Dynamical analysis of the cosmology of mass-varying massive gravity

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Abstract

We study cosmological evolutions of the generalized model of nonlinear massive gravity in which the graviton mass is given by a rolling scalar field and is varying along time. By performing dynamical analysis, we derive the critical points of this system and study their stabilities. These critical points can be classified into two categories depending on whether they are identical with the traditional ones obtained in General Relativity. We discuss the cosmological implication of relevant critical points.
I. INTRODUCTION

Initiated by Fierz and Pauli (FP) [1], it has been questioned for long time whether the graviton is allowed to acquire a mass and leads to a consistent covariant modification of General Relativity. At quadratic order the FP mass term is the only ghost-free term describing a gravitational theory containing five degrees of freedom [2], but this theory cannot recover linearized Einstein gravity in the limit of vanishing graviton mass, due to the existence of the van Dam-Veltman-Zakharov (vDVZ) discontinuity arising from the coupling between the longitude mode of the graviton and the trace of the energy momentum tensor [3, 4]. It was later noticed that this troublesome mode could be suppressed at macroscopic length scales due to the so-called Vainshtein mechanism [5]. However, these nonlinear terms, which are responsible for the suppression of vDVZ discontinuity, lead inevitably to the existence of the Boulware-Deser (BD) ghost [6] and therefore make the theory unstable [7–10].

Recently, a family of nonlinear extension on the massive gravity theory was constructed by de Rham, Gabadadze and Tolley (dRGT) [11, 12]. In this model the BD ghosts can be removed in the decoupling limit to all orders in perturbation theory through a systematic construction of a covariant nonlinear action [13–16] (see [17] for a review). As a consequence, the theoretical and phenomenological advantages of the dRGT model led to a wide investigation in the literature. For example, cosmological implications of the dRGT model are discussed in [18–39]; black holes and spherically symmetric solutions were analyzed in [40–50]; and connections to bi-metric gravity models were studied in [51–64].

Among these phenomenological studies, a generalized version of the dRGT model was constructed in Ref. [35] that the graviton mass can be determined by a rolling scalar field. This mass-varying massive gravity (MVMG) model was argued to be free of the BD ghosts as well through examining the constraint system in the Hamiltonian formulation. Thus, it is interesting to investigate the cosmological implications of this model, especially its late time evolution. In the present work we perform a phase-space and stability analysis of this model, and investigate the possible cosmological dynamics in a systematic way. This method was widely applied in the literature and was proven to be very powerful particularly in the study of dark energy physics [65–76] (see also Refs. [77, 78] for relevant reviews). In the present model the cosmological system shows a couple of critical points at late times, and they can
generally be classified into two categories depending on whether they are identical with the traditional ones obtained in General Relativity (GR). Although some of the critical points are able to be recovered in GR in the limit of vanishing graviton mass, their background dynamics are different since of the graviton potential. By performing stability analysis we find that the parameter space of the model is tightly constrained and the stable cosmological evolutions are quite trivial. Moreover, there exist some new critical points in the MVMG model which might be of theoretical interests, but the corresponding cosmological evolutions are basically ruled out by observations.

The present paper is organized as follows. In Section II, we briefly review the MVMG model and its cosmological equations of motion. Then we perform a detailed phase-space and stability analysis of this cosmological system and summarize the results in Section III. Finally, we conclude with a discussion in Section IV.

II. THE MVMG COSMOLOGY

To begin with, we briefly review the MVMG model constructed in [35]. This model requires the graviton mass to a function of another scalar field by introducing a nontrivial potential term $V(\psi)$ and thus it is able to vary along cosmic evolution. General Relativity can be automatically recovered when this scalar field $\psi$ sits at $V(\psi) = 0$. In order to better control dynamics of the scalar field, the model also includes an additional potential $W(\psi)$. Therefore, the complete action can be expressed as

$$ S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R + V(\psi) (U_2 + \alpha_3 U_3 + \alpha_4 U_4) - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - W(\psi) \right], \quad (1) $$

The $U_2$, $U_3$ and $U_4$ terms are the graviton potentials following the dRGT model, which are given by,

$$ U_2 = K_{[\mu} K_{\nu]} , \quad U_3 = K_{[\mu} K_{\nu} K_{\rho]} , \quad U_4 = K_{[\mu} K_{\nu} K_{\rho} K_{\sigma]} , \quad (2) $$

with

$$ K_\nu^\mu = \delta_\nu^\mu - \sqrt{g^{\mu\rho} f_{AB} \partial_\rho \phi^A \partial_\nu \phi^B} . \quad (3) $$

Moreover, $f_{AB}$ is a fiducial metric, which is often chosen as Minkowski $f_{AB} = \eta_{AB}$. The four $\phi^A(x)$ are Stäckelberg scalars introduced to restore general covariance under

$$ g_{\mu\nu}(x) \rightarrow \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\sigma}{\partial x_\mu} g_{\rho\sigma}(x) , \quad \phi^A(x) \rightarrow \phi^A(x) ; \quad x^\mu \rightarrow x'^\mu . \quad (4) $$
For the case \( f_{AB} = \eta_{AB} \), the Stückelberg scalars form Lorentz 4-vectors in the internal space. By performing the Hamiltonian constraint of the system it is argued that this model is still free of BD ghosts \([35]\).

In the following we take into account the regular matter component in the total action which minimally coupled to the gravitational system. Further, we consider the fiducial metric to be Minkowski

\[
f_{AB} = \eta_{AB},
\]

and assume that the dynamical and fiducial metrics are of diagonal forms simultaneously for simplicity. Specifically, we consider a flat Friedmann-Robertson-Walker (FRW) metric

\[
d^2s = -N(\tau)^2d\tau^2 + a(\tau)^2\delta_{ij}dx^idx^j,
\]

and for the Stückelberg fields we choose the ansatz

\[
\phi^0 = b(\tau), \quad \phi^i = a_0 x^i,
\]

where \( a_0 \) is constant.

Finally, the total action for the cosmological background reduces to

\[
S_T = \int d^4x \left[ -3M_P^2\frac{a^2a'}{N} + V(\psi)(u_2^F + \alpha_3 u_3^F + \alpha_4 u_4^F) + \frac{a^3}{2N}\dot{\psi}^2 - Na^3W(\psi) \right] + S_m,
\]

where

\[
\begin{align*}
u_2^F &= 3a(a - a_0)(2Na - a_0N - ab'), \\
u_3^F &= (a - a_0)^2(4Na - a_0N - 3ab'), \\
u_4^F &= (a - a_0)^3(N - b'),
\end{align*}
\]

and we have define \( \dot{\tau} = d/d\tau \). Variations of the total action \( S_T \) with respect to \( N \) and \( a \) lead to the two Friedmann equations

\[
\begin{align*}
3M_P^2H^2 &= \rho_{MG} + \rho_m, \\
-2M_P^2\dot{H} &= \rho_{MG} + p_{MG} + \rho_m + p_m,
\end{align*}
\]

respectively. In the above equations, \( \rho_m \) and \( p_m \) are the density and pressure for the matter component, respectively. The effective density and pressure are given by

\[
\begin{align*}
\rho_{MG} &= \frac{1}{2}\dot{\psi}^2 + W(\psi) + V(\psi)(X - 1)f_3(\alpha_i, X) + V(\psi)(X - 1)f_1(\alpha_i, X), \\
p_{MG} &= \frac{1}{2}\dot{\psi}^2 - W(\psi) - V(\psi)(X - 1)f_3(\alpha_i, X) - V(\psi)(\dot{b} - 1)f_1(\alpha_i, X),
\end{align*}
\]
with
\[ f_1(\alpha, X) = (3 - 2X) + \alpha_3(3 - X)(1 - X) + \alpha_4(1 - X)^2, \]
\[ f_2(\alpha, X) = (1 - X) + \alpha_3(1 - X)^2 + \frac{\alpha_4}{3}(1 - X)^3, \]
\[ f_3(\alpha, X) = (3 - X) + \alpha_3(1 - X), \]
and \( X \equiv a_0 / a \).

Moreover, variations of the total action with respect to \( b \) and \( \psi \) give rise to the following two equations of motion,
\[ V(\psi)H f_1(\alpha, X) + \dot{V}(\psi)f_2(\alpha, X) = 0, \]
\[ \ddot{\psi} + 3H\dot{\psi} + \frac{dW}{d\psi} + \frac{dV}{d\psi}[(X - 1)(f_3(\alpha, X) + f_1(\alpha, X)) + 3bf_2(\alpha, X)] = 0. \]
which will be frequently used in detailed calculations later. In addition, it is convenient to define an effective equation of state parameter for modified gravity terms as follows,
\[ w_{MG} = \frac{\rho_{MG}}{p_{MG}} = \frac{\frac{1}{2}\dot{\psi}^2 + W(\psi) + V(\psi)(X - 1)f_3(\alpha, X) + V(\psi)(X - 1)f_1(\alpha, X)}{\frac{1}{2}\dot{\psi}^2 - W(\psi) - V(\psi)(X - 1)f_3(\alpha, X) - V(\psi)(b - 1)f_1(\alpha, X)}, \]
and the total equation of state parameter is defined as:
\[ w_{tot} = \frac{\rho_{MG} + \rho_m}{p_{MG} + p_m} = -1 - \frac{2\dot{H}}{3H^2}. \]

III. DYNAMICAL FRAMEWORK OF MVMG COSMOLOGY

In this section we perform a detailed phase-space analysis of cosmic evolutions described by the MVMG model. Following the method extensively developed in [65–75] (see also [76] for a recent analysis in the frame of generalized Galileon cosmology), we first transform the dynamical system into the autonomous form.

A. Dynamics of the autonomous system

In general, for a dynamical system one can suitably choose a group of auxiliary variables, and the corresponding equation of motion can be expressed as a group of first-order differential equations, respectively. Namely, to illustrate the method of phase-space analysis, we consider the following two-variable dynamical system:
\[ \dot{x} = f(x, y), \quad \dot{y} = g(x, y). \]
The system is said to be autonomous if \( f \) and \( g \) do not contain explicit time-dependent terms. A point \((x_c, y_c)\) is said to be a critical point of the autonomous system if \( (f, g)\) \((x_c, y_c) = 0\).

One can check whether the system approaches one of the critical points or not by performing the stability analysis around the fixed points. Specifically, one can introduce \( \delta x \) and \( \delta y \) as small perturbations and expand the differential equations \((23)\) to the first order of \( \delta x \) and \( \delta y \) around the critical point, and then can derive out the following equations of motion,

\[
\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x=x_c, y=y_c)} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}.
\]

\( (24) \)

As a consequence, the general solution for the evolution of linear perturbations can be written as

\[
\delta x = c_1 e^{\mu_1 N} + c_2 e^{\mu_2 N},
\]

\( (25) \)

\[
\delta y = c_3 e^{\mu_1 N} + c_4 e^{\mu_2 N},
\]

\( (26) \)

where \( \mu_1 \) and \( \mu_2 \) are the two eigenvalues of matrix in the left hand side of Eq. \( (24) \), and \( c_1, c_2, c_3 \) and \( c_4 \) are constant coefficients. If \( \mu_1 < 0 \) and \( \mu_2 < 0 \), then the point is stable, which means the system could evolve to this fixed point eventually. The method can be extended to a system with many variables, a critical point is stable if the real parts of all the corresponding eigenvalues are negative.

In the model we consider, there are five dimensionless variables

\[
x_\rho = \sqrt{\frac{P_m}{3M_pH}}, \quad x_\psi = \frac{\dot{\psi}}{\sqrt{6M_pH}}, \quad x_W = \sqrt{\frac{W(\psi)}{3M_pH}}, \quad x_V = \sqrt{\frac{V(\psi)}{3M_pH}}, \quad x_a = \frac{a_0}{a}.
\]

\( (27) \)

Among them, \( x_a \) and \( x_\rho \) can be determined by background equations of motion as will be analyzed in this subsection. Thus the system only involves three independent variables.

Making use of these variables, one can reexpress the Friedmann equation as follows,

\[
1 = x_\rho^2 + x_\psi^2 + x_W^2 + x_V^2 (x_a - 1) (f_1(x_a) + f_3(x_a)).
\]

\( (28) \)

which now is a constraint equation.

Then, we particularly choose an exponential potential

\[
W(\psi) = \exp \left[-\frac{\lambda}{M_p}\psi\right],
\]

\( (29) \)
with \( \lambda > 0 \). Moreover, we parameterize the form of \( b \) as a linear function of cosmic time, which is given by

\[
  b = Bt \quad \text{with} \quad B > 0 ,
\]

so that this autonomous system is analytically solvable. In addition, we also assume the matter fluid satisfies a barotropic equation of state \( p_m = (\gamma - 1) \rho_m \), with \( \gamma \) being a constant and \( 0 < \gamma \leq 2 \). From equation (19), one can get

\[
  \dot{V}(\psi) = \frac{dV}{d\psi} \dot{\psi} = -\frac{V(\psi) H f_1(x_a)}{f_2(x_a)},
\]

and thus

\[
  \frac{dV}{d\psi} = -\frac{V H f_1(x_a)}{\psi f_2(x_a)} .
\]

if \( \dot{\psi} \neq 0 \). Note that, if \( \dot{\psi} = 0 \) the mass term \( V(\psi) \) is fixed and then the model would reduce to the dRGT version which has been shown in [19] that a flat FRW background is not allowed. Therefore, we will not consider this case in the present work. Moreover, using the auxiliary variables, one can transform the equations of motion to the following autonomous forms,

\[
  \frac{1}{H} \frac{d}{dt} x_\rho = \frac{3}{2} x_\rho \left( \gamma x_\rho^2 + 2 x_\psi^2 + f_1(x_a)(x_a - B) x_\psi^2 \right) - \frac{3}{2} \gamma x_\rho ,
\]

\[
  \frac{1}{H} \frac{d}{dt} x_\psi = \frac{3}{2} x_\psi \left( \gamma x_\rho^2 + 2 x_\psi^2 + f_1(x_a)x_\psi^2(x_a - B) \right) - 3 x_\psi + \frac{\lambda \sqrt{6}}{2} x_W^2 + \frac{x_\psi^2 f_1(x_a)}{2 x_\psi f_2(x_a)} ((x_a - 1)(f_3(x_a) + f_1(x_a)) + 3B f_2(x_a)) ,
\]

\[
  \frac{1}{H} \frac{d}{dt} x_W = \frac{3}{2} x_W \left( \gamma x_\rho^2 + 2 x_\psi^2 + f_1(x_a)x_\psi^2(x_a - B) \right) - \frac{\lambda \sqrt{6}}{2} x_\psi x_W ,
\]

\[
  \frac{1}{H} \frac{d}{dt} x_V = \frac{3}{2} x_V \left( \gamma x_\rho^2 + 2 x_\psi^2 + f_1(x_a)x_\psi^2(x_a - B) \right) - \frac{f_1(x_a)x_V}{2 f_2(x_a)} ,
\]

\[
  \frac{1}{H} \frac{d}{dt} x_a = -x_a .
\]

We restrict our discussion of the existence and stability of critical points to the expanding universes with \( H > 0 \). The critical points correspond to those fixed points where \( \frac{d}{dt} x_\rho = 0, \frac{d}{dt} x_\psi = 0, \frac{d}{dt} x_W = 0, \frac{d}{dt} x_V = 0 \) and \( \frac{d}{dt} x_a = 0 \), and there are self-similar solutions satisfying

\[
  \frac{\dot{H}}{H^2} = -\frac{3}{2} (\gamma x_\rho^2 + 2 x_\psi^2 + f_1(x_a)x_\psi^2(x_a - B)) .
\]

7
The equation (37) simply suggests $x_a = 0$. It can be shown that having the variable $x_a$ in the system always bring a eigenvalue of $-1$ and leave the other eigenvalues unchanged. Thus in the following, for simplicity we will take $x_a = 0$, which does not affect the final results.

By defining
\[ f_1 = 3 + 3\alpha_3 + a_4, f_2 = \frac{f_1}{3}, f_3 = 3 + \alpha_3, \]
and using $x_a = 0$, the constrain equation reduces to
\[ 1 = x_\rho^2 + x_\psi^2 + x_W^2 - x_V^2(f_1 + f_3). \] (40)

Further, we use the constrain equation to eliminate the variable $x_\rho$ and get
\[ x_\psi(\gamma(1 - x_W^2) + (2 - \gamma)x_\psi^2 + cx_V^2) - 2x_\psi + \frac{\lambda\sqrt{6}}{3}x_W^2 - d\frac{x_V^2}{x_\psi} = 0, \] (41)
\[ 3x_W(\gamma(1 - x_W^2) + (2 - \gamma)x_\psi^2 + cx_V^2) - \lambda\sqrt{6}x_\psi x_W = 0, \] (42)
\[ x_V(\gamma(1 - x_W^2) + (2 - \gamma)x_\psi^2 + cx_V^2) - x_V = 0, \] (43)
and the self-similar solutions reduce to
\[ \frac{\dot{H}}{H^2} = -\frac{3}{2}(\gamma(1 - x_W^2) + (2 - \gamma)x_\psi^2 + cx_V^2), \] (44)
where in order for convenience we have defined two dimensionless parameters $c = (f_1 + f_3)\gamma - Bf_1$ and $d = f_1(1 - B) + f_3$.

**B. Phase-space analysis and results**

We summarize the fixed points of this autonomous system and their stability analysis in Table I and Table II, respectively. In the following we discuss these solutions case by case.

From Table I, one may notice that the points are either of $x_V = 0$ or not. For those with $x_V = 0$ (which are the points (a1), (a2), (b), and (c), respectively), the final state of the universe corresponds to that the background evolution is mainly determined by the exponential potential instead of the graviton mass. These solutions are able to be recovered in the limit of vanishing graviton mass and thus are consistent with the standard results
Points that can be recovered in GR

| Label | $x_\psi$  | $x_W$ | $x_\mathcal{V}$ |
|-------|---------|------|---------------|
| (a1)  | 1       | 0    | 0             |
| (a2)  | -1      | 0    | 0             |
| (b)   | $\frac{\lambda}{\sqrt{6}}$ | $\sqrt{1 - \frac{\lambda^2}{6}}$ | 0 |
| (c)   | $\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$ | $\sqrt{\frac{3}{2}}\sqrt{(2 - \gamma)\gamma}$ | 0 |

Points that can’t be recovered in GR

| Label | $x_\psi$  | $x_W$ | $x_\mathcal{V}$ |
|-------|---------|------|---------------|
| (d)   | $\sqrt{\frac{3}{2}}\frac{1}{\lambda}$ | $\sqrt{x_\psi^2 + dx_\mathcal{V}^2}$ | $\sqrt{\frac{(\gamma-1)}{3}}\frac{3}{\lambda^2 - 1}$ |
| (e1)  | $\sqrt{\frac{d(\gamma-1)}{c+d(\gamma-2)}}$ | 0 | $\sqrt{\frac{1-\gamma}{c+d(\gamma-2)}}$ |
| (e2)  | $-\sqrt{\frac{d(\gamma-1)}{c+d(\gamma-2)}}$ | 0 | $\sqrt{\frac{1-\gamma}{c+d(\gamma-2)}}$ |

TABLE I: The critical points in a spatially flat FRW universe with exponential potentials in the MVMG cosmology.

Points that can be recovered in GR

| Label | Existence | Stability | Equation of state |
|-------|-----------|-----------|-------------------|
| (a1)  | all $\gamma$ and $\lambda$ | unstable | 1 |
| (a2)  | all $\gamma$ and $\lambda$ | unstable | 1 |
| (b)   | $\lambda^2 < 6$ | stable if $\lambda^2 < \min(3\gamma, 3)$ | $-1 + \frac{\lambda^2}{3}$ |
| (c)   | $\lambda^2 > 3\gamma$ | stable if $\gamma < 1$ | $-1 + \gamma$ |

Points that can’t be recovered in GR

| Label | Existence | Stability | Equation of state |
|-------|-----------|-----------|-------------------|
| (d)   | $\frac{(\gamma-1)}{c-d\gamma} \left( \frac{3}{\lambda^2} - 1 \right) > 0$ | stable if $\gamma > 1$, $\lambda^2 > 3$, $E > 0$ | 0 |
| (e1)  | $d > 0$, $\frac{1-\gamma}{c+d(\gamma-2)} > 0$ | stable if $\gamma > 1$, $\lambda^2 > \frac{3}{2} \left( 1 + \frac{c-d}{d(\gamma-1)} \right)$ | 0 |
| (e2)  | $d < 0$, $\frac{1-\gamma}{c+d(\gamma-2)} > 0$ | unstable | 0 |

TABLE II: The properties of the critical points. The definitions of $E$ is given in the text below.

obtained in GR. A further stability analysis suggests the points (a1) and (a2) are unstable. For the point (b), the solution is allowed if $\lambda^2 < 6$ and stable against perturbations when $\lambda^2 < \min(3\gamma, 3)$. For the point (c), the solution exists when $\lambda^2 > 3\gamma$ but is stable only if
\( \gamma < 1 \). We list the eigenvalues of these solutions in detail as follows.

Point (a1) is a kinetic-dominated solution. The linearization of the system yields three eigenvalues

\[ m_1 = \frac{3}{2}, m_2 = 6 - 3\gamma, m_3 = 3 - \frac{3}{2}\lambda. \tag{45} \]

The corresponding effective equation of state for the whole system approaches to \( w = 1 \) which implies a stiff fluid dominant phase at late times of the universe.

Point (a2) is a kinetic-dominated solution as well. The linearization of the system yields three eigenvalues

\[ m_1 = \frac{3}{2}, m_2 = 6 - 3\gamma, m_3 = 3 + \frac{3}{2}\lambda, \tag{46} \]

and correspondingly, the destiny of the universe is the same as Point (a1).

Point (b) is a scalar-dominated solution since \( x_{\psi}^2 + x_{W}^2 = 1 \), and only exists when \( \lambda^2 < 6 \). The linearization of the system yields three eigenvalues

\[ m_1 = \frac{1}{2}(\lambda^2 - 3), \quad m_2 = \frac{1}{2}(\lambda^2 - 6), \quad m_3 = \lambda^2 - 3\gamma. \tag{47} \]

The corresponding total effective equation of state is \( w_{tot} = -1 + \frac{\lambda^2}{3} \) which is always less than unity.

Point (c) is a solution depending on both the scalar field and the matter fluid, since \( x_{\psi}^2 + x_{W}^2 = \frac{3\gamma}{\lambda^2} \). The linearization of the system yields three eigenvalues

\[
\begin{align*}
m_1 &= \frac{3}{2}(\gamma - 1), \\
m_2 &= -\frac{3}{4}(2 - \gamma) \left(1 + \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2 - \gamma)}}\right), \\
m_3 &= -\frac{3}{4}(2 - \gamma) \left(1 - \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2 - \gamma)}}\right).
\end{align*} \tag{48} \]

The total effective equation of state is given by \( w_{tot} = -1 + \gamma \) in this solution.

Point (d) is a solution depending on both the scalar field and the graviton mass since \( x_{\psi}^2 + x_{W}^2 - x_{V}^2(f_1 + f_3) = 1 \), the linearization of the system yields three eigenvalues

\[
\begin{align*}
m_1 &= 3(1 - \gamma), \\
m_2 &= -E + \sqrt{3(3 - \lambda^2)x_{W}^2 + E^2}, \\
m_3 &= -E - \sqrt{3(3 - \lambda^2)x_{W}^2 + E^2}.
\end{align*} \tag{49} \]
where $E$ is a dimensionless parameter with $E = \frac{3}{4} + \frac{d(3-\lambda^2)(1-\gamma)}{2(c-d\gamma)}$.

Point (e1) is a solution determined by the kinetic term of the scalar field and the graviton mass since $x_W = 0$ at late times. Its linearization yields three eigenvalues

$$m_1 = -3$$
$$m_2 = 3(1-\gamma)$$
$$m_3 = \frac{3}{2} - \lambda \sqrt{\frac{3}{2} \sqrt{\frac{d(\gamma-1)}{c+d(\gamma-2)}}}.$$  

(50)

Point (e2) is similar to Point (e1) with its eigenvalues given by

$$m_1 = -3$$
$$m_2 = 3(1-\gamma)$$
$$m_3 = \frac{3}{2} + \lambda \sqrt{\frac{3}{2} \sqrt{\frac{d(\gamma-1)}{c+d(\gamma-2)}}}.$$  

(51)

It is clear that $m_3$ is positive, so the solution is unstable.

It is worth noticing that the total effective equation of state of Points (a1), (a2), (b), and (c) are the same as the massless case. This result shows that the effect of graviton mass does not contribute manifestly in the solutions corresponding to GR. Moreover, if we substitute the critical points (d), (e1) and (e2) into the Eq. (44), then we get $\dot{H}/H^2 = -3/2$ which implies a matter domination at late times. This conclusion suggests that the appearance of a scalar field dependent graviton mass could strongly fix the background dynamics of the universe which manifestly conflicts with the observational fact of late time acceleration.

We would like to point out that the existence of points (d), (e1) and (e2) does not conflict with the fact that massive gravity theory does not allow flat FRW cosmologies, the fixed value of $x_V$ does not mean that the graviton mass is fixed. What actually happens is that the graviton mass and $H$ both gradually approach 0 as $t \to \infty$ with the magnitude about $1/t^2$ and $1/t$ respectively.

Finally, we can conclude that if a MVMG model is of cosmological interest, then its model parameters have to satisfy either $\lambda^2 < \min(3\gamma, 3)$ (required by the stability of Point (b)) or $\gamma < 1$ (required by the stability of Point (c)). Specifically, the solution of Point (b) corresponds to that the final evolution of the universe is determined by the scalar field and the effect of graviton mass is totally negligible, and thus this solution is quite trivial. Moreover, the solution of Point (c) corresponds to that the destiny of the universe is determined by the
combined effects of the scalar field and the matter fluid. However, the stability of Point (c) requires the barotropic equation of state of matter fluid to be less than unity, which implies the corresponding matter fluid has to violate the strong energy condition and thus obviously conflict with cosmological observations.

IV. CONCLUSIONS

In the present paper we have studied the dynamical behavior of the MVMG cosmology. This model, due to the varying of the graviton mass, possesses plentiful phenomenological properties and consequently has attracted many attentions in the literature. Namely, it was shown to be able to violate the null energy condition effectively and thus could be applied to realize the phantom divide crossing [36]; further, it was also applied into early universe and a class of bouncing and oscillating solutions were reconstructed [37]. However, while phenomenological studies of this model is still proceeding extensively, it is necessary to investigate the phase space of the model and examine the stability of the critical points existing in cosmological trajectories. Thus we performed a detailed dynamical analysis of the MVMG model with a specifically chosen potential for the cosmic scalar field, namely an exponential potential. Our result reveals that there are mainly two types of critical points in this model. One type of critical points correspond to the case that they can recover the standard results in GR if the graviton mass is chosen to be vanishing; the other type of critical points then is discontinuous with GR in the massless limit. We analyze these points respectively and find that there are only two critical points which might be stable against perturbations and both two belong to the first type. However, one of these two solutions is difficult to accommodate with cosmological observations since its stability requires the matter components in the universe to violate strong energy condition. Eventually, there is only one viable solution in this model but the final state of the universe is completely determined by the cosmic scalar field and there is no effect of modified gravity. As a consequence, the MVMG cosmology severely degenerate with the standard cosmology based on GR.

We would like to point out that, although the graviton mass does not give rise to observable effects on the background evolution at late times, it may still leave signatures on cosmological perturbations and thus affect the large scale structure formation. Moreover, in
our investigation, we focused on a particularly chosen potential for the cosmic scalar field. It would be interesting to generalize the case to a much more generic potential and verify if the phase space of the viable solutions could be enlarged.

**Note added:** While this work was being finalized, we learned of a related work by G. Leon, E. N. Saridakis and J. Saavedra which will be appeared on arXiv [79]. Part of their content overlaps with ours, and their conclusions are similar as well while their focus is on a generalized structure of the phase space.

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