Invariant subspaces of generalized Hardy algebras associated with compact abelian group actions on $W^*$-algebras

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ABSTRACT. We consider an action of a compact abelian group whose dual is any subgroup of the additive group of real numbers (so, an archimedean linearly ordered group) or a direct product (or sum) of such groups on a $W^*$-algebra, $M$. We define the generalized Hardy subspace of the Hilbert space of a standard representation the algebra, and the Hardy subalgebra of analytic elements of $M$ with respect to the action. We find conditions in order that the Hardy algebra is a hereditarily reflexive algebra of operators. In particular if every non zero spectral subspace, contains a unitary operator, the condition is satisfied and therefore the Hardy algebra is hereditarily reflexive. This is the case if the action is the dual action on a crossed product, or an ergodic action, or, if, in some situations, the fixed point algebra is a factor.

1 Introduction

This paper is concerned with the study of invariant subspaces and reflexivity of operator algebras associated with compact group actions on $W^*$-algebras. Recall first the definition of a reflexive operator algebra.

Let $A \subset B(X)$ be a weakly closed algebra of operators on a Banach space $X$. Denote by $Lat(A)$ the lattice of closed subspaces of $X$ that are invariant for all operators $a \in A$. Let

$$algLat(A) = \{ b \in B(X) : bK \subset K \text{ for all } K \in Lat(A) \}.$$ 

The algebra $A$ is called reflexive if $A = algLat(A)$. Hence, a reflexive operator algebra is completely determined by the lattice of its invariant subspaces. An algebra $A \subset B(X)$ is called hereditarily reflexive if every unital weakly closed subalgebra of $A$ is reflexive. Sarason [19], proved two results: (1) every commutative von Neumann algebra is hereditarily reflexive and (2) the algebra of
analytic Toeplitz operators on the Hardy space $H^2(T)$ where $T$ is the the unit circle $T = \{ z \in \mathbb{C} : |z| = 1 \}$, is hereditarily reflexive. In [14] we extended this result in two directions: (1) to the case of $H^p(T), 1 < p < \infty$ and (2) to the not necessarily commutative case of non selfadjoint crossed products of finite von Neumann algebras by the semigroup $\mathbb{Z}_+$. Later, in [9], Kakariadis has considered the more general case of reduced $w^*$-semicrossed products and, among other results, he has extended the particular case of our reflexivity result in [14, Proposition 4.5,] for $p = 2$ to the semicrossed product setting [9, 2.10.H,]. This result was considered later by Helmer [5] in the context of $W^*$-correspondences [12]. Further, in [15], we studied a related problem in a more general setting than the crossed product or the reduced $w^*$semicrossed product considered in [14] and [9, 2.9] for the case of von Neumann algebras. We considered a $W^*$-dynamical system $(M, T, \alpha)$ where $T = \{ z \in \mathbb{C} : |z| = 1 \}$ is the circle group and $M$ is a $\sigma$–finite $W^*$-algebra. We constructed a standard covariant representation of the system on a certain Hilbert space, $H$, a generalized Hardy space, $H_+$ and the corresponding Hardy algebra $M_+ \subset B(H_+)$. We have shown that if the spectral subspace corresponding to the smallest positive element of the spectrum contains a unitary element, then, the algebra $M_+$ is reflexive. Actually, [15, Theorem 3.5.0] shows that if $M \subset B(H)$ is a $\sigma$-finite von Neumann algebra in its standard representation such that each spectral subspace contains a unitary operator (as is, in particular, the algebra of analytic Toeplitz operators considered by Sarason), then, $M_+ \subset B(H_+)$ is a reflexive operator algebra. Recently Bickerton and Kakariadis [2] have obtained results about reflexivity of algebras associated with actions of $\mathbb{Z}_d^d$ (the direct product of $d$ copies of $\mathbb{Z}_+$).

In this paper we make two significant steps towards solving the reflexivity problem of Hardy algebras associated to one-parameter dynamical systems $(M, \mathbb{R}, \alpha)$: 1. We consider a $W^*$-dynamical system $(M, G, \alpha)$ where $M$ is a von Neumann algebra in standard form, and $G$ is a compact abelian group whose dual is an arbitrary subgroup of $\mathbb{R}$ (possibly $\mathbb{R}$ itself with the discrete topology), 2. $G$ is a compact abelian group whose dual is a direct product of discrete groups with linear archimedean order. We also consider actions of compact abelian groups, $G$, whose duals, $\Gamma$, are direct products or direct sums of discrete groups with linearly archimedean order and consider the lattice order on $\Gamma$ (see for instance [3]). In Section 2.1. we define a standard covariant representation of the system that will be the framework for the rest of the paper. In the Corollary to Proposition 2.4. we show that every von Neumann algebra in standard form (in particular every maximal abelian von Neumann algebra) is hereditarily reflexive, thus extending the first result of Sarason mentioned above to every von Neumann algebra in its standard representation. In Section 3 we consider the case when the dual $\Gamma$ of $G$ has an archimedean linear order, or is a direct product of such groups, we define a generalized Hardy space $H_+ \subset H$ and a corresponding Hardy algebra $M_+ \subset B(H_+)$, where $H$ is the Hilbert space of the standard covariant representation of the system $(M, G, \alpha)$ and we prove that, in some conditions, including the conditions in [15], $M_+ \subset B(H_+)$ is hereditarily reflexive (in [15] we proved only reflexivity for the particular case when $\Gamma = \mathbb{Z}$). We do not assume as in [15] that $M$ is $\sigma$-finite. Also, if $\Gamma$ is an arbitrary archimedean linearly ordered discrete group, it can be any subgroup...
of $\mathbb{R}$ with the discrete topology, not only $\mathbb{Z}$ as in [15], [9], [14]. Examples include
the Hardy algebra of analytic Toeplitz operators, $H^\infty(T)$, the results in [15], $w^*$-crossed products by abelian archimedean ordered discrete groups or a direct
product of such groups, some reduced $w^*$-semicrossed products considered in [9], [2] and other situations as stated in the Corollaries 3.14., 3.15., 3.16. and
3.17.

2 Preliminary results and notations

2.1 Standard representations of $W^*$-algebras

In this section we review some concepts and results related to the standard
representation of a von Neumann algebra. Some of these results are certainly
known, but we did not find an exact reference for them. We provide proofs
of these results for the convenience of the reader. In Proposition 2.4. and its
Corollary we prove that every von Neumann algebra in its standard represent-
ation is hereditarily reflexive. In particular, every abelian von Neumann algebra
is hereditarily reflexive ([19, Theorem 1]).

Let $M$ be a $W^*$-algebra and let $\rho$ be a weight on the positive part, $M^+$, of
$M$, that is a mapping $\rho : M^+ \to [0, \infty) \cup \{\infty\}$ such that

$$\rho(m + n) = \rho(m) + \rho(n), m, n \in M^+$$

and

$$\rho(\lambda m) = \lambda \rho(m), m \in M^+, \lambda \in \mathbb{R}, \lambda \geq 0$$

with the convention $0 \cdot \infty = 0$. As it is customary ([8], [20]), denote

$$N_\rho = \{m \in M : \rho(m^*m) < \infty\}.$$

$$N_\rho = \{m \in M : \rho(m^*m) = 0\}.$$

$$F_\rho = \{m \in M^+ : \rho(m) < \infty\}.$$

$$M_\rho = \text{linear span of } F_\rho.$$

It is immediate that $N_\rho$ is a left ideal of $M$. The weight $\rho$ is called faithful if
$N_\rho = \{0\}$, normal if it is the sum of a family $\{\varphi_i\}$ of positive normal linear
functionals and semifinite if $M_\rho$ or, equivalently [20, 2.1.], $N_\rho$ is $w^*$- dense in
$M$.

Now let $M$ be a $W^*$-algebra, $M_0 \subset M$ a $W^*$-subalgebra, and $P_0 : M \to M_0$ a
$w^*$-continuous projection of norm 1 of $M$ onto $M_0$ which is, in addition, faithful
on the set of positive elements of $M$. Let $\rho_0$ be a faithful normal semifinite
weight on $M$. It is known that such a weight exists. Indeed, consider a family
$\{\varphi_i\}$ of positive normal linear functionals of $M_0$ such that their supports $\{p_i\}$
form a maximal family of mutually orthogonal projections of $M_0$, in particular $\sum p_i = I$. Then $\rho_0 = \sum \varphi_i$ is a faithful normal semifinite weight of $M_0$.

The following fact is stated in [19, Corollary 10.5] as a consequence of a theorem of Takesaki [20, Theorem 10.1.]. We present a short proof of this fact in our setting for the convenience of the reader.

**Lemma 2.1.** $\rho = \rho_0 \circ P_0$ is a faithful normal semifinite weight on $M$.

**Proof.** Since $\rho_0$ and $P_0$ are faithful, it follows that $\rho$ is faithful. Since $\rho_0$ is normal and $P_0$ is $w^*$-continuous, it follows that $\rho$ is normal. To prove that $\rho$ is semifinite, notice that from the definition of $N_\rho$ we have that $M N_{\rho_0} \subset N_\rho$, where $M N_{\rho_0}$ denotes the linear span of $\{ m n : m \in M, n \in N_{\rho_0} \}$. Since $N_{\rho_0}$ is $w^*$-dense in $M_0$ it follows that $M N_{\rho_0}$ and therefore $N_\rho$ is dense in $M$ so $\rho$ is semifinite.

By [8, Theorem 7.5.3.], there exists a faithful normal representation $\pi_\rho$ of $M$ on the completion $H_{\rho}$ of $N_\rho \subset M$ with respect to the inner product

$$\langle m, n \rangle = \rho(n^* m).$$

This representation is uniquely determined up to unitary equivalence and is, in that sense, independent of the choice of the weight $\rho_0$. We will use the version of Tomita-Takesaki Theorem from [8, Theorem 9.2.37.]. If $S$ is the conjugate linear operator defined on $N_\rho \cap N_\rho^*$ by $S(n) = n^*$, then $S$ is a preclosed densely defined operator on $H_{\rho}$ and its closure has the polar decomposition $J \Delta \frac{1}{2}$, in which $\Delta$ is an invertible positive operator and $J$ is a conjugate linear isometry acting on $H_{\rho}$ such that $J^2 = I$ and $J \pi_\rho(M) J = \pi_\rho(M)^\prime$ where $\pi_\rho(M)^\prime$ is the commutant of $\pi_\rho(M)$ in $B(H_{\rho})$. We have $N_{\rho_0} \subset N_\rho$, where, as above

$$N_{\rho_0} = \{ m \in M_0 : \rho_0(m^* m) < \infty \}.$$ 

and

$$N_\rho = \{ m \in M : \rho(m^* m) < \infty \}.$$ 

In the rest of the paper if $\rho$ is a faithful, normal semifinite weight on $M^+$ we will identify $\pi_\rho(M)$ with $M$ and will write $m$ instead of $\pi_\rho(m), m \in M$. We will call this representation the standard representation of $M$ and we will refer to the inclusion $M \subset B(H_{\rho})$ as the standard form of $M$. Also, we will denote $H_{\rho}$ by $H$ and the closure of $N_{\rho_0}$ in $H$ by $H_0$.

**Lemma 2.2.** The restriction of $P_0$ to $N_\rho$ extends to the orthogonal projection of $H$ onto $H_0$. 

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Proof. Clearly, \( P_0(N_{\rho_0}) = N_{\rho_0} \). We will prove next that \( P_0(N_{\rho}) \subset N_{\rho_0} \). Indeed, let \( n \in N_{\rho} \). Then \( n = P_0(n) + (I - P_0)(n) = n_0 + n_1 \). Since \( n \in N_{\rho} \), we have \( \rho(n^* n) < \infty \), so

\[
\rho(n^* n_0 + n_0^* n_1 + n_1^* n_0 + n_1^* n_1) = \rho_0(n_0^* n_0 + n_0^* n_1 + n_1^* n_0 + n_1^* n_1) = \rho_0(n_0^* n_0 + P_0(n_1^* n_0) + P_0(n_1^* n_1)) = \rho_0(n_0^* n_0 + P_0(n_1^* n_1)) < \infty.
\]

Therefore, \( \rho_0(n_0^* n_0) < \infty \) and we are done. On the other hand,

\[
\langle P_0(n), m \rangle = \langle n, P_0(m) \rangle.
\]

since both of the above terms equal \( \rho_0(P_0(m^*) P_0(n)) \), so \( P_0 \) is self adjoint. ■

**Lemma 2.3.** i) \( MH_0 \) is dense in \( H \) and \( H_0 \) is a separating set for \( M \) that is, if \( m \in M \) is such that \( m\xi_0 = 0 \) for all \( \xi_0 \in H_0 \), then \( m = 0 \). Here, \( MH_0 \)

denotes the linear span of \( \{ m \xi : m \in M, \xi \in H_0 \} \).

ii) \( M'H_0 \) is dense in \( H \) and \( H_0 \) is a separating set for \( M' \) that is, if \( m' \in M' \)

is such that \( m'\xi_0 = 0 \) for all \( \xi_0 \in H_0 \), then \( m' = 0 \).

**Proof.** i) We will prove that \( MH_0 \) is dense in \( N_{\rho} \subset H \) and, since \( N_{\rho} \) is dense in \( H \), the first part of i) will follow. Let \( x \in N_{\rho} \). Therefore, \( \rho(x^* x) = \rho_0(P_0(x^* x)) < \infty \). Since \( \rho_0 \) is a faithful normal semifinite weight on \( M_0 \) we can assume that \( \rho_0 \) is the sum of a family of normal positive linear functionals \( \{ \varphi_i \} \) on \( M_0 \) such that their supports \( \{ p_i \} \) form a maximal family of mutually orthogonal projections in \( M_0 \) and \( \sum p_i = I \). Since \( \rho_0(P_0(x^* x)) = \sum \varphi_i(P_0(x^* x)) < \infty \), it follows that the sumable family of positive numbers \( \{ \varphi_i(P_0(x^* x)) \} \) is at most countable, say \( \{ \varphi_i(P_0(x^* x)) \} \) with \( \rho(x^* x) = \rho_0(P_0(x^* x)) = \sum_{i=1}^{\infty} \varphi_i(P_0(x^* x)) < \infty \). Let \( q_n = \sum_{i=1}^{n} p_i \), where, for each \( i \in \mathbb{N} \), \( p_i \) is the support of \( \varphi_i \). Then, \( q_n \in N_{\rho_0} \subset H_0 \) and \( xq_n \in MN_{\rho_0} \subset MH_0 \) for every \( n \in \mathbb{N} \). Clearly, since \( p_i \) is the support of \( \varphi_i \) we have \( \varphi_i(y) = \varphi_i(p_i y) = \varphi_i(y p_i) \) for all \( i \in \mathbb{N} \), \( y \in M_0 \) and \( \varphi_i(q_n y) = \varphi_i(y) \) for every \( i, n \in \mathbb{N} \) with \( i \leq n \), \( y \in M_0 \) and \( \varphi_i(q_n x) = 0 \) if \( i > n \). We will show that \( \lim_{n \to \infty} xq_n = x \) in \( H \). Let \( \epsilon > 0 \). Then, there exists \( N = N_\epsilon > 0 \) such that \( \sum_{i=N+1}^{\infty} \varphi_i(P_0(x^* x)) < \epsilon^2 \). Therefore, if \( n \geq N \)

\[
\langle q_n x - x, q_n x - x \rangle = \rho((q_n x - x)^* (q_n x - x)) = \sum_{i=1}^{n} \varphi_i(P_0((q_n x - x)^* (q_n x - x))) +
\]

\[
+ \sum_{i=n+1}^{\infty} \varphi_i(P_0((q_n x - x)^* (q_n x - x)))
\]

Using the preceding observations, if \( i \leq n \), we get

\[
\varphi_i(P_0((q_n x - x)^* (q_n x - x))) = \]
\[ \varphi_i(q_nP_0(x^*x)q_n - q_nP_0(x^*x) - P_0(x^*x)q_n + P_0(x^*x)) = 0 \]

and, if \( i > n \geq N \)
\[ \varphi_i(P_0((q_nx - x)^*(q_nx - x)) = \varphi_i(P_0(x^*x)) \]

Hence \( (q_nx - x, q_nx - x) < \epsilon^2 \) and the claim is proven. To prove the second part, let \( m \in M \) such that \( m\xi_0 = 0 \) for every \( \xi_0 \in H_0 \), in particular, \( mn = 0 \) for every \( n \in N_{\rho_0} \subset H_0 \). Since \( N_{\rho_0} \) is \( \ast \)-dense in \( M_0 \subset M \) and \( I \in M_0 \), it follows that \( m = 0 \) and part i) is proven.

ii). Denote by \( K \) the closure of \( M'H_0 \). Then \( K \) is a closed subspace of \( H \) which is invariant for every \( m' \in M' \), so, the orthogonal projection, \( p \), of \( H \) on \( K \) commutes with \( M' \), and therefore \( p \in M \). Since \( H_0 \subset K \), it follows that \( (1 - p)H_0 = \{0\} \). Since by i) \( H_0 \) is a separating set for \( M \), we have \( 1 - p = 0 \) and thus \( K = H \). To prove the second part of ii), let \( m' \in M' \) be such that \( m'H_0 = \{0\} \). It follows that \( Mm'H_0 = \{0\} \), so \( m'MH_0 = \{0\} \). Since, by i) \( MH_0 \) is dense in \( H \), it follows that \( m' = 0 \).

Some of the statements in the next Proposition are probably known, but we did not find a reference for any of them.

**Proposition 2.4.** i) Let \( M \subset B(H) \) be a von Neumann algebra in standard form. Then, every normal linear functional, \( \varphi \), on \( M \) is a vector functional, that is, there exist \( \xi, \eta \in H \) such that \( \varphi(m) = \langle m\xi, \eta \rangle \), \( m \in M \).

ii) If \( N \subset B(H) \) is an abelian von Neumann algebra, not necessarily in standard form, then every normal linear functional on \( N \) is a vector functional.

iii) If \( M_0 \subset B(H_0) \) is a maximal abelian von Neumann algebra, then it is spatially isomorphic with its standard form.

**Proof.** i) Let \( p \in M \) be a countably decomposable projection. According to [8, 9.6.18.], the hypotheses of [8, 9.6.20.] are satisfied, so there exists \( \xi_0 \in H \) such that \( J\xi_0 = \xi_0 \) and \( M'\xi_0 = pH \). Therefore, every countably decomposable projection \( p \in M \) is a cyclic projection. Now, let \( \varphi \) be a normal linear functional on \( M \). By the polar decomposition of normal linear functionals [8, Theorem 7.3.2.], it is enough to prove the statement for normal positive linear functionals. Let \( \psi \) be a normal positive functional on \( M \) and its support (that is, \( p \) is the complement of the supremum of all projections \( q \in M \) for which \( \psi(q) = 0 \)). Then, \( p \) is countably decomposable, so by the previous arguments, \( p \) is a cyclic projection. Applying [8, Proposition 7.2.7.] it follows that \( \psi \) is a vector normal positive functional.

ii) Let \( \varphi \) be a normal linear functional on \( M \). As argued in i), using the polar decomposition of normal linear functionals it is enough to prove the statement in ii) for normal positive functionals. Let \( \psi \) be a normal positive functional on \( M \) and \( p \) its support which is a countably decomposable projection. Without loss of generality we can assume that \( p = I \). We will show that there exists a separating
vector, \( \xi_0 \in H \) for \( N \) and therefore, cyclic for \( N' \). Let \( \{ \xi_i : \xi_i \in H : i \in A \} \) be a maximal family of orthogonal unit vectors such that, the projections \( \{ p_i : p_i \in N : i \in A \} \) onto \( \{ H_i = N^i \xi_i \subset H : i \in A \} \) are mutually orthogonal, so \( H = \bigoplus_i H_i \). Since \( p = I \) is countably decomposable, it follows that the set \( A \) is at most countable. Suppose \( A \subseteq \mathbb{N} \). Let \( \xi_0 = \sum_{i \in A} \xi_i \), and let \( m \in \mathbb{N}^+ \) be such that \( m \xi_0 = 0 \). Hence \( \langle m \xi_0, \xi_i \rangle = 0, i \in A \). Then, since \( N \) is abelian, so \( N \subseteq N' \), it follows that \( \langle m \xi_i, \xi_i \rangle = 0 \) for every \( i \). Therefore, since \( m \in \mathbb{N}^+ \), it follows that \( m \xi_i = 0 \), so \( mH_i = \{ 0 \} \) for every \( i \), and thus \( m = 0 \). Hence \( \xi_0 \) is separating for \( N \). The statement ii) follows from [8, 7.2.7.]

iii) Let \( \{ p_i \} \) be a maximal family of mutually orthogonal countably decomposable projections of \( M_0 \) and \( \{ \varphi_i \} \) a family of positive linear functionals such that the support of \( \varphi_i \) is \( p_i \). Clearly \( \sum p_i = I \). Let \( \rho_0 = \sum \varphi_i \). Then \( \rho_0 \) is a faithful normal semifinite weight on \( M_0^+ \). By ii) for every \( \iota \) there exists \( \xi \in p_i H_0 \) such that \( \varphi_i(m) = \langle m \xi, \xi_i \rangle, m \in M_0 \). Obviously, \( \xi_i \) is a cyclic and separating vector of \( M p_i |_{p_i H_0} \) for every \( \iota \). It is also clear that \( \langle m_1 \xi_i, m_2 \xi_i \rangle = \varphi_i(m_2^* m_1), m_1, m_2 \in M_0 \), for every \( \iota \). If \( N_{\rho_0} \) is as above,

\[
N_{\rho_0} = \{ m \in M_0 : \rho_0(m^* m) < \infty \},
\]

then the mapping \( mp_i \rightarrow m \xi_i \) extends to a unitary operator from \( H_{\rho_0} \) to \( H_0 \) and we are done. ■

**Corollary**
i) Every von Neumann algebra in standard form is hereditarily reflexive.

ii) [19, Theorem 2] Every abelian von Neumann algebra, not necessarily in standard form is hereditarily reflexive.

**Proof.** i) Follows from Proposition 2.4. i) and [10, Theorem 3.5.].

ii) Follows from Proposition 2.4. ii) and [10 Theorem 3.5.]. ■

### 2.2 W*-dynamical systems with compact abelian groups

Let \( (M, G, \alpha) \) be a W*-dynamical system, where \( M \) is a W*-algebra, \( G \) is a compact abelian group with dual \( \Gamma \), and \( \alpha \) a faithful \( w^* \)-continuous action of \( G \) on \( M \), that is \( \alpha_g \neq id \) if \( g \neq 0 \), where \( id \) is the identity automorphism of \( M \) and the mapping \( g \rightarrow \varphi(\alpha_g(m)) \) for every \( m \in M \) and every \( \varphi \in M_* \), where \( M_* \) denotes the predual of \( M \). For each \( \gamma \in \Gamma \), denote by

\[
M_\gamma = \left\{ \int (g, \gamma) \alpha_g(m)dg : m \in M \right\},
\]

where the integral is taken in the \( w^* \)-topology. In particular, if \( \gamma = 0 \), \( M_0 \) is the fixed point algebra of the system. It can immediately be checked that

\[
M_\gamma = \{ m \in M : \alpha_g(m) = \langle g, \gamma \rangle m \}.
\]
It is clear that the mapping \( P_\gamma : M \rightarrow M_\gamma \) defined by \( P_\gamma(m) = \int (g, \gamma) \alpha_g(m) \, dg \) is a \( w^* \)-continuous projection of \( M \) onto the closed subspace \( M_\gamma \subset M \). In particular, \( P_0 \) is a \( w^* \)-continuous projection of \( M \) onto \( M_0 \) which is clearly faithful (on \( M^+ \)). It is well known that \( M \) is the \( w^* \)-closed linear span of \( \{ M_\gamma : \gamma \in \Gamma \} \). The Arveson spectrum of the action \( \alpha \) is, by definition ([1], [13])

\[
sp(\alpha) = \{ \gamma \in \Gamma : M_\gamma \neq \{0\} \}.
\]

**Lemma 2.5.** i) \( M_{-\gamma} = M_\gamma^* \), where \( M_\gamma^* = \{ m^* : m \in M_\gamma \} \) \( M_\gamma^* = \{ m^* : m \in M_\gamma \} \).

ii) \( M_{\gamma_1} M_{\gamma_2} \subset M_{\gamma_1 + \gamma_2} \) where \( M_{\gamma_1} M_{\gamma_2} \) is the linear span of \( \{ xy : x \in M_{\gamma_1}, y \in M_{\gamma_2} \} \).

iii) If \( m \in M_\gamma \) has polar decomposition \( m = u |x| \), then \( u \in M_\gamma \) and \( |x| \in M_0 \).

**Proof.** i) and ii) are obvious. iii) is a straightforward consequence of the uniqueness of the polar decomposition of \( m \). ■

Let \((M, G, \alpha)\) be as above, \( \rho_0 \) a faithful normal semifinite weight on \( M_0 \) and \( \rho = \rho_0 \circ P_0 \). Consider the corresponding normal faithful representation \( \pi_\rho \) on \( H_\rho \) and the Tomita-Takesaki operators \( S, J \) as in 2.1. above. As in 2.1. we will write \( H \) instead of \( H_\rho \) and \( M \) instead of \( \pi_\rho(M) \). In the case when \( \Gamma \) is a partially ordered group, this representation will allow us to construct a generalized Hardy space on which, in certain situations, the subalgebra of analytic elements of the system \((M, G, \alpha)\) is hereditarily reflexive.

For every \( g \in G \) define the unitary operator \( U_g \in B(H) \) as the unique extension of \( U_g(n) = \alpha_g(n), n \in N_\rho \) to \( H \). Then, since clearly, \( N_\rho \) is an \( \alpha \)-invariant left ideal of \( M \), it is straightforward to check that the group of unitary operators \( \{ U_g : g \in G \} \) implements the action \( \alpha \). Also, from the definition of \( S \) it follows that \( SU_g = U_gS \) and \( S^* U_g = U_g S^* \) for all \( g \in G \). Therefore, \( JU_g = U_gJ \), \( g \in G \). It follows that the group \( \{ U_g : g \in G \} \) implements an action \( \alpha ' \) of \( G \) on \( M' \), namely

\[
\alpha'(JmJ) = U_gJmJU^*_g = J\alpha_g(m)J.
\]

Similarly with the projections \( P_\gamma \) of \( M \) onto \( M_{\gamma' \gamma} \in \Gamma \) one can define the projections \( P'_{\gamma} \) of \( M' \) onto \( M_{\gamma'} \)

\[
P'_{\gamma}(x) = \int \overline{(g, \gamma)} \alpha'_g(x) \, dg.
\]

The proof of the following lemma is a straightforward application of the definitions.

**Lemma 2.6.** With the notations above, we have the following:

i) \( \alpha'_g(m') = U_g m' U^*_g \) is an action of \( G \) on \( M' \), where \( M' \) is the commutant of \( M \) in \( B(H) \).
ii) If \( g \in G \), then \( U_g \) commutes with \( J \), and \( J\alpha_g(m)J = \alpha'_g(JmJ), m \in M, g \in G \).

iii) \( (M')\gamma = JM_{-\gamma}J, \gamma \in \Gamma \).

iv) \( U_g(m'\xi) = \alpha'_g(m')\xi, \xi \in H_0, m' \in M' \).

v) \( \text{sp}(\alpha) = \text{sp}(\alpha') \).

We will need also the following

**Remark 2.7.** If \( M \) is a finite \( W^* \)-algebra, then \( M' \) is a finite \( W^* \)-algebra. This fact is immediate from the definition of the standard representations.

Let \( H_\gamma = \{ \int \langle g, \gamma \rangle U_g(\xi)dg : \xi \in H \} = \{ \xi \in H : U_g\xi = \langle g, \gamma \rangle \xi \} \). Then, the map \( P^H_\gamma \) from \( H \) to \( H_\gamma \) defined as follows

\[
P^H_\gamma(\xi) = \int \langle g, \gamma \rangle U_g(\xi)dg : \gamma \in \Gamma, \xi \in H.
\]

is an orthogonal projection of \( H \) onto the closed subspace \( H_\gamma \). Applying Lemma 2.2, we see that if \( \gamma = 0 \), the Hilbert subspace \( H_0 \subset H \) coincides with the Hilbert subspace \( H_0 \) considered in Section 2.1.

**Lemma 2.8.** i) If \( \gamma_1 \neq \gamma_2 \), then \( H_{\gamma_1} \) and \( H_{\gamma_2} \) are orthogonal.

ii) For every \( \gamma \in \text{sp}(\alpha) \), we have \( M_\gamma H_0 = H_\gamma \), where \( M_\gamma H_0 = \{ m\xi_0 : m \in M_\gamma, \xi_0 \in H_0 \} \).

iii) The direct sum of Hilbert spaces \( \sum H_\gamma \) equals \( H \).

iv) For every \( \gamma \in \text{sp}(\alpha) \) we have \( (M')_\gamma H_0 = H_\gamma \), where \( (M')_\gamma H_0 = \{ m'\xi_0 : m' \in (M')_\gamma, \xi_0 \in H_0 \} \).

v) For all \( \gamma, \gamma' \in \text{sp}(\alpha) \) we have \( M_\gamma H_{\gamma'} \subset H_{\gamma+\gamma'} \) and \( M'_{\gamma'} H_\gamma \subset H_{\gamma+\gamma'} \).

**Proof.** i) Let \( \xi \in H_{\gamma_1}, \eta \in H_{\gamma_2} \). Then, by definition, \( U_g(\xi) = \langle g, \gamma_1 \rangle \xi \) and \( U_g(\eta) = \langle g, \gamma_2 \rangle \eta \), for all \( g \in G \). Since the operators \( U_g \) are unitary, we have

\[
\langle \xi, \eta \rangle = \langle U_g\xi, U_g\eta \rangle = \langle g, \gamma_1 - \gamma_2 \rangle \langle \xi, \eta \rangle, g \in G.
\]

Hence, if \( \gamma_1 \neq \gamma_2 \) it follows that \( \langle \xi, \eta \rangle = 0 \).

ii) Since, by Lemma 2.3. i), \( H_0 \) is a cyclic set for \( M \), the subspace \( MH_0 = \{ m\xi_0 : m \in M, \xi_0 \in H_0 \} \) is dense in \( H \). Then, if \( P^H_\gamma \) and \( P_\gamma \) are the above projections, we have

\[
M_\gamma H_0 = P_\gamma(M)H_0 = \left\{ \int \langle g, \gamma \rangle \alpha_g(m)\xi_0dg : m \in M, \xi_0 \in H_0 \right\} = \left\{ \int \langle g, \gamma \rangle U_g(\xi)dg : \xi = m\xi_0, m \in M, \xi_0 \in H_0 \right\} =
\]


\( \{ P^H_\gamma (\xi) : \xi = m \xi_0, m \in M, \xi_0 \in H_0 \} \).

Since by Lemma 2.3. i) the subspace \( MH_0 \) is dense in \( H \) and \( P^H_\gamma \) is an orthogonal projection, the result stated in ii) follows.

iii) Let \( \eta \in H \) be such that \( \eta \perp H_\gamma \) for all \( \gamma \in \Gamma \). Since \( M \) is the \( w^* \)-closed linear span of \( \{ M_\gamma : \gamma \in \Gamma \} \), it follows from ii) that \( \eta \perp MH_0 \), so, since by Lemma 2.3. i) \( H_0 \) is cyclic for \( M \), it follows that \( \eta = 0 \).

iv) The proof is similar with that of ii) taking into account that, according to Lemma 2.3. ii), \( H_0 \) is cyclic for \( M' \) as well.

v) Immediate from definitions. ■

3 Hereditary reflexivity of generalized Hardy algebras

In this section we will construct the generalized Hardy space and the generalized Hardy algebra and prove the main results of this paper, Theorem 3.8. and Theorem 3.9.

Let \((M, G, \alpha)\) be a \( W^* \)-dynamical system with \( G \) compact abelian. Throughout this section we will assume that \( M \subset B(H) \) where \( H \) is the Hilbert space constructed in Section 2.1. Suppose, in addition, that \( \Gamma \) is an archimedean linearly ordered discrete group, or \( \Gamma \subseteq \prod_{i \in I} \Gamma_i \) is the direct product or the direct sum of archimedean linearly ordered (discrete) groups \( \Gamma_i \). If \((\Gamma_i)_+\) is the semigroup of non negative elements of \( \Gamma_i \), denote by \( \Gamma_+ = \prod_{i \in I} (\Gamma_i)_+ \). Then, \( \Gamma_+ \) is a sub semigroup of \( \Gamma \) such that \( \Gamma_+ \cap (-\Gamma_+) = \{0\} \) and \( \Gamma_- - \Gamma_+ = \Gamma \) so \( \Gamma_+ \) defines a partial order on \( \Gamma \), namely \( \gamma_1 \leq \gamma_2 \) if \( \gamma_2 - \gamma_1 \in \Gamma_+ \).

**Lemma 3.1.** Let \( \gamma_1, \gamma_2 \in \Gamma_+, \gamma_1 \neq \gamma_2 \). Then, either

i) \( \gamma_1, \gamma_2 \) are not comparable under the above order relation, or,

ii) \( \gamma_1 < \gamma_2 \), or,

iii) \( \gamma_1 > \gamma_2 \) and, in this case, there exists \( p \in \mathbb{N} \) such that either

iii a) \( \gamma_1 \geq p\gamma_2 \) and \( \gamma_1 < (p+1)\gamma_2 \), or,

iii b) \( \gamma_1 \geq p\gamma_2 \) and \( \gamma_1 \) and \( (p+1)\gamma_2 \) are not comparable.

**Proof.** Suppose that \( \gamma_1, \gamma_2 \) are comparable. Thus, either \( \gamma_1 < \gamma_2 \), or \( \gamma_1 > \gamma_2 \). Suppose that \( \gamma_1 > \gamma_2 \). Since \( \gamma_1, \gamma_2 \in \prod_{i \in I} (\Gamma_i)_+ \), we can write \( \gamma_1 = (\gamma_1^i)_{i \in I}, \gamma_2 = (\gamma_2^i)_{i \in I} \) with \( \gamma_1^i, \gamma_2^i \in (\Gamma_i)_+ \), so \( \gamma_1^i \geq \gamma_2^i \), \( i \in I \) and \( \gamma_1^{i_0} > \gamma_2^{i_0} \) for some \( i_0 \in I \). Since for every \( i \in I \), \( \Gamma_i \) has an archimedean order, there exists a largest \( p_i \in \mathbb{N} \) such that \( \gamma_1^i \geq p_i \gamma_2^i \). If \( L \subset \mathbb{N} \) is the set of non repeating \( p_i \)'s,
then, since $\mathbb{N}$ is well ordered, there exists $p = \min L$. Therefore, $\gamma_1 \geq p \gamma_2$. By the definition of $p \in \mathbb{N}$, $\gamma_1 \neq (p + 1) \gamma_2$, so either iii a) or iii b) must hold. ■

The following consequence of the above Lemma will be used

**Corollary 3.2.** Let $\gamma_0 \in \Gamma_+ \setminus \{0\}$ and $\gamma \in \Gamma_+$. Then, there exists $p \in \mathbb{Z}_+$ such that either $p \gamma_0 < \gamma < (p + 1) \gamma_0$ or $p \gamma_0 < \gamma$ and $\gamma$ is not comparable with $(p + 1) \gamma_0$.

**Proof.** If $\gamma < \gamma_0$ or $\gamma$ and $\gamma_0$ are not comparable, then $p = 0$ satisfies the conclusion. If $\gamma > \gamma_0$, the statement follows from Lemma 3.1. iii). ■

**Lemma 3.3.** If $\Gamma \subseteq \Pi_{i \in I} \Gamma_i$ is the direct product or the direct sum of linearly ordered discrete groups $\Gamma_i$, then $\Gamma$ is lattice ordered (see [3]), i.e. if $A = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a finite subset of $\Gamma$ then there exists inf $A$ and sup $A$ in $\Gamma$.

**Proof.** If $\gamma_1 = (\gamma_j^i), 1 \leq j \leq n$, let $\mu^i = \min \{\gamma_j^i : j = 1, 2, \ldots, n\}$ and $\nu^i = \max \{\gamma_j^i : j = 1, 2, \ldots, n\}$ for each $i \in I$. Then clearly inf $A = (\mu^i)_{i \in I}$ and sup $A = (\nu^i)_{i \in I}$. ■

If $(M, G, \alpha), M \subset B(H)$, $H_0, \hat{G} = \Gamma$ and $\Gamma_+$ are as above, define

$$H_+ = \sum_{\gamma \in \Gamma_+} H_\gamma$$

Let $p_+$ be the orthogonal projection of $H$ onto $H_+$. By Lemma 2.8. v), the (closed) subspace $H_+ \subset H$ is invariant for $\forall_{\gamma \geq 0} M_\gamma$, and for $\forall_{\gamma \geq 0} (M')_\gamma$. We will denote by $M_+$ the weak operator closure

$$M_+ = \overline{p_+ (\bigvee_{\gamma \in \Gamma_+} M_\gamma) p_+^{\text{wo}}}$$

in $B(H_+)$ and similarly

$$(M')_+ = \overline{p_+ (\bigvee_{\gamma \geq 0} (M')_\gamma) p_+^{\text{wo}}}$$

where $\bigvee_{\gamma \geq 0} M_\gamma$, is the algebra generated by $\{M_\gamma : \gamma \geq 0\}$ and $\bigvee_{\gamma \geq 0} (M')_\gamma$ is the algebra generated by $\{(M')_\gamma : \gamma \geq 0\}$. Then, we will call $H_+$ the generalized Hardy space and $M_+$ the generalized Hardy algebra of analytic elements of the dynamical system $(M, G, \alpha)$.

To prove hereditary reflexivity, we also need the following

**Lemma 3.4.** If $\psi$ is a weakly continuous functional on $M_+ \subset B(H_+)$, then $\psi$ is a vector functional (that is, there exist $\xi, \eta \in H_+$ such that $\psi(m) = \langle m \xi, \eta \rangle$ for all $m \in M_+$).

**Proof.** Since $\psi$ is weakly continuous, there exist $n \in \mathbb{N}$ and $\xi_i, \eta_i \in H_+, 1 \leq i \leq n$ such that $\psi(m_+) = \sum_i \langle m_+ \xi_i, \eta_i \rangle, m_+ \in M_+$. Now let $\hat{\psi}$ be the functional
on \( \widetilde{\psi}(b) = \sum \langle b \xi_i, \eta_i \rangle, b \in B(H) \). Since \( \xi_i, \eta_i \in H_+ \), it follows that 
\( \widetilde{\psi}(b) = \psi(p_+ bp_+), b \in B(H) \), where, as above, \( p_+ \) is the projection of \( H \) onto \( H_+ \). The restriction of \( \widetilde{\psi} \) to \( M \) is a normal linear functional of \( M \). Applying Proposition 2.4. to this restriction, it follows that there exist \( \xi, \eta \in H \) such that 
\( \widetilde{\psi}(m) = \langle m \xi, \eta \rangle, m \in M \). Since, as noticed before, \( \widetilde{\psi}(m) = \psi(p_+ mp_+) \), we can take \( \xi, \eta \in H_+ \). Therefore, in particular, \( \psi(m_\gamma) = \psi(m_\gamma) = \psi(p_+ mp_+) = \psi(m_\gamma) = \langle m_\gamma \xi, \eta \rangle \) for every \( m_\gamma \in M_\gamma, \gamma \in \Gamma_+ \). The definition of \( M_+ \) implies that \( \psi(m_+) = \langle m_+ \xi, \eta \rangle, m_+ \in M_+ \) and the proof is completed. \( \blacksquare \)

Loginov and Sul'man [10, Theorem 2.3.] have shown, in particular, that if a reflexive algebra satisfies the hypothesis of Lemma 3.4. then it is hereditarily reflexive, that is, all unital weakly closed subalgebras are reflexive. Under the name of super reflexivity this fact has been also considered in [4, Proposition 2.5. (1)].

In Theorem 3.8. below we will assume that \((M, G, \alpha)\) satisfies the following condition:

(C) For every \( \gamma \in sp(\alpha) \setminus \{0\} \) there exists an element \( u_\gamma \in M_\gamma \) such that \( u_\gamma^* u_\gamma = u_\gamma u_\gamma^* = e_\gamma \) where \( e_\gamma \) is a central projection of \( M \) and \( M_\gamma = M_0 u_\gamma \). For \( \gamma = 0 \) we will take \( u_0 = I \).

Examples of dynamical systems \((M, G, \alpha)\) satisfying this condition include the following:

a) If \( M \) is a finite W*-algebra and the center, \( Z(M_0) \), of \( M_0 \) is contained in the center, \( Z(M) \), of \( M \) [18]. This is the case, in particular, when \( M \) is a finite W*-algebra and \( M_0 \) is a factor (this case will be discussed in a more general context in part b)). The conditions \( M \) finite and \( Z(M_0) \subset Z(M) \) also hold if \( M = \bigoplus M_i \) and \((M_i, G, \alpha)\) is a finite W*-algebra and the fixed point algebra is a factor. Corollary 3.14. below will refer to these example.

b) If \( M \) is a semifinite injective von Neumann algebra such that \( M_0 \) is a factor, except when \( M \) is type III and \( M_0 \) is a type \( II_1 \) factor [21]. Thomsen has proved that in these cases, the action \( \alpha \) has full unitary spectrum, that is every nonzero spectral subspace contains unitary operators. This is the case, in particular, when \( M \) is a finite W*-algebra and \( M_0 \) is a factor. In particular, this latter situation occurs if \( \alpha \) is a prime action of the compact abelian group \( G \) on the hyperfinite type \( II_1 \) factor [6], [7], in particular if \( \alpha \) is ergodic. Recall that an action is called prime if the fixed point algebra is a factor. In particular if the action \( \alpha \) is faithful, then the all the examples in this part b) satisfy \( sp(\alpha) = \Gamma \). Corollary 3.15. below will refer to these examples. Also, the Condition (C) is satisfied if \( M \) is the crossed product of a von Neumann algebra \( M_0 \) by an abelian discrete group \( \Gamma \). Corollaries 3.16. and 3.17. will consider this case.

**Lemma 3.5.** Suppose that condition (C) is satisfied. Then
i) $M_\gamma = u_\gamma M_0$

ii) There exists an element $w_\gamma \in M' = JMJ$ such that $w_\gamma w_\gamma^* = w_\gamma^* w_\gamma = e_\gamma$, and $(M')_\gamma = (M')_0 w_\gamma = w_\gamma (M')_0$.

**Proof.** i) Clearly, if $x = mu_\gamma$ for some $\gamma$, then $x = me_\gamma u_\gamma = e_\gamma mu_\gamma = u_\gamma (u_\gamma^* mu_\gamma) \in u_\gamma M_0$ and conversely.

ii) Obviously, $w_\gamma = Ju_\gamma^* J$ satisfies the equality $w_\gamma w_\gamma^* = w_\gamma^* w_\gamma = Je_\gamma J$. Since $e_\gamma$ is a central projection of $M$, we can apply [8, 9.6.18.] to get $Je_\gamma J = e_\gamma$. ■

**Lemma 3.6.** Let $(M, G, \alpha), M \subset B(H)$ be a $W^*$-dynamical system with $G$ compact abelian as above. Then, if $\gamma, \gamma' \in \text{sp}(\alpha)$ and $e_\gamma e_{\gamma'} \neq 0$, we have $\gamma' - \gamma \in \text{sp}(\alpha)$ (therefore, $\gamma - \gamma' \in \text{sp}(\alpha)$) and $e_\gamma e_{\gamma'} \leq e_{\gamma - \gamma'}$.

**Proof.** Since $e_\gamma e_{\gamma'} \neq 0$, we have $u_\gamma^* u_{\gamma'} u_\gamma^* u_{\gamma'} \neq 0$ or $u_\gamma^* u_{\gamma'} \neq 0$. Hence, applying Lemma 2.5. ii), it follows that $M_{\gamma' - \gamma} \neq 0$, so $\gamma' - \gamma \in \text{sp}(\alpha)$. To prove the last statement of the lemma, notice that by Lemma 2.5. and Lemma 3.5.

$$e_\gamma e_{\gamma'} = u_\gamma u_{\gamma'}^* u_\gamma u_{\gamma'}^* = u_\gamma u_{\gamma'} - e_{\gamma'} u_\gamma^* u_{\gamma'} = u_\gamma u_{\gamma'} - e_{\gamma'} m_\gamma u_{\gamma'}$$

for some $m \in M_0$. Further, using repeatedly Lemma 3.5. we get

$$e_\gamma e_{\gamma'} = u_\gamma u_{\gamma'} - e_{\gamma'} u_\gamma^* u_{\gamma'} = u_\gamma u_{\gamma'} - e_{\gamma'} = e_{\gamma - \gamma'} m_\gamma u_{\gamma'}$$

$$= u_\gamma u_{\gamma'} (u_{\gamma'} - e_{\gamma'}) u_\gamma^* u_{\gamma'} = u_{\gamma - \gamma'} m_\gamma u_{\gamma'} = u_{\gamma - \gamma'} m_\gamma u_{\gamma'} = u_{\gamma - \gamma'} m_\gamma u_{\gamma'} = e_{\gamma - \gamma'} m_\gamma u_{\gamma'}$$

for some $m_1, m_2, m_3, m_4 \in M_0 \setminus \{0\}$, Therefore,

$$e_\gamma e_{\gamma'} = e_{\gamma - \gamma'} m_4 m_4 e_{\gamma - \gamma'} \leq \|m_4 m_4\| e_{\gamma - \gamma'}$$

So $e_\gamma e_{\gamma'} = e_{\gamma - \gamma'}$.

**Lemma 3.7.** Suppose that Condition (C) is satisfied. Then

i) $M_+^\alpha$ is the $w^*$-closed subalgebra of $B(H_+)$ generated by $M_0$ and \{w_\gamma : \gamma \in \text{sp}(\alpha), \gamma > 0\}.

ii) $(M_+)^\alpha$ is the $w^*$-closed subalgebra of $B(H_+)$ generated by $(M')_0$ and \{w_\gamma : \gamma \in \text{sp}(\alpha') = \text{sp}(\alpha), \gamma > 0\}.

**Proof.** Follows from Lemma 3.5. ■

We will prove next our results about reflexivity.

In Theorem 3.8. we assume that $\Gamma$ is archimedean linearly ordered and that Condition (C) is satisfied.

**Theorem 3.8.** Let $(M, G, \alpha)$ be such that $\hat{G} = \Gamma$ is archimedean linearly ordered and Condition (C) is satisfied. Then $M_+ \subset B(H_+)$ is hereditarily reflexive.
In Theorem 3.9, we assume that $\Gamma$ is a direct product (or a direct sum) of archimedean linearly ordered discrete groups, but we assume a stronger condition than Condition (C).

**Theorem 3.9.** Let $(M, G, \alpha)$ be such that $\mathcal{G} = \Gamma \subseteq \prod_{\gamma \in \Gamma_i} \Gamma_i$ is the direct product, or the direct sum, of archimedean linearly ordered discrete groups, $\Gamma_i$. Suppose that $\text{sp}(\alpha) = \Gamma$ and that for every $\gamma \in \text{sp}(\alpha) = \Gamma$, there exists a unitary operator $u_\gamma \in M_\gamma$. Then $M_+ \subset B(H_+)$ is hereditarily reflexive.

The proofs of these theorems will be given after some auxiliary results.

**Lemma 3.10.** Suppose that $\Gamma$ is archimedean linearly ordered and that Condition (C) is satisfied. Then
1) $(M_+)' = (M')_+$, where $(M_+)'$ denotes the commutant of $M_+$ in $B(H_+)$ and
2) $(M')_+' = M_+$

**Proof.** i) Let $x \in (M')_{\gamma_1}$ and $m \in M_{\gamma_2}, \gamma_1, \gamma_2 \in \text{sp}(\alpha) \cap \Gamma_+$. Then,
$$p_+xp_mp_+ = p_+xp_+mp_+ = p_+xmp_+ = p_+mp_+xp_+$$
so, $(M')_+ \subset (M_+)'$. To prove the converse inclusion, let $x \in (M_+)' \subset B(H_+)$. Consider the following dense subspace of $H_+$
$$H' = \left\{ \sum_{\gamma \in F} H_\gamma : F \subset \text{sp}(\alpha) \cap \Gamma_+ \text{ a finite subset} \right\}. $$
Then, clearly, the subspace
$$H'' = \text{lin} \{ u_\gamma \xi : \xi \in H', \gamma \in \text{sp}(\alpha) \} ,$$
where $u_\gamma$ is the partial isometry in Condition (C), is dense in $H$. Let $\eta = \sum_{i=1}^n u_{\gamma_i} \xi_i \in H''$. Without loss of generality, we will assume in the rest of this proof that $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$. If $u_{\gamma_i}u_{\gamma_j}^* = u_{\gamma_j}^*u_{\gamma_i} = e_{\gamma_i}, 1 \leq i \leq n$ are the central projections from Condition (C) above, then, a standard calculation in the commutative W*-algebra $Z(M)$, shows that
$$e_{\gamma_1} \vee e_{\gamma_2} \vee \ldots \vee e_{\gamma_n} = e_{\gamma_1} + \sum_{i=2}^n (1 - e_{\gamma_1}) \ldots (1 - e_{\gamma_{i-1}}) e_{\gamma_i}.$$  \hspace{1cm} (1)
where $e_{\gamma_1} \vee e_{\gamma_2} \vee \ldots \vee e_{\gamma_n} = \sup \{ e_{\gamma_i} : 1 \leq i \leq n \}$. Clearly, the terms of the above sum are mutually orthogonal central projections in $Z(M)$. So if $\eta = \sum_{i=1}^n u_{\gamma_i} \xi_i \in H''$ it follows that
$$\eta = e_{\gamma_1} \eta + \sum_{i=2}^n (1 - e_{\gamma_1}) \ldots (1 - e_{\gamma_{i-1}}) e_{\gamma_i} \eta.$$  \hspace{1cm} (2)
or,

\[ \eta = \sum_{i=1}^{n} p_i \eta \]  

where \( p_1 = e_{\gamma_1} \) and \( p_i = (1 - e_{\gamma_i}) \ldots (1 - e_{\gamma_{i-1}}) e_{\gamma_i}, \) \( 2 \leq i \leq n. \) Define the operator \( \hat{x} \) on \( H'' \) as follows

\[ \hat{x}(\sum_{i=1}^{n} u_{\gamma_i} \xi_i) = \sum_{i=1}^{n} u_{\gamma_i} x \xi_i. \]

We prove first that \( \hat{x} \) is well defined. Indeed, suppose that \( \sum_{i=1}^{n} u_{\gamma_i} \xi_i = 0. \) We will show that \( \sum_{i=1}^{n} u_{\gamma_i} x \xi_i = 0. \) Now, if \( \eta = \sum_{i=1}^{n} u_{\gamma_i} \xi_i = 0, \) it follows that

\[ p_i \eta = 0, \quad 1 \leq i \leq n. \]  

We must show that

\[ p_i \sum_{j=1}^{n} u_{\gamma_j} x \xi_j = 0, \quad 1 \leq i \leq n. \]  

These equalities imply that \( \sum_{i=1}^{n} u_{\gamma_i} x \xi_i = 0, \) so \( \hat{x} \) is well defined. Since \( p_i \eta = 0, \)

\[ \sum_{j=1}^{n} p_i u_{\gamma_j} \xi_j = \sum_{j=i}^{n} p_i u_{\gamma_j} p_i \xi_j = 0. \]

Thus, factoring out \( u_{\gamma_i}, \)

\[ u_{\gamma_i} \sum_{j=1}^{n} p_i u_{\gamma_j} p_i \xi_j = 0 \]

By multiplying the above equality by \( u_{\gamma_j}^{*}, \) and taking into account that \( p_i \leq e_{\gamma_i} \)

we get

\[ \sum_{j=i}^{n} p_i u_{\gamma_i}^{*} u_{\gamma_j} \xi_j = 0. \]

so

\[ p_i e_{\gamma_i} \xi_i + \sum_{j=i+1}^{n} p_i u_{\gamma_i}^{*} u_{\gamma_j} \xi_j = 0. \]

where \( u_{\gamma_i}^{*} u_{\gamma_j} \in M_{\gamma_j - \gamma_i}. \) Since \( \gamma_i \leq \gamma_j, \) if \( i \leq j \leq n, \) so, \( M_{\gamma_j - \gamma_i} \subset M_{+} \) and \( x \in (M_{+})', \) it follows that

\[ x(p_i e_{\gamma_i} \xi_i + \sum_{i=2}^{n} p_i u_{\gamma_i}^{*} u_{\gamma_j} \xi_j) = p_i e_{\gamma_i} x \xi_i + \sum_{j=i+1}^{n} p_i u_{\gamma_i}^{*} u_{\gamma_j} x \xi_j = 0. \]

By multiplying the above equality by \( u_{\gamma_i}, \) we get

\[ p_i u_{\gamma_i} x \xi_i + \sum_{j=i+1}^{n} p_i u_{\gamma_j} x \xi_j = p_i \sum_{j=i}^{n} u_{\gamma_j} x \xi_j = p_i \sum_{j=i}^{n} u_{\gamma_j} x \xi_j = 0. \]

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and this proves (4). Therefore, \( \hat{x} \) is well defined. From the definition of \( \hat{x} \) it follows that \( \hat{x}m = m\hat{x} \) on \( H^n \) for all \( m \in M \) and \( \hat{x}(\eta) = x(\eta) \) for every \( \eta \in H' \), so if, as we will prove, \( \hat{x} \) is bounded, it follows that \( \hat{x} \in M' \). Next we prove that the operator \( \hat{x} \) is bounded. Indeed, if, as above, \( \eta = \sum u_{\gamma_i}^e \xi_i \), then, using the equality (3) and the fact that \( \hat{x} \) commutes with \( p_i, 1 \leq i \leq n \), we have

\[
\| \hat{x}(\eta) \| = \| \hat{x} \left( \sum_{i=1}^{n} p_i \eta \right) \| = \left\| \sum_{i=1}^{n} p_i \hat{x}(p_i \eta) \right\| = \sqrt{\sum_{i=1}^{n} \| p_i \hat{x}(p_i \eta) \|^2}.
\]

Further, since \( \gamma_i \leq \gamma_j \), and \( p_j u_{\gamma_i} = 0 \) when \( i \leq j \) and \( x \in (M_+)' \) we have

\[
\| p_i \hat{x}(p_i \eta) \|^2 = \| p_i \hat{x}(\sum_{j=1}^{n} p_i u_{\gamma_j} \xi_j) \|^2 = \left\| p_i \sum_{j=1}^{n} p_i u_{\gamma_j} x \xi_j \right\|^2 \leq \left\| p_i \sum_{j=1}^{n} p_i u_{\gamma_j}^* u_{\gamma_j} x \xi_j \right\|^2 = \left\| p_i \sum_{j=1}^{n} p_i u_{\gamma_j}^* u_{\gamma_j} x \xi_j \right\|^2 \leq \| x \|^2 \left\| p_i \sum_{j=1}^{n} p_i u_{\gamma_j} \xi_j \right\|^2 = \| x \|^2 \| p_i \eta \|^2 = \| x \|^2 \| p_i \eta \|^2.
\]

Therefore, \( \| \hat{x}(\eta) \| \leq \| x \| \| \eta \| \) so \( \hat{x} \) is bounded. As noticed above, \( \hat{x} \in M' \) and, since obviously, \( p_+ \hat{x} p_+ = x \) it follows that \( x \in (M')_+ \) and we are done.

ii) follows from i) by replacing \( M \) with \( M' \).

The following version of Lemma 3.10. will be used in the proof of Theorem 3.9. Here and in Theorem 3.9 we assume that \( \Gamma \) is a direct product or a direct sum of archimedean linearly ordered discrete groups, but we also assume a stronger version of Condition (C), namely that for every \( \gamma \in \text{sp}(\alpha), M_\gamma \) contains a unitary operator and that the action is faithful (i.e. \( \alpha_\gamma = \text{id} \) implies \( g = 0 \)). These conditions imply that \( \text{sp}(\alpha) = \Gamma \) (see for instance [21 Lemma 2.2.]).

\[\text{Lemma 3.11.} \quad \text{Let } (M, G, \alpha) \text{ be such that } \hat{G} = \Gamma \text{ is a direct product, } \Gamma = \Pi_{\gamma \in \Gamma} \Gamma_\gamma, \text{ (or a direct sum) of archimedean linearly ordered discrete groups } \Gamma_\gamma. \quad \text{Suppose that } \alpha \text{ is faithful and for every } \gamma \in \text{sp}(\alpha), \text{ there exists a unitary operator } u_\gamma \in M_\gamma \text{ (as noticed above, these conditions imply that } \text{sp}(\alpha) = \Gamma \text{). Then}
\]

\[i) \quad (M_+)' = (M')_+, \text{ where } (M_+)' \text{ denotes the commutant of } M_+ \text{ in } B(H_+), \text{ and}
\]

\[ii) \quad ((M')_+)' = M_+.
\]

\[\text{Proof.} \quad i) \text{ As in the proof of the previous Lemma 3.10. it follows that } (M_+)' \subset (M_+)' \text{. To prove the opposite inclusion, let } x \in (M_+)' \text{. Further, let us}
\]
denote
\[ H' = \left\{ \sum_{\gamma \in F} H_\gamma : F \subset \Gamma_+ \text{ a finite subset} \right\} \]
and
\[ H'' = \text{lin} \left\{ u_\gamma \xi : \xi \in H', \gamma \in \Gamma \right\}. \]
Clearly \( H' \) is dense in \( H_+ \) and \( H'' \) is dense in \( H \). Define the linear operator \( \hat{x} \) on \( H'' \) as follows
\[ \hat{x} \left( \sum_{i=1}^{n} u_\gamma_i \xi_i \right) = \sum_{i=1}^{n} u_\gamma_i x_\xi_i. \]
We will prove first that \( \hat{x} \) is well defined. Suppose that
\[ \sum_{i=1}^{n} u_\gamma_i \xi_i = 0. \] (5)
Since, by Lemma 3.3., \( \Gamma \) is lattice ordered, let \( \nu = \inf \{ \gamma_i : 1 \leq i \leq n \} \in \Gamma \).
Since, by hypothesis \( sp(\alpha) = \Gamma \), we have \( \nu \in sp(\alpha) \). Let \( u_\nu \in M_\nu \) be a unitary operator as in the hypothesis. By multiplying (5) by \( u_\nu^* \) we get
\[ \sum_{i=1}^{n} u_\gamma_i - \nu \xi_i = 0. \]
Since \( \gamma_i - \nu \in \Gamma_+, i = 1, 2, ..., n \) and \( x \in (M_+)^\prime \) it follows that
\[ x \sum_{i=1}^{n} u_\nu^* u_\gamma_i \xi_i = \sum_{i=1}^{n} u_\nu^* u_\gamma_i x_\xi_i = 0. \]
so
\[ u_\nu \sum_{i=1}^{n} u_\nu^* u_\gamma_i x_\xi_i = \sum_{i=1}^{n} u_\nu^* u_\gamma_i x_\xi_i = \hat{x} \left( \sum_{i=1}^{n} u_\gamma_i \xi_i \right) = 0 \]
so \( \hat{x} \) is well defined. We will prove next that \( \hat{x} \) is continuous. Indeed
\[ \left\| \hat{x} \left( \sum_{i=1}^{n} u_\gamma_i \xi_i \right) \right\| = \left\| \sum_{i=1}^{n} u_\gamma_i x_\xi_i \right\| = \left\| u_\nu \sum_{i=1}^{n} u_\nu^* u_\gamma_i x_\xi_i \right\| = \]
\[ = \left\| \sum_{i=1}^{n} u_\nu^* u_\gamma_i x_\xi_i \right\| = \left\| x \sum_{i=1}^{n} u_\nu^* u_\gamma_i \xi_i \right\| \leq \|x\| \left\| \sum_{i=1}^{n} u_\nu^* u_\gamma_i \xi_i \right\| = \]
\[ \|x\| \left\| u_\nu \sum_{i=1}^{n} u_\gamma_i \xi_i \right\| = \|x\| \left\| \sum_{i=1}^{n} u_\gamma_i \xi_i \right\|. \]
so \( \hat{x} \) is continuous. As in the proof of the previous Lemma 3.10. we can see that \( \hat{x} \in M' \) and \( p_+ \hat{x} p_+ = x \), so \( x \in (M')_+ \).
ii) follows from i) by replacing \( M \) with \( M' \).
It is worth mentioning that the previous two Lemmas imply, in particular, that \((M_+)^n = M_+\) but we will not use this fact.

In the next two lemmas we will assume that \((M, G, \alpha)\) is a W*-dynamical system that satisfies Condition (C) and \(M\) is in standard form. We also assume that \(\Gamma\) is a direct product or sum of archimedean linearly ordered abelian discrete groups. If \(\gamma_0 \in sp(\alpha) \cap \Gamma_+ \setminus \{0\}\) and \(\gamma \in sp(\alpha) \cap \Gamma_+\) according to Corollary 3.2, there exists \(p \in \mathbb{Z}_+\) such that either \(p\gamma_0 \leq \gamma < (p+1)\gamma_0\) or \(p\gamma_0 \leq \gamma\) and \(\gamma\) is not comparable with \((p+1)\gamma_0\). If, in addition, \(e_\gamma e_{\gamma_0} \neq 0\) we will denote for every \(\lambda \in \mathbb{C}, |\lambda| < 1\)

\[
K_{\gamma_0, \gamma, \lambda} = \left\{ x(\lambda, \gamma_0, \gamma, \xi) = \sum_{n \geq 0} \lambda^n u_{\gamma_0}^{\alpha} u_{\gamma - p\gamma_0}^{\alpha} \xi : \xi \in e_{\gamma_0} H_0 \right\} \subset e_{\gamma_0} H_+.
\]

and

\[
L_{\gamma_0, \gamma, \lambda} = \left\{ y(\lambda, \gamma_0, \gamma, \xi) = \sum_{n \geq 0} \lambda^n u_{\gamma_0}^{\alpha} u_{\gamma - p\gamma_0}^{\alpha} \xi : \xi \in e_{\gamma_0} H_0 \right\} \subset e_{\gamma_0} H.
\]

**Lemma 3.12.** Let \((M, G, \alpha)\) be a W*-dynamical system that satisfies Condition (C). \(\Gamma\) and \(\gamma_0, \gamma \in sp(\alpha) \cap \Gamma_+\) be as above. Then,

\[
K_{\gamma_0} = \text{lin} \left\{ K_{\gamma_0, \gamma, \lambda} : \gamma \in sp(\alpha) \cap \Gamma_+, \lambda \in \mathbb{C}, |\lambda| < 1 \right\}
\]

is dense in \(e_{\gamma_0} H_+\).

**Proof.** Notice first that if \(\gamma_0, \gamma\) are as in the hypothesis of the lemma, then, by the definition of \(e_{\gamma_0}\) in the Condition (C), we have that \(p\gamma_0 \in sp(\alpha)\) for every \(p \in \mathbb{Z}_+\) and, by Lemma 3.6., \(\gamma - p\gamma_0 \in sp(\alpha)\), so \(u_{\gamma - p\gamma_0}\) exists, and thus the definition of \(K_{\gamma_0, \gamma, \lambda}\) in the hypothesis of the Lemma is consistent. Now, taking \(\lambda = 0\) in \(K_{\gamma_0, \gamma, \lambda}\), it follows that \(e_{\gamma_0} u_{\gamma - p\gamma_0} H_0 \subset K_{\gamma_0, \gamma} \subset K_{\gamma_0}\). In particular, for \(p = 0\) (so, when either \(0 \leq \gamma < \gamma_0\) or \(0 \leq \gamma\) and \(\gamma\) is not comparable with \(\gamma_0\)), we have \(e_{\gamma_0} u_{\gamma} H_0 = e_{\gamma} H_0 \subset K_{\gamma_0}\). We will prove that \(e_{\gamma_0} H_\gamma = u_{\gamma} e_{\gamma_0} H_0 \subset K_{\gamma_0}\) for every \(\gamma \in sp(\alpha) \cap \Gamma_+\). This fact will imply that \(\sum_{\gamma \in \Gamma_+} e_{\gamma_0} H_\gamma \subset K_{\gamma_0}\) and therefore, \(e_{\gamma_0} H_+ = \sum_{\gamma \in \Gamma_+} e_{\gamma_0} H_\gamma \subset K_{\gamma_0}\), so \(K_{\gamma_0}\) is dense in \(e_{\gamma_0} H_+\) as claimed. We will prove first that \(e_{\gamma_0} u_{\gamma - p\gamma_0} H_0 \subset K_{\gamma_0, \gamma}\). The case \(p = 0\) was proved above. Suppose that \(p > 0\). We will prove by induction on \(k\) that

\[
e_{\gamma_0} u_{\gamma_0}^k u_{\gamma - p\gamma_0} H_0 \subset K_{\gamma_0}
\]

for every \(k\), in particular for \(k = p\). If \(k = 0\), the above inclusion follows, as noticed at the beginning of this proof from the definition of \(K_{\gamma_0, \gamma}\) for \(\lambda = 0\). Suppose by induction that \(u_{\gamma_0}^l u_{\gamma - p\gamma_0} \xi \in K_{\gamma_0}\) for \(l = 0, 1, \ldots k - 1\). Then

\[
\sum_{n \geq k} \lambda^n u_{\gamma_0}^n u_{\gamma - p\gamma_0} \xi \in K_{\gamma_0}
\]
for every $\xi \in e_{\gamma_{0}} H_{0}, \lambda \in \mathbb{C}, |\lambda| < 1$. Thus

$$\lambda^{k} u^{k}_{\gamma_{0}} u_{\gamma - p \gamma_{0}} \xi + \sum_{n \geq k+1} \lambda^{n} w^{n}_{\gamma_{0}} u_{\gamma - p \gamma_{0}} \xi \in K_{\gamma_{0}}$$

By dividing the above relation by $\lambda^{k}$, $\lambda \neq 0$ and then taking the limit as $\lambda \to 0$, we get that $u^{k}_{\gamma_{0}} u_{\gamma - p \gamma_{0}} \xi \in K_{\gamma_{0}}$, so, in particular, $u^{p}_{\gamma_{0}} u_{\gamma - p \gamma_{0}} \xi \in K_{\gamma_{0}}$ for every $\xi \in H_{0}$. Since, obviously, $u^{p}_{\gamma_{0}} (u^{*}_{\gamma_{0}})^{p} = e_{\gamma_{0}}$ for $p > 0$ and, by Lemma 2.5. ii), $M_{-p \gamma_{0}} M_{\gamma} \subseteq M_{\gamma - p \gamma_{0}}$, it follows that

$$e_{\gamma_{0}} H_{\gamma} = e_{\gamma_{0}} u_{\gamma} H_{0} = u^{p}_{\gamma_{0}} (u^{*}_{\gamma_{0}})^{p} u_{\gamma} H_{0} \subset u^{p}_{\gamma_{0}} u_{\gamma - p \gamma_{0}} H_{0} \subset K_{\gamma_{0}}$$

and we are done. \(\blacksquare\)

The following lemma can be proven similarly with the previous Lemma 3.12.

**Lemma 3.13.** Let $(M, G, \alpha)$ be a $W^{*}$-dynamical system that satisfies Condition (C), $\Gamma$ and $\gamma_{0}, \gamma \in \text{sp}(\alpha) \cap \Gamma_{+}$ be as above. Then,

$$L_{\gamma_{0}} = \text{lin} \{L_{\gamma_{0}, \gamma, \lambda} : p \gamma_{0} \leq \gamma < (p+1) \gamma_{0} \text{ for some } p \in \mathbb{Z}_{+}, \lambda \in \mathbb{C}, |\lambda| < 1\}$$

is dense in $e_{\gamma_{0}} H_{+}$.

**Proof of Theorem 3.8.** We will prove first that $M_{+} \subset B(H_{+})$ is reflexive and then apply Lemma 3.4. and the subsequent discussion to infer that $M_{+}$ is hereditarily reflexive. Let $\gamma_{0} \in \text{sp}(\alpha) \cap \Gamma_{+}$. Since $u_{\gamma_{0}} \in M$, it follows that $u_{\gamma_{0}} f = f u_{\gamma_{0}}$ for every projection $f \in (M')_{0}$, in particular for every projection $f \in (M')_{0}$. By Lemma 2.8. iv), since $(M')_{\gamma} H_{0} = H_{\gamma}, \gamma \in \text{sp}(\alpha)$, we have, in particular that $(M')_{\gamma} p_{+} \subset B(H_{+})$. Therefore, for every projection $f \in (M')_{0}$, $f H_{+}$ belongs to $\text{Lat}(M_{+})$. Let $x \in \text{algLat}(M_{+}) \subset B(H_{+})$. We will prove that $x \in ((M')_{+})'$ and then, applying Lemma 3.11. ii) it will follow that $x \in M_{+}$, so $M_{+}$ is a reflexive operator algebra. The way to prove this fact is to use Lemma 3.14. to show that $x^{*} \in ((M')_{+})'$ and then, clearly, it will follow that $x \in ((M')_{+})' = M_{+}$. As noticed above, $f H_{+} \in \text{Lat} M_{+}$ for every projection $f \in (M')_{0}$, so $xf = fx$ and therefore $x^{*} f = f x^{*}$ for every projection $f \in (M')_{0}$. It follows that $x$ commutes with every element of $(M')_{0}$. We will prove next that $x w_{\gamma} = w_{\gamma} x$ for every $\gamma \in \text{sp}(\alpha') \cap \Gamma_{+} = \text{sp}(\alpha) \cap \Gamma_{+}$ and then apply Lemma 3.10. ii) to infer that $x \in ((M')_{+})' = M_{+}$.

To this end, let $\gamma_{0} \in \text{sp}(\alpha) \cap \Gamma_{+}$. If $\gamma_{0} = 0$, then as convened, $w_{\gamma} = w_{\gamma_{0}} = I$, so nothing to prove. Let $\gamma_{0} \in \text{sp}(\alpha) = \text{sp}(\alpha'), \gamma_{0} > 0$. Denote by $T^{*}_{u_{\gamma_{0}}}$ (respectively $T^{*}_{w_{\gamma_{0}}}$) the operator $u_{\gamma_{0}} \in M_{+}$ (respectively $w_{\gamma_{0}} \in (M')_{+}$) defined on $H_{+}$. Then the adjoints of $T^{*}_{u_{\gamma_{0}}}, T^{*}_{w_{\gamma_{0}}}$ on $H_{+}$ are

$$T^{*}_{u_{\gamma_{0}}} \xi_{\gamma} = 0 \text{ if } 0 \leq \gamma < \gamma_{0} \text{ and } T^{*}_{u_{\gamma_{0}}} \xi_{\gamma} = u^{*}_{\gamma_{0}} \xi_{\gamma} \text{ if } \gamma \geq \gamma_{0}.$$

and similarly

$$T^{*}_{w_{\gamma_{0}}} \xi_{\gamma} = 0 \text{ if } 0 \leq \gamma < \gamma_{0} \text{ and } T^{*}_{w_{\gamma_{0}}} \xi_{\gamma} = w^{*}_{\gamma_{0}} \xi_{\gamma} \text{ if } \gamma \geq \gamma_{0}.$$
Since $u_{\gamma_0}, w_{\gamma_0}$ (so $T_{u_{\gamma_0}}, T_{w_{\gamma_0}}$) commute, it follows that $T_{u_{\gamma_0}}^*, T_{w_{\gamma_0}}^*$ commute as well. Notice that if, for $\lambda \in \mathbb{C}, |\lambda| < 1$, we denote

$$\bar{L}_{\gamma_0,\lambda} = \{ \xi \in H_+ : T_{w_{\gamma_0}}^* \xi = \lambda \xi \}. $$

then, since $T_{w_{\gamma_0}}^*$ commutes with $(M_+)^*$, we have $\bar{L}_{\gamma_0,\gamma,\lambda} \in \text{Lat}(M_+)^*$ and, since $x \in \text{algLat}(M_+)$ it follows that $L_{\gamma_0,\gamma,\lambda} \in \text{Lat}(x^*)$ for every $\lambda \in \mathbb{C}, |\lambda| < 1$. Since $\bar{L}_{\gamma_0,\gamma,\lambda}$ consists of eigenvectors of $T_{w_{\gamma_0}}^*$ for the eigenvalue $\lambda$, then, it follows that $x^* T_{w_{\gamma_0}}^* \xi = T_{w_{\gamma_0}}^* x^* \xi$ for every $\xi \in \bar{L}_{\gamma_0,\lambda}$. On the other hand, it is clear that

$$L_{\gamma_0,\gamma,\lambda} \subset \bar{L}_{\gamma_0,\lambda}$$

where $L_{\gamma_0,\gamma,\lambda}$ is as in Lemma 3.13., so $x^* T_{w_{\gamma_0}}^* \xi = T_{w_{\gamma_0}}^* x^* \xi$ for every $\xi \in L_{\gamma_0,\gamma,\lambda}$. By Lemma 3.13.,

$$L_{\gamma_0} = \text{lin} \{ L_{\gamma_0,\gamma,\lambda} : p \gamma_0 \leq \gamma < (p + 1) \gamma_0 \text{ for some } p \in \mathbb{Z}_+, \lambda \in \mathbb{C}, |\lambda| < 1 \}$$

is dense in $e_{\gamma_0} H_+$, and therefore $x^* T_{w_{\gamma_0}}^* \xi = T_{w_{\gamma_0}}^* x^* \xi$ for every $\xi \in e_{\gamma_0} H_+$, so $x^*$ commutes with $T_{w_{\gamma_0}}^* \gamma_0 \in \text{sp}(\alpha) \cap \Gamma_+$ and therefore with $(M')_+^*$. Since $x^*$ commutes with $(M')_+^*$, it follows that $x$ commutes with $(M')_+$, so by Lemma 3.10. ii) $x \in M_+$ so $M_+$ is reflexive. Finally, by applying Lemma 3.4. and the discussion following it, we see that $M_+$ is hereditarily reflexive and we are done.

Corollary 3.14. below refers to the Example a) to Condition (C).

**Corollary 3.14.** Let $(M, G, \alpha)$ be a $W^*$-dynamical system with $G$ compact abelian and $M$ a finite $W^*$-algebra in standard form such that $Z(M_0) \subset Z(M)$. Suppose that the dual $\Gamma$ of $G$ has an archimedean linear order. Then $M_+ \subset B(H_+)$ is reflexive.

**Proof.** According to [18, Theorem 2.3.], if $M$ is finite and $Z(M_0) \subset Z(M)$, then the Condition (C) is satisfied and therefore the result follows from Theorem 3.8. ■

**Proof.** of Theorem 3.9. The proof is very similar with the proof of Theorem 3.8. The only modification is using Lemma 3.11 instead of Lemma 3.10. ■

The next Corollary refers to Examples b) to Condition (C).

**Corollary 3.15.** Let $(M, G, \alpha)$ be a $W^*$-dynamical system with $M$ an injective von Neumann algebra in standard form and $G$ a compact abelian group such that the dual $\Gamma$ of $G$ is a direct product (or a direct sum) of archimedean linearly ordered discrete groups. Suppose that $\alpha$ is prime and faithful and it is not the case that $M$ is of type III and $M_0$ is of type II_1. Then $M_+$ is hereditarily reflexive.
Since $\alpha$ is faithful, we have $sp(\alpha) = \Gamma$ (see for instance [21, Lemma 2.2]). By [21, Theorem 2.3.] each spectral subspace $M_\gamma$ contains a unitary operator. The conclusion of the Corollary follows from Theorem 3.9.

The concept of nonselfadjoint crossed product, or more generally that of w*-semicrossed product were defined in [3], [11], [16].

**Corollary 3.16.** Let $(M_0, \Gamma, \beta)$ be a W*-dynamical system such that $M_0 \subset B(H_0)$ is in standard form and $\Gamma$ is a discrete abelian group. Suppose that $\Gamma \subseteq \Pi \Gamma$ is a direct product (or a direct sum) of archimedean linearly ordered groups. Let $M = M_0 \times_\alpha \Gamma \subset B(l^2(\Gamma, H_0))$ be the corresponding crossed product. Then, the non selfadjoint crossed product $M_+ = M_0 \times_\beta \Gamma_+$ (i.e. the algebra of elements of $M$ with non negative spectrum) is a hereditarily reflexive operator algebra in $B(H_+)$ where $H_+ = l^2(\Gamma_+, H_0)$.

**Proof.** If $G$ denotes the (compact) dual of $\Gamma$, and $\alpha = \hat{\beta}$ is the dual action of $\beta$ on $M$, then, consider the canonical conditional expectation $P_0 : M \rightarrow M_0$. Since $M_0 \subset B(H_0)$ is in standard form, from Lemma 2.1. and the subsequent discussion it follows that $M \subset B(l^2(\Gamma, H_0))$ is in standard form and by the definition of the crossed product, $M_\gamma$ contains a unitary operator $u_\gamma$ for every $\gamma \in \Gamma$ and thus the W*-dynamical system $(M, G, \hat{\beta})$ satisfies the conditions of Corollary 3.15.

**Corollary 3.17.** Let $(M_0, \Gamma, \beta)$ be a W*-dynamical system such that $M_0 \subset B(H_0)$ is a maximal abelian von Neumann algebra and $\Gamma$ is a discrete abelian group. Suppose that $\Gamma \subseteq \Pi \Gamma$ is a direct product (or a direct sum) of archimedean linearly ordered groups. Let $M = M_0 \times_\alpha \Gamma \subset B(l^2(\Gamma, H_0))$ be the corresponding crossed product. Then, the non selfadjoint crossed product $M_+ = M_0 \times_\beta \Gamma_+$ is a hereditarily reflexive operator algebra in $B(H_+)$ where $H_+ = l^2(\Gamma_+, H_0)$.

**Proof.** Since $M_0 \subset B(H_0)$ is a maximal abelian von Neumann algebra, by Proposition 2.4. it is spatially isomorphic with its standard form, so the result follows from the previous Corollary 3.16.

In [2, Corollary 5.14.] it is stated that if $M_0 \subset B(H_0)$ is a maximal abelian von Neumann algebra and $\Gamma = \mathbb{Z}^d, d \in \mathbb{N}$, then $M_0 \times_\beta \Gamma_+$ is reflexive, so the Corollary 3.18. above extends that result by showing also hereditary reflexivity in the special case $\Gamma = \mathbb{Z}^d, d \in \mathbb{N}$.

**REFERENCES**

1. W. B. Arveson, The harmonic analysis of automorphism groups, operator algebras and applications, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, R. I., 1982.

2. R. T. Bickerton, and E. T. A. Kakariadis, Free multivariate w*-semicrossed products: reflexivity and the bicommutant property, Canad. J. Math. 70(2018), 1201–1235.
3. K. R. Davidson, A. H. Fuller and E. T. A. Kakariadis, Semicrossed Products of Operator Algebras by Semigroups, Memoirs of the AMS, Vol. 247, 2017.

4. D. W. Hadwin and E. A. Nordgren, Subalgebras of reflexive algebras. J. Operator Theory 7 (1982), 3–23.

5. L. Helmer, Reflexivity of non-commutative Hardy algebras, J. Funct. Anal. 272(2017), 2752–2794.

6. V. F. R. Jones, Prime actions of compact abelian groups on the hyperfinite type II₁ factor, J. Operator Theory, 9(1983), 181-186.

7. V. F. R. Jones and M. Takesaki, Actions of compact abelian groups on semifinite injective factors, Acta Math. 153 (1984), 213–258.

8. R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, Vol. II Advanced Theory, Academic Press 1986.

9. E. T. A. Kakariadis, Semicrossed products and reflexivity, J. Operator Theory, 67(2012), 379-395.

10. A I Loginov and V. S Šul’man, Hereditary and intermediate reflexivity of W*-algebras, Mathematics of the USSR-Izvestiya 9(1975), 1189-1202.

11. M. McAsey, P. S. Muhly, and K.-S. Saito, Nonselfadjoint crossed products (invariant subspaces and maximality), Trans. Amer. Math. Soc., 248(1979), 381–409.

12. P. S. Muhly and B. Solel, Hardy algebras, W*-correspondences and interpolation theory, Math. Ann. 330 (2004), 353–415.

13. G. K. Pedersen, C*-algebras and their automorphism groups, Academic Press 1979.

14. C. Peligrad, Reflexive operator algebras on noncommutative Hardy spaces, Math. Annalen, 253(1980), 165-175.

15. C. Peligrad, Invariant subspaces of algebras of analytic elements associated with periodic flows on von neumann algebras, Houston J. Math., 42(2016), 1331-1445.

16. J. R. Peters, Semicrossed products of C*-algebras. J. Funct. Anal., 59(1984), 498–534.

17. H. Radjavi and P. Rosenthal, Invariant subspaces, 2nd edition, Dover Publications, Mineola, New, York, 2003.

18. K.-S. Saito, Nonselfadjoint subalgebras associated with compact abelian group actions on finite von Neumann algebras, Tohoku Math. Journ. 34(1982), 485-494.

19. D. Sarason, Invariant subspaces and unstarred operator algebras, Pacific J. Math., 17(1966), 511-517.

20. S. Stratila, Modular theory in operator algebras, Bucharest; Abacus Press, Tunbridge Wells, 1981

21. K. Thomsen, Compact abelian prime actions on von Neumann algebras, Trans. Amer. Math. Soc., 315(1989), 255-273.