Stochastic Gross-Pitaevskii Equation for the Dynamical Thermalization of Bose-Einstein Condensates

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We present a theory for the description of energy relaxation in a nonequilibrium condensate of bosonic particles. The approach is based on coupling to a thermal bath of other particles (e.g., phonons in a crystal, or noncondensed atoms in a cold atom system), which are treated with a Monte Carlo type approach. Together with a full account of particle-particle interactions, dynamic driving, and particle loss, this offers a complete description of recent experiments in which Bose-Einstein condensates are seen to relax their energy as they propagate in real space and time. As an example, we apply the theory to the solid-state system of microcavity exciton polaritons, in which nonequilibrium effects are particularly prominent.

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Introduction.—Bose-Einstein condensates form when multiple bosonic particles relax their energy, through interaction with other particles (e.g., phonons), to collect into a low energy state. Examples in solid-state systems include: condensates of magnons [1, 2] in quantum spin gases; indirect excitons [3] in coupled quantum wells; and exciton-polaritons [4–6] in semiconductor microcavities. While Bose-Einstein condensation is conventionally thought of as a macroscopic occupation of the ground state in thermal equilibrium, an important issue is that thermal equilibrium in solid state systems is never perfectly achieved since the particles have finite lifetimes. The departure from thermal equilibrium is particularly pronounced in exciton-polariton systems, where a short lifetime of particles may make the system highly nonequilibrium [7]. Nevertheless, spontaneous spatial coherence has been observed [8], although not necessarily in the ground state of the system [9, 10]. The theoretical description of such condensates thus requires a kinetic approach.

In excitonic systems (exciton or exciton-polariton condensates) scattering with acoustic phonons offers a mechanism of energy relaxation. It may not offer complete thermalization, but the effect of energy relaxation is clearly seen in experiments where a potential gradient is present. The latter can be created by application of stress [11], optically [12] or structurally engineered [13]. In this situation, the processes of energy relaxation and condensate propagation down the gradient are closely connected. Aside offering a clear demonstration of energy relaxation, propagating condensates have recently demonstrated potential as optoelectronic transistors in exciton [3] and exciton-polariton systems [14, 15].

Theoretically, the dynamics of spatially homogeneous systems have been described accounting for the exciton-phonon interaction by means of a system of semiclassical Boltzmann equations [14, 15]. This approach, however, has serious drawbacks. First, the corresponding formalism is based on the assumption of full incoherence in the system, thus quantum states are supposed to be completely uncorrelated. However, as long as the condensate has been formed, this is no longer valid. As a consequence, a number of coherent phenomena such as the onset of superfluidity [16], bistability [17] and hysteresis cannot be described. Second, the Boltzmann equations can provide us with information about the occupation numbers in reciprocal (k-space) only, whereas the real space (z-space) behavior remains obscure.

On the other hand, if the processes of decoherence are fully accounted for, the state of interacting particles can be treated as a classical field described by the Gross-Pitaevskii equation [19], which can be modified for incoherent pumping [20]. Such an approach has been successful for the description of a variety of recent experiments [21, 22] in semiconductor microcavities, including, for example, experiments on the dynamics of vortices [22], spatial pattern formation [23] and spin textures [23, 24]. However, the Gross-Pitaevskii equation conserves the particle energy and thus does not account for phonon-assisted scattering. Recent models include energy relaxation in a phenomenological way within a classical stochastic field [25] or Gross-Pitaevskii type formalism [26]. While these models give results in agreement with experimental data, they operate with unknown phenomenological parameters.

In this Letter we introduce a microscopic theory for the description of energy relaxation in a coherent excitonic ensemble. The exciton/exciton-polariton field is coupled to a field representing phonons in the system, which is modeled using stochastic variables. For simplicity, we consider a resonant coherent injection of excitons/exciton-polaritons (models of non-resonant excitation, involving coupling to an exciton reservoir, have been considered elsewhere [17] and are compatible with our approach). We consider injection with zero in-plane momentum in a system containing a potential gradi-
The potential gradients accelerate the particles which undergo scattering with acoustic phonons as they propagate. The latter process leads to the energy dissipation and thermalization in the system. To give a complete description of the dynamics we fully account for exciton-exciton scattering and losses provided by finite lifetime.

We will use parameters corresponding to cavity exciton-polaritons – the system for which non-equilibrium effects are most clearly pronounced – although it should be noted that our formalism is applicable to the modelling of other systems in which bosons relax their energy through interactions with an incoherent gas of other particles. Another example of application of our theory would be the system of indirect excitons [3]. In the excitonic optoelectronic transistor (EXOT) [29, 30], electric fields introduce potential gradients and the control of the electric fields modulates the fluxes of excitons. Similar to the exciton-polariton case, the relaxation down potential gradients should be mediated by phonons, the process that has not been addressed theoretically so far to the best of our knowledge.

Theory.—For simplicity, we consider a 1D system, as in the case of microwires where there has been a recent experimental focus on energy relaxation of propagating condensates [11, 13]. We note however that the formalism is expected to be compatible with 2D systems as well. We introduce the quantum field operators for exciton-polaritons, \( \hat{\Psi}_x \) connected with annihilation operators in reciprocal \((k-)\) space by the Fourier transforms \((\mathcal{F} \) in short notation),

\[
\hat{a}_k = \mathcal{F}[\hat{\Psi}_x] = \frac{1}{\sqrt{N}} \sum_x \hat{\Psi}_x e^{-ikx}; \quad \hat{\Psi}_x = \mathcal{F}^{-1}[\hat{a}_k],
\]

where \( N \) is the discretization length.

The Hamiltonian of the system reads

\[
\hat{\mathcal{H}} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k + \sum_{\delta, \varphi} \mathcal{P}_x e^{i(\delta x - \varphi t)} \left( \hat{\Psi}_x^\dagger + \hat{\Psi}_x \right) + \sum_x \left( V_x \hat{\Psi}_x^\dagger \hat{\Psi}_x + \alpha \hat{\Psi}_x^\dagger \hat{\Psi}_x \hat{\Psi}_x^\dagger \hat{\Psi}_x \right) + \sum_{\delta, \varphi} \hbar \omega_0 \hat{b}_{\delta \varphi}^\dagger \hat{b}_{\delta \varphi} + \sum_{\delta, \varphi} G_{\delta \varphi} \hat{b}_{\delta \varphi}^\dagger \hat{a}_{k+\varphi, \delta} + G_{\delta \varphi}^* \hat{a}_{k-\varphi, \delta}^\dagger \hat{b}_{\delta \varphi}.
\]

The first two lines here correspond to coherent processes. \( E_k \) is the particle dispersion (which is non-parabolic for exciton-polaritons); the last term in the first line is the coherent pumping, where \( \mathcal{P}_x \) is the intensity of pump, \( k_p \) is the wavevector of the pump, corresponding to the inclination of an incident laser beam, and \( \hbar \omega_p \) is the pumping energy; \( V_x \) is the potential profile in \( x \)-space, \( \alpha \) is a constant describing the strength of particle-particle interactions.

The last line in the equation above corresponds to incoherent processes. To model the interaction with acoustic phonons, we introduce the Fröhlich Hamiltonian [10, 31], where the phonons described by operators \( \hat{b}_{\delta \varphi}^\dagger \hat{b}_{\delta \varphi} \) are considered to be three dimensional. The phonon wavevector is \( \delta \) and \( \varphi \) are unit vectors: \( \hat{e}_x \) is in the wire direction, \( \hat{e}_z \) is in the structure growth direction. The phonon dispersion relation, \( \hbar \omega_\delta \varphi = \hbar u \sqrt{q_x^2 + q_y^2 + q_z^2} \), is determined by the speed of sound, \( u \). \( G_{\delta \varphi} \) is the exciton-phonon interaction strength, whose calculation can be found elsewhere [31–35] (see also the detailed derivation of the formalism in the “Supplementary Materials” appended to this Letter).

Using the Heisenberg equations of motion for the operators, one can write the formal solution for the phonon field as following:

\[
\hat{b}_{\delta \varphi}(t) = \hat{b}_{\delta \varphi}(0)e^{-i\omega_{\delta \varphi}t} - i \frac{\hbar}{\omega_{\delta \varphi}} \int_0^t G_{\delta \varphi}^* \sum_k \hat{a}_{k+\varphi, \delta}^\dagger (\tau') \hat{a}_k (\tau') e^{-i\omega_{\delta \varphi}(t-\tau')} d\tau'.
\]

Remembering that phonons represent an incoherent thermal reservoir, we can replace the term \( \delta \hat{b}_{\delta \varphi}(0)e^{-i\omega_{\delta \varphi}t} \) by a stochastic classical variable \( b_{\delta \varphi}(t) \) (and a similar replacement can be made for the conjugate field). This represents the analogue of the Markov approximation within the Langevin approach, when phonons are assumed to have a randomly varying phase [32]. The stochastic variables are complex numbers with real and imaginary components normalized as follows,

\[
\langle b_{\delta \varphi}(t)b_{\delta \varphi}^* (t') \rangle = n_{\delta \varphi} \delta_{\delta \varphi} \delta(t-t'); \quad \langle b_{\delta \varphi}(t)b_{\delta \varphi}^* (t') \rangle = \langle b_{\delta \varphi}^*(t)b_{\delta \varphi} (t') \rangle = 0,
\]

where \( n_{\delta \varphi} \) is the number of phonons in the state with wavevector \( \delta \) determined by the temperature of the system.

The exciton-polariton field dynamics can then be determined solely by the exciton-polariton operators and the stochastic terms. Further, within the mean field approximation, the field operator \( \hat{\Psi}_x \) can be treated as a classical variable for condensed exciton-polaritons, \( \psi_x = \langle \hat{\Psi}_x \rangle \) (with the Fourier image \( \hat{\psi}_k \)). Then, physical observables are calculated over multiple realizations of the evolution dynamics with stochastic variables \( b_k(t) \). The corresponding equation of motion reads (see the Supplementary Material for the details of the derivation):

\[
i \hbar \frac{d\hat{\psi}_k}{dt} = \mathcal{F}^{-1} [E_k \hat{\psi}_k + \mathcal{S}_k(t)] + \left[ V_x + \alpha |\psi_x|^2 - \frac{\hbar \gamma}{2} \right] \psi_x + \mathcal{P}_x e^{ik_p x} e^{-i\omega_p t} + \sum_k \{ \mathcal{T}_- (t) + \mathcal{T}_k^* (t) \} e^{-ikx} \psi_x,
\]

where we introduced phenomenologically the decay term \( -i\hbar \gamma \psi_x^2 / 2 \) to account for the radiative decay of particles [19]. The constant \( \alpha \) describing polariton-polariton
interactions can be estimated as $[34]: \alpha \approx E_0 a_B^2 / (L_y \Delta x)$, where $L_y$ is the lateral size of the microwire and $\Delta x = L_x / N$ is the discretization unit. One can see that interaction with phonons leads to the appearance of two types of terms. First, one has a term

$$S_k(t) = \sum_{q_x} \psi_{k+q_x}(t) \left( \int_0^t A_{q_x}(t') K_0(t - t') dt' \right)$$

(5)

where $A_{q_x}(t) = \sum_{q_y} \psi_{k+q_x}^* (t) \psi_{k'}(t)$. The term is proportional to the cube of the polariton field and does not directly include a stochastic term. It can be thus interpreted as the term corresponding to the emission of phonons by a condensate stimulated by polariton density. The convolution integral is responsible for energy conservation. Note, that the function

$$K_{q_x}(t) = -\sum_{q_x, q_z} |G_{q_x}|^2 (e^{-i\omega q t} - e^{i\omega q t})$$

$$→ 2i \frac{L_x}{2\pi} \frac{a_B}{2m} \int \int |G(q)|^2 \sin[\omega(q)t] dq_x dq_z$$

(6)

is approximately independent of $q_x$ in the range of $q_x \in (-10^8, 10^8)$ $m^{-1}$, and thus in our calculations we put $K_{q_x}(t) \approx K_0(t)$.

The stochastic functions $T_{q_x}$ and $T_{q_x}^*$ in the last line of Eq. (5) are defined by the correlators:

$$\langle T_{q_x}^*(t) T_{q_x'}(t') \rangle = \sum_{q_y, q_z} |G_{q_x, q_y, q_z}|^2 n_{q_x, q_y, q_z} \delta_{q_x, q_x'} \delta(t - t')$$

$$\langle T_{q_x}(t) T_{q_x'}(t') \rangle = \langle T_{q_x}^*(t) T_{q_x'}^*(t') \rangle = 0.$$  

(7)

These thermal terms contain the phonon field and so are strongly temperature dependent. The proportionality of the thermal part to the first power of $\psi_x$ in Eq. (5) means that the scattering processes proceed at a rate proportional to the first power in exciton-polariton density. Consequently, this term corresponds to the absorption of the phonons by the polariton ensemble and their emission which is stimulated by final state phonon occupancy, but not exciton-polariton density. The latter processes are instead represented by the stimulated term (Eq. (5)) considered above.

**Results.**—We considered an InGaAlAs alloy-based microwire and in computations used the following set of parameters: speed of sound $u = 5370$ m/s [31], $\gamma = 1/18$ ps$^{-1}$. The function $V_x = V_0 - \beta x$ is the exciton-polariton potential in space, composed of: the potential defining the walls of the exciton-polariton wire, $V_0$ and a potential gradient $\beta = 9$ meV/nm. We consider a linear potential gradient as in the experiment of Ref. [13]. Our formalism would however be compatible with any potential shape and could be applied to, for example, the parabolic [2] or staircase [12] potentials studied recently in experiments. The exciton-polariton dispersion was calculated using a two oscillator model with cavity photon effective mass $4 \times 10^{-5}$ of the free electron mass, Rabi splitting 10 meV and exciton-photon detuning 2.5 meV at zero in-plane wavevector.

The main phenomenon we investigated was the relaxation of energy of exciton-polaritons (thermalization) caused by the interaction with acoustic phonons. Exciton polaritons were introduced to the system by localized coherent pump with energy coinciding with the bottom of lower polariton branch (zero detuning case) which guarantees that we are outside bistable regime.

Stochastic equation (5) can be solved numerically. The results of the modeling are presented in Figs. 1-2a-f. Figure 1 illustrates the propagation of particles created by a short pulse along the 1D wire. The pumping is switched on during the first 20 ps of the theoretical experiment, and then it is off and we observe the decay of intensity due to the finite particle lifetime. The quantity $|\psi_x|^2$ is depicted for different times: 20, 40, 60 and 80 ps ($t$ is the parameter). One can see that the wavepacket of exciton-polaritons propagates along the channel with time, disperses, and accumulates at the right hand side. Note, that reflection from the potential jump at the end of the wire produces pronounced interference fringes clearly visible in recent experiments [36].

Figure 2 illustrates the energy relaxation in the system for three time intervals: 0-50 ps, 50-100 ps and 100-150 ps correspondingly. It is clearly seen that exciton-polaritons pumped at $x = 0$, $k = 0$ (a,b) propagate along the potential slope losing their energy (c,d) and accumulate on the right-hand side of the wire (e) thus condensing at $k = 0$ in reciprocal space (f). The plots (a,c,e) snapshot the $x$-space dynamics of the system, the $k$-space dynamics is

![FIG. 1: Propagation of particles along the potential slope in the quantum wire due to phonon-assisted relaxation. The curves correspond to the particle concentration profile in $x$-space for different times: 40 (red), 60 (green) and 80 ps (blue). The particles are introduced by a coherent Gaussian pump of the duration of 20 ps of the theoretical experiment. Particles propagate along the wire (green and blue curves), suffering losses caused by their finite lifetime and reflecting from the end of the wire. The interference of the incoming and reflected waves produces fringes clearly visible at 80 ps.](image-url)
depicted in plots (b,d,f). The white dashed curves in the figures correspond to the potential profile (in \(x\)-space) and free polaritons dispersion (in \(k\)-space). It should be noted, that the relaxation with phonons is an efficient process, thus particles rapidly move towards the bottom of the slope and accumulate there; and their concentration near the pumping spot soon becomes lower than the concentration in the signal point.

**Discussion.**—Comparing with previous theoretical models aimed at description of energy relaxation in coherent condensates, we would like to stress that all the parameters, including phonon scattering rates, are calculated microscopically rather than being treated in phenomenological way. Furthermore our approach allows the possibility to treat large regions of direct and reciprocal space with reasonably low computational cost. This in contrast to the theory developed in Ref. [28] where the need to calculate an evolving particle spectrum in a coarse time window was extremely memory-demanding. In the approach of Ref. [27], the analysis was restricted to a single quantum state and the energy relaxation introduced phenomenologically. Very recently, in Ref. [37], an approach merging the Boltzmann equations with a Gross-Pitaevskii treatment was developed where Boltzmann scattering rates are dynamically calculated from the mean-field wavefunctions. The theory was successful in the description of micropillars with a small number of confined states, but it is not obvious how efficient the theory would be in the description of many different modes in a large reciprocal space.

Comparing our results to the Monte Carlo approach developed in Ref. [38], it should be noted that we are seeking to address a different problem. In Ref. [38], the focus was on how quantum fluctuations affect the coherence properties of parametric oscillators. Here the stochastic element describes fluctuations due to phonons and we are interested in how they can cause relaxation of propagating condensates. To address coherence properties, an approach based on a matrix of correlators would be more appropriate [39, 40]. Unfortunately, such an approach is very demanding computationally, requiring matrices of size \(N \times N\), where \(N\) represents the number of states in reciprocal space.

Finally, while we have considered the energy relaxation of exciton-polaritons via phonon interactions, we should note that another relaxation mechanism is provided by scattering processes involving hot excitons with large momentum [18]. Such hot excitons are typically created in non-resonant/incoherently pumped systems [36] but can be neglected under resonant coherent excitation which we consider in this manuscript. In principle, the description of scattering with high momentum excitons can be accommodated within our formalism. This would allow the theoretical study of the interplay between both exciton mediated and phonon mediated scattering processes in extended systems and is an important direction for future research.

**Conclusion.**—We have derived a stochastic Gross-Pitaevskii equation, where the energy relaxation of bosons is provided by coupling to an incoherent field, treated as stochastic variable. As an example, we applied the theory to the modeling of exciton-polaritons in a 1D microwire with a potential gradient. The partial thermalization of exciton-polaritons is observed, together with their trapping in the real space. The partial thermalization of exciton-polaritons is observed, together with their trapping in the real space. This result is of particular relevance to a variety of recent experiments in exciton and exciton-polariton systems where the energy relaxation of propagating Bose-Einstein condensates was reported.

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In the present supplemental appendix we derive the stochastic Gross-Pitaevskii-type equation, which accounts for the phonon-mediated relaxation. In the paper we focus on polariton propagation in a 1D wire provided by confinement in the lateral direction, thus the appendix also corresponds to the 1D case. It should be noted, however, that generalization of the formalism for the 2D case and for any system of interacting bosons is straightforward. We introduce the quantum field operators for polaritons, $\hat{\Psi}_x$ connected with annihilation operators in reciprocal ($k$-) space by the Fourier transforms ($\mathcal{F}$ in short notation),

$$\hat{\Psi}_x = \mathcal{F}^{-1}[\hat{a}_k] = \frac{1}{\sqrt{N}} \sum_k \hat{a}_k e^{i k x},$$

where $N$ is the discretization length.

The Hamiltonian of the system can be represented as a sum of two parts,

$$\hat{H} = \hat{H}_1 + \hat{H}_2.$$  \hspace{1cm} (10)

The first term $\hat{H}_1$ corresponds to coherent processes,

$$\hat{H}_1 = \sum_k E_k \hat{\Psi}_x \hat{\Psi}_x + \sum_x \left( V_x \hat{\Psi}_x \hat{\Psi}_x + \alpha \hat{\Psi}_x \hat{\Psi}_x \hat{\Psi}_x \right) + \sum_x \sqrt{N} \hat{P}_x e^{i (k_p x - \omega_p t)} \left( \hat{\Psi}_x \hat{\Psi}_x \right),$$

where $E_k$ is the (non-parabolic) polariton dispersion, $V_x$ is the potential profile in $x$-space, $\alpha$ is a constant describing the strength of polariton-polariton interactions. The last sum in the equation is the coherent pumping term. Here $\mathcal{P}_x$ is the intensity of pump, $k_p$ is the wavevector of the pump, corresponding to the inclination of an incident laser beam, and $\hbar \omega_p$ is the pumping energy.

To model the interaction with acoustic phonons, we introduce the Fröhlich-type Hamiltonian \[10, 31\]:

$$\hat{H}_2 = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \sum_{\mathbf{q}, \mathbf{k}} G_{\mathbf{q}} \hat{b}_{\mathbf{q}} \hat{\Psi}_x \hat{\Psi}_x + G_{\mathbf{q}}^* \hat{\Psi}_x^\dagger \hat{\Psi}_x \hat{\Psi}_x \hat{\Psi}_x,$$

where the phonons described by operators $\hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}}$, unlike polaritons, are considered to be three dimensional. The phonon wavevector is $q = \mathbf{q} a_x + \mathbf{q} a_y + \mathbf{q} a_z$, where $\mathbf{q} a_x, \mathbf{q} a_y$ and $\mathbf{q} a_z$ are unit vectors: $\mathbf{q} a_x$ is in the wire direction, $\mathbf{q} a_y$ is in the structure growth direction, and $\mathbf{q} a_z$ is perpendicular to $\mathbf{q} a_x$ and $\mathbf{q} a_y$. The phonon dispersion relation, $\omega_{\mathbf{q}} = \hbar u \sqrt{q_x^2 + q_y^2 + q_z^2}$, is determined by the speed of sound, $u$. $G_{\mathbf{q}}$ is the exciton-phonon interaction strength, which can be calculated using the standard electron - deformation potential interaction \[33, 35\]. As a result of elastic scattering of a crystall, there appears an area with relatively higher density and thus higher polarizability. This area attracts an electron efficiently acting as a potential well. The relative change in volume $V$ can me expressed through the vector of displacement $\mathbf{u}_d(\mathbf{r})$ of a point called the deformation vector:

$$\frac{\delta V}{V} = \nabla \cdot \mathbf{u}_d.$$ \hspace{1cm} (13)

Then, the expression for the electron energy can be written in form:

$$E^e[\mathbf{u}_d(\mathbf{r})] = E_0^e - d_e \nabla \cdot \mathbf{u}_d(\mathbf{r}),$$

where $E_0^e$ is the energy of electron in a not deformed crystall; the parameter $d_e$ is the constant of the deformation potential for an electron. For the lattice with one atom in an elementary cell, the vector of deformation can be written in form \[33\]:

$$\mathbf{u}_d(\mathbf{r}) = \sqrt{\frac{\hbar}{2 \rho N}} \sum_{\mathbf{q}} \frac{\mathbf{q}}{\sqrt{\omega_{\mathbf{q}}}} e^{i \mathbf{q} \cdot \mathbf{r}} \left( \hat{b}_{\mathbf{q}} + \hat{b}_{\mathbf{-q}}^\dagger \right),$$

where $\rho$ is the electron density, $N$ is the number of atoms in the elementary cell.
where $\zeta$ is a real unit vector; $\rho$ is the mean density of the semiconductor material. Putting Eq. (14) in (15), we obtain the scattering strength of an electron on a phonon:

$$G_{\bar{q}} = -i\epsilon \sqrt{\frac{\hbar |q|}{2\rho V a}} = -i\epsilon \sqrt{\frac{\hbar \omega_{\bar{q}}}{2\rho V a^2}}. \quad (16)$$

The same but opposite in sign expression is true for a hole. Combining them and introducing the overlap integrals, we finally come to the expression:

$$G_{\bar{q}} = i \sqrt{\frac{\hbar \omega_{\bar{q}}}{2\rho V a^2}} \left[ d_h I_{\parallel}^{(h)}(q_x, q_y) I_{\perp}^{(h)}(q_z) - d_h I_{\parallel}^{(h)}(q_x, q_y) I_{\perp}^{(h)}(q_z) \right], \quad (17)$$

where $d_h$ are the deformation potentials of the lattice induced by phonons at the points of location holes. The integrals $I_{\parallel}^{(h)}(q_x, q_y)$ and $I_{\perp}^{(h)}(q_z)$ are the overlap integrals of the phonon wavefunctions with the electron and hole wavefunctions, respectively, in the in-plane and growth directions and can be evaluated following Ref. [31, 33]:

$$I_{\parallel}^{(h)}(q_x, q_y) = \left[ 1 + \left( \frac{m_{h(c)}}{m_e + m_h} \sqrt{q_x^2 + q_y^2} a_B \right)^2 \right]^{-3/2}, \quad (18)$$

$$I_{\perp}^{(h)}(q_z) = \frac{\pi^2}{a_B q_z} \left( \frac{\pi^2}{2} - \left( \frac{q_z L_z}{2} \right)^2 \right) \sin \left( \frac{q_z L_z}{2} \right). \quad (19)$$

It should be noted that a peculiarity of exciton-polariton systems is the variation of the excitonic fraction of polaritons with the in-plane wavevector, $k$, given by the $k$-dependent Hopfield coefficient. Such a dependence would introduce a $k$-dependence of the non-linear polariton-polariton interaction term and polariton-phonon interaction term. So as not to make our theory too specific to the exciton-polariton case, we neglect such a dependence. This is nevertheless valid in exciton-polariton systems in cases where the excited polaritons have similar in-plane wavevectors. In the excitation scheme that we consider, exciton-polaritons always have a small in-plane wavevector, due to the energy relaxation that limits their kinetic energy. Under such an assumption, the difference in a theory of excitons and of exciton-polaritons lies in the calculation of the non-linear interaction strength, $\alpha$, and of the phonon scattering coefficient, $G_{\bar{q}}$. The nonlinear constant $\alpha$ should be multiplied by $X^2$ and $G$ by $X$, where $X \approx 1/2$ for low energy exciton-polaritons and 1 for excitons.

In cases where one wishes to describe exciton-polaritons with very different in-plane wavevectors, a theory could be based on writing independent equations for the evolution of coupled excitons and photons. In such a case, the interaction of excitons with phonons is of the same form as written in the current theory.

The polariton field dynamics is given by:

$$\frac{d\hat{\Psi}_x}{dt} = \frac{i}{\hbar} \left[ \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2, \hat{\Psi}_x \right], \quad (20)$$

where the effect of the acoustic phonons is given by the term

$$\frac{i}{\hbar} \left[ \hat{\mathcal{H}}_2, \hat{\Psi}_x \right] = -\frac{i}{\hbar} \sum_{\bar{q}} \left( G_{-q_x,0,q_y,q_z} \hat{b}_{-q_x,0,q_y,q_z} + G_{q_x,0,q_y,q_z} \hat{b}_{q_x,0,q_y,q_z} \right) \hat{\Psi}_x e^{-i q_x x}. \quad (21)$$

The evolution of the acoustic phonons is determined by the equation

$$\frac{d\hat{b}_{\bar{q}}}{dt} = \frac{i}{\hbar} \left[ \hat{\mathcal{H}}_2, \hat{b}_{\bar{q}} \right] = -i\omega_{\bar{q}} \hat{b}_{\bar{q}} - \frac{i}{\hbar} G_{\bar{q}} \sum_{k'} \hat{\alpha}_{k' + q_x} a_{k'}. \quad (22)$$

The formal solution of Eq. (22) reads

$$\hat{b}_{\bar{q}}(t) = \hat{b}_{\bar{q}}(0) e^{-i\omega_{\bar{q}} t} - \frac{i}{\hbar} G_{\bar{q}} \int_0^t e^{-i\omega_{\bar{q}} (t-t')} A_{\bar{q}}(t') dt', \quad (23)$$
where
\[ A_{q_x}(t) = \sum_{k'} \hat{a}_{k' + q_x}(t) \hat{a}_{k'}(t) . \] (24)

Substitution of this expression into Eq. (21) gives:
\[
\begin{align*}
\frac{i}{\hbar} [\hat{H}_2, \hat{\psi}_x] &= -\frac{i}{\hbar} \hat{\psi}_x \sum_q (G_q \hat{b}_{-q_x, q_y, q_z}(0)e^{-i\omega_q t} + G_q^* \hat{b}_{-q_x, q_y, q_z}^*(0)e^{i\omega_q t}) e^{-\gamma_q x} \\
&\quad - \frac{1}{\hbar^2} \hat{\psi}_x \sum_q |G_q|^2 e^{-\gamma_q x} \left\{ \int_0^t e^{-i\omega_q (t-t')} A_{q_x}(t') dt' - \int_0^t e^{i\omega_q (t-t')} A_{q_x}(t') dt' \right\},
\end{align*}
\] (25)

where we noted that \( G_{-q_x, q_y, q_z} = G_{q_x, q_y, q_z} \) and \( \omega_{-q_x, q_y, q_z} = \omega_{q_x, q_y, q_z} \). Remembering that phonons represent an incoherent thermal reservoir, we can replace the terms \( \hat{b}_{-q_x, q_y, q_z}(0)e^{-i\omega_q t} \) and \( \hat{b}_{-q_x, q_y, q_z}^*(0)e^{i\omega_q t} \) by stochastic classical variables \( b_{-q_x, q_y, q_z}(t) \) and \( b_{-q_x, q_y, q_z}^*(t) \), respectively. This approximation is the analogue of the Markov approximation within the Langevin approach, when phonons are assumed to have a randomly varying phase. The stochastic variables are complex numbers with real and imaginary components drawn from a normal (Gaussian) distribution, normalized as follows,
\[
\begin{align*}
\langle b_{-q_x}^*(t)b_{q_y}(t') \rangle &= n_q \delta_{qq} \delta(t-t') ; \\
\langle b_{-q_x}^*(t)b_{q_y}^*(t') \rangle &= \langle b_{q_x}^*(t)b_{q_y}(t') \rangle = 0 ,
\end{align*}
\] (26)

where \( n_q \) is the number of phonons in the state with wavevector \( \vec{q} \) determined by the temperature of the system. The summation of the stochastic terms over \( q_y \) and \( q_z \) in Eq. (25) can be made by noting that the sum of two normally distributed stochastic variables again gives a normally distributed stochastic variable,
\[
\begin{align*}
\sum_{q_y, q_z} G_{-q_x, q_y, q_z} b_{-q_x, q_y, q_z}(t) &= \mathcal{T}_{-q_x}(t) ; \\
\sum_{q_y, q_z} G_{q_x, q_y, q_z} b_{q_x, q_y, q_z}^*(t) &= \mathcal{T}_{q_x}^*(t) ,
\end{align*}
\] (28)

where \( \mathcal{T}_{q_x} \) and \( \mathcal{T}_{q_x}^* \) represent the temperature dependent or thermal part of polariton scattering on phonons. They are defined by the correlators:
\[
\begin{align*}
\langle \mathcal{T}_{q_x}^*(t)\mathcal{T}_{q_x}^*(t') \rangle &= \sum_{q_y, q_z} |G_{q_x, q_y, q_z}|^2 n_{q_y, q_z} \delta_{q_x, q_x} \delta(t-t') , \\
\langle \mathcal{T}_{q_x}(t)\mathcal{T}_{q_x}(t') \rangle &= \langle \mathcal{T}_{q_x}^*(t)\mathcal{T}_{q_x}^*(t') \rangle = 0 .
\end{align*}
\] (30)

The time integrals in Eq. (25) can be simplified using the fact that the function
\[
\mathcal{K}_{q_x}(t) = -\sum_{q_y, q_z} |G_{q_y, q_z}|^2 \langle e^{-i\omega_q t} - e^{i\omega_q t} \rangle \rightarrow -\sum_{q_y, q_z} \frac{L_x a_p \omega_q}{2\pi} \int |G(\vec{q})|^2 \sin[\omega(\vec{q})t] dq_y dq_z ,
\] (32)

entering Eq. (21), is approximately independent of \( q_x \) in the range of \( q_x \in (-10^8, 10^8) \) m\(^{-1}\), and thus \( \mathcal{K}_{q_x}(t) \approx \mathcal{K}_0(t) \).

Further, within the Monte Carlo approach, the field operator \( \hat{\psi}_x \) can be treated as a classical variable for condensed polaritons, \( \psi_x = \langle \hat{\psi}_x \rangle \) (with the Fourier image \( \psi_k \)). Then, physical observables are calculated over multiple realizations of the evolution dynamics with stochastic variable \( b_k(t) \). We make a last notation
\[
S_k(t) = \sum_{q_x} \psi_{k+q_x}(t) \left( \int_0^t A_{q_x}(t') \mathcal{K}_0(t-t') dt' \right) ,
\] (33)

and finally obtain the stochastic Gross-Pitaevskii - type equation:
\[
\frac{i\hbar}{\gamma} \frac{d\psi_x}{dt} = \mathcal{F}^{-1} [E_k \psi_k + S_k(t)] + \left[ V_x - \frac{\hbar^2}{2} + \alpha |\psi_x|^2 \right] \psi_x + P_x e^{ik_x x} e^{-i\omega_x t} + \sum_k \{ \mathcal{T}_{-k}(t) + \mathcal{T}_k(t) \} e^{-i\kappa x} \psi_x ,
\] (34)
where we introduced the decay term $-i\hbar\gamma\psi_x/2$ to account for the radiative decay of particles \[^{19}\]. The function $S_k(t)$ is identified as the stimulated part of the phonon-mediated relaxation, where the convolution integral is responsible for energy conservation during the density-stimulated phonon-assisted process of energy relaxation.

Equation \[(34)\] is the main result of the Letter. It represents the dissipative GP equation and can be applied to various bosonic systems. As an example of application, here we present the figure which illustrates energy relaxation in a parabolic trap (see Fig. \[3\]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Relaxation of energy of exciton-polaritons in the parabolic trap in the regime of cw excitation in a quantum wire due to phonon-assisted processes in $x$- and $k$-space for the time range: 0-50 ps. White curves show the Harmonic-oscillator like potential profile in the real space and free exciton-polariton dispersion in the reciprocal space used in calculations. Coherent pumping represents a Gaussian in $x$-space centered around $x = 0$ \(\mu\)m. The inclination angle of the pump in the units of wavevector is $k_p \approx -0.9 \ \mu m^{-1}$, the energy $\hbar\omega_p = 0.5$ meV. Polaritons created at the pumping spot, first, propagate until they meet the potential profile and then along the potential gradient and accumulate in its bottom (a). The $k$-space behavior shows the decrease of the condensate energy (b).}
\end{figure}

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