Fermat-holonomic congruences

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9th March 2000

Abstract

Fermat-holonomic congruences are proposed as a weaker substitute for the too restrictive class of Born-rigid motions. The definition is expressed as a set of differential equations. Integrability conditions and Cauchy data are studied.

1 Introduction

The relevance of rigid motions in Newtonian mechanics basically stems from the following facts: (i) they model the motions of ideal rigid bodies, and also the behaviour of real rigid bodies at the first approximation level, (ii) they provide a definition of strains which, in elasticity theory, determine stresses, and (iii) they describe the relative motion of two Newtonian reference frames, i.e., those frames where Newton’s laws of mechanics hold, provided that inertial forces are taken into account.

A common feature of Newtonian rigid motions is that each one is unambiguously determined by the giving of the trajectory of one point together with the motion’s vorticity along that line (that is, the angular velocity).

The relativistic generalization of rigid motions needs to be formulated in terms of a spacetime manifold \((\mathcal{V}_4, g)\). A motion is then defined by a 3-parameter congruence of timelike worldlines, \(C\),

\[
x^\alpha(t) = \varphi^\alpha(t, y^1, y^2, y^3)
\]

where \(y^1, y^2, y^3\) are the parameters. In its turn, the congruence \(C\) is determined by its unit timelike velocity field

\[
u^\alpha(x), \quad \text{with} \quad g_{\mu\nu}u^\mu u^\nu = -1
\]
The stationary space $\mathcal{E}_3$ for the congruence is the quotient space, where cosets are worldlines in $\mathcal{C}$. The Fermat tensor $\hat{g}_{\alpha\beta}$

$$\hat{g}_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_{\beta}$$

(3)
yields the infinitesimal radar distance

$$d\hat{l}_R^2 = \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

(4)
between two neighbour worldlines. This quantity is not usually constant along a worldline in $\mathcal{C}$. Hence, it does not define an infinitesimal distance on $\mathcal{E}_3$.

Only in case that the Born-rigidity condition $\Sigma$

$$\Sigma_{\alpha\beta} = \mathcal{L}(u) \hat{g}_{\alpha\beta} = 0$$

(5)
holds, $g_{\alpha\beta}$ defines a Riemannian metric on $\mathcal{E}_3$.

The above condition $\Sigma$ consists of six independent first order partial differential equations with three independent unknowns, namely, $u^i$, $i = 1, 2, 3$ (since $u^4$ can be obtained from (2)), just like in the Newtonian case.

The class of Born-rigid motions would generalize Newtonian rigid motions also because some spatial distance between points in space is conserved. Unfortunately, the Herglotz-Noether theorem $\Sigma$ states that, even in Minkowski spacetime, the class of Born-rigid motions is narrower than sought. Indeed, motions combining arbitrary acceleration and rotation are excluded from this class.

Nevertheless, this shortness should not be surprising. Indeed, six first order partial differential equations for only three unknown functions unavoidably entail integrability conditions, which yield additional equations. The latter will lead to new integrability conditions, and so on. The process of completing the partial differential system $\Sigma$ ends up with a set of equations which is too restrictive for our expectations (namely, six degrees of freedom or six arbitrary functions of time).

In a recent work $\Sigma$ by one of us, 2-parameter congruences in a $(2+1)$-dimensional spacetime, $\mathcal{V}_3$, were considered as a simplification where the condition $\Sigma$ is still too restrictive (three partial differential equations for two unknowns: $u^1(x)$ and $u^2(x)$). Then, the vanishing of shear:

$$\sigma_{\alpha\beta} \equiv \Sigma_{\alpha\beta} - \frac{1}{2} \Sigma^\mu_{\mu} \hat{g}_{\alpha\beta} ; \quad \alpha, \beta = 0, 1, 2.$$  
(6)
was advanced as a candidate to substitute the condition of Born-rigidity.

The condition of vanishing shear (or, equivalently, conformal rigidity) reads:

$$\sigma_{\alpha\beta} = 0 \quad \alpha, \beta = 0, 1, 2$$

and yields two independent partial differential equations. Indeed, the six equations (7) are constrained by four relations:

$$g^{\mu\nu} \sigma_{\mu\nu} = 0 \quad \text{and} \quad \sigma_{\alpha\mu} u^\mu = 0, \quad \alpha = 0, 1, 2.$$ 

Since the number of unknown functions is also two, the condition (7) can be dealt by standard methods of solution of partial differential systems. Namely, a non-characteristic surface $S_2 \subset V_3$ and Cauchy data on it must be given so that a unique solution of (7) in a neighbourhood of $S_2$ is determined. The surface $S_2$ could be, for instance, a 1-parameter subcongruence.

In general, the amount of Cauchy data is much larger than our desideratum, namely, one worldline and the vorticity of the congruence in that line. We have however derived a way of getting a congruence out of a part of it.

In the particular case that one of the worldlines in the congruence is a geodesic, and the (2+1)-spacetime is flat, reference [4] goes a little further: given the congruence’s vorticity on the geodesic and assuming that strain vanishes on that worldline, the conformal rigidity condition (7) then determines a unique 2-parameter congruence. The latter would be useful to model a disk whose center is at rest (or in uniform motion), that spins at an arbitrary angular speed, and that remains as rigid as possible.

Another remarkable result in [4] is that it exists a flat, rigid, spatial metric, $g_{\alpha\beta}$, which is conformal to the Fermat tensor, $\hat{g}_{\alpha\beta}$.

The fact that the class of conformally rigid congruences in a (2+1)-spacetime is “wide enough” recalls the well know Gauss theorem [7]:

Any Riemannian 2-dimensional space can be conformally mapped into a flat space.

This suggests us a way to extend the results derived in (2+1)-spacetimes to (3+1)-spacetimes, namely, to inspire the formulation of “meta”-rigidity\(^1\)\(^2\).

\(^1\)This condition has been introduced in reference [4] as an enhancement of Einstein’s equivalence principle and named geodesic equivalence principle [4].

\(^2\)The word “meta”-rigidity was coined in [4] to generically refer to any relativistic extension of the notion of rigidity.
conditions in some extension of Gauss theorem to Riemannian 3-manifolds. One instance of the latter is [8], where Walberer’s theorem [9] is taken as the starting point. In the present paper we shall consider the following

**Theorem 1 (Riemann)** [10] Let \((M, g)\) be a Riemannian 3-manifold. They exist local charts of mutually orthogonal coordinates. Moreover, this can be done in a number of ways.

This means that six functions, namely, three coordinates \(y^i\) and three factors \(f_i\), \(i = 1, 2, 3\) can be locally found such that the metric coefficients in this local coordinates are

\[
g_{ij}(y) = f_i^2(y)\delta_{ij} \tag{8}
\]

that is, the Riemannian metric locally admits an orthogonal basis which is holonomic.

This result suggests us the following

**Definition 1** A congruence is said to be Fermat-holonomic iff its Fermat tensor \(\hat{g}_{\alpha\beta}\) admits an orthogonal basis which is holonomic.

That is, six functions exist: \(y^i(x), f_i(x), i = 1, 2, 3\) such that:

\[
\hat{g}_{\mu\nu}(x)dx^\mu \otimes dx^\nu = f_i^2(x)\delta_{ij}dy^i \otimes dy^j \tag{9}
\]

the summation convention is understood throughout the paper unless the contrary is explicitly indicated (if one of the repeated indices is in brackets, the convention is suspended in that formula). Greek indices run from 1 to 4 and lattin indices from 1 to 3.

Section 2 is devoted to develop some geometrical properties of Fermat-holonomic congruences, and in section 3 the existence of these congruence is discussed and posed as a Cauchy problem for a partial differential system. The method is somewhat similar to that used in proving the existence of orthogonal triples of coordinates in a Riemannian 3-manifold (it has been specially inspiring the reading of reference [11]). In section 4 a given 2-congruence is used as the Cauchy hypersurface for the aforementioned partial differential system, and the problem of getting a Fermat-holonomic congruence out of one of its parts (namely, a 2-parameter subcongruence) is studied. The kinematical meaning of the Cauchy data is also analysed. We must finally insist in the purely local validity of the results here derived. No global aspect of spacetimes has been considered.
2 Fermat-holonomic congruences

Let $\mathcal{C}$ be a Fermat-holonomic 3-parameter congruence and let $u(x)$ be the unit tangent vector. According to Definition 1, the Fermat tensor, $\hat{g}$, can be written as in equation (9). Consider the differential 1-forms:

$$\omega^i = f_i dy^i$$  \hspace{1cm} (10)

We thus have:

$$\hat{g} = \delta_{ij} \omega^i \otimes \omega^j$$  \hspace{1cm} (11)

Moreover, since $\hat{g}$ is orthogonal to $u$, we have that

$$i(u)\omega^l = 0$$  \hspace{1cm} (12)

As a consequence, the functions $y^i$ are constant along any worldline in the congruence:

$$u(y^i) = 0$$

Let us now introduce the differential 1-form:

$$\omega^4 \equiv -g(u, \omega) = u_\alpha(x)dx^\alpha.$$  \hspace{1cm} (13)

From (3) and (11) it follows that

$$g = \hat{g} - \omega^4 \otimes \omega^4 \equiv \eta_{\alpha\beta} \omega^\alpha \omega^\beta$$

By definition [equation (10)] the 1-forms $\omega^i$ must be integrable or, equivalently, they must satisfy:

$$d\omega^i \wedge \omega^i = 0$$  \hspace{1cm} (14)

As a result we have thus proved the following

**Theorem 2** Let $(\mathcal{V}_4, g)$ be a spacetime, $\mathcal{C}$ a Fermat-holonomic 3-parameter congruence and $u$ the unit velocity vector. There exist three integrable 1-forms $\omega^i, i = 1, 2, 3$ such that completed with $\omega^4 \equiv -g(u, \omega)$, yield a $g$-orthonormal basis.

The converse theorem can be easily proved too.
Theorem 3  If \( \{\omega^a\}_{a=1..4} \) is a g-orthonormal thetrad such that \( \omega^i, i = 1, 2, 3 \) are spacelike and integrable, then the integral curves of \( u \), the vector that results from raising the index in \( \omega^4 \), form a 3-parameter Fermat-holonomic congruence.

We shall now present some geometric properties of Fermat-holonomic congruences.

Proposition 1  Let \( \omega^l \in \Lambda^1(V_4) \), \( l = 1, 2, 3 \), be the orthonormal set fulfilling conditions (12), (13) and (14) above. Then
\[
\mathcal{L}(u)\omega^l \wedge \omega^l = 0 \quad (15)
\]
Proof: Using (12) and (14) we can write:
\[
\mathcal{L}(u)\omega^l \wedge \omega^l = [i(u)d\omega^l + d(i(u)\omega^l)] \wedge \omega^l = i(u)d\omega^l \wedge \omega^l
\]
\[
= i(u)[d\omega^l \wedge \omega^l] - d\omega^l \wedge i(u)\omega^l = 0 \quad (16)
\]

Proposition 2  The strain rate tensor \( \Sigma \equiv \mathcal{L}(u)\hat{g} \) has \( \omega^i, i = 1, 2, 3 \) as principal directions. Furthermore, the same holds for any of its Lie derivatives along the congruence: \( \Sigma^{(n)} \equiv \mathcal{L}(u)^n\Sigma = \mathcal{L}(u)^{n+1}\hat{g} \)

Proof: As a consequence of proposition 1, they exist three functions \( \phi_l \), \( l = 1, 2, 3 \) such that \( \mathcal{L}(u)\omega^l = \phi_l\omega^l \). Hence
\[
\Sigma \equiv \mathcal{L}(u)\hat{g} = 2\phi_i\delta_{ij}\omega^i \otimes \omega^j .
\]
The second statement, concerning \( \Sigma^{(n)} \), is easily shown by induction. □

A sort of converse result is the following

Proposition 3  If \( \mathcal{L}(u)\Sigma \) and \( \Sigma \) diagonalize in the same g-orthonormal basis, then it exists an orthonormal set \( \omega^l \in \Lambda^1(V_4) \), such that
\[
\mathcal{L}(u)\omega^l \wedge \omega^l = 0 \quad (17)
\]
Proof: According to the hypothesis there exist three 1-forms \( \rho^i, i = 1, 2, 3 \) such that

\[
\hat{g} = \delta_{ij} \rho^i \otimes \rho^j, \quad \Sigma = 2\phi_i \delta_{ij} \rho^i \otimes \rho^j
\]  
(18)

\[
\mathcal{L}(u) \Sigma = 2\psi_i \delta_{ij} \rho^i \otimes \rho^j
\]  
(19)

These \( \rho^i \)'s are orthogonal to \( u \), and the same holds for \( \mathcal{L}(u) \rho^i \), hence:

\[
\mathcal{L}(u) \rho^j = A^j_i \rho^k
\]  
(20)

The latter can be used to calculate the Lie derivatives of (18):

\[
\Sigma = \mathcal{L}(u) \hat{g} = (A^r_i \delta_{rj} + A^r_j \delta_{ri}) \rho^i \otimes \rho^j
\]
\[
\mathcal{L}(u) \Sigma = 2(\dot{\phi}_i \delta_{ij} + \phi_i \delta_{jr} A^r_j + \phi_j \delta_{jr} A^r_i) \rho^i \otimes \rho^j
\]

which compared with (18) and (19) yield:

\[
(A^i_i + A^i_j) = 2\phi_{(i)} \delta_{ij}
\]  
(21)

\[
\dot{\phi}_{(i)} \delta_{ij} + \phi_{(i)} A^j_i + \phi_{(j)} A^i_j = \psi_{(i)} \delta_{ij}
\]  
(22)

From (21) we have that:

\[
A^i_i = \phi_i, \quad A^j_i = -A^i_j, \quad i \neq j
\]  
(23)

which substituted in equation (22) yields:

for \( i = j \):  
\[
\dot{\phi}_i + 2\phi_i^2 = \psi_i
\]

for \( i \neq j \):  
\[
(\dot{\phi}_{(i)} - \dot{\phi}_{(j)}) A^j_i = 0
\]  
(24)

Now three cases must be considered according to the degeneracy of the eigenvalues of \( \Sigma \).

a) In the case \( \phi_1 \neq \phi_2 \neq \phi_3 \) the set \( \{ \rho^i \} \) is unambiguously defined. Eq.(24) implies \( A^j_i = 0, i \neq j \). Then taking (23) and (21) into account we obtain \( \mathcal{L}(u) \rho^i = \phi_{(i)} \rho^i \) and equation (17) follows for \( \omega^i = \rho^i \).

b) In the case \( \phi_1 = \phi_2 \neq \phi_3 \) from (23) and (24), we obtain:

\[
A^1_i = \phi_i, \quad i = 1, 2, 3; \quad A^2_1 = -A^1_2, \quad A^3_a = -A^a_3 = 0, \quad a = 1, 2
\]  
(25)
which introduced in (20) yield:
\[
\mathcal{L}(u)\rho^3 = \phi(3)\rho^3 \quad ; \quad \mathcal{L}(u)\rho^a = A^a_{\ b}\rho^b \quad a, b = 1, 2
\] (26)

In the present case, however, the set
\[
\omega^3 = \rho^3 \quad \text{and} \quad \omega^a = R^a_{\ b}\rho^b \quad a, b = 1, 2
\]
with \((R^a_{\ b}) \in O(2)\) is also a set of eigenvectors for \(\Sigma\). Now, an orthogonal matrix \((R^a_{\ b})\) can be found such that \(\mathcal{L}(u)\omega^a = \phi\omega^a\), with \(\phi_1 = \phi_2 = \phi\).

Indeed, since
\[
\mathcal{L}(u)\omega^a = [\mathcal{L}(u)R^a_{\ b}(R^{-1})^b_c + (RAR^{-1})^a_c]\omega^c
\]
it is enough to require
\[
\mathcal{L}(u)R^a_{\ b} = R^a_{\ c}(-A^c_{\ b} + \phi\delta^c_{\ b})
\]
which has many solutions \((R^a_{\ b}) \in O(2)\) because, by equation (25),
\[-A^c_{\ b} + \phi\delta^c_{\ b}\]
is skewsymmetric.

c) The completely degenerate case \(\phi_1 = \phi_2 = \phi_3\) can be handled in a similar way as case (b). □

**Theorem 4** \(\mathcal{L}(u)\Sigma\) and \(\Sigma\) diagonalize in a common \(g\)-orthonormal basis if, and only if, three functions \(A\), \(B\) and \(C\), exist such that
\[
\mathcal{L}(u)\Sigma = A\hat{g} + B\Sigma + C\Sigma^2
\]
(27)
where \(\Sigma^2_{\ a\beta} \equiv \Sigma_{\ a\mu}\Sigma^\mu_{\ \beta}\). Moreover if two among the eigenvalues of \(\Sigma\) are equal, then \(C = 0\) can be taken, and in the completely degenerate case, \(B = C = 0\) can be taken.

**Proof:**
\((\Rightarrow):\) Since \(u\) is orthogonal to both \(\Sigma\) and \(\mathcal{L}(u)\Sigma\), we shall have that \(\omega^4 \equiv -g(u, \cdot)\) is in the common orthogonal basis \(\{\omega^a\}\). Thus, expressions similar to (18) and (19) hold. Hence to prove (27) amounts to solve the linear system:
\[
A + 2\phi_i B + 4\phi_i^2 C = 2\psi_i
\]
(28)
for the unknowns $A$, $B$ and $C$. The determinant is:

$$\Delta = 8(\phi_2 - \phi_1)(\phi_2 - \phi_3)(\phi_3 - \phi_1).$$

In the non-degenerate case $\Delta \neq 0$ and (28) has a unique solution.

If $\phi_1 = \phi_2 \neq \phi_3$, only the equations for $l = 2$ and 3 in (28) are independent, there are infinitely many solutions and $C$ can be arbitrarily chosen. In particular, $C = 0$.

Finally, in the completely degenerate case, (28) has rank 1, hence it admits infinitely many solutions, and $B$ and $C$ are arbitrary.

$(\Leftarrow)$: Assume that (27) holds. Since $\Sigma$ diagonalize in a $g$-orthonormal basis, we substitute (18) in (27) and it follows immediately that $\mathcal{L}(u)\Sigma$ diagonalize in the same $g$-orthonormal basis.

**Theorem 5** If $\Sigma$, $\mathcal{L}(u)\Sigma$ and $\mathcal{L}(u)^2\Sigma$ diagonalize in a common $g$-orthonormal basis, then $\hat{g}$, $\Sigma$, $\mathcal{L}(u)\Sigma$ and $\mathcal{L}(u)^2\Sigma$ are linearly dependent. (Hence, the congruence is non-generic [12].)

**Proof**: By theorem 4, they exist $A$, $B$ and $C$ such that (27) holds. Taking the Lie derivative on both sides, using that $\Sigma = \mathcal{L}(u)\hat{g}$ and equation (27) itself, and taking into account that the minimal polynomial for $\Sigma \beta$ has at most degree 3, we arrive at:

$$\mathcal{L}(u)^2\Sigma = A'\hat{g} + B'\Sigma + C''\Sigma^2$$

where $A'$, $B'$ and $C''$ are some suitable functions.

If $C = 0$, then (27) already proves the theorem.

If, on the contrary $C \neq 0$, we can derive $\Sigma^2$ from (27) and substitute it into (28), so arriving at:

$$\mathcal{L}(u)^2\Sigma = \left(A' - \frac{A}{C}\right)\hat{g} + \left(B' - \frac{B}{C}\right)\Sigma + \frac{C'}{C}\mathcal{L}(u)\Sigma$$

which ends the proof. $\square$

### 3 Existence of Fermat-holonomic congruences

According to the theorems 2 and 3 in section 2, proving the existence of Fermat-holonomic congruences is equivalent to prove the existence of a $g$-orthonormal basis $\{\omega^\alpha\}$ such that:
(i) $\omega^i$ is spacelike, and

(ii) $d\omega^i \wedge \omega^i = 0$ (14)

The dual thetrad will be denoted $\{e_\alpha\}$ and the commutation relations:

$$[e_\alpha, e_\beta] = C^\mu_{\alpha\beta} e_\mu \quad d\omega^\alpha = -\frac{1}{2} C^\alpha_{\mu\nu} \omega^\mu \wedge \omega^\nu$$

(30)

using the latter, (14) can be written as:

$$-\frac{1}{2} C^i_{\mu\nu} \omega^\mu \wedge \omega^\nu \wedge \omega^i = 0 \quad i = 1, 2, 3$$

(31)

which in turn is equivalent to:

$$\begin{align*}
C^i_{jk} &= 0 & i \neq j \neq k \\
C^i_{ij} &= 0 & i \neq j
\end{align*}$$

(32)

Taking into account the relationship between $C^\gamma_{\alpha\beta}$ and the Riemannian connexion coefficients in an orthonormal frame [13], equations (32) are equivalent to:

$$\begin{align*}
\gamma^i_{jk} &= 0 & i \neq j \neq k \\
C^i_{ij} &= \gamma^i_{ij} - \gamma^i_{j4} = 0 & i \neq j
\end{align*}$$

(33)

(34)

For a Riemannian connexion in an orthonormal frame, it holds:

$$\gamma^i_{\alpha k} = -\gamma^k_{\alpha i} \quad , \quad \gamma^4_{\alpha k} = \gamma^k_{\alpha 4} \quad \alpha = 1...4, \quad i, k = 1, 2, 3$$

where the signature (+ + + −) has been used. Hence, at most three among the equations (33) are independent.

Since

$$\gamma^\gamma_{\alpha\beta} = (\omega^\gamma, e_\alpha \nabla e_\beta) = \eta^{\gamma\nu} g(e_\nu, e_\alpha \nabla e_\beta)$$

(35)

equations (33) and (34) yield a first-order partial differential system of nine equations where the unknown is the orthonormal frame $\{e_\alpha\}$.
3.1 The Cauchy problem

Given a hypersurface \( S_0 \), consider a 1-parameter family of hypersurfaces, \( S_\lambda \), containing it. Let \( n \) be the unit orthogonal vector field which, by construction, is hypersurface orthogonal, i.e., \( \nu = g(n, \cdot) \) is an integrable 1-form.

Relatively to \( n \), each vector \( e_\alpha \) of the sought frame can be decomposed in an orthogonal part \( e_\alpha^\top \) (which is tangent to the hypersurfaces \( S_\lambda \)) and a parallel part, namely:

\[
e_\alpha = e_\alpha^\top + n_\alpha n
\]

where \( n_\alpha \equiv g(e_\alpha, n) \).

Substituting this decomposition in (33) and (34), taking into account equation (35) and the fact that the connexion is Riemannian, we respectively arrive at:

\[
\begin{align*}
\gamma^i_{jk} &= n_j W_{ik} + g(e_i, e_j^\top \nabla e_k) = 0 & i \neq j \neq k \\
C^i_{4k} &= n_4 W_{ik} - n_k W_{i4} + H_{ik} = 0 & i \neq k
\end{align*}
\]

where

\[
W_{\alpha\beta} \equiv g(e_\alpha, n^\top e_\beta)
\]

is skewsymmetric, and

\[
H_{ik} \equiv g(e_i, [e_4^\top, e_k^\top]) + n_i \left( e_4^\top (n_k) - e_k^\top (n_4) \right) + n_k g(e_i, e_4^\top n) - n_4 g(e_i, e_k^\top n)
\]

only depends on the unknowns and their derivatives along directions that are tangential to \( S_\lambda \).

If, and only if, \( n_j \neq 0 \), the nine equations (37) and (38) can be solved for the six independent components of \( W_{\alpha\beta} \). A straight manipulation yields:

\[
\begin{align*}
W_{ik} &= -\frac{1}{n_j} g(e_i, e_j^\top \nabla e_k) & i \neq j \neq k \\
W_{i4} &= \frac{1}{n_k} H_{ik} - \frac{n_4}{n_k n_j} g(e_i, e_j^\top e_k) & i \neq j \neq k
\end{align*}
\]

Notice that for each value of \( i = 1, 2, 3 \) there are two ways of choosing \( j \neq k \) in equation (42). This comes from the fact that the system (37)–(38) is overdetermined. Hence, three subsidiary conditions follow:

\[
S_i \equiv \frac{1}{n_k} H_{ik} - \frac{1}{n_j} H_{ij} - \frac{n_4}{n_k n_j} g(e_i, e_j^\top e_k - e_k^\top e_j) = 0 & i \neq j \neq k
\]
these conditions only depend on “tangential derivatives” of the unknowns.

**Lemma 1** Given an orthonormal thetrad \( \{ \tau_\alpha \} \) on \( S_0 \), such that \( g(n, \tau_i) \neq 0 \), it exists a neighbourhood \( U \) of \( S_0 \) and an orthonormal thetrad \( \{ e_\alpha \} \) which is a solution of

\[
W_{ik} = -\frac{1}{n_j} g(e_i, e_j \nabla e_k)
\]

\[
W_{i4} = \frac{1}{n_k} H_{ik} - \frac{n_4}{n_k n_j} g(e_i, e_j \nabla e_k)
\]

where \( (ikj) \) is a cyclic permutation of \( (123) \) and such that \( e_\alpha = \tau_\alpha \) on \( S_0 \).

**Proof:** Let \( \{ e_\alpha \} \) be a given orthonormal frame. In terms of it the unknown frame \( \{ e_\alpha \} \) can be written as:

\[
e_\alpha = L^\mu_\alpha \tilde{e}_\mu
\]

where \( (L^\beta_\alpha) \) is a Lorentz matrix valued function.

Substituting the latter into equation (43) we have:

\[
W_{\alpha\beta} = L^\mu_\alpha g(\tilde{e}_\mu, n^\nu (L^\nu_\beta \tilde{e}_\nu))
\]

Then, after a straightforward calculation, we arrive at:

\[
nL^\beta_\alpha = L^\beta\mu W_{\mu\alpha} - \eta^{\alpha\beta} L^\rho_\alpha g(\tilde{e}_\mu, n^\nu \tilde{e}_\rho)
\]

where indices are raised by contraction with \( \eta^{\alpha\beta} \) and \( nL^b_\alpha \) is the directional derivative along \( n \).

The second term in the right hand side is known and, since \( g(n, \tilde{e}_i) \neq 0 \) on \( S_0 \), equation (44) yield \( W_{\alpha\beta} \) on \( S_0 \) as a function of \( e_\alpha \) and their “tangential” derivatives (i.e. \( L^\beta_\alpha \) and their “tangential” derivatives). Hence, the Cauchy-Kowalevski theorem [14] can be invoked to end the proof. \( \square \)

**Lemma 2** Let \( \{ e_\alpha \} \) be an orthonormal thetrad, which is a solution of (44) and fulfills the subsidiary conditions (43) on \( S_0 \). Then (43) holds in the neighbourhood \( U \) of \( S_0 \) where \( \{ e_\alpha \} \) is defined.

**Proof:** Since \( \{ e_\alpha \} \) is a solution of (44),

\[
C^i_{jk} = 0 \quad , \quad C^i_{4k} = 0 \quad C^i_{4j} = -n_j S_i
\]
where \((ikj)\) is a cyclic permutation of \((123)\) and \(S_i\) is defined in (43).

The commutation coefficients \(C^\mu_{\alpha\beta}\) satisfy the Jacobi like identity:

\[
\eta^{\rho\mu\nu}(e_{\rho}C^\beta_{\mu
u} - C^\beta_{\sigma\nu}C^\sigma_{\mu\rho}) = 0 \quad (47)
\]

Taking \(\alpha = \beta = i\) in (47) and using (46) we arrive at:

\[
e_kC^i_{4j} + C^i_{4j}(C^{(j)}_{k(j)} + C^4_{k4} - C^{(i)}_{k(i)}) = 0, \quad \varepsilon_{ikj} = 1
\]

and taking the decomposition (36) into account, and the fact that \(n_k \neq 0\) we have

\[
n(C^i_{4j}) + \frac{1}{n_k}[e_k^\top C^i_{4j} + A_kC^i_{4j}] = 0, \quad (48)
\]

with \(A_k = C^{(j)}_{k(j)} + C^4_{k4} - C^{(i)}_{k(i)}\).

The latter can be considered as a partial differential system on \(C^i_{4j}\). In case that \(n_k \neq 0\), the hypersurface \(S_0\) is non-characteristic, and the Cauchy-Kowalevski theorem states that (48) has a unique solution, which for \(S_i = 0\) on \(S_0\), ensures that \(C^i_{4j} = 0\) in a neighbourhood of \(S_0\) and, equivalently, \(S_i = 0\) in a neighbourhood of \(S_0\).

Summarizing, we have shown the following

**Theorem 6**

Given an orthonormal tetrad \(\{e_\alpha\}\) on \(S_0\), such that:

(i) \(g(n, e_i) \neq 0\) and

(ii) \(S_i = 0\) on \(S_0\)

it exists an orthonormal frame \(\{e_\alpha\}\) in a neighbourhood \(U\) of \(S_0\), such that:

(i) it is a solution of (44) and (43), and

(ii) \(e_\alpha = \bar{\tau}_{\alpha}\) on \(S_0\)

Thus, the congruence generated by \(u = e_4\) is Fermat-holonomic.

The proof is straightforward from lemmas 1 and 2.

\(\square\)
4 Getting a Fermat-holonomic 3-congruence out of a given 2-congruence

Let \( S_0 \) be the hypersurface spanned by a given 2-congruence of worldline \( s \) and let \( \pi \) be the unit velocity vector. (Hereafter, a bar over a symbol indicates that we are only considering the values of that object on \( S_0 \).)

4.1 Are the subsidiary conditions consistent?

We are interested in completing an orthonormal thetrad \( \{ \varepsilon_\alpha \} \) on \( S_0 \), such that:

\[ \varepsilon_4 = \pi, \quad n_i \equiv g(\varepsilon_i, \pi) \neq 0, \quad \text{and} \quad S_i = 0 \text{ on } S_0, \]

where \( n \) is the unit vector orthogonal to \( S_0 \).

For each \( \varepsilon_i \), we consider the decomposition (36):

\[ \varepsilon_i = \varepsilon_i^\top + n_i \pi \]

and take the unit vector

\[ \hat{a}_i \equiv \frac{\varepsilon_i^\top}{|\varepsilon_i^\top|} \]  

(49)

Furthermore, we can consider the combinations:

\[ b_i = \varepsilon_i^{jk} n_j \varepsilon_k \]

(50)

and define the unit vector \( \hat{b}_i \equiv b_i / |b_i| \). It is straightforward to see that \( \{ \pi, \pi, b_i, \hat{a}_i \} \) is an orthonormal thetrad at any point in \( S_0 \).

We also have that

\[ g(\hat{a}_i, \hat{a}_j) = \frac{1}{|\varepsilon_i^\top| |\varepsilon_j^\top|} g(\varepsilon_i^\top, \varepsilon_j^\top) = \frac{1}{|\varepsilon_i^\top| |\varepsilon_j^\top|} (\delta_{ij} - n_i n_j) \]

Now, since \( n_i \neq 0 \), then \( |n_j| < 1 \) and it follows that:

\[ \forall i \neq j \ |g(\hat{a}_i, \hat{a}_j)| < 1 \quad \text{and} \quad g(\hat{a}_i, \hat{a}_j) \neq 0 \]  

(51)

Thus, for any triad \( \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) in \( T_x \mathcal{V}_4, \ x \in S_0 \), such that: \( g(\varepsilon_i, \pi) = 0 \), and \( g(\varepsilon_i, \pi) = n_i \neq 0 \), we can obtain a triad \( \hat{a}_i \in T_x S_0, \ i = 1, 2, 3 \), such that \( g(\hat{a}_i, \pi) = 0 \) and that (51) holds.

The converse result is the following
Theorem 7  Given a triad $\hat{\alpha}_i \in T_xS_0$, $i = 1, 2, 3$, such that $g(\hat{\alpha}_i, \pi) = 0$ and fulfills (51), a triad $\{\tau_i\}$ can be obtained such that $g(\tau_i, \pi) \neq 0$ and that $\{\tau_1, \tau_2, \tau_3, \tau_4 = \pi\}$ is an orthonormal frame.

Proof: We must find $A_i$ and $B_i$ such that

$$\tau_i = A_i \hat{\alpha}_i + B_i \pi$$

since $\hat{\alpha}_i$, and $\pi$ are $u$-orthogonal, so will be $\tau_i$.

Now, the condition $g(\tau_i, \tau_j) = \delta_{ij}$ implies:

$$A_i^2 + B_i^2 = 1 \quad (52)$$

$$A_i A_j g(\hat{\alpha}_i, \hat{\alpha}_j) + B_i B_j = 0 \quad i \neq j \quad (53)$$

From which we easily obtain:

$$B_i = A_i \sqrt{-g(\hat{\alpha}_i, \hat{\alpha}_j) g(\hat{\alpha}_i, \hat{\alpha}_k) \over g(\hat{\alpha}_j, \hat{\alpha}_k)} \quad i \neq j \neq k \quad (54)$$

and

$$A_i = \sqrt{-g(\hat{\alpha}_i, \hat{\alpha}_j) \over g(\hat{\alpha}_j, \hat{\alpha}_k) - g(\hat{\alpha}_i, \hat{\alpha}_j) \cdot g(\hat{\alpha}_i, \hat{\alpha}_k)} \quad i \neq j \neq k \quad (55)$$

The right hand side of (54) is well defined because, by the hypothesis, the inequalities (51) hold.

The denominator in (53) neither vanishes, as a consequence of (51) too. Indeed, denoting by $\varphi_{ij}$ the angle between $\hat{\alpha}_i$ and $\hat{\alpha}_j$, and taking into account that $|\varphi_{ij}| + |\varphi_{jk}| + |\varphi_{ki}| = 2\pi$, $i \neq j \neq k$ this denominator reads:

$$\cos(\varphi_{jk}) - \cos(\varphi_{ij}) \cdot \cos(\varphi_{ik}) = \sin(\varphi_{ij}) \cdot \sin(\varphi_{ik}) \quad i \neq j \neq k$$

which does not vanish because $|g(\hat{\alpha}_i, \hat{\alpha}_j)| = |\cos(\varphi_{ij})| < 1$, $\forall i \neq j$. \hfill \Box

We shall attempt to find $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ such that the triad $\{\tau_i\}$ so reconstructed fulfill the subsidiary condition (14). It is easily seen that $S_i$ can be written as:

$$S_i = \frac{1}{n_j} g(e_i, [e_4, e_j]) - \frac{1}{n_k} g(e_i, [e_4, e_k]) = - \frac{1}{n_j n_k} g(e_i, [e_4, b_i])$$

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(where $(ijk)$ is a cyclic permutation of $(123)$) on the hypersurface $S_0$, that is:

$$S_i = - \frac{1}{\pi_j \pi_k} g(\tau_i, [\pi, \overline{b}_i])$$

Taking now into account that $\pi$ and $\overline{b}_i$ are tangent to $S_0$, (hence, $[\pi, \overline{b}_i]$ is tangent too), we can write:

$$-\pi_j \pi_k S_i = g(\overline{\pi}_i, [\pi, \overline{b}_i]) = |\overline{\pi}_i| \cdot |\overline{b}_i| g(\overline{a}_i, [\pi, \overline{b}_i])$$

so that the subsidiary conditions $S_i = 0$ are equivalent to

$$g(\overline{a}_i, [\pi, \overline{b}_i]) = 0 \quad (56)$$

Let $\pi$ and $\overline{m}$ be two unit vector fields such that $\{\pi, \overline{m}, \overline{v}\}$ is an orthonormal basis on the tangent space of $S_0$. In terms of this basis, we can write:

$$\begin{align*}
\overline{b}_i &= \cos \theta_i \overline{m} + \sin \theta_i \overline{v} \\
\overline{a}_i &= -\sin \theta_i \overline{m} + \cos \theta_i \overline{v}
\end{align*} \quad (57)$$

and the conditions (56) lead to:

$$\dot{\theta} + \frac{1}{4} \sin 2\theta_i (\Sigma_{ve} - \Sigma_{mm}) + \frac{1}{2} \cos 2\theta_i \Sigma_{mv} + \frac{1}{2} (g(\overline{v}, [\pi, \overline{m}]) - g(\overline{m}, [\pi, \overline{v}])) = 0 \quad (58)$$

which is an ordinary differential equation for each $\theta_i, i = 1, 2, 3$, and has a solution for every initial data $\theta^0_i$ given in a submanifold $\mathcal{M} \subset S_0$, such that it is nowhere tangent to $\pi$.

Summarizing, we have thus proved that given a 2-congruence, it spans a hypersurface $S_0$ on which Cauchy data $\{\tau_1, \tau_2, \tau_3, \tau_4 = \pi\}$ can be found fulfilling the subsidiary conditions (43). Moreover, this can be done in an infinite number of ways.

### 4.2 The kinematical meaning of the Cauchy data

In the particular case considered in this section, where the Cauchy hypersurface $S_0$ is spanned by a 2-congruence, we shall analyse the kinematical significance of the Cauchy data $\{\overline{v}_i\}_{i=1,2,3}$. 
According to Proposition 2, the latter is a principal basis for $\Sigma$ (i.e., the values of the strain rate of the Fermat-holonomic 3-congruence. We shall see how $\{\tau_i\}_{i=1,2,3}$ determine $\Sigma$.

Indeed, let $\pi$ be the unit vector normal to $S_0$ and $\pi \equiv g(\pi, \omega)$. $\Sigma$ can be written as:

$$\Sigma = \Sigma^0 + \pi \otimes \pi + \pi \otimes \pi + \pi \otimes \pi \otimes \pi \tag{59}$$

where:

$$\Sigma^0(\pi, \omega) = \Sigma^0(\pi, \omega) = 0 \quad \text{and} \quad \langle \pi, \pi \rangle = 0 .$$

The condition that $\tau_i$ is a principal vector then reads:

$$\exists \lambda_i \text{ such that } \Sigma(\tau_i, \omega) = \lambda_i \omega_i$$

which, using (59) and considering the parallel and orthogonal parts, respectively yields:

$$\langle \pi, \tau_i \rangle = \frac{\pi_i}{\lambda_i} \pi_i \tag{60}$$

$$\Sigma^0(\tau_i^T, \omega) + \pi_i \pi = \lambda_i \hat{g}(\tau_i^T, \omega) \tag{61}$$

Now, taking into account (36) and expressions like

$$\pi \cdot \tau_l = \pi \quad \text{and} \quad \sum_{i=1}^{3}(\pi')^2 = 1 ,$$

after a short calculation, we arrive at:

$$\pi = -\sum_{j \neq l} \frac{1}{\eta_j} \Sigma^0(\tau_j^T, \tau_l^T) \tau_l^T , \quad \text{where} \quad \pi = \hat{g}(\pi, \omega) \tag{62}$$

$$\bar{s} = 2 \Sigma^0(\epsilon_j^T, \epsilon_l^T) \delta_{lj} + \sum_{j \neq l} \frac{1}{\eta_j \eta_l} \Sigma^0(\epsilon_j^T, \epsilon_l^T) \tag{63}$$

$$\lambda_i = \Sigma^0(\epsilon_i^T, \epsilon_i^T) \delta_{ij} + \frac{1}{\eta_i \eta_k} \Sigma^0(\epsilon_j^T, \epsilon_l^T) \tag{64}$$

Let us now see what does $\Sigma^0$ mean. Given any couple of vector fields, $\tau$ and $\bar{w}$, that are tangent to $S_0$, we have that:

$$\Sigma^0(\tau, \bar{w}) = \Sigma(\tau, \bar{w}) = \mathcal{L}(u) \hat{g}(\tau, \bar{w})$$

$$= \tau \left( \hat{g}(\tau, \bar{w}) \right) - (\hat{g}(\tau, \bar{w}), \bar{w}) - (\hat{g}(\tau, \bar{w}), \bar{w})$$

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Now, since $S_0$ is a submanifold, $[\pi, \nu]$ and $[\pi, \nu]$ are also tangent to $S_0$. Hence, the Fermat tensor $\tilde{g}$ on the right hand side can be replaced by its restriction to $S_0$, namely, $\tilde{g}^0$. So that,
\[
\Sigma^0 = \mathcal{L}(\pi)\tilde{g}^0
\]
Thus, $\Sigma^0$ is the strain rate of the given 2-congruence, $C_2$ in the Riemannian submanifold $(S_0, \tilde{g}^0)$.

We have so far shown the following

**Proposition 4** The Cauchy data $(S_0, \bar{e}_1, \bar{e}_2, \bar{e}_3, \e_4 = \nu)$ determine $\Sigma$, i.e., the values on $S_0$ of the strain rate of the Fermat-holonomic congruence obtained as a solution of the partial differential system (37)–(38).

The converse is true only if all the eigenvalues of $\Sigma$ restricted to spacelike vectors have multiplicity 1. Indeed, if some eigenvalue of $\Sigma$ has multiplicity greater than 1, then the eigenvectors are determined up to a rotation.

### 4.3 Uniqueness

From Proposition 4 it seems to follow that, given the 2-congruence $C_2$ and $\Sigma$, there is a unique Fermat-holonomic 3-congruence that includes $C_2$ and its strain rate takes the values $\Sigma$ on the submanifold $S_0$. We shall now see that, although this is the generic case, it is not always true.

Once $C_2$ is given, $\Sigma$ is no more arbitrary. Indeed, the first block $\Sigma^0$ in the decomposition (59) is already determined by $C_2$. Hence, it is enough to give the 1-form $\pi$ and the scalar $s$ on $S_0$, and use (59) to determine $\Sigma$. Then, a basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of spacelike eigenvectors of $\Sigma$ can be chosen.

In order that $(S_0, \bar{e}_1, \bar{e}_2, \bar{e}_3, \e_4 = \bar{\nu})$ is a suitable set of Cauchy data for the partial differential system (57)–(58) it is necessary that $\pi_i \neq 0$, $i = 1, 2, 3$. This is equivalent to require that no $\bar{e}_i$ is tangent to $S_0$.

Now, for $\bar{e}_i$ to be tangent to $S_0$ we should have that:
\[
\langle \pi, \bar{e}_i \rangle = 0 \quad \text{and} \quad \Sigma^0(\bar{e}_i, \omega) = \lambda_i \tilde{g}^0(\bar{e}_i, \omega),
\]
where (50) and (51) have been used. That is, $\bar{e}_i$ is a principal vector of $\Sigma^0$ and $\pi$ is orthogonal to $\bar{e}_i$. Hence, $\pi$ itself should be an eigenvector of $\Sigma^0$.

Thus, avoiding to choose $\bar{\nu}$ among the eigenvectors of $\Sigma^0$ guarantees that $\pi_i \neq 0$, $i = 1, 2, 3$. Of course, this is not possible when $\Sigma^0$ is a multiple of $\tilde{g}^0$.

So that, several possibilities occur:
(A) Either $\Sigma$ has three simple eigenvalues and therefore the eigenvectors \(\{\tau_i\}_{i=1,2,3}\) are uniquely determined, in which case:

(A.1) either $\tau_i \neq 0$, $i = 1, 2, 3$, and $(S_0, \tau_1, \tau_2, \tau_3, \tau_4 = \tau)$ is a suitable set of Cauchy data,

(A.2) or some $\tau_i = 0$, and $S_0$ is characteristic.

(B) Or some eigenvalue of $\Sigma$ is not simple. Then, the basis of eigenvectors is determined up to a rotation.

Therefore, only in the case (A.1) $C_2$ and $\Sigma$ determine a unique Fermat-holonomic 3-congruence.

5 Conclusion and outlook

Looking for a less restrictive substitute for Born’s relativistic definition of rigid motion, we have suggested the definition of Fermat-holonomic motion, namely, a 3-parameter congruence of timelike worldlines which admits an adapted system of coordinates $(t, y^1, y^2, y^3)$, such that the hypersurfaces $y^i =$constant, $i = 1, 2, 3$, are mutually orthogonal (i.e., the Fermat tensor is diagonal: $\hat{g}_{ij}(t, y) = 0$ whenever $i \neq j$).

We have expressed this condition as a partial differential system and analysed the Cauchy problem. We have proved — theorem 6 — that, given a 2-parameter congruence $C_2$ and a triad of spatial vectors $\{\tau_i\}_{i=1,2,3}$ on $S_0$ (the track of $C_2$), there is a unique Fermat-holonomic 3-congruence, $C_3$, containing $C_2$ and admitting an adapted coordinate system such that the spatial coordinate curves are tangent to $\tau_i$ on $S_0$.

Moreover, the given directions $\{\tau_i\}$ are related to the eigenvectors of the strain rate tensor $\Sigma$ and, except in the case that the shear of $C_3$ vanishes on $S_0$, we have shown that $C_2$ together with $\Sigma$ determine $C_3$ in a neighbourhood of $S_0$. Hence, a Fermat-holonomic congruence is determined by “a part of it”.

The definition of Fermat-holonomic congruences has been devised as an extension to a (3+1)-spacetime of the shear-free congruences studied in reference 3 in a similar context, for the simplified problem of a (2+1)-spacetime. There, we went further and, adding symmetry arguments and the so called
geodesic equivalence principle \cite{4}, \cite{5}, a unique shear-free congruence was obtained out of a worldline and the angular velocity on it, just like in the case of Newtonian rigid motions. In a forthcoming paper we shall try to supplement in a similar way (symmetries plus geodesic equivalence principle) the general results that have been derived here, to model an arbitrary rotational motion with a fixed point.

6 Acknowledgments

One of us is indebted to M. A. Garcia Bonilla for helpful suggestions and comments. This work is partly supported by DIGICyT, contract no.PPB96-0384 and by Institut d’Estudis Catalans (S.C.F.).

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