Probability Density Function Analysis of SNR for Time-Reversed Transmission System

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Abstract. The time reversal (TR) transmission technique can harvest energy from the surrounding environment by exploiting multipath, and focus the signal energy which would have otherwise been lost in most existing communication modes on the desired user with a relatively low system complexity. In this paper, we derived the probability density function (PDF) of the received signal-to-noise ratio (SNR) in a single antenna communication system. Simulation results show the PDF curve is correct.

1 Introduction
In the transmission process, radio waves will be reflected, scattered and diffracted due to environmental factors such as buildings, trees, and topography fluctuations. The transmitter sends a signal to the receiver along different paths with different delays so that the received signal is distorted in amplitude and phase, causing multipath fading and inter-symbol interference (ISI). Although multipath transmission causes signal fading, it can also be used. Time Reversal (TR) is a technique that utilizes multipath propagation. TR, combined with ultra-wideband processing technology, can gather signal energy of multi-path at the desired destinations with lower complexity processing [1, 2]. When the bandwidth is large enough, by performing TR preprocessing on the signal, a large number of multi-path naturally existing in the scattering environment can be utilized to form virtual transmission antennas, which can achieve similar transmission performance of multiple input single output (MISO) [3]. The study analyzed the gains of the average received signal energy and the average interference power of the TR transmission system compared with the direct transmission system and conducted a comparative analysis with the Rake receiver in [4]. The expression of the average signal-to-interference-plus-noise ratio (SINR) derived in TR multi-user downlink system, and the author of [5] simulated the achievable sum rate and outage probability, but, he did not give theoretical analysis results. The paper of [6] analyzes the average SNR of intended and unintended users in a distributed TR transmission scheme. Compared with the direct transmission, line of sight (LOS), distributed beamforming (BF), and maximum gain path BF, the SNR performance of the transmission scheme based on distributed TR is effectively improved.

Due to the complex composition of the received signal at the receiver in the TR system, the analysis of the performance is also complicated. Existing works mainly analyzes the average (mathematical expectation) of performance indicators (such as SNR) As far as we know, there is no literature to analyze the probability distribution of received signal power and received SNR in the TR system. This paper deduces the probability distribution of received signal power and received SNR in TR system on Rayleigh fading channel.
2 System Model

This article studies single-antenna TR systems. In this paper, we assume that the channel noise is additive white Gaussian noise (AWGN). The channel is a fading wireless multipath channel, and the paths are independent of each other. The channel impulse response (CIR) in the discrete time domain can be expressed as

\[ h[n] = \sum_{l=0}^{L-1} h_l \delta[n-l] \]  

(1)

where \( L \) is the number of channel paths, and \( h_l \) is the fading coefficient of the \( l \)-th path of the channel, namely the coefficient of the \( l \)-th tap of CIR. In the Rayleigh channel, the channel coefficient \( h_l \) is a circular symmetric complex Gaussian random variable with zero mean and variance

\[ \mathbb{E}[h_l^2] = \eta e^{-\frac{\nu_l}{\sigma_l^2}} = \sigma_l^2, \quad 0 \leq l \leq L-1 \]  

(2)

where \( T_s \) is the sampling period, and \( \sigma_l \) is the delay spread of the channel. \( e^{-\frac{\nu_l}{\sigma_l^2}} \) is the small-scale fading coefficient of the path, and \( \eta \) is the large-scale fading coefficient. When the time reversal technology is applied, the receiver firstly sends a pilot signal to the transmitting end. Then the transmitting end estimates the impulse response of the channel according to the pilot signal, and use the normalized time-reversed conjugate signals as the impulse response of the pre-filter filter, i.e.,

\[ g[n] = h^*[L-1-n]/\sqrt{\sum_{l=0}^{L-1} |h_l|^2} \]  

(3)

where superscript * represents complex conjugate operation. The preprocessing filter is used as part of the equivalent channel in this system, and the equivalent CIR is

\[ h_{eq}[n] = (g * h)[n] = \sum_{l=0}^{L-1} h_l \cdot g_{l+n}, \quad n = 0, 1, L, 2L-1 \]  

(4)

Denote the sending symbol as \( x[n] \). Let the power of the transmitted signal be \( P_x \). \( x[n] \) passes through the pre-filter and the multipath channel, and the received signal at the receiver can be expressed as

\[ y[n] = x * g * h[n] + z[n] = \sum_{i=0}^{2L-1} h_{eq}[i] x[n-i] + z[n] \]  

\[ = h_{eq}[L-1] x[n-L-1] + \sum_{i=L-1}^{2L-2} h_{eq}[i] x[n-i] + z[n] \]  

(5)

where \( z[n] \) is AWGN with zero mean and variance. The first part in the right of the last equal of (5) is the signal, the second part is ISI, and the last part is noise.

3 PDF Analysis of SNR

3.1 PDF of the Signal Power

According to Equation (5), the power of the received signal can be obtained as

\[ S = \left| h_{eq}[L-1] x[n-L-1] \right|^2 = P_x \left( \sum_{i=0}^{2L-1} |h_i|^2 \right) \]  

(6)
The PDF of signal power is deduced first. Note that the random variable \( |h_l|^2 \) is \( U_l, l = 0, 1, L - 1 \), the signal power can be rewritten as \( S = P_X \sum_{l=0}^{L-1} U_l. \) Since the real part and the imaginary part of \( h_l \) are independent and identically distributed Gaussian random variables, according to the literature [7], \( U_l \) obeys the exponential distribution, whose PDF can be expressed as

\[
f_{U_l}(u_l) = \frac{1}{\sigma_l^2} \exp\left(-\frac{u_l}{\sigma_l^2}\right)
\]

(7)

Since \( h_l (l = 0, 1, L - 1) \) is independent of each other, \( U_l \) is independent of each other too. The characteristic function (CF) of \( U_l \) is shown in the following:

\[
\psi_{U_l}(\omega) = \int_{-\infty}^{\infty} e^{j\omega u_l} f_{U_l}(u_l) du_l = \frac{1}{1 - j\omega\sigma_l^2}
\]

(8)

Let \( V = \sum_{l=0}^{L-1} U_l. \) Because the CF of the sum of multiple independent random variables is the product of CF for each random variable [8], the CF of \( V \) is

\[
\psi_{V}(\omega) = \prod_{l=0}^{L-1} \psi_{U_l}(\omega) = \prod_{l=0}^{L-1} \frac{1}{1 - j\omega\sigma_l^2}
\]

(9)

In order to obtain the PDF of \( V \), we expand the right part of the equation (9) using partial fractional expansion method. Then we can get

\[
\psi_{V}(\omega) = \sum_{l=0}^{L-1} K_i \prod_{m \neq i} 1 - j\omega\sigma_m^2
\]

(10)

where \( K_i = \prod_{m \neq i} \frac{\sigma_m^2}{\sigma_i^2 - \sigma_m^2}. \) Furthermore, we can obtain the PDF of \( V \):

\[
f_{V}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_{V}(\omega) e^{-j\omega v} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \prod_{l=0}^{L-1} \frac{K_i}{1 - j\omega\sigma_l^2} \right) e^{-j\omega v} d\omega
\]

(11)

\[
= \sum_{l=0}^{L-1} \frac{K_i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 - j\omega\sigma_l^2} e^{-j\omega v} d\omega = \sum_{l=0}^{L-1} \frac{K_i}{\sigma_l^2} \exp\left(-\frac{v}{\sigma_l^2}\right)
\]

The PDF of signal power \( S = P_X V \) can be expressed as

\[
f_S(s) = f_V\left(\frac{s}{P_X}\right) \frac{1}{P_X} = \sum_{l=0}^{L-1} \frac{K_i}{P_X\sigma_l^2} \exp\left(-\frac{s}{P_X\sigma_l^2}\right)
\]

(12)

3.2 PDF of the SNR

It is assumed there is a perfect equalizer to eliminate any ISI. Hence, we ignore the ISI of the SINR and instead derive PDF and CDF of the SNR only. Received SNR can be expressed as

\[
\text{SNR} = \frac{S}{P_{\text{noise}}} = \gamma
\]

(13)

The PDF of \( S \) is derived from the PDF of \( V \). Similarly, the PDF of the received SNR can be deduced from the PDF of \( S \), which can be expressed as

\[
f_{\gamma}(\gamma) = f_S(P_{\text{noise}} \cdot \gamma) \cdot P_{\text{noise}} = \sum_{l=0}^{L-1} \frac{\sigma_l^2 K_i}{P_X\sigma_l^2} \exp\left(-\frac{\sigma_l^2 \gamma}{P_X\sigma_l^2}\right)
\]

(14)
4 Simulation Analysis

In the simulation, we assume the bandwidth is $B=50\text{MHz}$, the number of paths is $L=50$, and the sampling period $T_s=1/B$. Furthermore, the fading coefficient of each path follows a complex Gaussian distribution with a zero mean and a variance shown in equation (2). The large-scale fading coefficient of this system is set to $\eta = 10^{-9}$. What is more, the noise power of the channel is assumed to be $-120\text{dBW}$. We set the delay spread of the channel is $\sigma_\tau = 5/B$, the signal transmission power is $P_X=1\text{mW}$. During simulation, a total of $10^6$ channel samples were randomly generated. We first calculate the SNR values of each sample based on the expression of the signal power; then we count the ratio of the number of samples in each cell to the total number of samples in the interval $[0, 15]$ with a step of $\Delta=0.25$; finally, the resulting value is divided by the step size to get the simulation value of PDF of SNR, which is shown in the histogram of Fig 1. The theoretical value of PDF is calculated according to equation (14), which is also depicted in Fig 1. It can be seen that the theoretical value matches the simulation value.

![Figure 1](image1.png)

**Figure 1.** The PDF of the received SNR ($P_X=1\text{mW}$ and $\sigma_\tau = 5/B$).

When the transmission power is constant, we increase the delay spread of the channel. We count the values of SNR in the interval $[0, 20]$ with a step of $\Delta=0.5$. The PDF of SNR is plotted in Figure 2. It can be seen that the two curves are almost completely coincident, and the PDF curve of the TR system spreads to high SNR with the increase of the delay spread of the channel.

![Figure 2](image2.png)

**Figure 2.** The PDF of the received SNR ($P_X=1\text{mW}$ and $\sigma_\tau = 10/B$).
We consider the effect of the transmission power on the distribution of SNR. Set $P_x=5\text{mW}$. The delay spreads of the channel are $\sigma_r = 5/B$ and $\sigma_r = 10/B$. Fig 3 and 4 plot PDFs in different delay spread of the channel. As the transmit power increases, the PDF spreads at a high SNR.

![Figure 3. The PDF of the received SNR ($P_x=5\text{mW}$ and $\sigma_r = 5/B$).](image)

![Figure 4. The PDF of the received SNR ($P_x=5\text{mW}$ and $\sigma_r = 10/B$).](image)

5 Conclusion
This paper analyzes the performances of SNR of single-antenna TR systems in Rayleigh fading channels. We derive the distribution of received signal power and SNR in the system, using the corresponding relationship between the CF and the PDF, and get the expression of the PDF of the received SNR. Finally, we verify the correctness of the derivation through the simulation.

6 References
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