A Sharp Bound for the Growth of Minimal Graphs

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Abstract
We consider minimal graphs \( u = u(x, y) > 0 \) over unbounded domains \( D \subset \mathbb{R}^2 \) bounded by a Jordan arc \( \gamma \) on which \( u = 0 \). We prove a sort of reverse Phragmén-Lindelöf theorem by showing that if \( D \) contains a sector

\[
S_\lambda = \{(r, \theta) = (-\lambda/2 < \theta < \lambda/2), \quad \pi < \lambda \leq 2\pi,\]

then the rate of growth is at most \( r^{\pi/\lambda} \).

Keywords Minimal surface · Harmonic mapping · Asymptotics

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1 Introduction

Let \( D \) be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

\[
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0
\]

(1.1)

with

\[
u = 0 \quad \text{on } \partial D \quad \text{and} \quad u > 0 \quad \text{in } D.
\]

(1.2)
We shall study the constraints on growth of non-trivial solutions to (1.1) and (1.2) as determined by the maximum

\[ M(r) = \sup u(x, y), \]

where the sup is taken over the values \( r = \sqrt{x^2 + y^2} \) and \( (x, y) \in D \).

The methods of this paper extend the results of [6], where the following is proved.

**Theorem A** Suppose \( D \) is a simply connected domain whose boundary is a Jordan arc, and \( D \) contains a sector \( S_\lambda = \{ z : |\arg z| < \lambda / 2 \} \), with \( \pi < \lambda \leq 2\pi \). With \( M(r) \) defined as above, if \( u \) satisfies (1.1) and (1.2) in \( D \), then there exist positive constants \( K \) and \( R \) such that

\[ M(r) \leq Kr, \quad r > R. \] (1.3)

As in Theorem A above, throughout this paper we shall use complex notation for convenience.

Results regarding upper and lower bounds for the growth of solutions to (1.1) and (1.2) are rather scarce and fragmented. To begin with, the first relevant theorem in this direction was proved by Nitsche [8, p. 256] who observed that there are no solutions to (1.1) and (1.2) with \( D \) being contained in a sector of opening strictly less than \( \pi \). For domains contained in a half plane, but not contained in any such sector, there are solutions to (1.1) and (1.2) with differing growth rates given in [6].

For angles \( \lambda \geq \pi \), in terms of the order \( \rho \) of \( u \) defined by

\[ \rho = \lim_{r \to \infty} \sup \frac{\log M(r)}{\log r}, \]

it follows by using the module estimates of Miklyukov [7] as in [11] that if \( D \) omits a sector of opening \( 2\pi - \alpha, \pi \leq \alpha \leq 2\pi \), the omitted set in the case \( \alpha = 2\pi \) being a line, then the order \( \rho \) of any non-trivial solution to (1.1) and (1.2) is at least \( \pi / \alpha \). More precisely, the results in [11] are phrased in terms of the asymptotic angle \( \beta \) defined as follows.

Let \( \Theta(r) \) be the angular measure of the set \( D \cap \{|z| = r\} \) if \( D \cap \{|z| = r\} \neq \emptyset \), and \( +\infty \) otherwise. Let

\[ \beta = \lim_{r \to \infty} \sup \Theta(r). \]

With this definition, the lower bound is given by

**Theorem B** Let \( D \) be an unbounded domain whose boundary \( \partial D \) is a piecewise differentiable arc, and suppose that \( u \) satisfies (1.1) and (1.2). If \( \beta \geq \pi \), then \( \rho \geq \pi / \beta \).

Regarding upper bounds, it was conjectured [12] that solutions to (1.1) and (1.2) in general have at most exponential growth, and this is achieved by the horizontal catenoid. In [12] the following is proved.
Theorem C If \( u \) satisfies (1.1) and (1.2) in a domain \( D \) contained in a half plane and bounded by an unbounded Jordan arc, then
\[
Cr \leq M(r) \leq e^{Cr} \quad r > r_0, 
\]
for some positive constants \( C \) and \( r_0 \).

The main result of this paper is the following bound for the order \( \rho \) of solutions when \( D \) contains a large sector.

**Theorem 1.1** Let \( D \) be a simply connected domain whose boundary is a Jordan arc, and \( D \) contains a sector \( S_\lambda = \{ z : |\arg z| < \lambda / 2 \} \), with \( \pi < \lambda \leq 2\pi \). If \( u \) satisfies (1.1) and (1.2) in \( D \), then \( \rho \leq \pi / \lambda \).

The examples given in [6] show that the theorem is sharp. Further details regarding those prototypes can be found in [13].

Note that Theorem B and Theorem 1.1 taken together imply that if \( D \) is bounded by a piecewise differentiable arc and is asymptotic to a sector \( S_\lambda \) with \( \lambda > \pi \), then the order of \( u \) will in fact be equal to \( \pi / \lambda \).

## 2 Preliminaries

Let \( u(z) \) be a solution to (1.1) and (1.2) over a simply connected domain \( D \). We shall make use of the parametrization of the surface given by \( u \) in isothermal coordinates using Weierstrass functions \((x(\zeta), y(\zeta), U(\zeta))\) with \( \zeta \) in the right half plane \( H \) and \( U(\zeta) = u(f(\zeta)) \), where
\[
z = f(\zeta) = x(\zeta) + iy(\zeta), \quad \zeta \in H. \tag{2.1}
\]

Then \( f(\zeta) \) is univalent and harmonic, and since \( H \) is simply connected it can be written in the form
\[
f(\zeta) = h(\zeta) + \overline{g(\zeta)} \tag{2.2}
\]
where \( h(\zeta) \) and \( g(\zeta) \) are analytic in \( H \),
\[
| h'(\zeta) | > | g'(\zeta) |,
\]
and
\[
U(\zeta) = 2 \operatorname{Re} \left( i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta \right). \tag{2.3}
\]
(cf. [3]).

Now, \( U(\zeta) \) is harmonic and in (2.3) can be taken as positive in \( H \) and vanishing on \( \partial H \). Thus (cf. [10, p. 151]),
\[
U(\zeta) = C \, \operatorname{Re} \zeta,
\]
for some positive constant \( C \).
where $C$ is a positive constant. This with (2.3) gives

$$h'(\xi)g'(\xi) = -\frac{C^2}{4}.$$  

We may reparametrize for convenience, replacing $f(\xi)$ and $U(\xi)$ by $f(2\xi/C)$ and $U(2\xi/C)$. Continuing to use $\xi$ as the preferred variable, this means we may assume that

$$h'(\xi)g'(\xi) = -1 \quad \text{and} \quad U(\xi) = 2 \Re \xi.$$  

In order to estimate the function $f(\xi)$ in (2.2), we shall use the following lemma on quasiconformal mappings from [2] (see [1, Lem. 5.8]).

**Lemma A** Suppose $\varphi$ is quasiconformal in the plane such that $\varphi(\infty) = \infty$, and the dilatation

$$\mu(z) = \frac{\varphi(z)}{\varphi'(z)}$$

satisfies

$$\int_0^{2\pi} \left| \mu(re^{i\theta}) \right| d\theta \to 0 \quad \text{as} \quad r \to \infty.$$  

Then, in any fixed annulus $A(R) = \{ R^{-1} \leq |z| \leq R \}, \ R > 1,

$$\frac{\varphi(tz)}{\varphi(t)} \to z$$

uniformly in $A(R)$ as $0 < t \to \infty$. In particular,

$$|\varphi(z)| = |z|^{(1+o(1))} \quad \text{as} \quad z \to \infty.$$  

**Remark** For our later applications of Lemma A, note that if $r > 0$ and $a$ and $b$ are in $(0, 2\pi)$, then

$$\frac{\varphi(tre^{ia})}{\varphi(tre^{ib})} = \frac{\varphi(tre^{ia})}{\varphi(t)} \frac{\varphi(t)}{\varphi(tre^{ib})} \to \frac{re^{ia}}{re^{ib}} = e^{i(a-b)}$$

as $t \to \infty$, uniformly in $A(R)$.

At the last stage we shall need a barrier argument based on the following [4, p. 827].

**Lemma B** Let $u(z)$ be a solution to the minimal surface equation over a domain $\Omega$ of the form $S_\lambda \setminus E$ ($\lambda < \pi$) with $u(z) = 0$ on $\partial E$ and $u(z) \leq ax^m + b$, $0 < m < 1$, $a, b \geq 0$, for $z = x + iy \in \partial S_\lambda$. Then $u(z) \leq ax^m + b$ in $\Omega$.

**Proof** Let $T_1 = S_\lambda \cap \{ z : \Re z < 1 \}$ and $A > 0$. Then, there exists [5, p. 322] a solution $V_{1,A}(z)$ to the minimal surface equation over $T_1$ with values $A\lambda$ on
where \( A \) and let \( \zeta \) and subsets of \( S \).

**Lemma 2.1** Let \( u \) be as in the statement of the lemma and take a fixed \( x_0 > 0 \). Then, for \( z \in \partial S_\lambda \),

\[
u(z) \leq a \left( x_0^m + mx_0^{m-1}(x - x_0) \right) + b. \tag{2.6}\]

Since \( u(z) = 0 \) on \( \partial E \), it follows that

\[
u(z) \leq V_{R,A}(z) + B, \quad z \in \Omega \cap \{ \text{Re } z < R \}, \tag{2.7}\]

where \( A = amx_0^{m-1} \), and \( B = ax_0^m(1 - m) + b \). Letting \( R \to \infty \) in \( (2.7) \), it then follows that \( (2.6) \) holds in \( \Omega \). Thus for any \( z = x_0 + iy \) in \( \Omega \) we have

\[
u(x_0 + iy) \leq ax_0^m + b,
\]

and since \( x_0 \) was arbitrary, the lemma is proved. \( \square \)

As a final preliminary lemma, we need the following qualitative growth estimate.

**Lemma 2.1** Let \( u(z) \) be a solution to \( (1.1) \) and \( (1.2) \) over a domain \( D \) containing a sector \( S_\lambda \) with \( \lambda > \pi \), and \( f(\xi), h(\xi), g(\xi), \) and \( U(\xi) \) be as in \( (2.2) \) and \( (2.3) \) corresponding to \( u(z) \). Then, for any proper subsector \( S_{\lambda'} \) with \( \pi < \lambda' < \lambda \) and \( D_{\lambda'} = f^{-1}(S_{\lambda'}) \),

\[
h'(\xi) \to \infty \text{ as } \xi \to \infty
\]

uniformly for \( \xi \in D_{\lambda'} \).

**Proof** Let \( f(\xi), U(\xi) \) be as above. So, \( u(f(\xi)) = U(\xi) = 2 \text{Re } \xi \).

Let \( P_\alpha = \{ \xi : \text{Re } e^{i\alpha} f(\xi) > 0 \}, |\alpha| < \lambda/2 - \pi/2 \), and introduce a new variable \( \tilde{\xi} \), and let \( \xi = \psi_0(\tilde{\xi}) \) be a conformal map from the right half plane \( H = \{ \xi : \text{Re } z > 0 \} \) onto \( P_0 \) with \( \psi_0(\infty) = \infty \).

Define

\[
\begin{align*}
\tilde{f}(\tilde{\xi}) &= f(\psi_0(\tilde{\xi})) \\
\tilde{g}(\tilde{\xi}) &= g(\psi_0(\tilde{\xi})) \\
\tilde{h}(\tilde{\xi}) &= h(\psi_0(\tilde{\xi}))
\end{align*}
\]

Then \( \tilde{f} \) is a harmonic map

\[
\tilde{f}(\tilde{\xi}) = \tilde{h}(\tilde{\xi}) + \overline{\tilde{g}(\tilde{\xi})}, \quad \tilde{\xi} \in H,
\]
satisfying
\[ |\tilde{h}'(\tilde{\zeta})| > |\tilde{g}'(\tilde{\zeta})|, \quad \tilde{\zeta} \in H. \quad (2.8) \]

Note \( \tilde{F}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta}) \) is an analytic function with the same real part as \( \tilde{f} \). Then \( \text{Re} \ F \) is positive in \( H \) and vanishes on \( \partial H \), and therefore (see \cite[p. 151]{10})
\[ \tilde{F}(\tilde{\zeta}) = k\tilde{\zeta} + ik_0 \implies \tilde{F}'(\tilde{\zeta}) = k, \quad k > 0, \]
that is,
\[ \tilde{h}'(\tilde{\zeta}) + \tilde{g}'(\tilde{\zeta}) = k > 0. \quad (2.9) \]

Now,
\[ \tilde{h}'(\tilde{\zeta}) = h'(\psi_0(\tilde{\zeta})) \cdot \psi_0'(\tilde{\zeta}), \quad (2.10) \]
and by (2.4),
\[ \tilde{g}'(\tilde{\zeta}) = -\frac{\psi_0'(\tilde{\zeta})}{h'(\psi_0(\tilde{\zeta}))} = -\frac{\psi_0'(\tilde{\zeta})^2}{h'(\tilde{\zeta})}. \quad (2.11) \]

Combining this with (2.9) we have
\[ k = \frac{\psi_0'(\tilde{\zeta})^2}{h'(\tilde{\zeta})} \]
which implies
\[ \tilde{h}'(\tilde{\zeta})^2 - k\tilde{h}'(\tilde{\zeta}) - \psi_0'(\tilde{\zeta})^2 = 0. \]

Thus, with the principal branch of the square root,
\[ \tilde{h}'(\tilde{\zeta}) = \frac{k \pm \sqrt{k^2 + 4\psi_0'(\tilde{\zeta})^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) = \frac{-2\psi_0'(\tilde{\zeta})^2}{k \pm \sqrt{k^2 + 4\psi_0'(\tilde{\zeta})^2}}. \quad (2.12) \]

Since \( \psi_0(\tilde{\zeta}) \) is a conformal map with \( \text{Re} \ \psi_0(\tilde{\zeta}) > 0 \) in \( H \), there exists a real constant \( 0 \leq c < \infty \) such that in any sector \( S_\beta = \{ \tilde{\zeta} : |\arg \tilde{\zeta}| < \beta \} \) with \( 0 < \beta < \pi/2 \), \( \psi_0'(\tilde{\zeta}) \to c \) uniformly as \( \tilde{\zeta} \to \infty \) in \( S_\beta \) (see \cite[p. 152]{10}). Thus,
\[ \tilde{h}'(\tilde{\zeta}) \to \frac{k \pm \sqrt{k^2 + 4c^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{k \pm \sqrt{k^2 + 4c^2}}, \quad \text{as} \ \frac{\tilde{\zeta}}{c} \to 1. \quad (2.13) \]

If \( c > 0 \), a simple calculation with (2.13) shows that if the minus sign in (2.13) were to hold, this would contradict (2.8). In case \( c = 0 \), with a minus sign in (2.13), this would imply that \( \tilde{h}'(\tilde{\zeta}) \to 0 \). However, (2.8) and (2.9) show that this is not possible. Thus, (2.13) becomes
\[ \tilde{h}'(\tilde{\zeta}) \to \frac{k + \sqrt{k^2 + 4c^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{k + \sqrt{k^2 + 4c^2}}, \quad \text{as} \ \frac{\tilde{\zeta}}{c} \to 1. \quad (2.14) \]
**Case 1:** $\psi_0'(\tilde{\zeta}) \to c > 0$ as $\tilde{\zeta} \to \infty$ in $S_\beta$.

Using (2.14) we have

$$\tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta}) = \left[ k \text{Re} \, \zeta + i \sqrt{k^2 + 4c^2} \, \text{Im} \, \tilde{\zeta} \right] (1 + o(1))$$

(2.15)

for $\tilde{\zeta} \to \infty$ uniformly in $S_\beta$. From this it follows that

$$f(\zeta) = \left[ k \frac{\text{Re} \, \zeta}{c} + i \sqrt{k^2 + 4c^2} \, \frac{\text{Im} \, \zeta}{c} \right] (1 + o(1))$$

(2.16)

uniformly as $\zeta \to \infty$ in proper subsectors of $H$. Therefore, $P_0$ is asymptotically the half plane $H$, in the sense that for any $0 < \delta < \pi$, we have $P_0 \supset S_\delta \cap \{ |\zeta| > R \}$ for large $R$.

By (2.1) and (2.4), the graph of the minimal surface is given parametrically by

$$(\text{Re} \, f(\zeta), \text{Im} \, f(\zeta), 2 \text{Re} \, \zeta).$$

Using (2.16) we then have that the surface is asymptotic to a plane as $\zeta \to \infty$ in proper subsectors of $H$.

Consider now $P_\alpha$ as above with $\alpha \neq 0$ and let $\psi_\alpha(\tilde{\zeta})$ be a conformal map from the right half plane $H = \{ \tilde{\zeta} : \text{Re} \, \tilde{\zeta} > 0 \}$ onto $P_\alpha$ with $\psi_\alpha(\infty) = \infty$. In this case we define

$$\begin{cases} \tilde{f}_\alpha(\tilde{\zeta}) = e^{i\alpha} f(\psi_\alpha(\tilde{\zeta})) \\ \tilde{g}_\alpha(\tilde{\zeta}) = e^{-i\alpha} g(\psi_\alpha(\tilde{\zeta})) \\ \tilde{h}_\alpha(\tilde{\zeta}) = e^{i\alpha} h(\psi_\alpha(\tilde{\zeta})) \end{cases}$$

(2.17)

Proceeding in analogy with (2.9)–(2.16), we have

$$\tilde{h}_\alpha'(\tilde{\zeta})^2 - k_\alpha \tilde{h}_\alpha'(\tilde{\zeta}) - \psi_\alpha'(\tilde{\zeta})^2 = 0$$

(2.18)

and with the principal branch of the square root,

$$\tilde{h}_\alpha'(\tilde{\zeta}) = \frac{k_\alpha + \sqrt{k_\alpha^2 + 4\psi_\alpha'(\tilde{\zeta})^2}}{2}, \quad \tilde{g}_\alpha'(\tilde{\zeta}) = \frac{-2\psi_\alpha'(\tilde{\zeta})^2}{k_\alpha + \sqrt{k_\alpha^2 + 4\psi_\alpha'(\tilde{\zeta})^2}}$$

(2.19)

in $H$, where the minus sign in the roots of (2.18) is eliminated as before.

Again, $\psi_\alpha'(\tilde{\zeta}) \to c_\alpha \geq 0$ as $\tilde{\zeta} \to \infty \in S_\beta$. We wish to show that when $c_0 > 0$ then $c_\alpha > 0$.

Suppose that $c_\alpha = 0$. We reflect $\tilde{f}_\alpha$ to the left half $\tilde{\zeta}$ plane and note that the $\beta$ corresponding to the sectors $S_\beta$ can approach $\pi/2$. It then follows from (2.19) that Lemma A applies to $\tilde{f}_\alpha(\tilde{\zeta})$. To apply Lemma A, we note that the dilatations in $S_\beta$ tend to 0, and since $\tilde{f}_\alpha$ is sense preserving, the dilatations are less than 1 outside. So (2.5) applies.

Then for $\epsilon > 0$ (see Remark following Lemma A), the image $\tilde{f}_\alpha(S_\beta)$ covers $S_{\pi - \epsilon} \cap \{ |z| > R \}$ if $\beta$ is sufficiently close to $\pi/2$ and $R$ sufficiently large. From this and (2.15) it follows that if $Q = Q_{\alpha,\beta} = \tilde{f}(S_\beta) \cap e^{-i\alpha}\tilde{f}_\alpha(S_\beta)$, then for $\beta$ close to $\pi/2$ and all large $R$, the intersection $Q \cap \{ |z| = R \}$ is non-empty.
From the original analysis of $P_0$, we find that for points $\zeta \in f^{-1}(Q)$, (2.10) and (2.14) imply that $h'(\zeta)$ remains bounded.

On the other hand, from the analysis of $P_{\alpha}$, it follows from (2.17) and (2.19) that $g'(\zeta)/h'(\zeta) \to 0$ as $\zeta \to \infty$ and $\zeta \in f^{-1}(Q)$. This with (2.4) implies that $h'(\zeta)$ is unbounded, a contradiction.

Therefore, it must be that $c_{\alpha} > 0$ also, and as in the case of $P_0$ above, the graph above $f(P_{\alpha})$ must be asymptotically a plane. Since $f(P_0)$ and $f(P_{\alpha})$ overlap, these graphs must be asymptotically the same plane, and since $f(P_0) \cup f(P_{\alpha})$ extends outside a half plane with $u(z) > 0$, we obtain a contradiction. We conclude that Case 1 cannot occur.

**Case 2:** $\psi'(\tilde{\zeta}) \to 0$ as $\tilde{\zeta} \to \infty$ in $S_{\beta}$.

As in Case 1 above, Lemma A can be used to show that for $\epsilon > 0$, the image $\tilde{f}(S_{\beta})$ covers $S_{\pi - \epsilon} \cap \{ z : |z| > R \}$ for large $R$ if $\beta$ is sufficiently close to $\pi/2$. Using an argument similar to Case 1, we can then deduce that the $c_{\alpha}$ corresponding to each $P_{\alpha}$ ($|\alpha| < \lambda/2 - \pi/2$) must also be 0. Since the part of $S_{\lambda'}$ of large modulus and $\pi < \lambda' < \lambda$, can be covered by $f_{\alpha}(S_{\beta}) \cup f(S_{\beta}) \cup f_{-\alpha}(S_{\beta})$

for some $0 < \alpha < \lambda/2 - \pi/2$ and large $R$, it follows from (2.17), and (2.19) that $g'(\zeta)/h'(\zeta) \to 0$. This with (2.4) implies that $h'(\zeta) \to \infty$ uniformly as $\zeta \to \infty$ with $\zeta \in f^{-1}(S_{\lambda'})$. 

\[3 \text{ Proof of Theorem 1.1} \]

**Proof** For fixed $\lambda$, let $f_1(\zeta)$ denote the function in (2.1) corresponding to a solution to (1.1) and (1.2) over a domain $D$ containing $S_{\lambda}$. Then for $\lambda'$ such that $\pi < \lambda' < \lambda$ we define $f_2(\zeta) = \zeta^{\lambda'/\pi} + 1$. Let $\tilde{S}_{\lambda'} = f_2(H)$ and $\tilde{H} = f_1^{-1}(\tilde{S}_{\lambda'})$. Then if $\psi(\zeta)$ is a $1 - 1$ conformal mapping of $H$ onto $\tilde{H}$ with $\psi(\infty) = \infty$, it follows that $f_1(\psi(H)) = f_2(H)$ and there exists an orientation preserving homeomorphism $\varphi : H \to H$ with $\varphi(\infty) = \infty$ such that

\[ f_1(\psi(\zeta)) = f_2(\varphi(\zeta)), \quad \zeta \in H. \quad (3.1) \]

Differentiating (3.1) with respect to $\zeta$ and $\tilde{\zeta}$, and using the first equality in (2.4) we obtain

\[ \psi'(\zeta) h_1'(\psi(\zeta)) = \varphi_\zeta(\zeta) f_2'(\varphi(\zeta)) \quad (3.2) \]

and

\[ -\frac{\psi'(\zeta)}{h_1'(\psi(\zeta))} = \varphi_{\tilde{\zeta}}(\zeta) f_2'(\varphi(\zeta)). \quad (3.3) \]

Dividing (3.3) by (3.2) we have

\[ \frac{1}{|h_1'(\psi(\zeta))|^2} = \left| \frac{\varphi_{\tilde{\zeta}}(\zeta)}{\varphi_\zeta(\zeta)} \right|. \quad (3.4) \]

Now, $\psi(\zeta) \to \infty$ as $\zeta \to \infty$ in $H$, so by Lemma 2.1 it follows that the left side of (3.4) tends to 0. It therefore follows from (3.4) and the fact that $\varphi$ is a sense preserving
differentiable homeomorphism, that \( \varphi \) is quasiconformal in \( H \) and that the dilatation of \( \varphi \) satisfies

\[
\left| \frac{\varphi_\zeta'(\zeta)}{\varphi_\zeta(\zeta)} \right| \to 0 \quad (3.5)
\]

uniformly as \( \zeta \to \infty \) in proper subsectors of \( H \).

The mapping \( \varphi \) can then be extended by reflection to a quasiconformal mapping of the complex plane onto the complex plane with (3.5) still in force. As in the proof of Lemma 2.1, Lemma A is applicable to \( \varphi \). Further, by the symmetry of the reflection, the conclusion of Lemma A can be improved to

\[
\varphi \left( r e^{i\theta} \right) = r^{(1+o(1))} e^{i(\theta+o(1))}
\]

so that

\[
f_1 \left( \psi \left( r e^{i\theta} \right) \right) = f_2 \left( \varphi \left( r e^{i\theta} \right) \right) = r^{(\lambda'/\pi+o(1))} e^{i(\lambda'/\pi+o(1))}, \quad \text{as } \zeta = r e^{i\theta} \to \infty, \quad \zeta \in H.
\]

From this we see that, given any \( \lambda'' \) such that \( \pi < \lambda'' < \lambda' \), there is a proper sector \( \Sigma_{\lambda''} \) in \( H \) such that \( f_1(\psi(\Sigma_{\lambda''})) \) covers \( S_{\lambda''} \cap \{|z| > R\} \) for large \( R \). But \( \psi(\zeta) \) is a conformal mapping of \( H \) into \( H \), so \( \psi'(\zeta) \to k \) as \( \zeta \to \infty \) in \( \Sigma_{\lambda''} \) for some \( k \geq 0 \) (cf. [10, p. 151]). Combining this with (2.4) we conclude that for sufficiently large \( z \),

\[
u(z) < |z|^{(\pi/\lambda'+o(1))}, \quad z \in S_{\lambda''}. \quad (3.6)
\]

The boundary of the sector \( S_{\lambda''} \) on which (3.6) holds forms an angle in the left half plane of opening less than \( \pi \). On the boundary of \( D \) in the left half plane \( u(z) = 0 \). Therefore, Lemma B applied to the sector of opening \( 2\pi - \lambda'' \) centered on the negative real axis, with \( E \) being the complement of \( D \), tells us that (3.6) holds in \( D \setminus S_{\lambda''} \). Thus we see that (3.6) holds on all of \( D \). Since \( \lambda' \) can be taken arbitrarily close to \( \lambda \) in (3.6), the proof is complete. \( \square \)

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