An equilibrium problem on the sphere with two equal charges

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Abstract

We study the equilibrium measure on the two dimensional sphere in the presence of an external field generated by two equal point charges. The support of the equilibrium measure is known as the droplet. Brauchart et al. showed that the complement of the droplet consists of two spherical caps when the charges are small. When the charges are bigger the droplet becomes simply connected and we prove that the boundary of the droplet is mapped by stereographic projection to an ellipse in the plane.

Moreover, we compute a mother body for the droplet that we derive from an equilibrium problem with a weakly admissible external field on the real line.

1 Introduction and statement of results

1.1 Main result

Let us denote by $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ the two dimensional unit sphere, where $\| \cdot \|$ stands for the usual Euclidean norm. We use $\lambda$ for the Lebesgue measure on $S^2$, normalized so that $\lambda(S^2) = 1$, and for a closed subset $D \subset S^2$ we write $\lambda_D$ for the Lebesgue measure restricted to $D$.

An external field is a function $Q : S^2 \to \mathbb{R} \cup \{+\infty\}$ that is lower semicontinuous and finite at least on a set of positive logarithmic capacity. The weighted logarithmic energy in the presence of $Q$ of a probability measure $\mu$ on $S^2$ is given by

$$I_Q[\mu] = \iint_{S^2 \times S^2} \log \frac{1}{\|x - y\|} d\mu(x) d\mu(y) + 2 \int_{S^2} Q(x) d\mu(x). \quad (1.1)$$

Then $I_Q$ represents the total energy of a large system of mutually repelling particles with the external field $Q$ acting on them. It is a well known result (see [10, Theorem 1.2], but the proof is the same as in the case of a weighted logarithmic energy problem in the complex plane [21]) that there is a unique probability measure $\mu_Q$, called the equilibrium measure, that minimizes (1.1) among all probability measures on $S^2$.

The equilibrium measure is characterized by Frostman-type [11] variational conditions which say that for some constant $\ell_Q$,

$$\begin{cases} Q + U^{\mu_Q} \leq \ell_Q & \text{on } \text{supp}(\mu_Q), \\ Q + U^{\mu_Q} \geq \ell_Q & \text{quasi-everywhere on } S^2, \end{cases} \quad (1.2)$$
where quasi-everywhere means up to a set of zero logarithmic capacity, and $U^{\mu Q}$ is the logarithmic potential, which for general positive measures $\sigma$ is defined by

$$U^{\sigma}(x) = \int_{S^2} \log \frac{1}{\|x - y\|} d\sigma(y).$$

(1.3)

If there is no exceptional set of zero logarithmic capacity then (1.2) takes the form

$$\begin{align*}
Q + U^{\mu Q} &= \ell_Q \quad \text{on } \text{supp}(\mu Q), \\
Q + U^{\mu Q} &\geq \ell_Q \quad \text{on } S^2,
\end{align*}$$

(1.4)

and this will be the situation for the external fields considered in this paper.

The precise determination of the equilibrium measure or its support is usually a very hard problem for a general (and even for a not too trivial) choice of $Q$. The problem becomes somewhat more tractable if we consider external fields that are logarithmic potentials (1.3) of highly concentrated measures $\sigma$. We shall refer to $\sigma$ as a charge distribution and we use $\mu_\sigma$ to denote the equilibrium measure in the external field $U^\sigma$. We are going to determine $\mu_\sigma$ explicitly for the external field

$$Q(x) = U^\sigma(x), \quad \sigma = a\delta_{p_1} + a\delta_{p_2}$$

(1.5)

with $p_1, p_2 \in S^2$ and $a > 0$.

For discrete charge distributions $\sigma$ like the one in (1.5) it is known that

$$\text{supp}(\mu_\sigma) \cap \text{supp}(\sigma) = \emptyset$$

(1.6)

and whenever (1.6) holds the equilibrium measure takes the form

$$\mu_\sigma = \lambda(D)^{-1} \lambda_D,$$

(1.7)

where $D = \text{supp}(\mu_\sigma)$ is some compact set called the droplet. In other words, $\mu_\sigma$ has constant density with respect to Lebesgue measure on its support and therefore the droplet $D$ is all one needs to know in order to solve the equilibrium problem. See \cite{13} Theorem 11\footnote{It is Theorem 6.3 in the preprint \texttt{arXiv:1605.03102}} for a general version of this statement in terms of partial balayage on compact manifolds. Moreover, it follows from the general theory that

$$\lambda(D) = \frac{1}{1 + m(\sigma)},$$

(1.8)

where $m(\sigma) = \int d\sigma$ is the total mass of $\sigma$. For ease of reference we give the (simple) proof of (1.7) and (1.8) in the appendix A.

The case $\sigma = a\delta_p$, where $a > 0$ and $p \in S^2$, was studied in \cite{9}. For this $\sigma$ the droplet is the complement of a spherical cap $B$ centered at $p$ with area

$$\lambda(B) = \frac{a}{1 + a}.$$
A clever choice of the parameter $a$ leads in that article to a bound on the separation of minimal logarithmic energy points. A similar technique is employed in [10] and [4] to prove bounds on the separation of minimal Riesz $s$–energy points on the $d$–dimensional sphere $\mathbb{S}^d$, and in [7] to bound the separation of minimal Green energy points on compact manifolds. In [4] the authors consider the case where

$$\sigma = \sum_{j=1}^{m} a_j \delta_{p_j}, \quad \text{with } a_j > 0 \quad \text{and } p_j \in \mathbb{S}^2 \quad \text{for every } j,$$

is a combination of point masses (not only for the logarithmic case, but also for the Riesz interaction). They prove that if the $a_j$ are sufficiently small, then

$$D = \mathbb{S}^2 \setminus \left( \bigcup_{j=1}^{m} B_j \right) \quad (1.9)$$

where $B_j$ is the (open) spherical cap centered at the point $p_j$ with area

$$\lambda(B_j) = \frac{a_j}{1 + a_1 + \cdots + a_m}.$$ 

The result (1.9) holds if and only if the spherical caps are mutually disjoint.

The equilibrium problem becomes more complicated (and maybe more interesting) when some of the $a_j$ grow large enough so that two (or more) of the spherical caps start to overlap. In such a case the complement of the droplet is no longer a union of spherical caps, but some larger set as in Figure 1.
The goal of this paper is to study the symmetric instance of two point charges, i.e.,
\[ \sigma = a_1 \delta_{p_1} + a_2 \delta_{p_2} \] with \( a_1 = a_2 = a \) but the charge \( a \) can be arbitrarily large. In other words, for us
\[ Q(x) = U^{\sigma}(x) = a \log \frac{1}{\|x - p_1\|} + a \log \frac{1}{\|x - p_2\|}, \quad a > 0, \tag{1.10} \]
where \( p_1 \) and \( p_2 \) are two distinct points on \( S^2 \).

We are going to use the stereographic projection \( \phi : S^2 \to \mathbb{C} \cup \{\infty\} \) given by
\[ \phi(x_1, x_2, x_3) = \begin{cases} \frac{x_1 + ix_2}{1-x_3}, & \text{for } (x_1, x_2, x_3) \in S^2, x_3 \neq 1, \\ \infty, & \text{for } (x_1, x_2, x_3) = (0, 0, 1). \end{cases} \]
We assume, without loss of generality, that \( p_1 \) and \( p_2 \) are mapped to two purely imaginary complex numbers
\[ \phi(b_1) = i b, \quad \phi(p_2) = -i b, \quad \text{with } b \geq 1. \tag{1.11} \]
We achieve (1.11) by rotating the sphere such that \( p_1 = (0, x_2, x_3) \) and \( p_2 = (0, -x_2, x_3) \) are symmetric with respect to the north pole and \( x_3 \geq 0 \).

We consider \( b \) fixed and study the effect on the droplet when \( a \) varies. In this setting the critical value is
\[ a_{cr} = (b^2 - 1)^{-1} \tag{1.12} \]
in the sense that \( a \leq a_{cr} \) corresponds to the situation when the charge \( a \) is small enough so that the droplet is the complement of two spherical caps (see [4, Theorem 1]). For \( a > a_{cr} \) we find the surprising result that the stereographic projection of the droplet is an ellipse, and this is our main result.

**Theorem 1.1.** Let \( Q \) be given by (1.10) where \( p_1 \) and \( p_2 \) are two points on the unit sphere, that are mapped by stereographic projection to \( i b \) and \( -i b \), respectively, with \( b \geq 1 \). Suppose \( a > a_{cr} = (b^2 - 1)^{-1} \). Let \( D \) be the support of the equilibrium measure and \( \phi(D) = \Omega \). Then \( \Omega \) is the compact region in the complex plane enclosed by the ellipse with equation
\[ \frac{2(b^2a^2 - a - 1)}{b^2 + 1} x^2 + \frac{2(b^2a^2 + a + 1)}{b^2 - 1} y^2 = 1. \tag{1.13} \]

We are going to prove Theorem 1.1 by explicitly verifying the variational conditions (1.4). Namely, if \( \Omega \) is the region enclosed by the ellipse (1.13), and \( D = \phi^{-1}(\Omega) \) then we prove for some constant \( \ell_a \),
\[
(1 + 2a)U^{\lambda_D}(x) + a \log \frac{1}{\|x - p_1\|} + a \log \frac{1}{\|x - p_2\|} \begin{cases} = \ell_a & \text{on } D, \\ \geq \ell_a & \text{on } S^2. \end{cases} \tag{1.14}
\]
For \( a = a_{cr} \) the equation (1.13) reduces to \( \frac{4b^2}{(b^2 - 1)^2} y^2 = 1 \) and the domain \( \Omega \) tends to the horizontal strip
\[ \Omega_{cr} = \left\{ z \in \mathbb{C} : |\text{Im } z| \leq \frac{b^2 - 1}{2b} \right\}. \tag{1.15} \]
as \( a \to a_{cr}^+ \). Then \( D_{cr} = \phi^{-1}(\Omega_{cr}) \) is the complement of two spherical caps centered at \( p_1 \) and \( p_2 \) that are tangent at the north pole. The conditions (1.14) hold in this critical case, and this follows from the results in [4].
1.2 A dual weighted energy problem

We are able to compute the droplet $D$ because of duality between $(\sigma, D)$ and $(\sigma^*, D^*)$ where

$$D^* = S^2 \setminus D$$

and $\sigma^*$ is a measure on $D$ that will be such that

$$\mu_{\sigma^*} = \lambda(D^*)^{-1} \lambda_{D^*},$$

which is the relation dual to (1.7). In addition, the measure $\sigma^*$ will be highly concentrated, although not a finite combination of point masses. It will be supported on $\phi^{-1}(\mathbb{R} \cup \{\infty\})$ which is the great circle on $S^2$ containing those points of $S^2$ that have equal distance to $p_1$ and $p_2$, see Figure 1, where the support of $\sigma^*$ is represented by a dashed line inside the droplet. Because of (1.8) and $\lambda(D^*) = 1 - \lambda(D)$, we will have

$$m(\sigma) = 2a, \quad \lambda(D) = \frac{1}{1 + 2a},$$

$$m(\sigma^*) = \frac{1}{2a}, \quad \lambda(D^*) = \frac{2a}{1 + 2a}.$$ (1.17)

The measure $\sigma^*$ is related to a mother body (sometimes called potential theoretic skeleton [14]) for the domain $D$ in the sense that the relations (1.18)–(1.19) from the next theorem hold. The identity (1.18) specifies that, in the complement of $D$, the two probability measures $2a\sigma^*$ and $(1 + 2a)\lambda_D$ have the same logarithmic potentials, up to a constant.

**Theorem 1.2.** Let $a > 0$ be arbitrary. There is a positive measure $\sigma^*$ supported on $\phi^{-1}(\mathbb{R} \cup \infty)$ with total mass $m(\sigma^*) = \frac{1}{2a}$ such that for some constant $m$,

$$U^{\lambda_D}(x) = U^{\frac{2a}{1 + 2a}\sigma^*}(x) + m \quad \text{if } x \in S^2 \setminus D,$$

$$U^{\lambda_D}(x) \leq U^{\frac{2a}{1 + 2a}\sigma^*}(x) + m \quad \text{if } x \in D.$$ (1.18)

The measure $\sigma^*$ is given by

$$\sigma^* = \frac{1}{2a} (\phi^{-1})_* (\mu_V),$$ (1.20)

where $\mu_V$ is a probability measure on $\mathbb{R}$ that will be described in Theorem 1.6 below. Thus $2a\sigma^*$ is the pushforward by the inverse stereographic projection $\phi^{-1}$ of $\mu_V$.

The properties (1.18)–(1.19) of $\sigma^*$ express that the complement of the droplet solves the weighted energy problem for the measure $\sigma^*$.

**Corollary 1.3.** We have $\mu_{\sigma^*} = \lambda(D^*)^{-1} \lambda_{D^*}$. That is, if we set $Q^*(x) = U^\sigma(x)$ and $D^* = S^2 \setminus D$, then the probability measure $\mu_{\sigma^*} = \lambda(D^*)^{-1} \lambda_{D^*}$ minimizes the weighted energy functional $I_{Q^*}[\mu]$. 


Proof. From (1.18), (1.17), and the fact that \( U^\lambda = \ell_0 \) is constant on \( S^2 \), we find

\[
U^{\sigma^*} + U^{(D^*)^{-1}\lambda_{D^*}} = \frac{1 + 2a}{2a} (U^\lambda - m) + \frac{1 + 2a}{2a} U^{\lambda_{D^*}} \quad \text{on } D^* \quad (1.21)
\]

while (1.19) implies that inequality \( \geq \) in (1.21) holds on \( D^* \). Thus \( \lambda(D^*)^{-1}\lambda_{D^*} \) satisfies the variational conditions (1.4) associated with the weighted energy problem with external field \( U^{\sigma^*} \) and the corollary follows. \( \square \)

Remark 1.4. For \( \sigma \) and \( \lambda_{D^*} \), we have the similar relations, for some constant \( m^* \),

\[
U^{\lambda_{D^*}}(x) = U^{\frac{1}{1+2a}\sigma}(x) + m^* \quad \text{if } x \in S^2 \setminus D^*, \quad (1.22)
\]

\[
U^{\lambda_{D^*}}(x) \leq U^{\frac{1}{1+2a}\sigma}(x) + m^* \quad \text{if } x \in D^*, \quad (1.23)
\]

which follows from (1.14) with similar arguments as in the proof of Corollary 1.3. Thus

\[
\frac{1}{1 + 2a\sigma} = \frac{a}{1 + 2a}\delta_{p_1} + \frac{a}{1 + 2a}\delta_{p_2}
\]

is a mother body for the complement of the droplet.

Remark 1.5. The complement of the droplet can be shown to be a quadrature domain for subharmonic functions. This is, for every function \( \varphi \) continuous in \( S^2 \setminus D \) and subharmonic in \( S^2 \setminus D \), we have

\[
\int_{S^2 \setminus D} \varphi \, d\lambda \geq \frac{a}{1 + 2a} \varphi(p_1) + \frac{a}{1 + 2a} \varphi(p_2). \quad (1.24)
\]

In particular, if \( \varphi \) is harmonic in \( S^2 \setminus D \), then (1.24) becomes an equality. The inequality (1.24) for subharmonic functions is essentially a consequence of (1.22)–(1.23).

Likewise, it is a consequence of (1.18)–(1.19) that the droplet itself is also a quadrature domain in a generalized sense, namely

\[
\int_D \varphi \, d\lambda \geq \frac{2a}{1 + 2a} \int_{\phi^{-1}(\mathbb{R})} \varphi \, d\sigma^* \quad \text{for every } \varphi \text{ continuous in } D \text{ and subharmonic in the interior of } D.
\]

1.3 A weakly admissible external field on \( \mathbb{R} \)

We obtain \( \sigma^* \) from a weakly admissible external field on the real line. For measures \( \mu \) on the real line we continue to use the notation as in (1.1) and (1.3) for their weighted logarithmic energy and logarithmic potential, where now of course the integration is over the real line and \( \|x - y\| \) is simply the absolute value \( |x - y| \).

The equilibrium measure \( \mu_V \) in the external field \( V : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) is the unique minimizer of the energy functional

\[
I_V[\mu] = \int_{\mathbb{R} \times \mathbb{R}} \log \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y) + 2 \int_{\mathbb{R}} V(x) \, d\mu(x)
\]

among probability measures on \( \mathbb{R} \). We continue to use \( a_{cr} = (b^2 - 1)^{-1} \) as in (1.12).
Theorem 1.6. For $a > 0$ and $b \geq 1$, the equilibrium measure in the external field

$$V(x) = \frac{1}{2} a \log \left(x^2 + b^{-2}\right) - \frac{a}{2} \log \left(x^2 + b^2\right)$$  \hspace{1cm} (1.25)$$
on the $\mathbb{R}$ is the probability measure $\mu_V$ on $\mathbb{R}$, which is described as follows:

(i) If $a \leq a_{cr}$, then $\mu_V$ is the probability measure supported on the full real line with density

$$\frac{d\mu_V}{dx} = \left(1 + \frac{a - b^2 a}{\pi b(x^2 + b^2)(x^2 + b^{-2})}\right), \quad x \in \mathbb{R}. \hspace{1cm} (1.26)$$

(ii) If $a > a_{cr}$, then $\mu_V$ is the probability measure supported on $[-A, A] \subset \mathbb{R}$ with density

$$\frac{d\mu_V}{dx} = \frac{\sqrt{C} \sqrt{A^2 - x^2}}{\pi (x^2 + b^2)(x^2 + b^{-2})}, \quad x \in [-A, A], \hspace{1cm} (1.27)$$

where

$$A = \frac{b \sqrt{1 + 2a}}{\sqrt{b^2 a + a + 1 \sqrt{b^2 a - a - 1}}}, \hspace{1cm} (1.28)$$

$$C = \frac{(b^4 - 1)(b^2 a + a + 1)(b^2 a - a - 1)}{b^4}. \hspace{1cm} (1.28)$$

In case $a > a_{cr}$ it can be checked that $\pm A$ where $A$ is given by $A$ are the foci of the ellipse $A$. If $a = a_{cr}$, then $\mu_V$ reduces to

$$\frac{d\mu_V}{dx} = \frac{b^2 + 1}{\pi b(x^2 + b^2)(x^2 + b^{-2})}, \quad x \in \mathbb{R}, \hspace{1cm} (1.29)$$

and it can be checked that this is the limit of $\mu_V$ as $a \to a_{cr}^+$. The external field $V(x)$ is weakly admissible, since $V(x) - \frac{1}{2} \log(1 + x^2)$ has a finite limit as $|x| \to \infty$. For admissible external fields this limit is $+\infty$, and this is the main focus of the monograph [21] and many other works. The study of weakly admissible external fields started with Simeonov [22], see also the more recent papers [15], [20].

The external fields $V(x)$ belong to the external fields studied in [20]. The transition at $a_{cr} = (b^2 - 1)^{-1}$ is contained in [20, Theorem 3.14], where it is shown (in a different, but equivalent, setup) that the support of $\mu_V$ is a bounded interval $[-A, A]$ if and only if $a > a_{cr}$ and the support is the full real line otherwise. In the latter case, the equilibrium measure is a combination of balayage measures, namely

$$\mu_V = (1 + a)\widehat{\delta}_{ib^{-1}} \quad - a\widehat{\delta}_{ib} \quad \text{in case } a \leq a_{cr}, \hspace{1cm} (1.30)$$

where $\widehat{\delta}_{ic} = \text{Bal}(\delta_{ic}, \mathbb{R})$ is the balayage of the point mass at $ic, c > 0$, to the real line. It is known that this balayage has a (rescaled) Cauchy density

$$\widehat{d\delta}_{ic}(x) = \frac{c}{\pi(x^2 + c^2)} dx$$

and this indeed leads to the density $\mu_V$ for the equilibrium measure $\mu_V$ in case $a \leq a_{cr}$.

For $a > a_{cr}$ the density of the right-hand side of $\mu_V$ becomes negative near infinity and it is no longer the equilibrium measure.
1.4 Overview

The rest of the paper is organized as follows.

The precritical case $a \leq a_{cr}$ is discussed in section 2. Theorem 1.2 is a new result in this case as well, and it will be instructive to see from its proof how the equilibrium problem with external field $V$ arises in this case.

In section 3 we prove Theorem 1.6 about the equilibrium measure $\mu_V$ in particular for the case $a > a_{cr}$. We use the technique of Schiffer variations that has been used before, for example in [18], but we give a detailed exposition.

To prepare for the proof of Theorems 1.1 and 1.2 in case $a > a_{cr}$, we first collect a number of auxiliary results in section 4 about the transformation of logarithmic potentials under stereographic projection. It allows us to reformulate Theorems 1.1 and 1.2 in terms of equalities and inequalities for the logarithmic potentials of $\mu_V$ and a measure $\mu_\Omega$ on $\Omega$, see Proposition 4.1.

In sections 5 and 6 we prove these equalities and inequalities. The proof of equalities in section 5 makes use of a function $S(z)$ that we construct out of the equilibrium measure $\mu_V$ and that we call the spherical Schwarz function for the ellipse, since $\partial \Omega$ will be characterized by the equation

$$\frac{\bar{z}}{1 + |z|^2} = S(z),$$

see (5.2). The proof is by a reduction to the usual Schwarz function, but it is largely computational and we do not have a conceptual proof why the connection should hold. However, it is the key result that connects $\mu_V$ with the ellipse $\partial \Omega$. The equalities of Proposition 4.1 then follow from standard calculations around Schwarz functions where we use the complex Green’s theorem

$$\frac{1}{\pi} \int_{\Omega} f(z) dA(z) = \frac{1}{2\pi i} \oint_{\partial \Omega} \frac{\partial f}{\partial \bar{z}} dz,$$

where $dA$ denotes Lebesgue measure in the plane, to transform integrals over $\Omega$ to integrals over its boundary.

In section 5 we prove the inequalities in Proposition 4.1. Here we use a dynamical picture that could be of independent interest. We analyze the way that $\mu_\Omega$ and $\mu_V$ evolve in terms of a time parameter $t = \frac{1}{1 + 2a}$. Denoting the $t$-dependent quantities by $\Omega(t)$ and $V(t)$, we find that both $(t\mu_\Omega(t))_t$ and $(t\mu_V(t))_t$ are increasing families of measures. By taking derivatives with respect to $t$ we find measures $\rho_t = \frac{\partial}{\partial t}(t\mu_\Omega(t))$ and $\omega_t = \frac{\partial}{\partial t}(t\mu_V(t))$ that determine the evolution. This is similar to work of Buyarov and Rakhmanov [5] who considered such derivatives for equilibrium measures with varying masses.

The measure $\rho_t$ is supported on $\partial \Omega(t)$ and describes the growth of the ellipse which is comparable to Laplacian growth, see e.g. [14]. In our situation the projections $\pm ib$ of the points $p_1$ and $p_2$ act as repellers for the growth of the droplet in a somewhat similar way as in [1] for usual Laplacian growth.

As in [1] our model also allows a natural discretization that can be analyzed by polynomials that are orthogonal with respect to a measure on the complex plane, and the orthogonality can be rewritten as non-Hermitian orthogonality on a contour. We plan to come back to this in a separate publication.
2 The case $a \leq a_{cr}$: proof of Theorem 1.2

While our main interest is in the case $a > a_{cr}$ we first discuss the case $a \leq a_{cr}$. Theorem 1.6 follows from the results of [20] as already discussed after the statement of the theorem. In particular the measure $\mu_V$ is the combination of balayage measures [1.30].

Proof of Theorem 1.2 in case $a \leq a_{cr}$. In case $a \leq a_{cr}$ we know that $D = \mathbb{S}^2 \setminus (B_1 \cup B_2)$ where $B_1$ and $B_2$ are spherical caps, centered at $p_1$ and $p_2$ respectively and $\lambda(B_1) = \lambda(B_2) = \frac{a}{1 + 2a}$. The geodesic radius of $B_1$ and $B_2$ is thus

$$r_a = \arccos \left(1 - \frac{2a}{1 + 2a}\right).$$

We start by noting that

$$U^{\lambda_D}(x) = \int_{\mathbb{S}^2 \setminus (B_1 \cup B_2)} \log \frac{1}{\|x - y\|} d\lambda(y)$$

$$= \int_{\mathbb{S}^2 \setminus B_1} \log \frac{1}{\|x - y\|} d\lambda(y) - \int_{B_2} \log \frac{1}{\|x - y\|} d\lambda(y)$$

$$= \int_{\mathbb{S}^2 \setminus B_1} \log \frac{1}{\|x - y\|} d\lambda(y) - \int_{B_2} \log \frac{1}{\|x - y\|} d\lambda(y)$$

since $\lambda(\partial B_1) = 0$. Then we recall the following mean value property for the logarithmic potential (see for instance [2, Proposition 3.2]): If $B = B(p, r)$ is an open spherical cap centered at $p \in \mathbb{S}^2$ with geodesic radius $r \in (0, \pi)$, then

$$\int_B \log \frac{1}{\|x - y\|} d\lambda(y) = \lambda(B)U^{\delta_p}(x) + c(r), \quad x \in \mathbb{S}^2 \setminus B,$$  

(2.2)

where $c(r) = (1 + \cos(r)) \log \cos \frac{r}{2} - \frac{\cos r}{2} + \frac{1}{2}$ is a constant depending on $r$ only. It also follows from [2, Proposition 3.2] (after a little calculation) that

$$\int_B \log \frac{1}{\|x - y\|} d\lambda(y) < \lambda(B)U^{\delta_p}(x) + c(r), \quad x \in B.$$  

(2.3)

We apply (2.2) to $B_2 = B(p_2, r_a)$ and (2.2)–(2.3) to $\mathbb{S}^2 \setminus B_1 = B(-p_1, \pi - r_a)$, where $-p_1$ is the antipodal point to $p_1$. It follows that

$$U^{\lambda_D}(x) \begin{cases} 
= h_1(x), & x \in B_1, \\
\leq h_1(x), & x \in D,
\end{cases}$$

(2.4)

where

$$h_1(x) = \lambda(\mathbb{S}^2 \setminus \overline{B_1})U^{\delta_{-p_1}}(x) - \lambda(B_2)U^{\delta_{p_2}}(x) + c(\pi - r_a) - c(r_a),$$

$$= \frac{a}{1 + 2a} U^{\delta_{-p_1}}(x) - \frac{a}{1 + 2a} U^{\delta_{p_2}}(x) + c(\pi - r_a) - c(r_a).$$

(2.5)
Similarly,

\[ U^{\lambda_D}(x) \begin{cases} = h_2(x), & x \in B_2, \\ \leq h_2(x), & x \in D, \end{cases} \] (2.6)

with

\[ h_2(x) = \frac{1 + a}{1 + 2a} U^{\delta_{p_2}}(x) - \frac{a}{1 + 2a} U^{\delta_{p_1}}(x) + c(\pi - r_a) - c(r_a). \] (2.7)

Let \( \gamma \subset S^2 \) be the great circle containing all those points that are equidistant to \( p_1 \) and \( p_2 \). Then \( \gamma \) separates the sphere into two closed hemispheres \( H_1 \) and \( H_2 \) where \( H_1 \) contains \( p_1 \) and \( -p_2 \) and \( H_2 \) contains \( -p_1 \) and \( p_2 \). The balayage of a measure \( \sigma \) supported on one of the hemispheres onto \( \gamma \) is the unique measure \( \hat{\sigma} \) supported on \( \gamma \) with \( m(\hat{\sigma}) = m(\sigma) \) such that \( U^{\hat{\sigma}} - U^{\sigma} \) is constant on the other hemisphere. Thus, if

\[ \eta = \frac{1 + a}{1 + 2a} \delta_{-p_2} - \frac{a}{1 + 2a} \delta_{p_1}, \] (2.8)

then it follows from (2.4) and (2.5) that, for some constant \( m \),

\[ U^{\lambda_D}(x) \begin{cases} = U^\eta(x) + m, & x \in B_1, \\ \leq U^\eta(x) + m, & x \in H_1 \cap D. \end{cases} \] (2.9)

Because of symmetry (\( p_1 \) and \( p_2 \) lie symmetric with respect to \( \gamma \), and so do \( -p_1 \) and \( -p_2 \), which gives \( \delta_{\pm p_1} = \delta_{\pm p_2} \)), we also have

\[ \eta = \frac{1 + a}{1 + 2a} \delta_{-p_1} - \frac{a}{1 + 2a} \delta_{p_2}, \]

and by (2.6) and (2.7)

\[ U^{\lambda_D}(x) \begin{cases} = U^\eta(x) + m, & x \in B_2, \\ \leq U^\eta(x) + m, & x \in H_2 \cap D. \end{cases} \] (2.10)

with the same constant \( m \).

Then putting

\[ \sigma^* = \frac{1 + 2a}{2a} \eta \] (2.11)

we see from (2.9) and (2.10) that (1.18) and (1.19) are satisfied. Applying the stereographic projection to (2.8) we find, since \( \phi(-p_2) = \bar{ib}^{-1} \) and \( \phi(p_1) = ib \), and \( \gamma \) is mapped to \( \mathbb{R} \),

\[ (1 + 2a)\phi_a(\eta) = (1 + a) \text{Bal} (\delta_{\bar{ib}^{-1}}, \mathbb{R}) - a \text{Bal} (\delta_{ib}, \mathbb{R}) \]

which according to (1.30) is equal to the equilibrium measure \( \mu_V \), since \( a \leq a_{cr} \). Then (1.20) follows, and since \( \mu_V \) is a probability measure it also follows that \( \sigma^* \) is a positive measure supported on \( \gamma = \phi^{-1}(\mathbb{R} \cup \{\infty\}) \) with total mass \( \frac{1}{2a} \).

This completes the proof of Theorem 1.2 in the case \( a \leq a_{cr} \). \( \square \)
3 Proof of Theorem 1.6

Theorem 1.6 is proved by Orive, Sánchez Lara, and Wielonsky [20] in case \( a \leq a_{cr} \), as already noted. They also showed that for \( a > a_{cr} \) the support of \( \mu_V \) is an interval \([-A, A]\). Our task here is to compute \( A \) and prove that the density of \( \mu_V \) on \([-A, A]\) is equal to (1.27).

The external field (1.25) extends to an analytic function in a neighborhood of the real line that is also denoted by \( V \). Its derivative is a rational function that we consider in the full complex plane.

The following lemma is not new. It is a variation of Theorem 1.34 in [8] which deals with equilibrium problems on a finite interval instead of the full real line. The proof in [8] is based on an analysis of weighted Fekete points. The proof below uses the technique of Schiffer variations, which was used for example in [17] for equilibrium problems on the real line. The technique is also useful for equilibrium problems on contours in the complex plane [13], [16] as a way to construct contours with the so-called S-property [12], see also [10] and references therein. Since we feel it deserves to be better known we give a self-contained exposition.

Lemma 3.1. With

\[
R(z) := \left[ V'(z) \right]^2 - 2 \int_{\mathbb{R}} \frac{V'(z) - V'(s)}{z - s} d\mu_V(s) \tag{3.1}
\]

the following hold:

(a) \( R \) is a rational function with double poles at \( \pm ib \) and \( \pm ib^{-1} \) and

\[
R(z) = \left[ \int_{\mathbb{R}} \frac{d\mu_V(s)}{z - s} - V'(z) \right]^2, \quad z \in \mathbb{C} \setminus \text{supp}(\mu_V). \tag{3.2}
\]

(b) The equilibrium measure has support

\[
\text{supp} \mu_V = \{x \in \mathbb{R} : R(x) < 0\} \tag{3.3}
\]

and density

\[
\frac{d\mu_V(x)}{dx} = \frac{1}{\pi} \sqrt{R^{-}(x)}, \quad x \in \text{supp} \mu_V, \tag{3.4}
\]

where \( R^{-} = \max(0, -R) \) denotes the negative part of \( R \).

Proof. (a) By (1.25) we have

\[
V'(z) = \frac{1 + a}{z^2 + b^{-2}} - \frac{a}{z^2 + b^2} \tag{3.5}
\]

and then it is clear from (3.1) that \( R \) is a rational function with double poles at \( z = \pm ib \) and \( z = \pm ib^{-1} \). To prove (3.2), we apply Schiffer variations which goes as follows. Let \( h : \mathbb{R} \to \mathbb{R} \) be a bounded \( C^2 \) function and following [13] Lemma 3.1, we consider the family of measures \( \{\mu_{\varepsilon}\}_{\varepsilon \in \mathbb{R}} \), given by their action on a continuous function \( f \) by

\[
\int_{\mathbb{R}} f(s) d\mu_\varepsilon(s) = \int_{\mathbb{R}} f(s + \varepsilon h(s)) d\mu_0(s).
\]
Since $\mu_0 = \mu_V$ and each $\mu_\epsilon$ is a probability measure it follows that $\epsilon \mapsto I_V[\mu_\epsilon]$ has its minimum at $\epsilon = 0$. Observe that

\[
\int \int \log \frac{1}{|t - s|} d\mu_\epsilon(t) d\mu_\epsilon(s) = - \int \int \log |t - s + \epsilon[h(t) - h(s)]| d\mu_\epsilon(t) d\mu_\epsilon(s)
\]

\[
= - \int \int \left( \log |t - s| + \log \left| 1 + \epsilon \frac{h(t) - h(s)}{t - s} \right| \right) d\mu_\epsilon(t) d\mu_\epsilon(s)
\]

\[
= \int \int \log \frac{1}{|t - s|} d\mu_V(t) d\mu_V(s)
\]

\[
- \epsilon \int \int_{\mathbb{R}} \frac{h(t) - h(s)}{t - s} d\mu_V(t) d\mu_V(s) + o(\epsilon),
\]

as $\epsilon \to 0$, and

\[
2 \int V(s) d\mu_\epsilon(s) = \int_{\mathbb{R}} V(s + \epsilon h(s)) d\mu_V(s)
\]

\[
= 2 \int \left( V(s) + \epsilon h(s)V'(s) + O(\epsilon^2) \right) d\mu_V(s)
\]

\[
= 2 \int V(s) d\mu_V(s) + 2 \epsilon \int_{\mathbb{R}} h(s)V'(s) d\mu_V(s) + o(\epsilon),
\]

as $\epsilon \to 0$. Since $I[\mu_\epsilon] + 2 \int V d\mu_\epsilon$ has its minimum at $\epsilon = 0$, it follows from the above that

\[
\int \int \frac{h(t) - h(s)}{t - s} d\mu_V(t) d\mu_V(s) = 2 \int h(s)V'(s) d\mu_V(s).
\]

(3.6)

By taking real and imaginary parts separately, the identity (3.6) is also satisfied for a complex valued $h : \mathbb{R} \to \mathbb{C}$. In particular, taking

\[
h(s) = \frac{1}{z - s}
\]

for some $z \in \mathbb{C} \setminus \mathbb{R}$, we have that

\[
\left( \int \frac{d\mu_V(s)}{z - s} \right)^2 = 2 \int \frac{V'(s)}{z - s} d\mu_V(s)
\]

\[
= 2V'(z) \int \frac{d\mu_V(s)}{z - s} - 2 \int \frac{V'(z) - V'(s)}{z - s} d\mu_V(s).
\]

(3.7)

After bringing the first term on the right to the left, and completing the square we obtain (3.2) in view of the definition (3.1) of $R$.

(b) Observe from the square in (3.2) that $R(x) \geq 0$ for every $x \in \mathbb{R} \setminus \text{supp}(\mu_V)$, and therefore we certainly have

\[
\text{supp}(\mu_V) \subset \{ x \in \mathbb{R} : R(x) < 0 \}.
\]

Let us denote by

\[
F_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(s)}{z - s}
\]
the Stieltjes transform of a measure $\mu$. If $z$ is in a neighborhood of the real line where $V(z)$ is analytic, then (3.2) reads
\[
[F_{\mu V}(z) - V'(z)]^2 = R(z),
\]
from which we obtain
\[
F_{\mu V}(z) = V'(z) - R(z)^{1/2},
\]
where $R(z)^{1/2}$ denotes the appropriate analytic square root of $R(z)$. Since $\mu_V$ has no atoms and $V'(z)$ is real for real $z$, we can apply the Perron–Stieltjes inversion formula to (3.9) for an interval $[x_1, x_2]$ with $x_1 < x_2$. We obtain that
\[
\mu_V([x_1, x_2]) = \lim_{\delta \to 0^+} \int_{x_1}^{x_2} \text{Im} F_{\mu V}(x - i\delta)^{1/2} \, dx
= \lim_{\delta \to 0^+} \int_{x_1}^{x_2} \text{Im} R(x - i\delta)^{1/2} \, dx
= \frac{1}{\pi} \int_{x_1}^{x_2} \text{Im} R_{\text{sc}}^{1/2}(x) \, dx.
\]
(3.10)
Only the values $x \in [x_1, x_2]$ with $R(x) < 0$ contribute to the integral in (3.10), which gives (3.3) and then (3.4) follows from (3.10) as well.

We are now ready for the proof of Theorem 1.6.

**Proof of Theorem 1.6.** We consider $a > a_{cr}$, since the case $a \leq a_{cr}$ already follows from [20], as already noted.

From (3.1) and (3.5) we obtain $R(z) = O(z^{-4})$ as $z \to \infty$. Since $R$ has double poles at $\pm ib$ and $\pm ib^{-1}$ (and no other poles), we conclude that
\[
R(z) = \frac{pz^4 + qz^2 + r}{(z^2 + b^2)^2(z^2 + b^{-2})^2}
\]
for certain real coefficients $p$, $q$ and $r$. Thus $R$ has at most four zeros in $\mathbb{C}$. We are in the situation that the support of $\mu_V$ is a finite interval $[-A, A]$. We then see from (3.3) that $R$ has a sign change at $\pm A$, which means that $R$ has a zero of odd order at $\pm A$. The zeros are actually simple, since there are no more than four zeros in total (counting multiplicities), and by symmetry, the orders of the zeros in $A$ and $-A$ are the same.

There are at most two remaining zeros of $R$ in $\mathbb{C}$. Any zero in $\mathbb{C} \setminus \text{supp}(\mu_V)$ has even order because of the square in the formula (3.2). By symmetry, with any such zero, its negative and its complex conjugate are also zeros of the same order. So there are no zeros in $\mathbb{C} \setminus \text{supp}(\mu_V)$. Since
\[
\text{supp}(\mu_V) = [-A, A] = \{x \in \mathbb{R} : R(x) < 0\}
\]
any zero in $(-A, A)$ must be of even order. This rules out any additional zero, except maybe a double zero at $z = 0$. But then by (3.4) the equilibrium density behaves as $\sim c|x|$ as $x \to 0$, for some $c > 0$, and this is not possible. Indeed, if the support of the equilibrium measure is known then the density can be recovered by solving a singular integral equation.
and from the explicit formulas as in [21, Chapter IV, Theorem 3.2] one sees that the density is real analytic in the interior of its support, for a real analytic external field.

Thus, apart from $-A$, there are no other zeros of $R$ in $\mathbb{C}$, and it follows that $p = 0$, and

$$R(z) = \frac{C(z^2 - A^2)}{(z^2 + b^2)^2(z^2 + b^{-2})^2} \quad (3.11)$$

for certain positive constant $C > 0$. From (3.8) we find,

$$\lim_{z \to \pm ib} (z^2 + b^2)^2 R(z) = \lim_{z \to \pm ib} (z^2 + b^2)^2 V'(z)^2,$$

$$\lim_{z \to \pm ib^{-1}} (z^2 + b^{-2})^2 R(z) = \lim_{z \to \pm ib^{-1}} (z^2 + b^{-2})^2 V'(z)^2, \quad (3.12)$$

which in view of (3.11) and (3.5) leads to the equations

$$C(-b^2 - A^2) (-b^2 + b^{-2}) = -a^2 b^2,$$

$$C(-b^{-2} - A^2) (-b^{-2} + b^2) = -(1 + a)^2 b^{-2}.$$ 

This simplifies to two equations for $C$ and $A^2$, namely

$$Cb^4 + CA^2 b^2 = a^2 (b^4 - 1)^2,$$

$$Cb^4 + CA^2 b^6 = (1 + a)^2 (b^4 - 1)^2,$$

with solution (1.28), since $A > 0$. This completes the proof of Theorem 1.6 in case $a > a_{cr}$.

4 Preparation for the proofs of Theorem 1.1 and 1.2 in case $a > a_{cr}$

4.1 Stereographic projection

The pushforward $\phi_\ast(\mu)$ of a measure $\mu$ on $S^2$ is the measure on $\mathbb{C} \cup \{\infty\}$ that assigns to a Borel set $B$ the value $\phi_\ast(\mu)(B) = \mu(\phi^{-1}(B))$. The pushforward of the normalized Lebesgue measure is

$$\phi_\ast(\lambda) = \frac{dA(z)}{\pi(1 + |z|^2)^2} \quad (4.1)$$

where $dA$ denotes the two dimensional Lebesgue measure in the plane, and so for every $D$

$$\phi_\ast(\lambda_D) = \frac{dA(z)}{\pi(1 + |z|^2)^2} \bigg|_\Omega,$$ 

where $\Omega = \phi(D). \quad (4.2)$

We will need to know how logarithmic potentials and weighted logarithmic energies transform under the stereographic projection. This can be computed from the basic formula for the transformation of distances. If $x, y \in S^2$, $z = \phi(x)$, $w = \phi(y)$, then

$$\|x - y\| = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}.$$
with proper modification in case $z$ or $w$ is at infinity. For a measure $\mu$ on $\mathbb{S}^2$ with pushforward measure $\phi_*\mu$ we thus have if $z = \phi(x)$,

$$U^\mu(x) = \int \log \frac{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}{2|z - w|} d\phi_\ast \mu(w)$$

$$= U^{\phi_* \mu}(x) + \frac{m(\mu)}{2} \log(1 + |z|^2) + \frac{1}{2} \int \log(1 + |w|^2) d\phi_* \mu(w) - m(\mu) \log 2,$$  \hspace{1cm} \text{(4.3)}

and

$$I[\mu] = \iint \log \frac{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}{2|z - w|} d\phi_* \mu(z) d\phi_* \mu(w)$$

$$= I[\phi_* \mu] + m(\mu) \int \log(1 + |z|^2) d\phi_* \mu(z) - m(\mu)^2 \log 2.$$ \hspace{1cm} \text{(4.4)}

Hence, in the presence of an external field $Q$ we have (for a probability measure $\mu$)

$$I_Q[\mu] = I_{\hat{Q}}[\phi_* \mu] - \log 2,$$ \hspace{1cm} \text{(4.5)}

with

$$\hat{Q}(z) = Q(\phi^{-1}(z)) + \frac{1}{2} \log(1 + |z|^2).$$ \hspace{1cm} \text{(4.6)}

If $\mu$ is the equilibrium measure with external field $Q$ on $\mathbb{S}^2$ then $\phi_* \mu$ is the equilibrium measure with external field $\hat{Q}$ on $\mathbb{C}$. Conversely, if $\mu_V$ is the equilibrium measure with external field $V$ on $\mathbb{C}$ (or on $\mathbb{R}$) then $(\phi^{-1})_* \mu_V$ is the equilibrium measure with external field

$$V \circ \phi - \frac{1}{2} \log (1 + \phi^2) \text{ on } \mathbb{S}^2 \text{ (or on } \phi^{-1}(\mathbb{R})).$$

### 4.2 Transformation of the theorems to $\mathbb{C}$

Since we prefer to do the calculations in $\mathbb{C}$ rather than on $\mathbb{S}^2$, we first transform Theorems 1.1 and 1.2 to statements about logarithmic potentials in the complex plane. We continue with the case $a > a_{cr}$.

Let $\partial \Omega$ be the ellipse from Theorem 1.1 and define $\mu_\Omega$ on $\Omega$ by

$$d\mu_\Omega(z) := \frac{1 + 2a}{\pi} \frac{dA(z)}{(1 + |z|^2)^2} \bigg|_\Omega,$$ \hspace{1cm} \text{(4.7)}

where $dA(z)$ is the usual area measure on $\mathbb{C}$.

Theorems 1.1 and 1.2 for $a > a_{cr}$ then follow from the following

**Proposition 4.1.** Let $a > a_{cr}$. Then $\mu_\Omega$ is a probability measure on $\Omega$ whose logarithmic potential satisfies

$$U^{\mu_\Omega}(z) \begin{cases} U^{\mu_V}(z), & z \in \mathbb{C} \setminus \Omega, \\ \leq U^{\mu_V}(z), & z \in \mathbb{C} \end{cases},$$ \hspace{1cm} \text{(4.8)}

and, with some constant $c$,

$$U^{\mu_\Omega}(z) + a \log \frac{1}{|z^2 + b^2|} + \frac{1 + 2a}{2} \log (1 + |z|^2) \begin{cases} = c, & z \in \Omega, \\ \geq c, & z \in \mathbb{C}. \end{cases}$$ \hspace{1cm} \text{(4.9)}
The proof of Proposition 4.1 is in sections 5 and 6 below. To see how Theorems 1.1 and 1.2 follow from Proposition 4.1, we put $D = \phi^{-1}(\Omega)$. Under inverse stereographic projection, the measure $\mu_\Omega$ transforms by (4.2) and (4.7) to

$$(\phi^{-1})_*(\mu_\Omega) = (1 + 2a)\lambda_D.$$  

Then $\lambda(D) = (1 + 2a)^{-1}$ since according to Proposition 4.1 $\mu_\Omega$ is a probability measure and then (4.10) is a probability measure as well.

By (4.3) we have if $x \in S^2$ and $z = \phi(x)$,

$$(1 + 2a)U^\lambda_D(x) = U^{\mu_\Omega}(z) + \frac{1}{2} \log(1 + |z|^2) + C_1$$

for some constant $C_1$, and also since $\phi_*(\sigma) = a(\delta_\theta + \delta_{-\theta})$,

$$U^\sigma(x) = aU^{\delta_\theta + \delta_{-\theta}}(z) + a \log(1 + |z|^2) + C_2$$

$$= a \log \frac{1}{|z^2 + b^2|} + a \log(1 + |z|^2) + C_2,$$

with another constant $C_2$. Combining (4.11) and (4.12) with (4.9) we obtain (4.14) which shows that $\mu_\sigma = (1 + 2a)\lambda_D$ and so indeed $D = \phi^{-1}(\Omega)$ is the droplet as claimed in Theorem 1.1.

Theorem 1.2 follows similarly from (4.8). We use (1.20) to define $\sigma^*$ in terms of $\mu_\Omega$. Then $\sigma^*$ is a measure on $\phi^{-1}(\mathbb{R})$ with total mass $m(\sigma) = \frac{1}{2a}$. Transforming the equality and inequality in (4.8) back to the sphere by means of (4.3) we obtain (1.18) and (1.19) as required for Theorem 1.2.

## 5 Proof of the equalities in Proposition 4.1

In this section we prove the equalities from Proposition 4.1. The inequalities will be dealt with in the next section.

### 5.1 Spherical Schwarz function

For the proof of the equalities we make use of the function

$$S(z) = \frac{1}{1 + 2a} \left[ \frac{2az}{z^2 + b^2} + \int \frac{d\mu_\Omega(s)}{z - s} \right]$$

that we call the *spherical Schwarz function* because of the following property.

**Proposition 5.1.** The ellipse $\partial \Omega$ is characterized by the equation

$$\partial \Omega : \frac{\bar{z}}{1 + |z|^2} = S(z)$$
Proof. If in the equation for an ellipse \( \frac{x^2}{p^2} + \frac{y^2}{q^2} = 1, \) we write \( x = \frac{z + \bar{z}}{2}, \) \( y = \frac{z - \bar{z}}{2i}, \) and solve for \( \bar{z}, \) then we obtain

\[
\bar{z} = S_0(z), \quad S_0(z) = \frac{(p^2 + q^2)z - 2pq(z^2 - r^2)^{1/2}}{r^2},
\]

(5.3)

where \( r^2 = p^2 - q^2, \) as the equation for the ellipse. Thus \( S_0 \) is the well-known usual Schwarz function for the ellipse. For the ellipse (1.13) from Theorem 1.1 we have

\[
p^2 = \frac{b^2 + 1}{2(b^2 a - a - 1)}, \quad q^2 = \frac{b^2 - 1}{2(b^2 a + a + 1)}, \quad r^2 = A^2,
\]

(5.4)

with \( A \) as in (1.28).

From (5.3) we find

\[
\partial \Omega : \frac{\bar{z}}{1 + |z|^2} = \frac{S_0(z)}{1 + zS_0(z)}
\]

and to obtain (5.2) we will have to verify that

\[
\frac{S_0(z)}{1 + zS_0(z)} = S(z).
\]

(5.5)

The identity (5.5) can be checked by straightforward calculations. By (1.13), (3.9), (3.5), and (3.11), we have

\[
(1 + 2a)S(z) = \frac{2az}{z^2 + b^2} + V'(z) - R(z)^{1/2}
\]

\[
= \frac{az}{z^2 + b^2} + \frac{(1 + a)z}{z^2 + b^2} - \frac{\sqrt{C(z^2 - A^2)^{1/2}}}{(z^2 + b^2)(z^2 + b^2)}.
\]

(5.6)

Hence both \( S(z) \) and \( S_0(z) \) are of the form \( P(z) + Q(z)(z^2 - A^2)^{1/2} \) with rational \( P \) and \( Q. \) To prove (5.5) we expand \((1 + zS_0(z))S(z)\) into this form and verify that it is equal to \( S_0(z) \) from (5.3) with parameters (5.4). We also need that \( A \) and \( C \) are given by (1.28). It all fits rather miraculously, and we obtain (5.5) and then also the proposition.

5.2 About the identity (5.5)

The identity (5.5) is rather surprising, and indeed it is more special than it may seem at first sight. The identity implies of course that the zeros and poles of both sides are the same. From (5.6) it follows that \( S \) has poles at \( z = \pm ib, \) while poles of the left-hand side of (5.5) can only come from zeros of \( z \mapsto 1 + zS_0(z). \) It is not immediate that \( z = \pm ib \) is a zero of \( z \mapsto 1 + zS_0(z), \) but it can be verified by explicit calculation.

Extending this idea, we can in fact prove (5.5) by examining the zeros and poles of \( S(z), \) \( S_0(z) \) and \( 1 + zS_0(z) \) on the two sheeted Riemann surface associated with \( w^2 = z^2 - A^2. \) Both \( S \) and \( S_0 \) have analytic continuation to the second sheet of this Riemann surface, just by taking the different sign of the square roots in (5.3) and (5.6). Then the following can be checked:
(1) $S_0$ has simple zeros at the points $z = \pm i \frac{2pq}{r}$ on the first sheet of the Riemann surface, simple poles at the two points at infinity, and no other zeros or poles.

(2) $z \mapsto 1 + zS_0(z)$ has double poles at the two points at infinity, four simple zeros at the points

$$
\pm i \sqrt{p^2 + q^2 + 2p^2q^2 + 2pq\sqrt{1 + p^2\sqrt{1 + q^2}}} \quad \text{on the first sheet,}
$$

$$
\pm i \sqrt{p^2 + q^2 + 2p^2q^2 - 2pq\sqrt{1 + p^2\sqrt{1 + q^2}}} \quad \text{on the second sheet,}
$$

and no other zeros or poles.

(3) $S$ has four simple zeros and four simple poles. The poles are at $\pm ib$ on the first sheet and $\pm ib^{-1}$ on the second sheet. The zeros are at

$$
\pm i \frac{\sqrt{b^4 - 1}}{b\sqrt{1 + 2a}} \quad \text{on the first sheet,}
$$

and at the two points at infinity.

It follows from (1) and (2) that

(4) $z \mapsto \frac{S_0(z)}{1 + zS_0(z)}$ has simple zeros at $\pm i \frac{2pq}{r}$ on the first sheet, two simple zeros at the points at infinity, and four simple poles at the points in (5.7).

This agrees with the zeros and poles from (3) provided that

$$
\frac{\sqrt{b^4 - 1}}{b\sqrt{1 + 2a}} = \frac{2pq}{r},
$$

$$
b^2 = \frac{p^2 + q^2 + 2p^2q^2 + 2pq\sqrt{1 + p^2\sqrt{1 + q^2}}}{r^2},
$$

$$
b^{-2} = \frac{p^2 + q^2 + 2p^2q^2 - 2pq\sqrt{1 + p^2\sqrt{1 + q^2}}}{r^2},
$$

and these identities are indeed consequences of the formulas (5.4) for $p$, $q$ and $r$.

Thus both sides of (5.5) are meromorphic functions on the compact Riemann surface with the same zeros and poles, and as a result their ratio is a constant. The constant is one, since both functions behave as $z^{-1} + O(z^{-2})$ as $z \to \infty$, as is easy to check from (5.1) and (5.3).

5.3 Proof of the equalities

**Lemma 5.2.** $\mu_\Omega$ is a probability measure on $\Omega$ whose Stieltjes transform satisfies

$$
\int_\Omega \frac{d\mu_\Omega(s)}{z - s} = \int_\mathbb{R} \frac{d\mu_\nu(s)}{z - s}, \quad z \in \mathbb{C} \setminus \Omega, \quad (5.8)
$$

$$
\int_\Omega \frac{d\mu_\Omega(s)}{z - s} = -\frac{2az}{z^2 + b^2} + \frac{(1 + 2a)\bar{z}}{1 + |z|^2}, \quad z \in \Omega. \quad (5.9)
$$
Proof. Consider \( z \in \mathbb{C} \setminus \Omega \) first. Then, by the definition \([4.7]\) of \( \mu_\Omega \) and Green’s Theorem in the complex plane \([1.31]\),

\[
\int_\Omega \frac{d\mu_\Omega(s)}{z - s} = \frac{1 + 2a}{\pi} \int_\Omega \frac{dA(s)}{(z - s)(1 + |s|^2)^2}
\]

\[
= \frac{1 + 2a}{2\pi i} \oint_{\partial \Omega} \frac{\bar{s}}{(z - s)(1 + |s|^2)} ds.
\]

Here we use the property \([5.2]\) of the spherical Schwarz function, and we find

\[
\int_\Omega \frac{d\mu_\Omega(s)}{z - s} = 1 + 2a \int_{\partial \Omega} \frac{S(s)}{z - s} ds
\]

(5.10)

to which we apply the Residue Theorem for the exterior region as in the proof of \([5.8]\), but now there is no contribution from infinity. Then \([5.8]\) follows because of \([5.1]\).

Letting \( z \to \infty \) in \([5.8]\) we find \( m(\mu_\Omega) = m(\mu_V) = 1 \), and therefore \( \mu_\Omega \) is a probability measure, as it is clearly positive from \([4.7]\). It remains to prove \([5.9]\).

Let \( z \in \Omega \setminus \partial \Omega \) and let \( r > 0 \) be such that the open disk \( D_r(z) = \{ w \in \mathbb{C} : |z - w| < r \} \) is contained in \( \Omega \setminus \partial \Omega \). Then, by the definition of \( \mu_\Omega \) in \([4.7]\) and the complex Green’s Theorem \([1.31]\),

\[
\int_{\Omega \setminus D_r(z)} \frac{d\mu_\Omega(s)}{z - s} = \frac{1 + 2a}{\pi} \int_{\Omega \setminus D_r(z)} \frac{dA(s)}{(z - s)(1 + |s|^2)^2}
\]

\[
= \frac{1 + 2a}{2\pi i} \oint_{\partial D_r(z)} \frac{\bar{s}}{(z - s)(1 + |s|^2)} ds - \frac{1 + 2a}{2\pi i} \oint_{\partial D_r(z)} \frac{s}{(z - s)(1 + |s|^2)} ds.
\]

The integral over \( \partial \Omega \) is evaluated using the spherical Schwarz function and the Residue Theorem for the exterior region as in the proof of \([5.8]\), but now there is no contribution from the pole at \( s = z \). We find

\[
\frac{1 + 2a}{2\pi i} \oint_{\partial \Omega} \frac{\bar{s}}{(z - s)(1 + |s|^2)} ds = \frac{1 + 2a}{2\pi i} \oint_{\partial \Omega} \frac{S(s)}{z - s} ds = \frac{2az}{z^2 + b^2}
\]

from the residues at \( s = \pm ib \). In the integral over the circle \( \partial D_r(z) \) we write \( s = z + re^{i\theta} \),

\[
- \frac{1 + 2a}{2\pi i} \oint_{\partial D_r(z)} \frac{\bar{s}}{(z - s)(1 + |s|^2)} ds = \frac{1 + 2a}{2\pi} \int_0^{2\pi} \frac{\bar{z} + re^{-i\theta}}{1 + |z + re^{i\theta}|^2} d\theta
\]

\[
= \frac{(1 + 2a)|z|}{1 + |z|^2} + \mathcal{O}(r) \quad \text{as } r \to 0^+.
\]

Letting \( r \to 0^+ \) we find \([5.9]\) for \( z \in \Omega \setminus \partial \Omega \), and then by continuity also for \( z \in \partial \Omega \). \( \square \)
After integration we obtain from Lemma 5.2 the desired equalities. In the proof we use that for any measure \( \mu \) on \( \mathbb{C} \) one has

\[
-2 \frac{\partial}{\partial z} U^\mu(z) = \int \frac{d\mu(s)}{z - s}, \quad z \in \mathbb{C} \setminus \text{supp}(\mu). \tag{5.11}
\]

**Proof of the equalities in (4.8) and (4.9).** In view of (5.11) we find from (5.8) that \( \frac{\partial}{\partial z} U^{\mu_\Omega} = \frac{\partial}{\partial z} U^{\mu_V} \) on \( \mathbb{C} \setminus \Omega \). Since both \( U^{\mu_\Omega} \) and \( U^{\mu_V} \) are real-valued,

\[
\frac{\partial}{\partial \bar{z}} U^{\mu_\Omega}(z) = \frac{\partial}{\partial \bar{z}} U^{\mu_\Omega}(z) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} U^{\mu_V}(z) = \frac{\partial}{\partial \bar{z}} U^{\mu_V}(z).
\]

and also the \( \bar{z} \)-derivatives coincide. We conclude that \( U^{\mu_\Omega} - U^{\mu_V} \) is constant on \( \mathbb{C} \setminus \Omega \). As both \( \mu_V \) and \( \mu_\Omega \) are probability measures, we have \( \lim_{z \to \infty} (U^{\mu_\Omega}(z) - U^{\mu_V}(z)) = 0 \). Therefore the constant is zero, and the equality in (4.8) for \( z \in \mathbb{C} \setminus \Omega \) follows.

Applying \(-2 \frac{\partial}{\partial z}\) to the left-hand side of (4.9) we obtain by (5.11)

\[
\int d\mu_\Omega(s) \frac{2az}{z^2 + b^2} - \frac{(1 + 2a)\bar{z}}{1 + |z|^2},
\]

which is 0 for \( z \in \Omega \), by (5.9). Also the \( \bar{z} \)-derivative vanishes in \( \Omega \), since the left-hand side of (4.9) is real-valued, and the equality in (4.9) for \( z \in \Omega \), follows for some integration constant \( c \).

\[
\square
\]

### 6 Proof of the inequalities in Proposition 4.1

In this section we prove the inequalities in Proposition 4.1. For the proof we introduce a dynamical picture in which we introduce a time-like parameter \( t \) and we investigate how the ellipses \( \Omega \) and the measures \( \mu_\Omega \) and \( \mu_V \) evolve with \( t \).

As before we consider \( b > 1 \) to be fixed and \( a > a_{cr} = (b^2 - 1)^{-1} \) will vary. To \( a \) we associate the parameter

\[
t_a = t = \frac{1}{1 + 2a}. \tag{6.1}
\]

Since \( \mu_\Omega \) is a probability measure we see from (4.7) that

\[
t = \int_{\Omega} \frac{dA(z)}{\pi(1 + |z|^2)^2}. \tag{6.2}
\]

If \( a \) decreases from \( \infty \) to \( a_{cr} \) then \( t \) increases from 0 to

\[
t_{cr} = \frac{b^2 - 1}{b^2 + 1} < 1. \tag{6.3}
\]
6.1 Family of measures \( \rho_t \)

For each \( t \in (0, t_{cr}) \), we use \( \Omega(t) \) to denote the region enclosed by the ellipse (1.13) with \( a = \frac{1-t}{2t} \), and (6.1) holds.

As \( t \) increases, \( a \) decreases and, since \( b > 1 \), the coefficients of \( x^2 \) and \( y^2 \) in (1.13) also decrease. Hence the two semi-axes of the ellipse increase as \( t \) increases in a strictly monotone and continuous way, starting from zero at \( t = 0 \). Therefore, we can write \( \Omega(t) \) as a disjoint union of ellipses

\[
\Omega(t) = \bigcup_{s \in [0,t]} \partial\Omega(s),
\]

where we set \( \Omega(0) := \{(0,0)\} \).

For each \( t \leq t_{cr} \), we have the measure \( \mu_{\Omega(t)} \) and by (4.7)

\[
td\mu_{\Omega(t)}(z) = \frac{dA(z)}{\pi(1+|z|^2)^2} |\Omega(t)|.
\]

Thus \( (t\mu_{\Omega(t)})_t \) is an increasing family of measures. Then the derivative

\[
\rho_t = \frac{\partial}{\partial t} (t\mu_{\Omega(t)}) = \lim_{h \to 0} \frac{(t+h)\mu_{\Omega(t+h)} - t\mu_{\Omega(t)}}{h}
\]

exists for almost every \( t \in (0, t_{cr}) \) (by general theory as in [5]), where the limit is in the weak* sense. In our case the limit in (6.5) exists for every \( t \in (0, t_{cr}) \) and defines a probability measure \( \rho_t \) that is supported on \( \partial\Omega(t) \). The co-area formula, see e.g. [6, Chapter IV.1, Theorem 1], provides an explicit expression for \( \rho_t \)

\[
d\rho_t(z) = \frac{1}{\pi |\text{grad} u(z)| (1+|z|^2)^2} d\nu_t(z), \quad z \in \partial\Omega(t),
\]

in terms of the arclength \( \nu_t \) on \( \partial\Omega(t) \), and \( u : \Omega(t_{cr}) \to \mathbb{R}^+ \) is the mapping that assigns \( u(z) = t \) to \( z \in \partial\Omega(t) \). We will not use the formula (6.6).

We recover \( \mu_{\Omega(t)} \) by integration of (6.5) (i.e., the Fundamental Theorem of Integral Calculus)

\[
t\mu_{\Omega(t)} = \int_0^t \rho_s ds.
\]

In particular

\[
tU^{\mu_{\Omega(t)}}(z) = \int_0^t U^{\rho_s}(z) ds, \quad z \in \mathbb{C}.
\]

**Lemma 6.1.** For every \( t \in (0, t_{cr}) \) we have that \( \rho_t \) is the balayage of \( \frac{1}{2}(\delta_{ib} + \delta_{-ib}) \) onto \( \Omega(t) \).

**Proof.** From (4.9) with \( t = \frac{1}{1+2a} \), we have with a constant \( c = c(t) \) that will depend on \( t \),

\[
tU^{\mu_{\Omega(t)}}(z) + \frac{1-t}{2} \log \frac{1}{|z|^2 + b^2} + \frac{1}{2} \log(1+|z|^2) = c(t), \quad z \in \Omega(t).
\]
Comparing this with (6.12) we see that
\[
\omega \text{ which is clearly positive, and it gives part (a) of Lemma 6.2. Part (b) also follows, where}
\]
with
\[
C_t \text{ on the parameter } t \text{.}
\]
Next we obtain a similar decomposition of \( \mu \).

6.2 Family of measures \( \omega_t \)

The property (6.9) characterizes the balayage measure, and the lemma follows.

Taking the derivative of (6.8) with respect to \( t \), we obtain
\[
U^{\rho_t}(z) = \frac{1}{2} U^{\delta_{ib} + \delta_{-ib}}(z) + c'(t), \quad z \in \Omega(t),
\] (6.9)
since
\[
U^{\delta_{ib} + \delta_{-ib}}(z) = \log \frac{1}{|z - ib|} + \log \frac{1}{|z + ib|} = \log \frac{1}{|z^2 + b^2|}.
\]
The property (6.9) characterizes the balayage measure, and the lemma follows.

**Proof.** The proof is by calculation. Let \( 0 < t < t_{cr} \). By (1.27) and (1.28) we have
\[
\pi(x^2 + b^2)(x^2 + b^2) \frac{d\mu_{V(t)}}{dx} = \sqrt{C(t)} \sqrt{A(t)^2 - x^2}, \quad x \in [-A(t), A(t)],
\]
with \( C(t) = \frac{b^2 - 1}{b^2} \frac{b^4 (1 - t)^2 - (1 + t)^2}{4t^2 (1 - t)^2} \) and \( A(t)^2 = \frac{b^4 t^2}{b^4 (1 - t)^2 - (1 + t)^2} \). Then for \( x \in (-A(t), A(t)) \),
\[
\frac{d}{dt} \left( t \frac{d\mu_{V(t)}}{dx} \right) = \frac{1}{2\pi \sqrt{A(t)^2 - x^2}} \left[ \frac{b \sqrt{A(t)^2 + b^2}}{x^2 + b^2} + \frac{b^{-1} \sqrt{A(t)^2 + b^{-2}}}{x^2 + b^{-2}} \right],
\] (6.12)
which is clearly positive, and it gives part (a) of Lemma 6.2. Part (b) also follows, where \( \omega_t \) is the probability measure with density (6.12) on \( [-A(t), A(t)] \).

The balayage of a point mass \( \delta_{ic}, c > 0 \), onto the real interval \( [-A, A] \) has the density
\[
\frac{d}{dx} \text{Bal}(\delta_{ic}, [-A, A]) = \frac{c \sqrt{A^2 + c^2}}{\pi (x^2 + c^2) \sqrt{A^2 - x^2}}.
\]
Comparing this with (6.12) we see that \( \omega_t \) is indeed the average of balayage measures as claimed in part (c).
It follows from part (b) of Lemma 6.2 that for every $t \in (0, t_{cr})$,

$$t \mu_{N(t)} = \int_0^t \omega_s ds$$

and in particular

$$t U^{\mu_{N(t)}}(z) = \int_0^t U^{\omega_s}(z) ds, \quad z \in \mathbb{C}. \quad (6.13)$$

which is the analogue of (6.7).

6.3 Proof of inequality in (4.8)

We let $t_a = \frac{1}{1 + 2a}$ so that $\Omega = \Omega(t_a)$ and $V = V(t_a)$.

Let $t < t_a$. Then by (4.8)

$$U^{\mu_{\Omega(t)}}(z) = U^{\mu_{V(t)}}(z), \quad z \in \mathbb{C} \setminus \Omega(t).$$

By (6.7) and (6.13) we then have

$$\int_0^t U^{\rho_s}(z) ds = \int_0^t U^{\omega_s}(z) ds, \quad z \in \mathbb{C} \setminus \Omega(t).$$

Taking the derivative with respect to $t$, we find

$$U^{\rho_t}(z) = U^{\omega_t}(z),$$

for $z \in \mathbb{C} \setminus \Omega(t)$, and by continuity of the logarithmic potentials also for $z \in \partial \Omega(t)$. Since $\rho_t$ is a probability measure on $\partial \Omega(t)$, we conclude that $\rho_t$ is the balayage of $\omega_t$ onto $\partial \Omega(t)$.

In the interior of $\Omega(t)$, the logarithmic potential $U^{\rho_t}$ is harmonic, since $\rho_t$ is supported on $\partial \Omega(t)$, while $U^{\omega_t}$ is superharmonic. Thus $U^{\omega_t} - U^{\rho_t}$ is superharmonic in the interior and zero on the boundary of $\Omega(t)$. By the Minimum Principle for superharmonic functions it then follows that $U^{\rho_t} \leq U^{\omega_t}$ on $\Omega(t)$, and then in fact on all of $\mathbb{C}$.

Then integrating the inequality from 0 to $t_a$ and using (6.7) and (6.13) we obtain the inequality in (4.8).

6.4 Proof of inequality in (4.9)

Let again $t_a = \frac{1}{1 + 2a}$ so that $\Omega = \Omega(t_a)$. The inequality (4.9) comes down to proving

$$t U^{\mu_{\Omega(t_a)}}(z) + \frac{1 - t}{2} \log \frac{1}{|z^2 + b^2|} + \frac{1}{2} \log(1 + |z|^2) \geq c(t), \quad z \in \mathbb{C}, \quad (6.14)$$

for $t = t_a$, where the constant $c(t)$ is such that equality holds on $\Omega(t)$, see also (6.8).

As $t \to t_{cr}$ we recall that $\Omega(t)$ tends to the horizontal strip (1.15) and

$$\mu_{\Omega(t)} \to \mu_{\Omega(t_{cr})} = \frac{dA(z)}{\pi(1 + |z|^2)^2} \bigg|_{\Omega_{cr}}.$$

The inequality (6.14) with $t = t_{cr}$ follows from the work of Brauchart et al. [3]
Recall from (6.9) that we have
\[ U^\rho_t(z) = \frac{1}{2} \log \frac{1}{|z^2 + b^2|} + c'(t), \quad z \in \Omega(t). \]
Outside \( \Omega(t) \) we have inequality
\[ U^\rho_t(z) \leq \frac{1}{2} \log \frac{1}{|z^2 + b^2|} + c'(t), \quad z \in \mathbb{C} \setminus \Omega(t), \tag{6.15} \]
as can be seen from the Minimum Principle for superharmonic functions applied to the function \( z \mapsto \frac{1}{2} \log \frac{1}{|z^2 + b^2|} \) on \( \mathbb{C} \setminus \Omega(t) \).

Then we have by (6.7)
\[
t_a U^\mu_{\Omega(t_a)}(z) = t_c U^\mu_{\Omega(t_c)}(z) - \int_{t_a}^{t_c} U^\rho_t(z) dt.
\]
Using (6.15) and the fact that (6.14) holds for \( t_c \), we can estimate
\[
t_a U^\mu_{\Omega(t_a)}(z) \geq -\frac{1 - t_c}{2} \log \frac{1}{|z^2 + b^2|} - \frac{1}{2} \log(1 + |z|^2) + c(t_c) - \int_{t_a}^{t_c} \left( \frac{1}{2} \log \frac{1}{|z^2 + b^2|} + c'(t) \right) dt
\]
which is (6.14) with \( t = t_a \).

A Proof of (1.7) and (1.8)

**Lemma A.1.** Suppose \( \sigma \) is a measure on \( S^2 \) such that (1.6) hold. Let \( D = \text{supp}(\mu_\sigma) \). Then (1.7) and (1.8) hold.

**Proof.** It is well-known that for any measure \( \mu \) on \( S^2 \),
\[
\frac{1}{2\pi} \Delta U^\mu = \mu - m(\mu)\lambda
\]
in the distributional sense, where we use \( \Delta = -\text{div} \text{ grad} \) for the Riemannian Laplacian on \( S^2 \). We apply this to \( \sigma \) and \( \mu_\sigma \) to find
\[
\frac{1}{2\pi} \Delta U^\sigma = \sigma - m(\sigma)\lambda, \quad \frac{1}{2\pi} \Delta U^{\mu_\sigma} = \mu_\sigma - \lambda, \tag{A.1}
\]
By the variational conditions (1.4) the sum \( U^\sigma + U^{\mu_\sigma} \) is constant on the interior \( \text{int}(D) \), and therefore \( \Delta(U^\sigma + U^{\mu_\sigma}) = 0 \) on \( \text{int}(D) \). Thus by (A.1)
\[
\sigma - m(\sigma)\lambda + \mu_\sigma - \lambda = 0 \quad \text{on } \text{int}(D).
\]
Since \( \text{supp}(\sigma) \cap D = \emptyset \) this implies that \( \mu_\sigma = (1 + m(\sigma))\lambda \) on \( \text{int}(D) \). From [13, Theorem 11] (or Theorem 6.3 in the arXiv version) it follows that \( \lambda(\partial D) = 0 \) and hence \( \mu_\sigma = (1 + m(\sigma))\lambda \) on \( D \). Since \( \mu_\sigma \) is a probability measure on \( D \) we then have
\[
\mu_\sigma = (1 + m(\sigma))\lambda_D
\]
and \( 1 = m(\mu_\sigma) = (1 + m(\sigma))\lambda(D) \). Then (1.7) and (1.8) follow. \( \square \)
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