Nonminimal curvature-matter coupled wormholes with matter satisfying the null energy condition

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Recently, exact solutions of wormhole geometries supported by a nonminimal curvature-matter coupling were found, where the nonminimal coupling minimizes the violation of the null energy condition of normal matter at the throat. In this brief report, we present a solution where normal matter satisfies the energy conditions at the throat and it is the higher order curvature derivatives of the nonminimal coupling that is responsible for the null energy condition violation, and consequently for supporting the respective wormhole geometries. For simplicity, we consider a linear $R$ nonminimal curvature-matter coupling and an explicit monotonically increasing function for the energy density. Thus, the solution found is not asymptotically flat, but may in principle be matched to an exterior vacuum solution.

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Introduction: Recently, $f(R)$ modified theories of gravity have received a great deal of attention [1], in particular, to account for the late-time cosmic acceleration [2]. An interesting generalization of $f(R)$ gravity involves the inclusion of an explicit coupling between the curvature scalar and matter [3]. The latter coupling imposes the nonconservation of the stress-energy tensor, and consequently implies nongeodesic motion [3].

The action for a nonminimal curvature-matter coupling in $f(R)$ modified theories of gravity, considered in this work [3], is given by

$$S = \int \left\{ \frac{1}{2} f_1(R) [1 + \lambda f_2(R)] \mathcal{L}_m \right\} \sqrt{-g} \, d^4x , \quad (1)$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the Ricci scalar $R$ and $\mathcal{L}_m$ is the Lagrangian density corresponding to matter. Note that the coupling constant $\lambda$ characterizes the strength of the interaction between $f_2(R)$ and the matter Lagrangian.

Varying the action with respect to the metric $g_{\mu \nu}$ yields the field equations, given by

$$F_1(R) R_{\mu \nu} - \frac{1}{2} f_1(R) g_{\mu \nu} - \nabla_\mu \nabla_\nu F_1(R) + g_{\mu \nu} \square F_1(R)$$
$$= -2\lambda F_2(R) \mathcal{L}_m R_{\mu \nu} + 2\lambda (\nabla_\mu \nabla_\nu - g_{\mu \nu} \square) \mathcal{L}_m F_2(R)$$
$$+ [1 + \lambda f_2(R)] T_{\mu \nu}^{(m)} , \quad (2)$$

with the definition $F_1(R) = df_1(R)/dR$.

$f(R)$ gravity with the nonminimal curvature-matter coupling has been explored in different astrophysical and cosmological contexts [4], and in particular has also been applied in the context of wormhole physics [5]. In the latter case, it was shown that the nonminimal coupling minimizes the violation of the null energy condition (NEC) of normal matter at the throat. In the present paper, we further extend this analysis and find an exact wormhole solution, where normal matter satisfies the NEC at the throat, and it is the higher order curvature derivatives of the nonminimal coupling that are responsible for the NEC violation, and consequently for supporting the respective wormhole geometries.

Nonminimal curvature-matter coupled wormholes: Consider the following wormhole metric [6]

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) . \quad (3)$$

The redshift function $\Phi(r)$ must be finite everywhere to avoid the presence of event horizons [6]. In the analysis outlined below, we consider $\Phi' = 0$, which simplifies the calculations considerably, and provides interesting exact wormhole solutions. The shape function $b(r)$ obeys several conditions, namely, the flaring out condition of the throat, given by $(b - b')/b^2 > 0$ [6], and at the throat $b(r_0) = r = r_0$, the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. It is these restrictions that impose the NEC violation in classical general relativity. Another condition that needs to be satisfied is $1 - b(r)/r > 0$.

For simplicity, throughout this paper, we consider the specific case of $f_1(R) = f_2(R) = R$, and we choose the Lagrangian form of $\mathcal{L}_m = -\rho(r)$. The gravitational field equation (2) can be expressed in the following form $G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = T_{\mu \nu}^{\text{eff}}$, where the effective stress-energy tensor is given by

$$T_{\mu \nu}^{\text{eff}} = (1 + \lambda R) T_{\mu \nu}^{(m)} + 2\lambda \rho R_{\mu \nu} - (\nabla_\mu \nabla_\nu - g_{\mu \nu} \square) \rho . \quad (4)$$

The curvature scalar, $R$, for the wormhole metric (3), taking into account $\Phi' = 0$, is given by $R = 2b'/r^2$.

Thus, the gravitational field equations are given by the following relationships

$$2\lambda \rho''(b - r) + \lambda \rho'(rb' + 3b - 4r) + \rho(r'' + 2\lambda b') - b' = 0 , \quad (5)$$
$$4\lambda \rho'(b - r) + 2\lambda \rho(b - b') - r\rho(r'' + 2\lambda b') - b = 0 . \quad (6)$$

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which becomes singular when
\[ \lambda = -r_0^2 \rho_0 \pm \sqrt{r_0^2 \rho_0 (8 + r_0^2 \rho_0)} \frac{1}{4 \rho_0}. \] (15)

Thus, the nonsingular regimes lie outside the regions defined by Eq. (15) and the condition imposed by \( b_0 < 1 \), i.e., \( \lambda = 1/(3\rho_0) \), which are depicted in Figure 1.

The regions where the condition \((1 - b/r) \geq 0\) (which can be expressed as \((r - b) \geq 0\)) is satisfied are plotted in Fig. 2, for the values \( \rho_0 = 1 \), \( r_0 = 1 \). The left plot depicted is for the region in \([-r_0^2 \rho_0 + r_0^2 \rho_0 (8 + r_0^2 \rho_0)]^{1/2}/(4\rho_0) \) < \( \lambda < 2 \). The right plot depicted is for the region \(-2 < \lambda \leq -r_0^2 \rho_0 \pm \sqrt{r_0^2 \rho_0 (8 + r_0^2 \rho_0)}[1/2]/(4\rho_0) \). Note that for these regions, one cannot obtain the general relativistic limit, as one necessarily crosses the singular region depicted by the curves in Fig. 1.

We analyze now the regions where the flaring-out condition, \( b'(r_0) < 1 \), and the NEC at the throat, \((\rho + \rho)_{r=r_0} = 0\)
normal matter can be minimized as in Ref. [5].

\[ \lambda = \frac{1}{3r_0^2} \left( \frac{\rho}{\rho_0} + \frac{\rho}{\rho_0} \right) \]

where the upper (lower) solid curve depicts the positive (negative) sign, respectively.

\[ p_r \mid r_0 > 0 \]

are satisfied. Figure 3 depicts these regions for the domain \(-2 < \lambda < \frac{1}{3r_0^2} \left( \frac{\rho}{\rho_0} + \frac{\rho}{\rho_0} \right) \)

and \(0 < \rho_0 < 1/3\), and \( \lambda > 0 \). One can smoothly run to the GR limit, i.e., \( \lambda \to 0 \), due to the presence of singular regions. We emphasize that it is extremely difficult to obtain exact solutions due to the nonlinearity of the field equations. However, in principle it is possible to find solutions of asymptotically flat wormhole geometries for a general \( f(R) \) function, which tend continuously to the general relativistic regime through a reconstruction method, where one imposes a specific wormhole geometry and finally finds the form of \( f(R) \), much in the spirit of [8]. Work along these lines is presently underway.

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Conclusion: In this work we have considered wormhole geometries supported by a nonminimal curvature-matter coupling in a generalized \( f(R) \) modified theory of gravity. For simplicity, we considered a linear \( R \) nonminimal curvature-matter coupling and an explicit monotonically increasing function for the energy density. The wormhole solution obtained is not asymptotically flat, but may in principle be matched to an exterior vacuum solution. This brief report extends previous work [5], in that normal matter satisfies the null energy condition at the throat, and it is the higher order curvature derivatives that is responsible for the null energy condition violation, and consequently for supporting the respective wormhole geometries. A drawback for the solution obtained is that one cannot regain the general relativistic regime, i.e., \( \lambda \to 0 \), due to the presence of singular regions. We emphasize that it is extremely difficult to obtain exact solutions due to the nonlinearity of the field equations. However, in principle it is possible to find solutions of asymptotically flat wormhole geometries for a general \( f(R) \) function, which tend continuously to the general relativistic regime through a reconstruction method, where one imposes a specific wormhole geometry and finally finds the form of \( f(R) \), much in the spirit of [8]. Work along these lines is presently underway.

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FIG. 2: The shape function restriction $r - b(r) > 0$ is plotted for the values $\rho_0 = 1$, $r_0 = 1$. The left plot is given for the range $[-r_0^2 \rho_0 + [r_0^2 \rho_0(8 + r_0^2 \rho_0)]^{1/2}] / (4 \rho_0) < \lambda < 2$. The right plot is given for the range $-2 < \lambda < [-r_0^2 \rho_0 \pm [r_0^2 \rho_0(8 + r_0^2 \rho_0)]^{1/2}] / (4 \rho_0)$.

FIG. 3: Plots for the regions $[-r_0^2 \rho_0 + [r_0^2 \rho_0(8 + r_0^2 \rho_0)]^{1/2}] / (4 \rho_0) < \lambda < 2$ and $0.3 < \rho_0 < 1$: The left plot depicts the regions of $b'(r_0) < 1$; the right plot depicts the region satisfying the NEC at the throat, i.e., $(\rho + p_r)|_{r_0} > 0$.

FIG. 4: Plots for the regions $-2 < \lambda < [-r_0^2 \rho_0 + [r_0^2 \rho_0(8 + r_0^2 \rho_0)]^{1/2}] / (4 \rho_0)$ and $0.2 < \rho_0 < 2$: The left plot depicts the regions of $b'(r_0) < 1$; the right plot depicts the region satisfying the NEC at the throat, i.e., $(\rho + p_r)|_{r_0} > 0$. 