On numerical invariants for knots in the solid torus

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Abstract: We define some new numerical invariants for knots with zero winding number in the solid torus. These invariants explore some geometric features of knots embedded in the solid torus. We characterize these invariants and interpret them on the level of the knot projection. We also find some relations among some of these invariants. Moreover, we give lower bounds for some of these invariants using Vassiliev invariants of type one. We connect our invariants to the bridge number in the solid torus. We give a lower bound and an upper bound of the wrap of a knot in the solid torus in terms of our new invariants.

Keywords: Bridge number of a knot, Solid torus, Universal cover, Vassiliev invariants

MSC: 57M27

1 Introduction

Numerical invariants for knots and links have been used for understanding different geometric features of knots and links. They are usually non-negative integers, which are first defined for knot or link diagrams. Then for such an integer-valued function on diagrams to produce a knot or link invariant, the minimal possible number over all diagrams is taken, which results in a unique number. This number is then a numerical invariant of the knot or the link type that gives a specific type of information about the knot or link.

In [1], Milnor defined the crookedness of a knot as follows. Let \( u(t) \) be a vector-valued function parametrizing a given knot \( K \) in \( \mathbb{R}^3 \). Let \( v \) be a non-zero vector in \( \mathbb{R}^3 \). The minimal number of local maxima of the function \( u(t) \cdot v \) over all such vectors \( v \) and all possible parametrizations of \( K \) within its knot type is the crookedness of the knot \( K \) which we denote by \( m(K) \). On the level of the projection of the knot, the crookedness of the knot \( K \) is the minimal number of maxima over all diagrams of the knot and over all directions of non-zero vectors of \( \mathbb{R}^2 \).

An important numerical invariant for knots and links in \( \mathbb{R}^3 \) is the bridge number that was introduced in [2]. The bridge number for a knot \( K \), denoted by \( b(K) \), is the least possible number of bridges of all diagrams of a given knot or link, where a bridge is a maximal arc of the diagram that crosses over the other strands. At least one over crossing is required. By convention the unknotted circle has bridge number equal to one. In [2] Schubert proved that the bridge number of a knot is equal to the crookedness of the knot. We write this fact as

\[ m(K) = b(K). \]

In this research, we define some new numerical invariants for knots in \( KST_0 \); that is for knots with zero winding number in the solid torus. These invariants explore some geometric features of knots embedded in the solid torus that might not exist or be relevant when these knots are embedded in \( \mathbb{R}^3 \). We characterize these invariants and interpret them on the level of the knot projection. We also find some relations among some of these invariants. Moreover, we...
give lower bounds for some of these invariants using Vassiliev invariants of type one. Such bounds help in computing some of these numerical invariants that are usually easy to define and hard to compute. We define different types of crookedness; namely angular crookedness $A(K)$, radial crookedness $R(K)$, and height crookedness $H(K)$. We connect height crookedness to the bridge number in the solid torus $B(K)$.

A well-known numerical invariant for a knot $K$ in the solid torus is the wrap $w(K)$; that is the minimal number of intersections between the knot and a meridional disk in the solid torus. In 1989 Hoste and Przytycki [3] defined a two variable polynomial invariant for one-trivial dichromatic links, which is also an invariant for knots in the solid torus. In fact their invariant generalizes Jones polynomial. They conjectured in [3] that the wrap $w(K)$ is equal to the highest power of the variable $t$ in their polynomial. This is known in the literature as the wrapping conjecture.

In this research we give an upper bound of the wrap $w(K)$ in terms of the radial crookedness $R(K)$ and we give a lower bound of the wrap in terms of the length $L(K)$, which is also a new invariant we define in this article.

In section two, we give basic concepts and terminology, which we use in the later sections. In section three, we define the angular crookedness $A(K)$ for a knot in $KST_0$. We give interpretations and characterizations for this invariant and we give a lower bound for this invariant using Vassiliev invariants of type one. In section four, we define the length of a knot in $KST_0$; then we give a relation between the wrap and the length. We also give a lower bound of the length using type-one invariants. In section five we define the radial crookedness $R(K)$, the height crookedness $H(K)$, and the bridge number $B(K)$ for a knot $K$ in $KST_0$. We connect height crookedness to the bridge number $B(K)$, and the radial crookedness $R(K)$ to the wrap $w(K)$. Some relevant examples are also given.

## 2 Basic concepts and terminology

The following construction of lifting of a knot with zero winding number in the solid torus into the universal cover is the same as that we used in [4].

Let $ST$ be the closed solid torus defined by

$$ST = \{(x, y, z) : \frac{1}{4} \leq x^2 + y^2 \leq \frac{9}{4}, -\frac{1}{2} \leq z \leq \frac{1}{2}\}.$$  

Consider the universal cover of $ST$, denoted by $\tilde{ST}$ and viewed as the infinite closed solid cylinder

$$\tilde{ST} = \{(x, y, z) : -\infty < x < \infty, \frac{1}{2} \leq y \leq \frac{3}{2}, -\frac{1}{2} \leq z \leq \frac{1}{2}\}.$$  

Let the covering map $p : \tilde{ST} \to ST$ be given by $p(x, y, z) = (x \cos(2\pi x), y \sin(2\pi x), z)$. Consider a knot $K$ represented by a smooth simple closed curve $f : I \to ST$ with winding number $w_f = 0$ in $ST$. Let the knot $K$ be based at the point $(0, 1, 0)$; that is $f(0) = (0, 1, 0)$ in $ST$. For each $n \in \mathbb{Z}$, let $a_n \in \tilde{ST}$ be the point $(n, 1, 0)$. Then, by the Unique Lifting Theorem, there exists a unique path $\tilde{f}_n : [0, 1] \to \tilde{ST}$ such that $p \circ \tilde{f}_n = f$ and $\tilde{f}_n(0) = a = (n, 1, 0)$.

Note that the lift $\tilde{f}_n$ is a closed path, because $f$ has zero winding number $(0 = w_f = \tilde{f}_n(1) - \tilde{f}_n(0))$ implies that $\tilde{f}_n(1) = \tilde{f}_n(0)$. So for each $n \in \mathbb{Z}$ we get a knot $K_n$ in the infinite cylinder $\tilde{ST}$ represented by $\tilde{f}_n$ as in the following figure.

Let $K \in KST_0$ be an oriented knot, and $n \in \mathbb{N}$, then

$$\beta_n(K) = \text{the linking number of the two-component link } \{K_0, K_n\}.$$
is a Vassiliev invariant of order one. For the proof see [4]. It turns out that $\beta_n(K)$ can be computed from the punctured diagram of the knot $K$ as in the following formula

$$\beta_n(K) = \frac{1}{2} \sum_c \text{sign}(c)$$

(where the sum is taken over all crossings $c$, with winding numbers $\{n, -n\}$ of the two lobes at $c$). In fact, this sequence of invariants turns out to be universal in the sense that if this sequence of invariants fails to distinguish two given knots in $KST_0$, then any Vassiliev invariant of order one will fail too.

Let $F_K(x) = \sum_n \beta_n(K) x^n$, then the invariant $F_K(x)$ is a polynomial of degree $m = \max \{n : \beta_n(K) \neq 0\}$. Therefore, the leading coefficient of $F_K(x)$ is $\beta_m(K)$.

Note that the $\frac{1}{2}$ in the definition of $\beta_n(K)$ or in the linking number defining $\beta_n(K)$ implies that the crossings of winding numbers $\{n, -n\}; n \neq 0$ occur in pairs in any diagram of the knot $K$.

### 3 Angular crookedness for a knot in $KST_0$

We define the first invariant for knots in $KST_0$ as follows:

**Definition 3.1.** Let the knot $K \in KST_0$ be represented by a smooth simple closed curve $g : I \to ST$ using cylindrical coordinates as $g(t) = (r(t), \theta(t), z(t))$. Then the angular crookedness of $K$ denoted by $A(K)$ is given by the least possible number of local maxima of $\theta(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.

**Theorem 3.2.** Let $K \in KST_0$ be represented by $f(t) = (x(t), y(t), z(t))$ such that $\tilde{f}_0(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$. Then $A(K)$ equals the least possible number of local maxima of $\tilde{x}(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.

**Proof.** Note that

$$\left(p \circ \tilde{f}_0\right)(t) = f(t),$$

which implies that

$$p(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) = (x(t), y(t), z(t)).$$

By definition of the covering map $p$, we get

$$(\tilde{y}(t) \cos(2\pi \tilde{x}(t)), \tilde{y}(t) \sin(2\pi \tilde{x}(t)), \tilde{z}(t)) = (x(t), y(t), z(t)).$$

Therefore,

$$(\tilde{y}(t) \cos(2\pi \tilde{x}(t)), \tilde{y}(t) \sin(2\pi \tilde{x}(t)), \tilde{z}(t)) = (r(t) \cos \theta(t), r(t) \sin \theta(t), z(t)).$$

Hence

$$\tilde{y}(t) \cos(2\pi \tilde{x}(t)) = r(t) \cos \theta(t),$$

and

$$\tilde{y}(t) \sin(2\pi \tilde{x}(t)) = r(t) \sin \theta(t).$$

Therefore,

$$\tan(2\pi \tilde{x}(t)) = \tan \theta(t), \text{ or } \cot(2\pi \tilde{x}(t)) = \cot \theta(t),$$

which, by monotonicity of the tangent and cotangent functions, implies that $\tilde{x}(t)$ has a local maximum if and only if $\theta(t)$ has a local maximum. This completes the proof.
Remark 3.3. On the level of the projection of the knot, the angular crookedness $A(K)$ of the knot $K$ is the minimal number of maxima in the counter-clockwise direction over all punctured diagrams of the knot.

Example 3.4. As an example, in the following figure the punctured diagram to the left is not a minimal one for $A(K)$, because it involves three angular local maxima. However, the punctured diagram in the middle is minimal as this diagram has two angular local maxima, and we will show later that $A(K) = 2$ for this knot.

Remark 3.5. On the level of the lifted diagram of the knot, the angular crookedness $A(K)$ of the knot $K$ is the minimal number of maxima in the positive $x$-axis direction of only the lifted diagram of $K_0$ that results from a minimal diagram of $A(K)$. Therefore, it should be realized that we can not deform $K_0$ to get a lower number of local maxima. See the minimal and lifted diagram in the previous example.

Lemma 3.6. Let $K \in KST_0$, and $K_0$ be the $0^{th}$ component in the lifted diagram of $K$ viewed as a diagram of a knot in $\mathbb{R}^3$, then $A(K) \geq m(K_0) (= b(K_0))$.

Proof. This is true because $A(K)$ is the number of local maxima of one single diagram of the knot represented by the diagram $K_0$ in $\mathbb{R}^3$, and in one direction only, that is the positive $x$-axis, while $m(K_0)$ is the minimal number of local maxima over all diagrams in $\mathbb{R}^3$ equivalent to $K_0$ and over all directions.

Note that by this lemma and Theorem 3.2 we have

Corollary 3.7. $A(K) \geq 1$.

Next, we investigate a lower bound for the angular crookedness of a knot using type-one Vassiliev invariants.

Theorem 3.8. For a knot $K \in KST_0$, we have

$$A(K) \geq \max_n |\beta_n(K)|.$$  

Proof. Let $K \in KST_0$ be represented by $f(t) = (x(t), y(t), z(t))$ such that $\overline{f}_0(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t))$, and let $D$ be any diagram of $K$. By Theorem 3.2, $A(K)$ is determined by the number of local maxima of $\overline{x}(t)$. Recall that

$\beta_n(K) =$ the linking number of the two–component link $\{K_0, K_n\}$.

Let $\max_n |\beta_n(K)| = l$, then there are $2|l|$ crossings of the same sign in $D$, which are either all positive or all negative. This lifts to $2|l|$ crossings of the same sign between the two components $K_0$ and $K_n$ in the diagram of the two-component link $\{K_0, K_n\}$. Arrange the crossings from left to right as follows $a_{|l|}, \ldots, a_2, a_1, b_1, b_2, \ldots, b_{|l|}$. Consider the pair $a_1, b_1$, then we are in a situation like one of these in the following figure.
Note that in each case $\bar{\chi}(t)$ has at least one local maximum.

If we consider the two pairs of crossings $a_1, b_1$ and $a_2, b_2$, then we are in a situation like one of these in the following figure, with the appropriate choice of under or over crossings and directions, so that the four crossings have the same sign.

Observe that in each case $\bar{\chi}(t)$ has at least two local maxima.

Obviously, the idea is that every strand of $K_0$ going upwards must connect to a strand going downwards, producing at least one local maximum by a simple application of the Mean Value Theorem on $\bar{\chi}(t)$.

Similarly, considering all the pairs $a_{|l|}, a_2, a_1, b_1, b_2, \ldots, b_{|l|}$, then $\bar{\chi}(t)$ has at least $|l|$ local maxima. Therefore

$$A(K) \geq \max_n |\beta_n(K)|.$$ 

In the following example, we use the last theorem to compute $A(K)$ for a given knot.

**Example 3.9.** Consider the knot with punctured diagram in the following figure.

Note that $\beta_1(K) = 2$, and $\beta_i(K) = 0$, for $i \geq 2$. Therefore $\max_n |\beta_n(K)| = 2$. Moreover, this is a diagram $D$, with $A(D) = 2$. Therefore, $A(K) = 2$.

**Theorem 3.10.** For every $n \in \mathbb{N}$, there exists a knot $K \in KST_0$ such that $A(K) = n$.

**Proof.** The idea of the proof is involved in the previous example, that is by considering the knot resulting from replacing the two full twists by $n$ full twists in the knot diagram of the previous example. Next, we give an example of a knot at which $A(K)$ is strictly greater than $\max_n |\beta_n(K)|$.

**Example 3.11.** Consider the knot with punctured diagram and $O_h$ component lifted diagram as in the following figure.

Note that, by Lemma 3.6, $A(K) \geq 2$, because the trefoil is a rational knot (i.e. a knot with bridge number equal to two). See [5–7]. But this diagram $K_0$ involves two local maxima in the positive $x$-axis direction. Therefore $A(K) = 2$ for this knot.
However, 
\[ \beta_1(K) = 1, \text{ and } \beta_i(K) = 0, \text{ for } i \geq 2. \]

Therefore, 
\[ \max_n |\beta_n(K)| = 1. \]

4 The length of a knot in \( KST_0 \)

We define another numerical invariant for a knot \( K \in KST_0 \). Each crossing \( c \) in a given punctured diagram \( D \) of \( K \) has two lobes of winding numbers \( \{n, -n\}, n \geq 0 \). In this case we will say that the crossing \( c \) is of length \( n \). Let the length of the diagram \( D \) be given by the maximum of the lengths of the finitely many crossings in the diagram \( D \); that is
\[ L(D) = \max_c \{n : n \text{ is the length of the crossing } c \}. \]

**Definition 4.1.** For a knot \( K \in KST_0 \), the length of \( K \) denoted by \( L(K) \) is the least possible integer \( L(D) \) over all possible diagrams \( D \) of \( K \).

Recall that for a knot \( K \in KST_0 \), the wrap \( w(K) \) is the minimal number of intersections between the knot and a meridinal disk in the solid torus. We have the following important inequality between the length \( L(K) \), and the wrap \( w(K) \).

**Theorem 4.2.** For a knot \( K \in KST_0 \), the following inequality holds
\[ w(K) \geq 2L(K). \]

**Proof.** Note that if \( L(K) = 0 \), then the inequality is trivially true, as \( w(K) \geq 0 \). Assume that \( L(K) = n; n > 0 \), then in any diagram \( D \) of \( K \), there is a crossing \( c \) of length \( n \). Note that the two lobes at the crossing \( c \) are disjoint and they have winding numbers of \( n \) and \( -n \). Therefore, each of the two lobes intersects any meridional disk in the solid torus at least \( n \) times. Therefore, \( w(K) \geq 2n = 2L(K) \).

**Theorem 4.3.** Let \( K \in KST_0 \), and \( m = \deg(F_K(x)) \), where \( F_K(x) = \sum_n \beta_n(K)x^n \), then
\[ L(K) \geq m. \]

**Proof.** Let \( D \) be any punctured diagram of \( K \). As \( \deg(F_K(x)) = m \), we have \( \beta_m(K) \neq 0 \). Therefore, there is a crossing \( c_0 \) in \( D \) of length \( m \). However, since
\[ L(D) = \max_c \{n : n \text{ is the length of the crossing } c \}, \]
we have
\[ L(D) \geq m. \]
As \( D \) is an arbitrary punctured diagram of \( K \), we have
\[ L(K) \geq m. \]

**Example 4.4.** Consider the knot with punctured diagram in Example 3.9. Note that
\[ F_K(x) = 2x \Rightarrow \deg(F_K(x)) = 1. \]

Moreover, this is a diagram \( D \), with \( L(D) = 1 \). Therefore, \( L(K) = 1 \).
The bridge number and radial and height crookedness for a knot in $KST_0$

**Definition 5.1.** Let the knot $K \in KST_0$ be represented by a smooth simple closed curve $g : I \rightarrow ST$ using cylindrical coordinates as $g(t) = (r(t), \theta(t), z(t))$. Then

1. the radial crookedness of $K$ denoted by $R(K)$ is given by the least possible number of local maxima of $r(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.
2. the height crookedness of $K$ denoted by $H(K)$ is given by the least possible number of local maxima of $z(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.

**Theorem 5.2.** Let $K \in KST_0$ be represented by $f(t) = (x(t), y(t), z(t))$ such that $\tilde{f}_0(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$. Then

1. $R(K)$ equals the least possible number of local maxima of $\tilde{y}(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.
2. $H(K)$ equals the least possible number of local maxima of $\tilde{z}(t)$ over all possible functions $f$ representing knots with knot diagrams isotopy equivalent to $K$.

**Proof.** From the proof of the Theorem 3.2, we have

$$\tilde{y}(t) \cos(2\pi \tilde{x}(t)) = r(t) \cos \theta(t),$$

and

$$\tilde{y}(t) \sin(2\pi \tilde{x}(t)) = r(t) \sin \theta(t).$$

Therefore,

$$\tan(2\pi \tilde{x}(t)) = \tan \theta(t),$$

or

$$\cot(2\pi \tilde{x}(t)) = \cot \theta(t),$$

which implies that

$$2\pi \tilde{x}(t) = \theta(t),$$

or

$$2\pi \tilde{x}(t) = \theta(t) + \pi.$$

Therefore,

$$\tilde{y}(t) = r(t)$$

or

$$\tilde{y}(t) = -r(t).$$

However, as $r(t) > 0$, and $\tilde{y}(t) \geq \frac{1}{2}$, we have

$$\tilde{y}(t) = r(t),$$

and hence $\tilde{y}(t)$ has a local maximum if and only if $r(t)$ has a local maximum. This finishes the proof of part (1).

Part (2) is obvious as $\tilde{z}(t) = z(t)$.

**Remark 5.3.** On the level of the projection of the knot, the radial crookedness of the knot $K$ is the minimal number of maxima of the distance between a point on the knot diagram and the puncture over all punctured diagrams of the knot, or the minimal number of maxima in the $y$-direction over all lifted diagrams of the knot.

**Remark 5.4.** On the level of the projection of the knot, the height crookedness of the knot $K$ is the minimal number of maxima in the $z$-direction over all punctured diagrams of the knot, or the minimal number of maxima in the $z$-direction over all lifted diagrams of the knot. Note that this number is harder to see as the $z$-direction is perpendicular to the plane of projection. However, this number would have a special importance, as it is related to the bridge number for knots in $KST_0$.

**Definition 5.5.** For a knot $K \in KST_0$, let the natural bridge number of $K$, denoted by $B(K)$, be defined by the least number of bridges of a punctured diagram of $K$ over all punctured diagrams of $K$. 
In [2], Schubert proved that the bridge number of a knot in $\mathbb{R}^3$ is equal to the crookedness of the knot. Now we show that the natural bridge number of a knot $K \in KST_0$ is equal to the height crookedness of the knot.

**Theorem 5.6.** For a knot $K \in KST_0$, we have

$$B(K) = H(K).$$

**Proof.** Let $D$ be a punctured diagram of the knot $K$. A diagram is the union of two disjoint sets. One set is the set of bridges, which are pairwise disjoint maximal arcs that can not be embedded entirely in the punctured plane of projection. The other set is the set of maximal arcs that can be entirely embedded in the punctured plane of projection. See the following figure, where the bold arcs are the bridges.

Therefore, each bridge has at least one local maximum in the $z$-direction, because the bridge overpasses all the time with endpoints in the plane. Moreover, every local maximum in the $z$-direction of $D$ occurs on a bridge, because the other arcs are embedded entirely in the plane. Therefore, for any punctured diagram $D$, we have

$$H(D) \geq B(D).$$

Hence,

$$H(K) \geq B(K).$$

On the other hand, let $D^*$ be a punctured diagram of $K$ of minimal number of local maxima in the $z$-direction. Let $H(D^*) = n$. Suppose that $B(K) < n$, then there is a diagram $D^*$ such that $B(D^*) < n$. This means that there is a bridge that has more than one local maximum. But this bridge can be curved so that it has one local maximum in the $z$-direction. This contradicts the minimality of number of local maxima of $D^*$. Therefore,

$$B(K) \geq H(D^*) = H(K).$$

By the two inequalities, we have

$$B(K) = H(K).$$

We also have the following important inequality between the radial crookedness $R(K)$, and the wrap $w(K)$.

**Theorem 5.7.** For a knot $K \in KST_0$, the following inequality holds

$$R(K) \geq w(K).$$

**Proof.** Let $D$ be any punctured diagram with wrap $w(D) = n$ of a knot $K$. Obviously, the diagram $D$ has at least one local radial maximum on each arc making the wrap of $D$. Therefore $R(D) \geq w(D)$. Hence $R(K) \geq w(K)$.

**Remark 5.8.** Some of the invariants defined in this article can be generalized to arbitrary knots in the solid torus with any winding number. However, we can not obtain lifted knot diagrams, when the winding number is other than zero.
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