Gács–Kučera’s Theorem Revisited by Levin

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Abstract

Leonid Levin [12] published a new (and very nice) proof of Gács–Kučera’s theorem that occupies only a few lines when presented in his style. We try to explain more details and discuss the connection of this proof with image randomness theorems, making explicit some result (see Proposition 4) that is implicit in [12].

1 The Gács–Kučera Theorem

The Gács–Kučera theorem says that every (infinite) sequence of zeros and ones is Turing-reducible to some Martin-Löf random sequence. In other words, for every sequence \( \alpha \) there exist a Martin-Löf random (with respect to the uniform Bernoulli distribution) sequence \( \omega \) and a computable mapping \( T \) of the space of binary sequences to itself such that \( T \omega = \alpha \).

Let us comment on the notions used in the statement. The notion of Martin-Löf random sequence was defined by Martin-Löf in 1966 [14] and since then has become a standard notion in algorithmic randomness (see, e.g., [13, 6, 15, 16]).

To define computable mappings, consider an oracle machine has read access to an infinite bit sequence on the input tape and writes bits sequentially on the output tape. It computes a mapping \( T \) with domain \( \Omega \) (the space of all infinite binary sequences) and codomain \( \Sigma \) (the space of all finite and infinite sequences). Mappings that correspond to oracle machines are called computable.

The proofs given by Gács and Kučera (as well as their subsequent improvements that limit the number of input bits needed to produce \( n \) output bits) all follow the same scheme: given \( \alpha \), we construct some random sequence \( \omega \) and some ad hoc mapping \( T \) that maps \( \omega \) to \( \alpha \) but has no meaning outside this context. However, there is an alternative approach. There is an image randomness theorem (see, e.g., [16], or [4] for more general version) that says that for a computable mapping \( T \) that maps a uniform Bernoulli distribution \( P \) on \( \Omega \) to some probability distribution \( Q \) on \( \Omega \) (this means that \( T(\omega) \) is infinite for \( P \)-almost all \( \omega \)), every sequence that is random with respect to \( Q = T(P) \) is a \( T \)-image of some \( P \)-random sequence. The alternative plan is to find some extension of this result to the case where the image measure \( T(P) \) is a semimeasure, and then derive the Gács–Kučera theorem from this extended version.

Let us explain this plan and the notion of a semimeasure in more details. Consider an arbitrary computable mapping \( T: \Omega \to \Sigma \). Applying \( T \) to a random uniformly distributed point \( \omega \) in \( \Omega \), we get some random variable \( \xi = T(\omega) \) with values in \( \Sigma \). For every string \( x \) we may consider the probability of the event “\( \xi \) starts with \( x \)”. We get some non-negative real function \( M_T(x) \) defined on all strings. Obviously, \( M_T(\Lambda) = 1 \) for the empty string \( \Lambda \), and

\[
M_T(x) \geq M_T(x0) + M_T(x1)
\]  

\[ (*) \]
for every string $x$. The last inequality may not be an equality; the difference

$$M_T(x) - M_T(x_0) - M_T(x_1)$$

is the probability of the event “$\xi$ is finite and equals $x$” (since it corresponds to the cases when $x$ is a prefix of $\xi$ but neither $x_0$ nor $x_1$ are).

The non-negative functions $M$ on strings that have value 1 on the empty string and satisfy the inequality (1), are called semimeasures and correspond to arbitrary probability distributions on $\Sigma$; we often use the same letter for the probability distribution on $\Sigma$ and the corresponding function on strings. One may ask which semimeasures can be obtained as $M_T$ for a computable mapping $T$, as described above. This question was answered by Levin in 1970 [17]: the function $M$ should be lower semicomputable. This means that the set of pairs $\langle r, x \rangle$ where $x$ is a string and $r$ is a rational number smaller than $M(x)$, is (computably) enumerable.

In the same paper Levin noted that there is a maximal (up to $O(1)$-factor) function in the class of lower semicomputable semimeasures; it is now called the continuous a priori probability. More precisely, there are many such functions that differ by $O(1)$-factors; we fix one of them and denote it by $M(x)$. The continuous a priori probability $M(x)$ can be used to characterize randomness with respect to computable measures on $\Omega$. Let $Q$ is a computable measure on $\Omega$, i.e., a lower semicomputable semimeasure such that all finite strings have probability zero:

$$Q(x) = Q(x_0) + Q(x_1)$$

for all $x$. Levin proved in [10] that a sequence $\alpha$ is Martin-Löf random with respect to $Q$ if and only if the ratio $M(x)/Q(x)$ is bounded for all prefixes $x$ of $\alpha$. The intuitive meaning: $\alpha$ is not random with respect to $Q$ if some prefixes of $\alpha$ are much more likely according to a priori distribution $M$ than according to $Q$ (the ratio is unbounded).

Combining this characterization of randomness with the image randomness theorem mentioned above, we come to the following result.

**Proposition 1.** Let $P$ be the uniform Bernoulli distribution on $\Omega$. Let $T: \Omega \to \Sigma$ be a computable mapping such that $T(\omega)$ is infinite for $P$-almost $\omega$, and let $Q = T(P)$ be the image distribution. Then the following properties of an infinite sequence $\alpha$ are equivalent:

- $M(x)/Q(x)$ is bounded for all prefixes $x$ of $\alpha$;
- $\alpha = T(\omega)$ for some Martin-Löf random (with respect to $P$) sequence $\omega$.

If we could get rid of the restriction that $T(\omega)$ is infinite for almost all $\omega$, thus extending this proposition to the case when $Q$ is a semimeasure, the Gács–Kučera theorem will follow from this generalization. Indeed, the semimeasure $M$ (being semicomputable) can be represented as $T(P)$ where $T$ is some computable mapping of $\Omega$ to $\Sigma$. Then $M(x)/Q(x) = 1$ is bounded everywhere, and this (hypothetical) generalization would imply that every sequence $\alpha$ is $T(\omega)$ for some random $\omega$.

## 2 Levin’s approach

However, this plan does not work: as found in [5], this generalized statement is not true. Levin [12] found a way to overcome these difficulties by using a less general statement. Namely, he made the following three observations\footnote{We first provide them in a simplified form that is enough for Gács–Kučera theorem; see the next section for discussion.} (Propositions 2–4):

1. We first provide them in a simplified form that is enough for Gács–Kučera theorem; see the next section for discussion.
**Proposition 2.** There exists a maximal (up to $O(1)$-factor) lower semicomputable semimeasure whose values are finite binary fractions having at most $2n + O(1)$ bits for strings of length $n$.

**Proposition 3.** Every semimeasure $Q$ whose values on $n$-bit strings are finite binary fractions of length at most $2n + O(1)$, is an image of the uniform distribution $P$ by a computable mapping $T : \Omega \to \Sigma$ that uses only first $2n + O(1)$ input bits to produce $n$ output bits.

The claim means that the $n$ first bits of $T(\omega)$ depend only on $2n + O(1)$ bits of $\omega$. In particular, if $T(\omega)$ has length less than $n$, then the same is true for all $T(\omega')$ if $\omega'$ and $\omega$ have the same first $2n$ bits.

**Proposition 4.** Let $T : \Omega \to \Sigma$ be a computable mapping that uses only first $2n + O(1)$ input bits to produce $n$ output bits. Let $P$ be the uniform Bernoulli distribution on $\Omega$ and let $Q = T(P)$ be the image semimeasure. If the ratio $M(x)/Q(x)$ is bounded for all prefixes of some infinite sequence $\alpha$, then $\alpha = T(\omega)$ for some Martin-Löf random (with respect to $P$) sequence $\omega$.

**Proof of Proposition 2.** We may perform rounding and replace each value on $n$-bit string by the maximal $2n + c$-bit binary fraction that is strictly smaller than this value. Here $c$ is some constant that will be chosen later. This procedure gives a lower semicomputable function (here it is important that we use strict inequalities) with required granularity, but this function may not be a semimeasure. The property

$$Q(x) \geq Q(x0) + Q(x1)$$

may be violated after rounding. Indeed, while the right-hand side may only decrease after rounding (and this is not a problem), the left-hand side also may decrease making the inequality false. The change in the left-hand side is at most $2^{-2n-c}$ (where $n = |x|$), so a small safety margin is enough: if

$$Q(x) \geq Q(x0) + Q(x1) + 2^{-2n-c},$$

then after rounding we have a semimeasure. (We should also change the value on the empty string to 1, but this change does not violate the other requirements.)

To guarantee this safety margin, we add to the maximal semimeasure $M$ some other semimeasure with desired safety margin. (Then we have to divide the sum by 2 to get a semimeasure, but this only increases $c$ by 1.) For example, the semimeasure $Q(x) = 2^{-2|x|}$ has safety margin $2^{-2n} - 2 \cdot 2^{-2n-2} = 2^{-2n-1}$ for $n$-bit strings, so for $c \geq 1$ this is enough.

We need also to check that our semimeasure remains maximal after rounding. Indeed, the safety margin is bigger than the granularity, and the combined effect of the increase and rounding can only increase the value of the semimeasure.

**Proof of Proposition 3.** Here we have to recall the construction of the mapping for a given semimeasure (see [17] or [16] Section 5.1 for details). This construction is performed in terms of space allocation. Assume that some semimeasure $Q$ is given. For every string $x$, we allocate to $x$ some subset of $\Omega$ that consists of cones (a cone $\Omega_x$ is a set of all infinite sequences with a given finite prefix $y$) and has total measure $Q(x)$. If $Q(x)$ is a multiple of $2^{-2n-c}$, we may use cones $\Omega_y$ of size $2^{-2n-c}$ (with $|y| = 2n + c$) when allocating space to $x$. Since $2n + c$ is a monotone function of $n$, for the children of $x$ we may use smaller cones inside the cones allocated to $x$. In this way we get a mapping where $n$ output bits are determined by $2n + c$ input bits: the preimage of $x$ is the union of cones allocated to $x$, and all these cones have size $2^{-2n-c}$, so only first $2n + c$ bits matter.

**Proof of Proposition 4.** This proposition can be proven in different ways; for example, one may adapt the argument from [16] Section 5.9.3. Levin gives a much simpler argument, but it uses the
characterization of Martin-Löf randomness in terms of expectation-bounded randomness tests that were introduced by Levin in [11] and then studied in [7] (and probably are not so well known as they deserved to be). We recall their definition and properties that we need (see [16] Section 3.5) and [3] for a detailed exposition.

We consider lower semicomputable non-negative real functions on \( \Omega \) (the infinite value \( +\infty \) is also allowed). The notion of lower semicomputability is an effective version of lower semi-continuity: function \( f \) is lower semicomputable if the set of pairs \( \{ \langle r, \omega \rangle : r < f(\omega) \} \) (where \( r \) is a rational number and \( \omega \in \Omega \)) is an effectively open set in \( \mathbb{Q} \times \Omega \) (a union of an enumerable family of basic open sets of the form \( (p, q) \times \Omega_u \), where \( p < q \) are rational numbers and \( u \) is a binary string).

The sum of two lower semicomputable functions is lower semicomputable; the same is true for an (effectively given) series of semicomputable functions. A average-bounded randomness test is a lower semicomputable function whose integral (in other words, average, or expected value) over \( \Omega \) (with uniform measure \( P \)) is finite. Taking a mix of all average-bounded randomness tests, we get a maximal, or universal average-bounded randomness test. It is defined up to \( O(1) \)-factors; we denote it by \( t \) and may assume that \( E_P t < 1 \) (the expectation is less than 1). Now Martin-Löf random sequences can be equivalently defined as sequences \( \omega \in \Omega \) such that \( t(\omega) < \infty \).

Every non-negative integrable function on \( \Omega \) (with finite integral) is a density function of some measure on \( \Omega \), so we construct a measure \( \tau \) and let \( \tau(X) = \int_X t(\omega) \, dP(\omega) \). The measure \( \tau(\Omega) \) of the entire space \( \Omega \) is \( \int_\Omega t < 1 \). We can transform this measure into a lower semicomputable semimeasure by artificially declaring \( \tau(A) = 1 \) for the corresponding function on strings.²

Now we are ready to prove Proposition [3]. In addition to \( Q = T(P) \) we consider the image measure \( M' = T(\tau) \). It is easy to check that this measure also can be extended to a lower semicomputable semimeasure, and therefore it is does not exceed \( cM \) for some constant \( c \).

Recall our assumption: the ratio \( M'(x)/Q(x) \) is bounded for all prefixes \( x \) of \( \alpha \). Let \( c' \) be this bound, so \( M(x) \leq c'Q(x) \) for all prefixes \( x \) of \( \alpha \). Combining this with the previous inequality, we conclude that \( M' = T(\tau) \leq cM \leq cc'Q \). Now let us look more closely at the values of \( M'(x) \) and \( Q(x) \) for some \( x \) that is a prefix of \( \alpha \). Since \( M' = T(\tau) \) and \( Q = T(P) \), both values are (different) measures of the set \( U = T^{-1}(\Omega_x) \), the set of all sequences that are mapped by \( T \) into some extensions of \( x \). The first is \( \tau(U) = \int_U t(\omega) \, dP(\omega) \), the second is just \( P(U) \). The inequality \( M' \leq cc'Q \) implies that there exists a sequence \( \omega \in U \) (in other words, \( T(\omega) \) starts with \( x \)) such that \( t(\omega_x) \leq cc' \). We denote this sequence by \( \omega_x \), indicating that this sequence depends on \( x \) (while constants \( c \) and \( c' \) do not depend on \( x \)).

Taking longer and prefixes \( x \) of \( \alpha \), we get an infinite sequence of corresponding \( \omega_x \). We know that \( t(\omega_x) \leq cc' \) for all \( \omega_x \), and that \( T(\omega_x) \) starts with \( x \). Let \( x_i \) be the prefix \( \alpha_i \) that has length \( i \), and let \( \omega_i = \omega_{x_i} \) be the corresponding sequence. Using compactness, we take a limit point \( \omega \) of the sequence \( \omega_i \). Let us show that \( \omega \) is a sequence that we need: a random sequence such that \( T(\omega) = \alpha \). Indeed, \( \omega \) is random since the set of all sequences \( \xi \) such that \( t(\xi) \leq cc' \) is closed (since \( t \) is lower semicomputable and therefore lower semicontinuous). And for every prefix \( x \) the corresponding set \( U = T^{-1}(\Omega_x) \) is not only open (as it will be for any computable mapping) but also closed due to our assumption (\( n \) output bits are determined by \( 2n + O(1) \) input bits), and contains all \( \omega_x \) starting from \( i = |x| \), and therefore all limit points of the sequence \( \omega_{x_0}, \omega_{x_1}, \ldots \). Proposition [3] is proven.

²This semimeasure has a natural interpretation. Namely, the exact expression (see [2], [3]) for \( t \) says that it corresponds (up to a \( O(1) \)-factor) to the following random process: we first generate a finite string according to discrete a priori probability and then start adding random bits. But this is not important for the argument.
3 Remarks

Stronger versions

We formulated Propositions 2–4 in the simplest form that is enough for Gács–Kučera theorem. Levin’s formulations are stronger.

Proposition 2. Here we may easily replace 2n by n + d(n) where d(n) is a computable function such that \( \sum 2^{-d(n)} \leq 1 \). For that we use the safety margin \( 2^{-n-d(n)} \) for strings of length \( n \), and the total additional weight is \( \sum 2^{-d(n)} \), since we have \( 2^n \) strings of length \( n \) with weight \( 2^{-n-d(n)} \) each. Moreover, Levin notes that we may use \( m(x) = 2^{-K(x)} \) as safety margin for string \( x \), and round \( M(x) \) up to \( K(x) + O(1) \) bits, where \( K(x) \) is the prefix complexity of \( x \). In other words, we add to \( M(x) \) the semimeasure \( M'(x) = \sum_{x \leq y} m(y) \), where the sum is taken over all string \( y \) that have \( x \) as a prefix.\(^3\) Note that in this case the granularity for \( x \) cannot be computed in advance, we have to use the current upper bound for \( K(x) \) instead when providing approximation from below to the rounded value. When \( K(x) \) decreases, we need to reconsider the rounding, and this could lead to a decrease in the rounded value. However, this does not happen, since at the same time the overhead (also defined in terms of prefix complexity) increases accordingly.

For Proposition 3 we may replace \( 2n + O(1) \) by any computable non-decreasing function. The computability is needed to make the construction effective, and the function should be non-decreasing since we need to allocate intervals for \( x_0 \) and \( x_1 \) inside the intervals already allocated for \( x \). (It seems that this requirement is not mentioned in the last paragraph of Section 7.1, but it is not clear how we can proceed without it.) Moreover, we may use any computable function \( t(x) \), depending on the string \( x \), not only on its length, if \( t(x) \leq t(x_0) \) and \( t(x) \leq t(x_1) \) for all strings \( x \).

In Proposition 4 we need the preimage \( T^{-1}(\Omega_x) \) to be an clopen (closed and open) subset of \( \Omega \), i.e., the finite union of cones. The size of these cones does not matter. Let us note also that Proposition 4 is formulated in [12] only for universal semimeasure, and not for arbitrary semimeasure \( Q \) that is an output distribution of a mapping \( T \) with the above-mentioned property, but the argument can be applied to the general case without changes.

Combining these observations, Levin gets the improved version of Gács–Kučera theorem: if \( t(x) \) is a computable upper bound for \( K(x) \) such that \( t(x) \leq t(x_0) \) and \( t(x) \leq t(x_1) \) for all \( x \), then for every infinite sequence \( \alpha \) there exists a Martin-Löf random sequence \( \omega \) and a machine \( M \) that computes \( \alpha \) given oracle \( \omega \) and uses at most \( t(x) + O(1) \) bits of \( \omega \) when producing some prefix \( x \) of \( \alpha \). This covers some results obtained by Barmpalias and Lewis-Pye, see [1 2] (with rather complicated proofs). The paper [2] is entitled “Compression of data streams down to their information content”; indeed, Gács–Kučera theorem with restrictions on bit usage can be interpreted as follows: we want to compress all prefixes of \( \alpha \) into strings that are prefixes of each other (and form together some random string \( \omega \)). Levin’s argument shows, in a sense, that an adapted version of arithmetic compression can be used to achieve compression almost up to \( K(x) \) (more precisely, up to \( t(x) \) where \( t \) is monotone computable upper bound for \( K \)).

Partially continuous transformations

This notion (used by Levin in [12]) is a generalization of the notion of an oracle machine, and a notion of a (monotone) computable operation defined by him in [10]. Informally speaking, an oracle machine uses the bits from the oracle tape to produce output bits: the more it knows about the input, the more output bits are produced. If we want to consider finite and infinite inputs, we arrive to the notion of a (monotone) computable operation. Now Levin makes the

\(^3\)We already mentioned this semimeasure when discussing \( \tau \), see above.
next generalization step: a partially continuous transformation provides more information about
the output if given more information about the input. Such a transformation is an effectively
closed set in \( \Omega \times \Omega \) and can be defined by a computable sequence of statements of the following
form: “if the input sequence belongs to this cone, then the output sequence cannot belong to that
cone”. In this way we can do more than just specify output bits sequentially: transformation may
say, for example, that 7th bit is zero or that 5th and 16th bits are equal without claiming anything
about other bits.

The notion of partially continuous transformation is natural in itself, but is not strictly nec-
cessary for the argument: we may consider only oracle machines (as usual) and semimeasures
defined on finite strings (and not on arbitrary clopen subsets of \( \Omega \)).

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