MINIFOLDS AND PHANTOMS

SERGEY GALKIN, LUDMIL KATZARKOV, ANTON MELLIT, EVGENY SHINDER

Abstract. A minifold is a smooth projective $n$-dimensional variety such that its bounded derived category of coherent sheaves $D^b(X)$ admits a semi-orthogonal decomposition into an exceptional collection of $n + 1$ exceptional objects. In this paper we classify minifolds of dimension $n \leq 4$.

We discuss the structure of the derived category of fake projective spaces and conjecture that under some conditions they admit a quasi-phantom subcategory.

1. Introduction.

The question of homological characterization of projective spaces goes back to Severi, and the pioneering work of Hirzebruch–Kodaira [19]. Beautiful results have been obtained by Kobayashi–Ochiai [25], Yau [43], Fujita [15], Libgober–Wood [31].

Among smooth projective varieties of given dimension projective spaces have the smallest cohomology groups. We call a smooth projective variety a $\mathbb{Q}$-homology projective space if it has the same Hodge numbers as a projective space. Any odd-dimensional quadric is an example of $\mathbb{Q}$-homology projective space. We call an $n$-dimensional $\mathbb{Q}$-homology projective space of general type a fake projective space if in addition it has the same Hilbert polynomial (with respect to canonical line bundle) as $\mathbb{P}^n$. Any fake projective plane is simply a $\mathbb{Q}$-homology plane of general type, since Hodge numbers of a surface determine its Hilbert polynomial. On the level of realizations over $\mathbb{C}$, e.g. from the point of view of the Hodge structure, fake projective spaces are identical to projective spaces, however the study of their $K$-theory, motive or derived category meets cohomological subtleties.

The first example of a fake projective plane was constructed by Mumford [34] using $p$-adic uniformization developed by Drinfeld [14] and Mustafin [35]. From the point of view of complex geometry fake projective planes have been studied by Aubin [2] and Yau [43], who proved that any such surface $S$ is uniformized by a complex ball, hence by Mostow’s rigidity theorem $S$ is determined by its fundamental group $\pi_1(S)$ uniquely up to complex conjugation. Further Klingler and Yeung [24, 44] proved that $\pi_1(S)$ is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$. Finally such groups have been classified by Cartwright–Steger [12] and Prasad–Yeung [39], so all fake projective planes fit into 100 isomorphism classes.

Fake projective fourspaces were introduced and studied by Prasad and Yeung in [40].

Date: May 21, 2013.

This work was partially supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, Grant-in-Aid for Scientific Research (10554503) from Japan Society for Promotion of Science, and AG Laboratory NRU-HSE, RF government grant, ag. 11.G34.31.0023. S. G. and L. K. were funded by grants NSF DMS0600800, NSF FRG DMS-0652633, NSF FRG DMS-0854977, NSF DMS-0854977, NSF DMS-0901330, grants FWF P 24572-N25 and FWF P20778, and an ERC grant — GEMIS. E. S. has been supported by the Max-Planck-Institut für Mathematik, SFB 45 Bonn-Essen-Mainz grant and the Hausdorff Center, Bonn. S. G. and A. M. would like to thank ICTP for inviting them to the “School and Conference on Modular Forms and Mock Modular Forms and their Applications in Arithmetic, Geometry and Physics” to Trieste in March 2011, where this collaboration started, and IPMU for A. M.’s visit in November 2011, when the first version of this paper was completed.
In this paper we take a different perspective that started with a seminal discovery of *full exceptional collections* by Belinson [3], Kapranov [21], Bondal and Orlov [10] with their students Kuznetsov, Razin, Samokhin (see [29]): they found out that all known to them examples of Fano $\mathbb{Q}$-homology projective spaces admit a full exceptional collection of vector bundles. They put a conjecture that gives a homological characterization of projective spaces based on derived categories, and in this paper we prove it in Theorem 1.1(3).

We call an $n$-dimensional smooth complex projective variety a *minifold* if it has a full exceptional collection of minimal possible length $n + 1$ in its bounded derived category of coherent sheaves. A minifold is necessarily a $\mathbb{Q}$-homology projective space. Projective spaces and odd-dimensional quadrics are examples of minifolds: this follows from seminal work of Beilinson [3] and Kapranov [21].

It follows from work of Bondal, Bondal–Polishchuk and Positselski [8, 11, 38], that if a minifold $X$ is not Fano then all full exceptional collections on it are not strict and consist not of pure sheaves. In fact it is expected that all minifolds are Fano.

We now formulate our main theorem, which gives a classification of minifolds in dimension less or equal than 4 (with one-dimensional case being trivial).

**Theorem 1.1.**

1) The only two-dimensional minifold is $\mathbb{P}^2$.

2) The minifolds of dimension 3 are: the projective space $\mathbb{P}^3$ the quadric $Q^3$, the del Pezzo quintic threefold $V_5$ a six-dimensional family of Fano threefolds $V_{22}$.

3) The only four-dimensional Fano minifold is $\mathbb{P}^4$.

**Remark 1.2.** We also show that except for $\mathbb{P}^4$ the only possible minifolds of dimension 4 are non-arithmetic fake projective fourfolds, which presumably do not exist [40] (paragraph 4 and section 8.4).

In Section 2 we recall the necessary definitions and facts. In particular in Proposition 2.1 we recall that varieties admitting full exceptional collections have Tate motives with rational coefficients [33] and outline a straightforward proof of that fact. Section 2 finishes with the proof of Theorem 1.1.

In Section 3 we then give a conjecture that under some conditions fake projective $n$-spaces admit non-full exceptional collections of length $n + 1$ and thus have quasi-phantoms in their derived categories (Conjecture 3.1 and its Corollary 3.2).

It took a long way for the paper to take its present form. We would like to thank our friends and colleagues with whom we had fruitful discussions on the topic.

We thank Denis Auroux, Alexey Bondal, Alessio Corti, Alexander Efimov, Sergey Gorchinskiy, Jeremiah Heller, Daniel Huybrechts, Umut Isik, Maxim Kontsevich, Alexander Kuznetsov, Dmitri Orlov, Dmitri Panov, Tony Pantev, Konstantin Shramov, Vadim Vologodsky, and Alexander Voronov for their useful suggestions, references and careful proofreading. We thank Gopal Prasad, Sai Kee Yeung, Philippe Eyssidieux and Bruno Klingler and for answering our questions about fake projective planes and fourspaces.
2. Minifolds

An exceptional collection of length \( r \) on a smooth projective variety \( X/\mathbb{C} \) is a sequence of objects \( E_1, \ldots, E_r \) in the bounded derived category of coherent sheaves \( D^b(X) \) such that \( \text{Hom}(E_j, E_i[k]) = 0 \) for all \( j > i \) and \( k \in \mathbb{Z} \), and moreover each object \( E_i \) is exceptional, that is spaces \( \text{Hom}(E_i, E_i[k]) \) vanish for all \( k \) except for one-dimensional spaces \( \text{Hom}(E_i, E_i) \). An exceptional collection is called full if the smallest triangulated subcategory which contains it, coincides with \( D^b(X) \).

**Proposition 2.1.** Assume that \( X \) admits a full exceptional collection of length \( r \). Then:

1. The Chow motive of \( X \) with rational coefficients is a direct sum of \( r \) Tate motives \( \mathbb{L}^j \). In particular, all cohomology classes on \( X \) are algebraic. (For the definition and properties of Chow motives see 32.)
2. \( H^{p,q}(X) = 0 \) for \( p \neq q \) and \( \chi(X) = \sum h^{p,p}(X) = r \).
3. \( \text{Pic}(X) \) is a free abelian group of finite rank. Moreover the first Chern class map gives an isomorphism \( \text{Pic}(X) \cong H^2(X, \mathbb{Z}) \).
4. \( H_1(X, \mathbb{Z}) = 0 \).
5. The Grothendieck group \( K_0(X) = K_0(D^b(X)) \) is free of rank \( r \) and the bilinear Euler pairing

\[
\chi(E, F) = \sum_i (-1)^i \dim \text{Hom}(E, F[i])
\]

is non-degenerate and unimodular. Classes \([E_i]\) of exceptional objects form a semi-orthonormal basis in \( K_0(X) \). (By a semiorthonormal basis we mean a basis \((e_i)_{i=0}^n\) such that \( \chi(e_j, e_i) = 0 \), \( j > i \) and \( \chi(e_i, e_i) = 1 \).)

**Proof.** Most of the claims are well-known. (1) is proved in 33 using the language of non-commutative motives and in 18 using \( K \)-motives. We give a direct proof of (1) using the ideas developed in 37 (see also 4) for the sake of completeness.

First observe that the structure sheaf of the diagonal \( O_\Delta \) in the derived category \( D^b(X \times X) \) lies in the full triangulated subcategory generated by the objects \( p_1^*F_1 \otimes p_2^*F_2 \). This can be deduced from the standard fact that if \( E_1, \ldots, E_r \) is a full exceptional collection on \( X \), then \( p_1^*E_i \otimes p_2^*E_j \) forms a full exceptional collection on \( X \times X \) 34, 41.

It follows that the class of the diagonal \([O_\Delta] \in K_0(X \times X) \) has a decomposition

\[
[O_\Delta] = \sum_j p_1^*[\mathcal{F}_j] \cdot p_2^*[\mathcal{G}_j],
\]

for some sheaves \( \mathcal{F}_j, \mathcal{G}_j \) on \( X \).

Applying the Chern character to (2.1) and using the Grothendieck-Riemann-Roch formula

\[
\text{ch}(O_\Delta) = [\Delta] \cdot p_2^*td(X)
\]

we obtain an analogous decomposition for the class of the diagonal \([\Delta] \in CH^*(X \times X)_\mathbb{Q} \):

\[
[\Delta] = \sum_j p_1^*[\alpha_j] \cdot p_2^*[\beta_j],
\]

for some classes \( \alpha_j, \beta_j \in CH^*(X)_\mathbb{Q} \). We may assume that \( \alpha_j \) are homogenous, say \( \alpha_j \in CH^{\alpha_j}(X)_\mathbb{Q} \) and hence \( \beta_j \in CH^{\dim(X) - \alpha_j}(X)_\mathbb{Q} \).
We claim that the set \( \{ \alpha_j \} \) spans \( CH^*(X)_\mathbb{Q} \). Indeed for any \( \alpha \in CH^*(X)_\mathbb{Q} \) we have

\[
\alpha = p_1^*([\Delta] \cdot p_2^* \alpha) = p_1^*((\sum_j p_1^* \alpha_j \cdot p_2^* \beta_j) \cdot p_2^* \alpha) = \sum_j \alpha_j \cdot p_1^*(p_2^*(\beta_j \cdot \alpha)) = \sum_j \langle \beta_j, \alpha \rangle \alpha_j.
\]

(2.3)

Here we use the notation \( \langle \alpha, \beta \rangle \) for the bilinear form \( \deg(\alpha \cdot \beta) \).

We may assume that \( \{ \alpha_j \} \) are linearly independent, that is form a homogeneous basis of \( CH^*(X)_\mathbb{Q} \). From the formula (2.3) we see that \( \{ \beta_j \} \) is a dual basis.

We now define an isomorphism \( M(X) \cong \bigoplus_j \mathbb{L}^a_j \). By definition of the morphisms in the category of Chow motives we have

\[
\text{Hom}(\mathbb{L}^a, M(X)) = CH^a(M)
\]
\[
\text{Hom}(M(X), \mathbb{L}^a) = CH^{\dim(X)-a}(M)
\]

Therefore the set \( \{ \alpha_j \} \) determines a morphism of motives

\[
\Phi: \bigoplus_j \mathbb{L}^a_j \to M(X)
\]

and the \( \{ \beta_j \} \) determines a morphism in the opposite direction

\[
\Psi: M(X) \to \bigoplus_j \mathbb{L}^a_j.
\]

The composition \( \Psi \circ \Phi \) is equal to identity due to the fact that \( \{ \alpha_j \} \) and \( \{ \beta_j \} \) are dual bases. The composition \( \Phi \circ \Psi \) is equal to identity because of the decomposition (2.2).

By taking Hodge realization (1) implies (2). Alternatively, we can deduce (2) from Hochschild-Kostant-Rosenberg theorem

\[
HH_i(D^b(X)) \cong \bigoplus_{p-q=i} H^{p,q}(X)
\]

and additivity of Hochschild homology for semiorthogonal decompositions [30].

The fact that \( \text{Pic}(X) \) is free follows from (5) and Lemma 2.2 below. The isomorphism \( \text{Pic}(X) \cong H^2(X, \mathbb{Z}) \) comes from the exponential long exact sequence and (2). \( \text{Pic}(X) \) is of finite rank since it is isomorphic to \( H^2(X, \mathbb{Z}) \).

To prove (4) note that the Universal Coefficient Theorem implies that we have a non-canonical isomorphism

\[
H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{rk} \oplus H_1(X, \mathbb{Z})^{\text{tors}}
\]

which by (3) implies that \( H_1(X, \mathbb{Z}) \) must be torsion-free as well. On the other hand \( h^{1,0}(X) = 0 \) and hence \( H_1(X, \mathbb{Z}) = 0 \).

(5) follows easily from definitions.

\[\square\]

**Lemma 2.2.** Let \( X \) be a smooth algebraic variety such that \( K_0(X) \) has no \( p \)-torsion. Then \( \text{Pic}(X) \) has no \( p \)-torsion.
Lemma 2.5. Let $X$ be a minifold. Then

- either $X$ is a Fano variety i.e. the anticanonical line bundle $\omega_X^c = \det T_X$ is ample
- or canonical line bundle $\omega_X = \det T_X^*$ is ample

In particular, the variety $X$ is uniquely determined by $\mathcal{D}^b(X)$.

Proof. We first note that $\omega_X$ is not trivial, since $h^0(\omega_X) = h^n,0(X) = 0$ by Proposition 2.1(2). By Proposition 2.1(3), $Pic(X)$ is torsion free; hence the class of $\omega_X$ in $Pic(X) \cong H^2(X, \mathbb{Q}) = \mathbb{Q}$ is non-zero. Therefore either $\omega_X$ or $\omega_X^c$ is ample. Now the Bondal-Orlov [10] reconstruction theorem implies the last statement. □

Remark 2.6. If we weaken the assumption from ”projective” to ”proper” in the definition of a minifold, we still get the same class of varieties. Indeed, if $X$ is a proper smooth variety of dimension $n$ with a full exceptional collection of length $n + 1$ we can still deduce that $\omega_X$ or its dual is ample, in particular that $X$ is projective as follows.

From [31] Theorem 3 it follows that for a compact complex $n$-dimensional manifold, the Chern number $c_1c_{n-1}$ is determined by Hirzebruch $\chi$-genera $\chi_y$ and hence by the Hodge numbers.
Thus we have $c_1 c_{n-1}[X] = c_1 c_{n-1}[\mathbb{P}^n] = \frac{n(n+1)^2}{2} \neq 0$. Since the Kleiman-Mori cone of effective one-cycles modulo numerical equivalence $N_1(X) \subset H^2(X, \mathbb{R})$ is one dimensional (that is because $H^2(X, \mathbb{R})$ itself is one dimensional by Proposition 2.1(2) which still holds under the assumption that $X$ is proper), Kleiman’s criterion for ampleness implies that either $\omega_X$ or its dual is ample.

The rest of this section is devoted to proof of Theorem 1.1. In view of Lemma 2.5, the proof consists of classifying Fano minifolds and showing that there is no minifolds among varieties of general type.

We start in dimension 2. The only del Pezzo surface with Picard number one is a projective plane.

On the other hand it is known that fake projective planes have non-vanishing torsion first homology group, Theorem 10.1. Hence by Proposition 2.1(4) there is no minifold of general type of dimension 2.

Let us consider Fano threefolds. By Proposition 2.1(2) conditions $b_2(X) = 1$ and $b_3(X) = 0$ are necessary for a minifold. Such Fano threefolds were classified by Iskovskikh [20] into four deformation types: the projective space $\mathbb{P}^3$, the quadric $Q^3$, the del Pezzo quintic threefold $V_5$, and a family of Fano threefolds $V_{22}$.

All these varieties are known to admit an exceptional collection of length 4 by results of Beilinson, Kapranov, Orlov and Kuznetsov respectively [3], [21], [36], [28].

It is easy to see that 3-dimensional $\mathbb{Q}$-homology varieties of general type do not exist. Indeed $K_X$ ample implies that $c_1(X)^3$ is negative, but by Todd’s theorem $c_1(X)c_2(X) = 24$. This contradicts to Yau’s inequality $c_1(X)^3 \geq \frac{8}{3} c_2(X) c_1(X)$ [43].

According to Wilson [42] and Yeung [45] there are three alternatives for a $\mathbb{Q}$-homology projective fourspace $X$: either $X$ is $\mathbb{P}^4$, or $X$ is a fake projective fourspace, or $X$ has Hilbert polynomial $\chi(\omega_X^{-1}) = 1 + \frac{25}{8}(l+1)(3l^2 + 3l + 2)$ and Chern numbers $[c_1^4, c_2^2, c_1 c_3, c_4] = [225, 150, 100, 50, 5]$. In what follows the varieties of the latter type are named Wilson’s fourfolds.

There are some known examples of fake projective fourfolds, but it is not known whether any Wilson’s fourfold actually exist.

In what follows we show that (possibly non-existent) Wilson’s fourfolds do not satisfy conditions of Proposition 2.1(5), and hence do not admit a full exceptional collection. In order to do that we relate the Grothendieck group of a minifold to its Hilbert polynomial.

We need a simple Lemma from linear algebra.

**Lemma 2.7.** Let $P(x) = \sum_{j=0}^{n} p_j x^j \in K[x]$ be a polynomial of degree $\leq n$ with coefficients in a field $K$ of characteristic zero and let $A_P$ be the $(n+1) \times (n+1)$-matrix with coefficients $a_{i,j} = P(j-i)$. Then we have

$$det(A_P) = (n! p_n)^{n+1}.$$  

In particular the matrix $A_P$ is non-degenerate if and only if $d e g P = n$.

**Proof.** It suffices to prove the statement for algebraic closure $\bar{K}$ of $K$, we thus assume $K$ to be algebraically closed.

We first prove that

$$det(A_P) = 0 \iff p_n = 0.$$  

(2.4)
Indeed if $\deg(P(x)) < n$, then $n+1$ polynomials $P(x), P(x+1), \ldots, P(x+n)$ are linearly dependent which makes the columns of $A_P$ linearly dependent, thus $\det(A_P) = 0$. On the other hand, it is easy to see that if $\deg(P(x)) = n$, then

$$P(x), P(x+1), \ldots, P(x+n)$$

form a basis of the space of polynomials of degree $\leq n$, and $A_P$ is a matrix of an invertible linear transformation $P \mapsto (P(0), P(-1), \ldots, P(-n)) \in K^{n+1}$ in this basis, hence $\det(A_P) \neq 0$.

Let $F(p_0, p_1, \ldots, p_n) = \det(A_P)$. Since the entries of the matrix $A_P$ are linear forms in $p_0, p_1, \ldots, p_n$, it follows that $F$ is homogeneous in $p_i$’s of degree $n + 1$. Then (2.4) says that the support of the degree $n + 1$ hypersurface $F = 0$ in $\mathbb{P}^n$ is contained in the hyperplane $p_n = 0$. Therefore

$$F(p_0, p_1, \ldots, p_n) = C_n \cdot p_n^{n+1}$$

for some constant $C_n \in K$. In particular $\det(A_P)$ takes the same value $C_n$ for any monic polynomial $P(x)$ of degree $n$.

Let $P_0(x) = (x+1) \cdot (x+2) \cdots (x+n)$. Then the matrix $A_{P_0}$ is upper triangular with all diagonal entries equal to $n!$:

$$C_n = \det(A_{P_0}) = (n!)^{n+1}$$

The result now is the combination of (2.5) and (2.6) $\square$

**Proposition 2.8.** Let $X$ be a minifold. Let $O(1) = \det(T_X)$ be the anticanonical bundle, $\deg(X)$ be the anticanonical degree $c_1(X)^n$ and $P_X(k) = \chi(O(k))$ be the Hilbert polynomial. Consider a sublattice $\Lambda \subset K_0(X)$ spanned by

$$[O], [O(1)], \ldots, [O(n)].$$

Then the Euler pairing restricted to $\Lambda$ is non-degenerate, that is classes $[O], [O(1)], \ldots, [O(n)]$ are linearly independent in $K_0(X)$ and $\Lambda$ is a sublattice in $K_0(X)$ of full rank. Furthermore, $\Lambda$ admits a semi-orthonormal basis over the ring $\mathbb{Z}[\frac{1}{\deg(X)}]$ and hence modulo any prime $p$ that does not divide $\deg(X)$.

**Proof.** Let $A_X$ denote the matrix of the pairing on $\Lambda$, that is a matrix with entries $a_{i,j} = \chi(O(i), O(j)) = P_X(j - i)$.

We apply Lemma 2.7 to $P = P_X$, the Hilbert polynomial. Its top coefficient is equal to $p_n = \frac{\deg(X)}{n!}$; therefore $\det(A_X) = \deg(X) \neq 0$ is the anticanonical degree and the pairing on $\Lambda$ is non-degenerate.

The inclusion $\Lambda \subset K_0(X)$ becomes an isomorphism after inverting $\det(A_X) = \deg(X)$. Indeed let $e_j, j = 0, \ldots, n$ be a basis in $K_0(X)$ and write

$$[O(i)] = \sum G_{j,i} e_j, \ 0 \leq i \leq n.$$ 

The matrix $G^{-t} A_X G^{-1}$ is unimodular, hence $\deg(X) = \det(G)^2$, and after inverting $\deg(X)$, $G$ becomes invertible.

Since $K_0(X)$ admits a semiorthonormal basis by assumption and Proposition 2.15, the same holds for $\Lambda \otimes \mathbb{Z}[\frac{1}{\deg(X)}] \square$

Let $P_X$ be the Hilbert polynomial of Wilson’s fourfolds and $A_X$ be the $5 \times 5$-matrix $(A_X)_{i,j} = P_X(j - i)$. Consider their residues modulo two: $A_X = A_X \mod 2, (A_X)_{i,j} = P_X(j - i).$
In the Proof of Proposition 2.8 we showed that the determinant of matrix $A_X$ equals $\text{deg}(X)^{n+1} = 225^5 = 15^{10}$, hence the assumption that $X$ is a minifold would imply that $A_X$ admits a semiorthonormal basis modulo all primes $p \neq 3, 5$, in particular this would imply that $\overline{A}_X$ has a semiorthonormal basis.

Entries of $A_X$ and $\overline{A}_X$ are determined by values $P(n)$ for $0 \leq n \leq 4$ (that we tabulate) and Serre duality $P(n) = P(-1 - n)$:

| n  | 0  | 1  | 2  | 3  | 4  |
|----|----|----|----|----|----|
| $P(n)$ | 1  | 51 | 376| 1426| 3876|
| $P(n) \mod 2$ | 1  | 1  | 0  | 0  | 0  |

$$
\overline{A}_X = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
$$

The following Lemma gives a contradiction, from which we see that a Wilson fourfold $X$ cannot be a minifold.

**Lemma 2.9.** Let $(u, v) \mapsto u^t \overline{A}_X v$ be the bilinear form on a vector space $V = \mathbb{F}_2^5$ given by the matrix $\overline{A}_X$. There is no basis $e_1, e_2, e_3, e_4, e_5$ of $V$ such that $(e_i, e_j) = 0$ for $i > j$ and $(e_i, e_i) = 1$.

**Proof.** We begin by making a few remarks.

1. Let $S := \overline{A}_X^{-1} \overline{A}_X^t$ be an automorphism of $V$. In fact $S$ is induced by the Serre functor $S_X = \otimes \omega_X[\dim X]$ on $\mathcal{D}^b(X)$ [8, 9]. $S$ satisfies $(u, v) = (v, Su)$ for all $u, v$, so it preserves $\overline{A}_X$, i.e. $(u, v) = (Su, Sv)$, equivalently $S^t \overline{A}_X S = \overline{A}_X$. We have

$$
S = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

and $S$ has order 8 because the value of $P(n) \mod 2$ depends only on $n \mod 8$.

2. There are precisely 12 vectors $x$ such that $(x, x) = 1$. Indeed $(x, x) = 1$ if and only if the point $x$ does not lie on quadric $Q = \{x|(x, x) = 0\}$. The quadric $Q$ has a unique singular point in $\mathbb{P}(V)$ so it has 19 points over $\mathbb{F}_2$ and its complement has 12 points. These twelve points form two orbits under the action of $S$. One orbit of length 8 is generated by $a_1 := (1, 0, 0, 0, 0)^t$, another orbit of length 4 is generated by $b_1 := (1, 0, 1, 0, 0)^t$.

3. If a basis $e_1, e_2, \ldots, e_5$ is semi-orthonormal, then for each $i$ ($1 \leq i \leq 4$) the basis obtained by replacing $e_i, e_{i+1}$ with $e_{i+1}, e_i + e_{i+1}(e_i, e_{i+1})$ is also semi-orthonormal. This transformation corresponds to mutations of exceptional collections [8, 9].
Denote \( a_i = S^{i-1}a_1 \), \( b_i = S^{i-1}b_1 \) and \( c = (a_1, \ldots, a_8, b_1, \ldots, b_4) \). The following matrix has \((c_i, c_j)\) on position \( i, j \):

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Assume there exists a semi-orthonormal basis. Then all of its vectors must be from the set \( \{a_i\} \cup \{b_i\} \). Since there are only 4 vectors in \( \{b_i\} \), at least one of the basis vectors must be from \( \{a_i\} \). Applying \( S \) if necessary we may assume that this vector is \( a_1 \). Applying the transformation (3) we can obtain a semi-orthonormal basis with \( a_1 \) on the first position.

Any remaining basis vector \( x \) must satisfy \((x, a_1) = 0\). Looking at the first column of the matrix of \((c_i, c_j)\) we see that the remaining basis vectors must be from the set \( \{a_3, a_4, a_5, a_6, b_1, b_2\} \). Let \( x \) be the second basis vector. Then any vector \( y \) out of the remaining 3 basis vectors must satisfy \((y, x) = 0\). However, trying for \( x \) each of the \( \{a_3, a_4, a_5, a_6, b_1, b_2\} \) we see that there are only 2 choices remaining for \( y \). This is a contradiction. □

We also can prove that there is no minifolds among arithmetic fake projective fourspaces. This goes similarly to dimension 2 case: Prasad and Yeung proved that for an arithmetic fake projective fourspace the first homology group \( H_1(X, \mathbb{Z}) \) is non-zero [10], Theorem 4. Therefore by Proposition 2.1(3) these fourfolds are not minifolds.

3. Quasi-phantoms in fake projective spaces

Fake projective spaces seem to be very similar and yet very different from ordinary projective spaces. We propose the following conjecture.

**Conjecture 3.1.** Assume that \( X \) is an \( n \)-dimensional fake projective space with canonical class divisible by \((n+1)\) i.e. \( \omega_X = \mathcal{O}(n+1) \) for a line bundle \( \mathcal{O}(1) \). Then \( X \) has an exceptional collection of length \( n+1 \). More precisely the collection of line bundles \( \mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n) \) is an exceptional collection.

We call an admissible subcategory \( \mathcal{A} \subset D^b(X) \) a **quasi-phantom** if \( HH(A) = 0 \).

**Corollary 3.2.** Fake projective spaces as in Conjecture admit a quasi-phantom admissible subcategories in their derived categories \( D^b(X) \).

**Proof.** Assume that \( \mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n) \) is an exceptional collection, and consider its right orthogonal \( \mathcal{A} \). By results of Bondal and Kapranov [8, 9] the category \( \mathcal{A} \) is admissible, and thus we have a semi-orthogonal decomposition:

\[
D^b(X) = \langle \mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n), \mathcal{A} \rangle.
\]
Note that all $\mathcal{O}(i)$ are vector bundles, so this exceptional collection could not be full: if $\mathcal{O}(i)$ would be a full collection then by [11] (Theorem 3.4) or [38] (see proof of main theorem) manifold $X$ would be Fano, which contradicts to general type assumption. Finally, Hochschild homology is additive for semi-orthogonal decompositions [30], so $\dim HH(q)(\mathcal{A}) = 0$ that is $\mathcal{A}$ is a quasi-phantom.

Remark 3.3. 1. A statement analogous to Conjecture 3.1 holds for some fake del Pezzo surfaces of degrees one, six and eight [6, 1, 17, 7].

2. Fake projective planes with properties as in Conjecture 3.1 are constructed in [39], 10.4.

Choose $\mathcal{O}(1)$ such that $\mathcal{O}(3) = \omega_X$. Then by the Riemann-Roch theorem the Hilbert polynomial is given by

$$\chi(\mathcal{O}(k)) = \frac{(k-1)(k-2)}{2}.$$ 

Therefore the collection $E_* = (\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2))$ is at least numerically exceptional, that is

$$\chi(E_j, E_i) = 0, \ j > i.$$ 

In addition we have

$$H^0(S, \mathcal{O}(1)) = H^0(S, \mathcal{O}(3)) = 0.$$ 

Furthermore it follows from Serre duality that a necessary and sufficient condition for $E_*$ to be exceptional is vanishing of the space of the global sections $H^0(S, \mathcal{O}(2))$. We have not been able to verify the latter condition.

3. More generally our definition of an $n$-dimensional fake projective space includes that its Hilbert polynomial is the same that of a $\mathbb{P}^n$. It follows that if we assume $\omega_X = \mathcal{O}(n + 1)$, then we have

$$\chi(\mathcal{O}(k)) = (-1)^n \frac{(k-1)(k-2) \ldots (k-n)}{n!},$$ 

so that $k = 1, \ldots, n$ are the roots of $\chi$, and the collection $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ is numerically exceptional.

4. G. Prasad and S.-K. Yeung informed us that the assumption $\omega_X = \mathcal{O}(5)$ is known to be true for the four arithmetic fake projective fourspaces constructed in [40].

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Sergei Galkin, University of Vienna
sergey.galkin@phystech.edu

Ludmil Katzarkov, University of Miami and University of Vienna
lkatzark@math.uci.edu

Anton Mellit, ICTP
mellit@gmail.com

Evgeny Shinder, Max-Planck-Institut für Mathematik
evgenyshinder2011@u.northwestern.edu