Weak measurement of the optical polarization, chirality and orbital angular momentum via metasurface with polarization filtering

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Abstract
We propose a metasurface to evaluate the polarization, chirality and orbital angular momentum of the incident light. By employing the weak measurement mechanism, an aperture with a set orientation is utilized to filter the incident polarization components and enhance the polarization contrast. We demonstrate theoretically and numerically that the mechanism can detect the weak quantities of the incident field by evaluating the orientation angle of the azimuthal polarized component in the back focal plane or imaging plane. Using the aperture as the cell of metasurface, the enhancement can also be realized in the imaging plane. Moreover, by constructing a phase gradient through a couple of mirror symmetric apertures in the cell, the metasurface can produce orbit induced rotation in the imaging plane. Our results could find wide application in the polarization, chirality detection and orbital angular momentum communications.

1. Introduction

The electromagnetic wave has several degrees of freedom, such as intensity, phase, polarization, chirality and orbital angular momentum (OAM) [1], which play the important roles in light–matter interactions. Light can be coupled to surface plasmon polaritons (SPPs), which arise from the collective oscillation of free electrons at the interface between a metal and dielectric media. However, the transmission fields through the subwavelength apertures are extremely sensitive to the properties of the incident light [2, 3]. While the partial loss of the information from the incident beam will not cause the transmission process to disappear entirely, it does render the aperture unable to distinguish the properties of the incident light, especially when the rotation of the linear polarization state is very small, the weak chirality or the order of OAM [4, 5]. For example, the spin state of weakly elliptically polarized light, which is an important degree-of-freedom of an optical field and thus fundamental to many developments in modern physics and device applications, cannot be recognized using a conventional coupling configuration [6–10].

Recently, much attention has been paid to the determination of molecular localization and orientation using microscope systems, as this crucial for single-molecule tracking and wide-field super-resolution imaging [11–13]. Fluorescence molecules, which act as oscillating electric dipoles, can produce a characteristic anisotropic radiation distribution in the back focal plane (BFP). It seems that the shape of the images that appear in a microscope is highly reliant on the orientation of the molecule being imaged. In this paper, we theoretically analyze an SPP microscopic system to enhance the weak inclination of the linear polarization or the chirality by employing the concept of the weak measurement [14]. Firstly, the system consists of an isolated aperture in a multilayer isolation-metal-isolation (IMI) structure is analyzed mathematically. By choosing the orientation of aperture, the incident polarized state can be modulated and the contrast of the two transverse field components can be enhanced by several times, which is the point of the weak measurement. The transmission field was collected by a high-numerical aperture (NA) objective and imaged. In theory, we deduced a relation between the incident light and the orientation of the azimuthal component of the electric field in the BFP, which could be utilized to precisely reconstruct the incident linear polarized state or chirality. Subsequently, by embedding the
aperture in the cell of periodic structure, we propose a metasurface to exhibit the enhancement of the orientation angle for the incident weak linear polarization and chirality. Finally, we demonstrate that the method can also be utilized to determine the orders of the OAM. This proposed mechanism potentially has a broad range of applications for polarization measurements, chiral and OAM detection.

2. Theoretical investigation of single aperture

Firstly, we investigate the enhancement of orientation angle for the microscopic system with an isolated aperture. The schematic diagram is shown in figure 1. We assumed that the incident light propagates along the z-axis. By defining \((\hat{x}, \hat{y}, \hat{z})\) as the basic vectors of the laboratory Cartesian frame, the vector of a monochromatic electric field can be expressed as \(E^i = (\varepsilon_x, \varepsilon_y, 0)^T\), where the superscript denotes the Cartesian basis and the symbol T stands for the transpose of the vector matrix. Here, the ratio \(\varepsilon_x / \varepsilon_y\) is defined to the polarization contrast and employed to evaluate the incident polarized state. A real \(\varepsilon_x / \varepsilon_y\) indicates the rotation of a linear polarization state, while an imaginary \(\varepsilon_x / \varepsilon_y\) denotes the elliptical polarization of the incident light.

By employing the vectorial focusing theorem, the spectrum of the focused field in the spherical angles \((\theta_1, \varphi_1)\) is given by geometric rotational transformation [15].

\[
E^C_{\text{lem}}(\theta_1, \varphi_1) \propto \sqrt{\cos \theta_1} M^C_{\text{lem}}(\theta_1, \varphi_1) \vec{E}^C_i.
\]

(1)

Here \(\sqrt{\cos \theta_1}\) is the apodization factor. Generally, the real-space focused field is given by the two-dimensional Fourier integral over the spectrum \(E^C_{\text{lem}}(\theta_1, \varphi_1)\). The components of the focused fields can be found in figures 1(b)–(d) as \(\varepsilon_x / \varepsilon_y = 0, \varepsilon_x / \varepsilon_y = 0.022\) and \(\varepsilon_x / \varepsilon_y = -0.022\), respectively. All physical quantities are normalized to the maximal intensity.

A straight aperture is placed in the center of the focused beams to interact with the focused field. Considering the weak inclination of the linear polarized state or chirality \(|\varepsilon_x / \varepsilon_y| \ll 1\), the orientation angle of the aperture is arranged to be \(\phi_{\text{apt}} \approx 90^\circ\). By using \(\sin \phi_{\text{apt}} \approx 1\) and \(\cos \phi_{\text{apt}} \approx 0\), the electric field around the aperture can be expressed as

\[
E^C_{\text{apt}}(\theta_1, \phi_{\text{apt}}) \approx \sqrt{\cos \theta_1} (\varepsilon_x, \varepsilon_y \cos \theta_1, \varepsilon_y \sin \theta_1)^T.
\]

(2)

which indicates an enhancement of the polarization contrast by \(1 / \cos \theta_1\) in the focusing. The pre-selection operation of weak measurement is achieved by the metal aperture. Because the aperture can be seemed as a polarizer and filters the polarized component orthogonal to it, the interaction between the aperture and \(E^C_{\text{apt}}\) can be described by the projection of the field \(\hat{\mathbf{P}} E^C_{\text{apt}}\) onto the aperture [14]

\[
\hat{\mathbf{P}} E^C_{\text{apt}} \approx \sqrt{\cos \theta_1} (\varepsilon_x, \varepsilon_y \cos \theta_1, \varepsilon_y \sin \theta_1)^T,
\]

(3)

where \(\hat{\mathbf{P}}\) is the projection operator. The field \(\hat{\mathbf{P}} E^C_{\text{apt}}\) is the pre-selected state of weak measurement and can be regarded as the renewed source for the microscopic system. Generally, the interaction between an IMI structure and optical field is described by the Lippmann-Schwinger integral equation [16–22].
attached to the scattering emission. However, if incident light vanishes, the phase of the radial electric polarization rotation in the incident light. Moreover, if the dimensional integral along the aperture. Furthermore, because the contribution of the longitudinal component of the Lippmann-Schwinger integral equation can be approximated to be the one-dimensional integral of the scattering field is weaker than that of the transverse components, we tentatively ignore the longitudinal component and provide a simple expression of the scattering field in the BFP:

$$E^C_{r}(r) = k^2 \int_{\text{apt}} dr_{\text{apt}} \Delta \varepsilon(r_{\text{apt}}) \hat{G}_r(r, r_{\text{apt}}) \hat{E}^C_{r}(r_{\text{apt}}),$$

where \( E^C_{r}(r_{\text{apt}}) \) is the projection of \( E^C_{p} \) onto the metal surface. \( k \) gives the wave number of the scattering space; \( \Delta \varepsilon(r_{\text{apt}}) \) indicates the difference between the permittivity of air and the metal; \( \hat{G}_r(r, r_{\text{apt}}) \) is the Green’s dyad, which propagates the field from point \( r_{\text{apt}} \) in the aperture to point \( r \) of the scattering plane.

After interacting with the aperture, the field is partially scattered and collected via an immersion objective lens (NA = 1.49), which is attached to the back side of the IMI substrate. As the scattering is far enough from the source, the Green’s dyad \( \hat{G}_r(r, r_{\text{apt}}) \) for the layered structure is given by the function in [21]. Here, the basis vector \( \hat{n}_s \) is defined by the spherical coordinates \((\theta_s, \varphi_s)\) attached to the scattering field. Actually, the scattering of light to the BFP can be considered as the inverse transform of above-mentioned focusing process. Thus, the electric field in the BFP can be expressed as

$$E^C_{\text{BFP}}(r_{\text{BFP}}) = [\hat{N}^C_{\text{IMI}}(\theta_s, \varphi_s)]^{-1} E^C_{r}(r).$$

For comparison, the intensities (phases) of the radial and azimuthal component in the BFP are respectively shown in figures 2(a) and (c) (figures 2(b) and (d)) with the ratio \( \varepsilon_x/\varepsilon_y \) equal to 0, which indicates incident y-polarized light. Here, the orientation angle of the aperture was fixed at 93°. When the x-component of the incident light vanishes, the phase of the radial electric field is along the x-axis and that of the azimuthal electric field is in the direction of the y-axis, which is in accordance with the results of the surface plasmon coupled emission. However, if \( \varepsilon_x/\varepsilon_y \) increases to 0.022, which means a slight rotation of the linear polarization, the electric fields in the BFP are also rotated. This phenomenon presents an enhanced measurement of the weak polarization rotation in the incident light. Moreover, if \( \varepsilon_x/\varepsilon_y \) is 0.022i, which denotes an introduction of optical chirality, a vortex phase appears in the field components of the BFP. However, although the helical phase is rotated versus the change of Im(\( \varepsilon_x/\varepsilon_y \)), this rotation cannot easily be recognized by ordinary technology.
The field components in the imaging plane can be calculated through the vectorial focusing theorem. The azimuthal components of the images are summarized in figures 3(a) and (b) as \( \varepsilon_x/\varepsilon_y \) equal to 0.022 and 0.022i, respectively. It is obvious that the orientation of the azimuthal components of the image tilts as \( \varepsilon_x/\varepsilon_y \) is real. Thus, the polarization ratio of the incident light can be evaluated for real \( \varepsilon_x/\varepsilon_y \). However, the field components of the image are thoroughly different when \( \varepsilon_x/\varepsilon_y \) is imaginary. Because the orientation of these components are closely oriented along the y-direction.

Currently, the azimuthal field component is extracted to measure the orientation and localization of single molecules [12, 13]. We will demonstrate that this method can also be utilized to evaluate incident light. Thus, a non-polarizing beam-splitter is placed in the BFP to separate the scattering field into two parts: one that passes through an azimuthal polarizer acted as a post-selection operation to extract the azimuthal component of the electric field, while the other is focused by an objective lens into the imaging plane after passing through an azimuthal polarizer. We can flexibly extract the desired electric field component of the image to determine the orientation of the incident light. With the \( \varepsilon_x/\varepsilon_y \) given by 0.022 and \(-0.022\), the azimuthal components of the images are shown in figures 3(c) and (d). The orientation angle of the image is approximately 135° when \( \varepsilon_x/\varepsilon_y = 0.022 \), whereas the image is along 45° for \( \varepsilon_x/\varepsilon_y = -0.022 \). However, if \( \varepsilon_x/\varepsilon_y \) is imaginary, an additional vortex phase is added to the azimuthal component electric field in the BFP, which divides the electric field into two lobes. The centroids of these two lobes can indicate the chirality of the incident elliptical polarization. The resulting images are shown in figures 3(e) and (f) for \( \varepsilon_x/\varepsilon_y \) or 0.022i and \(-0.022i\), respectively. Thus, it can be concluded that we can employ the system to determine the optical polarization or chirality in the imaging plane immediately by carefully choosing the orientation angle of the aperture.

We will consider the relation between the polarization contrast and the azimuthal component in the BFP, which can be utilized to accurately reconstruct the incident light. Through equations (4) and (5), a proximate expression can be deduced. Theoretically, transition from the Cartesian to cylindrical basis is realized via unitary transformation. Thus, the radial and the azimuthal component of the electric field in the BFP is given by

\[
\begin{align*}
\hat{E}_{\text{r}} &= \delta r_{\text{apt}} \left[ -\varepsilon_x \sin \phi_{\text{apt}} \cos \varphi_{\text{r}} + \varepsilon_y \cos \theta_1 \cos \phi_{\text{apt}} \sin \varphi_{\text{r}} \right] \cos \theta_1 \Phi^2_{\text{r}}, \\
\hat{E}_{\varphi} &= \delta r_{\text{apt}} \left[ \varepsilon_x \sin \phi_{\text{apt}} \sin \varphi_{\text{r}} + \varepsilon_y \cos \theta_1 \cos \phi_{\text{apt}} \cos \varphi_{\text{r}} \right] \Phi^3_{\varphi},
\end{align*}
\]

Here the real space field components need an additional two-dimensional Fourier integral. Note that the potential \( \Phi^2_{\text{r}} \) is related to the radial transmission coefficient while the \( \Phi^3_{\varphi} \) is determined by the azimuthal transmission coefficient [21]. If the length of the aperture is much larger than the width of the aperture, the aperture can be considered as a nanometer antenna and the integrals in equation (6) can be approximatively...
ignored for the case of $r_{ap} < \lambda$ [2]. There gradients in the azimuthal direction can be expressed as

\[
\frac{\partial \vec{E}_{\text{real}}}{\partial \varphi_{\text{BFP}}} \propto \{ \varepsilon_x \sin \phi_{ap} \sin \varphi - \varepsilon_y \cos \theta_1 \cos \phi_{ap} \cos \varphi \} \cos^3 \varphi \Phi_f^2
\]

\[
\frac{\partial \vec{E}_{\text{imag}}}{\partial \varphi_{\text{BFP}}} \propto \{ \varepsilon_x \sin \phi_{ap} \cos \varphi - \varepsilon_y \cos \theta_1 \cos \phi_{ap} \sin \varphi \} \Phi_f \sin^3 \varphi,
\]

(7)

The orientation of the azimuthal component is relative to the null point of the gradient. Therefore, we can attain a helpful expression:

\[
\tan \varphi = \frac{\varepsilon_x}{\varepsilon_y \cos \theta_1} \tan (\phi_{ap}).
\]

(8)

From the expression, we obtain several conclusions. First, if the aperture is absent, the orientation angle of $\vec{E}_{\text{BFP}}$ is linearly relative to the incident polarization. Second, compared to the incident field, the contrast of both incident components is enhanced by the focusing system by a factor of $1 / \cos \theta_1$. Third, through the interaction with the aperture the polarization ratio is amplified by a factor of $\tan (\phi_{ap}) \gg 1$ when the orientation angle of the aperture $\phi_{ap} \approx 90^\circ$. Thus, the polarization contrast of the incident light can be deduced through the measurement of the orientation angle of $\vec{E}_{\text{BFP}}$. Finally, when the x-component of the incident light is in-phase with the y-component, the azimuthal component in the BFP is rotated correspondingly to the change of the real space. While the x-component of the incident light is out-of-phase with the y-component, the phase of the azimuthal component in the BFP is rotated correspondingly to any change of the Fourier space.

The orientation angle of the azimuthal component in the BFP versus $\varepsilon_x / \varepsilon_y$ is summarized in figures 4(a) and (c). It is obvious that the orientation angle increases as $\varepsilon_x / \varepsilon_y$ increases. Meanwhile, the ratio $\varepsilon_x / \varepsilon_y$ can be deduced through equation (8). First, by choosing the value of $\theta_1$ to be 0, the reconstructed values of $\varepsilon_x / \varepsilon_y$ are given by figures 4(b) and (d). From the data, it can be seen that the reconstructed values match the original values precisely for different angles of the aperture. Furthermore, it can be found that $\varepsilon_x / \varepsilon_y$ can be reconstructed with a same value of $\theta_1$ for different inclination angles, while arbitrary incident light components can be deduced by the rotation of the metallic aperture, whose precision is determined by the orientation angle of the aperture.

3. Metasurface for polarization, chirality and OAM measurement

Although the isolated aperture can present determined relation between the incident wave and the field in the BFP, the aperture need to be exactly placed in the center of the incident focused field, which limits the application of the mechanism. This sensitive local variation of polarization state in focused field was researched.
by several groups to achieve the nanoscale position metrology [23–25]. Thus, a periodic metasurface is designed by employing the former structure: the aperture is placed in the center of each cell and the period in the transverse direction is equal to the surface plasmon wavelength $\lambda_{sp}$ as shown in figure 5(a). This period is chosen to enhance the SPP field coupling and strengthen the weak quantity amplification. By extracting the azimuthal component in the imaging field, the orientation angles versus the incident polarization contrast are shown in figures 5(b) and (c) when the ratio are real and imaginary, respectively. Note that the incident focused system is abandoned here. Thus, it does not need to consider the problem of alignment. From the former analysis, although the incident focusing system can enhance the weak quantity of the incident light by $1/\cos \theta_1$, it does not affect the property of weak measurement. The weak oblique linear polarization or chirality can also be enhanced by the metasurface when the orientation angle of the aperture is close to the y-direction. However, the enhancement decreases comparing with that of an isolated aperture, which is in accord with our theoretical prediction. Moreover, the alteration of the sign in the incident polarization contrast leads to the alteration of the sign of the orientation angle in the image, which is not shown here.

Finally, we introduce a phenomenon of orbit induce rotation in the image field, which has potential application in the detection and demodulation of the various OAM. In the past decade, the researchers proposed the grating and coordinate transformation to detect the OAM. As shown in figure 6(a), a metasurface with the cell being a couple of the mirror symmetric apertures is designed to detect the OAM. When the OAM is absent, the field is concentrated in the center of the imaging plane, which indicates an orientation angle of zero [figure 6(b)]. While the OAM is added, the images are divided into two lobes in the imaging plane in the imaging plane. Thus, the orientation angle can be calculated by the two lobes [figures 6(c) and (d)]. In theory, if the weak phase difference between the two apertures is sufficiently small, the vortex phase can be expressed as $\exp (\text{im} \Delta \phi) \approx 1 + \text{im} \Delta \phi$. Thus, the difference of the orientation angles between the adjacent OAM is $\Delta \phi$. By choosing $\phi_{apt} = 93^\circ$ and $\varepsilon_y/\varepsilon_x = 0.02$, the calculated orientation angle of image versus the order of OAM is given in figure 6(e). By carefully examining the figure 6(e), the phase difference is approximate $6^\circ$, which is in accord with the former theoretical prediction.

4. Conclusion

In conclusion, we theoretically investigated the SPP microscopy system to evaluate the incident optical polarization. The polarization ratio could be enhanced by the proposed system and measured accurately. An isolated aperture with an arranged orientation was placed on the metal surface to filter the focused field of the incident polarization state. Through the azimuthal polarization filtering technology in the back focal plane, the direction of an oblique polarization could be distinguished. Remarkably, by adding an additional vortex phase to the azimuthal component field in the back focal plane for an elliptical polarization incidence, the chirality of
elliptically polarized state could be recognized by examining the centroid of the divided lobes of the image. The relation between the incident polarization state and the azimuthal component field in the back focal plane is given mathematically, which can be utilized to reconstruct the incident polarization state faultlessly. Such results could be widely exploited in the field of polarization or chirality detection.

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Appendices

Firstly, considering the uppermost focusing system by the geometric rotational transformation in the adiabatic approximation, we can obtain the spectrum of focusing.

By employing the two-dimensional Fourier-type integral, we can obtain the focused field with the cylindrical coordinate basis in the metal surface [1]:

Figure 6. (a) Cell of the designed metasurface. The transverse pixel is \( \lambda_p \). A couple of the mirror symmetric apertures with the same dimensions as the former analysis is placed in the center of the cell. (b) The image of incident light with OAM \( l = 0 \); (c) The image of incident light with OAM \( l = +3 \); (d) The image of incident light with OAM \( l = -3 \); (e) The orientation angle of the image versus the orders of OAM as \( \varphi_{opt} = 93^\circ \).
Here \( \Gamma_{\text{apt}}(\rho, \phi) \) is the potential of the focused field is reduced by \( \rho \cdot \phi \) and the metal surface.

From the equation, it is obvious that the x-component of the focused field is enhanced by \( (1 + \cos \theta_1) \rho_0 + (1 - \cos \theta_1) \rho_2 \), while the y-component of the focused field is reduced by \( (1 + \cos \theta_1) \rho_0 - (1 - \cos \theta_1) \rho_2 \), which indicate an enhancement of the weak component in the incident light. Generally, \( (1 - \cos \theta_1) > 0 \) and \( \rho_2 > 0 \).

The interaction between the aperture and the focused field \( \vec{E}_{\text{apt}} \) can be described by the projection of the field \( \vec{E}_{\text{apt}} \) onto the aperture, the field after interacting with the aperture is [4]

\[
\vec{E}_{\text{p}} = (-\sin \phi_{\text{apt}} \vec{E}_{\text{apt},x}, \cos \phi_{\text{apt}} \vec{E}_{\text{apt},y}, 0)^T.
\] (A.5)

The field \( \vec{E}_{\text{p}} \) can be considered as the renewed source for the following microscopic system. The interaction between an IMI structure and optical field is described by the Lippmann-Schwinger integral equation, and the scattering field below the IMI can be described in the spherical coordinates \((\theta, \phi, \zeta)\) by [5]:

\[
\vec{E}_{\text{p}}(\mathbf{r}_s) = k_s^2 \int_{\text{apt}} d\mathbf{r}_{\text{apt}} \Delta s(\mathbf{r}_{\text{apt}}) \hat{G}_s(\mathbf{r}_s, \mathbf{r}_{\text{apt}}) \vec{E}_{\text{p}}(\mathbf{r}_{\text{apt}}).
\] (A.6)

Here \( k_s \) is the wavevector in the scattering and \( \Delta s(\mathbf{r}_{\text{apt}}) \) indicates the difference between the permittivity of air and the metal. \( \vec{E}_{\text{p}} \) is the Green’s dyad, which propagates the field from point \( \mathbf{r}_{\text{apt}} \) to point \( \mathbf{r}_s \); \( \mathbf{r}_{\text{apt}} \) and \( \mathbf{r}_s \) are the coordinates attached to the aperture and scattering field. The Green’s dyad can be expressed as [27]

\[
\lim_{|\mathbf{r}| \to \infty} \hat{G}_s(\mathbf{r}_s, \mathbf{r}_{\text{apt}}) = \frac{\exp(i k_s |\mathbf{r}|)}{4 \pi |\mathbf{r}|} \int \exp(-i k_s |\mathbf{r}_s - \mathbf{r}_{\text{apt}}|) \left[ \begin{array}{c}
\cos^2 \varphi_1 \cos^2 \varphi_1 \Phi_1^1 \\
+ \sin^2 \varphi_1 \Phi_1^2 \\
\sin \varphi_1 \cos \varphi_1 \cos \varphi_1 \Phi_1^3 \\
- \sin \varphi_1 \cos \varphi_1 \Phi_1^3 \\
- \cos \varphi_1 \sin \varphi_1 \cos \varphi_1 \Phi_1^4 \\
- \sin \varphi_1 \sin \varphi_1 \cos \varphi_1 \Phi_1^4 \\
\end{array} \right] \left[ \begin{array}{c}
\sin \varphi_2 \cos \varphi_2 \cos \varphi_2 \Phi_2^1 \\
- \sin \varphi_2 \cos \varphi_2 \Phi_2^3 \\
\sin^2 \varphi_2 \cos \varphi_2 \Phi_2^4 \\
+ \cos^2 \varphi_2 \Phi_2^4 \\
- \sin \varphi_2 \sin \varphi_2 \cos \varphi_2 \Phi_2^4 \\
- \sin \varphi_2 \sin \varphi_2 \Phi_2^4 \\
\end{array} \right] (A.7)
\]

Here the expressions of the potentials \( \Phi_1^i, \Phi_2^i \) and \( \Phi_3^i \) can be found in [6]. Note here that \( \Phi_1^1 \) and \( \Phi_2^3 \) are related to the radial Fresnel transmission coefficient \( t' \) while \( \Phi_1^2 \) is determined by the azimuthal Fresnel transmission coefficient \( t' \). Moreover, the details to calculate the Fresnel transmission can be found in [22]. In the equation (7), \( \mathbf{r}_{\text{apt}} \) is the projection of \( \mathbf{r} \) to the metal surface.
Thus, the scattering field can be calculated to be

\[
\mathbf{E}_s^C(r) = \frac{\exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} k_0^2 \Delta \varepsilon(r_{ape}) \int_{\text{d}r_{ape}} \exp(-ik,\mathbf{n}_s, \mathbf{r}'_{ape}) \Gamma_s^C
\]

\[
\Gamma_s^C = \left\{ \begin{array}{l}
- \sin \phi_{ape} \mathbf{E}_{ape,x}^C \left[ \cos^2 \varphi_s \cos^2 \psi_s \Phi_s^2 + \sin^2 \varphi_s \Phi_s^2 \right] + \cos \phi_{ape} \mathbf{E}_{ape,y}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 - \sin \psi_s \cos \Phi_s^2 \right] \\
- \sin \phi_{ape} \mathbf{E}_{ape,x}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 + \sin \psi_s \cos \Phi_s^2 \right] + \cos \phi_{ape} \mathbf{E}_{ape,y}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 + \cos \psi_s \Phi_s^2 \right]
\end{array} \right\} \mathbf{e}_z
\]

\text{(A.8)}

The transformation from the scattering light to the field in the back focal plane (BFP) can be regarded as the inverse transform of above-mentioned focusing process. Therefore, the electric field in the BFP can be expressed as

\[
\mathbf{E}_{\text{BFP}}^C(r) = \frac{n_{\text{oil}}}{n_{\text{BFP}}} \left( \mathbf{M}_{\text{om}}^C(\theta, \varphi) \right) \mathbf{E}_s^C(r)
\]

\[
= \frac{n_{\text{oil}}}{n_{\text{BFP}}} \left( \cos^2 \varphi_s \cos \psi_s + \sin^2 \varphi_s (\cos \psi_s - 1) \sin \varphi_s \cos \phi_s - \sin \theta \sin \varphi_s \right) \mathbf{E}_{\text{ap}}^C(r).
\]

\text{(A.9)}

Here \(n_{\text{oil}}\) and \(n_{\text{BFP}}\) are the refractive indices of oil-immersion and the BFP, respectively. The expression \(\sqrt{1/\cos \theta}\) is introduced to ensure the conservation of the energy flow. From the equation (A.9), we can obtain the expressions of the electric fields in the BFP as:

\[
\mathbf{E}_{\text{BFP,x}}^C(r) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \frac{n_{\text{oil}}}{n_{\text{BFP}}} \left( \cos^2 \varphi_s \cos \psi_s + \sin^2 \varphi_s (\cos \psi_s - 1) \sin \varphi_s \cos \phi_s - \sin \theta \sin \varphi_s \right) \mathbf{E}_{\text{ap}}^C(r)
\]

\[
\mathbf{E}_{\text{BFP,y}}^C(r) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \frac{n_{\text{oil}}}{n_{\text{BFP}}} \left( \cos \phi_{ape} \mathbf{E}_{\text{ap},x}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 + \sin \psi_s \Phi_s^2 \right] + \cos \phi_{ape} \mathbf{E}_{\text{ap},y}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 - \sin \psi_s \cos \Phi_s^2 \right] \right) \mathbf{e}_y
\]

\[
\mathbf{E}_{\text{BFP,z}}^C(r) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \frac{n_{\text{oil}}}{n_{\text{BFP}}} \left( \sin \phi_{ape} \mathbf{E}_{\text{ap},x}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 - \sin \psi_s \cos \Phi_s^2 \right] + \cos \phi_{ape} \mathbf{E}_{\text{ap},y}^C \left[ \sin \varphi_s \cos \varphi_s \Phi_s^2 + \cos \psi_s \Phi_s^2 \right] \right) \mathbf{e}_z
\]

\text{(A.10)}

By the transition from the Cartesian coordinates to cylindrical coordinates, we can obtain the electric field in cylindrical coordinates \((r_{\text{BFP}}, \varphi_{\text{BFP}}, z)\):

\[
\mathbf{E}_{\text{BFP,x}}^C(r_{\text{BFP}}) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \left( \cos \phi_{ape} \mathbf{E}_{\text{ap},x}^C \left[ \sin \varphi_s \cos \Phi_s^2 + \sin \psi_s \Phi_s^2 \right] - \sin \phi_{ape} \mathbf{E}_{\text{ap},y}^C \cos \varphi_s \Phi_s^2 \right)
\]

\[
\mathbf{E}_{\text{BFP,y}}^C(r_{\text{BFP}}) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \left( \sin \phi_{ape} \mathbf{E}_{\text{ap},x}^C \left[ \sin \varphi_s \cos \Phi_s^2 + \sin \psi_s \Phi_s^2 \right] - \sin \phi_{ape} \mathbf{E}_{\text{ap},x}^C \cos \varphi_s \Phi_s^2 \right)
\]

\[
\mathbf{E}_{\text{BFP,z}}^C(r_{\text{BFP}}) = \Delta \varepsilon(r_{ape}) \frac{k_0^2 \exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} \left( \sin \phi_{ape} \mathbf{E}_{\text{ap},x}^C \left[ \sin \varphi_s \cos \Phi_s^2 + \cos \psi_s \Phi_s^2 \right] \Phi_s^2 \right)
\]

\text{(A.11)}

If the length of the aperture is much larger than the width of the aperture, the aperture can be considered as a nanometer antenna and the integrals in equation (A.8) can be approximatively ignored for the case of \(r_{ape} \ll \lambda\). Then, the field in the BFP can be approximatively expressed as
where we have

\[ E_{\text{BFP},r}^{{f}}(r_{\text{BFP}}) \approx r_{\text{ap}}(\cos \phi_{\text{ap}} \sin \varphi_{f} E_{\text{ap},x}^{C} - \sin \phi_{\text{ap}} \cos \varphi_{f} E_{\text{ap},y}^{C}) \cos \phi_{f} \Phi_{f}^{2} \]

\[ E_{\text{BFP},\varphi}^{{f}}(r_{\text{BFP}}) \approx r_{\text{ap}}(\sin \phi_{\text{ap}} \sin \varphi_{f} E_{\text{ap},x}^{C} + \cos \phi_{\text{ap}} \cos \varphi_{f} E_{\text{ap},y}^{C}) \Phi_{f}^{2} \]

\[ E_{\text{BFP}}^{{f}}(r_{\text{BFP}}) = 0. \]  

(A.12)

The orientation of the azimuthal component is relative to the null point of the gradient.

\[ \frac{\partial E_{\text{BFP},r}^{\varphi}}{\partial \varphi_{\text{BFP}}} \approx r_{\text{ap}} \sin \phi_{\text{ap}} \cos \varphi_{f} E_{\text{ap},x}^{C} - \cos \phi_{\text{ap}} \sin \varphi_{f} E_{\text{ap},y}^{C} \Phi_{f}^{3}. \]  

(A.13)

and

\[ \tan \varphi_{f} = \frac{E_{\text{ap},x}^{C}}{E_{\text{ap},y}^{C}} \tan \phi_{f}. \]  

(A.14)

The equation (A.14) is in accord with the equation (A.10) in the manuscript and is the mainly result of manuscript. The equation was utilized to reconstruct the polarization of incident light.

Subsequently, the filtering mechanism in the BFP can be expressed mathematically as

\[ E_{\text{BFP}}^{\varphi}(r_{\text{BFP}}) = \tilde{F} E_{\text{BFP}}^{\varphi}(r_{\text{BFP}}), \]  

(A.15)

where \( \tilde{F} \) indicates the filter operation in the BFP mentioned in the manuscript, such as the azimuthal polarization filtering, etc.

Finally, the field of image can be calculated by employing the vectorial focusing theorem:

\[ E_{\text{image}} = \frac{i k_{3} f_{s} e^{-i k_{3} f_{s}}}{2 \pi} \int \int_{\theta_{\text{BFP}}} d \theta_{\text{BFP}} d \varphi_{\text{BFP}} \sin \theta_{\text{BFP}} \cos \theta_{\text{BFP}} e^{i k_{3} sz} \cos \theta_{\text{ap}} e^{i k_{s} sz} \sin \theta_{\text{ap}} \cos \theta_{\text{ap}} \sin \theta_{\text{ap}} \Gamma_{\text{BFP}}. \]  

(A.16)

where we have

\[ \Gamma_{\text{BFP}} = \tilde{F} E_{\text{BFP},r}^{\varphi} \left( \begin{array}{c} \cos \theta_{\text{BFP}} \cos \varphi_{\text{BFP}} \\ \cos \theta_{\text{BFP}} \sin \varphi_{\text{BFP}} \\ \sin \theta_{\text{BFP}} \end{array} \right) + \tilde{F} E_{\text{BFP},\varphi}^{\varphi} \left( \begin{array}{c} -\sin \varphi_{\text{BFP}} \\ \cos \varphi_{\text{BFP}} \\ 0 \end{array} \right) \]  

(A.17)

Here \( k_{3} \) and \( f_{s} \) are the wavevector and focal length in the imaging space, respectively. \( \phi_{\text{BFP}}, \varphi_{\text{BFP}} \) is the spherical coordinates attached to the focal point of the imaging objective. Generally, we take \( \varphi_{\text{BFP}} = \varphi_{f} \) throughout the paper.

The equation (A.16) can be utilized to calculate the imaging field.

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