{-1}) \text{ that we can be confident that the data overwhelms the priors, since it is extremely unlikely that } H_0 \text{ is drawn to any particular value by an attractor.}

3 LIKELIHOODS AND CONSTRAINTS ON } \Omega K

The } Planck \text{ preference for closed universes has been pointed out in previous } Planck \text{ papers (Planck Collaboration et al. 2014, 2016, 2018a) and is closely related to the preference for the } Planck \text{ temperature spectra to favour more lensing than expected in the base } \Lambda CDM \text{ model (quantified by the phenomenological } A_L \text{ parameter (Calabrese et al. 2008)) since both effects are caused by the same features in the } Planck \text{ temperature power spectrum in the multipole range } \ell \sim 1200 - 1500 \text{. PCP18 also pointed out that when } Planck \text{ high multipole polarization spectra were included, the } P_{11k} \text{ TTTEE polarization likelihood pulls } A_L \text{ and } \Omega K \text{ away from the base } \Lambda CDM \text{ model with a higher significance than our own CamSpec TTTEE likelihood. The posterior for the parameter } \Omega K \text{ is therefore sensitive to both choice of prior and to the likelihood implementation. PCP18 discussed the possibility that both the } \Omega K \text{ and } A_L \text{ ‘tensions’ were a result of statistical fluctuations. Following the completion of PCP18, we investigated this possibility in detail (Efstathiou & Gratton 2019, hereafter EG) by constructing a } Planck \text{ likelihood (which we refer to as the } 12.5hM\text{cln likelihood) using more sky in temperature and polarization than in the } Planck \text{ CamSpec likelihood reported in PCP18. The construction of the } 12.5hM\text{cln likelihood is discussed at length in EG, to which we refer the reader for further details and for tests of the consistency of the TE and EE polarization spectra. Increasing the sky area reduced the ‘tensions’ in } \Omega K \text{ and } A_L \text{, as expected if they were caused by statistical fluctuations. Furthermore, we demonstrated that the features in the temperature power spectrum that drive these tensions are repeatable to high accuracy between the } 217 \times 217, 143 \times 217 \text{ and } 143 \times 143 \text{ GHz temperature cross-spectra. It therefore seems unlikely that the } \Omega K \text{ and } A_L \text{ results are influenced by systematic errors in the } Planck \text{ data. In addition, the polarization spectra are essentially neutral with respect to the parameters } \Omega K \text{ and } A_L.\]

In this paper, we use the 12.5hMcln likelihood at } \ell \geq 30, \text{ as described in EG, together with the 2018 } Planck \text{ temperature and polarization likelihoods at } \ell < 30 \text{ as described in Planck Collaboration et al. (2019). The notation for these data combinations follows that of EG, thus TT-TEEEN denotes the full 12.5hMcln likelihood at } \ell \geq 30 \text{ combining the temperature-temperature (TT), temperature-polarization (TE) and polarization-polarization (EE) cross-spectra; TT or TE denotes use of only the temperature-}

\[^3\text{As in the } Planck \text{ papers we refer to the six parameter } \Lambda CDM \text{ model (spatially flat, power law scalar adiabatic fluctuations, cosmological constant) as the base } \Lambda CDM \text{ model.}\]
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Figure 1. 68% and 95% contours on $\Omega_K$ and $H_0$ assuming a flat prior in $\Omega_K$ over the range $-0.3 < \Omega_K < 0.3$: (a) Planck data alone; (b) Planck data combined with the Pantheon supernova sample; (c) Planck data combined with BAO.

The result in (6c) is essentially identical to the constraint of eqn. (1) derived in PCP18. We interpret these results as extremely strong evidence that our Universe is nearly spatially flat. Furthermore, in the context of inflationary scenarios, these results show that our Universe has firmly locked on to the inflationary attractor, disfavouring models of incomplete inflation with a limited numbers of e-foldings ($N \approx N_\star$). This is a highly non-trivial result.

Di Valentino et al. (2019) argue that observational evidence for a closed Universe would present a crisis for cosmology. We agree with this conclusion. If the Universe were indeed closed with a value of $\Omega_K \approx -0.04$ then one would have to argue that unexpected new physics or systematics in the Planck lensing data, supernovae and BAO all act in the same way to favour $\Omega_K = 0$. Since these data sets are independent of each other and respond to different physics (supernovae and BAO test the background cosmology, while lensing tests theory at the level of perturbations) this is extraordinarily unlikely. It is much more plausible that these additional datasets break the geometrical degeneracy leading to values of $\Omega_K$ that are closer to the truth. The fact that all three datasets favour $\Omega_K = 0$ provides powerful evidence that our Universe is nearly spatially flat.

Another possibility is that the tendency for Planck power spectra to favour closed Universes is caused by systematic errors in the Planck likelihoods and/or Planck data. As discussed above, it is certainly true that different likelihood implementations lead to different results, with the Planck likelihood favouring closed Universes more strongly than our own CamSpec likelihood. We have discussed the construction of the CamSpec likelihood in great detail in EG and have argued that our methodology is robust and gives reasonable $\chi^2$ values for the polarization spectra, unlike Planck

4 All chains were produced using COSMOMC (see https://cosmologist.info/cosmomc/) using the CAMB Boltzmann code (see https://camb.info/readme.html) in exactly the same way as described in PCP18.

5 Note also that CMB lensing measurements from the South Pole Telescope agree well with the Planck lensing measurements and strongly favour a spatially flat Universe when combined with Planck (Bianchini et al. 2020).
Planck Collaboration et al. (2019, for further details). However, for readers interested in spatial curvature, whether Plik or CamSpec is the more reliable likelihood is irrelevant because differences between Planck likelihoods are overwhelmed when Planck data are combined with BAO. This is why the estimates of equs. (1) and (6d) agree so precisely.

The final question to consider is whether there is a statistical inconsistency, i.e. if we allow $\Omega_K$ to vary, are the fits to the Planck power spectra so much better than the fits to the base $\Lambda$CDM model to suggest systematics or new physics? We have already argued that the posterior distributions for $\Omega_K$ should not be interpreted as probability distributions because of their sensitivity to priors. Likewise, evidence ratios can give misleading results because of sensitivity to priors (Efstathiou 2008). Since the models are nested, we can answer this question in a definitive and particularly simple way, independent of priors, by looking at differences in $\chi^2$ values, i.e. likelihood ratios\(^6\). Table 3 lists values of $\chi^2 = -2\ln L$ for the best fit cosmology for the base $\Lambda$CDM cosmology with $\Omega_K = 0$ and for the best fit when $\Omega_K$ is allowed to vary as an additional parameter. We have decomposed the likelihood into the various components: the Commander temperature likelihood at $\ell < 30$ (denoted ‘lowl’), the SimAll polarization likelihood at $\ell < 30$ (denoted ‘lowE’) and the CamSpec likelihood at $\ell \geq 30$.

The overall reduction in $\chi^2$ is about 6, split roughly equally between the lowl likelihood and CamSpec. (The lowE likelihood is neutral to the addition of $\Omega_K$.) Adding $\Omega_K$ as a parameter reduces the CamSpec $\chi^2$ values by 2.13 (TT) and 3.02 (TTTEEE). These are very modest changes and are consistent with the conclusion of EG that the base $\Lambda$CDM model provides essentially a perfect fit to the Planck power spectra at $\ell \geq 30$ as judged by $\chi^2$ statistics. The improvement in the fits to the low multipole likelihood is a consequence of the low amplitudes of the low multipoles (including the quadrupole) relative to the predictions of the base $\Lambda$CDM model noted in previous Planck papers (see e.g. Planck Collaboration et al. 2014). There is, however, an additional subtlety involved in interpreting the low multipoles. In CAMB the power spectrum in non-flat models is written as

\[
P(k) = \frac{(q^2 - 4K)^2}{q^2 - K} k^{(n_s - 1)},
\]

where $q = \sqrt{k^2 + K}$, which is a highly specific assumption on how the fluctuation spectrum extends to scales greater than the curvature scale. This form leads to a suppression of the low multipoles in closed models (see Efstathiou 2003) and can compensate with other parameters to reduce the $\chi^2$ of the lowl likelihood. Since eq (7) is not based on any specific theory, we should not assign much weight to the reduction in $\chi^2$ in the lowl likelihood. A more reasonable statistical approach would be to add additional parameters to describe the fluctuations on scales greater than the curvature scale. If we exclude the lowl likelihood entirely, the best fits to CamSpec+lowE are shifted slightly towards spatially flat universes with minimum $\chi^2$ values as listed in the last two lines of Table 3. The overall shifts in $\chi^2$ in CamSpec are small and very similar to those when lowl is included. The fits to the high multipole data from Planck therefore are barely improved if curvature is added as an additional parameter to base $\Lambda$CDM.

Since many researchers are more comfortable with ptes than likelihood ratios, we can translate as follows (Wilks 1938). Assume that $\Delta \chi^2_{min}$ is drawn from a $\chi^2$ distribution with 1 degree of freedom, the total change in $\chi^2$ from low and high multipoles suggests that the Planck data, excluding Planck lensing, favour $\Omega_K < 0$ with a pte of about 1.6% (or a Gaussian 2.1$\sigma$). However, note the caveat above concerning low multipoles in temperature. If we exclude the TT spectrum at $\ell < 30$, the pte rises to about 7% (or a Gaussian 1.6$\sigma$). These numbers are very different from the pte of 0.034% (3.4$\sigma$) quoted by Di Valentino et al. (2019).

\[\text{Table 1. } \chi^2 \text{ values for best fit cosmologies with and without curvature.}\]

| Likelihood         | base $\Omega_K$ | $\Delta \chi^2_{min}$ |
|-------------------|------------------|-----------------------|
| (fits with lowl)   | $\chi^2_{min}$  | $\Delta \chi^2_{min}$ |
| lowl TT ($\ell < 30$) | 23.73            | -2.60                  |
| lowl E ($\ell < 30$) | 396.35           | -0.78                  |
| CamSpec ($\ell \geq 30$) | 5491.42         | -2.13                  |
| (fits without lowl)  | $\chi^2_{min}$  | $\Delta \chi^2_{min}$ |
| lowl E ($\ell < 30$) | 395.57           | +0.08                  |
| CamSpec ($\ell \geq 30$) | 5491.02         | -1.82                  |

\[\text{6 Other statistical measures for model selection are discussed by e.g. Liddle (2004, 2007); Handley & Lemos (2019).}\]

4 CONCLUSIONS

The geometry of the Universe is a question of fundamental importance to cosmology. We have argued that the claims in Di Valentino et al. (2019) that Planck data strongly favour closed Universes at high significance are a consequence of using the Plik TTTEEE likelihood which differs from the CamSpec likelihood and ignoring the importance of priors. There is no good reason to assume a uniform prior on $\Omega_K$ and so the posterior for $\Omega_K$ and ptes derived from it should not be over-interpreted. We have presented results from a
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new Planck likelihood that shows a weak and statistically insignificant pull towards closed universes. This tendency is overwhelmed when the Planck likelihood is combined with other types of data that break the geometrical degeneracy. Combining Planck power spectrum measurements with any one of Planck CMB lensing, Type Ia supernovae or BAO data, favours a spatially flat universe. The strongest constraint (equ. 6e) shows that the Universe is spatially flat to a precision of $\sim 0.0018$, in agreement with the results in PCP18. This is a profound result for inflationary cosmology. If inflation is indeed the solution to the flatness problem, the observations show that the Universe must have firmly locked on to the $\Omega_K = 0$ attractor. Models of incomplete inflation, with e-foldings $N \sim N_*$, are disfavoured by observations.

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