We argue that lattice calculations of the $\eta'$ mass in QCD with $N_c = 2$ colors performed at non-zero baryon chemical potential can be used to study the mechanism responsible for the mass of the $\eta'$. QCD with two colors is an ideal laboratory because it exhibits confinement, chiral symmetry breaking and a would-be $U(1)_A$ Goldstone boson at all densities. Since the instanton density and the confinement scale vary with density in a very different way, instantons are clearly distinguishable from other possible mechanisms. There is an instanton prediction for the $\eta'$ mass at large density that can be compared to lattice results. The density dependence of the instanton contribution is a simple consequence of the integer topological charge carried by the instanton. We also argue that $N_c = 3$ color QCD at finite isospin density can be used in order to study the origin of OZI-violation in the scalar sector.

\section{I. INTRODUCTION}

The $U(1)_A$ puzzle in QCD is related to the absence of a ninth Goldstone boson connected to the spontaneous breakdown of the $U(1)_A$ chiral symmetry \cite{1}. It was realized soon after the discovery of QCD that the $U(1)_A$ symmetry of the QCD lagrangian is anomalous, but it was also noted that the divergence of the $U(1)_A$ current is itself a total divergence \cite{2}. Superficially, it would then seem that the $U(1)_A$ anomaly is not sufficient to remove the $U(1)_A$ Goldstone boson. The puzzle was resolved after Belavin, Polyakov, Schwartz and Tyupkin discovered topological structures, instantons, in QCD \cite{3}. 't Hooft showed that instantons lead to the violation of axial charge \cite{4}, and that instantons induce an effective $(2N_f)$-fermion operator which contributes to the $\eta'$ meson mass \cite{5}.

Since then, lattice QCD calculations have identified instantons, verified the presence of fermion zero modes, and established their relation to the mass of the $\eta'$ \cite{6}. Also, the instanton liquid model was expanded into a phenomenologically successful description of chiral symmetry breaking and the $U(1)_A$ anomaly \cite{7, 8}. Unfortunately, we cannot compute the instanton contribution to the $\eta'$ mass from first principles. As a consequence, there are still speculations that the $\eta'$ mass is related to structures with fractional topological charge that do not appear in the classical limit, or that the $\eta'$ mass is in some way related to confinement \cite{9}. The latter suggestion was first made by Kogut and Susskind prior to the discovery of instantons in QCD \cite{10}.

Even if we cannot do a parameter-free calculation of the $\eta'$ mass in QCD we can still try to distinguish different mechanisms for generating the $\eta'$ mass by their scaling behavior. In QCD, of course, any contribution to the $\eta'$ mass has to be proportional to the QCD scale parameter $\Lambda_{QCD}$. Witten suggested that the number of colors $N_c$ could be used as a parameter \cite{11}. He argued that the instanton contribution scales as $\exp(-N_c)$ whereas effects related to confinement give $m_{\eta'}^2 \sim 1/N_c$. However, the relation $m_{\eta'}^2 \sim \exp(-N_c)$ is only correct for very small instantons, $\rho \ll \Lambda_{QCD}^{-1}$. Indeed, we recently argued that the instanton contribution to the $\eta'$ mass also scales as $m_{\eta'}^2 \sim 1/N_c$ \cite{12}.

Another possibility is to use the temperature $T$ as a parameter. The density of instantons is expected to be suppressed by a large power of $T$, $(N/V) \sim T^{b-4}$, where $b = 11N_c/3 - 2N_f/3$ is the first coefficient of the QCD beta function \cite{13, 14, 15, 16}. Effects related to confinement or topological objects other than instantons would presumably have a different dependence on temperature. The problem with this idea is that the suppression of instanton effects is related to perturbative color screening. This implies that the power law suppression only applies if the temperature is larger than the critical temperature $T_c$ for chiral symmetry restoration \cite{17}. In order to have rigorous theoretical control over instanton effects we have to consider the quark gluon plasma phase instead of the hadronic phase.

In this note we suggest using the baryon chemical potential $\mu$ as a parameter. We shall argue that this parameter is more useful than the number of colors or the temperature because there is chiral symmetry breaking and a hadronic phase for all values of $\mu$, and there is theoretical control over both the instanton contribution to the $\eta'$ mass and the scale of confinement effects. We first formulate our proposal in terms of $SU(2)$ gauge theory at finite baryon density. We then show that a similar situation arises in $SU(3)$ gauge theory at finite isospin density. Both of these theories can be studied with lattice algorithms that are available today.

We should note that many of the features that we shall discuss also apply, with some modifications, to $SU(3)$ gauge theory at finite baryon density \cite{18}. However, this theory cannot be studied on the lattice at present, and we shall not discuss it in detail.

\section{II. QCD WITH TWO COLORS}

Let us summarize some of the salient features of $SU(2)$ gauge theory at zero and non-zero baryon chemical potential \cite{19, 20, 21, 22, 23}. For simplicity, we will con-
centrate on $N_f = 2$ flavors. $SU(2)$ gauge theory has a meson spectrum which is very similar to $SU(3)$ QCD. Baryons, on the other hand, are bosons rather than fermions and their spectrum is determined as compared to $N_c = 3$ QCD. Because the $SU(2)$ gauge group is pseudo-real, there is a Pauli-Gürsey symmetry which relates quarks and anti-quarks. This symmetry mixes the quark-anti-quark condensate $\langle \bar{q}q \rangle$ with the diquark condensate $\langle e^{ib}q^aT C\gamma_5\tau_2q^b \rangle$. Here, $a, b$ are color indices and $\tau_2$ is the anti-symmetric $SU(2)_F$ flavor matrix. As a result the chiral symmetry group is $SU(2)$ rather than $SU(2)_L \times SU(2)_R$.

Let us consider chiral symmetry breaking in the presence of a small source term

$$\mathcal{L}_s = m\bar{q}^a q^a + \frac{i}{2} (e^{ib}q^a T C\gamma_5\tau_2q^b + h.c.).$$

Chiral symmetry is broken according to $SU(4) \rightarrow Sp(4)$ in the low-temperature and low-density phase for both $j = 0, m = 0$ and $m = 0, j = 0$. In the case $m \neq 0$ the order parameter is the quark-anti-quark condensate $\langle \bar{q}^a q^a \rangle$. There are 5 Goldstone bosons, three pions $\pi$, the scalar diquark $S$ and the scalar anti-diquark $\bar{S}$. Because of the $U(1)_A$ anomaly the would-be singlet Goldstone boson, the $\eta'$, is heavy. In the case $j \neq 0$ the order parameter is the diquark condensate $\langle q^a T C\gamma_5\tau_2 q^b \rangle$. Again there are 5 Goldstone bosons, three pions $\pi$, the sigma $\sigma$ and the scalar diquark $S$. The would-be singlet Goldstone boson, the pseudoscalar diquark $P$, is heavy.

For $\mu = 0$ all directions of the source term $(m, j) = m_0(\cos(\alpha), \sin(\alpha))$ are physically equivalent. In particular, the masses of the pseudoscalar diquark $P$ in the diquark phase is equal to the mass of the $\eta'$ in quark-anti-quark condensed phase. If the baryon chemical potential is non-zero the $SU(4)$ symmetry is broken explicitly. In the following, we shall consider the diquark phase $m = 0, j \rightarrow 0$. In this case, the phase diagram is particularly simple, see Fig. 1. Chiral symmetry is spontaneously broken at $T = \mu = 0$. There is a critical temperature $T_c$ such that chiral symmetry is restored for $T > T_c$. However, most likely, there is no phase transition as a function of the chemical potential for $T < T_c$. This, of course, is the main feature that distinguishes the $SU(2)$ theory at non-zero $\mu$ as a laboratory for studying the $U(1)_A$ anomaly.

The effective lagrangian for the singlet pseudoscalar Goldstone boson is

$$\mathcal{L} = f_P^2 \left[ (\partial \phi)^2 - v^2 (\partial \phi)^2 \right] - V(\phi).$$

The decay constant and Goldstone boson velocity can be determined in perturbation theory. At leading order, the result is $[24]$ $f_P^2 = \frac{\mu^2}{8\pi^2}, \quad v^2 = \frac{1}{3}$. (3)

The potential $V(\phi)$ receives contributions from instantons. We find $V(\phi) = -A_P \cos(\phi + \theta)$ where $\theta$ is the QCD theta angle. If the chemical potential is big, $\mu \gg \Lambda_{QCD}$, large instantons are suppressed and the coefficient $A$ can be determined in perturbation theory. We find $[24, 25]$ $A_P = C_{2,2} 6\pi^4 \left[ \frac{4\pi}{g} \right] \left( \frac{\mu^2}{2\pi^2} \right)^2 \left( \frac{8\pi^2}{g^2} \right)^4 \left( \frac{\Lambda}{\mu} \right)^8 \Lambda^{-2}$ (4)

with

$$C_{N_c, N_f} = \frac{0.466 \exp(-1.679 N_c) 1.34^{N_f}}{(N_c - 1)! (N_c - 2)!}. \quad (5)$$

At large $\mu$ the gap $\Delta$ can also be determined in perturbation theory. We get $[20, 27, 28, 29]$ $\Delta = 512 \pi^4 b_0 \mu g^{-5} \exp \left( \frac{-2\pi^2}{g(\mu)} \right), \quad (6)$

where the parameter $b_0 = \exp(-(\pi^2 + 4)(N_c - 1)/16)$ controls the size of non-Fermi liquid effects $[34, 31]$. Using eqn. (3) we can determine the mass of the pseudoscalar Goldstone boson. We have

$$m_P^2 = \frac{A_P}{2 f_P^2}.$$ (7)

The result has the structure of the Witten-Veneziano relation where $A_P$ plays the role of the topological susceptibility. We note, however, that $\chi_{top} = 0$ as expected for a theory with massless fermions. In fact, $A_P$ governs local, not global, fluctuations of the topological charge. For a dilute gas of instantons $A_P = (N/V)$ where $(N/V)$ is the density of instantons. This relation is exact in the limit of large baryon density. We also note that in the...
instanton liquid model \( A_P \approx (N/V) \) is very well satisfied even at zero baryon density. We observe that equ. (1) implies

\[
m_P \sim \Lambda \left( \frac{\Delta}{\Lambda} \right)^2 \left( \frac{\Lambda}{\mu} \right)^3 \left[ \log \left( \frac{\Lambda}{\Delta} \right) \right]^{5/2}.
\]  

(8)

As expected, the instanton contribution to the \( U(1)_A \) mass is suppressed at large baryon density. The power law is directly related to the topological charge of the instanton. Contributions from instantons with charge two or larger are suppressed by additional powers of \((\Lambda_{QCD}/\mu)\). By the same token we expect the contribution of hypothetical objects with fractional charge to dominate over instanton effects at large \( \mu \).

In practice, lattice calculations have to be carried out at finite diquark source or non-zero quark mass (or both). If the baryon density is small the pseudoscalar diquark and \( \eta' \) meson are very heavy and the effect of a non-zero quark mass is small. At large \( \mu \), however, the quark mass contribution is more important. If \( \mu \gg \Lambda_{QCD} \) the quark mass contribution to the effective potential can be computed in perturbation theory. Using the methods described in [32] we find

\[
V_m(\phi) = -\frac{4\Delta^2}{3\pi^2} \det(M)e^{-i\phi} + h.c.,
\]  

(9)

where \( M \) is the mass matrix. We observe that the topological susceptibility does not vanish if the quark mass is non-zero, \( \chi_{(\phi)} \sim \det(M) \Delta^2 \). The contribution to the mass of the would-be \( U(1)_A \) Goldstone boson is

\[
m_P^2 = \frac{32}{3} \frac{\Delta^2}{\mu^2} m_u m_d.
\]  

(10)

In the limit \( \mu \to \infty \) the \( m_q \neq 0 \) contribution will eventually dominate over the instanton contribution. However, even for quark masses as large as \( m_u = m_d = 40 \text{ MeV} \) the instanton contribution is expected to dominate for chemical potentials that can be achieved on the lattice.

We now comment on possible effects related to confinement [10, 33, 34]. At large baryon density the gap in the fermion spectrum is much larger than the QCD scale parameter, \( \Delta \gg \Lambda_{QCD} \). Since the diquark condensate is a color singlet, there is neither screening nor a Higgs effect operating at scales below \( \Delta \). As a consequence, we expect the color \( SU(2) \) to be confined at all densities. If there are effects related to confinement that contribute to the mass of the \( \eta' \), then these effects should persist at all densities.

We do not know how to compute confinement related contributions to the mass of the \( \eta' \). However, if these effects exist then they should be governed by the \( SU(2) \) confinement scale. Rischke et al. observed that the pure glue theory below \( \Delta \) is characterized by a non-trivial electric polarizability and magnetic permeability [35]. The effective action is

\[
S_{\text{eff}} = \frac{1}{2g^2} \int d^4 x \left( \epsilon \bar{E}^a \cdot \bar{E}^a - \frac{1}{\Lambda} \bar{B}^a \cdot \bar{B}^a \right),
\]  

(11)

where we have suppressed higher order terms that are suppressed by \((g/\Lambda)\). The main feature of the effective action equ. (11) is that the electric polarizability \( \epsilon \approx 1 + g^2 \mu^2/(18\pi^2 \Delta^2) \) is very large in the limit \( \mu \to \infty \). As a consequence, the \( SU(2) \) pure gauge theory described by equ. (11) is confined, but the confinement scale is
exponentially small \[35, 36\]

\[
\Lambda_{\text{conf}} \sim \Delta \exp \left( -\frac{8\pi^2}{g(\mu)^2} \frac{3\sqrt{\epsilon}}{22} \right). \tag{12}
\]

If the mass of the would-be \(U(1)_A\) Goldstone boson is dominated by confinement effects then we expect that \(m_P f_P^2 \sim \sigma^2 \Lambda_{\text{conf}}^4\). In Fig. 2 we compare this scaling relation with the instanton prediction for the pseudoscalar Goldstone boson mass. Of course we do not know the constant of proportionality relating \(m_P f_P\) and the string tension \(\sigma\). In Fig. 2 we have arbitrarily scaled \(m_P \sim \Lambda_{\text{conf}}^2 f_P^2\) to match the instanton prediction at \(\mu \approx 500\) MeV. We find that the instanton and confinement related contributions clearly scale differently. We also observe that the instanton contribution will always dominate at large \(\mu\).

We should note that the effective theory described by equ. (11) is not covariant. As a consequence, the timelike and space-like string tensions are not the same. We can restore covariance by rescaling the time coordinate like and space-like string tensions are not the same. We should also note that the estimate equ. (12) of the confinement scale only has exponential accuracy, so that we cannot reliably predict possible factors \(\sqrt{\epsilon}\) in the pre-exponent.

### III. QCD AT FINITE ISOSPIN DENSITY

The large mass of the \(\eta'\) implies that violations of the OZI rule in the pseudoscalar meson sector are substantial. However, the \(\eta'\) sector is not the only channel in which OZI violation is large. In particular, the OZI violating mass difference between the isovector-scalar \(a_0\) and isoscalar-scalar \(\sigma\) meson is almost as large as the \(\eta' - \pi\) splitting. We have argued that the \(a_0 - \sigma\) splitting is also dominated by instantons \[37, 38\]. In this section we show that this idea can be checked in \(N_c = 3\) QCD at finite isospin density.

In QCD with \(N_c = 3\) colors and \(N_f = 2\) flavors chiral symmetry is broken according to \(SU(2)_L \times SU(2)_R\). If the quark mass is non-zero there are three almost massless pions and a heavy would-be \(U(1)_A\) Goldstone boson, the \(\eta'\). There is a relativistic light scalar-isoscalar meson, the \(\sigma\), and a heavy scalar-isovector meson, the \(a_0\), which is close in mass to the \(\eta'\).

The effect of a non-zero isospin chemical potential term \(\mu_I (u^i u - d^i d)\) was studied in \[38\]. If \(\mu_I > 0\) the isospin chemical potential favors up quarks over down quarks. As a result, the mass of the positive pion is reduced. If \(\mu_I > m_\pi/2\) pion condensation takes place and the chiral order parameter starts to rotate from the \((\bar{q}q)\) direction to the \((\bar{q}i\gamma_5 \xi^+ q)\) direction. There is one exact Goldstone boson, the \(\pi^+\), and two heavy pions.

At large isospin density there is a \(U(1)_A\) would-be Goldstone boson, the \(a_0^+\). The effective lagrangian for the \(a_0^+\) is identical to the effective lagrangian for the \(\eta'\) in \(N_c = 2\) QCD, see equ. (3). If \(\mu_I > \Lambda_{\text{QCD}}\) the parameters in the effective lagrangian can be computed in perturbation theory. The decay constant and Goldstone boson velocity are given by

\[
f_S^2 = \left( \frac{3\mu_I}{16\pi^2} \right), \quad v^2 = \frac{1}{3}. \tag{13}
\]

The instanton contribution to the effective potential is given by \(A_S \cos(\phi + \theta)\) with

\[
A_S = C_{3.2} \frac{16\pi^4 \Gamma(\frac{3}{2})}{235/6} \Phi_S^2 \left( \frac{8\pi^2}{g^2} \right)^6 \left( \frac{\Lambda}{\mu_I} \right)^{35/3} \Lambda^{-2}, \tag{14}
\]

where the gap \(M\) and superfluid density \(\Phi_S\) are given by

\[
M = 512\pi q \delta g \mu g^{-5} \exp \left( -\frac{3\pi^2}{2g(\mu_I)} \right), \tag{15}
\]

\[
\Phi_S = \frac{3\pi}{g} M \left( \frac{\mu_I}{2\pi g} \right). \tag{16}
\]

The mass of the \(a_0^+\) satisfies a Witten-Veneziano type relation

\[
m_{a_0}^2 = \frac{A_S}{2f_S^2}. \tag{17}
\]

Again, \(A_S\) is related to the density of instantons. At large \(\mu_I\) the \(a_0^+\) mass scales as

\[
m_{a_0} \sim \Lambda \left( \frac{M}{\Lambda} \right) \left( \frac{\Lambda}{\mu_I} \right)^{29/6} \left[ \log \left( \frac{\mu_I}{\Lambda} \right) \right]^{7/2}. \tag{18}
\]

The difference in the power law suppression as compared to equ. (18) is related to the difference between the beta functions for \(N_c = 2, 3\).

Finally, we observe that the pion condensate is a color singlet and the color \(SU(3)\) is expected to be confined at large isospin density. The confinement scale is given by

\[
\Lambda_{\text{conf}} \sim \Delta \exp \left( -\frac{8\pi^2}{g(\mu_I)^2} \frac{3\sqrt{\epsilon}}{11} \right). \tag{19}
\]

In Fig. 2b we plot the instanton prediction for the mass of the \(a_0^+\) at large isospin density. Again, we compare the instanton prediction to the scaling relation \(m_{a_0} \sim \sigma/f_S\). We observe that the instanton contribution is more strongly suppressed as compared to the \(N_c = 2\) result. As noted below equ. (18), this is related to the fact that the beta function is larger. The scale \(\sigma/f_S\), on the other hand, drops off more slowly as compared to the \(N_c = 2\) case. This is a consequence of the fact that the gap is larger and as the result the polarizability is smaller. As asymptotically large isospin density, however, the \(a_0\) mass is again dominated by instantons.
IV. SUMMARY

We have argued that SU(2) QCD at finite baryon density and SU(3) QCD at finite isospin density can be used in order to study the mechanism for generating the mass of the would-be $U(1)_A$ Goldstone boson in QCD. The main point is that there is a hadronic phase at all densities, and that both the instanton contribution and the confinement scale are calculable at large chemical potential. Furthermore, both SU(2) QCD at non-zero baryon chemical potential and SU(3) QCD at non-zero isospin density can be studied on the lattice with presently available methods [33, 40].

We would like to mention some possible difficulties with our proposal. We have assumed that there is no phase transition along the finite baryon chemical potential axis in the phase diagram. This assumption is based on the observation that the phenomenologically established symmetries of the low density phase agree with the calculated symmetries of the high density phase. This means that a phase transition is not required, but of course a transition is not forbidden either. The question of whether or not there is a phase transition can be studied using lattice simulations. We should note that even if there is transition that separates the low and high-density phase, we can still study the question whether the mechanism for generating the mass of the $U(1)_A$ Goldstone boson is the same in both phases.

We have used perturbation theory in order to compute the gap, the instanton density, and the confinement scale at non-zero chemical potential. We do not know how reliable leading order perturbation theory is for baryon chemical potentials that can be achieved in lattice calculations. Indeed, it was argued that the perturbative expansion for the gap converges very slowly [41] and that the instanton contribution is very sensitive to the value of the QCD scale parameter [27]. However, the power law behavior of the $U(1)_A$ Goldstone boson mass is quite robust and a simple reflection of the topological charge carried by the instanton. Furthermore, the question whether the $U(1)_A$ Goldstone boson mass scales with the instanton density, $m_i^2 f_i^2 \sim (N/V)$, or the string tension, $m_s^2 f_s^2 \sim \sigma^2$, can be answered directly from the lattice data, without resort to perturbation theory.

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[1] S. Weinberg, Phys. Rev. D 11, 3583 (1975).
[2] H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. B 47, 365 (1973).
[3] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, Phys. Lett. B 59, 85 (1975).
[4] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976).
[5] G. ’t Hooft, Phys. Rept. 142, 203 (1986).
[6] R. G. Edwards, Nucl. Phys. Proc. Suppl. 106, 38 (2002) [hep-lat/0111009]; M. Garcia Perez, Nucl. Phys. Proc. Suppl. 94, 27 (2001) [hep-lat/0011026]; M. Tepfer, Nucl. Phys. Proc. Suppl. 83, 146 (2000) [hep-lat/9909124], and references therein.
[7] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. 70, 323 (1998) [hep-ph/9610451].
[8] D. Diakonov, Proceedings of the International School of Physics, 'Enrico Fermi', Course 80: Selected Topics (1998) [hep-ph/9610451].
[9] For a recent example, see N. Isgur and H. B. Thacker, Phys. Rev. D 51, 197 (1995).
[33] M. R. Frank and T. Meissner, Phys. Rev. C 57, 345 (1998) [hep-ph/9703270].
[34] R. Alkofer and L. von Smekal, Phys. Rept. 353, 281 (2001) [hep-ph/0007355].
[35] D. H. Rischke, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 87, 062001 (2001) [hep-ph/0011373].
[36] F. Sannino, N. Marchal and W. Schafer, Phys. Rev. D 66, 016007 (2002) [hep-ph/0202248].
[37] T. Schäfer and E. V. Shuryak, preprint, [hep-lat/0005025].
[38] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 86, 592 (2001) [hep-ph/0005225].
[39] A. Nakamura, Phys. Lett. B 149, 391 (1984); E. Dagotto, F. Karsch and A. Moreo, Phys. Lett. B 169, 421 (1986);
S. Hands, J. B. Kogut, M. P. Lombardo and S. E. Morrison, Nucl. Phys. B 558, 327 (1999) [hep-lat/9902034];
B. Alles, M. D’Elia, M. P. Lombardo and M. Pepe, Nucl. Phys. Proc. Suppl. 94, 441 (2001) [hep-lat/0010068];
J. B. Kogut, D. K. Sinclair, S. J. Hands and S. E. Morrison, Phys. Rev. D 64, 094505 (2001) [hep-lat/0105026];
J. B. Kogut, D. Toublan and D. K. Sinclair, Nucl. Phys. B 642, 181 (2002) [hep-lat/0205013].
[40] J. B. Kogut and D. K. Sinclair, Phys. Rev. D 66, 034505 (2002) [hep-lat/0202028].
[41] K. Rajagopal and E. Shuster, Phys. Rev. D 62, 085007 (2000) [hep-ph/0004074].