Possible enhancement of high frequency gravitational waves

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Abstract
We study the tensor perturbations in a class of non-local, purely gravitational models which naturally end inflation in a distinctive phase of oscillations with slight and short violations of the weak energy condition. We find the usual generic form for the tensor power spectrum. The presence of the oscillatory phase leads to an enhancement of gravitational waves with frequencies somewhat less than $10^{10}$ Hz.

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Introduction

During the inflationary era infrared virtual gravitons are ripped out of the vacuum by the accelerated expansion of spacetime [1]. We have proposed a mechanism through which the back-reaction to this process can slow the inflationary expansion rate [2]. The idea is that the energy density of the newly produced gravitons comes not just from their bare kinetic energy but also from the interactions they have with other gravitons. Because gravity is attractive, these interactions act to slow inflation. If one imagines the process occurring on a closed spatial manifold such as $T^3$ it is easy to show that the energy density of this gravitational self-interaction grows as the exponential of twice the number of e-foldings, assuming all the gravitons on a 3-surface were in interaction with one another [3]. That would make the manifold suffer gravitational collapse after only about 14 e-foldings. Of course this estimate ignores causality; any given graviton only feels the potentials from the gravitons which are visible within its past light cone. The actual growth of the interaction energy is therefore like the number of e-foldings, so the system requires something like $10^6$ e-foldings of inflation to evolve itself to the point of gravitational collapse. However, it seems inevitable that this effect must eventually stop inflation if nothing else supervenes.

The mechanism just sketched provides a wonderful way of resolving two perplexing problems:
• explaining why the dimensionless product of Newton’s constant $G$ and the cosmological constant $\Lambda$ is about $10^{-122}$; and

• finding a natural model of inflation.

Our solution to the first problem is to deny its premise: we do not believe $\Lambda$ is actually small. It only seems to be small when one infers its value from the current expansion rate using classical gravity, which ignores the quantum gravitational screening mechanism. To be specific, we believe $G/\Lambda \lesssim 10^{-6}$ is just small enough to justify using the semi-classical approximation, and that this is what started primordial inflation, without the need for a scalar inflaton. This gives a very attractive model of inflation because

• it dispenses with the need to assume that the initial configuration of the inflaton is homogeneous over more than a Hubble volume since the cosmological constant is homogeneous over all space;

• the long period of inflation required for back-reaction to build up dispenses with the problem of having to fine tune the inflaton potential to make inflation last long enough;

• the screening mechanism dispenses with the need to fine tune the inflaton potential so that inflation ends with nearly zero cosmological constant;

• the fact that inflation is driven by the bare cosmological constant, rather than a scalar inflaton, dispenses with the need to invent a new, otherwise undetected field; and

• the fact that this model contains only the single dimensionless parameter $G/\Lambda$ means that it makes unique predictions, unlike scalar-driven inflation, which can be tuned to give almost any result for the scalar power spectrum.

However, there is a severe problem of tractability when it comes to using this model. The bare kinetic energy of inflationary gravitons is a one-loop effect, so the interaction energy we seek does not appear until two-loop order. Two-loop results are not easy to compute even for simple theories on flat space background; this one requires quantum gravity on de Sitter background. In fact, the graviton one-point function has been evaluated at two-loop order and it does seem to support the relaxation mechanism [4], but this computation required a year’s labor and its interpretation is open to debate owing to the difficulty of formulating an invariant definition of relaxation [5]. Furthermore, at the same time as the two-loop effect finally becomes significant, the effects of higher loops also become significant. So one requires a non-perturbative resummation technique to evolve into the late time regime at which interesting predictions can be made.

We believe it may be possible to derive such a non-perturbative resummation technique by extending the stochastic method which Starobinsky devised for the same purpose in scalar potential models [6, 7]. However, generalizing this technique to gravity is a difficult problem [8]. This paper is part of a parallel effort [9] which is based on the idea of guessing the most cosmologically significant part of the effective field equations of quantum gravity. While there is no chance of guessing the full effective field equations, it might be possible to guess just enough to correctly describe the evolution of the scale factor $a(t)$ for a homogeneous and isotropic geometry, using what we know from perturbation theory about how the back-reaction effect scales.

An important feature of any model which relaxes the cosmological constant is non-locality. Non-local models of cosmology have been much studied [3, 10] because they can avoid the problem that de Sitter must be a solution for any local, stable theory, and because non-local couplings between different times can ease fine tuning problems. In a previous paper [9], we proposed a phenomenological model which can provide evolution beyond the perturbation
theory. In one sentence, we constructed an effective conserved stress–energy tensor $T_{\mu\nu}[g]$ which modifies the gravitational equations of motion:\(^3\)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}[g]$$

and which, we hope, contains the most cosmologically significant part of the full effective quantum gravitational equations.

What form to guess for $T_{\mu\nu}[g]$ was motivated by what we seek to do, and by what we know from perturbation theory. We seek to describe cosmology, which implies homogeneous and isotropic geometries. When specialized to such a geometry the full effective stress tensor must take the perfect fluid form and we lose nothing by assuming that generally

$$T_{\mu\nu}[g] = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (2)$$

The relation between $p[g], \rho[g]$ and $u_\mu[g]$ is heavily constrained by stress–energy conservation, but it is possible to specify one function for free. It turns out to be computationally simplest to take this free function to be the pressure \(^9\). We further require the pressure to be an ordinary function of some non-local scalar which grows like the number of e-foldings when specialized to de Sitter. The simplest choice for such a scalar is the inverse d’Alembertian acting on the Ricci scalar \(^3\). If the pressure is to grow the way we know it does from perturbation theory \(^11\), and to eventually end inflation, then one is lead to the form \(^9\)

$$p[g](x) = \frac{1}{\Lambda^2} f[-G\Lambda X(x)], \quad X \equiv \sqrt{-g} R, \quad (3)$$

where the function $f$ grows without bound and satisfies

$$f[-G\Lambda X] = -G\Lambda X + O((G\Lambda)^3), \quad (4)$$

and where the scalar d’Alembertian:

$$\square \equiv \frac{1}{\sqrt{-g}} \partial_{\mu}(g^{\mu\nu}\sqrt{-g}\partial_{\nu}), \quad (5)$$

is defined with retarded boundary conditions. The induced energy density $\rho[g]$ and 4-velocity $u_\mu[g]$ are determined, up to their initial value data, from stress–energy conservation:

$$D_{\mu}T_{\mu\nu} = 0. \quad (6)$$

The 4-velocity was chosen to be timelike and normalized:

$$g^{\mu\nu}u_\mu u_\nu = -1 \implies u^\mu u_{\mu\nu} = 0. \quad (7)$$

The homogeneous and isotropic evolution\(^4\) of this model—using a combination of numerical and analytical methods—revealed the following basic features\(^5\).

(i) After the onset and during the era of inflation, the source $X(t)$ grows while the curvature scalar $R(t)$ and Hubble parameter $H(t)$ decrease.

(ii) Inflationary evolution dominates roughly until we reach a critical point $X_{cr}$ defined by

$$1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] \equiv 0. \quad (8)$$

\(^3\) Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has a spacelike signature and our curvature tensor equals $R_{\mu\nu}^\alpha \equiv \Gamma^\alpha_{\mu\rho} + \Gamma^\alpha_{\rho\mu} - \Gamma^\alpha_{\mu\nu}. \quad (\alpha \leftrightarrow \nu).$ The initial Hubble parameter is $3H_0^2 = \Lambda$. We restrict our analysis to scales $M = (\Lambda/8\pi G)^{\frac{1}{2}}$ below the Planck mass $M_{Pl} = G^{-\frac{1}{2}}$ so that the dimensionless coupling constant $G\Lambda = (M/M_{Pl})^4$ of the theory is small.

\(^4\) The line element in co-moving coordinates is $ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}$. In terms of the scale factor $a$, the Hubble parameter equals $H(t) = \ddot{a}a^{-1}$ and the deceleration parameter equals $q(t) = -a\ddot{a}a^{-2} = -1 - H\dot{H}a^{-2} = -1 + \epsilon(t)$.

\(^5\) In \(^9\), our analytical results were obtained for any function $f$ satisfying (4) and growing without bound, our numerical results for the choice $f(x) = \exp(x) - 1$. 

3
(iii) The epoch of inflation ends close to but before the universe evolves to the critical time. This is most directly seen from the deceleration parameter since initially $q(t = 0) = -1$ while at criticality $q(t = t_{cr}) = +\frac{1}{2}$.

(iv) Oscillations in $R(t)$ become significant as we approach the end of inflation; they are centered around $R = 0$, their frequency equals

$$\omega = GA H_0 \sqrt{2\pi f_{cr}}$$

and their envelope is linearly falling with time.

(v) During the oscillations era, although there is net expansion, the oscillations of $H(t)$ take it to small negative values for short time intervals—a feature conducive to rapid reheating; those of $\dot{H}(t)$ take it to positive values for about half the time; and those of $a(t)$ are centered around a linear increase with time.

Furthermore, this simple model can be improved to account for the requirement to naturally produce negative pressure of the correct magnitude during the current epoch [12]. However, for the purposes of this paper such an improvement is inconsequential since we shall be concerned with the period about the end of inflation where the simple model and its improvement are indistinguishable.

A novel feature of this class of models is the existence of an oscillatory regime of short duration which commences toward the very end of the inflationary era. During this period $\dot{H}(t)$ is positive about half the time, which represents a violation of the weak energy condition. Such a violation cannot occur in classically stable theories [13] but it can be driven by quantum effects of the type we seek to model without endangering stability.

A simple, fully worked-out example is provided by a massless, minimally coupled scalar with a $\phi^4$ potential on the non-dynamical de Sitter background. For that model, inflationary particle production tends to push the scalar up its potential, which of course increases the vacuum energy and leads to a violation of the weak energy condition. That has been confirmed in a fully renormalized computation of the expectation value of the model’s stress tensor at two-loop order [14]. That there is no instability was confirmed by a fully renormalized computation of the scalar self-mass-squared at two-loop order, which was then used to solve the linearized effective field equations [15]. As might be expected, pushing the scalar up its potential makes it develop a small mass, which actually makes the system more stable, not less. And a fully non-perturbative analysis by Starobinsky and Yokoyama confirms that the system approaches a static limit [16].

Our own model is an attempt to model the most cosmologically significant features of the inflationary production of gravitons, so it shares many features with the simpler $\phi^4$ model. Of course it is quite a bit more difficult to derive comparably powerful results for quantum gravity. However, we have been able to show that there are no tachyonic modes [17].

The oscillatory phase and the associated weak energy condition violation is a very distinctive feature of our model and cannot occur in classical, scalar-driven inflation. The purpose of this study is to determine whether this oscillatory regime leaves its signature on the observable tensor power spectrum. We shall, therefore, obtain the amplitude and frequency of two kinds of gravitational waves and examine their evolution under the expansion history that this class of models predicts. The first kind of waves is now of cosmological scale and originated during inflation while the second kind was on the verge of experiencing first horizon crossing when the epoch of oscillations began. Because we shall be relating scales from the very early universe to current measurements, we first focus on presenting the basic equations and relevant relations, and then we apply them for our purpose.

* The more complicated analysis of the scalar power spectrum that this class of non-local cosmological models predicts has been done elsewhere [17].
The set-up

The analysis of tensor perturbations in this class of models is much simpler than that of scalar perturbations [17]. The reason is that—unlike the case of scalar perturbations—the non-local nature of the model does not alter the basic equation which tensor perturbations \( h_{ij}^{TT} \) satisfy at linearized order:

\[
\left[ \frac{\partial^2}{\partial t^2} + 3H(t) \frac{\partial}{\partial t} - \nabla^2 \frac{1}{a^2(t)} \right] h_{ij}^{TT}(t, x) = 0.
\] (10)

Therefore, up to sub-dominant corrections coming from the exact form of the mode functions before and after first horizon crossing, the resulting power spectrum \( \Delta^2(\eta) \) will have the usual form

\[
\Delta^2(\eta) \simeq \frac{16}{\pi} GH^2(\eta),
\] (11)

where the Hubble parameter \( H \) is evaluated at the time \( \eta_k \) of the first horizon crossing of the mode with the wavenumber \( k \):

\[
k = H(\eta_k)a(\eta_k).
\] (12)

Moreover, the tensor spectral index \( n_T \) is defined as

\[
n_T \equiv \frac{d}{d \ln k} \ln \left[ \Delta^2(k) \right],
\] (13)

and for the power spectrum (11) equals

\[
n_T \simeq -2\epsilon(\eta_k) \left( 1 - \epsilon(\eta_k) \right) \simeq -2\epsilon(\eta_k).
\] (14)

where the last relation assumes that \( \epsilon(\eta_k) \ll 1 \). It is apparent that the knowledge of the relevant scale factor \( a(t) \) suffices to compute the tensor power spectrum and spectral index.

Assumptions about the expansion history

For the purposes of this paper we divide cosmological history into three epochs.

Primordial inflation

The most convenient time parameter for the epoch of primordial inflation is the number \( N \) of e-foldings before criticality:

\[
a(t) \equiv a_{cr} e^{-N}.
\] (15)

The important cosmological parameters during this phase are [17]

\[
H^2(t) \simeq \frac{1}{9} \omega^2 \left( 4N + \frac{4}{3} \right),
\] (16)

\[
\epsilon(t) \simeq \frac{2}{4N + \frac{4}{3}}.
\] (17)

At the end of inflation the scale factor is about \( a_{cr} \) and the Hubble parameter is about \( \omega \).
The distinctive feature of our model is the epoch of oscillations. The most convenient time parameter during this era is the co-moving time after criticality:

$$\Delta t \equiv t - t_{cr}. \quad (18)$$

The important cosmological parameters during this phase are [9]

$$a(t) \simeq a_{cr} \left[ \frac{\omega}{\Delta t} + 1 + \sqrt{2} (\cos(\omega \Delta t) - 1) \right], \quad (19)$$

$$H(t) \simeq \frac{\omega [1 - \sqrt{2} \sin(\omega \Delta t)]}{\omega \Delta t + (1 - \sqrt{2}) + \sqrt{2} \cos(\omega \Delta t)}, \quad (20)$$

$$\epsilon(t) \simeq \frac{\sqrt{2} [\omega \Delta t + (1 - \sqrt{2}) \cos(\omega \Delta t)] + 3 - 2 \sqrt{2} \sin(\omega \Delta t)}{[1 - \sqrt{2} \sin(\omega \Delta t)]^2}. \quad (21)$$

The epoch of oscillations is terminated by the flow of energy density to the matter sector from the vast reservoir of super-horizon scalar modes, all of which begin to oscillate with frequency $\omega$. As explained in our analysis of scalar perturbations [17], we believe these oscillations should lead to very rapid reheating. Let us call the number of oscillatory e-foldings $\Delta N$. Then, at the end of the oscillation era

$$a \simeq a_{cr} e^{\Delta N}, \quad H \simeq \omega e^{-\Delta N}. \quad (22)$$

Before proceeding, it is worth reviewing the argument—in section 6 of [17]—for why the oscillatory phase should lead to rapid reheating. First, note that causality imposes absolutely no obstacle to the decay, eventually into radiation, of a super-horizon mode which is oscillating at a frequency comparable to the Hubble scale. Indeed, the usual mechanism of reheating relies on precisely such a decay of the inflaton zero mode, which has infinite frequency$^7$. Second, note that the quantum gravitational back-reaction to inflation, which our non-local field equations seek to model, requires a very long period of nearly de Sitter inflation before enough inflationary gravitons have been created so that their accumulated interaction is enough to slow inflation. One can estimate the number of inflationary e-foldings to be about $(G / \Lambda_1)^{-1} \gtrsim 10^6$. Third, this long period of inflation means that the number density of modes which have undergone first horizon crossing is staggering [17]:

$$n \sim \frac{H^3}{3 \pi^2} \exp \left( \frac{3}{GA} \right) \gtrsim H^3 \times 10^{6}. \quad (23)$$

Finally, recall that in our model, all of these super-horizon modes begin oscillating at about the same time. In the face of a number density such as (23) it makes little sense to attempt to estimate the rate at which the disturbance of a single oscillating mode would lead to the production of relativistic matter. As long as that rate is non-zero—and the universal character of the gravitational coupling ensures it is non-zero—the vast number of modes which participate must make reheating almost instantaneous.

$\Lambda CDM$

The $\Lambda CDM$ cosmology after the epoch of oscillations is standard, and we do not require explicit forms for the three geometrical parameters. To compare quantities from the first two eras with their redshifted descendants at present time it is useful to express the energy density

$^7$ See section 5.5 of the text by Mukhanov [18].
\( \rho_R \) at the onset of the \( \Lambda \)CDM epoch in terms of the reheating temperature \( T_R \) and the number of relativistic species:

\[
\rho_R \approx \frac{3c^2\omega^2 e^{-2\Delta N}}{8\pi G} \lesssim n \times \frac{\pi^2 (k_B T_R)^4}{30 \ell^3}.
\]  

(24)

The current energy density \( \rho_{\text{now}} \) can be written in terms of its tiny radiation fraction \( \Omega_r \approx 8.5 \times 10^{-4} \) and the corresponding temperature \( T_{\text{now}} \approx 2.726 \) K of that radiation [19]:

\[
\rho_{\text{now}} = \frac{3c^2 H_{\text{now}}}{8\pi G} \approx \frac{2}{\Omega_r} \times \frac{\pi^2 (k_B T_{\text{now}})^4}{30 \ell^3}.
\]

(25)

Dividing (25) by (24) gives a relation between current conditions and those prevailing at the end of inflation:

\[
\left( \frac{T_{\text{now}}}{T_R} \right)^4 \approx \frac{n}{\Omega_r} 2^{2\Delta N} \left( \frac{H_{\text{now}}}{\omega} \right)^2.
\]

(26)

We define \( N_{\text{now}} \) as the number of e-foldings from criticality to the present time. Using the relation (26) and

\[
T_{\text{now}} \approx \frac{a_R}{a_{\text{now}}} \approx \frac{a_{\text{cr}} e^{\Delta N}}{a_{\text{now}}},
\]

we see that \( N_{\text{now}} \) equals

\[
N_{\text{now}} \approx \Delta N + \ln \left[ \frac{T_R}{T_{\text{now}}} \right] = \frac{1}{2} \ln \left[ \frac{\omega}{H_{\text{now}}} \right] + \frac{1}{2} \Delta N - \frac{1}{4} \ln \left[ 2n \Omega_r \right].
\]

(28)

We shall later show that the measured value of the scalar power spectrum \( \Delta^2 \), and the current limit on the tensor-to-scalar ratio \( r \), together imply the restriction \( \omega \lesssim 10^{55} H_{\text{now}} \). Hence we conclude that

\[
N_{\text{now}} \lesssim 63 + \frac{1}{2} \Delta N.
\]  

(29)

The argument that \( \omega \lesssim 10^{55} H_{\text{now}} \) results from comparing expression (11) with the current bound on the tensor contribution to the quadrupole moment. To make this comparison we must solve the following problem concerning the relation between late times and early times.

**Given a physical wave number \( K_{\text{now}} \) at the current time, find the e-folding \( N_{\text{hor}} \) when it experienced first horizon crossing during inflation.** To solve this problem, we first use the horizon crossing condition (12) to express \( K_{\text{now}} \) in terms of \( N_{\text{hor}} \):

\[
K_{\text{now}} = \frac{k}{a_{\text{now}}} = \frac{k}{a(t)} \times \frac{a(t)}{a_{\text{cr}}} \times \frac{a_{\text{cr}}}{a_{\text{now}}} \approx \frac{1}{3} \omega \sqrt{4N \ell^4 + 4} \times e^{-N_{\text{hor}}} \times e^{-N_{\text{now}}}.
\]

(30)

Now invert (31) to solve for \( N_{\text{hor}} \):

\[
N_{\text{hor}} \approx \ln \left[ \frac{\omega K_{\text{now}}}{K_{\text{cr}}} \right] - N_{\text{now}} + \frac{1}{2} \ln \left[ \frac{4}{9} N + \frac{4}{27} \right]
\]

\[
\approx \frac{1}{2} \ln \left[ \frac{\omega H_{\text{now}}}{c^2 K_{\text{now}}} \right] - \frac{1}{2} \Delta N + \frac{1}{4} \ln \left[ 2n \Omega_r \right] + \frac{1}{2} \ln \left( \frac{\omega H_{\text{now}}}{c^2 K_{\text{now}}} \right).
\]

(32)

8 The inversion was done under the assumption that the Hubble parameter at the end of inflation is much bigger than its present value: \( \omega H_{\text{now}} \gg 1 \).
For the $\ell$th partial wave contribution to the anisotropies of the cosmic ray microwave background, the corresponding number $N_\ell$ of e-foldings before the end of inflation is

$$K_{\text{now}} \approx \frac{\ell}{2} \times \frac{H_{\text{now}}}{c} \implies N_\ell \approx -\ln \left( \frac{\ell}{2} \right) + \frac{1}{2} \ln \left( \frac{\omega}{H_{\text{now}}} \right) - \frac{1}{2} \Delta N + \frac{1}{4} \ln(2\pi \Omega_\ell) + \frac{1}{2} \ln \left[ \frac{2}{9} \ln \left( \frac{\omega}{H_{\text{now}}} \right) \right].$$

The restriction $\omega \lesssim 10^{55} H_{\text{now}}$ then implies

$$N_\ell \lesssim 65 - \frac{1}{2} \ln \left( \frac{\ell}{2} \right) - \frac{1}{2} \Delta N.$$  

We cannot hope to detect a signal outside the range $2 \leq \ell \lesssim 100$, so the interesting values of $N_\ell$ lie within a band of only four e-foldings.

We now deduce the restriction on $\omega$ coming from the measured value of the scalar power spectrum $\Delta^2_s$ [19]:

$$\Delta^2_s(k_0) \approx 2.44 \times 10^{-9}, \quad k_0 \equiv 0.0002 \text{ (Mpc)}^{-1},$$

and the 95% confidence bound on the tensor-to-scalar ratio: $r(k_0) \lesssim 0.22$ [19]. Employing expressions (11) and (20) we get

$$\Delta^2_s(k_0) = r(k_0)\Delta^2_h(k_0) \approx \frac{16}{9\pi} G \omega^2 \left( 4N_0 + \frac{4}{3} \right)$$

$$\lesssim [0.22] \times [2.44 \times 10^{-9}].$$

Now the wave number $k_0$ and its associated number of e-foldings $N_0$ correspond to the $\ell = 2$ partial wave so that—under the assumption that the Hubble parameter at the end of inflation is much bigger than its present value ($\omega H_{\text{now}}^{-1} \gg 1$) and that the duration of the oscillations era is very short ($\Delta N < 10$)—equation (34) implies $N_0 = N_{\ell=2} \gtrsim 60$. Thus, expressions (37)–(38) reduce to $\omega \sqrt{G} \lesssim 2 \times 10^{-6}$ and when we convert to Hz we get

$$\omega \lesssim 2 \times 10^{-6} \sqrt{\frac{c^5}{Gh}} \approx 3.7 \times 10^{37} \text{ Hz} \implies \omega \lesssim 10^{55} H_{\text{now}},$$

where we used $H_{\text{now}} \approx 3.2 \times 10^{-18} \text{ Hz}$ for the current value of the Hubble parameter.

At this stage it is natural to consider a second problem which is in some ways the inverse of the first.

**Given a physical wave number $K_N$ from the epoch of inflation, find its physical wave number now:** To achieve this, we express the current physical wave number in terms of $K_N$ and $N$:

$$K_{\text{now}} = \frac{k}{a_{\text{now}}} = \frac{k}{a(t)} \times \frac{a(t)}{a_{ct}} \times \frac{a_{ct}}{a_{\text{now}}} = K_N \times e^{-N} \times e^{-N_{\text{now}}}$$

$$\approx \sqrt{\frac{K_N^2 H_{\text{now}}}{\omega}} \left( \frac{n\Omega_\ell}{2} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \Delta N},$$

where in the last step we used (28) for $N_{\text{now}}$. An important special case is the oscillation frequency $f_{\text{peak}}$ for the wave vector $K_N = \frac{\omega}{c}$ at $N = 0$. In this situation, (40) gives

$$f_{\text{peak}} \equiv \frac{1}{2\pi} \left( \frac{c k}{a_{\text{now}}} \right)_{K_N = \frac{\omega}{c}} \approx \frac{\sqrt{\omega H_{\text{now}}/2\pi}}{2\pi} e^{-\frac{1}{2} \Delta N}.$$
Imposing the restriction $\omega \lesssim 10^{55} H_{\text{now}}$ and using the current value of $H_{\text{now}} \approx 71 \text{km/s/Mpc}^{-1} \approx 3.2 \times 10^{-18} \text{Hz}$ [19] implies

$$f_{\text{peak}} \lesssim (10^9 \text{Hz}) e^{-\frac{1}{2}\Delta N}.$$  (42)

Therefore, the late time descendant of the gravitational waves produced by the phase of oscillations are unobservable in the cosmic microwave background and have relevance only for high frequency direct detectors.

**Overview of gravitational waves in the oscillating regime**

In terms of the mode functions $u(t, k)$ the basic equation (10) takes the form

$$\ddot{u} + 3H\dot{u} + \frac{k^2}{a^2}u = 0.$$  (43)

We do not possess exact forms for the two linearly independent solutions during the oscillatory regime. Even if we had these solutions, we would not know the linear combination of them that gives ‘the’ mode function $u(t,k)$, which we define to be the coefficient of the annihilation operator in the free field expansion of the graviton. It makes sense to first develop a reasonable approximation for the linearly independent solutions and then consider which combination of them occurs in the actual mode function $u(t,k)$. In approximating the solutions it also makes sense to first include the effect of the overall linear expansion—for which exact solutions exist—and then numerically superimpose the effect of the oscillations.

**The case of linear expansion**

During the oscillatory epoch, the scale factor (19) consists of a linear expansion plus an oscillatory term which causes the Hubble parameter (20) to become negative for brief periods. Because we wish to quantify the potential enhancement from these periods of negative $H(t)$, it is useful to factor out the behavior that would arise from the linear growth, without the oscillatory term

$$\bar{a}(t) = a_{\text{cr}}[1 + \omega \Delta t].$$  (44)

Then, the Hubble parameter $\bar{H}$ can be expressed in terms of the scale factor $\bar{a}$ as follows:

$$\bar{H}(t) = \frac{\omega}{1 + \omega \Delta t} = \frac{\omega a_{\text{cr}}}{\bar{a}(t)}.$$  (45)

The canonically normalized, Bunch–Davies mode function for the linear expansion is [20]

$$\bar{u}(t, k) = \frac{1}{\sqrt{2\omega^2 k^2 - \omega^2 a_{\text{cr}}}} \times \frac{1}{\bar{a}(t)} \exp \left( -i \sqrt{\frac{\omega^2 k^2}{\omega^2 a_{\text{cr}}^2} - 1} \ln \left[ \frac{\bar{a}(t)}{a_{\text{cr}}} \right] \right).$$  (46)

Since the product $\bar{H}(t) \times \bar{a}(t)$ is constant, there is no horizon crossing during linear expansion. Modes which are sub-horizon at criticality ($k > \omega a_{\text{cr}}$) remain sub-horizon, and modes which are super-horizon at criticality ($k < \omega a_{\text{cr}}$) also remain super-horizon. One can see from expression (46) that sub-horizon mode functions oscillate and redshift, with the period of oscillation also redshifting. By contrast, super-horizon mode functions fall off like

$$\bar{u}_{\text{super}}(t, k) \sim \left( \frac{a_{\text{cr}}}{\bar{a}(t)} \right)^{1+\sqrt{1 - \frac{\omega^2 k^2}{\omega^2 a_{\text{cr}}^2}}}. $$  (47)
Initial conditions

We still have to include the effect of oscillations, which can only be done numerically. That defines a mode function \( \tilde{u}(t, k) \) which obeys equation (43) with the full oscillating geometry (19)–(20). We construct these mode functions to agree initially with those of the fictitious phase of linear expansion:

\[
\tilde{u}(t_{cr}, k) = \bar{u}(t_{cr}, k),
\]

(48)

The actual mode function \( u(t,k) \)—by which we mean the coefficient of the annihilation operator in the free field expansion—is neither \( \tilde{u}(t, k) \) nor \( \tilde{u}^*(t, k) \), but rather a linear combination of the two solutions:

\[
u(t, k) = \alpha \tilde{u}(t, k) + \beta \tilde{u}^*(t, k).
\]

(49)

We can solve for the combination coefficients in terms of the values of \( u(t,k) \) and its first derivative at criticality:

\[
\alpha = -i a_{cr}^3 \left[ \frac{k}{H_{N_{hor}} a_{cr}} \right]^2 \left[ 1 + i k H_{N_{hor}} a_{cr} \right] \frac{u(t_{cr}, k) \tilde{u}(t_{cr}, k) - \tilde{u}(t_{cr}, k) u(t_{cr}, k)}{u(t_{cr}, k) \tilde{u}(t_{cr}, k) - \tilde{u}(t_{cr}, k) u(t_{cr}, k)}.
\]

(50)

\[
\beta = -i a_{cr}^3 \left[ \frac{k}{H_{N_{hor}} a_{cr}} \right]^2 \left[ 1 + i k H_{N_{hor}} a_{cr} \right] \frac{\tilde{u}(t_{cr}, k) u(t_{cr}, k) - u(t_{cr}, k) \tilde{u}(t_{cr}, k)}{u(t_{cr}, k) \tilde{u}(t_{cr}, k) - \tilde{u}(t_{cr}, k) u(t_{cr}, k)}.
\]

(51)

Although we do not know precisely what these values are, some reasonable guesses can be made. For example, a far super-horizon mode, which experienced first horizon crossing \( N_{hor} \approx 60 \), implies

\[
\frac{k}{H_{N_{hor}} a_{cr}} \sim 10^{-26}.
\]

(52)

Recall that the potentially observable modes in the cosmic microwave background correspond to \( N_{hor} \approx 60 \), which implies

\[
\frac{k}{H_{N_{hor}} a_{cr}} \sim 10^{-26}.
\]

(53)

There is absolutely no point in retaining such small numbers. So during the oscillatory phase after criticality, the mode function of a cosmologically observable wave number would be unchanged from (52), for all practical purposes. That had to be true because, for far super-horizon wave numbers, (43) simplifies to

\[
\ddot{u}(t, k) + 3 H(t) \dot{u}(t, k) \approx 0
\]

and \( u(t, k) = constant \) remains a solution—indeed of \( a(t) \)—for as long as it is valid to neglect the last term of (43).

Gravitational waves enhancement

Now let us consider the effect of adding the oscillatory term to the scale factor. It is intuitively obvious that one gets a significant response at resonance; then, the natural time scale of the mode function is close to the inverse of the oscillatory frequency \( \omega_0 \). Whether or not this occurs depends upon two things: the wave number \( k \) and the values of \( u(t,k) \) and \( \dot{u}(t, k) \) at the start of the oscillatory period. The reason the initial condition matters is that there are always two linearly independent solutions to the mode equation and they can have vastly different natural time scales. There are three interesting wave number regimes:

\[9\] The Wronskian of the two solutions is \( uu^* - u^* u = i a_{cr}^3 \).
The far super-horizon, with $ck \ll \omega a_{cr}$. From expression (47) it is evident that, without the oscillatory term, super-horizon modes fall off with time scales:

$$T = \frac{\omega^{-1}}{1 \pm \sqrt{1 - \frac{ck^2}{\omega^2 a_{cr}^2}}}.$$  \hspace{1cm} (56)

For $ck \ll \omega a_{cr}$ one of these is—within a factor of 2—close to $\omega$ while the other is vastly longer. Numerical analysis shows—see figure 1—that the oscillations amplify the solution with the shorter time scale by about a factor of 4. As might be expected, the solution with the longer time scale experiences no significant amplification. Because the natural initial conditions (52)–(53) imply that the mode enters the oscillatory epoch almost entirely in the long time scale solution, the effect is that far super-horizon modes experience no significant enhancement from the oscillation.

The far sub-horizon, with $ck \gg \omega a_{cr}$. Far sub-horizon modes also receive no substantial enhancement, but for a different reason. For far sub-horizon modes the natural frequencies of both solutions are about $ck a_{cr}^{-1}$ which is much bigger than $\omega$, so neither solution experiences much enhancement and it does not matter much what the initial condition is.

The near-horizon, with $ck \approx \omega a_{cr}$. As one might expect, it is the near-horizon modes which experience the greatest enhancement. Figures 2 and 3 present numerical simulations for the case of $ck = \frac{10}{11} \omega a_{cr}$, giving the ratios of the actual mode functions—evolved with the oscillatory term—compared with the solution (46) which starts from the same initial condition but is evolved without the oscillatory term. In the near-horizon regime one expects both solutions to be present with about the same amplitude, so a reasonable estimate of the total enhancement is by adding the two solutions in quadrature and taking the ratio with and without the oscillatory term:

$$Q = \frac{\tilde{u}(t, k)}{\bar{u}(t, k)}. \hspace{1cm} (57)$$

The enhancement factor $Q1$ is associated with the real part: $Q1 = \text{Re}[\tilde{u}(t, k)] \div \text{Re}[\bar{u}(t, k)]$ while $Q2$ with the imaginary part: $Q2 = \text{Im}[\tilde{u}(t, k)] \div \text{Im}[\bar{u}(t, k)].$
From figure 4 one can see that the enhancement factor is about $Q \approx 10$. By comparison with the Hubble parameter—see figure 5—we see that almost all the enhancement is derived from the first oscillation. This is important because, as we explained, the phase of oscillations is likely to be short. The fact that almost all the enhancement occurs during the first oscillation means that the effect is reliable even if only one oscillation occurs.

Figures 6 and 7 give the total enhancement factor for the cases of $ck = 2\omega a_t$, and $ck = 5\omega a_t$, respectively. Note that the enhancement falls off rapidly as one moves away from resonance. Note also that the enhancement factor oscillates.

Modes which are slightly super-horizon are quite similar to those which are slightly sub-horizon. Figure 8 gives the total enhancement factor for the case of $ck = \frac{3}{100}\omega a_t$. However, decreasing the wave number much more rapidly reaches the factor of 4 enhancement which is concentrated on the solution that is not likely to be present. Figures 9 and 10 give the behaviors for $ck = \frac{1}{2}\omega a_t$ and $ck = \frac{1}{4}\omega a_t$, respectively.
Enhanced waves energy density and frequency

It remains to estimate the current energy density and frequency of gravitons which are produced during the epoch of oscillations. Suppose we regard the enhancement factor as $Q = 10$ for modes within the range $\frac{1}{2} \omega_{a cr} < c k < \frac{3}{2} \omega_{a cr}$, and zero outside this band. This is superimposed on the mode functions (46) of linear expansion. A reasonable estimate for the extra physical energy—above the 0-point value of $\frac{1}{2} \hbar c k \times a^{-1}(t)$—in a single wave vector $k$ within the band of enhancement is

$$E(t_{cr}, k) \sim \frac{1}{a_{cr}} \hbar c^2 k^4 |Q \tilde{u}(t_{cr}, k)|^2 = \frac{Q^2 \hbar c^2 k^2}{2 \omega_{a cr} \sqrt{1 - (ck/a_{cr})^2}}.$$  

(58)
The associated energy density comes from integrating the extra physical energy (58) over all the modes whose wave numbers are within the band of enhancement:

\[
\rho_{gw}(t_{cr}) \sim \frac{1}{a_{cr}^3} \int \frac{d^3k}{(2\pi)^3} \theta \left( \frac{3}{2} \omega a_{cr} - ck \right) \theta \left( ck - \frac{2}{3} \omega a_{cr} \right) E(t_{cr}, k) \tag{59}
\]

\[
\sim \frac{Q^2 \hbar \omega^4}{4\pi^2 c^3}. \tag{60}
\]

Perhaps of more relevance for gravity waves detectors is the amount of energy density per unit frequency since such detectors are only sensitive in certain frequency bands. We can estimate the energy density per angular frequency at the critical time by simply not performing the radial integration over \( k \) in expression (60):

\[
\frac{d\rho_{gw}(t_{cr})}{dck} \sim \frac{100}{4\pi^2 c^3 a_{cr}} \frac{\hbar \omega^3}{\left( \frac{\omega}{\omega_{cr}} \right)^4 \left( \frac{c k}{\omega_{cr}} \right)^4 \sqrt{1 - \left( \frac{c k}{\omega_{cr}} \right)^2}}. \tag{61}
\]
To convert this to the current epoch we note that these gravitons are sub-horizon after the phase of oscillations, so their energy density redshifts like radiation:

$$\rho_{gw}(t_{\text{now}}) = \left(\frac{a_{\text{cr}}}{a_{\text{now}}}\right)^4 \rho_{gw}(t_{\text{cr}}).$$

(62)

We also take note of the relations between the ordinary (not angular) current frequency and peak frequency:

$$f_{\text{now}} = \frac{ck}{2\pi a_{\text{now}}}, \quad f_{\text{peak}} = \frac{\omega a_{\text{cr}}}{2\pi a_{\text{now}}},$$

(63)

and we divide out a factor of the current critical density to obtain the fraction in gravity waves:

$$\Omega_{gw} \equiv \frac{\rho_{gw}(t_{\text{now}})}{\rho_{\text{cr}}(t_{\text{now}})} \div \left(\frac{3c^2 H_{\text{now}}^2}{8\pi G}\right).$$

(64)
Figure 10. The enhancement factor $Q$ versus co-moving time (in units of $\omega^{-1}$) for a near-horizon mode with $ck = 0.2 \times \omega a_{cr}$.

The enhancement factor is

$$Q = \frac{4 \times 10^{-10}}{\sqrt{1 - \left(\frac{f_{\text{now}}}{f_{\text{peak}}}\right)^2}}.$$  

Using the restrictions (29) and (39) gives

$$Q \sim \exp \left[-4 \left(\frac{f_{\text{now}}}{f_{\text{peak}}} - 1\right)^2\right].$$

A rough model for the $Q$-factor supported by our numerical analysis is

$$Q \sim \frac{1}{10} \exp \left[-4 \left(\frac{f_{\text{now}}}{f_{\text{peak}}} - 1\right)^2\right].$$

and it is for this choice of the $Q$-factor that figure 11 gives a plot of $[d\Omega_{gw}/d(f_{\text{now}}/f_{\text{peak}})]$ as a function of $[f_{\text{now}}/f_{\text{peak}}]$.

From figure 11 it is evident that the signal is highly peaked at the frequency $f_{\text{peak}} \sim 10^9$ Hz, and is negligible at significantly different frequencies. It would be challenging to detect gravitational radiation at such high frequencies but detectors in that range have been proposed [21]. As noted in the text, the phase of oscillations does not affect modes which experienced first horizon crossing more than a few e-foldings before the end of inflation. The wavelength of our effect is $\lambda_{\text{peak}} = c f_{\text{peak}}^{-1} \gtrsim 0.3$ m, whereas the smallest scale feature which is currently observed in the cosmic microwave radiation is about $10^{22}$ m [22]. Our model does not change either how matter couples to gravity or the propagation of linearized gravitons, so it has no effect on the spin-down rate of the binary pulsars. The gravity waves we predict will certainly distort how pulsar light propagates, but the short wavelength again seems to preclude a detectable effect. LIGO is not sensitive above frequencies of 7000 Hz, which is far too low. The situation is even worse with LISA's high frequency cutoff of 0.1 Hz.
Figure 11. Fraction of the current energy density in gravity waves from our signal per frequency.

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