SHIMURA CURVES EMBEDDED IN IGUSA’S THREEFOLD

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Abstract. Let $\mathcal{O}$ be a maximal order in a totally indefinite quaternion algebra over a totally real number field. In this note we study the locus $Q_\mathcal{O}$ of quaternionic multiplication by $\mathcal{O}$ in the moduli space $A_g$ of principally polarized abelian varieties of even dimension $g$ with particular emphasis in the two-dimensional case. We describe $Q_\mathcal{O}$ as a union of Atkin-Lehner quotients of Shimura varieties and we compute the number of irreducible components of $Q_\mathcal{O}$ in terms of class numbers of CM-fields.

Introduction

Let $A$ be an abelian variety over a fixed algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers. By Poincaré’s Decomposition Theorem, the algebra of endomorphisms $\text{End}(A) \otimes \mathbb{Q}$ of $A$ decomposes as a direct sum

$$\text{End}(A) \otimes \mathbb{Q} \simeq \bigoplus_{i=1}^s M_{n_i}(B_i)$$

of matrix algebras over a division algebra $B_i$.

The ranks $[B_i : \mathbb{Q}]$ are bounded in terms of the dimension of $A$ and, by a classical theorem of Albert, the algebras $B_i$ are isomorphic to either a totally real field, a quaternion algebra over a totally real field or a division algebra over a CM-field (cf. [8], [7]).

We will focus our attention on abelian varieties with totally indefinite quaternionic multiplication. More precisely, let $F$ be a totally real number field of degree $[F : \mathbb{Q}] = n \geq 1$, let $R_F$ be its ring of integers and let $\vartheta_F$ denote the different ideal of $F$ over $\mathbb{Q}$. We also let $F^*_+ = F^*_+ \cap F^*_+$.

Let $B$ be a totally indefinite division quaternion algebra over $F$, i.e. a division algebra of rank 4 over $F$ such that $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{i=1}^n M_2(\mathbb{R})$, and let $\text{disc}(B)$ denote the reduced discriminant ideal of $B$. We assume for simplicity that $\vartheta_F$ and $\text{disc}(B)$ are coprime ideals of $F$. Since $B$ is totally indefinite and division, it follows from [10] that $\text{disc}(B) = \vartheta_1 \cdots \vartheta_{2r}$ for pairwise different prime ideals $\vartheta_i$ of $F$ and $r \geq 1$.

We shall denote by $n = n_{B/F}$ and $\text{tr} = \text{tr}_{B/F}$ the reduced norm and reduced trace on $B$, respectively. For any subset $O$ of $B$, we shall write $O_0 = \{\beta \in O : \text{tr}(\beta) = 0\}$

Partially supported by a grant FPI from Ministerio de Educación y Ciencia and by Ministerio de Ciencia y Tecnología BFM2000-0627

1991 Mathematics Subject Classification. 11G18, 14G35.

Key words and phrases. Shimura variety, moduli space, abelian variety, quaternion algebra.
for the set of pure quaternions of $O$. We shall also use the notation $O_+ = \{\beta \in O : n(\beta) \in F^*_+\}$ and $O^1 = \{\beta \in O : n(\beta) = 1\}$.

Finally, let $O$ be a maximal order in $B$ and let $\vartheta(O) = \{\mu \in O : \text{disc}(B)|n(\mu)\}$ denote the reduced different of $O$ over $R_F$. This is a two-sided ideal of $O$ such that $n(\vartheta(O)) \cdot R_F = \text{disc}(B)$.

Let $A_g/\mathbb{Q}$ be the moduli space of principally polarized abelian varieties of dimension $g = 2n$. We let $[(A, \mathcal{L})]$ denote the isomorphism class of a principally polarized abelian variety $(A, \mathcal{L})$ regarded as a closed point in $A_g$.

It is our aim to investigate the nature of the quaternionic locus $Q_O \subset A_g(\mathbb{C})$ of isomorphism classes of complex principally polarized abelian varieties $[(A, \mathcal{L})]$ of dimension $g$ such that $\text{End}(A) \supseteq O$.

A reformulation of Proposition 6.2 in [10] yields that the quaternionic locus $Q_O$ is nonempty if and only if $\text{disc}(B)$ is a totally positive principal ideal of $F$. In consequence, for the sake of clarity of the exposition, we will assume throughout these notes that the narrow class number of $F$ is $h_+(F) = 1$. This automatically implies the nonemptiness of $Q_O$.

Acknowledgements. I am grateful to A. Arenas for some useful comments and correspondence.

1. ABELIAN VARIETIES WITH QUATERNIONIC MULTIPLICATION

As a first step in the study of the quaternionic locus $Q_O$ in the moduli space $A_g$, it is necessary to understand the geometry of the objects that $Q_O$ parametrizes. Let us review some of the results that were accomplished in [10] in this direction.

**Definition 1.1.** An abelian variety with quaternionic multiplication by $O$ over $\overline{\mathbb{Q}}$ is an abelian variety $A/\overline{\mathbb{Q}}$ such that $\text{End}(A) \simeq O$ and $\dim(A) = 2[F : \mathbb{Q}] = 2n$.

Let $A$ be an abelian variety with quaternionic multiplication by $O$ over $\overline{\mathbb{Q}}$ and let $\text{NS}(A)$ denote the free $\mathbb{Z}$-module of rank $3n = 3g/2$ of line bundles on $A$ up to algebraic equivalence. We say that two line bundles $\mathcal{L}, \mathcal{L}' \in \text{NS}(A)$ are isomorphic if there exists an automorphism $\alpha \in \text{Aut}(A) \simeq O^*$ such that $\mathcal{L}' = \alpha^*(\mathcal{L})$.

**Theorem 1.2.** Let $A/\overline{\mathbb{Q}}$ be an abelian variety with quaternion multiplication by $O$ and let $\iota : O \simeq \text{End}(A)$ be a fixed isomorphism of rings. Then, there is an isomorphism of groups

\[
\text{NS}(A) \xrightarrow{\sim} \vartheta(O)_0,
\]

\[
\mathcal{L} \mapsto c_1(\mathcal{L}) = \mu
\]

between the Néron-Severi group of $A$ and the group of pure quaternions of the reduced different $\vartheta(O)$ of $O$. 
Moreover, for any two non trivial line bundles \( L, L' \in \text{NS}(A) \), let \( \mu = c_1(L) \) and \( \mu' = c_1(L') \). Then we have that

1. \( L \cong L' \) if and only if there exists \( \alpha \in \mathcal{O}^* \) such that \( \mu' = \overline{\alpha}\mu\alpha \).

2. \( \deg(L) = \deg(\varphi_L : A \to \hat{A})^{1/2} = N_{\mathbb{F}/\mathbb{Q}}(n(\mu)/D) \).

3. \( L \) is a polarization if and only if \( n(\mu) \in \mathcal{F}^* \) and \( \mu \) is ample (cf. [10], §5).

4. The Rosati involution on \( \mathcal{O} \cong \text{End}(A) \) with respect to \( L \) is

\[
\circ : \mathcal{O} \to \mathcal{O} \\
\beta \mapsto \mu^{-1}\overline{\beta}\mu
\]

Let us write the isomorphism \( c_1 \) in more explicit terms. Fix an immersion of \( \mathbb{Q} \) into the field \( \mathbb{C} \) of complex numbers and let \( A(\mathbb{C}) = V/\Lambda \) for some complex vector space \( V \) and a lattice \( \Lambda \). Upon the choice of an isomorphism \( \iota : \mathcal{O} \cong \text{End}(A) \), the lattice \( \Lambda \) is naturally a left \( \mathcal{O} \)-module and, since \( h(F) = h_+(F) = 1 \), by a theorem of Eichler [2] we know that \( \Lambda \cong \mathcal{O} \).

By the Appell-Humbert Theorem, a line bundle \( L \in \text{NS}(A) \) on \( A \) can be regarded as a Riemann form \( E : \Lambda \times \Lambda \to \mathbb{Z} \) on \( V \). Let us identify \( \Lambda = \mathcal{O} \) through a fixed isomorphism and let \( t \in F^* \) be any generator of the principal ideal \( \vartheta_F \). Then, the inverse isomorphism \( c_{1}^{-1} : \vartheta(\mathcal{O})_0 \xrightarrow{\sim} \text{NS}(A) \) maps a pure quaternion \( \mu \) to the Riemann form \( E_{\mu} : \mathcal{O} \times \mathcal{O} \to \mathbb{Z}, (\beta_1, \beta_2) \mapsto -\text{tr}_{\mathbb{F}/\mathbb{Q}}(\text{tr}(\mu\beta_1\overline{\beta}_2)/n(\mu)) \).

Remark 1.3. The isomorphism \( c_1 \) is canonical in the sense that it does not depend on the choice of any polarization on \( A \). However, we warn the reader that it does depend on the choice of the isomorphism \( \iota : \mathcal{O} \cong \text{End}(A) \).

Theorem 1.4. Let \( A \) be an abelian variety over \( \mathbb{Q} \) with quaternionic multiplication by \( \mathcal{O} \). Let \( D \in F^*_+ \) be a totally positive generator of \( \text{disc}(B) \).

Then, \( A \) is principally polarizable and the number of isomorphism classes of principal polarizations on \( A \) is

\[
\pi_0(A) = \frac{1}{2} \sum_S h(S),
\]

where \( S \) runs among the set of orders in the CM-field \( F(\sqrt{-D}) \) that contain \( R_F[\sqrt{-D}] \) and \( h(S) \) denotes its class number.

2. Shimura varieties

As in the preceding sections, let \( B \) be a totally indefinite division quaternion algebra over a totally real field \( F \) of trivial narrow class number. Let \( \hat{D} \in F^*_+ \) be a totally positive generator of \( \text{disc}(B) \).

Definition 2.1. A principally polarized maximal order of \( B \) is a pair \((\mathcal{O}, \mu)\) where \( \mathcal{O} \subset B \) is a maximal order and \( \mu \in \mathcal{O} \) is a pure quaternion such that \( \mu^2 + uD = 0 \) for some \( u \in R_{F,+}^* \).
Attached to a principally polarized maximal order \((\mathcal{O}, \mu)\) there is the following moduli problem over \(\mathbb{Q}\): classifying isomorphism classes of triplets \((A, \iota, \mathcal{L})\) given by

- An abelian variety \(A\) of dimension \(g = 2n\).
- A ring homomorphism \(\iota: \mathcal{O} \hookrightarrow \text{End}(A)\).
- A principal polarization \(\mathcal{L}\) on \(A\) such that

\[
\iota(\beta) \circ = \iota(\mu^{-1} \beta \mu)
\]

for any \(\beta \in \mathcal{O}\), where \(\circ: \text{End}(A) \to \text{End}(A)\) is the Rosati involution with respect to \(\mathcal{L}\).

A triplet \((A, \iota, \mathcal{L})\) will be referred to as a polarized abelian variety with multiplication by \(\mathcal{O}\). Two triplets \((A_1, \iota_1, \mathcal{L}_1), (A_2, \iota_2, \mathcal{L}_2)\) are isomorphic if there exists an isomorphism \(\alpha \in \text{Hom}(A_1, A_2)\) such that \(\alpha \iota_1(\beta) = \iota_2(\beta) \alpha\) for any \(\beta \in \mathcal{O}\) and \(\alpha^*(\mathcal{L}_2) = \mathcal{L}_1 \in \text{NS}(A_1)\). Note also that, since a priori there is no canonical structure of \(R_F\)-algebra on \(\text{End}(A)\), the immersion \(\iota: \mathcal{O} \hookrightarrow \text{End}(A)\) is just a homomorphism of rings.

As it was proved by Shimura, the corresponding moduli functor is coarsely represented by an irreducible and reduced quasi-projective scheme \(X_\mu/\mathbb{Q}\) over \(\mathbb{Q}\) and of dimension \(n = [F: \mathbb{Q}]\). Moreover, since \(B\) is division, the Shimura variety \(X_\mu\) is complete (cf. [14], [15]).

Let \(\mathfrak{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}\) denote the upper half plane. Complex analytically, the manifold \(X_\mu(\mathbb{C})\) can be described independently of the choice of \(\mu\) as the quotient

\[
\mathcal{O}^1 \backslash \mathfrak{H}^n \simeq X_\mu(\mathbb{C})
\]

of the symmetric space \(\mathfrak{H}^n\) by the action of the group \(\mathcal{O}^1\) regarded suitably as a discontinuous subgroup of \(\text{SL}_2(\mathbb{R})^n\). See [11] for details.

In addition, we are also interested in the Hilbert modular reduced scheme \(H_F/\mathbb{Q}\) that coarsely represents the functor attached to the moduli problem of classifying principally polarized abelian varieties \(A\) of dimension \(g\) together with an homomorphism \(R_F \hookrightarrow \text{End}(A)\). The Hilbert modular variety \(H_F\) has dimension \(3n\) and \(H_F(\mathbb{C})\) is the quotient of \(n\) copies of the three-dimensional Siegel half space \(\mathfrak{H}_2\) by a suitable discontinuous group (cf. [14], [7]).

Notice that, when \(F = \mathbb{Q}\), \(H_F = A_2\) is Igusa’s three-fold, the moduli space of principally polarized abelian surfaces.

There are natural morphisms

\[
\pi: \quad X_\mu \quad \xrightarrow{\pi_F} \quad H_F \quad \quad \quad \rightarrow \quad A_g
\]

\[
(A, \iota, \mathcal{L}) \quad \mapsto \quad (A, \iota\mid_{R_F}, \mathcal{L}) \quad \mapsto \quad (A, \mathcal{L})
\]
from the Shimura variety $\mathfrak{X}_\mu$ to the Hilbert modular variety $H_F$ and the moduli space $A_g$ that consist of gradually forgetting the quaternionic endomorphism structure. These morphisms are representable, proper and defined over the field $\mathbb{Q}$ of rational numbers.

As it will be convenient for our purposes in the rest of this paper, we introduce the variety $\tilde{\mathfrak{X}}_\mu = \pi(\mathfrak{X}_\mu)$ to be the image of the Shimura variety $\mathfrak{X}_\mu$ attached to a pure quaternion $\mu \in \mathcal{O}$ with $\mu^2 + uD = 0$ for some $u \in R_{F,+}$ in the moduli space $A_g$ by the forgetful map $\pi$. It is important to remark that although the complex analytical structure of $\mathfrak{X}_\mu$ does not depend on the choice of $\mu$, the construction of the forgetful map $\pi$ and the subvariety $\tilde{\mathfrak{X}}_\mu$ of $A_g$ do.

The varieties $\tilde{\mathfrak{X}}_\mu$ are reduced, irreducible, complete and possibly singular schemes over $\mathbb{Q}$ of dimension $n$. The set of singularities of $\tilde{\mathfrak{X}}_\mu$ is a finite set and all the singularities are of quotient type (cf. [3] for the terminology).

3. The birational class of the forgetful maps

It is our aim now to describe the forgetful maps $\pi : \mathfrak{X}_\mu \rightarrow H_F \rightarrow A_g$ as the projection of the Shimura varieties $\mathfrak{X}_\mu$ onto their quotient by suitable Atkin-Lehner groups up to a birational equivalence. The following groups of Atkin-Lehner involutions were introduced in [12]. We keep the notations and assumptions of the Introduction.

**Definition 3.1.** The Atkin-Lehner group $W$ of the maximal order $\mathcal{O}$ is

$$W = \text{Norm}_{B,+}(\mathcal{O})/F^* \cdot \mathcal{O}^1.$$  

It was shown in [11] that $W \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^r \times \mathbb{Z}/2\mathbb{Z}$, where $2r$ is the number of ramified prime ideals of $B$.

**Definition 3.2.** Let $(\mathcal{O}, \mu)$ be a principally polarized maximal order in $B$. A twist of $(\mathcal{O}, \mu)$ is an element $\chi \in \mathcal{O} \cap \text{Norm}_{B,+}(\mathcal{O})$ such that $\chi^2 + n(\chi) = 0$ and $\mu \chi = -\chi \mu$.

In other words, a twist of $(\mathcal{O}, \mu)$ is a pure quaternion $\chi \in \mathcal{O} \cap \text{Norm}_{B,+}(\mathcal{O})$ such that

$$B = F + F_\mu + F_\chi + F_{\mu \chi} = (\frac{-uD, -n(\chi)}{F}).$$

We say that a principally polarized maximal order $(\mathcal{O}, \mu)$ in $B$ is twisting if it admits some twist $\chi$ in $\mathcal{O}$. We say that a maximal order $\mathcal{O}$ is twisting if there exists $\mu \in \mathcal{O}$ such that $(\mathcal{O}, \mu)$ is a twisting principally polarized order. Finally, we say that $B$ is twisting if there exists a twisting maximal order $\mathcal{O}$ in $B$. Note that $B$ is twisting if and only if $B \simeq (\frac{-uD, m}{F})$ for some $u \in R_{F,+}$ and $m \in F^*$ such that $m|D$.

**Definition 3.3.** A twisting involution $\omega \in W$ of $(\mathcal{O}, \mu)$ is an Atkin-Lehner involution such that $[\omega] = [\chi] \in W$ is represented in $B^*$ by a twist $\chi$ of $(\mathcal{O}, \mu)$. 
We let $V_0 = V_0(\mathcal{O}, \mu)$ denote the subgroup of $W$ generated by the twisting involutions of $(\mathcal{O}, \mu)$. For a principally polarized maximal order $(\mathcal{O}, \mu)$, let $R_\mu = F(\mu) \cap \mathcal{O}$ and let $\Omega = \Omega(R_\mu) = \{ \xi \in R_\mu : \xi^f = 1, f \geq 1 \}$ denote the finite group of roots of unity in the CM-quadratic order $R_\mu$ over $R_F$.

**Definition 3.4.** The *stable group* of $(\mathcal{O}, \mu)$ is the subgroup

$$W_0 = U_0 \cdot V_0$$

of $W$ generated by

$$U_0 = U_0(\mathcal{O}, \mu) = \text{Norm}_{F(\mu)^*}(\mathcal{O})/F^* \cdot \Omega(R_\mu),$$

and the group $V_0$ of twisting involutions of $(\mathcal{O}, \mu)$.

As it was also shown in [11], for any principally polarized maximal order $(\mathcal{O}, \mu)$, there are natural monomorphisms of groups $V_0 \subseteq W_0 \subseteq W \subseteq \text{Aut}_\mathbb{Q}(\mathcal{X}_\mu) \subseteq \text{Aut}_\mathbb{Q}(\mathcal{X}_\mu \otimes \bar{\mathbb{Q}})$. The question whether the two latter immersions are actually isomorphisms was studied by the author for the case of Shimura curves in [9].

The following was proved in [11].

**Theorem 3.5.** Let $(\mathcal{O}, \mu)$ be a principally polarized maximal order in $B$ and let $\mathcal{X}_\mu$ be the Shimura variety attached to it. Then there is a commutative diagram of finite maps

$$\mathcal{X}_\mu \xymatrix{ \ar[r]^-{\pi_F} & \mathcal{H}_F \ar[d]_{b_F}^\sim \ar@/_{1.5pc}/[dd] \ar@/^1.5pc/[dd] \ar[r]^-{b_F^{-1}} & \mathcal{X}_\mu/W_0, \ar@/_{1.5pc}/[dd] \ar@/^1.5pc/[dd] }$$

where $\mathcal{X}_\mu \to \mathcal{X}_\mu/W_0$ is the natural projection and $b_F : \mathcal{X}_\mu/W_0 \to \pi_F(\mathcal{X}_\mu)$ is a birational morphism from $\mathcal{X}_\mu/W_0$ onto the image of $\mathcal{X}_B$ in $\mathcal{H}_F$.

The domain of definition of $b_F^{-1}$ is $\pi_F(\mathcal{X}_\mu) \setminus \mathcal{T}_F$, where $\mathcal{T}_F$ is a finite set (of Heegner points).

### 4. The Quaternionic Locus

As above, we let $\mathcal{O}$ be a maximal order in a totally indefinite division quaternion algebra $B$ over a totally real field $F$ of trivial narrow class number and we fix a generator $D \in F_+^*$ of disc($B$).

It is the aim of this section to use the preceding results to study the quaternionic locus $Q_\mathcal{O}$ in $\mathcal{A}_g(\mathbb{C})$. As we mentioned in the Introduction, since $h_+^*(F) = 1$, the set $Q_\mathcal{O}$ is not empty.

**Definition 4.1.** A Heegner point in $Q_\mathcal{O}$ is an isomorphism class $[(A, \mathcal{L})]$ of a principally polarized abelian variety such that $\text{End}(A) \supseteq \mathcal{O}$. 
According to the Definitions \([14]\) and \([15]\) we note that \(Q_\mathcal{O}\) is the disjoint union of the set of principally polarized abelian varieties \([\mathcal{A}, \mathcal{L}]\) with quaternionic multiplication by \(\mathcal{O}\) and the set of Heegner points. The former is a subset of \(Q_\mathcal{O}\) whose closure with respect to the analytical topology is \(Q_\mathcal{O}\) itself. The latter is also a dense but discrete subset of \(Q_\mathcal{O}\) (cf. \([15]\)).

In order to understand the nature of the locus \(Q_\mathcal{O}\), we observe that for any principally polarized pair \((\mathcal{O}, \mu)\), the set \(\tilde{\mathbf{X}}_\mu(\mathbb{C})\) of complex points of the Shimura variety \(\tilde{\mathbf{X}}_\mu/\mathbb{Q}\) attached to \((\mathcal{O}, \mu)\) sits inside \(Q_\mathcal{O}\).

**Proposition 4.2.** Let \(\mu, \mu' \in \mathcal{O}_0\) be two pure quaternions such that \(n(\mu) = uD\) and \(n(\mu') = u'D\) for some units \(u, u' \in R_{F^+}^*\). If \(\tilde{\mathbf{X}}_\mu(\mathbb{C})\) and \(\tilde{\mathbf{X}}_{\mu'}(\mathbb{C})\) are different subvarieties of \(\mathcal{A}_g(\mathbb{C})\), then \(\tilde{\mathbf{X}}_\mu(\mathbb{C}) \cap \tilde{\mathbf{X}}_{\mu'}(\mathbb{C})\) is a finite set of Heegner points.

**Proof.** Assume that the isomorphism class \([\mathcal{A}, \mathcal{L}]\) of a principally polarized abelian variety falls at the intersection of \(\tilde{\mathbf{X}}_\mu\) and \(\tilde{\mathbf{X}}_{\mu'}\) in \(\mathcal{A}_g\). Write \([\mathcal{A}, \mathcal{L}] = \pi([\mathcal{A}, \iota, \mathcal{L}]) = \pi([\mathcal{A}', \iota', \mathcal{L}'])\) as the image by \(\pi\) of points in \(\mathbf{X}_\mu\) and \(\mathbf{X}_{\mu'}\), respectively. Since \([\mathcal{A}, \mathcal{L}] = ([\mathcal{A}', \mathcal{L}'])\) \(\in Q_\mathcal{O}\), we can identify the pair \((\mathcal{A}, \mathcal{L}) = (\mathcal{A}', \mathcal{L}')\) through a fixed isomorphism of polarized abelian varieties.

Let us assume that \([\mathcal{A}, \mathcal{L}] = [\mathcal{A}', \mathcal{L}']\) was not a Heegner point. Then \(\iota: \mathcal{O} \simeq \text{End}(\mathcal{A})\) would be an isomorphism of rings such that \(c_1(\mathcal{L}) = \mu\). We then would have by Theorem \([12]\) \(\S 4\), that \(c_1(\mathcal{L}) = c_1(\mathcal{L}') = \mu = \mu'\) up to multiplication by elements in \(F^*\). Since \(\tilde{\mathbf{X}}_\mu = \tilde{\mathbf{X}}_{u\mu}\) for all units \(u \in R_{F^+}^*\), this would contradict the statement. Since the set of Heegner points in \(\mathcal{A}_g(\mathbb{C})\) is discrete, we conclude that any two irreducible components of \(Q_\mathcal{O}\) meet at a finite set of Heegner points. \(\square\)

**Proposition 4.3.**

1. The locus \(Q_\mathcal{O}\) is the set of complex points \(Q_\mathcal{O}(\mathbb{C})\) of a reduced complete subscheme \(Q_\mathcal{O}\) of \(\mathcal{A}_g\) defined over \(\mathbb{Q}\).

2. Let \(\rho(\mathcal{O})\) be the number of absolutely irreducible components of \(Q_\mathcal{O}\). Then there exist quaternions \(\mu_k \in \mathcal{O}_0\) with \(\mu_k^2 + u_kD = 0\) for \(u_k \in R_{F^+}^*\), \(1 \leq k \leq \rho(\mathcal{O})\), such that

\[
Q_\mathcal{O} = \bigcup \tilde{\mathbf{X}}_{\mu_k}.
\]

is the decomposition of \(Q_\mathcal{O}\) into irreducible components.

**Proof.** Let \([\mathcal{A}, \mathcal{L}] \in Q_\mathcal{O}\) be the isomorphism class of a complex principally polarized abelian variety such that \(\text{End}(\mathcal{A}) \simeq \mathcal{O}\). Fix an isomorphism \(\iota: \mathcal{O} \simeq \text{End}(\mathcal{A})\). By Theorem \([12]\) \(\S 4\), the Rosati involution with respect to \(\mathcal{L}\) on \(\mathcal{O}\) must be of the form \(\theta_\mu: \mathcal{O} \to \mathcal{O}, \beta \mapsto \mu^{-1}\beta\mu\) for some \(\mu \in \mathcal{O}\) with \(\mu^2 + uD = 0\), \(u \in R_{F^+}^*\). Thus \([\mathcal{A}, \mathcal{L}] = \pi([\mathcal{A}, \iota, \mathcal{L}])\) \(\in \tilde{\mathbf{X}}_\mu(\mathbb{C})\), namely the set of complex points on a reduced, irreducible, complete and possibly singular scheme over \(\mathbb{Q}\) (cf. \([14]\), \([15]\)). Since the set of Heegner points \([\mathcal{A}, \mathcal{L}] \in \tilde{\mathbf{X}}_\mu(\mathbb{C})\) is a discrete set which lies on the Zariski closure of its complement, we conclude that \(Q_\mathcal{O}\) is the union of the
Shimura varieties $\tilde{X}_\mu(C)$ as $\mu$ varies among pure quaternions satisfying the above properties.

Let us now show that $Q_O$ is actually covered by finitely many pairwise different Shimura varieties. Let $A/C$ be an arbitrary abelian variety with quaternionic multiplication by $O$ and fix an isomorphism $\iota : O \simeq \text{End}(A)$.

Let $(O, \mu)$ be any principally polarized pair. Since $h_+(F) = 1$, there exists a unit $u \in R_F^*$ such that $u\mu$ is an ample quaternion in the sense of [10], §5. Let $L \in \text{NS}(A)$ be the line bundle on $A$ such that $c_1(L)^{-1} = u\mu$. From Theorem 1.2 it follows that $L$ is a principal polarization on $A$ such that the isomorphism class of the triplet $(A, \iota, L)$ corresponds to a closed point in $X_\mu(C)$ and hence $[A, L] \in \tilde{X}_\mu$.

Since, by Proposition 4.2, the intersection points of two different Shimura varieties $\tilde{X}_\mu(C)$ and $\tilde{X}_{\mu'}(C)$ in $\mathcal{A}_{g}(C)$ are Heegner points, this shows that for every irreducible component of $Q_O$ there exists at least one principal polarization $L$ on $A$ such that $[A, L]$ lies on it. Consequently, the number $\pi_0(A)$ of isomorphism classes of principal polarizations on $A$ is an upper bound for the number $\rho(O)$ of irreducible components of $Q_O$. Since, by Theorem 1.4 the number $\pi_0(A)$ is a finite number, this yields the proof of the proposition. $\square$

In view of Proposition 4.3 it is natural to pose the following

**Question 4.4.** What is the number $\rho(O)$ of irreducible components of $Q_O$? When is $Q_O$ irreducible?

5. **The Distribution of Principal Polarizations on an Abelian Variety in $Q_O$**

Let us relate Question 4.4 to the following problem. In Theorem 1.2 we computed the number $\pi_0(A)$ of principal polarizations on an abelian variety $A$ with quaternion multiplication by $O$ as the finite sum of relative class numbers of suitable orders in the CM-fields $F(\sqrt{-uD})$ for $u \in R_F^*/R_F^{*2}$. This has the following modular interpretation:

Let $L_1, \ldots, L_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on $A$. Then the pairwise nonisomorphic principally polarized abelian varieties $[(A, L_1)], \ldots, [(A, L_{\pi_0(A)})]$ correspond to all closed points in $Q_O$ whose underlying abelian variety is isomorphic to $A$. We then naturally ask the following

**Question 5.1.** Let $A$ be an abelian variety with quaternionic multiplication by $O$. How are the distinct principal polarizations $[(A, L_j)]$ distributed among the irreducible components $\tilde{X}_{\mu_k}$ of $Q_O$?

It turns out that the two questions above are related. The linking ingredient is provided by the definition below, which establishes a slightly coarser equivalence relationship on polarizations than the one considered in Theorem 1.2 §1.
Definition 5.2. Let $A$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{Q}$.

1. Two polarizations $\mathcal{L}$ and $\mathcal{L}'$ on $A$ are weakly isomorphic if $c_1(\mathcal{L}) \simeq mc_1(\mathcal{L}') \in \text{NS}(A)$ for some $m \in F_+^*$. We shall denote it $\mathcal{L} \simeq_w \mathcal{L}'$.

2. Two principal polarizations $\mathcal{L}$ and $\mathcal{L}'$ on $A$ are Atkin-Lehner isogenous, denoted $\mathcal{L} \sim \mathcal{L}'$, if there is an isogeny $\omega \in \mathcal{O} \cap \text{Norm}_{B_+}(\mathcal{O})$ of $A$ such that

$$\omega^*(\mathcal{L}) \simeq_w \mathcal{L}'$$

We note that there is a closed relationship between the above definition and the modular interpretation of the Atkin-Lehner group $W$ given in [11].

Definition 5.3. Let $A$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{Q}$. We let $\hat{\Pi}_0(A)$ be the set of principal polarizations on $A$ up to Atkin-Lehner isogeny and we let $\hat{\pi}_0(A) = \sharp\hat{\Pi}_0(A)$ denote its cardinality.

Theorem 5.4 (Distribution of principal polarizations). Let $A$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{Q}$ and let $\mathcal{L}_1, ..., \mathcal{L}_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on $A$.

Then, two closed points $[A, \mathcal{L}_1]$ and $[A, \mathcal{L}_j]$ lie on the same irreducible component of $\mathcal{Q}_\mathcal{O}$ if and only if the polarizations $\mathcal{L}_i$ and $\mathcal{L}_j$ are Atkin-Lehner isogenous.

Proof. We know from Proposition 4.8 that any irreducible component of $\mathcal{Q}_\mathcal{O}$ is $\hat{\mathcal{X}}_\mu$ for some principally polarized pair $(\mathcal{O}, \mu)$. We single out and fix one of these.

Let $\mathcal{L}$ be a principal polarization on $A$ such that $[(A, \mathcal{L})]$ lies on $\hat{\mathcal{X}}_\mu$ and let $\mathcal{L}'$ be a second principal polarization on $A$. We claim that $[A, \mathcal{L}'] \in \hat{\mathcal{X}}_\mu$ if and only if there exists $\omega \in \mathcal{O} \cap \text{Norm}_{B_+}(\mathcal{O})$ such that $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic.

Assume first that $\mathcal{L}' \simeq_w \omega^*(\mathcal{L})$ for some $\omega \in \mathcal{O} \cap \text{Norm}_{B_+}(\mathcal{O})$. This amounts to saying that $\omega c_1(\mathcal{L}) \omega = mc_1(\mathcal{L}')$ for some $m \in F_+^*$. Since both $\omega^*(\mathcal{L})$ and $\mathcal{L}'$ are polarizations, we deduce from Theorem 4.2 §3, that $m \in F_+^*$. Moreover, since $\mathcal{L}$ and $\mathcal{L}'$ are principal, we obtain from Theorem 4.2 §2, that $m = un(\omega)$ for some $u \in R_{F,+}$.

Note that $(A, \iota_\omega, \mathcal{L}')$ is a principally polarized abelian variety with quaternionic multiplication such that the Rosati involution that $\mathcal{L}'$ induces on $\iota_\omega(\mathcal{O})$ is $\theta_\mu$. Indeed, this follows because $\iota_\omega(\beta)^{\circ \omega'} = \iota((\omega^{-1} \beta \omega))^{\circ \omega'} = \iota((\omega^{-1} \mu \omega)^{-1} \omega^{-1} \beta \omega (\omega^{-1} \mu \omega))^{-1} = \iota_\omega(\mu^{-1} \beta \mu)$. This shows that, if $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic for some $\omega \in \mathcal{O} \cap \text{Norm}_{B_+}(\mathcal{O})$, then $[A, \mathcal{L}'] \in \hat{\mathcal{X}}_\mu$.

Conversely, let us assume that $[A, \mathcal{L}'] \in \hat{\mathcal{X}}_\mu$. Let $\iota' : \mathcal{O} \hookrightarrow \text{End}(A)$ be such that $[A, \iota', \mathcal{L}'] \in \hat{\mathcal{X}}_{\mathcal{O}X_\mu, \theta_\mu}$. By the Skolem-Noether Theorem, it holds that $\iota' = \omega^{-1} \iota_\omega$ for some $\omega \in \text{Norm}_{B_+}(\mathcal{O})$; we can assume that $\omega \in \mathcal{O}$ by suitably scaling it. Since it holds that $\iota_\omega(\beta)^{\circ \omega'} = \iota_\omega(\mu^{-1} \beta \mu)$ for any $\beta \in \mathcal{O}$, we have that $c_1(\mathcal{L}') = u \omega^{-1} c_1(\mathcal{L}) \omega$ for some $u \in R_F^*$ such that $un(\omega) \in F_+^*$. Since $n(\mathcal{O}^*) = R_F^*$, we can find $\alpha \in \mathcal{O}^*$ with reduced norm $n(\alpha) = u^{-1}$ and thus $\omega \alpha \in B_+^*$.  


Let $\mathcal{L}_{\omega\alpha}$ be the polarization on $A$ such that $c_1(\mathcal{L}_{\omega\alpha}) = \frac{\omega}{n(\omega)}c_1((\omega\alpha)^*(\mathcal{L}))$. The automorphism $\alpha \in \mathcal{O}^* = \text{Aut}(A)$ induces an isomorphism between the polarizations $\mathcal{L}_{\omega\alpha}$ and $\mathcal{L'}$, since $c_1(\alpha^*(\mathcal{L}')) = \overline{\alpha}(\omega^{-1}c_1(\mathcal{L})\omega)\alpha = c_1(\mathcal{L}_{\omega\alpha})$. Hence $\mathcal{L}'$ is weakly isomorphic to $(\omega\alpha)^*(\mathcal{L})$. This concludes our claim above and also proves the theorem. □

**Corollary 5.5.** The number of irreducible components of $Q_{\mathcal{O}}$ is

$$\rho(\mathcal{O}) = \tilde{\pi}_0(A),$$

independently of the choice of $A$.

For any irreducible component $\tilde{\mathcal{X}}_{\mu_k}$ of $Q_{\mathcal{O}}$, let $\Pi_{\mathcal{O}}^{(k)}(A) \subset \Pi_0(A)$ denote the set of isomorphism classes of the classes of principal polarizations lying on $\tilde{\mathcal{X}}_{\mu_k}$.

As another immediate consequence of Theorem 5.4, the following corollary establishes an internal structure on the set $\Pi_0(A)$. Roughly, it asserts that $\Pi_0(A) = \bigcup_{k=1}^{\rho(\mathcal{O})} \Pi_{\mathcal{O}}^{(k)}(A)$ is the disjoint union of the sets $\Pi_{\mathcal{O}}^{(k)}(A)$, which are equipped with a free and transitive action of a 2-torsion finite abelian group.

**Corollary 5.6.** Let $A$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Let $\tilde{\mathcal{X}}_{\mu_k}$ be an irreducible component of $Q_{\mathcal{O}}$ and let $W_{\mathcal{O}}^{(k)} \subseteq W$ be the stable subgroup attached to the polarized order $(\mathcal{O}, \mu_k)$.

Then there is a free and transitive action of $W/W_{\mathcal{O}}^{(k)}$ on the Atkin-Lehner isogeny class $\Pi_{\mathcal{O}}^{(k)}(A)$ of principal polarizations lying on $\tilde{\mathcal{X}}_{\mu_k}$.

In the case of a non twisting maximal order $\mathcal{O}$, we have that the stable group $W_{\mathcal{O}}(\mathcal{O}, \mu)$ attached to a principally polarized pair $(\mathcal{O}, \mu)$ is $U_{\mathcal{O}}(\mathcal{O}, \mu)$. The following corollary follows from the proof of Lemma 2.10 in [12].

**Corollary 5.7.** Let $\mathcal{O}$ be a non twisting maximal order in $B$ and assume that, for any $u \in R_{F,+}^*$, any primitive root of unity of odd order in the CM-field $F(\sqrt{-uD})$ is contained in the order $R_{F}[\sqrt{-uD}]$.

Let $A$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Then the distinct isomorphism classes of principally polarized abelian varieties $[(A, \mathcal{L}_1)], \ldots, [(A, \mathcal{L}_{\rho(\mathcal{O})})]$ are equidistributed among the $\rho(\mathcal{O})$ irreducible components of $Q_{\mathcal{O}}$.

In particular, it then holds that

$$\pi_0(A) = \frac{|W|}{|W_{\mathcal{O}}|} \cdot \rho(\mathcal{O}).$$

6. **Shimura curves embedded in Igusa’s threefold**

The whole picture becomes particularly neat when we consider the simplest case of quaternion algebras over $\mathbb{Q}$. Let then $B$ be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$ and let $\mathcal{O}$ be a maximal order in $B$. Since
$h(Q) = 1$, there is a single choice of $O$ up to conjugation by $B^*$. Moreover, all left ideals of $O$ are principal and hence isomorphic to $O$ as left $O$-modules.

Let $A$ be a complex abelian surface with quaternionic multiplication by $O$. By Theorem 1.4, $A$ is principally polarizable and the number of isomorphism classes of principal polarizations on $A$ is

$$\pi_0(A) = \frac{h(-D)}{2},$$

where, for any nonzero squarefree integer $d$, we write

$$h(d) = \begin{cases} h(4d) + h(d) & \text{if } d \equiv 1 \mod 4, \\ h(4d) & \text{otherwise.} \end{cases}$$

For any integral element $\mu \in O$ such that $\mu^2 + D = 0$, let now $X_\mu$ be the Shimura curve that coarsely represents the functor which classifies principally polarized abelian surfaces $(A, \iota, L)$ with quaternionic multiplication by $O$ such that the Rosati involution with respect to $L$ on $O$ is $\iota_\mu$. This is an algebraic curve over $Q$ whose isomorphism class does not actually depend on the choice of the quaternion $\mu$, but only on the discriminant $D$ (cf. [15]). Hence, it is usual to simply denote this isomorphism class as $X_D$.

Let $W = \{\omega_m : m|D\} \simeq (\mathbb{Z}/2\mathbb{Z})^{2r}$ be the Atkin-Lehner group attached to $O$ in Section 3. We know that $W \subseteq \text{Aut}_Q(X_D)$ is a subgroup of the group of automorphisms of the Shimura curve $X_D$.

Let now $A_2$ be Igusa’s three-fold, the moduli space of principally polarized abelian surfaces. By the work of Igusa (cf. [3]), it is an affine scheme over $Q$ that contains, as a Zariski open and dense subset, the moduli space $M_2$ of curves of genus 2, immersed in $A_2$ via the Torelli embedding.

Sitting in $A_2$ there is the quaternionic locus $Q_O$ of isomorphism classes of principally polarized abelian surfaces $[(A, L)]$ such that $\text{End}(A) \supseteq O$. Since all maximal orders $O$ in $B$ are pairwise conjugate, the quaternionic locus $Q_O$ does not actually depend on the choice of $O$ and we may simply denote it by $Q_O = Q_D$.

As explained in Section 2 and 3, there are forgetful finite morphisms $\pi : X_\mu \to Q_D \subset A_2$ which map the Shimura curve $X_\mu$ onto an irreducible component $\tilde{X}_\mu$ of $Q_D$. We insist on the fact that the image $\tilde{X}_\mu \subset Q_D$ does depend on the choice of the quaternion $\mu$.

Let us now compare the non twisting and twisting case, respectively. We first assume that

$$B \not\equiv \left(\frac{-D, m}{Q}\right)$$
for all positive divisors $m|D$ of $D$. Then all principally polarized pairs $(\mathcal{O}, \mu)$ in $B$ are non twisting and the stable subgroup attached to $(\mathcal{O}, \mu)$ is

$$W_0 = U_0 = \langle \omega_D \rangle \subset W,$$

independently of the choice of $\mu$. By Theorem 3.5, we deduce that any irreducible component $\tilde{\mathcal{X}}_\mu$ of $Q_D$ is birationally equivalent to the Atkin-Lehner quotient $\mathcal{X}_D/\langle \omega_D \rangle$ and thus the quaternionic locus $Q_D$ in $A_2$ is the union of pairwise birationally equivalent Shimura curves $\tilde{\mathcal{X}}_{\mu_1}, \ldots, \tilde{\mathcal{X}}_{\mu_{\rho(\mathcal{O})}}$, meeting at a finite set of Heegner points.

Moreover, for any abelian surface $A$ with quaternionic multiplication by $\mathcal{O}$, it follows from Theorem 5.4 that the closed points $\{(A, \mathcal{L}_j)\}_{j=1}^{\pi_0(A)}$ are equidistributed among the $\rho(\mathcal{O})$ irreducible components of $Q_D$. In addition, Corollary 5.7 ensures that

$$|W/W_0| = 2^{2r-1}|\pi_0(A)|,$$

as genus theory for binary quadratic forms already predicts.

Finally, we obtain that the number of irreducible components of $Q_D$ in the non twisting case is

$$\rho(\mathcal{O}) = \frac{\tilde{h}(-D)}{2^{2r}}.$$

On the other hand, let us assume that

$$B \simeq \left( \frac{-D, m}{\mathbb{Q}} \right)$$

for some $m|D$. In this case, there can be two different birational classes of irreducible components on $Q_D$. Indeed, the assumption means that there exist pure quaternions $\mu \in \mathcal{O}$, $\mu^2 + D = 0$, such that $(\mathcal{O}, \mu)$ is a twisting principally polarized maximal order. Then

$$W_0(\mathcal{O}, \mu) = \langle \omega_m, \omega_D \rangle$$

and $\tilde{\mathcal{X}}_\mu$ is birationally equivalent to $\mathcal{X}_D/\langle \omega_m, \omega_D \rangle$. We may refer to $\tilde{\mathcal{X}}_\mu$ as a twisting irreducible component of $Q_D$.

In addition to these, there may exist non twisting polarized orders $(\mathcal{O}, \mu)$ such that the corresponding irreducible components $\tilde{\mathcal{X}}_\mu$ of $Q_D$ are birationally equivalent to $\mathcal{X}_D/\langle \omega_D \rangle$. We may refer to these as the non twisting irreducible components of $Q_D$.

We then have the following lower and upper bounds for the number of irreducible components of $Q_D$:

$$\frac{\tilde{h}(-D)}{2^{2r}} < \rho(\mathcal{O}) \leq \frac{\tilde{h}(-D)}{2^{2r-1}}.$$
Summing up, we obtain the following

**Theorem 6.1.** Let $B$ be an indefinite division quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$. Then, the quaternionic locus $Q_D$ in $A_2$ is irreducible if and only if

$$h(-D) = \begin{cases} 2^{2r-1} & \text{if } B \simeq \left(\frac{-D,m}{\mathbb{Q}}\right) \text{ for some } m|D, \\ 2^{2r} & \text{otherwise.} \end{cases}$$

**Proof.** If $B$ is not a twisting quaternion algebra, we already know from the above that the number of irreducible components of $Q_D$ is $\frac{h(-D)}{2^{2r}}$. Hence, in this case, the quaternionic locus of discriminant $D$ in $A_2$ is irreducible if and only if $h(-D) = 2^{2r}$. If on the other hand $B$ is twisting, it follows from the above inequalities that $Q_D$ is irreducible if and only if $h(-D) = 2^{2r-1}$. $\blacksquare$

In view of Theorem 6.1, there arises a closed relationship between the irreducibility of the quaternionic locus in Igusa’s threefold and the genus theory of integral binary quadratic forms and the classical *numeri idonei* studied by Euler, Schinzel and others. We refer the reader to [1] and [13] for the latter.

### 7. Hashimoto-Murabayashi’s families

As the simplest examples to be considered, let $B_6$ and $B_{10}$ be the rational quaternion algebras of discriminant $D = 2 \cdot 3 = 6$ and $2 \cdot 5 = 10$, respectively. Hashimoto and Murabayashi [4] exhibited two families of principally polarized abelian surfaces with quaternionic multiplication by a maximal order in these quaternion algebras. Namely, let

$$C^{(6)}_{(s,t)} : Y^2 = X(X^4 + PX^3 + QX^2 + RX + 1)$$

be the family of curves with

$$P = 2s + 2t, \quad Q = \frac{(1 + 2t^2)(11 - 28t^2 + 8t^4)}{3(1 - t^2)(1 - 4t^2)}, \quad R = 2s - 2t$$

over the base curve

$$g^{(6)}(t, s) = 4s^2t^2 - s^2 + t^2 + 2 = 0.$$ 

And let

$$C^{(10)}_{(s,t)} : Y^2 = X(P^2X^4 + P^2(1 + R)X^3 + PQX^2 + P(1 - R)X + 1)$$

be the family of curves with

$$P = \frac{4(2t + 1)(t^2 - t - 1)}{(t - 1)^2}, \quad Q = \frac{(t^2 + 1)(t^4 + 8t^3 - 10t^2 - 8t + 1)}{t(t - 1)^2(t + 1)^2}$$
and 

$$R = \frac{(t - 1)s}{t(t + 1)(2t + 1)}$$

over the base curve

$$g^{(10)}(t, s) = s^2 - t(t - 2)(2t + 1) = 0.$$ 

Let \( J_{(6)}^{(6)} = \text{Jac}(C_{(s,t)}^{(6)}) \) and \( J_{(10)}^{(10)} = \text{Jac}(C_{(s,t)}^{(10)}) \) be the Jacobian surfaces of the fibres of the families of curves above respectively. It was proved in [4] that their ring of endomorphisms contain a maximal order in \( B_6 \) and \( B_{10} \), respectively.

Both \( B_6 = \left( -\frac{6}{4}, 2 \right) \mathbb{Q} \) and \( B_{10} = \left( -\frac{10}{4}, 2 \right) \mathbb{Q} \) are twisting quaternion algebras. Moreover, it turns out from our formula for \( \pi_0(A) \) in the above section that any abelian surface \( A \) with quaternionic multiplication by a maximal order in either \( B_6 \) or \( B_{10} \) admits a single isomorphism class of principal polarizations. This implies that \( \rho(B_6) = \pi_0(A) = \pi_0(A) = 1 \) and \( \rho(B_{10}) = \pi_0(A) = \pi_0(A) = 1 \), respectively.

Moreover, the Shimura curves \( X_6/\mathbb{Q} \) and \( X_{10}/\mathbb{Q} \) have genus 0, although they are not isomorphic to \( \mathbb{P}^1_{\mathbb{Q}} \) because there are no rational points on them. However, it is easily seen that \( X_6/W_0 = X_6/W \simeq \mathbb{P}^1_{\mathbb{Q}} \) and \( X_{10}/W_0 = X_{10}/W \simeq \mathbb{P}^1_{\mathbb{Q}} \), respectively.

As it is observed in [4], the base curves \( g^{(6)} \) and \( g^{(10)} \) are curves of genus 1 and not of genus 0 as it should be expected. This is explained by the fact that there are obvious isomorphisms between the fibres of the families \( C^{(6)} \) and \( C^{(10)} \), respectively.

Ibukiyama, Katsura and Oort [5] proved that the supersingular locus in \( \mathcal{A}_2/\mathbb{Q} \) is irreducible if and only if \( p \leq 11 \). As a corollary to their work, Hashimoto and Murabayashi obtained that the reduction mod 3 and 5 of the family of Jacobian surfaces with quaternionic multiplication by \( B_6 \) and \( B_{10} \) respectively yield the single irreducible component of the supersingular locus in \( \mathcal{A}_2/\overline{\mathbb{F}}_3 \) and \( \mathcal{A}_2/\overline{\mathbb{F}}_5 \) respectively.

The following statement may be considered as a lift to characteristic 0 of these results.

**Theorem 7.1.**

1. The quaternionic locus \( \mathcal{Q}_6 \) in \( \mathcal{A}_2/\mathbb{Q} \) is absolutely irreducible and birationally equivalent to \( \mathbb{P}^1_{\mathbb{Q}} \) over \( \mathbb{Q} \). A universal family over \( \mathbb{Q} \) is given by Hashimoto-Murabayashi’s family \( C^{(6)} \).
2. The quaternionic locus \( \mathcal{Q}_{10} \) in \( \mathcal{A}_2/\mathbb{Q} \) is absolutely irreducible and birationally equivalent to \( \mathbb{P}^1_{\mathbb{Q}} \) over \( \mathbb{Q} \). A universal family over \( \mathbb{Q} \) is given by Hashimoto-Murabayashi’s family \( C^{(10)} \).

**Proof.** This follows from Theorem 6.1 and the discussion above. \( \Box \)

In particular, we obtain from Theorem 7.1 that every principally polarized abelian surface \( (A, \mathcal{L}) \) over \( \mathbb{Q} \) with quaternionic multiplication by a maximal order of discriminant 6 or 10 is isomorphic over \( \mathbb{Q} \) to the Jacobian variety of one of the curves \( C^{(6)}_{(s,t)} \) or \( C^{(10)}_{(s,t)} \), except for finitely many degenerate cases.
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