Universality in the physics of cold atoms with large scattering length*

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Effective field theories exploit a separation of scales in physical systems in order to perform systematically improvable, model-independent calculations. They are ideally suited to describe universal aspects of a wide range of physical systems. I will discuss recent applications of effective field theory to cold atomic and molecular few-body systems with large scattering length.

1. INTRODUCTION

The Effective Field Theory (EFT) approach provides a powerful framework that exploits the separation of scales in physical systems. Only low-energy (or long-range) degrees of freedom are included explicitly, with the rest parametrized in terms of the most general local (contact) interactions. This procedure exploits the fact that a low-energy probe of momentum $k$ cannot resolve structures on scales smaller than $R \sim 1/k$. (Note that $\hbar = c = 1$ in this talk.) Using renormalization, the influence of short-distance physics on low-energy observables is captured in a small number of low-energy constants. Thus, the EFT describes universal low-energy physics independent of detailed assumptions about the short-distance dynamics. All physical observables can be described in a controlled expansion in powers of $kl$, where $l$ is the characteristic low-energy length scale of the system. The size of $l$ depends on the system under consideration: for a finite range potential, e.g., it is given by the range of the potential. For the systems discussed here, $l$ is of the order of the effective range $r_e$.

In this talk, I will focus on applications of EFT to few-body systems with large S-wave scattering length $a \gg l$. For a generic system, the scattering length is of the same order of magnitude as the low-energy length scale $l$. Only a very specific choice of the parameters in the underlying theory (a so-called fine tuning) will generate a large scattering length. Systems with large scattering length can be found in many areas of physics. Examples are the S-wave scattering of nucleons and of $^4$He atoms. For alkali atoms close to a Feshbach resonance, $a$ can be tuned experimentally by adjusting an external magnetic field.

2. THREE-BODY SYSTEM WITH LARGE SCATTERING LENGTH

In this section, we give a very brief review of the EFT for few-body systems with large scattering length $a$. We will focus on S-waves (For more details, see Ref. [1]).

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For typical momenta \( k \sim 1/a \), the EFT expansion is in powers of \( r_e/a \) so that higher order corrections are suppressed by powers of \( r_e/a \). The leading order corresponds to \( r_e = 0 \). We consider a 2-body system of nonrelativistic bosonic atoms with large scattering length \( a \) and mass \( m \). At sufficiently low energies, the most general Lagrangian for S-wave interactions may be written as:

\[
\mathcal{L} = \psi^\dagger \left( i \partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 - \frac{D_0}{6} (\psi^\dagger \psi)^3 + \ldots ,
\]

where the \( C_0 \) and \( D_0 \) are nonderivative 2- and 3-body interaction terms, respectively. The strength of the \( C_0 \) term is determined by the scattering length \( a \), while \( D_0 \) depends on a 3-body parameter to be introduced below. The dots represent higher-order derivative terms which are suppressed at low-energies. For momenta \( k \) of the order of the inverse scattering length \( 1/a \), the problem is nonperturbative in \( ka \). The exact 2-body scattering amplitude can be obtained analytically by summing the so-called bubble diagrams with the \( C_0 \) interaction term. The \( D_0 \) term does not contribute to 2-body observables. After renormalization, the resulting amplitude reproduces the leading order of the well-known effective range expansion for the atom-atom scattering amplitude:

\[
f_{AA}(k) = \left( \frac{1}{a} - ik \right)^{-1},
\]

where the total energy is \( E = k^2/m \). If \( a > 0 \), \( f_{AA} \) has a pole at \( k = i/a \) corresponding to a shallow dimer with binding energy \( B_2 = 1/(ma^2) \). Higher-order derivative interactions are perturbative and give the momentum-dependent terms in the effective range expansion.

We now turn to the 3-body system. Here, it is useful to introduce an auxiliary field for the two-atom state (see Ref. [1] for details). At leading order, the atom-dimer scattering amplitude is given by the integral equation shown in Fig. 1. A solid line indicates a single atom and a double line indicates an interacting two-atom state (including rescattering corrections). The integral equation contains contributions from both the 2-body and the 3-body force terms. The inhomogeneous term is given by the first two diagrams on the right-hand side: the one-atom exchange diagram and the 3-body force. The integral equation simply sums these diagrams to all orders. After projecting onto S-waves, we obtain the equation

\[
\mathcal{T}(k, p; E) = \frac{16}{3a} M(k, p; E) + \frac{4}{\pi} \int_0^\Lambda dq \frac{q^2 M(q, p; E)}{-1/a + \sqrt{3q^2/4 - mE - i\epsilon}} \mathcal{T}(k, q; E),
\]

for the off-shell atom-dimer scattering amplitude with the inhomogeneous term

\[
M(k, p; E) = \frac{1}{2pk} \ln \left( \frac{p^2 + pk + k^2 - mE}{p^2 - pk + k^2 - mE} \right) + \frac{H(\Lambda)}{\Lambda^2}.
\]

Figure 1. Integral equation for the atom-dimer scattering amplitude. Single (double) line indicates single atom (two-atom state).
The logarithmic term is the S-wave projected one-atom exchange, while the term proportional to $H(\Lambda)$ comes from the 3-body force. The physical atom-dimer scattering amplitude $f_{AD}$ is given by the solution $T$ evaluated at the on-shell point: $f_{AD}(k) = T(k, k; E)$ where $E = 3k^2/(4m) - 1/(ma^2)$. The 3-body binding energies $B_3$ are given by those values of $E$ for which the homogeneous version of Eq. (2) has a nontrivial solution.

Note that an ultraviolet cutoff $\Lambda$ has been introduced in (2). This cutoff is required to insure that Eq. (2) has a unique solution. All physical observables, however, must be invariant under changes of the cutoff, which determines the behavior of $H$ as a function of $\Lambda$ [1]:

$$H(\Lambda) = \frac{\cos[\ln(\Lambda/\Lambda_*) + \arctan s_0]}{\cos[\ln(\Lambda/\Lambda_*) - \arctan s_0]} ,$$

where $s_0 = 1.00624$ is a transcendental number and $\Lambda_*$ is a 3-body parameter introduced by dimensional transmutation. This parameter cannot be predicted by the EFT and must be taken from experiment. Note that $H(\Lambda)$ is periodic and runs on a limit cycle. When $\Lambda$ is increased by a factor of $\exp(\pi/s_0) \approx 22.7$, $H(\Lambda)$ returns to its original value.

In summary, two parameters are required in the 3-body system at leading order in $r_e/a$: the scattering length $a$ (or the dimer binding energy $B_2$) and the 3-body parameter $\Lambda_*$ [1]. The EFT reproduces the universal aspects of the 3-body system that were first derived by Efimov [2]. These include the accumulation of infinitely many 3-body bound states (so-called Efimov states) at threshold as $a \to \pm \infty$. As we will demonstrate in the next section, EFT is a very efficient calculational tool to calculate those properties.

### 3. APPLICATION TO COLD ATOMS AND BEC

In this section, we discuss some applications of the EFT to systems of cold atoms and BEC’s.

First we turn to the 3-body system of $^4$He atoms, where $a/r_e \approx 15$. Both the $^4$He dimer and trimer were observed. The complete 3-body bound state spectrum in leading order in the EFT follows from solving Eq. (2). It can also be parametrized in terms of a universal function $\Delta$ of the angle $\xi = a\sqrt{mB_3}$ and a 3-body parameter [2]. The function $\Delta(\xi)$ was recently calculated in EFT [3]. Unfortunately, no 3-body observables have been measured for $^4$He atoms up to now, so that $\Lambda_*$ can not be determined from experiment. However, extensive bound state calculations with modern potentials exist. We have used recent calculations by Motovilov et al. [4], to obtain the dimer binding energy and the 3-body parameter $\Lambda_*$ from the calculated binding energy of the trimer excited state $B_3^{(1)}$. We have then used this input to calculate the trimer ground state energy $B_3^{(0)}$ for four different $^4$He potentials [5]. The results are shown in Table 1. The predicted ground state binding energy $B_3^{(0)}$ agrees well with the direct calculation of Motovilov et al. [4]. This demonstrates that both the excited and ground state of the $^4$He trimer are Efimov states.

Universality also constrains 3-body scattering observables. For example the atom-dimer scattering length can be expressed in terms of $a$ and $\Lambda_*$ as [1][2]

$$a_{AD} = a (1.46 - 2.15 \tan[s_0 \ln(a\Lambda_*) + 0.09]) (1 + \mathcal{O}(r_e/a)) , \quad a > 0 .$$
Table 1: Universality for $^4$He atoms. The first three columns show the calculated binding energies of the dimer and the trimer ground and excited states for four modern $^4$He potentials [4]. The last two columns show the extracted value of $a\Lambda_*$ and the leading order EFT prediction for the trimer ground state [5].

| Potential     | $B_2$ | $B_3^{(0)}$ | $B_3^{(1)}$ | $a\Lambda_*$ | $B_3^{(0)}$ (pred.) |
|---------------|-------|-------------|-------------|--------------|---------------------|
| HFDHE2        | 0.830 | 116.7       | 1.67        | 1.258        | 118                 |
| HFD-B         | 1.685 | 132.5       | 2.74        | 0.922        | 138                 |
| LM2M2         | 1.303 | 125.9       | 2.28        | 1.033        | 130                 |
| TTY           | 1.310 | 125.8       | 2.28        | 1.025        | 129                 |

where the numerical constants were calculated in EFT. Using the value of $\Lambda_*$ extracted from the excited state binding energy, a good agreement with $a_{AD}$ from the direct calculation of Ref. [4] is obtained [5].

Universality is also manifest in universal scaling functions. In the left panel of Fig. 2 we display the scaling function relating $B_3^{(1)}/B_2$ to $B_3^{(0)}/B_2$ [5]. The data points give various calculations using modern $^4$He potentials while the solid line gives the universality prediction from EFT. Different points on this line correspond to different values of $\Lambda_*$. The small deviations of the potential calculations from the universal curve are mainly due to effective range corrections and can be calculated at next-to-leading order in EFT. The calculation corresponding to the data point far off the universal curve can easily be identified as problematic.

![Figure 2](https://example.com/figure2.png)

Figure 2. Left panel: The scaling function $B_3^{(1)}/B_2$ vs. $B_3^{(0)}/B_2$ for $^4$He atoms. Right panel: The loss rate coefficient $K_3$ in $^{85}$Rb as a function of the external magnetic field for different values of $\Lambda_*$. Solid (dashed) line gives contributions of shallow (deep) bound states. Data points are from Ref. [7].
Next we turn to 3-body recombination, which is the process when three atoms scatter to form a dimer and the third atom balances energy and momentum. This is one of the main loss processes for trapped atoms and condensates of atoms near a Feshbach resonance. The event rate can be parametrized as \( \nu = \alpha \rho^3 \), where \( \rho \) is the density of the atoms and \( \alpha \) is the recombination constant. At threshold, the leading order EFT result for the contribution to \( \alpha \) from recombination into the shallow dimer is \[ \alpha = 67.1 \frac{a^4}{m} \sin^2[s_0 \ln(a\Lambda_*) + 0.19] \left(1 + \mathcal{O}(r_e/a)\right), \quad a > 0. \] (6)

If deeply bound dimers are present, there are additional contributions from recombination into the deep bound states. If \( a < 0 \), recombination can only go into the deep states.

In the right panel of Fig. 2, we show the 3-body loss rate coefficient \( K_3 = 3\alpha \) from Ref. [6] for two different values of \( \Lambda_* \) compared to experimental data for 3-body losses in a cold gas of \(^{85}\text{Rb}\) atoms near a Feshbach resonance at \( B = 155 \text{ G} \) [7]. The value \( \Lambda_* = 5/\text{a.u.} \) seems to be preferred by the data.

4. SUMMARY & OUTLOOK

The renormalization of the EFT for 3-body systems with large scattering length requires a 3-body force at leading order [11]. The renormalization group evolution of the 3-body force is governed by a limit cycle. Via dimensional transmutation, the 3-body force introduces a dimensionful parameter, \( \Lambda_* \), that parametrizes universal relations between different 3-body observables.

The EFT is very general and can be applied to many physical systems ranging from the 3-body system of \(^4\text{He}\) atoms [5], to 3-body recombination and dimer deactivation in cold atomic gases/BEC’s [6,8], to the triton and hypertriton in nuclear physics [9].

Future challenges include the extension of the EFT to the four-body system, the stability and phase structure of cold atomic gases/BEC’s close to a Feshbach resonance [10], and the possibility of coexisting condensates of atoms, dimers, and trimers [11].

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