CLASS OF INTEGRALS AND APPLICATIONS OF FRACTIONAL KINETIC EQUATION WITH THE GENERALIZED MULTI-INDEX BESSEL FUNCTION

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Abstract. In this article, we have investigated certain definite integrals and various integral transforms of the generalized multi-index Bessel function, such as Euler transform, Laplace transform, Whittaker transform, K-transform and Fourier transforms. Also, found the applications of the problem on fractional kinetic equation pertaining to the generalized multi-index Bessel function using the Sumudu transform technique. Mittage-Leffler function is used to express the results of the solutions of fractional kinetic equation as well as its special cases. The results obtained are significance in applied problems of science, engineering and technology.

1. Introduction and Preliminaries. Fractional calculus is a division of mathematics which take care of the generalization of integration as well as differentiation to arbitrary (non-integer) order. Now a days, the recent research outputs are indicating that, the dimension of fractional order calculus (FOC) is tuned into the practical applications on science and engineering. Also, it is found that, the fractional calculus describes in models and respective other physical phenomena (see [6, 16, 18, 19, 22, 23, 24, 25, 26, 37, 38, 39, 45, 47, 48, 49, 50]).

Recently, Nisar et al. [33] was defined the generalized multi-index Bessel function (GMBF) as follow:

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\[ J_{(e_j),\gamma}^{(c)} [z] = \sum_{n=0}^{\infty} \prod_{j=1}^{m} \frac{c^n (\gamma)_{kn}}{\Gamma (\varepsilon_j n + \zeta_j + 1)} \frac{z^n}{n!}, \quad (m \in \mathbb{N}) \] (1)

for \( \varepsilon_j, \zeta_j, \gamma, b, c \in \mathbb{C} \) \((j = 1, 2, ..., m)\) be such that \( \sum_{j=1}^{m} \Re (\varepsilon_j) > \max \{0, \Re (k) - 1\} \) \(k > 0; \Re (\zeta_j) > 0\) and \( \Re (\gamma) > 0\).

where \((\gamma)_n\) denote the Pochhammer symbol defined in [41] as follow:

\[ (\gamma)_n = \frac{\Gamma (\gamma + n)}{\Gamma (\gamma)}, \quad (\gamma \in \mathbb{C} \setminus \{0\}, n = 0; \gamma \in \mathbb{C} \setminus \{0\}, \gamma \in \mathbb{C} \). \] (2)

Here, we consider some special cases of (1) following as:

(i) If we set \( c = b = 1 \) and the conditions on equation (1) are satisfied, at that time the GMBF breaks to

\[ J_{(e_j),\gamma}^{(c)} [z] = \sum_{n=0}^{\infty} \prod_{j=1}^{m} \frac{(\gamma)_{kn}}{\Gamma (\varepsilon_j n + \zeta_j + 1)} \frac{z^n}{n!}, \quad (m \in \mathbb{N}). \] (3)

(ii) If we put \( c = 1, b = -1 \) and the conditions on equation (1) are satisfied, then GMBF reduced to Saxena and Nishimoto [36] as

\[ J_{(e_j),\gamma}^{(c)} [z] = \sum_{n=0}^{\infty} \prod_{j=1}^{m} \frac{(-1)^n (\gamma)_{kn}}{\Gamma (\varepsilon_j n + \zeta_j + 1)} \frac{z^n}{n!}, \quad (m \in \mathbb{N}). \] (4)

(iii) When \( c = -1, b = 1 \) and the conditions on equation (1) are satisfied, then GMBF reduced to the result due to Choi and Agarwal [7] as

\[ J_{(e_j),\gamma}^{(c)} [z] = \sum_{n=0}^{\infty} \prod_{j=1}^{m} \frac{(\gamma)_{k0}}{\Gamma (\varepsilon_j n + \zeta_j + 1)} (-1)^n \frac{z^n}{n!}, \quad (m \in \mathbb{N}). \] (5)

The Watugala [51, 52] was introduced Sumudu transform in the form below:

\[ G (z) = \mathcal{S} [f (t); z] = \int_{0}^{\infty} e^{-zt} f (zt) dt, \quad z \in (\tau_1, \tau_2). \] (6)

The Wiman [53] was introduced Mittage-Leffler (M-L) function as:

\[ E_{\varepsilon,\zeta} (z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma (\varepsilon n + \zeta)}, \quad (z, \varepsilon, \zeta \in \mathbb{C}; \Re (\varepsilon) > 0, \Re (\zeta) > 0). \] (7)

2. Integrals of GMBF. For our purpose, we recall the following formulas which are defined by Qureshi et al. ([34], p.77, eq. (54)-(56)) as:

\[ \int_{0}^{\infty} \left[ \left( \sigma x + \frac{\beta}{x} \right)^2 + \delta \right]^{\zeta-1} dx = \frac{\sqrt{\pi}}{2 \sigma (4 \sigma \beta + \delta)^{\zeta+1/2}} \frac{\Gamma (\zeta + 1/2)}{\Gamma (\zeta + 1)}, \quad (\sigma > 0; \beta \geq 0; \delta + 4 \sigma \beta > 0; (\Re (\zeta) + 1/2) > 0). \] (8)

\[ \int_{0}^{\infty} \frac{1}{x^2} \left[ \left( \sigma x + \frac{\beta}{x} \right)^2 + \delta \right]^{\zeta-1} dx = \frac{\sqrt{\pi}}{2 \beta (4 \sigma \beta + \delta)^{\zeta+1/2}} \frac{\Gamma (\zeta + 1/2)}{\Gamma (\zeta + 1)}, \quad (\sigma \geq 0; \beta > 0; \delta + 4 \sigma \beta > 0; (\Re (\zeta) + 1/2) > 0). \] (9)
Also, we recall the definition of the Fox-Wright function \( p \). The following formulas ([40], p.75) will be also needed in our further investigation:

\[
\int_0^\infty \left( \sigma + \frac{\beta}{x} \right)^\delta 
\left( \frac{\sigma x + \beta}{x} \right)^2 \right)^{-\zeta - 1} \, dx = \frac{\sqrt{\pi}}{(4\sigma\beta + \delta)^{\zeta + 1/2}} \frac{\Gamma \left( \zeta + \frac{1}{2} \right)}{\Gamma (\zeta + 1)}. \tag{10}
\]

\( \sigma > 0; \beta > 0; \delta + 4\sigma\beta > 0; \Re(\zeta) > 1/2 \).

The following formulas ([40], p.75) will be also needed in our further investigation:

\[
(1 - x)^{\sigma + \beta - \delta} \, _2F_1 (2\sigma, 2\beta, 2\delta; x) = \sum_{r=0}^{\infty} \sigma_r \, x^r, \tag{11}
\]

and

\[
_2F_1 (\delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; x) \, _2F_1 (\sigma, \beta, \delta + \frac{1}{2}; x) = \sum_{r=0}^{\infty} \frac{(\delta)_r}{(\delta + \frac{1}{2})_r} \sigma_r \, x^r. \tag{12}
\]

Also, we recall the definition of the Fox-Wright function \( p,\Psi_q(z) \) \( p, q \in \mathbb{N}_0 \) (see, more details, [13, 55]):

\[
p,\Psi_q \left[ \left( \varepsilon_1, C_1 \right), \ldots, \left( \varepsilon_p, C_p \right) \right] \left( \beta_1, D_1 \right), \ldots, \left( \beta_q, D_q \right) \right] \left( z \right) = \sum_{n=0}^{\infty} \frac{\Gamma(\varepsilon_1 + C_1n) \cdots \Gamma(\varepsilon_p + C_pn)}{\Gamma(\beta_1 + D_1n) \cdots \Gamma(\beta_q + D_qn)} \frac{z^n}{n!} \tag{13}
\]

\( C_i \in \mathbb{R}^+ \) \( i = 1, \ldots, p \); \( D_j \in \mathbb{R}^+ \) \( i = 1, \ldots, q \); \( 1 + \sum_{i=1}^{q} D_j - \sum_{i=1}^{p} C_i > 0 \),

where convergence condition in appropriate way to holds true:

\[
|z| < \nabla := \left( \prod_{i=1}^{p} C_i \right) \cdot \left( \prod_{j=1}^{q} D_j \right).
\]

Since several centuries, extraordinary properties and various applications have been found in mathematical physics, computational mathematics and applied mathematics. Under numerous definite integral, one or more than one variable of special functions of image formulas are important from the setting of reference of the value of these significance’s in the valuation of extended integrals, applied in physics and in a various area of engineering. Inspired fundamentally by miscellaneous claims of these results, a numeral of integrals involving Srivastava’s polynomials and different of special functions have been established by numerous authors, see [29, 30, 43, 44]. Recently, via integrals (1)–(3), certain generalized Gradshteyn–Ryzhil type integrals have been established, see [3, 9, 42].

Here, we establish three theorems as:

**Theorem 2.1.** Let \( \varepsilon_1, v_1, c, b \in \mathbb{C} \) \( j = 1, 2, \ldots, m \), \( \sigma > 0; \beta > 0; \delta + 4\sigma\beta > 0; \Re(\lambda) + 1/2 > 0 - \frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2}; \) be such that \( \sum_{j=1}^{m} \Re (\varepsilon_j) > \max \{ 0, \Re (k) - 1 \}; k > 0; \Re (v_j) > 0 \) and \( \Re (\gamma) > 0 \), then the following integral formula holds:

\[
\int_0^\infty \zeta^{-\lambda - 1} \, _2F_1 (\sigma, \beta, \delta + \frac{1}{2}; \zeta) \, _2F_1 (\delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; \zeta) \, J_{(\varepsilon_j),m,k} (z)^c \frac{z}{\zeta} \, dx
\]

\[
= \frac{\sqrt{\pi}}{2\sigma (4\sigma\beta + \delta)^{\lambda + 1/2} \Gamma (\gamma)} \sum_{r=0}^{\infty} \frac{(\delta)_r}{(\delta + \frac{1}{2})_r} \frac{\sigma_r}{(4\sigma\beta + \delta)^{-r}}
\]
Proof. By virtue of equation (1), (8) and (12), we get the following

\[ \text{Theorem 2.2.} \]

In similar line of proof of Theorem 2.1, we can prove Theorem 2.2 and 2.3 as follows:

**Theorem 2.2.** Suppose \( \varepsilon_j, \zeta_j, \gamma, c, b \in \mathbb{C} \ (j = 1, 2, \ldots, m) \), \( \Re(\zeta_j) > 0, \Re(\gamma) > 0, \sum_{j=1}^{m} \Re(\varepsilon_j) > \max \{0, \Re(k)-1\}; k > 0 \), thereupon following integral formula holds appropriate:

\[
\int_0^\infty \mathbb{Z}^{-\lambda-1} 2F_1 (\sigma, \beta, \delta + \frac{1}{2}; \mathbb{Z}) 2F_1 (\delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; \mathbb{Z}) \mathcal{J}_{(\varepsilon_j)m, k, b}^{(\zeta_j)m, \gamma, c} \left[ \mathbb{Z}^{-1} \right] dx
\]

\[ = \frac{\sqrt{\pi}}{2\beta(4\sigma\beta + \delta)^{\lambda + 1/2} \Gamma(\gamma)} \sum_{r=0}^{\infty} \frac{(\delta)_r \sigma_r}{(\delta + \frac{1}{2})_r (4\sigma\beta + \delta)^{-r}} \times 2^{\psi_{m+1}} \left[ \left( \frac{\gamma, k}{(\lambda - r + \frac{1}{2}, 1)_{j=1}}, \lambda - r + 1, 1 \right) \frac{cz}{4\sigma\beta + \delta} \right]. \]

provided \( \mathbb{Z} = (\sigma x + \frac{\beta}{2})^2 + \delta \), such that \( \sigma \geq 0; \beta > 0; \delta + 4\sigma\beta > 0; \Re(\lambda) > -1/2 \) with \( -\frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2} \).
Theorem 2.3. Assume that $\varepsilon_j, \varsigma_j, \gamma, c, b \in \mathbb{C}$ ($j = 1, 2, ..., m$), $\Re(\varepsilon_j) > 0$, $\Re(\gamma) > 0$; $\sigma > 0$; $\beta > 0$; $\delta + 4\sigma\beta > 0$, $\Re(\lambda) > -1/2$, $-\frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2}$ be such that

$$\sum_{j=1}^{m} \Re(\varepsilon_j) > \max \{ 0, \Re(k) - 1 \}; k > 0, \text{and} \ \Re = (\sigma x + \frac{\beta}{x})^2 + \delta,$$

the subsequent integral formula holds:

$$\int_0^\infty \left( a + \frac{b}{x^2} \right) \Re^{-\lambda-1} \frac{\Re_{\varepsilon_j}^{(\varsigma_j)} \gamma, c}{z \Re^{-1}} \ dx$$

$$= \frac{\sqrt{\pi}}{(4\sigma\beta + \delta)^{\lambda+1/2}} \left( \frac{(\delta)^r}{r} \right) \frac{(4\sigma\beta + \delta)^{-\frac{1}{2}r}}{\Re^r}$$

$$\times 2\psi_{m+1} \left[ \left( \gamma, k \right), \left( \lambda - r + \frac{1}{2}, 1 \right) \left( \varsigma_j + \frac{\beta + 1}{2}, \varepsilon_j \right) \right]_{n=1}^{m} \left( \lambda - r + 1, 1 \right) \left( \frac{cz}{4\sigma\beta + \delta} \right). \quad (16)$$

Now on using $c = 1$ and $b = -1$, as special cases - I, the integral of Theorem 2.1 - 2.3 will give the following Corollaries 1 - 3 respectively as below:

**Corollary 1.** Let $\varepsilon_j, \varsigma_j, \gamma \in \mathbb{C}$; ($j = 1, ..., m$), $\sigma > 0$; $\beta \geq 0$; $\delta + 4\sigma\beta > 0$ and the condition of Theorem 2.1 be satisfied, the subsequent integral representation holds:

$$\int_0^\infty \Re^{-\lambda-1} \frac{\Re_{\varepsilon_j}^{(\varsigma_j)} \gamma, c}{z \Re^{-1}} \ dx

= \frac{\sqrt{\pi}}{2\sigma (4\sigma\beta + \delta)^{\lambda+1/2}} \left( \frac{(\delta)^r}{r} \right) \frac{(4\sigma\beta + \delta)^{-\frac{1}{2}r}}{\Re^r}

\times 2\psi_{m+1} \left[ \left( \gamma, k \right), \left( \lambda - r + \frac{1}{2}, 1 \right) \left( \varsigma_j, \varepsilon_j \right) \right]_{n=1}^{m} \left( \lambda - r + 1, 1 \right) \left( \frac{z}{4\sigma\beta + \delta} \right). \quad (17)$$

**Corollary 2.** Suppose that $\varepsilon_j, \varsigma_j, \gamma \in \mathbb{C}$; ($j = 1, ..., m$), $k > 0$, $\Re(\varepsilon_j) > 0$, $\Re(\gamma) > 0$, with satisfied condition of Theorem 2.2, then the consequent result:

$$\int_0^\infty \frac{1}{x^2} \Re^{-\lambda-1} \frac{\Re_{\varepsilon_j}^{(\varsigma_j)} \gamma, c}{z \Re^{-1}} \ dx

= \frac{\sqrt{\pi}}{2\beta (4\sigma\beta + \delta)^{\lambda+1/2}} \left( \frac{(\delta)^r}{r} \right) \frac{(4\sigma\beta + \delta)^{-\frac{1}{2}r}}{\Re^r}

\times 2\psi_{m+1} \left[ \left( \gamma, k \right), \left( \lambda - r + \frac{1}{2}, 1 \right) \left( \varsigma_j, \varepsilon_j \right) \right]_{n=1}^{m} \left( \lambda - r + 1, 1 \right) \left( \frac{z}{4\sigma\beta + \delta} \right). \quad (18)$$

**Corollary 3.** Assume that $\varepsilon_j, \varsigma_j, \gamma \in \mathbb{C}$ ($j = 1, ..., m$), $\sigma > 0$; $\beta > 0$; $\delta + 4\sigma\beta > 0$ and the condition of Theorem 2.3 be satisfied, the subsequent integral formula holds:

$$\int_0^\infty \left( a + \frac{b}{x^2} \right) \Re^{-\lambda-1} \frac{\Re_{\varepsilon_j}^{(\varsigma_j)} \gamma, c}{z \Re^{-1}} \ dx$$

$$= \frac{\sqrt{\pi}}{(4\sigma\beta + \delta)^{\lambda+1/2}} \left( \frac{(\delta)^r}{r} \right) \frac{(4\sigma\beta + \delta)^{-\frac{1}{2}r}}{\Re^r} \frac{(\sigma)^r}{r}$$

$$\times 2\psi_{m+1} \left[ \left( \gamma, k \right), \left( \lambda - r + \frac{1}{2}, 1 \right) \left( \varsigma_j, \varepsilon_j \right) \right]_{n=1}^{m} \left( \lambda - r + 1, 1 \right) \left( \frac{z}{4\sigma\beta + \delta} \right). \quad (19)$$
For the special cases - II, on using \( b = 1, c = -1 \) in the integral of Theorem 2.1 - 2.3 will provide the following Corollaries 4 - 6 respectively:

**Corollary 4.** Let \( \varepsilon_j, \gamma_j, \gamma, \in \mathbb{C}; (j = 1, \ldots, m), \sigma > 0; \beta > 0; \delta + 4\sigma\beta > 0; (\Re(\lambda) + 1/2) > 0 - \frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2}; \) be such that \( \sum^m_{j=1} \Re(\varepsilon_j) > \max \{0, \Re(k) - 1\}; \) \( k > 0; \Re(\gamma) > 0 and \Re(\gamma) > 0, \) the following integral holds true:

\[
\int_0^\infty \Im^{-\lambda-1} 2F_1 \left( \sigma, \beta, \delta + \frac{1}{2}; 3 \right) 2F_1 \left( \delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; 3 \right) \frac{z}{\Im} dx \\
= \frac{\sqrt{\pi}}{2\sigma(4\sigma\beta + \delta)^{\lambda+1/2} \Gamma(\gamma)} \sum_{r=0}^{\infty} (\delta)_r (4\sigma\beta + \delta)^{-r} \frac{\sigma_r}{r} \times 2\psi^{m+1} \left[ \frac{(\gamma,k), (\lambda-r+\frac{1}{2},1)}{(\gamma_j+\varepsilon_j)_{j=1}^m}, (\lambda-r+1,1) \right] \frac{-z}{4\sigma\beta + \delta}. \tag{19}
\]

**Corollary 5.** Assume that \( \varepsilon_j, \gamma_j, \gamma, \in \mathbb{C}; (j = 1, \ldots, m), \sigma \geq 0; \beta > 0; \delta + 4\sigma\beta > 0; (\Re(\lambda) > -1/2 \) with \( -\frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2}; k > 0, \Re(\gamma) > 0, \Re(\gamma) > 0, \sum^m_{j=1} \Re(\varepsilon_j) > \max \{0, \Re(k) - 1\}; \) thereupon following integral formula holds appropriate:

\[
\int_0^\infty \frac{1}{x^2} \Im^{-\lambda-1} 2F_1 \left( \sigma, \beta, \delta + \frac{1}{2}; 3 \right) 2F_1 \left( \delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; 3 \right) \\
\times \mathcal{F}^{(\gamma)}_{(\gamma_j), m,k} \left[ x \Im^{-1} \right] dx \\
= \frac{\sqrt{\pi}}{2\beta(4\sigma\beta + \delta)^{\lambda+1/2} \Gamma(\gamma)} \sum_{r=0}^{\infty} (\delta)_r (4\sigma\beta + \delta)^{-r} \frac{\sigma_r}{r} \times 2\psi^{m+1} \left[ \frac{(\gamma,k), (\lambda-r+\frac{1}{2},1)}{(\gamma_j+\varepsilon_j)_{j=1}^m}, (\lambda-r+1,1) \right] \frac{-z}{4\sigma\beta + \delta}. \tag{20}
\]

**Corollary 6.** Assume that \( \varepsilon_j, \gamma_j, \gamma, \in \mathbb{C}; (j = 1, \ldots, m), \sigma > 0; \beta > 0; \delta + 4\sigma\beta > 0; \Re(\lambda) > -1/2, -\frac{1}{2} < (\sigma - \beta - \delta) < \frac{1}{2}; \) be such that \( \sum^m_{j=1} \Re(\varepsilon_j) > \max \{0, \Re(k) - 1\}; \) \( \Re(\gamma) > 0, \Re(\gamma) > 0, k > 0, \) the subsequent integral formula holds:

\[
\int_0^\infty \left( a + \frac{b}{x^2} \right) \Im^{-\lambda-1} 2F_1 \left( \sigma, \beta, \delta + \frac{1}{2}; 3 \right) 2F_1 \left( \delta - \sigma, \delta - \beta, \delta + \frac{1}{2}; 3 \right) \\
\times \mathcal{F}^{(\gamma)}_{(\gamma_j), m,k} \left[ x \Im^{-1} \right] dx \\
= \frac{\sqrt{\pi}}{(4\sigma\beta + \delta)^{\lambda+1/2} \Gamma(\gamma)} \sum_{r=0}^{\infty} (\delta)_r (4\sigma\beta + \delta)^{-r} \frac{\sigma_r}{r} \times 2\psi^{m+1} \left[ \frac{(\gamma,k), (\lambda-r+\frac{1}{2},1)}{(\gamma_j+\varepsilon_j)_{j=1}^m}, (\lambda-r+1,1) \right] \frac{-z}{4\sigma\beta + \delta}. \tag{21}
\]
3. Integral transforms of generalized multi-index Bessel function. Here, we recall the following results which are required in the study as follows:

The Euler transform of a function $f(z)$ is defined in [11] as:

$$B \{ f(z); l, m \} = \int_0^1 z^{l-1} (1-z)^{m-1} f(z) \, dz \quad l, m \in \mathbb{C}, \Re (l) > 0, \Re (m) > 0.$$

(23)

The Laplace transform of a function $f(z)$, denoted by $F(s)$, is defined in [11] by the equation

$$F(s) = (Lf)(s) = L \{ f(z) ; s \} = \int_0^\infty e^{-sz} f(z) \, dz \quad \Re (s) > 0;$$

(24)

on condition that the integral (24) is convergent, for $z > 0$; the function $f(z)$ is continuous and of exponential order as $z \to \infty$, (24) can be written as

$$F(s) = L \{ f(z) ; s \} \quad \text{or} \quad f(z) = L^{-1} \{ F(s) ; z \}.$$  

(25)

In 1962, Whittakar and Watson [54] introduced the Whittakar transform defined by

$$\int_0^\infty e^{-\frac{1}{2}z\rho^{-1}} W_{\tau,\omega}(z) \, dz = \left( \frac{1}{2} + w + \rho \right) \Gamma \left( \frac{1}{2} - w + \rho \right) \Gamma \left( 1 - \tau + \rho \right)$$

(26)

where $\Re (\rho \pm \omega) > -1/2$ and $W_{\tau,\omega}(z)$ is the Whittakar function defined in [12].

$$W_{\omega,\rho}(z) = \frac{\Gamma (-2\omega)}{\Gamma (\frac{1}{2} - \tau - \omega)} \frac{1}{M_{\tau,\omega} (z)} + \frac{\Gamma (2\omega)}{\Gamma (\frac{1}{2} - \tau + \omega)} M_{\tau,-\omega} (z)$$

(27)

where $M_{\tau,\omega} (z)$ is defined by

$$M_{\tau,\omega} (z) = z^{1/2+\omega} e^{-1/2z} I_1 \frac{1}{2 + \omega - \tau; 2\omega + 1; z}.$$  

(28)

In 1954, Erdélyi et al. [12], the K-transform is defined by the integral equation

$$\Re [ f(z) ; q ] = g[q; v] = \int_0^\infty (qz)^{1/2} K_v (q, z) f(z) \, dz$$

(29)

where $\Re (q) > 0; K_v (z)$ is the Bessel function of the second kind defined by ([12], p.332)

$$K_v (z) = \left( \frac{\pi}{2z} \right)^{1/2} W_{0,v} (2z)$$

where $W_{0,v} (.)$ is the Whittaker function defined in equation (27).

We will use a relation given by Mathai et al. ([28], p. 54, eq. 2.37) for evaluating the integrals as:

$$\int_0^\infty z^{\beta-1} K_v (az) \, dz = 2^{\beta-2} a^{-\beta} \Gamma \left( \frac{\beta \pm v}{2} \right); \quad \Re (\beta \pm v) > 0, \Re (a) > 0.$$  

(30)

In 1980, Namia [32] introduced the fractional Fourier transform of order $\varepsilon$, with range $0 < \varepsilon \leq 1$; defined in the form:

$$\tilde{u}_\varepsilon (\omega) = \mathbb{F}_\varepsilon [u] (\omega) = \int_R e^{i\omega (t/\varepsilon)} u(t) \, dt$$

(31)

when $\varepsilon = 1$, equation (31) reduces to the conventional Fourier transforms and also some of its properties can be found in [27].
This section deals with the analysis of the Euler transform (ET), Laplace transform (LT), Whittaker transform (WT), K-transform (K-T) and fractional Fourier transforms (FFT) of the GMBF defined by (1).

**Theorem 3.1. (Euler Transform).** If $k > 0$, $\sigma > 0$, $\varepsilon_j$, $\varsigma_j$, $\gamma$, $c$, $b$ ($j = 1, \ldots, m$) are complex numbers with $\sum_{j=0}^{m} \Re (\varepsilon_j) > \max \Re \{0; -1 + \Re (k)\} + \Re (\varsigma_j) + \Re (\gamma) > 0$, then

\[
\int_0^1 z^{r-1} (1-z)^{s-1} \mathcal{J}^{(\varsigma)}_{m,k,b}(x z^\gamma) \, dz = \frac{\Gamma(s) \Gamma(\gamma)}{\Gamma(\varsigma)} 2^{\psi_{m+1}} \left[ \frac{(\gamma,k)}{(r,\sigma)} \frac{(\varsigma_j+b+1)}{(r+s,\sigma)} \right] \cdot (32)
\]

**Proof.** Applying generalized multi-index Bessel function (1), on the left hand side of the theorem, we obtain

\[
I_1 = \int_0^1 z^{r-1} (1-z)^{s-1} \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!} (x z^\gamma)^n \, dz
\]

By applying definition of Beta function, we have

\[
I_1 = \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!} \int_0^1 z^{r+\sigma n-1} (1-z)^{s-1} \, dz
\]

\[
= \frac{\Gamma(s)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + k n)}{\Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2})} \frac{(c x)^n}{\Gamma(r + \sigma n)} \left. \frac{(\gamma,k)}{(r,\sigma)} \frac{(\varsigma_j+b+1)}{(r+s,\sigma)} \right| \cdot (32)
\]

In accordance with Eq. (13), we obtain the result (32). This completes the proof of the theorem.

**Theorem 3.2. (Laplace Transform).** If $k > 0$, $\sigma > 0$, $\varepsilon_j$, $\varsigma_j$, $\gamma$, $c$, $b$, $r$ ($j = 1, \ldots, m$) are complex numbers with $\sum_{j=0}^{m} \Re (\varepsilon_j) > \max \Re \{0; -1 + \Re (k)\} + \Re (\varsigma_j) + \Re (\gamma) > 0$, $\Re (r) > 0$ and $\left| \frac{\varepsilon_j}{\varsigma_j} \right| < 1$, then

\[
\int_0^\infty z^{r-1} e^{-sz} \mathcal{J}^{(\varsigma)}_{m,k,b}(x z^\gamma) \, dz = \frac{s^{-r}}{\Gamma(\gamma)} 2^{\psi_{m}} \left[ \frac{(\gamma,k)}{(r,\sigma)} \frac{(\varsigma_j+b+1)}{(r+s,\sigma)} \right] \cdot (33)
\]

**Proof.** The left-hand side of Eq. (33) denoted by $I_2$, Using the definition of generalized multi-index Bessel function Eq. (1), we have

\[
I_2 = \int_0^\infty z^{r-1} e^{-sz} \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!} (x z^\gamma)^n \, dz
\]

\[
= \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!} \int_0^\infty z^{r+\sigma n-1} e^{-sz} \, dz
\]

\[
= \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!} \left. \frac{(z^{r+\sigma n-1})}{(r+\sigma n)} \right| \cdot (33)
\]
Now using the definition of Laplace transform, we have

\[ I_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{n!} \frac{\Gamma(r + \sigma n)}{\Gamma\left(\frac{b+1}{2} + \varepsilon_j n + \varepsilon_j n\right)} \left(\frac{c x}{s^\sigma}\right)^n, \]

In accordance with the Eq. (13), we arrive at the desired result (33).

**Theorem 3.3.** (Whittaker Transform). If \( k > 0, \sigma > 0, \varepsilon_j, \gamma, c, b, (j = 1, \ldots, m) \) are complex numbers with \( \Re(\lambda + \omega) > -1/2, \sum_{n=0}^{\infty} \Re(\varepsilon_j) > \max \Re\{0; -1 + \Re(k)\} \), \( \Re(\varepsilon_j) > 0, \Re(\gamma) > 0, \Re(c) > \Re(\omega) - 1/2, \Re(\beta) > 0 \), then

\[
\int_0^\infty z^{\lambda-1} e^{-\frac{z}{2}} W_{\tau,\omega}(\beta z) J^{(\varepsilon_j)_{m}}n,_{k,b} (x z^\sigma) \, dz = \frac{\beta^{-\lambda-\sigma}}{\Gamma(\gamma)} 3^{\psi_m+1} \left| \begin{array}{c} (\gamma, k), (\frac{1}{2} + \omega + \lambda, \sigma), (\frac{1}{2} - \omega + \lambda, \sigma) \\ (\varepsilon_j + \frac{b+1}{2}, \varepsilon_j) \end{array} \right| \frac{c x}{s^\sigma}. \tag{34}
\]

**Proof.** The left-hand side of Eq. (34) denoted by \( I_3 \). Let \( \beta z = y \), after interchanging the integration and summation, we obtain

\[ I_3 = \beta^{-\lambda-\sigma} \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{n!} \int_0^\infty y^{\lambda+\sigma n-1} e^{-\frac{y}{2}} W_{\tau,\omega}(y) \, dy, \]

Now using the Whittaker Transformation (26), we arrive at

\[ I_3 = \beta^{-\lambda-\sigma} \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{n!} \int_0^\infty y^{\lambda+\sigma n} e^{-\frac{y}{2}} W_{\tau,\omega}(y) \, dy. \]

In accordance with the Eq. (13), we arrive at the required result (34).

**Theorem 3.4.** (k-Transform). If \( k > 0, \sigma > 0, \varepsilon_j, \gamma, c, b, \lambda(j = 1, \ldots, m) \) are complex numbers with \( \sum_{j=0}^{\infty} \Re(\varepsilon_j) > \max \Re\{0; -1 + \Re(k)\} \), \( \Re(\varepsilon_j) > 0, \Re(\gamma) > 0 \) and \( \Re(\lambda + \tau) > 0, \Re(\omega) > 0 \), then the following result holds true;

\[
\int_0^\infty z^{\lambda-1} K_\tau(\omega z) J^{(\varepsilon_j)_{m}}_{n,2, c} (x z^\sigma) \, dz = \frac{2\lambda-2-\omega-\lambda}{\Gamma(\gamma)} 3^{\psi_m} \left| \begin{array}{c} (\gamma, k), (\frac{\lambda+\tau}{2}, \frac{\gamma}{2}), (\frac{\lambda-\tau}{2}, \frac{\gamma}{2}) \\ (\varepsilon_j + \frac{b+1}{2}, \varepsilon_j) \end{array} \right| \frac{c x}{2-\sigma^2} \tag{35}
\]

**Proof.** Using the Eq. (1) in the left-hand side of Eq. (35) denoted by \( I_4 \), then we obtain

\[ I_4 = \int_0^\infty z^{\lambda-1} K_\tau(\omega z) \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{n!} (x z^\sigma)^n \, dz, \]

By important simplification, this equation becomes

\[ I_4 = \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{n!} \int_0^\infty z^{\lambda+\sigma n} e^{-\frac{z}{2}} W_{\tau,\omega}(z) \, dz, \]

Then applying formula (30), we arrive at

\[ I_4 = \frac{2\lambda-2-\omega-\lambda}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(c x)^n (\gamma)_{kn}}{n!} \int_0^\infty z^{\lambda+\sigma n} e^{-\frac{z}{2}} W_{\tau,\omega}(z) \, dz, \]

Using the definition of Wright hypergeometric function (13), we arrive at the desired result (35).
Theorem 3.5. (Fourier Transform). If $k > 0$, $\varepsilon_j, \varsigma_j, \gamma, c, b, (j = 1, \ldots, m)$ are complex numbers with $\sum_{j=0}^{m} \Re(\varepsilon_j) > \max \{0; -1 + \Re(k)\}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0$ for FFT of order $\xi$ of the generalized multi-index Bessel function $z < 0$, is given by

$$
\Im \left[ J_{(\varepsilon_j)_{m}, \gamma, c} (z) \right] (\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{-n}(\xi - 1 - n)^{(\gamma - 1)}(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!}{\Gamma(n+1)(\gamma - 1)n!} \int_{0}^{\infty} e^{-\rho^2 n} d\rho
$$

where $\xi > 0$, $\omega > 0$.

Proof. By applying equation (1) and (31), then the left hand side of equation (36) gives

$$
\Im \left[ J_{(\varepsilon_j)_{m}, \gamma, c} (z) \right] (\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{-n}(\xi - 1 - n)^{(\gamma - 1)}(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!}{\Gamma(n+1)(\gamma - 1)n!} \int_{0}^{\infty} e^{-\rho^2 n} d\rho
$$

By setting $i\omega^{1/\xi} z = -\rho$, then

$$
\Im \left[ J_{(\varepsilon_j)_{m}, \gamma, c} (z) \right] (\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{-n}(\xi - 1 - n)^{(\gamma - 1)}(\varepsilon_j n + \varsigma_j + \frac{b+1}{2}) n!}{\Gamma(n+1)(\gamma - 1)n!} \int_{0}^{\infty} e^{-\rho^2 n} d\rho
$$

This completes the proof of the Theorem. $\Box$

Now, by putting $c = 1$ and $b = -1$ in the integral of Theorems 3.1 - 3.5 will give the following Corollaries 7 - 11 respectively as follow:

Corollary 7. Suppose that $k > 0$, $\varepsilon_j, \varsigma_j, \gamma, \beta, (j = 1, \ldots, m)$ are complex numbers with $\sum_{j=0}^{m} \Re(\varepsilon_j) > \max \{0; -1 + \Re(k)\}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(r) > 0$ and $\Re(s) > 0$, then the following result holds true;

$$
\int_{0}^{1} z^{r-1}(1 - z)^{s-1} J_{(\varepsilon_j, \varsigma_j)} (x z^2) dz = \frac{\Gamma(s)}{\Gamma(\gamma)} 2\psi_{m+1} \left[ \frac{(\gamma, k, (r, \sigma), (\varsigma_j, \varepsilon_j, s))}{m \sum_{j=1}^{m}} \right] x^2.
$$

Corollary 8. Assume that $k > 0$, $\varphi > 0$, $\varepsilon_j, \varsigma_j, \gamma, r (j = 1, \ldots, m)$ are complex numbers with $\sum_{j=0}^{m} \Re(\varepsilon_j) > \max \{0; -1 + \Re(k)\}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(r) > 0$ and $\Re(s) < 1$ then the following result holds true;

$$
\int_{0}^{\infty} z^{r-1} e^{-s z} J_{(\varepsilon_j, \varsigma_j)} (x z^2) dz = \frac{s^{-\sigma}}{\Gamma(\gamma)} 2\psi_{m+1} \left[ \frac{(\gamma, k, (r, \sigma), (\varsigma_j, \varepsilon_j, s))}{m \sum_{j=1}^{m}} \right] x^2.
$$

Corollary 9. Let $k > 0$, $\gamma > 0$, $\varepsilon_j, \varsigma_j, \gamma (j = 1, \ldots, m)$ are complex numbers with $\Re(\lambda + \sigma) > -1/2$, $\sum_{j=0}^{m} \Re(\varepsilon_j) > \max \{0; -1 + \Re(k)\}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(r) > \Re(\omega) - 1/2, \Re(\beta) > 0$, then the following result holds true;

$$
\int_{0}^{\infty} z^{r-1} e^{-s z} W_{(\varepsilon_j, \varsigma_j), m} (x z^2) dz = \frac{\beta^{-\lambda - \sigma}}{\Gamma(\gamma)} 2\psi_{m+1} \left[ \frac{(\gamma, k, (\frac{1}{2} + \omega + \lambda, \sigma), (\frac{1}{2} - \omega + \lambda, \sigma))}{m \sum_{j=1}^{m}} \right] x^2.
$$
Corollary 10. Suppose that \( k > 0, \sigma > 0, \varepsilon_j, \varsigma_j, \gamma, \lambda(j = 1, ..., m) \) are complex numbers with \( \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0 \) and \( \Re(\lambda \pm \tau) > 0, \Re(\omega) > 0 \), then the following result holds true;
\[
\int_{0}^{\infty} z^{\lambda-1} K_{\tau}(\omega z) J_{(\varepsilon_j, \varsigma_j)_{m}}(x z^\sigma) \, dz
\]
\[
= \frac{2^{\lambda-2} \omega^{-\lambda}}{\Gamma(\gamma)} 3^{\psi_{m}} \left[ (\gamma, k), \left( \begin{array}{c} \frac{1}{2} + \omega + \lambda, \sigma \\ \varsigma + 1, \varepsilon_j \end{array} \right) \right] \frac{-cx}{2^\sigma \omega^{\sigma}}. \tag{40}
\]

Corollary 11. Let \( k > 0, \varepsilon_j, \varsigma_j, \gamma(j = 1, ..., m) \) are complex numbers with \( \Re(\varsigma_j) > 0, \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\gamma) > 0 \) for FFT of order \( \xi \) of the generalized multi-index Bessel function \( z < 0 \), is given by
\[
\mathcal{R} \xi \left[ J_{(\varepsilon_j, \varsigma_j)_{m}}(z) \right] (\omega) = \sum_{n=0}^{\infty} i^{-n-1} \omega^{-(n+1)/\xi} (-1)^{n} (\gamma)_{kn} \Gamma(n+1) (-c)^n \prod_{j=1}^{m} \Gamma(\varepsilon_j + \varsigma_j) n! \tag{41}
\]
where \( \xi > 0, \omega > 0 \).

Further, By using \( b = 1, c = -1 \) in the integral of Theorems 3.1 - 3.5 will give the following Corollaries 12 - 16 respectively:

Corollary 12. Suppose that \( k > 0, \varepsilon_j, \varsigma_j, \gamma, \sigma(j = 1, ..., m) \) are complex numbers with \( \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(\sigma) > 0 \) and \( \Re(s) > 0 \), then the following result holds true;
\[
\int_{0}^{1} z^{r-1} (1-z)^{s-1} J_{(\varsigma_j)_{m}}^{(r)}(x z^\sigma) \, dz
\]
\[
= \frac{\Gamma(s)}{\Gamma(\gamma)} 2^{\psi_{m+1}} \left[ (\gamma, k), \left( \begin{array}{c} (r, \sigma) \\ \varsigma_j + 1, \varepsilon_j \end{array} \right) \right] - \frac{x}{s^\sigma}. \tag{42}
\]

Corollary 13. Suppose that \( k > 0, \sigma > 0, \varepsilon_j, \varsigma_j, \gamma(j = 1, ..., m) \) are complex numbers with \( \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(\sigma) > 0 \) and \( |z| < 1 \) then the following result holds true;
\[
\int_{0}^{\infty} z^{r-1} e^{-x z} J_{(\varsigma_j)_{m}}^{(r)}(x z^\sigma) \, dz = \frac{\Gamma(s)}{\Gamma(\gamma)} 2^{\psi_{m}} \left[ (\gamma, k), \left( \begin{array}{c} (r + s, \sigma) \\ \varsigma_j + 1, \varepsilon_j \end{array} \right) \right] - \frac{x}{s^\sigma}. \tag{43}
\]

Corollary 14. Assume that \( k > 0, \sigma > 0, \varepsilon_j, \varsigma_j, \gamma(j = 1, ..., m) \) are complex numbers with \( \Re(\lambda \pm \tau) > -1/2, \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0, \Re(\sigma) > 0, \Re(e) > \Re(\omega) - 1/2, \Re(\beta) > 0 \), then the following result holds true;
\[
\int_{0}^{\infty} z^{\lambda-1} e^{-\frac{b}{z} W_{\tau, \omega}(\beta z)} J_{(\varsigma_j)_{m}}^{(r)}(x z^\sigma) \, dz
\]
\[
= \frac{\beta^{-\lambda-\sigma}}{\Gamma(\gamma)} 3^{\psi_{m+1}} \left[ (\gamma, k), \left( \begin{array}{c} \frac{1}{2} + \omega + \lambda, \sigma \\ \varsigma_j + 1, \varepsilon_j \end{array} \right) \right] \frac{-cx}{2^\sigma \omega^{\sigma}}. \tag{44}
\]

Corollary 15. Let \( k > 0, \sigma > 0, \varepsilon_j, \varsigma_j, \gamma, \lambda(j = 1, ..., m) \) are complex numbers with \( \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0; -1 + \Re(k) \}, \Re(\varsigma_j) > 0, \Re(\gamma) > 0 \) and \( \Re(\lambda \pm \tau) > 0, \Re(\omega) > 0 \), then the following result holds true;
\[
\int_{0}^{\infty} z^{\lambda-1} K_{\tau}(\omega z) J_{(\varsigma_j)_{m}}^{(r)}(x z^\sigma) \, dz
\]
\[
= \frac{2^{\lambda-2} \omega^{-\lambda}}{\Gamma(\gamma)} 3^{\psi_{m}} \left[ (\gamma, k), \left( \begin{array}{c} \frac{1}{2} + \omega + \lambda, \sigma \\ \varsigma_j + 1, \varepsilon_j \end{array} \right) \right] \frac{-cx}{2^\sigma \omega^{\sigma}}. \tag{45}
\]
Corollary 16. If $k > 0$, $\varepsilon_j, \varsigma_j, \gamma, (j = 1, \ldots, m)$ are complex numbers with $\Re(\varsigma_j) > 0$, $\sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re\{0, -1 + \Re(k)\}$, $\Re(\gamma) > 0$ for FFT of order $\xi$ of the generalized multi-index Bessel function $z < 0$, is given by

$$
\exists \xi \left[ J^{(\varepsilon_{(j)}m, \gamma)}_{(\varsigma_{(j)}m, k)}(z) \right] (\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{i^{n+1} \omega^{(n+1)/\xi} (-1)^{n} (\gamma)_{kn} \Gamma(n+1)} \left( c_{n} \right)^{n} \prod_{j=1}^{m} \Gamma(\epsilon_{j} n + \varsigma_{j} + 1) n!
$$

(46)

where $\xi > 0$, $\omega > 0$.

4. Fractional kinetic equations. In a recent years, many researcher are working on the solution of the fractional kinetic equations due to their significance in the area of applied sciences like as mathematical physics, dynamical systems, astrophysics and control systems. Certain physical phenomenon only determine by the kinetic equations of fractional order, therefore, a huge number of research articles in the applications direction have been published in the literature (see [1, 2, 4, 5, 8, 10, 14, 15, 20, 21, 46]). The fractional differential equation of the quantities $\mathcal{N}(t)$ (approximate reaction which is dependent upon time), $d$ (destruction rate) and $p$ (production rate), established by Haubold and Mathai [15] is

$$
\frac{d\mathcal{N}}{dt} = -d(\mathcal{N}(t)) + p(\mathcal{N}(t))
$$

(47)

where $\mathcal{N}(t - t^*) = \mathcal{N}_i(t^*)$ for $t^* > 0$; when spatial fluctuation or inhomognities in quantity $\mathcal{N}(t)$ are neglected, equation (47) will become the subsequent differential equation,

$$
\frac{d\mathcal{N}_i}{dt} = -c_i\mathcal{N}_i(t)
$$

(48)

together with the initiative condition that $\mathcal{N}_i(t = 0) = \mathcal{N}_0$ is the number density of index $i$ at time $t = 0$; constant $c_i > 0$, accepted as standard kinetic equation (see: Kourganoff [17])

Further, solution of equation (48) is given by

$$
\mathcal{N}_i(t) = \mathcal{N}_0 e^{-c_i t}.
$$

(49)

Alternatively, if we omit the index $i$ and integrate the typical kinetic equation (48), we obtain

$$
\mathcal{N}(t) - \mathcal{N}_0 = c_i 0 D_t^{-1} \mathcal{N}(t)
$$

(50)

where $0 D_t^{-1}$ is the standard integral operator.

The generalization of fractional equation (50) is defined by Haubold and Mathai [15] in the form:

$$
\mathcal{N}(t) - \mathcal{N}_0 = c_i 0 D_t^{-v} \mathcal{N}(t)
$$

(51)

where $0 D_t^{-v}$ is the widely called Riemann–Liouville (R-L) fractional integral operator defined in Ross [31] and Samko et al [35] as:

$$
0 D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t - u)^{v-1} f(u) \, du, \quad \Re(v) > 0
$$

(52)

after simplification of the fractional kinetic equation (51), we arrive at

$$
\mathcal{N}(t) = \mathcal{N}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(vk + 1)} (ct)^{vk}.
$$

(53)
5. Solution of generalized fractional kinetic equations. Recently, Suthar et al. [46] obtained solutions of fractional kinetic equation associated with the generalized multi-index Bessel function using the Laplace transform technique. In this part, by using sumudu transform technique, we shall analysis an alternate solution technique for solving the generalized fractional kinetic equations by considering GMBF.

Theorem 5.1. If \( d > 0, v > 0, \varepsilon_j, \gamma, c, b \in \mathbb{C} \) (\( j = 1, \ldots, m \)) and \( 0 \neq \alpha \neq d \) such that \( \sum_{j=0}^{m} \Re(\varepsilon_j) > \max \Re \{ 0, -1 + \Re(k) \} \) : \( k > 0, \Re(\gamma) > 0, \Re(\gamma) > 0 \); then solution of the equation

\[
N(t) - N_0 \mathcal{J}_{m, c}^{(\varepsilon_j), m} \cdot (t^v) = -a^v D_t^{-v} N(t)
\]

is presented by the subsequent formula

\[
N(t) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} E_v vn (-a^v t^n)
\]

Proof. The Sumudu transform of Riemann-Liouville integral operator is given by

\[ S \{ D_t^{-\mu} \}; u \} = u^\mu G(u) \]

Now applying the Sumudu transform on both sides of (54); using (1) and (56), we obtain

\[
S \{ N(t) \}; u \} = N_0 S \left\{ \mathcal{J}_{m, c}^{(\varepsilon_j), m} \cdot (d^v t^v); u \right\} - a^v S \{ D_t^{-v} N(t); u \},
\]

\[
\mathcal{N}^* (u) = N_0 \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} dt - a^v u^v N^* (u),
\]

\[
\mathcal{N}^* (u) + a^v u^v N^* (u) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} \int_0^\infty e^{-t} (ut)^n dt,
\]

since \( S(t^\mu) = u^\mu - \Gamma(\mu) \), we get

\[
\mathcal{N}^* (u) (1 + u^v a^v) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} \Gamma(vn + 1),
\]

\[
\mathcal{N}^* (u) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} \left\{ u^n \sum_{s=0}^{\infty} \frac{[-(au)^v]^s}{s!} \right\},
\]

Taking both side the Sumudu inverse transform of equation (61), we get

\[
S^{-1} (\mathcal{N}^* (u)) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(vn + 1) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} \int_0^\infty e^{-t} (ut)^n dt S^{-1} \left\{ \sum_{s=0}^{\infty} \frac{(-1)^s a^v v(n+s)}{v(n+s)} \right\},
\]

now, applying the standard results \( S^{-1} (u^v; t) = t^{v-1} (\Gamma(v)) \) on (62) with some elementary mathematical manipulation, we obtain

\[
N(t) = N_0 \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_k n \Gamma(1 + vn) - 1 (t^n d^n)^n}{n!} \frac{(d^n t^n)^n}{n!} \left\{ \sum_{s=0}^{\infty} \frac{(-1)^s a^v v(n+s)}{v(n+s)} \right\},
\]

interpreting equation (63) in outline of (7), we show up the aimed result.

\[ \square \]
6. Special Cases. The formulas (4)-(5) are obtained by putting the particular value in Theorem 5.1 to making the Corollaries 17 - 18 respectively.

Corollary 17. If \(v > 0, d > 0 \varepsilon_j, \varsigma_j, \gamma \in \mathbb{C} \ (j = 1, \cdots, m)\), \(k > 0\), \(\Re(\gamma) > 0\) and \(a \neq d\) so that \(\sum_{j=0}^m \Re(\varepsilon_j) > \max\{0; -1 + \Re(k)\}; \Re(\varsigma_j) > 0\), thereupon for the solution of the equation

\[
N(t) - N_0 \mathcal{J}_{(\varepsilon_j, \varsigma_j)}^{k}(d^v t^v) = -a^v_0 D_t^{-v}N(t)
\]

the following results holds

\[
N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} \Gamma (vn + 1)}{\prod_{j=1}^{m} \Gamma (\varepsilon_j n + \varsigma_j)} \left(\frac{(t^v d^v)^n}{t^n!}\right) E_{v,vn} (-a^v_0 t^v) .
\]

Proof. By applying the Sumudu transform and its inversion as similar to the above theorem, we can easily obtain the required solution.

Corollary 18. If \(d > 0, v > 0, \varepsilon_j, \varsigma_j, \gamma \in \mathbb{C} \ (j = 1, \cdots, m)\), and \(a \neq d\) such that \(\Re(\gamma) > 0\), \(\sum_{j=0}^m \Re(\varepsilon_j) > \max\{0; -1 + \Re(k)\}; k > 0, \Re(\varsigma_j) > 0\), next the generalized fractional kinetic equation

\[
N(t) - N_0 \mathcal{J}_{(\varepsilon_j, \varsigma_j)}^{m,k}(d^v t^v) = -a^v_0 D_t^{-v}N(t)
\]

is given by

\[
N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} \Gamma (vn + 1)}{\prod_{j=1}^{m} \Gamma (\varepsilon_j n + \varsigma_j + 1)} \left(\frac{(-t^v d^v)^n}{t^n!}\right) E_{v,vn} (-a^v_0 t^v) .
\]

Proof. By using the Sumudu transform and its inversion as similar to the above theorem, we can easily obtain the desired solution.

7. Concluding Remarks. In the current paper, we have investigated three definite integrals of Gradshteyn-Ryzhik type and certain integral transforms of the GMBF. Alternate solution of the generalized fractional kinetic equations, using the Sumudu transform techniques have also been discussed. By suitably specializing the values of the parameters of GMBF (1), our main results can yield several integrals and solutions of generalized fractional kinetic equations corresponding the Bessel function, Bessel-Maitland function and multi-index Mittag-Leffler function, etc. Therefore, the investigated results in this paper would at once give many new results involving diverse field of special functions occurring in the problems of mathematical physics, astrophysics and engineering.

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