Notes on the Dipole Coordinate System

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Abstract
A strong magnetic field can make it advantageous to work in a coordinate system aligned with dipolar field lines. This monograph collects the formulas for some of the most frequently used expressions and operations in dipole coordinates.

In some physical systems (e.g., the terrestrial ionosphere or the solar corona) the magnetic field can impose a strong anisotropy by restricting transport processes perpendicular to the lines of force. If the field is approximately dipolar it can be useful to work in an aligned coordinate system even though the vector operations are somewhat more complicated than in a Cartesian or spherical polar representation. Here we present several of the formulas that frequently occur when working with vectors in dipolar coordinates. Although some of the results have previously appeared in the literature [2, 4], others seem to be new.

There are many possible (related) choices for the dipolar coordinates. The right-handed orthogonal system considered here, \((q, p, \phi)\), is defined in terms of the usual spherical polar coordinates, \((r, \theta, \phi)\), by

\[
q = \frac{\cos \theta}{r^2} \quad p = \frac{r}{\sin^2 \theta} \quad \phi = \phi
\] (1)

\(p\) is constant along a dipolar field line while \(q\) parameterizes the displacement parallel to the field: \(q = 0\) at the equator, \(q \to -\infty\) as \(\theta \to \pi\) and \(q \to +\infty\) as \(\theta \to 0\).

1 Inverse Transformation
The inversion of (1) — i.e., finding \((r, \theta)\) given \((q, p)\) — involves the solution of a non-trivial equation. Substituting for \(r\) gives \(q^2 r^4 + \frac{r}{p} - 1 = 0\) (2)

Descartes’s rule of signs states that (2) has exactly one positive, real root and, as has been previously noted [3], since (2) is a quartic this root has an algebraic representation. To be useful in numerical models however such a solution has to be expressed in a computationally stable form. Define the auxiliary quantities

\[
\alpha = \frac{256}{27} q^2 p^4 \quad \beta = (1 + \sqrt{1 + \alpha})^{2/3} \quad \gamma = \sqrt[3]{\alpha}
\] (3)

and

\[
\mu = \frac{1}{2} \left( \frac{\beta^2 + \beta \gamma + \gamma^2}{\beta} \right)^{3/2}
\] (4)

Then the positive real root of (2) is

\[
r = \frac{4\mu}{(1 + \mu)(1 + \sqrt{2\mu - 1})} p
\] (5)

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Since $\mu \geq 1$ this formulation eliminates the possibility of a catastrophic cancellation between terms. To complete the inversion recall that $\sin^2 \theta = r/p$. Although (5) makes it possible, in principle, to express formulas in terms of either $r$ and $\theta$ or $q$ and $p$, it is usually simpler to use the former representation.

## 2 Coordinate Derivatives

In this section we ignore the $\phi$ coordinate. The partial derivatives of the dipolar coordinates with respect to the spherical polar coordinates are

\[
\begin{align*}
\frac{\partial q}{\partial r} &= -\frac{2 \cos \theta}{r^3} & \frac{\partial p}{\partial r} &= \frac{1}{\sin^2 \theta} \\
\frac{\partial q}{\partial \theta} &= -\frac{\sin \theta}{r^2} & \frac{\partial p}{\partial \theta} &= -\frac{2r \cos \theta}{\sin^3 \theta}
\end{align*}
\]

The Jacobian is then

\[
\frac{\partial(q, p)}{\partial(r, \theta)} = \frac{\delta^2}{r^2 \sin^3 \theta}
\]

where $\delta(\theta) = \sqrt{1 + 3 \cos^2 \theta}$. The derivatives of the spherical polar coordinates with respect to the dipolar coordinates are

\[
\begin{align*}
\frac{\partial r}{\partial q} &= -\frac{2r^3 \cos \theta}{\delta^2} & \frac{\partial r}{\partial p} &= \frac{\sin^4 \theta}{\delta^2} \\
\frac{\partial \theta}{\partial q} &= -\frac{r^2 \sin \theta}{\delta^2} & \frac{\partial \theta}{\partial p} &= -\frac{2 \cos \theta \sin \theta}{r \delta^2}
\end{align*}
\]

A few second derivatives are also occasionally useful

\[
\begin{align*}
\frac{\partial^2 r}{\partial q^2} &= -\frac{2r^5}{\delta^6}(1 - 10 \cos^2 \theta - 15 \cos^4 \theta) & \frac{\partial^2 r}{\partial p^2} &= -\frac{4 \sin^6 \theta \cos^2 \theta}{r \delta^4}(5 + 3 \cos^2 \theta) \\
\frac{\partial^2 \theta}{\partial q^2} &= \frac{r^4 \sin \theta \cos \theta}{\delta^6}(11 + 9 \cos^2 \theta) & \frac{\partial^2 \theta}{\partial p^2} &= -\frac{2 \sin^5 \theta \cos \theta}{r^2 \delta^6}(1 - 16 \cos^2 \theta - 9 \cos^4 \theta)
\end{align*}
\]

## 3 Unit Vectors

The expressions for the dipolar and spherical polar unit vectors in the alternate coordinate system are

\[
\hat{\mathbf{q}} = -\frac{2 \cos \theta}{\delta} \hat{\mathbf{r}} - \frac{\sin \theta}{\delta} \hat{\mathbf{\theta}} \quad \hat{\mathbf{p}} = \frac{\sin \theta}{\delta} \hat{\mathbf{r}} - \frac{2 \cos \theta}{\delta} \hat{\mathbf{\theta}}
\]

\[
\hat{\mathbf{r}} = -\frac{2 \cos \theta}{\delta} \hat{\mathbf{q}} + \frac{\sin \theta}{\delta} \hat{\mathbf{p}} \quad \hat{\mathbf{\theta}} = -\frac{\sin \theta}{\delta} \hat{\mathbf{q}} - \frac{2 \cos \theta}{\delta} \hat{\mathbf{p}}
\]

### 3.1 First Derivatives of Unit Vectors

\[
\begin{align*}
\frac{\partial \hat{\mathbf{r}}}{\partial q} &= \hat{\mathbf{q}} \frac{\partial \theta}{\partial q} & \frac{\partial \hat{\mathbf{r}}}{\partial p} &= \hat{\mathbf{p}} \frac{\partial \theta}{\partial p} & \frac{\partial \hat{\mathbf{r}}}{\partial \phi} &= \hat{\mathbf{\phi}} \sin \theta \\
\frac{\partial \hat{\mathbf{\theta}}}{\partial q} &= -\hat{\mathbf{r}} \frac{\partial \theta}{\partial q} & \frac{\partial \hat{\mathbf{\theta}}}{\partial p} &= -\hat{\mathbf{r}} \frac{\partial \theta}{\partial p} & \frac{\partial \hat{\mathbf{\theta}}}{\partial \phi} &= \hat{\mathbf{\phi}} \cos \theta \\
\frac{\partial \hat{\mathbf{\phi}}}{\partial q} &= 0 & \frac{\partial \hat{\mathbf{\phi}}}{\partial p} &= 0 & \frac{\partial \hat{\mathbf{\phi}}}{\partial \phi} &= -\hat{\mathbf{r}} \sin \theta - \hat{\mathbf{\theta}} \cos \theta
\end{align*}
\]
The Christoffel symbols of the second kind, as defined by Arfken [1], are given by the formula
\( \Gamma^q_{ij} = \frac{1}{2} g^{km} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \)

where \( g^{ii} = 1/g_{ii} \). In matrix form they are
\[
\Gamma^q = \begin{pmatrix}
-3 r^2 \cos \theta \delta \frac{\partial}{\partial \theta} 
+ 3(1 + \cos^2 \theta) \frac{\partial}{\partial q} 
& \frac{3 \sin^4 \theta}{\delta^2} \left(1 + \cos^2 \theta\right) 
& 3 \sin \theta \cos \theta \\
\frac{3 \sin^4 \theta}{\delta^2} \left(1 + \cos^2 \theta\right) 
& \frac{6 \sin^2 \theta \cos \theta}{\delta^2 \cos \theta} 
& 0 \\
0 
& 0 
& \frac{3 \sin \theta \cos \theta}{\delta^2}
\end{pmatrix}
\]
\[ \Gamma^p = \begin{pmatrix} \frac{3r^5}{\delta^4 \sin^2 \theta} (1 + \cos^2 \theta) & -\frac{6r^2 \cos \theta}{\delta^4} (1 + \cos^2 \theta) & 0 \\ -\frac{6r^2 \cos \theta}{\delta^4} (1 + \cos^2 \theta) & -\frac{12 \sin^2 \theta \cos^2 \theta}{r \delta^4} (1 + \cos^2 \theta) & 0 \\ 0 & 0 & \frac{r}{\sin^2 \theta} (1 + \cos^2 \theta) \end{pmatrix} \] (32)

\[ \Gamma^\phi = \begin{pmatrix} 0 & 0 & -\frac{3r^2 \cos \theta}{\delta^2} \\ 0 & \frac{\sin^2 \theta}{r \delta^2} (1 - 3 \cos^2 \theta) & 0 \\ -\frac{3r^2 \cos \theta}{\delta^2} \frac{\sin^2 \theta}{r \delta^2} (1 - 3 \cos^2 \theta) & 0 \end{pmatrix} \] (33)

5 Vector Operations

The differential operators can be derived from the metric tensor. In what follows \( f \) is a scalar, \( \mathbf{A} \) and \( \mathbf{B} \) are vectors, and \( \mathbf{T} \) is a tensor.

5.1 Gradient

\[ \nabla f = \hat{q} \frac{\delta}{r^3} \frac{\partial f}{\partial q} + \hat{p} \frac{\delta}{\sin^2 \theta} \frac{\partial f}{\partial p} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \] (34)

5.2 Divergence

\[ \nabla \cdot \mathbf{A} = \frac{\delta^2}{r^6} \frac{\partial}{\partial q} \left( \frac{r^3}{\delta} A_q \right) + \frac{\delta^2}{r^4 \sin^2 \theta} \frac{\partial}{\partial p} \left( \frac{r^4 \sin \theta}{\delta} A_p \right) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \] (35)

which can also be written as either

\[ \nabla \cdot \mathbf{A} = \frac{\delta^2}{r^6} \frac{\partial}{\partial q} \left( \frac{r^3}{\delta} A_q \right) + \frac{\delta^2}{\sin^2 \theta} \frac{\partial^2}{\partial p} \left( \sin^3 \theta \frac{A_p}{\delta} \right) + \frac{4}{r \delta \sin \theta} A_p + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \] (36)

or

\[ \nabla \cdot \mathbf{A} = \frac{\delta}{r^3} \frac{\partial A_q}{\partial q} - \frac{3 \cos \theta}{r^3 \delta^3} (3 + 5 \cos^2 \theta) A_q + \frac{\delta}{\sin \theta} \frac{\partial A_p}{\partial p} + \frac{4}{r \delta^3 \sin \theta} A_p + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \] (37)

5.3 Curl

\[ \nabla \times \mathbf{A} = \hat{q} \frac{1}{r \sin \theta} \left[ \frac{\delta}{\sin^3 \theta} \frac{\partial}{\partial p} \left( r \sin \theta A_\phi \right) - \frac{\partial A_p}{\partial \phi} \right] \]

\[ + \hat{p} \frac{1}{r \sin \theta} \left[ \frac{\partial A_q}{\partial \phi} - \frac{\delta}{r^3} \frac{\partial}{\partial q} \left( r \sin \theta A_\phi \right) \right] \]

\[ + \hat{\phi} \frac{\delta^2}{r^3 \sin^3 \theta} \left[ \frac{\partial}{\partial q} \left( \frac{\sin^3 \theta}{\delta} A_p \right) - \frac{\partial}{\partial p} \left( \frac{r^3}{\delta} A_q \right) \right] \] (38)

which is equivalent to

\[ \nabla \times \mathbf{A} = \hat{q} \left[ \frac{\delta}{\sin^3 \theta} \frac{\partial}{\partial p} \left( r \sin \theta A_\phi \right) - \frac{1}{r \sin \theta} \frac{\partial A_p}{\partial \phi} + \frac{1 - 3 \cos^2 \theta}{r \delta \sin \theta} A_q \right] \]

\[ + \hat{p} \left[ \frac{1}{r \sin \theta} \frac{\partial A_q}{\partial \phi} - \frac{\delta}{r^3} \frac{\partial}{\partial q} A_\phi + \frac{3 \cos \theta}{r \delta} A_p \right] \]

\[ + \hat{\phi} \left[ \frac{\delta}{r^3} \frac{\partial A_p}{\partial q} - \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) A_p - \frac{\delta}{\sin^3 \theta} \frac{\partial A_q}{\partial p} + \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) A_q \right] \] (39)
5.4 Scalar Laplacian

\[
\nabla^2 f = \frac{\delta^2 \partial^2 f}{r^6 \partial q^2} + \frac{\delta^2 \partial}{r^4 \sin^4 \theta \partial p} \left( \frac{r^4 \partial f}{\sin^2 \theta \partial p} \right) + \frac{1}{r^2 \sin^2 \theta \partial \phi^2}
\]

which can also be written as

\[
\nabla^2 f = \frac{\delta^2 \partial^2 f}{r^6 \partial q^2} + \frac{\delta^2 \partial^2 f}{\sin^6 \theta \partial p^2} + \frac{4}{r^4 \sin^4 \theta \partial p} + \frac{1}{r^2 \sin^2 \theta \partial \phi^2}
\]

5.5 Vector Laplacian

\[
\nabla^2 A = \hat{q} \left[ \nabla^2 A_q + \frac{6 \sin \theta}{r \delta^2} (1 + \cos^2 \theta) \frac{\partial A_p}{\partial q} + \frac{12 \cos \theta}{r \delta^2 \sin \delta} (1 + \cos^2 \theta) \frac{\partial A_p}{\partial p} + \frac{6 \cos \theta}{r \delta^2 \sin \delta} \frac{\partial A_\phi}{\partial \phi} \right] - \frac{9}{r \delta^2} (1 + 3 \cos^2 \theta + 4 \cos^4 \theta) A_q - \frac{12 \cos \theta}{r \delta^2 \sin \delta} (1 + 3 \cos^4 \theta) A_p
\]

\[
+ \hat{\rho} \left[ \nabla^2 A_p - \frac{6 \sin \theta}{r \delta^2} (1 + \cos^2 \theta) \frac{\partial A_q}{\partial q} - \frac{12 \cos \theta}{r \delta^2 \sin \delta} (1 + \cos^2 \theta) \frac{\partial A_q}{\partial p} - \frac{2}{r \delta^2 \sin \theta} (1 - 3 \cos^2 \theta) \frac{\partial A_\phi}{\partial \phi} \right]
\]

\[
+ \hat{\phi} \left[ \nabla^2 A_\phi - \frac{6 \cos \theta}{r \delta^2 \sin \theta} \frac{\partial A_q}{\partial \phi} + \frac{2}{r \delta^2 \sin \theta} (1 - 3 \cos^2 \theta) \frac{\partial A_p}{\partial \phi} - \frac{1}{r \delta^2 \sin \theta} \left( 1 - 3 \cos^2 \theta \right) A_q \right]
\]

5.6 Directional Derivative

\[
(A \cdot \nabla)B = \hat{q} \left[ A \cdot B_q + \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) A_q B_q + \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) A_p B_p + \frac{3 \cos \theta}{r \delta^3} A_\phi B_\phi \right]
\]

\[
+ \hat{\rho} \left[ A \cdot B_p - \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) A_q B_q - \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) A_p B_q \right]
\]

\[
- \frac{1}{r \delta \sin \theta} (1 - 3 \cos^2 \theta) A_\phi B_\phi \]

\[
+ \hat{\phi} \left[ A \cdot B_q - \frac{3 \cos \theta}{r \delta} A_q B_q + \frac{1}{r \delta \sin \theta} (1 - 3 \cos^2 \theta) A_q B_q \right]
\]

5.7 Divergence of a Tensor

\[
\nabla \cdot T = \hat{q} \left[ \nabla \cdot (T_{qq} \hat{q} + T_{pq} \hat{p} + T_{q\phi} \hat{\phi}) + \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) T_{pq} + \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) T_{pp} + \frac{3 \cos \theta}{r \delta} T_{q\phi} \right]
\]

\[
+ \hat{\rho} \left[ \nabla \cdot (T_{qp} \hat{q} + T_{pp} \hat{p} + T_{p\phi} \hat{\phi}) - \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) T_{pp} - \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) T_{pp} \right]
\]

\[
- \frac{1}{r \delta \sin \theta} (1 - 3 \cos^2 \theta) T_{q\phi} \]

\[
+ \hat{\phi} \left[ \nabla \cdot (T_{q\phi} \hat{q} + T_{p\phi} \hat{p} + T_{\phi\phi} \hat{\phi}) - \frac{3 \cos \theta}{r \delta} T_{q\phi} + \frac{1}{r \delta \sin \theta} (1 - 3 \cos^2 \theta) T_{q\phi} \right]
\]

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