Some Integrable Systems in Nonlinear Quantum Optics

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In the paper we investigate the theory of quantum optical systems. As an application we integrate and describe the quantum optical systems which are generically related to the classical orthogonal polynomials. The family of coherent states related to these systems is constructed and described. Some applications are also presented.

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I. INTRODUCTION

Quantum optics affords a big amount of very interesting physical phenomena having important application at the same time. Our aim is to formulate the theory of these phenomena and elucidate their connection with the theory of orthogonal polynomials. This allows us to use the last one for the rigorous integration of some nonlinear quantum optical models describing the interaction of the finite number of modes of electromagnetic field with nonlinear medium. Let us mention, that in quantum optical literature, see e.g. [3], [5], [13], [14], [15] the solutions of the models of this type are usually approached by approximative or semiclassical methods.

In Section 2 we deduce from natural and not restrictive assumptions, the general form, see (2.35), of the Hamiltonian $H_I$, describing the interaction of the finite number of modes of electric field with the matter. Later on, in Section 3, we investigate the quantum reduction method which allows us to describe the quantum optical systems by the use of the theory of orthogonal polynomials (see [2], [4], [12]). We also show that the reduced system is related to some quantum algebras, see relations (3.27). These algebras were investigated in [21], where their relations to the theory of special and $q$-special functions was shown.

Having spectral measure of the interaction Hamiltonian, which is for example the case if the model under consideration is related to the classical orthogonal polynomials or their $q$-deformation, see [22], [12], we can introduce spectral coherent states. They are direct generalizations of Glauber coherent states (corresponding to the Hermite polynomials) and squeezed states. In Section 4, we show that spectral coherent states admit the holomorphic representation for the Hamiltonians under consideration which supplements the spectral and Fock representations. This simplifies remarkably the calculation of many important physical characteristics of the described system.

The spectral coherent states should have some physical meaning which needs the deeper understanding. In any case, they give the link of quantum optical systems with complex analytic and symplectic geometry. This opens the application of coherent states method, investigated in [20], [21] to the problems of the theory of quantum optics.

In Section 5, we give complete solution of the quantum systems, see (5.3), related to the classical orthogonal polynomials.

Finally, in Section 6, we present the physical interpretation of the Hamiltonian given by (2.37) as the parametric modulator, which includes as special cases such quantum optical systems as nondegenerate parametric amplifier and the frequency up-converter, see [27].

At the end we express our conviction that the proposed method will be helpful in better understanding of quantum optical problems.

II. QUANTUM ELECTROMAGNETIC FIELD IN NONLINEAR MEDIUM

Nonlinear optics deals with phenomena that occur as a consequence of the modification of the optical properties of a material system in the presence of light. Practically, only laser light is sufficiently intensive to produce the measurable effects. By an optical nonlinearity we mean that the dipole moment per unit volume, or polarization $\vec{P}$, of a material system depends in nonlinear way, upon the strength of the applied electromagnetic field.

As many authors [5], [23] we assume that $\vec{P}$ depends only on the electric part $\vec{E}$ of the electromagnetic field $(\vec{E}, \vec{B})$ i.e. $\vec{P} = \vec{P}[\vec{E}]$. We assume moreover that this dependence is a functional one. Thus, in most general case we can write

$$\vec{P}[\vec{E}](t, \vec{r}) = \varepsilon_0 \int dN \int dt_1 d^3\vec{r}_1 \ldots dt_N d^3\vec{r}_N \times T_{(N)}(t, \vec{r}, t_1, \vec{r}_1, \ldots, t_N, \vec{r}_N) \left( \vec{E}(t_1, \vec{r}_1), \ldots, \vec{E}(t_N, \vec{r}_N) \right),$$  \hspace{1cm} (2.1)
where the vector valued $N$-linear map $\vec{T}_N$, is called in optical literature $N$-th response tensor of the medium $\mathcal{N}$.

The time-invariance principle, which says that the dynamical properties of the system are assumed to be unchanged by a translation of the time origin, leads to

$$\vec{T}_N(t, \vec{r}, t_1, \vec{r}_1, \ldots, t_N, \vec{r}_N) =: \vec{R}_N(t, t_1 - t, \vec{r}_1, t_2 - t, \vec{r}_2, \ldots, t_N - t, \vec{r}_N).$$  

(2.2)

The interaction of the electromagnetic field $(\vec{E}, \vec{B})$ with a nonlinear medium characterized by polarization $\vec{P}$ can be described by the source-free Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B},$$

$$\nabla \times \vec{B} = \mu_0 \frac{\partial}{\partial t} (\varepsilon_0 \vec{E} + \vec{P}[\vec{E}]),$$

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}[\vec{E}]) = 0,$$

$$\nabla \cdot \vec{B} = 0.$$  

(2.3)

Therefore the divergence of the Poynting vector $\frac{1}{\mu_0} \vec{E} \times \vec{B}$ takes the form

$$\frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B}) = -\frac{\partial}{\partial t} \left( \frac{1}{2 \mu_0} \vec{B}^2 + \frac{\varepsilon_0}{2} \vec{E}^2 \right) - \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} [\vec{E}],$$

(2.4)

where the quantity

$$u_0 := \frac{1}{2 \mu_0} \vec{B}^2 + \frac{\varepsilon_0}{2} \vec{E}^2$$

(2.5)

is the energy density of the free electromagnetic field. Analogously, we define the interaction energy density $u_1$ by the equation

$$-\frac{\partial u_1}{\partial t} := \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} [\vec{E}].$$

(2.6)

The energy density

$$u(t, \vec{r}) := u_0(t, \vec{r}) + u_1(t, \vec{r})$$

(2.7)

determines the Hamiltonian $H$ of our system

$$H = H_0 + H_1,$$

(2.8)

where

$$H_0 = \int u_0(t, \vec{r}) \, d^3 \vec{r}$$

(2.9)

is the Hamiltonian of the free electromagnetic field and the Hamiltonian

$$H_1 = \int u_1(t, \vec{r}) \, d^3 \vec{r}$$

(2.10)

describes the interaction of electric field $\vec{E}$ with the medium under consideration.

In order to obtain an explicit formula for $u_1$ let us consider the electromagnetic field potential $\vec{A}$

$$\vec{A}(t, \vec{r}) = \sum_{\lambda} \int d^3 \vec{k} \left[ e_{k,\lambda} A_k(\vec{k}) e^{i(\omega_k t - \vec{k} \cdot \vec{r})} + e_{k,\lambda}^* A_k^*(\vec{k}) e^{-i(\omega_k t - \vec{k} \cdot \vec{r})} \right],$$

(2.11)

expressed in terms of Fourier modes, where the index $\lambda \in \{1,2\}$ labels the polarization of the field, which is described by the pair of unit vectors $\vec{e}_{k,1}$ and $\vec{e}_{k,2}$ orthogonal to the wave vector $\vec{k}$ (we choose the Coulomb gauge $\nabla \cdot \vec{A} = 0$). Here we do not specify the form of the dispersion relation, so we assume that $\omega_k$ is any function of $|\vec{k}|$. In this gauge we have

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}.$$  

(2.12)

Let us introduce the following simplifying notation

$$e_{k,\lambda}^\sigma := \begin{cases} -i \omega_k e_{k,\lambda} & \text{for } \sigma = 1 \\ i \omega_k e_{k,\lambda} & \text{for } \sigma = -1 \end{cases}$$

(2.13)

$$A_k^\sigma(\vec{k}) := \begin{cases} A_k(\vec{k}) & \text{for } \sigma = 1 \\ A_k^*(\vec{k}) & \text{for } \sigma = -1 \end{cases}.$$  

(2.14)

We have now

$$\vec{E}(t, \vec{r}) = \sum_{\lambda, \sigma} \int e_{k,\lambda}^\sigma \vec{A}_k^\sigma(\vec{k}) e^{i(\omega_k t - \vec{k} \cdot \vec{r})} \, d^3 \vec{k},$$

(2.15)

and, therefore, (2.1) becomes

$$\vec{P}[\vec{E}](t, \vec{r}) = \sum_{N=0}^{\infty} \sum_{\sigma_1, \lambda_1} \cdots \sum_{\sigma_N, \lambda_N} \int d^3 \vec{k}_1 \cdots d^3 \vec{k}_N \chi_N \left( \vec{r}, \sigma_1, \vec{k}_1, \omega_k_1, \ldots, \sigma_N, \vec{k}_N, \omega_k_N \right) \left( \vec{e}_{k_1,\lambda_1}, \ldots, \vec{e}_{k_N,\lambda_N} \right) \right) \times e^{i \sum_{r=1}^N \sigma_r \omega_k r} A_{\lambda_1}^\sigma(\vec{k}_1) \cdots A_{\lambda_N}^\sigma(\vec{k}_N),$$

(2.16)

where $\chi_N \left( \vec{r}, \sigma_1, \vec{k}_1, \omega_k_1, \ldots, \sigma_N, \vec{k}_N, \omega_k_N \right)$ is the $N$-th susceptibility tensor $\mathcal{N}$ defined by

$$\chi_N \left( \vec{r}, \sigma_1, \vec{k}_1, \omega_k_1, \ldots, \sigma_N, \vec{k}_N, \omega_k_N \right) = \int \vec{R}_N \left( \vec{r}, \tau_1, \vec{r}_1, \ldots, \tau_N, \vec{r}_N \right) e^{i \sum_{r=1}^N \sigma_r \omega_k r} e^{-i \sum_{r=1}^N \sigma_r \vec{k}_r \cdot \vec{r}} \, d \tau_1 \, d^3 \vec{r}_1 \cdots d \tau_N \, d^3 \vec{r}_N.$$  

(2.17)
Inserting (2.16) into (2.6) we find up to additive constant that
\[
u_1(t, \bar{r}) = \sum_{N=0}^{\infty} \sum_{\sigma_0, \lambda_0} \sum_{\sigma_1, \lambda_1} \cdots \sum_{\sigma_N, \lambda_N} \int d^3 \bar{k}_1 \cdots d^3 \bar{k}_N e^{\frac{\sigma_0}{\sigma_{k_0, \lambda_0}}} \cdot \bar{\chi}_0(N) \left( \bar{r}, \sigma_1, \bar{k}_1, \omega_{k_1}, \ldots, \sigma_N, \bar{k}_N, \omega_{k_N} \right) \left( e^{\sigma_{k_1, \lambda_1}}, \ldots, e^{\sigma_{k_N, \lambda_N}} \right) \times \sum_{r=1}^{N} e^{i \sum_{s=0}^{N} \sigma_{r, \omega_{k_r}}} e^{-i \sigma_0 \bar{k}_0 \cdot \bar{r}} A_{\lambda_0}^{\sigma_0}(\bar{k}_0) A_{\lambda_1}^{\sigma_1}(\bar{k}_1) \cdots A_{\lambda_N}^{\sigma_N}(\bar{k}_N). \tag{2.18}
\]

In the quantization procedure the classical quantities \(A_{\lambda}^{\sigma}(\bar{k})\) in (2.11) are replaced by the operators
\[
a_{k, \lambda}^{\sigma} := \begin{cases} a_{k, \lambda}^{\sigma} & \text{for } \sigma = 1 \\ a_{k, \lambda}^{\sigma'} & \text{for } \sigma = -1 \end{cases}, \tag{2.19}
\]
which satisfy the commutation relations of a free quantum field:
\[
\left[ a_{k, \lambda}^{\sigma}, a_{k', \lambda'}^{\sigma'} \right] = \delta_{\lambda, \lambda'} \delta(k - k') \left( 1 - \sigma \sigma' \right). \tag{2.20}
\]
The products \(A_{\lambda_0}^{\sigma_0}(\bar{k}_0) \cdots A_{\lambda_N}^{\sigma_N}(\bar{k}_N)\) in (2.11) and (2.18) we obtain
\[
H = H_0 + \sum_{N=0}^{\infty} \sum_{\sigma_0, \lambda_0} \cdots \sum_{\sigma_N, \lambda_N} \int d^3 \bar{k}_0 \cdots d^3 \bar{k}_N : a_{k_0, \lambda_0}^{\sigma_0} \cdots a_{k_N, \lambda_N}^{\sigma_N} : e^{it \sum_{s=0}^{N} \sigma_{r, \omega_{k_r}}} e^{-i \sigma_0 \bar{k}_0 \cdot \bar{r}} A_{\lambda_0}^{\sigma_0}(\bar{k}_0) A_{\lambda_1}^{\sigma_1}(\bar{k}_1) \cdots A_{\lambda_N}^{\sigma_N}(\bar{k}_N). \tag{2.22}
\]
where
\[
\bar{\chi}_0(N) \left( \sigma_0, \bar{k}_0, \omega_{k_0}, \sigma_1, \bar{k}_1, \omega_{k_1}, \ldots, \sigma_N, \bar{k}_N, \omega_{k_N} \right) := \sum_{r=1}^{N} \sum_{s=0}^{N} \sigma_{r, \omega_{k_r}} \int \bar{\chi}_0(N) \left( \bar{r}, \sigma_1, \bar{k}_1, \omega_{k_1}, \ldots, \sigma_N, \bar{k}_N, \omega_{k_N} \right) e^{-i \sigma_0 \bar{k}_0 \cdot \bar{r}} d^3 \bar{r}. \tag{2.23}
\]

Using the commutation relations (2.22) one can prove that
\[
e^{-i H_0 t} a_{k, \lambda}^{\sigma} e^{i H_0 t} = e^{i \sigma \omega_{k, \lambda}} a_{k, \lambda}^{\sigma}. \tag{2.24}
\]

Hence the Hamiltonian (2.8) becomes
\[
H = H_0 + e^{-i H_0 t} H_f e^{i H_0 t}, \tag{2.25}
\]
where due to (2.22)
\[
H_f = \sum_{N=0}^{\infty} \sum_{\sigma_0, \lambda_0} \cdots \sum_{\sigma_N, \lambda_N} \int d^3 \bar{k}_0 \cdots d^3 \bar{k}_N : a_{k_0, \lambda_0}^{\sigma_0} \cdots a_{k_N, \lambda_N}^{\sigma_N} : e^{i \sigma_0 \bar{k}_0 \cdot \bar{r}} \bar{\chi}_0(N) \left( \sigma_0, \bar{k}_0, \omega_{k_0}, \sigma_1, \bar{k}_1, \omega_{k_1}, \ldots, \sigma_N, \bar{k}_N, \omega_{k_N} \right) \left( e^{\sigma_{k_1, \lambda_1}}, \ldots, e^{\sigma_{k_N, \lambda_N}} \right). \tag{2.26}
\]

does not depend on time, and therefore the solution of the Schrödinger equation
\[
i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \tag{2.27}
\]
is given by
\[
|\psi(t)\rangle = e^{-i H_0 t} e^{-i H_f t} |\psi(0)\rangle. \tag{2.28}
\]

The operator
\[ U_0(t) := e^{-iH_0 t} \] (2.29)
is the free electromagnetic field evolution operator. The operator
\[ U_I(t) := e^{-iH_I t} \] (2.30)
is the evolution operator of the system in the interaction picture.

For the real models in quantum optics one assumes that the system under consideration contains a finite number of modes of electric field (see [8, 13, 14, 23]). This means that the label \((\vec{k}, \lambda)\) in (2.21) and (2.22) takes a finite number of values
\[ (\vec{k}, \lambda) \equiv j \in \{0, 1, \ldots, M\} \] (2.31)
and the integrals are reduced to finite sums over \(j\):
\[ H_0 = \sum_j \omega_j a_j^\dagger a_j \] (2.32)
\[ H_I = \sum_{N=0}^{\infty} \sum_{\sigma_0, \lambda_0} \ldots \sum_{\sigma_M, \lambda_M} : a_{\sigma_0,j_0}^\sigma \ldots a_{\sigma_M,j_N}^\Sigma : \]
\[ \times e^{\sigma_{j_0}} \cdot \Theta_N(\sigma_0, \omega_{j_0}, \ldots, \sigma_N, \omega_{j_N}) \left( \tilde{e}_{j_1}^\sigma, \ldots, \tilde{e}_{j_N}^\Sigma \right). \] (2.33)
In this case the Hamiltonian (2.33) can be transformed into the form which is more useful for our aims. It is defined by the exchange of the normal ordering of the annihilation and creation operators into the one which we will call boson-number ordering in the sequel.

In order to define the **boson-number ordering** let us introduce the following notation for creation and annihilation operators
\[ a_l := \begin{cases} 
  a_l & \text{for } l = 1, 2, \ldots \\
  1 & \text{for } l = 0 \\
  (a^*)^{-l} & \text{for } l = -1, -2, \ldots 
\end{cases} \] (2.34)
A product of \(m\) annihilation and \(n\) creation operators being in the same mode, is said to be boson-number ordered if it is of the form \(P(a^* a)^{m-n}\), where \(P\) is a polynomial.

Changing the normal ordering in each term of the Hamiltonian (2.33) to the boson-number ordering \(P(a^*_0 a_0, \ldots, a^*_M a_M) a^{l_0}_0 \ldots a^{l_M}_M\) where \(P\) is a polynomial of \(M + 1\) variables and collecting the terms with the same factor \(a^{l_0}_0 \ldots a^{l_M}_M\) we obtain
\[ H_I = \sum_{l_0, \ldots, l_M \in \mathbb{Z}} g_{l_0, \ldots, l_M} (a^*_0 a_0, \ldots, a^*_M a_M) a^{l_0}_0 \ldots a^{l_M}_M, \] (2.35)
where \(g_{l_0, \ldots, l_M}\) are functions of \((M + 1)\)-variables dependent on \(\Theta_N(\sigma)\). The Hamiltonian \(H_I\) is a symmetric operator if
\[ [g_{l_0, \ldots, l_M} (a^*_0 a_0, \ldots, a^*_M a_M)]^* = g_{-l_0, \ldots, -l_M} (a^*_0 a_0 - l_0, \ldots, a^*_M a_M - l_M). \] (2.36)
In the next sections we restrict our considerations to the Hamiltonians of the form
\[ H_I = \hbar (a^*_0 a_0, \ldots, a^*_M a_M) + g (a^*_0 a_0, \ldots, a^*_M a_M) a^{l_0}_0 \ldots a^{l_M}_M + h.c. \] (2.37)
Such form of the Hamiltonian is strictly related to the theory of orthogonal polynomials. The physical interpretation of this Hamiltonian is given in the Paragraph 6.A.

**III. REDUCTION OF THE HAMILTONIAN**

In this section, we briefly describe the decomposition of the Hilbert space \(\mathcal{H}\) spanned by elements of the orthonormal Fock basis
\[ \mathcal{F} = \left\{ |n_0, \ldots, n_M \rangle := \frac{1}{\sqrt{n_0! \ldots n_M!}} (a^*_0)^{n_0} \ldots (a^*_M)^{n_M} |0\rangle, \quad n_0, \ldots, n_M \in \mathbb{N} \cup \{0\} \right\} \] (3.1)
into invariant subspaces of the operators \(H_0\) and \(H_I\).

The method of this decomposition is presented in details in [22]. In such a way we obtain the reduction of the Hamiltonian \(H\).

The invariant subspaces of \(H_I\) are obtained in two steps. The first step is related to some family of integrals of motion; the second one is related to a family of pseudo-vacuum vectors.

Let us start with a few definitions:
\[ A_i := g (a^*_0 a_0, \ldots, a^*_M a_M) a^{l_0}_0 \ldots a^{l_M}_M \] (3.2)
\[ \mathcal{A}_i = \mathcal{A}_i^* := \sum_{j=0}^{M} \alpha_{ij} a^*_j a_j, \quad i = 0, 1, \ldots, M, \] (3.3)
where $\alpha = (\alpha_{ij})$ is a real $(M + 1) \times (M + 1)$-matrix satisfying the conditions
\begin{equation}
\det \alpha \neq 0,
\end{equation}
\begin{equation}
\sum_{j=0}^{M} \alpha_{ij} l_j = \delta_{ij}.
\end{equation}

The invertibility of the matrix $\alpha$ allows one to express the boson-number operators $a_j^* a_i$ by $A_j$, which gives
\begin{equation}
H_0 = \sum_{j=0}^{M} \gamma_j A_j
\end{equation}
with real constants $\gamma_j$ determined by the matrix $\alpha$. In particular we have
\begin{equation}
\gamma_0 = \sum_{i=0}^{M} \omega_j l_j.
\end{equation}
Additionally
\begin{equation}
H_I = H_d(A_0, A_1, \ldots, A_M) + A + A^*
\end{equation}
with $H_d$ uniquely determined by the function $h$ and the matrix $\alpha$. Using the canonical commutation relations for creation and annihilation operators one obtain
\begin{align}
AA^* &= \mathcal{G}(A_0, A_1, \ldots, A_M), \\
A^* &= \mathcal{G}(A_0 - 1, A_1, \ldots, A_M), \\
[A_0, A] &= -A, \quad [A_0, A^*] = A^*, \\
[A_j, A] &= 0, \quad j = 1, \ldots, M, \\
[A_i, A_j] &= 0, \quad i, j = 0, \ldots, M
\end{align}
with the nonnegative function $\mathcal{G}$ uniquely determined by $g$ and $\alpha$.

Direct calculations gives
\begin{equation}
[A_j, H_0] = [A_j, H_I] = 0 \quad j = 1, 2, \ldots, M.
\end{equation}
which means that operators $A_1, A_2, \ldots, A_M$ are integrals of motion.

In order to reduce $H_0$ and $H_I$ to the common eigenstate of integrals of motion let us notice that the operators $A^* A$, $A A^*$, $A_0, \ldots, A_M$ are diagonal in the Fock basis $B_F$. This, in particular, means that each vector $|n_0, \ldots, n_M\rangle \in B_F$ is the eigenvector of the operators $A_j$, $j = 0, \ldots, M$, with eigenvalues given by:
\begin{equation}
\lambda_j = \sum_{i=0}^{M} \alpha_{ji} n_i.
\end{equation}

Moreover, the operators $A_0, \ldots, A_M$ form a system of commuting independent observables. In such a way we can use the sequences of eigenvalues $(\lambda_0, \lambda_1, \ldots, \lambda_M)$ as a new parametrization $\{|\lambda_0, \lambda_1, \ldots, \lambda_M\rangle\}$ of the Fock basis elements. So we obtain
\begin{equation}
A_j |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle = \lambda_j |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle, \quad j = 0, \ldots, M.
\end{equation}

Since $[A_0, A] = -A$ then, from (3.10) and (3.9) we have
\begin{align}
A |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle &= \sqrt{\mathcal{G}}(\lambda_0 - 1, \lambda_1, \ldots, \lambda_M) |\lambda_0 - 1, \lambda_1, \ldots, \lambda_M\rangle, \\
A^* |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle &= \sqrt{\mathcal{G}}(\lambda_0, \lambda_1, \ldots, \lambda_M) |\lambda_0 + 1, \lambda_1, \ldots, \lambda_M\rangle.
\end{align}

It is clear that the subspace $H_{\lambda_1, \ldots, \lambda_M}$ of the Fock space $H$ spanned by the eigenvectors $|\lambda_0, \lambda_1, \ldots, \lambda_M\rangle$ with fixed $\lambda_1, \ldots, \lambda_M$ is $H_0$ and $H_I$-invariant and $dim H_{\lambda_1, \ldots, \lambda_M} = \infty$ if and only if all $l_j$ in (3.2) are nonnegative. The problem of integration of the system (2.25) is reduced to integration of the system described by the reduced Hamiltonian
\begin{align}
H_{0,\text{red}} &:= \gamma_0 A_0 + \sum_{j=1}^{M} \gamma_j A_j, \\
H_{I,\text{red}} &:= H_d(A_0, \lambda_1, \ldots, \lambda_M) + A + A^*
\end{align}
and therefore (up to additive constant)
\begin{equation}
H_{\text{red}} = \gamma_0 A_0 + e^{-i\alpha_A t} H_{I,\text{red}} e^{i\alpha_A t}.
\end{equation}

Now we go to the next step of the reduction. In order to make it let us define the pseudo-vacuum vector as such vector $|\lambda_0, \lambda_1, \ldots, \lambda_M\rangle$ from the Fock basis in $H_{\lambda_1, \ldots, \lambda_M}$ which is annihilated by the operator $A$, i.e.
\begin{equation}
A |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle = 0
\end{equation}
or equivalently
\begin{equation}
G(\lambda_0 - 1, \lambda_1, \ldots, \lambda_M) = 0.
\end{equation}

In [22] it was shown that the set $\{\lambda_0, \ldots, \lambda_M\}$ of solutions of (3.22) is nonempty if in the definition (3.2) any $l_i$, $i = 0, 1, \ldots, M$, is greater then zero.

Now, if for simplicity, we introduce the notation
\begin{align}
|n\rangle &:= |\lambda_0, \lambda_1, \ldots, \lambda_M\rangle, \\
b(n) &:= \sqrt{\mathcal{G}}(\lambda_0, \lambda_1, \ldots, \lambda_M), \\
N &:= A_0 - \lambda_{0, l},
\end{align}
then
\begin{align}
N|n\rangle &= n|n\rangle, \\
A|n\rangle &= b(n)|n - 1\rangle, \\
A^*|n\rangle &= b(n + 1)|n + 1\rangle.
\end{align}

Thus we obtain that the space
\begin{equation}
\mathcal{F} := \text{span} \{ |n\rangle, \quad n = 0, 1, \ldots \}
\end{equation}
is the irreducible representation space for the algebra $\mathcal{A}_{\text{red}}$ generated by the operators $N, A$ and $A^*$, which satisfy the relations:
\begin{align}
[N, A] &= -A, \quad [N, A^*] = A^*, \\
A^* A &= b^2(N), \\
AA^* &= b^2(N + 1).
\end{align}
These algebras were investigated in [21]. The question when the dimension of $\mathcal{F}$ is finite or infinite was discussed in detail in [22]. Here we assume that $\dim \mathcal{F} = \infty$. After restriction to $\mathcal{F}$, the Hamiltonians \( H_{0,\text{red}} \) \( H_{1,\text{red}} \) belongs to $\mathcal{A}_{\text{red}}$ and take the form (up to additive constant)

\[
H_{0,\text{red}} = \gamma_0 N, \quad H_{1,\text{red}} = h(N) + A + A^*,
\]

where $h(N) := H_d(N + \lambda_0, \lambda_1, \ldots, \lambda_M)$. Thus the operators $H_{0,\text{red}}, H_{1,\text{red}}$ and consequently $H_{\text{red}}$ belong to the algebra $\mathcal{A}_{\text{red}}$. In the Fock basis \{ $|n\rangle$, $n = 0, 1, \ldots$ \} the operator $H_{1,\text{red}}$ assumes the three diagonal (Jacobi) form:

\[
H_{1,\text{red}}|n\rangle = (N + 1)|n\rangle + \gamma_0|n-1\rangle + (N+1)|n+1\rangle,
\]

where $H_{0,\text{red}}$ is diagonal

\[
H_{0,\text{red}}|n\rangle = \gamma_0 n|n\rangle.
\]

From now on we restrict our consideration to the space $\mathcal{F}$. In particular we restrict all operators discussed above to $\mathcal{F}$ and omit the index red for simplicity.

The evolution of the system given by \( \mathcal{H}_{1,\text{red}} \) now takes the form

\[
|\psi(t)\rangle = e^{-i\gamma_0 N t} e^{-iH_{1,\text{red}} t}|\psi(0)\rangle
\]

and therefore for any operator $F$ we have

\[
\langle \psi(t)|F|\psi(t)\rangle = \langle \psi(0)|e^{iH_{1,\text{red}} t}F e^{-iH_{1,\text{red}} t}|\psi(0)\rangle. \tag{3.30}
\]

For a special, but interesting case this formula simplifies. Namely,

\[
\langle \psi(t)|f(N)|\psi(t)\rangle = \langle \psi(0)|e^{iH_{1,\text{red}} t}f(N)e^{-iH_{1,\text{red}} t}|\psi(0)\rangle, \quad \tag{3.31}
\]

\[
\langle \psi(t)|f(A)|\psi(t)\rangle = \langle \psi(0)|e^{H_{1,\text{red}} t}f(A)e^{-H_{1,\text{red}} t}|\psi(0)\rangle, \quad \tag{3.32}
\]

\[
\langle \psi(t)|H|\psi(t)\rangle = \gamma_0 \langle \psi(0)|e^{H_{1,\text{red}} t}N e^{-H_{1,\text{red}} t}|\psi(0)\rangle + \langle \psi(0)|H_{1,\text{red}}|\psi(0)\rangle, \quad \tag{3.33}
\]

where $f$ is an analytic function.

The next section is devoted to the detailed study of the operator $H_I$ and 1-parameter group $e^{-iH_I t}$ generated by it.

## IV. SPECTRAL AND COHERENT STATES REPRESENTATIONS

The operators $H_I$ of the type \( \mathcal{H}_{1,\text{red}} \) are very well known in the theory of orthogonal polynomials [1], [2], [3]. They are symmetric in $\mathcal{F}$ and, by \( \mathcal{H}_{1,\text{red}} \), have a dense domain which consists of finite linear combinations of elements of the Fock basis. The deficiency indices of $H_I$ are $(0,0)$ or $(1,1)$. One can prove that if \( \sum_{n=1}^{\infty} \frac{1}{\langle \psi(n)|\psi(n)\rangle} = \infty \), then the operator $H_I$ has deficiency indices $(0,0)$, which is equivalent to its essential selfadjointness.

From now on we will assume that the deficiency indices of $H_I$ are $(0,0)$. Hence $H_I$ admits a unique selfadjoint extension, which will be denoted by the same symbol. Moreover, $H_I$ has simple spectrum. This fact allows us to identify the Fock space $\mathcal{F}$ with the Hilbert space of square integrable functions $L^2(\mathbb{R}, d\sigma)$ of real variable $\omega \in \mathbb{R}$. The measure $d\sigma$ is determined by the spectral measure $dE$ of the hamiltonian $H_I$ and is defined by the formula

\[
d\sigma(\omega) := \langle 0|dE(\omega)|0\rangle. \tag{4.1}
\]

Additionally one can prove that polynomials $\{ \langle \omega^n \rangle \}_{n=0}^{\infty}$ form a linearly dense subset in $L^2(\mathbb{R}, d\sigma)$. After the Gram-Schmidt orthonormalization of the basis $\{ \langle \omega^n \rangle \}_{n=0}^{\infty}$ we obtain an orthonormal set $\{ P_n \}_{n=0}^{\infty}$ in $L^2(\mathbb{R}, d\sigma)$ called the orthonormal polynomial system. Notice that $\deg P_n = n$.

The unitary isomorphism $U : \mathcal{F} \rightarrow L^2(\mathbb{R}, d\sigma)$ of Hilbert spaces is given by

\[
U|\psi\rangle := \sum_{n=0}^{\infty} \langle n|\psi\rangle P_n. \tag{4.2}
\]

According to the spectral theorem and \( \mathcal{H}_{1,\text{red}} \) one has

\[
(U \circ f(H_I) \circ U^{-1})|\psi\rangle = f(\omega)|\psi\rangle, \quad \tag{4.3}
\]

for $\psi \in L^2(\mathbb{R}, d\sigma)$ and any measurable function $f$. By \( \mathcal{H}_I \) and \( \mathcal{H}_{1,\text{red}} \) the expression \( \mathcal{H}_{1,\text{red}} \) converts into the three-term recurrence formula

\[
\omega P_n(\omega) = h(n) P_n(\omega) + b(n) P_{n-1}(\omega) + b(n+1) P_{n+1}(\omega), \quad \tag{4.4}
\]

for the system of orthonormal polynomials $\{ P_n \}_{n=0}^{\infty}$. So by the spectral theorem in the notion of this orthonormal system, we have:

\[
\langle m|f(H_I)|n\rangle = \int f(\omega) P_m(\omega) P_n(\omega) d\sigma(\omega). \tag{4.5}
\]

In particular, for $f(H_I) = \frac{1}{\gamma_0} H_I^k$ we obtain the moments $\mu_k$ of the measure \( \mu_0 \) of \( \mathcal{H}_I \):

\[
\mu_k := \int \omega^k d\sigma(\omega) = \frac{1}{\gamma_0^k} \langle 0| H_I^k |0\rangle. \tag{4.6}
\]

Similarly, for $f(H_I) = \frac{1}{\gamma_0^k} H_I^k$ we obtain the absolute moments $|\mu_k|$ of \( \mu_0 \):

\[
|\mu_k| := \int |\omega|^k d\sigma(\omega) = \frac{1}{\gamma_0^{2k}} \langle 0| |H_I|^k |0\rangle. \tag{4.7}
\]
For the case under consideration the moments \( \{ \mu_k \}_{k=0}^{\infty} \) determine \( d\sigma \) in the unique way \([1]\).

From (4.3) and (4.5) one obtains that the evolution operator \( e^{-iH_1 t}, \ t \in \mathbb{R} \), in the Hilbert space \( L^2(\mathbb{R}, d\sigma) \) is given by
\[
(U \circ e^{-iH_1 t} \circ U^{-1}) \psi(\omega) = e^{-i\omega t} \psi(\omega)
\] (4.8)
and its mean value in the vacuum is realized by the characteristic function
\[
\hat{\sigma}(t) := \int e^{-i\omega t} d\sigma(\omega) = \frac{1}{P_0}(0) e^{-iH_1 t} (0)
\] (4.9)
of the measure \( d\sigma \), compare with \([1]\).

After these preliminary remarks we will show that apart from realizations of the Hamiltonian \( H_1 \) in the Fock space \( F \) and in the Hilbert space \( L^2(\mathbb{R}, d\sigma) \) it is useful and natural to consider its realization in some Hilbert space which consists of square integrable holomorphic functions defined on an open subset of complex plane. To do it let us first prove the following

**Lemma IV.1** Let us assume that absolute moments \( |\mu_n| \) are finite for all \( n \in \mathbb{N} \cup \{0\} \) and they satisfy the condition
\[
\lim_{n \to \infty} \sqrt[n]{|\mu_n|} = \frac{1}{e R} < +\infty.
\] (4.10)

Then there exists a maximal strip in \( \mathbb{C} \), which is open, connected and invariant under the one-parameter group of translations
\[
T_{tz} := z + t, \ t \in \mathbb{R},
\] (4.11)
such that the characteristic function \( \hat{\sigma}(t) \) can be holomorphically extended to it.

The maximality of the strip means that \( \hat{\sigma}(t) \) cannot be extended to a larger set with the same properties.

**Proof:** We prove firstly that characteristic function \( \hat{\sigma} \) is analytic on the strip \( |Im z| < R \). One has
\[
\frac{d^n}{dt^n} \hat{\sigma}(t) = (-i)^n \int e^{-it\omega} \omega^n d\sigma(\omega)
\] (4.12)
for \( n = 0, 1, \ldots \). In order to prove (4.12) one proceeds by induction. The equality \([\ref{1}][4.12]\) is valid for \( n = 0 \). Let us assume that it is true for \( n \), then
\[
\frac{d^{n+1}}{dt^{n+1}} \hat{\sigma}(t) = \lim_{n \to \infty} \int e^{-i(t+h)\omega} - e^{-it\omega} \omega^n d\sigma(\omega)
\]
\[
= \lim_{n \to \infty} \int e^{-i\omega} - \frac{1}{h} e^{-it\omega} \omega^n d\sigma(\omega)
\]
\[
= -i \int e^{-it\omega} \omega^{n+1} d\sigma(\omega).
\] (4.13)
Since
\[
\left| \frac{e^{-ih\omega} - 1}{h} \omega^n \right| \leq |\omega|^{n+1}
\] (4.14)
and \( |\mu_n| \) is finite for \( n = 0, 1, \ldots \), we were able to use Lebesgue theorem in \([\ref{1}][4.13]\). For \( h \in \mathbb{R} \) we have the estimate
\[
\left| \hat{\sigma}(t+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} \frac{d^k}{dt^k} \hat{\sigma}(t) \right|
\]
\[
= \left| \hat{\sigma}(t + h) - \int \left( e^{-i(t+h)\omega} - \sum_{k=0}^{n-1} \frac{(-i\omega)^k}{k!} \omega^n \right) d\sigma(\omega) \right|
\]
\[
\leq \frac{1}{n} \int |\omega|^n d\sigma(\omega) = \frac{|\omega|^n}{n!} |\mu_n|,
\] (4.15)
where for the last inequality we used
\[
e^{-ih} - \sum_{k=0}^{n-1} \frac{(-ih)^k}{k!} \leq \frac{|\omega|^n}{n!}.
\] (4.16)

By the Cauchy criterion and Stirling formula
\[
n! = \sqrt{2\pi n} n^n e^{-n} e^{\Theta(n)},
\] (4.17)
where \( \Theta(n) < \frac{1}{12n} \), the series \( \sum_{n=0}^{\infty} \frac{|\mu_n|^n}{n!} \) is convergent for \( |h| < R \). This and \([\ref{1}][4.13]\) imply that Taylor expansion
\[
\hat{\sigma}(t+z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{d^k}{dt^k} \hat{\sigma}(t)
\] (4.18)
is convergent for \( |z| < R \). We have proved the analyticity of \( \hat{\sigma} \) on the strip \( |Im z| < R \). So there exist a nonempty, maximal strip \( \{ z \in \mathbb{C} : 2 r < |Im z| < 2 s \} \), such that the characteristic function \( \hat{\sigma}(t) \) can be holomorphically extended to it. We have shown, moreover that this strip contains the real axis i.e. \( -\infty < 2r < 0 < 2s < +\infty \).

QED

Let us consider a "half" of the strip \( \{ z \in \mathbb{C} : 2r < |Im z| < 2s \} \) i.e.
\[
\Sigma := \{ z \in \mathbb{C} : r < |Im z| < s \},
\] (4.19)
As a consequence of the Lemma \([\ref{1}][IV.1]\) we can formulate the following

**Proposition IV.1** Under the assumptions of Lemma \([\ref{1}][IV.1]\) the map
\[
\tilde{K} : \Sigma \ni z \mapsto e^{-iz \cdot} \in L^2(\mathbb{R}, d\sigma)
\] (4.20)
is holomorphic and its image \( \tilde{K}(\Sigma) \) is linearly dense in \( L^2(\mathbb{R}, d\sigma) \).

**Proof:** In order to see that the function \( e^{-iz \cdot} \) belongs to \( L^2(\mathbb{R}, d\sigma) \) let us notice that
\[
\int |e^{-iz \omega}|^2 d\sigma(\omega) = \int e^{-i(z - \bar{z}) \omega} d\sigma(\omega)
\]
\[
= \hat{\sigma}(z - \bar{z}) < +\infty
\] (4.21)
for $(z - \bar{z}) \in \{z \in \mathbb{C} : 2r < Im z < 2s \}$. Thus in the basis $(P_n)_{n=0}^{\infty}$ we have
\[ \tilde{K}(z) = e^{-iz} = \sum_{n=0}^{\infty} \tilde{\sigma}_n(z)P_n(t), \]
(4.22)
where the coefficients functions $\tilde{\sigma}_n$ are holomorphic extensions of
\[ \tilde{\sigma}_n(t) := \int e^{-it\omega}P_n(\omega)\,d\sigma(\omega) \]
(4.23)
on the whole strip $\Sigma$. Thus the map $\tilde{K}$ is complex analytic map of the strip $\Sigma$ into Hilbert space $L^2(\mathbb{R}, d\sigma)$.

In order to show that $K(\Sigma)$ is linearly dense in $L^2(\mathbb{R}, d\sigma)$ let us notice that the monomials
\[ \omega^n = i^n \frac{d^n}{dz^n} \tilde{K}(\omega) \mid_{z=0}, \]
(4.24)
where $n = 0, 1, \ldots$, belong to the linear closure of $\tilde{K}(\Sigma)$. Since, they form linearly dense subset of $L^2(\mathbb{R}, d\sigma)$ and the same property is shared by $K(\Sigma)$. QED

Combining (1.2) with (4.22) we obtain a holomorphic map
\[ K := U^{-1} \circ \tilde{K} : \Sigma \ni z \mapsto |z| := \sum_{n=0}^{\infty} \tilde{\sigma}_n(z)|n\rangle \in \mathcal{F} \]
(4.25)
of $\Sigma$ into Fock space $\mathcal{F}$. Following [20], [21] we shall call $K : \Sigma \rightarrow \mathcal{F}$ the coherent states map related to the quantum system described by the Hamiltonian $H_I$. The states $|z\rangle$, where $z \in \Sigma$, will be called spectral coherent states. The coherent states map has nice physical properties and, as we will show later, it is useful for the calculations of physical characteristics of the system.

By the formulae
\[ \Omega := \frac{1}{i} \frac{\partial^2}{\partial z \partial \bar{z}} (\log \tilde{\sigma}(z - \bar{z})) \, dz \wedge d\bar{z} \]
\[ = - \frac{1}{2} \frac{d^2}{dy^2} (\log \tilde{\sigma}(2iy)) \, dx \wedge dy, \]
(4.26)
where $z = x + iy$, we will define the symplectic form $\Omega$ on $\Sigma$. Using the mean value function
\[ \langle H_I \rangle_z := \frac{\langle z|H_I|z\rangle}{\langle z|z\rangle} = - \frac{1}{2} \frac{d}{dy} \log \tilde{\sigma}(2iy) \]
(4.27)
of the Hamiltonian in spectral coherent states $|z\rangle, z \in \Sigma$, we define the classical Hamiltonian system
\[ X_{(H_I)}^\Sigma \Omega = d\langle H_I \rangle_z \]
(4.28)
on the symplectic manifold $(\Sigma, \Omega)$. The Hamiltonian flow, tangent to the vector field $X_{(H_I)}^\Sigma$, is given by (1.11). Let us denote by $\mathcal{C}(\mathcal{F})$ the complex projective Hilbert space modelled on the Fock space $\mathcal{F}$. Let $\Omega_{FS}$ denote Fubini-Study (1,1)–form on $\mathcal{C}(\mathcal{F}),$(for the definition of $\Omega_{FS}$ consult for example [21]). The form $\Omega_{FS}$ is closed and nonsingular. So, $(\mathcal{C}(\mathcal{F}),\Omega_{FS})$ can be considered as a symplectic manifold which can be interpreted as the quantum phase space of the system described by the Hamiltonian $H_I$.

**Proposition IV.2** The projectivization $\mathcal{K} : \Sigma \rightarrow \mathcal{CP}(\mathcal{F}), \mathcal{K}(z) := \mathbb{C}|z\rangle$ of the coherent states map $\mathbb{C}|z\rangle$ is the holomorphic symplectic map, i.e.
\[ \mathcal{K}^*\Omega_{FS} = \Omega \]
(4.29)
and the diagram
\[ \begin{array}{ccc}
\Sigma & \xrightarrow{K} & \mathcal{CP}(\mathcal{F}) \\
T_t \uparrow & & \uparrow e^{-iH_I t} \\
\Sigma & \xrightarrow{K} & \mathcal{CP}(\mathcal{F})
\end{array} \]
(4.30)
is commutative for any $t \in \mathbb{R}$.

**Proof**
The equality (4.29) can be checked by direct calculation. The commutativity of the diagram (4.30) follows from (4.22) by the use of the formulae for quantum (1.8) and classical (4.11) evolution of the system. QED

Recapitulating: we see that the coherent states map symplectically the classical phase space $\Sigma$ of the system $(\Sigma, \Omega, (H_I)_z)$ into the quantum phase space $\mathcal{CP}(\mathcal{F})$ of the system $(\mathcal{C}(\mathcal{F}), \Omega_{FS}, H_I)$. It is equivariant with respect to the classical and the quantum flows. The mean value function $\langle H_I \rangle_z$ of the quantum Hamiltonian $H_I$ give the classical Hamiltonian of the system. So, the above picture is analogous to the one related to the harmonic oscillator (see [24]). For the general theory of quantization and description of physical systems in terms of the coherent states map see [21]. The model of the physical system considered here gives an important and interesting example illustrating the theory which was developed in [21].

Let us define spectral annihilation operator $\mathbf{a}$ by the condition
\[ \mathbf{a}|z\rangle = z|z\rangle, \]
(4.31)
which means that $\mathbf{a}$ has the spectral coherent states $|z\rangle$ as eigenvectors with eigenvalues $z \in \Sigma$. It is defined on the dense linear domain, spanned by spectral coherent states. The representation in $L^2(\mathbb{R}, d\sigma)$ is given by
\[ (U \circ \mathbf{a} \circ U^{-1})^t(\omega) \equiv \frac{d}{d\omega} \psi(\omega). \]
(4.32)
The domain $D(U \circ \mathbf{a} \circ U^{-1})$ is given as a vector space of all polynomials.

According to Proposition IV.1, the spectral coherent states form a linearly dense subset in $\mathcal{F}$. Hence one can define anti-linear monomorphism $\overline{U}$
\[ \mathcal{F} \ni |\psi\rangle \mapsto \overline{U}|\psi\rangle := (\psi|K(\cdot)) \in \mathcal{O}(\Sigma) \]
(4.33)
of the Fock space $\mathcal{F}$ into vector space $\mathcal{O}(\Sigma)$ of holomorphic functions on $\Sigma$. In such a way we obtain the third realization of Hilbert space of states, this time as the space of holomorphic functions $\overline{U}(\mathcal{F}) \subset \mathcal{O}(\Sigma)$ with the scalar product defined by
\[ \langle \Phi|\Psi \rangle \equiv \langle \overline{U}(|\Phi\rangle), \overline{U}(|\Psi\rangle) \rangle := \langle \psi|\phi \rangle, \]
(4.34)
where $\Psi = \overline{U}|\psi\rangle$, $\Phi = \overline{U}|\phi\rangle$. 
Proposition IV.3 Let the measure
\[ d\mu(\bar{z}, z) = \mu(y) dx dy, \] (4.35)
on \Sigma, \ (z = x + iy) be such that the weight function \( \mu \) satisfies
\[ \frac{d\sigma}{d\omega}(\omega) \int_r^s dy \mu(y) e^{2y\omega} = 1 \] (4.36)
for \( \omega \in \text{supp} \, d\sigma \).
Then the scalar product (4.34) can be expressed by the integral
\[ \langle \psi | \phi \rangle = \int_{\Sigma} \Psi(z) \Phi(z) d\mu(\bar{z}, z). \] (4.37)
Moreover, the kernel function
\[ \langle z | v \rangle = \hat{\sigma}(v - \bar{z}) \] (4.38)
is a reproducing kernel function with respect to the measure (4.35), i.e.
\[ \Psi(v) = \int_{\Sigma} \hat{\sigma}(v - \bar{z}) \Psi(z) d\mu(\bar{z}, z) \] (4.39)
for any \( \Psi \in \mathcal{U}(\mathcal{F}) \).

Proof: In order to prove that (4.37) and (4.39) are valid for \( d\mu(\bar{z}, z) \) given by (4.35-4.36) let us observe that \( d\mu(\bar{z}, z) \) has the form (4.35) since the kernel \( \hat{\sigma}(\cdot - \bar{z}) \) is invariant with respect to the one-parameter group of translation (4.11). Hence we have
\[
\int_{\Sigma} \hat{\sigma}(v - \bar{z}) \hat{\sigma}(z - \bar{w}) d\mu(\bar{z}, z) = \int d\sigma(\omega) \int d\sigma(\omega') \int_{-\infty}^{\infty} dx \int_r^s \mu(y) dy e^{-i\nu\omega + i\bar{w}\omega'} e^{-iz(\omega' - \omega)} e^{iy(\omega + \omega')} \\
= \int d\sigma(\omega) \int d\sigma(\omega') \delta(\omega - \omega') \int_r^s d\mu(y) \, e^{iy(\omega') e^{-i\nu\omega + i\bar{w}\omega'}} \\
= \int d\sigma(\omega) \int d\sigma(\omega + \tau) \delta(\tau) \, d\tau \int_r^s d\mu(y) \, e^{iy(2\omega + \tau)} e^{-i(v - \bar{w})\omega} e^{i\bar{w}\tau} \\
= \int d\sigma(\omega) \frac{d\sigma}{d\omega}(\omega) \int_r^s d\mu(y) \, e^{2y\omega} e^{-i(v - \bar{w})\omega}. \] (4.40)

If \( \mu \) satisfies (4.36) then (4.40) takes the form
\[
\int_{\Sigma} \hat{\sigma}(v - \bar{z}) \hat{\sigma}(z - \bar{w}) d\mu(\bar{z}, z) = \hat{\sigma}(v - \bar{w}), \] (4.41)
which the reproducing property, QED.

In the sequel, let us assume that \( \mathcal{U}(\mathcal{F}) = L^2(\Sigma, d\mu) \), where \( L^2(\Sigma, d\mu) \) denotes the Hilbert space of holomorphic functions which are square-integrable with respect to \( d\mu \) on \( \Sigma \). Due to this assumption \( \mathcal{U} \) is an anti-unitary map and the holomorphic functions
\[
\mathcal{U} | n \rangle = | n | z \rangle = \hat{\sigma}_n(z), \quad n = 0, 1, \ldots \] (4.42)
form an orthonormal basis in \( L^2(\Sigma, d\mu) \).

One has the commutative diagram
\[
\begin{array}{c}
\mathcal{F} \\
\mathcal{U} \\
\downarrow \mathcal{U}^{-1} \\
L^2(\mathbb{R}, d\sigma) \\
\downarrow \mathcal{U}^{-1} \\
L^2(\Sigma, d\mu)
\end{array}
\] (4.43)
where the anti-unitary map \( \mathcal{U} \circ \mathcal{U}^{-1} \) is given by
\[
(\mathcal{U} \circ \mathcal{U}^{-1} \psi)(z) = \int e^{-iz\omega} \psi(\omega) \, d\sigma(\omega), \] (4.44)
where \( \psi \in L^2(\mathbb{R}, d\sigma) \). Thus the Hamiltonian is given by
\[
(\mathcal{U} \circ \mathcal{H}_I \circ \mathcal{U}^{-1}) \psi(z) \equiv i \frac{d}{dz} \psi(z) \] (4.45)
and is defined on the domain \( D(\mathcal{U} \circ \mathcal{H}_I \circ \mathcal{U}^{-1}) = \{ \psi \in L^2(\Sigma, d\mu) : \frac{d}{dz} \psi \in L^2(\Sigma, d\mu) \} \).

In terms of the Hilbert space \( L^2(\Sigma, d\mu) \) it is possible to find an explicit form of the creation operator \( \alpha^* \), i.e. Hermitian conjugate of the spectral annihilation operator \( \alpha \) defined by (4.31). We will call \( \alpha^* \) the spectral creation operator. Using (4.31) and (4.33), we obtain
\[
(\mathcal{U} \circ \alpha^* \circ \mathcal{U}^{-1} \psi)(z) \equiv z\psi(z) \] (4.46)
Thus we see that the domain of \( \mathcal{U} \circ \alpha^* \circ \mathcal{U}^{-1} \) is given by
\[
D(\mathcal{U} \circ \alpha^* \circ \mathcal{U}^{-1}) = \{ \psi \in L^2(\Sigma, d\mu) : z\psi \in L^2(\Sigma, d\mu) \}. \] (4.47)
Taking into the account the above considerations let us notice that the operator \( \alpha^* \) is described explicitly in the \( L^2(\Sigma, d\mu) \)-realization and the operator \( \alpha \) is explicitly given in \( L^2(\mathbb{R}, d\sigma) \)-realization. They satisfy the
canonical commutation relations
\[ [H_f, \alpha] = [H_f, \alpha^*] = i \] (4.48)
with the Hamiltonian \( H_f \), giving
\[ [H_f, \alpha - \alpha^*] = 0 \] (4.49)
i.e. the operator \( \alpha - \alpha^* \) is an integral of motion for the system under consideration.

From a physical point of view (see (3.34, 3.35, 3.36)) it is important to describe the time evolution in interaction picture of the system, i.e. \( e^{-iH_1t} \psi(0) \). To do this let us introduce the following notation for the matrix elements of \( e^{-iH_1t} \)
\[ \tilde{\sigma}_{m,n}(t) := \langle m | e^{-iH_1t} | n \rangle = \int e^{-i\omega t} P_n(\omega) d\sigma(\omega). \] (4.50)
Note that
\[ \tilde{\sigma}_{m,n}(t) = P_n(i \frac{d}{dt}) \tilde{\sigma}_{n}(t), \] (4.51)
where \( \tilde{\sigma}_{n}(t) \) are given by (4.23) and they satisfy
\[ \tilde{\sigma}_{n}(t) = \frac{1}{P_0} \langle n | e^{-iH_1t} | 0 \rangle = P_n(i \frac{d}{dt}) \tilde{\sigma}(t). \] (4.52)
The interaction evolution in the space \( \mathcal{F} \) is thus given by
\[ e^{-iH_1t} \psi = \sum_{m,n=0}^{\infty} \langle m | \psi \rangle \tilde{\sigma}_{m,n}(t) | n \rangle, \] (4.53)
while in the space \( L^2(\mathbb{R}, d\sigma) \) the evolution is described by (4.8). In \( L^2(\Sigma, d\mu) \)-realization we have
\[ (U \circ e^{-iH_1t} \circ U^{-1}) \psi(z) = \Psi(z + t). \] (4.54)
As a consequence of (4.50) and (4.51) we obtain the relation
\[ \tilde{\sigma}_{m,n}(z_1 + z_2) = \sum_k \tilde{\sigma}_{m,k}(z_1) \tilde{\sigma}_{k,n}(z_2), \] (4.55)
which for \( m = 0 \) can be expressed in the form
\[ \tilde{\sigma}_{n}(z_1 + z_2) = \sum_k \tilde{\sigma}_{k,n}(z_1) \tilde{\sigma}_{k,z_2}. \] (4.56)
Moreover, putting \( m = n = 0 \) in (4.55) we obtain the formulae (4.38) for the reproducing kernel.

At the end of Section 3 it was shown (see (3.34), (3.35), (3.36)) that the quantities
\[ \langle \psi(0) | e^{iH_1t} F e^{-iH_1t} | \psi(0) \rangle \] (4.57)
plays an important role if we consider the expectation values of the operator \( F \) on the time evolving state \( | \psi(t) \rangle \) (see (2.28)).

The variety of the realizations of our model, namely, the \( \mathcal{F}, L^2(\mathbb{R}, d\sigma) \) and \( L^2(\Sigma, d\mu) \) representations allow us to give three equivalent formulae on (4.57)
\[ \langle \psi(0) | e^{iH_1t} F e^{-iH_1t} | \psi(0) \rangle = \sum_{m,n,k,l} \langle \psi(0) | m \rangle \tilde{\sigma}_{m,n}^*(t) \langle n | F k \rangle \tilde{\sigma}_{k,l}(t) \langle l | \psi(0) \rangle \]
\[ = \int e^{-i\omega t} \overline{\psi(\omega)} (U \circ F \circ U^{-1}) (e^{i\omega t} \psi(\omega)) d\sigma(\omega) \]
\[ = \int_{\Sigma} \Psi(z + t) (U \circ F \circ U^{-1}) (\Psi(z)) \overline{\sigma(z, \tau)} d\mu(z, \tau), \]
where \( \psi = U|\psi(0)\rangle \) and \( \Psi = U|\psi(0)\rangle \). In this way we have a very strong instrument for calculations of many physical characteristics of the system under consideration.

In particular we have
\[ e^{iH_1t} \alpha e^{-iH_1t} = \alpha + t \] (4.59)
and therefore
\[ \langle \psi(0) | e^{iH_1t} \alpha e^{-iH_1t} | \psi(0) \rangle = \langle \psi(0) | \alpha \psi(0) \rangle + t \langle \psi(0) | \psi(0) \rangle. \] (4.60)

V. INTEGRABLE SYSTEMS RELATED TO CLASSICAL ORTHOGONAL POLYNOMIALS

Here we shall investigate the classes of the physical systems with Hamiltonians of the form (4.36) with the coefficients \( b(n) \) and \( h(n) \) given in table (4.36). The three classes of Hamiltonian operators are related to Hermite, Laguerre and Jacobi polynomials. We choose one mode case for simplicity and the circumstances which make the reduction not necessary. Then the Hamiltonians are expressed in terms of usual creation and annihilation operators in the following form:

\[ H_f^{Her} := -\frac{a_0}{a_1} + \sqrt{-\frac{b_0}{a_1}} (\alpha + \alpha^*), \] (5.1a)
\[ H_f^{Lag} := -\frac{b_1}{a_1} \mu - \frac{b_0}{b_1} - \frac{2b_1}{a_1} \alpha^* \alpha - \frac{b_1}{a_1} \sqrt{\alpha^* \alpha + \mu} \alpha - \frac{b_1}{a_1} \sqrt{\alpha^* \alpha + \mu + 1} \alpha^*, \] (5.1b)
\[ H_{1}^{Jac} := \frac{2a^*a(a+b)(\mu+\nu-1)+2(a^*a)^2(a+b)-2b\mu-2a\nu+\mu\nu(a+b)+b\mu^2+a\nu^2}{(\mu+\nu-2+2a^*a)(\mu+\nu+2a^*a)} \]
\[ +(b-a)\sqrt{\frac{(\mu+a^*a)(\nu+a^*a)(\mu+\nu+a^*a-1)}{(\mu+\nu+2a^*a-1)(\mu+\nu+2a^*a)^2(\mu+\nu+2a^*a+1)}} a \]
\[ +(b-a)\sqrt{\frac{(\mu+a^*a+1)(\nu+a^*a+1)(\mu+\nu+a^*a)}{(\mu+\nu+2a^*a+1)(\mu+\nu+2a^*a+2)^2(\mu+\nu+2a^*a+3)}} a^*. \] (5.1c)

The ranges of the parameters \( \mu, \nu, b_0, \) and \( a_1 \) are chosen such that the operators are well defined and are essentially self-adjoint. In \( L^2(\mathbb{R}, d\sigma) \) (i.e. spectral) representation the formulae (3.30) lead to three-therm recurrence relation (4.4) (see also (A.10)).

From Pearson equation (see (A.5) and table I) we obtain the expressions for measures:

\[ d\sigma^{Herc}(\omega) = C e^{\frac{\alpha_1}{\alpha_2}(\omega+\omega^*)^2} \] (5.2a)
for \( \omega \in \mathbb{R} \),

\[ d\sigma^{Log}(\omega) = C\left(\omega + \frac{b_0}{b_1}\right)^{\mu-1} e^{\frac{\alpha_1}{\alpha_2} \omega} d\omega \] (5.2b)
for \( \omega \in \left(-\frac{b_0}{b_1}, \infty\right) \),

\[ d\sigma^{Jac}(\omega) = C(\omega-a)^{\mu-1}(b-\omega)^{\nu-1} d\omega \] (5.2c)
for \( \omega \in (a,b) \).

In the holomorphic representation \( L^2(\Sigma, d\mu) \) all Hamiltonians act as derivations: \( i\nabla \) (see formulae (4.43)) but the difference between the systems is hidden in the reproducing measures \( d\mu(z,z) = \mu(y) dxdy \), \( z = x + iy \), and the choice of the domain \( \Sigma \). The general case is described in Proposition IV.3. Here we solve equation (1.30) for \( \mu(y) \) in the special class, namely: continuous functions except, possibly finite number of points in every compact subset. The discontinuity points are assumed to be of first kind. Let us summarize the results in the following:

**H) Hermite case:** \( \Sigma = \mathbb{C} \) and

\[ \mu^{Herc}(y) = \frac{1}{C} e^{-\frac{\alpha_1}{\alpha_2}(\omega-\frac{1}{\alpha_2})} \sqrt{\frac{\alpha_1}{2\beta_0\pi}} e^{-\frac{\alpha_1}{\alpha_2}(y-\omega^*)^2} \] (5.3a)

**L) Laguerre case:**

\[ \Sigma = \{z = x + iy \in \mathbb{C} : y < -\frac{a_1}{2b_1}\} \] and for \( \mu > 1 \)

\[ \mu^{Log}(y) = \frac{2}{C} e^{\frac{b_0a_1}{\alpha_1}} \left(-2y - \frac{a_1}{b_1}\right)^{-\mu-2} e^{\frac{2b_0y}{\alpha_1}} \] (5.3b)

For \( \mu = 1 \) we obtain an isomorphism of \( L^2(\Sigma, d\mu) \) with \( H^2(D, d\lambda) \) - the Hardy class of functions on the unit disc \( D \subset \mathbb{C} \) with the measure \( d\lambda \) supported on the circle \( \partial D = \{e^{i\varphi} : \varphi \in [0,2\pi]\} \) and given by

\[ d\lambda = \frac{1}{1-\sin \varphi} d\varphi. \] (5.4)

**J) Jacobi case:** \( \Sigma = \mathbb{C} \) and for \( \mu + \nu > 3 \)

\[ \mu^{Jac}(y) = \begin{cases} \frac{2}{C} \left(-\frac{2y}{b-a}\right)^{\mu-\frac{3}{2}} & \text{for } y > 0 \\ \frac{2}{C} \left(-\frac{2y}{b-a}\right)^{\mu-\frac{1}{2}} & \text{for } y < 0 \end{cases} \] (5.3c)

where \( W_{\kappa,\lambda}(z) \) are confluent hypergeometric Whittaker’s functions (for definition see [4]). This formula simplifies in the case \( \mu = \nu \) corresponding to the Gegenbauer polynomials. The following statement is true for a larger domain of parameter \( \mu \), namely for \( \mu > 1 \),

\[ \mu^{Geg}(y) = \frac{2}{C} \left(-\frac{2y}{b-a}\right)^{\mu-\frac{3}{2}} \times e^{-(b+a)y} \frac{1}{\Gamma(\mu-1)\sqrt{\pi}} K_{\mu-\frac{3}{2}}((a-b)y) \] (5.5)
with $K_n(z)$ being the modified Bessel functions (for definition see (4.5)).

For all three cases one can find the explicit form of matrix elements of propagator (4.5). Because of the relations (4.52) and (4.51), we should display the characteristic functions (4.9) first:

\[
\hat{\sigma}^{H_{\text{er}}}(z) = C \sqrt{-\pi} \frac{2b_0}{a_1} e^{\frac{b_0^2}{a_1}} e^{\frac{b_0}{a_1} (z + \frac{a_0}{b_0})^2}, \quad (5.6a)
\]

\[
\hat{\sigma}^{Lag}(z) = C \Gamma(\mu) e^{-\frac{a_0}{b_0} \mu} \left( -\frac{z}{i} - \frac{a_1}{b_1} \right)^{-\mu} e^{\frac{a_0}{b_0} iz}, \quad (5.6b)
\]

\[
\hat{\sigma}^{Jac}(z) \equiv \hat{\sigma}(z; \mu, \nu) = C \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)}
\times (b - a)^{\mu + \nu - 1} e^{-iaz} F_1(\mu + \nu; (a - b)iz). \quad (5.6c)
\]

The symbols $\hat{\sigma}^{Jac}(z; \mu, \nu)$ are introduced in order to simplify the next formulæ. Using the Rodrigues formula (see (4.8)) we obtain the explicit form of $\hat{\sigma}_n(z)$:

\[
\hat{\sigma}^{H_{\text{er}}}_n(z) = e^{\frac{b_0}{a_1} (ib_0z)^n} \hat{\sigma}^{H_{\text{er}}}(z), \quad (5.7a)
\]

\[
\hat{\sigma}^{Lag}_n(z) = e^{\frac{b_0}{a_1} (ib_0z)^n} \hat{\sigma}^{Lag}(z) e^{\frac{b_0}{a_1} iz}, \quad (5.7b)
\]

\[
\hat{\sigma}^{Jac}_n(z) = e^{\frac{b_0}{a_1} (ib_0z)^n} \hat{\sigma}^{Jac}(z; \mu + n, \nu + n). \quad (5.7c)
\]

After a simple but tedious calculation we find:

\[
\hat{\sigma}^{H_{\text{er}}}_{m,n}(z) = e^{\frac{b_0}{a_1} (z + \frac{a_0}{b_0})^2} e^{\frac{a_0}{b_0} iz} \sqrt{\frac{(b_0/a_1)^{m+n}}{m!n!} \sum_{k=0}^{\min\{m,n\}} \frac{(a_1/b_1)^k}{(m-k)!(n-k)k!}} z^{-2k}, \quad (5.8a)
\]

\[
\hat{\sigma}^{Lag}_{m,n}(z) = e^{\frac{b_0}{a_1} (ib_0z)^n} \hat{\sigma}^{Lag}(z) \sum_{k=0}^{\min\{m,n\}} \frac{i a_1}{b_1 z + ia_1} \left( \frac{a_1}{b_1 z + ia_1} \right)^k 2F_1 \left( \mu + k, -\nu; \frac{a_1}{ia_1 + b_1 z} \right), \quad (5.8b)
\]

\[
\hat{\sigma}^{Jac}_{m,n}(z) = e^{\frac{b_0}{a_1} (ib_0z)^n} (-b_0)^{m+n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+l} \Gamma(\mu + m) \Gamma(\nu + n) \Gamma(\mu + n - l) \Gamma(\nu + k) \Gamma(\nu + l) \times \hat{\sigma}^{Jac}(z; \mu + m + n - k - l, \nu + k + l). \quad (5.8c)
\]

The physical quantities which are of great importance are: the Hamiltonians $H$ and $H_I$, the creation $a^*$ and annihilation $a$ operators, and the occupation number operator $\mathbf{N} = a^* a$. In our case the operators $A$, $A^*$ are also important. They can be interpreted as the cluster annihilation and cluster creation operators. Similarly, the operators $a$ and $a^*$ which are related to the spectral coherent states map (4.22) are interesting, too. Their physical meaning is partially explained by the commutation relations (4.48). They are related to $H_I$, $a$ and $a^*$ in the following way:

\[
\alpha^{H_{\text{er}}} = -\frac{i a_1}{\sqrt{-a_1 b_0}} a, \quad (5.9a)
\]

\[
(b_0 + b_1 H^{Lag}) \alpha^{Lag} = -ib_1 a^* a + i \sqrt{b_1^2 (a^* a + \mu)} a, \quad (5.9b)
\]

In the spectral representation of $H_I$ the operator $\alpha$ is given for all the cases by $i \frac{d}{d\omega}$ but the conjugates are given by different formulœs:

\[
(\alpha^{H_{\text{er}}}^*) = -i \left( \frac{a_1}{b_0} \omega + \frac{a_0}{b_0} + \frac{d}{d\omega} \right), \quad (5.10a)
\]

\[
(a - H^{Jac}) (b - H^{Jac}) \alpha^{Jac} = i a^* a \frac{b(2\mu - \nu + a^* a - 1) + a(2\nu - \mu + a^* a - 1)}{\mu + 2a^* a - 2} - i H^{Jac} a^* a
\]

\[
- i \frac{(b - a)(-\mu - \nu + a^* a - 1)(\mu + a^* a)(\nu + a^* a)}{|b_2| (\mu + \nu + 2a^* a - 1)^2 (\mu + \nu + 2a^* a)^3} \sqrt{\frac{\mu + \nu + 2a^* a + 1}{\mu + \nu + a^* a - 1}} a. \quad (5.9c)
\]
\begin{align}
    (\alpha^{Lag})^* &= -i \left( \frac{a_1 \omega + a_0 - b_1}{b_1 \omega + b_0} + \frac{d}{d\omega} \right) \quad (5.10b) \\
    \text{for } \mu > 1,
    \end{align}

\begin{align}
    (\alpha^{Jac})^* &= -i \left( \frac{a_1 + 2b_2 \omega + a_0 - b_2(a + b)}{b_2(\omega - a)(b - \omega)} + \frac{d}{d\omega} \right) \\
    \text{for } \mu, \nu > 1.
    \end{align}

In the holomorphic representation the operator $\alpha^*$ is given by \((4.46)\), i.e. as the operator of multiplication by the argument $z$. The operators $\alpha^{Her}$, $\alpha^{Lag}$ and $\alpha^{Jac}$ are pseudodifferential ones and we shall not express them explicitly here.

The occupation number operators $N$ defined by \((4.22)\) take in the spectral representation the following form:

\begin{align}
    N^{Her} &= \left( \omega + \frac{a_0}{a_1} \right) \frac{d}{d\omega} + \frac{b_0}{a_1} \frac{d^2}{d\omega^2}, \\
    N^{Lag} &= \left( \omega + \frac{a_0}{a_1} \right) \frac{d}{d\omega} + \left( \frac{b_1}{a_1} \omega + \frac{b_0}{a_1} \right) \frac{d^2}{d\omega^2}. \\
    \text{For the Jacobi case } N^{Jac} \text{ we are able to write down only the relation}
    \end{align}

\begin{align}
    N^{Jac} (N^{Jac} - \mu - \nu - 1) &= (\omega - a)(b - \omega) \frac{d^2}{d\omega^2} \\
    &+ [(\nu - \mu)\omega + \mu b + \nu a] \frac{d}{d\omega}. \quad (5.11c)
    \end{align}

In the holomorphic representation $N$ can be expressed as:

\begin{align}
    N^{Her} &= -\frac{b_0}{a_1} \left( z - i \frac{a_0}{b_0} \right) z + z \frac{d}{dz}, \\
    N^{Lag} &= i \left( \frac{b_1}{a_1} \mu + \frac{b_0}{b_1} + i \frac{b_0}{a_1} \right) z \\
    &+ \left( 1 + \frac{b_1}{a_1} i \frac{z}{d} \right) z \frac{d}{dz}. \quad (5.12b)
    \end{align}

Now, we will present the expectation values on the following states, interesting from the physical point of view:

\begin{align}
    \langle (N^{Her}(t))^l \rangle_z &= e^{\mu z + t^2} \left( -\frac{b_0}{a_1} \right) \left| z + t \right|^2 \Gamma_{\mu - 1} \left( \frac{2, \ldots, 2}{\mu, \ldots, 1}, -\frac{b_0}{a_1} \right) \left| z + t \right|^2, \\
    \langle (N^{Lag}(t))^l \rangle_z &= \left( -\frac{2 a_1 + i \omega - \bar{z}}{a_1^2 + 2 i \omega} \right)^\mu \left| z + t \right|^2 \mu \Gamma_{\mu - 1} \left( 1, \ldots, 1, \mu + 1 \right) \left| z + t \right|^2, \\
    \langle (N^{Jac}(t))^l \rangle_z &= \frac{1}{\bar{\sigma}^{Jac}(z - \bar{z})} \sum_{n=0}^{\infty} c_{\mu}^{Jac} \left| b_2(z + t) \right|^{2n} \left| \bar{\sigma}^{Jac}(z + t; \mu + n, \nu + n) \right|^{2n}. \quad (5.19c)
    \end{align}
We give now the formulae for the correlation functions:

\[
\langle a^* r(t)a^r(t) \rangle_n = \sum_{m=0}^{\infty} \sigma_{m,n+r}(t) \frac{(m+r)!}{m!} \frac{(m+s)!}{m!},
\]

\[
\langle a^* r(t)a^r(t) \rangle_\zeta = e^{-|\zeta|^2} \sum_{k,m=0}^{\infty} \tilde{\sigma}_{m,n+r}(t) \frac{(n+r)!}{n!} \frac{(n+s)!}{n!},
\]

\[
\langle a^* r(t)a^r(t) \rangle_z = \frac{1}{\sigma(z-z)} \sum_{n=0}^{\infty} \tilde{\sigma}_{n+r}(z+t) \frac{(n+r)!}{n!} \frac{(n+s)!}{n!}.
\]

Replacing the creation and annihilation operators \(a^*, a\) by the cluster creation and the cluster annihilation operators \(A^*, A\) we obtain the functions which by analogy will be called the cluster correlation functions:

\[
\langle A^* r(t)A^r(t) \rangle_n = \sum_{k,l=0}^{\infty} b(k+r)b(k+1)b(k+s)b(k+1) \sigma_{n,k+r}(t) \sigma_{k,s+l}(t),
\]

\[
\langle A^* r(t)A^r(t) \rangle_\zeta = e^{-|\zeta|^2} \sum_{k,m,l=0}^{\infty} \tilde{\sigma}_{m,n+r}(t) \frac{(n+r)!}{n!} \frac{(n+s)!}{n!},
\]

\[
\langle A^* r(t)A^r(t) \rangle_z = \frac{1}{\sigma(z-z)} \sum_{k=0}^{\infty} b(k+r)b(k+1)b(k+s)b(k+1) \tilde{\sigma}_{k+r}(z+t) \tilde{\sigma}_{k+s}(z+t).
\]

The time evolution of \(a\) is given by \([5.59]\). This allows us to express the time dependence of \(\langle a^l(t) \rangle_\psi\), (where \(\psi\) is an arbitrary state and \(l \in \mathbb{N}\), in terms of the mean values of some powers \(a \equiv a(0)\) acting on the state \(\psi\):

\[
\langle a^l(t) \rangle_\psi = \sum_{k=0}^{l} \binom{l}{k} t^k \langle a^{l-k} \rangle_\psi.
\]

As a consequence we conclude that the dispersion \((\Delta a(t))_\psi = \sqrt{\langle \psi | [\hat{a}^2(t) - \langle \hat{a}(t) \rangle_\psi^2] | \psi \rangle}\) of the operator \(a(t)\) in an arbitrary state \(|\psi\rangle\) does not depend on time

\[
(\Delta a(t))_\psi = (\Delta a)_\psi.
\]

The following expectations take especially simple form:

\[
\langle a^l(t) \rangle_n = t^l,
\]

\[
\langle a^l(t) \rangle_z = (z+t)^l.
\]

Let us now see what will happen when the intensity of electromagnetic field is sufficiently large (the light of a strong laser). This corresponds to the limit of the large \(a\) in the Hamiltonian \([5.34]\) (see table \(1\) too). We get the following strong-field Hamiltonians \(H_s\):

\[
H_s^{Her} = \sqrt{-\frac{b_0}{a_1}} (a + a^*),
\]

\[
H_s^{Lag} = -2\frac{b_1}{a_1} a^* a
\]

\[
-\frac{b_1}{a_1} (\sqrt{a^2 a + 1} a + \sqrt{a^* a + 2} a^*),
\]

\[
H_s^{Jac} = \frac{a + b}{2} + \frac{b - a}{4} \left( \frac{1}{\sqrt{a^2 a + 1}} a + \frac{1}{\sqrt{a^* a + 2}} a^* \right).
\]

These Hamiltonians belong to the respective families given by \([5.1]\). They are obtained in the Hermite case by putting \(a_0 = 0\) in \([5.1a]\), in the Laguerre case by putting \(\mu = 1\) and \(b_0 = -\frac{b_1}{a_1}\) in \([5.1b]\), and in the Jacobi case by putting \(\mu = \nu = \frac{3}{2}\) in \([5.1c]\).

Let us recall the definition of the phase operator \(\hat{\phi}\) \([23]\):

\[
\exp (i \hat{\phi}) := (a^* a + 1)^{-\frac{1}{2}} a,
\]

\[
\exp (-i \hat{\phi}) := a^* (a^* a + 1)^{-\frac{1}{2}},
\]

\[
\cos (\hat{\phi}) := \frac{1}{2} \left( \exp (i \hat{\phi}) + \exp (-i \hat{\phi}) \right).
\]
We can now rewrite (5.30) and (5.30b):
\begin{align}
H_{s}^{\text{Lag}} &= -\frac{b_{1}}{a_{1}} \left( 2a^{*}a + 2a^{*}a \cos (\phi) + \exp (i\phi) \right) \quad (5.34) \\
H_{s}^{\text{Jac}} &= \frac{a + b}{2} + \frac{b - a}{2} \cos (\phi). \quad (5.35)
\end{align}
So, in the Jacobi case in the strong-field limit, the Hamiltonian tends, up to a constant, to the cosine of the phase operator. This subcase does not depend on the choice of the ranges of the parameters $\mu$, $\nu$.

VI. A PHYSICAL REMARKS

A. Parametric modulator

In order to present some physical interpretations of the Hamiltonian (2.25) with $H_{j}$ given by (2.37), let us rewrite it in the following form

\begin{align}
\mathbf{H}_{I} &= \sum_{j=0}^{M} \omega_{j}a_{j}^{*}a_{j} + h(a_{j}^{*}a_{0}, \ldots, a_{j}^{*}a_{M}) \\
&\quad + \left( e^{i\sum_{j=0}^{M} \omega_{j}a_{j}^{*}a_{j}} (g(a_{0}^{*}a_{0}, \ldots, a_{M}^{*}a_{M})a_{0}^{l_{0}} \ldots a_{M}^{l_{M}} + [ e^{i\sum_{j=0}^{M} \omega_{j}a_{j}^{*}a_{j}} (g(a_{0}^{*}a_{0}, \ldots, a_{M}^{*}a_{M})a_{0}^{l_{0}} \ldots a_{M}^{l_{M}})]^{*} \right). \quad (6.1)
\end{align}

The first term, which is linear in photon number operators describes the free field. The second term, which is an arbitrary function of these operators may be treated as a generalization of the Kerr medium description, where $H_{I} = \frac{\chi}{2} ( (a^{*}a)^{2} - a^{*}a) $, where $\chi$ is proportional to the third-order nonlinear susceptibility. [23]. The terms of the type $h(a_{0}^{*}a_{0}, \ldots, a_{M}^{*}a_{M})$, after the appropriate choice of the function $h$, play an important role in the theory of the nondemolition measurement [27, 19] and in the description of many other phenomena e.g. the optical bistability effect [4].

The last term in (6.1) one can interpreted as a general form of the parametric modulator Hamiltonian. To motivate this interpretation let us recall the form of the Hamiltonian of nondegenerate parametric amplifier [27, 13]:
\begin{align}
\mathbf{H} &= \omega_{0}a_{0}^{*}a_{0} + \omega_{1}a_{1}^{*}a_{1} \\
&\quad + ig \left( e^{2i\omega_{0}t}a_{0}a_{1} - (e^{2i\omega_{0}t}a_{0}a_{1})^{*} \right). \quad (6.2)
\end{align}
This Hamiltonian describes the case when the classical pump mode at frequency $2\omega$ interacts in a nonlinear optical medium with two modes at frequency $\omega_{0}$ and $\omega_{1}$, such that $\omega_{0} + \omega_{1} = 2\omega$. If the system starts in an initial Gaussian 2-photon coherent state $|\xi_{0}G_{1}\rangle$, the mean photon number in 0-mode after time $t$ is
\begin{align}
\langle a_{0}^{\dagger}(t)a_{0}(t) \rangle = |\xi_{0}\cosh gt + \xi_{1}^{*}\sinh gt|^{2} + \sinh^{2} gt, \quad (6.3)
\end{align}
hence this mode is amplified. The next example is the Hamiltonian for the frequency up-converter (27):
\begin{align}
\mathbf{H} &= \omega_{0}a_{0}^{*}a_{0} + \omega_{1}a_{1}^{*}a_{1} + \kappa \left( e^{i\omega t}a_{0}^{*}a_{1} + e^{-i\omega t}a_{0}a_{1}^{*} \right), \quad (6.4)
\end{align}
where $\omega = \omega_{1} - \omega_{0}$.

It is easy to compare (6.1) with (6.2) and with (6.4) and conclude that our Hamiltonian is a natural generalization of that describing parametric amplification. In order to understand that in general (6.1) describes not only amplification but also modulation let us notice that due to (3.3) we can express the mean values $\langle a_{j}^{\dagger}(t)a_{j}(t) \rangle, j = 0, \ldots, M$ in terms of the mean values of the operators $A_{0}(t), \ldots, A_{M}(t)$. But $A_{1}(t), \ldots, A_{M}(t)$ are the integrals of the motion, so if our system starts at the initial state from the reduced subspace $\mathcal{F}$ (see (3.26)) we obtain
\begin{align}
\langle a_{j}^{\dagger}(t)a_{j}(t) \rangle = l_{j}\langle A_{0}(t) \rangle + \beta_{j}, \quad (6.5)
\end{align}
where the constant $\beta_{j}$ are uniquely determined by $\lambda_{1}, \ldots, \lambda_{M}$ and the matrix $\kappa$. This means that the mean photon number in each mode is a linear function of $\langle A_{0}(t) \rangle$ or, in other words, the strength of the light in each mode is modulated by the function $\langle A_{0}(t) \rangle$. The modulation of the $j$-th mode depends on the exponent $l_{j}$.

As an example of the modulation function $\langle A_{0}(t) \rangle$ let us consider the situation when after reduction, we obtain the case corresponding to Laguerre polynomials and the initial state is the spectral coherent state $|z\rangle$. From (5.19b) we obtain
\begin{align}
\langle A_{0}(t) \rangle_{z} = E|z + t|^{2} + F, \quad (6.6)
\end{align}
where the real constants $E, F$ depend on $\mu, a_{1}, b_{1}$ and $\lambda_{0,1}$. In this example the modulation function is of parabolic shape. This means, that in some interval of time we have the amplification and dumping of the light signal in others.
B. Generalized squeezed states

The special cases of the interaction evolution operators $e^{-iHt}$ are the unitary displacement operators $U(\zeta) = \exp(\zeta a^* - \zeta a)$, $\zeta \in \mathbb{C}$, (6.7) the unitary squeeze operators $S(z) = \exp(\bar{z}a^2 - za^2)$, $z \in \mathbb{C}$ (6.8) and the unitary two-mode squeeze operators $T(\xi) = \exp(\bar{\xi}a_0a_1 - \xi a_0^*a_1^*)$, $\xi \in \mathbb{C}$. (6.9)

This means that the Glauber coherent states and the squeezed states are special cases of the spectral coherent states defined in Section 4. In such a way, the two concepts of the notion of the coherent states meet each other in our framework. The first one, presented in [24], [16], [6], [8], is related to the minimalization of the suitable uncertainty relations. The second one, presented in [21], [20] is based on the symplectic embedding of the classical phase space of the system into the quantum phase space (equipped with the Fubbini-Study symplectic form).

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Appendix

Here we present some facts from the theory of classical polynomials.

Let us consider a pair of real polynomials $(A(\omega), B(\omega))$ of degree not greater than one and two, respectively

$$A(\omega) := a_1 \omega + a_0, \quad a_i \in \mathbb{R},$$

$$B(\omega) := b_2 \omega^2 + b_1 \omega + b_0, \quad b_i \in \mathbb{R}. \quad (A.1)$$

The Pearson equation associated with $(A(\omega), B(\omega))$ on the interval $(a, b) \subset \mathbb{R}$ ($-\infty < a < b < +\infty$) is the differential equation for the weight function $\rho$:

$$\frac{d}{d\omega} (\rho B) = \rho A \quad (A.3)$$

with the boundary conditions

$$\rho(a) B(a) = 0 = \rho(b) B(b). \quad (A.4)$$

Each family of classical orthogonal polynomials $\{P_n\}$ can be obtained by the Gram-Schmidt orthogonalization of the basis $\{\omega^n\}_{n=0}^{\infty}$ in $L^2(\mathbb{R}, d\sigma)$, where

$$d\sigma(\omega) := \begin{cases} \frac{\rho(\omega)}{\rho} d\omega & \omega < a \\ 0 & a \leq \omega \leq b \\ \frac{1}{\rho(\omega)} d\omega & \omega > b \end{cases} \quad (A.5)$$

and $\rho$ satisfies the Pearson equation with appropriately chosen polynomials $(A(\omega), B(\omega))$. Namely if $\text{deg} B(\omega) = 0$ (i.e. $b_2 = b_1 = 0$) then we obtain the Hermite polynomials; if $\text{deg} B(\omega) = 1$ (i.e. $b_2 = 0$, $b_1 \neq 0$), we obtain the Laguerre polynomials and if $\text{deg} B(\omega) = 2$ (i.e. $b_2 \neq 0$), we obtain the Jacobi polynomials. In the last case the boundary conditions (A.4) hold if and only if if $a$ and $b$ are roots of $B(\omega)$. For solution of the Pearson equation in these cases see table I. Additional conditions enforced on $A(\omega)$ by (A.4) are presented in this table too.

By straightforward calculation one can prove that the family of polynomials $\left\{ \frac{d^k}{d\omega^k} P_n(\omega) \right\}_{n=0}^{\infty}$, $k \in \mathbb{N}$, is orthogonal in the space $L^2(\mathbb{R}, d\sigma(\omega))$, where

$$d\sigma(\omega) := B^k(\omega) d\sigma(\omega). \quad (A.6)$$

The weight function $\rho(\omega)$ satisfies the Pearson equation on the interval $(a, b)$ associated with $(A(k)(\omega), B(\omega))$, where

$$A(k)(\omega) := A(\omega) + k \frac{dB(\omega)}{d\omega}. \quad (A.7)$$

**Proposition A.1** For a given Pearson data i.e. a pair $(A(\omega), B(\omega))$ on $(a, b) \subset \mathbb{R}$ the following statements are equivalent:

A. $\{P_n(\omega)\}_{n=0}^{\infty}$ form an orthonormal system in $L^2(\mathbb{R}, d\sigma)$.

B. The polynomials are given by Rodrigues’ formula

$$P_n(\omega) = c_n \frac{1}{\rho(\omega)} \frac{d^n}{d\omega^n} (\rho(\omega) B^n(\omega)), \quad (A.8)$$

where $c_n$ is the normalising constant. (see table IV)

C. The polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ satisfy the differential equation

$$\left( A(\omega) \frac{d}{d\omega} + B(\omega) \frac{d^2}{d\omega^2} \right) P_n(\omega) = \lambda_n P_n(\omega), \quad (A.9)$$

where $\lambda_n = a_1 n + b_2 (n-1)$.

D. The polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ are related by the three-term recurrence formula (for $h(n)$ and $b(n)$ see table IV)

$$\omega P_n(\omega) = h(n) P_n(\omega) + b(n) P_{n-1}(\omega) + (n+1) P_{n+1}(\omega) \quad (A.10)$$

with the initial condition

$$P_0(\omega) \equiv \text{const} = \left[ \int d\sigma(\omega) \right]^{-\frac{1}{2}}. \quad (A.11)$$
TABLE I:

| Pearson data | Weight function | Additionally conitions |
|--------------|-----------------|------------------------|
| $A(\omega)$  | $B(\omega)$     | $(a, b)$ $\rho(\omega)$ | $\rho^{Her}(\omega) = C \frac{a}{b} \frac{a_n \omega + a_n}{(n+1)}^2$ $C > 0$, $\frac{a_n}{b} < 0$ |
| $A^{Log}(\omega) = a_1 \omega$ | $B^{Log}(\omega) = b_0 \omega$ | $(-\infty, \infty)$ $\rho^{Log}(\omega) = C (\omega + \frac{b_0}{b_1})^{\mu - 1} e^{\frac{a_n}{b_1} \omega}$ $C > 0$, $\frac{a_n}{b_1} < 0$, $\mu := \frac{a n + b_0 \omega}{b_1} > 0$ |
| $A^{Jac}(\omega) = a_1 \omega$ | $B^{Jac}(\omega) = b_2 (\omega - a)(b - \omega)$ | $(a, b)$ $\rho^{Jac}(\omega) = C (\omega - a)^{\mu - 1} (b - \omega)^{\nu - 1}$ $C > 0$, $a < b$, $b_2 > 0$, $\mu := \frac{a_n + b_2 (a - b)}{b_2 (b - a)} > 0$, $\nu := \frac{b_2 + a_n}{b_2 (b - n)} > 0$ |

TABLE II:

cn — in Rodrigues’ formula

\[
c_n^{Her} = \left( C n! \left( -a_1 b_0 \right) ^n \sqrt{-n \frac{b_0}{a_1}} \right) ^{-\frac{1}{2}}
\]

\[
c_n^{Log} = \left( C n! \left( -a_1 b_1 \right) ^n \left( -b_1 \right) ^{\mu + n} \Gamma (\mu + n) e^{-\frac{a n b_0}{b_1}} \right) ^{-\frac{1}{2}}
\]

\[
c_n^{Jac} = \left( C n! b_2 ^{2n} (b - a) ^{\mu + \nu + 2n - 1} \frac{\Gamma (\mu + n) \Gamma (\nu + n)}{(\mu + \nu + 2n - 1)! (\mu + \nu + n - 1)!} \right) ^{-\frac{1}{2}}
\]

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### TABLE III:

| $b(n)$ | $h(n)$ |
|--------|--------|
| $b^{Her}(n) = \sqrt{-\frac{b_0}{a_1}n}$ | $h^{Her}(n) = -\frac{2n}{a_1}$ |
| $b^{Las}(n) = \frac{b_1}{a_1} \sqrt{n(n+\mu-1)}$ | $h^{Las}(n) = -\frac{b_1}{a_1} (2n+\mu) - \frac{b_0}{a_1}$ |
| $b^{Jac}(n) = (b-a) \sqrt{\frac{n(\mu+n-1)(\nu+n-1)(\mu+\nu+n-2)}{(\mu+\nu+2n-3)(\mu+\nu+2n-2)^2(\mu+\nu+2n-1)}}$ | $h^{Jac}(n) = \frac{2n(a+b)(\mu+\nu-1)+2n^2(a+b)-2\mu^2-2\nu^2+\mu\nu(b^2+a^2)}{(\mu+\nu+2n-2)(\mu+\nu+2n)}$ |

**Attention** The necessary condition $b(0) = 0$ is automatically satisfied with the exception of the Jacobi case for $\mu = \nu = \frac{1}{2}$ and $\mu = \nu = \frac{3}{2}$ where we must put it additionally.