What is faster—light or gravity?

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Abstract

General relativity lacks the notion of the speed of gravity. This is inconvenient, and the current paper is aimed at filling this gap. To that end I introduce the concept of the ‘alternative’ and argue that its variation, called the ‘superluminal alternative’, describes exactly what one understands by the ‘superluminal gravitational signal’. Another, closely related, object called the ‘semi-superluminal alternative’ corresponds to the situation in which a massive (and therefore gravitating) body reaches its destination sooner than a photon would if the latter were sent instead of the body. I prove that in general relativity constrained by the condition that only globally hyperbolic space-times are allowed, (1) semi-superluminal alternatives are absent, and (2) under some natural conditions and conventions, admissible superluminal alternatives are absent too.

Keywords: speed, gravity, superluminal

1. Introduction

The goal of this paper is to compare the speed of gravity with the speed of light within general relativity. In this section we discuss a major obstacle in achieving this goal, which is the lack of a suitable—that is, physically motivated but rigorous—definition of the ‘speed of gravity’ in the general case (i.e., say, beyond the linearized theory). Without such a definition any answer to the question posed in the title of this paper is obviously meaningless, but the reason for the lack of such a definition is quite valid: the Universe according to relativity is a ‘motionless’, ‘unchanging’ four-dimensional object, and gravity is just its shape. But what can be called the speed of a shape? What is the ‘speed of being a ball’?

Still, there are situations in which it would be convenient to be able to assign a speed to gravity, or at least to be able to tell whether it is greater or less than the speed of light. Consider, for example, an observer orbiting a red giant. Suppose one day the events $s$, $a$, and $b$ happen: $s$ is the star exploding as a supernova, $a$ is the observer seeing the explosion, and $b$ is the observer’s equipment showing that the local geometry has drastically changed as a
result of the same explosion. The sought-for definitions must allow one to say that the propagation of gravity was superluminal if $b \prec a$ (written also $b \in I^-(a)$ or $a \in I^+(b)$), i.e., if there is a piecewise timelike future-directed curve from $a$ to $b$. To put it slightly more mathematically, let us write $x \prec y$ for ‘$x$ gravitationally affects $y$’ or, interchangeably, ‘$x$ is a (gravitational) cause of $y$’. Then our task is to define the ‘gravitational cause’ so that the gravitational signaling will be recognized as superluminal if and only if there is a pair of points $s, b$ such that $s \prec b$ even though $s \notin b$, where $s \approx b$; or $s \in J^-(b)$ means ‘there is a piecewise nonspacelike future-directed curve from $s$ to $b$’. By saying so we, of course, have not solved the problem but have made it clearer, or so it seems. All one needs now is to put forward an intuitively acceptable criterion for whether an event acts gravitationally upon another. (In the example with the supernova this fact was hidden in the words ‘a result of the same explosion’.)

Probably the simplest step along these lines is to declare that $s \prec b$ when and only when the two events can be connected by a piecewise smooth curve—called a gravitational signal—defined by a condition imposed on its velocity. For example, the tangent to the curve might be required to be null, or to obey some constraints involving velocities often mentioned in discussing superluminal signaling by material fields: phase velocity, group velocity, or the velocity of transport of energy. For the gravitational field, however, this approach does not work. In the general case it is even hard to define those quantities, but there is also a more serious reason for the failure. Let us turn for a moment to material fields.

**Example 1.** (I) Consider a Minkowski space with a field $f$ in it which obeys only the equation
\[
\square f = \sum_{k=1}^{K} \delta(x - x_k - v_k t),
\]
where the constants $K, x_k, v_k$ are free parameters of the theory. It is not that simple to justify any particular definition of the signal in this case (hereafter we will touch on that). What is clear in advance, however, is that any reasonable definition of signals must be satisfied, in particular, by the future-directed lightlike broken lines. (Otherwise, one will have to reinterpret the entire special relativity, with its thought experiments involving essentially the same field.) Similarly, spacelike separated points must prove to be causally disconnected. So, for instance, the point $b_N$ with the coordinates $t = 1, x = 1, y = N, z = 0$ is affected by the origin of the coordinates when $N = 0$, but not when $N = 1$.

It is noteworthy that such a choice of cause–effect relationship makes the interpretation of the lines $x = x_k + v_k t$ with $v_k < 1$ and with $v_k > 1$ strikingly different. Although the former geodesics are just the world lines of ordinary pointlike charges (the zero acceleration may mean that their masses are very large), the latter do not correspond to any particles at all. Each point of such a line is causally disconnected from all others. So instead of propagation of a particle we have a process which takes place independently at every point of the line (cf. a light spot running along a remote surface [1]) and which consists in $f$ infinitely growing prior to such a point and falling immediately after its occurrence. (II) Consider now a theory in which the field $f$ obeys the wave equation but also is subject to an additional condition of the periodicity in the $y$ direction:
\[
f(t, x, y + 1, z) = f(t, x, y, z).
\]
Now whatever information about the origin of coordinates \( o \) can be inferred from the values at \( b_0 \) of the field and its derivatives \( f_{\mu...} (b_0) \), exactly the same information will be available to an observer who measures \( f_{\mu...} (b_N) \), \( N \neq 0 \). So we have to conclude that in this theory \( o \ll b_N \; \forall N \).

There are no reasons whatsoever to believe in periodic fields. The example is cited only to demonstrate that (i) equations of motion alone cannot determine the causal structure of a theory; correspondingly, none of the aforementioned velocities can serve as the signal speed, and (ii) two events \( (o \ll b_2, \text{ for instance}) \) can be causally related \( (o \ll b_2), \) even though they are not connected by a signal understood as a curve \( \sigma(\tau) \), such that

\[
\sigma(0) = o, \quad \sigma(1) = b_2, \quad \sigma(\tau_1) \ll \sigma(\tau_2) \quad \text{at} \; \tau_1 < \tau_2.
\]

In this sense the relation \( \ll \) is not quite local.

From the foregoing it appears that one ought to abandon (at this stage, at least) the concept of the signal and to define the relation \( \ll \) directly from its physical meaning, in the spirit of the preceding examples. It appears that the notion of ‘cause’ will be satisfactorily captured by a relation \( \ll \) if the latter has the following properties:

P1. \( \ll \) is a partial order relation. Indeed, it must be transitive (since the cause of a cause is obviously a cause), reflexive (it is just a matter of convention and we choose the analogy with the relation \( \equiv \)), and antisymmetric (an event different from \( a \) cannot be both a cause and an effect of \( a \)).

P2. if \( a \ll b \), then there exists a set \( S \) such that

(a) \( S \) determines \( f(b) \) in the sense that the values taken in \( S \) by the field and its derivatives \( f_{\mu...} (x) , x \in S \) fix uniquely the value \( f(b) \).

(b) If \( A \) is a neighborhood of \( a \), then \( S - A \) does not determine \( f(b) \).

The requirement P2 is justified by the fact that it is an embodiment of the idea that

P2*. Any change in the effect is produced only by a change in some of its causes.

The relation \( \ll \) is not defined uniquely by those properties; for example, the relation \( \gg \) defined by the equivalence

\[
a \gg b \iff b \ll a
\]

presumably also possesses them. To fix the non-uniqueness one may need an additional convention, which is not surprising: different definitions of the causal order within a given theory account for different views on what is freely specifiable in that theory.

One might wish the above formulated definition to be more strict, but by and large it seems adequate in discussing causal properties of matter fields. It may be expected that in the gravitational case the cause–effect relationship can be introduced in the same manner; one only must take \( f \) to be the metric. Presumably, it is this conviction that suggests the following simple resolution of the problems considered in this paper: ‘The solution [to the Einstein equations] obtained depends, at a point \( x \), only on the initial data within the hypercone of light rays […] with vertex \( x \), that is, on the relativistic past of that point. This result confirms the relativistic causality principle as well as the fact that gravitation propagates with the speed of light’ [2]. The flaw in this resolution is that ‘is fixed as a solution to a differential equation by the data within a set \( S \)’ and ‘is caused only by points of \( S \)’ are not the same. In other words, the causality relation in the gravitational case may not obey P2. This is, in particular, because
the principle P2* does not apply to the metric. The point is that while for a material field \( f \) it is quite clear what ‘a change in \( f(p) \)’ is, there is no such thing as a ‘change in the metric at \( p \)’. Indeed, in considering a spacetime \((M_1, g_1)\) one can give a precise meaning to the words ‘the geometry of a set \( V_1 \subseteq M_1 \) has changed’: they mean that we consider another spacetime \((M_2, g_2)\) and state that there is a set \( V_2 \subseteq M_2 \) and an isometry \( \phi \) which maps \( M_1 - V_1 \) to \( M_2 - V_2 \) but which cannot be extended to an isometry mapping the entire \( M_1 \) to \( M_2 \). However, that change cannot be resolved into pointwise changes: there is no way, in the general case, to have a particular \( p_2 \in V_2 \) correspond to each \( p_1 \in V_1 \) (note that \( V_2 \) even need not be diffeomorphic to \( V_1 \)) so as to compare \( g_2(p_2) \) to \( g_1(p_1) \) and thus to find out whether the metric in \( p_1 \) has changed.

It is clear from the foregoing that there is no easy way of introducing the relation \( \preceq \). Therefore we take a completely different approach.

2. Alternatives

In this section we formulate conditions which, being imposed on a pair of spacetimes \( M_1 \) and \( M_2 \), allow one to speak of that pair as describing two different extensions of a common prehistory. (In the example which opens the paper this prehistory would include the life of the red giant prior to the explosion \( s \).) That will enable us to translate the question of whether relativity (in a broad sense) admits superluminality of any kind into the question of when the difference between such \( M_1 \) and \( M_2 \) is attributable to a certain event and its consequences [3].

Definition 2. A pair of pointed inextendible spacetimes \((M_k, g_k, s_k), k = 1, 2\) is called an alternative if there exists a pair of open connected past sets \( N_k \supset J^-(s_k) - s_k \) and an isometry \( \phi \) which maps \( N_1 \) to \( N_2 \) and \( J^-(s_1) - s_1 \) to \( J^-(s_2) - s_2 \). (All matter fields in \( N_1 \) and \( N_2 \) are assumed to be tensors related by the same \( \phi \).)

Notation 3. For a given alternative the pair \( N_k, \phi \) need not be unique. Let \( \{N^*_{k}, \phi^*\} \) be the family of all such pairs. By \((N^*_{k}, \phi^*)\) we will denote its maximal element, that is, one which is, not ‘smaller’ than any other:

\[
\exists \alpha_0: \quad N^*_{1} \subset N^{\alpha_0}_{1}, \quad \phi^* = \phi^{\alpha_0}|_{N^*_{1}}.
\]

Correspondingly, \( N^*_{2} \equiv \phi^*(N^*_{1}) \).

The existence of \((N^*_{k}, \phi^*)\) follows from Zorn’s lemma since the open subsets of \( M_1 \) and \( M_2 \) are partially ordered by inclusion

\[
A \subset B \iff A \subseteq B,
\]

and with such an ordering every chain \( \ldots \subseteq A_1 \subseteq A_2 \subseteq \ldots \) has an upper bound \( \bigcup_i A_i \subseteq M_{1,2} \).

Comment 4. It is the regions \( N^*_{k} \subseteq M_k, k = 1, 2 \) that describe the mentioned prehistory. The requirement that they be isometric is obvious. It is also obvious why both of them must be past sets. (It is evident that two spacetimes do not describe the same region of the Universe if their inhabitants differ in remembrances.) As was explained in the introduction, our main interest is actually non-isometric regions of \( M_k \), and we need \( N^*_{k} \) only as a tool for outlining those regions. That is why we require \( N^*_k \) to be connected and maximal. Finally, the points
describe the event (the star explosion in the mentioned example) responsible for splitting the evolution of the Universe into the two branches.

**Definition 5.** The sets $\mathcal{F}_k \equiv \text{Bd } N^*_k, \ k = 1, 2$ will be termed fronts. A front $\mathcal{F}_k$ is superluminal if $\mathcal{F}_k \not\subset J^+ (s_k)$.

Being the boundary of a past set, a front is a closed, embedded, achronal three-dimensional $C^1$ submanifold [4, proposition 6.3.1]. At either $k$ the front $\mathcal{F}_k$ bounds the region $M_k \setminus N^*_k$, in which, loosely speaking, the remembrances (concerning the gravitational or material fields) of every observer differ from what they would remember if some other event happened in $s$. We interpret such a difference as evidence that the mentioned observer received a signal from $s$. Correspondingly, when such an observer is located out of $J^+ (s_k)$ the signal is superluminal; hence our definition.

The concept of an alternative is quite rough. In the general case it does not make it possible to assign a specific speed to a ‘gravitational signal’ if by the latter a front is understood. Even the source of the signal cannot be determined uniquely: the same pair of spacetimes can satisfy the definition of alternative with different choices of points $s_k$. Nevertheless, it allows one to formulate a necessary condition for calling the speed of gravity superluminal. Namely, in considering a particular theory (i.e., a set of material fields and their relationship to the geometry of the spacetime), let us single out a class of admissible alternatives by which the alternatives are understood consisting of spacetimes $M_{1,2}$ such that they are equally possible in that theory and differ only by the events $s_k$ and by the events which we agree to recognize as consequences of $s_k$ (not as consequences of some primordial difference in the spacetimes). If none of the admissible alternatives has a superluminal front, we will acknowledge that the speed of gravity in this theory is bounded by the speed of light.

3. Superluminal gravitational signals in GR

Let us adopt the convention that an alternative $(M_k, g_k, s_k)$ in which both spacetimes are globally hyperbolic is admissible only if there are Cauchy surfaces $S_k \subset M_k$ such that

$$s_k \in S_k, \quad S_2 - s_2 = \phi_b (S_1 - s_1)$$

and the values of material fields (and their derivatives, if necessary) in any $p \in S_1$ are the same as in $\phi_b (p)$. Such a criterion does not look far-fetched, for if there is no such pair of Cauchy surfaces, why should one regard the difference in $M_1$ and $M_2$ as ensuing from what happened in $s$ and its consequences, cf. [5]? It rather must be acknowledged as primordial.

The global hyperbolicity of $M_{1,2}$ implies the equality

$$M_k - J^+ (s_k) = \left[ J^- (s_k) - s_k \right] \bigcup D (S_k - s_k),$$

where $D(X)$ denotes the Cauchy domain of the set $X \subset M_k$ (i.e., the set of all points $p$ of $M_k$ such that every inextendible nonspacelike curve through $p$ meets $X$). By the existence and uniqueness theorem, equality of the initial data fixed at initial 3-surfaces implies isometry of the corresponding Cauchy domains. So if an alternative is admissible, $D (S_k - s_k)$ are isometric and hence $N^*_k$ (which by definition include $J^- (s_k) - s_k$) include also $M_k - J^+ (s_k)$. Thus, neither of the fronts is superluminal. In this sense general relativity does prohibit superluminal propagation of the gravitational field: under the above formulated assumptions the speed of a gravitational signal does not exceed the speed of light.
Remark 6. The approach developed in this paper is suitable for other geometric theories as well. For example, to analyze the signaling in a theory dealing only with the causal relations between events, not with the entire metric, it suffices to replace the words \textit{an isometry} $\phi$ in definition 2 with \textit{a conformal isometry} $\phi$. Correspondingly, one might be interested in a theory which considers only Ricci flat (i.e., empty, if the Einstein equations are imposed) spacetimes. The proposition proven in [6] and reformulated in terms of alternatives says that in such a theory superluminal alternatives turn out to be prohibited if an alternative is admissible only when in one of its spacetimes the Weyl tensor vanishes to the future from a Cauchy surface through $s$.

It is important that the aforementioned existence and uniqueness theorem is proven only under some ‘physically justified’ assumptions regarding the properties of the right-hand side of the Einstein equation. A possible set of such assumptions is listed, for example, in [4], and one of them is that the stress–energy tensor is at most a polynomial in $g^{ab}$. (The corresponding assumption in [7] allows the tensor to include also the first derivatives of the metric.) But those assumptions are known to fail in some physically interesting situations. In particular, vacuum polarization typically leads to the appearance, in the right-hand side of the Einstein equations, of terms containing second derivatives of the metric. This suggests that strong semiclassical effects like those expected in the early Universe, or near black hole horizons, may lead to superluminal propagation of gravity.

4. ‘Semi-superluminal’ alternatives

The fact that a single event is associated with two fronts, either in its own spacetime, has a quite nontrivial consequence because they do not need to be superluminal both \textit{at once}.

Definition 7. An alternative is called \textit{superluminal} if both its fronts are superluminal and \textit{semi-superluminal} if only one is.

Suppose, in a world $M_1$, a photon is sent from the Earth (we denote this event $s_1$) to arrive at a distant star at some moment $\tau_1$ by the clock of that star. Let, further, $M_2$ be the world which was initially the same as $M_1$ (whether it was the same may depend on what theory we are using for our analysis of the situation), but from which instead of the photon a mighty spaceship is sent to the star (the start of the spaceship is $s_2$). On its way to the star the spaceship warps and tears the spacetime by exploding passing stars, merging binary black holes and triggering other imaginable powerful processes. Assuming that no superluminal (‘tachyonic’) matter is involved, the spaceship arrives at the star later than the photon emitted in $s_2$, but nevertheless it is imaginable that its arrival time $\tau_2$ is \textit{less} than $\tau_1$. Thus, the speed of the spaceship in one world ($M_2$) would exceed the speed of light in another ($M_1$), which would not contradict the non-tachyonic nature of the spaceship. Nor would such a flight break the ‘light barrier’ in $M_1$; the inequality $\tau_2 < \tau_1$ does imply that the front $F_1$ is superluminal, but no \textit{material signal} in $M_1$ corresponds to that front. In particular, there is no spaceship in \textit{that} spacetime associated with $F_1$. It is such a pair of worlds $M_{1,2}$ that we call a semi-superluminal alternative. A theory admitting such alternatives allows superluminal signaling without tachyons.

Example 8. Let $M_1$ be a Minkowski plane and $s_1$ be its point with the coordinates $t = -3/2$, $x = -1$. Let, further, $M_2$ be the spacetime obtained by removing the segments $t \in [-1, 1]$. 
\[ x = \pm 1 \] from another Minkowski plane and gluing the left/right bank of either cut to the right/left bank of the other one. The differences between \( M_1 \) and \( M_2 \) are confined, in a sense, to the future of the points \( t = -1, x = -1 \) and \( t = -1, x = 1 \); see figure 1. Speaking more formally, \( N_1^* \) is the complement to the union of two future cones with the vertices at those two points. That \( N_1^* \) is maximal indeed is clear from the fact that any larger past set would contain a past-directed timelike curve \( \lambda \) terminating at one of the mentioned vertices, whereas \( \phi(\lambda) \) cannot have a past end point (because of the singularity). Evidently, \( \phi \subset +J^s(s_1) \), so \( \phi \) is superluminal. At the same time the surface \( \phi \subset M_1 \) does not correspond to any signal in \( M_1 \) (see preceding). And the front \( \phi \) is not superluminal, whence we conclude that the alternative \( (M_1, g_k, s_k) \) is semi-superluminal. Although the spaceship reaches the destination sooner than the photon shown in figure 1, the photon belongs to another Universe. In its own Universe \( M_2 \) the spaceship moves on a timelike curve, in full agreement with its non-tachyonic nature.

A flaw in the alternative just considered is that the difference between \( M_1 \) and \( M_2 \) is too exotic. One cannot say today whether a ‘topology change’ of that kind (if possible at all) can be attributable to something that takes place in \( s_{k,2} \). Unfortunately, this is a general rule: as the following proposition shows, the spacetimes of a semi-superluminal alternative cannot be ‘too nice’.

**Proposition 9.** The spacetimes \( M_1 \) and \( M_2 \) of a semi-superluminal alternative \( (M_1, g_k, s_k) \) cannot both be globally hyperbolic.

**Remark 10.** Note the difference between this proposition and the statement proven earlier to the effect that within general relativity the spacetimes of a superluminal alternative cannot both be globally hyperbolic. The former, in contrast with the latter, states a purely kinematical fact depending neither on the Einstein equations nor on criteria of admissibility of alternatives. Essentially, that fact is just a geometrical property of globally hyperbolic spacetimes.

**Proof.** Suppose that the front \( \phi \) is superluminal. Then some of its points must be separated from the—closed by the global hyperbolicity of \( M_1 \); see proposition 6.6.1 of [4]—set \( J^*(s_i) \),

![Figure 1](image.png)

(a) The world \( M_1 \). (b) The world \( M_2 \). The shaded region is the causal future of \( s_2 \), and the dashed broken line is the front \( \phi \), which bounds \( N_2^* \).
that is, there must be a point \( p \) (see figure 2) such that
\[
\mathcal{I} \cap \mathcal{I} = \emptyset \quad \forall r < \tilde{r},
\]
where \( \tilde{r} \) is a constant and \( B_r \) is a coordinate ball of radius \( r \) centered at \( p \).

To find for a given \( j \) a curve \( \mu_j \) of the type just mentioned, pick a pair of points
\[
b_j \in \left( \mathcal{F}_1 \cap B_{r_j} \right) \quad \text{and} \quad c_j \in \mathcal{F}_2,
\]
such that for any their neighborhoods \( U_j \supset b_j \) and \( V_j \supset c_j \) it is true that
\[
\phi \left( N_i \cap U_j \right) \cap V_j \neq \emptyset. \quad (*)
\]
To see that such pairs always exist, note that otherwise the maximal—by hypothesis—spacetime \( M_2 \) would have an extension \( M_2^{\text{ext}} \equiv B_{\tilde{r}} \cup \phi' M_2 \), where \( \phi' \) is the restriction of \( \phi \) to a connected component of \( N_i \cap B_{\tilde{r}} \). (Obviously, \( M_2^{\text{ext}} \) is a smooth connected pseudo-Riemannian manifold containing \( M_2 \) as a proper subset. So it is an extension of \( M_2 \) if it is Hausdorff, i.e., if there are no points \( b_j, c_j \).

Now assume that \( \mathcal{F}_2 \) is not superluminal. Then \( c_j \) being a point of \( \mathcal{F}_2 \) must lie in \( \overline{J^+(s_2)} \), and hence in the (closed) set \( J^+(s_2) \) too. Thus (recall that \( a_j \prec s_1 \), whence \( \phi(a_j) \prec s_2 \)) a pair \( a_j, c_j \) can be found such that

Figure 2. The hatched regions are \( N_i \cap B_{r_j} \) and its image under \( \phi \), respectively. The ball \( B_{\tilde{r}} \) bounded by the dashed line lies, by hypothesis, outside \( J^+(s_1) \). But this contradicts the fact that the curves \( \phi^{-1}(a_j) \) must converge to a future-directed curve from \( s_1 \) to \( p \).
By proposition 4.5.10 of [4] it follows that $\phi(a_j) \prec c_j$. Hence there is a neighborhood of $c_j$ which lies entirely in the open—by [8 lemma 14.3]—set $I^*(\phi(a_j))$. And according to (*) that neighborhood contains points of $\phi(N^+_j \cap B_{r_j})$. So there also must exist points $d_j$:

$$\phi(a_j) \prec d_j, \quad d_j \in \phi\left(N^+_j \cap B_{r_j}\right) \subset N^+_j.$$ 

The last inclusion coupled with the fact that $N^+_j$ is a past set means that the timelike curve $\lambda_j$ connecting $\phi(a_j)$ with $d_j$ lies entirely in $N^+_j$ and thus defines the curve $\mu_j \equiv \phi^{-1}(\lambda_j)$. The latter possesses all the desired properties: it is timelike, it starts in $a_j$, and it ends in $B_{r_j}$.

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