ASYMPTOTICS OF HEIGHT CHANGE ON TOROIDAL TEMPERLEYAN DIMER MODELS

JULIEN DUBÉDAT AND REZA GHEISSARI

Abstract. The dimer model is an exactly solvable model of planar statistical mechanics. In its critical phase, various aspects of its scaling limit are known to be described by the Gaussian free field. For periodic graphs, criticality is a condition on the spectral curve of the model, determined by the edge weights [10]; isoradial graphs constitute another class of critical dimer models, in which the edge weights are determined by the local geometry.

In the present article, we consider another class of graphs: general Temperleyan graphs, i.e. graphs arising in the (generalized) Temperley bijection between spanning trees and dimer models. Under a minimal assumption - viz. that the underlying random walk converges to Brownian motion - we show that the natural topological observable on macroscopic tori (the winding or height change of the dimer configuration) converges in law to its universal limit.

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1. Introduction

The dimer model is a classical and extensively studied model of statistical mechanics (see e.g. the survey [14]). Given an underlying graph $\mathcal{G}$, a dimer configuration (or perfect matching) is a subset of vertex-disjoint edges covering the graph; we consider here only the case where $\mathcal{G}$ is bipartite. For planar graphs (or, more generally, graphs on surfaces), the model is exactly solvable, in the sense that its partition function can be represented as the determinant of a modified adjacency matrix, the Kasteleyn matrix [10].
Exact solvability has allowed for a detailed analysis of large scale behavior of the dimer model. Following Thurston, a dimer configuration on \( G \) may be associated to a height function on \( G^* \); this gives a way to think about scaling limits of the dimer model (as the mesh of the graph goes to zero). When \( G \) is embedded on a torus, the height function is additively multivalued (i.e. picks up an additive constant when tracked along a non-contractible cycle on the torus) and can be decomposed into two components: an affine multivalued part (height change), and a scalar single-valued part.

For a microscopically periodic graph, [16] shows the existence of three phases: a deterministic solid phase, an exponentially decorrelated gaseous phase, and a “critical” liquid phase. We will focus here on that critical phase.

For critical dimer models, one expects the height fluctuations to be universal and described by a (compactified) free field. For specific lattices, Kenyon showed convergence of scalar fluctuations on the square lattice to the Gaussian free field (GFF) in [12], and Boutillier and de Tilière established convergence of the height change on the hexagonal lattice in [3]. So far, universality results on dimers have focussed on two classes of graphs.

The first class consists of periodic graphs (where a microscopic finite fundamental domain is repeated many times to fill the plane or a macroscopic torus). The scalar fluctuations are shown to converge to a GFF in [16] (for a proper choice of embedding); recently, [18] showed that the height change on tori also converges to its universal limit.

The second class consists of isoradial graphs (see [13]) derived from a lozenge tiling. These are not necessarily periodic and have an additional Yang-Baxter–type solvability. Universality of scalar fluctuations in the plane is shown in [6]. In the more restrictive set-up of so-called Temperleyan isoradial graphs, the full distribution of the height on tori is obtained in [8] (the scalar fluctuations and height change are asymptotically independent).

A natural question is whether universality can be extended to other classes of graphs; a difficulty is that critical behavior is highly sensitive to edge weights, as exemplified by [5]. In the present article, we undertake the analysis of dimers on Temperleyan graphs. This is a class of graph \( G \) derived from a generic graph \( \Gamma \), in such a way that dimer configurations on \( G \) correspond to spanning trees on \( \Gamma \) (generalized Temperley bijection, [17]); \( G \) is obtained by superimposing \( \Gamma \) with its dual \( \Gamma^* \).

Since spanning trees can be derived from random walks (Wilson’s algorithm, [25]), a natural (and essentially minimal) criticality condition is that the underlying random walk (RW) on \( \Gamma \) converges, up to time change, to Brownian motion (BM). Under this assumption, Yadin and Yehudayoff [26] showed convergence of the Loop-Erased Random Walk (LERW) to SLE\(_2\) (Schramm-Loewner Evolution with \( \kappa = 2 \)), extending celebrated work of Lawler, Schramm, Werner on regular lattices [20].

Since dimers on \( G \) correspond to Uniform Spanning Trees on \( \Gamma \), and USTs can be generated from LERWs ([25]), this suggests universality of dimers when the underlying RW converges to BM. Remark also that the dimer height can be expressed in terms of windings of LERWs ([17]). However at this stage it is unclear how to implement rigorously this heuristic.

In the present article, we thus focus on the height change of dimers on a torus, when the underlying random walk converges to BM; our main result, Theorem 5,
establishes that this quantity is indeed asymptotically universal. The method is based not on Kasteleyn enumeration, but on Temperley’s bijection and Forman’s formula [9], a deformation of the Matrix-Tree Theorem enumerating spanning trees in terms of Laplacian determinants.

Some components of the argument subsist in higher genus, in particular the universality of ratios of Laplacian determinants (Proposition 3). This in turn gives information on the homology of root cycles of cycle-rooted spanning forests. However, the topology of these cycles and their relation with dimers is more complicated than in genus 1, and it is unclear whether there is a concise description of a limiting topological observable in that case. Remark also that more information can be extracted from Kenyon’s vector bundle Laplacian [15].

2. Set-up and notation

2.1. Basic structures.

2.1.1. Underlying graphs. We start from a (finite) graph $\Gamma = (V, E)$ embedded on a torus $\Sigma \overset{\text{def}}{=} \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, where the modulus $\tau$ of the torus is in the upper half-plane: $\Im \tau > 0$. The embedded edges do not cross (except at their endpoints) and bound faces homeomorphic to disks.

The graph $\Gamma$ is endowed with nonnegative edge weights (conductances) on oriented edges ($c(xy) \geq 0$, where $(xy)$ is an oriented edge); we do not assume $c(xy) = c(yx)$.

Let $\Gamma^* = (V^*, E^*)$ be the dual graph of $\Gamma$, so that faces of $\Gamma$ are in bijection with vertices of $\Gamma^*$ and vice versa: two vertices in $\Gamma^*$ are adjacent if they correspond to faces of $\Gamma$ sharing a common boundary edge. All edges on $\Gamma^*$ are assigned weight 1.

Finally we consider the graph $\mathcal{G}$ obtained by superimposing $\Gamma$ and $\Gamma^*$. More precisely, $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G})$ is the bipartite graph whose black vertices $V_\mathcal{G}^B$ are in bijection with $V \sqcup V^*$ and white vertices $V_\mathcal{G}^W$ are in bijection with $E^* \simeq E$. See Figure 2.1

![Figure 2.1. Left: portion of $\Gamma$ (solid) and its dual $\Gamma^*$ (dashed). Right: bipartite graph $\mathcal{G}$](image)

Edges in $\mathcal{G}$ are of two types: $(v, (vv'))$ and $(f, (ff'))$ where $v, v'$ are adjacent in $\Gamma$ and $f, f'$ are adjacent in $\Gamma^*$. In other words, each edge in $\mathcal{G}$ corresponds to an oriented edge in $\Gamma$ or $\Gamma^*$. Notice that $\mathcal{G}$ is a quadrangulation, i.e. all its faces have degree 4. We call such a graph $\mathcal{G}$ Temperleyan.
One may associate weights to edges of $G$ in the following way:

\[ w(v, (vv')) = c(vv') \quad v, v' \in V, v \sim v' \]

\[ w(f, (ff')) = 1 \quad f, f' \in V^*, f \sim f' \]

Notice than some edges may have zero weight.

Remark that by Euler’s formula for $\Gamma$,

\[ |V^B_G| - |V^W_G| = |V| - |E| + |V^*| = 2 - 2g \]

where $g$ is the genus of $\Sigma$; in particular $G$ is balanced iff $\Sigma$ is a torus.

2.1.2. Spanning forests. A vector field (without zeroes) on $\Gamma$, in the sense of Forman, is an assignment $Y : V \to V$ s.t. $v \sim Y(v)$ for all $v$. Since $\Gamma$ is finite, $(Y^k(v))_{k \geq 0}$ is eventually periodic for any $v$. It is then easy to see that the data of a vector field is equivalent to that of a cycle-rooted spanning forest (CRSF), in the sense of Kenyon, viz. a spanning subgraph in which each connected component contains exactly one cycle, which we take to be oriented. Other edges are oriented towards the unique cycle (“root”) in their connected component. We also only consider vector fields/CRSF where all the root cycles are noncontractible in $\Sigma$ (“incompressible”).

Given a CRSF $F$ on $\Gamma$, one can construct a dual CRSF $F^*$ on $\Gamma^*$, s.t. each $w \in V^W_G \simeq E \simeq E^*$ is crossed either by an edge of $F$ or of $F^*$. If $F$ has $k$ root cycles, they are in the same homology class (up to sign); $F^*$ also has $k$ root cycles in the same class and is specified by $F$ up to the orientation of these $k$ cycles, which may be chosen freely.

2.1.3. Dimers. A dimer configuration or perfect matching on $G$ is a subset $m$ of edges of $G$ s.t. each vertex in $V_G$ is an endpoint of exactly one edge in $m$. The weight of a dimer configuration is the product of the weights of edges it contains:

\[ w(m) = \prod_{e \in m} w(e) \]

There is then a natural probability measure on the space of dimer configurations, viz. the Boltzmann measure given by

\[ \mu\{m\} = \frac{\prod_{e \in m} w(e)}{Z} \]

where the partition function is

\[ Z = \sum_m \prod_{e \in m} w(e). \]

Following Thurston, associated to a dimer configuration on $G$, one can define a height function on the dual graph $G^*$.

The height function is defined as follows (see [14]): Given a perfect matching $m$ define a closed 1-form $\omega_m$ (an antisymmetric function on oriented edges) on the dual graph $G^*$ given by

\[ \omega_m((bw)^*) = \begin{cases} 1 & \text{if } b \text{ and } w \text{ are matched, oriented from black to white} \\ 0 & \text{otherwise} \end{cases} \]

Here $(bw)^*$ is the edge of $G^*$ dual to $(bw)$, oriented so that $((bw), (bw)^*)$ is a direct frame.

Then $d\omega_m$ is 1 on black vertices, $-1$ on white vertices; here $(d\omega_m)(f)$ is the sum of $\omega_m(e)$ over edges $e$ bounding the face $f$ of $G^*$, taken counterclockwise.
Consider a reference matching \( m_0 \). Let \( \omega_0 \) be the corresponding 1-form. Then \( d(\omega - \omega_0) = 0 \).

Consequently, locally we can write \( d(\omega_m - \omega_0) = dh \) (i.e. \( (\omega_m - \omega_0)(vv') = h(v') - h(v) \)), where \( h \) is a height function defined on \( \mathcal{G}^* \) (given up to an additive constant).

Because the torus has non-trivial homology, \( h \) need not be defined as a function on all of \( \mathcal{G}^* \), but merely as an additively multivalued function (i.e. picking an additive constant when traced along a non-contractible cycle). One may also lift \( \mathcal{G}^* \) to the universal cover \( \mathbb{C} \) of \( \Sigma \), and then \( h \) defines an additively quasi-periodic function.

A standard basis of the homology group \( H_1(\Sigma, \mathbb{Z}) \) is given by the \( A \)-cycle \( t \mapsto t \) and the \( B \)-cycle \( t \mapsto t \tau \), \( t \in \mathbb{R}/\mathbb{Z} \).

2.1.4. Temperley’s bijection. There is a 1-1 correspondence between pairs \((F, F^*)\) of dual (oriented) CRSFs on \( \Gamma, \Gamma^* \) and perfect matchings \( m \) on \( \mathcal{G} \), the generalized Temperley’s bijection [17] (which we present here in the toroidal setting). Set

\[
(\nu, (\nu')) \in m \iff (\nu') \in F \quad \nu, \nu' \in V, \nu \sim \nu'
\]

\[
(f, (ff')) \in m \iff (ff') \in F^* \quad f, f' \in V^*, f \sim f'
\]

The weight of a CRSF (resp. of a perfect matching) is the product of the weights of edges it contains. Recall that edge weights on \( \Gamma^*, \mathcal{G} \) are derived from those on \( \Gamma \). Then Temperley’s bijection is also weight-preserving. See Figure 2.2.

Let us discuss the relation between the height periods defined in (2.1) and the pair of CRSFs \((F, F^*)\) corresponding to the matching \( m \).

Let \( [F] + [F^*] \in H_1(\Sigma, \mathbb{Z}) \) be its homology class, defined as the sum of the homology classes of the oriented root cycles of \( F, F^* \). Trivially this is an even class (i.e. in \( 2H_1(\Sigma, \mathbb{Z}) \)), and we define

\[
[m] = \frac{1}{2} ([F] + [F^*]) \in H_1(\Sigma, \mathbb{Z})
\]
Hence, given $\mathbf{m}$ and the corresponding pair $(F,F^*)$ we defined an element of $H_1(\Sigma,\mathbb{Z})^\vee$ in (2.2) and an element of $H_1(\Sigma,\mathbb{Z})$.

There is a canonical identification $H_1(\Sigma,\mathbb{Z})^\vee \simeq H_1(\Sigma,\mathbb{Z})$ via the intersection pairing $\cdot$, the antisymmetric bilinear form such that $[A] \cdot [B] = 1$ ($[\gamma_1] \cdot [\gamma_2]$ counts the intersections of $\gamma_1$ and $\gamma_2$ with a sign depending on orientation).

Without loss of generality, we may assume that the root cycles are homotopic to $\pm A$, so that $[\mathbf{m}] = k[A]$, $k \in \mathbb{Z}$. By choosing a representative of $[A]$ running on $G^*$ along a root of $F$, one gets immediately $\int_A \omega_\mathbf{m} = 0$. Then one chooses a representative of $[B]$ running along branches of $F$. A more careful examination shows that $\int_B \omega_\mathbf{m} = k$.

In conclusion we have

$$\int_{\gamma} \omega_\mathbf{m} = [\mathbf{m}] \cdot [\gamma]$$

so that the points of view of (2.2) and (2.3) are identified via the intersection pairing.

2.2. Convergence. The discussion is so far purely combinatorial, and is for instance invariant under deformation of the embedding. We now introduce an essentially minimal condition that guarantees “criticality” of a sequence of Temperleyan graphs.

Recall that the oriented edges of $\Gamma = (V,E)$ are assigned non-negative weights (conductances). Given these weights, one can introduce a Laplacian $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ by setting

$$(\Delta f)(x) = \sum_{y \in V : y \sim x} c(xy)(f(x) - f(y))$$

where $\sim$ denotes adjacency (using the positive Laplacian convention). Then $-\Delta$ is the generator of a continuous-time random walk $(X_t)_{t \geq 0}$, where the conductances give the jump rates.

To avoid trivialities, we assume that the graph is irreducible in the sense that any two vertices are joined by a chain of edges with positive weights.
Alternatively, one may consider its discrete-time skeleton, the Markov chain 
\((X_n)_{n \in \mathbb{N}}\) specified by
\[ P(X_{n+1} = y|X_n = x) = \frac{c(xy)}{\sum_{y' \sim x} c(xy')} \]

The objects of interest to us - harmonic functions, harmonic measures, Poisson
operators, etc. - are invariant under time change, so that the distinction between
discrete and continuous time is essentially moot.

The mesh size is defined as \(\delta := \sup_{x, y \in V; x \sim y} |x - y|\). We will consider a sequence
of graphs \((\Gamma_\delta)_{\delta > 0}\) indexed by mesh size, where \(\delta\) goes to zero along some sequence.

The dependence of various objects on \(\delta\) will be omitted when there is no ambiguity.

The key (and essentially minimal) assumption is that, as the mesh goes to zero,
the corresponding random walk converges weakly, up to time change, to Brown-
ian motion on \(\Sigma\). Among several equivalent specific formulations, we choose the
following (essentially as in [26]).

Let \(\tilde{X}_\delta\) be the continuous process obtained from the random walk \(X_\delta\) on \(\Gamma_\delta\) by
linear interpolation between jump times. Let \(x_\delta \in V_\delta\), the starting vertex, be such
that \(x_\delta \to 0\). Let \(\sigma\) be the time of first exit of, say, 
\([-1, 1] + \tau[−1, 1]\) (the union of
four fundamental domains; any set containing a fundamental domain in its interior
would work). Let \(\mu_\delta\) be the law induced on \(C([0, 1], \mathbb{C})\) by \(t \mapsto \tilde{X}_\delta(t\sigma)\). The state
space \(C([0, 1], \mathbb{C})\) is metrized by uniform convergence up to time reparametrization,
viz.
\[ d(\gamma_1, \gamma_2) = \inf_{\phi: [0, 1] \to [0, 1]} \|\gamma_1 - \gamma_2 \circ \phi\|_{\infty} \]
where the infimum is taken over increasing homeomorphisms \(\phi : [0, 1] \to [0, 1]\). This
turns \(C([0, 1], \mathbb{C})\) (quotiented by reparametrization) into a Polish space. Now let \(\mu\)
be the law induced by \((B_{t\sigma})_{0 \leq t \leq 1}\), where \(B\) is a standard Brownian motion. The
condition is then simply that
\[ \mu_\delta \text{ converges to } \mu \text{ weakly.} \]

This condition is for instance satisfied for simple random walk on supercritical
percolation clusters [2], among other models of random walks in random environ-
ments. Recall also from [17] that dimers on the hexagonal lattice correspond to a
periodic, non-reversible random walk.

Using planarity, one may check that the condition \(2.4\) is independent of the
choice of starting point. We denote by \(P_{x, \delta}\) the law of the RW on \(\Gamma_\delta\), simply denoted
by \(X\), started from (the point on \(V(\Gamma_\delta)\) closest to) \(x\). Similarly, \(P^x\) denotes the law
of standard BM started from \(x\).

For our purposes, a key consequence of the weak convergence condition is the
convergence of harmonic measure. For \(D \subset \Sigma\) an open set (with Jordan boundary,
say), \(J \subset \partial D\) and \(x \in D\), we denote
\[ \text{Harm}_\delta(x, J, D) = P_{x, \delta}(X \text{ exits } D \text{ on } J) \]
\[ \text{Harm}(x, J, D) = P^x(B \text{ exits } D \text{ on } J) \]

The following is a weaker version of Lemma 4.8 in [26].

**Lemma 1.** Fix \(\eta \in (0, 1)\) and \(\varepsilon > 0\). There is \(\delta_0\) such that for any \(\delta \leq \delta_0\), \(x \in \mathbb{C}\),
any simply connected Jordan domain \(D\) with \(B(x, \eta) \subset D \subset B(x, \eta^{-1})\) and arc
\(J \subset \partial D\), we have
\[ |\text{Harm}_\delta(x, J, D) - \text{Harm}(x, J, D)| \leq \varepsilon \]
Remark that \((2.4)\) does not guarantee the type of \(C^1\) convergence of Green’s functions which has been instrumental in the asymptotic analysis of dimers since \([11]\).

### 2.3. Twisted Laplacians

A graph \(\Gamma = (V,E)\) on \(\Sigma = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})\) lifts to a graph \(\tilde{\Gamma} = (\tilde{V}, \tilde{E})\) on the universal cover \(\mathbb{C}\), and the Laplacian on \(\Gamma\) lifts to an operator \(\tilde{\Delta} : \mathbb{C}^{\tilde{V}} \to \mathbb{C}^{\tilde{V}}\).

The fundamental group operates on \(\mathbb{C}^{\tilde{V}}\) by translations, and that action commutes with \(\tilde{\Delta}\). Consequently \(\tilde{\Delta}\) stabilizes eigenspaces of translations.

Fix a character \(\chi : \pi_1(\Sigma) \to U\), \(U\) the unit circle (equivalently, choose \(\chi(A)\) and \(\chi(B)\) in \(U\)), and set

\[ (\mathbb{C}^{\tilde{V}})_\chi = \{ \phi \in \mathbb{C}^{\tilde{V}} : \forall \gamma \in \pi_1(\Sigma), \phi(\cdot + \gamma) = \chi(\gamma) \phi(\cdot) \} \]

We denote by \(\Delta_\chi : \left(\mathbb{C}^{\tilde{V}}\right)_\chi \to \left(\mathbb{C}^{\tilde{V}}\right)_\chi\) the restriction of \(\tilde{\Delta}\) to \(\left(\mathbb{C}^{\tilde{V}}\right)_\chi\). Remark that for the trivial character \(\chi = 1\), there is a natural identification \(\Delta_1 \simeq \Delta : \mathbb{C}^V \to \mathbb{C}^V\).

An equivalent formulation \([15]\) identifies \(\Delta_\chi\) with the Laplacian operating on the flat line bundle with monodromy given by the character \(\chi\).

Concretely, one can fix a fundamental domain for \(\Sigma\) in \(\mathbb{C}\) corresponding to an \(A\)- and \(B\)- cycle (simple paths on the dual graph to \(\tilde{\Gamma}\)). For \(v \in \tilde{V}\) in this fundamental domain, set \(e_v(v') = \delta_{v,v'}\chi(v' - v)\). This gives a basis of \(\left(\mathbb{C}^{\tilde{V}}\right)_\chi\). Relative to this basis, the matrix of \(\Delta_\chi\) differs from the matrix of \(\Delta\) on entries corresponding to edges crossing the cycles. More precisely,

\[ (\Delta_\chi)(v,v') = -c(v,v')\chi^\pm(A) \]

if \((vv')\) crosses \(A\), with \(\pm = 1\) if \((A,(vv'))\) is directly oriented and \(-1\) otherwise; the corresponding statement holds for the \(B\)-cycle, and all other entries of the matrix are unchanged.

From the maximum principle it is easy to see that \(\Delta_\chi\) has trivial kernel (equivalently, \(\det(\Delta_\chi) \neq 0\)) iff \(\chi\) is non-trivial.

The Matrix-Tree theorem states that the determinant (more precisely, a cofactor) of the discrete Laplacian enumerates (weighted) spanning trees on \(\Gamma\). The following is a twisted version of the Matrix-Tree Theorem, due to Forman \([9]\) and, independently, Kenyon \([15]\).

**Theorem 2.** Let \(\Gamma = (V,E,c)\) be a weighted graph on the torus \(\Sigma\), \(\chi : \pi_1(\Sigma) \to U\) a character, and \(\Delta_\chi\) the Laplacian on \(\Gamma\). Then

\[
\det(\Delta_\chi) = \sum_{F \text{ CRSF}} \left( \prod_{e \in F} c(e) \prod_{\gamma \text{ cycle in } F} (1 - \chi(\gamma)) \right)
\]

In the reversible case \((c(vv') = c(v'v)\) for all \(v,v')\), this becomes

\[
\sum_{F \text{ CRSF}} \left( \prod_{e \in F} c(e) \prod_{\Gamma \text{ cycle in } F} (2 - \chi(\gamma) - \chi(\gamma)^{-1}) \right)
\]

where the sum is over unoriented CRSFs. There are also extensions of this result \([15]\), which will not be of use here.
3. Statement of Main Results

The goal is to show convergence of the distribution of $[m]$ to an explicit, universal limit under the general assumption that the underlying random walk on $\Gamma_\delta$ converges to Brownian motion (2.4) as the mesh goes to zero (along some sequence). The argument has three essentially independent components.

The assumption (2.4) is essentially a condition on convergence of harmonic measure (Lemma 1). Our first task is to show that ratios of Laplacian determinants can be expressed in terms of harmonic measure and are consequently universal in the small mesh limit, resulting in

**Proposition 3.** Let $\chi, \chi': \pi_1(\Sigma) \to U$ be two non-trivial characters. Then

$$\frac{\det \Delta^{\delta'}_\chi}{\det \Delta^{\delta}_\chi} \xrightarrow{\delta \to 0} \frac{h(\chi')}{h(\chi)}$$

where the RHS does not depend on the sequence $(\Gamma_\delta)$ satisfying (2.4).

Given universality, in order to identify the righthand side we can work on graphs of our choosing. We pick square lattices, for which we have not only convergence in the weak sense of (2.4) (up to time change), but also precise heat kernel asymptotics. This leads to

**Proposition 4.** Fix $\chi, \chi'$ non-trivial unitary characters of $\pi_1(\Sigma)$ and take $\Gamma_\delta = \delta Z^2/(\mathbb{Z} + \tau_\delta \mathbb{Z})$ where $\tau_\delta \in \delta \mathbb{Z}^2$ and $\tau_\delta = \tau + o(1)$, $\delta^{-1} \in \mathbb{N}$. Then as $\delta \to 0$,

$$\frac{\det(\Delta^{\delta'}_\chi)}{\det(\Delta^{\delta}_\chi)} \xrightarrow{\delta \to 0} \frac{T'(\chi')^2}{T(\chi)^2}$$

where

$$T(\chi) = \left| \eta(\tau)^{-1} e^{i\pi v^2 \tau} \theta(u - v\tau|\tau) \right|$$

and $\chi(m\tau + n) = \exp(2i\pi (mu + nv))$.

On the square lattice, we take all (nearest neighbor) conductances to be equal, so that the corresponding random walk is a simple random walk (see Section 5 for definitions and conventions for $\theta$ functions).

Finally, we need to identify the limiting distribution of $[m]$ from the limit of Laplacian determinants; this is performed in

**Theorem 5.** As the mesh $\delta$ goes to zero, the law induced by $[m]$ on $H_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z} + \tau \mathbb{Z}$ converges to the discrete Gaussian distribution $\mathbb{P}_0$ specified by

$$\mathbb{P}_0\{r\tau + s\} \propto \exp \left( -\frac{\pi}{23\tau} |r\tau + s|^2 \right)$$

for $r, s \in \mathbb{Z}$.

There is a natural interpretation of the limiting distribution in terms of compactified free field (see e.g. the discussion in [8]). In terms of dimer height, $[m] = r\tau + s$ corresponds to a height variation of $-r$ (resp. $s$) along the $A$ (resp. $B$ cycle).

The ground state (additively multivalued function with said periods and minimal Dirichlet energy) lifted to $\mathbb{C}$ is the affine function

$$z \mapsto -r \left( \Re(z) - \frac{\Re_\tau}{3\tau} \Im(z) \right) + s \frac{\Im(z)}{3\tau} = \Im \left( z(r\tau + s) \right)$$
with Dirichlet energy

$$\text{Area}(\Sigma) \left( \frac{|r\bar{\tau} + s|}{3\tau} \right)^2 = \frac{|r\tau + s|^2}{3\tau}$$

The theorem is thus consistent with a conjectured convergence of the height field towards a universal compactified free field limit under (2.4).

4. Convergence of Laplacian determinants

Our goal in this section is to prove Proposition 3, i.e. that ratios of Laplacian determinants have a universal limit under (2.4). We start with some intuitive justification of that fact in terms of loop measures. Then we proceed to express these ratios in terms of series built from harmonic measures. Uniform convergence of these series is justified by a contraction argument, and term-wise convergence follows from Lemma 1, which is sufficient to conclude.

4.1. Interpretation in terms of loop measure. Let $\chi, \chi' : \pi_1(\Sigma) \to \mathbb{U}$ be two nontrivial characters s.t. $\chi(A) = \chi'(A)$ and $\chi(B) \neq \chi'(B)$. Our goal is to show that

$$\frac{\det(\Delta_{\chi'})}{\det(\Delta_{\chi})}$$

is universal in the small mesh limit.

We start with an informal argument, aimed at the reader familiar with loop measures [21, 22].

First we observe that the ratio (4.1) is unchanged when replacing the conductances $c(xy)$ by the normalized conductances:

$$c'(xy) = \frac{c(xy)}{\sum_{y'y \sim x} c(xy')}$$

so that we may assume $\sum_{y'y \sim x} c(xy) = 1$ for all $x \in \Gamma$. Then $\Delta = \text{Id} - P$ where $P$ is the Markov chain transition matrix for the discrete-time random walk on $\Gamma$.

Consider the space of rooted loops on $\Gamma$, i.e. paths on $\Gamma$ of type

$$\ell = (v_0, v_1, \ldots, v_{n-1}, v_n = v_0)$$

for arbitrary length $|\ell| \overset{def}{=} n$. This space is endowed with the (rooted) loop measure $\mu^{\text{loop}}$ specified by

$$\mu^{\text{loop}}((v_0, \ldots, v_n = v_0)) = \frac{1}{n} c(v_0 v_1) \ldots c(v_{n-1} v_0)$$

Remark that a loop $\ell$ induces an homology class $[\ell] \in H_1(\Sigma, \mathbb{Z})$. One possible approach to universality of the ratio (4.1) is based on the following formal identity

$$\frac{\det(\Delta_{\chi'})}{\det(\Delta_{\chi})} = \exp \left( - \int (\chi'(|\ell|) - \chi(|\ell|)) d\mu^{\text{loop}}(\ell) \right)$$

A difficulty is that the integral is not absolutely convergent (due to long loops); this can be remediated by introducing a small, positive killing rate and letting it go to zero (this gives a meaning to the RHS and it can then be checked that it equals the LHS). Working on bridge measures seems also inconvenient under (2.4).

Since that weak convergence condition is essentially a condition on convergence of harmonic measure, we work instead with somewhat less probabilistic but more
robust Poisson operators. The general idea is to enumerate loops as they travel 
back and forth between macroscopically distant cycles on the torus.

4.2. Poisson operators. For now we work on a fixed graph $\Gamma$ with mesh $\delta$. Let $\gamma_1$ and $\gamma_2$ be two disjoint simple cycles on $\Gamma$ homotopic to the $A$-cycle and denote $\gamma = \gamma_1 \sqcup \gamma_2$ (for levity, we omit the dependence on $\delta$ when there is no ambiguity). In the small mesh limit, one may think of $\gamma_1$ and $\gamma_2$ as $[0, 1]$ and $\frac{\pi}{2} + [0, 1]$ up to $o(1)$. We define Poisson operators $R^\delta_\chi, Q^\delta_\chi$ by

$$R^\delta_\chi : (C^{\gamma_1})_\chi \to (C^V)_\chi$$

$$f \mapsto (x \mapsto (R^\delta_\chi f)(x) = E^\delta_x (f(X_{\sigma_1})))$$

and symmetrically

$$Q^\delta_\chi : (C^{\gamma_2})_\chi \to (C^V)_\chi$$

$$f \mapsto (x \mapsto (Q^\delta_\chi f)(x) = E^\delta_x (f(X_{\sigma_2})))$$

where $\sigma_i$ is the first hitting time of $\gamma_i$, $i = 1, 2$. Remark that, since $f$ is multivalued, $f(X_{\sigma_i})$ depends not only on the exit position $X_{\sigma_i}$ but also on the “horizontal” winding number of the path up to $\sigma_i$.

Let us consider the operator $\phi_\chi : (C^V)_\chi \to (C^V)_\chi$ given by

$$f \mapsto R^\delta_\chi f|_{\gamma_1} + Q^\delta_\chi f|_{\gamma_2} + f|_{V \setminus \gamma}$$

It has a block decomposition

$$\phi_\chi = \begin{pmatrix}
Id & (R^\delta_\chi f)|_{\gamma_2} & 0 \\
(Q^\delta_\chi f)|_{\gamma_1} & Id & 0 \\
* & * & Id
\end{pmatrix}$$

corresponding to the partition $V = \gamma_1 \sqcup \gamma_2 \sqcup (V \setminus \gamma)$.

Let us denote by $N_1$ the “Neumann-jump operator” on $(C^{\gamma_1})_\chi$ given by

$$N_1 f = (\Delta^\delta_\chi (R^\delta_\chi f)|_{\gamma_1})$$

and $N_2$ operating on $(C^{\gamma_2})_\chi$ is defined similarly. Then

$$\Delta^\delta_\chi \phi_\chi = \begin{pmatrix}
N_1 & 0 & * \\
0 & N_2 & * \\
0 & 0 & (\Delta^\delta_\chi)|_{V \setminus \gamma}
\end{pmatrix}$$

The bottom right last block corresponds to a Laplacian with Dirichlet conditions on $V \setminus \gamma$ and depends on $\chi$ only through $\chi(A)$ and not $\chi(B)$.

Finally, if $\psi_\gamma$ is the operator on $(C^V)_\chi$ given by

$$\psi_\gamma f = (R^\delta_\chi f|_{\gamma_1}) + f|_{V \setminus \gamma_1}$$

we have

$$\psi_\chi = \begin{pmatrix}
Id & 0 \\
* & Id
\end{pmatrix}$$

$$\Delta^\delta_\chi \psi_\chi = \begin{pmatrix}
N_1 & * \\
0 & (\Delta^\delta_\chi)|_{V \setminus \gamma_1}
\end{pmatrix}$$
where the bottom right block does not depend on $\chi(B)$, in the block decomposition corresponding to the partition $V = \gamma_1 \sqcup (V \setminus \gamma_1)$. We conclude that

$$\frac{\det(\Delta_\delta^\chi)}{\det(N_i)}$$

does not depend on $\chi(B)$ (for $i = 1, 2$), and neither does

$$\frac{\det(\Delta_\delta^\phi \chi)}{\det(N_1) \det(N_2)}$$

Together with the classical identity

$$\det(\text{Id} - BA) = \det\left(\begin{array}{cc} \text{Id} & A \\ B & \text{Id} \end{array}\right) = \det(\text{Id} - AB)$$

this yields the

**Lemma 6.** If $S_\chi^\delta$ is the operator on $(C^\gamma_1)_\chi$ given by

$$S_\chi^\delta f = (Q_\delta^\chi((R_\delta^\chi f)|_{\gamma_2}))|_{\gamma_1}$$

then

$$\frac{\det(\Delta_\delta^\chi)}{\det(\Delta_\delta^\phi \chi)} = \frac{\det(\text{Id} - S_\chi^\delta)}{\det(\text{Id} - S_\phi^\chi)}$$

whenever $\chi$ is non-trivial and $\chi(A) = \chi'(A)$.

Concretely, if $\sigma_{21}$ is the time of the first visit to $\gamma_1$ after the first visit to $\gamma_2$,

$$(S_\chi^\delta f)(x) = E_\delta^\chi(f(X_{\sigma_{21}}))$$

where $f$ is understood as a $\chi$-multivalued function on the universal cover.

Let us sketch an alternative proof of Lemma 6 closer to Proposition 2.2 in [7].

Let $c_{\chi}(xy) = c(xy)$ for a generic edge $(xy)$ of $\Gamma$, and $c_{\chi}(xy) = \chi(B)c(xy)$ if $(xy)$ traverses an $A$-cycle bounding a fundamental domain with direct orientation, and likewise for other edges crossing the boundary of a fundamental domain, so that $\Delta_\chi$ may be identified with $(c_{\chi}(xy))_{x,y \in V}$. Then we have the expansion

$$-\log \det(\Delta_\delta^\chi) = \lim_{\varepsilon \downarrow 0} \sum_{k \geq 1} \frac{(1 - \varepsilon)^k}{k} \sum_{v_0, \ldots, v_{k-1} \in V} c_{\chi}(v_0v_1) \ldots c_{\chi}(v_{k-1}v_0)$$

Then one can decompose each summand (corresponding to a rooted loop on $\Gamma$) between successive visits to $\gamma_1$ and $\gamma_2$; some rather tedious bookkeeping leads to Lemma 6.

### 4.3. Contractions.

Given Lemma 6, we are now concerned with the convergence of

$$\det(\text{Id} - S_\chi^\delta)$$

along a suitable sequence of graphs $(\Gamma_\delta)$ on $\Sigma$ (the mesh $\delta$ going to zero along some sequence), where $S_\chi^\delta$ is given by [4.3]. We assume that $\gamma_1 = \gamma_1^\delta$ (resp. $\gamma_2$) is a simple cycle on $\Gamma_\delta$ within $o(1)$ of $[0, 1]$ (resp. $\frac{\tau_2}{2} + [0, 1]$), in the sense of uniform convergence up to reparametrization.

From [4.3], it is obvious that $\|S_\chi^\delta\| \leq 1$. Here $\|\cdot\|$ denotes the $L^\infty$ operator norm. From the maximum principle, one can also argue that $\|S_\chi^\delta\|_{\infty} < 1$ for fixed mesh $\delta$. In order to control expansions of $\det(\text{Id} - S_\chi^\delta)$ in the small mesh limit, we need an operator norm estimate uniform in $\delta$. 

Lemma 7. Fix $\chi : \pi_1(\Sigma) \to \mathbb{U}$ non-trivial. There is $\varepsilon = \varepsilon(\chi) > 0$ such that for $\delta$ small enough, $S^\delta_\chi : (\mathbb{C}_{\chi}^\infty) \to (\mathbb{C}_{\chi}^\infty)$ is a $(1 - \varepsilon)$-contraction:

$$\|S^\delta_\chi\|_\infty \leq 1 - \varepsilon$$

Proof. For simplicity of exposition we treat the case $\chi(B) \neq 1$, $\tau$ pure imaginary, the general case being similar. Let $f \in L^\infty(\gamma_1)$ with $\|f\|_\infty \leq 1$. For $x \in \gamma_1$, we have

$$(S^\delta_\chi f)(x) = E^\delta_\chi(f(X_{\sigma^{21}_x}))$$

and want to show $|S^\delta_\chi f(x)| \leq 1 - \varepsilon$. Take a rectangle $R = [\frac{\varepsilon}{\tau}, \frac{3\varepsilon}{\tau}] \times [\frac{3\varepsilon}{\tau}, 3\tau]$. By the harmonic measure estimate of Lemma 1, the probability that the RW started from $\omega$ exits $R$ on the top side is bounded away from zero for small $\delta$.

For any $x \in \gamma_1$ (say with $\Re(x) \in [\frac{3}{4}, 1]$), one can find (again by Lemma 1) a polygonal ($L$-shaped) domain $U_x$ with a boundary arc $J_x$ s.t.:

1. $B(x, \eta) \subset U_x \subset B(x, \eta^{-1})$ for some positive $\eta$ independent of $x$;
2. $\text{Harm}_x(J_x, U_x)$ is bounded away from zero (uniformly in $x$ and in $\delta$ small enough);
3. any path started from $x$ exiting $U_x$ on $J_x$ intersects any path started from $\tau/2$ exiting $R$ on top.

Similarly, one can find a polygonal domain $U'_x$ with a boundary arc $J'_x$ s.t. any path started from $x$ exiting $U'_x$ on $J'_x$ intersects any path started from $-\tau/2$ exiting $R - \tau$ on top (also satisfying (1)-(2)), and crosses $-\frac{\tau}{2} + [0, 1]$.

One can sample a RW started from $x$ as follows. First sample a RW $Y$ started from $\tau$ and stopped when exiting $R$. Then run a RW $X$ started from $x$ up to first intersection of $Y$ or $Y - \tau$; then follow $Y$ or $Y - \tau$. This shows that there are two events $E, E'$ with probability bounded away from zero and a measure preserving correspondence $\omega \in E \mapsto \omega' \in E'$ s.t. $X_{\sigma^{21}_x}(\omega) = X_{\sigma^{21}_x}(\omega') + \tau$ (on the universal cover). See Figure 4.1.

Thus

$$|(S^\delta_\chi f)(x)| \leq P_x(E)\chi(B) + 1 + 1 - 2P_x(E) \leq 1 - \varepsilon$$

as claimed (since $|\chi(B) + 1| < 2$).

Alternatively, one could reason by contradiction, by extracting a subsequence $f_\delta \in L^\infty(\gamma_1)$ (the mesh $\delta$ goes to zero along some subsequence) with $\|f_\delta\|_\infty \leq 1$, $\|S^\delta_\chi f_\delta\|_\infty \nearrow 1$, and then a uniformly convergent subsequence; for this we need a Harnack estimate in lieu of Lemma 1.

4.4. Iterated traces. We now turn to the convergence of iterated traces $\text{Tr}((S^\delta_\chi)^k)$ as $\delta \searrow 0$, for $k \geq 1$ fixed. At some general level, $S^\delta_\chi$ is built from discrete harmonic measure, which converges (Lemma 1) to (continuous) harmonic measure. We proceed to check that this is enough to ensure convergence of the iterated traces to a universal limit.

In the continuum, we can define a natural limiting operator (compare with (4.3)) by

$$(S^\delta_\chi f)(x) = E^\delta(f(B_{\sigma^{21}_x}))$$
which can be decomposed as $S_{\chi} = Q_{\chi} R_{\chi}$ with (compare with (4.2))

$R_{\chi} : L_1^\infty(\gamma_1) \to L_1^\infty(\gamma_2) : (R_{\chi} f)(x) = E^x(f(B_{\sigma_1}))$

$Q_{\chi} : L_1^\infty(\gamma_2) \to L_1^\infty(\gamma_1) : (Q_{\chi} f)(x) = E^x(f(B_{\sigma_2}))$

where we take $\gamma_1 = [0, 1]$, $\gamma_2 = \frac{\tau}{2} + [0, 1]$ for concreteness. Here $L_1^\infty(\gamma_i)$ designates measurable bounded $\chi$-multivalued functions on $\gamma_i$ with natural $L_1^\infty$ norm ($\chi$ is unitary).

We may also write it as an integral kernel operator

$$(R_{\chi} f)(x) = \int_{\gamma_1} f(y) R_{\chi}(x, y) dy$$

and likewise for $Q_{\chi}$. Plainly, $R_{\chi}$ (as a kernel) is bicontinuous and $S_{\chi}$ is trace-class.

From Lemma 1 it is easy to see that $R_{\chi}^k$ converges to $R_{\chi}$ in the following weak sense: if $f$ continuous on $\gamma_i$, $f_{\delta}$ the restriction to $\gamma_i^\delta$ of a continuous extension of $f$, then $R_{\chi}^k f_{\delta}$ converges to $R_{\chi} f$ pointwise on $\gamma_2$. In order to get convergence of the trace we need additional continuity estimates (of the discrete harmonic measure w.r.t. the starting point).

**Lemma 8.** If $\gamma_i^\delta$ is within $o(1)$ of $\gamma_i$, $i = 1, 2$, and $k \geq 1$, then

$$\Tr((S_{\chi}^\delta)^k) \xrightarrow{\delta \to 0} \Tr((S_{\chi})^k)$$

**Proof.** We treat the case $k = 1$, the general case being similar. The discrete trace may be written as

$$\Tr(S_{\chi}^\delta) = \sum_{x \in \gamma_1^\delta} \sum_{y \in \gamma_2^\delta} Q_{\chi}^\delta(x, y) R_{\chi}^\delta(y, x)$$
and the continuous trace (in the sense of trace-class operators, see e.g. [24]) is
\[ \text{Tr}(S(x)) = \int_{\gamma_1} \int_{\gamma_2} Q(x, y) R(x, y) dx dy \]

Here we look at \( \gamma_i \) as closed cycles on \( \Sigma \), \( i = 1, 2 \). Let \( U(a, b) \subset \mathbb{C} \) denote the horizontal strip
\[ U(a, b) = \{ z \in \mathbb{C} : a < \Re z < b \} \]
and \( t = \frac{3\pi}{2} \).

If we lift to the universal cover, we get an expression of type
\[ \text{Tr}(S(x)) = \sum_{m \in \mathbb{Z}} \int_{[0, 1]} \int_{\frac{1}{2} + [0, 1]} \chi(A)^m \text{Harm}(x - m, dy, U(-t, t)) \text{Harm}(y, dx, U(0, 2t)) \]
\[ + \sum_{m \in \mathbb{Z}} \int_{\tau + [0, 1]} \int_{\frac{1}{2} + [0, 1]} \ldots + \sum_{m \in \mathbb{Z}} \int_{-\tau + [0, 1]} \int_{\frac{1}{2} + [0, 1]} \ldots \]
so that everything is expressed in terms of harmonic measure in strips (and likewise for \( \text{Tr}(S(x)) \)). Up to a small (uniformly in \( \delta \)) error, one may replace the \( \sum_{m \in \mathbb{Z}} \) by finite truncated sums and the strips by long rectangles.

We are left with proving convergence of a term of type
\[ \sum_{x \in \gamma_1^i} \sum_{y \in \gamma_2^j} \text{Harm}_{3}(x - m, \{ y \}, R) \text{Harm}_{3}(y, \{ x \}, R + it) \]
where \( R \) is a long rectangle and \( m \in \mathbb{Z} \) is fixed. For simplicity of notation set \( m = 0 \). From the convergence of Poisson kernels (Lemma 1.2 of [26]), it follows that for any \( \varepsilon > 0 \), for \( L \) large enough,
\[ | \text{Harm}_{3}(x', \{ y \}, R) - \text{Harm}_{3}(x, \{ y \}, R) | \leq \varepsilon \]
uniformly in \( x, x' \in \gamma_1 \), \( |x - x'| \leq L^{-1}, y \in \partial R \) and \( \delta \) small enough. (By contrast, notice that \( y \mapsto \log \text{Harm}_{3}(x, \{ y \}, R) \) is typically highly oscillatory). Let us partition \( \gamma_1^1, \gamma_2^2 \) into \( O(L) \) intervals \( I_1^i, J_2^j \) of diameter \( \leq L^{-1} \); and pick a point \( x_i^i, y_j^j \) in any such interval. We may assume \( (I_1^i)_i \) converges to a partition \( (I_i)_i \) of \( \gamma_1 \), and similarly for \( (J_2^j)_j \). Then
\[ \sum_{x \in \gamma_1^i} \sum_{y \in \gamma_2^j} \text{Harm}_{3}(x - m, \{ y \}, R) \text{Harm}_{3}(y, \{ x \}, R + it) \]
\[ = \left( \sum_{i,j} \text{Harm}_{3}(x_i^i, J_2^j, R) \text{Harm}_{3}(y_j^j, I_1^i, R + it) \right) (1 + O(\varepsilon)) \]
By Lemma [1] the sum in the RHS converges (for fixed \( L \), as \( \delta \searrow 0 \)) to
\[ \sum_{i,j} \text{Harm}(x_i, J_j, R) \text{Harm}(y_j, I_i, R + it) \]
Moreover, we can pick \( L \) large enough so that this sum is within \( \varepsilon \) of
\[ \int_{[0, 1]} \int_{\frac{1}{2} + [0, 1]} \text{Harm}(x, dy, R) \text{Harm}(y, dx, R + it) \]
by Riemann sum approximation and the Harnack estimate (4.5). For general \( k \), one expresses \( \text{Tr}(S^k) \) as a sum of \( 4^k (2k) \)-fold integrals. This concludes the argument. \( \square \)

4.5. Conclusion. We may now complete the proof of Proposition 3.

Proof of Proposition 3. By Lemma 6,

\[
\frac{\det \Delta_\chi}{\det \Delta_\chi'} = \frac{\det(\text{Id} - S^\delta_\chi)}{\det(\text{Id} - S^\delta_\chi')}
\]

If \( \chi(A) = \chi'(A) \).

Lemma 7 justifies the expansion

\[(4.6) \quad - \log \det(\text{Id} - S^\delta) = \sum_{k \geq 1} \frac{1}{k} \text{Tr}((S^\delta)^k)\]

where the sum converges exponentially fast.

Consider \( S^\delta_\chi \) as an operator \( L^1(\gamma^\delta_{11}, \mu_\delta) \rightarrow L^\infty(\gamma^\delta_{12}) \), where \( \mu_\delta \) is the harmonic measure on \( \gamma^\delta_{12} \) seen from a reference point on \( \gamma^\delta_{12} \), say. From Lemma 1.2 in [26], we see that for any \( x, x' \in \gamma^\delta_{12}, y \in \gamma^\delta_1 \)

\[
\frac{\text{Harm}_\delta(x', \{y\}, \Gamma_\delta \setminus \gamma^\delta_1)}{\text{Harm}_\delta(x, \{y\}, \Gamma_\delta \setminus \gamma^\delta_1)} \leq C
\]

for some constant \( C \) independent of \( \delta \). It follows that \( \|S^\delta_\chi\|_{L^1 \rightarrow L^\infty} \leq C \), and that \( \|S^\delta_\chi\|^k_{L^1 \rightarrow L^\infty} \) decays exponentially uniformly in \( \delta \).

It is easy to check that for any (finite dimensional) operator \( S \), \( |\text{Tr}(S)| \leq \|S\|_{L^1(\mu) \rightarrow L^\infty} \cdot \|\mu\|_{TV} \) (\( \|\cdot\|_{TV} \) denotes the total variation).

Consequently, the convergence of the sum in (4.6) is uniform in \( \delta \). Lemma 8 gives term-wise convergence. This gives the result with the condition \( \chi(A) = \chi'(A) \) (eg upon setting \( h(\chi) = \det(\text{Id} - S_\chi)/\det(\text{Id} - S_{\chi_0}) \) for a reference character \( \chi_0 \)). Applying this twice (with an affine change of coordinates or equivalently a change of homology basis \( (A, B) \rightarrow (B, -A) \) to exchange the roles of the \( A \) and \( B \) cycles) gives the general case. \( \square \)

5. The case of the square lattice

In the light of Proposition 3, in order to identify the limit it is enough to work with a fixed sequence of graphs. For \( \tau \) pure imaginary, we could use hexagonal lattices and the spectral arguments of [3], or [18]; for general \( \tau \), and so-called isoradial lattices, we could use the variational argument of [8]. Another possible approach, involving more machinery, is based on relating the Fredholm determinant \( \det(\text{Id} - S_\chi) \) (where \( S_\chi \) is the trace-class operator of (4.4)) to \( \zeta \)-regularized determinants and use [23].

For the sake of variety and self-containedness, we give (yet) another approach for the square lattice, based on heat kernel convergence (which is not available under the general assumption (2.4)).

So let us take \( \delta^{-1} \in \mathbb{N} \), and

\[
\Gamma_\delta = (\delta \mathbb{Z}^2) / (\mathbb{Z} + \tau_\delta \mathbb{Z})
\]

where \( \tau_\delta = \tau + o(1) \) (one could also fix \( \tau_\delta \) and apply a small affine distortion to the square lattice; we shall omit the dependence of \( \tau \) on \( \delta \) from now on). Let \( \Lambda \)
denote the lattice \((\mathbb{Z} + \tau \mathbb{Z}) \simeq \pi_1(\Sigma)\) (so that \((1, \tau)\) corresponds to \((|A|, |B|)\)). The weights are, say, \(\frac{1}{4}\) on all edges (nearest neighbors), so that the corresponding RW is a simple random walk and \(\Delta\) is the discrete Laplacian:

\[
(\Delta^\delta f)(z) = f(z) - \frac{1}{4} \left( f(z + \delta) + f(z - \delta) + f(z + i\delta) + f(z - i\delta) \right)
\]

Let \(\chi : \pi_1(\Sigma) \rightarrow \mathbb{U}\) a non-trivial character. Then \(\Delta^\delta \chi\) has a complete set of eigenvectors of type
\[
f(x + iy) = \exp(i(\alpha x + \beta y))
\]
where \(e^{i\alpha} = \chi(1), e^{i(\alpha \Re \tau + \beta \Im \tau)} = \chi(\tau)\). It follows that, by writing \(\Delta^\delta \chi = \text{Id} - P^\delta\chi\), the eigenvalues of \(P^\delta\chi\) have modulus < 1 (one can also obtain \(\|P^\delta\chi\|_\infty < 1\) using the maximum principle). This justifies the expansion

\[
(5.1) \quad -\log \det(\Delta^\delta \chi) = \sum_{k \geq 1} \frac{1}{k} \text{Tr}((P^\delta\chi)_k)
\]

(with convergence for fixed \(\delta\) but not uniformly in \(\delta\)). Concretely, if \(x,y\) in a fundamental domain and \(P^\delta\chi\) denotes the transition matrix for SRW on \(\delta \mathbb{Z}^2\), we have

\[
(P^\delta\chi)_k(x,y) = \sum_{\ell \in \Lambda} \chi(\ell)(P^\delta\chi)_k(x,y + \ell)
\]

We shall need a few standard estimates on \(P^\delta\chi\). Let \(t = k/n^2\), \((P_t)_{t \geq 0}\) be the heat kernel, the semigroup of standard Brownian motion on \(\mathbb{C}\) and \(p_t\) its density:

\[
p_t(x,y) = \frac{1}{2\pi t} \exp\left(-\frac{|y-x|^2}{2t}\right)
\]

With \(\delta = 1/n\) and \(n\) is even (we take \(2k,n\) even in order to avoid essentially notational issues due to parity), we have the Local Central Limit theorem

\[
(5.2) \quad (P^\delta)_{2k}(x,y) = \frac{2}{n^2} p_t(x,y)(1 + o(1)) \quad \text{if} \quad n|y-x| = o(k^{3/4})
\]

\[
= \frac{2}{n^2} p_t(x,y)(1 + O(1)) \quad \text{if} \quad n|y-x| = O(k^{3/4})
\]

which is valid in a diffusive (i.e. \(|y-x| = O(\sqrt{t})\)) and up to a moderate deviation regime. See e.g. Proposition 2.5.3 in [19] (using the usual trick of projecting the 2d SRW on the diagonals to obtain independent 1d SRWs).

At larger scales, one can use the following large deviation estimate

\[
(5.3) \quad (P^\delta)_{2k}(x,y) = O(\exp(-c(|x-y|n^2/k)))
\]

which follows e.g. from the exponential Chebychev inequality (\(c\) is a positive constant).

Finally, we shall need a first difference estimate (see e.g. Theorem 2.3.6 in [19]):

\[
(5.4) \quad (P^\delta)_{2k}(x,y+1) - (P^\delta)_{2k}(x,y) = O(n^{-2} t^{-3/2})
\]
From \([5.1]\), we may write

\[
- \log \frac{\det(\Delta^\delta_{\chi'})}{\det(\Delta^\delta_{\chi})} = \sum_{k \geq 1} \frac{1}{k} \left( \text{Tr}((P^\delta)^k) - \text{Tr}((P^\delta)^k) \right)
\]

\[
= \sum_{k \geq 1} \frac{1}{k} \sum_{x \in \Sigma} \sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell))(P^\delta)^k(x, x + \ell)
\]

\[
= \sum_{k \geq 1} \frac{1}{2k} \sum_{x \in \Sigma} \sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell))(P^\delta)^{2k}(x, x + \ell)
\]

Remark that \(\chi(\ell) - \chi'(\ell) = 0\) when \(\ell = 0\), so that the summand vanishes for \(k = o(n)\). The last line follows from assuming \(n\) even, \(n\tau_3 \in 2(\mathbb{Z} + i\mathbb{Z})\).

We can discuss convergence of the sum in \([5.5]\) as \(n \to \infty\) for small, intermediate, and long times.

1. For short times, the large deviation estimate \([5.3]\) guarantees that

\[
\sum_{k \leq \lfloor n^{3/2} \rfloor} (\ldots) = \sum_{k \leq \lfloor n^{3/2} \rfloor} O(k^{-1}n^2 \exp(-c'n^2/k))
\]

which goes to zero as \(n \to \infty\) (\(c' > 0\) is a positive constant).

2. For intermediate times, say \([n^{3/2}] < k \leq [Tn^2]\) with \(T\) large but independent of \(n\), the random walk quantities converge to their natural continuous counterpart by the LCLT \([5.2]\):

\[
\sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell))(P^\delta)^{2k}(x, x + \ell) = \frac{2}{n^2} \sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell)) \frac{e^{-|\ell|^2/2t}}{2\pi t} + o(1)
\]

where \(t = 2k/n^2\). (More precisely, we use the LCLT for \(n|\ell| = o(k^{3/4})\) and the large deviation estimate \([5.3]\) otherwise).

Remark that \(|\Gamma_3| \sim n^2 \tau_3\). It follows that

\[
\sum_{\lfloor n^{3/2} \rfloor < k \leq \lfloor Tn^2 \rfloor} (\ldots) \sim 3\tau \int_0^T \left( \sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell)) e^{-|\ell|^2/2t} \right) \frac{dt}{2\pi t^2} + o(1)
\]

The convergence of the integral as \(T \to \infty\) is not immediately apparent but can be justified using e.g. the Poisson summation formula, see Section 4 of \(\text{[23]}\).

3. Finally we need to control \(\sum_{\ell \geq \lfloor Tn^2 \rfloor} (\ldots)\) uniformly in \(n\). For such a \(k\), let \(B_k\) (resp. \(B'_k\)) be a box of diameter \(O(t^{1/2+\varepsilon})\) (resp. \(O(n^{-1}k^{3/4})\)) around the origin, so that \(B_k \subset B'_k\).

Inside \(\Lambda \cap B_k\) (which contains \(O(t^{1+2\varepsilon})\) points), we use an Abel summation by part argument. Assume \(\eta = \chi(1) \neq 1\) (otherwise reason on \(\chi(\tau)\)). Then

\[
\chi(\ell) = (\eta - 1)^{-1}(\chi(\ell - 1) - \chi(\ell))
\]

so that

\[
\sum_{\Lambda \cap B_k} \chi(\ell)(P^\delta)^{2k}(x, x + \ell) = (1 - \eta)^{-1} \sum_{\ell \in \Lambda \cap B_k} \chi(\ell) \left( (P^\delta)^{2k}(x, y + \ell + 1) - (P^\delta)^{2k}(x, y + \ell) \right) + \text{(boundary terms)}
\]
Then from (5.4), we obtain
\[ \sum_{\Lambda \cap B_k} \Lambda \cap B_k \chi(\ell)(P^k)^2(x, x + \ell) = O(t^{1 + 2\epsilon} n^{-2} t^{-3/2}) \]
(the much smaller contribution of boundary terms is handled by (5.3)).

In \( B_k' \setminus B_k \), we use the LCLT (5.2) to obtain
\[ \sum_{\Lambda \cap (B_k' \setminus B_k)} \Lambda \cap (B_k' \setminus B_k) \chi(\ell)(P^k)^2(x, x + \ell) = O\left( \sum_{m \in \left[ \frac{k^{3/4}}{n} \right]} m^{k-1} \exp(-c'm^2/t) \right) = O\left( n^{-2} \exp(-c't^{2\epsilon}) \right) \]
for some \( c' > 0 \) (depending on \( \tau \)).

Finally, outside \( B_k' \) we have by (5.3)
\[ \sum_{\Lambda \cap (B_k')} \chi(\ell)(P^k)^2(x, x + \ell) = O\left( \sum_{m \geq \left[ \frac{k^{3/4}}{n} \right]} m^{k-1} \exp(-c'm^2/t) \right) = O\left( n^{-2} \exp(-c't^{2\epsilon}) \right) \]
for some \( c' > 0 \).

Consequently,
\[ \frac{1}{2k} \sum_{x \in \Sigma} \sum_{\Lambda} (\ldots) = O(t^{2\epsilon - 1/2}) + O(\exp(-c't^{2\epsilon}) + O(k \exp(-c'\sqrt{k})) \]
and
\[ \sum_{k > [Tn^{2}]^2} (\ldots) = O\left( \sum_{k > [Tn^{2}]} k^{-1} (kn^{-2})^{2\epsilon - 1/2} \right) + O\left( \sum_{k > [Tn^{2}]} \exp(-c'\sqrt{k}) \right) \]
\[ = O(T^{2\epsilon - \frac{1}{2}}) + o(1) \]
with \( o(1) \) going to zero as \( Tn^{2} \) goes to infinity.

In conclusion we have
\[ \frac{\det(\Delta^\delta \chi)}{\det(\Delta^\delta \chi')} \rightarrow \exp\left( -\Im \tau \int_0^\infty \frac{dt}{2\pi t^2} \sum_{\ell \in \Lambda} (\chi(\ell) - \chi'(\ell)) e^{-|\ell|^2/2t} \right) \]
This is a classical quantity, which is evaluated in Theorem 4.1 of [23] (as a reformulation of Kronecker’s second limit formula in analytic number theory), as we now explain.

Let \( T(\chi) \) be the analytic torsion of the unitary bundle associated with the character \( \chi \) ([23]), for our purposes we can take as definition
\[ T(\chi) = \exp\left( -\frac{1}{2} \int_0^\infty \frac{dt}{t} \int_{\Lambda} \sum_{\ell \in \Lambda} \chi(\ell)p_t(x, x + \ell)dA(x) \right) \]
\[ = \exp\left( -\frac{\Im \tau}{2} \int_0^\infty \frac{dt}{2\pi t^2} \sum_{\ell \in \Lambda} \chi(\ell)e^{-|\ell|^2/2t} \right) \]
and we have established
\[ \frac{\det(\Delta^\delta \chi)}{\det(\Delta^\delta \chi')} \rightarrow \frac{T(\chi')^2}{T(\chi)^2} \]
Remark that $T(\chi)^2$ is also the $\zeta$-regularized determinant of $\Delta_{\chi}$ (the continuous Laplacian operating on $\chi$-multivalued function).

Let $u, v \in [0, 1)$ be s.t. $\chi(m\tau + n) = \exp(2i\pi(mu + nv))$. Then Theorem 4.1 of [23] provides the following evaluation of $T(\chi)$:

$$T(\chi) = |\eta(\tau)^{-1}e^{i\pi u^2 \tau}\theta(u - v|\tau)|$$

where $\eta$ is the Dedekind $\eta$ function ($q = e^{i\pi \tau}$):

$$\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n})$$

and $\theta(\cdot|\tau)$ is the odd $\theta$ function (conventions as in [4]):

$$\theta(w|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2 e^{(2n+1)i\pi w}} = q^{1/6} \eta(q)2\sin(\pi w) \prod_{n=1}^{\infty} (1 - q^{2n}e^{2i\pi w})(1 - q^{2n}e^{-2i\pi w})$$

This completes the proof of Proposition 4.

6. Identification of the limiting distribution

Based on Propositions 3 and 4, we can now identify the limiting distribution of $[m]$, which encodes the winding of roots in pairs of dual CRSFs, or equivalently the height change of the dimer configuration along cycles. The argument is rather similar to Section 5.2 of [8] and Section 4.C of [1]; however we reason here based on Temperley’s bijection and Forman’s formula, rather than Kasteleyn’s determinantal enumeration of dimers as in [8].

Our goal is to identify the limiting distribution of $[m]$ in the small mesh limit (recall (2.3)).

Let $k \geq 1$, $\gamma$ a primitive element of $H_1(\Sigma, \mathbb{Z})$ (i.e. $[\gamma] = [mA] + [nB]$, $\gcd(m, n) = 1$). As is well-known, any (non-contractible) simple cycle is primitive. Let $Z_{k_+, k_-}[\gamma]$ be the partition function (i.e. the sum of the weights) of oriented CRSFs on $\Gamma$ with exactly $k_+ \geq 0$ (resp. $k_-$) cycles with class $[\gamma]$ (resp. $-[\gamma]$), $\gamma$ a primitive cycle (so that $Z_{k_+, k_-}[\gamma] = Z_{k_-, k_+,-[\gamma]}$).

Given $F$ a CRSF on $\Gamma$ counted in $Z_{k_+, k_-}[\gamma]$, there are $2^{k_+ + k_-}$ dual CRSFs $F^*$ on $\Gamma^*$, corresponding to possible choices of orientations of the root cycles. Recall that by construction, the conductances on $\Gamma^*$ are all 1.

Let $Z_{k}[\gamma]$ denote the dimer partition function on $\mathcal{G}$ restricted to $[m] = k[\gamma]$, i.e.

$$Z_{k}[\gamma] = \sum_{m:[m] = k[\gamma]} w(m)$$

Since dimer configurations are in measure-preserving bijection with pairs $(F, F^*)$, we have

$$Z_{k}[\gamma] = \sum_{k_+, k_- \geq 0} Z_{k_+, k_-}[\gamma] \binom{k_+ + k_-}{k_+ - k_+}$$

for $k \neq 0$: indeed, if $F^*$ has $k_+ - k_-$ cycles in the class $[\gamma]$ (resp. $-[\gamma]$), then $[m] = k[\gamma]$. 


We may look at \( k[\gamma] \mapsto Z_k[\gamma] \) as a function on the abelian group \( H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^2 \) (with finite support for a given \( \Gamma \)). Its Fourier-Pontryagin transform is:

\[
\hat{Z}(\rho) = Z_0 + \sum_{k>0, \gamma} \rho(\gamma)^k Z_k[\gamma] = \frac{1}{2} \sum_{k,k_+ \geq 0, [\gamma]} Z_{k_+,k_-}[\gamma] (1 + \rho(\gamma))^{k_+} (1 + \rho(\gamma)^{-1})^{k_-}
\]

(6.2)

where \( \rho : H_1(\Sigma, \mathbb{Z}) \to \mathbb{U} \) is a unitary character and the sums are over primitive classes \([\gamma]\); the second line follows from (6.1) and the binomial formula.

In particular, if \( 1 \) denotes the trivial character, \( \hat{Z}(1) = Z \) is the partition function of the model, and

\[
\frac{\hat{Z}(\rho)}{Z(1)} = \mathbb{E}(\rho([m]))
\]

is the characteristic function of \([m]\) (\( \mathbb{E} \) denotes the expectation under the dimer measure).

From Forman’s formula (Theorem 2) we have

\[
\text{det}(\Delta_\chi) = \frac{1}{2} \sum_{k, k_- \geq 0, [\gamma]} Z_{k_+, k_-}[\gamma] (1 - \chi(\gamma))^{k_+} (1 - \chi(\gamma)^{-1})^{k_-}
\]

(6.3)

Observe that there are four characters \( \epsilon \) of \( H_1(\Sigma, \mathbb{Z}) \) with \( \epsilon^2 = 1 \) (\( \epsilon([A]) = \pm 1 \) and \( \epsilon([B]) = \pm 1 \)). If \( \chi \) is any character, \( \sum_{\epsilon} (\epsilon \chi)(\gamma) = 0 \) unless \( \gamma = 2\gamma' \) for some \( \gamma' \), in which case \( \sum_{\epsilon} (\epsilon \chi)(\gamma) = 4 \chi(\gamma) \). Moreover, for \( \gamma \) primitive,

\[
-\chi^k(\gamma) + \frac{1}{2} \sum_{\epsilon \epsilon^2 = 1} (\epsilon \chi)^k(\gamma) = \begin{cases} \chi^k(\gamma) & \text{if } k \text{ even} \\ -\chi^k(\gamma) & \text{if } k \text{ odd} \end{cases}
\]

(6.4)

The expressions (6.2), (6.4) show that the expansions of \( \hat{Z}(\chi) \), \( \text{det}(\Delta_\chi) \) in powers of \( \chi(\gamma) \) agree on even powers and are opposite on odd powers. It follows that:

\[
\sum_{\epsilon} \hat{Z}(\epsilon \chi) = \sum_{\epsilon} \text{det}(\Delta_{\epsilon \chi})
\]

(6.5)

and that

\[
\hat{Z}(\chi) = -\text{det}(\Delta_\chi) + \frac{1}{2} \sum_{\epsilon} \text{det}(\Delta_{\epsilon \chi})
\]

(6.6)

Parameterizing characters by \( \chi(r\tau + s) = \exp(2i\pi ru + 2i\pi vs), (u, v) \in [0, 1)^2 \), the convergence result of Proposition 4 shows that

\[
\text{det}(\Delta_\chi) = (c_3 + o(1)) \left( |e^{i\pi s^2 \theta(v - u\tau \theta)^2} | \right)
\]

as the mesh \( \delta \searrow 0 \), for \( \chi \) non-trivial (where the positive sequence \( c_3 \) does not depend on \( \chi \)). Remark that \( \text{det}(\Delta_\chi) = 0 \) at \( \chi = 1 \) since constant functions are then in the kernel. It follows that the characteristic function (6.3) of \([m]\) converges pointwise as the mesh \( \delta \) goes to zero. We now want to identify the limiting distribution.
Let us compute the Fourier transform of $\chi \mapsto h(\chi) = |e^{i\pi \nu^2 \tau} \theta(v - u \tau)|^2$. First we write
\[
h(\chi) = e^{-2\pi \nu^2 \tau} \sum_{m,n} (-1)^{m+n} q^{(m+\frac{1}{2})^2} \bar{q}^{(n+\frac{1}{2})^2} e^{i\pi (2m+1)u - i\pi (2n+1)\bar{v}} \\
= \sum_{r,m} (-1)^r e^{2\pi r u} q^{(m+\frac{1}{2})^2 - v(2m+1) + v^2} \bar{q}^{(m-r+\frac{1}{2})^2 - v(2m-2r+1) + v^2} \\
= \sum_{r,m} (-1)^r e^{2\pi r u} q^{(v-m-\frac{1}{2})^2} \bar{q}^{(v-m+r-\frac{1}{2})^2}
\]

Applying the Poisson summation formula
\[
\sum_{m \in \mathbb{Z}} \phi(v + m) = \sum_{s \in \mathbb{Z}} e^{2i\pi s v} \hat{\phi}(s)
\]

to the rapidly decaying function $\phi(v) = q^{(v-\frac{1}{2})^2} \bar{q}^{(v-r-\frac{1}{2})^2} = |q|^2 (v+\frac{r-1}{2})^2 + |\bar{q}|^2 e^{-2\pi \Re(r) v (v+\frac{r-1}{2})}$,

such that:
\[
\hat{\phi}(s) = \int_{-\infty}^{\infty} \phi(t) e^{-2\pi s t} dt = |q|^2 \exp(-2\pi s \frac{r-1}{2}) \int_{-\infty}^{\infty} \exp(-2\pi 3 \tau t^2 - 2i\pi (r\Re(t) + s)t) dt \\
= |q|^2 |-1|^{s(r-1)} \frac{1}{\sqrt{2\pi \tau}} \exp\left(-\frac{\pi}{2} (r\Re(t) + s)^2 / 3 \tau\right) = (-1)^{s(r-1)} \frac{1}{\sqrt{2\pi \tau}} \exp\left(-\frac{\pi}{2 3 \tau} |r\Re(t) + s|^2\right)
\]

one obtains:
\[
h(\chi) = \sum_{r,s \in \mathbb{Z}} \chi(r\tau + s)(-1)^{(s-1)(r-1)+1} \frac{1}{\sqrt{2\pi \tau}} \exp\left(-\frac{\pi}{2 3 \tau} |r\Re + s|^2\right)
\]

The sign in the summand is $-1$ unless $r, s$ are both even; remark that as in \[6.5\] \[6.6\]

\[-1 + \frac{1}{2} \sum_{r,s \in \mathbb{Z}} \epsilon(r\tau + s) = \left\{ \begin{array}{ll} 1 & \text{if } r = s = 0 \mod 2 \\ -1 & \text{otherwise} \end{array} \right.\]

Comparing with \[6.6\] gives
\[
\lim_{s \rightarrow 0} c_s^{-1} \hat{Z}(\chi) = |\eta(\tau)|^{-2} \sum_{r,s} \chi(r\tau + s) \frac{1}{\sqrt{2\pi \tau}} \exp\left(-\frac{\pi}{2 3 \tau} |r\Re + s|^2\right)
\]

Pointwise convergence of the characteristic function \[6.3\] yields convergence in law for $|m|$. This concludes the proof of Theorem \[5\]

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**DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY.** 2990 BROADWAY, NEW YORK, NY 10027, USA.

E-mail address: dubedat@math.columbia.edu

**DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY.** 2990 BROADWAY, NEW YORK, NY 10027, USA.

E-mail address: rg2696@columbia.edu

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