Non-local elasticity theory as a continuous limit of 3D networks of pointwise interacting masses
E. Khruslov, M. Goncharenko

Small oscillations of an elastic system of point masses (particles) with a nonlocal interaction are considered. We study the asymptotic behavior of the system, when number of particles tends to infinity, and the distances between them and the forces of interaction tends to zero. The first term of the asymptotic is described by the homogenized system of equations, which is a nonlocal model of oscillations of elastic medium.

Introduction
The progress in development of new materials and the modelling of nanostructures caused the emergence of nonlinear elasticity theories (see, for example, [1], [2], [3]). Classical local theory is based on the concept of contact interaction and it can not explain some observed experimental phenomena. Therefore, it is necessary to take into account the long-range interaction between the particles of the material and this leads to the nonlocal elasticity theory.

The nonlocal elasticity theory can be traced back to the works of Kröner, who formulated the continual theory of elastic materials with long-range interaction forces ([4], [5]). At present, the nonlocal mechanics of the elastic continuum is treated with two different approaches: the gradient elasticity theory (weak nonlocality) and the integral nonlocal theory (strong nonlocality).

The first approach is related to the study of the gradients of the strain tensors. It leads to models with spatial derivatives of order more than 2 ([6] - [8]). The main difficulties in using this model are the setting of boundary conditions for the corresponding boundary value problems (see [9]).

The second approach has been developed almost independently. The nonlocal interaction here is represented in the form of a convolution integral
of the deformation tensor with a kernel that depends on the distance between the particles of the elastic material. This approach leads to models described by integro-differential equations ([10] - [13]).

The correctness of these continuum models of nonlocal elasticity theory depends on the effectiveness of long-range molecular forces in the material. Therefore, a natural approach to their justification is the so-called microstructural approach, which is studying discrete elastic systems (lattice models). This approach has been used mainly in physical works ([14] - [18]). Apparently, one of the first mathematical works, in which the system of equations of the local elasticity theory was derived using the microstructural approach, was [19]. The short-range interactions between particles were considered. Only the nearest particles interact in the system. The asymptotic behavior of the oscillations of such a system was investigated when the distances between the nearest neighbors and the forces of interaction between them tend to zero. A homogenized system of differential equations describing the leading term of the asymptotic was obtained. This system is a continuum model of the local theory of elasticity. In this work the method based on the studying of the asymptotic behavior of the system, when the scale of the microstructure tends to zero, was applied. This approach is the basis for the homogenization of partial differential equations ([20] - [22]).

We apply this approach of homogenization to study the asymptotic behavior of the oscillations of an elastic system of point masses (particles) with a nonlocal interaction. It is assumed that the system depends on the small parameter $\varepsilon$. More precisely, the distance between the nearest neighbors is of the order $O(\varepsilon)$, and the long-range forces are of order $O(\varepsilon^6)$. It is proved that the main term of the asymptotic is described by a homogenized system of integro-differential equations. The integral term is a convolution of the difference of the displacements of the elastic medium at various points with some kernel. Note that such a system differs from the continual model of Eringen, where the convolution of the deformation tensor with the kernel is taken. A similar system of integro-differential equations was proposed earlier
(without justification) in [23] as a variant of the integral elasticity theory and was used to calculate steel plates. The indicated order of interactions in the system corresponds to the integral (and not gradient) elasticity theory.

1 Statement of the problem

We consider a system $M_\varepsilon$ of interacting point masses (we will call them particles) in a fixed bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial \Omega$. It is assumed that this system depends on the small parameter $\varepsilon > 0$. The total number of particles in the system is $O(\varepsilon^{-3})$ and the distances between the nearest particles are of order $O(\varepsilon)$. We denote by $x^i_\varepsilon$ $(i = 1, \ldots, N_\varepsilon)$ the positions of the particles in the equilibrium state of the system $M_\varepsilon$, and we denote by $u^i_\varepsilon = u^i_\varepsilon(t)$ the displacements of particles relative to their equilibrium positions $x^i_\varepsilon$.

The potential energy for small variation of the system $M_\varepsilon$ from the equilibrium position is determined by the equality

$$H_\varepsilon(u_\varepsilon) = H_0 + \frac{1}{2} \sum_{i,j=1}^{N_\varepsilon} \langle E^{ij}_\varepsilon(u^i_\varepsilon - u^j_\varepsilon), (u^i_\varepsilon - u^j_\varepsilon) \rangle, \quad H_0 = \text{const},$$

(1.1)

where $u_\varepsilon = \{u^1_\varepsilon, \ldots, u^{N_\varepsilon}_\varepsilon \}$, parentheses $\langle, \rangle$ denote the scalar product in $\mathbb{R}^3$, and $E^{ij}_\varepsilon$ are symmetric nonnegative matrices of the pair interaction between the $i$-th and $j$-th particles. If the particles interact through the central elastic forces (for example, they connected by elastic springs), then the matrices $E^{ij}_\varepsilon$ satisfy the equalities

$$E^{ij}_\varepsilon u = K^{ij}_\varepsilon \langle u, e^{ij}_\varepsilon \rangle e^{ij}_\varepsilon,$$

(1.2)

where $e^{ij}_\varepsilon = (x^i_\varepsilon - x^j_\varepsilon)|x^i_\varepsilon - x^j_\varepsilon|^{-1}$ is the unit vector of direction between the $i$-th and $j$-th particles and the coefficient $K^{ij}_\varepsilon$ characterizes the intensity of interaction (stiffness of springs).

The coefficient $K^{ij}_\varepsilon$ depends on the distances $|x^i_\varepsilon - x^j_\varepsilon|$ between particles. Generally speaking, it can be zero if the corresponding pair of particles does
not interact with each other. In this paper we assume that the coefficient $K_{ij}^\varepsilon$ is defined by formula

$$K_{ij}^\varepsilon = \varepsilon^6 \left[ K(|x_i^\varepsilon - x_j^\varepsilon|) + \frac{K_{ij}^\varepsilon}{|x_i^\varepsilon - x_j^\varepsilon|} \varphi \left( \frac{|x_i^\varepsilon - x_j^\varepsilon|}{\varepsilon} \right) \right] A_{ij}^\varepsilon, \quad (1.3)$$

where $K(r), \varphi(r) \in C([0, L]), K(r) \geq 0, \varphi(r) = 1$ as $r \leq \alpha$ and $\varphi(r) = 0$ as $r \geq \beta$ ($0 < \alpha < \beta < L = \text{diam} \Omega$); $A_{ij}^\varepsilon = 1$ (for interacting pairs of particles) and $A_{ij}^\varepsilon = 0$ (for noninteracting pairs of particles), $a_0 \leq K_{ij}^\varepsilon \leq A_0$.

The formula above simulates a weak interaction (of the order $O(\varepsilon^6)$) between not very close particles ($|x_i^\varepsilon - x_j^\varepsilon| > \beta \varepsilon$) and stronger interaction $O(\varepsilon)$ between close ones ($|x_i^\varepsilon - x_j^\varepsilon| < \alpha \varepsilon$) (see Figure 1). This type of interaction is characteristic for some intermolecular forces (for example, van der Waals forces).

![Figure 1:](image)

The interaction energy of the system $M_\varepsilon$ - (1.1) - (1.3) is invariant under rotations and shear. Therefore, the equilibrium state $(x_1^\varepsilon, \ldots, x_N^\varepsilon)$ of the system is not isolated: rotations and shifts are allowed. To exclude this we fix the part of the particles $M_\varepsilon^0 \subseteq M_\varepsilon$ on the boundary $\partial \Omega$ (at the corresponding points $x_i^\varepsilon \in \partial \Omega, u_i^\varepsilon = 0$). We assume the following conditions hold.

I. The condition of "$\varepsilon$-net" on the boundary $\partial \Omega$. The set $M_\varepsilon^0$ of particles assigned to $\partial \Omega$ is an $\varepsilon$-net for $\partial \Omega$. It is clear that the number of such particles is $N_\varepsilon^0 = O(\varepsilon^{-2}) \ll N_\varepsilon$

II. The triangulation condition.
Let $\Gamma_{\varepsilon}$ be a graph with vertices at points $x_i^\varepsilon$ and edges $(x_i^\varepsilon, x_j^\varepsilon)$ ($i, j = 1, \ldots, N, i \neq j$). Assume that for any $\varepsilon > 0$ there exists a subgraph $\Gamma'_{\varepsilon} \subset \Gamma_{\varepsilon}$ with the same set of vertices $M_{\varepsilon}$ and edges of length $|x_i^\varepsilon - x_j^\varepsilon| = d_{ij}^\varepsilon$ ($0 < d_1 \leq d_{ij}^\varepsilon < d_2$), that correspond to the interaction coefficients $K_{ij}^{\varepsilon} = k_{ij}^{\varepsilon}$ ($0 < a \leq k_{ij}^{\varepsilon} \leq A$). The subgraph $\Gamma'_{\varepsilon}$ triangulates the domain $\Omega$. The volumes $|P_{\alpha}^\varepsilon|$ of the corresponding simplexes of the triangulation $P_{\alpha}^\varepsilon$ ($\alpha = 1, \ldots, \hat{N}_{\varepsilon}$) satisfy the inequality $|P_{\alpha}^\varepsilon| > C_{\varepsilon}^3$ ($C > 0$).

Under these conditions, the equilibrium state $(x_1^\varepsilon, \ldots, x_{N_{\varepsilon}}^\varepsilon)$ is isolated. In the small neighborhood of $(x_1^\varepsilon, \ldots, x_{N_{\varepsilon}}^\varepsilon)$ the nonstationary oscillations of the system $M_{\varepsilon}$ are described by the following problem

$$m_i^\varepsilon \ddot{u}_i^\varepsilon = -\nabla u_i^\varepsilon H_{\varepsilon}(u_1^\varepsilon, \ldots, u_{N_{\varepsilon}}^\varepsilon), \quad x_i^\varepsilon \in \Omega, t > 0,$$

$$u_i^\varepsilon(t) = 0, \quad x_i^\varepsilon \in \partial\Omega, t > 0,$$

$$u_i^\varepsilon(0) = a_i^\varepsilon, \quad \dot{u}_i^\varepsilon(0) = b_i^\varepsilon, \quad i = 1, \ldots, N_{\varepsilon}, \quad (1.4) \quad (1.5) \quad (1.6)$$

where $m_i^\varepsilon$ is a mass of $i$-th particle, $a_i^\varepsilon$ are the given initial displacements of the particles, $b_i^\varepsilon$ are the given initial velocities ($a_i^\varepsilon = 0, b_i^\varepsilon = 0$ when $x_i^\varepsilon \in \partial\Omega$). There exists a unique solution $\{u_{\varepsilon}\} = \{u_1^\varepsilon, \ldots, u_{N_{\varepsilon}}^\varepsilon\}$ of this problem. The main goal of the paper is to study the asymptotic behavior of the solution as $\varepsilon \to 0$. We obtain a homogenized system of equations. This system describes the leading term of the asymptotic and is a macroscopic model of the oscillation of an elastic medium with a nonlocal interaction.

2 Quantitative characteristics of the system of interacting particles and formulation of main result

We denote by $K_h^\varepsilon = K(x, h)$ cubes with centers at points $x \in \Omega$ and sides of length $h$ with a fixed orientation. It is assumed that $0 < \varepsilon \ll h \ll 1$ and the cube $K_h^\varepsilon$ contains a large number of particles (of order $O\left(\frac{h^3}{\varepsilon^3}\right)$). Consider
the following functional of the symmetric tensor $T = \{T_{np}\}_{n,p=1}^3$:

$$H_{K^x_h}(T) = \inf_{v_\varepsilon} \left\{ \frac{1}{2} \sum_{i,j \in K^x_h} \frac{1}{\varepsilon^2} E_{ij}(v^i_\varepsilon - v^j_\varepsilon), (v^i_\varepsilon - v^j_\varepsilon) + \sum_i \left( v^i_\varepsilon - \sum_{n,p} \psi^{np}(x^n_i T_{np}) \right)^2 \right\}.$$  \hspace{1cm} (2.1)

The sum $\sum_{K^x_h}$ consist of particles $x^n \in K^x_h$ and inf is taken over displacements $v_\varepsilon = \{v^n_i, i = 1, \ldots, N_\varepsilon\}$ of these particles. The vector function $\psi^{np}(x)$ is defined by equality $\psi^{np}(x) = \frac{1}{2}(x^n e^p + x^p e^n)$, and $\gamma$ is an arbitrary penalty parameter: $0 < \gamma < 2$.

The functional $H_{K^x_h}(T)$ is quadratic and we can rewrite it in the form

$$H_{K^x_h}(T) = \sum_{n,p,q,r=1}^3 a_{npqr}(x; \varepsilon, h; \gamma) T_{np} T_{qr},$$  \hspace{1cm} (2.2)

where $a_{npqr}(x; \varepsilon, h; \gamma)$ are the components of the symmetric tensor of 4-th rank in $\mathbb{R}^3$: $a_{npqr} = a_{qrnp} = a_{pnqr} = \ldots$. This tensor is a mesoscopic ($0 < \varepsilon \ll h \ll 1$) characteristic of the concentration of the short-range interaction energy in a neighborhood of the point $x \in \Omega$.

Assume that the limits

$$\lim_{h \to 0} \lim_{\varepsilon \to 0} \frac{a_{npqr}(x; \varepsilon, h; \gamma)}{h^3} = \lim_{h \to 0} \lim_{\varepsilon \to 0} \frac{a_{npqr}(x; \varepsilon, h; \gamma)}{h^3} = a_{npqr}(x)$$  \hspace{1cm} (2.3)

exist.

Remark. Formally, the limit tensor $\{a_{npqr}(x)\}_{n,p,q,r=1}^3$ must depend on the parameter $\gamma$ and the orientation of the cubes $K(x, h)$. But the main result and the example in Section 6 show that the limiting tensor $\{a_{npqr}(x)\}_{n,p,q,r=1}^3$ does not depend on the parameter $\gamma$ and the orientation of the cubes $K(x, h)$.

Let $\rho_\varepsilon(x) \in L_\infty(\Omega)$ be a density of the distribution of particles masses and let $\varphi_\varepsilon(x, y) \in L_\infty(\Omega \times \Omega)$ be a function of the distribution of the pairs
of particles in $\Omega \times \Omega$ with long-range interaction. We will denote by $V^i_\varepsilon$ $(i = 1, \ldots, N_\varepsilon)$ Voronoi cells of a set of points $x^i_\varepsilon \in \Omega$

$$V^i_\varepsilon = \bigcap_{j=1}^{N_\varepsilon} \{ x \in \Omega : |x - x^i_\varepsilon| < |x - x^j_\varepsilon| \},$$

$|V^i_\varepsilon|$ denotes the volume of the cell and $\chi^i_\varepsilon(x)$ is a characteristic function of the cell. Assume that

$$\rho_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} \frac{m^i_\varepsilon}{|V^i_\varepsilon|} \chi^i_\varepsilon(x),$$

(2.4)

$$\varphi_\varepsilon(x, y) = \varepsilon^6 \sum_{i,j=1}^{N_\varepsilon} \frac{A^{ij}_\varepsilon}{|V^i_\varepsilon||V^j_\varepsilon|} \chi^i_\varepsilon(x) \chi^j_\varepsilon(y)$$

(2.5)

where $m^i_\varepsilon$ are the masses of the particles, $A^{ij}_\varepsilon$ are the elements of the adjacency matrix $A_\varepsilon = \{A^{ij}_\varepsilon\}^{N_\varepsilon}_{i,j=1}$ of the complete graph $\Gamma_\varepsilon$ for the system $M_\varepsilon$ (see (1.3)).

Suppose, that for any $i = 1, \ldots, N_\varepsilon$

$$m^i_\varepsilon = m^i \varepsilon^3 \quad (0 < m_1 \leq m^i \leq m_2 < \infty).$$

(2.6)

By the triangulation condition II $|V^i_\varepsilon| = c^i_\varepsilon \varepsilon^3$ $(0 < C_1 \leq c^i \leq C_2 < \infty)$, and the estimates $\|\rho_\varepsilon\|_{L_\infty(\Omega)} < C$, $\|\varphi_\varepsilon\|_{L_\infty(\Omega \times \Omega)} < C$ are valid uniformly with respect to $\varepsilon$. Hence the set of functions $\{\rho_\varepsilon(x), \varepsilon > 0\}$ is *-weakly compact in $L_\infty(\Omega)$ and the set $\{\varphi_\varepsilon(x, y), \varepsilon > 0\}$ is *-weakly compact in $L_\infty(\Omega \times \Omega)$ (see [21], [22]).

We assume that

$$\rho_\varepsilon(x) \rightharpoonup \rho(x) \quad \text{*-weakly in } L_\infty(\Omega),$$

(2.7)

$$\varphi_\varepsilon(x, y) \rightharpoonup \varphi(x, y) \quad \text{*-weakly in } L_\infty(\Omega \times \Omega),$$

(2.8)

as $\varepsilon \to 0$. Here $\rho(x) > 0$ and $\varphi(x, y) \geq 0$ are the functions in $L_\infty(\Omega)$ and $L_\infty(\Omega \times \Omega)$ respectively.

For each discrete function $u_\varepsilon(x) = \{u^1_\varepsilon, \ldots, u^{N_\varepsilon}_\varepsilon\}$ that defined at the points $x^i_\varepsilon$: $u_\varepsilon(x^i_\varepsilon) = u^i_\varepsilon$ we will match the vector function $u_\varepsilon(x) \in L_\infty(\Omega)$ by the
formula
\[ \tilde{u}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} u^i_\varepsilon \chi^i_\varepsilon(x). \]  

The vector-functions \( \tilde{a}_\varepsilon(x) \in L_\infty(\Omega) \), \( \tilde{b}_\varepsilon(x) \in L_\infty(\Omega) \) correspond to the initial data \( \{a^1_\varepsilon, ..., a^{N_\varepsilon}_\varepsilon\} \) and \( \{b^1_\varepsilon, ..., b^{N_\varepsilon}_\varepsilon\} \). The vector-function \( \tilde{u}(x,t) \in L_\infty(\Omega \times [0,T]) \) \( \forall T > 0 \) correspond to the solution \( \{u^1_\varepsilon(t), ..., u^{N_\varepsilon}_\varepsilon(t)\} \) of the problem.

We assume that
\[ \tilde{a}_\varepsilon(x) \to a(x), \quad \tilde{b}_\varepsilon(x) \to b(x) \quad \text{in} \quad L^2(\Omega), \]  

as \( \varepsilon \to 0 \). Here \( a(x) \) and \( b(x) \) are the vector functions from \( \mathcal{W}^1_2(\Omega) \). Suppose that the inequality
\[ \sum_{i,j=1}^{N_\varepsilon} \langle E^i_\varepsilon(a^j_\varepsilon - a^j_\varepsilon)(a^j_\varepsilon - a^j_\varepsilon) \rangle < C \]  

holds uniformly with respect to \( \varepsilon \).

Now we can formulate the main result.

**Theorem 2.1.** Let the system of interacting particles \( M_\varepsilon \) with the interaction energy (1.1) - (1.3) and the masses \( m^i_\varepsilon \) (2.6) be located in \( \tilde{\Omega} \) and conditions I and II are fulfilled. Suppose that conditions (2.3), (2.7), (2.8) and (2.10), (2.11) hold as \( \varepsilon \to 0 \). Then the vector function \( \tilde{u}_\varepsilon(x,t) \) constructed by (2.9) using the solution \( u_\varepsilon(t) = \{u^1_\varepsilon(t), ..., u^{N_\varepsilon}_\varepsilon(t)\} \) of the problem (1.4) - (1.6) converges in \( L_2(\Omega \times [0,T]) \) as \( \varepsilon \to 0 \) to the solution \( u(x,t) \) of the following initial-boundary value problem

\[ \rho(x) \frac{\partial^2 u}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} \{a_{n p q r}(x) e_{n p}[u] e^r\} + \int_{\Omega} G(x,y)(u(x,t) - u(y,t))dy = 0, \quad x \in \Omega, t > 0, \]  

\[ u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \]  

(2.12)

(2.13)
\( u(x, 0) = a(x), \quad \frac{\partial u}{\partial t}(x, 0) = b(x). \) (2.14)

Here \( e_{np}[u] = \frac{1}{2} \left( \frac{\partial u}{\partial x_p} + \frac{\partial u}{\partial x_n} \right) \) are the components of the elasticity tensor, \( e^r \) is the unit vector of \( x_r \) axis, and the elements of the matrix \( G(x, y) \) are defined by

\[
G_{kl}(x, y) = \frac{K(|x - y|)\varphi(x, y)}{|x - y|^2} (x_k - y_k)(x_l - y_l).
\]

The proof of the theorem is carried out in Sections 4, 5. In remainder of this Section we give the main ideas of the prove. By the Laplace transform in time we reduce in Section 4 the initial problem to a stationary problem with a spectral parameter \( \lambda \) (\( \text{Re}\lambda > 0 \)). We formulate the variational formulation of the problem for real \( \lambda > 0 \). Then we study the asymptotic behavior of its solution as \( \varepsilon \to 0 \) and obtain the homogenized equation. Using the Vitali’s theorem we investigate in Section 5 analytic properties of solutions of the initial and homogenized stationary problems on \( \lambda \) for \( \text{Re}\lambda > 0 \). We prove the convergence of the solutions and, finally, we prove the convergence of solutions of the original non-stationary problem (1.3) - (1.6) to the solution of the homogenized problem (2.11) - (2.13) with the help of the inverse Laplace transform.

3 Auxiliary propositions

Let us denote by \( L_i^{\varepsilon}(x) \) a continuous function in \( \mathbb{R}^3 \) that is linear in every simplex \( P_{k\varepsilon}^\alpha \) (condition of triangulation II) and \( L_i^{\varepsilon}(x^j_\varepsilon) = \delta_{ij} \) at \( x_i^j_\varepsilon \). It is clear that it is non-zero only in simplexes with vertices \( x_i^j_\varepsilon \).

Using this function we construct a piecewise linear spline \( \hat{u}_\varepsilon(x) \) to interpolate the given discrete vector-function \( u_\varepsilon = \{u_1^1_\varepsilon, \ldots, u_N^N_\varepsilon\} \):

\[
\hat{u}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} u_i^i_\varepsilon L_i^{\varepsilon}(x), \quad (3.1)
\]

where \( u_i^i_\varepsilon = u_\varepsilon(x_i^i_\varepsilon) \).
In what follows, we assume that \( u^i_\varepsilon = 0 \) for \( x^i_\varepsilon \in \partial \Omega \). Then, \( \hat{u}_\varepsilon(x) \in \overset{\circ}{W}^1_2(\Omega) \) for any \( \varepsilon > 0 \) if the domain \( \Omega \) is convex. If \( \Omega \) is not convex, then \( \hat{u}_\varepsilon(x) \in \overset{\circ}{W}^1_2(\Omega_\delta) \) for sufficiently small \( \varepsilon \leq \varepsilon(\delta) \). Here \( \Omega_\delta \) is a domain in \( \mathbb{R}^3 \) such that \( \Omega \subset \Omega_\delta \) and \( \text{dist}(\partial \Omega, \partial \Omega_\delta) = \delta \) (\( \forall \delta > 0 \)). This statement follows from conditions I, II, and smoothness of the \( \partial \Omega \).

**Lemma 3.1.** Let us construct vector-functions \( \tilde{u}_\varepsilon(x) \) and \( \hat{u}_\varepsilon(x) \) by formulas \( (2.9) \) and \( (3.1) \) for the same set of vectors \( (u^1_\varepsilon, ..., u^N_\varepsilon) \) \( (u^i_\varepsilon = 0 \) for \( x^i_\varepsilon \in \partial \Omega) \). If the inequality

\[ \| \hat{u}_\varepsilon \|_{W^1_2(\Omega)} < C, \]

holds uniformly with respect to \( \varepsilon \), then

\[ \| \hat{u}_\varepsilon - \tilde{u}_\varepsilon \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

**Proof.** Denote by \( v_\varepsilon(x) = \hat{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x) \). Let \( V^i_\varepsilon \) be the Voronoi cell at the point \( x^i_\varepsilon \), and \( P^\alpha_\varepsilon \) be a simplex with vertex at the point \( x^i_\varepsilon \) (see condition of triangulation II). By \( (3.1) \) with \( x \in V^i_\varepsilon \cap P^\alpha_\varepsilon \), we get

\[ |\nabla v_\varepsilon(x)|^2 = |\nabla \hat{u}_\varepsilon(x)|^2 \equiv |M^\alpha_\varepsilon|^2 = \text{const}. \quad (3.2) \]

Taking into account \( v_\varepsilon(x^i_\varepsilon) = 0 \), we obtain

\[ v_\varepsilon(x) = \int_0^{\frac{|x-x^i_\varepsilon|}{R_\varepsilon}} \frac{\partial v_\varepsilon}{\partial r}(x^i_\varepsilon + r(x-x^i_\varepsilon))dr, \quad x \in V^i_\varepsilon \cap P^\alpha_\varepsilon. \]

By this equality, condition II and \( (3.2) \), we have

\[ |v_\varepsilon(x)|^2 \leq C\varepsilon^2 |M^\alpha_\varepsilon|^2, \quad x \in V^i_\varepsilon \cap P^\alpha_\varepsilon \]

and, consequently

\[ \int_{\Omega} |v_\varepsilon(x)|^2 dx = \sum_{i,j} \int_{V^i_\varepsilon \cap P^\alpha_\varepsilon} |v_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \sum_{i,\alpha} \int_{V^i_\varepsilon \cap P^\alpha_\varepsilon} |M^\alpha_\varepsilon|^2 dx. \]
Thus, according to (3.2) the inequality
\[
\int_\Omega |v_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_\Omega |\nabla \hat{u}_\varepsilon|^2 dx,
\]
holds, which establishes the assertion of the lemma.

Consider the function \( G_{\varepsilon kl}(x, y) \in L_\infty(\Omega \times \Omega) \) \((k, l = 1, 2, 3)\)
\[
G_{\varepsilon kl}(x, y) = \varepsilon^6 \sum_{i,j=1}^{N} \frac{K(|x^i_\varepsilon - x^j_\varepsilon|)e^{ij}_{\varepsilon k}e^{ij}_{\varepsilon l}A^i_\varepsilon \chi(x)^i_\varepsilon(x)\chi(x)\chi(y)^j_\varepsilon(y)},
\]
(3.3)
where \( e^{ij}_{\varepsilon k} \) are k-th components of the vectors \( e^{ij}_\varepsilon = (x^i_\varepsilon - x^j_\varepsilon)|x^i_\varepsilon - x^j_\varepsilon|^{-1} \).

**Lemma 3.2.** Let conditions (2.8) hold, then the function \( G_{\varepsilon kl}(x, y) \) converges to the function
\[
G_{kl}(x, y) = \frac{K(|x - y|)\varphi(x, y)}{|x - y|^2} (x_k - y_k)(x_l - y_l).
\]
\( * \)-weakly in \( L_\infty(\Omega \times \Omega) \) as \( \varepsilon \to 0 \).

**Proof.** Let \( f(x, y) \) be an arbitrary function in \( L_1(\Omega \times \Omega) \). By (3.3), we write
\[
\int_\Omega \int_\Omega G_{\varepsilon kl}(x, y)f(x, y)dxdy = \]
\[
= \int_\Omega \int_\Omega \left( \varepsilon^6 \sum_{i,j=1}^{N} \frac{A^i_\varepsilon}{|V^i_\varepsilon||V^j_\varepsilon|} \chi^i_\varepsilon(x)\chi^j_\varepsilon(y) \right) R_{kl}(x, y)f(x, y)dxdy +
\]
\[
+ \int_\Omega \int_\Omega \sum_{i,j=1}^{N} \frac{(R_{kl}(x^i_\varepsilon, x^j_\varepsilon - R_{kl}(x, y))}\frac{A^i_\varepsilon}{|V^i_\varepsilon||V^j_\varepsilon|} \chi^i_\varepsilon(x)\chi^j_\varepsilon(y)f(x, y)dxdy +
\]
\[
+ \int_\Omega \int_\Omega \sum_{\beta \varepsilon < |x^i_\varepsilon - x^j_\varepsilon| \leq \delta} \frac{(R_{kl}(x^i_\varepsilon, x^j_\varepsilon - R_{kl}(x, y))\frac{A^i_\varepsilon}{|V^i_\varepsilon||V^j_\varepsilon|} \chi^i_\varepsilon(x)\chi^j_\varepsilon(y)f(x, y)dxdy =
\]
\[
= I_{kl}^1 + I_{kl}^2(\delta) + I_{kl}^3(\delta).
\]
(3.5)
Here
\[ R_{kl}(x, y) = K(x, y) \frac{x_k - y_k)(x_l - y_l)}{|x - y|^2}, \tag{3.6} \]
and \( \delta \) is an arbitrary number \( \delta \gg \varepsilon \).

Since \( R_{kl}(x, y)f(x, y) \in L_\infty(\Omega \times \Omega) \),
\[ \lim_{\varepsilon \to 0} I_{kl}^1(\varepsilon) = \int_\Omega \int_\Omega G_{kl}(x, y)f(x, y)dx dy. \tag{3.7} \]
This follows from (2.5), (2.8), and (3.4), (3.6).

As \( f(x, y) \in L_1(\Omega \times \Omega) \), the function \( R_{kl}(x, y) \) is continuous for \(|x - y| > \delta > 0\), and \( \text{diam} V^i_\varepsilon \leq C \varepsilon, |V^i_\varepsilon| \geq C_2 \varepsilon^3 \) (condition II). We have
\[ \lim_{\varepsilon \to 0} I_{kl}^2(\varepsilon) = 0, \tag{3.8} \]
and
\[ \lim_{\delta \to 0} \lim_{\varepsilon \to 0} I_{kl}^2(\varepsilon) = 0 \tag{3.9} \]
for any fixed \( \delta > 0 \).

From (3.5)-(3.9) follows the assertion of the lemma.

The following lemma plays a fundamental role in studying the compactness of discrete vector-valued functions. The same role plays the well-known Korn inequality for the functions in \( W^{1, 2}_2(\Omega) \) cite 24.

**Lemma 3.3** (discrete analogue of the Korn inequality). Let conditions I and II hold. Then
\[ \sum_{i,j} E^i_{ij} [u^i_{z} - u^j_{z}], [u^i_{z} - u^j_{z}] \geq C\| \hat{u}_{z} \|^2_{W^{1, 2}_2(\Omega)} \geq C_1 \left( \varepsilon \sum_{i,j} |u^i_{z} - u^j_{z}|^2 + \varepsilon^3 \sum_{i} |u^i_{z}|^2 \right), \]
for any discrete function \( u_{z}(x) \) defined at points \( x^i_{z} \) by \( u_{z}(x^i_{z}) = u^i_{z} \) \( i = 1, ..., N_{z} \), and \( u^i_{z} = 0 \) for \( x^i_{z} \in \partial \Omega \). Here \( E^i_{ij} \) are the pair interaction matrices (see (1.1)-(1.3)); the sum \( \sum_{i,j} \) is taken over all \((i,j)\) of the edges \((x^i_{z}, x^j_{z})\) of triangulation subgraph \( \Gamma^i_{z} \), and \( C, C_1 > 0 \) are the constants that do not depend on \( \varepsilon \).
The proof of lemma 3.3 is carried out in [19].

Next lemma establish the estimates of the solution \( \{ v_\varepsilon^i \} \) of the problem (2.1). We give this lemma without proof. For more details we refer to [19].

**Lemma 3.4.** Let conditions (2.2) hold. Then

\[
\sum_{K_h}^{i,j} \langle E_{\varepsilon}^{ij} (v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j) \rangle = O(h^3),
\]

\[
\sum_{i}^{K_h \setminus K_{x_1}} \left| v_\varepsilon^i - \sum_{n,p} \psi_{np}(x_\varepsilon^i) T_{np} \right|^2 \leq O(h^{5+\gamma}),
\]

\[
\sum_{i}^{K_h \setminus K_{x_1}} \langle E_{\varepsilon}^{ij} (v_\varepsilon^i - v_\varepsilon^j), (v_\varepsilon^i - v_\varepsilon^j) \rangle = o(h),
\]

\[
\sum_{i}^{K_h \setminus K_{x_1}} \left| v_\varepsilon^i - \sum_{n,p} \psi_{np}(x_\varepsilon^i) T_{np} \right|^2 = o(h^{5+\gamma}),
\]

where \( v_\varepsilon^i \) is a solution of the problem (2.1); \( h^1 = h - 2h^{1+\gamma/2}, \gamma > 0 \) and \( \varepsilon \leq \hat{\varepsilon}(h) \).

4 Variational formulation of the problem and asymptotic behavior of the solution

as \( \varepsilon \to 0 \)

By Laplace transform we convert the function \( u_\varepsilon^i(t) \) of a real variable \( t \) to the function of a complex variable \( \lambda \):

\[
u_\varepsilon^i(t) \to u_\varepsilon^i(\lambda) = \int_0^\infty u_\varepsilon^i(t)e^{-\lambda t} dt, \quad i = 1, ..., N, \text{Re}\lambda > 0.
\]

Applying the Laplace transform to the problem (1.4)-(1.6) and taking into account (1.1), we get the stationary problem for \( u_\varepsilon(\lambda) = \{ u_\varepsilon^1(\lambda), ..., u_\varepsilon^N(\lambda) \} \)
with a spectral parameter $\lambda$

$$\lambda^2 m_\varepsilon^i u_\varepsilon^i(\lambda) + \sum_{j=1}^{N_\varepsilon} E_\varepsilon^{ij}(u_\varepsilon^j(\lambda) - u_\varepsilon^i(\lambda)) = m_\varepsilon^i f_\varepsilon^i(\lambda), \quad x_\varepsilon^i \in \Omega,$$

$$u_\varepsilon^i(\lambda) = 0, \quad x_\varepsilon^i \in \partial \Omega,$$

(4.1)

where $f_\varepsilon^i(\lambda) = \lambda a_\varepsilon^i + b_\varepsilon^i, \ i = 1, \ldots, N_\varepsilon$.

This problem has a unique solution for all $\lambda \in \mathbb{C}$, except the finite number of the spectrum points $\lambda = \pm i\mu k$ ($\mu_\varepsilon k > 0$, $k = 1, \ldots, N_\varepsilon' < N_\varepsilon$). For $\lambda = 0$, the problem describes the equilibrium elastic system under the action of the forces $m_\varepsilon^i f_\varepsilon^i$.

Solution $u_\varepsilon = \{u_\varepsilon^1, \ldots, u_\varepsilon^{N_\varepsilon}\}$ of the problem (4.1) for $\lambda^2 \geq 0$ minimizes the functional

$$\Phi_\varepsilon[v_\varepsilon] = \frac{1}{2} \sum_{i,j=1}^{N_\varepsilon} \langle E_\varepsilon^{ij}[v_\varepsilon^i - v_\varepsilon^j], [v_\varepsilon^i - v_\varepsilon^j] \rangle + \lambda^2 \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |v_\varepsilon^i|^2 - \frac{1}{2} \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i \langle f_\varepsilon^i, v_\varepsilon^i \rangle$$

(4.2)

in the space $\overset{\circ}{J}_\varepsilon$ of discrete vector-functions $v_\varepsilon(x) = \{v_\varepsilon^1, \ldots, v_\varepsilon^{N_\varepsilon}\}$ that equal 0 on $\partial \Omega$: $v_\varepsilon^i = 0$ when $x_\varepsilon^i \in \partial \Omega$. Thus, the vector-function $u_\varepsilon = \{u_\varepsilon^1, \ldots, u_\varepsilon^{N_\varepsilon}\}$ is the solution of the minimization problem

$$\Phi_\varepsilon[u_\varepsilon] = \min_{v_\varepsilon \in \overset{\circ}{J}_\varepsilon} \Phi_\varepsilon[v_\varepsilon].$$

(4.3)

To describe the asymptotic behavior of $u_\varepsilon$ as $\varepsilon \to 0$, we introduce in $W_0^1(\Omega)$ the functional

$$\Phi[v] = \int \sum_{n,p,q,r=1}^{3} a_{n p q r}(x) e_{n p}[v] e_{q r}[v] dx + \lambda^2 \int \rho(x) |v|^2 dx +$$

$$+ \frac{1}{2} \int \int \langle G(x, y)(v(x) - v(y), (v(x) - v(y)) dx dy - 2 \int \rho(x) (f, v) dx. $$

(4.4)

Here $e_{n p}[v] = \frac{1}{2} \left[ \frac{\partial v}{\partial x_n} + \frac{\partial v}{\partial x_p} \right]$, tensor $\{a_{n p q r}(x)\}_{n,p,q,r=1}^{3}$ is given by (2.3), functions $\rho(x)$ and $\varphi(x, y)$ are defined by (2.7) and (2.8), vector-function $f(x) =$
$\lambda a(x) + b(x)$ is given by (2.10), and the elements of the matrix $G(x, y)$ are defined by (3.4).

Consider the minimization problem

$$
\Phi[u] = \min_{w \in W^1_2(\Omega)} \Phi[w].
$$

(4.5)

**Theorem 4.1.** Let conditions I, II, (1.2), (1.3) hold and let the limits (2.3), (2.7), (2.8) exist as $\varepsilon \to 0$. Then the vector-function $\tilde{u}_\varepsilon(x)$ constructed by (2.9) on the solution $u_\varepsilon = \{u_\varepsilon^1, ..., u_\varepsilon^N\}$ of the minimization problem (4.3) converges in $L^2(\Omega)$ to the solution of the minimization problem (4.5).

**Proof.** Taking into account that $\{0\} \in \mathcal{J}_\varepsilon$ and $\Phi_\varepsilon[0] = 0$, we get the inequality

$$
N_\varepsilon \sum_{i,j=1}^{N_\varepsilon} \langle E_{i,j}^\varepsilon [u_\varepsilon^i - w_\varepsilon^j], [u_\varepsilon^i - w_\varepsilon^j] \rangle + 2\lambda^2 \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |u_\varepsilon^i|^2 \leq 4 \left\{ \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |f_\varepsilon^i|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |u_\varepsilon^i|^2 \right\}^{1/2}.
$$

(4.6)

From (2.4), (2.7), (2.10) and condition II it follows that

$$
\sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |f_\varepsilon^i|^2 \leq C(|\lambda|^2 + 1),
$$

(4.7)

where $C$ does not depend on $\varepsilon$.

We construct the vector-function $\hat{u}_\varepsilon(x)$ by (3.1) where $u_\varepsilon = \{u_\varepsilon^1, ..., u_\varepsilon^N\}$ is the solution of (4.3). By inequalities (4.6), (4.7) and lemma 3.3 we get

$$
\|\hat{u}_\varepsilon\|_{W^1_2(\Omega)} \leq C.
$$

(4.8)

The inequality is satisfied uniformly with respect to $\varepsilon$.

Thus, the set of vector-functions $\{\hat{u}_\varepsilon(x), \varepsilon > 0\}$ is a weakly compact set in $W^1_2(\Omega)$. We can extract a subsequence $\{\hat{u}_{\varepsilon_k}(x), \varepsilon_k \to 0\}$ converges to
the vector-function \( u(x) \in W^1_2(\Omega) \) weakly in \( W^1_2(\Omega) \) and strongly in \( L_q(\Omega) \) \((q \leq 6)\).

By (2.9) we construct the subsequence \( \{\tilde{u}_{\varepsilon_k}(x), \varepsilon_k \to 0\} \) for the set of vector-functions \( \{u^1_{\varepsilon_k}, \ldots, u^N_{\varepsilon_k}\} \). According to lemma 3.1 and (4.8) the subsequence converges to \( u(x) \) in \( L_q(\Omega) \). Let us prove that \( u(x) \) minimizes (4.5). For this purpose we rewrite the functional \( \Phi_\varepsilon \) (4.2) in the form:

\[
\Phi_\varepsilon[v_\varepsilon] = \Phi_{1\varepsilon}[v_\varepsilon] + \Phi_{2\varepsilon}[v_\varepsilon],
\]

where

\[
\Phi_{1\varepsilon}[v_\varepsilon] = \frac{1}{2} \sum_{i,j=1}^{N_\varepsilon} \langle E^{ij}_\varepsilon(v^i_\varepsilon - v^j_\varepsilon), (v^i_\varepsilon - v^j_\varepsilon) \rangle,
\]

\[
\Phi_{2\varepsilon}[v_\varepsilon] = \frac{1}{2} \sum_{\|x^i_\varepsilon - x^j_\varepsilon\| \geq \beta \varepsilon} \langle E^{ij}_\varepsilon(v^i_\varepsilon - v^j_\varepsilon), (v^i_\varepsilon - v^j_\varepsilon) \rangle + \lambda \sum_{i=1}^{N_\varepsilon} m^i_\varepsilon |v^i_\varepsilon|^2 - 2 \sum_{i=1}^{N_\varepsilon} m^i_\varepsilon \langle f^i_\varepsilon, v^i_\varepsilon \rangle.
\]

Strong interactions between nearby particles are included in functional \( \Phi_{1\varepsilon} \). According to (1.2), (1.3) this interaction has the order \( O(\varepsilon) \) (see [19]). Recalling that \( \hat{u}_\varepsilon \to u \) converges weakly in \( W^1_2(\Omega) \) as \( \varepsilon = \varepsilon_k \to 0 \), and taking into account (2.2), we get the lower bound for \( \Phi_{1\varepsilon} \) by the same method as in [19]:

\[
\lim_{\varepsilon = \varepsilon_k \to 0} \Phi_{1\varepsilon}[\tilde{u}_\varepsilon] \geq \Phi_1[u] = \int_{\Omega} \sum_{n,p,q,r=1}^3 a_{npqr}(x)e_{np}[u]e_{qr}[u]dx.
\]
By (1.2), (1.3), (2.4), (3.5) we can rewrite $\Phi_2[\tilde{u}_\varepsilon]$ in the form:

$$
\Phi_2[\tilde{u}_\varepsilon] = \int_\Omega \int_\Omega \sum_{k,l=1}^3 G_{klt}(\tilde{u}_{\varepsilon k}(x) - \tilde{u}_{\varepsilon k}(y))(\tilde{u}_{\varepsilon l}(x) - \tilde{u}_{\varepsilon l}(y))dxdy + \\
+ \lambda^2 \int_\Omega \rho(x)|\tilde{u}_\varepsilon|^2dx - 2 \int_\Omega \rho(x)\langle \tilde{f}_\varepsilon, \tilde{u}_\varepsilon \rangle dx, 
$$

where

$$
\tilde{f}_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} f_i^\varepsilon \chi_i^\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} (\lambda a_i^\varepsilon + b_i^\varepsilon) \chi_i^\varepsilon(x),
$$

and $\tilde{u}_{\varepsilon k}(x)$ is the $k$-th component of the vector-function $\tilde{u}_\varepsilon(x)$ (2.4).

Since $\tilde{u}_\varepsilon \to u$ in $L_2(\Omega)$ as $\varepsilon = \varepsilon_k \to 0$, we have

$$
(\tilde{u}_{\varepsilon k}(x) - \tilde{u}_{\varepsilon k}(y))(\tilde{u}_{\varepsilon l}(x) - \tilde{u}_{\varepsilon l}(y)) \to (u_k(x) - u_k(y))(u_l(x) - u_l(y)) \text{ in } L_2(\Omega \times \Omega)
$$

and by (2.10)

$$
\langle \tilde{f}_\varepsilon, \tilde{u}_\varepsilon \rangle \to \langle f, u \rangle \text{ in } L_1(\Omega),
$$

where $f(x) = \lambda a(x) + b(x)$.

From the above, by lemma 3.2, (2.7), and (4.13), we obtain

$$
\lim_{\varepsilon = \varepsilon_k \to 0} \Phi_2[\tilde{u}_\varepsilon] = \Phi_2[u] = \\
= \int_\Omega \int_\Omega \langle G(x,y)(u(x) - u(y)), (u(x) - u(y)) \rangle dxdy + \\
+ \lambda^2 \int_\Omega \rho(x)u^2(x)dx - 2 \int_\Omega \rho(x)\langle f(x), u(x) \rangle dx.
$$

(4.14)

On account of (4.9), (4.13), (4.14), we get the lower bound for $\Phi_\varepsilon[\tilde{u}_\varepsilon]$:

$$
\lim_{\varepsilon = \varepsilon_k \to 0} \Phi_\varepsilon[\tilde{u}_\varepsilon] \geq \Phi[u],
$$

(4.15)

where $u(x)$ is a limit in $L_2(\Omega)$ of the vector-functions $\tilde{u}_\varepsilon(x)$, and the functional $\Phi[u] = \Phi_1[u] + \Phi_2[u]$ is defined by (4.4).
In order to get the upper bound, we introduce the test vector-function $w_{\varepsilon} = (w_{\varepsilon}^1, ..., w_{\varepsilon}^N)$ in $J_\varepsilon$ for the problem (4.3). To this end, we cover $\Omega$ by cubes $K_h^\alpha = K(x^\alpha, h)$ with centers at the points $x^\alpha$ and sides of length $h$. The centers of the cubes form a cubic lattice with period $h - h^{1+\gamma/2}$ ($0 < \gamma < 2$). By this covering we construct the partition of unity $\varphi_\alpha(x)$. Namely, the set of functions with the following properties:

$$\varphi_\alpha(x) \in C^2_0(K_h^\alpha), \quad \sum_\alpha \varphi_\alpha(x) = 1, \quad \varphi_\alpha(x) = 0 \text{ when } x \notin K_h^\alpha, \quad \varphi_\alpha(x) = 1 \text{ when } x \in K_h^\alpha \setminus \bigcup_{\beta \neq \alpha} K_h^\beta;$$

$$|\nabla \varphi_\alpha(x)| \leq Ch^{-1-\gamma/2}.$$

Let $w(x)$ be an arbitrary vector-functions in $C^2(\Omega)$ with the compact support in $\Omega$. Define

$$w_{\varepsilon}^i = \sum_\alpha \left( w(x^\alpha) + \sum_{n,p=1}^3 (e_{np}[w(x^\alpha)]) v_{\varepsilon}^{anp}(x_\varepsilon^i) + \omega_{np}[w(x^\alpha)] \varphi_{\varepsilon}^{np}(x_\varepsilon^i - x^\alpha) \right) \varphi_\alpha(x_\varepsilon^i), \quad i = 1, ..., N_\varepsilon. \quad (4.16)$$

Here $v_{\varepsilon}^{anp}(x_\varepsilon^i)$ is a minimizer of the functional (2.1) in the cube $K_h^\alpha$ for $T = T_{ik}^{anp}$ ($T_{ik}^{anp} = \delta_{in} \delta_{pk}$); $e_{np}[w]$, $\omega_{np}[w]$ are a symmetric and antisymmetric parts of the tensor $\nabla w$; $\varphi_{\varepsilon}^{np}(x) = \frac{1}{2}(x_ne^p - x_pe^n)$.

Using the properties of the discrete vector-functions $v_{\varepsilon}^{anp}$ (see lemma 3.4), the properties of the partition of unity $\{\varphi_\alpha(x)\}$, and by (2.2) we get

$$\lim_{h \to 0} \lim_{\varepsilon \to 0} \Phi_1[w_{\varepsilon}] \leq \Phi_1[w], \quad (4.17)$$

in the same manner as in [19]. The functional $\Phi_1$ is defined by (4.12).

To estimate $\Phi_2[w_{\varepsilon}]$, we use the following equality for the vector-functions $w(x) \in C^2_0(\Omega)$ for $x \in K_h^\alpha$

$$w(x) = w(x^\alpha) + \sum_{n,p} e_{np}[w(x^\alpha)] \psi_{np}(x - x^\alpha) + \omega_{np}[w(x^\alpha)] \varphi_{np}(x - x^\alpha) + O(h^2).$$

Substituting this equality in (4.16) and applying lemma 3.4, we conclude

$$\lim_{h \to 0} \lim_{\varepsilon \to 0} \|w_{\varepsilon} - w\|^2_{L^2(\Omega)} = 0. \quad (4.18)$$
Taking into account convergence (2.7), (2.10), and lemma 3.2, we get

\[
\lim_{h \to 0} \lim_{\varepsilon \to 0} \Phi_{2\varepsilon}[w_{\varepsilon h}] = \Phi_2[w],
\]

where \( \Phi_2 \) is defined by (4.14).

Thus, by (4.9), (4.17), (4.19)

\[
\lim_{h \to 0} \lim_{\varepsilon \to 0} \Phi_{\varepsilon}[w_{\varepsilon h}] \leq \Phi[w].
\]

Recalling that \( u_{\varepsilon} \) is the minimizer of \( \Phi_{\varepsilon} \) in \( J_{\varepsilon} \) for sufficient small \( h \) (\( \varepsilon < \hat{\varepsilon}(h) \)) and \( w_{\varepsilon h} \in J_{\varepsilon} \) we can write

\[
\lim_{h \to 0} \lim_{\varepsilon \to 0} \Phi_{\varepsilon}[u_{\varepsilon}] \leq \Phi[w].
\]

Combining (4.15) and (4.20) we obtain

\[
\Phi[u] \leq \Phi[w], \quad w \in C^2_0(\Omega).
\]

The inequality is valid for any vector-function \( w \in W^2_1(\Omega) \) due to the continuity of the functional \( \Phi[w] \) in \( W^2_1(\Omega) \). Thus, the limit \( u(x) \) of the vector-functions \( \tilde{u}_\varepsilon(x) \) by subsequence \( \varepsilon = \varepsilon_k \to 0 \) is a solution of minimizing problem (4.5). Consequently, \( u(x) \) is a weak solution of the following boundary value problem

\[
\begin{align*}
\sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_q} \{a_{npqr}(x)e_{np}[u]e_r\} + \lambda^2 \rho(x)u + \\
+ \int_{\Omega} (G(x,y)(u(x) - u(y)))dy = \lambda a(x) + b(x), \quad x \in \Omega, \\
u(x) = 0, \quad x \in \partial\Omega.
\end{align*}
\]

Since \( \lambda \geq 0 \), function \( \rho(x) \) and matrix-function \( G(x,y) \) are non-negative, and tensor \( \{a_{npqr}(x)\}_{n,p,q,r=1}^{3} \) is positive definite, the problem have a unique solution. Thus, theorem 4.1 is proved.
5 The convergence of solutions of the problem (4.1) to the solution of the problem (4.21) for complex $\lambda$

1. Consider the problem (4.1) for complex $\lambda$ in semiaxis $\text{Re}\lambda > 0$. Denote by $L_\varepsilon$ a Hilbert space of finite sets of $N_\varepsilon$ 3-components complex vectors defined in $x^i_\varepsilon \in \bar{\Omega}$: $u_\varepsilon = \{u^1_\varepsilon, ..., u^{N_\varepsilon}_\varepsilon\}$. If $x^i_\varepsilon \in \partial\Omega$ then $u^i_\varepsilon = 0$. Define a scalar product in $L_\varepsilon$:

$$(u_\varepsilon, w_\varepsilon)_\varepsilon = \sum_{i=1}^{N_\varepsilon} (u^i_\varepsilon, w^i_\varepsilon) m^i_\varepsilon,$$

where $m^i_\varepsilon$ is a mass of the point $x^i_\varepsilon$. By parenthesis $(\cdot, \cdot)$ we denote the scalar product in $\mathbb{R}^3$. The bar denotes the complex conjugation. The corresponding norm is denoted by $\|u_\varepsilon\|_\varepsilon = (u_\varepsilon, \bar{u}_\varepsilon)^{1/2}$

Consider in $L_\varepsilon$ a linear operator $A_\varepsilon : L_\varepsilon \to L_\varepsilon$:

$$(A_\varepsilon u_\varepsilon)_i = \begin{cases} \frac{1}{m^i_\varepsilon} \sum_{i=1}^{N_\varepsilon} E^{ij}_\varepsilon (u^i_\varepsilon - u^j_\varepsilon), & x^i_\varepsilon \in \Omega, \\ 0, & x^i_\varepsilon \in \partial\Omega. \end{cases} \quad (5.1)$$

From (2.5), (1.2), (1.3) it follows that $A_\varepsilon$ is a bounded selfadjoint operator in $L_\varepsilon$. By lemma 3.3 $A_\varepsilon$ is positive definite operator uniformly with respect to $\varepsilon$:

$$(A_\varepsilon u_\varepsilon, u_\varepsilon)_\varepsilon = \sum_{i,j=1}^{N_\varepsilon} (E^{ij}_\varepsilon (u^i_\varepsilon - u^j_\varepsilon), u^i_\varepsilon - u^j_\varepsilon) \geq \alpha \|u_\varepsilon\|^2_\varepsilon, \quad (\alpha > 0). \quad (5.2)$$

Let us rewrite the problem (4.1) in operator form in $L_\varepsilon$:

$$A_\varepsilon u_\varepsilon + \lambda^2 u_\varepsilon = \lambda a_\varepsilon + b_\varepsilon. \quad (5.3)$$

By the indicated properties of operator $A_\varepsilon$ its resolvent is a meromorphic operator function of the parameter $\tau = \lambda^2$ with poles on the negative semiaxis $\tau < 0$. Hence the solution $u_\varepsilon = u_\varepsilon(\lambda)$ of (5.3) is a holomorphic function of $\lambda$ in the half-plane $\text{Re}\lambda > 0$. Multiplying (5.3) on $\bar{u}_\varepsilon$ and separating the
imaginary and real parts, taking into account (2.5) and (2.10), we obtain the estimate for \( u \varepsilon \) in half-plane \( \Re \lambda > \sigma \) \( (\forall \sigma > 0) \) uniform with respect to \( \varepsilon \):
\[
\|u \varepsilon\|_{\varepsilon} \leq C \quad (C = C(\sigma) < \infty).
\]
This implies that the vector-function \( \tilde{u}_\varepsilon = u \varepsilon(x, \lambda) \) defined by (2.9) is a holomorphic function in \( \Re \lambda > \sigma \) \( (\forall \sigma > 0) \).
Moreover, \( \tilde{u}_\varepsilon \) is bounded in the norm of \( L^2(\Omega) \) uniformly with respect to \( \varepsilon \):
\[
\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} \leq C < \infty.
\]
(5.4)

2. We now turn to the problem (4.21). Denote by \( L^2(\Omega, \rho) \) a Hilbert space of a complex-valued vector-functions in \( L^2(\Omega) \) with a weight \( \rho(x) > 0 \).
The scalar product in \( L^2(\Omega, \rho) \) we define by
\[
(u, w)_\rho = \int_\Omega u(x)\overline{w(x)}\rho(x)dx.
\]
Consider a sesquilinear form defined on the set of vector-valued functions \( C_0(\Omega) \) that is dense in \( L^2(\Omega, \rho) \)
\[
\hat{A}(u, w) = \frac{1}{\rho} \int_\Omega \sum_{n,p,q,r=1}^3 a_{npqr}e_{np}[u]e_{qr}[\bar{w}]dx + \frac{1}{2\rho} \int_\Omega \langle G(x, y)[u(x) - u(y)], [\bar{w}(x) - \bar{w}(y)] \rangle dxdy.
\]
The form generates in \( L^2(\Omega, \rho) \) a selfadjoint operator \( A \), that is due to the properties of the elasticity tensor \( \{a_{npqr}\}_{n,p,q,r=1}^3 \) and the long-range matrix \( G(x, y) \) [21]. The equality
\[
(Au, u)_\rho = \int_\Omega \sum_{n,p,q,r=1}^3 a_{npqr}(x)|e_{np}[u]|^2dx + \frac{1}{2} \int_\Omega \langle G(x, y)[u(x) - u(y)], [\bar{u}(x) - \bar{u}(y)] \rangle dxdy,
\]
is valid. From the Korn’s inequality it follows that
\[
(Au, u)_\rho \geq C\|u\|_{W^2_0(\Omega)}^2 \quad (C > 0).
\]
(5.5)
This inequality implies that the operator \( A \) is positive definite and has a completely continuous inverse operator. Now we can rewrite the problem (4.21) in operator form:
\[
Au + \lambda^2 u = \lambda a + b.
\]
(5.6)
The properties of the operator $A$ implies that equation (5.6) has a solution $u(x)$ for complex $\lambda$ ($\text{Re}\lambda > 0$) and this solution is a holomorphic function of $\lambda$ satisfying the inequality

$$\|u\|_\rho < C.$$  

3. By theorem 2.1 the vector-function $\tilde{u}_\varepsilon(x, \lambda)$ converges in $L_2(\Omega)$ for $\lambda > 0$ to the solution $u(x, \lambda)$ of the problem (4.21) (or equation (5.6)) as $\varepsilon \to 0$. Moreover, the set of vector-functions $\{\tilde{u}_\varepsilon, \varepsilon > 0\}$ is bounded by norm in $L_2(\Omega)$ uniformly with respect to $\varepsilon$ in the half-plane $\text{Re}\lambda > \sigma$ ($\forall \sigma > 0$). Therefore, using Vitali’s theorem and taking into account that $u(x, \lambda)$ is holomorphic, we get the following assertion.

**Theorem 5.1.** Let assumptions I, II, (2.2), (2.6), (2.9) hold. Let construct the function $\tilde{u}_\varepsilon(x, \lambda)$ by (2.9) on the solution of the problem (4.1). Then vector-function $\tilde{u}_\varepsilon(x, \lambda)$ converges in $L_2(\Omega)$ to the solution $u(x, \lambda)$ of equation (5.6) (or the problem (4.21)) uniformly with respect to complex $\lambda$ from the half-plane $\text{Re}\lambda > \sigma$ ($\forall \sigma > 0$).

6 The end of the proof of the main theorem

By the definition (5.1) of operator $A_\varepsilon$ the problem (1.4) – (1.6) is representable in $L_\varepsilon$ in the form:

$$\ddot{u}_\varepsilon + A_\varepsilon u_\varepsilon = 0,$$

$$u_\varepsilon(0) = a_\varepsilon, \quad \dot{u}_\varepsilon(0) = b_\varepsilon. \quad (6.1)$$

From this on account of (5.1), we have

$$\|\dot{u}_\varepsilon\|_\varepsilon^2 + \sum_{i,j=1}^{N_\varepsilon} \langle E_{\varepsilon}^{ij}(u_i^\varepsilon - u_j^\varepsilon), (u_i^\varepsilon - u_j^\varepsilon) \rangle = \|b_\varepsilon\|_\varepsilon^2 + \sum_{i,j=1}^{N_\varepsilon} \langle E_{\varepsilon}^{ij}(a_i^\varepsilon - a_j^\varepsilon), (a_i^\varepsilon - a_j^\varepsilon) \rangle.$$ 

The equality above with the discrete Korn’s inequality, the properties of $E_{\varepsilon}^{ij}$ and $m_\varepsilon^i$, and (2.9), (2.10) implies inequality:

$$\int_{\Omega_T} \left\{ \left( \frac{\partial \hat{u}_\varepsilon}{\partial t} \right)^2 + |\nabla \hat{u}_\varepsilon|^2 \right\} dxdt \leq CT \quad (\forall T > 0),$$
where \( \hat{u}_\varepsilon(x,t) \) is a spline vector-function, defined by (3.1), and \( C \) does not depend on \( \varepsilon \).

Thus the set of vector-functions \( \{ \hat{u}_\varepsilon, \varepsilon \to 0 \} \) is bounded in \( W^1_2(\Omega_T) \) uniformly with respect to \( \varepsilon \). We can extract a subsequence \( \{ \hat{u}_\varepsilon, \varepsilon = \varepsilon_k \to 0 \} \) converges weakly in \( W^1_2(\Omega_T) \) to a function \( u(x,t) \in W^1_2(\Omega_T) \) (and by embedding theorem converges strongly in \( L_q(\Omega \times (0,T)) \) \( (q \leq 4) \) and for almost all \( t \in (0,T] \) converges strongly in \( L_2(\Omega) \)). By the above and lemma 3.1 we conclude that the piecewise-constant vector-functions \( \tilde{u}_\varepsilon(x,t) \) defined by (2.9) converges in \( L_4(\Omega) \) and \( L_2(\Omega) \) to \( u(x,t) \) for almost all \( t \in [0,T] \) as \( \varepsilon = \varepsilon_k \to 0 \).

Let us prove that the function \( u(x,t) \) is a solution of the problem (2.11) – (2.13). By definition of operator \( A \) this problem can be written in operator form:

\[
\ddot{u} + Au = 0 \\
u(0) = a, \quad \dot{u} = b.
\]  

(6.2)

The solution \( u_\varepsilon(x,t) \) of the problem (1.4)- (1.6) is an inverse Laplace transform of the solution \( u_\varepsilon(x,\lambda) \) of the problem (4.1)

\[
u_\varepsilon(x,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} u_\varepsilon(x,\lambda)d\lambda, \quad \sigma > 0.
\]

Thus we have

\[
\tilde{u}_\varepsilon(x,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \tilde{u}_\varepsilon(x,\lambda)d\lambda, \quad (6.3)
\]

where \( \tilde{u}(x,t) \) and \( \tilde{u}(x,\lambda) \) are defined by (2.9). Multiply the equality above by \( \psi(x)\varphi(t) \), where \( \psi(x) \in L_2(\Omega) \), \( \varphi(t) \in C^2_0(0,T] \), and integrate over \( \Omega_T \). Changing the integration order and integrating on \( t \) by parts, we obtain

\[
\int_{\Omega_T} \tilde{u}_\varepsilon(x,t)\psi(x)\varphi(t)dxdt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\lambda t}}{\lambda^2} \left( \int_{\Omega_T} \tilde{u}_\varepsilon(x,\lambda)\psi(x) \frac{\partial^2 \varphi}{\partial t^2} dxdt \right) d\lambda.
\]

Note that the integrals on \( \lambda \) in the right-hand side converge absolutely, due to (5.4).
Let us pass to the limit in the equation above as $\varepsilon = \varepsilon_k \to 0$. Passing to the limit we take into account that $\tilde{u}_{\varepsilon_k}(x, t)$ converges to $u(x, t)$ in $L_2(\Omega)$, and $\tilde{u}_{\varepsilon_k}(x, \lambda)$ converges to the solution $u(x, \lambda)$ of the equation (5.6) in $L_2(\Omega)$ uniformly on the compacts $\Lambda$ from the half-plane $\mathrm{Re} \lambda > 0$ (see theorem 2).

We get

$$\int_{\Omega_T} u(x, t)\psi(x)\varphi(t)dxdt = \int_{\Omega_T} \left\{ \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} u(x, \lambda e^{\lambda t})d\lambda \right\} \psi(x)\varphi(t)dxdt.$$ 

Since the linear combination of the functions $\psi(x)\varphi(t)$ form a dense set in $L_2(\Omega_T)$ then $u(x, t)$ is a solution of the problem (6.2). By the properties of operator $A$, this problem has a unique solution. Thus $\tilde{u}(x, t)$ converges to $u(x, t)$ in $L_2(\Omega_T)$ as $\varepsilon \to 0$. The theorem 1.1 is proved.

7 Periodic structure

We now consider the concrete case when the conditions of theorem 1.1 are satisfied and the elastic tensor $\{a_{npqr}(x)\}$ and the matrix-function $G(x, y)$ are computed explicitly.

Suppose that the points $x_i^\varepsilon$ of the equilibrium state of the system are located periodically. They form a cubic lattice with a period $\varepsilon$. Each point $x_i^\varepsilon$ interacts with the tops of the cube $x_j^\varepsilon$. The points $x_i^\varepsilon, x_j^\varepsilon$ belong to the same cube. For clarity, we can assume that the interaction is carried out by elastic springs. The stiffness of the springs (the elasticity coefficient in Hooke’s law) directed along the edges of the cubes is $k_1\varepsilon^2$; directed along the diagonals of the faces of the cubes is $k_2\varepsilon$; and directed along the diagonals of the cubes is $k_3\varepsilon^2$ (Fig. 2). This is a strong short-range interaction.

The corresponding coefficients of interaction (1.2) $K_{ij}^{\varepsilon}$ have an order $O(\varepsilon)$ $K_{ij}^{\varepsilon} = k_1\varepsilon, 4k_2\varepsilon, 9k_3\varepsilon$.

Let us assume that there exist a long-range interaction. Each point $x_i^\varepsilon$ interacts with the points $x_j^\varepsilon$ of the cubic sublattice $\{x_j^\varepsilon\}^{(i)}$ with the period
Nε (∃N ∈ Z, N ≥ 2). This is a weak interaction and

$$K^{ij}_ε = \varepsilon^6 K |x^i_ε - x^j_ε|,$$

where $K(r)$ is a non-negative function (see (1.3)).

The system of the points $x^i_ε$ satisfies the triangulation condition II. The corresponding interaction is described by (1.2), where $\alpha = \sqrt{3}, \beta = 2 K_{ij} = k_1, k_2, k_3; \ A_{ij} = 1$ only for $|x^i_ε - x^j_ε| = \varepsilon, \sqrt{2}\varepsilon, \sqrt{3}\varepsilon$ and for $|x^i_ε - x^j_ε| = \sqrt{k^2 + l^2 + m^2 N^3 \varepsilon}$ ($k, l, m = 1, 2, 3, \ldots$).

By (2.3) the limit dense $\varphi(x, y)$ is equal to $\frac{1}{N^3}$. Therefore, by (3.4)

$$G_{kl}(x, y) = \frac{K(|x - y|)|x_k - y_k|(x_l - y_l)}{N^3|x - y|^2}. \quad (7.1)$$

The components of elasticity tensor for this system are calculated in [19]. They are determined by formulas:

$$a_{nnmn} = k_1 + \frac{2 k_2}{\sqrt{2}} + \frac{4 k_3}{3 \sqrt{3}}, \quad a_{nnpp} = a_{npnp} = \frac{k_2}{\sqrt{2}} + \frac{4 k_3}{3 \sqrt{3}} \quad (n \neq p)$$

and $a_{nppq} = 0$ in other cases.

Remark. If we take $k_1 = \frac{k_2}{\sqrt{2}} + \frac{8 k_3}{3 \sqrt{3}},$ then the components of limiting elasticity tensor are satisfying the condition: $a_{nnmn} = 2 a_{npmp} + a_{nppp}$ and the
limit model of elastic system is isotropic. The equation (2.12) has a form:

$$\frac{\partial^2 u}{\partial t^2} - a\Delta u + 2a\nabla \text{div} u + \int_{\Omega} G(x,y)(u(x) - u(y))dy = 0,$$

where $a = a_{npp} = a_{npn}$, and the elements of matrix $G(x,y)$ are defined by (7.1).

References

[1] Liew K.M., Zhang Y., Zhang L.W. Nonlocal elasticity theory for graphene modeling and simulation: prospects and challenges. Journal of modeling in Mechanics and Materials, 2017.

[2] Eringen A.C. Non-local polar field models. Academic Press, New York, 1996.

[3] Gopalakrishnan S., Narendar S. Wave Propagation in Nanostructures. Nonlocal Continuum Mechanics Formulations. Springer, 2013.

[4] Kröner E. Elasticity Theory of Materials with Long Range Cohesive Forces. Int. J. Solids and Structures, 1967, vol. 3, pp. 731-742.

[5] Kröner E. Problem of non-locality in the mechanics of solids: review on present status. In Proceedings of the Conference of Fundamental Aspects of Dislocation Theory. p.729-736.

[6] Eringen, A.C. Non-local polar elastic continua. Int. J. Eng. Sci. 10, 1972, p. 1-16.

[7] Eringen, A.C., Edelen D.G.B. On non-local elasticity. Int. J. Eng. Sci. 10, 1972, p. 233-248.

[8] Aifantis E.C. Gradient effects at macro, micro and nano scales. Journal of the Mechanical Behavior of Materials, 5, 1994, p. 355-375.
[9] Polizzotto C. Non local elasticity and related variational principles. International Journal of Solids and Structure, 38, 2001, p.7359-7380.

[10] Krumhansl, J.A., 1963. Generalized continuum field representation for lattice vibrations. In: Wallis, R.F. (Ed.), Lattice Dynamics, Proc. of Int. Conference. Pergamon Press, London.

[11] Kunin, I.A., 1982. Elastic Media with Microstructure I. Springer-Verlag, Berlin

[12] Di Paola, M., Failla, G., Zingales, M., 2009. Physically-based approach to the mechanics of strong non-local elasticity theory. Journal of Elasticity 97 (2), 103–130.

[13] Di Paola, M., Pirrotta, A., Zingales, M., 2010. Mechanically-based approach to nonlocal elasticity: variational principles. International Journal of Solids and Structures 47 (5), 539–548.

[14] R. D. Mindlin, Theories of elastic continua and crystal lattice theories, Chapter 3, pp. 312–320 in Mechanics of generalized continua (Stuttgart, 1967), edited by E. Kroner, Springer, Berlin, 1968.

[15] Born M, Huang K. Dynamical Theory of Crystal Lattices. Oxford University Press: London, 1954

[16] Tarasov V. E., General lattice model of gradient elasticity, Mod. Phys. Lett. B 28:7 (2014), article 1450054

[17] Tarasov V. E., Three-dimensional lattice models with long-range interactions of GrunwaldLetnikov type for fractional generalization of gradient elasticity, Meccanica (Milano) 51:1 (2016), 125138.

[18] Silling S.A. Reformulation of elasticity theory for discontinuities and long-range forces. Journal of Mechanics and Physics of Solids, 48, 2000, p.175-209.
[19] Berezhnyy M., Berlyand L. Continuum limit for three-dimensional mass-spring network and discret Korn’s inequality. Journal of the mech. and Phys. of Solids, 54 (2006), pp. 635-669.

[20] Marchenko V.A. and Khruslov E.Ya., Homogenization of Partial Differential Equations. Birkhäuser, Boston, Basel, Berlin, 2006.

[21] Sanchez-Palencia E. Non-homogeneous media and vibration theory. – Springer-Verlag Berlin Heidelberg, 1980. – 400 c.

[22] Piatnitski A.L., Chechkin G.A, Shamaev A.S. Homogenization. Methods and applications. – Novosibirsk, 2007, 250 p.

[23] M. Di Paola, G. Failla, M. Zingales. The mechanically-based approach to 3D non-local linear elasticity theory: long-range central interactions. International Journal of Solids and Structures, 47, 2010, p. 2347-2358.

[24] Oleinik O.A., Shamaev A.S., Yosifian G.A. Mathematical problems in elasticity and Homogenization. – North-Holland, 1992, 250p.

[25] Markushevich A.I. The theory of analytical functions. – Moscow: Mir Publishers, 1983, 423p.