Monogamy deficit for quantum correlations in multipartite quantum system

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(Dated: February 20, 2013)

We introduce the concept of monogamy deficit for quantum correlation by combining together two types of monogamy inequalities depending on different measurement sides. For tripartite pure state, we demonstrate a relation which connects two types of monogamy inequalities for quantum discord. By using this relation, we get an unified physical interpretation for these two monogamy deficit. In addition, we find an interesting fact that there is a general monogamy condition for several quantum correlations for tripartite pure states. We then provide a necessary and sufficient condition for the establishment of one kind of monogamy inequality for tripartite mixed state and generalize it to multipartite quantum state.

PACS numbers: 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum correlations, such as entanglement and quantum discord, are assumed to be resources in quantum information processing and are different from classical correlations. On the other hand, in general, entanglement and discord are different from each other. Previous studies focus on entanglement which is a special quantum correlation enabling fascinating quantum information tasks such as superdense coding [1], teleportation [2], quantum cryptography [3], remote-state preparation [4] and so on. However, some quantum applications superior than their classical counterparts are found with vanishing or negligible entanglement [5–7]. In this sense, entanglement seems not capture all the quantum features of quantum correlations. So other measures of quantum correlations are proposed. Among those measures that in general go beyond entanglement, quantum discord is a widely accepted one in recent years [8–11]. The analytic results of quantum discord and its physical meaning are studied extensively, for example, in Refs. [7, 11–14]. The experiments about quantum discord are also implemented [15, 16]. Quantum discord can also be generalized to the multipartite situation [17, 20]. For more results, see a recent review paper [10].

There are many fundamental differences between classical correlation and quantum correlations. One of them is the shareability of correlation among many parties. Generally speaking, classical correlation can be freely shared among many parties, while quantum ones do not have this property. For example, for tripartite pure state, if two parties are highly entangled, they cannot have a large amount of entanglement shared with a third one. The limits on the shareability of quantum correlations are described by monogamy inequalities. Much progresses have already been made about the monogamy properties of various quantum correlations [22–26]. As one application, the monogamy property of quantum correlations also play a fundamental role for the security of the quantum key distributions [27, 37]. Some known monogamy properties of entanglement measure are, for example, concurrence and squashed entanglement [26, 31, 34, 35].

However, the monogamy relation does not always be satisfied by the quantum correlations. So it is necessary to know when a specified quantum correlation can satisfy this property. Concerning about quantum discord, in general, it does not satisfy this nature [36]. However, it may have some interesting applications in case the monogamy condition is satisfied [28, 29]. We should note that there are two types of monogamy inequalities for quantum discord since it is asymmetric depending on the measurement side for a bipartite state [33]. A necessary and sufficient condition for one type of monogamy relation satisfying is given where only one side of measurement is studied [36]. A natural question is then that does there exist an analogous property for another class of monogamy with measurement taken on a different side? In this paper, we first provide a relation between those two types of monogamy conditions for tripartite pure state. Then we give a necessary and sufficient condition for the holding of the second type of monogamy relation and further generalize the result to the N partite system. In particular, those two types of monogamy relations are generally studied independently. Our result that two monogamy inequalities can be combined together by introducing monogamy deficit provides a new, in general, more complete viewpoint. This can enlighten much research both on quantum correlation and monogamy property.

II. THE MONOGAMY DEFICIT FOR PURE STATE

A. The connection of two types of monogamy deficit

Quantum discord is defined as the difference between mutual information, which is accepted to be the total correlation, and maximum classical mutual information [8, 9]

\[
D^+(\rho_{AB}) = I(\rho_{AB}) - I^+(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) + \min_{\rho_B} \sum_i p_i S(\rho_{Bi}),
\]

(1)
where arrow “→” means measurement on ‘A’ and “←” means measurement on ‘B’, M\(_i\) represent POVM measurement performed on A for a bipartite state \(\rho_{AB}\). So quantum discord is considered describing the quantumness of correlations. In this paper, we mainly use the same notations as those in Ref. [36]. We use the left arrow (“←”) and the right arrow (“→”) to distinguish the side of the measurement. Also we have notations, \(\tilde{I}(\rho_{AB}) = S(\rho_A) - S(\rho_{AB})\), and \(\tilde{I}(\rho_{AB}) = S(\rho_{BC}) - S(\rho_B)\), here \(S(\sigma) = -\text{tr}(\sigma \log_2 \sigma)\) is the von Neumann entropy of a density matrix \(\sigma\). By those definitions presented above, we know that quantum discord in general should be asymmetric and depend on the measurement side, which can be either A or B. It is understandable that those two definitions possess different fundamental properties.

Recently, two kinds of monogamy inequalities have been studied in Refs. [32, 36] and [33]. For a tripartite state \(\rho_{ABC}\), by combining two monogamy inequalities together, we define two kinds of monogamy deficit of quantum discord, \(\Delta_{DA}^\rightarrow = D^\rightarrow(\rho_{ABC}) - D^\rightarrow(\rho_{AB}) - D^\rightarrow(\rho_{AC})\), \(\Delta_{DA}^\leftarrow = D^\leftarrow(\rho_{ABC}) - D^\leftarrow(\rho_{AB}) - D^\leftarrow(\rho_{AC})\). For simplicity, denote \(\Delta_{DA}^\rightarrow\) the first term \(D^\rightarrow(\rho_{ABC})\) involves a positive operator valued measurement (POVM) performed on B and C, and the other involve measurements only on A.

Because of the asymmetry of quantum discord, the above two monogamy deficit are apparently quite different. In this paper, however, we find that there is a relation between them, which means that the monogamy relations on one of them provide some limits on another. We first have a following observation. For two kinds of monogamy deficit \(\Delta_{DA}^\rightarrow, \Delta_{DA}^\leftarrow\) of an arbitrary tripartite pure state \(\rho_{ABC}\), we find,

\[
\Delta_{DA}^\rightarrow = \frac{1}{2}(\Delta_{DA}^\rightarrow + \Delta_{DA}^\leftarrow), \\
\Delta_{DA}^\leftarrow = \Delta_{DA}^\rightarrow + \Delta_{DA}^\leftarrow - \Delta_{DA}^\rightarrow. 
\]

The proof of those two relations can be the following. For simplicity, denote \(S(\rho_{BA})\) as the optimal conditional entropy of \(I^\rightarrow(\rho_{BA})\) after the measurement which defined as \(\min_{p_{\{\lambda\}}} \sum_{\lambda} p_{\lambda} S(\rho_{AB})\), that is \(I^\rightarrow(\rho_{BA}) = S(\rho_B) - \min_{p_{\{\lambda\}}} \sum_{\lambda} p_{\lambda} S(\rho_{AB}) = S(\rho_B) - S(\rho_{AB})\). Using the Koashi-Winter formula [34], we have \(S(\rho_{BA}) = E(\rho_{BC})\), where \(E(\rho_{BC})\) means the entanglement of formation for a bipartite state \(\rho_{BC}\). Generally, for any tripartite pure state \(|\Psi\rangle_{A'A''A'''}\), we have \(S(\rho_{A'A''A'''}) = E(\rho_{A'A''A'''}), \) where \(A', A'', A'''\) correspond to any permutations of A, B, C. Further more, we find,

\[
D^\rightarrow(\rho_{A'A''}) = \tilde{I}(\rho_{A'A''}) - (S(\rho_A) - S(\rho_{A'A''})) = S(\rho_A) - S(\rho_{A''}) + E(\rho_{A'A''}), \\
D^\leftarrow(\rho_{A'A''}) = \tilde{I}(\rho_{A'A''}) - (S(\rho_{A''}) - S(\rho_{A''}')) = S(\rho_{A''}) - S(\rho_{A'}) + E(\rho_{A'A''}).
\]

Inserting (4) into (2,3), we have that

\[
\Delta_{DA}^\rightarrow = S(\rho_A) - E(\rho_{AC}) - E(\rho_{AB}), \\
\Delta_{DA}^\leftarrow = S(\rho_B) + S(\rho_C) - S(\rho_A) - 2E(\rho_{BC}).
\]

Similarly, \(\Delta_{DB}^\rightarrow, \Delta_{DB}^\leftarrow\) and \(\Delta_{DC}^\rightarrow, \Delta_{DC}^\leftarrow\) can be obtained by permutating the indices of (2 and (3). Combining these results, we have (4, 5), which completes the proof.

The above relations are interesting. They tell us that the two kinds of monogamy inequalities which was studied previously [32, 33, 36] actually are not independent. We find that \(\Delta_{DA}^\rightarrow\), which is the defined monogamy deficit having a coherent measurement taken on two parties B and C, is precisely equal to the arithmetic mean of \(\Delta_{DA}^\rightarrow\) in which the measurements are only performed individually on B and C. To be explicit, the measurement for left hand side (l.h.s.) of equality (4) is a coherent measurement on “BC” while on right hand side (r.h.s.), local measurements on “B” and “C” are performed. In Ref. [32], a transition from satisfying the monogamy inequality to violation of monogamy inequality is given, where the positive or negative of \(\Delta_{DA}^\rightarrow\) are studied. By the definition of monogamy deficit \(\Delta_{DA}^\rightarrow = D^\rightarrow(\rho_{ABC}) - D^\rightarrow(\rho_{AB}) - D^\rightarrow(\rho_{AC})\), it is apparent that a coherent measurement on “BC” is necessary. Here our result (4) shows that instead of a coherent measurement, local measurements individually on “B” and “C” can be performed to find this conclusion. We remark that local operation is much easier to be implemented than coherent measurement. Those results reveal the hidden relationship in monogamy deficit for quantum discord where the coherent measurement is replaced by local measurements.

As an application of the above relation, for tripartite pure states, we provide an unified physical significance for these two monogamy deficit. To see this, we first consider the equivalent expression of the second monogamy deficit. According to the results in [36] and [33], we have \(I^\rightarrow(\rho_{BA}) - D^\rightarrow(\rho_{BA}) = I^\rightarrow(\rho_{BC}) - D^\rightarrow(\rho_{BC}) = \tilde{I}(\rho_{AC}) - 2E(\rho_{AC})\) and \(\tilde{I}(\rho_{AC}) - 2E(\rho_{AC})\). Combing the two equations, we have that

\[
\Delta_{DA}^\rightarrow = I^\rightarrow(\rho_{BA}) - D^\rightarrow(\rho_{BA}) = I^\rightarrow(\rho_{BC}) - D^\rightarrow(\rho_{BC}).
\]

Where \(I^\rightarrow(\rho_{BA})\) represents the classical correlation, \(D^\rightarrow(\rho_{BA})\) represents quantum discord. By exchanging the subscript, we have \(\Delta_{DA}^\leftarrow = I^\rightarrow(\rho_{CA}) - D^\rightarrow(\rho_{CA}) = I^\rightarrow(\rho_{CB}) - D^\rightarrow(\rho_{CB})\). It tells us that this monogamy deficit is equivalent to the difference between classical correlation and quantum correlation. Since the classical correlation can be regarded as locally accessible mutual information (LAMI) [39], while the quantum correlation can be seen as locally inaccessible mutual information (LIMI). In this sense, this monogamy deficit tells us that how much mutual information can be extracted from a tripartite pure state by using local measurement on one party. To be more explicit, the monogamy inequality holds if and only if more than half of the mutual information between AC or BC can be accessed through local measurement performed on C.

Now we can give a similar equivalent expression of \(\Delta_{DA}^\leftarrow\). In order to achieve this purpose, we first define the average of classical correlation and discord. As we all know, the discord and classical correlation are asymmetry quantities. By using the asymmetry of \(I^\rightarrow(\rho_{BA})\) and \(I^\rightarrow(\rho_{AB})\), we define the average of classical correlation \(\bar{I}_{\{A\}(I)}\) and the average discord
\[ \tilde{\omega}_{AB}(d) \] [39],
\[ \tilde{\omega}_{AB}^+(d) = \frac{1}{2}(I^-(\rho_{BA}) + I^-(\rho_{AB})), \]
\[ \tilde{\omega}_{AB}^+(d) = \frac{1}{2}(D^-(\rho_{BA}) + D^-(\rho_{AB})). \]

From this definition, combing Eq. (4) and (5), we have
\[ \Delta^{-}_{B} = \frac{1}{2}(\Delta^{-}_{A} + \Delta^{-}_{B}) \]
\[ = \frac{1}{2}(I^-(\rho_{AB}) - D^-(\rho_{AB}) + I^-(\rho_{BA}) - D^-(\rho_{BA})) \]
\[ = \tilde{\omega}_{AB}^-(d) - \tilde{\omega}_{AB}^+(d). \] (9)

This formula means that the monogamy deficit which needs a coherent measurement performed on two parties A and B is equivalent to \( \tilde{\omega}_{AB}^-(d) - \tilde{\omega}_{AB}^+(d) \), which is the difference between the average of classical correlations and quantum correlations where local measurements are made individually on A and B. In other words, according to previous view, this monogamy deficit represents our ability to extract the mutual information by performing local measurements on the two parties. According to the above definition, simply we have the relation, \( \tilde{\omega}_{AB}^+(d) + \tilde{\omega}_{AB}^-(d) = I(\rho_{AB}) \). Which means that the monogamy inequality holds if and only if we can acquire at least half of the mutual information through local measurements in the average sense.

The Eq. (9) presented above can also be used as a criterion to check whether a given tripartite pure state belongs to GHZ class or W class state under stochastic local operations and classical communication (SLOCC). According to the results in Ref. [34], we have that a tripartite pure state belongs to GHZ class state if and only if \( \Delta^{-}_{C} \geq 0 \), and if \( \tilde{\omega}_{AB}^-(d) \) is less than LAMI in the average sense when local measurements are performed on A and B. While a tripartite pure state belongs to W class state if and only if LAMI is less than LIMI in the average sense when local measurements are performed on A and B. In addition, a tripartite pure state belongs to GHZ class or W class depends on whether one can acquire no less than half of the mutual information through local measurements in the average sense.

As a short summary for previous discussion, we provide an unified view for two kinds of monogamy inequalities. That is, both of the monogamy inequalities hold only if one can extract more than half of mutual information by using local measurement.

### B. Monogamy deficit for discord and other quantum correlations

The squashed entanglement is an entanglement monotone for bipartite quantum states introduced by Christandl and Winter [31]. For bipartite state \( \rho_{AB} \), the squashed entanglement is given by
\[ E_{sq}(\rho_{AB}) = \frac{1}{2} \text{inf} \tilde{I}(\rho_{A:BC}). \]

Here \( \tilde{I}(\rho_{A:BC}) \) is the conditional mutual information of \( \rho_{ABC} \) with respect to particle C (see Eq. (17)), \( \rho_{ABC} \) is the extension of \( \rho_{AB} \) and the infimum is taken over all extensions of \( \rho_{AB} \) such that \( \rho_{AB} = T_{C} \rho_{ABC} \). The squashed entanglement has many important properties [34]. For example, (1) The squashed entanglement is upper bounded by entanglement of formation. (2) For any tripartite state \( \rho_{ABC} \), we have \( E_{sq}(\rho_{ABC}) + E_{sq}(\rho_{AC}) \leq E_{sq}(\rho_{ABC}) \).

By using the property (2), we can generalize the concept of monogamy deficit for squashed entanglement. We define the monogamy deficit for squashed entanglement as
\[ \Delta(E_{sq}) = E_{sq}(\rho_{C:AB}) - E_{sq}(\rho_{CA}) - E_{sq}(\rho_{CB}). \] (10)

Similarly, one can also define the monogamy deficit \( \Delta_{E_{sq}} = E(\rho_{C:AB}) - E(\rho_{CA}) - E(\rho_{CB}) \) for the entanglement of formation.

For the monogamy deficit for squared entanglement, we can have the following result. For any tripartite pure state \( \rho_{ABC} \), we may observe,
\[ \Delta(E_{sq}) \geq \max \{ \Delta^{-}_{C}, 0 \} = \max \{ \tilde{\omega}_{AB}^-(d) - \tilde{\omega}_{AB}^+(d), 0 \}. \] (11)

The proof of this relation is presented below. By the above property (2) of monogamy deficit for squared entanglement, we have \( \Delta(E_{sq}) \geq 0 \), and the second equality is given by Eq. (9). We now only need to prove \( \Delta(E_{sq}) \geq \Delta_{E_{sq}} \) and \( \Delta_{E_{sq}} = \Delta^{-}_{D} \). For tripartite pure state, \( E_{sq}(\rho_{C:AB}) = E(\rho_{C:AB}) \), thus we have
\[ \Delta(E_{sq}) = E_{sq}(\rho_{C:AB}) - E_{sq}(\rho_{CA}) - E_{sq}(\rho_{CB}) \]
\[ = E(\rho_{C:AB}) - E(\rho_{CA}) - E(\rho_{CB}) \]
\[ \geq E(\rho_{C:AB}) - E(\rho_{CA}) - E(\rho_{CB}) \]
\[ = \Delta(E_{sq}). \]

For tripartite pure state, we have \( E(\rho_{C:AB}) = D^{-}(\rho_{C:AB}) = S(\rho_{CA}) + E(\rho_{CB}) \) in Ref. [30], thus we have \( \Delta_{E_{sq}} = \Delta^{-}_{D} \). That is \( \Delta(E_{sq}) \geq \Delta_{E_{sq}} = \Delta^{-}_{D} \). Now we know that Eq. (11) is true.

The quantum work deficit is an important information-theoretic measure of quantum correlation introduced by Oppenheim et al. [41]. For an arbitrary bipartite state \( \rho_{AB} \), the quantum work-deficit is defined as
\[ \Delta(\rho_{AB}) = I_{S}(\rho_{AB}) - I(\rho_{AB}). \] (12)

where \( I_{S}(\rho_{AB}) \) represents the thermodynamic “work” that can be extracted from \( \rho_{AB} \) by “closed local operations”, \( I_{S}(\rho_{AB}) \) represents the thermodynamic “work” that can be extracted from \( \rho_{AB} \) by closed local operation and classical communication (CLOCC) [28]. Further more, the one side work deficit \( \Delta^{-}(\rho_{AB}) (\Delta^{-}(\rho_{AB})) \) means that CLOCC is restricted on projection measurements at one particle A (B). According to Ref. [28], the one side work deficit is lower bounded by quantum discord, that is
\[ D^{-}(\rho_{AB}) \leq \Delta^{-}(\rho_{AB}), \]
\[ D^-(\rho_{AB}) \leq \Delta^{-}(\rho_{AB}). \]

Similar to quantum discord and squashed entanglement, we provide the definition of the monogamy deficit for work
The two kinds of monogamy deficit for work deficit are given as,
\[
\Delta_{\Delta_A} = \Delta_{\Delta}(\rho_{AB}) - \Delta_{\Delta}(\rho_{AB}) - \Delta_{\Delta}(\rho_{AC}).
\]
(13)
\[
\Delta_{\Delta_{\Delta}} = \Delta_{\Delta}(\rho_{ABC}) - \Delta_{\Delta}(\rho_{AB}) - \Delta_{\Delta}(\rho_{AC}).
\]
(14)
The first definition involves a POVM coherently performed on \( B \) and \( C \) together, and the other involves measurements only on \( A \).

For the monogamy deficit for work deficit, we present the following observations. For any tripartite pure state \( \rho_{ABC} \), we have that
\[
\Delta_{\Delta_{\Delta}} \leq \Delta_{\Delta_{D_{\Delta}}} \leq \Delta_{(E_{\text{meas}})}.
\]
(15)
\[
\Delta_{\Delta_{\Delta}} \leq \Delta_{\Delta_{D_{\Delta}}}.
\]
(16)
The correctness of these observations are presented below. The inequality \( \Delta_{\Delta_{\Delta}} \leq \Delta_{(E_{\text{meas}})} \) is from (11). For tripartite pure state \( \rho_{ABC} \), we have \( \Delta_{\Delta}(\rho_{ABC}) = D_{\Delta}(\rho_{ABC}) = S(\rho_A) \), \( D_{\Delta}(\rho_{AB}) \leq \Delta_{\Delta}(\rho_{AB}) \), \( D_{\Delta}(\rho_{AC}) \leq \Delta_{\Delta}(\rho_{AC}) \) (24). Which implies that \( \Delta_{\Delta_{\Delta}} \leq \Delta_{\Delta_{D_{\Delta}}} \). Similarly, we can prove (16).

The physical interpretation of Eq. (15) can be like the following. The monogamy property for work deficit implies the monogamy property for quantum discord, entanglement of formation and squashed entanglement for any pure state. In this case, we have \( \Delta_{\Delta_{\Delta}} \leq \Delta_{S_{BC}(i)} \). In addition, we can extract more than half of the mutual information between BC in the average sense through local measurements. Combing (8) and (16), we have \( \Delta_{\Delta_{\Delta}} \geq 0 \) which implies one can extract at least half of the mutual information between \( AB \) or \( AC \) by using local measurement of \( A \).

III. NECESSARY AND SUFFICIENT CRITERIA FOR NON-NEGATIVE MONOGAMY DEFICIT \( \Delta_{\Delta_{\Delta}} \)

As is shown in (36), a necessary and sufficient condition for discord to be monogamous is \( \Delta_{\Delta_{\Delta}} \geq 0 \), see Eq. (4) for definition. Since we present two kinds of monogamy deficit, an interesting question is what does it means if the measurement is taken on another side, \( \Delta_{\Delta_{\Delta}} \geq 0 \)? In this section, we will consider this question and prove a similar necessary and sufficient condition for the second kinds of monogamy inequality. We first present some definitions about mutual information, conditional mutual information with respect to a single particle \( A \).

For a tripartite state \( \rho_{ABC} \), the unmeasured conditional mutual information with respect to particle \( A \) is given as,
\[
\overline{I}_A(\rho_{BC,i}) = S(\rho_{BA}) + S(\rho_{CA}) - S(\rho_{BC,i}).
\]
(17)
and the interrogated conditional mutual information with respect to particle \( A \) is,
\[
I_A(\rho_{BC,i}) = S(\rho_{BA}) + S(\rho_{CA}) - S(\rho_{BC,i}).
\]
(18)
By the strong subadditivity of von Neumann entropy, we know, \( \overline{I}_A(\rho_{ABC}) \geq 0, I_A(\rho_{BC,i}) \geq 0 \), both are non-negative.

We next propose the concept of interaction information. The (unmeasured) interaction information \( \overline{I}_A(\rho_{ABC}) = \overline{I}_A(\rho_{BC,i}) - \overline{I}(\rho_{BC}) \). By simple calculating, one may observe that \( I_A(\rho_{ABC}) = S(\rho_{AB}) + S(\rho_{AC}) - \overline{I}(\rho_{ABC}) = \overline{I}(\rho_{ABC}) \), this is the interaction information defined in Ref. [36].

For the state \( \rho_{ABC} \) and a given measurement \( \{ M_i \} \), an interrogated interaction information with respect to \( A \) is given as,
\[
I_A(\rho_{ABC}) = I_A(\rho_{BC,i}) - I_A(\rho_{BC},\{ M_i \}).
\]
(19)
Since \( I_A(\rho_{BC},\{ M_i \}) \) do not have particle \( A \), we have \( I_A(\rho_{BC},\{ M_i \}) \) do not involve any measurement. Given a tripartite quantum state \( \rho_{ABC} \), \( I_A(\rho_{ABC}) \) represents the interaction information with respect to \( A \), which is defined in (19). For this definition (19), the first term of the right hand side is the conditional mutual information of \( B, C \) when \( A \) is present and measured, the second term is the mutual information of \( BC \) where \( A \) is absent. Here, \( I_A(\rho_{ABC}) \) measures the effect on the amount of correlation shared between \( B \) and \( C \) by measuring \( A \). A positive interaction information with respect to \( A \) means the presentation of \( A \) can enhance the total correlation between \( B \) and \( C \), while negative interaction information with respect to \( A \) means the presentation of \( A \) inhibits the total correlation between \( B \) and \( C \). \( I_A(\rho_{ABC}) \) has the similar property as \( I(\rho_{ABC}) \) proposed in Ref. [36] and can be read as a necessary and sufficient criteria for a monogamy inequality. We next have the following theorem.

Theorem 1. For any state \( \rho_{ABC} \), \( D_{\Delta}(\rho_{AB}) + D_{\Delta}(\rho_{AC}) \leq D_{\Delta}(\rho_{ABC}) \) if and only if the interrogated interaction information with respect to \( A \) is less than or equal to the unmeasured interaction information with respect to \( A \).

Proof. We only need to calculate the monogamy deficit \( \Delta_{\Delta_{\Delta}} \),
\[
\Delta_{\Delta_{\Delta}} = D_{\Delta}(\rho_{ABC}) - D_{\Delta}(\rho_{AB}) - D_{\Delta}(\rho_{AC})
\]
\[
= S(\rho_{BC,i}) - S(\rho_{BC}) - (S(\rho_{BA}) - S(\rho_{BA}))
\]
\[
= S(\rho_{BC,i}) - S(\rho_{BA}) - S(\rho_{BC}) + I(\rho_{BC})
\]
\[
= I_A(\rho_{ABC}) - I_A(\rho_{ABC},\{ M_i \}).
\]
(20)
From (20), we have \( \Delta_{\Delta_{\Delta}} \geq 0 \) if and only if \( \overline{I}_A(\rho_{ABC}) \) \( \geq I_A(\rho_{ABC},\{ M_i \}) \) which completes the proof.

For pure state, we have \( \overline{I}_A(\rho_{ABC}) = 0 \), and the monogamy deficit of quantum discord is equivalent to the non-positivity of the interrogated information with respect to \( A \).

To see a transition from violation to observation of monogamy, we consider a family of states [32],
\[
|\psi(\varphi, \varepsilon)\rangle = \sqrt{p}\varphi(000) + \sqrt{p(1-\varepsilon)}|111\rangle
\]
\[
+ \sqrt{1-p/2}|(101) + |110\rangle\rangle.
\]
(21)
Note that \( |\psi(\varphi, \varepsilon)\rangle \) is the maximally entangled W state \( \sqrt{2}(|000\rangle + |101\rangle + |110\rangle) \), while \( |\psi(1, \varphi)\rangle \) is the GHZ state.
FIG. 1. (Color online) the monogamy deficit $\Delta^\epsilon_{D_{A}}$ for $|\tilde{\psi}(p,\epsilon)\rangle$, $\Delta^\epsilon_{D_{A}}$ as a function of $p$ for different values of $\epsilon$ (see the main text). States are monogamy when the respective curves are positive. Red dashed line is for $\epsilon = 0.5$, blue dotted line is for $\epsilon = 0.75$, blue solid line is for $\epsilon = 1$. The monogamy deficit is decreasing with the increasing of $\epsilon$. All of the three lines are very close to each other when $\Delta^\epsilon_{D_{A}}$ is polygamy and the critical point from polygamy to monogamy is almost identical for them. When $\epsilon \to 1$, $|\tilde{\psi}(p,\epsilon)\rangle$ approach to W states, in this case, when $p \to 1$, $\Delta^\epsilon_{D_{A}} \to 0$. The blue solid line is increased first and then decreased.

\[ \sqrt{2}(|000\rangle + |111\rangle). \] In Fig. 1, $\Delta^\epsilon_{D_{A}} = D^{-}(\rho_{AB}) - D^{-}(\rho_{BB}) - D^{-}(\rho_{AC})$ is plotted as a function of $p$ for different values of $\epsilon$. From this figure, we can show that the interrogated information with respect to $A$ can be positive or negative for tripartite pure state. The $I(\rho_{ABC}|_{M^A})$ is increasing with the increasing of $\epsilon$. All of the three lines are very close to each other when $I(\rho_{ABC}|_{M^A})$ is positive and the critical point from positive to negative is almost identical for them. Especially for the W state, when $p$ approaches to 1, the $I(\rho_{ABC}|_{M^A})$ approaches to zero.

As an application of the necessary and sufficient conditions, we find an interesting equivalent expression of entanglement of formation for tripartite pure states.

For tripartite pure states, it is shown that $\Delta^\epsilon_{D_{A}} = I(\rho_{ABC}) - 2E(\rho_{ABC})$. From the previous discussion, we have the formula, $\Delta^\epsilon_{D_{A}} = I(\rho_{BCA}) - I(\rho_{BCA})$, which holds for general tripartite mixed states. When we consider the case of pure states, the above two expressions should be equal. That is to say, in this case, $I(\rho_{ABC}) - 2E(\rho_{ABC}) = I(\rho_{BCA}) - I(\rho_{BCA})$. At the same time, it is easy to show that $\Delta^\epsilon_{D_{A}} = S(\rho_{BCA}) = S(\rho_{BCA}) - S(\rho_{CA}) - S(\rho_{BC}) + S(\rho_{BCA}) = I(\rho_{ABC})$. So we have $E(\rho_{ABC}) = \frac{1}{2}I(\rho_{BCA})$. Thus the interrogated conditional mutual information with respect to $A$ is twice of the entanglement of formation for state $\rho_{BC}$.

IV. NECESSARY AND SUFFICIENT CRITERIA FOR NON-NEGATIVE MONOGAMY DEFICIT FOR MULTIPARTITE SYSTEM.

In this section, we generalize our result to multipartite system. We give a necessary and sufficient condition for $\Delta^\epsilon_{D_{A}} \geq 0$, where the monogamy deficit is defined for multipartite state, $\Delta^\epsilon_{D_{A}} = D^{-}(\rho_{A|S_{\Delta}}) - D^{-}(\rho_{A|S_{\Delta}}) - D^{-}(\rho_{A|S_{\Delta}}) - D^{-}(\rho_{A|S_{\Delta}})$. In order to consider this question, similar as the tripartite state, we next present some definitions about mutual information, conditional mutual information with respect to a single particle $A_1$.

For a $N$-partite state $\rho_{A_1 \cdots A_N}$, the unmeasured conditional mutual information with respect to particle $A_1$ is given as $I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}}) = S(\rho_{A_1}) + S(\rho_{A_2}) - S(\rho_{A_1, A_2})$. The interrogated conditional mutual information with respect to particle $A_1$ is $I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}}) = S(\rho_{A_1}) + S(\rho_{A_2}) - S(\rho_{A_1, A_2})$. By the strong subadditivity of von Neumann entropy, we have $I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}}) = I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}})$ and $I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}})$ are both non-negative.

We define the concept of interaction information with respect to $A_1$. The (unmeasured) interaction information is defined as, $I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}}) = I_{A_1}(\rho_{A_1|\{A_2 \cdots A_N\}})$. For the state $\rho_{A_1 A_2|\{A_2 \cdots A_N\}}$ and a given measurement $\{M^A_{i}\}$, an interrogated interaction information with respect to $A_1$ is given as $I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}|_{M^A_{i}}) = I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}|_{M^A_{i}})$, where the suffix $I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}|_{M^A_{i}})$ is used to indicate the measurements on $A_1$.

Similar as the above calculating, we find that $I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}) = S(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}) + S(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}) = I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}})$, which is the interaction information we have defined. Since $I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}})$ do not have particle $A_1$, we have $I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}}|_{M^A_{i}}) = I_{A_1}(\rho_{A_1, A_2|\{A_2 \cdots A_N\}})$, which also does not involve any measurement as in tripartite case.

Now we can give the necessary and sufficient condition for $\Delta^\epsilon_{D_{A}} \geq 0$ is no less than zero. We have the following theorem.

**Theorem 2.** For any $\rho_{A_1 \cdots A_N}$, $\Delta^\epsilon_{D_{A}} \geq 0$ if and only if the interrogated interaction information with respect to $A_1$ being less than or equal to the unmeasured interaction information with respect to $A_1$. 
Proof. Here, we only need to calculate the monogamy deficit, 
\[
\Delta_{D_{A(k)}}^{-} = D^{-}(\rho_{A_{1}A_{2}...A_{N}}) - D^{-}(\rho_{A_{1}A_{2}}) - D^{-}(\rho_{A_{1}A_{3}}) \\
\hspace{1cm} - \cdots - D^{-}(\rho_{A_{1}A_{N}}) \\
\hspace{1cm} = [\Gamma_{A_{1}}(\rho_{A_{2}A_{3}...A_{N}A_{k}}) - \Gamma_{A_{1}}(\rho_{A_{2}A_{3}...A_{N}A_{k}})] \\
\hspace{1cm} + [\Gamma_{A_{2}}(\rho_{A_{1}A_{3}...A_{N}A_{k}}) - \Gamma_{A_{2}}(\rho_{A_{1}A_{3}...A_{N}A_{k}})] \\
\hspace{1cm} + \cdots + [\Gamma_{A_{N}}(\rho_{A_{1}A_{2}...A_{N-1}A_{k}}) - \Gamma_{A_{N}}(\rho_{A_{1}A_{2}...A_{N-1}A_{k}})].
\]
\[
= \sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}}) - \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}})] \\
\hspace{1cm} - \sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}})], M^{A_{k}}. \tag{22}
\]
From Eq. (22), we have \(\Delta_{D_{A(k)}}^{-} \geq 0\) if and only if 
\[
\sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}}) \geq \sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}})], M^{A_{k}}.
\]
Similarly, we can also get a necessary and sufficient condition for 
\[
\Delta_{D_{A(k)}}^{-} = 
D^{-}(\rho_{A_{1}A_{2}...A_{N}}) - D^{-}(\rho_{A_{1}A_{2}}) - D^{-}(\rho_{A_{1}A_{3}}) \\
\hspace{1cm} - \cdots - D^{-}(\rho_{A_{1}A_{N}}) \\
\hspace{1cm} = \sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}}) - \Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}})) \tag{23}
\]
where there is no measurement contained in \(\Gamma_{A_{k}}(\rho_{A_{1}A_{k}A_{k+1}...A_{N}})\), while local measurements \(M^{A_{k}}\)

\[m = 2, \ldots, N\) and coherent measurements \(\{M^{A_{k,N-k}}\}\) 
\((k = 2, \ldots, N - 1)\) contained in \(I(\rho_{A_{1}A_{k}(A_{k+1}...A_{N})})\). \(\square\)

From the above proof, we have \(\Delta_{D_{A(k)}}^{-} \geq 0\) if and only if 
\[
\sum_{k=2}^{N-1} \Gamma_{A_{k}}(\rho_{A_{1}A_{k}(A_{k+1}...A_{N})}) \geq \sum_{k=2}^{N-1} I(\rho_{A_{1}A_{k}(A_{k+1}...A_{N})}).
\]

V. SUMMARY AND DISCUSSION.

We have introduced the concept of monogamy deficit by combining together the monogamy inequalities of quantum correlation for multipartite quantum system. Although two types of monogamy inequalities seem very different on their measurement sides, based on the concept of monogamy deficit, we have observed a relation between them. Using this relation, we obtain a unified physical interpretation for these two monogamy deficit. In addition, we find an interesting fact that there exists a general monogamy condition for several quantum correlations for tripartite pure states. By using the concept of interaction information with respect to one particle, we prove that the necessary and sufficient condition for the quantum correlation being monogamous is that the interrogated interaction information with respect to one particle is less than or equal to the unmeasured interaction information. Our result can be generalized to \(N\) partite system and may have applications in quantum information processing.

ACKNOWLEDGMENTS

We thank L. Chen for useful comments. This work is supported by “973” program (2010CB922904) and NSFC (11075126, 11031005, 11175248).

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