I. INTRODUCTION

Turbulence in clouds increases the relative velocity of small rain droplets and enhances the formation of larger droplets and raindrops. This intuitive phenomenon is observed in numerical simulation of models based on particles in fluids, with various degrees of idealization; see for instance [17]. Looking for macroscopic models of particle density, typically of Smoluchowski type, the leading proposal has been the modification of collision rates, from the case of laminar or steady flow to the turbulent case; the literature is very large, see for instance [8, 15]. Here we advance a different explanation, based on eddy diffusion in the velocity component.

We consider a particle-fluid model of rain formation subject to a turbulent velocity idealizing the latter by a random field and we perform two scaling limits. The first one, a limit in the particle number, leads to a stochastic Smoluchowski-type system, with the turbulent velocity field still described by a noise stochastic process. The second scaling limit, in parameters of the noise moving in the direction of small scale turbulence, leads to a final deterministic equation with eddy dissipation in the velocity variable. Then we simulate this equation numerically and show the degree of influence of turbulence on rain formation.

The physical intuition, beyond the specific mathematical derivation proposed here, is similar to the classical one of Boussinesq [16], that small-scale turbulence enhances dissipation and viscosity. However, the additional viscous term proposed by such theories is in the space variable, whereas the idea is that fluid particles move so erratically to produce effects similar to the molecular motion. Here, in the case of rain droplets, it is essential to consider them as inertial particles embedded into the fluid, subject to Stokes force. This is the origin of the fact that the additional viscous term is in the velocity component. This passage from space to velocity diffusion is essential: no improvement in raindrop formation is observed if a diffusion in position is included into the Smoluchowski equations.

II. THE MICROSCOPIC MODEL

The model used below will be of Smoluchowski type with random transport. However, the description of its microscopic origin may help. Call $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, the space domain of the system, occupied by the fluid and by small rain droplets. The number $N(t)$ of droplets changes in time due to coalescence. Droplet motion is described in a Newtonian way by position and velocity $(x^i(t), v^i(t))$, $i = 1, ..., N(t)$. Droplets have masses $m^i(t)$ taking values in the natural numbers $\{1, 2, ..., M\}$, limited by the threshold $M$ with the following meaning: when particles of mass $> M$ are created by coalescence of smaller ones, they become raindrops and fall down, namely they exit the system. During the intermittent between a collision and the next one, the motion is given by

$$\frac{dx^i}{dt} = v^i, \quad \frac{d m^i}{dt} = \alpha \left( m^i \right)^{1/d} (U(t, x^i) - v^i)$$

where $U(t, x)$ is the fluid velocity; we adopt a Stokes law for the particle-fluid interaction and denote by $\alpha$ a positive constant (including the dynamic viscosity coefficient of the fluid), the term $\left( m^i \right)^{1/d}$ playing the role of the radius of the particle.

The rule of coalescence is crucial, see [8, 9, 11, 13, 14]. There are two typical mathematical models: one is based on deterministic coalescence, the other on probability rates. The first one is easier to describe: when two particles meet, they become a new single particle with mass given by the sum of the masses and momentum given by conservation of momentum. For mathematical investigation of the macroscopic limit, this scheme is usually more difficult. Easier is thinking in terms of rate of coalescence: when two particles are below a certain small distance one from the other, they have a certain probability per unit of time to become a new single particle, with the mass and momentum law as above. The kernels in Smoluchowski equations are the macroscopic footprint of rates.

But the model based on rates has a flaw precisely in connection with the turbulence background we want to investigate here. Since coalescence happens due to a probability per unit of time, if the time spent by two particles, at the prescribed distance of potential coalescence, is small, the probability that their encounter leads...
to coalescence is smaller. This is in sharp contrast with the deterministic model where coalescence always happens, at a certain distance, independently of the time spent nearby. In other words, in the model based on rates, coalescence is facilitated by slow motion, which is false in practice and goes in the opposite direction of understanding whether turbulence enhances coalescence.

To avoid this bias towards slow motion, of say particles $i$ and $j$, we insert in their coalescence rate the factor $|v_i^*-v_j^*|$. This factor multiplied by the time spent nearby is constant, in the average, hence the probability of coalescence is roughly constant. Finally, since the probability of coalescence should depend on the particle surface, main factor involved in the collision, we multiply the rate by the surface factor

$$s(m^i, m^j) = \left((m^i)^{1/d} + (m^j)^{1/d}\right)^{d-1}.$$

### III. THE SMULUCHOWSKI-TYPE MODEL

A rigorous study of the link between the microscopic model and the macroscopic one is under investigation, following [3 17 19] where similar models have been already treated. However, following the mean field paradigm we may safely choose the following macroscopic model as a good one for the density evolution.

Denote by $f_m(t, x, v)$, $m \in \{1, 2, \ldots, M\}$, the density of droplets of mass $m$ at position $x \in D$ having velocity $v \in \mathbb{R}^d$. Then (dropping the time variable) the density satisfies

$$\frac{\partial f_m(x, v)}{\partial t} + \text{div}_v (vf_m(x, v)) + c(m) \text{div}_v ((U(t, x) - v)f_m(x, v)) = Q_m^+ - Q_m^-,$$

where $c(m) = \alpha m^{(1-d)/d}$ and $Q_m^+$ and $Q_m^-$ are the two collision terms given by

$$Q_m^+ = \sum_{n=1}^{m-1} \int_{\{n'v' + (m-n)v'' = mv\}} s(n, m-n, |v' - v''|) \cdot f_n(t, x, v') f_{m-n}(t, x, v'') dv' dv''$$

$$Q_m^- = 2 f_m(x, v) \sum_{n=1}^{m} \int s(n, m, |v - v'| f_n(x, v') dv'.$$

Crucial is the kernel $|v' - v''|$, as described above. The first collision term describes the amount of new particles of mass $m$ created by collision of smaller ones, with the momentum conservation rule $n v' + (m-n) v'' = m v$. The second collision term gives us the percentage of the density $f_m(x, v)$ of particles of mass $m$ which disappears by coalescence into larger particles.

### IV. STOCHASTIC MODEL OF TURBULENT VELOCITY FIELD

Similarly to a large body of simplified modeling of passive scalars, we consider a model of velocity fluid which is delta-correlated in time, namely a white noise with suitable space dependence. We may write

$$U(t, x) dt = \sum_{k \in K} \sigma_k(x) dW_t^k$$

where $\sigma_k(x)$ are smooth divergence free deterministic vector fields on $D$ and $W_t^k$ are independent one-dimensional Brownian motions; $K$ is a finite index set (or countable, with some care on summability assumptions). In this case the term $c(m)U(t, x) \cdot \nabla_v f_m(x, v)$ must be interpreted as a Stratonovich integral (still written here in differential form for sake of clarity)

$$c(m) \sum_{k \in K} \sigma_k(x) \cdot \nabla_v f_m(x, v) \circ dW_t^k.$$ 

By the rules of stochastic calculus, it is given by an Itô-Stratonovich corrector plus an Itô integral; precisely, the previous term is given by

$$-\frac{c^2(m)}{2} \sum_{k \in K} \sigma_k(x) \cdot \nabla_v (\sigma_k(x) \cdot \nabla_v f_m(x, v)) dt + dM(t, x, v)$$

where $M(t, x)$ is a (local) martingale, the Itô term. The Itô-Stratonovich corrector takes also the form

$$-\frac{c^2(m)}{2} \text{div}_v (Q(x, x) \nabla_v f_m(x, v)) dt$$

where $Q(x, y)$ is the matrix-valued function given by the space-covariance function of the noise

$$Q(x, y) = \sum_{k \in K} \sigma_k(x) \otimes \sigma_k(y).$$

Summarizing, the stochastic model, in Itô form, is

$$df_m(x, v) + (v \cdot \nabla_x f_m(x, v) - c(m) \text{div}_v (vf_m(x, v))) dt$$

$$-\frac{c^2(m)}{2} \text{div}_v (Q(x, x) \nabla_v f_m(x, v)) dt$$

$$= (Q^+_m - Q^-_m) dt + dM(t, x, v).$$

Also for later reference, let us mention an example of noise, introduced by R. Kraichnan [20 21], relevant to our analysis. For the sake of simplicity of exposition, assume we are in full space $\mathbb{R}^d$, but modifications in other geometries are possible. Its covariance function is space-homogeneous, $Q(x, y) = Q(x - y)$, with the form

$$Q(z) = \sigma^2 k_0 \int_{k_0 \leq |k| < k_1} \frac{1}{|k|^{d+c+\zeta}} \left( I - \frac{k \otimes k}{|k|^2} \right) dk.$$

The case $\zeta > 0$ includes Kolmogorov 41 case $\zeta = 4/3$. In this case, take $k_1 = +\infty$. Then $Q(0) = C \sigma^2$ where the constant $C$ is given by $\int_{1 \leq |k| < \infty} \frac{1}{|k|^{d+c+\zeta}} \left( I - \frac{k \otimes k}{|k|^2} \right) dk$. 

$$Q(x, y) = \sum_{k \in K} \sigma_k(x) \otimes \sigma_k(y).$$

$$s(m^i, m^j) = \left((m^i)^{1/d} + (m^j)^{1/d}\right)^{d-1}.$$

$$\frac{\partial f_m(x, v)}{\partial t} + \text{div}_v (vf_m(x, v)) + c(m) \text{div}_v ((U(t, x) - v)f_m(x, v)) = Q_m^+ - Q_m^-,$$

$$Q_m^+ = \sum_{n=1}^{m-1} \int_{\{n'v' + (m-n)v'' = mv\}} s(n, m-n, |v' - v''|) \cdot f_n(t, x, v') f_{m-n}(t, x, v'') dv' dv''$$

$$Q_m^- = 2 f_m(x, v) \sum_{n=1}^{m} \int s(n, m, |v - v'| f_n(x, v') dv'.$$
V. THE DETERMINISTIC SCALING LIMIT

Following [22–24], we may consider small-scale turbulent velocity fields depending on a scaling parameter and take their scaling limit. In the case of Kraichnan model above, choose

\[ k_0 = k_0^N \to \infty \]

The result \( Q(0) = C \sigma^2 \) is independent of \( N \), so that the Itô-Stratonovich corrector becomes equal to \( \sigma^2 \Delta_x f_m(x, \nu) \); and simultaneously we may have that the Itô term goes to zero. The final equation is deterministic, and precisely given by

\[
\frac{\partial f_m(x, \nu)}{\partial t} + \nu \cdot \nabla_x f_m(x, \nu) - c(m) \text{div}_\nu (\nu f_m(x, \nu)) - \frac{c^2(m) \sigma^2}{2} \Delta_x f_m(x, \nu) = Q_m^+ - Q_m^-.
\]

Now, for sake of numerical simplicity, we assume that all densities are uniform in \( x \). Then we have

\[
\frac{\partial f_m(\nu)}{\partial t} - c(m) \text{div}_\nu (\nu f_m(\nu)) - \frac{c^2(m) \sigma^2}{2} \Delta_v f_m(\nu) = Q_m^+ - Q_m^- \quad (1)
\]

where now the collision terms \( Q_m^+ - Q_m^- \) include only functions of \( \nu \). This is our final equation for the density of droplets. It is parametrized by \( \sigma^2 \), the intensity of noise covariance which, in the approximation of this white noise model, corresponds to the concept of turbulence kinetic energy.

VI. NUMERICAL RESULTS

To understand the effect of the turbulent velocity field on coagulation, we identify and build on a key quantity, \( \mathcal{M}_t^+(t) \) below, which is essentially the first moment of the mass in the system at time \( t \). Since \( M < \infty \) in our model, eventually all masses leave the system, hence we measure the efficiency of coagulation by looking at how fast this first moment decays in time, with respect to different values of \( \sigma \).

A. Total mass

To this end, we define

\[
\mathcal{M}_t^+(t) := \sum_{m=1}^M m \int f_m(t, \nu) d\nu,
\]

which we also call “total mass” for simplicity. Analyzing the nonlinearity of our PDE, we notice that

\[
\sum_{m=1}^M \int m(Q_m^+ - Q_m^-) d\nu \leq 0, \quad \forall t
\]

implying that \( d\mathcal{M}_t^+(t)/dt \leq 0 \), that is, the function \( \mathcal{M}_t^+(t) \) is non-increasing in time. Moreover, for the infinite system \( M = \infty \), equality is achieved in \( \mathcal{M}_t^+(t) \), hence we see that the mass deficiency in the finite system is not lost at all and it is simply sent to higher order \( (\sim M) \) of mass-type densities.

Indeed, in view of the form of the negative part of coagulation operator \( Q_m^- \), every coagulation at the level of \( f_m \), with \( m + n > M \), represents a decrease in mass that, ideally, increases the density \( f_{m+n} \) that is outside of our system. Therefore, the more and faster the quantity \( \mathcal{M}_t^+(t) \) decreases over time, the faster and richer the coagulation to higher mass-type is achieved.

To this end, we perform a numerical simulation of the system \( \mathcal{M}_t^+(t) \) for dimension \( d = 1 \), maximal mass level \( M = 1 \) and time window \([0, 2]\), with a semi-implicit method to compute its solutions. Thanks to the fast decay to zero as \( |\nu| \to \infty \) of the solution, we truncate the velocity variable in the range \( \nu \in [-20, 20] \) both for the numerical integration of the nonlinearity and for the total mass \( \mathcal{M}_t^+(t) \).

In Figure 1 we plot the function \( \mathcal{M}_t^+(t) \) for different values of the turbulence parameter \( \sigma^2 \) that range from 0.05 to 10, which we refer to as the non-turbulent case, to 10, which represents an intense eddy diffusivity. It shows a faster decay correlated to the increase of turbulence, and a speedup coagulation process.

![FIG. 1: Decay of \( \mathcal{M}_t^+(t) \) for \( t \in [0, 1] \), with maximal mass level \( M = 1 \), initial density \( f_1(0, \nu) \) of mass \( m = 1 \) concentrated on the set \( \nu \in [-1/2, 1/2] \). The parameter \( \sigma^2 \) ranges from a sample in the set \( 0.05 \) to \( 10 \) (around 30 points). A visible increase in coagulation is present at the increase of \( \sigma^2 \).](image-url)
believe that this is dimension-independent.

The second quantity we consider is closely linked to the enhanced coagulation due to turbulence we just established and gives more quantitative information. Consider the same numerical setting as above, and we estimate a decay law that links the first time that the total mass $M^T_1(t)$ drops below a certain level, to the turbulence parameter $\sigma$. Specifically, let

$$m^T_0 := \inf_{t \in [0,T]} M^T_1(t)$$

and define a sequence of “barrier exit times” $(\tau_\sigma)_{\sigma \geq 0}$

$$\tau_\sigma := \inf \{ t \geq 0, M^T_1(t) \leq m^T_0 \} \wedge T.$$ 

Since $t \mapsto M^T_1(t)$ is decreasing, we have that $\tau_0 = T$. Since $M^T_1(t)$ decays faster as $\sigma$ increases, $\sigma \mapsto \tau_\sigma$ should be decreasing. We see from Figures 3a and 3b that it is indeed the case, and the curve exhibits a power like decay, with an asymptotic limit to zero. In Figure 3a, we performed a log-log plot and regression taking $T = 1$ and it yields $\tau_\sigma \sim \sigma^{-2/3}$ (where $\sim$ denotes proportional to), whereas the same analysis in Figure 3b taking $T = 2$ and considering only those exit times that are in the interval $[1,2]$ yields $\tau_\sigma \sim \sigma^{-1}$.

We conjecture that the function $[2]$ can be expressed as

$$M^T_1(t) \sim \frac{1}{A_d(\sigma)t + M^T_0 (0)^{-1}},$$

for some function $A_d$ that depends on dimension $d$, and that $A_1(\sigma) \sim \sigma$. A rough explanation of the numerical findings may be the following one, that will be explored more closely in a future work, since - as shown below - our understanding is still incomplete. When $M = 1$, the density $f(t,v)$ of the unique level $m = 1$ satisfies the identity

$$\frac{d}{dt} \int f(t,v) dv = - \int |v - v'| f(t,v) f(t,v') dv dv'$$

because the differential terms cancel by integration by parts. Assume that, at least after a transient time (confirmed by Figure 2), up to a small approximation,

$$f(t,v) \sim \alpha(t) f_0(v)$$

denoting the decay of $f(t,v)$ is self-similar [25]. Then (up to approximation) $\alpha' = -\sigma_0 \alpha^2$ where

$$\sigma_0 = \int \int |w - w'| f_0(w) f_0(w') dw dw'$$

which is an average variation of velocity under $f_0$, namely

$$\alpha(t) \sim \frac{1}{\sigma_0 t + C}$$

after an initial transient. Moreover, speculating that the standard deviation of $f_0$ should be of order $\sigma$ (since the dispersion produced by the linear differential operator is proportional to $\sigma$), we expect that $\sigma_0$ increases linearly with $\sigma$. The numerical results of Figures 3a and 3b show that this looks the trend for sufficiently large time but for a short time another power, $\sigma^{2/3}$, emerges, that should be understood.

VII. CONCLUSION

In this article, we presented a new kinetic model of a modified Smoluchowski PDE system with discrete and finite mass levels, that exploits small scale turbulence and eddy diffusion in velocity to enhance coagulation. We presented the derivation of the PDE system from a particle-fluid model subjected to a transport-type noise, and we analyzed numerically the behavior of its solutions. We showed that coagulation efficiency increases steadily with the increase of turbulence and, moreover, a power-law decay in time and in the turbulence parameter is present.
(a) A plot of the barrier exit time $\tau_{\sigma}$ with respect to the turbulence parameter $\sigma$, and the corresponding log-log regression in the time window $[0, 1]$ yields $\tau_{\sigma} \sim \sigma^{-2/3}$.

(b) A plot of the barrier exit time $\tau_{\sigma}$ with respect to the turbulence parameter $\sigma$, and the corresponding log-log regression in the time window $[0, 2]$, taking into consideration only those exit times in the interval $[1, 2]$, yields $\tau_{\sigma} \sim \sigma^{-1}$.

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