Zeno Effect for Bohmian Trajectories: The Unfolding of the Metatron

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Abstract

We analyse the track of an α-particle passing through a cloud chamber using the Bohm theory and show that the resulting classical track has its origins in the quantum Zeno effect. By assuming the ionised gas molecules reveal the positions of the α-particle on its trajectory and using these positions in a short time propagator technique developed by de Gosson, we show it is the failure of the quantum potential to develop quickly enough that leads to the appearance of the classical trajectory. Bohm and Hiley have already argued that it is this failure of the quantum potential to develop appropriately that prevents an Auger electron from undergoing a transition if continuously watched. This allows us to conclude that, in general, it is the suppression of the quantum potential that accounts for the quantum Zeno effect.

1 Introduction

Einstein writes to Bohm in 1954,

I am glad that you are deeply immersed seeking an objective description of the phenomena and that you feel the task is much more difficult as you felt hitherto. You should not be depressed by the enormity of the problem. If God had created the world his primary worry was certainly not to make its understanding easy for us. I feel it strongly since fifty years.[12]

When David Bohm completed his book, “Quantum Theory” [4], which was an attempt to present a clear account of Bohr’s actual position, he
became dissatisfied with the overall approach \[6\]. The reason for this dissatisfaction was the fact that the theory had no place in it for an adequate notion of an independent actuality, that is of an actual movement or activity by which one physical state could pass over into another.

In a meeting with Einstein, ostensibly to discuss the content of his book, the conversation eventually turned to the possibility of whether a deterministic extension of quantum mechanics could be found. Later while exploring the WKB approximation, Bohm realised that this approximation was giving an essentially deterministic approach. Surely by merely truncating a series, one cannot turn a probabilistic theory into a deterministic theory. Thus by retaining all the terms in the series, Bohm found that one could, indeed, obtain what looked like a deterministic description of quantum phenomena. To carry this through, he had to assume that a quantum particle actually had a well defined but unknown position and momentum and followed a well-defined trajectory. This assumption does not violate the uncertainty principle since that principle merely states it is not possible to measure simultaneously the position and momentum and says nothing about whether the particle actually has a simultaneous position and momentum.

In the simple approach to the Bohm model, the Schrödinger equation is split into its real and imaginary parts with the real part showing its close relationship to the classical Hamilton-Jacobi theory. The only difference being the appearance of an additional term which can be regarded as a new quality of energy, called the ‘quantum potential energy’. It is the properties of this energy that enables us to account for all quantum phenomena such as, for example, the two-slit interference effect where the trajectories are shown to undergo a non-classical behaviour \[38\], bunching to produce the observed fringe pattern.

Since the Schrödinger equation still plays a defining role in the Bohm theory, one of the key problems is to understand how the quantum potential becomes suppressed to produce the classical world. This topic was discussed in some detail from the global point of view in chapter 8 of Bohm and Hiley \[9\]. In this paper we want to re-consider this problem from an alternate point of view, focusing on an analysis of a mathematically rigorous short time propagator technique originally developed by de Gosson \[17, 18\] to determine the quantum trajectory of a charged particle as it passes through a gas. What we have in mind here is an $\alpha$-particle passing through a cloud chamber, which we know leaves a track that is essentially classical. How does this approach produce a classical trajectory in this case?

As the $\alpha$-particle travels through the gas it leaves a trail of ions in its wake and these ions are assumed to mark the track taken by the $\alpha$-particle.
The ions can be regarded as revealing the positions of the $\alpha$-particle as it moves through the gas so that in a sense the particle is being “continuously watched” or “monitored”. If we analyse this process from the Bohm point of view, we find that the Bohm trajectory is a classical trajectory. Thus, in a sense, continuous observation “dequantizes” quantum trajectories. This property is, of course, essentially an example of the quantum Zeno effect, which has been shown to inhibit the decay of unstable quantum systems when under continuous observation (see [9, 13, 21, 22]).

The idea lying behind the Bohm approach (Bohm and Hiley [9], Hiley [25], Hiley and collaborators [28, 29], Holland [30]) is the following: let $\Psi = \Psi(r, t)$ be a wavefunction solution of Schrödinger’s equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r)\right] \Psi.$$  

Writing $\Psi$ in polar form $\sqrt{\rho} e^{iS/\hbar}$ Schrödinger’s equation is equivalent to the coupled systems of partial differential equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla_r S)^2}{2m} + V(r) + Q^\Psi(r, t) = 0 \quad (1.1)$$

where

$$Q^\Psi = -\frac{\hbar^2}{2m} \frac{\nabla^2_r |\Psi|}{|\Psi|}. \quad (1.2)$$

is Bohm’s quantum potential (equation (1.1) is thus mathematically a Hamilton-Jacobi equation), and

$$\frac{\partial \rho}{\partial t} + \nabla_r \left(\rho \frac{\nabla_r S}{m}\right) = 0 \quad (1.3)$$

which is an equation of continuity that ensures the conservation of probability. The trajectory of the particle is determined by the equation

$$m \frac{d \mathbf{r}^\Psi}{dt} = \nabla_r S(\mathbf{r}^\Psi, t), \quad \mathbf{r}^\Psi(t_0) = \mathbf{r}_0 \quad (1.4)$$

where $\mathbf{r}_0$ is the initial position.

The simple derivation of equations (1.1) to (1.3) obscures a deeper mathematical relation between the Hilbert space formalism of quantum mechanics and the Hamiltonian flows of classical mechanics. This exact relationship has been derived in very general terms by de Gosson and Hiley [19], a paper that generalises the earlier work of de Gosson [15, 16, 18]. Specifically
what we show is that there is a one-to-one and onto correspondence between Hamiltonian flows generated by a Hamiltonian $H$ and the strongly continuous unitary one-parameter groups satisfying Schrödinger’s equation with Hamiltonian operator $\tilde{H} = H(x, -ih\nabla_x, t)$ obtained from $H$ by Weyl quantisation. This relation exploits the metaplectic representation of the underlying symplectic structure [15, 16, 18]. It is the metaplectic structure that gives rise to the quantum properties. Since the classical and quantum motions are related but different, it was proposed in de Gosson [17] to call the object that obeys the Bohmian law of motion (1.4) a metatron.

We choose this term rather than the usual term ‘particle’, because we are talking about an excitation induced by the metaplectic representation of the underlying Hamiltonian evolution, rather than a classical object. Indeed a deeper investigation suggests that the metatron is more like an invariant feature of an underlying extended process, which elsewhere we have argued that the term quantum blob [20] may be more suggestive. However in this paper it is sufficient to regard it as a particle-like object.

We have frequently been asked the question “Did Bohm believe that there was an actual classical point-like particle following these quantum trajectories?” The answer is a definite ‘No’! For Bohm there was no solid ‘particle’ either, but instead, at the fundamental level, there was a basic process or activity which left a ‘track’ in, for example, the cloud chamber. Thus the track could be explained by the enfolding and unfolding of an invariant form in the overall underlying process [7].

Thus rather than seeing the track as the continuous movement of a material particle, it can be regarded as the continuity of a “quasi-local, semi-stable autonomous form” evolving within this unfolding process [25]. This is what we call the metatron.

The question we will answer here is the following:

What will the trajectory be if we continuously monitor the metatron?

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1 We have deliberately chosen the word ‘monitored’ and avoided the word ‘measurement’ because ‘measurement’ in a quantum context means, following von Neumann’s Process 1 [37], collapsing the wave function of the $\alpha$-particle into a position eigenfunction. Such a process would produce a very different trajectory. In the case we are considering here what is actually being measured is the position of the ions and then only after the $\alpha$-particle has left the chamber. After each gas atom is ionised, the wave function of the $\alpha$-particle takes the form $f(\theta)e^{ik|r|}/|r|$. (See Mott [33] and Bell [3] for a more detailed treatment). In our approach the information contained in this wave function in reflected in the Hamiltonian flow $f_{t_1, t_2}$ used in Section 3.
2 Bohmian Trajectories Are Hamiltonian

Let us start with the particular case where the metatron is initially localized at a point. In this case the Bohm trajectory is Hamiltonian, a point that we explain in section 2.2 (the general case is slightly more subtle; we refer to the papers by Holland [31, 32] for a thorough discussion of the interpretation of Bohmian trajectories from the Hamiltonian point of view).

We will consider systems of $N$ material particles with the same mass $m$, and work in generalized coordinates $x = (x_1, ..., x_n)$ and $p = (p_1, ..., p_n)$, $n = 3N$. Suppose that this system is sharply localized at a point $x_0 = (x_{1,0}, ..., x_{n,0})$ at time $t_0$. The classical Hamiltonian function is

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

(2.1)

hence the organising field of this system is the solution of the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla_x^2 + V(x) \right] \Psi, \quad \Psi(x, t_0) = \delta(x - x_0)$$

(2.2)

where $\nabla_x$ is the $n$-dimensional gradient in the variables $x_1, ..., x_n$. The function $\Psi$ is thus just the propagator $G(x, x_0; t, t_0)$ of the Schrödinger equation. We write $G$ in polar form

$$G(x, x_0; t, t_0) = \sqrt{\rho(x, x_0; t, t_0)} e^{\frac{i}{\hbar} S(x, x_0; t, t_0)}.$$

The equation of motion (1.4) is in this case

$$m\dot{x} = \nabla_x S(x^{\Psi}, x_0; t, t_0), \quad x^{\Psi}(t_0) = x_0.$$ 

(2.3)

2.1 Short-time estimates

We are going to give a short-time estimate for the function $S$. The interest of this estimate is two-fold: it will not only allow us to give a precise statement of the Zeno effect for Bohmian trajectories, but it will also allow us to prove in detail the Hamiltonian character of these trajectories.

We will assume that the potential $V$ is at least twice continuously differentiable in the variables $x_1, ..., x_n$.

In [17], Chapter 7, we established the following short-time formulas for $t - t_0 \to 0$ (a similar formula has been obtained in [33, 44, 40, 41]):

$$S(x, x_0; t, t_0) = \sum_{j=1}^{n} \frac{m(x_j - x_{0,j})^2}{2(t - t_0)} - \tilde{V}(x, x_0)(t - t_0) + O((t - t_0)^2)$$

(2.4)
where $\bar{V}(x, x')$ is the average value of the potential on the line segment $[x', x]$:  
\begin{equation}
\bar{V}(x, x_0) = \int_0^1 V(\lambda x + (1 - \lambda)x_0) d\lambda.
\end{equation}

We observe that the quantum potential is absent from formula (2.4); we would actually have obtained the same approximation if we had replaced $S$ with the solution to the classical Hamilton–Jacobi equation  
\begin{equation}
\frac{\partial S_{cl}}{\partial t} + \frac{(\nabla x S)^2}{2m} + V(x) = 0
\end{equation}
while $S$ is actually a solution of the quantum Hamilton–Jacobi equation  
\begin{equation}
\frac{\partial S}{\partial t} + \frac{(\nabla x S)^2}{2m} + V(x) + Q\Psi(x, t) = 0.
\end{equation}
How can this be? The reason is that if we replace the propagator $G(x, x_0; t, t_0)$ by its “classical” approximation  
\begin{equation}
G_{cl}(x, x_0; t, t_0) = \sqrt{\rho_{cl}(x, x_0; t, t_0)} e^{i \frac{\hbar}{2m} S_{cl}(x, x_0; t, t_0)}
\end{equation}
where $\rho_{cl}$ is the Van Vleck density (i.e. the determinant of the matrix of second derivatives of $S_{cl}$) then we have  
\begin{align*}
G(x, x_0; t, t_0) - G_{cl}(x, x_0; t, t_0) &= O((t - t_0)^2) 
\end{align*}
(cf. Lemma 241 in [17]) from which follows that  
\begin{align*}
- \frac{\hbar^2}{2m} \frac{\nabla^2_x G}{G} - \left( - \frac{\hbar^2}{2m} \right) \frac{\nabla^2_x G_{cl}}{G_{cl}} &= O((t - t_0)^2);
\end{align*}
the difference between these two terms, $O((t - t_0)^2)$, is thus absorbed by the corresponding term in (2.4). [We take the opportunity to remark that when the potential $V(x)$ is quadratic in the position variables $x_1, ..., x_n$ then $G_{cl} = G$; we will come back to this relation later in section 3.1].

Moreover, formula (2.4) can be twice continuously differentiated with respect to the variables $x_j$ and $x_{0,j}$. It follows that the second derivatives of $S$ are given by  
\begin{align*}
\frac{\partial^2 S}{\partial x_j \partial x_{0,k}} &= \frac{m}{t - t_0} \delta_{jk} + O(t - t_0)
\end{align*}
and hence the Hessian matrix $S_{x,x_0}$ (i.e. the matrix of mixed second derivatives) satisfies

$$\det(S_{x,x_0}) = \left(\frac{m}{t-t_0}\right)^n + O(t-t_0). \quad (2.7)$$

Formula (2.4) is the key to the following important asymptotic version of Bohm’s equation (2.3):

$$\dot{x}^\Psi = \frac{x^\Psi - x_0}{t-t_0} - \frac{1}{2m} \nabla_x V(x_0)(t-t_0) + O((t-t_0)^2). \quad (2.8)$$

Let us prove this formula. Using the expansion (2.4), formula (2.3) becomes

$$\dot{x}^\Psi = \frac{x^\Psi - x_0}{t-t_0} - \frac{1}{m} \nabla_x \tilde{V}(x^\Psi, x_0)(t-t_0) + O((t-t_0)^2). \quad (2.9)$$

Let us show that

$$\nabla_x \tilde{V}(x^\Psi, x_0) = \frac{1}{2} \nabla_x V(x_0) + O(t-t_0); \quad (2.10)$$

this will complete the proof of formula (2.8). We first note that (2.9) implies in particular that

$$\dot{x}^\Psi = \frac{x^\Psi - x_0}{t-t_0} + O(t-t_0)$$

and thus $x^\Psi$ is given by

$$x^\Psi(t) = x_0 + \frac{p_0}{m}(t-t_0) + O((t-t_0)^2) \quad (2.11)$$

where $p_0$ is an arbitrary constant vector. In particular we have $O(x^\Psi - x_0) = O(t-t_0)$ and hence

$$\nabla_x \tilde{V}(x^\Psi, x_0) = \nabla_x \tilde{V}(x_0, x_0) + O(x^\Psi - x_0)$$

$$= \nabla_x \tilde{V}(x_0, x_0) + O(t-t_0)$$

from which it follows that

$$\nabla_x \tilde{V}(x^\Psi, x_0) = \int_0^1 \lambda \nabla_x V(\lambda x_0 + (1-\lambda)x_0)d\lambda + O(t-t_0)$$

$$= \frac{1}{2} \nabla_x V(x_0) + O(t-t_0)$$

which is precisely the estimate (2.10).
2.2 The Hamiltonian character of Bohmian trajectories

Let \( \mathbf{p}_0 = (p_{1,0}, ..., p_{n,0}) \) be an arbitrary momentum vector, and set

\[
\mathbf{p}_0 = -\nabla_{x_0} S(x, x_0; t, t_0). \tag{2.12}
\]

In view of formula (2.7), the Hessian of \( S \) in the variables \( x \) and \( x_0 \) is invertible for small values of \( t \), hence the implicit function theorem implies that (2.12) determines a function \( x = x(t) \) (depending on \( x_0 \) and \( t_0 \) viewed as parameters), defined by

\[
\mathbf{p}_0 = -\nabla_{x_0} S(x(t), x_0; t, t_0). \tag{2.13}
\]

Setting

\[
\mathbf{p}(t) = \nabla_x S(x(t), x_0; t, t_0) \tag{2.14}
\]

we claim that the functions \( x(t) \) and \( p(t) \) thus defined are solutions of the Hamilton equations

\[
\dot{x} = \nabla_p H^\Psi(x, p, t) \quad \dot{p} = -\nabla_x H^\Psi(x, p, t) \tag{2.15}
\]

and that we have \( x(t_0) = x_0, p(t_0) = p_0 \). We are actually going to use classical Hamilton–Jacobi theory (see [2, 14, 17, 18] or any introductory text on analytical mechanics). For notational simplicity we assume that \( n = 1 \). The function \( S \) satisfies the equation

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V(x) - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} = 0; \tag{2.16}
\]

introducing the quantum potential

\[
Q^\Psi = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \tag{2.17}
\]

we set \( H^\Psi = H + Q^\Psi \) so that (2.16) is just the quantum Hamilton–Jacobi equation

\[
\frac{\partial S}{\partial t} + H^\Psi \left( x, \frac{\partial S}{\partial x}, t \right) = 0. \tag{2.18}
\]

Differentiating the latter with respect to \( p = \partial S/\partial x \) yields, using the chain rule,

\[
\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x_0 \partial x} = 0 \tag{2.19}
\]
and differentiating the equation (2.13) with respect to time yields
\[
\frac{\partial^2 S}{\partial x \partial x_0} + \frac{\partial^2 S}{\partial x \partial x_0} \dot{x} = 0.
\] (2.20)

Subtracting (2.20) from (2.19) we get
\[
\frac{\partial^2 S}{\partial x \partial x_0} \left( \frac{\partial H^\Psi}{\partial p} - \dot{x} \right) = 0
\]
which produces the first Hamilton equation (2.15) since it is assumed that we have \(\frac{\partial^2 S}{\partial x \partial x_0} \neq 0\). Let us next show that the second Hamilton equation (2.15) is satisfied as well. For this we differentiate the quantum Hamilton–Jacobi equation (2.18) with respect to \(x\), which yields
\[
\frac{\partial^2 S}{\partial x \partial t} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x^2} = 0.
\] (2.21)

Differentiating the equality (2.14) with respect to \(t\) we get
\[
\frac{\partial^2 S}{\partial t \partial x} = -\dot{p}(t) - \frac{\partial^2 S}{\partial x^2} \dot{x}
\] (2.22)
and hence the equation (2.21) can be rewritten
\[
-\dot{p}(t) - \frac{\partial^2 S}{\partial x^2} \dot{x} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x^2} = 0.
\]

Taking into account the relation \(\dot{x} = \frac{\partial H^\Psi}{\partial p}\) established above we have
\[
-\dot{p}(t) - \frac{\partial H^\Psi}{\partial x} = 0
\]
which is precisely the second Hamilton equation (2.15). There remains to show that we have \(x(t_0) = x_0\) and \(p(t_0) = p_0\). Recall that \(x(t)\) is defined by the implicit equation
\[
p_0 = -\nabla_{x_0} S(x(t), x_0; t, t_0)
\] (equation (2.13)); in view of the short-time estimate (2.4) this means that we have
\[
p_0 = m(x(t) - x_0) \frac{1}{t - t_0} + O(t - t_0)
\]
and hence we must have \(\lim_{t \to t_0} x(t) = x(t_0) = x_0\). This also implies that \(p_0 = m \dot{x}(t_0) = p(t_0)\).

In conclusion we have thus shown that:
Bohm’s equation of motion (2.3) is equivalent to Hamilton’s equations (2.15).

To complete our discussion, we make two important observations:

- Even when the Hamiltonian function $H$ does not depend explicitly on time, the function $H^\Psi = H + Q^\Psi$ is usually time-dependent (because the quantum potential generally is), so the flow $(f^\Psi_t)$ it determines does not inherit the usual group property $f_t f_{t'} = f_{t+t'}$ of the flow determined by the classical Hamiltonian $H$. One has instead to use the “time-dependent flow” $(f^\Psi_{t,t'})$, which has a groupoid property in the sense that $f^\Psi_{t,t'} f^\Psi_{t',t''} = f^\Psi_{t,t''}$.

- The time-dependent flow $(f^\Psi_{t,t'})$ consists of canonical transformations; that is, the Jacobian matrix of $f^\Psi_{t,t'}$ calculated at any point $(x, p)$ where it is defined by a symplectic matrix. This is an immediate consequence of the fact discussed above, namely, that the flow determined by any Hamiltonian function has this property.

We have seen that the Bohmian trajectory for a particle initially sharply localized at a point $x_0$ is Hamiltonian, and in fact governed by the Hamilton equations (2.15):

$$\dot{x} = \nabla_p H^\Psi(x, p, t), \quad \dot{p} = -\nabla_x H^\Psi(x, p, t).$$  \hspace{1cm} (2.23)

The discussion of short-time solutions of Bohm’s equation of motion allows us to give approximations to the solution. First, the solutions of the equation \(\dot{x} = \nabla_p H^\Psi(x, p, t)\) are given by the simple relation

$$x^\Psi(t) = x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2)$$

as was already noticed in (2.11). Then we proved that the momentum $p^\Psi(t) = m \dot{x}^\Psi(t)$ is given by equation (2.23):

$$m \dot{x}^\Psi(t) = \frac{m(x^\Psi(t) - x_0)}{t - t_0} - \frac{1}{2} \nabla_x V(x_0)(t - t_0) + O((t - t_0)^2).$$  \hspace{1cm} (2.24)

However we cannot solve this equation by inserting the value of $x^\Psi(t)$ above since this would lead to an estimate modulo $O(t - t_0)$ not $O((t - t_0)^2)$. What we do is the following: differentiating both sides of the equation (2.24) with respect to $t$ we get

$$\ddot{x}^\Psi(t) = \frac{x^\Psi(t) - x_0}{(t - t_0)^2} + \frac{\dot{x}^\Psi(t)}{t - t_0} - \frac{1}{2m} \nabla_x V(x_0) + O(t - t_0)$$
that is, replacing $\dot{x}^\Psi(t)$ by the value given by (2.24),

$$\dot{p}^\Psi(t) = m\ddot{x}^\Psi(t) = -\nabla x V(x_0) + O(t - t_0).$$

Solving this equation we get

$$p^\Psi(t) = p_0 - \nabla x V(x_0)(t - t_0) + O((t - t_0)^2).$$

Summarizing, the solutions of the Hamilton equations (2.23) for $H^\Psi = H + Q^\Psi$ are given by

$$x^\Psi(t) = x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2) \quad (2.25)$$

$$p^\Psi(t) = p_0 - \nabla x V(x_0)(t - t_0) + O((t - t_0)^2). \quad (2.26)$$

The observant reader will have noticed that (up to the error term $O((t - t_0)^2)$) there is no trace of the quantum potential $Q^\Psi$ in these short-time formulas. Had we replaced the function $H^\Psi$ with the classical Hamiltonian $H$ we would actually have obtained exactly the same solutions, up to the $O((t - t_0)^2)$ term.

3 Bohmian Zeno Effect

3.1 The case of quadratic potentials

Here is an easy case; it is in fact so easy that it is slightly misleading: the Bohmian trajectories are here classical trajectories from the beginning, because the quantum potential vanishes.

Let us assume that the potential $V(x)$ is a quadratic form in the position variables, that is

$$V(x) = \frac{1}{2} M x \cdot x$$

where $M$ is a symmetric matrix. Using the theory of the metaplectic representation [15, 16, 17, 18] it is well-known that the propagator $G$ is given by the formula

$$G(x, x_0; t, t_0) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{i m(t, t_0)} e^{\frac{i}{\hbar} W(x, x_0; t, t_0)} \quad (3.1)$$

where $W(x, x_0; t, t_0)$ is Hamilton’s two-point characteristic function (see e.g. [2, 14]): it is a quadratic form

$$W = \frac{1}{2} P x \cdot x - L x \cdot x_0 + \frac{1}{2} B x_0 \cdot x_0$$
where $P = P(t, t_0)$ and $B = B(t, t_0)$ are symmetric matrices and $L = L(t, t_0)$ is invertible; viewed as function of $x$ it satisfies the Hamilton–Jacobi equation
\[
\frac{\partial W}{\partial t} + \frac{(\nabla_x W)^2}{2m} + \frac{1}{2} M x \cdot x.
\]
Moreover, $m(t, t_0)$ in equation (3.1) is an integer (“Maslov index”) and $\rho(t, t_0)$ is the determinant of $L = L(t, t_0)$ (the Van Vleck density). Since $m(t, t_0)$ and $\rho(t, t_0)$ do not depend on $x$, it follows that the quantum potential $Q^\Psi$ determined by the propagator (3.1) is zero. Since we have $H^\Psi = H + Q^\Psi$, we see immediately that the quantum motion is perfectly classical in this case: the quantum equations of motion (2.15) reduce to the ordinary Hamilton equations
\[
\dot{x} = \frac{p}{m}, \quad \dot{p} = -M x
\] which can be easily integrated: in particular the flow $(f_t)$ they determine is a true flow (because $H = H^\Psi$ is time-independent) and consists of symplectic matrices ([2, 17, 18, 14]). In fact,
\[
f_t = e^{tx}, \quad X = \begin{pmatrix} 0_{n \times n} & \frac{1}{m} I_{n \times n} \\ -M & 0_{n \times n} \end{pmatrix}.
\]
Thus, in the case of quadratic potentials the Bohmian trajectories associated with the propagator are the usual Hamilton trajectories associated with the classical Hamiltonian function of the problem.

Suppose now that we monitor “continuously” (in the sense discussed in the introduction) the time evolution of the metatron—which is so far “quantum”—and try to find out what effect this interaction has on the trajectory. In the example of the cloud chamber, let $\Delta t$ be the time between successive ionisations. From the mathematical point of view we will assume the limit $\Delta t \rightarrow 0$ exists and that it is continuous and smooth. In other words we are neglecting the reaction of the ion formation on the $\alpha$-particle, an assumption that Mott [35] also makes. Thus we can assign at every point a velocity vector.

Let us choose a time interval $[0, t]$ (typically $t = 1$ s) and subdivide it in a sequence of $N$ intervals
\[
[0, \Delta t] [\Delta t, 2\Delta t] [2\Delta t, 3\Delta t] \cdots [(N - 1)\Delta t, N\Delta t]
\] with $\Delta t = t/N$; the integer $N$ is assumed to be very large (for instance $N \approx 10^6 - 10^8$). Assume that at time $t_0 = 0$ the particle is at a point $x_0$ and after after time $\Delta t$ it is at $x_1$; its momentum is $p_1$ and we have
\[(x_1, p_1) = f_{\Delta t}(x_0, p_0).\] We now repeat the procedure, replacing \(x_0\) by \(x_1\); since the trajectory is assumed to be smooth, the initial momentum will be \(p_1\) and after time \(\Delta t\) the particle will be at \(x_2\) with momentum \(p_2\) such that \((x_2, p_2) = f_{\Delta t}(x_1, p_1) = f_{\Delta t}f_{\Delta t}(x_0, p_0)\). Repeating the same process until time \(t = N\Delta t\) we find a series of points in space which the particle takes as positions one after another. Since the trajectory is assumed to be smooth, the initial momentum will be \(p_1\) and after time \(\Delta t\) the particle will be at \(x_2\) with momentum \(p_2\) such that \((x_2, p_2) = f_{\Delta t}(x_1, p_1) = f_{\Delta t}f_{\Delta t}(x_0, p_0)\). Repeating the same process until time \(t = N\Delta t\) we find a series of points in space which the particle takes as positions one after another.

\[f_{\Delta t}f_{\Delta t}f_{\Delta t} = f_{3\Delta t}f_{\Delta t}(x_0, p_0) = f_{3\Delta t}(x_0, p_0) = f_{4\Delta t}(x_0, p_0).\]

But in view of the group property \(f_t f_{t'} = f_{t+t'}\) of the flow we have \((f_{\Delta t})^N = f_{N\Delta t} = f_t\) and hence \((x_N, p_N) = f_t(x_0, p_0)\). The observed Bohmian trajectory is thus the classical trajectory predicted by Hamilton’s equations.

### 3.2 The general case

In generalizing the discussion above to arbitrary potentials, \(V(x)\), there are two difficulties. The first is that we do not have exact equations for the Bohmian trajectory, but only short-time approximations. The second is that the Hamilton equations for \(\dot{x}^\Psi\) and \(\dot{p}^\Psi\) no longer determine a flow having a group property because the Hamiltonian \(H^\Psi\) is time-dependent. Nevertheless the material we have developed so far is actually sufficient to show that the observed trajectory is the classical one.

The key will be the theory of Lie–Trotter algorithms which is a powerful method for constructing exact solutions from short-time estimates. The method goes back to early work of Trotter [43] elaborating on Sophus Lie’s proof of the exponential matrix formula \(e^{A+B} = \lim_{N \to \infty} (e^{A/N}e^{B/N})^N\); see Chorin et al. [11] for a detailed and rigorous study; we have summarized the main ideas in the Appendix B of [17]; also see Nelson [36]. (We mention that there exists an operator variant of this procedure, called the Trotter–Kato formula.)

Let us begin by introducing some notation. We have seen that the datum of the propagator \(G_0 = G(x, x_0; t, t_0)\) determines a quantum potential \(Q^\Psi\) and thus Hamilton equations (2.15) associated with \(H^\Psi = H + Q^\Psi\). We now choose \(t_0 = 0\) and denote the corresponding quantum potential by \(Q^0\) and set \(H^0 = H + Q^0\). After time \(\Delta t\) the position of the particle is at \(x_1\). The future evolution of the particle is now governed by the new propagator \(G_1 = G(x, x_1; t, t_0)\), leading to a new quantum potential \(Q^1\) and to a new Hamiltonian \(H^1\); repeating this until time \(t\) we thus have a sequence of points \(x_0, x_1, ..., x_N = x\) and a corresponding sequence of Hamiltonian functions \(H^0, H^1, ..., H^N\) determined by the quantum potentials \(Q^0, Q^1, ..., Q^N\).

\[\text{In conformity with W. Heisenberg's statement: “By path we understand a series of points in space which the electron takes as ‘positions’ one after another” [23].}\]
denote by \((f_0^0, t_0, t_1), (f_1^1, t_1, t_2), \ldots, (f_{N-1}^{N-1}, t_{N-1}, t_N)\) the time dependent flows determined by the Hamiltonian functions \(H^0, H^1, \ldots, H^N\); we have set here \(t_1 = t_0 + \Delta t, t_2 = t_1 + \Delta t\) and so on.

Repeating the procedure explained in the case of quadratic potentials, we get in this case a sequence of successive equalities

\[
(x_1, p_1) = f_{t_1, t_0}^0 (x_0, p_0) \\
(x_2, p_2) = f_{t_2, t_1}^1 (x_1, p_1) \\
\ldots \\
(x, p) = f_{t_N, t_{N-1}}^{N-1} (x_{N-1}, p_{N-1})
\]

which implies that the final position \(x = x_N\) at time \(t\) is expressed in terms of the initial point \(x_0\) by the formula

\[
(x, p) = f_{t_N, t_{N-1}}^{N-1} \cdots f_{t_1, t_0}^1 f_{t_0}^0 (x_0, p_0).
\]

Denote now by \((g_0^0, t_0), (g_1^1, t_1), \ldots, (g_{N-1}^{N-1}, t_{N-1})\) the approximate flows determined by the equations

\[
(x_1, p_1) = (x_0 + \frac{p_0}{m} \Delta t, p_0 - \nabla_x V(x_0) \Delta t) \\
(x_2, p_2) = (x_1 + \frac{p_1}{m} \Delta t, p_1 - \nabla_x V(x_1) \Delta t) \\
\ldots \\
(x, p) = (x_{N-1} + \frac{p_{N-1}}{m} \Delta t, p_{N-1} - \nabla_x V(x_{N-1}) \Delta t).
\]

Invoking the Lie–Trotter formula, the sequence of estimates

\[
f_{t_{k-1}, t_k}^k (x_{k-1}, p_{k-1}) - g_{t_{k-1}, t_k}^k (x_{k-1}, p_{k-1}) = O(\Delta t^2)
\]

implies that we have

\[
\lim_{N \to \infty} g_{t_N, t_{N-1}}^{N-1} \cdots g_{t_2, t_1}^1 g_{t_1, t_0}^0 (x_0, p_0) = \lim_{N \to \infty} f_{t_N, t_{N-1}}^{N-1} \cdots f_{t_2, t_1}^1 f_{t_1, t_0}^0 (x_0, p_0)
\]

The argument goes as follows (for a detailed proof see [17]): since we have \(g_{t_{k-1}, t_k}^k = f_{t_{k-1}, t_k}^k + O(\Delta t^2)\) the product is approximated by

\[
g_{t_N, t_{N-1}}^{N-1} \cdots g_{t_2, t_1}^1 g_{t_1, t_0}^0 = f_{t_N, t_{N-1}}^{N-1} \cdots f_{t_2, t_1}^1 f_{t_1, t_0}^0 + NO(\Delta t^2)
\]

and since \(\Delta t = t/N\) we have \(NO(\Delta t^2) = O(\Delta t)\) which goes to zero when \(N \to \infty\).
Now, recall our remark that the quantum potential is absent from the approximate flows $g_{k,t_{k-1}}^k$; using again the Lie–Trotter formula together with short-time approximations to the Hamiltonian flow $(f_t)$ determined by the classical Hamiltonian $H$, we get

$$
\lim_{N \to \infty} g_{t_{N-1},t_{N-1}}^N \cdots g_{t_1,t_1}^1 g_{t_0,t_0}^0(x_0,p_0) = f_t
$$

and hence

$$
\lim_{N \to \infty} f_{t_{N-1},t_{N-1}}^N \cdots f_{t_1,t_1}^1 f_{t_0,t_0}^0(x_0,p_0) = f_t
$$

which shows that the trajectory is the classical one.

4 Conclusion.

In this paper we have shown how a detailed mathematical examination of the deeper symplectic structure that underlies the Bohm approach predicts that if a quantum particle is monitored continuously in the way we have suggested, it will follow a classical trajectory.

The idea lying behind this result becomes clear once one realises that it is the appearance of the quantum potential energy that distinguishes quantum behaviour from classical behaviour. Indeed this is very obvious if we examine equation (1.1) and compare it with the classical Hamilton-Jacobi equation. The essential difference is the appearance of the term, $Q\Psi$, in equation (1.1). This means that when $Q\Psi$ is negligible compared with the kinetic energy, the equation simply reduces to the classical Hamilton-Jacobi equation.

In this paper we have shown that the suppression of the quantum potential is possible if the successive positions of the particle can be defined in a short enough time. To see this, we must examine equations (2.25) and (2.26), which are exact to $O((\Delta t)^2)$. Notice that there is no quantum potential present in either equation. Only when we allow higher order terms does the quantum potential appear. Thus it is possible to obtain information of succession of positions in a short enough time without deflecting the particle significantly, then equation (3.3) shows that no quantum potential will appear and the trajectory will be a classical trajectory. In other words the quantum Zeno effect arises because $Q\Psi$ is prevented from contributing to the process.

Another illustration of how continuous observations of a different kind can give rise to a quantum Zeno effect has already been given in Bohm and Hiley [9]. They considered the transition of an Auger-like particle and showed that the perturbed wave function, which is proportional to $\Delta t$ for
times less that $1/\Delta E$, ($\Delta E$ is the energy released in the transition) will never become large and therefore cannot make a significant contribution to the quantum potential necessary for the transition to occur. Thus again for the reason that no transition will take place is the vanishing of the quantum potential.

Our discussion shows that the Bohm model has a very different way of arriving at the classical limit than the prevailing view based on decoherence. In our view the main difficulty in using decoherence is that it merely destroys the off-diagonal elements of the density matrix but it does not explain how the classical equations of motion arise. It continues to describe classical objects using wave functions, a criticism that has already been made by Primas [39].

The mathematics we have used in this paper is a further example of how the relation between the symplectic and metaplectic representations discussed in our earlier paper [19] holds a further clue of the relationship between the quantum and classical domains. It is when the global properties of the covering (metaplectic) group become unimportant that the classical world emerges. As has been pointed out by Hiley [26] [27], the Bohm approach has a close relationship to the Moyal approach. This supplements the work of de Gosson [18] who shows exactly how the Wigner-Moyal transformation is related to the mathematical structure we are exploiting here. The Moyal approach involves a deformed Poisson algebra from which the classical limit emerges in a very simple way, namely, in those situations where the deformation parameter can be considered to be small which is essentially similar to neglecting the quantum potential.

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