ON THE LOCAL COHOMOLOGY OF MODULAR INVARIANTS

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Abstract. We compute some numerical invariants of local cohomology of the ring of invariants by a finite group, mainly in the modular case. Also, we present some applications. In particular, we study Cohen-Macaulay property of modular invariants from the viewpoints of depth, Serre’s condition and the relevant generalizations (e.g., the Buchsbaum property, etc). The situation in the local case is different from the global case.

1. Introduction

There are a lot of research papers on local cohomology modules and on modular invariant theory. Also, there are some links between these. For example, Ellingsrud and Skjelbred [8] (resp. Fogarty [10]) applied beautiful sort of local cohomology arguments to compute bounds on the depth of modular invariant rings. We start by a new presentation of a result of Kemper [18] (see Proposition 3.2). Then, we observe that:

Observation 1.1. Let $G \to GL(n, \mathbb{F})$ be a representation of a finite group $G$ and denote the invariant ring by $R$. Let $a \triangleleft h R$ be such that $H^i_a(R)$ is of finite length for all $i < \text{cd}(a)$. Then $H^i_a(R) = 0$ for all $i < \text{cd}(a)$. In particular, grade($a$, $R$) = ht($a$) = cd($a$) = $f_a(R)$.

In dimension 4 we can talk a little more (see Corollary 3.9). Almkvist and Fossum proved that $R := \mathbb{F}_2[x_1, \ldots, x_4]/\mathbb{Z}/4\mathbb{Z}$ does not satisfy Serre’s condition $S(3)$. This is the Bertin’s famous ring which is not Cohen-Macaulay. In particular, we can recover the following funny result of Larry Smith:

Corollary 1.2. Let $G \to GL(n, \mathbb{F})$ be a representation of a finite group $G$ and denote the invariant ring by $R$. If $R$ is $S(n-1)$, then $R$ is Cohen-Macaulay. In particular, if $n = 3$ then $\mathbb{F}[x, y, z]^G$ is Cohen-Macaulay.

Also, the following is a strengthening a result of Hartshorne and Ogus:

Corollary 1.3. Let $G \to GL(5, \mathbb{F}_p)$ be a representation of a finite $p$-group $G$ and denote the invariant ring by $R$. If $R$ is $S(3)$, then $R$ is Cohen-Macaulay (and hence Gorenstein).

The situation in local case is quite different: Despite of Corollary 1.2, Fogarty constructed a regular local ring $R$ of dimension 3 and of prime characteristic equipped with a wild action of a finite group $G$ such that depth($R^G$) = 2 (see [11]). By applying Fogarty’s idea we present the following result:

Observation 1.4. Let $R$ be a local ring of characteristic $p > 0$. Let $G$ be a cyclic group of order $p^n h$, $n > 0$, acting wildly on $R$. The following assertions hold:

i) If $R$ is generalized Cohen-Macaulay, then $R^G$ is generalized Cohen-Macaulay.


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ii) If $\text{depth}(R) > 1$, then $\text{ht}(\text{im}(R \xrightarrow{t} R^G)) = \dim R$.

Let $G \to \text{GL}(n, \mathbb{F}_p)$ be a representation of a finite group. It is known that $\text{ht}(\text{im}(R \xrightarrow{t} R^G)) < \dim(\mathbb{F}_p[V]^G)$. This result is due to Feshbach (see [22, Theorem 6.4.7]). Observation 1.1 shows that this is not the case for the wild actions over local rings of depth bigger than one. It may be nice to characterize cases for which $\text{im}(R \xrightarrow{t} R^G)$ is the maximal ideal. Here, we present a sample:

**Corollary 1.5.** Let $R$ be a 3-dimensional Cohen-Macaulay local ring of characteristic 2. Let $G$ be a group of order 2 acting wildly on $R$. Then $(R^G, n)$ is Buchsbaum if and only if $\text{im}(\text{tr}) = n$.

Also, we extend and simplify some results of Ellingsrud and Skjelbred by elementary methods:

**Corollary 1.6.** Let $(R, m)$ be a local domain and $G$ a finite group acting wildly on $R$.

i) If $R$ is $S(2)$ then $R^{G}$ is $S(2)$.

ii) Suppose $\text{depth}(R) > 1$ and $\dim R > 2$. Then $R^G$ is not $S(n)$ for all $n > 2$.

2. Preliminary

Subsection 2.A: A quick review of local cohomology. Let $R$ be any commutative ring with an ideal $a$ with a generating set $a := a_1, \ldots, a_r$. By $H^i_a(M)$, we mean the $i$-th cohomology of the Čech complex of a module $M$ with respect to $a$. This is independent of the choose of the generating set. For simplicity, we denote it by $H^i_a(M)$. Recall that $\text{cd}(a, M)$ is defined to be the spermium of $i$’s such that $H^i_a(M) \neq 0$. We set $\text{grade}(a, M) := \inf \{ i : H^i_a(M) \neq 0 \}$. We put $\text{depth}(M) := \text{grade}(m, M)$ when $m$ is the unique (graded) maximal ideal. We say $R$ satisfies Serre’s condition $S(n)$ if $\text{depth}(R_p) \geq \min\{n, \dim R_p\}$ for all $p \in \text{Spec}(R)$. Recall that $f_a(M) := \inf \{ i : H^i_a(M) \text{ is not finitely generated} \}$. Here, is the relations between these:

$$\text{grade}(a, R) \leq f_a(R) \leq \text{ht}(a) \leq \text{cd}(a) \leq \dim R,$$

provided $R$ is noetherian. The term local means noetherian and quasi-local.

**Fact 2.1.** The following assertions hold:

i) Let $(A, m)$ be a local ring. Then $H^{\dim A}_m(A)$ is not finitely generated.

ii) (Grothendieck’s finiteness theorem) Let $A$ be a quotient of a noetherian regular ring. Then

$$f_a(A) = \inf \{ \text{depth}(A_a) + \text{ht}(\frac{a}{q}) : q \in \text{Spec}(A) \setminus V(a) \}.$$

iii) (Hartshorne-Lichtenbaum vanishing theorem) Let $A$ be analytically irreducible local ring and $\dim(A/a) > 0$. Then $H^{\dim A}_a(A) = 0$.

For more details on local cohomology modules see the books [15], [14] and [6].

Subsection 2.B: A quick review of local algebra. A finite sequence $\underline{x} := x_1, \ldots, x_r$ of elements of $(A, m)$ is called weak regular sequence if $x_i$ is a nonzero-divisor on $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, r$. Then depth$(A)$ coincides with the supremum of the lengths of all weak regular sequences contained in $m$. Recall that a local ring $(A, m)$ is called Cohen-Macaulay if depth $A = \dim A$. Also, $(A, m)$ is called generalized Cohen-Macaulay if $i(H^i_m(A)) < \infty$ for all $i < \dim A$. Recall that a sequence $x_1, \ldots, x_r \subset m$ is called a weak sequence if $m((x_1, \ldots, x_{i-1}) : x_i) \subseteq (x_1, \ldots, x_{i-1})$ for all $i$. The ring $R$ is called Buchsbaum if every system of parameters is a weak sequence. This implies that $n H^i_m(R) = 0$ for all $i < \dim R^G$. This
property is called quasi-Buchsbaum. In general quasi-Buchsbaum rings are not Buchsbaum. Despite of this, there are cases for which we can deduce the Buchsbaum property from quasi-Buchsbaum:

Fact 2.2. Suppose $R$ is quasi-Buchsbaum and that $H^i_m(R) = 0$ for all $i \neq \text{depth}(R)$ and $i \neq \text{dim}(R)$. Then is Buchsbaum.

For more details on Buchsbaum rings (resp. local algebra), see the book [25] (resp. [20]).

Subsection 2.C: A quick review of invariant theory. Let $A := \bigoplus_{n \geq 0} A_n$ be the polynomial algebra of a field $F$. Suppose $G$ acts on $A$ by degree-preserving homomorphisms. This means that $g(A_n) \subseteq A_n$ for all $g \in G$ and $n \in \mathbb{N}$. Then

$$R := A^G = \{ a \in A : g(a) = a \ \forall g \in G \}$$

is the (graded) ring of invariants. The notation $a \triangleleft_R R$ stands for homogeneous ideal of $R$ and $m := \bigoplus_{n > 0} R_n$ is the irrelevant ideal. Also, if $G$ is a finite group, then by the very first result of invariant theory, $R$ is finitely generated as an $R_0$-algebra (see [22, Theorem 2.1.4]). This remarkable result is due to Emmy Noether. If $|G|$ is invertible in $R$ the action is called non-modular. If $|G|$ is not invertible in $R$ the action is called modular.

Fact 2.3. Let $S$ be a normal domain and $G \subset \text{Aut}(S)$ be any group. Then $S^G$ is normal.

For more details on invariant theory of finite groups see the book [22] by Neusel and Smith.

3. The global results

Discussion 3.1. (Kemper) Let $G \to GL(V, F)$ be a representation of a finite group $G$ and denote the invariant ring by $R$. By Cohen-Macaulay defect we mean $\text{CM.def}(R) := \dim(R) - \text{depth}(R)$. For each $m$, let $\text{Loc}(\text{CM.def} > m) := \{ p : \text{CM.def}(R_p) > m \}$. In view of [18, Proposition 3.1],

$$0 < \dim(\text{Loc}(\text{CM.def} > m)) < \dim(V) - m - 1,$$

provided $\text{CM.def}(R) > m$.

Set $X := \text{Spec}(-)$ and denote the punctured spectrum by $\hat{X} := X \setminus \{ m \}$. In general, it is not true to extend a property $P$ from $\hat{X}$ to $X$. For example, there are 3-dimensional normal rings that are not Cohen-Macaulay. In particular, $P := \text{Cohen-Macaulay}$ does not extend from $\hat{X}$ to $X$.

Proposition 3.2. Let $G \to GL(n, F)$ be a representation of a finite group $G$ and denote the invariant ring by $R$. The following are equivalent:

(i) $R$ is Cohen-Macaulay,
(ii) $R$ is Buchsbaum,
(iii) $R$ is quasi-Buchsbaum,
(iv) $R$ is generalized Cohen-Macaulay,
(v) $R$ is Cohen-Macaulay over the punctured spectrum.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are true over any commutative noetherian rings. The implications (ii) $\Rightarrow$ (i) is due to Kemper. Let us repeat its argument to deduce (i) from (v):

Suppose $R$ is generalized Cohen-Macaulay. We are going to show it is $R$ is Cohen-Macaulay. Suppose on the contradiction that $R$ is not Cohen-Macaulay. Then CM.def($R$) $> 0$. We apply Discussion 3.4 for $m = 0$ to find a prime $p$ of height $n - 1$ such that $p \in \text{Loc}(\text{CM.def} > 0)$. This means that $R_p$ is not Cohen-Macaulay. This contradicts the Cohen-Macaulay property over $\hat{X}$. □

The module version of Proposition 3.2 is not true: Let $G$ be the trivial group act on $R := k[x, y, z, w]$.

So, $R^G = R$. In view of [22] there is a prime ideal $p$ of height two such that $H^0_m(R/p) = R/m$. Set $M := R/p$. Since $p$ is prime, $H^0_m(R/p) = 0$. Note that $\dim M = 2$. So, $M$ is generalized Cohen-Macaulay. Clearly, $M$ is not Cohen-Macaulay.

Corollary 3.3. Let $G \rightarrow \text{GL}(n, \mathbb{F})$ be a representation of a finite group $G$ and denote the invariant ring by $R$. If $R$ is $S(n - 1)$, then $R$ is Cohen-Macaulay. In particular, if $n = 3$ then $\mathbb{F}[x, y, z]^G$ is Cohen-Macaulay. □

Proof. Suppose on the contradiction that $R$ is not Cohen-Macaulay, e.g., CM.def($R$) $> 0$. In the light of Discussion 3.4 $\dim (\text{Loc}(\text{CM.def} > 0)) \neq 0$. There is $p$ of height $n - 1$ such that CM.def($R_p$) $> 0$ and so depth($R_p$) $< n - 1 = \min\{n - 1, \dim R_p\}$. This contradicts $S(n - 1)$.

Now we prove the particular case: By a folklore result of Serre, noetherian normal rings are $S(2)$ (see [23] Theorem 23.8). The claim follows by the first part. □

Recall that $G$ is called $p$-group if $|G| = p^i$ for some $i$.

Corollary 3.4. Let $G \rightarrow \text{GL}(n, \mathbb{F}_p)$ be a representation of a finite $p$-group $G$ and denote the invariant ring by $R$. Suppose $n < 6$. If $R$ is $S(3)$, then $R$ is Cohen-Macaulay and hence Gorenstein.

Proof. We deal with the case $n = 5$. Suppose on the contradiction that $R$ is not Cohen-Macaulay. In view of [23] Proposition 5.6.10, depth($R$) $\geq 3$. Suppose first that depth($R$) $= 3$. Then CM.def($R$) $= 2$. In the light of Discussion 3.4 $\dim (\text{Loc}(\text{CM.def} > 1)) \neq 0$. There is $p$ of height $4$ such that CM.def($R_p$) $> 1$. Apply S(2) along this to see $2 \leq \text{depth}(R_p) \leq 4 - 2 = 2$, i.e., depth($R_p$) $= 2 < \min\{3, \dim R_p\}$. This contradicts $S(3)$. Suppose now that depth($R$) $\geq 4$. Since $R$ is not Cohen-Macaulay, we deduce that depth($R$) $= 4$. Since $R$ is $S(3)$ we have depth($R_p$) $\geq \frac{\text{ht}(p)}{2} + 1$ for all $p \in \hat{X}$. Since depth($R$) $= 4$ we have depth($R_m$) $\geq \frac{\text{ht}(m)}{2} + 1$. In sum,

$\text{depth}(R_p) \geq \frac{\text{ht}(p)}{2} + 1 \geq \min\{\frac{\text{ht}(p)}{2} + 1, \dim R_p\}$

for all $p \in \text{Spec}(R_m)$. Hartshorne and Ogus call this the $C$-property. Now we use [15] Corollary 1.8:

8Smith’s argument is as follows: First, he reduces to the prime characteristic case by applying the famous theorem of Hochster and Eagon. Second, he uses the following result: If $F[X, . . . ]^{Syl_p(G)}$ is Cohen-Macaulay, then $F[X, . . . ]^{G}$ is Cohen-Macaulay (this is the main result of “Rings of invariants and $p$-Sylow subgroups” by Campbell, Hughes, and Pollack). Finally, he applies the mentioned result of Ellingsrud and Skjelbred to deduce that depth($F[X, Y, Z]^{Syl_p(G)}$) $\geq 3$. 
Fact A) Let $A$ be a local factorial ring which is quotient of a regular local ring with C-property. Then $A$ is Cohen-Macaulay.

Recall that invariant rings of $p$-groups are UFD (see [22 Corollary 1.7.4]). Also, recall that a domain is UFD if and only if its height one prime ideals are principal. From this, $R_m$ is UFD. By Fact A), we get that $R_m$ is Cohen-Macaulay. Thus, $R$ is Cohen-Macaulay. This contradiction completes the proof. □

A ring $R$ is called almost Cohen-Macaulay if grade($p, R$) = depth($pR_p, R_p$) for every $p \in \text{Spec}(R)$. We use the following characterization of almost Cohen-Macaulay rings: $R$ is an almost Cohen-Macaulay ring if and only if ht($p$) \leq 1 + grade($p, R$) for every $p \in \text{Spec}(R)$.

**Corollary 3.5.** Let $G \to GL(n, \mathbb{F})$ be a representation of a finite group $G$ and denote the invariant ring by $R$. The following are equivalent:

(i) $R$ is almost Cohen-Macaulay,

(ii) $R$ is almost Cohen-Macaulay over the punctured spectrum.

In particular, if $G \to GL(4, \mathbb{F})$ then $R$ is almost Cohen-Macaulay and depth($R$) \geq 3.

**Proof.** Suppose $R$ is almost Cohen-Macaulay. This is clear from definition that $R$ is almost Cohen-Macaulay over $\hat{X}$. Conversely, suppose $R$ is almost Cohen-Macaulay over $\hat{X}$. On may find easily that depth($R_m$) = depth($R$) and dim($R_m$) = dim($R$). Suppose on the contrary that $R$ is not almost Cohen-Macaulay. One may read this as CM. def($R$) > 1. In view of Discussion 3.1, dim ($\text{Loc}($CM. def $> 1)) \neq 0$. In particular, there is a height $n - 1$ prime ideal $p$ such that dim($R_p$) − depth($R_p$) > 1. This contradicts almost Cohen-Macaulayness over $\hat{X}$.

Any four-dimensional normal domain is almost Cohen-Macaulay over the punctured spectrum. This follows by Serre’s characterization of normality. Conjugate this along with the first item to check almost Cohen-Macaulayness of $R$. In particular, depth($R$) \geq dim $R − 1 = 3$. □

Here, $\ell(−)$ is the length function.

**Proposition 3.6.** Let $G \to GL(n, \mathbb{F})$ be a representation of a finite group $G$ and denote the invariant ring by $R$. Let $a <_{\text{min}} R$ be such that $\ell(H^i_a(R)) < \infty$ for all $i < \text{cd}(a)$. Then $H^i_a(R) = 0$ for all $i < \text{cd}(a)$.

In particular, grade($a, R$) = ht($a$) = cd($a$) = $f_a(R)$.

**Proof.** We have dim $R = n$. Note that grade($a, R$) \leq ht($a$) \leq cd($a, R$) \leq dim $R$. We claim that ht($a$) = cd($a$): Indeed, let $p \in \text{min}(a)$ of same height as of $a$. By Fact 2.1 $H^\text{dim}(R_p)_{R_p} (p)$ is not finitely generated. Since $H^\text{ht}(p)_{R_p} (R_p) \simeq H^\text{ht}(a)_{R_p} (R_p) \simeq (H^\text{ht}(a)_{R_p} (R_p))_p$ we get that $H^\text{ht}(a)_{R_p} (R_p)$ is not finitely generated. We look

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*Let $R := \mathbb{F}[x_1, \ldots, x_4]^{G/\mathbb{Z}}$. It may be worth to note that computing depth of $R$ was a challenging problem. In their paper Fossum and Griffith wrote that "many hours of calculations, using several hundred sheets of paper, have convinced the authors that the depth of Bertin’s example is 3." (see page 194 of: Complete local factorial rings which are not Cohen-Macaulay in characteristic $p$, Ann. Sci. Ecole Norm. Sup. (4) 8 (1975), 189-199). Now, we know that there are at least four different arguments for this: The first proof of this is in [1] Proposition 2.3. They worked with Bertin’s example, directly. The second one is in [8 Corollaire 3.3]. This is more general than the first one; They work with indecomposable action of $\mathbb{Z}/p^n\mathbb{Z}$. The third one is Corollary 3.5 which works for any 4-dimensional invariant rings. The fourth one is in the book [22 Proposition 5.6.10]. This is very strong: for an invariant ring $A$ by a finite group we have depth($A$) \geq \text{min}(3, \text{dim } A)$. Concerning depth of Bertin’s ring, I feel that there is a typographical error in the book "polynomial invariants of finite groups" by Benson.*
at our assumption to observe that $ht(a) = cd(a)$, as claimed. In sum, $grade(a, R) \leq ht(a) = cd(a, R)$. In particular, if $grade(a, R) = ht(a)$, then by definition of grade, $H^i_a(R) = 0$ for all $i < cd(a)$.  This yields the claim in the Cohen-Macaulay case. Let $m \geq 0$ be such that $dim(R) - depth R = m + 1$. We bring the following fact:

Fact A): Let $M$ be any graded $R$-module (not necessarily finitely generated) such that $r := grade(a, M) < depth M$. Then $H^r_a(M)$ is not artinian. This stated in [2] without any proof. We use induction on $r$ to prove it. Let $r = 0$. If $H^0_a(M)$ were be artinian, then we should have $0 \neq H^0_a(M) \cong H^0_m(H^0_a(M)) \cong H^0_m(M)$. In particular, $H^0_m(M) \neq 0$. That is $depth(M) = 0$. This contradicts $r < depth M$. Now, assume that $r > 0$. Recall that $grade(a, M) := \inf \{ i : H^i_a(M) \neq 0 \}$. In particular, $H^0_a(M) = 0$. Let $E$ be the graded injective envelope of $M$ and $N := E/M$ (such a thing exists. For more details see [6] §12). Recall that $M \subset E$ is graded-essential if $M \cap F \neq 0$ for every non-zero graded submodule $F \subset E$.  Let $e \in H^0_a(E)$ by any graded element. Suppose $e$ is not zero. Since the extension $M \subset E$ is essential, there is a homogeneous element $r \in R$ such that $0 \neq re \in M$.

Clearly, $re \in H^0_a(M) = 0$. This contradiction says that $H^0_a(E) = 0$. Similarly, $H^0_m(E) = 0$. From $0 \to M \to E \to N \to 0$ we obtain that $H^0_a(N) \cong H^1_a(M)$ and $H^0_m(N) \cong H^{i+1}_m(M)$ for all $i \geq 0$.

Hence $grade(a, N) = grade(a, M) - 1$ and $depth N = depth M - 1$. The claim follows by the induction hypothesis.

Suppose first that $ht(a) = n$. Since local cohomology modules are invariant under taking radical we may assume that $a$ is radical. Hence $a = m$. Therefore, the desired claim is in Proposition 3.2. If $grade(a) < depth R = n - m - 1$, then the claim follows by Fact A). In particular, the claim is true whenever $ht(a) < n - m - 1$. Suppose now that $ht(a) = n - m - 1$. In view of Fact A we may and do assume that $grade(a, R) = ht(a)$. In particular, $grade(a, R) = ht(a) = cd(a)$. Therefore, the claim follows by definition of grade. Finally, suppose that $n - m \leq ht(a) \leq n - 1$. Since $\ell(H^i_a(R)) < \infty$ for all $i \leq cd(a)$ it follows from definition that $f_a(R) = ht(a)$. We are going to use Grothendieck’s finiteness theorem:

$$f_a(R) = \inf \{ depth(R_q) + ht(\frac{a+q}{q}) : q \in Spec(R) \setminus V(a) \} \quad (\ast)$$

Claim B): Let $q$ be any prime ideal of height $n - 1$. Then $CM. def(R_q) \leq m - 1$.

Indeed, suppose first that $q \in Spec(R) \setminus V(a)$. Since $q$ is 1-dimensional, we have $ht(\frac{a+q}{q}) = dim(\frac{R}{q}) = 1$. In the light of $(\ast)$ we observe that

$$1 + depth(R_q) = depth(R_q) + ht(\frac{a+q}{q}) \geq f_a(R) = ht(a) \geq n - m.$$

Conclude by this that $depth(R_q) \geq n - m + 1$. Hence

$$dim(R_q) - depth(R_q) \leq (n - 1) - n + m - 1 = m - 2.$$

Second, suppose that $q \in V(a)$. In view of our assumption we observe for all $i < cd(a) = ht(a)$ that $H^i_{aR_q}(R_q) \cong H^i_a(R)_{aq} = 0$. This implies that $ht(a) \leq grade(a_q, R_q)$. We have

$$n - m \leq ht(a) \leq grade(a_q, R_q) \leq depth(R_q),$$

because grade becomes larger with respect to inclusion. This yields that

$$CM. def(R_q) = dim(R_q) - depth(R_q) \leq (n - 1) - n + m = m - 1.$$

In both cases we showed $CM. def(R_q) \leq m - 1$. This completes the proof of Claim B).
Recall that CM.def($R$) = $m + 1 > m$. By Discussion 3.11 we have $0 < \dim(\text{Loc}(\text{CM.def} > m))$. In particular, there is $q \in \text{Spec}(R)$ of height $n - 1$ such that CM.def($R_q$) > $m$. This is a contradiction with Claim B). This contradiction shows that any ideal $\mathfrak{a}$ of height in the range $n - m \leq \text{ht}(\mathfrak{a}) \leq n - 1$ disregards the hypothesis of the lemma.

To see the particular case we remarked that $\text{grade}(\mathfrak{a}, R) \leq f_\mathfrak{a}(R) \leq \text{ht}(\mathfrak{a}) \leq \text{cd}(\mathfrak{a})$. By the first part, $\text{grade}(\mathfrak{a}, R) = \text{cd}(\mathfrak{a})$. From these $\text{grade}(\mathfrak{a}, R) = \text{cd}(\mathfrak{a}) = f_\mathfrak{a}(R)$. This is what we want to prove. □

One can not replace the finite length assumption with artinian in the above lemma: It is enough to look at a non Cohen-Macaulay invariant ring $R$ and recall that $H^n_m(R)$ is artinian for any $i$.

**Lemma 3.7.** Let $M$ be an artinian module and let $x \in M$. Then $M_x = 0$.

**Proof.** Suppose first that $M$ is of finite length an let $n = \ell(M) + 1$. Then $x^n M = 0$. From this $\frac{M}{x M} = 0$ for all $m \in M$. So, $M_x = 0$. In general, write $M \cong \varinjlim M_i$ where $M_i \subset M$ is finitely generated. Then $\ell(M_i) < \infty$. Recall that $M_x \cong \varinjlim M_i x_i$. Since $(M_i)_x = 0$, we get that $M_x = 0$. □

A Krull domain with torsion classical group is called almost factorial.

**Fact 3.8.** (Samuel) Let $G \rightarrow \text{GL}(n, \mathbb{F})$ be a representation of a finite group $G$. Denote the invariant ring by $R$. Then $R$ is almost factorial.

**Proof.** Recall that classical group of $R$ is a subgroup of $\text{Hom}(G, \mathbb{F})$. Let $f \in \text{Hom}(G, \mathbb{F})$ and $g \in G$. By definition, $f(g) = f(g^{\text{def}}) = f(1_G) = 1_{\mathbb{F}}$. Also, a noetherian normal domain is a Krull domain. So, $R$ is almost factorial. □

**Corollary 3.9.** Let $G \rightarrow \text{GL}(n, \mathbb{F})$ be a representation of a finite group $G$. Denote the invariant ring by $R$. Let $\mathfrak{a} \triangleleft_R R$ be such that $\ell(H^n_i(R)) < \infty$ for some $i$. Then $H^n_i(R) = 0$.

**Proof.** Recall from Corollary 3.6 that $4 = \dim R \geq \text{depth} R \geq 3$. First we deal with the case $\text{ht}(\mathfrak{a}) = 4$. Since local cohomology modules are invariant under taking radical we may assume $\mathfrak{a}$ is radical. Hence $\mathfrak{a} = m$. Recall that $H^4_m(R)$ is not finitely generated. Then, the only crucial $H^j_m(R)$ is $H^3_m(R)$. Suppose it is of finite length. From this we observe that $H^j_m(R)$ is of finite length for all $j < \dim R$. In view of Proposition 3.2 $H^n_m(R) = 0$ for all $j < \dim R$, as claimed.

Without loss of the generality we assume that $\text{ht}(\mathfrak{a}) < 4$. Recall that $R$ is a normal domain. Note that $m \in \text{Supp}(H^j_n(\mathfrak{a}(R)))$. Also, if a local ring of an algebraic variety is normal, then it is analytically irreducible. By Hartshorne-Lichtenbaum vanishing theorem we deduce that $\text{cd}(\mathfrak{a}, R) < 4$.

In this paragraph we deal with the case $\text{ht}(\mathfrak{a}) = 3$. Recall that $3 = \text{ht}(\mathfrak{a}) \leq \text{cd}(\mathfrak{a}, R) < 4$, i.e., $\text{ht}(\mathfrak{a}) = \text{cd}(\mathfrak{a}, R)$. If $\text{grade}(\mathfrak{a}, R) = 3$ then $i = 3$. This is not the case, because $H^3_m(R)$ is not finitely generated for $j = \text{cd}(\mathfrak{a}, R)$. If $\text{grade}(\mathfrak{a}, R) = 2$ then $i = 2$. This case excluded by Fact A) in Proposition 3.5. Then we may assume that $\text{grade}(\mathfrak{a}, R) = 1$. Let us again revisit Fact A) in Proposition 3.5. This allow us to assume that $i = 2$. We pick $x$ such that $m = \text{rad}(\mathfrak{a} + x R)$. There is the following long exact sequence of local cohomology modules (see 3 Proposition 8.1.2])

$$\cdots \rightarrow H^{j-1}_{\mathfrak{a}R}(R_x) \rightarrow H^j_m(R) \rightarrow H^j_{\mathfrak{a}R}(R) \rightarrow H^j_{\mathfrak{a}R_x}(R_x) \rightarrow \cdots .$$

Recall that $H^j_{\mathfrak{a}R_x}(R_x) \cong H^j_{\mathfrak{a}R}(R)_x$. In view of Lemma 3.7 $H^2_{\mathfrak{a}R}(R)_x = 0$. Hence $H^3_m(R) \rightarrow H^2_{\mathfrak{a}}(R) \rightarrow 0$. Recall that $\text{depth}(R) \geq 3$. By definition of depth, $H^3_m(R) = 0$. In view of $H^3_m(R) \rightarrow H^2_{\mathfrak{a}}(R) \rightarrow 0$ we get that $H^2_{\mathfrak{a}}(R) = 0$ as desired.
Next, we assume that ht(\(a\)) = 2. Recall that 2 = ht(\(a\)) ≤ cd(\(a, R\)) < 4. Note that \(H^2_a(R)\) is not finitely generated for \(j = cd(\(a, R\))\) and for \(j = ht(\(a\)). From this we conclude that \(i = 1\). Recall that grade(\(a, R\)) ≤ ht(\(a\)). If grade(\(a, R\)) = 2 the claim holds by definition of grade. Then we may assume that grade(\(a, R\)) = 1 = i. In particular, grade(\(a, R\)) = 1 < depth \(R\). This case excluded by Fact A) in Proposition 3.6.

Finally, we assume that ht(\(a\)) = 1. Recall from Fact 3.8 that \(R\) is almost factorial. This is well-known by a result of Stroch that any height one radical ideal over almost factorial is principal up to radical (see e.g. [12, Proposition 6.8]). Thus, \(H^1_a(R)\) = 0. It remains to note that \(H^2_a(R)\) is not finitely generated. □

Example 3.10. Let \(R := \mathbb{F}_2[x_1, \ldots, x_4]^{\mathbb{Z}/4\mathbb{Z}}\) via the assignments \(x_i \mapsto x_{i+1}\) for \(i < 4\) and \(x_4 \mapsto x_1\). Let \(a := (\sum x_1 x_3 + x_2 x_4, \sum_{i<j<k} x_i x_j x_k)\). Then \(\ell(H^2_a(R)) = \infty\). Also, \(\ell(H^3_m(R)) = \infty\).

Proof. Bertin proved that \(R\) is not Cohen-Macaulay by showing that the parameter sequence \(\sum x_i x_3 + x_2 x_4, \sum_{i<j<k} x_i x_j x_k\) is not regular sequence. From this we see that grade(\(a, R\)) = 2 = 3 = ht(\(a\)). In view of Corollary 3.4 \(\ell(H^2_a(R)) = \infty\). Recall from Corollary 3.9 that depth(\(R\)) = 3. Again, Corollary 3.9 implies that \(\ell(H^3_m(R)) = \infty\). □

It may be interesting to give an explicit chain of submodules of \(H^3_m(R)\) (resp. \(H^2_a(R)\)) with simple factors. We remark that "depth(\(-\)) = r" is not enough to deduce S(r). We use the Bertin’s example:

\(\text{depth}(\(R\)) = 3\) but \(\text{but } R \text{ is not } S(3)\).

4. THE LOCAL RESULTS

The reader may have to skip the next two items.

Remark 4.1. The main difference between local case and global case of invariant rings by a finite group is the following:

i) Nagata constructed a zero-dimensional noetherian \(k\)-algebra \(R\) and a finite group \(G\) of automorphisms of \(R\) such that \(R^G\) is non-noetherian (see [11, Introduction]).

ii) Here is a useful criterion: If the \(R\)-module of Kähler differentials for \(R/pR\) over \(k\) is finitely generated for all primes \(p\) dividing the order of \(G\), then \(R^G\) is noetherian (see [11, main result]).

Discussion 4.2. Here is a quick review of the notion of non-noetherian grade.

i) The classical grade of \(a\) on a module \(M\), denoted by c.grade(\(a, M\)), is the supremum lengths of all weak regular sequences on \(M\) contained in \(a\). In the case that \(a\) is finitely generated by generating set \(\mathcal{G} := x_1, \ldots, x_r\), the Čech grade of \(a\) on \(M\) is defined by

\[\text{Č. grade}_R(\(a, M\)) := \inf \{i \in \mathbb{N} \cup \{0\} | H^i_a(M) \neq 0\} .\]

For not necessarily finitely generated ideal \(a\) the Čech grade of \(a\) on \(M\) is defined

\[\text{Č. grade}_R(\(a, M\)) := \sup \{\text{Č. grade}_R(\(b, M\)) : b \in \Sigma\} ,\]

where \(\Sigma\) is the family of all finitely generated subideals \(b\) of \(a\). Recall that Č. grade\(_R(\(a, M\)) \geq\) grade\(_R(\(a, M\)). The notation Č. depth(\(R\)) stands for Č. grade\(_R(\(m, R\)) when (\(R, m\)) is a quasilocal ring.

ii) The Cohen-Macaulay property of non-noetherian invariant rings investigated in [3] and [8].
The following unifies (and extends) \[10\] Proposition 2 and \[19\] Proposition 2.

**Fact 4.3.** Let \((R, \mathfrak{m})\) be a local ring (resp. integral domain) and let \(G\) be a finite group acting on \(R\). Let \(a\) and \(b\) be in \(R^G\) be such that they \(R\)-sequence. Then \(a\) and \(b\) is an \(R^G\)-sequence. In particular, if \(\text{depth}(R) \geq 2\) then \(\mathcal{C}. \text{depth}(R^G) \geq 2\).

**Proof.** Clearly, \(a\) is regular over \(R^G\). Let \(r \in R^G\) be such that \(rb = ac\) for some \(c \in R^G\). Since \(a\) and \(b\) is an \(R\)-sequence, there is \(d \in R\) such that \(r = ad\). Then \(a(db - c) = 0\). By this, \(db = c\). Let \(g \in G\). Then \(g(d)b = c = db\), i.e., \((g(d) - d) = 0\). Recall that a permutation of regular sequences is regular (this needs both local and noetherian assumption). Since \(b\) is regular, \(g(d) = d\). Thus \(d \in R^G\). By definition, \(a\) and \(b\) is an \(R^G\)-sequence.

For the particular case, let \(x \in \mathfrak{m}\) be a regular element. Let \(n(x) := \prod_{g \in G} g(x)\). Clearly, \(n(x) \in R^G\). One has \(g(x)\) is \(R\)-regular. (If not then there is \(y\) such that \(g(x)y = 0\). Apply \(g^{-1}\) to this to see \(xg^{-1}(y) = 0\). Since \(x\) is regular, \(g^{-1}(y) = 0\). So \(y = 0\) and claim follows.) Since product of regular elements is regular, we see that \(n(x)\) is regular. Note that length of all maximal \(R\)-sequences are the same (this needs both local and noetherian assumption). There is \(y \in \mathfrak{m}\) which is regular over \(\overline{R} := R/\mathfrak{m}R\). Clearly, \(g(n(x)R) \subset n(x)R\). Thus \(G\) acts on \(\overline{R}\). Similarly, \(n(y)\) is regular over \(\overline{R}\). Set \(a := n(x)\) and \(b := n(y)\). By the first part, \(a\) and \(b\) is an \(R^G\)-sequence. By Fact \(2.4\) \(R^G\) is quasilocal. So, \(\mathcal{C}. \text{depth}(R^G) \geq 2\) as claimed. \(\square\)

**Proposition 4.4.** Let \((R, \mathfrak{m})\) be a 3-dimensional local ring and let \(G\) be a finite group acting on \(R\). If \(R\) is \(S(2)\) then \(R^G\) is \(S(2)\).

**Proof.** Since \(R\) is \(S(2)\), \(\text{depth}(R) \geq \min\{2, \dim(R)\}\), i.e., \(\dim(R) \leq \text{depth}(R) + 1\). This means that \(R\) is almost Cohen-Macaulay. In the light of the almost Cohen-Macaulay property we see that

\[
\text{grade}(P, R) = \text{depth}(PRP) \geq \min\{2, \dim(R)\} \quad \forall P \in \text{Spec}(R) \quad (\ast)
\]

Let \(\mathfrak{p}\) be any prime ideal in \(R^G\). It is well-known from \[7\] Page 324 that \(S^{-1}(R^G) = (S^{-1}R)^G\) for any multiplicative closed subset \(S\) of \(R^G\). Apply this for \(S := R^G \setminus \mathfrak{p}\). So, \((R^G)_\mathfrak{p} \cong (R_\mathfrak{p})_\mathfrak{p}\). Recall that the integral extension preserves Krull’s dimension (see \[20\] Ex. 9.2)). This shows that

\[
\dim R_\mathfrak{p} = \dim((R_\mathfrak{p})^G) = \dim((R^G)_\mathfrak{p}) = \text{ht}(\mathfrak{p}) \quad (+)
\]

Let \(P \in \text{Spec}(R)\) be such that \(\text{ht}(S^{-1}P) = \dim R_\mathfrak{p}\). Since \(P \cap S = \emptyset\) we get that \(P \cap R^G \subset \mathfrak{p}\). Recall that localization (resp. lying over) does not increase height (see \[20\] Ex. 9.8)). Thus,

\[
\text{ht}(\mathfrak{p}) \overset{(\ast)}{=} \dim R_\mathfrak{p} = \text{ht}(S^{-1}P) \leq \text{ht}(P) \leq \text{ht}(\mathfrak{p}).
\]

That is \(\text{ht}(\mathfrak{p}) = \text{ht}(P)\). Suppose first that \(\text{ht}(\mathfrak{p}) = 1\). So, \(\text{ht}(P) = 1\). We are going to use \((\ast)\). Let \(a\) be an \(R\)-regular in \(P\). Clearly, \(n(a)\) is an \(R^G\)-regular in \(P \cap R^G \subset \mathfrak{p}\). Suppose now that \(\text{ht}(\mathfrak{p}) > 1\). This implies that there is a regular sequence \(a, b\) in \(P\). We observed that \(n(a)\) is regular. Note that length of all maximal \(R\)-sequences in \(P\) is the same. There is \(y \in P\) which is regular over \(\overline{R} := R/\mathfrak{m}R\). Similarly, \(n(y) \in P \cap R^G \subset \mathfrak{p}\) is regular over \(\overline{R}\). In particular, there is an \(R\)-sequence of length two in \(P \cap R^G\). In view of the above fact we see there is a \(R^G\)-regular sequence of length two in \(\mathfrak{p}\). Regular sequence behave nicely with respect to localization. That is

\[
\mathcal{C}. \text{grade}(\mathfrak{p}R^G_\mathfrak{p}, R^G_\mathfrak{p}) \geq \mathcal{C}. \text{grade}(\mathfrak{p}, R^G) \geq \min\{2, \dim(R^G)_\mathfrak{p}\}.
\]
From this we see that $R^G$ is S(2).

\[\square\]

**Remark 4.5.** The 3-dimensional assumption were used to deduce that $R$ is almost Cohen-Macaulay. In particular, we have: Let $(R, \mathfrak{m})$ be an almost Cohen-Macaulay local ring and let $G$ be a finite group acting on $R$. If $R$ is S(2) then $R^G$ is S(2).

From now on we assume $(R, \mathfrak{m}, k)$ is a local ring of characteristic $p > 0$ and $G$ is a cyclic group of order $p^n$, $n > 0$, acting on $R$.

**Definition 4.6.** Following Fogarty, we say $G$ acts wildly on $R$ if the following three properties hold: i) $G$ acts trivially on $k$, ii) $G$ acts freely on $X := \text{Spec}(R)$, and iii) $R^G$ is noetherian and $R$ is a finite $R^G$-module.

The following extends and simplifies [8 Corollaire 2.4].

**Proposition 4.7.** Let $(R, \mathfrak{m})$ be a local domain and $G$ a finite group acting wildly on $R$.

i) If $R$ is S(2) then $R^G$ is S(2).

ii) Suppose $\text{depth}(R) > 1$ and $\text{dim} R > 2$. Then $R^G$ is not S(n) for all $n > 2$.

We prove the first item by the weaker assumption: the extension $R^G \to R$ is locally flat and the assumption given by Definition 4.6(iii).

**Proof.** By Fact 2.3 and Definition 4.6(iii) we see $(R^G, n)$ is local.

i) The case $\text{dim}(R) \leq 3$ is true without any condition, see Proposition 4.4. We assume that $\text{dim} R > 3$. Let $p$ be any prime ideal in $R^G$ of height bigger than 1. In view of Fact 2.3 we may and do assume that $p \neq n$. Recall that $(R^G)_p \cong (R_p)^G$. Since $G$ acts freely, the extension $R^G \to R$ is étale for all $q \neq n$. Conclude by this that $(R_p)^G \cong (R^G)_p \to R_p$ is flat. Going down holds for flat extensions (see [20 Theorem 9.5]). In view of [20 Ex 9.8] and [20 Ex 9.9] we have

$$\text{ht}(p) = \text{ht}(R^G)_p = \text{ht}(Q) \quad (+)$$

for any $Q \in \text{Spec}(R_p)$ lying over $p(R^G)_p \in \text{Spec}((R^G)_p)$.

Let $P \in \text{Spec}(R)$ be such that $S^{-1}P$ is the maximal ideal of $R_p$, where $S := R \setminus p$. Since $P \cap S = 0$ we get that $P \cap R^G \subset P$. Suppose on the contrary that $P \cap R^G \not\subset p$. By going-up [20 Theorem 9.4(i)] and incomparability property [20 Theorem 9.3(ii)], there is $P \subset Q \in \text{Spec}(R)$ such that $Q \cap R^G = p$. Then $S^{-1}P \subset S^{-1}Q$. Since $S^{-1}P$ is maximal we get to a contradiction. Hence $P \cap R^G = p$. We are going to apply [20 Ex. 9.3]: there are only a finite number of prime ideals lying over $p$. In particular, $R_p$ is semilocal. Let $\text{max}(R_p) := \{Q_1, \ldots, Q_n\}$. Since $R_p$ is an integral domain and semilocal we have $\text{Jac}(R_p) \neq 0$. Let $0 \neq x \in \text{Jac}(R_p)$. Suppose on the contradiction that $\text{Jac}(R_p) \subset \bigcup_{q \in \text{Ass}(R_p)} q$. In the light of prime avoidance $\bigcap_{i=1}^n Q_i = \text{Jac}(R_p) \subset q$ for some $q \in \text{Ass}(R_p)$. Since the intersection index is finite and $q$ is prime we have $Q_i \subset q$ for some $i$. Thus $Q_i = q$, because $Q_i$ is maximal. Suppose $Q_i = S^{-1}P_i$. Then, $(R_p/(x))Q_i \cong (R/(x))P_i$ is of zero depth. Therefore, $\text{depth}(R_{P_i}) = 1$. We denote this by (\dag). Recall that

$$\text{ht}(p) \overset{(\dag)}{=} \text{ht}(S^{-1}P_i) \leq \text{ht}(P_i) \leq \text{ht}(p).$$

That is $1 < \text{ht}(p) = \text{ht}(P_i)$. This contradicts the S(2) property of $R$, see (\dag). This contradiction says that $\text{Jac}(R_p) \not\subset \bigcup_{q \in \text{Ass}(R_p)} q$. Let $y \in \text{Jac}(R_p) \setminus \bigcup_{q \in \text{Ass}(R_p)} q$. We proved that $x$ and $y$ is an $R_p$-sequence in $\text{Jac}(R_p)$. 


Clearly, \( a := n(x) \) is regular. Note that length of all maximal \( R_p \)-sequences in the Jacobson radical are the same. There is \( z \in \text{Jac}(R_p) \) which is regular over \( R := R_p / aR_p \). Similarly, \( b := n(z) \in \text{Jac}(R_p) \cap (R_p)^G \) is regular over \( R_p \). In the light of Fact 4.3 we see there is an \( (R_p)^G \)-regular sequence of length two in \( \text{Jac}(R_p) \cap (R_p)^G \subset \mathfrak{p} \). From this we see that

\[
\text{grade}(p R_p^G, R_p^G) \geq \min \{ 2, \dim(R^G)_p \},
\]

and so \( R^G \) is \( S(2) \).

ii) Since \( \text{depth}(R) > 1 \) we have \( \text{depth}(R^G) \leq 2 \), see [10] Proposition 4. Let \( n > 2 \). Combine this with the assumption \( \dim R > 2 \) to see \( \text{depth}(R^G) < \min \{ \dim(R^G)_n, n \} \). By definition, \( R^G \) is not \( S(n) \).

**Discussion 4.8.** Let \( R \) be a generalized Cohen-Macaulay local ring of characteristic \( p > 0 \) and of dimension \( d > 1 \). We assume that \( R^G \) is noetherian and \( R \) is a finite \( R^G \)-module. Recall that \( R^G \) acts on \( A = H^0(\tilde{X}, \mathcal{O}_X) \) and \( H^i(G, A) \) is finitely generated as an \( R^G \)-module.

**Proof.** We look at \( 0 \to \frac{R}{I_m(R)} \to A \to H^1_m(R) \to 0 \) (+) to see \( A \) is finitely generated as an \( R \)-module. Recall that \( R \) is a finite \( R^G \)-module. From this \( A \) is finitely generated as an \( R^G \)-module. Thus, \( H^0(G, A) = A^G \) is finitely generated as an \( R^G \)-module. To show finiteness of higher group cohomology, we denote the elements of trace zero in \( A \) by \( E \). The set \( E \) is an \( R^G \)-submodule of \( A \). Indeed, recall that \( A = H^0(\tilde{X}, \mathcal{O}_X) = \lim \limits_{\to} \text{Hom}_R(m^n, R) \). Let \( \mathfrak{g} \in G \) and \( n \in \mathbb{N} \). Take \( f : m^n \to R \) and define \( g.f : m^n \to R \) by the role \( (g.f)(x) := g(f(g^{-1}_n(x))) \) where \( x \in m^n \). This defines the action of \( G \) on \( \lim \limits_{\to} \text{Hom}_R(m^n, R) \). Let \( r \in R^G \) and \( f \in E \). We need to show \( rf \in E \), i.e., \( tr(rf) = 0 \). To this end, recall that

\[
g.(rf)(x) = g\left((rf)(g^{-1}_n(x))\right) = g\left((rf)(g^{-1}_{n-1}(x))\right) = g(r)g\left(f(g^{-1}_n(x))\right) = rg\left(f(g^{-1}_n(x))\right) = rg.(f(x)).
\]

We showed \( g.(rf) = r(g.f) \). We denote this property by (*). Recall that \( tr(f) = 0 \). By definition,

\[
tr(rf) := \sum_{g \in G} g.(rf) = \sum_{g \in G} r(g.f) = r \sum_{g \in G} g.f = r \cdot tr(f) = 0.
\]

The set \( F := (g.r - r : r \in A) \) is an \( R^G \)-submodule of \( A \). Indeed, let \( a \in R^G \) and let \( (g.f) - f \in F \) for some \( f \in A \). First,

\[
a(g.f)(x) = ag.f(g^{-1}(x)) = g(a)g(f(g^{-1}(x))) = g.(a.f)(g^{-1}(x)).
\]

From this

\[
(a(g.f) - f)(x) = g\left((a.f)(g^{-1}(x))\right) - (a.f)(x) = g.(a.f)(g^{-1}(x)).
\]

That is \( a(g.f) - f = g.(a.f) - a.f \in F \). Keep in mind that \( G \) is cyclic. Recall that (see e.g. [22] page 178):}

\[
H^i(G, A) = \begin{cases} 
A^G & \text{if } i = 0 \\
\frac{\ker(tr.A \to A^G)}{(g(a) - a)} & \text{if } i \in 2\mathbb{N} + 1 \\
\frac{A^G}{tr(A)} & \text{if } i \in 2\mathbb{N}
\end{cases}
\]

Since \( H^{2n+1}(G, A) \simeq E/F, H^{2n+1}(G, A) \) is finitely generated as an \( R^G \)-module.

The set \( tr(A) \) is an \( R^G \)-submodule of \( A^G \). Indeed, let \( r \in R^G \) and \( f \in tr(A) \). Let \( F \in A \) be such that \( f = tr(F) \). In view of (*) we have \( g.(rf) = r(g.f) \). By definition,

\[
tr(rf) := \sum_{g \in G} g.(rf) = \sum_{g \in G} r(g.f) = r \sum_{g \in G} g.f = r \cdot tr(F) = rf.
\]
So, \( r f \in \text{im}(\text{tr} : A \to A^G) \), as claimed. From this we observe that \( H^{2n}(G, A) = \frac{A^G}{(n)} \) is finitely generated as an \( R^G \)-module.

\[ \square \]

**Discussion 4.9.** Adopt the assumption of Discussion 4.8 and let \( L_i := H^i_m(R) \). Then \( L_i \) is a \( G \)-module and \( H^i(G, L_i) \) is finitely generated as an \( R^G \)-module for all \( i < d \).

**Proof.** The assumptions implies that \( (R^G, n) \) is local and that \( \text{rad}(nR) = m \). Let \( x = x_1, \ldots, x_d \) be a system of parameter in \( n \). This shows that \( \text{rad}(xR) = m \). By definition, \( H^i_m(R) = H^i_m(R) \). Let \( g : R \to R \) be an element of \( G \) and \( y \in R^G \). The assignment \( r/y^n \mapsto g(r)/y^n \) induces an \( R^G \)-algebra automorphism which we denote it again by \( g : R \to R_y \). This induces an \( R^G \)-isomorphism of the Čech complexes \( g : C^\bullet(x, R) \to C^\bullet(x, R) \). Conclude by this that there is an \( R^G \)-isomorphisms \( g : H^i(C^\bullet(x, R)) \to H^i(C^\bullet(x, R)) \), and so there is an \( R^G \)-isomorphisms \( g : L_i \to L_i \). Recall that:

\[
H^i(G, L_i) = \begin{cases} 
H^i_m(R)^G & \text{if } j = 0 \\
\frac{\ker(\text{tr} : L_i \to L^G)}{(g(r) - r)(\ell) \in L_i)} & \text{if } j \in 2N + 1 \\
L^G_i / \text{tr}(L_i) & \text{if } j \in 2N.
\end{cases}
\]

Let \( \ell \in L_i \) and \( r \in R \). Since \( g(r\ell) = g(r)g(\ell) \) we see that \( \text{tr}(L_i) \) is an \( R^G \)-submodule of \( L^G_i \). Similarly, \( \ker(\text{tr} : L_i \to L^G_i) \) and \( (g(\ell) - \ell) \in L_i \) are \( R^G \)-submodule of \( L^G_i \). Since \( R \) is generalized Cohen-Macaulay, \( L_i \) is of finite length as an \( R \)-module for all \( i < d \). Thus \( L_i \) is finite as an \( R^G \)-module for all \( i < d \). From this \( L^G_i / \text{tr}(L_i) \) is finitely generated as an \( R^G \)-module. Similarly, \( \frac{\ker(\text{tr} : L_i \to L^G_i)}{(g(a) - a)} \) is finitely generated as an \( R^G \)-module.

The origin source of the next result is [13] by Grothendieck. There are a lot of works motivated by this. For our’s propose, Fogarty’s exposition is useful. If the reader is not family with this technology we suggest to look at the friendly approach by Larry Smith [23].

**Fact 4.10.** (Grothendieck, Ellingsrud-Skjelbred, Fogarty, L. Smith, etcetera) Let \( (R, m, k) \) be a local ring of characteristic \( p > 0 \) and \( G \) is a cyclic group of order \( p^n \), \( n > 0 \), acting wildly on \( R \). By \( \hat{\text{−}} \) we mean the punctured spectrum. In view of [10] Proposition 4 there is spectral sequence

\[
H^q \left( G, H^p(\text{Spec}(\hat{R}), \mathcal{O}_{\text{Spec}(\hat{R})}) \right) \Rightarrow H^{p+q}(\hat{Y}, \mathcal{O}_{\text{Spec}(R^G)}).
\]

**Proposition 4.11.** Let \( (R, m, k) \) be a \( d \)-dimensional local ring of prime characteristic \( p \). Let \( G \) be a cyclic group of order \( p^n \), \( n > 0 \), acting wildly on \( R \). The following assertions hold:

a) If \( R \) is generalized Cohen-Macaulay, then \( R^G \) is generalized Cohen-Macaulay.

b) If \( \text{depth}(R) > 1 \), then \( \text{ht}(\text{im}(R \to R^G)) = \text{dim} R \). Also, \( \text{im}(\text{tr}) = m \) provided \( (R^G, n) \) is quasi-Buchsbaum and \( \text{depth}(R) > 2 \).

**Proof.** The assumptions implies that \( (R^G, n) \) is local. Let \( Y := \text{Spec}(R^G) \), \( X := \text{Spec}(R) \) and recall that \( \hat{\cdot} \) stands for the punctured spectrum. In the light of Fact 4.10 we see \( H^q(G, H^p(\hat{X}, \mathcal{O}_X)) \Rightarrow H^{p+q}(\hat{Y}, \mathcal{O}_Y) \).

1) By definition \( H^i_m(R) \) is finitely generated for all \( i < \text{dim} R \). We look at the exact sequence of finitely generated \( R^G \)-modules \( 0 \to R^G \to R \to R/R^G \to 0 \). This induces the following exact sequence:

\[
0 \to H^0_m(R^G) \to H^0_m(R) \to H^0_m(R/R^G) \to H^1_m(R^G) \to H^1_m(R).
\]

By independence theorem, \( H^0_m(R) \cong H^0_m(R) \) as an \( R^G \)-module. Also, \( H^0_m(R) \cong H^0_m(R) \), because \( \text{rad}(nR) = m \). Since \( H^0_m(R) \) is finitely generated as an \( R \)-module and that \( R \) is a finite \( R^G \)-module
we have $H^0_m(R)$ is finitely generated as an $R^G$-module. From $0 \to H^0_n(R^G) \to H^0_m(R)$ we see $H^0_n(R^G)$ is finitely generated. We may assume $d > 1$. In view of Discussion 4.8 and Discussion 4.9 cohomology groups $H^0(G, H^p(\tilde{X}, \mathcal{O}_X))$ are finitely generated as an $R^G$-module for all $p < d - 1$ and all $q$, because $H^p(\tilde{X}, \mathcal{O}_X) \simeq H^{n+1}_m(R)$ if $p > 0$. For $p + q < d - 1$ the $E_{q,p}^r$ is finitely generated as an $R^G$-module. The same thing holds for all of its sub-quotients. Thus, $E_{\infty,p}^r$ is finitely generated as an $R^G$-module. By definition, there is a chain

$$0 = H^{n+1} \subseteq H^n \subseteq \cdots \subseteq H^0 := H^{p+q}(\tilde{Y}, \mathcal{O}_Y)$$

such that $H^i / H^{i+1} \cong E_{\infty,i}$ for all $i = 0, \ldots, n$. It follows that $H^{p+q}(\tilde{Y}, \mathcal{O}_Y)$ is finitely generated as an $R^G$-module provided $p + q < d - 1$. Thus, $H^i_n(R^G)$ is finitely generated as an $R^G$-module for all $i < d$.

b) Since $\text{depth}(R) > 1$ we have $H^0_n(R^G) = H^1_m(R) = 0$. From this $H^0(\tilde{X}, \mathcal{O}_X) \simeq R$ (see (+) in Discussion 4.8). We use the five-term exact sequence of the spectral sequence $H^0(G, R) \Rightarrow H^{p+q}(\tilde{Y}, \mathcal{O}_Y)$:

$$0 \to H^1(\tilde{Y}, \mathcal{O}_Y) \to H^1(\tilde{X}, \mathcal{O}_X)^G \to H^2(\tilde{Y}, \mathcal{O}_Y) \to H^2(\tilde{Y}, \mathcal{O}_Y).$$

Recall that $H^1_n(R^G) = H^2(\tilde{Y}, \mathcal{O}_Y)$ is artinian. Also $H^1(\tilde{X}, \mathcal{O}_X) = H^2_m(R)$ is artinian. Conclude by this that $H^2(G, R)$ is artinian. Since $H^2(G, R) = \frac{G}{\text{im}(tr)}$ we get that

$$\text{ht}(\text{im}(R \to R^G)) = \dim R.$$

Suppose $R^G$ is quasi-Buchsbaum and $\text{depth}(R) > 2$. Then $H^1(\tilde{X}, \mathcal{O}_X)^G = 0$. In view of Definition 4.8 i), Fogarty remarked that $1 \notin \text{im}(tr)$. This implies that $H^2(G, R) \neq 0$. We conclude from the quasi-Buchsbaum property that $\mathfrak{n} H^2(\tilde{Y}, \mathcal{O}_Y) = \mathfrak{n} H^2_m(R^G) = 0$. Therefore, its nonzero submodule $H^2(G, R) = \frac{G}{\text{im}(tr)}$ is simple. From this we get that $\mathfrak{n} = \text{im}(R \to R^G)$, as claimed. □

Corollary 4.12. Let $(R, m, k)$ be a 3-dimensional Cohen-Macaulay local ring of characteristic 2. Suppose $G$ is of order two and acts wildly on $R$. Then $(R^G, \mathfrak{n})$ is Buchsbaum if and only if $\text{im}(tr) = \mathfrak{n}$.

Proof. By Fact 4.3 $\text{depth}(R^G) \geq 2$. Let $r \in \text{ker}(tr : R \to R^G)$. Then $0 = \text{tr}(r) = \sum_{h \in G} h(a) = r + g(r)$ where $g$ is a generator of $G$. Thus, $g(r) = -r = r$ because $\text{char}(R) = 2$. We conclude that $\text{ker}(tr) = R^G$. Also,

$$(g(r) - r)_{r \in R} = (g(r) + r)_{r \in R} = \text{im}(tr).$$

Recall from Proposition 4.11 that

$$H^2_n(R^G) \simeq H^1(G, R) \simeq \frac{\text{ker}(tr : R \to R^G)}{(g(r) - r)} \simeq \frac{R^G}{\text{im}(tr)}.$$  

Suppose first that $R^G$ is Buchsbaum. In particular, $\mathfrak{n} H^2_n(R^G) = 0$. From this $\text{im}(tr) = \mathfrak{n}$. Conversely, assume $\text{im}(tr) = \mathfrak{n}$. Then $\mathfrak{n} H^2_i(R^G) = 0$ for all $i < \text{dim } R^G$. By definition, $R^G$ is quasi-Buchsbaum. Recall that $H^2_n(R^G) = 0$ for all $i \neq \text{dim } R^G$ and $i \neq \text{depth}(R^G)$. In view of Fact 2.2 we see $R^G$ is Buchsbaum. □

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