THE EVOLUTION OF A PRIMORDIAL GALACTIC MAGNETIC FIELD

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ABSTRACT

We consider the hypothesis that galactic magnetic fields are primordial. We also discuss the various objections to this hypothesis. To test this hypothesis properly we assume that there was a magnetic field present in the galactic plasma before the galaxy formed and explore how such a field would evolve assuming a specific model for the interstellar medium in the galactic disk. After the galactic disk formed, the lines of force thread through it and remain connected to the external cosmic medium. They enter through one side of the disk, proceed horizontally a distance \(l\) in the disk, and then leave through the other side. We find that the lines of force are stretched by the differential rotation of the galactic disk, amplifying the toroidal component of the field and increasing \(l\). When the magnetic field is strong enough, it produces ambipolar velocities that try to lift the line out of the galactic disk but in opposite directions on different parts of the line. The result is that, instead of the line being expelled from the disk, its horizontal length \(l\) is shortened, both in the radial and in the toroidal direction. This leads to a reduction of the rate of horizontal stretching and finally a reduction in the magnetic field strength. After a sufficient time, the magnetic field at all points goes through this reduction and the field strength approaches a universal function of time. This function is slowly decreasing and only depends on the ambipolar properties of the interstellar medium. At any given time the magnetic field is toroidal and has the same strength everywhere. On the other hand, it turns out that its direction varies rapidly with radius, changing sign every 100 parsecs or so. However, if the initial cosmic magnetic field is not uniform, the areas of one sign of the toroidal field dominate over the other. The resulting field has a net Faraday rotation. If such a field were observed with low resolution in an external galaxy, then the field would appear toroidal in between the spiral arms. The spiral density wave would turn it so that the lines appear to trace out the spiral arm, although the apparent lines really are the sum of pieces of magnetic lines as they cross the disk. They do not necessarily extend very far along the arms. We contend that this model of the magnetic field, which arises naturally from a primordial origin, can fit the observations as well as other models for the magnetic field, such as those arising from the mean field dynamo theory. Finally, because the field lines are topologically threaded through the disk, they cannot be expelled from the disk. This counters the objection against the primordial origin, namely that such a field could not survive very long in the galaxy.

Subject headings: galaxies: evolution — galaxies: kinematics and dynamics — galaxies: magnetic fields — MHD

1. INTRODUCTION

There are two schools of thought as to the origin of the galactic magnetic field. The first school holds that, after formation of the Galactic disk, there was a very weak seed field of order \(10^{-17}\) G or so, and that this field was amplified to its present strength by dynamo action driven by interstellar turbulence. The description of how this happens is well known (Ruzmaikin, Shukurov, & Sokoloff 1988) and is summarized by the mean field dynamo theory. The second school of thought concerning the origin of the galactic field is that it is primordial. We take as the definition of a primordial origin for the magnetic field that it is toroidal and that the lines of force thread through it and remain connected to the external cosmic medium. This model for the origin and evolution of a primordial magnetic field is very simple. We first start with a large ball of plasma, with mass equal to the galactic mass, but with radius much larger than the current size of our Galaxy (Fig. 1a). We assume that the field lines filling this sphere is uniform and makes a finite angle \(\theta\) with the rotation axis of the galaxy. Then we allow the ball to evolve during the entire lifetime of the galaxy up to the present time. The purpose of this paper is to carry out a careful calculation of this evolution. To do this we assume a definite model for the material in the interstellar medium and calculate how a primordial magnetic field would evolve in such a model. To make the calculation tractable we include only the simplest physics in this model. We include the differential rotation of the interstellar medium about the galactic center and the ambipolar motions of the ionized component of the interstellar medium through the neutral component. We also assume the bulk of the interstellar medium is in the form of uniformly distributed dark clouds in which the ionization of the interstellar medium is quite low. We attempt to decide the plausibility or implausibility of the primordial origin by comparing the observations of the present-day field with the field that results from our calculation.

1.1. Our Model

The model for the origin and evolution of a primordial magnetic field that we consider is very simple. We first start with a large ball of plasma, with mass equal to the galactic mass, but with radius much larger than the current size of our Galaxy (Fig. 1a). We first assume that the magnetic field filling this sphere is uniform and makes a finite angle \(\theta\) with the rotation axis of the galaxy. Then we allow the ball to...
collapse to a sphere the size of the galaxy (Fig. 1b) and finally to a disk the thickness of the galactic disk (Fig. 1c). We assume the first collapse is radial and uniform, while the second collapse into the disk is linear and one-dimensional.

During the time when the galaxy contracts uniformly into the disk along the \( z \) direction, where \( z \) is in the direction of \( \Omega \), we ignore any rotation. Then the resulting magnetic field configuration is as in Figure 1c. The horizontal component of the magnetic field has been amplified by the large compressional factor, while the vertical component is unchanged, so that the resulting field is nearly parallel to the galactic disk. At this stage, some lines enter the disk from the top and leave from the top, e.g., line a. Some lines enter from the bottom and leave through the top, e.g., lines b and c. Finally, some lines enter from the bottom and leave through the bottom, e.g., line d. It turns out that lines such as lines a and d are eventually expelled from the disk by ambipolar diffusion, so that we ignore them. Now we set the disk into differential rotation at time \( t = 0 \).

We have neglected any differential rotation during the collapse to the disk. If the magnetic field is weak enough at this time that no significant ambipolar diffusion occurs during this phase, then the effect of the differential rotation can be included simply by changing the origin of time in a way that the total number of rotations is included. This is the case if the rotation as a function of mass during this phase is identical with that after the disk forms. If it is different, then the situation is more complicated and the precise initial conditions of the magnetic field would be more difficult to determine. For example, the magnetic field will no longer be uniform in the vertical direction across the disk. However, the subsequent behavior is expected to smooth out any such irregularities, and only the mean radial component of the initial field averaged over the thickness of the disk at any given point will be important in the long time behavior of the galactic magnetic field. Moreover, the results of this paper are local so that this generalization can be carried out independently at every radius of the galactic disk. Further, it will become clear that the conclusion that a primordial field need not evolve to a field that is inconsistent with the observations will still be qualitatively correct. Since the actual pattern of rotation during collapse is not known, we have not attempted to carry out such a generalization at this time. It would not be difficult to modify our code to take into account any such pattern of rotation during the collapsing phase of the galaxy.

1.2. Results of the Model

We here summarize the conclusions found from the detailed analysis of our model, given in the body of this paper.

Initially, after the disk forms, all the lines have a horizontal component that is larger than the vertical component by a factor equal to the radius of the disk divided by its thickness \( \approx R/D \approx 100 \). This is the case if the initial angle \( \alpha \) was of order 45° or at least not near 0 or 90°.
As a result of the neglect of rotation in the collapse, the horizontal component of all the field lines is in a single direction, the $x$ direction, say, so that

$$B = B_i \hat{x} + B_i(D \tan \alpha/R) \hat{z}$$

and

$$= B_i \cos \theta_i \hat{r} - B_i \sin \theta_i \hat{\theta} + B_i(D \tan \alpha/R) \hat{z}. \quad (1)$$

After $t = 0$ the differential rotation of the disk stretches the radial magnetic field into the toroidal direction so that, following a given fluid element with initial angle $\theta_1$, one has

$$B = B_i \cos \theta \hat{r} + \left[ B_i \left( r \frac{d\Omega}{dr} \right) \cos \theta_1 - B_i \sin \theta_1 \right] \hat{\theta}$$

$$+ B_i(D \tan \alpha/R) \hat{z}. \quad (2)$$

After a few rotations the second component dominates the first and third components. It is seen that the total magnetic field strength grows linearly with time. After the toroidal magnetic field becomes strong enough, the magnetic force on the ionized part of the disk forces it through the neutral component, primarily in the $z$ direction.

Consider a single line of force. If we turn off the differential rotation for a moment, the $z$ motion steepens the line of force but does not change the vertical field component. This leads to a shortening of the line, both in the radial direction and in the azimuthal direction (see Fig. 1d). Now we let the differential rotation continue. Because the radial component of the magnetic field is reduced, the toroidal component increases more slowly. Eventually, there comes a time when the radial field is small enough that the shortening motions in the azimuthal direction are stronger than the stretching motion, and the azimuthal component of the magnetic field actually decreases even in the presence of differential rotation. From this time on, the magnetic field strength decreases at a rate such that the vertical ambipolar velocity is just enough to move the plasma a distance approximately equal to the thickness of the galactic disk, in the time $t$, i.e.,

$$v_D t = D. \quad (3)$$

The quantity $v_D$ is essentially proportional to the average of the square of the magnetic field strength. Therefore, for a uniform partially ionized plasma we have

$$v_D \approx \left( \frac{1}{\rho_i} \right) \frac{B^2}{8\pi D}, \quad (4)$$

where $D$ is the half-thickness of the disk, $\rho_i$ is the ion mass density, and $v$ is the ion-neutral collision rate.

Thus, for times long compared to the time for the magnetic field to grow by stretching and saturate by ambipolar diffusion one finds that

$$B \approx D \sqrt{v \rho_i / t}. \quad (5)$$

That is, the magnetic field strength approaches a saturated time behavior independent of its initial value. The saturated time behavior only depends on the ambipolar diffusion properties of the interstellar medium and on the time $t$. However, the time to reach saturation does depend on the initial value of the magnetic field strength.

The qualitative behavior is shown in Figure 2, where the dependence of $B$ on time for different initial values is shown. For a very weak initial radial component of the field, saturation is not reached during a Hubble time. However, for fields substantially larger than the critical initial field strength for reaching saturation, the final saturated field is independent of the initial radial field.

In the interstellar medium, ambipolar diffusion is not well modeled by diffusion through a uniform plasma. In fact, the bulk of the mass of the interstellar medium is in dark clouds, in which the degree of ionization is very low. Also, the outward magnetic field is concentrated in the volume of the clouds. As a result, the ambipolar diffusion velocity is more accurately given by the formula

$$v_D = \frac{B^2(1 + \beta/\alpha)}{8\pi \rho_i v f D}, \quad (6)$$

where $f$ is the filling factor for the clouds, $\rho_i$ is the effective ion density in the clouds, $v$ is the effective ion-neutral collision rate in the clouds, and $\beta/\alpha$ is the ratio of cosmic-ray pressure to magnetic pressure.

The model for the interstellar medium, which we employ to study the magnetic field evolution, is sketched in Figure 3. The magnetic field is anchored in the clouds for which the gravitational mass holds the magnetic field in the galactic disk. The magnetic field lines bow up in between the clouds. As a result, the magnetic field is concentrated in the volume of the clouds. A derivation of equation (6) based on this model is given in § 5.

Taking plausible values for the properties of the clouds, one finds that the critical initial value of the magnetic field for it to reach saturation in a Hubble time is about $10^{-8}$ G.
Further, the saturated value of the field at \( t = 10^{10} \) yr is estimated to be 2 \( \mu G \). At this field strength the time to diffuse across the disk is \( D/v_g \approx 3 \times 10^9 \) yr, which is of order of the lifetime of the disk.

Finally, we consider the structure of the saturated field. The time evolution of the magnetic field refers to the field in the rotating frame. Thus, if we wish to find the toroidal field at time \( t \) and at \( \theta = 0 \) we need to know the initial radial component of the field at \( \theta_1 = -\Omega r \). Thus, assuming that the field reaches a saturation value of \( B'_S \), one has

\[
B_\theta = \pm B_S , \tag{7}
\]

where the sign is plus or minus according to the sign of

\[
\cos \left[ -\Omega r (t) \right] . \tag{8}
\]

Since \( \Omega \) depends on \( r \), we see that \( B_\theta \) changes sign over a distance \( \Delta r \) such that \( (\Delta \Omega) r = (\Delta r d\Omega/dr) t \approx \pi, \) i.e.,

\[
\frac{\Delta r}{r} = \frac{-\pi}{r d\Omega/dr} t \approx \frac{\pi}{|\Omega| t} . \tag{9}
\]

Taking the rotation period of the galaxy to be 200 million years, one finds for \( t = 10^{10} \) yr, \( \Delta r \approx 100 \) pc.

Therefore, because the field saturates to a constant field strength, it is predicted that a uniform initial field that has a value greater than \( 10^{-8} \) G immediately after the collapse to the disk leads to a toroidal field that varies with \( r \), at fixed \( \theta \), as a square wave with a reversal in sign every 100 pc or so. Such a field structure would produce no net Faraday rotation and would contradict observations.

However, this result can be traced to our assumption that the initial magnetic field in Figure 1 was entirely uniform. If we suppose it were not uniform, then, after collapse, the radial field would not be purely sinusoidal (see Fig. 4). In fact, one expects a behavior more like

\[
B_R(t = 0) = B_0 \cos \theta + \epsilon \cos 2\theta , \tag{10}
\]

where \( \epsilon \) is an index of the nonuniformity. As an example, for moderately small \( \epsilon \), \( \cos \theta + \epsilon \cos 2\theta \) is positive for

\[
-\frac{\pi}{2} + \epsilon < \theta < \frac{\pi}{2} - \epsilon \tag{11}
\]

and negative for

\[
\frac{\pi}{2} - \epsilon < \theta < \frac{3\pi}{2} + \epsilon . \tag{12}
\]

Therefore the radial extent in which \( B_\theta(t) \) is negative is larger than the radial extent in which it is positive by a factor

\[
\frac{\pi - 2\epsilon}{\pi + 2\epsilon} . \tag{13}
\]

The resulting field is positive over a smaller range of \( r \) than the range in which it is negative. Consequently, in such a magnetic field there would be a net Faraday rotation of polarized radio sources. We note that it is easily possible that the regions in which the radial field is initially weaker can end up as regions that dominate in flux over those regions that come from regions in which the radial field was stronger! If one averages this field over regions much larger than 100 pc, then the resulting mean field is smooth and axisymmetric. This result is contrary to the generally held belief that a primordial field should lead to a field with bisymmetric symmetry (Sofue, Fujimoto, & Wielebinski 1986). This result alone shows that a more careful treatment of the evolution of the primordial field, such as that discussed above, leads to quite different results from those commonly assumed. (However, as mentioned above, including the effect of the spiral density wave will produce magnetic field lines in the spiral arms that will appear to have bisymmetrical shape.)

We emphasize that, although the actual magnetic field derived in our model is a tightly wound spiral, the magnetic field averaged over a moderate size scale appears to be smooth and axisymmetric.
The outline of the rest of the paper is as follows:

In § 2 an analytic model is developed to demonstrate the properties described in this introduction. In § 3 a more precise one-dimensional numerical simulation is carried out that confirms the evolution of the field in the z direction given in § 2. In § 4 it is shown that the three-dimensional equations for the evolution of the field can be reduced to two independent variables. These are z and an angular coordinate \( \omega = \theta - \Omega(r)t \) that is constant along the spirals generated by the differential rotation of the galaxy. A numerical simulation of the resulting differential equations is carried out. It is shown that after a long time the resulting magnetic field does evolve locally in the same way as is given in §§ 2 and 3. In addition, it varies in radius as a square wave with uneven lobes. In § 5 the astrophysics of the interstellar medium clouds is discussed and a derivation is given of equation (6) for the effective mean ambipolar motion of the field in the disk. In § 6, the implications of this model for the evolution of a magnetic field of primordial origin are given, and the bearing of these implications on the various criticisms of the primordial field hypothesis are discussed.

2. LOCAL THEORY: ANALYTIC

In this and the next section we wish to consider the local behavior of the magnetic field following a fluid element that moves with the galactic rotation. If the ambipolar diffusion were strictly in the z direction, the evolution of the field in a given fluid element would be independent of its behavior in other fluid elements at different \( r \) or \( \theta \). It would only be affected by the differential rotation of the galaxy and by ambipolar velocity in the z direction. Thus, in particular, we could imagine the magnetic field at different values of \( \theta \), behaving in an identical manner. In other words, we could replace the general problem by an axisymmetric one.

Let

\[
B = B_r(r, z)\hat{\mathbf{r}} + B_\theta(r, z)\hat{\mathbf{\theta}} + B_z(r, z)\hat{\mathbf{z}} .
\]

(14)

We neglect the radial velocity and let the ambipolar velocity be only in the \( \hat{\mathbf{z}} \) direction and proportional to the \( \hat{\mathbf{z}} \) derivative of the magnetic field strength squared. We also neglect any turbulent velocity. The only velocities that we consider are the differential rotation of the galaxy, \( \Omega(r) \), and the ambipolar diffusion velocity of the ions.

Then

\[
v = r\Omega(r)\hat{\mathbf{\theta}} + v_z\hat{\mathbf{z}} ,
\]

(15)

where

\[
v_z = -K\frac{\partial B^2}{8\pi \partial z},
\]

(16)

and

\[
K = \frac{(1 + \beta/\alpha)}{\rho_ifv},
\]

(17)

where, as in § 1, \( \beta/\alpha \) is the ratio of the cosmic-ray pressure to the magnetic pressure, \( \rho_i \) is the ion density, and \( v \) is the ion-neutral collision rate in the clouds. The equation for the evolution of the magnetic field is

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) ,
\]

(18)

or, in components,

\[
\frac{\partial B_r}{\partial t} = \frac{\partial}{\partial z} (v_z B_z) ,
\]

(19)

\[
\frac{\partial B_\theta}{\partial t} = \frac{\partial}{\partial z} (v_z B_z) + \frac{d\Omega}{dr} B_r ,
\]

(20)

\[
\frac{\partial B_z}{\partial t} = \frac{\partial v_z}{\partial r} B_r .
\]

(21)

We integrate these equations numerically in § 3. They are too difficult to handle analytically. To treat them approximately, we make the assumption that for all times \( B_r \) and \( B_\theta \) vary parabolically in z, i.e.,

\[
B_r = B_0^r (1 - z^2/D^2) ,
\]

(22)

\[
B_\theta = B_0^\theta (1 - z^2/D^2) ,
\]

(23)

where \( B_r^0 \) and \( B_\theta^0 \) are functions of time. We apply equations (19) and (20) at \( z = 0 \) and \( r = r_0 \) and solve for \( B_r^0 \) and \( B_\theta^0 \) as functions of \( t \).

Thus, making use of equations (16), (18), and (19), we find

\[
\frac{\partial B_r}{\partial t} = -\frac{v_D}{D} B_r ,
\]

(24)

\[
\frac{\partial B_\theta}{\partial t} = -\frac{v_D}{D} B_\theta + \left( \frac{d\Omega}{dr} \right)_r B_r ,
\]

(25)

\[
v_D = \frac{K}{2\pi D} (B_r^2 + B_\theta^2) ,
\]

(26)

where all quantities are evaluated at \( z = 0 \) and \( r = r_0 \) so we drop the superscripts on \( B_r \) and \( B_\theta \). Since for galactic rotation \( r\Omega \) is essentially constant, we have \( (d\Omega/dr)_r = -\Omega(r_0) = -\Omega_0 \).

Initially \( B_r \) and \( B_\theta \) are of the same order of magnitude. Also, we expect \( B_\theta \) to grow, by stretching, to a value much greater than its initial value before ambipolar diffusion becomes important. Thus, for simplicity, we make the choice of initial conditions \( B_r = B_1 \) and \( B_\theta = 0 \), where \( B_1 \) is the initial value for the radial component of the field. According to equation (2), \( B_1 = B_1 \cos \theta \) for a fluid element starting at \( \theta \).

If initially \( v_D/D < \Omega \), then according to equation (25), we expect \( B_\theta \) to at first grow linearly in time. Then by equation (26), \( v_D \) increases quadratically in time until \( v_D/D \approx \Omega \), and the ambipolar velocity begins to affect the evolution of \( B_\theta \) and \( B_r \). If the initial field is small, then the contribution of \( B_r \) to the ambipolar diffusion velocity is never important. In this case we can write

\[
v_D = v_{D1}\left( \frac{B_\theta^2}{B_1^2} \right) ,
\]

(27)

where \( v_{D1} = KB_1^2/(2\pi D) \). The solution to the differential equation (24) and equation (25) with \( v_D \) given by equation (27) is

\[
B_r = \frac{B_1}{[1 + (2v_{D1}/3D)\Omega^2t^3]^{1/2}} ,
\]

(28)

\[
B_\theta = \frac{B_1\Omega t}{[1 + (2v_{D1}/3D)\Omega^2t^3]^{1/2}} ,
\]

(29)
and

$$v_D = \left( \frac{v_{D1}}{D} \right) \frac{\Omega^2 t^2}{1 + (2v_{D1}/3D)\Omega^2 t^3}.$$  (30)

(The above solution, eqs. [28] and [29], of eqs. [24], [25], and [27], is derived explicitly in Appendix A. However, it can be shown by direct substitution that it is a solution of eqs. [24], [25], and [27].)

From these equations we see that for

$$t \ll \left( \frac{3D}{2v_{D1}\Omega^2} \right)^{1/3},$$  (31)

$B_\phi$ is unchanged, $B_\theta$ increases as $\Omega t$, and $\int v_D dt \ll D$, so that up to this time ambipolar diffusion carries the plasma only a small fraction of the disk thickness.

On the other hand, if

$$t \gg \left( \frac{3D}{2v_{D1}\Omega^2} \right)^{1/3},$$  (32)

then

$$B_\phi \approx B_1 \sqrt{\frac{3D}{2v_{D1}\Omega^2}} = D \sqrt{\frac{6\pi}{Kt}} = D \sqrt{\frac{6\pi \rho_i v}{(1 + \beta/\Omega)^2}},$$  (33)

and $B_\phi$ is independent of both the initial value of $B_\phi$ and of $\Omega$. It depends only on the ambipolar diffusion properties of the interstellar medium.

For all $t$, we have

$$\frac{B_\theta}{B_\phi} = \frac{1}{\Omega t},$$  (34)

so for $t = 10^{10}$ yr, $B_\theta = B_\phi/300$ and the field becomes strongly toroidal.

The question arises as to how strong $B_1$ must be in order that the saturated solution, equation (33), is reached. Taking $t = t_H$, the Hubble time, and making use of the expression for $v_{D1}$, one finds from equation (32) that for saturation to be reached we must have

$$B_1 > \frac{D}{t_H^{3/2}} \sqrt{\frac{4\pi \rho_i v}{3(1 + \beta/\Omega)}}.$$  (35)

Making use of the numbers derived in § 5 one finds that if $B_\phi$ is saturated at $t = t_H$, then

$$B_\phi = \frac{D}{100 \text{ pc}} \left( \frac{10^{10} \text{ yr}}{t_h} \right)^{1/2} \left( \frac{6 \times 10^{-4}}{n_i n_0} \right)^{1/2} 1.9 \times 10^{-6} \text{ G},$$  (36)

where $n_i$ is the ion density in the clouds, assumed to be ionized carbon, and $n_0$ is the mean hydrogen density in the interstellar medium. The densities are in cgs units.

For saturation, the critical value for $B_1$ is

$$B_{1\text{crit}} = \frac{D}{100 \text{ pc}} \left( \frac{10^{10} \text{ yr}}{t_h} \right)^{3/2} \left( \frac{6 \times 10^{-4}}{n_i n_0} \right)^{1/2} 4 \times 10^{-9} \text{ G},$$  (37)

where the densities are in cgs units.

Hence, for the above properties of the clouds and for an initial radial field greater than the critical value, the magnetic field at $t = t_H$ saturates at about the currently observed value. Such a field arises from compression if, when the galaxy was a sphere of radius 10 kpc, the magnetic field strength was greater than $10^{-10}$ G. If before this the virialized radius of the protosphere was 100 kpc, then in this sphere the comoving initial value for the cosmic field strength had to be greater than $10^{-12}$ G. (That is, if the cosmic field filled all space then it had to be so strong that the present value of the magnetic field in intergalactic space must now be $10^{-12}$ G.) There are good reasons to believe that a magnetic field stronger than this minimum value could have been generated by turbulence in the protogalaxy during its collapse to this virialized radius (Kulsrud et al. 1997). However, this field would be local to the galaxy and not fill all space.

Finally, one expects that $B_\phi \approx B_1/100$ from the initial compression into the disk. If we take $B_1$ to be unaffected by the differential stretching and by the ambipolar diffusion velocity, we can derive an expression for the length of a line of force:

$$\frac{rd\theta}{dz} = \frac{B_\phi}{B_1} = \frac{100\Omega t}{[1 + (2v_{D1}/3D)\Omega^2 t^3]^{1/2}} \approx 100 \frac{3D}{v_{D1} t},$$  (38)

or

$$r\Delta\theta = \left( \frac{2.5 \times 10^{-6} \text{ G}}{B_1} \right) 10 \text{ kpc}.$$  (39)

Thus, for example, if $B_1 = 4.5 \times 10^{-7}$ G, then a line of force passing through the solar position in the galaxy, would stretch once around the galaxy. Stronger initial fields lead to shorter lines of force. If the initial field is stronger than 2 $\mu$G, then it is possible that cosmic rays can escape along the lines of force into the halo during their average lifetime.

3. ONE-DIMENSIONAL NUMERICAL SIMULATION

It is clear that any gradient of the magnetic field strength will lead to ambipolar diffusion. However, gradients in the angular direction are weak so we may neglect ambipolar diffusion in the angular direction. Similarly, the gradients in the radial direction are weak at first, although, as pointed out in the introduction, the magnetic field ends up reversing rapidly in the toroidal direction, so eventually radial ambipolar diffusion becomes as important as vertical ambipolar diffusion.

In this section, we restrict ourselves to gradients only in the $\xi$ direction. In this case, the axisymmetric approximation is valid, and the relevant equations are equations (19) and (20), where $v_z$ is given by equation (16),

$$v_z = -K_{\xi} \frac{\partial B^2}{\partial \xi},$$  (40)

In the previous section, these equations were reduced to zero dimensions by the Ansatz that $B_\phi$ and $B_\theta$ were parabolic in $z$ according to equations (22) and (23), and the basic equations were applied only at $z = 0$. In the present section we treat these equations numerically for all $z$ with $|z| < D$ and drop the parabolic assumption. We treat the disk as uniform so that $\rho_i$ and $v$ are taken as constants.

We need boundary conditions at $z = \pm D$. We expect $B$ to be quite small outside the disk, $|z| > D$. (This is because we suppose that neutrals are absent in the halo and $v = 0$, so that the flow velocity $v_z$ becomes very large. Since flux is
conserved, \( vB \) must be a constant in a steady state and the magnetic field must be very small.) In order to match smoothly to the outer region, we first assume that \( \rho_i \) is a constant for all \( z \) and is very small. In addition, we assume that \( v \) is constant for \( |z| < D \) and decreases rapidly to zero in a narrow region, \( D < |z| < D + \Delta \). Then because \( v \approx 0 \) in the halo region, \( |z| > D \), \( v_i \) becomes so large that inertia is important.

The equation for \( v \) should read

\[
\rho_i \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} \right) = - \frac{\partial B^2/8\pi}{\partial z} - \rho_i v' v_z ,
\]

where we have set \( \rho_i v' = (\rho_i v_f)(1 + \beta/v_f) \). Because the time evolution is over the Hubble time \( t_H \), \( \frac{\partial v}{\partial t} \ll v \frac{\partial v}{\partial z} \), i.e., \( v/D \) is very large compared to \( 1/t_H \). Thus, we drop the partial time derivative.

The equations for \( B_z \) and \( B_\theta \) are the same as in the disk. However, because \( v \) is large, the divergence terms \( \partial(v_z B)/\partial z \) and \( \partial(v_\theta B_\theta)/\partial z \) are much larger than the other terms in the region near \( |z| = D \) and in the halo. Thus, we have

\[
\frac{1}{\partial z} (v_z B) = 0 ,
\]

where \( B = (B_z^2 + B_\theta^2)^{1/2} \) is the magnitude of the magnetic field. This means that

\[
vB = \Phi = \text{const} ,
\]

where \( \Phi \) is the rate of flow of magnetic flux, which is a constant in these regions. (We drop the subscript on \( \epsilon \)).

Dividing equation (41) by \( \rho_i \) and dropping the partial time derivative, we get

\[
\frac{1}{\partial z} \left( v^2 + \frac{B^2}{4\pi \rho_i} \right) = -2\nu v' .
\]

Combining this equation with equation (43), we have

\[
\frac{1}{\partial z} \left( v^2 + \frac{\Phi^2}{4\pi \rho_i v'^2} \right) = -2\nu v' ,
\]

or

\[
\frac{d(v^3 + \Phi^2/4\pi \rho_i v'^3)}{v} = -2\nu dz ,
\]

or

\[
- \int_0^z \left( \frac{\Phi^2}{4\pi \rho_i v'^2} - 1 \right) dv = - \int_v^0 v' dz .
\]

For \( z \) larger than \( D + \Delta \), the right-hand side becomes a constant, so that \( v^3 \to \Phi^2/4\pi \rho_i \). As \( z \) decreases below \( D \), the right-hand side becomes linear in \( z \), and we find that for \( z \) far enough into the disk, the inertial term becomes negligible. Hence,

\[
v^3 \approx \frac{\Phi^2}{4\pi \rho_i v(D - z)} ,
\]

and \( v \) approaches a small value. This result breaks down when \( v \) is small enough that the other terms in equations (19) and (20) become important.

Thus, the connection of the main part of the disk to the halo is given by equation (47),

\[
\int_{v_z}^{v} \left( \frac{\Phi^2}{4\pi \rho_i v'^2} - 1 \right) dv = - \int_{v_z}^{0} v' dz ,
\]

where \( v_z = (\Phi^2/4\pi \rho_i)^{1/4} \). Transforming this equation to one for \( B \), we have

\[
\int_{B_z}^{B} \left( \frac{B^2}{4\pi \rho_i \Phi^2} - 1 \right) dB = - \int_{v_z}^{0} v' dz ,
\]

where \( B_z = \Phi/v_z = (4\pi \rho_i \Phi^2)^{-1/4} \) and \( z_z \) is the value of \( z \) for which \( B = B_z \). We write \( \int_{v_z}^{0} v' dz = v'_0 \Delta' \), where on the right-hand side \( v'_0 \) denotes the constant value of \( v' \) in the disk. Then for \( z \) far enough into the disk, equation (50) reduces to

\[
\frac{B^3}{3(4\pi \rho_i)} \approx (D - \epsilon - z)v_0 ,
\]

where

\[
\epsilon = \Delta' + 2v_0/3v_0 .
\]

The quantity \( \Delta' \) is small by assumption. If we estimate \( \Phi \) by \( B_0 v_{\text{diff}} \), where \( v_{\text{diff}} \) is the order of the ambipolar diffusion velocity at the center of the disk, then \( v_0 \approx [B_0 v_{\text{diff}}/(4\pi \rho_i)^{1/2}]^{1/2} \approx B_0/(4\pi \rho_i)^{1/2} \), so that \( \epsilon \) is much smaller than the distance an Alfvén wave can propagate in the ion-neutral collision time. This is clearly a microscopic distance compared to the thickness of the disk, so that we may neglect it.

In summary, equation (50) implies that the inner solution essentially vanishes at \( z \approx \pm D \), so that we may take our boundary condition to be \( B_z = B_\theta = 0 \) at \( z = D \). Given this solution, the above analysis shows how to continue it smoothly into the halo.

Equations (19) and (20) can be made dimensionless by a proper transformation. We choose a transformation that is consistent with that employed in the next section. We define a unit of time \( t_0 \) by

\[
\Omega_0 t_0 = R/D ,
\]

where \( R \) is the radius of the Sun's galactic orbit and \( \Omega_0 = \Omega(R) \) is its angular velocity. We next choose a unit of magnetic field \( B_0 \) to satisfy

\[
KB_0^2 = D/4\pi \frac{t_0}{D} .
\]

For such a field the ambipolar diffusion velocity is such that the field lines cross the disk in a time \( t_0 \). We then set

\[
t = t_0 t' ,
\]

\[
B_z = (D/R)B_0 B'_0 ,
\]

\[
B_\theta = B_0 B'_\theta ,
\]

\[
z = Dz' .
\]

For a cloud ion density of \( 6 \times 10^{-4} \text{cm}^{-3} \) and a mean interstellar medium density of \( n_0 = 10 \text{ cm}^{-3} \) and \( B_0 = 2.85 \times 10^{-6} \text{ G} \) (see §5). In addition, \( D = 100 \text{ pc} \), and \( t_0 = 3 \times 10^4 \text{ yr} \). The Hubble time in these dimensionless units is about 3.

The details of the numerical simulation are given in Howard (1996). The dimensionless equations to be solved...
are
\[
\begin{align*}
\frac{\partial B_r}{\partial t'} &= \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial B^2}{\partial z} B_r \right), \\
\frac{\partial B_\theta}{\partial t'} &= \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial B^2}{\partial z} B_\theta \right) - B_r.
\end{align*}
\]

Our dimensionless boundary condition is \( B_r = B_\theta = 0 \) at \( z = \pm 1 \).

The results of the integration of these equations are shown in Figures 5 and 6. The initial profiles of \( B_r \) and \( B_\theta \) are parabolic.

In Figure 5 the initial value of \( B_r \) is \( B_r = 0.01 \), so that \( B_r = 2.85 \times 10^{-8} \text{ G} \). It is seen that the qualitative behavior is the same as that described in §§ 1 and 2. The field at first grows and then relaxes back to a decaying solution. Figure 6 gives the time evolution of \( B_\theta \) at \( z = 0 \) for the set of initial conditions, \( B_r = B_r \cos \theta \), where \( \theta \) runs from zero to 180° in increments of 15°. It is seen that the curves all tend to the same asymptotic behavior and are similar to those in Figure 2. As in Figure 2, the field from weaker initial values of \( B_r \) takes longer times to reach its peak value and the saturation curve. These results are similar to those which are represented by equations (28) and (29).

It is seen in Figure 5 that, except near the edge, the profile remains similar to a parabola. Near the edge the cube root behavior of equation (51) is also evident. This cube root behavior can be understood since the flux \( v_z B \) is roughly constant, and therefore \( (\partial B^2/\partial z)B \approx (\partial/\partial z)(1 - z)^{2/3}(1 - z)^{1/3} \approx (1 - z)^{1} \approx \text{constant} \).

The most important feature in Figures 5 and 6 is the saturation of \( B \) to the envelope curve. This saturation occurs if the initial value of \( B_r \) is larger than 0.01. Thus, even in our more precise calculations, information on the initial \( B \) is lost, and the final value of \( B \) at fixed time depends only on the initial sign of \( B_r \).

4. TWO-DIMENSIONAL ANALYSIS

In the last section we treated the evolution of the magnetic field under the influence of differential rotation and ambipolar diffusion in a one-dimensional approximation. Only the \( z \) component of the ambipolar diffusion was retained. In this approximation the magnetic field in each column of fluid at \( r, \theta \) evolves independently of any other \( r, \theta \) column of fluid.

In early times, since the galactic disk is thin compared to its radius, this is a good approximation because the horizontal gradients of \( B^2 \) are small compared to the vertical gradients. However, because of the differential rotation of the galaxy, fluid elements at different radii that were initially very far from each other are brought much closer together, and the horizontal gradients are increased until they become comparable to the vertical gradients.
For example, two fluid elements that were initially on opposite sides of the galaxy, and at a difference in radius of 100 pc, will be brought to a position on the same radius after about 50 rotations. Since the evolution of the field in these two fluid elements is very different, we expect the gradient of $B^2$ in the radial direction to become as large as that in the vertical direction. As a consequence, ambipolar drift velocities in the horizontal direction can be expected to be as large as those in the vertical direction. On the other hand, two fluid elements at the same radius, but initially far apart, remain far apart. The ambipolar motions in the $\theta$ direction should remain small.

Thus, we expect that, at first, only ambipolar $z$ motions are important for the evolution of the magnetic field, but eventually the radial ambipolar motions will also become important, although not the angular ambipolar motions. That is, we expect the problem to become two-dimensional.

In order to demonstrate this evolution properly, we introduce a new independent variable

$$ u = \theta - \Omega(r)t $$ (58)

If ambipolar diffusion is neglected, this variable is just the initial angular position of a fluid element that is at the position $r$, $\theta$, $z$ at time $t$. In the absence of ambipolar diffusion the variables $r$, $u$, and $z$ are constant, following a given fluid element. They would be the Lagrangian variables if only rotational motion is considered. It is appropriate to describe the evolution of the magnetic field components $B_r$, $B_\theta$, and $B_z$ in terms of these variables.

Inspection of equation (58) reveals that when $\Omega t$ is large, $u$ varies rapidly with $r$ at fixed $\theta$, in agreement with the above qualitative discussion. Changing $r$ by only a small amount will change the initial angular position by $\pi$. Thus, we expect that the components of $B$ will vary finitely with $u$, but only slowly with $r$ for fixed $u$. The surfaces of constant $u$ are tightly wrapped spirals. Thus, the behavior of the field should be finite in $r$, $u$, and $z$, but only gradients with respect to $u$ and $z$ should be important.

To see this, let us first write the total velocity, $v = w + \Omega r \theta$, where $w$ is the ambipolar velocity. Next, let us derive the equations for the components $B_r$, $B_\theta$, $B_z$ and $w_r$, $w_\theta$, $w_z$ in terms of the Eulerian variables $r$, $\theta$ and $z$:

$$ \frac{\partial B_r}{\partial t} = (B \cdot \nabla)w_r - (w \cdot \nabla)B_r - B_r (\nabla \cdot w) - \Omega \frac{\partial B_r}{\partial \theta}, $$ (59)

$$ \frac{\partial B_\theta}{\partial t} = (B \cdot \nabla)w_\theta - \frac{B_\theta w_r}{r} (w \cdot \nabla)B_r + \frac{B_r w_\theta}{r} + \frac{w_r}{r} B_\theta - \frac{w_\theta}{r} B_r, $$ (60)

$$ \frac{\partial B_z}{\partial t} = (B \cdot \nabla)w_z - (w \cdot \nabla)B_z - \frac{\partial B_z}{\partial \theta} - B_r (\nabla \cdot w), $$ (61)

where the ambipolar velocities are

$$ w_r = -K \frac{\partial B^2/8\pi}{\partial r}, \quad w_\theta = -K \frac{\partial B^2/8\pi}{r \partial \theta}, $$

$$ w_z = -K \frac{\partial B^2/8\pi}{\partial z}, $$ (62)

and

$$ \mathbf{\nabla} \cdot \mathbf{w} = \frac{\partial w_r}{\partial r} + \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} + \frac{\partial w_z}{\partial z}. $$ (63)

Finally, let us transform these equations to the new coordinates $r$, $u$, $z$. In doing this we assume that the galactic rotation velocity $\Omega$ is a constant.

The result of the transformation is

$$ \frac{\partial B_r}{\partial t} = \frac{B_r}{r} \frac{\partial w_r}{\partial u} + B_z \frac{\partial w_z}{\partial z}, $$

$$ -w_r \frac{\partial B_r}{\partial r} - \frac{w_\theta}{r} \frac{\partial B_\theta}{\partial u} - \frac{w_r}{r} \frac{\partial B_\theta}{\partial z} - w_z \frac{\partial B_z}{\partial z} $$

$$ -B_r \left( \frac{w_r}{r} + \frac{\partial w_z}{\partial z} \right), $$ (64)

$$ \frac{\partial B_\theta}{\partial t} = B_r \frac{\partial w_\theta}{\partial r} + \frac{\Omega t}{r} B_r \frac{\partial w_\theta}{\partial u} $$

$$ + B_z \frac{\partial w_\theta}{\partial z} - \frac{w_r}{r} \frac{\partial B_\theta}{\partial r} - \frac{w_\theta}{r} \frac{\partial B_\theta}{\partial u} - \frac{w_z}{r} \frac{\partial B_\theta}{\partial z}, $$ (65)

$$ \frac{\partial B_z}{\partial t} = B_r \frac{\partial w_z}{\partial r} + B_\theta \frac{\partial w_z}{\partial u} + \frac{\Omega t}{r} B_r \frac{\partial w_z}{\partial u} $$

$$ - w_r \frac{\partial B_z}{\partial r} - \frac{w_\theta}{r} \frac{\partial B_z}{\partial u} - \frac{w_z}{r} \frac{\partial B_z}{\partial u} $$

$$ - B_r \frac{\partial B_z}{\partial z} - B_\theta \frac{\partial B_z}{\partial z} $$

$$ - \frac{\Omega t}{r} B_r \frac{\partial B_z}{\partial u} - B_z \frac{w_r}{r} - B_z \frac{\partial w_z}{\partial u}, $$ (66)

$$ w_r = -K \frac{\partial B^2/8\pi}{\partial r} - K \frac{\partial B^2/8\pi}{\partial u}, $$ (67)

$$ w_\theta = -K \frac{\partial B^2/8\pi}{\partial u}, $$ (68)

$$ w_z = -K \frac{\partial B^2/8\pi}{\partial z}. $$ (69)

Only a few of these terms are important. To see this let us introduce dimensionless variables for the velocity and field components. (We will denote the dimensionless variables by primes.) We will choose these variables based on our one-dimensional results, as follows:

The unit of length for the $z$ variable is the galactic disk thickness $D$.

$$ z = Dz'. $$ (70)

The variation of quantities with $r$ is finite over the distance $R$ the radius of the Sun’s orbit in the galaxy, so we set

$$ r = Rr'. $$ (71)
The variable $u$ is already dimensionless and quantities vary finitely with it, so we leave it unchanged.

The unit of time $t_0$ should be of order of the age of the disk. During this time the number of radians through which the galaxy rotates, $\Omega t$ is of the same order of magnitude as the ratio $R/D$, so for analytic convenience we choose $t_0$ so that

$$\Omega_0 t_0 = R/D ,$$

(72)

and set

$$t = t_0 t' .$$

(73)

If we take $R/D = 100$, and $\Omega_0 = 2 \pi / (2 \times 10^8 \text{ yr})$, then $t_0 = 3 \times 10^9 \text{ yr}$.

It is natural to choose the unit for $B_0$, $B_0$ as that field whose $z$ gradient produces an ambipolar $z$ velocity of order $D/t_0$ that is an average velocity near that which would be produced by the saturated field. Thus, we choose $B_0$ so that

$$KB_0^2 / (4 \pi D) = D/t_0 ,$$

(74)

and set

$$B_0 = B_0 B_0 .$$

(75)

(Note that the definitions in these units are consistent with those of § 3.)

In most cases of interest to us, $B_r$ and $B_z$ are much smaller than $B_0$. (In general, $B_r$ is initially small compared to $B_0$, but it grows by a factor of $R/D$ to up to the saturated value of about $B_0$.) Thus, we set

$$B_r = (D/R)B_0 B_r .$$

(76)

The vertical field $B_z$ starts out even weaker than this magnitude since the initial horizontal components of the magnetic field were amplified by the initial compression which formed the disk. However, the $B_z$ field is amplified up to the size of the $B_z$ field by the shear of the radial ambipolar velocity acting on $B_z$. We thus transform $B_z$ by

$$B_z = (D/R)B_0 B_z .$$

(77)

The ambipolar velocities $w_z$, $w_\theta$, and $w_z$ arise from gradients of the magnetic field in the corresponding directions. Thus, after $B_0$ has been amplified by stretching to be of order $B_0$, $w_z \approx KB_0^2 / D$. Similarly, after the differential rotation has acted to reduce the scale of the variation of $B^2$ in the radial direction, $w_\theta$ becomes of order $w_z$. However, the scale of variation in the $\theta$ direction remains of order $R$ over the age of the galaxy so that $w_\theta \approx KB_0^2 / R$. Thus, we change the $w$ components to $w'$ components by

$$w_r = (D/t_0)w_r' ,$$

$$w_\theta = (D^2/Rt_0)w_\theta' ,$$

$$w_z = (D/t_0)w_z' .$$

(78)

Now if we transform equations (64)–(69) by the change of variables given by equations (70)–(78) and clear the dimensional factors $t_0$, $B_0$, etc., from the left-hand side, we find that the terms on the right-hand side are either independent of dimensional units entirely or are proportional to powers of $(D/R) \ll 1$. The full equations are given in Appendix B. Dropping these “smaller” terms proportional to a power of $D/R$ greater than zero, we find that the equations for the dimensionless variables, to lowest order in $D/R$ are

$$\frac{\partial B_r'}{\partial t'} = - \frac{1}{r^2} w_r' \frac{\partial B_r'}{\partial u} - \frac{\partial (w_r' B_r)}{\partial z'} ,$$

(79)

$$\frac{\partial B_\theta'}{\partial t'} = - \frac{r'}{r^2} \frac{\partial B_r'}{\partial u} - \frac{\partial (w_r' B_\theta)}{\partial z'} - \frac{r'}{r^2} B_r' \frac{\partial w_r'}{\partial u} ,$$

(80)

$$\frac{\partial B_z'}{\partial t'} = \frac{B_r'}{r'} \frac{\partial w_r'}{\partial u} + \frac{t' w_r'}{r^2} \frac{\partial B_r'}{\partial u} - \frac{w_z'}{r^2} \frac{\partial B_z'}{\partial u} - \frac{t' w_r'}{r^2} \frac{\partial B_\theta'}{\partial u} - \frac{\partial (w_r' B_\theta)}{\partial z'} - \frac{t' w_r'}{r^2} \frac{\partial w_r'}{\partial u} ,$$

(81)

$$w_r' = - \frac{t' \frac{\partial B_r'}{\partial u}}{2r^2} ,$$

(82)

$$w_\theta' = - \frac{1}{2} \frac{\partial B_\theta'}{\partial u} ,$$

(83)

$$w_z' = - \frac{1}{2} \frac{\partial B_z'}{\partial u} .$$

(84)

Note that $w_\theta$ does not occur in the equations for the evolution of the $B^\prime$ components. Also, the $r'$ derivatives are absent from these lower order equations. The initial conditions on $B^\prime(t', u, z'; r')$ are

$$B_r^\prime(0, u, z', r') = B_r(0, r'R, u, z'D) ,$$

$$B_\theta^\prime(0, u, z', r') = B_\theta(0, r'R, u, z'D) ,$$

$$B_z^\prime(0, u, z', r') = B_z(0, r'R, u, z'D) .$$

(85)

These transformations are formal, but they enable us to correctly drop the terms whose effect is small. Once these terms are dropped, the equations reduce to two-dimensional equations, which are more easily handled numerically. Although we have assumed that the dimensionless variables are originally of order unity, it may be the case that they differ substantially from unity. However, an examination of the various possible relevant cases leads to the conviction that all the important terms have been kept as well as other terms which are, perhaps, unimportant. For example, the initial value of $B_0^\prime$ is much smaller than unity. However, because of the shearing terms in equation (80) (the last term) $B^\prime_0$ grows to finite order when $t'$ becomes of order unity, so during the later stages of the galactic disk, $B^\prime_0$ is of order unity.

Many of the terms in equations (79)–(81) have an obvious significance. The second term on the right-hand side of the $B_r^\prime$ equation is the vertical decompression term present in the one-dimensional simulation. The first term is the radial decompression term, which only becomes important when $t' \approx 1$, and the wrapping up has made the radial ambipolar diffusion important.

Similarly, there is a $z$ decompressional term in the $B_z^\prime$ equation, but, of course, no radial decompression term. There is a term representing the effect of shear on the toroidal field in increasing the radial component, and a similar term resulting from the action of shear on the $B_z^\prime$ component. These shear terms would be small if $B_0^\prime$ were of order of $B_0$, or if $B_r^\prime$ were much smaller than $B_0$, which is the
The results of the two-dimensional numerical simulation, for the case of an initial uniform magnetic field, $B_0$, is shown as function of $u$ at $z = 0$ for different times. That the integral of $B_0$ is zero is evident. This field will not lead to any Faraday rotation. The labeling of the curves is the same as in Fig. 5.

Further, $u$ does not occur explicitly in the differential equations. The initial conditions given by equation (85) do involve u and are periodic in it. Therefore, the magnetic field components remain periodic in $u$ for all $t'$.

Let us consider the behavior of such a solution, periodic in $u$, in the neighborhood of the Sun, $r' = 1$, at fixed $z'$, say $z' = 0$. Transform the solution back to $r$, $\theta$ coordinates. For fixed $r$ and $t$, the solution is periodic in $\theta$, e.g., $B_\theta(r, \theta) = B_\theta[r, u + \Omega(r)t']$. Moreover, for fixed $\theta$ we can write

$$u = \theta - \Omega(r)t = \theta - \frac{\Delta r}{r} \Omega_0 t$$

$$= \theta - \frac{\Delta r}{r} t' \frac{R}{D} = \theta - \frac{\Delta r t'}{D}, \quad (86)$$

so

$$\Delta r = \theta - \frac{D}{r'} u.$$  

Thus, for fixed $\theta$, $r$ changes by an amount $2\pi D/r'$ when $u$ changes by its periodic length $2\pi$. Since $r'$ changes by a small amount $\approx 2\pi D / R t'$, we may ignore the dependence of the solution for the components of $B$ on $r$ and the components of $B$ are nearly periodic in $r$ (at fixed $\theta$). However, because of the actual dependence of the solution on $r'$, as a parameter in the equations, the amplitude and phase (as well as the shape) of the periodic solution do change slowly when one goes a distance comparable with the radius of the galaxy.

Equations (79)-(84) were integrated numerically. The details of the integration are discussed in Howard (1995, 1996), where most of the results are presented. Initial conditions were set by starting with a cosmic field before compression into the disk, and then calculating the resultant fields. In this paper we present the results for two initial cases. Only the results for the integrations at $r = R$, the radius of the galactic solar orbit, are included. The variation of as a function of $u$, $z = 0$, is plotted in for the case that the initial cosmic field was uniform. The antisymmetry in $u$ is evident, and it is clear, after transforming the field to be a function of $r$ as the independent variable by equation (87) that no Faraday rotation would be produced by this field. The same result for the case when the initial cosmic field was nonuniform is shown in Figure 8. [The initial cosmic field was chosen so after compression into the disk the horizontal field was $B = B_0(0.5 + \chi x + y)$.] The resulting saturated field is not antisymmetric and does not
average out in $u$. It also would not average out when transformed to be a function of $r$ and would produce a Faraday rotation. The variation of $B_0$ with $z$ at $u = 0$ up to a time of 9 Gyr is shown in Figure 9. It has the parabolic shape found in §3. The variation of $B_0$ with time at the point $u = 0, z = 0$ is given in Figure 10 for the two cases. The results are also similar to those of Figure 2 derived from the simple parabolic approximation.

5. AMBIPOLAR DIFFUSION IN THE INTERSTELLAR MEDIUM

We now consider the averaged equations for the magnetic field, taking into account the interstellar clouds. The bulk of interstellar matter is in the form of diffuse clouds and molecular clouds. Because the properties of the molecular clouds are not very well known, we make the simplifying assumption that essentially all the interstellar matter is in diffuse clouds, with a small amount of matter in the intercloud region. In describing the clouds, we make use of properties given by Spitzer (1968, p. 85).

We assume that all the clouds are identical. We further include the cosmic ray pressure, and the magnetic pressure, in the intercloud region but neglect the pressure of the intercloud matter. Then the cosmic rays and the magnetic fields are held in the disk against their outward pressures by the weight of the clouds in the gravitational field of the stars. (See Fig. 3.)

Now the force due to the magnetic and cosmic-ray pressure gradients is exerted only on the ionized matter in the clouds, while the gravitational force is exerted mainly on the neutrals in the clouds, since the fraction of ionization in the clouds is generally very low. Thus, these contrary forces pull the ions through the neutrals with ambipolar diffusion velocity $v_D$. The frictional force between the ions and neutrals is proportional to $v_D$. By equating the magnetic plus the cosmic-ray force to the frictional force, we can obtain the mean ambipolar velocity in the clouds. Now we assume that the cosmic-ray pressure $p_R$ is related to the magnetic pressure $B^2/8\pi$ by the factor $\beta/\alpha$ (Spitzer 1968, pp. 177–181). We take $\beta/\alpha$ independent of time and space. This is plausible since, when the magnetic field is strong, we expect the cosmic-ray confinement to be better and therefore the cosmic-ray pressure to be larger.

The mean vertical force per unit volume produced by the magnetic field strength gradients and cosmic-ray pressure gradients is

$$F = -(1 + \beta/\alpha) \frac{\partial B^2/8\pi}{\partial z}. \quad (88)$$

But this force is counterbalanced by the gravitational force on neutrals in the clouds, which occupy a fractional volume equal to the filling factor $f$, times the total volume.

Thus, the force per unit volume on the ions in the clouds is

$$F_{\text{cloud}} = \frac{F}{f} = -(1 + \beta/\alpha) \frac{1}{f} \frac{\partial B^2/8\pi}{\partial z}. \quad (89)$$

This force produces an ambipolar velocity, $v_D$, of the ions relative to the neutrals such that
we get
\[ dpAxL = LdpvDt , \]  
(96)
or
\[ \frac{\Delta x}{t} = v_D . \]  
(97)

This effective velocity, averaged over many cloud lifetimes, is the only velocity for the magnetic fields lines that makes sense. It is equal to the velocity, \( v_D \), of the ions in the cloud material.

So far, we have made the simplifying assumption that the clouds are stationary, which is of course not the case. As the clouds move through the interstellar medium they stretch the lines, and additional tension forces and ambipolar velocities arise. However, these velocities are always directed toward the mean position of the cloud material, and thus they tend to average to zero. The ambipolar velocity which we have calculated above, under the assumption of stationary clouds, actually gives the rate of displacement of the lines of force relative to the mean position of the cloud. It is a secular velocity, and it is the only velocity which really counts.

We assume that in the diffuse interstellar clouds only carbon is ionized, so that \( m_{\text{eff}}^2 = m_H \). We choose \( \langle \sigma v \rangle = 2 \times 10^{-6} \text{ cm}^2 \text{ s}^{-1} \) (Spitzer 1978, p. 4). If the filling factor of the clouds is \( f = 0.1 \) and if the mean interstellar density of hydrogen is \( n_0 \approx 1.0 \text{ cm}^{-3} \), then the density in the clouds is \( n_c \approx 10 \text{ cm}^{-3} \). The cosmic abundance of carbon is \( 3 \times 10^{-4} \) (Allen 1963). We assume that this abundance is depleted by \( \beta_0 \approx 0.2 \) (Spitzer 1978, p. 25), the rest of the carbon being locked up in grains. We further take \( \beta_0/\beta = 2 \). Then comparing equation (40) with equation (93) gives

\[ K = \frac{(1 + \beta/\beta_0)}{n_1 m_{\text{eff}}^2 \langle \sigma v \rangle n_0} = 1.5 \times 10^{36} , \]  
(98)
in cgs units. Taking \( D = 100 \text{ pc} \), we get from equation (54)

\[ B_0 = 2.85 \times 10^{-6} \text{ G} . \]  
(99)

We take \( t_0 = 3 \times 10^9 \text{ yr} \). We see that if \( t' = 3 \), corresponding to an age for the galactic disk of \( 9 \times 10^9 \text{ yr} \), then the present value of the magnetic field from Figure 8, should be \( 1.5 \times 10^{-6} \text{ G} \).

This value depends on our assumptions concerning the properties of the clouds. In particular, if the hydrogen in the interstellar clouds is partly ionized by low energy cosmic rays penetrating the clouds, then the ambipolar diffusion will be slower, and \( K \) will be smaller, leading to a larger value for \( B_0 \). The consequence of this is, that the initially dimensionless magnetic field \( B_0 \) will be smaller, so that it would take longer to reach saturation. However, the saturated value will be larger.

6. CONCLUSION

It was Fermi (1949) who first proposed that the galactic magnetic field was of primordial origin. Piddington (1957, 1964, 1967, 1981) in a number of papers suggested how this might actually have happened. However, he supposed the magnetic field strong enough to influence the collapse. One of the first problems with the primordial origin is the wrapup problem. It was pointed out by Hoyle & Ireland (1960, 1961) and by Ök et al. (1964) that the field would wrap up into a spiral similar to the spiral arms in only two
or three galactic rotations. It was supposed that this is the natural shape of the magnetic field lines. If the winding up continued for the 50 or so rotations of the galaxy, the field lines would reverse in direction every 100 pc. This seemed absurd. The various attempts to get around this problem involved magnetic fields which were strong enough to control the flow and also strong radial outflows. Several earlier attempts to resolve this problem by one of the authors are given in Kulskrud (1986, 1989, 1990). Several forceful arguments against the primordial origin were advanced by Parker. The arguments were that the field would be expelled from the galaxy, either by ambipolar diffusion (Parker 1968) or by rapid turbulent diffusion (Parker 1973a, 1973b).

In this paper we have reexamined the hypothesis that there may have been a primordial field. To explore this hypothesis we constructed a simplified model of the interstellar medium in order to investigate how the magnetic field would evolve. We then compared the results of our calculations with current observations to judge the plausibility of the hypothesis. The two essential ingredients of this model are the differential rotation of the interstellar medium and the motion of the field lines through the interstellar medium produced by the ambipolar diffusion of the ionized component of the plasma which is driven through the neutrals by magnetic and cosmic-ray pressure gradients. The effect of turbulent motions is assumed to average out. Also no large-scale mean field dynamo action on the magnetic field has been included in our model. (This model differs from that of Piddington in that the magnetic field is too weak to effect the galactic rotation of the interstellar medium, and also, ambipolar diffusion is included.)

The consequences of the model were investigated by an approximate analytical calculation in § 2, and by more detailed numerical simulations in §§ 3 and 4. These simulations confirmed the results of the approximate analysis of § 2.

The basic results are:

1. To first approximation, the magnetic field evolves locally following a rotating fluid element. It first grows by stretching the radial component of the magnetic field into the toroidal direction. When the field becomes strong enough, the line commences to shorten because of the vertical motions produced by ambipolar diffusion. This reduces the radial component, and therefore the stretching. After a certain time, the field strength saturates and starts to decrease as the reciprocal square root of time. This asymptotic behavior is determined only by the ambipolar diffusion properties of the clouds. Thus, the field strength everywhere approaches the same value at a given time. At the present time, this value is estimated to be in the range of a few microgauss. This saturated value is independent of the initial value which the field had when the disk first formed, provided that the initial value of the field strength is greater than $10^{-8}$ G. The extent of each magnetic field line in the toroidal direction also saturates, but its length in the disk does depend on the initial value.

2. The direction of the toroidal field in any given fluid element depends on the sign of the initial radial component. Since this sign varies with position, and since differential rotation mixes these positions, it turns out that the resulting toroidal field varies rapidly with radius along a fixed radial direction. The toroidal field changes direction on a scale of 100 pc. Because the saturated field strength is nearly constant in magnitude, the toroidal field strength as a function of $r$ at fixed $\theta$ varies as a square wave. However, the lobes of this square wave need not be equal since the regions of one initial sign of $B_r$ may be larger than those of the other sign. In this case, the model predicts a toroidal field that would produce a net Faraday rotation in radio sources such as pulsars or polarized extragalactic radio sources, in spite of its rapid variation in sign.

It is the prevailing belief that the galactic magnetic field does not reverse in radius on small scales. In fact, this belief is grounded in an analysis of Faraday rotation measure of pulsars (Hamilton & Lynn 1987). In analyzing these rotation measures Rand & Kulkarni (1989) employed various simple models of the galactic field. In every one of these models, the magnetic field reversed only on large scales of order 1 kpc. These models, which only allowed a slow variation of $B$, led to results that were consistent with the observations, and thus supported the general belief in reversal only on large scales. However, when this analysis was carried out, there was no apparent reason to exclude a model in which the field varied rapidly in radius on scales of order of 100 pc. However, such a model was not included in the analysis of the rotation measures, and thus it was not tested.

In short, the prevailing belief that the galactic magnetic field is of constant sign on a large scale actually resulted from the assumption that the field was a large-scale field, and from the consistency of this assumed model with observations. Therefore, this conclusion has not been rigorously demonstrated. A magnetic field, such as that arrived at from our model, which has a rapid variation of the field with radius, would lead to fluctuations from the mean. Indeed, such fluctuations are in the data and are attributed to a general isotropic random magnetic field in addition to the mean field.

We feel the nonuniform square wave could probably fit the observations equally as well as the other models, so that our model can also be shown to be consistent with observations. We have not yet demonstrated quantitative consistency with observations, but we hope to carry out this task in the future.

It is interesting to compare the wrapup problem of the galactic magnetic field with that of the spiral arms. The spiral arm phenomena is a density wave that propagates through the differentially rotating galactic disk, in the radial as well as the azimuthal direction, and therefore the arms need not wrap up as strongly as the magnetic field. If the field lines are frozen into the plasma, as we assume to be the case in this paper, then it is conclusive that they must wrap up tightly, reversing in direction on a short radial scale. Their direction, however, is modified as they cross the spiral arms, by motions associated with the spiral arms, but not directly considered in this paper. As a result of these motions, the field should be parallel to the spiral arms in the spiral arm regions.

3. The magnetic field observed in our Galaxy and in other galaxies should actually be the average of the true detailed magnetic field averaged over regions in space larger than those regions over which we find our field to vary (several hundred parsecs at least). Thus, under averaging the field of our model would actually appear as an axisymmetric toroidal magnetic field. An analysis of the various galactic magnetic fields has been carried out which has made the Ansatz that the origin of the field can be distinguished as primor-
Thus, objection (a) does not defeat our model for a field of primordial origin.

With respect to objection (b) our model predicts field lines which thread through the disk, entering on one edge and leaving on the other. Thus, it is impossible for a vertical ambipolar motion or turbulent mixing to expel lines of force. This result counters objection (b).

As for Parker’s third objection, there are now a number of proposals for the creation of magnetic fields during the era of inflation (see for example Ratra (1991, 1992a, 1992b)). It is expected that any magnetic fields created on small scales, less than 1 Mpc, would be removed by damping during the era just prior to recombination (A. Olinto, 1997, private communication). Thus, if this were the origin of a primordial field, then the assumption that the field is uniform on the scale larger than the protogalaxy is plausible. However, there is a recent proposal that the primordial magnetic fields are produced by the turbulence generated during the formation stage of the galaxy (Kulsrud et al. 1997). Such magnetic fields would be quite nonuniform after they have collapsed into the disk. Our paper addresses both of these limiting cases, although for definiteness in the numerical calculation only the case of a magnetic field which is initially has a uniform gradient is considered. However, the method that we apply to the evolution of a primordial magnetic field are sufficiently general that it is possible to apply them to initial magnetic fields with more structure. This is because the evolution of the magnetic field is essentially a local problem. It is planned to carry out the investigation of such fields in conjunction with the detailed discussion of their origin.

Two observations that should test our model are (1) The magnetic field in other galaxies should be observed to be toroidal in between the arms. (2) A reanalysis of the pulsar rotation measures as well as the depolarization of the background radio emission should be carried out on the basis of our model. At least some of the the very large extra fluctuations in the rotation measures could be produced by the rapid reversals of our model. On the other hand, these reversals should not produce any depolarization of the background radio emission.

To summarize, we have shown that a careful analysis of the evolution of a primordial field throws new light on the way one should view the primordial field hypothesis. The resulting field to be expected from a primordial origin differs considerably from the generally accepted picture of the galactic magnetic field in that it reverses rapidly. However, a closer examination of the observations shows that such a magnetic field is also in agreement with observations and that it is not so easy to rule out a primordial origin for the galactic magnetic field.

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APPENDIX A

In this appendix we give the derivation of the solution given in equations (28)-(30) to the differential equations (24), (25), and (27) of § 2. Let us drop the θ subscript in  Bθ. Then the equations to be solved are

\[
\frac{dB_r}{dt} = -\frac{v_D}{D} B_r, \\
\frac{dB}{dt} = -\frac{v_D}{D} B - \Omega B_r, \\
v_D \frac{D}{D} = kB^2. (A1)
\]

Define ξ by

\[
\frac{d\xi}{dt} = \frac{v_D}{D} = kB^2. (A2)
\]
Then
\[ \frac{dB_r}{d\xi} = -B_r, \]  
(A3)

or
\[ B_r = B_1 e^{-\xi}, \]  
(A4)

and
\[ \frac{dB}{d\xi} = -B - \frac{\Omega}{kB^2} = -B - \frac{C}{B^2} e^{-\xi}, \]  
(A5)

where \( C = \Omega B_1 / k \) is a constant.

Multiplying by \( e^\xi \), we get
\[ \frac{d}{d\xi} (B e^\xi) = \frac{C}{B^2} = \frac{C e^{2\xi}}{(Be^\xi)^2}. \]  
(A6)

Now, \( B \) is negative. Let \( y = -Be^\xi \). Then
\[ \frac{dy}{d\xi} = \frac{C e^{2\xi}}{y^2}. \]  
(A7)

Integrating and assuming the initial value of \( y \) is small we get
\[ \frac{y^3}{3} = \frac{C}{2} (e^{2\xi} - 1), \]  
(A8)

or
\[ y = \left( \frac{3C}{2} \right)^{1/3} (e^{2\xi} - 1)^{1/3}. \]  
(A9)

Now,
\[ e^{2\xi} \frac{d\xi}{dt} = kB^2 e^{2\xi} = ky^2. \]  
(A10)

Next, let \( e^{2\xi} = \eta \). Then
\[ \frac{d\eta}{dt} = \frac{2k}{3} \left( \frac{3C}{2} \right)^{2/3} (\eta - 1)^{2/3}, \]  
(A11)

or
\[ \frac{d\eta}{(\eta - 1)^{2/3}} = \frac{2k}{3} \left( \frac{3C}{2} \right)^{2/3} dt. \]  
(A12)

Integrating this equation we have
\[ (\eta - 1)^{1/3} = \frac{2}{3} k \left( \frac{3}{2} C \right)^{2/3} t. \]  
(A13)

Now expanding this equation and making use of the definition of \( \eta \) we have
\[ e^{2\xi} = 1 + \frac{3}{2} C^2 k^2 t^3. \]  
(A14)

The toroidal field \( B = ye^{-\xi} \), and \( y \) is given by equation (A9) so
\[ B^2 = y^2 e^{-2\xi} = \left( \frac{3C}{2} \right)^{2/3} \left( e^{2\xi} - 1 \right)^{2/3} \]  
(A15)

or
\[ B^2 = \frac{C^2 k^2 t^3}{1 + (2/3) C^2 k^2 t^3}. \]  
(A16)
APPENDIX B

In this appendix we give the full equations for $dB_z/dt$, $dB_\theta/dt$, and $dB_r/dt$ including all the terms in $R/D$. They result from the transformation of equations (64)-(69) to dimensionless variables by making use of equations (70)-(77).

For convenience, the terms are listed in exactly the same order as in the original dimensional equations. The resulting equations are

$$\frac{\partial B_z}{\partial t} = \frac{B_r}{r^2} \frac{\partial w_r}{\partial z} - \frac{B_z}{r^2} \frac{\partial w_r}{\partial z}$$

$$- \left( \frac{D}{R} \right) \frac{w_r}{r^2} \frac{\partial B_z}{\partial r} - \left( \frac{D}{R} \right)^2 \frac{w_\theta}{r^2} \frac{\partial B_z}{\partial u} - \left( \frac{D}{R} \right)^2 \frac{w_r}{r^2} \frac{\partial B_z}{\partial u} - \frac{w_r}{r^2} \frac{\partial B_z}{\partial r}$$

$$- \left( \frac{D}{R} \right) \frac{w_r}{r^2} B_z \frac{\partial w_\theta}{\partial r} - \left( \frac{D}{R} \right)^2 \frac{w_\theta}{r^2} \frac{\partial B_z}{\partial u}$$

$$\frac{\partial B_\theta}{\partial t} = \left( \frac{D}{R} \right) B_r \frac{\partial w_r}{\partial r} + \left( \frac{D}{R} \right)^2 B_r \frac{\partial w_\theta}{\partial r} - \frac{t'}{r^2} B_\theta \frac{\partial w_r}{\partial r} - B_\theta \frac{\partial w_\theta}{\partial r}$$

$$\frac{\partial B_r}{\partial t} = \left( \frac{D}{R} \right) B_r \frac{\partial w_r}{\partial r} + B_\theta \frac{\partial w_\theta}{\partial r} + \frac{t'}{r^2} B_r \frac{\partial w_r}{\partial r}$$

$$- \left( \frac{D}{R} \right) w_\theta \frac{\partial B_r}{\partial u} - \left( \frac{D}{R} \right)^2 w_\theta \frac{\partial B_r}{\partial u} - \frac{t'}{r^2} w_r \frac{\partial B_r}{\partial u}$$

$$- \frac{w_r}{r^2} \frac{\partial B_r}{\partial u} - \left( \frac{D}{R} \right) B_z \frac{\partial w_r}{\partial r} - \left( \frac{D}{R} \right) B_z \frac{\partial w_\theta}{\partial r}$$

$$- \frac{t'}{r^2} B_z \frac{\partial w_r}{\partial r} - \left( \frac{D}{R} \right) B_z \frac{\partial w_\theta}{\partial r} - \left( \frac{D}{R} \right)^2 B_z \frac{\partial w_r}{\partial u} - \left( \frac{D}{R} \right)^2 B_z \frac{\partial w_\theta}{\partial u}$$

(B1)

(B2)

(B3)

The equations for the components of $w$ are

$$w_r' = - \frac{t'}{r^2} \frac{\partial B^2}{\partial u} + \left( \frac{D}{R} \right) \frac{\partial B^2}{\partial r}$$

$$w_\theta' = - \frac{1}{r^2} \frac{\partial B^2}{\partial u}$$

$$w_z' = - \frac{\partial B^2}{\partial z}$$

(B4)

In these equations $B^2$ stands for

$$B^2 = B_\theta^2 + \left( \frac{D}{R} \right)^2 (B_r^2 + B_z^2)$$

(B5)

We do not expand the expression for the components of $w'$ out fully.

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