Finite Basis Problem for words with at most two non-linear variables

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Abstract

Let $\mathcal{A}$ be an alphabet and $W$ be a set of words in the free monoid $\mathcal{A}^*$. Let $S(W)$ denote the Rees quotient over the ideal of $\mathcal{A}^*$ consisting of all words that are not subwords of words in $W$. We call a set of words $W$ finitely based if the monoid $S(W)$ is finitely based.

We find a simple algorithm that recognizes finitely based words among words with at most two non-linear variables.

1 Introduction

An algebra is said to be finitely based (FB) if there is a finite subset of its identities from which all of its identities may be deduced. Otherwise, an algebra is said to be non-finitely based (NFB). The famous Tarski’s Finite Basis Problem asks if there is an algorithm to decide when a finite algebra is finitely based. In 1996, R. McKenzie [7] solved this problem in the negative showing that the classes of FB and inherently not finitely based finite algebras are recursively inseparable. (A locally finite algebra is said to be inherently not finitely based (INFB) if any locally finite variety containing it is NFB.)

It is still unknown whether the set of FB finite semigroups is recursive although a very large volume of work is devoted to this problem (see the surveys [15, 16]). In contrast with McKenzie’s result, a powerful description of the INFB finite semigroups has been obtained by M. Sapir [9, 10]. These results show that we need to concentrate on NFB finite semigroups that are not INFB.

In 1976, M. Sapir suggested to concentrate on the class of monoids of the form $S(W)$. (A monoid is a semigroup with an identity element.) Monoids of the form $S(W)$ are defined as follows.

Let $\mathcal{A}$ be an alphabet and $W$ be a set of words in the free monoid $\mathcal{A}^*$. Let $S(W)$ denote the Rees quotient over the ideal of $\mathcal{A}^*$ consisting of all words that are not subwords of words in $W$. For each set of words $W$, the semigroup $S(W)$ is a monoid with zero whose nonzero elements are the subwords of words in $W$. Evidently, $S(W)$ is finite if and only if $W$ is finite.
The identities of these semigroups have been of interest since P. Perkins showed that $S\{abtba, atbab, abab, aat\}$ was NFB. It was one of the first examples of a finite NFB semigroup. It is clear from the results of [9, 10] that a semigroup of the form $S(W)$ is never INFB. It is shown in [3] that the class of monoids of the form $S(W)$ is as “bad” with respect to the finite basis property as the class of all finite semigroups. In particular, the set of FB semigroups and the set of NFB semigroups in this class are not closed under taking direct products, and there exists an infinite chain of varieties generated by such semigroups where FB and NFB varieties alternate.

We call a set of words $W$ finitely based if the monoid $S(W)$ is finitely based. In this paper we study the following problem.

**Question 1.** [13, M. Sapir] Is the set of finite finitely based sets of words recursive?

A partial answer to Question 1 is contained in [11, Theorem 5.1]. That theorem says that a word $U$ in a two-letter alphabet $\{a, b\}$ is FB if and only if $U$ is of the form $a^n b^m$ or $a^n ba^m$ for some $n, m \geq 0$ modulo renaming $a$ and $b$. If a variable $t$ occurs exactly once in a word $u$ then we say that $t$ is linear in $u$. If a variable $x$ occurs more than once in a word $u$ then we say that $x$ is non-linear in $u$. In this article, we generalize Theorem 5.1 in [11] into an algorithm which given a word $U$ with at most two non-linear variables, decides whether $U$ is finitely based or nor.

A word $u$ is said to be an isoterm for a semigroup $S$ if $S$ does not satisfy any nontrivial identity of the form $u \approx v$. The notion of an isoterm was introduced by Perkins in [8] and has proved to be crucial for understanding the difference between finitely based and non-finitely based semigroups. According to [9], a finite semigroup $S$ is INFB iff every Zimin word ($Z_1 = x_1, \ldots, Z_{k+1} = Z_k x_{k+1} z_k, \ldots$) is an isoterm for $S$ iff the word $Z_k$ is an isoterm for $S$ where $k = |S|^2$. We use $\text{var}S$ to denote the variety generated by semigroup $S$. A semigroup $S$ is said to be hereditary finitely based (HFB) if every subvariety of $\text{var}S$ is finitely based. It is proved in [2] that a finite aperiodic semigroup with central idempotents is HFB if and only if the word $Z_2 = xtx$ is not an isoterm for $S$.

It is not a surprise that the notion of an isoterm plays a crucial role in this article as well. First, we prove (see Theorem 6.4 below) that a word $U$ with at most two non-linear variables is FB if and only if the monoid $S(\{U\})$ belongs to certain intervals in the lattice of semigroup varieties. The end-points of these intervals are generated by monoids of the form $S(W)$. Theorem 6.4 can be easily transformed into Theorem 6.5 which says that $U$ is FB if and only if certain words are isoters for $S(\{U\})$ and certain words are not. Finally, in Theorem 7.6 we present our algorithm in a computation-free form. This work was inspired by the article [17] where all finitely based words with two non-linear 2-occurring variables are described.
2 Identities of monoids of the form $S(W)$ and a quasi-order on sets of words

Throughout this article, elements of a countable alphabet $\mathcal{A}$ are called variables and elements of the free monoid $\mathcal{A}^*$ are called words. We use $\epsilon$ to denote the empty word. If $\mathcal{X}$ is a set of variables then we write $u(\mathcal{X})$ to refer to the word obtained from $u$ by deleting all occurrences of all variables that are not in $\mathcal{X}$ and say that the word $u$ deletes to the word $u(\mathcal{X})$. If $\mathcal{X} = \{y_1, \ldots, y_k\} \cup \mathcal{Y}$ for some variables $y_1, \ldots, y_k$ and a set of variables $\mathcal{Y}$ then instead of $u(\{y_1, \ldots, y_k\} \cup \mathcal{Y})$ we simply write $u(y_1, \ldots, y_k, \mathcal{Y})$. We say that a set of variables $\mathcal{X}$ is stable in an identity $u \approx v$ if $u(\mathcal{X}) = v(\mathcal{X})$. Otherwise, we say that set $\mathcal{X}$ is unstable in $u \approx v$. In particular, a variable $x$ is stable in $u \approx v$ if and only if it occurs the same number of times in $u$ and $v$. An identity $u \approx v$ is called balanced if every variable is stable in $u \approx v$. A substitution $\Theta : \mathcal{A} \to \mathcal{A}^*$ is a homomorphism of the free monoid $\mathcal{A}^*$. We use $W^e$ to denote the closure of a set of words $W$ under taking subwords. If a semigroup $S$ satisfies all identities in a set $\Sigma$ then we write $S \models \Sigma$.

Lemma 2.1. [11, Lemma 2.5] Let $W = W^e$ be a set of words and $u \approx v$ be a balanced identity. Suppose that for every pair of variables $\{x, y\}$ unstable in $u \approx v$ and every substitution $\Theta : \mathcal{A} \to \mathcal{A}^*$ such that $\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)$, neither $\Theta(u)$ nor $\Theta(v)$ belongs to $W$. Then $S(W) \models u \approx v$.

Lemma 2.2. [2, Lemma 3.3] Let $W$ be a set of words and $S$ be a monoid. Then each word in $W$ is an isoterm for $S$ if and only if var($S$) contains $S(W)$.

If $W$ and $W'$ are two sets of words then we write $W \preceq W'$ if for any monoid $S$ each word in $W'$ is an isoterm for $S$ whenever each word in $W$ is an isoterm for $S$. It is easy to see that the relation $\preceq$ is reflexive and transitive, i.e. it is a quasi-order on sets of words. If $W \preceq W' \preceq W$ then we write $W \sim W'$. We say that two sets of words $W$ and $W'$ are equationaly equivalent if the monoids $S(W)$ and $S(W')$ satisfy the same identities. The following proposition shows that if we identify sets of words modulo $\sim$ then we obtain an ordered set antiisomorphic to the set of all varieties of the form var$S(W)$ ordered under inclusion. In particular, two sets of words $W$ and $W'$ are equationaly equivalent if and only if $W \sim W'$.

Proposition 2.3. For two sets of words $W$ and $W'$ the following conditions are equivalent.

(i) $W \preceq W'$.
(ii) Each word in $W'$ is an isoterm for $S(W)$.
(iii) var$S(W)$ contains $S(W')$.

Proof. (i) $\rightarrow$ (ii) Since each word $w \in W$ is an isoterm for $S(W)$, each word $w' \in W'$ is also an isoterm for $S(W)$.

(ii) $\rightarrow$ (iii) Since each word $w' \in W'$ is an isoterm for $S(W)$, Lemma 2.2 implies that the variety generated by $S(W')$ contains $S(W')$. 

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(iii) \( \rightarrow \) (i) Let \( S \) be a monoid so that each word \( w \in W \) is an isoterms for \( S \). Since the variety generated by \( S(W) \) contains \( S(W') \), Lemma 2.2 implies that each word \( w' \in W' \) is an isoterms for \( S(W) \).

The relations \( \leq \) and \( \sim \) we extend to individual words. Say, if \( u \) and \( v \) are two words then \( u \sim v \) means \( \{u\} \sim \{v\} \). Also, if \( W \) is a set of words and \( u \) is a word then \( W \leq u \) means \( W \leq \{u\} \). We use \( \langle W \downarrow \rangle \) to denote the closure of \( W \) under going down in order \( \leq \) and \( \langle W \uparrow \rangle \) to denote the closure of \( W \) under going up in order \( \leq \). It is easy to see that \( W \subseteq W^c \subseteq \langle W \uparrow \rangle \) and \( W \sim W^c \sim \langle W \uparrow \rangle \). If \( W \) is finite then \( W^c \) is also finite. On the other hand, if, set \( W^c \) contains \( ab \) then set \( \langle W \uparrow \rangle \) is always finite, because it contains the words \( t_1 t_2 \ldots t_n \) for arbitrary \( n > 0 \). If \( L \) and \( N \) are two sets of words then we define \( L/N := \{u \in L|\forall n \in N, u \not\sim n\} \). In other words, \( L/N \) is the largest subset \( W \) of \( L \) so that \( \text{var} S(W) \) contains none of \( S(\{n\}) \) for any \( n \in N \). In particular, \( \mathfrak{A}^*/N \) is the set of all words which do not belong to \( \langle N \downarrow \rangle \). So, the free monoid \( \mathfrak{A}^* \) is a disjoint union of \( \mathfrak{A}^*/N \) and \( \langle N \downarrow \rangle \).

Evidently, \( L/N = (\mathfrak{A}^*/N) \cap L \).

**Lemma 2.4.** Let \( L = L^c \) and \( N \) be sets of words and \( u \approx v \) be a balanced identity. Suppose that for every pair of variables \( \{x, y\} \) unstable in \( u \approx v \) and every substitution \( \Theta : \mathfrak{A} \rightarrow \mathfrak{A}^* \) so that \( \Theta(x) \) contains some \( a \in \mathfrak{A} \) and \( \Theta(y) \) contains \( b \neq a \), each of the following conditions is satisfied.

(i) If \( \Theta(u) \in L \) then \( \Theta(u) \leq n \) for some \( n \in N \).

(ii) If \( \Theta(v) \in L \) then \( \Theta(v) \leq n \) for some \( n \in N \).

Then \( S(L/N) \models u \approx v \).

**Proof.** Let \( \{x, y\} \) be a pair of variables unstable in \( u \sim v \) and \( \Theta : \mathfrak{A} \rightarrow \mathfrak{A}^* \) be a substitution such that \( \Theta(y)\Theta(x) \neq \Theta(x)\Theta(y) \). Since \( \Theta(x)\Theta(y) \neq \Theta(y)\Theta(x) \), the word \( \Theta(x) \) contains some letter \( a \) and the word \( \Theta(y) \) contains \( b \neq a \).

If \( \Theta(u) \in L/N \) then by Condition (i) we have that \( \Theta(u) \leq n \) for some \( n \in N \). To avoid a contradiction to the definition of \( L/N \), we conclude that \( \Theta(u) \) does not belong to \( L/N \). A similar argument shows that \( \Theta(v) \) does not belong to \( L/N \). Therefore, Lemma 2.1 implies that \( S(L/N) \models u \approx v \). \(\square\)

### 3 Some NFB intervals between sets of block-2-simple words

If a variable \( t \) occurs exactly once in a word \( u \) then we say that \( t \) is *linear* in \( u \). If a variable \( x \) occurs more than once in a word \( u \) then we say that \( x \) is *non-linear* in \( u \). We reserve letter \( t \) with or without subscripts to denote linear variables. If we use letter \( t \) several times in a word, we assume that different occurrences of \( t \) represent distinct linear variables. A *block* of a word \( u \) is a maximal subword of \( u \) that does not contain any linear letters of \( u \). For \( n \geq 0 \), a word \( u \) is called *block-\( n \)-simple* if each block of \( u \) depends on at most \( n \) variables. For example, the word \( aabbab_1bcct_2caca \) is block-2-simple. Evidently, every word is block-\( n \)-simple for some \( n \geq 0 \). It is also
evident, that a word \( u \) is block-0-simple if and only if \( u = t_1t_2 \ldots t_k \) for some \( k \geq 0 \). The following two facts can be easily verified and are needed only to prove Lemma 3.3.

**Fact 3.1.** Let \( u \) be a word that contains only letters \( a \) and \( b \). If \( u \) contains an occurrence of \( a \) that precedes an occurrence of \( b \) then \( u \) contains \( ab \) as a subword.

**Fact 3.2.** (i) For any set of words \( W \) we have \( W \preceq \ ytxy \) if and only if \( W \) contains a word of the form \( abPab \) for some possibly empty word \( P \) and some distinct letters \( a \) and \( b \).

(ii) If \( W \) is a set of block-2-simple words then \( W \preceq \ ytxy \) if and only if \( W \) contains a word of the form \( abPba \) for some possibly empty word \( P \) and some distinct letters \( a \) and \( b \).

As in [3], the words \( x_1x_2 \ldots x_n \) and \( x_nx_{n-1} \ldots x_1 \) are denoted by \( [Xn] \) and \( [nX] \) respectively. If \( \pi \) is a permutation then we denote \([Xk\pi] = x_1x_2\pi \ldots x_{kn}. The word \( x_1y_1x_2y_2 \ldots x_ny_n \) is denoted by \([XYn] \). We use \( U \) (\( V \)) denote the word obtained from a word \( U \) by inserting a linear letter after (before) each occurrence of each variable in \( U \). For example, \([Zn] = z_1z_2t \ldots z_n \).

Row 1 in Table 1.

| set \( I \) | identity \( U_n \approx V_n \) for \( n > 1 \) | set \( N \) |
|-------------|----------------------------------|-------------|
| \( xytxy \)  | \([XYn][Yn][Xn] \approx [Yn][Xn][Yn] \) | \( xytxy \) |
| \( xytyx \)  | \( y[Xn][y][yX] \approx [Xn][y][nX]y \) | \( xytxy \) |
| \( xtytxy \) | \( [Xn][Xn\rho] \approx [Xn\rho][Xn] \) | \( xtytxy \) |
| \( xxyy \)   | \( [YPn]^t[x][ZQn][xy][PRn][y][QRn] \approx \) | \( xxyy \)           |
| \( xtyty \)  | \( [ZPn]^t[x][ZQn][xy][PRn][y'][QRn] \approx \) | \( xxyy \)           |
| \( xxyy \)   | \( xtyxy \approx [Xy][xty]^t[n][A] \approx [yx][An][xty][n][A] \) | \( xxyy \) |
| \( xxyy \)   | \( \{yytx^kx^d[y][0 < d < k] \} \approx [y][xty]^t[n][A] \approx [yx][An][xty][n][A] \) | \( xxyy \) |

Table 1: Seven NFB intervals \([I, B^2/N]\)

**Lemma 3.3.** If \( B^2 \) is the set of all block-2-simple words and \( N \) is one of the seven finite sets of words listed in Table 1, then \( S(B^2/N) \models \{ U_n \approx V_n \} \) for \( n > 1 \), where the identity \( U_n \approx V_n \) is taken from the same row of Table 1 as set \( N \).

**Proof.** Row 1 in Table 1. Each unstable pairs of variables in \( U_n \approx V_n \) is of the form \( \{x_i, y_j\} \) for some \( 1 \leq i, j \leq n \). If \( \{x_i, y_j\} \) is an unstable pair in \( U_n \approx V_n \), then \( U_n \) deletes to \( x_iy_jy_jx_i \). Let \( \Theta : \mathfrak{A} \rightarrow \mathfrak{A}^* \) be a substitution so that \( \Theta(x_i) \) contains some letter \( a \) and \( \Theta(y_j) \) contains \( b \neq a \). If \( \Theta(U_n) \) is a block-2-simple word, then by Fact 3.1 \( \Theta([XYn]) \) contains \( ab \) as a subword. Similarly, \( \Theta([Yn][Xn]) \) contains \( ba \) as a subword. Then \( \Theta(U_n) \) contains a subword \( abPba \) for some possibly empty empty...
Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(x(t))$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, the word $\Theta([Xn])$ contains $ab$ as a subword. Similarly, $\Theta([Xn\rho])$ contains either $ab$ or $ba$ as a subword. So, $\Theta(U_n)$ contains a subword $abPab$ for some possibly empty word $P$. Then by Fact 3.2, we have $\Theta(U_n) \preceq xytyx$. By symmetric arguments, we show that if $\Theta(V_n)$ is a block-2-simple word then $\Theta(V_n) \preceq xytyx$.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xytyx\})$ satisfies the identity $\{U_n \approx V_n|n > 1\}$ in Row 2 of Table 1.

Row 2 in Table 1. The only unstable pairs of variables in $U_n \approx V_n$ are $\{x_i, y_j\}$, $i = 1, \ldots, n$. Fix some $1 \leq i < j \leq n$. Let $\Theta$ be a substitution so that $\Theta(x_i)$ contains some letter $a$ and $\Theta(x_j)$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, the word $\Theta([Xn])$ contains $ab$ as a subword. Similarly, $\Theta([Xn\rho])$ contains either $ab$ or $ba$ as a subword. So, $\Theta(U_n)$ contains a subword $abPab$ or $abPba$ for some possibly empty word $P$. Then by Fact 3.2, we have that either $\Theta(U_n) \preceq xytyx$ or $\Theta(U_n) \preceq xytyx$.

By symmetric arguments, we show that if $\Theta(V_n)$ is a block-2-simple word then $\Theta(V_n) \preceq xytyx$.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xytyx\})$ satisfies the identity $\{U_n \approx V_n|n > 1\}$ in Row 3 of Table 1.

Row 3 in Table 1. Let $\{x_i, x_j\}$, $1 \leq i < j \leq n$ be an unstable pair of variables in $U_n \approx V_n$. Let $\Theta$ be a substitution so that $\Theta(x_i(t))$ contains some letter $a$ and $\Theta(x_j(t))$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, the word $\Theta([Xn])$ contains $ab$ as a subword. Similarly, $\Theta([Xn\rho])$ contains either $ab$ or $ba$ as a subword. So, $\Theta(U_n)$ contains a subword $abPab$ or $abPba$ for some possibly empty word $P$. Then by Fact 3.2, we have that either $\Theta(U_n) \preceq xytyx$ or $\Theta(U_n) \preceq xytyx$.

By symmetric arguments, we show that if $\Theta(V_n)$ is a block-2-simple word then $\Theta(V_n) \preceq xytyx$.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xytyx\})$ satisfies the identity $\{U_n \approx V_n|n > 1\}$ in Row 4 of Table 1.

Row 4 in Table 1. The only unstable pair of variables in $U_n \approx V_n$ is $\{x, y\}$. Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(y)$ contains $b \neq a$.

First we suppose that $\Theta(U_n)$ is a block-2-simple word. If $\Theta([ZQn])$ contains $b$ or $\Theta([PRn])$ contains $a$ then by Fact 3.1, the word $\Theta(U_n)$ contains a subword $abCba$ for some possibly empty word $C$. Therefore, $\Theta(U_n) \preceq xytyx$ by Fact 3.2. If $\Theta([ZQn]) = a^k$ for some $k \geq 0$ and $\Theta([PRn]) = b^q$ for some $q \geq 0$ then $\Theta(U_n) \preceq xxyy$.

Now we suppose that $\Theta(U_n)$ is a block-2-simple word. Then, in view of Fact 3.1, the word $\Theta([ZQn]y[PRn]y)$ contains a subword $abCba$ for some possibly empty word $C$. Therefore, $\Theta(U_n) \preceq xytyx$ by Fact 3.2.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xxyy, xytyx\})$ satisfies the identity $\{U_n \approx V_n|n > 1\}$ in Row 5 of Table 1.

Row 5 in Table 1. The only unstable pair of variables in $U_n \approx V_n$ is $\{x, y\}$. Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(y)$ contains $b \neq a$.

If $\Theta(U_n)$ is a block-2-simple word then by using similar arguments one can show that $\Theta(U_n) \preceq xytyx$.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xytyx\})$ satisfies the identity $\{U_n \approx V_n|n > 1\}$ in Row 5 of Table 1.

Row 6 in Table 1. The only unstable pair of variables in $U_n \approx V_n$ is $\{x, y\}$.  


Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(y)$ contains letter $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, the content of $\Theta(xy[An]yx)$ is $\{a, b\}$. Now it is easy to see that modulo renaming letters the word $\Theta(xy[An]yx)$ contains either $ababa$ or $ab^ma$ for some $m > 1$ as a subword. Therefore, $\Theta(U_n) \leq xyxyx$ or $\Theta(U_n) \leq xy^m x$ for some $m > 1$. If $\Theta(V_n)$ is a block-2-simple word, then by symmetry $\Theta(V_n) \leq xyxyx$ or $\Theta(V_n) \leq xy^m x$ for some $m > 1$.

Lemma 2.4 implies that for each $n > 1$, monoid $S(B^2/\{xyxyx\} \cup \{xy^m x \mid m > 1\})$ satisfies the identity $\{U_n \approx V_n \mid n > 1\}$ in Row 6 of Table 1.

Row 7. Fix some $m > 2$. Each unstable pairs of variables in $U_n \approx V_n$ is of the form $\{x, y\}$ or $\{x, z\}$ or $\{x, p_i\}$ for some $1 \leq i \leq n$.

Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(y)$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, $\Theta(x^{m-1} y p^2 \ldots p^2 z x)$ contains $a V b a$ as a subword for some word $V \in \{a, b\}^+$. Then $\Theta(U_n) \leq b t a b a \in N$.

Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(z)$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, $\Theta(x^{m-1} y p^2 \ldots p^2 z x)$ contains $a b V a$ as a subword for some word $V \in \{a, b\}^*$. Then $\Theta(U_n) \leq a b t a b \in N$.

Let $\Theta$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(p_i)$ contains $b \neq a$. If $\Theta(U_n)$ is a block-2-simple word, then by Fact 3.1, $\Theta(x^{m-1} y p^2 \ldots p^2 z x)$ contains $a b V b a$ as a subword for some word $V \in \{a, b\}^*$. Then $\Theta(U_n) \leq a b t a b \leq x t y t x y \in N$.

Lemma 2.4 implies that for each $m > 2$ and $n > 1$, monoid $S(B^2/\{x^m y t y, x t y t x y, x t y t x y\})$ satisfies the identity $\{U_n \approx V_n \mid n > 1\}$ in Row 7 of Table 1. \qed

If $W_1 \subseteq W_2$ are sets of words then we use $[W_1, W_2]$ to refer to the interval between $\text{var}(W_1)$ and $\text{var}(W_2)$ in the lattice of all semigroup varieties.

**Theorem 3.4.** Every monoid in each of the following intervals is NFB:

(i) $\{\{x t y t x y\}, B^2/\{x t y t x y\}\}$;

(ii) $\{\{x t y t x y\}, B^2/\{x t y t x y\}\}$;

(iii) $\{\{x t y t x y, x t y t x y\}, B^2/\{x t y t x y, x t y t x y\}\}$;

(iv) $\{\{x t y t x y\}, B^2/\{x t y t x y\}\}$;

(v) $\{\{x t y t x y, x t y t x y\}, B^2/\{x t y t x y, x t y t x y\}\}$;

(vi) $\{\{x t y t x y, x t y t x y, x t y t x y\}, B^2/\{x t y t x y, x t y t x y, x t y t x y\}\}$;

(vii) $\{\{x t y t x y, x t y t x y, x t y t x y\}, B^2/\{x t y t x y, x t y t x y, x t y t x y\}\}$.

$k > 2$.

**Proof.** (i) If $S \in \{\{x t y t x y\}, B^2/\{x t y t x y\}\}$ then by Lemma 3.3 for each $n > 1$, $S$ satisfies the identity $U_n \approx V_n$ in Row 1 of Table 1. Since the word $x t y t x y$ is an isoterm for $S$, monoid $S$ is NFB by Lemma 4.4 in [3].

(ii) If $S \in \{\{x t y t x y\}, B^2/\{x t y t x y\}\}$ then by Lemma 3.3 for each $n > 1$, $S$ satisfies the identity $U_n \approx V_n$ in Row 2 of Table 1. Since the word $x t y t x y$ is an isoterm for $S$, monoid $S$ is NFB by Lemma 5.2 in [11].

(iii) If $S \in \{\{x t y t x y, x t y t x y\}, B^2/\{x t y t x y, x t y t x y\}\}$ then by Lemma 3.3 for each $n > 1$, $S$ satisfies the identity $U_n \approx V_n$ in Row 3 of Table 1. Since the words $x t y t x y$ and $x t y t x y$ are isotems for $S$, monoid $S$ is NFB by Lemma 5.4 in [2].
(iv) If \( S \in \{xtxyty\}, B^2/\{xyxy, ytxy\} \) then by Lemma 3.3 for each \( n > 1 \), \( S \) satisfies the identity \( U_n \approx V_n \) in Row 4 of Table I. Since the word \( xtxyty \) is an isoterm for \( S \), monoid \( S \) is NFB by Theorem 4.6 (row 3 in Table I) in [12].

(v) If \( S \in \{xxxy, ytxytxy\}, B^2/\{xytxy, ytxytxy\} \) then by Lemma 3.3 for each \( n > 1 \), \( S \) satisfies the identity \( U_n \approx V_n \) in Row 5 of Table I. Since the words \( xxxy \) and \( ytxytxy \) are isoterms for \( S \), monoid \( S \) is NFB by Theorem 4.6 (row 4 in Table I) in [12].

(vi) If \( S \in \{xtxyty, ytxytxy, ytxyty\}, B^2/\{xytxy, ytxytxy\} \) then by Lemma 3.3 for each \( n > 1 \), \( S \) satisfies the identity \( U_n \approx V_n \) in Row 6 of Table I. Since the words \( xtxyty \), \( ytxytxy \) and \( ytxyty \) are isoterms for \( S \), monoid \( S \) is NFB by Theorem 4.6 (row 5 in Table I) in [12].

(vii) If \( S \in \{xxyy\} \cup \{ytyx^dtx^{k-d}, x^{k-d}tx^dty\} \) for some \( k > 2 \), then by Lemma 3.3 for each \( n > 1 \), \( S \) satisfies the identity \( U_n \approx V_n \) in Row 7 of Table I. Since all the words in \( \{xxyy\} \cup \{ytyx^dtx^{k-d}, x^{k-d}tx^dty\} \) are isoterms for \( S \), monoid \( S \) is NFB by Theorem 4.6 (row 8 in Table I) in [12].

Theorem 3.4(i)-(iii) immediately implies the following.

**Corollary 3.5.** Let \( W \) be a set of block-2-simple words so that \( W \subseteq \{xytxy, ytxytxy\} \). Then either \( W \) is NFB or \( W \subseteq \{xytxy, ytxytxy\} \).

Corollary 3.5 and Theorem 3.4(vi) immediately implies the following.

**Corollary 3.6.** Let \( W \) be a set of block-2-simple words so that \( W \subseteq \{xytxy, ytxytxy, xtxyty\} \). Then either \( W \) is NFB or \( W \subseteq xyxyx \) or \( W \subseteq xy^m x \) for some \( m > 1 \).

**Corollary 3.7.** Let \( S \) be a monoid so that \( S \in \{xtxyty, B^2/\{xytxy\}\} \) or \( S \in \{xtxyty, B^2/\{ytxy\}\} \). Then either \( S \) is NFB or \( S \in \{xxyy, B^2/\{xytxy, ytxytxy\}\} \).

**Proof.** Notice that \( xytxy \subseteq \{xytxy, ytxytxy\} \) and \( ytxytxy \subseteq \{ytxytxy, ytxy\} \). Since one of the words \( \{ytxytxy, ytxy\} \) is not an isoterm for \( S \), neither \( xytxy \) nor \( ytxytxy \) is an isoterm for \( S \). Theorem 3.4(iv) implies that either \( S \) is NFB or the word \( xxyy \) is an isoterm for \( S \). Now Theorem 3.4(v) implies that either \( S \) is NFB or the word \( xytxytxy \) is not an isoterm for \( S \). The dual argument shows that either \( S \) is NFB or the word \( xytxytxy \) is not an isoterm for \( S \). Therefore, \( S \in \{xxyy, B^2/\{xytxy, ytxytxy\}\} \). □

## 4 Words with two non-linear letters and long blocks are NFB

Let \( u \) be a word containing the letters \( a \) and \( b \). Following Definition 2.4 in [1], we use \( \tilde{u} \) to denote the word obtained from \( u \) by replacing all maximal subwords of \( u \) not containing the letters \( a \) or \( b \) by linear letters and by replacing all subwords of the form \( ab \) by words of the form \( atb \), where \( t \) is a linear letter.
Lemma 4.1. [1] Theorem 2.7] Let \( w = w_1 a^{\alpha_1} b^{\beta_1} w_2 a^{\alpha_2} p b^{\beta_2} w_3 \) be a word such that \( a \) and \( b \) are letters, \( p, w_1, w_2 \) and \( w_3 \) are possibly empty words and \( \alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \) are maximal. If both \( w \) and \( xytz \) are isoterms for a monoid \( S \) and for each \( n > 0 \) the word \( w = \tilde{w}_1 a^{\alpha_1} [Xn] b^{\beta_1} \tilde{w}_2 a^{\alpha_2} t [nX] b^{\beta_2} \tilde{w}_3 \) is not an isoterms for \( S \), then \( S \) is NFB.

[Theorem 2.7] in [1] is a modified and generalized version of Lemma 5.3 in [11] that we used to show that a long word in two letters is NFB. Now we are going to use [1] Theorem 2.7] to show that a word with a long block in two letters is NFB.

Lemma 4.2. Let \( U \) be a word such that all letters in \( U \) other than \( a \) and \( b \) are linear. If \( U \) contains a subword \( ab^\beta a \) for some \( \beta > 1 \) then \( U \) is NFB.

Proof. If \( U \) does not contain any occurrence of \( b \) outside of the word \( ab^\beta a \) then \( U \) is NFB by Corollary 3.3. So, without loss of generality we assume that \( U = u_1 a^{\alpha_1} b^{\beta_1} a^{\alpha_2} v b^{\beta_2} u_2 \) or \( U = u_1 a^{\alpha_1} b^{\beta_1} a^{\alpha_2} w a^{\alpha_3} v b^{\beta_2} u_2 \).

In both cases, \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 > 0 \), \( \alpha_1 \) and \( \beta_1 \) are maximal, \( u_1, v \) and \( u_2 \) are possibly empty words so that if \( v \) is not empty then \( v \) contains only linear letters. The word \( w \) starts and begins with a linear letter and does contain any occurrences of \( b \). Each possibility can be handled by using Lemma 4.1 in a similar way.

If \( U = u_1 a^{\alpha_1} b^{\beta_1} a^{\alpha_2} v b^{\beta_2} u_2 \), then we use Lemma 4.1 for \( w_1 = u_1, \ w_2 = 1, \ p = v \) and \( w_3 = u_2 \) and show that for each \( n > 0 \) monoid \( S = S(\{U\}) \) satisfies the following identity:

\[
\begin{align*}
\mathbf{u}_n &= \mathbf{\tilde{u}}_1 x^{\alpha_1}[An] y^{\beta_1} x^{\alpha_2} t_1 [nA] t_2 y x^{\alpha_3} \mathbf{\tilde{u}}_2 = v_n,
\end{align*}
\]

where \( u_1 \) and \( u_2 \) are written in \( x \) and \( y \) instead of \( a \) and \( b \).

If \( U = u_1 a^{\alpha_1} b^{\beta_1} a^{\alpha_2} w a^{\alpha_3} v b^{\beta_2} u_2 \), then we use Lemma 4.1 for \( w_1 = u_1, \ w_2 = a^{\alpha_2} w, \ p = v \) and \( w_3 = u_2 \) and show that for each \( n > 0 \) semigroup \( S = S(\{U\}) \) satisfies the following identity:

\[
\begin{align*}
\mathbf{u}_n &= \mathbf{\tilde{u}}_1 x^{\alpha_1}[An] y^{\beta_1} x^{\alpha_2} w x^{\alpha_3} t_1 [nA] t_2 y x^{\alpha_3} \mathbf{\tilde{u}}_2 = v_n,
\end{align*}
\]

where \( u_1 \) and \( u_2 \) are written in \( x \) and \( y \) instead of \( a \) and \( b \).

Notice that for each \( n > 0 \), \( \{x, y\} \) is the only unstable pair of variables in \( u_n \approx v_n \). Let \( \Theta : \mathcal{A} \to \mathcal{A}^* \) be a substitution so that \( \Theta(x) \Theta(y) \neq \Theta(y) \Theta(x) \). Then \( \Theta(x) \) contains, say, \( a \) and \( \Theta(y) \) contains \( b \) or visa versa.

Let \( m \) denote the total number of occurrences of non-linear letters \( (a \text{ and } b) \) in \( U \). Notice that \( x \) occurs in \( u_n \) and \( v_n \) the same number of times as \( a \) in \( U \) and the number of occurrences of \( y \) in \( u_n \) and \( v_n \) is one less than the number of occurrences of \( b \) in \( U \). If \( \Theta([An]) \neq \epsilon \) then \( \Theta(u_n) \) ( \( \Theta(v_n) \)) contains at least \( m + 1 \) occurrences of non-linear letters. Therefore, we can assume that \( \Theta([An]) = \Theta([nA]) = \epsilon \).

If \( \Theta(x) \) contains \( a \) then \( \Theta(x) = a \) and \( \Theta(y) = b \). But then neither \( \Theta(u_n) \) nor \( \Theta(v_n) \) has the word \( b^\beta \) between \( a^{\alpha_1} \) and \( a^{\alpha_2} \).

If \( \Theta(x) \) contains \( b \) then \( \Theta(x) = b \) and \( \Theta(y) = a \). In this case \( \text{occ}_U(a) = \text{occ}_{u_n}(x) = \text{occ}_{v_n}(x) \leq \text{occ}_U(b) \) and \( \text{occ}_U(b) = \text{occ}_{u_n}(y) + 1 = \text{occ}_{v_n}(y) + 1 \leq \text{occ}_U(a) + 1 \). So, either \( \text{occ}_U(b) = \text{occ}_U(a) \) or \( \text{occ}_U(b) = \text{occ}_U(a) + 1 \).

If \( \text{occ}_U(b) = \text{occ}_U(a) \) then \( \text{occ}_{u_n}(x) = \text{occ}_{v_n}(x) = \text{occ}_{u_n}(y) + 1 = \text{occ}_{v_n}(y) + 1 \), then:

(i) The image of no letter other than \( x \) contains \( b \).
4.1. satisfies the identity $S$ letters, the word $\Theta(v)$ nor suffix of $U$ contain that $\Theta(u)$ once.

Proof. We have that $a > q > 0$ so that $a$ contain $n$ once. Let $\Theta(u_n)$ and $\Theta(v_n)$ be a word such that all letters in $U$ other than $a$ and $b$ are linear. If $\Theta(u_n)$ (a, b) nor $\Theta(v_n)$ (a, b) is a subword of $U$, then $\Theta(u_n)$ (a, b) can be neither prefix nor suffix of $U(a, b)$. Since $\Theta(v_n)(a, b)$ and $U(a, b)$ start and end with different letters, the word $\Theta(u_n)(a, b)$ can be neither prefix nor suffix of $U(a, b)$.

Overall, neither $\Theta(u_n)$ nor $\Theta(v_n)$ is a subword of $U$. By Lemma 2.1 monoid $S$ satisfies the identity $u_n \approx v_n$ for each $n > 0$. Therefore, $U$ is NFB by Lemma 4.1.

Lemma 4.3. Let $U$ be a word such that all letters in $U$ other than $a$ and $b$ are linear. If $U$ contains a subword ababa then $U$ is NFB.

Proof. We have that $U = u_1a^pbaba^q u_2$ for some possibly empty words $u_1$ and $u_2$ so that $p, q > 0$ are maximal. We use Lemma 4.1 for $w_1 = u_1$, $w_2 = p = \epsilon$ and $w_3 = a^q u_2$.

Let us check that for each $n > 0$ monoid $S = S(\{U\})$ satisfies the following identity:

$u_n = \tilde{u}_1 x^p [nA] x t_1 [nA] ^{2} x y x^q \tilde{u}_2 \approx \tilde{u}_1 x^p [nA] x t_1 [nA] ^{2} x y \tilde{u}_2 = v_n$, where $u_1$ and $u_2$ are written in $x$ and $y$ instead of $a$ and $b$.

Notice that for each $n > 0$, $\{x, y\}$ is the only unstable pair of variables in $u_n \approx v_n$. Let $\Theta: \mathcal{A} \rightarrow \mathcal{A}^*$ be a substitution so that $\Theta(x)\Theta(y) \neq \Theta(y)\Theta(x)$. Then $\Theta(x)$ contains, say, $a$ and $\Theta(y)$ contains $b$ or visa versa.

Case 1: Letter $b$ occurs twice in $U$.

Notice that letter $a$ occurs $m \geq 3$ times in $U$. Since letter $x$ occurs $m$ times in $u_n$ and $v_n$, we have that $\Theta(x) = a$. If $\Theta([nA]) \neq \epsilon$ then $\Theta([nA])$ and $\Theta([nA])$ must contain $b$ and $\Theta(u_n)$ ( $\Theta(v_n)$ ) would contain at least three $b$-s. Therefore, $\Theta([nA]) = \Theta([nA]) = \epsilon$. But then neither $\Theta(u_n)$ nor $\Theta(v_n)$ has a $b$ between $a^p$ and $a$.

Case 2: Letter $b$ occurs at least three times in $U$.

Let $m$ denote the total number of occurrences of non-linear letters ($a$ and $b$) in $U$. Notice that $x$ occurs in $u_n$ and $v_n$ the same number of times as $a$ in $U$ and the number of occurrences of $y$ in $u_n$ and $v_n$ is one less than the number of occurrences of $b$ in $U$. If $\Theta([nA]) \neq \epsilon$ then $\Theta(u_n)$ ( $\Theta(v_n)$ ) contains at least $m + 1$ occurrences of non-linear letters. Therefore, we can assume that $\Theta([nA]) = \Theta([nA]) = \epsilon$.

If $\Theta(x)$ contains $a$ then $\Theta(x) = a$. But then neither $\Theta(u_n)$ nor $\Theta(v_n)$ has a $b$ between $a^p$ and $a$. 

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If $\Theta(x)$ contains $b$ then $\Theta(x) = b$ and $\Theta(y) = a$. In this case $occ_U(a) = occ_{\text{lin}}(a) = occ_{\text{non-lin}}(x) \leq occ_U(b)$ and $occ_U(b) = occ_{\text{lin}}(y) + 1 = occ_{\text{non-lin}}(y) + 1 \leq occ_U(a) + 1$. So, either $occ_U(b) = occ_U(a)$ or $occ_U(b) = occ_U(a) + 1$.

If $occ_U(b) = occ_U(a) = occ_{\text{lin}}(x) = occ_{\text{non-lin}}(y) + 1 = occ_{\text{non-lin}}(y) + 1$, then:

(i) The image of no letter other than $x$ contains $b$. 

(ii) In addition to $\Theta(y)$ the image of at most one linear letter in $u_n$ may contain $a$ once.

If $occ_U(b) = occ_U(a) + 1 = occ_{\text{lin}}(x) + 1 = occ_{\text{non-lin}}(y) + 1 = occ_{\text{non-lin}}(y) + 1$, then:

(i) The image of no other letter than $y$ contains $a$.

(ii) In addition to $\Theta(x)$ the image of at most one linear letter in $u_n$ may contain $b$ once.

If $\Theta(u_n)$ ($\Theta(v_n)$) is a subword of $U$ then in view of Conditions (i)-(ii) we have that $\Theta(u_n)(a, b)$ ($\Theta(v_n)(a, b)$) is a prefix or suffix of $U(a, b)$. Since $\Theta(u_n)(a, b)$ and $U(a, b)$ start and end with different letters, the word $\Theta(u_n)(a, b)$ can be neither prefix nor suffix of $U(a, b)$. Since $\Theta(v_n)(a, b)$ and $U(a, b)$ start with different letters, the word $\Theta(v_n)(a, b)$ can not be a prefix of $U(a, b)$. If the word $u_2$ contains some non-linear letters, then the words $\Theta(v_n)(a, b)$ and $U(a, b)$ end with different letters and consequently, $\Theta(v_n)(a, b)$ can not be a suffix of $U(a, b)$. If the word $u_2$ does not contain any linear letters, then the word $\Theta(v_n)(a, b)$ ends with $b^{p+1+q}a$ but the word $U(a, b)$ ends with $aba^q$ and consequently, the word $\Theta(v_n)(a, b)$ is not a suffix of $U(a, b)$.

Overall, neither $\Theta(u_n)$ nor $\Theta(v_n)$ is a subword of $U$. By Lemma 2.1 monoid $S$ satisfies the identity $u_n \approx v_n$ for each $n > 0$. Therefore, $U$ is NFB by Lemma 4.1.

5 All solutions to the equations in two variables in the free monoid $\mathcal{A}^*$ and some identities with two non-linear variables

Lemma 5.1. [4, Lemma 5.1, Sect. 11] Let $w_1$ and $w_2$ be some non-empty words. If $uw_1 = w_2u$ for some $u \in \mathcal{A}^*$ then for some words $X, Y \in \mathcal{A}^*$ and $k \geq 0$ we have that $w_1 = XY$, $w_2 = YX$ and $u = (XY)^kX$.

We use $|u|$ to denote the length of a word $u$. The next lemma generalizes Corollary 5.3 in [4] that says that two words in a free monoid commute if and only if they are powers of the same word.

Theorem 5.2. Let $u \approx v$ be a non-trivial identity in two variables $x$ and $y$ and $\Theta : \mathcal{A} \to \mathcal{A}^*$ be a substitution. If $\Theta(u) = \Theta(v)$ then both $\Theta(x)$ and $\Theta(y)$ are powers of the same word.

Proof. We prove it by induction on the maximal length of $\Theta(x)$ and $\Theta(y)$. Without loss of generality we may assume that $|\Theta(x)| \leq |\Theta(y)|$. The statement is obvious if
\(|\Theta(y)| = 1\) and we assume that \(|\Theta(y)| = n > 1\). If \(|\Theta(x)| = |\Theta(y)|\) then \(\Theta(x) = \Theta(y)\) and we are done. So, we may assume that \(|\Theta(x)| < |\Theta(y)|\). If \(\Theta(x)\) is not a prefix (suffix) of \(\Theta(y)\) then \(\Theta(u)\) and \(\Theta(v)\) have different prefixes (suffixes) and consequently, \(\Theta(u) \neq \Theta(v)\). If \(\Theta(x)\) is the empty word then we are done. So, we may assume that \(\Theta(x)\) is a proper prefix (suffix) of \(\Theta(y)\), i.e. \(\Theta(y) = \Theta(x)w_1 = w_2\Theta(x)\) for some non-empty words \(w_1, w_2\). Then by Lemma 5.1 we have that \(\Theta(x) = (XY)^kX\) and \(\Theta(y) = (XY)^{k+1}X\) for some \(k \geq 0\) and some words \(X, Y \in A^*\). If one of the words \(X\) or \(Y\) is empty then we are done. So, we may assume that \(0 < |X|, |Y| < n\). If \(\Gamma(x) = (xy)^kx\) and \(\Gamma(y) = (xy)^{k+1}x\) then the identity \(\Gamma(u) \approx \Gamma(v)\) is non-trivial. If \(\Delta(x) = X\) and \(\Delta(y) = Y\) then \(\Delta_\Theta = \Theta\) on \(\{x, y\}^*\). If we apply the induction hypothesis to the identity \(\Gamma(u) \approx \Gamma(v)\) and substitution \(\Delta\), we get that both \(X\) and \(Y\) are powers of the same word. Consequently, \(\Theta(x)\) and \(\Theta(y)\) are powers of the same word. □

We use \(\operatorname{Lin}(u)\) to denote the set of all linear variables in a word \(u\). An identity \(u \approx v\) is called block-balanced if for each variable \(x \in A\), we have \(u(x, \operatorname{Lin}(u)) = v(x, \operatorname{Lin}(u))\). Evidently, an identity \(u \approx v\) is block-balanced if and only if it is balanced, the order of linear letters is the same in \(u\) and \(v\) and each block in \(u\) is a permutation of the corresponding block in \(v\).

**Corollary 5.3.** Let \(W = W^c\) be a set of words and \(u \approx v\) be a non-trivial block-balanced identity with two non-linear variables \(x \neq y\). Then \(S(W) \models u \approx v\) if and only if for every substitution \(\Theta : A \to A^*\) so that \(\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)\), neither \(\Theta(u)\) nor \(\Theta(v)\) belongs to \(W\).

**Proof.** (i) \(\rightarrow\) (ii) Follows immediately from Lemma 2.1.

(ii) \(\rightarrow\) (i) Suppose that \(S(W) \models u \approx v\). To obtain a contradiction, let us assume that for some substitution \(\Theta : A \to A^*\) so that \(\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)\), say, the word \(\Theta(u)\) belongs to \(W\). Since the pair \(\{x, y\}\) is unstable in a block-balanced identity \(u \approx v\), for some corresponding blocks \(B\) in \(u\) and \(B'\) in \(v\) we have that \(B = B(x, y) \neq B'(x, y) = B'\). Since \(\Theta(u)\) belongs to \(W\) and \(S(W) \models u \approx v\) we have that \(\Theta(u) = \Theta(v)\). Since the identity \(u \approx v\) is block-balanced, we have that \(\Theta(B) = \Theta(B')\). Since \(\Theta(x)\Theta(y) \neq \Theta(y)\Theta(x)\), Theorem 5.2 implies that \(B = B(x, y) = B'(x, y) = B'\). To avoid a contradiction, we conclude that for every substitution \(\Theta : A \to A^*\) so that \(\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)\), neither \(\Theta(u)\) nor \(\Theta(v)\) belongs to \(W\).

We say that a pair of variables \(\{x, y\}\) is \(b\)-unstable in a word \(u\) with respect to a semigroup \(S\) if \(S\) satisfies a block-balanced identity of the form \(u \approx v\) so that \(u(x, y) \neq v(x, y)\). Otherwise, we say that \(\{x, y\}\) is \(b\)-stable in \(u\) with respect to \(S\).

**Corollary 5.4.** Let \(W = W^c\) be a set of words and \(u\) be a word with exactly two non-linear variables \(x\) and \(y\). Suppose that one can find a substitution \(\Theta : A \to A^*\) so that \(\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)\) and \(\Theta(u) \in W\). Then the pair \(\{x, y\}\) is \(b\)-stable in \(u\) with respect to \(S(W)\).
Now we consider some special block-balanced identities with two non-linear variables.

**Fact 5.5.** [12, Fact 3.1] If $xtx$ is an isoterm for a monoid $S$, then

(i) the words $xt_1x_2y$ and $x_1xt_2y$ can only form an identity of $S$ with each other;
(ii) the words $xyt_1xt_2y$ and $yxt_1xt_2y$ can only form an identity of $S$ with each other;
(iii) the words $xt_1yt_2x$ and $x_1yt_2yx$ can only form an identity of $S$ with each other.

We also need the following generalization of Fact 5.5(i).

**Fact 5.6.** If for some $m > 0$ the word $x^m t x$ is an isoterm for a monoid $S$, then the words $x^m t_1 x y t_2 y$ and $x^m t_1 y x t_2 y$ can only form an identity of $S$ with each other.

Facts 5.5 and 5.6 immediately imply the following.

**Fact 5.7.** For each $m > 0$ we have:

\[
x^m t_1 x y t_2 y \sim x^m t_1 y x t_2 y, \quad x y t_1 x t_2 y \sim y x t_1 x t_2 y \quad \text{and} \quad x t_1 y t_2 x \sim x t_1 y t_2 y x.
\]

**Lemma 5.8.** For each $m > 0$ we have \( S(\mathbb{A}^*/\{xytxy, x^m txyty\}) \models \{xytxy \approx yxtxty, x^m txyty \approx x^m txyty\}. \)

**Proof.** The desired statement is an immediate consequence of Lemma 2.4 and the following claim.

**Claim 1.** Let $m > 0$ and $\Theta : \mathbb{A} \to \mathbb{A}^*$ be a substitution so that $\Theta(x)$ contains some letter $a$ and $\Theta(y)$ contains $a \neq b$. Then

(i) $\Theta(xytxy) \preceq xytxy, \Theta(yxtxy) \preceq xytxy$;
(ii) $\Theta(x^m txyty) \preceq x^m txyty$ or $\Theta(x^m txyty) \preceq x^m txyty$;
(iii) $\Theta(x^m txyty) \preceq x^m txyty$ or $\Theta(x^m txyty) \preceq x^m txyty$.

**Proof.** (i) If $\Theta(x)$ is not a power of $a$ or $\Theta(y)$ is not a power of $b$ then $\Theta(xytxy) \preceq xytxy < xytxy$. If $\Theta(x)$ is a power of $a$ and $\Theta(y)$ is a power of $b$ then we still have $\Theta(xytxy) \preceq xytxy$. Similarly, $\Theta(yxtxy) \preceq xytxy$.

(ii) If $\Theta(x)$ is not a power of $a$ or $\Theta(y)$ is not a power of $b$ then $\Theta(x^m txyty) \preceq x^m txyty < x^m txyty$. If $\Theta(x)$ is a power of $a$ and $\Theta(y)$ is a power of $b$ then we have $\Theta(x^m txyty) \preceq x^m txyty$.

(iii) Similarly to Part (ii), if $\Theta(x)$ is not a power of $a$ or $\Theta(y)$ is not a power of $b$ then $\Theta(x^m txyty) \preceq x^m txyty < x^m txyty$. If $\Theta(x)$ is a power of $a$ and $\Theta(y)$ is a power of $b$ then we have $\Theta(x^m txyty) \preceq x^m txyty$.

\[\square\]
6 An algorithm for recognizing finitely based words among words with at most two non-linear variables

The identities $xt_1xyt_2y \approx xt_1yx_1ty$ and $yxt_1xt_2y \approx yxt_1xt_2y$ we denote respectively by $\sigma_\mu$, $\sigma_1$ and $\sigma_2$. Notice that the identities $\sigma_1$ and $\sigma_2$ are dual to each other.

**Lemma 6.1.** [13, Theorem 1.1] Every monoid that satisfies the identities $\sigma_1$ and $\sigma_\mu$ is finitely based.

A word that contains at most one non-linear variable is called almost-linear.

**Lemma 6.2.** [13, Theorem 4.3] Let $S$ be a monoid so that for some $m > 1$, the word $x^m y^m$ is an isoterm for $S$ and $\sigma = \{ \sigma_1, \sigma_2, x^m tyxty \approx x^m tyxty \}$. Suppose also that for each $1 < k \leq m$, $S$ satisfies each of the following dual conditions:

(i) If for some almost-linear word $Ax$ with $\text{occ}_A(x) > 0$ the pair $\{ x, y \}$ is $b$-unstable in $Axy^k$ with respect to $S$ then $S$ satisfies the identity $Axy^{k-1}ty \approx Ayx^{k-2}ty$;

(ii) If for some almost-linear word $yB$ with $\text{occ}_B(y) > 0$ the pair $\{ x, y \}$ is $b$-unstable in $x^byB$ with respect to $S$ then $S$ satisfies the identity $xtx^{k-1}yB \approx xtx^{k-2}yxB$.

Then $S$ is finitely based.

**Lemma 6.3.** Let $U$ be a word with two non-linear letters so that $S(\{ U \}) \in \{ \{ x^m y^m \}, \mathfrak{A}^*/\{ xtytxy, xtytxy \} \}$ for some $m > 1$. Then

(i) modulo renaming letters, $U = Ct_1a^\beta b^\gamma t_2B$ for some possible empty almost-linear words $Ct_1 = C(a, \text{Lin}(C))t_1$ and $t_2B = t_2B(b, \text{Lin}(B))$ so that $\min(\alpha, \beta) \geq m > 1$;

(ii) $S(\{ U \})$ satisfies Conditions (i) and (ii) in Lemma 6.2.

**Proof.** (i) This easily follows from the fact that $U \not\approx x^m y^m$ but neither $xtytxy$ nor $xytxty$ is an isoterm for $S(\{ U \})$. See also Theorem 7.3 for a more general statement.

(ii) Since Conditions (i) and (ii) in Theorem 6.2 are dual, we check only Condition (i). Let $1 < k \leq m$ and $Ax$ be an almost-linear word with $\text{occ}_A(x) > 0$ so that the pair $\{ x, y \}$ is $b$-unstable in $Axy^k$ with respect to $S(\{ U \})$. Let $\Theta : \mathfrak{A} \to \mathfrak{A}^*$ be a substitution so that $\Theta(y)\Theta(x) \neq \Theta(x)\Theta(y)$. Evidently, $\Theta(Ayx^{k-2}ty)$ can not be a subword of $U$. If $\Theta(Ayx^{k-1}ty)$ is a subword of $U = Ct_1a^\alpha b^\beta t_2B$, then $\Theta(x) = a^l$, $\Theta(y) = b^r$ for some $l, r > 0$ and the word $\Theta(Ax)$ is a suffix of $Ct_1a^\alpha$. Let $\Theta'$ be a substitution which coincides with $\Theta$ on all variables other than $y$ and $\Theta'(y) = b$. Since $k \leq \beta$, the word $\Theta'(Axy^k)$ is a subword of $U$. Then by Corollary 5.4 the pair $\{ x, y \}$ is $b$-stable in $Axy^k$ with respect to $S(\{ U \})$. To avoid a contradiction, we conclude that the word $\Theta(Axy^{k-1}ty)$ is not a subword of $U$. So, by Lemma 2.1, we have that $S(\{ U \}) \models Axy^{k-1}ty \approx Ayx^{k-2}ty$. This means that $S(\{ U \})$ satisfies Condition (i) in Theorem 6.2. 

$\blacksquare$
Theorem 6.4. A word $U$ with at most two non-linear variables is finitely based if and only if the monoid $S(\{U\})$ belongs to one of the following dual intervals:

(i) $[\emptyset, \mathbb{A}^*/\{xytxy, xtxty\}]$;
(ii) $[\emptyset, \mathbb{A}^*/\{xtytxy, xtxty\}]$,
or, for some $m > 1$, $S(\{U\})$ belongs to one of the following dual intervals:
(iii) $\{(x^m y^m), \mathbb{A}^*/\{xtytxy, xtxty, x^mtxty\}\}$;
(iv) $\{(x^m y^m), \mathbb{A}^*/\{xtytxy, xtxty, xtxty^m\}\}$.

Proof. Denote $S = S(\{U\})$. If $S \in [\emptyset, \mathbb{A}^*/\{xytxy, xtxty\}]$ then by Lemma 5.8 we have $S \models \{\sigma_1, \sigma_\mu\}$. Consequently, $S$ is finitely based by Lemma 6.1. If $S \in [\emptyset, \mathbb{A}^*/\{xtytxy, xtxty\}]$ then $S$ is finitely based by dual arguments.

If $S \in [(x^m y^m), \mathbb{A}^*/\{xtytxy, xtxty, x^mtxty\}]$ then by Lemma 5.8 we have $S \models \{\sigma_1, \sigma_2, x^mxtxy \approx x^mtytxy \}$. By Lemma 6.3, $S$ satisfies Conditions (i) and (ii) in Lemma 6.2. Therefore, $S$ is finitely based by Lemma 6.2. If $S \in [(x^m y^m), \mathbb{A}^*/\{xtytxy, xtxty, xtxty^m\}]$ then $S$ is finitely based by dual arguments.

Suppose that $S$ does not belong to any of these intervals and consider two cases.

Case 1: $U \leq \{xyttxy, xtxty\}$.

In view of Lemma 3.3, we may assume that $U \leq \{xyttxy, xtxty\}$. Since $U$ is a single word with two non-linear letters, the condition $U \leq \{xyttxy, xtxty\}$ implies that $U \leq xtxty$. Then by Lemma 3.6, the word $U$ contains either $abab$ or $ab^3a$ for some $\beta > 1$ as a subword. Lemmas 3.1 and 3.3 show that $U$ is NFB in each of these cases. So, if $U \leq \{xyttxy, xtxty\}$ then $U$ is NFB.

Case 2: $S \in \{xtytxy, B^2/\{xyttxy\}\}$ or $S \in \{xtytxy, B^2/\{xtxty\}\}$. In view of Lemma 3.7, we may assume that $S \in \{xxyy, B^2/\{xyttxy, xtxty\}\}$. Let $m$ be the maximal so that $U \leq x^m y^m$. Since $U \leq xxyy$, we have $m > 1$. Since $S$ belongs neither to $\{x^m y^m, \mathbb{A}^*/\{xtytxy, xtxty, x^mtxty\}\}$ nor to $\{x^m y^m, \mathbb{A}^*/\{xtytxy, xtxty, xtxty^m\}\}$, we have $U \leq \{yttxxm, x^m xty\}$. Now Lemma 6.3 (i) implies that $U \leq \{yttxxm, x^m+1ytc, x^m+1-dtxty\}[0 < d \leq m]$ but one of the words $\{x^m+1yty, ytxy^m\}$ is not an isoterms for $S$. Theorem 3.4 (vii) or its dual implies that $S$ is non-finitely based. \qed

Here is a reformulation of Theorem 6.4 in terms of isoterms.

Theorem 6.5. Let $U$ be a word with at most two non-linear variables and $m$ be the maximum so that $U \leq x^m y^m$. Then $S(\{U\})$ is finitely based if and only if $U$ satisfies each of the following conditions:

(i) At least two of the words $\{xyttxy, xtytxy, xtxty\}$ are not isoterms for $S(\{U\})$;
(ii) If $U \leq xtxty$ then $m > 1$ and one of the words $\{ytxxm, x^m xty\}$ is not an isoterms for $S(\{U\})$. 


7 Syntactic description of the isoterms for certain varieties and a computation-free way to recognize FB words among words with at most two non-linear variables

We use \( u_i x \) to refer to the \( i^{th} \) from the left occurrence of variable \( x \) in a word \( u \). We use \( \ell_u x \) to refer to the last occurrence of \( x \) in \( u \). The set \( \text{OccSet}(u) = \{ i_u x \mid x \in \mathcal{A}, 1 \leq i \leq \text{occ}_u(x) \} \) of all occurrences of all variables in \( u \) is called the occurrence set of \( u \). As in [13], with each subset \( \Sigma \) of \( \{ \sigma_1, \sigma_\mu, \sigma_2 \} \) we associate an assignment of two Types to all pairs of occurrences of distinct non-linear variables in all words as follows. We say that each pair of occurrences of two distinct non-linear variables in each word is \( \{ \sigma_1, \sigma_\mu, \sigma_2 \}\)-good. If \( \Sigma \) is a proper subset of \( \{ \sigma_1, \sigma_\mu, \sigma_2 \} \), then we say that a pair of occurrences of distinct non-linear variables is \( \Sigma\)-good if it is not declared to be \( \Sigma\-bad \) in the following definition.

**Definition 7.1.** If \( \{ c, d \} \subseteq \text{OccSet}(u) \) is a pair of occurrences of two distinct non-linear variables \( x \neq y \) in a word \( u \) then

(i) pair \( \{ c, d \} \) is \( \{ \sigma_1, \sigma_2 \}\)-bad if \( \{ c, d \} = \{ 1_u x, 1_u y \} \);

(ii) pair \( \{ c, d \} \) is \( \{ \sigma_1, \sigma_\mu \}\)-bad if \( \{ c, d \} = \{ \ell_u x, \ell_u y \} \);

(iii) pair \( \{ c, d \} \) is \( \{ \sigma_1, \sigma_2 \}\)-bad if \( \{ c, d \} = \{ i_u x, \ell_u y \} \).

(iv) pair \( \{ c, d \} \) is \( \sigma_\mu\)-bad if \( \{ c, d \} = \{ 1_u x, 1_u y \} \) or \( \{ c, d \} = \{ \ell_u x, \ell_u y \} \);

(v) pair \( \{ c, d \} \) is \( \sigma_\ell\)-bad if \( c = 1_u x \) or \( d = 1_u y \);

(vi) pair \( \{ c, d \} \) is \( \sigma_1\)-bad if \( c = \ell_u x \) or \( d = \ell_u y \).

We denote the set of all left sides of identities from \( \Sigma \) by \( L_\Sigma \) and the set of all right sides of identities from \( \Sigma \) by \( R_\Sigma \).

**Lemma 7.2.** If \( S \) is a monoid so that \( xtx \) is an isoterm for \( S \) and \( \Sigma \subseteq \{ \sigma_1, \sigma_\mu, \sigma_2 \} \) then the following conditions are equivalent:

(i) \( S \models \Sigma \);

(ii) if a word \( u \) is an isoterm for \( S \) then each adjacent pair of occurrences of two distinct non-linear variables in \( u \) is \( \Sigma\-bad \);

(iii) no word in \( L_\Sigma \) is an isoterm for \( S \);

(iv) no word in \( R_\Sigma \) is an isoterm for \( S \).

**Proof.** (i) \( \to \) (ii) Suppose that \( u \) contains a \( \Sigma\)-good adjacent pair \( \{ c, d \} \subseteq \text{OccSet}(u) \) of occurrences of two distinct non-linear variables. Then one of the identities in \( \Sigma^\delta \) is applicable to \( u \). Therefore, \( S \models u \approx v \) so that the word \( v \) is obtained from \( u \) by swapping \( c \) and \( d \). This contradicts the fact that \( u \) is an isoterm for \( S \). So, we must assume that every adjacent pair of occurrences of two distinct non-linear variables in \( u \) is \( \Sigma\-bad \).

(ii) \( \to \) (iii) follows from the fact that the only adjacent pair of occurrences of two distinct non-linear variables in each word in \( L_\Sigma \) is \( \Sigma\-good \).

(iii) \( \to \) (iv) follows immediately from Fact 5.7.

(iv) \( \to \) (i) follows immediately from Fact 5.7.
Together with Definition 7.1, the following statement gives us explicit syntactic descriptions of monoids of the form $S(W)$ contained in the seven varieties defined by non-empty subsets of $\{\sigma_1, \sigma_\mu, \sigma_2\}$.

**Theorem 7.3.** If $W$ is a set of words and $\Sigma \subseteq \{\sigma_1, \sigma_\mu, \sigma_2\}$ then the following conditions are equivalent:

(i) $S(W) \models \Sigma$;

(ii) every adjacent pair of occurrences of two distinct non-linear variables in each word in $W$ is $\Sigma$-bad;

(iii) for each $u \in L_\Sigma$ we have $W \not\preceq u$;

(iv) for each $u \in R_\Sigma$ we have $W \not\preceq u$.

**Proof.** If $W \preceq xtx$ then the equivalence of the four conditions follows from Lemma 7.2. The property that the word $xtx$ is an isoterm for $S$ is only used in the proof of Lemma 7.2 for verifying the implication (iv) $\rightarrow$ (i). Let us verify this implication when $S = S(W)$ and the word $xtx$ is not an isoterm for $S$. If the word $xtx$ is not an isoterm for $S$ then every word in $W$ is block-1-simple. Consequently, $S \models \{\sigma_1, \sigma_\mu, \sigma_2\}$ by Lemma 2.1. □

If we take $\Sigma = \{\sigma_\mu, \sigma_1, \sigma_2\}$ then Theorem 7.3 immediately implies the following.

**Corollary 7.4.** $S(W) \models \{\sigma_\mu, \sigma_1, \sigma_2\}$ if and only if each word in $W$ is block-1-simple.

We say that a set of words $W$ is hereditary finitely based with respect to the order $\preceq$ if every set of words $W'$ with the property $W \preceq W'$ is finitely based.

**Corollary 7.5.** A set of words $W$ is hereditary finitely based with respect to the order $\preceq$ if and only if it satisfies one of the following dual conditions:

(i) every adjacent pair of occurrences of two non-linear variables $x \neq y$ in each word in $W$ is of the form $\{1u_x, 1u_y\}$;

(ii) every adjacent pair of occurrences of two non-linear variables $x \neq y$ in each word in $W$ is of the form $\{elu_x, elu_y\}$.

In particular, every set of block-1-simple words is finitely based.

**Proof.** If $W$ satisfies Condition (ii), then by Theorem 7.3 and Definition 7.1 (ii), $S(W) \models \{\sigma_1, \sigma_\mu\}$. If $W'$ is a set of words with the property $W \preceq W'$, then by Proposition 2.3, $S(W') \models \{\sigma_1, \sigma_\mu\}$. By the result of Lee (Lemma 6.1), $S(W')$ is finitely based. If $W$ satisfies Condition (i), then $W$ is hereditary finitely based by dual arguments.

Evidently, $W$ satisfies both Conditions (i) and (ii) if and only if each word in $W$ is block-1-simple.

Now suppose that $W$ satisfies neither Conditions (i) nor Condition (ii). Then by Theorem 7.3 and Definition 7.1 (i)-(ii), either $W \preceq xtxty$ or $W \preceq \{xtytx, xtytxy\}$. By the result of Jackson from [2], both $S(\{xtytx\})$ and $S(\{xtytx, xtytxy\})$ are non-finitely based. □
The next theorem gives us a computation-free way to recognize FB words among words with at most two non-linear variables.

**Theorem 7.6.** Let $U$ be a word with at most two non-linear letters $a$ and $b$. Then the word $U$ is finitely based if and only if $U$ is either block-1-simple or contains a single block $B$ so that $B \not\in \{a^n, b^n|n \geq 0\}$ and satisfies one of the following conditions modulo renaming $a$ and $b$.

(i) $B = ab^m$ for some $m > 0$ and $B$ is the first non-empty block of $U$.

(ii) $B = b^ma$ for some $m > 0$ and $B$ is the last non-empty block of $U$.

(iii) $B = a^nb^m$ so that $\min(n,m) = k > 1$, $U$ contains no $a$ to the right of $B$, no $b$ to the left of $B$ and one of the words $\{aU_1abU_2b^k, a^kU_1abU_2b\}$ is not a subword of $U$ for any $U_1, U_2 \in \mathcal{A}$.

**Proof.** Denote $S = S(\{U\})$. In view of Corollary 7.5 and Theorem 7.3, the word $U$ is hereditary finitely based with respect to the order $\preceq$ if $U$ is block-1-simple or contains a single block $B$ so that $B \not\in \{a^n, b^n|n \geq 0\}$ and satisfies either Condition (i) or Condition (ii) if $S \in \{0, \mathcal{A}^*/\{xytxyt, xtxyty\}\}$ or $S \in \{0, \mathcal{A}^*/\{xtxty, xtytxy\}\}$.

Theorem 7.3 for $\Sigma = \{\sigma_1, \sigma_2\}$ and Definition 7.1(iii) imply that

$S \in \{x^ky^k, \mathcal{A}^*/\{xtytxy, xytxt, x^kytxy\}\}$ or $S \in \{x^ky^k, \mathcal{A}^*/\{xytxty, xtytxy, x^kytxy\}\}$

for some $k > 1$ if and only if $U$ contains a single block $B$ so that $B \not\in \{a^n, b^n|n \geq 0\}$ and satisfies Condition (iii).

The rest follows from Theorem 6.4. □

Following Definition 5.1 in [2], we say that a word $U$ is **hereditary finitely based** (HFB) if each subword of $U$ is finitely based. Evidently, if a word $U$ is hereditary finitely based with respect to the order $\preceq$, then $U$ is HFB in the sense of Definition 5.1 in [2].

**Corollary 7.7.** A word $U$ with at most two non-linear variables is FB if and only if $U$ is HFB.

**Proof.** According to Theorem 7.6 if $U$ is finitely based then either $U$ is block-1-simple or $U$ contains a single adjacent pair $\{c, d\} \subseteq \text{OccSet}(U)$ of occurrences of $a$ and $b$ so that either $\{c, d\} = \{1u\sigma_1, 1u\sigma_2\}$ or $\{c, d\} = \{\ell\sigma_1, \ell\sigma_2\}$ or $\{c, d\} = \{\ell\sigma_1, \ell\sigma_2\}$. If $\{c, d\} = \{1u\sigma_1, 1u\sigma_2\}$ or $\{c, d\} = \{\ell\sigma_1, \ell\sigma_2\}$ then by Corollary 7.5, the word $U$ is hereditary finitely based with respect to the order $\preceq$, and consequently, is HFB.

If $\{c, d\} = \{\ell\sigma_1, \ell\sigma_2\}$ then $U$ satisfies Condition (iii) in Theorem 7.6. Then each subword of $U$ is either block-simple or also satisfies Condition (iii). In any case the word $U$ is HFB. □

**Example 7.8.** Let $U = aat_1aabbt_2bb$. Then

(i) the set $\{U, a^ib^i\}$ is FB.

(ii) the word $U$ is NFB.

(iii) each subword of $U$ is FB.
Proof. (i) The set \( \{ U, a^4b^4 \} \) is finitely based by Lemma 6.2 or by Theorem 4.2 (ii) in [13].

(ii) The word \( U \) is NFB by Theorem 6.4 because \( U \not\preceq \{ xxtxyty, ytyxtxt \} \).

(iii) The word \( V = aataaabbb \) is FB by Theorem 6.4 because the word \( aU_1abU_2bb \) is not a subword of \( V \) for any \( U_1, U_2 \in \mathbb{A}^* \). So, each subword of \( U \) is FB by symmetry and Corollary 7.7. \( \square \)

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