Equivariant $K$-theory of GKM bundles

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Received: 31 January 2012 / Accepted: 27 April 2012 / Published online: 17 May 2012
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Abstract Given a fiber bundle of GKM spaces, $\pi : M \to B$, we analyze the structure of the equivariant $K$-ring of $M$ as a module over the equivariant $K$-ring of $B$ by translating the fiber bundle, $\pi$, into a fiber bundle of GKM graphs and constructing, by combinatorial techniques, a basis of this module consisting of $K$-classes which are invariant under the natural holonomy action on the $K$-ring of $M$ of the fundamental group of the GKM graph of $B$. We also discuss the implications of this result for fiber bundles $\pi : M \to B$ where $M$ and $B$ are generalized partial flag varieties and show how our GKM description of the equivariant $K$-ring of a homogeneous GKM space is related to the Kostant–Kumar description of this ring.

Keywords Equivariant $K$-theory · Equivariant fiber bundles · GKM manifolds · Flag manifolds

Mathematics Subject Classification Primary 55R91 · Secondary 19L47 · 05C90

1 Introduction

Let $T$ be an $n$-torus and $M$ a compact $T$ manifold. The action of $T$ on $M$ is said to be GKM if $M^T$ is finite and if, in addition, for every codimension one subtorus, $T'$, of $T$ the connected
components of $M^T \Gamma$ are of dimension at most 2. An implication of this assumption is that these fixed-point components are either isolated points or diffeomorphic copies of $S^2$ with its standard $S^1$ action, and a convenient way of encoding this fixed-point data is by means of the GKM graph, $\Gamma$, of $M$. By definition, the $S^2$'s above are the edges of this graph, the points on the fixed-point set, $M^T$, are the vertices of this graph and two vertices are joined by an edge if they are the fixed points for the $S^1$ action on the $S^2$ representing that edge. Moreover, to keep track of which $T^\Gamma$'s correspond to which edges, one defines a labeling function $\alpha$ on the set of oriented edges of $\Gamma$ with values in the weight lattice of $T$. This function (the “axial function” of $\Gamma$) assigns to each oriented edge the weight of the isotropy action of $T$ on the tangent space to the north pole of the $S^2$ corresponding to this edge. (The orientation of this $S^2$ enables one to distinguish the north pole from the south pole.)

The concept of GKM space is due to Goresky–Kottwitz–MacPherson, who showed that the equivariant cohomology ring, $H_T(M)$, can be computed from $(\Gamma, \alpha)$ (see [1]). Then Allen Knutson and Ionel Rosu (see [6]) proved the much harder result that this is also true for the equivariant $K$-theory ring, $KT(M)$. (We will give a graph theoretic description of this ring in Sect. 2 below.)

Suppose now that $M$ and $B$ are GKM manifolds and $\pi: M \to B$ a $T$-equivariant fiber bundle. Then the ring $H_T(M)$ becomes a module over the ring $H_T(B)$ and in [2] we analyzed this module structure from a combinatorial perspective by showing that the fiber bundle $\pi$ of manifolds gives rise to a fiber bundle, $\pi: \Gamma_M \to \Gamma_B$ of GKM graphs, and showing that the salient module structure is encoded in this graph fiber bundle. In this article, we will prove analogous results in $K$-theory. More explicitly in the first part of this article (from Sects. 2 to 4) we will review the basic facts about GKM graphs, the notion of “fiber bundle” for these graphs and the definition of the $K$-theory ring of a graph-axial function pair, and we will also discuss an important class of examples: the graphs associated with GKM spaces of the form $M = G/P$, where $G$ is a complex reductive Lie group and $P$ a parabolic subgroup of $G$. The main results of this paper are discussed in Sects. 5 and 6. In Sect. 5 we prove that for a fiber bundle of GKM graphs $(\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ a set of elements $c_1, \ldots, c_k$ of $K_\alpha(\Gamma)$ is a free set of generators of $K_\alpha(\Gamma)$ as a module over $K_{\alpha_B}(\Gamma_B)$, providing their restrictions to $K_{\alpha_p}(\Gamma_p)$ are a basis for $K_{\alpha_p}(\Gamma_p)$, where $\Gamma_p$ is the graph theoretical fiber of $(\Gamma, \alpha)$ over a vertex $p$ of $\Gamma_B$. Then in Sect. 6 we describe an important class of such generators. One property of a GKM fiber bundle is a holonomy action of the fundamental group of $\Gamma_B$ on the fiber and we show how a collection of holonomy invariant generating classes $c_1', \ldots, c'_k$ of $K_{\alpha_p}(\Gamma_p)$ extend canonically to a free set of generators $c_1, \ldots, c_k$ of $K_\alpha(\Gamma)$. In Sects. 7 and 8 we describe how these results apply to concrete examples: special cases of the $G/P$ examples mentioned above. Finally in Sect. 9 we relate our GKM description of the equivariant $K$-ring, $KT(G/P)$, to a concise and elegant alternative description of this ring by Bertram Kostant and Shrawan Kumar in [7], and analyze from their perspective the fiber bundle $G/B \to G/P$.

To conclude these introductory remarks we would like to thank Tudor Ratiu for his support and encouragement and Tara Holm for helpful suggestions about the relations between GKM and Kostant–Kumar.

2 $K$-theory of integral GKM graphs

Let $\Gamma = (V, E)$ be a $d$-valent graph, where $V$ is the set of vertices, and $E$ the set of oriented edges; for every edge $e \in E$ from $p$ to $q$, we denote by $\overline{e}$ the edge from $q$ to $p$. Let $i : E \to V$ (respectively, $i : E \to V$) be the map which assigns to each oriented edge $e$ its
initial (respectively, terminal) vertex (so $i(e) = t(\overline{e})$ and $t(e) = i(\overline{e})$); for every $p \in V$ let $E_p$ be the set of edges whose initial vertex is $p$.

Let $T$ be an $n$-dimensional torus; we define a “$T$-action on $\Gamma$” by the following recipe (see [5]).

**Definition 2.1** Let $e = (p, q)$ be an oriented edge in $E$. Then a connection along $e$ is a bijection $\nabla_e : E_p \to E_q$ such that $\nabla_e(e) = \overline{e}$. A connection on the graph $\Gamma$ is a family $\nabla = \{\nabla_e\}_{e \in E}$ satisfying $\nabla_T = \nabla_{e}^{-1}$ for every $e \in \Gamma$.

Let $t^*$ be the dual of the Lie algebra of $T$ and $\mathbb{Z}_T^*$ its weight lattice.

**Definition 2.2** Let $\nabla$ be a connection on $\Gamma$. A $\nabla$-compatible integral axial function on $\Gamma$ is a map $\alpha : E \to \mathbb{Z}_T^*$ satisfying the following conditions:

1. $\alpha(\overline{e}) = -\alpha(e)$;
2. for every $p \in V$ the vectors $\{\alpha(e) \mid e \in E_p\}$ are pairwise linearly independent;
3. for every edge $e = (p, q)$ and every $e' \in E_p$ we have

$$\alpha(\nabla_e(e')) - \alpha(e') = m(e, e')\alpha(e),$$

where $m(e, e')$ is an integer which depends on $e$ and $e'$.

An integral axial function on $\Gamma$ is a map $\alpha : E \to \mathbb{Z}_T^*$ which is $\nabla$-compatible, for some connection $\nabla$ on $\Gamma$.

**Definition 2.3** An integral GKM graph is a pair $(\Gamma, \alpha)$ consisting of a regular graph $\Gamma$ and an integral axial function $\alpha : E \to \mathbb{Z}_T^*$.

**Remark 2.4** The graphs we described in the introduction are examples of such graphs. In particular condition 2 in Definition 2.2 is a consequence of the fact that, for every codimension one subgroup of $T$, its fixed-point components are of dimension at most two, and condition 3 a consequence of the fact that this subgroup acts trivially on the tangent bundles of these component.

Observe that an integral GKM graph is a particular case of an abstract GKM graph, as defined in [2]; here we require $\alpha$ to take values in $\mathbb{Z}_T^*$ rather than in $t^*$, and in Definition 2.2 (3) we require $\alpha(\nabla_e(e')) - \alpha(e')$ to be an integer multiple of $\alpha(e)$, for every $e = (p, q) \in E$ and $e' \in E_p$. The necessity of these integrality properties will be clear from the definition of $T$-action on $\Gamma$. Let $R(T)$ be the representation ring of $T$; notice that $R(T)$ can be identified with the character ring of $T$, i.e., with the ring of finite sums

$$\sum_{k} m_k e^{2\pi \sqrt{-1}\alpha_k},$$

where the $m_k$’s are integers and $\alpha_k \in \mathbb{Z}_T^*$. So giving an axial function $\alpha : E \to \mathbb{Z}_T^*$ is equivalent to giving a map which assigns to each edge $e \in E$ a one dimensional representation $\rho_e$, whose character $\chi_e : T \to S^1$ is given by

$$\chi_e \left(e^{2\pi \sqrt{-1}\xi}\right) = e^{2\pi \sqrt{-1}\alpha(e)(\xi)}.$$  

For every $e \in E$, let $T_e = \ker(\chi_e)$, and consider the restriction map

$$r_e : R(T) \to R(T_e).$$
Then for every vertex $p \in V$, we also obtain a $d$-dimensional representation

$$v_p \simeq \bigoplus_{e \in E_p} \rho_e$$

which, by Definition 2.2 (3) satisfies

$$r_e(v_{i(e)}) \simeq r_e(v_{t(e)}). \quad (2.2)$$

So an integral axial function $\alpha : E \rightarrow \mathbb{Z}_T^*$ defines a one dimensional representation $\rho_e$ for every edge $e \in E$ and for every $p \in V$ a $d$-dimensional representation $v_p$ satisfying (2.2); this is what we refer to as a $T$-action on $\Gamma$.

**Remark 2.5** Henceforth in this article all GKM graphs will be, unless otherwise specified, integral GKM graphs.

We will now define the $K$-ring $K_\alpha(\Gamma)$ of $(\Gamma, \alpha)$. As we remarked in the introduction, Knutson and Rosu have proved that if $(\Gamma, \alpha)$ is the GKM graph associated to a GKM manifold $M$, then

$$K_\alpha(\Gamma) \simeq K_T(M),$$

where $K_T(M)$ is the equivariant $K$-theory ring of $M$ (cf. [6]).

Let Maps($V, R(T)$) be the ring of maps which assign to each vertex $p \in V$ a representation of $T$. Following the argument in [6], we define a subring of Maps($V, R(T)$), called the ring of $K$-classes of $(\Gamma, \alpha)$.

**Definition 2.6** Let $f$ be an element of Maps($V, R(T)$). Then $f$ is a $K$-class of $(\Gamma, \alpha)$ if for every edge $e = (p, q) \in E$

$$r_e(f(p)) = r_e(f(q)). \quad (2.3)$$

Observe that using the identification of $R(T)$ with the ring of finite sums (2.1), condition (2.3) is equivalent to saying that for every $e = (p, q) \in E$

$$f(p) - f(q) = \beta \left(1 - e^{2\pi \sqrt{-1} \alpha(e)}\right), \quad (2.4)$$

for some $\beta$ in $R(T)$.

If $f$ and $g$ are two $K$-classes, then also $f + g$ and $fg$ are; so the set of $K$-classes is a subring of Maps($V, R(T)$).

**Definition 2.7** The $K$-ring of $(\Gamma, \alpha)$, denoted by $K_\alpha(\Gamma)$, is the subring of Maps($V, R(T)$) consisting of all the $K$-classes.

### 3 GKM fiber bundles

Let $(\Gamma_1, \alpha_1)$ and $(\Gamma_2, \alpha_2)$ be GKM graphs, where $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$, $\alpha_1 : E_1 \rightarrow \mathbb{Z}_{T_1}^* \subset t_1^*$ and $\alpha_2 : E_2 \rightarrow \mathbb{Z}_{T_2}^* \subset t_2^*$.

**Definition 3.1** An **isomorphism of GKM graphs** $(\Gamma_1, \alpha_1)$ and $(\Gamma_2, \alpha_2)$ is a pair $(\Phi, \Psi)$, where $\Phi : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism of graphs, and $\Psi : t_1^* \rightarrow t_2^*$ is an isomorphism of linear spaces such that $\Psi(\mathbb{Z}_{T_1}^*) = \mathbb{Z}_{T_2}^*$, and for every edge $(p, q)$ of $\Gamma_1$ we have $\alpha_2(\Phi(p), \Phi(q)) = \Psi(\alpha_1(p, q))$. 
By definition, if \((\Phi, \Psi)\) is an isomorphism of GKM graphs, the diagram

\[
\begin{array}{c}
E_1 \\
\downarrow \alpha_1 \downarrow \Psi_{|Z^*_T_1} \\
Z^*_T_1 \\
\downarrow \alpha_2 \\
Z^*_T_2 \\
\end{array} \xrightarrow{\Phi} \begin{array}{c}
E_2 \\
\downarrow \Psi_{|Z^*_T_2} \\
Z^*_T_2 \\
\end{array}
\]  

(3.1)

commutes. We can extend the map \(\Psi\) to be a ring homomorphism from \(R(T_1)\) to \(R(T_2)\) using the identification (2.1) and defining \(\Psi \left( e^{2\pi \sqrt{-1}\alpha} \right) = e^{2\pi \sqrt{-1}\Psi(\alpha)}\), for every \(\alpha \in \mathbb{Z}^*_T_1\).

Given \(f \in \text{Maps}(V_2, R(T_2))\), let \(\Upsilon^*(f) \in \text{Maps}(V_1, R(T_1))\) be the map defined by \(\Upsilon^*(f)(p) = \Psi^{-1}(f(\Phi(p)))\). From the commutativity of the diagram (3.1) it’s easy to see that if \(f \in K_{\alpha_2}(\Gamma_2)\) then \(\Upsilon^*(f) \in K_{\alpha_1}(\Gamma_1)\), and \(\Upsilon^*\) defines an isomorphism between the two \(K\)-rings.

We are now ready to define the main combinatorial objects of this paper, GKM fiber bundles. Let \((\Gamma, \alpha)\) and \((\Gamma_B, \alpha_B)\) be two GKM graphs, with \(\alpha\) and \(\alpha_B\) having images in the same weight lattice \(\mathbb{Z}^*_T\). Let \(\pi : \Gamma = (V, E) \rightarrow \Gamma = (V_B, E_B)\) be a surjective morphism of graphs. By that we mean that \(\pi\) maps the vertices of \(\Gamma\) onto the vertices of \(\Gamma\), such that, for every edge \(e = (p, q)\) of \(\Gamma\), either \(\pi(p) = \pi(q)\) (in which case \(e\) is called \(\text{vertical}\)), or \(\pi(p), \pi(q)\) is an edge of \(\Gamma_B\) (in which case \(e\) is called \(\text{horizontal}\)). Such a morphism of graphs induces a map \((d\pi)_p : H_p \rightarrow E_{\pi(p)}\) from the set of horizontal edges at \(p \in V\) to the set of all edges starting at \(\pi(p) \in V_B\). The first condition we impose for \(\pi\) to be a GKM fiber bundle is the following:

1: For all vertices \(p \in V\), \((d\pi)_p : H_p \rightarrow E_{\pi(p)}\) is a bijection compatible with the axial functions:

\[
\alpha_B((d\pi)_p(e)) = \alpha(e),
\]

for all \(e = (p, q) \in H_p\).

The second condition has to do with the connections on \(\Gamma\) and \(\Gamma_B\).

2: The connection along edges of \(\Gamma\) moves horizontal edges to horizontal edges, and vertical edges to vertical edges. Moreover, the restriction of the connection of \(\Gamma\) to horizontal edges is compatible with the connection on \(\Gamma_B\).

For every vertex \(p \in \Gamma_B\), let \(V_p = \pi^{-1}(p) \subset V\) and \(\Gamma_p\) the induced subgraph of \(\Gamma\) with vertex set \(V_p\). If the map \(\pi\) satisfies condition 1, then, for every edge \(e = (p, q)\) of \(\Gamma_B\), it induces a bijection \(\Phi_{p,q} : V_p \rightarrow V_q\) by \(\Phi_{p,q}(p') = q'\) if and only if \((p, q) = (d\pi)_p(p', q')\).

3: For every edge \((p, q) \in \Gamma_B\), \(\Phi_{p,q} : \Gamma_p \rightarrow \Gamma_q\) is an isomorphism of graphs compatible with the connection \(\nabla\) on \(\Gamma\) in the following sense: for every lift \(e' = (p_1, q_1)\) of \(e = (p, q)\) at \(p_1\) and every edge \(e'' = (p_1, p_2)\) of \(\Gamma_p\) the connection along the horizontal edge \((p_1, q_1)\) moves the vertical edge \((p_1, p_2)\) to the vertical edge \((q_1, q_2)\), where \(q_i = \Phi_{p,q}(p_i), i = 1, 2\).

We can endow \(\Gamma_p\) with a GKM structure, which is just the restriction of the GKM structure of \((\Gamma, \alpha)\) to \(\Gamma_p\). The axial function on \(\Gamma_p\) is the restriction of \(\alpha : E \rightarrow \mathbb{Z}^*_T\) to the edges of \(\Gamma_p\); we refer to it as \(\alpha_p\), and it takes values in \(v^*_p\), the subspace of \(v^*\) generated by values of axial functions \(\alpha(e)\), for edges \(e \in E\). The next condition we impose on \(\pi\) is the following:
4: For every edge \((p, q)\) of \(\Gamma_B\), there exists an isomorphism of GKM graphs
\[
\Upsilon_{p, q} = (\Phi_{p, q}, \Psi_{p, q}): (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q).
\]
By property (3) of Definition 2.2 and the fact that \(\alpha_B(e) = \alpha(e')\) we have
\[
\alpha(\Phi_{p, q}(p_1), \Phi_{p, q}(p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e),
\]
where \(m'(p_1, p_2)\) is an integer; so by the commutativity of (3.1) we have
\[
\Psi_{p, q}(\alpha(p_1, p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e).
\]
Condition (3.3) defines a map \(m': E_p \to \mathbb{Z}\), where \(E_p\) is the set of edges of \(\Gamma_p\). The next condition is a strengthening of this.

5: There exists a function \(m: v^*_p \cap \mathbb{Z}_T^* \to \mathbb{Z}\) such that
\[
\Psi_{p, q}(x) = x + m(x)\alpha_B(p, q).
\]
Observe that if \(\Gamma_B\) is connected, then all the fibers of \(\pi\) are isomorphic as GKM graphs. More precisely, for any two vertices \(p, q \in V_B\) let \(\gamma\) be a path in \(\Gamma_B\) from \(p\) to \(q\), i.e., \(\gamma: p = p_0 \to p_1 \to \cdots \to p_m = q\). Then the map
\[
\Upsilon_{\gamma} = \Upsilon_{p_{m-1}, p_m} \circ \cdots \circ \Upsilon_{p_0, p_1}: (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q).
\]
defines an isomorphism between the GKM graphs \((\Gamma_p, \alpha_p)\) and \((\Gamma_q, \alpha_q)\). As observed before, this isomorphism restricts to an isomorphism between the two \(K\)-rings, \(\Upsilon_{\gamma}^*: K_{\alpha_q}(\Gamma_q) \to K_{\alpha_p}(\Gamma_p)\), which is not an isomorphism of \(R(T)\)-modules, unless the linear isomorphism
\[
\Psi_{\gamma} = \Psi_{p_0, p_1} \circ \cdots \circ \Psi_{p_{m-1}, p_m}: v^*_q \to v^*_p
\]
is the identity.

Let \(\Omega(p)\) be the set of all loops in \(\Gamma_B\) that start and end at \(p\), i.e., the set of paths \(\gamma: p_0 \to p_1 \to \cdots \to p_{m-1} \to p_m\) such that \(p_0 = p_m = p\). Then every such \(\gamma\) determines a GKM isomorphism \(\Upsilon_{\gamma}\) of the fiber \((\Gamma_p, \alpha_p)\). The holonomy group of the fiber \((\Gamma_p, \alpha_p)\) is the subgroup of the GKM isomorphisms of the fiber, \(\text{Aut}(\Gamma_p, \alpha_p)\), given by
\[
\text{Hol}_\pi(\Gamma_p) = \{\Upsilon_{\gamma} \mid \gamma \in \Omega(p)\} \leq \text{Aut}(\Gamma_p, \alpha_p).
\]

4 Flag manifolds as GKM fiber bundles

In this section, we will discuss some important examples of GKM fiber bundles coming from generalized partial flag varieties.

Let \(G\) be a complex semisimple Lie group, \(\mathfrak{g}\) its Lie algebra, \(\mathfrak{h} \subset \mathfrak{g}\) a Cartan subalgebra and \(t \subset \mathfrak{h}\) a compact real form; let \(T\) be the compact torus whose Lie algebra is \(t\). Let \(\Delta \subset \mathbb{Z}_T^* \subset t^*\) be the set of roots and
\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha
\]
the Cartan decomposition of \(\mathfrak{g}\). Let \(\Delta_0 = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta\) be a choice of simple roots and \(\Delta^+\) the corresponding positive roots. The set of positive roots determines a Borel subalgebra \(\mathfrak{b} \subset \mathfrak{g}\) given by
\[
\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.
\]
Let $\Sigma \subset \Delta_0$ be a subset of simple roots, and $B \leq P(\Sigma) \leq G$, where $B$ is the Borel subgroup whose Lie algebra is $b$ and $P(\Sigma)$ the parabolic subgroup of $G$ corresponding to $\Sigma$. Then $M = G/P(\Sigma)$ is a (generalized) partial flag manifold. In particular, if $\Sigma = \emptyset$, then $P(\emptyset) = B$, and we refer to $M = G/B$ as the (generalized) complete flag manifold.

The torus $T$ with Lie algebra $\mathfrak{t}$ acts on $M = G/P(\Sigma)$ by left multiplication on $G$; this action determines a GKM structure on $M$ with GKM graph $(\Gamma, \alpha)$. In fact, let $W$ be the Weyl group of $\mathfrak{g}$ and $W(\Sigma)$ the subgroup of $W$ generated by reflections across the simple roots in $\Sigma$, and $(\Sigma)$ the positive roots which can be written as a linear combination of the roots in $\Sigma$. Then the vertices of $\Gamma$, corresponding to the $T$-fixed points, are in bijection with the right cosets

$$W/W(\Sigma) = \{vW(\Sigma) \mid v \in W\} = \{[v] \mid v \in W\},$$

where $[v] = vW(\Sigma)$ is the right $W(\Sigma)$-coset containing $v$. Two vertices $[v]$ and $[w]$ are joined by an edge if and only if there exists $\beta \in \Delta^+ \setminus \langle \Sigma \rangle$ such that $[v] = [ws_\beta]$; moreover the axial function $\alpha$ on the edge $e = ([v], [ws_\beta])$ is given by $\alpha([v], [ws_\beta]) = v\beta$; and it’s easy to see that this label is well defined. Moreover, given an edge $e' = ([v], [ws_\beta])$ starting at $[v]$, a natural choice of a connection along $e$ is given by $\nabla_e e' = ([v], [ws_\beta], [ws_\beta s_\beta])$. Then

$$\alpha(\nabla_e e') - \alpha(e') = s_\beta w_\beta \beta' - v\beta' = s_\beta v_\beta \beta' - v_\beta = m(e, e')v_\beta,$$

where $m(e, e')$ is a Cartan integer; so this connection satisfies property (3) of Definition 2.2.

Consider the natural $T$-equivariant projection $\pi : G/B \to G/P(\Sigma)$; this map induces a projection map on the corresponding GKM graphs $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ given by $\pi(w) = [w]$ for every $w \in W$, where $[w] = wW(\Sigma)$. As we proved in [2, section 4.3], $\pi$ is a GKM fiber bundle. It is in fact a balanced bundle, see [3]). Moreover, let $\Gamma_0$ be the GKM graph of the fiber containing the identity element of $W$; then

$$\text{Hol}_\pi(\Gamma_0) \simeq W(\Sigma).$$

The key ingredient in the proof of this is the identification of the weights of the $T$ action on the tangent space to the identity coset $p_0$ of $G/P$ with the complement of $\langle \Sigma \rangle$ in $\Delta^+$. Namely for any such root $\alpha$ let $w \in W$ be the Weyl group element $w : t \to t$ associated with the reflection in the hyperplane $\alpha = 0$. Then for the edge $e = (p_0, wp_0)$ of the GKM graph of $G/P$ the GKM isomorphism of $\Gamma_{p_0}$ onto $\Gamma_p$ is the isomorphism $\Phi_{p_0, p}$ associated with the left action of $w$ on $G/B$ and the $T$ automorphism (3.4) is just the action of $w$ on $t$ given by compositions of reflections in the hyperplane $\alpha_B(p, q) = 0$ in $t$, for some horizontal edge $(p, q)$. (These results are also a byproduct of the identification of the GKM model of $K_T(G/P)$ with the Kostant–Kumar model which we will describe in Sect. 9).

5 $K$-theory of GKM fiber bundles

Given a GKM fiber bundle $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$, we will describe the $K$-ring of $(\Gamma, \alpha)$ in terms of the $K$-ring of $(\Gamma_B, \alpha_B)$. In the proof of the main theorem we will need the following technical lemma.

Lemma 5.1 Let $\alpha$ and $\beta$ be linearly independent weights in $t^e$, and $P$ an element of $R(T)$. If $1 - e^{2\pi \sqrt{1} \alpha}$ divides $\left(1 - e^{2\pi \sqrt{1} \beta}\right) P$ then $1 - e^{2\pi \sqrt{1} \alpha}$ divides $P$.

Proof Let $\alpha = m\alpha_1$, for some $m \in \mathbb{Z}$, where $\alpha_1$ is a primitive element of the weight lattice in $t^e$. We can complete $\alpha_1$ to a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ of the lattice. Let $x_j = e^{2\pi \sqrt{1} \alpha_j}$ for
all \( j = 1, \ldots, d \).

Then by hypothesis \( (1 - x_1^n) \) divides \( (1 - x_i^nQ(x_2, \ldots, x_d))P(x_1, \ldots, x_d) \) for some non-constant polynomial \( Q(x_2, \ldots, x_d) \). Consider an element \( \xi \in \mathbb{t} \otimes \mathbb{C} \) such that \( x_1(\xi)^m = 1 \); then \( (1 - x_1^nQ(x_2, \ldots, x_d))P(x_1, \ldots, x_d)(\xi) = 0 \). Since in general \( (1 - x_1^nQ(x_2, \ldots, x_d))(\xi) \neq 0 \), this implies that \( (1 - x_1^n) \) divides \( P(x_1, \ldots, x_d) \). \( \square \)

For every \( K \)-class \( f : V_B \to R(T) \), define the pull-back \( \pi^*(f) : V \to R(T) \) by \( \pi^*(f)(q) = f(\pi(q)) \). It’s easy to check that \( \pi^*(f) \) is a \( K \)-class on \( (\Gamma, \alpha) \). So \( K_\alpha \) contains \( K_\alpha^B \) as a subring, and the map \( \pi^* : K_\alpha B(\Gamma_B) \to K_{\alpha}(\Gamma) \) gives \( K_{\alpha}(\Gamma) \) the structure of a \( K_{\alpha B}(\Gamma_B) \)-module.

**Definition 5.2** A \( K \)-class \( h \in K_{\alpha}(\Gamma) \) is called basic if \( h \in \pi^*(K_\alpha B(\Gamma_B)) \).

We denote the subring of basic \( K \)-classes by \( (K_\alpha(\Gamma))_{bas} \), clearly we have

\[
(K_\alpha(\Gamma))_{bas} \simeq K_{\alpha B}(\Gamma_B).
\]

**Theorem 5.3** Let \( \pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B) \) be a GKM fiber bundle, and let \( c_1, \ldots, c_m \) be \( K \)-classes on \( \Gamma \) such that for every \( p \in V_B \) the restriction of these classes to the fiber \( \Gamma_p = \pi^{-1}(p) \) form a basis for the \( K \)-ring of the fiber. Then, \( K_{\alpha B}(\Gamma_B) \)-modules, \( K_{\alpha}(\Gamma) \) is isomorphic to the free \( K_{\alpha B}(\Gamma_B) \)-module on \( c_1, \ldots, c_m \).

**Proof** First of all, observe that any linear combination of \( c_1, \ldots, c_m \) with coefficients in \( (K_\alpha(\Gamma))_{bas} \) is an element of \( K_{\alpha B}(\Gamma_B) \). Now we want to prove that the \( c_i \)'s are independent over \( K_{\alpha B}(\Gamma_B) \). In order to prove so, let \( \sum_{k=1}^m \beta_k c_k = 0 \) for some \( \beta_1, \ldots, \beta_m \in (K_\alpha(\Gamma))_{bas} \). Let \( \Gamma_p = \pi^{-1}(p) \) denote the fiber over \( p \in B \), \( \iota_p : \Gamma_p \to \Gamma \) the inclusion, and \( \iota_p^* : K_\alpha(\Gamma) \to K_{\alpha}(\Gamma_p) \) the restriction to the \( K \)-theory of the fiber. Then \( \sum_{k=1}^m \iota_p^*(\beta_k c_k) = 0 \) for all \( p \in B \). Since the \( \beta_k \)'s are basic \( K \)-classes, \( \iota_p^*(\beta_k) \) is just an element of \( R(T) \) for all \( k \). But by assumption \( \{\iota_p^*(c_1), \ldots, \iota_p^*(c_m)\} \) is a basis of \( K_{\alpha B}(\Gamma_p) \); so \( \iota_p^*(\beta_k) = 0 \) for all \( k = 1, \ldots, m \), for all \( p \in \Gamma_B \), which implies that \( \beta_k = 0 \) for all \( k \). We need to prove that the free \( K_{\alpha B}(\Gamma_B) \)-module generated by \( c_1, \ldots, c_m \) is \( K_{\alpha}(\Gamma) \).

Let \( c \in K_{\alpha}(\Gamma) \). Since the classes \( \iota_p^*(c_1), \ldots, \iota_p^*(c_m) \) are a basis for \( K_{\alpha}(\Gamma_p) \), there exist \( \beta_1, \ldots, \beta_m \in \text{Maps}(B, R(T)) \) such that

\[
c = \sum_{k=1}^m \beta_k c_k;
\]

we need to prove that the \( \beta_k \)'s belong to \( (K_\alpha(\Gamma))_{bas} \) for all \( k \). In order to prove this, it is sufficient to show that (2.4) is satisfied for every edge \( (p, q) \) of \( \Gamma_B \). Let \( e' = (p', q') \) be the lift of \( (p, q) \) at \( p' \in \Gamma_p \). Then

\[
c(q') - c(p') = \sum_{k=1}^m (\beta_k(q) c_k(q') - \beta_k(p) c_k(p'))
\]

\[= \sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') + \sum_{k=1}^m \beta_k(q) (c_k(q') - c_k(p')).\]

Since \( c, c_1, \ldots, c_m \) belong to \( K_{\alpha}(\Gamma) \), by (2.4) the differences \( c(q') - c(p') \), \( c_k(q') - c_k(p') \) are multiples of \( (1 - e^{2\pi \sqrt{-1} \alpha(e')}) \), for all \( k = 1, \ldots, m \). Therefore, for all \( p' \in \Gamma_p \),

\[
\sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') = (1 - e^{2\pi \sqrt{-1} \alpha(e')}) \eta(p'),
\]

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where \( \eta(p') \in R(T) \). We will show that \( \eta : \Gamma_p \to R(T) \) belongs to \( K_\alpha(\Gamma_p) \).

If \( p' \) and \( p'' \) are vertices in \( \Gamma_p \), joined by an edge \((p', p'')\), then

\[
\sum_{k=1}^{m} (\beta_k(q) - \beta_k(p))(c_k(p'') - c_k(p')) = \left(1 - e^{2\pi \sqrt{-1} \alpha(e')}\right)(\eta(p'') - \eta(p')).
\]

Each \( c_k \) is a \( K \)-class on \( \Gamma \), so \( c_k(p'') - c_k(p') \) is a multiple of \( \left(1 - e^{2\pi \sqrt{-1} \alpha(p', p'')}\right) \), for all \( k = 1, \ldots, m \). Then \( \left(1 - e^{2\pi \sqrt{-1} \alpha(p', p'')}\right) \) divides \( \left(1 - e^{2\pi \sqrt{-1} \alpha(e')}\right) \) and so \( \alpha(e') \) divides \( \eta(p'') - \eta(p') \), and so \( \eta \) is a \( K \)-class on \( \Gamma_p \).

Since the classes \( t_p^* c_1, \ldots, t_p^* c_m \) are a basis for \( K_\alpha(\Gamma_p) \) there exist \( Q_1, \ldots, Q_m \in R(T) \) such that

\[
\eta = \sum_{k=1}^{m} Q_k t_p^* c_k.
\]

Then

\[
\sum_{k=1}^{m} \left(\beta_k(q) - \beta_k(p) - Q_k \left(1 - e^{2\pi \sqrt{-1} \alpha(e')}\right)\right) t_p^* c_k = 0.
\]

But \( t_p^* c_1, \ldots, t_p^* c_m \) are linearly independent over \( R(T) \), so

\[
\beta_k(q) - \beta_k(p) = Q_k \left(1 - e^{2\pi \sqrt{-1} \alpha(e')}\right).
\]

Since \( \alpha(e') = \alpha(p', q') = \alpha_B(p, q) \), this implies that \( \beta_k \in K_{\alpha_B}(\Gamma_B) \). Therefore every \( K \)-class on \( \Gamma \) can be written as a linear combination of classes \( c_1, \ldots, c_m \), with coefficients in \( K_{\alpha_B}(\Gamma_B) \).

\[ \square \]

6 Invariant classes

Let \( \pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B) \) be a GKM fiber bundle, and let \((\Gamma_p, \alpha_p)\) be one of its fibers. We say that \( f \in K_{\alpha_p}(\Gamma_p) \) is an invariant class if \( \Upsilon_{\gamma}(f) = f \) for every \( \Upsilon_\gamma \in \text{Hol}_\pi(\Gamma_p) \). We denote by \((K_{\alpha_p}(\Gamma_p))^{\text{Hol}}\) the subring of \( K_{\alpha_p}(\Gamma_p) \) given by invariant classes.

Given any such class \( f \in (K_{\alpha_p}(\Gamma_p))^{\text{Hol}} \), we can extend it to be an element of \( \text{Maps} : V \to R(T) \) by the following recipe: let \( q \) be a vertex of \( \Gamma_B \), and \( \gamma \) a path in \( \Gamma_B \) from \( q \) to \( p \). Let \( \Upsilon_\gamma : (\Gamma_q, \alpha_q) \to (\Gamma_p, \alpha_p) \) be the isomorphism of GKM graphs associated to \( \gamma \); then, as we observed before, \( \Upsilon_{\gamma}(f) \) defines an element of \( K_{\alpha_q}(\Gamma_q) \). Notice that the invariance of \( f \) implies that \( \Upsilon_{\gamma}(f) \) only depends on the end-points of \( \gamma \); so we denote \( \Upsilon_{\gamma}(f) \) by \( f_q \).

**Proposition 6.1** Let \( c \in \text{Maps}(V, R(T)) \) be the map defined by \( c(q') = f_{\pi(q')}(q') \) for any \( q' \in V \). Then \( c \in K_\alpha(\Gamma) \).

**Proof** Since \( c \) is a class on each fiber, it is sufficient to check the compatibility Condition (2.4) on horizontal edges; let \( e = (p, q) \) be an edge of \( \Gamma_B \), and let \( e' = (p', q') \) be its lift at \( p' \in V \).
If \( c(p') = f_p(p') = \sum_k n_k e^{2\pi \sqrt{-1} \alpha_k} \), where \( \alpha_k \in \mathbb{Z}^*_+ \cap v_p^* \) and \( n_k \in \mathbb{Z} \) for all \( k \), then

\[
c(p') - c(q') = f_p(p') - f_q(q') = f_p(p') - \Psi_e(f_p(p')) = \sum_k n_k \left( e^{2\pi \sqrt{-1} \alpha_k} - e^{2\pi \sqrt{-1} \Psi_e(\alpha_k)} \right).
\]

By definition of GKM fiber bundle, \( \Psi_e(\alpha_k) = \alpha_k + c(\alpha_k) \alpha_B(p, q) \) for every \( k \), where \( c(\alpha_k) \) is an integer and \( \alpha_B(p, q) = \alpha(p', q') \). So

\[
e^{2\pi \sqrt{-1} \alpha_k} - e^{2\pi \sqrt{-1} \Psi_e(\alpha_k)} = e^{2\pi \sqrt{-1} \alpha_k} \left( 1 - e^{2\pi \sqrt{-1} c(\alpha_k) \alpha_B(p, q)} \right),
\]
and it’s easy to see that

\[
\left( 1 - e^{2\pi \sqrt{-1} c(\alpha_k) \alpha_B(p, q)} \right) = \beta_k \left( 1 - e^{2\pi \sqrt{-1} \alpha_B(p, q)} \right) \text{ for some } \beta_k \in R(T), \text{ for every } k. \quad \square
\]

7 Classes on projective spaces

Let \( T = (S^1)^n \) be the compact torus of dimension \( n \), with Lie algebra \( t = \mathbb{R}^n \), and let \( \{y_1, \ldots, y_n\} \) be the basis of \( t^* \simeq \mathbb{R}^n \) dual to the canonical basis of \( \mathbb{R}^n \). Let \( \{e_1, \ldots, e_n\} \) be the canonical basis of \( \mathbb{C}^n \). The torus \( T \) acts componentwise on \( \mathbb{C}^n \) by

\[
(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1z_1, \ldots, t_nz_n).
\] (7.1)

This action induces a GKM action of \( T \) on \( \mathbb{C}P^{n-1} \), and the GKM graph is \( \Gamma = \mathcal{K}_n \), the complete graph on \( n \) vertices labelled by \( [n] = \{1, \ldots, n\} \). The axial function \( \alpha \) on the edge \((i, j)\) is \( y_i - y_j \), for every \( i \neq j \). Let \( \mathcal{S} = \mathbb{Z}[y_1, \ldots, y_n], (\mathcal{S}) \) the field of fractions of \( \mathcal{S} \), \( \mathcal{M} = \text{Maps}([n], \mathcal{S}) \), and

\[
H_\alpha(\Gamma) = \{ f \in \mathcal{M} \mid f(j) - f(k) \in (y_j - y_k)\mathcal{S}, \text{ for all } j \neq k \}.
\]

Then \( H_\alpha(\Gamma) \) is an \( \mathcal{S} \)-subalgebra of \( \mathcal{M} \). Let \( \int_\Gamma : \mathcal{M} \to (\mathcal{S}) \) be the map

\[
\int_\Gamma f = \sum_{k=1}^n f(k) \prod_{j \neq k} (y_k - y_j).
\]

**Proposition 7.1** Let \( f \in \mathcal{M} \). Then \( f \in H_\alpha(\Gamma) \) if and only if \( \int_\Gamma f \in \mathcal{S} \).

**Proof** We have

\[
\int_\Gamma f = \sum_{k=1}^n f(k) \prod_{j \neq k} (y_k - y_j) = \frac{P}{\prod_{j < k} (y_j - y_k)},
\]
where \( P \in \mathcal{S} \). The factors in the denominator are distinct and relatively prime, hence \( \int_\Gamma f \in \mathcal{S} \) if and only if all factors in the denominator divide \( P \).

The factor \( y_j - y_k \) comes from

\[
\frac{f(k)}{\prod_{i \neq k} (y_i - y_j)} + \frac{f(j)}{\prod_{i \neq j} (y_i - y_j)} = \frac{f(j) - f(k)}{(y_j - y_k) \prod_{i \neq j, k} (y_j - y_i)} + \frac{f(k) \left( \prod_{i \neq k, j} (y_k - y_i) - \prod_{i \neq j, k} (y_j - y_i) \right)}{(y_j - y_k) \prod_{i \neq j, k} (y_j - y_i)}.
\]

But \( y_j - y_k \) divides the numerator of the second fraction, hence \( y_j - y_k \) divides \( P \) if and only if it divides \( f(j) - f(k) \). \quad \square
The permutation group $S_n$ acts on $\mathbb{S}$ by permuting variables, and that action induces an action on $H_\alpha(\Gamma)$ by

$$(w \cdot f)(j) = w^{-1} \cdot f(w(j)).$$

We say that a class $f \in H_\alpha(\Gamma)$ is $S_n$-invariant if $w \cdot f = f$ for every $w \in S_n$, i.e.,

$$f(w(j)) = w \cdot f(j) \text{ for every } w \in S_n.$$ 

The goal of this section is to construct bases of the $\mathbb{S}$-module $H_\alpha(\Gamma)$ consisting of $S_n$-invariant classes, and to give explicit formulas for the coordinates of a given class in those bases.

Let $\phi : [n] \to \mathbb{S}$, $\phi(j) = y_j$ for all $1 \leq j \leq n$. Then $\phi$ is an $S_n$-invariant class in $H_\alpha(\Gamma)$. For $1 \leq k \leq n$, let $f_k = \phi^k - 1$. Then $f_1, f_2, \ldots, f_n$ are $\mathbb{S}$-linearly independent invariant classes.

For $0 \leq j \leq n$, let $s_j$ be the $j$th elementary symmetric polynomial in the variables $y_1, \ldots, y_n$. Then $s_0 = 1, s_1 = y_1 + \cdots + y_n, s_2 = y_1 y_2 + y_1 y_3 + \cdots + y_{n-1} y_n, \ldots, s_n = y_1 y_2 \cdots y_n$. For $1 \leq k \leq n$, let

$$g_k = f_k - s_1 f_{k-1} + s_2 f_{k-2} - \cdots + (-1)^{k-1} s_{k-1} f_1.$$

Then $g_1, g_2, \ldots, g_n$ are invariant classes and the transition matrix from the $f$’s to the $g$’s is triangular with ones on the diagonal, hence it is invertible over $\mathbb{S}$. Therefore the classes $g_1, \ldots, g_n$ are also $\mathbb{S}$-linearly independent.

Let $\langle ., . \rangle : H_\alpha(\Gamma) \times H_\alpha(\Gamma) \to \mathbb{S}$ be the pairing

$$\langle f, g \rangle = \int_{\Gamma} f g.$$

**Theorem 7.2** The sets of classes $\{f_1, \ldots, f_n\}$ and $\{g_n, \ldots, g_1\}$ are dual to each other:

$$\langle f_j, g_{n-k+1} \rangle = \delta_{jk},$$

for all $1 \leq j, k \leq n$.

**Proof** We have $\int_{\Gamma} \phi^k = 0$ for all $0 \leq k \leq n-2$ and $\int_{\Gamma} \phi^{n-1} = 1$. Moreover

$$\phi^n - s_1 \phi^{n-1} + s_2 \phi^{n-2} - \cdots + (-1)^n s_n \phi^0 = 0.$$

Let $1 \leq j, k \leq n$. Then

$$f_j g_{n-k+1} = \phi^{j-1} \left( \phi^{n-k} - s_1 \phi^{n-k-1} + \cdots + (-1)^{n-k} s_{n-k} \phi^0 \right).$$

If $j < k$, then

$$f_j g_{n-k+1} = \text{ a combination of powers of } \phi \text{ at most } n - 2,$$

and then $\langle f_j, g_{n-k+1} \rangle = 0$.

If $j = k$, then

$$f_k g_{n-k+1} = \phi^{n-1} + \text{ a combination of powers of } \phi \text{ at most } n - 2,$$

hence $\langle f_k, g_{n-k+1} \rangle = \int_{\Gamma} \phi^{n-1} = 1$. 

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If $j > k$, then
\[
 f_j g_{n-k+1} = \phi^{j-k-1} \left( \phi^n - s_1 \phi^{n-1} + \cdots + (-1)^{n-k}s_{n-k} \phi^k \right)
 = -\phi^{j-k-1} \left( (-1)^{n-k-1}s_{n-k+1} \phi^{k-1} + \cdots + (-1)^n s_n \phi^0 \right)
 = a \text{ a combination of powers of } \phi \text{ at most } n - 2,
\]
and then $\langle f_j, g_{n-k+1} \rangle = 0$. \hfill \Box

The following is an immediate consequence of this result.

**Corollary 7.3** The sets $\{f_1, f_2, \ldots, f_n\}$ and $\{g_1, g_2, \ldots, g_n\}$ are dual bases of the $S$-module $H\alpha(\Gamma)$, and both bases consist of invariant classes. Moreover, if $h \in H\alpha(\Gamma)$ and
\[
 h = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = b_1 g_1 + b_2 g_2 + \cdots + b_n g_n,
\]
then
\[
 a_k = \langle g_{n-k+1}, h \rangle \quad \text{and} \quad b_k = \langle f_{n-k+1}, h \rangle.
\]

The entire discussion above extends naturally to $K$-theory. Let $z_j = e^{2\pi \sqrt{-1} y_j}$ for $j = 1, \ldots, n$. If $y_1, \ldots, y_n$ denote a basis of $\mathbb{Z}_T^n$, we have that
\[
 R(T) = \mathbb{Z} \left[ z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1} \right].
\]

Let $\psi: S \to R(T)$ be the injective ring morphism determined by $\psi(y_j) = z_j$, for $j = 1, \ldots, n$. Its image is $\psi(S) = R_+(T) = \mathbb{Z}[z_1, \ldots, z_n]$. Let
\[
 K\alpha(\Gamma) = \{g: [n] \to R(T) \mid f(j) - f(k) \in (z_j - z_k)R(T), \text{ for all } j \neq k\}.
\]

Then
\[
 \Phi: H\alpha(\Gamma) \to K\alpha(\Gamma), \quad \Phi(f)(j) = \psi(f(j))
\]
is an injective morphism of rings and
\[
 \Phi(qf) = \psi(q) \Phi(f)
\]
for all $f \in H\alpha(\Gamma)$ and $q \in S$. The image of $\Phi$ is
\[
 \text{im}\Phi = \{g \in K\alpha(\Gamma) \mid \text{im}(g) \subset R_+(T)\}.
\]

Let $\nu = \Phi(\phi)$; then $\nu: [n] \to R(T), \nu(j) = z_j$.

The symmetric group $S_n$ acts on $R(T)$ by simultaneously permuting the variables $z$ and $z^{-1}$, and that action induces an action on $K\alpha(\Gamma)$. The $K$-class $\nu$ is invariant, and so are its powers.

**Proposition 7.4** The invariant classes $\{1, v, \ldots, v^{n-1}\}$ form a basis of $K\alpha(\Gamma)$ over $R(T)$.

**Proof** A Vandermonde determinant argument shows that the classes are independent. If $g \in K\alpha(\Gamma)$, then there exists an invertible element $u \in R(T)$ such that $ug \in \text{im}\Phi$. Let $ug = \Phi(f)$, with $f \in H\alpha(\Gamma)$. If
\[
 f = a_0 \phi^0 + \cdots + a_{n-1} \phi^{n-1},
\]
then
\[
 g = u^{-1} \psi(a_0) v^0 + \cdots + u^{-1} \psi(a_{n-1}) v^{n-1}
\]
and therefore the classes also generate $K\alpha(\Gamma)$. \hfill \Box
8 Invariant bases on flag manifolds

In this section, we use the same technique as above to produce a basis of invariant $K$-classes on the variety of complete flags in $\mathbb{C}^{n+1}$.

As in the previous section, let $T = (S^1)^{n+1}$ act componentwise on $\mathbb{C}^{n+1}$ by

$$(t_1, \ldots, t_{n+1}) \cdot (z_1, \ldots, z_{n+1}) = (t_1z_1, \ldots, t_{n+1}z_{n+1}),$$

and let $x_1, \ldots, x_{n+1} \in \mathbb{Z}_+^n$ be the weights. This action induces a GKM action both on the flag manifold $Fl(\mathbb{C}^{n+1})$ and $\mathbb{C}^n$.

The GKM graph $(\Gamma, \alpha)$ associated to $Fl(\mathbb{C}^{n+1})$ is the permutahedron: its vertices are in bijection with the elements of $S_{n+1}$, the group of permutations on $n + 1$ elements, and there exists an edge $e$ between two vertices $\sigma$ and $\sigma'$ if and only if $\sigma$ and $\sigma'$ differ by a transposition, i.e., $\sigma' = \sigma(i, j)$, for some $1 \leq i < j \leq n + 1$; the axial function is given by $\alpha(\sigma, \sigma') = x_{\sigma(i)} - x_{\sigma'(i)}$.

The group $S_{n+1}$ acts on the GKM graph by left multiplication on its vertices, and on $t^\sigma$ by

$$\sigma \cdot x_i = x_{\sigma(i)}.$$  \hspace{1cm} (8.1)

Using the identification (2.1), this action determines an action on $R(T)$ by defining

$$\sigma \cdot \left(e^{2\pi \sqrt{-1} x_i}\right) = e^{2\pi \sqrt{-1} x_{\sigma(i)}}$$

and then extending it to the elements of $R(T)$ in the natural way. Using the fiber bundle construction introduced in this paper we will now produce a basis of the $K$-ring of $(\Gamma, \alpha)$ composed of $S_{n+1}$-invariant classes, i.e., elements $f \in K_\alpha(\Gamma)$ satisfying

$$f(u) = u \cdot f(id) \quad \text{for every} \quad u \in S_{n+1}.$$  \hspace{1cm} (8.2)

The main idea in the construction of our invariant basis of $K$-classes for $Fl(\mathbb{C}^{n+1})$ is to use the natural projection $\pi$ of $Fl(\mathbb{C}^{n+1})$ to $\mathbb{C}P^n$ with fiber $Fl(\mathbb{C}^n)$ to construct these classes by an induction argument on $n$.

As we saw in Sect. 7, the GKM graph $(\Gamma_B, \alpha_B)$ associated to $\mathbb{C}P^n$ is $(K_{n+1}, \alpha_B)$, where $\alpha_B(i, j) = x_i - x_j$ for every $1 \leq i \neq j \leq n + 1$. The projection $\pi : Fl(\mathbb{C}^{n+1}) \rightarrow \mathbb{C}P^n$ can be described in a simple way in terms of the vertices of the GKM graphs; in fact $\pi : (\Gamma, \alpha) \rightarrow (\Gamma_B, \alpha_B)$ is simply given by $\pi(\sigma) = \sigma(1)$, for every $\sigma \in S_{n+1}$.

Let $z_i$ be $e^{2\pi \sqrt{-1} x_i}$ for every $i = 1, \ldots, n + 1$ and $v : [n + 1] \rightarrow R(T)$ be invariant the $K$-class given by $v(i) = z_i$ for $i = 1, \ldots, n + 1$. By Proposition 7.4, the classes $1, v, \ldots, v^n$ form an invariant basis of $K_{\alpha_B}(K_{n+1})$, and they lift to $S_{n+1}$-invariant basic classes on $\Gamma$.

Let $\Gamma_{n+1}$ be the fiber over $\mathbb{C} \cdot e_{n+1}$, where $\{e_1, \ldots, e_{n+1}\}$ denotes the canonical basis of $\mathbb{C}^{n+1}$, the holonomy group on this fiber is isomorphic to $S_n$ viewed as the subgroup of $S_{n+1}$ which leaves the element $n + 1$ fixed. Once we construct a basis of $K_\alpha(\Gamma_{n+1})$ consisting of holonomy invariant classes, we can extend those to a set of classes of $K_\alpha(\Gamma)$, which, together with the invariant basic classes constructed above, will generate a basis of $K_\alpha(\Gamma)$ as a module over $R(T)$. We will show that if we start with a natural choice for the base of the induction, then the global classes generated by this means are indeed $S_{n+1}$ invariant.

Letting $I = [i_1, \ldots, i_n]$ be a multi-index of non-negative integers, define

$$z^I = z_1^{i_1}z_2^{i_2}\cdots z_n^{i_n}$$

and let $C_I = C_T(z^I) : S_{n+1} \rightarrow R(T)$ be the element defined by
\[ C_I(\sigma) = \sigma \cdot z^I \] for every \( \sigma \in S_{n+1} \);

it’s easy to verify that \( C_I \) is an invariant class of \( K_\alpha(\Gamma) \).

**Theorem 8.1** Let

\[ A_n = \{ I = [i_1, \ldots, i_n] \mid 0 \leq i_1 \leq n, 0 \leq i_2 \leq n - 1, \ldots, 0 \leq i_n \leq 1 \}. \]

Then the set

\[ \{ C_I = C_T(z^I) \mid I \in A_n \} \]

is an invariant basis of \( K_\alpha(\Gamma) \) as an \( R(T) \)-module.

**Proof** As mentioned above, the proof is by induction. Let \( n = 2 \), then the fiber bundle \( \pi : Fl(\mathbb{C}^3) \to \mathbb{CP}^2 \) is a \( \mathbb{CP}^1 \)-bundle. Let \( p = \mathbb{C} \cdot e_3 \in \mathbb{CP}^2 \) be the one dimensional subspace generated by the third vector in the canonical basis of \( \mathbb{C}^3 \). Then the fiber over \( p \) is a copy of \( \mathbb{CP}^1 \) and the invariant classes \( C_{[0]} \) and \( C_{[1]} \) form a basis of the \( K \)-ring of the fiber. We can extend these classes using transition maps between fibers, thus obtaining \( S_3 \)-invariant \( K \)-classes \( C_{[0,0]} \) and \( C_{[0,1]} \) on \( Fl(\mathbb{C}^3) \). By Proposition 7.4, the invariant classes \( 1, v, v^2 \) form a basis of the \( K \)-ring of the base, and they lift to \( S_3 \)-invariant basic classes \( C_{[1,0]} \) and \( C_{[2,0]} \). By Theorem 5.3, the \( K \)-ring of \( Fl(\mathbb{C}^3) \) is freely generated over \( R(T) \) by the invariant classes \( C_{[0,0]}, C_{[0,1]}, C_{[1,0]}, C_{[1,1]}, C_{[2,0]} \) and \( C_{[2,1]} \).

The general statement follows by repeating inductively the same argument, since at each stage the fiber of \( \pi : Fl(\mathbb{C}^m+1) \to \mathbb{CP}^m \) is a copy of \( Fl(\mathbb{C}^m) \).

**Remark 8.2** This argument can be adapted to give a basis of invariant \( K \)-classes for the generalized flag variety of type \( C_n \). Let \( \alpha_i = x_i - x_{i+1} \) for \( i = 1, \ldots, n-1 \) and \( \alpha_n = 2x_n \) be a choice of simple roots of type \( C_n \). The corresponding Weyl group \( W \) is the group of signed permutations of \( n \) elements. Let \( \Sigma = \{ \alpha_2, \ldots, \alpha_n \} \), then \( G/P(\Sigma) \) is a GKM manifold diffeomorphic to a complex projective space \( \mathbb{CP}^{2n-1} \); its GKM graph is a complete graph \( K_{2n} \) whose vertices can be identified with the set \( \{ \pm 1, \ldots, \pm n \} \) and the axial function \( \alpha \) is simply given by \( \alpha(\pm i, \pm j) = \pm x_i \mp x_j \), for every edge \( (\pm i, \pm j) \) of the GKM graph. Observe that the procedure in Sect. 7 can be used here to produce a \( W \)-invariant basis of \( K_\alpha(K_{2n}) \).

In fact it is sufficient to let \( y_1 = x_1, \ldots, y_n = x_n, y_{n+1} = -x_1, y_{2n} = -x_n \); the basis of Proposition 7.4 is \( S_{2n} \)-invariant, and hence in particular \( W \)-invariant.

Let \( G/B \) be the generic co-adjoint orbit of type \( C_n \), and \( (\Gamma, \alpha) \) its GKM graph. If we consider the natural projection \( G/B \to G/P(\Sigma) \), the fiber is diffeomorphic to a generic co-adjoint orbit of type \( C_{n-1} \). Hence, we can repeat the inductive argument used in type \( A_n \) to produce a \( W \)-invariant basis of \( K_\alpha(\Gamma) \).

**9 The Kostant–Kumar description**

The manifolds in Sect. 4 are also describable in terms of compact groups. Namely, if we let \( G_0 \) be the compact form of \( G \) and \( K \) the maximal compact subgroup of \( P \), then \( M = G/P = G_0/K \). Moreover, \( W = W_{G_0} \) and \( W(\Sigma) = W_K, W_{G_0}, \) and \( W_K \) being the Weyl groups of \( G_0 \) and \( K \), so \( M^T = W_{G_0}/W_K \). A fundamental theorem in equivariant \( K \)-theory is the Kostant–Kumar theorem, which asserts that \( K_T(M) \) is isomorphic to the tensor product

\[ R^{W_K} \otimes R^W R, \]

(9.1)
where $R$ is the character ring $R(T)$, and $R^{WK}$ and $R^W$ are the subrings of $W_K$ and $W$-invariant elements in $R$. This description of $K_T(M)$ generalizes to $K$-theory the well-known Borel description of the equivariant cohomology ring $H_T(M)$ as the tensor product

$$S(t^*)^{WK} \otimes_{S(t^*)^W} S(t^*)$$

(9.2)

and in [4] the authors showed how to reconcile this description with the GKM description of $H_T(M)$. Mutatis mutandi, their arguments work as well in $K$-theory and we will give below a brief description of the $K$-theoretic version of their theorem.

Let $\Gamma$ be the GKM graph of $M$. As we pointed out above, $M^T = W/W_K$, so the vertices of $\Gamma$ are the elements of $W/W_K$. Now let $f \otimes g$ be a decomposable element of the tensor product (9.1). Then one gets an $R$-valued function, $k(f \otimes g)$, on $W/W_K$ by setting

$$k(f \otimes g)(w_{WK}) = wfg.$$  

(9.3)

One can show that this defines a ring morphism, $k$, of the ring (9.1) into the ring Maps($M^T$, $R$), and in fact that this ring morphism is a bijection of the ring (9.1) onto $K_\alpha(\Gamma)$. (For the proof of the analogous assertions in cohomology see Sect. 2.4 of [4].) Moreover the action of $W$ on $K_T(M)$ becomes, under this isomorphism, the action

$$w(f_1 \otimes f_2) = f_1 \otimes wf_2$$

(9.4)

defined on the ring (9.1), so the ring of $W$-invariant elements in $K_T(M)$ gets identified with the tensor product $R^{WK} \otimes_{R^W} R^W$, which is just the ring $R^{WK}$ itself. Finally we note that if $M$ is the generalized flag variety, $G/B = G_0/T$, (9.1) becomes the tensor product

$$R \otimes_{R^W} R$$

(9.5)

and the ring of $W$-invariant elements in $K_T(M)$ becomes $R$. Moreover, if $\pi$ is the fibration $G_0/T \to G_0/K$, the fiber $F$ over the identity coset of $G_0/K$ is $K/T$; so $K_T(F)$ is the tensor product $R \otimes_{R^{WK}} R$, and the subring of $W_K$-invariant elements in $K_T(F)$ is $R$ which, as we saw above, is also the ring of $W_G$-invariant elements in $K_T(G_0/T)$ which is also (see Sect. 6) the ring of invariant elements associated with the fibration $G_0/T \to G_0/K$. Thus, most of the features of our GKM description of the fibration $G_0/T \to G_0/K$ have simple interpretations in terms of this Kostant–Kumar model.

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