Temporal localized structures in photonic crystal fibre resonators and their spontaneous symmetry-breaking instability

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We investigate analytically and numerically the formation of temporal localized structures (TLSs) in an all photonic crystal fibre resonator. These dissipative structures consist of isolated or randomly distributed peaks in a uniform background of the intensity profile. The number of peaks and their temporal distribution are determined solely by the initial conditions. They exhibit multistability behaviour for a finite range of parameters. A weakly nonlinear analysis is performed in the neighbourhood of the first threshold associated with the modulational instability. We consider the regime where the instability is not degenerate. We show that fourth-order dispersion affects the threshold associated with the formation of bright TLSs. We estimate both analytically and numerically the linear and nonlinear corrections to the velocity of moving temporal structures induced by spontaneous broken reflection symmetry mediated by third-order dispersion. Finally, we show that third-order dispersion affects the threshold associated with the moving TLSs.

1. Introduction

Driven all-fibre cavities constitute a basic configuration in nonlinear fibre optics. More specifically, experimental studies have demonstrated that, when these cavities are pumped by a continuous wave, they exhibit spontaneously self-organized temporal structures in
the form of trains of short pulses with well-defined repetition rate [1,2]. The theoretical prediction of this phenomenon was carried out in the seminal paper by Lugiato & Lefever (LL model, [3]). The break-up of the continuous wave into trains of pulses is attributed to competition between the following phenomena: (i) a nonlinear mechanism that originates from the intensity-dependent refractive index (Kerr effect), which tends to amplify the field intensity locally, (ii) a chromatic dispersion, which on the contrary tends to restore uniformity, and (iii) dissipation.

Besides a pulse train distribution, temporal localized structures (TLSs) are found in a well-defined region of parameters called the pinning zone. In this regime, the system exhibits the coexistence two states: the uniform background and the train of pulses of light that emerges from subcritical modulational instability [4]. In the same way as in the temporal regime, where the break-up of the continuous wave into trains of pulses results from the interplay between nonlinearity and group-velocity dispersion, in spatial cavities the competition between nonlinearity and diffraction induces the formation of two-dimensional localized structures [4–7]. When dispersion and diffraction have comparable influences, their competition with nonlinearity could produce varieties of three-dimensional periodic and localized ‘light bullet patterns’ [8–13]. These structures consist of regular three-dimensional lattices of bright light bullets travelling at the group velocity of light in the material.

TLSs are often called dissipative solitons or cavity solitons. Experimental observation of TLSs in an all-fibre nonlinear cavity has stimulated further interest in this field [14]. Nowadays, TLSs in standard silica optical fibres is an active field due to the maturity of fibre technology and the possible applications as an ideal support for bits in an optical buffer that could be used for all-optical storage, all-optical reshaping and wavelength conversion [14]. Recently, experimental study has revealed that the interaction between two TLSs is ultra-weak [15]. For the high-intensity regime, temporal cavity solitons could exhibit self-pulsation or chaotic behaviour [16–19]. Front propagation and switching waves between the two stable homogeneous steady states (HSSs) have been investigated experimentally and theoretically in nonlinear all-fibre cavities subjected to injection [20]. More recently, the study of front propagation into an unstable state has revealed that, during time evolution, the velocity of the propagating front evolves according to a universal power law [21].

In all the above-mentioned studies, dispersion is limited to cavities with group-velocity dispersion restricted to second order. However, when an optical cavity is operating close to the zero-dispersion wavelength, high-order chromatic dispersion effects could play an important role in the dynamics of the photonic crystal fibre (PCF) [22–24]. A PCF permits a high degree of control of the dispersion curve and allows previously inaccessible parameter regimes to be explored [25,26]. The inclusion of fourth-order dispersion in the description of all-fibre cavities permits the modulational instability to have a finite domain of existence delimited by two pump power values [27] and allows for the stabilization of dark TLSs [28]. Recently, it has been shown that the combined influence of third- and fourth-order dispersion induced a motion of dark localized structures in the high-intensity regime [29]. These moving solutions involved an asymmetric odd or even number of dips, which coexist for finite values of the input field intensity. When the spatio-temporal dynamics of PCF cavities is ruled by ultra-short pulses, multistability as well as spontaneous breaking in pulse-shape symmetry have been observed [30]. More recently, the role of third-order dispersion in the dynamics of bright TLSs in micro-ring resonators pumped in the proximity of the zero of the group-velocity dispersion has been reported [31].

In this paper, it is our aim to analyse analytically the influence of high-order dispersion on the nonlinear dynamical properties of bright TLSs in an all-PCF cavity. We derive a normal form in the vicinity of the first modulational instability threshold by taking into account second-, third- and fourth-order dispersion. The weakly nonlinear analysis allows us to determine the threshold associated with the emergence of TLSs. In addition, we show that third-order dispersion corrects the linear velocity and pushes TLSs to move with a constant velocity. This regular drift is induced by a broken reflection symmetry mediated by third-order dispersion.
The paper is organized as follows. After briefly introducing the model of a PCF cavity, we provide a summary of a linear stability analysis of HSSs in §2. We present a weakly nonlinear analysis and the estimation of the nonlinear velocity associated with the moving periodic temporal structures in §3. Stationary and moving TLSs are studied in §4. We conclude in §5.

2. Model equation

We consider a single-mode PCF cavity pumped by a continuous wave of power $S^2$. Propagation of light inside the fibre is governed by the nonlinear Schrödinger equation [32]. The use of PCFs allows one to expand the propagation constant up to fourth order in a Taylor series. This equation is supplemented by appropriate resonator boundary conditions. The nonlinear Schrödinger equation combined with the boundary conditions leads to the equation that characterizes the propagation of light along the cavity, which is described by the generalized Lugiato–Lefever model with non-dimensional variables as [27]

$$\frac{\partial E}{\partial t} = S - (1 + i\Delta)E + i|E|^2E - iB_2\frac{\partial^2 E}{\partial \tau^2} + B_3\frac{\partial^3 E}{\partial \tau^3} + iB_4\frac{\partial^4 E}{\partial \tau^4}, \quad (2.1)$$

where $E$ is the slowly varying envelope of the electric field propagating inside the cavity. The time $t$ is the slow time scale that describes the evolution of the field envelope $E$ from one cavity round trip to another. The coefficients $B_{2,3,4}$ account for the second-, third- and fourth-order chromatic dispersion, respectively; $\tau$ is the fast time in the reference frame moving with the group velocity of the light; and $i|E|^2E$ corresponds to nonlinearity described only by the Kerr effect, because we consider that the pulse width is larger than 1 ps. In this case, we can neglect Raman scattering. Finally, $\Delta$ is the cavity detuning. The model equation (2.1) is valid in the limit of high cavity finesse and nonlinear phase shift, and losses have to be smaller than unity. We assume that the optical field maintains its polarization as it propagates along the fibre. Note that the LL equation without higher-order dispersions has a broader applicability than a fibre or spatial resonator. It has been shown that the LL model could describe the Kerr-comb evolution in whispering-gallery-mode resonators, where $t$ is the time and variable $\tau$ is the azimuthal angle [33,34]. Indeed, they show that, at low threshold, wide-span combs can emerge as the well-known TLSs reported in [4].

The HSSs of equation (2.1) satisfy $S = [1 + i(\Delta - |E_s|^2)]E_s$. Obviously, high-order dispersion does not affect these solutions; they are thus identical to the ones of the LL model [3]. The linear stability analysis of the HSS with respect to finite frequency perturbations of the form $\exp(\lambda t - i\Omega \tau)$ yields the eigenvalue

$$\lambda = -1 + iB_3\Omega^3 \pm \sqrt{I_s^2 - (-\Delta + 2I_s + B_2\Omega^2 + B_4\Omega^4)^2},$$

where $I_s = |E_s|^2$ corresponds to the uniform intensity background of the light. The thresholds associated with modulational instability are $I_m1 = 1$ and $I_m2 = [2\kappa + \sqrt{\kappa^2 - 3}]/3$, with $\kappa = B_2^2/(4B_4) + \Delta$. In the monostable case ($\Delta < \sqrt{3}$), the primary instability threshold is degenerate: two critical frequencies appear spontaneously and simultaneously, $\Omega_1^2$ and $\Omega_2^2$, where $\Omega_{i,s}^2 = [-B_2 \pm \sqrt{B_2^2 + 4B_4(\Delta - 2)}]/2B_4$. At the threshold $I_m2$, a new, large, critical frequency appears, $\Omega^2 = -B_2/2B_4$. The existence of two thresholds associated with the modulational instability allows the instability domain to be bounded as shown in figure 1.

The linear stability analysis shows that third-order dispersion affects neither the threshold nor the frequency associated with the periodic train of pulses. This analysis provides a linear velocity for temporal dissipative structures. The linear velocity reads

$$V_l = \frac{\partial \text{Im}(\lambda)}{\partial \Omega} = \frac{3B_3\sqrt{(2 - \Delta)B_4}}{B_4}. \quad (2.2)$$

This simple expression indicates that, in the absence of third-order dispersion, $B_3 = 0$, the trains of temporal pulses are motionless, $V_l = 0$. The third-order dispersion pushes the train of temporal pulses to move along the $\tau$-direction, as we shall see in §3.
and its real and imaginary parts as instability, we use a weakly nonlinear analysis. To this end, we decompose the electric field into

To evaluate the nonlinear solutions that emerge from the first threshold of the modulational instability threshold. Next, we expand, in powers of \( \epsilon \)

The solvability condition at the third order of \( \epsilon \)

where \( \mathbf{X} \) and the wavenumber of the fastest growing frequency:

\[
\Delta = 2 - \frac{B_2^2}{4B_4} \quad \text{and} \quad \Omega_c^4 = -\frac{B_2^2}{4B_4}. \tag{3.1}
\]

To evaluate the nonlinear solutions that emerge from the first threshold of the modulational instability, we use a weakly nonlinear analysis. To this end, we decompose the electric field into its real and imaginary parts as \( E = X_1 + iX_2 \), and we introduce the deviation \( X_{1,2} = X_{1,2s} + u(t, \tau) \). We explore the vicinity of the first threshold associated with the modulational instability. We choose a small parameter \( \epsilon^2 = S - S_{1m} \), which measures the distance from the modulational instability threshold. Next, we expand, in powers of \( \epsilon \), the input field \( S \), the variables \( u_1 \) and \( u_2 \) and the homogeneous stationary solutions \( X_{1s} \) and \( X_{2s} \), i.e.

\[
S = S_{1m} + \epsilon p_1 + \epsilon^2 p_2 + \cdots, \tag{3.2}
\]

\[
u_{1,2} = \epsilon u_{10,20} + \epsilon^2 u_{11,12} + \epsilon^3 u_{21,22} + \cdots, \tag{3.3}
\]

\[
X_{1s} = X_{1m} + \epsilon a_1 + \epsilon^2 a_2 + \cdots, \tag{3.4}
\]

and

\[
X_{2s} = X_{2m} + \epsilon b_1 + \epsilon^2 b_2 + \cdots, \tag{3.5}
\]

where \( X_{1m} = 1/S_{1m} \) and \( X_{2m} = (1 - \Delta)/S_{1m} \) are the values of the real and imaginary parts at the threshold associated with the modulational instability, with \( S_{1m}^2 = [1 + (\Delta - 1)^2] \). We also introduce a slow time \( T = \epsilon^2 t \). The solutions at the leading order in \( \epsilon \)

\[
\nu_{10,20} = \left[ 1, \frac{(2 - \Delta)}{\Delta} \right] A_1 \exp[i(\Omega_c \tau + \phi T)] + \text{c.c.,} \tag{3.6}
\]

where \([1, (2 - \Delta)/\Delta]\) is the eigenvector of the linearized operator at the modulational instability point, c.c. denotes the complex conjugate and \( \phi \) is the phase. The solvability condition at the leading order imposes the condition that \( p_1 = 0 \).

The solvability condition at the third order of \( \epsilon \) yields the following amplitude equation for the wavenumber of the fastest growing frequency:

\[
\frac{1}{2S_{1m}^2} \frac{\partial A}{\partial t} = \frac{S - S_{1m}}{S_{1m}(2 - \Delta)^2} A - (f_1(\Delta) + i f_2(\Delta)) |A|^2 A, \tag{3.7}
\]

where

\[
f_1(\Delta) = \frac{ac + bd}{c^2 + d^2} \quad \text{and} \quad f_2(\Delta) = \frac{bc - ad}{c^2 + d^2}. \tag{3.8}
\]

Figure 1. Marginal stability curves for HSSs representing the instability regions in the plane \((|E|^2, \Omega_s^2)\) for a fixed value of the detuning parameter \( \Delta = 1.3 \): (a) \( B_4 = 0 \) and \( B_2 = -1 \); (b) \( B_4 = 0.5 \) and \( B_2 = -1.5 \); and (c) \( B_4 = 0.5 \) and \( B_2 = -1.1832 \).
with

\[
a = 2(200B_2^2(2\Delta - 3)\Omega^6 - (\Delta - 2)^2(342\Delta - 521)),
\]

\[
b = 20B_3(\Delta^2 - 12\Delta + 16)\Omega^3,
\]

\[
c = - (\Delta - 2)^2\Delta^2(10B_3\Omega^3 - 9\Delta + 18)(10B_3\Omega^3 + 9\Delta - 18)
\]

and

\[
d = 20B_3(\Delta - 2)^2\Delta^2\Omega^3.
\]

The amplitude equation (3.7) is expressed in terms of the amplitude $A = \epsilon A_1$. For given values of $B_3$ and $B_4$ such that $f_1(\Delta) > 0$, for any input field $S > S_{1m}$, the nonlinear solutions are stable, and this bifurcation is called supercritical. While, if $f_1(\Delta) < 0$, the nonlinear solutions bifurcate subcritically and are unstable when $S < S_{1m}$.

Next, the nonlinear analysis allows one to calculate the amplitude and the phase of the periodic solutions that emerge from supercritical bifurcation. Assuming that $A = A_1 \exp[i(\phi T + \Omega \tau)]$, we obtain

\[
|A_1|^2 = \frac{1}{f_1(\Delta)} \frac{\epsilon^2}{S_{1m}(2 - \Delta)^2}
\]

and

\[
\phi = -2S_{1m}^2 f_2(\Delta)|A_1|^2.
\]

According to these results, it is obvious that, in addition to the parameters of the system ($S$, $B_2$, $B_3$, $B_4$ and $\Delta$), the distance from the modulational instability threshold plays an important role in the dynamics of periodic solutions. Both amplitude $|A_1|^2$ and phase $\phi$ are proportional to this distance. The nonlinear phase $\phi$ is caused by third-order dispersion. When taking into account the nonlinear correction, the velocity takes the form

\[
v = \frac{3B_3\sqrt{(2 - \Delta)B_4}}{B_4} + \frac{\partial \phi}{\partial \Omega}
\]

and

\[
v = \frac{3B_3\sqrt{(2 - \Delta)B_4}}{B_4} - h(\Delta, B_4)(S - S_{1m}),
\]

where $h$ is the velocity correction function depending on $\Delta$ and $B_4$ and proportional to $B_3$ and $S$. The velocity of moving periodic solutions as a function of the third-order dispersion coefficient is shown in figure 2a. The linear velocity equation (2.2) of the periodic structures is affected by third-order dispersion. This nonlinear correction can increase or decrease the velocity depending on the value of $B_3$ and the distance from the threshold of instability. To check this result, we numerically integrate equation (2.1) with periodic boundary conditions. The numerical results are plotted together with the analytical expression of the velocity. The comparison between the numerical results and the analytical ones is close as shown in figure 2a. The $\tau - t$ map of figure 2b(i) describes the time evolution of periodic structures in the absence of third-order dispersion. When third-order dispersion is neglected, the structure is always stationary. However, in the presence of third-order dispersion, the temporal structures that propagate inside the cavity undergo a drift from a round-cavity trip to another with a well-defined velocity as shown in the $\tau - t$ map of figure 2b(ii).

### 4. Subcritical modulational instability and temporal localized structures

Localized structures are usually excited in the pinning region involving the HSSs and the periodic dissipative structures [4,5]. Therefore, the occurrence of a subcritical modulational instability is often the prerequisite condition for the emergence of TLSs. A large body of literature exists on the study of localized structures in biology, chemistry, physics and mathematics (see some of the overviews on this issue [35–37]). This field is now attracting growing interest in optics because
of its potential application in information technology. In particular, they could be used for all-optical storage, all-optical reshaping and wavelength conversion [14]. The aim of this section is twofold: (i) to determine, through a weakly nonlinear analysis performed in §3, the threshold associated with the formation of bright TLSs in the absence of third-order dispersion; and (ii) to study the role of third-order dispersion that breaks the reflection symmetry ($\tau \rightarrow -\tau$), and leads to the formation of moving TLSs. Equation (2.1) admits a variety of temporal TLSs [28,29]. These solutions exhibit a complex homoclinic snaking type of bifurcation as shown in [28]. This means that the system exhibits a high degree of multistability for a finite range of parameters often called the pinning region. There exist an infinite number of stable TLSs, each of them characterized by either an odd number or an even number of peaks or dips. The configuration that maximizes the number of peaks or dips in the pattern corresponds to trains of short pulses with well-defined repetition rate. An example of motionless TLSs having a single peak and a single dip is displayed in the $\tau$–$t$ map of figure 3a(i). This figure is obtained for $B_3 = 0$. This solution is symmetric as shown in the cross section along the $\tau$ coordinate (cf. figure 3b(i)). When taking into account the third-order dispersion $B_3 \neq 0$, a single peak TLS exhibits a spontaneous motion as shown in the $\tau$–$t$ map of figure 3b(ii). The cross section along the $\tau$ coordinate shows an asymmetry in the intensity profile of the intracavity field (figure 3b(ii)). They are obtained by numerical integration of (2.1) with periodic boundary conditions.

The weakly nonlinear theory presented in §3 cannot describe TLSs, because it does not take into account the non-adiabatic effects that involve the fast temporal scales responsible for the stabilization of TLSs [38]. The inclusion of amended terms in the amplitude equations can capture this dynamics [39,40]. However, the weakly nonlinear analysis provides information about the threshold associated with the appearance of bright temporal localized structures. Indeed, the sign of $f_1(\Delta)$ provides information about the nature of the bifurcation that can exist in this resonator. An explicit expression for $f_1(\Delta)$ is given by equation (3.8). When $f_1(\Delta) > 0$, the bifurcation is supercritical and the train of periodic solutions emerges beyond the modulational instability threshold. However, when $f_1(\Delta) < 0$, the bifurcation is subcritical and periodic solutions can exist.
Figure 3. (a) The $\tau-t$ maps showing the time evolution of bright TLSs that emerge from subcritical bifurcation: (i) $B_3 = 0$, (ii) $B_3 = 0.12$. (b) (i) Stationary bright TLS, $B_3 = 0$, (ii) single moving bright soliton, $B_3 = 0.12$. Parameters are $B_2 = -0.7483$, $B_4 = 0.5$, $\Delta = 1.72$ and $S = 1.228$.

Figure 4. (a) Variation of $f_1(\Delta)$ as a function of $\Delta$; the threshold is shifted from $\Delta_1 = 1.367$ when $B_4 = 0$ to $\Delta_2 = 1.523$ when $B_4 \neq 0$, $B_3 = 0$. (b) Variation of $\Delta$ threshold ($f_1(\Delta) = 0$) according to $B_3$, $B_4 \neq 0$.

below the modulational instability threshold. In this case, there exists a hysteresis loop involving the coexistence of the HSS and the periodic solution, which are both linearly stable. In this region, there exists a pinning zone for which TLSs are stable. The condition $f_1(\Delta) = 0$ gives the threshold for the appearance of TLS. The plot of the function $f_1(\Delta)$ shows that, even when $B_3 = 0$, the threshold associated with the formation of TLSs is shifted with respect to the detuning parameter, i.e. $\Delta = 521/342 \approx 1.523$, as shown in figure 4a. Indeed, when $B_4 = B_3 = 0$, we recover the classical condition for the inversion of the bifurcation derived by Lugano & Lefever, $\Delta = 41/30 \approx 1.367$ [3]. For $B_3 = 0$ and $B_4 = 0$, transition from zone I to zone II indicates the change in the nature of the bifurcation, i.e. from super- to subcritical modulational instability (figure 4a). When $B_3 = 0$ and $B_4 \neq 0$, transition from super- to subcritical modulational instability occurs between zone II and zone III as shown in figure 4a. When taking into account third- and fourth-order dispersion ($B_3 \neq 0$ and $B_4 \neq 0$), the real part of the coefficient of the nonlinear term in the amplitude equation $f_1$ depends on the third-order dispersion. In figure 4b, we plot the function $f_1 = 0$ in the plane ($\Delta$, $B_3$). The solid line in this figure indicates the threshold associated with the appearance of
TLSs. We can then see from figures 3 and 4 that higher-order dispersion induced a spontaneous symmetry-breaking instability and allows bright TLSs to appear for larger intensity of the injected beam.

It has been shown that dark TLSs exhibit a homoclinic snaking type of instability [28]. In the rest of this paper, we will show that the same type of behaviour occurs for a moving bright TLS when taking into account both third and fourth orders of dispersion ($B_3 \neq 0$ and $B_4 \neq 0$). The homoclinic nature of these solutions implies that, for a given set of control parameters, the number and the temporal distribution of both bright and dark TLSs immersed in the bulk of the HSSs are determined only by the initial condition. TLSs may, therefore, be used for signal processing, since the addition or the removal of a TLS simply means the change from one solution to another. Note that the same model equation (2.1) using fourth-order diffraction instead of dispersion has previously also been proved to support higher-order spatial effects on bright spatial solitons [41,42]. Moving TLSs involving multipeak solutions are shown in figure 5. They are obtained for the parameter values as the single-peak TLS of figure 3.

5. Conclusion

In conclusion, we have studied the impact of the effects of high orders of dispersion on the dynamics of TLSs in a PCF resonator pumped by a continuous wave. Both bright and dark TLSs are possible. Without fourth-order dispersion, dark localized structures do not exist. They consist of asymmetric moving peaks or dips in a uniform background of the intensity profile. The number of moving localized peak structures and their temporal distribution are determined solely by the initial conditions. We have focused the analysis on bright temporal localized structures. We have characterized this motion by computing the velocity of bright TLSs. The weakly nonlinear analysis in the vicinity of the first threshold associated with the modulational instability is performed. This analysis shows first that the threshold associated with the TLSs is shifted from $\Delta = 41/30 \approx 1.367$ to $\Delta = 521/342 \approx 1.523$. Second, the weakly nonlinear analysis allows one to estimate the linear and the nonlinear velocity associated with the moving TLSs. Numerical simulations of the governing model for an all photonic crystal fibre resonator are performed. The numerical solutions are in close agreement with the analytical predictions. Our study confirms the possibility of reducing the size of TLSs close to the zero-dispersion wavelength by using PCFs.

Figure 5. Moving multi-peak localized structures with (i) two, (ii) three and (iii) four peaks. They are asymmetric solutions since the third-order dispersion breaks the reflection symmetry ($\tau \to -\tau$). (a) The $\tau - t$ maps showing the time evolution of bright moving solitons associated with the respective panels in (b). They are obtained numerically by integrating equation (2.1) with periodic boundary conditions. Parameters are the same as figure 3.
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