Global solutions and large time behavior for some Oldroyd-B type models in $\mathbb{R}^2$

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Abstract

In this paper, we are concerned with global solutions to the co-rotation Oldroyd-B type model and large time behavior for the general Oldroyd-B type model. We first establish the energy estimate and B-K-M criterion for the 2-D co-rotation Oldroyd-B type model. Then, we obtain global solutions with large data in Sobolev space by proving the boundedness of vorticity. As a corollary, we prove the global existence of corresponding Hooke model near equilibrium. Furthermore, we present the global existence for the 2-D co-rotation Oldroyd-B type model in critical Besov space by a refined estimate in Besov spaces with index 0. Finally, we study large time behaviour for the general Oldroyd-B type model. Applying the Fourier splitting method, we prove the $H^1$ decay rate for global solutions constructed by T. M. Elgindi and F. Rousset in [9].

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1 Introduction

In this paper we investigate the following general Oldroyd-B type model (with \( \nu = 0 \)):

\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \text{div} \, \tau + \nu \Delta u, & \text{div} \, u = 0, \\
\partial_t \tau + u \cdot \nabla \tau + \alpha \tau + Q(\nabla u, \tau) = \alpha D(u) + \mu \Delta \tau, \\
u|_{t=0} = u_0, & \tau|_{t=0} = \tau_0.
\end{cases}
\]

(1.1)

In (1.1), \( u(t,x) \) denotes the velocity of the polymeric liquid, \( \tau(t,x) \) represents the symmetric tensor of constrains and \( P \) is the pressure. The parameters \( \alpha, \mu \) and \( \nu \) are nonnegative and \( \alpha > 0 \). Moreover, \( Q(\nabla u, \tau) = \tau \Omega - \Omega \tau + b(D(u)\tau + \tau D(u)) \), with \( b \in [-1, 1] \), the vorticity tensor \( \Omega = \frac{\nabla u - (\nabla u)^T}{2} \) and the deformation tensor \( D(u) = \frac{\nabla u + (\nabla u)^T}{2} \). For more explanations on the modeling, one can refer to [33, 10, 9].

Taking \( b = 1 \) and \( \alpha = 2 \), then the general Oldroyd-B type model (1.1) can be derived from the following micro-macro model [24, 14] with Hooke potential \( U = \frac{1}{2}|q|^2 \), \( \int_{\mathbb{R}^d} \psi dq = \int_{\mathbb{R}^d} \psi_0 dq = 1 \) and the drag term \( \sigma(u) = \nabla u \) :

\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \text{div} \, \tau + \nu \Delta u, & \text{div} \, u = 0, \\
\psi_t + u \cdot \nabla \psi = \text{div}_q [-\sigma(u) \cdot q \psi + \frac{a}{2} \nabla_q \psi + \frac{a}{2} \nabla_q U\psi] + \mu \Delta \psi, \\
\tau_{ij} = \int_{\mathbb{R}^d}(q_{ij} Uq)\psi dq - Id, \\
u|_{t=0} = u_0, & \psi|_{t=0} = \psi_0.
\end{cases}
\]

(1.2)

In (1.2), the polymer particles are described by the distribution function \( \psi(t,x,q) \). Here the polymer elongation \( q \) satisfies \( q \in \mathbb{R}^2 \), which means that the extensibility of the polymers is infinite and \( x \in \mathbb{R}^2 \). \( \tau \) is an extra-stress tensor which generated by the polymer particles effect. In general, \( \sigma(u) = \nabla u \). For the co-rotation case, \( \sigma(u) = \Omega \).

When \( \int_{\mathbb{R}^d} \psi_0 dq = 1 \), the following co-rotation Oldroyd-B type model can be derived from the
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micro-macro model (1.2) with \( \nu = 0, \mathcal{U} = \frac{1}{2} |q|^2 \) and \( \sigma(u) = \Omega \):

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P &= \text{div} \tau, \quad \text{div} u = 0, \\
\partial_t \tau + u \cdot \nabla \tau + a \tau + Q(\Omega, \tau) &= \mu \Delta \tau, \\
u|_{t=0} = u_0, \quad \tau|_{t=0} = \tau_0.
\end{aligned}
\]

(1.3)

Notice that the equations (1.3) reduces to the well-known Euler equation by taking \( \tau = 0 \). However, taking \( \tau = 0 \) in (1.1), then we have \( Du = 0 \), which implies \( u = 0 \) in Sobolev spaces. The observation reveals the essential difference between (1.1) and (1.3).

1.1. The Oldroyd-B type model

T. M. Elgindi and F. Rousset [9] first proved global regularity for the 2-D Oldroyd-B type models (1.1) with \( \nu = 0 \). Later on, T. M. Elgindi and J. Liu [8] obtained global strong solutions of the 3-D case under the assumption that initial data is sufficiently small.

Taking \( \nu > 0 \) and \( \mu = 0 \) in (1.1), we obtain the classical Oldroyd-B model. In [13], C. Guillopé, and J. C. Saut first showed that the Oldroyd-B model admits a unique global strong solution in Sobolev spaces. The \( L^p \)-setting was given by E. Fernández-Cara, F. Guillén and R. Ortega [11]. The weak solutions of the Oldroyd-B model were proved by P. L. Lions and N. Masmoudi [25] for the case \( b = 0 \). Notice that the problem for the case \( b \neq 0 \) is still open, see [27, 28]. Later on, J. Y. Chemin and N. Masmoudi [6] proved the existence and uniqueness of strong solutions in homogenous Besov spaces with critical index of regularity. Optimal decay rates for solutions to the 3-D Oldroyd-B model were obtained by M. Hieber, H. Wen and R. Zi [15]. An approach based on the deformation tensor can be found in [20, 19, 22, 23, 39].

1.2. The Hooke model

Let \( \nu, \mu > 0 \). The construction of global weak solutions for micro-macro systems was considered in [2, 3, 4, 5, 36, 38]. Recently, global existence and uniqueness of a large class of initial data for the diffusive 2D models was proved in [18]. It’s worthy mentioning that the so-called moments \((u, M_{n, b})\) considered in [18] are strong solutions with macroscopic variables \((t, x)\) while \( \psi \) is nonnegative measures on \( \mathbb{R}^2 \times \mathbb{R}^2 \) merely. Let \( \nu > 0, \mu = 0 \). The local existence of micro-macro systems were proved by many researchers in different settings, see [34, 7]. F. Lin, C. Liu and P. Zhang [24] studied the incompressible micro-macro polymeric system and proved global existence near equilibrium with some assumptions on the potential \( \mathcal{U} \). Global regularity for the 2-D co-rotation Hooke dumbbell model was proved by N. Masmoudi, P. Zhang, and Z. Zhang [30]. Long time behavior for the 3-D micro-macro polymeric system was considered by L. He and P. Zhang [14].

1.3. Main results

Global well-posedness with \( d = 2 \) and long time behavior for polymeric models were noticed by F. Lin [21] and N. Masmoudi [29]. To our best knowledge, global well-posedness for the Oldroyd-B type model (1.3) and large time behaviour for (1.1) have not been studied yet. In this paper, we first study about global solutions for (1.3) with large data in \( H^s \). The proof is based on the bootstrap argument in [37]. To prove global existence, we derive the energy estimate and B-K-M criterion for (1.3) in \( H^s \).
The main difficult in the proof is to prove the boundedness of vorticity from \(1.3\). Motivated by [9], we can cancel \(\text{div} \, \tau \) and \(\Delta \tau \) by virtue of the structural trick \(\Gamma = \Omega - R \tau\) where \(R = -(\Delta)^{-1} \text{curl} \, \text{div} \). However, for \(1.3\), there is no dissipation term in the equation of \(\Gamma\) for the lack of \(D(u)\). We thus fail to use the bootstrap argument as in [9]. Fortunately, the disappearance of \(D(u)\) leads to exponential dissipation for \(\tau\) in \(H^1\). The effect of exponential dissipation of \(\tau\) is essential in the estimation of \(\Gamma\). We finish the proof of global existence with large data in \(H^s\) by deriving the \(L^\infty\) estimate for \(\Omega\). To our best knowledge, there is still no any global existence result of the Hooke models \(1.2\) with \(\nu = 0\). As a corollary of Theorem 1.1, we prove the global existence of \(1.2\) with large data in \(H^s\). It’s worth mentioning that the estimate of \(\langle \rho \rangle^n \nabla \eta g\) in \(L^\infty(L^2)\) is essential in the proof of global existence.

Furthermore, we establish local existence for \(1.3\) in \(B^0_{\infty,1} \times B^0_{\infty,1}\) and present the global existence with large data in \((H^1 \cap B^1_{\infty,1}) \times (H^1 \cap B^0_{\infty,1})\). The proof of the global existence is based on the refined estimate in Besov space with index 0 and the \(H^1 \times (H^1 \cap L^\infty)\) boundedness for \((u, \tau)\). By virtue of the refined estimate in Besov spaces with index 0, the authors [1] prove the global well-posedness for the Euler equation in the borderline case and obtain the exponential growth estimate of vorticity in \(B^0_{\infty,1}\). Considering the global existence for \(1.3\) in critical Besov space, the main difficult for us is to estimate external force in the equation of \(\Gamma\). We find that exponential dissipation for \(\tau\) can prevent the exponential growth of external force. Thus we obtain the exponential growth estimate of \(\Gamma\) in \(B^0_{\infty,1}\), which implies the global existence for \(1.3\). Finally, we study about large time behaviour for \(1.1\) with large data. Since the structural trick \(\Gamma\) transfer dissipation from \(\tau\) to \(u\) for \(1.1\), we obtain the dissipation energy estimate for \((u, \tau)\) which is useful to prove large time behaviour. For any \(l \in \mathbb{N}^+\), we get initial time decay rate \(\ln^{-1}(\epsilon + t)\) for \((u, \tau)\) in \(L^2\) by the Fourier splitting method, see [35] [26]. By virtue of the time weighted energy estimate and the logarithmic decay rate, then we improve the time decay rate to \((1 + t)^{-\frac{\lambda}{2}}\).

Our main results can be stated as follows:

**Theorem 1.1** (Global well-posedness in Sobolev space). Let \(d = 2\) and \(s > 2\). Assume that \(a > 0\), \(\mu > 0\) and \(\kappa = \min\{a, \mu\}\). Let \((u, \tau)\) be a strong solution of \(1.3\) with the initial data \((u_0, \tau_0) \in H^s\). Then there exists some sufficiently small constant \(c\) such that if

\[
\|\nabla u_0\|_{L^2} \leq c\kappa, \quad \|\tau_0\|_{H^1} \leq c(a\mu)^{\frac{3}{2}} \kappa,
\]

and

\[
\|\Gamma_0\|_{L^\infty} \leq ca\mu, \quad \|\tau_0\|_{H^1}^2 \leq \frac{c^2 a\kappa(\mu + 1)\mu}{\ln(C + \|\nabla \phi_0\|_{H^s})}, \quad \|\tau_0\|_{H^1} \leq c^2 \lambda,
\]

where \(\lambda = \min\{a^2, a^{\frac{3}{2}}\mu, (a\mu)^{\frac{3}{2}}, a, \mu, a\mu^{\frac{3}{2}}, \mu^2\}\), then the system \(1.3\) admits a unique global strong solution \((u, \tau) \in C([0, \infty); H^s)\).

**Remark 1.2.** Let \(\phi_0(x) = A(x_2 e^{-|x|^2}, -x_1 e^{-|x|^2})^T\) and \(\varphi_0(x) = A e^{-|x|^2} I_d\). Consider \(u_0 = \varepsilon \phi_0(\varepsilon x)\) and \(\tau_0 = \varepsilon^2 \varphi_0(\varepsilon x)\), then we can verify that \(\|\nabla u_0\| = 0\). We infer that \(\|u_0\|_{L^2} = \|\phi_0\|_{L^2}\) and \(\|\tau_0\|_{L^2} = \varepsilon \|\varphi_0\|_{L^2}\). Moreover, we deduce that \(\|u_0\|_{H^s} = \varepsilon^s \|\phi_0\|_{H^s}\) and \(\|\tau_0\|_{H^s} = \varepsilon^{s+1} \|\varphi_0\|_{H^s}\), for any \(\varepsilon > 0\). Finally, we can construct large initial data in \(H^s\) which satisfies \(1.4\) and \(1.5\) by taking \(A\) sufficiently large and \(\varepsilon\) small enough.

**Remark 1.3.** For any \(a\) and \(\mu\), the system \(1.3\) reduces to the well-known Euler equation by taking \(\tau = 0\). In this case, the parameters \(a\) and \(\mu\) in Theorem 1.1 can be regarded as infinity, which means that our results cover the global existence for the 2-D Euler equation in Sobolev spaces \(H^s\).
Remark 1.4. Notice that equations (1.3) contain more solutions than equations (1.2). In the Corollary 3.6 we establish the connection between the solutions \((u, \tau)\) of (1.3) constructed in Theorem 1.7 and the solutions \((u, \psi)\) of (1.2).

Theorem 1.5 (Global well-posedness in critical Besov space). Let \(d = 2\). Assume that \(a > 0\), \(\mu > 0\) and \(\beta = \min\{a, \mu, \beta\}\). Let \((u, \tau)\) be a strong solution of (1.3) with the initial data \((u_0, \tau_0) \in (H^1 \cap B^1_{\infty,1}) \times (H^1 \cap B^0_{\infty,1})\). Then there exists some sufficiently small constant \(c\) such that

\[
(1.6) \quad \|\nabla u_0, \tau_0\|_{B^4_{\infty,1}} \leq c\beta,
\]

and

\[
(1.7) \quad \|\tau_0\|_{L^4} \leq c\beta \min\{\mu^{1/2}, 1\}, \quad H_0(\|\tau_0\|_{B^4_{\infty,1}} + \|\tau_0\|_{L^4}) \leq c\beta \min\{\mu^{1/2}, \mu, a\mu, \beta\},
\]

where \(H_0 = \|(u_0, \tau_0)\|_{H^1}^2 e^{\frac{a}{2}\|\tau_0\|_{L^1}^2 + 3\mu^{1/2}\|\tau_0\|_{L^2}^2}\), then the system (1.3) admits a global strong solution \((u, \tau) \in C([0, \infty); (H^1 \cap B^1_{\infty,1}) \times (H^1 \cap B^0_{\infty,1}))\).

Remark 1.6. Let \(h(x) = \sum_{k \geq 1} \frac{1}{2^k} h_k(x)\) with \(h_k(x)\) given by the Fourier transform \(\hat{h}_k(\xi) = i\varphi(2^{-k}\xi)\), where \(\varphi\) is given in Proposition 2.2. Let \(\phi_0(x) = (x_2 e^{-|x|^2}, -x_1 e^{-|x|^2})^T\) and \(\varphi_0(x) = h(x) Id\). Set \(\varepsilon \in (0, 1)\) and positive integer \(N\) such that \(\sum_{k \geq 1} \frac{1}{2^k} \approx \varepsilon^{-1} \|\varphi\|_{H^2}^2\). Consider \(u_0 = e^{-\frac{i}{2}\varepsilon^2\varphi_0(\varepsilon x)}\) and \(\tau_0 = e^{10\varphi_0(x)}\), then we can verify that \(\text{div} u_0 = 0\). We infer that \(\|u_0\|_{H^1} = \varepsilon^{-\frac{1}{2}} \|\varphi_0\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\phi_0\|_{H^1}\) and \(\|\tau_0\|_{L^2} = \sum_{k \geq 1} \frac{\varepsilon^{10}}{2^k} \|h_k(x)\|_{L^2}^2 \lesssim \varepsilon^{10}\). Moreover, we obtain \(\|\tau_0\|_{H^1} = \varepsilon^{10} \|\phi_0\|_{B^2_{\infty,2}}^2 + \sum_{k \geq 1} \varepsilon^{10} \|\phi_k(x)\|_{L^2}^2 = \varepsilon^{10} \|\varphi\|_{H^2}^2 \approx \varepsilon^{-1}\). Furthermore, we deduce that \(\|u_0\|_{B^4_{\infty,1}} \approx \varepsilon^{-\frac{1}{2}} \|\phi_0\|_{B^4_{\infty,1}}, \quad \|\nabla u_0\|_{B^0_{\infty,1}} \approx \varepsilon^{\frac{1}{2}} \|\phi_0\|_{B^1_{\infty,1}}\) and \(\|\tau_0\|_{B^4_{\infty,1}} \lesssim \sum_{k \geq 1} 2^{k+2} \|\varphi\|_{L^1} \varepsilon^{10} \lesssim \varepsilon^{10}\). Finally, we can construct large initial data in \((H^1 \cap B^1_{\infty,1}) \times (H^1 \cap B^0_{\infty,1})\) by taking \(\varepsilon\) small enough. It should be underlined that the initial data \(\|u_0\|_{H^1 \cap B^0_{\infty,1}}\) and \(\|\tau_0\|_{H^1}\) is large.

Remark 1.7. For any \(a\) and \(\mu\), the system (1.3) reduces to the well-known Euler equation by taking \(\tau = 0\). In this case, the parameters \(a\) and \(\mu\) in Theorem 1.7 can be regarded as infinity, which means that our results cover the global existence for the 2-D Euler equation in critical Besov space \(B^1_{\infty,1}\).

Theorem 1.8 (Large time behaviour). Let \((u, \tau)\) be a strong solution of (1.3) with the initial data \((u_0, \tau_0)\) under the condition in Theorem 5.7. In addition, if \((u_0, \tau_0) \in L^1\), then there exists \(C > 0\) such that for every \(t > 0\) we have

\[
(1.8) \quad \|(u, \tau)\|_{H^1} \leq C(1 + t)^{-\frac{1}{2}}.
\]

Remark 1.9. Notice that Theorem 1.7 don’t provide any information for the global solution of (1.1) in \(L^\infty([0, \infty); H^s)\) with some large initial data. However, by virtue of the Fourier splitting method and the time weighted energy estimate, we can prove the large time behaviour by taking full advantage of the \(H^1\) energy estimation (5.2) and the low-frequency assumption \((u_0, \tau_0) \in L^1\). The proof does not involve the higher derivative, which is useful in studying large time behaviour of global solutions with some initial data.
Remark 1.10. The conclusions in Theorem 1.8 and Theorems 1.1, 1.5 reveal the essential difference between \((1.1)\) and \((1.3)\). More precisely, the solutions \((u, \tau)\) of \((1.2)\) with \(\nu = 0\) decay in \(H^1\), while the solutions \(u\) of \((1.3)\) are bounded in \(H^1\). Moreover, \(u\) conserve in \(L^2\) whenever \(\tau = 0\). Such observation reflects the obstacle of global approximation in \(H^s\) between \((1.1)\) and \((1.3)\).

The paper is organized as follows. In Section 2 we introduce some notations and give some preliminaries which will be used in the sequel. In Section 3 we prove that the 2-D co-rotation Oldroyd-B type model admits a unique global strong solution in Sobolev space. As a corollary, we prove the global existence of the Hooke models near equilibrium. In Section 4 we prove that the 2-D co-rotation Oldroyd-B type model admits a global strong solution in critical Besov space. In Section 5 we study the \(H^1\) decay of global solutions to the general Oldroyd-B type model for \(d = 2\) by virtue of the Fourier splitting method.

2 Preliminaries

In this section we introduce some notations and useful lemmas which will be used in the sequel.

Let \(\psi_\infty(q) = \frac{e^{-|q|^2}}{\int_{\mathbb{R}^d} e^{-|q|^2} dq}\) For \(p \geq 1\), we denote by \(L^p\) the space
\[
L^p = \{ \psi \mid \|\psi\|_{L^p} = \left( \int_{\mathbb{R}^d} |\psi|^p dq \right)^{1/p} < \infty \}.
\]

We will use the notation \(L^p_\rho(L^p')\) to denote \(L^p(\mathbb{R}^d; L^p')\):
\[
L^p_\rho(L^p') = \{ \psi \mid \|\psi\|_{L^p_\rho(L^p')} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\psi|^p dq \right)^{1/p} dx \right)^{1/p} < \infty \}.
\]

We now recall the Littlewood-Paley decomposition theory and Besov spaces.

Proposition 2.1. \([1]\) Let \(\mathcal{C}\) be the annulus \(\{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \}\). There exist radial functions \(\chi\) and \(\varphi\), valued in the interval \([0, 1]\), belonging respectively to \(D(B(0, \frac{3}{4}))\) and \(D(\mathcal{C})\), and such that

\[
\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,
\]

\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,
\]

\[
|j - j'| \geq 2 \Rightarrow \text{Supp} \varphi(2^{-j} \cdot) \cap \text{Supp} \varphi(2^{-j'} \cdot) = \emptyset,
\]

\[
j \geq 1 \Rightarrow \text{Supp} \chi(\cdot) \cap \text{Supp} \varphi(2^{-j} \cdot) = \emptyset.
\]

The set \(\tilde{\mathcal{C}} = B(0, \frac{3}{4}) + \mathcal{C}\) is an annulus, and we have
\[
|j - j'| \geq 5 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \tilde{\mathcal{C}} = \emptyset.
\]

Further, we have

\[
\forall \xi \in \mathbb{R}^d, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1,
\]

\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1.
\]
$\mathcal{F}$ represents the Fourier transform and its inverse is denoted by $\mathcal{F}^{-1}$. Let $u$ be a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$. For all $j \in \mathbb{Z}$, define

$$\Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi (\xi))^2 u, \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \text{ if } j \geq 0,$$

with $S_j u = \sum_{j' < j} \Delta_{j'} u$.

Then the Littlewood-Paley decomposition is given as follows:

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } \mathcal{S}'(\mathbb{R}^d).$$

Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}$ and $B^s_{p,r}(\mathcal{L}^q)$ is defined by

$$B^s_{p,r} = \{ u \in \mathcal{S}' : \|u\|_{B^s_{p,r}} = \| (2^j \|\Delta_j u\|_{L^p})_j \|_{l^r(\mathbb{Z})} < \infty \},$$

and

$$B^s_{p,r}(\mathcal{L}^q) = \{ \phi \in \mathcal{S}' : \|\phi\|_{B^s_{p,r}(\mathcal{L}^q)} = \| (2^j \|\Delta_j \phi\|_{\mathcal{L}^q})_j \|_{l^r(\mathbb{Z})} < \infty \}.$$

The transport-diffusion equation is given as follows:

$$\begin{align*}
\begin{cases}
f_t + v \cdot \nabla f - v \Delta f = g, & \text{div } v = 0, \\
f(0, x) = f_0(x), & x \in \mathbb{R}^d, \ t > 0.
\end{cases}
\end{align*}$$

(2.1)

**Lemma 2.2.** Let $1 \leq p_1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $s \geq -1 - d \min \left( \frac{1}{p_1}, \frac{1}{p} \right)$. There exists a constant $C$ and $1 \leq \rho_1 \leq \rho \leq \infty$ such that for all solutions $f \in L^\infty([0, T]; B^s_{p,r})$ of (2.1) with initial data $f_0$ in $B^s_{p,r}$, and $g$ in $L^\rho([0, T]; B^{s-2+\frac{\rho}{n}}_{p,r})$, we have

$$\nu \frac{p}{p_1} \| f(t) \|_{L^p_{p_1} B^{s+\frac{2}{p}}_{p,r}} \leq C \frac{e^{C(1+\nu T)^{\frac{1}{\rho}}}}{V_{p_1}(T)} \left( (1 + \nu T)^{\frac{1}{\rho} - \frac{1}{p_1}} \| f_0 \|_{B^s_{p,r}} + (1 + \nu T)^{1 + \frac{1}{\rho} - \frac{1}{p_1} - \frac{1}{\rho}} \| g \|_{L^\rho_{p_1} B^{s-2+\frac{\rho}{n}}_{p,r}} \right),$$

with

$$V_{p_1}(t) = \begin{cases} 
\| \nabla v \|_{B^{s+\frac{2}{p}}_{p_1}}, & \text{if } s < 1 + \frac{d}{p_1} \\
\| \nabla v \|_{B^{s-1+\frac{2}{p_1}}_{p_1}}, & \text{if } s > 1 + \frac{d}{p_1} \text{ or } (s = 1 + \frac{d}{p_1}, \ r = 1).
\end{cases}$$

The following refined estimate in Besov spaces with index 0 is crucial to estimate $\Gamma$.

**Lemma 2.3.** Assume that $v$ is divergence-free and that $f$ satisfies (2.1) with $\nu = 0$. There exists a constant $C$, depending only on $d$, such that for all $1 \leq p, r \leq \infty$ and $t \in [0, T]$, we have

$$\| f \|_{L^\infty_t(B^0_{p,r})} \leq C (\| f_0 \|_{B^0_{p,r}} + \| g \|_{L^1_t(B^0_{p,r})}) (1 + V(t))$$

with $V(t) = C \int_0^t \| \nabla v \|_{L^\infty} ds$.

We have the following product laws:

**Lemma 2.4.** For any $\epsilon > 0$, there exists $C > 0$ such that

$$\| u v \|_{B^{0,1}_\infty} \leq C (\| u \|_{L^\infty} \| v \|_{B^{0,1}_\infty} + \| u \|_{B^{0,1}_\infty} \| v \|_{B^{0,\infty}_\infty}).$$
We introduce the following lemma to describes the action of the heat equation.

**Lemma 2.5.** [1] Let $C$ be an annulus. Positive constants $c$ and $C$ exist such that for any $p \in [1, +\infty]$ and any couple $(t, \lambda)$ of positive real numbers, we have

\[(2.4) \quad \text{Supp}\, u \subset \lambda C \Rightarrow \|e^{t\lambda^2}u\|_{L^p} \leq Ce^{-ct}\|u\|_{L^p}.
\]

The following commutator lemma is useful to estimate $\Gamma$.

**Lemma 2.6.** [2,12,16] Let $u = 0$ and $R = \Delta^{-1}\text{curl div}$. Then we have

1. There exists a constant $C$ such that
   \[(2.5) \quad \|R\|_{H^s_{\Theta, \infty}} \leq C\|\tau\|_{L^\infty}.
   \]
2. For every $(p, r) \in [2, \infty) \times [1, \infty]$, there exists a constant $C = C(p, r)$ such that
   \[(2.6) \quad \|R, u \cdot \nabla\tau\|_{B^p_{\infty,r}} \leq C\|\nabla u\|_{L^p}(\|\tau\|_{B^p_{\infty,r}} + \|\tau\|_{L^p}).
   \]
3. For every $(r, p) \in [1, \infty] \times (1, \infty)$ and $\varepsilon > 0$, there exists a constant $C = C(r, \varepsilon)$ such that
   \[(2.7) \quad \|R, u \cdot \nabla\tau\|_{B^p_{\infty,r}} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^p})(\|\tau\|_{B^p_{\infty,r}} + \|\tau\|_{L^p}).
   \]

The following lemma is the Gagliardo-Nirenberg inequality of Sobolev type.

**Lemma 2.7.** [7] Let $d \geq 2$, $p \in [2, +\infty)$ and $0 \leq s, s_1 \leq s_2$, then there exists a constant $C$ such that

\[\|\Lambda^s f\|_{L^p} \leq C\|\Lambda^{s_1} f\|^{1 - \theta}_{L^2} \|\Lambda^{s_2} f\|^{\theta}_{L^2},\]

where $0 \leq \theta \leq 1$ and $\theta$ satisfy

\[s + d\left(\frac{1}{2} - \frac{1}{p}\right) = s_1(1 - \theta) + \theta s_2.
\]

Note that we require that $0 < \theta < 1$, $0 \leq s_1 \leq s$, when $p = \infty$.

We introduce a commutator lemma.

**Lemma 2.8.** [17] Let $s \geq 1$, $p, p_1, p_4 \in (1, \infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, then we have

\[(2.8) \quad \|[[\Lambda^s, f]g]\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|\nabla f\|_{L^{p_3}}\|\Lambda^{s_1} g\|_{L^{p_4}}),
\]
and

\[(2.9) \quad \|[[\Lambda^s, f]g]\|_{L^p(L^2)} \leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}(L^2)} + \|\nabla f\|_{L^{p_3}}\|\Lambda^{s_1} g\|_{L^{p_4}(L^2)}).
\]

The following lemma will be useful in the proof of the global existence for the Hooke models.

**Lemma 2.9.** [7] Assume $g \in H^s(L^2)$ with $\int_{\mathbb{R}^d} g\psi d\omega = 0$, then there exists a constant $C$ such that

\[(2.10) \quad \|\nabla q\mathcal{G} g\|_{H^s(L^2)} + \|q g\|_{H^s(L^2)} \leq C\|\nabla g\|_{H^s(L^2)},
\]
and

\[(2.11) \quad \|q\nabla q\mathcal{G} g\|_{H^s(L^2)} + \|q^2 g\|_{H^s(L^2)} \leq C\| q\nabla g\|_{H^s(L^2)}.
\]

Moreover,

\[(2.12) \quad \|q\nabla q\mathcal{G} \nabla q g\|_{L^2(L^2)} + \|q^2 \nabla q g\|_{L^2(L^2)} \leq C(\|q\| \nabla q \nabla q g\|_{L^2(L^2)} + \| \nabla q g\|_{L^2(L^2)}).
\]
The following lemma is about Calderon-Zygmund operator.

Lemma 2.10. [1, 9] (1) For any $a \in [1, \infty)$ and $b \in [1, \infty]$, there exists positive constant $C$ such that

$$\|\Delta_{-1} \nabla v\|_{L^\infty} \leq C \min\{\|\Omega\|_{L^a}, \|v\|_{L^b}\}.$$  \hspace{1cm} (2.13)

(2) For all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, there exists a constant $C'$ such that

$$\|(Id - \Delta_{-1}) R f\|_{B^s_{p,r}} \leq C' \|f\|_{B^s_{p,r}}.$$  \hspace{1cm} (2.14)

From Lemma 2.10, we immediately infer the following estimate:

Lemma 2.11. [1, 9] There exists positive constant $C$ such that

$$\|u\|_{L^\infty} \leq C (\|u\|_{L^2} + \|\Omega\|_{L^\infty}).$$  \hspace{1cm} (2.15)

We introduce an interpolation inequality.

Lemma 2.12. Let $s > \frac{d}{2}$. Then there exist $C > 0$ such that

$$\|u\|_{L^\infty} \leq \|u\|_{B^s_{\infty,1}} \leq C \|u\|_{B^s_{\infty,\infty}} \ln(e + \|u\|_{H^s}) + C.$$  \hspace{1cm} \hspace{0.5cm}

Proof. According to the Littlewood-Paley decomposition theory, we have

$$\|u\|_{L^\infty} \leq \|u\|_{B^s_{\infty,1}} = \sum_{-1 \leq j \leq N} \|\Delta_j u\|_{L^\infty} + \sum_{j \geq N} \|\Delta_j u\|_{L^\infty},$$

for integer $N > 0$ which will be chosen later on. There exist $C > 0$ such that

$$\sum_{-1 \leq j \leq N} \|\Delta_j u\|_{L^\infty} \leq NC \|u\|_{B^0_{\infty,\infty}},$$

and

$$\sum_{j \geq N} \|\Delta_j u\|_{L^\infty} \leq C 2^{-N(s - \frac{d}{2})} \|u\|_{H^s}.$$  \hspace{1.5cm} \hspace{0.5cm}

Consider $N = \left[\frac{\ln(\ln(1 + \|u\|_{H^s}))}{s - \frac{d}{2}}\right] + 1$, then we complete the proof of Lemma 2.12. \hfill \Box

3 Global solutions for co-rotation case in Sobolev space

In this section, we are concerned with global solutions to the co-rotation Oldroyd-B type model in Sobolev space. We divide it into three steps to prove Theorem 1.1.

3.1 Energy estimate

From now on, we derive the energy estimate which is useful to prove global existence. We prove conservation laws and boundness for (1.3) in the following propositions.

Proposition 3.1. Set $p \in [2, \infty]$. Suppose $(u, \tau)$ is a smooth solution to (1.3) with $\tau_0$ in $L^p$. Then we obtain

$$\|\tau\|_{L^p} \leq \|\tau_0\|_{L^p} e^{-at}.$$  \hspace{1cm} (3.1)
Proof. Let \( \hat{\tau}^{ij} = \tau^{ij} e^{at} \), we infer from (3.2) that
\[
\partial_t \hat{\tau}^{ij} + u \cdot \nabla \hat{\tau}^{ij} + Q(\Omega^{ik}, \hat{\tau}^{kj}) = \Delta \hat{\tau}^{ij}.
\]
Applying inner product with \( \hat{\tau}^{ij} |\hat{\tau}|^{p-2} \) to (3.2) and summing up \( i, j \), we get
\[
\frac{1}{p} \frac{d}{dt} \| \hat{\tau} \|_{L^p} = \sum_{i,j=1}^{3} \int_{\Omega} \hat{\tau}^{ij} |\hat{\tau}|^{p-2} \Delta \hat{\tau}^{ij} dx.
\]
Notice that
\[
\int_{\Omega} \hat{\tau}^{ij} |\hat{\tau}|^{p-2} \Delta \hat{\tau}^{ij} dx = - \int_{\Omega} \nabla^k \hat{\tau}^{ij} |\hat{\tau}|^{p-2} \nabla^k \hat{\tau}^{ij} dx - \int_{\Omega} \tau^{ij} \hat{\tau}^{ij} \nabla^k \hat{\tau}^{ij} |\hat{\tau}|^{p-4} \nabla^k \hat{\tau}^{ij} dx
\]
\[
= - \int_{\Omega} (\nabla^k \hat{\tau}^{ij})^2 |\hat{\tau}|^{p-2} dx - \frac{1}{4} \int_{\Omega} (\nabla^k (\hat{\tau}^{ij})^2)^2 |\hat{\tau}|^{p-4} dx.
\]
According to (3.3) and (3.4), we obtain
\[
\frac{1}{p} \frac{d}{dt} \| \hat{\tau} \|_{L^p} \leq 0,
\]
which implies that
\[
\| \hat{\tau} \|_{L^p} \leq \| \tau_0 \|_{L^p}.
\]
We thus complete the proof of Proposition 3.1.

Proposition 3.2. Let \((u, \tau) \in C([0, T]; H^s) \times C([0, T]; H^s) \cap L^2([0, T]; H^{s+1}) \) be a solution for (1.3). Then we obtain
\[
\|u\|_{L^2} \leq \|u_0\|_{L^2} + (4 \mu a)^{-\frac{1}{2}} \| \tau_0 \|_{L^2}, \quad e^{2at} \| \tau \|_{L^2}^2 + 2\mu \int_0^t e^{2as} \| \nabla \tau \|_{L^2}^2 ds = \| \tau_0 \|_{L^2}^2.
\]
Moreover, for any \( t \in [0, T] \), if \( \| \nabla u(t) \|_{L^2} \leq 4 \kappa c \kappa \) with \( \kappa = \min \{a, \mu\} \) and sufficiently small constant \( c \), then we obtain
\[
e^{at} \| \tau \|_{H^1}^2 + \mu \int_0^t e^{as} \| \nabla \tau \|_{H^1}^2 ds \leq \| \tau_0 \|_{H^1}^2,
\]
and
\[
\| \nabla u \|_{L^2} \leq \| \nabla u_0 \|_{L^2} + (\mu a)^{-\frac{1}{2}} \| \tau_0 \|_{H^1}.
\]
Proof. Firstly, we consider the \( L^2 \) estimate of \((u, \tau)\). Taking the \( L^2 \) inner product with \( \tau \) to (1.3) and we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \tau \|_{L^2}^2 + a \| \tau \|_{L^2}^2 + \mu \| \nabla \tau \|_{L^2}^2 = 0,
\]
which implies that
\[
e^{2at} \| \tau \|_{L^2}^2 + 2\mu \int_0^t e^{2as} \| \nabla \tau \|_{L^2}^2 ds = \| \tau_0 \|_{L^2}^2.
\]
Taking the \( L^2 \) inner product with \( u \) to (1.3) and we obtain
\[
\frac{d}{dt} \| u \|_{L^2} \leq \| \nabla \tau \|_{L^2}.
\]
Integrating (3.12) over $[0, t]$ with $s$, we deduce that

$$\|u\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|\nabla\tau\|_{L^2} ds$$

$$\leq \|u_0\|_{L^2} + (\int_0^t e^{2as}\|\nabla\tau\|_{L^2}^2 ds)^{\frac{1}{2}} (\int_0^t e^{-2as} ds)^{\frac{1}{2}}$$

$$\leq \|u_0\|_{L^2} + (4\mu a)^{-\frac{1}{2}} \|\tau_0\|_{L^2}.$$  

Notice that $(u, \tau)$ are bound in $L^2$. Taking the $L^2$ inner product with $\Delta\tau$ to (1.3) and using Lemma 2.7, we have

$$\frac{1}{2} d dt \|\nabla\tau\|_{L^2}^2 + a \|\nabla\tau\|_{L^2}^2 + \mu \|\nabla\tau\|_{L^2}^2 = -\langle u \cdot \nabla\tau, \Delta\tau \rangle + \langle Q(\Omega, \tau), \Delta\tau \rangle$$

$$\leq C \|\nabla u\|_{L^2} \|\nabla\tau\|_{L^2} \|\nabla^2\tau\|_{L^2} + C \|\Omega\|_{L^2} (\|\tau\|_{L^2} + \|\nabla^2\tau\|_{L^2}) \|\Delta\tau\|_{L^2}.$$  

Adding up (3.10) and (3.13), we infer that

$$\frac{1}{2} \frac{d}{dt} \|\tau\|_{H^1}^2 + a \|\tau\|_{H^1}^2 + \mu \|\nabla\tau\|_{H^1}^2 \leq C (\|\nabla u\|_{L^2} \|\nabla\tau\|_{L^2} + \|\Omega\|_{L^2} \|\tau\|_{H^2}) \|\nabla^2\tau\|_{L^2}.$$  

Assume that $\|\nabla u\|_{L^2} \leq 4\epsilon \min\{a, \mu\}$ with sufficiently small constant $c$, then we obtain

$$\frac{d}{dt} \|\tau\|_{H^1}^2 + a \|\tau\|_{H^1}^2 + \mu \|\nabla\tau\|_{H^1}^2 \leq 0,$$

which implies that

$$e^{at} \|\tau\|_{H^1}^2 + \mu \int_0^t e^{as} \|\nabla\tau\|_{H^1}^2 ds \leq \|\tau_0\|_{H^1}^2.$$  

We now consider the $L^2$ estimate of $\nabla u$. Taking the $L^2$ inner product with $\Delta u$ to (1.3), we can deduce that $\langle u \cdot \nabla u, \Delta u \rangle = 0$ with $d = 2$ and $\text{div} u = 0$. Then we have

$$\frac{d}{dt} \|\nabla u\|_{L^2} \leq \|\nabla^2\tau\|_{L^2}.$$  

Integrating (3.17) over $[0, t]$ with $s$ and using (3.16), we deduce that

$$\|\nabla u\|_{L^2} \leq \|\nabla u_0\|_{L^2} + \int_0^t \|\nabla^2\tau\|_{L^2} ds \leq \|\nabla u_0\|_{L^2} + (\mu a)^{-\frac{1}{2}} \|\tau_0\|_{H^1}.$$  

Combining (3.11) and (3.16), we complete the proof of Proposition 3.2.  

**Corollary 3.3.** Under the conditions in Proposition 3.2, we have the following estimates:

$$\left\{ \begin{array}{l}
\int_0^t \|\tau\|_{H^1} ds \leq a^{-\frac{1}{2}} \|\tau_0\|_{H^1}, \\
\int_0^t \|\tau\|_{H^2} ds \leq (a^{-\frac{1}{2}} + (\mu a)^{-\frac{1}{2}}) \|\tau_0\|_{H^1}, \\
\int_0^t \|\tau\|_{H^3} ds \leq (a^{-1} + \mu^{-1}) \|\tau_0\|_{H^1}.
\end{array} \right.$$  

**Proof.** Using (3.15) and (3.16), we can deduce that

$$\int_0^t \|\tau\|_{H^1} ds \leq \left( \int_0^t e^{-as} ds \right)^{\frac{1}{2}} \|\tau_0\|_{H^1} \leq a^{-\frac{1}{2}} \|\tau_0\|_{H^1},$$

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and
\[ \int_0^t \| \tau \|^2_{H^2} ds \leq \int_0^t \| \tau \|^2_{H^1} ds + \int_0^t \| \nabla^2 \tau \|^2_{L^2} ds \leq (a^{-1} + \mu^{-1}) \| \tau_0 \|^2_{H^1} . \]

Similarly, we have
\[ \int_0^t \| \tau \|^2_{H^2} ds \leq \int_0^t \| \tau \|^2_{L^2} ds + \left( \int_0^t e^{-as} ds \right) \frac{1}{2} \left( \int_0^t e^{as} \| \nabla \tau \|^2_{H^1} ds \right) \frac{1}{2} \leq (a^{-\frac{1}{2}} + (\mu a)^{-\frac{1}{2}}) \| \tau_0 \|^2_{H^1} . \]

\[ \square \]

3.2. B-K-M Criterion

In Proposition 3.2, it’s clear that \( u \) is merely bound in \( L^2 \) while \( \tau \) decays exponentially in \( L^2 \). Then we can state a blow-up criterion for (1.3) which depends on \( \| \Omega \|_{L^\infty} \) in the following proposition.

**Proposition 3.4.** Assume that \( d = 2, s > 2, a > 0 \) and \( \mu > 0 \). Let \((u, \tau)\) be a strong solution of (1.3) with the initial data \((u_0, \tau_0) \in H^s\). If \( T^* \) is the maximal existence time, then the solution blows up in finite time \( T^* < \infty \) if and only if

\[ \int_0^{T^*} \| \Omega(t) \|^2_{L^\infty} dt = \infty. \]

**Proof.** Applying \( \Lambda^s \) to (1.3), taking the \( L^2 \) inner product with \( \Lambda^s u \) and using Lemma 2.8 we have

\[ \frac{1}{2} \frac{d}{dt} \| \Lambda^s u \|^2_{L^2} = -\langle \Lambda^s (u \cdot \nabla u), \Lambda^s u \rangle + \langle \text{div} \Lambda^s \tau, \Lambda^s u \rangle \]

\[ \leq \| \nabla u \|_{L^\infty} \| \Lambda^s u \|^2_{L^2} + C_\mu \| \Lambda^s u \|^2_{L^2} + \frac{\mu}{4} \| \nabla \Lambda^s \tau \|^2_{L^2} , \]

where \( C_\mu = \frac{\mu}{a} \). Applying \( \Lambda^s \) to (1.3), taking the \( L^2 \) inner product with \( \Lambda^s \tau \) and using Lemmas 2.7, 2.8 we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \Lambda^s \tau \|^2_{L^2} + a \| \Lambda^s \tau \|^2_{L^2} + \mu \| \nabla \Lambda^s \tau \|^2_{L^2} = -\langle [\Lambda^s, u] \tau, \nabla \Lambda^s \tau \rangle + \langle \Lambda^s Q(\Omega, \tau), \Lambda^s \tau \rangle \]

\[ \leq C_\mu \| \tau \|^2_{L^2} \| \Lambda^s u \|^2_{L^2} + C_\mu \| \Lambda^s u \|^2_{L^2} \| \nabla \Lambda^s \tau \|^2_{L^2} + C_\mu \| \Lambda^{s-1} Q(\Omega, \tau) \|^2_{L^2} + \frac{\mu}{4} \| \nabla \Lambda^s \tau \|^2_{L^2} \]

\[ \leq C_\mu \| \tau \|^2_{L^2} \| \Lambda^s u \|^2_{L^2} + C_\mu \| \| u \|_{L^\infty} + \| \Omega \|_{L^\infty} \| \tau \|^2_{H^s} + \frac{\mu}{4} \| \nabla \Lambda^s \tau \|^2_{L^2} . \]

We infer from (3.10), (3.12), (3.20) and (3.21) that

\[ \frac{1}{2} \frac{d}{dt} \| (u, \tau) \|^2_{H^s} \leq \| \nabla u \|_{L^\infty} + C_\mu \| \tau \|^2_{L^2} + C_\mu \| u \|^2_{H^s} + C_\mu (\| u \|_{L^\infty} + \| \Omega \|_{L^\infty}) \| \tau \|^2_{H^s} , \]

which implies that

\[ \| (u, \tau) \|^2_{H^s} \leq \| (u_0, \tau_0) \|^2_{H^s} + \int_0^t \| \nabla u \|_{L^\infty} + C_\mu \| \tau \|^2_{L^2} + C_\mu \| u \|^2_{H^s} ds \]

\[ + \int_0^t C_\mu (\| u \|^2_{L^\infty} + \| \Omega \|^2_{L^\infty}) \| \tau \|^2_{H^s} ds . \]

Applying Gronwall’s inequality, we deduce that

\[ C + \| (u, \tau) \|^2_{H^s} \leq (C + \| (u_0, \tau_0) \|^2_{H^s}) e^{\int_0^t \| \nabla u \|_{L^\infty} + C_\mu (\| \tau \|^2_{L^\infty} + \| u \|^2_{L^\infty} + \| \Omega \|^2_{L^\infty} + 1) ds} . \]
According to Lemma 2.12 we have
\[
\| \nabla u \|_{L^\infty} \leq C \| \nabla u \|_{B^0_{\infty, \infty}} \ln(C + \| u \|_{H^s}^2) + C.
\]
By virtue of (3.24) and Propositions 3.1, 3.2, then we have (3.27)
\[
\ln(C + \| (u, \tau) \|_{H^s}^2) \leq \ln(C + \| (u_0, \tau_0) \|_{H^s}^2) + \int_0^t C_\mu \| \tau \|_{L^\infty}^2 + \| u \|_{L^\infty}^2 + \| \Omega \|_{L^\infty}^2 + 1) \, ds
\]
Applying Gronwall’s inequality to (3.20), we infer that
\[
\ln(C + \| (u, \tau) \|_{H^s}^2) \leq \ln(C + \| (u_0, \tau_0) \|_{H^s}^2) + C t e^{\int_0^t \| \nabla u \|_{B^0_{\infty, \infty}} \, ds} + C t e^{\int_0^t \| \tau \|_{L^\infty}^2 + \| u \|_{L^\infty}^2 + \| \Omega \|_{L^\infty}^2 + 1) \, ds}
\]
Assume that \( T^* < \infty \) and \( \int_0^{T^*} \| \Omega(t) \|_{L^\infty}^2 \, dt < \infty \). By virtue of Lemmas 2.10, 2.11 we obtain
\[
\| \nabla u \|_{B^0_{\infty, \infty}} + \| u \|_{L^\infty} \leq C(\| u \|_{L^2} + \| \Omega(t) \|_{L^\infty})
\]
According to (3.27) and Propositions 3.1, 3.2 then we have \((u, \tau) \in L^\infty([0, T^*); H^s)\), which contradicts the assumption that \( T^* \) is the maximal existence time.

**Remark 3.5.** We can deduce that
\[
\| [\Lambda^s, u] \tau \|_{L^2}^2 \leq C_\mu \| \tau \|_{L^\infty}^2 \| \Lambda^s u \|_{L^2}^2 + C_\mu \| \nabla u \|_{L^2}^2 \| \Lambda^{s-1} \tau \|_{L^2}^2
\]
\[
\leq C_\mu \| \tau \|_{L^\infty}^2 \| \Lambda^s u \|_{L^2}^2 + C_\mu \| \Omega \|_{L^2}^2 \| \tau \|_{H^s}
\]
\[
\leq C_\mu \| \tau \|_{L^\infty}^2 \| \Lambda^s u \|_{L^2}^2 + C_\mu (\| \nabla u \|_{L^2}^2 + \| \Omega \|_{L^2}^2) \| \tau \|_{H^s}^2.
\]
One can see that (3.27) can be rewritten as
\[
\ln(C + \| (u, \tau) \|_{H^s}^2) \leq (\ln(C + \| (u_0, \tau_0) \|_{H^s}^2) + C t e^{\int_0^t \| \nabla u \|_{B^0_{\infty, \infty}} \, ds} + C t e^{\int_0^t \| \nabla u \|_{B^0_{\infty, \infty}} \, ds} \int_0^t C_\mu (\| \tau \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \Omega \|_{L^2}^2 + 1) \, ds,
\]
which is of significance in the proof of Theorem 1.1.

### 3.3. Global Solutions

#### 3.3.1 The Oldroyd-B type model

**The proof of Theorem 1.1:**
The proof of the local well-posedness of (1.3) is standard. We thus omit it and present the result here. For any \( T < T^* \), we have
\[
u \in C([0, T]; H^s), \quad \tau \in C([0, T]; H^s) \cap L^2([0, T]; H^{s+1}).
\]
To get the global existence, the key point is to obtain the uniform estimate of \( \| \Omega \|_{L^\infty} \). However, due to the linear term \( \nabla \times div \tau \), it is difficult to get the global estimate of \( \| \Omega \|_{L^\infty} \) from the following equation
\[
\frac{d}{dt} \Omega + u \cdot \nabla \Omega = \nabla \times div \tau.
\]
Motivated by [9], we can cancel $\nabla \times div \tau$ with the dissipation term $\Delta \tau$. Define
\[ \Gamma = \mu \Omega - R \tau, \quad R = \Delta^{-1} curl \ div. \]

Since $RDu = \Omega$, we obtain
\[
\frac{d}{dt} \Gamma + u \cdot \nabla \Gamma = aR \tau + RQ(\Omega, \tau) + [R, u \cdot \nabla] \tau = \sum_{i=1}^{3} F_i.
\]

(3.30)

Different from [9], there is no damping phenomenon for $\Gamma$ or $\Omega$. It seems impossible to expect the global existence even in small initial data case. However, the disappearance of $D(u)$ leads to exponential dissipation for $\tau$ in $H^1$, which is useful to estimate $\Gamma$ in $L^\infty$. Assume that
\[
\|\nabla u(t)\|_{L^2} \leq 4\varepsilon, \quad \|\Gamma(t)\|_{L^\infty} \leq 4a\mu,
\]
for any $t \in [0, T]$. By Proposition 3.2 and the condition (1.3), we deduce that $\|\nabla u(t)\|_{L^2} \leq 2\varepsilon$ for any $t \in [0, T]$. Then we focus on $\|\Gamma\|_{L^\infty}$. According to (3.30), we obtain
\[
\|\Gamma\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty} + \sum_{i=1}^{3} \int_0^t \|F_i\|_{L^\infty} ds.
\]

(3.32)

From Lemma 2.10 we have
\[
\|F_1\|_{L^\infty} \leq a\|\Delta^{-1} R \tau\|_{L^\infty} + a\| (Id - \Delta^{-1}) R \tau\|_{L^\infty}
\leq Ca\|\tau\|_{L^2} + Ca\|\tau\|_{B^{2,1}_{2,1}}
\leq Ca\|\tau\|_{H^2}.
\]

(3.33)

Applying Lemmas 2.1, 2.10 and 2.12 we get
\[
\|F_2\|_{L^\infty} \leq \|\Delta^{-1} RQ(\Omega, \tau)\|_{L^\infty} + \| (Id - \Delta^{-1}) RQ(\Omega, \tau)\|_{L^\infty}
\leq C\|Q(\Omega, \tau)\|_{L^2} + C\|Q(\Omega, \tau)\|_{B^{2,1}_{2,1}}
\leq C\|\nabla u\|_{L^2} \|\tau\|_{H^2} + C\|\Omega\|_{B^{2,1}_{2,1}} \|\tau\|_{H^2}
\leq C\|\nabla u\|_{L^2} \|\tau\|_{H^2} + C\|\Omega\|_{L^\infty} \ln(C + \|u\|_{H^2}) \|\tau\|_{H^2} + C\|\tau\|_{H^2}.
\]

(3.34)

From Lemma 2.6 we obtain
\[
\|F_3\|_{L^\infty} \leq C(\|\Omega\|_{L^2} + \|\Omega\|_{L^\infty}) \|\tau\|_{H^2}
\leq C\|\nabla u\|_{L^2} \|\tau\|_{H^2} + C\|\Gamma\|_{L^\infty} \|\tau\|_{H^2} + C\|\tau\|_{H^2}
\leq C\|\nabla u\|_{L^2} \|\tau\|_{H^2} + C\|\Gamma\|_{L^\infty} \|\tau\|_{H^2} + C\|\tau\|_{H^2}.
\]

(3.35)

Plugging (3.33) into (3.32), we deduce from (1.4), (3.31) and Corollary 3.3 that
\[
\|\Gamma\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty} + C(1 + a)(1 + 2\mu) \|\tau\|_{H^2} + C\|\nabla u\|_{L^2} \|\tau\|_{H^2} + C\|\Gamma\|_{L^\infty} \|\tau\|_{H^2} + C_\mu \|\tau\|_{H^2}^2
+ C\|\tau\|_{H^2} \ln(C + \|u\|_{H^2}) + C\|\Gamma\|_{L^\infty} \|\tau\|_{H^2} \ln(C + \|u\|_{H^2}) ds
\leq \|\Gamma_0\|_{L^\infty} + C(1 + a)(a^\frac{1}{\mu} + (a\mu)^{-\frac{1}{\mu}}) \|\tau_0\|_{H^2} + C\mu \int_0^t \|\tau\|_{H^2}^2 \ln(C + \|u\|_{H^2}) ds
\]

(3.36)
+ C\mu \int_0^t \|\Gamma\|_{L^\infty} \|\tau\|_{H^2} \ln(C + \|u\|_{H^s}) ds.

By (1.5), we get

\begin{equation}
\|\Gamma\|_{L^\infty} \leq \frac{3}{2} ca\mu + C\mu \int_0^t \|\tau\|^2_{H^2} \ln(C + \|u\|_{H^s}) ds + C\mu \int_0^t \|\Gamma\|_{L^\infty} \|\tau\|_{H^2} \ln(C + \|u\|_{H^s}) ds,
\end{equation}

where we using the condition \(\|\tau_0\|_{H^1} \leq c^2 \lambda\) with

\[ \lambda = \min\{a^{\frac{1}{4}} \mu, a^{\frac{3}{4}} \mu, (a\mu)^{\frac{3}{4}}, (a\mu)^{\frac{5}{4}}, a, \mu, a\mu, a\mu^2, a\mu^3, a\mu^4, \mu^2\}, \]

According to Lemma 2.10 we obtain

\begin{equation}
\|\nabla u\|_{L^0} \leq \|\nabla u\|_{L^2} + \|\Omega\|_{L^\infty},
\end{equation}

and

\begin{equation}
\int_0^t \|\nabla u\|_{H^\infty, \infty} ds \leq C \int_0^t \|\nabla u\|_{L^2} + \|\Omega\|_{L^\infty} ds
\end{equation}

\[ \leq \frac{a}{8} t + C\mu \int_0^t \|\tau\|_{H^2} ds \]

\[ \leq \frac{a}{8} t + C\mu (a^{-\frac{1}{4}} + (a\mu)^{-\frac{1}{4}}) \|\tau_0\|_{H^1} \]

\[ \leq \frac{a}{8} t + C. \]

By (3.38), (3.38) and (3.39), we deduce that

\begin{equation}
\ln(C + \|u(t, \tau)\|_{H^2}^2) \leq C e^{\frac{a}{2} t} [\ln(C + \|u_0, \tau_0\|_{H^s}^2) + t (1 + c^2 k + c^2 a^2 + \mu^{-1}) + c^2 a + c^4 (a + \mu)]
\end{equation}

\[ \leq C e^{\frac{a}{2} t} \ln(C + \|\nabla(a_0, \tau_0)\|_{H^s}^2) + (a\mu)^{-1} + a^{-1} + a + \mu] \]

\[ = A_0 e^{\frac{a}{2} t}, \]

where \(A_0 = C \ln(C + \|u_0, \tau_0\|_{H^s}^2) + (a\mu)^{-1} + a^{-1} + a + \mu\). Plugging (3.40) into (3.37), using (1.5) and applying Proposition 3.2, we obtain

\begin{equation}
\|\Gamma\|_{L^\infty} \leq \frac{3}{2} ca\mu + C\mu \int_0^t \|\tau\|^2_{H^2} A_0 e^{\frac{a}{2} t} ds + C\mu \int_0^t \|\Gamma\|_{L^\infty} \|\tau\|_{H^2} A_0 e^{\frac{a}{2} t} ds
\end{equation}

\[ \leq \frac{3}{2} ca\mu + C\mu (\mu^{-1} + a^{-1}) \|\tau_0\|_{H^1} A_0 + (\mu^{-1} + a^{-1}) \|\tau_0\|_{H^1} A_0 \]

\[ \leq 2ca\mu + (C\mu + 1) (\mu^{-1} + a^{-1}) \|\tau_0\|^2_{H^1} \ln(C + \|u_0, \tau_0\|_{H^s}^2)
\]

\[ \leq 3ca\mu, \]

which implies that

\[ \|\Gamma\|_{L^\infty((0, T^*) \times L^\infty)} \leq 3ca\mu. \]

According to Propositions 3.2 and 3.4 we can deduce that \(T^* = +\infty\). We thus complete the proof of Theorem 1.1.
3.3.2 The Hooke model

Taking $\psi = (g + 1)\psi_0$ with $\psi_0 = e^{-\frac{1}{2}q^2}$ and $\nu = 0$, $a = 2$, $\mu = 1$ in (3.4), we obtain

$$
\begin{aligned}
\frac{d}{dt} u + u \cdot \nabla u + \nabla P &= \text{div} \tau, \quad \text{div} u = 0, \\
\frac{d}{dt} g + u \cdot \nabla g + \frac{1}{\psi_0} \nabla_q \cdot (\Omega q g \psi_0) - \Delta g &= \frac{1}{\psi_0} \nabla_q \cdot (\nabla_q g \psi_0).
\end{aligned}
$$

Let $q = \sqrt{1 + q^2}$. Global well-posedness for the Hooke model (3.42) is considered in the following corollary. Firstly, we establish a new estimate of $\langle q \rangle^n \nabla^m g$ in $L^\infty(L^2)$. Then, we obtain the smallness of $\|\Omega\|_{L^\infty}$ under the condition (3.43) by virtue of the corresponding Ordoñez-B model (3.4). Finally, we derive the global estimate for $\|u\|_{H^s} + \|\langle q \rangle g\|^2_{L^2(L^2)} + \|\langle q \rangle \nabla_q g\|^2_{H^{s-1}(L^2)}$, which implies the global existence of the Hooke model considered.

**Corollary 3.6.** Let $(u, g)$ be a strong solution of (3.42) with the initial data $(u_0, g_0) \in H^s \times H^s(L^2)$ and $(\langle q \rangle g_0, \langle q \rangle \nabla_q g_0, \langle q \rangle \nabla^2_q g_0) \in L^\infty(L^2)$. Let $\int_{\mathbb{R}^2} g_0 \psi_\infty dq = 0$ and $(u_0, \tau_0)$ satisfies the conditions in Theorem 3.7. In addition, if

$$
\|g_0\|_{B^{s-1}_2(L^2)} + \|(u_0, \tau_0)\|_{L^2} \|g_0\|_{L^\infty(L^2)} < \epsilon,
$$

for some positive $\epsilon$ sufficiently small, then the Hooke model (3.42) admits a unique global strong solution $(u, g) \in C([0, \infty); H^s \times H^s(L^2))$.

To begin with, we establish a new estimate of $\|\langle q \rangle^n \nabla^m g\|_{L^\infty(L^2)}$ in the following lemma.

**Lemma 3.7.** Let $(u, g)$ be a strong solution of (3.42) with the initial data $(u_0, g_0) \in H^s \times H^s(L^2)$ and $(\langle q \rangle g_0, \langle q \rangle \nabla_q g_0, \langle q \rangle \nabla^2_q g_0) \in L^\infty(L^2)$. Let $\int_{\mathbb{R}^2} g_0 \psi_\infty dq = 0$. There exists positive constant $C$ such that

$$
\|\langle q \rangle g\|_{L^\infty(L^2)} + \|\langle q \rangle \nabla_q g\|_{L^\infty(L^2)} + \|\langle q \rangle^2 \nabla_q g\|_{L^\infty(L^2)} + \|\langle q \rangle \nabla^2_q g\|_{L^\infty(L^2)} \leq C e^{C t}.
$$

**Proof.** Firstly, we have $\|g\|_{L^\infty(L^2)} \leq \|g_0\|_{L^\infty(L^2)}$ by noticing that the term $\frac{1}{\psi_0} \nabla_q \cdot (\Omega q g \psi_0)$ would vanish since the antisymmetry of $\Omega$ and $\int_{\mathbb{R}^2} g \psi_\infty dq = 0$. More details can refer to [30].

For $\|\langle q \rangle g\|_{L^\infty(L^2)}$, taking $L^2$ inner product with $\langle q \rangle^2 g$ to (3.42), we infer that

$$
\begin{aligned}
\frac{d}{dt} \|\langle q \rangle g\|^2_{L^2} &+ \frac{1}{2} u \cdot \nabla \|\langle q \rangle g\|^2_{L^2} - \frac{1}{2} \Delta \|\langle q \rangle g\|^2_{L^2} + \frac{1}{2} \|\langle q \rangle \nabla g\|^2_{L^2} + \|\langle q \rangle \nabla_q g\|^2_{L^2} \\
&= - \int_{\mathbb{R}^2} \frac{1}{\psi_\infty} \nabla_q \cdot (\Omega q g \psi_\infty) \langle q \rangle^2 g^2 \psi_\infty dq + \int_{\mathbb{R}^2} g^2 \psi_\infty dq - \int_{\mathbb{R}^2} q^2 g^2 \psi_\infty dq.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{1}{\psi_\infty} \nabla_q \cdot (\Omega q g \psi_\infty) \langle q \rangle^2 g^2 \psi_\infty dq &= - \int_{\mathbb{R}^2} \Omega q g \psi_\infty \cdot (2 q g + \langle q \rangle^2 \nabla_q g) dq \\
&= - \frac{1}{2} \int_{\mathbb{R}^2} \Omega q \psi_\infty \langle q \rangle^2 \nabla_q g^2 dq \\
&= - \frac{1}{2} \int_{\mathbb{R}^2} \Omega \delta_k (\langle q \rangle^2 + q_k q_k) g^2 \psi_\infty dq \\
&= 0,
\end{aligned}
$$

we deduce that for any $p \geq 2$,

$$
\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|\langle q \rangle g\|^p_{L^p(L^2)} &\leq C \|\langle q \rangle g\|^p_{L^p(L^2)},
\end{aligned}
$$
which implies \( \| \langle q \rangle g \|_{L^\infty(\mathbb{R}^2)} \leq C e^{C t} \). Similarly, for \( \| \langle q \rangle^n \nabla g \|_{L^\infty(\mathbb{R}^2)} \), \( n \in \{1, 2\} \), we have

\[
\int_{\mathbb{R}^2} \nabla_q \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\Omega q g \psi_\infty) \right) \langle q \rangle^n \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \nabla_q \left( \Omega^{ik} q_k \nabla^k g - \Omega^{ik} q_k q_i g \right) \langle q \rangle^n \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \left( \Omega^{ik} \delta^i_k \nabla^i g + \Omega^{ik} q_k \nabla^i g - \Omega^{ik} (\delta^i_k q_i + \delta^i_k q_k) - \Omega^{ik} q_k q_i \nabla^i g \right) \langle q \rangle^n \nabla_q g \psi_\infty dq \\
= 0.
\]

We deduce from Lemma 2.9 that

\[
\int_{\mathbb{R}^2} \nabla_q \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\nabla_q g \psi_\infty) \right) \langle q \rangle^{2n} \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \nabla_q \left( \Omega^{ik} q_k \nabla^k g \right) \langle q \rangle^{2n} \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \nabla_q \left( \Omega^{ik} q_k \nabla^k g \right) \langle q \rangle^{2n} \nabla_q g dq - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 \\
= -2n \int_{\mathbb{R}^2} \nabla_q \nabla_q g \langle q \rangle^{2(n-1)} \nabla_q g dq - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 \\
\leq C \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 - \| \langle q \rangle^{2n} \nabla_q g \|_{L^2}^2.
\]

Taking \( L^2 \) inner product with \( \langle q \rangle^{2n} \nabla_q g \) to \( 3.41 \), we infer that

\[
\frac{1}{2} \frac{d}{dt} \| \langle q \rangle^{n} \nabla_q g \|_{L^2}^2 + \frac{1}{2} \nabla \| \langle q \rangle^{n} \nabla_q g \|_{L^2}^2 - \frac{1}{2} \Delta \| \langle q \rangle^{n} \nabla_q g \|_{L^2}^2 \leq C \| \langle q \rangle^n \nabla_q g \|_{L^2}^2.
\]

Therefore we deduce that for any \( p \geq 2 \),

\[
\frac{1}{p} \frac{d}{dt} \| \langle q \rangle^{n} \nabla_q g \|_{L^p(\mathbb{R}^2)}^p \leq C \| \langle q \rangle^{n} \nabla_q g \|_{L^p(\mathbb{R}^2)}^p,
\]

which implies \( \| \langle q \rangle^{n} \nabla_q g \|_{L^\infty(\mathbb{R}^2)} \leq C e^{C t} \). For \( \| \langle q \rangle^n \nabla^2 g \|_{L^\infty(\mathbb{R}^2)} \), we have

\[
\int_{\mathbb{R}^2} \nabla_q^2 \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\Omega q g \psi_\infty) \right) \langle q \rangle^{2n} \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \nabla_q \left( \Omega^{ik} q_k \nabla^k g \right) \langle q \rangle^{2n} \nabla_q g \psi_\infty dq \\
= \int_{\mathbb{R}^2} \nabla_q \left( \Omega^{ik} q_k \nabla^k g \right) \langle q \rangle^{2n} \nabla_q g dq - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 \\
= -2n \int_{\mathbb{R}^2} \nabla_q \nabla_q g \langle q \rangle^{2(n-1)} \nabla_q g dq - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 - \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 \\
\leq C \| \langle q \rangle^n \nabla_q g \|_{L^2}^2 - \| \langle q \rangle^{2n} \nabla_q g \|_{L^2}^2.
\]

and

\[
\int_{\mathbb{R}^2} \left[ \nabla_q^2 \left( \frac{1}{\psi_\infty} \nabla_q \cdot (\nabla_q g \psi_\infty) \right) \right] \langle q \rangle^{2n} \nabla_q g \psi_\infty dq = \int_{\mathbb{R}^2} \left[ \frac{1}{\psi_\infty} \nabla_q \cdot (\nabla_q \nabla^2 g \psi_\infty) \right] - \nabla_q^2 g \langle q \rangle^{2n} \nabla_q g \psi_\infty dq \\
= \| \nabla_q^2 g \|_{L^2}^2 - \| \langle q \rangle \nabla_q \nabla^2 g \|_{L^2}^2 - 2 \| \langle q \rangle \nabla_q^2 g \|_{L^2}^2.
\]
Then we deduce that for any $p \geq 2$,
\[
\frac{1}{p} \frac{d}{dt} \| q \|_{L^p(\mathcal{L}^2)}^p \leq C \| q \|_{L^p(\mathcal{L}^2)}^p,
\]
which implies $\| q \|_{L^\infty(\mathcal{L}^2)} \leq Ce^{Ct}$. We thus complete the proof of Lemma 3.7.

By virtue of Theorem 1.1 we obtain the global existence of $u$. The following lemma is about the global existence of $g$.

**Lemma 3.8.** Let $(u, \tau)$ be a strong solution of (1.3) considered in Theorem 1.1. Then for any $\sigma > 0$, there exist positive constant $\varepsilon$ small enough such that if
\[
\| g_0 \|_{B^0_{\infty,1}(\mathcal{L}^2)} + \| (u_0, \tau_0) \|_{L^2} \| g_0 \|_{L^\infty(\mathcal{L}^2)} < \varepsilon,
\]
then $\| \Omega \|_{L^\infty} < \sigma$.

**Proof.** By virtue of Lemma 2.10 we deduce that
\[
\| \Omega \|_{L^\infty} \leq \| \Gamma \|_{L^\infty} + \| R \tau \|_{L^\infty} \leq \| \Gamma \|_{L^\infty} + \| \tau \|_{B^0_{\infty,1}}.
\]
It’s follows from the proofs of Theorem 1.1 that
\[
\| \Gamma \|_{L^\infty} \leq C \varepsilon < \frac{\sigma}{2},
\]
provided $\varepsilon < \frac{\sigma}{2C}$. We need to prove $\| \tau \|_{B^0_{\infty,1}} < \frac{\sigma}{2}$. Applying $\Delta_j$ to (1.3) with $j \geq 1$ yields
\[
\partial_t \Delta_j \tau + \Delta_j \tau + \Delta_j Q(\Omega, \tau) = \Delta \Delta_j \tau - \Delta_j (u \cdot \nabla \tau).
\]
Therefore
\[
\Delta_j \tau = e^{-t(1+\Delta_j)} \Delta_j \tau_0 - \int_0^t e^{-(t-s)} e^{(t-s)\Delta} (\Delta_j Q(\Omega, \tau) + \Delta_j (u \cdot \nabla \tau)) ds.
\]
According to Lemma 2.15 we infer that
\[
\| \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \Delta_j Q(\Omega, \tau) ds \|_{L^\infty} \leq \int_0^t e^{-2j(t-s)} \| \Delta_j Q(\Omega, \tau) \|_{L^\infty} ds \leq \int_0^t e^{-2j(t-s)} \| \Delta_j Q(\Omega, \tau) \|_{L^2} ds \leq \int_0^t e^{-2j(t-s)} 2^{2j} \| \nabla u \|_{L^2} \| \tau \|_{L^\infty} ds.
\]
Similarly, by virtue of $\text{div} \ u = 0$, we have
\[
\| \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \Delta_j (u \cdot \nabla \tau) ds \|_{L^\infty} \leq \int_0^t e^{-2j(t-s)} 2^{2j} \| \Delta_j (u \cdot \tau) \|_{L^\infty} ds \leq \int_0^t e^{-2j(t-s)} 2^{2j} \| u \otimes \tau \|_{L^\infty} ds \leq \int_0^t e^{-2j(t-s)} 2^{2j} \| u \|_{H^1} \| \tau \|_{L^\infty} ds.
\]
Notice that

\[
\|\tau\|_{L^\infty} \leq C\|g\|_{L^\infty(L^2)} \leq C\|g_0\|_{L^\infty(L^2)},
\]

and

\[
\sup_{t \geq 0} \sum_{j \in \mathbb{N}} \int_0^t e^{-2^{2j}(t-s)} 2^{2j} ds \leq C.
\]

According to (3.50)-(3.55) and Proposition 3.2, we deduce that

\[
\|\tau\|_{B^0_{\infty,1}} \leq \|\tau_0\|_{B^0_{\infty,1}} + C(\|\nabla u\|_{L^\infty([0,T],L^2)} + \|(u_0,\tau_0)\|_{L^2})\|g_0\|_{L^\infty(L^2)}
\]

\[
\leq C(\epsilon + \epsilon^2) < \frac{\sigma}{2}.
\]

We thus complete the proof of Lemma 3.8. \(\square\)

**The proof of Corollary 3.6:**

By virtue of the lemmas above, we finally obtain the global well-posedness of \((u, g)\). Taking \(L^2(L^2)\) inner product with \(g\) to (3.42), we deduce that

\[
\frac{d}{dt}\|g\|_{L^2(L^2)}^2 + \|\nabla g\|_{L^2(L^2)}^2 + \|\nabla g\|_{L^2(L^2)}^2 = 0.
\]

Applying \(\Lambda^s\) to (3.42) and taking \(L^2(L^2)\) inner product with \(\Lambda^s g\), we obtain

\[
\frac{d}{dt}\|\Lambda^s g\|_{L^2(L^2)}^2 + \|\Lambda^{s+1} g\|_{L^2(L^2)}^2 + \|\nabla \Lambda^s g\|_{L^2(L^2)}^2
\]

\[
\leq C\|u\|_{L^\infty}^2 \|\Lambda^s g\|_{L^2(L^2)}^2 + C\|u\|_{H^s}^2 \|g\|_{L^\infty(L^2)}^2 + \epsilon \|\Lambda^{s+1} g\|_{L^2(L^2)}^2
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Lambda^s(\frac{1}{\psi_0} \nabla g \cdot (\Omega q g \psi_\infty)) \Lambda^s(\psi_0 dq).
\]

According to Lemma 2.8, we deduce that

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} \Lambda^s(\frac{1}{\psi_0} \nabla g \cdot (\Omega q g \psi_\infty)) \Lambda^s(\psi_0 dq)
\]

\[
\leq \int_{\mathbb{R}^2} \|\Lambda^{s-1}(\Omega q \nabla g \cdot (\Omega q g \psi_\infty))\|_{L^2} \|\Lambda^{s+1} g\|_{L^2(L^\infty)} dq
\]

\[
\leq C\|\Omega\|_{L^\infty}^2 \|\nabla \Lambda^{s-1} g\|_{L^2(L^\infty)}^2 + C\|u\|_{H^s}^2 \|\nabla g\|_{L^\infty(L^2)}^2 + \epsilon \|\Lambda^{s+1} g\|_{L^2(L^2)}^2,
\]

which implies that

\[
\frac{d}{dt}\|\Lambda^s g\|_{L^2(L^2)}^2 + \|\Lambda^{s+1} g\|_{L^2(L^2)}^2 + \|\nabla \Lambda^s g\|_{L^2(L^2)}^2 \leq C\|u\|_{L^\infty}^2 \|\Lambda^s g\|_{L^2(L^2)}^2
\]

\[
+ C\|\Omega\|_{L^\infty}^2 \|\nabla \Lambda^{s-1} g\|_{L^2(L^\infty)}^2 + C\|u\|_{H^s}^2 \|\nabla g\|_{L^\infty(L^2)}^2 + \epsilon \|\Lambda^{s+1} g\|_{L^2(L^2)}^2(\|\nabla g\|_{L^\infty(L^2)}^2).
\]

The appearance of the term \(\|\nabla g\|_{H^{-1}(L^2)}^2\) force us to consider mixed derivative estimates which have been used in [30] and [17]. Applying \(\Lambda^m\) to (3.42) with \(m \in \{0, s\}\) and taking \(L^2(L^2)\) inner product with \(\langle q \rangle^2 \Lambda^m g\), we infer

\[
\frac{d}{dt}\|\langle q \rangle g\|_{L^2(L^2)}^2 + \|\langle q \rangle \nabla g\|_{L^2(L^2)}^2 + \|\langle q \rangle \nabla g\|_{L^2(L^2)}^2 \leq C\|g\|_{L^2(L^2)}^2,
\]
and

\[
\begin{align*}
\frac{d}{dt} \| \langle q \rangle \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2 + \| \langle q \rangle \Lambda^{s+1} g \|_{L^2(\mathcal{L}^2)}^2 + \| \langle q \rangle \nabla_q \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2 & \leq C \| u \|_{L^\infty}^2 \| \langle q \rangle \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2
\end{align*}
\]

+ \| \langle q \rangle \nabla_q \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2 + \| \langle q \rangle \nabla_q \Lambda^{s-1} g \|_{L^2(\mathcal{L}^2)}^2 + C \| u \|_{L^\infty}^2 \| \langle q \rangle \nabla_q \Lambda^{s-1} g \|_{L^2(\mathcal{L}^2)}^2
\]

+ \| \langle q \rangle \nabla_q \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2 + \| \langle q \rangle \nabla_q \Lambda^{s-1} g \|_{L^2(\mathcal{L}^2)}^2 + C \| u \|_{L^\infty}^2 \| \langle q \rangle \nabla_q \Lambda^{s-1} g \|_{L^2(\mathcal{L}^2)}^2 + C \| \langle q \rangle \nabla_q \Lambda^s g \|_{L^2(\mathcal{L}^2)}^2 + C \| \langle q \rangle \nabla_q \Lambda^{s-1} g \|_{L^2(\mathcal{L}^2)}^2.
\]

Together with Lemma 2.11 and the following estimate

\[
\frac{d}{dt} \| u \|_{H^s}^2 \leq C (\| \nabla u \|_{L^\infty} + 1) \| u \|_{H^s}^2 + C \| g \|_{H^{s+1}(\mathcal{L}^2)}^2.
\]

According to (3.52)–(3.64), Lemma 3.8 and Gronwall’s inequality, we deduce that

\[
\begin{align*}
\| u \|_{H^s} + \| \langle q \rangle g \|_{H^s(\mathcal{L}^2)}^2 + \| \langle q \rangle \nabla_q g \|_{H^{s-1}(\mathcal{L}^2)}^2 & \leq C (\| u_0 \|_{H^s}^2 + \| \langle q \rangle g_0 \|_{H^s(\mathcal{L}^2)}^2 + \| \langle q \rangle \nabla_q g_0 \|_{H^{s-1}(\mathcal{L}^2)}^2) e^{\frac{t^*}{2} \| u \|_{L^\infty} + \| \nabla u \|_{L^\infty} dt'}.
\end{align*}
\]

According to Theorem 2.11, we finish the proof of Corollary 3.6. \qed

**Remark 3.9.** The main difficulty for the global estimates of (3.42) is that once we stop the growth of regularity in \( x \) by \( \Delta \psi \), we can not stop the growth of power \( \langle q \rangle \) by \( \psi \) at the same time. It is worth mentioning that the estimate of \( \| \langle q \rangle \nabla_q g \|_{H^{s-1}(\mathcal{L}^2)} \) instead of \( \| \langle q \rangle \nabla_q g \|_{H^s(\mathcal{L}^2)} \) enable us to stop the growth of power \( \langle q \rangle \) caused by the term \( \frac{1}{q} \nabla_q \cdot (\Omega g \psi_{\infty}) \). As a result, we obtain the prior estimate of \( \| \langle q \rangle g \|_{H^s(\mathcal{L}^2)}^2 \) and \( \| \langle q \rangle \nabla_q g \|_{H^{s-1}(\mathcal{L}^2)}^2 \).

**Remark 3.10.** The estimate of \( \langle q \rangle^n \nabla_q g \) in \( L^\infty(\mathcal{L}^2) \) and the smallness of \( \| \Omega \|_{L^\infty} \) are significant in the proof of Corollary 3.6. The global existence of (3.42) for arbitrary initial data and the global existence of the non-co-rotation Hooke model are interesting problems. We are going to study these problems in the future.

4 Global solutions for co-rotation case in critical Besov space

In this section, we are concerned with global solutions to the co-rotation Oldroyd-B type model in critical Besov space. We divide it into three steps to prove Theorem 3.5.
4.1. Energy estimates

From now on, we prove the boundness for (1.3) in the following propositions.

**Proposition 4.1.** Suppose \((u, \tau)\) is a smooth solution to (1.3) with \(u_0 \in H^1\) and \(\tau_0 \in H^1 \cap L^\infty\). Then we obtain

\[
\|\tau\|_{L^2_t L^\infty_x}^2 + \frac{1}{\mu} \|\nabla \tau\|_{L^2_t L^\infty_x}^2 \leq \left( \|u_0\|_{L^2_x}^2 + \|\tau_0\|_{L^2_x}^2 \right) e^{\frac{\mu}{\alpha} + \frac{\mu}{\alpha} \|\tau_0\|_{L^\infty_x}^2 + 3\mu^{-2} \|\tau_0\|_{L^2_x}^2}.
\]

Moreover, we get

\[
\|u, \tau\|_{H^2}^2 \leq H_0,
\]

where \(H_0 = \|(u_0, \tau_0)\|_{\mu, \tau}^2 e^{\frac{\mu}{\alpha} + \frac{\mu}{\alpha} \|\tau_0\|_{L^\infty_x}^2 + 3\mu^{-2} \|\tau_0\|_{L^2_x}^2}.
\]

**Proof.** Taking the \(L^2\) inner product with \(\Delta \tau\) to (1.3) and using Lemma 2.7, we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \tau\|_{L^2}^2 + \mu \|\nabla^2 \tau\|_{L^2}^2 \leq \|u \cdot \nabla \tau, \Delta \tau\| + (Q(\Omega, \tau), \Delta \tau) \leq \|\nabla u\|_{L^2} \|\nabla \tau\|_{L^2} + \|Q(\Omega, \tau)\|_{L^\infty} \|\Delta \tau\|_{L^2} \leq \frac{3}{\mu} \|\nabla u\|_{L^2}^2 (\|\nabla \tau\|_{L^2}^2 + \|\tau\|_{L^\infty}^2) + \frac{2\mu}{3} \|\nabla^2 \tau\|_{L^2}^2.
\]

Consider \(\tilde{\tau}^i = \tau^j e^{\tilde{\tau}^j}\), we infer from (4.3) that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{\tau}\|_{L^2}^2 + \mu \|\nabla^2 \tilde{\tau}\|_{L^2}^2 \leq \frac{3}{\mu} \|\nabla \tilde{\tau}\|_{L^2}^2 (\|\nabla \tilde{\tau}\|_{L^2}^2 + \|\tilde{\tau}\|_{L^\infty}^2) + \frac{2\mu}{3} \|\nabla^2 \tilde{\tau}\|_{L^2}^2.
\]

Taking the \(L^2\) inner product with \(\Delta \tau\) to (1.3), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq \frac{3}{\mu} e^{-at} \|\nabla u\|_{L^2}^2 + \frac{\mu}{3} \|\nabla^2 \tilde{\tau}\|_{L^2}^2.
\]

Combining (4.4) and (4.5), we deduce that

\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla \tilde{\tau}\|_{L^2}^2) \leq \frac{3}{\mu} \|\nabla u\|_{L^2}^2 (e^{-at} + \|\nabla \tilde{\tau}\|_{L^2}^2 + \|\tilde{\tau}\|_{L^\infty}^2).
\]

Applying Gronwall’s inequality to (4.6) and using propositions 3.1 3.2 we obtain

\[
\|\nabla u\|_{L^2}^2 + \|\nabla \tilde{\tau}\|_{L^2}^2 \leq \left( \|u_0\|_{L^2_x}^2 + \|\tau_0\|_{L^2_x}^2 \right) e^{\frac{\mu}{\alpha} + \frac{\mu}{\alpha} \|\tau_0\|_{L^\infty_x}^2 + 3\mu^{-2} \|\tau_0\|_{L^2_x}^2} ds + \frac{\mu}{\alpha} \|\tau_0\|_{L^\infty_x}^2 + \frac{\mu}{\alpha} \int_0^t e^{3\mu^{-2} \|\tau_0\|_{L^2_x}^2} \|\nabla \tau\|_{L^2}^2 ds \leq \left( \|u_0\|_{L^2_x}^2 + \|\tau_0\|_{L^2_x}^2 \right) e^{\frac{\mu}{\alpha} + \frac{\mu}{\alpha} \|\tau_0\|_{L^\infty_x}^2 + 3\mu^{-2} \|\tau_0\|_{L^2_x}^2},
\]

Combining (4.7) and (4.5), we finish the proof of Proposition 4.1.

Then, we consider time integrability of \(\tau\). The exponential weight in the following proposition is the key to estimating \(\Gamma\) in \(B_{\infty,1}^0\).

**Proposition 4.2.** Suppose \((u, \tau)\) is a smooth solution to (1.3) with \(u_0 \in H^1\) and \(\tau_0 \in H^1 \cap L^\infty\). Then we obtain

\[
\|e^{\tilde{\tau}^j}\|_{L^2_t B_{\infty,\infty}^\frac{1}{2}} \leq B_0,
\]

where \(B_0 = \mu^{-\frac{1}{4}} \|\tau\|_{L^4}^4 + \mu^{-\frac{1}{4}} a^{-\frac{1}{4}} H_0^{-\frac{1}{4}} \|\tau_0\|_{L^\infty}^4\).
Proof. Set \( \tilde{\tau}^{ij} = \tau^{ij} e^{at} \), we infer from (4.3)
\[
(4.9) \quad \partial_t \tilde{\tau}^{ij} + u \cdot \nabla \tilde{\tau}^{ij} + Q(\Omega^{ik}, \tilde{\tau}^{kj}) = \mu \Delta \tilde{\tau}^{ij}.
\]
Applying \( \Delta_j \) to (4.9), we obtain
\[
(4.10) \quad \partial_t \Delta_j \tilde{\tau}^{ij} - \mu \Delta_j \Delta \tilde{\tau}^{ij} = -\Delta_j \text{div}(u \tilde{\tau}^{ij}) - \Delta_j Q(\Omega^{ik}, \tilde{\tau}^{kj}),
\]
which implies
\[
(4.11) \quad \Delta_j e^{a(t-s)} \tilde{\tau}^{ij} = \Delta_j \tau_0^{ij} e^{\mu t} - \int_0^t e^{\mu(t-s)} \Delta_j \text{div}(u \tilde{\tau}^{ij}) ds.
\]
Applying Minkowski’s inequality and Proposition 3.1, we deduce
\[
(4.12) \quad \left( \int_0^t 2^j \| \tau_0^{ij} e^{\frac{\mu t}{2}} e^{\mu \Delta (t - s)} \|_{L^\infty} ds \right)^{\frac{1}{2}} \leq \left( \int_0^t e^{-\mu s} \| \Delta_j \text{div}(u \tilde{\tau}^{ij}) \|_{L^\infty} ds \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^t e^{-\mu s} \| \Delta_j Q(\Omega^{ik}, \tilde{\tau}^{kj}) \|_{L^\infty} ds \right)^{\frac{1}{2}}
\]
\[
\leq \mu^{-\frac{1}{2}} \| \tau_0 \|_{L^4}.
\]
By virtue of Minkowski’s inequality and Proposition 3.1, we infer
\[
(4.13) \quad \left( \int_0^t 2^j \| \int_0^s e^{-\frac{\mu s}{2}} e^{\mu (s-t) \Delta (\Delta_j \text{div}(u \tilde{\tau}^{ij}) + \Delta_j Q(\Omega^{ik}, \tilde{\tau}^{kj}))} ds \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^t e^{-\mu s} \left( \int_0^s 2^j e^{-\mu 2^j (s-t)} \| \Delta_j Q(\Omega^{ik}, \tilde{\tau}^{kj}) \|_{L^\infty} ds \right)^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^t e^{-\mu s} \left( \int_0^s 2^j e^{-\mu 2^j (s-t)} \| u \|_{L^4} \| \tilde{\tau}^{ij} \|_{L^\infty} + 2^j \| \Omega^{ik} \|_{L^2} \| \tilde{\tau}^{kj} \|_{L^\infty} ds \right)^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^t e^{-\mu s} \left( \int_0^s 2^j e^{-\mu 2^j (s-t)} \| u \|_{H^1} \| \tilde{\tau}^{ij} \|_{L^\infty} ds \right)^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \| u \|_{H^1} \| \tilde{\tau}^{ij} \|_{L^\infty} \left( \int_0^t e^{-\mu s} \left( \int_0^s 2^j e^{-\mu 2^j (s-t)} ds \right)^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \mu^{-\frac{1}{2}} H_{0}^{\frac{1}{2}} \| \tau_0 \|_{L^\infty}.
\]
According to (4.11)-(4.13), we deduce that
\[
(4.14) \quad \| e^{\frac{\mu t}{2}} \tau \|_{L^2 B^{\frac{1}{2}}_{\infty, \infty}} \leq \mu^{-\frac{1}{2}} \| \tau_0 \|_{L^4} + \mu^{-1} a^{-\frac{1}{2}} H_{0}^{\frac{1}{2}} \| \tau_0 \|_{L^\infty}.
\]
We thus complete the proof of Proposition 4.2 \( \square \)

4.2. Local well-posedness

**Proposition 4.3.** Let \((u_0, \tau_0) \in B^1_{\infty, 1} \times B^0_{\infty, 1} \). There exists a time \( T > 0 \) such that (1.3) has a solution \((u, \tau) \in L^\infty([0, T); B^1_{\infty, 1} \times B^0_{\infty, 1})\).
Proof. Since \( \text{div } u = 0 \), we have
\[
\Delta p = \text{div} \text{ div } (\tau - u \otimes u),
\]
which implies
\[
\nabla p = \nabla \Delta^{-1} \text{div} \text{ div } (\tau - u \otimes u).
\]
Applying \( \Delta_j \) to (1.3), we obtain
\[
\frac{\partial}{\partial t} \Delta_j u + u \cdot \nabla \Delta_j u = -[\Delta_j, \nabla] u + \Delta_j \text{div} \tau - \nabla \Delta^{-1} \text{div} \text{ div } (\tau - u \otimes u)
\]
Integrating (4.17) over \([0,T]\), we infer
\[
\|\Delta_j u\|_{L^\infty} \leq \|\Delta_j u_0\|_{L^\infty} + \int_0^T \|[\Delta_j, \nabla] u\|_{L^\infty}
\]
\[
+ 2^j \|\Delta_j \tau\|_{L^\infty} + \|\nabla \Delta^{-1} \text{div} \text{ div } (\tau - u \otimes u)\|_{L^\infty} dt.
\]
By virtue of Lemmas 2.6 and 2.10, we obtain
\[
\|\nabla \Delta^{-1} \text{div} \text{ div } \tau\|_{B^{s_1}_{\infty,1}} \leq C\|\tau\|_{B^{s_1}_{\infty,1}}.
\]
Hence, we deduce from (1.13) and (4.19) that
\[
\|u\|_{L^p_t B^{s_1}_{\infty,1}} \leq \|u_0\|_{B^{s_1}_{\infty,1}} + C(T\|u\|_{L^p_t B^{s_1}_{\infty,1}} + \int_0^T \|\tau\|_{B^{s_1}_{\infty,1}} dt).
\]
Applying \( \Delta_j \) to (1.3), we obtain
\[
\Delta_j \tau = e^{t(\mu a_0 - t)} \Delta_j \tau_0 + \int_0^t e^{(t-s)(\mu a_0 - t)} (\Delta_j (Q(\tau, \tau)) + \text{div} \Delta_j (u \otimes \tau)) ds.
\]
By virtue of Lemmas 2.5 and 2.10, we infer that
\[
\|\tau\|_{L^p_t B^{s_1}_{\infty,1}} + \|\tau\|_{L^p_t B^{s_1}_{\infty,1}} \leq \|\tau_0\|_{B^{s_1}_{\infty,1}} + \int_0^T \|Q(\tau, \tau)\|_{B^{s_1}_{\infty,1}} ds + \int_0^T \|u \otimes \tau\|_{B^{s_1}_{\infty,1}} ds
\]
\[
\leq \|\tau_0\|_{B^{s_1}_{\infty,1}} + C\|u\|_{L^p_t B^{s_1}_{\infty,1}} (T\|\tau\|_{L^p_t B^{s_1}_{\infty,1}} + T^\frac{1}{p}\|\tau\|_{L^p_t B^{s_1}_{\infty,1}}).
\]
Notice that
\[
\|\tau\|_{L^p_t B^{s_1}_{\infty,1}} \leq \|\tau\|_{L^p_t B^{s_1}_{\infty,1}} \|\tau\|_{L^p_t B^{s_1}_{\infty,1}}.
\]
According to (4.20), (4.22) and (4.23) we obtain
\[
\|u\|_{L^p_t B^{s_1}_{\infty,1}} + \|\tau\|_{L^p_t B^{s_1}_{\infty,1}} \leq 6C(\|u_0\|_{B^{s_1}_{\infty,1}} + \|\tau_0\|_{B^{s_1}_{\infty,1}}).
\]
Applying Gronwall’s inequality to (4.31), we deduce that

\[ T = \min\{1, \frac{1}{36C^3(\|u_0\|_{B^1_{\infty,1}} + \|\tau_0\|_{B^0_{\infty,1}})} \cdot \frac{1}{296C^6(\|u_0\|_{B^1_{\infty,1}} + \|\tau_0\|_{B^0_{\infty,1}})^2} \}. \]

Plugging (4.25), (4.26) into (4.24) leads to

\[ \|u\|_{L^\infty_T B^1_{\infty,1}} + \|\tau\|_{L^\infty_T B^1_{\infty,1}} + \|\tau\|_{L^1_T B^2_{\infty,1}} < 6C(\|u_0\|_{B^1_{\infty,1}} + \|\tau_0\|_{B^0_{\infty,1}}). \]

We thus complete the proof of Proposition 4.3.

**Proposition 4.4.** Assume that \( d = 2 \). Let \((u, \tau)\) be a strong solution of (1.3) with the initial data \((u_0, \tau_0) \in (H^1 \cap B^1_{\infty,1}) \times (H^1 \cap B^0_{\infty,1})\). If \( T^* \) is the maximal existence time, then the solution blows up in finite time \( T^* < \infty \) if and only if

\[ \int_0^{T^*} \|\Omega\|_{B^0_{\infty,1}} \, dt = \infty. \]

**Proof.** According to Bony’s decomposition, we obtain

\[ \int_0^T \|Q(\Omega, \tau)\|_{B^0_{\infty,1}} \, dt \leq \int_0^T (\|\Omega\|_{L^\infty} + \|\tau\|_{L^\infty} + \|\tau\|_{H^1})(\|u\|_{B^1_{\infty,1}} + \|\tau\|_{B^0_{\infty,1}}) \, ds, \]

By virtue of Lemma 2.2 and Proposition 4.1, we deduce from (1.3) and (4.20) that

\[ \|\tau\|_{L^\infty_T B^0_{\infty,1}} + \|\tau\|_{L^1_T B^1_{\infty,1}} \leq Ce^{\int_0^T \|\Omega\|_{B^0_{\infty,1}} \, dt} \left(\|\tau_0\|_{B^1_{\infty,1}} + \int_0^T \|Q(\Omega, \tau)\|_{B^0_{\infty,1}} \, dt \right) \]

\[ \leq Ce^{\int_0^T \|\Omega\|_{B^0_{\infty,1}} \, dt} \left(\|\tau_0\|_{B^1_{\infty,1}} + \int_0^T (C + \|\Omega\|_{L^\infty})(\|u\|_{B^1_{\infty,1}} + \|\tau\|_{B^0_{\infty,1}}) \, dt \right). \]

According to Lemma 2.2 and (4.30), we infer that

\[ \|u\|_{L^\infty_T B^1_{\infty,1}} + \|\tau\|_{L^\infty_T B^0_{\infty,1}} \leq Ce^{\int_0^T \|\Omega\|_{B^0_{\infty,1}} \, dt} \left(\|u_0\|_{B^1_{\infty,1}} + \|\tau_0\|_{B^1_{\infty,1}} + \int_0^T (C + \|\Omega\|_{B^0_{\infty,1}})(\|u\|_{B^1_{\infty,1}} + \|\tau\|_{B^0_{\infty,1}}) \, dt \right). \]

Applying Gronwall’s inequality to (4.31), we deduce that

\[ \|u\|_{L^\infty_T B^1_{\infty,1}} + \|\tau\|_{L^\infty_T B^0_{\infty,1}} \leq C \left(\|u_0\|_{B^1_{\infty,1}} + \|\tau_0\|_{B^0_{\infty,1}} \right) e^{\int_0^T \|\Omega\|_{B^0_{\infty,1}} \, dt}, \]

Assume that \( T^* < \infty \) and \( \int_0^{T^*} \|\Omega(t)\|_{B^0_{\infty,1}} \, dt < \infty \). By virtue of Proposition 4.3 and (4.32), we infer that the solution can be continued beyond \([0, T^*)\), which contradicts the assumption that \( T^* \) is the maximal existence time.

### 4.3. Global well-posedness

**The proof of Theorem 1.5:**

Notice that

\[ \int_0^T \|\Omega\|_{B^0_{\infty,1}} \, dt \leq \int_0^T \|\Gamma\|_{B^0_{\infty,1}} \, dt + \int_0^T \|\tau\|_{B^0_{\infty,1}} \, dt. \]
We infer from Proposition 2.6 and Proposition 2.10 that the estimate of $\|\Gamma\|_{B^a_{\infty,1}}$ will finish the proof of the global existence for (1.3). Recall that

$$\frac{d}{dt} \Gamma + u \cdot \nabla \Gamma = R \tau + RQ(\Omega, \tau) + [R, u \cdot \nabla] \tau = \sum_{i=1}^{3} F_i,$$  \hspace{1cm} (4.34)

Note that

$$E_0 = H_0(\|\tau_0\|_{B^a_{\infty,1}} + \|\tau_0\|_{L^4}) \text{ and } D_0 = \|(\nabla u_0, \tau_0)\|_{B^a_{\infty,1}}.$$  \hspace{1cm} (4.35)

Suppose $\forall \ t \in [0, T)$, we have

$$\|\Gamma\|_{B^a_{\infty,1}} \leq c_1 a^{\frac{2}{3}} e^{\frac{2}{3}t},$$  \hspace{1cm} (4.36)

for some $c_1$ small enough. Applying Lemma 2.3 to (4.34), we obtain

$$\|\Delta_1 \nabla u\|_{L^\infty} + \|(I - \Delta_1) \nabla u\|_{L^\infty} ds \leq \int_0^t \|\Gamma\|_{B^a_{\infty,1}} ds + \int_0^t \|\tau\|_{B^a_{\infty,1}} ds \leq C(a^{\frac{4}{3}} e^{\frac{4}{3}t} + a^{\frac{3}{2}} B_0).$$  \hspace{1cm} (4.37)

According to Lemmas 2.6 and 2.10 we have

$$\int_0^t |\nabla u|_{L^\infty} ds \leq \int_0^t |\nabla u|_{L^\infty} ds + \int_0^t |(I - \Delta_1) \nabla u|_{L^\infty} ds \leq \int_0^t \|\Omega\|_{B^a_{\infty,1}} ds \leq \int_0^t \|\Gamma\|_{B^a_{\infty,1}} ds + \int_0^t \|\tau\|_{B^a_{\infty,1}} ds \leq C(a^{\frac{4}{3}} e^{\frac{4}{3}t} + a^{\frac{3}{2}} B_0).$$  \hspace{1cm} (4.38)

Using the conditions (1.6) and (1.7), we get

$$\|\tau\|_{L^4} \leq c_1 a^{\frac{2}{3}} \eta, \ D_0 \leq c_1 \gamma \text{ and } E_0 \leq c_1 a^{\frac{2}{3}} \eta^2,$$  \hspace{1cm} (4.39)

where $\gamma = \min\{a^{\frac{2}{3}}, a\}$ and $\eta = \min\{a^{\frac{2}{3}}, a^\frac{1}{3}\}$. Then we have

$$B_0 = \mu^{-\frac{1}{2}} \|\tau_0\|_{L^4} + \mu^{-1} a^{-\frac{1}{2}} R H_0^{\frac{1}{2}} \|\tau_0\|_{L^\infty} \leq c_1 \eta + a^{-\frac{1}{2}} \mu D_0 \leq 2c_1 \eta.$$  \hspace{1cm} (4.40)

By virtue of (4.38) and (4.40), we obtain $1 + \int_0^t |\nabla u|_{L^\infty} ds \leq C a^{\frac{4}{3}} e^{\frac{4}{3}t}$, which implies that

$$\|\Gamma_0\|_{B^a_{\infty,1}}(1 + \int_0^t |\nabla u|_{L^\infty} ds) \leq CD_0 + C a^{\frac{4}{3}} e^{\frac{4}{3}t} \leq \frac{c_1}{10} a^{\frac{4}{3}} e^{\frac{4}{3}t}.$$  \hspace{1cm} (4.41)

According to Lemmas 2.6, 2.10 and proposition 4.2, we deduce that

$$\int_0^t |F_i|_{B^a_{\infty,1}} ds \leq \int_0^t |\Delta_1 R \tau|_{L^\infty} + \|(I - \Delta_1) R \tau\|_{B^a_{\infty,1}} ds \leq \int_0^t |\tau|_{B^a_{\infty,1}} ds.$$  \hspace{1cm} (4.42)
According to (4.46), Propositions 4.2 and 3.1, we get

\[
(4.45) \quad (\int_0^t e^{a_s \|\tau\|_{B^\frac{1}{2}_{2,\infty}}} ds)^{\frac{1}{t}} \leq a^{-\frac{1}{\beta}} B_0.
\]

Thus we infer from (4.40) and (4.42) that

\[
(4.43) \quad \int_0^t \| F_1 \|_{B^\alpha_{2,1}} ds (1 + \int_0^t \| \nabla u \|_{L^\infty} ds) \leq C a^{-\frac{1}{\beta}} B_0 (1 + a^{-\frac{1}{\beta}} e^{\frac{4}{\beta} t}) \leq \frac{C_1}{10} a^{\frac{1}{\beta}} e^{\frac{4}{\beta} t}.
\]

By virtue of Lemmas 2.6, 2.10 and proposition 3.2, we obtain

\[
(4.44) \quad \int_0^t \| F_2 \|_{B^\alpha_{2,1}} ds \leq \int_0^t \| Q(\Omega, \tau) \|_{B^\alpha_{2,1}} ds
\]

\[
\leq C \int_0^t \| \Omega \|_{B^\alpha_{2,1}} \| \tau \|_{B^\frac{1}{2}_{2,\infty}} ds
\]

\[
\leq C \int_0^t \| \Gamma \|_{B^\alpha_{2,1}} \| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| R \tau \|_{B^\alpha_{2,1}} \| \tau \|_{B^\frac{1}{2}_{2,\infty}} ds
\]

\[
\leq C (a^{-\frac{1}{\beta}} B_0 + B_0^2).
\]

Then we deduce from (4.40) and (4.44) that

\[
(4.45) \quad \int_0^t \| F_2 \|_{B^\alpha_{2,1}} ds (1 + \int_0^t \| \nabla u \|_{L^\infty} ds) \leq C (a^{-\frac{1}{\beta}} B_0 + B_0^2) (1 + a^{-\frac{1}{\beta}} e^{\frac{4}{\beta} t}) \leq \frac{C_1}{10} a^{\frac{1}{\beta}} e^{\frac{4}{\beta} t}.
\]

We infer from Lemma 2.6 that

\[
(4.46) \quad \| [R, u \cdot \nabla] \tau \|_{B^\alpha_{2,1}} \leq C (\| \Omega \|_{L^\infty} + \| \Omega \|_{L^1}) (\| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| \tau \|_{L^1})
\]

\[
\leq C (\| \Omega \|_{L^\infty} + \| u \|_{H^1}) (\| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| \tau \|_{L^1})
\]

\[
\leq C (\| \Gamma \|_{L^\infty} + \| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| u \|_{H^1}) (\| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| \tau \|_{L^1}).
\]

According to (4.46), Propositions 4.2 and 5.1, we get

\[
(4.47) \quad \int_0^t \| F_3 \|_{B^\alpha_{2,1}} ds \leq \int_0^t C (\| \Omega \|_{L^\infty} + \| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| u \|_{H^1}) (\| \tau \|_{B^\frac{1}{2}_{2,\infty}} + \| \tau \|_{L^1})
\]

\[
\leq C (a^{-\frac{1}{\beta}} B_0 + B_0^2 + a^{-\frac{1}{2}} \| \tau_0 \|_{L^1} + a^{-\frac{1}{2}} \| \tau_0 \|_{L^1} B_0
\]

\[+ a^{-\frac{1}{2}} H_0^\frac{1}{2} B_0 + a^{-1} H_0^1 \| \tau_0 \|_{L^1}).
\]

Using the conditions (1.17), we have \( \| \tau_0 \|_{L^1} \leq c \min \{ a^{\frac{1}{2}}, a^{2} \} \). Then we deduce from (4.47) that

\[
(4.48) \quad \int_0^t \| F_3 \|_{B^\alpha_{2,1}} ds (1 + \int_0^t \| \nabla u \|_{L^\infty} ds) \leq \frac{C_1}{10} a^{\frac{1}{2}} e^{\frac{4}{\beta} t} + C (a^{-\frac{1}{2}} H_0^\frac{1}{2} B_0 + a^{-1} H_0^1 \| \tau_0 \|_{L^1}) (1 + a^{-\frac{1}{\beta}} e^{\frac{4}{\beta} t})
\]

\[
\leq \frac{C_1}{10} a^{\frac{1}{2}} e^{\frac{4}{\beta} t} + (\mu^{-\frac{1}{2}} E_0^\frac{1}{2} + a^{-1} \mu^{-1} E_0 + E_0^\frac{3}{2}) (1 + a^{-\frac{1}{\beta}} e^{\frac{4}{\beta} t})
\]

\[
\leq \frac{C_1}{5} a^{\frac{1}{2}} e^{\frac{4}{\beta} t},
\]

where we use the condition \( E_0 \leq c \gamma \min \{ \mu, a, \gamma \} \). Combining above estimates for (4.37), we infer

\[
(4.49) \quad \| \Gamma \|_{B^\alpha_{2,1}} \leq \frac{C_1}{2} a^{\frac{1}{2}} e^{\frac{4}{\beta} t},
\]

which implies that \( T^* = +\infty \). We thus complete the proof of Theorem 1.5.
5 Large time behavior for the general Oldroyd-B type model

In this section we consider large time behavior of global solutions for (1.1) in $H^1$. For simplify, the parameters in (1.1) will be taken as the constant 1.

For the reader’s convenience, we first recall the following theorem.

**Theorem 5.1.** [9] Let $d = 2$ and $s > 2$. Assume that $a > 0$ and $\mu > 0$. Let $(u, \tau)$ be a strong solution of (1.2) with the initial data $(u_0, \tau_0) \in H^s$. Then, there exists some sufficiently small constant $\delta$ such that if

\[
\|(u_0, \tau_0)\|_{H^1} + \|\omega_0, \tau_0\|_{B^\infty_{\infty,1}} \leq \delta, \quad \omega_0 = \text{curl } u_0,
\]

then the system (1.3) admits a unique global strong solution $(u, \tau) \in C([0, \infty); H^s)$. Moreover, the energy estimation for $(u, \tau, \Gamma)$ with $\Gamma = \Omega - R\tau$ implies

\[
\frac{d}{dt}(u, \tau)\|_{H^1} + \|\nabla u\|_{L^2}^2 + \|\tau\|_{H^2}^2 \leq 0.
\]

Motivated by [14] and [20], we can cancel $\text{div} \, \tau$ in Fourier space and prove the following initial time decay rate of $(u, \tau)$ in $H^1$ by the Fourier splitting method and the bootstrap argument.

**Proposition 5.2.** Under the condition in Theorem 1.8 Then there exists $C > 0$ such that for any $l \in N$ and $t > 0$, we have

\[
\|(u, \tau)\|_{H^1} \leq C \ln^{-l}(e + t).
\]

**Proof.** Let $S_0(t) = \{\xi : f(t)|\xi|^2 \leq 2C_2f'(t)\}$ with $C_2$ large enough. According to Theorem 5.1, we have

\[
\frac{d}{dt}[f(t) \|(u, \tau)\|_{H^1}^2 + C_2f'(t)\|u\|_{L^2}^2 + f(t)\|\tau\|_{H^2}^2 \leq C_2f'(t)\int_{S_0(t)} |\hat{u}|^2 d\xi + 2f'(t)\|\nabla u\|_{L^2}^2,
\]

for some $t > 0$ sufficiently large. Applying Fourier transformation to (1.1), we obtain

\[
\left\{
\begin{array}{l}
\frac{d}{dt}\hat{u} + i\xi^T \mathcal{F}(u \otimes u) + i\xi^T \hat{\tau}, \\
\frac{d}{dt}\hat{\tau} + \mathcal{F}(u \cdot \nabla \tau) + i|\xi|^2 \hat{\tau} + \mathcal{F}Q(\nabla u, \tau) = \frac{i}{2}(\xi \otimes \hat{u} + \hat{u} \otimes \xi).
\end{array}
\right.
\]

Multiplying (5.5) by $(\hat{u}, \hat{\tau})$ and taking the real part, we deduce that

\[
\frac{1}{2} \frac{d}{dt}|\hat{u}|^2 = \mathcal{R}e[-i\xi^T \mathcal{F}(u \otimes u)\hat{u} + i\xi^T \hat{\tau}\hat{\tau}],
\]

and

\[
\frac{1}{2} \frac{d}{dt}|\hat{\tau}|^2 + |\hat{\tau}|^2 + |\xi|^2 |\hat{\tau}|^2 = \mathcal{R}e[\mathcal{F}(u \cdot \nabla \tau) : \hat{\tau} - \mathcal{F}Q(\nabla u, \tau) : \hat{\tau} + \frac{i}{2}(\xi \otimes \hat{u} + \hat{u} \otimes \xi) : \hat{\tau}].
\]

Since $\tau$ is symmetric, we have

\[
\mathcal{R}e[i\xi^T \hat{\tau}\hat{u} + \frac{i}{2}(\xi \otimes \hat{u} + \hat{u} \otimes \xi) : \hat{\tau}] = 0,
\]

which implies that

\[
\frac{1}{2} \frac{d}{dt}(|\hat{u}|^2 + |\hat{\tau}|^2) + |\hat{\tau}|^2 + |\xi|^2 |\hat{\tau}|^2 = \mathcal{R}e[-i\xi^T \mathcal{F}(u \otimes u)\hat{u} - \mathcal{F}(u \cdot \nabla \tau) : \hat{\tau} - \mathcal{F}Q(\nabla u, \tau) : \hat{\tau}]
\]

inserting (5.6) to (5.7), then we have

\[
\frac{1}{2} \frac{d}{dt}(|\hat{u}|^2 + |\hat{\tau}|^2) + |\hat{\tau}|^2 + |\xi|^2 |\hat{\tau}|^2 = \mathcal{R}e[-i\xi^T \mathcal{F}(u \otimes u)\hat{u} - \mathcal{F}(u \cdot \nabla \tau) : \hat{\tau} - \mathcal{F}Q(\nabla u, \tau) : \hat{\tau}]
\]
\[ \leq |\xi||\mathcal{F}(u \otimes u)||\tilde{u}| + |\mathcal{F}(u \cdot \nabla \tau)|^2 + |\mathcal{F}Q(\nabla u, \tau)|^2 + |\tilde{\tau}|^2. \]

Let \( f(t) = \ln^3(e + t) \). According to Theorem 5.1, we have

\[
\int_{S_0(t)} |\tilde{u}|^2 + |\tilde{\tau}|^2 d\xi \leq \|(u_0, \tau_0)\|^2_{L^2} + \frac{f'(t)}{f(t)} + \int_0^t \int_{S_0(t)} |\xi||\mathcal{F}(u \otimes u)||\tilde{u}|d\xi ds
\]

\[
+ \int_0^t \int_{S_0(t)} |\mathcal{F}(u \cdot \nabla \tau)|^2 + |\mathcal{F}Q(\nabla u, \tau)|^2 d\xi ds
\]

\[
\leq C \frac{f'(t)}{f(t)} + \int_0^t \|u\|^2_{L^2} \int_{S_0(t)} |\xi|^2 d\xi + \frac{f'(t)}{f(t)} \int_0^t \|\nabla u\|^2_{L^2} \|\tau\|^2_{L^2} + \|u\|^2_{L^2} \|\nabla \tau\|^2_{L^2} ds
\]

\[
\leq C \frac{f'(t)}{f(t)} + C \frac{f'(t)}{f(t)} \int_0^t \|u\|^2_{L^2} ds
\]

\[
\leq C \frac{f'(t)}{f(t)} + C \frac{f'(t)}{f(t)} + C \frac{f'(t)}{f(t)} (1 + t)
\]

\[
\leq C \ln^{-1}(e + t).
\]

This together with (5.4) and (5.2) ensures that

\[
f(t)\|(u, \tau)\|^2_{H^1} \leq C + C \int_0^t f'(s) \ln^{-1}(e + t) ds + C \int_0^t f'(s)\|\nabla u\|^2_{L^2} ds
\]

\[
\leq C \ln^2(e + t),
\]

which implies

\[
(5.10) \quad \|(u, \tau)\|^2_{H^1} \leq \ln^{-1}(e + t).
\]

We prove (5.10) by induction. Assume that

\[
(5.11) \quad \|(u, \tau)\|^2_{H^1} \leq \ln^{-t}(e + t).
\]

Let \( f(t) = \ln^{t+3}(e + t) \). Using (5.11), we can deduce that

\[
\int_{S_0(t)} |\tilde{u}|^2 + |\tilde{\tau}|^2 d\xi \leq C \frac{f'(t)}{f(t)} + \frac{f'(t)}{f(t)} \int_0^t \|u\|^2_{L^2} ds
\]

\[
\leq C \ln^{-\frac{3}{2}t-1}(e + t).
\]

This together with (5.4) and (5.2) ensures that

\[
f(t)\|(u, \tau)\|^2_{H^1} \leq C + C \int_0^t f'(s) \ln^{-\frac{3}{2}t-1}(e + t) ds + C \int_0^t f'(s)\|\nabla u\|^2_{L^2} ds
\]

\[
\leq C \ln^2(e + t),
\]

which implies that

\[
\|(u, \tau)\|^2_{H^1} \leq C \ln^{-t-1}(e + t).
\]

We thus complete the proof of Proposition 5.2. \qed
The proof of Theorem 1.8:  
Now we are going to improve initial time decay rate in Proposition 5.2. Let \( S(t) = \{ \xi \mid \| \xi \|^2 \leq C_2(1 + t)^{-1} \} \) with sufficiently large \( C_2 > 0 \) and \( t > 0 \). According to Theorem 5.1, we obtain

\[
\frac{d}{dt} \left( \| u(t, \tau) \|^2_{H^1} \right) + C_2(1 + t)^{-1} \| u(t, \tau) \|^2_{H^2} \leq C(1 + t)^{-1} \int_{S(t)} \| \tilde{u} \|^2 d\xi,
\]

which implies that

\[
\frac{d}{dt} \left( (1 + t)^2 \| (u, \tau) \|^2_{H^1} \right) + \frac{1}{2} C_2(1 + t)^2 \| u \|^2_{L^2} + \frac{1}{2} (1 + t)^2 \| \tau \|^2_{H^2} \leq C(1 + t) \int_{S(t)} \| \tilde{u} \|^2 d\xi,
\]

Integrating (5.12) over \( S(t) \times [0, t] \) with \( (\xi, s) \) and according to Theorem 5.1, we can deduce that

\[
\int_{S(t)} \| \tilde{u} \|^2 + \| \tilde{\tau} \|^2 d\xi \leq \frac{C}{1 + t} + \int_0^t \int_{S(t)} \| \xi \| | \mathcal{F}(u \otimes u) | | \tilde{u} \| + | \mathcal{F}(u \cdot \nabla \tau) |^2 d\xi ds
\]

\[
\leq \frac{C}{1 + t} + \frac{C}{1 + t} \int_0^t \| u \|^3_{L^2} ds.
\]

Together with (5.12) and (5.13), we infer that

\[
(1 + t)^2 \| (u, \tau) \|^2_{H^1} \leq \| (u_0, \tau_0) \|^2_{H^1} + C(1 + t) + C \int_0^t \int_0^s \| u \|^3_{L^2} ds' ds + C \int_0^t (1 + s) \| \nabla u \|^2_{L^2} ds
\]

\[
\leq C(1 + t) + C(1 + t) \int_0^t \| u \|^3_{L^2} ds + C \int_0^t \| (u, \tau) \|^2_{H^1} ds
\]

\[
\leq C(1 + t) + C(1 + t) \int_0^t \| u \|^3_{L^2} ds.
\]

Let \( M(t) = \sup_{s \in [0, t]} (1 + s) \| (u, \tau) \|^2_{H^1} \). Using (5.14) and (5.3) with \( t = 2 \), we obtain

\[
M(t) \leq C + C \int_0^t M(s)(1 + s)^{-1} \ln^{-2}(e + t) ds,
\]

Applying Gronwall's inequality to (5.15), we get

\[
M(t) \leq Ce^{C \int_0^t (1 + s)^{-1} \ln^{-2}(e + t) ds} \leq C,
\]

which implies that

\[
\| u \|^2_{H^1} + \| \tau \|^2_{H^1} \leq (1 + t)^{-1}.
\]

We thus complete the proof of Theorem 1.8.  \( \Box \)

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